Demazure Characters and Affine Fusion Rules

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Abstract: The Demazure character formula is applied to the Verlinde formula for affine fusion rules. We follow Littelmann’s derivation of a generalized Littlewood-Richardson rule from Demazure characters. A combinatorial rule for affine fusions does not result, however. Only a modified version of the Littlewood-Richardson rule is obtained that computes an (old) upper bound on the fusion coefficients of affine $A_r$ algebras. We argue that this is because the characters of simple Lie algebras appear in this treatment, instead of the corresponding affine characters. The Bruhat order on the affine Weyl group must be implicated in any combinatorial rule for affine fusions; the Bruhat order on subgroups of this group (such as the finite Weyl group) does not suffice.

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1. Introduction

The fusion coefficients \( N_{n,l,m}^{n} \in \mathbb{Z}_{\geq 0} \) of a rational conformal field theory (RCFT) may be defined by [1]

\[
\frac{S_{\ell,p}}{S_{0,p}} \frac{S_{m,p}}{S_{0,p}} = \sum_{n \in \Phi} N_{n,l,m}^{n} \frac{S_{n,p}}{S_{0,p}}, \quad \forall p \in \Phi. \tag{1.1}
\]

Here the indices \( \ell, m, n, p, \ldots \in \Phi \) label the primary fields of the RCFT, with the index 0 specifying the identity field. \( S_{i,j} \) denotes an element of the matrix describing the transformation of the characters of the primary fields under the change of torus modulus \( \tau \rightarrow -1/\tau \).

We will restrict attention to those RCFTs realizing an affine Kac-Moody algebra \( X_{r,k} \) that is the central extension at fixed level \( k \) of the loop algebra of the simple Lie algebra \( X_{r} \) of rank \( r \). These are often called Wess-Zumino-Witten (WZW) models [2].

If the chiral algebra is not extended, the primary fields of such theories are in one-to-one correspondence with the integrable highest-weight representations of \( X_{r,k} \). The set of highest weights of these affine representations can therefore be used to label the primary fields. Alternatively, we can label the primary fields using the following set of dominant weights of \( X_{r} \):

\[
P_{\geq}(X_{r,k}) := \left\{ \lambda = \sum_{i=1}^{r} \lambda_{i}\omega^{i} \mid \lambda_{i} \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{r} \lambda_{i}a_{i}^{\vee} \leq k \right\}. \tag{1.2}
\]

\( \omega^{i} \) denotes the \( i \)-th fundamental weight of \( X_{r} \), and \( a_{i}^{\vee} \) is the corresponding co-mark, so that \( 1 + \sum_{i=1}^{r} a_{i}^{\vee} = h^{\vee} \), the dual Coxeter number. \( F(X_{r}) \) will stand for the set of fundamental weights of \( X_{r} \). The fusion rules of WZW models may be written as

\[
\frac{S_{\lambda,\zeta}}{S_{0,\zeta}} \frac{S_{\mu,\zeta}}{S_{0,\zeta}} = \sum_{\nu \in P_{\geq}(X_{r,k})} N_{\lambda,\mu}^{\nu} \frac{S_{\nu,\zeta}}{S_{0,\zeta}}, \quad \forall \zeta \in P_{\geq}(X_{r,k}). \tag{1.3}
\]

For short, we will refer to these as affine fusion rules.

The ratios of matrix elements of \( S \) appearing in (1.3) are intimately related to the characters of representations of \( X_{r} \) [3]. Let \( R(\lambda) \) be the integrable representation of \( X_{r} \) with highest weight \( \lambda \), and let \( P(\lambda) \) denote the set of its weights. If

\[
P(X_{r}) := \left\{ \lambda = \sum_{i=1}^{r} \lambda_{i}\omega^{i} \mid \lambda_{i} \in \mathbb{Z} \right\} \tag{1.4}
\]
is the set of integral weights of $X_r$, we have $P(\lambda) \subset P(X_r)$. Let $e^\lambda$ denote the formal exponential of the weight $\lambda$, with the property $e^\lambda e^\mu = e^{\lambda+\mu}$. The formal character of $R(\lambda)$ may be written as

$$
\text{ch}_\lambda := \sum_{\sigma \in P(\lambda)} \text{mult}_\lambda(\sigma) \, e^\sigma,
$$

where $\text{mult}_\lambda(\sigma)$ is the multiplicity of the weight $\sigma$ in the representation $R(\lambda)$, and $\rho = \sum_{i=1}^r \omega^i$ is the Weyl vector.

Define

$$
e^\sigma(\zeta) := \exp \left[ -\frac{2\pi i}{k+h^\vee} \sigma \cdot (\zeta + \rho) \right].
$$

Kac and Peterson showed that

$$
\frac{S_{\lambda,\zeta}}{S_{0,\zeta}} = \text{ch}_\lambda(\zeta), \quad (\lambda, \zeta \in P_{\geq}(X_{r,k})) .
$$

This remarkable result allows us to relate the affine fusion coefficients $N_{\lambda,\mu}^\nu$ to the coefficients appearing in the decomposition of a tensor product of two representations of $X_r$. Let

$$
P_{\geq}(X_r) := \left\{ \lambda = \sum_{i=1}^r \lambda_i \omega^i \mid \lambda_i \in \mathbb{Z}_{\geq 0} \right\}
$$

be the set of dominant weights of $X_r$. Then, if the tensor product coefficients $T_{\lambda,\mu}^\nu$ are defined by

$$
R(\lambda) \otimes R(\mu) = \bigoplus_{\nu \in P_{\geq}(X_r)} T_{\lambda,\mu}^\nu R(\nu) ,
$$

the corresponding characters obey

$$
\text{ch}_\lambda \text{ch}_\mu = \sum_{\nu \in P_{\geq}(X_r)} T_{\lambda,\mu}^\nu \text{ch}_\nu .
$$

The Weyl character formula is

$$
\text{ch}_\lambda = \frac{\sum_{w \in W} (\det w) \, e^{w(\lambda+\rho)}}{\sum_{w \in W} (\det w) \, e^{w\rho}} ,
$$

where $W$ is the Weyl group of $X_r$. Therefore,

$$
\text{ch}_\lambda(\zeta) = \frac{\sum_{w \in W} (\det w) \, \zeta^{w(\lambda+\rho)}}{\sum_{w \in W} (\det w) \, \zeta^{w\rho}} ,
$$

where we have defined

$$
\zeta^\sigma := e^\sigma(\zeta)
$$

for $\sigma \in \mathbb{C}$.
for notational convenience. The Weyl formula makes manifest the Weyl symmetry of the characters:

\[ \text{ch}_\lambda(\zeta) = (\det w) \text{ch}_{w.\lambda}(\zeta) = \text{ch}_\lambda(w.\zeta), \quad \forall w \in W \ (\zeta \in P_{\mathbb{R}}(X_r)) , \]  

where \( w.\lambda := w(\lambda + \rho) - \rho \) denotes the shifted action of the Weyl group element \( w \), and \( P_{\mathbb{R}}(X_r) := \{ \lambda = \sum_{i=1}^r \lambda_i \omega_i \ | \lambda_i \in \mathbb{R} \} \). When \( \zeta \in P(X_r) \), this symmetry can be extended to include the elements of the Weyl group of \( X_{r,k} \), the so-called affine Weyl group \( \hat{W} \):

\[ \text{ch}_\lambda(\zeta) = (\det w) \text{ch}_{w.\lambda}(\zeta) = \text{ch}_\lambda(w.\zeta), \quad \forall w \in \hat{W} \ (\zeta \in P(X_r)) . \]  

