GRÖBNER CELLS OF PUNCTUAL HILBERT SCHEMES IN DIMENSION TWO

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Abstract. We begin with a comprehensive discussion of the punctual Hilbert scheme of the regular two-dimensional local ring in terms of the Gröbner cells. These schemes are the most degenerate fibers of the Grothendieck-Deligne norm map (the Hilbert-Chow morphism), playing an important role in the study of Hilbert schemes of smooth surfaces. They are generally singular, but their Gröbner cells are affine spaces; they admit an explicit parametrization due to Conca and Valla. We use this to obtain the Gröbner decomposition of compactified Jacobians of plane curve singularities, which is non-trivial even for the generalized Jacobians (principal ideals only). One of the application is the topological invariance of certain variants of compactified Jacobians and the corresponding motivic superpolynomials for analytic deformations of quasi-homogenous plane curve singularities and some similar families.

Key words: Hilbert schemes, affine plane, Grothendieck-Deligne map, Gröbner cells, zeta functions, plane curve singularities.

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1. Introduction

We begin with a discussion of the Hilbert scheme $H^{(n)}$ of the local ring $\mathbb{C}[[x, y]]$ in terms of Gröbner cells $C_\lambda$, providing all details. Here $n$ is the codimension (the length) of the ideals in $\mathbb{C}[[x, y]]$ and $\lambda$ is a partition of $n$. The cellular decomposition of $H^{(n)}$ was obtained [ES] via [B-B] from the analysis of the action of the maximal torus in $SL(3, \mathbb{C})$ in the tangent spaces of $\text{Hilb}^{(n)}(P^2_\mathbb{C})$ at the corresponding fixed points, which are monomial ideals. Our starting point is an entirely local definition of the Gröbner cells of $H^{(n)}$ and their explicit parametrization following [CV], Theorem 3.3 ($i = 2$). The Gröbner decomposition is an important tool in the study of $H^{(n)}$; it can be potentially used for any isolated surface singularities, not only quasi-homogeneous.

We note that the definition of the Gröbner cells generally requires some valuation of $\mathbb{C}[x, y]$ or $\mathbb{C}[[x, y]]$. The dependence on the ratio of the valuations of $x$ and $y$ can be interpreted as some wall-crossing.

This decomposition induces that of compactified Jacobians of plane curve singularities under the natural embeddings, which is non-trivial even for the generalized Jacobians, which are for principal fractional ideals instead of all of them. Combined with that in terms of the so-called Piontkowski strata, it can be presumably used to obtain the super-duality for the motivic superpolynomials from [ChP1]. This duality was conjectured in [Ch] to coincide with the functional equation of the Galkin-Stöhr zeta functions for any plane curve singularities; this is related to motivic theory of $H^{(n)}$. See [Sto] and also [MY, MS, ORS].

The topological invariance of these superpolynomials is a significant part of the conjectures in [ChP1, Ch]. We prove a stronger fact for the deformations of quasi-homogeneous singularities and some similar families. Namely, $\tilde{\text{Jac}}^*$ from Lemma 3.1, closely related to the compactified Jacobians, are topological invariants for such families.
1.1. Punctual Hilbert schemes. The Hilbert scheme $H^{(n)}$ of $\mathbb{C}[[x, y]]$ is defined as a scheme of all its ideals of codimension $n$. Equivalently, it can be introduced as the fiber of the Grothendieck-Deligne norm map (the Hilbert-Chow morphism) $\pi_n : \text{Hilb}^{(n)}(\mathbb{C}^2) \to S^n(\mathbb{C}^2)$ over the point $nO$ in the symmetric power $S^n(\mathbb{C}^2)$ of the affine plane $\mathbb{C}^2$; see [Del]. Here $O = \{x = 0, y = 0\}$, $\text{Hilb}^{(n)}(\mathbb{C}^2)$ is the Hilbert scheme formed by ideals $I \subset \mathbb{C}[x, y]$ of codimension $n$. This fiber is called the punctual Hilbert scheme; some authors call them “local punctual”.

Let $(x, y)_c = (e^a x, e^b y)$ be the action of the torus $\mathbb{C}^* \ni \eta$ in $\mathbb{C}^2$, depending on $u, v \in \mathbb{R}_+$. Assuming that $0 < u < v$ and $\epsilon > 0$, the limit $I^0 = \lim_{\epsilon \to 0} I_\eta$ is well-defined and is a monomial ideal in the same $\text{Hilb}^{(n)}(\mathbb{C}^2)$. Monomial ideals are those linearly generated by $x^a y^b$. Combinatorially, we can obtain $I^0$ simply by taking from any $f \in I$ its top monomial $f^0$ under the following lexicographic ordering: $1 < y < y^2 < \cdots < x < xy < xy^2 < \cdots$.

The monomial ideals are fully determined by the partitions $\lambda = \{m_1 \geq m_2 \geq \cdots \geq m_\ell > 0\}$, where $\sum_{i=1}^\ell m_i = n$. Let $I_\lambda$ be linearly generated by $x^a y^b$ such that $\{a, b\} \in \mathbb{Z}_+^2 \setminus \lambda$, where we represent $\lambda$ as the set $\lambda' \overset{\text{def}}{=} \{(i, j) \in \mathbb{Z}_+^2 \mid 0 \leq i < \ell, 0 \leq j < m_{i+1}\}$. Equivalently, $x^a y^b \in I_\lambda$ if and only if $a \geq i^\circ$ and $b \geq j^\circ$ for at least one corner $\{i^\circ, j^\circ\}$ of $\mathbb{Z}_+^2 \setminus \lambda'$. Any monomial ideal is $I_\lambda$ for some $\lambda$.

For a given partition $\lambda \vdash n$, the Gröbner cell $\text{Gr}_\lambda$ of $\text{Hilb}^{(n)}(\mathbb{C}^2)$ is defined as follows: $\text{Gr}_\lambda \overset{\text{def}}{=} \{I \in \text{Hilb}^{(n)}(\mathbb{C}^2) \mid I^0 = I_\lambda\}$. They form a cellular decomposition of $\text{Hilb}^{(n)}(\mathbb{C}^2)$ in the sense of Fulton.

Using the embedding $H^{(n)} \subset \text{Hilb}^{(n)}(\mathbb{C}^2)$, let $\text{Gr}_\lambda^0 \overset{\text{def}}{=} \text{Gr}_\lambda \cap H^{(n)}$. There is an entirely local definition of the Gröbner cells for $H^{(n)}$. Namely, we switch the order of $x$ and $y$ (now $y > x$), and take the lowest monomials instead of the top ones used in the construction of $I^0$. The notation is $I_0$ instead of $I^0$ throughout the paper. The corresponding strata of $H^{(n)}$ will be denoted by $C_\lambda$. This definition is compatible with the passage to the completion $\mathbb{C}[[x, y]]$ of $\mathbb{C}[x, y]$ at $(x = 0 = y)$, i.e. it is indeed local.

There is an embeddings $C_\lambda \hookrightarrow \text{Gr}_\lambda$, which results in the identification $C_\lambda \simeq \text{Gr}_\lambda^0$. We make this very explicit. Note that the cell decomposition of $H^{(n)}$ is obtained in [ES] without any reference to $\text{Hilb}^{(n)}(\mathbb{C}^2)$. It is some part of the decomposition of $\text{Hilb}^{(n)}(P_2^2)$, and there is some Poincaré duality between $H^{(n)}$ and $\text{Hilb}^{(n)}(\mathbb{C}^2)$ (see below).
1.2. Some basic facts. In contrast to $\text{Hilb}^{(n)}(\mathbb{C}^2)$, the scheme $H^{(n)} = \pi_n^{-1}(nO)$ is projective and generally not smooth. It is irreducible due to J. Briancon [Bri] and of dimension $n - 1$; see also [Ia]. Furthermore, it is a complete intersection and reduced, so it is Cohen-Macaulay. This is due to M. Haiman; see Proposition 2.10 in [Ha1].

Here $\mathbb{C}^2$ can be replaced by any smooth quasi-projective surface $X$; the corresponding $\text{Hilb}^{(n)}(X)$ formed by subschemes in $X$ supported in one (any) point is isomorphic to $H^{(n)}$, i.e. to that for $\mathbb{C}^2$. The fibers $\pi_n^{-1}(nP)$ are the most degenerate ones; knowing them is sufficient to calculate any fibers of $\pi_n$ for any $X$.

Namely, let $\mu = \{n_1 \geq n_2 \geq \cdots \geq n_r > 0\}$ be a partition of $n$, $P_1, \ldots, P_r$ be pairwise distinct points. We set $D_\mu = \sum_{i=1}^r n_i P_i$. Then $\pi_n^{-1}(D_\mu)$ is naturally isomorphic to the product of $\pi_n^{-1}(n_iO)$ over $1 \leq i \leq r$. We arrive at the fibration of $\text{Hilb}^{(n)}(X)$ with respect to the standard stratification of $S^n(X)$ with the strata $\mathcal{D}_\mu \overset{\text{def}}{=} \{D_\mu\}$. The latter are unramified covers of $\prod S^r(X)$ minus the diagonals, i.e. they are smooth. The fibration of $\text{Hilb}^{(n)}(X)$ corresponding to $\pi_r$ is locally trivial upon the restriction to any $\mathcal{D}_\mu$.

From $\dim H^{(n)} = n - 1$, we obtain that $\dim \pi_n^{-1}(D_\mu) = \sum_{i=1}^r (n_i - 1) = n - r$, i.e. it is $|\mu|$ minus the length of $\mu$. This formula can be possibly connected with the formula $\dim C_\lambda = n - \ell(\lambda)$ (below) via some deformation procedure. The fibers $H^{(n)}$ are the key in the theory of $\text{Hilb}^{(n)}(\mathbb{C}^2)$ and its various applications; see e.g. [Ha2]. The explicit parametrization of $C_\lambda$, can be helpful here.

Importantly, we do not need the action of $\mathbb{C}^*$ in the definition of $C_\lambda$, and it is entirely local. It can be extended to more general isolated surface singularity, not only quasi-homogeneous. An explicit parametrization of the corresponding cells can be involved, but the direct calculations follow the same lines as for $\mathbb{C}[[x, y]]$ and are doable for relatively simple surface singularities.

Using the smoothness of $\text{Hilb}^{(n)}(\mathbb{C}^2)$ (J. Fogarty) and considering its tangent space at $I_\lambda$, the general result of A. Bialynicki-Birula [B-B] gives that $Gr_\lambda$ is an affine space. Here the definition of $I^0$ via the action of $\mathbb{C}^*$ is used; this action can be calculated explicitly in the tangent space at $I_\lambda$. For instance, it gives that the dimension of $Gr_\lambda$ is $n + m(\lambda)$, where $m(\lambda) = m_1$. See [ES], Theorem 1.1, (iii), [CV], Theorem 3.3 ($i = 2$), and also [Nak1, Nak2, MO] for different aspects.
and generalizations. Note that the embedding \( C_\lambda \hookrightarrow Gr_\lambda \) is from the space of dimension \( n - \ell(\lambda) \) to the one of dimension \( n + m(\lambda) \).

Generally, \( [B-B] \) cannot be used for \( H(n) \) since it is not smooth. However, G. Ellingsrud and S. Strømme obtain a cellular decomposition of \( H(n) \) as part of that of \( Hilb(n)(P^2) \), which is smooth. The cells are affine spaces, which follows from \( [B-B] \). They calculate Betti numbers of \( H(n) \) in their Theorem 1.1, \((iv)\), which gives the number of cells and their dimensions. We note that taking here the \( n\)-row (in our notations) as \( \lambda \) readily gives the irreducibility of \( H(n) \); see Corollary 1.2 in [ES]. Namely, the closure of the corresponding \( C_{\square} \) of dimension \( n - 1 \), the big cell, is the whole \( H(n) \).

### 1.3. Hilbert-type zetas

The classical Hasse-Weil zeta has the following presentation:
\[
Z(X,t) = \sum_{n=0}^{\infty} t^n |S^n(X)(\mathbb{F}_q)|.
\]
This formula holds for any varieties \( X \) over \( \mathbb{F}_q \) and their symmetric powers \( S^n(X) \). Here \( X \) can be singular; then 0-cycles over \( \mathbb{F}_q \) must be counted instead of the points. See e.g. [Mus].

Importantly, the Betti numbers \( b_i(X) \), the ranks of Borel-Moore \( i \)th homology, are related to the classical zeta-function for smooth \( X \). Algebraically, they occur as the degrees of the polynomials in \( t \) for the contributions of the corresponding cohomology in Weil’s formula.

The knowledge of the cell decomposition of \( H(n) \) generally can be used to calculate the Hilbert-type zeta-functions of any quasi-projective smooth surfaces \( X \), which are defined as follows. We replace \( S^n(X) \) by \( Hilb(n) \), setting \( Z(X,t) = \sum_{n=0}^{\infty} t^n |Hilb(n)(X)(\mathbb{F}_q)| \). In its motivic counterpart, the cardinality \( |Hilb(n)(X)(\mathbb{F}_q)| \) is replaced by the class of \( Hilb(n)(X) \) in the Grothendieck ring of varieties over the basic field; the count of \( \mathbb{F}_q \)-points is then considered as the counting motivic measure.

Let us outline the general way of calculating such \( Z \)-functions based on Weil conjectures. The Hasse-Weil zeta \( Z(X,t) \) gives all \( |S^n(X)(\mathbb{F}_q)| \), which is sufficient to calculate all \( |D_\mu(\mathbb{F}_q)| \) using the inclusion-exclusion principle. Then we can apply the formulas for \( |H(n)(\mathbb{F}_q)| \): each cell \( A^n \) of \( X \) results in \( q^m \). This gives \( Z(X,t) \). We will provide an example of such a calculation below. This is related to the way L. Göttche obtained his well-known formula from [Got1, Got2] in terms of Betti numbers of Hilbert schemes of smooth quasi-projective surfaces \( X \).

A significant part of the paper is devoted to the connections with the plane curve singularities. For any element \( P(x,y) \) of \( I \in H(n) \), this
ideal is the inverse image of some ideal in $\mathcal{R} = \mathbb{C}[[x, y]]/(P(x, y))$ of codimension $n$. Any such $I$ has a canonical "first Gröbner generator" $P_1(x, y)$, the one with the minimal possible $x^a(a > 0)$ and other monomials in it "from" the boxes of $\lambda$. We fix $P(x, y)$ and identify the subset $\{I \in C_\lambda \mid P_1(x, y) = P(x, y)\}$ with some subset of the compactified Jacobian $\overline{\text{Jac}_\mathcal{R}}$ of $\mathcal{R}$. Cf. Theorem 3.6, (ii). Using the parametrization of $C_\lambda$, this subset is given by explicit equations.

We call such subsets Gröbner strata of $\overline{\text{Jac}_\mathcal{R}}$, which stratification can be related to [MY, MS]. The strata can be singular as schemes, but they are affine spaces topologically in many cases; see (3.27) and around. This decomposition can be quite non-trivial even in the case of the generalized Jacobian, $\text{Jac}_\mathcal{R} \subset \overline{\text{Jac}_\mathcal{R}}$, the group of invertible $\mathcal{R}$-submodules in the normalization ring of $\mathcal{R}$.

Example. Let us show how to calculate $|Hilb^3(\mathbb{F}_q)|$ for $X = \mathbb{A}^2$ using this approach. We need the formulas $|\text{Gr}_\lambda(\mathbb{F}_q)| = q^{n+m(\lambda)}, |C_\lambda(\mathbb{F}_q)| = q^n - \ell(\lambda)$, and the number of $\mathbb{F}_q$-points of the fiber in $Hilb^{(m)}$ over $n_1 P_1 + \cdots + n_r P_r \in S^n(X)$ over $\mathbb{F}_q$ for $\mu = \{n_1 \geq n_2 \geq \cdots \geq n_r > 0\} \vdash n$ and pairwise distinct points $P_1, \ldots, P_r$. The latter equals $\prod_{i=1}^r |H^{(m_i)}(\mathbb{F}_q)|$.

We already know that $|Hilb^3(\mathbb{F}_q)| = \sum_{\lambda \vdash 3} |\text{Gr}_\lambda(\mathbb{F}_q)| = q^6 + q^5 + q^4$. Let us obtain this quantity using the approach via $S^n(X)$, which is generally applicable to any smooth surfaces $X$ over $\mathbb{F}_q$. Using the formula $|H^{(m)}(\mathbb{F}_q)| = \sum_{\lambda \vdash m} |C_\lambda(\mathbb{F}_q)|$, we obtain: $|H^{(1)}(\mathbb{F}_q)| = 1$, $|H^{(2)}(\mathbb{F}_q)| = 1 + q$, $|H^{(3)}(\mathbb{F}_q)| = 1 + q + q^2$.

Finally, showing the source of the terms in $\langle \cdots \rangle$:

\[
|Hilb^3(\mathbb{F}_q)| = (1 + q + q^2) q^2 \langle \mu = \Box \Box \rangle + (1 + q) q^2 (q^2 - 1) \langle \mu = \Box \rangle + q^2 (q^2 - 1) \langle \mu = \Box^2 \rangle
\]

\[
+ \frac{q^2 (q^2 - 1)(q^2 - 2)}{6} \langle \mu = \Box, P_i \in \mathbb{F}_q \rangle + q^2 (q^4 - q^2) \langle \mu = \Box \rangle
\]

\[
\langle P_1 \in \mathbb{F}_q, P_{2,3} \in \mathbb{F}_q \setminus \mathbb{F}_q \rangle + \frac{(q^6 - q^2)}{3} \langle P_i \in \mathbb{F}_q^3 \setminus \mathbb{F}_q \rangle = q^6 + q^5 + q^4.
\]

1.4. Toward functional equation. Inspired by the theory of plane curve singularities, it was expected in [Ch] that $L$-functions of reasonably good isolated surface singularities over $\mathbb{F}_q$ depend on $q$ uniformly, which property is called “strong polynomial count”, and satisfy the functional equation. These $L$–functions are infinite products in contrast to those for plane curve singularities, so the functional equation will be with “infinite” scaling factors. We will provide examples below.
Here only Hilbert-type zeta-functions make sense to consider; symmetric powers of a singularity (just a point) are meaningless.

