Classes of Nonnegative Quadratic Optimization Problems with Exact Copositive Relaxation*

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May 2, 2018

Abstract

In this paper we establish exact copositive relaxation in the sense that the optimal values of a quadratically constrained quadratic minimization problem with nonnegative variables and its copositive relaxation are equal under a general geometric condition. We then show that exact copositive relaxation holds for broad classes of nonnegative quadratic minimization problems under verifiable conditions which satisfy the geometric condition. These classes of problems include nonnegative uniform quadratic optimization problems, nonnegative extended trust-region problems, and problems involving $Z$-matrices. As a consequence we present a copositive reformulation for the problem of finding the smallest radius ball enclosing a given nonnegative intersection of balls. We provide various examples illustrating our results.

1 Introduction

A copositive program is a conic linear program where a linear function is minimized over the cone of copositive matrices subject to linear constraints [5, 6, 7] and its dual is known as a completely positive program. These programs have been extensively studied in the framework of relaxation schemes for solving optimization problems [1, 6, 8, 10, 11, 12, 14, 20]. Especially, the copositive and completely positive programming relaxation approaches proved to be powerful tools for examining and solving classes of challenging quadratic optimization problems [5, 6, 14]. For comprehensive surveys of copositive optimization and its applications, we refer the readers to Bomze [5], Burer [12, and Dür [14]. Recently, completely positive relaxations have also been given for more general quadratic optimization problems with conic constraints in [1].

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In this paper we examine the quadratically constrained quadratic global minimization problems with nonnegative variables of the form

\[
(P_1) \quad \inf_{x \in \mathbb{R}^n} x^T Ax + b^T x + c
\]

s.t. \( x^T A_i x + b_i^T x + c_i \leq 0, \ i = 0, 1, ..., m, \ x_j \geq 0, \ j = 1, 2, \ldots, n, \)

where \( A, A_i \) are \((n \times n)\) symmetric matrices and \( b, b_i \in \mathbb{R}^n, \ c, c_i \in \mathbb{R}, \ i = 0, 1, ..., m. \) The copositive (resp. completely positive) relaxation of \((P_1)\) is said to be exact whenever the optimal values of \((P_1)\) and its copositive (resp., completely positive) relaxation problem are equal. Recently, Bomze [6] established that the semi-Lagrange dual for a problem of the form \((P_1)\) with additional linear equalities is equivalent to a natural copositive relaxation. This copositive relaxation possesses nice hierarchies of tractable conic bounds tighter than the usual Lagrangian dual bounds.

Unfortunately, this copositive relaxation is not always exact (see Example 2.2). Some sufficient conditions for exact copositive relaxation have been given using a generalized Karush-Kuhn-Tucker condition and the copositivity of the related slack matrix [6, Theorem 5.1]. However, finding classes of quadratic optimization problems that possess exact copositive relaxation has become a critical question in the area of copositive optimization from the point of view of applications. In this paper, we answer this question by providing various classes of nonnegative quadratic optimization problems \((P_1)\) that enjoy exact copositive relaxation under suitable conditions.

We make the following fundamentally important contributions to copositive optimization and nonconvex quadratic optimization:

(i) Under a geometric condition, we first establish that exact copositive relaxation of \((P_1)\) and the semi-Lagrangian duality hold where the problem \((P_1)\) is not required to attain its infimum. In the case where the problem \((P_1)\) attains its infimum, we show that our geometric condition guarantees related known conditions for exact copositive relaxation.

(ii) We provide suitable conditions under which nonnegative uniform quadratic optimization problems, where the Hessian of a quadratic constraint function is a scalar multiple of the Hessian of the objective function, exhibit exact copositive relaxation. As a consequence we present a copositive reformulation for the problem of finding the smallest radius ball enclosing a given nonnegative intersection of balls [3].

(iii) We also establish that exact copositive relaxation holds for nonnegative extended trust-region problems under easily verifiable conditions, satisfying the geometric property.

(iv) Finally, under a strict interior point condition, we present exact copositive relaxation for \((P_1)\) where the matrices \( A \) and \( A_i, i = 0, 1, \ldots, m \) are Z-matrices [17].

The outline of the paper is as follows. Section 2 presents semi-Lagrangian duality and copositive relaxation results. Section 3 establishes min-max semi-Lagrangian duality and copositive relaxation, where the optimal value, \( \inf(P_1) \), is attained. Section 4 provides copositive relaxation results for nonnegative uniform quadratic optimization problems with an application. Section 5 examines nonnegative extended trust-region problems. Section 6 studies the model problem \((P_1)\) with Z-matrices. Finally, Section 7 provides concluding statements with comments on future work.
2 Semi-Lagrangian Duality & Copositive Relaxation

Consider the following nonconvex quadratic optimization problem:

\[
\begin{align*}
(P_1) \quad & \inf_{x} x^T A x + b^T x + c \\
\text{s.t.} & \quad x \in \mathbb{R}^n, \ x^T A_i x + b_i^T x + c_i \leq 0, \ i = 0, 1, \ldots, m,
\end{align*}
\]

where \(A, A_i \in S^n\) are \((n \times n)\) symmetric matrices and \(b, b_i \in \mathbb{R}^n, c, c_i \in \mathbb{R}, i = 0, 1, \ldots, m\). Here \(S^n\) denotes the set of \((n \times n)\) symmetric matrices. In the sequel we assume \(\inf(P_1) > -\infty\), where \(\inf(P_1)\) is not assumed to be attained.

The problem \((P_1)\) can be rewritten as follows:

\[
\begin{align*}
\inf_{X \in C} & \quad \text{Tr}(HX) \\
\text{s.t.} & \quad \text{Tr}(H_i X) \leq 0, \ i = 0, 1, \ldots, m, \\
& \quad \text{Tr}(J_0 X) = 1, \ \text{rank}(X) = 1,
\end{align*}
\]

where \(C := \text{conv}\{\bar{x}\bar{x}^T : \bar{x} \in \mathbb{R}^{n+1}_+\}\) is the so-called cone of completely positive matrices,

\[
H := \begin{pmatrix} c & b^T/2 \\ b/2 & A \end{pmatrix} \quad \text{and} \quad H_i = \begin{pmatrix} c_i & b_i^T/2 \\ b_i/2 & A_i \end{pmatrix}, \ i = 0, 1, \ldots, m.
\]

We note that \(C\) is a full-dimensional closed convex pointed cone, and its dual is the so-called copositive cone defined by

\[
C^* := \{Q = Q^T \in \mathbb{R}^{n+1} \mid Q \text{ is copositive}\}.
\]

Recall that a symmetric matrix \(Q \in S^n\) is said to be copositive (resp., strictly copositive) if \(x^T Q x \geq 0\) for all \(x \in \mathbb{R}^n_+\) (resp., \(x^T Q x > 0\) for all \(x \in \mathbb{R}^n_+ \setminus \{0\}\)).

By removing the rank one constraint, we get the completely positive relaxation of \((P_1)\):

\[
(CP_1) \quad \inf_{X \in \mathcal{C}} \text{Tr}(HX) \quad \text{s.t.} \quad \text{Tr}(H_i X) \leq 0, \ i = 0, 1, \ldots, m, \\
\text{Tr}(J_0 X) = 1,
\]

where \(J_0 := e_0 e_0^T\) with \(e_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}\). The conic dual of \((CP_1)\) is called the copositive relaxation of \((P_1)\) defined as follows:

\[
(CP^*_1) \quad \sup \{ y_0 : Z_+(y) \in C^* , \ y = (y_0, u) \in \mathbb{R} \times \mathbb{R}^{m+1}_+ \},
\]

where

\[
Z_+(y) := H + \sum_{i=0}^m u_i H_i - y_0 J_0 = \begin{pmatrix} c + \sum_{i=0}^m u_i c_i - y_0 & (b + \sum_{i=0}^m u_i b_i)^T/2 \\ (b + \sum_{i=0}^m u_i b_i)/2 & A + \sum_{i=0}^m u_i A_i \end{pmatrix}.
\]

Recently, Bomze [6] has shown that the optimal value of \((CP^*_1)\) is equal to the optimal value of the following semi-Lagrangian dual of \((P_1)\):

\[
(D_1) \quad \sup_u \Theta(u) \quad \text{s.t.} \quad u \in \mathbb{R}^{m+1}_+,
\]
where \( \Theta(u) \) is given by \( \Theta(u) := \inf_{x \in \mathbb{R}^n_+} L(x, u) \) with \( L(x, u) := f(x) + \sum_{i=0}^{m} u_i g_i(x) \) and \( f(x) = x^T A x + b^T x + c \), \( g_i(x) = x^T A_i x + b_i^T x + c_i, i = 0, 1, ..., m \). So, by the strong convex separation theorem, there exists \( u \) such that

\[
\inf(P_1) \geq \inf(CP_1) \geq \sup(CP_1^*) = \sup(D_1).
\]

(1)

The copositive relaxation is said to be exact whenever \( \inf(P_1) = \sup(CP_1^*) \).

We first establish the exact copositive relaxation for \( P_1 \) under a geometric condition.

**Theorem 2.1 (Exact copositive relaxation with a geometric condition).** For problem \( P_1 \), suppose that \( \mathcal{A}_{P_1} := \{(g_0(x), g_1(x), ..., g_m(x), f(x)) : x \in \mathbb{R}^n_+ \} + \text{int} \mathbb{R}^{n+2}_+ \) is convex, and the Slater condition is fulfilled, that is, there exists \( \hat{x} \in \mathbb{R}^n_+ \) with \( g_i(\hat{x}) < 0, i = 0, 1, ..., m \). Then, we have

\[
\inf(P_1) = \inf(CP_1) = \max(CP_1^*).
\]

**Proof.** Let \( \mu := \inf(P_1) \). Then \( \mu \) is finite, due to the existence of a feasible point for \( P_1 \). Since \( \mathcal{A}_{P_1} \) is convex by our assumption, it follows that

\[
\{(g_0(x), g_1(x), ..., g_m(x), f(x) - \mu) : x \in \mathbb{R}^n_+ \} + \text{int} \mathbb{R}^{n+2}_+ = \mathcal{A}_{P_1} - \{0_{\mathbb{R}^{m+1}}\} \times \{\mu\}
\]

is also convex. Moreover, since \( \mu \) is the optimal value of \( P_1 \), we have

\[
(0, 0) \not\in \mathcal{A}_{P_1} - \{0_{\mathbb{R}^{m+1}}\} \times \{\mu\}.
\]

So, by the strong convex separation theorem, there exists \( (u_0, u_1, ..., u_m, u_{m+1}) \in \mathbb{R}^{m+2}_+ \backslash \{0\} \) such that

\[
\sum_{i=0}^{m+1} u_i v_i \geq 0 \quad \text{for all } v = (v_0, v_1, ..., v_m, v_{m+1}) \in \mathcal{A}_{P_1} - \{0_{\mathbb{R}^{m+1}}\} \times \{\mu\}.
\]

This implies that \( (u_0, u_1, ..., u_{m+1}) \in \mathbb{R}^{m+2}_+ \backslash \{0\} \) and

\[
\sum_{i=0}^{m} u_i g_i(x) + u_{m+1}(f(x) - \mu) \geq 0 \quad \text{for all } x \in \mathbb{R}^n_+.
\]

Taking into account that Slater condition holds, we have \( u_{m+1} > 0 \), and thus, we may assume without loss of generality that \( u_{m+1} = 1 \). Consequently, for \( u := (u_0, u_1, ..., u_m) \),

\[
L(x, u) \geq \mu \quad \text{for all } x \in \mathbb{R}^n_+.
\]

The latter shows \( \max(D_1) \geq \inf(P_1) \). Combining this with (1) gives us that

\[
\inf(P_1) = \inf(CP_1) = \sup(CP_1^*).
\]

On the other hand, since the Slater condition holds, by [6, Theorem 4.3], there exists \( X \in \text{int}(C) \) with \( \text{Tr}(J_0 X) = 1 \) and \( \text{Tr}(H_i X) < 0, i = 0, 1, ..., m \). The latter ensures the attainability of problem \( CP_1^* \). Therefore, we obtain the desired conclusion. \( \square \)
In the preceding theorem, we provided a geometric condition which ensures the exactness of copositive relaxation. As we will see later in Sections 4-6, this geometric condition can be satisfied easily for specially structured optimization problems such as uniform quadratic programs, extended trust region problems over nonnegative constraints and quadratic optimization problem with $Z$-matrix structure.

