On star and biclique edge-colorings

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Abstract

A biclique of $G$ is a maximal set of vertices that induces a complete bipartite subgraph $K_{p,q}$ of $G$ with at least one edge, and a star of a graph $G$ is a maximal set of vertices that induces a complete bipartite graph $K_{1,q}$. A biclique (resp. star) edge-coloring is a coloring of the edges of a graph with no monochromatic bicliques (resp. stars). We prove that the problem of determining whether a graph $G$ has a biclique (resp. star) edge-coloring using two colors is NP-hard. Furthermore, we describe polynomial time algorithms for the problem in restricted classes: $K_3$-free graphs, chordal bipartite graphs, powers of paths, and powers of cycles.

Keywords: star edge-coloring; biclique edge-coloring; NP-hard

1. Introduction

Bicliques have been studied in several contexts such as telecommunications (Gualandi et al., 2013; Faure et al., 2014) and bioinformatics (Zhang et al., 2014). In the graph theory, the biclique vertex-coloring problem was proposed by Groshaus et al. (2014) and it is attracting much attention in recent works (Macêdo Filho et al., 2012, 2015). In the present work, we address the edge-coloring version of the biclique coloring problem, that is, we investigate how to color the edges of a graph in such a way that no biclique is monochromatic. We also address a variant of the problem in which the stars cannot be monochromatic, the so-called star edge-coloring problem. We prove that the problem of determining whether a graph has a biclique (resp. star) edge-coloring using two colors
is NP-hard. Observe that the NP-hardness of biclique (resp. star) edge-coloring using two colors does not imply that the problem remains NP-hard if three or more colors are allowed. Indeed, for the analogous problem of clique vertex-coloring, the classical NP-hardness result of Kratochvıl and Tuza (2002) refers to clique vertex-coloring using precisely two colors; only almost a decade later, Marx (2011) proved the $\Sigma_2 P$-completeness of clique $k$-vertex-coloring for any $k \geq 2$. The NP-hardness of biclique (resp. star) edge-coloring using two colors motivates the study of these two problems in restricted graph classes. The present work proposes some techniques that allow us to obtain biclique edge-colorings and star edge-colorings of graphs in the following classes: $K_3$-free graphs, chordal bipartite graphs, powers of paths, and powers of cycles. Coloring problems have been largely studied in these classes, see, for example, Cerioli and Posner (2012), Dabrowski et al. (2012), Campos and de Mello (2007), Luiz et al. (2015), and Macêdo Filho et al. (2015).

Let $G = (V, E)$ be a simple graph with order $n = |V|$ vertices and $m = |E|$ edges. A biclique of $G$ is a maximal set of vertices that induces a complete bipartite subgraph $K_p,q$ of $G$ with at least one edge; and a star of a graph $G$ is a maximal set of vertices that induces a complete bipartite graph $K_{1,q}$. A biclique edge-coloring of $G$ is a function $C'_b$ that associates a color to each edge of $G$ such that no biclique with at least two edges is monochromatic. If the function $C'_b$ uses at most $c$ colors, we say that $C'_b$ is a biclique $c$-edge-coloring. The biclique chromatic index of $G$ is the least $c$ for which $G$ has a biclique $c$-edge-coloring. Similarly, a star edge-coloring of $G$ is a function $C'_s$ that associates a color to each edge of $G$ such that no star with at least two edges is monochromatic. If the function $C'_s$ uses at most $c$ colors, we say that $C'_s$ is a star $c$-edge-coloring. The star chromatic index of $G$ is the least $c$ for which $G$ has a star $c$-edge-coloring.

The paper is organized as follows. In Section 2, we prove that the BICLIQUE 2-EDGE-COLORING and the STAR 2-EDGE-COLORING problems are NP-hard. We observe that the construction of the particular instance in this last proof has a polynomial amount of bicliques, so the STAR 2-EDGE-COLORING problem is NP-complete for the class of graphs in which each vertex belongs to at most one $K_3$, have degree at most 3, and are $C_4$-free, case in which we can define the precise complexity of the problems. In Section 3, we investigate the problems in the class of $K_3$-free graphs and determine its star chromatic index in polynomial time. Finally, in Section 4, we construct biclique edge-colorings and star edge-colorings for chordal bipartite graphs, powers of cycles, and powers of paths.