An expected connection with the \( q \)-deformations of the classical \( L \)-functions from Number Theory is touched upon in [Ch]; this is quite a motivation, but very preliminary by now.

The definition of the local zeta-function of any singularity ring \( \mathcal{R} \) is straightforward: \( Z_{\mathcal{R}}(t) \overset{\text{def}}{=} \sum_{n=0}^{\infty} t^n|H_{\mathcal{R}}^{(n)}(\mathbb{F}_q)| \), where \( H_{\mathcal{R}}^{(n)} \) is a scheme of ideals in \( \mathcal{R} \) of codimension \( n \). and \( \mathcal{R} \) must be at least Gorenstein. The latter can be insufficient: it is expected that the surface isolated singularities corresponding to Seifert 3-folds (as their links) constitute a natural class. The uniform dependence on \( q \) is an important test, what is called ”strong polynomial \( q \)-growth” of \( |H_{\mathcal{R}}^{(n)}(\mathbb{F}_q)| \).

The rings \( \mathcal{R} \) are initially over \( \mathbb{C} \), so we need to consider them over proper extensions of \( \mathbb{Z} \), and then switch to \( \mathbb{F}_q \), where \( q = p^m \) assuming that primes \( p \) are of ”good reduction”; almost all \( p \) are such. This passage to \( \mathbb{F}_q \) is sufficiently well understood for curve singularities. Then we switch to the \( L \)-functions; the functional equation is expected only for them.

**Plane curve singularities.** For the rings \( \mathcal{R} \) of Gorenstein curve singularities, one has: \( L_{\mathcal{R}} \overset{\text{def}}{=} (1 - t)Z_{\mathcal{R}}(t) \), which is the Galkin-Stöhr \( L \)-function. It is a polynomial in terms of \( t \); see [Sto]. The multiplication by \( (1 - t) \) is a counterpart of the multiplication by \( (1 - qt)(1 - qt) \) for smooth projective curves.

Conjecture 4.5 from [Ch] states that \( L_{\mathcal{R}} = H_{\mathcal{R}}(qt, t) \) for plane curve singularities \( \mathcal{R} \), where the motivic superpolynomial \( H_{\mathcal{R}}(q, t) \) is as follows. We consider \( \mathcal{R} \) as a subring of \( \mathbb{F}_q[[z]] \), where \( z \) is the uniformization parameter; so \( \mathcal{R} \) is an arbitrary subring in \( \mathcal{O} \) with two generators and such that its fields of rationals coincide for \( \mathcal{R} \) and \( \mathbb{F}_q[[z]] \).

Then \( H_{\mathcal{R}}(q, t) \overset{\text{def}}{=} \sum_M t^{dim_{\mathbb{Q}}(\mathbb{F}_q[[z]])/M} \), where the summation is over \( \mathcal{R} \)-submodules \( M \subset \mathbb{F}_q[[z]] \) such that \( M\mathbb{F}_q[[z]] = \mathbb{F}_q[[z]] \).

Generally, the coincidence \( L_{\mathcal{R}} = H_{\mathcal{R}}(qt, t) \) does not hold for non-planar Gorenstein curve singularities, so the usage of Punctual Hilbert schemes \( H^{(n)} \) and similar objects seems inevitable here.

The substitution \( q \mapsto qt \) requires an assumption that \( H_{\mathcal{R}}(q, t) \) is a polynomial in terms of \( q \), conjectured for any plane curve singularities. Then a conjectural relation with the DAHA superpolynomials from [ChP1] gives that \( H_{\mathcal{R}} \) are topological invariants, i.e. depend only on
the corresponding valuation semigroup $\Gamma_R$; see Section 3.1. This will follow from our considerations for some families of $R$.

**Surface singularities.** For surface singularities, the division of $Z_R(t)$ by $Z_O(t)$ for $O = \mathbb{C}[[x, y]]$ can be expected, but this can be more involved than this. Anyway, the objective is to have the functional equation for $L_R(t)$ with respect to the substitution $t \mapsto 1/(q^2 t)$.

To explain which functional equation can be expected here, let us reproduce the Göttsche formula:

$$\sum_{n \geq 0} \sum_{i \geq 0} (-1)^n b_i(Hilb^{(n)}(X)) q^{i/2} t^n = \prod_{k \geq 1} \prod_{j=0}^{4} (1 - q^{k-1+j/2} t^k)^{-1} (-1)^{j+1} b_j(X)$$

for the Betti numbers of $X$ and $Hilb^{(n)}(X)$. Using the Poincaré duality for smooth projective $X$: $b_j(X) = b_{4-j}(X)$ and the right-hand side satisfy a formal functional equation upon the substitution $t \mapsto 1/(q^2 t)$ from the classical functional equation for surfaces. This of course holds up to some infinite (!) monomial in terms of $q^{1/2}$ and $t$, necessary to get rid of the denominators in the binomials. Such an “infinite rescaling” will not be addressed in this paper.

Here $q$ is treated as a free parameter. Let us assume that all $Hilb^{(n)}(X)$ have cellular decompositions; see e.g. Proposition 1.5 from [ES]. Then the left-hand side above coincides with $Z(X, t)$, based on counting the $\mathbb{F}_q$-points. Without this assumption, we have generally two different approaches, the modular one and its geometric counterpart based on the Borel-Moore homology (homology with closed support) or other kinds of (co)homology. The same functional equation is expected for either one, however the tools for its verification will be very different. Both approaches can be potentially used for $L$-functions of (reasonably good) isolated surface singularities.

**Back to $\mathbb{C}^2$.** For $X = \mathbb{A}^2$ considered over $\mathbb{F}_q$, one has:

$$Z(X, t) = \sum_{n=0}^{\infty} t^n |Hilb^{(n)}(X)(\mathbb{F}_q)| = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} q^{n+m(\lambda)} = \prod_{i=1}^{\infty} (1 - q^{i+1} t^i)^{-1}.$$  

See [ES] and Remark 4.7 in [KR]. The formula for the local Hilbert-type zeta of $R = \mathbb{F}_q [[x, y]]$ is

$$Z_R(t) = \sum_{n=0}^{\infty} t^n |H^{(n)}(\mathbb{F}_q)| = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} = \prod_{i=1}^{\infty} (1 - q^{i-1} t^i)^{-1}.$$
The similarity of the latter with that for $\mathbb{A}^2$ is not accidental. Following [ES], let us use the decomposition $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. Accordingly, $Z(\mathbb{P}^2, t)$ is the product of the corresponding zetas for the 0-dimensional subschemes in $\mathbb{P}^2$ supported in $\mathbb{A}^2, \mathbb{A}^1, \mathbb{A}^0$. The products above are for $\mathbb{A}^2$ and $\mathbb{A}^0$. They can be readily seen in the Göttsche formula for $\mathbb{P}^2$: the products corresponding to $b_0$ and $b_4$. Thus they are dual to each other with respect to the Poincaré duality, which is here the formal substitution $t \mapsto 1/(q^2 t)$ followed by the rescaling, the multiplication by an infinite $q, t$–monomial. The product for $\mathbb{A}^1$ is in terms of $q^i t^i$; see [ES]. It corresponds to $b_2(X)$; all odd Betti numbers are 0.

1.5. Some perspectives. Let us mention here a series of papers on the generating function for Euler numbers of Hilbert schemes of points on a simple singularities $\mathbb{C}^2/\Gamma$ for finite subgroups $\Gamma \subset SL(2, \mathbb{C})$. The conjecture by A. Gyenge, A. Némethi, and B. Szendrői, was that it is the character of the corresponding Kac-Moody basic representation, where its torus variable is evaluated at a proper root of unity of order related to the Coxeter number. It was checked in [DS, Tod] in type $A$, and for $D, E$ in [GNS1, GNS2, Nak3]. See Theorem 1 from the last reference for an exact statement. This is directly related to our generating function, but we need the refined version of this theorem in terms of the Betti numbers. Also, we focus on the local theory of isolated surface singularities, i.e. on punctual Hilbert schemes.

Another equally important direction is the passage to instantons, i.e. to torsion-free sheaves of any ranks instead of ideals in the definition of Hilbert schemes. Here we have a solid theory in any ranks for plane curve singularities from [ChP2], though with quite a few conjectures. Even the cases of affine plane and its local version for $\mathcal{R} = \mathbb{C}[[x, y]]$ are quite interesting, directly related to the Nekrasov instanton sums. More generally, such theory must be associated with arbitrary Young diagrams; the sheaves of rank $r$ correspond to the $r$-column. The functional equation then includes the transposition of the Young diagrams.

The generalization to arbitrary Young diagrams is done by now for a different DAHA-based approach, which conjecturally gives the same superpolynomials as those from the motivic theory in [ChP1, ChP2] (for columns). Also, let us at least mention the connections of the compactified Jacobians to affine Springer fibers; see e.g. [Yun]. For this, we make $a = 0, t = 1$ in the superpolynomials, so the functional equation, which requires $t$, generally can not be seen. The parameter
a, which we do not introduce in this paper, is associated with complete \textit{flags} of the modules and the ideals, related to the \textit{nested Hilbert schemes}. Importantly, \( a \mapsto a \) in the functional equation.

The totally local theory has many advantages; torsion-free bundles over singular curves and surfaces are a difficult topic in classical algebraic geometry. In the case of local isolated singularities, the corresponding definitions are actually no different from those in [ChP2] in dimension one. Presumably the local approach captures many features of the theory of instanton sums and related directions. It is important that the \( \mathcal{Z} \)-function of \( \mathbb{C}[[x, y]] \) in terms of \( \{H^{(n)}\} \), is closely related to that of \( \mathbb{C}^2 \); it is certainly no simpler in spite of its local nature.

A related direction is the \textit{combinatorial wall-crossing}. One can generalize the Gröbner decomposition, coupling it with the valuations of \( \mathbb{C}[[x, y]] \). For instance, let \( \text{val}_{r, s}(x^a y^b) = ra + sb \) for relatively prime numbers \( r, s \in \mathbb{Z}_+ \). \( \text{val}_{r, s}(p) \) be the minimum of valuations of monomials in a polynomial \( p(x, y) \). Then we do the following. First, we group and order the monomials in any \( f \in \mathbb{C}[[x, y]] \) with respect to \( \text{val}_{r, s} \). Second, we find the minimal monomial in the groups with coinciding valuations for the Gröbner ordering \( \{x^\infty < y\} \).

For \( r = 0, s = 1 \), this gives the standard definition; the case \( r = 1, s = 0 \) corresponds to the switch of \( x \) and \( y \). Given a valuation, the corresponding Gröbner strata provide some basic elements in Borel-Moore homology of \( H^{(n)} \). The construction depends on \( r/s \), so we have \textit{connection matrices}. We will not develop this in the present paper, but some relations to the singularities \( \{x^s = y^r\} \) will be discussed.

Generally, allowing the valuations to be \( \infty \), we naturally arrive at plane curve singularities; namely, \( \mathcal{R} = \mathbb{C}[[x, y]]/\{f \in \mathbb{C}[[x, y]] \mid \text{val}(f) = \infty\} \). The semigroup of all valuation gives its topological type in the unibranch case. In a sense, \textit{tropical geometry} is when 0 is allowed.

All these and related directions obviously require as constructive theory of \( H^{(n)} \) as possible, which is the subject of the present paper.

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2. Gröbner cells

We will begin with some connections between Gröbner cells for \( \mathbb{C}[x, y] \) and entirely local ones for \( \mathbb{C}[[x, y]] \). Then we will adjust the construction from [CV] for the former ring to the latter.

2.1. Basic definitions. As in the Introduction, \( \text{Hilb}^{(n)}(\mathbb{C}^2) \) is defined as the scheme of ideals \( I \subset \mathbb{C}[x, y] \) such that \( \dim \mathbb{C}[x, y]/I = n \). We have two standard lexicographic orderings:

\[
\begin{align*}
\{ y^\infty < x \} & : 1 < y < y^2 < \cdots < x < xy < xy^2 < \cdots , \\
\{ x^\infty < y \} & : 1 < x < x^2 < \cdots < y < yx < yx^2 < \cdots .
\end{align*}
\]

For any \( f \in \mathbb{C}[x, y] \) let \( f^0 \) be its maximal monomial \( x^a y^b \) with respect to \( \{ y^\infty < x \} \). We will mainly need below \( f^0 \) defined as the minimal monomial of \( f \) with respect to \( \{ x^\infty < y \} \) from (2.2). Obviously,

\[
(x^a y^b + y^{b+1}f)^0 = x^a y^b = (x^a y^b + y^{b+1}f)_0 \text{ if } \deg_x(f) < a
\]

for any \( f \in \mathbb{C}[x, y], a, b \geq 0 \).

For the ideals \( I \subset \mathbb{C}[x, y] \), we set:

\[
I^0 \overset{\text{def}}{=} \{ f^0 \mid f \in I \} \text{ and } I_0 \overset{\text{def}}{=} \{ f_0 \mid f \in I \},
\]

which are monomial ideals by construction, which means, as in the Introduction, that they are linearly generated by \( x^a y^b \).

An arbitrary monomial ideal coincides with one of \( I_\lambda \) for the partition \( \lambda \vdash n \) defined as follows. Let \( \lambda = \{ m_1 \geq m_2 \geq \cdots \geq m_\ell > 0 \} \), where \( \sum_{i=1}^\ell m_i = n \). We set \( \ell(\lambda) = \ell \), which is called the length of \( \lambda \), and \( m(\lambda) \overset{\text{def}}{=} m_\ell \). As in the Introduction:

\[
\lambda' \overset{\text{def}}{=} \{ (i, j) \in \mathbb{Z}^2_+ \mid 0 \leq i < \ell, 0 \leq j < m_{i+1} \},
\]

and \( I_\lambda = \oplus_{a,b} \mathbb{C} x^a y^b \), where \( \{a, b\} \in \mathbb{Z}^2_+ \setminus \lambda' \).

Equivalently, \( x^a y^b \in I_\lambda \) if and only if \( a \geq i^o \) and \( b \geq j^o \) for at least one corner \( \{i^o, j^o\} \) of \( \mathbb{Z}^2_+ \setminus \lambda' \).

Any \( I \in \text{Hilb}^{(n)}(\mathbb{C}^2) \) must contain pure polynomials \( f(x) \) and \( g(y) \) in terms of \( x \) and \( y \) of (nonzero) degree no greater than \( n \). The ideals \( I \) containing the monomials \( x^n \) and \( y^n \) form the (local) punctual Hilbert scheme \( H^{(n)} \). More systematically:

**Definition 2.1.** The punctual Hilbert scheme is a subscheme formed by \( I \in \text{Hilb}^{(n)}(\mathbb{C}^2) \), satisfying one of the following equivalent conditions:
(a) $I$ contains $x^N, y^N$ for sufficiently large $N$, which implies that $N = n$ can be taken,

(b) $H^{(n)} = \{ I \in \text{Hilb}^{(n)}(\mathbb{C}^2) \mid m^n \subset I \}$ for the maximal ideal $m = x\mathbb{C}[x,y] + y\mathbb{C}[x,y]$ of $(0,0)$ in $\mathbb{C}[x,y]$. \hfill \Box

The actual conductor $C(I)$ of such $I$, defined as the greatest monomial ideal it contains, can be of course larger than $m^n$. So only a lower bound for $C(I)$ is in this definition. For the sake of completeness, let us check the equivalence of (a) and (b).

First of all, $I \in H^{(n)}$ contains a sufficiently large power of $m$ and can be considered naturally as a module over $\mathbb{C}[[x,y]]$, which we will do constantly. Note that the definition of $I_0$ is compatible with the passage to $\mathbb{C}[[x,y]]$, since we take the smallest monomials here. The ring $\mathbb{C}[x,y]/I$ is local artinian and the image of $m_I$ of $m$ in this ring is its maximal ideal. If $m^{k+1}_I = m^k_I$ for some $k > 0$, then $m^k_I = \{0\}$ by the Nakayama Lemma. However, such a repetition must occur no later than at $k = n$, since the length of the chain of consecutive $m^k_I$ cannot be greater than $\dim \mathbb{C}[x,y]/I = n$.

Similarly, let us provide the following lemma and its justification: we want to clarify in full here the relation between ”global” and ”local”.

**Lemma 2.2.** The ideals $I^0$ for any $I \in \text{Hilb}^{(n)}(\mathbb{C}^2)$ and $I_0$ for any $I \in H^{(n)}$ belong to $H^{(n)}$, i.e. $\dim \mathbb{C}[x,y]/I^0 = n$ and, correspondingly, $\dim \mathbb{C}[x,y]/I_0 = n$.

**Proof.** Let $I_\lambda$ be $I^0$ or $I_0$. First of all, if $I^0 = I_\lambda$, then any nonzero linear combination of $x^iy^j$ for $\{i,j\} \in \lambda'$ is nonzero modulo $I$; otherwise $I^0$ would correspond to a smaller partition. Here the (linear) ordering of monomials can be arbitrary, so the same argument works if $I_0 = I_\lambda$ for any $I \in \text{Hilb}^{(n)}$, not only those from $H^{(n)}$. Thus $\dim \mathbb{C}[x,y]/I^0 \leq n$, and the same holds for $I_0$. Let us check that such $\{x^iy^j\}$ linearly generate $\mathbb{C}[x,y]$ modulo $I$.

For the ordering $\{y^\infty < x\}$, any monomial can be represented modulo $\{x^iy^j\}$ above and $I$ as a sum of strictly smaller monomials. Then we continue by induction.

In the case of $\{x^\infty < y\}$ and $I_0$, it will be a sum of strictly bigger monomials, so we need an additional argument. Namely, we use that the condition $I \in H^{(n)}$ implies that all sufficiently big monomials belong to $I$. \hfill \Box
The following proposition is actually a reformulation of the lemma.

**Proposition 2.3.** (i) Let \( I^0 = I_\lambda \) for an arbitrary \( I \in \text{Hilb}^{(n)}(\mathbb{C}^2) \) or \( I_0 = I_\lambda \) for an arbitrary \( I \in H^{(n)} \). Then \( |\lambda| = n \), and the images of \( x^iy^j \) for \( \{i,j\} \in \lambda' \) form a basis of \( \mathbb{C}[x,y]/I \).

