Exactly solvable strings in Minkowski spacetime

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Abstract
We study the integrability of the equations of motion for the Nambu–Goto strings with a cohomogeneity-one symmetry in Minkowski spacetime. A cohomogeneity-one string has a world surface which is tangent to a Killing vector field. By virtue of the Killing vector, the equations of motion reduce to the geodesic equation in the orbit space. Cohomogeneity-one strings are classified into seven classes (types I to VII). We investigate the integrability of the geodesic equations for all the classes and find that the geodesic equations are integrable. For types I to VI, the integrability comes from the existence of Killing vectors on the orbit space which are the projections of Killing vectors on Minkowski spacetime. For type VII, the integrability is related to a projected Killing vector and a nontrivial Killing tensor on the orbit space. We also find that the geodesic equations of all types are exactly solvable, and show the solutions.

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1. Introduction
Cosmic strings are topological defects which are produced when the \(U(1)\) symmetry breaks down in the unified theories. Such a symmetry breaking is supposed to have occurred at the early stage of the Universe. If the existence of the cosmic strings is confirmed, it is a strong evidence of vacuum phase transition in the Universe. Besides the interests from the unified theories, cosmic strings have been studied in the context of cosmology since they were proposed as possible seeds of the structure formation in the Universe [1, 2]. However, this scenario was rejected due to conflict with the precise observational data of cosmic microwave backgrounds [3, 4].
Recently, cosmic strings have gathered much attention in the context of the superstring theories, since fundamental strings and other string-like solitons such as D-strings could exist in the Universe as cosmic strings [5]. In the brane inflation models, these cosmic superstrings are produced at the end of inflation [6, 7] and stretched by the expansion of the Universe. The detection of the cosmic superstrings will give a strong evidence of the superstring theories.

One of the main differences between the gauge-theoretic cosmic strings and the cosmic superstrings is in the reconnection probabilities. For the former strings, the reconnection probability is almost one. Therefore, when a long gauge-theoretic string intersects with itself, it breaks up into a closed string and an open string which is shortened by the reconnection. Closed strings cannot exist stably since they decay by radiating gravitational waves. Then, most of the gauge-theoretic strings could not survive in the Universe. For the cosmic superstrings, it is clarified that the reconnection probability may be much suppressed [8]. If it is true, the cosmic superstrings can survive in the Universe and stay in isolation. Since they rarely interact with each other, they must have gone through some relaxation process such as gravitational radiations, and then, they must be in stationary motions.

The stationary string has a world surface which is tangent to a timelike Killing vector field. The existence of the tangent Killing vector reduces the equations of motion to ordinary differential equations, which are much more tractable than partial differential equations. Stationary strings in various spacetimes have been studied [9–20], and many non-trivial solutions were found even in Minkowski spacetime [9–11, 16, 18, 20].

The notion of stationary string is generalized to that of a cohomogeneity-one string. A cohomogeneity-one string is defined as a string whose world surface is tangent to a Killing vector field on the spacetime which is not restricted to being timelike. If the Killing vector is timelike, the cohomogeneity-one string is stationary. The cohomogeneity-one string is characterized by the tangent Killing vector. When the spacetime admits multiple independent Killing vectors, there are infinitely many Killing vectors in the form of linear combinations. Correspondingly, infinitely many cohomogeneity-one strings are possible. However, we do not have to distinguish all of them. For example, in Minkowski spacetime, we can identify two stationary rotating strings which have equal angular velocity and different rotational axes, e.g. the x-axis and y-axis. This identification is generalized as follows: two strings are equivalent if their world surfaces, say $\Sigma_1$ and $\Sigma_2$, are mapped by an isometry $\varphi$,

$$\Sigma_2 = \varphi(\Sigma_1).$$

In the case of cohomogeneity-one strings, we can identify the strings if the isometry $\varphi$ sends the tangent Killing vector $\xi_1$ which defines the cohomogeneity-one property of $\Sigma_1$ to the Killing vector $\xi_2$ which defines that of $\Sigma_2$:

$$\xi_2 = \varphi_* \xi_1.$$  \hspace{1cm} (2)

In Minkowski spacetime, the Killing vectors are classified into seven families (types I to VII) under identification by isometries. Therefore, cohomogeneity-one strings fall into seven families [21]. The type I family includes stationary rotating strings. The Nambu–Goto equations of motion for this class are exactly solved and various configurations are found [9–11, 16, 18, 20]. By using the exact solutions, the energy–momentum tensors are calculated and the properties of the stationary rotating strings are clarified [18]. The gravitational perturbations are also studied in detail and the wave form of the gravitational waves are obtained [22]. The classification of cohomogeneity-one strings corresponding to [21] is found also in the anti-de Sitter spacetime in five dimensions [19].

