HOMOMORPHISMS BETWEEN VERMA MODULES OF SIMPLE LIE SUPeralgebras

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Abstract. For certain positive even roots $\gamma$ of a simple Lie superalgebra of type BDFG, we prove the existence of a nonzero homomorphism between Verma modules $M(s,\lambda) \rightarrow M(\lambda)$. This is a super generalization of a classic theorem of Verma.

1. Introduction

Let $g$ be a complex semisimple Lie algebra and $M(\lambda)$ the Verma module of highest weight $\lambda - \rho$, where $\rho$ is the Weyl vector. Denote by $s_\gamma$ reflection about the positive root $\gamma$. Suppose that $s_\gamma \lambda \leq \lambda$. Then, Verma’s theorem (cf. [H, §4.7]) states that there is a nonzero homomorphism $M(s,\lambda) \rightarrow M(\lambda)$ or, equivalently, there is a singular vector of weight $s,\lambda - \rho$ in $M(\lambda)$. As Verma modules also play a prominent role in the theory of Lie superalgebras, a generalization of Verma’s theorem to the situation of Lie superalgebras has long been desired.

There are some partial results in this direction due to Musson. Take $g$ to be a basic Lie superalgebra over $C$, where we choose to consider $\mathfrak{gl}(m|n)$ in type A, with a simple system containing at most one isotropic odd root. Denote the Weyl group of $g$ by $W$ and denote the subgroup generated by reflections about nonisotropic simple roots by $W'$. Let $\gamma$ be a positive even root in the $W'$-orbit of a positive simple root, and let $M(\lambda)$ be the Verma module of highest weight $\lambda - \rho$, where $\rho$ is the Weyl vector. Musson [M, Theorem 9.2.6] has adapted a traditional proof of Verma’s theorem by Shapovalov [S] to show that there exists a nonzero homomorphism $M(s,\lambda) \rightarrow M(\lambda)$, when $s,\lambda < \lambda$. This gives a complete solution for $\mathfrak{gl}(m|n)$, as $W = W'$ and every positive even root is in the $W$-orbit of an even simple root. Furthermore, for $\lambda - \rho$ typical, Musson [M, Corollary 9.3.6] extended the above result to any positive $\gamma$ that is neither isotropic nor twice another root.

In types other than A and for $\lambda$ general (i.e. no typicality conditions), there are positive even roots to which Musson’s result does not apply. Cheng and Wang (see [CW1, Lemma 2.5]) completed the solution in type $D(2|1;\zeta)$, by developing an explicit formula for a singular vector corresponding to the only positive even root not in the $W'$-orbit of an even simple root. This was inspired by a prior construction in [KW] in the setting of affine superalgebras. The success of this case led Cheng and Wang to conjecture (see [CW1, Introduction]) that a similar construction would work for other basic Lie superalgebras.

In this paper, we find six such candidate singular vectors, each corresponding to a positive even root not in the $W'$-orbit of an even simple root: two for $\mathfrak{osp}(2n+1|2m), n \geq 1$, one for the standard simple system and one for the opposite simple system, two for $\mathfrak{osp}(2m|2m), n \geq 2$, one for the standard simple system and one for the opposite simple system, and one each for $F(3|1)$ and $G(3)$, for their standard simple systems. For example, in the case of $\mathfrak{osp}(2n+1|2m), n \geq 1$, standard simple system, we construct a vector of weight $s_{2m}\lambda - \rho$ in $M(\lambda)$, where $\lambda - \rho$ need not be typical.
It is not obvious that the candidate singular vectors are nonzero, as their formulas are not written in a PBW basis of $U(n^-)$, and a significant part of the paper is spent proving this fact. In each case, the proof amounts to showing that if we write the vector in a particular PBW basis, there is a basis element that can be seen to occur with nonzero coefficient in the vector. This allows us to avoid finding a closed formula for the vector in the standard PBW basis, as Cheng and Wang did in [CW1 Lemma 2.5], which would be difficult in the cases dealt with here. Notably, unlike in the classical case, the homomorphisms determined by the singular vectors associated to even reflections are not necessarily embeddings (see Remark 4.3).

In the $F(3|1)$ and $G(3)$ cases, these singular vectors correspond to the only positive even roots not in the $W'$-orbit of an even simple root, so that completes these cases. For the other cases, we adapt the proof of [M, Theorem 9.2.6] to obtain the result for all roots in the $W'$-orbit of the root corresponding to the singular vector. This completes the case of $\text{osp}(2n|2m)$, $n \geq 2$ with the opposite simple system. In the other three cases, there remain $W'$-orbits of positive even roots for which the super analogue of Verma’s theorem remains open.

The paper is organized as follows. In §2 we introduce the basic terminology, notations and concepts that will be used throughout the paper as well as the formulas for the candidate singular vectors. In §3 we prove that the candidate singular vectors are, in fact, singular. In §4 we adapt the proof of [M, Theorem 9.2.6] to obtain the desired homomorphisms for all roots in the $W'$-orbit of the root corresponding to the singular vector.

Acknowledgement. The author would like to thank his advisor, Weiqiang Wang, for formulating the conjecture on which this paper is based and for his discussions and advice.

2. The Preliminaries and Formulas for Singular Vectors

In this section, we set up notation and introduce some basic ideas and constructions, (cf. [CW2]). We also write down formulas for the vectors that we will prove to be singular in §3.

Let $g$ be a basic Lie superalgebra over $\mathbb{C}$ with simple system $\Pi$ and triangular decomposition

$$
g = n^- \oplus \mathfrak{h} \oplus n^+,
$$

where $n^\pm$ can be expressed as a direct sum of root spaces: $n^\pm = \bigoplus_{\alpha \in \Phi^\pm} g_\alpha$. Here, $\Phi^+$ (resp. $\Phi^-$) are the positive roots (resp. negative roots). We also write $\mathfrak{b}^\pm = n^\pm \oplus \mathfrak{h}$. We have a decomposition of the positive roots into even and odd roots respectively: $\Phi^+ = \Phi^+_0 \cup \Phi^+_1$. Write $W$ for the Weyl group of $g$ and $W'$ for the subgroup of $W$ generated by reflections about nonisotropic simple roots.

