Excision boundary conditions for the conformal metric

Gregory B. Cook
Department of Physics, Wake Forest University, Winston-Salem, North Carolina 27109

Thomas W. Baumgarte
Department of Physics and Astronomy, Bowdoin College, Brunswick, Maine 04011
(Dated: October 24, 2008)

Shibata, Uryū and Friedman recently suggested a new decomposition of Einstein’s equations that is useful for constructing initial data. In contrast to previous decompositions, the conformal metric is no longer treated as a freely-specifiable variable, but rather is determined as a solution to the field equations. The new set of freely-specifiable variables includes only time-derivatives of metric quantities, which makes this decomposition very attractive for the construction of quasiequilibrium solutions. To date, this new formalism has only been used for binary neutron stars. Applications involving black holes require new boundary conditions for the conformal metric on the domain boundaries. In this paper we demonstrate how these boundary conditions follow naturally from the conformal geometry of the boundary surfaces and the inherent gauge freedom of the conformal metric.

PACS numbers: 04.20.-q,04.20.Cv,04.25.dg,04.25.D-

I. INTRODUCTION

Most numerical relativity applications solve Einstein’s field equations with the help of a 3+1 decomposition, which slices the spacetime into a foliation of spatial hypersurfaces and splits the equations into a set of constraint equations and a set of evolution equations [1, 2]. The constraint equations restrict the geometry of each hypersurface, representing an instant of constant coordinate time, while the evolution equations determine how the geometry changes from one hypersurface to the next. In its simplest form, constructing initial data for an evolution calculation therefore requires finding a solution to the constraint equations.

The Einstein constraints form a set of four equations – one in the Hamiltonian constraint and three in the momentum (or vector) constraint – and therefore can determine only four of the initial-data variables; the remaining variables are freely specifiable and have to be chosen independently before the constraint equations can be solved. A decomposition of the constraint equations separates the freely specifiable variables from the constrained ones. Given a particular decomposition, the construction of the initial data then entails making well-motivated choices for the freely-specifiable variables that encode the physical characteristics of the system one wishes to model.

Most decompositions of the constraint equations conformally decompose the spatial metric (see Sec. II A for details) and treat the conformally related metric as a freely-specifiable variable, meaning that this “background” geometry can be chosen arbitrarily. In most applications, the conformally related metric is then chosen to be flat. This choice simplifies the equations dramatically, and while it may encode a certain amount of physically unrealistic gravitational radiation in the initial data – which often manifests itself as “junk radiation” when the data are evolved – it is rarely clear a priori how to make a better choice.

An attractive alternative was recently proposed by Shibata, Uryū and Friedman (3; hereafter SUF). In their new initial-data decomposition, the conformal metric is no longer treated as a freely specifiable variable, and is instead determined during the solution of the equations. Aside from the trace of the extrinsic curvature, the new set of freely specifiable variables includes only time derivatives of the metric and extrinsic curvature. For the construction of equilibrium or quasiequilibrium data it is much more natural to specify the time derivative of metric quantities rather than the quantities themselves, which makes this new decomposition very appealing.

So far, this formalism has been used only for binary neutron stars [4]. When solving the constraint equations for black holes, the black hole interior is often excited to avoid singularities, which requires suitable boundary conditions on the black hole horizons (see 5, 6, 7, 8, 9, 10, 11). In the context of constrained evolution, excision boundary conditions for all variables, including the spatial metric, were explored in Ref. 12. Because the spatial metric is evolved in this case, and the spacelike boundaries used have no incoming characteristics, no boundary conditions on the spatial metric were needed. However, since the new SUF formalism treats the spatial conformal metric as a constrained variable rather than as either a freely-specifiable metric or an evolved quantity – and hence as a solution of an elliptic equation – we need to provide suitable boundary conditions for the conformal metric on black-hole excision boundaries before this formalism can be applied to black...
holes. In this paper we demonstrate how these boundary conditions can be formulated quite naturally in terms of the conformal geometry of the excision boundaries and the inherent gauge freedom in the conformally related metric.

We found it useful to develop this formalism in terms of a reference metric approach. While this approach is not new to general relativity, it has not been used widely within the numerical relativity community. We will therefore present a full description of the reference metric approach and write all of the Einstein evolution and constraint equations in terms of this formalism.

Finally, we must keep in mind that for coupled nonlinear elliptic equations, it is not always clear whether or not a given set of boundary conditions are independent and lead to a well-posed elliptic system. In Refs. [9] and [11], two different physically motivated boundary conditions were developed for the lapse for use with the extended conformal thin-sandwich equations (see Sec. II B). When combined with the boundary conditions for the remaining elliptic variables, however, neither condition gave rise to an independent set of boundary conditions, and hence left the system degenerate (see Refs. [6] and [8], two different physically motivated boundary conditions for the conformal metric, we have not yet shown that they lead to a well-posed elliptic system.

This article is organized as follows. We provide an overview of the problem in Sec. II and derive the reference metric approach in Sec. III. We then derive the boundary conditions on the conformally related metric in Sec. IV and briefly summarize in Sec. V. We also include a complete list of Einstein’s equations in the reference metric approach in App. A.

II. OVERVIEW

A. 3 + 1 and conformal decompositions

We begin with the spacetime metric written in the general 3 + 1 form

$$ds^2 = -d\tau^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

which treats the full spacetime as a foliation of spacelike slices with timelike unit normal vector $n^\mu$. In Eq. (1), $\alpha$ is the scalar lapse of time, $\beta^i$ is the shift vector, and $\gamma_{ij}$ is the metric of a spatial slice. In terms of this decomposition, the vacuum Einstein equations are written as

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha \left[ R_{ij} - 2K_{ik}K^k + KK_{ij} \right] + \beta^l \nabla_l K_{ij} + 2K_{ikj} \beta^k,$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + 2\nabla_i(\beta_j)$$

$$0 = \bar{R} + K^2 - K_{ij}K^{ij}$$

$$0 = \nabla_j(K^{ij} - \gamma^{ij}K)$$

Equations (1) and (2) are, respectively, the Hamiltonian and momentum constraint equations, and Eq. (2) is the evolution equation for the extrinsic curvature $K_{ij}$ which is itself defined by Eq. (3). The covariant derivative compatible with the spatial metric $\gamma_{ij}$ is written as $\nabla_i$, the Ricci tensor and scalar curvature as $\bar{R}$, the Ricci tensor and scalar curvature as $\bar{R}_{ij}$ and $\bar{R}$, and $\partial_t$ is the derivative along the time vector

$$t^\mu = \alpha n^\mu + \beta^\mu.$$  

Finally, it is often convenient to decompose the extrinsic curvature into its trace $K \equiv \gamma^{ij}K_{ij}$ and trace-free parts

$$A_{ij} \equiv K_{ij} - \frac{1}{3} \gamma_{ij}K.$$  

It is also useful to conformally decompose the spatial metric and other quantities. In particular, we will make use of the conformal factor $\psi$ which allows us to define the conformal metric $\bar{\gamma}_{ij}$, conformal trace-free extrinsic curvature $A_{ij}$ and conformal lapse $\bar{\alpha}$ via

