Non-formally integrable centers admitting an algebraic inverse integrating factor

A. Algaba, N. Fuentes, C. García, M. Reyes

Department of Mathematics. Faculty of Experimental Sciences.
Avda. Tres de Marzo s/n, 21071 Huelva, Spain

Abstract

We study the existence of a class of inverse integrating factor for a family of non-formally integrable systems, in general, whose lowest-degree quasi-homogeneous term is a Hamiltonian vector field. Once the existence of an inverse integrating factor is established, we characterize the systems having a center. Among others, we characterize the centers of the systems whose lowest-degree quasi-homogeneous term is \((-y^3, x^3)^T\) with an algebraic inverse integrating factor.

Keywords: Nonlinear differential systems, Inverse integrating factor, Integrability problem, Degenerate center problem

1. Introduction and statement of the main results.

One of the classic problems in the qualitative theory of the planar analytic systems is to characterize when a monodromic point (singular point which is surrounded by orbits of the system) is a center or a focus. This problem, so-called center problem, has been solved theoretically for a nondegenerate singular point (systems whose linear part evaluated at singular point has two imaginary eigenvalues non-zero) and for the nilpotent case. Nowadays, the problem remains still unsolved for the remaining case, i.e. the systems with linear part identically zero at singular point, so-called degenerate singular point.

One of the main tools used for characterizing the nondegenerate and nilpotent centers has been the computation of a normal form, see Poincaré [15], Moussu [12]. It is not strange to think that a possible solution might be given by means of the theory of normal forms for the degenerate case.

Another problem related to the center problem is, once the monodromy is established, to determine the existence of an analytic first integral. So, for instance, for a nondegenerate singular point, the analytic integrability and center problems are equivalent. Otherwise, the existence of a first integral is a sufficient condition but it is not necessary for the singular point to be a center.

Email address: column@ahu.es (M. Reyes)
In this context, the existence of an integrating factor or an inverse integrating factor enable us to provide information about both center and integrability problems.

For more details about the relevance of the presence of an inverse integrating factor in a neighborhood of a singular point see [8, 9, 10] and references therein.

In this paper mainly we focus on the problem of characterizing, by means of the theory of normal forms, when a system has an inverse integrating factor in a neighborhood of the singular point. Once the existence of an inverse integrating factor and the monodromy of the origin have been established, we determine if the origin is either a center or a focus.

We consider an autonomous system
\[ \dot{x} = F(x) = (P(x), Q(x))^T, \quad x \in \mathbb{C}^2, \] (1)

where \( F \) is a formal planar vector field defined in a neighborhood of the origin \( U \subset \mathbb{C}^2 \) having a singular point at the origin, i.e., \( F(0) = 0 \) and \( P, Q \in \mathbb{C}[[x, y]] \) (algebra of the power series in \( x \) and \( y \) with coefficient in \( \mathbb{C} \)).

A non-null \( C^1 \) class function \( V \) is an inverse integrating factor of system (1) (or also of \( F \)) on \( U \) if satisfies the linear partial differential equation
\[ L_F V = \text{div}(F)V, \]
being \( L_F V := P \partial V/\partial x + Q \partial V/\partial y \), the Lie derivative of \( V \) respect to \( F \), and \( \text{div}(F) := \partial P/\partial x + \partial Q/\partial y \), the divergence of \( F \). This name for \( V \) comes from the fact that \( V^{-1} \) defines on \( U \setminus \{ V = 0 \} \) an integrating factor of system (1), i.e. \( F/V \) is divergence-free. So, if system (1) has an formal inverse integrating factor \( V \) then it is formally integrable on \( U \setminus \{ V = 0 \} \). For more details about the relation between the integrability and the inverse integrating factor see [6, 7].

We are interested in characterizing degenerate systems which have an algebraic inverse integrating factor over \( \mathbb{C}((x, y)) \) (which will be named AIIF) where \( \mathbb{C}((x, y)) \) denotes the quotient field of the algebra of the power series \( \mathbb{C}[[x, y]] \). In this sense, the only results we know are Walcher [14] where is claimed its existence for non-degenerate cusp nilpotent singularity, and Albega et. al. [4] where is characterized all nilpotent systems having an AIIF.

Given \( t = (t_1, t_2) \) non-null with \( t_1 \) and \( t_2 \) non-negative integer numbers without common factors, we will denote by \( \mathcal{P}^t_k \) to the vector space of quasi-homogeneous polynomials of type \( t \) and degree \( k \), i.e.
\[ \mathcal{P}^t_k = \{ f \in \mathbb{C}[x, y] : f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y) \}, \]
and by
\[ \mathcal{Q}^t_k = \{ F = (P, Q)^T : P \in \mathcal{P}^t_{k+t_1}, \ Q \in \mathcal{P}^t_{k+t_2} \} \]
to the vector space of the quasi-homogeneous polynomial vector fields of type \( t \) and degree \( k \). Any vector field can be expanded into quasi-homogeneous terms of type \( t \) of successive degrees. Thus, the vector field \( F \) can be written in the form
\[ F = F_r + F_{r+1} + \cdots, \]
for some \( r \in \mathbb{Z} \), where \( \mathbf{F}_j = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j \) and \( \mathbf{F}_r \neq 0 \). Such expansions will be expressed as
\[
\mathbf{F} = \mathbf{F}_r + \text{q.h.h.o.t.,}
\]
where “q-h.h.o.t.” means “quasi-homogeneous higher order terms.”

If we select the type \( t = (1, 1) \), we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees.

Given \( h \in \mathcal{P}_{r+|t|} \), we define the linear operator
\[
\ell_j : \mathcal{P}_{j-r} \longrightarrow \mathcal{P}_j
\]
\[
\mu_{j-r} \longrightarrow \ell_j(\mu_{j-r}) := \frac{\partial h}{\partial x} \frac{\partial \mu_{j-r}}{\partial y} - \frac{\partial h}{\partial y} \frac{\partial \mu_{j-r}}{\partial x},
\]
(2)
(Poisson bracket of \( h \) and \( \mu_{j-r} \)) and denote by \( \text{Cor}(\ell_j) \) a complementary subspace to the range of the linear operator \( \ell_j \).

We also define \( \mathcal{P}_{r+|t|} \) as the set of all \( h \in \mathcal{P}_{r+|t|} \) satisfying:

H1 the factorization of \( h \) on \( \mathbb{C}[x, y] \) has only simple factors,

H2 \( h \mathcal{P}_{r+|t|} \) is a complementary subspace to the range of \( \ell_{r+|t|+j} \) for all \( j \).

In this paper, fixed \( h \in \mathcal{P}_{r+|t|} \), we deal with the systems of the form
\[
\dot{x} = X_h + \text{q.h.h.o.t.},
\]
(3)
where \( X_h := (-\partial h / \partial y, \partial h / \partial x)^T \in \mathcal{Q}_r \), i.e. a class of systems which can be considered as perturbations of a Hamiltonian system whose Hamiltonian function \( h \) is a quasi-homogeneous function.

This class of systems is a wide family and contains, among others, to the non-degenerate saddle \( (h = xy) \), linear center \( (h = x^2 + y^2) \), the nilpotent systems of the form \( (\dot{x}, \dot{y}) = (y, ax^n)^T + \text{q.h.h.o.t} \) with \( a \neq 0 \) \( (h = \frac{n}{n+1}x^{n+1} - \frac{y^2}{2}) \).