Let us now present a numerical example verifying Theorem 2.1 where the optimal value of the primal problem is not attained.

**Example 2.1 (Exact copositive relaxation without attainment of the optimal value of the primal problem)** Consider the two-dimensional quadratic optimization problem

$$
\begin{align*}
\inf_{x} & \quad x_1 \\
\text{s.t.} & \quad 1 - x_1 x_2 \leq 0, \\
& \quad x_1, x_2 \geq 0.
\end{align*}
$$

Let $f(x) = x_1$ and $g_0(x) = 1 - x_1 x_2$. We see that $g_0(\hat{x}) < 0$ for $\hat{x} := (1, 2) \in \mathbb{R}_+^2$, that is, the Slater condition is satisfied, the optimal value of $(E_1)$ is $\inf(E_1) = 0$, and problem $(E_1)$ has no solution. We next show that the set

$$
\mathcal{A}_{E_1} := \{(x_1, 1 - x_2 x_2) \mid (x_1, x_2) \in \mathbb{R}_+^2\} + \text{int}\mathbb{R}_+
$$

is convex. To do this, take any $u = (u_1, u_2) \in \mathcal{A}_{E_1}$, $v = (v_1, v_2) \in \mathcal{A}_{E_1}$ and $t \in (0, 1)$. By definition of $\mathcal{A}_{E_1}$, there exist $(x_1^u, x_2^u) \in \mathbb{R}_+^2$ and $(x_1^v, x_2^v) \in \mathbb{R}_2^2$ such that

$$
\begin{align*}
x_1^u &< u_1, \quad x_2^v < v_1, \\
1 - x_1^u x_2^u &< u_2, \\
1 - x_1^v x_2^v &< v_2.
\end{align*}
$$

Let $x_1^0 := (1 - t)x_1^u + tx_1^v$. If $x_1^0 = 0$, then $x_1^0 = x_1^v = 0$. So, for $x_2^0 := (1 - t)x_2^u + tx_2^v \in \mathbb{R}_+$, we have $x_1^0 < (1 - t)u_1 + tv_1$, $1 - x_1^0 x_2^0 < (1 - t)u_2 + tv_2$, showing that $(1 - t)u + tv \in \mathcal{A}_{E_1}$. If $x_1^0 \neq 0$, then $x_1^0 > 0$, due to $(x_1^u, x_1^v) \in \mathbb{R}_+^2$. Thus we can choose $x_2^0 \geq 0$ sufficiently large such that $1 - x_1^0 x_2^0 < (1 - t)u_2 + tv_2$. Noting that $x_1^0 < (1 - t)u_1 + tv_1$, the latter implies $(1 - t)u + tv \in \mathcal{A}_{E_1}$. Therefore, $\mathcal{A}_{E_1}$ is a convex set. We see that

$$
H = \begin{pmatrix}
0 & 1/2 & 0 \\
1/2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

and $H_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1/2 \\
0 & -1/2 & 0
\end{pmatrix}$.

So, the completely positive relaxation of $(E_1)$ is the following problem

$$
\inf_{X \in \mathcal{C}} \ X_{12}
$$

$$
\text{s.t.} \quad 1 - X_{23} \leq 0, \\
X_{11} = 1,
$$

while the copositive relaxation of $(E_1)$ can be written as

$$
\sup \{ y_0 : Z_+(y) \in \mathcal{C}^*, \quad y = (y_0, u_0) \in \mathbb{R} \times \mathbb{R}_+ \},
$$

5
where
\[ Z_+(y) := H + u_0 H_0 - y_0 J_0 = \begin{pmatrix} u_0 - y_0 & 1/2 & 0 \\ 1/2 & 0 & -u_0/2 \\ 0 & -u_0/2 & 0 \end{pmatrix}. \]

For each \( n \in \mathbb{N} \), let \( \tilde{x} := \left[ 1 \ 1/n \right]^T \). Then, for any \( y = (y_0, u_0) \in \mathbb{R} \times \mathbb{R}_+ \) feasible for \((CE^*_1)\), we have \( \tilde{x}^T Z_+(y) \tilde{x} = \frac{1}{n} - y_0 \geq 0 \). This implies \( \sup(CE^*_1) \leq 0 \). On the other hand, \( y := (0, 0) \) is a feasible point of \((CE^*_1)\). Hence, \( \inf(E_1) = \inf(CE_1) = \max(CE^*_1) = 0 \).

Next, we present a simple one-dimensional example illustrating that the copositive relaxation may not be exact when our geometric condition is not satisfied.

**Example 2.2 (Failure of exact copositive relaxation without the required convexity assumption)** Consider the one-dimensional nonconvex quadratic optimization problem
\[
\begin{align*}
\text{(E2)} & \quad \inf -x^2 \\
& \quad \text{s.t. } x - 1 \leq 0, \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Proof. Since \( Z_+(\tilde{y}) \in \mathcal{C}^* \) and \( \tilde{y} := (f(\tilde{x}), \tilde{u}) \in \mathbb{R} \times \mathbb{R}_{+}^{m+1} \), \( \tilde{y} \) is a feasible point for \((CP_1^*)\). This implies that \( \sup(CP_1^*) \geq f(\tilde{x}) \). On the other hand, \( f(\tilde{x}) \geq \inf(P_1) \) due to the feasibility of \( \tilde{x} \). Therefore, \( \min(P_1) = \max(CP_1^*) \) and \( \tilde{x} \) is a global optimal solution to \((P_1)\). Furthermore, we see that \( \bar{X} := \left( \frac{1}{\bar{x}} \right) \left( \frac{1}{\bar{x}} \right)^T \) is a global solution to \((CP_1)\). So, the desired conclusion follows. \( \square \)

Recall from [5], that \((x, u) \in \mathbb{R}^n \times \mathbb{R}_{+}^{m+1} \) is said to be a generalized KKT pair for \((P_1)\) if \( x \) is a feasible point, \( u_i g_i(x) = 0 \) for all \( i = 0, 1, ..., m \), and \( x_j (2(A + \sum_{i=0}^{m} u_i A_i)x + b + \sum_{i=0}^{m} u_i b_i)_j = 0 \) for all \( j = 1, 2, ..., n \).

The following theorem shows the links between Theorem 2.1 and the corresponding result of [6, Theorem 5.1] where the implication \((ii) \Rightarrow (iv)\) was established while the converse was justified under additional assumptions.

**Theorem 3.1 (Min-max semi-Lagrangian duality).** Let \( x^* \) be feasible for \((P_1)\). Consider the following statements: (i) \( x^* \) is a global optimal solution to \((P_1)\), \( \mathcal{A}_{P_1} \) is convex and the Slater condition holds;

(ii) There exists \( u^* \in \mathbb{R}_{+}^{m+1} \) such that \( (x^*, u^*) \) is a generalized KKT pair satisfying \( \tilde{Z} \) is a feasible point for \((CP_1^*)\) for \( y^* = (f(x^*), u^*) \);

(iii) There exists \( u^* \in \mathbb{R}_{+}^{m+1} \) such that \( Z_+(y^*) \in \mathcal{C}^* \) for \( y^* = (f(x^*), u^*) \);

(iv) \( \min(P_1) = \min(CP_1) = \max(CP_1^*) \) and \( x^* \) is a global optimal solution to \((P_1)\).

Then we have \((i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)\).

**Proof.** We start with \([(i) \Rightarrow (ii)]\). Let \( x^* \) be a global optimal solution to \((P_1)\). By Theorem 2.1, there exists \( u^* \in \mathbb{R}_{+}^{m+1} \) such that \( \Theta(u^*) = \max(D_1) = \inf(P_1) = f(x^*) \). So, we have

\[
L(x, u^*) = f(x) + \sum_{i=0}^{m} u_i^* g_i(x) \geq f(x^*) \quad \text{for all} \quad x \in \mathbb{R}^n.
\]

This guarantees \( u_i^* g_i(x^*) = 0 \), \( Z_+(y^*) \in \mathcal{C}^* \) for \( y^* = (f(x^*), u^*) \), and hence \( x^* \) is a global optimal minimizer of \( L(\cdot, u^*) \) on \( \mathbb{R}^n_+ \). By the first-order optimality condition, there exists \( v \in \mathbb{R}^n_+ \) such that \( v = \nabla_x L(x^*, u^*) \) and \( x_j^* v_j = 0 \) for all \( j = 1, ..., n \). The latter implies

\[
x_j^* (2(A + \sum_{i=0}^{m} u_i A_i)x + b + \sum_{i=0}^{m} u_i b_i)_j = 0 \quad \text{for all} \quad j = 1, 2, ..., n.
\]

Therefore, \((x^*, u^*)\) is a generalized KKT pair satisfying \( Z_+(y^*) \in \mathcal{C}^* \) for \( y^* = (f(x^*), u^*) \).

Since \([(ii) \Rightarrow (iii)]\) is obvious while \([(iii) \Rightarrow (iv)]\) is due to Lemma 3.1, it remains to show \([(iv) \Rightarrow (ii)]\). Suppose that \((iv)\) holds. Then there exists \( u^* \in \mathbb{R}_{+}^{m+1} \) such that

\[
\Theta(u^*) = \max(D_1) = \inf(P_1) = f(x^*).
\]

So, from the proof of \([(i) \Rightarrow (ii)]\) we see that \((ii)\) is valid. Thus, the equivalence of \([(i)-(iv)]\) follows. \( \square \)

**Remark 3.1 (Links between the solutions of \((CP_1)\) and \((P_1)\))** Note that for a completely positive \((l \times l)\) matrix \( X \), its cp-rank \((cf. \ [14])\) is defined as the smallest positive integer \( r \) such
that $X = \sum_{i=1}^{n} x_i x_i^T$ for some $x_i \in \mathbb{R}_+^n$. We now see how a solution of the completely positive program (CP) can be related to a solution of (P) via cp-rank.

Let $X^*$ be a global solution of (CP) with cp-rank one. Then $X^* := \begin{pmatrix} 1 \\ x^* \end{pmatrix} \begin{pmatrix} 1 \\ x^* \end{pmatrix}^T$ and $x^*$ is a global optimal solution to (P). To see this, we note, by construction of (CP), that $X_{11}^* = 1$ and hence $X^* := \begin{pmatrix} 1 \\ x^* \end{pmatrix} \begin{pmatrix} 1 \\ x^* \end{pmatrix}^T$ for some $x^* \in \mathbb{R}_+^n$. Moreover,

$$\text{Tr}(HX^*) = (x^*)^T A x^* + b^T x^* + c$$

and

$$\text{Tr}(H_i X^*) = (x^*)^T A_i x^* + b_i^T x^* + c_i, \ i = 0, \ldots, m.$$

This shows that $x^*$ is feasible for (P) and $\min(P_1) \leq (x^*)^T A x^* + b^T x^* + c = \min(CP_1)$. This together with $\min(P_1) \geq \min(CP_1)$ shows that $x^*$ is a global solution of (P).

As a corollary, we show that our geometric condition always holds for the nonconvex homogeneous quadratic program (QP)

$$\min \{ x^T A x : x \in \mathbb{R}_+^n, (\sum_{i=1}^{n} x_i)^2 - 1 \leq 0 \},$$

where $A$ is an $(n \times n)$ symmetric matrix.

**Corollary 3.1 (Homogeneous QP satisfying the geometric condition)** For problem (P) with $m = 0$, $f(x) = x^T A x$ with $A \in S^n$, and $g_0(x) = (\sum_{i=1}^{n} x_i)^2 - 1$, we have

$$\min(P_1) = \min(CP_1) = \max(CP_1).$$

**Proof.** It is clear that a global solution $x^*$ exists and the Slater condition holds for the quadratic optimization problem $\min \{ x^T A x : x \in \mathbb{R}_+^n, (\sum_{i=1}^{n} x_i)^2 - 1 \leq 0 \}$. From the preceding theorem, it suffices to verify that $A_{P_1} := \{(g_0(x), f(x)) : x \in \mathbb{R}_+^n \}$ is convex. To see this, we first note that $A_{P_1} = (-1, 0) + K$ where $K = \{( (\sum_{i=1}^{n} x_i)^2, x^T A x) : x \in \mathbb{R}_+^n \} + \mathbb{R}_+^2$. So, we only need to show $K$ is convex. Denote $\alpha^* := \min \{ x^T A x : x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i = 1 \}$.

