Braided Hopf Crossed Modules Through Simplicial Structures

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Abstract

Any simplicial Hopf algebra involves 2n different projections between the Hopf algebras $H_n, H_{n-1}$ for each $n \geq 1$. The word projection, here meaning a tuple $\partial: H_n \to H_{n-1}$ and $i: H_{n-1} \to H_n$ of Hopf algebra morphisms, such that $\partial i = \text{id}$. Given a Hopf algebra projection $(\partial: I \to H, i)$ in a braided monoidal category $\mathcal{C}$, one can obtain a new Hopf algebra structure living in the category of Yetter-Drinfeld modules over $H$, due to Radford’s theorem. The underlying set of this Hopf algebra is obtained by an equalizer which only defines a sub-algebra (not a sub-coalgebra) of $I$ in $\mathcal{C}$. In fact, this is a braided Hopf algebra since the category of Yetter-Drinfeld modules over a Hopf algebra with an invertible antipode is braided monoidal. To apply Radford’s theorem in a simplicial Hopf algebra successively, we require some extra functorial properties of Yetter-Drinfeld modules. Furthermore, this allows us to model Majid’s braided Hopf crossed module notion from the perspective of a simplicial structure.

Keywords Hopf algebra, braided monoidal category, Yetter-Drinfeld module, simplicial object, braided Hopf crossed module.

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1 Introduction

A Hopf algebra [37] can be considered as an abstraction of the group algebra (of a group) and the universal enveloping algebra (of a Lie algebra). In other words, considering the group-like and primitive elements in a Hopf algebra, we have the functors:

\[
\begin{align*}
\ast_{gl}: \{\text{Hopf Algebras}\} & \rightarrow \{\text{Groups}\}, \\
\text{Prim}: \{\text{Hopf Algebras}\} & \rightarrow \{\text{Lie Algebras}\}.
\end{align*}
\]

(1)

There exists a wide variety of variations of Hopf algebras by relaxing its properties or adding some extra structures; for instance, quasi-Hopf algebras [13], quasi-triangular Hopf algebras [28] and quantum groups [28, 29], etc. The original definition of Hopf algebras is given in the category of vector spaces. However, we know that the category of vector spaces is a trivial example of (symmetric) braided monoidal categories. As a consequence of this categorical relationship, a more general definition of Hopf algebras in a braided monoidal category is given in [27].

Suppose that, we have a Hopf algebra projection \((\partial: I \rightarrow H, i)\) in the category of vector spaces (i.e. there exists a Hopf algebra map \(i: I \rightarrow H\) such that \(d i = \text{id}_H\)). One can obtain a Yetter-Drinfeld module [41] over \(H\) where the action and coaction is defined by taking the Hopf algebra morphisms \(\partial\) and \(i\) into account. The category of such Yetter-Drinfeld modules will be denoted by \(\mathcal{YD}(H)\). Moreover, there exists a Hopf algebra structure \(B\) (subset of \(I\)) living in \(\mathcal{YD}(H)\). The underlying set of the Hopf algebra \(B\) is categorically given by an equalizer which is only a sub-algebra of \(I\). In fact, this is a braided Hopf algebra since the category of Yetter-Drinfeld modules over a Hopf algebra with an invertible antipode is braided monoidal. On the other hand, considering the underlying set of \(B\), we have a Hopf algebra structure \(B \otimes H\) in the category of vector spaces (not in \(\mathcal{YD}(H)\)) comes equipped with the smash product and smash coproduct. Furthermore, we have the isomorphism of Hopf algebras \(I \cong B \otimes I\). This is called “Radford’s theorem” which is introduced in [35] and extended in [26]. Inspired by the categorical relationship between vector spaces and braided monoidal categories, it is proven in [5] that, Radford’s Theorem is also valid in any braided monoidal category \(\mathcal{C}\) with equalizers.

The notion of strict 2-group is a strict vertical categorification of that of group. Equivalently, a strict 2-group is the internal category in the category of groups. The essential example of a strict 2-group is called crossed module [40]. In other words, they can be seen as a way to encode a strict group [12]. For more details on crossed modules especially from the topological, algebraic and geometric point of view, we refer [10, 20, 31]. Moreover, the notion and some applications of crossed modules are extended to various algebraic structures such as monoids [6, 7], Lie algebras [17], groupoids [11], groups with operations [33], modified categories of interest [9], shelves (via generalized Yetter-Drinfeld modules) [24], etc. More recently, motivated from the group theoretical analogy, Majid defines strict quantum 2-groups in [25]. Through encoding that of strict quantum 2-groups, his paper also includes two different approaches to crossed module notion for Hopf algebras, as follows:
• Hopf crossed modules:
  - They are given by a Hopf algebra morphism in the category of vector spaces.
  - They unify the Lie algebra crossed modules and group crossed modules with the following functors (between the categories of crossed modules of Lie algebras, Hopf algebras and groups):

\[
\begin{align*}
\{\text{XLie}\} & \xrightarrow{\text{Prim}} \{\text{XHopf}\} \\
\{\text{XGrp}\} & \xrightarrow{\text{Prim}} \{\text{XGrp}\}.
\end{align*}
\]

which are induced from \([1]\). For more details about the functorial relationships between various algebraic categories of crossed modules, see \([19]\).

- Categorically, they are equivalent to the certain case of simplicial Hopf algebras given in \([18]\), if we impose the cocommutativity condition.

- We also know that the category of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian. From this point of view, it is proven in \([21]\) that Majid’s crossed module definition is coherent in the sense of internal crossed modules in semi-abelian categories \([22]\).

• Braided Hopf crossed modules:
  - They are given by a twisted Hopf algebra map: briefly, it is not a Hopf algebra morphism in a certain category.
  - The codomain of braided Hopf crossed modules are still vector spaces, while the domains are now Yetter-Drinfeld modules.
  - The main motivation to introduce this notion is to generalize the characterization of Hopf crossed modules.

When we forget the cocommutativity case, we notice that Majid’s first crossed module notation does not make a contact with a simplicial structure in the sense of \([18]\). Because, the structure \(B\) we mentioned in the Radford’s theorem is:

- a Hopf algebra living in the base category (vector spaces), if the Hopf algebras are cocommutative,
- a Hopf algebra living in the category of Yetter-Drinfeld modules, in general case (without the cocommutativity condition).

So, it gives us an idea that, Majid’s braided Hopf crossed module notion would be the correct one to model crossed modules in the sense of simplicial structures and also to discover its other well-known group theoretical properties in the category of Hopf algebras. For this aim, in this paper, we first recall Radford’s theorem in a braided monoidal category \(\mathcal{C}\). Afterwards, we see that a Hopf algebra living in the braided monoidal category \(\mathcal{YD}_\mathcal{C}(H)\) can also be converted into a Hopf algebra living in the braided monoidal category \(\mathcal{YD}_\mathcal{C}(I)\), for a given Hopf algebra projection \((\partial: I \to H, i)\). This property will be used to obtain new projections in induced categories in a simplicial Hopf algebra. Consequently, it gives rise to a connection between braided Hopf crossed modules and simplicial Hopf algebras.

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2 Preliminaries

2.1 Braided monoidal categories

A braided monoidal category \cite{23} (or braided tensor category) is a monoidal category \( \mathcal{C} \) equipped with a braiding map (natural isomorphism) \( R_{A,B} : A \otimes B \rightarrow B \otimes A \) for each objects of \( \mathcal{C} \), satisfying the following hexagon diagrams:

\[
\begin{align*}
A \otimes (B \otimes C) & \xrightarrow{\alpha_{A,B,C}} \alpha_{B,C,A} (B \otimes C) \otimes A \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} B \otimes (C \otimes A) \\
A \otimes (B \otimes C) & \xrightarrow{\alpha_{A,B,C}^{-1}} (B \otimes C) \otimes A \\
A \otimes (C \otimes B) & \xrightarrow{\alpha_{A,C,B}^{-1}} (A \otimes C) \otimes B
\end{align*}
\]

where \( \alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C) \) denotes the associator.

Any symmetric monoidal category is a braided monoidal category. Therefore, the category of vector spaces is a braided monoidal category with a trivial braiding \( R_{V,W} : v \otimes w \in V \otimes W \mapsto w \otimes v \in W \otimes V \).

As another example with a non-trivial braiding, suppose that \( G \) is a group. Then, the category of crossed \( G \)-sets forms a braided monoidal category where the braiding is given by using the group action.

2.2 Hopf algebraic conventions

In this subsection, we recall some Hopf algebraic notions in a braided monoidal category \( \mathcal{C} \) from \cite{27,39} which generalizes the ordinary Hopf algebras \cite{28} (i.e. Hopf algebras in the category of vector spaces). The most important examples are the braided Hopf algebras living in the category of Yetter-Drinfeld modules \cite{24}, and the Nichols algebras \cite{4}.

