Pion and kaon role in $\tau$ decays, $(g_\mu - 2)$, the running of $\alpha_{QED}$ and the muonium hyperfine splitting

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Abstract

We make use of recent accurate results obtained for the pion and kaon vector form factors within a chiral unitary approach in order to calculate the decay widths of the $\tau$ lepton to these mesons and also to evaluate the contribution of this two mesons to the anomalous magnetic moment of the muon, the running of the fine structure constant and the muonium hyperfine splitting.

1 Introduction

In this paper we apply the pion and kaon vector form factors of reference [1] to calculate the tau decay to these mesons and also to study their contributions to the hadronic part of the anomalous magnetic moment of the muon, the running of the effective structure constant and the muonium hyperfine splitting. All these applications should be viewed as a complement of ref. [1] where a coupled-channel non-perturbative chiral approach was used to calculate the final state interactions in the pion and kaon vector form factors.

The decay of the $\tau$ lepton into a tau neutrino plus hadrons provides a unique framework to study low energy QCD. The mass of this lepton, about 1.8 GeV, allows the application of perturbative QCD to study inclusive decays, and in fact offers the possibility to measure the strong coupling constant $\alpha_s(\mu)$ at the low scale $\mu = m_\tau$ [2]. However, the calculation of exclusive semileptonic decays is not possible nowadays within perturbative QCD. One can then try to apply chiral perturbation theory [3], but its range of applicability is far below $m_\tau$. This means that it is possible to calculate differential decay rates at low energies, but predictions of integrated rates are not possible within standard $\chi PT$. A study of the differential decay rates of $\tau$ decaying to two and three pions was done in ref. [1] using standard $\chi PT$. The more problematic resonance region in the two meson decay mode has been theoretically studied using vector meson dominance [4, 5] or unitary...
techniques [9]. Here we will try to apply the form factors obtained in ref. [1] to calculate the decay rates of the decays $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ and $\tau^- \rightarrow K^- K^0 \nu_\tau$. The fact that in these decay modes only the $I = 1$ current is involved is useful since from the two pion decay mode data one can extract the pion form factor free from the $I = 0$ contamination due to the $\rho - \omega$ mixing.

The other observables to be studied in this paper are the anomalous magnetic moment of the muon $a_\mu \equiv (g_\mu - 2)/2$, the running of the effective coupling constant of QED and the muonium hyperfine splitting. The anomalous magnetic moment of the muon provides one of the most precise tests of the Standard Model (SM). The evaluation of $a_\mu$ within the SM has three sources: QED, electroweak (EW) and hadronic. Recently, the E821 BNL experiment obtained an experimental value which presented a 2.6 $\sigma$ deviation from the SM calculation. This discrepancy can be interpreted as a signal of new physics and it has stimulated a lot of publications in the last year. However, recent reevaluations [8, 9, 10] of the pseudoscalar pole contribution to $a_\mu$, correcting a mistake in its sign, reduce the discrepancy to 1.6 $\sigma$. The different SM contributions and also some possible sources of discrepancy based on supersymmetric loop effects are scrutinized in [11]. The hadronic contribution provides the main source of error in the confrontation between theory and experiment. In fact, a first principle QCD calculation is not available, and one has to resort to the use of dispersion relations relating the anomaly to the cross section for production of hadrons in $e^+ e^-$ annihilations. There is a large amount of such kind of analysis (see [12] and references therein). They find that the hadronic contribution is dominated by the low energy regime, being the $\rho$ contribution about a 72% of the total hadronic effect (the weight of the different energy regions can be seen for instance in figure 6 of ref. [13]). Our aim here is to study the pion and kaon contribution to the hadronic part of $a_\mu$. The pion contribution to this magnitude and to the running of the effective structure coupling is studied also in ref. [14] making use of elastic unitarity in the two pion channel.

