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Decomposition Algorithm for Median Graph of Triangulation of a Bordered 2D Surface

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Abstract
This paper develops an algorithm that identifies and decomposes a median graph of a triangulation of a 2-dimensional (2D) oriented bordered surface and in addition restores all corresponding triangulation whenever they exist. The algorithm is based on the consecutive simplification of the given graph by reducing degrees of its nodes. From the paper [2], it is known that such graphs can not have nodes of degrees above 8. Neighborhood of nodes of degrees 8, 7, 6, 5, and 4 are consecutively simplified. Then, a criterion is provided to identify median graphs with nodes of degrees at most 3. As a byproduct, we produce an algorithm that is more effective than previous known to determine quivers of finite mutation type of size greater than 10.

1 Introduction

Triangulations of 2D surface are instrumental for in many math theories. To mention a few, they are used in [1] to construct coordinates on Teichmüller Space. Also, in [2], the authors uses triangulations of a 2D surface to construct a cluster algebra. Moreover, they described a principal way to determine if a cluster algebra is originated from a surface triangulation. The method uses the idea of block decomposition of the median graph of triangulation.

Block decomposition also plays an important role in determining the mutation class of a quiver. A quiver is defined as a finite oriented multi-graph without loops and 2-cycles. Based on the cluster algebra constructed in [2], a seed $(f, B)$ is defined, where $f$ is a collection of $n$ algebraically independent rational functions of $n$ variables and $B$ is a skew-symmetrizable matrix. Cluster algebra formalism introduces a certain operation on seeds. This operation is called mutation, see [3] Definition 2.1. Two seeds are said to be mutation equivalent if one is obtained from the other by a sequence of seed mutation. A mutation class is the collection of mutation-equivalent seeds. In [2], the authors prove that the mutation class of any block-decomposable quiver is finite.

In this paper, we provide a combinatorial algorithm that determines if a given graph is decomposable. Moreover, the algorithm also determines all corresponding triangulations for decomposable graphs.

A block is a directed graph that is isomorphic to one of the graphs shown in Figure 1. They are categorized as one of the following: type I (spike), II (triangle), IIIa (infork), IIIb (outfork), IV (diamond), and V (square). The nodes marked by unfilled circles are called outlets or white nodes. The nodes marked by filled circles are called dead ends or black nodes. A directed graph $G$ is called block decomposable or simply decomposable if it can be obtained from disjoint blocks as a result of the following gluing rules: (See [2] for definition.)

1. Two white nodes of two different blocks can be identified. As a result, the graph becomes a union of two parts. The common node becomes black. A white node can not be identified with another node of the same block. See Figure 3 [5].

2. A black node can not be identified with any other node.
Table 1: Blocks

3. If an edge \( a = x \to y \) with two white nodes \((x, y)\) is glued to another edge \( b = p \to q \) with two white nodes \((p, q)\) in the following way: \( x \) is glued to \( p \) and \( y \) is glued to \( q \), then a multi-edge is formed, and the nodes \( x = p, y = q \) become black. (Figure 1)

4. If an edge \( a = x \to y \) with two white nodes \( x, y \) is glued to another edge \( b = q \to p \) in the following way: \( x \) is glued to \( p \) and \( y \) is glued to \( q \), then both edges are removed after gluing, the nodes \( x = p, y = q \) become black. We say that edges annihilate each other. (Figure 2)

Remark 1. By design, a block-decomposable graph has no loop and all edge multiplicities are 1 or 2.

Remark 2. Note that in the algorithm, the color of a vertex is not specified in the original graph, and is determined only for vertices of blocks of a specified block decomposition. (There might be several ways to decompose a graph. Hence, a vertex may have different colors in different decompositions, see Figure 3, 4, and 5.)

We will assume in the following discussion that \( G \) is a finite oriented multi-graph without loops and 2-cycles.

**Proposition.** A graph \( G \) without isolated nodes is decomposable if and only if every disjoint connected component is decomposable.
Proof. It’s suffice to show that annihilating an edge in a connected graph generates a connected graph. Since we can only annihilated edges in a spike, triangle or diamond block, and before annihilating an edge, both of its endpoint must be white. Denote these two endpoints by \( x, y \).

- Suppose \( x, y \) are endpoints of a spike. Notice that the original graph must be a single spike. If we annihilate the edge by gluing a triangle, \( x, y \) will be connected via the third node of the triangle. If we annihilate the edge by gluing a diamond, \( x, y \) will be connected via the remaining nodes of the diamond. If we annihilate it by gluing a spike, the new graph will consist only two nodes and no edge. Contradiction.

- Suppose \( x, y \) are endpoints of a triangle. If we annihilate the edge \( xy \) by gluing a spike or diamond, the remaining two edges of the triangle can not be annihilated, and \( x, y \) will still be connected via the third node of the triangular block. If we glue another triangle, there are two cases: we can annihilate only one edge, namely \( xy \). Then \( x, y \) will still be connected via the third node. We can also annihilate the whole triangle when all three nodes are white. In this case, the original graph is a single triangular block and the new graph is simply three nodes. This is again a contradiction.

- Suppose \( x, y \) are endpoints of a diamond. Since none of the boundary edges can be annihilated, after gluing a spike or triangle or another diamond to edge \( xy \), \( x, y \) will still be connected.

According to the above proposition, if \( G \) is decomposable, we may break connectivity in a graph only in two trivial cases. In both cases, the resulting graph contains isolated nodes. On the other hand, in a decomposable graph, isolated nodes can only be obtained in the above manner. In particular, the decomposition of subset of \( G \) consisting isolated nodes is independent with the rest of \( G \). Therefore, we can assume from now on that the graph is connected.

Notice that the highest degree of any node in a block is 4. If \( G \) is decomposable, the highest degree of any node in \( G \) does not exceed 8.

In the algorithm, for every graph, the neighborhoods of nodes of highest degree (at most 8) are simplified. The neighborhoods of all such nodes are analyzed and replaced by simpler ones so that the degrees of these nodes decrease. After the neighborhoods of nodes of degree 8 are exhausted, proceed to the nodes of degree 7, then, 6,5 and 4. This paper proves that all replacements are reversible: the original graph is decomposable if and only if the new graph is. At last, we give a theorem that identifies decomposable graphs containing nodes of degree at most 3.

2 Simplification on Nodes of Degree 8,7,6 and 5

In this section we show how to replace the neighborhood of certain node by an equivalent one which decreases the degree of this node. As a result, the nodes of degree larger than four are eliminated, consecutively.

2.1 Nodes of Degree 8

A node \( o \) of degree 8 in decomposable graph \( G \) can only be obtained by gluing a square with another square. (See Figure 6) The result is a disjoint connected component. Otherwise, \( G \) is undecomposable.

2.2 Nodes of Degree 7

If \( o \) is a node of degree 7 in a decomposable graph \( G \), it must have resulted from gluing together a diamond and a square, see Figure 7. The neighborhood is replaced by the one in Figure 8. The Lemma 1 shows that this replacement is reversible.
Lemma 1. Suppose the neighborhood of node o is as in Figure 8. nodes x, p and nodes y, p are disconnected. If G is decomposable, the neighborhood can only be decomposed into a triangle and a spike.

Proof. It is necessary to show that b, c, d form a triangular block in the decomposition and a comes from a spike block.

Assume there is a decomposition, I claim that the block containing b must be a triangle. Suppose that the claim is false.

1. Suppose that b comes from a fork. Since both edges in a fork contain black endpoints, they can not be annihilated. Thus, the fork containing b must also contain a or c. However, the directions of a and c are not compatible with the directions of edges in any fork block. Therefore, b can not be a part of a fork.

2. Suppose b comes from a square block. Since at least one endpoint of any edge in a square block is black, none of the edges can be annihilated. Thus, the degree of any corner node is 3, and the central node has degree at least 4. Since the degree of node o is 3, it can only be one of the corner node in the square. Moreover, since nodes x and p are not connected, they must both corner nodes on the same diagonal. Therefore, node y must be the central node. Hence nodes y and p must be connected, which is a contradiction. Hence b can not come from a square.

3. Suppose that b comes from a diamond. If the diamond does not contain c or d, then it is necessary to glue d and c together. Since the only white nodes are the endpoints of mid-edge, b must be the mid-edge of the diamond. Suppose the diamond does not contain c. The edges a, d must be both contained in the diamond since the degree of node o is 3. Hence nodes x, p must be connected, which is a contradiction. Suppose the diamond does not contain d, then after gluing d to node o, the degree of o must be at least 4. This again leads to a contradiction. So the diamond must contain d and c. The directions of b, c suggest that both b, c are in the upper or lower triangle of the diamond. This forces d to be contained in the diamond. Notice that node x is not connected with p, the other half of the diamond is annihilated. This is again a contradiction. So the diamond can not contain c. To conclude, b is not contained in a diamond block.

4. Suppose b comes from a spike, a, d must come from the same block. Thus, they form a fork. Then, c can not be attached. This proves the claim.

Now, the only option is that b comes from a triangular block △1. If a also comes from △1, the third edge in △1 should be annihilated by another edge, denoted by e. Moreover, both c and e must be obtained from the same block. Taking into account direction of edges, this block must be a triangle △2. Note the third edge of △2 is annihilated by an edge f incident to node y, so f and d must come from the same block. Again, considering the directions of edges, this block must also be a triangle △3. Therefore, the third edge of △3 must be a. This contradicts to the assumption that a is an edge of block △1. Therefore a is not contained in △1 and this triangle is formed by b, c, d. This forces a to be a spike block.

