Research article

Extended incomplete Riemann-Liouville fractional integral operators and related special functions

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Abstract: In this study, we introduce the extended incomplete versions of the Riemann-Liouville (R-L) fractional integral operators and investigate their analytical properties rigorously. More precisely, we investigate their transformation properties in $L_1$ and $L_\infty$ spaces, and we observe that the extended incomplete fractional calculus operators can be used in the analysis of a wider class of functions than the extended fractional calculus operator. Moreover, by considering the concept of analytical continuation, definitions for extended incomplete R-L fractional derivatives are given and therefore the full fractional calculus model has been completed for each complex order. Then the extended incomplete $\tau$-Gauss, confluent and Appell’s hypergeometric functions are introduced by means of the extended incomplete beta functions and some of their properties such as integral representations and their relations with the extended R-L fractional calculus has been given. As a particular advantage of the new fractional integral operators, some generating relations of linear and bilinear type for extended incomplete $\tau$-hypergeometric functions have been derived.

Keywords: extended incomplete hypergeometric functions; extended incomplete Appell’s functions; incomplete fractional calculus; generating functions

1. Introduction

There is a close relationship between the special functions and fractional calculus, which is a branch gaining popularity, especially in the last decades, because of its potential usefulness in real world applications (see [1–10]). Different definitions of fractional calculus have been introduced in the literature, each of which has their own advantages or disadvantages. One direction of research has been to add
more parameters, for instance the Erdelyi-Kober fractional model, and another direction is to consider some analytic functions in the kernel of fractional calculus operators such as the Prabhakar model.

Recently, the extended R-L fractional integral of order $\mu$ [11] was defined by

$$D^\mu_z (f(z)) = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z - t)^{-\mu-1} \exp\left(\frac{-pz^2}{t(z - t)}\right) dt, \quad \text{Re}(\mu) < 0, \quad \text{Re}(\rho) > 0,$$

and it has been shown that these operators are useful in the analysis of certain extensions of special functions defined in [12–14].

Another interesting approach in generalizing fractional calculus is given in the papers [15, 16], where the authors, instead of integrating over a full interval $[0, z]$, introduced two integral operators by separating the interval by a variable $yz$ ($0 < y < 1$), therefore this approach provides a general definition of fractional integrals, in which the singular and nonsingular parts of the integral can be separated. More precisely they introduced the operators

$$D^\mu_\gamma [f(z); y] = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y f(uz)(1 - u)^{-\mu-1} du, \quad \text{Re}(\mu) < 0 \quad (1.1)$$

and

$$D^\mu_\gamma [f(z); y] = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_y^1 f(uz)(1 - u)^{-\mu-1} du, \quad \text{Re}(\mu) < 0. \quad (1.2)$$

The incomplete fractional integrals and derivatives have been subjected to an in-depth analysis in the papers [15–18].

On the other hand, in recent papers such as [11–14, 19–28], several extensions of the well-known special functions have been considered, many of which have close relationships with fractional calculus. Also we should refer the recent surveys on the transcendental functions with their connections between the fractional calculus [29, 30].

Very recently, in order to introduce a different variant of incomplete Gauss hypergeometric functions which is more suitable for the fractional calculus results as well, the authors introduced the incomplete Pochhammer ratios as follows [15]:

$$[b, c; y]_n := \frac{B_y(b + n, c - b)}{B(c - b, b)}$$

and

$$[b, c; y]_n := \frac{B_{1-y}(c - b, b + n)}{B(c - b, b)}$$

where $0 \leq y < 1$ and

$$B_y(x, z) = \int_0^y t^{x-1}(1 - t)^{z-1} dt, \quad \text{Re}(x) > 0, \text{Re}(z) > 0, \quad 0 < y < 1. \quad (1.3)$$

is the incomplete beta function.

They defined the incomplete Gauss hypergeometric functions as follows:

$$_{2}F_1(a, [b, c; y]; x) := \sum_{n=0}^{\infty} (a)_n [b, c; y]_n \frac{x^n}{n!}, \quad (1.4)$$
and

\[ 2F_1(a, \{b, c; y\} ; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b, c)_n}{n!} x^n. \] (1.5)

Several properties of these functions were obtained, such as integral representations, derivative formulae, transformation formulae, and recurrence relations. Also, the incomplete Appell’s functions were defined and expressed using integral representations. It should be mentioned that, in a recent paper, the incomplete Gauss hypergeometric function was used in the derivation of some new estimates for the generalized Simpson’s quadrature rule [31].

One of the generalisations of Gauss hypergeometric function was defined by Chaudhry [13, 14]

\[ F_p(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b + n, c - b)_n z^n}{B(b, c - b) n!}, \quad p \geq 0, \quad \text{Re}(c) > \text{Re}(b) > 0 \] (1.6)

where

\[ B_p(x, y) := \int_0^1 t^{x-1} (1 - t)^{y-1} \exp \left[-\frac{pt}{t(1-t)}\right] dt, \quad \text{Re}(p) > 0, \quad \text{Re}(x) > 0, \quad \text{Re}(y) > 0. \] (1.7)

is the extended beta function. On the other hand, the incomplete version of the extended beta function has been defined as

\[ B_y(x, z; p) = \int_0^y t^{x-1} (1 - t)^{y-1} \exp \left(-\frac{pt}{t(1-t)}\right) dt, \quad \text{Re}(p) > 0, \quad 0 \leq y < 1 \] (1.8)

and investigated in [14]. Some applications, where these functions are used, can be found in [14].

We saw from the above discussion that both structures of generalisations by adding new parameters or incompletifications are interesting topics of study with potential real world applications. By combining both forms of generalisations, we are able to construct new functions and operators which have the advantages of both the incomplete versions and the parametric versions. Therefore in this paper, in Sections 2 and 3 we combine these generalisations and investigate them thoroughly. More precisely in Section 2, we introduce the extended incomplete versions of the Riemann-Liouville (R-L) fractional integral operator and investigate their transformation properties in \( L_1 \) and \( L_\infty \) spaces. We observe that the extended incomplete fractional calculus operators can be used in the analysis of a wider class of functions than the extended fractional calculus operator. Moreover, by considering the concept of analytical continuation, definitions for extended incomplete R-L fractional derivatives are given and therefore the full fractional calculus model has been completed for each complex order. In Section 3, similar treatment has been considered to introduce extended incomplete \( \tau \)-Gauss, confluent and Appell’s hypergeometric functions. Some of their properties such as integral representations and their relations with the extended R-L fractional calculus has been given. In the last section, one particular advantage of the new fractional integral operators has been exhibited by deriving some generating relations of linear and bilinear type for extended incomplete \( \tau \)-hypergeometric functions.
2. Extended incomplete R-L fractional integral operator

The extended incomplete R-L fractional integral operators \( D_{z}^{\mu,p} [f(z); y] \) and \( D_{z}^{\mu,p} [f(z); y] \) are introduced by

\[
D_{z}^{\mu,p} [f(z); y] := \frac{\Gamma(-\mu)}{\Gamma(-\mu)} \int_{0}^{y} f(uz)(1 - u)^{-\mu - 1} \exp\left(\frac{-p}{u(1 - u)}\right) du, \quad \text{Re}(\mu) < 0, \quad \text{Re}(p) > 0, \quad 0 \leq y \leq 1,
\]
and

\[
D_{z}^{\mu,p} [f(z); y] := \frac{\Gamma(-\mu)}{\Gamma(-\mu)} \int_{y}^{z} f(uz)(1 - u)^{-\mu - 1} \exp\left(\frac{-p}{u(1 - u)}\right) du, \quad \text{Re}(\mu) < 0, \quad \text{Re}(p) > 0, \quad 0 \leq y \leq 1.
\]

(2.1)

(2.2)

Setting \( p \to 0 \) in (2.1) and (2.2), we obtain the incomplete R-L fractional integral operators which are defined in (1.1) and (1.2), respectively. These extended incomplete R-L fractional integral operators satisfy the following decomposition formula:

\[
D_{z}^{\mu,p} [f(z); y] + D_{z}^{\mu,p} [f(z); y] = D_{z}^{\mu,p} (f(z)).
\]

We start the analytical investigation of these operators by considering their transformation properties:

**Theorem 2.1.** Let \( A > 0, \ 0 < y < 1 \) and \( \text{Re}(\mu) > 0 \). Then \( D_{z}^{\mu,p} [.; y] : L_{1} [0, yA] \to L_{1} [0, A] \).

