Abstract. Let \( n \geq 2 \). We introduce the notion of \( n \)-representations of quivers, and we explicitly provide concrete examples of \( 2 \)-representations of quivers. We establish the categories of \( n \)-representations and investigate kernels and cokernels in the categories of \( n \)-representations of quivers. Further, we construct them in terms of kernels and cokernels of morphisms in the usual categories of quiver representations. We show that every morphism in the categories of \( n \)-representations has a canonical decomposition. Most importantly, we prove that the categories of \( n \)-representations of quivers are \( k \)-linear abelian categories.

1. Introduction

The notions of quiver and their representation can be traced back to 1972 when they were introduced by Gabriel [13]. Since then, it has been studied as a vibrant subject with a strong linkage with many other mathematics areas. This comes from the modern approach that quiver representations theory suggests. Due to its inherent combinatorial flavor, this theory has recently been largely studied as extremely important theory with connections to many theories, such as associative algebra, combinatorics, algebraic topology, algebraic geometry, quantum groups, Hopf algebras, tensor categories. Further, it bridges the gap between combinatorics and category theory, and this simply comes from the well-known fact that there is a forgetful functor, which has a left adjoint, from the category of small categories to the category of quivers. It turns out that it gives “new techniques, both of combinatorial, geometrical and categorical nature.” [8, p. ix].

The interaction area of quivers representation theory with other branches of mathematics can be significantly extended by introducing a generalization of this theory. However, suggesting a useful generalization needs to be done carefully because not all generalizations are capable of supporting our goal of finding a generalization that plays a successful role in developing this theory.

Furthermore, generalizing the notion of an object (or objects) with structures can be done with no compatibility condition between these structures, or with a compatibility condition between them. For instance, bitopological spaces can be regarded as a generalization of the notion topological spaces. A bitopological space, introduced by Kelly in [14], is a triple \((X, \tau, \tau')\), where \( X \) is a set equipped with two arbitrary topologies \( \tau, \tau' \) [9, p. ix]. Obviously, this definition does not require any compatibility condition between \( \tau, \tau' \). However, it is still very important, and indeed chapter VII in [9, p. 318-384] is totally devoted for applications of bitopologies.

On the other hand, there is an another kind of generalization involved with compatibility condition. For example, the concept of corings is a generalization of that of coalgebras, and it involves certain compatibility conditions. The compatibility conditions are substantially helpful in characterizing

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and describing many notions.

The notion of $n$-representations of quivers can be introduced as a generalization with a compatibility condition. We start with 2-representations of quivers and inductively define $n$-representations of quivers. Then we mainly concentrate our study on 2-representations of quivers because they roughly give a complete description of $n$-representations of quivers which can be established inductively. We alternatively and preferably call 2-representations of quivers birepresentations of quivers.

Birepresentations of quivers are fundamentally different from representations of biquivers introduced by Sergeichuk in [24, p. 237].

The main goal of this paper is to introduce the concept of $n$-representations of quivers and set up the basic notions of this concept. Further, we mainly establish the categories of $n$-representations of quivers and show that these categories are abelian.

As a part of our next paper, we will show that $n$-representations of quivers can be identified as representations of certain quivers. We will intentionally not use this observation in this paper since this allows us to explore a more explicit description and characterization for $n$-representations of quivers without using the categorical perspective description of being “essentially the same”.

The sections of this paper can be summarized in the following setting.

In Section 2, we give some detailed background on quiver representations and few categorical notions that we need for the next sections.

In Section 3, we introduce the notion of $n$-representations of quivers, we explicitly give concrete examples of birepresentations of quivers. In addition, we establish the categories of $n$-representations of quivers.

In Section 4, we investigate the kernels and cokernels in the categories of $n$-representations of quivers. We also construct them in terms of kernels and cokernels in the usual categories of quiver representations corresponding to each component.

In Section 5, we show that the morphisms in the categories of $n$-representations of quivers have canonical decomposition. We also show that each hom set in these categories is equipped with a structure of an abelian group such that composition of morphisms is biadditive with respect to this structure. We end the paper by showing that the categories of $n$-representations of quivers are abelian.

2. Preliminaries

Throughout this paper $k$ is an algebraically closed field, $n \geq 2$, and $Q$, $Q'$, $Q_1$, $Q_2$, ..., $Q_n$ are quivers. We also denote $kQ$ the path algebra of $Q$. Unless otherwise specified, we will consider only finite, connected, and acyclic quivers.

Let $A$ be a (locally small) category and $A$, $B$ objects in $A$. We denote by $A(A, B)$ the set of all morphisms from $A$ to $B$.

Let $A$, $B$ be categories. Following [18, p. 74], the product category $A \times B$ is the category whose objects are all pairs of the form $(A, B)$, where $A$ is an object of $A$ and $B$ an object of $B$. An arrow is a pair $(f, g) : (A, B) \to (A', B')$, where $f : A \to A'$ is an arrow of $A$ and $g : B \to B'$ is an arrow of $B$. The identity arrow for $A \times B$ is $(id_A, id_B)$ and composition is defined component-wise, so

---

1 A directed graph with usual and dashed arrows will be called a biquiver. Its representation is given by assigning to each vertex a complex vector space, to each usual arrow a linear mapping, and to each dashed arrow a semilinear mapping [24, p. 237].
\((f, g)(f', g') = (ff', gg')\). There is a projective functor \(P_1 : A \times B \to A\) defined by \(P_1(A, B) = A\) and \(P_1(f, g) = f\). Similarly, we have a projective functor \(P_2 : A \times B \to B\) defined by \(P_2(A, B) = B\) and \(P_2(f, g) = g\). For the fundamental concepts of category theory, we refer to \([15, 17, 4, 21, 1, 7, 12, 20, 19]\).

Following \([22]\), a quiver \(Q = (Q_1, Q_0, s, t)\) consists of
- \(Q_1\) a set of vertices,
- \(Q_0\) a set of arrows,
- \(s : Q_0 \to Q_1\), a map from arrows to vertices, mapping an arrow to its starting point,
- \(t : Q_0 \to Q_1\), a map from arrows to vertices, mapping an arrow to its terminal point.

We will represent an element \(\alpha \in Q_1\) by drawing an arrow from its starting point \(s(\alpha)\) to its endpoint \(t(\alpha)\) as follows: \(s(\alpha) \xrightarrow{\alpha} t(\alpha)\).

A representation \(M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}\) of a quiver \(Q\) is a collection of \(k\)-vector spaces \(M_i\) one for each vertex \(i \in Q_1\), and a collection of \(k\)-linear maps \(\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}\) one for each arrow \(\alpha \in Q_0\).