Remark 3. After the original neighborhood is replaced by the one in Figure 8, assume the new graph is not decomposable. This means that if the lower triangle and spike described in Lemma 1 are removed, the rest is not decomposable. Therefore, in the original graph, after the original neighborhood of o is removed, the graph is undecomposable. However, in this case, the neighborhood of o can only be obtained from gluing a square and a diamond. Hence the original graph is non-decomposable. This proves that the replacement is reversible. Moreover, all decomposition of the original graph are in 1-1 correspondence with decomposition of the new one.

2.3 Nodes of Degree 6

If o is a node in G of degree 6, there are three cases:
1. The neighborhood of $o$ comes from a triangle and a square block. (Figure 9) Then replace it by the one in Figure 11. Lemma 1 shows that this replacement is reversible.

2. The neighborhood of $o$ comes from a fork block (infork or outfork) and a square block. (Figure 10). Then the neighborhood is a disjoint connected component, otherwise the graph is undecomposable.

3. The neighborhood of $o$ comes from two diamonds, as illustrated in Figure 12. Note that in Figure 12A, if node $p$ and node $q$ are glued together, the result, if decomposable, is a disjoint connected component, see Figure 13A. Otherwise, $G$ is undecomposable. If $p$, $q$ are not glued together, the neighborhood is then replaced by the graph in Figure 14. Lemma 1 shows that this replacement is reversible. On the other hand, gluing nodes $p$, $q$ together in Figure 12B, the mid-edges are annihilated, as seen in Figure 13B. Hence, the degree of $o$ must be 4. This contradicts the fact that node $o$ has degree 6. In this case, the neighborhood is replaced by Figure 14.

**Corollary 1.** Figure 11 can only be decomposed as a triangle plus a spike plus a triangle.

**Proof.** By Lemma 1, the lower part of Figure 11 can only be obtained from gluing a spike and a triangle. For the upper part, the two edges incident to $o$ must come from the same block. Judging by their directions, the block can only be a triangle. Note that the third edge of this triangle may be annihilated, as indicated by dashed line in Figure 11.

**Remark 4.** By the similar argument as in Remark 3, this replacement is reversible.
If node $o$ has degree 5, there are three cases:

1. The neighborhood of $o$ comes from a spike and a square, see Figure 15. In this case, we replace it with the neighborhood in Figure 8. According to Lemma 1, the replacement is reversible.

2. The neighborhood of $o$ comes from a fork and a diamond. See Figure 16. Note that the direction of the fork and the diamond can change, so there are 4 subcases. In all of these cases replace the neighborhoods by the one in Figure 18. The replacement is reversible due to Lemma 1 and Remark 3.

3. The neighborhood of $o$ comes from a triangle and a diamond. See Figure 17. Similarly, note that the orientation of both the triangle and the diamond can also be reversed, there are 4 possible neighborhoods in this case. Up to direction reversions, the neighborhood is replaced by the one in Figure 19. Lemma 1 and Corollary 1 ensure that this replacement is reversible.

### 3 Simplification on Nodes of Degree 4

After simplification in the previous section, assume now that all nodes in graph $G$ have degrees at most 4. In this section we shall denote the node in consideration by $o$ and the nodes connected to it. we call them boundary nodes. Note that taking into account directions of edges incident to $o$ we can distinguish the following three cases:

- **A**: 4 outward edges or 4 inward edges.
- **B**: 3 outward edges + 1 inward edge or 3 inward edges + 1 outward edge.
- **C**: 2 outward edges + 2 inward edges.

We shall consider all the situations above cases by case.
3.1 4 outward edges or 4 inward edges.

Without loss of generality, assume that there are 4 edges directed outwards. If the graph is decomposable, there is only one case. Namely, the neighborhood is obtained by the gluing of two forks as below.

3.2 Three outward edges + one inward edge or three inward edges + one outward edge

Without loss of generality, assume that o is incident to three outward edges and one inward edge.

Assume that the only distinct from o node p that is incident to the incoming edge has degree at least two.

1. The inward edge can not be obtained from a fork. To show this, we use contradiction, suppose it’s contained in a fork block, o must be the white node in this block. Therefore, the other inward edge must be incident to o and can not be annihilated. This contradicts the fact that there is only one inward edge incident to o. Note that this argument is still true even if p has degree one.

2. Suppose the inward edge comes from a square. Since the degree of o is 4, it must be the center of the square and all four edges are contained in the same square. This is impossible since none of the edges in a square can be annihilated and the central node of a square is incident to at least two inward edges and two outward edges.

3. Assume the inward edge is a part of a triangular block. Suppose this triangle does not contain any of the remaining three outward edges. Then the other edge of the triangle which is incident to o is
annihilated by another edge, denoted as $e$. In this case, $e$ and the remaining three outward edges must come from the same block. It can only be a square with central node $o$. On the other hand, $o$ is incident to three outward edges and one inward edge. This is a contradiction. Therefore, the triangle must contain one of the outward edges. This forces the remaining two outward edges to be in the same block. This block must be a fork. See Figure 21.

4. Assume that the inward edge comes from a diamond. Node $o$ must be a white node in the diamond. Judging by the directions of the remaining edges, two of them must be boundary edges of the same diamond, see Figure 22.

5. Suppose that the inward edge comes from a spike, then the remaining three outward edges come from the same block. However there is no block that contains three outward edges incident to the same node. Hence in this case, the graph is undecomposable.

For Figure 21, replace the neighborhood with the one in Figure 11. According to Corollary 1 and Remark 3 this replacement is reversible. Denote the new graph as $G'$ and the original graph as $G$. $G'$ has one less node of degree 4. For Figure 22, lemmas 2 and 3 show that it has a reversible replacement.

**Lemma 2.** If $y, w$ are not connected by an edge, Figure 24 has only one possible decomposition, shown in Figure 25.

**Proof.** Consider edge $a$. We claim that it comes from a spike block. We only need to rule out all other possibilities.

Suppose $a$ comes from a fork, the other edge of the same fork can not be annihilated since it has a black endpoint. Hence it must be edge $b$ or $c$. Assume it is $b$, then degree of $b$ must be one. This is a contradiction.

Suppose $a$ comes from a square. Since the degree of $o$ is 4, $o$ must be the center of the square, which means edges $b, c, f$ are contained in the same square block. This is a contradiction since $o$ must be incident to at least two outward edges and two inward edges.

Suppose $a$ is contained in a diamond. The degree of node $o$ suggests that $o$ is a white node in the diamond block containing $a$. Since the boundary edges of a diamond can not be annihilated, two of $a, b, c, f$ must be boundary edges. Judging by the directions, the boundary edges can only be $\{a, b\}$, $\{a, c\}$ or $\{b, c\}$. If $\{a, b\}$ are two boundary edges, then $d$ must be contained in the same diamond. This means node $w$ must be connected to node $y$. This contradicts our assumption. The situation is similar if $\{a, c\}$ are two boundary edges.
edges. If \(\{b, c\}\) are two boundary edges of the diamond. Then \(a\) is the mid-edge of the same diamond block. Therefore nodes \(p, q\) must be connected with node \(w\), and they must be black. However edge \(d, e\) are incident to them. This is a contradiction.

Suppose \(a\) comes from a triangular block. If this triangle does not contain edge \(f\), the other edge of the same triangle which is incident to \(o\) must be annihilated by another edge, denoted as \(h\). So \(b, c, f, h\) come from the same block. It must be a square. However, none of the edges in a square can be annihilated. This contradicts to the fact that \(h\) is annihilated. If the triangle contains \(f\), then \(b, c\) must come from the same block. It must be a fork or a diamond. If it is the latter, the mid-edge must be annihilated. But \(o\) is already a black node once the triangle and diamond are glued together. Contradiction. If it is a fork, degree \(p\) must be 1. This is again a contradiction.

To sum up, \(a\) must be a single spike.

Now \(b, c, f\) come from the same block. This forces the block to be a diamond.

**Lemma 3.** If \(w\) is connected to \(y\) in Figure 24, then the decomposable graph must have a disjoint connected components shown in Figure 27.

Proof. According to the previous lemma, there are two possibilities. Either \(b, c, d, e, f\) form a diamond and \(a, h\) come from two spikes, or, \(a, b, d, f, h\) form a diamond and \(e, c\) come from two spikes. In either case, the neighborhood is a disjoint connected component. Figure 27 illustrates the first case. To see the second case, one only needs to change the labeling of the edges in Figure 27.

**Remark 5.** The replacement of Figure 25 ↔ Figure 26 is reversible.

Assume now the node incident to the edge directed inwards has degree one.

1. Suppose the inward edge comes from a spike. We show that the remaining three edges can not come from one block, and this contradicts decomposability. Indeed, there is no block that contains node of degree 3 that is with three outward edges.

2. Suppose the inward edge comes from a fork. Since degree of \(o\) is 4, it must be the white node in the fork. Hence one of the remaining edges is contained in the same fork. However their directions are inconsistent with a fork. This is a contradiction.

