Braiding with Majorana lattices: Groundstate degeneracy and supersymmetry

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Majorana-based topological qubits are expected to exploit the nonabelian braiding statistics of Majorana modes in topological superconductors to realize fault-tolerant topological quantum computation. Scalable qubit designs require several Majorana modes localized on quantum wires networks, with braiding operations relying on the presence of the groundstate degeneracy of the topologically nontrivial superconducting phase. However, this degeneracy is lifted due to the hybridization between Majorana modes localized at a finite distance. Here, we describe a braiding protocol in a trijunction where each branch consists of a lattice of Majorana modes overlapping at a finite distance. We find that the energy splitting between the groundstate and the lowest-energy state decreases exponentially with the number of Majorana modes if the system is in its topologically nontrivial regime. This result does not rely on the specific braiding geometry and on the details of the braiding scheme but is a consequence of the supersymmetry and nontrivial topology of the effective low-energy Hamiltonian describing the Majorana lattice.

I. INTRODUCTION

Topologically-protected Majorana zero modes localized at the boundaries of the nontrivial phase of a one-dimensional (1D) topological superconductor [1–3] are expected to exhibit nonabelian exchange statistics, which can be exploited to realize the quantum gates of a topological quantum computer [4–6]. Motivated by their potential impact in the field of quantum computing, there is an ongoing experimental effort to realize Majorana modes in realistic and scalable devices, including semiconducting nanowires with strong spin-orbit coupling proximitized by conventional superconductors, semiconductor-superconductor planar heterostructures, and in arrays of magnetic adatoms on a conventional superconductor substrate (see Refs. [7, 8]). In principle, the nonabelian braiding statistics can be probed by adiabatically exchanging Majorana modes in physical space in a planar heterostructures, and in arrays of magnetic adatoms on a conventional superconductor substrate (see Refs. [7, 8]). In the real world, however, Majorana modes gain a finite but exponentially small energy splitting, restoring the groundstate degeneracy in the limit of infinitely many Majorana modes.

II. TRIJUNCTION WITH MAJORANA LATTICES

We consider a T-shaped or Y-shaped trijunction with a lattice of 2^N Majorana modes on each branch. In order to obtain a lattice of regularly-spaced Majorana modes, one can employ periodical magnetic textures induced by nanomagnets [16, 17]. In this setup, the boundary between trivial and nontrivial segments can be smoothly controlled by rotating an applied field, inducing the adiabatic translation of the Majorana modes, i.e., and adiabatic “Majorana pump” [16, 17]. Alternatively, regularly-spaced Majorana modes can be produced in the presence of arrays of tunable spin-valves [26] or magnetic tunnel junctions [27, 28], or periodic arrays of electric gates [9, 29, 30]. In these configurations, by switching on and off the spin-valves, magnetic tunnel
can be described as the unitary braiding operators $U$ from the upper left and lower branches, and from the upper right and lower branches. If performed adiabatically, these processes branch) crosses the junction into the right branch. At the end of the day, the Majorana modes which were sitting close to the branches in the upward direction, such that one Majorana mode from the lower branch (which originally started in the upper left branch) crosses the center of the junction into the left branch. Finally, we slide the Majorana lattices on the upper right and lower branches in the leftward direction, such that one Majorana mode from the right branch crosses the center of the junction, reaching the lower branch. We then slide in Ref. 31. As a first step, we slide the Majorana lattices on the upper left and lower branches in the downward direction, such that one Majorana mode from the right branch, or electric gates, one can drive different segments of the wire in and out of the topologically nontrivial phase to obtain a regular lattice of Majorana modes localized at the boundaries between trivial and nontrivial segments. In this "piano keyboard" setup [9, 29, 30], the Majorana modes can slide by slowly rearranging the magnetic texture or the electric gates configurations, as shown in Fig. 1(a). Alternatively, one can consider a Y-junction in planar Josephson junctions formed by depositing superconducting islands on top of a topological insulator, with Majorana modes localized at the superconducting vortex cores, which can be moved by applied currents, voltages, or phase differences [31].

