FINITE INTERPOLATION WITH
MINIMUM UNIFORM NORM IN $\mathbb{C}^n$

Eric Amar and Pascal J. Thomas
Université Bordeaux-I
Université Paul Sabatier

Abstract. Given a finite sequence $a := \{a_1, \ldots, a_N\}$ in a domain $\Omega \subset \mathbb{C}^n$, and complex scalars $v := \{v_1, \ldots, v_N\}$, consider the classical extremal problem of finding the smallest uniform norm of a holomorphic function verifying $f(a_j) = v_j$ for all $j$. We show that the modulus of the solutions to this problem must approach its least upper bound along a subset of the boundary of the domain large enough to contain the support of a measure whose hull contains a subset of the original $a$ large enough to force the same minimum norm. Furthermore, all the solutions must agree on a variety which also contains this hull. An example is given to show that the inclusions can be strict.

0. Introduction and statement of results...

Given a finite sequence $a := \{a_1, \ldots, a_N\}$ in a domain $\Omega \subset \mathbb{C}^n$, and complex scalars $v := \{v_1, \ldots, v_N\}$, consider the classical extremal problem:

\[
\inf \{ \|f\|_{\infty} : f \in H^\infty(\Omega), f(a_j) = v_j, 1 \leq j \leq N \} =: m.
\]

It will be convenient to consider the data $v$ as lying in $H^\infty(\Omega)/I_a$, where $H^\infty(\Omega)$ is the algebra of functions holomorphic and bounded on $\Omega$, and $I_a$ is the ideal associated to $a$,

\[
I_a := \{f \in H^\infty(\Omega) : f(a_j) = 0, j = 1, \ldots, N\}.
\]

Then $m = \|v\|_{H^\infty(\Omega)/I_a}$, by the definition of the quotient norm. By Montel’s Theorem, we know that this problem always admits an extremal function $f$, i.e. a representative of the class $v$ such that $\|f\|_{\infty} = \|v\|_{H^\infty(\Omega)/I_a}$.

When $n = 1$ and $\Omega = \mathbb{D}$, it is a classical fact that $f$ is unique, and indeed given by a constant multiple of a Blaschke product of degree $N - 1$. In particular, it is holomorphic in a neighborhood of $\widehat{\mathbb{D}}$ and of constant modulus on $\partial\mathbb{D}$, and the same properties hold when $\mathbb{D}$ is replaced by any bounded domain $\Omega$ with smooth boundary in the complex plane, and in even more general one-dimensional cases (see [Gr], or for instance [Gm] and references therein).

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Those properties cannot hold in higher dimension. Consider the simple example where \( a = (\alpha, 0) \subset \mathbb{D} \times \{0\} \subset \mathbb{B}^n \), the unit ball of \( \mathbb{C}^n \), with \( \alpha := \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{D} \).

Then, given any solution \( f \in H^\infty(\mathbb{B}^n) \) to the problem (1), the function \( f_0 \in H^\infty(\mathbb{D}) \) defined by restriction (\( f_0(\zeta) := f(\zeta, 0) \)) will be a solution to the classical extremal problem in the disk, therefore \( \|v\|_{H^\infty(\mathbb{B}^n)/I_n} \geq \|v\|_{H^\infty(\mathbb{D})/I_n} \). But given any solution \( f_0 \) to the problem in the disk, the trivial extension \( f(z) := f_0(z_1) \) will solve the problem in the ball, thus \( \|v\|_{H^\infty(\mathbb{B}^n)/I_n} = \|v\|_{H^\infty(\mathbb{D})/I_n} = \|f_0\|_\infty \), and we have a solution given by a constant multiple of a Blaschke product in \( z_1 \) which has modulus \( \|v\|_{H^\infty(\mathbb{B}^n)/I_n} \) on \( \partial \mathbb{D} \times \{0\} \) and strictly less than \( \|v\|_{H^\infty(\mathbb{B}^n)/I_n} \) elsewhere on \( \partial \mathbb{B}^n \).

Furthermore, this solution is not unique. Indeed, using for instance the fact that, on the unit circle, \( |f_0|^2 \) can only have critical points where \( f_0' \) vanishes, so that in this case the non-vanishing of any further derivative at that point would imply a local violation of the maximum principle, one sees easily that \( f_0' \) does not vanish anywhere on the circle, and thus there exists a \( \gamma > 0 \) such that \( |f_0(\zeta)| \leq \|f_0\|_\infty - \gamma(1 - |\zeta|^2) \) for all \( \zeta \in \mathbb{D} \). On the other hand, if \( (z_1, z') \in \mathbb{B}^n \subset \mathbb{C} \times \mathbb{C}^{n-1} \), then \( |z'|^2 < 1 - |z|^2 \). Therefore any function of the form \( g(z_1, z') := f_0(z_1) + \gamma h(z) \), where \( |h(z)| \leq |z'|^2 \), will provide another solution to the problem.

We are interested in the relationship between our extremal problem, notably the sequence \( a \), and the subsets of \( \partial \Omega \) where its solutions reach their maximum modulus \( \|v\|_{H^\infty(\Omega)/I_n} \).

