Landau’s quasi-particle mapping: Fermi liquid approach and Luttinger liquid behavior

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A continuous unitary transformation is introduced which realizes Landau’s mapping of the elementary excitations (quasi-particles) of an interacting Fermi liquid system to those of the system without interaction. The conservation of the number of quasi-particles is important. The transformation is performed numerically for a one-dimensional system, i.e. the worst case for a Fermi liquid approach. Yet evidence for Luttinger liquid behavior is found. Such an approach may open a route to a unified description of Fermi and Luttinger liquids on all energy scales.

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The behavior of strongly correlated electron systems is a fundamental issue with many intriguing but unresolved questions[1,2]. Two antithetic concepts for non-symmetry-broken phases govern the discussion: the Fermi liquid (FL)[3] and the Luttinger liquid (LL)[4]. The FL, well-established in three-dimensional (3D) systems, is based on Landau’s conjecture that the low-lying excitations of interacting fermion systems can be connected continuously to those of the non-interacting Fermi gas: there is a smooth mapping between the quasi-particles of the interacting and of the non-interacting system. The LL, well-established in one dimension (1D), is based on the observation that collective particle-hole excitations exhaust all degrees of freedom. These collective excitations are bosonic and thus completely different from the non-interacting fermionic excitations. In two dimensions (2D) – relevant for the understanding for high-temperature superconductivity[1,3] – the situation is marginal[4] due to the occurrence of logarithmic factors. Here we report a non-perturbative prescription based on quasi-particle conservation to construct Landau’s mapping. Evidence is found that LL behavior is retrieved with our method.

FL theory is most successful as phenomenological theory based on the energy functional[3]

\[ E = \sum_{k} \varepsilon(k) \delta n_k + \sum_{k,q} f(k,q) \delta n_k \delta n_q \]  

(1)

where \( k, q \) are momenta close to the Fermi surface, \( \delta n_k \) the density of quasi-particles – particles above the Fermi sea, holes below, \( \varepsilon(k) \) the kinetic energy and \( f(k,q) \) is Landau’s interaction function. We highlight that Eq. 1 implies the conservation of the number of quasi-particles. Microscopic derivations of Eq. 1 are based on perturbation theory[3]. Ordinary electrons or holes are good quasi-particles on the Fermi surface \( k = k_F \) due to the restricted scattering phase space. The quasi-particle weight \( 0 < Z \leq 1 \) quantifies to which extent, i.e. to which fraction, real electrons or holes overlap with true quasi-particles. Moreover, the renormalization group (RG) has been used in the last decade to give FL theory a firm basis[5,6]. Additionally, RG approaches[7,8] are recently used to understand the 2D Hubbard models in the context of high-\( T_c \) superconductivity[9,10]. All these approaches are conceptually based on diagrammatic perturbation theory in the interaction strength. They do not employ or construct Landau’s mapping except on the Fermi surface. Quasi-particle conservation does not appear.

We use continuous unitary transformations (CUT)[11,12] to treat interacting Fermi systems

\[ H = E_0 + \sum_{k} \varepsilon(k) : n_k : + \sum_{K,p,q} \Gamma(K,p,q) : c_{K+p}^\dagger c_{K-p} c_{qK+q} : \]  

(2)

where \( : \) stands for normal ordering (in particular : \( n_k := \delta n_k \) and \( \Gamma(K,p,q) \) is the vertex function encoding all scattering processes induced by the interaction. We define a CUT which maps the Hamiltonian [3] to an effective one \( H_{\text{eff}} \) which conserves the total number of quasi-particles \( Q := \sum_k \text{sign}(k) - k_F : n_k ; \) i.e. \( \{ Q, H_{\text{eff}} \} = 0 \). To this end, the flow parameter \( \ell \) is used to denote intermediate Hamiltonians \( H(\ell) \). Starting point is \( H(\ell = 0) = H; \) end point is \( H(\ell = \infty) = H_{\text{eff}}. \) The transformation is defined by

\[ \partial_\ell H = [\eta(\ell), H(\ell)] \]  