$$\bar{\gamma}_{ij} \equiv \psi^4 \gamma_{ij}$$

$$A_{ij} \equiv \psi^{-2} A_{ij}$$

$$\bar{\alpha} \equiv \psi^6 \alpha.$$  

In terms of these conformal variables and the covariant derivative $\nabla_i$, Ricci tensor $\bar{R}_{ij}$ and scalar curvature $\bar{R}$ compatible with the conformal metric $\bar{\gamma}_{ij}$, we can write the various constraint and evolution equations as

$$0 = \nabla^j \psi - \frac{1}{\psi} \bar{R}_{ij} - \frac{1}{2} \bar{\gamma}_{ij} \bar{R} + \frac{1}{6} \bar{\psi}^{-1} A_{ij}$$

$$0 = \nabla_j \bar{A}_{ij} - \frac{4}{3} \psi^6 \bar{\gamma}^{ij} \nabla_i \bar{K}$$

$$\partial_t \bar{\gamma}_{ij} = -2 \dot{\bar{\alpha}} \bar{A}_{ij} + (\bar{K}[\psi^{-4} \beta])_{ij}$$

$$\partial_t \bar{A}_{ij} = \psi^8 \bar{\alpha} (\bar{R}_{ij} - \frac{1}{3} \bar{\gamma}^{ij} \bar{R})$$

$$- \left[ \nabla^i \nabla_j - \frac{1}{3} \gamma^{ij} \nabla^2 \right] (\psi^8 \bar{\alpha})$$

$$+ 8 \psi^8 \bar{\alpha} \left[ \psi (\psi^4 \bar{\gamma}^i) \nabla_k \ln(\psi^7 \bar{\alpha}) \right]^{-1} \left[ \nabla^k (\bar{\psi} \ln(\psi^7 \bar{\alpha})) \right]$$

$$+ \frac{\beta^k}{\bar{\psi}^6} \nabla_k \bar{A}_{ij} + \bar{A}_{ij} \nabla_k \beta^k + \bar{A}_{ij} \bar{\gamma}^{ij} \beta^k$$

$$- \bar{A}_{ij} \dot{\bar{\gamma}} \bar{\gamma}^{ij}$$

$$\partial_t \bar{K} = -\bar{\psi}^{-7} \nabla^2 (\psi^4 \bar{\alpha}) + \frac{1}{6} \psi \bar{\alpha} \bar{R}_{ij} + \frac{1}{6} \bar{\psi}^6 \bar{\alpha} K^2$$

$$+ \frac{7}{8} \bar{\psi}^{-6} \bar{\alpha} \bar{A}_{ij} \bar{A}_{ij} + 3 \beta^k \nabla_k \bar{K}$$

$$\partial_t \bar{\psi} = \frac{3}{8} \psi \bar{\psi} \bar{\alpha} \bar{K}$$

Here the conformal longitudinal derivative of a 1-form $(\bar{L})_{ij}$ is defined by

$$(\bar{L}V)_{ij} = 2\nabla_i V_j - \frac{2}{3} \gamma_{ij} \gamma^{kl} \nabla_k V_l,$$

and we note that the term $\psi^{-4}$ appears in its argument in Eq. (13) because the index of the shift $\beta^i$ has to be lowered with the physical spatial metric $\gamma_{ij}$ rather than with the conformal metric $\bar{\gamma}_{ij}$.
B. Current initial data formulations

The most commonly used decompositions of the constraint equations are the conformal transverse-traceless and the conformal thin-sandwich decompositions (see, e.g., [14, 15, 16] for reviews).

The conformal transverse-traceless decomposition solves the Hamiltonian and momentum constraints, Eqs. (11) and (12), and fixes the conformal metric $\tilde{\gamma}_{ij}$ as well as the mean curvature $K$ and the transverse-traceless parts of $A_{ij}$ as the freely-specifiable variables. If the conformal metric is chosen to be flat, $\gamma_{ij} = f_{ij}$, and the initial hypersurface to be maximal, so that $K = 0$, the momentum constraints decouple from the Hamiltonian constraint and become linear. Vacuum solutions describing one or multiple black holes, known as Bowen-York solutions [17], can then be found analytically and form the basis for both puncture [18, 19, 20] and conformal-imaging solutions [17, 21] to the Hamiltonian constraint.

In the conformal thin-sandwich decomposition [22] the evolution equation for the conformal metric, Eq. (13), is used to replace the transverse-traceless parts of $A_{ij}$ with the time derivative of the conformally related metric in the set of freely-specifiable variables. Effectively, we can solve Eq. (13) for $A_{ij}$ and then insert this equation into every occurrence of $A_{ij}$ in the other equations. We now fix the trace of the extrinsic curvature $K$, the conformal metric $\tilde{\gamma}_{ij}$, its time derivative $\partial_t \tilde{\gamma}_{ij}$, as well as the conformal lapse $\tilde{\alpha}$ as the freely specifiable data and solve the Hamiltonian constraint (I1) for the conformal factor $\psi$ and the momentum constraint (I2), via (I3), for the shift $\beta^i$.

The extended conformal thin-sandwich decomposition [23] incorporates the time derivative of the trace of the extrinsic curvature, Eq. (15). If we take $\partial_t K$ as specified, this equation yields an elliptic equation for the conformal lapse. Essentially, we now replace the conformal lapse with $\partial_t \tilde{\alpha}$ in the subset of the freely specifiable variables, and we obtain one additional elliptic equation which must now be solved for the conformal lapse. While this change of perspective results in an additional elliptic equation which must be solved in order to construct initial data, this added burden is offset by the fact that the set of freely specifiable data now takes on a particularly attractive form. Specifically, we now specify the conformal metric and its time derivative, along with the trace of the extrinsic curvature and its time derivative. In many situations, in particular as mentioned before for the construction of equilibrium or quasiequilibrium initial data, it is advantageous to specify the time derivative of some quantity rather than some other field. (See [24, 25, 26, 27] for a discussion of the uniqueness issue in the extended conformal thin-sandwich formalism.)

C. Determining the conformal metric

In all of the initial-data methods sketched out in Sec. II B the conformal metric is taken as part of the subset of freely-specifiable initial data. Roughly speaking, the five degrees of freedom that are fixed by specifying the conformal metric include the initial choice of the spatial gauge and two dynamical degrees of freedom. Therefore, fixing the conformal metric strongly affects the initial gravitational radiation content of the initial data.

In most cases, the conformal metric is chosen to be flat. But even when it is not chosen to be flat, in all but the most trivial cases, the chosen metric is not completely compatible with the physics that one wishes to build into the initial data. The result is that undesired “junk” gravitational radiation is built into the initial data. While this defect in the data is usually small and has little effect on the gross physics one wishes to simulate, it can have a significant effect on detailed comparisons with post-Newtonian methods [28], and may impact future parameter estimation efforts.