There are two main reasons for imposing that \( h \) belongs to \( \mathcal{P}_{r+|t|} \). On the one hand, if \( \text{H1} \) holds, a cyclicity of the co-ranges of the operators \( \ell_j \) appears. Concretely,
\[
\text{Cor}(\ell_{j+r+|t|}) = h \text{Cor}(\ell_j), \quad \text{for all } j > r \text{ with } \mathcal{P}_{j-r} \neq \{0\},
\]
(4)
see \( \text{[3]} \). Algaba et. al. \( \text{[2]} \) provide an orbital equivalent normal form up any order for the system \( \text{[3]} \). This normal form is
\[
\dot{x} = X_h + X_g + \mu D_0,
\]
(5)
(we have denoted \( D_0 := (t_1 x, t_2 y)^T \in \mathcal{Q}_0 \) being \( g = \sum_{j \geq 1} g_{r+|t|+j} \in \text{Cor}(\ell_{r+|t|+j}) \setminus h \mathcal{P}_{r+|t|+j} \) for \( j \leq r \) or \( j > r \) such that \( \mathcal{P}_{j-r} = \{0\} \), and \( \mu = \sum_{j > r} \mu_j, \mu_j \in \text{Cor}(\ell_j) \).

Moreover, if \( \mu_j \equiv 0 \), for all \( j \), then system \( \text{[3]} \) is formally orbital equivalent to a Hamiltonian system and, in such case, it is a formally integrable system. Otherwise, from Algaba et al. \( \text{[3]} \), the system is non-formally integrable.
On the other hand, the condition $H2$ on $h$ implies that $g \equiv 0$, i.e. in this paper we limit to studying the systems whose normal form is a perturbation of a Hamiltonian vector field with dissipative vector fields. The following theorem summarizes the above results.

**Theorem 1** ([2, 3]). We consider system (3) with $h \in \mathcal{F}_{r+|t|}$. It holds that:

1. System (3) is formally orbital equivalent to $\dot{x} = X_h + \mu D_0$, with $\mu = \sum_{j>r} \mu_j$ with $\mu_j \in \text{Cor}(\ell_j)$.
2. System (3) is formally integrable if and only if it is formally orbital equivalent to $\dot{x} = X_h$.

The main result of this paper is stated in the next theorem.

**Theorem 2.** System (3) with $h \in \mathcal{F}_{r+|t|}$ has an AIIF (algebraic inverse integrating factor over $\mathbb{C}((x, y))$) if and only if it is formally orbital equivalent either to $\dot{x} = X_h$ (formally integrable system) or to system (6)

$$
(\dot{x}, \dot{y})^T = X_h + \mu_{r+N} D_0,
$$

with $N$ a natural number and $\mu_{r+N} \in \text{Cor}(\ell_{r+N}) \setminus \{0\}$ (non-formally integrable system). Moreover, the AIIF is $(h + g\cdot h, \text{h.o.t.})^{1+N/(r+|t|)}$, up to a multiplicative constant.

As a consequence, it has the main result of Algaba et al. [2].

**Corollary 1.** Under the conditions of Theorem 2, system (3) has a formal inverse integrating factor (it belongs to $\mathbb{C}[[x, y]]$, algebra of the power series in $x$ and $y$ with coefficient in $\mathbb{C}$) if and only if it is formally orbital equivalent either to $\dot{x} = X_h$ (formally integrable system) or to system (2) with $N$ a multiple of $r + |t|$ (non-formally integrable system).

**Remark 1.** From [3, Theorem 3.19], system (3) is formally integrable if and only if it is formally orbital equivalent to $\dot{x} = X_h$, i.e. there exist a diffeomorphism $\Phi$ and a function $\eta$ on $U \subset \mathbb{C}^2$ with $\det D\Phi$ has no zero on $U$ and $\eta(0) \neq 0$, such that $\Phi_\ast(\eta F) = X_h$, where we have denoted as $\Phi_\ast$ to the push-forward defined by $\Phi$. As $X_h$ is a Hamiltonian vector field, $f(h)$ is a first integral for any $f$ non-constant. In particular, it is an inverse integrating factor. So, the pull-back $\Phi^\ast$ brings $f(h)$ to the inverse integrating factor of system (3), $V = f(h + \cdots + \cdots)$, i.e. it is not unique. Also, if $f(0) \neq 0$, $V$ would be a formal inverse integrating factor with $V(0, 0) \neq 0$.

We study the monodromic and center problems of system (3). For the monodromy problem, it has the following result.

**Proposition 3.** The origin of system (3) with $h \in \mathcal{F}_{r+|t|}$ is monodromic if and only if $h$ is only zero at the origin.
We note that if the origin is a monodromic point and the system is formally integrable, then the origin is a center. Last on, we state the result which gives title to this work where it characterizes the centers of the non-formally integrable systems \( (3) \) having an AIIF.

**Theorem 4.** We assume that the origin of system \( (3) \) with \( h \in \mathbb{T}_{r+1+j} \) is monodromic and it is formally orbital equivalent to the non-formally integrable system \( (6) \). Then, the origin is:

1. a center, if \( I = 0 \),
2. an unstable focus, if \( \text{sig}(h)I > 0 \),
3. a stable focus, if \( \text{sig}(h)I < 0 \),

being \( I = \int_{h=\text{sig}(h)} \mu_{r+1+j} \).

2. Some examples and applications

In this section we show several families of systems \( (3) \) with \( h \in \mathbb{T}_{r+1+j} \) where the origin is or not monodromic. For the non-monodromic case, we determine the systems with an AIIF. For the monodromic case, we also characterize the centers admitting an AIIF.

In order to determine if a quasi-homogeneous function holds the condition \( H_2 \), we need to describe the sets \( P_k^t \) of quasi-homogeneous polynomials according the type \( t = (t_1, t_2) \). The following result provides bases for these spaces.

**Lemma 5.** Fixed \( t = (t_1, t_2) \), it has that:

1. \( P_0^t = \text{span}\{1\} \).
2. if \( t_1 = 1 \), for every \( t_2 \geq 1 \), the sets \( P_k^t \) are non-trivial spaces for all \( k \),
3. \( P_k^t = \{0\} \), if \( k \notin I^t \),
4. if \( k > t_1 t_2 - |t| \), then \( k \in I^t \), i.e. \( P_k^t \) is a non-trivial space.
5. \( P_k^t = \text{span}\{x^{k_1+i_2(k_3-j)}y^{k_2+1+j} : j = 0, \ldots, k_3\} \), if \( k \in I^t \setminus \{0\} \),

being \( I^t = \{k = k_1 t_1 + k_2 t_2 + k_3 t_1 t_2 \in \mathbb{N} : k_1, k_2, k_3 \in \mathbb{N}, k_1 < t_2, k_2 < t_1\} \).