Another magnitude to study in this paper is the effect of pions and kaons in the running of the effective coupling of QED. A good knowledge of this parameter is crucial in precision physics since it is one of the basic input parameters of the SM. The vacuum polarization effects are responsible for a partial screening of the fine structure constant in the Thomson limit, while at higher energies the strength of the electromagnetic interaction increases. Vacuum polarization is dominated by the QED contribution which is very well known. Again, the problem is that the calculation of the low energy contributions of the loop of two quarks cannot be performed within perturbative QCD, and one has to resort again to dispersion relations and the analysis of $e^+ e^-$ data. Recently the different hadronic contributions have been reevaluated with such techniques [13, 14].

Finally we also study in section 5 the effect of pions and kaons in the
muonium hyperfine splitting. The hyperfine structure of two-body systems has attracted the interest of both experimentalists and theoreticians since it provides precise tests of bound-state QED and accurate determinations of fundamental constants like the muon to electron mass ratio. Theoretically the case of muonium is interesting since it has not the problem of proton structure present in other two-body atomic systems like the hydrogen, and it has been measured very precisely despite the short muon lifetime. As in the former observables, the hadronic contributions are the main theoretical problem and are evaluated through an analysis of $e^+e^-$ data using dispersion relations. Recent analysis can be found in [15, 16]. In the first reference the muon to electron mass ratio is also determined in very good agreement with the experimental value.

2 Pion and kaon vector form factors

In this section we briefly review the calculation of the pion and kaon vector form factors done in ref. [1]. These form factors are the necessary information to estimate the contributions of the aforementioned mesons to the muon anomalous magnetic moment, the running of the effective fine structure constant and the muonium hyperfine splitting, and to calculate the branching ratios of the $\tau$ decay to such mesons. The approach of reference [1] calculates the final state interaction corrections to the tree level amplitudes, calculated from lowest order $\chi PT$ [3] and from the inclusion of explicit resonance fields [17], while matching with the $\chi PT$ vector form factors calculated at next-to-leading order [3]. In [1] it is shown that starting from the unitarity of the $S$-matrix for definite isospin and using matrix notation (since the pion and kaon channels couple in $I = 1$) it is possible to write the following equation for the form factor $F(s)$:

$$\text{Im} F^I(s) = \tilde{Q}(s)^{-1} \cdot T^I(s) \cdot \frac{Q(s)}{8\pi \sqrt{s}} \cdot \tilde{Q}(s) \cdot F^{I^*}(s)$$  \hspace{1cm} (1)

where $Q(s)_{ij} = \sqrt{s/4 - m_i^2} \delta_{ij}$, $\tilde{Q}(s)_{ij} = \sqrt{s/4 - m_i^2} \delta_{ij}$, and $T(s)$ is a matrix containing the meson-meson scattering amplitudes. In the $I = 1$ channel we have pions, kaons and the $\rho$ resonance, while in the $I = 0$ channel we have kaons and the $\omega$ and $\phi$ resonances.

If one uses in the former equation the $T$-matrix expression provided by the $N/D$ method adapted to the chiral framework (see ref. [18]), it is possible to show that the form factor must have the form:

$$F^I(s) = [1 + \tilde{Q}(s)^{-1} \cdot K^I(s) \cdot \tilde{Q}(s) \cdot g^I(s)]^{-1} \cdot R^I(s)$$  \hspace{1cm} (2)
where $K^I(s)$ is a matrix collecting the tree level meson-meson scattering amplitudes, $g^I(s)$ is the diagonal matrix given by the loop with two meson propagators (see ref. [18]) and $R^I(s)$ is a vector whose components are functions free of any cut.

To fix these unknown functions we take a look at the large $N_c$ limit of eq. (2). In this limit loop physics is suppressed, therefore from eq. (2) we find that $F^I_{N_c^{leading}}(s) = R^I_{N_c^{leading}}(s) = F^I_{t}(s)$, where $F^I_{t}(s)$ is the tree level form factor. Once the $N_c$-leading part of $R^I(s)$ is known only its subleading part remains to be fixed. This part should be a polynomial since it has no cuts and the poles coming from the resonances are already included in the leading part. If we further require that the form factors of eq. (2) vanish in the limit $s \to \infty$ we find that the polynomials are in fact constants.

