RELATIVE ENTROPY DIMENSION OF TOPOLOGICAL DYNAMICAL SYSTEMS

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Abstract. We introduce the notion of relative topological entropy dimension to classify the different intermediate levels of relative complexity for factor maps. By considering the dimension or “density” of special class of sequences along which the entropy is encountered, we provide equivalent definitions of relative entropy dimension. As applications, we investigate the corresponding localization theory and obtain a disjointness theorem involving relative entropy dimension.

1. Introduction. Since entropy was introduced by Kolmogorov from information theory, it has played an important role in the study of dynamical systems. Shannon’s entropy (information entropy) is known to be the average information content in information science. Entropy (resp. relative entropy) is an important conjugate invariant for a dynamical system (resp. a factor map). Since zero entropy systems make up a dense \( G_\delta \) subset of all homeomorphisms, there are several kinds of works about conjugate invariants of zero entropy systems, for example, sequence entropy [19, 15], maximal pattern entropy [13], entropy dimension [2, 6, 3] and Slow entropy [17]. But there is no relative conjugate invariants for zero entropy factor maps. We are attempting to systematically study relative zero-entropy invariants, especially relative invariants that can classify the different intermediate levels of relative complexity. Entropy dimension for a topological dynamical system first introduced by M. De Carvalho in [2] measures the superpolynomial, but subexponential growth rate of the number of open sets that cover the space out of the sequence of iterated open covers. S. Ferenczi and K. K. Park have introduced the entropy dimension in [6] to measure the complexity of entropy zero measurable dynamics. It measures the growth rate of \( H(\sqrt[n]{\sum_{i=0}^{n-1} T^{-i} P}) \). D. Dou, W. Huang and K. K. Park in [3] introduced the notion of the dimension (upper and lower) of a subset of \( \mathbb{Z} \) with density 0. They used the dimension of a special class of sequences which were called entropy generating sequences to measure the complexity of a topological system and showed that the topological entropy dimension can be computed through the dimensions of entropy generating sequences. We would like to define these notions for the relative setting and study their properties.

In positive entropy systems, the “independence” point of view enables people to have a better understanding how the complexity of a system produces. One can

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find an infinite subset \( W \subset \mathbb{Z}_+ \) in the case of positive entropy such that along the sequence \( W \), the symbolic names are “independent” and there exists \( c > 0 \) with \( \lim \inf_{n \to \infty} \frac{|W \cap [1,n]|}{n} > c \). We prove that if a system \((X,T)\) relevant to a factor map has positive relative entropy dimension, then there exists an relative entropy generating sequence \( S \subset \mathbb{Z}_+ \) which is a union of disjoint finite sets along which the dynamics are “independent”. Given a sequence \( S \subset \mathbb{Z}_+ \), we define the dimension of the sequence and show that the relative entropy dimension of the system is the supremum of the dimensions of the relative entropy generating sequences.

H. Furstenberg in [7] first introduced the concept of disjointness to characterize the difference of dynamical behavior between two systems. Two well-known examples are that in measurable dynamics \( K \)-mixing systems are disjoint from ergodic zero entropy systems and that weak mixing systems are disjoint from group rotations. In the case of topological settings, these properties are explored in [1, 10, 12, 11]. D. Dou, W. Huang and K. K. Park in [3] introduced the notion of the dimension set \( D(X,T) \subset [0,1] \) of a zero entropy topological system \((X,T)\) to measure the various levels of topological complexity of subexponential growth rate. They investigated the property of disjointness within entropy zero systems via the dimension set and proved that the relative entropy dimension of the system is the supremum of the dimensions of the relative entropy generating sequences.

The paper is organized as follows. In Sect. 2, we give the definitions and some basic properties of relative entropy dimension. In Sect. 3, we consider the dimensions of the relative entropy generating sequence and investigate the interrelations among these dimensions. In Sect. 4, we give the notions of relative dimension tuples and dimension sets and prove disjointness theorem with respect to the relative dimension set.

2. Relative entropy dimension. In this paper, a topological dynamical system (TDS, for short) is a pair \((X,T)\), where \( X \) is a compact metric space endowed with a self-homeomorphism \( T \). Before we introduce the notion of relative entropy dimension for a TDS, we recall some definitions. Given a TDS \((X,T)\), denote by \( \mathcal{C}_X \) the set of finite covers of \( X \) and \( \mathcal{C}_X^0 \) the set of finite open covers of \( X \). Given two covers \( U, V \in \mathcal{C}_X \), we say that \( U \) is finer than \( V \) (\( U \succeq V \)) if for every \( U \in U \) there is a set \( V \in V \) that \( U \subseteq V \). Let \( U \cap V = \{ U \cap V : U \in U, V \in V \} \). Clearly, \( U \cup V \supseteq U \) and \( U \cap V \supseteq V \). Given integers \( m \leq n \) and a cover \( U \in \mathcal{C}_X \) let \( U_m^n = \bigvee_{i=m}^n T^{-i} U \).

Let \((X,T)\) and \((Y,S)\) be two TDSs. Suppose that \((Y,S)\) is a factor of \((X,T)\) in the sense that there exists a continuous surjective map \( \pi : (X,T) \to (Y,S) \) such that \( \pi \circ T = S \circ \pi \). The map \( \pi \) is called a factor map from \( X \) to \( Y \).