2. Star and biclique 2-edge-colorings are NP-hard

In this section, we prove that both BICLIQUE 2-EDGE-COLORING (2-BEC) and STAR 2-EDGE-COLORING (2-SEC) problems are NP-hard by reducing the NP-hard problem NOT-ALL-EQUAL 3-SATISFIABILITY (Schaefer, 1978) to 2-BEC problem (Fig. 1).

These two decision problems are defined as follows:

**NOT-ALL-EQUAL 3-SATISFIABILITY (NAE 3-SAT).** Instance: Set $X = \{x_1, \ldots, x_n\}$ of Boolean variables, collection $C = \{c_1, \ldots, c_m\}$ of clauses over $X$ such that each clause $c_i \in C$ has $|c_i| = 3$. Question: Is there a truth assignment for $X$ such that each clause in $C$ has at least one true literal and at least one false literal?

**BICLIQUE 2-EDGE-COLORING (2-BEC).** Instance: Graph $G = (V, E)$. Question: Does $G$ admits a 2-BEC?
Theorem 1. The 2-bec problem is NP-hard.

Proof. In order to reduce the NOT-ALL-EQUAL 3-SATISFIABILITY to the 2-bec problem, we need to construct, in polynomial time, a particular instance \( G = (V, E) \) of 2-bec problem from a generic instance \( (X, C) \) of NOT-ALL-EQUAL 3-SATISFIABILITY, such that \( C \) is satisfiable if, and only if, \( G = (V, E) \) admits a biclique 2-edge-coloring. First, we construct a particular instance \( G = (V, E) \) of 2-bec described next; second, we prove that every graph \( G \) that admits a biclique 2-edge-coloring defines a not-all-equal truth assignment for \( (X, C) \) (Lemma 1); third, we prove that every not-all-equal truth assignment for \( (X, C) \) defines biclique 2-edge-coloring for graph \( G \) (Lemma 2). □

Construction of particular instance. Let \( (X, C) \) be a generic instance of NAE 3-SAT such that \( X = \{x_1, \ldots, x_n\} \) is the variable set and \( C = \{c_1, \ldots, c_m\} \) is a collection of clauses, where \( c^j = (l^j_1, l^j_2, l^j_3) \) and \(|c^j| = 3\).

For each variable \( x_i \), we have a gadget \( X_i \) such that \( V(X_i) = \bigcup_{t=1}^{k} \{y^t_i, z^t_i, x^t_i\} \cup \{v^j, v^{j'}\} \) and \( E(X_i) = \bigcup_{t=1}^{k-1} \{y^t_i z_{t+1}^i, z^t_i x^t_i, x^t_i y^{t+1}_i, z_{t+1}^i y^{t+1}_i\} \cup \{y_{k-1}^i z_k^i, z_k^i x_k^i, x_k^i y_k^i, v^j, v^{j'}, v^j v^{j'}, v^{j'} v^j\} \), where \( k \) is the number of occurrences of literal corresponding to \( x_i \) or \( \bar{x}_i \) in \( C \). We note that for each \( 1 \leq i \leq n \), the number of vertices of \( X_i \) is \( 3k + 2 \) (e.g., see Fig. 2).

For each clause \( c^j = (l^j_1, l^j_2, l^j_3) \) we have one clause vertex \( c_j \). For each \( 1 \leq j \leq m \) and \( d \in \{1, 2, 3\} \), if \( l^j_d \) is equal to variable \( x_i \), then we have edge \( c_j x^t_i \), where \( t \) is one of the \( k \) vertices \( x^t_i \) in \( X_i \) with no edge to some clause vertex \( c \). Otherwise, if \( l^j_d \) is equal to \( \bar{x}_i \) then we add vertex \( x^t_i \) and edges \( \{x^t_i x^t_{i'}, c_j x^t_{i'}\} \), again \( t \) is one of the \( k \) vertices \( x^t_i \) in \( X_i \) with no edge to some clause vertex \( c \). Note that the constructed graph is \( C_4 \)-free, so that any biclique edge-coloring is a star edge-coloring and vice versa.