The studies of the type I family show that exact solutions are useful for investigating the gravitational phenomena such as gravitational lensing and gravitational waves, which are indispensable to verifying the existence of cosmic strings. Furthermore, exact solutions
provide us with a deeper insight into the string dynamics. In this paper, we clarify the integrability of the remaining families (types II to VII) to complete exact solutions of cohomogeneity-one strings in Minkowski spacetime. We assume that the motions of the strings are governed by the Nambu–Goto action.

We emphasize that the integrability above means exact solvability (up to quadrature) of the embedding of the string’s world surface into the spacetime and should be clearly distinguished from the well-known ‘integrability’ of classical strings which means the existence of an infinite number of conserved quantities. To be concrete, when we work in the conformal gauge, the Nambu–Goto equations in Minkowski spacetime reduce to the wave equations in the two-dimensional flat spacetime supplemented with constraint equations. Though general solutions of the former can be easily obtained, the latter are not solvable in general because they are nonlinear partial differential equations. In the case of cohomogeneity-one strings, the equations of motion reduce to geodesic equations in a curved space, where the metric of the reduced space depends on the symmetry of the world surface [19, 21]. Even in this case, however, it is still non-trivial whether the geodesic equations on the metric are integrable or not.

In the next section, we review the equations of motion for the cohomogeneity-one strings and argue the integrability. In section 3, we solve the equations of motion and give the solutions in closed form. We conclude the work in the final section.

2. Integrability of cohomogeneity-one strings

A trajectory of the string is a two-dimensional surface, say \( \Sigma \), embedded in the spacetime \( (\mathcal{M}, g) \). We denote the embedding as

\[\zeta^a \mapsto x^\mu = x^\mu(\zeta^a),\]  

where \( \zeta^a(\zeta^0, \zeta^1) \) are the coordinates on \( \Sigma \) and \( x^\mu (\mu = 0, 1, 2, 3) \) are the coordinates in \( \mathcal{M} \). The Nambu–Goto action is written as

\[S = \int_\Sigma \sqrt{\gamma} \, d^2 \zeta,\]  

where \( \gamma \) is the determinant of the metric \( \gamma_{ab} \) induced on \( \Sigma \) which is given by

\[\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^a} \frac{\partial x^\nu}{\partial \zeta^b}.\]  

Let us consider the case that the spacetime \( \mathcal{M} \) admits a Killing vector field \( \xi \). We denote the one-parameter isometry group generated by \( \xi \) as \( H \). The group action of \( H \) on \( \mathcal{M} \) generates orbits of \( H \) or the integral curves of \( \xi \). If the string world surface \( \Sigma \) is foliated by the orbits of \( H \), the string is called cohomogeneity-one associated with the Killing vector \( \xi \). It is obvious that \( \Sigma \) is tangent to \( \xi \).

When we identify the points in \( \mathcal{M} \) which are connected by the action of \( H \), we have the orbit space \( \mathcal{O} := \mathcal{M}/H \). Under this identification, the cohomogeneity-one world surface \( \Sigma \) becomes a curve in \( \mathcal{O} \). This curve is shown to be a spacelike geodesic in \( \mathcal{O} \) with the norm-weighted metric

\[\tilde{h}_{\mu\nu} = -fh_{\mu\nu},\]  

where \( f \) is the squared norm of \( \xi \), and \( h_{\mu\nu} \) is a naturally induced metric on \( \mathcal{O} \):

\[h_{\mu\nu} = g_{\mu\nu} - \xi_\mu \xi_\nu/f.\]  

