Quasi-Equivalence of Width and Depth of Neural Networks

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Quasi-Equivalence of Width and Depth of Neural Networks

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Abstract

While classic studies proved that wide networks allow universal approximation, recent research and successes of deep learning demonstrate the power of the network depth. Based on a symmetric consideration, we investigate if the design of artificial neural networks should have a directional preference, and what the mechanism of interaction is between the width and depth of a network. We address this fundamental question by establishing a quasi-equivalence between the width and depth of ReLU networks. Specifically, we formulate a transformation from an arbitrary ReLU network to a wide network and a deep network for either regression or classification so that an essentially same capability of the original network can be implemented. That is, a deep regression/classification ReLU network has a wide equivalent, and vice versa, subject to an arbitrarily small error. Interestingly, the quasi-equivalence between wide and deep classification ReLU networks is a data-driven version of the De Morgan law.

1 Introduction

Recently, deep learning [20, 12] has become the mainstream approach of machine learning and achieved the state-of-the-art performance in many important tasks [7, 19, 3, 35]. One of the key reasons that accounts for the successes of deep learning is the increased depth, which allows a hierarchical representation of features. There are a number of papers dedicated to explaining why deep networks are better than shallow ones. Encouraging progresses have been made along this direction. The idea to show the superiority of deep networks is basically to find a special family of functions that are very hard to be approximated by a shallow network but easy to be approximated by a deep network, or that a deep network can express more complicated functions than those represented as a wide network using some complexity measure [34, 20, 6, 11, 22, 27, 1]. For example, in [11] a special class of radial functions was constructed so that a one-hidden-layer network needs to use an exponential number of neurons to obtain a good approximation, but a two-hidden-layer network only requires a polynomial number of neurons for the same purpose. With the number of linear regions as the complexity measure, Montufar et al. [27] showed that the number of linear regions grows exponentially with the depth of a network but only polynomially with the width of a network. In [11], a topological measure was utilized to characterize the complexity of functions. It was shown that deep networks can represent functions of a much higher complexity than what the shallow counterparts can.

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Although both theoretical insights and real-world applications suggest that deep networks are better than shallow ones, intuitively speaking, a wide network and a deep network should be complementary. Indeed, the effects of network width are also increasingly recognized. In the last eighties, a one-hidden-layer network with sufficiently many neurons was shown to have a universal approximation ability \cite{14,16}. Clearly, an unlimited increment in either width and depth can offer a sufficient representation ability. In recent years, the term ‘wide/broad learning’ was coined to complement deep learning. In Cheng et al. \cite{4}, a wide network and a deep network were conjugated to realize the memorization (wide network) and generalization (deep network) in a recommender. Challenged by the long training time and large numbers of parameters of deep networks, Chen et al. \cite{2} proposed to use a broad random vector functional-link neural network for broad learning. Zagoruyko and Komodakis \cite{38} designed a novel architecture in which the depth of a residual network is decreased while the width of a residual network is increased, producing a far better performance than that of commonly used thin and deep networks.

Furthermore, width-bounded universal approximators were developed \cite{25,24,13} in analogy to the depth-bounded universal approximators. The core technique in prototyping universal approximator is to construct a function over a local and tiny hypercube and aggregate these functions. Evidently, width-bounded and depth-bounded universal approximators can be utilized to transform a general ReLU network so as to elaborate equivalence of width and depth. However, this is inefficient and impractical as the function expressed by a ReLU network is a piecewise continuous function over polytopes \cite{5}. A better strategy is to represent a function over a simplex as a building block, since a polytope is naturally decomposed as a collection of simplices.

Here, we demonstrate via analysis and synthesis that the width and depth of neural networks are actually quasi-equivalent through either a width- or depth-oriented configuration of modularized network constructions. In this study, we focus on regression and classification, which are two essential machine learning tasks to predict continuous quantities and discrete class labels respectively. It is underlined that these two kinds of tasks are closely related. On one hand, sufficiently many discrete labels can be used to approximate a piecewise continuous function. On the other hand, discrete labels can be represented as indicator functions, which are a special case of piecewise continuous functions. We first elaborate quasi-equivalence of networks for regression problems by constructing a transformation of an arbitrary ReLU network to both a wide network and a deep network, thereby verifying a general quasi-equivalence of the width and depth of networks. Our constructive scheme is largely based on the fact that a ReLU network partitions a space into polytopes \cite{5}. This enables us to have a simplicial complex for the space and thus to establish a quasi-equivalence of networks using the essential building blocks, fan-shape functions, in the form of modularized ReLU networks. We further extend our discussion on the quasi-equivalence to classification networks. Our idea is to apply the proposed fan-shape function approximation to indicator functions. Specially, we combine with the wisdom from the De Morgan law:

$$A_1 \lor A_2 \cdots \lor A_n = \neg \left( (\neg A_1) \wedge (\neg A_2) \cdots \wedge (\neg A_n) \right),$$

where $A_i$ is a propositional rule, and such rules are disjoint, and interpret the quasi-equivalence of classification networks as the network-based De Morgan law by regarding an indicator function as a rule over a simplicial complex.

Our main contribution is the establishment of the width-depth quasi-equivalence of neural networks by prototyping the transformation of a general network into its wide and deep counterparts respectively. We summarize our main results in Table 1.

### Table 1: Network structures and complexities through transformation of regression and classification networks.

| Network Type | Network | Width | Depth |
|--------------|---------|-------|-------|
| Transform Regression Networks (Theorem 2) | Wide Network | $O [D(D + 1)(2^D - 1)M]$ | $D + 1$ |
| Transform Classification Networks (Theorem 4) | Wide Network | $O [(D + 1)M]$ | 2 |
|                         | Deep Network | $(D + 1)D^2$ | $O [(D + 2)M]$ |
|                         | Deep Network | $D + 1$ | $O [(D + 1)M]$ |

To put our contributions in perspective, we would like to mention relevant studies. Kawaguchi et al. \cite{18} analyzed the effect of width and depth on the quality of local minima. They showed that the quality of local minima improves toward the global minima as depth and width becomes larger. Daniely et al. \cite{8} shed light on
the duality between neural networks and compositional kernel Hilbert spaces using an acyclic graph that can succinctly describe neural networks and compositional kernels in a unified framework. From the physical point of view, Georgiev [15] discussed the duality of observables (for example, image pixels) and observations in machine learning. In addition, the importance of width was implicated in light of the neural tangent kernel (NTK) [17], and we discuss the width of neural networks as related to NTK in the Supplementary Information I. To our best knowledge, this work is the first that reveals the width-depth quasi-equivalence of neural networks.

2 Preliminaries

For convenience, we use \( \sigma(x) = \max\{0, x\} \) to denote the ReLU functions. We call a network using the ReLU activation function as a ReLU network. We call a network using the binary step activation function

\[
z(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}
\]

as a binary step network. We mainly discuss ReLU networks in this work, thus all networks in the rest of this paper are referred as ReLU networks unless otherwise specified.

**Definition 1.** A regression network is a network with continuous outputs, while a classification network is a network with categorized outputs (for example, \{0, 1, \cdots, 9\} for digit recognition). In this study, we investigate a classification network with binary labels without loss of generality.

**Definition 2.** Denote the length of one route between a neuron and the input as the number of affine operations. We define that two neurons are in the same layer if they have the routes sharing the same length. Then, the width is defined as the maximum number of neurons in a layer, whereas the depth is defined as the length of the longest route in the network.

**Definition 3** (Simplicial complex). A \( D \)-simplex \( S \) is a \( D \)-dimensional convex hull provided by convex combinations of \( D + 1 \) affinely independent vectors \( \{v_i\}_{i=0}^D \subset \mathbb{R}^D \). In other words, \( S = \left\{ \sum_{i=0}^D \xi_i v_i \mid \xi_i \geq 0, \sum_{i=0}^D \xi_i = 1 \right\} \). The convex hull of any subset of \( \{v_i\}_{i=0}^D \) is called a face of \( S \). A simplicial complex \( S = \bigcup_\alpha S_\alpha \) is composed of a set of simplices \( \{S_\alpha\} \) satisfying: 1) every face of a simplex from \( S \) is also in \( S \); 2) the non-empty intersection of any two simplices \( S_1, S_2 \in S \) is a face of both \( S_1 \) and \( S_2 \).

