The Einstein-Maxwell Equations, Kähler Metrics, and Hermitian Geometry

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Abstract

Any constant-scalar-curvature Kähler (escK) metric on a complex surface may be viewed as a solution of the Einstein-Maxwell equations, and this produces solutions of these equations on any 4-manifold that arises as a compact complex surface with $b_1$ even. It is shown, however, that not all solutions of the Einstein-Maxwell equations on such manifolds arise in this way; new examples can be constructed by means of conformally Kähler geometry.

Let $M$ be a smooth compact oriented 4-manifold, equipped with a Riemannian metric $h$ and a real-valued 2-form $F$. One then says that the triple $(M, h, F)$ satisfies the Einstein-Maxwell equations if the relations

\begin{align*}
    dF &= 0 \quad (1) \\
    d \star F &= 0 \quad (2) \\
    \left[ r + F \circ F \right]_0 &= 0 \quad (3)
\end{align*}

all hold, where $r$ is the Ricci tensor of $g$, the subscript $[\ ]_0$ indicates the trace-free part with respect to $g$, and the symmetric tensor $(F \circ F)_{jk} = F_j^\ell F_{\ell k}$

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is obtained by composing $F$ with itself as an endomorphism of $TM$. If $M$ is compact, equations (1–2) can be unambiguously summarized as saying that $F$ is a harmonic 2-form, but (3) is less familiar to differential geometers. In physics, these equations represent the interaction of a gravitational field $h$ and an electromagnetic field $F$; physicists would call these the “Euclidean Einstein-Maxwell equations with cosmological constant,” embellishing the terminology to emphasize that we have taken $h$ to be a Riemannian metric rather than a Lorentzian one, and that we are (implicitly) allowing the scalar curvature to be an arbitrary constant, rather than requiring it to vanish.

While relativists understand these equations as the Euler-Lagrange equations of a suitable Lagrangian, it was pointed out in [21] that they also arise from a non-traditional variational problem that, while apparently unfamiliar to physicists, is of immediate interest in Riemannian geometry. Indeed, suppose that $M^4$ is compact, let $[\omega] \in H^2(M, \mathbb{R})$ be a fixed cohomology class with $[\omega]^2 > 0$, and let $\mathcal{G}_{[\omega]}$ be the Fréchet manifold of smooth Riemannian metrics $g$ on $M$ for which the harmonic representative $\omega$ of $[\omega]$ is self-dual with respect to the relevant metric $g$. Letting $s$ denote the scalar curvature of a Riemannian metric, we can then consider the Einstein-Hilbert functional

$$g \mapsto \int_M s_g \, d\mu_g \sqrt{\int_M d\mu_g}$$

as a functional on $\mathcal{G}_{[\omega]}$, rather than as a functional on the space of all Riemannian metrics. A metric $h \in \mathcal{G}_{[\omega]}$ is then a critical point for this problem if and only if there is a harmonic 2-form $F$ with self-dual part $F^+ \in [\omega]$ such that the pair $(h, F)$ solves (1–3). Similarly, the critical points of the Calabi-type functional

$$g \mapsto \int_M s_g^2 \, d\mu_g$$

on $\mathcal{G}_{[\omega]}$ are either Einstein-Maxwell or scalar-flat.

It is now worth emphasizing that equations (1–3) imply that the scalar curvature of $h$ is constant. Indeed, if $M$ is compact, this can be deduced from the fact that if $h$ belongs to $\mathcal{G}_{[\omega]}$, so does its entire conformal class; and the restriction of (4) to a conformal class is exactly the functional used in the Yamabe problem to identify metrics of constant scalar curvature. However, this fact about (1–3) can also be inferred directly, via a local calculation. Indeed, a contraction of the second Bianchi identity tells us that

$$\nabla \cdot \hat{\tau} = \frac{1}{4} ds$$
in dimension 4, where \( \hat{r} \) is the trace-free Ricci tensor, and where \( s \) is the scalar curvature. On the other hand, (3) can be rewritten as

\[
\hat{r} = -2F^+ \circ F^-,
\]

where \( F^\pm = \frac{1}{2}[F \pm \ast F] \) denotes the self-dual or anti-self-dual part of \( F \), depending on the sign. Since

\[
\nabla \cdot (F^+ \circ F^-) = (\nabla \cdot F^+) \circ F^- + F^+ \circ (\nabla \cdot F^-),
\]

we therefore see that \( ds = 0 \) if \( F^+ \) and \( F^- \) are both co-closed, and the latter stipulation is exactly equivalent to equations (1–2).

The author’s main point in [21] was that constant-scalar-curvature Kähler (cscK) metrics on complex surfaces \((M^4, J)\) can be considered as solutions of the Einstein-Maxwell equations; moreover, when \( M \) is compact and the scalar curvature is non-positive, such solutions are actually \textit{minima} of (5) on \( \mathcal{G}[\omega] \), rather than just critical points.