Let \( H = (H, \nabla, \eta, \Delta, \epsilon, S) \) be a Hopf algebra in \( \mathcal{C} \). That means:

- \((H, \nabla, \eta)\) is a unital associative \( \kappa \)-algebra. Thus

\[
\begin{align*}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes \text{id}} H \otimes H \\
\text{id} \otimes \nabla & \xrightarrow{\text{id} \otimes \nabla} H \otimes H \\
H \otimes H & \xrightarrow{\nabla} H
\end{align*}
\]

- \((H, \Delta, \epsilon)\) is a counital coassociative \( \kappa \)-coalgebra. Thus

\[
\begin{align*}
H \otimes H & \xrightarrow{\Delta} H \otimes H \\
\Delta \otimes \text{id} & \xrightarrow{\Delta \otimes \text{id}} H \otimes H \otimes H \\
H \otimes H & \xrightarrow{\text{id} \otimes \epsilon} H \otimes \kappa \\
\kappa \otimes H & \xrightarrow{\epsilon \otimes \text{id}} H \otimes H \otimes H
\end{align*}
\]
• $(H, \nabla, \eta, \Delta, \epsilon)$ is a bialgebra; i.e. $\eta$, $\nabla$ are coalgebra morphisms, and $\epsilon$, $\Delta$ are algebra morphisms. Thus

\[
\begin{array}{cccc}
H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H & \xrightarrow{id \otimes R \otimes id} & H \otimes H \otimes H \\
\nabla & \downarrow & \Delta & \downarrow & \nabla \\
H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{id \otimes R \otimes id} & H \otimes H \\
\kappa & \xrightarrow{\eta} & H & \xrightarrow{\epsilon} & H & \xrightarrow{\kappa} & H \\
\eta \otimes \eta & \downarrow & \Delta & \downarrow & \epsilon \otimes \epsilon & \downarrow & \eta & \downarrow & \kappa \\
H \otimes H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow{\epsilon \otimes \epsilon} & H \otimes H & \xrightarrow{\kappa} & H \\
\end{array}
\]

(2)

• There exists an (inverse-like) anti-homomorphism $S : H \to H$ at the level of algebra and coalgebra, satisfying:

\[
\begin{array}{cccc}
H \otimes H & \xrightarrow{id \otimes S} & H \otimes H \\
\Delta & \downarrow & \nabla \uparrow & \kappa \downarrow & \eta \uparrow & \kappa \\
H & \xrightarrow{\epsilon} & H & \xrightarrow{\kappa} & H & \xrightarrow{id \otimes S} & H \otimes H \\
\end{array}
\]

(3)

which is called an antipode.

**Remark 1** Moreover, we have the following properties and conventions:

• A Hopf algebra $H$ is said to be cocommutative, iff $\Delta = R \circ \Delta$.

• A Hopf algebra morphism is a bialgebra morphism compatible with the antipode.

• We use Sweedler’s notation [37] for coproducts; namely $\Delta(x) = \sum_{(x)} x' \otimes x''$, for all $x \in H$.

• We use the notation $\nabla(x \otimes y) = xy$ in the calculations, for the sake of simplicity.

• Let $\mathcal{C}$ be the category of vector spaces and $H$ be a Hopf algebra in $\mathcal{C}$. An element $x \in H$ is said to be “primitive”, if $\Delta(x) = x \otimes 1 + 1 \otimes x$; and “group like”, if $\Delta(x) = x \otimes x$. If $x$ is group-like then $\epsilon(x) = 1$ and $S(x) = x^{-1}$; and if $x$ is primitive then $\epsilon(x) = 0$ and $S(x) = -x$. The set of primitive elements $\text{Prim}(H)$ defines a Lie algebra, and the set of group-like elements $\text{Gl}(H)$ defines a group. Thus we have the functors:

\[
\begin{array}{ccc}
\{\text{Lie Algebras}\} & \xrightarrow{\text{Prim}} & \{\text{Hopf Algebras}\} \\
\text{Gl} & \xrightarrow{} & \{\text{Groups}\} \\
\end{array}
\]

From this point of view, Hopf algebras can be thought as a unification of groups and Lie algebras.

• Let $H$ be a Hopf algebra. A sub-Hopf algebra $A \subseteq H$ is a sub-object $A$, such that $\nabla(A \otimes A) \subseteq A$, $\Delta(A) \subseteq A \otimes A$ and $\eta(A) \subseteq A$.

• The category of Hopf algebras has a zero object $\kappa$, see [33]. We have unique morphisms $\eta_A : \kappa \to A$ and $\epsilon_A : A \to \kappa$. The zero morphism between two Hopf algebras $A$ and $B$ is therefore $\zeta_{A,B} = \zeta : A \to B$, where $z_{A,B} = \eta_B \circ \epsilon_A$, thus $\zeta_{A,B}(x) = \epsilon(x)1_B$. 

5
2.3 Yetter-Drinfeld modules

In this section, we recall the category of Yetter-Drinfeld modules which is defined via Hopf algebra actions and coactions. Moreover, this category is an essential example of a braided monoidal category.

2.3.1 Modules and comodules

**Definition 2 (module)** Let $H$ be a bialgebra. $V$ is said to be a $H$-module (on the left) if we are given a bilinear map $\rho$ called (left) action:

$$\rho: (x, v) \in H \times V \mapsto x \triangleright v \in V,$$

such that, for all $x, y \in H$ and each $v \in V$, we have:

$$(xy) \triangleright v = x \triangleright (y \triangleright v), \quad 1_H \triangleright v = v,$$

where the first condition means the following diagram commutes:

$$\begin{align*}
H \otimes H \otimes V &\xrightarrow{\nabla \otimes \text{id}} H \otimes V \\
\text{id} \otimes \rho &\downarrow \\
H \otimes V &\xrightarrow{\rho} V
\end{align*}$$

**Definition 3 (comodule)** Let $H$ be a bialgebra. $V$ is said to be a (left) $H$-comodule, if we are given a linear map $\phi: V \rightarrow H \otimes V$ called (left) coaction:

$$\phi: v \in V \mapsto \sum_{[v]} v_H \otimes v_V \in H \otimes V,$$

such that, for all $v \in V$, we have:

$$\sum_{(vH)} \sum_{[v]} v'_H \otimes v''_H \otimes v_V = \sum_{[vH]} \sum_{[vV]} v_H \otimes (v_H)_H \otimes (v_V)_V,$$

namely the diagram commutes:

$$\begin{align*}
V &\xrightarrow{\phi} H \otimes V \\
\phi &\downarrow \\
H \otimes V &\xrightarrow{\Delta \otimes \text{id}} H \otimes H \otimes V
\end{align*}$$

Analogously, one can define the right (co)action and therefore right (co)modules. However, we only use left (co)action and left $H$-(co)module conventions throughout this paper.

2.3.2 Yetter-Drinfeld modules

**Definition 4** Let $H$ be a bialgebra. We say that $V$ is a left-left Yetter-Drinfeld module, if:

- $V$ is a left $H$-module,
- $V$ is a left $H$-comodule,

\footnote{From now on, we simply call it a Yetter-Drinfeld module, since we only use left (co)actions. In the literature, Yetter-Drinfeld modules are also called “crossed bimodules” and “Yang-Baxter modules” [24].}
The following compatibility condition holds:

\[
\sum\sum (x'v_H \otimes x'' \triangleright v_V) = \sum\sum (x' \triangleright v)_H x'' \otimes (x' \triangleright v)_V,
\]

which is to say, the following diagram commutes:

\[
\begin{array}{ccc}
H \otimes V & \xrightarrow{\Delta \otimes \text{id}} & H \otimes H \otimes V \\
\downarrow \Delta \otimes \text{id} & & \downarrow \text{id} \otimes \Delta \otimes \text{id} \\
H \otimes V \otimes H & \xrightarrow{\text{id} \otimes \mathcal{R} \otimes \text{id}} & H \otimes H \otimes H \otimes V \\
\downarrow \rho \otimes \text{id} & & \downarrow \nabla \otimes \rho \\
V \otimes H & \xrightarrow{\phi \otimes \text{id}} & V \otimes V \\
\downarrow \rho \otimes \text{id} & & \downarrow \phi \otimes \text{id} \\
H \otimes V \otimes H & \xrightarrow{\text{id} \otimes \mathcal{R}} & H \otimes H \otimes V \\
\downarrow \nabla \otimes \rho & & \downarrow \phi \otimes \text{id} \\
H \otimes V & \xrightarrow{\mathcal{R}'} & H \otimes \mathcal{R} \otimes V
\end{array}
\]

\[
(6)
\]

\section*{Notation.} The category of Yetter-Drinfeld modules over a Hopf algebra \(H\) in \(\mathcal{C}\) will be denoted by \(\mathcal{YD}_\mathcal{C}(H)\). Especially, if \(\mathcal{C}\) is the category of vector spaces, we simply denote it by \(\mathcal{YD}(H)\).

\section*{Lemma 5} Suppose that \(H\) has an invertible antipode. Then, \(\mathcal{YD}_\mathcal{C}(H)\) is braided. Here, given two Yetter-Drinfeld modules \(V\) and \(W\), we have the braiding:

\[
\mathcal{R}': v \otimes w \in V \otimes W \mapsto \sum_{[v]} v_H \triangleright w \otimes v_V \in W \otimes V,
\]

which is obtained by the following diagram:

\[
\begin{array}{ccc}
V \otimes W & \xrightarrow{\phi \otimes \text{id}} & H \otimes V \otimes W \\
\downarrow \mathcal{R}' & & \downarrow \text{id} \otimes \mathcal{R} \\
W \otimes V & \xrightarrow{\rho \otimes \text{id}} & H \otimes W \otimes V
\end{array}
\]

\[
(8)
\]

\section*{Remark 6} If the antipode is not invertible, \((8)\) is called a “prebraiding” yielding a prebraided monoidal category in which the braiding map is not an isomorphism. For a detailed discussion about the alternative conditions to make \(\mathcal{YD}_\mathcal{C}(H)\) into a braided monoidal category, see [36]. For this reason, all Hopf algebras will be considered to have an invertible antipode in the rest of the paper.