For each $\alpha \in \Phi^+$, fix root vectors $e_\alpha \in g_\alpha$ and $f_\alpha = e_{-\alpha} \in g_{-\alpha}$. Let $\rho$ be the Weyl vector, which is given by

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+_0} \alpha - \frac{1}{2} \sum_{\beta \in \Phi^+_1} \beta.
$$

Let $\langle -, - \rangle$ be the standard bilinear pairing on $\mathfrak{h}^* \times \mathfrak{h}$, and let $\langle -, - \rangle$ be a non-degenerate, $W$-invariant bilinear pairing on $\mathfrak{h}$ (cf. [CW2 Theorem 1.18]). For a non-isotropic root $\beta$, ...
define $h_\beta$ to be the coroot of $\beta$, i.e., the unique element of $\mathfrak{h}$ such that $\langle \mu, h_\beta \rangle = \frac{2(\mu, \beta)}{(\beta, \beta)}$ for all $\mu \in \mathfrak{h}^*$. We will be concerned with a particular positive even root $\gamma$. Let $\lambda \in \mathfrak{h}^*$ be any weight such that
\[(2.1) \quad N := \langle \lambda, h_\gamma \rangle \in \mathbb{Z}_{>0}.\]
Denote by $M(\lambda)$ the Verma module of $\mathfrak{g}$ of highest weight $\lambda - \rho$ with highest weight vector $v^+$. It is defined as
\[M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \mathbb{C}_{\lambda - \rho},\]
where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $\mathbb{C}_{\lambda - \rho}$ is the 1 dimensional $\mathfrak{b}^+$-module upon which $\mathfrak{n}^+$ acts as 0 and $\mathfrak{h}$ acts by the weight $\lambda - \rho$.

Below we list the relevant Lie superalgebras and their simple systems along with the corresponding Dynkin diagrams, $\Phi^+_0$, $\Phi^+_1$, $\gamma$, and $\rho$. We also introduce a vector, $u$, which we will show is singular of weight $\lambda - \rho - N\gamma$ (see (2.1)) in the next section.

For $\Phi^+_0$ in (2.1) below, we have that
\[1 \leq i < j \leq m, 1 \leq p \leq m, 1 \leq k < l \leq n, 1 \leq q \leq n.\]

2.1. $\mathfrak{g} = \mathfrak{osp}(2n + 1|2m)$, $n \geq 1$, I.
\[\Pi = \{\delta_i - \delta_{i+1}, \delta_m - \varepsilon_1, \varepsilon_j - \varepsilon_{j+1}, \varepsilon_n | 1 \leq i \leq m - 1, 1 \leq j \leq n - 1\},\]
\[\Phi^+_0 = \{\delta_i \pm \delta_j, 2\delta_p, \varepsilon_k \pm \varepsilon_l, \varepsilon_q\}, \Phi^+_1 = \{\delta_p, \delta_p \pm \varepsilon_q\},\]
\[\gamma = 2\delta_m, \quad \rho = \sum_{i=1}^m (m - n - i + \frac{1}{2})\delta_i + \sum_{j=1}^n (n - j + \frac{1}{2})\varepsilon_j.\]

We define a vector
\[u := \prod_{i=1}^n (c_{\delta_{m-i} \varepsilon_i} c_{\delta_{m-i} \varepsilon_i}) f_{2\delta_m}^{N+n} v^+,\]
which we will later prove is singular of weight $\lambda - \rho - 2N\delta_m$ (see (2.1)).

2.2. $\mathfrak{g} = \mathfrak{osp}(2n + 1|2m)$, $n \geq 1$, II.
\[\Pi = \{\varepsilon_j - \varepsilon_{j+1}, \varepsilon_n - \delta_1, \delta_i - \delta_{i+1}, \delta_m | 1 \leq i \leq m - 1, 1 \leq j \leq n - 1\},\]
\[\Phi^+_0 = \{\delta_i \pm \delta_j, 2\delta_p, \varepsilon_k \pm \varepsilon_l, \varepsilon_q\}, \Phi^+_1 = \{\delta_p, \varepsilon_q \pm \delta_p\},\]
\[\gamma = \varepsilon_n, \quad \rho = \sum_{i=1}^m (m - i + \frac{1}{2})\delta_i + \sum_{j=1}^n (n - m - j + \frac{1}{2})\varepsilon_j.\]
We define a vector
\[ u := \prod_{i=1}^{m} (e_{\epsilon_n-\delta_i}e_{\epsilon_n+\delta_i})f_{\epsilon_n}^{N+2m}u^+, \]
which we will later prove is singular of weight \( \lambda - \rho - N\epsilon_n \) (see (2.1)).

We will be using the explicit choice of root vectors in \( g = \text{osp}(2n+1|2m) \) of [CW2] Section 1.2.4 only for the \([2.2]\) case of the proof of Theorem 3.3 and for Remark 4.3. We provide the relevant formulas below:

\[ [e_{\epsilon_n+\delta_m}, f_{\epsilon_n}] = -e_{-\delta_m}, \quad [e_{\epsilon_n-\delta_m}, f_{\epsilon_n}] = -e_{-\delta_m}, \quad [e_{\delta_m}, f_{\epsilon_n}] = e_{\delta_m-\epsilon_n}, \]
\[ [e_{-\delta_m}, f_{\epsilon_n}] = e_{-\delta_m+\epsilon_n}, \quad [e_{\epsilon_n+\delta_m}, e_{-\epsilon_n}+\delta_m] = \frac{1}{2}(h_{\delta_m} + h_{\epsilon_n}), \quad [e_{\epsilon_n-\delta_m}, e_{\delta_m}] = -e_{\epsilon_n}, \]
\[ [e_{\delta_m}, e_{-\delta_m}] = \frac{1}{2}h_{\delta_m}, \quad [e_{\delta_m}, e_{-\delta_m-\epsilon_n}] = -f_{\epsilon_n}, \quad [e_{-\epsilon_n+\delta_m}, e_{-\delta_m}] = f_{\epsilon_n}. \]

2.3. \( g = \text{osp}(2n|2m), \ n \geq 2, \text{ I.} \)
\[ \Pi = \{ \delta_i - \delta_{i+1}, \delta_m - \epsilon_1, \epsilon_j - \epsilon_{j+1}, \epsilon_{n-1} + \epsilon_n \} \quad 1 \leq i \leq m-1, 1 \leq j \leq n-1 \}, \]
\[ \Phi_0^+ = \{ \delta_i \pm \delta_j, 2\delta_p, \epsilon_k \pm \epsilon_l \}, \Phi_1^+ = \{ \delta_p \pm \epsilon_q \}, \]
\[ \gamma = 2\delta_m, \]
\[ \rho = \sum_{i=1}^{m} (n-i+1)\delta_i + \sum_{j=1}^{n} (n-j)\epsilon_j. \]

We define a vector
\[ u := \prod_{i=1}^{n} (e_{\delta_m-\epsilon_i}e_{\delta_m+\epsilon_i})f_{2\delta_m}^{N+n}u^+, \]
which we will later prove is singular of weight \( \lambda - \rho - 2N\delta_m \).