Let \( S(X_r) \) signify the set of simple roots of \( X_r \). \( W \) is generated by the primitive reflections \( r_\alpha \),

\[ r_\alpha \lambda := \lambda - (\lambda \cdot \alpha^\vee) \alpha, \quad \alpha \in S(X_r), \]  

where \( \alpha^\vee \) is the simple co-root dual to the simple root \( \alpha \). \( \hat{W} \) has one additional generator \( r_{\alpha_0} \) corresponding to the 0-th affine simple root \( \alpha_0 = \delta - \theta \). Here \( \theta \) is the highest root of \( X_r \), and \( \delta \) is the imaginary root of the affine algebra that yields \( \delta \cdot \lambda = k \) for any level \( k \) weight \( \lambda \) (and \( \delta \cdot \beta = 0 \) for any root \( \beta \)). The action of \( r_{\alpha_0} \) in (1.15) is therefore

\[ r_{\alpha_0} \lambda = r_{\theta} \lambda + (k + h^\vee)(\theta - \delta) . \]  

Since \( \zeta \) corresponds to a level \( k \) weight, and \( \rho \) to a level \( h^\vee \) one, we have \( \zeta^\delta = 1 \). Then the identity

\[ \zeta^{r_{\alpha_0} \lambda} = \zeta^{r_{\theta} \lambda + (k + h^\vee)\theta} = \zeta^{r_{\theta} \lambda} \]  

leads to the full affine Weyl invariance of (1.15).

Comparing equations (1.3) and (1.10), taking into account the Kac-Peterson relation \[3\] and the affine Weyl symmetry (1.15), one finds \[3\] \[4\] \[5\] \[6\] : 

\[ N_{\lambda,\mu}^{\nu} = \sum_{w \in \hat{W}} (\det w) T_{\lambda,\mu}^{w.\nu} . \]  

Using the well-known connection between multiplicities and tensor product coefficients,

\[ T_{\lambda,\mu}^{\nu} = \sum_{w \in W} (\det w) \text{mult}_\mu(w.\nu - \lambda) , \]  

we can also write

\[ N_{\lambda,\mu}^{\nu} = \sum_{w \in \hat{W}} (\det w) \text{mult}_\mu(w.\nu - \lambda) . \]
These formulas make it clear that the affine fusion coefficients are integers, while it is not at all evident from the form of the matrix \( S \) in (1.3). They do not, however, confirm that the fusion coefficients are non-negative integers, as must be.

That the tensor product coefficients are non-negative is made manifest by rules for their computation that reduce to counting certain objects, usually Young tableaux (see [8], for example). The most famous is the Littlewood-Richardson rule that computes the tensor product coefficients of the simple Lie algebra \( A_r \). Such rules do not involve over-counting and cancellations, and will be called combinatorial rules.

Algorithms like the Littlewood-Richardson rule involve Young tableaux. The action of the affine Weyl group on the Young tableaux for classical simple Lie algebras can be simply described. So, (1.19) allows the combinatorial methods for computing tensor products to be adapted to the computation of affine fusion rules [9]. The adaptation is not combinatorial, however; the factor \( \det w = \pm 1 \) in (1.19) means the adapted methods involve over-counting and cancellations.

Can one find a purely combinatorial method for the computation of affine fusion rules? To the best of our knowledge, no such rule is known\(^1\). We believe that such a rule, if it exists, would find wide application, much as the original Littlewood-Richardson rule has. More importantly, however, it would provide a constructive mathematical proof that fusion coefficients are non-negative integers, something that is obvious and necessary from a physical point of view. It may also deepen our understanding of affine fusions; perhaps it will provide a definition of affine fusion expressed solely in terms of representations, analogous to that for simple Lie algebra tensor products.

We may try to use (1.19) as a guide to a combinatorial rule. The problem with (1.19), however, is the factor \( \det w = \pm 1 \), inherited from the Weyl character formula (1.11). So, perhaps a non-negative character formula can lead to a combinatorial rule. The Demazure character formula [10][11] is such a non-negative formula, and Littelmann [12] has derived the Littlewood-Richardson rule and generalizations from it. In this work, we adapt his methods to the computation of affine fusion rules, in an attempt to find a purely combinatorial rule. Although our results are only valid for \( X_r = A_r \), we use universal language appropriate to all simple Lie algebras (and their untwisted affine analogues) whenever possible. This reflects our hope that a universal result, valid for all \( X_{r,k} \), will eventually be found.

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\(^1\) Of course, for sufficiently large level \( k \), when the fusion coefficient coincides with the corresponding tensor product coefficient, we can use the combinatorial rules for tensor products.
As far as we know, this work is the first direct application of Demazure character formulas to affine fusions. In light of Littelmann’s results, this is a natural attempt at finding a combinatorial rule for affine fusions. Unfortunately, no such rule is obtained. Our effort does make it clear why, however. We hope it will therefore help as a stepping stone to a combinatorial rule for affine fusions.

2. Demazure Character Formulas

Introduce the primitive Demazure operators \( D_\alpha \):

\[
D_\alpha(e^\mu) := \frac{e^\mu - e^{r_\alpha \cdot \mu}}{1 - e^{-\alpha}},
\]

for all \( \alpha \in S(X_r) \). These linear operators are associated with the primitive reflections of the Weyl group \( W \) of \( X_r \), and are defined for all \( \zeta \in P_\geq(X_r) \), and \( \mu \in P(X_r) \). More explicitly,

\[
D_\alpha(e^\mu) = \begin{cases} 
  e^\mu + \cdots + e^{\mu-(\mu \cdot \alpha^\vee) \alpha} & , \mu \cdot \alpha^\vee \geq 0 ; \\
  0 & , \mu \cdot \alpha^\vee = -1 ; \\
  -e^\mu + \alpha - \cdots - e^{\mu-(\mu \cdot \alpha^\vee-1) \alpha} & , \mu \cdot \alpha^\vee \leq -2 .
\end{cases}
\]

A Demazure operator can also be associated to every element of the Weyl group of \( X_r \). Suppose \( w = r_{\beta_1} r_{\beta_2} \cdots r_{\beta_\ell} \), with all \( \beta_i \in S(X_r) \), is a decomposition of \( w \in W \). Such a decomposition is said to have length \( \ell \). For fixed \( w \in W \), any decomposition of minimum length is called a reduced decomposition. The length \( \ell(w) \) of any reduced decomposition of \( w \in W \) is known as the length of \( w \). Now, if \( w = r_{\beta_1} r_{\beta_2} \cdots r_{\beta_\ell} \) is a reduced decomposition of \( w \in W \), then we define

\[
D_w := D_{\beta_1} \circ D_{\beta_2} \circ \cdots \circ D_{\beta_\ell}.
\]

