Wavenumber selection in pattern forming systems

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Wavenumber selection in pattern forming systems remains a long-standing puzzle in physics. Previous studies have shown that external noise is a possible mechanism for wavenumber selection. We conduct an extensive numerical study of the noisy stabilized Kuramoto Sivashinsky equation. We use a fast spectral method of integration, which enables us to investigate long-time behavior for large system sizes that could not be investigated by earlier work. We find that a state with a unique wavenumber has the highest probability of occurring at very long times. We also find that this state is independent of the strength of the noise and initial conditions, thus making a convincing case for the role of noise as a mechanism of state selection.

I. INTRODUCTION

This work addresses the question of wavenumber selection in pattern forming systems. Pattern forming systems are characterized by the emergence of a band of spatially periodic steady states, as a particular quantity called the control parameter is varied [1]. Examples of pattern forming systems in physics are Rayleigh-Bénard convection [1] and directional solidification [2]. Although a large number of periodicities are mathematically allowed for such systems, experiments and simulations of realistic physical systems have repeatedly shown that only a narrow range of periodicities is realized in practice. The tendency of a system to prefer a narrow set of states out of many possible states, irrespective of the initial conditions is known as wavenumber selection. The physical mechanism that causes this is, as yet, unknown. Various computational studies [3–5] have hinted that the presence of additive noise is one of the mechanisms by which wavenumber selection takes place. This can be understood heuristically with the help of a simple dynamical system that evolves in a relaxational manner, i.e. it minimizes a potential energy. This potential energy can, in principle, have many local minima. In the absence of noise, the system would evolve to one of these local minima, depending on the initial condition and stay in this state indefinitely. However, in the presence of noise, the system would escape from any local minima and eventually end up in the global minimum of the potential energy, where it would then spend most of its time, provided the noise strength is small. While this simple example of a potential system illustrates wavenumber selection, most of the real systems that have exhibited wavenumber selection in the presence of noise do not have a potential energy function. What causes non-potential systems to prefer one wavenumber over all others is an unsolved theoretical question.

In this paper, we want to study wavenumber selection in a model known as the stabilized Kuramoto Sivashinsky equation [6]. Our reason for choosing this model is that it exhibits rich nonlinear behavior and a band of spatially periodic steady states, while being relatively simple and one-dimensional, which makes simulating it easy. Noise induced wavenumber selection in this model has been studied before using direct numerical simulation [5] and path integral methods [7]. The noise selected state for small system sizes and a limited range of control parameter values was determined in [5]. It is of considerable interest to extend their analysis to larger system sizes, since true selection of a unique final state can only occur in the thermodynamic limit. Our aim in this work is to determine the noise selected wavenumber for large system sizes and for a wide range of control parameter values. This paper describes our attempts to do so and is organized as follows: Section II introduces the mathematical formalism used to study pattern forming systems and describes the SKS model. Section III briefly summarizes previous work done on wavenumber selection in this model. Section IV describes our computational method in detail and Section V shows some of our results and their interpretation. Finally, Section VI discusses some of the drawbacks of our computational methods and touches on potential improvements that will be the focus of future work.

II. PATTERN FORMATION AND THE SKS MODEL

Pattern forming systems such as the ones mentioned above are represented by nonlinear partial differential equations that govern the evolution of a given quantity of interest. A typical equation for such a system would look like,

\begin{equation}
\partial_t u(x,t) = \hat{L}_p u(x,t) + \hat{N}[u(x,t)]
\end{equation}

