Semi-relativistic effects in spin-1/2 quantum plasmas

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Abstract. Emerging possibilities for creating and studying novel plasma regimes, e.g. relativistic plasmas and dense systems, in a controlled laboratory environment also require new modeling tools for such systems. This motivates theoretical studies of the kinetic theory governing the dynamics of plasmas for which both relativistic and quantum effects occur simultaneously. Here, we investigate relativistic corrections to the Pauli Hamiltonian in the context of a scalar kinetic theory for spin-1/2 quantum plasmas. In particular, we formulate a quantum kinetic theory for the collective motion of electrons that takes into account effects such as spin–orbit coupling and Zitterbewegung. We discuss the implications and possible applications of our findings.

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1. Introduction

Plasmas, in their full generality, constitute a highly complex class of physical systems, ranging from classical tenuous plasmas, in e.g. fluorescent lighting, to dense, strongly coupled systems, such as QCD plasmas. The large span of plasma systems implies that a wide variety of theoretical methods have been developed for their treatment. Even so, there are general principles that remain the common features of different plasma systems. Therefore, the methods used for one plasma system can in some cases be transferred to another plasma type, sometimes leading to new insights. One such example is the transfer of techniques for treating nonlinearities in classical plasmas to quantum mechanical plasmas. The latter, often termed quantum plasmas (see, e.g., [1–4]), to lowest order contain corrections due to the classical regime in terms of nonlocal terms, related to the tunneling aspects of the electron (into quantum plasmas, the ions are most often treated classically). Such tunneling effects can be incorporated in both kinetic and fluid descriptions of the collective electron motion [5, 6]. Such collective tunneling effects may, e.g., lead to nanoscale limitations in plasmonic devices [7] and bound states near moving test charges in plasmas [8]. Another mean-field effect that may be added to the dynamics of classical plasmas concerns the electron spin, i.e. the possibility for large-scale magnetization of plasmas [9]. Such magnetization switching is known to be able to give new non-trivial features, such as metamaterial properties, allowing for, e.g., new soliton modes [10, 11]. Moreover, the inclusion of electron spin in the collective dynamics can be done through either a fluid or a kinetic approach [12]. Furthermore, the inclusion of spin in the dynamics of a quantum plasma points in the direction of relativistic effects in such plasma systems, e.g. collective spin–orbit coupling. These types of effects have recently been shown to play an important role in the interaction between magnetic materials and lasers [13]. It is the aim of the present work to extend previous work to the weakly relativistic regime.

As indicated above, the dynamics of plasmas under extreme conditions is an important and integral part of many current and upcoming experimental facilities, and such investigations therefore constitute a very active research field. In particular, laboratory plasmas, such as laser-generated plasmas, are currently presenting a possibility of studying previously unattainable plasma density regimes. It is well known that in the high-density regime [14] quantum effects start playing a role for, e.g., the dispersive properties of plasma waves [15–17]. In nature, such dense relativistic plasmas can be found in planetary interiors and in stars [18]. Moreover, relativistic contributions to such plasma dynamics are under many circumstances very important [19]. Thus, when the parameters take values characteristic of the quantum relativistic regime, one needs to consider more complex dynamical models in order to obtain accurate descriptions of a host of phenomena. A canonical starting point for dealing with high-density effects in a perturbative relativistic regime is offered by the quantum kinetic approach, here based on the Dirac description. Effects that can be included in such a perturbative model include spin dynamics, spin–orbit coupling and Zitterbewegung. These examples have a close connection to the non-perturbative relativistic quantum regime, in which e.g. pair production [20–23] and other nonlinear quantum vacuum effects [24] become pronounced.

Dense plasmas, and in particular short-time scale phenomena therein, have been successfully studied using Green’s function techniques, such as the Kadanoff–Baym kinetic equations [25] (see also [26–30] for similar approaches and [31] for an overview and examples from femtosecond laser physics) or QED effects in kinetic plasma theory [32, 33]. Other methods in quantum kinetic theories invoke the Wigner functions and gauge-invariant Wigner
functions to study high-energy phenomena in dense plasmas, such as pair production in the Schwinger limit [34], dynamical screening under non-equilibrium conditions in lasers [35], the Kadanoff–Baym equations applied to strong laser pulses [36] or harmonics generation of a coherent field in lasers [37] or quantum transport [38]. Although the Kadanoff–Baym equations, with the Green and Wigner function techniques, and similar indeed give ample opportunity to treat a wide variety of systems, their generality also makes simplifying assumptions necessary, and under certain circumstances a mean-field model that retains memory effects and nonlocal structures can be an adequate approximation [18]. In particular, the mean-field approach is well suited outside the regime of strong coupling effects. Here, we will be interested in phenomena in plasmas that are not strongly coupled, but still in regimes where a classical plasma description is not fully adequate. Here we stress that a large number of different dimensionless parameters (see, e.g., [39, 40] for a more complete discussion) are needed to give a thorough description of various plasma regimes. However, much insight can be gained by considering a simple density–temperature plot. In figure 1 a schematic view of the parameter regime of interest for our study is presented, adopted from [41]. It should be noted that the quantum regime identified in figure 1 fills a somewhat larger part of the parameter space than the regime in focus here,
as we also need (weakly) relativistic effects to be of significance. Typically, this means that we may need slightly higher densities than for the pure quantum regime. However, as we will see in the following sections, several different parameters play a role, and the situation is complicated by the presence of a magnetic field as well as by the appearance of resonances.

In section 2, we recall the Foldy–Wouthuysen (FW) transformation for particles in external fields and, as a result, we obtain a semi-relativistic Hamiltonian. We then go on to define the scalar quasi-distribution function for spin-1/2 particles in section 3. In section 4, we then derive the corresponding evolution equation for the distribution function. The evolution equation clearly depicts the importance of different terms of the relativistic expansion. A comparison of our results to previous studies is made, and we discuss the applicability of our equation as well as the interpretation of the variables involved. In section 5, our theory is illustrated by means of two examples from linearized theory, and finally, in section 6, our main conclusions are summarized.