3. The inward edge can not be obtained from a diamond since every node in a diamond has degree at least 2.

4. The same argument shows that the inward edge doesn’t come from a square.

5. If the inward edge is obtained from a triangle, then by arguments as in Lemma 2, the triangle must contain one of the remaining outward edges. The only possible decomposition is shown in Figure 28. The dashed edge can only be annihilated by a spike. Otherwise, the degree of the node will be greater than one. In this case, the neighborhood is a disjoint connected component.
3.3 Two outward edges + two inward edges

Here we will distinguish cases by the number of boundary nodes of degree at least 2. Denote the number of such nodes by \( n \). For example, if \( n = 0 \), it is a 4-star.

3.3.1 \( n=0 \)

The neighborhood can be constructed from gluing two forks, as shown in Figure 28. Also, it can be constructed from gluing two triangles, each has one edge annihilated. It must be a disjoint connected component, otherwise, \( G \) is undecomposable.

3.3.2 \( n=1 \)

Without loss of generality, assume the node 1 incident to an outward edge has degree at least 2. (Figure 29).

1. Edge \( a \) does not come from a fork since the degrees of both nodes 1 and \( o \) are at least 2.

2. Suppose \( a \) comes from a diamond. Since degrees of nodes 2,3,4 are all 1, they cannot be contained in the same diamond. So node \( o \) is a white node of the diamond before attaching edge \( b,c,d \). Hence at least one boundary edge in the diamond must be annihilated, which is impossible.

3. Suppose \( a \) comes from a square. If \( o \) is the central node of the square, edges \( b,c,d \) must be contained in the same square. Hence the remaining 4 nodes must be corner nodes. Thus, they all have degree 3. This is a contradiction since only node 1 has degree more than 1. So \( o \) is a corner node of the square. But then the degree of node \( o \) must be 3. This contradicts the fact that degree of \( o \) is 4.

4. If \( a \) comes from a spike block, \( b,c,d \) must come from the same block, which must be a diamond. Hence, edge \( d \) is the mid-edge. However the degree of node 3 is 1, this is impossible since no boundary edge in a diamond can be annihilated.

5. Assume that \( a \) comes from a triangle \( \triangle \). If the other edge of \( \triangle \) incident to \( o \) is not \( b \) or \( c \), that edge must be annihilated by another one denoted as \( e \), as shown in Figure 30. Thus, \( b,c,d,e \) come from the same block. It can only be a square. But the degrees of nodes 2,3,4 are all 1, which is impossible for nodes in a square block. So either \( b \) or \( c \) is contained in the same triangle. Assume that it is \( b \). Notice that node 2 has degree 1. So the edge in \( \triangle \) that connects node 1 and 2 is annihilated by another edge, denoted as \( f \). If \( f \) comes from a spike, the degree of node 1 must be 1 after gluing. This is a contradiction. If \( f \) comes from a triangle or a diamond, the degree of node 2 has degree at least 2 after gluing. This is also a contradiction.

To conclude, when \( n = 1 \), the graph is undecomposable.

3.3.3 \( n=2 \)

In this case, only two boundary nodes have degree at least 2.
Case 1. Assume that the edges incident to the boundary nodes of degree at least 2 have the same direction. Without loss of generality we assume that both are directed outwards. (nodes 1 and 4 in Figure 30 have degree at least 2.) First, suppose either a or d is a single spike, the remaining three edges must come from the same blocks, which can only be a diamond or a square. However the degrees of nodes 2,3 are both 1. This is impossible. Second, neither of the edges a or d can be obtained from a fork since both of its two endpoints have degree at least 2. Third, Suppose a comes from a diamond. Then b, c must also be contained in the diamond. In this case, nodes 1 and 2 must be connected. This means the degree of node 2 is at least 2, which leads to a contradiction. Next, suppose a or d comes from a square, then all four edges must be contained in the same square. However, the degrees of node 2,3 are both one. This is again a contradiction. Last of all, assume a, b come from the same triangular block and c, d come from another triangular block. Since node 2 has degree 1, the third edge in the triangle containing edges a, b is annihilated, as discussed in the case when n = 2, this is a contradiction. So in this case, the graph is not decomposable.

Case 2. o is connected to the boundary nodes of degree at least 2 by two edges. Denote the edge directed inwards by a and the one directed outwards by b. (Figure 31).

1. For the same reason as in Case 1, neither a nor b comes from a spike block.

2. Neither a nor b comes from a fork since both endpoints have degrees at least 2.

3. Suppose a comes from a diamond. Since the degree of o is 4, it must be a white node in the diamond. Since no boundary edge in a diamond can be annihilated, b, c must be boundary edges in the same diamond. Then, nodes 1 and 3 must be connected. But degree of node 3 is 1 and the boundary edge can not be annihilated. This is a contradiction.

4. Suppose a comes from a square block. Since degree of o is 4, it must be the central node of the square. Since none of the edges in a square can be annihilated, a, b, c, d must all be contained in the same square. But the degrees of node 3,4 are one. Contradiction.

5. Assume a comes from a triangle △. Suppose this triangle does not contain b or c. The other edge in this triangle that is incident to o must be annihilated by an edge, denoted by e. Hence edges b, c, d, e must be contained in a square block. However none of the edges in a square block can be annihilated. Therefore, the triangle must contain either b or c. Using similar arguments as in section 3.3.2, c is not contained in △. So a, b are contained in △. Then we replace the neighborhood with the one in Figure 33. The replacement operation is reversible by Corollary 1.

3.3.4 n=3

Without loss of generality, assume node 1 is incident to the edge directed outwards (denoted as a), and it has degree one, see Figure 33.
1. Suppose that \( a \) comes from a single spike. The remaining edges \( b, c, d \) must come from the same block. The only possible situation is that they come from a diamond (Figure 36). Since \( \text{deg}(1) = 1 \), nodes 1, 4 are not connected. Lemma 34 below shows that this neighborhood can be replaced by the one in Figure 34. This replacement is reversible according to Lemma 1.

2. Suppose that \( a \) comes from a fork. Since \( \text{deg}(o) = 4 \), \( o \) must be the white node in the fork. Then \( d \) is also contained in the same fork. Hence, node 4 must have degree 1. This contradicts the fact that the degree of node 4 at least 2. So \( a \) does not come from a fork.

3. Assume \( a \) comes from a triangle. According to the argument in section 3.2, this triangle must contain edge \( b \) or \( c \). Assume that the triangle contains \( a, b \). Since the degree of node 1 is one and the degree of node 2 is at least two, we obtain a contradiction by arguments from section 3.3.2.

4. Suppose that \( a \) comes from a diamond, then the degree of node 1 must be at least 2, which is a contradiction.

5. Suppose that \( a \) comes from a square block. This block must also contain edges \( b, c, d \) since none of the edges in a square can be annihilated. Moreover, \( o \) is the central block of the square. This means node 1 must have degree 3. This is a contradiction.

### 3.3.5 n=4

In this section, we assume all four boundary nodes have degree at least 2. (Figure 37). We focus our discussion on edge \( a \). By the symmetry of the neighborhood, we can carry the same argument to any of edges \( b, c, d \).

1. Edge \( a \) does not come from a fork since both of its endpoints have degrees at least 2.

2. Assume that \( a \) comes from a triangle. Similar to argument in section 3.2, the triangle must contain \( b \) or \( c \). Assume \( b \) is contained in this triangle. Then \( c \) and \( d \) must come from the same block.
• If this block is a diamond, judging by their directions, one of edges $c, d$ (assume it is $d$) must be the mid-edge. Thus, besides $c$, there is another boundary edge incident to $o$ that comes from the same diamond. Hence, the degree of $o$ is at least 5. This contradicts our assumption.

• Assume $c, d$ come from a triangle, as shown in Figure 38. We replace the neighborhood by the one in Figure 39.

Remark 6. Notice that in order to perform replacement, it is necessary to determine whether $a, b$ or $a, c$ are in the same triangle. This will be discussed later.

3. Suppose that $a$ comes from a diamond. Since the degree of $o$ is 4, it must be a white node of the diamond. Judging by the directions of edges, there are three cases.

- $a$ is the mid-edge and $b, c$ are the boundary edges. In this case, we obtain a neighborhood as shown in Figure 41. We will discuss it later in this section.
- $a, d$ are the boundary edges and $b$ or $c$ is the mid-edge. We get the neighborhood shown in Figure 42.
- $a, d$ are the boundary edges, and the mid-edge is annihilated by another edge $e$. So $b, c, e$ come from the same block. It must be a diamond with mid-edge $e$, see Figure 40. In this case, the neighborhood is a disjoint connected component.

4. If $a$ comes from a spike, $b, c, d$ must come from the same block. Hence, this block must be a diamond, see Figure 36.

5. Suppose $a$ comes from a square, then $a, b, c, f$ must all be contained in the same square. Thus, the neighborhood is the square itself. Since the degree of $o$ is 4, the neighborhood is a disjoint connected component.
Note that Figure 41, 42, 36 represent the same neighborhood except for edge labeling. For the sake of convenience, we relabel the edges as in Figure 43. Note that the degree of node 1 is at least 2. If 1 is not connected to 4, the only possible decomposition is the one shown in Figure 45 (See Lemma 4). We apply the replacement as in Figure 26. If nodes 1, 4 are connected by an edge directed from 1 to 4, Lemma 5 shows that there exists a decomposition as in Figure 46. Thus, we can apply the replacement as in Figure 48. The following lemmas show that our choices of replacements for the neighborhood in Figure 43 are reversible.