It is desirable to find a way to construct initial data in which the conformal metric is not fixed a priori, but is constructed in a way that is consistent with the physics one wishes to simulate and that eliminates, or at least reduces, the junk radiation. Following in the spirit of the extended conformal-thin-sandwich approach, SUF noticed that the equation for the time derivative of the trace-free part of the extrinsic curvature, Eq. (14), can be written as an elliptic equation for the conformal metric if the spatial gauge is imposed in a suitable way and if we take $\partial_t A^i$ as freely specified data. In close analogy to going from the conformal transverse-traceless to the extended conformal thin-sandwich decomposition, namely using the evolution equation for the conformal metric to replace $A_{ij}$ with $\partial_t \tilde{\gamma}_{ij}$, and the evolution equation for the trace of the extrinsic curvature to replace $\tilde{\alpha}$ with $\partial_t K$ as freely specifiable variables, we now use the evolution equation for the extrinsic curvature to replace $\tilde{\gamma}_{ij}$ with $\partial_t A_{ij}$. This formalism is very attractive since it is again more natural to make choices for time derivatives of functions than for the functions themselves.

To date, this approach for constructing initial data has been implemented successfully only for binary neutron stars [4]. The main goal of this paper is to develop the formalism necessary for applications to black-hole initial data where the black-hole interiors are excised from the computational domain. When black-hole initial data are computed using excision methods, boundary conditions for all of the initial data that are determined by elliptic equations must be applied on the excision boundaries. For the case of the extended conformal-thin-sandwich initial data, the boundary conditions have been worked out and thoroughly tested [2, 6, 7, 8]. In the context of the new formalism of SUF, however, additional boundary conditions are required for the conformal metric.
III. REFERENCE METRIC APPROACH

Our goal is to derive a system of equations that will determine the conformal metric together with some other quantities. Many of the operators that need to be inverted to construct the conformal metric depend on the conformal metric themselves. It is therefore convenient (though certainly not necessary) to use a reference metric approach wherein, in addition to the conformal metric $\tilde{g}_{ij}$, we also associate some appropriate fixed metric with our solution manifold. We can then formulate the operators in terms of this fixed metric, which greatly simplifies the inversion of the operators. This approach is not new (cf Refs. [29, 30]), but is not widely used within the numerical relativity community. A notable exception is [31], whose formalism shares many elements with ours. We will give a basic outline of the approach in this Section, and list the complete set of Einstein’s equations in Appendix A.

A. Basic outline

We assume that our initial data hypersurface is represented at a basic level by a manifold with coordinates and coordinate maps defined everywhere. We associate with this manifold two metrics which are not necessarily the same: $g_{ij}$ and $\hat{g}_{ij}$. Each metric has an inverse and covariant derivative such that

$$g_{jk}g^{ik} = \delta^i_j \quad \text{and} \quad \hat{g}_{jk}\hat{g}^{ik} = \delta^i_j \quad (18)$$

$$\nabla_k g_{ij} = 0 \quad \text{and} \quad \nabla_k \hat{g}_{ij} = 0. \quad (19)$$

The difference between two connections is a tensor which can be written as

$$\delta \Gamma^k_{ij} = \Gamma^k_{ij} - \hat{\Gamma}^k_{ij} \quad (20)$$

$$= \frac{1}{2}g^{kl} \left[ \nabla_l g_{ij} + \nabla_j g_{il} - \nabla_i g_{lj} \right].$$

The differences between the Riemann and Ricci tensors for each metric can be written as

$$\delta R^\ell_{ijk} \equiv R^\ell_{ijk} - \hat{R}^\ell_{ijk} \quad (21)$$

$$= \nabla_j \delta \Gamma^\ell_{ki} - \nabla_i \delta \Gamma^\ell_{kj} + \delta \Gamma^m_{ki} \delta \Gamma^\ell_{mj} - \delta \Gamma^m_{kj} \delta \Gamma^\ell_{mi}$$

$$\delta R_{ij} \equiv R_{ij} - \hat{R}_{ij} \quad (22)$$

$$= \nabla_j \delta \Gamma^\ell_{ji} - \nabla_i \delta \Gamma^\ell_{j} + \delta \Gamma^m_{ji} \delta \Gamma^\ell_{mi} - \delta \Gamma^m_{ji} \delta \Gamma^\ell_{mj}.$$

The latter can be rewritten as

$$\delta R_{ij} = -\dot{R}_{ij} - g^{km}g_{kl}\ddot{R}_{jmk} - \frac{1}{2}g^{km}\nabla_l \nabla_m g_{ij} \quad (23)$$

$$\quad + \nabla_i (\hat{g}_{jk}g^{lm}\delta \Gamma_{lm}^{kj}) - g_{pk}g^{lm}\delta \Gamma_{lm}^{kp}$$

$$\quad + \frac{1}{2}g^{lp}g^{mp}\left\{ (\nabla_i g_{np})\nabla_l g_{jm} + (\nabla_j g_{np})\nabla_l g_{in} + (\nabla_l g_{in})\nabla_m g_{jp} - (\nabla_l g_{jm})\nabla_p g_{in} \right\}$$

$$\quad + \frac{1}{2}\nabla_i (\nabla_l g_{mp})\nabla_j g_{mn}. \quad (24)$$

As it turns out, this form is particularly useful because all second derivatives of the $g_{ij}$ have now been absorbed into only two terms. The first of these two terms, $g^{lm}\nabla_l \nabla_m g_{ij}$, forms an elliptic operator acting on $g_{ij}$ as long as both $g_{ij}$ and $\hat{g}_{ij}$ are sufficiently well behaved, while the second of the two terms, $\nabla_i (\hat{g}_{jk}g^{lm}\delta \Gamma_{lm}^{kj})$, can be eliminated by virtue of a suitable gauge choice, as we will discuss in the next section.

B. Reference metric and gauge choice

We will assume that we are dealing with one or more black holes in an asymptotically flat initial-data hypersurface. In this case, the solution domain is topologically $E^4$ with a “ball” cut out for each excised black hole interior. It is therefore appropriate to take the reference metric $\hat{g}_{ij}$ to be a flat metric $\delta_{ij}$ (not necessarily in Cartesian coordinate). For consistency with our previous notation, the “other” metric $g_{ij}$ will be the conformal metric we wish to determine, $\gamma_{ij}$. To make the notation as clear as possible, we will denote the covariant derivative compatible with the flat reference metric as $\nabla_k$, and so the difference of connections becomes

$$\delta \Gamma^k_{ij} \equiv \hat{\Gamma}^k_{ij} - \Gamma^k_{ij} \quad (24)$$

$$= \frac{1}{2}\gamma^{kl} \left[ \nabla_l \gamma_{ij} + \nabla_j \gamma_{il} - \nabla_i \gamma_{lj} \right].$$

We next impose a spatial gauge condition by setting the contractions of the connection coefficients equal to some predetermined gauge source functions $V^k$, such as $\gamma_{ij}$. This is possible because, as will be shown in Appendix A, the determinant of the conformal metric as either $\gamma_{ij}$ or $\hat{\gamma}_{ij}$. One possible choice is $V^k = 0$, in which case we would obtain a “spatial conformal harmonic gauge”, but we will leave $V^k$ arbitrary for generality (compare also the “generalized Dirac gauge” of [31]). With the gauge source functions $V^k$ given as specified functions of the coordinates, we see that the only remaining second-order term in Eq. (23) is the first one, bringing the conformal Ricci tensor into a familiar elliptic form. This or similar properties of the Ricci tensor have often been utilized before, both in a four-dimensional...
(e.g. [32, 33, 34, 35, 36, 37, 38]) and a three-dimensional context (e.g. [30, 31, 33, 39, 40]). An attractive feature of the reference metric approach is the fact that differences of connection coefficients are tensors, so that the gauge source functions $V^k$ become vectors and take a gauge-invariant meaning.