Table 1 shows the sets \( \mathbb{N} \setminus I^t \), that is, the degrees \( l \) such that \( P_l^t \) is a trivial set, for \( t_2 \leq 5 \).

| \mathbb{N} \setminus I^{(1, t_2)} | \mathbb{N} \setminus I^{(2, 3)} | \mathbb{N} \setminus I^{(2, 5)} | \mathbb{N} \setminus I^{(3, 4)} | \mathbb{N} \setminus I^{(3, 5)} |
|---|---|---|---|---|
| \{1\} | \{1\} | \{1, 3\} | \{1, 2, 5\} | \{1, 2, 4, 7\} |
| \{1, 2, 3, 6, 7, 11\} |

**Remark 2.** By \( (4) \), the condition \( H_2 \), by assuming \( H_1 \), is equivalent to \( hP_j^t \) is a complementary subspace to the range of \( \ell_{r+1+j} \) for all \( j \leq r \), or \( j > r \) satisfying \( P_{j-r}^t = \{0\} \), that is, \( H_2 \) holds if it satisfies a finite number of conditions.
Remark 3. From above lemma, the number of trivial spaces $\mathcal{P}_{t}$ is a finite number, and by (4), only it is enough the computation of a certain number of co-ranges, concretely, from $r + 1$ to $n_0 + r + |t| - 1$ (with $n_0 := 1 + r$ if $\mathbb{N} \setminus \mathcal{T}$ is an empty set, or $n_0 := 1 + r + \max\{\mathbb{N} \setminus \mathcal{T}\}$, otherwise) for obtaining the normal form of (3). So, if $h \in \mathcal{P}_{r+|t|}$, the normal form (3) provided in Algaba et. al. [2] is

\[ \dot{x} = X_h + \sum_{j=r+1}^{n_0+r+|t|-1} \eta_j^{(0)} D_0 + \sum_{i=1}^{\infty} \sum_{j=0}^{r+|t|-1} \eta_j^{(i)} h^i D_0, \tag{7} \]

with $\eta_j^{(i)} \in \text{Cor}(\ell_j)$. Moreover $r + 1 \leq n_0 \leq r + 1 + \max\{0, t_1 t_2 - |t|\}$.

A) Perturbations of Hamiltonian quadratic systems. These systems can be written as

\[ (\dot{x}, \dot{y})^T = X_h + q \text{- h.o.t. } h = ax^3 + bx^2 y + cxy^2 + dy^3. \tag{8} \]

That is, $t = (1, 1)$ and $r = 1$.

From Proposition 3, the origin of these systems is non-monodromic. We focus on our study in characterizing the systems (8) with an AIIF.

For $d \neq 0$, without loss of generality, we can assume $c = 0$ and $d = 1$, the polynomial $h$ has only simple factors if $27a^2 + 4b^3 \neq 0$, and by Lemma 5, the sets $\mathcal{P}_j$ are non-trivial spaces for all $j$. Table 2 shows the range and co-range of the operator $\ell_j$, $j = 2, 3, 4$ for system (8) with $d \neq 0$. It is easy to check that $h \in \mathcal{P}_3^{(1,1)}$.

| $\ell_2$ | $\text{Range}(\ell_2) = \text{span}\{-bx^2 - 3y^2, 3ax^2 + 2bxy\}$ |
|---------|---------------------------------------------------------------|
|         | If $a \neq 0$, $\text{Cor}(\ell_2) = \text{span}\{xy\}$. If $a = 0$, $\text{Cor}(\ell_2) = \text{span}\{x^2\}$ |
| $\ell_3$ | $\text{Range}(\ell_3) = \text{span}\{-2bx^3 - 6xy^2, 6ax^3 + 4bx^2 y - 3h, 6ax^2 y + 4bxy^2\}$ |
|         | $\text{Cor}(\ell_3) = \text{span}\{h\}$ |
| $\ell_4$ | $\text{Range}(\ell_4) = \text{span}\{3bx^4 + 9x^2 y^2, -9ax^4 - 6bx^3 y + 6xh, -9ax^3 y - 6bx^2 y^2 + 3yh\}$ |
|         | $\text{Cor}(\ell_4) = \text{span}\{xh, yh\}$ |

The normal form (7) of system (8) becomes

\[ (\dot{x}, \dot{y})^T = (-bx^2 - 3y^2, 3ax^2 + 2bxy)^T + \sum_{j=0}^{n_0+r+|t|} f_j(x, y, h) h^i D_0, \tag{9} \]

with $D_0 = (x, y)^T$ and $f_j \in \text{span}\{h, xh, yh, xyh\}$ if $a \neq 0$, or $f_j \in \text{span}\{h, xh, yh, x^2 h\}$ if $a = 0$.

Applying Theorem 2 we get the following result.
Theorem 6. System (5) has an AIIF if and only if is formally orbital equivalent to one of the following systems:

1. $\dot{x} = X_h$. It admits an AIIF of the form $g(h+q\cdot h.o.t.)$ with $g$ any nonzero function. In particular, there are inverse integrating factors nonzero at the origin.

2. $\dot{x} = X_h + \alpha_3 h^j D_0$, $\alpha_3 \neq 0, j \geq 1$. The AIIF is $(h + q\cdot h.o.t.)^{j+2/3}$.

3. $\dot{x} = X_h + (\alpha_{3j+1}^x + \beta_{3j+1}^y) h^j D_0$, $(\alpha_{3j+1}, \beta_{3j+1}) \neq (0,0), j \geq 1$. The AIIF is $(h + q\cdot h.o.t.)^{j+2}$, i.e. it is a formal inverse integrating factor.

4. $\dot{x} = X_h + \alpha_{3j+2} xy^2 h^j D_0$ if $a \neq 0$, or $\dot{x} = X_h + \alpha_{3j+2} x^2 h^j D_0$ if $a = 0$ with $\alpha_{3j+2} \neq 0, j \geq 1$. The AIIF is $(h + q\cdot h.o.t.)^{j+4/3}$.

B) Perturbations of nilpotent Hamiltonian systems. We consider the nilpotent systems whose quasi-homogeneous expansion is of the form

$$\begin{align*}
(\dot{x}, \dot{y})^T & = (y, sx^n)^T + q\cdot h.o.t. \quad \sigma = \pm 1.
\end{align*}
$$

From Proposition 3 the origin is not monodromic if and only if $n$ even, or $n$ odd and $\sigma = 1$.

Algaba et al. [2] give the following result, by characterizing the systems (11) which admit an AIIF.

Theorem 7. System (11) has an AIIF if and only if it is formally orbital equivalent to

$$\begin{align*}
(\dot{x}, \dot{y})^T & = (y, sx^n)^T + \alpha_M^{(L)} x^M h^L f(h) D_0,
\end{align*}
$$

with $h = 2sx^{n+1} - (n+1)y^2$, $D_0 = (2x, (n+1)y)^T$, $\alpha_M^{(L)}$ a real number, $f$ a function with $f(0) = 1$, $L$ a non-negative integer, and $M \in \{0,1,\ldots,n-1\}$ if $L > 0$ or $M \in \lfloor (n+1)/2 \rfloor, \ldots, n-1\}$ if $L = 0$.

Moreover, if $\alpha_M^{(L)} \neq 0$, then the system (11) is not formally integrable, and if it admits an AIIF, the AIIF is $(h + q\cdot h.o.t.) \frac{2(M+n+3)}{2(n+1)} + L$, up to a multiplicative constant. Otherwise, if $\alpha_M^{(L)} = 0$, system (11) is formally integrable.

If $n$ is even, system (11) has a formal inverse integrating factor if and only if $\alpha_M^{(L)} = 0$, since otherwise the number $\frac{2(M+n+3)}{2(n+1)}$ is non-integer and hence the inverse integrating factor is not formal. Therefore, as a consequence of Theorems 6 and 7 it has the following result provided in Algaba et al. [3].

