New Normality on Generalized Topological Spaces

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Abstract. A space $X$ is named a $\pi p$-normal if for each closed set $F$ and each $\pi$-closed set $F'$ in $X$ with $F \cap F' = \emptyset$, there are $p$-open sets $U$ and $V$ of $X$ with $U \cap V = \emptyset$ whereas $F \subseteq U$ and $F' \subseteq V$. Our work studies and discusses a new kind of normality in generalized topological spaces. We define $\vartheta \pi p$-normal, $\vartheta$-mildly normal, $\vartheta$-almost normal, $\vartheta p$-normal, $\vartheta$-mildly $p$-normal, $\vartheta$-almost $p$-normal and $\vartheta \pi$-normal space, and we discuss some of their properties.

1. Introduction and Preliminaries
In our paper a topological space will be denoted by $X$ and the closure and the interior of $F$ in $X$ by $\text{cl}(F)$ and $\text{int}(F)$ respectively. The concept of $\pi p$-normal topological Spaces was introduced and considered by Sadeq (2012) [1].

We recall some concepts and notations defined previously by another authors. A subset $F$ of a space $X$ is said to be pre-open (briefly $p$-open) (resp. semi-open, $\alpha$-open) if $F \subseteq \text{int}(\text{cl}(F))$ (resp. $F \subseteq \text{cl}(\text{int}(F))$, $F \subseteq \text{int}(\text{cl}(\text{int}(F)))$) [2]. The complement of a $p$-open (resp. semi-open , $\alpha$-open) set is called $p$-closed [3] (resp. semi-closed[2], $\alpha$-closed[4]). The following two definitions are taken from[2], preclosure of $F$ is $\text{pcl}(F) = \bigcap\{F' : F'$ is a preclosed set containing $F\}$, preinterior of $F$ is $\text{pint}(F) = \bigcup\{F' : F'$ is preopen set contained in $F\}$. According to [5], a subset $F$ of a topological space $X$ is called regular open or an open domain if $F = \text{int}(\text{cl}(F))$, or equivalently if it is the interior of some closed set. A set $F$ is said to be a regular closed or a closed domain if its complement is an open domain, or equivalently, if $F = \text{cl}(\text{int}(F))$. The finite union of open domain sets in $X$ is said to be $\pi$-open. The complement of a $\pi$-open set is $\pi$-closed, or equivalently, a $\pi$-closed set is a finite intersection of closed domain subsets, [6]. The following diagram shows the relationship among closed sets:

\[
\text{closed domain} \Rightarrow \pi \text{-closed} \Rightarrow \text{closed} \Rightarrow \alpha \text{-closed} \Rightarrow p \text{-closed}
\]

On the other hand, the notions of normal, almost normal and mildly normal spaces were introduced by sharma[7], Singal[8] and Singal[9] respectively. A space $X$ is called normal if every pair of disjoint closed sets are contained in disjoint open sets and it is called almost normal if every pair of disjoint closed sets one of which is closed domain are contained in disjoint open sets and it is said to be mildly normal if every pair of disjoint closed domain sets are contained in disjoint open sets.
In 1995 Paul and Bhattacharyya[10] introduced the concept of $p$–normal space, a topological space $X$ is said to be $p$–normal if any disjoint closed sets $F$ and $F'$ of $X$ are contained in two disjoint $p$–open sets $U$ and $V$ of $X$. While in 1991 Navalagi [2] introduced the concepts of mildly $p$-normal and almost $p$-normal, a space $X$ is said to be an almost $p$–normal if any disjoint closed sets $F$ and $F'$ of $X$, one of which is closed domain, are contained in two disjoint $p$–open sets $U$ and $V$ of $X$, and it is said to be a mildly $p$–normal if any disjoint closed domain sets $F$ and $F'$ of $X$ are contained in two disjoint $p$–open sets $U$ and $V$ of $X$. Kalantan[11], introduced the definitions of $\pi$–normal and quasi-normal spaces, a space $X$ is called a $\pi$–normal if any two disjoint closed subsets $F$ and $F'$ of $X$, one of which is $\pi$–closed are contained in disjoint open sets, and it is called quasi-normal if any two disjoint $\pi$–closed subsets $F$ and $F'$ of $X$ there exist two open disjoint subsets $U$ and $V$ of $X$ such that $F \subseteq U$ and $F' \subseteq V$.

In 2012 the notion of $\pi p$-normal space was introduced by Sadeq [1]. A space $X$ is said to be $\pi p$–normal if for every pair of disjoint closed sets $F$ and $F'$ of $X$, one of which is $\pi$–closed, are contained in two disjoint $p$–open sets $U$ and $V$ of $X$. The following diagrams shows the relationship among normal spaces:

\[
\text{normal } \Rightarrow \pi \text{– normal } \Rightarrow \text{almost normal } \Rightarrow \text{mildly normal}
\]

\[
\text{normal } \Rightarrow \pi \text{– normal } \Rightarrow \text{quasi– normal } \Rightarrow \text{mildly normal}
\]

\[
\text{normal } \Rightarrow p \text{– normal } \Rightarrow \pi p \text{– normal } \Rightarrow \text{almost } p \text{– normal}
\]

\[
\text{normal } \Rightarrow \pi \text{– normal } \Rightarrow \pi p \text{– normal}
\]

Recently many topological concepts have been modified to give new concepts in the structure of generalized topological spaces. In this section we review some definitions in generalized topological spaces and we review some modified open sets and modified normal spaces, we begin with the definition of the generalized topological space.