A representation \(M\) is called \textit{finite-dimensional} if each vector space \(M_i\) is finite-dimensional.

Let \(Q\) be a quiver and let \(M = (M_i, \varphi_\alpha), M' = (M'_i, \varphi'_\alpha)\) be two representations of \(Q\). A \textit{morphism} of representations \(f : M \to M'\) is a collection \((f_i)_{i \in Q_0}\) of \(k\)-linear maps \(f_i : M_i \to M'_i\) such that for each arrow \(s(\alpha) \xrightarrow{\alpha} t(\alpha)\) in \(Q_1\) the diagram

\[
\begin{array}{ccc}
M_{s(\alpha)} & \xrightarrow{\phi_\alpha} & M_{t(\alpha)} \\
\downarrow f_{s(\alpha)} & & \downarrow f_{t(\alpha)} \\
M'_{s(\alpha)} & \xrightarrow{\phi'_\alpha} & M'_{t(\alpha)}
\end{array}
\]

commutes.

A morphism of representations \(f = (f_i) : M \to M'\) is an isomorphism if each \(f_i\) is bijective. The class of all representations that are isomorphic to a given representation \(M\) is called the \textit{isoclass} of \(M\).

This gives rise to define a category \(\text{Rep}_k(Q)\) of \(k\)-linear representations of \(Q\). We denote by \(\text{rep}_k(Q)\) the full subcategory of \(\text{Rep}_k(Q)\) consisting of the finite dimensional representations.

Given two representations \(M = (M_i, \phi_\alpha)\) and \(M' = (M'_i, \phi'_\alpha)\) of \(Q\), the representation

\[
M \oplus M' = (M_i \oplus M'_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \phi'_\alpha \end{bmatrix})
\]

is the \textit{direct sum} of \(M\) and \(M'\) in \(\text{Rep}_k(Q)\) \([2]\ p. 71\).

A nonzero representation of a quiver \(Q\) is said to be \textit{indecomposable} if it is not isomorphic to a direct sum of two nonzero representations \([11]\ p. 21\).

We will need the following propositions.

**Proposition 2.1.** \([2]\ p. 70\] Let \(Q\) be a finite quiver. Then \(\text{Rep}_k(Q)\) and \(\text{rep}_k(Q)\) are \(k\)-linear abelian categories.

**Proposition 2.2.** \([2]\ p. 74\] Let \(Q\) be a finite, connected, and acyclic quiver. There exists an equivalence of categories \(\text{Mod}_kQ \simeq \text{Rep}_k(Q)\) that restricts to an equivalence \(\text{mod}_kQ \simeq \text{rep}_k(Q)\), where \(kQ\) is the path algebra of \(Q\). \(\text{Mod}_kQ\) denotes the category of right \(kQ\)-modules, and \(\text{mod}_kQ\) denotes the full subcategory of \(\text{Mod}_kQ\) consisting of the finitely generated right \(kQ\)-modules.
This is a very brief review of the basic concepts involved with our work. For the basic notions of quiver representations theory, we refer the reader to [2], [22], [3], [5], [11], [6], [26].

3. n-representations of Quivers: Basic Concepts

Let \( Q = (Q_0, Q_1, s, t) \), \( Q' = (Q'_0, Q'_1, s', t') \) be quivers.

**Definition 3.1.** A 2-representation of \((Q, Q')\) (or a birepresentation of \((Q, Q')\)) is a triple \( \bar{M} = ((M_i, \phi_i), (M'_j, \phi'_j), (\psi^\alpha_{ij}))_{i,j \in Q_0, \alpha \in Q_1} \), where \((M_i, \phi_i), (M'_j, \phi'_j)\) are representations of \(Q, Q'\) respectively, and \((\psi^\alpha_{ij})\) is a collection of \(k\)-linear maps \(\psi^\alpha_{ij} : M_i \to M'_j\), one for each pair of arrows \((\alpha, \beta) \in Q_1 \times Q'_1\).

Unless confusion is possible, we denote a birepresentation simply by \( \bar{M} = (M, M', \psi) \). Next, we inductively define \(n\)-representations for any integer \(n \geq 2\).

For any \( m \in \{2, \ldots, n\} \), let \( Q_m = (Q^{(m)}_0, Q^{(m)}_1, s^{(m)}, t^{(m)}) \) be a quiver. A \(n\)-representation of \((Q_1, Q_2, \ldots, Q_n)\) is \((2n - 1)\)-tuple \( V = (V^{(1)}, V^{(2)}, \ldots, V^{(n)}, \psi_1, \psi_2, \ldots, \psi_{n-1}) \), where for every \( m \in \{1, 2, \ldots, n\} \), \( V^{(m)} \) is a representation of \( Q_m \), and \((\psi^{\gamma(m)}_{m, \gamma(m-1)})\) is a collection of \(k\)-linear maps

\[
\psi^{\gamma(m)}_{m, \gamma(m-1)} : V^{(m-1)}_{t^{(m)}(\gamma(m-1))} \to V^{(m)}_{s^{(m)}(\gamma(m))},
\]

one for each pair of arrows \((\gamma^{(m-1)}, \gamma^{(m)}) \in Q^{(m-1)}_1 \times Q^{(m)}_1\) and \(m \in \{2, \ldots, n\}\).

**Remark 3.2.**

(i) When no confusion is possible, we simply write \(s, t\) instead of \(s', t'\) respectively, and for every \(m \in \{1, 2, \ldots, n\}\), we write \(s, t\) instead of \(s^{(m)}, t^{(m)}\) respectively.

(ii) It is clear that if \((V^{(1)}, V^{(2)}, \ldots, V^{(n)}, \psi_1, \psi_2, \ldots, \psi_{n-1})\) is an \(n\)-representation of \((Q_1, Q_2, \ldots, Q_n)\), then \((V^{(1)}, V^{(2)}, \ldots, V^{(n-1)}, \psi_1, \psi_2, \ldots, \psi_{n-2})\) is an \((n-1)\)-representation of \((Q_1, Q_2, \ldots, Q_{n-1})\) for every integer \(n \geq 2\).

(iii) Part (ii) implies that for any integer \(n > 2\), \(n\)-representations roughly inherit all the properties and the universal constructions that \((n-1)\)-representations have. Thus, we mostly focus on studying birepresentations since they can be regarded as a mirror in which one can see a clear description of \(n\)-representations for any integer \(n > 2\).