Let \( \pi : (X,T) \to (Y,S) \) be a factor map and \( U \in \mathcal{C}_X^0 \). For \( E \subseteq X \), let \( N(U,E) \) denote the number of the sets in a subcover of \( U \) which covers \( E \) with smallest
Let \( D(T, U, \alpha|\pi) \) denote the relative upper entropy dimension of \( T \) relevant to \( \pi \). Similarly, \( D(T, U, \alpha|\pi) \) does not decrease as \( \alpha \) decreases, and \( D(T, U, \alpha|\pi) \) is trivial if \( \alpha \) is arbitrary. Here, we define the relative upper entropy dimension of \( T \) relevant to \( \pi \) by

\[
D(T, U|\pi) = \inf\{\alpha \geq 0 : D(T, U, \alpha|\pi) = 0\} = \sup\{\alpha \geq 0 : D(T, U, \alpha|\pi) = \infty\}. 
\]

Similarly, \( D(T, U, \alpha|\pi) \) does not decrease as \( \alpha \) decreases, and \( D(T, U, \alpha|\pi) \) is trivial if \( \alpha \) is arbitrary. Here, we define the relative upper entropy dimension of \( T \) relevant to \( \pi \) by

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\[
D(T, U|\pi) = \inf\{\alpha \geq 0 : D(T, U, \alpha|\pi) = 0\} = \sup\{\alpha \geq 0 : D(T, U, \alpha|\pi) = \infty\}. 
\]

If \( D(T, U|\pi) = D(T, U|\pi) = \alpha \), then we say \( U \) has relative entropy dimension \( \alpha \). Clearly \( 0 \leq D(T, U|\pi) \leq D(T, U|\pi) \leq 1 \) if \( h(T, U|\pi) > 0 \), then the relative entropy dimension of \( U \) is equal to 1.

**Definition 2.1.** Let \( \pi : (X, T) \to (Y, S) \) be a factor map between TDSs. The relative upper (resp. lower) entropy dimension of TDS \((X, T)\) relevant to \( \pi \) is

\[
D(T|\pi) = \sup\{D(T, U|\pi) : U \in C^\circ_X\} \quad \text{resp.} \quad D(T|\pi) = \sup\{D(T, U|\pi) : U \in C^\circ_X\}. 
\]

It is clear that \( 0 \leq D(T|\pi) \leq D(T|\pi) \leq 1 \). When \( D(T|\pi) = D(T|\pi) \), we just call the common value the relative entropy dimension of \((X, T)\) relevant to \( \pi \), denoted by \( D(T|\pi) \). When \((Y, S)\) is a trivial system, we recover the classical entropy dimension (see [3]), and in this case we shall omit the restriction on \( \pi \). The following two propositions are the basic properties of relative entropy dimension.

**Proposition 2.2.** Let \( \pi : (X, T) \to (Y, S) \) be a factor map between TDSs and \( U, V \in C^\circ_X \).

1. If \( U \leq V \), then \( D(T, U|\pi) \leq D(T, V|\pi) \) and \( D(T, U|\pi) \leq D(T, V|\pi) \).
2. For any \( 0 \leq m \leq n \), \( D(T, U|\pi) = D(T, U^n|\pi) \) and \( D(T, U|\pi) = D(T, U^m|\pi) \).
3. \( D(T, U \lor V|\pi) = \max\{D(T, U|\pi), D(T, V|\pi)\} \).
4. \( \max\{D(T, U|\pi), D(T, V|\pi)\} \leq D(T, U \lor V|\pi) \leq \max\{D(T, U|\pi), D(T, V|\pi), \min\{D(T, U|\pi), D(T, V|\pi)\}\} \).

**Proof.** (1) and (2) are clear. For (3), if \( \alpha > \max\{D(T, U|\pi), D(T, V|\pi)\} \), then

\[
\limsup_{n \to \infty} \frac{\log N(U^n|\pi)}{n^\alpha} = 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{\log N(V^n|\pi)}{n^\alpha} = 0.
\]

Hence

\[
\limsup_{n \to \infty} \frac{\log N(U \lor V|\pi)}{n^\alpha} \leq \limsup_{n \to \infty} \frac{\log N(U^n|\pi)}{n^\alpha} + \frac{\log N(V^n|\pi)}{n^\alpha} = 0.
\]

This implies \( D(T, U \lor V|\pi) < \alpha \). Since \( \alpha \) is arbitrary, we have

\[
D(T, U \lor V|\pi) \leq \max\{D(T, U|\pi), D(T, V|\pi)\}.
\]

By (1) we have the result.
For (4), from (1) the first inequality is obvious.

If \( \alpha > \max \{ D(T, U|\pi), D(T, V|\pi), \min \{ \overline{D}(T, U|\pi), \overline{D}(T, V|\pi) \} \} \), without loss of generality we assume \( \alpha > \overline{D}(T, U|\pi) \) and \( \alpha > D(T, V|\pi) \). Then

\[
\limsup_{n \to \infty} \frac{\log N(U_0^{n-1}|\pi)}{n^\alpha} = 0 \quad \text{and} \quad \liminf_{n \to \infty} \frac{\log N(V_0^{n-1}|\pi)}{n^\alpha} = 0.
\]

Hence

\[
\liminf_{n \to \infty} \frac{\log N((U \lor V)^{n-1}|\pi)}{n^\alpha} \leq \liminf_{n \to \infty} \frac{\log N(U_0^{n-1}|\pi)}{n^\alpha} + \log N(V_0^{n-1}|\pi) + \liminf_{n \to \infty} \frac{\log N(V_0^{n-1}|\pi)}{n^\alpha} = 0.
\]

This implies \( D(T, U \lor V|\pi) < \alpha \). Since \( \alpha \) is arbitrary, we have \( D(T, U \lor V|\pi) \leq \max \{ D(T, U|\pi), D(T, V|\pi), \min \{ \overline{D}(T, U|\pi), \overline{D}(T, V|\pi) \} \} \)

As a direct application of Proposition 2.2, we have

**Proposition 2.3.** Let \( \pi: (X, T) \to (Y, S) \) be a factor map between TDSs. If \( \{ U_n \} \) is a sequence of finite open covers of \( X \) with \( \lim_{n \to +\infty} \operatorname{diam}(U_n) = 0 \), then

\[
\lim_{n \to \infty} \overline{D}(T, U_n|\pi) = \overline{D}(T|\pi) \quad \text{and} \quad \lim_{n \to \infty} D(T, U_n|\pi) = D(T|\pi).
\]

In particular, if \( U \) is a generating open cover of \( X \) (i.e. \( \lim_{n \to \infty} \operatorname{diam}(V_i^{n-1}T^{-i}U) = 0 \)), then \( \overline{D}(T, U|\pi) = \overline{D}(T|\pi) \) and \( D(T, U|\pi) = D(T|\pi) \).