2. The high-order corrections

The Dirac Hamiltonian can be written in the form

\[ \hat{H} = \beta mc^2 + \Omega_e + \Omega_o, \]  

(1)

where we have the even \((\beta \hat{\Omega_e} = \Omega_e \beta)\) operator

\[ \Omega_e = q \phi, \]  

(2)

and the odd \((\beta \hat{\Omega_o} = -\Omega_o \beta)\) operator

\[ \Omega_o = c \hat{\alpha} \cdot (\hat{p} - q \hat{A}), \]  

(3)

where \(\beta\) and \(\alpha\) are the 4 \times 4 Dirac matrices, \(m\) is the electron mass, \(q\) is the charge \((q = -e\) for an electron), \(c\) is the speed of light, \(\hat{p}\) is the momentum operator and \(\phi\) and \(A\) are the scalar and vector potentials, respectively.

The odd operators in the Dirac Hamiltonian couple the positive and negative energy states of the Dirac bi-spinor. For the purpose of obtaining a perturbative expansion in the parameter \(E/mc^2\), where \(E\) is the typical energy associated with the second and third terms in (1), we assume that the first term in (1) is large compared to these terms. The consecutive application of the unitary FW \([42]\) transformation

\[ \hat{H} \rightarrow \exp \left( \beta \Omega_o / 2mc^2 \right) \hat{H} \exp \left( \beta \Omega_o / 2mc^2 \right) \]  

(4)

yields a new Hamiltonian of the form (1) in which the new odd operators are of order \(1/mc^2\). Performing this transformation \(n\) times yields terms up to order \((mc^2)^{-n}\). This gives a separation of the positive and negative energy states up to an arbitrary order \(n\) in \(1/mc^2\).

Applying this transformation four times gives the following Hamiltonian for positive energy states with only even operators \([43]\):

\[ \hat{H} = mc^2 + q \phi + \frac{1}{2m} \left( \hat{p} - \frac{q}{c} A \right)^2 - \frac{q \hbar}{2mc} \sigma \cdot B + \frac{\hbar^2 q}{8m^2c^2} \nabla \cdot E - \frac{\hbar q}{4m^2c^2} \sigma \cdot \left[ E \times \left( \hat{p} - \frac{q}{c} A \right) \right] \]  

\[ - \frac{i \hbar^2 q}{8m^2c^2} \sigma \cdot \nabla \times E + \frac{1}{8m^3c^2} \left( \hat{p} - \frac{q}{c} A \right)^4, \]  

(5)

where \(\sigma\) denotes a vector containing the 2 \times 2 Pauli matrices, \(\hbar\) is the reduced Planck constant, \(q\) is the charge, \(m\) is the mass, \(B\) and \(E\) are the magnetic and electric fields and \(\phi\) and \(A\)
are the corresponding potentials in the Coulomb gauge. We see that the first four terms constitute the Pauli Hamiltonian, whereas the remaining terms are higher-order corrections. In particular, the sixth and seventh terms together give Thomas precession and spin–orbit coupling, while the fifth and eight terms are the Darwin term and the so-called mass–velocity correction term, respectively.

In equation (5) as well as in the Dirac theory we started from, the value of the spin $g$-factor is exactly 2. When applying the resulting theory, in section 5, we will use the QED corrected value of $g \simeq 2.00232$, however. In spite of the smallness of the modification it turns out that this correction is important, as the applications of our theory are very sensitive to the exact value of the $g$-factor. In fact, the sensitivity of the kinetic theory to the value of $g$ has already been seen in [44]. This may suggest that for consistency, the Hamiltonian for QED corrections should be added to equation (5). Such an approach would indeed modify the $g$-value to the desired one in the theory presented below, but the augmented Hamiltonian would also add several new terms into the evolution equation for the electrons. Those extra terms are at least smaller than those kept by a factor of order $(g - 2)$, however. Thus the main effect from QED in the regime of study is the modification of the value of the $g$-factor as compared to the Dirac theory. As a consequence, the contributions from QED (see, e.g., [45] for QED corrections to the Dirac Hamiltonian) besides modifying the $g$-value will not be included here.

3. The gauge-invariant Stratonovich–Wigner function

The extended phase-space scalar kinetic model is obtained using Hamiltonian (5). Following [41], we are able to construct a gauge invariant scalar kinetic theory using a density matrix description for a spin-1/2 particle.

The basis states are $|x, \alpha \rangle = |x \rangle \otimes |\alpha \rangle$, where $|x \rangle$ is a state with position $x$ and $|\alpha \rangle$ is the state with spin-up $\alpha = 1$ or spin-down $\alpha = 2$. As a starting point of this model, we use the spinor state $\psi(x, \alpha, t) = \langle x, \alpha | \psi \rangle$ which fulfills the dynamical equation $i \hbar \frac{\partial}{\partial t} \psi(x, \alpha, t) = \hat{H} \psi(x, \alpha, t)$, with the Hamiltonian (5).

With the spinors, we can define the density matrix as

$$\rho_{\alpha \beta}(x, y, t) = \langle x, \alpha | \rho | y, \beta \rangle = \sum_i p_i \psi_i(x, \alpha, t) \psi_i^\dagger(y, \beta, t),$$

(6)

where $p_i$ is the probability of having a state $\psi_i$. The density matrix fulfills the von Neumann equation

$$i \hbar \frac{\partial \rho}{\partial t} = [\hat{H}, \rho].$$

(7)

Once the density matrix has been defined, we can define the Wigner–Stratonovich transform [47] as

$$W_{\alpha \beta}(x, p, t) = \int \frac{d^3z}{(2\pi \hbar)^3} \exp \left[ -\frac{i}{\hbar} z \cdot \Phi \right] \rho_{\alpha \beta}(x + \frac{z}{2}, x - \frac{z}{2}, t),$$

(8)

where the phase

$$\Phi = p - \frac{q}{c} \int_{-1/2}^{1/2} d\eta A(x + \eta z, t)$$

(9)
is used to ensure gauge invariance of the resulting distribution function. The Wigner–Stratonovich transform has the property that it must be taken separately for each component of the $2 \times 2$ density matrix.