(3a)

\[ \eta_{\ell,i,j} = \text{sign}(Q_{i,i} - Q_{j,j}) H_{i,j}(\ell) \]  

(3b)

where (3a) with sign(0) = 0 defines the infinitesimal generator \( \eta \). The matrix elements \( \eta_{\ell,i,j} \) and \( H_{i,j}(\ell) \) are given in an eigen-basis of \( Q \). The flow stops for \( \{ Q, H \} = 0 \) since then \( \eta = 0 \). This represents indeed the fix point at \( \ell = \infty \) as can be shown generally under certain assumptions[9,12]. The choice (3b) eliminates all parts of \( H \) changing the number of quasi-particles. The block band diagonality in \( H(\ell)[11,12] \) is retained which means here that the number of quasi-particles is changed only by \( 0, \pm 2, \pm 4 \). Even the 3- or more-particle terms do not change it by \( \pm 6 \) or higher. It is very important that (3) is renormalizing since it suppresses matrix elements between states with very different energies more strongly.
than those between similar energies \[\Gamma(0,2\Gamma)\]. Since approximations have to be made to solve (3) this renormalizing property is essential in order not to lose important low-energy physics.

Let us assume that we are able to solve Eqs. (3) exactly and that the resulting flow converges. Then the above defined \(\text{CUT}\) represents Landau’s mapping of quasi-particles. The conservation \([Q,H_{\text{eff}}]=0\) allows to describe all excitations by a \(\text{finite}\) number of quasi-particles. This is one of the crucial points of the present work. The conservation of the number of quasi-particles ensures that terms in \(H_{\text{eff}}\) consisting of 6, 8, or more fermionic operators become relevant only if three or more quasi-particles are present in the system. The terms made from 2 and 4 fermionic operators constitute the 1-particle and the 2-particle parts of the effective Hamiltonian. They represent the dispersion and the effective interaction (cf. Ref. [16]). The basic idea of FL theory is that at low energies only the 1-particle and the 2-particle parts are relevant even though 3-particle terms are present. Comparing Eq. 1 and Eq. 2 at \(\ell=\infty\) the vertex function at zero momentum transfer \((p=q)\) can be identified with Landau’s interaction function \(f(K-p,K+p) = \Gamma(K,p,p)\). For non-zero momentum transfer \(\Gamma(K,p,q)\) can be interpreted as general quasi-particle scattering amplitude. A precise one-to-one correspondence, however, to the quantities known from conventional FL theory \([4]\) cannot be established since the \(\text{CUT}\) keeps all interactions local in time so that the vertex function \(\Gamma\) in Eq. 2 has no frequency dependence.

So the \(\text{CUT}\) defined in Eq. (3) allows to compute Landau’s interaction function \(f\) directly for a microscopic model. But there is a caveat. Terms comprising six and more fermionic operators do not vanish and they influence the 1- and 2-particle terms for \(0<\ell<\infty\) because the quasi-particle conservation is achieved only at the end of the transformation \(\ell=\infty\). The restriction to the scattering processes labelled by the vertex function \(\Gamma\) constitutes an approximation. It is related to a 1-loop RG calculation for the interaction vertex or, equivalently, to the summation of the parquet diagrams \([24]\). Recently, such RG computations are performed by numerous groups \([13–15,22]\) for 2D Hubbard models, for which results by a \(\text{CUT}\) exist, too. There are three particularly attractive features of the \(\text{CUT}\) approach \([3]\). First, it is local in the flow parameter \(\ell\). Second, there is no need to take a retardation into account, i.e. no frequency dependence occurs because the unitarity of the transformation ensures that all intermediate Hamiltonians stay hermitian. This is a conceptual advantage over the RG procedures used so far since the book-keeping related to another dimension, namely time, is avoided. Third, and most importantly, a \(\text{CUT}\) keeps states a \(\text{all}\) energies. No information is lost. This major conceptual advancement makes it possible to address also non-universal behavior like the full frequency dependence of spectral densities \([24]\).