**Proof.** Fix \( 0 < y < 1 \) choose any \( f \in L_{1} [0, yA].\) For \( z \in [0, A], \) since \( \text{Re}(p) > 0, \) we can write

\[
|D_{z}^{\mu,p} [f(z); y]| \leq \frac{1}{\Gamma(\mu)} \int_{0}^{y} \left| f(t)(z - t)^{-\mu - 1} \exp\left(\frac{-p}{t(z - t)}\right)\right| dt
\]

\[
\leq \sup_{z \in [0, y]} \int_{0}^{y} \left| f(t)(z - t)^{-\mu - 1} \exp\left(\frac{-p}{t(z - t)}\right)\right| dt
\]

\[
\leq \sup_{z \in [0, y]} \left| f(t)(z - t)^{-\mu - 1} \right| \frac{1}{\Gamma(\mu)} \int_{0}^{y} |f(t)| dt
\]

\[
= \left\{ \begin{array}{ll} \frac{(z-y)\text{Re}(\mu) - 1}{\Gamma(\mu)} \|f\|_{L_{1}[0,yA]}, & 0 < \text{Re}(\mu) < 1 \\ \frac{\text{Re}(\mu)}{\Gamma(\mu)} \|f\|_{L_{1}[0,yA]}, & \text{Re}(\mu) > 1. \end{array} \right.
\]

Integrating both sides of this inequality over \( z \in [0, A], \) we get

\[
\|D_{z}^{\mu,p} [f; y]\|_{L_{1}[0,A]} \leq \left\{ \begin{array}{ll} \frac{(1-y)\text{Re}(\mu) - 1}{\Gamma(\mu)} \|f\|_{L_{1}[0,yA]}, & 0 < \text{Re}(\mu) < 1 \\ \frac{\text{Re}(\mu)}{\Gamma(\mu)} \|f\|_{L_{1}[0,yA]}, & \text{Re}(\mu) > 1. \end{array} \right.
\]

Thus the proof is completed.

**Theorem 2.2.** Let \( A > 0, \ 0 < y < 1 \) and \( \text{Re}(\mu) > 1. \) Then \( D_{z}^{\mu,p} [.; y] : L_{1} [0, A] \to L_{1} [0, A]. \)
Proof. Fix $0 < y < 1$ choose any $f \in L_1[0, A]$. For $z \in [0, A]$, since $\text{Re}(p) > 0$ we have

$$
|D_{z}^{-\mu, p}[f(z), y]| \leq \frac{1}{|\Gamma(\mu)|} \int_{y}^{z} \left| f(t)(z - t)^{\mu - 1} \exp \left( -\frac{p z^2}{t(z - t)} \right) \right| dt
$$

$$
\leq \left[ \sup_{[y, z]} (z - t)^{\text{Re}(\mu) - 1} \exp \left( -\frac{\text{Re}(p) z^2}{t(z - t)} \right) \right] \frac{1}{|\Gamma(\mu)|} \int_{y}^{z} |f(t)| dt
$$

$$
\leq \frac{(z - y)^{\text{Re}(\mu) - 1}}{|\Gamma(\mu)|} \| f \|_{L_1[0, A]}. 
$$

Integrating both sides of this inequality over $z \in [0, A]$, we get

$$
\| D_{z}^{-\mu, p}[f; y] \|_{L_1[0, A]} \leq \frac{(1 - y)^{\text{Re}(\mu) - 1} A^{\text{Re}(\mu)}}{|\Gamma(\mu)| \text{Re}(\mu)} \| f \|_{L_1[0, A]}.
$$

Whence the result. \(\square\)

Therefore, using the above two Theorems, we can give the following definitions.

**Definition 2.3.** Let $A > 0$, $0 < y < 1$ and $\text{Re}(\mu) > 0$. Then for all $f \in L_1[0, yA]$ the $\mu$ th order extended incomplete lower fractional integral is defined by

$$
_{0}D_{z}^{-\mu, p}[f(z); y] := \frac{z^\mu}{\Gamma(\mu)} \int_{0}^{y} f(u)(1 - u)^{-\mu - 1} \exp \left( \frac{-p}{u(1 - u)} \right) du.
$$

**Definition 2.4.** Let $A > 0$, $0 < y < 1$ and $\text{Re}(\mu) > 1$. Then for all $f \in L_1[0, A]$ the $\mu$ th order extended incomplete upper fractional integral is defined by

$$
_{0}D_{z}^{\mu, p}[f(z); y] := \frac{z^\mu}{\Gamma(\mu)} \int_{y}^{1} f(u)(1 - u)^{-\mu - 1} \exp \left( \frac{-p}{u(1 - u)} \right) du.
$$

There is a gap in the definition of the extended incomplete upper fractional integral for the case $0 < \text{Re}(\mu) \leq 1$. In order to fill this gap, in the following theorem, we consider the operators in the space $L_\infty$.

**Theorem 2.5.** Let $A > 0$, $0 < y < 1$ and $\text{Re}(\mu) > 0$. Then we have

$$
D_{z}^{-\mu, p}[.; y] : L_\infty[0, yA] \rightarrow L_\infty[0, A]
$$

and

$$
D_{z}^{\mu, p}[.; y] : L_\infty[0, A] \rightarrow L_\infty[0, A].
$$

Proof. For any $z \in [0, A]$, since $\text{Re}(p) > 0$, we have

$$
D_{z}^{\mu, p}[f(z); y] \leq \frac{1}{|\Gamma(\mu)|} \int_{0}^{\infty} |f(t)| (z - t)^{\mu - 1} \exp \left( -\frac{p z^2}{t(x - t)} \right) dt
$$

$$
\leq \frac{1}{|\Gamma(\mu)|} \text{es sup}_{[0, yA]} |f| \int_{0}^{\infty} (z - t)^{\mu - 1} dt
$$

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\[ A^{\Re \mu} \left( 1 - (1 - y)^{\Re \mu} \right) \frac{1}{\Re \mu [\Gamma (\mu)]} \| f \|_{L_\infty [0,A]} \cdot \]

Taking essup over all \( z \in [0,A] \) on both sides of the inequality, we complete the proof of the first statement.

Since \( \Re (p) > 0 \), we have for any \( x \in [0,A] \) that

\[ |D_{\mu}^{\mu,p} \cdot (f(z); y)| \leq \frac{1}{\Gamma (\mu)} \int_{0}^{z} f(t) \left| (z - t)^{\mu - 1} \exp \left( \frac{-pz^2}{t(z-t)} \right) \right| dt \]

\[ \leq \frac{1}{\Gamma (\mu)} \text{essup}_{[0,A]} f(z) \int_{0}^{z} (z - t)^{\Re \mu - 1} dt \]

\[ \leq \frac{A^{\Re \mu} (1 - y)^{\Re \mu}}{\Re \mu [\Gamma (\mu)]} \| f \|_{L_\infty [0,A]} \cdot \]

Taking essup over all \( z \in [0,A] \) on both sides of the inequality, we complete the proof of the second statement. \( \square \)

Using the above Theorem, in the following we give definition of the \( \mu \) th order extended incomplete upper fractional integrals for the case \( 0 < \Re (\mu) \leq 1 \).

**Definition 2.6.** Let \( A > 0 \), \( 0 < y < 1 \) and \( 0 < \Re (\mu) \leq 1 \). Then for all \( f \in L_\infty [0,A] \), the \( \mu \) th order extended incomplete upper fractional integral is defined by

\[ D_{\mu}^{\mu,p} (f(z); y) := \frac{e^z}{\Gamma (\mu)} \int_{0}^{z} f(t)(1 - u)^{\mu - 1} \exp \left( \frac{-p}{u(1-u)} \right) du. \]

**Remark 2.7.** It should be remarked that the transformation properties of

\[ D_{\mu}^{\mu,p} (f(z)) = \frac{1}{\Gamma (\mu)} \int_{0}^{\infty} f(t)(z - t)^{\mu - 1} \exp \left( \frac{-pz^2}{t(z-t)} \right) dt, \]

has not been investigated and it can be easily proved by majorizing the exponential term that \( D_{\mu}^{\mu,p} (\cdot) : L_1 [0,A] \to L_1 [0,A] \) and \( D_{\mu}^{\mu,p} (\cdot) : L_\infty [0,A] \to L_\infty [0,A] \) for \( \Re (\mu) > 0 \). Therefore it is clear that the extended incomplete fractional calculus operators can be used in the analysis of a wider class of functions than the extended fractional calculus operator \( D_{\mu}^{\mu,p} \).

The extended incomplete R-L fractional integral operators \( D_{\mu}^{\mu,p} [f(z); y] \) and \( D_{\mu}^{\mu,p} [f(z); y] \) are defined in the case \( \Re (\mu) > 0 \). In order to extend the domain of \( \mu \) to \( \Re (\mu) \leq 0 \), and by this way defining the corresponding derivative operators, we consider the concept of analytic continuation in \( \mu \). The following theorems will be crucial in this respect.