Different approaches to constructing a kinetic theory from the Wigner–Stratonovich transformation were discussed in [41]. Following [41], we define here a scalar distribution function $f(x, p, s, t)$ in the extended phase space\(^3\) where $s$ is a vector of unit length. This distribution function satisfies that

$$f(x, s, t) = \int d^3 p f(x, p, s, t)$$

(10)

gives the probability of finding the particle at position $x$ with spin-up in the direction of $s$, and

$$f(p, s, t) = \int d^3 x f(x, p, s, t)$$

(11)

gives the probability of finding the particle with momentum $p$ with spin-up in the direction of $s$. Using the Wigner–Stratonovich transformation, the scalar distribution function will be defined as [41]

$$f(x, p, s, t) = \frac{1}{4\pi} \sum_{\alpha, \beta = 1}^2 (1 + s \cdot \sigma)^{\alpha\beta} W^{\beta\alpha}(x, p, t)$$

$$= \frac{1}{4\pi} \text{tr}(1 + s \cdot \sigma) W(x, p, t),$$

(12)

where $\text{tr}$ denotes the trace over the spin indices. We recall that the expectation value polarization density is now given by

$$\langle \sigma \rangle(x, t) = \text{tr}[\sigma \rho(x, y, t)] = 3 \int d^3 p d^2 s f(x, p, s, t)s,$$

(13)

where we stress the need for factor 3. This follows from the form of the transformation (12) and is needed to compensate for the quantum mechanical smearing of the distribution function in spin space. Furthermore, it should be stressed that the independent spin variable $s$ constructed in (12) generates the rest frame expression for the spin. In our theory, which is only weakly relativistic, this has limited consequences. The relation between the rest frame spin $s$ and the spatial part of the spin four-vector $S$ is given by $S = s + [\gamma^2/(\gamma + 1)](v \cdot s)v/c^2$ [48], where the kinematic quantities (i.e. the gamma factor $\gamma$ and the velocity $v$) can be expressed in terms of $p$ and $s$ (see below). Since our weakly relativistic theory presented here is concerned only with spin-dependent terms up to order $v/c$, the difference between $S$ and $s$ may be overlooked for the most part, e.g. when computing the magnetization current density.

4. The evolution equation for the scalar distribution function

Using the above formalism we obtain a fully gauge-invariant Vlasov-like evolution equation for charged particles. One of the most basic quantum effects is the tendency for the wave function to spread out. In the non-relativistic version of the theory [41], this effect ends up in the operators $\sin(h \nabla_s \cdot \nabla_p)$ and $\cos(h \nabla_s \cdot \nabla_p)$ acting on the fields and the distribution function, where the

\(^3\) Note that the extended phase space of our theory [41] is not itself a phase space, since no angular position variable is present. Instead it is a regular phase space $(x, p)$ plus a spin variable.
operators can be defined through the trigonometric Taylor expansions\(^4\). In our present theory, we will view the spatial gradient operator \(\nabla_s\) as a small parameter and drop terms of order \(\nabla_s^2\) or smaller, which means dropping particle dispersive effects, that are smaller by a factor of order \(\delta^2\) where \(\delta\) is the characteristic de Broglie wavelength over the macroscopic scale length. The other approximation made is to only account for weakly relativistic effects, as described above. This implies that only terms up to first order in the velocity are kept and that the gamma factor is set to unity. The evolution equation is found using transformations (12) on evolution equation (7), which together with the above approximations results in

\[
0 = \frac{\partial f}{\partial t} + \left\{ \frac{p}{m} + \frac{\mu}{2mc} E \times (s + \nabla_s) \right\} \cdot \nabla_s f + q \left( E + \frac{1}{c} \left\{ \frac{p}{m} + \frac{\mu}{2mc} E \times (s + \nabla_s) \right\} \right) \times B \cdot \nabla_p f \\
+ \frac{2\mu}{\hbar} s \times \left( B - \frac{p \times E}{2mc} \right) \cdot \nabla_s f + \mu \left( s + \nabla_s \right) \cdot \partial_i \left( B - \frac{p \times E}{2mc} \right) \partial_p^i f \\
- \frac{\hbar^2 q}{8m^2 c^2} \partial_s \left( \nabla \cdot E \right) \partial_p^i f,
\]

where \(p\) is the momentum (which is related to the velocity through the spin; see below) and \(\mu = \hbar q/2mc\) (or \(\mu = \hbar q/4mc\)).

Evolution equation (14) has three new effects compared to the equation in [41] for spin-1/2 particles. The first one is the Thomas precession effect where the previous theory [41] is extended by the substitution \(B \rightarrow B - p \times E/2mc\) in the fourth and fifth terms of equation (14). This effect comes from the spin–orbit coupling contribution in Hamiltonian (5) and therefore it is directly coupled with the evolution of the spin. The second new effect is the last term which is associated with the Darwin term. This term introduces the Zitterbewegung effect of the electron, and is the only contribution proportional to \(\hbar^2\). The third effect is seen in the velocity–momentum relation, which is highlighted in the second and third terms. In equation (14) the term in \(\{\}\) brackets resembles a velocity which has been modified by the spin, which will be discussed in some detail below. Finally, we point out that the factor in front of \(\nabla_s f\) in the third term is indeed given by \(\text{d}s/\text{d}t\) [48], i.e. the laboratory rate of change of the rest frame value of the spin. Thus we note that equation (14) is consistent with the interpretation of \(s\) as the rest frame variable for the spin. Thus when calculating the contribution to the magnetization \(\text{d}M\) from the spin as \(\text{d}M = 3\mu f(x, p, s, t)s\text{d}^3p\text{d}s\), we obtain the magnetization in the particle rest frame, i.e. in a frame moving with a velocity \(\hat{v}\) (whose relation to the momentum will be considered below) with respect to the laboratory frame. Since the velocity is only weakly relativistic, the difference from the laboratory frame value is small, but there will still be a consequence when the spin polarization current is considered, as will be seen below.