It makes sense to label these operators with the Weyl group element \( w \), since they do not depend on which reduced decomposition is used. It is important, however, that a reduced decomposition be used in the definition of \( D_w \); we have \( r_{\alpha}^2 = id \), but

\[
D_\alpha \left( D_\alpha(e^\mu) \right) = D_\alpha(e^\mu).
\]

This last relation follows from the more general one:

\[
D_\alpha \left( e^\chi \ D_\alpha(e^\mu) \right) = D_\alpha(e^\chi) \ D_\alpha(e^\mu).
\]
The Weyl group of any simple Lie algebra contains a unique element of maximum length. If \( w_L \) symbolizes the longest element of the Weyl group \( W \), we have the following Demazure character formula:

\[
\text{ch}_\lambda = D_{w_L}(e^\lambda) .
\]  

(2.6)

It is also useful to introduce other primitive Demazure operators related to the \( D_\alpha \):

\[
d_\alpha(e^\mu) := (D_\alpha - \text{id})(e^\mu) = \frac{e^{\mu-\alpha} - e^{r_\alpha \cdot \mu}}{1 - e^{-\alpha}} ,
\]  

(2.7)

for all \( \alpha \in S(X_r) \). More explicitly,

\[
d_\alpha(e^\mu) = \begin{cases} 
  e^{\mu-\alpha} + \ldots + e^{\mu-(\mu \cdot \alpha^\vee)\alpha}, & \mu \cdot \alpha^\vee \geq 1 ; \\
  0, & \mu \cdot \alpha^\vee = 0 ; \\
  -e^{\mu} - \ldots - e^{\mu-(\mu \cdot \alpha^\vee+1)\alpha}, & \mu \cdot \alpha^\vee \leq -1 .
\end{cases}
\]  

(2.8)

These operators can also be generalized to \( d_w \), for each \( w \in W \setminus \text{id} \), in a way similar to that defining the \( D_w \). For later convenience, we also define \( d_{\text{id}}(e^\lambda) := e^\lambda \).

To write other Demazure character formulas, we need the Bruhat partial order on the elements of the Weyl group \( W \). Let \( r_\alpha \) be the Weyl reflection across the hyperplane in weight space normal to the root \( \alpha \), i.e. \( r_\alpha \lambda = \lambda - (\lambda \cdot \alpha^\vee)\alpha \). The length \( \ell(w) \) of a Weyl group element \( w \) is defined as the number of primitive reflections in a reduced decomposition of the element. For \( v_1, v_2 \in W \), we write \( v_1 \leftarrow v_2 \) if and only if both \( v_1 = r_\alpha v_2 \) for some positive root \( \alpha \), and \( \ell(v_1) = \ell(v_2) + 1 \). Then for \( w, \tilde{w} \in W \), the Bruhat order is defined by \( w \succ \tilde{w} \) if \( w \leftarrow v_1 \leftarrow v_2 \leftarrow \cdots \leftarrow v_n \leftarrow \tilde{w} \), for some \( v_1, v_2, \ldots, v_n \in W \).

Another formula for the character of the representation \( R(\lambda) \) is

\[
\text{ch}_\lambda = \sum_{v \in W} d_v(e^\lambda) .
\]  

(2.9)

A generalization is

\[
D_w(e^\lambda) = \sum_{v \preceq w} d_v(e^\lambda) .
\]  

(2.10)

The formula

\[
D_\alpha (d_{r_\alpha v}(e^\mu)) = d_{r_\alpha v}(e^\mu) + d_v(e^\mu) , \text{ if } r_\alpha v \prec v , \ \alpha \in S(X_r) ,
\]  

(2.11)

makes explicit the relation between the two types of Demazure operators. Using (2.7) this translates into

\[
d_v(e^\mu) = d_\alpha \circ d_{r_\alpha v}(e^\mu) , \text{ if } r_\alpha v \prec v , \ \alpha \in S(X_r) .
\]  

(2.12)
3. Tableaux and the Weyl Group

In order to relate the Demazure formulas to the Littlewood-Richardson rule, the Weyl group must be related to Young tableaux and standard tableaux. This is done by recognizing certain sequences of elements of \( W \) as the appropriate generalizations of standard tableaux. The so-called minimal defining chains (MDCs) of Weyl group elements are in one-to-one correspondence with certain vectors of fixed weight that form a basis of the irreducible representation \( R(\lambda) \) of \( X_r \). For the algebra \( A_r \), it is well known that such vectors are in one-to-one correspondence with standard tableaux. MDCs are what work for all simple Lie algebras \( X_r \).

Suppose \( \mu \) is the highest weight of an integrable representation \( R(\mu) \) of \( X_r \). Such a weight can be expressed as a sum of fundamental weights \( \mu = \nu^1 + \nu^2 + \ldots + \nu^n \), where \( \nu^j \in F(X_r) \), \( 1 \leq j \leq n \). If we fix an order \( > \) on the elements of \( F(X_r) \), so that we can also impose \( \nu^i \geq \nu^{i+1} \) on the sum, the expression is unique. Now consider a sequence \( c = (w_1, w_2, \ldots, w_n) \) of \( n \) Weyl group elements \( w_i \in W \). The sequence, or chain, is a defining chain if it respects the Bruhat order: \( w_1 \leq w_2 \leq \cdots \leq w_n \). A sequence of weights of fundamental representations corresponding to each defining chain and weight \( \mu \) is given by

\[
P_\mu(c) := (w_1 \nu^1, w_2 \nu^2, \ldots, w_n \nu^n).
\]

Different defining chains, however, lead to the same weight-sequence. A unique \textit{minimal defining chain} (MDC) can be associated with the weight sequence: \( c = (w_1, \ldots, w_n) \) is a minimal defining chain if for any other defining chain \( \tilde{c} = (\tilde{w}_1, \ldots, \tilde{w}_n) \) satisfying \( P_\mu(c) = P_\mu(\tilde{c}) \), we have \( w_j \preceq \tilde{w}_j \), for all \( 1 \leq j \leq n \).

Define the \( \mu \)-weight \( p_\mu[c] \) of the MDC \( c \) as

\[
p_\mu[c] = p_\mu[(w_1, \ldots, w_n)] := w_1 \nu^1 + \ldots + w_n \nu^n.
\]

Let \( C_{\mu}[\sigma] \) be the set of MDCs with \( \mu \)-weight \( \sigma \), and let \( C_{\mu}(w) \) denote the set of MDCs with last element equal to \( w \in W \). The weights of a representation \( R(\mu) \) and those of MDCs are related:

\[
\text{mult}_\mu(\sigma) = |C_{\mu}[\sigma]|,
\]

so that

\[
\text{ch}_\mu = \sum_{\sigma \in P(\mu)} \sum_{c \in C_{\mu}[\sigma]} e^{p_\mu[c]}.
\]
Also relevant here is the connection with Demazure operators:

\[ d_w(e^\lambda) = \sum_{c \in \mathcal{C}(\lambda)} e^{\lambda[c]} . \] (3.5)