\section*{Definition 7 (Adjoint Action)} Let \(A\) be a Hopf algebra. The left and right adjoint actions of \(A\) on itself are:

\[
a \triangleright a_{ad} b = \sum_{(a)} a'bS(a''), \quad b \triangleleft a_{ad} a = \sum_{(a)} S(a')ba''.
\]

\[
(10)
\]

\section*{Proposition 8} Suppose that \(\partial\): \(I \to H\) is an algebra morphism. Then \(H\) has a natural \(I\)-module algebra (Definition 14) structure with the action \(\rho\): \(I \otimes H \to H\) given by:

\[
\begin{array}{ccc}
I \otimes H & \xrightarrow{\partial} & H \\
\downarrow (\partial \otimes \text{id}) & & \downarrow \rho_{ad} \\
H \otimes H & \xrightarrow{\rho} & H \otimes H
\end{array}
\]
Consequently, every Hopf algebra $A$ turns into an $A$-module algebra. On the other hand, note that:

$$\Delta(a \triangleright_{ad} b) = \sum_{(a)} \sum_{(b)} a'b' S(a''') \otimes a''b'' S(a'''),$$

$$\Delta(b \triangleleft_{ad} a) = \sum_{(a)} \sum_{(b)} S(a'') b' a''' \otimes S(a') b'' a'''.$$

From here we can see that neither adjoint action ought to turn, in general, $A$ into a module coalgebra over itself. This is however the case if $A$ is cocommutative.

**Definition 9 (Adjoint Coaction)** The left and right adjoint coactions of a Hopf algebra on itself are:

$$\phi(y) = \sum_{(y)} y' S(y'') \otimes y'', \quad \phi(y) = \sum_{(y)} y'' \otimes S(y') y''' ,$$

which are obtained in a dual way to adjoint actions as:

![Diagram](image)

**Proposition 10** Suppose that $\partial: I \rightarrow H$ is a coalgebra morphism. Then $H$ has a natural $I$-comodule coalgebra (Definition 14) structure with the coaction $\rho: I \rightarrow I \otimes H$ given by:

$$\xymatrix{ I \ar[r]^{\rho} \ar[d]_{(\Delta)} & I \otimes H \ar[d]_{(\Delta)} \\ I \otimes I \ar[r]_{(\iota \otimes \id)} & I \otimes I }$$

Consequently, every Hopf algebra $A$ turns into an $A$-comodule coalgebra.

**Lemma 11** Let $H$ be a Hopf algebra. Then $H$ itself is a Yetter-Drinfeld module with the adjoint action $\partial$ and regular coaction $\phi = \Delta$.

The previous construction has a natural extension by considering Hopf algebra projections as follows:

**Definition 12 (Hopf algebra projection)** A Hopf algebra projection $(\partial: I \rightarrow H, i)$ is given by a Hopf algebra morphism $\partial$ together with another Hopf algebra morphism $i: H \rightarrow I$ (an inclusion) such that $\partial i = \id_H$.

**Lemma 13** Let $(\partial: I \rightarrow H, i)$ be a Hopf algebra projection. Then, $I$ has a natural Yetter-Drinfeld module structure over $H$ with the action $\rho: H \otimes I \rightarrow I$ and coaction $\phi: I \rightarrow H \otimes I$ given by:

![Diagram](image)
2.3.3 Braided Hopf algebras in \( \mathcal{YD}(H) \)

The main idea of this part is to obtain a Hopf algebra structure living in the category of Yetter-Drinfeld modules. For this reason, we follow the categorical meaning of braided Hopf algebras [4, 35]. However, there is also a non-categorical definition of braided Hopf algebras given in [38].

**Definition 14** Let \( H \) be a Hopf algebra. A braided Hopf algebra living in \( \mathcal{YD}_\kappa(H) \) is given by:

1. An object \( A \) of \( \mathcal{YD}_\kappa(H) \).
2. A unital associative algebra structure \( (A, \nabla, \eta) \), making it
   - a \( H \)-module algebra with:
     \[
     x \triangleright (ab) = \sum_x (x' \triangleright a) (x'' \triangleright b), \quad \text{and} \quad x \triangleright 1_A = \epsilon(x)1_A.
     \]
   That means, \( A \) is an \( H \)-module and also a \( \kappa \)-algebra, and \( \nabla, \eta \) are module maps.

   \[
   \begin{array}{c}
   \begin{array}{ccc}
   H \otimes A \otimes A & \xrightarrow{id \otimes \nabla} & H \otimes A \\
   \Delta \otimes \text{id} & \downarrow & \nabla \\
   H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes R \otimes \text{id}} & H \otimes A \otimes H \otimes A
   \end{array}
   \end{array}
   \]

3. A counital coassociative coalgebra structure \( (A, \Delta, \epsilon) \) with the notation of coproduct:
   \[
   \Delta(a) = \sum_{[a]} a \otimes a
   \]
   making it
   - a \( H \)-module coalgebra with:
     \[
     \sum_{[x \triangleright a]} x \triangleright a \otimes x \triangleright a = \sum_{(x)} \sum_{[a]} x' \triangleright a \otimes x'' \triangleright a \quad \text{and} \quad \epsilon(x \triangleright a) = \epsilon(x)\epsilon(a).
     \]
   That means, \( A \) is an \( H \)-module and also a \( \kappa \)-coalgebra, and \( \Delta, \epsilon \) are module maps.

   \[
   \begin{array}{c}
   \begin{array}{ccc}
   H \otimes A & \xrightarrow{\rho \otimes \text{id}} & A \\
   \Delta & \downarrow & \Delta \\
   H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes R \otimes \text{id}} & H \otimes A \otimes H \otimes A
   \end{array}
   \end{array}
   \]
• a $H$-comodule coalgebra with:

$$\sum_{[a]} \sum_{[a_A]} a_H \otimes a_A \otimes a_A = \sum_{[a]} \sum_{[a_A]} a_H a_A \otimes a_A$$

and

$$\sum_{[a]} a_H e(a_A) = e(a) 1_A.$$

That means, $A$ is an $H$-comodule and also a $\kappa$-coalgebra, and $\Delta, \epsilon$ are comodule maps.

4. A unital algebra morphisms $\Delta : A \to A \otimes_H A$ and $\epsilon : A \to \kappa$. Here the product in $A \otimes_H A$ is:

$$(a \otimes b)(c \otimes d) = \sum_{[b]} (a b_H \triangleright c) \otimes (b_A d),$$

namely:

5. An antipode $S : A \to A$ satisfying the usual properties given in [8].

Notation. A braided Hopf algebra $A$ living in the category of Yetter-Drinfeld modules over $H$ will be called a $\mathcal{YD}(H)$-Hopf algebra.

Definition 15 Let $A$ be a $\mathcal{YD}(G)$-Hopf algebra, and $B$ be a $\mathcal{YD}(H)$-Hopf algebra. A (braided) map between $t : A \to B$ consists of a Hopf algebra morphism $r : G \to H$ such that $t$ is:

• an algebra morphism (to preserve products),
• a coalgebra morphism (to preserve coproducts),
• a Yetter-Drinfeld module morphism to preserve the actions and coactions; namely the diagrams below commute:

2.3.4 More on module algebras

The following is well known and a proof appears in [28].

Theorem 16 (Majid) If $I$ is an $H$-module algebra with the action $\rho$, then we can define an algebra $I \otimes_{\rho} H$ with the underlying vector space $I \otimes H$, with product:

$$(u \otimes x)(v \otimes y) = \sum_{(x')} (u x' \triangleright v) \otimes (x'y),$$

where $u, v \in I$ and $x, y \in H$. 

and identity $1_I \otimes 1_H$. If $H$ is an $H$-module coalgebra, additionally satisfying the following compatibility condition:

$$
\sum_{(x)} x' \otimes (x'' \triangleright v) = \sum_{(x)} x'' \otimes (x' \triangleright v), \text{ for each } x \in H \text{ and } v \in I,
$$

(17)

then we have a bialgebra structure living in $I \otimes \rho H$ with coproduct:

$$
\Delta(u \otimes x) = \sum_{(u),(x)} (u' \otimes x') \otimes (u'' \otimes x''), \text{ where } x \in H \text{ and } u \in I,
$$

and:

$$
\epsilon(u \otimes x) = \epsilon(u) \otimes \epsilon(x), \text{ where } x \in H \text{ and } u \in I.
$$

If, in addition, $H$ and $I$ are Hopf algebras, then $I \otimes \rho H$ is a Hopf algebra with antipode:

$$
S(u \otimes x) = (1_I \otimes S(x)) (S(v) \otimes 1_H), \text{ where } x \in H \text{ and } u \in I.
$$

3 A Functorial Approach to Radford’s Theorem

We fix a braided monoidal category with equalizers $\mathcal{C}$ in the rest of the paper.