Lemma 1. If the particular instance \( G = (V, E) \) of 2-bec admits a biclique 2-edge-coloring, then there exists an NAE truth assignment that satisfies \( (X, C) \).
Fig. 2. Instance $G = (V, E)$ of 2-\textsc{bec} obtained from the satisfiable instance of 3-\textsc{sat}: $I = (X; C) = (\{x_1, x_2, x_3, x_4\}, \{(x_1 \lor x_2 \lor x_3), (x_1 \lor \bar{x}_2 \lor x_4)\})$.

Proof. The truth assignment is defined based on the following property (we refer to Fig. 2): Each gadget $X_i$ contains a unique hole $H_i$, which is odd. Moreover, $X_i$ has the property that all $P_3$s are bicliques, except for $\{v^i, y^i_1, z^i_k\}$, which is included in the biclique $\{v^i, v'^i, y^i_1, z^i_k\}$. It follows that if in some biclique 2-edge-coloring (with colors 1,2) the edge $v^i y^i_1$ has color $\lambda \in \{1, 2\}$, then every second edge along the hole $H_i$ must have color $\lambda$ too, up to edge $v^i z^i_k$, and consequently $v^i v'^i$ must have color $3 - \lambda$; moreover, all other edges of $X_i$ must have color $3 - \lambda$.

Assume that $G$ admits a biclique 2-edge-coloring. Define a truth assignment as follows: the value of variable $x_i$ is True if the edge $v^i v'^i$ is colored 1 and is False if the edge $v^i v'^i$ is colored 2. The truth assignment is valid because each clause is the center of a star that is a biclique, hence not all incident edges have the same color and, equivalently, not all literals have the same truth value. □

The converse of Lemma 1 is given next by Lemma 2.

Lemma 2. If there exists a not-all-equal truth assignment that satisfies $(X, C)$, then the particular instance $G = (V, E)$ of 2-\textsc{bec} admits a biclique 2-edge-coloring.

Proof. Assume that $(X, C)$ has a not-all-equal truth assignment. We color each pendant edge incident to $X_i$ with color 1 if $x_i$ is True; and with color 2 if $x_i$ is False. Thus, we extend the coloring to each $X_i$ as in Fig. 2. Finally, we color each edge incident to $x_j^i$ with the color distinct from the color of the pendant edge incident to $X_j$, and each remaining edge—corresponding to negative literals—receives the unique available color. The coloring is valid because it corresponds to a not-all-equal truth assignment, so that not all colors incident to a clause are equal, and the stars centered in the clause vertices are bicolored. □

Since the particular instance of 2-\textsc{bec} does not contain $C_4$s, we obtain the following result.

Corollary 1. The 2-\textsc{sec} problem is NP-hard.

Furthermore, since a graph $G$ resulting from the construction above is such that each vertex belongs to at most one $K_3$, $G$ is $C_4$-free and has degree at most 3; we obtain one more result.

Corollary 2. The 2-\textsc{bec} and 2-\textsc{sec} problems are NP-complete for the class of graphs $G$ such that each vertex belongs to at most one $K_3$, $G$ is $C_4$-free and has degree at most 3.
3. On the star chromatic index of $K_3$-free graphs

In this section, we determine the star chromatic index of odd cycles and of all connected $K_3$-free graphs in polynomial time. The first result is an immediate consequence of the well-known result of edge-coloring of cycles.

**Theorem 2.** If $G$ is a chordless cycle $C_n$ with odd $n \geq 5$ vertices, then the star chromatic index of $G$ is equal to 3. Furthermore, if $G$ is a connected $K_3$-free graph that is not isomorphic to $C_n$, for odd $n \geq 5$, then the star chromatic index of $G$ is equal to 2.

**Proof.** Since $G$ is $K_3$-free, every vertex with all its neighbors form an induced star. If $G$ has a vertex with degree 1 (leaf), construct a depth-first search tree starting on a leaf of $G$. If $G$ has no leaf, then $G$ has a cycle, and let $C = (v_1, v_2, \ldots, v_k, v_1)$ be an induced cycle of $G$ such that $v_k$ has a neighbor not in $C$ (note that such a cycle always exists). Construct a depth-first search tree on $G$ with $v_1$ as a root and choosing vertices $v_2, v_3, \ldots, v_k$ in that order.