Let \( (X, T) \) be a TDS. A cover \( \{ U, V \} \) of \( X \) which consists of two non-dense open sets of \( X \) is called a standard cover of \( X \). Denote by \( C_X^s \) the set of all standard covers of \( X \). The following proposition shows that the relative upper entropy dimension with respect to standard covers determine the relative upper entropy dimension of the system.

**Proposition 2.4.** Let \( \pi: (X, T) \to (Y, S) \) be a factor map between TDSs. Then

\[
\overline{D}(T|\pi) = \sup \{ \overline{D}(T, W|\pi) : W \in C_X^s \}.
\]

**Proof.** We follow the argument in the proof of Proposition 1 in [1]. Since \( C_X^s \supset C_X^s \), we have that \( \overline{D}(T|\pi) \geq \sup \{ \overline{D}(T, W|\pi) : W \in C_X^s \} \). Conversely, let \( U \in C_X^s \). Then there exist \( U_i = \{ U_i, V_i \} \in C_X^s \) (\( i = 1, 2, \cdots, k \)) such that \( \bigvee_{i=1}^k U_i \supseteq U \) (see the proof of Proposition 1 in [1]). Note that \( \overline{D}(T, U_i|\pi) \leq \overline{D}(T, \bigvee_{i=1}^k U_i|\pi) = \max \{ \overline{D}(T, U_i|\pi) : i = 1, 2, \cdots, k \} \), hence there exists \( r \in \{ 1, 2, \cdots, k \} \) such that \( \overline{D}(T, U_r|\pi) \geq \overline{D}(T, U|\pi) \).

Now it is easy to shrink \( U_r = \{ U_r, V_r \} \) into a standard cover \( W \) with \( \overline{D}(T, W|\pi) \geq \overline{D}(T, U_r|\pi) \). First if \( U_r \) is dense, then \( V_r \) contains some closed ball \( F = \{ y : d(x, y) \leq \epsilon \} \) with \( \epsilon > 0 \) and \( x \in U_r \). Let \( W_1 = U_r \setminus F \). Since \( F \) has nonempty interior, the set \( W_1 \) is not dense and \( \{ W_1, V_r \} \) is an open cover of \( X \) as \( V_r \) contains \( F \). Since this cover is finer than \( U_r \), \( \overline{D}(T, \{ W_1, V_r \}) \geq \overline{D}(T, U_r) \). By doing the same with \( V_r \), if \( V_r \) is dense, one obtains a standard cover \( W = \{ W_1, W_2 \} \) such that \( \overline{D}(T, W|\pi) \geq \overline{D}(T, U|\pi) \). This implies \( \sup \{ \overline{D}(T, W|\pi) : W \in C_X^s \} \geq \overline{D}(T|\pi) \) since \( U \) is arbitrary.
3. Relative entropy dimension via entropy generating sequences. Let $S = \{s_1 < s_2 < \cdots\}$ be an increasing sequence of integers. For $\tau \geq 0$, we define
\[
\overline{D}(S, \tau) = \limsup_{n \to \infty} \frac{n}{(s_n)\tau} \quad \text{and} \quad \underline{D}(S, \tau) = \liminf_{n \to \infty} \frac{n}{(s_n)\tau}.
\]
It is clear that $\overline{D}(S, \tau)$ does not decrease as $\tau$ decreases and $\overline{D}(S, \tau) \notin \{0, +\infty\}$ for at most one $\tau \geq 0$. We define the upper dimension of $S$ by
\[
\overline{D}(S) = \inf\{\tau \geq 0 : \overline{D}(S, \tau) = \infty\} = \sup\{\tau \geq 0 : \overline{D}(S, \tau) = \infty\}.
\]
Similarly, we define the lower dimension of $S$ by
\[
\underline{D}(S) = \inf\{\tau \geq 0 : \underline{D}(S, \tau) = 0\} = \sup\{\tau \geq 0 : \underline{D}(S, \tau) = 0\}.
\]
Clearly $0 \leq \underline{D}(S) \leq \overline{D}(S) \leq 1$. When $\overline{D}(S) = \underline{D}(S) = \tau$, we say the sequence $S$ has dimension $\tau$.

In the following, we will investigate the dimension of a special kind of sequence, which we call the relative entropy generating sequence.

Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $U \in C^0_X$. We say an increasing sequence of integers $S = \{s_1 < s_2 < \cdots\}$ is a relative entropy generating sequence of $U$ relevant to $\pi$ if
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} U|\pi) > 0.
\]
Denote by $E(T, U|\pi)$ the set of all relative entropy generating sequences of $U$ relevant to $\pi$ and by $P(T, U|\pi)$ the set of sequence $S = \{s_1 < s_2 < \cdots\}$ of $\mathbb{Z}_+$ with the property that
\[
h^\mathcal{S}_0(T, U|\pi) := \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} U|\pi) > 0.
\]
In other words, $P(T, U|\pi)$ is the set of increasing sequence of integers along which $U$ has positive relative entropy.