**Theorem 2.8.** Let \( A > 0 \), \( 0 < y < 1 \), \( \Re (\mu) > 2 \). For all \( f, z^{-1} f \in L_1 [0, yA] \), we have

\[ \frac{d}{dz} \left( D_{\mu}^{\mu,p} [f(z); y] \right) = \frac{y f(y z)(z - yz)^{\mu - 1} \exp \left( \frac{-p}{y(z-y)} \right)}{\Gamma (\mu)} + D_{\mu}^{\mu,p} [f(z); y] \]

\[ + \frac{pz}{(\mu - 1)(\mu - 2)} D_{\mu}^{\mu,p} [f(z); y] \]
For all \( f, z^{-1}f \in L_1[0,A] \), we have

\[
\frac{d}{dz} \left(D_{z}^{-\mu,p}[f(z);y]\right) = \frac{1}{\Gamma(\mu)} \int_{0}^{y} f(t)(z-t)^{-\mu-1} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
+ (\mu - 1) \int_{0}^{y} f(t)(z-t)^{-\mu-2} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
+ p \int_{0}^{y} f(t)(z-t)^{-\mu-3} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
- p \int_{0}^{y} t^{-1} f(t)(z-t)^{-\mu-2} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
= \frac{y(z - y)^{\mu-1} \exp\left(-\frac{p}{y(1-y)}\right) f(yz)}{\Gamma(\mu)} + D_{z}^{-\mu,p}[f(z);y]
\]

\[
+ \frac{pz}{(\mu - 1)(\mu - 2)} D_{z}^{2-\mu,p}[f(z);y] - \frac{pz}{\mu - 1} D_{z}^{1-\mu,p}[z^{-1}f(z);y].
\]

**Proof.** Firstly, using the usual technique of differentiation under the integral sign, we have

\[
\frac{d}{dz} \left(D_{z}^{-\mu,p}(f(z);y)\right) = \frac{1}{\Gamma(\mu)} \int_{0}^{y} f(t)(z-t)^{\mu-1} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
+ (\mu - 1) \int_{0}^{y} f(t)(z-t)^{-\mu-2} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
+ p \int_{0}^{y} f(t)(z-t)^{-\mu-3} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
- p \int_{0}^{y} t^{-1} f(t)(z-t)^{-\mu-2} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
= \frac{y(z - y)^{\mu-1} \exp\left(-\frac{p}{y(1-y)}\right) f(yz)}{\Gamma(\mu)} + D_{z}^{-\mu,p}[f(z);y]
\]

\[
+ \frac{pz}{(\mu - 1)(\mu - 2)} D_{z}^{2-\mu,p}[f(z);y] - \frac{pz}{\mu - 1} D_{z}^{1-\mu,p}[z^{-1}f(z);y].
\]

Secondly, taking derivative from the definition of the upper incomplete integral operator, we get

\[
\frac{d}{dz} \left(D_{z}^{-\mu,p}(f(z);y)\right) = \frac{1}{\Gamma(\mu)} \frac{d}{dz} \left(\int_{y}^{z} f(t)(z-t)^{\mu-1} \exp\left(-\frac{p}{t(z-t)}\right) dt\right)
\]

\[
= \frac{1}{\Gamma(\mu)} \left[ -y f(yz)(z - yy^{1-y})^{\mu-1} \exp\left(-\frac{p}{y(1-y)}\right) dt \right]
\]

\[
+ (\mu - 1) \int_{y}^{z} f(t)(z-t)^{-\mu-2} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
+ p \int_{y}^{z} f(t)(z-t)^{-\mu-3} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
- p \int_{y}^{z} t^{-1} f(t)(z-t)^{-\mu-2} \exp\left(-\frac{p}{t(z-t)}\right) dt
\]

\[
= \frac{-y f(yz)(z - yy^{1-y})^{\mu-1} \exp\left(-\frac{p}{y(1-y)}\right)}{\Gamma(\mu)} + D_{z}^{-\mu,p}(f(z);y)
\]
Example 2.12. Let \( \Re(z) < R \) and has a power series expansion \( f(z) = \sum_{n=0}^{\infty} c_n z^n \), then for \( \Re(\lambda) > 0 \) and \( \Re(p) > 0 \) we have

\[
D_{z}^{\mu,p}[z^{\lambda-1} f(z); y] = \frac{z^{1-\mu}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_{\lambda+y} (\lambda + n, -\mu; p) z^n,
\]

(2.3)

and

\[
D_{z}^{\mu,p}[z^{\lambda-1} f(z); y] = \frac{z^{1-\mu}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_{\lambda-y} (-\mu, \lambda + n; p) z^n.
\]

Remark 2.11. It is important to mention that the definition of \( D_{z}^{\mu,p} [f(z); y] \) given in (10) does not require the condition \( \Re(\mu) < 0 \) since the interval of integration in this definition is \([0, y]\) with \( 0 < y < 1 \). Therefore the formula (10) is valid for all \( \mu \in \mathbb{C} \).

Example 2.12. Let \( \Re(\lambda) > -1, \ \Re(\mu) < 0 \) and \( \Re(p) > 0 \). Then

\[
D_{z}^{\mu,p}[z^{\lambda}, y] = \frac{B_{\lambda}(\lambda + 1, -\mu; p)}{\Gamma(-\mu)} z^{\lambda-\mu}.
\]
Proof. Since the function is analytic in the given disc, its series expansion is uniformly convergent. Using the relation in the above Example,

\[ D_\mu^\mu [z^{\lambda}, y] = \frac{B_\mu (\lambda + 1, -\mu; p)}{\Gamma (-\mu)} z^{\lambda-\mu}, \]

(Re(\lambda) > -1, Re(p) > 0),

we get

\[ D_\mu^\mu [z^{\lambda-1} f(z); y] = \sum_{n=0}^{\infty} a_n D_\mu^\mu [z^{\lambda+n-1}; y] \]

\[ = \sum_{n=0}^{\infty} a_n \left( \frac{z^{-\mu}}{\Gamma (-\mu)} \int_{0}^{y} (u z)^{\lambda+n-1} (1-u)^{-\mu-1} \exp \left( \frac{-p}{u(1-u)} \right) du \right) \]

\[ = \frac{z^{-\mu-1}}{\Gamma (-\mu)} \sum_{n=0}^{\infty} a_n B_\mu (\lambda + n - \mu; p) z^n. \]

Similarly,

\[ D_\mu^\mu [z^{\lambda-1} f(z); y] = \sum_{n=0}^{\infty} a_n D_\mu^\mu [z^{\lambda+n-1}; y] \]

\[ = \sum_{n=0}^{\infty} a_n \left( \frac{z^{-\mu}}{\Gamma (-\mu)} \int_{y}^{1} (u z)^{\lambda+n-1} (1-u)^{-\mu-1} \exp \left( \frac{-p}{u(1-u)} \right) du \right) \]

\[ = \frac{z^{-\mu-1}}{\Gamma (-\mu)} \sum_{n=0}^{\infty} a_n B_{1-y} (-\mu, \lambda + n; p) z^n, \]

using uniform convergence of the series and absolute convergence of the integral under the given conditions. \qed

Now we consider the Mellin transform of the extended incomplete beta function. For Re(s) > 0, we have

\[ M \left( B_\mu (x, z; p) : p \to s \right) = \int_{0}^{\infty} p^{s-1} \left( \int_{0}^{y} t^{s-1} (1-t)^{s-1} \exp \left( \frac{-p}{t(1-t)} \right) dt \right) dp \]

\[ = \Gamma(s) B_\mu (x + s, z + s). \]

Therefore, from the inverse Mellin transform, we have

\[ B_\mu (x, z; p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) B_\mu (x + s, z + s) p^{-s} ds. \]

Using this result, we have

\[ D_\mu^\mu [z^{\lambda-1} f(z); y] = \frac{z^{\lambda-\mu-1}}{\Gamma (-\mu)} \sum_{n=0}^{\infty} a_n B_\mu (\lambda + n - \mu; p) z^n. \]

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where we have interchanged the contour integral and the series by considering that the function is
analytic in the given disc and Re(s) > 0, Re(p) > 0, Re(\lambda) > 0.

Now we consider the case \( p \to 0 \), which gives

\[
D^u_z \left[ f(z); y \right] = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y f(uz)(1 - u)^{\mu - 1} du,
\]

the incomplete lower R-L fractional integral operator. Recalling the incomplete beta ratio

\[
I_\alpha(p, q) = \frac{B(p, q)}{B(p, q)} = \frac{1}{B(p, q)} \int_0^y \tau^{\alpha - 1} (1 - \tau)^{q - 1} d\tau,
\]

the following complex contour representation of this function was given in [32] for \( p < 1, p + q > 0, 0 < d < 1; \)

\[
I_\alpha(p, q) = \frac{y^p (1 - y)^q}{2\pi i} \int_{d - i\infty}^{d + i\infty} \tau^{-\alpha} (1 - \tau)^{q - 1} \frac{d\tau}{\tau - x}.
\]

The condition \( p < 1 \), which is important for the evaluation of the contour integral, can be cancelled by
using the analytic continuation principle. Using the above integral representation, we have for \( \lambda - \mu > 0 \) that

\[
D^u_z \left[ z^{\lambda - 1} f(z); y \right] = \frac{z^{\lambda - 1 - \mu} \Gamma(-\mu)}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_\lambda(\lambda + n, -\mu) z^n
\]

\[
= \frac{z^{\lambda - 1 - \mu}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_\lambda(\lambda + n, -\mu) B(\lambda + n, -\mu) z^n
\]

\[
= \frac{z^{\lambda - 1 - \mu}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n I_\lambda(\lambda + n, -\mu) B(\lambda + n, -\mu) z^n
\]

\[
= \frac{z^{\lambda - 1 - \mu}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} \frac{a_n y^{\lambda + n} (1 - y)^{-\mu}}{2\pi i} \int_{d - i\infty}^{d + i\infty} \tau^{-\lambda - n} (1 - \tau)^{-\mu} \frac{d\tau}{\tau - x} B(\lambda + n, -\mu) z^n
\]

\[
= \frac{z^{\lambda - 1 - \mu} y^{\lambda} (1 - y)^{-\mu}}{2\pi i \Gamma(-\mu)} \int_{d - i\infty}^{d + i\infty} \tau^{-\lambda} (1 - \tau)^{-\mu} \sum_{n=0}^{\infty} a_n B(\lambda + n, -\mu) \left( \frac{z}{\tau} \right)^n d\tau
\]

\[
= \frac{z^{\lambda - 1 - \mu} y^{\lambda} (1 - y)^{-\mu}}{2\pi i B(\lambda, -\mu)} \int_{d - i\infty}^{d + i\infty} \tau^{-\lambda} \frac{1}{x - \tau} \sum_{n=0}^{\infty} a_n \left( \frac{\lambda}{\lambda - \mu} \right)_n \left( \frac{z}{\tau} \right)^n d\tau.
\]

