MACROSCOPIC REGULARITY FOR THE RELATIVISTIC BOLTZMANN EQUATION WITH INITIAL SINGULARITIES

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Abstract. In this paper, it is proved that the macroscopic parts of the relativistic Boltzmann equation will be continuous, even though the macroscopic components are discontinuity initially. The Lorentz transformation plays an important role to prove the continuity of nonlinear term.

1. Introduction. The relativistic Boltzmann equation is written as

\[ p^\mu \partial_\mu F = C(F, F), \]

where \( F(t, x, p) \) is a distribution function for the gas particles at time \( t > 0 \), position \( x \in \Omega = \mathbb{R}^3 \) or \( T^3 \), and particle velocity \( p \in \mathbb{R}^3 \). The collision operator \( C(F, F) \) takes the following bilinear form

\[ C(F_1, F_2) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W(p, q, p', q') [F_1(p')F_2(q') - F_1(p)F_2(q)] \frac{dp'}{p_0'} \frac{dq'}{q_0'} \frac{dq}{q_0} \]

Here the translation rate \( W(p, q, p', q') \) is given by

\[ W(p, q, p', q') = c_2 s \sigma(g, \theta) \delta^4(p^\mu + q^\mu - p'^\mu - q'^\mu), \]

where \( \sigma(g, \theta) \) is the scattering kernel which measures the interactions between particles and Dirac function \( \delta^4 \) is the delta function of four variables. The constant \( c \) is the light speed, we normalize \( c \) to be 1 for simplicity of presentation. The relativistic momentum of a particle is denoted by \( p^\mu, \mu = 0, 1, 2, 3 \). We raise and lower the indices with the Minkowski metric \( g^{\mu\nu} = g_{\mu\nu} \), where \( g_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \). The signature of the metric is \( (-, +, +, +) \). For \( p \in \mathbb{R}^3 \), we write \( p^\mu = (p_0, p) \), where \( p_0 = \sqrt{|p|^2 + 1} \) represents the energy of a relativistic particle with velocity \( p \). Using

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Einstein convection of implicit summation over repeated indices, then the Lorentz inner product is given by
\[ p^\mu q_\mu = -p_0q_0 + \sum_{i=1}^{3} p_i q_i. \]
The streaming term of relativistic Boltzmann equation is given by
\[ p^\mu \partial_\mu = p_0 \partial_t + p \cdot \nabla_x. \]
Then the Boltzmann equation (1.1) becomes
\[ \partial_t F + \hat{p} \cdot \nabla_x F = Q(F, F). \quad (4) \]
with collision operator
\[ Q(F, F) = \frac{1}{p_0} C(F, F). \quad \text{And } \hat{p} \text{ is the normalized particle velocity which is given by} \]
\[ \hat{p} = \frac{p}{p_0} = \frac{p}{\sqrt{1 + |p|^2}}. \]
The steady solutions of this model are the well known Jüttner solution, also known as the relativistic Maxwellian, i.e.,
\[ J(p) = e^{-\frac{cp}{p_0}} \frac{1}{4\pi e^{\frac{c^2p^2}{2p_0}}}, \]
where \( K_2(z) \) is the Bessel function
\[ K_2(z) = \frac{z^2}{2\pi} \int_1^\infty e^{-zt} (t^2 - 1)^2 dt, \]
\( T \) is the temperature and \( k_B \) is the Boltzmann constant. Throughout this paper, we normalize all the physical constants to be one, including the speed of light. Then the normalized relativistic Maxwellian becomes
\[ J(p) = \frac{1}{4\pi} e^{-p_0}. \quad (5) \]
Now, we define the quantity \( s \), which is the square of the energy in the “center of momentum” system, \( p + q = 0 \), as
\[ s = s(p^\mu, q^\mu) = -(p^\mu + q^\mu)(p_\mu + q_\mu) = -2(p^\mu q_\mu - 1). \quad (6) \]
The relative momentum is denoted
\[ g^2 = g^2(p^\mu, q^\mu) = (p^\mu - q^\mu)(p_\mu - q_\mu) = -2(p^\mu q_\mu + 1) \geq 0. \quad (7) \]
It is easy to check that \( s = g^2 + 4 \). Conservations of momentum and energy for collisions are expressed as
\[ \begin{cases} p + q = p' + q', \\ p_0 + q_0 = p_0' + q_0'. \end{cases} \quad (8) \]
The scattering angle \( \theta \) is defined by
\[ \cos \theta = \frac{(p^\mu - q^\mu)(p'_\mu - q'_\mu)}{g^2}. \]
This angle is well defined under (8), see also [9]. Using the Lorentz transformations as described in [18], we can obtain
\[ Q(F_1, F_2) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\sigma(g, \theta) F_1(p') F_2(q') d\omega dq - \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\sigma(g, \theta) F_1(p) F_2(q) d\omega dq \]
\[ := Q_+(F_1, F_2) - Q_-(F_1, F_2), \quad (9) \]
where \( \nu = \nu(p,q) \) is the Moller velocity
\[
\nu = \nu(p,q) := 2\sqrt{\frac{p}{p_0} - \frac{q}{q_0} - \frac{p}{p_0} \times \frac{q}{q_0}^2} = \frac{g \sqrt{s}}{p_0 q_0} \leq 1. \tag{10}
\]
The post-collisional momentum in (1.8) satisfies
\[
\begin{align*}
p' &= \frac{1}{2}(p + q) + \frac{g}{2}\left(\omega + (\tilde{\gamma} - 1)(p + q) \frac{q + q'}{a + q} \right), \\
q' &= \frac{1}{2}(p + q) - \frac{g}{2}\left(\omega + (\tilde{\gamma} - 1)(p + q) \frac{q + q'}{a + q} \right).
\end{align*}
\]
where \( \tilde{\gamma} = \frac{p_0 q_0}{\sqrt{s}} \). And the energies are given by
\[
\begin{align*}
p_0' &= \frac{1}{2}(p_0 + q_0) + \frac{g}{2\sqrt{s}}(p + q) \cdot \omega, \\
q_0' &= \frac{1}{2}(p_0 + q_0) - \frac{g}{2\sqrt{s}}(p + q) \cdot \omega.
\end{align*}
\]
Throughout this paper, we consider both the hard and the soft potentials with angular cut-off, i.e.,

(1) for soft potentials,
\[
\frac{g}{\sqrt{s}} g^{-b} \sigma_0(\theta) \leq \sigma(g, \theta) \leq g^{-b} \sigma_0(\theta), \tag{11}
\]
where \( b, \gamma \) satisfies \( b \in (0, \min\{4, 4 + \gamma\}), \gamma > -2 \). Here we assume that \( \sigma_0(\theta) \leq \sin^{-1}(\theta) \), and \( \sigma_0(\theta) \) is non-zero on a set of positive measure.

(2) For hard potentials,
\[
\frac{g}{\sqrt{s}} g^{-b} \sigma_0(\theta) \leq \sigma(g, \theta) \leq (g^a + g^{-b}) \sigma_0(\theta), \tag{12}
\]
where \( \gamma > -2, a \in [0, 2 + \gamma], b \in [0, \min\{4, 4 + \gamma\}] \). We impose the relativistic Boltzmann equation (1) with an initial condition
\[
F(t, x, p)|_{t=0} = F_0(x, p) \geq 0. \tag{13}
\]

There are many mathematical investigations in the Boltzmann equation and relativistic Boltzmann equation. In 1940, Lichnerowicz-Marrot [16] derived the relativistic Boltzmann equation which is a fundamental model for relativistic particles whose speed is comparable to the speed of light. In 1992, Dudyński and Ekiel-Jeżewska [4] obtained the global existence of the Diperna-Lions renormalized solution of the relativistic Boltzmann equation by using their causality results [5, 6] for the case of large initial data. Dudyński and Ekiel-Jeżewska [8, 7] studied the linearized relativistic Boltzmann equation.

For small initial data, there are lots of results on the existence and uniqueness of global solutions to the relativistic Boltzmann equation. The local existence and uniqueness were firstly investigated by Bichteler [1] in the \( L^\infty \) framework under smallness conditions on the initial data. In 1993, Glassey and Strauss [10] proved the global existence of smooth solution on the torus for the relativistic Boltzmann equation, and in the meanwhile obtained the exponential decay rate was also obtained. It is noted that they [10] considered only the hard potential cases. In 1995, it was further extended to the Cauchy problem [11]. In 2006, Hsiao and Yu [14] reduced the restrictions on the cross-section of [10], but still fell into the case of hard potentials. In 2006, Glassey [12] constructed a global continuous mild solution to relativistic Boltzmann equation near vacuum. In 2010, Strain [17] proved the global existence of \( L^\infty \)-mild solution to the relativistic Boltzmann equation in n torus for
the soft potentials. In 2016, by using $L^1_tL^\infty_p \cap L^\infty_{x,\phi}$ method, Wang [19] obtained the
global existence of $L^\infty$-mild solution to relativistic Boltzmann equation even though
the initial data may have large amplitude, which greatly extended the result [17].

It is an interesting problem to consider the macroscopic regularity of the Boltz-
mann equation. For the case with angular cut-off, it is believed that the ini-
tial singularities propagate in time since the Boltzmann equation is hyperbolic.
This property was proved by Boudin-Desvillettes [2] with propagation of Sobolev
$H^{\frac{3}{2}}$ singularity in the case near vacuum, later by Duan-Li-Yang [3] in the case
near global Maxwellian. In fact, the famous averaging lemma [13] shows that
$\int_{\mathbb{R}^3} F(t,x,p)\varphi(p)dp \in H^{\frac{3}{2}}(t,x)$ for any smooth compact support function $\varphi(p)$.
This indicates that the macroscopic components like density, momentum and total
energy probably have $H^{\frac{3}{2}}(t,x)$ regularity. However more regularity is not known
from the average lemma. Recently, Huang-Wang [15] proved that the macroscopic
parts of the Boltzmann equation are continuous for any positive time even though it
is discontinuous at the initial time.

The purpose of this paper is to investigate the regularity of macroscopic quantities
of solutions to the relativistic Boltzmann equation (1) with angular cut-off even
initial singularities are contained. For this, we firstly define a weight function

$$w(p) = e^{\alpha p_0}, \ \alpha > 0.$$  \hspace{1cm} (14)

We remark that we have to choose the exponential weight (14) but not some poly-
nomial weight. This is mainly due to that the polynomial weight is not compatible
with Lorentz transformation.