**Definition 3.1.** Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $U \in C^0_X$. We define
\[
\overline{D}_e(T, U|\pi) = \begin{cases} \sup_{S \in E(T, U|\pi)} \overline{D}(S) & \text{if } E(T, U|\pi) \neq \emptyset \\ 0 & \text{if } E(T, U|\pi) = \emptyset \end{cases},
\]
\[
\underline{D}_p(T, U|\pi) = \begin{cases} \sup_{S \in P(T, U|\pi)} \underline{D}(S) & \text{if } P(T, U|\pi) \neq \emptyset \\ 0 & \text{if } P(T, U|\pi) = \emptyset \end{cases}.
\]
Similarly, we can define $\overline{D}_e(T, U|\pi)$ and $\underline{D}_p(T, U|\pi)$ by changing the upper dimension into the lower dimension.

**Definition 3.2.** Let $(X, T)$ be a TDS. We define
\[
\overline{D}_e(T|\pi) = \sup_{U \in C^0_X} \overline{D}_e(T, U|\pi), \quad \underline{D}_p(T|\pi) = \sup_{U \in C^0_X} \underline{D}_p(T, U|\pi).
\]
Similarly, we can define $\overline{D}_e(T|\pi)$ and $\underline{D}_p(T|\pi)$.

In the following, we investigate the interrelations among these dimensions.
Proposition 3.3. Let \( \pi : (X, T) \to (Y, S) \) be a factor map between TDSs and \( \mathcal{U} \in \mathcal{C}_X^\pi \). Then

\[
\mathcal{D}_e(T, \mathcal{U}|\pi) \leq \mathcal{D}_e(T, \mathcal{U}|\pi) \leq \mathcal{D}_o(T, \mathcal{U}|\pi) \leq \mathcal{D}(T, \mathcal{U}|\pi).
\]

Proof. 1). \( \mathcal{D}_e(T, \mathcal{U}) \leq \mathcal{D}_e(T, \mathcal{U}) \) is obvious by the definition.

2). To show \( \mathcal{D}_e(T, \mathcal{U}|\pi) \leq \mathcal{D}_o(T, \mathcal{U}|\pi) \), it is sufficient to assume that \( \mathcal{D}_e(T, \mathcal{U}|\pi) > 0 \).

Let \( \tau \in (0, \mathcal{D}_e(T, \mathcal{U}|\pi)) \) be given. There exists \( S = \{s_1 < s_2 \cdots \} \in \mathcal{E}(T, \mathcal{U}|\pi) \) with \( \mathcal{D}(S) > \tau \), i.e. \( \limsup_{n \to +\infty} \frac{n}{s_n^\tau} = +\infty \). Hence

\[
\limsup_{n \to +\infty} \frac{n}{n + s_n^\tau} = 1. \tag{3.1}
\]

Let \( F = S \cup \{\lceil n^\tau \rceil : n \in \mathbb{N}\} \).

Clearly \( \mathcal{D}(F) \geq \tau \). Let \( F = \{t_1 < t_2 \cdots \} \). Then for each \( n \in \mathbb{N} \) there exists a unique \( m(n) \in \mathbb{N} \) such that \( s_n = t_{m(n)} \). Since \( \{s_1, s_2, \cdots , s_n\} \subseteq \{t_1, t_2, \cdots , t_{m(n)}\} \subseteq \{s_1, s_2, \cdots , s_n\} \cup \{\lceil k^\tau \rceil : k \leq s_n^\tau\} \), we have \( n \leq m(n) \leq n + s_n^\tau \). Combining this with (3.1), we get

\[
\limsup_{n \to +\infty} \frac{n}{m(n)} = 1. \tag{3.2}
\]

Now we have

\[
\limsup_{m \to +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^m T^{-t_i} \mathcal{U}|\pi)}{m} \geq \limsup_{n \to +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^{m(n)} T^{-t_i} \mathcal{U}|\pi)}{m(n)}
\]

\[
\geq \limsup_{n \to +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U}|\pi)}{n} \cdot \frac{n}{m(n)}
\]

\[
\geq \liminf_{n \to +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U}|\pi)}{n} \cdot (\limsup_{n \to +\infty} \frac{n}{m(n)})
\]

\[
= \liminf_{n \to +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^n T^{-s_i} \mathcal{U}|\pi)}{n} \quad \text{(by (3.2))}
\]

\[
> 0 \quad \text{(since } S \in \mathcal{E}(T, \mathcal{U}|\pi)) \text{.}
\]

This implies \( F \in \mathcal{P}(T, \mathcal{U}|\pi) \). Hence \( \mathcal{D}_o(T, \mathcal{U}|\pi) \geq \mathcal{D}(F) \geq \tau \). Since \( \tau \) is arbitrary, we have \( \mathcal{D}_e(T, \mathcal{U}) \leq \mathcal{D}_o(T, \mathcal{U}) \).

3). If \( \mathcal{D}_o(T, \mathcal{U}|\pi) \leq \mathcal{D}(T, \mathcal{U}|\pi) \) were not true, then there would exist \( \tau \in (0, 1) \) such that \( \mathcal{D}_o(T, \mathcal{U}|\pi) > \tau > \mathcal{D}(T, \mathcal{U}|\pi) \). On the one hand, we have

\[
\limsup_{m \to +\infty} \frac{\log \mathcal{N}(\bigvee_{i=1}^m T^{-t_i} \mathcal{U}|\pi)}{m^\tau} = 0 \tag{3.3}
\]

since \( \tau > \mathcal{D}(T, \mathcal{U}|\pi) \), and on the other hand, since \( \mathcal{D}_o(T, \mathcal{U}|\pi) > \tau \) there exists \( S = \{s_1 < s_2 \cdots \} \in \mathcal{P}(T, \mathcal{U}|\pi) \) with \( \mathcal{D}(S) > \tau \), i.e. \( \liminf_{n \to +\infty} \frac{n}{s_n^\tau} = +\infty \).
Hence there exists \( c > 0 \) such that \( \frac{1}{n} \geq c \) for all sufficiently large \( n \). Now
\[
\limsup_{m \to +\infty} \frac{\log N(\bigvee_{i=1}^{m} T^{-i}U | \pi)}{m^2} \geq \limsup_{n \to +\infty} \frac{\log N(\bigvee_{i=1}^{n} T^{-i}U | \pi)}{s_n} \geq \limsup_{n \to +\infty} \frac{\log N(\bigvee_{i=1}^{n} T^{-s}U | \pi)}{n} \geq \frac{\log N(\bigvee_{i=1}^{n} T^{-s}U | \pi)}{s_n}
\]
which contradicts (3.3).