Therefore, the equations of motion for the cohomogeneity-one string are reduced to the geodesic equations on \( (\mathcal{O}, \tilde{h}) \).
Integrability of the geodesic equations is related to the existence of Killing vectors and Killing tensors. Let $K^\mu$ be a Killing vector field on $(\mathcal{O}, \tilde{h})$, which satisfies the Killing equations
\begin{equation}
\nabla_\mu K_\nu = 0,
\end{equation}
where $\nabla_\mu$ denotes the covariant derivative with respect to $\tilde{h}$. For a tangent vector $u^\mu$ of a geodesic, $K_\mu u^\mu$ is conserved along the geodesic. Let $K_{\mu\nu}$ be a Killing tensor field on $(\mathcal{O}, \tilde{h})$, which is symmetric and satisfies the Killing equations
\begin{equation}
\nabla_\mu (K_\nu \xi_\lambda) = 0,
\end{equation}
$K_{\mu\nu} u^\mu u^\nu$ is also conserved along the geodesic. If the geodesic has enough such conserved quantities which commute with each other, the geodesic equations are integrable. In the case of geodesics in $(\mathcal{O}, \tilde{h})$, where $\dim \mathcal{O} = 3$, two conserved quantities are required for the integrability. Therefore, if $(\mathcal{O}, \tilde{h})$ admits two or more Killing vectors and Killing tensors, geodesic equations are integrable.

In the orbit space $(\mathcal{O}, \tilde{h})$, we can find such Killing vectors without solving the Killing equations. Let us consider a Killing vector $X$ in $(\mathcal{M}, g)$ which commutes with $\xi$. We can easily find that the projection of $X$, say $\pi_* X$, where $\pi : \mathcal{M} \to \mathcal{O}$ is the projection, is a Killing vector in $(\mathcal{O}, \tilde{h})$:
\begin{align}
L_{\pi_* X} \tilde{h}_{\mu\nu} &= L_X (-f g_{\mu\nu} + \xi_\mu \xi_\nu) = L_X \left( (-g_{\rho\sigma} g_{\mu\nu} + g_{\mu\rho} g_{\sigma\nu}) \xi^\rho \xi^\sigma \right) \\
&= (-g_{\rho\sigma} g_{\mu\nu} + g_{\mu\rho} g_{\sigma\nu}) \{ [X, \xi]^\rho \xi^\sigma + \xi^\rho [X, \xi]^\sigma \} = 0.
\end{align}
Killing vectors which commute with $\xi$ constitute a Lie subalgebra, called centralizer of $\xi$ which we denote $\mathcal{C}(\xi)$. Let $X, Y \in \mathcal{C}(\xi)$ commute with each other. We can show that the projections of them on $\mathcal{O}$ also commute:
\begin{equation}
[\pi_* X, \pi_* Y] = \pi_* [X, Y] = 0.
\end{equation}
Therefore, if there are two or more linearly independent and commuting Killing vectors in $\mathcal{C}(\xi)$ except for $\xi$ itself, $(\mathcal{O}, \tilde{h})$ inherits the same number of commuting Killing vectors, and then the geodesic equations in $(\mathcal{O}, \tilde{h})$ are integrable.

In Minkowski spacetime, all of the cohomogeneity-one strings are classified into seven families (types I to VII). For each type, we list the Killing vector $\xi$, basis of $\mathcal{C}(\xi)$ and the number of commuting basis of $\mathcal{C}(\xi)$ except for $\xi$ in table 1. For types I to VI, there are more than two commuting Killing vectors in $\mathcal{C}(\xi)$; hence, the geodesic equations in $(\mathcal{O}, \tilde{h})$ are integrable. As shown in the next section, the equations of motions for these strings are not only integrable but also solved exactly. For the strings of type VII, there is only one Killing vector in $\mathcal{C}(\xi)$. Nevertheless, the geodesic equations are solved exactly. This is due to the existence of a Killing tensor in $(\mathcal{O}, \tilde{h})$. We also solve the geodesic equation exactly.

3. Solutions of cohomogeneity-one strings in Minkowski spacetime

3.1. Type I

The tangent Killing vector of this class is given as
\begin{equation}
\xi = P_t + a L_z, \quad (a : \text{const.})
\end{equation}
where $P_t$ is the Killing vector of time translation and $L_z$ is that of rotation around the $z$-axis.