This article, however, will focus on another class of solutions, suggested by a recent paper of Apostolov, Calderbank and Gauduchon [1]. We begin with a definition that is ostensibly much weaker than theirs:

\textbf{Definition 1} Let \( J \) be an integrable almost-complex structure on \( M \), thus making \( (M^4, J) \) into a complex surface. We will say that a solution \((h, F)\) of the Einstein-Maxwell equations (1–3) on \((M, J)\) is strongly Hermitian if \( h \) and \( F \) both are invariant under the action of \( J \):

\[
h = h(J\cdot, J\cdot),
\]

\[
F = F(J\cdot, J\cdot).
\]

Our first main result, proved in §2 below, asserts that, aside from a well-understood exceptional case, solutions of this type are in fact \textit{globally} of the type studied locally by Apostolov-Calderbank-Gauduchon [1]:

\textbf{Theorem A} Let \((h, F)\) be a strongly Hermitian solution of the Einstein-Maxwell equations on a (connected) complex surface \((M^4, J)\). Then either \( h \) is Einstein and anti-self-dual, or else there is a \( J \)-compatible Kähler metric \( g \) on \( M \) and a real holomorphy potential \( f > 0 \) on \((M, J, g)\) such that \( h = f^{-2}g \) has constant scalar curvature, and such that \( F^+ \) is a constant times the Kähler form \( \omega \) of \( g \). Conversely, if \((M^4, g, J)\) is a Kähler manifold and if \( f > 0 \) is a real holomorphy potential such that \( h = f^{-2}g \) has constant scalar curvature, then there is a unique harmonic 2-form \( F \) on \( M \) with \( F^+ = \omega \) such that \((h, F)\) solves the Einstein-Maxwell equations.
The above result does not require that $M$ be compact, or that $h$ be complete. However, the statement can be simplified in the compact case:

**Theorem B** Let $(h, F)$ be a strongly Hermitian solution of the Einstein-Maxwell equations on a compact complex surface $(M^4, J)$. Then there is a Kähler metric $g$ on $(M, J)$, together with a holomorphy potential $f > 0$ such that $h = f^{-2}g$, and such that $F^+$ is a constant times the Kähler form $\omega$ of $g$.

One consequence is that any compact complex surface that admits a non-Kähler, strongly Hermitian solution of the Einstein-Maxwell equations must be rational or ruled; cf. §4 below for details. While it is beyond the scope of the present paper to try to classify the rational or ruled surfaces which do actually admit such solutions, we will construct a non-trivial example in §3 that demonstrates that such solutions really do exist on at least one rational ruled surface:

**Theorem C** Let $[\omega]$ be a Kähler class on $(M, J) = \mathbb{CP}^1 \times \mathbb{CP}^1$ for which the area of one factor $\mathbb{CP}^1$ is more than double the area of the other. Then $[\omega]$ contains pairs of Kähler metrics which engender two geometrically distinct solutions of the Einstein-Maxwell equations (1–3) via Theorem A.

In other words, if $[\omega](\mathcal{G}_1) > 2[\omega](\mathcal{G}_2)$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the homology classes of the two factor $\mathbb{CP}^1$’s, there are at least two different orbits of the action of

$$\text{Aut}_0(\mathbb{CP}^1 \times \mathbb{CP}^1) = PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$$

on metrics in the given Kähler class which engender strongly Hermitian solutions of the Einstein-Maxwell equations; moreover, the Hermitian metrics $h$ arising from these two different orbits can be distinguished from each other by their curvature properties. This indicates that the uniqueness theorems for cscK metrics on compact complex manifolds [11, 14] unfortunately do not generalize to Einstein-Maxwell metrics in any obvious manner.

These new examples have some curious incidental properties that almost seem like an invitation to revisit difficult open questions regarding the Yamabe problem on $S^2 \times S^2$. A discussion of these issues and other open problems can be found in the concluding section of this article.
1 Preliminaries

Let us now discuss some basic facts needed to provide a solid foundation for the rest of the article. We begin with a different characterization of solutions of the Einstein-Maxwell equations; cf. [1, Proposition 5].

**Proposition 1** Let \((M, h)\) be an oriented Riemannian 4-manifold, and let \(F\) be a real-valued 2-form on \(M\). Let \(s\) and \(\hat{r}\) respectively denote the scalar curvature and trace-free Ricci curvature of \(h\). Let \(Y \subset M\) be the (possibly empty) open set where \(F^+ \neq 0\). Then \((h, F)\) solves the Einstein-Maxwell equations (1–3) on \(Y\) iff the following conditions all hold there:

\[
\begin{align*}
    dF^+ &= 0 \quad (6) \\
    s &= \text{const} \quad (7) \\
    \hat{r} &= -2F^+ \circ F^- \quad (8)
\end{align*}
\]

**Proof.** The endomorphisms of the tangent bundle corresponding to self-dual or anti-self-dual forms of length \(\sqrt{2}\) are almost-complex structures; those arising from \(\Lambda^+\) moreover commute with those arising from \(\Lambda^-\), and the composition of any such pair of almost-complex structures is trace-free. Thus \(F^+ \circ F^+\) and \(F^- \circ F^-\) are multiples of \(h\), and \(F^+ \circ F^- = F^- \circ F^+\). Hence \([F \circ F]_0 = 2F^+ \circ F^-,\) and (3) is therefore algebraically equivalent to (8). On the other hand, we have already seen that (1–2) \(\implies (6),\) and we saw in the introduction that (1–3) \(\implies (7).\) Thus (1–3) \(\implies (6–8)\) on any open set where they hold.