Theorem 8. System (11) with $n$ even, has a formal inverse integrating factor if and only if it is formally integrable.

Now, we analyze the center problem for system (11) admitting an AIIF. We assume that the origin is monodromic, i.e. $n$ odd ($n = 2m - 1$) and $\sigma = -1$. These systems are

$$\begin{align*}
(\dot{x}, \dot{y})^T & = (y, -x^{2m-1})^T + q\cdot h.o.t., \quad m \geq 1,
\end{align*}
$$

The first quasi-homogeneous term of the right-hand side of (12) is $X_h \in Q_{m-1}$ with $t = (1, m)$, $h = \frac{1}{2m} x^{2m} + \frac{1}{2} my^2 \in P_{2m}$.

We get the following result which characterizes the centers of the systems (12) having an AIIF.

7
Theorem 9. We assume that system (12) has an AIIF. Then, the origin is a center if and only if it is formally orbital equivalent to a system invariant to the symmetry \((x, y, t) \rightarrow (-x, y, -t)\).

**Proof of Theorem 9.** From Theorem 7 if system (12) has an AIIF then it is formally orbital equivalent either to \((\dot{x}, \dot{y})^T = (y, -x^{2m-1})^T\) which is a center, or to
\[
(\dot{x}, \dot{y})^T = (y, -x^{2m-1})^T + Ax^M h^L f(h)D_0,
\]
(13)
with \(D_0 = (x, my)^T\), \(A\) a real number non-zero, \(f\) a function con \(f(0) = 1\), \(L\) a non-negative integer, and \(M \in \{0, 1, \ldots, 2m - 2\}\) if \(L > 0\) or \(M \in \{m, m + 1, \ldots, 2m - 2\}\) if \(L = 0\).

By applying Theorem 2, we obtain a further reduction of the normal form (13) of system (12), it which consists in assuming \(f(h)\) identically one.

In order to get the centers, it is enough to compute the integral \(I\) given by Theorem 4. In this case \(I = A\int_0^T C s^M(\theta) d\theta\), where \((Cs(\theta), Sn(\theta))^T\) is the solution of the initial value problem
\[
\frac{dx}{d\theta} = X_h(x), \quad x(0) = (1, 0)^T,
\]
and \(T\) is a minimal period of both functions.

It is known that the integral \(I\) is different from zero if and only if \(M\) even. So, we arrive to a system invariant to the symmetry \((x, y, t) \rightarrow (-x, y, -t)\).

The sufficient condition is trivial. □

We study the form of the AIIF’s of system (12). For \(n = 2m - 1\), the number \(2M + n + 3\) \(2n+1\) is natural if \(M = (2k - 1)m - 1\) with \(k\) natural. By imposing that \(M \leq 2m - 2\), it has that \(k = 1\) and \(M = m - 1\). So, we have the following result.

**Theorem 10.** The origin of the system (12) is a non-formally integrable center admitting a formal inverse integrating factor if and only if system (13) is formally orbital equivalent to
\[
(\dot{x}, \dot{y})^T = (y, -x^{4k-1})^T + Ax^{2k-1} h^L D_0, \quad L \geq 1, \quad A \neq 0.
\]
(14)
Consequently, the centers of systems (12) having an AIIF, are formally orbital equivalent to time-reversible systems but no all of them have a formal inverse integrating factor.

C) **Quadratic nilpotent generalized systems.** We consider the degenerate systems of the form
\[
(\dot{x}, \dot{y})^T = (y^2 + \sum_{j \geq 3} P_j(x, y), \sum_{j \geq 3} Q_j(x, y))^T,
\]
(15)
with \(P_j\) and \(Q_j\) homogeneous polynomials of degree \(j\) and \(Q_3(1, 0) \neq 0\) (without loss of generality, we can assume \(Q_3(1, 0) = 1\)). We write \(P_j(x, y) =\)
\[ \sum_{j=m+n} a_{mn} x^m y^n, \quad Q_j(x,y) = \sum_{j=m+n} b_{mn} x^m y^n. \] The quasi-homogeneous expansion with respect to \( t = (3,4) \) of system (15) is of the form
\[ (\dot{x}, \dot{y})^T = (y^2, x^3)^T + q\text{-h.o.t.}, \] i.e., system (3) for \( r = 5, \quad h = x^4/4 - y^3/3. \)

From Proposition 3, the origin of these systems is non-monodromic since \( h \) does not preserve the sign. So, we focus on our study in characterizing the systems (8) with an AIIF.

Note that \( h \) has only simple factors, \( n_0 = 11 \) (see Table 1). Table 3 shows the range and co-range of \( \ell_j \) for \( 6 \leq j \leq 22 \) and \( j \in \mathbb{N} \setminus \mathbb{I}_t \).

Table 3: Range and co-range of operator \( \ell_j \) for system (15)

\[
\begin{align*}
\text{Range}(\ell_6) &= \text{span}\{0\}, \quad \text{Cor}(\ell_6) = \text{span}\{x^2\} \\
\text{Range}(\ell_7) &= \text{span}\{0\}, \quad \text{Cor}(\ell_7) = \text{span}\{xy\} \\
\text{Range}(\ell_8) &= \text{span}\{y^2\}, \quad \text{Cor}(\ell_8) = \text{span}\{0\} \\
\text{Range}(\ell_9) &= \text{span}\{x^3\}, \quad \text{Cor}(\ell_9) = \text{span}\{0\} \\
\text{Range}(\ell_{10}) &= \text{span}\{0\}, \quad \text{Cor}(\ell_{10}) = \text{span}\{x^2y\} \\
\text{Range}(\ell_{11}) &= \text{span}\{xy^2\}, \quad \text{Cor}(\ell_{11}) = \text{span}\{0\} \\
\text{Range}(\ell_{12}) &= \text{span}\{7x^4 - 12h\}, \quad \text{Cor}(\ell_{12}) = \text{span}\{h\} \\
\text{Range}(\ell_{13}) &= \text{span}\{x^3y\}, \quad \text{Cor}(\ell_{13}) = \{0\} \\
\text{Range}(\ell_{14}) &= \text{span}\{x^2y^2\}, \quad \text{Cor}(\ell_{14}) = \{0\} \\
\text{Range}(\ell_{15}) &= \text{span}\{x^3 - 6xh\}, \quad \text{Cor}(\ell_{15}) = \text{span}\{xh\} \\
\text{Range}(\ell_{16}) &= \text{span}\{11x^4y - 12yh\}, \quad \text{Cor}(\ell_{16}) = \text{span}\{yh\} \\
\text{Range}(\ell_{17}) &= \text{span}\{x^3y^2\}, \quad \text{Cor}(\ell_{17}) = \{0\} \\
\text{Range}(\ell_{18}) &= \text{span}\{13x^6 - 36x^2h\}, \quad \text{Cor}(\ell_{18}) = \text{span}\{x^2h\} \\
\text{Range}(\ell_{19}) &= \text{span}\{7x^5 - 12xyh\}, \quad \text{Cor}(\ell_{19}) = \text{span}\{xyh\} \\
\text{Range}(\ell_{22}) &= \text{span}\{17x^6y - 9x^2yh\}, \quad \text{Cor}(\ell_{22}) = \text{span}\{x^2yh\}
\end{align*}
\]