**Definition.**

For any function \( f \in H^\infty(\Omega) \), let

\[
M(f) := \{ \xi \in \partial \Omega : \lim \sup_{z \to \xi, z \in \Omega} |f(z)| = \|f\|_\infty \}.
\]

It will be useful to highlight those subproblems of the original problem which yield the same extremal norm.

**Definition.** We say that \( a' \) defines a sufficient subproblem of (1) if and only if \( \|v\|_{H^\infty(\Omega)/I_{n'}} = \|v\|_{H^\infty(\Omega)/I_n} \).

We say that a problem \( (a, v) \) is minimal when it does not contain any strictly smaller sufficient subproblem.

Note that this definition depends on the values \( v \). That \( a' \) be sufficient implies of course that the points of \( a \setminus a' \) are “inactive constraints”, in the sense that removing them will not change the extremum we are looking for. Note however that the converse is not true: it is quite possible to have problems \( (a, v) \), every constraint \( (a_i, v_i) \) of which is inactive, but of course removing them all (resp. all but one) would lead to a problem without constraints whose solution is \( \infty \) (resp. the modulus of the remaining \( v_j \)). Take for instance three points in the disk and values at those three points of a Möbius automorphism of the disk. Then any pair of points will provide a minimal sufficient subproblem.

We denote by \( A(\Omega) \) the algebra of functions holomorphic on \( \Omega \) and continuous on \( \overline{\Omega} \). For any compact set \( \overline{K} \subset \Omega \), the \( A(\Omega) \)-hull is defined to be

\[
\hat{K}_{A(\Omega)} := \{ z \in \mathbb{C}^n : \forall F \in A(\Omega), |F(z)| \leq \max_{\overline{K}} |F| \}.
\]

In the case where \( \overline{\Omega} \) has a neighborhood basis of Runge domains (for instance when \( \Omega \) is convex), then we can replace \( A(\Omega) \) by \( C_n[\mathbb{Z}] \), the set of all (holomorphic) polynomials in \( n \) variables, and we just get the polynomial hull, denoted by \( \hat{K} \).
We will restrict attention to open sets where bounded holomorphic functions are well approximated by functions continuous up to the boundary, in the following sense.

**Definition.** We say that $\Omega$ has property (A) if and only if $\Omega$ is a bounded domain and for any $g \in H^\infty(\Omega)$, there exists a sequence $\{g_n\} \subset A(\Omega)$ such that for any open set $U$, $\lim_{n \to \infty} \|g_n\|_{L^\infty(U)} = \|g\|_{L^\infty(U)}$, and $g_n \to g$ uniformly on compacta of $\Omega$.

This property holds in particular when $\Omega$ is convex and bounded (use dilations).

**Theorem 1.**
Suppose that $\Omega$ has property (A). Let $f \in H^\infty(\Omega)$, and $\|f\|_\infty = \|v\|_{H^\infty(\Omega)/I_a}$. There exists a sufficient subproblem $(a', v|_{a'})$ such that $[M(f)]^\wedge_{A(\Omega)} \supset a'$.

In particular, if all the points of $a$ are active constraints, then $[M(f)]^\wedge_{A(\Omega)} \supset a$, and in general $[M(f)]^\wedge_{A(\Omega)} \cap a \neq \emptyset$.

Notice that it follows from the maximum principle that, when $v$ is not constant, a subsequence $a'$ giving a sufficient subproblem must contain at least two points. In the case of the example given above, $M(f) \supset \partial D \times \{0\}$ and $[M(f)]^\wedge \supset \overline{D} \times \{0\}$. In fact there is always a single set $M(f_0)$ contained in all the $M(f)$, for any $f$ solution to the problem.

**Lemma 2.** Given any $a$ and $v$ as above, there exists a holomorphic solution $f_0$ to the problem (1) such that

$$M(f_0) = \bigcap_{f \in H^\infty(\Omega); f(a) = v, \|f\|_\infty = \|v\|_{H^\infty(\Omega)/I_a}} M(f).$$

In the case of the example, $M(f_0) = \partial D \times \{0\}$. Theorem 1 says that $M(f_0)$ cannot be too small. We give some well-known consequences in the case when $\Omega = \mathbb{B}^n$.

**Corollary 3.**

1. The set $M(f_0)$ has positive (possibly infinite) length.
2. The set $M(f_0)$ cannot be a peak-interpolation set.
3. If $M(f_0) \subset \partial D \times \{0\}$, then $M(f_0) = \partial D \times \{0\}$.

**Proof.** By applying an automorphism of the ball, we may assume that $0 \in [M(f_0)]^\wedge \cap a$. Then [Fo] and [La] show that the length of $M(f_0)$ is at least $2\pi$, so remains positive after applying the inverse automorphism.

For (ii), see [Ru]; (iii) is elementary.

**Representing measures**

Let $\mu$ be a Borel measure on $\overline{\Omega}$. We define the hull of $\mu$, $\mathcal{E}_\mu \subset \overline{\Omega}$ by $z \in \mathcal{E}_\mu$ if and only if there exists a measure $\nu_z$, absolutely continuous with respect to $\mu$, which is a representing measure for $z$, i.e. for any $f \in A(\Omega)$, $f(z) = \int_{\overline{\Omega}} f d\nu_z$.

