A Simple Proof of a Congruence for a Series Involving the Little $q$-Jacobi Polynomials

Dedicated to Professor George E. Andrews on the occasion of his 80th birthday

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Abstract. We give a simple and a more explicit proof of a mod 4 congruence for a series involving the little $q$-Jacobi polynomials which arose in a recent study of a certain restricted overpartition function.

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1. Introduction

In [3], Andrews, Schultz, Yee and the author studied the overpartition function $p_\omega(n)$, namely, the number of overpartitions of $n$ such that all odd parts are less than twice the smallest part, and in which the smallest part is always overlined. In the same paper, they obtained a representation for the generating function of $p_\omega(n)$ in terms of a $3\phi_2$ basic hypergeometric series and an infinite series involving the little $q$-Jacobi polynomials. The latter are given by [2, Equation (3.1)]

$$p_n(x;\alpha,\beta:q) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n \cdots (a_{r+1};q)_n}{(q;q)_n(b_1;q)_n \cdots (b_r;q)_n} z^n,$$

where the basic hypergeometric series $\phi_{r+1}$ is defined by

$$\phi_{r+1}(a_1, a_2, \ldots, a_{r+1}; b_1, b_2, \ldots, b_r; q, z) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n \cdots (a_{r+1};q)_n}{(q;q)_n(b_1;q)_n \cdots (b_r;q)_n} z^n,$$

and where we use the notation

$$(A;q)_0 = 1; \quad (A;q)_n = (1 - A)(1 - Aq) \cdots (1 - Aq^{n-1}), \quad n \geq 1,$$

$$(A;q)_1 = \lim_{n \to 1} (A;q)_n \quad (|q| < 1).$$
The precise representation for the generating function of $\overline{\mathcal{P}}_\omega(n)$ obtained in [3] is as follows.

**Theorem 1.1.** The following identity holds for $|q| < 1$:

$$
\overline{\mathcal{P}}_\omega(q) := \sum_{n=1}^{\infty} \overline{\mathcal{P}}_\omega(n) q^n
= -\frac{1}{2} \left( \frac{q}{1-q} \right) \left( \frac{q^2}{1-q^2} \right) \varphi_2 \left( \frac{1}{q}, \frac{1}{q^2} ; q, q \right)
+ \frac{\left( \frac{q}{1-q} \right)}{\left( \frac{q}{1-q} \right)} \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(1 + q^{2n})^2} p_2n(-1; q^{-2n+1}, -1 : q). \tag{1.2}
$$

Later, Bringmann, Jennings-Shaffer and Mahlburg [4, Theorem 1.1] showed that

$$
\overline{\mathcal{P}}_\omega(q) + \frac{1}{4} - \frac{\eta(4\tau)}{2\eta(2\tau)^2},
$$

where $q = e^{2\pi i \tau}$ and $\eta(\tau)$ is the Dedekind eta function, can be completed to a function $\hat{\mathcal{P}}_\omega(\tau)$, which transforms like a weight 1 modular form. They called the function

$$
\hat{\mathcal{P}}_\omega(q) + \frac{1}{4} - \frac{\eta(4\tau)}{2\eta(2\tau)^2}
$$
a higher depth mock modular form.

While the series involving the little $q$-Jacobi polynomials in Theorem 1.1 itself looks formidable, it was shown in [3, Theorem 1.3] that it is a simple $q$-product modulo 4. The mod 4 congruence proved in there is given below.

**Theorem 1.2.** The following congruence holds:

$$
\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(1 + q^{2n})^2} p_2n(-1; q^{-2n+1}, -1 : q) \equiv \frac{1}{2} \left( \frac{q; q^2}{1-q^2} \right) \varphi_2 \left( \frac{1}{q}, \frac{1}{q^2} ; q, q \right) \pmod{4}. \tag{1.3}
$$

The proof of this congruence in [3] is beautiful but somewhat involved. The objective of this short note is to give a very simple proof of it. In fact, we derive it as a corollary of the following result.

