On links of certain semiprime ideals of a noetherian ring

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Abstract
In this paper we prove our main theorem, namely, theorem (8), which states that a link \( Q \to P \), of prime ideals \( Q \) and \( P \) of a noetherian ring \( R \) that are \( \sigma \)-semistable with respect to a fixed automorphism \( \sigma \) of \( R \), induces a link \( Q^0 \to P^0 \) of the semiprime ideals \( Q^0 \) and \( P^0 \) of the ring \( R \), where \( Q^0 \) and \( P^0 \) are the largest \( \sigma \)-invariant or \( \sigma \)-stable ideals contained in the prime ideals \( Q \) and \( P \). We also prove a converse to this theorem.
Introduction

In this paper we study the links of certain semiprime ideals of a noetherian ring R. Following the definition in [1], page 178, or from [3], we recall, there is a link from a prime ideal Q to a prime ideal P of a noetherian ring R (written as $Q \rightarrow P$), if there is an ideal A of R such that $QP \leq A < Q \cap P$ and the R- bi-module $Q \cap P / A$ is torsion free as a left $R/Q$ - module and as a right $R/P$ - module. Also following [2], we define the $\sigma$- invariant or the $\sigma$- stable part $I^0$ of an ideal I of R that is $\sigma$- semistable with respect to a fixed automorphism $\sigma$ of R as the largest $\sigma$- invariant ideal contained in I. This definition immediately yields the fact that if Q and P are prime ideals of a Noetherian ring R that are $\sigma$- semistable with respect to a fixed automorphism $\sigma$ of R, then $Q^0$ and $P^0$ are $\sigma$- invariant semiprime ideals of R. Following the above definition of a link of prime ideals as defined in [1] or [3], we define what we mean by the existence of a link between the semi prime ideals $Q^0$ and $P^0$ of a noetherian ring R, where $Q^0$ and $P^0$ are the $\sigma$- invariant parts of the prime ideals Q and P that are assumed to be semistable with respect to an automorphism $\sigma$ of R. With these definitions in mind we prove our main theorem, namely, theorem (8), which
states that if R is a Noetherian ring, then a link $Q \to P$ of prime ideals $Q$ and $P$ of $R$, that are $\sigma$-semistable with respect to an automorphism $\sigma$ of the ring $R$, induces a link $Q^0 \to P^0$ of the semiprime ideals $Q^0$ and $P^0$ of $R$. We also prove a converse to this theorem.

**Definitions and Notation**: To make mention of the source of our reference we state that throughout this paper we adhere to and adapt, as convenience and relevance permits, the notation and definitions of [1],[2], or [3] respectively. Thus, for example, for an ideal $I$ of a noetherian ring $R$, which is $\sigma$-semistable with respect to a fixed automorphism $\sigma$ of $R$, we denote by $I^0$, as in [2], lemma(6.9.9), the largest $\sigma$-invariant ideal contained in $I$ and call it the $\sigma$-invariant part of $I$. For a right module or a bimodule $M$ over a ring we denote by $|M|$ the right Krull dimension of the module $M$ whenever this dimension exists. We must mention here that for a bimodule $M$ over a ring $R$, we will again use the symbol $|M|$ to denote only the right Krull dimension of $M$ unless otherwise stated. For the basic definition and results on Krull dimension we refer the reader to [1]. For a few more words about the terminology in this paper we mention
that a ring \( R \) is noetherian means that \( R \) is a left as well as right noetherian ring. If \( R \) is a ring and \( M \) is a right \( R \) module then we denote by \( \text{Spec} R \), the set of prime ideals of \( R \). Moreover \( r\text{-ann}.T \) denotes the right annihilator of a subset \( T \) of \( M \) and \( l\text{-ann}.T \) denotes the left annihilator of a left subset \( T \) of \( W \) in case \( W \) is a left \( R \) module. For two subsets \( A \) and \( B \) of a given set, \( A \leq B \) means \( B \) contains \( A \) and \( A < B \) denotes \( A \leq B \) but \( A \neq B \). For an ideal \( A \) of \( R \), \( c(A) \) denotes the set of elements of \( R \) that are regular modulo the ideal \( A \). Finally we mention that throughout all our rings are with identity element and all our modules are unitary.