Definition 1.1[12]
Let $X$ be a nonempty set. A collection $\mathcal{G}$ of subsets of $X$ is called a generalized topology (in brief, $GT$) on $X$ if $\emptyset$ belongs to $\mathcal{G}$ and the arbitrary unions of elements of $\mathcal{G}$ is an element in $\mathcal{G}$. $(X, \mathcal{G})$ is called generalized topological space (in brief, $GT_S$).

Every set in $\mathcal{G}$ is called $\mathcal{G}$-open while the complement of $\mathcal{G}$-open is called $\mathcal{G}$-closed. The largest $\mathcal{G}$-open set contained in a set $F$ is called the interior of $F$ and is denoted by $i_\mathcal{G}(F)$ whereas the smallest $\mathcal{G}$-closed set containing $F$ is called the closure of $F$ and is denoted by $c_\mathcal{G}(F)$. A generalized topology $\mathcal{G}$ is said to be strong [13] if $X \in \mathcal{G}$, accordingly, $(X, \mathcal{G})$ is called strong generalized topological space (in brief, $SGT_S$).

Definition 1.2[14]
A subset $A$ of a $GT_S(X, \mathcal{G})$ is called

(i) $\mathcal{G}$-semi-open if $F \subseteq c_\mathcal{G}(i_\mathcal{G}(F))$, and its complement is $\mathcal{G}$-semi-closed.

(ii) $\mathcal{G}$-$\alpha$-open if $F \subseteq i_\mathcal{G}(c_\mathcal{G}(F))$, and its complement is $\mathcal{G}$-$\alpha$-closed.

(iii) $\mathcal{G}$-$\alpha$-open if $F \subseteq i_\mathcal{G}(c_\mathcal{G}(i_\mathcal{G}(F)))$, and its complement is $\mathcal{G}$-$\alpha$-closed.

The intersection of all $\mathcal{G}$-$\alpha$-closed sets containing $F$ is called $\mathcal{G}$-$\alpha$-closure of $F$ and denoted by $pc_\mathcal{G}(F)$. The $\mathcal{G}$-$\alpha$-interior of $F$, denoted by $pi_\mathcal{G}(F)$, is defined to be the union of all $\mathcal{G}$-$\alpha$-open sets contained in $F$.

Definition 1.3[15]
Let $A$ be a subset of a $GT_S(X, \mathcal{G})$. Then $A$ is said to be $\mathcal{G}$-regular open or $\mathcal{G}$-open domain if $F = i_\mathcal{G}(c_\mathcal{G}(F))$. The complement of a $\mathcal{G}$-open domain is called $\mathcal{G}$-closed domain.
Sarsak [16], define \( \vartheta \)-closed domain as a subset \( A \) of \( GTS(X, \vartheta) \) that fulfills the condition \( F = c_\vartheta(I_\vartheta(F)) \).

Definition 1.4[17] A subset \( F \) of \( GTS(X, \vartheta) \) is said to be \( \vartheta \)-\( \pi \)-open if \( F \) is the union of finitely many \( \vartheta \)-open domain sets, and it is called \( \vartheta \)-\( \pi \)-closed if its complement is a \( \vartheta \)-\( \pi \)-open, or equivalently if \( F \) is the intersection of finitely many \( \vartheta \)-closed domain sets.

The relationship among all above sets can be illustrated by the following schemes:

\[
\begin{align*}
\vartheta \text{-closed domain} & \Rightarrow \vartheta \text{-}\pi \text{-closed} \Rightarrow \vartheta \text{-closed} \Rightarrow \vartheta \text{-}\alpha \text{-closed} \Rightarrow \vartheta \text{-} p \text{-closed} \\
\vartheta \text{-open} & \Rightarrow \vartheta \text{-}\alpha \text{-open} \Rightarrow \vartheta \text{-semi-open} \\
\vartheta \text{-closed} & \Rightarrow \vartheta \text{-}\alpha \text{-closed} \Rightarrow \vartheta \text{-semi-closed} \\
\vartheta \text{-open domain} & \Leftrightarrow \vartheta \text{-open and} \vartheta \text{-semi-closed} \Leftrightarrow \vartheta \text{-}\alpha \text{-open and} \vartheta \text{-semi-closed} \\
\vartheta \text{-closed domain} & \Leftrightarrow \vartheta \text{-closed and} \vartheta \text{-semi-open} \Leftrightarrow \vartheta \text{-}\alpha \text{-closed and} \vartheta \text{-semi-open} \\
\vartheta \text{-p-open} & \text{ and \vartheta \text{-p-closed}}
\end{align*}
\]

Properties of normal GT's were discussed in ( [18], [19], [20], [21], [22], [23] ). A GT\( \vartheta \) is normal iff, whenever \( F \) and \( F' \) are \( \vartheta \)-closed sets such that \( F \cap F' = \emptyset \), there exists \( \vartheta \)-open sets \( U \) and \( V \) satisfying \( F \subseteq U, F' \subseteq V \) and \( U \cap V = \emptyset \).