**Proposition 1.** Suppose that \( f(x) \) is a function represented by a ReLU network, then \( f \) is a piecewise linear function that splits the space into polytopes, where each polytope can be filled with a number of simplices.

**Proof.** Because of the affine transform and the piecewise linear function ReLU, \( f(x) \) will partition the space into polytopes, each of which is defined by a linear function. A feedforward ReLU network partitions the space into convex polytopes [5], whereas a ReLU network with shortcuts may create non-convex polytopes. Nevertheless, those non-convex polytopes are de facto obtained by convex polytopes because a ReLU network with shortcuts can be taken as the combination of several branches of feedforward networks. Clearly, the polytopes by \( f(x) \) can be filled with a number of simplices [9]. \( \square \)

**Definition 4.** We define the complexity \( M \) of the function class represented by ReLU networks as the minimum number of simplices that are needed to cover each and every polytope to support the network.

**Definition 5.** We define a wide network and a deep network as follows. Let us assume a class of functions that can be sufficiently complex and yet can be represented by a network. When such a function becomes increasingly complex, the structure of this network must be also increasingly complex, depending on the complexity of the function. We call a network wide if its width is larger than its depth by at least an order of magnitude in \( M \); i.e. \( O(M^{\alpha+1}) \) vs \( O(M^\alpha) \), where \( M \) is the complexity measure and \( \alpha > 0 \). Similarly, we call a network deep if its depth is larger than its width by at least an order of magnitude in \( M \).

It is underscored that we use two different concepts: the complexity of the function class represented by networks and the structural complexity of a network. The former is to measure the complexity of the function, while the latter measures the topological structure of a network. In our transformation scheme, the structures of constructs/networks are determined by the complexity of the function of interest.
Definition 6. We call a wide network $N_1 : \Omega \to \mathbb{R}$ is equivalent to a deep network $N_2 : \Omega \to \mathbb{R}$ if $N_1(x) = N_2(x), \forall x \in \Omega$. We call a wide network $N_1$ is $\delta$-equivalent to a deep network $N_2$, if there is $\delta > 0$, $m(\{|x \in \Omega | N_1(x) \neq N_2(x)\}) < \delta$, where $m$ is a measurement defined on $\Omega$.

3 Quasi-equivalence of Width and Depth of Networks

This section describes the main contribution of our paper. We formulate the transformation from an arbitrary ReLU network to a wide network and a deep network respectively using a network-based building block to represent a linear function over a simplex, integrating such building blocks to represent any piecewise linear function over polytopes, thereby elaborating a general equivalence of the width and depth of networks. Particularly, a regression ReLU network is converted into a wide ReLU network and a deep ReLU network (Theorem 2), while a classification ReLU network is turned into a wide binary step network and a deep binary step network respectively (Theorem 4). In the regression networks, the transformation of a univariate network is rather different from that of a multivariate network. The equivalence for the wide and deep networks in the univariate case is precise, whereas the multivariate wide and deep networks can be made approximately equivalent up to an arbitrary small error relative to the performance of the original network. This is why we term such an equivalence as a quasi-equivalence.

In the classification networks, we provide a structurally more symmetric scheme, where the wide and deep networks employ a binary step function as activation functions. Because the binary step activation is suitable for the discontinuous nature of indicator functions, the resultant wide and deep networks are much more compact than results without using the binary step function.

3.1 Regression Networks

The sketch of transforming a regression ReLU network is that we first construct either a wide modular network or a deep modular network to represent the corresponding function over each and every simplex, then we aggregate the results into deep or wide networks in series or parallel respectively to represent the original network well.

Theorem 1 (Quasi-equivalence of Univariate Regression Networks). Given any ReLU network $f : [-B, B] \to \mathbb{R}$ with one dimensional input and output variables. There is a wide ReLU network $H_1 : [-B, B] \to \mathbb{R}$ and a deep ReLU network $H_2 : [-B, B] \to \mathbb{R}$, such that $f(x) = H_1(x) = H_2(x), \forall x \in [-B, B]$.

Our main result is formally summarized as the following quasi-equivalence theorem for multivariate case.

Theorem 2 (Quasi-equivalence of Multivariate Regression Networks). Suppose that the representation of an arbitrary ReLU network is $h : [-B, B]^D \to \mathbb{R}$, for any $\delta > 0$, there exist a wide ReLU network $H_1$ of width $O[D(D + 1)(2^D - 1)M]$ and depth $D + 1$, and also a deep ReLU network $H_2$ of width $(D + 1)^2D^2$ and depth $O[(D + 2)M]$, where $M$ is the minimum number of simplices to cover the polytopes to support $h$, satisfying that

$$m\left(\mathbf{x} \mid h(\mathbf{x}) \neq H_1(\mathbf{x})\right) < \delta$$

$$m\left(\mathbf{x} \mid h(\mathbf{x}) \neq H_2(\mathbf{x})\right) < \delta,$$

where $m(\cdot)$ is the standard measure in $[-B, B]^D$.

We defer the proof of Theorem 1 to Supplementary Information II, and split the proof of Theorem 2 into the two-dimensional case (more intuitive) and the general case in Appendix A for better readability.

The key idea to represent a linear function over a simplex is to construct high-dimensional fan-shaped functions that are supported in fan-shaped domains, and to use these constructs to eliminate non-zero functional values outside the simplex of interest. This is a new and local way to represent a piecewise linear function over polytopes. In contrast, there is a global way to represent piecewise linear functions [36]. Specifically, for every piecewise linear function $f : \mathbb{R}^m \to \mathbb{R}$, there exists a finite set of linear functions $g_1, \cdots, g_m$ on the intersections of hyperplanes $T_1, \cdots, T_P$ such that $f = \sum_{p=1}^{P} s_p(\max_{i \in T_p} g_i)$, where $s_p \in \{-1, +1\}, p = 1, \cdots, P$. However, due to its unboundedness, the global representation of a piecewise linear function is problematic in representing a function over polytopes that make a non-convex region, which is nevertheless typical for manifold learning. We highlight the construction of fan-shaped functions, which opens a new door for high-dimensional
piecewise function representation. Particularly, the employment of fan-shaped functions will enable a neural network to express a manifold more efficiently and effectively.

Since such a fan-shaped function is a basic building block in our construction of wide and deep equivalent networks, let us explain it in a two-dimensional case for easy visualization. An essential building block expressed by a network in Figure [a] is an approximate fan-shaped function:

\[
F(x) = \sigma \circ (h_1(x) - \mu \sigma \circ h_2(x)),
\]

where \( h_1(x) = p_1^{(1)} x_1 + p_2^{(1)} x_2 + r^{(1)} \), and \( h_2(x) = p_1^{(2)} x_1 + p_2^{(2)} x_2 + r^{(2)} \) are provided by two linearly independent vectors \( \{p_1^{(1)}, p_2^{(1)}\}, \{p_1^{(2)}, p_2^{(2)}\} \), and \( \mu \) is a positive controlling factor. Eq. [3] is a ReLU network of depth=2 and width=2 according to our width-depth definition. As illustrated in Figure [b], the piecewise linear domains of \( F(x) \) contain three boundaries and four polytopes (two of which only allow zero value of \( F \)).

