Abstract

We study the probabilities with which chordal Schramm-Loewner Evolutions (SLE) visit small neighborhoods of boundary points. We find explicit formulas for general chordal SLE boundary visiting probability amplitudes, also known as SLE boundary zig-zags or order refined SLE multi-point Green’s functions on the boundary. Remarkably, an exact answer can be found to this important SLE question for an arbitrarily large number of marked points. The main technique employed is a spin chain - Coulomb gas correspondence between tensor product representations of a quantum group and functions given by Dotsenko-Fateev type integrals. We show how to express these integral formulas in terms of regularized real integrals, and we discuss their numerical evaluation.

The results are universal in the sense that apart from an overall multiplicative constant the same formula gives the amplitude for many different formulations of the SLE boundary visit problem. The formula also applies to renormalized boundary visit probabilities for interfaces in critical lattice models of statistical mechanics: we compare the results with numerical simulations of percolation, loop-erased random walk, and Fortuin-Kasteleyn random cluster models at $Q = 2$ and $Q = 3$, and find good agreement.
## Contents

1 Introduction
   1.1 The SLE curves
   1.2 The chordal SLE boundary visits
   1.3 Organization of the article

2 The problem: partial differential equations and asymptotics
   2.1 Differential equations for boundary visit amplitudes
   2.2 Asymptotics for boundary visit amplitudes

3 Quantum group and integral formulas
   3.1 Coulomb gas integrals
      3.1.1 Standard Coulomb gas integrals and their properties
      3.1.2 Spin chain - Coulomb gas basis functions
   3.2 Quantum group
      3.2.1 Definition of the quantum group
      3.2.2 Representations of the quantum group
   3.3 Spin chain - Coulomb gas correspondence
      3.3.1 Definition of the correspondence
      3.3.2 Asymptotics via the correspondence
      3.3.3 Highest weight vectors and closed integration surfaces
   3.4 Linear problem in quantum group representations
   3.5 Solutions in terms of quantum group representations
      3.5.1 One-point solutions
      3.5.2 Two-point solutions
      3.5.3 Three-point solutions
      3.5.4 Four-point solutions
      3.5.5 Well-posedness of the problem

4 Regularized real integrals and evaluation of the formulas
   4.1 Transformation to real integration contours
   4.2 Solutions in terms of real integrals
      4.2.1 One-point solutions
      4.2.2 Two-point solutions
   4.3 Divergences of the real integrals

5 Notions of SLE boundary visits and applications
   5.1 Definition of chordal SLE in half-plane
   5.2 Conformal covariance of boundary visit amplitudes
   5.3 Different definitions of SLE boundary visits
   5.4 Applications of the results and universal and non-universal aspects
      5.4.1 Boundary visit probabilities for interfaces in lattice models
      5.4.2 Covariant measure of SLE on the boundary
      5.4.3 Conditioned SLE and first visit point recursion for the zig-zag amplitudes

6 Comparisons with lattice model simulations
   6.1 Lattice model interfaces
      6.1.1 Relevant domains and conformal maps
      6.1.2 Loop-erased random walk
      6.1.3 Percolation
      6.1.4 FK-model
1 Introduction

1.1 The SLE curves

Schramm-Loewner evolutions (SLE) are conformally invariant random fractal curves in the plane, whose most important characteristics are determined by one parameter \( \kappa > 0 \). They were introduced by Oded Schramm [Sch00] as the only plausible candidates for the scaling limits of random interfaces in statistical mechanics models that are expected to display conformal invariance, with different models corresponding to different values of the parameter \( \kappa \).\(^1\) Proofs that interfaces in various critical lattice models do converge to SLEs in the scaling limit have been obtained for example in [Smi01,LSW04,SS05,Smi06,CN07,Zha08,Sm10a,HK13,Izy13,CDCH+13].

The fundamental example of SLEs is the chordal SLE\( _\kappa \) [LSW01,RS05]. For a given simply connected domain \( \Lambda \subset \mathbb{C} \) with two marked boundary points \( a, b \in \partial \Lambda \), the chordal SLE\( _\kappa \) in \( \Lambda \) from \( a \) to \( b \) is an oriented but unparametrized random curve \( \gamma \) in the closure of \( \Lambda \) starting from \( a \) and ending at \( b \). Its two characterizing properties are conformal invariance and domain Markov property:

- Conformal invariance states that the image of a chordal SLE under a conformal map is a chordal SLE in the image domain.

- Domain Markov property states that given an initial segment of a chordal SLE, the conditional law of the continuation is a chordal SLE in the remaining subdomain.

Some features of SLEs vary continuously in \( \kappa \), notably the Hausdorff dimension of the fractal curve is given by \( d_H(\gamma) = 1 + \frac{\kappa}{8} \) for \( 0 < \kappa \leq 8 \) [Bef08]. On the other hand, some qualitative properties of SLEs show abrupt phase transitions with respect to the parameter \( \kappa \). For the present purposes, it is important to distinguish the following three phases [RS05]:

\(^1\)Figure 1.1 shows two SLE curves. Examples of interfaces in lattice models are shown in Figures 6.1, 6.2, and 6.3 on pages 34, 35, and 36 respectively.
1 INTRODUCTION

Figure 1.1: Chordal SLE$_\kappa$ is a random fractal curve. For $\kappa \leq 4$ the curve is simple and does not touch boundary, and for $4 < \kappa < 8$ the curve has double points and touches the boundary on a random Cantor set. The two pictures show chordal SLE$_\kappa$ in the upper half-plane $\mathbb{H}$ from 0 to $\infty$ — in the left picture $\kappa = 3$, and the right picture $\kappa = 6$.

0 $\kappa \leq 4$: The chordal SLE$_\kappa$ is a simple curve, i.e., the curve does not have double points, see Figure 1.1 (left). The curve does not touch the boundary $\partial \Lambda$ of the domain except at the starting point $a$ and the end point $b$. The curve avoids any given point $z \in \Lambda$ of the domain with probability one.

4 $\kappa < 8$: The chordal SLE$_\kappa$ is a non self-traversing curve with double points, see Figure 1.1 (right). The intersection of the curve with the boundary $\partial \Lambda$ of the domain is a random Cantor set. The curve still avoids any given point $z \in \Lambda \setminus \{a, b\}$ of the domain or of its boundary with probability one.

8 $\leq \kappa$: The chordal SLE$_\kappa$ is a space-filling curve; any point $z \in \Lambda$ of the domain is on the curve.

The behavior in the case $\kappa \geq 8$ is somewhat pathological. No interfaces in statistical mechanics models are expected to correspond to $\kappa > 8$.

In this article we restrict our attention to the cases $0 < \kappa < 8$.

1.2 The chordal SLE boundary visits

The main goal in this article is to solve the probability with which the chordal SLE visits small neighborhoods of given boundary points. Partial answers to similar questions have been obtained in [BB03a, SZ10, AS08, AS09].

It is easiest to illustrate the question in the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\},$$

with the chordal SLE$_\kappa$ curve $\gamma$ starting from the origin and ending at infinity. Denote the half-disk of radius $\epsilon > 0$ centered at a boundary point $y \in \mathbb{R} \subset \partial \mathbb{H}$ by

$$B_\epsilon(y) = \{z \in \mathbb{H} \mid |z - y| < \epsilon\}.$$

Given points $y_1, y_2, \ldots, y_N \in \mathbb{R}$ and radii $\epsilon_1, \epsilon_2, \ldots, \epsilon_N > 0$, the probability that the curve $\gamma$ visits all of $B_{\epsilon_j}(y_j)$, $j = 1, 2, \ldots, N$, should tend to zero as a power law as the radius $\epsilon_j$ is taken small. More precisely, the scaling exponent of the power law is

$$h = \frac{8 - \kappa}{\kappa} \quad (1.1)$$

In the borderline case $\kappa = 8$, the (space-filling) chordal SLE$_8$ curve is the scaling limit of the Peano curve of the uniform spanning tree [LSW04].
Figure 1.2: A schematic illustration of the boundary zig-zag studied in this article: the chordal SLE\(_\kappa\) curve in the upper half-plane \(\mathbb{H}\) starts from \(x\) and visits small neighborhoods of boundary points \(y_1, y_2, \ldots, y_N\).

(see Appendix A), and we are interested in the limit

\[
C^{(N)}_{(\mathbb{H};0,\infty)}(y_1, y_2, \ldots, y_N) = \lim_{\varepsilon_1, \ldots, \varepsilon_N \searrow 0} \frac{1}{\varepsilon_1 \cdots \varepsilon_N} P \left[ \gamma \cap B_{\varepsilon_j}(y_j) \neq \emptyset \text{ for } j = 1, 2, \ldots, N \right]
\]

(1.2)

of probabilities of events illustrated schematically in Figure 1.2. In the spirit of [Law10, LS11, AKL12, LW13, LZ13], it is appropriate to call the limit (1.2) an SLE boundary Green’s function. We emphasize that one could choose a different definition of the boundary visit, and yet, independently of the precise formulation, the answer remains universal apart from a multiplicative constant which depends on the details of the formulation.

Recalling that \(\gamma\) is an oriented curve, we may even specify the order of the boundary visits, i.e., require that the curve \(\gamma\) first reaches the chosen small neighborhood of \(y_1\), then the neighborhood of \(y_2\) and so on until reaching the neighborhood of \(y_N\). The order refinement of the SLE boundary Green’s function is the limit

\[
P^{(N)}_{(\mathbb{H};0,\infty)}(y_1, y_2, \ldots, y_N) = \lim_{\varepsilon_1, \ldots, \varepsilon_N \searrow 0} \frac{1}{\varepsilon_1 \cdots \varepsilon_N} P \left[ \tau_{y_1;\varepsilon_1} < \tau_{y_2;\varepsilon_2} < \cdots < \tau_{y_N;\varepsilon_N} < \infty \right],
\]

(1.3)

where any increasing parametrization \(t \mapsto \gamma_t\) of the curve \(\gamma\) is chosen, and we denote by

\[
\tau_{y_j;\varepsilon_j} = \inf \{ t \geq 0 \mid \gamma_t \in B_{\varepsilon_j}(y_j) \}
\]

(1.4)

the stopping time at which the curve \(\gamma\) first reaches the \(\varepsilon_j\)-neighborhood of \(y_j\). Obviously one can recover the complete correlation function \(C^{(N)}_{(\mathbb{H};0,\infty)}\) from the ordered ones \(P^{(N)}_{(\mathbb{H};0,\infty)}\) by summing over all possible orders of visits  

\[
C^{(N)}_{(\mathbb{H};0,\infty)}(y_1, y_2, \ldots, y_N) = \sum_{\sigma \in S_N} P^{(N)}_{(\mathbb{H};0,\infty)}(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(N)}).
\]

\(^3\text{In fact in the sum we only need those permutations which respect the order of positive } y_j\text{'s and reverse the order of negative } y_j\text{'s, otherwise the curve essentially disconnects its future passage to a point that it would need to visit later.}\)
In the general form with the order of visits specified, the question of finding the asymptotic amplitudes of the visiting probabilities of chordal SLE$_\kappa$ was posed in [BB03a], where these quantities were called “(boundary) zig-zag probabilities”.

Depending on the details of the precise formulation of the boundary visit question, one would obtain a different non-universal multiplicative constant in the SLE boundary Green’s function (1.2) and its order refinement (1.3). We therefore prefer to use a generic notation for a quantity of this type, for which we are free to choose a more convenient multiplicative normalization. We also prefer to make explicit the dependence of the question on the starting point $x \in \mathbb{R}$ of the chordal SLE$_\kappa$ curve, but the end point of the curve will always be kept at infinity. In the rest of this article,

$$\zeta^{(N)}(x; y_1, y_2, \ldots, y_N)$$

denotes a (boundary) zig-zag amplitude, which is proportional to any of the interpretations (see Sections 5.3 and 5.4) of the order refined boundary visit question. In particular we have

$$P^{(N)}_{(x;0,\infty)}(y_1, y_2, \ldots, y_N) = \text{const.} \times \zeta^{(N)}(0; y_1, y_2, \ldots, y_N).$$

Similarly, we denote by

$$\chi^{(N)}(x; y_1, y_2, \ldots, y_N)$$

a complete (boundary) correlation function, so that in particular

$$C^{(N)}_{(x;0,\infty)}(y_1, y_2, \ldots, y_N) = \text{const.} \times \chi^{(N)}(0; y_1, y_2, \ldots, y_N),$$

with the same proportionality constant.

Explicit formulas for the above types of quantities are known in the following two special cases:

- The one-point function ($N = 1$) behaves simply as a power law, as follows immediately from the invariance under dilatations $z \mapsto \lambda z$ ($\lambda > 0$) of the chordal SLE$_\kappa$ in $(\mathbb{H}; 0, \infty)$

$$\zeta^{(1)}(x; y_1) = \chi^{(1)}(x; y_1) \propto |y_1 - x|^{-h} = |y_1 - x|^{1 - \frac{\kappa}{2}}. \quad (1.5)$$

- The two-point function when $y_1$ and $y_2$ are on the same side of the starting point (either $x < y_1 < y_2$ or $y_2 < y_1 < x$) is given by a hypergeometric function [SZ10] (see also [BB03a])

$$\zeta^{(2)}(x; y_1, y_2) = \chi^{(2)}(x; y_1, y_2) \propto |y_1 - x|^{1 - \frac{\kappa}{2}} |y_2 - y_1|^{1 - \frac{\kappa}{2}} \times {}_2F_1 \left( \frac{4}{\kappa}; \frac{8}{\kappa}; \frac{8}{\kappa}; \frac{y_2 - y_1}{y_2 - x} \right). \quad (1.6)$$

In this article we present a method for finding the solutions in the general case. We write down a system of partial differential equations (PDEs) motivated by conformal field theory (CFT) for the quantities of interest, $\zeta^{(N)}$ and $\chi^{(N)}$. Our solutions for them are written in terms of Coulomb gas integrals [DF84] and are found by quantum group calculations. This technique is developed in the present article and in [KP13], we call it the spin chain - Coulomb gauge correspondence. Our primary goal here is to find the explicit formulas and show their wide applicability: the functions $\zeta^{(N)}$ and $\chi^{(N)}$ answer various formulations of boundary visit questions for SLEs as well as for interfaces in lattice models. We also compare the results to numerical simulations of various lattice models, and outline a strategy of proof that our formulas give the (order refined) SLE boundary Green’s functions.

We emphasize that it is very rarely possible to find the exact solution for an SLE problem involving a large number of marked points — the few existing solutions to such problems rely on finding tricks that appear particular to each problem [Hag09, HD08, SZK09, SK11, SKFZ11, BI12, AKL12, FKZ12, FK12, FK13] The key technique that enables us to find the exact solution here is the spin chain - Coulomb gauge correspondence. It provides a systematic method to solve a quite general class of SLE problems.

\footnote{In contrast, it is almost routine to answer chordal SLE questions which involve only two boundary points or one bulk}
1.3 Organization of the article

The rest of the article is organized as follows.

In Section 2, we formulate the PDE problem which we solve in the subsequent sections to find the zig-zag amplitudes $\zeta^{(N)}$ and the complete correlation functions $\chi^{(N)}$:

- The functions $\zeta^{(N)}$ and $\chi^{(N)}$ are conformally covariant.
- The functions $\zeta^{(N)}$ and $\chi^{(N)}$ satisfy a second order PDE and $N$ third order PDEs.
- The boundary conditions depend on the order of visits: they are written in terms of asymptotic behaviors of $\zeta^{(N)}$ and their inhomogeneous terms involve the $\zeta^{(N-1)}$ in a recursive manner.

In Section 3, we discuss the spin chain - Coulomb gas correspondence, by which the PDE problem is translated to a linear problem in representations of a quantum group:

- We associate functions defined by Coulomb gas integrals to vectors in a finite-dimensional tensor product representation of the quantum group $U_q(sl_2)$.
- The functions associated to highest weight vectors are solutions to the partial differential equations of Section 2 and for particular highest weights they also have the correct conformal covariance.
- Projections to subrepresentations in consecutive tensorands determine the asymptotic behaviors of the functions.
- There are unique highest weight vectors of the correct highest weights whose subrepresentation projections correspond to the boundary conditions imposed on the zig-zag amplitudes $\zeta^{(N)}$.

In Section 4, the integrals obtained in the spin chain - Coulomb gas correspondence are rewritten as regularized real integrals. Different regularizations are considered and are all shown to give the same results. The transformation to real integrals concretely exhibits the needed closed homology properties of our solutions.

In Section 5, we discuss basic properties, applications, interpretations, and universality of the SLE boundary visit question and outline a strategy of proof.

In Section 6, we compare our formula numerically to simulations of lattice models of statistical mechanics. We study random interfaces in percolation, random cluster model, and loop-erased random walk. We perform computer simulations of them and collect frequencies of multi-point boundary visits of the interfaces, and compare renormalized frequencies to the zig-zag amplitudes $\zeta^{(N)}$.

We conclude the article by discussion and outlook in Section 7.

The article is complemented with several appendices. Appendix A provides two derivations of the value of the scaling exponent (1.1), and a derivation of the second order PDE. Appendix B contains relevant background on conformal field theory. Our normalization conventions for some quantum group representations and some explicit four-point solutions are contained in Appendix C. Numerical evaluation of the integrals of Sections 3 and 4 is treated in Appendix D.
2 The problem: partial differential equations and asymptotics

We find the boundary visit amplitudes $\zeta^{(N)}$ and $\chi^{(N)}$ by solving a PDE problem. The system of partial
differential equations is given below in Section 2.1. This part is the same for $\chi^{(N)}$ and for $\zeta^{(N)}$, and moreover
the system is the same for all boundary zig-zag amplitudes corresponding to different orders of visits to
the same set of points. The results will be different, however, as each of the functions satisfies different
boundary conditions, detailed in Section 2.2.

2.1 Differential equations for boundary visit amplitudes

The linear homogeneous system of PDEs below contains essentially three different types of partial differential
equations — all of them can be argued to hold by conformal field theory (see Appendix B.2), but from the
point of view of SLE analysis, the argument leading to each of them is different. For $\zeta^{(N)}$ the system reads:

\[
\begin{align*}
\left[ \frac{\partial}{\partial x} + \sum_{j=1}^{N} \frac{\partial}{\partial y_j} \right] \zeta^{(N)}(x; y_1, \ldots, y_N) &= 0 \\
\left[ \frac{x}{\partial x} + \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j} - Nh \right] \zeta^{(N)}(x; y_1, \ldots, y_N) &= 0 \\
\left[ \frac{\partial^2}{\partial x^2} - \frac{4}{\kappa} \mathcal{L}_{-2} \right] \zeta^{(N)}(x; y_1, \ldots, y_N) &= 0 \\
\left[ \frac{\partial^3}{\partial y_j^3} - \frac{16}{\kappa} \mathcal{L}^{(j)} \frac{\partial}{\partial y_j} + \frac{8(8 - \kappa)}{\kappa^2} \mathcal{L}^{(j)}_{-3} \right] \zeta^{(N)}(x; y_1, \ldots, y_N) &= 0 \quad (j = 1, 2, \ldots, N),
\end{align*}
\]

where

\[
\mathcal{L}_{-2} = \sum_{k=1}^{N} \left( \frac{-1}{y_k - x} \frac{\partial}{\partial y_k} + \frac{h}{(y_k - y_j)^2} \right)
\]

and

\[
\mathcal{L}^{(j)} = \frac{-1}{(x - y_j)^{n-1}} \frac{\partial}{\partial x} + \frac{(n - 1)\delta}{(x - y_j)^n} + \sum_{k \neq j} \left( \frac{-1}{(y_k - y_j)^{n-1}} \frac{\partial}{\partial y_k} + \frac{(n - 1)h}{(y_k - y_j)^n} \right).
\]

The first order PDEs (2.1) and (2.2) express the translation invariance and homogeneity of the amplitudes. More general conformal covariance of the answer will be discussed in Section 2.2 and again from a
conformal field theory point of view in Appendix B.1. The second order PDE (2.3) can be interpreted either
in terms of the SLE process as the statement of a local martingale property of the answer, see Appendix A.3
or in terms of conformal field theory as a conformal Ward identity associated to a second order degeneracy
of the boundary field located at $x$, as will be discussed in Appendix B.2. The $N$ third order PDEs (2.3) are
similarly the conformal Ward identities associated to third order degeneracies of the boundary fields
located at $y_j$, $j = 1, 2, \ldots, N$, see Appendix B.2. Unlike for the first and second order equations we do not
know how to explain the third order equations by SLE analysis directly. The validity of these equations for
the SLE boundary visit amplitudes would need to be established by first finding the explicit answer, which
is the main task in the present article, and then proving that it gives the SLE boundary Green’s function
following the strategy that will be outlined in Section 5.4.3.\(^5\)

\(^5\)Given that this proposed route to Equations (2.3) is somewhat indirect, one may wonder if more direct hints of these third
order differential equations exist. To this end, recall that for $N = 1$ and $N = 2$ the explicit zig-zag amplitudes (1.5) and (1.6)
can in any case be found by routine SLE calculations. For these already known functions $\zeta^{(1)}$ and $\zeta^{(2)}$, then, one may simply
check the validity of the third order equations, which conformal field theory predicts.
2.2 Asymptotics for boundary visit amplitudes

The system of differential equations of Section 2.1 has a large space of solutions. To pin down the correct solution we need boundary conditions, which will be specified in the form of asymptotic behavior of the boundary zig-zag amplitudes. Considerations of the possible asymptotics allowed by conformal field theory can be found in Appendix B.3. The particular requirements that finally specify the solutions are given below.

Consider the question of visiting the neighborhoods of \( y_1, y_2, \ldots, y_N \) in this order. Some notation and terminology is needed to conveniently describe the specific asymptotics of \( \zeta^{(N)} \) in this case. We say that points \( y_j \) such that \( y_j < x \) are on the left and points \( y_j \) such that \( x < y_j \) are on the right. We say that the points are in an outwards increasing order if for any \( y_j, y_k \) on the left we have that \( j < k \) implies \( y_k < y_j \) and for any \( y_j, y_k \) on the right we have that \( j < k \) implies \( y_j < y_k \), in other words that among points on the same side, the point further away from starting point is visited later.