On the other hand, recall that, by the contracted Bianchi identity, (7) is equivalent to the condition \(\nabla \cdot \hat{r} = 0.\) Thus (6–8) tell us that

\[
0 = -\frac{1}{2} \nabla \cdot \hat{r} = (\nabla \cdot F^+) \circ F^- + F^+ \circ (\nabla \cdot F^-)
\]

so \(F^-\) is co-closed, and hence, by anti-self-duality, closed on the open set \(Y\) where \(F^+ \neq 0.\) It therefore follows that \(F = F^+ + F^-\) is both closed and co-closed. Thus (6–8) \(\iff\) (1–3) on \(Y,\) as claimed.

Our next result depends on the notion of a **holomorphy potential**. This concept, already implicit in the work of Matsushima [24] and Lichnerowicz [5].
was eventually codified by Calabi and others \[6, 9\]. A complex valued function \( f : M \to \mathbb{C} \) on a Kähler manifold \((M, g, J)\) is called a holomorphy potential if the \((1, 0)\) component of its gradient is a holomorphic vector field:

\[
\nabla_\mu \nabla^\nu f = 0.
\]

By lowering an index, this is equivalent to saying that the \(\otimes^2 \Lambda^{0,1}\) component of its Hessian vanishes:

\[
\nabla_\mu \nabla_\nu f = 0.
\]

In the special case that \( f \) is real, this is equivalent to saying that its Hessian belongs to \(\Lambda^{1,0} \otimes \Lambda^{0,1}\), or in other words that \(\text{Hess} f\) is \(J\)-invariant. In this real case, something even more remarkable happens. First of all, \( \xi = J \nabla f \) is the imaginary part of a holomorphic vector field, and so the flow of \( \xi \) preserves \( J \). At the same time, \( \xi \) is the symplectic gradient of a function, so its flow also preserves the Kähler form \( \omega \). But since \( g = \omega(\cdot, J\cdot) \), this implies that the flow of \( \xi \) also preserves \( g \). In other words, \( \xi = J \nabla f \) is a Killing field whenever \( f \) is a real holomorphy potential, and this assumption on a real-valued function \( f \) is equivalent to requiring that \(\text{Hess} f\) be a \(J\)-invariant symmetric tensor.

**Proposition 2** Let \((h, F)\) be a strongly Hermitian solution of the Einstein-Maxwell equations on a complex surface \((M^4, J)\). Let \( f = 2^{-1/4} |F^+|^{1/2} \), and let \( Y \) be the (possibly empty) open set where \( f \neq 0 \). Then \( g = f^2 h \) is a Kähler metric on \( Y \), and \( f \) is a real holomorphy potential on \((Y, g)\), possessing the additional property that \( h = f^{-2} g \) has constant scalar curvature.

**Proof.** On a Hermitian surface, the \(J\)-invariant 2-forms are given by

\[
\Lambda^{1,1} = \mathbb{R} \omega_h \oplus \Lambda^- ,
\]

so the self-dual part \( F^+ \) of any \(J\)-invariant 2-form \( F \) is a function times the associated 2-form \( \omega_h = h(J\cdot, \cdot) \) of our metric. If, near any point where \( F^+ \) is non-zero, we rescale our Hermitian metric so as to give it pointwise norm \( \sqrt{2} \), then, the self-dual 2-form \( \pm F^+ \) is the associated 2-form of a new metric \( g \) which is conformal to \( h \); and if \( F^+ \) is closed, the resulting metric \( g \) is then Kähler. This is exactly realized by setting \( g = f^2 h \).

On the other hand, if \( F^+ \) is a multiple of \( \omega_h \), \( F^+ \circ F^- \) is a multiple of \( F^-(J\cdot, \cdot) \), so, given that \((h, F)\) is a strongly Hermitian solution of the
Einstein-Maxwell equations, (8) implies that $\hat{\mathfrak{r}}_h$ is $J$-invariant. However, on the set where $f \neq 0$, the function $f$ is smooth and $g$ is defined, the trace-free Ricci tensors of the two metrics are related [6] by

$$\hat{\mathfrak{r}}_h = \hat{\mathfrak{r}}_g + 2f^{-1}\text{Hess}_g f \quad (9)$$

where the trace-free Hessian of $f$ is computed with respect to $g$. Since both $g$ and $h$ have $J$-invariant Ricci tensors, it follows that $\nabla\nabla f$ is $J$-invariant. Hence $f$ is a real positive holomorphy potential on $(Y, g)$, and $h = f^{-2}g$ has constant scalar curvature by [7].