where \(u(x,t)\) is a field that represents the quantity of interest, \(\hat{L}_p\) is a linear differential operator acting on \(u(x,t)\) and \(\hat{N}[u(x,t)]\) is a nonlinear operator. The subscript \(p\) on the linear operator indicates that it depends on the control parameter \(p\). As an example, in the case of directional solidification, the quantity of interest \(u(x,t)\) would be the position of the interface between the liquid and solid phases. The trivial solution (or base state) \(u_0(x,t) = 0\) is generally a stationary state of these equations. However, this solution is stable only for a certain range of values of \(p\). After \(p\) crosses a critical value \(p_c\), the
trivial, spatially uniform state becomes unstable to periodic perturbations and a band of stable, periodic steady states emerges. To determine when the uniform state becomes unstable, we imagine adding to the uniform base state \( u_b(x, t) = 0 \) the perturbation \( \delta u \sim e^{iqx+\sigma(q)t} \). This perturbation is periodic in space with wavenumber \( q \) and grows with time at a rate \( \sigma \). We then substitute \( u(x, t) = u_b(x, t) + \delta u \) in Eq. (1), retain only terms linear in \( \delta u \) and derive an expression for the growth rate \( \sigma \) as a function of \( q \). If, for a given value of \( p \), \( \text{Re}(\sigma(q)) \) is negative, that means that the perturbations with those wavenumbers decay exponentially with time and hence the uniform base state \( u_b \) is stable to those perturbations. However, if \( \text{Re}(\sigma(q)) \) is positive, then perturbations with wavenumber \( q \) grow exponentially with time, implying that the base state is unstable to them. Of course, in practice, the unbounded exponential growth of these perturbations is balanced by the nonlinear terms that we neglected.

To make these ideas more concrete, we illustrate the above steps for the deterministic SKS model. The SKS equation is given by,

\[
\partial_t u(x, t) = -\alpha u(x, t) - \partial_x^2 u(x, t) - \partial_x^4 u(x, t) + (\partial_x u(x, t))^2 \tag{2}
\]

Here, \( \alpha \) plays the role of the control parameter, and \( u(x, t) \) is a dimensionless field of dimensional space-time variables. This equation is used to describe directional solidification and the Burton-Cabrera-Frank model of terrace growth [8]. The trivial solution \( u(x, t) = 0 \) is one of the steady states of this equation. We now add to this solution a small perturbation of the form \( \delta u \sim e^{iqx+\sigma(q)t} \). (We restrict ourselves to the case where \( \sigma \) is purely real.) Substituting into Eq. (2) and linearizing about the zero state gives,

\[
\sigma = -\alpha + q^2 - q^4 \tag{3}
\]

From this, we see that the growth rate is non-negative for \( 1/2 - \sqrt{1/4 - \alpha} \leq q^2 \leq 1/2 + \sqrt{1/4 - \alpha} \), as shown in Fig. [1]

Hence, if \( \alpha \leq 1/4 \), there exists a band of modes that grow exponentially with time. For these modes, the state \( u(x, t) = 0 \) is unstable. By solving the equation \( \partial_q \sigma = 0 \), we see that the growth rate is maximum for \( q = q_c = 1/\sqrt{2} \), for all \( \alpha \). Therefore, for \( \alpha \leq 1/4 = \alpha_c \), a band of periodic steady states exists, with wavenumbers given by,

\[
1/2 - \sqrt{1/4 - \alpha} \leq q^2 \leq 1/2 + \sqrt{1/4 - \alpha}, \quad \alpha \leq \alpha_c \tag{4}
\]

The states with these wavenumbers may themselves be unstable to long wavelength periodic perturbations [1], hence, in practice, one observes a band of periodic states that is narrower than suggested by Eq. (4) and is called the Eckhaus stable band. Wavenumbers within this band are stable to long wavelength periodic perturbations.

### III. Previous Work

Obeid et. al. [5] is one of the very few works that investigates noise induced wavenumber selection in the SKS equation. The SKS equation with noise is,

\[
\partial_t u(x, t) = -\alpha u(x, t) - \partial_x^2 u(x, t) - \partial_x^4 u(x, t) + (\partial_x u(x, t))^2 + \zeta(x, t) \tag{5}
\]

where \( \zeta(x, t) \) is an additive Gaussian noise satisfying,

\[
\langle \zeta(x, t) \rangle = 0 \tag{6}
\]

and

\[
\langle \zeta(x, t)\zeta(x', t') \rangle = 2\pi \delta(x - x')\delta(t - t') \tag{7}
\]

Here \( \varepsilon \) is the noise strength. Eq. (5) was discretized using a simple finite difference scheme and the time integration was performed using the explicit forward Euler method. They tried various values of \( \alpha \) between 0.17 and 0.24 and noise strength ranging from \( 10^{-5} \) to \( 10^{-3} \). Their system size was \( N = 1024 \) lattice points with lattice spacing \( \Delta x = 0.5 \) and periodic boundary conditions were imposed on the system. For \( \alpha = 0.24 \), they found that the state with \( q = 0.6995 \) was most stable, in the sense that the system could not be knocked out of this state for \( 10^8 \) time steps, with noise strength up to \( \varepsilon = 5 \times 10^{-4} \).