In 2012 Bishwambhar[24], defined almost normal and mildly normal spaces, in generalized topological spaces, by modifying the condition related to the open sets in the original definition. As we show below.

Definition 1.5[24]
Let \( \vartheta \) be a GT on a topological space \( X \). Then

1. \( X \) is named almost \( \vartheta \)-normal if \( F \) is a closed set and \( B \) is a closed domain set in \( X \) with \( F \cap F' = \emptyset \), , then there are two disjoint \( \vartheta \)-open sets \( U \) and \( V \) such that \( F \subseteq U \) and \( F' \subseteq V \).

2. \( X \) is named mildly \( \vartheta \)-normal if \( A \) and \( B \) are closed domain sets in \( X \) with \( F \cap F' = \emptyset \), , then there are two disjoint \( \vartheta \)-open sets \( U \) and \( V \) such that \( F \subseteq U \) and \( F' \subseteq V \).

We will denote the class of \( \vartheta \)-closed (resp. \( \vartheta \)-\( p \)-open, \( \vartheta \)-\( p \)-closed, \( \vartheta \)-\( \pi \)-open, \( \vartheta \)-\( \pi \)-closed, \( \vartheta \)-semi-open, \( \vartheta \)-semi-closed, \( \vartheta \)-\( \alpha \)-open, \( \vartheta \)-\( \alpha \)-closed, \( \vartheta \)-open domain, \( \vartheta \)-closed domain) sets in \( X \) by \( \vartheta' \) (resp. \( \vartheta PO(X) \), \( \vartheta PC(X) \), \( \vartheta \pi O(X) \), \( \vartheta \pi C(X) \), \( \vartheta SO(X) \), \( \vartheta SC(X) \), \( \vartheta \alpha O(X) \), \( \vartheta \alpha C(X) \), \( \vartheta RO(X) \), \( \vartheta RC(X) \)).

2. \( \vartheta \pi p \)–Normal Generalized Topological Spaces
In this section, another definitions about almost normal and mildly normal spaces, in generalized topological spaces, will be given by modifying the conditions related to open and closed sets in the original definition, called \( \vartheta \)-almost normal and \( \vartheta \)-mildly normal. By the same way, \( p \)-normal,
almost p−normal, mildly p−normal, π−normal, quasi normal and πp−normal spaces, in generalized topological spaces will be defined, called θ−p−normal, θ−almost p−normal, θ−mildly p−normal, θπ−normal, θ−quasi normal and θπp−normal, and the relationship among them will be discussed, as we show in the following section.

Definition 2.1
A GTS(X, θ) is called
1. θ−almost normal if ∀F ∈ θ′ and ∀F′ ∈ θRC(X) with F ∩ F′ = ∅, ∃U, V ∈ θ ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
2. θ−mildly normal if for any two sets F, F′ ∈ θRC(X) with F ∩ F′ = ∅ there are two sets U, V ∈ θ ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
3. θπ−normal if ∀F ∈ θ′ and ∀F′ ∈ θπC(X) with F ∩ F′ = ∅, ∃U, V ∈ θ ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
4. θ−quasi normal if for any two sets F, F′ ∈ θπC(X) with F ∩ F′ = ∅ there are two sets U, V ∈ θ ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
5. θ−p−normal if for any two sets F, F′ ∈ θ′ with F ∩ F′ = ∅ there are two sets U, V ∈ θPO(X) ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
6. θ′−almost p−normal if ∀F ∈ θ′ and ∀F′ ∈ θPO(X) ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
7. θ′−mildly p−normal if for any two sets F, F′ ∈ θRC(X) with F ∩ F′ = ∅ there are two sets U, V ∈ θPO(X) ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
8. θπp−normal if ∀F ∈ θ′ and ∀F′ ∈ θπC(X) with F ∩ F′ = ∅, ∃U, V ∈ θPO(X) ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.
9. θ−quasi p−normal if for any two sets F, F′ ∈ θπC(X) with F ∩ F′ = ∅ there are two sets U, V ∈ θPO(X) ∋ F ⊆ U and F′ ⊆ V and U ∩ V = ∅.

Remark 2.2
The following trends are true for the generalized topological space (X, θ).

θ−normal ⇒ θπ−normal ⇒ θ−almost normal ⇒ θ−mildly normal

These directions are not reversible, as shown in the following example

Example 2.3
Let ℝ be the set of all real numbers, and let θ = {ℝ, ∅, {0}} ∪ {ℝ \ {x}: x ≠ 0}, then θ′ = {ℝ, ∅, ℝ \ {0}} ∪ {{x}: x ≠ 0}. θRC(X) = {ℝ, ∅}, and θπC(X) = {ℝ, ∅}.

Note that (ℝ, θ) is not θ−normal because for any two nonempty disjoint θ−closed sets in ℝ, there are no two disjoint θ−open sets containing them. While it is θπ−normal, θ−almost normal and θ−mildly normal.

Remark 2.4
The following outline is correct for the generalized topological space (X, θ).

θ−p−normal ⇒ θπp−normal ⇒ θ−almost p−normal ⇒ θ−mildly p−normal

The inverse of the above directions is generally not correct as shown by the following examples.