Let \( k \geq 2 \) and \( B \) be a nonempty finite subset of \( \mathbb{Z}_+ \). Assume \( U \) is the cover of \( \{0, 1, \ldots, k\}^B = \prod_{z \in B} \{0, 1, \ldots, k\} \) consisting of subsets of the form \( \prod_{z \in B} \{i_z\}^c \), where \( 1 \leq i_z \leq k \) and \( \{i_z\}^c = \{0, 1, \ldots, k\} \setminus \{i_z\} \) for each \( z \in B \). For \( S \subseteq \{0, 1, \ldots, k\}^B \) we let \( C_S \) denote the minimal cardinality of subcovers of \( U \) one needs to cover \( S \). Note that we shall use natural logarithms unless we explicitly indicate otherwise.

Let \( (X, T) \) be a TDS. Let \( A_1, A_2, \ldots, A_k \) be \( k \)-subsets of \( X \) and \( W \subseteq \mathbb{Z}_+ \). We say \( \{A_1, A_2, \ldots, A_k\} \) is independent along \( W \) relevant to \( \pi \), if there exists \( y \in Y \) such that for any \( s \in \{1, 2, \ldots, k\}^W \) we have \( \bigcap_{w \in W} T^{-w}A_{s(w)} \bigcap \pi^{-1}(y) \neq \emptyset \) (see [18]).

**Lemma 3.4.** Let \( \pi : (X, T) \to (Y, S) \) be a factor map between TDSs, and let \( A_1, A_2 \) be two disjoint non-empty closed subsets of \( X \), \( U = \{A_1^c, A_2^c\} \). For \( \tau \in (0, 1) \), \( 0 < \eta < \tau \) and \( c > 0 \) there exists \( N \in \mathbb{N} \) (depending on \( \tau, \eta, c \)) such that if a finite subset \( B \) of \( \mathbb{Z}_+ \) satisfying \( |B| \geq N \) and \( N(\bigvee_{z \in B} T^{-z}U | \pi) > e^{c|B|^\tau} \), then there exists \( W \subseteq B \) with \( |W| \geq |B|^\eta \) such that \( \{A_1, A_2\} \) is independent along \( W \) relevant to \( \pi \).

The proof of this lemma is completely similar to the proof of Lemma 3.7 in [3]. With the help of above lemma, we have the following theorem.

**Theorem 3.5.** Let \( \pi : (X, T) \to (Y, S) \) be a factor map between TDSs, \( A_1, A_2 \) be two disjoint non-empty closed subsets of a TDS \( (X, T) \) and \( U = \{A_1^c, A_2^c\} \). Then there exists a sequence \( F \in \mathcal{E}(T, U | \pi) \) such that \( \mathcal{D}(F) = \mathcal{D}(T, U | \pi) \) when \( \mathcal{E}(T, U | \pi) \) is nonempty.

**Proof.** If \( \mathcal{D}(T, U | \pi) = 0 \) , then by Proposition 3.3, any sequence in \( \mathcal{E}(T, U | \pi) \) will have upper dimension zero.

Now assume \( \mathcal{D}(T, U | \pi) > 0 \) and let \( \{\tau_j\} \subset (0, \mathcal{D}(T, U | \pi)) \) be a sequence of strictly increasing real number such that \( \lim_{j \to +\infty} \tau_j = \mathcal{D}(T, U | \pi) \). Then we can choose \( a > 0 \) so that
\[
\limsup_{n \to +\infty} \frac{1}{n^{\tau_j}} \log N(\bigvee_{i=1}^{n} T^{-i}U | \pi) > a \quad \text{for each } j \in \mathbb{N}.
\]

Let \( \tau_j - 1 < \eta_j < \tau_j \) for \( j \in \mathbb{N} \). By Lemma 3.4, there exists \( N_j \in \mathbb{N} \) such that for every finite set \( B \) with \( |B| \geq N_j \) and \( N(\bigvee_{z \in B} T^{-z}U | \pi) \geq e^{c|B|^{\eta_j}} \), we can find \( W \subseteq B \) with \( |W| \geq |B|^\eta_j \) and \( \{A_1, A_2\} \) which is independent along \( W \) relevant to \( \pi \).

Take \( 1 = n_1 < n_2 < \cdots \) such that \( (n_{j+1} - n_j)^{\eta_j} \geq jn_j + N_j \) and
\[
N(\bigvee_{i=n_j+1}^{n_{j+1}} T^{-i}U | \pi) \geq e^{c(\tau_j+1)-n_j})^{\eta_j}
\]
for each $j \in \mathbb{N}$. Now for each $j \in \mathbb{N}$ there exists $W_j \subseteq \{n_j + 1, n_j + 2, \ldots, n_{j+1}\}$ with $|W_j| \geq (n_{j+1} - n_j)^n$, $y_j \in Y$ and $\{A_1, A_2\}$ which is independent along $W_j$ relevant to $\pi$, i.e., for any $s \in \{1, 2\}^{W_j}$, $\cap_{w \in W_j} T^{-w}A_{s(w)} \cap \pi^{-1}(y_j) \neq \emptyset$.

