

SUPPORT AND VANISHING FOR NON-NOETHERIAN RINGS AND TENSOR TRIANGULATED CATEGORIES

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Abstract. We define and characterise small support for complexes over non-Noetherian rings and in this context prove a vanishing theorem for modules. Our definition of support makes sense for any rigidly compactly generated tensor triangulated category. Working in this generality, we establish basic properties of support and investigate when it detects vanishing. We use pointless topology to relate support, the topology of the Balmer spectrum, and the structure of the idempotent Bousfield lattice.

1. Introduction

In this article, we propose the following definition.

Definition 1.1. For an arbitrary commutative ring $R$ and a complex $M$ of $R$-modules, we say $p \in \text{Spec } R$ is in $\text{supp } M$ if for every finite subset $\underline{x} = x_1, \ldots, x_n \in p$, the cohomology

$$H^iK^\infty(\underline{x}: M_p) = H^i((R \to R_{x_1}) \otimes \cdots \otimes (R \to R_{x_n}) \otimes M_p) \neq 0$$

does not vanish for some $i \in \mathbb{Z}$.

We justify this definition with the following result, which is Theorem 5.5 in the text.

Theorem 1.2. Let $M$ be a complex over a commutative ring $R$. Suppose that either

- $H^i(M) = 0$ for $i \ll 0$, e.g. $M$ is a module
- or the prime ideals of $R$ satisfy the descending chain condition.

Then $\text{supp } M = \emptyset$ if and only if $M = 0$.

In [11], assuming $R$ is commutative Noetherian ring and $M$ a complex of $R$-modules, Foxby defined $\text{supp } M$ as the primes $p$ such that $M \otimes^L k(p) \neq 0$ where $k(p)$ is the residue field $R_p/pR_p$. In this work, Foxby showed that this support detected vanishing, i.e. $\text{supp } M = \emptyset$ if and only if $M = 0$. By [12] (2.1) and (4.1), our definitions of support coincide in the Noetherian case.

Support is a powerful tool for triangulated categories. In [28], Neeman classified the localising subcategories of $\text{D}(R)$ in terms of support. In [1], Benson, Iyengar, and Krause construct a support theory in a compactly generated triangulated category acted upon by a Noetherian ring. In [6] they used this support theory to classify the localising subcategories of the stable category of modular group representations.
For non-Noetherian rings, Foxby’s theory of supports breaks down: there are modules $M$ such that $M \otimes k(p) = 0$ for all $p \in \text{Spec } R$; see Example 5.7. Moreover, Neeman’s classification of localising subcategories fails spectacularly for non-Noetherian rings; see [30] or more generally [10].

In [2] Balmer and Favi define support in certain rigidly compactly generated tensor triangulated categories. Their support takes values in the Balmer spectrum of the compact objects, and their definition is valid whenever this space is topologically Noetherian. Greg Stevenson studied this support in [38] and applied these results to the derived category of an absolutely flat ring in [39] and [42]. These results suggest that even though Neeman’s classification fails, support detects some semblance of order in the localising subcategories.

Following the example of Balmer, Favi, and Stevenson, we study support in rigidly compactly generated tensor triangulated categories. We introduce the following definition. See Section 3 for relevant definitions.

**Definition 1.3.** Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category. For a prime in the Balmer spectrum $p \in \text{Spc } \mathcal{T}^c$ and an object $T \in \mathcal{T}$, we say $p \in \text{supp } T$ if for every Thomason subset $V \subseteq \text{Spc } \mathcal{T}$ with $p \in V$,

$$\Gamma V T_p \neq 0.$$  

We call $\mathcal{T}$ supportive if support detects vanishing, i.e. $\text{supp } T = \emptyset$ if and only if $T = 0$.

When $\mathcal{T} = D(R)$, this definition specialises to Definition 1.1; see Lemma 5.2. When $\text{Spc } \mathcal{T}^c$ is topologically Noetherian, our definition coincides with Balmer and Favi’s in [2].

Unfortunately, we do not know if $\mathcal{T}$ is always supportive. But when it is, our support exhibits strong properties and behaves similarly to the support developed by Benson, Iyengar, and Krause in [5]. See Theorems 4.2 and 4.7. Moreover, our support is the only reasonable support function taking values in $\text{Spec } R$; see Theorem 4.8. We summarise these results below.

**Theorem 1.4.** Let $T \in \mathcal{T}$, and consider the following properties of a function $s: \mathcal{T} \rightarrow \{ \text{Subsets of } \text{Spc } \mathcal{T}^c \}$.

1. **Vanishing**: $s(T) = \emptyset$ if and only if $T = 0$.
2. **Local**: If $V \subseteq \text{Spc } \mathcal{T}^c$ is Thomason and $T$ is $V$-local, then $s(T) \cap V = \emptyset$.
3. **Big Support**: $s(T) \subseteq \text{Supp } T$ for any compact object $T \in \mathcal{T}^c$.
4. **Orthogonality**: For any $S \in \mathcal{T}$, if there is a Thomason subset $V \subseteq \text{Spc } \mathcal{T}^c$ such that $s(T) \subseteq V$ and $s(S) \cap V = \emptyset$, then $\text{Hom}_\mathcal{T}(T, S) = 0$.
5. **Exactness**: For any exact triangle $S \rightarrow T \rightarrow S' \rightarrow$ in $\mathcal{T}$,

$$s(T) \subseteq s(S) \cup s(S').$$

6. **Separation**: For any Thomason subset $V \subseteq \text{Spc } \mathcal{T}^c$, there is an exact triangle

$$T' \rightarrow T \rightarrow T'' \rightarrow$$

with $s(T') \subseteq V$ and $s(T'') \cap V = \emptyset$.

If $\mathcal{T}$ is supportive, then $\text{supp}$ satisfies all of these properties. Moreover, if a function $s$ satisfies all of these properties, then $s = \text{supp}$ and $\mathcal{T}$ is supportive.
We know of may instances where \( T \) is supportive and none where it is not. The following result is Corollary 6.4, Theorem 7.9 and Theorem 5.5.

**Theorem 1.5.** A rigidly compactly generated tensor triangulated category \( T \) is supportive in the following cases.

1. The idempotent Bousfield lattice of \( T \) is a spatial frame.
2. The Hochster dual of the Balmer spectrum \( \text{Spc} T^c \) is weakly scattered: for every Thomason set \( U \subseteq \text{Spc} T^c \), there is a point \( p \notin U \) and a Thomason subset \( V \) such that \( \{ p \} \subseteq V \cap U^c = \downarrow p \).
3. \( T = D(R) \) with \( R \) a commutative ring satisfying DCC on prime ideals.

In Section 2 and Section 3 we discuss preliminary material, including the basics of spectral spaces, pointless topology, the Balmer spectrum, and localisation functors. In Section 4 we give the definition of support, discuss its various properties, and prove Theorem 1.4. In Section 5 we specialise to the case \( T = D(R) \) with \( R \) a commutative ring, proving Theorem 1.2. In Section 6 we relate support to the Bousfield lattice, and characterise in Theorem 6.3 the supportive condition using pointless topology. In Section 7 we study topological conditions on the Balmer spectrum which imply supportive. In Section 8 we pose several questions.

We close this section by establishing some conventions. Triangulated categories and their subcategories will generally be denoted with curly letters such as \( T \) while their objects will be noted with roman capital letters such as \( T \). Rings, modules and complexes will also be denoted in roman capital letters, and their elements in lowercase roman letters. Topological spaces and their subsets will be denoted with bold capital roman fonts and their points with lowercase gothic fonts, e.g. \( p \in X \). Continuous functions will be denoted with bold lowercase letters like \( f \). Lattices will be denoted with blackboard bold fonts like \( X \), and their elements in lower case e.g. \( x \in X \). Greek letters will denote lattice homomorphisms.

## 2. Topological preliminaries

### 2.1. Point-set topology.

**Definition 2.1.** A space \( X \) is called spectral if

(a) \( X \) is \( T_0 \)
(b) every irreducible closed set \( V \subseteq X \) has a generic point, i.e. there is a \( p \in V \) such that \( V = \overline{p} \)
(c) the quasi-compact open subsets of \( X \) are a basis
(d) the intersection of any two quasi-compact open sets is again quasi-compact open
(e) the space \( X \) is quasi-compact.

Spectral spaces were introduced by Hochster, where he gave the following classification.

**Theorem 2.2 (15).** A space \( X \) is spectral if and only if there exists a commutative ring \( R \) such that \( \text{Spec} R \cong X \).

Any spectral space has another important topology.

**Definition 2.3.** Let \( X \) be a spectral space.
(1) A subset $V \subseteq X$ is Thomason if it is the union of complements of quasi-compact open sets.

(2) The quasi-compact open sets of a spectral space form a closed base, and the topology that they define is called the Hochster dual of $X$ and denoted $X^\dagger$. The open subsets of $X^\dagger$ are the Thomason subsets of $X$.

**Theorem 2.4** ([15, Proposition 8]). For any spectral space $X$, the Hochster dual $X^\dagger$ is also spectral. Moreover, there is a natural homeomorphism $X \cong (X^\dagger)^\dagger$.

The points in a spectral space $X$ are partially ordered: for $p, q \in X$ we say that $p \subseteq q$ if $q \in \bar{p}$. If Spec $R \cong X$ for a commutative ring $R$, this partial order is induced by the inclusion of prime ideals in $R$.

**Example 2.5.** Let $X$ be a spectral space and $p \in X$ be a point. Define

$Z(p) = \{q \in X | q \nsubseteq p\}$.

This is the largest Thomason subset of $X$ not containing $p$. Therefore, the closure of $p$ in the Hochster dual is

$\downarrow p = \{q \in \text{Spec } R | q \subseteq p\}$.

To see that $Z(p)$ is Thomason, assume that $X = \text{Spec } R$ and $p \subseteq R$ is a prime ideal for a commutative ring $R$. Then

$Z(p) = \bigcup_{x \notin p} V(x)$.

The complement of each $V(x)$ is quasi-compact, proving the claim.

Recall the following definitions from point-set topology.

**Definition 2.6.** Let $X$ be an arbitrary topological space.

(1) Denoted by $\text{skula}(X)$, the Skula or front topology on $X$ is the weakest topology where every open set of $X$ is open and closed.

(2) Let $f_X : \text{skula}(X) \rightarrow X$ be the set theoretic identity function.

(3) The space $X$ satisfies the separation axiom $T_{\frac{1}{2}}$ if for every point $p \in X$ there is an open set $V \subseteq X$ and a closed set $U \subseteq X$ such that $\{p\} = V \cap U$.

The Skula topology on $X$ is discrete if and only if it is $T_{\frac{1}{2}}$; for every point $p \in X$.

2.2. **Pointless topology.** In this section we discuss the basics of pointless topology. The reader should refer to [20] or [33] for further reading.

**Definition 2.7.**

(1) A complete lattice is a partially ordered set $L$ such that every subset admits both a supremum and an infimum, i.e. a join denoted $\vee$ and a meet denoted $\wedge$. In particular every frame has a maximum and minimum denoted by $1_L$ and $0_L$ respectively, or more often simply 1 and 0.

(2) A frame $X$ is a complete lattice such that for every $x \in X$ and $\{y_i\} \subseteq F$ satisfies

$x \wedge \bigvee_i y_i = \bigvee_i x \wedge y_i$.

(3) A map $\varphi : X \rightarrow Y$ between frames is a frame homomorphism if

(a) $\varphi$ preserves the maximum and minimum, i.e.

$\varphi(1_X) = 1_Y \quad \varphi(0_X) = 0_Y$
(b) $\varphi$ preserves arbitrary joins, i.e. every $\{x_i\} \subseteq X$ satisfies

$$\varphi \left( \bigvee_i x_i \right) = \bigvee_i \varphi(x_i)$$

(c) $\varphi$ preserves finite meets, i.e. every $x, y \in X$ satisfies

$$\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$$

(4) Let $\text{ Frm}$ be the category of frames and frame homomorphisms.

The following is the critical example of a frame.

**Example 2.8.** For a topological space $X$, let $F(X)$ be the open sets of $X$ partially ordered by inclusion. It is easy to check that $F(X)$ is a frame where joins are unions and finite meets are intersections. Given a continuous function $g: X \to Y$, the induced map

$$F(g) = g^{-1}: F(Y) \to F(X)$$

is a frame homomorphism. Thus we have defined a contravariant functor

$$F: \text{ Top} \to \text{ Frm}.$$ We say that a frame is *spatial* or has enough points if it is isomorphic to a frame in the image of $F$.

**Definition 2.9.** Let $X$ be a frame.

(1) An element $p \in X$ is meet irreducible or prime if for any $x, y \in X$

$$x \wedge y \leq p$$

implies either $x \leq p$ or $y \leq p$.

(2) Let $\text{ Spc} X$ be the set of meet irreducible elements of $X$.

(3) For any $x \in X$, let $D(x) \subseteq \text{ Spc} X$ be the meet irreducible $p$ such that $x \not\leq p$.