While we have arranged to make $\pm F^+$ into the Kähler form $\omega_g$ of the conformally related Kähler metric $g$, the reader might be right to worry that the sign might be different on different connected components of $Y$. However, assuming that $M$ is connected, this turns out to be impossible. Indeed, by a theorem of Bär [14], the zero locus of the closed and co-closed form $F^+$ has Hausdorff dimension $\leq 2$, and, as we will show in the next section, this means it cannot disconnect $M^4$. Since the Einstein-Maxwell equations are invariant under $F \to -F$, we will therefore eventually be entitled to assume that $\omega = +F^+$, at the modest price of perhaps changing the sign of $F$.

Finally, we conclude this section with a partial converse of the above result:

**Proposition 3** Let $f : M \to \mathbb{R}^+$ be a positive holomorphy potential on a Kähler surface $(M^4, g, J)$ with Kähler form $\omega = g(J\cdot, \cdot)$. If $h = f^{-2}g$ has constant scalar curvature, then there is a unique 2-form $F$ with $F^+ = \omega$ such that $(h, F)$ is a strongly Hermitian solution of the Einstein-Maxwell equations.

**Proof.** Since $f$ is a positive function with $J$-invariant Hessian, [9] guarantees that the trace-free Ricci curvature of $h = f^{-2}g$ is $J$-invariant, and so can be uniquely written as $\hat{\mathfrak{r}}_h = \varphi(\cdot, J\cdot)$ for a unique $\varphi \in \Lambda^-$. Setting $F^+ = \omega$ and $F^- = \frac{1}{2}f^{-2}\varphi$ then produces a solution of (6) and (8), and this choice of $F^- \in \Lambda^-$ is moreover the only one that satisfies (8) in conjunction with $F^+ = \omega$. This ansatz thus solves (6–8) iff (7) is satisfied, and therefore solves (1–3) iff $h$ has constant scalar curvature. □
The First Main Theorems

The main worry aroused by the results in the previous section is that the construction breaks down at the zero locus of the 2-form $F^+$. Fortunately, however, this worry turns out to be largely misplaced. To get around this problem, we will develop a sequence of lemmata inspired by an argument of Derdziński [12], now carefully implemented by using a fundamental result on zero sets of generalized harmonic spinors due to Bär [4]. To do the job properly, we begin with a regularity result:

**Proposition 4** Let $(h, F)$ be a solution of the Einstein-Maxwell equations (1–3) on a smooth 4-manifold $M$. Suppose that, in some coordinate atlas, $h$ is of class $C^{2,\alpha}$ for some $\alpha > 0$, and that $F$ is of class $C^{1}$. Then $h$ and $F$ are $C^{\infty}$ in harmonic coordinates.

**Proof.** By the results of DeTurck and Kazdan [13], we can pass to harmonic coordinates without losing any regularity, and if, in harmonic coordinates, $h$ is of class $C^{k,\alpha}$, with Ricci tensor $r$ of class $C^{k,\alpha}$, then $h$ is actually of class $C^{k+2,\alpha}$. On the other hand, since $F$ is in the kernel of $d + d^*$, elliptic regularity [25] implies that if $h$ is of class $C^{k,\alpha}$, then $F$ is of class $C^{k,\alpha}$, too. However, equation (3) tells us that $r = -[F \circ F]_0 + \lambda h$ for some constant $\lambda$. Thus, if $h$ is of class $C^{k,\alpha}$, $r$ is also of class $C^{k,\alpha}$, so $h$ is actually $C^{k+2,\alpha}$. It therefore follows by induction ("bootstrapping") that $h$ and $F$ are both smooth in harmonic coordinates.  

In particular, the implicit assumption of smoothness used throughout §1 can now be seen to have been perfectly justified.

**Lemma 1** Let $(X, h)$ be a $C^3$ Riemannian $n$-manifold, and let $Z \subset X$ be a closed subset of Hausdorff dimension $< (n - 1)$. Let $Y = X - Z$ be the complement of $Z$, and suppose that $\xi$ is a Killing field on $(Y, h)$. Then $\xi$ extends to $(X, h)$ as a Killing vector field $\hat{\xi}$.