Let $\Omega: I \to H$ be a Hopf algebra morphism in $\mathcal{C}$. Put:

$$
\text{RKer}_\mathcal{C}(\Omega) = \{ v \in I : \sum_{(v)} v' \otimes \Omega(v'') = v \otimes 1 \}.
$$

(18)

We know from [3] that:

- $\text{RKer}_\mathcal{C}(\Omega)$ only defines a subalgebra of $A$ in $\mathcal{C}$, with the identity. Moreover, it is a left coideal, i.e. $\Delta(\text{REqual}(f,g)) \subseteq A \otimes \text{REqual}(f,g)$.

- $\text{RKer}_\mathcal{C}(\Omega)$ is not the actual kernel of $\Omega$ in the category of Hopf algebras.

- $\text{RKer}_\mathcal{C}(\Omega)$ is not closed under the antipode, since $S(\text{RKer}_\mathcal{C}(\Omega)) \subseteq \text{LKer}_\mathcal{C}(\Omega)$, where we put:

$$
\text{LKer}_\mathcal{C}(\Omega) = \{ v \in I : \sum_{(v)} \Omega(v') \otimes v'' = 1 \otimes v \}.
$$

- $\text{RKer}_\mathcal{C}(\Omega)$ is invariant under the (left) adjoint action.

- In fact, $\text{RKer}_\mathcal{C}(\Omega)$ is given by $\text{REqual}_\mathcal{C}(f, \eta_H \circ \epsilon_I)$ where we use:

$$
\text{REqual}(f,g) = \left\{ a \in A : \sum_{(a)} a' \otimes f(a'') = \sum_{(a)} a' \otimes g(a'') \right\},
$$

- However, more general case of $\text{RKer}_\mathcal{C}(\Omega)$ is given by:

$$
\text{CKer}_\mathcal{C}(\Omega) = \left\{ a \in A : \sum_{(a)} a' \otimes f(a'') \otimes a''' = \sum_{(a)} a' \otimes 1 \otimes a'' \right\},
$$

that correctly defines the kernel in the category of Hopf algebras. Remark that, we have:

$$
\text{RKer}_\mathcal{C}(\Omega) = \text{CKer}_\mathcal{C}(\Omega) = \text{LKer}_\mathcal{C}(\Omega),
$$

if the category is cocommutative.

For more details on the categorical properties of Hopf algebras, we refer [1, 2, 14, 15].

---

2 The compatibility condition (17) is used to prove that the coproduct of $I \otimes \rho H$ is an algebra morphism.
3.1 Radford’s Theorem

Radford/Majid’s theorem gives an idea to put a Hopf algebra structure on (18) when it is obtained from a Hopf algebra projection. The method was introduced in [27, 35] in the category of vector spaces. Afterwards, it is generalized to any braided monoidal category in [5].

3.1.1 \text{RKer}_\mathcal{C}(\partial) \text{ as a braided Hopf algebra}

Remark 17 Let \((\partial: I \to H, i)\) be a Hopf algebra projection in \(\mathcal{C}\). Then, \(I\) has a natural Yetter-Drinfeld module structure over \(H\), from Lemma 13. Since \(\text{RKer}_\mathcal{C}(\Omega)\) is invariant under the (left) adjoint action of \(I\), it follows that \(\text{RKer}_\mathcal{C}(\Omega)\) is, itself, a Yetter-Drinfeld module over \(H\).

Notation. We fix \(B = \text{RKer}_\mathcal{C}(\partial)\) for a given Hopf algebra morphism \(\partial: I \to H\) in the rest of this section.

Definition 18 For a given Hopf algebra projection \((\partial: I \to H, i)\) we define maps \(f, g: I \to I\) as follows:

\[
\begin{align*}
I & \xrightarrow{\Delta} I \otimes I & I & \xrightarrow{\Delta} I \\
\xrightarrow{\text{id} \otimes i \partial S} I \otimes I & & \xrightarrow{i \partial S} I \otimes I
\end{align*}
\]

Remark 19 Both \(f, g\) take their values in \(B = \text{RKer}_\mathcal{C}(\partial)\). Moreover, \(f|_B = \text{id}_B\), therefore \(f^2 = f\). On the other hand, we have:

\[
f(v i(x)) = f(v) \text{ and } f(i(x) \triangleright_{ad} v) = f(i(x) v) = x \triangleright_{ad} f(v),
\]

and also \(g f = g\). Therefore:

\[
\sum_{(v')} f(v') g(v'') = \epsilon(v), \quad \sum_{(v')} g(f(v') v'') = \epsilon(v), \quad \sum_{(v')} f(v') i(\partial(v'')) = v.
\]

Lemma 20 Let \((\partial: I \to H, i)\) be a Hopf algebra projection in \(\mathcal{C}\). Following the previous constructions, \(I\) forms a \(\mathcal{YD}_\mathcal{C}(H)\)-Hopf algebra structure with the following new coproduct and antipode:

\[
\Delta = (f \otimes \text{id}) \circ \Delta, \quad S = g.
\]

In general, we know that \(B = \text{RKer}_\mathcal{C}(\partial)\) is not a sub-Hopf algebra of \(I\) in \(\mathcal{C}\). However, since both \(f\) and \(g\) take values in \(B\), and \(B\) is invariant under the adjoint action, \(B\) forms a sub-\(\mathcal{YD}_\mathcal{C}(H)\)-Hopf algebra of \(I\). Consequently, \(B\) is also a \(\mathcal{YD}_\mathcal{C}(H)\)-Hopf algebra.

Definition 21 (Braided adjoint action) Suppose that \(A\) is a \(\mathcal{YD}_\mathcal{C}(H)\)-Hopf algebra. The braided adjoint action of \(A\) on itself is given by:

\[
a \triangleright_{bad} b = \sum_{[a]} \sum_{[b]} a \triangleright_{H} (a \triangleright b) \triangleright_{A} S(a \triangleright_{A} b).
\]

Notice that, this is obtained in a same diagrammatic way to normal adjoint action given in [12]. However, the braiding map is not trivial in the category \(\mathcal{YD}(H)\) which makes the difference in the formulae.

\footnote{These maps are not Hopf algebra morphisms. Moreover, we call them “kernel generator” maps.}
3.1.2 Bosonisation

**Lemma 22** Let $A$ be any $\mathcal{YD}_C(H)$-Hopf algebra. We have a Hopf algebra $A \otimes H$ with the underlying tensor product $A \otimes H$, and with:

\[(a \otimes x)(b \otimes y) = \sum_{(x)} (a x' \triangleright b) \otimes (x'' y),\]

\[\Delta(a \otimes x) = \sum_{[a]} \sum_{[x]} a \otimes a^H x' \otimes a^A x'',\]

\[S(a \otimes x) = (1 \otimes S(x))(S(a) \otimes 1).\]

The smash product and the smash coproduct given above is categorically defined in a dual way to each other diagrammatically by:

![Diagram](image)

**Theorem 23** Let $(\partial: I \to H, i)$ be a Hopf algebra projection in $C$. Then the pair of morphisms:

\[\Psi: v \in I \mapsto \sum_{(v)} f(v') \otimes \partial(v'') \in B \otimes H.\]

\[\Phi: a \otimes x \in B \otimes H \mapsto a i(x) \in I\]

are mutually inverse, defining the isomorphism of Hopf algebras $I \cong B \otimes H$ in $C$.

**Corollary 24** Therefore, any $\mathcal{YD}_C(H)$-Hopf algebra can arise from a Hopf algebra projection.

3.2 Interchanging Yetter-Drinfeld modules

**Theorem 25** Suppose that we have a Hopf algebra projection $(\partial: I \to H, i)$ in $C$, and $B$ is any Yetter-Drinfeld Module over $H$. That means, we already have an action and coaction of $H$ on $B$ given by:

\[\rho: H \otimes B \to B,\]

\[\phi: B \to H \otimes B.\]

Now, let us define $\bar{\rho}: I \otimes B \to B$ and $\bar{\phi}: B \to I \otimes B$ as follows:

\[\begin{align*}
I \otimes B \xrightarrow{\bar{\rho}} B \xrightarrow{\rho} H \otimes B \\
B \xrightarrow{\bar{\phi}} I \otimes B \xrightarrow{\phi} H \otimes B
\end{align*}\]

With this (induced) action and coaction, $B$ also defines a Yetter-Drinfeld module over $I$. 

13
Proof. First of all, it is clear that $1_I \triangleright b = b$ and $\phi(1) = 1_I \otimes 1_B$ for all $b \in B$ by definition, since $\partial$ and $i$ are (Hopf) algebra morphisms. Moreover:

- $B$ is an $I$-module: In the diagram below, top square commutes since $\partial$ is a Hopf algebra map. Also, it is easy to see that the bottom-left square also commutes. Moreover, since $B$ is an $H$-module, bottom-right square commutes, regarding (4). Consequently, the outer diagram commutes that proves $B$ is an $I$-module.

- $B$ is an $I$-comodule: In the diagram below, right square commutes since $i$ is a Hopf algebra map, and also the bottom-left square already commutes. Since $B$ is an $H$-comodule, top-left square commutes, regarding (5). Consequently, the outer diagram commutes that proves $B$ is an $I$-comodule.