In order to back up the above general results we performed a numerical calculation for a tight-binding model of interacting fermions without spin on a chain with average filling \(n=1/2\). The bare dispersion \(\epsilon_0(k)\) reads \(-\cos(k)\). The interaction is taken to be a nearest-neighbor repulsion implying \(\Gamma_0(K,p,q) := V\sin(p)\sin(q)\) \([23]\). This model is equivalent to an anisotropic, antiferromagnetic Heisenberg spin chain with \(S=1/2\) via the Jordan-Wigner transformation \([28]\). It is exactly solvable so that much is known about its properties \([27,28]\). The system is metallic for \(V \leq 1\). For larger values the discrete translational invariance is broken and a gap is opened making the system insulating: here we do not consider this phase. The metallic phase itself is a generic realization of a LL \([23,30]\). The Luttinger parameter can be deduced from the comparison of exactly known quantities like velocities \([31]\) to the predictions of bosonization \([29,30]\). So the power-law behavior in the momentum distribution at the Fermi edge is known quantitatively \([30,31]\). This power-law behavior and the absence of a jump in the momentum distribution is taken as signature for the absence of quasi-particles and the prevalence of bosonic modes. Thus the ansatz \([4]\) in terms of fermions may seem inappropriate. Yet this is not the case.

![FIG. 1. Cuts of \(\Gamma(K,p,q)\) in Eq. (2) for \(K = \pi/4\) and \(p,q \in [-\pi,\pi]\): (a) beginning \((\ell = 0)\), (b) middle \((\ell = 0.5)\), (c) end of transformation \((\ell = \infty)\). Black lines stand for processes changing the number of quasi-particles by 0; dark grey for changes by \pm 4; light grey for changes by \pm 2.](image_url)
We use the parametrization in Eq. 2 with $\ell$-dependent energy constant $E_0$, dispersion $\epsilon_\ell$, and vertex function $\Gamma$. In the limit of an infinite system the starting values are $\Gamma(\ell = 0) = \Gamma_0$, $\epsilon(\ell = 0) = -(1 + 2V/\pi)\cos(\ell)$, and $E_0(\ell = 0) = -(1 + V/2)/\pi)N$, where $N$ is the system size. The deviations from the bare values result from the normal-ordering assumed in Eq. 2. The Eqs. 3 and 4 form a closed set of equations once the normal-ordered 3-particle terms engendered on the r.h.s. of Eq. 3a are omitted (22). We treat the resulting set of functional differential equations by discretization, i.e. we solve the equations for finite systems. Up to about $N = 50$ sites could be treated well numerically, which amounts up to a set of several thousand differential equations. In Fig. 1, cuts of the vertex function $\Gamma$ for a given total momentum $K$ illustrate the transformation at work. Different grey scales encode the action of the scattering process on the number of quasi-particles. Clearly, the processes that change the quasi-particle number are gradually suppressed (light and dark grey) on $\ell \rightarrow \infty$ so that at $\ell = \infty$ only the processes (black) remain which leave the number of quasi-particles unchanged.

In Fig. 2a the ground state energy beyond the absolute and linear term in $V$, i.e. the correlation part, is depicted. The approximate CUT result agrees well with the exact result. Only for $V \gtrsim 0.75$ a deviation becomes discernible; beyond $V_{si} \approx 0.94$ the approximate CUT treatment breaks down because of the omission of the 3-particle terms. This omission leads to a spurious instability (subscript $si$). Scaling the interaction $V_{si}$ where the spurious breakdown occurs to infinite system size we find $V_{si}(N = \infty) \approx 0.75$ which shows that there is a finite range of interactions where the CUT restricted to the ansatz (2) works.

