Axiomatic Definition of Limit of Real-valued Functions

Abstract We present a new way of organizing the few mathematical statements which form introduction to Calculus: the epsilon-delta characterization of the limit is now derived from four simple, intuitive and frequently used statements, which we choose as axioms.

Introduction

Several concepts of analysis, like real numbers and the Lebesgue measure, have both axiomatic and non-axiomatic definitions. Apparently, until now the limit concept had no axiomatic definition; in this note we present such an axiomatic definition for the limit of real-valued functions. The definition does not extend to limits in topological spaces. (The situation is somewhat different with metric spaces since a function \( \varphi(x) \) with values in a metric space \((M,d)\) converges to a point \( y \) in \( M \) if and only if the real-valued function \( f(x) = d(\varphi(x), y) \) converges to zero).

One difference between the previous definitions and the new one is that now the limit is defined as a mapping, previously as a relation (between real-valued functions and real numbers). Because of that we have now to define also the domain of the mapping, i.e., the class \( F \) of convergent functions. In this note we define \( F \) axiomatically; it is possible however to combine the two axioms that we use to define the limit with a non-axiomatic definition of the class \( F \) (as we did in [1]), or to define \( F \) simply as the largest class of functions on which a limit exists (a limit satisfying our two axioms), but we think that the axiomatic definition of \( F \) provides more direct path into Calculus.

There are numerous kinds of limits in Calculus: limits of sequences of real numbers; of real-valued functions of a real variable \( x \) as \( x \) tends to \( x_0^- \), to \( x_0^+ \), to \( x_0 \), to \( \infty \), ...; various limits of real-valued functions on \( \mathbb{R}^n \), on a metric space; limits of Riemann and Riemann-Stieltjes sums. In textbooks usually the limit is defined and the basic limit facts are listed for each case separately; we shall in this note present the axiomatic definition for the left-hand limit of the real valued function \( f \) at the point \( x_0 \) and establish the equivalence with the standard epsilon-delta definition; these statements are easily adapted to the right-hand limit of \( f \) at \( x_0 \), or to the limit as \( x \) tends to infinity, or to minus infinity. However the adaptation may not be obvious for the two-sided limit, or for the limit of functions defined on \( \mathbb{R}^n \) or on a metric space. In the appendix we shall see that this adaptation is easy and natural if the axioms are formulated for net convergence. However, since the concept of nets is found mostly in advanced texts ([4],[7],[10],[11]), (where it is used for the study of certain topological spaces) and only rarely in introductory analysis texts ([2],[5],[6],[8],[9]) (where it is used to unify different kinds of limits which appear in Calculus), we did not want to
assume that the reader is familiar with net convergence and therefore nets appear only in the appendix, which, we hope, would be understandable even to a reader who meets here the net convergence for the first time.

**Axioms and their immediate consequences**

The two axioms defining the mapping “left-hand limit at \( x_0 \)” (denoted \( \lim_{x \to x_0^-} \)) and the two axioms defining the domain of the mapping, i.e. defining real-valued functions that “converge as \( x \to x_0^- \)”, are interconnected.

The limit axioms are

(1) (Constants axiom) If \( f(x) = c \) for all \( x < x_0 \), then \( \lim_{x \to x_0^-} f(x) = c. \)

(2) (Inequality axiom) If the functions \( f \) and \( g \) converge as \( x \to x_0^- \) and if \( \lim_{x \to x_0^-} f(x) < \lim_{x \to x_0^-} g(x) \), then there exists \( a < x_0 \) such that \( f(x) < g(x) \) for \( a < x < x_0 \).

The inequality axiom is crucial, it plays here the role that the functional equation of the exponential function has in elementary analysis, or the role that countable additivity has in measure theory. By raising this well-known statement to the piedestal of an axiom, we did uncouple the traditional epsilon and delta.

Immediate consequences of the limit axioms:

By contradiction from (2) we derive

(3) (Inequality theorem) If the functions \( f \) and \( g \) converge as \( x \to x_0^- \) and if there exists \( a < x_0 \) such that \( f(x) \leq g(x) \) for \( a < x < x_0 \) then \( \lim_{x \to x_0^-} f(x) \leq \lim_{x \to x_0^-} g(x) \).

From (1) and (2) we obtain

(4)(i) If \( \lim_{x \to x_0^-} f(x) = l \) and \( l' < l \) then there exists \( a' < x_0 \) such that \( l' < f(x) \) for \( a' \leq x < x_0 \).

(ii) If \( \lim_{x \to x_0^-} f(x) = l \) and \( l < l'' \) then there exists \( a'' < x \) such that \( f(x) < l'' \) for \( a'' \leq x < x_0 \).

