Schauder estimates for degenerate Lévy Ornstein-Uhlenbeck operators

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Abstract

We establish global Schauder estimates for integro-partial differential equations (IPDE) driven by a possibly degenerate Lévy Ornstein-Uhlenbeck operator, both in the elliptic and parabolic setting, using some suitable anisotropic Hölder spaces. The class of operators we consider is composed by a linear drift plus a Lévy operator that is comparable, in a suitable sense, with a possibly truncated stable operator. It includes for example, the relativistic, the tempered, the layered or the Lamperti stable operators. Our method does not assume neither the symmetry of the Lévy operator nor the invariance for dilations of the linear part of the operator. Thanks to our estimates, we prove in addition the well-posedness of the considered IPDE in suitable functional spaces. In the final section, we extend some of these results to more general operators involving non-linear, space-time dependent drifts.

Keywords: Schauder estimates, degenerate IPDEs, Lévy Ornstein-Uhlenbeck Operators.

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1 Introduction

Fixed an integer \(N\) in \(\mathbb{N}\), we consider the following integro-partial differential operator of Ornstein-Uhlenbeck type:

\[
\mathcal{L}_{\text{ou}} := \mathcal{L} + \langle Ax, D_x \rangle \quad \text{on} \quad \mathbb{R}^N,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the Euclidean inner product on \(\mathbb{R}^N\), \(A\) is a matrix in \(\mathbb{R}^N \otimes \mathbb{R}^N\) and \(\mathcal{L}\) is a possibly degenerate, Lévy operator acting non-degenerately only on a subspace of \(\mathbb{R}^N\). We are interested in showing the well-posedness and the associated Schauder estimates for elliptic and parabolic equations involving the operator \(\mathcal{L}_{\text{ou}}\) and with coefficients in a generalized family of Hölder spaces.

We only assume that \(A\) satisfies a natural controllability assumption, the so-called Kalman rank condition (condition \([K]\) below), and that the operator \(\mathcal{L}\) is comparable, in a suitable sense, to a non-degenerate, truncated \(\alpha\)-stable operator on the same subspace of \(\mathbb{R}^N\), for some \(\alpha < 2\) (condition \([ND]\) below).

The topic of Schauder estimates for Ornstein-Uhlenbeck operators has been widely studied in the last decades, especially in the diffusive, local setting, i.e. when \(\mathcal{L} = \frac{1}{2} \text{Tr}(QD_x^2)\) for some suitable matrix \(Q\), and it is now quite well-understood. See e.g. [DPL95], [Lun97], [GT01], [DFP06], [KP10], [CdRHM18]. On the other hand, a literature on the topic for the pure jump, non-local framework has been developed only in the recent years ([Bas09], [DK13], [BK15], [ROS16], [FRRO17], [IJIS18], [CdRMP20], [Küh19], but mainly in the non-degenerate, \(\alpha\)-stable setting, i.e. when \(\mathcal{L} = \Delta_x^{\alpha/2}\) is the fractional Laplacian on \(\mathbb{R}^N\) or similar. Up to the best of our knowledge, the only two articles dealing with the degenerate, non-local framework (if \(\mathcal{L} = \Delta_x^{\alpha/2}\) acts non-degenerately only on a sub-space of \(\mathbb{R}^N\)) are [HPZ19], that takes into account the kinetics dynamics (\(N = 2d\)), and [Mar20], for the general chain. In order to use [HPZ19] or [Mar20] for our operator (1.1), we would need to impose the additional strong assumption of invariance for dilations of the matrix \(A\).

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Even if the Ornstein-Uhlenbeck operator is usually seen as a "toy model" for more general operators with space-time dependent, non-linear coefficients, we highlight that they appear naturally in various scientific contexts: for example in physics, for the analysis of anomalous diffusions phenomena or for Hamiltonian models in a turbulent regime (see e.g. [BMB01], [CPKM05] and the references therein) or in mathematical finance and econometrics (see e.g. [Bro01], [BNs01]).

The interest in Schauder estimates involving this type of operator also follows from the natural application which consists in establishing the well-posedness of stochastic differential equations (SDE) driven by Lévy processes and the associated stochastic control theory. See e.g. [FM83], [CdRM20], [HWZ20].

Under our assumptions, we have been able to consider more general Lévy operators not usually included in the literature, such as the relativistic stable process, the layered stable process or the Lamperti one (see Paragraph "Main Operators Considered" below for details). Moreover, we do not require the operator \( L \) to be symmetric.

Here, we only mention one important example that satisfies our hypothesis, the Ornstein-Uhlenbeck operator on \( \mathbb{R}^2 \) driven by the relativistic fractional Laplacian \( \Delta^{\alpha/2} \) and acting only on the first component:

\[
x_1(D_{x_1}\phi(x) + D_{x_2}\phi(x)) + \text{p.v.} \int_{\mathbb{R}} \left[ \frac{1}{\sqrt{2\pi}} \frac{e^{-|z|^2/2}}{|z|^{d+\alpha}} \right] \phi \left( \frac{x_1+x_2}{x_2} \right) - \phi \left( \frac{x_1}{x_2} \right) dx_2 = (Ax, D_x\phi(x)) + \mathcal{L}\phi(x) \tag{1.2}
\]

where \( x = (x_1, x_2) \) in \( \mathbb{R}^2 \). Such an example is included in the framework of Equation (1.1) considering

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

This operator appears naturally as a fractional generalization of the relativistic Schrödinger operator (see [Ryz02] for more details).

We remark that example (1.2) cannot be considered in [HPZ19] or in our previous work [Mar20]. Indeed, the matrix \( A_0 \) is not "dilation-invariant" (see example 2.1 below) and thus, it cannot be rewritten in the form used in [Mar20] (see also [LP94] Proposition 2.2 for a more thorough explanation). Furthermore, operators like the relativistic fractional Laplacian cannot be treated in [HPZ19] or [Mar20] that indeed have taken into account only stable-like operators on \( \mathbb{R}^N \). Another useful advantage of our technique is that we do not need anymore the symmetry of the Lévy measure \( \nu \) which was, again, a key assumption in [Mar20].

More in details, given an integer \( d \leq N \) and a matrix \( B \) in \( \mathbb{R}^N \otimes \mathbb{R}^d \) such that \( \text{rank}(B) = d \), we consider a family of operators \( \mathcal{L} \) that can be represented for any sufficiently regular function \( \phi : \mathbb{R}^N \to \mathbb{R} \) as

\[
\mathcal{L}\phi(x) := \frac{1}{2} \text{Tr}(BQ(B^*D_x^2\phi(x))) + \langle Bb, D_x\phi(x) \rangle + \int_{\mathbb{R}^d} \left[ \phi(x+Bz) - \phi(x) - (D_x\phi(x), Bz) \mathbf{1}_{B(0,1)}(z) \right] \nu(dz), \tag{1.3}
\]

where \( b \) is a vector in \( \mathbb{R}^d \), \( Q \) is a symmetric, non-negative definite matrix in \( \mathbb{R}^d \otimes \mathbb{R}^d \) and \( \nu \) is a Lévy measure on \( \mathbb{R}^d_0 := \mathbb{R}^d \smallsetminus \{0\} \), i.e. a \( \sigma \)-finite measure on \( \mathcal{B}(\mathbb{R}^d_0) \), the Borel \( \sigma \)-algebra on \( \mathbb{R}^d_0 \), such that \( \int (1 + |z|^2) \nu(dz) \) is finite. We then suppose \( \nu \) to satisfy the following non-degeneracy condition:

[ND] there exists \( r_0 > 0 \), \( \alpha \) in \( (0, 2) \) and a finite, non-degenerate measure \( \mu \) on the unit sphere \( S^d-1 \) such that

\[
\nu(C) \geq \int_0^{r_0} \int_{S^d-1} \mathbf{1}_C(r\theta) \mu(d\theta) \frac{dr}{r^{d+\alpha}}, \quad C \in \mathcal{B}(\mathbb{R}^d_0).
\]

We recall that a measure \( \mu \) on \( \mathbb{R}^d \) is non-degenerate if there exists a constant \( \eta \geq 1 \) such that

\[
\eta^{-1}|p|^{\alpha} \leq \int_{S^d-1} |p \cdot s|^{\alpha} \mu(ds) \leq \eta |p|^{\alpha}, \quad p \in \mathbb{R}^d, \tag{1.4}
\]

where "\( \cdot \)" stands for the inner product on the smaller space \( \mathbb{R}^d \). Since any \( \alpha \)-stable Lévy measure \( \nu_\alpha \) can be decomposed into a spherical part \( \mu \) on \( S^d-1 \) and a radial part \( r^{-1+\alpha}dr \) (see e.g. Theorem 14.3 in [Sat13]), assumption [ND] roughly states that the Lévy measure of the integro-differential part of \( \mathcal{L} \) is bounded from below by the Lévy measure of a possibly truncated, \( \alpha \)-stable operator on \( \mathbb{R}^d \).

It is assumed moreover that the matrices \( A, B \) satisfy the following Kalman condition:

[K] it holds that \( N = \text{rank}[B, AB, \ldots, A^{N-1}B] \),

where \( [B, AB, \ldots, A^{N-1}B] \) is the matrix in \( \mathbb{R}^N \otimes \mathbb{R}^dN \) whose columns are \( B, AB, \ldots, A^{N-1}B \).

Such an assumption is equivalent, in the linear framework, to the Hörmander condition (see [Hör77]) on the

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commutators, ensuring the hypoellipticity of the operator $\partial_t - \mathcal{L}^{ou}$. Moreover, condition $[K]$ is well-known in control theory (see e.g. [Zab92], [PZ09]).

Mathematical Outline. In this paper, we aim at establishing global Schauder estimates for equations involving the operator $\mathcal{L}^{ou}$ on $\mathbb{R}^N$, both in the elliptic and parabolic settings. Namely, we consider for a fixed $\lambda > 0$ the following elliptic equation:

$$\lambda u(x) - \mathcal{L}^{ou} u(x) = g(x), \quad x \in \mathbb{R}^N,$$

and, for a fixed time horizon $T > 0$, the following parabolic Cauchy problem:

$$\begin{cases}
\partial_t u(t, x) = \mathcal{L}^{ou} u(t, x) + f(t, x), & (t, x) \in (0, T) \times \mathbb{R}^N; \\
u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}$$

where $f, g, u_0$ are given functions. Since our aim is to show optimal regularity results in Hölder spaces, we will assume for the elliptic case (Equation (1.5)) that the source $g$ belongs to a suitable anisotropic Hölder space $C^{\alpha,\beta}_{b,d}(\mathbb{R}^N)$ for some $\beta$ in $(0, 1)$, where the Hölder exponent depends on the 'direction' considered. The space $C^{\alpha,\beta}_{b,d}(\mathbb{R}^N)$ can be understood as composed by the bounded functions on $\mathbb{R}^N$ that are Hölder continuous with respect to a distance $\rho$ somehow induced by the operator $\mathcal{L}^{ou}$. We refer to Section 2 for a detailed exposition of such an argument but we highlight already that the above mentioned distance $d$ can be seen as a generalization of the classical parabolic distance, adapted to our degenerate, non-local framework. It is precisely assumption $[K]$, or equivalently the hypoellipticity of $\partial_t + \mathcal{L}^{ou}$, that ensures the existence of such a distance $d$ and gives it its anisotropic nature. Roughly speaking, it allows the smoothing effect of the Lévy operator $\mathcal{L}$ acting non-degenerately only on some components, say $B\mathbb{R}^N$, to spread in the whole space $\mathbb{R}^N$, even if with lower regularizing properties.

Concerning the parabolic problem (1.6), we assume similarly that $u_0$ is in $C^{\alpha,\beta}_{b,d}(\mathbb{R}^N)$ and that $f(t, \cdot)$ is in $C^{\alpha,\beta}_{b,d}(\mathbb{R}^N)$, uniformly in $t \in (0, T)$. The typical estimates we want to prove can be stated in the parabolic setting in the following way: there exists a constant $C$, depending only on the parameters of the model, such that any distributional solution $u$ of the Cauchy problem (1.6) satisfies

$$\|u\|_{L^\infty(C^{\alpha,\beta}_{b,d})} \leq C \left[\|u_0\|_{C^{\alpha,\beta}_{b,d}} + \|f\|_{L^\infty(C^{\alpha,\beta}_{b,d})}\right].$$

As a by-product of the Schauder Estimates ($\mathcal{S}$), we will obtain the well-posedness of the Cauchy problem (1.6) in the space $L^\infty(0, T; C^{\alpha,\beta}_{b,d}(\mathbb{R}^N))$, once the existence of a solution is established. The additional regularity for the solution $u$ with respect to the source $f$ reflects the appearance of a smoothing effect associated with $\mathcal{L}^{ou}$ of order $\alpha$, as it is expected by condition $[ND]$. It can be seen as a generalization of the 'standard' parabolic bootstrap to our degenerate, non-local setting. We highlight that the parabolic bootstrap in ($\mathcal{S}$) is precisely derived from the non-degenerate stable-like part in $\mathcal{L}$ (lowest regularizing effect in the operator).

To show our result, we will follow the semi-group approach as firstly introduced in [DPL95], which became afterwards a very robust tool to study Schauder estimates in a wide variety of frameworks ([LM97], [Lor05], [Sai07], [Pri09], [Pri12], [DK13], [KK15], [CDRH18], [Küh19]). The main idea is to consider the Markov transition semigroup $P_t$ associated with $\mathcal{L}^{ou}$ and then, in the elliptic case, to use the Laplace transform formula in order to represent the unique distributional solution $u$ of Equation (1.5) as:

$$u(x) = \int_0^\infty e^{-\lambda t} [P_t g](x) dt =: \int_0^\infty e^{-\lambda t} P_t g(x) dt, \quad x \in \mathbb{R}^N.$$

In the parabolic setting, we exploit instead the variation of constants (or Duhamel) formula in order to show a similar representation for the weak solution of the Cauchy problem (1.6):

$$u(t, x) = P_t u_0(x) + \int_0^t \left[P_{t-s} f(s, \cdot)\right](x) ds =: P_t u_0(x) + \int_0^t P_{t-s} f(s, x) ds, \quad t \in [0, T], \quad x \in \mathbb{R}^N.$$

In order to prove global regularity estimates for the solutions, the crucial point is to understand the action of the operator $P_t$ on the anisotropic Hölder spaces. In particular, we will show in Corollary 5 the continuity of $P_t$ as an operator from $C^{\beta}_{b,d}\left(\mathbb{R}^N\right)$ to $C^{\gamma}_{b,d}\left(\mathbb{R}^N\right)$ for $\beta < \gamma$ and, more precisely, that it holds:

$$\|P_t \phi\|_{C^{\gamma}_{b,d}} \leq C \|\phi\|_{C^{\beta}_{b,d}} \left(1 + t^{-\frac{\gamma - \beta}{\alpha}}\right), \quad t > 0.$$  

(1.7)
The above estimate can be obtained through interpolation techniques (see Equation (4.8)), once sharp controls in supremum norm (Theorem 4 below) are established for the spatial derivatives of $P_t\phi$ when $\phi \in C_c^{\alpha,\beta}(\mathbb{R}^N)$. We think that such an estimate (1.7) and the controls in Theorem 4 can be of independent interest and used also beyond our scope in other contexts.

We face here two main difficulties to overcome. While in the gaussian setting, $L^\infty$-estimates of this type have been established exploiting, for example, explicit formulas for the density of the semigroup $P_t$ ([Lun97]), a priori controls of Bernstein type combined with interpolation methods ([Lor05] and [Sai07], when $n = 2$ in (2.2) below) or probabilistic representations of the semigroup $P_t$, allowing Malliavin calculus ([Pri09]), we cannot rely on these techniques in our non-local framework, mainly due to the lower integrability properties for $P_t$. Instead, we are going to use a \textit{perturbative approach} which consists in considering the Lévy operator $\mathcal{L}$ as a perturbation, in a suitable sense, of an $\alpha$-stable operator, at least for the associated small jumps. Indeed, we can 'decompose' the operator $\mathcal{L}$ in a smoother part, $\mathcal{L}^\alpha$, whose Lévy measure is given by

$$\mu(d\theta)\frac{1_{(0,r_0)}(r)}{r^{1+\alpha}}dr$$

and a remainder part. It is precisely condition $[\text{ND}]$ that allows such a decomposition, since it ensures the positivity of the Lévy measure

$$d\nu - d\mu\frac{1_{(0,r_0)}(r)}{r^{1+\alpha}}dr$$

associated with the remainder term. The main difference with the previous techniques in the diffusive setting is that we will work mainly on the truncated $\alpha$-stable contribution $\mathcal{L}^\alpha$, being the remainder term only bounded.

Following [SSW12], we will establish that the Hartman-Winter condition holds, ensuring the existence of a smooth density for the semigroup associated with $\mathcal{L}^\alpha$ and then, the required gradient estimates. Indeed, assumption $[\text{ND}]$ roughly states that the small jump contributions of $\nu$, the ones responsible for the creation of a density, are controlled from below by an $\alpha$-stable measure, whose absolute continuity is well-known in our framework.