**Lemma 4.** If $\mu$ is supported on the (closed) set $K$, that is, if $\mu(\overline{\Omega} \setminus K) = 0$, then $\mathcal{E}_\mu \subset \mathcal{K}_{A(\Omega)}$.

**Proof.** For any $z \in \mathcal{E}_{\mu}$, the measure $\nu_z$ given by the above definition is also supported on $K$. For any $f \in A(\Omega)$,

$$|f(z)| = \left| \int_{\overline{\Omega}} f(\zeta) d\nu_z(\zeta) \right| = \left| \int_K f(\zeta) d\nu_z(\zeta) \right| \leq \sup_K |f||\nu| = \sup_K |f|.$$
With the help of the above lemma, Theorem 1 is a consequence of the following result.

**Theorem 5.** Let \( f_0 \) be a solution to the extremal problem (1). There exists \( a' \) giving a sufficient subproblem of \((a,v)\) and a Borel measure \( \mu \) on \( \overline{\Omega} \), supported on \( M(f_0) \), such that \( a' \subset E_\mu \).

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1. **Proof of Theorem 5.**

The methods needed to prove Theorem 5 rely on concepts developed long ago by the first-named author [Am1], and recently put to use to study Pick-Nevanlinna problems in several variables [Am2]. First we reduce ourselves to \( A(\Omega) \). When needed, we will write \( J_\alpha := I_\alpha \cap A(\Omega) \).

**Lemma 6.** For any finite sequence \( a \) and values \( v \), \( \| v \|_{H^\infty(\Omega) / I_\alpha} = \| v \|_{A(\Omega) / J_\alpha} \).

**Proof.** Take any \( f \in H^\infty \) such that \( f(a_j) = v_j \), \( 1 \leq j \leq N \) and \( \| f \|_\infty = \| v \|_{H^\infty(\Omega) / I_\alpha} \). If \( \{ f_n \} \) is the sequence given by property \((A)\), \( \sup_n |f - f_n| \to 0 \) as \( n \to \infty \). Let \( L_n \) be the Lagrange polynomial interpolating the values \( (f - f_n)(a_j) \) at the points \( a_j \); then \( f_n + L_n \) provide representatives of the class of \( f \) (i.e. \( (f_n + L_n) - f \in I_\alpha \) and \( \| f_n + L_n \|_\infty \to \| f \|_\infty \), so \( \| v \|_{A(\Omega) / J_\alpha} \leq \| v \|_{H^\infty(\Omega) / I_\alpha} \), and the reverse inequality is trivial.

Now to avoid trivialities, suppose that \( v \neq 0 \), and let \( g \) be a representative of \( v \) in \( A(\Omega) \). Then, by Hahn-Banach’s Theorem, there exists \( \ell \in (A(\Omega) / J_\alpha)^* \) a continuous linear form such that \( \ell(v) = \| v \|_{A(\Omega) / J_\alpha} \), and \( \| \ell \| = 1 \); equivalently, we may consider that \( \ell \in A(\Omega)^*, \ell|_{I_\alpha} = 0, \ell(g) = \| v \|_{A(\Omega) / J_\alpha} \), and \( \| \ell \| = 1 \).

Since \( A(\Omega) \) can be considered as a subspace of \( C(\partial \Omega) \), there is a measure \( \nu \) on \( \partial \Omega \) which represents \( \ell \), with \( \| \nu \| = 1 \). Write \( d\nu = \theta d\mu \), where \( \mu \) is a probability measure and \( \| \theta \| = 1 \) \( \mu \)-a.e.

**Proposition 7.** Let \( f_0 \in H^\infty(\Omega) \) be a solution of the problem (1), and \( m = \| v \|_{H^\infty(\Omega) / I_\alpha} = \| v \|_{A(\Omega) / J_\alpha} \). Then there exists \( F^* \in L^\infty(\mu) \) (defined \( \mu \)-almost everywhere), such that \( |F^*| = m, \mu \)-almost everywhere, and \( \{ \zeta \in \partial \Omega : |F^*(\zeta)| = m \} \subset M(f_0) \).

Notice that we do not prove that \( F^* \) represents the boundary values of \( f_0 \).

**Proof.** Take a sequence \( F_n := f_n + L_n \) as in the proof of Lemma 6. By weak* compactness of the unit ball of \( L^\infty(\partial \Omega) \), \( \{ F_n \} \) admits a subsequence which converges weakly to some \( F^* \in L^\infty(\partial \Omega) \), \( \| F^* \|_\infty \leq m \). Furthermore,

\[
\int F^* \theta d\mu = \lim_{n \to \infty} \int F_n \theta d\mu = \lim_{n \to \infty} \ell(F_n) = \ell(g) = m ,
\]

so we must have \( F^*(\zeta) \theta(\zeta) = m \) for \( \mu \)-a.e. \( \zeta \). This proves the first assertion about \( F^* \), and reduces the second one to proving that \( \mu \) is supported on the (closed) set \( M(f) \).