- We check the compatibility condition: In the diagram below, the inner square commutes since $B$ is a YD-module over $H$, see (7). It is readily checked that all other small diagrams commute. Therefore, the outer diagram commutes that proves the required compatibility.
Consequently:

**Corollary 26** For a given Hopf algebra projection \((\partial : I \rightarrow H, i)\) in \(\mathcal{C}\) and any object \(B\) of \(\mathcal{YD}_\mathcal{C}(H)\), we proved that \(B\) can be converted to an object of \(\mathcal{YD}_\mathcal{C}(I)\). Therefore, we obtain a monoidal functor:

\[ F : \mathcal{YD}_\mathcal{C}(H) \rightarrow \mathcal{YD}_\mathcal{C}(I). \]

**Proposition 27** The functor \(F\) is braided monoidal, since it is identical on the braiding.

**Proof.** Recalling the braiding \(\mathcal{R}'\) in \(\mathcal{YD}_\mathcal{C}(H)\) from (20), we write \(F(\mathcal{R}')\) as follows:

\[
\begin{align*}
V \otimes W & \xrightarrow{\phi \otimes id} I \otimes V \otimes W \\
\mathcal{F}(\mathcal{R}') & \\
W \otimes V & \xleftarrow{\mathcal{R} \otimes id} I \otimes W \otimes V
\end{align*}
\]
which is (by definition):

\[
\begin{array}{c}
V \otimes W \xrightarrow{\phi \otimes \text{id}} H \otimes V \otimes W \xrightarrow{\text{id} \otimes \text{id}} I \otimes V \otimes W \\
W \otimes V \xrightarrow{\rho \otimes \text{id}} H \otimes W \otimes V \xrightarrow{\theta \otimes \text{id} \otimes \text{id}} I \otimes W \otimes V
\end{array}
\]

and using the fact that \(\partial i = \text{id}\), we get:

\[
\begin{array}{c}
V \otimes W \xrightarrow{\phi \otimes \text{id}} H \otimes V \otimes W \\
W \otimes V \xrightarrow{\rho \otimes \text{id}} H \otimes W \otimes V
\end{array}
\]

(24)

\[\]

**Proposition 28** Suppose that we have a braided Hopf algebra structure \(A\) living in \(\mathcal{YD}_\mathcal{C}(H)\). Then we have a natural Hopf algebra structure \(F(A)\) living in \(\mathcal{YD}_\mathcal{C}(I)\).

**Proof.** In fact, the functor \(F\) only changes the module structure, i.e. actions and coactions. Therefore, it does not change the object \(A\) and Hopf algebra operations on it. The most crucial point here is, the braiding is used in the compatibility law (2). However, we also proved that the braiding is preserved (identical) under \(\mathcal{H}\). □

Consequently, we can give the following:

**Corollary 29** Suppose that we have:

- A Hopf algebra projection \((\partial: I \to H, i)\) in a braided monoidal category \(\mathcal{C}\),
- A Hopf algebra \(B\) living in \(\mathcal{YD}_\mathcal{C}(H)\).

Then we obtain a Hopf algebra structure \(F(B)\) living in \(\mathcal{YD}_\mathcal{C}(I)\).

## 4 Radford’s Theorem in a Simplicial Structure

In this section, we apply Radford’s theorem to a simplicial Hopf algebra given in a braided monoidal category \(\mathcal{C}\), based on the previous section.

### 4.1 Simplicial Hopf algebras

A simplicial Hopf algebra \(\mathcal{H}\) is a simplicial set in the category of Hopf algebras. In other words, it is a collection of Hopf algebras \(H_n\) \((n \in \mathbb{N})\) in \(\mathcal{C}\) together with Hopf algebra morphisms:

\[
\begin{align*}
\delta^i_n &: H_n \to H_{n-1}, & 0 \leq i \leq n \\
\sigma^1_{n+1} &: H_n \to H_{n+1}, & 0 \leq j \leq n
\end{align*}
\]
which are called faces and degeneracies, respectively\footnote{To avoid overloaded notation, we will not use superscripts for faces and degeneracies.}. These morphisms are required to satisfy the following axioms, called “simplicial identities”:

\begin{align*}
(1) \quad d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\
(2) \quad s_i s_j &= s_{j+1} s_i \quad \text{if } i \leq j \\
(3a) \quad d_i s_j &= s_{j-1} d_i \quad \text{if } i < j \\
(3b) \quad d_j s_j &= d_{j+1} s_j = \text{id} \\
(3c) \quad d_i s_j &= s_{j-1} d_i \quad \text{if } i > j + 1
\end{align*}

A simplicial Hopf algebra can be pictured as:

\[\mathcal{H} = \begin{array}{c}
H_3 \overset{d_2}{\rightarrow} H_2 \overset{d_1}{\rightarrow} H_1 \overset{s_0}{\leftarrow} H_0 \\
\end{array}\]

**Remark 30** In a simplicial Hopf algebra structure, we can obtain various Hopf algebra projections using the simplicial identity (3b), such as:

\[\left( d_1 : H_1 \rightarrow H_0, s_0 \right), \quad \left( d_2 : H_2 \rightarrow H_1, s_1 \right), \quad \left( d_2 : H_3 \rightarrow H_2, s_2 \right), \quad \text{etc.}\]

Obviously, there exist \(2n\) different Hopf algebra projections coming from (3b) between \(H_n\) and \(H_{n-1}\).

**Notation.** Suppose that we have a Hopf algebra projection:

\[\left( \partial_j : H_n \rightarrow H_{n-1}, s_k \right),\]

in a braided monoidal category \(\mathcal{D}\). We put:

\[A_{j,k}^n = \text{RKer}_{\mathcal{D}}(\partial_j : H_n \rightarrow H_{n-1}),\]

where the kernel generators (19) will be denoted by:

\[f_{j,k}^n = \nabla (\text{id} \otimes s_k d_j S) \Delta, \quad g_{j,k}^n = \nabla (s_k d_j \otimes S) \Delta,\]

and following (25), the coproduct of \(A_{j,k}^n\) becomes:

\[\Delta_{A_{j,k}^n} = (f_{j,k}^n \otimes \text{id}) \Delta \varepsilon,\]

and the antipode of \(A_{j,k}^n\) is given by \(g_{j,k}^n\) as usual.

**Example 31** *In a simplicial Hopf algebra, we have:*

\[A_{0,0}^2 = \text{RKer}_{\mathcal{D}}(d_0 : H_2 \rightarrow H_1),\]

*which is obtained from the Hopf algebra projection \((d_0 : H_2 \rightarrow H_1, s_0)\).*

**4.2 Applying to dimension one and dimension two**

Consider the second part of the simplicial Hopf algebra, namely:

\[H_2 \overset{d_1}{\rightarrow} H_1\]

We know that:
• $A_{0,0}^1 \subset H_1$ is the braided Hopf algebra living in $\mathcal{YD}_C(H_0)$, which is obtained from the Hopf algebra projection $(d_0: H_1 \to H_0, s_0)$.

• $A_{0,0}^2 \subset H_2$ is the braided Hopf algebra living in $\mathcal{YD}_C(H_1)$, which is obtained from the Hopf algebra projection $(d_0: H_2 \to H_1, s_0)$.

Remark 32 We already know from \cite{472} that, $A_{0,0}^1$ also has a braided Hopf algebra structure in $\mathcal{YD}_C(H_1)$, considering the Hopf algebra projection $(d_0: H_1 \to H_0, s_0)$.

Question. Considering the Hopf algebra morphisms \cite{28} given in $\mathcal{C}$, is it possible to obtain another induced Hopf algebra projection in $\mathcal{YD}_C(H)$ between $A_{0,0}^2$ and $A_{0,0}^1$?

Idea. We have two candidates for it, namely:

$$\begin{array}{c}
A_{0,0}^2 \xrightarrow{d_1} A_{0,0}^1 \quad \text{or} \quad A_{0,0}^2 \xleftarrow{s_1} A_{0,0}^1,
\end{array}$$

where the morphisms are the restrictions\cite{1}.

Lemma 33 The following diagrams commute:

$$\begin{array}{cc}
H_2 \xrightarrow{d_2} H_1 & H_2 \xleftarrow{s_1} H_1 \\
A_{0,0}^2 \xrightarrow{d_2} A_{0,0}^1 & A_{0,0}^2 \xleftarrow{s_1} A_{0,0}^1
\end{array}$$

In other words, the kernel generator maps $f_{0,0}^*$ and $g_{0,0}^*$ are compatible with $d_2$ and $i_1$ in $\mathcal{C}$.

Proof. For the $f_{0,0}^*$ case, we have (by using simplicial identitites):

$$
\begin{align*}
f_{0,0}^1 d_2 &= \nabla (\text{id} \otimes s_0 d_0 S) \Delta d_2 \\
&= \nabla (d_2 \otimes s_0 d_0 S d_2) \Delta \\
&= \nabla (d_2 \otimes s_0 d_0 d_2 S) \Delta \\
&= \nabla (d_2 \otimes s_0 d_0 d_1 S) \Delta \\
&= \nabla (d_2 \otimes s_0 d_0 d_1 S) \Delta \\
&= d_2 \nabla (\text{id} \otimes s_0 d_0 S) \Delta \\
&= d_2 f_{0,0}^2 \\
\end{align*}
$$

Same simplicial identities also proves the $g_{0,0}^*$ case. \hfill \blacksquare

Remark 34 However, $d_1$ is not compatible with $f_{0,0}^*$ and $g_{0,0}^*$.