For any fixed $(t,x) \in (0, +\infty) \times \Omega$, we assume the initial data $F_0$ satisfies

$$\|wF_0\|_{L^\infty_{x,p}} < +\infty \quad \text{and} \quad \int_{\mathbb{R}^3} |F_0(x + \delta - \hat{p}(t + \eta), p) - F_0(x - \hat{p}t, p)|dp \to 0, \ \text{as} \ \eta, \delta \to 0. \hspace{1cm} (15)$$

We note that there is a large class of initial data satisfying (15). For example, we can choose

$$F_0(x,p) = \rho_0(x)(1 + P(p)) \exp \left( - \frac{p_0}{2} \right), \hspace{1cm} (16)$$

where $|P(p)| \leq C(1 + |p|)^k$ holds for some positive constant $k \geq 0$, which is allowed
to be discontinuous for $p \in \mathbb{R}^3$, $\rho_0(x)$ may also be discontinuous in $x \in \mathbb{R}^3$ and
satisfies

$$0 \leq \rho_0(x) \leq \hat{C} < +\infty, \ \|\rho_0 - 1\|_{L^l} < +\infty, \ \text{for some} \ l \in [1, \infty). \hspace{1cm} (17)$$

One can check that $F_0(x,v)$ in (16) satisfies the condition (15), see appendix for
details.

The solutions considered in this paper are in the following space $X$:

Near relativistic Maxwellian $J$, $F(t,x,p) \geq 0$ for a.e. $(t,x,p) \in (0, +\infty) \times \Omega \times \mathbb{R}^3$
with $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$,

$$w(\cdot)[F(\cdot,\cdot,\cdot) - J(\cdot)] \in L^{\frac{m}{m+1}}(0,T;L^2_{x,p}) \quad \text{and} \quad \hspace{1cm} (18)$$

$$w(\cdot)F(\cdot,\cdot,\cdot) \in L^m(0,T;L^\infty_{x,p}) \ \text{for} \ m > 2;$$

Near vacuum, $F(t,x,p) \geq 0$ for a.e. $(t,x,p) \in (0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$w(\cdot)F(\cdot,\cdot,\cdot) \in L^{\frac{m}{m+1}}(0,T;L^2_{x,p}) \quad \text{and} \quad \hspace{1cm} (19)$$

$$w(\cdot)F(\cdot,\cdot,\cdot) \in L^m(0,T;L^\infty_{x,p}) \ \text{for} \ m > 2.
One can check that $X$ is not empty. For example, in the case of near relativistic Maxwellian, the global $L^\infty$-mild solutions constructed in [17, 19] indeed belong to the space $X$. Especially, the solution of [19] allows large amplitude initial data. For the sake of completeness, we shall write down the results in [19] in appendix.

Then our main results are as follows:

**Theorem 1.1.** For any given $0 < \alpha \ll 1$, let the initial data $F_0$ satisfy (15). Let $F(t) \in X$ be the mild solution to the relativistic Boltzmann equation (4), (13), then the macroscopic components of solutions $F(t)$ are continuous functions of $(x, t) \in \Omega \times (0, +\infty)$.

Moreover, if the initial data $F_0$ further satisfies

$$
\sup_{(x,t) \in \Omega \times [t_1, T]} \int_{\mathbb{R}^3_{loc}} |F_0(x + \delta - \hat{p}(t + \eta), p) - F_0(x - \hat{p}t, p)|dp \to 0, \text{ as } \eta, \delta \to 0. \quad (20)
$$

where $t_1$ and $T$ are any given times with $0 < t_1 < T < +\infty$, then the macroscopic components of solutions $F(t)$ are uniformly continuous functions of $(x, t) \in \Omega \times [t_1, T]$.

A few remarks are in order.

**Remark 1.** It is noted that the condition (20) is slightly stronger than (15). Indeed, the example given in (16) satisfies (20) for any $0 < t_1 < T < +\infty$, see the appendix below.

**Remark 2.** It is noted that the solutions constructed in [17, 19], belong to $X$. Moreover, the solutions of [17, 19] allow the initial macroscopic components to be discontinuous. If their initial data further satisfy (15), then the macroscopic quantities of these solutions are continuous in $(x, t)$. Thus, Theorems 1.1 are self-contained in this sense.

**Remark 3.** We have to use the exponential weight function $w(p) = e^{\alpha p_0}$, $0 < \alpha \ll 1$ but not a polynomial weight. Indeed, if we use the polynomial weight $w(p) = p_0^\beta$, $\beta > 0$, then we will meet difficulty when we utilize Lorentz transformation to treat the nonlinear term $Q_+(F, F)$. Especially, we can not bound the key quantity $A(p, q)$ in (52).

It is noted that the relativistic Boltzmann equation is essentially transport, the solution propagates along the direction of normalized velocity $\hat{p}$. Thus the integral with respect to $p$ can be translated to $x$ by changing variables. The continuity of macroscopic components is then derived from the continuity of translations on $L^2_{x,p}$. Indeed, the changing of variable is the main difficulty in the relativistic Boltzmann equation due to the complicated form of collision kernel. To realize the changing of variable, one has to use the Lorentz transformation, and this is why we need the exponential weight (14).

**Notations.** Throughout this paper, $C$ denotes a generic positive constant which may depend on $a, b, \gamma, \alpha$ and may vary from line to line. $C(t)$ denotes the generic positive continuous function depending on time $t > 0$ and $\gamma$ which also may vary from line to line. $\| \cdot \|_{L^2}$ denotes the standard $L^2(\Omega_x \times \mathbb{R}^3_p)$-norm, and $\| \cdot \|_{L^\infty}$ denotes the $L^\infty(\Omega_x \times \mathbb{R}^3_p)$-norm.
2. Proof of the main Theorem. Let $F(t, x, p)$ be the mild solution of the relativistic Boltzmann equation (4), then for $(t, x, p) \in (0, +\infty) \times \Omega \times \mathbb{R}^3$, one has

$$
F(t, x, p) = F_0(x - \dot{p}t, p) e^{-\int_0^t g(\tau; t, x, p) d\tau} + \int_0^t e^{\int_0^\tau g(\tau'; t, x, p) d\tau'} Q_+(F, F)(s, x - \tilde{p}(t - s), p) ds
$$

$$
:= I_1(t, x, p) + I_2(t, x, p),
$$

(21)

where

$$
g(\tau; t, x, p) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\delta \sigma(g, \theta) F(\tau, x - \tilde{p}(t - \tau), q) d\omega dq \geq 0.
$$

(22)

2.1. Proof of Theorem 1.1 for $\Omega = \mathbb{R}^3$. Part I. Continuity of Macroscopic Components:

Let $\Omega = \mathbb{R}^3$, for any fixed $(t, x) \in (0, +\infty) \times \mathbb{R}^3$, let $\eta \in \mathbb{R}$, $\delta \in \mathbb{R}^3$ with $|\eta| \leq \min\{1, \frac{5}{2}\}$ and $|\delta| \leq 1$. The continuity of density $\rho(x, t)$ in $(x, t)$ is equivalent to show that

$$
\left| \int_{\mathbb{R}^3} F(t + \eta, x + \delta, p) - F(t, x, p) dp \right| \to 0 \text{ as } \eta, \delta \to 0.
$$

That is, for any $\varepsilon > 0$, we need to prove that there exists $\chi > 0$, which may depend on $(x, t)$ and $\varepsilon$, such that if $|\eta| \leq \chi$ and $|\delta| \leq \chi$,

$$
\left| \int_{\mathbb{R}^3} F(t + \eta, x + \delta, p) - F(t, x, p) dp \right| \leq \varepsilon.
$$

We assume $\eta \geq 0$ without loss of generality and denote $t_\eta = t + \eta$ for notation simplicity. We divide the proof into two parts. The proof is based on the initial condition (15) and the continuity of translations on $L^1_{t,x,p}$.

Firstly, we introduce a lemma in [19], which will be used frequently later.

Lemma 2.1 ([19]). Let (11), (12) hold, and $\gamma > -2, a \in [0, 2 + \gamma], b \in [0, \min\{4, 2 + \gamma\})$. For any fixed $c > 0$ and $1 \leq d < \min\left\{\frac{2}{\max\{2 - \gamma, 1\}}, \frac{3}{\max\{b - 1, 1\}}\right\}$, it holds that

$$
\int_{\mathbb{R}^3} |\nu_\delta \sigma(g, \theta)|^d e^{-c \rho_0} d\omega dq \equiv \begin{cases} 
(p_0^2)^d, & \text{for hard potentials}, \\
(p_0^2)^d, & \text{for soft potentials}.
\end{cases}
$$

Estimation on $\int_{\mathbb{R}^3} I_1(t, x, p) dp$: Firstly, we note that

$$
H_1 := \left| \int_{\mathbb{R}^3} (I_{t_\eta}(t, x + \delta, p) - I_1(t, x, p)) dp \right|
$$

$$
\leq \int_{\mathbb{R}^3} F_0(x + \delta - \tilde{p}t_\eta, p) - F_0(x - \tilde{p}t, p) dp
$$

$$
+ \int_{\mathbb{R}^3} F_0(x - \tilde{p}t, p) \int_t^{t_\eta} g(\tau; t_\eta, x + \delta, p) d\tau dp
$$

$$
+ \int_0^t \int_{\mathbb{R}^3} F_0(x - \tilde{p}t, p) g(\tau; t_\eta, x + \delta, p) - g(\tau; t, x, p) d\rho dt d\tau
$$

$$
:= H_{11} + H_{12} + H_{13}.
$$

(23)

Denote $\tilde{F}(\eta, \delta, p) := F_0(x + \delta - \tilde{p}t_\eta, p) - F_0(x - \tilde{p}t, p)$, then it follows from (15) that

$$
H_{11} \leq \int_{p_0 \geq N} |\tilde{F}(\eta, \delta, p)| dp + \int_{p_0 \leq N} |\tilde{F}(\eta, \delta, p)| dp
$$

$$
+ \int_0^t \int_{\mathbb{R}^3} F_0(x - \tilde{p}t, p) g(\tau; t_\eta, x + \delta, p) - g(\tau; t, x, p) d\rho dt d\tau
$$

$$
:= H_{11} + H_{12} + H_{13}.
$$

(23)
\[
\leq C \|wF_0\|_{L^\infty} \int_{p_0 \geq N} e^{-\alpha_0 \nu} dp + \int_{p_0 \leq N} \left| \hat{F}(\eta, \delta, p) \right| dp \\
\leq C e^{-\alpha N} + \int_{p_0 \leq N} \left| \hat{F}(\eta, \delta, p) \right| dp.
\] (24)