(ii) For \( I_\lambda \) as in (i), let \( \{i^o,j^o\} \) be the corners of \( \mathbb{Z}_+^2 \setminus \lambda' \). Then \( I \) is generated as an ideal by the elements \( f_{ij} \) such that \( f_{ij}^o - x^iy^j \) is a linear combination of \( x^iy^j \) for \( \{i,j\} \in \lambda' \).

(iii) Moreover, \( f_{ij} \) for \( i^o,j^o \) from (ii) is unique such; it can contain \( x^iy^j \) only if \( i < i^o \) or \( i = i^o \& j < j^o \). Furthermore, here \( i < i^o \) and \( j > j^o \) must hold in the case of \( I_0 = I_\lambda \) for \( I \in H^{(n)} \).

The definition of \( I_0 \) is compatible with the completion of the ideals at \((x = 0, y = 0)\). To see this, let \( \tilde{I} \) and \( \tilde{I}_0 \) be the completions with respect to \( m \) of an ideal \( I \subset \mathbb{C}[x,y] \) and \( I_0 \) naturally embedded into \( \mathbb{C}[[x,y]] \). Here \( \tilde{I}_0 \) is simply \( I_0 \) where infinite sums of its monomials are allowed. Then \((\tilde{I})_0\), which is defined by picking the smallest monomials for the same ordering (2.2), coincides with \( \tilde{I}_0 \).

Since \( \dim \mathbb{C}[x,y]/I = \dim \mathbb{C}[[x,y]]/\tilde{I} \) if and only if \( I \in H^{(n)} \), we obtain the following reformulation of Definition 2.1.

**Lemma 2.4.** Among all ideals \( I \) in \( \mathbb{C}[[x,y]] \), the ideals \( I \in H^{(n)} \) are characterized by the condition \( I_0 \in H^{(n)} \), which is the relation \( \dim \mathbb{C}[x,y]/I_0 = n \). \( \square \)

**Proposition 2.5.** (i) Let us assume that an ideal \( I \in \mathbb{C}[x,y] \) of finite codimension has generators \( \{f_i\} \) (as an ideal) such that \( (f_i)_0 = (f_i) \) and \( I_0 \) is the linear span of \( \{(x^ay^bf_i)_0 \mid a,b \geq 0, \forall i\} \). Then \( I^0 = I_0 \).

(ii) When \( I \in H^{(n)} \) the generators \( \{f_{ij}^o\} \) from Proposition 2.3 satisfy the conditions for \( \{f_i\} \) above. Thus \( I^0 = I_0 \) for such \( I \). Vice versa, the conditions from (i) imply that \( I \in H^{(n)} \).

Proof of (i). Obviously \( (x^ay^bf_i)_0 = x^ay^b(f_i)_0 = (x^ay^bf_i) \) for any \( a,b \), so \( I_0 \subset I^0 \). The problem can only be with linear combinations \( g = \sum c_{ab}x^ay^bf_i \in I \), which potentially can have \( g^0 \) smaller than \( \max\{x^ay^bf_i \mid c_{ab} \neq 0\} \) with respect to \( y^\infty < x \). Let \( g = x^iy^j + \sum c_{ab}x^ay^b \) with \( g^0 = x^iy^j \) and nonzero \( c_{ab} \). Then either \( a < i \) for any \( j \), or \( a = i \& b < j \). One has \( g_0 = (g')_0 \) for \( g' = x^iy^j + \sum c'_{ab}x^ay^b \), where \( c'_{ab} = c_{ab} \) when \( b < j (\forall j) \) or \( b = j \& a < i \), and \( c'_{ab} = 0 \) otherwise; use the definition of \( g_0 \). Intersecting the inequalities for \( a,b \), we obtain that \( g' = \sum_{a \leq i, b \leq j} c'_{ab}x^ay^b \), where \( c'_{ij} = 1 \); see (2.3). Therefore
if \( g^0 = x^i y^j \) does not belong to \( I_0 \), then \( (g')_0 \not\in I_0 \), which is a contradiction. We use here that if \( \{i,j\} \in \lambda' \) for \( I_0 = I_\lambda \), then the whole rectangle from \( \{0,0\} \) to \( \{i,j\} \) belongs to \( \lambda' \).

**Proof of (ii).** Using (i) and Lemma 2.4, we obtain that \( I_0 = I^0 \) combined with \( \dim C[x,y]/I^0 = n \) gives that \( I_0 \in H^{(n)} \). Without using (i), the direct reasoning is as follows.

Let \( I_0 = I_\lambda \) and \( f = f_{0m} = y^m \) for \( m = m(\lambda) \) in the notations from Proposition 2.3. We assume that \( n > 0 \), so \( m > 0 \). The partition \( \lambda \) is then nonempty due to part (i). Indeed, it is empty only if at least one of \( f_i \) in any system of generators has a nonzero constant term. However this is impossible unless \( I = \mathbb{C}[x,y] \) due to the condition \( (f')_0 = (f')^0 \).

Now let us take the generator \( g \) such that \( (g)_0 = x^\ell \); it must exist. Here \( g = x^\ell + yp(x,y) \) and \( \deg_x p < \ell \) due to \( (g)_0 = g^0 \), but we will not need this inequality. Then \( y^{m-1}g = x^\ell y^{m-1} \mod (y^m) \) and \( x^\ell y^{m-1} \in I \). Next, \( x^\ell y^{m-2}g = x^{2\ell} y^{m-2} \mod (x^\ell y^{m-1}) \) gives \( x^{2\ell} y^{m-2} \in I \), and so on. Thus \( x^m \ell \in I \) and we can use Definition 2.1.

Finally, for partitions \( \lambda \vdash n \), Gröbner schemes are:

\begin{equation}
Gr_\lambda = \{ I \in \text{Hilb}^{(n)}(\mathbb{C}^2) \mid I^0 = I_\lambda \}, \quad Gr^0_\lambda = Gr_\lambda \cap H^{(n)},
\end{equation}

\begin{equation}
C_\lambda = \{ I \in H^{(n)} \mid I_0 = I_\lambda \} \cong \{ \tilde{I} \subset \mathbb{C}[[x,y]] \mid \tilde{I}_0 = (I_\lambda)^\gamma \}. 
\end{equation}

In (2.7), we identify ideals \( I \subset \mathbb{C}[x,y] \) with their completions \( \tilde{I} \subset \mathbb{C}[[x,y]] \) and \( H^n \) with a scheme of ideals \( \tilde{I} \subset \mathbb{C}[[x,y]] \) of codimension \( n \). Note that the relation \( \tilde{I}_0 = (I_\lambda)^\gamma \), where the latter is the completion of \( I_\lambda \), automatically results in \( \dim \mathbb{C}[[x,y]]/\tilde{I} = n \). This makes the definitions of \( H^{(n)} \) and \( C_\lambda \) entirely local, canonically equivalent to the ones in terms of \( \mathbb{C}[x,y] \).

Proposition 2.5, provides that \( I_0 = I^0 \) for \( I \in H^{(n)} \); moreover, \( \dim C[x,y]/I_0 = n \) for \( I \in \text{Hilb}^{(n)}(\mathbb{C}^2) \) results in \( I \in H^{(n)} \) due to Lemma 2.4. We obtain that \( Gr^0_\lambda = C_\lambda \) for any partition \( \lambda \). This is somewhat unexpected because of quite different definitions of \( (f)_0 \) and \( (f)^0 \). So it suffices to use only \( C_\lambda \), which we will do from now on.

2.2. **Two examples of C-cells.** Let us provide a direct calculation of \( C_\lambda \) in a typical example. Generally, the machinery of syzigies can be used here; see [ES, CV, KR]. We take \( \lambda = \{3,3,2,1\} \); i.e. it is of order \( |\lambda| = 9 \), of length \( \ell(\lambda) = 4 \) and with \( m(\lambda) = 3 \). The monomials associated with the corresponding boxes of \( \lambda' \) are shown in Figure 1. The monomials without framing are for the *corners* of \( \mathbb{Z}^2 \setminus \lambda' \).
Accordingly, the ideals $I \in C_{\lambda}$, which are $I \subset H^{(9)}$ such that $I_0 = I_{\lambda}$, are generated (as ideals in $\mathbb{C}[x, y]$) by the polynomials:

$$f_1 = x^4 + C_{21}^1 x^2 y + C_{11}^1 x y + C_{01}^1 y + C_{12}^1 x y^2 + C_{02}^1 y^2,$$

$$f_2 = x^3 y + C_{12}^1 x y^2 + C_{02}^1 y^2,$$

and

$$f_3 = x^2 y^2, f_4 = y^3,$$

where we set $f_1 = f_{03}, f_2 = f_{22}, f_3 = f_{31}, f_4 = f_{40}$

in the notation $f_i \circ j$ from Proposition 2.3. There are 7 $C$-parameters here, but $\text{dim} C_{\lambda} = |\lambda| - \ell(\lambda) = 9 - 4 = 5$ due to Theorem 2.7 below. So there must be 2 relations. Let us find them. One has:

$$y f_1 = x^4 y + C_{11}^1 x y^2 + C_{01}^1 y^2 \text{ mod } (f_3, f_4),$$

$$x f_1 = x^5 + C_{21}^1 x^2 y + C_{11}^1 x y + C_{01}^1 y + C_{12}^1 x y^2 + C_{02}^1 y^2,$$

$$+ C_{11}^1 x^2 y + C_{01}^1 x y + C_{02}^1 x y^2 \text{ mod } (f_3),$$

and $y f_1 - x f_2 = (C_{11}^1 - C_{02}^1) x y^2 + C_{01}^1 y^2 \text{ mod } (f_4)$.

In the last binomial, $y^2$ is the minimal monomial. Thus $C_{01}^1 = 0$, since $y^2$ belongs to (the boxes of) $\lambda'$. Then $C_{11}^1 - C_{02}^1$ must vanish too, since $x y^2$ belongs to $\lambda'$. Finally, the relations are: $C_{01}^1 = 0 = C_{11}^1 - C_{02}^1$.

The following example is the most involved for the partitions with $|\lambda| = 5$. Let $\lambda = \{4, 1\}$. From Figure 2, we obtain the following generators for any $I \in C_{\lambda}$:

$$f_1 = f_{20} = x^2 + C_{01}^1 y + C_{02}^1 y^2 + C_{03}^1 y^3,$$

$$f_2 = f_{11} = x y + C_{02}^1 y^2 + C_{03}^1 y^3, f_3 = f_{04} = y^4.$$
One has: $I \ni y^2 f_2 = xy^3 \mod (f_3)$; so $xy^3 \in I$. Using this, $xy f_2 = x^2 y^2 \mod (f_3, xy^3)$, and therefore $x^2 y^2 \in I$. Now:

$$y^2 f_1 = x^2 y^2 + C_{01}^1 y^3 + C_{02}^1 y^4 = C_{01}^1 y^3 \mod (x^2 y^2, f_3).$$

We conclude that $C_{01}^1 = 0$, since $y^3$ belongs to (the boxes of) $\lambda'$. Using now that $C_{01}^1 = 0$, we arrive at the $2 \times 2$-system for $xy^2, y^3$:

$$y f_2 = xy^2 + C_{02}^2 y^3 \mod (f_3),$$
$$x f_2 - y f_1 = C_{02}^2 x y^2 - C_{02}^1 y^3 \mod (f_3, xy^3).$$

It gives that $C_{02}^1 + (C_{02}^2)^2 = 0$; otherwise $y^3$ would belong to $I$. Summarizing, $C_\lambda$ is obtained from $\mathbb{C}^5$ by imposing $C_{01}^1 = 0, C_{02}^1 = -(C_{02}^1)^2$. So it is an affine space of dimension $3 = |\lambda| - \ell(\lambda)$.

It is of interest to calculate the conductor $C(I)$ of $I$. One has $x^2 f_1 = x^4 + C_{02}^1 x^2 y^2 \in I$ due to $C_{01}^1 = 0$. Since $x^2 y^2 \in I$, we obtain that $x^4 \in I$. Finally, the generators of $C(I)$ are $y^4, xy^3, x^2 y^2, x^3 y, x^4$, i.e. it is $m^4$ for generic $C$-parameters. When $C_{01}^1 = 0 = C_{02}^1$, it will also contain $xy$ for $C_{03}^2 = 0$, and $x^2$ for $C_{13}^2 = 0$.

The disadvantage of this direct approach is that it does not generally provide that $C_\lambda$ are affine spaces. However it can be used in any ranks and for arbitrary isolated surface singularities, at least those with local rings belonging to $\mathbb{C}[[x, y]]$, where Gröbner schemes can be readily defined. Actually, the number of variables can be here greater than 2.

**Another example.** The following example will be needed later; it is actually "simpler" than the one before. Let $\lambda = \{4, 2\}$. From Figure 3, we obtain the following generators of $I \in C_\lambda$:

\begin{equation}
(2.10) \quad f_1 = f_{20} = x^2 + C_{11}^1 x y + C_{01}^1 y + C_{02}^1 y^2 + C_{03}^1 y^3,
\end{equation}
\begin{equation}
\quad f_2 = f_{11} = xy^2 + C_{02}^2 y^3, \quad f_3 = f_{04} = y^4.
\end{equation}
Figure 3. $\lambda = \{4, 2\}$

One has: $I \ni yf_2 = xy^3 \mod (f_3)$; so $xy^3 \in I$. Using this, $xf_2 = x^2y^2 + C_{02}^1xy^3 \mod (f_3, xy^3)$, so $x^2y^2 \in I$. Now:

$$y^2f_1 = x^2y^2 + C_{01}^1y^3 + C_{11}^1xy^3 + C_{02}^1y^4 = C_{01}^1y^3 \mod (xy^3, x^2y^2, f_3).$$

We conclude that $C_{01}^1 = 0$, since $y^3$ belongs to (the boxes of) $\lambda'$, and that $C_\lambda$ is an affine space of dimension $4 = |\lambda| - \ell(\lambda)$.

2.3. The parametrization. It is generally not true that $\{f_1, yj\}$ from Proposition 2.3 constitute a minimal set of generators of $I$ as an ideal. The simplest example is $\begin{array}{c} \text{Figure 1} \end{array}$. One has: $f_1 = x^2 + cy, f_2 = xy, f_3 = y^2$, and $yf_1 - xf_2 = cy^2$. Even if they are such, the number of the corresponding coefficients of $x^i y^j$ in their decompositions (for all corners) is generally significantly greater than the dimension of the Gröbner cells $Gr_{\lambda}, C_{\lambda}$; they are affine spaces. Let us address this.

Following the parametrization of $Gr_{\lambda}$ from [CV], we will provide an explicit parametrization of $C_{\lambda}$. A priori, the definition of $C_{\lambda}$ is very different from that for $Gr_{\lambda}$: (2.2) is used instead of (2.1) and the leading term in $f_0$ is the minimal monomial, not the maximal one as in $f'$. Furthermore, $C_{\lambda}$ of dimension $n - \ell(\lambda)$ is embedded in $Gr_{\lambda}$ of dimension $n + m(\lambda)$; changing $-\ell(\lambda)$ to $+m(\lambda)$ is some combinatorial challenge too. Theorem 1.1 from [ES] provides the dimensions of cells in $Hilb^{(n)}(\mathbb{P}^2)$ and $H^{(n)}$, but not the embedding above. They use [B-B], which approach generally does not provide explicit embeddings of subschemes. Following Section 3.1 from [KR], let us reproduce the description of $Gr_{\lambda}$ from [CV].

Given $\lambda = \{m_1 \geq m_2 \geq \cdots \geq m_\ell > 0\}$ such that $|\lambda| = \sum_{i=1}^\ell m_i = n$, we set $d_1 = m_\ell, d_2 = m_{\ell-1} - m_\ell, \ldots, d_\ell = m_1 - m_2$. I.e. nonzero $d_i$ are the lengths of the horizontal segments in the corresponding Young diagram starting with the bottom. The $d$-set in Figure 1 is $\{1, 1, 1, 0\}$; it is $\{1, 3\}$ in Figure 2. The construction is in terms of the following...
polynomials in $y$:
\begin{align}
\{p_i(y), 1 \leq i \leq \ell \mid \deg p_i < d_i\}, \quad \text{and} \\
\{p_{i,j}(y), 1 \leq i \leq j \leq \ell \mid \deg p_{i,j} < d_i\}.
\end{align}

Their coefficients will be the free parameters of $Gr_{\lambda}$. The following matrix of size $(\ell + 1) \times \ell$ is the key:

$$
T_{\lambda} = \begin{pmatrix}
y^{d_1} + p_1 & 0 & 0 & \ldots & 0 & 0 \\
p_{1,1} - x & y^{d_2} + p_2 & 0 & \ldots & 0 & 0 \\
p_{1,2} & p_{2,2} - x & y^{d_3} + p_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_{1,\ell-2} & p_{2,\ell-2} & p_{3,\ell-2} & \ldots & y^{d_{\ell-1}} + p_{\ell-1} & 0 \\
p_{1,\ell-1} & p_{2,\ell-1} & p_{3,\ell-1} & \ldots & p_{\ell-1,\ell-1} - x & y^{d_{\ell}} + p_{\ell} \\
p_{1,\ell} & p_{2,\ell} & p_{3,\ell} & \ldots & p_{\ell-1,\ell} & p_{\ell,\ell} - x
\end{pmatrix}.
$$

\textbf{Theorem 2.6.} [CV]. Given $\lambda \vdash n$ and an arbitrary set of polynomials from (2.11), the ideal $I$ generated by the $\ell \times \ell$-minors of $T_{\lambda}$ belongs to $Gr_{\lambda}$. Any $I \in Gr_{\lambda}$ can be represented in this form for a unique set of $p$-polynomials. In particular, $Gr_{\lambda}$ is an affine space of dimension $|\lambda| + m(\lambda) = n + m_1$. \hfill \square

The reference is [CV], Theorem 3.3. The next theorem is the case $(i = 2)$ of this theorem. This reduction is of importance to us; we adjust it to what we will need and provide its complete justification (mostly following [CV]).