Lemma 35 We have the following Hopf algebra projection in $\mathcal{YD}_C(H_1)$:

$$A_{0,0}^2 \xrightarrow{d_1} A_{0,0}^1.$$ \hfill (29)

Proof. It is clear that $d_2, i_1$ are well-defined since the simplicial identities (1) and (3a). They define algebra morphisms since $A_{0,0}^2$ and $A_{0,0}^1$ are sub algebras of $H_2$ and $H_1$, respectively. Moreover:

\footnote{We always use same notation for the restricted cases of face and degeneracy morphisms in the rest of the paper.}
• $d_2, i_1$ are coalgebra (therefore bialgebra) morphisms, since:

$$\Delta_{A_{1,0}} d_2 = (f_{0,0}^1 \otimes \text{id}) \Delta_{d_2}$$

$$= (f_{0,0}^1 \otimes \text{id}) (d_2 \otimes d_2) \Delta_{i_1}$$

$$= (d_2 \otimes d_2) (f_{0,0}^1 \otimes \text{id}) \Delta_{i_1}$$  \text{(Lemma 33)}

$$= (d_2 \otimes d_2) \Delta_{A_{0,0}^1}$$

that proves $d_2$ is a coalgebra morphism. Similarly, $s_1$ defines a coalgebra morphism as well.

• It is also straightforward from Lemma 33 that, $d_2$ and $s_1$ are compatible with the antipode, by using the fact that $S^* = g_{0,0}^*$. That proves $d_2$ and $s_1$ Hopf algebra morphisms in $\mathcal{YD}(H_1)$. Therefore, (29) defines a Hopf algebra projection in $\mathcal{YD}(H_1)$. □

Remark 36 Since $d_1$ does not commute with $f_{0,0}^*$ and $g_{0,0}^*$, it does not define a Hopf algebra morphism between $A_{0,0}^0 \rightarrow A_{1,0}^1$ in $\mathcal{YD}(H_1)$. Therefore we can not obtain any Hopf algebra projection through $d_1$.

Corollary 37 Consequently, if we apply Radford’s Theorem to (29) in $\mathcal{YD}(H_1)$, we get the braided Hopf algebra:

$$A_{2,1}^2 = \text{RKer}_{\mathcal{YD}(H_1)} (d_2: A_{0,0}^2 \rightarrow A_{0,0}^1),$$

which is living in the (induced) braided monoidal category $\mathcal{YD}_{\mathcal{YD}(H_1)} (A_{0,0}^1)$.

Remark 38 Note that, $A_{2,1}^2 \subset A_{0,0}^2$ at the subalgebra level.

Remark 39 We can summarize the above constructions as follows:

\[
\begin{array}{c}
A_{0,0}^2 & \xrightarrow{d_2} & A_{1,0}^1 \\
\downarrow & & \downarrow \\
A_{2,1}^2 & \xrightarrow{s_1} & A_{0,0}^2
\end{array}
\]

\[
\begin{array}{cccccccc}
H_2 & & & & & & & H_0 \\
\downarrow & & & & & & & \downarrow \\
H_1 & & & & & & & H_0 \\
\downarrow & & & & & & & \downarrow \\
A_{1,0}^1 & & & & & & & H_0 \otimes H_0 \\
\downarrow & & & & & & & \downarrow \\
A_{0,0}^1 & & & & & & & H_0 \otimes H_0
\end{array}
\]

with expressing that, each level consists of Hopf algebras (projections) in different categories.

Lemma 40 Let us define $\partial_1: A_{0,0}^1 \rightarrow H_0$ as the restriction of $d_1$. Then, $\partial_1$ defines a twisted Hopf algebra map, i.e. the following diagram commutes:

\[
\begin{array}{cccccccc}
A_{1,0}^1 & \xrightarrow{\partial_1} & H_0 & \xrightarrow{\Delta_{i_1}} & H_0 \otimes H_0 \\
\downarrow & & \downarrow & & \downarrow \\
\Delta_{A_{0,0}^1} & & \nabla \otimes \partial_1 & & \nabla \otimes \partial_1 \\
A_{0,0}^1 \otimes A_{0,0}^1 & \xrightarrow{\partial_1 \otimes \rho} & H_0 \otimes H_0 \otimes A_{0,0}^1
\end{array}
\]
Proof. By using the simplicial identities, we have:

\[
(\nabla \otimes d_1)(d_1 \otimes \rho) \Delta_{A_{1,0}}(a) = (\nabla \otimes d_1)(d_1 \otimes \rho)(f_{0,0}^1 \otimes \text{id}) \Delta_{H_0}(a)
\]

\[
= (\nabla \otimes d_1)(d_1 \otimes \rho)(f_{0,0}^1 \otimes \text{id}) \sum_{(a)} a' \otimes a''
\]

\[
= (\nabla \otimes d_1) \sum_{(a)} d_1 f_{0,0}^1(a') \otimes \rho(a'')
\]

\[
= (\nabla \otimes d_1) \sum_{(a)} d_1 f_{0,0}^1(a') \otimes d_0(a'') \otimes a'''
\]

\[
= (\nabla \otimes d_1) \sum_{(a)} d_1(a') d_1 s_0 d_0 S(a'') \otimes d_0(a''') \otimes a'''
\]

\[
= (\nabla \otimes d_1) \sum_{(a)} d_1(a') d_0 S(a'') \otimes d_0(a''') \otimes a'''
\]

\[
= (\text{id} \otimes d_1) \sum_{(a)} d_1(a') \otimes d_0 S(a'') \otimes d_0(a''') \otimes a'''
\]

\[
= (\text{id} \otimes d_1) \sum_{(a)} d_1(a') \otimes a''
\]

\[
= \sum_{(a)} d_1(a') \otimes d_1(a'') = \sum_{(d_1(a'))} d_1(a') \otimes d_1(a'')
\]

\[
= \Delta_{H_0} d_1(a),
\]

for all \(a \in A_{1,0}\), which proves that the diagram above commutes. \(\blacksquare\)

Remark 41 If we similarly define \(\partial_1 : A_{2,1}^2 \to A_{0,0}^1\) as the restriction of \(d_1 : H_2 \to H_1\), then \(\partial_1\) defines only an algebra morphism, not a (twisted) Hopf algebra map.

4.3 A problem for higher dimensions

Now, let us consider the first three parts of the simplicial Hopf algebra. Then, apply Radford’s Theorem step by step analogously to §4.2 and see what happens.

Remark 42 In the diagram above, the restriction \(s_2 : A_{2,1}^3 \to A_{2,1}^3\) is not well-defined, therefore \(s_2\) does not define a (Hopf) algebra morphism.
Consequently, it is not possible to apply Radford’s Theorem after the third level of the process since we do not have any candidate to obtain a Hopf algebra projection between $A^3_{2,1}$ and $A^2_{2,1}$ except $d_3 s_2$, in $\mathcal{YD}(\mathcal{YD}_c(H_2))(A^2_{0,0})$.

5 Application: Braided Hopf Crossed Modules

In this section, our main is to understand braided Hopf crossed modules from the point of simplicial structures. But first, let us briefly recall the definition of braided Hopf crossed modules [25].

5.1 Braided Hopf crossed modules

**Definition 43** Let $H$ be a Hopf algebra and $I$ be a braided Hopf algebra living in $\mathcal{YD}(H)$. The “twisted Hopf algebra map” $\partial: I \to H$, namely an algebra morphism obeys:

$$\Delta(\partial(x)) = \sum_{(h)} \sum_{[h]} \partial(x') \partial(x') \otimes \partial(x''),$$

is called a “braided Hopf crossed module” if it further satisfies:

- $\partial(h \triangleright x) = \sum_{(h)} h' \partial(x) S(h'')$,
- $\partial(x) \triangleright y = \sum_{(x)} \sum_{[x]} x' \ (x_H \triangleright y) S(x'')$,

for all $x, y \in I$ and $h \in H$. Notice that, the right hand side of the second condition is the braided adjoint action in $\mathcal{YD}(H)$, given in (21). On the other hand, without the last condition, we call it a braided Hopf pre-crossed module.

5.2 Generating the elements of $A^2_{2,1}$

In this subsection, we will calculate a specific type of elements in a simplicial Hopf algebra $\mathcal{H}$ which is defined over the category of vector spaces. Afterwards, this specific type of elements will lead us to discover the relationship between simplicial Hopf algebra and Majid’s braided Hopf crossed module definition. The following idea and terminology firstly defined and also applied in [16, 32] for the case of groups. They call it “iterated Peiffer pairings” to construct higher level Peiffer elements to model higher dimensional categorical structures. However, in this paper, we slightly modify the idea to fit our construction.

Let $\mathcal{H}$ be a simplicial Hopf algebra and recall the constructions given in [30]. Then, for all $x, y \in A^1_{0,0}$, we have the following diagram to construct an element of $A^2_{2,1}$.

$$\begin{align*}
A^1_{0,0} \times A^1_{0,0} &\xrightarrow{F_{(1)}} A^2_{2,1} \\
A^1_{0,0} \times A^1_{0,0} &\xrightarrow{s_0 \times s_1} A^2_{2,1}
\end{align*}$$

$\text{(31)}$

---

$^6$Working in the category of vector spaces will allow us to have combinatorial calculations. On the other hand, our main aim is to make contact with Majid’s braided crossed module notion [25] where the base category is vector spaces. For the general case of such crossed modules, see [3].