**Lemma 4.** In Figure 43, assume 4 is neither connected to 1 nor coincide with 1. Nodes 1, 2, nodes 1, 3 are disconnected. If the graph $G$ is decomposable, then the neighborhood of o can be decomposed as in Figure 45.

**Proof.**

1. Suppose that a comes from a fork. Since the degree of o is 4, o must be the white node in the fork. Thus, $f$ is contained in the same fork. Then node 4 must have degree 1. This is a contradiction.

2. Suppose that a comes from a triangle, denoted as $\triangle$. Then there are two cases:
   - **Case 1:** $\triangle$ contains neither $b$ nor $c$;
   - **Case 2:** $\triangle$ contains either $b$ or $c$

   In case 1, consider node o in Figure 43. The other edge in $\triangle$ that is incident to o is annihilated by another edge, denoted as $e$. Hence $b, c, f, e$ come from the same block, which can only be a square block. However, none of the edges in a square can be annihilated. Therefore, case 1 is impossible. In case 2, assume that $\triangle$ contains $b$. The third edge in $\triangle$ is annihilated by another edge, denoted as $\lambda$. (See Figure 44). $\lambda$ and $d$ must come from the same block. It can only be a diamond or a triangle. If it is a diamond, $\lambda$ must be the mid-edge. So node 4 is black. However, edges $f, e$ need to be glued to 4. This is a contradiction. So both $b, \lambda$ belong to a triangle. Since 4 is not connected to 1, the edge $\lambda$ in this triangle must be annihilated by another edge $h$. So $h, f, e$ come from the same block, which must be a diamond and $h$ is the mid-edge. This means that nodes o and 1 are connected by a boundary edge of this diamond. Thus, degree of o is at least 5. This contradicts to the assumption that $\deg(o)=4$.

3. Suppose that a comes from a diamond. Since the degree of o is 4, it must be a white node in the diamond. Since the boundary edges can not be annihilated. Judging by the directions of the edges, there are only two possible cases:
   - $a$ is the mid-edge and $b, c$ are two boundary edges of the diamond.
• $a, f$ are the boundary edges and one of $b, c$ is mid-edge.

In either cases, $1,2$ must be connected by a boundary edge and it can not be annihilated. This is a contradiction.

4. Suppose that $a$ comes from a square, then $a, b, c, f$ must all be contained in the same square. Thus, the neighborhood is the square. Moreover, nodes $1,2$ must be connected. This is a contradiction.

5. Suppose edge $a$ comes from a spike. Then $b, c, f$ come from the same block. This forces the block to be a diamond. See Figure 45

Lemma 5. In Figure 43, assume that $4$ is connected to $1$ by an edge directed from $1$ to $4$. If the graph is decomposable, nodes $1,2$ and node $1,3$ are disconnected. Then the degree of $4$ is $4$ and the degree of $1$ is $2$.

Proof. The argument differs from the previous one only in the place when $a$ is assumed to come from a triangle. Notice that $4$ is connected to $1$. If $a, b$ comes from a triangular block $\triangle$, the edge $41$ must come from another block. This block can be a triangle or a diamond. Thus, $e, f$ must both come from the other block, which can not be a diamond since this will force the degree of node $4$ to be $5$. Recall that we already simplified all nodes of decomposable graph so that the degree of any node does not exceed $4$. Thus, this block containing $e, f$ must be a triangle. The corresponding decomposition is shown in Figure 46. In this case, if degree of node $3$ is at least $3$, there is another edge incident to it. The neighborhood can be replaced by the one in Figure 48. It is trivial that if $G$ is decomposable, so is $G'$. The converse statement follows from Lemma 1.

Remark 7. Note that we have found all possible decomposition of the neighborhoods of nodes with degree $4$. Except some cases when the neighborhood is a disjoint connected component, we want to identify which decompositions the considered neighborhood can have. To be more specific, to determine decomposition of the neighborhood of a node $o$ we want to use only the information that can be directly derived from the graph:

• How is node $o$ connected to the boundary nodes? We want to check the direction of the edges connecting node $o$ and its boundary nodes.
• How are the boundary nodes connected to each other? We want to check if and how some of the boundary nodes are connected to each other.

• If necessary, we want to check if there is any other node that is connected to the boundary nodes, and how are they connected.

This method will be discussed in detail in the next section.

Remark 8. If node 4 coincides with node 1, we have a neighborhood as in Figure 49. In this case, we need to examine nodes p, q.

• If both nodes have degree two, then there are two possible decomposition. Namely, a diamond plus a spike or two triangles.

• If at least one of p, q has degree more than two, then it must come from gluing two triangles.

Figure 49 shows the decomposable neighborhood. All cases other than the above two give an undecomposable graph.

4 Identification when n=4

In the previous section, we have found all possible neighborhoods and possible decomposition of node of degree 4. In order to perform proper replacement, we need to identify the neighborhood by examining the boundary nodes of node o. For example, in the situation when the neighborhood may come from two triangles, in order to choose proper replacement, we must determine whether a, b or a, c are in the same triangle. Also, in some other cases, we need to determine which one of the four edges comes from a spike, the remaining three edges then come from a diamond.

To determine decomposition, we must consider connectivity between the boundary nodes. First of all, we need to consider decompositions depending on how nodes 3, 4 are connected to node 1.

4.1 Node 1 is Connected to Node 4 and 3

Assume nodes 1, 4 are connected by an edge denoted by λ and nodes 1, 3 are connected by an edge denoted by γ. Let us consider directions of a, d, λ and a, c, γ. More exactly, we check if λ is directed from node 1 to node 4 and if γ is directed from node 1 to 3. (See Figure 50)

Suppose λ is directed from node 2 to node 1 and γ is directed from node 3 to 1, then neither a, b, λ nor a, c, γ come from a triangle. Assume a comes from a spike, then b, c, d come from the same block which must be a diamond. Hence, nodes 2, 4 must be connected and node 2 is a black node before λ is attached. In this case,
node 1 must coincide with node 4. But the directions of $\lambda, \gamma$ are prescribed by the decomposition. Hence, the graph is undecomposable. If $a$ comes from a triangle, the triangle must also contain $b, c, \lambda, \gamma$. Again, their directions do not fit in a diamond block. To conclude, if $a, b, \lambda$ or $a, c, \gamma$ cannot form a triangular block, the graph is undecomposable.

Suppose $a, b, \lambda$ have the same direction setup as a triangular block, and $a, c, \gamma$ don’t. We claim that if the graph is decomposable, then $a, b, \lambda$ must come from a triangular block. Suppose the contrary. Notice that $a, c, \gamma$ do not come from a triangular block. Edge $a$ comes either from a spike or from a diamond. In the first case, $b, c, d$ come from the same block which must be a diamond, and node 2 is connected only to nodes 4 and $o$. Since node 2 is connected to 1, node 1 must coincide with node 4. But then the direction of $\gamma$ does not match the direction of the corresponding edge in a diamond. This is a contradiction. If $a$ comes from a diamond block, the block can contain $b, c, \lambda, \gamma$ or $b, d, \lambda$ or $c, d, \gamma$. But none of this cases has the directions that match with a diamond block. This again leads to a contradiction.

Suppose $\lambda = \overrightarrow{12}$ and $\gamma = \overrightarrow{13}$. If $a$ comes from a spike, then $\lambda, \gamma$ come from the same block. This block can be a fork or a diamond. But the former is impossible since the degree of node 2 is 2. If it is the latter, $b, c$ must be boundary edges of this diamond. Thus, the mid-edge must connect node 1 and $o$. This forces node 1 to coincide with node 4, as shown in Figure 49. Suppose that $a$ come from a triangle. There are two possibilities. First, $a, b, d$ come from the same triangle. Second, $a, b, c$ come from the same diamond. If it is the former, node 1 is already black before $\gamma$ is glued. This is impossible. Suppose it is the latter. Notice that node 2, 3 and 3, 4 are disconnected unless nodes 1 coincide with node 4.

**Lemma 6.** Suppose there is an edge directed from node 1 to node 2 and an edge directed from node 1 to node 3, nodes 2, 3 and node 3, 4 are disconnected, both nodes 3 and 4 have degree 2. If the graph $G$ is decomposable, then there is a decomposition of $G$ in which $a, b, c, \lambda, \gamma$ come from the same diamond.

**Proof.** Suppose that it is false.

1. If $a$ comes from a spike, then $b, c, d$ come from the same block which must be a diamond. Thus, node 3 is a black node of the diamond and $\gamma$ cannot be attached. Contradiction.

2. If $a$ comes from a triangle, the block could contain either edges $b, \lambda$ or edges $c, \gamma$.

   In the former case, edges $c, d$ come from the same triangle $\Delta$. Since nodes 3, 4 are disconnected the third edge of $\Delta$ is annihilated by another edge, denoted as $\tau$. Hence $\tau, \gamma$ come from the same triangle, and node 1 is connected to 4. If $G$ is decomposable, the neighborhood is as in Figure 51. Moreover, degree of node 1 is 4 and degree of node 4 is 2. Notice that the degree of node 2 is 2, the neighborhood is a disjoint connected component. In this case, it can also be decomposed as a diamond containing $a, b, c, \lambda, \gamma$ plus two spikes. In the latter case, the triangle which contains $a$ also contains $c, \gamma$. By the same argument as above, if $G$ is decomposable, the neighborhood of $o$ is as in Figure 51. In this case, it’s a disjoint connected component, and it can be obtained by gluing two spikes $\overrightarrow{14}, d$ to a diamond that contains $a, b, c, \lambda, \gamma$. 17
3. Suppose $a$ comes from a diamond. According to the assumption, $a, b, c, \lambda, \gamma$ do not come from the same diamond. Therefore, the diamond containing $a$ must contain $b, d$ with $b$ as its mid-edge. Then node 1 is black and $\gamma$ cannot be attached. Contradiction.