We divide the discussion into two cases.

Suppose that $\alpha^* \geq 0$. Then $x^T A x \geq 0$ for all $x \in \mathbb{R}_+^n$. This shows that $\{( (\sum_{i=1}^{n} x_i)^2, x^T A x) : x \in \mathbb{R}_+^n \} \subseteq \mathbb{R}_+^2$ and so, $K \subseteq \mathbb{R}_+^2$. This shows that $K = \mathbb{R}_+^2$ which is convex.

Suppose that $\alpha^* < 0$. We verify the convexity of $K$ by showing that

$$K = \{(t, \alpha^* t) : t \geq 0 \} + \mathbb{R}_+^2.$$

To see this, let $(u, v) \in K$. Then, there exists $x \in \mathbb{R}_+^n$ such that $(\sum_{i=1}^{n} x_i)^2 < u$ and $x^T A x < v$. Let $t = (\sum_{i=1}^{n} x_i)^2 \geq 0$. Then, the definition of $\alpha^*$ entails that $x^T A x \geq \alpha^* t$. So, $t < u$ and $\alpha^* t < u$, that is, $(u, v) \in \{(t, \alpha^* t) : t \geq 0 \} + \mathbb{R}_+^2$. Thus, $K \subseteq \{(t, \alpha^* t) : t \geq 0 \} + \mathbb{R}_+^2$. On the other hand, let $(u, v) \in \{(t, \alpha^* t) : t \geq 0 \} + \mathbb{R}_+^2$. Then, there exists $t \geq 0$ such that $t < u$ and $\alpha^* t < v$. Let $x^* \in \mathbb{R}_+^n$ be a solution of $\min \{ x^T A x : x \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i = 1 \}$ and let $z = (z_1, \ldots, z_n)$ with $z_i = \sqrt{t} x_i$. Then, $z \in \mathbb{R}_+^n, (\sum_{i=1}^{n} z_i)^2 = t$ and $z^T A z = t(x^*)^T A x^* = \alpha^* t$. So, $(\sum_{i=1}^{n} z_i)^2 < u$ and $z^T A z < v$, and hence $(u, v) \in K$. Thus, the reverse inclusion also holds. \qed
4 Uniform QPs with Nonnegativity Constraints

In this section, we study the copositive relaxation of the following quadratic program:

\[
(P_2) \quad \inf \ x^T Ax + b^T x + c \quad \text{s.t.} \quad x \in \mathbb{R}^n, \quad a_i x^T Ax + b_i^T x + c_i \leq 0, \quad i = 0, 1, \ldots, m,
\]

where \( A \in S^n, \ b, b_i \in \mathbb{R}^n, \ c, c_i \in \mathbb{R}, \ i = 0, 1, \ldots, m, \ a_i \in \mathbb{R}, \ i = 1, \ldots, m. \)

The specific feature of \((P_2)\) is that each Hessian matrix of the constraint function is different from the one of the objective function only by a multiple constant. For this problem, its completely positive relaxation reads as follows

\[
(CP_2) \quad \inf_{X \in \mathcal{C}} \text{Tr}(HX) \quad \text{s.t.} \quad \text{Tr}(H_i X) \leq 0, \quad i = 0, 1, \ldots, m, \quad \text{Tr}(J_0 X) = 1,
\]

while its copositive relaxation can be written as follows:

\[
(CP_2^*) \quad \sup \{ y_0 : Z_+(y) \in \mathcal{C}^*, \ y = (y_0, u) \in \mathbb{R} \times \mathbb{R}^{n+1}_+ \},
\]

where \( H := \begin{pmatrix} c & b^T/2 \\ b/2 & A \end{pmatrix}, \ H_i := \begin{pmatrix} c_i & b_i^T/2 \\ b_i/2 & a_i A \end{pmatrix}, \ i = 0, 1, \ldots, m, \) and

\[
Z_+(y) := H + \sum_{i=0}^m u_i H_i - y_0 J_0 = \begin{pmatrix} c + \sum_{i=0}^m u_i c_i - y_0 & (b + \sum_{i=0}^m u_i b_i)^T/2 \\ (b + \sum_{i=0}^m u_i b_i)/2 & 1 + \sum_{i=0}^m u_i a_i A \end{pmatrix}.
\]

The following result provides some sufficient conditions for the exactness of the copositive relaxation of \((P_2)\).

**Theorem 4.1 (Exact copositive relaxation for uniform QPs).** For problem \((P_2)\), suppose that there exists \( \hat{x} \in \mathbb{R}^n_+ \) such that \( g_i(\hat{x}) < 0 \) for all \( i = 0, 1, \ldots, m \). Suppose further that one of the following conditions is satisfied:

(i) \( A \) is a positive semidefinite matrix having some eigenvector \( d \in \mathbb{R}^n_+ \) corresponding to a nonzero eigenvalue, with \((b_i - \alpha_i b)^T d = 0\) for all \( i = 0, 1, \ldots, m\);

(ii) \(-A\) is a positive semidefinite matrix having some eigenvector \( d \in \mathbb{R}^n_+ \) corresponding to a nonzero eigenvalue, with \((b_i - \alpha_i b)^T d = 0\) for all \( i = 0, 1, \ldots, m\);

(iii) \( A \) has eigenvectors \( d \in \mathbb{R}^n_+ \) and \( \hat{d} \in -\mathbb{R}^n_+ \) corresponding to a positive eigenvalue and a negative eigenvalue of \( A \), respectively, with \((b_i - \alpha_i b)^T d = 0, (b_i - \alpha_i b)^T \hat{d} = 0\) for all \( i = 0, 1, \ldots, m\).

Then, we have

\[
\inf(P_2) = \inf(CP_2) = \max(CP_2^*).
\]

**Proof.** We first show that the set \( \mathcal{A}_{P_2} := \{(g_0(x), g_1(x), \ldots, g_m(x), f(x)) : x \in \mathbb{R}^n_+\} + \text{int} \mathbb{R}^{n+2}_+ \) is convex, where \( f(x) = x^T Ax + b^T x + c \) and \( g_i(x) = a_i x^T Ax + b_i^T x + c_i \). To do this, let

\[
\Omega := \{((b_0 - \alpha_0 b)^T x + c_0 - \alpha_0 c, \ldots, (b_m - \alpha_m b)^T x + c_m - \alpha_m c, x^T Ax + b^T x + c) : x \in \mathbb{R}^n_+\}.
\]
Take any \( u = (u_0, \ldots, u_m, u_{m+1}) \in \Omega, \) \( v = (v_0, v_1, \ldots, v_m, v_{m+1}) \in \Omega, \) and \( \lambda \in (0, 1). \) Then there exist \( x_u \in \mathbb{R}_+^n \) and \( x_v \in \mathbb{R}_+^n \) such that
\[
u_i = (b_i - \alpha_i b)^T x_u + c_i - \alpha_i c, \quad i = 0, 1, \ldots, m, \]
and
\[
u_i = (b_i - \alpha_i b)^T x_v + c_i - \alpha_i c, \quad i = 0, 1, \ldots, m.
\]

**Case 1.** Suppose \((i)\) holds. Let \( d \in \mathbb{R}_+^n \) be an eigenvector corresponding to a nonzero eigenvalue of \( A \) with \((b_i - \alpha_i b)^T d = 0\) for all \( i = 0, 1, 2, \ldots, m, \) and let \( z_0 := z + t d \) for \( t \in \mathbb{R}. \) Define the function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
\varphi(t) := z^T A z_t + b^T z_t + c \quad \text{for} \quad t \in \mathbb{R},
\]
where \( z := \lambda x_u + (1 - \lambda) x_v. \) Obviously, \( \varphi \) is a continuous function. Moreover, since \( A \) is positively semidefinite and \( d \) is an eigenvector of \( A \) corresponding to a nonzero eigenvalue of \( A, \) we have
\[
\varphi(0) = z^T A z + b^T z + c = \lambda (x_u^T Ax_u + b^T x_u + c) + (1 - \lambda)(x_v^T Ax_v + b^T x_v + c) = \lambda u_{m+1} + (1 - \lambda) v_{m+1},
\]
and \( \lim_{t \to +\infty} \varphi(t) = +\infty. \) By the intermediate value theorem, there exists \( t_0 \in \mathbb{R}_+ \) such that
\[
z_t = z_A z_{t_0} + b^T z_{t_0} + c = \varphi(t_0) = \lambda u_{m+1} + (1 - \lambda) v_{m+1}.
\]
Note that \((b_i - \alpha_i b)^T d = 0\) for all \( i = 0, 1, 2, \ldots, m, \) \( t_0 \in \mathbb{R}_+, \) and \( d \in \mathbb{R}_+^n. \) So we have that \( z_t \in \mathbb{R}_+, \)
\[
z_t = z_A z_{t_0} + b^T z_{t_0} + c = \lambda u_{m+1} + (1 - \lambda) v_{m+1}
\]
and
\[
(b_i - \alpha_i b)^T z_{t_0} + c_i - \alpha_i c = \lambda u_i + (1 - \lambda) v_i \quad \text{for} \quad i = 0, 1, \ldots, m.
\]
This implies \((1 - \lambda) u + \lambda v \in \Omega, \) and thus \( \Omega \) is convex.

**Case 2.** Suppose \((ii)\) holds. Then, according to Case 1, the set
\[
\tilde{\Omega} := \left\{ (-b_0 - \alpha_0 b)^T x - (c_0 - \alpha_0 c), \ldots, -(b_m - \alpha_m b)^T x - (c_m - \alpha_m c), x^T (A x - b^T x) : x \in \mathbb{R}_+^n \right\}
\]
is convex. On the other hand, \( \Omega = -\tilde{\Omega}. \) Therefore, \( \Omega \) is a convex set.

**Case 3.** Suppose \((iii)\) holds. Let \( d \) and \( \hat{d} \) be two eigenvectors with the properties given in the condition \((iii).\) Consider the function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
\varphi(t) := z^T A z_t + b^T z_t + c \quad \text{for} \quad t \in \mathbb{R},
\]
where
\[
z_t := \begin{cases} z + t d & \text{for} \quad t \in \mathbb{R}_+ \\ z + t \hat{d} & \text{for} \quad t \in \mathbb{R}_- \end{cases}
\]
and \( z := \lambda x_u + (1 - \lambda) x_v. \) We see that \( \varphi \) is continuous, \( \lim_{t \to +\infty} \varphi(t) = +\infty \) and \( \lim_{t \to -\infty} \varphi(t) = -\infty. \) By the intermediate value theorem, there exists \( t_0 \in \mathbb{R} \) such that
\[
z_t = z_A z_{t_0} + b^T z_{t_0} + c = \varphi(t_0) = \lambda u_{m+1} + (1 - \lambda) v_{m+1}.
\]
So, similarly to what have been done in Case 1, the convexity of Ω follows.

On the other hand, we have \( A_P^2 = L(Ω) + \text{int}\mathbb{R}^{m+2}_+ \), where \( L : \mathbb{R}^{m+2} \to \mathbb{R}^{m+2} \) is the linear mapping defined by

\[
L(y) := (y_0 + \alpha_0 y_{m+1}, y_1 + \alpha_1 y_{m+1}, \ldots, y_m + \alpha_m y_{m+1}) \quad \text{for} \quad y = (y_0, \ldots, y_{m+1}) \in \mathbb{R}^{m+2}.
\]

Therefore, \( A_P^2 \) is convex. So, by Theorem 2.1, we get the desired conclusion. □

**Remark 4.1** Problem \( (P_2) \) can be rewritten as problem \( (QP) \) studied in [3] by replacing the sign constraint \( x \in \mathbb{R}^n_+ \) with inequality constraints. However, one cannot directly obtain a semidefinite relaxation by [3, Corollary 2.1], due to the violation of the condition on the restriction of the number of constraints.