For convenience, given a linear function \( \ell(x) = c_1 x_1 + c_2 x_2 + c_3 \), we write \( \ell^- = \{x \in \mathbb{R}^2 \mid \ell(x) < 0\} \) and \( \ell^+ = \{x \in \mathbb{R}^2 \mid \ell(x) \geq 0\} \). Thus, we can write \( \Omega_1 = h_1^+ \cap h_2^- \) and \( \Omega_2 = (h_1 - \mu h_2)^- \cap h_2^+ \). There are three properties of \( F(x) \). First, the common edge shared by \( \Omega_1 \) and \( \Omega_2 \) is \( h_2(x) = 0 \). Second, the size of \( \Omega_2 \) is adjustable by controlling \( \mu \). Note that \( h_1(x) - \mu h_2(x) = 0 \) can move very close to \( h_2(x) = 0 \) as \( \mu \to \infty \), which makes \( \Omega_2 \) negligible. In the limiting case, the support of \( F(x) \) converges to the fan-shaped domain \( \Omega_1 \).

Because \( h_1(x) - \mu h_2(x) = 0 \) is almost parallel to \( h_2(x) = 0 \) when \( \mu \) is big enough, we approximate the area of \( \Omega_2 \) as the product of the length of \( h_2(x) = 0 \) within \([-B, B]^2\) and the distance between two lines, which yields \(|\Omega_2| \leq 2\sqrt{2} B/\mu \). Third, the function \( F \) over the fan-shaped area \( \Omega_1 \) is \( h_1 \).

Remark 1: The number of polytopes \( N_\mu(\leq M \text{ by definition}) \) is widely used as a measure of the complexity of a function class represented by deep ReLU networks [27]. Empirical bounds of \( N_\mu \) in a feedforward ReLU network were estimated in [22,23,31], where one result in [22] states that let \( n_1 > D, i = 1, \ldots, L, \) be the number of neurons in the \( i \)-th layer, and \( N \) be the total number of neurons, \( N_\mu \) is lower bounded by \( \left( \prod_{i=1}^{L-1} \left( \frac{n_i}{D} \right)^D \right)^{\sum_{j=0}^{D} \binom{n_j}{n_k}} \) and upper bounded by \( 2^N \). It is conceivable that as a ReLU network of interest partitions a space into more and more polytopes, the number of needed simplices will go increasingly larger, and the width of \( H_1(x) \) and the depth of \( H_2(x) \) will dominate. Since the width of \( H_1(x) \) is higher than its depth by an order of magnitude in terms of \( M \), and the depth of \( H_2(x) \) is higher than its width also by an order of magnitude in a similar way, we conclude that \( H_1(x) \) is a wide network, while \( H_2(x) \) is a deep network.

3.2 Classification Networks

A classification neural network can be interpreted as a disjoint rule-based system \( \overline{A_1 \lor A_2 \cdots \lor A_n} \) by splitting the representation of a neural network into many decision polytopes: IF (input \( \in \) certain polytope), THEN (input belongs to some class). Furthermore, each rule is a local function supported over a decision region. We abuse \( A_i \) as a rule or a function interchangeably, and then \( A_i \lor A_j = A_i + A_j, i \neq j \). Given such a rule system, the De Morgan law holds:

\[
A_1 \lor A_2 \cdots \lor A_n = \neg \left( \neg A_1 \land \neg A_2 \cdots \land \neg A_n \right).
\]

Considering a binary label and viewing each \( A_i \) as a rule over a hypercube, we give an intuitive example in Figure [2] where we construct a deep network to realize a logic union of propositional rules (the left hand side of the De Morgan law holds).
Morgan law) and a wide network that realizes the negation of the logic intersection of those rules after negation (the right hand side of the De Morgan law). As a result, the constructed deep and wide networks are equivalent by the De Morgan law. In the Supplementary Information III and IV, we further discuss more details of such constructions and more results obtained by re-expressing the De Morgan law.

Figure 2: The width and depth equivalence in light of the De Morgan equivalence. In this construction, a deep network to implement \( A_1 \lor A_2 \lor A_n \) using a trapezoid function and a wide version to implement \( \lnot \left( \lnot A_1 \land \cdots \lnot A_n \right) \) using the trap-like function. \((-)^+\) denotes ReLU.

However in practice, each rule may not be as simple as an indicator function over a hypercube but it can be defined on a rather complicated domain. Since a ReLU network splits the space into polytopes, it is reasonable to assume that each rule is guided by a polytope which can be associated with a simplicial structure. We further write each rule \( A_i \) by an indicator function \( g_i(x) \) over a simplicial complex \( S_i \) in a bounded domain:

\[
g_i(x) = \begin{cases} 
1, & \text{if } x \in S_i \\
0, & \text{if } x \in S_i^c.
\end{cases}
\]

Then, we can use techniques similar to that used in Section 3.2 to construct indicator functions. The difference is that a classification ReLU network represents a piecewise constant function over polytopes. Viewing this piecewise constant function as a special case of the piecewise linear function and applying the same techniques in proving Theorem 2, we are able to directly build the equivalence for classification networks (Theorem 3). However, since the output is discrete, we can have a simplified construction of \( H_1(x) \) and \( H_2(x) \) by introducing a binary step function. The utility of binary step functions can greatly improve the efficiency of constructing a simplex, where all the regions outside the simplex can be eliminated at once. As a consequence, the dependence of width of \( H_1(x) \) on \( D \) can be reduced to the same level as that of the depth of \( H_2(x) \), moving towards a simplified variant of quasi-equivalence of classification networks (Theorem 4), and the proof of Theorem 4 is put into Appendix B.

**Theorem 3** (Quasi-equivalence of Classification Networks). Without loss of generality, we assume a multivariate input and a binary output. Suppose that the representation of an arbitrary ReLU network is \( h : [-B, B]^D \to \{0, 1\} \), for any \( \delta > 0 \), there exist a wide ReLU network \( H_1 \) of width \( \mathcal{O}[D(D + 1)(2^D - 1)|M|] \) and depth \( D + 1 \), and also a deep ReLU network \( H_2 \) of width \( (D + 1)D^2 \) and depth \( \mathcal{O}[(D + 2)|M|] \), where \( M \) is the minimum number of simplices to cover the polytopes to support \( h \), satisfying that

\[
\begin{align*}
&\mathbb{P}\left( x \mid h(x) \neq H_1(x) \right) < \delta \\
&\mathbb{P}\left( x \mid h(x) \neq H_2(x) \right) < \delta,
\end{align*}
\]
where \( m(\cdot) \) is the standard measure in \([-B, B]^D\).

**Proof.** The key is to regard the classification network as a special case of regression network. Then, applying the construction techniques used in the proof of Theorem 2 will lead to that for any \( \delta > 0 \),

\[
m(x | h(x) \neq H_1(x)) < \delta \quad \text{and} \quad m(x | h(x) \neq H_2(x)) < \delta.
\]

which verifies the correctness of Theorem 3.

**Theorem 4** (Simplified Quasi-equivalence of Classification Networks). Without loss of generality, we assume a multivariate input and a binary output. Suppose that the representation of an arbitrary ReLU network is \( h : [-B, B]^D \rightarrow \{0, 1\} \), for any \( \delta > 0 \), there exist a wide binary step network \( H_1 \) of width \( O[(D + 1)M] \) and depth 2, and also a deep binary step network \( H_2 \) of width \( (D + 1) \) and depth \( O[(D + 1)M] \), where \( M \) is the minimum number of simplices to cover the polytopes to support \( h \), satisfying that

\[
h(x) = H_1(x)
\]

\[
m(x | h(x) \neq H_2(x)) < \delta,
\]

where \( m(\cdot) \) is the standard measure in \([-B, B]^D\).

**Remark 2:** The De Morgan equivalence in Figure 2 and Theorems 3 and 4 can be summarized respectively as

\[
H_2(x) \simeq A_1 \lor A_2 \cdots \lor A_M
\]

\[
= \neg((\neg A_1) \land (\neg A_2) \cdots \land (\neg A_M)) \simeq H_1(x),
\]

when the rules are based on hypercubes, and

\[
H_2(x) \simeq A_1 \lor A_2 \cdots \lor A_M \simeq H_1(x),
\]

when rules are based on simplices.