**Proof.** Let $q \in Z$ be any point, and let $U \subset M$ be a geodesically convex neighborhood of $q$. Since $Z$ has $n$-dimensional measure zero, it has empty interior, and it follows that $U - Z$ is non-empty. Thus, there exists some $p \in U$ which belongs to the complement $Y$ of $Z$; and since $Z$ is closed, some small metric ball $B_{2\epsilon}(p)$ is also contained in $U - Z$. Let $S \approx S^{n-1}$ be the
unit sphere in $T_p M$, and let $\Pi : (U - \{p\}) \to S$ be the $C^2$ map which sends $x \in (U - \{p\})$ to the initial unit tangent vector of the geodesic segment $\overline{px}$. Since $\Pi$ is Lipschitz, the Hausdorff dimension of $\Pi(Z \cap U)$ is also $< (n - 1)$, so almost every geodesic through $p$ misses $Z$. Now recall that the restriction of a Killing field $\xi$ to any geodesic $\gamma$ solves Jacobi's equation, since the flow of $\xi$ sends $\gamma$ to a family of geodesics. Let us therefore define a $C^1$ vector field $\hat{\xi}$ on $U - \{p\}$, as the unique family of solutions of Jacobi's equation along geodesics radiating from $p$, with the same initial values and initial (radial) derivatives as $\xi$ along the sphere $S_\varepsilon(p) = \partial B_\varepsilon(p)$. Then $\xi$ and $\hat{\xi}$ are both $C^1$ on $U - Z - \{p\}$, and agree on $W$, where $W \subset (U - Z - \{p\})$ is the union of all the geodesics radiating from $p$ which miss $X$. However, $W$ is dense in $U - Z - \{p\}$, since its complement $\Pi^{-1}[\Pi(Z \cap U)]$ is of $n$-dimensional measure zero. Because two continuous vector fields on $U - Z - \{p\}$ which agree on a dense set must be equal, it follows that $\hat{\xi} = \xi$ on $U - Z - \{p\}$. We can therefore extend $\xi$ across $Z$ as $\hat{\xi}$, and extend $\hat{\xi}$ across $p$ as $\xi$. Moreover, since the $C^1$ vector field $\hat{\xi}$ solves Killing's equation on the open dense set $U - Z - \{p\}$ where it coincides with $\xi$, it actually solves Killing's equation everywhere. Finally, since the intersection of two geodesically convex sets is geodesically convex, any two such local extension of $\xi$ across $Z$ agree on the overlap, and we can therefore consistently extend $\xi$ to $M$ as a Killing field $\hat{\xi}$.

The above argument also establishes a minor noteworthy point:

**Lemma 2** Let $X$ be a smooth connected $n$-manifold, and let $Z \subset X$ be a closed subset of Hausdorff dimension $< (n - 1)$. Then $X - Z$ is path connected. Moreover, for any metric $h$ on $X$, the Riemannian distance in $(X - Z, h)$ is just the restriction of the Riemannian distance from $(X, h)$.

**Proof.** In any small geodesically convex ball $U$, a dense set of points of $U - Z$ can be reached from any $p \in U - Z$ by following distance-minimizing geodesics that avoid $Z$. We can therefore reach any $q \in U - Z$ from $p \in U - Z$ by following a broken geodesic path $\overline{pq_1} \cup \overline{q_1q}$ in $U - Z$, where $q_1 \to q$, and the length of such a path is then arbitrarily close to the Riemannian distance from $p$ to $q$ in $X$. Any piecewise geodesic path in $X$ joining two points in $X - Z$ can therefore be approximated by piecewise geodesic paths in $X - Z$ of essentially the same length.
Lemma 3  Let $(M^n, J, h)$, $n = 2m \geq 4$, be a Hermitian manifold with $J$-invariant Ricci tensor, and suppose that $f$ is a continuous non-negative function such that $g = f^2 h$ is a Kähler metric on the open subset $Y$ where $f$ is non-zero. Assume that $h$ and $g$ are at least $C^2$ in complex coordinates, and that $h$ is at least $C^3$ in some $C^2$-compatible coordinates. Suppose, moreover, that $Z = M - Y = f^{-1}(0)$ has Hausdorff dimension $< (n - 1)$. Then $Z = \emptyset$, $f$ is everywhere positive, and $(M, J, h)$ is globally conformally Kähler.

Proof. The trace-free Ricci tensors of the conformally related metrics $h$ and $g$ are related \cite{6} by

$$\hat{\mathring{r}}_h = \hat{\mathring{r}}_g + (n - 2) f^{-1} \text{Hess}_0 f$$

where the trace-free Hessian of $f$ is computed with respect to $g$. Since the Ricci tensors of both metrics are $J$-invariant, it follows that the component of $\text{Hess}_f$ in $\odot^2 \Lambda^{0,1}$ vanishes, so that $f$ is a real holomorphy potential on $(Y, g)$, and its symplectic gradient $\xi = J \text{grad}_g f$ is therefore a Killing field. However, $\xi f = 0$, so the flow of $\xi$ preserves not only $g$, but also $h = f^{-2} g$; that is, $\xi$ is a Killing field for $h$, defined on the complement of a closed set $Z$ of Hausdorff dimension $< (n - 1)$. Thus $\xi$ extends across $Z$ as a Killing field by Lemma 1, and the vector field $-J \xi = \text{grad}_g f$ therefore extends across $Z$, too. However, $\text{grad}_g f = f^{-2} \text{grad}_h f = -\text{grad}_h f^{-1}$, so, by lowering an index, it follows that the 1-form $\hat{\phi} = h(J \xi, \cdot) = d(f^{-1})$ extends across $Z$ as a 1-form $\hat{\phi} = h(J \hat{\xi}, \cdot)$ which is at least $C^1$. On the other hand, the 1-form $\phi$ satisfies $d\phi = 0$ on open the dense set $Y \subset M$, so $d\hat{\phi} = 0$. If $U$ is a geodesically convex neighborhood of some $q \in Z \subset M$, the Poincaré lemma therefore tells us that $\hat{\phi} = du$ for some $C^2$ function $u$ on $U$, since $U$ is contractible. On the other hand, Lemma 2 guarantees that $U - Z$ is path connected. However, $d(u - f^{-1}) = \hat{\phi} - \phi = 0$ on $U - Z$, and since $U - Z$ is connected, it follows that $u - f^{-1}$ is constant. Thus $f^{-1}$ extends across $q \in Z$ as $u + \text{const}$. However, since $f(q) = 0$, this is a contradiction. We are therefore forced to conclude that $Z = \emptyset$, and that $f > 0$ on all of $M$. In particular, $h = f^{-2} g$ is globally conformally Kähler. \hfill \blacksquare