The boundary visit amplitude vanishes unless that points are in an outwards increasing order — a visit to a small neighborhood of a point further away on the same side almost disconnects the future passage of the curve to the point that would need to be visited later.

It is convenient to use a separate ordering for the points on the left and right. Denote therefore \( y_1^-, \ldots, y_L^- \) the points on the left in a decreasing order (in the order of visits) and \( y_1^+, \ldots, y_R^+ \) the points on the right in an increasing order (in the order of visits). The following notation makes the arguments of the zig-zag amplitude appear in the same order as they are on the real axis,

\[
\zeta_\omega(y_L^-, \ldots, y_1^-; x; y_1^+, \ldots, y_R^+) = \zeta^{(N)}(x; y_1, y_2, \ldots, y_N),
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_N) \in \{+, -\}^N \) is a sequence of “\( \pm \)”-symbols specifying the sequence of sides of the visits in the sense that \( \omega_j = - \) (resp. \( \omega_j = + \)) if \( y_j \) is on the left (resp. on the right). If we fix the number \( L \) of points on the left and the number \( R \) of points on the right, \( N = L + R \), then the number of different outwards increasing orders is \( \binom{N}{L} \), corresponding to the choices of \( \omega \in \{+, -\}^N \) with \( L \) “\(-\)”-symbols and \( R \) “\(+\)”-symbols. The complete correlation function \( \chi^{(N)} \) is the sum of these \( \binom{N}{L} \) zig-zag amplitudes. In the particular case when all the points are on the same side, the complete correlation function coincides with the zig-zag amplitude.

The specific asymptotics depend on the order of visits, and to describe them we need a few separate cases.

We claim that for any outwards increasing order \( \omega \) the boundary zig-zag amplitude \( \zeta_\omega \) satisfies the asymptotics conditions given below, and that up to a multiplicative constant these asymptotics determine all \( \zeta^{(N)} \). The conditions are intuitive in view of the possibilities listed in Appendix B.3. They state that the order of magnitude of the amplitude is larger if successively visited points are close and smaller if non-successively visited points are close, and in the former case the leading asymptotic is proportional to an \((N-1)\)-point function, where the two close-by points are replaced by a single point. Moreover, they state that the leading behavior when successively visited points are close-by is given by the \((N-1)\)-point function with the two close-by points replaced by just one. The eventual justification will be a proof, discussed in Section 5.4.3 of the formula obtained by solving the PDE system with these conditions.

- **Asymptotics for successively visited points:** If \( y_j \) and \( y_{j+1} \) are successively visited points on the same side, then

\[
\lim_{y_j, y_{j+1} \to y'} \frac{1}{|y_{j+1} - y_j|^{1 - \frac{\omega}{2}}} \zeta^{(N)}(x; y_1, \ldots, y_j, y_{j+1}, \ldots, y_N) = \text{const.} \times \zeta^{(N-1)}(x; y_1, \ldots, y_{j-1}, y', y_{j+2}, y_{j+3}, \ldots, y_N). \tag{2.5}
\]

\(^6\)Rigorous estimates of the appropriate SLE probabilities are of the type considered, e.g., in Bef88 LW13. The present situation is somewhat easier.
• Asymptotics for non-successively visited points: If \( y_j \) and \( y_k \) are non-successively visited consecutive points on the same side, then
\[
\lim_{y_j, y_k \to y'} \frac{1}{|y_k - y_j|^{1-\frac{D}{2}}} \zeta^{(N)}(x; y_1, y_2, \ldots, y_N) = 0. \tag{2.6}
\]

• Asymptotics for the first points on the left and right: For the first point \( y_1 \) to be visited we have
\[
\lim_{x, y_1 \to x'} \frac{1}{|y_1 - x|^{1-\frac{D}{2}}} \zeta^{(N)}(x; y_1, y_2, \ldots, y_N) = \text{const.} \times \zeta^{(N-1)}(x'; y_2, y_3, \ldots, y_N). \tag{2.7}
\]

For the first point on the opposite side, i.e., for \( y_1^\pm \neq y_1 \), we have
\[
\lim_{x, y_1^\pm \to x'} \frac{1}{|y_1^\pm - x|^{1-\frac{D}{2}}} \zeta^{(N)}(x; y_1, y_2, \ldots, y_N) = 0. \tag{2.8}
\]

Our choice of normalization of \( \zeta^{(N)} \) and \( \chi^{(N)} \) will be determined recursively by fixing the constant appearing in Equation (2.7), see Section 3.4. Once this natural choice is made, the different \( N \)-point functions \( \zeta, \chi \) obtain correct relative normalizations, with the universal ratios referred to in Section 5.4. The constant appearing in Equation (2.5) gets automatically fixed as well.

3 Quantum group and integral formulas

3.1 Coulomb gas integrals

The main tool that allows us to solve the PDE problem of Section 2 and therefore to find the explicit formula for the SLE boundary visit amplitudes is a spin chain - Coulomb gas correspondence. In this article, for the sake of concreteness, we describe only the case relevant to the problem of boundary visit amplitudes — a more general treatment can be found in [KP13].

3.1.1 Standard Coulomb gas integrals and their properties

The Coulomb gas formalism of conformal field theory, or Dotsenko-Fateev integrals [DF84], are a way of producing solutions to systems of differential equations of the type of Section 2.1 by integrating an auxiliary function, which in our case takes the form
\[
f_\ell^{(N)}(x; y_1, y_2, \ldots, y_N; w_1, w_2, \ldots, w_\ell) = \prod_{j=1}^N (y_j - x)^{\frac{D}{2}} \times \prod_{1 \leq j < k \leq N} (y_k - y_j)^{\frac{D}{2}} \times \prod_{s=1}^\ell (w_s - x)^{-\frac{D}{2}} \times \prod_{j=1}^N \prod_{s=1}^\ell (w_s - y_j)^{-\frac{D}{2}} \times \prod_{1 \leq s < r \leq \ell} (w_r - w_s)^{\frac{D}{2}}. \tag{3.1}
\]

Consider the function
\[
F(x; y_1, \ldots, y_N) = \int_{\Gamma} f_\ell^{(N)}(x; y_1, \ldots, y_N; w_1, \ldots, w_\ell) dw_1 \cdots dw_\ell, \tag{3.2}
\]
where \( \Gamma \) is a closed \( \ell \)-surface avoiding the points \( x, y_1, \ldots, y_N \). The integral of course only depends on the homotopy type of the contour \( \Gamma \). The function is defined such that while the contour \( \Gamma \) of the \( w \)-variables may depend on the positions of \( x, y_1, \ldots, y_N \), the choice is locally constant. One then observes:

• translation invariance: \( F \) satisfies Equation (2.1).
The translation invariance follows immediately from the translation invariance of the integrand $f^{(N)}_{\ell}$ by considering a shift of the variables $x, y_1, \ldots, y_N$ small enough so that the integration contour $\Gamma$ can be kept constant, and then the same shift of the integration contour, which now does not change the homotopy type. The scaling covariance is shown similarly, starting with scaling close enough to identity. The relevant scaling covariance of the integrand reads

$$f^{(N)}_{\ell}(\lambda x; \lambda y_1, \ldots; \lambda w_1, \ldots) = \lambda^\Delta f^{(N)}_{\ell}(x; y_1, \ldots; w_1, \ldots)$$

and an extra factor $\lambda^\ell$ comes from the change of variables in the integration.

The second and third order differential equations rely more crucially on the fact that the integration surface $\Gamma$ is closed. One again starts from a property satisfied by the integrand alone. Starting from the second order equation, let

$$D_{1,2} = \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} + \sum_{j=1}^N \left( \frac{2}{y_j - x} \frac{\partial}{\partial y_j} - \frac{2h}{(y_j - x)^2} \right)$$

be the differential operator we want to show annihilates $F$. It is a matter of straightforward verification to see that the integrand satisfies

$$\left[D_{1,2} + \sum_{s=1}^\ell \left( \frac{2}{w_s - x} \frac{\partial}{\partial w_s} - \frac{2}{(w_s - x)^2} \right) \right] f^{(N)}_{\ell}(x; y_1, \ldots; w_1, \ldots) = 0$$

and to notice that this can also be read as

$$D_{1,2} f^{(N)}_{\ell}(x; y_1, \ldots; w_1, \ldots) = -2 \sum_{s=1}^\ell \frac{\partial}{\partial w_s} \left( \frac{1}{w_s - x} \times f^{(N)}_{\ell}(x; y_1, \ldots; w_1, \ldots) \right).$$

Thus when acting on $F$ by the differential operator $D_{1,2}$, we may take the operator inside the integral, and rewrite the integrand as a sum of total derivatives. The integral of these vanish because the contour was assumed to be closed. Hence one gets the second order differential equation for $F$. The third order differential equations are shown to hold similarly.

### 3.1.2 Spin chain - Coulomb gas basis functions

Our solution will eventually be of the form (3.2), with $\ell = N$. As in [KPI3], we need to unveil an underlying quantum group structure, which will be useful for calculations, and in particular crucial for dealing with the asymptotics. For this purpose, we introduce the functions

$$\varphi_{t_L^-, \ldots, t_R^+}(y_L, \ldots, y_R; x; y_1, y_2, \ldots, y_d)$$

indexed by $t_j^+ \in \{0, 1\}$ and $d \in \{0, 1\}$, which are defined by the integrals

$$\varphi_{t_L^-, \ldots, t_R^+}(y_L, \ldots, y_R) = \int_{\Gamma_{t_L^-, \ldots, t_R^+}} f^{(N)}_{\ell}(y_L, \ldots, y_R; w_1, \ldots, w_\ell) \, dw_1 \cdots dw_\ell, \quad (3.3)$$

where:
Figure 3.1: The integration contours of the $w_j$-variables in $\Gamma_{t_L-1,...,t_1-d:t_1^+,...:t_R^+}$ and the point (marked by red circles) where the integrand is rephased to be positive.

- The integration surface $\Gamma_{t_L-1,...,t_1-d:t_1^+,...:t_R^+}$ is shown in Figure 3.1. The dimension of the integration surface, i.e., the number of integration variables $w_s$, is $\ell = d + \sum_{j=1}^L t_j^- + \sum_{j=1}^R t_j^+$. In the functions appearing in our final answer this will always be $\ell = N$. The contour of each integration variable $w_s$ is a loop based at an anchor point $z_0$ to the left of all of the variables, and the loop encircles one of the points in the positive direction. The loops of the first $t_L^-$ variables encircle the point $y_L^-$, the next $t_{L-1}^-$ variables encircle the point $y_{L-1}^-$ and so on. The loops encircling different points avoid each other so that the contours to a point further on the right go below.

- The integrand $f^\circ_{t_L^-,...,t_1-d:t_1^+,...:t_R^+}$ is a rephased branch of the integrand $f^{(N)}_\ell$ defined in Equation (3.1): we multiply by a suitable complex number of modulus one to make $f^\circ_{t_L^-,...,t_1-d:t_1^+,...:t_R^+}$ real and positive at the point where each of the integration variables is on the real axis to the right of the point it encircles, see Figure 3.1.

We make the following remarks about the role and properties of the above functions:

- Individually the surfaces $\Gamma_{t_L^-,...,t_1-d:t_1^+,...:t_R^+}$ are not closed, but our solution will be a linear combination which is closed in the appropriate homology \cite{FW91}.

- The individual functions $\varphi_{t_L^-,...,t_1-d:t_1^+,...:t_R^+}$ depend also on the point $z_0$ where the loops in $\Gamma_{t_L^-,...,t_1-d:t_1^+,...:t_R^+}$ are anchored. This dependence will cancel in the final answer — the cancellation will be shown concretely in Section 4, and a proof of this property in a general setup is given in \cite{KP13}.

In the spin chain - Coulomb gas correspondence defined in Section 3.3.1 we will make basis vectors in a quantum group representation correspond to the functions $\varphi_{t_L^-,...,t_1-d:t_1^+,...:t_R^+}$. In Sections 3.3.2 and 3.3.3 we explain how straightforward quantum group calculations will allow us to decide about the asymptotics of the functions as well as the closedness of the surfaces in an appropriate homology — see also \cite{FW91,KP13}.

3.2 Quantum group

We need to recall some facts and fix some notation for the quantum group $U_q(\mathfrak{sl}_2)$. It should be thought of as a deformation of (the universal enveloping algebra of) the Lie algebra $\mathfrak{sl}_2$, with a deformation parameter $q$ — with a suitable normalization when $q \to 1$ one recovers $\mathfrak{sl}_2$ from the definitions we give below.

We let $q = e^{i\pi i/\kappa}$, and assume that $\kappa$ is generic in the sense that $\kappa \notin \mathbb{Q}[\sqrt{m}]$ We define the $q$-integers $[m]_q$.

\footnote{For irrational $\kappa$ the parameter $q$ is not a root of unity, and the representation theory of the quantum group is semisimple. For the SLE boundary visit amplitudes, we may in the end argue by continuity in the parameter $\kappa$.}
3 QUANTUM GROUP AND INTEGRAL FORMULAS

(for \( m \in \mathbb{Z} \))

\[
[m] := \frac{q^m - q^{-m}}{q - q^{-1}}
\]

Since we assume \( \kappa \notin \mathbb{Q} \), all \( q \)-integers \([m]\) with \( m \neq 0 \) are non-zero.

3.2.1 Definition of the quantum group

The quantum group \( \mathcal{U}_q(\mathfrak{sl}_2) \) is the algebra over \( \mathbb{C} \) with generators \( E, F, K, K^{-1} \) and relations

\[
KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quadKF = q^{-2}FK,
\]

\[
EF - FE = \frac{1}{q - q^{-1}} (K - K^{-1}).
\]

Moreover, \( \mathcal{U}_q(\mathfrak{sl}_2) \) is equipped with the unique Hopf algebra structure such that the coproducts of the generators are

\[
\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.
\]

The coproduct \( \Delta \) determines the action of the quantum group in tensor product \( V \otimes V' \) of two representations \( V \) and \( V' \), for example \( E.(v \otimes v') = E.v \otimes K.v' + v \otimes E.v' \). The tensor product of representations is then associative but not commutative: multiple tensor products are well defined, for example \( (V \otimes V') \otimes V'' \cong V \otimes (V' \otimes V'') \), but the order of the tensorands is important.

3.2.2 Representations of the quantum group

The quantum group \( \mathcal{U}_q(\mathfrak{sl}_2) \) is semisimple (for \( q \) not a root of unity) in the sense that any finite dimensional representation is the direct sum of its irreducible subrepresentations. In fact, the representation theory essentially just deforms that of \( \mathfrak{sl}_2 \). For any \( d \in \mathbb{N} \), there exists a \( d \)-dimensional irreducible representation \( V_d \) with a basis \( e_0, e_1, e_2, \ldots, e_{d-1} \) such that the action of the generators on the basis vectors is given by

\[
K.e_j = q^{d-1-2j} e_j
\]

\[
F.e_j = e_{j+1}
\]

\[
E.e_j = [j][d-j] e_{j-1} \quad \text{(with interpretation } e_{d} = 0) \]

This representation \( V_d \) is the appropriate deformation of the \( d \)-dimensional irreducible of \( \mathfrak{sl}_2 \) ("the spin-d/2 representation"). The tensor products of \( V_d \) decompose according to the (quantum) Clebsch-Gordan formula

\[
V_{d_1} \otimes V_{d_2} \cong V_{d_1+d_2-1} \oplus V_{d_1+d_2-3} \oplus \cdots \oplus V_{|d_1-d_2|+1},
\]

and our calculations will require some specific cases of this decomposition to be made explicit.

The one-dimensional irreducible \( V_1 \cong \mathbb{C} \) is the trivial representation, it acts as a neutral element of the tensor products: for any representation \( V \) we have the isomorphisms \( V_1 \otimes V \cong V \cong V \otimes V_1 \). This allows us to omit \( V_1 \) in tensor products, when needed.

3.3 Spin chain - Coulomb gas correspondence

3.3.1 Definition of the correspondence

With the above preparations we can now define the correspondence. The spin chain - Coulomb gas correspondence linearly associates to vectors

\[
v \in V_3^R \otimes V_2 \otimes V_3^L
\]
in a tensor product of representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ a function, so that for the natural tensor product basis vectors the associated functions are those defined in Section 3.1.2.

$$e_{t_R}^+ \otimes \cdots \otimes e_{t_1}^+ \otimes e_d \otimes e_{t_1}^- \otimes \cdots \otimes e_{t_L}^- \mapsto \varphi_{t_L^- \cdots t_1^- \cdots t_R^+}.$$  

Note that in our convention, the order of the variables of the function is the reverse of the order of the corresponding factors in the tensor product.

### 3.3.2 Asymptotics via the correspondence

A key property of the spin chain - Coulomb gas correspondence is that the asymptotics of the functions can be straightforwardly read from the projections to subrepresentations of the corresponding vectors in $V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes L}$. We use the conventions of Appendix C.1 in particular the various $\pi^{(d)}$ below are projections to various subrepresentations of $V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes L}$, and $\tilde{\pi}^{(d)}$ are similar projections together with identifications of these subrepresentations with other appropriate representations.

Let $v \in V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes L}$ and let $\varphi$ be the function associated to $v$ by the correspondence of Section 3.3.1

The correspondence of asymptotics and subrepresentations is stated precisely in the following:

- Consider two consecutive points $y_m^\pm, y_{m+1}^\pm$ on the right or left (superscript “+” or “−”, respectively).

  - Suppose that $v$ is in the singlet of the components corresponding to $y_m^\pm, y_{m+1}^\pm$, that is $v = \pi^{(1)}_{\pm;m}(v)$.
    Then as $y_m^+, y_{m+1}^+ \to y'$, we have
    $$\varphi(x; y_1, \ldots, y_N) \sim B_1 \times |y_{m+1}^+ - y_m^+|^{-\frac{2}{\kappa}} \times \varphi^{(1)}(x; y_1, \ldots, y_N),$$
    where the variables $y_m^+, y_{m+1}^+$ have been removed from the right side, the function $\varphi^{(1)}$ is the function of two variables less associated to the vector $\pi^{(1)}_{\pm;m}(v)$ interpreted as a vector in either $V_3^{\otimes (R-2)} \otimes V_2 \otimes V_3^{\otimes L}$ or $V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes (L-2)}$, and the constant is the generalized beta-function\textsuperscript{8}.
    $$B_1 = \int_0^1 dw_1 \int_{w_1}^1 dw_2 \ w_1^{-\frac{\kappa}{8}} w_2^{-\frac{\kappa}{8}} (w_2 - w_1)^{\frac{\kappa}{8}} (1 - w_1)^{-\frac{\kappa}{8}} (1 - w_2)^{-\frac{\kappa}{8}}.$$

  - Suppose that $v$ is in the triplet of the components corresponding to $y_m^\pm, y_{m+1}^\pm$, that is $v = \pi^{(3)}_{\pm;m}(v)$.
    Then as $y_m^+, y_{m+1}^+ \to y'$, we have
    $$\varphi(x; y_1, \ldots, y_N) \sim B_3 \times |y_{m+1}^+ - y_m^+|^{-\frac{1}{\kappa}} \times \varphi^{(3)}(x; y_1, \ldots, y', \ldots, y_N),$$
    where on the right hand side the two variables $y_m^+, y_{m+1}^+$ have been removed and replaced by one $y'$, the function $\varphi^{(3)}$ is the function of one variable less associated to the vector $\pi^{(3)}_{\pm;m}(v)$ interpreted as a vector in either $V_3^{\otimes (R-1)} \otimes V_2 \otimes V_3^{\otimes L}$ or $V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes (L-1)}$ and the constant is the beta-function
    $$B_3 = \int_0^1 dw \ w^{-\frac{\kappa}{8}} (1 - w)^{-\frac{\kappa}{8}} = \frac{\Gamma\left(\frac{\kappa-\kappa}{8}\kappa\right)}{\Gamma\left(2\frac{\kappa-\kappa}{8}\kappa\right)}.$$

  - Suppose that $v$ is in the quintuplet of the components corresponding to $y_m^\pm, y_{m+1}^\pm$, that is $v = \pi^{(5)}_{\pm;m}(v)$.
    Then as $y_m^+, y_{m+1}^+ \to y'$, we have
    $$\varphi(x; y_1, \ldots, y_N) \sim |y_{m+1}^\pm - y_m^\pm|^{-\frac{\kappa}{8}} \times \varphi^{(5)}(x; y_1, \ldots, y', \ldots, y_N),$$
    where on the right hand side the two variables $y_m^+, y_{m+1}^+$ have been removed and replaced by one $y'$, and the function $\varphi^{(5)}$ could also be written explicitly (see [KP13] for details).

\textsuperscript{8}The integrals here are convergent for $\kappa > 8$, whereas for the relevant parameter range $\kappa \in (0, 8) \setminus \mathbb{Q}$ they need to be regularized as discussed in Section 3.1. Careful examination of our approach shows that we get well defined integrals along Pochhammer-type contours, but in fact many reasonable regularization procedures yield the same results.
• Consider the point $x$ and the first point $y_1^\pm$ on the right or left (superscript “+” or “−”, respectively).