**Example 3.3.** Let \(Q, Q'\) be the following quivers

(3.1)

\[
\begin{align*}
Q & : & 1 & \longrightarrow & 2 \\
Q' & : & 3 & \leftarrow & 4 \\
& & 2
\end{align*}
\]
and consider the following:

\[
\begin{array}{c}
\begin{array}{cc}
\text{M} & \text{M'}
\end{array}
\end{array}
\]

Then \( M \) (respectively \( M' \)) is a representation of \( Q \) ((respectively \( Q' \)). The following are birepresentations of \((Q,Q')\).

Definition 3.4. Let \( \bar{V} = (V,V',\psi) \), \( \bar{W} = (W,W',\psi') \) be birepresentations of \((Q,Q')\). Write \( V = (V_i,\phi_\alpha) \), \( V' = (V'_i,\mu_\beta) \), \( W = (W_i,\phi'_\alpha) \), \( W' = (W'_i,\mu'_\beta) \). A morphism of birepresentations \( \bar{f} : \bar{V} \to \bar{W} \) is a pair \( \bar{f} = (f,f') \), where \( f = (f_i) : (V_i,\phi_\alpha) \to (W_i,\phi'_\alpha) \), \( f' = (f'_i) : (V'_i,\mu_\beta) \to (W'_i,\mu'_\beta) \).
are morphisms in \( \text{Rep}_k(Q), \text{Rep}_k(Q') \) respectively such that the following diagram commutes.

\[
\begin{array}{c}
\vdots
\end{array}
\]

The composition of two maps \((f, f')\) and \((g, g')\) can be depicted as the following diagram.

\[
\begin{array}{c}
\vdots
\end{array}
\]

In general, if \( \bar{V} = (V^{(1)}, V^{(2)}, ..., V^{(n)}, \psi_1, \psi_2, ..., \psi_{n-1}) \), \( W = (W^{(1)}, W^{(2)}, ..., W^{(n)}, \psi'_1, \psi'_2, ..., \psi'_{n-1}) \) are \( n \)-representations of \((Q_1, Q_2, ..., Q_n)\), then a morphism of \( n \)-representations \( \bar{f} : \bar{V} \to \bar{W} \) is \( n \)-tuple \( f = (f^{(1)}, f^{(2)}, ..., f^{(n-1)}) \), where

\[
f^{(m)} = (f^{(m)}_{ij}) : (V^{(m)}_{ij}, \phi^{(m)}_{ij}) \to (W^{(m)}_{ij}, \mu^{(m)}_{ij}),
\]

is a morphism in \( \text{Rep}_k(Q_m) \) for any \( m \in \{2, ..., n\} \), and for each pair of arrows \((\gamma^{(m-1)}, \gamma^{(m)}) \in Q^{(m-1)}_i \times Q^{(m)}_i\) the following diagram is commutative.
A morphism of \( \text{Rep}(Q,Q') \) consists of \( B \) of the categories of birepresentations of quivers to build a bicategory. Indeed, there is a bicategory \( \mathcal{B} \) for every \( V \) \( (3.7) \)

for every \( m \in \{2, \ldots, n\} \).

A morphism of \( n \)-representations can be depicted as:

\[
\begin{array}{ccc}
V^{(m-1)} \ & \psi^{(m)} & V^{(m)} \\
\downarrow f^{(m-1)} & & \downarrow f^{(m)} \\
W^{(m-1)} \ & \psi' \gamma^{(m)} & W^{(m)}
\end{array}
\]

(3.8)

Remark 3.5.

(i) The above definition gives rise to form a category \( \text{Rep}_{(Q,Q')} \) of \( k \)-linear birepresentations of \( (Q,Q') \). We denote by \( \text{rep}_{Q,Q'} \) the full subcategory of \( \text{Rep}_{(Q,Q')} \) consisting of the finite dimensional birepresentations. Similarly, it also creates a category \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_n)} \) of \( n \)-representations. We denote \( \text{rep}_{(Q_1, Q_2, \ldots, Q_n)} \) the full subcategory of \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_n)} \) consisting of the finite dimensional \( n \)-representations.

(ii) For any \( m \in \{2, \ldots, n\} \), let \( Q_m = (Q_0^{(m)}, Q_1^{(m)}, s^{(m)}, t^{(m)}) \) be a quiver and fix \( j \in \{2, \ldots, n\} \). Let \( \Upsilon_{\text{rep}_k(Q_j)} \) be the subcategory of \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_n)} \) whose objects are \((2n-1)\)-tuples \( \bar{X} = (0,0, \ldots, V^{(j)}, 0, \ldots, 0, \psi_1, \psi_2, \ldots, \psi_{n-1}) \), where \( V^{(j)} \) is a representation of \( Q_j \), and \( \psi_{\gamma^{(m)}} = 0 \) for every pair of arrows \( (\gamma^{(m-1)}, \gamma^{(m)}) \in Q^{(m-1)} \times Q^{(m)} \) and \( m \in \{2, \ldots, n\} \). Then \( \Upsilon_{\text{rep}_k(Q_j)} \) is clearly a full subcategory of \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_n)} \). Notably, we have an equivalence of categories \( \Upsilon_{\text{rep}_k(Q_j)} \simeq \text{Rep}_k(Q_j) \), and thus by Proposition 2.2, we have \( \text{Rep}_k(Q_j) \simeq \Upsilon_{\text{rep}_k(Q_j)} \simeq \text{Mod} kQ_j \). It turns out that the category \( \text{Rep}_k(Q_j) \) and \( \text{Mod} kQ_j \) can be identified as full subcategories of \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_n)} \).

The category \( \Upsilon_{\text{rep}_k(Q_j)} \) has a full subcategory \( \Upsilon_{\text{rep}_k(Q_j)} \) when we restrict the objects on the finite dimensional representations. Therefore, we also have \( \text{rep}_k(Q_j) \simeq \Upsilon_{\text{rep}_k(Q_j)} \simeq \text{mod} kQ_j \).

Remark 3.6. Let \( B_0 \) be the class of all quivers. One might consider the class \( B_0 \) and full subcategories of the categories of birepresentations of quivers to build a bicategory. Indeed, there is a bicategory \( \mathcal{B} \) consists of

- the objects or the 0-cells of \( \mathcal{B} \) are simply the elements of \( B_0 \)
• for each \(Q, Q' \in B_0\), we have \(\mathcal{B}(Q, Q') = \text{Rep}_k(Q) \times \text{Rep}_k(Q')\), whose objects are the 1-cells of \(\mathcal{B}\), and whose morphisms are the 2-cells of \(\mathcal{B}\).