In the case \( X_r = A_r \), the usual standard tableaux are recovered as follows. Let \( \{ e_i \mid i = 1, \ldots, r + 1; e_i \cdot e_j = \delta_{i,j} \} \) be an orthonormal basis of \( \mathbb{R}^{r+1} \). The fundamental weights can be chosen to be \( \omega^i = e_1 + e_2 + \ldots + e_i - i\psi/(r + 1) \), with \( \psi = \sum_{i=1}^{r+1} e_i \). The Weyl group of \( A_r \) acts as the group of permutations on the \( e_i \). Therefore the weights \( w_j \nu^j \) (\( w_j \in \mathcal{W}, \nu^j \in F(A_r) \)) in the weight sequence \( P_\mu(c) = (w_1 \nu^1, w_2 \nu^2, \ldots, w_n \nu^n) \) of a MDC \( c \) can be associated with a column of boxes, with numbers from 1 to \( r + 1 \) in each, increasing down the column. A box \[ \text{2} \], say, corresponds to a summand \( e_2 \) in the expression for the weight, modulo multiples of \( \psi \). If the fundamental weights of \( A_r \) are ordered \( \omega^1 < \omega^2 < \cdots < \omega^r \), then if the columns of boxes corresponding to the weights in the weight-sequence of a MDC are assembled into a tableaux, the columns will have heights that do not increase from left to right. Furthermore, the Bruhat order imposed on a MDC ensures that the numbers in the rows of the tableau also do not increase from left to right. But these are precisely the defining properties of a standard tableau for \( A_r \). (The further minimal property of the MDCs is necessary for (3.3) and (3.5).)

The MDCs, corresponding weight sequences and standard tableaux associated with vectors in the \( A_2 \) representation \( R(2,1) \) of highest weight \( 2\omega^1 + \omega^2 =: (2,1) \) are given as an example, in Table 1. There we use the shorthand notation \( r_{\alpha_i} =: r_i \).

If a standard tableau corresponds to a weight-sequence \( P_\mu(c) \) of the MDC \( c \), then we say that it has shape \( \mu \). The shape of a standard tableau is determined by the configuration of its boxes, and not by the numbers therein. If the numbers are removed from a standard tableau of shape \( \mu \), a Young tableau of shape \( \mu \) results. Young tableaux also enter into the Littlewood-Richardson rule.

It will be useful to define subchains obtained from other chains by keeping the last \( i \) elements on the right, and also by keeping the first \( n - i \) elements on the left. If \( c = (w_1, w_2, \ldots, w_n) \) is a defining chain, we define

\[ R_i(c) := (w_{n-i+1}, w_{n-i+2}, \ldots, w_n), \quad \text{and} \quad L_i(c) := (w_1, w_2, \ldots, w_{n-i}) . \] (3.6)

These are sometimes called semi-standard tableaux, when standard tableaux designate those with numbers increasing in both rows and columns.
| MDC     | weight sequence          | tableau          | weight |
|---------|--------------------------|------------------|--------|
| (id, id, id) | ((0, 1), (1, 0), (1, 0)) | 1 1 1 2/3       | (2, 1) |
| (id, id, r₁) | ((0, 1), (1, 0), (−1, 1)) | 1 1 2 2/3          | (0, 2) |
| (id, r₁, r₁) | ((0, 1), (−1, 1), (−1, 1)) | 1 2 2 2/3          | (−2, 3) |
| (r₂, r₂, r₂) | ((1, −1), (1, 0), (1, 0)) | 1 1 1 3/3        | (3, −1) |
| (id, id, r₂r₁) | ((0, 1), (1, 0), (0, −1)) | 1 1 3 2/3          | (1, 0) |
| (r₂, r₂, r₁r₂) | ((1, −1), (1, 0), (−1, 1)) | 1 1 2 3/3          | (0, 0) |
| (r₂r₁, r₂r₁, r₁r₂) | ((−1, 0), (−1, 1), (−1, 1)) | 2 2 2 3/3          | (−3, 2) |
| (r₂, r₂, r₂r₁) | ((1, −1), (1, 0), (0, −1)) | 1 1 3 3/3          | (2, −2) |
| (r₂, r₁r₂, r₂r₁r₂) | ((1, −1), (−1, 1), (0, −1)) | 1 2 3 3/3          | (0, −1) |
| (id, r₂r₁, r₂r₁) | ((0, 1), (0, −1), (0, −1)) | 1 3 3 2/3          | (0, −1) |
| (r₁r₂, r₁r₂, r₂r₁r₂) | ((−1, 0), (−1, 1), (0, −1)) | 2 2 3 3/3          | (−2, 0) |
| (r₂, r₂r₁r₂, r₂r₁r₂) | ((1, −1), (0, −1), (0, −1)) | 1 3 3 3/3          | (1, −3) |
| (r₁r₂, r₂r₁r₂, r₂r₁r₂) | ((−1, 0), (0, −1), (0, −1)) | 2 3 3 3/3          | (−1, −2) |

Table 1. Minimal defining chains and corresponding standard tableaux for the $A₂$ representation of highest weight $2\omega₁ + \omega₂ =: (2, 1)$.

If $\mu = \nu^1 + \nu^2 + \ldots + \nu^n \in P_\geq(X_r)$, we also define

$$R_i(\mu) := \nu^{n-i+1} + \nu^{n-i+2} + \ldots + \nu^n, \text{ and } L_i(\mu) := \nu^1 + \nu^2 + \ldots + \nu^{n-i}. \quad (3.7)$$
4. Littelmann’s Generalization of the Classical Littlewood-Richardson Rule

From (1.10) and the Weyl character formula (1.11), we have

\[
\sum_{w \in W} (\det w)e^{w(\lambda + \rho)}c_{\mu} = \sum_{\nu \in P_\geq(X_r)} T_{\lambda,\mu}^{\nu} \sum_{w \in W} (\det w)e^{w(\nu + \rho)}. \tag{4.1}
\]

Defining the linear operator \( \Theta \) by

\[
\Theta(e^\lambda) := \sum_{w \in W} (\det w)e^{w(\lambda + \rho)}, \tag{4.2}
\]

this can be rewritten as

\[
\Theta(e^{\lambda}c_{\mu}) = \Theta \left( \sum_{\nu \in P_\geq(X_r)} T_{\lambda,\mu}^{\nu}e^{\nu} \right). \tag{4.3}
\]

What Littelmann showed was that the left hand side simplifies because it contains terms of the form

\[
\Theta(e^\kappa d_\alpha(e^\gamma)) = 0, \quad \text{for } r_\alpha v < v \in W, \quad \kappa \cdot \alpha^\vee = 0, \quad \alpha \in S(X_r). \tag{4.4}
\]

To see why these contributions vanish, first note that by the definition (4.2), we get

\[
\Theta(e^{w \cdot \gamma}) = (\det w) \Theta(e^\gamma), \quad \forall w \in W, \tag{4.5}
\]

so that

\[
\Theta(D_\alpha(e^\gamma)) = \Theta(e^\gamma), \quad \forall \alpha \in S(X_r). \tag{4.6}
\]