Example 2.5
Let X = {a, b, c}, θ = {X, ∅, {a}, {a, b}, {a, c}}, θ′ = {X, ∅, {b, c}, {c}, {b}}, θPO(X) = {X, ∅, {a}, {a, b}, {a, c}}, θRC(X) = {X, ∅}, θπC(X) = {X, ∅}.

Note that (X, θ) is not θ−p−normal because {c} and {b} are two disjoint θ−closed sets in X, but there are no two disjoint θ−p−open containing them. On the other hand, we see that (X, θ) is θπp−normal, θ−almost p−normal and θ−mildly p−normal.
Example 2.6
Take the set of all real numbers $\mathbb{R}$ with $\theta = \{ U \subseteq X: \forall x \in U \ \exists N \in \eta_x \ \exists N \subseteq U \}$, where $\{\eta_x\}_{x \in \mathbb{R}}$ is defined as follows:

For each $x \in \mathbb{Q}'$ (the set of all irrational numbers) put a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Q}$ (the set of all rational numbers) such that $x_n \to x$ (where the convergence is taken in $\mathbb{R}$ with the usual topology), let $A_n(x) = \{x_k: k \geq n\}$, and let $U_n(x) = \{x\} \cup A_n(x)$. For $x \in \mathbb{Q}'$, $\eta_x = \{U_n(x): n \in \mathbb{N}\}$ and for $x \in \mathbb{Q}$, $\eta_x = \{\{x\}\}$. Then $(\mathbb{R}, \theta)$ is $\theta$ - almost normal and $\theta$ - almost $p$ - normal, but it is neither $\theta \pi$ - normal nor $\theta \pi p$ - normal.

Remark 2.7
Diagram below is true for the generalized topological space $(X, \theta)$.

$\theta$ - normal $\Rightarrow$ $\theta \pi$ - normal $\Rightarrow$ $\theta$ - quasi normal

These implications are not reversible as shown by the following example.

Example 2.8
Let $P = \{(x, y): x, y \in \mathbb{R}, y > 0\}$ and $L = \{(x, 0): x \in \mathbb{R}\}$ be a subspace of $\mathbb{R}^2$ with the usual topology, and let $X = P \cup L$.

For $z \in P$, $\eta_z$ = any basic open disc around $z$ contained in $P$ with its usual Euclidean topology.

For $z \in L$, $\eta_z$ = $\{z\} \cup (P \cap D)$, where $D$ is any open disc around $z$ in the plane $\mathbb{R}^2$ with its usual Euclidean topology.

Let $\theta = \{ U \subseteq X: \forall z \in U \ \exists N \in \eta_x \ \exists N \subseteq U \}$, then $(X, \theta)$ is neither $\theta$ - normal nor $\theta \pi$ - normal, while it is a $\theta$ - quasi normal.

Remark 2.9
A condition "every $\theta$ - closed must be $\theta$ - semi - open" can be added in Remarks 2.2, 2.4 and 2.7 to achieve opposite directions.

Proposition 2.10
Every $\theta \pi$ - normal space is a $\theta \pi p$ - normal space.

Proof:
Let $(X, \theta)$ be $\theta \pi$ - normal space, we have to show that $X$ is $\theta \pi p$ - normal space, let $F$ be $\theta$ - closed set and $F'$ be $\theta$ - $\pi$ - closed set of $X$ with $F \cap F' = \emptyset$, by definition of $\theta \pi$ - normal we get two disjoint $\theta$ - open sets $U$ and $V$ such that $F \subseteq U$ and $F' \subseteq V$. Since every $\theta$ - open set is $\theta$ - $p$ - open, so we got what we needed.

Remark 2.11
The opposite direction of Proposition 2.10 is generally incorrect. Note the generalized topological space in Example 2.8 is a $\theta \pi p$ - normal space, but not $\theta \pi$ - normal.

Theorem 2.12
For a generalized topological space $(X, \theta)$, the following are equivalent:

a) $(X, \theta)$ is $\theta \pi p$ - normal.

b) $\forall U, V \in \theta$, one of which is $\theta$ - $\pi$ - open, $U \cup V = X$, there are $\theta$ - $p$ - closed sets $F$ and $F'$ so that $F \subseteq U, F' \subseteq V$ and $F \cup F' = X$.

c) $\forall F \in \theta'$ and $\forall U \in \partial \theta O(X)$ with $F \subseteq U$, there is a $\theta$ - $p$ - open set $V$ so that $F \subseteq V \subseteq p_c \theta (V) \subseteq U$.

Proof:
If (a) Then (b)
Let \( U \) and \( V \) be \( \theta \)-open sets in \( X \), such that \( V \) is \( \theta - \pi \)-open and \( U \cup V = X \), consequently we have \( X \setminus U \) and \( X \setminus V \) are \( \theta \)-closed sets in a \( \theta \)-normal GTS \( (X, \theta) \) with \( X \setminus V \theta - \pi \)-closed and \( X \setminus U \cap X \setminus V = \emptyset \), so by Definition 2.1, no. 8 there exist two disjoint \( \theta - p \)-open sets \( G \) and \( H \) such that \( X \setminus U \subseteq G \) and \( X \setminus V \subseteq H \). Let \( F = X \setminus G \) and \( F' = X \setminus H \), then \( F \) and \( F' \) are \( \theta - p \)-closed sets such that \( F \subseteq U, F' \subseteq V \) and \( F \cup F' = X \).