Let us give two examples of uniform QPs, exhibiting exact copositive relaxation and satisfying our assumptions of Theorem 4.1.

**Example 4.1 (Convex objective function & nonconvex feasible set).** Consider the following quadratic optimization problem

\[
(E_3) \quad \begin{array}{ll}
\inf & x_1^2 + 2x_1x_2 + x_2^2 + x_1 + x_2 \\
\text{s.t.} & -2x_1^2 - 4x_1x_2 - 2x_2^2 - x_1 - 3x_2 + 5 \leq 0, \quad x = (x_1, x_2) \in \mathbb{R}_+^2.
\end{array}
\]

Let \( f(x) := x_1^2 + 2x_1x_2 + x_2^2 + x_1 + x_2 \) and \( g_0(x) := -2x_1^2 - 4x_1x_2 - 2x_2^2 - x_1 - 3x_2 + 5 \). For \( A := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), \( b := (1, 1) \), \( b_0 := (-1, -3) \), \( c := 0 \), \( c_0 := 5 \), \( \alpha_0 := -2 \), we have \( f(x) = x^TAx + b^Tx + c \) and \( g_0(x) = \alpha_0 x^TAx + b_0^Tx + c_0 \). This shows that \( (E_3) \) is a special case of \( (P_2) \). We see that

\[
H = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1 & 1 \\ 1/2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad H_0 = \begin{pmatrix} 5 & -1/2 & -3/2 \\ -1/2 & -2 & -2 \\ -3/2 & -2 & -2 \end{pmatrix}.
\]

Therefore, the completely positive relaxation of \( (E_3) \) is the following problem

\[
(CE_3) \quad \begin{array}{ll}
\inf & X_{12} + X_{13} + X_{22} + 2X_{23} + X_{33} \\
\text{s.t.} & 5 - X_{12} - 3X_{13} - 2X_{22} - 4X_{23} - 2X_{33} \leq 0, \\
& X_{11} = 1,
\end{array}
\]

while the copositive relaxation of \( (E_3) \) reads as follows

\[
(CE_3') \quad \sup \{ y_0 : Z_+(y) \in C^* ; y = (y_0, u_0) \in \mathbb{R} \times \mathbb{R}_+ \},
\]

where

\[
Z_+(y) := H + u_0H_0 - y_0J_0 = \begin{pmatrix} 5u_0 - y_0 & 1/2 - u_0/2 & 1/2 - 3u_0/2 \\ 1/2 - u_0/2 & 1 - 2u_0 & 1 - 2u_0 \\ 1/2 - 3u_0/2 & 1 - 2u_0 & 1 - 2u_0 \end{pmatrix}.
\]

We note that \( A \in S^2_+ \), \( \det(A - \lambda I) = \lambda(\lambda - 2) \), and the eigenspace corresponding to the eigenvalue \( \lambda = 2 \) is given by \( \ker(A - 2I) = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 \} \). So, \( (b_0 - \alpha_0b)^Td = 0 \)

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for \( d := (1, 1) \in (\mathbb{R}^3_+ \setminus \{0\}) \cap \ker(A - 2I) \). On the other hand, \( g_0(\tilde{x}) < 0 \) for \( \tilde{x} := (0, 2) \in \mathbb{R}^2_+ \), and \( \inf(E_1P_2) \) is finite. Thus, the assumption of Theorem 4.1 is fulfilled. Therefore, by Theorem 4.1, \( \inf(E_3) = \inf(CE_3) = \max(CE_3^*) \). We next directly check this confirmation. Let \((x_1, x_2)\) be feasible for the problem \((E_3)\). Then \((x_1, x_2) \in \mathbb{R}^2_+\) and
\[
-2(x_1 + x_2)^2 - 3(x_1 + x_2) + 5 \leq -2(x_1 + x_2)^2 - 3(x_1 + x_2) + 2x_1 + 5 = g_0(x) \leq 0.
\]
This implies \( x_1 + x_2 \geq 1 \) and \( f(x) = (x_1 + x_2)^2 + (x_1 + x_2) \geq 2 \). On the other hand, \( f(\bar{x}) = 2 \) where \( \bar{x} := (0, 1) \) is feasible for \((CE_3)\). So, we have \( \min(CE_3) = 2 \). Let \( \hat{x} = [1 \ 0 \ 1]^T \in \mathbb{R}^3_+ \). Then, if \( y = (y_0, u_0) \) is feasible for \((CE_3^*)\), then we have \( \hat{x}^T Z_+(y) \hat{x} = 2 - y_0 \geq 0, u_0 \geq 0 \). This shows \( y_0 \leq 2 \). Thus \( \sup(CE_3^*) \leq 2 \). On the other hand, for \( \bar{y} = (2, 3/7) \in \mathbb{R} \times \mathbb{R}_+ \) we have
\[
Z_+(\bar{y}) = \begin{pmatrix}
\frac{1}{7} & 2/7 & -1/7 \\
2/7 & 1/7 & 1/7 \\
-1/7 & 1/7 & 1/7
\end{pmatrix}.
\]
For each \( \hat{x} = (a, b, c) \in \mathbb{R}^3_+ \), direct computation shows that
\[
\hat{x}^T Z_+(\bar{y}) \bar{x} = \frac{1}{7} [(a - c)^2 + b^2 + 4ab + 2bc] \geq 0.
\]
This implies that \( Z_+(\bar{y}) \in C^* \). So, \( \bar{y} \) is a global optimal solution to \((CE_3^*)\) and \( \max(CE_3^*) = 2 = \min(CE_3) \).

**Example 4.2 (Non-convex and non-concave objective function & Non-convex feasible set).** Consider the following optimization problem:
\[
\begin{align*}
\inf_{x \in \mathbb{R}^3_+} & \quad x_1^3 + x_2^2 - x_3^2 + 2x_1x_2 + x_1 + x_2 \\
\text{s.t.} & \quad x_1^3 + x_2^2 - x_3^2 - x_1 + x_2 + 1 \leq 0, \\
& \quad -x_1^2 - x_2^2 + x_3^2 + x_1 - x_2 - 5 \leq 0, \\
& \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3_+.
\end{align*}
\]
Let \( f(x) := x_1^3 + x_2^2 - x_3^2 + 2x_1x_2 + x_1 + x_2, g_0(x) := x_1^3 + x_2^2 - x_3^2 - x_1 + x_2 + 1, \) and \( g_1(x) := -x_1^2 - x_2^2 + x_3^2 + x_1 - x_2 - 5. \) These functions can be written as follows \( f(x) = x^TAx + b^Tx + c \) and \( g_i(x) = \alpha_0x^TAx + b_i^Tx + c_i, \) \( i = 0, 1, \) where \( A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, b = (1, 1, 0), b_0 = (-1, 1, 0), \) \( b_1 = (1, -1, 0), c = 0, c_0 = 1, c_1 = -5, \alpha_0 = 1, \alpha_1 = -1. \) Hence \((E_4)\) is also a special case of \((P_2)\). We see that
\[
H = \begin{pmatrix}
0 & 1/2 & 1/2 & 0 \\
1/2 & 1 & 1 & 0 \\
1/2 & 1 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad H_0 = \begin{pmatrix}
1 & -1/2 & 1/2 & 0 \\
-1/2 & 1 & 0 & 0 \\
1/2 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad H_1 = \begin{pmatrix}
-5 & 1/2 & -1/2 & 0 \\
1/2 & -1 & 0 & 0 \\
-1/2 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Therefore, the completely positive relaxation of \((E_4)\) is the following problem
\[
\begin{align*}
\inf_{x \in \mathbb{R}^3_+} & \quad X_{12} + X_{13} + X_{22} + 2X_{23} + X_{33} - X_{44} \\
\text{s.t.} & \quad 1 - X_{12} + X_{13} + X_{22} + X_{23} - X_{44} \leq 0, \\
& \quad -5 + X_{12} - X_{13} - X_{22} - X_{33} + X_{44} \leq 0, \\
& \quad X_{11} = 1,
\end{align*}
\]
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while the copositive relaxation of \((E_4)\) is the following problem
\[
(CE_4^*) \quad \sup \{ y_0 : Z_+(y) \in C^*, \quad y = (y_0, u_0) \in \mathbb{R} \times \mathbb{R}_+^2 \},
\]
where
\[
Z_+(y) := H + u_0 H_0 + u_1 H_1 - y_0 J_0
\]
\[
= \begin{pmatrix}
  u_0 - 5u_1 - y_0 & 1/2 - u_0/2 + u_1/2 & 1/2 + u_0/2 - u_1/2 & 0 \\
 1/2 - u_0/2 + u_1/2 & 1 + u_0 - u_1 & 1 & 0 \\
1/2 + u_0/2 - u_1/2 & 0 & 1 + u_0 - u_1 & 0 \\
0 & 0 & 0 & -1 - u_0 + u_1
\end{pmatrix}.
\]

We note that \(A \in S^3_+, \det(A - \lambda I) = \lambda(\lambda + 1)(2 - \lambda)\), the eigenspace corresponding to the eigenvalue \(\lambda = 2\) is given by
\[
\ker(A - 2I) = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2, x_3 = 0 \},
\]
while the eigenspace corresponding to the eigenvalue \(\lambda = -1\) is given by
\[
\ker(A + I) = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0 \}.
\]
So, for \(d := (1, 1, 0)\) and \(\tilde{d} := (0, 0, -1)\), we have that \(d \in \ker(A - 2I) \cap \mathbb{R}_+^3\), \(\tilde{d} \in \ker(A + I) \cap \mathbb{R}_+^3\), and \((b_i - \alpha_i b)^T d = 0, (b_i - \alpha_i b)^T \tilde{d} = 0\) for all \(i = 0, 1\). On the other hand, \(g_0(\tilde{x}) < 0\) for \(\tilde{x} := (0, 0, 2) \in \mathbb{R}_+^3\), and \(\inf(E_4)\) is finite. Consequently, the assumption of Theorem 4.1 is satisfied. So, by Theorem 4.1, \(\inf(E_4) = \inf(CE_4) = \max(CE_4^*)\). We next directly verify this confirmation. For any feasible point \(x = (x_1, x_2, x_3)\) of \((E_4)\), we have
\[
f(x) = (x_1^2 + x_2^2 - x_3^2) + 2x_1 x_2 + x_1 + x_2 \\
\geq (x_1 - x_2 - 5) + 2x_1 x_2 + x_1 + x_2 \\
= -5 + 2x_1 x_2 + 2x_1 \\
\geq -5.
\]

On the other hand, \(\bar{x} := (0, 0, \sqrt{5})\) is a feasible point of \((E_4)\) satisfying \(f(\bar{x}) = -5\). Hence \(\min(E_4) = -5\). Let \(\bar{x} = [1 \ 0 \ 0 \ 0 \ \sqrt{5}]^T\). Then, for each \(y = (y_0, u_0, u_1) \in \mathbb{R} \times \mathbb{R}_+^2\), \(\bar{x}^T Z_+(y) \bar{x} = -u_0 - y_0 - 5 \geq 0\). This implies \(y_0 \leq -5\). So, we have \(\sup(CE_4^*) \leq -5\). On the other hand, direct verification shows that, for \(\bar{y} = (-5, 0, 1) \in \mathbb{R} \times \mathbb{R}_+^2\),
\[
Z_+(\bar{y}) = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix} \in C^*.
\]
This shows that \(\bar{y}\) is feasible for \((CE_4^*)\). Therefore, \(\max(CE_4^*) = -5 = \min(E_4)\). Taking into accounts that \(\max(CE_4^*) \leq \inf(E_4) \leq \min(E_4)\), we obtain the desired conclusion.