To prevent a possible confusion while comparing this theorem with that in Section 3.1 of [KR], let us calculate the dimension of $Gr_{\lambda}$. It is

$$(\ell + 1)d_1 + (\ell + 1)d_2 + \ldots + 2d_\ell = \sum_{i=1}^{\ell} (\ell - i + 1)d_i + \sum_{i=1}^{\ell} d_i = n + m_1.$$

It suffices to take here only the minors where lines $\{\ell + 1 - i^o\}$ are removed from $T_{\lambda}$ for the corners $\{i^o, j^o\}$ of $\mathbb{Z}_+^2 \setminus \lambda'$. Given $i^o$, the corresponding minor will be up to proportionality a unique element $f_{i^o j^o} \in I$ such that $f = x^{i^o}y^{j^o} + \sum_{i,j} C_{ij} x^iy^j$, where $\{i, j\} \in \lambda'$ and either $i < i^o$ or $i = i^o$ & $j < j^o$. See Proposition 2.3, (iii).

These elements form a set of generators of $I$. However, the coefficients $C_{ij}$ are not arbitrary at all. They must satisfy algebraic relations to ensure that $I^0 = I_{\lambda}$, which are "resolved" in the construction of the theorem. The counterpart of this theorem for $C_{\lambda}$ is as follows.
Theorem 2.7. [CV]. (a) Given a partition $\lambda \vdash n$, let $p_i = 0$ in (2.11) and also $p_{i,j}(y = 0) = 0$. Moreover, for any segment $[a, b]$ such that $d_{a-1} \neq 0$ and $\{d_a = 0, d_{a+1} = 0, \ldots, d_b = 0\}$, we additionally impose the relations $p_{a-1,j}(y = 0) = 0$ for $a \leq j \leq b$.

(i) Then the $\ell \times \ell$-minors of $T_\lambda$ generate an ideal $I$ from $C_\lambda$. For a corner $\{i^o, j^o\} \in \mathbb{Z}_+^2 \setminus \lambda'$, let $T_\lambda^{(\ell-i^o+1)}$ be the minor of $T_\lambda$ where the line $(\ell - i^o + 1)$ is omitted. Then $T_\lambda^{(\ell-i^o+1)}$ is $(-1)^{i^o}$ times $f_{i^o j^o} = x^{i^o} y^{j^o} + \sum_{i,j} C_{ij} x^i y^j$ from Proposition 2.3, (iii). The latter is unique in $I$ subject to $\{i, j\} \in \lambda'$, and the inequalities $i < i^o, j > j^o$.

(ii) Any ideal $I \in H^{(n)}$ can be obtained this way, and the corresponding set of polynomials $\{p_{i,j}\}$ subject to the conditions above is uniquely determined by $I$. In particular, $C_\lambda$ is an affine space of dimension $|\lambda| - \ell(\lambda) = n - \ell$, it is naturally embedded into $Gr_\lambda$, and its image is $Gr_0^0$ defined in (2.6).

Proof. Let us begin with the calculation of the dimension of the set of $p$-polynomials. It is, indeed:

$$\sum_{d_i \neq 0} ((l - i + 1)d_i - 1) - |\{d_i = 0\}| = n - \ell,$$

where $|\{d_i = 0\}|$ counts the additional conditions imposed when $d_i = 0$. Compare with $\dim Gr_\lambda = n + m_1$; note that $+m_1$ is "replaced" by $-\ell$. We will set $d_0 = 0$ and $p_0 = 0$ later on. See [CV], Corollary 3.1 $(i = 2)$.

Due to Propositions 2.3 and 2.5, the following property is necessary for the minors $T_\lambda^{(i)}$, which are determinants of $T_\lambda$ in Theorem 2.6 without line $i$ $(1 \leq i \leq \ell + 1)$ and under the assumption that $\ell - i + 1$ is $i^o$ from some corner $\{i^o, j^o\}$ of $\mathbb{Z}_+^2 \setminus \lambda'$.

The property which must hold is that

$$(2.12)\quad T_\lambda^{(i)} = \pm x^{\ell - i + 1} y^{d_0 + \ldots + d_{i-1}} \sum_{a,b} C_{ab} x^a y^b, \quad \text{where}$$

$$\{a, b\} \in \lambda' \quad \text{and} \quad a < \ell - i + 1, \ b > d_0 + \ldots + d_{i-1},$$

for some $C_{ab}$ if and only if the conditions $p_i = 0$ and the other ones from $(a)$ are imposed. I.e. that these conditions for $p_i$ are necessary and sufficient for the inequalities for $a, b$ in (2.12).

Recall that $\ell - i + 1$ is some $i^o$ only when $d_i \neq 0$ or when $i = \ell + 1$. Provided $(a)$, if $\ell - i' + 1$ is not assumed to be $i^o$, then $T_\lambda^{(\ell')}$ is $\pm x^c$ multiplied by $T_\lambda^{(i)}$ from (2.12) for a corner $i^o = \ell - i + 1$ such that
$d_i \neq 0$, $d_{i+1} = 0$, \ldots, $d_{i+c} = 0$. So $c$ is the distance from $i'$ to the greatest possible $i$ such that $d_i \neq 0$, $i \leq i'$.

Generally, $T^{(i)}_\lambda = \prod_{j=0}^{i-1} (y^{d_j} + p_j) \det T^{[i]}_\lambda$, where $1 \leq i \leq \ell + 1$, for

$$T^{[i]}_\lambda = \begin{pmatrix}
p_{i,i} - x & y^{d_{i+1}} + p_{i+1} & 0 & \cdots & 0 \\
p_{i,i+1} & p_{i+1,i+1} - x & y^{d_{i+2}} + p_{i+2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{i,\ell-2} & p_{i+1,\ell-2} & \cdots & y^{d_{\ell-1}} + p_{\ell-1} & 0 \\
p_{i,\ell-1} & p_{i+1,\ell-1} & \cdots & p_{\ell-1,\ell-1} - x & y^{d_{\ell}} + p_{\ell} \\
p_{i,\ell} & p_{i+1,\ell} & \cdots & p_{\ell-1,\ell} & p_{\ell,\ell} - x
\end{pmatrix}.$$

Since the structure of these matrices is uniform with respect to $i$, only $T^{[1]}_\lambda$ are sufficient to consider. Let us make this exact. We set $\delta_0 = 0$, $\delta_i = |\{1 \leq j < \ell - i + 1 \mid m_j = m_{\ell-i+1}\}|$ for $1 \leq i \leq \ell$. Let $\lambda[i]$ be the partition given by $\{m_1 - m_{\ell-i+1}, m_2 - m_{\ell-i+1}, \ldots, m_{\ell-i} - m_{\ell-i+1}\}$ where we omit the last $\delta_i$ zeros. Geometrically, we remove the first $i$ columns from the diagram describing $\lambda$. Then for $1 \leq i \leq \ell + 1$:

\begin{equation}
\det T^{[i]}_\lambda = (-x)^{\delta_{i-1}} \det T^{[1]}_{\lambda[i-1]},
\end{equation}

and

$$T^{(i)}_\lambda = (-x)^{\delta_{i-1}} \prod_{j=0}^{i-1} (y^{d_j} + p_j) \det T^{[1]}_{\lambda[i-1]},$$

where the indices of $p$-polynomials in $T^{[1]}_{\lambda[i-1]}$ must be as in $T^{[i]}_\lambda$ where the first $\delta_i$ columns and rows are deleted; here $\lambda[0] \overset{\text{def}}{=} \lambda$.

We can now proceed by induction with respect to $n = |\lambda|$, where the case $n = 1$ is obvious. This is helpful but not actually needed below. Let us check that the conditions from (o) are necessary for $I \in H^{(0)}$.

First of all, without any induction, $T^{(\ell+1)}_\lambda = \prod_{j=0}^{\ell} (y^{d_j} + p_j)$ is $y^{m_1}$ if and only if all $p_j$ are zero. Let us now assume that the entries of $T^{[1]}_\lambda$ with the shift of indices as in (2.13) satisfy the conditions from (o). Then $p_{1,1}$ must be divisible by $y$, since otherwise $T^{[1]}_\lambda$ will contain an $x$-monomial of degree smaller than $\ell$ because of the contribution of the diagonal. Here the usage of the induction is not necessary too.

The last check concerns the additional conditions addressing the columns where $d_i = 0$ for some $i$. Let $d_{i+1} = 0$ and $d_i \neq 0$, i.e.
\( \ell - i + 1 \) is \( i^\circ \) for some corner. Then \( T_\lambda^{(i)} \) must have the smallest monomial \( x^{\ell-i+1}y^{d_{i+1}+\ldots+d_{i-1}} \) due to (2.12). However, if \( p_{i,i+1} \) has a nonzero constant term, then it must contain \( x^{\ell-i+1}y^{d_{i+1}+\ldots+d_{i-1}} \), which is smaller with respect to the ordering \( \{ x^\infty < y \} \). This is impossible.

More generally, let \( d_i \neq 0, d_{i+1} = 0 = \ldots = d_{i+r}, d_{i+r+1} \neq 0 \). We set \( A_\lambda^{[i]} = T_\lambda^{[i]}(y=0), a_{i,j} = p_{i,j}(y=0) \); we set \( r = 3 \) as in the picture.

Using what we have already checked, the matrix \( A_\lambda^{[i]} \) is as follows:

\[
A_\lambda^{[i]} = \begin{pmatrix}
-x & 1 & 0 & 0 & \cdots & 0 \\
a_{i,i+1} & -x & 1 & 0 & \cdots & 0 \\
a_{i,i+2} & 0 & -x & 1 & \cdots & 0 \\
a_{i,i+3} & 0 & 0 & -x & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{i,\ell} & 0 & 0 & 0 & a_{i+4,\ell} & \cdots & -x
\end{pmatrix}.
\]

The (bold) entry \( \{4, 5\} \) equals 0 because it is \( y^{d_{i+r+1}} = y^{d_{i+r}} \) at \( y=0 \). Thus, the determinant of \( A_\lambda^{[i]} \) is the product of the determinant of its upper principal \( 4 \times 4 \)–block and the determinant of the principal block starting with the entry at \( \{5, 5\} \). The former determinant is classical:

\[
x^4 - (a_1 x^2 + a_2 x + a_3) \text{ for } a_j = a_{i,i+j}.
\]

Unless \( a_1 = a_2 = a_3 = 0 \), the final product cannot be a pure \( x \)–monomial, as it is supposed to be. This gives the required.

These arguments can be equally used in the opposite direction; they actually give that conditions \( (o) \) are not only necessary but sufficient too, i.e. equivalent to the summation restrictions in (2.12). A direct deduction of these restrictions from \( (o) \) is not difficult as well. Let us emphasize that we rely in this proof on Theorem 2.6, which provides that no polynomials strictly within (the boxes of) \( \lambda' \) can occur in \( I \) generated by \( T_\lambda^{(i)} \). Since it holds for \( T_\lambda \), then of course it is true under any specializations of the coefficients of \( p \)–polynomials.

\[ \square \]

3. Plane curve singularities

3.1. Compactified Jacobians. We will define their Gröbner decomposition. Let us consider only unibranch plane curve singularities. They are subrings \( R \subset \mathbb{C}[[z]] \) in terms of the uniformizing parameter \( z \) that have 2 generators, \( x \) and \( y \), and have \( \mathbb{C}((z)) \) as the field of fractions. We set \( \nu_z(u z^a + \sum_{i>a} c_i z^i) = a \) for \( u \neq 0 \), and define the
 valuation semigroup $\Gamma = \Gamma_{\mathcal{R}} = \{ \nu_z(f) \mid f \in \mathcal{R} \}$. See here and below [PS, GP] or [Ch, ChP1]. One has:

\[(3.14) \quad \delta \overset{\text{def}}{=} \dim_\mathbb{C} \mathbb{C}[[z]]/\mathcal{R} = |\mathbb{Z}_+ \setminus \Gamma|, \quad \text{and} \quad \mathcal{R} \supset (z^{2\delta}).\]

From now on, $(z^m) \overset{\text{def}}{=} \mathbb{C}[[z]]z^m$ for $m \in \mathbb{Z}_+$; it is a principal ideal in $\mathbb{C}[[z]]$, not in $\mathcal{R}$.

The compactified Jacobian of $\mathcal{R}$, denoted by $\text{Jac}_\mathcal{R}$ or simply by $\text{Jac}$ is a projective reduced irreducible scheme defined as follows:

\[(3.15) \quad \text{Jac} \overset{\text{def}}{=} \{ \mathcal{R} - \text{submodules } M \subset \mathbb{C}[[z]] \mid \dim_\mathbb{C} \mathbb{C}[[z]]/M = \delta \}.\]

As in [PS], the structure of projective variety is due to the following:

\[(3.16) \quad \text{for any } M \in \text{Jac}, \quad M \supset (z^{2\delta}) = \mathbb{C}[[z]]z^{2\delta}.\]

An important invariant of an $\mathcal{R}$–module $M \subset \mathbb{C}((z))$ is $\Delta(M) \overset{\text{def}}{=} \{ \nu_z(v) \mid v \in M \}$, for $M \subset \mathbb{C}[[z]]$, which we will always assume below unless stated otherwise, and for $k \in \mathbb{Z}_+$:

\[\dim \mathbb{C}[[z]]/z^k M = k + \dim \mathbb{C}[[z]]/M, \quad \dim \mathbb{C}[[z]]/M = |\mathbb{Z}_+ \setminus \Delta(M)|.\]

Let $M_* \overset{\text{def}}{=} z^{-v}M$ for $v = v(\Delta) \overset{\text{def}}{=} \min \Delta$, and $\Delta_*(M) = \Delta(M) - v$. The latter is a $\Gamma$–module, which means by definition that $\Gamma + \Delta \subset \Delta$. One has: $\Delta_*(M) = \Delta(M_*)$.

The image of the map $M \mapsto M_*$ is the set of all standard modules in $\mathbb{C}[[z]]$, which are those containing some elements in $1 + z\mathbb{C}[[z]]$. So $v = 0$ for such modules and $M_* = M$ if and only if $M$ is standard. Equivalently, $M$ is standard if $M\mathbb{C}[[z]] = \mathbb{C}[[z]]$ or if $\Delta[M]$ is standard, where a $\Gamma$-module $\Delta$ is called standard if $0 \in \Delta \subset \mathbb{Z}_+$. The generalized Jacobian $\text{Jac} \subset \text{Jac}$ is formed by all invertible modules, i.e. $M_\phi = \mathcal{R}\phi$ for $\phi \in 1 + z\mathbb{C}[[z]]$. Equivalently, invertible modules are such that $\Delta(M) = \Gamma$. The generalized Jacobian is a group with respect to the multiplication of the generators $\phi$. It is an affine space of dimension $\delta$; the simplest parametrization is as follows: $\phi = 1 + \sum g \phi g z^g$, where $g \in \mathbb{Z}_+ \setminus \Gamma, \phi g \in \mathbb{C}$. Its closure is the whole $\overline{\text{Jac}}$. Let $\text{Jac}^* \overset{\text{def}}{=} \{ M = M_* \}$, i.e. it is formed set-theoretically by all standard $\mathcal{R}$–modules. It is a disjoint union of quasi-projective schemes $\text{Jac}^{(d)} \overset{\text{def}}{=} \{ M = M_* \mid \dim C[[z]]/M = d \}$; note that $\text{Jac} = \text{Jac}^{(\delta)}$. 
We will also use the duality (also called reciprocity). For an $R$-module $M \in \mathbb{C}((z))$, let $M^* \overset{\text{def}}{=} \{ f \in \mathbb{C}((z)) \mid fM \in R \}$. It is an $R$-module. One has: $(M^*)^* = M$, $\mathbb{C}[[z]]^* = (z^{2\delta})$, and $(R)^* = R$.

Let us assume that $M = M_\bullet$, i.e. that $M$ is standard. Then $(M^*)^* \in \mathbb{C}[[z]]$, since $M$ contains $1 + z(\cdot)$, and it belongs to $R$ if and only if $M$ contains $R$. The latter gives that $\Delta(M^*) \subset \Gamma$, which is another defining property of standard $M$:

$$M = M_\bullet \iff \Delta(M^*) \subset \Gamma.$$ 

Indeed, if $\Delta(M^*) \subset \Gamma$ then $\Gamma \subset \Delta(M)$, and therefore $M$ is standard.

The following holds for standard $M$ and for any Gorenstein rings $R \subset \mathbb{C}[[z]]$, not only for the rings of plane curve singularities we consider in this paper; see [PS] and [GM2]. Let $\Delta = \Delta(M)$. Then $\Delta(M^*) = \Delta^* \overset{\text{def}}{=} \{ p \in \mathbb{Z} \mid p + \Delta \subset \Gamma \} \subset \mathbb{Z}_+$ and

$$\Delta^* = (2\delta - 1) - (\mathbb{Z} \setminus \Delta) = (2\delta - 1 - (\mathbb{Z}_+ \setminus \Delta)) \cup (2\delta + \mathbb{Z}_+),$$

where $v + X = \{ v + x \mid x \in X \}$ for $X \subset \mathbb{Z} \ni v$. It gives that $M^\vee \overset{\text{def}}{=} z^{c(M) - 2\delta}M^*$ is again a standard module for the conductor $c(M) = c(\Delta)$, which is defined as $\min\{ c \mid (z^c) \subset M \} = \min\{ c \mid c + \mathbb{Z}_+ \subset \Delta \}$. We call $M^\vee$ the standard dual of $M$. The conductor is obviously $g_{\text{top}}(\Delta) + 1$ for the top element $g_{\text{top}}$ in $\mathbb{Z}_+ \setminus \Delta$. Thus $\Delta((M^*)_\bullet) = \Delta^\vee \overset{\text{def}}{=} (c(\Delta) - 1 - (\mathbb{Z}_+ \setminus \Delta)) \cup (2\delta + \mathbb{Z}_+)$ for a standard $M$. We call $\Delta$ selfdual or, more exactly, standard selfdual if $\Delta = \Delta^\vee$.

Applications. The minimal embeddings $fM \subset R$ are as follows. Let

$$u_{\text{min}} = \min\{ u \mid f_u M \subset R \text{ for } f_u - z^u \in (z^{u+1}) \}. \quad \text{(3.18)}$$

Then $u_{\text{min}} = \min \Delta^* = 2\delta - 1 - g_{\text{top}}(\Delta) = 2\delta - c(M)$ due to (3.17), which holds for any $R$-modules $M$, not only standard.