In our case, the conformal Ricci tensor and conformal Ricci scalar take the form

$$\bar{R}^{ij} = \frac{1}{2}\Delta \gamma^{ij} + \bar{\nabla}^i \bar{V}^j - \frac{1}{2} \bar{B}^{ij} - \frac{1}{4} \bar{C}^{ij} - \frac{1}{4} \bar{D}^{ij} + \bar{E}^{ij}, \quad (26)$$

$$\bar{R} = -\frac{1}{2}\Delta \ln(\det(\gamma)) + \bar{\nabla}_k V^k + \frac{1}{2} \bar{\nabla}^k \left[ \bar{C}^{kl} - \frac{1}{2} \bar{B}^{kl} \right]$$

(compare Eqs. (44) – (47) in [31]). Here $\Delta$ is a generalized Laplacian defined by

$$\Delta \equiv \bar{\nabla}^k \bar{\nabla}_k. \quad (28)$$

and the symmetric tensors $\bar{B}^{ij}, \bar{C}^{ij}, \bar{D}^{ij},$ and $\bar{E}^{ij}$ are quadratic combinations of first-derivatives of the conformal metric defined by

$$\bar{B}^{ij} \equiv \bar{\nabla}_m \bar{\nabla}_n \bar{\gamma}^{ij} = \bar{\nabla}^i \bar{\nabla}^j, \quad (29)$$

$$\bar{C}^{ij} \equiv \left( \bar{\nabla}_k \bar{\gamma}^{ij} \right) \bar{\nabla}_k = \bar{C}^{ij}, \quad (30)$$

$$\bar{D}^{ij} \equiv \bar{\nabla}_m \bar{\nabla}_n \bar{\gamma}^{ij} = \bar{\nabla}^i \bar{\nabla}^j = \bar{D}^{ij}, \quad (31)$$

$$\bar{E}^{ij} \equiv \bar{\nabla}_m \bar{\nabla}_n \bar{\gamma}^{ij} = \bar{\nabla}^i \bar{\nabla}^j = \bar{E}^{ij}. \quad (32)$$

We note that the derivatives of the gauge source functions have been written in terms of the conformal covariant derivative $\bar{\nabla}_k$, and not the flat covariant derivative $\nabla_k$. Also, in the future, we plan to solve directly for the inverse conformal metric $\tilde{\gamma}^{ij}$ and so the conformal Ricci tensor and the various derivatives of the metric are written with the indices raised (compare [31]). We list the complete set of Einstein’s equations in the reference metric form in Appendix [A].

IV. BOUNDARY CONDITIONS

Our goal in this section is to determine the proper boundary conditions for the inverse conformal metric $\tilde{\gamma}^{ij}$ required on the excision boundaries when we solve an elliptic equation for this variable. We begin with some simple counting arguments.

The conformal metric $\tilde{\gamma}^{ij}$ is a member of a conformal equivalence class of metrics which each have five independent degrees of freedom. To fix upon a particular member of this class, we can fix the determinant of the conformal metric $\det(\tilde{\gamma})$. As mentioned previously, these five degrees of freedom are usually thought of as containing the two dynamical degrees of freedom and the three spatial gauge degrees of freedom in the metric. Of course, this splitting is not clean and, in general, they mix with the sixth constrained (or longitudinal) degree of freedom of the full spatial metric $\tilde{\gamma}_{ij}$. However, this rough splitting suggests how we should fix the boundary conditions.

First, the excision surface is a closed 2-surface with $S^2$ topology. The metric on any such surface is conformally equivalent to the unit sphere. Since the metric induced on the excision surface by the physical metric $\gamma_{ij}$ will be conformally equivalent to a spherical metric, it seems that we should be able to demand that the metric induced on the excision surface by the conformal metric $\tilde{\gamma}_{ij}$ be a spherical metric. In fact, this can be generalized to the statement that we have the freedom to fix the metric induced on the excision surface by the conformal metric $\tilde{\gamma}_{ij}$ to that of any metric with $S^2$ topology. A 2-metric has three degrees of freedom, but because of the conformal equivalence of all 2-metrics, specifying the induced metric really only fixes two degrees of freedom.

Having fixed two of the five degrees of freedom of $\tilde{\gamma}_{ij}$ on the excision surface leaves three more boundary conditions that must be specified. Since the conformal metric contains three gauge degrees of freedom, it seems natural that these three remaining boundary conditions should result from the gauge conditions [25].

A. 2 + 1 decomposition

To rigorously define and clearly understand the new boundary conditions, it is most convenient to rewrite the conformal metric in terms of a 2 + 1 decomposition adapted to the excision surface. We define on the solution manifold a foliation of concentric surfaces with topology $S^2$ in the neighborhood of the excision boundary. Each surface is defined as a level surface of a scalar function $r(x^i)$ such that $r = r_0$ defines the excision surface. Independently of which metric we are using, we have a normal 1-form for the excision surface given by

$$\partial_i r(x^j)|_{r_0}. \quad (33)$$

In terms of each metric ($\tilde{\gamma}_{ij}$ and $f_{ij}$) we can normalize this 1-form and define a unit-normal vector and unit-normal 1-form. In terms of the unit-normals to the excision surface, we can then define projection operators associated with each metric and a corresponding induced metric on the surface defined by $r = r_0$. For the conformal metric $\tilde{\gamma}_{ij}$, we will denote its induced metric on the excision surface as $\tilde{h}_{AB}$, and for the flat metric $f_{ij}$ we will denote the induced metric as $S_{AB}$. Here and in the following, upper-case Latin indices denote adapted coordinates in the excision surface.

1. Conformal metric

Focusing first on the 2+1 decomposition of the conformal metric we write the three-dimensional line interval as

$$\tilde{ds}^2 = \Lambda^2 dr^2 + \tilde{h}_{AB}(dx^A + \lambda^A dr)(dx^B + \lambda^B dr), \quad (34)$$
and note that the determinant is given by
\[ \det \tilde{\gamma} = \Lambda^2 \det \hat{h}. \]  
(35)

Evidently, \( \Lambda \) and \( \Lambda^A \) play the role of the lapse \( \alpha \) and shift \( \beta^i \) in the more familiar 3+1 decompositions. In “matrix” form we write the conformal metric as
\[ \tilde{\gamma}_{ij} = \begin{bmatrix} \Lambda^2 & \Lambda \gamma_{iC} \alpha^C & \Lambda \gamma_{iB} \\ \Lambda \gamma_{iC} \alpha^C & \lambda_A & \lambda_B \\ \Lambda \gamma_{iB} & \lambda_B & \hat{h}_{AB} \end{bmatrix}, \]  
(36)
where \( \lambda_A \equiv \hat{h}_{AB} \lambda^B \), and its inverse as
\[ \tilde{\gamma}^{ij} = \begin{bmatrix} \Lambda^{-2} & -\Lambda^{-2} \lambda^B \\ -\Lambda^{-2} \lambda^A & \hat{h}^{AB} + \Lambda^{-2} \lambda^A \lambda^B \end{bmatrix}. \]  
(37)

The unit-normal 1-form is given by
\[ \hat{s}_i \equiv \Lambda \partial_i r(x^i)|_{r_0} = \Lambda[1, \vec{0}], \]  
(38)
and the unit-normal vector by
\[ \vec{s}^i = \tilde{\gamma}^{ij} \hat{s}_j = \Lambda^{-1}[1, -\lambda^A]. \]  
(39)
Similarly, the “radial” vector is given by
\[ r^i = \Lambda \vec{s}^i + \lambda^i = [1, \vec{0}], \]  
(40)
where the 3-vector \( \lambda^i \equiv [0, \lambda^A] \).