We are now ready to prove our first main results.

Theorem 1  Let $(h, F)$ be a strongly Hermitian solution of the Einstein-Maxwell equations on a complex surface $(M^4, J)$, and assume that $h$ is not Einstein. Also, for some $\alpha > 0$, suppose that $h$ is of differentiability class
\( C^{2,\alpha} \) and that \( F \) is at least \( C^1 \), relative to the complex atlas of \((M,J)\). Then there is a \( C^\infty \) Kähler metric \( g \) on \((M,J)\) and a \( C^\infty \) positive holomorphy potential \( f \) on \((M,J,g)\) such that \( h = f^{-2}g \), and such that \( F^+ \) is a constant times the Kähler form \( \omega \) of \( g \).

**Proof.** First notice that \( h \) is smooth in harmonic coordinates by Proposition 4, and we can therefore invoke the work of Bär [4]. The self-dual 2-form \( F^+ \) is in the kernel of the Dirac-type operator \( d + d^* \) on \((M,h)\), and cannot be identically zero, because otherwise [8] would force \( h \) to be Einstein, contrary to our assumptions. Thus Bär’s theorem [4] asserts that the zero locus \( Z \) of \( F^+ \) has Hausdorff codimension \( \geq 2 \). It follows that \( f = 2^{-1/4}|F^+|^{1/2} \) and \( g = f^2h \) fulfill the hypotheses of Lemma 3, since elliptic regularity guarantees that \( F \) is at least \( C^{2,\alpha} \) in complex coordinates. Hence \( Z = f^{-1}(0) \) is empty, and that \((M,h)\) is globally conformally Kähler, with \( h = f^{-2}g \) for a Kähler metric \( g \) and a positive holomorphy potential \( f \).

It only remains to show that \( g \) and \( f \) are actually smooth in the complex coordinate atlas of \((M,J)\). Initially, we know that \( g = f^2h \) is of class \( C^{2,\alpha} \) in this setting. However, since \( \nabla^{1,0} f \) is a holomorphic vector field, we also know that if \( g \) is of class \( C^{k,\alpha} \), then \( f \) is of class \( C^{k+1,\alpha} \). Moreover, \( h \) has constant scalar curvature, and the Yamabe equation therefore tells us that the scalar curvature of \( g \) can be written as \((\text{const} \cdot f^{-2} - 6f \Delta_g f^{-1})\), which is therefore of class \( C^{k-1,\alpha} \). On the other hand, the scalar curvature of our Kähler metric \( g \) can be written in complex coordinates as \( \Delta_g \log V \), where the function \( V \) represents the volume form of \( g \) in a complex coordinate system. Hence \( V \) is of class \( C^{k+1,\alpha} \). But elliptic regularity for the Monge-Ampère equation [3] then tells us that if \( V \) is of class \( C^{k+1,\alpha} \), so is the Kähler metric \( g \). In short, if \( g \) is \( C^{k,\alpha} \), it is \( C^{k+1,\alpha} \). By induction, \( g \) is therefore smooth in complex coordinates, and it then follows that \( f, h, \) and \( F \) are therefore smooth in these coordinates, too.

To prove Theorems A and B it thus suffices to address the case in which \( h \) is Einstein. However, this case follows [19] from the work of Derdziński [12], together with the Riemannian Goldberg-Sachs theorem [2, 15] and results of Boyer [8] on compact anti-self-dual Hermitian manifolds. Here one of Derdziński’s important discoveries is that either \( W_+ \) is nowhere zero or must vanish identically. We note in passing that Lemma 3, with \( f = (24|W_+|^2)^{1/3} \), also now provides a clear and watertight way of establishing this point.
3 Some Compact Examples