In the conventional cylindrical coordinate $(\dot{t}, \rho, \dot{\phi}, z)$ of Minkowski spacetime, $\xi$ is written as
\begin{equation}
\xi = \partial_t + a \partial_\phi.
\end{equation}
Table 1. Inherited symmetry \( C(\xi) \) on \((O, \tilde{h})\). \( P_\mu \) \((\mu = t, x, y, z)\) is the generator of translation for the \( \mu \)-direction. \( L_i \) \((i = x, y, z)\) are the generators of rotation around the \( i \)-axis. \( K_i \) \((i = x, y, z)\) are the generators of Lorentz boosts for the \( i \)-directions. \( n \) is the number of commuting basis in \( C(\xi) \).

| Type | Tangent Killing vector \( \xi \) | Basis of \( C(\xi) \) | \( n \) |
|------|-------------------------------|-----------------|---|
| I    | \( P_t + a L_z \) \((a \neq 0)\) | \( P_t, P_z, L_z \) | 2 |
| I    | \( P_t \)                       | \( P_t, P_x, P_y, P_z, L_x, L_y, L_z \) | 3 |
| I    | \( L_z \)                       | \( P_t, P_x, L_x, K_z \) | 3 |
| II   | \( (P_t + P_z) + a L_z \) \((a \neq 0)\) | \( P_t, P_z, L_z \) | 2 |
| II   | \( P_t + P_z \)                 | \( P_t, P_x, P_y, P_z, K_x + L_x, K_y - L_y, L_z \) | 3 |
| III  | \( P_t + a L_z \) \((a \neq 0)\) | \( P_t, P_z, L_z \) | 2 |
| III  | \( P_t \)                       | \( P_t, P_x, P_y, P_z, L_x, K_x \) | 3 |
| IV   | \( P_t + a (K_y + L_z) \)       | \( P_t - P_x, P_y, P_z + a (K_y - L_z), K_y + L_z \) | 2 |
| V    | \( P_t + a K_y \) \((a \neq 0)\) | \( P_t, P_y, K_y \) | 2 |
| V    | \( K_y \)                       | \( P_t, P_x, L_x, K_y \) | 2 |
| VI   | \( K_z + L_y + a P_t \) \((a \neq 0)\) | \( K_z + L_y + a P_t, P_y - P_x, P_z \) | 2 |
| VII  | \( K_z + a L_y \) \((a \neq 0)\) | \( L_z, K_z \) | 1 |

and the norm of \( \xi \) is

\[
f = |\xi|^2 = -(1 - a^2 \tilde{\rho}^2). \tag{14}\]

Then, \( \xi \) is timelike in \( \tilde{\rho} < 1/|a| \) and spacelike in \( \tilde{\rho} > 1/|a| \). The surface \( \tilde{\rho} = 1/|a| \) is called light cylinder. Cohomogeneity-one strings of type I inside the light cylinder are the stationary rotating strings. The constant \( a \) represents the angular velocity of the rotation.

Here, we introduce a coordinate \((t, \tilde{\rho}, \tilde{\phi}, z) = (\tilde{t}, \tilde{\rho}, \tilde{\phi} - a \tilde{t}, \tilde{z})\) so that \( \xi = \partial_t \), i.e. one of the coordinates, say \( t \), is a coordinate along the orbits of \( H \) which is generated by \( \xi \). In the new coordinate, the spacetime metric is

\[
g = -(1 - a^2 \rho^2) \, dt^2 + 2a \rho^2 \, dt \, d\phi + d\rho^2 + \rho^2 \, d\phi^2 + dz^2 \tag{15}\]

and the norm of \( \xi \) is

\[
f = -(1 - a^2 \rho^2). \tag{16}\]

Then, the norm-weighted metric on the orbit space is calculated as

\[
\tilde{h} = -fg + \xi \bar{\xi} = (1 - a^2 \rho^2)(d\rho^2 + dz^2) + \rho^2 \, d\phi^2. \tag{17}\]

We solve the geodesic equations in \((O, \tilde{h})\) with the action

\[
S = \int \left( \frac{\mathcal{L}}{N} + N \right) \, d\sigma, \tag{18}\]

\[
\mathcal{L} = (1 - a^2 \rho^2)(\rho^2 - z^2) + a^2 \rho^2 \, d\phi^2, \tag{19}\]

where \( \sigma \) is a parameter of the geodesic curve, \( N \) is a function of \( \sigma \) and the prime denotes the derivative with respect to \( \sigma \). The action (18) is invariant under the transformations

\[
\sigma \mapsto \tilde{\sigma} = \tilde{\sigma}(\sigma), \tag{20}\]

\[
N \mapsto \tilde{N} = \frac{d\sigma}{d\tilde{\sigma}} \bar{N}. \tag{21}\]
Therefore, the function \( N \) determines the parametrization of the geodesic curve. We should note that even though we fix the functional form of \( N \), there remains a residual freedom of the parametrization:

\[
\sigma \mapsto \tilde{\sigma} = \pm \sigma + \sigma_0. \tag{22}
\]