2.4. \( g = \text{osp}(2n|2m), \ n \geq 2, \text{ II.} \)
\[ \Pi = \{ \epsilon_j - \epsilon_{j+1}, \epsilon_n - \delta_1, 1 \leq i \leq m - 1, 1 \leq j \leq n - 1 \}, \]
\[ \Phi_0^+ = \{ \delta_i \pm \delta_j, 2\delta_p, \epsilon_k \pm \epsilon_l \}, \Phi_1^+ = \{ \epsilon_q \pm \delta_p \}, \]
\[ \gamma = \epsilon_{n-1} + \epsilon_n, \]
\[ \rho = \sum_{i=1}^{m} (m-i+1)\delta_i + \sum_{j=1}^{n} (n-m-j)\epsilon_j. \]
We define a vector
\[ u := \prod_{i=1}^{m} (e_{\epsilon_n-\delta_i} e_{\epsilon_n+\delta_i}) \prod_{i=1}^{m} (e_{\epsilon_{n-1}-\delta_i} e_{\epsilon_{n-1}+\delta_i}) f^{N+2m}_{\epsilon_{n-1}+\epsilon_n} v^+, \]
which we will later prove is singular of weight \( \lambda - \rho - N(\epsilon_{n-1} + \epsilon_n) \).

2.5. \( g = F(3|1) \).
\[
\Pi = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3, \frac{1}{2}(\delta - \epsilon_1 - \epsilon_2 - \epsilon_3) \},
\]
\[
\Phi^-_0 = \{ \delta, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq 3 \}, \Phi^+_1 = \{ \frac{1}{2}(\delta \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \},
\]
\[
\gamma = \delta,
\]
\[
\rho = \frac{1}{2}(-3\delta + 5\epsilon_1 + 3\epsilon_2 + \epsilon_3).
\]
We introduce some notation for \( g = F(3|1) \). An odd root is denoted by the ordered tuple of signs appearing in it. For example \( \frac{1}{2}(\delta - \epsilon_1 - \epsilon_2 + \epsilon_3) \) is denoted by \(+−−+\).

We define a vector
\[ u := e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} e_{+−−} f_{\delta}^{N+4} v^+, \]
which we will later prove is singular of weight \( \lambda - \rho - N\delta \).

2.6. \( g = G(3) \).
\[
\Pi = \{ \epsilon_2 - \epsilon_1, \epsilon_1, \delta + \epsilon_3 \}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0,
\]
\[
\Phi^-_0 = \{ 2\delta, \epsilon_1, -\epsilon_3, \epsilon_2 - \epsilon_1, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3 \}, \Phi^+_1 = \{ \delta, \delta \pm \epsilon_i \mid 1 \leq i \leq 3 \},
\]
\[
\gamma = 2\delta,
\]
\[
\rho = -\frac{5}{2}\delta + 2\epsilon_1 + 3\epsilon_2.
\]
We define a vector
\[ u := e_{\delta-\epsilon_1} e_{\delta+\epsilon_1} e_{\delta-\epsilon_2} e_{\delta+\epsilon_2} e_{\delta-\epsilon_3} e_{\delta+\epsilon_3} f_{2\delta}^{N+3} v^+, \]
which we will later prove is singular of weight \( \lambda - \rho - 2N\delta \).
3. The vectors $u$ are singular

Recall the positive even roots $\gamma$ and vectors $u$ defined in \S 2.1 - 2.6. In this section, we will prove that the vectors $u$ are singular, which is equivalent to proving the existence of a nonzero homomorphism $M(s, \lambda) \to M(\lambda)$.

**Lemma 3.1.** Changing the order of the product of positive odd root vectors in the formulas for $u$ given in \S 2.1 - 2.6 changes $u$ by a factor of $\pm 1$.

**Proof.** In the cases of \S 2.1 and \S 2.3 among the odd root vectors in the formula for $u$, the only one failing to anti-commute with $e_{\delta_m \pm \varepsilon_k}$ is $e_{\delta_m \pm \varepsilon_k}$. Instead $[e_{\delta_m \pm \varepsilon_k}, e_{\delta_m \pm \varepsilon_k}]$ is a multiple of $e_{2\delta_m}$. Let $u'$ be any vector obtained by rearranging the terms of the product. Then, $u'$ is equal to $\pm u$ plus a linear combination of terms of the form

$$v := \prod_{i=1}^{k-1} (e_{\delta_m - \varepsilon_i} e_{\delta_m + \varepsilon_i}) \prod_{i=k+1}^{n} (e_{\delta_m - \varepsilon_i} e_{\delta_m + \varepsilon_i}) e_{2\delta_m} f_{2\delta_m}^{N+n} v^+.$$

In the case of \S 2.3, we have by $\mathfrak{sl}_2$ relations,

$$e_{2\delta_m} f_{2\delta_m}^{N+n} v^+ = f_{2\delta_m}^{N+n} e_{2\delta_m} v^+ + (N + n) f_{2\delta_m}^{N+n-1}(h_{2\delta_m} - (N + n - 1)) v^+
= (N + n) f_{2\delta_m}^{N+n-1}(\langle \lambda - \rho, h_{2\delta_m} \rangle - (N + n - 1)) v^+ = 0.$$

Hence, $v = 0$ and $u' = \pm u$. The cases of \S 2.2, \S 2.4 and \S 2.5 are similar.

In the case of \S 2.1, $v$ is a multiple of the following vector:

$$v' := \prod_{i=1}^{k-1} (e_{\delta_m - \varepsilon_i} e_{\delta_m + \varepsilon_i}) \prod_{i=k+1}^{n} (e_{\delta_m - \varepsilon_i} e_{\delta_m + \varepsilon_i}) e_{2\delta_m}^2 f_{2\delta_m}^{2N+2n} v^+.$$

By $\mathfrak{sl}_2$ relations, we have

$$e_{\delta_m} f_{\delta_m}^{2N+2n} v^+ = f_{\delta_m}^{2N+2n} e_{\delta_m} v^+ + (2N + 2n) f_{\delta_m}^{2N+2n-1}(h_{\delta_m} - (2N + 2n - 1)) v^+
= (2N + 2n) f_{\delta_m}^{2N+2n-1}(\langle \lambda - \rho, h_{\delta_m} \rangle - (2N + 2n - 1)) v^+ = 0.$$

Hence, $v' = 0$ and $u' = \pm u$. The case of \S 2.6 is similar. □

**Lemma 3.2.** The vectors $u$ defined in \S 2.1 - 2.6 are nonzero.

**Proof.** In each case, we will proceed as follows. We will fix a basis of $M(\lambda)$, and describe a sequence of elements of this basis $v_k$, and corresponding vectors $u_k$, such that $u_K = v_K$, where $K$ is the largest index in each case and $u_l = u$, where $l$ is the smallest index. Then, we will show that the coefficient of $v_k$ in $u_k$ is a nonzero multiple of the coefficient of $v_{k+1}$ in $u_{k+1}$ for each $k$, proving the lemma.