• for each \(Q, Q', Q'' \in B_0\), a composition functor

\[\mathcal{F} : \text{Rep}_k(Q') \times \text{Rep}_k(Q'') \times \text{Rep}_k(Q) \to \text{Rep}_k(Q) \times \text{Rep}_k(Q'')\]

defined by:

\[\mathcal{F}((N', N''), (M, M')) = (M, N''), \quad \mathcal{F}((g', g''), (f, f')) = (f, g'')\] on 1-cells

\[\quad (M, M'), (N'', N')\] and 2-cells \((f, f'), (g', g'')\).

• for any \(Q \in B_0\) and for each \((M, M') \in \mathcal{B}(Q, Q)\), we have \(\mathcal{F}((M, M'), (M, M')) = (M, M')\) and \(\mathcal{F}((M', M'), (M, M')) = (M, M').\) Furthermore, for any 2-cell \((f, f')\), we have \(\mathcal{F}((f', f'), (f, f')) = (f', f')\) and \(\mathcal{F}((f, f'), (f, f)) = (f, f').\) Thus, the identity and the unit coherence axioms hold.

The rest of bicategories axioms are obviously satisfied. For each \(Q, Q' \in B_0\), let \(\Xi_{Q,Q'}\) be the full subcategory of \(\text{Rep}_{Q,Q'}\), whose objects are the triples \((X, X', \Psi)\), where \((X, X') \in \text{Rep}_k(Q) \times \text{Rep}_k(Q')\) and \(\Psi_{\alpha\beta} = 0\) for every pair of arrows \((\alpha, \beta) \in Q \times Q'\), and whose morphisms are usual morphisms of birepresentations between them. Clearly, \(\Xi_{Q,Q'} \cong \text{Rep}_k(Q) \times \text{Rep}_k(Q')\) for any \(Q, Q' \in B_0\). Thus, by considering the class \(B_0\) and the full subcategories described above of the birepresentations categories of quivers, we can always build a bicategory as above.

Obviously, the discussion above implies that for each \(Q, Q' \in B_0\), the product category \(\text{Rep}_k(Q) \times \text{Rep}_k(Q')\) can be viewed as a full subcategory of \(\text{Rep}_{Q,Q'}\). Further, it implies that the product category \(\text{Rep}_k(Q_1) \times \text{Rep}_k(Q_2) \times \ldots \times \text{Rep}_k(Q_n)\) can be viewed as a full subcategory of \(\text{Rep}_{Q_1,Q_2,\ldots,Q_n}\), where \(Q_1, Q_2, \ldots, Q_n \in B_0\) and \(n \geq 2\).

We also have the same analogue if we replace \(\text{Rep}_{Q_1,Q_2,\ldots,Q_n}\), by \(\text{rep}_{Q_1,Q_2,\ldots,Q_n}\), and \(\text{Rep}_k(Q_1), \text{Rep}_k(Q_2), \ldots, \text{Rep}_k(Q_n)\) by \(\text{rep}_k(Q_1), \text{rep}_k(Q_2), \ldots, \text{rep}_k(Q_n)\) respectively. For the basic notions of bicategories, we refer the reader to [16].

**Example 3.7.** Let \(Q, Q'\) be the quivers defined in Example 3.3 and consider the following:

\[
\begin{align*}
&V \\
&\begin{array}{c}
        \downarrow 1 \\
        k \quad 0 \\
        \downarrow 1 \\
        \quad k^2 \\
\end{array} \\
&\begin{array}{c}
        \downarrow 1 \\
        0 \\
        \downarrow 1 \\
\end{array} \\
&V' \\
&\begin{array}{c}
        \downarrow 1 \\
        k \\
\end{array} \\
\end{align*}
\]

\[
\begin{align*}
&W \\
&\begin{array}{c}
        \downarrow 1 \\
        k \\
\end{array} \\
\end{align*}
\]

\[
\begin{align*}
&(3.9) \\
&V' \\
&\begin{array}{c}
        \downarrow 1 \\
        k \\
\end{array} \\
&W' \\
&\begin{array}{c}
        \downarrow 1 \\
        k \\
\end{array} \\
\end{align*}
\]

Then \(V, V'\) (respectively \(W, W'\)) are representations of \(Q\) (respectively \(Q'\)) [22]. Furthermore, it is straightforward to verify that \(\text{Rep}_k(Q)(V, W) \cong k^2\) and \(\text{Rep}_k(Q)(V', W') \cong k\). We refer the reader
to [22] for more details. Consider the following.

(3.11) \( \vec{V} = (V, V', \psi) \)

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

(3.12) \( \vec{V} = (V', V, \psi') \)

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

(3.13) \( \vec{W} = (W, W', \psi') \)

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]
(3.14) 

$$ \textbf{\overline{W}} = (W', W, \psi') $$

Then $\textbf{\overline{V}}$, $\textbf{\overline{W}}$ are birepresentations of $(Q, Q')$, and $\textbf{\overline{V}}$, $\textbf{\overline{W}}$ are birepresentations of $(Q', Q)$. To compute $\text{Rep}_{Q, Q'}(\textbf{\overline{V}}, \textbf{\overline{W}})$, consider the following diagram.
The commuting squares give the relations
\begin{equation}
\begin{pmatrix}
a & c \\
0 & d \\
0 & c + d + e \\
0 & e \\
0 & a \\
0 & b
\end{pmatrix} = \begin{pmatrix}
a & c \\
b & d \\
c & d + e \\
e & l \\
e & a \\
0 & b
\end{pmatrix} = 0.
\end{equation}
Hence, we obtain \( \text{Rep}_{(Q,Q')}((\bar{V},\bar{W})) \cong k \).

We leave it to the reader to compute \( \text{Rep}_{(Q,Q')}((\bar{W},\bar{V})), \text{Rep}_{(Q',Q)}((\bar{W},\bar{V})) \text{ and } \text{Rep}_{(Q',Q')}((\bar{W},\bar{V})) \).