The variation with respect to \( \phi \) leads to a conserved quantity related to the \( \phi \)-independence of \( \tilde{h} \):

\[
\frac{\rho^2 \phi'}{N} = L \tag{const}. \tag{23}
\]

We also have a conserved quantity related to the \( z \)-independence of \( \tilde{h} \):

\[
\frac{1 - a^2 \rho^2}{N} z' = P \tag{const}. \tag{24}
\]

The other variations lead to

\[
(1 - a^2 \rho^2) (\rho^2 + z^2) + \rho^2 \phi'^2 = N^2, \tag{25}
\]

\[
\left(1 - a^2 \rho^2 \right) \left( -a^2 \rho (\rho^2 + z^2) + \rho \phi'^2 \right) = 0. \tag{26}
\]

By fixing the parametrization freedom as

\[
N = 1 - a^2 \rho^2, \tag{27}
\]

we obtain

\[
\rho^2 = 1 - P^2 + a^2 L^2 - a^2 \rho^2 \frac{L^2}{\rho^2}. \tag{28}
\]

This equation is readily integrated as

\[
a^2 \rho^2 (\sigma) = \alpha + \beta \cos 2a(\sigma + \sigma_0), \tag{29}
\]

\[
\alpha := \frac{1 - P^2 + a^2 L^2}{2} \geq 0, \tag{30}
\]

\[
\beta := \sqrt{\alpha^2 - a^2 L^2}, \tag{31}
\]

where \( \sigma_0 \) is an integration constant. We can set \( \sigma_0 \) to zero by using the residual reparametrization freedom (22). Then, the solution is written as

\[
a^2 \rho^2 (\sigma) = \alpha + \beta \cos 2a \sigma. \tag{32}
\]

Using the solution, we can solve (23) and (24) as

\[
\phi (\sigma) = -a^2 L \sigma + \tan^{-1} \left[ \frac{aL}{\alpha + \beta} \tan a \sigma \right] + \phi_0, \tag{33}
\]

\[
z (\sigma) = P \sigma + z_0, \tag{34}
\]

where \( \phi_0 \) and \( z_0 \) are constants.

The string solution, i.e. embedding of the world surface \((\tau, \sigma) \mapsto (t, \rho, \phi, z)\), is given by (32), (33), (34) and \( t = \tau \). The solution has four integration constants: \( P, L, \phi_0 \) and \( z_0 \). \( P \) and \( L \) determine the shape of the string. However, \( \phi_0 \) and \( z_0 \) have no physical meaning because we can identify the solution of \( \phi_0 \neq 0 \) and \( z_0 \neq 0 \) with that of \( \phi_0 = z_0 = 0 \) by the isometries in \((M, g)\):

\[
\phi \mapsto \phi + \phi_0, \tag{35}
\]

\[
z \mapsto z + z_0, \tag{36}
\]

where we should remember that the spacetime metric (15) does not depend on \( \phi \) and \( z \).
3.2. Types II–VI

For types II–VI, we can reduce the equations of motion to the geodesic equations in the orbit space and solve them in the same manner as in the case of type I. We summarize the results in Table 2.

3.3. Type VII

The tangent Killing vector of type VII string is
\[ \xi = K_z + a L_z, \] (37)
where \( K_z \) is a Killing vector of the Lorentz boost along the \( z \)-axis. As in the case of type I, we introduce a coordinate suitable for the reduction. In order to find such a coordinate, we use a combination of the Rindler coordinate and a cylindrical rotating coordinate in the form
\[ (\tilde{t}, \tilde{\rho}, \tilde{\phi}, \tilde{z}) = (t \cosh z, \rho, \phi + az, t \sinh z) \] (38)
such that \( \xi \) is written as \( \partial_t \). Since this coordinate covers only the part of Minkowski spacetime where \( K_z \) is timelike, we use another coordinate
\[ (\tilde{t}, \tilde{\rho}, \tilde{\phi}, \tilde{z}) = (t \cosh z, \rho, \phi + az, t \sinh z) \] (39)
in the spacelike regions of \( K_z \). In this coordinate, \( \xi \) is written as \( \partial_z \).