Let \( \zeta \in \partial \Omega \setminus M(f) \) and \( \psi \in C(\partial \Omega, [0,1]) \) such that \( \supp \psi \subset \overline{B}(\zeta,r) \cap \partial \Omega \subset \partial \Omega \setminus M(f) \), \( \psi \equiv 1 \) on \( \overline{B}(\zeta,r/2) \). Then by definition of \( M(f) \), there exists \( \varepsilon > 0 \) such that

\[
\max_{\overline{B}(\zeta,r) \cap \partial \Omega} |f| \leq m - 2\varepsilon ;
\]
for $n$ large enough, by property (A), $\max_{B(\zeta, r) \cap \partial \Omega} |F_n| \leq m - \varepsilon$, so

$$\left| \int \psi F_n \theta d\mu \right| \leq (m - \varepsilon) \int \psi d\mu,$$

thus

$$\left| \int \psi F^* \theta d\mu \right| \leq (m - \varepsilon) \int \psi d\mu,$$

which implies $\int \psi d\mu = 0$, thus $\mu(B(\zeta, r/2)) = 0$, q.e.d.

**A representation of $H^2(\mu)$**

Let $H^2(\mu)$ be the closure in $L^2(\mu)$ of $A(\Omega)$. Let $b \in \Omega$. Then $J_b \subset H^2(\mu)$; let $e_b \in J_b^\perp \subset H^2(\mu)$. Observe that $\dim J_b^\perp \leq 1$.

For all $f \in A(\Omega)$, $\langle f, e_b \rangle = f(b) \langle \theta, e_b \rangle$. So either $J_b$ is dense in $H^2(\mu)$ and $e_b = 0$, or if we can find some $e_b \neq 0$, $k_b := (\langle \theta, e_b \rangle)^{-1} e_b$ is a reproducing kernel for the point $b$. This proves the following.

**Lemma 8.** If $J_b^\perp \neq \{0\}$, then $b \in E_\mu$.

**Definition.** For any $f \in A(\Omega)$, we let $\pi^\mu(f)$ be the (antilinear) map from $H^2(\mu)$ to itself given by

$$\langle \pi^\mu(f)h, k \rangle := \langle h, f k \rangle$$

for any $h, k \in H^2(\mu)$.

**Lemma 9.** (see [Am2]).

$\pi^\mu$ is an antilinear representation of $A(\Omega)$ and $\|\pi^\mu(f)\| \leq \|f\|_\infty$.

If $e_b \in J_b^\perp$, $\pi^\mu(f)(e_b) = f(b)e_b$.

Now for $s \subset \Omega$ let $E_s := \operatorname{span}\{e_b, e_b \in J_b^\perp, b \in s\} = J_s^\perp$, and $\pi^\mu_s$ denote the restriction of $\pi^\mu$ to $E_s$. The above definitions could be made for any measure $\mu$, but here we will be using the fact that the construction of $\mu$ depended on a solution of the problem (1).

**Proposition 10.**

For any $g \in A(\Omega)$ representing the given values $v \in A(\Omega)/J_a$, $\|\pi^\mu_a(g)\| = \|g\|_{A(\Omega)/J_a} = \|g\|_{A(\Omega)/J_{a'}}$, where $a' := \{a_j \in a : J_{a_j}^\perp \neq \{0\}\}$.

**Proof.**

Observe first, to avoid trivialities, that $a'$ cannot be empty, otherwise $E_a$ would be reduced to $\{0\}$, and $I_a$ would be dense in $H^\infty$, which is impossible when $v \neq 0$, because then we’d have solutions of the problem (1) with arbitrarily small norm.

By the definition of $a'$, $E_a = E_{a'}$, thus $\pi^\mu_a = \pi^\mu_{a'}$. Applying this to the same function $g$, we get the same operator norms, so it will be enough to prove the first equality to complete the proof.

Let $f$ be any function in $A(\Omega)$, $h \in J_a$. Since the map only depends on the values of $f$ on $a$, $\pi^\mu_a(f + h) = \pi^\mu_a(f)$. Thus

$$\|\pi^\mu_a(f)\|_{op} = \|\pi^\mu_a(f + h)\|_{op} \leq \|\pi^\mu(f + h)\|_{op} \leq \|f + h\|_{\infty},$$

and passing to the infimum we get $\|\pi^\mu_a(f)\|_{op} \leq \|f\|_{A(\Omega)/J_a}$. So $\|\pi^\mu_a(g)\|_{op} \leq m$. 


Conversely, given $g$, take $F^*$ as in Proposition 7. Again denote $m = \|v\|_{A(\Omega)/J_a}$. Then, since $F^*$ is obtained as a limit of holomorphic functions, $F^* \in H^\infty(\mu) \subset H^2(\mu)$, and $\|F^*\|_{H^2(\mu)} = m$ (because its modulus is constant $\mu$-a.e.).

For any $h \in J_a$,

$$\langle h, F^* \rangle = \int h F^* d\mu = \int hm\theta d\nu = m \int h d\nu = m\ell(h) = 0,$$

by definition of $\ell$. Thus $F^* \in J_a^\perp = E_a$.

We can then test $\pi_\mu(a)(g)$ on $F^*$:

$$\langle \pi_\mu(a)(g)(F^*), \nu \rangle = \langle F^*, g \rangle = m\ell(g) = m^2.$$

This proves the required inequality.