**Definition 3.8.** Let \( \bar{V} = (V,V',\psi), \bar{W} = (W,W',\psi') \) be birepresentations of \( (Q,Q') \). Write \( V = (V_i,\phi_i), V' = (V_i',\phi_i'), W = (W_i,\mu_i), W' = (W_i',\mu_i') \). Then

\begin{equation}
\bar{V} \oplus \bar{W} = ((V_i \oplus W_i, \begin{bmatrix}
\phi_i & 0 \\
0 & \mu_i
\end{bmatrix}), (V_i' \oplus W_i', \begin{bmatrix}
\phi_i' & 0 \\
0 & \mu_i'
\end{bmatrix}), \begin{bmatrix}
\psi_i & 0 \\
0 & \psi_i'
\end{bmatrix}),
\end{equation}

where \( (V_i \oplus W_i, \begin{bmatrix}
\phi_i & 0 \\
0 & \mu_i
\end{bmatrix}), (V_i' \oplus W_i', \begin{bmatrix}
\phi_i' & 0 \\
0 & \mu_i'
\end{bmatrix}) \) are the direct sums of \( (V_i,\phi_i), (W_i,\mu_i) \) and \( (V_i',\phi_i'), (W_i',\mu_i') \) in \( \text{Rep}_k(Q), \text{Rep}_k(Q') \) respectively, is a birepresentation of \( (Q,Q') \) called the **direct sum** of \( \bar{V}, \bar{W} \) (in \( \text{Rep}_{(Q,Q')} \)).

Similarly, direct sums in \( \text{Rep}_{(Q_1,Q_2,...,Q_n)} \) can be defined.

**Example 3.9.** Consider the birepresentations in Example 3.7. Then the direct sum \( \bar{V} \oplus \bar{W} \) is the birepresentation

\begin{equation}
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\end{equation}

**Definition 3.10.** A birepresentation \( \bar{V} \in \text{Rep}_{(Q,Q')} \) is called **indecomposable** if \( M \neq 0 \) and \( M \) cannot be written as a direct sum of two nonzero birepresentations, that is, whenever \( \bar{M} \cong \bar{L} \oplus \bar{N} \) with \( \bar{L}, \bar{N} \in \text{Rep}_{(Q,Q')} \), then \( \bar{L} = 0 \) or \( \bar{N} = 0 \).

**Example 3.11.** Consider the birepresentations in Example 3.7. The birepresentation \( \bar{V} \) is indecomposable, but the birepresentation \( \bar{W} \) is not.
The above example also shows that if $\tilde{W} = ((W_i, \phi_\alpha), (W'_i, \phi'_\beta), (\psi^\alpha_\beta))$ is birepresentation of $(Q, Q')$ such that the representations $(W_i, \phi_\alpha)$ and $(W'_i, \phi'_\beta)$ are indecomposable in $Rep_k(Q)$, $Rep_k(Q')$ respectively, then $\tilde{W}$ need not be indecomposable. The proof of the following proposition is straightforward.

**Proposition 3.12.** Let $\tilde{V} = ((V_i, \phi_\alpha), (V'_i, \phi'_\beta), (\psi^\alpha_\beta)) \in Rep_{(Q, Q')}$ be an indecomposable birepresentation, then the representations $(W_i, \phi_\alpha)$ and $(W'_i, \phi'_\beta)$ are indecomposable in $Rep_k(Q)$, $Rep_k(Q')$ respectively.

**Proof.**

4. **Construction For Kernels and Cokernels in $Rep_{(Q, Q')}$**

Following [25, p. 49], let $C$ be a category with zero object and $f : A \rightarrow B$ a morphism in $C$.

(i) A morphism $i : K \rightarrow A$ is called a kernel of $f$ if $if = 0$ and, for every morphism $g : D \rightarrow A$ with $gf = 0$, there is a unique morphism $h : D \rightarrow K$ with $hi = g$, i.e. the diagram is commutative.

\[
\begin{array}{ccc}
D & \rightarrow & A \\
\downarrow^{g} & & \downarrow^{f} \\
K & \rightarrow & B \\
\uparrow^{i} & & \uparrow^{h}
\end{array}
\]

(ii) A morphism $p : B \rightarrow C$ is called a cokernel of $f$ if $fp = 0$ and, for every $g : B \rightarrow D$ with $fg = 0$, there is a unique morphism $h : C \rightarrow D$ with $ph = g$, i.e. the diagram is commutative.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^{f} & & \downarrow^{p} \\
D & \rightarrow & C \\
\uparrow^{g} & & \uparrow^{h}
\end{array}
\]

**Remark 4.1.** [25, p. 50] Let $R$ be a ring with unity and $R \rightarrow Mod$ (resp $Mod \rightarrow R$) be category of left $R$-modules (resp right $R$-modules), and let $f : M \rightarrow N$ be a homomorphism in $R \rightarrow Mod$ (resp $Mod \rightarrow R$). Then

(i) The inclusion $i : Ker \ f \rightarrow M$ is a kernel of $f$ in $R \rightarrow Mod$ (resp $Mod \rightarrow R$) because if $g : L \rightarrow M$ is given with $fg = 0$, then $g(L) \subseteq Ker \ f$. Clearly, we can define $g' : L \rightarrow Ker \ f$, $x \mapsto g(x)$ to be the unique morphism with $ig' = g$.

(ii) The projection $p : N \rightarrow Coker \ f = N/f(M)$ is a cokernel of $f$ in $R \rightarrow Mod$ (resp $Mod \rightarrow R$) because if $h : N \rightarrow L$ is given with $hf = 0$, then $Im \ f \subseteq Ker \ h$. Thus, by using the First Isomorphism Theorem, we can define $h' : Coker \ f \rightarrow L$, $n + f(M) \mapsto h(n)$ to be the unique morphism with $h'p = h$. 

Let $\bar{f} = (f, f') : \bar{V} = (V, V', \psi) \to \bar{W} = (W, W', \psi')$ be a morphism in $Rep_{(Q, Q')}$. From Proposition 2.1, the kernels of $f, f'$ exist in $Rep_k(Q), Rep_k(Q')$ respectively. Let $(\kappa, \zeta)$, $(\kappa', \zeta')$ be the kernels of $f, f'$ in $Rep_k(Q), Rep_k(Q')$ respectively. Write $\kappa = (\kappa_i, \chi_{\alpha})$, $\zeta = (\zeta_{\alpha})$, $\kappa' = (\kappa'_{i'}, \nu_{\beta})$, $\zeta' = (\zeta'_{\beta})$ and consider the following diagram.

For all $\alpha \in Q_1$, $\beta \in Q'_1$, define $\xi_{\alpha}^{\beta} : \kappa_{t(\alpha)} \to \kappa'_{s(\alpha)}$ to be the restriction of $\psi_{\alpha}^{\beta}$. Then $\xi_{\alpha}^{\beta}$ is well defined for all $\alpha \in Q_1$, $\beta \in Q'_1$ since for all $x \in \kappa_{t(\alpha)}$, we have $f'_{s(\beta)} \psi_{\beta}^{\alpha}(x) = \psi_{\beta}^{\alpha} f_t(\alpha)(x) = 0$. Thus, $\psi_{\beta}^{\alpha}(x) \in \kappa'_{s(\alpha)}$, and it follows that $\xi_{\alpha}^{\beta}$ is well defined.