**Lemma 2.10** ([33, II.4.1, II.4.3.1]). The set $\text{ Spc} X$ is a topological space whose open sets are of the form $D(x)$. Moreover, we have actually defined a contravariant functor $\text{ Spc}: \text{ Frm} \to \text{ Top}$.

A space is sober if it is homeomorphic to $\text{ Spc} X$ for a frame $X$. Sober spaces are ubiquitous. Indeed, a space is sober if and only if it is $T_0$ and every irreducible closed set has a generic point, see [20, II.1.7]. Thus every Hausdorff and spectral space is sober; see [20, II.1.6] and Definition 2.1. On the other hand, the maximal ideal spectrum of $C[x, y]$ is not sober.

Let $\text{ Spt}$ and $\text{ Sob}$ denote the full subcategories of $\text{ Frm}$ and $\text{ Top}$ respectively consisting of spatial frames and sober spaces.

**Theorem 2.11** ([20, II.1.7 Corollary and II.1.5 and II.1.6]).

(1) The functors $F$ and $\text{ Spc}$ restrict to an equivalence of categories

$$\text{ Sob} \cong \text{ Spt}^{op}.$$

(2) For every frame $X$, there is a natural frame homomorphism

$$\lambda_X: X \to F(\text{ Spc} X).$$

Moreover, $X$ is spatial if and only if $\lambda_X$ is an isomorphism.
(3) For any space $X$ there is a natural continuous function

$$\ell_X : X \to \text{Spc} \mathbb{F}(X).$$

Moreover, $X$ is sober if and only if $\ell_X$ is an isomorphism.

Thus, sober spaces are completely described by their associated frames, and spatial frames completely describe their associated space.

3. Tensor triangulated preliminaries

3.1. The Balmer spectrum. Let $(\mathcal{T}, \Sigma, \otimes, 1)$ be an essentially small tensor triangulated category. This means that $\mathcal{T}$ is a triangulated category with shift functor $\Sigma$ and a closed symmetric monoidal product $\otimes$. Thus we assume the following: there is a tensor unit $1 \in \mathcal{T}$; $S \otimes T \cong T \otimes S$ for all $S, T \in \mathcal{T}$; and that $\otimes$ is exact. Lastly, we assume that the isomorphism classes of $\mathcal{T}$ form a set.

Definition 3.1.

(1) A subcategory $\mathcal{I} \subseteq \mathcal{T}$ is a thick tensor ideal if
   
   (a) $\mathcal{I}$ is sub triangulated i.e. contains 0 and is closed under mapping cones
   
   (b) $\mathcal{I}$ is closed under direct summands
   
   (c) for every $S \in \mathcal{I}$ and $T \in \mathcal{T}$, $S \otimes T \in \mathcal{I}$.

(2) A thick tensor ideal $\mathcal{I} \subseteq \mathcal{T}$ is radical if $T \otimes \cdots \otimes T \in \mathcal{I}$ implies $T \in \mathcal{I}$.

(3) A thick tensor ideal $\mathcal{P} \subseteq \mathcal{T}$ is prime if $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$ implies either $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$ for any thick tensor ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{T}$. Equivalently, prime thick tensor ideals are the meet irreducible elements of the lattice of radical thick tensor ideals.

(4) For an object $T \in \mathcal{T}$, let $\text{Supp} T$ denote the set of primes which do not contain $T$.

(5) Let $\text{Spc} \mathcal{T}$ be the set of prime thick tensor ideals of $\mathcal{T}$. We consider the weakest topology such that $\text{Supp} T$ is closed for all $T \in \mathcal{T}$. This topological space is called the Balmer Spectrum.

The prototypical example is taken from considering $\mathcal{T} = \text{Perf}(R)$ the perfect complexes over a commutative ring $R$. In this case, $\text{Spc} \mathcal{T}$ is homeomorphic to $\text{Spec} R$. The construction above is given by Balmer in [1].

Theorem 3.2.

(1) The Balmer spectrum $\text{Spc} \mathcal{T}$ is always spectral.

(2) The support function $\text{Supp}$ gives a bijection between the radical tensor ideals of $\mathcal{T}$ and the Thomason subsets of $\text{Spc} \mathcal{T}$.

Statement (1) is in [9]. Statement (2) is in [1]. We can reinterpret this theorem using the topological language of the previous section.

Definition 3.3.

(1) Let $\mathbb{T}(\mathcal{T}^c)$ be the set of radical thick tensor ideals partially ordered by inclusion.

(2) Let $\text{HSp} \mathcal{T}$ denote the Hochster dual of $\text{Spec} \mathcal{T}$.

By Theorem 3.2 the lattice $\mathbb{T}(\mathcal{T}^c)$ is isomorphic to the lattice of Thomason subsets, and so we freely confuse the two. But the latter is the collection of open sets of $\text{HSp} \mathcal{T}$. So $\mathbb{T}(\mathcal{T}^c) \cong \mathbb{F}(\text{HSp} \mathcal{T})$ and thus is a spatial frame. Moreover,
since $\text{HSpc}\, T$ is sober, $\text{HSpc}\, T \cong \text{Spc}\, T(\mathcal{C})$. See [24] or [24] for a more thorough discussion.

The situation is simplified if $T$ satisfies a technical condition called rigid. Since we will not use the condition itself, and only its consequences we refer the reader to [40, Definition 1.3].

**Lemma 3.4** ([40] Remark 1.18]). If $T$ is rigid, then every thick tensor ideal is radical. In this case, $T(\mathcal{C})$ is the lattice of thick tensor ideals.

3.2. **Localisation.** Let $(T, \Sigma, 1, \otimes)$ be a rigidly compactly generated tensor triangulated category. This means that $T$ is generated by its compact objects $T^c$ and that this category is essentially small, tensor closed, and is rigid. Furthermore, we assume that the unit 1 is compact. See [2, Section 1.1] for a discussion of these hypotheses. The following are examples of such categories.

- The category $T = D(R)$ for a commutative ring $R$.
- The stable homotopy category.
- The stable module category of $kG$ modules for $G$ a group whose order is divisible by the characteristic of $k$.

**Definition 3.5.**

1. A subcategory $\mathcal{L} \subseteq T$ is localising if it is thick and closed under arbitrary set-indexed coproducts. A localising subcategory $\mathcal{L}$ is a tensor ideal if for every $T \in \mathcal{L}$ and $S \in T$, $S \otimes T$ is in $\mathcal{L}$.
2. For any Thomason subset $V \subseteq \text{Spc}\, T^c$, let $T_V$ be the generated by the compact object $C \in T^c$ such that $\text{Supp}\, C \subseteq V$.

We call $T_V$ the $V$-acyclic objects of $T$.

3. An object $T \in T$ is $V$-local if $\text{Hom}_T (T_V, T) = 0$ or equivalently $\text{Hom}_T (T_V^c, T) = 0$.

**Theorem 3.6.** Let $V \subseteq \text{Spc}\, T^c$ be a Thomason subset. There exist triangulated coproducts preserving functors

$$\Gamma_V : T \rightarrow T \quad L_V : T \rightarrow T$$

and natural transformations

$$\Gamma_V \xrightarrow{\gamma} \text{id} \xrightarrow{\lambda} L_V \xrightarrow{\eta} \Sigma \Gamma_V$$

which have the following properties.

1. For every $T \in T$, the triangle

$$\Gamma_V T \xrightarrow{\gamma} T \xrightarrow{\lambda} L_V T \xrightarrow{\eta} \Sigma \Gamma_V T$$

is exact.
2. The natural transformations

$$\Gamma_V \gamma^V : \Gamma_V \Gamma_V \rightarrow \Gamma_V \quad L_V \lambda^V : L_V \rightarrow L_V L_V$$

are isomorphisms.
3. The $V$-acyclic objects are equal to

$$T_V = \text{Im}(\Gamma_V) = \ker L_V.$$  

Moreover, $\Gamma_V$ is the identity on this category.
(4) The $V$-local objects are precisely the categories 
$$\ker \Gamma_V = \text{Im}(L_V).$$

(5) The following functors are isomorphic 
$$\Gamma_V \cong - \otimes L_V, \quad L_V \cong - \otimes L_V 1.$$

(6) If $V' \subseteq \text{Spc } T^c$ is another Thomason subset, then the following functors 
are isomorphic 
$$\Gamma_{V \cap V'} \cong \Gamma_V \Gamma_V, \quad L_{V \cup V'} \cong L_V L_V'.$$

(7) For Thomason subsets $V, U, V', U' \subseteq \text{Spc } T^c$ such that 
$$V \cap U^c = V' \cap U'^c$$
there is an isomorphism of functors 
$$\Gamma_V L_U \cong \Gamma_{V'} L_{U'}.$$

Proof. The category of $V$-acyclic objects is smashing; see [27] or [18, Theorem 3.3.3]. Statements (1)-(5) are standard properties of smashing localisation; see [25, 4.9.1, 10.1, 4.11.1] and [18, Definition 3.3.2]. Statements (6) and (7) are [2, Theorem 5.18] and [2, Corollary 7.5] respectively.

We end this section with some notation.

**Definition 3.7.** Recall from Example 2.5 that for any prime $p \in \text{Spc } T$, the set $Z(p)$ is Thomason. Let $T$ be an object in $T$.

1. Set $T_p = L_{Z(p)}$.
2. Let $\text{Supp } T$ be the primes $p \in \text{Spc } T^c$ such that $T_p \neq 0$.

**Remark 3.8.** For every Thomason subset $V \subseteq \text{Spc } T^c$, the category $\mathcal{T}_V^c$ is the tensor ideal of $T^c$ corresponding to $V$ in Theorem 3.2 (2). Furthermore, the categories $\mathcal{T}_{Z(p)}^c$ are the prime tensor ideals. These ideals are also the meet irreducible elements of $\mathcal{T}(T^c)$.

### 4. Support

In this section, $T$ is a rigidly compactly generated tensor triangulated category.

#### 4.1. Defining support.

**Definition 4.1.**

1. For a $T \in T$ and $p \in \text{Spc } T^c$, let $p \in \text{supp } T$ if for all Thomason subsets $V, U \subseteq \text{Spc } T^c$, with $p \in V \cap U^c$

$$\Gamma_V L_U T \neq 0.$$ 

2. The localising space $\text{Lspc } T$ will be the set $\text{Spc } T^c$ with the topology generated by $V \cap U^c$ with $V, U \subseteq \text{Spc } T^c$ Thomason. In short 

$$\text{Lspc } T = \text{skula}(\text{HSpc } T^c).$$

We will refer to this topology as the localising topology.

**Theorem 4.2.** Suppose $T$ is a rigidly compactly generated tensor triangulated category. The following are true.
(1) For an exact triangle

\[ S \to T \to S' \to \]

we have

\[ \text{supp} T \subseteq \text{supp} S \cap \text{supp} S' \]

(2) If \( T \cong S \oplus S' \) in \( \mathcal{T} \), then

\[ \text{supp} T = \text{supp} S \cup \text{supp} S' \]

(3) For any Thomason subset \( V \subseteq \text{Spc} \mathcal{T}^c \) and \( T \in \mathcal{T} \), the following hold.
   (a) \( \text{supp} \Gamma_V T = \text{supp} T \cap V \)
   (b) \( \text{supp} L_V T = \text{supp} T \cap V^c \)
   (c) There is an exact triangle

\[ T' \to T \to T'' \to \]

such that

\[ \text{supp} T' = \text{supp} T \cap V \quad \text{supp} T'' = \text{supp} T \cap V^c. \]

(4) For any object \( T \in \mathcal{T} \), we have \( \text{supp} T \subseteq \text{Supp} T \). Equality holds when \( T \) is compact.

(5) \( \text{supp} T \) is always closed in \( \text{Lspc} \mathcal{T} \).

(6) For any localising subcategory \( \mathcal{L} \subseteq \mathcal{T} \), the set

\[ \text{supp} \mathcal{L} = \bigcup_{T \in \mathcal{L}} \text{supp} T \]

is closed in \( \text{Lspc} \mathcal{T} \).

We devote the rest of this section to proving this theorem.

**Lemma 4.3.** Consider Thomason subsets \( V, U, V', U' \subseteq \text{Spc} \mathcal{T}^c \) such that

\[ V' \subseteq V \quad U'^c \subseteq U^c. \]

If \( \Gamma_V L_UT = 0 \) then \( \Gamma_V L' UT = 0 \). In particular \( p \in \text{supp} T \) if and only if for all Thomason subsets \( V \subseteq \text{Spc} \mathcal{T}^c \) with \( p \in V \)

\[ \Gamma_V T_p \neq 0. \]

**Proof.** By Theorem 3.6 (6), the hypotheses imply that \( \Gamma_V = \Gamma_V. \Gamma_V \) and \( L' U' = L' U'. \)

Therefore, we have

\[ \Gamma_V L' U'T = \Gamma_V L' U'. \Gamma_V L' U'T = 0. \]

\[ \square \]

**Proof of Theorem 4.2 (1) and (2).** Consider an exact triangle

\[ S \to T \to S' \to \]

and suppose that \( p \notin \text{supp} S \) and \( p \notin \text{supp} S' \). Then there exists Thomason subsets \( V, U, X, W \) such that

\[ p \in V \cap U^c \quad p \in X \cap W^c \]

\[ \Gamma_V L' U'S = 0 \quad \Gamma_X L' W'S' = 0. \]

By the previous Lemma, we know that

\[ \Gamma_V \cap L' (U \cup X)^c S = \Gamma_V \cap L' (U \cup X)^c S' = 0. \]
Therefore, \( p \notin \text{supp} T \), proving Theorem 1.2 (1).