On the other hand, we will have to deal with the degeneracy of the operator $\mathcal{L}$, that acts non-degenerately, through the embedding matrix $B$, only on a subspace of dimension $d$. It will be managed adapting the reasonings firstly appeared in [HM16]. Namely, we will show that the semigroup associated with the Ornstein-Uhlenbeck operator $\mathcal{L}^{ou}$ coincides with a non-degenerate one but "multiplied" by a time-dependent matrix that precisely takes into account the original degeneracy of the operator (see definition of matrix $M_t$ in Section 2.1).

**Main Operators Considered.** We conclude this introduction showing that assumption $[\text{ND}]$ applies to a large class of Lévy operators on $\mathbb{R}^d$. As already pointed out in [SSW12], it is satisfied by any Lévy measure $\nu$ that can be decomposed in polar coordinates as

$$\nu(C) = \int_0^\infty \int_{S^{d-1}} 1_C(r\theta)Q(r,\theta) \mu(d\theta) \frac{dr}{r^{1+\alpha}}, \quad C \in \mathcal{B}(R^d_0),$$

for a finite, non-degenerate (in the sense of Equation (1.4)), measure $\mu$ on $S^{d-1}$ and a Borel function $Q: (0,\infty) \times S^{d-1} \rightarrow \mathbb{R}$ such that there exists $r_0 > 0$ so that

$$Q(r,\theta) \geq c > 0, \quad \text{a.e. in } [0,r_0] \times S^{d-1}.$$

In particular, assumption $[\text{ND}]$ holds for the following families of "stable-like" examples with $\alpha \in (0,2)$:

1. Stable operator [Sat13]:

   $$Q(r,\theta) = 1;$$

2. Truncated stable operator with $r_0 > 0$ [KS08]:

   $$Q(r,\theta) = 1_{(0,r_0)}(r);$$

3. Layered stable operator with $\beta$ in $(0,2)$ and $r_0 > 0$ [HK07]:

   $$Q(r,\theta) = 1_{(0,r_0)}(r) + 1_{(r_0,\infty)}(r)r^{-\beta};$$
4. Tempered stable operator [Ros07]:

\[ Q(\gamma, \theta) \text{ completely monotone, } Q(0, \theta) > 0 \text{ and } Q(\infty, \theta) = 0 \text{ a.e. in } S^{d-1}; \]

5. Relativistic stable operator [CMS90], [BMeR99]:

\[ Q(r, \theta) = (1 + r)^{(d+\alpha-1)/2} e^{-r}; \]

6. Lamperti stable operator with \( f: S^{d-1} \to \mathbb{R} \) such that \( \sup f(\theta) < 1 + \alpha \) [CPP10]:

\[ Q(r, \theta) = e^{f(\theta)} \left( \frac{r}{e^r - 1} \right)^{1+\alpha}. \]

**Organization of Paper.** The article is organized as follows. Section 2 introduces some useful notations and then, the anisotropic distance \( d \) induced by the dynamics as well as Zygmund-Hölder spaces associated with such a distance. In Section 3, we are going to show some analytical properties of the semigroup \( P_t \) generated by \( \mathcal{L}^{ou} \), such as the existence of a smooth density and, at least for small times, some controls for its derivatives. Section 4 is then dedicated to different estimates in the \( L^\infty \)-norm for \( P_t f \) and its spatial derivatives, involving the supremum or the Hölder norm of the function \( f \). In particular, we show here the continuity of \( P_t \) as an operator between anisotropic Zygmund-Hölder spaces. In Section 5, we use the controls established in the previous parts in order to prove the elliptic Schauder estimates and show that Equation (1.5) has a unique weak solution. Similarly, we establish the weak well-posedness of the Cauchy problem (1.6) as well as the associated parabolic Schauder estimates. In the final section of the article, we briefly explain some possible extensions of the previous results to non-linear, space-time dependent operators.

## 2 Geometry of the Dynamics

In this section, we are going to choose the right functional space "in which" to state our Schauder estimates. The idea is to construct an Hölder space \( C^d \) with such a distance. In Section 2.1, we are going to choose the right functional space "in which" to state our Schauder estimates. The idea is to construct an Hölder space \( C^d \) with such a distance. In Section 2.1, we are going to choose the right functional space "in which" to state our Schauder estimates.

### 2.1 The distance associated with the Dynamics

To construct the suitable distance \( d \), we start noticing that the Kalman rank condition [K] allows us to denote

\[ n := \min\{ r \in \mathbb{N} : N = \text{rank} \{ B, AB, \ldots, A^{r-1} B \} \}. \]

Clearly, \( n \) is in \([1, N]\), where \([\cdot, \cdot]\) denotes the set of all the integers in the interval, and \( n = 1 \) if and only if \( d = N \), i.e. if the dynamics is non-degenerate.

As done in [Lun97], the space \( \mathbb{R}^N \) will be decomposed with respect to the family of linear operators \( B, AB, \ldots, A^{n-1} B \). We start defining the family \( \{ V_h : h \in [1, n] \} \) of subspaces of \( \mathbb{R}^N \) through

\[ V_h := \begin{cases} \text{Im}(B), & \text{if } h = 1, \\ \bigoplus_{k=1}^h \text{Im}(A^{k-1}B), & \text{otherwise}. \end{cases} \]

It is easy to notice that \( V_h \neq V_k \) if \( k \neq h \) and \( V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{R}^N \). We can then construct iteratively the family \( \{ E_h : h \in [1, n] \} \) of orthogonal projections from \( \mathbb{R}^N \) as

\[ E_h := \begin{cases} \text{projection on } V_1, & \text{if } h = 1; \\ \text{projection on } (V_{h-1})^\perp \cap V_h, & \text{otherwise}. \end{cases} \]
With a small abuse of notation, we will identify the projection operators $E_h$ with the corresponding matrices in $\mathbb{R}^N \otimes \mathbb{R}^N$. It is clear that $\dim E_1(\mathbb{R}^N) = d$. Let us then denote $d_1 := d$ and 
\[ d_h := \dim E_h(\mathbb{R}^N), \quad \text{for } h > 1. \]

We can define now the distance $d$ through the decomposition $\mathbb{R}^N = \bigoplus_{h=1}^n E_h(\mathbb{R}^N)$ as
\[ d(x, x') := \sum_{h=1}^n |E_h(x - x')|^{-1/h}. \]

The above distance can be seen as a generalization of the usual Euclidean distance when $n = 1$ (non-degenerate dynamics) as well as an extension of the standard parabolic distance for $\alpha = 2$. It is important to highlight that it does not induce a norm since it lacks of linear homogeneity. The anisotropic distance $d$ can be understood direction-wise: we firstly fix a "direction" $h$ in $[1, n]$ and then calculate the standard Euclidean distance on the associated subspace $E_h(\mathbb{R}^N)$, but scaled according to the dilation of the system in that direction. We conclude summing the contributions associated with each component. The choice of such a dilation will be discussed thoroughly in the example at the end of this section.

As emphasized by the result from Lanconelli and Polidoro recalled below (cf. [LP94], Proposition 2.1), the decomposition of $\mathbb{R}^N$ with respect to the projections $\{E_h : h \in [1, n]\}$ determines a particular structure of the matrices $A$ and $B$. It will be often exploited in the following.

**Theorem 1** ([LP94]). Let $\{e_i : i \in [1, N]\}$ be an orthonormal basis consisting of generators of $\{E_h(\mathbb{R}^N) : h \in [1, n]\}$. Then, the matrices $A$ and $B$ have the following form:

\[
B = \begin{pmatrix} B_0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} * & * & \ldots & \ldots & * \\ A_2 & * & \ldots & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & A_3 & * & \ldots & * \\ \vdots & \vdots & \ldots & \ldots & * \end{pmatrix}
\]

(2.3)

where $B_0$ is a non-degenerate matrix in $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1}$ and $A_h$ are matrices in $\mathbb{R}^{d_h} \otimes \mathbb{R}^{d_h-1}$ with $\text{rank}(A_h) = d_h$ for any $h$ in $[2, n]$. Moreover, $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$.

Applying a change of variables if necessary, we will assume from this point further to have fixed such a canonical basis $\{e_i : i \in [1, N]\}$. For notational simplicity, we denote by $I_h$, $h \in [1, n]$, the family of indexes $i$ in $[1, N]$ such that $\{e_i : i \in I_h\}$ spans $E_h(\mathbb{R}^N)$.

The particular structure of $A$ and $B$ given by Theorem 1 allows us to decompose accurately the exponential $e^{tA}$ of the matrix $A$ in order to make the intrinsic scale of the system appear. Further on, we will consider fixed a time-dependent matrix $M_t$ on $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

\[ M_t := \text{diag}(I_{d_1 \times d_1}, tI_{d_2 \times d_2}, \ldots, t^{n-1}I_{d_n \times d_n}), \quad t \geq 0. \]

**Lemma 1.** There exists a time-dependent matrix $\{R_t : t \in [0, 1]\}$ in $\mathbb{R}^N \otimes \mathbb{R}^N$ such that

\[
e^{tA}M_tR_t = M_tR_t, \quad t \in [0, 1].
\]

Moreover, there exists a constant $C > 0$ such that for any $t$ in $[0, 1],$

- any $l, h$ in $[1, n]$ and any $\theta$ in $\mathbb{S}^{N-1}$, it holds that
  \[
  |E_l e^{tA}E_h \theta| \leq \begin{cases} Cl^{-h}, & \text{if } l \geq h \\
  Ct, & \text{if } l < h.
  \end{cases}
  \]

- any $\theta$ in $\mathbb{S}^{d-1}$, it holds that
  \[
  |R_tB\theta| \geq C^{-1}.
  \]
Proof. By definition of the matrix exponential, we know that

\[ E_1 e^{tA} E_h = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_1 A^k E_h. \]  

(2.5)

Using now the representation of \( A \) given by Theorem 1, it is easy to check that \( E_1 A^k E_h = 0 \) for \( k < l - h \) (when \( l - h \) is non-negative). Thus, for \( l \geq h \), it holds that

\[ |E_1 e^{tA} E_h \theta| = \frac{1}{\|E(h)\|} \sum_{k=h-l-1}^{\infty} \frac{t^k}{k!} |E_1 A^k E_h \theta| \leq Ct^{l-h}, \]

where we exploited that \( t \) is in \([0,1]\) and \( |\theta| = 1 \). Assuming instead that \( l < h \), it is clear that \( E_t I_{N \times N} E_h \) vanishes. We can then write that

\[ |E_1 e^{tA} E_h \theta| = \sum_{k=0}^{\infty} \frac{t^k}{k!} |E_1 A^k E_h \theta| \leq Ct, \]

using again that \( t \) is in \([0,1]\) and \( |\theta| = 1 \). To show the other control, we highlight that the matrix \( M_t \) is not invertible in \( t = 0 \) and for this reason, we define the time-dependent matrix \( R_t \) as

\[ R_t := \begin{cases} I_{N \times N}, & \text{if } t = 0; \\ M_t^{-1} e^{tA} M_t, & \text{if } t \in (0,1]. \end{cases} \]

We could have also defined \( R_t := (\tilde{R}_t^s)_{s=1} \) where \( \tilde{R}_t^s \) solves the following ODE:

\[ \begin{aligned} \partial_s \tilde{R}_t^s &= M_t^{-1} t A M_t \tilde{R}_t^s, & \text{on } (0,1], \\ \tilde{R}_0^s &= I_{N \times N}. \end{aligned} \]

Equivalently, \( \tilde{R}_t^s \) is the resolvent matrix associated with \( M_t^{-1} t A M_t \), whose sub-diagonal entries are 'macroscopic' from the structure of \( A \) and \( M_t \).

It follows immediately that Equation (2.4) holds. Moreover, we notice that

\[ |R_t B \theta| \geq |E_1 R_t B \theta| = |E_1 e^{tA} E_1 B \theta|. \]

Remembering the definition of matrix exponential (Equation (2.5) with \( l = h = 1 \)), we use now that

\[ E_1 A^k E_1 = (E_1 A E_1)^k = (A_{1,1})^k E_1, \]

where in the last expression the multiplication is meant block-wise, in order to conclude that

\[ |R_t B \theta| \geq |e^{tA_{1,1}} B_0 \theta|. \]

Using that \( e^{tA_{1,1}} B_0 \) is non-degenerate and continuous in time and that \( \theta \) is in \( S^{d-1} \), it is easy to conclude. \( \square \)

We conclude this sub-section with a simpler example taken from [HMP19]. We hope that it will help the reader to understand the introduction of the anisotropic distance \( d \).

**Example.** Fixed \( N = 2d, n = 2 \) and \( d = d_1 = d_2 \), we consider the following operator:

\[ \mathcal{L}_\alpha = \Delta_x^\frac{\alpha}{2} + x_1 \cdot \nabla x_2 \quad \text{on } \mathbb{R}^{2d}, \]

where \((x_1, x_2) \in \mathbb{R}^{2d} \) and \( \Delta_x^\frac{\alpha}{2} \) is the fractional Laplacian with respect to \( x_1 \). In our framework, it is associated with the matrixes

\[ A := \begin{pmatrix} I_{d \times d} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} I_{d \times d} \\ 0 \end{pmatrix}. \]
The operator $L^{ou}_\alpha$ can be seen as a generalization of the classical Kolmogorov example (see e.g. [Kol34]) to our non-local setting.

In order to understand how the system typically behaves, we search for a dilation

$$\delta_\lambda: [0, \infty) \times \mathbb{R}^{2d} \rightarrow [0, \infty) \times \mathbb{R}^{2d}$$

which is invariant for the considered dynamics, i.e. a dilation that transforms solutions of the equation

$$\partial_t u(t, x) - L^{ou}_\alpha u(t, x) = 0 \quad \text{on} \ (0, \infty) \times \mathbb{R}^{2d}$$

into other solutions of the same equation.

Due to the structure of $A$ and the $\alpha$-stability of $\Delta^{\frac{\alpha}{2}}$, we can consider for any fixed $\lambda > 0$, the following

$$\delta_\lambda(t, x_1, x_2) := (\lambda^\alpha t, \lambda x_1, \lambda^{1 + \alpha} x_2).$$

It then holds that

$$\left(\partial_t - L^{ou}_\alpha\right) u = 0 \quad \Rightarrow \quad \left(\partial_t - L^{ou}_\alpha\right) (u \circ \delta_\lambda) = 0.$$

Introducing now the complete time-space distance $d_P$ on $[0, \infty) \times \mathbb{R}^{2d}$ given by

$$d_P((t, x), (s, x')) := |s - t|^\frac{\alpha}{2} + d(x, x'),$$

we notice that it is homogenous with respect to the dilation $\delta_\lambda$, so that

$$d_P(\delta_\lambda(t, x); \delta_\lambda(s, x')) = \lambda d_P((t, x); (s, x')).$$

Precisely, the exponents appearing in Equation (2.6) are those which make each space-component homogenous to the characteristic time scale $t^{\frac{1}{\alpha}}$. From a more probabilistic point of view, the exponents in Equation (2.6), can be related to the characteristic time scales of the iterated integrals of an $\alpha$-stable process. It can be easily seen from the example, noticing that the operator $L^{ou}_\alpha$ corresponds to the generator of an isotropic $\alpha$-stable process and its time integral.

Going back to the general setting, the appearance of this kind of phenomena is due essentially to the particular structure of the matrix $A$ (cf. Theorem 1) that allows the smoothing effect of the operator $L$, acting only on the first "component" given by $B_0$, to propagate into the system.

### 2.2 Anisotropic Zygmund-Hölder spaces

We are now ready to define the Zygmund-Hölder spaces $C^{q}_{D, \delta}(\mathbb{R}^N)$ with respect to the distance $d$. We start recalling some useful notations we will need below.

Given a function $f: \mathbb{R}^N \rightarrow \mathbb{R}$, we denote by $D^1 f(x)$, $D^2 f(x)$ and $D^3 f(x)$ the first, second and third Fréchet derivative of $f$ at a point $x$ in $\mathbb{R}^N$ respectively, when they exist. For simplicity, we will identify $D^3 f(x)$ as a 3-tensor so that $[D^3 f(x)](u, v)$ is a vector in $\mathbb{R}^N$ for any $u, v$ in $\mathbb{R}^N$. Moreover, fixed $h$ in $[1, n]$, we will denote by $D_{E_h} f(x)$ the gradient of $f$ at $x$ along the direction $E_h(\mathbb{R}^N)$. Namely,

$$D_{E_h} f(x) := E_h D f(x).$$

A similar notation will be used for the higher derivatives, too.

Given $X, Y$ two real Banach spaces, $\mathcal{L}(X, Y)$ will represent the family of linear continuous operators between $X$ and $Y$.