  - Suppose that $v$ is in the doublet of the components corresponding to $x, y_1^\pm$, that is $v = \pi_\pm^{(2)}(v)$.
  Then as $x, y_1^\pm \rightarrow x'$, we have
    \[ \varphi(x; y_1, \ldots, y_N) \sim B_2 \times |y_1^\pm - x'|^{1 - \frac{\kappa}{2}} \times \varphi^{(2)}(x'; \ldots, y_N), \]
  where on the right hand side the two variables $x, y_1^\pm$ have been removed and replaced by one $x'$, the function \( \varphi^{(2)} \) is the function of one variable less associated to the vector \( \hat{\pi}_\pm^{(2)}(v) \) interpreted as a vector in either \( V_3^{\otimes(R-1)} \otimes V_2 \otimes V_3^{\otimes L} \) or \( V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes(L-1)} \) and the constant is the beta-function
    \[ B_2 = \int_0^1 dw \ w^{-\frac{\kappa}{4}} (1 - w)^{-\frac{\kappa}{2}} = \frac{\Gamma(\frac{\kappa-4}{\kappa}) \Gamma(\frac{\kappa+2}{\kappa})}{\Gamma(2\frac{\kappa-2}{\kappa})}. \] (3.4)
  - Suppose that $v$ is in the quadruplet of the components corresponding to $x, y_1^\pm$, that is $v = \pi_\pm^{(4)}(v)$.
  Then as $x, y_1^\pm \rightarrow x'$, we have
    \[ \varphi(x; y_1, \ldots, y_N) \sim |y_1^\pm - x'|^{\frac{\kappa}{2}} \times \varphi^{(4)}(x'; \ldots, y_N), \]
  where on the right hand side the two variables $x, y_1^\pm$ have been removed and replaced by one $x'$, and the function \( \varphi^{(4)} \) could also be written explicitly (see [KP13] for details).

For a general $v \in V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes L}$ the asymptotics of $\varphi$ are obtained by the above formulas and linearity.

The statements are proved by straightforward manipulations of the integrals, which are done in a more general setup in [KP13]. Indeed, when the vector $v$ is of the supposed form, we know from Appendix C.1 explicitly how its two consecutive tensor components must be related. Considering the different possibilities for $y_1^\pm, y_{m+1}^\pm$ namely $v = \pi_{\pm;m}^{(5)}(v), v = \pi_{\pm;m}^{(3)}(v), v = \pi_{\pm;m}^{(1)}(v)$ one manages to rearrange zero, one, or two integration variables on contours between the points $y_m^\pm$ and $y_{m+1}^\pm$ so that the contours of the rest of the integration variables remain away from these points. Then extracting the asymptotics becomes easy: first of all there is a factor $|y_m^\pm - y_{m+1}^\pm|^{\frac{\kappa}{2}}$ in the integrand, and secondly the integral over the contours between the points $y_m^\pm$ and $y_{m+1}^\pm$ can be rescaled to produce (modulo error terms that can be neglected in the limit $y_m^\pm, y_{m+1}^\pm \rightarrow y'$) a generalized beta-function times a power law $|y_m^\pm - y_{m+1}^\pm|^{\Delta_i}$ with $\Delta_i = l + \frac{\kappa}{4}(\frac{2l-1}{2} - 2l)$ according to the number $l = 0, 1, 2$ of integration variables on contours between the points $y_m^\pm$ and $y_{m+1}^\pm$.

For the rest of the integrations, we may combine the factors in the integrand containing the variables $y_m^\pm, y_{m+1}^\pm$ or any of the integration variables between them, and we get a function of the same type, with fewer variables. The different possibilities for $x, y_1^\pm$ are treated in an entirely parallel fashion.

### 3.3.3 Highest weight vectors and closed integration surfaces

For fundamental properties of the Dotsenko-Fateev functions in Section 3.1.1 it was important that the integration surface $\Gamma$ was closed in an appropriate homology related to the multivalued integrand (3.3), see [FW91]. Our basis functions $\varphi_{\Gamma t_0^\ldots t_n^k}$ for the spin chain - Coulomb gas correspondence, introduced in Section 3.1.2 are obtained by integrals along the contours $\Gamma_{t_0^\ldots t_n^k}$ of Figure 3.1 which do not constitute a closed surface. Remarkably, however, Felder and Wieczorkowski [FW91] showed that if the vector $v$ is annihilated by the quantum group generator $E$, i.e., if $v$ is a sum of highest weight vectors of subrepresentations, then the homology class of the associated linear combination of $\Gamma_{t_0^\ldots t_n^k}$ is closed. Less abstractly, this can be viewed as a generalization of the manipulations of the integrals we described in the end of Section 3.3.2 and we exhibit this property very concretely by transforming the integrals to integrals along the real axis in Section 4.1.

Importantly, if $v \in V_3^{\otimes R} \otimes V_2 \otimes V_3^{\otimes L}$ satisfies $E.v = 0$, then the associated function $\varphi$ has the following properties:
• The function $\varphi$ does not depend on the choice of the anchor point $z_0$ of the contours $\Gamma_{t^+_L, ..., t^+_R}$.

• The function $\varphi$ satisfies the second order differential equation (2.3).

• The function $\varphi$ satisfies the third order differential equations (2.4).

### 3.4 Linear problem in quantum group representations

Recall that we are looking for solutions to the partial differential equations (2.1), (2.2), (2.3), (2.4), with boundary conditions specified in terms of the asymptotics (2.5), (2.6), (2.7), (2.8). We will produce the solution by the spin chain - Coulomb gas correspondence of Section 3.3.1: we will find a vector $v$ so that the associated function $\varphi$ solves the problem.

More precisely, for all order specifications $\omega \in \{+,-\}^N$ with $R$ “+”-symbols and $L$ “−”-symbols, we want vectors $v^{(N)}_\omega \in V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L$ such that the function associated to $v_\omega$ by the spin chain - Coulomb gas correspondence is the boundary zig-zag amplitude $\zeta_\omega(y_L, ..., y_1; x; y^+_1, ..., y^+_R)$. This will be achieved if the vectors $v_\omega$ satisfy the following:

- **Highest weight vector of a doublet subrepresentation:**
  
  \[ E.v^{(N)}_\omega = 0 \]
  \[ K.v^{(N)}_\omega = q v^{(N)}_\omega. \]

- **Projections to singlet and triplet for successively visited points:** If $y^+_m$ and $y^+_m+1$ are successively visited points on the same side ($y^+_m = y_j$ and $y^+_m+1 = y_{j+1}$), then
  
  \[ \pi^{(1)}_{\pm;m}(v^{(N)}_\omega) = 0 \]
  \[ \pi^{(3)}_{\pm;m}(v^{(N)}_\omega) = \text{const.} \times v^{(N-1)}_\omega, \]

  where $\omega' = (\omega_1, \omega_2, ..., \omega_j, -1, \omega_j, \omega_j+1, \omega_j+3, ..., \omega_N)$.

- **Projections to singlet and triplet for non-successively visited points:** If $y^+_m$ and $y^+_m+1$ are non-successively visited consecutive points on the same side ($y^+_m = y_j$ and $y^+_m+1 = y_k$ with $k-j > 1$), then
  
  \[ \pi^{(1)}_{\pm;m}(v^{(N)}_\omega) = 0 \]
  \[ \pi^{(3)}_{\pm;m}(v^{(N)}_\omega) = 0. \]

- **Projections to doublet for the first points on the left and right:** Let $\pm$ denote the side of the first visited point, $y_1 = y^+_1$, and $\mp$ the opposite side. For the first visited point the condition is
  
  \[ \pi^{(2)}_{\pm}(v^{(N)}_\omega) = \text{const.} \times v^{(N-1)}_\omega, \]

  where $\omega' = (\omega_2, \omega_3, ..., \omega_N)$. For the first point on the opposite side the condition is
  
  \[ \pi^{(2)}_{\mp}(v^{(N)}_\omega) = 0. \]

By the closed integration surface considerations of Section 3.3.3, Equation (3.5) guarantees that the function associated to $v_\omega$ is independent of the anchor point and satisfies the PDEs (2.3), (2.4). The translation invariance (2.1) is then obvious. Equation (3.6) guarantees that the associated function is a linear combination of $\varphi_{t^+_L, ..., t^+_R, d, t^+_1, ..., t^+_R}$ with $d + \sum_j t^+_j + \sum_j t^-_j = N$, and therefore by the results of Section 3.1.1 it
has the correct scaling covariance \((2.2)\). Finally, by the asymptotics properties of Section \(3.3.2\) we see that Equations \((3.7)\), \((3.8)\), \((3.9)\), \((3.10)\) guarantee \((2.5)\), \((2.6)\), \((2.7)\), \((2.8)\), respectively.

As for the choice of multiplicative normalization, we first make an explicit choice for the cases \(N = 1\) in Section \(3.5.1\). The rest of the multiplicative factors are fixed recursively in \(N\), by requiring that the constant appearing on the right hand side of Equation \((3.9)\) is equal to one. This corresponds to fixing the multiplicative constant in Equation \((2.7)\) to the value \(B_2\) given in \((3.4)\).

### 3.5 Solutions in terms of quantum group representations

A priori, the system of equations \((3.5)\), \((3.6)\), \((3.7)\), \((3.8)\), \((3.9)\), \((3.10)\) given in Section \(3.4\) is a linear algebra problem in the \(2 \times 3^N\)-dimensional tensor product space \(V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L\). The first two equations \((3.5)\), \((3.6)\) reduce this ambient dimension in a well understood way: their meaning is that \(v_\omega\) is a highest weight vector of a subrepresentation of dimension two in the tensor product. We have

\[
\dim \left( \text{Ker}(E) \cap \text{Ker}(K - q) \right) = m_N,
\]

where \(m_N\) is the multiplicity of \(V_2\) in the semisimple decomposition of the tensor product, determined recursively by the formula of Section \(3.2.2\). For small \(N\) the multiplicities are

| \(N\) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | ⋯ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(m_N\) | 1  | 2  | 4  | 9  | 21 | 51 | 127| 323| 835| 2188| 5798| 15511 | ⋯ |

and for \(N\) large they behave as

\[
m_N \sim 3 \sqrt{\frac{3}{4\pi}} \times N^{-\frac{3}{2}} 3^N.
\]

Superficially the system of Section \(3.4\) still seems overdetermined, but we find that in each case the solution space is one dimensional, so up to multiplicative normalizations the solutions are unique.

Next we give the explicit solutions to the system of equations for a few small values of \(N\).

#### 3.5.1 One-point solutions

There are two separate states that we need to solve, \(v_-(1) \in V_2 \otimes V_3\) for a visit on the left \((y_1 < x)\), and \(v_+(1) \in V_3 \otimes V_2\) for a visit on the right \((x < y_1)\). The solutions, unique up to normalization, are

\[
v_-(1) = \frac{q^4}{1 - q^4} e_0 \otimes e_1 - \frac{q}{1 - q^2} e_1 \otimes e_0 \quad (3.11)
\]

\[
v_+(1) = \frac{q^2}{1 - q^2} e_0 \otimes e_1 - \frac{q^2}{1 - q^4} e_1 \otimes e_0. \quad (3.12)
\]

The normalization above has been chosen such that the corresponding functions both are equal to

\[
\zeta^{(1)}(x; y_1) = B_2 |y_1 - x|^{1-\frac{3}{2}},
\]

where the constant \(B_2\) is given by \((3.4)\) (in particular both functions take positive real values). The calculation of the corresponding integrals is discussed in more detail in Section \(4.2.1\).
3 QUANTUM GROUP AND INTEGRAL FORMULAS

3.5.2 Two-point solutions

There are four separate states that we need to solve for,

\[ v^{(2)}_{-} \in V_2 \otimes V_3 \otimes V_3 \quad (y_2 < y_1 < x) \]
\[ v^{(2)}_{+} \in V_3 \otimes V_2 \otimes V_3 \quad (y_1 < x < y_2) \]
\[ v^{(2)}_{t} \in V_3 \otimes V_2 \otimes V_3 \quad (y_2 < x < y_1) \]
\[ v^{(2)}_{++} \in V_3 \otimes V_3 \otimes V_2 \quad (x < y_1 < y_2). \]

For the normalization of the states, we use the asymptotics as \(|y_1 - x| \to 0\), i.e., we fix the constant in either (2.7) or (3.9).

The solutions, unique with the chosen normalization, read

\[ v^{(2)}_{+} = \frac{q^4(1 + q^2 + q^4)}{(1 - q^4)^2(1 + q^4)} \left( (q^2 + q^4) e_{011} - e_{020} - (1 + q^2) e_{101} - (1 - q^2) e_{110} + e_{200} \right) \]
\[ v^{(2)}_{-} = \frac{q^3(1 + q^2 + q^4)}{(1 - q^4)^2(1 + q^4)} \left( q^4 e_{002} + (q^5 - q^3) e_{011} - q^3 e_{020} - q^2 e_{101} - q^4 e_{110} + (1 + q^2) e_{110} \right) \]
\[ v^{(2)}_{t} = \frac{q^3(1 + q^2 + q^4)}{(1 - q^4)^2(1 + q^4)} \left( q^4 e_{002} + q^5 e_{011} - q^3 e_{101} - q^4 e_{110} + \frac{1 + q^2 + q^4}{1 + q^2} e_{200} \right) \]
\[ v^{(2)}_{++} = \frac{q^3(1 + q^2 + q^4)}{(1 - q^4)^2(1 + q^4)} \left( \frac{q^4(1 + q^2 + q^4)}{1 + q^2} e_{002} - q e_{011} - q^3 e_{101} + e_{110} + \frac{q^2}{1 + q^2} e_{200} \right), \]

where we use the shorthand notation \( e_{i_2 t_1 d} = e_{i_2} \otimes e_{t_1} \otimes e_d \in V_3 \otimes V_3 \otimes V_2 \) in the first case, and similarly for the rest.

3.5.3 Three-point solutions

For \( N = 3 \) there are eight separate states that we need to solve for. For brevity, in the formulas below, we factor out the constant

\[ C_3 = \frac{q^5 (q^4 + q^2 + 1)^2}{(q^4 - 1)^3 (q^{12} + q^{10} + 2q^8 + 2q^6 + 2q^4 + q^2 + 1)}. \]

Then, with a shorthand notation similar to above, the unique normalized solutions are

\[ v^{(3)}_{+++} = C_3 \times \left( - (q^6 + 2q^4 + 2q^2 + 1) q^3 e_{0021} - (q^2 + 1) (q^6 + q^4 - 1) q^3 e_{0111} + (q^2 + 1)^2 q^4 e_{0120} \right. \]
\[ + (q^2 + 1)^2 q^3 e_{0201} + (q^6 - q^2 - 1) q e_{0210} + (q^3 + q)^3 e_{101} + (q^6 - q^2 - 1) q e_{1020} \]
\[ + (q^2 + 1) (q^6 - q^2 - 1) q e_{1101} + (q^6 - q^2 - 1) q e_{1120} - (q^6 + 2q^4 + 2q^2 + 1) q e_{2001} - (q^6 + q^4 - 1) q e_{2010} + (q^2 + 1)^2 q e_{2100} \right) \]
\[ v^{(3)}_{++-} = C_3 \times \left( - (q^4 + q^2 + 1) q^4 e_{0012} + \left( - q^8 + q^2 + \frac{1}{q^2 + 1} - 1 \right) e_{0102} - (q^6 + q^4 - 1) q^5 e_{0111} \right. \]
\[ + (q^7 + q^5) e_{0201} + (q^6 + q^4) e_{0210} + (q^2 + 1)^2 q^5 e_{101} \]
\[ + (q^6 - q^2 - 1) q^3 e_{1101} + (q^6 - q^2 - 1) q^4 e_{1110} - (q^4 + q^2 + 1) q^4 e_{1200} \]
\[ - (q^6 + q^4 + 1) q^3 e_{2001} - (q^4 + q^2 + 1) q^4 e_{2010} + \frac{(q^4 + q^2 + 1)^2 q e_{2100}}{q^2 + 1} \right). \]
\( v_{++}^{(3)} = C_3 \times \left( - (q^4 + 1) (q^4 + q^2 + 1) q^2 e_{0012} + \left( q^4 + \frac{q^2}{q^2 + 1} \right) e_{0102} + (q^7 + q^5 + q^3) e_{0111} \right.
\quad - q e_{0201} - q^2 e_{0210} + \left( q^6 + q^2 + \frac{1}{q^2 + 1} - 1 \right) e_{1002} + (q^9 + q^7 + q^5) e_{1011} \right.
\quad - (q^2 + 1) q^3 e_{1101} - (q^2 + 1) q^4 e_{1110} + \left( q^2 + \frac{1}{q^2 + 1} \right) e_{1200} - q^5 e_{2001} \bigg) + q^6 \left( - e_{2010} \right) + \left( q^4 + \frac{q^2}{q^2 + 1} \right) e_{2100} \bigg) \\
\left( - (q^2 + 1) q^6 e_{0012} + \frac{- (q^{10} + q^6 + q^4) e_{0021}}{q^2 + 1} - (q^2 + 1) q^7 e_{0102} \right.
\quad + (q^{11} + q^7 + q^9) e_{0111} + (q^9 + q^7 + q^5) e_{0120} + (q^7 + q^5 + q^3) e_{1002} \right.
\quad + (q^6 + q^4 - 1) q^3 e_{1011} - (q^2 + 1) q^4 e_{1020} + (q^6 + q^4 - 1) q^4 e_{1101} - (q^2 + 1) q^4 e_{1110} \bigg) \bigg) \\
\left( - (q^4 + q^2 + 1) q^4 e_{0012} - (q^4 + q^2 + 1) q^6 e_{0102} + (q^2 + 1)^2 q^3 e_{0111} \right.
\quad + (q^7 + q^5) e_{0201} + (q^7 - q^2 - 1) e_{0210} + \left( q^5 + q^3 + q^2 \right) e_{1002} + (q^6 - q^2 - 1) q e_{1011} \right.
\quad + (q^6 - q^2 - 1) q^3 e_{1101} + (-q^6 - q^4 + 1) e_{1110} + \left( \frac{q^2}{q^2 + 1} - q^6 \right) e_{1200} \bigg) \bigg) \\
\left( - (q^4 + q^2 + 1) q^3 e_{2001} + (q^2 + 1) e_{2010} + (q^4 + q^2) e_{2100} \right)
\bigg) \\
\left( - (q^{10} + q^8 + q^6) e_{0012} - \frac{(q^4 + q^2 + 1) q^8 e_{0021}}{q^2 + 1} + q^3 e_{0102} \right.
\quad + (q^7 + q^5) e_{0111} + q^7 e_{0120} + q^5 e_{1002} + (q^9 + q^7) e_{1011} + q^9 e_{1020} \right.
\quad - (q^4 + q^2 + 1) q^2 e_{1101} - (q^4 + q^2 + 1) q^4 e_{1110} - \frac{(q^8 + q^6 + q^4) e_{2001}}{q^2 + 1} \bigg) \bigg) \\
\left( - \frac{(q^{10} + q^8 + q^6) e_{2010}}{q^2 + 1} + \left( q^7 + q^5 + 2q^3 + q + \frac{1}{q} \right) e_{2100} \right)
\bigg) \\
\left( - \frac{(q^4 + q^2 + 1)^2 q^4 e_{0012}}{q^2 + 1} + (q^6 + q^4 + q^2) e_{0021} + (q^5 + q^3 + q) e_{0102} \right.
\quad + (q^6 + q^4 - 1) q e_{0111} - (q^2 + 1) q e_{0120} + (q^7 + q^5 + q^3) e_{1002} + (q^6 + q^4 - 1) q^3 e_{1011} \right.
\quad - (q^2 + 1) q^3 e_{1020} - (q^3 + q^2) e_{1101} + (-q^6 + q^2 + 1) e_{1110} - (q^2 + 1) q^4 e_{2001} \bigg) \bigg) \\
\left( - \frac{(q^8 + q^4 + q^2) e_{2010}}{q^2 + 1} + (q^5 + q^3 + q) e_{2100} \right)
corresponding (one-dimensional) integral thus becomes a sum of integrals over the real line. Extending this 
the base point and on the real line. We get two contours (one from both the lower and the upper edges of the loop) between 
that as any of the integration variables

\[ \sim |w_j - y_L|^{1-8/k}. \]

Thus the resulting integrals will be convergent if \( \kappa > 8 \). Therefore, let us first assume that \( \kappa > 8 \). We will discuss the divergences in Section 4.3.

When \( \kappa > 8 \), a loop contour enclosing, for example, \( y_L^- \) can be divided into \( 2(L - k + 1) \) subcontours on the real line. We get two contours (one from both the lower and the upper edges of the loop) between the base point and \( y_L^- \) as well as between all consecutive pairs \{ \( y_j^-, y_{j+1}^- \) \} with \( j = k, \ldots, L - 1 \). The corresponding (one-dimensional) integral thus becomes a sum of integrals over the real line. Extending this

\[ v^{(3)} = C_3 \times \left( -q^2 + 1 \right)^2 q^5 e_{0012} + \left( -q^9 + q^5 + q^3 \right) e_{0021} + \left( -q^9 + q^5 + q^3 \right) e_{0102} \]

\[ + \left( -q^8 + q^6 + 2q^4 + 6q^2 - 1 \right) q^3 e_{0111} + \left( q^6 - q^2 - 1 \right) q^3 e_{0120} + \left( q^6 - q^2 - 1 \right) q^3 e_{0201} \]

\[ - \left( q^2 + 1 \right)^2 q^3 e_{0210} + \left( q^2 + 1 \right) (q^2 + q^2 + 1) q^2 e_{1002} + \left( q^2 + 1 \right) (q^6 + q^4 - 1) q^2 e_{1011} \]

\[ - \left( q^3 + q \right)^2 e_{1020} - \left( q^2 + 1 \right)^3 q^2 e_{1101} - \left( q^2 + 1 \right) (q^6 - q^2 - 1) e_{1110} + \left( q^2 + 1 \right) (q^4 + q^2 + 1) e_{1200} \]

3.5.4 Four-point solutions

For \( N = 4 \) there are sixteen separate states that we need to solve for. The solutions are again unique (with 
the chosen normalization). In Appendix C.2 we include the results for those vectors that have been used 
in the plots of Figure 6.9.

3.5.5 Well-posedness of the problem

The linear problem of Section 3.4 is well-posed: one always finds solutions and they are unique (with 
the chosen normalization). In Appendix C.2 we include the results for those vectors that have been used 
in the plots of Figure 6.9.