3.3.1. Timelike regions of \( K_z \). We take the coordinate (38) in the timelike regions of \( K_z \).

With respect to the coordinate, the spacetime metric is written as
\[ g = -(z^2 - a^2 \rho^2) dt^2 + 2a \rho^2 d\phi dt + d\rho^2 + \rho^2 d\phi^2 + dz^2, \] (40)
and the norm of the Killing vector \( \xi \) is
\[ f = -(z^2 - a^2 \rho^2). \] (41)

Then, the metric \( \tilde{h} \) on \( O \) is given as
\[ \tilde{h} = (z^2 - a^2 \rho^2)(d\rho^2 + dz^2) + z^2 \rho^2 d\phi^2. \] (42)

This metric admits a manifest Killing vector \( \partial_\phi \) and an irreducible Killing tensor
\[ K = a^2 \rho^2 (z^2 - a^2 \rho^2) dz^2 + z^2 (z^2 - a^2 \rho^2) d\rho^2 + z^2 \rho^2 (z^2 + a^2 \rho^2) d\phi^2. \] (43)

In order to solve the geodesic equations, we start from the action (18) with
\[ L = (z^2 - a^2 \rho^2)(\rho^2 + z^2) + z^2 \rho^2 \phi^2. \] (44)

The existence of the Killing vector and the Killing tensor ensures two conserved quantities, say \( L \) and \( C \) respectively,
\[ L = z^2 \rho^2 \phi'/N, \] (45)
\[ 2C = a^2 \rho^2 (z^2 - a^2 \rho^2) \left( \frac{z'}{N} \right)^2 + z^2 (z^2 - a^2 \rho^2) \left( \frac{\rho'}{N} \right)^2 + z^2 \rho^2 (z^2 + a^2 \rho^2) \left( \frac{\phi'}{N} \right)^2. \] (46)

Here, we should note that the prime does not represent the differentiation with an affine parameter. The geodesic tangent with an affine parameter is written as \( \kappa' / N \). We solve (45), (46) and the constraint equation
\[ (z^2 - a^2 \rho^2)(\rho^2 + z^2) + z^2 \rho^2 \phi^2 = N^2. \] (47)
We show tangent Killing vectors $\xi$, coordinates for the reduction, orbit space metrics $\tilde{h}$ and solutions of the geodesic equations. $P$, $Q$ and $L$ are the conserved quantities related to the Killing vectors of $\tilde{h}$, and $C$ is that related to the Killing tensor of $\tilde{h}$. $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ and $(\tilde{t}, \tilde{p}, \tilde{q}, \tilde{z})$ are the Cartesian coordinate and cylindrical coordinate of Minkowski spacetime, respectively.