As another application, let $J^1 \overset{\text{def}}{=} \{ M^1 \mid M^1 \supset R \}$, so the modules $M^1 \subset \mathbb{C}[[z]]$ here are automatically standard; $J^1$ is a disjoint union of projective schemes with fixed dim $\mathbb{C}[[z]]/M^1$. One has:

$$\text{Jac}^\bullet = (U^1/U_1^1) J^1 \text{ for } U^1 \overset{\text{def}}{=} 1 + z\mathbb{C}[[z]], \text{ and } U_1^1 = U^1 \cap R. \quad \text{(3.19)}$$

Indeed, one can obtain any standard $M$ as $\phi M^1$ for proper $M^1$ and invertible $\phi \in \mathbb{C}[[z]]$. So $J^1$ is a certain skeleton of $\text{Jac}^\bullet$:
Lemma 3.1. Let \( \widehat{\text{Jac}} \) \( \overset{\text{def}}{=} \{(M, \phi) \mid M = M_\bullet, \phi \in (U^1 \cap M)/U^1_R \} \). Then \( \widehat{\text{Jac}} \) is isomorphic as a scheme to \( J^1 \times (U^1/U^1_R) \), and \( \text{Jac}^{(d)} \) \( \overset{\text{def}}{=} \{(M, \phi) \mid M \in \text{Jac}^{(d)} \} \simeq \text{Jac}^{(d)} \times \mathbb{C}^{d-d} \) (as schemes).

Proof. The action \((M, \phi) \mapsto f(M, \phi) = (fM, f\phi)\) of \( f \in U^1/U^1_R \) on the pairs \((M, \phi)\) is free because it is free at the \( \phi \)-component. Then \((M, \phi) = \phi(M/\phi, 1)\), where \( M/\phi \) contains 1 and belongs to \( J^1 \).

To finish the proof, we pick \( f_g \in R \) for \( g \in \Gamma \) such that \( f_g = z^g \mod (z^{g+1}) \) in \( \mathbb{C}[[z]] \). Let \( \Delta = \Delta(M) \) for a standard \( M \), \(|Z_+ \setminus \Delta| = d\) and \( \Delta \setminus \Gamma = \{g_1 < g_2 < \ldots < g_{d-\Lambda}\} \). Then any \( \phi = 1 + \sum g \phi_g z^g \) in \((M, \phi)\) can be represented (modulo \( U^1_R \)) as \( \phi = 1 + \sum_i \phi_i z^{g_i} + \sum g \phi_z z^g \).

We proceed here by induction. Let \( \phi_g \neq 0 \) for the minimal such \( g \in \Gamma \setminus \{0\} \). Then \( \psi = \phi(1 - \phi_g f_g) \) has \( \psi_h = 0 \) for any \( h \leq g \) in \( \Gamma \setminus \{0\} \).

The coefficients \( \phi_i \) determine \( \phi \) uniquely. Indeed, \( \psi_g = c \) for \( \psi = \phi(1 + cf_g) \), any \( c \in \mathbb{C}^* \) and \( g \in \Gamma \setminus \{0\} \). Any \( \phi_i \in \mathbb{C} \) can occur here:

\[ M \text{ modulo } \{\sum g \phi_z z^g\} \text{ has a basis } z^g \text{ for } g \in \Delta. \]

Thus, \( \widehat{\text{Jac}}^{(d)} \simeq \text{Jac}^{(d)} \times \mathbb{C}^{d-d} \), where \( \phi_i \) are the coordinates of the latter factor. \( \square \)

Choosing \( x, y \). Changing the parameter \( z \), we can assume that the generators of \( \mathcal{R} \) are \( y = z^a, x = z^b h(z), h = 1 + \sum_j u_j z^j \) for some \( u_j \), and \( 1 < a < b \). Here the total \( gcd \) of \( a, b \) combined with the degrees of monomials in \( h \) must be 1. Also, \( b \) can be assumed the smallest in \( \Gamma \) non-proportional to \( a \), and \( z^j \) with \( j + b \not\in \Lambda \) are sufficient. The sharp version of the latter restriction is that \( h = 1 + z^{a-b} + \sum_j u_j z^j \), where \( j + b \not\in \Lambda \) for \( \lambda \overset{\text{def}}{=} \min \{\Lambda \setminus \Gamma\} \), the Zariski invariant, and

\[ (3.20) \quad \Gamma' \overset{\text{def}}{=} (\Gamma \setminus 0) - a \subset \Lambda' \overset{\text{def}}{=} \nu_z(\mathcal{R} z^{-a-1} + \mathcal{R}(d(z^b h)/dz)) + 1-a. \]

The latter is the \( \Gamma \)-module of "Kähler differentials", shifted to make it a standard one. See e.g. [HH].

Let \( P(x,y) = \text{res}(z^a - y, z^b h(z) - x; z) \), the resultant with respect to \( z \). Then \( P(z^b h(z), z^a) = 0 \) and \((-1)^a P = x^a + (-1)^{(a+1)(b+c)} y^{b+c} + \sum_{i+j \geq a, i+j \geq a} d_{ij} x^i y^j \) where \( i + j \geq a \), which is direct from the definition of the resultant. Also, \((-1)^a P = \prod_{i=0}^{a-1} (x - (\omega^i z)^b h(\omega^i z)) \) for \( \omega = \exp(\frac{2\pi i}{a}) \). The polynomial \( P \) is irreducible.

Lemma 3.2. Let \( P \) be as above, \( d_{ij} \neq 0 \) and \( i = a - v \). Then

\[ j \in \mathcal{J}_v \overset{\text{def}}{=} \left\{ \frac{vb + r_1 + \ldots + r_v}{a} \in \mathbb{N} \mid 0 \leq r_k \leq c, 1 \leq k \leq v \right\}. \]
where \( r \) are 0 or from the set of indices such that \( u_v \neq 0 \) in \( h(z) \).
Setting \( \epsilon_v = \min J_v - vb/a \geq 0 \), one has: \( ib + ja = \nu_2(x^iy^j) \geq a(b + \epsilon_v) \).

**Proof.** Consider \( x \) as an element of \( \mathbb{C}(y^{1/a}) \): \( x = y^{b/a}h(z \mapsto y^{1/a}) \). It generates this field over \( \mathbb{C}(y) \) by our assumptions. The coefficient \( d_{a-v,j} \)

is then the sum of the traces in the field extension \( \mathbb{C}(y^{1/a})/\mathbb{C}(y) \) of the elements \( \mathbb{C} \)-proportional to \( y^{v(b/a+(r_1+\ldots+r_v))/a} \), where \( 0 \leq r_k \leq c \). The exponent here must be an integer, which provides that \( j \geq \epsilon_v + vb/a \).

So \( ib + ja \geq (a-v)b + (\epsilon_v p + vb/a)a = a(b + \epsilon_v) \).

For instance, let us consider the simplest unibranch plane curve singularity that is not quasi-homogeneous, which is for the ring \( \mathcal{R} = \mathbb{C}[[y^2, x = z^6 + z^7]] \). Here one has: \( a = 4, b = 6, c = 1 \) and \( P = x^4 - 2x^2y^3 - 4xy^5 + y^6 - y^7 \). Indeed, \( J_1 = \emptyset \), i.e. \( x^3 \) does not appear, \( J_2 = \{(12 + \{0, 1, 2\})/4\} \cap \mathbb{N} = \{3\} \), which corresponds to \( x^3y^3 \), \( J_3 = \{(18 + \{0, 1, 2, 3\})/4\} \cap \mathbb{N} = \{5\} \), and \( J_4 = \{(24 + \{0, 1, 2, 3, 4\})/4\} \cap \mathbb{N} = \{6, 7\} \), which gives \( y^6, y^7 \). The exact calculation of the coefficients of \( P \) using the \( J \)-sets is straightforward. For generic \( \{d_{i,j}\} \), the estimates for \( j \) from the Lemma can be reached. The structure of \( P(x, y) \) is of importance in what will follow.

Note that the monomial ideal in \( \mathbb{C}[[x, y]] \) linearly generated by \( x^iy^j \)

such that \( \nu_2(x^iy^j) \geq ab \), which contains \( P(x, y) \), maps into \( (z^{2\delta}) \) if \( 2\delta \leq ab \); the latter holds when \( gcd(a, b) = 1 \) and in some other cases.

### 3.2. Topological invariance

In the classical geometry of smooth projective curves, they can be “recovered” from \( \widetilde{\text{Jac}} \) supplied with the polarization divisor. We will check that \( \widetilde{\text{Jac}}^* \) from Lemma 3.1 is a topological invariant of \( \mathcal{R} \) for some basic families. Also, some additional structures of \( \widetilde{\text{Jac}} \) allow the “extraction” of the equation of the initial singularity from it; see Proposition 3.5, (i).

The topological type of \( \mathcal{R} \) is given by the isotopy type of its link: the intersection of \( \{(x, y) \mid P(x, y) = 0\} \) with sufficiently small \( S^3 \) centered at \((0, 0)\). The semigroup \( \Gamma \) fully determines it, which is a classical fact; see [Za]. The topological invariance of \( \widetilde{\text{Jac}}^* \) as a scheme is generally subtle. If it holds, then it readily gives the topological invariance of the motivic superpolynomial of \( \mathcal{R} \) conjectured in [ChP1]; the demonstration is below.

**Proposition 3.3.** Let \( \widetilde{C}_R \) be the full preimage of \( (z^{2d}) \subset \mathcal{R} \) in \( \mathbb{C}[[x, y]] \),

i.e. \( \widetilde{C}_R = \mathbb{C}[[x, y]]P(x, y) + \mathbb{C}[[x, y]](z^{2d})' \) for any set-theoretical lift of \( (z^{2d})' \) of \( (z^{2d}) \) to \( \mathbb{C}[[x, y]] \). We assume that for some family of rings...
\( \mathcal{R}, \Gamma \) is fixed and \( \tilde{C}_\mathcal{R} \) is constant considered up to automorphisms of the ring \( \mathbb{C}[[x, y]] \). Then the schemes \( \tilde{\text{Jac}}^* \) and \( \tilde{\text{Jac}}^{(d)} \) from Lemma 3.1 considered up to isomorphisms are constant. Here \( \tilde{\text{Jac}}^{(d)} \) for \( 0 \leq d \leq g \) with the fibers isomorphic to \( \mathbb{C}^{d-d} \) due to this lemma.

Proof. One has: \( \tilde{\text{Jac}}^* = \{(M, \phi) \mid \phi \in (U^1/U_R^1) \cap (M/U_R^1)\} \) for standard \( M \); it is a disjoint union of \( \tilde{\text{Jac}}^{(d)} \) for \( d = |Z_+ \setminus \Delta(M)| \). Here \( U^1 = 1+z\mathbb{C}[[z]] \), \( U_R^1 = U^1 \cap \mathcal{R} \) and \( U^1/U_R^1 \simeq \mathbb{C}^d \) (as spaces). One has: \( \tilde{\text{Jac}}^* \simeq J^1 \times (U^1/U_R^1) \simeq J_1 \times (U^1/U^1_R) \), where \( J^1 = \{M = M_* \mid 1 \in M\} \) and \( J_1 \defeq \{ z^{2\delta} \mathbb{C}[[z]] \subset M_* \subset \mathcal{R} \} \); we use the duality map \( M \mapsto M^* \), which identifies \( J^1 \) with \( J_1 \). By sending ideals \( M^* \) to their full lifts \( \tilde{M}^* \) in \( \mathbb{C}[[x, y]] \), we identify \( J_1 \) with the scheme \( \{ \text{ideals } \tilde{I} \subset \mathbb{C}[[x, y]] \mid \tilde{C}_\mathcal{R} \subset \tilde{I} \} \). Such \( \tilde{I} \) are exactly \( \tilde{M}^* \) by construction. Thus, \( \Gamma \) and \( \tilde{C}_\mathcal{R} \) completely determine \( \tilde{\text{Jac}}^* \) (up to isomorphisms).

Now fix \( \Delta = \Delta(M) \) for standard \( \Delta = \Delta(M) \). From (3.17): \( \Delta^* = (2\delta - 1 - (Z_+ \setminus \Delta)) \cup (2\delta + Z_+) \), and the corresponding \( d^* \) equals \( 2\delta - d \). Then \( \tilde{J}_1[\Delta] = J^1[\Delta] \times (U^1/U_R^1) \simeq J_1[\Delta^*] \times (U^1/U_R^1) \) under the map \( \ast \). Here \( J^1[\Delta] = J[\Delta] \cap J^1 \) and \( J_1[\Delta^*] = J[\Delta^*] \cap J_1 \). Therefore \( \tilde{\text{Jac}}^{(d)} \) is the product of \( (U^1/U_R^1) \) and the scheme \( \{M^* \in J_1 \mid |Z_+ \setminus \Delta(M)| = 2\delta - d \} = \{ \tilde{C}_\mathcal{R} \subset \text{ideals } \tilde{I} \subset \mathbb{C}[[x, y]] \mid \dim \mathbb{C}[[x, y]]/\tilde{I} = 2\delta - d \} \), i.e. constant. We use that \( \dim \mathbb{C}[[x, y]]/\tilde{M}^* = \dim \mathcal{R}/M^* = 2\delta - d \), where \( \tilde{M}^* \) is the full lift (preimage) of the ideal \( M^* \subset \mathcal{R} \).

This proposition can be extended to the flags of standard modules and ideals from [ChP1]; we will omit the details. Examples of "constant" \( \tilde{C}_\mathcal{R} \) will be provided below. When \( \tilde{C}_\mathcal{R} \) is a monomial ideal in \( \mathbb{C}[[x, y]] \), [B-B] can be used to prove that \( \tilde{\text{Jac}}[\Delta] \) are affine cells.

Motivic superpolynomials. They are defined for the rings \( \mathcal{R} \subset \mathbb{F}_q[[z]] \) over \( \mathbb{F}_q \) (the field with \( q \) elements): \( \mathcal{H}_\mathcal{R}(q, t) \defeq \sum_M t^{\dim_q(\mathbb{F}_q[[z]]/M)} \), where the summation is over standard \( M \), \( \mathcal{R} \)-submodules \( M \subset \mathbb{F}_q[[z]] \) such that \( M\mathbb{F}_q[[z]] = \mathbb{F}_q[[z]] \). The ring \( \mathcal{R} \) is supposed to be with 2 generators and with the field of fractions \( \mathbb{F}_q[[z]] \). For any \( \mathcal{R} \) over \( \mathbb{C} \), we can define it over \( \mathbb{Z} \) and then over \( \mathbb{F}_q \) for any \( q = p^m \) for sufficiently general prime \( p \) (apart from finitely many of them) with the same valuation semigroup \( \Gamma \). The preservation of \( \Gamma \) is the weakest possible definition
of places $p$ of good reduction. The $\Gamma$-module of Kähler differentials (see above) must be also assumed to remain unchanged upon the passage from $\mathbb{C}$ to $\mathbb{F}_q$ for some considerations.

For $\widetilde{\text{Jac}}^\bullet$, we naturally set: $\widetilde{\mathcal{H}}_\mathcal{R}(q,t) \overset{\text{def}}{=} \sum_{(M,\phi)} t^{\dim \mathbb{F}_q[[z]]/M}$, where the summation is over $(M,\phi)$ in $\widetilde{\text{Jac}}^\bullet$ defined over $\mathbb{F}_q$. Due to Lemma 3.1, which holds for rings $\mathcal{R}$ over $\mathbb{F}_q$, the number of $\phi$ for a given $M$ equals $q^{\delta-d}$ for $d = \dim \mathbb{F}_q[[z]]/M$. Thus, $\widetilde{\mathcal{H}}_\mathcal{R}(q,t) = q^\delta \mathcal{H}_\mathcal{R}(q,t/q)$, and we see that the usage of $\widetilde{\text{Jac}}^\bullet$ is sufficient to obtain $\mathcal{H}_\mathcal{R}(q,t)$. The number of $\mathbb{F}_q$-points of $\widetilde{\text{Jac}}^\bullet$ depends only on $\Gamma$ if $\tilde{C}_\mathcal{R}$ depends only on $\Gamma$ due to Proposition 3.3; the latter holds over $\mathbb{F}_q$ (if $p$ is a place of good reduction). Thus $\mathcal{H}_\mathcal{R}(q,t)$ is a topological invariant for such $\mathcal{R}$.

**Semigroups with 2, 3 generators.** The assumption of the proposition is sufficiently explicit for such semigroups $\Gamma$. First of all, let $\Gamma^{(2)} = \{a,b\}$ for $1 < a < b \in \mathbb{Z}$ such that $\gcd(a,b) = 1$. It is for $\mathcal{R}^{(2)} = \mathbb{C}[[y = z^a, x = z^b h(z)]]$ and any $h(z) = 1 + z(\ldots)$ as above. By $(\ldots)$, we mean here the span over $\mathbb{Z}_+$.

Now we consider the case of 3 generators. Let $1 < a = vm < b = vn$ for $v = \gcd(a,b) > 1$, $\mathcal{R} = \mathbb{C}[[y = z^vm, x = z^vn h(z)]]$, where $h(z) = 1 + C z^p + \mod (z^{p+1})$ for $C \neq 0$, and $b+p$ is the first $z$-exponent in $z^b h(z)$ that is not a linear combination over $\mathbb{Z}_+$ of $a,b$. Then $x^m - y^n = mC z^{vmn+p} \mod (z^{vmn+p+1})$ and therefore $vmn+p \in \Gamma$ is the smallest element not in the $\mathbb{Z}_+$-span of $a,b$.

Let $\gcd(p,v) = 1$, which is necessary and sufficient to ensure that $\Gamma$ has 3 generators: they will be $vm, vn, vmn + p$. We will denote such ring $\mathcal{R}$ and the semigroup $\Gamma = \langle vm, vn, vmn+p \rangle$ by $\mathcal{R}^{(3)}, \Gamma^{(3)}$. In this case, $vn + p$ is the Zariski invariant of $\mathcal{R}$. Here we can (and will) assume that $h(z) = 1 + z^p + uz^{p+\kappa} \mod (z^{p+\kappa+1})$ for some $u$, where $\kappa \overset{\text{def}}{=} \text{Min}(\mathbb{Z}_+ \setminus \Lambda') \cap (p+\mathbb{Z}_+)$ for $\Lambda'$ from (3.20). We set $\kappa = 0$ if this intersection is empty; then $\Gamma$ fully determines the singularity.