**Main Theorem**

To prove our main theorem we first define the \( \sigma \)-invariant (or the \( \sigma \)-stable) part of an ideal \( I \) that is \( \sigma \)-semistable with respect to an automorphism \( \sigma \) of the ring \( R \). Next as mentioned in the introduction, following the definition of a link between two prime ideals of a noetherian ring \( R \) as given in [3], for example, we give the definition of a link \( Q^o \to P^o \) where \( Q^o \) and \( P^o \) are the \( \sigma \) - invariant parts of the prime ideals \( Q \) and \( P \) of a Noetherian ring \( R \) which are \( \sigma \) - semi stable with respect to a fixed automorphism \( \sigma \) of \( R \). We then describe when these links exist.
**Definition (1):** Let R be a noetherian ring and let I be an ideal of R. Then following the definition (6.9.8) of [2], we say I is $\sigma$-semi-stable or $\sigma$-semi-invariant ideal of R if there exists an integer $n \geq 1$ such that $\sigma^n(I) = I$. I is said to be $\sigma$ stable or $\sigma$ invariant ideal of R if $\sigma(I) = I$.

With this definition we prove the following result below.

**Proposition (2):** Let R be a Noetherian ring with an automorphism $\sigma$ of R. Then the following hold,

a) If I is a $\sigma$-semi stable ideal of R with $\sigma^n(I) = I, n \geq 1$ and if $I^\sigma = I \cap \sigma(I) \cap \sigma^2(I) \cap \sigma^3(I) \ldots \cap \sigma^{n-1}(I)$, than $I^\sigma$ is the largest $\sigma$-invariant ideal of R contained in I. In case I is a prime ideal of R, then $I^\sigma$ is a $\sigma$ invariant semi prime ideal of R.

b) If I, J are $\sigma$-semistable ideals of R, then there exists a common integer $k \geq 1$ such that $\sigma^k(I) = I$ and $\sigma^k(J) = J$.

Prof: a) The proof of (a) is obvious (see; e.g.;[2], Lemma (6.9.9)(i)).

b) For the proof of (b) note that if m and n integers ($m, n \geq 1$) such that $\sigma^m(I) = I$ and $\sigma^n(J) = J$, then $k = mn$ implies that $\sigma^k(I) = I$ and $\sigma^k(J) = J$.

**Notation (3):** For an ideal I of R which is $\sigma$-semi stable with respect to an automorphism $\sigma$ of R we will denote by $I^\sigma$ the $\sigma$
invariant ideal of proposition (2). We also call $I^\sigma$ the $\sigma$ invariant part of the ideal $I$.

**Definition (4):** Let $R$ be a Noetherian ring and let $\sigma$ be an automorphism of $R$. Let $P, Q \in \text{Spec} (R)$ be prime ideals such that $P$ and $Q$ are $\sigma$-semistable prime ideals of $R$. Let $P^\sigma$ and $Q^\sigma$ be the $\sigma$-invariant parts of $P$ and $Q$ respectively. Then following the definition of a link between two prime ideals of a noetherian ring $R$ as given, for example, in [3], we say that a link $Q^\sigma \to P^\sigma$ between the semi prime ideals $Q^\sigma$ and $P^\sigma$ exists if there exists a non zero bi-module $Q^0 \cap P^0/A$ with $A$ an ideal of $R$ such that $Q^\sigma P^\sigma \leq A < Q^\sigma \cap P^\sigma$ and $Q^0 \cap P^0/A$ is a left $R/Q^0$ torsionfree module and a right $R/P^0$ torsionfree module.

We now prove the result below regarding links of prime ideals of a noetherian ring whose use will become apparent as we proceed to prove our main theorem.

**Proposition (5):** Let $R$ be a Noetherian ring with

$P, Q \in \text{Spec. } R$. Let $Q \to P$ be a link between the prime ideals $Q$ and $P$. Then there exists a linking bi-module $Q \cap P/B$ with $QP \leq B < Q \cap P$ such that $B$ is the unique minimal ideal such that a link $Q \to P$ exists via the ideal $B$.

**Proof:** Consider the set $S = \{A_i/A_i \text{ are ideals of } R \text{ with } QP \leq A < Q \cap P \text{ and such that a link } Q \to P \text{ exists via the ideal } A_i\}$. Let $B = \cap A_i$. Now consider the bi-module, $Q \cap P/B$. 