The induced metric on the excision surface is given by
\[ \tilde{h}_{ij} = \tilde{\gamma}_{ij} - \hat{s}_i \hat{s}_j, \]  
(41)
and we define the extrinsic curvature of the excision surface as
\[ \tilde{H}_{ij} \equiv \tilde{h}_k^{ij} \tilde{h}^k_{\ell} \tilde{\nabla}_{\ell} \hat{h}. \]  
(42)

The covariant derivative compatible with the induced metric \( \hat{h}_{AB} \) will be denoted \( \tilde{D}_C \). Because we are using adapted coordinates, the Lie derivative in the radial direction \( \mathcal{L}_r \) is equivalent to the partial derivative in the radial direction,
\[ \mathcal{L}_r \equiv \frac{\partial}{\partial r} = r^i \partial_i. \]  
(43)

Finally, we note that
\[ \partial_r \hat{h}_{AB} = 2 \Lambda \tilde{H}_{AB} + \tilde{D}_A \lambda_B + \tilde{D}_B \lambda_A, \]  
(44)
which follows from the definition of the extrinsic curvature.

Since our gauge conditions \( \Box = 0 \) are defined in terms of the difference between the conformal and flat connections, it is useful to list the connections associated with the conformal metric in terms of the 2 + 1 variables.

\[ \tilde{\Gamma}^{rr} = \partial_r \ln \Lambda + \Lambda^A \tilde{D}_A \ln \Lambda - \frac{1}{\Lambda} \Lambda^A \lambda^B \hat{h}_{AB}, \]  
(45a)
\[ \tilde{\Gamma}^{rA} = \tilde{D}_A \ln \Lambda - \frac{1}{\Lambda} \Lambda^B \hat{h}_{AB}, \]  
(45b)
\[ \tilde{\Gamma}^{AB} = -\frac{1}{\Lambda} \hat{h}_{AB}, \]  
(45c)
\[ \tilde{\Gamma}^{A}_r = \partial_r \lambda^A - \Lambda^2 \tilde{D}^A \ln \Lambda + 2 \Lambda \hat{h}^{AB} \tilde{D}_B \lambda_A + \lambda^B \tilde{D}_B \lambda^A - \lambda^A \partial_r \hat{h}_{AB}, \]  
(45d)
\[ \tilde{\Gamma}^{A}_B = -\lambda^A \tilde{\Gamma}^{AB} + \hat{h}^{AC} \tilde{D}_B \lambda_C, \]  
(45e)
\[ \tilde{\Gamma}^{A}_{BC} = -\lambda^A \tilde{\Gamma}^{BC} + \hat{\Gamma}^{A}_{BC}, \]  
(45f)
where \( \hat{\Gamma}^{A}_{BC} \) is the usual connection constructed from the induced conformal metric \( \hat{h}_{AB} \).

2. Flat metric

The 2 + 1 decomposition of the flat metric \( f_{ij} \) follows precisely the same form as that for the conformal metric as outlined above in Sec. [V A]. We simply need to define new variables for the analogs of \( \Lambda, \lambda^A \), and \( \hat{h}_{AB} \). While it is possible to proceed in full generality, we will restrict ourselves to the case where we take the flat metric analog of \( \Lambda \to 1 \) and \( \lambda^A \to 0 \) so that we have
\[ f_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & S_{AB} \end{bmatrix} \]  
(46)
and
\[ f^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & S^{AB} \end{bmatrix}. \]  
(47)

Specifying the induced metric \( S_{AB} \) is sufficient to fix the coordinate transformations between the coordinates on the 3-dimensional initial-data manifold and the 2 + 1 adapted coordinates. For example, the simplest choice is to let \( S_{AB} \) be the standard metric for a sphere of radius \( r \). Then, if the initial-data manifold uses standard Cartesian coordinates, the coordinate transformations are well known.

To proceed, we will need to define the covariant derivative compatible with \( S_{AB} \). We will denote this derivative as \( \nabla_A \). Finally, we can write the connections associated with our flat metric as
\[ \tilde{\Gamma}^{r}_r = 0, \]  
(48a)
\[ \tilde{\Gamma}^{r}_A = 0, \]  
(48b)
\[ \tilde{\Gamma}^{r}_{AB} = -\frac{1}{2} \partial_r S_{AB}, \]  
(48c)
\[ \tilde{\Gamma}^{A}_r = 0, \]  
(48d)
\[ \tilde{\Gamma}^{A}_B = \frac{1}{2} S^{AC} \partial_r S_{BC}, \]  
(48e)
\[ \tilde{\Gamma}^{A}_{BC} = \tilde{\Gamma}^{A}_{BC}, \]  
(48f)
where \( \tilde{\Gamma}^{A}_{BC} \) is the usual connection constructed from the induced flat metric \( S_{AB} \). Note that Eqs. [48c] and [48e] follow from Eqs. [45c] and [45e] by using Eq. [44].

B. Gauge boundary conditions

All the pieces are now in place to fully understand and rigorously define excision boundary conditions that can be used for the conformal metric. Recall from our discussion at the beginning of Sec. [IV] that to fix upon a unique member from the conformal equivalence class of conformal metrics \( \tilde{\gamma}_{ij} \), we may specify its determinant \( \det(\tilde{\gamma}) \) everywhere in the initial-data manifold. We choose to
demand that the determinants of the conformal metric and the reference metric are equal. In this paper, we are restricting ourselves to the case that the reference metric is flat, so we demand

$$\det \tilde{\gamma} = \det f.$$  \hspace{1cm} (49)

We also point the reader to Sec. II.B of Ref. [31] for a more in-depth discussion of this point.

Also recall that $\hat{h}_{AB}$ and $S_{AB}$, each having topology $S^2$, are both related to the standard metric of a unit-sphere by a conformal transformation and an appropriate coordinate transformation. Now, since we are expressing both $\hat{h}_{AB}$ and $S_{AB}$ in the same coordinates, we can demand that they are related to each other simply by a conformal transformation:

$$\hat{h}_{AB} = \Omega^2 (x^i) S_{AB}.$$  \hspace{1cm} (50)

It follows immediately from Eq. (35), its analog for the induced flat metric that $\det(f) = \det(S)$, and Eq. (50) that

$$\Omega^2 = \frac{1}{\Lambda}.$$  \hspace{1cm} (51)

Notice that because of Eqs. (50) and (51), the 2 + 1 version of the conformal metric can be expressed in terms of $\Lambda$, $\lambda^A$, and $S_{AB}$:

$$\tilde{d}^2 = \Lambda^2 r^2 + \frac{1}{\Lambda} S_{AB} (dx^A + \lambda^A dr)(dx^B + \lambda^B dr).$$  \hspace{1cm} (52)

These variables hold six degrees of freedom. But recall that they incorporate the restriction on the conformal equivalence class given in Eq. (19), so that there are really only five degrees of freedom being fixed by the choice of these variables.