4. If $a$ comes from a square, it must contain $b, c, d, \lambda, \gamma$. Moreover, nodes 2,4 and 3,4 must be connected. This contradicts our assumption.

To conclude, under the given assumption, $a, b, c, \lambda, \gamma$ come from the same diamond in one of the decomposition of $G$.

**Remark 9.** If nodes 2,3 and nodes 3,4 are connected, and degrees of nodes 2,3 are both two, node 4 must coincide with node 1. In this case, the neighborhood is shown in the Figure 49.

**Remark 10.** In the above situation, if $G$ is decomposable, we may have more than one decompositions. However, according to the proof of the lemma, there are more than one decomposition only when the neighborhood is a disjoint connected component. We list all such disjoint connected components in Figure 82. If the whole graph coincides with such a disjoint component from this list we know already all the possible decompositions and we don’t need to do simplifying replacements. On the other hand, if a decomposable graph does not coincide with any of the graphs in Figure 82 then the decomposition is unique.

**Remark 11.** The lemma above explains that by examining the connectivity of nodes 2,3 and nodes 3,4, we can tell if $a$ comes from a diamond. Moreover, if we can rule out the possibility that $a$ comes from a diamond and node 1 $\neq$ 4, we can furthermore check if the neighborhood comes from a square. In the following argument, assume we already rule out the possibility that $a, b, c, d$ comes from square, diamond or spike. This means, if the graph is decomposable, edges $a, b, c, d$ must come from two triangles.

In order to determine if $a, b$ or $a, c$ come from the same triangle, it is necessary to examine node 4.

Assume node 4 is connected to both nodes 3 and 2, see Figure 52. If nodes 3,4 are connected by an edge directed from node 3 to nodes 4, relabel the indices of nodes as the following: $4 \rightarrow 1, 3 \rightarrow 2, 2 \rightarrow 3$ and $1 \rightarrow 4$. Then apply the previous argument. It’s similar if nodes 2,4 are connected by an edge directed from node 2 to nodes 4. Hence, without loss of generality, assume edge 24 is directed from node 4 to 2 and edge 34 is directed from node 4 to 3. If there is no node (except for node o) which is connected to any of the nodes 1,2,3,4, then it is a disjoint connected component.

Suppose we can find a node $x \neq o$ that is connected to some of the nodes 1,2,3,4. We check if there is any node among 1,2,3,4 that is connected to $x$. Assume $x$ is connected to only one of 1,2,3,4. Without loss of generality, assume $x$ is connected to 1 by an edge denoted as $\tau$. (Notice that $x$ may be connected to nodes in the graph other than 1,2,3,4). In this case, if edges $a, b$ come from the same triangle, then edges
Figure 52:

c, d come from another triangle, denoted as $\triangle_1$. Moreover, $\tau, \gamma$ come from the same block, which must be a triangle $\triangle_2$. Because $x$ is only connected to one of nodes $1,2,3,4$, the third edge of $\triangle_2$ is annihilated by another edge, denoted as $\eta$. However, nothing can be attached to the node 3 after gluing $\triangle_2$ to $\triangle_1$. Contradiction. Hence, edges $a, c$ come from the same triangle and $b, d$ come from the same triangle. Therefore, edges $\tau$ and $\lambda$ come from the same block. This is impossible.

Assume that there is a node $x \neq o$ that is connected to only two nodes of $1,2,3,4$. Notice that $x$ may be connected to nodes other than $1,2,3,4$. Up to a relabeling of indices, there are two possible situations: Either $x$ is connected to nodes 1,4 or nodes 1,3.

First, suppose $x$ is connected to nodes 1,4 by edges $\tau, \eta$ respectively. If $a$ comes from a spike, then $\tau, \lambda, \gamma$ come from the same block that is a diamond. Judging by the directions of $\lambda, \gamma, \tau$ must be the mid-edge. Therefore, node $x$ must coincide with node $o$. This contradicts our assumption. Suppose $a$ comes from a diamond. Since the degrees of node 1 and $o$ are at least 4, $a$ must be the mid-edge. Therefore, the diamond must contain $\lambda, \gamma, b, d$. Moreover, the degrees of node 2,3 must be two. This is a contradiction.

We can also rule out the possibility that $a$ comes from a square since both its endpoints have degree 4. To conclude, $a$ must come from a triangle. Otherwise, $G$ is undecomposable.

Suppose $a, \tau$ come from the same triangle, then the third edge of this block is annihilated by another edge, denoted as $\delta$. Hence $\delta, \eta, b, c, d$ come from the same block. This is impossible. If $a, b$ come from the same triangle, then $c, d$ come from another triangle, denoted as $\triangle_1$. Thus, $\tau, \gamma$ come from the same block, which must be a triangle. Denote it as $\triangle_2$. Notice that its third edge is annihilated since $x$ is connected only to two of nodes $1,2,3,4$. But this is impossible since node 3 is already black after gluing $\triangle_1$ to $\triangle_2$. The similar argument shows that $a, c$ can not come from the same triangle. Thus, the graph is undecomposable when $x$ is connected only to nodes 1,4.

Assume that $x$ is connected to nodes 1,3 by edges $\tau, \eta$ respectively. If $a, c$ come from the same triangle, then $b, d$ come from another triangle, denoted as $\triangle_1$. Thus, $\tau, \lambda$ come from another block. This block can not be a diamond since $\lambda$ must then be the mid-edge and the boundary edge $\overline{22}$ is annihilated, which is impossible. Hence this block must be a triangle, denoted as $\triangle_2$. Since $x$ is connected only to two of nodes 1,2,3,4, the third edge of $\triangle_2$ is annihilated. However, node 2 is already black after gluing $\triangle_1$ to $\triangle_2$. This is a contradiction. Therefore if $G$ is decomposable, $a, b$ must come from the same triangle and $c, d$ come from another triangle. Apply the corresponding replacement as in Figure 39.

Assume $x$ is connected to at least three of nodes 1,2,3,4. Up to an index relabeling, there are two cases: Either $x$ is connected to nodes 1,2,3 or $x$ is connected to nodes 1,3,4.

Suppose $x$ is connected to nodes 1,3,4 by $\tau, \rho, \eta$ respectively. Assume that $a, b$ come from the same tri-
angle, then \( c, d \) come from one triangle too. Thus, \( \eta, \overline{\lambda} \) come from the same block, which must be a triangle. Thus, node \( x \) and node 2 must be connected according to previous argument. The argument is similar when \( a \) and \( c \) come from the same triangle. In both cases, the neighborhood is a disjoint connected component, see Figure 53. Otherwise, \( G \) is undecomposable.

Assume \( x \) is connected to nodes 1, 2, 3 by \( \tau, \xi, \rho \) respectively. Assume that \( a, b \) come from the same triangle, then \( c, d \) come from another triangle, denoted as \( \triangle_1 \). Therefore, \( \tau, \gamma, \rho \) must come from the same triangular block, denoted as \( \triangle_2 \). Notice that node \( x \) is black after gluing \( \triangle_1 \) to \( \triangle_2 \), node \( x \) and node 4 must be connected by the third edge of \( \triangle_2 \). Similarly, assuming that \( a, c \) come from the same triangle will also result in the same neighborhood. In this case, the neighborhood is a disjoint connected component, see Figure 53. Otherwise, the graph is undecomposable.

Suppose that node 4 is connected only to one of nodes 2, 3. Assume that node 4 is connected to node 3. In this case, if edge \( 34 \) is directed towards node 4, then edges \( c, d, 34 \) can not form a triangular block. This means edges \( b, d \), edges \( a, c \) must come from two triangle, otherwise, the graph is undecomposable. Since by assumption, nodes 2, 4 are not connected, the corresponding edge is annihilated by another edge, denoted as \( \eta \). Hence, \( \eta, \lambda \) come from the same block which must be a triangle. So node 4 is black before attaching edge \( 34 \). This is a contradiction. In this case, the graph is undecomposable.

If edge \( 34 \) is directed towards node 3, there are two possibilities: edges \( c, d, 34 \) form a triangular block or edges \( b, d \) come from a triangle \( \triangle \). Suppose it’s the latter case, the edge of \( \triangle \) that connects nodes 2, 4 is annihilated by another edge \( \tau \). This forces edges \( \tau, 34, \lambda \) to form a block. This is impossible. So edges \( c, d, 34 \) form a triangle. Otherwise, the graph is undecomposable. Hence, edges \( c, d, 34 \) form a triangular block. We apply the similar replacement as in Figure 53.

