Two-dimensional solitary wave solution to the quadratic-cubic nonlinear Schrödinger equation

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Abstract. The Madelung fluid transformation is applied to find the link between the modified Korteweg-de Vries and the quadratic-cubic nonlinear Schrödinger equation. The two-dimensional solitary wave solution of the quadratic-cubic nonlinear Schrödinger equation will be determined by the Petviashvili method. This solution will be used for the initial condition for the time evolution to study the stability analysis. The spectral method is applied for the time evolution.

1. Introduction
One of possible solutions of the nonlinear wave equation, which retain its shape as a result of the balance between dispersion and non-linearity, is known as a solitary wave. For the one-dimensional spatial dimension, there are two major types that the solitary wave solutions can be found. One is called the Korteweg-de Vries (KdV) equation,

\[ \phi_t + \phi \phi_x + \phi_{xxx} = 0. \tag{1} \]

where the subscripts \( x \) and \( t \) denote the differentiation with respect to space and time, respectively. It’s originally derived for a shallow water wave, (see e.g. [1] for further details). This equation can also be found in plasma in which describe a weakly nonlinear ion-acoustic waves [2]. Another type is well-known as the cubic nonlinear Schrödinger (cNLS) equation,

\[ i \phi_t + \phi_{xx} + |\phi|^2 \phi = 0. \tag{2} \]

which is the crucial equation to describe a traveling soliton along fiber optics [3, 4]. However, there is a transformation between (1) and (2) via the Madelung’s fluid transformation [5]. The idea of this transformation is the use of the following wave form,

\[ \phi(x, t) = \sqrt{\rho(x, t)} e^{i \Theta(x, t)}, \tag{3} \]

where \( \rho = |\phi|^2 \) is the Madelung’s fluid density and \( \Theta \) is a wave phase. For the modified nonlinear Schrödinger equation, we consider the quadratic-cubic nonlinear Schrödinger (qc-NLS) equation,

\[ i \phi_t + \phi_{xx} + \phi_{yy} + |\phi| \phi + |\phi|^2 \phi = 0, \tag{4} \]
and the modified KdV equation, which is expected to be related to (4), can be written as [6, 7, 8, 9],
\[ \rho_t + \rho \rho_x + \rho^{1/2} \rho_x + \rho_{xxx} = 0. \] (5)
This equation also governs a weakly nonlinear ion-acoustic wave as well as nonlinear dust ion-acoustic waves in plasma with some electrons are trapped on the ion-acoustic waves while some moves freely. The traveling wave solutions of KdV and cNLS are not difficult to determine but these might be more difficult for the higher dimensional solutions. The bending soliton solution of Zakharov-Kutnetsov (ZK) equation with the long-wavelength perturbation [10, 11],
\[ \phi_t + \phi \phi_x + \phi_{xxx} + \phi_{yyx} = 0, \]
is other methods to find the higher soliton solution. As well as the perturbed soliton solution of modified Kadomtsev-Petiashvili (KP) equation [12],
\[ (\phi_t + \phi^{1/2} \phi_x + \phi_{xxx})_x = \phi_{yy}, \]
which gives the 2D lump solution. We then find the higher soliton solution from the perturbed planed soliton but it sometimes might not be stable. However, if there is a suitable method to obtain 2D soliton solution directly, the higher soliton solution of other types of the nonlinear wave equation will be more studied. Before we move to the numerical method for the 2D solution, we will consider the Madelung’s fluid transformation for the connection between (4) and (5).

2. Madelung’s fluid transformation
We will repeat the original approach [5, 13] for (4). The main result of this transformation is to obtain the governed fluid equations. We first substitute (3) into (4). There are real and imaginary parts, the real part can be written as
\[ \frac{1}{2} \rho_t + (\rho V)_x = 0. \] (6)
This equation can be related to the continuity equation. For the imaginary part, we have
\[ V_t + 2VV_x = \frac{\partial}{\partial x} \rho^{1/2} + \frac{\partial}{\partial x} \left[ \frac{1}{\sqrt{\rho}} \frac{\partial^2}{\partial x^2} \sqrt{\rho} \right]. \] (7)
where \( V(x, t) = \frac{\partial \Theta}{\partial x} \) denotes the current velocity. This equation can be referred as the momentum equation. To solve for \( \rho \), the real part (6) will be multiplied by \( V \),
\[ \rho \left( \frac{\partial}{\partial t} V + 2V \frac{\partial}{\partial x} V \right) = -V \frac{\partial \rho}{\partial t} + \rho \frac{\partial V}{\partial t} - 2V^2 \frac{\partial \rho}{\partial x}. \] (8)
We will combine the momentum equation (7), and the modified real part (8), and integrate with respect to time,
\[ -V \frac{\partial \rho}{\partial t} + \rho \frac{\partial V}{\partial t} + 4 \left[ -c(t) + \frac{\partial}{\partial t} \int V dx \right] \frac{\partial}{\partial x} \rho 
= \left( \rho \frac{\partial}{\partial x} \sqrt{\rho} + 2\sqrt{\rho} \frac{\partial}{\partial x} \rho \right) + 3\rho \frac{\partial}{\partial x} \rho + \frac{\rho_{xxx}}{2} \]
where \( c(t) \) is a function of time. To obtain the soliton solution, the current velocity is assumed to be a constant, \( V = V_0 \),
\[ \Theta = V_0 x - 2c_0 t \quad \text{and} \quad c(t) = c_0 \]
The traveling wave equation can be written as
\[-V_0 \frac{\partial}{\partial t}\rho - 4c_0 \frac{\partial}{\partial x}\rho = \left( \rho \frac{\partial}{\partial x}\sqrt{\rho} + 2\sqrt{\rho} \frac{\partial}{\partial x}\rho \right) + 3\rho \frac{\partial}{\partial x}\rho + \frac{\rho_{xxx}}{2}\] (9)

If a new variable is introduced as \(\xi = x - 2V_0 t\), the stationary equation of (9) can be found as
\[(4V_0^2 - 8c_0) \frac{d}{d\xi}\rho = 3\sqrt{\rho} \frac{d}{d\xi}\rho + 6\rho \frac{d}{d\xi}\rho + \frac{d^3}{d\xi^3}\rho,\]
which is the modified KdV equation, (4). We will consider the numerical 2D solution of (4) by applying the iterative method.