Let $\bar{\kappa} = (\kappa, \kappa', \xi)$ and $\bar{\zeta} = (\zeta, \zeta')$. Then $\bar{\kappa} \in Rep_{(Q, Q')}$, and we have the following proposition.

**Proposition 4.2.** The pair $(\bar{\kappa}, \bar{\zeta})$ is the kernel of $\bar{f}$ in $Rep_{(Q, Q')}$. 

**Proof.** Let $\bar{\lambda} = (\lambda, \lambda') : \bar{N} = (N, N', \Psi) \to \bar{V} = (V, V', \psi)$ be a morphism in $Rep_{(Q, Q')}$ with $\bar{f} \bar{\lambda} = \bar{0}$, where $\bar{0}$ is the zero object in $Rep_{(Q, Q')}$. Consider the following diagram.
Since $(\kappa_i, \chi_\alpha), (\kappa'_i, \nu_\beta)$ are the kernels of $f, f'$ in $\text{Rep}_k(Q), \text{Rep}_k(Q')$ respectively, there exist unique morphisms $\tau : N \to \kappa$, $\tau' : N' \to \kappa'$ in $\text{Rep}_k(Q), \text{Rep}_k(Q')$ respectively making their respective subdiagrams commutative. So all we need is to show that $\bar{\tau} = (\tau, \tau') : \tilde{N} \to \tilde{\kappa}$ is a morphism in $\text{Rep}_{(Q, Q')}$. To check this, the constructions of the kernels of $f, f'$ in $\text{Rep}_k(Q), \text{Rep}_k(Q')$ respectively and Remark (4.1) imply that $\tau(x) = \lambda(x)$ and $\tau'(x') = \lambda'(x')$ for all $x \in N$ and $x' \in N'$ respectively. For any $x \in N_{t(\alpha)}, \alpha \in Q$, $\beta \in Q'$, we have

\[
\tau'_{s(\beta)} \Psi_\alpha^\beta(x) = \tau'_{s(\beta)}(\Psi_\beta^\alpha(x)) = \lambda'_{s(\beta)}(\Psi_\beta^\alpha(x)) = \lambda'_{s(\beta)}(\Psi_\beta^\alpha(x)) = \psi_{s(\beta)}^{\alpha}(\tau_{t(\alpha)}(x)) = \xi_{s(\beta)}^{\alpha}(\tau_{t(\alpha)}(x)) = \xi_{s(\beta)}^{\alpha}(\tau_{t(\alpha)}(x)) = \xi_{s(\beta)}^{\alpha}(\tau_{t(\alpha)}(x)) = \xi_{s(\beta)}^{\alpha}(\tau_{t(\alpha)}(x))
\]
Therefore, \( \bar{\tau} = (\tau, \tau') : \bar{N} \to \bar{k} \) is a morphism in \( \text{Rep}_{(Q, Q')} \), and thus \((\bar{k}, \bar{\zeta})\) is the kernel of \( \bar{f} \) in \( \text{Rep}_{(Q, Q')} \).

\[ \square \]

Using induction on \( n \) and the same procedure used above, we can show that kernels exist in the category \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_n)} \), and they can similarly be constructed.

Explicitly, let \( \bar{V} = (V(1), V(2), \ldots, V(n), \psi_1, \psi_2, \ldots, \psi_{n-1}) \), \( \bar{W} = (W(1), W(2), \ldots, W(n), \psi'_1, \psi'_2, \ldots, \psi'_{n-1}) \) be \( n \)-representations of \((Q_1, Q_2, \ldots, Q_n)\), and let \( \bar{f} : \bar{V} \to \bar{W} \) be a morphism of \( n \)-representations, where \( \underline{f} = (f^{(1)}, f^{(2)}, \ldots, f^{(n)}) \). For any \( m \in \{2, \ldots, n\} \), write

\[
f^{(m)} = (f^{(m)}_{\gamma(m)}) : (V^{(m)}_{\gamma(m)}), \phi^{(m)}_{\gamma(m)}) \to (W^{(m)}_{\gamma(m)}), \mu^{(m)}_{\gamma(m)}),
\]

By induction, the kernel of \( (f^{(1)}, f^{(2)}, \ldots, f^{(n-1)}) \) in \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_{n-1})} \) exists in \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_{n-1})} \). Let

\[
((\kappa^{(1)}, \kappa^{(2)}, \ldots, \kappa^{(n-1)}, \xi_1, \xi_2, \ldots, \xi_{n-2}), (\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(n-1)}))
\]

be the kernel of \( (f^{(1)}, f^{(2)}, \ldots, f^{(n-1)}) \) in \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_{n-1})} \).

For all \((\gamma^{(n-1)}, \gamma^{(n)}) \in Q^{(n-1)} \times Q^{(n)}\), define

\[
\xi^{(n)}_{\gamma^{(n-1)} \gamma^{(n)}} : \kappa^{(n-1)}_{\gamma^{(n-1)}} \to \kappa^{(n)}_{\gamma^{(n)}}
\]

to be the restriction of \( \xi^{(n-1)}_{\gamma^{(n-1)} \gamma^{(n-2)}} \). Then \( \xi^{(n)}_{\gamma^{(n)}} \) is well defined for all \((\gamma^{(n-1)}, \gamma^{(n)}) \in Q^{(n-1)} \times Q^{(n)}\) by using similar argument used for the case \( n = 2 \).

Let \( \underline{K} = (K^{(1)}, K^{(2)}, \ldots, K^{(n)}, \xi_1, \xi_2, \ldots, \xi_{n-1}), \underline{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(n)}) \) and \( \underline{f} = (f^{(1)}, f^{(2)}, \ldots, f^{(n)}) \), where \((K^{(n)}, \zeta^{(n)})\) is the kernel of \( f^{(n)} \). Using similar argument used in Proposition 4.2, gives the following proposition.