Eqn. (2.5) above then gives

\[
\Theta(e^\kappa D_\alpha(e^\gamma)) = \Theta(D_\alpha(e^\kappa D_\alpha(e^\gamma))) = \Theta(D_\alpha(e^\kappa)D_\alpha(e^\gamma)) = \Theta(D_\alpha(e^\kappa)e^\gamma), \tag{4.7}
\]

for any simple root \( \alpha \in S(X_r) \). This implies

\[
\Theta(e^\kappa d_\alpha(e^\gamma)) = \Theta(d_\alpha(e^\kappa)e^\gamma), \tag{4.8}
\]

by the definition (2.7) of the operators \( d_\alpha \). From (2.12) we then get

\[
\Theta(e^\kappa d_v(e^\gamma)) = \Theta(d_\alpha(e^\kappa)d_{r_\alpha v}(e^\gamma)), \quad \text{for } r_\alpha v < v. \tag{4.9}
\]

But since \( d_\alpha(e^\kappa) = 0 \) for \( \kappa \cdot \alpha^\vee = 0 \), (4.4) is proved.
The vanishing contributions to the tensor product $R(\lambda) \otimes R(\mu)$ encoded in (4.4) can be identified easily using the MDCs. From (3.4), the left hand side of (4.3) contains expressions of the form

$$\Theta \left( e^{\lambda + p_\mu[c]} \right) = \Theta \left( e^{[\lambda + p_{R_i(\mu)}[R_i(\nu)] + p_{L_i(\mu)}[L_i(\nu)]} \right),$$

where $1 \leq i \leq n$. Suppose that for the particular MDC under consideration

$$\lambda + p_{R_i(\mu)}[R_i(\nu)] \in P_{\geq}(X_r), \text{ for } i = 1, \ldots, \ell < n;$$

$$\lambda + p_{R_{\ell+1}[\mu]}(R_{\ell+1}(c)) \notin P_{\geq}(X_r).$$

For $X_r = A_r$, we have $w \nu \cdot \alpha^\vee \in \{-1, 0, 1\}$, for all $\alpha \in S(X_r)$ and $\nu \in F(X_r)$. With $c = (w_1, \ldots, w_n)$ and $p_\mu(c) = (w_1 \nu^1, \ldots, w_n \nu^n)$, (4.11) implies that $(\lambda + p_{R_i(\mu)}[R_i(\nu)]) \cdot \alpha^\vee = 0$ and $(\lambda + p_{R_{\ell+1}[\mu]}[R_{\ell+1}(c)]) \cdot \alpha^\vee = -1$, for some simple root $\alpha \in S(X_r)$. This means that $(w_{n-\ell} \nu^{\alpha-\ell}) \cdot \alpha^\vee = -1$ and that $r_\alpha w_{n-\ell} < w_{n-\ell}$.

By (3.3) then, the expression in (4.10) is contained in

$$\Theta \left( e^{[\lambda + p_{R_i(\mu)}[R_i(\nu)]} \mu \cdot \nu \cdot \alpha^\vee \right) = \Theta \left( e^{\mu - R_i(\mu)} \right),$$

which vanishes, by (4.4). The Bruhat order obeyed by the elements of minimal defining chains, and their connection with Demazure characters, ensures that all terms in (4.10) obeying (4.11) are present in the left hand side of (4.3), if one is.

A minimal defining chain $c$ is called $\lambda$-dominant if

$$\lambda + p_{R_i(\mu)}[R_i(\nu)] \in P_{\geq}(X_r), \text{ for all } 0 \leq i \leq n.$$

So, only the $\lambda$-dominant MDCs can contribute to the left hand side of (4.3). Define $C_{\lambda,\mu}[\sigma]$ to be the set of $\lambda$-dominant MDCs of $\mu$-weight $\sigma$. Then once the terms of the form (4.12) are eliminated from the left the hand side of (4.3), we are left with

$$\Theta \left( \sum_{\nu \in P_{\geq}(X_r)} \dim C_{\lambda,\mu}[\nu - \lambda] \ e^\nu \right) = \Theta \left( \sum_{\nu \in P_{\geq}(X_r)} T_{\lambda,\mu}^\nu e^\nu \right).$$

But it is easy to see that

$$\Theta \left( \sum_{\mu \in P_{\geq}(X_r)} a_\mu e^\mu \right) = \Theta \left( \sum_{\nu \in P_{\geq}(X_r)} b_\nu e^\nu \right) \Rightarrow a_\mu = b_\mu \ \forall \mu \in P_{\geq}(X_r) .$$
Littelmann’s generalization of the Littlewood-Richardson rule then follows:

\[ T_{\lambda, \mu}^\nu = |C_{\lambda, \mu}[\nu - \lambda]|. \] (4.16)

In the case \( X_r = A_r \), the classical Littlewood-Richardson rule is recovered. Let a standard tableau be called \( \lambda \)-dominant if the corresponding MDC is \( \lambda \)-dominant. To find the tensor product coefficient \( T_{\lambda, \mu}^\nu \), one simply counts the number of \( \lambda \)-dominant standard tableaux of shape \( \mu \) and weight \( \nu - \lambda \). These can be found by taking each standard tableau of shape \( \mu \) and weight \( \nu - \lambda \) and adding the weights of its columns, from right to left, to \( \lambda \). If the addition of the weight of one of the columns results in a non-dominant weight, no contribution results. If this does not happen, the standard tableau contributes 1 to \( T_{\lambda, \mu}^\nu \).

The rule is most easily implemented by adding the columns directly to the Young tableau of shape \( \lambda \), to form a mixed tableau. The shape of the mixed tableau is similar to the shape of a Young tableau. One takes the columns of the standard tableau, and any box \( \; \), e.g., to the 3rd row of the Young tableau, etc. If the addition in this manner of any column results in a mixed tableau of non-dominant shape, the standard tableau does not contribute.

For example, consider the decomposition of the tensor product of \( A_2 \) representations: \( R(2, 0) \otimes R(2, 1) = R(0, 3) \oplus R(1, 1) \oplus R(2, 2) \oplus R(3, 0) \oplus R(4, 1) \), where \( R(\lambda_1, \lambda_2) \) means \( R(\lambda) \) with \( \lambda = \lambda_1 \omega^1 + \lambda_2 \omega^2 \). We have \( T_{(2,0),(2,1)}^{(3,0)} = 1 \) while there are two standard tableaux of shape \((2, 1)\) and of the appropriate weight \((3, 0) - (2, 0) = (1, 0)\):

\[
\begin{array}{ccc}
1 & 1 & 2 \\
3 & & \\
\end{array}, \quad
\begin{array}{ccc}
1 & 1 & 1 \\
3 & & \\
\end{array}, \quad
\begin{array}{ccc}
2 & & \\
3 & & \\
\end{array}.
\] (4.17)

Only the first standard tableau contributes because by adding its columns, from right to left, to the Young tableau \( \begin{array}{ccc}
3 & & \\
\end{array} \) of shape \((2, 0)\), we obtain the following sequence of mixed tableaux:

\[
\begin{array}{ccc}
2 & & \\
\end{array}, \quad
\begin{array}{c}
1 \\
\end{array}, \quad
\begin{array}{c}
1 \\
\end{array}, \quad
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}.
\] (4.18)

each of which has dominant shape. In attempting to do the same for the second standard tableau, we immediately encounter

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\end{array}.
\] (4.19)

This mixed tableau is clearly not dominant, and so does not contribute to the tensor product.
Only five tableaux from Table 1 contribute to the tensor product $R(2,0) \otimes R(2,1)$. The rest must result in terms of the form (4.4). For example, the five tableaux

\[
\begin{array}{cccc}
1 & 1 & 3 & 2 \\
2 & 1 & 2 & 3 \\
3 & 1 & 1 & 3 \\
2 & 1 & 3 & 3 \\
3 & 1 & 3 & 3 \\
\end{array}
\]

(4.20)

furnish together such a form, with $\kappa = (2,0)$, $\gamma = (2,1)$, and $\nu = r_2r_1$.