Let us now construct some compact, non-Kähler examples of strongly Hermitian solutions of the Einstein-Maxwell equations. To do this, we will look for Kähler metrics $g$ on a product $\mathbb{CP}_1 \times \Sigma$, together with positive holomorphy potentials $f$ such that $h = f^{-2}g$ has constant scalar curvature. For simplicity, we will take $g$ to be the Riemannian product of an axisymmetric metric on $S^2 = \mathbb{CP}_1$ with a metric of constant scalar curvature $s_2 = c$ on $\Sigma$, and we will take our holomorphy potential $f$ to be the Hamiltonian for rotation of $S^2$ about the given axis, with period $2\pi$. Thus,

$$g = g_1 + g_2$$

where $(\Sigma, g_2)$ has constant scalar curvature $c \in \mathbb{R}$, and where the metric $g_1$ on $S^2$ can be written in cylindrical coordinates $(t, \theta) \in (a, b) \times (0, 2\pi]$ as

$$g_1 = \frac{dt^2}{\Psi(t)} + \Psi(t)d\theta^2$$

for some smooth positive function $\Psi(t)$. Here we have put the Kähler form

$$\omega_1 = dt \wedge d\theta$$

in Darboux coordinates, so that we may assume that our holomorphy potential is given by $f = t$ as long as we remember to insist that $b > a > 0$. The scalar curvature of $g$ is then given by

$$s_1 = \Delta_{g_1} \log \Psi = -\Psi''(t).$$

Thus the scalar curvature of $g$ is given by

$$s = s_1 + s_2 = c - \Psi''(t).$$

We now want to arrange that $h = f^{-2}g$ has constant scalar curvature, which is to say that

$$(6\Delta_g + s)f^{-1} = f^{-3}d$$

for a constant $d = s_h$. We may now rewrite this as

$$s = f^{-2}d - 6f\Delta f^{-1}$$
or in other words, as

\[ c - \Psi'' = \frac{d}{t^2} - 6t \Delta g_1 \left( \frac{1}{t} \right) \]

since the Hessian of \( f \) is trivial in the \( \Sigma \)-directions of our Riemannian product. Since

\[ \Delta g_1 \left( \frac{1}{t} \right) = \left( \frac{\Psi}{t^2} \right)' \]

the Yamabe equation (10) therefore reduces to the ODE

\[ c - \Psi'' = \frac{d}{t^2} - 6 \frac{\Psi'}{t} + 12 \frac{\Psi}{t^2}, \]

or equivalently

\[ t^2 \Psi'' - 6t \Psi' + 12 \Psi = ct^2 - d. \tag{11} \]

Since the linear operator

\[ y \mapsto t^2 y'' - 6ty' + 12y \]

acts on monomials by

\[ t^n \mapsto (n - 3)(n - 4)t^n \]

the general solution of equation (11) is therefore a quartic polynomial

\[ \Psi(t) = At^4 + Bt^3 + \frac{c}{2}t^2 - \frac{d}{12} \]

with \( \Psi'(0) = 0 \). Now, in order to get a metric on \( S^2 \), we need to impose the boundary conditions that

\[ \Psi(a) = \Psi(b) = 0, \quad \Psi'(a) = -\Psi'(b) = 2, \]

while remembering that we also must have \( \Psi'(0) = 0 \) and \( \Psi(t) > 0 \) on \( (a, b) \); for example, if we set \( t - a = r^2/2 \), these conditions guarantee that the metric takes the form \( (1 + O(r^2))dr^2 + (1 + O(r^2))r^2d\theta^2 \) for small positive \( t - a \). For \( 0 < a < b \), the unique such quartic polynomial is given by
Ψ(t) = \frac{(t - a)(t - b)}{a - b} \left[ 2 - \frac{(t - a)(t - b)}{ab} \right]. \tag{12}

One can then read off that
\[
\mathbf{d} = -12 \Psi(0) = \frac{12ab}{b - a} > 0
\]
and that
\[
\mathbf{c} = \Psi''(0) = \frac{2(a + b)^2}{(b - a)ab} > 0.
\]
In particular, the constant scalar curvature \( s_2 = \mathbf{c} \) of \( \Sigma \) must be positive, so \( \Sigma \) must be a 2-sphere \( \mathbb{C}P_1 \). Gauss-Bonnet moreover guarantees that the total area of \( \Sigma \) must be
\[
\omega(\Sigma) = \frac{4\pi(b - a)ab}{(a + b)^2}.
\]
By contrast, the total area of \( (\mathbb{C}P_1, g_1) \) is \( 4\pi(b - a) \). Thus, we have constructed a Kähler metric
\[
g = g_1 + g_2
\]
on \( \mathbb{C}P_1 \times \mathbb{C}P_1 \) which is not locally symmetric, but engenders a solution
\[
h = \frac{1}{t^2}g,
\]
\[
F^+ = \omega
\]
of positive scalar curvature \( \mathbf{d} = s_h \), and belongs to the Kähler class
\[
[\omega] = 4\pi(b - a) \left( \mathcal{G}_2 + \frac{\mathcal{G}_1}{1 + \frac{1}{2} \left( \frac{2}{b} + \frac{2}{a} \right)} \right) \in H^2(\mathbb{C}P_1 \times \mathbb{C}P_1),
\]
where $\mathcal{S}_1$ and $\mathcal{S}_2$ are the Poincaré duals of the first and second factors of $\mathbb{CP}_1 \times \mathbb{CP}_1$. As we allow $b > a > 0$ to vary, these sweep out all the Kähler classes for which the second factor has less than half the area of the first factor. In particular, the constant-scalar-curvature Kähler metric in such a class $[\omega]$, obtained by taking the Riemannian product of round 2-spheres of appropriate radii, is not the only Kähler metric in $[\omega]$ that engenders a solution of the Einstein-Maxwell equations. The solutions $(h, F)$ so engendered are moreover geometrically distinct; one family of solutions consists of symmetric spaces, whereas the metrics in the other family are not even locally symmetric. Theorem C now follows.