The formulas for the corresponding $2\delta$ are as follows:

$$(3.21) \quad 2\delta^{(2)} = (a-1)(b-1), \quad 2\delta^{(3)} = v^2 mn - v(n+m) + (v-1)p+1.$$ 

We use here formula (2.1) from [HH] for $2\delta$, which is for any $\Gamma$. By the way, the latter gives that the condition $\gcd(p,v) = 1$ implies that $\Gamma$ has exactly 3 generators. Otherwise the corresponding $2\delta$ would be smaller than $2\delta^{(3)}$ above, and other generators must be no smaller than $v(vmn+p) > 2\delta^{(3)}$, which is impossible (they cannot belong to $\Gamma^{(3)}$).
See [HH] for the (formal) analytic classification of the singularities with $a = 4, b = 9$ and the table for $a = 4$. For instance, $\mathcal{R}_a = \mathbb{C}[[y = z^4, x = z^9 + z^{10} + uz^{11}]$ are non-isomorphic for different $u \in \mathbb{C}$. Here $10$ is the Zariski invariant and $\Lambda' = \nu_z(\mathcal{R}z^{a-1} + \mathcal{R}(dx/dz)) + 1 - a = (0, 4, 5, 8, 9, 10, 12, 13, 14, 15, \ldots)$ for $u \neq 10$. We use that $f \overset{\text{def}}{=} \frac{dz^4}{dx/dz} - 9xz^3 = z^{17} + 2uz^{18}$ and $x(dx/dz) - 9yf = (19 - 18u)z^{18} \mod (z^{19})$. Only $11$ is missing after $10$, so adding the terms $z^{>11}$ to $x$ does not change the corresponding analytic type of $\mathcal{R}$. For $\nu > 1$, an example of $\mathcal{R}$ with continuous moduli is $a = 4, b = 10, p = 1$.

Let Ceiling $[e] = \min\{\mathbb{Z} \ni n \geq e\}$ in the next theorem. The polynomial $P$ is as above; it depends only on $a, b, p$ for any $\mathcal{R}^{(3)}$ with $\kappa = 0$.

**Theorem 3.4.** (i) The polynomial $P(x, y)$ is constant modulo the monomials with the $\nu_z$-valuations no smaller than $2\delta$ for any $\mathcal{R}^{(2)}$ and for $\mathcal{R}^{(3)}$ with $\kappa \geq 1$ under the following inequalities (for $p$):

\begin{equation}
(v - 1)p \leq a \cdot \text{Ceiling}[\frac{b + p + \kappa}{a}] + a - 1, \text{ or }
\end{equation}

\begin{equation}
(v - 2)p \leq a + b + \kappa - 1.
\end{equation}

This gives that $\tilde{C}_\mathcal{R}$ coincides with that for $y = z^a, x = z^b + z^{b+p}$, Proposition 3.3 is applicable, and $\text{Jac}^\bullet$ is a topological invariant of $\mathcal{R}$.

(ii) If $(b + p + \kappa)$ is replaced by $(b + p)$ in (3.22) and $a + b + \kappa - 1$ by $a + b - 1$ in (3.23), then $\tilde{C}_\mathcal{R}$ has a basis of eigenvectors for the action $x \mapsto v^b x, y \mapsto v^a y, v \in \mathbb{C}^\times$. Here the case $\kappa = 0$ (when $\mathcal{R}$ is determined by $\Gamma_\mathcal{R}$) is included. Furthermore, all monomials of $P$ belong to $\tilde{C}_\mathcal{R}$ if $(v - 1)p \leq a + b - 1$. Finally, if $p = 1$ for $\mathcal{R}^{(3)}$ or for any $\mathcal{R}^{(2)}$, then the ideal $\tilde{C}_\mathcal{R}$ is monomial (and depends only on $\Gamma_\mathcal{R}$).

**Proof.** (i) Let us examine the monomials in $P(x, y)$ with potentially non-constant coefficients, i.e. those depending on $u_j (j \geq p + \kappa)$ in the presentation $x = z^b + z^{b+p} + u_{p+\kappa}z^{b+p+\kappa} + \ldots$. A natural lower bound of the $\nu_z$-valuations for such monomials in $P(x, y)$ is $a(b + \epsilon)$, where $\epsilon \geq \text{Ceiling}[\frac{b+\kappa}{a}] - \frac{b}{a} \geq \frac{p+\kappa}{a}$; we use Lemma 3.2. This readily gives that (3.22) or its somewhat stronger version (3.23) provide that $P(x, y)$ is constant modulo $\tilde{C}_\mathcal{R}$. Thus, $P(x, y)$ coincides with $P_0(x, y)$ obtained for $x = z^b + z^{b+p}$ modulo the monomials with $\nu_z \geq 2\delta$.

To conclude (i), $\tilde{C}_\mathcal{R}$ is constant, since we can find at least one constant element $g_k \in \mathbb{C}[[x, y]]$ for any $k \geq 2\delta$ with $\nu_z(g_k) = k$. These
elements will be in \( \tilde{C}_R \) by the definition of the latter. Namely, one represents: \( k = \alpha a + \beta b + \gamma (vmn + p) \) with \( \alpha, \beta, \gamma \in \mathbb{Z}_+ \) for any \( k \geq 2\delta \) (\( \gamma = 0 \) for \( \mathcal{R}^{(2)} \)). Then we set \( g_k = y^\alpha x^\beta (x^m - y^n)^\gamma \). Together with the ideal \( \mathbb{C}[[x,y]] P_0(x,y) \), such elements \( g_k \) for \( k \geq 2\delta \) generate \( \tilde{C}_R \).

(ii) First of all, the inequality \( 2\delta \leq ab \), which provides that all monomials of \( P \) belong to \( \tilde{C}_R \), is equivalent to \( (v - 1)p \leq a + b - 1 \) from (ii). We use that \( 2\delta^{(3)} = (a - 1)(b - 1) + (v - 1)p \). Technically, we can set \( p = 1, v = 0 \) for \( \mathcal{R}^{(2)} \); obviously, \( 2\delta = (a - 1)(b - 1) < ab \) in this case.

Next, the first inequality from part (ii) gives that \( \tilde{C}_R \) is generated by the elements of valuations no smaller than \( 2\delta \) and the monomials of \( P \) of the valuation \( ab \) (only such are sufficient). This gives that \( g_k \) above are eigenvectors under the action from (ii).

The monomiality for \( p = 1 \) and \( \mathcal{R}^{(2)} \). Since \( \gamma \) used above (for \( g_k \)) is 0 for \( \mathcal{R}^{(2)} \), \( \tilde{C}_R \) is a monomial ideal in this case. Namely, it contains \( x^a, y^b \), which have the valuation \( ab \), and \( g_k \) are monomials for \( (a-1)(b-1) \leq k < ab \), which is sufficient. We note that they are unique monomials with \( \nu_z = k \) up to proportionality in this range of \( k \).

Let us consider \( \mathcal{R}^{(3)} \) with \( p = 1 \). One has: \( 2\delta/v = vmn - m - n + 1 \) for \( p = 1 \). Let \( (m - 1)(n - 1) = \alpha m + \beta n \) with \( \alpha, \beta \in \mathbb{Z}_+ \). We obtain that \( 2\delta/v = (\alpha + (v - 1)n)m + \beta n = \alpha m + (\beta + (v - 1)m)n \). Therefore, \( g_y = y^{\alpha+(v-1)n}x^\beta \) and \( g_x = y^{\alpha+(v-1)n}x^\beta \); both elements are with \( \nu_z(g) = 2\delta \). One has:

\[
g_x - g_y = y^{\alpha+(v-1)n}x^\beta (x^m - y^n) = x^\beta y^{\alpha+(v-1)n}((v-1)nz^2(\cdots)).
\]

Thus, \( g_x - g_y \) represents \( \nu_z = 2\delta + 1 \), where the monomials \( g_x \) and \( g_y \) are in \( \tilde{C}_R \). (This step is more involved in the nonzero characteristic.)

We note \((v-1)mn\) is the smallest number such that it has \( v \) different \( \mathbb{Z}_+ \)-representations in terms of \( m, n \) according to Theorem 4 from [BR]. It gives that some linear combinations of the monomials in \( \tilde{C}_R \) provide the valuations \( 2\delta + 1, \ldots, 2\delta + (v - 1) \), which is sufficient to finish the proof. However, a simpler argument works.

We represent \( 2\delta + q \) with \( q \geq 0 \) as \( \alpha'(vm) + \beta'(vn) + \gamma'(vmn + 1) \) with \( \gamma' = q \) mod \( v \). The last condition is necessary here. It is sufficient because \( v(vmn + 1) \) can be represented in terms of \( a, b \) over \( \mathbb{Z}_+ \); any number no smaller than \( v(m - 1)(n - 1) \) is such. Thus, \( \nu_z \) of any monomial in the product \( y^{\alpha'} x^\beta (x^m - y^n)^{\gamma'} \) is \( \alpha'(vm) + \beta'(vn) + \gamma'(vmn) = (2\delta + q) - \gamma' \geq 2\delta \). So they belong to the monomial part of \( \tilde{C}_R \). \( \square \)
An example of $p = 1$. Let us illustrate the theorem for $a = 6, b = 9, p = 1$, i.e. for $v = 3, m = 2, n = 3$ and $\mathcal{R} = \mathbb{C}[[y = z^{6}, x = z^{9}h(z)]]$. One can assume that $h(z) = 1 + z + \ldots$. Then $\nu_{\ast}(x^{2} - y^{3}) = 19, 2\delta = 42$, where

\[ \tilde{C}_{\mathcal{R}} = \langle \{y^{7}, x^{4}y, x^{2}y^{4}\}_{42}, \{y(x^{4} - y^{6}), y^{4}(x^{2} - y^{3})\}_{43}, \{y(x^{2} - y^{3})^{2}\}_{44}, \{x^{5}, x^{3}y^{3}, xy^{6}\}_{45}, \{x^{3}(x^{2} - y^{3})\}_{46}, \{x(x^{2} - y^{3})^{2}\}_{47}, \ldots \rangle; \]

we show only some elements for the corresponding $\nu = \nu_{\ast}$ (inside $\{\}_{\nu}$). Therefore: $\tilde{C}_{\mathcal{R}} = \langle y^{7}, x^{4}y, x^{2}y^{4}, x^{5}, x^{3}y^{3}, xy^{6} \rangle$ as a $\mathbb{C}[[x, y]]$-module. The corresponding $\lambda$ and $\lambda'$ for this monomial ideal are

\[ \lambda(\tilde{C}_{\mathcal{R}}) = \{7, 6, 4, 3, 1\}, \lambda' = \square_{\square_{\square}}. \]

The number of boxes is $\delta = 21$. For $x = z^{9} + z^{10}$, one has: $P = y^{10} - 6xy^{8} - y^{9} - 2x^{3}y^{5} + 3x^{2}y^{6} - 3x^{4}y^{3} + x^{6}$, where all monomials have their valuations no smaller than 54; so they are well inside $\tilde{C}_{\mathcal{R}}$.

An example of $p = 2$. The ring $\mathcal{R} = \mathbb{C}[[y = z^{6}, x = z^{9} + z^{11}]]$ is the simplest when some Piontkowski cells are non-affine spaces; see the Appendix to [ChP1]. In this case, $\Gamma = \langle 6, 9, 20 = vn + p \rangle$, $2\delta = (a - 1)(b - 1) + (v - 1)p = 44, ab = 54 > 2\delta = 44$, and $a + b - 1 = 14 \geq p(v - 1) = 4$. The latter inequality gives that all monomials of $P$ belong to $\tilde{C}_{\mathcal{R}}$. Let us provide $P(x, y)$ for this $\mathcal{R}$: $x^{6} - 3x^{4}y^{3} + 3x^{2}y^{6} - 6x^{2}y^{7} - y^{9} - 2y^{10} - y^{11}$. In this case, $\tilde{C}_{\mathcal{R}}$ is generated by $\{(x^{2} - y^{3})x^{2}y\}_{44}, \{x^{5}\}_{45}, \{(x^{2} - y^{3})^{2}\}_{45}, \{(x^{2} - y^{3})^{4}\}_{47}, \{y^{8}\}_{48}, \{(x^{2} - y^{3})^{2}x\}_{49}, \ldots ;$ we show the corresponding valuations. To obtain the other representatives, multiply those we provided by powers of $y$. This is not a monomial ideal.

Recall that always $P(x, y) \in \tilde{C}_{\mathcal{R}}$ by definition, and that this ideal generally depends on the coefficients of $P$ if $2\delta > ab$. Even if all monomials of $P$ belong to $\tilde{C}_{\mathcal{R}}$, its linear generators can involve the coefficients of $h(z)$. On the other hand, if Theorem 3.4 is not applicable, the ideal $\tilde{C}_{\mathcal{R}}$ can be still a topological invariant of $\mathcal{R}$ up to isomorphisms, as well as $\text{Jac}$. A significantly weaker propriety was conjectured in [ChP1]: that the motivic super-polynomials are always topological invariants (for plane curve singularities), i.e. depend only on $\Gamma_{\mathcal{R}}$.

We think that this theorem provides the main cases when one can obtain a reasonably simple connection between the refined invariants of plane curve singularities from [ChD] and Hilbert schemes of $\mathbb{C}^{2}$. We mean here mostly some possible generalizations of Theorem 1.1 from
[GN]. It is not surprising that their theorem was restricted to torus knots; in this case, \( \overline{C}_R \) is the simplest.

3.3. Employing the parametrization. Given any \( M \in \overline{Jac} \), we set \( M^e \overset{\text{def}}{=} z^{2\delta + e} M \), which is an ideal in \( R \) containing \( (z^{4\delta + e}) \) for \( e \in \mathbb{Z}_+ \). Recall that the conductor \( c(M) \) is no greater than \( \delta + |Z_+ \setminus \Delta(M)| \) for standard \( M = M_* \), i.e. \( M \supset (z^{\delta + |Z_+ \setminus \Delta(M)|}) \); see (3.14) and [PS]. Since \( \overline{Jac} = \{ z^{|\Delta(M)|} M \} \) for standard \( M \), the modules for the points of \( \overline{Jac} \) automatically contain \( (z^{2\delta}) \).

The embedding \( M^e \subset R \) can happen for \( e < 0 \) for some \( M \), which values of \( e \) will be allowed in the considerations below. For such \( e \):

\[
\dim_c R/M^e = \dim_c \mathbb{C}[[z]]/M^e - \delta = 2\delta + e.
\]

The (full) lift of \( M^e \) to the ideals from \( \mathbb{C}[[x, y]] \) is natural:

\[
(3.24) \quad \overline{I}^e(M) \overset{\text{def}}{=} \mathbb{C}[[x, y]] P(x, y) + \mathbb{C}[[x, y]] \overline{M}, \quad \overline{I}(M) \overset{\text{def}}{=} \overline{I}^{e=0}(M)
\]

for any set \( \overline{M} \subset \mathbb{C}[[x, y]] \) such that its image in \( R \) is \( M^e \). One has: \( \dim_c \mathbb{C}[[x, y]]/\overline{I}^e(M) = 2\delta + e \), and \( \overline{C}_R = \overline{I}(\mathbb{C}[[z]]) \) in this notation. We will use that \( x^i y^j \in \overline{I}^e M \) if \( bi + aj \geq 4\delta + e \).

Using the operation \( I \mapsto I_0 \) for the ideals \( I \subset \mathbb{C}[[x, y]] \) from the previous section, we set:

\[
\overline{I}_0^e(M) \overset{\text{def}}{=} (\overline{I}^e(M))_0 = I_{\overline{\lambda}^e}
\]

for the corresponding partition \( \overline{\lambda}^e = \overline{\lambda}^e(M) \) of order \( 2\delta + e \). Since \( P(x, y) \in \overline{I}^e(M) \), it has no greater than \( a \) lines, i.e. \( \ell(\overline{\lambda}) \leq a \). Given any \( R \)-module \( M \), the polynomial \( P(x, y) \) is uniquely determined by \( \overline{I}^e(M) \) for sufficiently large \( e \), which is part of the following proposition.

**Proposition 3.5.** (i) Given \( M \in \overline{Jac} \), the polynomial \( P(x, y) \) coincides with \( f_{a0} \) of \( \overline{I}^e(M) \) for sufficiently large \( e \). Combinatorially, this holds if and only if the diagram \( \overline{\lambda}^e \) for \( \overline{\lambda}^e(M) \) contains all monomials from \( P(x, y) - (-x)^a \), i.e. all boxes \( \{ i, j \} \) with \( d_{i,j} \neq 0 \) in (3.2).

(ii) For such \( e \), \( \overline{I}^{e+a}(M) = y\overline{I}^e(M) + \mathbb{C}[[x, y]] f_{a0} \), and the diagram for \( \overline{\lambda}^{e+a}(M) \) is obtained from \( \overline{\lambda}^e \) for \( e \) by adding one \( a \)-column at its beginning. In the presentation from Theorem 2.7, the polynomials \( p_{i,j} \) remain unchanged when \( e \mapsto e + a \), but the corresponding \( y^{d_1} \) must be replaced for \( e + a \) by \( y^{d_1+1} \). \( \square \)
The ideals $\widetilde{C}^{(e)}_R \overset{\text{def}}{=} \widetilde{I}^e(\mathbb{C}[z])$ are the key in the description of the $\overline{\text{Jac}}$ via $\mathbb{C}[x,y]$, including $\widetilde{C}^{(0)}_R = \widetilde{C}_R$ used in Proposition 3.3. Note that $e = 0$ is generally far from the "stabile values" of Proposition 3.5. However it is of importance to calculate the $\lambda$-partitions for $e = 0$ and for $\tilde{I}^e_M$, which is defined as $\tilde{I}^e(M)$ with the smallest possible $e$ for one or some $M$ providing that $z^{2\delta + e}M \subset \mathcal{R}$. The Piontkowski strata $\{M = M_\ast \mid \Delta(M) = \Delta\}$ are natural here to analyze.