For \( \alpha \) values farther from the critical value of 0.25, they were unable to pinpoint a particular state as being most stable. For \( \alpha = 0.2 \), the allowed wavenumbers are \( 0.589 \leq q \leq 0.767 \). They found that states with \( 0.650 \leq q \leq 0.712 \) all remained stable up to \( 10^8 \) time steps and noise strengths up to \( \varepsilon = 2 \times 10^{-3} \). Similarly, for \( \alpha = 0.17 \), states with \( 0.650 \leq q \leq 0.699 \) remain stable at \( 10^8 \) time steps and noise strengths up to \( \varepsilon = 2 \times 10^{-3} \). Increasing the noise strength made it hard to discern any periodic states and they concluded that very long computational times would be required to destroy the stability of the states. In summary, their simulations hinted that the noise does indeed prefer one wavenumber over others. However, they could conclusively demonstrate selection...
of a unique state only for $\alpha = 0.24$. A conclusive proof that state selection occurs would require integration over large system sizes and a wider range of control parameters.

A more recent study of state selection in the SKS equation was conducted by Qiao et al. [7]. They used the least action principle of Freidlin Wentzell theory [9] to calculate transition probabilities between pairs of periodic steady states for the SKS equation. For a stochastic process given by,

$$\dot{\Phi}(t) = f(\Phi) + \zeta(t)$$

the probability of a particular trajectory $\phi(t)$ defined over a time interval $[0, T]$ is proportional to

$$P_T[\phi(t)] \propto \exp[-S_T[\phi(t)]/\varepsilon]$$

Here, as before, $\varepsilon$ is the noise strength, defined by,

$$\langle \zeta(t)\zeta(t') \rangle = 2\varepsilon \delta(t - t')$$

and $S_T[\phi(t)]$ is the action, given by,

$$S_T[\phi(t)] = \frac{1}{2} \int_0^T \langle \dot{\phi}(t) - f(\phi) \rangle^2 dt$$

Qiao et al. computed the most likely paths entering and leaving successive steady states of the SKS equation by finding the actions and took differences of the corresponding actions to determine the escape rate from each steady state. They then used these escape rates to determine the selected wavenumber. With this method, they identified selected wavenumbers for a wide range of $\alpha$ values.

**IV. CALCULATING THE EMPIRICAL PROBABILITY DISTRIBUTION OF FINAL STATES**

Although Qiao et al. have devised a way to calculate the selected wavenumber, we wish to know if the same result can be obtained by direct integration of the equation of motion. There are two hurdles that must be overcome in order to study wavenumber selection in this way. The first is that farther from threshold, several neighboring states in the middle of the Eckhaus band become very stable [5]. In order to determine which one, if any, is the most stable, we have to induce transitions between these states, so that the state in which the system spends most of its time is the most stable state. Observing such transitions between highly stable states requires extremely long integration times, as noted in [5]. The second hurdle is that true state selection would only occur in the thermodynamic limit, which necessitates simulation of large systems. These two hurdles suggest that one looks for a fast and efficient integration algorithm. We decided to use a semi-implicit, Fourier spectral integration method instead of explicit time integration ensures that one can use significantly a larger time step without compromising on accuracy, thus yielding a much higher speed of integration. At the same time, using a Fourier spectral method to approximate spatial derivatives gives much higher accuracy than finite difference methods. Thus, the use of a semi-implicit Fourier spectral method enabled us to integrate Eq. [15] for long times and large system sizes. The general idea behind the semi-implicit Fourier method is as follows. Consider the following partial differential equation,

$$\frac{\partial u(x, t)}{\partial t} = \hat{L}u(x, t) + \hat{N}[u(x, t), \partial_x u(x, t), \partial_x^2 u(x, t), \ldots]$$