4 Concluding Remarks

We have just seen that not every strongly Hermitian Einstein-Maxwell solution on a compact complex surface is given by a cscK metric. This presents us with the intriguing problem of determining when such non-Kähler solutions exist. One hint is provided by the following easy result:

**Proposition 5** Let $(h, F)$ be a strongly Hermitian solution of the Einstein-Maxwell equations (1–3) on a compact complex surface $(M^4, J)$. Then either $h$ is a constant-scalar-curvature Kähler metric, or else $(M, J)$ is a rational or ruled surface.

**Proof.** If the holomorphy potential $f$ is non-constant, $\xi = J\nabla f$ is a non-trivial Killing field for $g$. The flow of $\xi$ is then a connected Abelian group of isometries of $(M, g)$, the closure of which is a torus subgroup of the isometry group which acts on $M$ with non-empty fixed-point set. However, the generators are also the real parts of holomorphic vector fields. If this torus has dimension $> 1$, one gets a non-trivial holomorphic section of $K^{-1}$ with non-empty zero locus by taking the wedge product of two independent holomorphic vector fields associated with the action. Otherwise, $\xi$ is a periodic vector field, and we obtain an embedded rational curve of non-negative self-intersection by taking the closure of generic orbit of the group generated by $\xi$ and $\nabla f = -J\xi$. Either way, it follows that the plurigenra $p_\ell(M, J) = h^0(M, \mathcal{O}(K^\ell)), \ell \in \mathbb{N}$, must all vanish, and, since $(M, J)$ also admits a Kähler metric $g$, surface classification [5, 16] tells us $(M, J)$ is rational or ruled.
Of course, the Einstein Hermitian metrics [10, 12, 26] on \(\mathbb{CP}_2\) and \(\mathbb{CP}_2\#2\mathbb{CP}_2\) provide two more examples, so it seems certain that the full story will turn out to be rich and interesting. But there is obviously an enormous gulf between the minuscule menagerie of currently known examples and the world of possibilities allowed by Proposition 5. I can only hope that some interested reader will feel motivated to construct some further examples!

The examples constructed in §3 have an intriguing feature that is also worth mentioning here. The constructed metrics \(h\) of course all have constant scalar curvature. The question is, which of them, if any, are Yamabe metrics. Recall that Yamabe’s program for proving the existence of constant-scalar-curvature metrics on compact manifolds involves minimizing the functional (10) (or the appropriate \(n\)-dimensional generalization) in any given conformal class; such metrics always exist [22, 27], and are called Yamabe metrics. But while every Yamabe metric has constant scalar curvature, not every constant-scalar-curvature metric is Yamabe. This complication only occurs when the scalar curvature is positive, as is the case for the examples in question. For precisely this reason, the Yamabe invariant of 4-manifolds like \(S^2 \times S^2\) remains unknown. Here, the Yamabe invariant \(\mathcal{Y}(M)\) of a 4-manifold is obtained by by taking the infimum of (10) in each conformal class, and then taking the supremum of these infima over all conformal classes; equivalently, it is the supremum of the scalar curvatures of all unit-volume Yamabe metrics on \(M\).

What is currently known [7, 18, 20, 17] strongly suggests that one should have

\[
12\pi \sqrt{2} = \mathcal{Y}(\mathbb{CP}_2) \leq \mathcal{Y}(S^2 \times S^2) \leq \mathcal{Y}(S^4) = 8\pi \sqrt{6},
\]

but here, for the moment, the lower bound is merely conjectural. On the other hand, the value of the functional (10) is easy to compute for the constructed metrics \(h = g/t^2\) on \(\mathbb{CP}_1 \times \mathbb{CP}_1\); namely, since the scalar curvature \(s_h\) is exactly the constant \(d\), one can easily show that

\[
s_h V^{1/2} = 8\pi \sqrt{6(a^2 + ab + b^2)}/(a + b).
\]

Intriguingly, as \(b/a\) ranges over the interval \((1, \infty)\), this exactly sweeps out the interval \((12\pi \sqrt{2}, 8\pi \sqrt{6})\) in question. Thus, if any of these metrics is Yamabe, the conjectural lower bound would be established; and if they are all Yamabe, the upper bound would be saturated!
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