**Copositive reformulation of a ball inclusion problem.** Let \(a, a_i \in \mathbb{R}^n\) and \(r_i \in \mathbb{R}_+\), \(i = 0, 1, ..., m\). Next, using Theorem 4.1, we will examine a copositivity reformulation of the problem: find a minimum radius ball with \(a\) enclosing the intersection \(\cap_{i=1}^m \mathbb{B}(a_i; r_i) \cap \mathbb{R}_+^l\)
(see Figure 1 below). That is, our problem is to find a smallest number \( \gamma \in \mathbb{R}_+ \) satisfying the ball-inclusion

\[
\bigcap_{i=0}^{m} B(a_i; r_i) \cap \mathbb{R}_+^n \subset B(a; \sqrt{\gamma}),
\]

which can be reformulated as follows

\[
(P_B) \quad \inf_{\gamma \in \mathbb{R}} \gamma \\
st. \quad \bigcap_{i=0}^{m} B(a_i; r_i) \cap \mathbb{R}_+^n \subset B(a; \sqrt{\gamma}), \quad \gamma \in \mathbb{R}_+.
\]

The figure illustrates a simple example for finding the minimum radius of a ball centered at \((0, -4)\) which encloses the intersection of two balls \((x-3)^2 + (y-1)^2 \leq 4\) and \((x-5)^2 + (y+1)^2 \leq 9\) as well

![Figure 1: A ball with minimum radius center at \((0, -4)\) enclosing the intersections of two balls \((x-3)^2 + (y-1)^2 \leq 4\) and \((x-5)^2 + (y+1)^2 \leq 9\) with \(\mathbb{R}_+^2\).](image)

To obtain the copositive program reformulation of smallest enclosing ball problem \((P_B)\), we need the following result, which shows how to reformulate the ball-inclusion as a linear copositive inequality.

**Corollary 4.1 (Copositive reformulation of ball-inclusion).** Let \(a, a_i \in \mathbb{R}^n\), \(\delta \in \mathbb{R}_+\), and \(r_i > 0\), \(i = 0, 1, \ldots, m\). Suppose there exists \(d \in \mathbb{R}_+^n \setminus \{0\}\) with \((a_i + a)^T d = 0\), and there exists \(\hat{x} \in \mathbb{R}_+^n\) with \(\|\hat{x} - a_i\| < r_i\), \(i = 0, 1, \ldots, m\). Then, the following assertions are equivalent:
(i) \( \bigcap_{i=0}^{m} B(a_i; r_i) \cap \mathbb{R}_+^n \subset B(a; \delta) \);

(ii) There exists \((\lambda_0, \lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^{m+1}\) such that

\[
\sum_{i=0}^{m} \lambda_i N_i - N \in C^*;
\]

where

\[
N = \begin{pmatrix} \|a\|^2 - \delta^2 & -a^T \\ -a & I \end{pmatrix}, \quad N_i = \begin{pmatrix} \|a_i\|^2 - r_i^2 & -a_i^T \\ -a_i & I \end{pmatrix}, \quad i = 0, 1, \ldots, m.
\]

**Proof.** We first note that \( \bigcap_{i=0}^{m} B(a_i; r_i) \cap \mathbb{R}_+^n \subset B(a; \delta) \) amounts to that

\[
\|x - a\|^2 \leq \delta^2 \quad \text{whenever} \quad x \in \mathbb{R}_+^n, \|x - a_i\|^2 \leq r_i^2, \quad i = 0, 1, \ldots, m. \quad (3)
\]

Suppose now that there exist \( \lambda_0, \lambda_1, \ldots, \lambda_m \geq 0 \) such that \( \sum_{i=0}^{m} \lambda_i N_i - N \in C^* \). We will show that (3) holds. To see this, take any \( x \in \mathbb{R}_+^n \) with \( \|x - a_i\|^2 \leq r_i^2, \quad i = 0, 1, \ldots, m \). We have

\[
\begin{pmatrix} 1 \\ x \end{pmatrix}^T \left( \sum_{i=0}^{m} \lambda_i N_i - N \right) \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0,
\]

or equivalently,

\[
\sum_{i=0}^{m} \lambda_i (\|x - a_i\|^2 - r_i^2) - (\|x - a\|^2 - \delta^2) \geq 0.
\]

This implies that (3) holds.

Conversely, suppose (3) is valid. Then we have

\[
\inf \{ \delta^2 - \|x - a\|^2 : x \in \mathbb{R}_+^n, \|x - a_i\|^2 \leq r_i^2, \quad i = 0, 1, \ldots, m \} \geq 0. \quad (4)
\]

Let \( f(x) := \delta^2 - \|x - a\|^2 \) and \( g_i(x) := \|x - a_i\|^2 - r_i^2, \quad i = 0, 1, \ldots, m \). We now consider the following problem

\[
\inf_x f(x) \quad \text{s.t.} \quad x \in \mathbb{R}_+^n, \quad g_i(x) \leq 0, \quad i = 0, 1, \ldots, m. \quad (5)
\]

Put \( A := -I_n, \quad b := 2a, \quad c := \delta^2 - \|a\|^2, \quad b_i := -2a_i, \quad c_i := \|a_i\|^2 - r_i^2, \quad \alpha_i = -1, \quad i = 0, 1, \ldots, m \). Then \( f(x) = x^T Ax + b^T x + c \) and \( g_i(x) = \alpha_i x^T Ax + b_i x + c_i, \quad i = 0, 1, \ldots, m \). This shows that problem (5) is a special case of (P2). Obviously, \( -A = I_n \) is positive definite and has one eigenvalue \( \lambda = 1 \) with the eigenspace \( \ker(A - \lambda I) = \mathbb{R}^n \). So, from our assumption it follows that \( -A \) is a positive semidefinite matrix having some eigenvector \( d \in \mathbb{R}_+^n \) corresponding to a nonzero eigenvalue, with \( (b_i - \alpha_i b)^T d = 0 \) for all \( i = 0, 1, \ldots, m \). Moreover, the optimal value of (5) is finite, and the Slater condition is satisfied. Hence, by Theorem 4.1, there exists \((y_0, \lambda) \in \mathbb{R} \times \mathbb{R}_+^{n+1}\) such that

\[
\sum_{i=0}^{m} \lambda_i N_i - N - y_0 J_0 \in C^*.
\]
and
\[ y_0 = \inf_x \{ f(x) : x \in \mathbb{R}^n_+ \text{, } g_i(x) \leq 0, \text{ } i = 0, 1, \ldots, m \}. \]

Taking (4) into account, we get \( y_0 \geq 0 \), and thus \( \sum_{i=0}^{m} \lambda_i N_i - N \in C^* \).

**Corollary 4.2 (Linear copositivity reformulation of \((P_B)\)).** Let \( a_i, a \in \mathbb{R}^n \) and \( r_i > 0, \) \( i = 0, 1, \ldots, m \). Suppose that \((a_i + a)^T d = 0\) for some \( d \in \mathbb{R}^n_+ \setminus \{0\}\), and there exists \( \hat{x} \in \mathbb{R}^n_+ \) such that \( \|\hat{x} - a_i\| < r_i, \) \( i = 0, 1, \ldots, m \). Then, \((P_B)\) is equivalent to the following problem:

\[
\begin{align*}
\inf_{\gamma \in \mathbb{R}} & \quad \gamma \\
\text{s.t.} & \quad \sum_{i=0}^{m} \lambda_i N_i - N \in C^* \\
& \quad (\gamma, \lambda_0, \lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{m+2},
\end{align*}
\]

where
\[
N = \begin{pmatrix}
\|a\|^2 - \gamma & -a^T \\
-a & I
\end{pmatrix}, \quad
N_i = \begin{pmatrix}
\|a_i\|^2 - r_i^2 & -a_i^T \\
-a_i & I
\end{pmatrix}, \quad i = 0, 1, \ldots, m.
\]

**Proof.** The desired conclusion follows from Corollary 4.1 by letting \( \delta = \sqrt{\gamma} \).

**Remark 4.2** In the preceding corollary, we established a linear copositive program reformulation for \((P_B)\) under the strict feasibility condition together with an additional regularity assumption \((a_i + a)^T d = 0, \) \( i = 0, 1, \ldots, m \), for some \( d \in \mathbb{R}^n_+ \setminus \{0\} \). We note that this additional regularity assumption can be easily verified by solving the feasibility problem of the following linear system

\[
\begin{cases}
\quad d \in \mathbb{R}^n_+, \quad \sum_{j=1}^{n} d_j = 1, \\
\quad (a_i + a)^T d = 0, \quad i = 0, 1, \ldots, m.
\end{cases}
\]

### 5 Nonnegative Extended Trust-region Problems

Consider the following extended trust-region problem (NNETP) with nonnegativity constraints:

\[
(P_3) \quad \inf_x x^T Ax + a^T x \\
\text{s.t.} \quad \|x - x_0\|^2 \leq \alpha, \\
\quad b_i^T x \leq \beta_i, \quad i = 1, \ldots, m, \\
\quad x \in \mathbb{R}^n_+.
\]

where \( A \in S^n, \) \( a, x_0, b_i \in \mathbb{R}^n, \) \( \beta_i \in \mathbb{R}, \) \( i = 1, \ldots, m, \) and \( \alpha \in (0, +\infty) \). Its completely positive relaxation is the following problem:

\[
(CP_3) \quad \inf_{X \in C} \text{Tr}(HX) \\
\text{s.t.} \quad \text{Tr}(H_iX) \leq 0, \quad i = 0, 1, \ldots, m, \\
\quad \text{Tr}(J_0X) = 1,
\]

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Theorem 5.1 (Exact copositive relaxation for NETPs).

For problem (P), there exists a nonzero vector \( v \) where

\[
Z \quad \text{if} \quad \exists \quad \text{such that}
\]

So, the copositive relaxation can be written as follows:

\[
(CP_3^+) \quad \sup \{ y_0 : Z_+(y) \in C^*, \; y = (y_0, u) \in R \times R_{+}^{m+1} \},
\]

where

\[
Z_+(y) := H + \sum_{i=0}^{m} u_i H_i - y_0 J_0 = \left( u_0(\|x_0\|^2 - \alpha) - \sum_{i=1}^{m} u_i \beta_i - y_0 \quad a^T/2 - u_0 x_0^T + \sum_{i=1}^{m} u_i b_i^T/2 \right)
\]

\[
\left( a/2 - u_0 x_0 + \sum_{i=1}^{m} u_i b_i/2 \quad A + u_0 I_n \right)
\]

In the following theorem, we provide a verifiable sufficient condition for the exact copositive relaxation for problem \((P_3)\).

**Theorem 5.1 (Exact copositive relaxation for NETPs).** For problem \((P_3)\), suppose that there exists a nonzero vector \( v \in R^n \) such that

\[
\begin{align*}
Av - \lambda_{\min}(A) v &= 0, \\
v &\geq 0, \quad b_i^T v = 0, \; i = 1, ..., m, \\
(a + 2\lambda_{\min}(A) x_0)^T v &\leq 0.
\end{align*}
\]

Then, we have

(i) \( \min(P_3) = \min(CP_3) = \sup(CP_3^+) \);

(ii) If, in addition, the Slater condition holds, that is, there exists \( \hat{x} \in R^n \) such that \( g_i(\hat{x}) < 0 \) for all \( i = 0, 1, ..., m \), then

\[
\min(P_3) = \min(CP_3) = \max(CP_3^+),
\]

where \( g_0(x) = \|x - x_0\|^2 - \alpha \) and \( g_i(x) = b_i^T x - \beta_i, \; i = 1, ..., m \);

**Proof.** Let \( V : R^{m+1} \to R \) be the optimal value function of \((P_3)\), that is,

\[
V(s_0, s_1, ..., s_m) := \begin{cases} 
\min_{x \in R^n} \{ f(x) : \|x - x_0\|^2 \leq \alpha + s_0, b_i^T x \leq \beta + s_i, i = 1, ..., m \} & \text{if } x \in D, \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( f(x) := x^T A x + a^T x \) is the objective function of \((P_3)\), and

\[
D := \{ (s_0, s_1, ..., s_m) \in R^{m+1} : \|x - x_0\|^2 \leq \alpha + s_0, b_i^T x \leq \beta + s_i, i = 1, ..., m, \text{ for some } x \in R^n \}.
\]

We first show that epi\( V \) is closed. Take any \( (s_0^k, ..., s_m^k, r^k) \in \text{epi} V \) with

\[
(s_0^k, ..., s_m^k, r^k) \to (s_0, ..., s_m, r) \in R^{m+2}.
\]
Then, there exists \( x^k \in \mathbb{R}_+^n \) such that
\[
f(x^k) \leq x^k, \quad \|x^k - x_0\|^2 \leq \alpha + s_0^k, \quad b_i^T x^k \leq \beta_i + s_i^k, \quad i = 1, \ldots, m.
\] (7)
This together with the convergence of \( \{s_0^k\} \) can ensure the boundedness of \( \{x^k\} \). So, replacing a subsequence if necessary, we may assume that \( x^k \to x \in \mathbb{R}_+^n \). Consequently, taking the limits in (7) as \( k \to \infty \) we get
\[
f(x) \leq r, \quad \|x - x_0\|^2 \leq \alpha + s_0, \quad b_i^T x \leq \beta_i + s_i, \quad i = 1, \ldots, m,
\]
showing that \((s_0, \ldots, s_m, r) \in \text{epi} V\). This implies the set \( \text{epi} V \) is closed.