3. Petviashvili method

The ground state of the nonlinear Schrödinger equation with the power-law potential had been proposed by Petviashvili [14] and revised by Lakoba and Yang [15]. The convergence of the method for the homogeneous equation has been shown by Pelinovsky and Stepanyants [16]. To determine the solution, we consider
\[\phi(x, y, t) = u(x, y)e^{i\mu t}\] (10)
where \(u(x, y) > 0\) and \(\mu\) is the propagation constant. After substitute (10) into (4), we obtain
\[-\mu u + u_{xx} + u_{yy} + u^2 + u^3 = 0.\]

A new variable will be introduced as
\[Mu = (\mu - \partial_{xx} - \partial_{yy})u = u^2 + u^3.\] (11)

To determine the steady solution, (11) will be iterated until it meets the steady solution, namely,
\[u_{i+1} = M^{-1}(u_i^2 + u_i^3),\] (12)
where \(i\) denotes a number of iteration from zero to any integer number. The result of (12) might be either zero or infinity. The key idea of Petviashvili method is to find the stabilized factor which maintain the result of the iteration without diverge to zero or infinity. We can now rewrite (12) as
\[u_{i+1} = S_i^\gamma M^{-1}(u_i^2 + u_i^3),\]
where \(\gamma\) is a constant and this stabilizing factor can be obtained by inner product of the previous value, \(u_i\),
\[S_i = \frac{\langle u_i, M u_i \rangle}{\langle u_i, u_i^2 + u_i^3 \rangle}.\]

To determine \(\gamma\), we consider the exact solution \(u(x, y) = O(1)\). At some iterations, the solution becomes \(u_i(x, y) = O(\epsilon)\), where \(\epsilon \ll 1\) or \(\epsilon \gg 1\). The order of magnitude of the stabilizing factor can be represented as
\[S_i = O(\epsilon^{-1}).\]

The order of magnitude of \((u(x, y))\) becomes
\[u_{i+1} = O(\epsilon^{-\gamma+2}).\] (13)
This implies that $\gamma = 2$. The Gaussian function will be used as the initial condition,

$$u_0(x, y) = 5e^{-(x^2+y^2)},$$

where the length in $x$ and $y$ spaces are $L_x = L_y = [-5, 5]$ and spacing for each space is 128 ($N_x = N_y = 128$). We choose $\mu = 3/2$ for the examination. Figure 1 shows how the initial Gaussian function approaches to its steady state solution for each step of iteration. It looks like the Gaussian function expands until it meets the steady state. To be confirm this structure we consider the second example of the initial condition,

$$u_0(x, y) = 5\text{sech}^2(\sqrt{x^2+y^2}) \cos x \sin y.$$  

The result is shown in Figure 2, there are a few steps between Figure 2(a)-(b), before it becomes the Gaussian-like function.

![Figure 1](image1.png)

**Figure 1.** Result of each iteration: (a) $i = 0$, (b) $i=10$, (c) $i = 20$, (d) $i = 30$, (e) $i = 40$, (f) $i = 52$.

We will use these results as the initial condition for the time evolution. We expect to get the same shape for the evolution.

### 4. Time evolution of a vortex soliton

We apply the Fourier transform to the spatial coordinates of (4),

$$\frac{d\phi}{dt} = -i \left( F^{-1}\left( (\xi^2 + \chi^2) F(\phi) \right) - |\phi|\phi - |\phi|^2\phi \right)$$

where $N_x$ and $N_y$ are the number of mesh points in the $x$ and $y$ direction. $(x_l, y_m) = (lL_x/N_x, mL_y/N_y)$, in which $L_x$ and $L_y$ are the lengths of the domain in the $x$ and $y$ directions. $(\xi_p, \chi_q) = 2\pi(p/L_x, q/L_y)$ for $p = 0, \ldots, N_x - 1$ and $q = 0, \ldots, N_y - 1$. The Runge-Kutta method [17] is applied to the time derivative, while the initial condition is the final step of the Petviashvili method. The time evolution of the stationary state will be shown in Figure 3. The initial condition is calculated for the stationary state, there are no any movement during the simulation. This result confirms that the results from the iterative method are the possible 2D solution of (4).
Figure 2. Result of each iteration: (a) $i = 0$, (b) $i=1$, (c) $i = 2$, (d) $i = 3$, (e) $i = 30$, (f) $i = 44$.

Figure 3. Time step evolution: (a) $t = 0$, (b) $t=2$, (c) $t = 4$, (d) $t = 6$, (e) $t = 8$, (f) $t = 10$.

5. Conclusion
The 2D solitary wave solution of the cqNLS can be obtained by the Petviashvili method. The time evolution is shown that the results are correct. We used the one example of the computational methods to calculate the vortex-like solution of the nonlinear wave equation. It’s also a good start to study some phenomena of the vortex solitons or some excited states of others nonlinear wave equations.

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