Next we need to consider the relation between velocity and momentum. In order to relate these variables we use the Heisenberg equation of motion for the velocity operator

\[
\dot{\hat{v}} = \frac{1}{i\hbar} [\hat{x}, \hat{H}].
\]

For Hamiltonian (5) we then obtain

\[
\dot{\hat{v}} = \frac{1}{m} \left( -i\hbar \nabla - \frac{q}{c} A \right) - \frac{\mu}{2mc} \sigma \times E,
\]

\(^4\) More generally, the final equation can be written as an integro-differential equation.
where we have neglected the last term in the Hamiltonian to simplify the equation slightly (see the discussion about the mass correction below). We recall that the Wigner transformation for an operator is multiplied by a factor \((2\pi \hbar)^3\), as compared with the Wigner transformation for the density matrix. Similarly, for the spin transformation, the transformation for operators comes with a factor 3. Taking this into account and calculating the Wigner and \(Q\) transformation of the operator above gives the final relation

\[
v = v(x, p, s, t) = \frac{p}{m} + 3\frac{\mu}{2mc}E \times s. \tag{17}\]

This is the function in extended phase space, which can be used to calculate the average velocity and current density of the plasma. An important question that arises is whether the current density based on velocity \((17)\) will give the free current density or if it corresponds to some other physical quantity, e.g. the total current density. This question will be addressed below, where we calculate the energy conservation law of our system, which confirms that the velocity in equation \((17)\) is indeed the variable corresponding to the free current density.

In a more general context, the relationship between momentum and velocity is non-trivial. For example, the spin–orbit coupling has been shown to arise as a Berry phase term \([49]\). For a further discussion of this interesting topic, see, e.g., \([50–53]\).

When obtaining the evolution equation \((14)\), we have not considered the effect of the mass–velocity correction term in order to obtain a more transparent formalism. This term will only produce a correction of the form \(p/m \rightarrow p/m(1 + p^2/2m^2c^2)\) in the second term. Although this term is of the same order in an expansion in \(1/c\), as compared to other terms that have been kept, we will not consider it, as the classical relativistic terms are already well known. Instead, we focus on the new effects introduced by the spin and the Zitterbewegung.

The dynamics of the distribution function given by the Vlasov equation \((14)\) is in the mean-field approximation coupled to the Maxwell equations in the form

\[
\nabla \cdot E = 4\pi \rho_T, \quad \nabla \times B = \frac{\partial E}{\partial t} + 4\pi J_T, \tag{18}\]

where the total charge density and the total current density are given by

\[
\rho_T = \rho_F + \nabla \cdot P, \quad J_T = J_F + \nabla \times M + \frac{\partial P}{\partial t}. \tag{19}\]

In the above expressions, the free charge density is

\[
\rho_F = q \int d\Omega f, \tag{20}\]

where \(d\Omega = d^3v d^2s\) is the integration measure performed over the three velocity variables and the two spin degrees of freedom. The spin vector has a fixed unit length and it is thus convenient to use spherical coordinates \((\varphi_s, \theta_s)\) to describe it. The free current density is given by

\[
J_F = q \int d\Omega \left( \frac{p}{m} + \frac{3\mu}{2mc}E \times s \right) f. \tag{21}\]

With these charge and current densities, the conservation of charge is obtained from \((14)\) to be \(\partial_t \rho_F + \nabla \cdot J_F = 0\). Furthermore, the magnetization \(M\) and the polarization \(P\) are both due to the spin and are calculated, respectively, as

\[
M = 3\mu \int d\Omega s f \tag{22}\]
\[ P = -3\mu \int d\Omega \frac{s \times p}{2mc} f. \]  

(23)

The system of Maxwell’s equations with magnetization (22) and polarization (23), and free current density (21), together with our main equation (14), satisfies an energy conservation law of the form

\[ \partial_t W + \nabla \cdot K = 0. \]  

(24)

Here, the total energy density \( W \) is given by

\[ W = \frac{1}{2} (|E|^2 + |B|^2) + \int d\Omega \left( \frac{p^2}{2m} - 3\mu s \cdot B \right) f, \]  

(25)

and the energy flux vector \( K \) is given by

\[ K = E \times (B - M) + \int d\Omega \left( \frac{p^2}{2m} + 3\mu \left( \frac{B - p \times E}{2mc} \right) \cdot s \right) v f. \]  

(26)

Apparently, the first terms in (25) constitute the electromagnetic field energy density, and the integral term is the combined kinetic and magnetic dipole energy densities. The first term of (26) is the Poynting vector, whereas the latter represents the combined flux of kinetic energy density and magnetic dipole energy. This energy conservation equation is a generalization of previous results for semi-classical theories for spin-1/2 plasmas [44]. It should be noted that although the theory presented here contains approximation, e.g. due to the weakly relativistic assumptions, the conservation law (24) is an exact property of the presented model.