If the lengths of the trivial and nontrivial segments are comparable to the Majorana localization length, there is a small overlap between the Majorana modes wavefunctions. In this case the low-energy Hamiltonian of the system can be obtained by projecting onto the subspace of Majorana operators, which in a trijunction configuration gives

$$H_{\text{eff}} = i\Gamma H^\dagger = i(u_{1,2}\gamma_{1,1}\gamma_{2,1} + u_{2,3}\gamma_{2,1}\gamma_{3,1} + u_{3,1}\gamma_{3,1}\gamma_{1,1}) + i\sum_{m=1}^{3}\sum_{n=1}^{2N-1} w_{m,n}\gamma_{m,n}\gamma_{m,n+1},$$

where $\gamma_{m,n}$ are the Majorana operators on each branch $m = 1, 2, 3$ and with $n = 1, 2, \ldots, 2N$ counting outward from the center, as shown in Fig. 1(b), $\Gamma = (\gamma_{1,1}, \gamma_{2,1}, \gamma_{3,1}, \ldots, \gamma_{1,2N}, \gamma_{2,2N}, \gamma_{3,2N})$ the vector of the Majorana operators, $u_{m,m'} \in \mathbb{R}$ the coupling between Majorana modes $\gamma_{m,1}$ and $\gamma_{m',1}$ at the center of the junction, and $w_{m,n} \in \mathbb{R}$ the couplings between contiguous Majorana modes $\gamma_{m,n}$ and $\gamma_{m,n+1}$ on each branch. Let us also assume that all the couplings can be written as $w_{m,n} = E_0 e^{i \theta_{m,n}/\xi_M}$ in terms of the Majorana localization length $\xi_M$ where $l_{m,n}$ is the distance between contiguous Majorana modes $\gamma_{m,n}$ and $\gamma_{m,n+1}$ on each branch, and $E_0$ a characteristic energy scale of the junction. We will later consider a simpler case where each branch forms a translational-invariant bipartite lattice with $w_{m,n} = w$ or $w_{m,n} = v$ for $n$ odd and even, respectively, and with $u_{1,2} = u_{2,3} = u_{3,1} = u$, as in Fig. 1(c).

### III. BRAIDING PROTOCOL

The Majorana modes can move along the three branches of the trijunction by manipulating magnetic textures or electric gates, as already mentioned [see Fig. 1(a)]. A simple braiding protocol which exchanges two Majorana modes at the center of the junction can be engineered by sliding the lattices of Majorana modes separately on the three different branches, as illustrated in Fig. 2. This braiding protocol is similar to the braiding protocol of Majorana modes in planar Josephson junctions introduced in Ref. 31. As a first step, we slide the Majorana lattices on the upper left and lower branches in the downward direction, such that one Majorana mode from the upper left branch crosses the center of the junction, reaching the lower branch. We then slide the Majorana lattices on the upper right and left branches in the leftward direction, such that one Majorana mode from the right branch crosses the center of the junction into the left branch. Finally, we slide the Majorana lattices on the upper right and lower branches in the upward direction, such that one Majorana mode from the lower branch (which originally started in the upper left branch) crosses the junction into the right branch. At the end of the day, the Majorana modes which were sitting close to the center of the junction on the upper left and right branches are exchanged. Analogously, one can exchange the Majorana modes from the upper left and lower branches, and from the upper right and lower branches. If performed adiabatically, these processes can be described as the unitary braiding operators $U_{m,m'} = \frac{1}{\sqrt{2}} (1 + \gamma_{m,1}\gamma_{m',1}) = \exp \left( \frac{i}{\xi_M} \gamma_{m,1}\gamma_{m',1} \right)$ with $m, m' \in \{1, 2, 3\}$, which correspond to the adiabatic exchange of the two Majorana modes $\gamma_{m,1}$ and $\gamma_{m',1}$ at the center of the junction.
SUSY multiplets with opposite fermion parity (the SUSY is spontaneously broken) [16–18]. The two degenerate groundstates of the contiguous modes $\gamma_{1,2,1}$ and $\gamma_{3,1}$, and depends only on the couplings $w_{m,2n-1}$ between the contiguous Majorana modes $\gamma_{m,2n-1}$ and $\gamma_{m,2n}$, but not on the couplings $w_{m,2n}$ of the contiguous modes $\gamma_{m,2n}$ and $\gamma_{m,2n+1}$. This mandates that, if the coupling between a single pair of contiguous Majorana modes $\gamma_{m,2n-1}$ and $\gamma_{m,2n}$ on one of the branches is zero, there exists at least one energy level which is exactly zero. Hence, one can recover an exactly degenerate groundstate by tuning a single parameter. The presence of a zero-energy level in a system of several Majorana modes coupled together is a manifestation of quantum mechanical SUSY [16, 17, 19].