If (b) Then (c)
Let \( F \) be a \( \theta \)-closed set and \( U \) be a \( \theta - \pi \)-open set so that \( F \subseteq U \), thus \( F \cap X \setminus U = \emptyset \), which leads to \( X \setminus F \cup U = X \), where \( X \setminus F \) is \( \theta \)-open. By (b) we can get \( \theta - p \)-closed sets \( F_1 \) and \( F_2 \) such that \( F_1 \subseteq X \setminus F, F_2 \subseteq U \) and \( F_1 \cup F_2 = X \). We have got \( F \subseteq X \setminus F_1 \) and \( X \setminus F_1 \cap X \setminus F_2 = \emptyset \) which results in \( X \setminus F_1 \subseteq F_2 \). Put \( V = X \setminus F_1 \), then \( F \subseteq V \subseteq F_2 \subseteq U \). We know that \( V \) is \( \theta \)-open and \( F \) is \( \theta - p \)-open set containing \( V \), but \( p\theta c_\theta (V) \) is the smallest \( \theta - p \)-closed set containing \( V \), therefore \( p\theta c_\theta (V) \subseteq F_2 \subseteq U \). Hence \( F \subseteq V \subseteq p\theta c_\theta (V) \subseteq U \).

If (c) Then (a)
Let \( F \) and \( F' \) be any \( \theta \)-closed sets of \( X \) with \( F' \) is \( \theta - \pi \)-open and \( F \cap F' = \emptyset \), so \( F \subseteq X \setminus F' \) which is \( \theta - \pi \)-open, then by (c) there is a \( \theta - p \)-open set \( V \) so that \( F \subseteq V \subseteq p\theta c_\theta (V) \subseteq X \setminus F' \). Let \( G = V \) and \( H = X \setminus p\theta c_\theta (V) \), then \( G, H \in \theta \partial \Omega (X) \) with \( G \cap H = \emptyset \) and \( F \subseteq G, F' \subseteq H \). Hence \( (X, \theta) \) is \( \theta \)-normal.

3. Preservation Theorems of \( \theta \)-Normality
Definition 3.1 [25]
A function \( f : (X, \theta_1) \rightarrow (Y, \theta_2) \) from a GTS \( (X, \theta_1) \) into a GTS \( (Y, \theta_2) \) is called

1. \( (\theta_1, \theta_2) \)-continuous if \( f^{-1}(H) \in \theta_1 \forall H \in \theta_2 \).
2. \( (\theta_1, \theta_2) \)-open if \( f(G) \in \theta_2 \forall G \in \theta_1 \).
3. \( (\theta_1, \theta_2) \)-\( R \)-irresolute if \( f^{-1}(H) \in \theta_1 \cap C(X) \forall H \in \theta_2 \cap C(Y) \).
4. Completely \( (\theta_1, \theta_2) \)-irresolute if \( f^{-1}(H) \in \theta_1 \cap C(X) \forall H \in \theta_2 \cap C(Y) \).
5. Almost \( (\theta_1, \theta_2) \)-closed if \( f(G) \in \theta_2 \cap C(V) \forall G \in \theta_1 \cap C(X) \).
6. \( (\theta_1, \theta_2) \)-closed if \( f(F) \in \theta_2 \forall F \in \theta_1 \) \[26\]

Theorem 3.2 [27]
A function \( f : (X, \theta_1) \rightarrow (Y, \theta_2) \) is called \( (\theta_1, \theta_2) \)-continuous if for every \( \theta_2 \)-closed set \( K \) in \( Y \), \( f^{-1}(K) \) is \( \theta_1 \)-closed in \( X \).

The following theorems show the functions that maintain the property of being \( \theta \)-normal, \( \theta \)-almost normal, \( \theta \)-mildly normal, \( \theta \pi \)-normal and \( \theta \)-quasi normal.

Theorem 3.3
Let \( f : (X, \theta_1) \rightarrow (Y, \theta_2) \) be a bijective \( (\theta_1, \theta_2) \)-continuous \( (\theta_1, \theta_2) \)-open function, if \( X \) is \( \theta_1 \)-normal, then \( Y \) is \( \theta_2 \)-normal.

Proof:
Let \( F \) and \( F' \in \theta_2 \) with \( F \cap F' = \emptyset \), by Theorem 2.3 we get that \( f^{-1}(F), f^{-1}(F') \in \theta_1 \) with \( f^{-1}(F \cap F') = \emptyset \). But \( X \) is \( \theta_1 \)-normal space, so there exists \( \theta_1 \)-open sets \( U \) and \( V \) satisfying \( f^{-1}(F) \subseteq U, f^{-1}(F') \subseteq V \) and \( U \cap V = \emptyset \).

Now we have \( f(U) \) and \( f(V) \) are \( \theta_2 \)-open sets in \( Y \) with \( F = f(f^{-1}(F)) \subseteq f(U) \) and \( F' = f(f^{-1}(F')) \subseteq f(V) \) and \( f(U) \cap f(V) = f(U \cap V) = \emptyset \). Hence \( Y \) is \( \theta_2 \)-normal space.

