Dissipative Stability Conditions for Linear Coupled Differential-Difference Systems via a Dynamical Constraints Approach

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Abstract

In this short note, we derive dissipative conditions with slack variables for a linear coupled differential-difference (CDDS) via constructing a Krasovskii functional. The approach can be interpreted as a generalization of the Finsler Lemma approach for standard LTI systems proposed previously in de Oliveira & Skelton (2001). We also show that the proposed slack variables scheme is equivalent to the approach based on directly substituting the system trajectory $\dot{x}(t)$, similar to the case of LTI system.

Keywords: Krasovskii functional, Coupled Differential-Difference System, Finsler Lemma, Projection Lemma.

1 Introduction

Many approaches concerning the stability analysis of time delay system Fridman (2014) have been proposed over the recent decades based on solving semidefinite programs derived via constructing Krasovskii functional. Most of such results Briat (2014) are obtained by directly substituting $\dot{x}(t)$ with the system trajectory expression during the process of constructions. In this note however, we will exploit the idea of Finsler Lemma (Projection Lemma) approach, originated from de Oliveira & Skelton (2001) concerning LTI systems, to derive dissipative conditions for a linear coupled differential-difference system Gu & Liu (2009) with a distributed delay. The dissipative conditions derived via Finsler Lemma in this note, which is also based on constructing a Krasovskii functional, can be considered as a generalization of the exiting results for LTI systems in de Oliveira & Skelton (2001). The application of Finsler Lemma with Projection Lemma avoids a direct substitution of $\dot{x}(t)$, which results in the introduction of extra matrix terms in the resulting dissipative conditions. Finally, we show that the dissipative conditions with slack variables are equivalent to the conditions derived via the approach of directly substituting $\dot{x}(t)$. For the advantage of having slack variables in dissipative conditions, see the examples in explained in de Oliveira & Skelton (2001) which can be also valid for the corresponding delay problems.

Notation

We define $\mathbb{T} := \{x \in \mathbb{R} : x \geq 0\}$ and $S^n := \{X \in \mathbb{R}^{n \times n} : X = X^\top\}$ and $\mathbb{R}_{[n]}^{n \times n} := \{X \in \mathbb{R}^{n \times n} : \text{rank}(X) = n\}$. The notations $\|x\|_q = (\sum_{i=1}^n |x_i|^q)^\frac{1}{q}$ and $\|f(\cdot)\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^\frac{1}{p}$ and $\|f(\cdot)\|_{pq} = \left(\int_{\mathbb{R}} \|f(t)\|_p^q dt\right)^\frac{1}{q}$ are the norms associated with $\mathbb{R}^n$ and Lebesgue integrable functions spaces $\mathbb{L}_p(\mathbb{R}; \mathbb{R})$ and $\mathbb{L}_{pq}(\mathbb{R}; \mathbb{R}^n)$, respectively. In addition, We use $\mathbf{C}^\bullet_p(A; \mathbb{R}^n)$ to denote the space of right piecewise continuous functions. Let $\mathbf{Sy}(X) := X + X^\top$ to be the sum of a matrix with its transpose. $\text{col}_{i=1}^n x_i := \left[\text{row}_{i=1}^n x_i^\top\right]^\top = [x_1^\top \cdots x_i^\top \cdots x_n^\top]^\top$ is defined a column vector containing a sequence of objects. Furthermore, let $x \lor y = \max(x, y)$ and $x \land y = \min(x, y)$. $O_{n \times n}$ is used to denote a $n \times n$ zero matrix with the abbreviation $O_n$, whereas $\mathbf{0}_n$ denotes a $n \times 1$ column vector. Finally, we assume the order of matrix operations as $\text{matrix (scalars) multiplications} > \circ > \odot > \text{matrix (scalar) additions}$.