Because all 2-metrics are conformally equivalent, specifying $S_{AB}$ can now clearly be seen as simply specifying the coordinates on the excision surface. The variables $\Lambda$ and $\lambda^A$ then determine how these coordinates propagate off of this boundary surface into the full initial-data hypersurface. These three degrees of freedom must be chosen so that the coordinates satisfy the gauge conditions in Eq. (25). In terms of our adapted coordinates, and making use of Eqs. (47) and (48), these three gauge conditions are

$$V^r = -\frac{1}{\sqrt{f}} \partial_r \left( \sqrt{f} \Lambda^2 \right) + \nabla_A (\Lambda^{-2} \lambda^A)$$

$$+ \Lambda \partial_r \ln \sqrt{f} + \frac{1}{2} \Lambda^{-2} \lambda^A \lambda^B \partial_r S_{AB},$$

$$V^A = \frac{1}{\sqrt{f}} \partial_r \left( \sqrt{f} \Lambda^{-2} \lambda^A \right) - \nabla_B (\Lambda S^{AB} + \Lambda^{-2} \lambda^A \lambda^B)$$

$$+ \Lambda^{-2} \lambda^B S^{AC} \partial_r S_{BC},$$  \hspace{1cm} (54)

and again for compactness we define $\det(f) \equiv f$ and $\det(S) \equiv S$.

As an explicit example, assume we are using Cartesian coordinates on the spatial hypersurface. For simplicity, consider the case where the gauge source functions are chosen to vanish ($V^i = 0$) and the excision boundary is chosen to be a coordinate sphere of radius $r$. Then the coordinates adapted to the excision surface are standard spherical coordinates, we have

$$S_{AB} = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (55)

and $\det(f) = \det(S) \Rightarrow r^4 \sin^2 \theta$, and the coordinate transformations between the three-dimensional Cartesian coordinates and our adapted coordinates are well known. The three gauge conditions then take the form

$$3^i \partial_i \Lambda = \frac{1}{r} (1 - \Lambda^3) - \frac{1}{2} \left( \nabla_A \lambda^A + \frac{1}{r} \lambda_A \lambda^A \right)$$

$$3^i \partial_i \lambda^A = \Lambda \nabla^A \Lambda - \frac{2}{r \Lambda} \left( 1 + \Lambda^3 + \frac{1}{2} \lambda_B \lambda^B \right) \lambda^A$$

$$+ \frac{1}{\Lambda} \lambda^B \lambda^C \Gamma^A_{BC}. \hspace{1cm} (57)$$

Note that these conditions are identically satisfied if the conformal metric is flat so that $\Lambda = 1$ and $\lambda^A = 0$.

In this example, we impose boundary conditions on the six Cartesian components of the conformal metric $\tilde{\gamma}_{ij}$ as follows. First we use standard Cartesian-to-spherical coordinate transformations to construct the three components of $\tilde{\gamma}_{ij} = \hat{h}_{AB}$. In a similar fashion, we construct the two components of $\lambda^A$ and the scalar $\Lambda$. The variables $\hat{h}_{AB}$, $\lambda^A$, and $\Lambda$ are then explicit functions of the six Cartesian components of the unknown conformal metric $\tilde{\gamma}_{ij}$. We then demand that $\hat{h}_{AB} = S_{AB} / \Lambda$ which constitute three Dirichlet boundary conditions. Finally, we impose the three normal-derivative boundary conditions given in Eqs. (56) and (57). Together, these form a coupled set of six equations for the six components of $\tilde{\gamma}_{ij}$. However, we emphasize that the choice of $S_{AB}$ guarantees that the global condition of $\det(\tilde{\gamma})$ is satisfied, and so we are really only constraining the five remaining degrees of freedom in the conformal metric.

Finally, we note that when implementing boundary conditions on the Cartesian components of the conformal metric, one would not actually use the boundary conditions as written in Eqs. (56) and (57). Instead, one would use the gauge conditions of Eq. (25) directly in terms of Cartesian components. The 2 + 1 decomposition we have derived is actually most useful in implementing the Dirichlet conditions, and in better understanding the nature of the boundary conditions.

V. SUMMARY AND DISCUSSION

In this paper we derive boundary conditions for the conformal metric that can be applied on black-hole excision boundaries. These boundary conditions are needed in the context of a new initial-data decomposition recently proposed by SUF, which provides an equation for
the conformal metric rather than treating it as a freely specifiable variable. This seems very attractive, since it avoids the need to make ad-hoc choices for the conformal metric – like conformal flatness – and instead computes the conformal metric as part of the solution.

Whenever black-hole initial data are constructed using excision methods, boundary conditions for all constrained data are required on the excision boundaries. In the conformal thin-sandwich decomposition, boundary conditions for the conformal factor \( \psi \) and shift \( \beta^i \) had a strong physical motivation that can be naturally viewed within the isolated-horizons framework \([5, 6, 8]\). Interestingly, this is not the case for the boundary conditions for the conformal metric.

It might seem that different choices for the induced metric \( S_{AB} \) should lead to physically different initial data. While we have emphasized the conformal equivalence of all closed 2-metrics, we must recognize that this is a statement about the intrinsic geometry of the 2-surface only. We must also consider the effect of the choice of the extrinsic curvature. Although we are restricted to choosing a boundary 2-surface with topology \( S^2 \), we have a wide range of choices for how this surface is embedded in the reference space. For example, we could choose the boundary to be either a coordinate sphere or a coordinate ellipse in a flat Cartesian space, each of which has a different extrinsic curvature. Note also that we can make this choice independently of the choice of the metric on the excision surface: for example, we could choose \( S_{AB} \) to be the metric of a unit sphere even on a coordinate ellipse. Both the choice for the shape of the 2-surface and for \( S_{AB} \) will affect the extrinsic curvature of the 2-surface as defined by Eq. (44). But, it is important to remember that these fix the conformal extrinsic curvature of the 2-surface, and not its physical extrinsic curvature.

The 2-surface’s influence on the dynamical degrees of freedom of our initial data are embodied in its physical extrinsic curvature via the sheer and expansion of the family of outgoing null rays passing through the boundary 2-surface. The physically motivated boundary conditions on the conformal factor \( \psi \) and shift \( \beta^i \) mentioned above are, in fact, obtained by demanding the sheer and expansion vanish on the boundary. These choices will certainly have an effect on the physical extrinsic curvature of the excision surface. Furthermore, the choice of the shape of the excision surface may have some affect on the dynamical degrees of freedom of the initial data, although the choice for \( S_{AB} \) should not.

As we demonstrate in this paper, it is natural to write the conformal metric on the excision surface in a 2+1 decomposition. The induced conformal metric on the excision surface must be related by suitable coordinate and conformal transformations to the metric on a unit sphere – making this choice therefore does not impose any physical restriction on the conformal metric. The remaining degrees of freedom of the conformal metric on the excision surface can be expressed as a radial “lapse” and “shift”, and can be determined from the gauge conditions that are imposed on the conformal metric. We therefore conclude that these boundary conditions affect only the gauge degrees of freedom of the conformal metric and in no way restrict the dynamical degrees of freedom of our initial-data solution.