**End of Proof of Theorem 5.** By Proposition 10, the subproblem defined by $a'$ is sufficient. Proposition 7 shows that the measure $\mu$ defined after Lemma 6 is supported on $\{\zeta : |F^*(\zeta)| = m\} \subset M(f_0)$. And by Lemma 8, the $a'$ we have obtained is included in $E_\mu$.

### 2. Questions of uniqueness.

**Definition.**

The uniqueness variety for the problem (1) is defined by

$$U(a,v) := \{z \in \mathbb{B}^n : \forall f, g \text{ solving (1)}, (f - g)(z) = 0\}.$$

Clearly, $U(a,v)$ is an analytic variety containing $a$.

**Proposition 11.** Whenever $\mu$ is chosen as in Theorem 5, $E_\mu \subset U(a,v)$.

**Proof.**

We reuse the notations of Lemma 6 and Proposition 7. Suppose $f_0$ and $\tilde{f}_0$ are distinct solutions to the problem (1). Take two sequences of functions in $A(\Omega)$, $\{f_n\}$ (resp. $\{\tilde{f}_n\}$) converging uniformly on compacta of $\Omega$ to $f_0$ (resp. $\tilde{f}_0$) and in $L^\infty(\mu)$ to $F^*$ (resp. $\tilde{F}^*$). The proof of Proposition 7 shows that in fact $F^* = \tilde{F}^*$ $\mu$-a.e.

Suppose $b \in E_\mu$. Then, denoting by $\nu_b$ a representing measure for $b$ that is absolutely continuous with respect to $\mu$,

$$f_0(b) - \tilde{f}_0(b) = \lim_{n \to \infty} \int (f_n(\zeta) - \tilde{f}_n(\zeta)) d\nu_b(\zeta) = 0$$

by the dominated convergence theorem.

**Examples.**

In the case of the example given in the introduction, $U(a,v) = \mathbb{D} \times \{0\} \subset \mathbb{B}^2$; we shall see presently that there are some cases when $U(a,v) = \Omega$, that is to say, the solution to the problem (1) is unique.

**Theorem 12.**

For $\Omega = \mathbb{B}^2$, there exists $a := \{a_1, \ldots, a_4\}$ and $v$ such that $M(f_0)$ is a 2-real-dimensional torus in $\partial \mathbb{B}^2$, and the solution to the problem (1) is unique.

The above theorem will reduce to a result about extension of inner functions from an analytic disk embedded into the ball $\mathbb{B}^2$. First we need a simple one-variable lemma.
Lemma 13.
Let $a := \{a_1, \ldots, a_N\}$ be distinct points in $\mathbb{D}$ and $B_{N-1}$ a Blaschke product of degree exactly equal to $N - 1$. Let $v_j := B_{N-1}(a_j)$, $1 \leq j \leq N$. Then $B_{N-1}$ is the unique solution to the extremal problem (1).

Proof.
It will be enough to show that $\|v\|_{H^\infty(\mathbb{D})/L_a} = 1$. The proof will proceed by induction. For $N = 1$, $B_{N-1}$ is a unimodular constant and the property is obvious. Suppose it is true for $N$, and consider $a := \{a_1, \ldots, a_N, a_{N+1}\}$.

For any $\alpha \in \mathbb{D}$, denote by $\varphi_\alpha$ the involutive automorphism of $\mathbb{D}$ which exchanges 0 and $\alpha$. Suppose $f \in H^\infty(\mathbb{D})$, $f(a) = v$, and $\|f\|_\infty < 1$. Let $g = \varphi_{B_{N}(a_N)} \circ \hat{f} \circ \varphi_{a_N}$. We have $g(0) = 0$, so $g(\zeta) = \zeta h(\zeta)$, with $\|h\|_\infty = \|g\|_\infty < 1$.

Set $a'_j := \varphi_{N}(a_j)$, $v'_j := \varphi_{B_{N}(a_N)}(B_{N}(a_j))$, $1 \leq j \leq N + 1$. We have $v'_j = \hat{B}(a'_j)$, $1 \leq j \leq N + 1$, where $B := \varphi_{B_{N}(a_N)} \circ B_{N} \circ \varphi_{a_N}$. This implies that $\hat{B}(\zeta) = \zeta B_{N-1}(\zeta)$, where $B_{N-1}$ is a Blaschke product of degree $N - 1$ exactly.

Now letting $v''_j := v'_j/a'_j$, $1 \leq j \leq N$, we have $v''_j = B_{N-1}(a'_j)$ and

$$\|v''\|_{H^\infty(\mathbb{D})/L_{a'}} \leq \|h\|_\infty < 1,$$

a contradiction with the inductive hypothesis.

From now on we are considering the disk embedded in the unit ball $\mathbb{B}^2$ of $\mathbb{C}^2$ given by $\varphi(\mathbb{D}) = \{\varphi(\zeta) : \zeta \in \mathbb{D}\}$ where $\varphi(\zeta) := \sqrt[2]{\zeta \zeta^2}$. Observe that $\varphi(\mathbb{D}) = \{(z_1, z_2) \in \mathbb{B}^2 : z_2 = \sqrt{2}z_1^2\}$.