At this moment there are several unknown parameters: the $R^I_{subleading}$, the $d^I_i$ parameters appearing in the $g^I(s)$ matrix (see refs. [1, 18]) and also the bare masses of the resonances. The bare masses of the resonances can be fixed by the requirements that the moduli of the scattering amplitudes have a maximum at the energy $\sqrt{s} = M_{phys}\text{emanation}$, and the rest of the parameters are fixed by matching our results with the ones of $\chi PT$ vector form factors at the one loop level in the chiral counting applied to both schemes. Finally, the isospin violation effects can be introduced via the $\rho-\omega$ mixing.

This method provides a good description of the pion and kaon form factors up to 1.2 GeV and compares very well with the two-loop $\chi PT$ prediction of [4, 19]. For higher energies the effect of other channels like $\omega\pi$, $4\pi$, etc., and other resonances ($\rho'$, $\rho''$, $\omega'$, $\phi'$, etc.) becomes relevant. The inclusion of these resonances is straightforward in the formalism, but since their masses and couplings are not well known they lead to a big number of free parameters.

It is worth saying that, although we use exactly the same expressions for the form factors as in [1], in the calculations here we use different values for some parameters. The value of $f = 87.4$ MeV used in [1] was calculated from eq. (7.14) of the last reference in [3], relating $f_\pi$ and $f$. Here we have recalculated that relation starting from eq. (7.13) of the former reference and using the experimental value of $\langle r^2 \rangle_\pi$, thus obtaining $f = 86.6 \pm 0.5$ MeV. For $F_V$ we use the value $F_V = 153 \pm 4$ MeV from the $\rho$ decay to $e^+e^-$. Finally, in [1] we used $G_V = 53$ MeV. This value was taken from an estimation of the chiral corrections in the vector form factor by means of vector meson dominance plus one loop $\chi PT$ corrections. Since our approach includes the former evaluation but also higher orders in the $\chi PT$ expansion we have fitted this value to better reproduce the experimental data of the pion form factor, finding a value of $G_V = 55.52 \pm 0.12$ MeV. The errors in the results quoted

\footnote{Calculated from lowest order $\chi PT$ plus $s$-channel vector resonance exchange contributions.}

\footnote{The tree level form factors are also evaluated using the lowest order $\chi PT$ Lagrangian plus the chiral resonance Lagrangian.}
in the following sections are obtained by summing in quadrature the errors from the uncertainties in each of the fundamental parameters $f$, $F_V$, $G_V$.

3 \( \tau \) decay to \( \pi^- \pi^0 \nu_\tau \) and to \( K^- K^0 \nu_\tau \)

In this section we apply the form factors calculated on the previous section to the calculation of the branching ratios of the decays \( \tau^- \to \pi^- \pi^0 \nu_\tau \) and \( \tau^- \to K^- K^0 \nu_\tau \). Let us start with \( \tau^- \to \pi^- \pi^0 \nu_\tau \). Its amplitude can be calculated using standard techniques and one has:

\[
\Gamma(\tau^- \to \pi^- \pi^0 \nu_\tau) = \frac{G_F^2 \cos^2 \theta_C}{384\pi^3} m_\tau^3 \int_{4m_\pi^2}^{m_\tau^2} dp \frac{dp}{m_\tau^2} \left( 1 - \frac{p^2}{m_\tau^2} \right)^2 \left( 1 + \frac{2p^2}{m_\tau^2} \right)^2 \\
\left( 1 - \frac{4m_\pi^2}{p^2} \right)^{3/2} |F_\pi(p^2)|^2
\]