In the case of \S 2.1 we take the basis to be any PBW basis of $M(\lambda)$ that contains the following vectors:

$$v_{a_{i,n-1,n-1}} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} \cdots e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i} e_{-\varepsilon_i} e_{\delta_m - \varepsilon_i} e_{\delta_m - \varepsilon_i}$$

of weight $\lambda - \rho - 2(N + k - 1)\delta_m$, where $a_{i,j,\pm} R \in \mathbb{Z}_{\geq 0}$ and $b_k, c_t \in \{0, 1\}$. In (3.3), $e_{\pm \varepsilon_i}$ can appear for any $i, j$ such that $k \leq i < j \leq n$. From left to right in (3.3), the sum of the
The above vectors play the same role as the vectors of the same name in the proof of the (3.4) here play the same role as those of the form (3.3) in the case of $i$ these terms does not matter.

We define the following:

$$u_k := \prod_{i=k}^{n} (e_{\delta_m-e_i} e_{\delta_m+e_i}) f^{N+n}_{2\delta_m} v^+,$$

$$u_{n+1} = v_{n+1} := f^{N+n}_{2\delta_m} v^+,$$

$$v_k := e_{\delta_n-e_{n-1}} \cdots e_{\delta_k+1+\delta_k} e_{-\delta_m-e_{n-1}} e_{\delta_m+e_k} f^{N+k-2}_{2\delta_m} v^+.$$

We suppose that for some $k$, $u_k$ can be written as a linear combination of elements of the form (3.3), and that the coefficient of $v_k$ in $u_k$ is nonzero. Clearly, these statements hold for $k = n + 1$. As $u_{k-1} = e_{\delta_m-e_{k-1}} e_{\delta_m+e_{k-1}} u_k$, it is easy to check that $v_{k-1}$ can only be obtained from the term $v_k$ among those of the form (3.3) by multiplying on the left by $e_{\delta_m-e_{k-1}} e_{\delta_m+e_{k-1}}$, by commuting $e_{\delta_m-e_{k-1}}$ with $e_{-\delta_m+e_k}$ and $e_{\delta_m+e_{k-1}}$ with $f^{N+k-2}_{2\delta_m}$. The exception is for $k = n + 1$, in which case both $e_{\delta_n-e_{n-1}}$ and $e_{\delta_m+e_{n-1}}$ are commuted with the $f_{2\delta_m}$. Hence, the coefficient of $v_{k-1}$ in $u_{k-1}$ is nonzero. It is also easy to check that multiplying on the left by $e_{\delta_m-e_{k-1}} e_{\delta_m+e_{k-1}}$ preserves the form (3.3) with $k$ replaced by $k - 1$. This completes the induction and the proof of the lemma in the case of §2.1.

The proof in the case of §2.2 is similar to the one in the case of §2.1. We take the basis to be any PBW basis consisting of the following vectors:

$$e^{-\delta_m} \cdots e^{b_{m-1},m,-} e_{\delta_m-e_{n-1}} \cdots e^{b_{k+1},k+1+\delta_k} e^{d_m}_{\delta_m+\delta_k} e^{d_k}_{\delta_n+\delta_k} f^R v^+$$

of weight $\lambda - \rho - (N + 2k - 2)\delta_n$, where the $a$’s, the $b$’s and $R$ are non-negative integers and the $c$’s and $d$’s are 0 or 1. In (3.4), $e_{\pm\delta_j - \delta_i}$ can appear for any $i, j$ such that $k \leq i < j \leq m$. From left to right in (3.4), the sum of the indices, i.e. $i + j$, of $e_{\pm\delta_j - \delta_i}$ is decreasing. Other than that, the fixed ordering of these terms in (3.4) does not matter. The terms of the form (3.3) here play the same role as those of the form (3.3) in the case of §2.1. We define the following:

$$u_k := \prod_{i=k}^{m} (e_{\delta_n-\delta_i} e_{\delta_n+\delta_i}) f^{N+2m}_{\delta_n} v^+,$$

$$u_{m+1} = v_{m+1} := f^{N+2m}_{\delta_n} v^+,$$

$$v_k := e^{-\delta_m} e_{\delta_m-e_{n-1}} \cdots e_{\delta_k+1+\delta_k} e_{-\delta_m-\delta_k} f^{N+2k-3}_{\delta_n} v^+.$$

The above vectors play the same role as the vectors of the same name in the proof of the §2.1 case. The vector $v_{k-1}$ is obtained from $v_k$ by left multiplication by $e_{\delta_n-\delta_{k-1}} e_{\delta_n+\delta_{k-1}}$ by commuting $e_{\delta_n+\delta_{k-1}}$ with the $f_{\delta_n}$ and $e_{\delta_n-\delta_{k-1}}$ with $e_{-\delta_n+\delta_k}$, except for $k = m + 1$, in which case both $e_{\delta_n+\delta_m}$ and $e_{\delta_n-\delta_m}$ are commuted with the $f_{\delta_n}$.