We firstly take \( N \) large such that \( C \leq \frac{\varepsilon}{12} \), for. On the other hand, from (15), there exists \( \chi_1 > 0 \) depending only on \((t, x)\) and \( \varepsilon \) such that if \( \eta + |\delta| \leq \chi_1 \), it holds that
\[
H_{11} \leq \frac{\varepsilon}{16}.
\] (25)

For \( H_{12} \), it follows from (22) that
\[
H_{12} \leq C \|wF_0\|_{L^\infty} \int_0^t \int_{R^3} \int_{S^2} w(p)^{-1} w(q)^{-1} \nu_\phi \sigma(g, \theta) \|wF(\tau)\|_{L^\infty} d\omega dq d\tau dp \\
\leq C \int_0^t \|wF(\tau)\|_{L^\infty} d\tau \int_0^t \int_{R^3} \int_{S^2} w(p)^{-1} \nu_\phi \sigma(g, \theta) w(q)^{-1} d\omega dp dq \\
\leq C \eta^{-1} \left( \int_0^t \|wF(\tau)\|_{L^\infty} d\tau \right) \frac{1}{\eta} \int_{R^3} w(p)^{-1} \max\{p_0^2, p_0^{-2}\} dp \\
\leq C(t) \eta^{-1} \frac{1}{\varepsilon} \to 0, \text{ as } \eta \to 0,
\] (26)

where we have used the fact that \( F \in X \) and the Lemma 2.1. Hence there exists \( \chi_2 > 0 \) depending on \( t, \varepsilon \) such that if \( \eta \leq \chi_2 \), it holds that
\[
H_{12} \leq \frac{\varepsilon}{16}.
\] (27)

For \( H_{13} \), it is noted that
\[
H_{13} \leq \|wF_0\|_{L^\infty} \int_0^t \int_{R^3} \int_{S^2} w(p)^{-1} w(q)^{-1} \nu_\phi \sigma(g, \theta) \\
\times \left| w(q) [F(\tau, x + \delta - \hat{p}(t_\eta - \tau), q) - F(\tau, x - \hat{p}(t - \tau), q)] \right| d\omega dq d\tau dp \\
\leq C \left\{ \int_0^t \int_{p_0 \geq N} \int_{p_0 < N} + \int_0^t \int_{p_0 < N} \right\} \left( \cdots \right) d\omega dq d\tau dp \\
:= H_{131} + H_{132} + H_{133}.
\] (28)

A direct calculation shows that
\[
H_{131} \leq C \int_0^t \|wF(\tau)\|_{L^\infty} \int_{R^3} \int_{S^2} w(p)^{-1} w(q)^{-1} \nu_\phi \sigma(g, \theta) d\omega dp dq d\tau \\
\leq C \|wF\|_{L^p_t(L^\infty_{x,p})} \eta^{-1} \frac{1}{\varepsilon} \leq \frac{\varepsilon}{32}, \text{ if } \lambda \text{ small enough.}
\] (29)

For \( H_{132} \), one has
\[
H_{132} \leq C \int_0^t \|wF(\tau)\|_{L^\infty} d\tau \int_{p_0 \geq N} w(p)^{-1} w(q)^{-1} \nu_\phi \sigma(g, \theta) d\omega dp dq \\
\leq C(t) \int_{p_0 \geq N} w(p)^{-1} \max\{p_0^2, p_0^{-2}\} dp \\
\leq C(t) e^{-\frac{\alpha N}{2}} \int_{R^3} w(p)^{-\frac{1}{2}} \max\{p_0^2, p_0^{-2}\} dp \\
\leq \frac{\varepsilon}{32}, \text{ if } N \text{ large enough.}
\] (30)
Using the change of variable $y$

For $H_{133}$, it follows from Lemma 2.1 that

$$H_{133} \leq C \int_0^{t-\lambda} \int_{p_0 < N} \int_{\mathbb{R}^3} w(p)^{-1} w(q)^{-1} \nu_p \sigma(g, \theta)$$

$$\times \left| w(q) [F(\tau, x + \delta - \hat{p}(t_\eta - \tau), q) - F(\tau, x - \hat{p}(t_\eta - \tau), q)] \right| \, dw \, dq \, dp \, d\tau$$

$$+ C \int_0^{t-\lambda} \int_{p_0 < N} \int_{\mathbb{R}^3} w(p)^{-1} w(q)^{-1} \nu_p \sigma(g, \theta)$$

$$\times \left| w(q) [F(\tau, x - \hat{p}(t_\eta - \tau), q) - F(\tau, x - \hat{p}(t - \tau), q)] \right| \, dw \, dq \, dp \, d\tau$$

$$:= H_{1331} + H_{1332}. \quad (31)$$

From (11), (12), we know $\gamma > -2, b \in [0, \min\{4, 4 + \gamma\})$, thus it holds that

$$1 < \min \left\{ \frac{2}{\max\{-\gamma, 1\}}, \frac{3}{\max\{b - 1, 1\}} \right\} < 2.$$

Choosing $l \in (1, \min\left\{ \frac{2}{\max\{-\gamma, 1\}}, \frac{3}{\max\{b - 1, 1\}} \right\}$, then one has

$$H_{1331} \leq C \int_0^{t-\lambda} \left\{ \int_{p_0 < N} \int_{\mathbb{R}^3 \times S^2} \left| w(p)^{-1} w(q)^{-1} \nu_p \sigma(g, \theta) \right|^l \, dw \, dq \, dp \right\} \frac{1}{l}$$

$$\times \left\{ \int_{p_0 < N} \int_{\mathbb{R}^3} \left| w(q) [F(\tau, y + \delta, q) - F(\tau, y, q)] \right|^l \, dq \, dp \right\} \frac{1}{l} \, d\tau$$

$$\leq C(t, N) \lambda^{\frac{2}{3}} \int_0^t \left\| w(F(\tau, \cdot + \delta, \cdot) - F(\tau, \cdot, \cdot)) \right\|_{L^2} \left\| w F(\tau) \right\|_{L^\infty} \, d\tau$$

$$\leq C(t, N) \lambda^{\frac{2}{3}} \left( \int_0^t \left\| \cdot \right\|_{L^m} \, d\tau \right)^{\frac{1}{m} - \frac{2}{m_t}} \left( \int_0^t \left\| \cdot \right\|_{L^m} \, d\tau \right)^{\frac{2(m-1)}{m_t}}$$

$$\leq C(t, N) \lambda^{\frac{2}{3}} \left( \int_0^t \left\| w(F(\tau, \cdot + \delta, \cdot) - F(\tau, \cdot, \cdot)) \right\|_{L^m} \, d\tau \right)^{\frac{2(m-1)}{m_t}}$$

$$\leq \frac{\varepsilon}{64}, \quad \text{if } \delta \text{ small enough,} \quad (32)$$

where $l_* = \frac{l}{1 - \lambda}$, and $\tilde{d} = 1 - \frac{1}{m} + \frac{4}{m_t} - \frac{2}{m_t} = (1 - \frac{1}{m})(1 - \frac{2}{m_t}) + \frac{2}{m_t} > 0$ due to $m > 2, l_* > 2$.

For $H_{1332}$, it is more complicated. By the definition of $X$, there exists a smooth compact support function $F^\varepsilon(\cdot, \cdot, \cdot)$ such that

$$\left( \int_0^t \left\| w(F(\tau, \cdot, \cdot) - F^\varepsilon(\tau, \cdot, \cdot)) \right\|_{L^2} \, d\tau \right)^{\frac{2(m-1)}{m_t}} \leq \frac{\lambda^{\frac{2}{3}} \varepsilon}{C_1}, \quad (33)$$
where $C_1$ will be chosen later. Using (33), we have

$$H_{1332} \leq C \int_0^{t-\lambda} \int_{p_0 \leq N} \int_{\mathbb{R}^3 \times S^2} w(p)^{-1} w(q)^{-1} \nu_\sigma(g, \theta)$$

$$\times \left\{ \left| w(q) [F(\tau, x - \hat{p}(t_\eta - \tau), q) - F^\varepsilon(\tau, x - \hat{p}(t_\eta - \tau), q)] \right| + \left| w(q) [F(\tau, x - \hat{p}(t - \tau), q) - F^\varepsilon(\tau, x - \hat{p}(t - \tau), q)] \right| + \left| w(q) [F^\varepsilon(\tau, x - \hat{p}(t_\eta - \tau), q) - F^\varepsilon(\tau, x - \hat{p}(t - \tau), q)] \right| \right\} \frac{d\omega dq dp dt}{\nu \sigma}$$

$$\leq C(t, N) \chi^2 \left( \int_0^t \| F(\tau, \cdot, \cdot) - F^\varepsilon(\tau, \cdot, \cdot) \|_{L^1} \right) \left( \int_0^t \| F^\varepsilon(\tau, y - \hat{\nu} \eta, q) - F^\varepsilon(\tau, y, q) \| \right)$$

$$\leq C(t, N) \frac{\varepsilon}{C_1} + \frac{\varepsilon}{128} \leq \frac{\varepsilon}{64}, \text{ if } \eta \text{ small enough},$$

(34)

where we have used the fact that

$$\lim_{\eta \to 0} C_{N, \varepsilon} t \sup_{|p| \leq N, (r, y, q) \in (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3} |w(q)[F^\varepsilon(r, y - \hat{\nu} \eta, q) - F^\varepsilon(r, y, q)]| = 0,$$

(35)

and $C_1$ is chosen so large that (33) holds. Thus, from (28)-(34), there exists $\chi_3 > 0$ depending on $(t, x)$ and $\varepsilon$ such that if $|\delta| + \eta \leq \chi_3$, it holds that

$$H_{13} \leq \frac{\varepsilon}{8}.$$  

(36)

Therefore, it follows from (23), (25), (27) and (36) that

$$\left| \int_{\mathbb{R}^3} \left\{ I_1(t + \eta, x + \delta, p) - I_1(t, x, p) \right\} dp \right| \leq \frac{\varepsilon}{2}, \text{ for } \eta + |\delta| \leq \min\{ \chi_1, \chi_2, \chi_3 \}.$$  