Note that if there are return edges to the root (there is no leaf in $G$ to be used as a root), then there is at least one return edge that is not from a leaf—the return edge $v_k v_1$.

Color the tree edges from level $i$ to level $i + 1$ with color $(i \mod 2) + 1$. If the root vertex is a leaf, then its star has just one edge. If the root vertex has degree greater than 1, then choose a return edge leaving from a vertex that is not a leaf and color it with color 2.

For each leaf $f$ with return edges, choose a return edge to color with a color different from the tree edge arriving in $f$. If $f$ has no return edge, the star of $f$ has just one edge. Therefore, every star has two colors or has only one edge. \hfill $\Box$

4. On the biclique chromatic index of some graphs

In this section, we determine the biclique chromatic index of all chordal bipartite graphs and an upper bound for the biclique chromatic index of powers of cycles and powers of paths.

A graph is **chordal bipartite** if it is bipartite and each cycle of length at least 6 has a chord. The proof of the next result is based on the property that these graphs have a bisimplicial elimination ordering. Note that if an edge $uv$ belongs to two distinct bicliques, then $N(u) \cup N(v)$ cannot be a biclique. Hence, a bisimplicial edge of the graph belongs to precisely one biclique. The coloring is obtained by an algorithm based on induction of the number of edges.

**Theorem 3.** Every chordal bipartite graph has biclique chromatic index 2.

**Proof.** Let $G = (V, E)$ be a chordal bipartite graph. We may assume $G$ connected. If $G$ has two edges, color one edge with color 1 and the other edge with color 2. Now, assume that $G$ has more than two edges and let $uv$ be a bisimplicial edge, such that $N(u) \cup N(v)$ induces a complete bipartite graph.

Note that the bicliques of $G \setminus uv$ are the same as the bicliques of $G$, except for $G[N(u) \cup N(v)]$, which is a biclique of $G$ that is not a biclique of $G \setminus uv$. By induction, graph $G \setminus uv$ has a biclique 2-edge-coloring, that is, an association of colors to the edges is in such a way that every biclique is 2-colored. We construct a biclique 2-edge-coloring of $G$ as follows. First, color the edges of $E \setminus \{uv\}$ as in a biclique 2-edge-coloring of $G \setminus uv$ using colors 1 and 2. If this coloring results in
A power of a cycle $C_n^k$, for $n, k \geq 1$ is a simple graph on $n$ vertices with $V(G) = \{v_1, \ldots, v_{n-1}\}$ and \{v_i, v_j\} $\in E(G)$ if, and only if, $\min((j - i) \mod n, (i - j) \mod n) \leq k$. Note that $C_n^1$ is the induced cycle $C_n$, and $C_n^k$ with $n \leq 2k + 1$ is the complete graph $K_n$. In a power of a cycle $C_n^k$, we take $(v_0, \ldots, v_{n-1})$ to be a cyclic order on the vertex set and we always perform arithmetic modulo $n$ on vertex indices. A power of a path $P_n^k$, for $k \geq 1$, is a simple graph on $n$ vertices with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and \{v_i, v_j\} $\in E(G)$ if, and only if, $|i - j| \leq k$. Note that $P_n^1$ is the induced path $P_n$, and $P_n^k$ with $n \leq k + 1$ is the complete graph $K_n$. In a power of a path $P_n^k$, we take $(v_0, \ldots, v_{n-1})$ to be a linear order on the vertex set.

In the following, we obtain an upper bound for the biclique chromatic index of both classes by analyzing the cases according to the number of vertices of the graph, as described next.

The bicliques of a power of a path $P_n^k$, $n > k + 1$, are precisely (Macêdo Filho et al., 2015):

- $K_2$ and $P_3$ bicliques, if $k + 2 \leq n \leq 2k$; and
- $P_3$ bicliques, if $n \geq 2k + 1$.

The bicliques of a power of a cycle $C_n^k$, $n > 2k + 1$, are precisely (Macêdo Filho et al., 2015):

- $C_4$ bicliques, if $2k + 2 \leq n \leq 3k + 1$;
- $P_3$ and $C_4$ bicliques, if $3k + 2 \leq n \leq 4k$; and
- $P_3$ bicliques, if $n \geq 4k + 1$.