2 Problem formulation

Consider a linear coupled differential-difference system (CDDS)

\[
\begin{align*}
\dot{x}(t) &= A_1 x(t) + A_2 y(t - r) + \int_{-\tau}^0 \dot{A}_3(\tau) y(t + \tau) d\tau + D_1 w(t), \\
y(t) &= A_4 x(t) + A_5 y(t - r),
\end{align*}
\]  

(1)

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z(t) = C_1 x(t) + C_2 y(t - r) + \int_0^t \dot{C}_3(\tau)y(t + \tau) d\tau + D_2 w(t),
\text{col}(x(0), y(0 + \cdot)) = \text{col}(\xi, \phi(\cdot)) \in \mathbb{R}^n \times \mathbb{C}_r^*([-r, 0) \times \mathbb{R}),
\]
where \(x(t) \in \mathbb{R}^n\) and \(y(t) \in \mathbb{R}^r\) are the solution of (1), \(w(\cdot) \in L_2(T; \mathbb{R})\) represents disturbance, \(z(t) \in \mathbb{R}^m\) is the regulated output. Furthermore, \(\xi \in \mathbb{R}^n\) and \(\phi(\cdot) \in \mathbb{C}_r^*([-r, 0) \times \mathbb{R})\) are the initial conditions with a known delay value \(r > 0\). The distributed delay term is \(F(\tau) = f(\tau) \otimes I_v\) with \(f(\tau) = \text{col}_{l=1}^d f_l(\tau) \in \mathbb{C}_r^1([-r, 0) \times \mathbb{R})^d\).

Moreover, we have \(\forall\) the following conditions are satisfied:

**Assumption 1.** There exist \(\text{col}_{l=1}^d f_l(\tau) = f(\cdot) \in \mathbb{C}_r^1([0; \mathbb{R}])^d\) with \(d \in \mathbb{N}_0\), and \(A_3 \in \mathbb{R}^{n \times r}, C_3 \in \mathbb{R}^{m \times r}\) with \(q = nd\) such that \(\forall r \in [-r, 0], \mathbb{R}^n \otimes A_3(\tau) = A_3 F(\tau)\) and \(\mathbb{R}^m \otimes C_3(\tau) = C_3 F(\tau)\). In addition, \(\{f_l(\cdot)\}_{l=1}^d\) are linearly independent and \(f(\cdot)\) satisfies

\[
\exists M \in \mathbb{R}^{d \times d}, \quad \frac{df(\tau)}{d\tau} = M f(\tau).
\]

**2.1 Preliminaries**

To prove the main results in this note, the following Lemmas and Definitions are required.

**Lemma 1.** For all \(\forall P \in \mathbb{R}^{p \times q}\) and \(Q \in \mathbb{R}^{n \times m}\), we have

\[
(P \otimes I_n)(I_q \otimes Q) = P \otimes Q = (I_p \otimes Q) (P \otimes I_m).
\]

Moreover, we have \(\forall X \in \mathbb{R}^{n \times n}, \forall Y \in \mathbb{R}^{n \times p}, \forall Z \in \mathbb{R}^{q \times r}\)

\[
(XY) \otimes Z = (X \otimes Z)(Y \otimes I_r).
\]

Proof. (3) and (4) are derived via the property of the Kronecker product: \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\).

The following Lemma 2 is a particular case of the Theorem 3 in Gu & Liu (2009).

**Lemma 2.** Given \(r > 0\), suppose a differential-difference system

\[
\dot{x}(t) = f(x(t), y(t + \cdot)), \quad y(t) = g(x(t), y(t + \cdot)), \quad f(0_n, 0_n(\cdot)) = 0_n, \quad g(0_n, 0_n(\cdot)) = 0_n(\cdot)
\]

satisfying the prerequisites in the Theorem 3 of Gu & Liu (2009), where \(y(t + \cdot) \in \mathbb{C}_r^1([-r, 0) \times \mathbb{R})\) and \(y(t) = g(x(t), y(t + \cdot))\) is uniformly input to state stable. Then the origin of \(A_3(\tau) = A_3 F(\tau)\) and \(\mathbb{R}^m \otimes C_3(\tau) = C_3 F(\tau)\). In addition, \(\{f_l(\cdot)\}_{l=1}^d\) are linearly independent and \(f(\cdot)\) satisfies

\[
\exists M \in \mathbb{R}^{d \times d}, \quad \frac{df(\tau)}{d\tau} = M f(\tau).
\]