In the following, $c$ or $C$ denote generic positive constants whose precise value is unimportant. They may change from line to line and they will depend only on the parameters given by the model and assumptions [ND], [K]. Namely, $d, N, A, B, \alpha, \nu, \gamma, \tau_0$ and $\mu$. Other dependencies that may occur will be explicitly specified.

Let us introduce now some function spaces we are going to use. We denote by $B_0(\mathbb{R}^N)$ the family of Borel measurable and bounded functions $f: \mathbb{R}^N \rightarrow \mathbb{R}$. It is a Banach space endowed with the supremum norm $\| \cdot \|_{\infty}$. We will consider also its closed subspace $C_0(\mathbb{R}^N)$ consisting of all the uniformly continuous functions. Fixed some $k$ in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\beta$ in $(0, 1]$, we follow Lunardi [Lun97] denoting the Zygmund-Hölder semi-norm for a function $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$ as

$$[\phi]_{C^{k+\beta}} := \begin{cases} 
\sup_{|\theta| = k} \sup_{x \neq y} \frac{|D^\theta \phi(x) - D^\theta \phi(y)|}{|x - y|^\beta}, & \text{if } \beta \neq 1; \\
\sup_{|\theta| = k} \sup_{x \neq y} \frac{|D^\theta \phi(x) + D^\theta \phi(y) - 2D^\theta \phi(x + y)|}{|x - y|^\beta}, & \text{if } \beta = 1.
\end{cases}$$

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Consequently, The Zygmund-Hölder space $C_{b}^{k+\beta}(\mathbb{R}^{N})$ is the family of functions $\phi: \mathbb{R}^{N} \to \mathbb{R}$ such that $\phi$ and its derivatives up to order $k$ are continuous and the norm

$$
\|\phi\|_{C_{b}^{k+\beta}} := \sum_{i=1}^{k} \sup_{|\theta|=i} \|D^{\theta}\phi\|_{L^{\infty}} + [\phi]_{C_{b}^{k+\beta}} \text{ is finite.}
$$

We can define now the anisotropic Zygmund-Hölder spaces associated with the distance $d$. Fixed $\gamma > 0$, the space $C_{\gamma}^{b,d}(\mathbb{R}^{N})$ is the family of functions $\phi: \mathbb{R}^{N} \to \mathbb{R}$ such that for any $h$ in $[1,n]$ and any $x_0$ in $\mathbb{R}^{N}$, the function

$$
z \in E_h(\mathbb{R}^{N}) \to \phi(x_0 + z) \in \mathbb{R} \text{ belongs to } C_{\gamma}^{b,h}(\mathbb{R}^{N}),
$$

with a norm bounded by a constant independent from $x_0$. It is endowed with the norm

$$
\|\phi\|_{C_{\gamma}^{b,d}} := \sum_{h=1}^{n} \sup_{x_0 \in \mathbb{R}^{N}} \|\phi(x_0 + \cdot)\|_{C_{\gamma}^{b,h}(\mathbb{R}^{N})}.
$$

We highlight that it is possible to recover the expected joint regularity for the partial derivatives, when they exist, as in the standard Hölder spaces. In such a case, they actually turn out to be Hölder continuous with respect to the distance $d$ with order one less than the function (See Lemma 2.1 in [Lun97] for more details).

It will be convenient in the following to consider an equivalent norm in the "standard" Hölder-Zygmund spaces $C_{\gamma}^{b}(E_h(\mathbb{R}^{N}))$ that does not take into account the derivatives with respect to the different directions. We suggest the interested reader to see [Lun97], Equation (2.2) or [Pri09] Lemma 2.1 for further details.

**Lemma 2.** Fixed $\gamma$ in $(0,3)$ and $h$ in $[1,n]$ and $\phi$ in $C_{\gamma}(E_h(\mathbb{R}^{N}))$, let us introduce

$$
\Delta_{x_0}^{3}\phi(z) := \phi(x_0 + 3z) - 3\phi(x_0 + 2z) + 3\phi(x_0 + z) - \phi(x_0), \quad x_0 \in \mathbb{R}^{N}; z \in E_h(\mathbb{R}^{N}).
$$

Then, $\phi$ is in $C_{\gamma}^{b}(E_h(\mathbb{R}^{N}))$ if and only if

$$
\sup_{x_0 \in \mathbb{R}^{N}} \sup_{z \in E_h(\mathbb{R}^{N}); z \neq 0} \frac{|\Delta_{x_0}^{3}\phi(z)|}{|z|^{\gamma}} < \infty.
$$

We conclude this subsection with a result concerning the interpolation between the anisotropic Zygmund-Hölder spaces $C_{\gamma}^{b,d}(\mathbb{R}^{N})$. We refer to Theorem 2.2 and Corollary 2.3 in [Lun97] for details.

**Theorem 2.** Let $r$ be in $(0,1)$ and $\beta, \gamma$ in $[0,\infty)$ such that $\beta \leq \gamma$. Then, it holds that

$$
\left(C_{\beta}^{r}(\mathbb{R}^{N}), C_{\gamma}^{b,d}(\mathbb{R}^{N})\right)_{r,\infty} = C_{\gamma}^{b,d}(\mathbb{R}^{N})
$$

with equivalent norms, where we have denoted for simplicity: $C_{\gamma}^{\infty}(\mathbb{R}^{N}) := C_{b}(\mathbb{R}^{N})$.

### 3 Smoothing Effect for Truncated Density

We present here some analytical properties of the semigroup generated by the operator $L^{au}$. Following [SSW12] and [SW12], we will show the existence of a smooth density for such a semigroup and its anisotropic smoothing effect, at least for small times.

Throughout this section, we consider fixed a stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_{t})_{t \geq 0}, \mathbb{P})$ satisfying the usual assumptions (see [App09], page 72). Let us then consider the (unique in law) Lévy process $\{Z_t\}_{t \geq 0}$ on $\mathbb{R}^{d}$ characterized by the Lévy symbol

$$
\Phi(p) = -ib \cdot p + \frac{1}{2}b \cdot Qp + \int_{\mathbb{R}^{d}} (1 - e^{i p \cdot z} + ip \cdot z 1_{B(0,1)}(z)) \nu(dz), \quad p \in \mathbb{R}^{d}.
$$

It is well-known by the Lévy-Kitchine formula (see [Jac01]), that the infinitesimal generator of the process $\{BZ_t\}_{t \geq 0}$ is then given by $L$ on $\mathbb{R}^{N}$.
Fixed $x$ in $\mathbb{R}^N$, we denote by $\{X_t\}_{t \geq 0}$ the $N$-dimensional Ornstein-Uhlenbeck process driven by $BZ_t$, i.e. the unique (strong) solution of the following stochastic differential equation:

$$
X_t = x + \int_0^t AX_s \, ds + BZ_t, \quad t \geq 0, \quad \mathbb{P}\text{-almost surely.}
$$

By the variation of constants method, it is easy to check that

$$
X_t = e^{tA}x + \int_0^t e^{(t-s)A}B \, dZ_s, \quad t \geq 0, \quad \mathbb{P}\text{-almost surely.} \tag{3.1}
$$

The transition semigroup associated with $\mathcal{L}^{ou}$ is then defined as the family $\{P_t: t \geq 0\}$ of linear contractions on $B_b(\mathbb{R}^N)$ given by

$$
P_t \phi(x) = \mathbb{E}\left[\phi(X_t)\right], \quad x \in \mathbb{R}^N, \quad \phi \in B_b(\mathbb{R}^N). \tag{3.2}
$$

We recall that $P_t$ is generated by $\mathcal{L}^{ou}$ in the sense that its infinitesimal generator $\mathcal{A}$ coincides with $\mathcal{L}^{ou}$ on $C_c^\infty(\mathbb{R}^N)$, the family of smooth functions with compact support.

The next result shows that the random part of $X_t$ (see Equation (3.3)) satisfies again the non-degeneracy assumption [ND], even if re-scaled with respect to the anisotropic structure of the dynamics.

**Proposition 1** (Decomposition). For any $t$ in $(0, 1]$, there exists a Lévy process $\{S^t_u\}_{u \geq 0}$ such that

$$
X_t \overset{\text{law}}{=} e^{tA}x + M_t S^t_t.
$$

Moreover, $\{S^t_u\}_{u \geq 0}$ satisfies assumption [ND] with same $\alpha$ as before.

**Proof.** For simplicity, we start denoting

$$
\Lambda_t := \int_0^t e^{(t-s)A}B \, dZ_s, \quad t > 0, \tag{3.3}
$$

so that $X_t = e^{tA}x + \Lambda_t$. To conclude, we need to construct a Lévy process $\{S^t_u\}_{u \geq 0}$ on $\mathbb{R}^N$ satisfying assumption [ND] and

$$
\Lambda_t \overset{\text{law}}{=} M_t S^t_t. \tag{3.4}
$$

To show the identity in law, we are going to reason in terms of the characteristic functions. By Lemma 2.2 in [SW12], we know that $\Lambda_t$ is an infinitely divisible random variable with associated Lévy symbol

$$
\Phi_{\Lambda_t}(\xi) := \int_0^t \Phi((e^{sA}B)^*\xi) \, ds, \quad \xi \in \mathbb{R}^N.
$$

Remembering the decomposition $e^{sA}B = e^{sA}M_sB = M_sR_sB$ from Lemma 1, we can now rewrite $\Phi_{\Lambda_t}$ as

$$
\Phi_{\Lambda_t}(\xi) = t \int_0^1 \Phi((e^{stA}B)^*\xi) \, ds = t \int_0^1 \Phi((R_sB)^*M_s\xi) \, ds.
$$

The above equality suggests us to define, for any $t$ in $(0, 1]$, the (unique in law) Lévy process $\{S^t_u\}_{u \geq 0}$ associated with the Lévy symbol

$$
\tilde{\Phi}(\xi) := \int_0^1 \Phi((R_sB)^*M_s\xi) \, ds, \quad \xi \in \mathbb{R}^N.
$$

It is not difficult to check that $\tilde{\Phi}$ is indeed a Lévy symbol associated with the Lévy triplet $(\tilde{Q}^t, \tilde{b}^t, \tilde{\nu}^t)$ given by

$$
\tilde{Q}^t = \int_0^1 M_sR_sBQ(M_sR_sB)^* \, ds; \tag{3.5}
$$

$$
\tilde{b}^t = \int_0^1 M_sR_sBb \, ds + \int_0^1 \int_{\mathbb{R}^d} M_sR_sB [1_{B(0,1)}(M_sR_sBz) - 1_{B(0,1)}(z)] \nu(dz) \, ds; \tag{3.6}
$$

$$
\tilde{\nu}^t(C) = \int_0^1 \nu((M_sR_sB)^{-1}C) \, ds, \quad C \in \mathcal{B}(\mathbb{R}^d_0). \tag{3.7}
$$
Since we have that
\[
E[e^{i(\xi, \Lambda_1)}] = e^{-\Phi_\Lambda\ell} = e^{-i\Phi'(M_\ell)} = E[e^{i(\xi, M_\ell S_1)]},
\]
it follows immediately that the identity (3.4) holds. It remains to show that the family of Lévy measure \(\tilde{\nu}^t: t \in (0, 1]\) satisfies the assumption [ND]. Recalling that condition [ND] is assumed to hold for \(\nu\), we know that
\[
\tilde{\nu}^t(C) = \int_0^1 \nu\left((M_\ell R_{s\theta} B)^{-1} C\right) ds \geq \int_0^1 \int_{S^{d-1}} \int_{C} (rM_\ell R_{s\theta} B \theta) \mu(d\theta) \frac{dr}{r^{1+\alpha}} ds,
\]
for any \(C \in \mathcal{B}(\mathbb{R}^d_0)\). Furthermore, it holds from Lemma 1 that
\[
\inf_{s \in (0, 1), \ell \in (0, 1], \theta \in S^{d-1}} |M_\ell R_{s\theta} B \theta| =: R_0 > 0.
\]
It allows us to define two functions \(l^t: [0, 1] \times S^{d-1} \to S^{N-1}\), \(m^t: [0, 1] \times S^{d-1} \to \mathbb{R}\), given by
\[
l^t(s, \theta) := \frac{M_\ell R_{s\theta} B \theta}{|M_\ell R_{s\theta} B \theta|} \quad \text{and} \quad m^t(s, \theta) := |M_\ell R_{s\theta} B \theta|.
\]
Using the Fubini theorem, we can now rewrite Equation (3.8) as
\[
\tilde{\nu}^t(C) \geq \int_0^1 \int_{S^{d-1}} \int_0^{R_0} \mathbb{I}_C(l^t(s, \theta)m^t(s, \theta) r) \frac{dr}{r^{1+\alpha}} \mu(d\theta) ds
\]
\[
= \int_0^1 \int_{S^{d-1}} \int_0^{m^t(s, \theta) R_0} \mathbb{I}_C(l^t(s, \theta) r) \frac{dr}{r^{1+\alpha}} [m^t(s, \theta)]^{\alpha} \mu(d\theta) ds.
\]
Exploiting again Control (3.9), we can conclude that
\[
\tilde{\nu}^t(C) \geq \int_0^1 \int_{S^{d-1}} \int_0^{R_0} \mathbb{I}_C(l^t(s, \theta) r) \frac{dr}{r^{1+\alpha}} \tilde{m}^t(ds, d\theta) = \int_0^{R_0} \int_{S^{d-1}} \mathbb{I}_C(\tilde{\nu}^t r) \tilde{m}^t(ds, d\theta) \frac{dr}{r^{1+\alpha}},
\]
where \(\tilde{m}^t(ds, d\theta)\) is a measure on \([0, 1] \times S^{d-1}\) given by
\[
\tilde{m}^t(ds, d\theta) := [m^t(s, \theta)]^{\alpha} \mu(d\theta) ds
\]
and \(\tilde{\nu}^t := (l^t), \tilde{m}^t\) is the measure \(\tilde{m}^t\) push-forwarded through \(l^t\) on \(S^{N-1}\). It is easy to check that the measure \(\tilde{\nu}^t\) is finite and non-degenerate in the sense of (1.4), replacing therein \(d\) by \(N\).

An immediate application of the above result is a first representation formula for the transition semigroup \(\{P_t: t \geq 0\}\) associated with the Ornstein-Uhlenbeck process \(\{X_t\}_{t \geq 0}\), at least for small times. Indeed, denoting by \(\mathbb{P}_X\) the law of a random variable \(X\), Equation (3.4) implies that for any \(\phi \in \mathcal{B}_b(\mathbb{R}^N)\), it holds that
\[
P_t \phi(x) = \int_{\mathbb{R}^N} \phi(e^{tA} x + y) \mathbb{P}_X(dy) = \int_{\mathbb{R}^N} \phi(e^{tA} x + M_{\ell t} y) \mathbb{P}_{S_1^t}(dy), \quad x \in \mathbb{R}^N, t \in (0, 1].
\]
Moreover, condition [ND] for \(\{S_1^t\}_{t \geq 0}\) allows us to decompose it into two components: a truncated, \(\alpha\)-stable part and a remainder one. Indeed, if we denote by \(\nu_\alpha^t\) the measure serving as lower bound to the Lévy measure \(\tilde{\nu}^t\) in (3.10), i.e.
\[
\nu_\alpha^t(C) := \int_0^{R_0} \int_{S^{d-1}} \mathbb{I}_C(r \theta) \tilde{\nu}^t(d\theta) \frac{dr}{r^{1+\alpha}}, \quad C \in \mathcal{B}(\mathbb{R}_0^d),
\]
we can consider \(\{Y_t^\alpha\}_{t \geq 0}\) the Lévy process on \(\mathbb{R}^N\) associated with the Lévy triplet \((0, 0, \nu_\alpha^t)\). We recall now a useful fact involving the Lévy symbol \(\Phi_\alpha^t\) of the process \(Y^\alpha\). The non-degeneracy of the measure \(\tilde{\nu}^t\) is equivalent to the existence of a constant \(C > 0\) such that
\[
\Phi_\alpha^t(\xi) \geq C|\xi|^\alpha, \quad \xi \in \mathbb{R}^N.
\]
A proof of this result can be found, for example, in [Pri12] p.424.
In order to apply the results in [SSW12], we are going to truncate the above process at the typical time scale for an \( \alpha \)-stable process. This is \( t^{1/\alpha} \) when considering the process at time \( t \) (cf. Example 2.1). Namely, we consider the family \( \{ \mathbb{P}^T_t \}_{t \geq 0} \) of infinitely divisible probabilities whose characteristic function has the form \( \Phi^T_t(\xi) := \exp[-\Phi^T_t(\xi)] \), where
\[
\Phi^T_t(\xi) := \int_{|z| \leq t^{1/\alpha}} \left[ 1 - e^{i(\xi \cdot z) + iz} \right] \nu^T_t(\xi, z) \, dz.
\]