Assume that nodes 3, 4 and nodes 2, 4 are disconnected. Then one edge of the triangular block that contains edge \( b \) is annihilated by another edge, denoted as \( \tau \). If \( a, b \) come from the same triangle, then \( \tau \) connects nodes 3, 4. Therefore, \( \gamma, \tau \) must form a triangle. This means that nodes 1, 4 must be connected and nodes 2, 3 are disconnected. Similarly, if \( \lambda, \tau \) must form a triangle, then \( \tau \) connects nodes 2, 4. Thus, nodes 2, 3 must be disconnected and nodes 1, 4 are connected. (Figure 54). Notice that in Case A, node 2 may have degree larger than two. And in Case B, node 3 may have degree larger than 2. Thus, it suffices to examine the degrees of nodes 2, 3 to determine whether edges \( a, b \) or edges \( a, c \) come from a triangular block, then apply the corresponding replacement. To be more precise, if degree of node 2 is at least 3, it is Case A; if degree of node 3 is at least 3, it is Case B; if both have degree 2, either decomposition is possible, and the neighborhood is a disjoint connected component.
4.2 Node 1 is Connected to Node 2,Disconnected from Node 3

Assume that nodes 1,2 are connected by edge $\lambda$ but nodes 1,3 are not connected. If $\lambda$ is directed from node 2 to node 1, then $a,b,\lambda$ do not form a triangular block. Moreover, $a$ does not come from a diamond. If $a$ comes from a spike, $b,c,d$ must come from a diamond and node 4 is connected only to nodes 2,3. Since by assumption, nodes 1,2 are connected, node 1 must coincide with node 4. This means nodes 1,3 are connected, and it contradicts our assumption. Therefore, $a$ must come from a triangle that does not contain $b,\lambda$. Hence the block must contain $c$. The third edge in that triangle is annihilated by another edge, denoted as $\tau$. Moreover, $\tau$ is directed from node 3 to node 1. Thus, $\tau,\lambda$ come from the same block. However, there is no such block with such directions. Hence in this case, the graph is undecomposable.

Assume that $\lambda$ is directed from node 1 to node 2.

If $a$ comes from a spike, $b,c,d$ come from the same block, which must be a diamond. Therefore, nodes 1,2 must be disconnected. This means node 1 coincides with nodes 4. Therefore nodes 1,3 are connected. Contradiction.

If $a$ comes from a diamond, there are two cases.

- 1,3 and nodes 3,4 are disconnected. The diamond contains $b,c,\lambda$. Nodes 1,3 must be connected. Contradiction.
- The diamond contains $b,d,\lambda$. Notice that in this case, nodes 2,4 are connected, node

**Lemma 7.** Assume that nodes 1,2 are connected by edge $\lambda$ directed from node 1 to node 2, nodes 1,3 are disconnected, nodes 2,4 are connected by an edge directed from node 4 to 2, the degrees of nodes 1,4 are two.

1. If $G$ is decomposable and nodes 2,3 is disconnected, then $a$ comes from a diamond containing $a,b,d,\lambda$ and $c$ comes from a spike.

2. Assume nodes 2,3 are connected by an edge directed from node 2 to 3:

   (a) If the degree of node 3 is two, then the neighborhood is a disjoint connected component.

   (b) If the degree of node 3 is at least three, then the graph is not decomposable.

**Proof.** 1: Suppose nodes 2,3 are disconnected and the statement is false. It is easy to rule out the possibility that $a$ comes from a square or a fork. If $a$ comes from a spike, $b,c,d$ comes from a diamond and node 2 is black. Thus, $\lambda$ can not be attached unless node 1 coincide with node 4. Thus degree of node 1 coincide with node 4 is 4. This contradicts the assumption that degree of node 1 is 2. Suppose $a$ comes from a diamond. Since the statement is false, the diamond must contain $a,b,c$. Hence node 1 must be connected to node 3. This is a contradiction to our assumption.

If $a,b$ come from a triangle, $c,d$ also come from a triangle denoted by $\triangle$. Thus, the third edge of $\triangle$ is annihilated by another edge, denoted as $\eta$. Hence $\eta,\overline{\triangle}$ come from the same block, which must be a triangle.
Thus, node 2,3 must be connected. Contradiction.

If a, c come from a triangle, then the third edge of this triangle is annihilated by another edge, again denoted as η. Hence η, λ must come from the same block, which must be a triangle. Hence nodes 2,3 must be connected. Contradiction.

2: Assume nodes 2,3 are connected by an edge directed from node 2 to 3. It suffices to check the cases when a comes from a triangle or a diamond. In the first case, by previous argument, all nodes in this neighborhood are already black. This proves (a). In the second case, the degree of node 3 is three. The only possibility to obtain a decomposition is to glue a triangle or diamond on nodes 2,3. In either case, the degree of node 2 is larger than 4. This contradicts the assumption of the section that the degree of any node of G is at most 4. Hence the graph is undecomposable. This proves (b).

Suppose node 1 or node 4 has degree at least three, then a doesn’t come from a diamond. Moreover, we can rule out the possibility that a comes from a square, since this will force node 1 to be connected to node 3 with an edge 1→3. Hence a must come from a triangle. There are two cases. In case A, edges a, b are in the same triangle, thus, edges c, d are in the same triangle. In this case, apply the same replacement as in Figure 49. In case B, edges a, c are in the same triangle denoted as Δ. Thus, edges b, d are in the same triangle. The edge in Δ that connects nodes 1,3 is annihilated by an edge denoted as λ. Thus, γ, λ come from the same block, and it must be a triangle. This means nodes 2,3 are connected. Moreover, in Case B, nodes 2,4 must be connected by an edge directed from node 4 to 2 and nodes 1,2,3 are black. Notice that nothing has been glued to node 4 yet, we use this to identify case B.

**Lemma 8.** Suppose that graph G is decomposable and node 1 is connected to node 2 by an oriented edge 1→2, nodes 1,3 are disconnected:

a Suppose nodes 2,4 and nodes 2,3 are connected. If degree of node 4 is at least 3, then a, c come from one triangle and b, d come from another triangle. If degree of node 1 is at least 3, then a, b come from the same triangle and c, d come from the same triangle. (Figure 49). If the degrees of both nodes 1 and 4 are 2, then the neighborhood is a disjoint connected component.

b If nodes 2,4 or 2,3 are disconnected, then a, b come from one triangle, c, d come from another triangle.

**Proof.** According to the previous argument, it suffices to prove part a. Suppose nodes 2,4 and nodes 2,3 are connected. If a, b come from one triangle, then c, d come from another triangle. Moreover, edges 24, 23 must come from the same block, which must be a triangle. The third edge of this triangle annihilates 34. Therefore, node 4 have degree 2. Thus, if the degree of node 4 is at least 3, a, c must come from one triangle. Otherwise, the graph is undecomposable. The rest of part a follows from the previous argument. □

### 4.3 Node 1 is Disconnected from Nodes 2,3

Assume that neither nodes 1,2 nor nodes 1,3 are connected. Without loss of generality, we can assume that neither nodes 3,4 nor nodes 2,4 are connected. Otherwise, we can relabel the indices of boundary nodes
and apply the previous argument.

Assume that \( a, b \) come from the same triangle. Then the third edge of it is annihilated by another edge, denoted as \( \lambda \). This edge \( \lambda \) can be a part of a spike, a triangle, or the mid-edge of a diamond block. If it comes from a triangle or a diamond, nodes 1, 2 must both be connected to another node \( x \).

Conversely, it is possible to determine whether \( a, b \) or \( a, c \) come from the same triangle by considering nodes connecting some of nodes 1, 2, 3, 4.

Suppose none of nodes 1, 2, 3, 4 is connected to any other node except for \( o \), then the neighborhood is a disjoint connected component.

Assume that nodes 1, 4 or 2, 3 are both connected to the same node \( x \). We assume that vertex \( x \) is distinct from nodes 1, 2, 3, 4, and \( o \). Without loss of generality, assume nodes 1, 4 are connected to \( x \). Denote \( \alpha = 1x \) and \( \beta = 4x \). If \( a, d \) come from the same block, it must be a diamond. Therefore, \( b, c \) come from another diamond. Since degree of \( o \) is 4, the mid-edges of these two diamonds annihilate each other. Then the neighborhood is a disjoint connected component, see Figure 10. If we rule out this case, \( a, d \) must come from two blocks. By the previous argument, neither \( a \) nor \( d \) comes from the a diamond. Thus, they must come from triangular blocks. Moreover, the triangle that contains \( a \) must contain \( c \) or \( b \). Without loss of generality, assume \( c \) is contained in such triangle \( \Delta \). Then the third side of \( \Delta \) is annihilated by another edge, denoted as \( \tau \). Similarly, \( b, d \) come from another triangle \( \Delta_1 \). The third edge of \( \Delta_1 \) is annihilated by an edge denoted as \( \eta \). Then \( \alpha, \tau \) come from the same block, which must be a triangle. If node 3 and \( x \) are not connected, the third edge of the triangle containing \( \alpha, \tau \) must be annihilated. This is impossible since node 3 is already black. Therefore, in this case, the graph is undecomposable. If nodes 3 and \( x \) are connected by an edge denoted as \( \gamma \), then \( \alpha, \tau, \gamma \) form a triangle if their directions match a triangle, otherwise, the graph is undecomposable. Notice that \( \beta, \eta \) must come from the same block. Thus, it must be a triangle if their directions match, otherwise, the graph is undecomposable. In this case, nodes \( x, 2 \) must be connected, and the neighborhood is a disjoint connected component, see Figure 50. In this case, there is an alternative decomposition, see Figure 57.

Suppose \( x \) is connected to nodes \( \{1, 2\} \) (resp. \( \{1, 3\}, \{3, 4\}, \{2, 4\} \)), we claim that \( a, b \) (resp. \( a, c, d, b, d \)) come from one triangle, therefore \( c, d \) (resp. \( b, d, a, b, a, c \)) must come from another triangle.