As above, we observe that \( h \in \mathcal{G}_{12}^{(3,4)}. \) So, the normal form (7) of system (15) becomes
\[ (\dot{x}, \dot{y})^T = (y^2, x^3)^T + \sum_{j \geq 0} f_j(x, y, h) h^j \mathbf{D}_0, \] with \( \mathbf{D}_0 = (3x, 4y)^T \) and \( f_j \in \text{span}\{x^2, xy, x^2y, h, xh, yh\}. \)

As a consequence of Theorem 2, we get the following result which characterizes the systems (15) with an AIIF.
Theorem 11. System (15) has an AIIF if and only if, it is formally orbital equivalent to one of the following systems:

1. \( \dot{x} = X_h \). The AIIF is \( g(h + q\text{-h.o.t.}) \) with \( g \) any nonzero function (in particular, there are inverse integrating factors nonzero at the origin).
2. \( \dot{x} = X_h + \alpha_{12j} + 6x^3h\mathcal{D}_0 \). The AIIF is \( (h + q\text{-h.o.t.})^{1+j+1/12} \).
3. \( \dot{x} = X_h + \alpha_{12j+7}xy^2h\mathcal{D}_0 \). The AIIF is \( (h + q\text{-h.o.t.})^{1+j+5/12} \).
4. \( \dot{x} = X_h + \alpha_{12j+10}x^2yh\mathcal{D}_0 \). The AIIF is \( (h + q\text{-h.o.t.})^{1+j+7/12} \).
5. \( \dot{x} = X_h + \alpha_{12j+11}xh^3\mathcal{D}_0 \). The AIIF is \( (h + q\text{-h.o.t.})^{1+j+11/12} \).
6. \( \dot{x} = X_h + \alpha_{12j+14}y^3\mathcal{D}_0 \). The AIIF is \( (h + q\text{-h.o.t.})^{1+j+11/12} \),

with \( \alpha_k \neq 0 \) and \( j \geq 0 \).

We claim that the AIIF’s of the non-formally integrable systems (15) are algebraic but no formal. Consequently, we get the following result.

Proposition 12. System (15) is formally integrable if and only if it admits a formal inverse integrating factor.

Next, we give necessary conditions for the existence of an AIIF for system (15). The first two coefficients of the right-hand side of (17) are

\[
\begin{align*}
\alpha_6 &= 3a_{30} + b_{21}, \\
\alpha_7 &= 13(a_{21} + b_{12}) + (3a_{30} + b_{21})(4a_{30} - 3b_{21}).
\end{align*}
\]

These coefficients of the quasi-homogeneous normal form have been obtained by using the procedure given in Algaba et al. (1).

From Theorem 11, we deduce the following result.

Proposition 13. System (15) with \( 3a_{30} + b_{21} \neq 0 \) is not formally integrable. Moreover, if it has an AIIF, then \( 13(a_{21} + b_{12}) + (3a_{30} + b_{21})(4a_{30} - 3b_{21}) = 0 \) and the AIIF is equal to \( (4y^3 - 3x^3 + q - h\text{-h.o.t.})^{13/12} \exp(u) \), for some series \( u \) which is unique up to an additive constant.

We study a particular case of systems (15). We consider the family of systems (15) with \( P_j = Q_j \equiv 0 \) for \( j > 3 \), and \( P_3(1,0) = 0 (a_{30} = 0) \), that is,

\[
(\dot{x}, \dot{y})^T = (y^2, x^3)^T + (a_{21}y^2 + a_{12}xy^2 + a_{03}y^3, b_{21}x^2 + b_{12}xy^2 + b_{03}y^3)^T. \tag{20}
\]

we get the following result.

Proposition 14. We assume that system (20) has an AIIF. It has that:

1. if \( b_{21} \neq 0 \), then \( 13(a_{21} + b_{12}) = 3b_{21}^2 \), (non-formally integrable case),

2. if \( b_{21} = 0 \), then system (20) has a formal inverse integrating factor (integrable case).

Moreover, in such a case, system (20) is one of the following systems

(a) \( b_{21} = a_{21} + b_{12} = a_{12} + 3b_{03} = 0 \), (Hamiltonian case).
(b) \( a_{21} = a_{03} = b_{21} = b_{12} = 0, a_{12} + 3b_{03} \neq 0 \), \textit{(non-Hamiltonian, not axis-reversible case)}.

**Proof of Proposition 14** First part follows from above proposition. We assume that \( b_{21} = 0 (a_{6} = 0) \). If \( a_{21} + b_{12} \neq 0 (\alpha_{7} \neq 0) \), it is easy to check that \( \alpha_{10}, \alpha_{12}, \alpha_{15} \) and \( \alpha_{16} \) are not zero simultaneously. Therefore, from Theorem 2, system (20) does not have an AIIF. Otherwise, \( a_{21} + b_{12} = 0 (\alpha_{6} = \alpha_{7} = 0) \). The coefficient \( \alpha_{10} \) is \( \alpha_{10} = (3b_{12} - 4a_{21})(a_{12} + 3b_{03}) \). If it is not zero, the following coefficients under the cancellation of the above ones are

\[
\begin{align*}
\alpha_{12} &= (a_{12} + 3b_{03})(98a_{03} + (3b_{12} - 4a_{21})^2), \\
\alpha_{15} &= (3b_{12} - 4a_{21})^2(a_{12} + 3b_{03})(5b_{03} - 4a_{12}), \\
\alpha_{16} &= (3b_{12} - 4a_{21})(a_{12} + 3b_{03})((289/1372)(3b_{12} - 4a_{21})^3 \\
& + (11/25)(a_{12} + 3b_{03})^2), \\
\alpha_{18} &= (3b_{12} - 4a_{21})^2(a_{12} + 3b_{03})^3.
\end{align*}
\]

Thus, \( \alpha_{18} \) is different from zero and therefore system (20) does not have an AIIF.

Otherwise, \( (3b_{12} - 4a_{21})(a_{12} + 3b_{03}) = 0 (\alpha_{6} = \alpha_{7} = \alpha_{10} = 0) \). If \( a_{12} + 3b_{03} = 0 \), system (20) is a Hamiltonian system whose Hamiltonian is a polynomial inverse integrating factor and a first integral. So, the system is formally integrable and it has a formal inverse integrating factor (family 2.(a)). If \( a_{12} + 3b_{03} \neq 0 \) and \( 3b_{12} - 4a_{21} = 0 \), it has that

\[
\begin{align*}
\alpha_{12} &= (a_{12} + 3b_{03})a_{03}, \\
\alpha_{15} &= a_{03}(a_{12} + 3b_{03})(11b_{03} - 8a_{12}), \\
\alpha_{16} &= a_{03}^2(a_{12} + 3b_{03}).
\end{align*}
\]