Corollary 3.4
The image of \( \theta_1 \)-normal space under the bijective \( (\theta_1, \theta_2) \)-continuous \( (\theta_1, \theta_2) \)-open function is \( \theta_2 \pi \)-normal space.
Let \( f : (X, \theta_1) \to (Y, \theta_2) \) be a bijective \((\theta_1, \theta_2)\)-continuous \((\theta_1, \theta_2)\)-open and \((\theta_1, \theta_2)\)-irresolute function, if \( X \) is \( \theta_1 \)-almost normal, then \( Y \) is \( \theta_2 \)-almost normal.

**Proof:**

Let \( F \) be \( \theta_2 \)-closed and \( F' \) be \( \theta_2 \)-closed domain in \( Y \) such that \( F \cap F' = \emptyset \), we get that \( f^{-1}(F) \) is \( \theta_1 \)-closed and \( f^{-1}(F') \) is \( \theta_2 \)-closed domain in \( X \) with \( f^{-1}(F) \cap f^{-1}(F') = \emptyset \). But \( X \) is \( \theta_1 \)-almost normal space, so there are \( \theta_1 \)-open sets \( U \) and \( V \) such that \( f^{-1}(F) \subseteq U, f^{-1}(F') \subseteq V \) and \( U \cap V = \emptyset \).

Now we have \( f(U) \) and \( f(V) \) are \( \theta_2 \)-open sets in \( Y \) with \( F = f(f^{-1}(F)) \subseteq f(U) \) and \( F' = f(f^{-1}(F')) \subseteq f(V) \) such that \( f(U) \cap f(V) = f(U \cap V) = \emptyset \). Hence \( Y \) is \( \theta_2 \)-almost normal space.

Theorem 3.6

Let \( f : (X, \theta_1) \to (Y, \theta_2) \) be a bijective \((\theta_1, \theta_2)\)-open \((\theta_1, \theta_2)\)-irresolute function, if \( X \) is \( \theta_1 \)-mildly normal, then \( Y \) is \( \theta_2 \)-mildly normal.

**Proof:**

Take \( F, F' \in \partial_2 \mathcal{R}C(Y) \) with \( F \cap F' = \emptyset \), we get that \( f^{-1}(F), f^{-1}(F') \in \partial_1 \mathcal{R} \mathcal{C}(X) \) with \( f^{-1}(F) \cap f^{-1}(F') = \emptyset \) and \( X \) is \( \theta_1 \)-mildly normal space, so there exists \( \theta_1 \)-open sets \( U \) and \( V \) such that \( f^{-1}(F) \subseteq U, f^{-1}(F') \subseteq V \) and \( U \cap V = \emptyset \).

Now we have \( f(U) \) and \( f(V) \) are \( \theta_2 \)-open sets in \( Y \) such that \( F = f(f^{-1}(F)) \subseteq f(U) \) and \( F' = f(f^{-1}(F')) \subseteq f(V) \) such that \( f(U) \cap f(V) = f(U \cap V) = \emptyset \). Hence \( Y \) is \( \theta_2 \)-mildly normal space.

Theorem 3.7

Let \( f : (X, \theta_1) \to (Y, \theta_2) \) be a bijective \((\theta_1, \theta_2)\)-continuous \((\theta_1, \theta_2)\)-open and Completely \((\theta_1, \theta_2)\)-irresolute function, if \( X \) is \( \theta_1 \pi - \text{normal} \), then \( Y \) is \( \theta_2 \pi - \text{normal} \).

**Proof:**

Let \( F \) be \( \theta_2 \)-closed and \( F' \) be \( \theta_2 \)-\( \pi \)-closed in \( Y \) with \( F \cap F' = \emptyset \), we get that \( f^{-1}(F) \) is \( \theta_1 \)-closed and \( f^{-1}(F') \) is \( \theta_1 \)-\( \pi \)-closed domain in \( X \) with \( f^{-1}(F) \cap f^{-1}(F') = \emptyset \) and \( X \) is \( \theta_1 \pi \)-normal space, so there exists \( \theta_1 \)-open sets \( U \) and \( V \) such that \( f^{-1}(F) \subseteq U, f^{-1}(F') \subseteq V \) and \( U \cap V = \emptyset \).

Now we have \( f(U) \) and \( f(V) \) are \( \theta_2 \)-open sets in \( Y \) whereas \( F = f(f^{-1}(F)) \subseteq f(U) \) and \( F' = f(f^{-1}(F')) \subseteq f(V) \) such that \( f(U) \cap f(V) = f(U \cap V) = \emptyset \). Hence \( Y \) is \( \theta_2 \pi - \text{normal} \).

Corollary 3.8

The image of \( \theta_1 \pi - \text{normal} \) space under the \((\theta_1, \theta_2)\)-continuous \((\theta_1, \theta_2)\)-open and Completely \((\theta_1, \theta_2)\)-irresolute function is \( \theta_2 \pi - \text{normal} \).