### 4 Discussions

**Equivalent Networks:** In a broader sense, our quasi-equivalence studies demonstrate that there are mutually equivalent networks in a practical sense (in principle, as accurately as needed). Such an equivalence between two networks implies that given any input, two networks produce essentially the same output. The network equivalence is useful in many ways, such as in network optimization. Although deep networks manifest superb power, their applications can be constrained, for example, when the application is time-critical. In that case, we can convert the deep network to a wide counterpart that can be executed at a high speed. A main goal of network optimization is to derive a compact network that maintain a high performance of a complicated network, through quantization \[37\], pruning \[21\], distillation \[28\], binarification \[29\], low-rank approximation \[40\], etc. We envision that the equivalence of a deep network and a wide network suggests a new direction of network optimization in a task-specific fashion. Ideally, a wide network is able to replace the well-trained deep network without compromising the performance. Since the processing steps of the wide network are paralleled, the wide network can be trained on a computing cluster with many machines, which facilitates the fast training. Because of its parallel nature, the inference time of the equivalent wide network is shorter than its deep counterpart \[39\].

**Width-Depth Correlation:** The class of partially separable multivariate functions \[23\] allows that every continuous \( n \)-variable function \( f \) on \([0, 1]^n\) can be in the \( L_1 \) sense represented as:

\[
\int_{(x_1, \cdots, x_n) \in [0, 1]^n} |f(x_1, \cdots, x_n) - \sum_{l=1}^L \prod_{i=1}^n \phi_l(x_i)| < \epsilon.
\]
where $\epsilon$ is an arbitrarily small positive number, $\phi_l$ is a continuous function, and $L$ is the number of products. In the Supplementary Information V and VI, we justify the suitability of the partially separable representation by showing its boundedness, and comparing it with other representations.

Further, we can correlate the width and depth of a network to the structure of a function to be approximated. Previously, our group designed the quadratic neuron [13] that replaces the inner product in a conventional neuron with a quadratic function. In a nutshell, each continuous function $\phi_l$ can be approximated by a polynomial of some degree. Based on the Algebraic Fundamental Theorem [33], each polynomial can be factorized as the product of quadratic terms, which can be appropriately represented by quadratic neurons. As a consequence, in such a quadratic representation scheme, the width and depth of a network structure must reflect the complexity of $\sum_{l=1}^{L} \prod_{i=1}^{n} \phi_l(x_i)$. In other words, they are controlled by the nature of a specific task. As the task becomes complicated, the width and depth must increase accordingly, and the combination of the width and depth is not unique. For more details, please see the Supplementary Information VII.

Effects of Width on Optimization, Generalization and VC dimension: In the Supplementary Information VIII, we illustrate the importance of width on optimization in the context of over-paramterization, kernel ridge regression, and NTK, and then report our findings that the existing generalization bounds and VC dimension results also shed light on the relationship of the width and depth for a given complexity of networks.

5 Conclusion

Inspired by a symmetric consideration and the De Morgan law, we have established the quasi-equivalence between the depth and width of ReLU neural networks for regression and classification tasks. Specifically, we have formulated the transformation from an arbitrary regression ReLU network to a wide ReLU network and a deep ReLU network respectively, and as a special case from an arbitrary classification ReLU network to a wide binary step network and a deep binary step network respectively. In a good sense, our proposed network reconfiguration scheme is a data-driven version of the De Morgan law in the context of binary classification. Clearly, more efforts are needed to realize the full potential of this quasi-equivalence theory in basic research and for real-world applications.

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We write three vertices of \( \sigma \) which can be represented by two neurons \( S \) the function using a ReLU network. To tackle this issue, we start from a linear function \( A \) can be easily verified that \( S \) clear that \( \ell \) function

\[ f = \begin{cases} a^\top x + b, & \text{if } x \in S \\ 0, & \text{if } x \in S^c \end{cases} \tag{12} \]

where \( a = (f(\mathbf{v}_1) - f(\mathbf{v}_0), f(\mathbf{v}_2) - f(\mathbf{v}_0)), b = f(\mathbf{v}_0) \). Our goal is to approximate the given piecewise linear function \( f \) over \( S \) so that we need to cancel \( f \) outside its domain. We first index the polytopes separated by three lines \( \ell_1(x) = 0, \ell_2(x) = 0, \) and \( \ell_3(x) = 0 \) as \( \mathcal{A}(x_1, x_2, x_3) = \ell_1^{x_1} \cap \ell_2^{x_2} \cap \ell_3^{x_3}, x_1, x_2, x_3 \in \{+, -, \}, i = 1, 2, 3. \) It is clear that \( S = \mathcal{A}(+, +, +). \) In addition, we use \( \nu \) to exclude a component. For instance, \( \mathcal{A}(x_1, +, +) = \ell_1^{x_1} \cap \ell_3^+ \). It can be easily verified that \( \mathcal{A}(x_1, +, +) \cap \mathcal{A}(x_1, -, +) \).

Representing \( f \) with a wide ReLU network: The discontinuity of \( f \) in \([12]\) is a major challenge of representing the function using a ReLU network. To tackle this issue, we start from a linear function \( f(\mathbf{x}) = a^\top \mathbf{x} + b, \forall \mathbf{x} \in \mathbb{R}^2 \), which can be represented by two neurons \( \sigma \circ f - \sigma \circ (-f) \). The key idea is to eliminate \( f \) over all polytopes outside \( S \). In other words, \( \hat{f} \) over three fan-shaped polytopes \( \mathcal{A}(\nu, +, +), \mathcal{A}(+, +, \nu), \) and \( \mathcal{A}(+,-,+) \) should be cancelled.
Figure 3: Quasi-equivalence analysis in 2D case. (a) The structure of the wide network to represent \( f \) over \( S \), where two neurons denote \( f \) over \([-B, B]^2\) and nine fan-shaped functions handle the polytopes outside \( S \). (b) The polytopes outside \( S \) comprise of three fan-shaped domains, on which \( f \) can be cancelled by three linearly independent fan-shaped functions. (c) The structure of the deep network to represent \( f \) over \( S \), where six building blocks represent three linearly independent functions over \( S \), and then these functions are aggregated to represent \( f \) over \( S \). (d) Allowing using more layers, a linear function over \( S \) can be obtained.

Let us take the polytope \( A^{(+, \vee, -)} \) as an example. Note that \( A^{(+, \vee, -)} \) has two boundaries \( \ell_1(x) = 0 \) and \( \ell_3(x) = 0 \) as illustrated in Figure 3(b). We choose a sufficiently large positive number \( \mu \) to construct the three fan-shaped functions:

\[
\begin{align*}
F^{(+, \vee, -)}_1(x_1, x_2) &= \sigma(x_1 - \mu \sigma(-x_1 - x_2 + 1)) \\
F^{(+, \vee, -)}_2(x_1, x_2) &= \sigma(x_1 - \eta x_2 - \mu \sigma(-x_1 - x_2 + 1)) \\
F^{(+, \vee, -)}_3(x_1, x_2) &= \sigma(x_1 - \eta - \mu \sigma(-x_1 - x_2 + 1)),
\end{align*}
\]

where the positive number \( \eta \) is chosen to be small enough such that the lines \( x_1 - \eta x_2 = 0 \) and \( x_1 - \eta = 0 \) are very close to \( x_1 = 0 \), then \( \min((x_1)^+ \cap (x_1 - \eta x_2)^-) < 2\sqrt{2B}\eta \) and \( \min((x_1)^+ \cap (x_1 - \eta)^-) < 2\sqrt{2B}\eta \).