For an individual $M$, it is actually more natural to allow here more relaxed embeddings $\phi_{2\delta + e'}M \subset \mathcal{R}$ for proper $\phi_{2\delta + e'} \in z^{2\delta + e'} + (z^{2\delta + e'} + 1)$. According to (3.18), one has: $\min\{e'\} = -c(M)$ for the conductor $c(M)$. This relaxation makes sense for families of $M$ too, assuming that $\text{Aut}(M)$ is fixed in this family; cf. [Sto].

Let us summarize what we obtained, combining our analysis with Theorem 2.7.

**Theorem 3.6.** For a ring $\mathcal{R} \subset \mathbb{C}[z]$ with the generators $x,y$ picked as in Lemma 3.2, let $\Gamma$, $\delta$, and the compactified Jacobian $\overline{\text{Jac}} \ni M$ be as above. Let $\pi^e(M) = \tilde{I}^e(M)$, which is the inverse image of $M^e = z^{2\delta + e}M$ in $\mathbb{C}[x,y]$, an ideal of codimension $2\delta + e$. Here $e \geq 0$ ensures that $M^e \subset \mathcal{R}$ for all $M \in \overline{\text{Jac}}$, but it can be negative for special families of $M$. As above: $\widetilde{C}^{(e)}_R = \tilde{I}^e(\mathbb{C}[z])$, $\widetilde{C}_R = \widetilde{C}^{(0)}_R$.

(i) The map $\pi^0$ establishes an isomorphism of schemes between $\overline{\text{Jac}}$ and $\pi^0(\overline{\text{Jac}}) \subset H^{(2\delta)}$. The latter coincides with the projective subscheme $\{I \in H^{(2\delta)} \mid P(x,y) \in I \subset \widetilde{C}_R\}$ of $H^{(2\delta)}$ for the equation $P(x,y)$ as above. Such ideals $I$ automatically contain $\tilde{C}^{(2\delta)}_R$, which is $(z^{4\delta})$ lifted to $\mathbb{C}[x,y]$, including all $x^iy^j$ such that $bi + aj \geq 4\delta$. Recall that $H^{(2\delta)}$ contains all monomials from $m^{2\delta}$, i.e. for $i + j \geq 2\delta$.

(ii) Let us fix $\lambda$ such that $|\lambda| = 2\delta + e$ and $\ell(\lambda) = a$, where $e$ is assumed to satisfy the stabilization conditions from Proposition 3.5. Then $I \in C_\lambda$ for the Gröbner cell $C_\lambda \subset H^{(2\delta + e)}$ can be presented as $\pi^e(M) = \tilde{I}^e(M)$ for some $M \in \overline{\text{Jac}}$ if and only if

(a) $I \subset \widetilde{C}^{(e)}_R = \tilde{I}^e(\mathbb{C}[z])$ and (b) $f_{a0} = T^{(1)}_\lambda = P(x,y)$.

Accordingly, $C_\lambda \cap \pi^e(\overline{\text{Jac}})$ is the scheme of common zeros of all polynomials $\{p_{ij}\}$ from Theorem 2.7.

(iii) Continuing, the generators $f_{i^0j^0}$ for the corners $\{i^0, j^0\}$ of $\lambda'$ with $j^0 > 0$ are as follows. We switch to $\lambda$ obtained from $\lambda$ by removing the
first column in \( \lambda' \) and consider the corresponding generators \( \hat{f} \), given by the minors \( T_{\lambda'} \) with the corresponding \( p \)–polynomials obtained from \( \{ p \} \) as follows: \( \hat{p}_{i,j} = p_{i,j+1} \), where \( j \geq 0 \). Then \( \{i^o, j^o-1\} \) constitute the set of corners of \( \hat{\lambda'} \) and \( f_{i^o,j^o} = y\hat{f}_{i^o,j^o-1} \) for \( j^o > 0 \). \( \square \)

3.4. Quasi-homogeneous singularities. They are for the rings \( \mathcal{R} = \mathbb{C}[x = z^r, y = z^s] \) for \( r > s > 0 \) such that \( \gcd(r,s) = 1 \). Then \( \delta = (r-1)(s-1)/2 \) and the lift of \( (z^{2\delta}) \) to \( \mathbb{C}[x,y] \) is

\[
\tilde{C} = \tilde{C}_R = \left\{ \sum_{i,j} c_{ij} x^i y^j \mid i, j \in \mathbb{Z}_+, ir + js \geq 2\delta \right\}.
\]

This is a monomial ideal, which dramatically simplifies the usage of punctual Hilbert schemes for \( \mathbb{C}[x,y] \) for the study of \( \text{Jac} \) and \( \text{Jac}^\bullet \).

The diagram \( \lambda' \) for the partition \( \lambda \) of \( \tilde{C} \) is formed by all boxes in the \( s \times r \)–rectangle below the diagonal connecting \( \{i = s-1, j = 0\} \) and \( \{i = 0, j = r-1\} \) in our standard presentation of diagrams. Their number is indeed \( \frac{(r-1)(s-1)}{2} \). As above, the boxes are numbered by their upper-left corners \( \{i,j\} \), where \( 0 \leq i \leq s-1, 0 \leq j \leq r-1 \).

Let us put the numbers \( (2\delta - 1) - ir - js \) in the corresponding boxes. See Figure 4 for \( r = 4, s = 3 \). Then we arrive at the interpretation of the Piontkowski \( \Delta \)-modules from \([\pi]\) in terms of the Dyke paths from \([GM1]\); see there Section 2.2 and Figure 1. Let us state it and connect it with \( \tilde{\lambda}^\min(M) \) for \( z \)-monomial \( \mathcal{R} \)-modules \( M \subset \mathbb{C}[z] \).

Recall that \( \Delta_\bullet(M) = \Delta(M_\bullet) = \Delta(M) - v \) for \( M_\bullet = z^{-v}M \), where \( v = \min \Delta(M) \), where \( \Delta(M) \) is the \( \Gamma \)-modules of \( M \): \( \Delta \subset \mathbb{Z} \) and \( \Gamma + \Delta \subset \Delta \). We always assume that \( \Delta \subset \mathbb{Z}_+ \). Any \( \Gamma \)-modules in the quasi-homogeneous case come from some \( M \in \mathbb{C}[[z]] \). They are fully described by their sets of gaps, which is \( \mathbb{Z}_+ \setminus \Delta \). Finally, \( \hat{\Delta} \overset{\text{def}}{=} \Delta_\bullet \setminus \Gamma \), which is the set of added gaps from \( \Gamma \) for \( \Delta_\bullet = \Delta - \min \Delta \).

We define Dyck paths as Young diagrams in this rectangle, which can be empty, above the anti-diagonal, i.e. with \( ir + sj < 2\delta \) in the \( \{i,j\} \)-presentation. The correspondence from \([GM1]\) is between the Dyck paths and standard \( \Delta \). Namely, the set of numbers \( 2\delta - 1 - ir - js \) calculated for the boxes of the corresponding Dyck path is \( \hat{\Delta} \). The next proposition follows from this interpretation.

**Proposition 3.7.** Given a \( z \)-monomial standard \( \mathcal{R} \)-module \( M \), let \( \lambda' \) be the diagram constructed from the partition \( \lambda = \tilde{\lambda}^\min(M) \). Then
it coincides with the Dyck path for the (standardization of the) dual module $M^*$, which is \( \{ f \in \mathcal{R} \mid fM \subset \mathcal{R} \} \).

Example: \( r = 4, s = 3 \). Then \( \mathcal{R} = \mathbb{C}[[x = z^4, y = z^3]] \), \( 2\delta = 6 \). We will calculate \( \lambda' \) from Proposition 3.7 for all \( z \)-monomial standard \( M \) and the corresponding diagrams \( \tilde{\lambda}' = \tilde{\lambda}'(z^{2\delta}M) \). Generally, the corresponding inverse images are not monomial in \( \mathbb{C}[[x, y]] \) for \( z \)-monomial modules \( M \); this is due to the presence of \( \mathbb{C}[[x, y]](x^3 - y^5) \) in the lifts. However they are monomial in this particular case.

First, \( z^6\mathbb{C}[[z]] \subset \mathcal{R} \) is the minimal embedding of \( M_{bt} \overset{\text{def}}{=} \mathbb{C}[[z]] \) into \( \mathcal{R} \). So \( \tilde{\lambda}_{bt} = \{1, 2, 5\} = \mathbb{Z}_+ \setminus \Gamma \), and \( \lambda'_{bt} = \begin{array}{c} \lambda_{bt} \end{array} = \begin{array}{c} \lambda'_{bt} \end{array} \). Second, \( \lambda' = 0 \), \( \tilde{\lambda}' = \begin{array}{c} \lambda' \end{array} \) for \( M_0 \overset{\text{def}}{=} \mathcal{R} \). The modules \( M_{bt}M_0 \) are standard selfdual.

Let \( M_1 \) be generated by \( 1, z \) over \( \mathcal{R} \). It is linearly generated by \( \{1, z, z^2, z^3, z^4, z^5, z^6 \ldots\} \), so \( \tilde{\Delta}(M_1) = \{1, 5\} \). Its standard dual is \( M_3 \) below. Here minimal \( v \) such that \( z^vM_1 \subset \mathcal{R} \) is 3. So \( z^3M_1 \) is the linear span \( \{z^3, z^4, z^6, z^7, \ldots\} = \{y, x, y^2, yx, \ldots\} \), and \( \lambda '_1 = \begin{array}{c} \lambda \end{array} , \tilde{\lambda}'_1 = \begin{array}{c} \lambda' \end{array} \). The latter is for \( y\{y, x, y^2, yx, \ldots\} \), which results in the additional 3-column.

The module \( M_2 \overset{\text{def}}{=} \langle 1, z^2 \rangle \) is the linear span of \( \{1, z^2, z^3, z^4, z^5, z^6 \ldots\} \). This module is standard self-dual. One has: \( \tilde{\Delta}(M_2) = \{2, 5\} \), \( v = 4 \) and \( z^4M_2 \) is the linear span of \( \{z^4, z^6, z^7, z^8, \ldots\} = \{z^4, z^6, z^7, z^8, \ldots\} \); for \( z^6M_2 \) it is \( \{y^2, x^2, y^3, y^2x, \ldots\} \). Thus \( \lambda'_2 = \begin{array}{c} \lambda \end{array} \) and \( \tilde{\lambda}'_2 = \begin{array}{c} \lambda' \end{array} \).

Finally, let \( M_3 \overset{\text{def}}{=} \langle 1, z^5 \rangle \); this conclude the list. It is the span of \( \{1, z^3, z^4, z^5, z^6 \ldots\} \). It coincides with the standard dual of \( M_1 \), which is \( \langle M_1' \rangle \). Then \( \tilde{\Delta}(M_3) = \{5\} \), \( v = 3 \) and \( z^3M_3 \) is the linear span of \( \{z^3, z^6, z^7, z^8, \ldots\} = \{y, y^2, yx, x^2, \ldots\} \). So \( \lambda'_3 = \begin{array}{c} \lambda \end{array} , \tilde{\lambda}'_3 = \begin{array}{c} \lambda' \end{array} \).

It is of interest to interpret combinatorially \( \check{t}(M) \) for any admissible \( e \). We do this in this section only for minimal \( e \) and \( e = 0 \). Generally non-trivial stratifications of \( \text{Jac} \) can be obtained this way, which are in a sense ”orthogonal” to the Piontkowski one in terms of \( \Gamma \)-modules \( \Delta \). The following stratification of \( \text{Jac} \), which is the Piontkowski stratum of \( \text{Jac} \) corresponding to \( \Delta = \Gamma \), is of particular interest.

We consider the invertible \( \mathcal{R} \)-modules belonging to various standard \( z \)-monomial modules. The stratification of \( \text{Jac} \) by the set-theoretical differences of the corresponding closures of these sets in \( \text{Jac} \) is presumably related to the \( q \leftrightarrow t^{-1} \) duality of the motivic superpolynomials.
The diagrams for $\overline{\text{Jac}}$. Continuing with the same $r = 4, s = 3$, let us provide a complete stratification of $\overline{\text{Jac}}$ in terms of Gröbner cells for $e = 0$. Given a standard $M$, let $M' \overset{\text{def}}{=} z^v M$ for $v = v_M = \delta - \dim \mathbb{C}[[z]]/M$. We will describe $\tilde{\lambda}(M')$ for the map $M' \mapsto \tilde{I}' = \pi^0(M') = I(M')$, where the latter is the lift of $z^{2\delta} M'$ to $\mathbb{C}[[x, y]]$ (the inverse image). Note that $\dim \mathbb{C}[[z]]/M' = \dim \tilde{C}/\tilde{I}'$. Generally,

$\pi^e : \{ M \mid z^{2\delta+e} M \subset R \} \sim \{ \tilde{C}^{(2\delta+e)} \subset \tilde{I} \subset \tilde{C}'(e) \mid P(x, y) \in \tilde{I}' \}$

for any $R$-modules $M \subset \mathbb{C}[[z]]$ and ideals $\tilde{I} \subset \mathbb{C}[[x, y]]$. One has here: $\dim \mathbb{C}[[z]]/M = e + \dim \tilde{C}/\tilde{I}$ for $\pi^e(M) = \tilde{I}$. Note that standard $M$ correspond to $\tilde{I}$ containing some (full) lifts of $\phi z^{2\delta+e}$ for $\phi \in 1 + z \mathbb{C}[[z]]$. For $\pi^{\{e=0\}}$ we have:

$\pi^0 : \overline{\text{Jac}} = \{ M' \} \sim \{ \tilde{C}^{(2\delta)} \subset \tilde{I} \subset \tilde{C} \mid P(x, y) \in \tilde{I}' \}.$

We have 5 standard $\Gamma$-modules $\Delta$ (containing 0) and the corresponding 5 families of standard modules $M \subset \mathbb{C}[[z]]$. Let us describe the partitions $\tilde{\lambda}$ and the diagrams $\tilde{\lambda}'$ for these families considered in $\overline{\text{Jac}}$.

**Total $M_{\text{tot}} = \mathbb{C}[[z]]$.** This module has $\tilde{\Delta} \overset{\text{def}}{=} \Delta \setminus \Gamma = \{ 1, 2, 5 \}$. It was considered after Proposition 3.7. We have: $v = 3, \tilde{I}(M_{\text{tot}}') = \{ y^3, y^2 x, y x^2, x^3, \ldots \}$ and $\tilde{\lambda}'_{\text{tot}} = \boxed{\text{condition}}$. This family is just one point.

**Family 0: invertibles.** The condition $\tilde{\Delta} = \emptyset$ is necessary and sufficient. These modules are $M = \phi R \subset \mathbb{C}[[z]]$, where $\phi = 1 + \alpha z + \beta z^2 + \gamma z^5$
and $\alpha, \beta, \gamma \subset \mathbb{C}$ give the parametrization of $Jac$. Here $M' = M$ and we need to lift $t^6 M$ to $\mathbb{C}[x, y]$. The result is $\tilde{I}'$ generated by the lift of $z^6 \phi = z^6 + \alpha z^7 + \beta z^8 + \gamma z^{11}$, which is $\varphi \stackrel{\text{def}}{=} y^2 + \alpha y x + \beta x^2 + \gamma x^2 y$, and by the lift of $(z^4 \delta)$, which is $\tilde{C}(6) = (y^4, x^3, y^3 x, y^2 x^2, y x^3, x^4, \ldots)$. Note that the latter contains $-P = x^3 - y^4$, a special feature of this example simplifying a bit the considerations. The cases are as follows.

$(0,i)$: $\beta \neq 0, \beta \neq \alpha^2$. Then $x^2, y^2 x$ belong to $\tilde{I}'_0$, the monomial ideal for $\tilde{I}'$. To see this compare $y \varphi$ and $x \varphi$ modulo $\tilde{C}(6)$. The corresponding $\tilde{\lambda}'_0$ is $\begin{bmatrix} \alpha \beta z \gamma \end{bmatrix}$. This sub-family is isomorphic to $\mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^*$ as a space.

$(0,ii)$: $\beta = 0, \alpha \neq 0$. Then the lowest Gröbner monomial of $y^2 \varphi = y^2 + \alpha y x + \gamma x^2 y$ is $y, x$. Since $x^3, y^4 \in \tilde{I}'$, we obtain: $\tilde{\lambda}'_0 = \begin{bmatrix} \alpha \gamma \end{bmatrix}$. As a space, this sub-family is isomorphic to $\mathbb{C} \times \mathbb{C}^*$.

$(0,iii)$: $\beta \neq 0, \beta = \alpha^2$. In this case $y^2 x \notin \tilde{I}'_0$ as in $(i)$. However, $y^3 \in \tilde{I}'_0$ due to $y \varphi - \alpha x \varphi = y^3$ mod $\tilde{C}(6)$. Thus $\tilde{\lambda}'_0 = \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$ and the corresponding space is $\mathbb{C} \times \mathbb{C}^*$.

$(0,iv)$: $\alpha = 0 = \beta, \gamma \neq 0$. Here $\tilde{\lambda}'_0 = \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$ and the space is $\mathbb{C}^*$. 

$(0,v)$: $\alpha = 0 = \beta = \gamma$. We lift $z^6 \mathcal{R}_1$, $\tilde{\lambda}'_0 = \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$, the space is one point.

The families below are those containing the monomial ideals $M_1, M_2$ and $M_3$, considered above.

**Family 1:** through $M_1$. Such $M$ are generated by $1 + \alpha z^2, z + \beta z^2$. One has: $\tilde{\Delta} = \{1, 5\}, v = |\tilde{\Delta}| = 2$. Accordingly, the lift of $z^6 M$ is generated by $x^2 + \alpha y^2 x, y^3 + \beta y^2 x$ modulo $\tilde{C}(6)$. The cases are:

$(1,i)$: $\beta \neq 0$. Then $y^2 x \in \tilde{I}'_0$ and $\tilde{\lambda}'_1 = \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$. The space is $\mathbb{C} \times \mathbb{C}^*$. The Gröbner $f$-generators will be $x^2 - \frac{\alpha}{\beta} y^3, y^2 x + \frac{1}{\beta} y^3$.