**Definition 1** (Dissipativity). Given \(r > 0\), a delay system

\[
\dot{x}(t) = f(x(t), y(t + \cdot), w(t)), \quad y(t) = g(x(t), y(t + \cdot)), \quad z(t) = h(x(t), y(t + \cdot), w(t)),
\]

is dissipative with respect to the supply rate function \(s(z(t), w(t))\), if there exists a differentiable functional \(\forall r_1, r_2) \in \mathbb{C}_r^1([-r, 0) \times \mathbb{R}) \to \mathbb{R}\) with \(\dot{\varphi}(\cdot, \phi(\cdot)) \leq -\epsilon_3 \|\phi(\cdot)\|_{\infty}^2\)

\[
\exists \epsilon_3 > 0, \forall \xi \in \mathbb{R}^n, \forall \phi(\cdot) \in \mathbb{C}_r^1([-r, 0) \times \mathbb{R}), \quad \dot{\varphi}(r, \xi, \phi(\cdot)) \leq -\epsilon_3 \|\xi\|_2^2
\]

where

\[
\dot{\varphi}(\cdot, \phi(\cdot)) := \frac{d}{dt} \varphi(x(t), y(t + \cdot)) \bigg|_{t=r, x(\tau) = \xi, y(\tau + \cdot) = \phi(\cdot)} - \frac{d}{dx} f(x) = \limsup_{\eta \downarrow 0} \frac{f(x + \eta) - f(x)}{\eta}.
\]

with \(\dot{x}(t)\) and \(y(t + \cdot)\) satisfying (5). Under the assumption that \(\forall r_1, r_2) \in \mathbb{C}_r^1([-r, 0) \times \mathbb{R}) \to \mathbb{R}\) is differentiable, (10) is equivalent to the original definition of dissipativity. (See Briat (2014) for the definition of dissipativity without delays)
To conduct dissipative analysis for (1), a quadratic supply function
\[ s(z(t), u(t)) = - \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}^T J \begin{bmatrix} z(t) \\ u(t) \end{bmatrix} \] with \( J = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix} \in S^{m+q}, \ J_1 \geq 0 \) (11)
is considered in this note, which is taken from Scherer et al. (1997).

To analyze the stability of the origin of (1), we apply the Krasovskii functional
\[ v(x(t), y(t + \cdot)) := \left[ \begin{array}{c} x(t) \\ \int_{-r}^{0} F(\tau)y(t + \tau) d\tau \end{array} \right] + \int_{-r}^{0} y^T(t + \tau) \left[ J + (r + \tau)U \right] y(t + \tau) d\tau \] to be constructed, where \( P \in S^{n+e} \) and \( S; U \in S^r \) and \( F(\tau) := f(\tau) \otimes I_\nu \) with \( f(\tau) \) which is given and defined in (1) satisfying Assumption 1.

3 Main results

3.1 Dissipative stability conditions without slack variables

In this subsection, we first present dissipative conditions constructed by directly substituting the expression of \( \dot{x}(t) \) during our derivation.

**Theorem 1.** Given \( J_1 > 0 \) in (11), the linear CDDS (1) is globally uniformly asymptotically stable at its origin and dissipative with respect to (11), if there exist \( P \in S^{n+e} \) and \( S; U \in S^r \) such that the following conditions hold,
\[ P + \begin{bmatrix} O_n \otimes (F \otimes S) \end{bmatrix} > 0, \quad S \geq 0, \quad U \geq 0 \] (13)
\[ \begin{bmatrix} -J_1^{-1} & \Sigma \\ * & \Phi \end{bmatrix} < 0, \] (14)
where \( \Gamma := [O_{\nu \times q} \quad A_4 \quad A_5 \quad O_{\nu \times q}] \) and \( \Sigma := [D_2 \quad C_1 \quad C_2 \quad C_3] \) and
\[ \Phi := Sy(HP\Theta) + \Gamma^T(S + rU) \Gamma + \left( J_4 \oplus O_n \oplus -S \oplus (F \otimes U) \right) + Sy \left( \begin{bmatrix} \Sigma^T J_2 \quad O_{(n+\nu) \times (n+\nu)} \end{bmatrix} \right). \] (15)

with
\[ H = \begin{bmatrix} O_{\nu \times q} & O_{\nu \times q} & O_{\nu \times q} & O_{\nu \times q} \\ J_n & O_{\nu \times q} & O_{\nu \times q} & O_{\nu \times q} \end{bmatrix}, \quad \Theta = \begin{bmatrix} D_1 & A_1 & A_2 & A_3 \\ O_{\nu \times q} & F(0)A_4 & F(0)A_5 - F(-r) & -M_1 \end{bmatrix}, \quad F^{-1} = \int_{-r}^{0} f(\tau) f^T(\tau) d\tau \] (16)