**Proof:**

Follows from Theorem 3.7 and Proposition 2.10

Theorem 3.9

Let \( f : (X, \theta_1) \to (Y, \theta_2) \) be a bijective \((\theta_1, \theta_2)\)-open Completely \((\theta_1, \theta_2)\)-irresolute function, if \( X \) is \( \theta_1 \)-\( quasi \text{ normal} \), then \( Y \) is \( \theta_2 \)-\( quasi \text{ normal} \).
Proof:
Let $F, F' \in \theta_2 \pi C(Y)$ with $F \cap F' = \emptyset$, we get that $f^{-1}(F), f^{-1}(F') \in \theta_2 \pi C(X)$ with $f^{-1}(F) \cap f^{-1}(F') = \emptyset$. But $X$ is $\theta_1 - quasi$ normal space, so there exists $\theta_1 - open$ sets $U$ and $V$ such that $f^{-1}(F) \subset U, f^{-1}(F') \subset V$ and $U \cap V = \emptyset$. Now we have $f(U)$ and $f(V)$ are $\theta_2 - open$ sets in $Y$ whereas $F = f(f^{-1}(F)) \subset f(U)$ and $F' = f(f^{-1}(F')) \subset f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Hence $Y \theta_2 - quasi$ normal space.

To indicate the type of functions that preserve the property of being $\theta - p - normal$, $\theta - mildly$ $p - normal$, $\theta \pi p - normal$ and $\theta - quasi p - normal$, we need the following definition.

Definition 3.10
A function $f : (X, \theta_1) \rightarrow (Y, \theta_2)$ from a GTS$(X, \theta_1)$ into a GTS$(Y, \theta_2)$ is called $(\theta_1, \theta_2) - M - preopen$ if $f(G) \in \theta_2 PO(Y) \forall G \in \theta_1 PO(X)$.

Theorem 3.11
Let $f : (X, \theta_1) \rightarrow (Y, \theta_2)$ be a bijective $(\theta_1, \theta_2) - continuous$ $(\theta_1, \theta_2) - M - preopen$ function, if $X$ is $\theta_1 - p - normal$, then $Y$ is $\theta_2 - p - normal$.

Proof:
Let $F$ and $F' \in \theta_2$ with $F \cap F' = \emptyset$, by Theorem 2.3 we get that $f^{-1}(F), f^{-1}(F') \in \theta_1$ with $f^{-1}(F) \cap f^{-1}(F') = \emptyset$. But $X$ is $\theta_1 - p - normal$ space, so there exists $\theta_1 - p - open$ sets $U$ and $V$ satisfying $f^{-1}(F) \subset U, f^{-1}(F') \subset V$ and $U \cap V = \emptyset$. Therefore $f(U)$ and $f(V)$ are $\theta_2 - p - open$ sets in $Y$ so that $F = f(f^{-1}(F)) \subset f(U)$ and $F' = f(f^{-1}(F')) \subset f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Hence $Y \theta_2 - p - normal$ space.

Corollary 3.12
The image of $\theta_1 - P - normal$ space under the bijective $(\theta_1, \theta_2) - continuous$ $(\theta_1, \theta_2) - M - preopen$ function is $\theta_2 \pi p - normal$ space.

Proof:
Follows from Theorem 3.11 and Remark 2.4.

Theorem 3.13
Let $f : (X, \theta_1) \rightarrow (Y, \theta_2)$ be a bijective $(\theta_1, \theta_2) - continuous$ $(\theta_1, \theta_2) - M - preopen$ and $(\theta_1, \theta_2) - R - irresolute$ function, if $X$ is $\theta_1 - almost$ $p - normal$, then $Y$ is $\theta_2 - almost$ $p - normal$.

Proof:
Let $F$ be $\theta_2 - closed$ and $F'$ be $\theta_2 - closed$ domain in $Y$ such that $F \cap F' = \emptyset$, we get that $f^{-1}(F)$ is $\theta_1 - closed$ and $f^{-1}(F')$ is $\theta_1 - closed$ domain in $X$ with $f^{-1}(F) \cap f^{-1}(F') = \emptyset$. But $X$ is $\theta_1 - almost$ $p - normal$ space, so there are $\theta_1 - p - open$ sets $U$ and $V$ such that $f^{-1}(F) \subset U, f^{-1}(F') \subset V$ and $U \cap V = \emptyset$.

Thus $f(U)$ and $f(V)$ are $\theta_2 - p - open$ sets in $Y$ whereas $F = f(f^{-1}(F)) \subset f(U)$ and $F' = f(f^{-1}(F')) \subset f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Hence $Y \theta_2 - almost$ $p - normal$ space.

Theorem 3.14
Let $f : (X, \theta_1) \rightarrow (Y, \theta_2)$ be a bijective $(\theta_1, \theta_2) - M - preopen$ $(\theta_1, \theta_2) - R - irresolute$ function, if $X$ is $\theta_1 - mildly$ $p - normal$, then $Y$ is $\theta_2 - mildly$ $p - normal$. 

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Proof:
Take $F$ and $F'$ be bijective on $\mathbb{R}$ with $F \cap F' = \emptyset$, we get that $f^{-1}(F), f^{-1}(F') \in \partial_2 R C(X)$ with $f^{-1}(F) \cap f^{-1}(F') = \emptyset$. But $X$ is $\partial_2$ mildly $p$-normal space, so there exists $\partial_1 - p -$ open sets $U$ and $V$ such that $f^{-1}(F) \subset U, f^{-1}(F') \subset V$ and $U \cap V = \emptyset$. This gives us that $f(U)$ and $f(V)$ are $\partial_2 - p$-open sets in $X$ whereas $F = f(f^{-1}(F)) \subset f(U)$ and $F' = f(f^{-1}(F')) \subset f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Hence $Y$ $\partial_2$ mildly $p$-normal space.