From (4) it follows obviously

(5) If \( f \) converges as \( x \to x_0^- \), then there exists \( a < x_0 \) such that \( f \) is bounded on the interval \([a, x_0)\), and

(6) If \( \epsilon > 0 \) and \( \lim_{x \to x_0^-} f(x) = l \), then there exists \( a < x_0 \) such that \( |f(x) - l| < \epsilon \) for \( a < x < x_0 \).

Finally, we can use (4) to prove the uniqueness of limit in the following sense:
(7) If \( \lim'_{x \to x_0^-} \) and \( \lim''_{x \to x_0^-} \) are two mappings satisfying axioms (1) and (2), having possibly different domains, and if \( f \) belongs to both domains, then \( \lim'_{x \to x_0^-} f = \lim''_{x \to x_0^-} f \).

The uniqueness in stronger sense (the fact that also the domains of the two mappings coincide) will follow from the epsilon-delta theorem.

The convergence axioms are

(8) (MB axiom). If there exists \( a < x_0 \) such that \( f \) is monotone and bounded on the interval \([a, x_0)\), then \( f(x) \) converges as \( x \to x_0^- \)

(9) (Sandwich axiom) Let \( a < x_0 \) and let the functions \( f, g \) and \( h \) satisfy \( f(x) \leq g(x) \leq h(x) \) on the interval \([a, x_0)\). If \( f \) and \( h \) converge as \( x \to x_0^- \) and if \( \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^-} h(x) \), then \( g \) converges as \( x \to x_0^- \).

An immediate consequence of (9) and (3) is

(10) If there exists \( a < x_0 \) such that \( f(x) = g(x) \) on the interval \([a, x_0)\), then if \( f \) converges as \( x \to x_0 \), so does \( g \), and \( \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^-} g(x) \).

Keeping unchanged the two limit axioms and slightly modifying the two convergence axioms we can define also the extended limit (limit with values in the extended real line \( \mathbb{R} \cup \{\infty\} \cup \{-\infty\} \)) and the real-valued functions that converge in the extended sense.

**Equivalence with the epsilon-delta definition**

Consider the statements:

(i) \( f(x) \) converges as \( x \to x_0^- \)

(ii) \( \lim_{x \to x_0^-} f(x) = l \)

(iii) for every \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) \) positive such that \( |f(x) - l| \leq \epsilon \) for \( x_0 - \delta \leq x < x_0 \).

Epsilon-delta theorem. Both (i) and (ii) hold if and only if (ii) holds.

Proof.

If (i) and (ii) hold, then so does (ii). This is statement (6) which was already proved. The converse is proved in two steps. First we show

(11) If both (i) and (ii) hold, then (iii) holds.

By (ii) we have

(12) \( l - \epsilon \leq f(x) \leq l + \epsilon \) on the interval \([x_0 - \delta, x_0)\) Since \( f(x) \) converges as \( x \to x_0^- \), we deduce from (12), (1) and (3) that \( l - \epsilon \leq \lim_{x \to x_0^-} f(x) \leq l + \epsilon \).
Since this holds for every positive \( \epsilon \) we conclude that \( \lim_{x \to x_0^-} f(x) = l \), which proves (11). The final step is to show (13) (ii) implies (i').

Let

\[(14) \quad M(x) = \sup\{f(t) | x \leq t < x_0\}, \quad m(x) = \inf\{f(t) | x \leq t < x_0\}.\]

Obviously both \( M \) and \( m \) are monotone functions, and -by (12)- they are both bounded on some interval \([a, x_0)\). By the MB axiom they both converge as \( x \to x_0^- \), in other words both satisfy (i'). It follows from (12) that they both satisfy (ii). So, by (11) both \( M \) and \( m \) satisfy (i'') and we have

\[(15) \quad \lim_{x \to x_0^-} M(x) = l \quad \text{and} \quad \lim_{x \to x_0^-} m(x) = l.\]

Observing that the definition (14) implies

\[(16) \quad m(x) \leq f(x) \leq M(x) \quad \text{for} \quad x < x_0,\]

we deduce from (15), (16) and the sandwich axiom that \( f(x) \) converges as \( x \to x_0^- \), which ends the proof.

Once the epsilon-delta theorem is established, it can be applied to obtain some other results as, for example,

(17) Theorem on convergence of the sum, product and quotient of two convergent functions.