Lemma 14. Suppose that $g \in \mathcal{H}(\mathbb{D})$ is an inner function (i.e. $|g(e^{i\theta})| = 1$ for all $\theta \in \mathbb{R}$), analytic in a neighborhood of the closed unit disk, such that there exists $\tilde{g} \in H^\infty(\mathbb{B}^2)$ with $\tilde{g}(\varphi(\zeta)) = g(\zeta)$ for all $\zeta \in \mathbb{D}$ and $\|\tilde{g}\|_\infty = \|g\|_\infty = 1$. Then $g'(0) = 0$, and if we write $g(\zeta) = \zeta h(\zeta)$, there exists $H$ holomorphic in $\mathbb{B}^2$ such that

$$\tilde{g}(z_1, z_2) = g(\sqrt{2}z_1) + (z_2 - \sqrt{2}z_1^2)\frac{\sqrt{2}}{3}h(\sqrt{2}z_1) + (z_2 - \sqrt{2}z_1^2)^2H(z_1, z_2).$$

Proof.
Step 1: Claim
For any differentiable function $f$ on the ball, set

$$Lf(z) := \left(\frac{\partial}{\partial z_1} f - \sqrt{2}z_1 \frac{\partial}{\partial z_2} f\right)(z).$$

Then for any $\zeta \in \mathbb{D}$, $L\tilde{g}(\varphi(\zeta)) = 0$.

This is to be compared with [Ru, Theorem 11.4.7].

If $\tilde{g}$ was assumed to be smooth in a neighborhood of $\mathbb{B}^2$, it would be enough to notice that for each $z \in \varphi(\partial \mathbb{D}) \subset \partial \mathbb{B}^2$, $L$ is a derivation along the complex tangent line to $\partial \mathbb{B}^2$ at $z$. Since $|\tilde{g}|$ is maximal on $\varphi(\partial \mathbb{D})$ with respect to $\partial \mathbb{B}^2$, its derivative $L\tilde{g}$ should vanish there. The slightly more intricate argument that follows merely extends this to the case where $|\tilde{g}|_\infty = 1$.

Notice first that since $g$ is smooth across the unit circle, its derivative is bounded in a neighborhood of it and we have $c_1 > 0$ such that $1 - |g(\zeta)|^2 \leq c_1(1 - |\zeta|^2)$ for all $\zeta \in \mathbb{D}$. Now consider the complex line $\mathcal{L}$ passing through the point $\varphi(\zeta_0)$ and
parallel to the vector $(1, -\zeta_0)$. Since when $|\zeta_0| = 1$ this is the complex tangent to $\partial \mathbb{B}^2$ at $\varphi(\zeta_0)$, there exists a $c_2 > 0$ such that the disk of center $\varphi(\zeta_0)$, of radius $c_2(1 - |\zeta_0|^2)^{1/2}$ along the line $L$ is contained in $\mathbb{B}^2$. Then the function

$$f(\zeta) := \tilde{g}(\varphi(\zeta_0) + c_2(1 - |\zeta_0|^2)^{1/2}\zeta(1, -\zeta_0))$$

is bounded by 1 in modulus on the unit disk, and Schwarz-Pick’s Lemma (see [Gr, Chap. I, Lemma 1.2]) shows that

$$|f'(0)| \leq 1 - |f(0)|^2 = 1 - |\tilde{g}(\varphi(\zeta_0))|^2 = 1 - |g(\zeta_0)|^2 \leq c_1(1 - |\zeta_0|^2),$$

and since $f'(0) = c_2(1 - |\zeta_0|^2)^{1/2} L\tilde{g}(\varphi(\zeta_0))$ (notice that along $\varphi(\mathbb{D})$, $z_1 \sqrt{2} = \zeta$), we have $|L\tilde{g}(\varphi(\zeta_0))| \leq C(1 - |\zeta_0|^2)^{1/2}$. Now $L\tilde{g}(\varphi(\zeta))$ is a holomorphic function on $\mathbb{D}$, so it must be identically zero, which proves the Claim.

Step 2.

Consider the change of variables

$$\begin{cases} w_1 = z_1 \\ w_2 = z_2 - \sqrt{2}z_3^2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} z_1 = w_1 \\ z_2 = w_2 + \sqrt{2}w_3^2 \end{cases}.$$ 

If we set $\tilde{g}_1(w_1, w_2) := \tilde{g}(z_1, z_2)$, we then have $L_1\tilde{g}_1(w_1, 0) = 0$, where $L_1\tilde{g}_1(w) = \left(\frac{\partial}{\partial w_1}\tilde{g}_1 - 3\sqrt{2}w_1\frac{\partial}{\partial w_2}\tilde{g}_1\right)(w)$.