We expect to obtain a rather good result for the total width in spite of not having a very good description of the pion form factor up to \( s = m_\tau^2 \) since the neutrino \( \nu_\tau \) carries in average a sizeable fraction of the energy and hence there is a smaller energy left for the \( \pi\pi \) system, and furthermore the form factor is dominated by the \( \rho \) resonance which is well described in our model. For energies close to the \( \tau \) mass the form factor is very small and the contribution of this region is not so relevant. In ref. [1] our pion form factor is compared to the experimental one obtained from an analysis of tau decay data. The result we get for the integrate width is:

\[
\Gamma(\tau^- \to \pi^- \pi^0 \nu_\tau) = (5.5 \pm 0.3) \cdot 10^{-10} \text{ MeV} \quad \text{BR} = 0.244 \pm 0.012
\]

to be compared to the PDG value \( \Gamma(\tau^- \to \pi^- \pi^0 \nu_\tau) / \Gamma^{\text{tot}}_\tau = (25.40 \pm 0.14)\% \) [20]. The agreement between this value and our calculation is remarkable.

Now we can study the decay of the \( \tau \) lepton to a pair of kaons. In this case we do not expect a priori such a good result. On one hand the threshold of production of kaons, around 1 GeV, where the \( I = 1 \) pion vector form factor calculated in ref. [1] begins to deviate from data. On the other hand, the \( \phi \) resonance, which dominates the kaon form factor for such energies, does not show up here because only the \( I = 1 \) part contributes. As pointed out in the previous section, our approach can be improved with the inclusion of more massive vector resonances, such as the \( \rho'(1450), \rho'' \) and so on [20]. However, one has to face then the problem that for these energies channels like \( 4\pi, \omega \pi \), etc... are no longer negligible and indeed dominate the width
of the latter resonances. As a result we should include more free parameters to describe these resonances, such as couplings, widths, etc..., which makes us decide not to include these resonances and see what comes out with our previous calculation. The width of the process is then given by:

\[
\Gamma(\tau^- \to K^- K^0 \nu_\tau) = \frac{G_F^2 \cos^2 \theta_c m_\tau^2}{768 \pi^3} \int_{4m_K^2}^{m_\tau^2} ds \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right)^2 \left(1 - \frac{4m_K^2}{s}\right)^{3/2} |F_K(s)|^2
\]  

(5)

We get the following result:

\[
\Gamma(\tau^- \to K^- K^0 \nu_\tau) = (2.8 \pm 0.4) \cdot 10^{-12} \text{ MeV} \quad \text{BR} = (1.25 \pm 0.13) \cdot 10^{-3} \]  

(6)

to be compared to the PDG value for the branching ratio of \((1.55 \pm 0.17) \cdot 10^{-3}\) [20]. Our result is a bit lower than the experimental one but still in agreement within errors.

We have also studied the invariant mass distribution of the \(K\bar{K}\) system. The results are plotted in fig 1, from were we can see that although the integrated width seems a bit low, our mass distribution is compatible with present data in shape and strength.

4 Anomalous magnetic moment of the muon

The anomalous magnetic moment of the muon can be calculated and measured with high accuracy, providing an extraordinary test of the electroweak theory. Any residual difference between the sum of the Standard Model (SM) contributions and the experimental value \(a_{\mu}^{exp}\) will be indicative of new physics. This observable is being widely studied nowadays and many works have appeared recently because the last experiment performed at Brookhaven [23] has obtained a value higher than most of the SM predictions:

\[
a_{\mu}^{exp} = 11659202(14)(6) \cdot 10^{-10} \]  

(7)

The theoretical calculation contains terms of different origin. We may write \(a_{\mu}^{th}\) as a sum of the QED, weak and hadronic contributions. It is important to assess accurately these contributions in order to see if a new contribution from extensions of the SM is needed.
Our aim here is to study the pion, charged kaon and neutral kaon contributions to the hadronic part of \( a_\mu \). The leading hadronic contribution to \( a_\mu \) is due to the photon vacuum polarization insertion into the diagram of the electromagnetic vertex of a muon, shown in fig 2. This contribution can be calculated in terms of the experimental cross section \( \sigma_{\text{had}}(e^+e^- \rightarrow \text{hadrons}) \) by using dispersion relations [24, 25, 26]:

\[
a_{\mu}^{\text{had}} = \left( \frac{\alpha(0)m_\mu}{3\pi} \right)^2 \int_{4m_\mu^2}^{\infty} \frac{ds}{s^2} R(s) \hat{K}(s), \quad \text{with} \quad R(s) = \frac{3s}{4\pi\alpha^2(s)} \sigma(e^+e^- \rightarrow \text{hadrons}) \tag{8}
\]
where the $\hat{K}(s)$ function is [27]:

\[
\hat{K}(s) = \frac{3s}{m_\mu^2} K(s) \quad \text{where}
\]

\[
K(s) = \frac{x^2}{2}(2 - x^2) + \frac{(1 + x^2)(1 + x)^2}{x^2} \left( \ln(1 + x) - x + \frac{x^2}{2} \right) + \frac{1 + x}{1 - x} x^2 \ln(x); \quad \text{with}
\]

\[
x = \frac{1 - \beta_\mu(s)}{1 + \beta_\mu(s)}; \quad \beta_\mu(s) = \sqrt{1 - 4m_\mu^2/s} \quad (9)
\]

Since $\hat{K}(s)$ grows smoothly, the integral is dominated by the low energy region.

We have now all the ingredients to calculate the pionic and kaonic contributions. The cross section $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$ is:

\[
\sigma(e^+e^- \rightarrow \pi^+\pi^-) = \frac{\pi\alpha^2 \sigma_\pi^3}{3s} |F_\pi(s)|^2 \quad (10)
\]

where $\sigma_\pi = \sqrt{1 - 4m_\pi^2/s}$. Finally the integral we have to calculate is:

\[
a_{\pi\pi}^\mu = \left( \frac{\alpha(0)m_\mu}{6\pi} \right)^2 \int_4^{\Lambda^2} \frac{ds}{s^2} \sigma_\pi^3 |F_\pi(s)|^2 \hat{K}(s) \quad (11)
\]

The results we get for different values of the cut off $\Lambda$ are given in table I. They are to be compared to results coming from experimental analysis in the region $0.320 \text{ GeV} \leq \sqrt{s} \leq 2.125 \text{ GeV}$: $(500.81 \pm 6.03) \times 10^{-10}$ [28] and $(510 \pm 5.3) \times 10^{-10}$ [29]. The agreement between our prediction and the experiment is remarkable. We also agree with the theoretical estimation done in [14]. The recent analysis of reference [12] gives for the pion contribution in the region $4m_\pi^2 \leq s \leq 0.8 \text{ GeV}^2$ a value of $a_{\pi\pi}^\mu = (479.46 \pm 6.07) \times 10^{-10}$, while in the same energy interval we find $(490 \pm 18) \times 10^{-10}$, in agreement with that reference.

The calculation for the kaons is analogous to the former one. We obtain for a value of the cut-off $\Lambda=1.2 \text{ GeV}$ the values of $a_{\mu K^+K^-} \times 10^{10} = 18.1 \pm 1.0$ and $a_{\mu K^0\bar{K}^0} \times 10^{10} = 10.7 \pm 0.6$.

The analysis of data done in ref. [28] gives the values $a_{\mu K^+K^-} \times 10^{10} = 4.30 \pm 0.58$ and $a_{\mu K^0\bar{K}^0} \times 10^{10} = 1.20 \pm 0.42$ in the region $1.055 \text{ GeV} \leq \sqrt{s} \leq 8$.
Table 1: Values of $a_{\mu}^{\pi\pi}$ obtained for different values of the cut off.

| $\Lambda$(GeV) | 1.1  | 1.2  | 1.3  | 1.4  | 2.1  |
|----------------|------|------|------|------|------|
| $a_{\mu}^{\pi\pi} \times 10^{10}$ | 512 ± 19 | 515 ± 20 | 516 ± 20 | 517 ± 20 | 518 ± 20 |