where $\hat{L}$ is a linear differential operator and $\hat{N}$ is a nonlinear functional of $u$ and its spatial derivatives. If we discretize space and denote the value of the field $u(x, t)$ at each grid point by $u_i(t)$, we get a system of ordinary differential equations,

$$\frac{du_i}{dt} = (\hat{L}u)_i + \hat{N}[u(x, t), \ldots]_i$$

Taking the Fourier transform of this equation gives,

$$\frac{d\tilde{u}_k}{dt} = (\hat{L}u)_k + \hat{N}_k[u(x, t), \ldots]$$

For the (deterministic) SKS equation,

$$\hat{L}u(x, t) = -\alpha u(x, t) - \partial_x^4 u(x, t)$$

and

$$\hat{N}[u(x, t), \partial_x u(x, t)] = (\partial_x^4 u(x, t))^2$$

The Fourier transform of the discretized field $u_i$ can be found using the Fast Fourier Transform (FFT). If $\tilde{u}_k$ denotes the $k$-th wave number component of the discrete Fourier transform of the field $u$, then the discrete Fourier transform of Eq. [15] is [11],

$$(\hat{L}u)_k = -\alpha \tilde{u}_k + (2\pi k/Nh)^2 \tilde{u}_k - (2\pi k/Nh)^4 \tilde{u}_k$$

Here, $N$ is the number of lattice points and is not to be confused with the nonlinear operator $N$. Evaluating the nonlinear term $\hat{N}_k[u(x, t), \ldots]$ in Fourier space directly is computationally expensive, since the Fourier transform of a product of functions is a convolution, which involves $O(N^2)$ computations. Therefore, we evaluated the nonlinear term in position space and then transformed it to Fourier space to do the integration. Putting all this together, we get,

$$\frac{d\tilde{u}_k}{dt} = -\alpha \tilde{u}_k + (2\pi k/Nh)^2 \tilde{u}_k - (2\pi k/Nh)^4 \tilde{u}_k + \tilde{N}_k[u(x, t), \ldots]$$

In order to solve the issue of small time step, we use a semi-implicit integration scheme. We integrate forward
in time by treating the linear terms implicitly and the nonlinear term explicitly [10]. Let \( u_{kj} \) be the value of \( u_{k} \) at time \( t_j \). Then we approximate the time derivative by \( \frac{du_k}{dt} \). Eq. (18) becomes,

\[
  u_{k}^{j+1} = \frac{u_{k}^{j} + \Delta t \tilde{N}_{k}^{j}}{1 - \Delta t \left[ -\alpha + \left( \frac{2\pi k}{L} \right)^2 \right]} \tag{19}
\]

With this semi-implicit scheme, we could increase the time step by a factor of about 50 compared to that in an explicit time integration scheme, without causing instabilities and without loss of accuracy. This resulted in a significant speed up in the algorithm. We still need to add the noise term. If \( \zeta_{kj} \) denotes the value of the noise term at time \( t_j \), Eq. (19) is modified to read,

\[
  u_{k}^{j+1} = \frac{u_{k}^{j} + \Delta t \tilde{N}_{k}^{j} + \left\{ \frac{2\pi N \Delta t \zeta_{kj}}{h} \right\}}{1 - \Delta t \left[ -\alpha + \left( \frac{2\pi k}{L} \right)^2 \right]} \tag{20}
\]

where \( \langle \zeta_{kj} \rangle = 0 \) and \( \langle \zeta_{kj} \zeta_{kj'} \rangle = \delta_{k,-k'} \delta_{jj'} \). The procedure for generating noise in Fourier space that satisfies equations Eqs. (6) and (7) is given in [12], Appendix B.