4 Regularized real integrals and evaluation of the formulas

4.1 Transformation to real integration contours

Let us then analyze further the integrals \( \phi_{t_2, \ldots, t_L^+; t_2^-, \ldots, t_R^-} dt_2^+ dt_3^+ \) 

\[ \text{given by the spin chain - Coulomb gas correspondence. Recall that the integral was defined in Section 3.1.2, where the integration surface } \Gamma \text{ consists of non-intersecting loop contours for each of the integration variables } w_s \text{ as depicted in Figure 3.1.} \]

First we shall describe a transformation of the contours which makes the integrands explicitly real in 
general, and examples will follow below. The procedure is, in principle, straightforward. We assume that 
the anchor point \( z_0 \) of the loop integrals lies on the real axis left of the points \( x \) and \( y_L^- \). (As stated in 
Section 3.3.3 and as we shall see below, the integrals of interest to us in the end are independent of this 
anchor point.) We can then deform the loop-shaped contours so that they follow the real line, starting from 
the innermost loops on the left and proceeding towards right.

There is, however, a complication as the integrals along the real axis may become singular. Notice 
that as any of the integration variables \( w_i \) approaches any of the points \( y_j^- \), the integrand behaves as 
\( \sim |w_i - y_j^-|^{1-8/k}. \) Thus the resulting integrals will be convergent if \( \kappa > 8 \). Therefore, let us first assume that \( \kappa > 8 \). We will discuss the divergences in Section 4.3.

When \( \kappa > 8 \), a loop contour enclosing, for example, \( y_L^- \) can be divided into \( 2(L - k + 1) \) subcontours on the real line. We get two contours (one from both the lower and the upper edges of the loop) between the base point and \( y_L^- \) as well as between all consecutive pairs \{ \( y_j^-, y_{j+1}^- \) \} with \( j = k, \ldots, L - 1 \). The corresponding (one-dimensional) integral thus becomes a sum of integrals over the real line. Extending this
procedure to the loops enclosing $x$ and the points $y_k^*$ right of $x$, each integral $\varphi_{\bar{\omega}^{-}, \ldots, \omega_{N}^{+}, \omega_{N}^{-}, \ldots, \omega_{1}^{-}}$ can be written as a linear combination of integrals having all integration contours on the real line.

In order to obtain the explicit linear combination of the integrals, the remaining and most non-trivial task is to calculate the phase factors which arise as the integrals is a multi-valued function. The phase convention for the integrand $f_{\rho_{\bar{\omega}^{-}, \ldots, \omega_{N}^{+}}}$ of (3.3) for the loop contours was defined by the red circles in Figure 3.1 and this convention leads to rather impractical branch choices for the integrand as the contours are transformed. We shall choose the phases for the contours along the real line as depicted by the red circles in Figure 4.1 where the integration contours have been deformed away from the real axis in order to make their multiplicity and the phase convention visible. Let us denote these integrals by $\hat{\rho}_{k_{L}^{-}, \ldots, k_{N}^{-}, k_{1}^{+}, k_{1}^{-}, \ldots, k_{R}^{+}}$, when the number of variables integrated from the anchor to $y_{L}^{-}$ is $k_{L}^{-}$, the number of variables integrated from $y_{L}^{-}$ to $y_{L-1}^{-}$ is $k_{L-1}^{-}$ and so on (we thus choose to index the integrals in terms of the rightmost points of the integration intervals). It is not worthwhile to write down a general formula for the phase factors which appear when expressing each $\varphi$ as a sum of the integrals $\hat{\rho}$, but it is straightforward to calculate them case by case as seen in the examples below. As the phase factors reflect the branch choices of the integrand in (3.2), they will be integer powers of $q = \exp(4\pi i/\kappa)$, possibly multiplied by $-1$ if the direction of integration needs to be reversed.

As the final step, we arrange the integration over each interval such that the integration variables have a fixed order. The natural phase convention in this case is that the integrand is real and positive. We denote these integrals by $\rho_{k_{L}^{-}, \ldots, k_{N}^{-}, k_{1}^{+}, k_{1}^{-}, \ldots, k_{R}^{+}}$. The integrals are over products of simplices of dimensions $k_{L}^{-}, \ldots, k_{1}^{+}; k_{1}^{-}, \ldots, k_{R}^{+}$, for example when $L = 0, R = N$ and $K = k + \sum_{j=1}^{N} k_{j}^{+}$ we have

$$
\rho_{k_{1}^{+}, k_{2}^{+}, \ldots, k_{N}^{+}}(x; y_{1}, \ldots, y_{N}) = \int \cdots \int \int_{x < \omega_{1} < \omega_{2} < \omega_{3} < \cdots < \omega_{k+1} < y_{1} < \omega_{k+2} < \cdots < \omega_{k+k_{1}} < y_{2} < \cdots < \omega_{k+k_{2}} < \cdots < \omega_{k+k_{N}} < y_{N}} \cdots dw_{1}dw_{2} \cdots dw_{K} |f_{N}^{(N)}(x; y_{1}, \ldots, y_{N}; w_{1}, \ldots, w_{N})|.
$$

It is easy to see that reordering gives a factor of $|k|! q^{-k(k-1)/2}$ for each interval with $k$ integrations, where $|k|! = \prod_{m=1}^{k} |m|$ is a $q$-factorial. Thus, we have

$$
\hat{\rho}_{k_{L}^{-}, \ldots, k_{N}^{-}, k_{1}^{+}, k_{1}^{-}, \ldots, k_{R}^{+}} = \prod_{j=1}^{L} [k_{j}^{-}]! q^{-k_{j}^{-}(k_{j}^{-}-1)/2} \times [k_{j}^{+}]! q^{-k_{j}^{+}(k_{j}^{+}-1)/2}
$$

$$
\times \prod_{j=1}^{R} [k_{j}^{+}]! q^{-k_{j}^{+}(k_{j}^{+}-1)/2} \times \rho_{k_{L}^{-}, \ldots, k_{N}^{-}, k_{1}^{+}, k_{1}^{-}, \ldots, k_{R}^{+}}.
$$
4.2 Solutions in terms of real integrals

Let us then calculate explicitly the solutions obtained in Section 3.5 for low numbers of boundary visits \( N \). We shall discuss in detail the case \( N = 1 \), and list the results for the solutions with a higher number of points. Case by case, we will check that the obtained solutions for the boundary zig-zag amplitudes satisfy the following two requirements:

- The integration contour \( \Gamma \) is closed, and therefore the solution is independent of the choice of the anchor point of the loop integrals. When the amplitude is expressed in terms of the integrals \( \rho_{k_L^-,k_1^-,\ldots,k_L^+} \) (or a similar \( \hat{\rho} \)), this will be clear as the solutions do not depend on the integrals which include integrations starting from the base point — we will only have terms with \( k_L^- = 0 \) (or if \( L = 0 \) then \( k_0^-=0 \)).
- The solution is real: when expressed in terms of the integrals \( \rho \), all coefficients will be real.

4.2.1 One-point solutions

We start from the \( N = 1 \) case where the single visit takes place right of the starting point \( x \). In this case we found that the state \( \nu_+^{(1)} \in V_3 \otimes V_2 \) in (3.12) which satisfies the constraints is

\[
\nu_+^{(1)} = \frac{q^2}{1 - q^2} e_0 \otimes e_1 - \frac{q^2}{1 - q^2} e_1 \otimes e_0.
\]

By the spin chain - Coulomb gas correspondence of Section 3.3.1, the zig-zag probability amplitude is given by

\[
\zeta_+^{(1)}(x; y_1^+) = \frac{q^2}{1 - q^2} \varphi_+^{1; 0}(x; y_1^+) - \frac{q^2}{1 - q^2} \varphi_+^{0; 1}(x; y_1^+).
\]

Let us then do the transformation to the integrals along the real line. The first term \( \varphi_+^{1; 0} \) has the loop integral encircling \( x \), which can only lead to integrals over the real line between the base point and \( x \), i.e., the integral \( \hat{\rho}_+^{1; 0} \). The phase factor from the lower edge of the loop is \( q = e^{i\pi/\kappa} \) (as the phase conventions of Figures 3.1 and 4.1 differ by a rotation of the integration variable \( w \) around \( x \) by the angle \( -\pi \)), whereas the phase factor for the upper edge of the loop is \( -q^{-1} \) (where the rotation is in the opposite direction, and the minus sign arises from reversing the direction of integration). Together,

\[
\varphi_+^{1; 0}(x; y_1^+) = \left( q - \frac{1}{q} \right) \hat{\rho}_+^{1; 0}(x; y_1^+).
\]

The other loop integral \( \varphi_+^{0; 1} \) breaks into four integrals along the intervals on the real axis, two integrals between the base point and \( x \), and two integrals between \( x \) and \( y_1^+ \). The phase factors can be calculated analogously to the case of \( \varphi_+^{1; 0} \), and they are integer powers of \( q \). We find that

\[
\varphi_+^{0; 1}(x; y_1^+) = \left( q^2 - \frac{1}{q^2} \right) \hat{\rho}_+^{0; 1}(x; y_1^+) + \left( q^3 - \frac{1}{q^3} \right) \hat{\rho}_+^{1; 0}(x; y_1^+).
\]

Substituting in these results, we get

\[
\zeta_+^{(1)}(x; y_1^+) = \hat{\rho}_+^{0; 1}(x; y_1^+) = \rho_+^{0; 1}(x; y_1^+).
\]

In particular, the contributions of the integral \( \hat{\rho}_+^{1; 0} \) cancel. The remaining integral \( \hat{\rho}_+^{0; 1} \) is independent of the anchor point of the loop contours, which shows that the contour \( \Gamma \) was closed. In this case there is only one integration variable, so trivially \( \hat{\rho}_+^{0; 1} = \rho_+^{0; 1} \). From the final expression we also see that the result is real.
When \( N = 1 \) the resulting integral can be calculated easily. Using the definitions from (3.2),
\[
\zeta_+^{(1)}(x; y_1^+) = (y_1^+ - x)^{4/\kappa} \int_x^{y_1^+} dw (w - x)^{-4/\kappa} (y_1^+ - w)^{-8/\kappa} = B_2 (y_1^+ - x)^{1 - 8/\kappa},
\]
where the constant is given by the same beta function \( B_2 = B \left( \frac{\kappa - 8}{\kappa}, \frac{\kappa - 4}{\kappa} \right) = \Gamma \left( \frac{\kappa - 4}{\kappa} \right) \Gamma \left( \frac{\kappa - 8}{\kappa} \right) / \Gamma \left( \frac{2 \kappa - 6}{\kappa} \right) \) as in Equation (3.4).

For comparison, let us also take a look at the case where the visit takes place left of \( x \). The state \( v_{-1}^{(1)} \in W_2 \otimes W_3 \) was given in 3.11 and by the correspondence we get the probability amplitude
\[
\zeta_-^{(1)}(y_1^-; x) = \frac{q^2}{1 - q^2} \varphi_{1;0}(y_1^-; x) - \frac{q}{1 - q^2} \varphi_{0;1}(y_1^-; x).
\]

The transformations to real integrals read in this case
\[
\varphi_{1;0}(y_1^-; x) = \left( q^2 - \frac{1}{q^2} \right) \hat{\rho}_{1;0}(y_1^-; x)
\]
\[
\varphi_{0;1}(y_1^-; x) = \left( q - \frac{1}{q} \right) \hat{\rho}_{0;1}(y_1^-; x) + (q^3 - q) \hat{\rho}_{1;0}(y_1^-; x).
\]
Inserting these gives again a simple result
\[
\zeta_-^{(1)}(y_1^-; x) = \hat{\rho}_{0;1}(y_1^-; x) = \rho_{0;1}(y_1^-; x).
\]
This evaluates to
\[
\zeta_-^{(1)}(y_1^-; x) = (x - y_1^-)^{4/\kappa} \int_{y_1^-}^x dw (w - y_1^-)^{-8/\kappa} (x - w)^{-4/\kappa} = B_2 (x - y_1^-)^{1 - 8/\kappa}.
\]

The results for the left and right side visits can be collected in the (well known) \( N = 1 \) probability amplitude already stated in Equation (1.5).
\[
\zeta^{(1)}(x; y_1) = \chi^{(1)}(x; y_1) = B_2 |y_1 - x|^{1 - \frac{8}{\kappa}},
\]
with our multiplicative normalization convention resulting in \( B_2 \) given in (3.4).

### 4.2.2 Two-point solutions

Let us start the discussion of the two-point solutions from the case where both visits take place on the right hand side. The relevant vector \( v_{++}^{(2)} \in V_3 \otimes V_3 \otimes V_2 \) reads
\[
v_{++}^{(2)} = \frac{q^4 (1 + q^2 + q^4)}{(1 - q^4)^2 (1 + q^4)} \left( (q^2 + q^4) e_{011} - e_{020} - (1 + q^2) e_{101} + (1 - q^2) e_{110} + e_{200} \right),
\]
where \( e_{t_2t_1t} \equiv e_{t_2} \otimes e_{t_1} \otimes e_d \). Thus the probability amplitude is
\[
\zeta_{++}^{(2)}(x; y_1^+, y_2^+) = \frac{q^4 (1 + q^2 + q^4)}{(1 - q^4)^2 (1 + q^4)} \left( (q^2 + q^4) \varphi_{0;1,1}(x; y_1^+, y_2^+) - \varphi_{0;2,0}(x; y_1^+, y_2^+) 
- (1 + q^2) \varphi_{1;0,1}(x; y_1^+, y_2^+) + (1 - q^2) \varphi_{1;1,0}(x; y_1^+, y_2^+) 
+ \varphi_{2;0,0}(x; y_1^+, y_2^+) \right).
\]
The transformation to real integrals is still straightforward albeit more involved, as one needs to take into account the phases related to the order of the integration variables. The number of terms is also larger, e.g., the integral \( \varphi_{0;2,0} \) breaks into 16 different terms (some of which immediately cancel against each other).
Collecting the results in the expression for the probability amplitude, however, there are again lots of simplifications:

\[
\zeta_{++}^{(2)}(x; y_1^+, y_2^+) = \frac{q^{-2} + 1 + q^2}{q^{-2} + q^2} \left( \rho_{0;0,2}(x; y_1^+, y_2^+) + \rho_{0;1,1}(x; y_1^+, y_2^+) \right).
\]

Again we notice that as the first index of all integrals is zero, the integration contour is closed. The probability amplitude is also real.

The amplitudes with other orderings of visits can be calculated similarly. The results can be collected as

\[
\zeta_{--}^{(2)}(y_1^-, y_2^-; x) = \frac{q^{-2} + 1 + q^2}{q^{-2} + q^2} \left( \rho_{0;0,2}(y_1^-, y_2^-; x) + \rho_{0;1,1}(y_1^-, y_2^-; x) \right)
\]

\[
\zeta_{-+}^{(2)}(y_1^-, x; y_1^+) = \frac{q^{-2} + 1 + q^2}{q^{-2} + q^2} \left( \rho_{0;2,0}(y_1^-, y_1^+; x) + \rho_{0;1,1}(y_1^-, y_1^+; x) \right)
\]

\[
\zeta_{++}^{(2)}(x; y_1^+, y_2^+) = \frac{q^{-2} + 1 + q^2}{q^{-2} + q^2} \left( \rho_{0;0,2}(x; y_1^+, y_2^+) + \rho_{0;1,1}(x; y_1^+, y_2^+) \right).
\]

One can check that

\[
\zeta_{++}^{(2)}(x, y_1, y_2) = B_2^2 \frac{\Gamma(16-\kappa)}{\Gamma(12-\kappa)} \frac{\Gamma(\frac{4}{\kappa})}{\Gamma(\frac{2}{\kappa})} (y_1 - x)^{1-\frac{4}{\kappa}} (y_2 - y_1)^{1-\frac{8}{\kappa}} \times 2F_1 \left( \frac{4}{\kappa}, \frac{\kappa - 8}{\kappa}; \frac{8}{\kappa}; \frac{y_2 - y_1}{y_2 - x} \right)
\]

and

\[
\zeta_{++}^{(2)}(y_2, x, y_1) = B_2^2 \frac{\Gamma(16-\kappa)}{\Gamma(12-\kappa)} \frac{\Gamma(\frac{8}{\kappa})}{\Gamma(\frac{12}{\kappa})} (x - y_2)^{\frac{4}{\kappa}} (y_1 - x)^{\frac{8}{\kappa}} (y_1 - y_2)^{2-\frac{16}{\kappa}} \times 2F_1 \left( \frac{8}{\kappa}, \frac{\kappa - 4}{\kappa}; \frac{12}{\kappa}; -\frac{x - y_2}{y_1 - x} \right),
\]

and that \(\zeta_{--}^{(2)}(y_2, y_1; x)\) and \(\zeta_{-+}^{(2)}(y_1; x; y_2)\) are given by the obvious reflection in the above formulas. In particular our formula for \(\zeta_{++}^{(2)}\) and \(\zeta_{--}^{(2)}\) agrees up to the choice of normalization with those given in [SZ10].

### 4.3 Divergences of the real integrals

As we mentioned above, the integrals over the real line contain divergences. The integrals converge for \(\kappa > 8\), but diverge when \(0 < \kappa \leq 8\), which is the range of most interesting values of \(\kappa\). There are several strategies to tame the divergences:

- **Analytic continuation.** We can first restrict to \(\kappa > 8\), where the integrals converge, and analytically continue the final expressions to smaller values of \(\kappa\) (as we were essentially doing above).

- **Cutoff regularization.** We can start from the final expressions involving real integrals, and introduce a small cutoff \(\varepsilon\) to regulate all divergent integrals. More precisely, we require that all integration
variables are further away than ε from any of the points x or yₖ. With this prescription, the results diverge as ε ↘ 0. All divergent terms are powers of ε, with the exponents depending on κ. They can be subtracted unambiguously at least for irrational values of κ. The final result is then obtained by taking ε ↘ 0 after subtracting the divergent counterterms. We will discuss the details below.

- **Pochhammer regularization.** We can also do the transformation to real integrals, which was described in Section 4.1 in a way that avoids the divergences. We first choose ε which is smaller than half of the separation of any two of the points yₖ or x. When deforming the loops into integrals over the real line, we replace the sections of contours on the real line, which are closer than ε to the points yₖ or x, by (semi-)circles having radii ε. In the end, this defines a generalization of the usual Pochhammer contour for the higher dimensional integrals.

It is easy to see that all of the above regularization schemes lead to the same final result. Let us sketch how this can be proven. First, the loop integrals converge for all values of κ > 8. The Pochhammer-regularization is obtained by modification of the contours, and no terms are dropped. Therefore it also gives the analytic continuation of the results to small values of κ, independently of the value of ε.

Second, the pieces of contour on the real line in the Pochhammer contours equal the cutoff regularized integrals over the real line. The integrals over the (semi-)circles can be expanded around ε = 0, and the terms which are divergent as ε ↘ 0 provide the counterterms for the cutoff regularization. For generic irrational κ the expansions contain no constant term. Therefore, taking ε ↘ 0, the Pochhammer-regularized result matches with the cutoff-regularized one for all values of κ for which the cutoff procedure could be defined unambiguously.

Let us then work out the details of the cutoff regularization, i.e., find a method to calculate the counterterms. We already pointed out that this can be done by studying the expansion of the contributions from the (semi-)circles to the Pochhammer-type integrals, but tracking the phases of these integrals is quite involved. It turns out to be easier to read off the divergent terms from the real integrals directly. We can first take κ > 8 and start from the integrals without any cutoff. Then we separate the “divergent” terms by dividing the integrations into several pieces, effectively introducing a “cutoff”.

Let us first discuss the generic framework in more detail. We shall also give an example below. We start from the integral ρ where all integrals are along the real line and the integrand is real. We divide the integrals over each of the real intervals into two pieces: the “regular” one where all integration variables are further away than ε from the endpoints, and the “divergent” one where one of the variables (either the first or the last one) is within ε from the endpoints. The basic idea is then to develop the divergent pieces as series at ε = 0.

For an N-point function, the highest possible divergence appears when all integration variables are within ε from different points yₖ. Taking into account the behavior of the integrand and the integration measure, such contribution is \( \sim \varepsilon^{N(1-8/\kappa)} \). Developing the integrand as series at ε = 0, and taking into account the contributions having divergent terms from \( n < N \) integrations, the generic divergent contribution has the power behavior

\[
\sim \varepsilon^{n(1-8/\kappa)} \varepsilon^k, \quad \text{where } n = 1, 2, \ldots, N \text{ and } k = 0, 1, 2, \ldots
\]  

(4.2)

All such terms can be in principle calculated by analyzing the divergent terms. Analytically continuing to κ < 8, terms with small k will be divergent as ε ↘ 0. (Alternatively, we could keep κ < 8 fixed from the start and work with two cutoffs.) Since we started from an integral that was independent of ε, these terms must cancel when all divergent and regular pieces are summed, and they are thus the required counterterms. How all of this works is best illustrated by considering an example.

---

9For the specific values of κ, where the counterterms of the cutoff procedure involve constants, cutoff regularization can be defined such that it matches with the other schemes. Equivalently we can, e.g., require that the counterterms are analytic in κ.
Let us discuss the \( N = 2 \) integral
\[
\rho_{0,0,2}(x; y_1, y_2) = \int_{y_1}^{y_2} \int_{w_1}^{w_2} dw_1 dw_2 \left[ \frac{(w_2 - w_1)(y_2 - y_1)}{(y_2 - w_1)(y_2 - w_2)(w_2 - y_1)(w_1 - y_1)} \right] \frac{\varepsilon}{\kappa} \times F(w_1, w_2; x; y_1, y_2),
\]
where \( x < y_1 < y_2 \) and we denoted by
\[
F(w_1, w_2; x; y_1, y_2) = \left[ \frac{(y_2 - x)(y_1 - x)}{(w_2 - x)(w_1 - x)} \right]^{\frac{\varepsilon}{\kappa}}
\]
the part which would be replaced by a more complicated function for a higher point integral having a similar structure, i.e., integral of two variables between consecutive points \( y_j \). The regular term is
\[
R = \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} dw_1 dw_2 \left[ \frac{(w_2 - w_1)(y_2 - y_1)}{(y_2 - w_1)(y_2 - w_2)(w_2 - y_1)(w_1 - y_1)} \right] \frac{\varepsilon}{\kappa} \times F(w_1, w_2; x; y_1, y_2)
\]
and the divergent terms can be written as
\[
D_1 + D_2 + D_3 + D_4 + D_5 = \left( \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \right) dw_1 dw_2 \\
\times \left[ \frac{(w_2 - w_1)(y_2 - y_1)}{(y_2 - w_1)(y_2 - w_2)(w_2 - y_1)(w_1 - y_1)} \right] ^{\frac{\varepsilon}{\kappa}} F(w_1, w_2; x; y_1, y_2),
\]
where the first two terms include one divergent piece of integration, and the last three include two pieces.

