Improved variational approach for the Cornell potential

V.V. Kudryashov and V.I. Reshetnyak

Institute of Physics, National Academy of Sciences of Belarus
68 Nezavisimosti Ave., 220072, Minsk, Belarus

The approximate radial wave functions for the Cornell potential describing quark-antiquark interaction are constructed in the framework of a variational method. The optimal values of the variational parameters are fixed by the fulfillment of the requirements of the virial theorem and the minimality condition for integral discrepancy. The results of calculation with the simple trial function are in a good agreement with exact numerical results.

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Many investigations have been devoted to the quantum mechanical non-relativistic description of quark-antiquark bound states with the Cornell potential [1] by means of different approximation procedures (see, e.g., [2, 3, 4, 5] and references therein) based on perturbation theory, the variational method and their combinations. Recently a new improved variational approach [6] was proposed. This method was applied to the radial Schrödinger equation with nonsingular power-law potentials $r^s$ with positive $s$ [7]. In the present work, we apply our approach to the Cornell potential with the Coulomb singularity.

Following [1] we study the radial Schrödinger equation in the dimensionless form

$$
\hat{H}\psi(r) = E\psi(r), \quad \hat{H} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r)
$$

for the Cornell (Coulomb plus linear) potential

$$
V(r) = -\frac{k}{r} + r.
$$

We use the simple functions [8]

$$
<r|\psi_{l,n}(a,b)> = \psi_{l,n}(a,b,r) = N\sqrt{a}G_{l,n}(b,ar)
$$

as the trial functions, where

$$
G_{l,n}(b,x) = x^{l+1}\exp(-bx)L_{n}^{2l+1}(2bx).
$$

Here $L_{n}^{2l+1}(2bx)$ is the Laguerre polynomial, $a$ and $b$ are variational parameters, $N$ is a normalization factor ($<\psi_{l,n}(a,b)|\psi_{l,n}(a,b)> = 1$). Note that these functions reproduce the exact results for the Coulomb potential if we choose $b = 1$. In order to fix the parameter values we consider two conditions to which the trial functions should satisfy.

The first is the virial theorem. The virial condition

$$
<\psi_{l,n}(a,b)| -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{1}{2}\frac{1}{r} - \frac{1}{2}r|\psi_{l,n}(a,b)> = 0
$$

leads to the following equation for $a$ :

$$
a^3 = ku_2(b)a^2 + u_3(b)
$$

where

$$
u_2(b) = \frac{\int_0^\infty G_{l,n}(b,x)x^{-1}dx}{2\int_0^\infty G_{l,n}(b,x)(-\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2})G_{l,n}(b,x)dx},
$$
\[ u_3(b) = \frac{\int_0^\infty G_{l,n}(b,x) dx}{2 \int_0^\infty G_{l,n}(b,x)(-\frac{d^2}{dx^2} + \frac{k}{x})G_{l,n}(b,x) dx}. \]

The exact solution of this equation allows us to express the parameter \( a \) via the parameter \( b \) :

\[
a_0(b) = \frac{1}{3} ku_2(b) + \left( \frac{k^2 u_2^3(b)}{27} + \frac{u_3(b)}{2} + \sqrt{\frac{k^2 u_2^3(b)u_3(b)}{27} + \frac{u_3^2(b)}{4}} \right)^{1/3} + \left( \frac{k^2 u_2^3(b)}{27} + \frac{u_3(b)}{2} - \sqrt{\frac{k^2 u_2^3(b)u_3(b)}{27} + \frac{u_3^2(b)}{4}} \right)^{1/3} \tag{7}
\]

Thus, the considered problem is transformed into a one-parameter problem. Note that the virial condition is equivalent to the usual condition

\[
\frac{\partial}{\partial a} <\psi_{l,n}(a,b)|\hat{\mathcal{H}}|\psi_{l,n}(a,b)> = 0. \tag{8}
\]

Denote the one-parameter trial functions with \( a = a_0(b) \) as

\[
< r | \psi_{l,n}^0(b) >= \psi_{l,n}^0(b,r) = < r | \psi_{l,n}(a_0(b),b) >= \psi_{l,n}(a_0(b),b,r).
\]

Then the energy has the form

\[
E_{l,n}^{(1)}(b) = < \psi_{l,n}^0(b)|\hat{\mathcal{H}}|\psi_{l,n}^0(b) >. \tag{9}
\]

In addition to the energy, expectation values of the squared Hamiltonian characterizing the goodness of the approximate eigenfunctions, can be calculated:

\[
E_{l,n}^{(2)}(b) = \sqrt{< \psi_{l,n}^0(b)|\hat{\mathcal{H}}^2|\psi_{l,n}^0(b) >}. \tag{10}
\]