(37)

**Estimation on** $\int_{\mathbb{R}^3} I_2(t, x, p) dp$: It is noted that

$$H_2 := \left| \int_{\mathbb{R}^3} I_2(t_\eta, x + \delta, p) - I_2(t, x, p) dp \right|$$

$$\leq \left| \int_{\mathbb{R}^3} I_2(t_\eta, x + \delta, p) - I_2(t_\eta, x, p) dp \right| + \left| \int_{\mathbb{R}^3} I_2(t_\eta, x, p) - I_2(t, x, p) dp \right|$$

$$:= H_{21} + H_{22}.$$  

(38)

For $H_{21}$, it follows from (21) that

$$H_{21} \leq \int_0^{t_\eta} \int_{\mathbb{R}^3} Q_+(F, F)(s, x + \delta - \hat{p}(t_\eta - s), p)$$

$$\int_s^{t_\eta} |g(\tau; t_\eta, x + \delta, p) - g(\tau; t_\eta, x, p)| d\tau dp ds$$

$$+ \int_0^{t_\eta} \int_{\mathbb{R}^3} \left| Q_+(F, F)(s, x + \delta - \hat{p}(t_\eta - s), p) - Q_+(F, F)(s, x - \hat{p}(t_\eta - s), p) \right| dp ds$$

$$:= H_{211} + H_{212}.$$  

(39)
Using Lemma 2.1, we have

$$Q_+(F,F)(s,y,p)$$

\[
\leq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g,\theta) w(p')^{-1} w(q')^{-1} |w(p') F(s,y,p')| \cdot |w(q') F(s,y,q')| d\omega dq
\]

\[
\leq \|w F(s)\|_{L^\infty}^2 \int_{\mathbb{R}^3 \times \mathbb{S}^2} w(p)^{-1} w(q)^{-1} \nu_\phi \sigma(g,\theta) d\omega dq
\]

\[
\leq C w(p)^{-1} \max\{p_0^2, p_0^{-\frac{1}{2}}\} \|w F(s)\|_{L^\infty}^2
\]

\[
\leq C w(p)^{-\frac{1}{2}} \|w F(s)\|_{L^\infty}^2,
\]

and we have used the fact that

\[w(p') w(q') = w(p) w(q),\]

Thus, it follows from (22) and (40) that

\[
H_{2111} \leq C(t) \int_{t_\lambda}^{t_\eta} \|w F(\tau)\|_{L^\infty} d\tau \int_{\mathbb{R}^3} w(p)^{-\frac{1}{2}} dp
\]

\[
\leq C(t)(\eta + \lambda)^{1-\frac{1}{2}} \leq \frac{\varepsilon}{64}, \quad \text{if} \quad \eta, \lambda \text{ small enough.} \tag{43}
\]

and

\[
H_{2112} \leq C \int_{0}^{t_\lambda} \|w F(\tau)\|_{L^\infty} d\tau \int_{p_0 N} w(p)^{-\frac{1}{2}} dp
\]

\[
\leq C(t) e^{-\frac{\tau}{2} N} \leq \frac{\varepsilon}{64}, \quad \text{if} \quad N \text{ large enough.} \tag{44}
\]

On the other hand, for any fixed $\lambda > 0$ and $N$, it follows from Lemma 2.1 that

\[
H_{2113} \leq C(t) \int_{0}^{t_\lambda} \left( \int_{p_0 N} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \{\nu_\phi \sigma(g,\theta) w(q)^{-1} w(p)^{-\frac{1}{2}}\}^{1} d\omega dq dp \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_{p_0 N} \int_{\mathbb{R}^3} |w(q)| [F(\tau, x + \delta - \hat{p}(t_\eta - \tau), q) - F(\tau, x - \hat{p}(t_\eta - \tau), q)]^{1} dq dp \right)^{\frac{1}{2}} d\tau
\]

\[
\leq C(t) \int_{0}^{t_\lambda} \left( \int_{p_0 N} \int_{\mathbb{R}^3} |w(q)| [F(\tau, x + \delta - \hat{p}(t_\eta - \tau), q)] dq dp \right)^{\frac{1}{2}} d\tau
\]
It follows from (30) that
\[ -\mathcal{F}(\tau, x - \hat{p}(t_\eta - \tau), \nu) \bigg|_{t_\nu}^{t_0} dq dp \bigg\}^2 d\tau \]
\[ \leq C(t)\lambda^{\frac{\lambda}{2}} (1 + N^2) \frac{1}{\pi^2} \left( \int_0^{t-\lambda} \|\omega(q)(\mathcal{F}(\tau, \cdot + \delta, \cdot) - \mathcal{F}(\tau, \cdot, \cdot))\|_{L^2}^{m-1} d\tau \right)^{\frac{2(m-1)}{m+1}} \]
\[ \to 0, \text{ as } \delta \to 0. \] (45)

For \( H_{212} \), we have
\[ H_{212} \leq \int_0^{t_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g, \theta) \mathcal{F}(s, x + \delta - \hat{p}(t_\eta - s), q') \]
\[ \times \left| \mathcal{F}(s, x + \delta - \hat{p}(t_\eta - s), p') - \mathcal{F}(s, x - \hat{p}(t_\eta - s), p') \right| dq dp dq dp ds \]
\[ + \int_0^{t_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g, \theta) \mathcal{F}(s, x - \hat{p}(t_\eta - s), p') \]
\[ \times \left| \mathcal{F}(s, x + \delta - p(t_\eta - s), q') - \mathcal{F}(s, x - \hat{p}(t_\eta - s), q') \right| dq dp dq dp ds \]
\[ := H_{2121} + H_{2122}. \] (46)

We only consider \( H_{2121} \), since \( H_{2122} \) can be treated similarly. It is noted that
\[ H_{2121} \leq \int_0^{t_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g, \theta) w(q')^{-1} w(p')^{-1} \|\mathcal{F}(s)\|_{L^\infty} \]
\[ \times \left| w(p') \mathcal{F}(s, x + \delta - \hat{p}(t_\eta - s), p') - \mathcal{F}(s, x - \hat{p}(t_\eta - s), p') \right| dq dp dq dp ds \]
\[ \leq C \int_0^{t-\lambda} \|\mathcal{F}(s)\|_{L^\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g, \theta) w(q')^{-1} w(p')^{-1} dq dp dq dp ds \]
\[ + C \int_0^{t-\lambda} \|\mathcal{F}(s)\|_{L^\infty} \int_{p_0 > N} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g, \theta) w(q')^{-1} w(p')^{-1} \]
\[ \times \left| w(p') \mathcal{F}(s, x + \delta - \hat{p}(t_\eta - s), p') - \mathcal{F}(s, x - \hat{p}(t_\eta - s), p') \right| dq dp dq dp ds \]
\[ + C \int_0^{t-\lambda} \|\mathcal{F}(s)\|_{L^\infty} \int_{p_0 \leq N} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g, \theta) w(q')^{-1} w(p')^{-1} \]
\[ \times \left| w(p') \mathcal{F}(s, x + \delta - \hat{p}(t_\eta - s), p') - \mathcal{F}(s, x - \hat{p}(t_\eta - s), p') \right| dq dp dq dp ds \]
\[ := H_{21211} + H_{21212} + H_{21213}. \] (47)

Noting (41), a direct calculation shows that
\[ H_{21211} \leq C(t)(\eta + \lambda)^{1 - \frac{3}{2}}, \text{ if } \lambda, \eta \text{ small enough.} \] (48)

It follows from (30) that
\[ H_{21212} \leq C \int_0^{t-\lambda} \|\mathcal{F}(s)\|_{L^\infty}^2 \int_{p_0 > N} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g, \theta) w(q')^{-1} w(p')^{-1} dq dp dq dp ds \]
\[ \leq C(t)N^{-\frac{3}{4}} \leq \frac{\varepsilon}{16}, \text{ if } N \text{ large enough.} \] (49)

To estimate \( H_{21213} \), we need to use the changing of variables to translate the continuity of macroscopic into the continuity of translations in \( L^2_{\omega, \nu} \). But it is difficult to realize this process since the collision kernel is very complicated in the relativistic Boltzmann equation. And it is in this part that we need the exponential velocity weight \( w(p) \) but not a polynomial weight.
Following [19], we firstly note that

\[
H_{2123} \leq \int_0^{t-\lambda} \|wF(s)\|_{L^\infty} \left\{ \int_{p_0 \leq N} \int_{\mathbb{R}^3} \int_{S^2} \left| \nu_{\phi}(g, \theta) \right| |w(q)|^{-1} |w(p)|^{-1} d\omega dp dq \right\}^{\frac{1}{r}} \\
\times \left\{ \int_{p_0 \leq N} \int_{\mathbb{R}^3} \int_{S^2} \left| w(p') \right| \left| F(s, x + \delta - \hat{p}(t\eta - s), p') - F(s, x - \hat{p}(t\eta - s), p') \right| \right\}^{\frac{1}{r}} \\
\times w(q)^{-1} |w(p)|^{-1} d\omega dp dq \right\}^{\frac{1}{r}} ds,
\]

where we have used Lemma 2.1 and the fact \( d\omega = \sqrt{2g} \delta(4)(p^\mu + q^\mu - p'^\mu - q'^\mu) \frac{dp' dq'}{p_0 q_0}. \)

Exchanging the variables \( p' \) and \( q \) in (50) and using (41), one obtains that

\[
H_{2123} \leq \int_0^{t-\lambda} \|wF(s)\|_{L^\infty} \left\{ \int_{p_0 \leq N} \int_{\mathbb{R}^3} \int_{S^2} \left( w(q)^{-1} w(p)^{-1} \sqrt{2g} \delta(4)(p^\mu + q^\mu - p'^\mu - q'^\mu) \frac{dp' dq'}{p_0 q_0} \right) \right\}^{\frac{1}{r}} ds
\]

\[
= \int_0^{t-\lambda} \|wF(s)\|_{L^\infty} \left\{ \int_{p_0 \leq N} \int_{\mathbb{R}^3} \left( w(q)^{-1} w(p)^{-1} \sqrt{2g} \delta(4)(p^\mu + q^\mu - p'^\mu - q'^\mu) \frac{dp' dq'}{p_0 q_0} \right) \right\}^{\frac{1}{r}} ds,
\]

where

\[
\begin{align*}
\hat{g}^2 &:= |g(p^\mu, p'^\mu)|^2 \equiv g^2 + \frac{1}{2} (p^\mu + q^\mu)(p_\mu + q_\mu - p'_\mu - q'_\mu) , \\
\hat{s} &:= 4 + \hat{g}^2, \quad \cos \theta = 1 - 2 \left( \frac{\hat{g}}{\sqrt{2}} \right)^2 , \\
A(p, q) &:= \int_{p_0 \leq N} \int_{\mathbb{R}^3} \frac{p'_0}{q'_0} \frac{w^{-\frac{1}{2}}}{w^{-\frac{1}{2}}(q') \sqrt{2g} \delta(4)(p^\mu + q^\mu - p'^\mu - q'^\mu) \frac{dp' dq'}{p_0 q_0} } .
\end{align*}
\]