Now we assume that \( T \cong S \oplus S' \). To prove Theorem 1.2 (2), observe that \( \Gamma \vee L_U T = 0 \) if and only if \( \Gamma \vee L_U S = 0 \) and \( \Gamma \vee L_U S' = 0 \) for any Thomason subsets \( V, U \subseteq \text{Spc} T \).

**Proof of Theorem 4.2 (3).** Let \( V \subseteq \text{Spc} T \) be a Thomason subset and let \( T \in \mathcal{T} \). The vanishing objects

\[
\Gamma_{\text{Spc} T \setminus L \Gamma} (\Gamma \vee T) = 0 \quad \Gamma \vee L_0 (L \Gamma T) = \Gamma \vee L \Gamma T = 0
\]

implies that \( \text{supp} \Gamma \vee T \subseteq V \) and \( \text{supp} L \Gamma T \subseteq V^c \).

The idempotent triangle

\[
\Gamma \vee T \rightarrow T \rightarrow L \Gamma T \rightarrow .
\]

and Theorem 1.2 (1) imply the containment

\[
\text{supp} \Gamma \vee T \subseteq \text{supp} T \cup \text{supp} L \Gamma T \subseteq \text{supp} T \cup V^c.
\]

Intersecting both sides with \( V \) yields

\[
\text{supp} \Gamma \vee T = \text{supp} T \cap V
\]

proving Theorem 1.2 (3b). A similar calculation proves Theorem 1.2 (3b). Setting \( T' = \Gamma \vee T \) and \( T'' = L \Gamma T \) proves Theorem 1.2 (3).

**Proof of Theorem 4.2 (4).** Let \( T \in \mathcal{T} \). If \( p \notin \text{Supp} T \), then

\[
T_p = \Gamma_{\text{Spc} T \setminus L Z_{(p)}} T = 0
\]

and so \( p \notin \text{supp} T \), which implies the containment \( \text{supp} T \subseteq \text{Supp} T \).

Now suppose \( T \) is compact. To show the reverse containment, suppose that \( p \notin \text{supp} T \). Then there exists a Thomason subset \( V \subseteq \text{Spc} T \) such that \( p \in V \) and \( \Gamma V T_p = 0 \). Let \( I \) be the thick tensor ideal of \( T \) generated by \( T \). Since \( T \) is rigid, \( I \) is automatically radical by Lemma 3.3. Since every \( S \in T^c \cap I \) satisfies

\[
S_p \cong \Gamma V S_p = 0
\]

we have

\[
T^c \cap I \subseteq T_{(p)}^c.
\]

Since \( p \in V \), we know that \( V \not\subseteq Z(p) \) and so \( T^c \not\subseteq T_{(p)}^c \) by Theorem 3.2 and Remark 3.5. Since \( T_{(p)}^c \) is a prime tensor ideal, we conclude that \( I \subseteq T_{(p)}^c \). Thus \( T_p = L Z_{(p)} \geq 0 \).

**Proof of Theorem 4.2 (5).** Suppose \( p \notin \text{supp} T \) for some \( T \in \mathcal{T} \). Then there exists Thomason subsets \( V, U \subseteq \text{Spc} T \) such that \( \Gamma \vee L_U T = 0 \) and \( p \in V \cap U^c \). Now consider any \( q \in V \cap U^c \). Then \( q \) is also not in \( \text{supp} T \). Therefore, there is an open neighbourhood of \( p \) in \( L \text{Spc} T \) which is not in \( \text{supp} T \). It follows that \( \text{supp} T \) is closed.

**Proof of Theorem 4.2 (6).** Let \( L \subseteq \mathcal{T} \) be a localising subcategory. For every \( p \in \text{supp} L \), choose some element \( T^p \) such that \( p \in \text{supp} T^p \). Set

\[
T = \coprod_{p \in \text{supp} L} T^p
\]
We claim that
\[ \text{supp } \mathcal{L} = \text{supp } \mathcal{T}. \]
Given the claim, the result follows from Theorem 4.2 (5). First, \( \text{supp } \mathcal{T} \subseteq \text{supp } \mathcal{L} \) since \( \mathcal{T} \in \mathcal{L} \). For any \( q \in \text{supp } \mathcal{L} \) and any Thomason closed subsets \( V, U \subseteq \text{Spc } \mathcal{T}^c \) with \( p \in V \cap U^c \), we have
\[
\Gamma_V L_U T = \Gamma_V L_U \prod_{p \in \text{supp } \mathcal{L}} T^p \cong \Gamma_V L_U T^q \sqcup \prod_{p \in \text{supp } \mathcal{L} \setminus \{q\}} \Gamma_V L_U T^p \neq 0.
\]
Thus \( p \in \text{supp } \mathcal{T} \). □

4.2. Visible points. In this section, we relate our support to that of Balmer, Favi, and Stevenson in [2, 38]. Following [38], a point \( p \) in a spectral space \( X \) is visible if there exists Thomason subsets \( V, U \subseteq X \) such that
\[ \{p\} = V \cap U^c. \]
This definition is more general than [2, Definition 7.9].

**Lemma 4.4.** The following are equivalent for a spectral space \( X \).

1. All points of \( X \) are visible.
2. The localising topology on \( X \) is discrete.
3. The Hochster dual \( X^\dagger \) is \( T_1 \)

Moreover, these conditions hold when \( X \) is Noetherian.

See Definition [2.6] to recall the oft forgotten separation axiom \( T_1 \).

**Proof.** The equivalence is straightforward. See [2, Proposition 7.13] for the last statement. □

Let \( \mathcal{T} \) be a rigidly compactly generated tensor triangulated category. When a prime \( p \in \text{Spc } \mathcal{T}^c \) is visible and write \( \{p\} = V \cap U^c \) for Thomason subsets \( V, U \). Define the **Rickard idempotent**
\[ \Gamma_p 1 = \Gamma_V 1 \otimes L_U 1. \]
By [2, Corollary 7.5], this definition is independent of the choice of \( V \) and \( U \). Rickard idempotents have appeared in [34, 5] and other works. When every prime is visible, then \( p \in \text{supp } \mathcal{T} \) if and only if
\[ \Gamma_p 1 \otimes \mathcal{T} \neq 0. \]
Thus our definition of support recovers [2, Definition 7.16] and [38, Definition 5.6]. Furthermore, Theorem 4.2 is a generalisation of [2, Theorem 7.17, Proposition 7.18].

For \( R \) a commutative Noetherian ring, \( \mathcal{T} = \text{D}(R) \) is a compactly generated tensor triangulated category. In this case, \( \text{Spc } \text{D}(R)^c = \text{Spec } R \) by the Hopkins Neeman theorem [16, 28]. Every point is visible in \( \text{Spec } R \) and so our support coincides with that of Balmer and Favi. Moreover by the work of Foxby and Iyengar in [11] and [12, 2.1 and 4.1], \( p \in \text{supp } M \) for \( M \in \text{D}(R) \) if and only if \( M \otimes^L k(p) \neq 0 \) where \( k(p) \) is the residue field at \( p \).
4.3. Detecting vanishing.

Definition 4.5. We call $\mathcal{T}$ supportive if $T = 0$ if and only if $\text{supp} T = \emptyset$.

Example 4.6. By [11, Lemma 2.6] and [28], $D(R)$ is supportive for all Noetherian commutative rings $R$.

Theorem 4.7. Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category. If $\mathcal{T}$ is supportive, the following are true.

1. For any Thomason subset $V \subseteq \text{Spc} \mathcal{T}^e$ and $T \in \mathcal{T}$, the following statements hold.
   - (a) $T$ is $V$-acyclic if and only if $\text{supp} T \subseteq V$
   - (b) $T$ is $V$-local if and only if $\text{supp} T \subseteq V^c$
   - (c) $\mathcal{T}_V = \{ T \in \mathcal{T} \mid \text{Supp} T \subseteq V \}$

2. If there exists a Thomason subset $V \subseteq \text{Spc} \mathcal{T}^e$ such that $\text{supp} S \subseteq V$ and $V \cap \text{supp} T = \emptyset$, then $\text{Hom}_\mathcal{T} (S, T) = 0$.

3. For any Thomason subsets $V, U \subseteq \text{Spc} \mathcal{T}^e$ and $T \in \mathcal{T}$, then $V \cap U^c \cap \text{supp} T = \emptyset$ if and only if $\Gamma \sum T \in \mathcal{U}$.

4. For any $X \subseteq \mathcal{T}$ let $\text{loc} \otimes X$ be the localising tensor ideal closure of $X$. Then $\text{supp} \otimes \mathcal{X} = \text{supp} \mathcal{X}$

5. For any close set $X \subseteq \text{Lspc} \mathcal{T}$, the category $\{ T \in \mathcal{T} \mid \text{supp} T \subseteq X \}$ is localising.

Proof of Theorem 4.7 (1). We prove Theorem 4.7 (1a). By Theorem 4.2 (1a), if $T$ is $V$-acyclic then $\text{supp} T \subseteq V$. We now prove the converse. If $\text{supp} T \subseteq V$, then by Theorem 4.2 (3), $\text{supp} L V T = \text{supp} T \cap V^c = \emptyset$.

Proof of Theorem 4.7 (2). If there exists a Thomason subset $V \subseteq \text{Spc} \mathcal{T}^e$ such that $\text{supp} S \subseteq V$ and $V \cap \text{supp} T = \emptyset$, then $S$ is $V$-acyclic and $T$ is $V$-local. Theorem 4.7 (1). It follows that $\text{Hom}_\mathcal{T} (S, T) = 0$.

Proof of Theorem 4.7 (3). This follows by applying Theorem 4.2 (3a) and (3b) and the supportive condition.
Proof of Theorem 4.7 (4). Let \( X \subseteq \mathcal{T} \) be a collection of objects. It is clear that \( \text{supp} X \subseteq \text{supp} \text{loc} X \). By Theorem 4.2 (6), we know that \( \text{supp} \text{loc}^\otimes X \) is closed, and so
\[
\text{supp} X \subseteq \text{supp} \text{loc}^\otimes X.
\]
Now take a prime \( p \notin \text{supp} X \). There exist Thomason subsets \( V, U \subseteq \text{Spc} \mathcal{T}^c \) such that \( p \in V \cap U^c \) and \( V \cap U^c \) is disjoint from \( \text{supp} X \). From Theorem 4.7 (3),
\[
\Gamma V L U X = 0.
\]
Since the kernel of \( \Gamma V L U \) is a tensor ideal, \( \Gamma V L U \text{loc}^\otimes X = 0 \). Therefore \( p \notin \text{supp} \text{loc}^\otimes X \).
\( \square \)

Proof of Theorem 4.7 (5). Let \( X \subseteq \text{Spc} \mathcal{T}^c \) be closed in the localising topology. Set
\[
\mathcal{L} = \{ T \in \mathcal{T} \mid \text{supp} T \subseteq X \}.
\]
Theorem 4.2 (1) shows that \( \mathcal{L} \) is thick. For a set \( \{ T_\alpha \} \subseteq \mathcal{L} \) Theorem 4.7 (4) implies that
\[
\text{supp} \coprod T_\alpha = \bigcup \text{supp} T_\alpha \subseteq X
\]
since \( X \) is closed. It is easy to check that \( \mathcal{L} \) is a tensor ideal. \( \square \)

4.4. Characterization of support. It is not clear if \( \mathcal{T} \) is always supportive. However, the following result tells us that if \( \text{supp} \) does not detect vanishing, then there is no other reasonable support function taking values in \( \text{Spc} \mathcal{T}^c \) that will.

**Theorem 4.8.** Consider the following properties of a function
\[
s : \mathcal{T} \to \{ \text{Subsets of} \ \text{Spc} \mathcal{T}^c \}.
\]
Let \( T \in \mathcal{T} \).

1. Vanishing: \( s(T) = \text{if and only if} \ T = 0 \).
2. Local: If \( V \subseteq \text{Spc} \mathcal{T}^c \) is Thomason and \( T \) is \( V \)-local, then \( s(T) \cap V = \emptyset \).
3. Big Support: \( s(T) \subseteq \text{Supp} \ T \) for any compact object \( T \in \mathcal{T}^c \).
4. Orthogonality: For any \( S \in \mathcal{T} \), if there is a Thomason subset \( V \subseteq \text{Spc} \mathcal{T}^c \) such that \( s(T) \subseteq V \) and \( s(S) \cap V = \emptyset \), then
\[
\text{Hom}_\mathcal{T} (T, S) = 0.
\]
5. Exactness: For any exact triangle \( S \to T \to S' \to \) in \( \mathcal{T} \),
\[
s(T) \subseteq s(S) \cup s(S').
\]
6. Separation: For any Thomason subset \( V \subseteq \text{Spc} \mathcal{T}^c \), there is an exact triangle
\[
T' \to T \to T'' \to
\]
with \( s(T') \subseteq V \) and \( s(T'') \cap V = \emptyset \).