2.055 GeV for $K^+K^-$ and 1.090 GeV $\leq \sqrt{s} \leq 2.055$ GeV for $K^0\bar{K}^0$. We have integrated also in these intervals of energies, finding $a_{\mu}^{K^+K^-} \times 10^{10} = 3.2 \pm 0.3$ and $a_{\mu}^{K^0\bar{K}^0} \times 10^{10} = 0.233 \pm 0.013$. Our values are lower but this is not strange since in this region of energies there are more resonances and channels that are not considered in our approach, so that our form factors are not too accurate for energies higher than 1.2 GeV, specially in the kaon case. In any case the biggest contribution comes from energies close to the $\phi$ peak, therefore it is more interesting to compare our results in the region where the $\phi$ resonance dominates the form factor. In reference [28] the contribution of the $\phi$ resonance to $a_{\mu}^{had}$ was studied in the region $1.000$ GeV $\leq \sqrt{s} \leq 1.055$ GeV, giving a value $a_{\mu}^{\phi} \times 10^{10} = 39.23 \pm 0.94$. This has the contribution not only from $K^+K^-$ and $K^0\bar{K}^0$ pairs but also from other decay channels of the $\phi$ as the $\pi^+\pi^-\pi^0$, $\rho\pi$, etc. The $K^+K^-$ branching ratio is 49.2% and the $K^0\bar{K}^0$ one is 33.8%. We have then that the total branching ratio of $\phi \rightarrow KK$ is 83%. Since these partial widths are also related to the form factor, we can make an estimation of the contribution of kaons in this range of energies by multiplying the $\phi$ contribution by 0.83, obtaining in this way a kaon contribution of approx. $32.6 \times 10^{-10}$. Carrying out the integral with our form factor in this region of energies we find $a_{\mu}^{K^+K^-} \times 10^{10} = 16.3 \pm 0.8$ and $a_{\mu}^{K^0\bar{K}^0} \times 10^{10} = 10.3 \pm 0.5$, which give a total kaon contribution of $(26.5 \pm 1.5) \times 10^{10}$, close to the former estimation. The fact that we do not get very good values in the case of the $\phi$ is not surprising since there are long standing problems related to these resonance like the $\Gamma(\phi \rightarrow K^+K^-)/\Gamma(\phi \rightarrow K^0\bar{K}^0)$ problem [31], and in the calculation of the widths the kaon vector form factor is also involved.

5. **$\pi$ and $K$ contributions to the running effective fine structure constant and the muonium hyperfine splitting.**

In this section we estimate the pion and kaon contributions to the running of the fine structure constant and to the muonium hyperfine splitting. The effective structure constant at scale $\sqrt{s}$ is given by:
Table 2: Values of $\Delta \alpha_{\pi^+\pi^-}(M_Z^2)$ obtained for different values of the cut off.

| $\Lambda$(GeV) | 1.1  | 1.2  | 1.3  | 1.4  | 2.1  |
|----------------|------|------|------|------|------|
| $\Delta \alpha_{\pi^+\pi^-}(M_Z^2) \times 10^4$ | 33.9 ± 1.4 | 34.3 ± 1.5 | 34.5 ± 1.5 | 34.7 ± 1.5 | 35.1 ± 1.6 |

The low energy contribution to the hadronic part cannot be calculated from perturbative QCD, but it can be related to the $e^+e^-$-annihilation data by using dispersion relations and the optical theorem [31, 32], as it is usually done in the case of $a_\mu$. The hadronic contribution to the photon vacuum polarization corresponds to the "blob" in fig. 3.

Figure 3: Hadronic contribution to photon vacuum polarization.

Using this equation we get the results shown in table 3. This results are to be compared with the result coming from experimental analysis in the region $0.320$ GeV $\leq \sqrt{s} \leq 2.125$ GeV: $(34.31 \pm 0.38) \times 10^{-4}$ [28]. As we can see, we have a good agreement with the experiment and also with the calculation done in ref. [14]. Also, in the region $4m_{\pi}^2 \leq s \leq 0.8$ GeV$^2$ we get $\Delta\alpha_{\pi^+\pi^-}(M_Z^2) = 31.7 \pm 1.3$, in agreement with the recent estimation of [15] $\Delta\alpha_{\pi^+\pi^-}(M_Z^2) = 31.45 \pm 0.23$.