| Type | $\xi = P_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{p}, \tilde{q}, \tilde{z}) = (t, \rho, \phi + at, z)$, $\tilde{h} = (1 - a^2 \rho^2)(d\rho^2 + dz^2) + \rho^2 d\phi^2$, $z(\sigma) = P\sigma, a^2 \rho^2(\sigma) = \alpha + \beta \cos 2\sigma, \phi(\sigma) = -a^2L\sigma + \tan^{-1}\left(\frac{\beta}{\alpha}\right)$, $\alpha := (1 - P^2 + a^2L^2)/2, \beta := \sqrt{a^2 - a^2L^2}$ |
|---|---|
| II | $\xi = P_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{p}, \tilde{q}, \tilde{z}) = (u + \frac{1}{\lambda}, \rho, a\rho + au, u - \frac{1}{\lambda})$, $\tilde{h} = du^2 - 2\rho^2 du d\phi - a^2 \rho^2 d\rho^2$, $v(\sigma) = aP\sigma, a^2 \rho^2(\sigma) = aP(\tilde{L} + \sqrt{\tilde{L}^2 - 1}) \cos 2\sigma$, $\phi(\sigma) = -aL\sigma + \tan^{-1}(1 - \sqrt{\tilde{L}^2 - 1}) \tan a\sigma$ |
| III | $\xi = P_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{p}, \tilde{q}, \tilde{z}) = (t, \rho, a\rho + az, z)$, $\tilde{h} = (1 + a^2 \rho^2)(dt^2 - d\rho^2 - \rho^2 d\phi^2)$, $t(\sigma) = Q\sigma, a^2 \rho^2(\sigma) = \alpha + \beta \cos 2\sigma, \phi(\sigma) = a^2L\sigma + \tan^{-1}\left(\frac{\beta}{\alpha}\right)$, $\alpha := (1 - Q^2 - a^2L^2)/2, \beta := \sqrt{a^2 - a^2L^2}$ |
| IV | $\xi = P_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (\frac{1}{2} \lambda x^2 + u + v, -\tilde{t} + u, a\lambda u, \lambda + w)$, $\tilde{h} = (1 + a^2u^2)(2du - du^2) - a^2 u^2 dw^2$, $u(\sigma) = aP\sigma, v(\sigma) = \frac{\lambda u^2 + \rho}{2a^2u^2} + \frac{1}{2a^2u^2}$, $w(\sigma) = Q\sigma$ |
| V | $\xi = P_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = (y \sinh at, x, y \cosh at, z + t)$, $\tilde{h} = a^2 y^2 dz^2 - (1 - a^2 y^2) (dx^2 + dy^2)$, $x(\sigma) = P\sigma, a^2 \rho^2(\sigma) = \alpha \pm \frac{1}{4} (e^{2\sigma} + \beta^2 e^{-2\sigma})$, $z(\sigma) = Q\sigma + \frac{1}{4} \ln \left(\frac{\epsilon^{a\sigma} + w(\sigma)}{\epsilon^{\sigma} + w(\sigma)}\right)$, $\alpha := (1 + P^2 + Q^2)/2, \beta := \sqrt{a^2 - Q^2}$ |
| VI | $\xi = P_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = \left(\frac{1}{2} \lambda x^2 + au \lambda + v, -\tilde{t} + u, \lambda \tilde{x}^2 + u, w\right)$, $\tilde{h} = (2au - 1)(du^2 + dw^2) + du^2$, $2au(\sigma) - 1 = \frac{\rho_0^2}{\rho_{\perp}^2} + a^2 (1 - P^2) \sigma^2$, $v(\sigma) = \frac{\rho_0^2}{\rho_{\perp}^2} \sigma + \frac{a^2 (1 - P^2)}{\rho_{\perp}^2}$, $w(\sigma) = Q\sigma$ |
| VII | $\xi = K_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{p}, \tilde{q}, \tilde{z}) = (\sinh t, \rho, \phi + at, \cosh t)$, $\tilde{h} = (z^2 - a^2 \rho^2)(dp^2 + dz^2) + z^2 \rho^2 d\phi^2$, $z^2(\sigma) = C \pm \frac{1}{2} (e^{2\sigma} + \beta^2 e^{-2\sigma})$, $a^2 \rho^2(\sigma) = C + \beta \cos 2\sigma (\sigma + \sigma_0)$, $\phi(\sigma) = \tan^{-1}\left(\frac{a}{1 + \rho_{\perp}^2} \tan a(\sigma + \sigma_0)\right) + \frac{1}{2} \ln \left(\frac{\rho_{\perp}^2 + a^2 (1 - P^2)}{\rho_{\perp}^2 (C - a^2 L^2)}\right)$, $\beta := \sqrt{C^2 - a^2L^2}$ |
| | $\xi = K_{\perp} + aL_{\perp}$, $(\tilde{t}, \tilde{p}, \tilde{q}, \tilde{z}) = (t \cosh z, \rho, \phi + az, t \sinh z)$, $\tilde{h} = (t^2 + a^2 \rho^2)(dt^2 - d\rho^2) - t^2 \rho^2 d\phi^2$, $t^2(\sigma) = -C + \frac{1}{2} (e^{2\sigma} + \beta^2 e^{-2\sigma})$, $a^2 \rho^2(\sigma) = C + \beta \cos 2\sigma (\sigma + \sigma_0)$, $\phi(\sigma) = \tan^{-1}\left(\frac{a}{1 + \rho_{\perp}^2} \tan a(\sigma + \sigma_0)\right) + \frac{1}{2} \ln \left(\frac{\rho_{\perp}^2 + a^2 (1 - P^2)}{\rho_{\perp}^2 (C - a^2 L^2)}\right)$, $\beta := \sqrt{C^2 - a^2L^2}$ |
By fixing the parametrization freedom as
\[ N = \frac{z^2 - a^2 \rho^2}{\rho^2}, \]  
we can separate the variables
\[ \rho^2 = 2C - \frac{L^2}{\rho^2} - a^2 \rho^2, \]  
(49)
\[ z^2 = -2C + z^2 + \frac{a^2 L^2}{z^2}, \]  
(50)
and solve the equations exactly as
\[ z^2(\sigma) = C \pm \frac{e^{2\sigma} + \beta^2 e^{-2\sigma}}{2}, \]  
(51)
\[ a^2 \rho^2(\sigma) = C + \beta \cos 2a(\sigma + \sigma_0), \]  
(52)
\[ \phi(\sigma) = \tan^{-1} \left\{ \frac{aL}{C + \beta} \tan a(\sigma + \sigma_0) \right\} + \frac{a}{2} \ln \left| \frac{e^{2\sigma} \pm (C + aL)}{e^{2\sigma} \pm (C - aL)} \right|, \]  
(53)
\[ \beta := \sqrt{C^2 - a^2 L^2}, \]  
(54)
where \( \sigma_0 \) is an integration constant.