The leading contribution from the divergent pieces is contained in the third term \( D_3 \), where \( |w_1 - y_1| < \varepsilon \) and \( |w_2 - y_2| < \varepsilon \), as the terms \( D_1 \) and \( D_5 \) are suppressed by the factor \((w_2 - w_1)^{8/\kappa}\). We denote the \( \mathcal{O}(\varepsilon) \) integration variables as \( \hat{w}_1 = w_1 - y_1 \) and \( \hat{w}_2 = y_2 - w_2 \). Developing at \( \varepsilon = 0 \) we find
\[
D_3 = \int_0^\varepsilon \int_0^\varepsilon d\hat{w}_1 d\hat{w}_2 \hat{w}_1^{-\kappa/\kappa} \hat{w}_2^{-\kappa/\kappa} \left[ F(y_1, y_2; x; y_1, y_2) \\
+ \hat{w}_1 \frac{\partial}{\partial w_1} F(w_1, y_2; x; y_1, y_2) \bigg|_{w_1 = y_1} - \hat{w}_2 \frac{\partial}{\partial w_2} F(y_1, w_2; x; y_1, y_2) \bigg|_{w_2 = y_2} + \mathcal{O}(\varepsilon^2) \right],
\]
where we wrote the terms of the expansions up to next-to-leading order, corresponding to \( k = 1 \) in \((4.2)\). Doing the integrals gives the counterterms
\[
D_3 = \frac{\varepsilon^{2(1-8/\kappa)}}{(1 - 8/\kappa)^2} \left[ F(y_1, y_2; x; y_1, y_2) + \frac{\varepsilon(1 - 8/\kappa)}{2(1 - 4/\kappa)} \right. \\
\times \left. \left( \frac{\partial}{\partial w_1} F(w_1, y_2; x; y_1, y_2) \bigg|_{w_1 = y_1} - \frac{\partial}{\partial w_2} F(y_1, w_2; x; y_1, y_2) \bigg|_{w_2 = y_2} \right) + \mathcal{O}(\varepsilon^2) \right]
\]
\[
= \frac{\varepsilon^{2(1-8/\kappa)}}{(1 - 8/\kappa)^2} \left[ 1 - \frac{2\varepsilon(1 - 8/\kappa)(y_2 - y_1)}{\kappa(1 - 4/\kappa)(y_2 - x)(y_1 - x)} + \mathcal{O}(\varepsilon^2) \right].
\]
As another example, let us consider the term \( D_1 \). Denoting again \( \hat{w}_1 = w_1 - y_1 \), we find

\[
D_1 = \int_0^\infty \int_{y_1 + \varepsilon}^{y_2 - \varepsilon} \frac{d\hat{w}_1}{d\hat{t}} d(w_2 - w_1)^{-8/\kappa} F(y_1, w_2; x; y_1, y_2) + \hat{w}_1 \left( \frac{\partial}{\partial w_1} F(w_1, w_2; x; y_1, y_2) \right)_{w_1 = y_1} - \frac{8(y_2 - w_2)}{\kappa(w_2 - y_1)(y_2 - y_1)} F(y_1, w_2; x; y_1, y_2) + O(\varepsilon^2)
\]

Thus rather nontrivial integrals remain in these counterterms. Notice that even though the explicit \( \varepsilon \)-factor which arises from the divergent pieces is of lower order than in \( D_3 \), the overall divergence is of the same order as the integral over \( w_2 \) also diverges for \( \varepsilon \to 0 \).

The calculation for \( D_2 \) is similar as for \( D_1 \). The terms \( D_4 \) and \( D_5 \) only contribute at \( O(\varepsilon^{2-8/\kappa}) \), and their calculation is rather involved. Actually we slightly cheated in the calculation of next-to-leading order terms for \( D_1 \): we replaced \( w_2 - w_1 \) by \( w_2 - y_1 \) even though this approximation fails when \( w_2 \) is close to the lower bound of its integration range. Corrections due to this approximation can be combined with the contributions from \( D_4 \).

5 Notions of SLE boundary visits and applications

In this section we give the definition of chordal SLE in the upper half-plane \( \mathbb{H} \), and give the conformal covariance rule to transport the boundary visit amplitudes from the half-plane to any other domain. We then consider alternative definitions of SLE boundary visits, and discuss applications of our main result.

5.1 Definition of chordal SLE in half-plane

By conformal invariance, it is sufficient to define the chordal \( \text{SLE}_\kappa \) in one reference domain with marked points. The upper half-plane \( \mathbb{H} \) with the starting point of the curve at 0 and the end point of the curve at \( \infty \) is the most common choice. The following definition also gives a convenient time parametrization for the curve. To define the chordal \( \text{SLE}_\kappa \) in \( \mathbb{H}; 0, \infty \), consider the Loewner chain

\[
g_0(z) = z, \quad \frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - X_t} \quad (\text{for } z \in \mathbb{H}) \tag{5.1}
\]

where the driving process \((X_t)_{t \geq 0}\) is taken to be

\[X_t = \sqrt{\kappa} B_t\]

a multiple of the standard Brownian motion \((B_t)_{t \geq 0}\) on the real line — the parameter \( \kappa \) gives the variance increment per unit time.

The hull \( K_t \) of the chordal \( \text{SLE}_\kappa \) at time \( t \) is the closure of the set of points \( z \in \mathbb{H} \) for which the solution to the Loewner differential equation, Equation (5.1), has ceased to exist by time \( t \). The hulls are growing compacts, \( K_s \subset K_t \) for \( s \leq t \). It can be shown [RS05] that the hulls are generated by a continuous curve \( \gamma : [0, \infty) \to \mathbb{H} \) in the sense that the unbounded component of the complement \( \mathbb{H} \setminus \gamma[0,t] \) of an initial segment up to time \( t \) coincides with the complement \( \mathbb{H} \setminus K_t \) of the hull. We think of the chordal \( \text{SLE}_\kappa \) simply as this random curve \( \gamma \).
5.2 Conformal covariance of boundary visit amplitudes

We content ourselves to writing down the solutions to the boundary visit question in the upper half-plane $\mathbb{H}$ for a chordal SLE$_{\kappa}$ from $x$ to $\infty$. The answer can be transported to other domains by conformal covariance as follows.

Let us denote by $\zeta^{(N)}(\Lambda,a,b)\left(y_1,\ldots,y_N\right)$ the boundary zig-zag amplitude for chordal SLE$_{\kappa}$ in domain $\Lambda$ from $a$ to $b$, defined in a similar manner as in the half-plane, when the points $y_1,\ldots,y_N \in \partial \Lambda$ are on smooth parts of the boundary of the domain. Consider the chordal SLE$_{\kappa}$ curve $\gamma$ in $(\Lambda; a, b)$, and a conformal map $f: \Lambda \to f(\Lambda)$. For boundary points $y \in \partial \Lambda$ at which $f'(y)$ exists, a neighborhood of $y$ of radius $\varepsilon$ is approximately mapped to a neighborhood of the image $f(y)$ and having radius $\varepsilon \times |f'(y)|$. The SLE curve itself is conformally invariant, that is, $f(\gamma)$ has the law of a chordal SLE$_{\kappa}$ in $(f(\Lambda); f(a), f(b))$. Correspondingly, after passing to the limit of small radii in the definition of the amplitude

$$
\lim_{\varepsilon \searrow 0} \left( \frac{1}{\prod_{j=1}^N \varepsilon_j^{h_j}} \times \mathbb{P} \left[ \text{SLE}_\kappa \text{ visits nbrhoods of } y_j \text{ of radii } \varepsilon_j \right] \right),
$$

we get that the boundary zig-zag amplitudes satisfy the following conformal covariance rule

$$
\zeta^{(N)}(\gamma;\Lambda,a,b)\left(y_1,\ldots,y_N\right) = \left( \prod_{j=1}^N |f'(y_j)|^{h_j} \right) \times \zeta^{(N)}(\gamma(f(\Lambda);f(a),f(b))\left(f(y_1),\ldots,f(y_N)\right),
$$

and similarly for the complete correlation functions $\chi^{(N)}(\Lambda,a,b)$.

Appendix B.1 discusses this conformal covariance from the viewpoint of conformal field theory.

5.3 Different definitions of SLE boundary visits

There are several formulations of boundary visits, and one expects many limits of the types of Equations (1.2) or (1.3) to exist. Consider for example the following alternative formulations:

- **Touching small boundary intervals (for $\kappa > 4$):** In the phase $\kappa > 4$, where the curve $\gamma$ can touch the boundary of the domain, a natural notion of reaching a neighborhood of a point $y_j \in \mathbb{R} \setminus \{x\} \subset \partial \mathbb{H}$ is that the curve $\gamma$ touches the boundary between the point $y_j$ and a point which is $\varepsilon_j$ further away from the starting point $x$ of the curve. If $y_j > x$ set $I_{\varepsilon_j}(y_j) = [y_j, y_j + \varepsilon_j]$ and if $y_j < x$ set $I_{\varepsilon_j}(y_j) = [y_j - \varepsilon_j, y_j]$. The corresponding boundary visit amplitude is given by the limit of

$$
\varepsilon_1^{-h_1} \cdots \varepsilon_N^{-h_N} \mathbb{P} \left[ \gamma \cap I_{\varepsilon_j}(y_j) \neq \emptyset \quad \forall j = 1,2,\ldots,N \right]
$$

as $\varepsilon_1,\ldots,\varepsilon_N \searrow 0$.

- **Reaching small conformal distances from the boundary points:** For $\Lambda \subset \subset \mathbb{C}$ a simply connected open domain and $z \in \Lambda$, define the conformal radius $\rho_\Lambda(z)$ such that if $f: \mathbb{D} \to \Lambda$ is a conformal map with $f(0) = z$, then $\rho_\Lambda(z) = |f'(0)|$. By Schwarz lemma and Kőbe $\frac{1}{4}$-theorem, $\rho_\Lambda(z)$ is comparable to the distance of $z$ to $\partial \Lambda$:

$$
\frac{1}{4} \rho_\Lambda(z) \leq \text{dist} \left( z, \partial \Lambda \right) \leq \rho_\Lambda(z).
$$

Now for $y_j \in \mathbb{R} \setminus \{x\} \subset \partial \mathbb{H}$, let $U_j$ be the (unique) connected component of $\mathbb{H} \setminus \gamma$ such that $y_j \in \partial U_j$. Join to $U_j$ its reflection across the real axis, to obtain a larger domain in which $y_j$ is an interior point — more precisely, let $V_j$ be the interior of the closure of $U_j \cup U_j^*$, where $U_j^* = \{ \bar{z} \mid z \in U_j \}$. The quantity $\rho_{\mathbb{H}\setminus\gamma}(y_j) = \rho_{V_j}(y_j)$ gives a conformally covariant notion of the distance of $y_j$ to $\gamma$ — recall
that $\frac{1}{4} \rho_{E \setminus \gamma}(y_j) \leq \mathrm{dist}(y_j, \gamma) \leq \rho_{E \setminus \gamma}(y_j)$. The corresponding boundary visit amplitude is given by the limit of

$$
\varepsilon_1^{-h} \cdots \varepsilon_N^{-h} P\left[\rho_{E \setminus \gamma}(y_j) < \varepsilon_j \quad \forall j = 1, 2, \ldots, N\right]
$$

as $\varepsilon_1, \ldots, \varepsilon_N \downarrow 0$.

One could give an endless list of possible formulations: it is essentially possible to define the notion of a boundary visit as the intersection of the curve with a small neighborhood of any imaginable shape. Each of the different formulations admits both a complete correlation function analogous to Equation (1.2) as exemplified in the two cases above, and an ordered zig-zag amplitude analogous to Equation (1.3). The formulations (5.3) and (5.4) are convenient for various reasons. In Appendix A we in particular present a derivation of the correct value of the scaling exponent $h = \frac{8 - \kappa}{\kappa}$ given in (1.1) based on each of them.

### 5.4 Applications of the results and universal and non-universal aspects

In Section 5.3 we have argued that the SLE boundary visit amplitudes describe the probabilities of events where the SLE trace comes close to marked boundary points, independent of the details of the definition of these events. In this section we mention further applications.

First, however, we emphasize that the details of the formulation or application affect a multiplicative constant in the answer, but not the functional shape of the zig-zag amplitude of these events. In this section we mention further applications.

Also some ratios of the multiplicative constants are universal: the most immediate example comes from

$$
\left[\sum_{\sigma \in S_N} \zeta^{(N)}(x; y_{\sigma(1)}, \ldots, y_{\sigma(N)})\right]^{\frac{1}{N}}
$$

for the complete correlation function as a sum over different orders of visits — for the formula to be meaningful, the ratios of the different multiplicative constants for a given $N$ have to be independent of the formulation.

In Section 2 we argued that the amplitudes $\zeta^{(N)}$ and $\chi^{(N)}$ are obtained as solutions to a system of linear partial differential equations and boundary conditions. Solutions to this linear homogeneous problem are at best fixed up to a multiplicative constant, and the above considerations explain that this is only natural.

#### 5.4.1 Boundary visit probabilities for interfaces in lattice models

The principal motivation for the introduction and study of SLEs is that these random curves are the scaling limits of interfaces in lattice models of statistical mechanics at criticality. The SLE zig-zag probabilities are closely related to the probabilities for an interface in a lattice model to pass through given boundary points. For some models these probabilities in turn have direct physical interpretations, for example the boundary visit probability of interface in Q-random cluster model ($Q$-FK model) gives a boundary magnetization in the $Q$-Potts model.

Also for lattice model interfaces, the exact meaning of passing through a boundary point involves some choices, and different choices lead to different non-universal constant factors. The idea, however, always is to consider the model on a lattice domain $\Lambda_\delta$ of small lattice mesh size $\delta$ so that $\Lambda_\delta$ approximates a given planar domain $\Lambda \subset \mathbb{C}$ as $\delta \downarrow 0$. One defines a boundary visit locally by requiring the lattice model interface to use for example a given edge or a given vertex near a marked point $y \in \partial \Lambda$ on the boundary.
The probabilities of thus visiting $N$ marked points on smooth parts of the boundary $\partial \Lambda$ are of order $\delta^{Nh}$, provided that also the lattice approximations to the boundary have a regular and consistent local structure as $\delta \to 0$. Thus the lattice mesh $\delta$ serves as a measure of the neighborhood size, and much like in (1.2), the limit of the lattice model interface probability renormalized by $\delta^{-Nh}$ should be given by $\zeta^{(N)}$ or $\chi^{(N)}$, correctly conformally transported to the domain $\Lambda$ by the conformal covariance rule of Section 5.2.

In Section 6 we discuss in more detail a few well-known lattice models and the details of the question of boundary visits of interfaces for them. We find that our formulas for $\zeta^{(N)}$ and $\chi^{(N)}$ are in very good agreement with the probabilities obtained from numerical simulations of these lattice models.

### 5.4.2 Covariant measure of SLE on the boundary

For lattice models, the most natural way of quantifying boundary proximity of an interface is by counting the number of boundary points visited by it, e.g., within a given boundary segment. In the scaling limit, the count must be renormalized properly by a power of the lattice spacing $\delta$: the probability to visit a given boundary point is of order $\delta^h$ and the expected number of boundary points visited in a segment of order $\delta^{h-1}$ (which diverges for $\kappa > 4$ and tends to zero for $\kappa < 4$).

The article [AS09] presents a construction of a covariant measure of SLEs on the boundary, which is the analogous boundary proximity count in the continuum. Roughly, this SLE boundary measure is constructed by studying a local martingale associated to the correlation function $\chi$. By construction this function $\chi$ then gives the density of the expectation of $\mu = \mu_{\infty,0,\infty}$ with respect to the Lebesgue measure on $\mathbb{R}$. The higher complete correlation functions $\chi^{(N)}$ of the present article should be the integral kernels for moments of the SLE boundary measure

$$
\expval{\varepsilon^{-N} \prod_{j=1}^{N} \mu([y_j, y_j + \varepsilon])} \sim \text{const} \times \chi^{(N)}(0; y_1, \ldots, y_N).
$$

In fact the proof [AS09] of non-triviality of the constructed SLE boundary measure employs the two-point function $\chi^{(2)}$, which had been found in [SZ10].

A convenient way to explicitly characterize a random measure is to give its Laplace transform. Denote briefly $\mu = \mu_{\infty,0,\infty}$. For a test function $\phi: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ let

$$
L(\phi) := \expval{\exp(-\int_0^\infty \phi d\mu)}
$$

be the Laplace transform of $\mu$ at $\phi$. For the sake of concreteness, consider $\phi$ supported on the positive real axis. Then the expansion of the Laplace transform around the zero function is given by

$$
L(\varepsilon \phi) = \expval{\exp(-\varepsilon \int_0^\infty \phi d\mu)}
$$

$$
= 1 - \varepsilon \expval{\int_0^\infty \phi(y) d\mu(y)} + \frac{\varepsilon^2}{2} \expval{\int_0^\infty \phi(y_1) \phi(y_2) d\mu(y_1) d\mu(y_2)} + \cdots
$$

$$
= 1 + \sum_{N=1}^{\infty} (-\varepsilon)^N c_N \int_0^\infty \cdots \int_0^\infty \phi(y_1) \cdots \phi(y_N) \chi^{(N)}(0; y_1, \ldots, y_N) dy_1 \cdots dy_N.
$$

The construction of [AS09] establishes that a unique (up to normalization) random measure satisfying the required abstract properties exists. The results of this article in principle give explicit formulas for the
random measure in terms of integral kernels for its moments or the power series expansion of its Laplace transform.

5.4.3 Conditioned SLE and first visit point recursion for the zig-zag amplitudes

Let us discuss one more interpretation of the results, which in fact also suggests a natural strategy of rigorous proof that our formulas give the order refined SLE Green’s functions on the boundary, as defined in Section 5.3 or alternatively in Section 5.3.3.

Consider conditioning the chordal SLE curve $\gamma$ to visit a boundary point $y$, for definiteness in $(\mathbb{H}, x, \infty)$ again. As such, this is a zero-probability event (for $\kappa < 8$), and one must perform a limiting procedure to properly define the conditioning: first condition on visiting $B_\varepsilon(y)$ and then let $\varepsilon \downarrow 0$. The conditioned curve can be described explicitly: its Radon-Nikodym derivative with respect to the ordinary chordal SLE is proportional to the indicator of the event of the visit, and in the limit $\varepsilon \downarrow 0$ we get a Girsanov transform of the ordinary chordal SLE

$$\frac{dP^{\text{cond.}}_{(\mathbb{H}, x, y, \infty)}}{dP_{(\mathbb{H}, x, \infty)}} \bigg|_{\mathcal{F}_t} \propto \chi^{(1)}_{\mathbb{H}\setminus K_\varepsilon}(\gamma(t); y) = |g'_1(y)|^8 \chi^{(1)}(g_1(x); g_t(y)).$$

This description of the conditioned curve is equivalent to the more familiar SLE$_{\kappa} (\rho)$ with $\rho = \kappa - 8$, i.e., the random Loewner chain (5.1) with driving process given by

$$X_0 = x, \quad dX_t = \sqrt{\kappa} dB_t + \frac{\rho}{X_t - g_t(y)} dt, \quad \text{where } \rho = \kappa - 8.$$ 

After the random time when the conditioned curve reaches $y$ (i.e., when $|X_t - g_t(y)| \rightarrow 0$), the curve will continue like an ordinary chordal SLE in the complement of the initial segment of the curve up to that time.

Using the one-point function $\chi^{(1)}$, one may thus describe the SLE conditioned to visit a given boundary point. Conditioning on visiting several points could be similarly done with our functions $\chi^{(N)}$ or $\zeta^{(N)}$. Below we will however turn the logic around, and see how our formulas could be rigorously proved using this conditioning.

The idea is to use the conditioning to reduce the $N$-point function question to an $(N - 1)$-point question. Namely, for the SLE curve $\gamma$ to make visits to $B_{\varepsilon_1}(y_1), \ldots, B_{\varepsilon_N}(y_N)$ in this order, it needs to make the first visit to $y_1$ by definition, and we may proceed by conditioning on this. We know, for example by considerations similar to Appendices A.1 or A.2, that the probability of this first visit is of order $\varepsilon_1^8 \chi^{(1)}(x; y_1)$, and we can describe the conditional law of the curve given this first visit essentially by the SLE$_{\kappa}(\rho)$ process above. After the time $\tau$ of the first visit, the curve is again a chordal SLE in the random domain $\mathbb{H} \setminus K_{\tau}$ at that time, and we would like it to visit the neighborhoods of the $N - 1$ remaining points $y_2, \ldots, y_N$. We may inductively assume that the $(N - 1)$-point visit formulas $\zeta^{(N-1)}(\gamma_{\tau}; y_2, \ldots, y_N)$ for chordal SLE have been established.