If \( a_{03} \neq 0 \), then \( \alpha_{12} \) and \( \alpha_{16} \) are different from zero. So, the existence of an AIIF arrives to \( a_{03} = 0 \), i.e. family 2.(b). It is straightforward to check that

\[
V = 1 + (a_{12} + 3b_{03})x + (3/2)b_{03}(a_{12} + 3b_{03})x^2 \\
- (1/2)b_{03}(a_{12} - 3b_{03})(a_{12} + 3b_{03})x^3 \\
+ (1/2)a_{12}b_{03}(-3b_{03} + 2a_{12})(a_{12} - 3b_{03})x^4 \\
- (1/2)b_{03}(-b_{03} + a_{12})(a_{12} - 3b_{03})(-3b_{03} + 2a_{12})y^3 \\
- (1/2)b_{03}(-b_{03} + a_{12})(a_{12} - 3b_{03})(-3b_{03} + 2a_{12})a_{12}xy^3,
\]

is a polynomial inverse integrating factor for family 2.(b) which is 1 at origin. Thus,

\[
H = -\int P/V \, dy + \int \left( Q/V + \frac{\partial}{\partial x} \int P/V \, dy \right) \, dx
\]

is a formal first integral defined in a neighborhood of the origin. Therefore, the system is formally integrable. \( \square \)
Remark 4. If there exists an AIIF of system (5) which does not have the form $(h + q\cdot h\cdot h\cdot o\cdot t.)^{1+\epsilon/(r+\epsilon)}$, up to a multiplicative constant, for a certain $j$, then the system is formally integrable. For instance, $V = (y^4/3 - x^4/4 - 3\lambda x^5)6/5$ is an inverse integrating factor of $(\dot{x}, \dot{y}) = (y^2 + 60\lambda xy^2, x^3 + 100\lambda y^3)^T$, and from Proposition 14, it is an formally integrable system.

Last on, we study the problem for the systems (15) given by
\[
(\dot{x}, \dot{y}) = (y^2, x^3)^T + (a_{30}x^3, b_{21}x^2y + b_{03}y^3)^T,
\]
with $a_{30} \neq 0$, (case $a_{30} = 0$, studied before). It has the following result.

Proposition 15. System (21), with $a_{30} \neq 0$, has an AIIF if and only if it satisfies:

1. $3a_{30} + b_{21} = b_{03} = 0$, (Hamiltonian system), or
2. $3b_{21} - 4a_{30} = 0$ and $b_{03} = 0$, (non-formally integrable system).

Moreover, in this case, the AIIF is $V = (4y^3 - 3x^4)^{13/12}$.

Proof of Proposition 15. We assume that system (21) with $a_{30} \neq 0$, has an AIIF. The first two coefficients of the quasi-homogeneous normal form of (21) are given by (18) and (19) for $a_{21} = b_{12} = 0$. Therefore, if $3a_{30} + b_{21} \neq 0$, it arrives to $4a_{30} - 3b_{21} = 0$. In such case, the following coefficient of the normal form is $\alpha_{10} = a_{30}^2b_{03}$. So, $b_{03} = 0$. It is easy to check that $V = (4y^3 - 3x^4)^{13/12}$ is an AIIF of the system.

Otherwise, $3a_{30} + b_{21} = 0$. In this case, $\alpha_6$ and $\alpha_7$ are zero and $\alpha_{10} = a_{30}^2b_{03}$. This arrives to $b_{03} = 0$, i.e. it is a Hamiltonian system. □

D) Systems of the form $(-y^3, x^3)^T + q\cdot h\cdot h\cdot o\cdot t$. The systems are
\[
\dot{x} = X_h + q\cdot h\cdot h\cdot o\cdot t.
\]
with $h = x^4/4 + y^4/4$, $t = (1, 1)$ and $r = 2$. From Lemma 5 the sets $P_j^0$ are non-trivial spaces for all $j$, hence $n_0 = 1 + r = 3$. So, in order to get a normal form, it is enough to compute the sets $\text{Cor}(E_j)$, $j = 3, 4, 5, 6$, which are given in Table 4.

We note that $h \in T_{1,(1,1)}^2$. A normal form of system (22) is
\[
(\dot{x}, \dot{y}) = (-y^3, x^3)^T + \sum_{j \geq 0} f_j(x, y, h)h^jD_0,
\]
with $D_0 = (x, y)^T$ and $f_j \in \text{span}\{x^2y, xy^2, h, x^2y^2, xh, yh, x^2h, xyh, x^2h\}$.

From Proposition 3 the origin is a monodromic singular point. In order to characterize the centers of system (23), it is necessary to compute the value of the integrals $I_{n,k} = \int_0^T C_n^k(\theta)S\theta^k(\theta)d\theta$, $n, k \in \{0, 1, 2\}$, being $g(\theta) = (C(\theta), S\theta(\theta))$, $\theta \in [0, T]$ a parameterization of the closed curve $h = 1$, where $(C(\theta), S\theta(\theta))^T$ is the solution of the initial value problem
\[
\begin{cases}
\frac{dC\theta}{d\theta} = -S\theta^2, \\
\frac{dS\theta}{d\theta} = C\theta^2,
\end{cases}
\]

12
with \((\text{Cs}(0), \text{Sn}(0)) = (1, 0)\), and \(T\) is a minimal period of both functions.

We cite some properties of these integrals. For the sake of shortness, we prefer to avoid its proof in this paper.

**Lemma 16.** For every \(n, k \geq 0\), it holds:

1. \(I_{2n+1, k} = I_{n, 2k+1} = 0\).
2. \(I_{2n+2, 2k+2} = \frac{(2n+1)(2k+1)}{(n+k+2)(n+k+1)} I_{2n, 2k}\).

So,

\[
I_{1, 0} = I_{0, 1} = I_{1, 1} = I_{2, 1} = I_{1, 2} = 0, \quad I_{2, 0} = I_{0, 2}, \quad I_{0, 0} = 8I_{2, 2}.
\]

Applying Theorems 2 and 4 we have the following result.

**Theorem 17.** System \((22)\) has an AIIF if and only if, it is formally orbital equivalent to

1. \(\dot{x} = X_h\). The AIIF is \(g(h)\) with \(g\) any nonzero function (in particular, there are inverse integrating factors nonzero at the origin). In this case, the origin is a center.

2. \(\dot{x} = X_h + (\alpha_{4j+3} x^2 y + \beta_{4j+3} x^2 y) h^j D_0\), \((\alpha_{4j+3}, \beta_{4j+3}) \neq (0, 0), j \geq 0\). The AIIF is \((h + q, h, h)\text{.o.t.})^{2+3j+1/4}\).

3. \(\dot{x} = X_h + (\alpha_{4j+4} x^2 y + \beta_{4j+4} x^2 y) h^j D_0\), \((\alpha_{4j+4}, \beta_{4j+4}) \neq (0, 0), j \geq 0\). The AIIF is \((h + q, h, h, h)\text{.o.t.})^{2+3j+1/2}\).

4. \(\dot{x} = X_h + (\alpha_{4j+5} x h + \beta_{4j+5} y h) h^j D_0\), \((\alpha_{4j+5}, \beta_{4j+5}) \neq (0, 0), j \geq 0\). The AIIF is \((h + q, h, h, h, h)\text{.o.t.})^{2+3j+1/4}\).