**Theorem 1.3.** For $|q| < 1$, we have

$$
\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(1 + q^{2n})^2} p_2n(-1; q^{-2n+1}, -1 : q) = \frac{1}{2} \left( \frac{q; q^2}{1-q^2} \right) + \frac{4q^2}{1+q} \sum_{n=0}^{\infty} \frac{(q^3; q^2)_n (-q)^n}{(1 + q^{2n+2})} \sum_{j=0}^{n} \frac{(-q; q^2)_{2j} q^{2j}}{(q^2; q^2)_{2j}}. \tag{1.4}
$$

The presence of 4 in front of the series on the right-hand side in the above equation immediately implies that Theorem 1.2 holds.
2. Proof of Theorem 1.3

Observe that from (1.1),

\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(-q; q^2)_n (1 + q^{2n})} p_{2n} (-1; q^{-2n-1}, -1 : q) = \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(-q; q^2)_n (1 + q^{2n})} \sum_{j=0}^{2n} \frac{(-1; q)_j}{(q; q)_j} (-q)^j. \tag{2.1}
\]

However, let us first consider

\[
A(q) := \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(-q; q^2)_n (1 + q^{2n})} \sum_{j=0}^{2n} \frac{(-1; q)_j}{(q; q)_j} q^j. \tag{2.2}
\]

The only difference in the series on the right-hand side of (2.1) and the series in (2.2) is the presence of \((-1)^j\) inside the finite sum in the former.

To simplify \(A(q)\), we start with a result of Alladi [1, p. 215, Equation (2.6)]:

\[
\frac{(abg; q)_n}{(bq; q)_n} = 1 + b(1 - a) \sum_{j=1}^{n} \frac{(abg; q)_{j-1} q^j}{(bq; q)_j}. \tag{2.3}
\]

Let \(a = -1, b = 1\) and replace \(n\) by \(2n\) so that

\[
\sum_{j=0}^{2n} \frac{(-1; q)_j q^j}{(q; q)_j} = \frac{(-q; q)_{2n}}{(q; q)_{2n}}. \tag{2.4}
\]

Substitute (2.4) in (2.2) to see that

\[
A(q) = \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(-q; q^2)_n (1 + q^{2n})} \frac{(-q; q)_{2n}}{(q; q)_{2n}}
\]

\[
= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-q^2; q^2)_n (-q)^n}{(q^2; q^2)_n}
\]

\[
= \frac{1}{2} (q; q^2)_{\infty}, \tag{2.5}
\]

where in the last step we used the \(q\)-binomial theorem

\[
\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}},
\]

valid for \(|z| < 1\) and \(|q| < 1\).

From (2.1) and (2.2),

\[
\sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(-q; q^2)_n (1 + q^{2n})} p_{2n} (-1; q^{-2n-1}, -1 : q) - A(q)
\]

\[
= \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(-q; q^2)_n (1 + q^{2n})} \sum_{j=0}^{2n} ((-1)^j - 1) \frac{(-1; q)_j q^j}{(q; q)_j}
\]
\[
= -2 \sum_{n=0}^{\infty} \frac{(q; q^2)_n (-q)^n}{(-q; q^2)_n (1 + q^{2n})} \sum_{j=1}^{n} \frac{(-1; q)_{2j-1} q^{2j-1}}{(q; q)_{2j-1}} \\
= \frac{4q^{2}}{(1 + q)} \sum_{n=0}^{\infty} \frac{(q^3; q^2)_n (-q)^n}{(-q^3; q^2)_n (1 + q^{2n+2})} \sum_{j=0}^{n} \frac{(-q; q)_{2j} q^{2j}}{(q^2; q)_{2j}}. \tag{2.6}
\]

Invoking (2.5), we see that the proof of Theorem 1.3 is complete.

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