Thus we need to be able to average the $(N - 1)$-point zig-zag amplitude $\zeta^{(N-1)}_{\mathbb{H}\setminus K_\tau}(\gamma_{\tau}; y_2, \ldots, y_N)$ over the randomness of the domain $\mathbb{H} \setminus K_{\tau}; \gamma_\tau, \infty$ that remains after the first visit. That will be achieved if we can construct a martingale for the conditioned SLE, whose value at the time $\tau$ is $\zeta^{(N-1)}_{\mathbb{H}\setminus K_\tau}(\gamma_{\tau}; y_2, \ldots, y_N)$. The key point is that such a martingale is constructed using the formula for $\zeta^{(N)}$ that we find in the present work — namely we set

$$M_t = \prod_{j=2}^{N} |g'_j(y_j)|^8 \times \frac{\zeta^{(N)}(X_t; g_1(y_1), \ldots, g_t(y_N))}{\chi^{(1)}(X_t; g_t(y_1))}.$$ 

This is a local martingale by the differential equations (A.3) that our $\zeta^{(N)}$ satisfies, and its value at time $\tau$ is the desired $(N - 1)$-point zig-zag amplitude in the random domain $\mathbb{H} \setminus K_{\tau}$ essentially by the asymptotics conditions (2.7) we impose on $\zeta^{(N)}$. What remains is to show that $(M_t)_{t \in [0, 1]}$ is a uniformly integrable martingale. This relies partly on prior estimates of SLE probabilities [Bef08, LW13] and on careful
control of the functions appearing in the spin chain - Coulomb gas correspondence of the present article
and in [KP13]. One also needs to control some approximations made, but roughly speaking at this stage
optional stopping for the martingale \((M_t)\) proves that \(\zeta(N)\) gives the \(N\)-point boundary zig-zag amplitude
or \(N\)-point order refined SLE Green’s function on the boundary.

Carrying out the proof with this strategy is the topic of a subsequent work in collaboration with Kon-
stantin Izyurov.

6 Comparisons with lattice model simulations

It is somewhat intricate and computationally demanding to obtain satisfactory computer simulations of
SLE curves [Ken07]. Therefore, comparing our results with direct numerics of SLEs would be difficult. A
more practical alternative is to simulate lattice models whose interfaces tend to SLEs in the scaling limit.
The boundary visits in such lattice models indeed constitute a natural interpretation and an important
physical application of our results, as discussed in Section 5.4.1. In the present section we elaborate on the
idea in the context of various lattice models. We discuss simulation of these models and their interfaces and
boundary visits of the interfaces. Finally, we compare the numerical results obtained from these simulations
to our solution presented in Sections 3 and 4.

On physical grounds it is completely natural to expect that the scaling limit of renormalized lattice
interface visit probabilities is proportional to the SLE Green’s functions \(\chi(N)\) and \(\zeta(N)\). We nevertheless
remark that even in models whose interface is rigorously known to converge to a chordal SLE in the scaling
limit (e.g., Sections 6.1.2, 6.1.3, 6.1.4 below), highly nontrivial additional mathematical work would be
needed to establish this. Actually, the validity of the physically unsurprising equivalence is highly sensitive
to the details of the lattice approximation of the domain boundary, and again even valid approximation
schemes lead to different non-universal proportionality constants. Incidentally, the equivalence of the two
formulations has been rigorously established for one case: one and two-point boundary visits of the FK-Ising
model interface (Section 6.1.4 below) on boundary segments parallel to coordinate axes — the boundary
visit probabilities (or equivalent boundary spin correlation functions) were used in [HK13] as a technique
to control the scaling limit of an interface in a dual model (the Ising model with particular boundary
conditions). Our simulation results below of course show a good match to our analytical solution, and thus
clearly support the physically expected equivalence of the formulations.

Let us still make general comments about the numerical comparison of simulation data with our main
results. Small lattice mesh sizes \(\delta\) are of course desirable to reduce finite size scaling effects, i.e., to obtain
better approximations to the conformally invariant scaling limit situation. As always, however, small
mesh size \(\delta\) or corresponding large size of the simulated system quickly increases needed computational
resources, particularly so in critical models that we are interested in. For our question, there is yet another
difficulty. With lattice mesh \(\delta\), the probability of having \(N\) boundary visits by the interface is of order \(\delta^{Nh}\),
where \(h = h_{1,3}(\kappa) = \frac{8-\kappa}{\kappa} > 0\) and \(\kappa\) depends on the model. We are thus interested in rare events, whose
probability further decreases with mesh size \(\delta\) and number of visit points \(N\), so in order to obtain acceptable
statistics, we need increasingly large numbers of samples. The trade-off between reducing finite size effects
and improving statistics is therefore a major issue. High values of the exponent \(h_{1,3}(\kappa)\), or correspondingly
models with small \(\kappa\) are the most problematic. We have simulated models corresponding to \(\kappa = 2\) (LERW,
Section 6.1.2), \(\kappa = \frac{24}{7}\), and \(\kappa = \frac{48}{31}\) (different FK-models, Section 6.1.4), and \(\kappa = 6\) (percolation, Section
6.1.3). In the most difficult case \(\kappa = 2\) we are essentially limited to \(N \leq 2\), and significant finite size effects
still remain in the data (see Figure 6.6). In the least problematic case \(\kappa = 6\), finite size effects can be made
reasonably small up to \(N = 4\) (see Figure 6.6). The issues in numerical evaluation of our analytical results
have been separately discussed in Appendix D, and we note that besides large \(N\), difficulties also arise due to
small \(\kappa\).
6.1 Lattice model interfaces

6.1.1 Relevant domains and conformal maps

We have simulated different statistical models in lattice approximations of domains of the simplest possible shapes: the square and the equilateral triangle. The frequencies of boundary visits of interfaces have been collected, and for comparison with our formulas they need to be transported to the half-plane $\mathbb{H}$ by conformal maps. The domains, lattice approximations, and the conformal maps are described below.

The unit square

$$S = \left\{ z \in \mathbb{C} \mid 0 < \Re(z) < 1, 0 < \Im(z) < 1 \right\}$$

will be discretized by a square lattice of small mesh size $\delta$: the vertex set is $S_\delta = \delta \mathbb{Z}^2 \cap \mathbb{S}$ and edges connect vertices at distance $\delta$. A conformal map $f_S: S \to \mathbb{H}$ from the square to the half-plane is a Jacobi elliptic sine function $sn$ composed with a Möbius transform, and our choice is

$$f_S(u) = \frac{sn((2u - 1)K;m) + 1}{sn((2u - 1)K;m) - 1/\sqrt{m}} \frac{sn(K;m) - 1/\sqrt{m}}{sn(K;m) + 1},$$

where $m$ is the elliptic modulus of square and $K = K(m)$ is the corresponding complete elliptic integral of the first kind. This choice is such that the lower left corner is mapped to the origin, the top right corner to infinity, and the bottom right and top left corners to $+1$ and $-1$, respectively.

The unit equilateral triangle

$$T = \left\{ z \in \mathbb{C} \mid -\frac{1}{2} < \Re(z) < \frac{1}{2}, 0 < \Im(z) < \frac{\sqrt{3}}{2} - \sqrt{3} |\Re(z)| \right\}$$

will be discretized by a fine triangular lattice. The small mesh size $\delta$ is the distance between its neighboring vertices, and $T_\delta$ denotes the set of such triangular lattice vertices in $\mathbb{T}$. A conformal map $f_T: T \to \mathbb{H}$ from the triangle to the half-plane is the inverse of a Schwarz-Christoffel map,

$$f_T^{-1}(z) = \frac{\Gamma(\frac{5}{6})}{\sqrt{\pi} \Gamma(\frac{1}{3})} \times \int_0^z (1 - w)^{-2/3}(1 + w)^{-2/3}dw.$$

The choice is such that $f_T$ maps the midpoint of the bottom side to the origin, and the left and right bottom corners to $-1$ and $+1$, respectively.

6.1.2 Loop-erased random walk

The loop-erased random walk (LERW) is a path obtained by performing loop erasure to a finite piece of a simple random walk. The conformal invariance of the scaling limit of interior-to-boundary LERW was shown in [LSW04]. Different LERW variants, including the one we study here, were proven to have conformally invariant scaling limits in [Zha08]. The scaling limit of the path we describe below is chordal SLE$_2$.

We consider the square lattice domain $S_\delta$, which approximates the unit square, as in Section 6.1.1. We send a simple random walk $(W_n)_{n=0}^\infty$ at the lower left corner $W_0 = \delta + i\delta$. We condition on the event that the walk exits the square via the upper right corner, and we denote the time of exit by $\tau$. The loop-erased random walk is the simple path $\gamma_\delta$ which is obtained from $(W_n)_{n=0}^{\tau - 1}$ by chronologically erasing all loops (sequences of consecutive steps which start and end at the same vertex). Figure 6.1 shows a realization of a LERW in $S_\delta$ with lattice mesh $\delta = \frac{1}{150}$. The figure also suggests that the loop-erased path is unlikely to come close to the boundary except at the two end points, indicating the difficulties of sampling boundary visits of this model with fine lattice mesh.
We define boundary visit as the event that the path $\gamma_\delta$ passes through a vertex $x$ at distance $\delta$ from the boundary $\partial S$ of the square. The behavior of the boundary visit probabilities should be

$$P[\gamma_\delta \text{ visits } x_1, x_2, \ldots, x_N] \approx \text{const.} \times \prod_{j=1}^{N} (|f'(x_j)|^\delta)^h \times \zeta^{(N)}(0; f(x_1), \ldots, f(x_N)),$$

(6.1)

where $h = h_{1,3}(2) = 3$ and $f = f_S : S \to \mathbb{H}$ is the conformal map from the unit square to the half-plane given in Section 6.1.1.

The simulation is done as follows: we sample a conditioned random walk using explicitly calculated transition probabilities, then perform the loop erasure of the random walk, and collect data of visited boundary points of the loop erasure. We correct the boundary visit frequencies obtained from the simulations by dividing by the factor $\prod_{j=1}^{N} (|f'_S(x_j)|^\delta)^h$ that appears in (6.1), and then compare with our SLE boundary visit amplitude $\zeta^{(N)}$ at $\kappa = 2$. Note that the probabilities decay as $\delta^{Nh}$ and due to the high value of the exponent $h = h_{1,3}(2) = 3$ it is very hard to obtain good statistics with a small mesh size, especially for higher $N$. Figures 6.4 and 6.6 present data from simulations with lattice mesh $\delta = \frac{1}{120}$ and $10^7$ realizations and with lattice mesh $\delta = \frac{1}{60}$ and $10^8$ realizations, respectively. The agreement with our analytical results is reasonable. The otherwise difficult small $\kappa$ turns out to have one advantage: the orders of magnitude of the visits in different pieces of the plot are rather different, and one notes in particular that the universal ratio of the boundary visit amplitudes with $y_2 < x = 0$ and $y_2 > y_1 = 1$ obtained by our method is undeniably correct — a single multiplicative constant has been fitted for the two pieces $\zeta_+^{(2)}$ and $\zeta_-^{(2)}$ in Figure 6.6.

6.1.3 Percolation

Percolation is an easily defined model of statistical physics, showing nevertheless interesting critical behavior. Its conformal invariance had been predicted in [LPPSA92], and impressive exact results had been
predicted using conformal field theory. The proof of conformal invariance of scaling limit of site percolation on triangular lattice was obtained by Smirnov in [Smi01], based on a formula found by Cardy [Car92]. The interface that we define below converges in the scaling limit to chordal SLE_6, see [Smi01, CN07].

We take a domain $T_\delta$ which is a triangular lattice approximation of an equilateral triangle as in Section 6.1.1. Triangular lattice site percolation with parameter $p \in (0,1)$ associates to each vertex of a domain in the triangular lattice (which we portray as a hexagon, a face of the dual lattice) a color: white with probability $p$ and black with probability $1-p$, independently. One studies questions concerning connected components of sites of one color at the critical parameter value $p_c = \frac{1}{2}$. We impose white boundary conditions on the left half of the boundary $\partial T_\delta \cap \{\Re(z) < 0\}$, and black on the right half $\partial T_\delta \cap \{\Re(z) > 0\}$.

There is a unique path $\gamma_\delta$ on the dual lattice from the midpoint of the bottom side of the triangle to the top vertex of the triangle, leaving white vertices on the left and black vertices on the right. This path, commonly called the percolation exploration path, is our interface. Figure 6.2 shows a realization of the exploration path in $T_\delta$ with lattice mesh $\delta = \frac{1}{40}$. Quite the contrary to Figure 6.1, here there is no shortage of places on the boundary that are visited by the path.

We define boundary visit as the event that the path $\gamma_\delta$ passes through the exteriormost corner $x$ of a hexagon next to the boundary layer. The behavior of the boundary visit probabilities should be given by Equation (6.1), where now $h = h_{1,3}(6) = \frac{1}{3}$ and $f = f_T : T \to \mathbb{H}$ is a conformal map from the triangle to the half-plane given in Section 6.1.1.

The simulation of percolation configurations hardly requires any comments. The only computationally intensive step is to extract the interface from the configuration. Another practical issue for high $N$, small $\delta$ and large number of samples is the storage of the obtained data of boundary visits. Once the data of boundary visit frequencies is collected, we again correct them by dividing by the factor $\prod_{j=1}^N (|f_T'(x_j)|\delta)^h$, and then compare with our SLE boundary visit amplitude $\zeta^{(N)}$ at $\kappa = 6$. Figures 6.4, 6.5, 6.7, and 6.9.
Figure 6.3: FK-model (random cluster model) interface closely follows the outer boundary of the cluster connected to the wired part of the boundary: the left and top sides.

present data for $N = 1, 2, 3, 4$, respectively, obtained from simulations with lattice mesh $\delta = \frac{1}{500}$ and $10^5$ realizations, with lattice mesh $\delta = \frac{1}{300}$ and $2 \times 10^6$ realizations, with lattice mesh $\delta = \frac{1}{180}$ and $10^6$ realizations, and with lattice mesh $\delta = \frac{1}{160}$ and $2 \times 10^8$ realizations, respectively. The agreement with our analytical results is nearly perfect. Note again that for any fixed $N$, only one multiplicative constant has been fitted, and the ratios of the magnitudes of boundary visit frequencies in different pieces of the plots are obtained from our results.

6.1.4 FK-model

The random cluster model (also called FK-model, named after Fortuin and Kasteleyn [FK72]) with parameters $(p, Q)$ is a generalization of bond percolation, which for integer values of $Q$ is closely related to the $Q$-Potts model. For $Q \in [0, 4]$ it is expected to undergo a continuous phase transition at the critical value $p = p_c(Q) = \sqrt{\frac{Q}{1+\sqrt{Q}}} m$ and behave conformally invariantly at the critical point. With Dobrushin boundary conditions, there is an interface somewhat analogous to the exploration path of percolation, which at the critical point is expected to converge in the scaling limit to (chordal) $\text{SLE}_\kappa$, where $\kappa = \kappa(Q) = \frac{4\pi}{\arccos(-\sqrt{Q}/2)}$.

The SLE scaling limit is rigorously known in two special cases: the case $Q = 2$ is known as the FK-Ising model and the techniques of [Smi06, Smi10a] led to a proof [CDCH+13], and the limiting case $Q = 0$ corresponds to the uniform spanning tree treated in [LSW04]. Figure 6.3 shows a realization of $Q = 4$ FK-model interface with lattice mesh $\delta = \frac{1}{180}$, together with the interface.

It is worth noticing that the probabilities of boundary visits of the interface can be used to express the boundary magnetization, and more generally boundary spin correlation functions of the Potts model,

\(^{10}\text{That this self-dual value is critical has been established in [BDC12] for } Q \geq 1.\)
with one of the boundary arcs having fixed spin. These exemplify some of the physical applications of the boundary visit problem.

For simulations in this article we restrict our attention to the values \( Q = 2 \) and \( Q = 3 \). Integer values of \( Q \) are convenient because there exists a Monte Carlo Markov chain by Swendsen and Wang, which does not suffer as much of critical slowing down as the more common Markov chains based on local updates [SW87]. This efficiency of simulation is important, because we need good statistics to get accurate information about the small probability events of multiple boundary visits. Swendsen-Wang algorithm works for all integer \( Q \), but for \( Q > 4 \) the model has a first order phase transition and does not exhibit conformal invariance. For \( Q = 4 \) the finite size corrections scale too badly for reliable simulations.

We define the model in the lattice approximation \( S_\delta \) of the unit square \( S \) given in Section 6.1.1. The random cluster model is a random subset \( \omega \) of edges of \( S_\delta \), with probability proportional to

\[
P_{(p,Q)}([\omega]) \propto \left( \frac{p}{1-p} \right)^{|\omega|} Q^{k(\omega)},
\]

where \( k(\omega) \) denotes the number of connected components (“clusters”) of the subgraph of \( S_\delta \) defined by all vertices and the edges \( \omega \). The appropriate Dobrushin boundary conditions amount to conditioning on the event that all edges of the left and top boundaries of the square are in \( \omega \). The interface \( \gamma_\delta \) is the path obtained as the boundary of the \( \frac{1}{\delta} \)-thickening of the component connected to the left and top, i.e., a path closely surrounding the “wired cluster”, see Figure 6.3.

The interface being defined on a lattice different from the square lattice, it is now natural to define boundary visits to points with half-lattice-unit coordinates. Moreover, the wiring of the boundary introduces some asymmetry in the definition. On the bottom we say that \( (x + \frac{1}{2})\delta \) is visited if the path goes outside the domain at \( (x + \frac{1}{2})\delta - \frac{1}{4}\delta \), and on the right a similar definition is used. On the left we say that \( i(y + \frac{1}{2})\delta \) is visited if the path comes to the point \( i(y + \frac{1}{2}) + \frac{1}{2}\delta \), and on the top a similar definition is used. These definitions are natural, as is illustrated by the figure of the interface. The behavior of the boundary visit probabilities should again be given by Equation (6.1), where now \( h = h_{1.3}(\frac{16}{\pi}) = \frac{1}{2} \) for \( Q = 2 \) and \( h = h_{1.3}(\frac{24}{\pi}) = \frac{3}{4} \) for \( Q = 3 \), and \( f = f_S : S \to \mathbb{H} \) is the conformal map from the unit square to the half-plane as in Section 6.1.1.

Our simulation runs the Swendsen-Wang Monte Carlo Markov chain and collects time averages of the boundary visiting frequencies obtained from the simulations by dividing by the factor \( \prod_{j=1}^{N} (|f_S(x_j)| \delta)^b \), and then compare with our SLE boundary visit amplitude \( \zeta(N) \) at \( \kappa = \kappa(Q) \). For \( N \leq 3 \) we get good enough statistics and the agreement with our analytical results is very good: Figures 6.4 and 6.5 show \( N = 1 \) and \( N = 2 \) data for both \( Q = 2 \) and \( Q = 3 \), with \( \delta = \frac{1}{100} \) and \( 10^5 \) samples in each case. We have included the plot of three-point boundary visit data in Figure 6.8 only for \( Q = 3 \) because the value of \( \kappa \) (\( \kappa = \frac{3}{2} \)) is sufficiently different from the case of percolation (\( \kappa = 6 \)) so that the shapes of the functions are clearly distinct (for this we use \( \delta = \frac{1}{100} \) and \( 5 \times 10^8 \) samples). We still point out how remarkably much is known of the FK-Ising case \( Q = 2 \), largely owing to the techniques of discrete complex analysis [Smi10a] [CS11] [CS12] [Smi10b]. This is the only lattice model for which the scaling limit of renormalized boundary visiting probabilities has in fact been proven to exist, and even the corresponding non-universal constants for \( N = 1 \) and \( N = 2 \) have been found explicitly [HK13].

The exact \( N = 1 \) formula reads for \( x \) away from the corners

\[
\frac{1}{\sqrt{\delta}} \sqrt{\frac{1}{|f_S(x)|}} P_{\text{FK-Ising}} \left[ \gamma_\delta \text{ visits } x \right] \xrightarrow{\delta \to 0} \sqrt{\frac{1 + \sqrt{2}}{2\pi}} \times |f_S(x)|^{-1/2}.
\]

We find excellent numerical agreement of the exponent value (best fit gives 0.499872 instead of \( \frac{1}{2} \)) and the non-universal multiplicative constant (best fit gives 0.618241 instead of \( \sqrt{\frac{1 + \sqrt{2}}{2\pi}} \approx 0.619866 \)).
Comparisons with Lattice Model Simulations

The solid line in the middle plot in Figure 6.5 uses this explicit non-universal multiplicative constant. This comparison to an exact scaling limit result gives a fair idea of the finite size effects present in the simulation data of the FK-Ising model, but one must remember that the finite size corrections scale differently for other models.

6.2 Simulation data and results of the comparison

Simulation data and corresponding plots of our analytical results are presented in Figures 6.4—6.9. The general conclusion is that the boundary visit probabilities of lattice model interfaces are in agreement with the predictions of type (6.1), where the amplitudes \( \zeta^{(N)} \) are given by our main results. The main source of numerical error is finite size effects.

Figure 6.4 shows one-point visit amplitudes on a log-log scale. The data from all models follows the power law \( \zeta^{(1)}(x; y) \propto |y - x|^{-h} \) over a range of scales. The slope \( h \) is so different for \( \kappa = 2 \) that we have included a separate plot for the LERW case. Particular finite size effects caused by error near the corners of the polygonal domain (triangle or square) are seen as bumps in the data. This effect diminishes for smaller \( \delta \), but it is visibly present in our data for all \( N \). We have centered the \( N = 1 \) data so that the bump appears in the middle of the plot. For \( N \geq 2 \) this error affects a part of the data points across the whole range of the plot, resulting in an apparent failure of a perfect data collapse seen as thickness of the data point cloud.

Figures 6.5 and 6.6 show two-point boundary visit data on a logarithmic scale both in the case where the points \( y_1, y_2 \) to be visited are on the same side and in the case where they are on different sides. We have scaled to the case \( y_1 = 1 \) and plotted as a function of \( y_2 \), so that ideally all data from a given model should collapse on the curve constructed from the two pieces \( \zeta^{(2)}_+(0; 1, y_2) \) (for \( y_2 > 1 \)) and \( \zeta^{(2)}_-(0; 1, y_2) \) (for \( y_2 < 0 \)). The same fitted multiplicative constant is used on both pieces for each model, and a clear agreement is observed in all cases. For the FK-Ising model case we have even avoided fitting, as we have been able to use the rare known explicit non-universal constant mentioned in Section 6.1.4. Data from all models show some finite size effects, and roughly these are worse for smaller \( \kappa \). The functional shape of all plots is nevertheless clearly correct. Again the shape for \( \kappa = 2 \) is so different from others that we have plotted it separately.