According to the aforementioned properties of fan-shaped functions, we approximately have

\[
\begin{align*}
F^{(+, \vee, -)}_1(x) &= x_1, \quad \forall x \in A^{(+, \vee, -)} \\
F^{(+, \vee, -)}_2(x) &= x_1 - \eta x_2, \quad \forall x \in A^{(+, \vee, -)} \setminus ((x_1)^+ \cap (x_1 - \eta x_2)^-) \\
F^{(+, \vee, -)}_3(x) &= x_1 - \eta, \quad \forall x \in A^{(+, \vee, -)} \setminus ((x_1)^+ \cap (x_1 - \eta)^-),
\end{align*}
\]

Let us find \( \omega_1^*, \omega_2^*, \omega_3^* \) by solving

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & -\eta & 0 \\
0 & 0 & -\eta
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}
= 
\begin{bmatrix}
a_1 \\
\sigma a_2 \\
b
\end{bmatrix}.
\]

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Then, the new function $F^{(+,\gamma,-)}(x) = \sum_{i} \omega_i^1 F_i^{(+,\gamma,-)}(x) + \sum_{i} \omega_i^2 F_i^{(+,\gamma,-)}(x) + \sum_{i} \omega_i^3 F_i^{(+,\gamma,-)}(x)$ satisfies
\[
\left( \{ x \in A^{(\gamma,-)} | \tilde{f}(x) \neq F^{(+,\gamma,-)}(x) \neq 0 \} \right) < 2\sqrt{2}B(2\eta + 3/\mu).
\]

Similarly, we can construct $F^{(\gamma,-,+)}$ and $F^{(-,\gamma,+)}$ to eliminate $\tilde{f}$ on $A^{(\gamma,-,+)}$ and $A^{(-,\gamma,+)}$ respectively. Finally, these fan-shaped functions are aggregated to form the following ReLU network $N_1$ (illustrated in Figure 3(a)):
\[
N_1(x) = \sigma \circ (\tilde{f}(x)) - \sigma \circ (-\tilde{f}(x)) + F^{(\gamma,-,+)}(x) + F^{(+,\gamma,-)}(x) + F^{(-,\gamma,+)}(x),
\]
where the width and depth of the network are $2 + 3 \times 3 = 20$ and $3$ respectively. In addition, due to the $9$ fan-shaped functions being utilized and the effect of the $\eta$, the total area of the regions suffering from errors is no more than
\[
2\sqrt{2}B(6\eta + 9/\mu).
\]

Therefore, for any $\delta > 0$, as long as we choose $\eta$ and $\mu$ satisfying
\[
0 < \eta, \mu < \frac{\delta}{2\sqrt{2}B(6 + 9)} = \frac{\delta}{30\sqrt{2}B},
\]
the constructed network $N_1$ will have
\[
\left( \{ x \in \mathbb{R}^2 | f(x) \neq N_1(x) \} \right) < \delta.
\]

According to Proposition 1, the network $h$ is piecewise linear and splits the space into polytopes. It is feasible to employ a number of simplices to fill the polytopes defined by $h$. Given that $M$ is the number of required simplices, by aggregating the network $N_1(x_1, x_2)$ concurrently, we have the following wide network:
\[
H_1(x) = \sum_{m=1}^{M} \sum_{m=1}^{\infty} N_1^{(m)}(x_1, x_2)
\]
where $N_1^{(m)}(x_1, x_2)$ represents the linear function over the $m^{th}$ simplex. Therefore, the constructed wide network $H_1(x)$ is of width $O(20M)$ and depth $3$. It is clear that the width $O(20M)$ of the wide network $H_1(x)$ dominates, as the number of needed simplices goes larger and larger.

Representing $f$ with a deep ReLU network: Allowing more layers in a network provides an alternate way to represent $f$. The fan-shaped functions remain to be used. The whole pipeline can be divided into two steps: (1) build a function over $S$; and (2) represent $f$ over $S$ by slightly moving one boundary of $S$ to create linear independent bases.

(1) Let $F(x_1, x_2) = \sigma \circ (x_1 - \mu_1 x_2)$ and $F'(x_1, x_2) = \sigma \circ (x_1 - \mu_2 x_2)$, both of which are approximately enclosed by boundaries $x_1 = 0$ and $x_2 = 0$. Therefore, the fan-shaped regions of $F(x_1, x_2)$ and $F'(x_1, x_2)$ almost overlap as $\nu$ is small. The negative sign for $x_2$ is to make sure that the fan-shaped region is $S$. To obtain the third boundary $\ell_3(x) = 0$ for building the simplex $S$, we stack one more layer with only one neuron to separate the fan-shaped region of $F(x_1, x_2)$ with the boundary $-x_1 - x_2 + 1 = 0$ as follows:
\[
N_2(x) = (\gamma_1 F(x) + \gamma_2 F'(x) + \gamma_3)^+, \quad (22)
\]
where $(\gamma_1, \gamma_2, \gamma_3)$ are roots of the following system of equations:
\[
\begin{cases}
\gamma_1 + \gamma_2 = -1 \\
-\nu \gamma_2 = -1 \\
\gamma_3 = 1.
\end{cases} \quad (23)
\]

Thus, $N_2(x_1, x_2)$ will represent the function $-x_1 - x_2 + 1$ over $S$ and zero in the rest area. The depth and width of $N_2(x_1, x_2)$ are $3$ and $4$ respectively. Similarly, due to the employment of the two fan-shaped functions and the effect of $\nu$, the area of the region with errors is estimated as
\[
2\sqrt{2}B(\nu + 2/\mu).
\]

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where \( \tilde{N} \) we have the following deep network:

\[
\ell = \ell_3 - \tau' x_1 + \ell_3 - \tau'' x_2.
\]

Repeating the procedure described in (1), for \( \ell_3 \) we construct the network \( N_3^2(x_1, x_2) \) that is \( \ell_3 - \tau' x_1 \) over \( \ell_3^+ \cap \ell_3^2 \cap (\ell_3')^+ \), while for \( \ell_1 \) we construct the network \( N_2^1(x_1, x_2) \) that is \( \ell_3 - \tau' x_1 \) over \( \ell_1^+ \cap \ell_2^2 \cap (\ell_1')^+ \). We set positive numbers \( \tau' \) and \( \tau'' \) small enough to have two triangular domains \( \ell_1^+ \cap \ell_2^2 \cap (\ell_1')^+ \) and \( \ell_1^+ \cap \ell_2^2 \cap (\ell_1')^+ \) almost identical with \( S \). In addition, let \( \tau' \) and \( \tau'' \) satisfy

\[
\begin{bmatrix}
-1 & -1 & -1-	au' \\
-1 & -1 & -1-	au'' \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\rho_1^0 \\
\rho_2^0 \\
\rho_3^0
\end{bmatrix} = \begin{bmatrix}
a_1 \\
a_2 \\
b
\end{bmatrix},
\]

where \( \rho_1^0, \rho_2^0, \rho_3^0 \) are solutions. As a consequence, the deep network illustrated in Figure 3(c): \( \tilde{N}_2(x) = \rho_1^0 N_2^2(x) + \rho_2^0 N_2^2(x) + \rho_3^0 N_2^2(x) \)

produces \( f \) on \( S \). The depth and width of the network are 4 and 12 respectively. Similarly, the area of the region with errors is bounded above by

\[
2\sqrt{2B} (3\omega + \tau' + \tau'' + 6/\mu)).
\]

Therefore, for any \( \delta > 0 \), if we choose

\[
0 < \nu, \tau', \tau'', 1/\mu < \frac{\delta}{2\sqrt{2B}(3 + 2)} = \frac{\delta}{22\sqrt{2B}}
\]

then the constructed network \( \tilde{N}_2 \) will satisfy

\[
m \left( \{x \in \mathbb{R}^2 | f(x) \neq \tilde{N}_2(x) \} \right) < \delta.
\]

Similarly, given that \( M \) is the number of required simplices, by stacking the network \( \tilde{N}_2(x_1, x_2) \) longitudinally, we have the following deep network:

\[
H_2(x) = \tilde{N}_2^{(M)}(x_1, x_2, \cdots, \tilde{N}_2^{(2)}(x_1, x_2, \tilde{N}_2(1)(x_1, x_2)) \cdots)),
\]

where \( \tilde{N}_2^{(m)}(x_1, x_2, t) = N_2^{(m)}(x_1, x_2) + t \), and \( N_2^{(m)}(x_1, x_2) \) represents the linear function also over the \( m \)th simplex. Therefore, the constructed deep network \( H_2(x) \) is of depth \( O(4M) \) and width 12. It is clear that the depth \( O(4M) \) of the deep network \( H_2(x) \) dominates.