The form of the Littlewood-Richardson rule just described differs from the original, in which the only numbered tableau that enters is

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 \\
\end{array}
\]

(4.21)

But the translation is simple, and is described explicitly in [13]. The version involving standard tableaux is the most convenient, since it can be directly related to vectors in representations of $X_r$.

5. Modified Littlewood-Richardson Rules for Fusions

In this section we will follow Littelmann’s derivation of a generalized Littlewood-Richardson rule [12], as outlined in the previous section, and attempt to find a combinatorial rule for affine fusions.

From (1.3) and (1.7), we can follow steps in the previous section to arrive at

\[
\Theta \left( e^{\lambda \cdot \text{ch}_\mu} \right)(\sigma) = \Theta \left( \sum_{\nu \in P_{\geq}(X_{r,k})} N_{\lambda,\mu}^{\nu} e^{\nu} \right)(\sigma),
\]

(5.1)

where $\lambda, \mu, \sigma$ are elements of $P_{\geq}(X_{r,k})$.

From (1.19) it is clear that the Weyl group $\hat{W}$ of $X_{r,k}$ must be implicated. Using the action (1.17) of the $\hat{W}$ generator $r_{\alpha_0}$ that is additional to the generators of $W$, it is simple to show that

\[
\Theta \left( e^{w \cdot \gamma} \right)(\sigma) = (\det w) \Theta \left( e^\gamma \right)(\sigma), \quad \forall w \in \hat{W},
\]

(5.2)

when $\sigma, \gamma \in P(X_r)$. But another ingredient in Littelmann’s derivation of a generalized Littlewood-Richardson rule is Demazure character formulas for $\text{ch}_\mu(\sigma)$ that involve the Bruhat order on $W$. We know of no way to write a similar formula for $\text{ch}_\mu(\sigma)$ involving the Bruhat order on the full affine Weyl group $\hat{W}$. Formulas involving the Bruhat order on subgroups of $\hat{W}$ can be written, however, as we now demonstrate.
In order to use the affine Weyl symmetry (5.2), the weights involved must have level \( k \): \( \delta \cdot \gamma = k \) is required. If two such weights are added, however, we get level \( 2k \). Let \( \omega^0 \) be the zeroth affine fundamental weight, so that \( \delta \cdot \omega^0 = 1 \). If \( \xi, \phi \) have level \( k \), i.e. \( \delta \cdot \phi = \delta \cdot \xi = k \), define \( \tilde{\phi} := \phi - k\omega^0 \). Then \( \delta \cdot (\xi + \tilde{\phi}) = k \). Notice that \( \sigma\omega^0 = 1 \), so that we are able to interpret the weight \( \mu \) in the left hand side of (5.1) above as \( \tilde{\mu} \), in order to exploit the affine symmetry (5.2).

With \( w = r_{\alpha_0} \) in (5.2), it is easy to show that

\[
\Theta \left( d_{\alpha_0}(e^\gamma) \right)(\sigma) = 0,
\]

for \( \delta \cdot \gamma = k \), with \( \gamma \in P(X_r) \), if \( d_{\alpha_0} \) is defined by (2.7) with \( \alpha = \alpha_0 \). From this

\[
\Theta \left( e^\gamma d_{\alpha_0}(e^{\tilde{\phi}}) \right)(\sigma) = 0, \quad \text{if} \quad \gamma \cdot \alpha_0 = k - \theta \cdot \gamma = 0
\]

follows, since \( \gamma + r_{\alpha_0} \tilde{\phi} = r_{\alpha_0} (\gamma + \tilde{\phi}) \) in that case. Finally then, if we define Demazure operators \( d_u \) for any \( u \in \hat{W} \) by the reduced decomposition of \( u \), we find

\[
\Theta \left( e^\gamma d_u(e^{\tilde{\phi}}) \right)(\sigma) = 0, \quad \text{if} \quad k - \theta \cdot \gamma = 0, \quad \text{and} \quad r_{\alpha_0} u \prec u ,
\]

where now \( \prec \) indicates the Bruhat order on \( \hat{W} \).

For \( X_r = A_r \), let \( W^m \) denote the subgroup of \( \hat{W} \) generated by all the affine primitive reflections except \( r_{\alpha_m} \):

\[
W^m = \langle r_{\alpha_i} \mid i \in \{0, 1, 2, \ldots, r\} \setminus m \rangle.
\]

So all \( W^m \cong W \), and \( W^0 = W \). Denote by \( x \in W \) the Weyl group element that permutes the simple roots of \( A_r \) and \( -\theta \) in a cyclic manner:

\[
x(-\theta, \alpha_1, \alpha_2, \ldots, \alpha_r) = (\alpha_1, \alpha_2, \ldots, \alpha_r, -\theta).
\]

Using these definitions we can write a modified Demazure character formula

\[
\operatorname{ch}_{\tilde{\mu}}(\sigma) = \sum_{v \in W^m} d_v(e^{x^m\tilde{\mu}})(\sigma),
\]

involving the subgroup \( W^m \) of the affine Weyl group \( \hat{W} \).

Defining chains and corresponding tableaux can also be associated to the affine subgroups \( W^m \). The highest weight \( \mu \) of an integrable representation \( R(\mu) \) can be expressed
uniquely as a sum \( \mu = x^m \nu_1 + x^m \nu_2 + \ldots + x^m \nu_n \), where \( \nu_i \in F(A_r) \), and the order \( \nu_i \geq \nu_{i+1} \) has been fixed. A sequence \( c = (w_1, w_2, \ldots, w_n) \) of elements of \( W^m \) can be associated with such a weight. If \( w_j \leq w_{j+1} \), the chain is a defining chain. If it is also the minimal defining chain corresponding to the following sequence of weights:

\[
P^\mu_m(c) := (w_1 x^m \nu_1, w_2 x^m \nu_2, \ldots, w_n x^m \nu_n), \tag{5.9}
\]

we will call it a \( m \)-MDC.