The function \( s \) satisfies all of these properties, if and only if \( s = \text{supp} \) and \( \mathcal{T} \) is supportive.

The proof is similar to that of [5, Theorem 5.15].
Proof. If \( T \) is supportive, then \( \text{supp} \) satisfies these properties by Theorem 1.2 and Theorem 4.7. Conversely, suppose \( s \) satisfies all of these properties. We need to show that \( s = \text{supp} \).

Let \( T \in T \), and consider the exact triangle 

\[ T' \to T \to T'' \to \]

guaranteed by (6). By (5) and (6), \( s(T') = s(T) \cap V \) and \( s(T'') = s(T) \cap V^c \).

We claim that 

\[ T' \cong \Gamma V T \quad \text{and} \quad T'' \cong L V T \]

for any Thomason subset \( V \subseteq \text{Spc} T^c \). Given the claim, \( p \notin s(T) \) if and only if there are Thomason subsets \( V, U \subseteq \text{Spc} T^c \) with \( p \in V \cap U^c \) such that 

\[ s(\Gamma V L U T) = \emptyset. \]

Therefore, the claim and (4) imply that \( s(T) = \text{supp} T \).

We now prove the claim. Let \( X \in T \) be a \( V \)-local object and let \( Y \in T^c \). Then by (2) and (3)

\[ s(X) \cap V = \emptyset \quad \text{and} \quad s(Y) \subseteq \text{Supp} Y \subseteq V. \]

Then (4) and (6) imply

\[ \text{Hom}_T (T', X) = 0 \quad \text{and} \quad \text{Hom}_T (Y, T'') = 0. \]

We conclude that \( T' \) is \( V \)-acyclic and \( T'' \) is \( V \)-local by [25, Proposition 4.10.1] and the definition of \( V \)-local.

By [25 Proposition 4.11.2], we have the following commutative diagram.

\[
\begin{array}{c}
T' \xrightarrow{f} T \xrightarrow{g} T'' \xrightarrow{\Sigma f} \Sigma T'
\end{array}
\]

\[
\begin{array}{c}
\Gamma V T \xrightarrow{f} T \xrightarrow{g} L V T \xrightarrow{\Sigma f} \Sigma \Gamma V T
\end{array}
\]

By the octahedral axiom, the cones

\[ \text{cone} g \cong \Sigma \text{cone} f \]

are isomorphic and thus both are \( V \)-acyclic and \( V \)-local. Hence the cones are zero, and so \( f \) and \( g \) are isomorphisms. \( \square \)

5. Commutative rings

5.1. Support. In this section, we specialise our theory of supports to the derived category \( D(R) \) for a commutative ring \( R \). For \( \underline{x} = x_1, \ldots, x_n \in R \), we write

\[
K_\infty (\underline{x}) = (R \to R_{x_1}) \otimes \cdots \otimes (R \to R_{x_n}) = 0 \to R \to \bigoplus_{1 \leq i \leq n} R_{x_i} \to \bigoplus_{1 \leq i < j \leq n} R_{x_i x_j} \to \cdots \to \bigoplus_{1 \leq i_1 < \cdots < i_n \leq n} R_{x_1 \cdots x_n} \to 0.
\]

Lemma 5.1.

(1) The category \( D(R) \) is a rigidly compactly generated tensor triangulated category whose tensor product is \( \otimes^L \).

(2) The compact objects of \( D(R) \) are the perfect complexes, i.e. complexes which are quasi-isomorphic to bounded complexes of finitely generated free modules. We will denote the perfect complexes by \( \text{Perf}(R) \).

(3) We have \( \text{Spc} \text{Perf}(R) = \text{Spec} R \).
(4) The closed sets of \( \text{Spec } R \) with quasi-compact complement are those of the form \( V(\underline{x}) \) with \( \underline{x} = x_1, \ldots, x_n \in R \). Thus, every Thomason set is a union of such sets. Moreover,

\[ \Gamma_{V(\underline{x})} R = K_\infty(\underline{x}). \]

When \( R \) is Noetherian this is \( R \Gamma_{\underline{x}} R \), the right derived torsion functor applied to \( R \).

(5) For any \( p \in \text{Spec } R \) and \( M \in D(R) \),

\[ L_{Z(p)} M = M_p. \]

Hence the notation in Definition \ref{def:Supportive} is unambiguous.

Proof. For (2), See \cite[Example 1.10,1.13]{29}. Statement (3) is the Hopkins, Neeman, Thomason theorem \cite{28, 16}, and \cite{43}. For (4), see \cite[Lemma 5.8]{13}, although the argument stems from the Noetherian case treated in \cite{14}.

We show (5). Any \( M \in D(R) \) satisfies \( M_p = 0 \), and thus \( L_{Z(p)} R_p \cong R_p \).

Furthermore, the complex

\[ 0 \to R \to R_p \to 0 \]

is a direct limit of complexes

\[ \Gamma_{V(\underline{x})} R = 0 \to R \to R_x \to 0 \]

with \( x \notin p \). Since \( L_{Z(p)} \Gamma_{V(\underline{x})} = 0 \), the first complex is in the kernel of \( L_{Z(p)} \). It follows that \( L_{Z(p)} R \cong L_{Z(p)} R_p \).

We can now restate our definition of support. In fact, for the reader whose interest only lies in commutative algebra, the following can be taken as a definition.

For a complex \( M \in D(R) \) and a sequence \( \underline{x} \in R \), set

\[ K_\infty(\underline{x}; M) = K_\infty(\underline{x}) \otimes M \quad \text{and} \quad \text{H}^i K_\infty(\underline{x}; M) = \text{H}^i(K_\infty(\underline{x}; M)). \]

Lemma 5.2. For a complex \( M \in D(R) \), a prime \( p \in \text{supp } M \) if and only if for every finite sequence \( \underline{x} = x_1, \ldots, x_n \in p \)

\[ \text{H}^i K_\infty(\underline{x}; M) \neq 0 \]

for some \( i \).

Proof. This result follows directly from Lemma \ref{lem:Supportive0} and Lemma \ref{lem:Supportive1} (4).

5.2. Detecting vanishing. We now discuss the supportive condition in \( D(R) \).

Definition 5.3. For a module \( M \in \text{Mod}(R) \), a prime \( p \in \text{Spec } R \) is weakly associated to a module \( M \) if there exists an element \( m \in M \) such that \( p \) is minimal amongst primes containing \( \text{ann}(m) \). Let \( \text{\text{Ass}}_R M \) denote the set of weakly associated primes of \( M \). Let \( | \text{\text{Ass}}_R M | \) be the minimal weakly associated primes.

The following result is an exercise in \cite[IV,1, Exercise 17]{7}. See \cite{26} for proofs.

Lemma 5.4. Let \( M \) be an \( R \)-module and \( p \in \text{Spec } R \) a prime.

(1) \( p \in \text{\text{Ass}}_R M \) if and only if \( pR_p \in \text{\text{Ass}}_R M \)

(2) \( M = 0 \) if and only if \( \text{\text{Ass}}_R M = \emptyset \)

(3) When \( R \) is Noetherian, then \( \text{\text{Ass}}_R M = \text{ass } M \)
(4) If $W \subseteq R$ then
\[ \tilde{\text{Ass}} \Gamma_W M = \{ p \in \tilde{\text{Ass}} M \mid p \cap W \neq \emptyset \} \]
where $\Gamma_W M$ is the $W$-torsion submodule of $M$.

(5) If $W \subseteq R$ is multiplicatively closed, then
\[ \tilde{\text{Ass}} M_W = \{ p \in \tilde{\text{Ass}} M \mid p \cap W = \emptyset \}. \]

### Theorem 5.5.

(1) Let $M \in D(R)$ be a complex of $R$-modules such that $H^{<n}(M) = 0$, e.g. $M$ is a module. Then
\[ \widetilde{\text{Ass}} R H^n(M) \subseteq \text{supp} M. \]
In particular, $M = 0$ if and only if $\text{supp} M = \emptyset$.

(2) For any complex $M \in D(R)$
\[ \min \tilde{\text{Ass}} R \bigoplus_{i \in \mathbb{Z}} H^i(M) \subseteq \text{supp} M. \]

(3) If the prime ideals of $R$ satisfy DCC, then $D(R)$ is supportive, i.e. $\text{supp} M = \emptyset$ if and only if $M = 0$.

Note, we will write $\text{supp}_R M$ for $\text{supp} M$ if there is any confusion over which ring we are computing the support.

**Proof of Theorem 5.5 (1).**
We may assume $n = 0$. Suppose $p \in \tilde{\text{Ass}} H^0(M)$. By Lemma 5.2 it suffices to show that $pR_p \in \text{supp}_R M_p$. Since $H^0(M)_p \cong H^0(M_p)$, Lemma 5.4 (1) implies that $pR_p \in \tilde{\text{Ass}} H^0(M_p)$. Therefore, we can assume that $R$ is local with maximal ideal $m$ and that $p = m$.

We claim that for every $x \in m$, the exact triangle $\Gamma_V(x)M \to M \to M_x \to \cdots$ yields
\[ \cdots \to 0 \to H^0 K_\infty(x : M) \to H^0(M) \to H^0(M)_x \to \cdots. \]
Therefore $H^{i<0} K_\infty(x : M) = 0$ and
\[ H^0 K_\infty(x : M) = H^0(K_\infty(x : M)) = \Gamma_x H^0(M). \]
By induction, for any $\mu = x_1, \ldots, x_n \in m$
\[ H^0 K_\infty(x : M) = H^0 K_\infty(x_n : K_\infty(x_1, \ldots, x_{n-1} : M)) = \Gamma_x H^0(M). \]
By Lemma 5.4 (4), $m \in \tilde{\text{Ass}} H^0(M)$, and so $H^0(M)$ does not vanish by Lemma 5.4 (2). □

**Proof of Theorem 5.5 (2) and (3).** We write $H(M)$ for $\bigoplus H^i(M)$. If the primes ideals of $R$ satisfy DCC, then $H(M)$ has a minimal weakly associated prime for all nonzero $M \in D(R)$, by Lemma 5.3 (2). Thus (2) implies (3).

We now show (2). Let $p \in \min \tilde{\text{Ass}} H(M)$. As in the proof of (1), $pR_p$ is a weakly associated prime of $H(M_p)$. By Lemma 5.4 (5), $pR_p$ is a minimal weakly associated prime. Thus, we assume that $R$ is a local ring with maximal ideal $m$, and that $p = m$.

Since $m$ is the only weakly associated prime of $H(M)$. In this case, for any element of $\mu \in H(M)$, $m$ is the only prime minimal over $\text{ann} \mu$, and so $\sqrt{\text{ann} \mu} = m$. 

\[ \]
Therefore every element of \( H(M) \) is \( m \)-torsion. It follows that \( H(M)_x = 0 \) for any \( x \in m \), and therefore

\[
K_\infty(x; M) = (R \to R_x) \otimes M \cong M.
\]

It follows that that \( K_\infty(x; M_m) \cong M \neq 0 \) for all finite sequences \( x \in M \), and so \( m \in \text{supp} M \).

Example 5.6. Set

\[
R = \frac{k[x_2, x_3, x_4, \cdots]}{(x_2^2, x_3^3, x_4^4, \cdots)}.
\]

In [30], Neeman shows that the collection of localising subcategories of \( D(R) \) is atrocious. However, the support theory is simple: \( \text{Spec} R \) has one prime ideal \( m = (x_2, x_3, x_4, \ldots) \) and so \( D(R) \) is supportive by the previous theorem. For \( M \in D(R) \), then \( \text{supp} M = \{m\} \) unless \( M \) is acyclic.

Example 5.7. We now give an example of a ring such that \( D(R) \) is supportive, but Foxby’s support does not detect vanishing. In [4], Example 5.34, Šťovíček describes a commutative ring \( R \) with the following properties. First, the ring \( R \) is a valuation domain with prime ideals 0 and \( m \). Furthermore, \( \text{Tor}_>0(k, k) = 0 \) where \( k = R/m \), then. Let \( Q \) be the quotient field of \( R \). Note that telescope conjecture fails for \( D(R) \) by [21]. Also \( D(R) \) is supportive since \( \text{Spec} R \) satisfies DCC on prime ideals.

Let \( M \) be the cokernel of the composition

\[
m \hookrightarrow R \twoheadrightarrow Q.
\]

We claim that \( M \otimes^L k(p) = 0 \) for all \( p \in \text{Spec} R \), i.e. Foxby’s support of \( M \) is empty. Indeed, \( M \) is quasi-isomorphic to the complex

\[
(R \to k) \otimes^L (R \to Q).
\]

Since \( R \) has only two prime ideals \( k(p) \) can either be \( k \) or \( Q \). However \( (R \to k) \otimes k \) and \( (R \to Q) \otimes^L Q \) are both acyclic, proving the claim.