**Proof.** It is obvious to see that (12) satisfies the property \( \exists \lambda, \eta > 0: \forall t \in T, \)
\[ v(x(t), y(t + \cdot)) \leq \left[ \int_{-r}^{0} F(\tau)y(t + \tau) d\tau \right] + \int_{-r}^{0} y^T(t + \tau)\lambda y(t + \tau) d\tau \]
\[ \leq \lambda \|x(t)\|_2^2 + \lambda \|y(t + \cdot)\|_{\infty|2}^2 \leq \lambda \|x(t)\|_2^2 + \lambda \|y(t + \cdot)\|_{\infty|2}^2 \]
\[ + \|x(t)\|_2^2 + \|y(t + \cdot)\|_{\infty|2}^2 + \int_{-r}^{0} \|x(t)\|_2^2 + \|y(t + \cdot)\|_{\infty|2}^2 \]
\[ \leq \lambda \|x(t)\|_2^2 + (\lambda + \eta) \|y(t + \cdot)\|_{\infty|2}^2 \leq \lambda \|x(t)\|_2^2 + (\lambda + \eta) \|y(t + \cdot)\|_{\infty|2}^2 \]
\[ \leq 2 (\lambda + \eta) \left( \|x(t)\|_2^2 \|y(t + \cdot)\|_{\infty|2}^2 \right)^2 \] (17)
which demonstrates that (12) satisfies
\[ \exists \epsilon_2 > 0: \forall t \in T, \quad v(x(t), y(t + \cdot)) \leq \epsilon_2 \left( \|x(t)\|_2 \|y(t + \cdot)\|_{\infty|2}^2 \right)^2 \] (18)

Applying the Lemma 5 in Feng & Nguyen (2016) to the integral term \( \int_{-r}^{0} y^T(t + \tau)S y(t + \tau) d\tau \) in (12), with the fact that \( y(t + \cdot) \in C^*_y([-r, 0); R^r) \subseteq L_2([-r, 0); R^r), \) yields
\[ \forall t \in T, \quad \int_{-r}^{0} y^T(t + \tau)S y(t + \tau) d\tau \geq \left( \int_{-r}^{0} F(\tau)y(t + \tau) d\tau \right)^T F \otimes S \int_{-r}^{0} F(\tau)y(t + \tau) d\tau. \] (19)