### V. RESULTS

Our aim is to compute the empirical probability distribution for the allowed periodic states and determine if the distribution has a peak at a particular wavenumber. If such a peak is present, it would support the hypothesis that noise is a possible mechanism of wavenumber selection. To do this, we used noise strengths that were around 0.1 % of the amplitude of the field \( u(x,t) \). This noise strength is quite large, but it allows the system to explore a large number of allowed states. We start the simulation with the system initially placed in one of the periodic states and then perturb it with noise. The noise causes the system to visit other steady states. We then count the number of times each state is visited and compute the fraction of the time spent in each state. The idea is that the system would spend the greatest fraction of time in the most stable or selected state. More formally, we calculate the time average of the indicator function for each state,

\[
  M_T(q) = \int_0^T 1_q(t) \, dt \tag{21}
\]

If the system is in a state with wavenumber \( k \) at time \( t \), then the indicator function for wavenumber \( q \) is defined by,

\[
  1_q(t) = \begin{cases} 
    1 & k = q \text{ at time } t \\
    0 & \text{otherwise} 
  \end{cases} \tag{22}
\]

This quantity gives the fraction of time spent in the state with wavenumber \( q \) and approaches the stationary probability distribution at very long times.

\[
  \lim_{T \to \infty} M_T(q) = P_{st}(q) \tag{23}
\]

Since wavenumber selection has already been demonstrated for a small system with control parameter \( \alpha = 0.24 \) in [5], we attempted to carry out the procedure described above for \( \alpha = 0.22 \). For this value of \( \alpha \), the wave numbers that are stable to perturbations lie within the range \( 0.6136 \leq q \leq 0.7486 \), [6]. We used periodic boundary conditions, which implies that out of the wave numbers in the above range, only those would appear in the simulation that satisfy,

\[
  q_n = \frac{2\pi n}{Nh} \tag{24}
\]

where \( n \) is an integer, \( N \) is the number of lattice points and \( h \) is the lattice spacing. Thus, the imposition of periodic boundary conditions limits the allowed wavenumbers to a discrete set given by Eq.(24), with \( n \) representing the number of cells in the solution. For all our simulations, the initial condition was of the form \( u_{in} \sim \sin(2\pi n x/Nh) \) with \( n \) being an integer. For the range \( 0.6136 \leq q \leq 0.7486 \), and \( N = 1024, h = 0.5 \), we have \( 50 \leq n \leq 61 \). Our time step was \( \Delta t = 0.3 \). We first used a small noise strength \( \varepsilon = 10^{-4} \) to determine which states were the most stable. We found that the states with \( n = 54, 55, 56 \) were all very stable and the system could not be knocked out of them with this noise strength. So now the problem was to determine which one of these three states was the most stable. We then turned up the noise strength gradually and found that at \( \varepsilon = 1.8 \times 10^{-3} \) the \( n = 54 \) and \( n = 56 \) states started becoming unstable, whereas the \( n = 55 \) (\( q = 0.6749 \)) state persisted. When we plotted the fraction of time spent in each state, the resulting histogram developed a peak at this state. This peak persists until the end of the integration (about \( 10^8 \) time steps), indicating that the system was spending most of its time in that state.

| Number of lattice points \( N \) | System Length \( (L = Nh) \) | Most visited state |
|----------------------------------|-----------------------------|-------------------|
| 1024                            | 512                         | 0.6749 (\( n = 55 \)) |
| 1600                            | 800                         | 0.6754 (\( n = 86 \)) |
| 2200                            | 1100                        | 0.6740 (\( n = 118 \)) |
| 3000                            | 1500                        | 0.6744 (\( n = 161 \)) |
| 4000                            | 2000                        | 0.6754 (\( n = 215 \)) |
The histogram also gets narrower with time, again indicating that the system is spending most of its time close to the $n = 55$ state. We ran a couple of simulations at $\varepsilon = 1.9 \times 10^{-3}$ and $\varepsilon = 3.9 \times 10^{-3}$ and again observed that after spending some time in $n = 54$ or $n = 56$, the system transitioned to $n = 55$ and spent the most amount of time there (see Fig. 2(a), left). In all three cases, the histogram became nearly stationary after around $10^8$ time steps. The observation that the most visited state remains the same in spite of increasing the noise strength is promising evidence of the selection of a unique state by noise. The most visited state was also found to be independent of initial condition.