**Theorem 4.** Every noncomplete power of a cycle has a biclique edge-coloring using at most four colors.

*Proof.* Let $G = C_n^k$ be a power of a cycle with $n \geq 2k + 2$. We show how to color $G$ in such a way that no induced $P_2$ is monochromatic, hence, no biclique of $G$ is monochromatic.

First define $[n/k]$ sets of vertices, each set having $k$ consecutive vertices of $G$, as follows:

- $B_1 = \{v_1, v_2, \ldots, v_k\}$
- $B_2 = \{v_{k+1}, v_{k+2}, \ldots, v_{k+k}\}$
- \ldots
- $B_i = \{v_{(i-1)k+1}, v_{(i-1)k+2}, \ldots, v_{(i-1)k+k}\}$
- \ldots
- $B_{[n/k]} = \{v_{(n/k)k+1}, v_{(n/k)k+2}, \ldots, v_{(n/k)k+k}\}$

if $n \neq 0 \mod k$, that is, if $[n/k] = [n/k] + 1$, then $B_{[n/k]+1} = V(G) \setminus (B_1 \cup B_2 \cup \ldots \cup B_{[n/k]})$ (note that this vertex set has size less than $k$).

Consider an auxiliary graph $G_B$ with vertices $b_1, \ldots, b_{[n/k]}$ corresponding, respectively, to the blocks $B_1, \ldots, B_{[n/k]}$ of $G$, in such a way that two vertices, $b_i$ and $b_j$, are adjacent in $G_B$, if there exists an edge in $G$ from a vertex of $B_i$ to a vertex of $B_j$. If vertex $b_{[n/k]+1}$ exists, then $G_B$ is composed by a cycle $b_1, \ldots, b_{[n/k]+1}$ having vertex $b_{[n/k]}$ adjacent to both $b_{[n/k]+1}$ and $b_1$. Note that if $n = 0 \mod k$, then $G_B$ is a cycle, and so it has maximum degree 2. If $n \neq 0 \mod k$, then $G_B$ has maximum degree 3. In what follows, we present how to construct a biclique 4-edge-coloring of $G$ from a 4-total-coloring of $G_B$. A $k$-total-coloring of a graph $G$ is an assignment of $k$ colors to
the elements (vertices and edges) of a graph, such that adjacent or incident elements have different colors, and a $k$-total-coloring of a graph with maximum degree $\Delta$, uses at least $\Delta + 1$ colors.

First, we prove that $G_b$ has a 4-total-coloring. If $n \leq 3k$ then $G_B$ is a $C_3$, which has a 4-total-coloring by coloring elements $b_1, b_1b_2, b_2, b_2b_3, b_3, b_3b_1$ with colors 1, 2, 3, 1, 4, 3, respectively. Hence, we consider the case $n > 3k$. If $n = 0 \mod k$, then $G_B$ is a cycle $b_1, b_2, \ldots, b_{\lfloor n/k \rfloor}$ and so it is easily 4-total-colorable. If $n \neq 0 \mod k$, then we color elements $b_{\lfloor n/k \rfloor+1}b_{\lfloor n/k \rfloor+1}, b_{\lfloor n/k \rfloor+1}b_{\lfloor n/k \rfloor+1}, b_{\lfloor n/k \rfloor+1}b_{\lfloor n/k \rfloor+1}, b_{\lfloor n/k \rfloor+1}b_{\lfloor n/k \rfloor+1}$ with colors 1, 2, 3, 4, 1, and extend the total coloring to the remaining elements using four colors.

We need some additional notations. An edge of $G$ whose endvertices belong to the same block $B_i$ is called an internal-$B_i$ edge. An edge of $G$ whose endvertices belong to distinct blocks $B_i$ and $B_j$ is called an external-$B_iB_j$ edge.

Now, we construct a biclique 4-edge-coloring of $G$ from a 4-total-coloring of $G_B$: For each block $B_i$ of $G$, each internal-$B_i$ edge of $G$ receives the same color as $b_i$ received in the 4-total-coloring of $G_B$; for each block $B_i$ of $G$, each external-$B_iB_j$ edge of $G$ receives the same color as $b_ib_j$ received in the 4-total-coloring of $G_B$.