Define the \( \mu \)-weight \( p_{\mu,m}[c] \) of the \( m \)-MDC \( c \) as

\[
p_{\mu,m}[c] = p_{\mu,m}[(w_1, w_2, \ldots, w_n)] := w_1 x^m \nu_1 + \ldots + w_n x^m \nu_n. \tag{5.10}
\]

Let \( C^m_\mu[\phi] \) be the set of \( m \)-MDCs with \( \mu \)-weight \( \phi \), and let \( C^m_\mu(w) \) denote the set of \( m \)-MDCs with last element equal to \( w \in W^m \). The weights of a representation \( R(\mu) \) and those of \( m \)-MDCs are related:

\[
\text{mult}_\mu(\phi) = |C^m_\mu[\phi]|, \tag{5.11}
\]

so that

\[
\text{ch}_\mu(\sigma) = \sum_{\phi \in P(\mu)} \sum_{c \in C^m_\mu[\phi]} e^{p_{\mu,m}[c]}(\sigma). \tag{5.12}
\]

The relevant connection with Demazure operators is given by

\[
d_w(e^\lambda)(\sigma) = \sum_{c \in C^w_\lambda(w)} e^{p_{\lambda,m}[c]}(\sigma). \tag{5.13}
\]

The previous two equations are valid for any \( \sigma \in P(A_r) \), and the previous three are consequences of the Weyl invariance of characters.

A minor variation of the usual standard tableaux associated to \( X_r = A_r \) can encode the \( m \)-MDCs. In terms of the orthonormal basis \( \{e_i\} \) of \( \mathbf{R}^{r+1} \) related to the usual standard tableaux, the action of \( x \in W \) is very simple:

\[
x(e_1, e_2, \ldots, e_{r+1}) = (e_2, e_3, \ldots, e_{r+1}, e_1). \tag{5.14}
\]

Using the same ordering of fundamental weights: \( \omega^r > \omega^{r-1} > \cdots > \omega^1 \), we can define \( m \)-standard tableaux as the numbered tableaux of shape \( \mu \) with numbers appearing in the order

\[
(m + 1, m + 2, \ldots, r + 1, 1, 2, \ldots, m), \tag{5.15}
\]
from left to right in its rows, and from top to bottom with no repetitions in its columns. Notice that 0-standard tableaux are the usual standard tableaux.

As an example, consider again the $A_2$ representation of highest weight $2\omega^1 + \omega^2 = (2,1)$. For $A_2$, $x = r_{\alpha_1}r_{\alpha_2}$, and the corresponding 1-MDCs, and 1-standard tableaux are shown in Table 2.

The derivation of a modified Littlewood-Richardson rule from (5.1),(5.5),(5.12) and (5.13) follows the derivation of the classical Littlewood-Richardson rule outlined in the previous section. To state the result, we define $c$, a $m$-MDC of length $n$, to be $(\lambda,k)$-dominant if
\[
\lambda + p_{R_i(\mu)} [R_i(c)] \in P_{\geq}(X_{r,k}) , \quad \text{for all } 0 \leq i \leq n .
\]
Let $C_{\lambda,\mu}^{m,k}[\sigma]$ denote the set of $(\lambda,k)$-dominant $m$-MDCs of $\mu$-weight $\sigma$. The modified Littlewood-Richardson rule for $A_{r,k}$ fusions is then:
\[
N_{\lambda,\mu}^\nu \leq |C_{\lambda,\mu}^{m,k}[\nu - \lambda]| \quad \forall \ 0 \leq m \leq r .
\]

Let a $m$-standard tableau be called $(\lambda,k)$-dominant if the corresponding $m$-MDC is $(\lambda,k)$-dominant. To treat the fusion coefficient $N_{\lambda,\mu}^\nu$ for $A_{r,k}$, one simply counts the number of $(\lambda,k)$-dominant $m$-standard tableaux of shape $\mu$ and weight $\nu - \lambda$. These can be found by taking each $m$-standard tableau of shape $\mu$ and weight $\nu - \lambda$ and adding the weights of its columns, from right to left, to $\lambda$. If the addition of the weight of one of the columns results in a non-dominant weight $\not\in P_{\geq}(A_{r,k})$, no contribution results. If this does not happen, the $m$-standard tableau contributes 1 to the upper bound on $N_{\lambda,\mu}^\nu$.

The modified rule is easily implemented by adding the columns directly to the Young tableau of shape $\lambda$, to form a mixed tableau. If the addition in this manner of any column results in a mixed tableau of shape $\not\in P_{\geq}(A_{r,k})$, the $m$-standard tableau does not contribute. As an example, consider the $A_{2,k=3}$ fusion rule: $R(2,0) \otimes_3 R(2,1) = R(1,1) \oplus R(0,3)$. Choosing $m = 1$ as in Table 2, we can explain $N_{(2,0),(2,1)}^{(3,0)} = 0$ (for $k = 3$). From Table 2 we read that the two 1-standard tableaux of shape $(2,1)$ and of the appropriate weight $(3,0) - (2,0) = (1,0)$ are
\[
\begin{array}{ccc}
2 & 3 & 1 \\
1 & & \\
3 & & \\
\end{array}, \quad \begin{array}{ccc}
2 & 1 & 1 \\
& 3 & \\
& & \\
\end{array}.
\]

---

3 Other tableaux, with orderings that are non-cyclic permutations of (5.15), can also be defined. Their relation to fusion coefficients, however, is not clear.
| 1-MDC                      | weight sequence                | tableau | weight |
|---------------------------|--------------------------------|---------|--------|
| $(id, id, id)$            | $((-1, 0), (-1, 1), (-1, 1))$ | 2 2 2 3 | $(-3, 2)$ |
| $(r_0, r_0, r_0)$         | $((0, 1), (-1, 1), (-1, 1))$  | 2 2 2 1 | $(-2, 3)$ |
| $(id, id, r_2)$           | $((-1, 0), (-1, 1), (0, -1))$ | 2 2 3 3 | $(-2, 0)$ |
| $(id, r_2, r_2)$          | $((-1, 0), (0, -1), (0, -1))$ | 2 3 3 3 | $(-1, -2)$ |
| $(r_0, r_0, r_2 r_0)$     | $((0, 1), (-1, 1), (0, -1))$  | 2 2 3 1 | $(0, -1)$ |
| $(id, id, r_0 r_2)$       | $((-1, 0), (-1, 1), (1, 0))$  | 2 2 1 3 | $(-1, 1)$ |
| $(id, r_2, r_0 r_2)$      | $((-1, 0), (0, -1), (1, 0))$  | 2 3 1 3 | $(0, -1)$ |
| $(r_0, r_2 r_0, r_2 r_0)$ | $((0, 1), (0, -1), (0, -1))$  | 2 3 3 1 | $(1, -3)$ |
| $(r_0, r_0, r_0 r_2)$     | $((0, 1), (-1, 1), (1, 0))$  | 2 2 1 1 | $(0, 2)$ |
| $(r_0, r_2 r_0, r_0 r_2 r_0)$ | $((0, 1), (0, -1), (1, 0))$ | 2 3 1 1 | $(1, 0)$ |
| $(id, r_0 r_2, r_0 r_2)$  | $((-1, 0), (1, 0), (1, 0))$  | 2 1 1 3 | $(1, 0)$ |
| $(r_2 r_0, r_2 r_0, r_0 r_2 r_0)$ | $((1, -1), (0, -1), (1, 0))$ | 3 3 1 1 | $(2, -2)$ |
| $(r_0, r_0 r_2, r_0 r_2)$ | $((0, 1), (1, 0), (1, 0))$  | 2 1 1 1 | $(2, 1)$ |
| $(r_2 r_0, r_0 r_2 r_0, r_0 r_2 r_0)$ | $((1, -1), (0, -1), (0, -1))$ | 3 1 1 1 | $(3, -1)$ |

**Table 2.** Minimal defining chains and corresponding 1-standard tableaux for the $A_2$ representation of highest weight $2\omega^1 + \omega^2 =: (2, 1)$, using the affine Weyl subgroup $W^1 = < r_0, r_2 >$.