$(1,ii)$: $\beta = 0$. Then $\tilde{\lambda}'_1 = \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$, and the space is $\mathbb{C}$.

**Family 2:** through $M_2$. Here $M$ are generated by $1 + \alpha z, z^2$, and $\Delta = \{2, 5\}, v = 2$. The module $z^6 M_2$ is generated by $x^2 + \alpha y^3, y^2 x$ modulo $\tilde{C}(6)$. Thus we have only one subcase here:

$(2,i)$: $\tilde{\lambda}'_1 = \begin{bmatrix} \alpha \beta \gamma \end{bmatrix}$, and the corresponding space is $\mathbb{C}$.

**Family 3:** through $M_3$. The modules $M$ are generated by $1 + \alpha z + \beta z^2, z^5$, and $\Delta = \{5\}, v = 1$; they also contain $z^3, z^4, z^5, \ldots$. So the
lift $\tilde{I}(M')$ of $z^7M$ has the generators $yx + \alpha x^2 + \beta y^3, y^2 x, y x^2$ modulo $\tilde{C}(6) = \langle y^4, x^3, y^3 x, y^2 x^2, y x^3, x^4, \ldots \rangle$. The cases are:

(3,i): $\alpha \neq 0$. Then $x^2 \in \tilde{I}(M'), \lambda = \overline{\lambda}_3$ and the space is $\mathbb{C} \times \mathbb{C}^*$.

(3,ii): $\alpha = 0$. Then $yx \in \tilde{I}(M'), \lambda = \overline{\lambda}_3$ and the space is $\mathbb{C}$.

Summary. The main purpose of this calculation is to decompose $\overline{Jac}$ using the Gröbner cells. E.g., the portion of $\pi^0(\overline{Jac})$ corresponding to $\lambda' = \overline{\lambda}$ for $\lambda = \{4, 2\}$ is $(\mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^*)_{0,i} \cup (\mathbb{C} \times \mathbb{C}^*)_{1,i} \cup (\mathbb{C}^*)_{2,i} \cup (\mathbb{C} \times \mathbb{C}^*)_{3,i}$, where the suffix shows the source of this contribution. In the Grothendieck ring $K_0(var/\mathbb{C})$, it is $\mathbb{C} \times (\mathbb{C} - pt)^2 + 2\mathbb{C}(\mathbb{C} - pt) + \mathbb{C} = \mathbb{C}^3 - 2\mathbb{C}^2 + \mathbb{C} + 2\mathbb{C}^2 - 2\mathbb{C} + \mathbb{C} = \mathbb{C}^3$.

This is the subset in the full Gröbner cell $C_\lambda$ in $H^0(6)$ of the ideals $\tilde{I}$ satisfying the embeddings

\[(3.27) \quad \tilde{C}(6) = \langle x^3, y^3 x, \ldots \rangle \subset \tilde{I} \subset \overline{C} = \langle y^2, y^2, x^2, \ldots \rangle.\]

Any ideals $\tilde{I} \in C_\lambda$ have generators: $f_{20} = x^2 + cxy + dy^2 + ey^3$, $f_{12} = xy^2 + gy^3 \in \tilde{I}$, and $f_{04} = y^4$. The term $y$ is missing here in the first generator for any $\tilde{I} \in H^0(6)$ corresponding to this diagram due to the equality $C_{01}^4 = 0$ from (2.10); see there and Figure 3. This is also granted due to $\tilde{I} \subset \overline{C}$.

Since $y f_{20}, y^3 \in \tilde{I}$, $x^2 y + cxy^2 + dy^3 \in \tilde{I}$. Using now that $x^3, xy^2 \in \tilde{I}$, we obtain that $xf_{20} \in \tilde{I}$ $\Rightarrow c x^2 y + dxy^2 \in \tilde{I}$. Thus, $(c^2 - d) x^2 y + cdy^3 \in \tilde{I}$. Combining it with $f_{12}$, we obtain that $cd = (c^2 - d)g$ is the equation of $\pi^0(\overline{Jac}) \cap C_\lambda$, where $e$ is arbitrary.

Note that $(c = 0, d = 0, f = 0)$ is a singular point of the surface \[\{cd = (c^2 - d)g\}. \] This relation reduces dim $C_\lambda$ from 4 = 6 to 2. It combines in one equality all relations above for $\alpha, \beta$ from different subcases. Namely, the equality $c = 0 = d$ (any $g, e$) corresponds to $(1, i) \& (2, i)$. When $c^2 - d \neq 0$ (any $e$), we obtain $(0, i) \& (3, i)$.

As another example, let us consider $\lambda' = \overline{\lambda}$. Then the intersection of $\pi^0(\overline{Jac})$ with the Gröbner cell $C_\lambda$ will be $(\mathbb{C} \times \mathbb{C}^*)_{0,ii} \cup (\mathbb{C})_{1,ii} = \mathbb{C}^2$ in the Grothendieck ring. Now the generators of $\tilde{I}'$ subject to (3.27) are $x^2 + cxy + dy^2 + ey^2$ and $y^3$, subject to the relation $d = c^2$. Also, \[\{pt\}_{tell} \cup (\mathbb{C}^*)_{0,iv} \] gives $\mathbb{C}$ in the case of $\overline{\lambda}$.
The monomiality from part (ii) of Theorem 3.4 combined with [B-B] provide some a priori reasons for these intersections to be (topologically) affine spaces.

Let us extend the first step of our calculation to \( a = 3 \) and \( b = m \).

**Proposition 3.8.** Let \( R = \mathbb{C}[x = z^m, y = z^3] \), \( 2\delta = 2(m-1) \), assuming that either (i) \( m = 2 + 3k \) or (ii) \( m = 1 + 3k \) for \( k \geq 1 \). The partition for generic standard invertible \( M \) for the Gröbner decomposition of Jac, the generalized Jacobian Jac, under \( \pi^0 \) will be denoted \( \tilde{\lambda}_0 \).

Then \( \tilde{\lambda}_0 = \{4k + 2, 2k\} \) in case (i) and the corresponding manifold is \( \mathbb{C}^* \times \mathbb{C}^{2-1} \). In case (ii), the partition is \( \{4k, 2k\} \). The corresponding manifold is \( \mathbb{C}^{\delta-2} \) times \( \mathbb{C}^2 \) minus a union of two different \( \mathbb{C}^1 \) inside it.

**Proof.** We represent the generators of the invertible modules constituting Jac as \( \phi = 1 + az + \beta z^2 + \gamma z^4 + \ldots \). Then \( 2\delta = 2(m-1) \) is \( m + 3k \) for \( m = 2 + 3k \) (case (i)) or \( 6k \) for \( m = 1 + 3k \) (case (ii)). Accordingly, \( \varphi = z^{2\delta} \phi = xy^k + \alpha y^{2k+1} + \beta x^2 + \gamma y^{2k+2} + \ldots \) in case (i) or \( \varphi = z^{2\delta} \phi = y^{2k} + \alpha xy^k + \beta x^2 + \gamma xy^{k+1} + \ldots \) in case (ii).

**Case (i).** The ideal \( \pi^0(z^{2\delta}) \) is linearly generated by the lift of \( z^{2\delta} \mathbb{C}[[z]] \) to \( \mathbb{C}[[x, y]] \), which is \( \{x^2y^{2k}, xy^{3k+1}, x^3y^k = y^{4k+2}, x^2y^{2k+1}, \ldots\} \). Thus it contains \( y^{4k+2} \) and the first row of \( \tilde{\lambda}_0 \) has at most \( 4k + 2 \) boxes. One has: \( x\varphi = x^2y^k + \alpha xy^{2k+1} + \beta x^3 + \gamma xy^{2k+2} + \ldots = -\frac{1}{\beta}(xy^k + \alpha y^{2k+1} + \gamma y^{2k+3} + \ldots)y^k + \alpha xy^{2k+1} + \beta y^m + \gamma xy^{2k+2} \). The smallest power of \( y \) here is in \( xy^{2k} \), which gives that the second row is at most with \( 2k \) boxes. However, \( 4k + 2 + 2k = 6k + 2 = 2\delta \), so \( \tilde{\lambda}_0 \) is exactly \( \{4k + 2, 2k\} \) in this case. The corresponding manifold is \( \mathbb{C}^* \times \mathbb{C}^{\delta-1} \).

**Case (ii).** The ideal \( \pi^0(z^{2\delta}) \), the lift of \( z^{2\delta} \mathbb{C}[[z]] \), is now linearly generated by \( \{y^{4k}, xy^{3k}, x^2y^{2k}, xy^{3k+1}, \ldots\} \). So it contains \( y^{4k} \) and the first row of \( \tilde{\lambda}_0 \) has at most \( 4k \) boxes. One has: \( x\varphi = xy^{2k} + \alpha x^2y^k + \beta x^3 + \gamma x^2y^{k+1} + \ldots = xy^{2k} - \frac{\alpha}{\beta}(y^{2k} + \alpha xy^k + \gamma xy^{k+1} + \ldots + \ldots)y^k + + \beta y^m + \beta y^m + \gamma x^2y^{k+1} \). The term with the smallest power of \( y \) is now \( (1 - \frac{2}{\alpha})xy^{2k} \), which gives that the second row is at most with \( 2k \) boxes provided that \( \beta \neq \alpha^2 \). However, \( 4k + 2k = 6k = 2\delta \), so \( \tilde{\lambda}_0 \) is exactly \( \{4k, 2k\} \) in this case. The corresponding manifold is isomorphic to \( \mathbb{C}^2 \setminus (\mathbb{C} \times pt \cup pt \times \mathbb{C}) \) multiplied by \( \mathbb{C}^{\delta-2} \).

Without going into all detail, let us provide the Young diagram \( \tilde{\lambda}' \) for generic points of Jac for the ring \( R = \mathbb{C}[x = z^5, y = z^4] \). It is with
First of all, $y^6 = t^{24} = t^{4k}$ and $y^6 \in \overline{I}$. Setting $\phi = z^{2\delta} = y^3 + \alpha xy^2 + \beta x^2 y + \gamma x^3 + \delta x^2 y^2 + \epsilon x^3 y + \nu x^3 y^2$, one assumes that $\gamma \neq 0$, which gives that $x^3$ belongs to the corresponding monomial $\overline{I}_0$. Considering $x\phi$, we obtain that $(\alpha - \beta x) x^2 y^2$ belongs to $\overline{I}_0$, so we assume next that $\beta x \neq \alpha y$ and obtain that $x^2 y^2 \in \overline{I}_0$. Similarly, $x^2 \phi$, $xy \phi$, and $y^2 \phi$ give that $x^2 y^3 + \alpha x^3 y^2$, $xy^4 + \alpha x^2 y^3 + \beta x^3 y^2$, $\alpha xy^4 + (\gamma - \beta \alpha) x^3 y^2 + y^5$ belong to $\overline{I}$. Combining the latter two elements, we obtain that $xy^4 \in \overline{I}$ if $\gamma \neq 2\alpha \beta + \alpha^3$.

For arbitrary relatively prime $b > a > 1$ and $\mathcal{R} = \mathbb{C}[[x = z^b, y = z^a]]$, the Young diagrams for $\pi^0(\overline{Jac})$ are of order $2\delta = (a - 1)(b - 1)$ and belong to the rectangle $a \times \kappa$ for $\kappa = Ceiling[4\delta/a]$. We use that the corresponding ideals contain $\pi^0(z^{2\delta}[z])$, so $y^\kappa$ belongs to all of them. The polynomial $x^a - y^b$ gives that the second dimension is $a$.

Recall, that $\overline{I} \subset \pi^0(\mathbb{C}[[z]])$, i.e. $\overline{\lambda}$ contains the diagram $\lambda_{ab}$ for the lift of $z^{2\delta}[z]$ to $\mathbb{C}[[x, y]]$. For instance, the minimal possible number of rows in $\overline{\lambda}$ is $Ceiling[2\delta/b] = a - 1$.

In examples, all such diagrams can be obtained from some modules $M$ in $\overline{Jac}$. Their total number of diagrams with $2\delta$ boxes in the rectangle $a \times \kappa$ equals the coefficient of $q^{2\delta}$ in $q^{\binom{a}{2}} \binom{a + \kappa}{a}_q$ for the $q$-binomial coefficients $\binom{n}{m}_q$. We need to diminish it by the number of such diagrams with $a - 2$ rows. Generally, this difference is greater than the rational slope Catalan number $\frac{\binom{a + b}{a}}{a+b}$, which gives the number of Pi-ontkowski cells. Recall that each such cell generally results in several diagrams.

If all such partitions $\overline{\lambda}$ can be obtained this way, then the connectivity of $\bigcup_{\overline{\lambda}} C_{\overline{\lambda}} \subset H^{(2\delta)}$ follows from that for $\overline{Jac}$, which can be of independent interest. Actually only the diagrams with the minimal possible number of rows, which is $a - 1$, are sufficient here to check, and we can use different orderings of $\{x^m y^n\}$, not only the one with $\{x^\infty < y\}$.

**Some perspectives.** Summarizing, we presented $\overline{Jac}$ above as the union of the intersections of its $\pi^0$-image with the proper Gröbner cells. The resulting intersections are homeomorphic to $\mathbb{C}^3, \mathbb{C}^2, \mathbb{C}^2, \mathbb{C}, pt$, i.e. the same as for the Piontkowski decomposition with respect to $\Delta(M)$. The list of cells must be the same because their multiplicities are Betti numbers of $\overline{Jac}$. Another, similar, approach is to consider the filtration of $\overline{Jac}$ in terms of closures of the sets of invertibles inside
monomial standard $M$ and the corresponding strata. Combined with
the Lusztig-Smelt-Piontkowski cells, it gives a justification of the super-
duality of the \textit{motivic superpolynomials} for quasi-homogeneous $R$, to
be discussed elsewhere. The Gröbner decomposition of $\pi^0(Jac)$ (Family
0: invertibles) is interesting in its own right, with possible relations to
[MY, MS].

Generally, we can decompose the images of $\pi^0(Jac)$ due to the pre-
sentation from (3.26) using [B-B] and the methods based on the stable
envelopes from [MO]. This is for the action of $\mathbb{C}^*$ for quasi-homogeneous
singularities. Actually, it suffices to assume here that $\tilde{C}_R$ and the im-
age of $\pi^0(Jac)$ due to the pre-
sentation from (3.26) are invariant with respect to the action
$x \mapsto u^b x$, $y \mapsto u^a y$ for $u \in \mathbb{C}^*$. The method from [B-B] provides then
that the corresponding ”cells” will be affine spaces. Actually, only the
$\mathbb{C}^*$-invariance of $\overline{Jac}$ and $Jac^*$ is sufficient. For instance, Theorem
3.4 (its end) provides this invariance when $\Gamma$ has 2 generators or 3 with
$p = 1$.

Using different orderings. There is another approach, adjusted directly
to quasi-homogeneous plane curve singularities. Its aim is to elimi-
nate the non-trivial combinatorics of the Gröbner decomposition of $Jac$
(Family 0) and other Lusztig-Smelt-Piontkowski cells. Given $a, b \in \mathbb{Z}_+$
such that $gcd(a, b) = 1$, one introduces the valuation $\nu(y) = a, \nu(x) = b$
in $\mathbb{C}[[x, y]]$. Then the \textit{weighted Gröbner basis} and the corresponding
$I_0$ are defined when we first order monomials with respect to $\nu$ (the
smallest $\nu$ first), and then with respect to our usual Gröbner ordering
$\{x^\infty < y\}$ if their $\nu$ coincide. This leads to a variant of the wall cross-
ing, where the ratio $a/b$ serves as the \textit{stability condition}. We will not
discuss here the corresponding theory; the approach from [B-B]-[MO]
can be used. Let us give one example.

We will calculate all $\tilde{\lambda}_\nu$ for $R = \mathbb{C}[[y = z^a, x = z^b]]$, and $Jac$ (Family
0) for the example above. Let $a = 3, b = 4$, $\nu = \nu_z$. The smallest
monomial in $\varphi = y^2 + \alpha xy + \beta x^2 + \gamma x^2 y$ will be then always $y^2$, and the
corresponding \textit{weighted $\tilde{\lambda}_\nu$} becomes $\text{[][]}$ for any $\alpha, \beta, \gamma$. Only \textit{monomial}
$M$ are sufficient to consider. Let us list all $\tilde{\lambda}_\nu$ for the corresponding families:

$$
\begin{align*}
\text{(tot)} & : \text{[][]}, \\
(0) & : \text{[][]}, \\
(1) & : \text{[][]}, \\
(2) & : \text{[][]}, \\
(3) & : \text{[]}
\end{align*}
$$
This is relatively straightforward for arbitrary quasi-homogeneous singularities. We represent $2\delta = (a - 1)(b - 1) = au + bv$ for $u, v \in \mathbb{Z}_+$. In the case of "the big cell" $\widetilde{Jac}$, the weighted $\lambda'_\nu$ will be the rectangle $a \times (b + u)$ without the corner $\{\{i, j\} \mid i \geq v \& j \geq u\}$. The corresponding monomial ideal is generated by $x^a, y^{b+u}, x^v y^u$. Note that the number of boxes of this diagram is $a(b+u)-(a-v)b = au+bv = 2\delta$, as it is supposed to be. The number of lines is always no greater than $a$, since $\pm P(x, y) = x^a + \ldots$ belongs to all ideals for $\pi^0(\widetilde{Jac})$. The smallest number of columns is $(b - 1) - \text{Floor}\left[\frac{b-1}{a}\right]$, which is for $M_0$. We see that every Piontkowski cell naturally maps to the corresponding (single) Gröbner cell under the ordering based on such $\nu$.

Thus the Piontkowski decomposition can be generally seen as a particular case of that based on [B-B], the theory of stable envelopes, and "localization theory". This is for quasi-homogeneous plane curve singularities, but can be possibly extended to those in Theorem 3.4. In Proposition 3.3, this gives the description of the scheme $\widetilde{Jac}[\Delta]$ entirely in terms of $\Gamma$ (for such singularities).

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