By considering (19) with (12), one can conclude that the feasible solution of (13) infers the existence of (12) satisfies (6) considering the right limit substitution \( t = \tau, x(\tau) = \xi, y(\tau + \cdot) = \phi(\cdot) \) and (18).
Now we start to derive the stability conditions inferring (7) and (10).
Differentiate \( v(x(t), y(t + \cdot)) \) alongside the trajectory of (1) and considering (11) and the relation
\[
\frac{d}{dt} \int_{t-r}^{t} F(\tau) y(\tau) \, d\tau = F(0) y(t) - F(-r) y(t - r) - (M \otimes I_n) \int_{t-r}^{0} F(\tau) y(t + \tau) \, d\tau = F(0) A_4 x(t) + [F(0) A_3 - F(-r)] y(t - r) - \tilde{M} \int_{0}^{\tau} F(\tau) y(t + \tau) \, d\tau,
\]
where \( \tilde{M} = M \otimes I_n \). Then we have
\[
\dot{v}(x(t), y(t + \cdot)) - s(z(t), w(t)) = \chi^T(t) S y (HP \Theta) x(t) + \chi^T(t) \left[ \Gamma^T (S + r U) \Gamma + \left( J_3 \oplus O_n \oplus -S \oplus O_q \right) \right] \chi(t)
\]
\[+ \chi^T(t) \left( \Sigma^T J_1 \Sigma + S y \left( [\Sigma^T J_2 \left( O_{n + n + r + q} \chi(n + n + r + q) \right)] \right) \right) \chi(t) - \int_{t-r}^{0} y^T(t + \tau) U y(t + \tau) \, d\tau,
\]
where \( \Gamma \) and \( \Sigma \) have been defined in the statements of Theorem 1 and
\[
\chi(t) := \text{col} \left( w(t), x(t), y(t - r), \int_{t-r}^{0} F(\tau) y(t + \tau) \, d\tau \right).
\]
Let \( U \succeq 0 \) and apply the Lemma 5 in Feng & Nguang (2016) to the integral \( \int_{t-r}^{0} y^T(t + \tau) U y(t + \tau) \, d\tau \) in (21) similar to (19). It produces
\[
\forall t \in \mathbb{T}, \int_{t-r}^{0} y^T(t + \tau) U y(t + \tau) \, d\tau \geq \left( \int_{t-r}^{0} F(\tau) y(t + \tau) \, d\tau \right)^T F \otimes U \int_{t-r}^{0} F(\tau) y(t + \tau) \, d\tau.
\]
Now considering (23) with (21), we have
\[
\forall t \in \mathbb{T}, \dot{v}(x(t), y(t + \cdot)) - s(z(t), w(t)) \leq \chi^T(t) \left( \Phi + \Sigma^T J_1 \Sigma \right) \chi(t),
\]
where \( \Phi \) and \( \Sigma \) have been defined in (15) and \( \chi(t) \) have been defined in (22). Based on the structure of (24), it is easy to see that if \( U \succeq 0 \) and
\[
\Phi + \Sigma^T J_1 \Sigma \prec 0,
\]
are satisfied then the dissipative inequality in (10) : \( \dot{v}(x(t), y(t + \cdot)) - s(z(t), w(t)) \leq 0 \) holds \( \forall t \in \mathbb{T} \). Furthermore, given \( J_1 \succ 0 \) with the structure of \( \Phi + \Sigma^T J_1 \Sigma \prec 0 \) and considering the properties of positive definite matrices, it is obvious that the feasible solution of (25) with \( U \succeq 0 \) infers the existence of (12) satisfying (10) and (7) considering the definition of (8).

On the other hand, given \( J_1 \succ 0 \), applying Schur complement to (25) enables one to conclude that given \( U \succeq 0 \), (25) holds if and only if (14) which is now a convex matrix inequality. Since \( U \succeq 0 \) is included in (13), thus one can conclude that the feasible solutions of (13)–(14) infer the existence of (12) satisfying (6), (7) and (10).

### 3.2 Dissipative stability conditions with slack variables

In this subsection, we derive dissipative conditions via the following Finsler and Projection Lemmas. The result can be considered as a generalization of the approach in de Oliveira & Skelton (2001) to handle delay systems. Furthermore, we prove that the conditions with slack variables are in fact equivalent to Theorem 1 in terms of feasibility.

**Lemma 3** (Finsler Lemma de Oliveira & Skelton (2001)). Given \( n; p, q \in \mathbb{N}, \Pi \in \mathbb{S}^n \), \( P \in \mathbb{R}^{p \times n}_{[q]} \) such that \( q < n \), then the following propositions are equivalent:
\[
x^T \Pi x < 0, \forall x \in \{ y \in \mathbb{R}^n \mid \{ 0 \} : P y = 0_m \} \quad (26)
\]
\[
\exists Y \in \mathbb{R}^{n \times p} : \Pi + S y (Y P)^T < 0, \quad (27)
\]
\[
P_\perp^T \Pi P_\perp < 0, \quad (28)
\]
where the columns of \( P_\perp \) contains any basis of the null space of \( P \).

**Lemma 4** (Projection Lemma). Feng & Nguang (2016) Given \( n; p, q \in \mathbb{N}, \Pi \in \mathbb{S}^n, P \in \mathbb{R}^{q \times n}, Q \in \mathbb{R}^{p \times n} \), there exists \( Y \in \mathbb{R}^{p \times q} \) such that the following two propositions are equivalent :
\[
\Pi + P^T Y^T Q + Q^T Y P < 0, \quad (29)
\]
\[
P_\perp^T \Pi P_\perp < 0 \text{ and } Q_\perp^T \Pi Q_\perp < 0, \quad (30)
\]
where \( P_\perp \) and \( Q_\perp \) are matrices in which the columns contain any basis of the null space of \( P \) and \( Q \), respectively.
Theorem 2. Given all the prerequisites in Theorem 1, (1) is globally uniformly asymptotically stable at its origin and dissipative with respect to (11), if there exist \( Y \in \mathbb{R}^{(2n+\nu+p+q)\times n} \) and \( P \in \mathbb{S}^{n+\nu} \) and \( S; U \in \mathbb{S}^\nu \) such that (13) and the following inequality hold,

\[
\begin{bmatrix}
-J_r^{-1} \tilde{\Sigma} \\
\Phi
\end{bmatrix} + S \begin{bmatrix}
0_{m \times n}^T \\
Y
\end{bmatrix} \begin{bmatrix}
O_{n \times m} \\
A - I_n
\end{bmatrix} < 0,
\]