Proof. \( \textbf{Theorem 2} \) \( D \geq 2 \) \( A \) \( D \)-simplex \( S \) is a \( D \)-dimensional convex hull provided by convex combinations of \( D + 1 \) affinely independent vectors \( \{v_i\}_{i=0}^D \subset \mathbb{R}^D \). In other words, \( S = \left\{ \sum_{i=0}^D \xi_i v_i | \xi_i \geq 0, \sum_{i=0}^D \xi_i = 1 \right\} \).

If we write \( V = (v_1 - v_0, v_2 - v_0, \cdots, v_D - v_0) \), then \( V \) is invertible, and \( S = \left\{ v_0 + Vx | x \in \Delta \right\} \) where \( \Delta = \left\{ x \in \mathbb{R}^D | x \geq 0, 1^T x \leq 1 \right\} \) is a template simplex in \( \mathbb{R}^D \). It is clear that the following one-to-one affine mapping between \( S \) and \( \Delta \) exists, which is

\[
T : S \to \Delta, p \mapsto T(p) = V^{-1}(p - v_0).
\]

Therefore, we only need to prove the statement on the special case that \( S = \Delta \).

We denote the domain of a network as \( \Omega = [-B, B]^D \). Given a linear function \( \ell(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + c_{n+1} \), we write \( \ell^- = \{ x \in \mathbb{R}^D | \ell(x) < 0 \} \) and \( \ell^+ = \{ x \in \mathbb{R}^D | \ell(x) \geq 0 \} \). \( S \) is enclosed by \( D + 1 \) hyperplanes provided by \( \ell_i(x) = x_i, i = 1, \cdots, D, \) and \( \ell_{D+1}(x) = -x_1 - \cdots - x_{D+1} = 1 \). We write \( D + 1 \) vertices of \( S \) as \( v_0 = (0, 0, \cdots, 0), v_1 = (1, 0, \cdots, 0), v_2 = (0, 1, \cdots, 0), \cdots, v_{D+1} = (0, \cdots, 0, 1) \). Then \( f : [-B, B]^D \to \mathbb{R} \) supported on \( S \) is provided as

\[
f(x) = \begin{cases}
  a^T x + b, & \text{if } x \in S \\
  0, & \text{if } x \in S^c,
\end{cases}
\]

where \( a = (f(v_1) - f(v_0), f(v_2) - f(v_0), \cdots, f(v_{D+1}) - f(v_0)) \), \( b = f(v_0) \). Our goal is to approximate the given piecewise linear function \( f \) using ReLU networks. We first index the polytopes separated by \( D + 1 \)}
hyperplanes $\ell_i(x) = 0, i = 1, \ldots, D + 1$ as $A^{(x_1, \ldots, x_D, x_{D+1})} = \ell_1^{x_1} \cap \ldots \cap \ell_i^{x_i} \cap \ldots \cap \ell_{D+1}^{x_{D+1}}, x_i \in \{+, -\}, i = 1, \ldots, D + 1$. It is clear to see that $S = A^{(+, +, \ldots, +)}$. In addition, we use $\lor$ to denote exclusion of certain component. For instance, $A^{(x_1, \lor x_2, \ldots, x_{D+1})} = \ell_1^{x_1} \cap \ell_2^{x_2} \cap \ldots \cap \ell_{D+1}^{x_{D+1}}$. It can be easily verified that

$$A^{(x_1, \lor x_2, \ldots, x_{D+1})} = A^{(x_1, +, x_3, \ldots, x_{D+1})} \cup A^{(x_1, -, x_3, \ldots, x_{D+1})}. \quad (33)$$

Please note that $A^{(-, +, \ldots, -)} = \emptyset$. Thus, $D + 1$ hyperplanes create in total $2^{D+1} - 1$ polytopes in the $\Omega$.

Now we recursively define an essential building block, a $D$-dimensional fan-shaped ReLU network $F_D(x)$:

$$\begin{cases}
F_1(x) = h_1(x) \\
F_{j+1}(x) = \sigma \circ (F_j(x) - \mu \sigma \circ h_{j+1}(x)), \quad j = 1, \ldots, D - 1,
\end{cases} \quad (34)$$

where the set of linear functions $\{h_k(x) = p_k^T x + r_k\}_{k=1}^D$ are provided by $D$ linearly independent vectors $\{p_k\}_{k=1}^D$, and $\mu$ is a large positive number ($\mu^2$ denotes $\mu$ with the power to $j$). Note that the network $F_D$ is of width $D$ and depth $D$. This network enjoys the following key characteristics: 1) As $\mu \to \infty$, the hyperplane $h_1 - \mu h_2 - \cdots - \mu^j h_{j+1} = 0$ is approximate to the hyperplane $h_{j+1} = 0$ as the term $\mu^j h_{j+1}$ dominates. Thus, the support of $F_D(x)$ converges to $h_1^T \cap h_2 \cap \cdots \cap h_D$ which is a $D$-dimensional fan-shaped function. 2) Let $C$ be the maximum area of hyperplanes in $[-B, B]^D$. Because the real boundary $h_1 - \mu h_2 - \cdots - \mu^j h_{j+1} = 0$ is almost parallel to the ideal boundary $h_{j+1} = 0$, the measure of the imprecise domain caused by $\mu^2$ is at most $C/\mu^j$, where $1/\mu^j$ is the approximate distance between the real and ideal boundaries. In total, the measure of the inaccurate region in building $F_D(x)$ is at most $C \sum_{j=1}^{D-1} 1/\mu^j \leq C/(\mu - 1)$. 3) The function over $D$-dimensional fan-shaped domain is $h_1^+$, since $(h_j)^+ = 0, j \geq 2$ over the $D$-dimensional fan-shaped domain.

Representing $f$ with a wide ReLU network: Discontinuity of $f$ in (32) is one of the major challenges of representing it using a ReLU network. To tackle this issue, we start from a linear function $\tilde{f}(x) = a^T x + b, \forall x \in \mathbb{R}^D$, which can be represented by two neurons $\sigma \circ f - \sigma \circ (-f)$. The key idea is to eliminate $f$ over all $2^{D+1} - 2$ polytopes outside $S$ using the $D$-dimensional fan-shaped functions.