The proof in the case of §2.3 is identical to the one in the case of §2.1 and so will be skipped.
The proof in the case of \( \text{§2.3} \) is similar to the one in the case of \( \text{§2.1} \). We take as a basis any PBW basis containing the following vectors:

\[
\begin{align*}
(3.5) \quad & e_{-2\delta_m} \ldots e_{-2\delta_2} e_{-\delta_m-\delta_{m-1}} \ldots e_{-\delta_1-\delta_0} e_{\delta_{k+1}-\delta_k} e_{\delta_{k+2}-\delta_{k+1}} \ldots e_{-\delta_1-\delta_0} e_{-\delta_{k+1}-\delta_k} e_{-\delta_{k+2}-\delta_{k+1}} \ldots e_{-\delta_1-\delta_0} e_{\delta_m} e_{-\varepsilon_n-\delta_n+\delta_{m+1}} e_{-\delta_{m+1}+\delta_{m+2}} \ldots \varepsilon_n \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad g \in \{0, 1\}.
\end{align*}
\]

The above vectors play the same role as the vectors of the same name in the proof of the \( \text{§2.1} \) case. Here \( v_{k-1} \) is obtained from \( v_k \) by left multiplication by \( e_{\varepsilon_n-\delta_n-1} e_{\varepsilon_n+\delta_n} e_{\varepsilon_n-1-\delta_n} e_{\varepsilon_n-2-\delta_n} \ldots e_{-\delta_n} e_{-\delta_n+1} e_{-\delta_n+2} \ldots e_{-\delta_n} e_{-\delta_n+1} e_{-\delta_n+2} \ldots e_{-\delta_n} \). The exception is for \( k = m + 1 \), in which case, \( e_{\varepsilon_n-\delta_n}, e_{\varepsilon_n+\delta_n} \) are commuted with the \( f_{\varepsilon_n+1} \) and \( e_{\varepsilon_n-\delta_n} \) is commuted with \( e_{-\varepsilon_n} \).
\[ u_0 := u, \]
\[ u_1 := e_{+---} e_{+---} e_{+---} e_{+---} e_{+---} e_{+---} f_{\delta}^{N+4} v^+, \]
\[ u_2 := e_{+---} e_{+---} e_{+---} e_{+---} e_{+---} e_{+---} f_{\delta}^{N+4} v^+, \]
\[ u_3 := e_{+---} e_{+---} e_{+---} e_{+---} e_{+---} e_{+---} f_{\delta}^{N+4} v^+, \]
\[ u_4 := e_{---} e_{---} e_{---} e_{---} e_{---} e_{---} f_{\delta}^{N+4} v^+, \]
\[ u_5 := e_{+---} e_{+---} e_{+---} e_{+---} f_{\delta}^{N+4} v^+, \]
\[ u_6 := e_{+---} e_{+---} f_{\delta}^{N+4} v^+, \]
\[ u_7 := e_{+---} f_{\delta}^{N+4} v^+, \]
\[ u_8 := f_{\delta}^{N+4} v^+. \]

We call the sum of the exponents of the negative root vectors other than \( f_\delta \) in a basis element its degree. Multiplying on the left by a positive odd root vector, degree can only increase and the exponent of \( f_\delta \) can only decrease by commuting the positive root vector past the \( f_\delta \). In such a case, the degree increases by one and the exponent of \( f_\delta \) decreases by one. Moreover, if a basis element, \( w' \), with greater exponent of \( f_\delta \) is gotten from one, \( w \), with lesser exponent of \( f_\delta \) by left multiplication by a positive odd root vector, then the sum of the degree and the exponent of \( f_\delta \) of \( w' \) is less than that of \( w \). Hence, for a basis element appearing in one of the \( u_i \), the sum of the degree and the exponent of \( f_\delta \) is less than or equal to \( N + 4 \). Note that this sum is exactly \( N + 4 \) for the \( u_i \).

Thus, the vector \( v_0 \) is obtained from terms of degree 4 with exponent of \( f_\delta \) equal to \( N \) or degree 5 with exponent of \( f_\delta \) equal to \( N - 1 \) in \( u_1 \) by applying \( e_{---} \). Because the term \( e_{---} \) does not appear in \( v_0 \), it must come from terms of degree 5. But, of the terms other than \( f_\delta \) appearing in \( v_0 \), only \( e_{---} \) lies in the image of \( \text{ad}_{e_{---}} \), so \( v_0 \) could only have come from \( v_1 \), with coefficient of \( v_0 \) in \( u \) a nonzero multiple of that of \( v_1 \) in \( u_1 \). Using similar arguments, one sees that the \( v_i \) term in \( u_i \) could only have come from the \( v_{i+1} \) term in \( u_{i+1} \) by applying the appropriate positive odd root vector, and that the coefficient of \( v_i \) in \( u_i \) is a nonzero multiple of that of \( v_{i+1} \) in \( u_{i+1} \). As the coefficient of \( v_8 \) in \( u_8 \) is 1, the coefficient of \( v_0 \) in \( u \) is nonzero, and, consequently, \( u \neq 0 \).

In the case of \([2,7]\) we consider \( u \) in the following basis for \( M(\lambda) \):

\[
(3.7) \quad e_{\pm \lambda}^a \ e_{\pm \lambda}^b \ e_{\pm \lambda}^c \ e_{\pm \lambda}^d \ e_{\pm \lambda}^e \ e_{\pm \lambda}^f \ e_{\pm \lambda}^g \ e_{\pm \lambda}^h \ e_{\pm \lambda}^i \ e_{\pm \lambda}^j \ e_{\pm \lambda}^k \ e_{\pm \lambda}^l \ e_{\pm \lambda}^m \ e_{\pm \lambda}^n \ e_{\pm \lambda}^o \ e_{\pm \lambda}^p \ e_{\pm \lambda}^q \ e_{\pm \lambda}^r \ e_{\pm \lambda}^s \ e_{\pm \lambda}^t \ e_{\pm \lambda}^u \ e_{\pm \lambda}^v \ e_{\pm \lambda}^w \ e_{\pm \lambda}^x \ e_{\pm \lambda}^y \ e_{\pm \lambda}^z \]

where \( a_i, b_j \) and \( R \in \mathbb{Z}_{\geq 0} \), and \( c_k, s \in \{0, 1\} \).

Consider the following basis elements \( v_i \) in \( u_i \):