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If we calculate such elements, we get:

\[
F_{(0)(1)}(x, y) = f_{2,1}^2 f_{0,0}^2 (s_0(x) \triangleright_{ad} s_1(y))
\]

\[
= f_{2,1}^2 \sum_{(s_0(x) \triangleright_{ad} s_1(x))} [s_0(x) \triangleright_{ad} s_1(y)]' s_0d_0(S[s_0(x) \triangleright_{ad} s_1(y)])''
\]

\[
= f_{2,1}^2 \sum_{(x)(y)} s_0(x') s_1(y') s_0S(x''') s_0d_0S \left(s_0(x'') s_1(y'') s_0S(x''')\right)
\]

\[
= f_{2,1}^2 \sum_{(x)(y)} s_0(x') s_1(y') s_0S(x''') S \left(s_0(x'') s_0d_0s_1(y'') s_0S(x''')\right)
\]

\[
= f_{2,1}^2 \sum_{(y)(x)} \sum_{s_0(x') s_1(y') s_0S(x''') S \left(s_0(x'') s_0d_0s_1(y'') s_0S(x''')\right)} \text{ since } \Delta(y) \subset H_1 \otimes RKerd(d_0)
\]

\[
= f_{2,1}^2 \sum_{s_0(x') s_1(y') s_0S(x''') S \left(s_0(x'') s_0S(x''')\right)}
\]

Here, we see that the element \((s_0(x) \triangleright_{ad} s_1(y))\) is also in \(A_{0,0}^2\). Now we need to calculate the element:

\[
f_{2,1}^2 \left(s_0(x) \triangleright_{ad} s_1(y)\right).
\]  

**Remark 44** To continue the calculation, we need to see what \(f_{2,1}^2\) is. We know from [26] that:

\[
f_{2,1}^2 = \nabla \left(id \otimes s_1 d_2 S_{A_{0,0}}^2\right) \Delta_{A_{0,0}^2},
\]

where:

\[
\Delta_{A_{0,0}^2} = (f_{0,0}^2 \otimes id) \Delta,
\]

from [27]. Therefore, in an explicit formula:

\[
f_{2,1}^2(a) = \nabla \left(id \otimes s_1 d_2 S_{A_{0,0}}^2\right) (f_{0,0}^2 \otimes id) \Delta
\]

\[
= \nabla \left(id \otimes s_1 d_2 S_{A_{0,0}}^2\right) \sum_{(a')} f_{0,0}^2 (a') \otimes a''
\]

\[
= \sum_{(a')} f_{0,0}^2 (a') s_1 d_2 g_{0,0}^2 (a'')
\]
\[= \sum_{(a)} a' s_0 d_0 S(a'') s_1 d_2 \left( s_0 d_0 (a''') S(a''') \right) \]
\[= \sum_{(a)} a' s_0 d_0 S(a'') s_1 d_2 s_0 d_0 (a'''') s_1 d_2 S(a''').\]

**Remark 45** We also need to obtain the clear formulae of:
\[
\left( \Delta \circ \Delta \right) \left( s_0(x) \triangledown d_1 s_1(y) \right).
\]

Since:
\[
\Delta(x \triangledown d_1 y) = \sum_{(x)(y)} x' y' S(x'''') \otimes x'' y'' S(x'''),
\]
we get:
\[
\Delta \left( \Delta(x \triangledown d_1 y) \right) = \sum_{(x)(y)} x' y' S(x^{viii}) \otimes x'' y'' S(x^{vii}) \otimes x''' y''' S(x^{vi}) \otimes x'''' y'''' S(x^v).
\]

Therefore:
\[
\left( \Delta \circ \Delta \right) \left( s_0(x) \triangledown d_1 s_1(y) \right) = \sum_{(x)(y)} s_0(x') s_1(y') s_0 S(x^{viii}) \otimes s_0(x'') s_1(y'') s_0 S(x^{vii}) \otimes s_0(x''') s_1(y'''') s_0 S(x^v) s_0 S(x^{vi}) \otimes s_0(x'''') s_1(y'''') s_0 S(x^v).
\]

If we put \( a = s_0(x) \triangledown d_1 s_1(y) \) in Remark 44 via Remark 45 we therefore obtain the result of (32) as follows:

\[
f'_1(a) = f'_1 \left( s_0(x) \triangledown d_1 s_1(y) \right) = \sum_{(x)(y)} s_0(x') s_1(y') s_0 S(x^{viii}) \otimes s_0(x'') s_1(y'') s_0 S(x^{vii}) \otimes s_0(x''') s_1(y'''') s_0 S(x^v) \otimes s_0 d_0 \left( s_0(x''') s_1(y'''') s_0 S(x^v) \right)
\]
\[= s_1 d_2 s_0 d_0 \left( s_0(x''') s_1(y'''') s_0 S(x^v) \right) s_1 d_2 S \left( s_0(x''') s_1(y'''') s_0 S(x^v) \right) \]
\[= \sum_{(x)(y)} s_0(x') s_1(y') s_0 S(x^{viii}) \otimes s_0 d_0 s_1(y'') s_0 S(x^{vii}) \otimes s_0 d_0 s_1(y''') s_0 S(x^v) \]
\[= \sum_{(x)(y)} s_0(x') s_1(y') s_0 d_0 s_1(y'') s_0 S(x''') s_1 d_2 s_0 S(x^{vii}) s_1 d_2 s_0 S(x^{vi}) s_1 d_2 s_0 S(x^v) \]
\[= \sum_{(x)(y)} s_0(x') s_1(y') s_0 d_0 s_1(y'') s_0 S(x''') s_1 d_2 s_0 (x'''') s_1 d_2 s_0 (x''') s_1 d_2 s_0 (x'''') s_1 d_2 s_0 (x'''') s_1 d_2 s_0 (x''') \]
\[\]
by using the fact that \( s_1 d_2 s_0 d_0 s_1 = s_1 d_2 d_0 s_1 s_1 = s_1 d_0 d_3 s_1 s_1 = s_1 d_0 d_3 s_2 s_1 = s_1 d_0 s_1 \). Moreover, we also used:
\[
\nabla \left( \sum_{(x)} (S(x))' \otimes S(S(x))'' \right) = \nabla \left( \sum_{(x)} S(x') \otimes S(S(x))' \right) = \epsilon(x) 1.
\]

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Corollary 46 If \( x, y \in A_{0,0}^1 \), then the element:

\[
F_{(0)(1)}(x, y) = \sum_{(x)(y)} s_0(x') s_1(y') s_0 d_0 s_1 S(y'') s_0 S(x''') s_1 d_2 s_0 (x'''') s_1 d_2 s_0 S(x'''')
\]

belongs to \( A_{2,1}^2 \), obtained from the diagram (31).

5.3 Braided Hopf crossed module through a simplicial structure

We know from Lemma 40 that, for a given simplicial Hopf algebra \( \mathcal{H} \), there exists a twisted Hopf algebra map:

\[
\partial_1 : A_{0,0}^1 \to H_0,
\]

in the sense of (39).

Theorem 47 Let \( \mathcal{H}' \) be a simplicial Hopf algebra such that \( A_{2,1}^2 \) is the zero object. Then (33) gives rise to a braided Hopf crossed module structure where the action of \( H_0 \) on \( A_{0,0}^1 \) is defined by the adjoint action via degeneracy morphism \( s_1 \), namely:

\[
h \triangleright x = s_0(h) \triangleright_{ad} x = \sum_{(h)} s_0(h') x s_0(S(h'')),
\]

for all \( h \in H_0 \) and \( x \in A_{0,0}^1 \).

Proof.

\( \blacklozenge \) Clearly, we have a “braided Hopf pre-crossed module” with:

\[
\partial_1(h \triangleright x) = d_1 \left( \sum_{(h)} s_0(h') x s_0(S(h'')) \right)
\]

\[
= \sum_{(h)} d_1 s_0(h') d_1(x) d_1 s_0(S(h''))
\]

\[
= \sum_{(h)} h' \partial(x) S(h''),
\]

by using simplicial identities.