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Clearly $QP \leq B < Q \cap P$. We make the following claim:

$Q \cap P / B$ is a right $R/P$–torsionfree module.

Proof of the claim: Suppose $Q \cap P / B$ is not right $R/P$ torsionfree module. Then there exists $f \in Q \cap P$ and $f \not\in B$ such that $fg \in B$, for some $g \in c(P)$. Now $f \not\in B$ implies that $f \not\in Aj$ for some $j$. Since $B \leq Aj$, so $fg \in B$ implies that $fg \in Aj$. Also $f \not\in Aj$ and $g \in c(P)$ then means that $Q \cap P / Aj$ is not a right $R/P$ torsionfree module which contradicts that $Aj \in S$. hence $Q \cap P / B$ is a right $R/P$ torsionfree module. This proves the claim. Similarly we can show that $Q \cap P / B$ is a left $Q/R$ torsionfree module. Hence $Q \to P$ is a link via the ideal $B$. Moreover it is clear that $B$ is the unique minimal ideal of $R$ such that $Q \to P$ is a link via the ideal $B$ because if $Q \to P$ is a link via an ideal $A$ of $R$, then obviously $A \in S$. which yields that $B \leq A$.

**Proposition (6):** Let $R$ be a noetherian ring. Let $Q, P \in \text{Spec. } R$, and let $Q \to P$ be a link via the unique minimal ideal $A$ of $R$. If $Q, P$ are $\sigma$- semistable prime ideals of $R$ for an automorphism $\sigma$ of $R$, then there exists an integer $n \geq 1$ such that $\sigma^n(p) = P$, $\sigma^n(Q) = Q$ and $\sigma^n(A) = A$.

Proof: First note that because of proposition (2) above there exists a common integer $n \geq 1$, such that $\sigma^n(p) = P$ and $\sigma^n(Q) =
Q. Now the link \( Q \rightarrow P \) via the ideal \( A \) implies that for any integer \( i \geq 0 \) there is a link \( \sigma^i(Q) \rightarrow \sigma^i(P) \) via the ideal \( \sigma^i(A) \). In particular \( \sigma^n(Q) \rightarrow \sigma^n(P) \) via \( \sigma^n(A) \) implies that there is a link \( Q \rightarrow P \) via the ideal \( \sigma^n(A) \). Hence by the hypothesis on \( A \), \( A \leq \sigma^n(A) \). We now work with the automorphism \( \sigma^{-1} \) and first observe that \( \sigma^{-n}(p) = P \) and \( \sigma^{-n}(Q) = Q \). The above argument now yields that \( A \subseteq \sigma^n(A) \) or \( \sigma^n(A) \subseteq A \). Thus we get that \( A = \sigma^n(A) \).

**Proposition (7):** Let \( R \) be a Noetherian ring. Let \( P \) and \( Q \) be prime ideals of \( R \) that are \( \sigma \)-semi stable with respect to an automorphism \( \sigma \) of \( R \) and let the prime ideal \( Q \) be linked to the prime ideal \( P \). Further (using proposition (5) and proposition (6) above) let \( Q \) be linked to \( P \) via a unique minimal ideal \( A \) and let \( m \) be the common integer such that \( \sigma^m(Q) = Q, \sigma^m(P) = P \) and \( \sigma^m(A) = A \). Let \( Q^0, P^0 \) and \( A^0 \) have the usual meaning. Then the following hold true:

1. The ring \( R \) has at most two prime ideals minimal over the ideal \( A \), namely, \( Q \) or \( P \). In case \( P \) is a minimal prime ideal over \( A \) such that \( |R/A| = |R/P| \), then \( P \) is also a prime ideal minimal over the ideal \( A^0 \).
2. If \( |R/Q| = |R/P| \), then both \( Q \) and \( P \) are minimal prime ideals over \( A \) as well as over the ideal \( A^0 \). Moreover in this case all
the components of the semiprime ideals $Q^0$ and $P^0$ are minimal prime ideals over $A^0$.