We can evaluate in the same way the kaon contribution. We obtain for a cut-off of $\Lambda = 1.2$ GeV the values $\Delta\alpha_{K^+K^-}(M_Z^2) = (2.43 \pm 0.13) \times 10^{-4}$ and $\Delta\alpha_{K^0\bar{K}^0}(M_Z^2) = (1.42 \pm 0.07) \times 10^{-4}$. The experimental data analysis of ref. [28] in the region $1.055$ GeV $\leq \sqrt{s} \leq 2.055$ GeV for $K^+K^-$ and

\begin{equation}
\alpha(s) = \frac{\alpha}{1 - \Delta\alpha(s)}
\end{equation}

where $\alpha$ is the fine structure constant and $\Delta\alpha$ is the photon vacuum polarization contribution. The hadronic contribution to the photon vacuum polarization corresponds to the "blob" in fig. 3.

\[ K_\alpha(s) = \frac{\pi}{\alpha M_Z^2} \]
1.090 GeV ≤ \sqrt{s} ≤ 2.125 GeV for \( K^0\bar{K}^0 \), where the \( \phi \) region is excluded, gives \( \Delta \alpha_{K^+K^-}(M_Z^2) = (0.85 \pm 0.10) \times 10^{-4} \) and \( \Delta \alpha_{K^0\bar{K}^0}(M_Z^2) = (0.23 \pm 0.08) \times 10^{-4} \) respectively. Carrying out the integration in the same energy regions we obtain \( \Delta \alpha_{K^+K^-}(M_Z^2) = (0.65 \pm 0.07) \times 10^{-4} \) and \( \Delta \alpha_{K^0\bar{K}^0}(M_Z^2) = (0.0392 \pm 0.0023) \times 10^{-4} \). As in the \( a_\mu \) case our results for these intervals of energies are not so good as in the pion case and the reasons are identical.

We can also try to make an estimation of the kaon contribution in the \( \phi \) region. Ref. [28] gives a \( \phi \) contribution of \( \Delta \alpha_{\phi}(M_Z^2) = (5.18 \pm 0.12) \times 10^{-4} \) for \( 1.000 \text{ GeV} \leq \sqrt{s} \leq 1.055 \text{ GeV} \), from where we estimate a value \( \Delta \alpha_{K^0\bar{K}^0}(M_Z^2) \simeq 4.3 \times 10^{-4} \). The integral in this region gives us \( \Delta \alpha_{K^+K^-}(M_Z^2) = (2.15 \pm 0.11) \times 10^{-4} \) and \( \Delta \alpha_{K^0\bar{K}^0}(M_Z^2) = (1.35 \pm 0.07) \times 10^{-4} \). Hence the sum, as in the \( a_\mu \) case, is a bit low.

Finally, we have studied also the contributions of pions and kaons to the muonium hyperfine splitting. The hadronic contribution to the hyperfine splitting of the muonium is given by the diagrams in figure 4. As in the previous cases, the hadronic blob must be evaluated and in order to do so one has to resort to dispersion relations.

Figure 4: Diagrams accounting for the hadronic contribution to the muonium hyperfine splitting.

The ground state hyperfine splitting is given also by eq. 8, but replacing there \( a_\mu^{\text{had}} \) by \( \Delta E^{\text{had}} \) in the LHS and the function \( K(s) \) (related to \( \tilde{K}(s) \) as established in eq. (9)) appearing there by (see [15, 16]):

\[
K_{\text{split}}(s) = \frac{16\alpha^4 m_R^3}{3m_\mu^2} \left\{ \left( \frac{s}{4m_\mu^2} + \frac{3}{2} \right) \log \frac{s}{m_\mu^2} - \frac{1}{2} - \left( 2 + \frac{s}{4m_\mu^2} \right) \beta_\mu(s) \log \frac{1 + \beta_\mu(s)}{1 - \beta_\mu(s)} \right\}
\]

(14)

where \( m_R \) is the reduced mass and \( \beta_\mu(s) \) is defined in eq. [15].