For any nonempty set $B \subseteq W_j$ and $s = (s(z))_{z \in B} \in \{1, 2\}^B$, we can find $x_s \in \cap_{z \in B} T^{-z}A_{s(z)} \cap \pi^{-1}(y_j)$. Let $X_B = \{x_s : s \in \{1, 2\}^B\}$. It is clear that for any $t \in \{1, 2\}^B$ we have $|\cap_{z \in B} T^{-z}A_{t(z)} \cap \pi^{-1}(y_j) \cap X_B| \leq (2 - 1)^{|B|} = 1$. Combining this fact with $|X_B| = 2^{|B|}$, we get

$$\mathcal{N}(\bigvee_{z \in B} T^{-z}U, \pi^{-1}(y_j)) \geq 2^{|B|} \text{ for any } B \subseteq W_j. \tag{3.4}$$

Put $F = \bigcup_{i=1}^\infty W_j$ and write $F = \{t_1 < t_2 < \cdots \}$. First for $n \in \mathbb{N}$ with $n \geq |W_1|$ there exists unique $k(n) \in \mathbb{N}$ such that $\sum_{i=1}^{k(n)} |W_i| \leq n < \sum_{i=1}^{k(n)+1} |W_i|$. Now

$$\mathcal{N}(\bigvee_{j=1}^n T^{-t_j}U|\pi) \geq \max\{\mathcal{N}(\bigvee_{w \in W_{k(n)}} T^{-w}U|\pi), \mathcal{N}(\bigvee_{w \in W_{k(n)+1}(\{t_1, \ldots, t_n\}) T^{-w}U|\pi)\}$$

$$\geq \max\{(2)^{|W_{k(n)}|}, (2)^{n-\sum_{i=1}^{k(n)} |W_i|}\} \quad \text{(by (3.4))}$$

$$\geq (2)^{n-\sum_{i=1}^{k(n)+1} |W_i|}. \quad \text{(by (3.4))}$$

Hence

$$\liminf_{n \to +\infty} \frac{1}{n} \log \mathcal{N}(\bigvee_{j=1}^n T^{-t_j}U|\pi) \geq \liminf_{n \to +\infty} \frac{n - \sum_{i=1}^{k(n)+1} |W_i|}{2n} \log 2$$

$$\geq \liminf_{n \to +\infty} \left(\frac{1}{2} - \frac{\sum_{i=1}^{k(n)+1} |W_i|}{2|W_{k(n)}|}\right) \cdot \log 2$$

$$\geq \liminf_{n \to +\infty} \left(\frac{1}{2} - \frac{\sum_{i=1}^{k(n)+1} (n_i + 1 - n_i)}{2|W_{k(n)}|}\right) \cdot \log 2$$

$$\geq \liminf_{n \to +\infty} \left(\frac{1}{2} - \frac{n_{k(n)}}{2(n_{k(n)+1} - n_{k(n)})}\right) \cdot \log 2$$

$$\geq \liminf_{n \to +\infty} \left(\frac{1}{2} - \frac{1}{2k(n)}\right) \cdot \log 2 = \frac{1}{2} \log 2 > 0.$$

This shows $F \in \mathcal{E}(T, U|\pi)$.

Note that

$$\limsup_{m \to +\infty} \frac{m}{t_m} \geq \limsup_{k \to +\infty} \frac{|W_1| + |W_2| + \cdots + |W_k|}{t_m^{\frac{1}{|W_1|} + |W_2| + \cdots + |W_k|}} \geq \limsup_{k \to +\infty} \frac{|W_k|}{n_{k+1}^{\frac{1}{n_k}}} \geq \limsup_{k \to +\infty} \frac{(n_k - n_{k+1})^n}{n_{k+1}^{n_k}} \geq 1,$$

we have $\overline{D}(F) \geq \eta_j$. Hence $\overline{D}(F) = \overline{D}(T, U|\pi)$. This finishes the proof. $\square$

**Theorem 3.6.** Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs. Then

1. $\overline{D}_e(T|\pi) = \overline{D}_p(T|\pi) = \overline{D}(T|\pi)$,
2. $\overline{D}_e(T|\pi) \leq \overline{D}(T|\pi) \leq \overline{D}(T|\pi) \leq \overline{D}_p(T|\pi)$. 


Proof. (1). By Proposition 3.3, we have $\overline{D}_c(T|\pi) \leq \overline{D}_p(T|\pi) \leq \overline{D}(T|\pi)$. Now it is sufficient to show that $\overline{D}(T|\pi) \leq \overline{D}_c(T|\pi)$. By Proposition 2.4 and Theorem 3.5, we have

$$\overline{D}(T|\pi) = \sup \{\overline{D}(T, W|\pi) : W \in C^\pi_X\} = \sup \{\overline{D}_c(T, W|\pi) : W \in C^\pi_X\} \leq \overline{D}_c(T|\pi).$$

(2). By Definition 3.1, it is clear that $\overline{D}(T, U|\pi) \leq \overline{D}(T, U|\pi)$. Now we are going to show $\overline{D}_c(T, U|\pi) \leq \overline{D}(T, U|\pi)$. For any $\tau \in (0, \overline{D}_c(T, U|\pi))$, there exists $S = \{s_1 < s_2 < \cdots \} \in \mathcal{E}(T, U|\pi)$ such that $\overline{D}(S) > \tau$, that is, $\liminf_{n \to +\infty} n^{-\tau} \geq +\infty$. Hence there is $c > 0$ such that $\frac{n}{s_n} \geq c^{-\tau}$, i.e. $s_n \leq cn^\tau$ for sufficiently large $n$.