For instance, let’s choose \( f(z) = a_0 F_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) \) with \( p \leq q \), which is an entire
function on the whole domain. From the above result, we can immediately write an elegant contour integral representation:
\[ D^\mu_{\tau} \left[ z^{\lambda-1} F_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \right] = \frac{z^{\lambda-\mu-1} y^{\lambda} (1 - y)^{-\mu}}{2\pi i \Gamma(-\mu)} \int_{d-i\infty}^{d+i\infty} \frac{\tau^{-\lambda}(1 - \tau)^{\mu}}{\tau - x} \times \rho F_q \left( \lambda, a_1, \ldots, a_p; b_1, \ldots, b_q, \lambda - \mu; \frac{z}{\tau y} \right) d\tau. \]

The heading levels should not be more than 4 levels. The font of heading and subheadings should be 12 point normal Times New Roman. The first letter of headings and subheadings should be capitalized.

3. Extended incomplete \( \tau \)-Gauss and confluent hypergeometric functions

The main aim of this section is to initiate the study of the extended incomplete \( \tau \)-hypergeometric type function and the extended incomplete \( \tau \)-Appell functions, where, as mentioned in the introduction, the investigation of the usual cases was a concern of the recent years. The second aim of the section is to make the preparation for the next section, where we obtain their generating relations. We should note here that the results obtained in Sections 3 and 4 are reduced to the incomplete versions in the case \( p \to 0 \), where the reduced results will be new for \( \tau \)-incomplete special functions discussed in these sections.

We shall introduce the extended incomplete \( \tau \)-Gauss and confluent hypergeometric functions in terms of the extended incomplete beta function \( B_{\tau}(x, z; p) \), as follows:

\[ 2 \mathcal{R}^{(\tau,p)}_1 \left[ z; y \right] = 2 \mathcal{R}^{(\tau,p)}_1 (a, [b, c]; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\tau}(b + \tau n, c - b; p) z^n}{B(c - b, b) n!}, \quad (\text{Re}(p) > 0, \tau > 0, |z| < 1, \text{Re}(c) > \text{Re}(b) > 0) \tag{3.1} \]

and

\[ 1 \mathcal{F}^{(\tau,p)}_1 \left[ z; y \right] = 1 \mathcal{F}^{(\tau,p)}_1 ([b, c]; z) = \sum_{n=0}^{\infty} \frac{B_{\tau}(b + \tau n, c - b; p) z^n}{B(c - b, b) n!}, \quad (\text{Re}(p) > 0, \tau > 0, |z| < 1, \text{Re}(c) > \text{Re}(b) > 0). \tag{3.2} \]

Remark 3.1. The special case of the definitions (3.1) and (3.2) when \( \tau = 1 \) and \( p = 0 \) are easily seen to reduce to the incomplete Gauss and confluent hypergeometric functions [15]:

\[ 2 \mathcal{F}_1(a, [b, c]; z) = \sum_{n=0}^{\infty} (a)_n [b, c; y]_n \frac{z^n}{n!} \]

and

\[ 1 \mathcal{F}_1([b, c]; z) = \sum_{n=0}^{\infty} [b, c; y]_n \frac{z^n}{n!}. \]

Also, it should be mentioned that in the special case of (3.1) and (3.2) when \( p \to 0 \), we arrive at the new definitions which can be called as the incomplete \( \tau \)-Gauss and confluent hypergeometric functions as follows:

\[ 2 \mathcal{R}^{\tau}_1 \left[ z; y \right] = 2 \mathcal{R}^{\tau}_1 (a, [b, c]; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_{\tau}(b + \tau n, c - b) z^n}{B(c - b, b) n!}, \quad (\text{Re}(p) > 0, \tau > 0, |z| < 1, \text{Re}(c) > \text{Re}(b) > 0). \tag{3.3} \]
(0 \leq y < 1, \tau > 0, \ |z| < 1, \ \text{Re}(c) > \text{Re}(b) > 0)

and

\begin{equation}
1\Phi_1^\tau[z; y] = 1\Phi_1^\tau([b, c; y]; z) = \sum_{n=0}^{\infty} \frac{B_n(b + \tau n, c - b) z^n}{B(c - b, b)} n!.
\end{equation}

(0 \leq y < 1, \tau > 0, \ |z| < 1, \ \text{Re}(c) > \text{Re}(b) > 0)

In the following propositions, we obtain integral representations and derivative formulas for incomplete \(\tau\)-Gauss and confluent hypergeometric functions.

**Proposition 3.2.** The extended incomplete \(\tau\)-Gauss hypergeometric function can be represented by an integral as follows:

\begin{equation}
R_{1}^{(r,p)}(a, [b, c; y]; z) = \frac{y^b}{B(c - b, b)} \int_0^1 u^{b-1}(1 - uy)^{c-b-1}(1 - (uy)^\tau z)^a \exp \left( \frac{-p}{uy(1 - uy)} \right) du,
\end{equation}

\(p > 0; \ p = 0 \ \text{and} \ |\arg(1 - z)| < \pi, \ \text{Re}(c) > \text{Re}(b) > 0.

**Proof.** Replacing the extended incomplete beta function in the definition (3.5) by its integral representation given by (1.8), we have

\begin{equation}
R_{1}^{(r,p)}(a, [b, c; y]; z) = \frac{1}{B(c - b, b)} \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!} \int_0^\infty t^{b+n-1}(1 - t)^{c-b-1} \exp \left( \frac{-p}{t(1 - t)} \right) dt.
\end{equation}

From the uniform convergence, summation and integration can be interchanged. Then, we have

\begin{align*}
R_{1}^{(r,p)}(a, [b, c; y]; z) &= \frac{1}{B(c - b, b)} \int_0^\infty t^{b+n-1}(1 - t)^{c-b-1} (1 - zt)^{-a} \exp \left( \frac{-p}{t(1 - t)} \right) dt \\
&= \frac{y^b}{B(c - b, b)} \int_0^1 u^{b-1}(1 - uy)^{c-b-1}(1 - (uy)^\tau z)^a \exp \left( \frac{-p}{uy(1 - uy)} \right) du.
\end{align*}

\(\square

**Corollary 3.3.** If \(p\) is set to 0 in the above proposition, we reach the result corresponding to the incomplete \(\tau\)-Gauss hypergeometric function which is given as follows:

\begin{equation}
R_{1}^{(r)}(a, [b, c; y]; z) = \frac{y^b}{B(c - b, b)} \int_0^1 u^{b-1}(1 - uy)^{c-b-1}(1 - (uy)^\tau z)^{-a} du,
\end{equation}

\(\text{Re}(c) > \text{Re}(b) > 0, \ |z| < 1.

**Proposition 3.4.** For the extended incomplete \(\tau\)-confluent hypergeometric function, we have the following integral representation:

\begin{equation}
1\Phi_1^{(r,p)}([b, c; y]; z) = \frac{y^b}{B(c - b, b)} \int_0^1 u^{b-1}(1 - uy)^{-b-1} \exp \left( \frac{-p}{uy(1 - uy) + (uy)^\tau z} \right) du.
\end{equation}

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Volume 30, Issue 5, 1723–1747.
Corollary 3.5. If we set $p = 0$ in the above theorem, we can give the corresponding result for the incomplete $\tau$-confluent hypergeometric function as follows:

$$1_{\phi}^\tau([b, c; y] ; z) = \frac{y^b}{B(c - b, b)} \int_0^1 u^{b-1} (1 - uy)^{c-b-1} e^{(uy)z} \, du.$$  

Proposition 3.6. The equation shown below holds true for the incomplete $\tau$-Gauss hypergeometric function:

$$\frac{d^n}{dz^n} \left[ 2{}_{R}^{\tau}(a, [b, c; y] ; z) \right] = \frac{(a)_n (b)_m}{(c)_n} 2{}_{R}^{\tau}(a + n, [b + \tau n, c + \tau n; y] ; z).$$  \hspace{1cm} (3.6)

Proof. Using (3.3), differentiating on both sides with respect to $z$, we get

$$\frac{d}{dz} \left[ 2{}_{R}^{\tau}(a, [b, c; y] ; z) \right] = \frac{a}{B(c - b, b)} \int_0^\gamma t^{b+\tau-1}(1 - t)^{c-b-1}(1 - zt^{-1})^{-a-1} \, dt$$

$$= \frac{a}{B(c - b, b)} \int_0^\gamma t^{(b+\tau)-1}(1 - t)^{(c+\tau)-(b+\tau)-1}(1 - zt^{-1})^{-(a+1)} \, dt$$

$$= \frac{a(b)_\tau}{(c)_z} \frac{1}{B(c - b, b + \tau)} \int_0^\gamma t^{(b+\tau)-1}(1 - t)^{(c+\tau)-(b+\tau)-1}(1 - zt^{-1})^{-(a+1)} \, dt.$$  

which is (3.6) for $n = 1$. Recursive application of this procedure yields the general result. \hfill $\square$

In a similar manner, we have the following.