\[
v_0 := e_{\pm \lambda}^a e_{\pm \lambda}^b e_{\pm \lambda}^c e_{\pm \lambda}^d f_{\delta}^{N-1} v^+, \]
\[
v_1 := e_{\pm \lambda}^a e_{\pm \lambda}^b e_{\pm \lambda}^c e_{\pm \lambda}^d f_{\delta}^{N-1} v^+, \]
\[
v_2 := e_{\pm \lambda}^a e_{\pm \lambda}^b e_{\pm \lambda}^c e_{\pm \lambda}^d f_{\delta}^{N-1} v^+, \]
\[
v_3 := e_{\pm \lambda}^a e_{\pm \lambda}^b e_{\pm \lambda}^c e_{\pm \lambda}^d f_{\delta}^{N-1} v^+, \]
\[
v_4 := e_{\pm \lambda}^a e_{\pm \lambda}^b e_{\pm \lambda}^c e_{\pm \lambda}^d f_{\delta}^{N-1} v^+, \]
\[
v_5 := e_{\pm \lambda}^a e_{\pm \lambda}^b e_{\pm \lambda}^c e_{\pm \lambda}^d f_{\delta}^{N-1} v^+, \]
\[
v_6 := f_{\delta}^{N+3} v^+. \]
We show that the coefficient of the basis element \( v_0 \) in \( u \) is nonzero, and hence that \( u \) is nonzero. This term is obtained by applying \( e_{\delta - \varepsilon_1} \) to \( u_1 \) written in the basis and reordering terms as necessary. Similarly to the proof of the case of \( \mathfrak{g} \), we say that the degree of a basis element is the sum of the exponents of the negative root vectors other than \( f_{2\delta} \) appearing in it. Multiplying on the left by a positive odd root vector other than \( e_\delta \), degree can only increase and the exponent of \( f_{2\delta} \) can only decrease by commuting the positive root vector past the \( f_{2\delta} \). This increases degree by one and also reduces the exponent of \( f_{2\delta} \) by one. Moreover, if a basis element, \( w' \), with greater exponent of \( f_{2\delta} \) is gotten from one, \( w \), with lesser exponent of \( f_{2\delta} \) by left multiplication by a positive odd root vector other than \( e_\delta \), then the sum of the degree and the exponent of \( f_{2\delta} \) of \( w' \) is less than that of \( w \). Hence, the sum of the degree and the exponent of \( f_{2\delta} \) is less than or equal to \( N + 3 \) for any basis element appearing in one of the \( u_i \). Note that this sum is exactly \( N + 3 \) for the \( v_i \).

Since \( v_0 \) does not contain the term \( e_{-\delta - \varepsilon_1} \), \( v_0 \) must have come from terms of degree 4 and exponent of \( f_{2\delta} \) equal to \( N + 1 \) in \( u_1 \) by applying \( e_{\delta - \varepsilon_1} \). Of the root vectors other than \( f_{2\delta} \) appearing in \( v_0 \), only \( e_{\varepsilon_3 - \varepsilon_1} \) lies in the image of \( ad_{e_{\delta - \varepsilon_1}} \), so the coefficient of \( v_0 \) in \( u \) is a non-zero multiple of the coefficient of \( v_1 \) in \( u_1 \). With similar reasoning, we see that the coefficient of \( v_i \) in \( u_i \) is a nonzero multiple of that \( v_{i+1} \) in \( u_{i+1} \). As the coefficient of \( v_6 \) in \( u_6 \) is 1, the coefficient of \( v_0 \) in \( u \) is non-zero, and hence \( u \neq 0 \).

Now that we know \( u \neq 0 \), we are in the position to prove that the vectors \( u \) are singular.

**Theorem 3.3.** The vectors \( u \) defined in \( \mathfrak{g}_1 - \mathfrak{g}_6 \) are singular.

**Proof.** We begin with the case of \( \mathfrak{g}_1 \). In all the cases, we check that \( e_\alpha u = 0 \), where \( \alpha \in \Pi \). The \( c_i \), \( d_k \), and \( g_k \) below denote scalars. First, note that \( \lambda - \rho - 2N\delta_m + \delta_i - \delta_{i+1} \) is not a lower weight than \( \lambda - \rho \). Hence, \( e_{\delta_i - \delta_{i+1}} u = 0 \). Also, it is clear that \( e_{\delta_m - \varepsilon_1} u = 0 \), as \( e_{\delta_m - \varepsilon_1}^2 = 0 \). We also have that

\[
(3.8) \quad e_{\varepsilon_k - \varepsilon_{k+1}} u = -c_1 \prod_{i=1}^{k-1} (e_{\delta_m - \varepsilon_1} e_{\delta_m + \varepsilon_i}) e_{\delta_m - \varepsilon_k} e_{\delta_m + \varepsilon_k} e_{\delta_m - \varepsilon_{k+1}} \prod_{i=k+2}^{n} (e_{\delta_m - \varepsilon_1} e_{\delta_m + \varepsilon_i}) f_{2\delta_m}^{N+n} u^+ 
\]

\[
+ c_2 \prod_{i=1}^{k-1} (e_{\delta_m - \varepsilon_1} e_{\delta_m + \varepsilon_i}) e_{\delta_m + \varepsilon_k} e_{\delta_m - \varepsilon_{k+1}} e_{\delta_m + \varepsilon_{k+1}} \prod_{i=k+2}^{n} (e_{\delta_m - \varepsilon_1} e_{\delta_m + \varepsilon_i}) f_{2\delta_m}^{N+n} u^+ = 0.
\]
The particular values of the scalars, $c_1$ and $c_2$, in (3.8) do not matter as each term is 0. This phenomenon will be observed throughout the proof. Finally, by $\mathfrak{sl}_2$ relations, we have that

$$e_{\varepsilon_n}u = c_3 \prod_{i=1}^{n-1} (e_{\delta_{m_i} - \varepsilon_i} e_{\delta_{m_i} + \varepsilon_i}) e_{\delta_{m_i} + \varepsilon_n} e_{\delta_{m_i}} f_{\delta_{m_i} + \varepsilon_n}^{N + n} v^+ = 0.$$  

This case of $\text{(2.2)}$ is similar to that of $\text{(2.1)}$ with the roles of the $\delta_i$ and $\varepsilon_k$ reversed. The one exception is that we will have to use a different method for $\delta_{m_i}$ from the one we used for $\varepsilon_n$, as $e_{\delta_{m_i}}$ does not commute with $f_{\varepsilon_n}$. Here, we use the commutation formulas of $\text{(2.2)}$. A tedious computation shows that

$$w := e_{\delta_{m_i}} e_{\varepsilon_{n-m}} e_{\varepsilon_n} + \delta_{m_i} f_{\delta_{m_i} + \varepsilon_n}^{N + 2m} v^+$$

(3.9)

$$= -A(N-2m-2)[a - b + (N-2m-3)/2] e_{-\varepsilon_n + \delta_{m_i}} f_{\varepsilon_n}^{N - 2m - 3} v^+,$$

where $A := \frac{1}{2}(N + 2m)(N + 2m - 1)$, $b := \langle \lambda - \rho - (N-2m-3)\varepsilon_n, \frac{1}{2}h\delta_{m_i} \rangle$, and

$$a := \langle \lambda - \rho - (N+2m-2)\varepsilon_n, \frac{1}{2}(h\varepsilon_n + h\delta_{m_i}) \rangle$$

(3.10)

$$= \frac{1}{2}[(\lambda, h\varepsilon_n) + (\lambda, h\delta_{m_i}) - (N-2m-2)(\varepsilon_n, h\varepsilon_n)] + 2\langle \lambda, h\delta_{m_i} \rangle$$

$$= b - (N + 2m - 3)/2,$$

so $w = 0$. Hence, we have that $e_{\delta_{m_i}}u = \prod_{i=1}^{m-n} (e_{\varepsilon_{n-m_i}} e_{\varepsilon_n} + \delta_{m_i}) w = 0$.