3.3.2. Spacelike regions of \( K_z \). With respect to the coordinate (39), the metric \( \tilde{h} \) on the orbit space is written as
\[ \tilde{h} = (t^2 + a^2 \rho^2)(dt^2 - a^2 \rho^2) - t^2 \rho^2 d\phi^2. \]  
(55)
This metric also admits a Killing vector \( \partial_\phi \) and a Killing tensor
\[ K = a^2 \rho^2(t^2 + a^2 \rho^2) dt^2 + t^2 \rho^2(t^2 - a^2 \rho^2) d\phi^2 + t^2(a^2 + a^2 \rho^2) d\rho^2. \]  
(56)
The Killing vector and Killing tensor ensure the existence of two conserved quantities, and then the geodesic equations are integrable. With a calculation similar to that used in deriving solutions (51), (52), (53), we obtain the exact solutions
\[ t^2(\sigma) = -C + \frac{1}{2} \left( e^{2\sigma} + \beta^2 e^{-2\sigma} \right), \]  
(57)
\[ a^2 \rho^2(\sigma) = C + \beta \cos 2a(\sigma + \sigma_0), \]  
(58)
\[ \phi(\sigma) = \tan^{-1} \left\{ \frac{aL}{C + \beta} \tan a(\sigma + \sigma_0) \right\} + \frac{a}{2} \ln \left| \frac{e^{2\sigma} \pm (C + aL)}{e^{2\sigma} \pm (C - aL)} \right|, \]  
(59)
\[ \beta := \sqrt{C^2 - a^2 L^2}, \]  
(60)
where \( \sigma_0 \) is an integration constant, \( L \) is a conserved quantity related to the Killing vector \( \partial_\phi \) and \( C \) is the one related to the Killing tensor \( K \).

4. Conclusion
We have shown that the Nambu–Goto equations of motion for all of the cohomogeneity-one strings in Minkowski spacetime \( (\mathcal{M}, g) \) are integrable. The cohomogeneity-one string is
a string whose world surface is tangent to a Killing vector field \( \xi \). The Killing vector \( \xi \) generates a one-parameter isometry group, say \( H \), which acts on the world surface. Then, the world surface has symmetry due to \( H \). By virtue of the symmetry on the world surface, the equations of motion reduce to the geodesic equations on the orbit space \( O := \mathcal{M}/H \) with a norm-weighted metric \( \tilde{h}_{\mu\nu} := -\xi^2 g_{\mu\nu} + \xi_\mu \xi_\nu \). We have investigated the integrability of these geodesic equations.

The integrability of the geodesic equations is related to the existence of Killing vectors. In the case of \( (O, \tilde{h}) \), we have shown that the projections of the Killing vectors in \( (\mathcal{M}, g) \) which commute with \( \xi \) are also Killing vectors in \( (O, \tilde{h}) \), i.e. the Killing vectors are inherited from \( (\mathcal{M}, g) \). We have focused on the number of these inherited Killing vectors.

For the cohomogeneity-one strings of types I to VI, we have found that there are more than two commuting Killing vectors in \( (O, \tilde{h}) \). The existence of two or more commuting Killing vectors guarantees the integrability of the geodesic equations in \( (O, \tilde{h}) \) because \( \dim O = 3 \). Then the geodesic equations for types I to VI are integrable. We have also found that the geodesic equations are solved exactly. The exact solutions are shown in table 2.

For the remaining cohomogeneity-one strings, i.e. type VII, there is only one inherited Killing vector. However, we have found a Killing tensor in \( (O, \tilde{h}) \). The existence of the Killing vector and the Killing tensor leads to two conserved quantities of the geodesic, and then the geodesic equations are integrable. We have also solved the geodesic equations exactly.

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