Assume that nodes 1, 2 are connected to node \( x \) and suppose \( a, b \) do not come from the same triangle in any of the possible decomposition of \( G \). Denote \( \alpha = 1x \), \( \beta = 2x \). Notice that from the previous argument, \( a \) does not come from a spike, a fork, a diamond or a square. So it must be contained in a triangular block \( \Delta \). If \( \Delta \) contains \( a \), then the third edge is annihilated by another edge, denoted as \( \tau \). Hence, \( \tau, b, c, d \) come
from the same block which must be a square, also nodes $o, x$ must be colored white in that block. This is impossible. Thus, $\triangle$ does not contain $\alpha$. So it must contain $c$. Then the third edge must also be annihilated by another edge, again denoted as $\tau$. In this case, $\tau, \alpha$ must come from the same block, which must be a triangle $\triangle_1$. If nodes $3$ and $x$ are connected, we will get a neighborhood similar as the one in Figure 56. As we already know, there is an alternative decomposition in which $a, b$ come from the same triangle. If nodes $3$ and $x$ are not connected, then the third edge of $\triangle_1$ is annihilated by another edge. However, node $3$ is already black after gluing $\triangle_1$. This means the third edge of $\triangle_1$ cannot be annihilated. This is a contradiction. Therefore, $a, b$ come from a same triangle in a decomposition of $G$. Otherwise, $G$ is not decomposable.

Remark 12. If $x$ is connected to nodes $1, 2$ and the graph is decomposable, except for one case when the neighborhood is a disjoint connected component, there is a unique decomposition in which $a, b$ comes from a triangular block and $c, d$ comes from another.

4.4 Summary

All possible neighborhoods of nodes with degree 4 are listed in Figure 58.

5 Simplification on Nodes of Degree 3

Assume the neighborhoods of nodes of degree at least 4 are all simplified, and every node in the graph has degree at most 3. We focus on the nodes of degree 3.

5.1 All edges have the same direction.

Without loss of generality, assume that they are all directed outwards. (Figure 59) Suppose one of them (denoted by $a$) comes from a triangle. Since $\text{deg}(o)=3$, and neither of the rest two edges comes from the same triangle, the incoming edge incident to $o$ in the same triangle must be annihilated. Denote this edge as $e$. Then $e$ must be annihilated by an outward edge from another block. This block must contain the remaining outward edges $b$ and $c$. But there is no such block. This is a contradiction. Hence edge $a$ is not contained in a triangular block. By symmetry, none of the three edges comes from a triangle.

If one of them comes from a fork, one of the remaining two edges must also belong to the same fork. Thus, the third edge must come from a single spike. (Figure 60) Otherwise, the graph is undecomposable. Replace the neighborhood with Figure 61. By Lemma 1, this replacement is reversible.

Remark 13. In order to apply the replacement, we need to identify which two edges come from a fork. This can be done by checking the degrees of boundary nodes. If one of the boundary nodes has degree more than 1, the corresponding edge must come from a spike and the remaining two edges form a fork. If all boundary nodes have degree 1, we have a disjoint connected component and the decomposition is non-unique. If at least two of the boundary nodes have degrees more than 1, the graph is undecomposable.

If one of the edges, denoted by $a$, comes from a diamond, denoted as ♦, then one of remaining, denoted by $b$, must come from the same diamond. Thus the third edge $c$ is not contained in the same block. Since the degree of $o$ is 3, the mid-edge in ♦ must be annihilated by another edge directed away from $o$, denoted as $e$. Thus, $c, e$ come from the same block. Since the degree of $o$ is 3, the block containing $c, e$ must be a triangle. However the directions of these two edges do not match a triangle. This is a contradiction.

To conclude, if all three edges incident to a node are all directed inwards or outwards and the graph is decomposable, the neighborhood must be obtained from gluing a fork and a spike.
5.2 Two outward edges + One inward edge

See Figure 62. Assume edge \( a \) is directed inwards with endpoints nodes \( 1 \) and \( o \). If \( a \) comes from a spike, the remaining two edges must come from a fork. If \( a \) comes from a diamond \( ♦ \), the block must contain at least \( b \) or \( c \) since only the mid-edge can be annihilated in a diamond. Assume \( b \) is contained in this block. The directions of \( a, b \) force \( c \) to be contained in \( ♦ \), as shown in Figure 63. Since all nodes in \( G \) has degree at most 3, this diamond must be a disjoint connected component. Otherwise, \( G \) is undecomposable.

Assume \( a \) comes from a triangle \( △ \), there are two cases:

**Case 1:** \( △ \) contains neither of \( b, c \). Then \( b, c \) must come from a diamond or a fork. In the latter, the remaining edge of \( △ \) that is incident to \( o \) must be annihilated. This forces \( o \) to be a black node even before \( b, c \) are attached. This is a contradiction. Therefore \( b, c \) come from a diamond \( ♦ \). Notice that the mid-edge in \( ♦ \) should be annihilated by an edge of \( △ \), as shown in Figure 64. Simplify the neighborhood by removing the diamond block and leaving the triangle \( △ \) containing \( a \). (See Figure 65)

**Case 2:** \( △ \) contains one of \( b, c \). Without loss of generality, assume it is \( b \). Then \( c \) must come from a spike. Let’s take deeper look at Case 2. Assume that the third edge of \( △ \) is \( d \). There are two possibilities.

\[ \text{Figure 65:} \]

- **a.** Edge \( d \) is annihilated in the graph.
- **b.** Edge \( d \) is not annihilated in the graph. (Figure 66)
Next, start with case **a**. There are three ways to annihilate \( d \).

**Case a1** Edge \( d \) is annihilated by a single spike. See Figure 67. Then replace this neighborhood by the one in Figure 68. By Lemma 1, this replacement is reversible.

\[
\begin{array}{ccc}
\text{Figure 66:} & \text{Figure 67:} & \text{Figure 68:} \\
\end{array}
\]

**Case a2** Edge \( d \) is annihilated by one edge of a triangle. See Figure 69. If \( p \) is connected to \( o \) via edge \( c \), then this graph forms a disjoint connected component. (Figure 70) Otherwise, \( G \) is undecomposable. If \( c \) does not connect \( p \), and there is nothing else connected to \( p \) (\( \text{deg}(p)=2 \)). Then we replace the neighborhood with the one in Figure 68. If \( c \) does not connect \( p \), and \( \text{deg}(p)=3 \), as shown in Figure 71. There are two cases:

- In Figure 71, suppose node 3 coincides with node \( x \), edge \( px \) is directed from \( x \) to \( p \) and \( \text{deg}(1)=2 \), the neighborhood in Figure 71 coincides with Figure 64. In this case, if graph \( G \) is decomposable, the neighborhood is a disjoint connected component.

- Suppose node 3 is not coincide with node \( x \). Then the neighborhood can replaced by Figure 72. It’s similar if edge \( px \) is directed from \( p \) to \( x \). This replacement it reversible by previous lemma.

\[
\begin{array}{ccc}
\text{Figure 69:} & \text{Figure 70:} & \\
\text{Figure 71:} & \text{Figure 72:} \\
\end{array}
\]

**Case a3** Assume \( d \) is annihilated by the mid-edge of a diamond, see Figure 73. Replace the neighborhood by the one in Figure 68 as well.
Next, let’s discuss case **b**. If $d$ is not annihilated, there are three subcases:

**b1** Both nodes 1,2 have degree two. In this case, the neighborhood must come from a spike and a triangle by Lemma \[1\].

**b2** One of the nodes 1,2 has degree two and the other one has degree three. Assume the degree of node 2 is two, and the degree of node 1 is three. In this case, the neighborhood is shown in Figure 74.

**b3** Both nodes 1,2 have degrees three. In this case, we count the number of nodes that are connected to nodes o and 1,2, denoted as $n$.

- $n = 3$, the only possible decomposable situation is Figure 77
- Suppose $n = 2$. One of the exterior nodes is connected to two of nodes o,1,2. Denote this node by $x$. If $x$ is connected to nodes 1,2 (resp. o,1 or o, 2), then the other exterior node is connected to nodes o (resp. node 2 or node 1). In this case, edges $x1, x2$ (resp. $xo, x1$ or $xo, x2$) come from two spikes and degree of $x$ must be two. (See Figure 75)

**Figure 73:**

**Figure 74:**

**Figure 75:**

- $n = 1$. Notice that we assume that the degree of nodes o,1,2 are all three. So there are two cases, as shown in Figure 76. Note that Figure 76A is undecomposable, so the only decomposable neighborhood is Figure 76B.

To sum up:
1. Every node in $G$ has degree at most 3.
2. Consider all nodes of degree 3. If all of them fall into the decomposable categories, then either the neighborhood form disjoint connected component that can be easily decomposed, or we can apply corresponding replacement. If graph $G$ contains any neighborhood (up to a direction reversion on edges) that is unlisted in Figure 78, the graph is not decomposable.