Since $\tilde{g}_1(w_1, 0) = g(\sqrt{2}w_1)$, we have $\frac{\partial}{\partial w_2}\tilde{g}_1(w_1, 0) = \sqrt{2}g'(\sqrt{2}w_1)$, and the above partial differential equation becomes

$$\sqrt{2}g'(\zeta) = 3\zeta \frac{\partial}{\partial w_2} \tilde{g}_1(\frac{\zeta}{\sqrt{2}}, 0),$$

which can be solved if and only if $g'(\zeta) = \zeta h(\zeta)$. We then have $\frac{\partial}{\partial w_2}\tilde{g}_1(w_1, 0) = \frac{\sqrt{2}}{3}h(\sqrt{2}w_1)$, and this provides the expansion of order 1 of $\tilde{g}_1$ near $\{w_2 = 0\}$, so

$$\tilde{g}_1(w_1, w_2) = g(\sqrt{2}w_1) + w_2 \frac{\sqrt{2}}{3}h(\sqrt{2}w_1) + w_2^2 H_1(w_1, w_2),$$

for some $H$ holomorphic on the image of $\mathbb{B}^2$ under the change of variables. Going back to the $(z_1, z_2)$ variables, we get the Lemma.

We now make a small aside to look into the problem of extending a family of simple functions (the monomials $\zeta^k$) from the analytic disk $\varphi(\mathbb{D})$ to $\mathbb{B}^2$ with the smallest possible $H^\infty$ norm.

**Lemma 15.**

(i) For any $\tilde{g} \in \mathcal{H}(\mathbb{B}^2)$ such that $\tilde{g}(\varphi(\zeta)) = \zeta$, $\|\tilde{g}\|_\infty \geq \sqrt{2}$, and this bound is attained by $\tilde{g}(z_1, z_2) = \sqrt{2}z_1$.

(ii) For any $k \neq 1$, there exists $\tilde{g}_k \in \mathcal{H}(\mathbb{B}^2)$ such that $\tilde{g}_k(\varphi(\zeta)) = \zeta^k$ and $\|\tilde{g}_k\|_\infty = 1$.

In particular, one can take $\tilde{g}_2(z_1, z_2) = \frac{2}{3}(z_1^2 + \sqrt{2}z_2)$, $\tilde{g}_3(z_1, z_2) = 2z_1z_2$.

**Remark.**
We know that the only analytic disks in the ball that allow the uniform norm-preserving extension of any bounded holomorphic function are the affine embeddings of $\mathbb{D}$ [St], [Su]; this is to be compared with Lempert’s result that the only disks which admit a holomorphic retraction are geodesic disks for the Kobayashi distance, i.e. in this instance affine disks once again [Le1], [Le2]. For our disk $\varphi(\mathbb{D})$, the above Lemma gives explicit examples of the functions which do or don’t admit norm-preserving extensions.

Proof.

(i) Since $\tilde{g}(\frac{\zeta}{\sqrt{2}}, \frac{\zeta^2}{2}) = \zeta$, $\frac{1}{\sqrt{2}} \partial_{\bar{z}_1} \tilde{g}(0,0) = 1$. Applying Schwarz’s Lemma, we get $\sup_{z \in \mathbb{D}} |\tilde{g}(z,0)| \geq \sqrt{2}$, whence the result.

(ii) When $g(\zeta) = \zeta^2$, $g'(\zeta) = 2\zeta$, $h(\zeta) = 2$, and setting $H = 0$, we find $\tilde{g}_2$. Checking the norm inequality for $(z_1, z_2) \in \partial \mathbb{B}^2$ is elementary.

In the same way, or by inspection, we find $\tilde{g}_3$. Given any integer $k \geq 2$, we can find two non negative integers $a, b$ such that $k = 2a + 3b$. We then set $\tilde{g}_k = \tilde{g}_2^a \tilde{g}_3^b$.

The next result will essentially complete the proof of Theorem 12.

**Lemma 16.**

The function $\tilde{g}_3$ in Lemma 14 is the only $\tilde{g} \in H^\infty(\mathbb{B}^2)$ such that $\tilde{g}(\varphi(\zeta)) = \zeta^3$ and $\|\tilde{g}\|_\infty = 1$.

**Proof.**

Let us first consider the simpler case where $\tilde{g}$ is holomorphic in a neighborhood of the closed ball. Then so is $H$ (the function obtained in Lemma 14).

For any $\nu \in [0, 2\pi]$, consider the map from the disk to the ball given by $\psi_{\nu}(\zeta) := \left(\frac{\zeta}{\sqrt{2}}, e^{i\nu} \frac{\zeta^2}{2}\right)$. Then, applying Lemma 14,

$$\tilde{g}(\psi_{\nu}(e^{i\theta})) = e^{i\nu} e^{3i\theta} + \frac{1}{2} e^{4i\theta} (e^{i\nu} - 1)^2 H(\psi_{\nu}(e^{i\theta})) .$$

A winding number argument then shows that $H$ must vanish at some point along the curve $\psi_{\nu}(\partial \mathbb{D})$. We will show that in fact $H$ is identically zero.

Now remove the additional assumption;

$$\tilde{g}(\psi_{\nu}(\zeta)) = e^{i\nu} \zeta^3 \left[1 + \frac{1}{2} \zeta (e^{i\nu} - 1)^2 H(\psi_{\nu}(\zeta)) \right] .$$

Set $f(\zeta) = \frac{1}{2} \zeta (e^{i\nu} - 1)^2 H(\psi_{\nu}(\zeta))$. This function can only be constant if $H = 0$. Suppose this is not the case.