Proposition 3.7. The equation shown below holds true for the incomplete $\tau$-Gauss hypergeometric function:

$$\frac{d^n}{dz^n} \left[ 1_{\phi}^\tau([b, c; y] ; z) \right] = \frac{(b)_m}{(c)_n} 1_{\phi}^\tau([b + \tau n, c + \tau n; y] ; z).$$

In the following theorem, we give expressions for the Mellin transforms of the extended incomplete $\tau$-Gauss hypergeometric function an expression which involves incomplete $\tau$-Gauss hypergeometric function.

Theorem 3.8. The extended incomplete $\tau$-hypergeometric function has a Mellin transform which can be written as follows:

$$\mathfrak{M} \left[ 2{}_{R}^{(\tau,p)}(a, [b, c; y] ; z) : p \to s \right] = \frac{\Gamma(s) B(c - b + s, b + s)}{B(c - b, b)} 2{}_{R}^{\tau}(a, [b + s, c + 2s; y] ; z).$$  \hspace{1cm} (3.7)

Proof. To get the Mellin transform, multiplying (3.5) by $p^{s-1}$ and integrate over the interval $[0, \infty)$ with respect to $p$ to get

$$\mathfrak{M} \left[ 2{}_{R}^{(\tau,p)}(a, [b, c; y] ; z) : p \to s \right] = \int_0^\infty p^{s-1} 2{}_{R}^{(\tau,p)}(a, [b, c; y] ; z) \, dp$$

$$= \frac{y^b}{B(c - b, b)} \int_0^1 u^{b-1} (1 - uy)^{c-b-1}(1 - z(uy)^{\tau})^{-a} \times \left[ \int_0^\infty p^{s-1} \exp \left( \frac{-p}{uy(1 - uy)} \right) \, dp \right] \, du.$$  \hspace{1cm} (3.8)
Setting $p = tuy(1 – uy)$ in (3.8),

$$
\int_0^\infty p^{t-1} \exp\left(\frac{-p}{uy(1 – uy)}\right) dp = \int_0^\infty t^{y-1}(uy)^x(1 – uy)^y \exp(-t) dt
$$

$$
= (uy)^y(1 – uy)^y \int_0^\infty t^{y-1} \exp(-t) dt
$$

Thus we get

$$
\Re\{_{2}R^{(\tau,p)}_1(a, [b, c; y]; z) : p \to s \}
$$

$$
= \frac{y^b + s \Gamma(s)}{B(c - b, b)} \int_0^1 u^{b+1}(1 – uy)^{c-b-1}(1 – z(uy))^{-a} du
$$

$$
= \frac{\Gamma(s) B(c – b + s, b + s)}{B(c – b)} \frac{(3.9) – (3.12)}{2} R^s_1(a, [b + s, c + 2s; y]; z).
$$

Remark 3.9. Setting $s = 1$ in (3.7), we get

$$
\int_0^\infty _2^{(\tau,p)}R_1(a, [b, c; y]; z) dp = \frac{b(c – b)}{c(c + 1)} _2^{R_1}R_1(a, [b + 1, c + 2; y]; z).
$$

By means of the extended incomplete beta function $B_{s}(x, z; p)$ stated by (1.8), we introduce the extended incomplete $\tau$–Appell’s functions as follows:

$$
F^{\tau,p}_1[\alpha, \beta; \mu; x, z; y] = \sum_{m,n=0}^\infty \frac{B_s(\lambda + \tau m + \tau n, \mu - \lambda; p)}{B(\mu - \lambda, \lambda)} (\alpha)_m (\beta)_n \frac{x^m z^n}{m! n!}, \quad \text{(3.9)}
$$

$$
F^{\tau,p}_1[\alpha, \beta, \lambda; x, z; y] = \sum_{m,n=0}^\infty \frac{B_{1-s}(\mu - \lambda, \lambda + \tau m + \tau n; p)}{B(\mu - \lambda, \lambda)} (\alpha)_m (\beta)_n \frac{x^m y^n}{m! n!}, \quad \text{(3.10)}
$$

where $\max(|x|, |z|) < 1$, and

$$
F^{\tau,p}_2[\alpha, \beta, \lambda, \gamma; \mu; x, z; y] = \sum_{m,n=0}^\infty (\alpha)_{n+m} \frac{B_s(\beta + \tau m, \gamma - \beta; p) B_s(\lambda + \tau n, \mu - \lambda; p)}{B(\gamma - \beta, \beta)} \frac{x^m z^n}{m! n!}, \quad \text{(3.11)}
$$

$$
F^{\tau,p}_2[\alpha, \beta, \lambda, \gamma; \mu; x, z; y] = \sum_{m,n=0}^\infty (\alpha)_{n+m} \frac{B_{1-s}(\gamma - \beta, \beta + \tau m; p) B_{1-s}(\mu - \lambda, \lambda + \tau n; p)}{B(\gamma - \beta, \beta)} \frac{x^m y^n}{m! n!}, \quad \text{(3.12)}
$$

where $|x| + |z| < 1$.

Remark 3.10. The special case of the definitions (3.9–3.12) when $\tau = 1$ and $p \to 0$ are easily seen to reduce to the incomplete Appell’s functions [15]:

$$
F_1[\alpha, \beta; \mu; x, z; y] := \sum_{m,n=0}^\infty [\alpha, \beta, \gamma]_{m+n}(\alpha)_{m+n} (\beta)_n \frac{x^m z^n}{m! n!}, \quad \max(|x|, |z|) < 1.
$$
and 
\[ F_1[\lambda, \alpha, \beta; \mu; x, z; y] := \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_n x^m z^n}{m! n!}, \quad \max(|\lambda|, |\mu|) < 1 \]

and 
\[ F_2[\alpha, \beta, \lambda; \gamma, \mu; x, z; y] := \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_n x^m z^n}{m! n!}, \quad |x| + |z| < 1 \]

and 
\[ F_2[\alpha, \beta, \lambda; \gamma, \mu; x, z; y] := \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_n x^m z^n}{m! n!}, \quad |x| + |z| < 1. \]

Also, in the case of \( p \rightarrow 0 \), we can give the incomplete versions of the \( \tau \)-Appell’s functions as follows:

\[ F_1^*[\lambda, \alpha, \beta; \mu; x, z; y] = \sum_{m,n=0}^{\infty} \frac{B_p(\lambda + \tau m, \mu - \lambda)}{B(\mu - \lambda)} (\alpha)_m (\beta)_n \frac{x^m z^n}{m! n!}, \quad (3.13) \]

\[ F_2^*[\lambda, \alpha, \beta; \mu; x, z; y] = \sum_{m,n=0}^{\infty} \frac{B_{1-\gamma}(\mu - \lambda, \lambda + \tau m + \tau n)}{B(\mu - \lambda)} (\alpha)_m (\beta)_n \frac{x^m z^n}{m! n!}, \quad (3.14) \]

\[ F_2^*[\alpha, \beta, \lambda; \gamma, \mu; x, z; y] = \sum_{m,n=0}^{\infty} \frac{B_y(\beta + \tau m, \gamma - \beta) B_y(\lambda + \tau n, \mu - \lambda) x^m z^n}{B(\gamma - \beta, \beta) B(\mu - \lambda)} \frac{m! n!}{m! n!}, \quad (3.15) \]

and

\[ F_2^*[\alpha, \beta, \lambda; \gamma, \mu; x, z; y] = \sum_{m,n=0}^{\infty} \frac{B_{1-\gamma}(\gamma - \beta, \beta + \tau m) B_{1-\gamma}(\mu - \lambda, \lambda + \tau n) x^m z^n}{B(\gamma - \beta, \beta) B(\mu - \lambda)} \frac{m! n!}{m! n!}, \quad (3.16) \]

and we call these functions as the incomplete \( \tau \)-Appell’s functions.

We can rewrite the series for incomplete \( \tau \)-Appell’s functions in terms of the incomplete \( \tau \)-Gauss hypergeometric functions, so that

\[ F_2^*[\alpha, \beta, \lambda; \gamma, \mu; x, z; y] = \sum_{n=0}^{\infty} (\alpha)_n \frac{B_y(\lambda + \tau n, \mu - \lambda)}{B(\mu - \lambda)} 2R^*_y(\alpha + n, [\beta, \gamma; y]; x) \frac{z^n}{n!}. \]

In the following proposition, integral representations of the extended incomplete \( \tau \)-Appell’s functions are given.