For $m \in \mathbb{N}$ with $m \geq s_1$, there exists a unique $n(m) \in \mathbb{N}$ such that $s_{n(m)} \leq m - 1 < s_{n(m)+1} \leq c(n(m)+1)^\tau$. Since $S \in \mathcal{E}(T, U|\pi)$, we have

$$\lim \inf_{m \to +\infty} \frac{1}{m^\tau} \log N\left( \bigvee_{i=0}^{m-1} T^{-i} U|\pi\right) \geq \lim \inf_{m \to +\infty} \frac{1}{c^\tau(n(m)+1)} \log N\left( \bigvee_{j=1}^{n(m)} T^{-\frac{j}{m}} U|\pi\right) \geq \lim \inf_{k \to +\infty} \frac{1}{k} \log N\left( \bigvee_{j=1}^{k} T^{-\frac{j}{m}} U|\pi\right) > 0.$$ 

This implies $\overline{D}(T, U|\pi) \geq \tau$. Finally as $\tau$ is arbitrary, we get $\overline{D}_c(T, U|\pi) \leq \overline{D}(T, U|\pi)$. Then by (1), $\overline{D}(T|\pi) = \overline{D}_p(T|\pi) \leq \overline{D}_p(T|\pi)$. This finishes the proof of (2). \hfill $\square$

4. Relative dimension tuples and dimension set. In this section, we will localize relative entropy dimension to obtain the notion of relative dimension tuples and dimension sets. Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $n \geq 2$. Set $X^{(n)} = X \times \cdots \times X$ ($n$-times), $T^{(n)} = T \times \cdots \times T$ ($n$-times), $\Delta_n(X) = \{(x_i) \in X^{(n)} | x_1 = \cdots = x_n\}$, the $n$-th diagonal of $X$.

Definition 4.1. Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $(x_i) \in X^{(n)} \setminus \Delta_n(X)$. The relative entropy dimension of $(x_i) \in X^{(n)}$ relevant to $\pi$ is

$$\overline{D}(x_1, \ldots, x_n|\pi) = \lim_{k \to +\infty} \overline{D}(T, U_k|\pi) \in [0, 1],$$

where $U_k = \{X \setminus B(x_1, \frac{1}{k}), \ldots, X \setminus B(x_n, \frac{1}{k})\}$. For $\alpha \in [0, 1]$, when $\overline{D}(x_1, \ldots, x_n|\alpha) = \alpha$, we will say $(x_i) \in \Delta_n(X)$ is an $\alpha$-relative dimension $n$-tuples relevant to $\pi$. The set of $\alpha$-relative dimension $n$-tuples relevant to $\pi$ of $(X, T)$ is denoted by $E_n^\alpha(T|\pi)$. We call the subset $\{\alpha \geq 0 : E_n^\alpha(T|\pi) \neq \emptyset\}$ of $[0, 1]$ the $n$-th relative dimension set of $(X, T)$ relevant to $\pi$ and denote it by $D_n(T|\pi)$. If $0 \notin D_n(T|\pi)$, we will say $(X, T)$ has strictly positive $n$-th relative entropy dimension.

Lemma 4.2. Let $\pi : (X, T) \to (Y, S)$ be a factor map between TDSs and $U = \{U_1, \ldots, U_n\} \in C^\pi_X$. Then there exists $x_i \in U_i$, $1 \leq i \leq n$ such that $\overline{D}(x_1, \ldots, x_n|\pi) \geq \overline{D}(T, U|\pi)$.

Proof. We follow the idea in the proof of Proposition 2 in [1].

If $U_i$ is a singleton then we put $U_i^1 = U_i$. If $U_i^1$ has at least two different points $y$ and $y'$, fix $\varepsilon_1 > 0$ with $\varepsilon_1 = \frac{d(y, y')}{2}$, and construct a cover of $U_i^1$ by open balls with radius $\varepsilon_1$ centered in $U_i^1$; call it $A$. Since $U_i^1$ is compact, there exist $A_1, A_2, \cdots, A_n \in A$ such that $\bigcup_{i=1}^n A_i \supseteq U_i^1$. Write $F_i = U_i^1 \cap \overline{A_i}$, $i = 1, 2, \cdots, n$. By the choice of $\varepsilon_1$, each closed set $F_i$ is a proper subset of $U_i^1$ and $\diam(F_i) \leq \frac{1}{2} \diam(U_i)$. Since
$\{U_1, \ldots, U_n\}$ is coarser than $\bigvee_{i=1}^{n} \{F_i, U_2, \ldots, U_n\}$, we have

$$\mathcal{D}(T, U|\pi) \leq \mathcal{D}(T, \bigvee_{i=1}^{n} \{F_i, U_2, \ldots, U_n\}|\pi) = \max_{1 \leq i \leq n} \mathcal{D}(T, \{F_i, U_2, \ldots, U_n\}|\pi),$$

where the last equality comes from Proposition 2.2 (3). Thus there exists $i_0 \in \{1, \ldots, u\}$ such that $\mathcal{D}(T, \{F_{i_0}, U_2, \ldots, U_n\}|\pi) \geq \mathcal{D}(T, U|\pi)$. We denote the set $F_{i_0}$ by $U_1$. Apply the same argument for $U_1$, and obtain $U_1 \cup U_2 < 1 < i \leq n$. Now we have found an open cover $U_i = \{U_1, U_2, \ldots, U_n\}$ coarser than $U$ and having the property that $\mathcal{D}(T, U_i|\pi) \geq \mathcal{D}(T, U|\pi)$ and $diam((U_i)^c) \geq \frac{1}{2}diam(U_i)$ for $i = 1, \ldots, n$.

Repeating the arguments, one gets $n$ decreasing sequences of nonempty closed sets $\{(U_i)^c\}, \ldots, \{U_n)^c\}$ such that

a. $diam((U_i)^c) \leq \frac{1}{2}diam((U_i)^c)$ for $i = 1, \ldots, n$ and $j = 0, 1, 2, \cdots$ (here let $U_0 = U_i$, $i = 1, \ldots, n$).

b. $\mathcal{D}(T, U_j|\pi) \geq \mathcal{D}(T, U|\pi)$ for each $j = 1, 2, \cdots$.