5. Linearized theory

In this section, we are going to study the influence of the spin–orbit coupling and of the Darwin term on linear wave propagation. For this purpose, we linearize the evolution equation (14), where the variables are separated into equilibrium and perturbed quantities (using the subindices 0 and 1, respectively, to denote them). Thus, the distribution function will be \( f = f_0 + f_1 \), and the electric and magnetic fields could be written as \( E = E_1 \) and \( B = B_0 + B_1 \), respectively. The evolution equation to linear order becomes

\[ \frac{\partial f_1}{\partial t} + \frac{p}{m} \cdot \nabla_x f_1 + \frac{q}{mc} p \times B_0 \cdot \nabla_p f_1 + \frac{2\mu}{\hbar} s \times B_0 \cdot \nabla_p f_1 + \mu \nabla_{xi} [B_0 \cdot (s + \nabla_s)] \nabla_{pi} f_1 \]

\[ = -qE_1 \cdot \nabla_p f_0 - \frac{\mu}{2mc} E_1 \times (s + \nabla_s) \cdot \nabla_x f_0 - \frac{q}{mc} p \times B_1 \cdot \nabla_p f_0 \]

\[ - \mu \nabla_{xi} [B_1 \cdot (s + \nabla_s)] \nabla_{pi} f_0 + \mu \nabla_{xi} [(p \times E_1) \cdot (s + \nabla_s)] \nabla_{pi} f_0 \]

\[ - \frac{q\mu}{2mc^2} [E_1 \times (s + \nabla_s)] \nabla_p f_0 - \frac{2\mu}{\hbar} s \times B_1 \cdot \nabla_s f_0 \]

\[ + \frac{\mu}{\hbar mc} s \times (p \times E_1) \cdot \nabla_s f_0 + \frac{\hbar^2 q}{8mc^2} \nabla_{xi} (\nabla \cdot E_1) \nabla_{pi} f_0. \]  

(27)

In the following, we only study electrostatic modes (e.g. \( B_1 = 0 \) in (27)) propagating along \( B_0 \), as this gives a good illustration of the contribution from the relativistic terms, which are due to the Zitterbewegung effect and the spin–orbit coupling.
5.1. **Darwin term contribution**

First we want to focus on the effect associated with Zitterbewegung. Zitterbewegung is a rapid oscillatory motion of the electron, which implies that if an instantaneous measurement of its velocity is performed, the result is the speed of light. The amplitude of the oscillatory motion is $x_{osc} \sim \hbar/2mc$ [43], which means that the electron cannot be localized, but is rapidly oscillating in a volume of the order of the cube of the Compton wavelength. Zitterbewegung is a quantum relativistic effect and is related to the particle–antiparticle nature of the Dirac theory and to the nature of the spin. At present, there is growing interest in the detection of effects such as Zitterbewegung [46].

The effect of Zitterbewegung of the electron is introduced in the last term of equation (27), the Darwin contribution, which represents the smeared out electrostatic potential field that the electron sees when it fluctuates over a distance $x_{osc}$.

For the sake of simplicity, we examine the dispersion relation of Langmuir waves in an unmagnetized plasma. In order to focus on the effects of the Darwin contribution, we consider a one-dimensional (1D) unperturbed momentum distribution and let $f_0 \to f_0(p_z^2)\delta(p_x)\delta(p_y)$ (for a more realistic 3D-momentum distribution, even the electrostatic unmagnetized case couples to the spin terms, as we will see in the next section). The total distribution function will then have the form $f(z, p_z, t) = f_0(p_z^2) + f_1(z, p_z, t) \exp(ikz - i\omega t)$, where $f_0$ and $f_1$ are the equilibrium and the perturbed distribution functions, respectively. Furthermore, we consider a homogeneous plasma and neglect the motion of the heavy ions. The perturbed electric field is longitudinal, i.e. $E_1 = \hat{z}E_1 \exp(ikz - i\omega t)$. Using the evolution equation (27), the perturbed distribution function is then related to the electric field amplitude by

$$f_1 = \frac{-iq E_1}{\omega - kp_z/m} \left(1 + \frac{\hbar^2 k^2}{8m^2c^2}\right) \partial f_0 / \partial p_z. \tag{28}$$

Combining equation (28) with the Poisson equation $\nabla \cdot E_1 = 4\pi q \int d\Omega f_1$, we obtain the dispersion relation

$$1 = \frac{\omega_p^2}{k^2} \left(1 + \frac{\hbar^2 k^2}{8m^2c^2}\right) \int_{-\infty}^{\infty} dp_z \frac{\hat{f}_0}{(p_z/m - \omega/k)^2}, \tag{29}$$

where the re-normalized distribution function $\hat{f}_0$ fulfills $\int_{-\infty}^{\infty} dp_z \hat{f}_0 = 1$. For phase velocities larger than the characteristic spread in $p_z/m$, we can Taylor expand the denominator, and write the dispersion relation as

$$\omega^2 = \omega_p^2 \left(1 + \frac{\hbar^2 k^2}{8m^2c^2}\right) \left(1 + \frac{k^2 \langle p_z^2 \rangle}{m^2\omega_p^2}\right), \tag{30}$$

where $\langle p_z^2 \rangle = \int_{-\infty}^{\infty} dp_z p_z^2 \hat{f}_0$ is the average of the squared momentum. Here, the Landau damping term has been dropped, since the resonance is assumed to lie in the tail of the distribution.

The term proportional to $\hbar^2 k^2$ is the Zitterbewegung contribution to electrostatic modes. We can compare the effect of Zitterbewegung motion with the dispersive effects due to the well-known Bohm potential [2]. The ratio of these two contributions is $2\omega_p^2/k^2$ and therefore there is a range $k < \omega_p/(c\sqrt{2})$ where Zitterbewegung motion, via the Darwin term, is larger than the Bohm potential effect. Provided the wave numbers are small, $\omega \simeq \omega_p$, and the dispersion
relation can be further approximated as

\[ \omega^2 = \omega_p^2 + k^2 \left( \frac{\langle p_z^2 \rangle}{m^2} + \frac{v_{Zitt}^2}{2} \right), \]  

(31)

where \( v_{Zitt} = \hbar \omega_p / 2mc \) can be understood as a velocity response of the plasma \( v_{Zitt} \sim x_{osc} \omega_p \) to the rapid oscillations of the Zitterbewegung motion. The term \( v_{Zitt}^2 k^2 \) comes from the fact that the electron sees a smeared out electrostatic potential, and therefore, a gradient of the electric field and a force of the order \( \hbar^2 \nabla (\nabla \cdot \mathbf{E}) / m^2c^2 \). Similar to the effects of the Fermi pressure (which gives a nonzero \( \langle p_z^2 \rangle \) even when the temperature goes to zero) this implies a non-zero group velocity of the electron plasma waves even in the cold case (see also figure 2 for a comparison between the Fermi statistics and the Zitterbewegung, relevant for equation (31) when \( T \to 0 \)).