**Proposition 3.11.** The extended incomplete \( \tau \)-Appell’s functions can be represented by an integral as follows:

\[ F_1^*[\lambda, \alpha, \beta; \mu; x, z; y] = \frac{y^\lambda}{B(\mu - \lambda)} \int_0^1 u^{\lambda-1} (1 - uy)^{\mu-1} (1 - x(uy)\tau)^{-\alpha} (1 - z(uy)\tau)^{-\beta} \]

\[ \times \exp\left(-\frac{p}{uy(1-uy)}\right) \, du, \quad p > 0; p = 0 \quad \text{and} \quad |\arg(1-y)| < \pi, \quad (3.17) \]

\[ |\arg(1-z)| < \pi, \quad Re(\mu) > Re(\lambda) > 0, \quad Re(\alpha) > 0, Re(\beta) > 0, \]
and

\[
F_2^{\tau_p}[\alpha, \beta, \lambda; \gamma, \mu; x, z; y] = \frac{y^{1-\beta}}{B(\gamma-\beta, \beta)B(\mu-\lambda, \lambda)} \int_0^1 \int_0^1 u^{\beta-1} (1 - uy)^{\gamma-\beta-1} v^{\lambda-1} (1 - vy)^{\mu-\lambda-1} \times (1 - (uy)^{\tau} x - (vy)^{\tau} z)^{-\alpha} \exp\left(\frac{-p}{uy(1-uy)}\right) \exp\left(\frac{-p}{vy(1-vy)}\right) dudv,
\]

\[p > 0; \quad p = 0 \text{ and } |\arg(1 - x - z)| < \pi, \quad Re(\mu) > Re(\lambda) > 0, \quad Re(\gamma) > Re(\beta) > 0, Re(\alpha) > 0. \tag{3.18}\]

**Proof.** Replacing the extended incomplete beta function in the definition (3.9) by its integral representation given by (1.8), then we have

\[
F_1^{\tau_p}[\alpha, \beta; \mu; x, z; y] = \frac{1}{B(\mu-\lambda, \lambda)} \sum_{m,n=0}^{\infty} (\alpha)_m (\beta)_n \frac{x^m z^n}{m! n!} \int_0^1 \int_0^1 t^{1+\tau m+n-1} (1 - t)^{\mu-\lambda-1} \exp\left(\frac{-p}{t(1-t)}\right) dt.
\]

From the uniform convergence condition, summation and integration can be swapped. Then, we get

\[
F_1^{\tau_p}[\alpha, \beta; \mu; x, z; y] = \frac{1}{B(\mu-\lambda, \lambda)} \int_0^1 t^{\lambda-1} (1 - t)^{\mu-\lambda-1} (1 - xt^{\tau})^{-\alpha} (1 - zt^{\tau})^{-\beta} \exp\left(\frac{-p}{t(1-t)}\right) dt
\]

\[
= \frac{y^\tau}{B(\mu-\lambda, \lambda)} \int_0^1 u^{\lambda-1} (1 - uy)^{\mu-\lambda-1} (1 - x(uy)^{\tau})^{-\alpha} (1 - z(uy)^{\tau})^{-\beta} \exp\left(\frac{-p}{uy(1-uy)}\right) du.
\]

Whence the result. In a similar manner, formula (3.18) can be proved. \qed

If \(p\) is set to 0 in the above proposition, we have the following corollary:

**Corollary 3.12.** The incomplete \(\tau\)-Appell’s functions can be represented by an integral as follows:

\[
F_1^{\tau}[\alpha, \beta; \mu; x, z; y] = \frac{y^\lambda}{B(\mu-\lambda, \lambda)} \int_0^1 u^{\lambda-1} (1 - uy)^{\mu-\lambda-1} (1 - x(uy)^{\tau})^{-\alpha} (1 - z(uy)^{\tau})^{-\beta} du,
\]

\[Re(\mu) > Re(\lambda) > 0, Re(\alpha) > 0, x \notin [1, \infty), z \notin [1, \infty), \tag{3.19}\]

and

\[
F_2^{\tau}[\alpha, \beta, \lambda; \gamma, \mu; x, z; y] = \frac{y^{1+\beta}}{B(\gamma-\beta, \beta)B(\mu-\lambda, \lambda)} \int_0^1 \int_0^1 u^{\beta-1} (1 - uy)^{\gamma-\beta-1} v^{\lambda-1} (1 - vy)^{\mu-\lambda-1} \times (1 - (uy)^{\tau} x - (vy)^{\tau} z)^{-\alpha} dudv,
\]

\[Re(\mu) > Re(\lambda) > 0, Re(\gamma) > Re(\beta) > 0, Re(\alpha) > 0, |\arg(1 - x - z)| < \pi. \tag{3.20}\]

Now we evaluate the following fractional derivative formulas, which we shall need them in the derivation of the generating functions in Section 4.
Proposition 3.13. Let $\Re(\mu) > \Re(\lambda) > 0$, $\Re(p) > 0$, $\Re(\alpha) > 0$, $\tau \in \mathbb{N}$ and $|z| < 1$. Then

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-z^\tau)^{-\alpha};y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \, _2F_1^{(\tau,p)}(\alpha, [\lambda, \mu; y]; z^\tau)
\]

and

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-z^\tau)^{-\alpha};y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \, 2R_1^{(\tau,p)}(\alpha, [\lambda, \mu; y]; z^\tau).
\]

Proof. We have

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-z^\tau)^{-\alpha};y] = \frac{z^{\mu-1}}{\Gamma(\mu - \lambda)} \int_0^y (uz)^{\lambda-1}(1-(uz)^\tau)^{-\alpha} (1-u)^{\mu-1} \exp\left(-\frac{p}{u(1-u)}\right) du
\]

\[
= \frac{z^{\mu-1}y^\lambda}{\Gamma(\mu - \lambda)} \int_0^1 (1-ty)^{\mu-1} (1-(ty)^\tau)^{-\alpha} \exp\left(-\frac{p}{ty(1-ty)}\right) dt.
\]

By (3.5), we can write

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-z^\tau)^{-\alpha};y] = \frac{z^{\mu-1}}{\Gamma(\mu - \lambda)} B(\lambda, \mu - \lambda) \, 2R_1^{(\tau,p)}(\alpha, [\lambda, \mu; y]; z^\tau)
\]

\[
= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \, 2R_1^{(\tau,p)}(\alpha, [\lambda, \mu; y]; z^\tau)
\]

Whence the result. In a similar manner, formula (3.21) can be proved. \(\Box\)

Proposition 3.14. Let $\Re(\mu) > \Re(\lambda) > 0$, $\Re(\beta) > 0$, $\Re(\alpha) > 0$, $\Re(p) > 0$, $\tau \in \mathbb{N}$; $|z| < \min\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)$. Then

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-az^\tau)^{-\alpha}(1-bz^\tau)^{-\beta};y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \, _2F_1^{(\tau,p)}[\lambda, \alpha, \beta; \mu; az^\tau, bz^\tau; y]
\]

and

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-az^\tau)^{-\alpha}(1-bz^\tau)^{-\beta};y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \, _2F_1^{(\tau,p)}[\lambda, \alpha, \beta; \mu; az^\tau, bz^\tau; y].
\]

Proof. Direct calculations yield

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-az^\tau)^{-\alpha}(1-bz^\tau)^{-\beta};y] = \frac{z^{\mu-1}}{\Gamma(\mu - \lambda)} \int_0^y (uz)^{\lambda-1}(1-a(uz)^\tau)^{-\alpha}(1-b(uz)^\tau)^{-\beta} (1-u)^{\mu-1} \exp\left(-\frac{p}{u(1-u)}\right) du
\]

\[
= \frac{z^{\mu-1}y^\lambda}{\Gamma(\mu - \lambda)} \int_0^1 (1-ty)^{\mu-1} (1-a(ty)^\tau)^{-\alpha} (1-b(ty)^\tau)^{-\beta} (1-ty)^{\mu-1} \times \exp\left(-\frac{p}{ty(1-ty)}\right) dt.
\]

By (3.17), we can write

\[
D_z^{1-\mu,p}[z^{\lambda-1}(1-az^\tau)^{-\alpha}(1-bz^\tau)^{-\beta};y] = \frac{z^{\mu-1}}{\Gamma(\mu - \lambda)} B(\lambda, \mu - \lambda) \, _2F_1^{(\tau,p)}[\lambda, \alpha, \beta; \mu; az^\tau, bz^\tau; y]
\]
\[= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{1}^{(\tau,p)} [\lambda, \alpha; \beta; \mu; az^\tau, b z^\tau; y].\]

Hence the proof is completed. In a similar manner, formula (3.22) can be proved. \(\square\)

**Proposition 3.15.** For \(\text{Re}(\mu) > \text{Re}(\lambda) > 0, \text{Re}(\beta) > 0, \text{Re}(\alpha) > 0, \text{Re}(p) > 0, \text{Re}(\gamma) > 0, \tau \in \mathbb{N}; \left|\frac{t}{1-z}\right| < 1 \text{ and } |t| + |z| < 1, \) we have

\[D_{z}^{k-\mu,p} \left[ z^{\lambda-1} (1 - z^\tau)^{-\alpha} \right] z R_{1}^{(\tau,p)} (\alpha, [\beta, \gamma; y]; \frac{t}{1 - z}) ] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{2}^{(\tau,p)} [\alpha, \beta, \lambda; \gamma, \mu; t, z^\tau; y].\]