5.3. Properties of support. In this section we show that support over non-Noetherian rings behaves similarly to support over Noetherian rings. Note that a careful examination of the proofs of Section 4.3 show that even though we do not have the supportive condition for all complexes, the results still hold for \( D(R) \).

Proposition 5.8. Let \( R \) be a ring, \( H^{<0}(M) \) and \( W \subseteq R \) be a multiplicatively closed subset. Then

\[
\text{supp} M_W = \{p \in \text{supp} M \mid p \cap W = 0 \}.
\]

Proof. For any \( p \) that does not intersect \( W \), we have \( M_p = (M_W)_p \) and so \( p \in \text{supp} M_W \) if and only if \( p \in \text{supp} M \). If there is a \( w \in p \cap W \), then the equality

\[
\Gamma_{V(w)}(M_W)_p = 0
\]

implies \( p \notin \text{supp} M \).

Proposition 5.9. Let \( f: R \to S \) be homomorphism of commutative rings, and \( M \in D(S) \).
(1) For any Thomason subset $\mathbf{V} \subseteq \text{Spec } R$, the morphism $af^{-1}(V)$ is Thomason. In particular, the $af$ is continuous in both the Hochster dual topology and the localising topology. Furthermore, the following are isomorphisms in $D(R)$

$$
\Gamma_{\mathbf{V}} M \cong \Gamma_{af^{-1}(\mathbf{V})} M \quad L \mathbf{V} M \cong L_{af^{-1}(\mathbf{V})} M.
$$

(2) All Thomason subsets $\mathbf{V}, \mathbf{U} \subseteq \text{Spec } R$ satisfy

$$
\operatorname{supp}_S M \cap af^{-1}(\mathbf{V} \cap \mathbf{U}^c) = \operatorname{supp}_S \Gamma_{af^{-1}(\mathbf{V})} L_{af^{-1}(\mathbf{U})} M.
$$

(3) If $\mathbb{H}^{\leq 0}(M)$, then

$$
a f(\operatorname{supp}_S M) = \operatorname{supp}_R M.
$$

Proof: For $x = x_1, \ldots, x_n \in R$, we have $af^{-1}(\mathbf{V}(x)) = \mathbf{V}(d(x))$. This proves the first statement of (1). Consider the idempotent triangle

$$
\Gamma_{\mathbf{V}(x)} R \rightarrow R \rightarrow L \mathbf{V}(x) R \rightarrow
$$

and apply $S \otimes^L \mathbf{.}$ By Lemma 5.4 and Theorem 3.6, we have

$$
\Gamma_{\mathbf{V}(x)} S \cong \Gamma_{(f(x))} S \cong \Gamma_{af^{-1}(\mathbf{V}(x))} S \quad L \mathbf{V}(x) S \cong L_{(f(x))} S \cong L_{af^{-1}(\mathbf{V}(x))} S.
$$

Statement (1) follows for $M = S$ now follows by taking the direct limit of the above functors over all $\mathbf{V}(x) \subseteq \mathbf{V}$. Note that this result applies to all

Statement (2) follows from (1) and Theorem 1.2. We now prove (3). Let $\mathfrak{p} \in \text{Spec } R$ and $\mathbf{V}, \mathbf{U} \in \text{Spec } R$ such that $\mathfrak{p} \in \mathbf{V} \cap \mathbf{U}^c$. Statement (2) and the supportive imply that the following are equivalent

$$
\Gamma_{\mathbf{V}} L_{\mathbf{U}} M = 0 \iff \operatorname{supp}_S \Gamma_{af^{-1}(\mathbf{V})} L_{af^{-1}(\mathbf{U})} M = \emptyset \iff \operatorname{supp}_S M \cap af^{-1}(\mathbf{V} \cap \mathbf{U}^c) = \emptyset.
$$

\ \square

Remark 5.10. Proposition 5.9 (3) holds for all $M \in D(S)$ if $D(S)$ is supportive. In particular, if $D(S)$ is supportive and $f : R \rightarrow S$ is faithfully flat, then $D(R)$ is supportive.

6. Support and the Bousfield Lattice

Again, let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category.

Definition 6.1.

(1) For any element $T \in \mathcal{T}$, set

$$
\mathcal{A}(T) = \{S \in \mathcal{T} \mid T \otimes S = 0\}.
$$

Any class of the form $\mathcal{A}(T)$ is called Bousfield.

(2) A Bousfield class $\mathcal{A}(T)$ is idempotent if $\mathcal{A}(T) = \mathcal{A}(T \otimes T)$.

(3) Let $\mathbb{D}(T)$ denote the collection of idempotent Bousfield classes. Order $\mathbb{D}(T)$ via reverse inclusion. The elements $\mathcal{A}(1)$ and $\mathcal{A}(0)$ are the maximum and minimum respectively.

For every pair of Thomason subsets $\mathbf{V}, \mathbf{U} \subseteq \text{Spec } \mathcal{T}^c$, the kernel of the functor $\Gamma_{\mathbf{V}} L_{\mathbf{U}}$ is precisely the Bousfield class $\mathcal{A}(\Gamma_{\mathbf{V}} L_{\mathbf{U}} 1)$ by Theorem 3.6. Moreover, since $\Gamma_{\mathbf{V}} L_{\mathbf{U}} 1$ is idempotent, this Bousfield class is also idempotent.

The first statement of the following theorem was originally proven by Ohkawa in [32] for the stable homotopy category. The second statement was also proven in this context by Bousfield in [8] and by Hovey and Palmieri in [17] Proposition
3.4. The following result was proven in our generality by Krause and Iyengar in [19, Theorem 3.1, Proposition 6.2].

**Theorem 6.2.** The class $\mathbb{D}(T)$ is a set. Moreover, it is a frame where arbitrary joins and finite meets are given by

$$\bigvee_i A(T_i) = \bigcap_i A(T_i) = A \left( \prod_i T_i \right) \quad A(T_1 \wedge \cdots \wedge T_n) = A(T_1 \otimes \cdots \otimes T_n)$$

Pointless topology now creeps into our theory. Suppose $\text{supp} T = \emptyset$. Then for every $p \in \text{Spc} \, T^c$ there is a pair of Thomason subsets $V_p, U_p$ with $p \in V_p \cap U_p^c$ with $\Gamma_{V_p}L_{U_p}T = 0$, or in other words with $T \in A(\Gamma_{V_p}V_{L_{U_p}1})$. Hence

$$T \in \bigvee_{p \in \text{Spc} \, T^c} A(\Gamma_{V_p}V_{L_{U_p}1}).$$

Now $\mathcal{T}$ is supportive precisely when the above Bousfield class always vanishes, i.e.

$$\bigvee_{p \in \text{Spc} \, T^c} A(\Gamma_{V_p}V_{L_{U_p}1}) = 0 = A(1) = A(\Gamma_{\text{Spc} \, T^c}L_\emptyset 1).$$

Compare this equality the union of basic open sets in $\text{Lspc} \, \mathcal{T}$

$$\bigcup_{p \in \text{Spc} \, T^c} V_p \cap U_p^c = \text{Spc} \, T^c \cap \emptyset^c.$$  

If two unions of basic open neighbourhoods in the Skula topology are set theoretically equal, do they define the same idempotent Bousfield class? If this is so, then $\mathcal{T}$ is supportive. As we will see in Theorem 6.3 the converse is also true.

Recall from Definition 3.3 and the following discussion that $\mathcal{T}(T^c)$ is both the frame of tensor ideals and the frame of Thomason sets of $\text{Spc} \, T^c$, i.e. the frame associated to the Hochster dual $\text{HSpc} \, T^c$. Also recall from Definition 2.6 that $\text{fHSpc} \, T^c : \text{skula}(\text{HSpc} \, T^c) \to \text{HSpc} \, T^c$ is the map induced by the identity function. Recall also that by definition $\text{skula}(\text{HSpc} \, T^c) = \text{Lspc} \, \mathcal{T}$.

**Theorem 6.3.** Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category.

1. Moreover, the assignment $V \mapsto A(\Gamma_{V}1)$ defines a frame homomorphism

$$\gamma_\mathcal{T} : \mathcal{T}(T^c) \to \mathbb{D}(T).$$

2. The following are equivalent.
   a. The category $\mathcal{T}$ is supportive.
   b. For any collections of Thomason subsets $\{V_i\}, \{U_i\}, \{V_j\}, \{U_j\} \subseteq \mathcal{T}(T^c)$, if the sets

$$\bigcup_i V_i \cap U_i^c = \bigcup_j V_j \cap U_j^c$$

are equal, then the following Bousfield classes coincide

$$A \left( \prod_i \Gamma_{V_i}L_{U_i}1 \right) = A \left( \prod_j \Gamma_{V_j}L_{U_j}1 \right).$$
(c) There exists a frame homomorphism $\eta$ which completes the following commutative diagram.

$$
\begin{array}{ccc}
T(T^c) & \xrightarrow{\gamma_T} & \mathbb{D}(T) \\
F_{HSpc} T^c & \xrightarrow{\sim} & \mathbb{F}(Lspc T) \\
\end{array}
$$

**Corollary 6.4.** A rigidly compactly generated tensor triangulated category $T$ is supportive if the idempotent Bousfield frame $\mathbb{D}(T)$ is spatial.

**Example 6.5.** Unfortunately, the idempotent Bousfield frame $\mathbb{D}(T)$ is not always spatial. Let $M$ be the Lebesgue measurable sets of the real numbers. Write $A \sim B$ for $A, B \in M$ if their symmetric difference has measure 0. The Boolean algebra $\mathbb{B} = M/\sim$ has no points and thus is a nonspatial frame. In [11], Stevenson constructs a triangulated category $T$ satisfying our usual assumptions such that $\mathbb{D}(T) = \mathbb{B}$.

Before we prove the corollary, we recall an easy lemma. This was observed in the stable homotopy category by Hovey and Palmieri in [17, Proposition 4.5].

**Lemma 6.6.** For a Thomason subset $V \in T(T^c)$, the Bousfield classes $A(\Gamma V 1)$ and $A(LV 1)$ are complements in $\mathbb{D}(T)$, i.e.

$$A(\Gamma V 1) \lor A(LV 1) = A(1) \quad A(\Gamma V 1) \land A(LV 1) = A(0).$$

**Proof.** First, note $A(\Gamma V 1) \land A(LV 1) = A(\Gamma V 1 \otimes LV 1) = A(0)$. Next, if $T$ is in $A(\Gamma V 1) \lor A(LV 1) = A(\Gamma V 1) \cap A(LV 1)$ then the idempotent triangle $\Gamma V T \to T \to LV T \to$ implies that $T = 0$. Hence this intersection is $\{0\} = A(1)$. \hfill $\Box$

**Proof of Corollary 6.4.** Apply the functor $Spc$ to the diagram in Theorem 6.3 [2] [2a].

$$
\begin{array}{ccc}
HSpc T^c & \xrightarrow{Spc \gamma_T} & Spc \mathbb{D}(T) \\
\downarrow f_{HSpc T^c} & & \downarrow \skula(HSpc T^c) \\
\end{array}
$$

Since $\mathbb{D}(T)$ is spatial, $\mathbb{D}(T)$ and the frame of open sets $F(Spc \mathbb{D}(T))$ are isomorphic. Thus to every Bousfield class $A(T)$, we associate the open set $\mathbb{D}(A(T))$.

By Lemma 6.6 the Bousfield classes $A(\Gamma V 1)$ and $A(LV 1)$ are complements. Thus their corresponding open sets $\mathbb{D}(A(\Gamma V 1))$ and $\mathbb{D}(A(LV 1))$ are set theoretic complements. In particular, the closed set $\mathbb{D}(A(\Gamma V 1))^c = \mathbb{D}(A(LV 1))$ is open. We conclude that

$$(Spc \gamma_T)^{-1}(V^c) = \left((Spc \gamma_T)^{-1}(V)\right)^c = \mathbb{D}(A(\Gamma V 1))^c = \mathbb{D}(A(LV 1))$$

is open, and thus $Spc \gamma_T$ is not only continuous with respect to the Hochster dual topology, $HSpc T^c$, but with respect to the localising topology $\skula(HSpc T^c)$.

Returning to the above diagram, we have actually just shown that there exists a continuous function $g: Spc \mathbb{D}(T) \to \skula(HSpc T^c)$ making the above diagram commute. Since $\mathbb{D}(T)$ is spatial, $F(Spc \mathbb{D}(T)) \cong \mathbb{D}(T)$ by Theorem 2.11. Thus $\eta = Fg$ satisfies the hypotheses of Theorem 6.3 [2] [2a]. \hfill $\Box$
Remark 6.7. The proof of Corollary 6.4 still holds if we only assume there exists a sub lattice \( X \subseteq D(T) \) such that
(a) \( X \) is a spatial frame
(b) the inclusion map \( X \hookrightarrow D(T) \) is a frame homomorphism
(c) for any Thomason set \( V \subseteq \Spc T^c \), the Bousfield classes \( \mathcal{A}(\Gamma V) \) and \( \mathcal{A}(L_V) \) are in \( X \).