We note that the effect of the Darwin term related to Zitterbewegung becomes important for high-density plasmas when \( \hbar \omega_p / mc^2 \) is not too much smaller than unity.

5.2. Spin–orbit coupling contribution

The spin–orbit effect of Hamiltonian (5) appears as the Thomas precession correction (see, e.g., [54]) of the magnetic field in the fourth and fifth terms of equation (14).

Although this contribution can introduce interesting corrections to different types of wave modes, in this work we are going to follow the previous spirit and we analyze the quantum corrections to Langmuir waves, this time in a magnetized plasma. These modes have been studied previously in [55] in a phenomenological relativistic formalism where the appropriate Thomas precession factor of 1/2 was not used. We consider again longitudinal electrostatic modes \( \mathbf{E}_1 = \hat{z} E_1 \exp(ikz - i\omega t) \), but now propagating along an external magnetic field \( \mathbf{B}_0 = B_0 \hat{z} \). As will be seen below, this will give an illustrative example of how spin–orbit coupling modifies the usual dispersion relation. The distribution function will be taken to be of the form \( f(z, \mathbf{p}, s, t) = f_0(p^2, \theta_\parallel) + f_1(z, \mathbf{p}, s, t) \exp(ikz - i\omega t) \). We use cylindrical coordinates for
the momentum, i.e. \( \mathbf{p} = \hat{x} p_{\perp} \cos \varphi_{z} + \hat{y} p_{\perp} \sin \varphi_{z} + \hat{z} \), with \( \mathbf{p}^2 = p_{\perp}^2 + p_{\parallel}^2 \). Furthermore, we use spherical coordinates for the spin, i.e. \( \mathbf{s} = \hat{x} \sin \theta_{s} \cos \varphi_{s} + \hat{y} \sin \theta_{s} \sin \varphi_{s} + \hat{z} \cos \theta_{s} \).

At first order, taking the Fourier analysis of the evolution equation (27), we have

\[
\left( \frac{\partial}{\partial t} + \frac{\mathbf{p} \cdot \nabla_{s}}{m} - \omega_{c} \frac{\partial}{\partial \varphi_{p}} - \omega_{c} \frac{\partial}{\partial \varphi_{s}} \right) f_{1} = -q E_{1} \left( 1 + \frac{\hbar^{2} k^{2}}{8m^{2} c^{2}} \right) \frac{\partial f_{0}}{\partial p_{z}} - \frac{i k \mu p_{\perp} E_{1}}{2mc} \left( \sin \theta_{s} + \cos \theta_{s} \frac{\partial}{\partial \theta_{s}} \right) \left( \cos \varphi_{p} \sin \varphi_{s} - \sin \varphi_{p} \cos \varphi_{s} \right) \frac{\partial f_{0}}{\partial p_{z}} + \frac{\mu}{\hbar mc} p_{\perp} E_{1} \left( \cos \varphi_{p} \cos \varphi_{s} + \sin \varphi_{p} \sin \varphi_{s} \right) \frac{\partial f_{0}}{\partial \theta_{s}} \frac{\partial f_{0}}{\partial \varphi_{s}} \frac{\partial f_{0}}{\partial p_{\parallel}} - \frac{q \mu B_{0} E_{1}}{2mc^{2}} \left( \sin \theta_{s} \cos \varphi_{s} + \cos \theta_{s} \cos \varphi_{s} \frac{\partial}{\partial \theta_{s}} \right) \left( \cos \varphi_{p} \frac{\partial f_{0}}{\partial p_{\parallel}} \right) - \frac{q \mu B_{0} E_{1}}{2mc^{2}} \left( \sin \theta_{s} \sin \varphi_{s} + \cos \theta_{s} \sin \varphi_{s} \frac{\partial}{\partial \theta_{s}} \right) \left( \sin \varphi_{p} \frac{\partial f_{0}}{\partial p_{\parallel}} \right),
\]

where \( \omega_{c} = q B_{0}/mc \) is the cyclotron frequency and \( \omega_{c} = (g/2) \omega_{c} \) is the spin precession frequency [44]. We note that the perturbed distribution function can be solved for in terms of the orthogonal eigenfunctions \( \psi_{n} \) to the right-hand side operator [44, 55]. Accordingly, we make the expansion

\[
f_{1} = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} g_{n,n'}(p_{\perp}, p_{\parallel}, \theta_{s}) \psi_{n}(p_{\perp}, \varphi_{s}) \exp(\im \varphi_{s}), \tag{33}
\]

where in general

\[
\psi_{n}(p_{\perp}, \varphi_{s}) = \frac{1}{\sqrt{2\pi}} \exp[-\im(n \varphi_{s} - (k_{\perp} p_{\perp}/\omega_{c}) \sin \varphi_{s})] = \frac{1}{\sqrt{2\pi}} \sum_{n'=\infty}^{\infty} J_{n'} \left( \frac{k_{\perp} p_{\perp}}{m \omega_{c}} \right) \exp(\im(n-n') \varphi_{s}),
\]

and \( J_{n'} \) is the Bessel function. However, in this case for the longitudinal mode with \( k_{\perp} = 0 \), then

\[
\psi_{n}(p_{\perp}, \varphi_{s}) = \frac{1}{\sqrt{2\pi}} \exp(\im \varphi_{s}).
\]