Let us consider the simpler case where each branch of a translational-invariant bipartite lattice with $w_{m,n} = w = E_0 e^{-l_w/\xi_M} > 0$ and $w_{m,n} = v = E_0 e^{-l_v/\xi_M} > 0$ where $l_w$ and $l_v$ are the distances between contiguous Majorana modes $\gamma_{m,n}$ and $\gamma_{m,n+1}$ with $n$ odd and even, respectively, and with $u_{1,2} = u_{2,3} = u_{3,1} = u$, as in Fig. 1(c). In this case the Hamiltonian in Eq. (1) can be written as

$$H_{\text{eff}} = i \Gamma H \Gamma^\dagger = i u \left( \gamma_{1,2,1} + \gamma_{2,1} \gamma_{3,1} + \gamma_{3,1} \gamma_{1,1} \right) + \frac{i}{3} \sum_{m=1}^{N} \left( \sum_{n=1}^{N-1} w_{m,2n-1} \gamma_{m,2n} + \sum_{n=1}^{N-1} v_{m,2n} \gamma_{m,2n+1} \right),$$

The geometric average of all positive energy levels in this case becomes $|\text{pf} H|^{1/3} = w$, which gives an upper bound to the energy splitting $\Delta E \leq w$. This seems to suggest that, in a trijunction with branches of fixed length $L = Nl$ with $l = l_w + l_v$ and $\Delta l = l_w - l_v$, the energy splitting would increase to a finite limit as $\Delta E \propto w = E_0 e^{-\Delta l/2\xi_M} e^{-L/2N\xi_M}$ with the number of Majorana modes $2N$. As anticipated, this is not always the case, as we will show below.

The trijunction exhibits two topologically distinct phases realized respectively for $|w| > |v|$ and $|v| > |w|$, which correspond to the trivial and nontrivial phases of the three Majorana lattices of the three branches of the trijunction. If the branches are decoupled $u = 0$, indeed, each branch can be described as a bipartite lattice of 0D Majorana modes with an effective $\mathbb{Z}_2$ topological invariant $P_{\text{eff}} = \text{sgn}(|w| - |v|)$ (see Refs. [16, 17]). In the topologically trivial phase $|w| > |v|$, the Majorana lattices exhibit two degenerate groundstates in the limit of infinite Majorana modes $N \to \infty$ (or, equivalently, in a finite lattice with periodic boundary conditions), which are SUSY multiplets with opposite fermion parity (the SUSY is spontaneously broken) [16–18]. The two degenerate groundstates

\[\text{FIG. 2. Braiding of two Majorana modes in a trijunction obtained by sliding the Majorana modes at the boundaries between trivial and nontrivial segments. We move the Majorana modes on the upper left and lower branches downwards, then move the modes on the upper right and branches leftwards, and finally, move the modes on the upper right and lower branches upwards.}\]
FIG. 3. Particle density $\rho$ of the lowest-energy state of a trijunction with branches of length $L = 20\xi_M$ with $2N = 40$ Majorana modes, plotted on a single branch and on the three branches, and corresponding energy splitting $\Delta E$ as a function of the number of Majorana modes $2N$. The data point corresponding to odd $\gamma_{m,2n-1}$ and even $\gamma_{m,2n}$ Majorana modes are grouped together. (a) Trivial phase $|w| > |v|$ ($w/v = 1.2$). The Majorana modes on odd and even sites hybridize along the entire width of the trijunction branches. The energy splitting between the lowest-energy state and the groundstates increases and becomes proportional to the overlap $w$ between contiguous Majorana modes $\gamma_{m,2n-1}$ and $\gamma_{m,2n}$, increasing to a finite limit as $\Delta E \propto w \propto e^{-L/2N\xi_M}$. The continuous line at a constant value is a guide for the eye. (b) Topological phase transition $|v| = |w|$. The Majorana modes on odd and even sites become partially decoupled forming two Majorana modes delocalized on the whole lattice. In the limit $N \to \infty$, SUSY mandates that the $\Delta E \to 0$, with each branch of the trijunction exhibiting two degenerate groundstates which are SUSY multiplets with opposite fermion parity. As verified numerically, the energy splitting decreases polynomially as $\Delta E \propto w/N$. The continuous line is the best fit for $\Delta E/w \propto 1/N$. (c) Nontrivial phase $|v| > |w|$ ($w/v = 0.8$). The Majorana modes on odd and even sites decouple and hybridize into left and right Majorana end modes exponentially localized respectively at the center and at the outer ends of the three branches, with localization length $\xi_{eff}$. The energy splitting decreases exponentially as $\Delta E \propto e^{-2N/\xi_{eff}}$. The continuous line is the best fit for $\log(\Delta E/w) \propto -2N/\xi_{eff}$.