More information on the underlying physics is provided by the the momentum distribution $n(k) := \langle n_k \rangle$ which can be computed by $n(k) = N\partial E/\partial \epsilon_k(k)$. A generic result is shown in Fig. 3 (crosses). At first glance, the data points towards a fairly large jump. But the points close to the Fermi surface bend strongly towards the value 1/2. Similar CUT results were obtained for a Tomonaga model (22). For a quantitative analysis we use a FL fit (33) and a LL fit (3) as shown in Eqs. 4a and 4b

$$n(k) - 1/2 \approx Z/2 + \Delta k(B \ln(\Delta k) + C)$$
$$n(k) - 1/2 \approx B\Delta k^\alpha + C\Delta k$$

with $\Delta k = |k - k_F|$; the parameters $Z_{k_F}, B, C$ and $\alpha, B, C$ are fitted. Checking the applicability of (33) on finite-size data obtained by bosonization for the Luttinger model (34) shows that least-square fits are most robust and provide good estimates within 5% accuracy for the exponent $\alpha$. In Fig. 3, the excellent agreement of the LL type fit and the much poorer agreement of the Fermi type fit are salient. This result is not affected by the precise fit procedure.

By numeric evaluation, we cannot prove the power law behavior. For instance, previous results (18), which rely on additional approximations in the transformation of the observable, yielded only a logarithmic correction. We find that our numerical data is better described by (33) than by a ln-type fit $B + a \ln(\Delta k) + C\Delta k$ just in the same way the data in Ref. (34) is better fitted by (33) than by the ln-type fit.

In addition, the exponents $\alpha$ in Fig. 2b obtained from the fits agree very well with the exact result. Thus the excellent agreement of the LL fit in Fig. 3 is not accidental. For $V < 0.6$ the approximated CUT result agrees even better with the exact result than does the straight application of bosonization. While the bosonization result is much too smooth the approximated CUT overestimates the exponent due to the neglect of the 3- and more-particle terms.
The very good agreements in Figs. 3 and 2b provide evidence that the CUT targeting on quasi-particle conservation captures the relevant LL behavior. This comes as a surprise since the concept of holes below and particles above the Fermi level as quasi-particles belongs in the first place to FL theory. This shows that a powerful RG procedure such as the CUTs allows one to stick to the FL type of quasi-particles. An analogous observation was made previously for spinons and triplets [24]. It is decisive to describe the multi-particle dynamics properly.

The validity of our mapping onto conserved fermionic quasi-particles is strongly supported by recent results [35,36] for the closely related quantum sine-Gordon model. By a similar CUT Kehrein mapped this strongly correlated model, which corresponds to an interacting fermionic model [57], for a wide range of parameters onto non-interacting fermions [38].

A description of LL behavior in terms of fermionic quasi-particles is intriguing since one can assume that the dimensional crossover to higher dimensional FL behavior is also captured: for $D > 1$, the quasi-particle description should work even better. Existing RG approaches provide good insight in the dimensional crossover in terms of effective low-energy models [29,1]. Renormalizing CUTs yield a quantitative description on all energy scales since no degrees of freedom are integrated out. Besides the advantages of the locality in the flow parameter $\ell$ and of the avoidance of retardation, this feature constitutes the main progress. In this way, spectral properties with full frequency dependence are accessible [24].

Summarizing, we presented a route based on continuous unitary transformations (CUT) to the explicit construction of Landau’s mapping of the excitations of the interacting system to the ones of the non-interacting system. The importance of quasi-particle conservation was highlighted. We constructed Landau’s mapping numerically in the worst case, namely for one-dimensional fermions. The results indicate that the CUT approach captures a finger-print of Luttinger liquids: power-law behavior in the momentum distribution. This finding suggests the CUT approach as method to describe the crossover from Fermi liquids to Luttinger liquids on all energy scales.

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