It remains to prove that there is no edge-monochromatic $P_3$ in $G$. Consider a set $\{v_1, v_2, v_3\}$ of vertices that induces a $P_3$ such that $v_1v_2$ and $v_2v_3$ are edges of $G$ and $v_1v_3$ is a nonedge of $G$. Note that $v_1$ and $v_3$ are not in the same block, for otherwise, they would be adjacent. There are three possible cases.

1. Vertices $v_1, v_2,$ and $v_3$ belong to distinct blocks $B_i, B_j,$ and $B_k$, respectively, of $G$—which implies that $b_i$ and $b_j$, resp. $b_j$ and $b_k$, are adjacent in $G_B$. In this case, edges $v_1v_2$ and $v_2v_3$ receive, resp., the color of $b_ib_j$ and $b_jb_k$ in $G_B$, which are distinct because $b_ib_j$ and $b_jb_k$ are adjacent edges of $G_B$.

2. Vertices $v_1$ and $v_2$ belong to the same block $B_i$—which implies that $v_3$ belongs to a block $B_j$ that is adjacent to $B_i$. In this case, edges $v_1v_2$ and $v_2v_3$ receive, resp., the color of $b_ib_j$ in $G_B$, which are distinct because $b_ib_j$ is incident to $b_i$ in $G_B$.

3. Vertices $v_2$ and $v_3$ belong to the same block $B_j$. This case is analogous to the previous case. $\square$

The above result provides an upper bound for the biclique chromatic index of powers of cycles. It is important to note that this upper bound is tight. Indeed, we could find a power of a cycle$^1$ whose biclique chromatic index is 4, namely, $C_{56}^{10}$. In addition, there exist powers of cycles with biclique chromatic index equal to 2 and 3. An example of a power of a cycle with biclique chromatic index equal to 2 is any graph $C_n^k$ with $2k + 2 \leq n \leq 3k + 1$. A valid coloring is constructed by defining set $B_1$ with $n-k$ consecutive vertices and set $B_2$ with the remaining $k$ consecutive vertices. In $B_1$, we color the edges, having both endvertices, with color 1 and the remaining edges with color 2 (we invite the reader to check that each $C_4$ contains at least one vertex from each color class and all bicliques are $C_4$s).

An example of a power of a cycle with biclique chromatic index equal to 3 is the graph $C_9^2$ with vertices $\{v_0, v_1, v_2, \ldots, v_8\}$. In fact, suppose that there exists a biclique 2-edge-coloring of $C_9^2$. Without loss of generality, start by assigning color 1 to edge $v_0v_1$, and so both edges $v_1v_3$ and $v_0v_7$ must be colored with color 2. This implies that edges $v_5v_7$ and $v_3v_5$ must have color 1, but

$^1$We run the small$k$ vertex coloring software (Culberson, 2000) over the graph $H$ whose vertices are the edges of $C_{56}^{10}$, such that two vertices are adjacent if the corresponding edges of $C_{56}^{10}$ are the edges of an induced $P_2$. The conversion is valid because all bicliques of $C_{56}^{10}$ are $P_2$. The software could not find any valid 3-coloring of $H$.
in this case, the biclique $v_3v_5v_7$ would be monochromatic, which is a contradiction. A biclique 3-edge-coloring is obtained by coloring edges $v_0v_1$, $v_3v_6$, $v_1v_8$, $v_0v_2$, $v_4v_6$, $v_4v_7$ with color 1, edges $v_2v_3$, $v_7v_8$, $v_1v_3$, $v_2v_4$, $v_6v_8$, $v_0v_7$ with color 2, and edges $v_1v_2$, $v_3v_4$, $v_4v_5$, $v_6v_7$, $v_0v_8$, $v_3v_5$ with color 3.

**Theorem 5.** Every noncomplete power of path has a biclique edge-coloring using at most four colors.

**Proof.** Every noncomplete power of a path $G$ is an induced subgraph of a power of a cycle $H$. By the proof of Theorem 4, $H$ has a 4-coloring of its edges in such a way that no induced $P_3$ is monochromatic. We claim that the restriction of this coloring to the edges of $G$ is a biclique 4-edge-coloring of $G$, because each induced $P_3$ of $G$ is an induced $P_3$ of $H$—therefore, it cannot be monochromatic by the coloring given to $H$. \[\square\]

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