As in the other cases, we have evaluated the pion contribution with different values of the cut-off, finding the results shown in table 5.

As we can see in table 5, the pion contribution barely depends on the value of the cut-off. Here we will compare with the analysis of the \( e^+e^- \) and \( \tau \) decay experimental data done in ref. [15] in the interval \( 4m_\pi^2 \leq s \leq 0.8 \text{ GeV}^2 \). \( \Delta \nu(Hz) = 152.9 \pm 1.8 \). Our prediction in the same energy region is \( \Delta \nu_{\pi\pi}(Hz) = 154 \pm 6 \), in perfect agreement with the former estimate. In the case of the kaons we will only give here the result obtained with a cut-off of \( \Lambda = 1.2 \text{ GeV} \) since this is the region in which the kaon form factor
Table 3: Values of $\Delta \nu_{\pi^+\pi^-}$ (Hz) obtained for different values of the cut off.

| $\Lambda$(GeV) | 1.1 | 1.2 | 1.3 | 1.4 | 2.1 |
|----------------|-----|-----|-----|-----|-----|
| $\Delta \nu_{\pi^+\pi^-}$(Hz) | 162 ± 6 | 163 ± 6 | 163 ± 6 | 164 ± 6 | 164 ± 6 |

is well reproduced. The values obtained are $\Delta \nu_{K^+K^-}$(Hz) = 6.2 ± 0.3 and $\Delta \nu_{K^0\bar{K}^0}$(Hz) = 3.7 ± 0.2, and we expect our results to be also a bit low compared to the data, as in the former cases when studying the kaon contribution.

6 Conclusions

We have applied the formalism developed in [1] to describe the pion and kaon vector form factors accounting for unitarity in coupled channels to calculate the contributions of these two mesons to the decay of the $\tau$ lepton and to the hadronic part of the anomalous magnetic moment of the muon, the running of the fine structure constant and the muonium hyperfine splitting at low energies where a calculation within perturbative QCD is not available. The evaluation of the branching ratios of the decays $\tau \to \pi^-\pi^0\nu_\tau$ and $\tau \to K^-K^0\nu_\tau$ are in good agreement with the experiment, although the calculations need a good description of the $I = 1$ form factors of the mesons up to 1.7 GeV, while the description of the form factors done in [1] is only good up to 1.2 GeV, due to the opening of more channels and the presence of more resonances at these energies that are not taken into account there. The previous agreement is found because the form factors are dominated by the $\rho$ resonance, having small values at high energies, and thus giving a larger weight to the low energy region where our description of the form factor is good.

We have also evaluated the pion and kaon contribution (in the low energy region) to the anomalous magnetic moment of the muon, the running of the QED effective coupling and the muonium hyperfine splitting. The results obtained for the pion are good when compared to the available experimental data analyses. However, in the case of the kaons our values are a bit lower than expected, and this can be due to the fact that the description of the kaon form factors employed here is done in the isospin limit, using an averaged value for the kaon mass (we use the physical values of $m_{K^+}$ and $m_{K^0}$ for phase space considerations). It is also worth noting that the kaon form factor is essential in the $\Gamma(\phi \to K^+K^-)/\Gamma(\phi \to K^0\bar{K}^0)$ problem which is not yet understood (see [30]).

We want also to stress that the rather large errors that we get in our
estimations are mainly due to the value of the $F_V$ parameter appearing in the chiral resonance Lagrangians \cite{17}, which is fixed by the $\rho \to e^+e^-$ decay. More accurate measurements of these quantity will thus be most interesting. Finally, it is worth saying that within the formalism of ref. \cite{1} the inclusion of another octet of vector resonances is straightforward, although it leads to a rather large amount of free parameters (new $F_V$ and $G_V$ parameters, bare masses and initial widths in the resonance propagators, since we do not take into account all the relevant channels at that energies), thus loosing the predictive power of the approach.

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