Using the distribution function (33) in equation (32) and then multiplying both sides by \( \psi_{n}^{*} e^{-\im \varphi_{s}} \) and integrating over \( \varphi_{s} \) and \( \varphi_{s} \), we find that the only terms that survive in the sum (33) are \( g_{0,0} \) and \( g_{\pm 1, \mp 1} \) [41, 44, 55]. Thus, we find that the solution for \( f_{1} \) is

\[
f_{1} = \left( \frac{-iq E_{1}(1 + \hbar^{2} k^{2}/8m^{2} c^{2})}{\omega - kp_{\parallel}/m} \right) \frac{\partial f_{0}}{\partial p_{z}} + \frac{i k \mu p_{\perp} E_{1}}{2mc} \left[ \frac{e^{i(\varphi_{s} - \varphi_{s})}}{\omega - \Delta \omega_{c} - kp_{\parallel}/m} + \frac{e^{i(\varphi_{s} - \varphi_{s})}}{\omega + \Delta \omega_{c} - kp_{\parallel}/m} \right] \frac{\partial f_{0}}{\partial \theta_{s}} + \frac{ik \mu p_{\perp} E_{1}}{4mc} \left[ \frac{e^{i(\varphi_{s} - \varphi_{s})}}{\omega - \Delta \omega_{c} - kp_{\parallel}/m} - \frac{e^{i(\varphi_{s} - \varphi_{s})}}{\omega + \Delta \omega_{c} - kp_{\parallel}/m} \right] \left( \sin \theta_{s} + \cos \theta_{s} \frac{\partial}{\partial \theta_{s}} \right) \frac{\partial f_{0}}{\partial \theta_{s}},
\]

where \( \Delta \omega_{c} = \omega_{c} - \omega_{c} \). The expression (37) combined with

\[
-\im \omega E_{1} = -4\pi J_{\parallel}
\]
is used to deduce the dispersion relation. Due to the dependence on the angles \( \varphi_s \) and \( \varphi_p \), the first term of (37) gives rise to a free current density, whereas the other terms give rise to a polarization current density. The magnetization current density vanishes identically. Combining (37) and (38) we find the dispersion relation

\[
\omega = - \left( \omega_p^2 + \frac{1}{2} v_{\text{ZM}} k^2 \right) \int d\Omega p_z \frac{\partial \tilde{f}_0}{\partial p_z} \frac{\omega - k p_z / m}{\omega - k p_z / m}
+ \frac{6 \pi^3 \mu^2 \omega}{\hbar m^2 c^2} \int d\Omega p_\perp \frac{\partial \tilde{f}_0}{\partial \theta_s} \sin \theta_s \left( \frac{1}{\omega - \Delta \omega_c - k p_z / m} - \frac{1}{\omega + \Delta \omega_c - k p_z / m} \right)
+ \frac{3 \mu^2 \omega k \pi^3}{m^2 c^2} \int d\Omega p_\perp \left( \sin \theta_s + \cos \theta_s \frac{\partial \tilde{f}_0}{\partial \theta_s} \right) \frac{\partial \tilde{f}_0}{\partial p_z}
\times \left( \frac{1}{\omega - \Delta \omega_c - k p_z / m} + \frac{1}{\omega + \Delta \omega_c - k p_z / m} \right),
\]

which is general for \( \tilde{f}_0 \) (where we have again used the distribution function re-normalized as \( \int \tilde{f}_0 d\Omega = 1 \)).

As an example, let us examine an equilibrium distribution function with the form of a Maxwellian distribution and a spin-dependent part [41]

\[
\tilde{f}_0(p^2, \theta_s) = \frac{1}{N_M} e^{-p^2/2m^2v_i^2} \left[ e^{\mu B_0/k_B T} (1 + \cos \theta_s) + e^{-\mu B_0/k_B T} (1 - \cos \theta_s) \right],
\]

where \( T \) is the temperature and \( k_B \) is the Boltzmann constant, and the normalization factor is \( N_M = 4 \pi (\pi m^2 v_i^2)^{3/2} \cosh(\mu B_0/k_B T) \). We note that expression (40) is the thermodynamic equilibrium distribution for a plasma of moderate density where the magnitude of the chemical potential is large\(^5\).

To simplify the integrals we will consider the frequency range where the wave frequency \( \omega \) is close to resonance with \( \Delta \omega_c \). In this case, we neglect in (39) the terms with the denominators \( 1/(\omega + \Delta \omega_c - p_z k / m) \) because they are small compared with the terms with denominators \( 1/(\omega - \Delta \omega_c - p_z k / m) \). We are also going to take the limit when \( \omega - \Delta \omega_c \gg p_z k / m \). Thus, using the equilibrium distribution function (40), Taylor expanding the denominators, keeping the dominant pole contribution (for long wavelengths) at the resonance \( \omega - \Delta \omega_c = p_z k / m \) and integrating in the expanded phase space, we finally find the dispersion relation for the Langmuir waves with spin–orbit coupling and Darwin effects

\[
\omega^2 \left\{ 1 + \frac{\hbar^2 \pi^2 \omega_p^4}{8 m^2 c^4} \left[ \frac{k^2 v_i^2}{(\omega - \Delta \omega_c)^2} + \frac{3 k^4 v_i^4}{2(\omega - \Delta \omega_c)^4} \right] + \frac{\hbar^2 \pi^2 \omega_p^2 v_i^2}{4 m c^4} \tanh \left( \frac{\mu B_0}{k_B T} \right) \right\}
\times \left\{ \frac{1}{\omega - \Delta \omega_c} + \frac{i \pi^{1/2} k v_i}{k v_i} \exp(-(\omega - \Delta \omega_c)^2/k^2 v_i^2) + \frac{k^2 v_i^2}{2(\omega - \Delta \omega_c)^2} \right\}
= \left( \omega_p^2 + \frac{1}{2} v_{\text{ZM}} k^2 \right) \left( 1 + \frac{3 k^4 v_i^4}{2 \omega^2} \right).
\]