**Proof.** Using Example 12 and (3.15), we have

\[D_{z}^{k-\mu,p} \left[ z^{\lambda-1} (1 - z^\tau)^{-\alpha} \right] z R_{1}^{(\tau,p)} (\alpha, [\beta, \gamma; y]; \frac{t}{1 - z}) ] = \frac{1}{B(\gamma - \beta, \beta)} D_{z}^{k-\mu,p} \left[ z^{\lambda-1} (1 - z^\tau)^{-\alpha} \right] \sum_{n=0}^{\infty} \frac{(\alpha)_n B_n (\beta + \tau n, \gamma - \beta; p)}{n!} \left( 1 - z^\tau \right)^{-\alpha - n}; y\]

\[= \frac{1}{B(\gamma - \beta, \beta)} \sum_{n,m=0}^{\infty} B_n (\beta + \tau n, \gamma - \beta; p) \frac{n}{n!} D_{z}^{\lambda-\mu,p} \left[ z^{\lambda+m-1} \right] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{2}^{(\tau,p)} [\alpha, \beta, \lambda; \gamma, \mu; t, z^\tau; y].\]

\(\square\)

In the case \(p \to 0,\) we list the consequences of the above propositions for the incomplete \(\tau\)-hypergeometric functions below:

**Corollary 3.16.** Let \(\text{Re}(\alpha) > 0, \text{Re}(\mu) > \text{Re}(\lambda) > 0, \tau \in \mathbb{N} \text{ and } |z| < 1. \) Then

\[D_{z}^{\lambda-\mu}[z^{\lambda-1} (1-z^\tau)^{-\alpha}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} z R_{1}^{\tau}(\alpha, [\lambda, \mu]; y^\tau).\]

**Corollary 3.17.** Let \(\text{Re}(\mu) > \text{Re}(\lambda) > 0, \text{Re}(\beta) > 0, \text{Re}(\alpha) > 0, \tau \in \mathbb{N}; \) \(|z| < \min\left(\frac{1}{\alpha}, \frac{1}{\beta}\right). \) Then

\[D_{z}^{\lambda-\mu}[z^{\lambda-1} (1-az^\tau)^{-\alpha} (1-bz^\tau)^{-\beta}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{1}^{\tau} [\lambda, \alpha, \beta; \mu; az^\tau, bz^\tau; y].\]

**Corollary 3.18.** For \(\text{Re}(\mu) > \text{Re}(\lambda) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0, \text{Re}(\alpha) > 0, \tau \in \mathbb{N}; \left|\frac{t}{1-z}\right| < 1 \text{ and } |t| + |z| < 1, \) we have

\[D_{z}^{k-\mu,p} \left[ z^{\alpha-1} (1 - z^\tau)^{-\alpha} \right] z R_{1}^{\tau}(\alpha, [\beta, \gamma; y]; \frac{t}{1 - z^\tau}) ] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{2}^{\tau} [\alpha, \beta, \lambda; \gamma, \mu; t, z^\tau; y].\]
4. Generating functions

Here we shall obtain linear and bilinear type generating relations for the extended incomplete \( \tau \)-hypergeometric functions.

**Theorem 4.1.** The extended incomplete \( \tau \)-hypergeometric function can be represented by a linear generating relation as follows:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} 2^{(\tau,p)}_1 \left( \alpha + n, [\alpha, \beta; y] ; z^\tau \right) t^n = (1 - t)^{-\lambda} 2^{(\tau,p)}_1 \left( \lambda, [\alpha, \beta; y] ; \frac{z^\tau}{1-t} \right)
\]

(4.1)

where \( \tau \in \mathbb{N}, |z| < \min\{1,|1-t|\} \).

**Proof.** By expanding as a binomial series, we have for \(|t| < |1-z^\tau|\) that

\[
(1 - z^\tau)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left( \frac{t}{1-z^\tau} \right)^n = [(1 - z^\tau) - t]^{-\lambda} = (1 - t)^{-\lambda} \left[ 1 - \frac{z^\tau}{1-t} \right]^{-\lambda}.
\]

Multiplying by \( z^{\alpha-1} \) on both sides and applying the extended incomplete fractional derivative operator \( D_z^{\alpha-\beta,p}[f(z); y] \) on both sides, we have

\[
D_z^{\alpha-\beta,p} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1 - z^\tau)^{-\lambda} \left( \frac{t}{1-z^\tau} \right)^n z^{\alpha-1}; y \right] = (1 - t)^{-\lambda} D_z^{\alpha-\beta,p} \left[ z^{\alpha-1} \left[ 1 - \frac{z^\tau}{1-t} \right]^{-\lambda}; y \right].
\]

Swapping the summation and integration, we get

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_z^{\alpha-\beta,p} \left[ z^{\alpha-1} (1 - z^\tau)^{-\lambda-n}; y \right] t^n = (1 - t)^{-\lambda} D_z^{\alpha-\beta,p} \left[ z^{\alpha-1} \left[ 1 - \frac{z^\tau}{1-t} \right]^{-\lambda}; y \right].
\]

Using Proposition 6, the result follows. \( \square \)

**Theorem 4.2.** The extended incomplete \( \tau \)-hypergeometric function can be represented by a linear generating relation as follows:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} 2^{(\tau,p)}_1 \left( \rho - n, [\alpha, \beta; y] ; z^\tau \right) t^n = (1 - t)^{-\lambda} F_1^{(\tau,p)} \left[ \alpha, \rho, \lambda; \beta; z^\tau, \frac{-z^\tau t}{1-t}; y \right]
\]

where \( \tau \in \mathbb{N}, |t| < \frac{1}{1+|z^\tau|} \).

**Proof.** By expanding as a binomial series, we have for \(|t| < |1-z^\tau|\) that

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1 - z^\tau)^{\rho-n} t^n = [1 - (1 - z^\tau) t]^{-\lambda} = (1 - t)^{-\lambda} \left[ 1 + \frac{z^\tau t}{1-t} \right]^{-\lambda}.
\]

Multiplying by \( z^{\alpha-1} (1 - z^\tau)^{-\rho} \) on both sides and applying the extended incomplete fractional derivative operator \( D_z^{\alpha-\beta,p}[f(z); y] \) on both sides, we have

\[
D_z^{\alpha-\beta,p} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\alpha-1} (1 - z^\tau)^{-\rho-n} t^n; y \right] = (1 - t)^{-\lambda} D_z^{\alpha-\beta,p} \left[ z^{\alpha-1} (1 - z^\tau)^{-\rho} \left[ 1 - \frac{-z^\tau t}{1-t} \right]^{-\lambda}; y \right].
\]
Interchanging the order, we get
\[
\sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{n!} D_{x}^{\gamma, \delta, p} \left[ x^{\gamma-1} (1 - x^\gamma)^{-\delta}; y \right] t^n = (1 - t)^{-\lambda} D_{x}^{\gamma, \delta, p} \left[ x^{\gamma-1} (1 - x^\gamma)^{-\delta}; \frac{-z t}{1 - t} \right] ; y .
\]

To get the desired result, we use Propositions 6 and 7.

\[ \square \]

**Theorem 4.3.** The extended incomplete \( \tau \)-hypergeometric function can be represented by a bilinear generating relation as follows:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{n!} 2 R_{1}^{(\tau, p)} (-n, [\gamma, \delta; y]; x^\tau) R_{1}^{(\tau, p)} (\lambda + n, [\alpha, \beta; y]; z^\tau) t^n = (1 - t)^{-\lambda} F_{2}^{(\tau, p)} \left[ \lambda, \alpha, \gamma; \beta, \delta; \frac{z^\tau}{1 - t}, \frac{-x^\tau t}{1 - t}; y \right]
\]

where \( \tau \in \mathbb{N}, |t| < \frac{\tau}{1+|\tau|} \) and \(|z| < 1\).

**Proof.** Starting from (4.2), we replace \( t \) with \((1 - x^\tau) t\), introduce a factor of \( x^{\gamma-1} \), and apply \( D_{x}^{\gamma, \delta, p} [f(x); y] \). Then we have

\[
D_{x}^{\gamma, \delta, p} \sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{n!} x^{\gamma-1} 2 R_{1}^{(\tau, p)} (\lambda + n, [\alpha, \beta; y]; z^\tau) (1 - x^\gamma)^{-\delta}; y \] 
\( = D_{x}^{\gamma, \delta, p} \left[ (1 - (1 - x^\tau) t)^{-\lambda} x^{\gamma-1} 2 R_{1}^{(\tau, p)} (\lambda, [\alpha, \beta; y]; \frac{z^\tau}{1 - (1 - x^\tau) t}; y \right].
\]

Provided that \(|z| < 1, |\frac{1}{1 - x^\tau}| < 1 \) and \(|\frac{\tau}{1+|\tau|}| < 1\), we can interchange the order to obtain:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{n!} D_{x}^{\gamma, \delta, p} \left[ x^{\gamma-1} (1 - x^\gamma)^{-\delta}; y \right] 2 R_{1}^{(\tau, p)} (\lambda + n, [\alpha, \beta; y]; z^\tau) 
\( = (1 - t)^{-\lambda} D_{x}^{\gamma, \delta, p} \left[ x^{\gamma-1} \left( 1 - \frac{-x^\tau t}{1 - t} \right)^{-\lambda} 2 R_{1}^{(\tau, p)} (\lambda, [\alpha, \beta; y]; \frac{z^\tau}{1 - \frac{x^\tau t}{1 - t}}); y \right].
\]

To get the result, we use Propositions 6 and 8.

\[ \square \]

In a similar manner, the linear and bilinear type generating relation can be given for incomplete \( \tau \)-hypergeometric function.