Define

\[
u = u(r) = \begin{cases} 
0, & \text{if } r < 0 , \\
1, & \text{if } r \geq 0 .
\end{cases}
\]

Let \( g := g(p^\mu, q^\mu) \) and \( \hat{s} := s(p^\mu, q^\mu) \), then the following identity [17, 19] holds

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} G(p^\mu, q^\mu|p'^\mu, q'^\mu) \frac{dp' dq'}{p_0 q_0} = 16 \int_{\mathbb{R}^4 \times \mathbb{R}^4} G(p^\mu, q^\mu|p'^\mu, q'^\mu) d\Theta(p^\mu, q^\mu),
\]

with

\[
d\Theta(p^\mu, q^\mu) = u(p_0 + q_0) u(s - 4) \delta(s - \hat{g}^2 - 4) \delta((p^\mu + q^\mu)(p_\mu - q_\mu)) dp^\mu dq^\mu ,
\]
Applying the identity (53) to (52), one has that
\[
A(p, q) = \frac{16}{q_0} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{\sqrt{s}}{2g} \delta^s(p^\mu + p^\nu - q^\mu - q^\nu) w^{-\frac{1}{2}}(p') w^{-\frac{1}{2}}(q') p_0' d\Theta(p^\mu', q^\nu')
\]
\[
\leq \frac{C}{q_0} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{\sqrt{s}}{2g} \delta^s(p^\mu + p^\nu - q^\mu - q^\nu) w^{-\frac{1}{2}}(q') d\Theta(p^\mu', q^\nu').
\]
We consider the change of variables
\[
\bar{p}^\mu = p^\mu + q^\nu, \quad \bar{q}^\mu = p^\mu - q^\nu,
\]
which yields immediately that
\[
A(p, q) \leq \frac{C}{q_0} \int_{\mathbb{R}^4} \frac{\sqrt{s}}{2g} \delta^s(p^\mu - q^\mu + \bar{q}^\mu) e^{-\frac{1}{2}(\bar{p}_0 - \bar{q}_0)} d\Theta(p^\mu', q^\nu')
\]
\[
\leq \frac{C}{q_0} \int_{\mathbb{R}^4} \frac{\sqrt{s}}{2g} e^{-\frac{1}{2}(\bar{p}_0 + p_0 - q_0)} u(\bar{p}_0) u(-\bar{p}_μ\bar{p}_μ - 4)
\]
\[
\delta(-\bar{p}_μ\bar{p}_μ - g^2 - 4) \delta(\bar{p}_μ(q_μ - p_μ)) d\bar{p}^μ.
\]
It is noted that \(u(p_0)\delta(-\bar{p}_μ\bar{p}_μ - g^2 - 4) = \frac{\delta(\bar{p}_0 - \sqrt{s^2 + |\bar{p}|^2})}{2\sqrt{s^2 + |\bar{p}|^2}}\), thus one has
\[
A(p, q)
\]
\[
\leq \frac{C}{q_0} \int_{\mathbb{R}^4} \frac{\sqrt{s}}{2g} e^{-\frac{1}{2}(p_0 - q_0)} \frac{\sqrt{s}}{2g} e^{-\frac{1}{2}\bar{p}_μ\bar{U}_μ}(\bar{p}_μ(q_μ - p_μ)) \frac{d\bar{p}_μ}{\bar{p}_0},
\]
(54)
where we have used the notations \(\bar{p}_0 = \sqrt{s^2 + |\bar{p}|^2}\), \(\bar{U}_μ = (-1, 0, 0, 0)^t\) and the fact
\(-\bar{p}_μ\bar{p}_μ - 4 = s - 4 \geq 0\). We introduce the Lorentz transformation \(\Lambda = (\Lambda^\mu_\nu)\) such that
\[
A_\nu = \Lambda^\mu_\nu(p_\mu + q_\nu) = (\sqrt{s}, 0, 0, 0), \quad B_\nu = -\Lambda^\mu_\nu(p_\mu - q_\mu) = (0, 0, 0, g).
\]
Indeed, Strain [18] gives details of the Lorentz transformation \(\Lambda = (\Lambda^\mu_\nu)\) such that
\[
\Lambda = \left(\begin{array}{cccc}
p_0 + q_0 \sqrt{s} & -p_1 + q_1 \sqrt{s} & -p_2 + q_2 \sqrt{s} & -p_3 + q_3 \sqrt{s} \\
\frac{2|p \times q|}{g\sqrt{s}} & \Lambda^1, & \Lambda^2, & \Lambda^3 \\
0, & \frac{|p \times q|}{|p \times q|} & \frac{(p \times q)_2}{|p \times q|} & \frac{(p \times q)_3}{|p \times q|} \\
\frac{p_0 - q_0}{g}, & \frac{p_1 - q_1}{g}, & \frac{p_2 - q_2}{g}, & -\frac{p_3 - q_3}{g}
\end{array}\right),
\]
where \(\Lambda^1, i = 1, 2, 3\) can be found in [18], we omit the details here.
Define \(U_\mu = \Lambda^\nu_\mu \bar{U}_\nu\), we have
\[
U_\mu = \left(\begin{array}{cccc}
-p_0 + q_0 \sqrt{s} & -2|p \times q| \sqrt{s} & 0, & -p_0 - q_0 \frac{g}{g}
\end{array}\right).
\]
Using the above Lorentz transformation, one can get that
\[
\int_{\mathbb{R}^3} \frac{\sqrt{s}}{2g} \delta(\bar{p}_μ(q_μ - p_μ)) e^{-\frac{1}{2}\bar{p}_μ\bar{U}_μ} \frac{d\bar{p}_μ}{\bar{p}_0} = \int_{\mathbb{R}^3} \frac{\sqrt{s}}{2g} \delta(\bar{p}_μ B_μ) e^{-\frac{1}{2}\bar{p}_μ\bar{U}_μ} \frac{d\bar{p}_μ}{\bar{p}_0},
\]
(55)
where we have used $\tilde{p}^\mu$ and $\frac{dp}{p_0}$ to be Lorentz invariants. Here, $\bar{g}_\Lambda$, $\bar{s}_\Lambda \geq 0$, $\bar{\theta}_\Lambda$ are given by

$$\bar{g}_\Lambda^2 = g^2 + \frac{1}{2} A^\mu (A_\mu - \tilde{p}_\mu) = g^2 + \frac{1}{2} \sqrt{s} (p_0 - \sqrt{s}),$$

$$\bar{s}_\Lambda = 4 + \bar{g}_\Lambda^2, \quad \cos \bar{\theta}_\Lambda = 1 - 2 \left( \frac{g}{\bar{g}_\Lambda} \right)^2. \tag{56}$$

To calculate (55), we use the polar coordinate

$$dp = |\tilde{p}|^2 \sin \psi d\psi d\varphi, \quad \tilde{p} = |\tilde{p}|(\sin \psi \cos \varphi, \sin \psi \sin \varphi, \cos \psi),$$

which, together with the fact $\tilde{p}^\mu B_\mu = \tilde{p}_3 g$, yields that

$$A(p, q) \leq \frac{C}{q_0} e^{-\frac{1}{2}(p_0 - q_0)} \int g^3 \frac{\bar{g}_\Lambda}{2\bar{g}_\Lambda} e^{-\frac{1}{2} \bar{p}_\Lambda U_\mu} \delta (\bar{p}^\mu B_\mu) \frac{d\bar{p}^\mu}{\bar{p}_0},$$

$$= \frac{C}{q_0} e^{-\frac{1}{2}(p_0 - q_0)} \int 2\pi \sin \psi d\psi \int 0^\infty \sqrt{\bar{g}_\Lambda e^{-\frac{1}{2} \bar{p}_0 U_\mu}} \cos \psi |\tilde{p}(d\tilde{p})| \frac{|\tilde{p}|^2 d|\tilde{p}|}{\bar{p}_0}, \tag{57}$$

where $I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \varphi} d\varphi$ is Bessel function. Denoting $z = |\tilde{p}|$, it follows from (56) that

$$g = \bar{g}_\Lambda \sqrt{\frac{1 - \cos \bar{\theta}_\Lambda}{2}} = \bar{g}_\Lambda \sin \frac{\bar{\theta}_\Lambda}{2}, \quad \bar{s}_\Lambda = \frac{1}{2} s + \frac{1}{2} s \sqrt{z^2/s + 1},$$

with

$$\sin \frac{\bar{\theta}_\Lambda}{2} = \frac{g}{\bar{g}_\Lambda} = \frac{\sqrt{2g}}{\sqrt{g^2 - 4 + s \sqrt{z^2/s + 1}}} \leq 1.$$

Set

$$i := \frac{p_0 + q_0}{2}, \quad j := \frac{|p \times q|}{g},$$

and note $\bar{p}_0 = \sqrt{s + z^2}$, one gets

$$A(p, q) \leq \frac{Ce^{-\frac{1}{2}(p_0 - q_0) \sqrt{s}}}{g^3 q_0} \int_0^\infty |1 + \sqrt{z^2/s + 1}| \frac{1}{2} \left[ \frac{1}{\sin \frac{\bar{\theta}_\Lambda}{2}} \right]^{-1} e^{-\frac{1}{2} \bar{p}_0 U_\mu} I_0 \left( \frac{\alpha j}{2 \sqrt{s}} \right) \frac{zd\bar{z}}{\sqrt{s + z^2}}$$

$$\leq \frac{Ce^{-\frac{1}{2}(p_0 - q_0) s}}{g^2 q_0} \int_0^\infty (1 + \sqrt{1 + y^2}) \frac{1}{2} (1 + y^2) \frac{1}{2} e^{-\frac{1}{2} \bar{p}_0 U_\mu} I_0 \left( \frac{\alpha j}{2} \right) dy$$

$$\leq \frac{Ce^{-\frac{1}{2}(p_0 - q_0) s}}{g^2 q_0} \int_0^\infty (1 + y^2)^{-\frac{1}{2}} e^{-\frac{1}{2} \bar{p}_0 U_\mu} I_0 \left( \frac{\alpha j}{2} \right) dy, \tag{58}$$
where we have denoted $y = \frac{z}{\sqrt{s}}$. Using Lemma 3.3, Lemma 3.4 in the appendix, we have

\[
A(p, q) \leq C_\alpha e^{-\frac{\sqrt{\alpha}}{4}(p_0-q_0)} \frac{i^\frac{1}{2} e^{-\frac{\sqrt{\alpha}}{2} \sqrt{i^2-j^2}}} {g^2 q_0 (i^2 - j^2)^{\frac{1}{4}}} \\
\leq C_\alpha \left( \frac{p_0 q_0}{|p \times q| + |p - q|^2} \right)^{\frac{1}{2}} e^{-\frac{\sqrt{\alpha}}{4} |p - q|} e^{-\frac{\sqrt{\alpha}}{4}(p_0-q_0)} \\
\leq C_\alpha \left( \frac{p_0 q_0}{|p \times q| + |p - q|^2} \right)^{\frac{1}{2}}.
\]

(59)

where we have used the fact $e^{-\frac{\sqrt{\alpha}}{4} |p - q|} e^{-\frac{\sqrt{\alpha}}{4}(p_0-q_0)} \leq 1$.