Let us use $A^{(+, +, \ldots, -)}$ and $A^{(+, +, \ldots, -)}$ to show how to cancel the function $\tilde{f}$ over the polytopes outside $S$. According to (33), $A^{(+, +, \ldots, -)}$ and $A^{(+, +, \ldots, -)}$ satisfy

$$A^{(+, +, \ldots, -)} = A^{(+, +, \ldots, -)} \cup A^{(+, +, \ldots, -)}, \quad (35)$$

where $A^{(+, +, \ldots, -)}$ is a $D$-dimensional fan-shaped domain. Without loss of generality, a number $D + 1$ of $D$-dimensional fan-shaped functions over $A^{(+, +, \ldots, -)}$ are needed as the group of linear independent bases to cancel $\tilde{f}$, where the $k^{th}$ fan-shaped function is constructed as

$$\begin{cases}
F_1^{(k)} = x_1 - \eta_k x_k \\
F_2^{(k)} = \sigma \circ (F_1^{(k)} - \mu \sigma \circ (-x_2)) \\
F_3^{(k)} = \sigma \circ (F_2^{(k)} - \mu^2 \sigma \circ (x_3)) \\
\vdots \\
F_D^{(k)} = \sigma \circ (F_{D-1}^{(k)} - \mu^{D-1} \sigma \circ (-x_1 - \cdots - x_{D+1})),
\end{cases} \quad (36)$$

where we let $x_{D+1} = 1$ for consistency, the negative sign for $x_2$ is to make sure that the fan-shaped region $\ell_1^+ \cap (-\ell_2) \cap \cdots \cap \ell_{D+1}$ of $F_D^{(k)}$ is $A^{(+, +, \ldots, -)}$, $\eta_1 = 0$, and $\eta_k = \eta, k = 2, \ldots, D + 1$ represents a small shift for $x_1 = 0$ such that $m((x_1)^+ \cap (x_1 - \eta_k x_k)^-) < C \eta_k$. The constructed function over $A^{(+, +, \ldots, -)}$ is

$$F_D^{(k)} = x_1 - \eta_k x_k, k = 1, \ldots, D + 1, \quad (37)$$

which is approximately over

$$\forall x \in A^{(+, +, \ldots, -)} \setminus ((x_1)^+ \cap (x_1 - \eta_k x_k)^-), \quad (38)$$
Let us find $\omega^*_1, \cdots, \omega^*_{D+1}$ by solving

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & -\eta & 0 & \cdots & 0 \\
0 & 0 & -\eta & \cdots & 0 \\
\vdots & & & & \vdots \\
0 & 0 & 0 & \cdots & -\eta
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\vdots \\
\omega_{D+1}
\end{bmatrix}
= 
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
\vdots \\
b
\end{bmatrix},
$$

(39)

and then the new function $F^{(+, +, \vee, \cdots, \vee)}(x) = \sum_{k=1}^{D+1} \omega^*_k F^{(k)}_D(x)$ satisfies that

$$
m\left\{x \in \mathcal{A}^{(+, +, \vee, \cdots, \vee)} | \tilde{f}(x) + F^{(+, +, \vee, \cdots, \vee)}(x) \neq 0\right\} \leq C(D\eta + \frac{D+1}{\mu - 1}).
$$

(40)

Similarly, we can construct other functions $F^{(+, +, \vee, \cdots, \vee)}(x), F^{(+, +, \vee, \cdots, \vee)}(x), \cdots$ to cancel $\tilde{f}$ over other polytopes. Finally, these $D$-dimensional fan-shaped functions are aggregated to form the following wide ReLU network $N_1(x)$:

$$
N_1(x) = \sigma \circ (\tilde{f}(x)) - \sigma \circ (-\tilde{f}(x)) + \sum_{k=1}^{2D-2\text{ terms}} F^{(+, +, \vee, \cdots, \vee)}_k(x) + \cdots,
$$

(41)

where the width and depth of the network are $D(D+1)(2^D - 1) + 2$ and $D+1$ respectively. In addition, because there are $2^D - 1$ polytopes being cancelled, the total area of the regions suffering from errors is no more than

$$
(2^D - 1)C(D\eta + \frac{D+1}{\mu - 1}).
$$

(42)

Therefore, for any $\delta > 0$, as long as we choose appropriate $\mu, \eta$ that fulfill

$$
0 < \frac{1}{\mu - 1}, \eta < \frac{\delta}{(2^D - 1)C(D+D+1)} = \frac{\delta}{(2^D - 1)C(2D + 1)},
$$

(43)

the constructed network $N_1(x)$ will have

$$
m\left\{x \in \mathbb{R}^D | f(x) \neq N_1(x)\right\} < \delta.
$$

(44)

Similarly, we can also aggregate the network $N_1(x)$ concurrently to obtain the following wide network:

$$
H_1(x) = \sum_{m=1}^{M} N_1^{(m)}(x),
$$

(45)

where $N_1^{(m)}(x)$ represents the linear function over the $m^{th}$ simplex. Therefore, the constructed wide network $H_1(x)$ is of width $O[D(D + 1)(2^D - 1)M]$ and depth $D + 1$.

Representing $f$ with a deep ReLU network: Allowing more layers in a network provides an alternate way to represent $f$. The fan-shaped functions remain to be used. The whole pipeline can be divided into two steps: (1) build a function over $S$; and (2) represent $f$ over $S$ by slightly moving one boundary of $S$ to create linear independent bases.

1. We construct a number $D$ of $D$-dimensional fan-shaped functions. Without loss of generality, the $k^{th}$ fan-shaped function is constructed as

$$
\begin{align*}
F_1^{(k)} &= x_1 - \nu_k x_k \\
F_2^{(k)} &= \sigma \circ (F_1^{(k)} - \mu^1 \sigma \circ (-x_2)) \\
&\vdots \\
F_D^{(k)} &= \sigma \circ (F_{D-1}^{(k)} - \mu^{D+1} \sigma \circ (-x_D)),
\end{align*}
$$

(46)
whose fan-shaped region is approximately \((\ell_1 - \nu_k x_k)^+ \cap (-\ell_2)^- \cap \cdots \cap (-\ell_D)^- = (\ell_1 - \nu_k x_k)^+ \cap \ell_2^+ \cap \cdots \cap \ell_D^+\), which almost overlaps with \(A^{(+,\ldots,+)} = \ell_1^+ \cap \ell_2^+ \cap \cdots \cap \ell_D^+\) as \(\nu_k\) becomes sufficiently small. The output of \(F_D^{(k)}\) is \(\tilde{x}_1 - \nu_k x_k, k = 1, \cdots, D\). To obtain the last boundary \(\ell_{D+1}(x) = -x_1 - \cdots - x_D + 1 = 0\) so as to construct the simplex \(S\), we stack one more layer with only one neuron as follows:

\[
N_2^1(x) = (\gamma_1^* F_D^{(1)} + \cdots + \gamma_D^* F_D^{(D)} + \gamma_{D+1}^*)^+,
\]

where \(\gamma_1^*, \cdots, \gamma_{D+1}^*\) are the roots of the following equation:

\[
\begin{align*}
(1 - \mu_1)\gamma_1 + \gamma_2 + \cdots + \gamma_D &= -1 \\
-\nu_2 \gamma_2 &= -1 \\
& \vdots \\
-\nu_D \gamma_D &= -1 \\
\gamma_{D+1} &= 1.
\end{align*}
\]

Thus, \(N_2^1(x)\) will approximately represent the linear function \(-x_1 - \cdots - x_D + 1\) over \(S\) and zero elsewhere. The depth and width of the network are \(D + 1\) and \(D^2\) respectively. Similarly, due to the employment of a number \(D\) of \(D\)-dimensional fan-shaped functions and the effect of \(\nu_k\), the area of the region with errors is estimated as

\[
CD/(\mu - 1) + C \sum_{k=1}^{\nu_k} \nu_k.
\]

(2) To acquire an arbitrary linear function, similarly we need \(D + 1\) linear independent functions as linear independent bases. Other than the one obtained in step (1), we further modify \(\ell_{D+1}\) a little bit \(D\) times to get \(\ell_{D+1}^l = \ell_{D+1} - \tau_l x_l, l = 1, \cdots, D\). Repeating the same procedure described in step (1), for \(\ell_{D+1}^l\), we can construct the network \(N_2^l(x)\) that is \(\ell_{D+1}^l = \tau_l x_l\) approximately over \(\ell_1^+ \cap \ell_2^+ \cap \cdots \cap (\ell_{D+1}^l)^+\), where \(\tau_l, l = 1, \cdots, D\) is small to render these domains almost identical to \(S\), and \(\tau_l, l = 1, \cdots, D\) satisfies that

\[
\begin{bmatrix}
-1 - \tau_1 & -1 & \cdots & -1 & -1 \\
-1 & -1 - \tau_2 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & -1 - \tau_D & -1 \\
1 & 1 & \cdots & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\rho_1^* \\
\rho_2^* \\
\vdots \\
\rho_D^* \\
\rho_{D+1}^*
\end{bmatrix}
= \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_D \\
b
\end{bmatrix},
\]

where \(\rho_1^*, \cdots, \rho_{D+1}^*\) are the roots. As a result, the deep network is built as

\[
N_2(x) = \rho_1^* N_2^1(x) + \rho_2^* N_2^2(x) + \cdots + \rho_{D+1}^* N_2^{D+1}(x),
\]

producing \(f\) on \(S\). The depth and width of \(N_2(x)\) are \(D + 2\) and \(D^2(D + 1)\) respectively. Similarly, the area of the region with errors is bounded above by

\[
CD(D + 1)/(\mu - 1) + C(D + 1) \sum_{k=1}^{\nu_k} \nu_k + C \sum_{l=1}^{D} \nu_l.
\]