On the other hand, since the measure \( \tilde{\nu}^t \) satisfies assumption [ND], we know that the remainder \( \tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})} \nu^T_0 \) is again a Le\'vy measure on \( \mathbb{R}^N \). Let \( \{ \pi_t \}_{t \geq 0} \) be the family of infinitely divisible probability associated with the following Lévy triplet:
\[
(\tilde{Q}^t, \tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})} \nu^T_0).
\]
It follows immediately that \( \mathbb{P}_{\mathcal{S}^t} = \mathbb{P}^T_t \ast \pi_t \) for any \( t > 0 \). We can now disintegrate the measure \( \mathbb{P}_{\mathcal{S}^t} \) in Equation (3.11) in order to obtain
\[
P_t(\phi(x)) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(e^{tA}x + M_t(y_1 + y_2)) \mathbb{P}^T_t(dy_1) \pi_t(dy_2). \tag{3.14}
\]

The next step is to use Proposition 2.3 in [SSW12] to show a smoothing effect for the family of truncated stable measures \( \{ \mathbb{P}^T_t : t \geq 0 \} \), at least for small times. Namely,

**Proposition 2.** Fixed \( m \) in \( \mathbb{N}_0 \), there exists \( T_0 := T_0(m) > 0 \) such that for any \( t \) in \( (0, T_0) \), the probability \( \mathbb{P}^T_t \) has a density \( \rho^T(t, \cdot) \) that is \( m \)-times continuously differentiable on \( \mathbb{R}^N \).
Moreover, for any \( \vartheta \) in \( \mathbb{N}^N \) such that \( |\vartheta| \leq m \), there exists a constant \( C := C(m, |\vartheta|) \) such that
\[
|D^\vartheta \rho^T(t, y)| \leq Ct^{-\|\vartheta\|/\alpha} \left( 1 + \frac{|y|}{t^{1/\alpha}} \right)^{|\vartheta| - m}, \quad t \in (0, T_0], \ y \in \mathbb{R}^N.
\]

**Proof.** The result follows immediately applying Proposition 2.3 in [SSW12]. To do so, we need to show that the Lévy symbol \( \Phi^m_0 \) of the process \( \{Y^m_0\}_{u \geq 0} \) satisfies the following assumptions:

- **Hartman-Wintner condition.** There exists \( T > 0 \) such that
  \[
  \liminf_{|\xi| \to \infty} \frac{\text{Re} \Phi^m_0(\xi)}{\ln(1 + |\xi|)} = \infty, \quad t \in (0, T];
  \]

- **Controllability condition.** There exist \( T > 0 \) and \( c > 0 \) such that
  \[
  \int_{\mathbb{R}^N} e^{-ct \text{Re} \Phi^m_0(\xi)} |\xi|^m \leq ct^{-\frac{m}{\alpha}}, \quad t \in (0, T].
  \]

In order to show that the above conditions hold, we fix \( T \leq 1 \) and we recall that the Lévy symbol \( \Phi^m_0 \) of \( Y_0 \), the truncated \( \alpha \)-stable process with Lévy measure introduced in (3.12), can be written through the Lévy-Kitchine formula as
\[
\Phi^m_0(\xi) = \int_{\mathbb{R}^N} (1 - e^{i(\xi \cdot z) + iz}) \nu^m_0(\xi, z) \, dz = \int_0^{R_0} \int_{\mathbb{S}^{N-1}} (1 - \cos(|\xi| r \theta)) \tilde{\mu}^m(d\theta) \frac{dr}{r^{1+\alpha}}.
\]
We have seen in Equation (3.13) that the non-degeneracy of \( \tilde{\mu}^m \) implies that \( \Phi^m_0(\xi) \geq C|\xi|^\alpha \). The Hartman-Wintner condition then follows immediately since
\[
\liminf_{|\xi| \to \infty} \frac{\text{Re} \Phi^m_0(\xi)}{\ln(1 + |\xi|)} \geq \liminf_{|\xi| \to \infty} \frac{c|\xi|^\alpha}{\ln(1 + |\xi|)} = \infty.
\]
To show instead the controllability assumption, let us firstly notice that
\[
e^{-ct \text{Re} \Phi^m_0(\xi)} \leq \begin{cases} 1, & \text{if } |\xi| \leq R; \\ e^{-cR|\xi|^\alpha}, & \text{if } |\xi| > R,
\end{cases}\]

\[
\int_{\mathbb{R}^N} e^{-ct \text{Re} \Phi^m_0(\xi)} |\xi|^m \leq ct^{-\frac{m}{\alpha}}, \quad t \in (0, T].
\]
for some $R > 0$. It then follows that 

$$\int_{\mathbb{R}^N} e^{-t\text{Re} \Phi_i(t)} |\xi|^m \, d\xi = \int_{|\xi| \leq R} |\xi|^m \, d\xi + \int_{|\xi| > R} e^{-c|\xi|^\alpha} |\xi|^m \, d\xi$$

$$\leq C + t^{-\frac{m+n}{\alpha}} \int_{|\xi| > t^{1/\alpha} R} e^{-c|\xi|^\alpha} |\xi|^m \, d\xi$$

$$\leq C + t^{-\frac{m+n}{\alpha}} \int_{\mathbb{R}^N} e^{-c|\xi|^\alpha} |\xi|^m \, d\xi$$

$$\leq Ct^{-\frac{m+n}{\alpha}} ,$$

where in the last step we used that $1 \leq t^{-\frac{m+n}{\alpha}}$.

\section{Estimates for Transition Semigroup}

The results in the previous section (Proposition 2 and Equation (3.14)) allow us to represent the semigroup $P_t$ of the Ornstein-Uhlenbeck process $\{X_t\}_{t \geq 0}$ as

$$P_t \phi(x) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(M_t(y_1 + y_2) + e^{tA} x) p^T(t, y_1) d_1 \pi_t(dy_1) d_2 , \quad x \in \mathbb{R}^N ,$$

(4.1)

at least for small time intervals.

Here, we will focus on estimates in $\| \cdot \|_\infty$-norm of the transition semigroup $\{P_t : t \geq 0\}$ given in Equation (3.2) and its derivatives. The main result in this section is Corollary 5 that shows the continuity of $P_t$ between anisotropic Zygmund-Hölder spaces. These controls will be fundamental in the next section to prove Schauder Estimates in the elliptic and parabolic settings.

As we will see in the following result, the derivatives of the semigroup $P_t$ with respect to a component $i$ in $I_h$ induces an additional time singularity of order $\frac{1+n(h-1)}{\alpha}$, corresponding to the intrinsic time scale of the considered component.

\textbf{Proposition 3.} Let $h, h', h''$ be in $[1, n]$ and $\phi$ in $B_0(\mathbb{R}^N)$. Then, there exists a constant $C > 0$ such that for any $i$ in $I_h$, any $j$ in $I_{h'}$, and any $k$ in $I_{h''}$, it holds that

$$\|D_{i,j,k} P_t \phi\|_\infty \leq C \|\phi\|_\infty (1 + t^{-\frac{1+n(h-1)}{\alpha}}), \quad t > 0 ;$$

(4.2)

$$\|D_i^2 P_t \phi\|_\infty \leq C \|\phi\|_\infty (1 + t^{-\frac{2+n(h-1)+h'}{\alpha}}), \quad t > 0 ;$$

(4.3)

$$\|D_i^3 P_t \phi\|_\infty \leq C \|\phi\|_\infty (1 + t^{-\frac{3+n(h-1)+h'+h''}{\alpha}}), \quad t > 0 .$$

(4.4)

\textbf{Proof.} We start fixing a time horizon $T := 1 \wedge T_0(N+4) > 0$, where $T_0(m)$ was defined in Proposition 2. Our choice of $N + 4$ is motivated by the fact that we consider derivatives up to order 3.

On the interval $(0,T)$, the representation formula (4.1) holds and $P_t \phi$ is three times differentiable for any $\phi$ in $B_0(\mathbb{R}^N)$. We are going to show only Estimate (4.2) since the controls for the higher derivatives can be obtained similarly.

Fixed $t \leq T$, let us consider $i$ in $I_h$ for some $h$ in $[1, n]$. When $t \leq T$, we recall from Equation (4.1) that, up to a change of variables, it holds that

$$|D_i P_t \phi(x)| = |D_i \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(M_t(y_1 + y_2) + e^{tA} x) p^T(t, y_1) d_1 \pi_t(dy_1) d_2| .$$

We can then move the derivative inside the integral and write that

$$|D_i P_t \phi(x)| = \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(M_t(y_1 + y_2))(\nabla p^T(t, y_1 - M_t^{-1} e^{tA} x)) d_1 \pi_t(dy_1) d_2 \right|$$

$$\leq |M_t^{-1} e^{tA} e_i| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi(M_t(y_1 + y_2))| |\nabla p^T(t, y_1 - M_t^{-1} e^{tA} x)| d_1 \pi_t(dy_1) d_2$$

(4.5)

$$\leq Ct^{-\left(h-1\right)} \|\phi\|_\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\nabla p^T(t, y_1)| d_1 \pi_t(dy_1) d_2 .$$
where in the last step we exploited Lemma 1 to control
\[ |M_t^{-1} e^{tA} e_i| \leq \sum_{k=1}^n |M_t^{-1} E_k e^{tA} E_k e_i| \leq C \left( \sum_{k=1}^{h-1} t^{k-h} t + \sum_{k=h}^{n} t^{-(k-1)h} \right) \leq C t^{-(h-1)}, \]
remembering that \( t \leq 1 \). We conclude the case \( t \leq T \) using the control on \( p^T \) (Proposition 2 with \( m = N+2 \)) to write that
\[ |D_r P_t \phi(x)| \leq C \| \phi \|_{\infty} \pi_t(\mathbb{R}^N) t^{-(h-1)} \int_{\mathbb{R}^N} t^{-\frac{\gamma}{\alpha}} \left( 1 + \frac{|y_1|}{t^{1/\alpha}} \right)^{-(N+1)} dy_1 \]
\[ \leq C \| \phi \|_{\infty} t^{-\frac{h-1}{\alpha}} \int_{\mathbb{R}^N} (1 + |z|)^{-N} dz \]  \( (4.6) \)
\[ \leq C \| \phi \|_{\infty} t^{-\frac{1+\alpha(h-1)}{\alpha}}. \]

Above, we used the change of variables \( z = t^{-1/\alpha} y_1 \). When \( t > T \), we can exploit the already proven controls for small times, the semigroup and the contraction properties of \( \{P_t: t \geq 0\} \) on \( B_0(\mathbb{R}^N) \) to write that
\[ \|D_r P_t \phi\|_{\infty} = \|D_r P_T(P_{t-T} \phi)\|_{\infty} \leq C_T \|P_{t-T} \phi\|_{\infty} \leq C \| \phi \|_{\infty}. \]  \( (4.7) \)
We have thus shown Control (4.2) for any \( t > 0 \).

The following interpolation inequality (see e.g. [Tri92])
\[ \| \phi \|_{C^\gamma_{b^1}} \leq C \| \phi \|_{L^r_{b}^{(1+\gamma)r_2}} \]  \( (4.8) \)
valid for \( 0 \leq \delta_1 < \delta_2, r \in (0, 1) \) and \( \phi \) in \( C^{\delta_2}(\mathbb{R}^N) \), allows us to extend easily the above result.

**Corollary 3.** Let \( \gamma \) be in \( [0, 1 + \alpha) \). Then, there exists a constant \( C > 0 \) such that
\[ \|P_t \|_{C_{b}^{\gamma}(C_{b,d})} \leq C \left( 1 + t^{-\frac{\gamma}{2}} \right), \quad t > 0. \]  \( (4.9) \)

**Proof.** Let us firstly assume that \( \gamma \) is in \( (0, 1] \). Remembering the definition of \( C_{b,d}^{\gamma} \)-norm in (2.7), we start fixing a point \( x_0 \) in \( \mathbb{R}^N \) and \( h \) in \( [2, n] \). Then, the contraction property of the semigroup implies that
\[ \|P_t \phi(x_0 + \cdot)_{E_b(\mathbb{R}^N)}\|_{\infty} \leq C \| \phi \|_{\infty}. \]
Moreover, Control (4.2) in Proposition 3 ensures that
\[ \|D_r P_t \phi(x_0 + \cdot)_{E_b(\mathbb{R}^N)}\|_{\infty} \leq C \| \phi \|_{\infty} \left( 1 + t^{-\frac{1+\alpha(h-1)}{\alpha}} \right). \]
It follows immediately that
\[ \|P_t \phi(x_0 + \cdot)_{E_b(\mathbb{R}^N)}\|_{C_b^1} \leq C \| \phi \|_{\infty} \left( 1 + t^{-\frac{1+\alpha(h-1)}{\alpha}} \right). \]
We can now apply the interpolation inequality (4.8) with \( \delta_1 = 0, \delta_2 = 1 \) and \( r = \gamma/(1 + \alpha(h - 1)) \) in order to obtain that
\[ \|P_t \phi(x_0 + \cdot)_{E_b(\mathbb{R}^N)}\|_{C_b^1} \leq C \|P_t \phi(x_0 + \cdot)_{E_b(\mathbb{R}^N)}\|_{C_b^1} \|P_t \phi(x_0 + \cdot)_{E_b(\mathbb{R}^N)}\|_{\infty}^{-r} \leq C \| \phi \|_{\infty} \left( 1 + t^{-\frac{\gamma}{2}} \right). \]
The argument is analogous for \( \gamma \) in \( (1, 3) \), considering only the case \( h = 0 \).

The next result allows us to extend the controls in Proposition 3 to functions in the anisotropic Zygmund-Hölder spaces. Roughly speaking, it states that the anisotropic \( \gamma \)-Hölder regularity induces a *homogeneous* gain in time of order \( \gamma/\alpha \) that can be used to weaken, at least partially, the time singularities associated with the derivatives. The general argument of proof will mimic the one of Proposition 3 even if, this time, we will need to make the Hölder modulus of \( \phi \) appear. It will be managed introducing an auxiliary function \( K \) (see Equation (4.15)).
Theorem 4. Let $h, h', h''$ be in $[1, n]$ and $\phi$ in $C^{\gamma}_{b,d}(\mathbb{R}^N)$ for some $\gamma$ in $[0, 1 + \alpha)$. Then, there exists a constant $C > 0$ such that for any $i$ in $I_h$, any $j$ in $I_{h'}$ and any $k$ in $I_{h''}$, it holds that

$$\|D_iP_t\phi\|_\infty \leq C\|\phi\|_{C^{\gamma}_{b,d}}(1 + t^{\frac{-(1+\alpha)(h-1)}{\alpha}}), \quad t > 0;$$

$$\|D^2_{i,j}P_t\phi\|_\infty \leq C\|\phi\|_{C^{\gamma}_{b,d}}(1 + t^{\frac{-(2+\alpha)(h'-1)}{\alpha}}), \quad t > 0;$$

$$\|D^3_{i,j,k}P_t\phi\|_\infty \leq C\|\phi\|_{C^{\gamma}_{b,d}}(1 + t^{\frac{-(3+\alpha)(h''-1)}{\alpha}}), \quad t > 0. \quad (4.12)$$

Proof. Similarly to Proposition 3, we start fixing a time horizon

$$T := 1 \land T_0(N + 6) > 0. \quad (4.13)$$

Then, Corollary 3 implies the Controls $P_t$ on $C^{\gamma}_{b,d}(\mathbb{R}^N)$, for any $t \geq T/2$. Indeed,

$$\|P_t\phi\|_{C^{\gamma}_{b,d}} \leq \|P\|_\infty(1 + t^{-\frac{\alpha}{\alpha}}) \leq CT\|\phi\|_{C^{\gamma}_{b,d}}. \quad (4.14)$$

The same argument shown in Equation (4.7) can now be applied to prove Controls (4.10) for $t > T$. Namely,

$$\|D_iP_t\phi\|_\infty = \|D_iP_{T/2}(P_{T-T/2}\phi)\|_\infty \leq CT\|P_{T-T/2}\phi\|_\infty \leq C\|\phi\|_\infty. \quad (4.15)$$

The same reasoning can be used for the higher derivatives, too.

When $t \leq T$, let us assume $\alpha > 1$, so that $1 + \alpha > 2$. The case $\alpha \leq 1$ can be handled similarly taking into account one less derivative. Moreover, we notice that we need to prove Controls (4.10)-(4.12) only for $\gamma$ in $(2, 1 + \alpha)$ thanks to interpolation techniques. Indeed, if we want, for example, to prove Estimates (4.10) for some $\gamma'$ in $(0, 2]$, we can use Theorem 2 to show that

$$\|D_iP_t\|_{L^\infty(C^{\gamma'}_{b,d}; B_h)} \leq (\|D_iP_{T/2}\|_{L^\infty(B_h)})^{1-\gamma'/\gamma}(\|D_iP_t\|_{L^\infty(C^{\gamma'}_{b,d}; B_h)})^{\gamma'/\gamma} \leq C(1 + t^{\frac{\gamma'-\gamma}{\alpha(1+h-1)})},$$

once we have proven Estimate (4.10) for $\gamma > 2$.