5. \(\dot{x} = X_h + (\alpha_{4j+6} x^2 h + \beta_{4j+6} x y h + \gamma_{4j+6} y^2 h) h^j D_0\), \((\alpha_{4j+6}, \beta_{4j+6}, \gamma_{4j+6}) \neq (0, 0, 0), j \geq 0\). The AIIF is \((h + q, h, h, h, h)\text{.o.t.})^{3+3j}\), i.e. it is formal. In this case, the origin is a center if and only if \(\alpha_{4j+6} + \gamma_{4j+6} = 0\).
We analyze the system
\[
(\dot{x}, \dot{y}) = (y^3, -x^3 + c_3x^2y^2 + c_4xy^3).
\] (24)

This system for \(c_3 = 1/2\) and \(c_4 = 0\) has been studied in Moussu [12] by showing that it is a degenerate analytic center without formal first integral and Giné and Peralta-Salas [11] have proved that the system does not admit a formal inverse integrating factor.

The first coefficients of the normal form (23) are \(\alpha_3 = 2c_3\), \(\beta_3 = 3c_4\). If both \(c_3\) and \(c_4\) are zero, the system is a Hamiltonian system whose first integral is \(h = x^4 + y^4\). Otherwise, the system is not formally integrable. Moreover,

- if \(c_3, c_4 \neq 0\), the coefficients of fourth order of the normal form are \(\alpha_4 = 0\) and \(\beta_4 = 2c_3c_4\). Thus, from Theorem 17, it does not have an AIIF,

- if \(c_3 = 0\) and \(c_4 \neq 0\), the coefficients of fourth order are zero but \(\alpha_5 = 6/5c_4^3\). And if \(c_3 \neq 0\) and \(c_4 = 0\), it has that \(\beta_5 = 16/45c_3^4\). Therefore, from Theorem 17, it does not have an AIIF.

Summarizing,

**Theorem 18.** System (24) with \((c_3, c_4) \neq (0, 0)\) does not admit an algebraic inverse integrating factor.

### 3. Proofs of the main results.

The following result will be used for the proof of Theorem 2 is an adjustment of Proposition 10 and Proposition 13 of [1]:

**Lemma 19.** Let system \(\dot{x} = X_h + \mu D_0\), where the factorization of \(h \in \mathbb{P}_r^t\) on \(\mathbb{C}[x,y]\) only has simple factors and \(\mu = \sum_{j \geq N} \mu_{r+j}\) with \(\mu_{r+j} \in \text{Cor}(\ell_j)\), for all \(j \geq N > 0\) and \(\mu_{r+N} \neq 0\). If \(V\) is an AIIF of the system, then \(V = (\sum_{j \geq 0} b_j h^j)^{(r+N+[t])/(r+|t|)}\), with \(b_1 = 1\), is the unique AIIF up to a multiplicative constant.

Moreover, the real numbers \(b_j\) verify the recursive relation

\[
0 = \sum_{i=0}^{j-1} \left[ \frac{N + (1+i)(r+|t|)}{r+|t|+N} - (j-i) \right] b_{j-i} h^{j-i} \mu_{r+N+i(r+|t|)}.
\] (25)

Furthermore, if \(\mu = \lambda f(h) + \nu\) with \(\lambda \in \text{Cor}(\ell_{r+N}) \setminus \{0\}\), \(f\) a scalar function, \(f(0) = 1\), and \(\nu = \sum_{j>N} \nu_{r+j}, \nu_{r+j} \in \text{Cor}(\ell_j)\), \(\nu_{r+N+i(r+|t|)} \equiv 0\) for all non-negative integer \(i\), then under these conditions, the system has an AIIF if and only if \(\nu \equiv 0\).

**Proof of Theorem 2** We prove the necessity. From Theorem 1, a normal form of system (3) is of the form \(\dot{x} = X_h + \mu D_0\), with \(\mu = \sum_{j \geq r} \mu_j, \mu_j \in \text{Cor}(\ell_j)\).

If \(\mu_j \equiv 0\), for all \(j\) then system (3) is formally orbital equivalent to a Hamiltonian system and, in such case, it is proved that system (3) has a formal inverse...
integrating factor.

Otherwise, let \( N = \min\{j, \mu_{r+j} \neq 0\} \). By \([2]\) Theorem 13\], system \([5]\) is formally orbital equivalent to \( \xi = X_h + \mu_{r+N}D_0 + \sum_{j>N} \tilde{\mu}_{r+j}D_0 \), with \( \tilde{\mu}_j \in \text{Cor}(\ell^{(2)}_j) \), a complementary subspace to the range of the linear operator \( \ell^{(2)}_k : \mathcal{P}^t_{k-r} \times \ker(\ell_{k-N}) \rightarrow \mathcal{P}^t_k \) defined by

\[
\ell^{(2)}_k(\mu_{k-r}, \alpha h^{(j)}) := \ell_k(\mu_{k-r}) + \alpha \mu_{r+N} h^{(j)} , \quad \text{if} \quad l_2 = 0,
\]

\[
\ell^{(2)}_k(\mu_{k-r}, 0) := \ell_k(\mu_{k-r}), \quad \text{if} \quad l_2 \neq 0.
\]

being \( k = r + N + l_1(r + |t|) + l_2 \), with \( 0 \leq l_2 < r + |t| \).

From Lemma \([19]\) \( V = (h + \sum_{j=1}^{l_1} b_j h^{(j+1)}(r+|t|)/(r+|t|)) / b_j \) with \( b_j \) verifying \((25)\). We see that \( b_j = 0 \) and \( \tilde{\mu}_{r+N+(j-1)(r+|t|)} = 0 \), for any \( j > 1 \). Indeed, we assume the contrary and let \( j_0 = \min\{j > 1 : b_j \neq 0\} \). By \((25)\), for \( j = j_0 \), it has that

\[
b_{j_0} h^{j_0-1} \mu_{r+N} - \frac{r+|t|}{r+|t|+N} h \tilde{\mu}_{r+N+(j_0-1)(r+|t|)} = 0.
\]

Consequently, \( b_{j_0} h^{j_0-1} \mu_{r+N} \in \text{Cor}(\ell^{(2)}_{r+N+(j_0-1)(r+|t|)}) \setminus \{0\} \), but also

\[
b_{j_0} h^{j_0-1} \mu_{r+N} = \ell^{(2)}_{r+N+(j_0-1)(r+|t|)}(0, b_{j_0} h^{j_0-1}),
\]

which is a contradiction. So, \( b_{j_0} = 0 \) and \( \mu_{r+N+(j-1)(r+|t|)} = 0 \), for all \( j > 1 \).

Applying Lemma \([19]\) for \( \lambda = \mu_{r+N} \), \( f(h) = 1 \) and \( \nu_{r+j} = \mu_{r+j} \), it has that \( \mu_{r+j} = 0 \), for any \( j > N \).

We prove that the condition is sufficient. If \( \mu_{r+N} \equiv 0 \), the polynomial \( h^m \) with \( m \) any natural number, is a polynomial first integral and, in particular, it is an inverse integrating factor. Thus, if we perform the transformation which brings \( \xi = X_h \) to the system \([3]\), then system \([3]\) admits an AIIF (in fact, it is formal) but it is not unique modulus a multiplicative constant.