**Corollary 4.4.** The incomplete \( \tau \)-hypergeometric function can be represented by a linear generating relation as follows:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{n!} 2 R_{1}^{(\tau, p)} (\lambda + n, [\alpha, \beta; y]; z^\tau) t^n = (1 - t)^{-\lambda} 2 R_{1}^{(\tau, p)} (\lambda, [\alpha, \beta; y]; \frac{z^\tau}{1 - t}).
\]

**Corollary 4.5.** The incomplete \( \tau \)-hypergeometric function can be represented by a linear generating relation as follows:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_{n}}{n!} 2 R_{1}^{(\tau, p)} (\rho - n, [\alpha, \beta; y]; z^\tau) t^n = (1 - t)^{-\lambda} F_{1}^{(\tau, p)} \left[ \alpha, \rho, \lambda; \beta, z^\tau, \frac{-z^\tau t}{1 - t}; y \right].
\]

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Corollary 4.6. The incomplete $\tau$-hypergeometric function can be represented by a bilinear generating relation as follows:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\lambda}{n!} z^n R_1^\tau (\alpha, \beta; z; t) = (1 - t)^{-\lambda} F_1^\alpha \left[ \lambda, \beta; \frac{z^\mu}{1 - t}; y \right].$$

Here we shall obtain linear and bilinear type generating relations for the extended incomplete $\tau$-hypergeometric functions.

Theorem 4.7. The extended incomplete $\tau$-hypergeometric function can be represented by a linear generating relation as follows:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\lambda}{n!} z^n R_1^\tau (\alpha, \beta; z; t) = (1 - t)^{-\lambda} z^\tau (1 - t)^{-\lambda} R_1^\tau (\lambda, \alpha, \beta; z; t).$$

where $\tau \in \mathbb{N}, |z| < \min\{1, |1 - t|\}$.

Proof. By expanding as a binomial series, we have for $|t| < |1 - z^\tau|$ that

$$(1 - z^\tau)^{-\lambda} \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\lambda}{n!} \left( \frac{t}{1 - z^\tau} \right)^n = (1 - z^\tau)^{-\lambda} = (1 - t)^{-\lambda} \left[ 1 - \frac{z^\tau}{1 - t} \right].$$

Multiplying by $z^{\alpha - 1}$ on both sides and applying the extended incomplete fractional derivative operator $D_z^{\alpha-\beta,p} [f(z); y]$ on both sides, we have

$$D_z^{\alpha-\beta,p} \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\lambda}{n!} (1 - z^\tau)^{-\lambda} \left[ 1 - \frac{z^\tau}{1 - t} \right]^\alpha = (1 - t)^{-\lambda} D_z^{\alpha-\beta,p} \left[ z^{\alpha-1} \left[ 1 - \frac{z^\tau}{1 - t} \right] \right]^\alpha.$$ 

Swapping the summation and integration, we get

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\lambda}{n!} D_z^{\alpha-\beta,p} \left[ z^{\alpha-1} (1 - z^\tau)^{-\lambda} \right]^\alpha = (1 - t)^{-\lambda} D_z^{\alpha-\beta,p} \left[ z^{\alpha-1} \left[ 1 - \frac{z^\tau}{1 - t} \right] \right]^\alpha.$$ 

Using Proposition 6, the result follows. 

Theorem 4.8. The extended incomplete $\tau$-hypergeometric function can be represented by a linear generating relation as follows:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\lambda}{n!} z^n R_1^\tau (\alpha, \beta; z; t) = (1 - t)^{-\lambda} F_1^\alpha \left[ \alpha, \beta; \frac{-z^\tau t}{1 - t}; y \right].$$

where $\tau \in \mathbb{N}, |t| < \frac{1}{1+|z^\tau|}$.

Proof. By expanding as a binomial series, we have for $|t| < |1 - z^\tau|$ that

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\lambda}{n!} z^n (1 - z^\tau)^{-\lambda} t^n = (1 - (1 - z^\tau)t)^{-\lambda} = (1 - t)^{-\lambda} \left[ 1 + \frac{z^\tau t}{1 - t} \right].$$
Multiplying by \( z^{\alpha-1}(1 - z^\tau)^{-\rho} \) on both sides and applying the extended incomplete fractional derivative operator \( D^{\alpha-\beta,p}_z[f(z); y] \) on both sides, we have

\[
D^{\alpha-\beta,p}_z \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\alpha-1}(1 - z^\tau)^{-\rho+n} t^n ; y \right] = (1 - t)^{-\lambda} D^{\alpha-\beta,p}_z \left[ z^{\alpha-1}(1 - z^\tau)^{-\rho} \left[ 1 - \frac{-z^\tau t}{1 - t} \right] ; y \right].
\]

Interchanging the order, we get

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D^{\alpha-\beta,p}_z \left[ z^{\alpha-1}(1 - z^\tau)^{-\rho+n} ; y \right] t^n = (1 - t)^{-\lambda} D^{\alpha-\beta,p}_z \left[ z^{\alpha-1}(1 - z^\tau)^{-\rho} \left[ 1 - \frac{-z^\tau t}{1 - t} \right] ; y \right].
\]

To get the desired result, we use Propositions 6 and 7.

**Theorem 4.9.** The extended incomplete \( \tau \)-hypergeometric function can be represented by a bilinear generating relation as follows:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \tau(t^{\alpha}, \underbrace{\sum_{n=0}^{\infty} (\lambda)_n \tau(t^{\alpha})}_{z}; \overbrace{x, \tau, \lambda, \gamma; \beta, \gamma}^{\alpha, \beta}; \overbrace{\lambda, \alpha, \beta, \gamma; \tau, \zeta}^{\alpha, \beta}; \overbrace{\frac{-t^\tau}{1 - t}; y}^{\alpha, \beta})\tau(t^{\alpha})
\]

where \( \tau \in \mathbb{N}, |t| < \frac{1 - |\alpha|}{|\tau|} \) and \( |z| < 1 \).

**Proof.** Starting from (4.2), we replace \( t \) with \( (1 - x^\tau)t \), introduce a factor of \( x^{\alpha-1} \), and apply \( D^{\alpha-\beta,p \alpha}_x[f(x); y] \). Then we have

\[
D^{\alpha-\beta,p}_x \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^{\alpha-1} \tau(t^{\alpha}) \left( \lambda + n, [\alpha, \beta; y] ; z^\tau \right) (1 - x^\tau)^n t^n ; y \right]
\]

\[
= D^{\alpha-\beta,p}_x \left[ (1 - (1 - x^\tau)t)^{-\lambda} x^{\alpha-1} \tau(t^{\alpha}) \left( \lambda + n, [\alpha, \beta; y] ; \frac{z^\tau}{1 - (1 - x^\tau)t} \right) t^n ; y \right].
\]

Provided that \( |z| < 1, \frac{1 - |\alpha|}{|\tau|} < 1 \) and \( \frac{z^\tau}{1 - (1 - x^\tau)t} < 1 \), we can interchange the order to obtain:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D^{\alpha-\beta,p}_x \left[ x^{\alpha-1} \left( 1 - x^\tau \right)^n t^n ; y \right] \tau(t^{\alpha}) \left( \lambda + n, [\alpha, \beta; y] ; z^\tau \right)
\]

\[
= (1 - t)^{-\lambda} \tau(t^{\alpha}) \left( \lambda, [\alpha, \beta; y] ; \frac{z^\tau}{1 - (1 - x^\tau)t} \right);
\]

To get the result, we use Propositions 6 and 8.

In a similar manner, the linear and bilinear type generating relation can be given for incomplete \( \tau \)-hypergeometric function.

**Corollary 4.10.** The incomplete \( \tau \)-hypergeometric function can be represented by a linear generating relation as follows:

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \tau(t^{\alpha}) \left( \lambda + n, [\alpha, \beta; y] ; z^\tau \right) t^n = (1 - t)^{-\lambda} \tau(t^{\alpha}) \left( \lambda, [\alpha, \beta; y] ; \frac{z^\tau}{1 - t} \right).
\]
Corollary 4.11. The incomplete $\tau$-hypergeometric function can be represented by a linear generating relation as follows:

$$
\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} 2R_1^\tau (\rho - n, [\alpha, \beta; y] ; z^\tau) t^n = (1 - t)^{-\lambda} F_1^\tau \left[ \alpha, \rho, \lambda; \beta; z^\tau, \frac{-z^\tau t}{1 - t}; y \right].
$$

Corollary 4.12. The incomplete $\tau$-hypergeometric function can be represented by a bilinear generating relation as follows:

$$
\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} 2R_1^\tau (-n, [\gamma, \delta; y] ; x^\tau) 2R_1^\tau (\lambda + n, [\alpha, \beta; y] ; z^\tau) t^n = (1 - t)^{-\lambda} F_2^\tau \left[ \lambda, \alpha, \gamma; \beta, \delta; \frac{z^\tau}{1 - t}, \frac{-x^\tau t}{1 - t}; y \right].
$$

Conflict of interest

The authors declare there is no conflicts of interest.

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