We are only going to show Control (4.10) for $t \leq T$ and $\gamma$ in $(2, 1 + \alpha)$. The estimates (4.11), (4.12) for higher derivatives can be obtained analogously.

Fixed $i$ in $I_h$ for some $h$ in $[1, n]$, we start writing from Equation (4.1) that

$$D_iP_t\phi(x) = D_i\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(M_t(y_1 + y_2) + e^{tA}x)p^{t}(t, y_1)dy_1 \pi_t(dy_2).$$

Moreover, we introduce the function $K: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ given by

$$K(y_1, y_2, \xi) := \phi(M_t(y_2 + \xi) + (D_i\phi(M_t(y_2 + \xi))E_1M_t(y_1) + \frac{1}{2}(D^2_i\phi(M_t(y_2 + \xi))E_1M_t(y_1), E_1M_t(y_1). \quad (4.15)$$

We then notice that the expression

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} p^{t}(t, y_1)K(y_1, y_2, \xi)dy_1 \pi_t(dy_2)$$

does not depend on $x$. Recalling $D_i$ stands for the derivative with respect to the variable $x_i$, we thus get that

$$D_i\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} p^{t}(t, y_1)K(y_1, y_2, \xi)dy_1 \pi_t(dy_2) = 0, \quad \xi \in \mathbb{R}^N.$$

This property will allow to use a cancellation argument in Equation (4.17) below, once we split the small jumps in the non-degenerate contributions and the other ones. It now follows that

$$D_iP_t\phi(x) = D_i\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} [\phi(M_t(y_1 + y_2) + e^{tA}x) - K(y_1, y_2, \xi)]p^{t}(t, y_1)dy_1 \pi_t(dy_2).$$

The same reasoning used in Equation (4.5) can be applied here to show that, for any (fixed) $\xi$ in $\mathbb{R}^N$, it holds that

$$|D_iP_t\phi(x)| \leq C t^{-(1-h)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\phi(M_t(y_1 + y_2) + e^{tA}x) - K(y_1, y_2, \xi)| \nabla p^{t}(t, y_1)|dy_1 \pi_t(dy_2). \quad (4.16)$$
We fix now $\xi = e^{tA}x$ and denoting for simplicity $z = e^{tA}x + M_t y_2$, we decompose the difference between absolute values as

$$\left| \phi(M_t y_1 + z) - K(y_1, y_2, e^{tA}x) \right| \leq \left| \phi(M_t y_1 + z) - \phi(E_1 M_t y_1 + z) \right| + \left| \phi(E_1 M_t y_1 + z) - K(y_1, y_2, e^{tA}x) \right| =: \Lambda_1 + \Lambda_2.$$  

(4.17)

While it is trivial to control the first term as

$$\Lambda_1 \leq \|\phi\|_{C^2_a} \sum_{k=2}^{n} |E_k M_t y_1|^{\alpha k - 1},$$  

(4.18)

the second component $\Lambda_2$ needs a Taylor expansion to write that

$$\Lambda_2 = \left| \phi(E_1 M_t y_1 + z) - \phi(z) - \langle D_t, \phi(z), E_1 M_t y_1 \rangle - \frac{1}{2} \langle D_t^2, \phi(z) E_1 M_t y_1, E_1 M_t y_1 \rangle \right|$$

$$= \left| \int_0^1 \langle D_t^2, \phi(z + \lambda E_1 M_t y_1) E_1 M_t y_1, E_1 M_t y_1 \rangle (1 - \lambda) \, d\lambda - \frac{1}{2} \langle D_t^2, \phi(z) E_1 M_t y_1, E_1 M_t y_1 \rangle \right|$$

(4.19)

$$\leq \|D_t^2 \phi\|_{C^{\gamma - 2}_a} |E_1 M_t y_1|^{\gamma - 2} |E_1 M_t y_1|^2$$

$$\leq \|\phi\|_{C^{\gamma}_a} |E_1 M_t y_1|^\gamma.$$

Going back to Expression (4.16) with Estimates (4.17), (4.18) and (4.19), we can show that

$$\left| D_t P_t \phi(x) \right| \leq C \|\phi\|_{C^{\gamma}_a} t^{-(h-1)} \sum_{k=1}^{n} \int_{\mathbb{R}^N} |E_k M_t y_1|^{\alpha k - 1} |\nabla p^1(t, y_1)| \, dy_1.$$  

The above expression allows us to conclude as in (4.6) using Proposition 2 with $m = N + 4$ and $|\beta| = 1$. Namely,

$$\left| D_t P_t \phi(x) \right| \leq C \|\phi\|_{C^{\gamma}_a} t^{-(h-1)} \sum_{k=1}^{n} \int_{\mathbb{R}^N} t^{\frac{N+1}{\alpha}} (1 + \frac{|y_1|}{t^{\alpha}})^{-(N+3)} |E_k M_t y_1|^{\alpha k - 1} \, dy_1$$

$$\leq C \|\phi\|_{C^{\gamma}_a} t^{\frac{2-(1+\alpha)(h-1)}{\alpha}} \sum_{k=1}^{n} \int_{\mathbb{R}^N} (1 + |z|)^{-(N+3)} |z|^{\frac{\gamma - \alpha}{\alpha k - 1}} \, dz$$

$$\leq C \|\phi\|_{C^{\gamma}_a} t^{\frac{2-(1+\alpha)(h-1)}{\alpha}},$$

where in the second step we used again the change of variable $z = y_1 t^{-1/\alpha}$. 

Next, we are going to use the controls in Theorem 4 to show the main result of this section. It states the continuity of the semigroup $P_t$ between anisotropic Zygmund-Hölder spaces at a cost of additional time singularities.

**Corollary 5.** Let $\beta, \gamma$ be in $[0, 1 + \alpha)$ such that $\beta \leq \gamma$. Then, there exists a constant $C > 0$ such that

$$\|P_t\|_{L_c(C^{\gamma}_{a,d}(\mathbb{R}^N), C^{\gamma}_{a,d}(\mathbb{R}^N))} \leq C (1 + t^{-\frac{\gamma - \alpha}{\alpha}}), \quad t > 0.$$  

(4.20)

**Proof.** It is enough to show the result only for $\gamma = \beta$ non-integer, thanks to interpolation techniques. Indeed, fixed $\beta < \gamma$, we can use Theorem 2 to show that

$$\|P_t\|_{L_c(C^\beta_{a,d}(\mathbb{R}^N), C^\beta_{a,d}(\mathbb{R}^N))} \leq \left( \|P_t\|_{L_c(C^\beta(\mathbb{R}^N), C^\beta_{a,d}(\mathbb{R}^N))} \right)^{1-\frac{\beta}{\gamma}} \left( \|P_t\|_{L_c(C^\beta_{a,d}(\mathbb{R}^N))} \right)^{\frac{\beta}{\gamma}}.$$  

On the other hand, if we fix $\gamma$ integer, we can take $\gamma'$ in $(\gamma, 1 + \alpha)$ non-integer such that Theorem 2 implies:

$$\|P_t\|_{L_c(C^\gamma_{a,d}(\mathbb{R}^N))} \leq \left( \|P_t\|_{L_c(C^\gamma(\mathbb{R}^N))} \right)^{1-\frac{\gamma}{\gamma'}} \left( \|P_t\|_{L_c(C^\gamma(\mathbb{R}^N))} \right)^{\frac{\gamma}{\gamma'}}.$$  

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The general result will then follows from the two above controls and Equation (4.9), once we have shown Estimate (4.20) for $\gamma = \beta$ non-integer.

Fixed again the time horizon $T$ given in (4.13), we start noticing that Control (4.20) for $t \geq T$ has already been shown in Equation (4.14).

To prove it when $t \leq T$, we are going to exploit the equivalent norm defined in (2.8) of Lemma 2. For this reason, we fix $h$ in $[1, n]$, a point $x_0$ in $\mathbb{R}^N$ and $z \neq 0$ in $E_h(\mathbb{R}^N)$ and we would like to show that

$$|\Delta_{x_0}^3 (P_t \phi)(z)| \leq C \|\phi\|_{C_{x_0}^3} |z|^{\frac{\gamma}{\gamma + n - 1}},$$

(4.21)

for some constant $C > 0$ independent from $x_0$. Before starting with the calculations, we highlight the presence of three different "regimes" appearing below. On the one hand, we will firstly consider a macroscopic regime appearing for $|z| \geq 1$. On the other hand, we will say that the off-diagonal regime holds if $\frac{t + \alpha(h - 1) - 1}{h} \leq |z| \leq 1$. It will mean in particular that the spatial distance is larger than the characteristic time-scale. Finally, a diagonal regime will be in force when $\frac{t + \alpha(h - 1) - 1}{h} \geq |z|$ and the spatial point will be instead smaller than the typical time-scale magnitude. While for the two first regimes, we are going to use the contraction property of the semigroup, the third regime will require to exploit the controls in Hölder norms given by Theorem 4.

As said above, Estimate (4.21) in the macroscopic regime (i.e. $|z| \geq 1$) follows immediately from the contraction property of $P_t$ on $B_0(\mathbb{R}^N)$. Indeed,

$$|\Delta_{x_0}^3 (P_t \phi)(z)| \leq C \|P_t \phi\|_{\infty} \leq C \|\phi\|_{C_{x_0}^3} |z|^{\frac{\gamma}{\gamma + n - 1}}.$$

(4.22)

For $\frac{t + \alpha(h - 1)}{h} \leq |z| \leq 1$ and $l$ in $[0, 3]$, we start noticing from Equation (4.1) that

$$P_t \phi(x_0 + lz) = \int_0^1 \int_{\mathbb{R}^N} \phi(M_t(y_1 + y_2) + e^{tA}(x_0 + lz))p^t(t, y_1)dy_1 \pi_t(dy_2)$$

$$= \int_0^1 \int_{\mathbb{R}^N} \phi(\xi_0 + e^{tA}z)p^t(t, y_1)dy_1 \pi_t(dy_2),$$

where we have denoted for simplicity $\xi_0 = M_t(y_1 + y_2) + e^{tA}x_0$. We can then exploit Lemma 1 to write that

$$|\Delta_{x_0}^3 (P_t \phi)(z)| \leq \int_0^1 \int_{\mathbb{R}^N} |\Delta_{x_0}^3 \phi(e^{tA}z)|p^t(t, y_1)dy_1 \pi_t(dy_2)$$

$$\leq \pi_t(\mathbb{R}^N) \|\phi\|_{C_{x_0}^3} \sum_{k=1}^n \left| E_k \right| \left| e^{tA}z \right|^{\frac{\gamma}{\gamma + n - 1}}$$

(4.23)

$$\leq C \|\phi\|_{C_{x_0}^3} \sum_{k=1}^{n-1} \left( t|z| \right)^{\frac{\gamma}{\gamma + n - 1}} + \sum_{k=n}^n \left( t^{k-1}|z| \right)^{\frac{\gamma}{\gamma + n - 1}}$$

$$\leq C \|\phi\|_{C_{x_0}^3} |z|^{\frac{\gamma}{\gamma + n - 1}}.$$
We can then conclude that
\[ |\Delta_{x_0}^\gamma (P_t \phi)(x)| \leq C \|\phi\|_{C^\gamma_{x t}} |x|^{\gamma - n - 1}. \]  
(4.25)

Going back to Controls (4.22), (4.23) and (4.25), we have thus proven Estimate (4.21) for any non-integer \( \gamma = \beta. \)

\[ \square \]

5 Elliptic and Parabolic Schauder Estimates

In this section, we use the controls shown before to prove Schauder Estimates both for the elliptic and the parabolic equation driven by the Ornstein-Uhlenbeck operator \( L^\text{ou}. \)

Fixed \( \lambda > 0 \) and \( g \) in \( C_b(\mathbb{R}^N) \), we say that a function \( u: \mathbb{R}^N \to \mathbb{R} \) is a weak solution of Elliptic Equation (1.5) if \( u \) is in \( C_b(\mathbb{R}^N) \) and for any \( \phi \) in \( C_c^\infty(\mathbb{R}^N) \) (i.e. smooth functions with compact support), it holds that

\[ \int_{\mathbb{R}^N} u(x) \left[ \lambda \phi(x) - (L^\text{ou})^* \phi(x) \right] dx = \int_{\mathbb{R}^N} \phi(x) g(x) dx, \]  
(5.1)

where \( (L^\text{ou})^* \) denotes the formal adjoint of \( L^\text{ou} \) on \( L^2(\mathbb{R}^N) \), i.e.

\[ (L^\text{ou})^* \phi(x) = L^* \phi(x) - \langle Ax, D_x \phi(x) \rangle - \text{Tr}(A) \phi(x), \quad (t, x) \in [0, T] \times \mathbb{R}^N, \]  
(5.2)

and \( L^* \) is the adjoint of the operator \( L \) on \( L^2(\mathbb{R}^N) \). It is well-known (see e.g. Section 4.2 in [App19]) that it can be represented for any \( \phi \) in \( C_c^\infty(\mathbb{R}^N) \) as

\[ L^* \phi(x) = \frac{1}{2} \text{Tr} \{ BQ B^* D^2 \phi(x) \} - \langle Bb, D \phi(x) \rangle + \int_{\mathbb{R}^N} \left[ \phi(x - Bz) - \phi(x) + \langle D \phi(x), Bz \rangle \mathbb{1}_{B(0,1)}(z) \right] \nu(dz). \]

We state now the main result for the elliptic case, ensuring the well-posedness (in a weak sense) for Equation (1.5).

**Theorem 6.** Fixed \( \lambda > 0 \), let \( g \) be in \( C_b(\mathbb{R}^N) \). Then, the function \( u: \mathbb{R}^N \to \mathbb{R} \) given by

\[ u(x) := \int_0^\infty e^{-\lambda t} P_t g(x) dt, \quad x \in \mathbb{R}^N, \]  
(5.3)

is the unique weak solution of Equation (1.5).

**Proof.** Existence. We are going to show that the function \( u \) given in Equation (5.3) is indeed a weak solution of the elliptic problem (1.5). It is straightforward to notice that \( u \) is in \( C_b(\mathbb{R}^N) \), thanks to the contraction property of \( P_t \) on \( C_b(\mathbb{R}^N) \). Fixed \( \phi \) in \( C_c^\infty(\mathbb{R}^N) \), we then use Fubini Theorem to write that

\[ \int_{\mathbb{R}^N} u(x) (L^\text{ou})^* \phi(x) dx = \lim_{\epsilon \to 0^+} \int_0^\infty \int_{\mathbb{R}^N} e^{-\lambda t} P_t g(x) (L^\text{ou})^* \phi(x) dx dt \]
\[ = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \int_0^\infty e^{-\lambda t} L^\text{ou} P_t g(x) \phi(x) dx dt, \]

where, in the last step, we exploited that \( P_t g \) is differentiable and bounded for \( t > 0 \) (Proposition 3). Since \( L^\text{ou} \) is the infinitesimal generator of the semigroup \( \{ P_t: t \geq 0 \} \), we know that \( \partial_t (P_t g) \) exists for any \( t > 0 \) and \( \partial_t (P_t g)(x) = L^\text{ou} P_t g(x) \) for any \( x \) in \( \mathbb{R}^N \). Integration by parts formula allows then to conclude that

\[ \int_{\mathbb{R}^N} u(x) (L^\text{ou})^* \phi(x) dx = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \phi(x) \int_0^\infty e^{-\lambda t} \partial_t P_t g(x) dt dx \]
\[ = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N} \left( -e^{-\lambda t} P_t g(x) + \lambda \int_0^\infty e^{-\lambda t} P_t g(x) dt \right) dx \]
\[ = \int_{\mathbb{R}^N} -g(x) \phi(x) dx + \int_{\mathbb{R}^N} \lambda u(x) \phi(x) dx. \]

**Uniqueness.** It is enough to show that any weak solution \( u \) of Equation (1.5) for \( g = 0 \) coincides with the zero function, i.e. \( u = 0 \). To do so, we fix a function \( \rho \) in \( C_c^\infty(\mathbb{R}^N) \) such that \( \|\rho\|_{L^1} = 1 \), \( 0 \leq |\rho| \leq 1 \) and we
then define the mollifier $\rho_m := m^N \rho(mx)$ for any $m$ in $\mathbb{N}$. Denoting now, for simplicity, $u_m := u \ast \rho_m$, we define the function

$$g_m(x) := \lambda u_m(x) - \mathcal{L}^0 u_m(x).$$

Using that $u$ is in $C_b(\mathbb{R}^N)$, it is easy to notice that $g_m$ is also in $C_b(\mathbb{R}^N)$ for any fixed $m$ in $\mathbb{N}$. Truncating the functions if necessary, we can assume that $u_m$ and $g_m$ are integrable with integrable Fourier transform so that we can apply the Fourier transform in Equation (5.4):

$$\lambda \hat{u}_m(\xi) - \mathcal{F}_x \{\mathcal{L}_0 u_m\}(\xi) = \hat{g}_m(\xi).$$