Noting $l \in (1, \min \left\{ \frac{2}{\max \{ -\gamma, 1 \}}, \frac{3}{\max \{ b - 1, 1 \}} \right\})$, we choose $1 < k < \min \left\{ \frac{3}{2}, \frac{2}{l} \right\}$, then it holds that

\[
1 - \frac{1}{m} \left( 2 - \frac{2}{l \ast k^*} \right) = \frac{m-1}{m} = (1 - \frac{2}{m}) (1 - \frac{2}{l \ast k^*}) > 0.
\]

Using the above relations, one obtains that

\[
H_{21213} \leq \int_0^{t-\lambda} \|wF(s)\|_{L^\infty} \left\{ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |w(q) - \frac{1}{2} w(p)| \frac{1}{2} A|^k dp dq \right\} \frac{1}{s} ds \\
\times \left\{ \int_{p_0 < N} \int_{\mathbb{R}^3} \left| w(q) [F(s, x + \delta - \hat{p}(t_\eta - s), q) - F(s, x - \hat{p}(t_\eta - s), q)] \right|^l \frac{1}{k^*} dp dq \right\} \frac{1}{s} ds \\
\leq C_N \int_0^t \|wF(s)\|_{L^\infty}^{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} \|w(\cdot)[F(\cdot, \cdot + \delta, \cdot) - F(\cdot, \cdot, \cdot)]\|_{L^2}^{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} ds \\
\leq C_N \left\{ \int_0^t \|wF(s)\|_{L^\infty}^{\frac{\lambda}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}}} \right\} \frac{1}{s} \left( \int_0^t ds \right)^{\frac{\lambda}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} - \frac{\lambda}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} \times \frac{m-1}{m}} \\
\times \left\{ \int_0^t ds \right\} \frac{1}{s} \left( \frac{1}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} - \frac{\lambda}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} \times \frac{m-1}{m} \right) ds \\
\leq C(t) \left\{ \int_0^t \|w(\cdot)[F(\cdot, \cdot + \delta, \cdot) - F(\cdot, \cdot, \cdot)]\|_{L^2}^{\frac{m}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} ds \right\} \frac{1}{s} \left( \frac{1}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} - \frac{\lambda}{2 - \frac{\lambda}{\max \{ -\gamma, 1 \}}} \times \frac{m-1}{m} \right) ds \to 0, \text{ as } \delta \to 0,
\]

(60)

where we have used

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |w(q) - \frac{1}{2} w(p)| A|^k dp dq \leq C < \infty,
\]

which is due to $k < \frac{\sqrt{\alpha}}{2}, (59)$ and Lemma 3.5.

Combining (39)-(49) and (60), there exists $\chi_4 > 0$ depending only on $(t, x)$ and $\varepsilon$ such that if $|\delta| + \eta \leq \chi_4$, it holds that

\[
H_{21} \leq \frac{\varepsilon}{16}.
\]

(61)
It remains to estimate \( H_{22} \). We first divide it into several parts.

\[
H_{22} \leq \int_t^{t+\eta} \int_{\mathbb{R}^3} Q_+(F,F)(s,x-\hat{\mu}(t_\eta-s),p)dpds \\
+ \int_0^t \int_{\mathbb{R}^3} Q_+(F,F)(s,x-\hat{\mu}(t_\eta-s),p) \int_t^{t+\eta} g(\tau; t_\eta, x, p)d\tau dpds \\
+ \int_0^t \int_{\mathbb{R}^3} Q_+(F,F)(s,x-\hat{\mu}(t_\eta-s),p) \int_s^t \left| g(\tau; t_\eta, x, p) - g(\tau; t, x, p) \right| d\tau dpds \\
+ \int_0^t \int_{\mathbb{R}^3} Q_+(F,F)(s,x-\hat{\mu}(t_\eta-s),p) - Q_+(F,F)(s,x-\hat{\mu}(t-s),p) | dpds \\
:= H_{221} + H_{222} + H_{223} + H_{224}. 
\]

(62)

A direct calculation shows that

\[
H_{221} \leq C \int_t^{t+\eta} \left\| wF(s) \right\|_{L^\infty}^2 ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} w(p')^{-1}w(q')^{-1} \nu_\phi \sigma(g,\theta) d\omega dp dq \\
\leq C(t) \eta^{1-\frac{3}{\eta}} \to 0, \text{ as } \eta \to 0, 
\]

(63)

and

\[
H_{222} \leq \int_0^t \int_{\mathbb{R}^3} w(p)^{-1} \max \{p_0^\frac{2}{3}, p_0^{-\frac{2}{3}}\} \left\| wF(s) \right\|_{L^\infty}^2 dpds \\
\times \int_t^{t+\eta} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \nu_\phi \sigma(g,\theta)w(q)^{-1} \left\| wF(\tau) \right\|_{L^\infty} d\omega dq d\tau \\
\leq C(t) \int_t^{t+\eta} \left\| wF(\tau) \right\|_{L^\infty} d\tau \leq C(t) \eta^{1-\frac{3}{\eta}} \to 0, \text{ as } \eta \to 0. 
\]

(64)

Using the similar arguments as in (28)-(36), if \( \eta \) is small enough, then we have that

\[
H_{223} \leq C \int_0^t \left\| wF(s) \right\|_{L^\infty}^2 ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \nu_\phi \sigma(g,\theta)w(q)^{-1} \left\| wF(\tau) \right\|_{L^\infty} d\omega dq dp \\
\times \left| w(q)[F(\tau,x-\hat{\mu}(t+\eta-\tau),q) - F(\tau,x-\hat{\mu}(t-\tau),q)] \right| d\omega dq d\tau \\
\leq C(t) \int_{t-\lambda}^t \left\| wF(\tau) \right\|_{L^\infty} d\tau + C(t) \int_{p_0 \leq N} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \nu_\phi \sigma(g,\theta)w(q)^{-1} w(p)^{-\frac{1}{2}} d\omega dq dp \\
+ C(t) \int_0^{t-\lambda} \int_{p_0 \leq N} \int_{\mathbb{R}^3 \times \mathbb{S}^2} \nu_\phi \sigma(g,\theta)w(q)^{-1} w(p)^{-\frac{1}{2}} \\
\times \left| w(q)[F(\tau,x-\hat{\mu}(t+\eta-\tau),q) - F(\tau,x-\hat{\mu}(t-\tau),q)] \right| d\omega dq d\tau \leq \frac{\varepsilon}{64}, 
\]

(65)

For \( H_{224} \), one notes that

\[
H_{224} \leq \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g,\theta) F(s,x-\hat{\mu}(t+\eta-s),q') \\
\times \left| F(s,x-\hat{\mu}(t+\eta-s),p') - F(s,x-\hat{\mu}(t-s),p') \right| d\omega dq dp ds \\
+ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \nu_\phi \sigma(g,\theta) F(s,x-\hat{\mu}(t-s),p') \\
\times \left| F(s,x-\hat{\mu}(t+\eta-s),q') - F(s,x-\hat{\mu}(t-s),q') \right| d\omega dq dp ds \\
:= H_{2241} + H_{2242}. 
\]

(66)
We only consider $H_{2241}$ since $H_{2242}$ can be bounded similarly. Indeed, by similar arguments as in

$$H_{2241} \leq C \int_{t-\lambda}^{t} \left\| wF(s) \right\|_{L^\infty}^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times S^2} \nu \sigma(g,\theta) w(p) w(q)^{-1} d\omega dq dp ds$$

$$+ \int_{0}^{t-\lambda} \left\| wF(s) \right\|_{L^\infty}^2 \int_{p_0 \geq N} \int_{\mathbb{R}^3 \times S^2} \nu \sigma(g,\theta) w(p) w(q)^{-1} d\omega dq dp$$

$$+ \int_{0}^{t-\lambda} \left\| wF(s) \right\|_{L^\infty} \int_{p_0 \leq N} \int_{\mathbb{R}^3 \times S^2} \nu \sigma(g,\theta) w(q') w(p')^{-1}$$

$$\times \left| w(p') [F(s, x - \dot{p}(t + \eta - s), p') - F(s, x - \dot{p}(t - s), p')] \right| d\omega dq dp ds$$

$$\leq \frac{\varepsilon}{64}. \quad (67)$$

Combining (62)-(67), there exists $\chi_5 > 0$ depending on $t$ and $\varepsilon$ such that if $\eta \leq \chi_5$, one has

$$H_{22} \leq \frac{\varepsilon}{32}. \quad (68)$$

Thus it follows from (38), (61) and (68) that

$$\left| \int_{\mathbb{R}^3} \left\{ I_2(t + \eta, x + \delta, p) - I_2(t, x, p) \right\} dp \right| \leq \frac{\varepsilon}{2}, \text{ for } \eta + |\delta| \leq \min\{\chi_4, \chi_5\}. \quad (69)$$

Taking $\chi := \min\{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$, then it follows from (21), (37) and (69), we have

$$\left| \int_{\mathbb{R}^3} \left\{ F(t + \eta, x + \delta, p) - F(t, x, p) \right\} dp \right| \leq \varepsilon, \text{ for } \eta + |\delta| \leq \chi. \quad (70)$$

That is, $\int_{\mathbb{R}^3} F(t, x, p) dp$ is continuous function of $(x, t) \in \mathbb{R}^3 \times (0, \infty)$. By similar arguments, one can prove that $\int_{\mathbb{R}^3} p F(t, x, p) dp$ and $\int_{\mathbb{R}^3} p_0 F(t, x, p) dp$ are also continuous functions of $(x, t) \in \mathbb{R}^3 \times (0, \infty)$. Thus we have proved that the macroscopic components of relativistic Boltzmann equation are continuous in $(x, t) \in \Omega \times (0, \infty)$.