(31)

where \( A = [D_1 A_1 A_2 A_3] \) and \( \tilde{\Gamma} := [\Gamma \ O_{n \times \nu}] \) and \( \tilde{\Sigma} := [\Sigma \ O_{m \times n}] \) with \( \Gamma, \Sigma \) defined in Theorem 1, and

\[
\tilde{\Phi} = S \begin{bmatrix}
H \\
O_{n \times (p+n)}
\end{bmatrix} P \begin{bmatrix}
O_{n \times q} & O_n & O_{n \times \nu} & O_{n \times \rho} & I_n \\
O_{\rho \times q} & F(0)A_4 & F(0)A_5 - F(-r) & -M & O_{\rho \times n}
\end{bmatrix} + \tilde{\Gamma}^T(S + rU) \tilde{\Gamma} + \left( J_3 \otimes O_n \oplus -S \oplus (-F \otimes U) \oplus O_n \right).
\]

(32)

with \( H \) and \( F \) defined in (16).

Proof. First of all, note that the conditions in (17) and (13) remain unchanged in this case as they can be derived without considering the configuration of (1). Unlike what has been presented in the proof of Theorem 1, \( \dot{v}(x(t), y(t + \cdot)) - s(z(t), w(t)) \) can be also formulated as

\[
\dot{v}(x(t), y(t + \cdot)) - s(z(t), w(t)) = \eta(t) \begin{bmatrix}
H \\
O_{n \times (p+n)}
\end{bmatrix} P \begin{bmatrix}
O_{n \times q} & O_n & O_{n \times \nu} & O_{n \times \rho} & I_n \\
O_{\rho \times q} & F(0)A_4 & F(0)A_5 - F(-r) & -M & O_{\rho \times n}
\end{bmatrix} \eta(t) + \eta(t) \begin{bmatrix}
\Sigma^T J_2 \\
O_{(2n+\nu+p+q)\times (2n+\nu+p)}
\end{bmatrix} \eta(t),
\]

(33)

where \( \Phi \) and \( \tilde{\Gamma} \) have been defined in the statement of Theorem 2 and \( \eta(t) := \text{col}\left(\chi(t), \dot{x}(t)\right) \) with \( \chi(t) \) defined in (22). Specifically, the information of (1) in (33) is characterized by the dynamical constraints \( \forall t \in \mathcal{E} : [\eta(t) : \forall t \in T, A - I_n \eta(t) = 0] \) where \( \dot{x}(t) \) is part of \( \eta(t) \).

Now applying (23) to (33) assuming \( U \geq 0 \) in (13) yields

\[
\forall t \in T, \forall \eta(t) \in \mathcal{E}, \dot{v}_1(x(t), y(t + \cdot)) - s(z(t), w(t)) \leq \eta(t) \begin{bmatrix}
\Phi + \tilde{\Sigma}^T J_1 \tilde{\Sigma}
\end{bmatrix} \eta(t),
\]

(34)

where \( \hat{\Phi} \) is defined in (32). Furthermore, since (26) and (28) are equivalent, one can conclude that

\[
\theta^T \begin{bmatrix}
\Phi + \tilde{\Sigma}^T J_1 \tilde{\Sigma}
\end{bmatrix} \theta < 0, \ \forall \theta \in \left\{ y \in \mathbb{R}^{2n+\nu+p+q} \setminus \{0\} : [A - I_n] y = 0 \right\}
\]

(35)

holds if and only if

\[
\exists Y \in \mathbb{R}^{(2n+\nu+p+q)\times n} : \tilde{\Phi} + \tilde{\Sigma}^T J_1 \tilde{\Sigma} + Y [A - I_n] < 0,
\]

(36)

where (35) infers the existence of (12) satisfying (7) and (10) considering the properties positive matrices and the right limit substitution \( t = \tau, x(\tau) = \xi, y(\tau + \cdot) = \phi(\cdot) \).