Figures 6.7 and 6.8 show three-point boundary visit data on a logarithmic scale for critical percolation and the critical \( Q = 3 \) FK-model, respectively. Data from percolation are still very well on the curves of our analytical results. In the \( Q = 3 \) FK-model the finite size effects are more apparent. Again, a single fitted multiplicative constant has been used for all pieces. In particular the several orders of magnitude difference of the boundary visit frequencies on the two sides of Figure 6.8 is in excellent agreement with our analytical results, even if, due to finite size effects, the data points otherwise only serve to give a sketchy idea of the shape of the function here.

Figure 6.9 shows four-point boundary visit data on a logarithmic scale for critical percolation. Both the numerical evaluation of our results \( \zeta^{(4)} \) and decent simulation results are starting to be computationally
Figure 6.4: Data of one-point boundary visit frequencies collected from simulations of lattice models. We have set $x = 0$ and plotted the conformally corrected frequency of visits as a function of $y_1$ on log-log scale. The solid lines are fitted power laws, in accordance with $\zeta^{(1)}(x, y_1) \propto |y_1 - x|^{\frac{1}{h_1}}$. The simulations are done in polygonal domains (triangle for percolation and square for the other models), and the bumps in the data in the middle of the plots are due to a corner of the polygonal domain.

Upper plot: percolation (top, blue), FK-Ising model (middle, red), FK model with $Q = 3$ (bottom, green)

Lower plot: loop-erased random walk
Figure 6.5: Data of two-point boundary visit frequencies collected from simulations of lattice models: percolation (top, blue), FK-Ising model (middle, red), FK-model with $Q = 3$ (bottom, green). We set $x = 0$, $y_1 = 1$ and plot the conformally corrected frequency as a function of $y_2$ on logarithmic scale. The solid curves are multiples of the two-point boundary visit amplitudes $\zeta^{(2)}(x; y_1, y_2)$, with the same multiplicative constant used for the two pieces: $\zeta_{++}(x; y_1, y_2)$ when $y_2 > 1$ and $\zeta_{+-}(y_2; x; y_1)$ when $y_2 < 0$. For FK-Ising we have used the known exact multiplicative constant from [HK13], for other models this non-universal constant is fitted to data.

Figure 6.6: Data of two-point boundary visit frequencies collected from simulations of LERW. We set $x = 0$, $y_1 = 1$ and plot the conformally corrected frequency as a function of $y_2$ on logarithmic scale. The solid curves are multiples of the two-point boundary visit amplitudes $\zeta^{(2)}(x; y_1, y_2)$, with again the fitted multiplicative constant being the same for the two pieces.
Figure 6.7: Data of three-point boundary visit frequencies collected from simulations of critical percolation. In the upper plot we set \( x = 0, y_1 = 1, y_3 = -1 \), and in the lower plot we set \( x = 0, y_1 = 1, y_3 = 2 \). In both plots the conformally corrected frequency is shown as a function of \( y_2 \) on logarithmic scale. The solid curves are multiples of the three-point boundary visit amplitudes \( \zeta^{(3)}(x; y_1, y_2, y_3) \) (that is, combinations of \( \zeta_{+-} \) and \( \zeta_{++} \) on the upper and of \( \zeta_{+-} \) and \( \zeta_{++} \) on the lower plot). The fitted multiplicative constant is again the same for all the different pieces.
Figure 6.8: Data of three-point boundary visit frequencies collected from simulations of FK random cluster model with $Q = 3$. In this plot we set $x = 0$, $y_1 = 1$, $y_3 = 2$. The plot shows conformally corrected frequency as a function of $y_2$ on logarithmic scale. The solid curves are multiples of the three-point boundary visit amplitudes $\zeta^{(3)}(x; y_1, y_2, y_3)$ (that is, combinations of $\zeta_{-+}$ and $\zeta_{+++}$). The fitted multiplicative constant is again the same for the different pieces.
Figure 6.9: Data of four-point boundary visit frequencies collected from simulations of critical percolation. On the upper plot we set $x = 0, y_1 = -1, y_2 = 1, y_4 = 2$ and plot as a function of $y_3$. On the lower plot we set $x = 0, y_1 = 1, y_2 = -1, y_4 = 2$ and plot as a function of $y_3$. The conformally corrected frequencies in both plots are on a logarithmic scale. The solid curves are multiples of the four-point boundary visit amplitudes $\zeta^{(3)}(x; y_1, y_2, y_3, y_4)$ (that is, $\zeta_{++++}$ on the upper and combinations of $\zeta_{+---}$ and $\zeta_{++-+}$ on the lower plot). The fitted multiplicative constant is again the same for all the different pieces.
very heavy — we have had to interpolate the analytical result from the calculations at the points shown on the plots. Nevertheless, the plot shows agreement of simulation data with our result.

7 Conclusions and outlook

We have presented a method based on quantum group calculations, which gives explicit solutions of the chordal SLE\(_\kappa\) boundary visit probability amplitudes \(\zeta^{(N)}\) and \(\chi^{(N)}\) for arbitrary numbers \(N\) of marked boundary points. The answers are expressed in terms of linear combinations of Coulomb gas integrals, and can be transformed to regularized real integrals. They give the universal answer to various formulations of the SLE boundary visit question, up to an overall non-universal constant, which depends on the formulation. In particular, they give the renormalized scaling limit boundary visit probabilities for lattice model interfaces.

Our results are obtained by solving a partial differential equation system with boundary conditions given recursively by the solutions with smaller number \(N\) of marked points. The system is suggested by plausible considerations of asymptotics, but we have not fully justified the use of this procedure. In an ongoing work with Konstantin Izyurov we plan to implement the strategy outlined in Section 5.4.3 to prove rigorously that the formulas obtained in the present article indeed give the SLE multi-point Green’s functions on the boundary.

The method we have used is an application of the spin chain - Coulomb gas correspondence presented in a more general setup in [KP13], and applied there to the problem of multiple SLE pure geometries and crossing probabilities. The method provides a systematic approach to a class of SLE and CFT problems depending on arbitrary numbers of marked points. It works directly only for irrational values of \(\kappa\), but for questions such as boundary visit amplitudes, one can naturally extend the final results to all \(\kappa\) by requiring continuity. It would be interesting to generalize the spin chain - Coulomb gas correspondence itself to rational values of \(\kappa\). This would presumably involve non-semisimple representation theory of the corresponding quantum group as well as results that correspond to logarithmic conformal field theory correlation functions.

It would be interesting to find also formulas for boundary visit probabilities for other variants of SLE, such as the radial SLE\(_\kappa\) and dipolar SLE\(_\kappa\), SLE\(_\kappa\)(\(\rho\)), or even more general variants. Finally, one of the most natural remaining open questions about Schramm-Loewner evolutions is the bulk analogue of the question answered in the present article: finding a formula for the multi-point Green’s function of the chordal SLE (for recent progress on this, see [RS05,Bef08,LS11,LW13]).
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A SLE derivations of the exponent and a PDE

A.1 Touching a small boundary interval

One can write down the exact solution for the probability of a chordal SLE to hit a boundary interval \([y, y + \varepsilon]\) (for \(y > x\)) and do the asymptotics as \(\varepsilon \searrow 0\), see, e.g., [BB03a,AS08]. We include the argument briefly here.

Assume that \(x < l < r\) and let \(P(x, l, r)\) be the probability that a chordal \(SLE_\kappa\) in the half-plane \(\mathbb{H}\) from \(x\) to \(\infty\) touches the interval \([l, r]\), and note that by translation and scaling invariance it can be reduced to a function of one variable,

\[
P(x, l, r) := P_{(\mathbb{H}; x, \infty)} [\gamma \cap [l, r] \neq \emptyset],
\]

\[
P(x, l, r) = p \left( \frac{l - x}{r - x} \right).
\]

By domain Markov property we create a martingale \((M_t)_{t \geq 0}\): we define \(M_t\) as the above probability conditionally on the knowledge of an initial segment \(\gamma[0, t]\)

\[
M_t = P_{(\mathbb{H}; x, \infty)} \left[ \gamma \cap [l, r] \neq \emptyset \mid \mathcal{F}_t \right] = P_{(H_t; \gamma(t), \infty)} [\gamma \cap [l, r] \neq \emptyset].
\]

By conformal invariance under the map \(g_t\) in (5.1) this can be written as

\[
M_t = P_{(\mathbb{H}; x_t, \infty)} [\gamma \cap [g_t(l), g_t(r)] \neq \emptyset] = P (X_t, g_t(l), g_t(r)).
\]

Stochastic calculus tells that for this to be a martingale, the drift term

\[
\frac{\kappa}{2} \frac{\partial^2}{\partial x^2} P + \frac{2}{l - x} \frac{\partial}{\partial l} P + \frac{2}{r - x} \frac{\partial}{\partial r} P
\]
in the Itô derivative must vanish. This is an ordinary differential equation for $p$,

$$p''(u) + \frac{-4 + (2\kappa - 4)u}{\kappa u(1 - u)} p'(u) = 0.$$ 

Integrating with the boundary conditions $p(0) = 1$, $p(1) = 0$ we obtain that (for $4 < \kappa < 8$)

$$P_{(\mathbb{H}; x, \infty)} \left[ \gamma \cap [l, r] \neq \emptyset \right] = \frac{4 \sqrt{\pi}}{2^{8/\kappa} \Gamma \left( \frac{8 - \kappa}{2\kappa} \right) \Gamma \left( \frac{\kappa - 4}{\kappa} \right)} \int_{\frac{1}{r - x}}^{1} u^{-\frac{\kappa}{2}} (1 - u)^{\frac{8 - \kappa}{2\kappa}} \, du.$$ 

From this exact answer we find that the probability of hitting a small interval of size $\varepsilon$ at $y$ scales as $\varepsilon^h$ with amplitude $|y - x|^{-h}$

$$P_{(\mathbb{H}; x, \infty)} \left[ \gamma \cap [y, y + \varepsilon] \neq \emptyset \right] \sim \varepsilon^{\frac{\kappa}{\kappa - 4}} \frac{4 \sqrt{\pi} \kappa}{(8 - \kappa) 2^{8/\kappa} \Gamma \left( \frac{8 - \kappa}{2\kappa} \right) \Gamma \left( \frac{\kappa - 4}{\kappa} \right)} (y - x)^{\frac{2 - \kappa}{\kappa}}. \tag{A.1}$$ 

Also the multiplicative constant in

$$\lim_{\varepsilon \searrow 0} (\varepsilon^{-h} \times P[\gamma \cap I(\varepsilon) \neq \emptyset]) = \text{const.} \times \zeta^{(1)}(x; y)$$

is explicit here, but it is given by a somewhat complicated expression, and such constants are in any case non-universal.

### A.2 Reaching a small conformal distance from boundary point

Another derivation of the scaling exponent is based on the notion of boundary visit defined in terms of conformal distance. Namely, one can find explicitly the asymptotics of the probability that the chordal SLE reaches a small conformal distance from a marked boundary point. The strategy is similar to the above, but the martingale argument leads to a parabolic partial differential equation, which we do not solve explicitly, but instead we just find the leading eigenvector and eigenvalue of the generator, and hence deduce the small neighborhood size asymptotics of solutions.

For the martingale argument we need to keep track of one more point, the rightmost point $r$ in the image of the SLE hull. Choose therefore $x < r < y$ and let $Q(x, r, y, s)$ be the probability that for a chordal SLE$_\kappa$ $\gamma$ in the half-plane $\mathbb{H}$ from $x$ to $\infty$ the conformal radius of $y$ in $\mathbb{H} \setminus \gamma \cup (-\infty, r])$ (with a Schwarz reflection as before) is at most $e^{-s}$. In the limit $r \searrow x$ this correctly measures the conformal distance to the curve $\gamma$ only. By translation and scaling invariance $Q$ can be reduced to a function of two variables,

$$Q(x, r, y, s) := P_{(\mathbb{H}; x, \infty)} \left[ p_{\mathbb{H}\setminus(\gamma \cup (-\infty, r])}(y) \leq e^{-s} \right], \quad Q(x, r, y, s) = q \left( \frac{r - x}{y - r}, s + \log(y - r) \right).$$

By domain Markov property we again create a martingale $(M_t)_{t \geq 0}$

$$M_t = P_{(\mathbb{H}; x, \infty)} \left[ p_{\mathbb{H}\setminus(\gamma \cup (-\infty, r])}(y) \leq e^{-s} \mid \mathcal{F}_t \right],$$

and by conformal invariance we write it as

$$M_t = P_{(\mathbb{H}; x, \infty)} \left[ p_{\mathbb{H}\setminus(\gamma \cup (-\infty, g_t(r)))}(g_t(y)) \leq e^{-s + \log|g_t'(y)|} \right] = Q \left( X_t, g_t(r), g_t(y), s - \log|g_t'(y)| \right).$$

For this to be a martingale, the Itô derivative drift term

$$\frac{\kappa}{2} \frac{\partial^2}{\partial x^2} Q + \frac{2}{r - x} \frac{\partial}{\partial r} Q + \frac{2}{y - x} \frac{\partial}{\partial y} Q + \frac{2}{(y - x)^2} \frac{\partial}{\partial s} Q$$
must vanish. This is a parabolic partial differential equation for \( q \),

\[
\left[ \frac{\partial}{\partial \sigma} - \mathcal{G} \right] q(\theta, \sigma) = 0 \quad \text{with generator} \quad \mathcal{G} = \frac{\kappa}{4} \frac{\partial}{\partial \sigma} (1 + \theta)^2 + (1 + \theta)(1 + 2\theta) \frac{\partial}{\partial \theta}.
\]

The asymptotics of small neighborhood size \( \varepsilon = e^{-s} \to 0 \) correspond to \( s \to +\infty \) and therefore \( \sigma \to +\infty \) in the above parabolic equation. In this limit the solution behaves like \( q(\theta, \sigma) \sim e^{\lambda_0 \sigma} q_0(\theta) \), where \( q_0 \) is the positive eigenvector and \( \lambda_0 \) the corresponding leading eigenvalue of the generator \( \mathcal{G} \). One finds explicitly

\[
q_0(\theta) = (1 + \theta)^{1 - \frac{\varepsilon h}{\kappa}}, \quad [\mathcal{G} q_0](\theta) = \left( 1 - \frac{8}{\kappa} \right) q_0(\theta), \quad \text{i.e.,} \quad \lambda_0 = 1 - \frac{8}{\kappa}.
\]

From this asymptotic we find that the probability of reaching a small conformal distance \( e^{-s} = \varepsilon \) at \( y \) scales as \( e^{\lambda_0 s} = \varepsilon^{\frac{h}{\kappa}} \) with the correct scaling exponent \( h = -\lambda_0 = \frac{8 - \kappa}{\kappa} \).

A.3 The second order PDE from stochastic calculus

Let \( \gamma \) be the chordal \( \text{SLE}_\kappa \) curve in \((\mathbb{H}; x, \infty)\) parametrized as in Section 5.1. By the domain Markov property, conditionally on an initial segment \( \gamma^- = \gamma|_{[0, T]} \) of the curve up to a stopping time \( T \), the rest of the curve \( \gamma^+ = \gamma|_{(T, \infty)} \) is a chordal \( \text{SLE}_\kappa \) in the domain \( \mathbb{H} \setminus K_T \) from the tip \( \gamma(T) \) of the initial segment to \( \infty \). Consider stopping times \( T \) smaller than the time at which any boundary visit happens. Then, conditionally on the initial segment \( \gamma^- \), the contribution to the boundary visit amplitude \( \zeta^{(N)}(x; y_1, \ldots, y_N) \) is \( \zeta^{(N)}_{\mathbb{H} \setminus K_T \gamma(T, \infty)}(y_1, y_2, \ldots, y_N) \). Using the conformal map \( g_T : \mathbb{H} \setminus K_T \to \mathbb{H} \) and conformal covariance of \( \zeta^{(N)} \), the conditional contribution equals

\[
M_T = \left( \prod_{j=1}^N g_T'(y_j)^{h} \right) \times \zeta^{(N)}(X_T; g_T(y_1), \ldots, g_T(y_N)). \quad (A.2)
\]

By construction, then, \((M_t)_{t \geq 0}\) is a local martingale. We can compute the Itô derivative of \( M_t \), and require that the drift term in it vanishes, leading to the second order partial differential equation

\[
\left[ \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} + \sum_{j=1}^N \left( \frac{2}{y_j - x} \frac{\partial}{\partial y_j} - \frac{2h}{(y_j - x)^2} \right) \right] \zeta^{(N)}(x; y_1, \ldots, y_N) = 0,
\]

which is Equation (2.3) in the PDE system of Section 2.1. The alternative explanation of this equation by conformal field theory is given in Appendix B.2.

B Conformal field theory considerations

B.1 Boundary visit amplitudes as conformal field theory correlation functions

From conformal field theory point of view, the boundary visit amplitudes are essentially correlation functions of boundary primary fields of conformal weights \( h \) in a conformal field theory with central charge \( c(\kappa) = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa} \), see [BB03a]. We remark that the value \( \{1.1\} \) is a conformal weight in the Kac table, \( h = h_{1,3}(\kappa) = \frac{8 - \kappa}{\kappa} \). This suggests the possibility of a degeneracy at grade three, which we below in Appendix B.2 argue to give rise to the third order PDEs (2.4).
The covariance rule (5.2) reflects the conformal transformation properties of primary fields. More precisely, the boundary zig-zag amplitude should be thought of as a ratio

$$\zeta^{(N)}(x; y_1, y_2, \ldots, y_N) = \frac{\langle \psi_{1,2}(x) \psi_{1,3}(y_1) \cdots \psi_{1,3}(y_N) \psi_{1,2}(\infty) \rangle}{\langle \psi_{1,2}(x) \psi_{1,2}(\infty) \rangle},$$

where:

- The numerator $$\langle \psi_{1,2}(x) \psi_{1,3}(y_1) \cdots \psi_{1,3}(y_N) \psi_{1,2}(\infty) \rangle$$ is a correlation function of $$N$$ boundary primary fields $$\psi_{1,3}$$ of conformal weight $$h = h_{1,3}(\kappa) = \frac{8 - \kappa}{\kappa}$$ located at $$y_1, y_2, \ldots, y_N$$, and two boundary primary fields $$\psi_{1,2}$$ of conformal weight $$\delta = h_{1,2}(\kappa) = \frac{4 - \kappa}{\kappa^2}$$ located at $$x$$ and $$\infty$$.

- The denominator $$\langle \psi_{1,2}(x) \psi_{1,2}(\infty) \rangle$$ is the correlation function of two boundary primary fields $$\psi_{1,2}$$ located at $$x$$ and $$\infty$$. This correlation function is in fact just a constant (independent of $$x$$), but the presence of the fields $$\psi_{1,2}$$ both in the numerator and denominator is the reason why the conformal covariance rule (5.2) does not contain a Jacobian factor $$|f'(x)|^\delta$$.

### B.2 Singular vectors and differential equations

From the point of view of conformal field theory, partial differential equations such as (2.3) and (2.4) are consequences of conformal Ward identities if the relevant boundary primary fields have vanishing descendants.

At the tip of the SLE curve, the boundary changing field is a primary field $$|\psi_{1,2}\rangle$$ of conformal weight $$\delta = h_{1,2}(\kappa) = \frac{4 - \kappa}{\kappa^2}$$, which has a vanishing descendant $$(L_{-1}^2 - \frac{4}{\kappa}L_{-2})|\psi_{1,2}\rangle = 0$$ at level 2 [BB03c, BB03b, BB04].

He associated conformal Ward identity is the second order PDE (2.3).

At the points to be visited by the SLE curve, the boundary fields are primaries $$|\psi_{1,3}\rangle$$ of conformal weights $$h = h_{1,3}(\kappa) = \frac{8 - \kappa}{\kappa}$$, and they have vanishing descendants

$$\left(L_{-1}^3 - \frac{16}{\kappa}L_{-2}L_{-1} + \frac{8(8 - \kappa)}{\kappa^2}L_{-3}\right)|\psi_{1,3}\rangle = 0$$

at level 3. The associated conformal Ward identities are the third order PDEs (2.4).

### B.3 Asymptotics from operator product expansions

Conformal field theory allows a finite number of different asymptotics as the distance of any two arguments of $$\zeta^{(N)}$$ or $$\chi^{(N)}$$ tends to zero. The reason is that the boundary primary field $$\psi_{1,2}(x)$$ is degenerate at level two [BB03c, BB03b, BB04], and similarly the boundary primary fields $$\psi_{1,3}(y_j)$$ are degenerate at level three [BB03a] (this level three degeneracy is not a priori granted, but it is suggested by known $$N = 1$$ and $$N = 2$$ cases and justified a posteriori by a proof of our formula). The degeneracies imply selection rules for the fusion of the corresponding fields. A fusion of primary fields located at $$z$$ and $$w$$, with respective conformal weights $$h(z)$$ and $$h(w)$$, to a field of conformal weight $$h(\infty)$$ and its descendants, leads to terms of the form

$$(z - w)^{h(\infty) - h(z) - h(w)} \times \text{reg.}$$

in the operator product expansion. Here and below, reg. stands for functions that are holomorphic and non-vanishing on the “diagonal” $$z = w$$. Taking into account the selection rules, conformal field theory suggests the following:

- **Possible asymptotics as two visit points approach each other**: The fusion of the fields at $$y_j$$ and $$y_k$$ may contain primary fields of weights $$h_{1,1} = 0$$, $$h_{1,3} = \frac{8 - \kappa}{\kappa}$$, $$h_{1,5} = \frac{2(12 - \kappa)}{\kappa}$$. Correspondingly the functions $$\zeta^{(N)}$$ and $$\chi^{(N)}$$ have the form

$$\langle y_j - y_k \rangle^{2(1 - \frac{\kappa}{4})} \times \text{reg.} + \langle y_j - y_k \rangle^{1 - \frac{\kappa}{2}} \times \text{reg.} + \langle y_j - y_k \rangle^{\frac{\kappa}{2}} \times \text{reg.}$$

as $$|y_k - y_j| \to 0$$. 

$$\zeta^{(N)}(x; y_1, y_2, \ldots, y_N) = \frac{\langle \psi_{1,2}(x) \psi_{1,3}(y_1) \cdots \psi_{1,3}(y_N) \psi_{1,2}(\infty) \rangle}{\langle \psi_{1,2}(x) \psi_{1,2}(\infty) \rangle},$$
C.1 Explicit normalization conventions for subrepresentations

In the spin chain - Coulomb gas correspondence, the asymptotics of the functions may be read off from projections to irreducible subrepresentations in consecutive tensorands. We again only state the needed results, and refer to [KP13] for the general case.