We next prove that \( \text{epi} V \) is convex. If \( A \) is positive semidefinite, then \( f, g_i, i = 0, 1, \ldots, m, \) are convex functions, and thus \( \text{epi} V \) is convex. Otherwise, we have \( \lambda_{\min}(A) < 0 \). We claim that for each \((s_0, s_1, \ldots, s_m) \in D\) the optimization problem
\[
\min_{x \in \mathbb{R}_+^n} \left\{ f(x) - \lambda_{\min}(A) \|x - x_0\|^2 : \|x - x_0\|^2 \leq \alpha + s_0, \quad b_i^T x \leq \beta_i + s_i, \quad i = 1, \ldots, m \right\}
\] (8)
has some optimal solution \( \bar{x} \in \mathbb{R}_+^n \) with \( \|\bar{x} - x_0\|^2 = \alpha + s_0 \). Granting this, we have
\[
\min_{x \in \mathbb{R}_+^n} \left\{ f(x) - \lambda_{\min}(A) \|x - x_0\|^2 : \|x - x_0\|^2 \leq \alpha + s_0, \quad b_i^T x \leq \beta_i + s_i, \quad i = 1, \ldots, m \right\}
= f(\bar{x}) - \lambda_{\min}(A)(\alpha + s_0)
\geq \min_{x \in \mathbb{R}_+^n} \left\{ f(x) : \|x - x_0\|^2 \leq \alpha + s_0, \quad b_i^T x \leq \beta_i + s_i, \quad i = 1, \ldots, m \right\} - \lambda_{\min}(A)(\alpha + s_0)
= \min_{x \in \mathbb{R}_+^n} \left\{ f(x) - \lambda_{\min}(A)(\alpha + s_0) : \|x - x_0\|^2 \leq \alpha + s_0, \quad b_i^T x \leq \beta_i + s_i, \quad i = 1, \ldots, m \right\}
\geq \min_{x \in \mathbb{R}_+^n} \left\{ f(x) - \lambda_{\min}(A) \|x - x_0\|^2 : \|x - x_0\|^2 \leq \alpha + s_0, \quad b_i^T x \leq \beta_i + s_i, \quad i = 1, \ldots, m \right\}.
\]

This implies that
\[
V(s_0, \ldots, s_m) = \min_{x \in \mathbb{R}_+^n} \left\{ f(x) - \lambda_{\min}(A) \|x - x_0\|^2 : \|x - x_0\|^2 \leq \alpha + s_0, \quad b_i^T x \leq \beta_i + s_i, \quad i = 1, \ldots, m \right\} + \lambda_{\min}(A)(\alpha + s_0).
\]

On the other hand, \( F(x) := f(x) - \lambda_{\min}(A) \|x - x_0\|^2 \) is a convex function. We deduce that \( V \) is a convex function, and thus \( \text{epi} V \) is convex. To end the proof, our remained task is to justify the above claim. Fix \((s_0, \ldots, s_m) \in D\) and suppose to the contrary that any optimal solution \( x^* \) of problem (8) satisfy \( \|x^* - x_0\|^2 < \alpha + s_0 \). Let \( v \in \mathbb{R}_+^n \setminus \{0\} \) be such that
\[
\begin{aligned}
Av - \lambda_{\min}(A)v &= 0, \\
v &\geq 0, \quad b_i^Tv = 0, \quad i = 1, \ldots, m, \\
(a + 2\lambda_{\min}(A)x_0)^Tv &\leq 0.
\end{aligned}
\]

We note that problem (8) always has at least one global optimal solution, due to the compactness of the feasible set and the continuity of the objective function. Let \( x^* \) be an arbitrary optimal solution of (8).
Case 1: \((a + 2\lambda_{\min}(A)x_0)^T v = 0\). Put \(x(t) := x^* + tv\). Since \(\|x^* - x_0\|^2 < \alpha + s_0\) and \(v \neq 0\), there exists \(t_0 > 0\) such that \(\|x(t_0) - x_0\|^2 = \alpha + s_0\). We see that \(x(t_0) \in \mathbb{R}^n_+\) and
\[
b_i^T x(t_0) = b_i^T (x^* + t_0v) = b_i^T x^* \leq \beta_i, \ i = 1, \ldots, m.
\]
Moreover,
\[
F(x(t_0)) = (x^* + t_0v)^T (A - \lambda_{\min}(A)I_n)(x^* + t_0v) + (a + 2\lambda_{\min}(A)x_0)^T (x^* + t_0v) - \lambda_{\min}(A)\|x_0\|^2 = F(x^*).
\]
This contradicts the assumption that every global optimal solution of problem (8) belongs to the open ball with center \(x_0\) and radius \(\alpha + s_0\).

Case 2: \((a + 2\lambda_{\min}(A)x_0)^T v < 0\). Since \(\|x^* - x_0\|^2 < \alpha + s_0\), there exists \(t_0 > 0\) such that \(\|x(t) - x_0\|^2 \leq \alpha + s_0\) for all \(t \in (0, t_0]\). We see that \(x(t_0) \in \mathbb{R}^n_+\), \(b_i^T x(t_0) = b_i^T (x^* + t_0v) = b_i^T x^* \leq \beta_i + s_i, \ i = 1, \ldots, m\), and
\[
F(x(t_0)) = (x^* + t_0v)^T (A - \lambda_{\min}(A)I_n)(x^* + t_0v) + (a + 2\lambda_{\min}(A)x_0)^T (x^* + t_0v) + (\gamma - \lambda_{\min}(A))\|x_0\|^2 < (x^*)^T (A - \lambda_{\min}(A)I_n)x^* + (a + 2\lambda_{\min}(A)x_0)^T x^* - \lambda_{\min}(A)\|x_0\|^2 = F(x^*).
\]
This contradicts the assumption that \(x^*\) is a global optimal solution of problem (8).

Consequently, \(\text{epi} V\) is a closed convex set. So, taking into account that \(V(0, 0, \ldots, 0) = \min(P_3)\) is finite, by [19, Theorem 4.1], we have \(\sup(D_3) = \min(P_3)\), where \((D_3)\) is the semi-Lagrangian dual of \((P_3)\). On the other hand, by [6, Theorem 4.1],
\[
\sup(D_3) = \sup(CP_3^*) \leq \inf(CP_3) \leq \inf((P_3)).
\]
Therefore, \(\sup(D_3) = \sup(CP_3^*) = \inf(CP_3) = \min(P_3)\). Furthermore, since the Hessian matrix of \(g_0\) is positive definite, it is strictly copositive, and thus, by [6, Theorem 4.3], the completely copositive relaxation problem \((CP_3)\) is attained, that is, \(\inf(CP_3) = \min(CP_3)\). So, we have
\[
\min(D_3) = \sup(CP_3^*) = \inf(CP_3) = \min(P_3).
\]

Secondly, suppose further that the Slater condition holds, i.e., there exists \(\hat{x} \in \mathbb{R}^n_+\) such that \(g_i(\hat{x}) < 0\) for all \(i = 0, 1, \ldots, m\). Put
\[
A_{P_3} := \{(f(x), g_0(x), g_1(x), \ldots, g_m(x)) : x \in \mathbb{R}^n_+\} + \text{int} \mathbb{R}^{m+1}.
\]
Since \(\text{epi} V = \{(g_0(x), g_1(x), \ldots, g_m(x), f(x)) : x \in \mathbb{R}^n_+\} + \text{int} \mathbb{R}^{m+1}\), we have \(A_{P_3} \subset \text{int}(\text{epi} V)\). To see the inverse inclusion, take any \((s_0, s_1, \ldots, s_m, r) \in \text{int}(\text{epi} V)\). Then there exists \(\varepsilon > 0\) such that \((s_0 - \varepsilon, s_1 - \varepsilon, \ldots, s_m - \varepsilon, r - \varepsilon) \in \text{epi} V\). Thus, for some \(x \in \mathbb{R}^n_+\),
\[
f(x) \leq r - \varepsilon < r \quad \text{and} \quad g_i(x) \leq s_i - \varepsilon < s_i, \ i = 0, 1, \ldots, m.
\]
This says that \((s_0, s_1, \ldots, s_m, r) \in A_{P_3}\). So, we have \(A_{P_3} = \text{int}(\text{epi} V)\). Taking into account that \(\text{epi} V\) is convex, we see that \(A_{P_3}\) is convex. Note that \((P_3)\) has a global optimal solution and thus \(\inf(P_3)\) is finite. Hence, from Theorem 2.1 it follows that \(\max(D_3) = \max(CP_3^*) = \inf(CP_3) = \min(P_3)\). Furthermore, if \(\bar{x}\) is a global optimal solution to \((P_3)\) then \((1_x)^T \bar{x} = x^T \bar{x}\) is a global optimal solution to \((CP_3)\). So, the second assertion follows. \(\square\)
Remark 5.1 The dimension condition given in [17, Definition 2.1] is strictly stronger than the existence of some nonzero solution $v$ of the following linear system:

\[
\begin{align*}
Av - \lambda_{\min}(A)v &= 0, \\
b_i^T v &= 0, \ i = 1, \ldots, m, \\
(a + 2\lambda_{\min}(A)x_0)^T v &\leq 0,
\end{align*}
\]

which can guarantee the validity of an exact semidefinite programming relaxation (even an exact second-order cone programming relaxation) for the extended trust-region problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n}\ x^T Ax + a^T x \\
\text{s.t.} \quad \|x - x_0\|^2 &\leq \alpha, \\
&\ b_i^T x \leq \beta_i, \ i = 1, \ldots, m;
\end{align*}
\]

see [17, 18]. However, this dimension condition (even its aforementioned consequence) fails whenever the constraint system contains the sign constraint $x \in \mathbb{R}_+^n$, treated as linear inequality constraints.

We provide an example verifying Theorem 5.1.