Theorem 3.15
Let $f : (X, \partial_1) \rightarrow (Y, \partial_2)$ be a bijective $(\partial_1, \partial_2)$-continuous $(\partial_1, \partial_2)$-$M$-preopen and Completely $(\partial_1, \partial_2)$-irresolute function, if $X$ is $\partial_1$ $p$-normal, then $Y$ is $\partial_2$ $p$-normal.

Proof:
Let $F$ be $\partial_2$-closed and $F'$ be $\partial_2$-$\pi$-closed in $Y$ with $F \cap F' = \emptyset$, we get that $f^{-1}(F)$ is $\partial_1$-$\pi$-closed and $f^{-1}(F')$ is $\partial_1$-$\pi$-closed domain in $X$ with $f^{-1}(F) \cap f^{-1}(F') = \emptyset$. But $X$ is $\partial_1$ $p$-normal space, so there are $\partial_1 - p$-open sets $U$ and $V$ such that $f^{-1}(F) \subset U, f^{-1}(F') \subset V$ and $U \cap V = \emptyset$. This implies $f(U)$ and $f(V)$ are $\partial_2 - p$-open sets in $Y$ whereas $F = f(f^{-1}(F)) \subset f(U)$ and $F' = f(f^{-1}(F')) \subset f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Hence $Y$ $\partial_2$ $p$-normal.

Corollary 3.16
The image of $\partial_1$ $p$-normal space under the bijective $(\partial_1, \partial_2)$-continuous $(\partial_1, \partial_2)$-$M$-preopen and Completely $(\partial_1, \partial_2)$-irresolute function is $\partial_2$-almost normal(resp. $\partial_2$-mildly normal, $\partial_2$-almost $p$-normal, $\partial_2$-mildly $p$-normal, $\partial_2$-quasi normal).

Proof:
Follows from Theorem 3.15 and Remarks 2.2, 2.4, 2.7 and Proposition 2.10.

Theorem 3.17
Let $f : (X, \partial_1) \rightarrow (Y, \partial_2)$ be a bijective $(\partial_1, \partial_2)$-$M$-preopen Completely $(\partial_1, \partial_2)$-irresolute function, if $X$ is $\partial_1$ quasi $p$-normal, then $Y$ is $\partial_2$ quasi $p$-normal.

Proof:
Let $F$ and $F'$ be bijective on $\mathbb{R}$ with $F \cap F' = \emptyset$, we get that $f^{-1}(F), f^{-1}(F') \in \partial_2 R C(X)$ with $f^{-1}(F) \cap f^{-1}(F') = \emptyset$. But $X$ is $\partial_1$ quasi $p$-normal space, so there are $\partial_1 - p$-open sets $U$ and $V$ whereas $f^{-1}(F) \subset U, f^{-1}(F') \subset V$ and $U \cap V = \emptyset$. We get $f(U)$ and $f(V)$ are $\partial_2 - p$-open sets in $Y$ such that $F = f(f^{-1}(F)) \subset f(U)$ and $F' = f(f^{-1}(F')) \subset f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Hence $Y$ $\partial_2$ quasi $p$-normal.

Definition 3.18
Let $(X, \partial_1)$ and $(Y, \partial_2)$ be two generalized topological spaces. A function $f : (X, \partial_1) \rightarrow (Y, \partial_2)$ is called $(\partial_1, \partial_2)$ quasi pre - irresolute if $f^{-1}(H) \in \partial_1 PO(X) \forall H \in \partial_2 PO(Y)$.

Theorem 3.19
Let $f : (X, \partial_1) \rightarrow (Y, \partial_2)$ be a bijective $(\partial_1, \partial_2)$-$\pi$-closed, Almost$(\partial_1, \partial_2)$-$\pi$-closed and $(\partial_1, \partial_2)$ pre - irresolute function, if $Y$ is $\partial_2$ $p$-normal, then $X$ is $\partial_1$ $p$-normal.

Proof:
Take $F \in \partial_1'$ and $F' \in \partial_1 R C(X)$ with $F \cap F' = \emptyset$, so $f(F)$ is $\partial_2$-$\pi$-closed and $f(F')$ be $\partial_2 - \pi$-closed in $Y$ such that $f(F) \cap f(F') = f(F \cap F') = \emptyset$. But $Y$ is $\partial_2$ $p$-normal space, therefor
we get two disjoint $\vartheta_2 - p$-open sets $U$ and $V$ in $Y$ such that $f(F) \subset U$, $f(F') \subset V$. Since $f$ is ($\vartheta_1$, $\vartheta_2$) pre irresolute function, so $f^{-1}(U)$ and $f^{-1}(V)$ are $\vartheta_2 - p$-open in $X$ with $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $F = f^{-1}(f(F)) \subset f^{-1}(U)$ and $F' = f^{-1}(f(F')) \subset f^{-1}(V)$. Hence $X$ is $\vartheta_1 \pi p$-normal space.

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