(As in some older textbooks, the epsilon-delta form of the definition may be used to prove (i) if the functions \( z' \) and \( z'' \) converge to zero, then so does \( z' + z'' \), (ii) the function \( L + z(x) \) converges to \( L \) if and only if the function \( z \) converges to zero, and (iii) if the function \( b \) is bounded, and the function \( z \) converges to zero, then the product \( bz \) converges to zero. From these three facts, by simple algebraic manipulations one obtains (17)).

**Presentation to beginners**

To build a course for beginners in which the limit would be introduced axiomatically one would need now to provide examples and exercises, to add some details and to decide which statements and proofs given above to place in some kind of appendix.

Obviously there are many ways such a course can be built, but whatever way it is done, we think that it would be useful before stating a definition of the limit to explicitly introduce the relation “\( f(x) \) is ultimately less than \( g(x) \) as \( x \) tends to ...”; perhaps to even introduce a provisional notation for that relation, like “\( f \prec g \) at \( x_0^- \)” (at \( x_0^+ \), at \( x_0 \), at \( \infty \), ...)

**Appendix**

There are two equivalent ways to unify different kinds of limits for real-valued functions - one can introduce a “filter” (or a “filter-base”) in the
domain \( X \) of the function \( f ((3],[10], [12]) \) or introduce a “direction” on \( X \) ((2], [5], [6], [8],[9],[10]).

The second approach is better suited for our axiomatic definition. A “direction” on a set \( X = \{x, y, z, \ldots\} \) is a binary relation on \( X \) which is transitive (\( x < y \) and \( y < z \) imply \( x < z \)), reflexive (\( x < x \) for every \( x \)) and has the property that for any two elements \( x \) and \( y \) in \( X \) there exists an element \( z \) in \( X \) such that \( x < z \) and \( y < z \). A directed set \( (X,<) \) is a set \( X \) with a direction; if \( a \in X \), then the set \( \{x|a < x\} \) is called a tail of the directed set \( (X,<) \); a real-valued function defined on a tail of a directed set is called a real-valued net on that directed set. A real-valued net \( f \) is monotone increasing (decreasing) if for some tail \( T \) and all \( x, y \) in \( T \) such that \( x < y \) we have \( f(x) \leq f(y)(f(x) \geq f(y)) \); it is monotone if it is monotone increasing or monotone decreasing; net \( f \) is bounded means that there exists a tail on which the function \( f \) is bounded. The net \( f \) converges to the real number \( l \) if for every \( \epsilon \) positive there exists a tail \( T = T(\epsilon) \) such that if \( x \) is in the tail \( T \) then \( |f(x) - l| < \epsilon \). This is the classical definition.

To formulate our axiomatic definition we introduce first a strict partial order relation \( -< \) between real-valued nets on the directed set \( (X,<) \) defined by \( f-< g \) if and only if there exists a tail \( T \) such that for all \( x \) in \( T \) we have \( f(x) < g(x) \) (similarly \( f-\leq g \) corresponds to \( f(x) \leq g(x) \)).

The limit and convergence axioms for nets on \( (X,<) \) are

(1') (Constants axiom) If \( f(x) = c \) for all \( x \), then \( \lim f = c \).

(2') (Inequality axiom) If the nets \( f \) and \( g \) converge and if \( \lim f < \lim g \), then \( f-< g \).

(8') (MB axiom). Monotone bounded nets are convergent.

(9') (Sandwich axiom) Let the nets \( f, g \) and \( h \) satisfy \( f-\leq g-\leq h \). If \( f \) and \( h \) converge and if \( \lim f = \lim h \), then \( g \) also converges.

On a given set \( X \) a statement of the form “as \( x \) tends to ...” denotes a direction. For example, if \( (X,d) \) is a metric space, and \( x_0 \in X \), then “as \( x \) tends to \( x_0 \)” denotes the direction defined on the set \( X\setminus\{x_0\} \) by \( x \prec y \) if and only if \( d(y,x_0) \leq d(x,x_0) \). A real-valued function \( f \) on the set \( X\setminus\{x_0\} \) is a net on the directed set \( (X\setminus\{x_0\},<) \). It is easy to see that the net \( f \) is monotone if and only if the function of the net \( f \) has the property “there exists a positive number \( h \) and a monotone function \( m \) on the interval \( (0,h] \) such that \( f(x) = m(d(x,x_0)) \) for \( x \) such that \( d(x,x_0) < h \).” So, if we want to adapt the MB axiom to convergence on metric spaces we have the choice of using nets or sacrificing simplicity of the axiom. This holds even in the case when the metric space is the real line, namely when directly defining the two-sided limit at \( x_0 \); obviously the simplicity can in that case be preserved without introducing nets, just defining the two-sided limit with the aid of the left and right limit.
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