We specifically make use of the tensor products

\[ V_3 \otimes V_3 \cong V_1 \oplus V_3 \oplus V_5 \]

and

\[ V_2 \otimes V_3 \cong V_2 \oplus V_4, \quad V_3 \otimes V_2 \cong V_2 \oplus V_4. \]

We will need projections to the irreducible subrepresentations. Note that if we want to identify the subrepresentations concretely with the irreducibles described in Section 3.2.2, we have to fix normalization factors. This corresponds to a choice of embedding of the irreducibles to the tensor products as subrepresentations.

For the former tensor product representation, \( V_3 \otimes V_3 \), we denote the projections to the three irreducible subrepresentations by \( \pi^{(d)} : V_3 \otimes V_3 \rightarrow V_d \subset V_3 \otimes V_3 \), where \( d \in \{1, 3, 5\} \). For the latter two, \( V_2 \otimes V_3 \) and \( V_3 \otimes V_2 \), we denote the projections to the two irreducible subrepresentations by \( \pi^{(d)} : V_2 \otimes V_3 \rightarrow V_d \subset V_2 \otimes V_3 \) and \( \pi^{(d)} : V_3 \otimes V_2 \rightarrow V_d \subset V_3 \otimes V_2 \), where \( d \in \{2, 4\} \). Although the same notation is used for these latter two different projections, the meaning should always be clear from the context.

Our embeddings of the irreducibles to the tensor products are the following. It is enough to specify the image of the highest weight vector \( e_0 \) in the tensor product, and our normalization choices are

\[ V_1 \mapsto V_3 \otimes V_3 : \quad e_0 \mapsto \frac{1}{(q^2 - q^{-2})^2} (e_0 \otimes e_2 - e_1 \otimes e_1 + q^{-2} e_2 \otimes e_0) \]

\[ V_3 \mapsto V_3 \otimes V_3 : \quad e_0 \mapsto \frac{1}{q^2 - q^{-2}} (-q^2 e_0 \otimes e_1 + e_1 \otimes e_0) \]

\[ V_5 \mapsto V_3 \otimes V_3 : \quad e_0 \mapsto e_0 \otimes e_0 \]

and

\[ V_2 \mapsto V_2 \otimes V_3 : \quad e_0 \mapsto \frac{q^4}{1 - q^4} e_0 \otimes e_1 - \frac{q}{1 - q^2} e_1 \otimes e_0 \]

\[ V_4 \mapsto V_2 \otimes V_3 : \quad e_0 \mapsto e_0 \otimes e_0 \]

and

\[ V_2 \mapsto V_3 \otimes V_2 : \quad e_0 \mapsto \frac{q^2}{1 - q^2} e_0 \otimes e_1 - \frac{q^2}{1 - q^2} e_1 \otimes e_0 \]

\[ V_4 \mapsto V_3 \otimes V_2 : \quad e_0 \mapsto e_0 \otimes e_0. \]
These choices of normalizing constants strike a compromise between simplicity of formulas for the quantum group representations and for the asymptotics of the corresponding functions treated in Section 3.3.2.

When an identification with a smaller tensor product is implied in a projection to subrepresentation, we indicate this with a hat: we thus define \( \hat{\pi}^{(1)} : V_3 \otimes V_3 \to \mathbb{C} \), \( \hat{\pi}^{(3)} : V_3 \otimes V_3 \to V_3 \), \( \hat{\pi}^{(2)} : V_2 \otimes V_3 \to V_2 \), and \( \hat{\pi}^{(2)} : V_3 \otimes V_2 \to V_2 \) with the identifications of the subrepresentations given above. We finally need to act on two consecutive components of the following big tensor product

\[
V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L.
\]

We define the following projections to a doublet subrepresentation in the tensor product of the doublet tensorand in the middle and a triplet on either side of it, according to the “\( \pm \)”-symbol

\[
\hat{\pi}^{(2)}_+ : V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L \to V_3^\otimes (R-1) \otimes V_2 \otimes V_3^\otimes L
\]

\[
\hat{\pi}^{(2)}_- : V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L \to V_3^\otimes R \otimes V_2 \otimes V_3^{(L-1)}
\]

Likewise, we define the following projections in two consecutive triplet factors (in the \( m \)th and \( (m+1) \)st factors on the left or on the right)

\[
\hat{\pi}^{(3)}_{+; m} : V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L \to V_3^\otimes (R-m-1) \otimes V_2 \otimes V_3^\otimes L
\]

\[
\hat{\pi}^{(3)}_{-; m} = \left( \text{id}_{V_3^m} \right) \otimes \hat{\pi}^{(3)} \otimes \left( \text{id}_{V_3^{L-m}} \right)
\]

\[
\hat{\pi}^{(3)}_{+; m} : V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L \to V_3^\otimes R \otimes V_2 \otimes V_3^{(L-1)}
\]

\[
\hat{\pi}^{(3)}_{-; m} = \left( \text{id}_{V_3^m} \right) \otimes \hat{\pi}^{(3)} \otimes \left( \text{id}_{V_3^{L-m}} \right)
\]

Finally, we also define the following projections in two consecutive triplet factors (in the \( m \)th and \( (m+1) \)st factors on the left or on the right)

\[
\hat{\pi}^{(1)}_{+; m} : V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L \to V_3^\otimes (R-2) \otimes V_2 \otimes V_3^\otimes L
\]

\[
\hat{\pi}^{(1)}_{-; m} = \left( \text{id}_{V_3^m} \right) \otimes \hat{\pi}^{(1)} \otimes \left( \text{id}_{V_3^{L-m}} \right)
\]

\[
\hat{\pi}^{(1)}_{+; m} : V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L \to V_3^\otimes R \otimes V_2 \otimes V_3^{(L-2)}
\]

\[
\hat{\pi}^{(1)}_{-; m} = \left( \text{id}_{V_3^m} \right) \otimes \hat{\pi}^{(1)} \otimes \left( \text{id}_{V_3^{L-m}} \right)
\]

The meaning of projections \( \pi^{(2)}_\pm, \pi^{(4)}_\pm, \pi^{(1)}_{\pm; m}, \pi^{(3)}_{\pm; m}, \pi^{(5)}_{\pm; m} \) acting on \( V_3^\otimes R \otimes V_2 \otimes V_3^\otimes L \) should now be self-evident, as there is no need for identifications.

### C.2 The quantum group solutions for some 4-point visits

For brevity, we factor out the constant

\[
C_4 = \frac{q^7 \left( q^4 + q^2 + 1 \right)^3}{(q^2 - 1)^4 (q^2 + 1)^5 (q^4 + 2q^2 + 1)^4 (q^4 + q^2 + 1)}.
\]

Then, with a shorthand notation similar to that in Sections 3.5.2 and 3.5.3, the normalized solutions are
$v^{(4)}_{\pm-+} = C_4 \times \left( q^2 + 1 \right) \left( q^4 + 1 \right) q^6 e_{00112} - q^4 e_{00202} - \left( q^2 + 1 \right) q^5 e_{00211} + q^4 e_{22000}$

$+ \left( q^2 + 1 \right) \left( q^4 + 1 \right) q^8 e_{01012} - \left( q^2 + 1 \right) q^6 e_{01102} - \left( q^2 + 1 \right)^2 q^7 e_{01111}$

$+ \left( q^5 + q^3 \right) e_{01211} \left( q^6 + q^4 \right) e_{01120} - q^8 e_{02002} - \left( q^2 + 1 \right) q^6 e_{02011} + q e_{20200}$

$+ \left( q^7 + q^5 \right) e_{02101} \left( q^8 + q^6 \right) e_{02110} + \left( -q^4 - 1 \right) e_{02200} + \left( q^4 + 1 \right) q^5 e_{12001}$

$- \left( q^2 + 1 \right) \left( q^5 + q^2 \right)^2 e_{10012} + \left( -q^8 + q^6 + q^2 \right) e_{10102} + \left( q^4 - 1 \right) q^6 e_{12100}$

$+ \left( -q^4 + q^7 + q^5 + q^3 \right) e_{10111} + \left( q^4 - 1 \right) q e_{10201} + \left( q^4 - 1 \right)^2 q e_{10210}$

$+ \left( -q^{10} + q^8 + q^4 \right) e_{11002} + \left( -q^{13} + q^9 + q^7 + q^5 \right) e_{11011} + \left( q^8 + q^4 \right) e_{20002}$

$+ \left( q^2 - 1 \right) \left( q^2 + 1 \right)^2 q e_{11101} + \left( q^2 - 1 \right) \left( q^2 + 1 \right)^2 q e_{11110} + \left( -q^6 - q^2 + 1 \right) e_{11200}$

$- \left( q^6 + q^2 - 1 \right) q^2 e_{12100} + \left( q^2 + 1 \right) \left( q^4 + 1 \right) q^5 e_{20011} - \left( q^2 + 1 \right) q^3 e_{20101}$

$- \left( q^2 + 1 \right) q^4 e_{20110} - \left( q^2 + 1 \right) q^5 e_{21001} - \left( q^2 + 1 \right) q^6 e_{21010} + \left( q^4 + q^2 \right) e_{21100}$

$\left( \frac{q^4 + 1}{q^2 + 1} \right) \left( q^4 + q^2 + 1 \right) q^5 e_{00022}$

$\left( \frac{q^4 + 1}{q^2 + 1} \right) \left( q^4 + q^2 + 1 \right) q^6 e_{00112} + \left( q^4 - q^9 \right) e_{21200}$

$- \left( q^4 + 1 \right) \left( q^4 + q^2 + 1 \right) q^2 e_{00202} - \left( q^4 + q^2 + 1 \right) q^4 e_{01012} + \left( q^2 - q^8 \right) e_{01021}$

$- \left( q^4 + q^2 + 1 \right) q^5 e_{01102} - \left( -q^4 + q^7 + q^5 + q^3 \right) e_{01111} + \left( q^7 + q^5 + q^3 \right) e_{01120}$

$+ \left( q^3 + \frac{q}{q^2 + 1} \right) e_{02002} + \left( q^4 + q^2 - 1 \right) q e_{02011} + \left( q^2 + \frac{1}{q^2 + 1} - 2 \right) q e_{02020}$

$+ \left( q^4 + q^2 - 1 \right) q^2 e_{02101} + \left( q^4 - q^2 - 1 \right) q^2 e_{02110} + \left( q^4 + q^2 + 1 \right) q^6 e_{10012}$

$+ \left( q^4 - q^{10} \right) e_{10021} - \left( q^4 + q^2 + 1 \right) q^7 e_{10102} + \left( q^2 - q^8 \right) e_{21010} + \left( q^6 + 2q^4 - 1 \right) q^3 e_{11011}$

$+ \left( q^9 + q^7 + q^5 \right) e_{10120} + \left( q^7 + q^5 + q^3 \right) e_{11020} - \left( q^8 + q^6 - q^2 - 1 \right) q^5 e_{10111}$

$+ \left( q^4 - q^2 - 1 \right) q^3 e_{11020} + \left( q^6 + 2q^4 - 1 \right) q^4 e_{11101} + \left( q^6 - 2q^2 - 1 \right) q^4 e_{11110}$

$- \left( q^4 + q^2 + 1 \right) q^2 e_{12001} + \left( -q^6 \right) e_{12010} + \left( q - q^7 \right) e_{12100} + \left( \frac{q^9 + q^7 + q^5}{q^2 + 1} \right) e_{20002}$

$- \left( -q^9 + q^7 + q^5 \right) e_{20020} - \left( q^4 + q^2 + 1 \right) q^4 e_{21001} + \left( q^4 + 1 \right) \left( q^4 + q^2 + 1 \right) q e_{22000}$

$+ \left( q^4 + q^2 - 1 \right) q^5 e_{20011} + \left( q^4 - q^2 - 1 \right) q^6 e_{20101} + \left( q^4 + q^2 - 1 \right) q^6 e_{20101}$
\[ v^{(4)}_{-1-1} = C_4 \times \left( q^2 + 1 \right) \left( q^4 + q^2 + 1 \right) q^6 e_{00112} + \left( q^4 + 1 \right) \left( q^4 + q^2 + 1 \right) q^2 e_{20002} \]

\[ + \left( q^8 + q^6 - q^2 - 1 \right) q^4 e_{01012} + \left( q^8 + q^6 - q^2 - 1 \right) q^6 e_{01102} + \left( q^8 + q^4 + q^2 \right) e_{22000} \]

\[ - \left( q^8 + q^6 - q^2 - 1 \right) q^4 e_{01201} + \left( q^8 + q^4 + q^2 \right) e_{02010} + \left( q^8 + q^6 - q^2 - 1 \right) q^2 e_{01210} - \left( q^8 + q^4 + q^2 \right) e_{02002} \]

\[ + \left( q^8 + q^6 - q^2 - 1 \right) q^4 e_{02001} + \left( q^8 + q^6 - q^2 - 1 \right) q^2 e_{10012} + \left( q^8 + q^6 - q^2 - 1 \right) q^4 e_{10010} \]

\[ + \left( q^8 + q^6 - q^2 - 1 \right) q^2 e_{10002} + \left( q^8 + q^6 - q^2 - 1 \right) q^4 e_{10010} + \left( q^8 + q^6 - q^2 - 1 \right) e_{10210} \]

\[ - \left( q^2 + 1 \right)^2 \left( q^4 - q^2 - 1 \right) q^3 e_{10111} - \left( q^2 + 1 \right)^2 \left( q^4 + q^2 - 1 \right) q^3 e_{01111} \]

\[ - \left( q^2 + 1 \right) \left( q^8 - 2q^6 - q^4 + 1 \right) q^3 e_{11101} + \left( q^2 + 1 \right) \left( q^8 - 2q^6 - q^4 + 1 \right) e_{11110} \]

D Numerical evaluation of the integrals

Let us then describe how the integral expressions can be evaluated numerically in practice. We have implemented two methods with symbolic computation software:

1. Direct evaluation of the complex loop integrals \( \varphi_{-1-1, \ldots, -1, \ldots, 1, \ldots, 1} \).

2. Evaluation of the (real) integrals \( \rho_{-1-1, \ldots, -1, \ldots, 1, \ldots, 1} \) by using the \( \varepsilon \)-regularization scheme described above.

Both of these approaches have advantages and disadvantages. The loop integrals are well defined as such for all values of \( \kappa \), but involve complex integrands and complicated numerical contours which slow down the integration. Real \( \varepsilon \)-regularized integrals are faster to evaluate, but one needs to add counterterms which also involve integrals, thus increasing the total number of integrations. In addition, the remaining \( \varepsilon \)-dependence of the result needs to be controlled.

In both methods, low values of \( \kappa \) are the most challenging. In the loop integrals, the variations in the absolute value of the integrand increase with decreasing \( \kappa \), leading to more and more precise cancellations between contributions from different sections of the integrations contours. In order to make the \( \varepsilon \)-regularization work, a larger number of counterterms is necessary at small \( \kappa \) than at values of \( \kappa \) close to \( \kappa = 8 \), which practically limits this method to \( \kappa \geq 4 \). As it turns out, probability amplitudes with \( N = 3 \) boundary visits are still relatively fast to evaluate, in particular when \( \kappa \) is close to eight, whereas it is already computationally demanding to evaluate the \( N = 4 \) amplitudes. For \( N = 3 \) the calculation of the loop integrals is the faster method. We have controlled the numerical errors by comparing the results obtained by the two methods for the final result of the probability amplitude.

D.1 Evaluation of the loop integrals

In order to evaluate the loop integrals, we first need to specify the integration contours. We choose the anchor point in the lower half plane. Each contour is chosen to be a combination of two straight lines and
an arc of a circle, with the center of the circle located at the encircled charge, and the lines being tangential to the circle (see Fig. D.1). The radii of the circles are chosen such that the minimum distance between any pair of charges is (approximately) maximized. The contours $w_k = c_k(s_k)$ are parametrized in terms of the real variables $s_k \in [0, 1]$, such that $w_k$ moves around the charge in the counterclockwise direction with increasing $s_k$. The parametrization can be chosen such that $c_k'(s_k)$ is continuous at the points where the arc joins with the lines.

The most tricky step is to write the multi-branched integrand in terms of the principal branches of the power functions such that it is an analytic function on the integration contours, and the phase convention of Fig. 3.1 is realized. By the principal branch we mean that

$$x^y = \exp(y \log(x)),$$

where the principal branch of the logarithm satisfies $\pi < 3\operatorname{Im} \log(x) \leq \pi$ for all complex $x \neq 0$. Let us denote by $\hat{s}_k$ the value of $s_k$ where $\operatorname{Im} w_k$ takes its largest value. It is then easy to check that the various terms of the integrand can be defined as follows.

- If the contours with indices $k_1$ and $k_2$ encircle two different charges $y_{j_1}$ and $y_{j_2}$, with $y_{j_1} < y_{j_2}$, we take

  $$\left(w_{k_2} - w_{k_1}\right)^{8/\kappa} = \exp\left(\frac{8}{\kappa} \log\left(w_{k_2} - w_{k_1}\right)\right).$$

  Similar definition holds when either of the contours is around $x$.

- If the contours with indices $k_1$ and $k_2$ encircle the same charge, with $c_{k_1}$ being the innermost contour, we take

  $$\left(w_{k_2} - w_{k_1}\right)^{8/\kappa} = \exp\left(\frac{8\pi}{\kappa} i + \frac{8}{\kappa} \log\left(w_{k_1} - w_{k_2}\right)\right) \quad \text{if } 0 \leq s_{k_2} \leq \hat{s}_{k_2}$$

  $$\left(w_{k_2} - w_{k_1}\right)^{8/\kappa} = \exp\left(-\frac{8}{\kappa} \log\left(w_{k_1} - w_{k_2}\right)\right) \quad \text{if } \hat{s}_{k_2} < s_{k_2} \leq 1.$$

- If the contour $c_k$ encircles $y_{j_1}$, we take for each $y_{j_2} \neq y_{j_1}$

  $$\left(w_k - y_{j_2}\right)^{-8/\kappa} = \exp\left(-\frac{8}{\kappa} \log\left(w_k - y_{j_2}\right)\right) \quad \text{if } y_{j_1} > y_{j_2}$$

  $$\left(y_{j_2} - w_k\right)^{-8/\kappa} = \exp\left(-\frac{8}{\kappa} \log\left(y_{j_2} - w_k\right)\right) \quad \text{if } y_{j_1} < y_{j_2},$$
and for the contribution from the charge \(y_j\), we use
\[
(w_k - y_j)^{-8/\kappa} = \exp \left( -\frac{8}{\kappa} \log(w_k - y_j) \right)
\]
if \(0 \leq s_k \leq \hat{s}_k\)
\[
(w_k - y_j)^{-8/\kappa} = \exp \left( -\frac{8\pi}{\kappa} - \frac{8}{\kappa} \log(y_j - w_k) \right)
\]
if \(\hat{s}_k < s_k \leq 1\).

The terms involving \(w_k\) and \(x\) are treated analogously.

The numerical integration can then be done after changing the integration variables to \(s_k\). It turns out that the integration on our symbolic computation software is often faster, if each of the contours is explicitly divided into the three pieces containing the two lines and the arc, and the contributions are integrated separately. It turns out that the probability amplitudes \(\zeta^{(N)}\) often have zeroes of poles at the rational values of \(\kappa\) of interest to us, but then one may just straightforwardly modify the normalizing constants. For example at \(\kappa = 6\), \(N = 3\) we can add a normalization factor \(\propto 1/(\kappa - 6)\) and study \(\zeta^{(3)}/(\kappa - 6)\) in the limit \(\kappa \to 6\). We have evaluate the amplitude for values of \(\kappa\) near the critical one, say, at \(\kappa = 6.05\) and \(\kappa = 5.95\), and estimated the amplitude at \(\kappa = 6\) as the average of the results. More elaborate fitting, as a function of \(\kappa\), can also be done.

D.2 Evaluation of the \(\varepsilon\)-regularized integrals

The most involved step in the evaluation of the \(\varepsilon\)-regularized integrals is the identification of the counterterms. In the previous section we already discussed how this can be done, and considered explicitly a simple example. Computation of the terms at higher \(N\) and to higher order in \(\varepsilon\) is in principle straightforward, but the complexity of the expressions grows relatively fast. We have written a code on symbolic computation software which automatically finds the counterterms for a given integral. All leading order terms in the expansion of the divergent terms at \(\varepsilon = 0\) [i.e., the \(k = 0\) terms \(\mathcal{O}(\varepsilon^{-n(8/\kappa-1)})\) in (4.2), with \(n = 1, 2, \ldots, N\)] and at least the leading divergence from the next-to-leading order term of the series [i.e., the terms \(\mathcal{O}(\varepsilon^{-(N-1)(8/\kappa-1)}\varepsilon)\)] are generated. Including these terms, the method converges for \(N = 2\) integrals when \(\kappa > 4\), and for \(N = 3\) integrals when \(\kappa > 16/3\). In practice the limits can be somewhat higher due to limited numerical precision.

After the counterterms have been identified, it is straightforward to evaluate the sum of the regularized integral and all counterterms for any fixed value of the cutoff \(\varepsilon\). Notice also that since we are not able to subtract counterterms to all orders, some dependence on \(\varepsilon\) remains, and we need to extrapolate the result down to \(\varepsilon = 0\). It is useful to calculate the amplitude at various values of \(\varepsilon\), and fit the remaining \(\varepsilon\)-dependence by using the highest order term which was not subtracted. Moreover, a similar interpolation as a function of \(\kappa\), as was described above for the loop integrals, is usually also required.

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