We remember in particular that the above operator $\mathcal{L}_0^a$ has an associated Lévy symbol $\Psi^a(\xi)$ and, following Section 3.3.2 in [App09], it holds that

$$\mathcal{F}_x \{\mathcal{L}_0^a u_m\}(\xi) = \Psi^a(\xi) \hat{u}_m(\xi).$$

We can then use it to show that $\tilde{u}_m$ is a classical solution of the following equation:

$$[\lambda - \Psi^a(\xi)] \tilde{u}_m(\xi) = \tilde{g}_m(\xi).$$

The above equation can be easily solved by direct calculation as

$$\tilde{u}_m(\xi) = \int_0^\infty e^{-\lambda t} e^{\Psi^a(\xi) t} \tilde{g}_m(\xi) \, ds.$$

In order to go back to $u_m$, we apply now the inverse Fourier transform to write that

$$u_m(x) = \int_0^\infty e^{-\lambda t} P_t g_m(x) \, dt.$$

We can then exploit the contraction property of the semigroup $P_t$ to show that $\|u_m\|_\infty \leq C \|g_m\|_\infty$. In order to conclude, we need to show that

$$\lim_{m \to \infty} \|g_m\|_\infty = 0. \tag{5.7}$$

We start noticing that, since $u$ is a weak solution of Equation (1.5) with $g = 0$, it holds that

$$g_m(x) = \int_{\mathbb{R}^N} u(y) \{\lambda \rho_m(x - y) - \mathcal{L} \rho_m(x - y)\} \, dy$$

$$= \int_{\mathbb{R}^N} u(y) \{\mathcal{L}^\ast \rho_m(x - y)(y) - \mathcal{L} \rho_m(x - y)\} \, dy + \langle A(x - y), D_x \rho_m(x - y) \rangle + \text{Tr}(A) \rho_m(x - y)$$

$$= R_m^1(x) + R_m^2(x) + R_m^3(x),$$

where we have denoted

$$R_m^1(x) := \int_{\mathbb{R}^N} u(y) \{\mathcal{L}^\ast \rho_m(x - y)(y) - \mathcal{L} \rho_m(x - y)\} \, dy;$$

$$R_m^2(x) := \int_{\mathbb{R}^N} u(y) \langle A(x - y), D_x \rho_m(x - y) \rangle \, dy;$$

$$R_m^3(x) := \int_{\mathbb{R}^N} u(y) \text{Tr}(A) \rho_m(x - y) \, dy.$$

On the one hand, it is easy to notice that $R_m^1 = 0$, since $\mathcal{L}^\ast \rho_m(x - y)(y) = \mathcal{L} \rho_m(x - y)\}$ for any $m$ in $\mathbb{N}$ and any $y$ in $\mathbb{R}^N$. Indeed, it holds that

$$\frac{1}{2} \text{Tr}(BQB^*D_x^2 \rho_m(x - y)) - \langle Bb, D_y \rho_m(x - y) \rangle = \frac{1}{2} \text{Tr}(BQB^*D_x^2 \rho_m(x - y)) + \langle Bb, D_x \rho_m(x - y) \rangle$$

and

$$\int_{\mathbb{R}^N} \rho_m(x - y + Bz) - \rho_m(x - y) + \langle D_y \rho_m(x - y), Bz \rangle \mathbf{1}_{B(0,1)}(z) \, \nu(dz)$$

$$= \int_{\mathbb{R}^N} \rho_m(x + Bz) - \rho_m(x - y) - \langle D_x \rho_m(x - y), Bz \rangle \mathbf{1}_{B(0,1)}(z) \, \nu(dz).$$
On the other hand, it can be checked (see e.g. [Pri09]) that \( \|R_m^2 + R_m^3\|_\infty \to 0 \) if \( m \) goes to infinity. Indeed, we firstly notice that \( R_m^3 \) converges, when \( m \) goes to infinity, to the function \( u \Tr(A) \), uniformly in \( x \). On the other hand, applying the change of variables \( y = x - z/m \) in \( R_m^2 \), we can obtain that
\[
R_m^2(x) = m \int_{\mathbb{R}^N} u(x - z/m)(A(z/m), D_x \rho(z)) \, dy.
\]
Letting \( m \) goes to infinity above, we can then conclude that \( R_m^2 \) converges to the function \(-u \Tr(A)\), uniformly in \( x \).

Let us deal now with the parabolic setting. Since we are working with evolution equations, the functions we consider will often depend on time, too. We denote for any \( t \in [0,T] \) the \( \mathbb{R}^N \) family of functions \( \phi \in B_b([0,T] \times \mathbb{R}^N) \) such that \( \phi(t,\cdot) \) is in \( C^\gamma_{b,d}(\mathbb{R}^N) \) at any fixed \( t \) and the norm
\[
\|\phi\|_{L^\infty(C^\gamma_{b,d})} := \sup_{t \in [0,T]} \|\phi(t,\cdot)\|_{C^\gamma_{b,d}} < \infty.
\]

We define now the notion of solution we are going to consider. Fixed \( T > 0 \), \( u_0 \) in \( C_b(\mathbb{R}^N) \) and \( f \) in \( L^\infty(0,T;C_b(\mathbb{R}^N)) \), we say that a function \( u : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N \) is a weak solution of the Cauchy problem (1.6) if \( u \) is in \( L^\infty(0,T;C_b(\mathbb{R}^N)) \) and for any \( \phi \) in \( C^\infty_c([0,T] \times \mathbb{R}^N) \), it holds that
\[
\int_{\mathbb{R}^N} u_0(x) \phi(0,x) \, dx + \int_0^T \int_{\mathbb{R}^N} u(t,x) \left[ \partial_t \phi(t,x) + (\mathcal{L}^{\text{au}}) \phi(t,x) \right] + f(t,x) \phi(t,x) \, dx \, dt = 0,
\]
where \( (\mathcal{L}^{\text{au}})^* \) denotes the formal adjoint of \( \mathcal{L}^{\text{au}} \) on \( L^2(\mathbb{R}^N) \) given in Equation (5.2).

Similarly to the elliptic setting, we show firstly the weak well-posedness of the Cauchy problem (1.6).

**Theorem 7.** Fixed \( T > 0 \), let \( u_0 \) be a function in \( C_b(\mathbb{R}^N) \) and \( f \) in \( L^\infty(0,T;C_b(\mathbb{R}^N)) \). Then, the function \( u : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N \) given by
\[
u(t,x) := P_t u_0(x) + \int_0^t P_{t-s} f(s,x) \, ds, \quad (t,x) \in [0,T] \times \mathbb{R}^N,
\]
is the unique weak solution of the Cauchy problem (1.6).

**Proof.** Existence. We start considering a 'regularized' version of the coefficients appearing in Equation (1.6). Namely, we consider a family \( \{u_{0,m}\}_{m \in \mathbb{N}} \) in \( C^\infty_c(\mathbb{R}^N) \) such that \( u_{0,m} \to u_0 \) uniformly in \( x \) and a family \( \{f_m\}_{m \in \mathbb{N}} \) in \( L^\infty(0,T;C^\infty_{b,d}(\mathbb{R}^N)) \) such that \( f_m \to f \) uniformly in \( t \) and \( x \). They can be obtained through standard mollification methods in space.

Fixed \( m \) in \( \mathbb{N} \), we denote now by \( u_m : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N \) the function given by
\[
u_m(t,x) := P_t u_{0,m}(x) + \int_0^t P_{t-s} f_m(s,x) \, ds, \quad t \in [0,T], x \in \mathbb{R}^N.
\]

On the one hand, we use again that \( \partial_t (P_t u_m)(t,x) = \mathcal{L}^{\text{au}} P_t u_m(t,x) \) for any \((t,x)\) in \([0,T] \times \mathbb{R}^N\) to check that \( u_m \) is indeed a classical solution of the 'regularized' Cauchy Problem:
\[
\begin{cases}
\partial_t u_m(t,x) = \mathcal{L}^{\text{au}} u_m(t,x) + f_m(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^N; \\
u_m(0,x) = u_{0,m}(x), \quad x \in \mathbb{R}^N.
\end{cases}
\]

On the other hand, we exploit the linearity and the continuity of the semigroup \( P_t \) on \( C_b(\mathbb{R}^N) \) to show that
\[
u_m = P_t u_{0,m}(x) + \int_0^t P_{t-s} f_m(s,x) \, ds \xrightarrow{m} P_t u_0(x) + \int_0^t P_{t-s} f(s,x) \, ds = u,
\]
uniformly in \( t \) and \( x \), where \( u \) is the function given in (5.9).

We fix now a test function \( \phi \) in \( C^\infty_c([0,T] \times \mathbb{R}^N) \) and we then notice that
\[
\int_0^T \int_{\mathbb{R}^N} \phi(t,y) (\partial_t - \mathcal{L}^{\text{au}}) u_m(t,y) \, dy \, dt = \int_0^T \int_{\mathbb{R}^N} \phi(t,y) f_m(t,y) \, dy \, dt.
\]
We can rewrite

\[- \int_0^T \int_{\mathbb{R}^N} \left( \partial_t + (\mathcal{L}^{ou})^* \right) \phi(t, y) u_m(t, y) \, dy \, dt = \int_{\mathbb{R}^N} \phi(0, y) u_{0,m}(y) \, dy + \int_0^T \int_{\mathbb{R}^N} \phi(t, y) f_m(t, y) \, dy \, dt, \tag{5.10}\]

where \((\mathcal{L}^{ou})^*\) denotes the formal adjoint of \(\mathcal{L}^{ou}\) on \(L^2(\mathbb{R}^N)\).

We would like now to go back to the solution \(u\), letting \(m\) go to infinity. We start rewriting the right-hand side term of (5.10) as \(R^1_m + R^2_m\), where

\[R^1_m := \int_{\mathbb{R}^N} \phi(0, y) u_{0,m}(y) \, dy; \]
\[R^2_m := \int_0^T \int_{\mathbb{R}^N} \phi(t, y) f_m(t, y) \, dy \, dt.\]

We can rewrite \(R^2_m\) as

\[R^2_m = \int_0^T \int_{\mathbb{R}^N} \phi(t, y) f(t, y) \, dy \, dt + \int_0^T \int_{\mathbb{R}^N} \phi(t, y) [f_m - f](t, y) \, dy \, dt.\]

Exploiting that, by assumption, \(f_m\) converges to \(f\) uniformly in \(t\) and \(x\), it is easy to see that the second contribution above converges to 0. A similar argument can be used to show that

\[\int_{\mathbb{R}^N} \phi(0, y) u_{0,m}(y) \, dy \xrightarrow{m} \int_{\mathbb{R}^N} \phi(0, y) u_0(y) \, dy.\]

On the other hand, we can rewrite the left-hand side of Equation (5.10) as

\[- \int_0^T \int_{\mathbb{R}^N} \left( \partial_t + (\mathcal{L}^{ou})^* \right) \phi(t, y) u_m(t, y) \, dy \, dt = - \int_0^T \int_{\mathbb{R}^N} \left( \partial_t + (\mathcal{L}^{ou})^* \right) \phi(t, y) u(t, y) \, dy \, dt + L^1_m + L^2_m + L^3_m,\]

where we have denoted

\[L^1_m := \int_0^T \int_{\mathbb{R}^N} \frac{1}{2} \text{Tr} (BQ B^* D_y^2 \phi(t, y)) + \langle Ay + Bh, D_y \phi(t, y) \rangle + \text{Tr}(A) \phi(t, y) \rangle [u_m - u](t, y) \, dy \, dt; \]
\[L^2_m := \int_0^T \int_{\mathbb{R}^N} \partial_t \phi(t, y) [u - u_m](t, y) \, dy \, dt; \]
\[L^3_m := \int_0^T \int_{\mathbb{R}^N} [u - u_m](t, y) \left[ \int_{\mathbb{R}^N} \phi(t, y - Bz) - \phi(t, y) + \langle D_y \phi(t, y), Bz \rangle 1_B(0,1) (z) \nu(dz) \right] \, dy \, dt. \tag{5.11}\]

To conclude, we need to show that the remainder \(L^1_m + L^2_m + L^3_m\) is negligible, if \(m\) goes to infinity. Exploiting that \(\phi\) has a compact support and that \(\|u_m - u\|_\infty \xrightarrow{m} 0\), it is easy to show that \(|L^1_m + L^2_m| \xrightarrow{m} 0\).

In order to control \(L^3_m\), we need firstly to decompose it as \(L^3_{m,1} + L^3_{m,2}\), where

\[L^3_{m,1} := \int_0^T \int_{\mathbb{R}^N} [u - u_m](t, y) \left[ \int_{|z| < 1} \phi(t, y - Bz) - \phi(t, y) + \langle D_y \phi(t, y), Bz \rangle \nu(dz) \right] \, dy \, dt; \]
\[L^3_{m,2} := \int_0^T \int_{\mathbb{R}^N} [u - u_m](t, y) \left[ \int_{|z| > 1} \phi(t, y - Bz) - \phi(t, y) \nu(dz) \right] \, dy \, dt.\]

The second term \(L^3_{m,2}\) can be controlled easily using the Fubini Theorem. Indeed, denoting by \(K\) the support of \(\phi\) and by \(\lambda\) the Lebesgue measure on \(\mathbb{R}^N\), we notice that

\[|L^3_{m,2}| \leq \|u - u_m\|_\infty \int_0^T \int_{|z| > 1} \int_{\mathbb{R}^N} |\phi(t, y - Bz) - \phi(t, y)| \, dy \, \nu(dz) \, dt \]
\[\leq C T 2 \lambda(K) \nu(B^*(0,1)) \|u - u_m\|_\infty.\]
Exploiting that $\nu(B^\circ(0,1))$ is finite since $\nu$ is a Lévy measure, we can then conclude that $|L^{3,2}_m|$ tends to zero if $m$ goes to infinity.

The argument for $L^{3,1}_m$ is similar but we need firstly to apply a Taylor expansion twice to make a term $|z|^2$ appear in the integral and exploit that $|z|^2 \nu(dz)$ is finite on $B(0,1)$.

**Uniqueness.** This proof will follow essentially the same arguments as for Theorem 6.

Let $u$ be any weak solution of Cauchy problem (1.6) with $u_0 = f = 0$. We are going to show that $u = 0$. We start considering a mollifying sequence $\{\rho_m\}_{m \in \mathbb{N}}$ in $C^\infty_c((0, T) \times \mathbb{R}^N)$. Denoting for simplicity $u_m(t, x) = u \ast \rho_m(t, x)$, we then notice that $u_m$ is continuously differentiable in time and that $u_m(0, x) = 0$. It makes sense to define now the function

$$f_m(t, x) := \partial_t u_m(t, x) - \mathcal{L}^{\alpha} u_m(t, x).$$

Moreover, we can truncate $f_m$ and $u_m$ if necessary, so that they are integrable with integrable Fourier transform. Then, the same reasoning in Equations (5.5), (5.6) allows us to write that

$$\begin{cases}
\partial_t \hat{u}_m(t, \xi) - \Psi^{\alpha}(\xi) \hat{u}_m(t, \xi) = \hat{f}_m(t, \xi), \\
\hat{u}_m(0, \xi) = 0.
\end{cases}$$

The above equation can be easily solved integrating in time, giving the following representation:

$$\hat{u}_m(t, \xi) = \int_0^t e^{(t-s)\Psi^{\alpha}(\xi)} \hat{f}_m(s, \xi) \, ds.$$

In order to go back to $u_m$, we apply now the inverse Fourier transform to write that

$$u_m(t, x) = \int_0^t P_t s f_m(s, x) \, ds.$$

The contraction property of $P_t$ allows us to conclude that $\|u_m\|_{\infty} \leq C \|f_m\|_{\infty}$. Letting $m$ go to zero, we obtain the desired result. Indeed, we can rely on the same reasonings used in the analogous elliptic case (Theorem 6) to show that

$$\lim_{m \to \infty} \|f_m\|_{\infty} = 0.$$  

The next two conclusive theorems provide the Schauder estimates both in the elliptic and in the parabolic setting.