\(^5\) More generally, for a plasma with high density where degeneracy effects are important, expression (40) can be replaced with the sum of two Fermi–Dirac-type distributions, where the Fermi energy is substituted according to \( E_F \rightarrow E_F + \mu B_0 \) for the spin-up distribution and \( E_F \rightarrow E_F - \mu B_0 \) for the spin-down species. The general thermodynamic equilibrium distribution in a homogeneous magnetized plasma also accounting for Landau quantization can be found in [41].
Due to the exponential factor $\exp(-(\omega - \Delta \omega_c)^2/k^2 v_t^2)$ we note that the damping contribution due to Landau damping is smaller than the other terms for long wavelengths $k v_t \ll |\omega - \Delta \omega_c|$. Furthermore, the coefficient in front of the terms with denominators $\propto (v_t)^2$ is typically small, except for very strong magnetic fields or high densities. Thus excluding such regimes, the frequency of the spin modes will be close to resonance, i.e. fulfill $\omega \approx \Delta \omega_c$ to lowest order. More specifically, the deviation from exact resonance is of the order $(\omega - \Delta \omega_c)/\Delta \omega_c \sim (\hbar \Delta \omega_c/mc^2) \tanh(\mu B_0/k_B T)$. We note that spin-induced modes with $\omega \approx \Delta \omega_c$ have already been found in [44] without the inclusion of spin–orbit coupling. However, it should be noted that the present wave mode is quite different from that found previously. In particular, the field is now completely electrostatic (whereas it was found to be completely electromagnetic in the previous case), and the present wave mode exists in the long-wavelength regime, whereas the former wave mode [44] was dependent on a short wavelength, i.e. of the order of the electron gyroradius or shorter. Finally, we note that even in the absence of an external magnetic field, a finite contribution from the electron spin remains, together with the Darwin contribution. In the absence of resonances we note that both these quantum contributions require a very high plasma density to be significant.

6. Conclusions

The starting point for our theory is the FW transformation, applied to the Dirac equation, which is used to pick the positive energy states. Using the Wigner–Stratonovich transformation (8) on the density matrix, together with a $Q$ transformation [41], a scalar distribution function (12) can be defined. In the weakly relativistic limit, the evolution equation is given by (14) to leading order in the expansion parameters $\hbar L^{-1}/p$ and $\mu B/mc^2$, where $L$ is a characteristic macroscopic scale length, $p$ a characteristic momentum of particles and $B$ a characteristic magnitude of the magnetic field. In addition to the magnetic dipole force and spin precession [41], which is included already in the Pauli Hamiltonian, equation (14) also contains spin–orbit interaction, including the Thomas factor, and also the contribution from the Darwin term. A complication in the (weakly) relativistic theory that should be noted is the relation between momentum and velocity (17), which is now dependent on the spin variable. Moreover, when closing the system, the polarization (23) associated with the spin must be included in the current and charge density (19), and the expression for the free current density (21) is affected due to the aforementioned relation (17). It should also be stressed that the spin variable used to define $f(x, p, s, t)$ refers to the rest-frame spin, which is convenient, as two spherical angles for the spin variables are sufficient.

In order to illustrate the theory, we have presented examples of electrostatic interaction in magnetized and non-magnetized plasmas. By picking a 1D unperturbed distribution function, the influence of the Darwin term, which is associated with Zitterbewegung, is highlighted in the case of a non-magnetized plasma. For a magnetized plasma, the spin–orbit terms lead to new types of resonances for electrostatic waves, which involve the combined effect of orbital and spin-precession motion.

The dimensionless parameters determining the magnitude of the spin–orbit contribution and the spin polarization current (as compared with the classical ones) that appears in equation (41) are given by $\hbar \omega_p/mc^2$, $\mu B_0/k_B T$ and (somewhat indirectly) by $\mu B_0/mc^2$. Due to the specific assumptions of the geometry (i.e. electrostatic waves, parallel propagation), this is
not always completely representative of the general case, but typically at least one of the above parameters needs to have an appreciable magnitude in case the spin–orbit effects need to be significant. On the other hand, due to the weakly relativistic assumptions, our theory becomes highly questionable when $\hbar \omega_p/\mu B_0/c^2 \rightarrow 1$ or $\mu B_0/mc^2 \rightarrow 1$. The situation is helped somewhat by resonance effects such as those displayed in equation (41), which means that the quantum effects can be appreciable also in a weakly relativistic regime when these parameters are well beyond unity, e.g. when $\hbar \omega_p/\mu B_0/c^2 \lesssim 10^{-3}$ or $\mu B_0/mc^2 \lesssim 10^{-3}$. In particular, an electrostatic wave with a frequency $\omega \approx \Delta \omega_k$ propagating parallel to the magnetic field exists only if spin–orbit interaction is included in the basic equations. Plasmas with suitable parameters exist in astrophysical scenarios, e.g. in white dwarf stars (where $\hbar \omega_p/mc^2 \sim 5 \times 10^{-2}$) or in the vicinity of strongly magnetized objects such as pulsars or magnetars (where $10^{-3} \lesssim \mu B_0/mc^2 \lesssim 1$). To some extent, highly compressed plasmas such as those in ICF schemes may also be of interest, where the highly compressed core may reach a parameter value of $\hbar \omega_p/mc^2 \sim 10^{-3}$. We also note that self-generated quasi-static magnetic fields in ICF scenarios may in some circumstances reach the level of a giga-Gauss [56, 57], in which case $\mu B/k_B T_F$ could approach or even exceed unity in the case that the plasma heating is limited, such as e.g. in the fast ignition scenario. However, due to Pauli blocking $\mu B/k_B T_F$ is the relevant parameter whenever $T < T_F$ (which determines the relative difference between spin-up and spin-down populations in thermodynamic equilibrium), and for the high plasma densities of ICF plasmas we have $\mu B/k_B T_F \lesssim 0.1$ even in the case of strong magnetic field generation. Finally, we point out that our discussion here solely concerns the description of the electrons. For the regime of ICF plasmas with $T < T_F$, the ions can generally be treated classically, but strong coupling effects for ions should typically be included.

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