As the second requirement imposed on the trial function, we select the requirement that integral discrepancy

\[
d_{l,n}(b) = \frac{< \psi_{l,n}^0(b)|\hat{\mathcal{H}}^2|\psi_{l,n}^0(b) >}{( < \psi_{l,n}^0(b)|\hat{\mathcal{H}}|\psi_{l,n}^0(b) >)^2} - 1, \tag{11}
\]

is minimum. The quantity \( d_{l,n}(b) \) characterizes goodness of the approximation and is equal to zero for an exact solution of the Schrödinger equation. The integral discrepancy is simply connected with the local discrepancy

\[
< r | D_{l,n}(b) >= D_{l,n}(b,r) = \frac{\hat{\mathcal{H}} \psi_{l,n}^0(b,r)}{< \psi_{l,n}^0(b,r)|\hat{\mathcal{H}}|\psi_{l,n}^0(b,r) >} - \psi_{l,n}^0(b,r) \tag{12}
\]

by relation

\[
d_{l,n}(b) = < D_{l,n}(b)|D_{l,n}(b) >. \tag{13}
\]

The corresponding equation for selection of parameter \( b \) is

\[
\frac{d}{db}d_{l,n}(b) = 0. \tag{14}
\]

Note that we must find the absolute minimum of \( d_{l,n}(b) \) corresponding to the minimal rotation of a trial vector under the action of the Hamiltonian in Hilbert space.

The minimality condition for integral discrepancy is alternative to the minimal sensitivity condition

\[
\frac{d}{db} E_{l,n}^{(1)}(b) = 0. \tag{15}
\]

There can be several stationary points in the case of the function \( E_{l,n}^{(1)}(b) \), and there is no method to select one of them. Besides, energy is not a preferred quantity in comparison with the other expectation values. It is also well
known, that calculation of the energy with increased accuracy does not always lead to the improvement of other characteristics.

We compare our results $E_{l,n}^{(1)}$ with the results $E_{l,n}^{vfm}$ of some variational method (VFM) [2] and with the results $E_{l,n}^{num}$ of numerical integration of the Schrödinger equation [1]. We also compare our values of the quantity

$$<v^2>_{l,n} = \int_0^\infty \left( \frac{d\psi_{l,n}}{dr} \right)^2 dr$$

(16)

with values from [1].

Table 1 shows that the proposed approximation gives fairly accurate results in the case of the purely linear potential. Table 2 demonstrates the efficiency of our approach in the case of the combined Coulomb plus linear potential.

For example we compare our variant (14) and usual variant (15) in the case $k = 1$, $l = 1$, $n = 1$:

$$E_{1,1}^{(1)}(14) = 4.4619, \quad E_{1,1}^{(2)}(14) = 4.4620; \quad <v^2>_{1,1}^{(1,2)}(14) = 1.1868 (a = 0.5439, b = 1.6769),$$

$$E_{1,1}^{(1)}(15) = 4.4583, \quad E_{1,1}^{(2)}(15) = 5.0658; \quad <v^2>_{1,1}^{(1,2)}(15) = 0.9905 (a = 1.3352, b = 0.9999)$$

while the numerical solution [1] gives $E_{1,1}^{num} = 4.4619, <v^2>^{num}_{1,1} = 1.1864$.

We see that in spite of the simplicity of the used trial functions, the optimal choice of the variational parameters leads to the satisfactory approximation. At the same time we assume that the approximation can be improved by means of some complication of the trial function reproducing the behavior at the origin more accurately.
[1] E. Eichten et al. Charmonium: The model. Phys. Rev. D17, 3090-3117 (1978).
[2] G. A. Arteca, F. M. Fernandez, E. A. Castro. Simple variational approaches to eigenvalues in quantum theory. J. Phys. A: Math. Gen. 20, 2221-2224 (1987).
[3] A. V. Turbiner. On eigenfunctions in quarkonium potential models (Perturbation theory and variational method). Yad. Fiz. 46, 204-218 (1987) (Russian).
[4] H. Ciftci, R. L. Hall, Q. D. Katadbeh. Coulomb plus power-law potentials in quantum mechanics. J. Phys. A: Math. Gen. 36, 7001-7008 (2003).
[5] F. M. Fernandez. Variationally improved perturbation theory for central-field models. Eur. J. Phys. 24, 289-296 (2003).
[6] V. V. Kudryashov, V. I. Reshetnyak. Modification of the perturbative-variational approach. Proc. of 9-th Annual Seminar "Nonlinear Phenomena in Complex Systems", eds. L. Babichev and V. Kuvshinov, Minsk, 2000, pp. 168-171.
[7] V. V. Kudryashov, V. I. Reshetnyak. Improved variational approach for radial wave functions. Nonlineae Dynamics and Applications, eds. L. Babichev and V. Kuvshinov, Minsk, 2007, pp. 81-84.
[8] H. Ciftci, E. Ateser and H. Koru. The power-law and logarithmic potentials. J.Phys.A: Math.Gen. 36, 3621-3628 (2003).
[9] V. V. Kudryashov, V. I. Reshetnyak. Minimization of relative discrepancy and a variational method for excited states. Foundation and Advances in Nonlinear Science, eds. V. Kuvshinov and G. Krylov, Minsk, 2004, pp. 43-47.
[10] I. D. Feranchuk et al. Operator method in the problem of quantum anharmonic oscillator. Ann. of Phys. 238, 370-440 (1995).