**Part II. Uniform Continuity of Macroscopic Components:**

For any fixed $0 < t_1 < T + \infty$, let $(x, t) \in \mathbb{R}^3 \times [t_1, T]$. The uniform continuity of density in $(x, t) \in \mathbb{R}^3 \times [t_1, T]$ is equivalent to prove that for any small $\varepsilon > 0$, there exists $\chi > 0$ depending only on $T$, $t_1^{-1}$ and $\varepsilon$ such that if $|\eta| + |\delta| \leq \chi$, it holds that

$$\left| \int_{\mathbb{R}^3} F(t + \eta, x + \delta, p) - F(t, x, p) dp \right| \leq \varepsilon. \quad (71)$$

Without loss of generality, we assume $\eta \geq 0$. Firstly, one notes that

$$K_1 := \left| \int_{\mathbb{R}^3} \left( I_1(t_\eta, x + \delta, p) - I_1(t, x, p) \right) dp \right|$$

$$\leq \int_{\mathbb{R}^3} \left| F_0(x + \eta - \dot{p} t_\eta, p) - F_0(x - \dot{p} t, p) \right| dp$$

$$+ \int_{\mathbb{R}^3} F_0(x - \dot{p} t, p) \left| \int_{t}^{t_\eta} g(\tau, t_\eta, x + \delta, p) d\tau \right| dp$$

$$+ \int_{0}^{t} \int_{\mathbb{R}^3} F_0(x - \dot{p} t, p) \left| g(\tau; t_\eta, x + \delta, p) - g(\tau; t, x, p) \right| dp d\tau$$

$$:= K_{11} + K_{12} + K_{13}.$$
By the same arguments as in (24), one has that
\[
K_{11} \leq \frac{\varepsilon}{32} + \sup_{(t,x) \in [t_1,T] \times \mathbb{R}^3} \int_{p_0 \leq N} \left| F_0(x + \delta - \hat{\nu}(t + \eta), p) - F_0(x - \hat{\nu}t, p) \right| dp. \tag{72}
\]
For the second term on the right hand side of (72), it follows from (20) that there exists \(\chi_1 > 0\) depending only on \(t_1\), \(T\) and \(\varepsilon\) such that if \(\eta + |\delta| \leq \chi_1\), one has that
\[
\sup_{(t,x) \in [t_1,T] \times \mathbb{R}^3} \int_{p_0 \leq N} \left| F_0(x + \delta - \hat{\nu}(t + \eta), p) - F_0(x - \hat{\nu}t, p) \right| dp \leq \frac{\varepsilon}{32},
\]
which, together with (72), yields that
\[
K_{11} \leq \frac{\varepsilon}{16}, \quad \text{if } \eta + |\delta| \leq \chi_1. \tag{73}
\]
On the other hand, by the same arguments as in (26)-(36), one can prove that there exists \(\chi_2 > 0\) depending only on \(t_1\), \(T\) and \(\varepsilon\) such that if \(\eta + |\delta| \leq \chi_2\), it holds that
\[
K_{12} + K_{13} \leq \frac{\varepsilon}{16}. \tag{74}
\]
Finally, by similar arguments as in (38)-(69), one can prove that there exists \(\chi_3 > 0\) depending only on \(t_1\), \(T\) and \(\varepsilon\) such that if \(|\delta| + \eta \leq \chi_3\), it holds that
\[
K_2 := \left| \int_{\mathbb{R}^3} I_2(t + \eta, x + \delta, p) - I_2(t, x, p) dp \right| \leq \frac{\varepsilon}{16}, \tag{75}
\]
Hence, taking \(\chi = \min\{\chi_1, \chi_2, \chi_3\}\), it follows from (73), (74) and (75) that
\[
\left| \int_{\mathbb{R}^3} F(t + \eta, x + \delta, p) - F(t, x, p) dp \right| \leq \varepsilon, \quad \text{if } \eta + |\delta| \leq \chi. \tag{76}
\]
Thus \(\int_{\mathbb{R}^3} F(t, x, p) dp\) is uniformly continuous in \((x, t) \in \mathbb{R}^3 \times [t_1, T]\). By similar arguments, one can prove that \(\int_{\mathbb{R}^3} p F(t, x, p) dp\) and \(\int_{\mathbb{R}^3} p_0 F(t, x, p) dp\) are also uniformly continuous in \((x, t) \in \mathbb{R}^3 \times [t_1, T]\). Therefore we proved that the macroscopic components of relativistic Boltzmann equation are uniformly continuous in \((x, t) \in \Omega \times [t_1, T]\). \hfill \Box

2.2. Proof of Theorem 1.1 for \(\Omega = \mathbb{T}^3\). For \(\Omega = \mathbb{T}^3\), most of the proof is similar to the case of \(\Omega = \mathbb{R}^3\). Here we only point out some differences. Firstly, in the period case, one should pay attention when using change variable \(x \mapsto y := x - \hat{\nu}(t_\eta - \tau)\).

For example, we consider the change of variable in \(H_{1331}\):
\[
H_{1331} := C \int_0^{t_1} \int_{p_0 < N} \int_{\mathbb{R}^3} \int_{S^2} \left| w(p)^{-1} w(q)^{-1} \nu_\sigma(g, \theta) w(q) |F(\tau, x - \hat{\nu}(t_\eta - \tau), q) - F(\tau, x - \hat{\nu}(t - \tau), q)| \right| d\omega dq dp d\tau
\]
\[
\leq C \int_0^{t_1} \int_{p_0 < N} \int_{\mathbb{R}^3 \times S^2} \left| w(p)^{-1} w(q)^{-1} \nu_\sigma(g, \theta) \right| d\omega dq dp d\tau
\]
\[
\times \left\{ \int_{p_0 < N} \int_{\mathbb{R}^3} \left| w(q) |F(\tau, x + \delta - \hat{\nu}(t_\eta - \tau), q) - F(\tau, x - \hat{\nu}(t_\eta - \tau), q)| \right| dq \right\}^{\frac{1}{2}} dp d\tau
\]
Using the change of variable \(y = x - \hat{\nu}(t_\eta - \tau)\) and Lemma 3.1 in the appendix, one can further bounded the above as
\[ H_{1331} \leq C(t) \lambda^{-\frac{2}{\lambda}} (1 + |N|^2)^{\frac{1}{2\lambda}} \int_0^{t-\lambda} \left( \int_{\mathbb{R}^3} w(p)^{-i} \max\{p_0 w, p_0 - \frac{p^2}{2} \} \, dp \right)^{\frac{1}{2}} \times \left\{ \int_{\Omega'(t)} \int_{\mathbb{R}^3} \left| w(q) \left[ F(\tau, y + \delta, q) - F(\tau, y, q) \right] \right|^{\frac{1}{2}} \, dq \right\} d\tau \]

\[ \leq C(t, N) \lambda^{-\frac{2}{\lambda}} \int_0^t \left\{ \int_{\Omega'(t)} \int_{\mathbb{R}^3} \left| w(q) \left[ F(\tau, y + \delta, q) - F(\tau, y, q) \right] \right|\, dq \right\} d\tau \]

where \( \Omega'(t) = \{ y : y = x - \hat{p}(t - \tau), \, |\hat{p}| \leq 1, t - \tau \geq \lambda > 0 \}. \) It is noted that \( |\Omega'(t)| \leq C(t) < \infty. \) Since \( F(\tau, y, q) \) is periodic function of \( y, \) then we can bound the above term as

\[ H_{1331} \]

\[ \leq C(t, N) \lambda^{-\frac{2}{\lambda}} \int_0^t \left\{ \int_{|y| \leq C(t)} \int_{\mathbb{R}^3} \left| w(q) \left[ F(\tau, y + \delta, q) - F(\tau, y, q) \right] \right| \, dq \right\} d\tau \]

\[ \leq C(t, N) \lambda^{-\frac{2}{\lambda}} \int_0^t \left( \int_{\Omega} \left| w(F(\tau, \cdot, \cdot) - F(\tau, \cdot, \cdot)) \right| L_2 \| wF(\tau) \|^{\frac{1}{2}} \, d\tau \right) \]

\[ \leq C(t, N) \lambda^{-\frac{2}{\lambda}} \left( \int_0^t \left\| w(F(\tau, \cdot, \cdot) - F(\tau, \cdot, \cdot)) \right\| L_2 \, d\tau \right)^{\frac{2(m-1)}{m+1}} \rightarrow 0, \text{ as } \delta \rightarrow 0. \] (77)

On the other hand, to bound \( H_{1332}, \) one can choose a smooth function \( F^\varepsilon(t, x, p) \) (may not has compact support) which is periodic in variable \( x \) such that

\[ \left( \int_0^t \left\| w(F(\tau, \cdot, \cdot) - F^\varepsilon(\tau, \cdot, \cdot)) \right\| L_2^{\frac{m}{m+1}} \, d\tau \right)^{\frac{2(m-1)}{m+1}} \leq \lambda^{-\frac{2}{\lambda, \varepsilon}}. \] (78)

Then, using (78) and by similar arguments in the case of \( \Omega = \mathbb{R}^3, \) one can prove the continuity of macroscopic components of solution to the relativistic Boltzmann equation. \( \square \)

3. Appendix A.

Lemma 3.1. For the changing variable \( y = x - \hat{p}(t - \tau), \) we have

\[ |\det \left( \frac{\partial y}{\partial p} \right) | = |t - \tau|^3 (1 + |p|^2)^{-\frac{3}{2}}. \]

Proof.