Example 5.1 Consider the optimization problem:

\[
\begin{align*}
\inf_{x} \quad & 2x_1x_2 - 2x_1x_3 - 2x_2x_3 \\
\text{s.t.} \quad & \|x\|^2 - 1 \leq 0, \\
&\ -x_1 - x_2 + x_3 - 1 \leq 0, \\
&\ x = (x_1, x_2, x_3) \in \mathbb{R}_+^3.
\end{align*}
\]

Let $f(x) := 2x_1x_2 - 2x_1x_3 - 2x_2x_3$, $g_0(x) := \|x\|^2 - 1$, $g_1(x) := -x_1 - x_2 + x_3 - 1$. These functions can be written as $f(x) = x^T Ax + a^T x$, $g_0(x) = \|x - x_0\|^2 - \alpha$, and $g_1(x) = b_i^T x - \beta_1$, where $a = x_0 = 0 \in \mathbb{R}^3$, $b_1 = (-1, -1, 1)$, $\beta_1 = 1$, $\alpha = 1$, and $A := \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. So, $(E_5)$ is a special case of $(P_3)$. We see that

\[
H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} -1 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}.
\]

Therefore, the completely positive relaxation of $(E_5)$ can be written as follows

\[
\begin{align*}
\inf_{X \in \mathcal{P}} \quad & 2X_{23} - 2X_{24} - 2X_{34} \\
\text{s.t.} \quad & X_{22} + X_{33} + X_{44} \leq 1, \\
&\ -X_{12} - X_{13} + X_{14} \leq 1, \\
&\ X_{11} = 1,
\end{align*}
\]
while the copositive relaxation of \((E_5)\) is the following problem
\[
(CE^*_5) \quad \sup \{ y_0 : Z_+(y) \in C^*, \ y = (y_0, u) \in \mathbb{R} \times \mathbb{R}^2_+ \},
\]
where
\[
Z_+(y) := H + u_0 H_0 + u_1 H_1 - y_0 J_0 = \begin{pmatrix}
-u_0 - u_1 - y_0 & -u_1/2 & -u_1/2 & u_1/2 \\
-u_1/2 & u_0 + u_1 & 1 & -1 \\
-u_1/2 & 1 & u_0 & -1 \\
u_1/2 & -1 & -1 & u_0
\end{pmatrix}.
\]
Note that \(A \in S^3\) and \(\det(A - \lambda I_3) = -(\lambda + 1)^2(\lambda - 2)\). Hence,
\[
\lambda_{\min}(A) = -1 \quad \text{and ker}(A - \lambda_{\min}(A) I_3) = \{v \in \mathbb{R}^3 : v_1 + v_2 - v_3 = 0\}.
\]
For \(v := (1, 1, 2) \in \mathbb{R}^n_+ \setminus \{0\}\), we have
\[
\begin{cases}
Av - \lambda_{\min}(A)v = 0, \\
v \geq 0, \ b^Tv = 0, \\
(a + 2\lambda_{\min}(A)x) v \leq 0.
\end{cases}
\]
Furthermore, \(g_i(\tilde{x}) < 0\), \(i = 0, 1\), where \(\tilde{x} := (0, 0, 0)\). Thus, the assumption of Theorem 5.1 is fulfilled. According to Theorem 5.1, \(\min(E_5) = \max(CE_5) = \max(CE^*_5)\). We next provide a direct verification of this confirmation. We see that
\[
f(x) = 2x_1 x_2 - 2x_1 x_3 - 2x_2 x_3 = (x_1 + x_2 - x_3)^2 - (x_1^2 + x_2^2 + x_3^2) \geq - (x_1^2 + x_2^2 + x_3^2) \geq -1,
\]
for all \(x \in \mathbb{R}^3_+\) with \(g_0(x) \leq 0\) and \(g_1(x) \leq 0\). Moreover, \(f(\tilde{x}) = -1\) where \(\tilde{x} := (0, 1/\sqrt{2}, 1/\sqrt{2}) \in \mathbb{R}^3_+\) is feasible for \((E_5)\). This implies \(\min(E_5) = -1\). Let \(\tilde{x} = [1 0 1/\sqrt{2} 1/\sqrt{2}]^T \in \mathbb{R}_+^4\). If \(y = (y_0, u)\) is feasible for \((EC^*_5)\), then \(\tilde{x}^T Z_+(y) \tilde{x} = -u_0 - 1 - y_0 \geq 0, u_0 \geq 0, u_1 \geq 0\). This shows \(y_0 \leq 1\). Thus \(\sup(EC^*_5) \leq -1\). For \(\bar{y} := (-1, 1, 0) \in \mathbb{R} \times \mathbb{R}^2_+\), we have
\[
Z_+(\bar{y}) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 \\
0 & 1 & 1 & -1 \\
0 & -1 & -1 & 1
\end{pmatrix}.
\]
Let \(\hat{x} = (a, b, c, d) \in \mathbb{R}_+^4\). Simple calculations show that \(\hat{x}^T Z_+(\bar{y}) \hat{x} = (a + b - c)^2 \geq 0\). Therefore, \(Z_+(\bar{y}) \in C^*\). So, \(\bar{y}\) is a global optimal solution to \((CE^*_5)\) and \(\max(CE^*_5) = -1 = \min(E_5)\). The latter implies \(\inf(CE_5) = -1\), due to the fact that \(\max(CE^*_5) \leq \inf(CE_5) \leq \min(E_5)\). Furthermore, it is easy to verify that \(X = [1 0 1/\sqrt{2} 1/\sqrt{2}]^T [1 0 1/\sqrt{2} 1/\sqrt{2}]\) is a global optimal solution to \((CE_5)\) with cp-rank one. Consequently, \(\max(CE_5) = \min(E_5) = \min(E_5) = \max(E_5)\) and \(x^* = [0 1/\sqrt{2} 1/\sqrt{2}]^T\) is a global solution for \((E_5)\).
6 Nonnegative Quadratic Programs with Z-matrices

Consider the following nonconvex quadratic program with \( Z \)-symmetric matrices:

\[
(P_4) \quad \inf x^T A x + b^T x + c \\
\text{s.t. } x \in \mathbb{R}_+^n, \ x^T A_i x + b_i^T x + c_i \leq 0, \ i = 0, 1, \ldots, m,
\]

where \( A, A_i \in S^{n \times n} \) are \( Z \)-symmetric matrices, i.e., they are the symmetric matrices having non-positive entries except those in the diagonal, and \( b, b_i \in \mathbb{R}^n, c, c_i \in \mathbb{R}, i = 0, 1, \ldots, m \). The \( Z \)-matrix arises naturally in solving Dirichlet problem numerically, and plays an important role in the theory of quadratic optimization [16] and linear complementary problem [15]. In the sequel, let \( f(x) := x^T A x + b^T x + c \) and \( g_i(x) := x^T A_i x + b_i^T x + c_i, i = 0, 1, \ldots, m \).

The completely positive relaxation of \((P_4)\) is defined as follows

\[
(CP_4) \quad \inf \{ \text{Tr}(HX) \} \\
\text{s.t. } \text{Tr}(H_i X) \leq 0, \ i = 0, 1, \ldots, m, \\
\text{Tr}(J_0 X) = 1,
\]

while the copositive relaxation of \((P_4)\) is given by

\[
(CP_4^*) \quad \sup \{ y_0 : Z_+(y) \in \mathcal{C}^*, y = (y_0, u) \in \mathbb{R} \times \mathbb{R}_+^{m+1} \},
\]

where

\[
Z_+(y) := H + \sum_{i=0}^m u_i H_i - y_0 J_0 = \begin{pmatrix} c + \sum_{i=0}^m u_i c_i - y_0 & (a + \sum_{i=0}^m u_i a_i)^T / 2 \\ (a + \sum_{i=0}^m u_i a_i) / 2 & A + \sum_{i=0}^m u_i A_i \end{pmatrix},
\]

\[
H := \begin{pmatrix} c & b^T / 2 \\ b / 2 & A \end{pmatrix}, \ H_i = \begin{pmatrix} c_i & b_i^T / 2 \\ b_i / 2 & A_i \end{pmatrix}, \ i = 0, 1, \ldots, m.
\]

Lemma 6.1 Let \( Q_i \in S^n \) be \( Z \)-symmetric matrices, \( i = 1, \ldots, m \). Then the set

\[
\mathcal{A} := \{(x^T Q_1 x, \ldots, x^T Q_m x) : x \in \mathbb{R}_+^n \} + \text{int} \mathbb{R}_+^m
\]

is convex.

Proof. Take any \( u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m) \in \mathcal{A} \) and \( \lambda \in (0, 1) \). By definition, there exist \( x_u, x_v \in \mathbb{R}_+^n \) such that

\[
\begin{align*}
x_u^T Q_i x_u &< u_i, \\
x_v^T Q_i x_v &< v_i, \ s = 1, \ldots, m.
\end{align*}
\]

Consider the point \( w := \lambda u + (1 - \lambda)v \). By [16, Theorem 5.1], there exists \( \hat{x} \in \mathbb{R}^n \) such that

\[
\hat{x}^T Q_i \hat{x} < \lambda u_i + (1 - \lambda)v_i, \ i = 1, \ldots, m.
\]
Let \( x_w := (\hat{x}_1, ..., \hat{x}_n) \in \mathbb{R}^n_+ \) and let \( a_{rs}^i \) be the \((r, s)\)-entry of \( A_i \), \( i = 1, ..., m, r = 1, ..., n, s = 1, ..., n \). Then, taking into account that \( a_{rs}^i \leq 0 \) for all \( r \neq s \), we have
\[
x_w^T Q_i x_w = x_w^T A_i x_w \leq \sum_r a_{rr}^i \hat{x}_r^2 + \sum_{r \neq s} a_{rs}^i \hat{x}_r \hat{x}_s < \lambda u_i + (1 - \lambda)v_i.
\]
So, \( x_w^T Q_i x_w < \lambda u_i + (1 - \lambda)v_i \) for all \( i = 1, ..., m \). Therefore,
\[
w = (\lambda u_1 + (1 - \lambda)v_1, ..., \lambda u_m + (1 - \lambda)v_m) \in \mathcal{A}.
\]
This justifies the convexity of \( \mathcal{A} \).

In contrast to a consequence of Dine's theorem [16], even in the case \( m = 2 \), the set \( \mathcal{A} := \{(x^T Q_1 x, ..., x^T Q_m x) : x \in \mathbb{R}^n_+ \} + \text{int}\mathbb{R}^m_+ \) may be nonconvex if the Z-matrix assumption is removed.

**Example 6.1** Let \( Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \), \( Q_2 = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \), \( f_1(x) := x^T Q_1 x = x_1^2 + 2x_1x_2 - x_2^2 \) and \( f_2(x) := x^T Q_2 x = -2x_1^2 + 2x_1x_2 + x_2^2 \). Then, \( \mathcal{A} := \{(f_1(x), f_2(x)) : x \in \mathbb{R}^2_+ \} + \text{int}\mathbb{R}^2_+ \) is nonconvex. Indeed, if \( \mathcal{A} \) is convex, then since \((0, 0) \not\in \mathcal{A}\), by the convex separation theorem, there exist (\( \lambda_1, \lambda_2 \) \( \in \mathbb{R}^2 \setminus \{(0, 0)\} \)) such that \( \lambda_1 u + \lambda_2 v \geq 0 \) for all \((u, v) \in \mathcal{A} \). Consequently, \( (\lambda_1, \lambda_2) \in \mathbb{R}^2_+ \setminus \{(0, 0)\} \) and \( \lambda_1 f_1(x) + \lambda_2 f_2(x) \geq 0 \) for all \( x \in \mathbb{R}^2_+ \). This contradicts [13, Counterexample 1.2]. So, \( \mathcal{A} \) is nonconvex. The reason is that \( Q_i, i = 1, 2 \), are not Z-matrices.

**Theorem 6.1 (Exact copositive relaxation for \( (P_i) \) under Z-matrix structure).** For problem \((P_i)\), suppose that \( H_i \) are Z-matrices, and there exists \( \hat{x} \in \mathbb{R}^n_+ \) such that \( g_i(\hat{x}) < 0 \) for all \( i = 0, 1, ..., m \). Then, we have
\[
\inf(P_i) = \inf(CP_i) = \max(CP^*_i).
\]

**Proof.** Let \( g(x) := f(x) - \bar{\mu}, \tilde{g}(t, x) := (t, x^T) \tilde{A}(t, x^T)^T \) and \( \tilde{g}_i(t, x) := (t, x^T) \tilde{A}_i(t, x^T)^T \)
for \( (t, x) \in \mathbb{R} \times \mathbb{R}^n \), where \( \bar{\mu} := \inf(P_i) \), \( \tilde{A} := \begin{pmatrix} c - \bar{\mu} & b^T/2 \\ b/2 & A \end{pmatrix} \), and \( \tilde{A}_i := \begin{pmatrix} c_i & b_i^T/2 \\ b_i/2 & A_i \end{pmatrix} \). Since \( H_i \) are Z-matrices, \( \tilde{A} \) and \( \tilde{A}_i \), \( i = 1, ..., m \), are also Z-matrices. By Lemma 6.1, the set \( \tilde{A} := \{(\tilde{g}_0(t, x), \tilde{g}_1(t, x), ..., \tilde{g}_m(t, x), \tilde{g}(t, x)) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \} + \text{int}\mathbb{R}^{m+2}_+ \) is convex. We now show that there exists no \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \) such that \( \tilde{g}(t, x) < 0, \tilde{g}_i(t, x) < 0, i = 1, ..., m \). Indeed, if this is not true, one can find \((t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \) with \( \tilde{g}(t_0, x_0) < 0 \) and \( \tilde{g}_i(t_0, x_0) < 0 \) for all \( i = 0, 1, ..., m \). If \( t_0 > 0 \) then \( f(t_0^{-1} x_0) - \bar{\mu} = g(t_0^{-1} x_0) = t_0^{-2} \tilde{g}(t_0, x_0) < 0, g_i(t_0^{-1} x_0) = t_0^{-2} \tilde{g}_i(t_0, x_0) < 0, i = 0, 1, ..., m \), and \( t_0^{-1} x_0 \in \mathbb{R}^n_+ \). This is absurd, since \( \bar{\mu} := \inf(P_i) \). If \( t_0 = 0 \), then \( x_0^T A x_0 < 0 \) and \( x_0^T A_i x_0 < 0 \), \( i = 0, 1, ..., m \). Hence, \( \lim_{\alpha \to +\infty} f(\alpha x_0) = -\infty \), and for each \( i = 1, ..., m \), \( \lim_{\alpha \to +\infty} g_i(\alpha x_0) = -\infty \).