Apply Schur complement to (36) with \( J_1 > 0 \). It shows the inequality in (36) is equivalent to

\[
\begin{bmatrix}
-J_r^{-1} \tilde{\Sigma} \\
\Phi
\end{bmatrix} + S \begin{bmatrix}
O_{n \times n} \\
Y
\end{bmatrix} \begin{bmatrix}
O_{n \times m} \\
A - I_n
\end{bmatrix} = \begin{bmatrix}
-J_r^{-1} \tilde{\Sigma} \\
\Phi
\end{bmatrix} + S \begin{bmatrix}
O_{m \times (2n+\nu+p+q)} \\
I_{2n+\nu+p+q}
\end{bmatrix} Y \begin{bmatrix}
O_{n \times m} \\
A - I_n
\end{bmatrix} < 0.
\]

(37)

By the equivalence between (30) and (29), we have (37) holds if and only if

\[
\begin{bmatrix}
O_{m \times (2n+\nu+p+q)} \\
I_{2n+\nu+p+q}
\end{bmatrix} \begin{bmatrix}
-J_r^{-1} \tilde{\Sigma} \\
\Phi
\end{bmatrix} [s] = \begin{bmatrix}
I_m \\
O_{m \times (2n+\nu+p+q)}
\end{bmatrix} \begin{bmatrix}
-J_r^{-1} \tilde{\Sigma} \\
\Phi
\end{bmatrix} [s] = -J_r^{-1} < 0
\]

(38)
and
\[
\begin{bmatrix}
J^{-1} \\
* \\
\Sigma \\
\Phi
\end{bmatrix} [O_{n \times m} \ A \ -I_n] = \begin{bmatrix}
J^{-1} \\
* \\
\Sigma \\
\Phi
\end{bmatrix} [I_{m+q+n+\nu+\rho}].
\]

(39)

Now realize that (38) holds given (39). Thus (39) is equivalent to (36). On the other hand, note that (39) gives
\[
\begin{bmatrix}
O_{(m+q) \times (2n+\nu+\rho)} \\
J^{-1} \\
\Sigma \\
\Phi
\end{bmatrix} \begin{bmatrix}
J^{-1} \\
* \\
\Sigma \\
\Phi
\end{bmatrix} [I_{m+q} \ O_{(m+q) \times (2n+\nu+\rho)}] = \begin{bmatrix}
J^{-1} \\
* \\
\Sigma \\
\Phi
\end{bmatrix} [J_3 + S_y(D^T_2 J_2)] < 0
\]

(40)

which also infers (38). By Lemma 4, one can conclude that (39) with (40) are equivalent to (31) which subsequently is equivalent to (37). This in fact shows that there are redundant slack variables in (37) which can be reduced into the form of (31). Thus (35) is equivalent to (31) which finishes the proof.

**Corollary 1.** Theorem 1 is equivalent to Theorem 2 in terms of feasibility.

**Proof.** Note that (14) can be reformulated into
\[
\begin{bmatrix}
J^{-1} \\
* \\
\Sigma \\
\Phi
\end{bmatrix} = \begin{bmatrix}
J^{-1} \\
* \\
\Sigma \\
\Phi
\end{bmatrix} [I_{m+q} \ O_{(m+q) \times (2n+\nu+\rho)}]
\]

(41)

considering all the structures of (15), (32) with A, Σ and Φ. Specifically, by letting P = \begin{bmatrix} P_1 \\
Q \\
R \end{bmatrix} in (12), in which
\[P_1 \in S^n, \ Q \in S^{n \times \nu} \ and \ R \in S^{\nu \times \nu},\]
(41) can be derived similarly as the equations (35)–(37) in Feng & Nguang (2016). Thus (39) is equal to (14). Since (39) is equivalent to (31), this finishes the proof.

**4 Conclusion**

Dissipative conditions for a linear CDDS have been derived based on a dynamical constraints approach. The result considered as a generalization of the Finsler Lemma approach in de Oliveira & Skelton (2001) towards delay related system. Moreover, we have shown that the dissipative conditions with slack variables are equivalent to the conditions derived by directly substituting \(\dot{x}(t)\) during the process of constructing a Krasovskii functional.

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