While the investigation of these boundary conditions was motivated by the need to provide excision boundary conditions for black-hole initial data, the fact that only general geometrical and gauge considerations are involved means that these boundary conditions can be applied to any topologically spherical boundary. These boundary conditions need not be applied at a black-hole horizon, and in particular these conditions can be applied without approximation on an outer boundary.

Finally, we repeat our warning from Sec. I that the set of boundary conditions we have derived have not yet been shown to be independent and give rise to a well-posed elliptic system. Because the set of elliptic equations are non-linear and coupled, it is very difficult (if even possible) to determine this analytically. Future numerical implementations will clarify this issue.

Acknowledgments

The authors are grateful to J. Isenberg, L. Lindblom, V. Moncrief, N. O’Murchadha, M. Scheel, and M. Shibata for illuminating discussions and are grateful to K. Thorne and L. Lindblom for their hospitality during the Caltech Visitors Program in the Numerical Simulation of Gravitational Wave Sources, during which work on this idea began. The authors gratefully acknowledge the hospitality at and support from the Kavli Institute of Theoretical Physics at the University of California Santa Barbara. This work was supported in part by NSF grants PHY-0555617 to Wake Forest University, and PHY-0456917 and PHY-0756514 to Bowdoin College. G.B.C. acknowledges support from the Z. Smith Reynolds Foundation.

APPENDIX A: EINSTEIN’S EQUATIONS IN REFERENCE METRIC FORM

For completeness, we include below the conformally decomposed 3+1 version of Einstein’s equations, expressed in a reference metric form, along with a few additional useful equations. In writing these equations, we assume that the
fundamental independent variables are
\[ \psi, \quad \tilde{\alpha}, \quad \beta^j, \quad \text{and} \quad \tilde{\gamma}^{ij}, \]  
while we take the following quantities to be freely specifiable
\[ \partial_t \tilde{\gamma}^{ij}, \quad \partial_t \tilde{A}^{ij}, \quad \partial_t K, \quad K, \quad \det \tilde{\gamma}, \quad \text{and} \quad V^i. \]

The conformal metric \( \tilde{\gamma}_{ij} \) and the trace-free conformal extrinsic curvature \( \tilde{A}^{ij} \) are auxiliary variables defined to simplify the equations. The conformal metric is derived from the conformal inverse metric \( \tilde{\gamma}^{ij} \) in the usual way, and the trace-free conformal extrinsic curvature is defined by Eq. (A5) below. Note that, as in the main text, derivatives of the gauge source functions \( V_i \) are taken using the conformal covariant derivative \( \nabla_i \) whereas derivatives of the main variables are taken using the flat covariant derivative \( \nabla_i \). We emphasize again that we fix the reference metric to be flat (although this can be generalized if needed) and therefore the Ricci tensor and scalar associated with the reference metric vanish.

The Hamiltonian constraint in Eq. (11) is an elliptic equation for the conformal factor and takes the form
\[ i\Delta \psi - \frac{1}{4} \psi^2 \tilde{\gamma}^{ij} (C^{ij} - \frac{1}{2} D^{ij}) - \frac{1}{12} \psi \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} \tilde{A}^{ik} \tilde{A}^{j\ell} = V^i \nabla_i \psi + \frac{1}{10} \psi \tilde{\gamma} \Delta \ln \tilde{\gamma}, \]  
and the momentum constraint in Eq. (12) becomes
\[ \nabla_j \tilde{A}^{ij} - \frac{1}{3} \psi \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} K + \frac{1}{2} \tilde{A}^{km} \tilde{\gamma}_{kl} \tilde{\gamma}_{mn} \tilde{\gamma}^{ij} \tilde{\gamma}^{lm} - \tilde{A}^{ik} \tilde{\gamma}_{jk} \tilde{\gamma}^{ij} = - \frac{1}{2} \tilde{A}^{ij} \nabla_i \ln \tilde{\gamma}. \]  

The trace-free conformal extrinsic curvature is defined in terms of the time derivative of the inverse-conformal metric
\[ \partial_t \tilde{\gamma}^{ij} = 2 \tilde{\alpha} \tilde{A}^{ij} - 2 \tilde{\gamma}^{k(\tilde{\gamma}_{kl} \beta^l)} + \frac{2}{3} \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} \beta^k + \beta^k \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} \ln \tilde{\gamma} \equiv - \tilde{u}^{ij}. \]  

Note the definition of \( \tilde{u}^{ij} \) made for convenience below. The evolution equation for the trace-free conformal extrinsic curvature Eq. (13) becomes an elliptic equation for the inverse conformal metric:
\[ \partial_t \tilde{A}^{ij} = \frac{1}{2} \psi^2 \tilde{\alpha} \left[ (\Delta \tilde{\gamma}^{ij} - B^{ij} - C^{ij} - \frac{1}{2} \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} C^{k\ell}) + 2 \tilde{C}^{ij} - \frac{1}{8} (D^{ij} - \frac{1}{4} \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} D^{k\ell}) \right] - \left( \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} \tilde{\gamma}^{km} \tilde{\gamma}^{lm} \tilde{\gamma}^{ij} \tilde{\gamma}^{lm} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \right) \nabla_k \ln \tilde{\gamma} \nabla_l \ln \tilde{\gamma} + \frac{1}{2} \psi^2 \tilde{\alpha} (L \nabla)^{ij} - \frac{1}{4} \tilde{\gamma}^{ij} V^k \nabla_k \ln \tilde{\gamma} + \frac{1}{3} \tilde{\gamma}^{ij} \beta^k \tilde{\gamma}_{kl} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl}, \]  
while the evolution equation for the trace of the extrinsic curvature Eq. (15) is an elliptic equation for the conformal lapse
\[ \partial_t K = - \psi^{-5} \Delta (\psi \tilde{\alpha}) + \frac{1}{10} \psi^2 \tilde{\alpha} \tilde{\gamma}^{ij} (C^{ij} - \frac{1}{2} D^{ij}) + \frac{3}{12} \psi^2 \tilde{\alpha} K^2 + \frac{1}{3} \psi \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} \tilde{A}^{ik} \tilde{A}^{j\ell} + \beta^i \nabla_i K \]  
+ \psi^{-5} V^i \nabla_i (\psi \tilde{\alpha}) + \frac{1}{6} \psi^2 \tilde{\alpha} V^i \nabla_i \Delta \ln \tilde{\gamma}. \]  

Finally, the evolution equation for the conformal factor becomes
\[ \partial_t \psi = \frac{1}{12} \psi \left( \nabla_i \beta^i + 6 \beta^i \nabla_i \ln \psi - \psi \tilde{\alpha} \tilde{K} \right) + \frac{1}{12} \psi \beta^i \nabla_i \ln \tilde{\gamma}. \]  