\[ \frac{\partial y_i}{\partial p_j} = -(t - \tau) \frac{\delta_{ij} \sqrt{1 + |p|^2} - p_i \cdot \sqrt{1 + |p|^2}}{1 + |p|^2} = -(t - \tau) \frac{\delta_{ij} (1 + |p|^2) - p_i p_j}{(1 + |p|^2)^{\frac{3}{2}}}. \]

where

\[ \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \]
Then we have that
\[
|\det(\frac{\partial y}{\partial p})|
= |t - \tau|^3(1 + |p|^2)^{-\frac{3}{2}} \times \left\{ \left(1 + |p|^2 - p_1^2\right)\left(1 + |p|^2 - p_2^2\right)\left(1 + |p|^2 - p_3^2\right) - (1 + |p|^2 - p_1^2p_2^2p_3^2 - (1 + |p|^2)p_1^2p_2^2 - (1 + |p|^2)p_1^2p_2^2 + p_1^2p_2^2p_3^2) \right\}
= |t - \tau|^3(1 + |p|^2)^{-\frac{3}{2}} \times \left\{ \left(1 + |p|^2 - p_1^2\right)\left(1 + |p|^2 - p_2^2\right)\left(1 + |p|^2 - p_3^2\right) - (1 + |p|^2)p_2^2p_3^2 - (1 + |p|^2)p_1^2p_3^2 + p_1^2p_2^2p_3^2 \right\}
= |t - \tau|^3(1 + |p|^2)^{-\frac{3}{2}} \times \left\{ (1 + |p|^2)^3 - (p_1^2 + p_2^2 + p_3^2)(1 + |p|^2)^2 \right\}
= |t - \tau|^3(1 + |p|^2)^{-\frac{3}{2}}.
\]

Lemma 3.2. The example given in (16) satisfies the condition (15) and (20).

Proof. Using (17), it is obvious that \( \|wF_0\|_{L^\infty} \leq C < \infty \). Next we prove the second part of (15). Given any \( N > 0 \), we only need to prove that for any small \( \varepsilon > 0 \) there exists \( \chi > 0 \) depending on \( t, \varepsilon \) such that if \( |\eta| + |\delta| \leq \chi \), it holds that
\[
\int_{|p_0| \leq N} |F_0(x + \delta - \hat{p}(t + \eta), p) - F_0(x - \hat{p}t, p)| dp \leq \varepsilon. \tag{79}
\]
For any fixed \( t > 0 \), it is noted that
\[
\int_{|p_0| \leq N} |F_0(x + \delta - \hat{p}(t + \eta), p) - F_0(x - \hat{p}t, p)| dp
\leq \int_{|p_0| \leq N} |F_0(x + \delta - \hat{p}(t + \eta), p) - F_0(x - \hat{p}(t + \eta), p)| dp
+ \int_{|p_0| \leq N} |F_0(x - \hat{p}(t + \eta), p) - F_0(x - \hat{p}t, p)| dp
:= L_1 + L_2.
\]
We assume \( \eta \geq 0 \) for simplicity of presentation. By changing \( y = x - \hat{p}(t + \eta) \) and using Lemma 3.1, we have
\[
L_1 \leq C \left\{ \int_{|p| \leq N} |\rho_0(x + \delta - \hat{p}t_\eta) - \rho_0(x - \hat{p}t_\eta)|^4 dp \right\}^{\frac{1}{4}}
\leq C_N (t + \eta)^{-\frac{3}{2}} \|\rho_0(\cdot + \delta) - \rho_0(\cdot)\|_{L^4(\Omega)}
\rightarrow 0, \text{ as } \eta, \delta \to 0,
\]
where we have used the continuity of translations on \( L^4 \). Thus there exists \( \chi_1 > 0 \) depending only on \( t > 0 \) and \( \varepsilon > 0 \) such that if \( |\eta| + |\delta| \leq \chi_1 \), it holds that
\[
L_1 \leq \frac{\varepsilon}{2}. \tag{80}
\]
For \( L_2 \), it is straightforward to get that
\[
L_2 \leq C \int_{|p_0| \leq N} |\rho_0(x - \hat{p}t - \hat{p}\eta) - \rho_0(x - \hat{p}t)| e^{-\frac{r_0}{4}} dp.
\]
Noting that $\rho_0 - 1 \in L^1$ and $0 \leq \rho_0(x) \leq \hat{C}$, there exists a smooth compact support function $\rho_0^\delta$ in the case $\Omega = \mathbb{R}^3$(or there exists a smooth periodic function $\rho^\delta x$) in the case $\Omega = \mathbb{T}^3$ such that 

$$\|\rho_0 - \rho_0^\delta\|_{L^1(\Omega)} \leq \frac{\varepsilon}{C_2},$$

where $C_2$ is a large constant chosen later. Hence one obtains that

$$\int_{\rho_0 \leq N} |(\rho_0(x - \hat{p}t - \hat{p}\eta) - \rho_0(x - \hat{p}t)|e^{-\eta} dp \leq \int_{\rho_0 \leq N} |\rho_0(x - \hat{p}t - \hat{p}\eta) - \rho^\delta_0(x - \hat{p}t)|e^{-\eta} dp \leq \frac{\varepsilon}{C_2},$$

provided $C_2$ large enough and $\eta$ small enough. Therefore, there exists $\chi_2 > 0$ depending on $t$ and $\varepsilon$ such that if $\eta \leq \chi_2$, it holds that

$$L_2 \leq \frac{\varepsilon}{2}.$$  

Combining (80) and (82), there exists $\chi = \min\{\chi_1, \chi_2\}$ depending on $t$ and $\varepsilon$ such that (79) holds provided $\eta + |\delta| \leq \chi$. Therefore the condition (15) holds for the example given in (16).

Finally, for any fixed $0 < t_1 < T < \infty$, from (80) and (82), it is not difficult to show that there exists $\chi > 0$ depending on $t_1^{-1}, T$ and $\varepsilon$ such that if $\eta + |\delta| \leq \chi$, it holds that

$$\sup_{(x, t) \in \mathbb{R}^3 \times [0, T]} \int_{\mathbb{R}^3_{loc}} |F_0(x + \delta - \hat{p}(t + \eta), p) - F_0(x - \hat{p}t, p)| dp \leq \varepsilon.$$

Thus the example given in (16) also satisfies the condition (20). □

In this paper, we need the following useful lemmas, whose proof can be found in [19].

**Lemma 3.3** ([19]). For $i, j$ defined as before, we have, for $\alpha \in [-2, 2]$, that

$$K_{\alpha}(i, j) := \int_0^\infty z(1 + z^2)^{-\delta} e^{-i\sqrt{1+z^2}} I_0(jz) dz \leq C \frac{j^{1+\frac{\alpha}{2}}}{(i^2 - j^2)^{1+\frac{\alpha}{2}}} e^{-\sqrt{i^2 - j^2}}.$$ □

**Lemma 3.4** ([19]). For any fixed $\theta \in \mathbb{R}$, it holds that

$$(p_0 + q_0)^\theta e^{-c|p - q|} \leq C(p_0q_0)^\theta e^{-\frac{\theta}{2}|p - q|},$$

and

$$(p_0q_0)^\theta e^{-c|p - q|} \leq C p_0^2 e^{-\frac{\theta}{2}|p - q|}.$$
Lemma 3.5 ([19]). Let $0 \leq \alpha < 3$, it holds that
\[
\int_{\mathbb{R}^3} \frac{e^{-|p-q|}}{|p \times q| + |p-q|} dq \leq \begin{cases} 
C_\alpha (1 + |p|)^{-\alpha}, & \text{for } 0 \leq \alpha < 2, \\
C(1 + |p|)^{-2} \ln(1 + |p|), & \text{for } \alpha = 2, \\
C_\alpha (1 + |p|)^{-2}, & \text{for } \alpha > 2.
\end{cases}
\]

4. Appendix B. We introduce the existence of $L^\infty$-mild solutions to relativistic Boltzmann equation near relativistic Maxwellian. Recently, Wang [19] obtained the global $L^\infty$-mild solution to the relativistic Boltzmann equation for a class of large initial data. To introduce the result of [19], we denote
\[
F \in \mathcal{E}_0 \cap L^3 \quad \text{and} \quad \beta > 0.
\]
It follows by a direct calculation that $\mathcal{E}(F(t)) \geq 0$ for all $t \geq 0$. Note, in particular, that $\mathcal{E}(F_0) \geq 0$ holds true for any function $F_0(x,p) \geq 0$.

Proposition 1 ([19]). Let $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$. Assume $b \in (0,2), \gamma > -\min\{\frac{3}{4}, 4 - 2b\}$ for soft potentials, and $\gamma > -\frac{1}{2}, a \in [0,2] \cap [0, \min\{2 + \gamma, 4 + 3\gamma\}], b \in [0,2)$ for hard potentials. For any given $\beta > 14$, $\hat{M} \geq 1$, suppose that the initial data $F_0$ satisfies $F_0(x,p) = J(p) + \sqrt{J(p)}f_0(x,p) \geq 0$ and $\|p_0^d f_0\|_{L^\infty} \leq \hat{M}$. There is a small constant $\epsilon_0 > 0$ depending on $a, b, \gamma, \beta, \hat{M}$ such that if
\[
\mathcal{E}(F_0) + \|f_0\|_{L_1^1 L_\infty^\infty} \leq \epsilon_0,
\]
the Boltzmann equation (4), (13) has a global unique mild solution $F(t, x, p) = J(p) + \sqrt{J(p)}f(t, x, p) \geq 0$ satisfying (83) and (84) and
\[
\|p_0^d f(t)\|_{L^\infty} \leq \tilde{C} \hat{M}^2,
\]
where the positive constant $\tilde{C}_1$ depends only on $a, b, \gamma, \beta$.

For $\Omega = \mathbb{T}^3$, from $L^\infty$-estimate (85), it is easy to know that $F(t, x, p) \in \mathcal{X}$ in the case near relativistic Maxwellian. On the other hand, for $\Omega = \mathbb{R}^3$, using (85) and the energy estimates, one can get easily that
\[
\|f(t)\|_{L^2(\Omega \times \mathbb{R}^3)} \leq C f_0, \quad t \geq 0,
\]
where $C$ depends only on the initial data. Thus we also have $F(t, x, p) \in \mathcal{X}$ in the case $\Omega = \mathbb{R}^3$. It is noted that the solution constructed in Proposition 1 allows initial macroscopic singularities.

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