We discuss the utility of this remark in Section 8.3.

The rest of the section is devoted to proving Theorem 6.3.

**Proof of Theorem 6.3 (1).** The frame homomorphism \( \gamma_T \) preserves extrema since \( \gamma_T(\Spc T) = \mathcal{A}(1) \) and \( \gamma_T(\emptyset) = \mathcal{A}(0) \). The frame homomorphism \( \gamma_T \) preserves finite meets since any Thomason subsets \( V, U \in T(T^c) \) satisfy
\[
\mathcal{A}(\Gamma V) \land \mathcal{A}(\Gamma U) = \mathcal{A}(\Gamma V \otimes \Gamma U) = \mathcal{A}(\Gamma V \cap \Gamma U) = \mathcal{A}(\Gamma V \cap \Gamma U)
\]
by Theorem 3.6.

We now check that \( \gamma_T \) preserves arbitrary joins. Let \( \{V_i\} \in T(T^c) \) be a family of Thomason subsets, and set \( V = \bigcup V_i \). Since
\[
\mathcal{A}(\Gamma V) \subseteq \bigcap_i \mathcal{A}(\Gamma V_i) = \bigvee_i \mathcal{A}(\Gamma V_i)
\]
we show the reverse containment.

Consider the following computation where \( \text{Thick}, \text{Thick}^\otimes, \text{Thick}^{\sqrt{\otimes}} \) respectively denote the thick closure, thick tensor ideal closure, and the thick radical tensor ideal closure;
\[
(\ast) \quad \text{Thick} \bigcup_i \mathcal{T} V_i^c = \text{Thick}^\otimes \bigcup_i \mathcal{T} V_i^c = \text{Thick}^{\sqrt{\otimes}} \bigcup_i \mathcal{T} V_i^c = \mathcal{T} V^c.
\]
The first equality holds because the thick closure of two tensor ideals is again a tensor ideal. The second follows from rigidity; see Lemma 3.4 The last equality is Theorem 3.2 (2).

Suppose \( T \) is an object in \( V_i, \mathcal{A}(\Gamma V_i, 1) \), i.e. suppose \( \Gamma V_i, T = 0 \) for all \( i \). Then \( \text{Hom}_T(\mathcal{T} V_i^c, T) = 0 \) for all \( i \) by Theorem 3.6 (1). Equation (\ast) implies that \( \text{Hom}_T(\mathcal{T} V_i^c, T) = 0 \). Hence \( T \in \mathcal{A}(\Gamma V) \) by Theorem 3.6 (1). \( \square \)

**Proof of Theorem 6.3 (2), (2a) \( \Leftrightarrow \) (2b).** Assume \( T \) is supportive. Suppose that \( \{V_i\}, \{U_i\}, \{V_j\}, \{U_j\} \subseteq T(T^c) \) are collections of Thomason subsets, and that
\[
\bigcup_i V_i \cap U_i^c = \bigcup_j V_j \cap U_j^c.
\]
Let \( T \in \mathcal{A}(\coprod_i \Gamma V_i, L_U, 1) \). For all \( i \)
\[
\Gamma V_i, L_U, 1 \otimes T = \Gamma V_i, L_U, T = 0.
\]
By Theorem 4.7 (3), this means that
\[
\text{supp} T \cap \left( \bigcup_i V_i \cap U_i^c \right) = \emptyset.
\]
But our assumption then implies \( \text{supp} T \cap V_j \cap U_j^c = \emptyset \) for all \( j \), and so
\[
\Gamma V_j, L_U, 1 \otimes T = \Gamma V_j, L_U, T = 0
\]
by Theorem 4.7 (3). Therefore \( T \) is in the class \( \mathcal{A}(\Gamma V, L_U, 1) \) for all \( j \).
Thus far we have shown that $A(\prod_i \Gamma_{V_i} L_{U_i}, 1) \subseteq A(\Gamma_{V_j} L_{U_j}, 1)$ for each $j$. We conclude that

$$A\left(\prod_i \Gamma_{V_i} L_{U_i}, 1\right) \subseteq \bigcap_j A(\Gamma_{V_j} L_{U_j}, 1) = A\left(\prod_j \Gamma_{V_j} L_{U_j}, 1\right).$$

Equality follows from symmetry. \hfill \Box

**Proof of Theorem 6.3 (2), (2b) $\Rightarrow$ (2c).** Assume (2b). Define the function

$$\eta: F(\text{skula}(H\text{Sp}c T^c)) \to D(T) \quad \text{by} \quad \bigcup_i V_i \cap U_i^c \mapsto A(\prod_i \Gamma_{V_i} L_{U_i}, 1).$$

This function is well defined by (2b). Clearly, $\eta$ makes the diagram in Theorem 6.3 commute.

We check that $\eta$ is a frame homomorphism. First $\eta$ preserves extrema.

$$\eta(\text{Sp}c T^c) = A(\Gamma_{\text{Sp}c T^c} 1) = A(1) \quad \eta(\emptyset) = A(\Gamma_{\emptyset} 1) = A(0)$$

Next, we check that $\eta$ preserves finite meets. The fourth equality is Theorem 3.6 (6).

$$\eta\left(\bigcup_i V_i \cap U_i^c\right) \wedge \eta\left(\bigcup_j V_j \cap U_j^c\right) = A\left(\prod_i \Gamma_{V_i} L_{U_i}, 1\right) \wedge A\left(\prod_j \Gamma_{V_j} L_{U_j}, 1\right)$$
$$= A\left(\prod_i \Gamma_{V_i} L_{U_i}, 1 \otimes \prod_j \Gamma_{V_j} L_{U_j}, 1\right)$$
$$= A\left(\prod_i \Gamma_{V_i} L_{U_i}, 1 \otimes \Gamma_{V_j} L_{U_j}, 1\right)$$
$$= A\left(\prod_i \Gamma_{V_i \cap U_j \cup U_j}, 1\right)$$
$$= \eta\left(\bigcup_i (V_i \cap V_j) \cap (U_i \cup U_j)^c\right)$$
$$= \eta\left(\bigcup_i V_i \cap U_i^c \cap \bigcup_j V_j \cap U_j^c\right)$$

A similar calculation shows that $\eta$ preserves arbitrary joins. \hfill \Box

**Proof of Theorem 6.3 (2), (2c) $\Rightarrow$ (2a).** Assume (2c). Every Thomason subset $V \in T(T^c)$ satisfies $\eta(V) = \gamma_T(V) = A(\Gamma_{V} 1)$. And so $\eta(V^c) = A(L_{V}, 1)$, by Lemma 6.6. Thus $\eta$ is the function defined in (2b).

Suppose $\text{supp} T = \emptyset$. Then for every $p \in \text{Sp}c T^c$ there exists a pair of Thomason subsets $V_p, U_p \in T(T^c)$ with $p \in V_p \cap U_p^c$ such that

$$\Gamma_{V_p} L_{U_p}, 1 \otimes T = \Gamma_{V_p} L_{U_p} T = 0.$$
It follows that
\[
T \in \bigcap_{p \in \text{Spc}^c} A(\Gamma_{V_p} L_{U_p}, 1) = A\left( \prod_{p \in \text{Spc}^c} \Gamma_{V_p} L_{U_p}, 1 \right).
\]
\[
= \eta \left( \bigcup_{p \in \text{Spc}^c} V_p \cap U_p^c \right) = \eta(\text{Spc}^c) = A(1) = \{0\}.
\]
Therefore \( T = 0 \).

7. Topology and support

7.1. The assembly of a frame.

Definition 7.1.

(1) An element \( \mathfrak{z} \) of a frame \( X \) is complemented if there exists an element \( \mathfrak{x}^c \in X \) such that \( \mathfrak{z} \land \mathfrak{x}^c = 0 \) and \( \mathfrak{z} \lor \mathfrak{x}^c = 1 \). We call \( \mathfrak{x}^c \) the complement of \( \mathfrak{z} \).

(2) A frame homomorphism \( \alpha : X \to Y \) is complemented if \( \alpha(\mathfrak{z}) \) is complemented in \( Y \) for every \( \mathfrak{z} \in X \).

(3) A nucleus of a frame \( X \) is a function \( \nu : X \to X \) satisfying the following.

(a) \( \mathfrak{x} \leq \nu(\mathfrak{x}) \) for all \( \mathfrak{x} \in X \)

(b) \( \mathfrak{x} \leq \mathfrak{y} \) implies \( \nu(\mathfrak{x}) \leq \nu(\mathfrak{y}) \) for all \( \mathfrak{x}, \mathfrak{y} \in X \)

(c) \( \nu(\mathfrak{x} \lor \mathfrak{y}) = \nu(\mathfrak{x}) \lor \nu(\mathfrak{y}) \) for all \( \mathfrak{x}, \mathfrak{y} \in X \)

(d) \( \nu(\mathfrak{x} \land \mathfrak{y}) = \nu(\mathfrak{x}) \land \nu(\mathfrak{y}) \) for all \( \mathfrak{x}, \mathfrak{y} \in X \)

(4) For a frame \( X \), let \( N(X) \) denote the set of Nuclei. We call \( N(X) \) the assembly of \( X \). The assembly is partially ordered in the following manner: for nuclei \( \nu, \mu \in N(X) \), we say that \( \nu \leq \mu \) if \( \nu(\mathfrak{x}) \leq \mu(\mathfrak{x}) \) for all \( \mathfrak{x} \in X \).

(5) For every element of a frame \( \mathfrak{x} \in X \), let \( \nu_\mathfrak{x} : X \to X \) be the nucleus defined by \( \mathfrak{y} \mapsto \mathfrak{y} \land \mathfrak{x} \).

The assembly has many different equivalent constructions; see [33] Chapters II, IV, VI for an overview.

Theorem 7.2. Let \( X \) be a frame.

(1) The assembly \( N(X) \) is a frame.

(2) The assignment \( \mathfrak{x} \mapsto \nu_\mathfrak{x} \) defines a complemented frame homomorphism \( \alpha_\mathfrak{x} : X \to N(X) \).

(3) For any complemented frame homomorphism \( \varphi : X \to Y \), there is a unique frame homomorphism \( \tilde{\varphi} : N(X) \to Y \) making the following diagram commute.

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow{\alpha_\mathfrak{x}} & & \downarrow{\exists! \tilde{\varphi}} \\
N(X) & & \\
\end{array}
\]

Proof. Statements (1) and (2) are [20] II.2.5 Proposition and [20] II.2.6 Lemma respectively. Statement (3) follows from the argument in [20] II.2.9 Corollary; see the unpublished notes [33] Theorem 5.4 for a complete proof. The result also follows from [33] IV.6.3.1,II.5.3. \( \square \)
Let $X$ be a topological space. There is a strong connection between the assembly and the Skula topology on $X$. Recall from Definition 2.6 that $f_X: \text{skula}(X) \to X$ is the continuous function induced by the identity.

**Corollary 7.3.** There is a unique frame homomorphism $\sigma_X$ making the following diagram commute.

$$
\begin{array}{ccc}
\mathbb{F}(X) & \xrightarrow{f_X} & \mathbb{F}(\text{skula}(X)) \\
\downarrow{\alpha_X} & & \downarrow{\sigma_X} \\
\mathbb{N}(\mathbb{F}(X)) & \xrightarrow{\sigma_X} & \mathbb{F}(\text{skula}(X))
\end{array}
$$

**Proof.** For every open set $V \subseteq X$, the elements $V$ and $V^c$ of the frame $\mathbb{F}(\text{skula}(X))$ are complements. Thus the result follows from Theorem 7.2 (3). \qed

For a continuous map $g: X \to Y$, the frame map $\mathbb{F}(g)$ is complemented if and only if for every open set $V \subseteq Y$ the preimage $g^{-1}(V)$ is clopen in $X$. Arguing as in Corollary 6.4, the function $f_X$ is universal amongst such “complemented” continuous functions whose domain is $X$. In this sense, the assembly of a frame is the pointless analogy of the Skula topology. However, the universal property in Theorem 7.2 makes the assembly a much more versatile object, as we will see in the next section.

Before we apply our results to tensor triangulated categories, we discuss when the assembly and $\mathbb{F}(\text{skula}(X))$ coincide.

**Definition 7.4.**

(1) For a space $X$ and a subspace $S \subseteq X$, a point $p \in S$ is **weakly isolated** if there exists an open set $V \subseteq X$ such that $p \in V \cap S \subseteq \bar{p}$.

(2) Let $X$ be a frame.
   (i) A prime $p \in X$ is **minimal** over $x$ if it is minimal amongst primes containing $x$. Let $\text{min}(x)$ denote the the minimal primes of $x$.
   (ii) A prime $p \in \text{min}(x)$ is **essential** over $x$ if
   $$
x = \bigwedge_{q \in \text{min}(x)} q \quad \text{and} \quad x \neq \bigwedge_{q \in \text{min}(x)} q_{q \neq p}
$$

**Theorem 7.5.** The following is equivalent for a topological space $X$.

(1) The map $\sigma_X$ is an isomorphism of frames.
(2) Every nonempty closed set has a weakly isolated point.
(3) Every closed set is the closure of its weakly isolated points.
(4) Every open set $V \subseteq X$ has an essential prime in the frame $\mathbb{F}(X)$.
(5) Every element in the frame $\mathbb{F}(X)$ is the meet of its essential primes.
(6) For every surjective frame homomorphism $\mathbb{F}(X) \to \mathbb{Y}$, the frame $\mathbb{Y}$ has enough points.