**Theorem 8 (Elliptic Schauder Estimates).** Fixed $\lambda > 0$ and $\beta$ in $(0, 1)$, let $g$ be in $C^{\alpha+\beta}_{b, d}(\mathbb{R}^N)$. Then, the unique solution $u$ of Equation (1.5) is in $C^\alpha_{b, d}(\mathbb{R}^N)$ and there exists a constant $C := C(\lambda) > 0$ such that

$$\|u\|_{C^{\alpha+\beta}_{b, d}} \leq C \|g\|_{C^\alpha_{b, d}}. \tag{5.13}$$

**Proof.** Thanks to Theorem 6, we know that the unique solution $u$ of the elliptic equation (1.5) is given in (5.3). In order to show that such a function $u$ satisfies Schauder estimates (5.13), we exploit again the equivalent norm defined in (2.8) of Lemma 2. Namely, we fix $h$ in $[1, n]$ and $x_0 \in \mathbb{R}^N$ and we show that

$$|\Delta_h^{2} u(z)| = \left| \int_0^\infty e^{-\lambda \Delta_h^{2} (P_t g)(z)} \, dt \right| \leq C \|g\|_{C^\alpha_{h, d}} |z|^{1+\min(\alpha, 1)} , \quad z \in E_h(\mathbb{R}^N),$$

for some constant $C > 0$ independent from $x_0$. For $|z| \geq 1$, it can be obtained easily from the contraction property of $P_t$ on $B_h(\mathbb{R}^N)$:

$$\left| \int_0^\infty e^{-\lambda \Delta_h^{2} (P_t g)(z)} \, dt \right| \leq C \|P_t g\|_{\infty} \leq C \|g\|_{\infty} |z|^{1+\alpha(\min(\alpha, 1))}. \tag{5.14}$$

When $|z| \leq 1$, we start fixing a transition time $t_0$ given by

$$t_0 = |z|^{1+\alpha(\min(\alpha, 1))}. \tag{5.15}$$
Notably, $t_0$ represents the transition time between the diagonal and the off-diagonal regime, accordingly to the intrinsic time scales of the system. We then decompose $\Delta^3_{x_0} u(z)$ as $R_1(z) + R_2(z)$, where

$$R_1(z) := \int_0^{t_0} e^{-\lambda t} \Delta^3_{x_0} (P_t g)(z) \, dt;$$

$$R_2(z) := \int_{t_0}^\infty e^{-\lambda t} \Delta^3_{x_0} (P_t g)(z) \, dt.$$  

The first component $R_1$ is controlled easily using Corollary 5 for $\beta = \gamma$. Indeed,

$$|R_1(z)| \leq \int_0^{t_0} |\Delta^3_{x_0} (P_t g)(z)| \, dt \leq \|P_t g\|_{C^\beta_{b,d}} |z|^{\beta(1-\alpha/b-\gamma/3)} \int_0^{t_0} dt \leq C \|g\|_{C^\beta_{b,d}} |z|^{\beta(1-\alpha/b-\gamma/3)}.$$  \hspace{1cm} (5.16)

On the other hand, the control for $R_2$ can be obtained following Equation (4.24) in order to write that

$$|R_2(z)| \leq C \|g\|_{C^\beta_{b,d}} |z|^{\beta} e^{-\lambda (1 + t \frac{\beta(1+\alpha/b-\gamma/3)}{\beta(1-\alpha/b-\gamma/3)})} dt$$

$$\leq C \|g\|_{C^\beta_{b,d}} |z|^{\beta} \lambda^{-1} + |z|^{\alpha/b(1+\alpha/b-\gamma/3)}$$

$$\leq C \|g\|_{C^\beta_{b,d}} |z|^{\gamma(\alpha/b-\gamma/3)},$$

where, in the last step, we exploited that $|z| \leq 1$. \hfill \Box

**Theorem 9** (Parabolic Schauder Estimates). *Fixed $T > 0$ and $\beta$ in $(0,1)$, let $u_0$ be in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ and $f$ in $L^\infty(0,T; C^\beta_{b,d}(\mathbb{R}^N))$. Then, the weak solution $u$ of Cauchy Problem (1.6) is in $L^\infty(0,T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^N))$ and there exists a constant $C := C(T) > 0$ such that

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C \left[ \|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})} \right].$$  \hspace{1cm} (5.18)

**Proof.** We are going to show that any function $u$ given by Equation (5.9) satisfies the Schauder Estimates (5.18). We start splitting the function $u$ in $u_1 + u_2$, where

$$u_1(t,x) := P_t u_0(x);$$

$$u_2(t,x) := \int_0^t P_{t-s} f(s,x) \, ds.$$  \hspace{1cm} (5.19, 5.20)

Corollary 5 allows then to control $u_1$ in the following way:

$$\|u_1\|_{L^\infty(C^{\alpha+\beta}_{b,d})} = \sup_{t \in (0,T]} \|P_t u_0\|_{C^{\alpha+\beta}_{b,d}} \leq C \|u_0\|_{C^{\alpha+\beta}_{b,d}}.$$  

In order to deal with the contribution $u_2$, we will follow essentially the same reasoning for the Schauder Estimates in the elliptic setting. Namely, we use again the equivalent norm defined in (2.8) of Lemma 2 in order to estimate

$$\|u_2\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C \|f\|_{L^\infty(C^\beta_{b,d})}.$$  

Fixed $h$ in $[1,n]$ and $x_0$ in $\mathbb{R}^N$, our aim is to show that

$$|\Delta^3_{x_0} u_2(z)| = \left| \int_0^t \Delta^3_{x_0} (P_t f)(s,z) \, ds \right| \leq C \|f\|_{L^\infty(C^{\alpha+\beta}_{b,d})} |z|^{\gamma(\alpha/b-\gamma/3)}, \quad z \in E_h(\mathbb{R}^N),$$

for some constant $C > 0$ independent from $x_0$. When $|z| \geq 1$, it can be obtained easily from the contraction property of $P_t$ on $C_0(\mathbb{R}^N)$ as in (5.14). For $|z| \leq 1$, we fix again the transition time $t_0$ given in (5.15) and we then decompose $\Delta^3_{x_0} u_2(t,z)$ as $\tilde{R}_1(t,z) + \tilde{R}_2(t,z)$, where

$$\tilde{R}_1(t,z) := \int_0^{t \wedge t_0} \Delta^3_{x_0} (P_t f)(t-s,z) \, ds;$$

$$\tilde{R}_2(t,z) := \int_{t \wedge t_0}^t \Delta^3_{x_0} (P_t f)(t-s,z) \, ds.$$  

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The first component $R_1$ can be controlled easily as in (5.16):

$$|\tilde{R}_1(t,z)| \leq \int_0^{t\wedge t_0} |\Delta_{x_0}^3 (P_s f)(t - s, z)| \, ds$$

$$\leq |z| |\frac{\sigma}{t^{\alpha - \frac{2}{3}}}| \int_0^{t\wedge t_0} \|P_s f(t - s, \cdot)\|_{C_{b,d}^\alpha} \, ds$$

$$\leq C \|f\|_{L^\infty(C_{b,d}^\alpha)} |z| |\frac{\sigma}{t^{\alpha - \frac{2}{3}}}|.$$ 

On the other hand, the control for $R_2$ is obtained following the same steps used in Equation (5.17). Namely,

$$|\tilde{R}_2(t,z)| \leq C \|f\|_{L^\infty(C_{b,d}^\alpha)} |z|^3 \int_0^{t\wedge t_0} \left(1 + s^\beta s (t \wedge t_0)^{\alpha - 1}\right) \, ds$$

$$\leq C \|f\|_{L^\infty(C_{b,d}^\alpha)} |z|^{\frac{\alpha + \beta}{\alpha - 1}}.$$ 

6 Extensions to Time Dependent Operators

In this final section, we would like to show some possible extensions of our method in order to include more general operators with non-linear, space-time dependent coefficients. Even in this framework, we will prove the well-posedness of the parabolic Cauchy problem and show the associated Schauder estimates.

Following [KP10], our first step is to consider a time-dependent Ornstein-Uhlenbeck operator of the following form:

$$L_t^\alpha \phi(t, x) := \frac{1}{2} \text{Tr}(B_t Q B_t; D^2 \phi(x)) + \langle A_t x, D \phi(x) \rangle + \int_{\mathbb{R}^d} \left[ \phi(x + B_t z) - \phi(x) - \langle D_x \phi(x), B_t z \rangle 1_{B(0,1)}(z) \right] \nu(dz),$$

where $B_t := B \sigma_0(t)$ and $A_t, \sigma_0(t)$ are two time-dependent matrixes in $\mathbb{R}^N \otimes \mathbb{R}^N$ and $\mathbb{R}^d \otimes \mathbb{R}^d$, respectively. From this point further, we assume that the matrices $A_t, \sigma_0(t)$ are measurable in time and that they satisfy the following conditions:

[tK] for any fixed $t$ in $[0,T]$, it holds that $N = \text{rank}[B, A_t B, \ldots, A_{t_0}^{N-1} B]$;

[B] the matrix $A_t$ is bounded in time, i.e. there exists a constant $\eta > 0$ such that

$$|A_t \xi| \leq \eta |\xi|, \quad \xi \in \mathbb{R}^N;$$

[UE] the matrix $\sigma_0$ is uniformly elliptic, i.e. it holds that

$$\eta^{-1} |\xi|^2 \leq \langle \sigma_0(t) \xi, \xi \rangle \leq \eta |\xi|^2, \quad (t, \xi) \in [0,T] \times \mathbb{R}^d.$$

It is important to highlight already that this new 'time-dependent" version [tK] of the Kalman rank condition [K] allows us to reproduce the same reasonings of Section 2. In particular, the anisotropic distance $d$ and the Zygmund-Hölder spaces $C_{b,d}^{\alpha, \beta}(\mathbb{R}^N)$ can be constructed under these assumptions, even if only at any fixed time $t$. A priori, the number of sub-divisions of the space $\mathbb{R}^N$ may change for different times, leading to consider a time-dependent $n(t)$ in Equation (2.2) and, consequently, time-dependent anisotropic distances and Hölder spaces. We will however drop the subscript in $t$ below since it does not add any difficulty in the arguments but it may damage the readability of the article.

**Proposition 4.** Let $u_0$ be in $C_b(\mathbb{R}^N)$ and $f$ in $L^\infty(0,T; C_b(\mathbb{R}^N))$. Then, there exists a unique solution $u: [0,T] \times \mathbb{R}^N \to \mathbb{R}$ of the following Cauchy problem:

$$\begin{cases}
\partial_t u(t,x) = L_t^\alpha u(t,x) + f(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^N; \\
u(0,x) = u_0(x), \quad x \in \mathbb{R}^N.
\end{cases}$$

(6.1)

Furthermore, if $u_0$ is in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^N)$ and $f$ in $L^\infty(0,T; C_{b,d}^{\beta}(\mathbb{R}^N))$, then $u$ is in $L^\infty(0,T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^N))$ and there exists a constant $C := C(T, \eta) > 0$ such that

$$\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq C[\|u_0\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^{\beta})}].$$

(6.2)
Proof. The proof of this result can be obtained mimicking the arguments already presented in the first part of the article with some slight modifications. The main difference is the introduction of the resolvent $R_{s,t}$ associated with the matrix $A_t$ in place of the matrix exponential $e^{tA}$. Namely, $R_{s,t}$ is a time-dependent matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ that is solution of the following ODE:

$$\begin{cases}
\partial_t R_{s,t} = A_t R_{s,t}, & t,s \in [0,T]; \\
R_{s,s} = \text{Id}_{N \times N}.
\end{cases}$$

(6.3)

As said before, Section 2 follows exactly in the same manner as above except for Lemma 1 (structure of the resolvent), whose proof can be found in [HM16]. Lemmas 5.1 and 5.2. The arguments in Section 3 and 4 can be applied again, even if the formulation of some objects presented there changes slightly. For example in Equation (3.1), the $N$-dimensional Ornstein-Uhlenbeck process $\{X_t\}_{t \geq 0}$ driven by $B_t Z_t$ should be now represented by

$$X_t = R_{0,t} x + \int_0^t R_{s,t} B_s dZ_s, \quad t \geq 0, x \in \mathbb{R}^N.$$ 

Finally in Section 5, the uniform ellipticity condition $[\text{UE}]$ of $\sigma_0(t)$ and the boundedness condition $[\text{B}]$ of $A_t$ allow us to control the remainder terms appearing in Equation (5.11) as done above and thus, to conclude as in Theorems 7 and 9.

Once we have shown our results for the time-dependent Ornstein-Uhlenbeck operator $\mathcal{L}_t^{\sigma_0}$, we add now a non-linearity to the problem, even if only dependent in time. Namely, we are interested in operators of the following form:

$$L_t \phi(t, x) := \mathcal{L}_t^{\sigma_0} \phi(t, x) + \{F_0(t), D_x \phi(x)\} - c_0(t) \phi(x), \quad (t, x) \in [0, T] \times \mathbb{R}^N,$$

(6.4)

where $c_0: [0, T] \to \mathbb{R}$ and $F_0: [0, T] \to \mathbb{R}^N$ are two functions. For any sufficiently regular function $\phi: [0, T] \to \mathbb{R}$, we are going to denote

$$\mathcal{T} \phi(t, x) := e^{-\int_0^t c_0(s) ds} \phi(t + \int_0^t F_0(s) ds), \quad (t, x) \in [0, T] \times \mathbb{R}^N.$$ 

(6.5)

We will see in the next result that the ‘operator’ $\mathcal{T}$ transforms solutions of the Cauchy problem associated with $\mathcal{L}_t^{\sigma_0}$ to solutions of the Cauchy problem driven by $L_t$, even if for a modified drift $\mathcal{T} f$.

Lemma 3. Fixed $T > 0$, let $u_0$ be in $C_b(\mathbb{R}^N)$, $f$ in $L^\infty([0, T]; C_b(\mathbb{R}^N))$ and $c_0$, $F_0$ in $C_b([0, T])$. Then, a function $u: [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a weak solution of Cauchy Problem (6.1) if and only if the function $v: [0, T] \times \mathbb{R}^N \to \mathbb{R}$ given by $v(t, x) = \mathcal{T} u(t, x)$ is a weak solution of the following Cauchy problem:

$$\begin{cases}
\partial_t u(t, x) = L_t u(t, x) + \mathcal{T} f(t, x), & (t, x) \in (0, T) \times \mathbb{R}^N; \\
u(0, x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}$$

(6.6)

In particular, there exists a unique weak solution of Cauchy Problem (6.6).

Proof. Given a weak solution $u$ of Cauchy problem (6.1), we are going to show that the function $v$ given in (6.5) is indeed a weak solution of Cauchy Problem (6.6). The inverse implication can be obtained in a similar manner and we will not prove it here.

By mollification if necessary, we can take two sequences $\{c_m\}_{m \in \mathbb{N}_0}$, $\{F_m\}_{m \in \mathbb{N}}$ in $C^\infty_c([0, T])$ such that $c_m \to c_0$ and $F_m \to F_0$ uniformly in $t$. Furthermore, we denote for simplicity

$$\tilde{c}_m(t) := \int_0^t c_m(s) ds; \quad \tilde{F}_m(t) := \int_0^t F_m(s) ds.$$ 

Given a test function $\phi$ in $C^\infty_c([0, T] \times \mathbb{R}^N)$, let us consider for any $m$ in $\mathbb{N}$, the following function

$$\psi_m(t, x) := e^{-\tilde{c}_m(t)} \phi(t, x - \tilde{F}_m(t)), \quad (t, x) \in [0, T] \times \mathbb{R}^N.$$
Since $\tilde{c}_m$ and $\tilde{F}_m$ are smooth and bounded, it is easy to check that $\psi_m$ is in $C^\infty_0([0, T) \times \mathbb{R}^N)$. We can then use $\psi_m$ in Equation (5.8) (with time-dependent $A_t$ and $B_t$) to show that
\[
\int_0^T \int_{\mathbb{R}^N} \left[ \partial_t + (L^\alpha_t)^* \right] \psi_m(t, y) u(t, y) + f(t, y) \, dy \, dt + \int_{\mathbb{R}^N} \psi_m(0, y) u_0(y) \, dy = 0.
\]
A direct calculation then show that $\psi_m(0, y) = \phi(0, y)$ and
\[
(\mathcal{L}^\alpha_t)^* \psi_m(t, y) = e^{-\tilde{c}_m(t)} (\mathcal{L}^\alpha_t)^* \phi(t, y - \tilde{F}_m(t));
\]
\[
\partial_t \psi_m(t, y) = e^{-\tilde{c}_m(t)} \left[ \partial_t \phi(t, y - \tilde{F}_m(t)) - (F_m(t), D_y \phi(t, y - \tilde{F}_m(t))) - c_m(t) \phi(t, y - \tilde{F}_m(t)) \right].
\]
The above calculations and a change of variable then imply that
\[
\int_0^T \int_{\mathbb{R}^N} \left[ \partial_t + (L^\alpha_t)^* \right] \phi(t, y) - (F_m(t), D_y \phi(t, y)) - c_m(t) \phi(t, y) \right] \mathcal{T}_m u(t, y) + \phi(t, y) \mathcal{T}_m f(t, y) \, dy \, dt + \int_{\mathbb{R}^N} u_0(y) \phi(0, y) \, dy = 0,
\]
where, analogously to Equation (6.5), we have denoted for any function $\varphi \colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$,
\[
\mathcal{T}_m \varphi(t, y) := e^{-\tilde{c}_m(t)} \varphi(t, y + \tilde{F}_m(t)).
\]
Following similar arguments exploited in the "existence" part in the proof of Theorem 7, i.e. exploiting the compact support of $\phi$ and the uniform convergence of the coefficients, it is possible to show that the above expression converges, when $m$ goes to infinity, to
\[
\int_0^T \int_{\mathbb{R}^N} \left[ \partial_t + (L^\alpha_t)^* \right] \phi(t, y) v(t, y) + \mathcal{T} f(t, y) \, dx \, dt + \int_{\mathbb{R}^N} \phi(0, x) u_0(x) \, dx = 0
\]
and thus, that $v$ is a weak solution of Cauchy problem (6.6).