We claim that for any $r \in [0, 1]$, there exists $\theta_r \in [0, 2\pi]$ such that $f(re^{i\theta_r}) > 0$. Indeed, there exists $\delta > 0$ such that this is true for all $r \in [0, \delta]$, by the Open Mapping Theorem, since $f(0) = 0$. Let $r_0$ be the largest number such that the conclusion of the claim holds for all $r \in [0, r_0]$. If $r_0 < 1$, since the winding number of the curve $f(re^{i\theta})$ around 0 is positive for $r$ small enough and can only change for a value of $r$ at which the curve goes through 0, there must be $0 < r_1 \leq r_0$ such that $f(r_1 e^{i\theta_r}) = 0$. Then there is $r_2 \in [0, r_1]$ such that $f(r_2 e^{i\theta_{r_2}})$ is maximal (we use the compactness of $f([0, r_1])$). But this violates the Open Mapping Theorem in a neighborhood of the point $r_2 e^{i\theta_{r_2}}$. 

By the same argument, we can see that the function which to $r$ associates the largest possible value $f(r e^{i \theta_r})$ cannot have a local maximum, and that (with a slight abuse of notation) $\limsup_{r \to 1^-} f(r e^{i \theta_r}) > 0$. This yields

$$\limsup_{r \to 1^-} |\tilde{g}(\psi_r(r e^{i \theta_r}))| \geq \limsup_{r \to 1^-} r^3 \left[1 + f(r e^{i \theta_r})\right] > 1,$$

a contradiction. Thus we must have $H \equiv 0$, q.e.d.

\textit{End of Proof of Theorem 12.}

Pick $a_j := \varphi(\zeta_j)$ where $\zeta_1, \ldots, \zeta_4$ are distinct points in the unit disk, and $v_j := \zeta_j^3$. By Lemma 12, any solution to the problem (1) must be equal to $\zeta^3$ at the point $\varphi(\zeta)$, for any $\zeta \in \mathbb{D}$, and $\|v\|_{H^\infty(B^2) / I_a} \geq 1$. By Lemma 15, $\|v\|_{H^\infty(B^2) / I_a} = 1$, and by Lemma 16, the solution to the problem is unique and assumes its maximum modulus on the set $\{|z_1| = |z_2| = \sqrt{2}/2\}$.

This example shows that the inclusions proved in Theorem 1 and Proposition 11 can be strict. Here $f_0(z) = 2z_1 z_2$, $\mathcal{U}(a, v) = \mathbb{H}^2$, and $M(f_0) = \{|z_1| = |z_2| = \sqrt{2}/2\}$, so $M(f_0)^\wedge = \{|z_1| \leq \sqrt{2}/2, |z_2| \leq \sqrt{2}/2\}$.

On the other hand, since $f_0 \in A(\mathbb{H}^2)$ already, we have $F^* = f_0|_{\partial B^2}$. It is elementary to see that the form $\ell$ can be represented by integration against a function along the boundary of the embedded disk $\varphi(\mathbb{D})$, so we may take $\mu = \varphi_*(\frac{1}{2\pi} d\theta)$, and $\mathcal{E}_\mu = \varphi(\mathbb{D})$. So we have the strict inclusions that we had announced.
FINITE INTERPOLATION WITH MINIMUM UNIFORM NORM IN $\mathbb{C}^n$

REFERENCES

[Am1] Amar E., *Suites d’interpolation dans le spectre d’une algèbre d’opérateurs*, Thèse, Université Paris XI, Orsay, 1977.

[Am2] Amar E., *Representation of Quotient Algebras*, preprint, 1997.

[Fo] Forstneric F., *The length of a set in the sphere whose polynomial hull contains the origin*, Indag. Math. (Proc. Kon. Ned. Akad. van Wetensch., new series) 3 (1992), 169-172.

[Gm] Gamelin, T. W., *Extremal problems in arbitrary domains*, Michigan Math. J. 20 (1973), 3-11.

[Gr] Garnett J., *Bounded analytic functions*, Academic Press, New York, 1981.

[La] Lawrence M., unpublished manuscript.

[Le1] Lempert L., *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. math. France 109 (1981), 427-474.

[Le2] Lempert L., *Holomorphic retracts and intrinsic metrics in convex domains*, Analysis Mathematica 8 (1982), 257-261.

[Ru] Rudin W., *Function theory in the unit ball of $\mathbb{C}^n$*, Springer Verlag, Berlin, 1980.

[St] Stanton C., *Embedding Riemann Surfaces in Polydisks*, Duke Math. J. 43 (1976), 791-796.

[Su] Suffridge T. J., *Common fixed points of commuting holomorphic maps of the hyperball*, Mich. Math. J. 21 (1974), 309-314.

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Eric Amar
U.F.R. Mathématiques
Université Bordeaux-I
351 cours de la Libération
33405 TALENCE, France
e-mail: eamar@math.u-bordeaux.fr

Pascal J. Thomas
Laboratoire Emile Picard
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse Cedex, France
e-mail: pthomas@cict.fr