**Remark 14.** We can reverse the directions of all edges to get another 14 neighborhoods in decomposable graph.

**Remark 15.** Note that the degree of node $o$ is not increased in any replacement.

In some of the above cases, the neighborhood of target node $o$ contains some other nodes of degree 3. The algorithm covers the analysis of the neighborhood of these nodes in the following manner:

- For neighborhood 2 in Figure 78 node $x$ has degree 3. The neighborhood of node $x$ is considered in the case derived from reversing the direction of edges in Figure 78. Similarly, the neighborhood of node $p$ in neighborhood 4, 6 and that of node 2 in 9 and 10 is covered by reversing the directions of the edges in corresponding pictures.

- The neighborhood of node $x$ in picture 5 of Figure 78 is covered by the one in picture 4. To be more specific, the neighborhood of $x$ in picture 5 is the neighborhood of $o$ in picture 4.

- Picture 9 is a part of picture 10. Note the replacement for picture 9 is the same as the replacement in 10. Therefore, the order of replacement does not affect the result of the algorithm.

- For node 1 in picture 13, its neighborhood is the same as the one of node $o$ in picture 12. Since we don’t apply any replacement for the neighborhood in picture 12, the order of examining nodes 1 and $o$ won’t affect the result of algorithm. Similarly for node 1 in picture 11.

### 5.3 Identify the Decomposition

In the previous section, we found all possible neighborhoods of nodes $o$ of degree 3 in a decomposable graph. In this section, we want to identify the neighborhood by checking two things:
Figure 78: All decomposable cases for degree 3
• The number of nodes (other than nodes $1,2,3$ and $o$) that are connected to some of nodes $1,2,3$. Denote the number of such nodes by $n$.

• The direction of edges connecting $o$, its boundary nodes and other nodes that are connected to nodes $1,2,3$

If all three edges incident to $o$ have the same direction, the only possible neighborhood in a decomposable graph comes from gluing a fork to a spike. Moreover, $n$ is 0 or 1. Suppose $n = 1$, there is only node that differs from $o$ and is connected to nodes $1,2,3$. Denote it by $x$. If $x$ is connected to node 1 (resp. 2 or 3), then edge $o_1$ (resp. $o_2$ or $o_3$) comes from a spike and edges $o_2, o_3$ (resp. $o_1, o_3$ or $o_1, o_2$) come from a fork.

We focus on the case when there is one edge going towards node $o$ and two edges going away from node $o$. Note that the remaining case is when there is one edge going away from node $o$ and two edges going towards node $o$. The latter case can be analyzed by reversing direction of all edges and using the following argument.

Assume node 1 is incident to the inward edge. Denote edges $o_1, o_2, o_3$ by $a, b, c$ respectively. By Figure 78, $n \leq 3$

Suppose $n = 0$. By previous discussion, if the graph is decomposable, we can only have neighborhoods as in Figure 79. After reversing all directions, we can get another four possible cases.

![Figure 79:]

Next, suppose $n = 1$. Denote this node by $x$

Assume $x$ is connected to all nodes $1,2,3$. There are two cases:

• If deg(1)=3, edges $b, c$ must come from the same diamond and edge $a$ comes from a triangle. See Figure 80 for directions of edges. Note that it’s exactly picture 4 in Figure 78.

• If deg(1)=2, the neighborhood is a disjoint connected component, there are two possible decomposition:
  1: $a$ comes from a triangular block and $b, c$ come from a diamond; 2: $a, b$ come from one triangular block, edges $2p$ and $1p$ come from another triangular block. edges $c$ and $3p$ come from two spikes.

Suppose $x$ is connected to exactly two of nodes $1,2,3$. There are three cases:

1. Suppose $x$ is connected to nodes $1,2$. The neighborhood can only be as picture 4 or 5 in Figure 78. Edges $a, b$ comes from the same triangle. Similarly, if $x$ is connected to nodes $1,3$, edges $a, c$ are in the same triangle.

2. Suppose $x$ is connected to nodes $2,3$. None of the pictures in Figure 78 contains such neighborhood. Hence the graph is undecomposable.
If $x$ is connected to exactly one of the nodes 1,2,3. First, suppose $x$ is connected to node 1 which is an endpoint of inward edge. Note that nodes 2 and 3 can not be connected to node 1. Use argument in the previous section, if the graph is decomposable, we conclude:

- If nodes 1,2 are connected, edges $a, b$ come from the same triangle and edge $c$ comes from a spike.
- If nodes 1,3 are connected, edges $a, c$ come from the same triangle and edge $b$ comes from a spike.
- If nodes 2,3 are both disconnected from node 1, edges $b, c$ come from the same fork and edge $a$ comes from a spike.

Next suppose $x$ is connected to node 2. If the graph is decomposable and node 1,2 are connected, then $a, b$ come from the same triangle. If nodes 1,2 are disconnected, then $a, c$ come from the same triangle. The criterion is similar if $x$ is connected only to node 3.

Next, suppose $n = 2$, denote these two corresponding nodes by $x, y$.

First, check if they are both connected to node 1. If it’s this case, neither node 2 or 3 can be connected to node 1. Moreover, according to the argument in previous section, we have two cases. 1: $a$ comes from a spike and $b, c$ comes from a fork; 2: $a$ comes from a triangle. In the second case, $x, y$ must both be connected to nodes 2 or 3 by edges with compatible directions, and edges $a, b$ (edges $a, c$) are in the same triangle. See Figure 75 picture 5.

If $x, y$ are not both connected to node 1, check if they are both connected to node 2. If so, edges $a, c$ can only be obtained from a triangle and $b$ comes from a spike. Since $n = 2$, there is no node other than $o$ that is connected to node 1 or 3. Therefore, the neighborhood is as the one in Figure 81. Note the neighborhood of $o$ is listed in picture 14 Figure 75. The argument is similar if $x, y$ are both connected to node 3.

Next suppose $x, y$ are not connected to the same node. If the graph is decomposable, there are the

```
following cases:
```
• $x$ is only connected to node 1 and $y$ only connected to node 2. In this case, nodes 1,2 must be connected and $a, b$ come from the same triangle. It’s similar if $x$ is only connected to node 1 and $y$ only connected to node 3.

• $x$ is only connected to nodes 1,2 and $y$ only connected to node 3. In this case, $a, b$ come from the same triangle. If nodes 1,2 are connected, we have neighborhood shown in Figure 78 picture 13. If nodes 1,2 are disconnected, the neighborhood is shown in picture 8 (node $p = x$).

• $x, y$ are connected to nodes 2,3 respectively. In addition, if nodes 1,2 are connected, then edges $a, b$ come from the same triangle. The neighborhood is as shown in picture 11. Similarly, if nodes 1,3 are connected, edges $a, c$ come from the same triangle. Notice that in this case, nodes 2 and 3 can not be connected to node 1 at the same time , neither can they both be disconnected at the same time.

Last of all, suppose $n = 3$. Denote three corresponding nodes by $x, y, z$. According to the argument in previous section, in this case, the graph $G$ is decomposable only if node 1 is connected to node 2 or 3, forming a triangle with the corresponding edges. See Figure 77.

**Theorem.** Assume that every node in $G$ has degree less than or equal to 3. If all nodes of degree 3 fall into the cases listed in Figure 78 (up to a reversion of edge directions), then $G$ is decomposable. Otherwise, $G$ is undecomposable.

**Proof.** Assume that all degree 3 nodes are from Figure 78 and all the necessary replacements have been applied. Except for picture 2 and 6, which don’t require replacement, the replacements for all neighborhoods in Figure 78 contain triangular blocks. Induction will be used based on that.

Apply the corresponding replacement for all graph in Figure 78 and get $G'$. Notice that according to the previous lemma, if $G'$ is decomposable, so is $G$. So besides separated connected components: Graph 2,6, all node of degree 3 in $G'$ are in the form of Figure 66. Remove the triangle as a block and use induction. After finitely many steps, all nodes have degree at most 2. This can be obtained by gluing finite many spikes. □

To conclude, if the graph $G$ is decomposable, we have exhausted all possible neighborhoods of any node of degree at least 3. Any undecomposable neighborhood forces the whole graph to be undecomposable. If none of these undecomposable neighborhoods is contained in the graph, we apply necessary replacement to those of degree 8,7,6,5 and 4 (in this exact order). These replacements reduce the degrees of nodes and simplify the graph. In every step, it is necessary to examine if any undecomposable neighborhood is contained in the new graph. It is possible that after a step of simplification, we obtain several connected components and the same algorithm can be applied to each component. Eventually the graph is reduced to the one with nodes of degree at most 3. The possible neighborhoods of nodes of degree 3 are listed in Section 5. By the last theorem, we can determine if such graph is decomposable. And lemmas are provided to show that all replacements are reversible. Thus, in this case, the original graph is decomposable.

**Remark 16.** In most cases, the decomposition is unique. However, as mentioned in the above argument, some neighborhood has non-unique decomposition. As shown in Figure 82 these neighborhoods are all disjoint connected components. Each of them has finite many possible decompositions. We can reverse the direction of each picture to obtain another 14 neighborhoods with non-unique decompositions.

**Remark 17.** At each stage of simplification, we apply replacements for at most as many as the number of nodes in the graph. Moreover, if the neighborhood of a node needs replacement in the algorithm, after applying the replacement, the degree of the considered node becomes 3. According to the algorithm, this means we apply replacement for at most once to the same nodes, that is, to reduce the degree to 3 if it’s not 3 in the original graph. The number of replacement less than the number of nodes in the graph. This is noticed by P.Tumarkin. In additoin, this algorithm provides a fast way to determine when a quiver of size larger than 10 has finite mutation type. (See [3] for detail.)
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