Proof:- First observe that it is given that $A$ is the unique minimal ideal of $R$ such that $Q \to P$ is a link via the ideal $A$. Also it is given that $Q$ and $P$ are $\sigma$- semistable prime ideals of $R$, hence, we get that there exists a common integer $m \geq 1$ such that $\sigma^m(Q) = Q$, $\sigma^m(p) = P$ and thus using proposition (6) above we have $\sigma^m(A) = A$. Let $A^0 = A \cap \sigma(A) \cap \ldots \cap \sigma^{m-1}(A)$. We now prove (1).

Proof of (1):- We prove $Q$ or $P$ is a minimal prime ideal over $A$. To see this we first note that since we are given that $Q \to P$ is a link via the ideal $A$ hence we have that $QP \leq A$ and this immediately implies that either $Q$ or $P$ is a prime ideal minimal over $A$. Now suppose that among the prime ideals minimal over $A$, $P$ is a prime ideal minimal over $A^0$, such that $|R/P| = |R/A|$. Then obviously since $|R/A| = |R/A^0|$, hence $P$ is a prime ideal minimal over the ideal $A^0$ as well.

Proof of (2):- Now we prove (2) under the assumption that $|R/Q| = |R/P|$. In this situation then it is clear that either $Q = P$ or $Q$ and $P$ are distinct incomparable prime ideals over the ideal $A$. Hence, since $QP \leq A$, it is not difficult to see that the set of minimal prime ideals of $R/A$ consists of the
prime ideals $\mathcal{Q}/\mathcal{A}$ and $\mathcal{P}/\mathcal{A}$. Assume that among the prime ideals of $\mathcal{R}$ minimal over the ideal $\mathcal{A}$, $\mathcal{P}$ is the prime ideal such that $|\mathcal{R}/\mathcal{P}| = |\mathcal{R}/\mathcal{A}| (= |\mathcal{R}/\mathcal{A}^0|)$ then by (1) above $\mathcal{P}$ is the prime ideal minimal over the ideal $\mathcal{A}^0$ as well. But it is given that $|\mathcal{R}/\mathcal{Q}| = |\mathcal{R}/\mathcal{P}|$. Hence again using (1) above we get that $\mathcal{Q}$ is also a prime ideal minimal over the ideal $\mathcal{A}^0$. The rest is obvious.

We are now ready to prove our main theorem.

**Theorem (8):** Let $\mathcal{R}$ be a Noetherian ring. Let $\mathcal{P}$ be a prime ideal of $\mathcal{R}$ that is $\sigma$-semi-stable with respect to an automorphism $\sigma$ of $\mathcal{R}$. Let $\mathcal{Q}$ be a $\sigma$-semistable prime ideal that is linked to $\mathcal{P}$. Then the link $\mathcal{Q} \rightarrow \mathcal{P}$ of the prime ideals $\mathcal{Q}$ and $\mathcal{P}$ induces a link $\mathcal{Q}^\circ \rightarrow \mathcal{P}^\circ$ of the semi prime ideals $\mathcal{Q}^\circ$ and $\mathcal{P}^\circ$ of $\mathcal{R}$.

**proof:** By proposition (5) above we choose the unique minimal ideal $\mathcal{A}$ of $\mathcal{R}$ such that $\mathcal{Q} \rightarrow \mathcal{P}$ is a link via the ideal $\mathcal{A}$. By proposition (2), since $\mathcal{Q}$ and $\mathcal{P}$ are $\sigma$-semi-stable prime ideals of $\mathcal{R}$ there exists a common integer $n \geq 1$ such that $\sigma^n(\mathcal{Q}) = \mathcal{Q}$, $\sigma^n(\mathcal{P}) = \mathcal{P}$ and in that case using proposition (6) we have $\sigma^n(\mathcal{A}) = \mathcal{A}$. Let $\mathcal{A}^\circ = \mathcal{A} \cap \sigma(\mathcal{A}) \cap \ldots \cap \sigma^{n-1}(\mathcal{A})$. We now prove that $\mathcal{Q}^\circ \rightarrow \mathcal{P}^\circ$ is a link via the ideal $\mathcal{A}^\circ$. 


For the proof we first assume by proposition (7) above that \( P \) is a minimal prime ideal over \( A \) as well over \( A^0 \).

We now make the claim: \( A^0 \neq Q^0 \cap P^0 \).