**Definition 7.6.** Any space satisfying the equivalent conditions above is called **weakly scattered**.

**Remark 7.7.**
(1) When $X$ is $T_0$, it is easy to check that (3) is equivalent to the following:
every nonempty closed set $U \subseteq X$ is the closure of some discrete subspace $S \subseteq X$.

(2) When $X$ is sober, it is easy to check that (5) equivalent to the following:
every surjective frame homomorphism $F(X) \to Y$ is induced by a frame isomorphism $F(S) \cong Y$ for some subspace $S \subseteq Y$. See [33, VI.2.2.1]

Proof of Theorem 7.5. In [31, Theorem 3.4, Corollary 3.7], it is shown that (1), (2), (4), (5), and (6) are equivalent. In [36, Theorem 4.4], it is shown that (1) and (3) are equivalent.

7.2. Support and the assembly. We now arrive to the point of our pointless machinery. Let $\mathcal{C}$ be a rigidly compactly generated tensor triangulated category. Recall the frame homomorphism $\gamma_\mathcal{C}: T(\mathcal{C}^c) \to \mathbb{D}(\mathcal{C})$ from Theorem 4.8. By Lemma 6.6, $\gamma_\mathcal{C}$ is complemented. Therefore, Theorem 7.2 (3) implies that there is a unique frame homomorphism $\tilde{\gamma}_\mathcal{C}$ making the following diagram commute.

By Theorem 6.3, $\mathcal{C}$ is supportive if and only if there exists a frame homomorphism $\eta$ making the following diagram commute.

By Theorem 7.5, $\sigma_{T(\mathcal{C}^c)}$ is an isomorphism if and only if the Hochster dual of $\text{Spc} \mathcal{T}^c$ is weakly scattered, motivating the following definition.

Definition 7.8. A spectral space $X$ is Hochster weakly scattered if its Hochster dual $X^\dagger$ is weakly scattered. By Example 2.5, $X$ is Hochster weakly scattered if and only if for every Thomason subset $U \subseteq X$, there exists a point $p \notin U$ and a Thomason subset $V \subseteq X$ such that $p \in V \cap U^c \subseteq \downarrow p$.

We have now proven the following result.

Theorem 7.9. A rigidly compactly generated tensor triangulated category $\mathcal{C}$ is supportive if $\text{Spc} \mathcal{T}^c$ is Hochster weakly scattered.

We now try to understand which spectral spaces are Hochster weakly scattered. First, we consider the dual question: when is a spectral space weakly scattered?
Definition 7.10. Let $R$ be a commutative ring and $I \subseteq R$ and ideal. A prime $p \in \text{Spec } R$ is an essential divisor of $I$ if $p$ is an essential prime of $\sqrt{I}$ in the frame of radical ideals. This means that $p \in \text{min } I$ but

$$\sqrt{I} \neq \bigcap_{p \in \text{min}(I), p \neq q} q$$

Lemma 7.11. The following are equivalent for a commutative ring $R$.

1. $\text{Spec } R$ is weakly scattered.
2. Every proper ideal has an essential prime divisor.
3. For every proper radical ideal $I \subseteq R$, there is a prime $p$ such that

$$\langle I : (I : p) \rangle = p.$$

4. Every radical ideal can be written as an irredundant intersection of primes.

In particular, a spectral space $X$ is Hochster weakly scattered if and only if $X^\dagger \cong \text{Spec } R$ for some commutative ring $R$ satisfying these hypotheses.

Proof. By [22, Lemma 1], any prime ideal satisfying the conclusion of (3) is an essential prime divisor of $I$. The rest of the result is [31, Theorem 4.1]. □

The following examples demonstrate Hochster weakly spectral spaces are tricky.

Even for a relatively nice non-Noetherian ring $R$, $\text{Spec } R$ need not be Hochster weakly scattered.

Example 7.12. Every Noetherian spectral space $X$ is weakly scattered. Indeed, for such a space, every closed set $V \subseteq X$ can be written as a union

$$V = V_1 \cup \cdots \cup V_n$$

of irreducible closed sets $V_i$. The generic point of each $V_i$ is weakly isolated in $V$.

Consider the Hochster dual $Y = X^\dagger$. The space $Y$ is Hochster weakly scattered, since $Y^\dagger = X$. Spaces of this form are rather bizarre. For instance

- every specialisation closed subset in $Y$ is closed
- the set of quasi-compact open sets of $Y$ satisfies DCC
- every quasi-compact open sets of $Y$ is a finite union irreducible open sets.

Example 7.13. Let $R = k[x_1, x_2, \cdots]$ with $k$ algebraically closed. Then $\text{Spec } R$ is not Hochster weakly scattered.

Proof. We claim that $(\text{Spec } R)^\dagger$ itself has no weakly isolated points. Let $p \in \text{Spec } R$ and $V \subseteq \text{Spec } R$ a Thomason subset with $p \in V$. We claim that $V$ is not contained in $\bar{p} = \downarrow p$, i.e. we must produce a prime $q \not\subseteq p$ which is in $V$. We may assume that $V = V(f_1, \ldots, f_s)$ with $f_1, \ldots, f_s \in R$. Each $f_i$ is a polynomial in a finite set of variables, and so there is an $n$ such that each $f_i$ is in $S = k[x_1, \ldots, x_n]$. Let $m \subseteq S$ be a maximal ideal containing the $f_i$. By the Nullstellensatz, after a suitable coordinate change, we may write $m = (x_1, \ldots, x_n)$. Now either $x_{n+1}$ or $x_{n+1} + 1$ is not in the ideal $p$. After again changing coordinates, we may assume $x_{n+1} \not\in p$. The ideal $q = (x_1, \ldots, x_n, x_{n+1}) \subseteq R$ is our desired prime. □
7.3. Hochster scattered spaces. In order to relate our theory to the work of Balmer, Favi, and Stevenson, we discuss a special class of weakly scattered spaces.

Definition 7.14. A space is scattered if every closed subset has an isolated point which is a point which is open in the subspace topology. A space is Hochster scattered if it is spectral and the Hochster dual is scattered.

Scattered space first arose in connection with the following famous construction.

Construction 7.15 (Cantor-Bendixon). Let $X$ be a topological space. A point in a subspace $p \in Y \subseteq X$ is isolated in $Y$ if it is open in the subspace topology. Equivalently, there is an open set $V \subseteq X$ such that $\{p\} = Y \cap V$. Note that isolated points are weakly isolated. Let $I(Y)$ denote the isolated points of $Y$. Note that this is an open subspace of $Y$.

Inductively, for every ordinal $\alpha$, we will define an open subspace $X_{\leq \alpha}$ and a closed subspace $X_{> \alpha}$. Set $X_{\leq 0} = \emptyset$ and $X_{> 0} = X$. Assuming these sets are defined for $\alpha$, set $X_{\leq \alpha + 1} = X_{\leq \alpha} \cup I(X_{> \alpha})$ and let $X_{> \alpha + 1}$ be its complement. These spaces are open and closed respectively. When $\alpha$ is a limit ordinal, set $X_{\leq \alpha} = \bigcup_{\beta < \alpha} X_{\leq \beta}$ and let $X_{> \alpha + 1}$ be its complement. These spaces are open and closed respectively.

The Cantor-Bendixon rank of $X$ is then the smallest ordinal $\alpha$ such that $X = X_{\leq \alpha}$. If no such ordinal exists, we say that $X$ has no Cantor-Bendixon rank.

Recall the definition of a visible point from Subsection 4.2.

Lemma 7.16. The following are equivalent for a $T_0$ space $X$.

1. $X$ is scattered
2. $X$ is weakly scattered and $T_1$
3. $X$ has Cantor-Bendixon rank
4. The assembly $\mathbb{N}(F(X))$ is a Boolean algebra

If a space is Hochster scattered, then all of its points are visible.

Proof. Statements (1) and (2) are equivalent by [33, Proposition VI.8.1.1]. The equivalence of (1) and (3) is an elementary exercise in topology. Statements (1) and (4) are equivalent by [35, Theorem 9]. By Lemma 4.4, if the Hochster dual of a spectral space is $T_1$, then all the points are visible, proving the last statement.

The following examples of Hochster scattered spaces were inspired by Stevenson’s work in [42]. Recall that the constructible topology of a spectral space $X$ is generated by the Hochster dual topology and the Zariski topology (the given topology on $X$).

Lemma 7.17. A spectral space $X$ is Hochster scattered if one of the following hold.

1. $X$ is Noetherian.
2. $X$ carries the constructible topology and has Cantor-Bendixon rank.

Proof. The closed points of any Noetherian spectral space are isolated in the Hochster dual. Since any closed subset of a Noetherian topological space is also a Noetherian spectral space, this proves (1). This is essentially the argument in [42, Lemma 4.3].
Assume (2). Then any Hochster closed set $U \subseteq X$ is Zariski closed. By Lemma 7.10 $X$ is scattered, so there exists a point $p$ such that $\{p\} = V \cap U$ for some Zariski open set $V$. We may assume that $V$ is quasi-compact open. We claim that $V^c$ is a quasi-compact clopen set, and thus $V$ is Hochster open. The closed set $V^c$ is open in the constructible topology and thus also in the Zariski topology. Thus $V^c$ is clopen. But clopen sets are automatically quasi-compact, completing the proof. □

Therefore if $T$ is a rigidly compactly generated tensor triangulated category, $T$ is supportive if $\text{Spc} T^c$ is one of the above spaces. Stevenson proved this in [42] assuming that $T$ has a monoidal model. A close examination of Stevenson’s paper implies a much stronger result.

**Theorem 7.18.** Suppose $T$ is a rigidly compactly generated tensor triangulated category and has monoidal model. Suppose further that $\text{Spc} T^c$ is Hochster scattered. Then $T$ satisfies the local-to-global principle, i.e. for every $\mathcal{T} \in T$

$$T \in \text{loc}^\otimes (\Gamma_p T \mid p \in \text{supp} T).$$

Proof. Let $X$ be the collection of all Hochster scattered spectral spaces. For every $X \in \mathcal{X}$, let $\dim_X X \to \text{Ord}$ denote the Cantor-Bendixon level computed in the Hochster dual. These functions are well defined by Lemma 7.10. The collection $D = \{\dim_X \mid X \in \mathcal{X}\}$ is what Stevenson calls a class of spectral dimension functions compatible with $\mathcal{X}$; see [42] Definition 3.5. In Theorem 3.7 and Lemma 3.13 of [42], Stevenson shows that $\mathcal{T}$ satisfies the local-to-global principle if $\text{Spc} T^c \in \mathcal{X}$ and $D$ is closed under spectral subspaces. However, an examination of the proof of Theorem 3.7 reveals that we only need to show that $\mathcal{X}$ is closed under complements of Thomason subsets.

So we now check that $\mathcal{X}$ has this property. Let $X \in \mathcal{X}$ and take a Thomason subset $V \subseteq X$. Its complement, $V^c$, is a spectral subspace and its Hochster dual is a subspace of $X^\dagger$. Since subspaces of scattered spaces are scattered, $(V^c)^\dagger$ is scattered, and so $V^c \in \mathcal{X}$. □

**Example 7.19.** Let $R$ be an absolutely flat ring that is not semi-Artinian. In [39, Theorem 4.7], Stevenson shows that local-to-global principle fails in $D(R)$, and so $\text{Spec} R$ cannot be Hochster scattered by Theorem 7.18. Furthermore, $\text{Spec} R$ is not Hochster weakly scattered: the Krull dimension of $R$ is 0, and so every prime is visible. However, $D(R)$ is supportive either by Stevenson’s arguments in loc.cit. or by Theorem 5.5. Therefore $\mathcal{T}$ supportive does not imply $\text{Spc} T^c$ is Hochster weakly scattered.

We summarise the following implications for a spectral space $X$

Noetherian $\implies$ Hochster scattered $\implies$ all points are visible.

Moreover, these implications are strict.

**Example 7.20.** Let $X = \mathbb{N} \cup \infty$. Let $V_n = [n, \infty]$ be the nontrivial open sets of $X$. The space $X$ is spectral but not Noetherian. The non-trivial Hochster open sets are of the form $[1, n]$ with $n \in \mathbb{N}$. In particular, $1$ is an open point in $X$, and $n + 1$ is isolated in the Hochster closed set $[n + 1, \infty]$. Thus $X$ is Hochster scattered.

The following example was communicated to the author by Bill Fleissner.
Example 7.21. Let $C$ denote the Cantor set. This space is spectral, and so let $X = C^\dagger$. Now $X^\dagger = C$, and so the Hochster dual of $X$ is $T_C$. Thus every point is visible by Lemma 4.4. However, $X$ is not Hochster scattered.