Thanks to the previous lemma, we are now able to show the Schauder estimates for the solution $v$ of the Cauchy problem (6.6) and, more importantly, without changing the constant $C$ appearing in Equation (6.2).

**Proposition 5.** Fixed $T > 0$ and $\beta$ in $(0, 1)$, let $u_0$ be in $C^{\alpha+\beta}_{b,d}([0, T]; \mathbb{R}^N)$, $f$ in $L^\infty([0, T]; C^{\beta}_{b}([0, T]; \mathbb{R}^N))$ and $c_0, F_0$ in $B_{\beta}([0, T])$. Then, the unique solution $v$ of Cauchy Problem (6.6) is in $L^\infty([0, T]; C^{\alpha+\beta}_{b,d}([0, T]; \mathbb{R}^N))$ and it holds that
\[
\|v\|_{L^\infty(C^{\alpha+\beta}_{b,d}([0, T]; \mathbb{R}^N))} \leq C \left[ \|u_0\|_{C^{\alpha+\beta}_{b,d}([0, T]; \mathbb{R}^N)} + \|f\|_{L^\infty(C^{\beta}_{b}([0, T]; \mathbb{R}^N))} \right],
\]
where $C := C(T, \eta) > 0$ is the same constant appearing in Theorem 9.

**Proof.** We start denoting for simplicity
\[
\tilde{c}_0(t) := \int_0^t c_0(s) \, ds \quad \text{and} \quad \tilde{F}_0(t) := \int_0^t F_0(s) \, ds.
\]
By Lemma 3, we know that if $v$ is a weak solution of Cauchy problem (6.6), then the function
\[
u(t, x) := e^{\tilde{c}_0(t)} v(t, x - \tilde{F}_0(t))
\]
is the weak solution of Cauchy problem (6.1) with $\tilde{f}$ instead of $f$, where
\[
\tilde{f}(t, x) := e^{\tilde{c}_0(t)} f(t, x - \tilde{F}_0(t)), \quad (t, x) \in (0, T) \times \mathbb{R}^N.
\]
Moreover, we have that $\tilde{f}$ is in $L^\infty(0, T; C^{\beta}_{b}([0, T]; \mathbb{R}^N))$. Considering, if necessary, a smaller time interval $[0, t]$ for some $t \leq T$, it is not difficult to check from Proposition 4 that
\[
\|e^{\tilde{c}_0(t)} v(t, \cdot - \tilde{F}_0(t))\|_{C^{\alpha+\beta}_{b,d}} \leq C \left[ \|u_0\|_{C^{\alpha+\beta}_{b,d}} + \sup_{s \in [0, t]} \|e^{\tilde{c}_0(s)} f(s, \cdot - \tilde{F}_0(t))\| \right].
\]
Using now the invariance of the Hölder norm under translations, we can show that
\[
\|v(t,\cdot)\|_{C^{\alpha+b,d}} \leq C[e^{-\epsilon_0(t)}\|u_0\|_{C^{\alpha+b,d}} + e^{-\epsilon_0(t)} \sup_{s \in [0,t]} \|e^{\epsilon_0(s)}f(s,\cdot)\|] \\
\leq C[\|u_0\|_{C^{\alpha+b,d}} + \sup_{s \in [0,t]} \|f(s,\cdot)\|],
\]
where in the last step we exploited that $\epsilon_0(t)$ is non-decreasing. Taking the supremum with respect to $t$ on both sides of the above inequality, we obtain our result. \hfill \Box

**Remark** (About space-time dependent coefficients). We briefly explain here how to extend the Schauder estimates (5.18) to a class of non-linear, space-time dependent operators, whose coefficients are only locally Hölder continuous in space and may be unbounded. Namely, we are interested in operators of the following form:
\[
L_{t,x} \phi(t,x) := \int_{\mathbb{R}^d} \phi(x + B\sigma(t,x)z) - \phi(x) - (D_x \phi(x), B\sigma(t,x)z) \mathbb{1}_{B(0,1)}(z) \nu(dz) + \langle F(t,x), D_x \phi(x) \rangle,
\]
where $B$ is as in (2.3) and $\sigma: [0,T] \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^d$, $F: [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ are two measurable functions such that $F(t,0)$ is locally bounded in time and $\sigma$ satisfies assumption [UE] at any fixed $(t,x)$ in $[0,T] \times \mathbb{R}^N$.

We would like now the operator $L_{t,x}$ to present a similar "dynamical" behaviour as above, i.e. the transmission of the smoothing effect of the Lévy operator to the degenerate components of the system; see Example 2.1.

For this reason, we suppose the following:

- the drift $F = (F_1, \ldots, F_n)$ is such that for any $i$ in $[1,n]$, $F_i$ depends only on time and on the last $n - (i - 2)$ components, i.e. $F_i(t,x_{i-1}, \ldots, x_n)$;
- the matrices $D_x F_i(t,x)$ have full rank $d_i$ at any fixed $(t,x)$ in $[0,T] \times \mathbb{R}^N$.

As said before, the functions $F$ and $\sigma$ are assumed to be only locally Hölder in space, uniformly in time. Namely, there exists a positive constant $K_0$ such that
\[
d(\sigma(t,x),\sigma(t,y)) \leq K_0 d^\beta(x,y); \quad d(F_i(t,x),F_i(t,y)) \leq K_0 d^{\beta + \gamma_i}(x,y)
\] (6.8)
for any $i$ in $[1,n]$, any $t$ in $[0,T]$ and any $x,y$ in $\mathbb{R}^N$ such that $d(x,y) \leq 1$, where
\[
\gamma_i := \begin{cases}
1 + \alpha(i-2), & \text{if } i > 1; \\
0, & \text{if } i = 1.
\end{cases}
\] (6.9)

We remark in particular that the function $F$ may be unbounded in space.

In order to recover Schauder-type estimates even in this framework, we can follow a perturbative method firstly introduced in [KP10] that allows to exploit the already proven results for time-dependent operators. Let us assume for the moment that $\sigma$ and $F$ are globally Hölder continuous in space, i.e. they satisfy (6.8) for any $x,y$ in $\mathbb{R}^N$. Informally speaking, the method links the operator $L_{t,x}$ with the space independent operator $L_t$ defined in (6.4), by "freezing" the coefficients of $L_{t,x}$ along a reference path $\theta: [0,T] \to \mathbb{R}^N$ given by
\[
\theta_t := x_0 + \int_{t_0}^t F(s,\theta_s) \, ds,
\]
for some $(t_0,x_0)$ in $[0,T] \times \mathbb{R}^N$. It is important to highlight that, since $F$ is only Hölder continuous, we need to fix one of the possible paths satisfying the above dynamics. We point out that the deterministic flow $\theta_t$ associated with the drift $F$ is introduced precisely to handle the possible unboundedness of $F$. We could then consider a proxy operator $L_t$ whose coefficients are given by $\sigma_0(t) := \sigma(t,\theta_t)$, $F_0(t) := F(t,\theta_t)$ and
\[
[A_i]_{i,j} = \begin{cases}
D_x F_i(t,\theta_t), & \text{if } j = i - 1; \\
0, & \text{otherwise}
\end{cases}
\]

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In particular, Theorem 5 assures the well-posedness and the Schauder estimates for the Cauchy problem associated with $L_t$. The final step of the proof would be to expand a solution $u$ of the Cauchy problem associated with $L_{t,x}$ around the proxy $L_t$ through a Duhamel-like formula and finally show that the expansion error only brings a negligible contribution so that the Schauder estimates still hold for the original problem. The a priori estimates for the expansion error are however quite involved (and they are the main reason why we have decided to not show here the complete proof), since they rely on some non-trivial controls in appropriate Besov norms.

In order to deal with coefficients that are only locally Hölder in space, we need in addition to introduce a 'localized' version of the above reasoning. It would be necessary to multiply a solution $u$ by a suitable bump function $\delta$ that localizes in space along the deterministic flow $\theta_t$ that characterizes the proxy. Namely, to fix a smooth function $\rho$ that is equal to 1 on $B(0,1/2)$ and vanishes outside $B(0,1)$ and define $\delta(t,x) := \rho(x-\theta_t)$. We would then follow the above method but with respect to the 'localized' solution

$$v(t,x) := \delta(t,x)u(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^N.$$ 

We suggest the interested reader to see [CdRHM18] for a detailed treatise of the argument in the degenerate diffusive setting, [CdRMP20] in the non-degenerate stable framework or [Mar20] for the precise assumptions on the coefficients.

References

[App09] D. Applebaum. \textit{Lévy processes and stochastic calculus}, volume 116 of \textit{Cambridge Studies in Advanced Mathematics}. Cambridge University Press, Cambridge, second edition, 2009.

[App19] D. Applebaum. \textit{Semigroups of linear operators}, volume 93 of \textit{London Mathematical Society Student Texts}. Cambridge University Press, Cambridge, 2019. With applications to analysis, probability and physics.

[Bas09] R. F. Bass. Regularity results for stable-like operators. \textit{J. Funct. Anal.}, 257(8):2693–2722, 2009.

[BK15] J. Bae and M. Kassmann. Schauder estimates in generalized Hölder spaces. 2015.

[BMB01] B. Baeumer, M. M. Meerschaert, and D. A. Benson. Operator lévy motion and multiscaling anomalous diffusion. \textit{Phys. Rev. E}, 63:021112, Jan 2001.

[BMer09] T. Byczkowski, J. Mal ecki, and M. Ryznar. Bessel potentials, hitting distributions and Green functions. \textit{Trans. Amer. Math. Soc.}, 361(9):4871–4900, 2009.

[BNS01] O. E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. \textit{J. R. Stat. Soc. Ser. B Stat. Methodol.}, 63(2):167–241, 2001.

[Bro01] P. J. Brockwell. Lévy-driven CARMA processes. volume 53, pages 113–124. 2001. Nonlinear non-Gaussian models and related filtering methods (Tokyo, 2000).

[CdRHM18] P.-É Chaudru de Raynal, I. Honoré, and S. Menozzi. Sharp Schauder estimates for some degenerate Kolmogorov equations. \textit{To appear in Ann. Sc. N. Sup. Pisa}, 2018.

[CdRM20] P.-É Chaudru de Raynal and S. Menozzi. Regularization effects of a noise propagating through a chain of differential equations: an almost sharp result. \textit{Transactions of AMS}, 2020.

[CdRMP20] P.-É. Chaudru de Raynal, S. Menozzi, and E. Priola. Schauder estimates for drifted fractional operators in the supercritical case. \textit{J. Funct. Anal.}, 278(8):108425, 57, 2020.

[CMS90] R. Carmona, W. C. Masters, and B. Simon. Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions. \textit{J. Funct. Anal.}, 91(1):117–142, 1990.

[CPKM05] J. H. Cushman, M. Park, N. Kleinfelter, and M. Moroni. Super-diffusion via Lévy lagrangian velocity processes. \textit{Geophysical Research Letters}, 32(19), 2005.
[CPP10] M.E. Caballero, J. Pardo, and J.L. Pérez. On the lamperti stable processes. *Probab. Math. Statist.*, 30:1–28, 03 2010.

[DFP06] M. Di Francesco and S. Polidoro. Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form. *Adv. Differential Equations*, 11(11):1261–1320, 2006.

[DK13] H. Dong and D. Kim. Schauder estimates for a class of non-local elliptic equations. *Discrete Contin. Dyn. Syst.*, 33(6):2319–2347, 2013.

[DPL95] G. Da Prato and A. Lunardi. On the Ornstein-Uhlenbeck operator in spaces of continuous functions. *J. Funct. Anal.*, 131(1):94–114, 1995.

[FM83] W. H. Fleming and S. K. Mitter. Optimal control and nonlinear filtering for nondegenerate diffusion processes. *Stochastics*, 8(1):63–77, 1982/83.

[FRRO17] X. Fernández-Real and X. Ros-Oton. Regularity theory for general stable operators: parabolic equations. *J. Funct. Anal.*, 272(10):4165–4221, 2017.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[HK07] C. Houdré and R. Kawai. On layered stable processes. *Bernoulli*, 13(1):252–278, 2007.

[HM16] L. Huang and S. Menozzi. A parametrix approach for some degenerate stable driven SDEs. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(4):1925–1975, 2016.

[HMP19] L. Huang, S. Menozzi, and E. Priola. $L^p$ estimates for degenerate non-local Kolmogorov operators. *J. Math. Pures Appl. (9)*, 121:162–215, 2019.

[Hör67] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[HPZ19] Z. Hao, X. Peng, and X. Zhang. Hörmander’s hypoelliptic theorem for nonlocal operators. *Journal of Theoretical Probability*, pages 1–47, 2019.

[HWZ20] Z. Hao, M. Wu, and X. Zhang. Schauder estimates for nonlocal kinetic equations and applications. *Journal de Mathématiques Pures et Appliquées*, 140:139 – 184, 2020.

[IJS18] C. Imbert, T. Jin, and R. Shvydkoy. Schauder estimates for an integro-differential equation with applications to a nonlocal Burgers equation. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(4):667–677, 2018.

[Ish16] Y. Ishikawa. *Stochastic calculus of variations*, volume 54 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, second edition, 2016. For jump processes.

[Jac01] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I*. Imperial College Press, London, 2001. Fourier analysis and semigroups.

[KK15] I. Kim and K.-H. Kim. A Hölder regularity theory for a class of non-local elliptic equations related to subordinate Brownian motions. *Potential Anal.*, 43(4):653–673, 2015.

[Kol34] A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35(1):116–117, 1934.

[KP10] N. V. Krylov and E. Priola. Elliptic and parabolic second-order PDEs with growing coefficients. *Comm. Partial Differential Equations*, 35(1):1–22, 2010.

[KS08] P. Kim and R. Song. Boundary behavior of harmonic functions for truncated stable processes. *J. Theoret. Probab.*, 21(2):287–321, 2008.

[Küh19] F. Kühn. Schauder estimates for equations associated with Lévy generators. *Integral Equations Operator Theory*, 91(2):Paper No. 10, 21, 2019.

[Lor05] L. Lorenzi. Schauder estimates for degenerate elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^N$. *Differential Integral Equations*, 18(5):531–566, 2005.
[LP94] E. Lanconelli and S. Polidoro. On a class of hypoelliptic evolution operators. Volume 52, pages 29–63. 1994. Partial differential equations, II (Turin, 1993).

[Lun97] A. Lunardi. Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24(1):133–164, 1997.

[Mar20] L. Marino. Schauder estimates for degenerate stable Kolmogorov equations. Bull. Sci. Math., 162:102885, 98, 2020.

[Pri09] E. Priola. Global Schauder estimates for a class of degenerate Kolmogorov equations. Studia Math., 194(2):117–153, 2009.

[Pri12] E. Priola. Pathwise uniqueness for singular SDEs driven by stable processes. Osaka J. Math., 49(2):421–447, 2012.

[PZ09] E. Priola and J. Zabczyk. Densities for Ornstein-Uhlenbeck processes with jumps. Bull. Lond. Math. Soc., 41(1):41–50, 2009.

[Ros07] J. Rosiński. Tempering stable processes. Stochastic Process. Appl., 117(6):677–707, 2007.

[ROS16] X. Ros-Oton and J. Serra. Regularity theory for general stable operators. J. Differential Equations, 260(12):8675–8715, 2016.

[Ryz02] M. Ryznar. Estimates of Green function for relativistic $\alpha$-stable process. Potential Anal., 17(1):1–23, 2002.

[Sai07] N. Saintier. Schauder estimates for degenerate elliptic and parabolic equations in $\mathbb{R}^N$ with Lipschitz drift. Differential Integral Equations, 20(4):397–428, 2007.

[Sat13] K. Sato. Lévy processes and infinitely divisible distributions, volume 68 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2013. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation.

[SSW12] R. L. Schilling, P. Sztonyk, and J. Wang. Coupling property and gradient estimates of Lévy processes via the symbol. Bernoulli, 18(4):1128–1149, 2012.

[SW12] R. L. Schilling and J. Wang. On the coupling property and the Liouville theorem for Ornstein-Uhlenbeck processes. J. Evol. Equ., 12(1):119–140, 2012.

[Tri92] H. Triebel. Theory of function spaces. II, volume 84 of Monographs in Mathematics. Birkhäuser Verlag, Basel, 1992.

[Zab92] J. Zabczyk. Mathematical control theory: an introduction. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.