Proof of the claim:- For the proof of this claim we assume that \( A^0 = Q^0 \cap P^0 \). Then \( A^0 = Q^0 \cap P^0 \leq A < Q \cap P \), implies, that in the semi prime ring \( R/A^0 \), the prime ideal \( Q/A^0 \) is linked to the minimal prime ideal \( P/A^0 \) via the ideal \( A/A^0 \). But this contradicts lemma (11.17) of \([1]\). This proves the claim.

Hence we must have \( A^0 < Q^0 \cap P^0 \).

We now show that \( Q^0 \rightarrow P^0 \) is a link via the ideal \( A^0 \). But first observe that \( Q \rightarrow P \) is a link via the ideal \( A \) implies also that \( \sigma^i(Q) \rightarrow \sigma^i(P) \) is a link via the ideal \( \sigma^i(A) \) for any integer \( i \geq 1 \).

We now make the claim: \( (Q^0 \cap P^0)/A^0 \) is a right \( R_{P^0} \) torsionfree module and a left \( R_{Q^0} \) torsionfree module.

Proof of the claim: Suppose first that \( (Q^0 \cap P^0)/A^0 \) is not a right \( R_{P^0} \) torsionfree module. Then there exists \( f \in Q^0 \cap P^0 \), \( f \not\in A^0 \) and a \( g \in c(P^0) \) such that \( fg \in A^0 \). Now \( f \not\in A^0 \) implies that there exists an integer \( m \geq 1 \) such that \( f \not\in \sigma^m(A) \). However, \( f \in Q^0 \cap P^0 \) implies that \( f \in \sigma^m(Q) \cap \sigma^m(P) \) and \( fg \in A^0 \) implies that \( fg \in \sigma^m(A) \). Observe that \( g \in c(P^0) \) means that \( g \in C[\sigma^m(P)] \). All this means that \( \sigma^m(Q)\cap\sigma^m(P)/\sigma^m(A) \) is not right \( R/\sigma^m(P) \) torsionfree module contradicting our earlier observation that
\( \sigma^m (Q) \to \sigma^m (P) \) is a right link via the ideal \( \sigma^m (A) \). Hence we must have that \( (Q^0 \cap P^0)/A^0 \) is a right \( \mathbb{R}_{/P^0} \) torsionfree module. Similarly we can show that \( (Q^0 \cap P^0)/A^0 \) is a left \( \mathbb{R}_{/Q^0} \) torsionfree module. Hence \( Q^0 \to P^0 \) is a link via the ideal \( A^0 \).

There is a converse to the above result which we shall state and prove now. We mention at the outset of this theorem that we shall adapt and mimic, the proof of theorem(11.2)of [1] and hence while doing this we will adapt(without further mention)as much as possible the terminology that is used in the proof of theorem (11.2) of [1].

**Theorem (9):** Let \( R \) be a Noetherian ring and let \( \sigma \) be an automorphism of \( R \). Let \( Q, P \) be \( \sigma \)- semi stable prime ideals of \( R \). Then a (right) link \( Q^0 \to P^0 \) of semi prime ideal \( Q^0 \) and \( P^0 \) of \( R \) implies that for any integer \( i \geq 0 \) there exists an integer \( j \geq 0 \) such that \( \sigma^j(Q) \to \sigma^j(P) \) is a (right) link.

**Proof:** As stated earlier we will give a sketch of the proof which is on the same lines as the proof of theorem (11.2) of [1]. Since we are given \( Q^0 \to P^0 \) is a (right) link via an ideal \( A \), thus following [1], theorem (11.2), we assume without loss of generality that \( A=0 \). So we may assume \( Q^0 P^0=0 \) and \( Q^0 \cap P^0 \) is a
nonzero torsionfree right $R/P^o$-module and a torsionfree left $R/Q^o$-module. Since $l\text{-ann.}(Q^o \cap P^o) = Q^o$, we conclude that $l\text{-ann}(Q^o) \leq Q^o$.