In the case of $\text{(2.3)}$ one argues as in the case of $\text{(2.1)}$ noting that the argument here for $\varepsilon_{n-1} + \varepsilon_n$ is similar to the one for $\varepsilon_i - \varepsilon_{i+1}$ in the case of $\text{(2.1)}$.

The case of $\text{(2.4)}$ is similar to that of $\text{(2.2)}$ except for the roots $2\delta_{m_i}$ and $\varepsilon_{n-1} - \varepsilon_n$. We have

$$e_{2\delta_{m_i}}u = c_4 \prod_{i=1}^{m-1} (e_{\varepsilon_{n-m_i}} e_{\varepsilon_n} + \delta_{m_i})^2 e_{\varepsilon_n + \delta_{m_i}} e_{\delta_{m_i}} f_{\delta_{m_i} + \varepsilon_n}^{N + 2m} v^+$$

(3.11)

$$+ c_5 \prod_{i=1}^{m-1} (e_{\varepsilon_{n-m_i}} e_{\varepsilon_n} + \delta_{m_i}) e_{\delta_{m_i}}^2 f_{\delta_{m_i} + \varepsilon_n}^{N + 2m} v^+ = 0,$$

(3.12)
In the case of \([2.3]\), we have that
\[
(3.13) \quad e_{\varepsilon_1-\varepsilon_2} u = c_\varepsilon e_{\varepsilon_1\varepsilon_2} e_{\varepsilon_3\varepsilon_4} e_{\varepsilon_5\varepsilon_6} f_{\delta}^N v^+ = 0.
\]
The case of \(e_{\varepsilon_2-\varepsilon_3}\) is entirely similar. We also have
\[
(3.14) \quad e_{\varepsilon_3} u = c_\varepsilon u + c_\varepsilon e_{\varepsilon_1\varepsilon_2} e_{\varepsilon_3\varepsilon_4} e_{\varepsilon_5\varepsilon_6} f_{\delta}^N v^+ = 0.
\]
and
\[
e_{\varepsilon_1-\varepsilon_2} u = e_{\varepsilon_1\varepsilon_2} e_{\varepsilon_3\varepsilon_4} e_{\varepsilon_5\varepsilon_6} f_{\delta}^N v^+ = 0.
\]
In the case of \([2.6]\) we have that
\[
(3.15) \quad e_{\varepsilon_1} u = c_\varepsilon e_{\varepsilon_1\varepsilon_2} e_{\varepsilon_3\varepsilon_4} e_{\varepsilon_5\varepsilon_6} f_{\delta}^N v^+ = 0.
\]
Similarly, we have that
\[
(3.16) \quad e_{\varepsilon_2-\varepsilon_1} u = c_\varepsilon e_{\varepsilon_1\varepsilon_2} e_{\varepsilon_3\varepsilon_4} e_{\varepsilon_5\varepsilon_6} f_{\delta}^N v^+ = 0.
\]
\[
(3.17) \quad e_{\varepsilon_2-\varepsilon_1} u = c_\varepsilon e_{\varepsilon_1\varepsilon_2} e_{\varepsilon_3\varepsilon_4} e_{\varepsilon_5\varepsilon_6} f_{\delta}^N v^+ = 0.
\]

Hence, the vectors \(u\) are singular.

\section{Homomorphisms between Verma Modules}

We now state the main theorem.

\begin{theorem}
Let \(\mu \in \mathfrak{h}^*\) be such that \(\langle \mu, h_\alpha \rangle \in \mathbb{Z}_{\geq 0}\), where \(\alpha\) is a positive even root. There exists a nonzero homomorphism \(M(s_\alpha \mu) \rightarrow M(\mu)\), in the following settings:

1. \(\alpha = 2\delta_p\), \(1 \leq p \leq m\) in the case of \([2.1]\).
2. \(\alpha = \varepsilon_q\), \(1 \leq q \leq n\) in the case of \([2.2]\).
3. \(\alpha = 2\delta_p\), \(1 \leq p \leq m\) in the case of \([2.3]\).
4. \(\alpha = \varepsilon_i + \varepsilon_j\), \(1 \leq i < j \leq n\) in the case of \([2.4]\).
5. \(\alpha = \delta\) in the case of \([2.5]\).
6. \(\alpha = 2\delta\) in the case of \([2.6]\).
\end{theorem}

\begin{proof}
Let \(C := \langle \mu, h_\alpha \rangle \in \mathbb{Z}_{> 0}\), the case of \(C = 0\) being trivial. We first consider \(\alpha = 2\delta_p\) in the case of \([2.1]\). Let \(\beta\) be a positive even root. Define a Shapovalov element, \(\theta_{\beta, C} \in U(\mathfrak{b}^-)^{-C\beta}\), to be an element such that \(\theta_{\beta, C} v^+ \in M(\nu)\) is singular for all \(v\) such that \(\langle \nu, h_\beta \rangle = C\). Note that this is less restrictive than some other definitions (see \[20\] \S 4.12 and \[21\] Chapter 9). The element of \(U(\mathfrak{n}^-)\) which \(\theta_{\beta, C}\) acts as on \(M(\mu)\) is written as \(\theta_{\beta, C}(\mu)\). Theorem \([3.3]\) shows the existence of a Shapovalov element \(\theta_{2\delta_m, C}\) for each positive integer \(C\).
\end{proof}
We will construct Shapovalov elements for the other $2\delta_i$ by downward induction on $i$, following the proof of [M, Theorem 9.2.6]. Musson’s argument itself follows a proof from [S] in the case of semisimple Lie algebras.