Therefore, for any \(\delta > 0\), if we choose \(\mu, \nu_k, \tau_l\) appropriately such that

\[
0 < 1/(\mu - 1), \nu_k, \tau_l < \frac{\delta}{C(D(D + 1) + (D + 1)D + D)},
\]

then the constructed network \(N_2(x)\) will satisfy

\[
\begin{align*}
\mathbb{m} \{ \{ x \in \mathbb{R}^D \mid f(x) \neq N_2(x) \} \} < \delta.
\end{align*}
\]

We stack the network \(N_2(x)\) longitudinally, we have the following deep network:

\[
H_2(x) = \tilde{N}_2^{(M)}(x, \cdots, \tilde{N}_2^{(2)}(x, \tilde{N}_2^{(1)}(x)), \cdots),
\]

where \(\tilde{N}_2^{(m)}(x, t) = N_2^{(m)}(x) + t\), and \(N_2^{(m)}(x)\) represents the linear function also over the \(m\)th simplex. Therefore, the constructed deep network \(H_2(x)\) is of depth \(O([D + 2]M)\) and width \((D + 1)D^2\).
7 Appendix B. Proof of Theorem 4

Proof. (Proof of Theorem 4) A $D$-simplex $S$ is a $D$-dimensional convex hull provided by convex combinations of $D + 1$ affinely independent vectors $\{v_i\}_{i=0}^D \subset \mathbb{R}^D$. In other words, $S = \left\{ \sum_{i=0}^D \xi_i v_i \mid \xi_i \geq 0, \sum_{i=0}^D \xi_i = 1 \right\}$.

Similarly, if we write $V = (v_1 - v_0, v_2 - v_0, \cdots, v_D - v_0)$, then $V$ is invertible, and $S = \{v_0 + Vx \mid x \in \Delta\}$ where $\Delta = \{x \in \mathbb{R}^D \mid x \geq 0, 1^T x \leq 1\}$ is a template simplex in $\mathbb{R}^D$. It is clear that the following one-to-one affine mapping between $S$ and $\Delta$ exists, which is

$$T : S \rightarrow \Delta, p \mapsto T(p) = V^{-1}(p - v_0). \quad (56)$$

Therefore, we only need to prove the statement on the special case that $S = \Delta$. Then, the indicator function $g : [-B, B]^D \rightarrow \{0, 1\}$ supported on $S$ is provided as

$$g(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \in S^c \end{cases}, \quad (57)$$

which is very suitable for a binary step activation function. The key point is that the function outside the simplex can be directly suppressed by a neuron with a binary step activation function.

![Binary Step Neuron](image)

Figure 4: A wide binary step network can more efficiently represent an indicator function over a simplex because a neuron with binary step activation is only needed to set the function to zero outside the simplex.

Representing $g$ with a wide binary step network: Let

$$\ell_i(x) = x_i, i = 1, \cdots, D$$
$$\ell_{D+1}(x) = -x_1 - x_2 - \cdots - x_D + 1, \quad (58)$$

then the constructed wide network is

$$N_1(x) = z \circ \left( \sum_{i=1}^{D+1} z \circ (\ell_i(x)) - D - 1 \right), \quad (59)$$

where the width is $D + 1$ and the depth is 2. It can be easily verified that $N_1(x) = g(x)$, since the functional value outside $S$ is cleared at the threshold of $D + 1$. As shown in Figure 4, a binary step wide network can more efficiently represent an indicator function over a simplex in 2D space.
By aggregating the network $N_1(x)$ concurrently, we have the following wide network:

$$H_1(x) = \sum_{i=1}^{M} N_1^{(i)}(x)$$  \(60\)

where $M$ is the number of required simplices to support $h$ and $N_1^{(i)}(x)$ represents the linear function over the $i^{th}$ simplex. Therefore, the constructed wide network $H_1(x)$ is of width $O((D+1)M)$ and depth 2.

Representing $g$ with a deep binary step network: We replace ReLU activations with binary step activations in a fan-shaped network

$$F_1 = x_1$$
$$F_2 = z \circ (F_1 - \mu z \circ (-x_2))$$
$$\vdots$$
$$F_D = z \circ (F_{D-1} - \mu^{D+1} z \circ (-x_D)),$$

which ends up with $F_D(x)$ that is approximately an indicator function over the region $\ell_1^+ \cup \ell_2^+ \cup \cdots \ell_D^+$. To gain the last boundary $\ell_{D+1}(x) = -x_1 - x_2 - \cdots - x_D + 1$, we stack one more layer as follows:

$$N_2(x) = z \circ (F_D(x) - \mu^{D+2} \circ \ell_{D+1}(x)),$$

where $N_2(x)$ is of width $D + 1$ and depth $D + 1$, and $N_2(x) \simeq g(x)$. The area with errors is bounded by $C/(\mu - 1)$.

For any $\delta > 0$, if we choose $\mu$ appropriately such that

$$0 < 1/(\mu - 1) < \frac{\delta}{C},$$

then the constructed network $N_2(x)$ will satisfy

$$m\left(\{x \in \mathbb{R}^D | g(x) \neq N_2(x)\}\right) < \delta.$$  \(64\)

Similarly, aggregating the network $N_2(x)$ longitudinally will result in the following deep network:

$$H_2(x) = \tilde{N}_2^{(M)}(x, \cdots, \tilde{N}_2^{(2)}(x, N_2^{(1)}(x)) \cdots),$$  \(65\)

where $\tilde{N}_2^{(i)}(x, t) = N_2^{(i)}(x) + t$, and $N_2^{(i)}(x)$ represents the linear function also over the $i^{th}$ simplex. Therefore, the constructed deep network $H_2(x)$ is of depth $O((D+1)M)$ and width $D + 1$. 

\[\square\]
Typical fan-shaped functions constructed by modularized networks to eliminate non-zero functional values outside the simplex of interest.

The width and depth equivalence in light of the DeMorgan equivalence. In this construction, a deep network to implement $A_1 \land A_2 \land \ldots \land A_n$ using a trapezoid function and a wide version to implement $\neg (\neg A_1) \land \neg (\neg A_2) \land \ldots \land \neg (\neg A_n)$ using the trap-like function. $(\cdot)^+$ denotes ReLU.
Figure 3

Quasi-equivalence analysis in 2D case. (a) The structure of the wide network to represent $f$ over $S$, where two neurons denote $f$ over $[-B,B]^2$ and nine fan-shaped functions handle the polytopes outside $S$. (b) The polytopes outside $S$ comprise of three fan-shaped domains, on which $f$ can be cancelled by three linearly independent fan-shaped functions. (c) The structure of the deep network to represent $f$ over $S$, where six building blocks represent three linearly independent functions over $S$, and then these functions are aggregated to represent $f$ over $S$. (d) Allowing using more layers, a linear function over $S$ can be obtained.
A wide binary step network can more efficiently represent an indicator function over a simplex because a neuron with binary step activation is only needed to set the function to zero outside the simplex.

Supplementary Files

This is a list of supplementary files associated with this preprint. Click to download.

- SupplementaryMaterial.pdf