By the above a), we have $\bigcap_{j=1}^{\infty} (U_i)^c = \{x_1\}, \ldots, \bigcap_{j=1}^{\infty} (U_n)^c = \{x_n\}$ for some $x_1, \ldots, x_n$ in $X$. Moreover, since $\{U_1, \ldots, U_n\}$ covers $X$ and $x_i \in (U_i)^c$, we have $(x_i)^c \in X^{(n)} \Delta_n(X)$.

Finally, we show $\mathcal{D}(x_1, \ldots, x_n|\pi) \geq \mathcal{D}(T, U|\pi)$. Since $(x_1)^c \in X^{(n)} \Delta_n(X)$, without loss of generality, we assume $x_1 \neq x_n$. There exists $k_0 \in \mathbb{N}$ such that $B(x_1, \frac{1}{k_0}) \cap B(x_n, \frac{1}{k_0}) = \emptyset$. For any $k \geq k_0$, there exists $j(k) \in \mathbb{N}$ large enough such that $(U_i)^c \subseteq B(x_i, \frac{1}{k})$ for $i = 1, \ldots, n$, and thus

$$\mathcal{D}(T, X \backslash B(x_1, \frac{1}{k}), \cdots, X \backslash B(x_n, \frac{1}{k})|\pi) \geq \mathcal{D}(T, U_{j(k)}|\pi) \geq \mathcal{D}(T, U|\pi),$$

where the last inequality comes from the above b). Hence we have

$$\mathcal{D}(x_1, \ldots, x_n|\pi) = \lim_{k \to \infty} \mathcal{D}(T, X \backslash B(x_1, \frac{1}{k}), \cdots, X \backslash B(x_n, \frac{1}{k})|\pi) \geq \mathcal{D}(T, U|\pi).$$

This finishes the proof of the lemma.

\[\square\]

Proposition 4.3. Let $\pi : (X, T) \to (Y, S)$ and $\pi' : (Z, R) \to (X, T)$ be two factor maps.

1. If $(z_1, \ldots, z_n) \in E^\alpha(Z, R|\pi')$ and $x_i = \pi'(z_i), (x_i)^c \in X^{(n)} \Delta_n(X)$, then $\mathcal{D}(x_1, \ldots, x_n|\pi) \geq \alpha$.

2. If $(x_1, \ldots, x_n) \in E^\alpha(X, T|\pi)$, then there exists $(z_1, \ldots, z_n) \in E^\alpha(Z, R|\pi')$ and $\pi(z_i) = x_i, i = 1, 2, \ldots, n$.

Proof. (1). If $(z_1, \ldots, z_n) \in E^\alpha(Z, R|\pi')$ and $x_i = \pi'(z_i), (x_i)^c \in X^{(n)} \Delta_n(X)$, then

$$\mathcal{D}(x_1, \ldots, x_n|\pi) = \lim_{k \to \infty} \mathcal{D}(T, X \backslash B(x_1, \frac{1}{k}), \cdots, X \backslash B(x_n, \frac{1}{k})|\pi)$$

$$= \lim_{k \to \infty} \mathcal{D}(R, (\pi')^{-1}\{X \backslash B(x_1, \frac{1}{k}), \cdots, X \backslash B(x_n, \frac{1}{k})\}|\pi')$$

$$\geq \lim_{m \to \infty} \mathcal{D}(Z \backslash B(z_1, \frac{1}{m}), \cdots, Z \backslash B(z_n, \frac{1}{m})|\pi)|\pi') = \mathcal{D}(z_1, \ldots, z_n|\pi').$$

(2). Let $(x_1, \ldots, x_n) \in E^\alpha(X, T|\pi)$. For any $k \in \mathbb{N}$ with $\frac{2}{k} < \max_{1 \leq i < j \leq n} d(x_i, x_j)$, let $U_k = \{X \backslash B(x_1, \frac{1}{k}), \cdots, X \backslash B(x_n, \frac{1}{k})\}$. Then there exists $(z_1, \ldots, z_n)$ in
Obviously, \( J \) is \( T \times R \) invariant. Now we show \( J' \) is closed. Let \( y_n \in Y \) and \( z_n \in J'(y_n) \) be such that \( y_n \to y_0 \) and \( z_n \to z_0 \). Since \( \pi_X \) is open, the map \( y \mapsto \pi_X^{-1}(y) \) is continuous. For any \( x_0 \in \pi_X^{-1}(y_0) \), the continuity implies that there exists \( x_n \in \pi_X^{-1}(y_n) \) with \( x_n \to x_0 \). Since \( z_n \in J(x_n) \) and the map \( x \mapsto J(x) \) is upper semi-continuous, \( z_0 = \lim z_n \in J(x_0) \). Since \( x_0 \) is arbitrary, \( z_0 \in J'(y_0) \). This
proves $J'$ is closed. Then by the previous claim, $J^*(y) \neq \emptyset$ for $y \in Y$. Moreover, 
\begin{equation}
\pi_1(J') = \bigcup_{y \in Y} \pi_X^{-1}(y) = X.
\end{equation}

Note that $\pi_2(J')$ is a closed $R$-invariant subset of $Z$ and $\pi_Z \pi_2(J') = \pi_X \pi_1(J') = \pi_X(X) = Y$. We deduce that $\pi_2(J') = Z$ by the minimality of $\pi_Z$. Since $\pi_2(J') = \bigcup_{y \in Y} J^*(y)$ and $J^*(y) \subset \pi_Z^{-1}(y)$ for all $y \in Y$. This implies $J' = J = X \times_Y Z$, i.e., $(X, T)$ and $(Z, R)$ are disjoint over $(Y, S)$.

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