Suppose that we have Shapovalov elements $\theta_{2\delta_i,C}$ for a fixed $i$ and all $C \in \mathbb{Z}_{>0}$. Write $\kappa = \delta_{i-1} - \delta_i$, and $\mu = s_\kappa \nu$. Suppose that $\langle \mu, h_\kappa \rangle = p \in \mathbb{Z}_{>0}$ and that $\langle \mu, h_{2\delta_i} \rangle = C$. We will show that there is a unique $\theta \in U(n^-)^{-2C\delta_{i-1}}$ such that

\[(4.1) \quad e^{p+2C}_\kappa \theta_{2\delta_i,C}(\mu) = \theta e^p_\kappa .\]

Write $L_{e^-\kappa}$ and $R_{e^-\kappa}$ for the endomorphisms of $U(n^-)$ given by left and right multiplication by $e^-\kappa$. Because $\text{ad}_{e^-\kappa}$ is locally nilpotent on $U(n^-)$, there exists a $k \in \mathbb{Z}_{>0}$ such that for all $l \in \mathbb{Z}_{\geq 0}$

\[e^l_{-\kappa} \theta_{2\delta_i,C}(\mu) = L^l_{e^-\kappa} \theta_{2\delta_i,C}(\mu) = (R_{e^-\kappa} + \text{ad}_{e^-\kappa})^l \theta_{2\delta_i,C}(\mu) = \sum_{i=0}^l \binom{l}{i} R_{e^-\kappa}^l \theta_{2\delta_i,C}(\mu) = \sum_{i=0}^k \binom{l}{i} \text{ad}_{e^-\kappa}^l \theta_{2\delta_i,C}(\mu) e^{l-i}_{-\kappa} .\]

Hence, for sufficiently large $l$, $e^l_{-\kappa} \theta_{2\delta_i,C}(\mu) v^+ \in U(n^-) e^p_{-\kappa} v^+ = M(\nu)$. By $\mathfrak{sl}_2$ relations,

\[e_\kappa e^l_{-\kappa} \theta_{2\delta_i,C}(\mu) v^+ = l e^{l-1}_{-\kappa} (h_\kappa - l + 1) \theta_{2\delta_i,C}(\mu) v^+ = l(p + 2C - l) e^{l-1}_{-\kappa} \theta_{2\delta_i,C}(\mu) v^+ .\]

This implies that $e^{p+2C}_\kappa \theta_{2\delta_i,C}(\mu) v^+ \in M(\nu)$. Since $M(\nu)$ is a free $U(n^-)$ module, there is a $\theta$ such that (4.1) holds, and because $e^-\kappa$ is not a zero divisor in $U(n^-)$, that $\theta$ is unique. Also, note that $\theta e^p_{-\kappa} v^+$ is singular.

Because, in (4.1), $\theta_\kappa, C$ is polynomial in $\mu$, and $\nu$ and $\mu$ are linearly related, $\theta$ is polynomial in $\nu$ such that $\langle \nu, h_\kappa \rangle$ is a negative integer, and $\langle \nu, h_{2\delta_{i-1}} \rangle = C$. As such $\nu$ form a Zariski dense subset of $\nu$ such that $\langle \nu, h_{2\delta_{i-1}} \rangle = C$, the $\theta$ lift to a Shapovalov element $\theta_{2\delta_{i-1},C} \in U(b^-)$, i.e. $\theta_{2\delta_{i-1},C}(\nu) = \theta$. This completes the induction and the proof of the theorem in the case of (2).

The proof of (3) in the case of (2.3) is entirely similar, and thus will be skipped.

The proof of (2) in the case of (2.2) is by downward induction on the index of $\varepsilon_i$ ($\varepsilon_i$ playing the role that $2\delta_i$ played in the proof of (1)), the case $i = n$ following immediately from Theorem 3.3. Moreover, $\varepsilon_{i-1} - \varepsilon_i$ plays the role of $\kappa$ here.

Finally, the proof of (4) in the case of (2.4) is by downward induction on the sum of the indices of $\varepsilon_i + \varepsilon_j$, ($\varepsilon_i + \varepsilon_j$ playing the role that $2\delta_i$ played in the proof of (1)) the case of $i + j = 2n - 1$, following immediately from Theorem 3.3. In this case $\varepsilon_{i-1} - \varepsilon_i$ or $\varepsilon_{j-1} - \varepsilon_j$ plays the role of $\kappa$.

Finally, (5) and (6) follow immediately from Theorem 3.3. □
Remark 4.2. Combining Theorem 4.1 with [M, Theorem 9.2.6], we see that the statement of Theorem 4.1 holds in the cases of §2.4, §2.5 and §2.6 for $\alpha$ any positive even root. However, in the cases of §2.1, §2.2 and §2.3, there remain $W'$-orbits not covered by Theorem 4.1: $\alpha = \delta_i + \delta_j, 1 \leq i < j \leq m$ in the cases of §2.1 and §2.3 and $\alpha = \epsilon_k + \epsilon_l, 1 \leq k < l \leq n$ and $\alpha = 2\delta_p, 1 \leq p \leq m$ in the case of §2.2.

Remark 4.3. A homomorphisms whose existence is given by one of the singular vectors $u$ defined in §2.1 - 2.6 is not necessarily an embedding by the following counterexample. We are in the case of §2.2, $\text{osp}(3|2)$ in particular. A tedious computation using the commutation formulas of §2.2 shows that

$$u = -\frac{(N+1)(N+2)}{2}(\lambda - \rho - N\epsilon_n, \frac{1}{2}(h\epsilon_n + h\delta_m))f^{N}u^+ + Ne_{\delta_m - \epsilon_n}e^{-\delta_m}f^{N-1}u^+ - \frac{(N-1)N}{2}e_{\delta_m - \epsilon_n}e^{-\delta_m - \epsilon_n}f^{N-2}u^+.$$  

For $\lambda$ such that $\langle \lambda - \rho - N\epsilon_n, \frac{1}{2}(h\epsilon_n + h\delta_m) \rangle = 0$, left multiplication by $e_{\delta_m - \epsilon_n}$ annihilates $u$, so that a homomorphism taking a highest weight vector of $M(s_{\epsilon_n}\lambda)$ to $u$ in $M(\lambda)$ is not an embedding.

Remark 4.4. Retain the notation of Theorem 4.1. In contrast to the semisimple Lie algebra setting, we do not know whether or not $\dim(\text{Hom}(M(s_{\alpha\mu}), M(\mu))) = 1$.

Remark 4.5. After Weiqiang Wang announced the results of this paper in an AMS meeting in 2017, Vera Serganova outlined to him an alternative approach of constructing singular vectors of a different form from ours using odd reflections. However, one would need to check, as we did for our candidate singular vectors in this paper, whether her candidate singular vectors are nonzero. It should also be noted that Serganova’s use of odd reflections resembles the approach of Musson’s [M, Corollary 9.3.6] super generalization of Verma’s theorem in the case of Verma modules of typical highest weight.

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