8. Questions

8.1. Supportiveness. Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category. This article raises the following question.

Question 1. Is $\mathcal{T}$ always supportive?

By our work in Section 6, supportiveness is a restriction on the idempotent Bousfield lattice. Suppose for instance, that $\text{Spc} \mathcal{T}$ is not Hochster weakly scattered and $\mathcal{T}$ is supportive. Then $\mathcal{D}(\mathcal{T})$ cannot be isomorphic to the assembly $\mathcal{N}(\mathcal{T}(T^c))$ by Theorem 6.3 and so $\mathcal{D}(\mathcal{T})$ cannot be any old frame. This may not seem very strong, but the Bousfield lattice is notoriously ill-behaved, see for example [30], [10], [39, Section 4], and [41]. Thus a negative answer to Question 1 is likely.

On the other hand, Theorem 1.5 describes three disparate cases where supportiveness hold, suggesting that supportiveness might be common.

Example 8.1. As we learned in Example 7.21, there exists a commutative ring $R$ such that $(\text{Spec} R)^\dagger$ is the Cantor set. Thus $\text{Spec} R$ is not Hochster weakly scattered. However, since the Cantor set has Krull dimension 0, so does $\text{Spec} R$. Thus $\mathcal{D}(R)$ is supportive by Theorem 5.5. This example generalises to any dimension 0 that is not Hochster weakly scattered.

This example shows that for commutative rings, $\mathcal{D}(R)$ is supportive not for topological reasons but for algebraic reasons.

Question 2. Is $\mathcal{D}(R)$ supportive for all commutative rings $R$?

8.2. Smashing subcategories. Assume as usual that $\mathcal{T}$ is rigidly compactly generated tensor triangulated category, but now assume that $\mathcal{T}$ has a monoidal model. Let $\mathcal{S}$ be the lattice of smashing subcategories. Recall that for every $\mathcal{S} \in \mathcal{S}$, there are coproduct preserving local cohomology and localisation functors $\Gamma_{\mathcal{S}}$ and $L_{\mathcal{S}}$ and an idempotent triangle

$$\Gamma_{\mathcal{S}}1 \to 1 \to L_{\mathcal{S}}1 \to .$$

See [2]. In [3], the authors prove that $\mathcal{S}$ is a frame, and moreover, by [2, Theorem 3.5] the assignment $\mathcal{S} \mapsto \mathcal{A}(\Gamma_{\mathcal{S}}1)$ defines a frame homomorphism $\varphi: \mathcal{S} \to \mathcal{D}(\mathcal{T})$. This homomorphism is complemented since $\mathcal{A}(\Gamma_{\mathcal{S}}1)$ and $\mathcal{A}(L_{\mathcal{S}}1)$ are complements in $\mathcal{D}(\mathcal{T})$.

Therefore we can apply the results of Theorem 7.2 concerning the assembly and the analysis of Section 7.2 extends to this general case. In particular we have the following commutative diagram of frame homomorphisms.

Now suppose for a moment that $\mathcal{S}$ is a spatial frame. Then the open sets of $\text{Spec} \mathcal{S}$ are in bijection with $\mathcal{S}$. For every $\mathcal{S} \in \mathcal{S}$, let $\mathcal{D}(\mathcal{S})$ be the associated open set.
Definition 8.2. Let \( T \in \mathcal{T} \). A point \( p \in \text{Spc} \mathbb{S} \) is in \( \text{supp}_{\text{smash}} T \) if for every \( S, S' \in \mathbb{S} \) with \( p \in D(S) \cap D(S')^c \), we have 
\[ \Gamma_S L_{S'} T \neq 0. \]
We say that \( T \) is smashing supportive if \( \text{supp}_{\text{smash}} T = \emptyset \) if and only if \( T = 0 \).

The main results of this paper, Theorems 4.2, 4.7, 4.8, 6.3, and 7.9 all generalise to the smashing context, motivating the following.

Question 3. Is the smashing frame \( \mathbb{S} \) spatial? When is \( T \) smashing supportive?

8.3. Subframes of the Bousfield lattice. In Remark 6.7 we observed that \( T \) is supportive if the image of \( \gamma_T : T(T^c) \to D(T) \) lies in a spatial frame containing \( A(LV1) \) for all \( V \in T(T^c) \). When can we apply this remark? Recall from Example 6.5 that not all Boolean algebras are spatial.

Question 4. Suppose \( B \) is a Boolean algebra and \( F \subseteq B \) is a spatial frame such that the inclusion map is a frame homomorphism. Let \( \overline{F} \subseteq B \) be the smallest subset of \( B \) closed under arbitrary joins, finite meets, and complements. Is the complete Boolean algebra \( \overline{F} \) always atomic?

Suppose Question 4 has an affirmative answer. Then \( T \) is supportive whenever \( D(T) \) is Boolean: let us explain. Since \( \gamma_T \) is injective, \( \text{Im}(\gamma_T) \) is a subframe. Thus, an affirmative answer to the question implies that \( \text{Im}(\gamma_T) \) is spatial satisfies the hypotheses of Remark 6.7. Hence \( T \) is supportive.

Occasionally, \( D(T) \) is indeed Boolean. For instance every Boolean algebra is \( D(T) \) for some \( T \) satisfying our usual assumptions by [41].

Lemma 8.3. If \( T \) has no nilpotent elements, i.e. \( T \otimes T = 0 \) implies \( T = 0 \), then every Bousfield class is idempotent and \( D(T) \) is Boolean.

Proof. First, if \( A(T) \neq A(T \otimes T) \), then there exists an \( X \in T \) such that \( T \otimes X \neq 0 \) but \( T \otimes T \otimes X = 0 \). It follows that \( T \otimes X \) is nilpotent. We now prove the second statement.

For a class \( X \subseteq T \), set 
\[ X^\perp = \{ T \in T \mid T \otimes X = 0 \ \forall X \in X \}. \]
Standard formalism implies that 
\[ X \subseteq X^\perp \perp \perp \quad \text{and} \quad X \subseteq Y \implies Y^\perp \subseteq X^\perp \]
allowing us to conclude
\[ X^\perp = X^\perp \perp \perp. \]
By definition, for a single object \( T \in T \),
\[ T^\perp = A(T). \]
Let \( A(T)^* \) be the pseudo-complement of an element \( A(T) \in D(T) \). We claim that \( A(T)^{**} = A(T) \). Since \( A(X) \in D(T) \) for every \( X \in T \), we compute 
\[ A(T)^* = \bigvee_{A(S) \in D(T)} A(S) = \bigvee_{S \in T} A(S) = \bigcap_{S \otimes T = 0} S^\perp = T^\perp = A(T)^{**}. \]
We now have 
\[ A(T)^{**} = A(T)^{**} = T^\perp \perp \perp = T^\perp = A(T) \]
where the second equality is the second to last equality of the previous calculation. \( \square \)
8.4. Localising tensor ideals. In Theorem 4.2 and Theorem 4.7, we saw that \( \text{supp} \) gives the following maps:

\[
\{ \text{Localising tensor ideals of } \mathcal{T} \} \xrightarrow{\text{supp}} \{ \text{Closed subsets of } \text{Lspc } \mathcal{T} \}
\]

\[
\{ \text{Closed subsets of } \text{Lspc } \mathcal{T} \} \xleftarrow{\text{supp}^{-1}} \{ \text{Localising tensor ideals of } \mathcal{T} \}
\]

**Question 5.** When are the localising tensor ideals in bijection with the closed subsets of \( \text{Lspc } \mathcal{T} \)?

The appearance of closed subsets is surprising. In Theorem 3.2, we saw that the radical tensor ideals of \( \mathcal{T}^c \) are in bijection with the open sets, and thus form a frame. However, if the closed subsets of \( \text{Lspc } \mathcal{T} \) classify the localising tensor ideals, then the localising tensor ideals form a co-frame, a lattice whose opposite lattice is a frame. This motivates the following question.

**Question 6.** Is the lattice of localising tensor ideals a co-frame? Equivalently, is the lattice of localising tensor ideals a frame when ordered by reverse inclusion?

This question has a positive answer when every localising tensor ideal is an idempotent Bousfield class, in which case \( D(\mathcal{T}) \) is the lattice in question.

It is not known whether or not the collection of localising tensor ideals is a set. But Question 6 is independent of such foundational concerns, since frames and coframes need not be sets.

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**References**

1. Paul Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. 588 (2005), 149–168.
2. Paul Balmer and Giordano Favi, *Generalized tensor idempotents and the telescope conjecture*, Proc. Lond. Math. Soc. (3) 102 (2011), no. 6, 1161–1185.
3. Paul Balmer, Henning Krause, and Greg Stevenson, *The frame of smashing tensor-ideals*, preprint (2017), arXiv:1701.05937.
4. Silvana Bazzoni and Jan Šťovíček, *Smashing localizations of rings of weak global dimension at most one*, Adv. Math. 305 (2017), 351–401.
5. David J. Benson, Srikanth B. Iyengar, and Henning Krause, *Local cohomology and support for triangulated categories*, Ann. Sci. Éc. Norm. Supér. (4) 41 (2008), no. 4, 573–619.
6. Stratifying modular representations of finite groups, Ann. of Math. (2) 174 (2011), no. 3, 1643–1684.
7. Nicolas Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1989, Translated from the French, Reprint of the 1972 edition.
8. A. K. Bousfield, *The Boolean algebra of spectra*, Comment. Math. Helv. 54 (1979), no. 3, 368–377.
9. Aslak Bakke Buan, Henning Krause, and Øyvind Solberg, *Support varieties: an ideal approach*, Homology Homotopy Appl. 9 (2007), no. 1, 45–74.
10. W. G. Dwyer and J. H. Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology Homotopy Appl. 10 (2008), no. 1, 413–436.
11. Hans-Bjørn Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra 15 (1979), no. 2, 149–172.
12. Hans-Bjørn Foxby and Srikanth Iyengar, *Depth and amplitude for unbounded complexes*, Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 119–137.

13. J. P. C. Greenlees, *Tate cohomology in axiomatic stable homotopy theory*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 149–176.

14. Robin Hartshorne, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin-New York, 1967.

15. M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142 (1969), 43–60.

16. Michael J. Hopkins, *Global methods in homotopy theory*, Homotopy theory (Durham, 1985), London Math. Soc. Lecture Note Ser., vol. 117, Cambridge Univ. Press, Cambridge, 1987, pp. 73–96.

17. Mark Hovey and John H. Palmieri, *The structure of the Bousfield lattice*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 175–196.

18. Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. 128 (1997), no. 610, x+114.

19. Srikanth B. Iyengar and Henning Krause, *The Bousfield lattice of a triangulated category and stratification*, Math. Z. 273 (2013), no. 3-4, 1215–1241.

20. Peter T. Johnstone, *Stone spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, 1982.

21. Bernhard Keller, *A remark on the generalized smashing conjecture*, Manuscripta Math. 84 (1994), no. 2, 193–198.

22. D. Kirby, *Closure operations on ideals and submodules*, J. London Math. Soc. 44 (1969), 283–291.

23. Joachim Kock, *Spectra, supports, and hochster duality*, HOCA T talk and letter to P. Balmer, G. Favi, and H. Krause (November 2007), http://mat.uab.cat/~kock/cat/spec.pdf.

24. Joachim Kock and Wolfgang Pitsch, *Hochster duality in derived categories and point-free reconstruction of schemes*, Trans. Amer. Math. Soc. 369 (2017), no. 1, 223–261.

25. Henning Krause, *Localization theory for triangulated categories*, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 161–235.

26. Jean Merker, *Idéaux faiblement associés*, Bull. Sci. Math. (2) 93 (1969), 15–21.

27. Haynes Miller, *Finite localizations*, Bol. Soc. Mat. Mexicana (2) 37 (1992), no. 1-2, 383–389, Papers in honor of José Adem (Spanish).

28. Amnon Neeman, *The chromatic tower for $D(R)$*, Topology 31 (1992), no. 3, 519–532, With an appendix by Marcel Bökstedt.

29. Jean Simmons, *Oddball Bousfield classes*, Topology 39 (2000), no. 5, 931–935.

30. Jean Simmons, *The assembly of a frame*, unpublished notes (2006), www.cs.man.ac.uk/~hsimmons/FRAMES/frames.html.

31. Greg Stevenson, *Support theory via actions of tensor triangulated categories*, J. Reine Angew. Math. 681 (2013), 219–254.
39. _____, Derived categories of absolutely flat rings, Homology Homotopy Appl. 16 (2014), no. 2, 45–64.
40. _____, A tour of support theory for triangulated categories through tensor triangular geometry, preprint (2016), arXiv:1601.03595.
41. _____, Complete boolean algebras are bousfield lattices, preprint (2017), arXiv:1707.06007.
42. _____, The local-to-global principle for triangulated categories via dimension functions, J. Algebra 473 (2017), 406–429.
43. R. W. Thomason, The classification of triangulated subcategories, Compositio Math. 105 (1997), no. 1, 1–27.