If we replace \( \tilde{A}^{ij} \) in the momentum constraint (A4) by its definition in Eq. (A5), we obtain an elliptic equation for the shift:
\[ i\Delta \beta^j + \nabla_j \nabla^j \nabla \tilde{\gamma}_{kl} \beta^k + \frac{1}{2} \psi \tilde{\gamma}^{ij} \tilde{\gamma}_{kl} \tilde{\gamma}^{km} \tilde{\gamma}^{lm} \tilde{\gamma}^{ij} \tilde{\gamma}^{lm} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \]  
\[ = V^i \nabla_j \beta^j - \frac{1}{3} V^i \nabla_i \beta^j - \beta^i \nabla_i V^j - \frac{1}{4} V^j V^i \nabla_j \ln \tilde{\gamma} - \frac{1}{4} (2 \tilde{A}^{ij} \nabla_j \ln \tilde{\gamma} + \frac{1}{3} \tilde{\gamma}^{ij} \nabla_j \nabla_k \beta^k) \nabla_k \ln \tilde{\gamma} - \frac{1}{3} \tilde{\gamma}^{ij} \nabla_j \nabla_k \beta^k \nabla_k \ln \tilde{\gamma} + \frac{1}{5} \tilde{\gamma}^{ij} (\nabla_k \beta^k) \nabla_j \ln \tilde{\gamma} - \frac{1}{3} \tilde{\gamma}^{ij} \beta^k \nabla_k \nabla_j \ln \tilde{\gamma} - \frac{1}{3} \tilde{\gamma}^{ij} \beta^k (\nabla_k \ln \tilde{\gamma}) \nabla_j \ln \tilde{\gamma}. \]  

In all of these equations terms of the form \( \nabla_i \ln \tilde{\gamma} \) appear. These terms are defined implicitly by
\[ \nabla_i \ln \tilde{\gamma} = \frac{1}{\tilde{\gamma}} \nabla_i \tilde{\gamma} = \partial_i \ln \left( \frac{\tilde{\gamma}}{\tilde{f}} \right). \]  

(A10)
Note that both $\det(\tilde{\gamma})$ and $\det(f)$ transform as scalar densities of weight 2, so their ratio is a simple scalar. If we choose $\det(\tilde{\gamma}) = \det(f)$ as we have in defining the boundary boundary conditions in Sec. IV, then we find that all terms involving derivatives of $\det(\tilde{\gamma})$ vanish.

Finally, we note that the Arnowit-Deser-Misner energy takes on the familiar form:

$$E_{ADM} = -\frac{1}{2\pi} \oint_{\infty} \nabla_i \psi \, d^2S^i + \frac{1}{16\pi} \oint_{\infty} (V^j - \frac{1}{2} \tilde{\gamma}^{ij} \nabla_i \ln \tilde{\gamma}) \, d^2S_j. \quad (A11)$$

[1] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), pp. 227–265.
[2] J. W. York, Jr., in *Sources of Gravitational Radiation*, edited by L. L. Smarr (Cambridge University Press, Cambridge, England, 1979), pp. 83–126.
[3] M. Shibata, K. Uryū, and J. L. Friedman, Phys. Rev. D 70, 044044 (2004).
[4] K. Uryū, F. Limousin, J. L. Friedman, E. Gourgoulhon, and M. Shibata, Phys. Rev. Lett. 97, 171101 (2006).
[5] G. B. Cook, Phys. Rev. D 65, 084003 (2002).
[6] G. B. Cook and H. P. Pfeiffer, Phys. Rev. D 70, 104016 (2004).
[7] M. Caudill, G. B. Cook, J. D. Grigsby, and H. P. Pfeiffer, Phys. Rev. D 74, 064011 (2006).
[8] J. L. Jaramillo, E. Gourgoulhon, and G. A. Mena Marugán, Phys. Rev. D 70, 124036 (2004).
[9] S. Dain, J. L. Jaramillo, and B. Krishnan, Phys. Rev. D 71, 064003 (2005).
[10] E. Gourgoulhon and J. L. Jaramillo, Phys. Rept. 423, 159 (2006).
[11] J. L. Jaramillo, M. Ansorg, and F. Limousin, Phys. Rev. D 75, 024019 (2007).
[12] J. L. Jaramillo, E. Gourgoulhon, I. Cordero-Carrion, and J. M. Ibáñez, Phys. Rev. D 77, 047501 (2008).
[13] S. Dain, Lect. Notes Phys. 692, 117 (2006).
[14] G. B. Cook, *Initial data for numerical relativity*, Article in online journal Living Reviews in Relativity (2000), http://www.livingreviews.org/lrr-2000-5.
[15] T. W. Baumgarte and S. L. Shapiro, Phys. Rept. 376, 41 (2003).
[16] H. P. Pfeiffer, J. Hyperbol. Diff. Equat. 2, 497 (2005).
[17] J. M. Bowen and J. W. York, Jr., Phys. Rev. D 21, 2047 (1980).
[18] S. Brandt and B. Brügmann, Phys. Rev. Lett. 78, 3606 (1997).
[19] R. Beig and N. Ó Murchadha, Class. Quantum Gravit. 11, 419 (1994).
[20] R. Beig and N. Ó Murchadha, Class. Quantum Gravit. 13, 739 (1996).
[21] G. B. Cook, Phys. Rev. D 44, 2983 (1991).
[22] J. W. York, Jr., Phys. Rev. Lett. 82, 1350 (1999).
[23] H. P. Pfeiffer and J. W. York, Jr., Phys. Rev. D 67, 044022 (2003).
[24] H. P. Pfeiffer and J. W. York, Jr., Phys. Rev. Lett. 95, 091101 (2005).
[25] T. W. Baumgarte, N. Ó Murchadha, and H. P. Pfeiffer, Phys. Rev. D 75, 044009 (2007).
[26] D. M. Walsh, Class. Quantum Gravit. 24, 1911 (2007).
[27] I. Cordero-Carrion, P. Cerda-Duran, H. Dimmelmeier, J. L. Jaramillo, J. Novak, and E. Gourgoulhon (2008), 0809.2325.
[28] M. Boyle, D. A. Brown, L. E. Kidder, A. H. Mroué, H. P. Pfeiffer, M. A. Scheel, G. B. Cook, and S. A. Teukolsky, Phys. Rev. D 76, 124038 (2007).
[29] P. Papadopoulos and C. F. Sopuerta, Phys. Rev. D 65, 044008 (2002).
[30] L. Andersson and V. Moncrief, Ann. Henri Poincaré 4, 1 (2003).
[31] S. Bonazzola, E. Gourgoulhon, P. Grandclément, and J. Novak, Phys. Rev. D 70, 104007 (2004).
[32] T. De Donder, *La gravifique einsteinienne* (Gauthier-Villars, Paris, 1921).
[33] C. Lanczos, Phys. Z. 23, 537 (1922).
[34] Y. Fourès-Bruhat, Acta. Math. 88, 141 (1952).
[35] A. E. Fischer and J. E. Marsden, Commun. Math. Phys. 28, 1 (1972).
[36] H. Friedrich, Commun. Math. Phys. 100, 525 (1985).
[37] D. Garfinkle, Phys. Rev. D 65, 044029 (2002).
[38] F. Pretorius, Phys. Rev. Lett. 95, 121101 (2005).
[39] M. Shibata and T. Nakamura, Phys. Rev. D 52, 5428 (1995).
[40] T. W. Baumgarte and S. L. Shapiro, Phys. Rev. D 59, 024007 (1998).