correspond to the vacuum and to counterpropagating and dispersive 1D Majorana modes delocalized on the whole length of the lattice [16, 17]. Hence, one expects that the energy splitting decreases with the number of Majorana modes, approaching zero for $N \to \infty$. We verify numerically that the energy splitting decreases polynomially with the inverse of the number of Majorana modes as $\Delta E \propto w/N$ for $|v| = |w|$, even for finite $|u| > 0$. In the topologically nontrivial phase $|v| > |w|$ instead, each branch exhibits two Majorana end modes given by the hybridization of the 0D Majorana modes in each branch $\tilde{\gamma}_L \propto \sum_j (w/v)^j \gamma_{m,2n-1}$ and $\tilde{\gamma}_R \propto \sum_j (w/v)^{N+1-n} \gamma_{m,2n}$ [16, 17]. The left and right Majorana end modes, given by the superposition of 0D Majorana modes on odd and even sites, are localized at the opposite ends, i.e., respectively at the center and at the outer ends of the trijunction, with effective localization length $\xi_{eff} = l/|\log |w/v|| = (l/|\Delta l|)\xi_M \geq \xi_M$. If the coupling between the three branches is finite $|u| > 0$, the three Majorana modes at the center hybridize into an unpaired Majorana mode at low energy [9]. The hybridization of the outer and inner Majorana modes yields a finite energy splitting $\Delta E \propto e^{-2N/\xi_{eff}}$. 


corresponding to the overlap of two end modes for each branch with localization length $\xi_{\text{eff}}$. Remarkably, the presence of the inner and outer Majorana end modes and the exponential scaling of the energy splitting does not depend on the coupling $u$ between the three 0D Majorana modes at the center of the trijunction. In particular, the numerical results indicate that $\Delta E \to 0$ in the limit $N \to \infty$ for $|v| = |w|$, even for finite $|u| > 0$. This suggests that SUSY is not fully broken by the coupling of the three different branches of the trijunction. However, it is an open question whether the trijunction topology further partially breaks the extended SUSY.

To numerically verify our findings, we diagonalize the Hamiltonian in Eq. (3) considering a trijunction with fixed length $L$ and overlaps $w = E_0 e^{-\Delta l/2\xi_{\text{eff}}}$ and $v = E_0 e^{-\Delta l/2\xi_{\text{eff}}}$ with fixed $\Delta l$. Figure 3 shows the particle density $\rho$ of the lowest-energy state and energy splitting $\Delta E$ as a function of the number of Majorana modes $2N$. In the trivial phase $|w| > |v|$ (i.e., $\Delta l > 0$), Majorana modes on odd and even sites hybridize along the trijunction, and the energy splitting increases to a finite limit as $\Delta E \propto w \propto e^{-L/2\xi_{\text{eff}}}$ [see Fig. 3(a)]. At the topological phase transition $|v| = |w|$ (i.e., $\Delta l = 0$), the Majorana modes on odd and even sites hybridize into two Majorana modes, and the energy splitting decreases polynomially as $\Delta E \propto w/N$ [see Fig. 3(b)]. In the nontrivial phase $|v| > |w|$ (i.e., $\Delta l < 0$), the Majorana modes on odd and even sites decouple and hybridize into left and right Majorana end modes, exponentially localized at the center and at the outer ends of the trijunction, with the energy splitting decreasing exponentially as $\Delta E \propto e^{-2N/\xi_{\text{eff}}}$ [see Fig. 3(c)]. In all these regimes, we numerically verified that the asymptotic behavior of the energy splitting is independent on the choice of the coupling $u$ between the Majorana modes at the center of the junction.

The scaling of the energy splitting fully characterizes the different regimes of the trijunction: Indeed, the energy splitting increases to a finite limit, decreases exponentially, or decreases polynomially, respectively, in the trivial, nontrivial, and at the topological phase transition. One can also expect that this asymptotic behavior is preserved when the translational invariance is broken, i.e., in the presence of random disorder in the coupling between contiguous Majorana modes.

V. CONCLUSIONS

In this work, we analyzed the groundstate properties and described a braiding protocol for a trijunction where each branch contains an array of Majorana modes overlapping at a finite distance. We found that, in this setup, the energy splitting between the groundstate and the lowest-energy many-body state decreases exponentially with the number of Majorana modes if the effective low-energy Hamiltonian of the Majorana lattices is topologically nontrivial. This result does not depend on the geometry of the junction and on the details of the braiding scheme, and suggests that, in the nontrivial regime, the more Majorana modes there are in a topological qubit, the smaller the energy splitting.

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