**Proposition 4.3.** The pair \((K, \zeta)\) is the kernel of \( f \) in \( \text{Rep}_{(Q_1, Q_2, \ldots, Q_n)} \).

**Proof.**

\[ \square \]

Let \( \bar{f} = (f, f') : \bar{V} = (V, V', \psi) \to \bar{W} = (W, W', \psi') \) be a morphism in \( \text{Rep}_{(Q, Q')} \). From Proposition 2.1, the cokernels of \( f, f' \) exist in \( \text{Rep}_K(Q) \), \( \text{Rep}_{K'}(Q') \) respectively. Let \((K, \eta), (K', \eta')\) be the cokernels of \( f, f' \) in \( \text{Rep}_K(Q) \), \( \text{Rep}_{K'}(Q') \) respectively. Write \( K = (K_i, \phi^i_{\alpha}), \eta = (\eta_{\alpha}), K' = (K'_{\check{\alpha}}, \check{\eta}^\beta_{\check{\alpha}}), \eta' = (\eta'_{\check{\beta}}) \) and Consider the following diagram.
For all \( \alpha \in Q_1, \beta \in Q'_1 \), define the \( k \)-linear map \( \psi'^{\alpha}_{\beta} : K_{t(\alpha)} \rightarrow K'_{s(\alpha)} \) by \( \psi'^{\alpha}_{\beta}(w + f_{t(\alpha)}(V_{t(\alpha)})) = \psi'^{\alpha}_{\beta}(w) + f'_{s(\beta)}(V'_{s(\beta)}) \) for all \( w \in W_{t(\alpha)} \). Then \( \psi'^{\alpha}_{\beta} \) is well defined for all \( \alpha \in Q_1, \beta \in Q'_1 \) since for all \( a, b \in W_{t(\alpha)} \) with \( a + f_{t(\alpha)}(V_{t(\alpha)}) = b + f_{t(\alpha)}(V_{t(\alpha)}) \), we have \( a - b \in f_{t(\alpha)}(V_{t(\alpha)}) \). Thus \( \psi'^{\alpha}_{\beta}(a) - \psi'^{\alpha}_{\beta}(b) = \psi'^{\alpha}_{\beta}(a - b) \in \psi'^{\alpha}_{\beta}(f_{t(\alpha)}(V_{t(\alpha)})) = f'_{s(\beta)}(V_{s(\alpha)}) \subseteq f'_{s(\beta)}(V'_{s(\beta)}) \). It follows that \( \psi'^{\alpha}_{\beta}(a + f'_{s(\beta)}(V'_{s(\alpha)})) = \psi'^{\alpha}_{\beta}(b + f'_{s(\beta)}(V'_{s(\alpha)})), \) and hence \( \psi'^{\alpha}_{\beta} \) is well defined.

Let \( \bar{K} = (K, K', \psi'^{\alpha}_{\beta}) \) and \( \bar{\eta} = (\eta, \eta') \). Then \( \bar{K} \in Rep_{(Q, Q')} \) and We have the following proposition.

**Proposition 4.4.** The pair \((\bar{K}, \bar{\eta})\) is the cokernel of \( \bar{f} \).

**Proof.** Let \( \bar{\gamma} : \bar{W} = (W, W', \psi) \rightarrow \bar{L} = (L, L', \Psi) \) be a morphism in \( Rep_{(Q, Q')} \) with \( \bar{\gamma}\bar{f} = 0 \) and consider the following diagram.
Since \((K_i, \phi_i), (K'_i, \mu_i)\) are the cokernels of \(f, f'\) in \(\text{Rep}_k(Q), \text{Rep}_k(Q')\) respectively, there exist unique morphisms \(\sigma : K \to L, \sigma' : K' \to L'\) in \(\text{Rep}_k(Q), \text{Rep}_k(Q')\) respectively making their respective diagrams commutative. So all we need is to show that \(\bar{\sigma} = (\sigma, \sigma') : \bar{K} \to \bar{L}\) is a morphism in \(\text{Rep}_{(Q,Q')}\). To show this, the constructions of the cokernels of \(f, f'\) in \(\text{Rep}_k(Q), \text{Rep}_k(Q')\), respectively, and Remark 4.1 imply that \(\sigma(x + f(V)) = \gamma(x)\) and \(\sigma'((x' + f'(V'))) = \gamma'(x')\) for all \(x \in W\) and \(x' \in W'\) respectively. For any \(w \in V_{t(\alpha)}\), we have

\[
\sigma'_{s(\beta)}\psi_\beta^\alpha(w + f_{t(\alpha)}(V_{t(\alpha)})) = \sigma'_{s(\beta)}(\psi_\beta^\alpha(w) + f'_{s(\beta)}(V_{s(\beta)}))
\]

(by the definition of \(\psi_\beta^\alpha\))

\[
= \gamma'_{s(\beta)}(\psi_\beta^\alpha(w))
\]

(since \(\sigma'((x' + f'(V'''))) = \gamma'(x')\) for all \(x' \in W'\))

\[
= \Psi_\beta^\alpha \gamma_{t(\alpha)}(w)
\]

(since \(\gamma\) is a morphism in \(\text{Rep}_{(Q,Q')}\))

\[
= \Psi_\beta^\alpha \sigma_{t(\alpha)} \eta_{s(\alpha)}(w)
\]

(since \(\gamma = \bar{\sigma}\eta\))

\[
= \Psi_\beta^\alpha \sigma_{t(\alpha)}(w + f_{t(\alpha)}(V_{t(\alpha)}))
\]

(by the definition of \(\eta\))

Therefore, \(\bar{\sigma} : \bar{K} \to \bar{L}\) is a morphism in \(\text{Rep}_{(Q,Q')}\), and thus \((\bar{K}, \bar{\eta})\) is the cokernel of \(\bar{f}\).

Using induction on \(n\), one can use the same procedure used above to show that cokernels exist in the category \(\text{Rep}_{(Q_1, Q_2, \ldots, Q_n)}\), and they can similarly be constructed.
Explicitly, let $\tilde{V} = (V^{(1)}, V^{(2)}, \ldots, V^{(n)}, \psi_1, \psi_2, \ldots, \psi_{n-1})$, $\tilde{W} = (W^{(1)}, W^{(2)}, \ldots, W^{(n)}, \psi'_1, \psi'_2, \ldots, \psi'_{n-1})$ be $n$-representations of $(Q_1, Q_2, \ldots, Q_n)$, and let $f : \tilde{V} \to \tilde{W}$ be a morphism of $n$-representations, where $f = (f^{(1)}, f^{(2)}, \ldots, f^{(n)})$. For any $m \in \{2, \ldots, n\}$, write

$$f^{(m)} = (f_{\gamma(n-1)}^{(m)}) : (V_{\gamma(n-1)}^{(m)}, \phi_{\gamma(n-1)}^{(m)}) \to (W_{\gamma(n-1)}^{(m)}, \mu_{\gamma(n-1)}^{(m)}).$$