We repeated these simulations for the same value of $\alpha$, i.e. $\alpha = 0.22$, but different system sizes and noise strengths. In each case, we observed that after initially visiting a narrow band of states, the system settled down in one of them. This state was independent of the noise strength and initial conditions. The largest system size we tried was $N = 4000$ lattice points, which gives us the most precise estimate of the selected wavenumber (see Fig. 2(a), right). For this size, the difference between two successive wavenumbers is the smallest and is equal to $\Delta q = 2\pi/Nh = 0.003$. The most visited states for $\alpha = 0.22$ are presented in Table II. Thus, we see that our best estimate for the selected wavenumber for $\alpha = 0.22$ is $q_s = 0.6754 \pm 0.0015$, corresponding to the case $N = 4000$. Next, we repeated the same process for values of $\alpha$ farther from the critical value $\alpha_c = 0.25$. As explained in [5], states in the middle of the Eckhaus band become more and more stable away from threshold. This made it necessary to use slightly larger noise strengths to destabilize the states. For $\alpha = 0.20$ and $N = 1024$, the Eckhaus stable wavenumbers are $0.5890 \leq q \leq 0.7608$ or $48 \leq n \leq 62$. For a range of noise strengths between $3 \times 10^{-3}$ and $6 \times 10^{-3}$ and different initial conditions, we found $n = 54$ ($q = 0.6627$) to be the most visited state (Fig. 2(b), left). For $N = 4000$ lattice points, we found $n = 209$ to be the most visited state, see Fig. 2(b), right. Results for various system sizes are shown in Table II. Finally, results for $\alpha = 0.17$ are shown in Table III and Fig. 2(c).

For the sake of completeness, we have also reproduced Obeid’s result for $\alpha = 0.24$ and extended it to larger sizes, as shown in Table IV. The power spectra at the end of integration for $\alpha = 0.22, 0.20$ and $0.17$ are shown in Fig. 3 and Fig. 4. Fig. 3 corresponds to low noise strengths and Fig. 4 corresponds to high noise strengths. These figures show that the final power spectra have prominent maxima at the most visited wavenumber, in spite of the large noise strengths we have used.

We also show plots of $u(x, t)$ at very long times in Fig. 5. It can be seen that the states visited by the system are not the actual steady states of the deterministic problem; the large noise strengths we use destroy the stationary states of the deterministic equation. However, these states are close to the deterministic steady states, in the sense that the power spectrum has a sharp peak at a particular wavenumber in the Eckhaus band. This can also be seen by counting the number of cells in the solution in Fig. 5. Finally, a plot of the selected wavenumbers for 4000 lattice points against $\alpha$ is shown in Fig. 6 with error bars representing the discretization error. It also shows the results obtained by Qiao et al for the same values of $\alpha$. We see that our results do not agree with theirs and the disagreement gets worse as we go farther from threshold. The reasons for the disagreement, except possibly the fact that they have used a different algorithm, are unknown at this point.

### A. Extension to thermodynamic limit

In order to obtain an estimate for the selected wavenumber for the thermodynamic limit (in which case a continuous band of wavenumbers is allowed), we used

| Number of lattice points $N$ | System Length ($L = Nh$) | Most visited state |
|-------------------------------|--------------------------|-------------------|
| 1024                          | 512                      | 0.6627 ($n = 54$) |
| 1600                          | 800                      | 0.6597 ($n = 84$) |
| 2200                          | 1100                     | 0.6569 ($n = 115$) |
| 3000                          | 1500                     | 0.6576 ($n = 157$) |
| 4000                          | 2000                     | 0.6566 ($n = 209$) |

| Table II: Most visited wavenumbers for various sizes, $\alpha = 0.20$. |

| Number of lattice points $N$ | System Length ($L = Nh$) | Most visited state |
|-------------------------------|--------------------------|-------------------|
| 1024                          | 512                      | 0.6381 ($n = 52$) |
| 1600                          | 800                      | 0.6361 ($n = 81$) |
| 2200                          | 1100                     | 0.6340 ($n = 111$) |
| 3000                          | 1500                     | 0.6367 ($n = 152$) |
| 4000                          | 2000                     | 0.6377 ($n = 203$) |

| Table III: Most visited wavenumbers for various sizes, $\alpha = 0.17$. |
(a) Empirical probability distribution for $\alpha = 0.22$ and various noise strengths and initial states, $n_{in}$. Left: $N = 1024$. Right: $N = 4000$.