\( \blacklozenge \) To make it a “braided Hopf crossed module”, we need to show that:

\[
\partial_1(x) \triangleright y = \sum_{[x]} x \triangleright_H (x H \triangleright y) S(x_A).
\]

The left hand side is:

\[
\partial_1(x) \triangleright y = d_1(x) \triangleright y
\]

\[
= s_0 d_1(x) \triangleright_{ad} y
\]

\[
= \sum_{(x)} s_0 d_1(x') y s_0(S(d_1(x''))),
\]

while on the right hand side, we have:
\[
\sum_{(x)} x_H \triangleright y \mathcal{S}(x_A) = \sum_{(x)} f(x') (x_H'' \triangleright y) \mathcal{S}(x_A')
\]
\[
= \sum_{(x)} f(x') (d_0(x'') \triangleright y) \mathcal{S}(x'')
\]
\[
= \sum_{(x)} f(x') \left( s_0 d_0(x'') \triangleright_{ad} y \right) \mathcal{S}(x'')
\]
\[
= \sum_{(x)} (x')' S s_0 d_0(x'') \left( s_0 d_0(x'') \triangleright_{ad} y \right) \mathcal{S}(x'')
\]
\[
= \sum_{(x)} x' S s_0 d_0(x'') \left( s_0 d_0(x'') \triangleright_{ad} y \right) \mathcal{S}(x'')
\]
\[
= \sum_{(x)} x' \epsilon(x'') y S s_0 d_0(x'') \mathcal{S}(x'')
\]
\[
= \sum_{(x)} x' y S s_0 d_0(x'') \mathcal{S}(x'')
\]
\[
= \sum_{(x)} x' y S s_0 d_0(x'') g(x'')
\]
\[
= \sum_{(x)} x' y S s_0 d_0(x'') s_0 d_0(x'') \mathcal{S}(x'')
\]
\[
= \sum_{(x)} x' y \epsilon(x'') \mathcal{S}(x'')
\]
\[
= \sum_{(x)} x' y \mathcal{S}(x'')
\]
\[
= x \triangleright_{ad} y
\]

which means braided adjoint action is equal to normal adjoint action in this case.\(^7\)

So, we need to show that:

\[
\sum_{(x)} s_0 d_1(x') y s_0 (S(d_1(x''))) = x \triangleright_{ad} y.
\]

Recall from Corollary\(^46\) that,

\[
t = \sum_{(x)(y)} s_0(x') s_1(y') s_0 d_0 s_1 S(y'') s_0 S(x'') s_1 d_2 s_0(x''') s_1 d_0 s_1(y''') s_1 S(y''') s_1 d_2 s_0 S(x'''),
\]

belongs to $A^2_{2,1}$.

On the other hand, we know that $A^2_{2,1}$ is fixed and trivial (i.e. zero object), consequently we have:

\[
d_1(t) = \epsilon(t) 1_{H_1}.
\]

\(^7\)Of course, this is not true in general. In fact, this is the consequence of simplicial identities and the definition of action and coaction.
Therefore:

\[ d_1(t) = d_1 \left( \sum_{(x)(y)} s_0(x') s_1(y') s_0 d_0 s_1 S(y'') s_0 S(x'') s_1 d_2 s_0(x'''') s_1 d_2 s_0(y''') s_1 d_2 s_0(x''') \right) \]

\[ = \sum_{(x)(y)} x' y' d_0 s_1 S(y'') S(x'') d_2 s_0(x''') d_0 s_1(y''') S(y''') d_2 s_0(x''') \]

is equal to:

\[ \epsilon(x) \epsilon(y) 1_{H_1} \]

Let \( f : H \otimes H \to H \) be:

\[ f(x \otimes y) = \sum_{(x)(y)} x' y' d_0 s_1 S(y'') S(x'') d_2 s_0(x''') d_0 s_1(y''') S(y''') d_2 s_0(x''') , \]

and therefore, we have:

\[ f(x \otimes y) = \epsilon(x) \epsilon(y) 1_{H_1} . \]

We can easily write,

\[ \sum_{(x)(y)} f(x' \otimes y') \otimes x'' \otimes y'' = \sum_{(x)(y)} (\epsilon(x') \epsilon(y') 1_{H_1}) \otimes x'' \otimes y''. \]

Moreover, by using the diagram:

\[ \begin{array}{ccc}
H \otimes H & \xrightarrow{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H \\
\downarrow \text{id} \otimes \tau \otimes \text{id} & & \downarrow \text{id} \otimes \text{id} \otimes \text{id} \\
H \otimes H \otimes H \otimes H & & H \otimes H \otimes H \otimes H \\
\downarrow f \otimes d_2 s_0 \otimes \text{id} & & \downarrow \nabla (\epsilon \otimes \epsilon) \otimes d_2 s_0 \otimes \text{id} \\
H \otimes H & \xleftarrow{m \otimes \text{id} \otimes \text{id}} & H \\
\end{array} \]

we can obtain:

\[ \sum_{(x)(y)} f(x' \otimes y') (d_2 s_0(x'') y'') = \sum_{(x)(y)} (\epsilon(x') \epsilon(y') 1_{H_1}) (d_2 s_0(x'') y'') \]  

(35)

The right hand side is equal to:

\[ d_2 s_0(x) y \]

while on the left hand side (after the calculations) we have:

\[ \sum_{(x)(y)} f(x' \otimes y') (d_2 s_0(x'') y'') \]

\[ = \sum_{(x)(y)} x' y' d_0 s_1 S(y'') S(x'') d_2 s_0(x''') d_0 s_1(y''') S(y''') d_2 s_0(x''') d_2 s_0(x'''') y''' \]  

(36)
Since \( y \in R\text{Ker}(d_0) \), we have

\[
\sum_{(y)} y' \otimes y'' \otimes y''' \subseteq H_1 \otimes H_1 \otimes R\text{Ker}(d_0).
\]

Therefore continuing with (36), we have:

\[
= \sum_{(x)(y)} x' y' d_0 s_1 S(y'') S(x'') d_2 s_0 (x''') d_0 s_1 (y''')
\]

\[
= \sum_{(x)(y)} x' y' s_0 d_0 S(y'') S(x'') d_2 s_0 (x''') s_0 d_0 (y''')
\]

\[
= \sum_{(x)(y)} x' y' s_0 d_0 S(y'') S(x'') d_2 s_0 (x''')
\]

Again, since \( y \in R\text{Ker}(d_0) \), we have \( \sum_{(y)} y' \otimes y'' \subseteq H_1 \otimes R\text{Ker}(d_0) \). Consequently:

\[
= \sum_{(x)(y)} x' y' s_0 d_0 S(y'') S(x'') d_2 s_0 (x''')
\]

\[
= \sum_{(x)} x' y' S(x'') d_2 s_0 (x''')
\]

So we have a new equality:

\[
\sum_{(x)} x' y' S(x'') d_2 s_0 (x''') = d_2 s_0 (x) y
\]

With the similar idea above, let \( g: H \otimes H \to H \) be:

\[
g(x \otimes y) = \sum_{(x)} x' y' S(x'') d_2 s_0 (x''')
\]

Therefore:

\[
g(x \otimes y) = d_2 s_0 (x) y
\]

that yields:

\[
\sum_{(x)} g(x' \otimes y) \otimes x'' = \sum_{(x)} (d_2 s_0 (x') \otimes x'')
\]

Again, consider the following diagram:
By using simplicial identities, on the right hand we have:
\[
\sum_{(x)} s_0 d_1 (x') y s_0 \left( S(d_1 (x'')) \right),
\]
and on the left hand side we get:
\[
g(x' \otimes y') d_2 s_0 S(x'') = x \triangleright_{\text{ad}} y,
\]
which satisfies the equality (34) and gives us the second condition of braided Hopf crossed modules. □

6 Conclusion

The Moore complex \[30\] of a simplicial group is a chain complex:
\[
N(G) = (\ldots \xrightarrow{d_{n+1}} N(G)_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} N(G)_2 \xrightarrow{d_1} N(G)_1 \xrightarrow{d_0} G_0)
\]
of groups, where \(N(G)_n = \bigcap_{i=0}^{n-1} \ker(d_i)\) at level \(n\), and the boundary morphisms \(d_n : N(G)_n \rightarrow N(G)_{n-1}\) are the restrictions of \(d_n : G_n \rightarrow G_{n-1}\). Moreover, \(N(G)\) defines a normal chain complex of groups, namely \(d_n(N(G)_n) \leq N(G)_{n-1}\), for all \(n \geq 1\). Thus, the Moore complex can be considered as the normalized chain complex of a simplicial group.

Based on this definition, we know that, the category of simplicial groups whose Moore complex is with length one, is equivalent to the category of group crossed modules \[34\] (for the monoid version, see \[8\]). The proof (for one direction) briefly contains the following functor: In such a simplicial group, the 2-truncation of (37), namely:
\[
\ker(d_0) \cap \ker(d_1) \xrightarrow{d_2} \ker(d_0) \xrightarrow{d_1} G_0,
\]
defines a group crossed module. The proof mainly uses the specific type of elements of \(\ker(d_0) \cap \ker(d_1)\) which is already trivial from the assumption.

However, if we slightly modify the definition of the Moore complex as:

- \(N(G)_n = \bigcap_{i=0}^{n-1} \ker(d_i)\) where \(i \neq 1\),
- put \(\partial_n : N(G)_n \rightarrow N(G)_{n-1}\) as the restrictions of \(d_1 : G_n \rightarrow G_{n-1}\),

then, we obtain an alternative definition of (37) (through simplicial identities). Moreover, we already checked that this alternative definition still involves a group crossed module analogously to (38) as follows:
\[
\ker(d_0) \cap \ker(d_2) \xrightarrow{d_1} \ker(d_0) \xrightarrow{d_1} G_0,
\]
after similar type of calculations. This gives an alternative functor from the category of that of simplicial groups to the category of group crossed modules.

Now, let us go back to the category of Hopf algebras. In fact, Theorem 47 generalizes the group theoretical case \[39\] to the category of Hopf algebras. However, we can not obtain an analogous version of (38) in the category of Hopf algebras due to Remark 36 which was one of the major problems we faced during our research. From this point of view, this paper can also be considered as the first serious approach to understand the Moore complex of simplicial Hopf algebras in the most general case.

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