In the case, \( \mu_{r+N} \neq 0 \), we see that \( V(h) = h^{r+|t|+N} \) is an AIIF of system \([6]\). Indeed, applying Euler theorem for quasi-homogeneous function, i.e. \( L_{D_0} f = sf \) with \( f \in \mathcal{P}^t_a \), it has that the Lie derivative of \( V \) respect to \( F = X_h + \mu_{r+N} D_0 \) is

\[
L_F V = V'(h)L_F h = (r + |t|)\mu_{r+N} h V'(h) = (r + |t| + N)\mu_{r+N} h^{r+|t|+N}/r+|t|,
\]

and

\[
\text{div}(F) = \text{div}(\mu_{r+N} D_0) = L_{D_0} \mu_{r+N} + |t|\mu_{r+N} = (r + |t| + N)\mu_{r+N}.
\]

So, \( L_F V - V \text{div}(F) = 0 \), that is, \( V \) is an AIIF of system \([6]\) (formal if \( N \) is a multiple of \( r + |t| \)). Thus, the system \([3]\) has the AIIF, \((h + q \text{-h.h.o.t.})^{r+|t|}/r+|t|\), up to a multiplicative constant. □
Proof of Proposition 3. As $h \in F_{r+|t|}$, we can write in a compact form $h = c \prod_{j=1}^{n} f_{j} \prod_{m=1}^{m} h_{j}$, where $f_{j} = (y^{j} - a_{j} x^{j})^{2} + b_{j}^{2} x^{2j}$, $j = 1, \ldots, m$ with $c, \lambda_{j}, a_{j}$ and $b_{j}$ real numbers and $\lambda_{j}, b_{j}$ non-zero, for all $j$.

We see the necessary condition. We assume the contrary one. Thus, $f_{j}$ is a factor of $h$. From Proposition 8 of [5], there exists an orbit of the system which leaves or enters at origin. Consequently, the origin is not monodromic.

On the other hand, the sufficient condition follows from Proposition 6 of [5]. □

Proof of Theorem 4. We assume that system (3) is formally orbital equivalent to system (6).

We also can assume that $h(x, y)$ is positive for all $(x, y) \neq (0, 0)$ since from Proposition 3 if the origin is monodromic $h$ preserves its sign and if $h$ is non-positive by changing the time $t$ by $-t$, $h$ becomes $-h$.

A parameterization of the closed curve $h = 1$ is $g(\theta) = (C_{s}(\theta), S_{n}(\theta))$, $\theta \in [0, T]$ where $(C_{s}(\theta), S_{n}(\theta))$ is the solution of the initial value problem

$$\frac{dx}{d\theta} = X_{h}(x), \quad x(0) = (1, 0)^{T},$$

and $T$ is a minimal period of both functions.

We consider the transformation

$$x = u^{t_{1}} C_{s}(\theta), \quad y = u^{t_{2}} S_{n}(\theta), \quad (26)$$

where $u > 0$, $\theta \in [0, T]$.

Differentiating (26) with respect to time, we get $\dot{x} = \frac{1}{u} D_{0} \dot{u} + \frac{1}{u} X_{h} \dot{\theta}$. From this, we obtain

$$\dot{x} \land X_{h} = \frac{1}{u} D_{0} \land X_{h} \dot{u}, \quad D_{0} \land \dot{x} = \frac{1}{u r} D_{0} \land X_{h} \dot{\theta} \quad (27)$$

We note that $D_{0} \land X_{h}(x, y) = \nabla h(x, y) \cdot D_{0} = (r + |t|) h(x, y) \neq 0$, for all $(x, y) \neq (0, 0)$.

For system (3) it has $\dot{x} \land X_{h} = u^{r+N} \mu_{r+N}(C_{s}(\theta), S_{n}(\theta)) D_{0} \land X_{h}$ and $D_{0} \land \dot{x} = \frac{1}{u^{r}} D_{0} \land X_{h}$. So, system (8) is

$$\left\{ \begin{array}{l}
\dot{u} = u^{r+N+1} \mu_{r+N}(C_{s}(\theta), S_{n}(\theta)), \\
\dot{\theta} = u^{r}.
\end{array} \right. \quad (28)$$

It can be further simplified by rescaling the time by $dt = \frac{1}{u^{r}} d\tau$, which yields

$$\left\{ \begin{array}{l}
u' = \frac{du}{d\tau} = u^{N+1} \mu_{r+N}(C_{s}(\theta), S_{n}(\theta)), \\
\theta' = \frac{d\theta}{d\tau} = 1.
\end{array} \right. \quad (29)$$

16
Finally, the change $z = -\frac{1}{N} u^N$ transforms the system into

$$
\begin{align*}
\begin{cases}
    z' &= \mu + N(C_s(\theta), S_n(\theta)), \\
    \theta' &= 1.
\end{cases}
\end{align*}
$$

(30)

Poincaré map for system (30) is $\Pi(z_0) = z(T, z_0) = z_0 + I$. From this, the result follows. □

**Acknowledgments.** This work has been partially supported by *Ministerio de Ciencia y Tecnología, Plan Nacional I+D+I* co-financed with FEDER funds, in the frame of the project MTM2010-20907-C02-02, and by *Consejería de Educación y Ciencia de la Junta de Andalucía* (FQM-276 and P12-FQM-1658).

**References**

[1] A. Algaba; E. Freire; E. Gamero; C. García, *Quasihomogeneous normal forms*, J. Comput. Appl. Math, 150, (2003), 193-216.

[2] A. Algaba; N. Fuentes; C. García; M. Reyes, *A class of non-integrable systems admitting an inverse integrating factor*, Journal Math. Anal. App. 420, (2014), 2, 1439-1454.

[3] A. Algaba; E. Gamero; C. García, *The integrability problem for a class of planar systems*, Nonlinearity 22, (2009), 2, 395-420.

[4] A. Algaba; C. García; M. Reyes, *Nilpotent systems admitting an algebraic inverse integrating factor over $\mathbb{C}((x, y))$*, Qualitative Theory of Dynamical Systems, 10, 2, 303-316, (2011).

[5] A. Algaba; C. García; M. Reyes, *Characterization of a monodromic singular point of a planar vector field*, Nonlinear Analysis, 74, 5402-5414, (2011).

[6] A. Algaba; C. García; M. Reyes, *Existence of an inverse integrating factor, center problem and integrability of a class of nilpotent systems*, Chaos Solitons & Fractals, 45, 869-878, (2012).

[7] J. Chavarriga; H. Giacomini; J. Giné; J. Llibre, *On the integrability of two-dimensional flows*, J. Differential Equations 157, (1999), 163-182.

[8] I. García; H. Giacomini; M. Grau, *The inverse integrating factor and the Poincaré map*, Trans. Amer. Math. Soc. 362 (2010), 7, 3591-3612.

[9] I. García; H. Giacomini; M. Grau, *Generalized Hopf Bifurcation for planar vector fields via the inverse integrating factor*, J. Dyn. Differ. Equat., 23, 2,(2011), 251-281.

[10] H. Giacomini; J. Llibre; M. Viano, *On the nonexistence, existence and uniqueness of limit cycles*, Nonlinearity 9, (1996), 501-516.
[11] J. Giné; D. Peralta-Salas, Existence of inverse integrating factors and Lie symmetries for degenerate planar centers, J. Differential Equations 252, (2012), 344-357.

[12] R. Moussu, Symétrie et forme normales des centres et foyers dégénérés, Ergodic Theory Dynam. Sys. 2 (1982), 241-251.

[13] H. Poincaré, Mémoire sur les courbes définies par les équations différentielles, J. Math. 37, (1881), 375-422.

[14] S. Walcher, Local integrating factors, J. Lie Theory 13, (2003), 279-289.