(b) Empirical probability distribution for $\alpha = 0.20$ and various noise strengths and initial states, $n_{in}$. Left: $N = 1024$. Right: $N = 4000$.

(c) Empirical probability distribution for $\alpha = 0.17$ and various noise strengths and initial states, $n_{in}$. Left: $N = 1024$. Right: $N = 4000$.

FIG. 2: Empirical probability distributions for various control parameter values and small and large system sizes.
FIG. 3: Long time power spectra for the three $\alpha$ values, for small and large system sizes. These plots correspond to the lower end of noise strengths studied.
(a) Long time power spectra for $\alpha = 0.22$, high noise case. Left: $N = 1024$. Right: $N = 4000$.

(b) Long time power spectra $\alpha = 0.2$, high noise case. Left: $N = 1024$. Right: $N = 4000$.

(c) Long time power spectra $\alpha = 0.17$, high noise case. Left: $N = 1024$. Right: $N = 4000$.

FIG. 4: Long time power spectra for the three $\alpha$ values, for small and large system sizes. These plots correspond to the higher end of noise strengths studied.
(a) Long time field configuration for $\alpha = 0.22$ and $N = 1024$. Left: Low noise case. Right: High noise case.

(b) Long time field configuration for $\alpha = 0.20$ and $N = 1024$. Left: Low noise case. Right: High noise case.

(c) Long time field configuration for $\alpha = 0.17$ and $N = 1024$. Left: Low noise case. Right: High noise case.

FIG. 5: Typical final configurations at the end of integration for various values of $\alpha$, for low and high noise strengths.
TABLE IV: Most stable wavenumbers for various sizes, $\alpha = 0.24$.

| Number of lattice points $N$ | System Length ($L = Nh$) | Most visited state |
|-----------------------------|---------------------------|-------------------|
| 1024                        | 512                       | 0.6995 ($n = 57$) |
| 1600                        | 800                       | 0.6990 ($n = 89$) |
| 2200                        | 1100                      | 0.6969 ($n = 122$) |
| 3000                        | 1500                      | 0.6953 ($n = 166$) |
| 4000                        | 2000                      | 0.6974 ($n = 222$) |

FIG. 6: Selected wavenumber as a function of control parameter. Solid lines represent our results for 4000 lattice points and dashed lines represent results from [7].

FIG. 7: Interpolating curve for $\alpha = 0.22$ with a maximum at $q_s = 0.6748$.

FIG. 8: Interpolating curve for $\alpha = 0.20$ with a maximum at $q_s = 0.6567$.

VI. CONCLUSIONS

In summary, our simulations show that a unique stationary state is indeed selected in the presence of noise, both close to and far from threshold. We have been able to determine the noise selected wavenumber for a range of control parameters and for progressively larger system sizes. Although the system never reached the exact stationary states of the deterministic problem because of the large noise strengths we had to use, the state of the system at all times was close to one of the deterministic states, as seen by the power spectra we obtained. Our simulations also show that the selected wavenumber is independent of noise strength. Of course, it would be desirable to observe state selection with noise strengths that do not destroy the deterministic steady states, but that would require extremely long simulations. However, based on our findings that the selected wavenumber is independent of noise strength, it is reasonable to conclude that the same wavenumber would be selected for noise strengths smaller than the ones we used. It should also be noted that even with relatively large noise, the...
TABLE V: Most visited or selected wavenumbers \( q_s \) in the thermodynamic limit.

| Control Parameter \( \alpha \) | \( q_s \) |
|-----------------------------|-------|
| 0.22                        | 0.6748|
| 0.20                        | 0.6567|
| 0.17                        | 0.6384|

FIG. 9: Interpolating curve for \( \alpha = 0.17 \) with a maximum at \( q_s = 0.6384 \).

The system did not visit states lying at the edges of the Eckhaus band. This is because these states are very unstable compared to those in the middle of the band. Therefore, the system did not sample all the available phase space during our simulations, leading to some inaccuracy in the stationary probability distributions we obtained. A potential direction for future research could be to calculate the probabilities for the outer states using techniques from rare event simulation. This would enable us to obtain much more accurate stationary probability distributions and hence a better qualitative picture of the relative stability of the various states.

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