Note that $l\text{-ann.}(P^o) = Q^o$ because $Q^o P^o = 0$. We first show that $Q^o$ is essential as a right ideal of $R$. To see this suppose $I$ is a nonzero right ideal of $R$. If $I Q^o = 0$, then $I \leq l\text{-ann.}(Q^o) \leq Q^o$. Hence $I \cap Q^o \neq 0$. Now clearly if $IQ^o \neq 0$, then obviously $I \cap Q^o \neq 0$. Hence $Q^o$ is essential as a right ideal of $R$. Next note that $Q^o \cap P^o$ is a torsionfree right $R/P^o$-module and since $Q^o/Q^o \cap P^o$ is isomorphic to a right ideal of $R/P^o$, thus $Q^o/Q^o \cap P^o$ is torsionfree as a right $R/P^o$-module hence $Q^o$ is torsionfree as a right $R/P^o$-module. Thus by proposition (6.18) of [1], $Q^o$ has an essential submodule isomorphic to a finite direct sum of uniform right ideals of $R/P^o$. Since $Q^o_R$ is essential as a right submodule of $R_R$, thus $E(Q^o)_R \approx E(R)_R$, where $E(Q^o)_R$ and $E(R)_R$ are the injective hulls of $Q^o$ and $R$ as right $R$ modules. Now since the components of the semiprime ring $R/P^o$ are isomorphic to each other, we must have that $E(R)_R \approx E^n_R$, where $E$ is the injective hull of a uniform right ideal of $R/P^o$ and $n = \text{rank}(R)_R$. Following the argument exactly as in proposition (6.23) of [1], we note that $E$ is independent of the choice of a uniform right ideal of $R/P^o$. Obviously $E$ has an essential submodule which is a torsionfree $R/P^o$-module. Thus $\text{ann.}_E(P^o)$ is torsionfree as a $R/P^o$ module. Since
l-\text{ann}_R(P^o) = Q^0, as seen in the first para above, it follows that 
Q^o = R \cap \text{ann}_{E(R)}(P^o). Hence, if W = \text{ann}_{E(R)}(P^o), then 
\frac{R}{Q^o} = R/R \cap W \cong W + R/W \leq E(R)/W \cong E^n/\text{ann}_{E}(P^o).

Now observe that \( E^n/\text{ann}_{E}(P^o) \cong (E/\text{ann}_{E}(P^o))^n \). Hence \( R/Q^o \) embeds in \( (E/\text{ann}_{E}(P^o))^n \). It thus follows that any uniform right ideal of \( R/P^o \) embeds in \( E/\text{ann}_{E}(P^o) \).

Now, let K be a submodule of E such that \( \text{ann}_{E}(P^o) < K \) and \( K/\text{ann}_{E}(P^o) \) is isomorphic to a uniform right ideal of \( R/Q^o \). Choose an element \( x \) in K not annihilated by \( P^o \). Let \( M = xR \) and let \( U = \text{ann}_M(P^o) \). Clearly \( M + U/U \) is isomorphic to a uniform right ideal of \( R/Q^0 \) and by an argument similar to cor.(6.20) of [1],

U is isomorphic to a uniform right ideal of \( R/P^o \). Since E is uniform, so is M, and it is clear from the definition of U that \( 0 < U < M \) is an affiliated series for M. As a consequence there exist integers \( m \) and \( n \) such that \( U \) is \( R/\sigma^m(P) \)-torsionfree and \( M/U \) is \( R/\sigma^n(Q) \)-torsionfree modules. Again by an argument similar to theorem (11.2) of [1], we get that \( \sigma^n(Q) \rightarrow \sigma^m(P) \). Now applying \( \sigma \) repeatedly to the link \( \sigma^n(Q) \rightarrow \sigma^m(P) \) and observing that \( Q \) is \( \sigma \)-semistable, we get that, for any integer \( i \geq 1 \) there is a link \( \sigma^i(Q) \rightarrow \sigma^j(P) \) for some \( j \geq 1 \). This completes the proof of the theorem.

Remark:- We remark that theorem(11.2)of [1] that we have used in the proof of theorem(9) above actually is proved in [1] to
give a characterization of the existence of links between two prime ideals in a noetherian ring and in fact this theorem tells how links really arise in the study of finitely generated modules over a noetherian ring. In this context, thus, our proof of theorem (9) given above is circular. We mention that a direct and a much easier proof of theorem (9), namely one, which does not use theorem (11.2) of [1], may be possible. But we have not verified the same.

References
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