By induction, the cokernel of $(f^{(1)}, f^{(2)}, \ldots, f^{(n-1)})$ in $\text{Rep}_{\gamma(n-1)}$ exists in $\text{Rep}_{\gamma(n-1)}$. Let

$$(\gamma^{(i)}(1), \gamma^{(i)}(2), \ldots, \gamma^{(i)}(n-1), \chi_1, \chi_2, \ldots, \chi_{n-2}), (\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n-1)})$$

be the cokernel of $(f^{(1)}, f^{(2)}, \ldots, f^{(n-1)})$ in $\text{Rep}_{\gamma(n-1)}$. For all $(\gamma^{(n-1)}, \gamma^{(n)}) \in Q^{(n-1)} \times Q^{(n)}$, define

$$\chi^{(n)}_{\gamma^{(n-1)}} : K^{(n-1)}_{\gamma^{(n-1)}} \to K^{(n)}_{\gamma^{(n)}}$$

by $\chi^{(n)}_{\gamma^{(n-1)}}(w + f_{\gamma^{(n-1)}}^{(i)}(V_{\gamma^{(n-1)}}^{(n)})) = \psi^{(n)}_{\gamma^{(n-1)}}(w) + f_{\gamma^{(n-1)}}^{(i)}(V_{\gamma^{(n-1)}}^{(n)})$ for all $w \in W_{\gamma^{(n-1)}}^{(n)}$. Then $\chi^{(n)}_{\gamma^{(n-1)}}$ is well defined for all $(\gamma^{(n-1)}, \gamma^{(n)}) \in Q^{(n-1)} \times Q^{(n)}$ by using similar argument used for the case $n = 2$.

Let $K = (K^{(1)}, K^{(2)}, \ldots, K^{(n)}, \chi_1, \chi_2, \ldots, \chi_{n-2}, \eta = (\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)})$ and $f = (f^{(1)}, f^{(2)}, \ldots, f^{(n)})$, where $(K^{(n)}, \eta^{(n)})$. Using similar argument used in Proposition 4.4 gives the following consequence.

\begin{proposition}
The pair $(K, \eta)$ is the cokernel of $f$ in $\text{Rep}_{\gamma(n-1)}$.
\end{proposition}

\begin{proof}
\end{proof}

5. Canonical Decomposition of Morphisms in $\text{Rep}_{\gamma(n-1)}$.

Following \cite{10}, p. 2, an additive category is a category $\mathcal{C}$ satisfying the following axioms:

(i) Every set $\mathcal{C}(X, Y)$ is equipped with a structure of an abelian group (written additively) such that composition of morphisms is biadditive with respect to this structure.

(ii) There exists a zero object $0 \in \mathcal{C}$ such that $\mathcal{C}(0, 0) = 0$.

(iii) (Existence of direct sums.) For any objects $X, X' \in \mathcal{C}$, the direct sum $X \oplus X' \in \mathcal{C}$.

Let $k$ be a field. An additive category $\mathcal{C}$ is said to be \textit{k-linear} if for any objects $X, Y \in \mathcal{C}(X, Y)$ is equipped with a structure of a vector space over $k$, such that composition of morphisms is $k$-linear. An \textit{abelian} category is an additive category $\mathcal{C}$ in which for every morphism $f : X \to Y$ there exists a sequence

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C$$

with the following properties:

(i) $ji = f$,

(ii) $(K, k) = \text{Ker}(f), \ (C, c) = \text{Coker}(f),$

(iii) $(I, i) = \text{Coker}(k), \ (I, j) = \text{Ker}(c).$
A sequence \([5.3]\) is called a **canonical decomposition** of \(f\).

Let \(\tilde{f} = (f, f') : \tilde{X} \to \tilde{Y}\) be a morphism in \(\text{Rep}_{(Q,Q')}\). It follows that \(f : X \to Y, \ f' : X' \to Y'\) are morphisms in \(\text{Rep}_k(Q), \text{Rep}_k(Q')\) respectively. From Proposition \([2.1]\) \(f : X \to Y, \ f' : X' \to Y'\) have the canonical decompositions

\[
\begin{align*}
K & \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} C \\
K' & \xrightarrow{k'} X' \xrightarrow{i'} I' \xrightarrow{j'} Y' \xrightarrow{c'} C'
\end{align*}
\]

in \(\text{Rep}_k(Q), \text{Rep}_k(Q')\) respectively. From Section \([5]\) kernels and cokernels exist in \(\text{Rep}_{(Q,Q')}\). It turns out that \(\tilde{f}\) has a canonical decomposition

\[
\begin{align*}
\tilde{K} & \xrightarrow{k} \tilde{X} \xrightarrow{\tilde{i}} I \xrightarrow{\tilde{j}} \tilde{Y} \xrightarrow{\tilde{c}} \tilde{C}
\end{align*}
\]

in \(\text{Rep}_{(Q,Q')}\), and this decomposition can explicitly be seen in the following commutative diagram.

This implies that any morphism \(f : \tilde{V} \to \tilde{W}\) of \(n\)-representations has a canonical decomposition in \(\text{Rep}_{(Q_1, Q_2, \ldots, Q_n)}\).

**Remark 5.1.** Let \(\tilde{f}, \tilde{g} : \tilde{V} \to \tilde{W}\) be morphisms in \(\text{Rep}_{(Q,Q')}\). Write \(\tilde{f} = (f, f'), \ \tilde{g} = (g, g')\), \(f = (f_i), \ g = (g_i), \ f' = (f_i'), \ g' = (g_i')\), \(\tilde{V} = (V, V', \psi), \ W = (W, W', \psi')\). Define \(\tilde{f} + \tilde{g} = (f + g, f' + g') = ((f_i + g_i), (f_i' + g_i'))\). Since \(\text{Rep}_k(Q)\) and \(\text{Rep}_k(Q')\) are abelian, the sets \(\text{Rep}_k(Q)(V, W)\), \(\text{Rep}_k(Q)(V', W')\) are equipped with a structure of an abelian group such that composition of morphisms is biadditive with respect to this structure \([2]\) p. 70. Since \(\tilde{f}, \tilde{g}\) are morphisms in \(\text{Rep}_{(Q,Q')}\) and since the category \(\text{Vec}_k\) is abelian, we have the following commutative diagram.
Thus, the set $\text{Rep}_{(Q,Q')}((\bar{V},\bar{W})$ is equipped with a structure of an abelian group such that composition of morphisms is biadditive with respect to the above structure.

We end the paper with the following crucial results.

**Theorem 5.2.** The category $\text{Rep}_{(Q,Q')}$ is a $k$-linear abelian category.

**Proof.** □

**Theorem 5.3.** The category $\text{Rep}_{(Q_1,Q_2,...,Q_n)}$ is a $k$-linear abelian category for any integer $n \geq 2$.

**Proof.** □

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