So, there exists some \( \alpha_0 > 0 \) such that \( f(\alpha_0 x_0) < 0 \) and \( g_i(\alpha_0 x_0) < 0 \) for all \( i = 0, 1, ..., m \). However, the latter leads to a contradiction, due to the facts that \( \alpha_0 x_0 \in \mathbb{R}^n_+ \) and \( \bar{\mu} := \inf(P_i) \). This shows that the system \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n_+, \hat{f}(t, x) < 0, \tilde{g}_i(t, x) < 0, i = 0, 1, ..., m, \)

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has no solution. Noting that \( \tilde{A} \) is convex, by the convex separation theorem, there exists \((\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_m, \bar{u}_{m+1}) \in \mathbb{R}_+^{m+2}\{0\} \) satisfying
\[
\sum_{i=0}^{m} \bar{u}_i \hat{g}_i(t, x) + \bar{u}_{m+1} \tilde{g}(t, x) \geq 0 \quad \text{for all} \ (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]
Thus, choosing \( t = 1 \), we see that there exists \((\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_m) \in \mathbb{R}_+^{m+2}\{0\} \) such that
\[
\sum_{i=0}^{m} \bar{u}_i g_i(x) + \bar{u}_{m+1} (f(x) - \tilde{\mu}) \geq 0 \quad \text{for all} \ x \in \mathbb{R}_+^n.
\]
So, by Slater condition, \( \bar{u}_{m+1} > 0 \) and thus we may assume \( \bar{u}_{m+1} = 1 \). Consequently, for \( \bar{u} := (\bar{u}_0, \ldots, \bar{u}_m) \), \( f(x) + \sum_{i=0}^{m} \bar{u}_i g_i(x) \geq \tilde{\mu} \) for all \( x \in \mathbb{R}_+^n \). This shows that \( \max(D_4) \geq \inf(P_4) \), where \( (D_4) \) denotes the semi-Lagrangian dual of \((P_4)\). Since \((P_4)\) satisfies the Slater condition, by [6, Theorem 4.3], \((CP_4)\) also satisfies the Slater condition. Thus, \((CP_4^*)\) has an optimal solution. Hence, by [6, Theorem 4.1], we obtain the desired conclusion. \( \square \)

Example 6.2 Consider the following quadratic programming problem:

\[
(E_6) \quad \begin{align*}
\text{inf} & \ -2x_1x_2 - x_1 - x_2 \\
\text{s.t.} & \ x_1^2 - 4x_1x_2 + 2x_2^2 - 2x_1 - x_2 - 1 \leq 0, \\
 & \ x_1^2 + x_2^2 - 1 \leq 0, \\
 & \ (x_1, x_2) \in \mathbb{R}_+^2.
\end{align*}
\]

Let \( f(x) := -2x_1x_2 - x_1 - x_2, g_0(x) := x_1^2 - 4x_1x_2 + 2x_2^2 - 2x_1 - x_2 - 1, \) and \( g_1(x) := x_1^2 + x_2^2 - 1. \) These functions can be rewritten as \( f(x) = x^T A x + b^T x + c, \) \( g_i(x) = x^T A_i x + b_i^T x + c_i, \) \( i = 0, 1, \) where \( b = (-1, -1), \) \( b_0 = (-2, -1), \) \( b_1 = (0, 0), \) \( c = 0, \) \( c_0 = -1, \) \( c_1 = -1, \) and \( A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \) \( A_0 = \begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}, \) \( A_1 = I_2. \) We see that
\[
H = \begin{pmatrix} 0 & -1/2 & -1/2 \\ -1/2 & 0 & -1 \\ -1/2 & -1 & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} -1 & -1 & -1/2 \\ -1 & -1 & -2 \\ -1 & -2 & 2 \end{pmatrix}, \quad \text{and} \quad H_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
are \( Z \)-matrices. This shows that \( (E_6) \) is a special case of \((P_4)\). The completely positive relaxation of \((E_6)\) is the following problem

\[
(CE_6) \quad \begin{align*}
\text{inf} & \ -X_{12} - X_{13} - 2X_{23} \\
\text{s.t.} & \ -1 - 2X_{12} - X_{13} + X_{22} - 4X_{23} + 2X_{33} \leq 0, \\
 & \ -1 + X_{22} + X_{33} \leq 0, \\
 & \ X_{11} = 1,
\end{align*}
\]
while the copositive relaxation of \((E_6)\) is the following problem

\[
(CE_6') \quad \sup \{ y_0 : Z_+(y) \in C^*, \ y = (y_0, u_0) \in \mathbb{R} \times \mathbb{R}_+^2 \},
\]

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where
\[ Z_+(y) := H + u_0H_0 + u_1H_1 - y_0J_0 \]
\[ = \begin{pmatrix}
- u_0 - u_1 - y_0 & -1/2 - u_0 & -1/2 - u_0/2 \\
-1/2 - u_0 & u_0 + u_1 & -1 - 2u_0 \\
-1/2 - u_0/2 & -1 - 2u_0 & 2u_0 + u_1 
\end{pmatrix}. \]

Since \( g_j(\hat{x}) < 0 \) for \( \hat{x} := (0, 0) \), and \( \inf(E_6) \) is finite, by Theorem 6.1,
\[ \inf(E_6) = \inf(CE_6) = \max(CE_6^e). \]

We next directly verify this conclusion. For any feasible point \( x = (x_1, x_2) \in \mathbb{R}^2 \) of \( (E_6) \), we see that
\[ f(x) = -2x_1x_2 - x_1 - x_2 \geq - (x_1^2 + x_2^2) - \sqrt{2}(x_1^2 + x_2^2) \geq - 1 - \sqrt{2}. \]

Furthermore, \( f(\bar{x}) = -1 - \sqrt{2} \) for \( \bar{x} := \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) feasible for \( (E_6) \). Thus, \( \min(E_6) = -1 - \sqrt{2} \).

Let \( \hat{x} := (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \) and let \( y = (y_0, u_0, u_1) \) be feasible for \( (CE_6^e) \). Then, we have \( \hat{x}^TZ_+(y)\hat{x} = -\frac{3}{2}(1 + \sqrt{2})u_0 - 1 - \sqrt{2} - y_0 \geq 0 \), \( u_0 \geq 0 \), \( u_1 \geq 0 \), which implies \( y_0 \leq -1 - \sqrt{2} \). Hence, \( \sup(CE_6^e) \leq -1 - \sqrt{2} \). For \( \bar{y} := (-1 - \sqrt{2}, 0, 1 + \frac{\sqrt{2}}{2}) \), we have
\[ Z_+(\bar{y}) = \begin{pmatrix}
\frac{\sqrt{2}}{2} & -1/2 & -1/2 \\
-1/2 + \frac{\sqrt{2}}{2} & 1 + \frac{\sqrt{2}}{2} & -1 \\
-1/2 & -1 & 1 + \frac{\sqrt{2}}{2} 
\end{pmatrix}. \]

For each \( \hat{x} = (a, b, c) \in \mathbb{R}_+^3 \), direct computation shows that
\[ \hat{x}^TZ_+(\bar{y})\hat{x} = \frac{\sqrt{2}}{2}a^2 - (b + c)a + (1 + \frac{\sqrt{2}}{2}))(b^2 + c^2) - 2bc \geq 0. \]

This implies \( Z_+(\bar{y}) \in \mathcal{C}^* \). Thus, \( \max(CE_6^e) = -1 - \sqrt{2} = \min(E_6) \). On the other hand, \( \max(CE_6^e) \leq \inf(C_6) \leq \min(E_6) \). So, we get the conclusion.

### 7 Conclusion

In this paper, we have provided various new classes of nonconvex quadratic optimization problems which enjoy exact copositive relaxation under suitable conditions. Our approach exploited a hidden convexity property of nonconvex quadratic inequality systems. The importance of the required convexity property of a quadratic optimization problem for exact copositive relaxation is illustrated by numerical examples. Our approach shows promise of extensions of exact copositive relaxation for quadratic optimization problems with conic constraints and for polynomial optimization problems. They will be investigated in a forthcoming study.
References

[1] L. Bai, J. E. Mitchell, J.-S. Pang, On conic QPCCs, conic QCQPs and completely positive programs, *Math. Program.* **159** (2016), 109-136.

[2] A. Beck, Quadratic matrix programming, *SIAM J. Optim.* **17** (2006), 1224-1238.

[3] A. Beck, On the convexity of a class of quadratic mappings and its application to the problem of finding the smallest ball enclosing a given intersection of balls, *J. Global Optim.* **39** (2007), 113-126.

[4] A. Beck, C. Yonina, Strong duality in nonconvex quadratic optimization with two quadratic constraints, *SIAM J. Optim.* **17** (2006), 844-860.

[5] I. M. Bomze, Copositive optimization-recent developments and applications, *European J. Oper. Res.* **216** (2012), 509-520.

[6] I. M. Bomze, Copositive relaxation beats Lagrangian dual bounds in quadratically and linearly constrained quadratic optimization problems, *SIAM J. Optim.* **25** (2015), 1249-1275.

[7] I. M. Bomze, M. Dür, E. de Klerk, C. Roos, A. Quist, T. Terlaky, On copositive programming and standard quadratic optimization problems, *J. Global Optim.* **18** (2000), 301-320.

[8] I. M. Bomze, V. Jeyakumar, G. Li, Extended trust region problems over one or two balls: exact (semi-)Langrangian relaxations, preprint, available at https://arxiv.org/abs/1702.08113.

[9] I. M. Bomze, W. Schachinger, Multi-standard quadratic optimization: interior point methods and cone programming reformulation, *Comput. Optim. Appl.* **45** (2010), 237-256.

[10] S. Burer, On the copositive representation of binary and continuous nonconvex quadratic programs, *Math. Program.* **120** (2009), 479-495.

[11] S. Burer, On the copositive programming. In: M. Anjos, J.-B. Lasserre (des.) Handbook on Semidefinite, Conic and Polynomial Optimization, *International Series in Operational Research and Management Science*. pp. 201-218. Springer, Berlin, 2012.

[12] S. Burer, A gentle, geometric introduction to copositive optimization, *Math. Program.* **151** (2015), 89-116.

[13] J.-P. Crouzeix, J.-E. Martnez-Legaz, A. Seeger, An alternative theorem for quadratic forms and extensions, *Linear Algebra Appl.* **215** (1995), 121-134.

[14] M. Dür, Copositive Programming - a Survey. Recent Advances in Optimization and its Applications in Engineering, 3-20, Springer, 2010.

[15] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, 1991, Cambridge University Press, United Kingdom.

[16] V. Jeyakumar, G. M. Lee and G. Li, Alternative theorems for quadratic inequality systems and global quadratic optimization, *SIAM J. Optim.* **20** (2009), 983-1001.

[17] V. Jeyakumar, G. Li, Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization, *Math. Program.* **147** (2014), 171-206.
[18] V. Jeyakumar, G. Li, Exact second-order cone programming relaxations for some minimax quadratic optimization problems, submitted to *SIAM J. Optim.* (revision under review).

[19] V. Jeyakumar and H. Wolkowicz, Zero duality gaps in infinite-dimensional programming, *J. Optim. Theory Appl.* 67 (1990), 87-108.

[20] A. J. Quist, E. De Klerk, C. Roos, T. Terlaky, Copositive relaxation for general quadratic programming, *Optim. Methods Softw.* 9 (1998), 185-208.