Neither of these tableaux contribute. The first does not, because by adding its columns, from right to left, to the Young tableau of shape $(2, 0)$, we obtain the following
sequence of mixed tableaux:

\[
\begin{array}{c}
\begin{array}{c}
1 \\
3 \\
111
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
111
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
2 \\
3 \\
111
\end{array}
\end{array},
\end{array}
\]

the second of which has shape \( \not\in P_{\geq}(A_{2,3}) \). For the second standard tableau, we find

\[
\begin{array}{c}
\begin{array}{c}
1 \\
111
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
111
\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
1 \\
3 \\
111
\end{array}
\end{array},
\end{array}
\]

Again, the second mixed tableau has a shape \( \not\in P_{\geq}(A_{2,3}) \), this time because it has a number of columns (of less than 3 boxes) greater than \( k = 3 \). In a similar fashion, the complete fusion rule is found correctly, at level 3 and all higher levels.

If we compare the non-combinatorial expressions for tensor product coefficients (1.21), and for fusion coefficients (1.20), it is clear that the modified Littlewood-Richardson rule (5.17) calculates the right hand side of

\[
N_{\lambda,\mu}^{\nu} \leq T_{\lambda,\mu}^{(m)\nu} := \sum_{w \in W^m} (\text{det} w) \text{ mult}_\mu (w.\nu - \lambda). \tag{5.21}
\]

We conjecture that

\[
N_{\lambda,\mu}^{\nu} = T_{\lambda,\mu}^{(m)\nu} \text{ when mult}_\mu (r_{\alpha_m},\nu - \lambda) = 0. \tag{5.22}
\]

It is not hard to find an example when the bound (5.17) is not saturated. Consider the \( A_{4,4} \) fusion \( R(1,1,1,1) \otimes_4 R(1,1,1,1) \), with decomposition containing four copies of \( R(1,1,1,1) \), so that \( N_{(1,1,1,1)}^{(1,1,1,1)} = 4 \). The modified Littlewood-Richardson rule gives \( T_{(1,1,1,1),(1,1,1,1)}^{(m)\nu} = 16, 6, 5, 6 \), for \( m = 0, 1, 2, 3, 4 \), respectively.

The \( A_{4,4} \) fusion coefficient \( N_{(1,1,1,1)}^{(1,1,1,1)} = 4 \) can be recovered from the Littlewood-Richardson rule, however, if level-rank duality [17] is used. But this fix is not general. If the level is increased by one, it fails: the \( A_{4,5} \) fusion coefficient \( N_{(1,1,1,1),(1,1,1,1)}^{(1,1,1,1)} = 14 \) cannot be recovered using (5.17) and level-rank duality. The problem is deeper than that, as we discuss in the next section.

We now show that the modified Littlewood-Richardson rule (5.17) gives a known upper bound on affine \( A_r \) fusion coefficients. Let \( J \) denote the diagram outer automorphism of \( P_{\geq}(A_{r,k}) \), with action

\[
J\lambda = J \left( \sum_{i=1}^{r} \lambda_i \omega^i \right) = \left( k - \sum_{i=1}^{r} \lambda_i \right) \omega^1 + \sum_{i=2}^{r} \lambda_{i-1} \omega^i. \tag{5.23}
\]
What we will show is that
\[ T^{\nu}_{\lambda,\mu} = T^{(0)\nu}_{\lambda,\mu} = T^{(m)J^m\nu}_{J^m\lambda,\mu}. \] (5.24)

From (1.20) we can write
\[ T^{(m)J^m\nu}_{J^m\lambda,\mu} = \sum_{w \in W^m} (\det w) \text{mult}_{\mu} (J^m \left((J^{-m} w J^m)(\nu + \rho) - (\lambda + \rho)\right)), \] (5.25)

using \(J^m \rho = \rho\). But for \(w \in W^m\), we have \(J^{-m} w J^m \in W\). Furthermore, if \(\sigma\) is any weight satisfying \(\sigma \cdot \delta = 0\), then \(J^m \sigma = x^m \sigma\). The Weyl invariance of the multiplicities then establishes (5.24).

The bound computed by (5.17) is just (5.21), or
\[ N^{\nu}_{\lambda,\mu} \leq T^{J^{-m}\nu}_{J^{-m}\lambda,\mu}. \] (5.26)

But it is well known that
\[ N^{\nu}_{\lambda,\mu} = N^{J^{-m-n}\nu}_{J^{-m-n}\lambda,\mu}, \] (5.27)

for all \(m, n\). And with \(N^{\nu}_{\lambda,\mu} \leq T^{\nu}_{\lambda,\mu}\), we obtain
\[ N^{\nu}_{\lambda,\mu} \leq T^{J^{-m-n}\nu}_{J^{-m-n}\lambda,\mu}. \] (5.28)

This last result is even stronger than the upper bound (5.21). It was originally conjectured to be saturated for all triples \(\lambda, \mu, \nu \in P_{\geq}(A_r, k)\), for some choice of \(m\) and \(n\).[6]

6. Discussion

The classical Littlewood-Richardson rule for \(A_r\) tensor products, and generalizations for other simple Lie algebras (see (4.16)), can be derived from Demazure character formulas, as Littelmann demonstrated [12]. Here we wrote Demazure character formulas (5.8) for ratios of elements of the modular \(S\) matrices of affine Kac-Moody algebras. These were then used in the Verlinde formula in an attempt to derive a combinatorial rule for affine \(A_r\) fusion rules.

The resulting modified Littlewood-Richardson rule (5.17), however, only provides an upper bound on the fusion coefficients. The basic reason is the use of the subgroups \(W^m\) of the affine Weyl group in the Demazure character formula (5.8), instead of the full affine Weyl group and its Bruhat order. The cancellations in (1.20) can be ordered.
systematically using the Bruhat order on the finite Weyl group $W$, so that a combinatorial rule results from (2.9) and (3.5). The affine Bruhat order is needed, however, to order all the cancellations in (1.21) in a similar way. But we were unable to write a character formula for the relevant quantities (1.7) that involves the Bruhat order on the full affine Weyl group.

Perhaps such a character formula can be written. On the other hand, a different derivation of the Littlewood-Richardson rule suggests it may be difficult. Let $R(\mu; \phi)$ denote the subspace of the representation $R(\mu)$ consisting of vectors of weight $\phi$. Then (see [18], Thm. 4, sect. 78, for example)

$$T^\nu_{\lambda, \mu} = \dim \left\{ V \in R(\mu; \nu - \lambda) \mid E(-\alpha_i)^{\nu_i+1}V = 0 \ \forall i \in \{1, 2, \ldots, r\} \right\}, \quad (6.1)$$

where $E(-\alpha_i)$ is the generator of $X_r$ in the Cartan-Weyl basis that decreases the weight of a vector by $\alpha_i$. So-called “good” bases for the representations of $X_r$ exist [19] that result in combinatorial rules when used in (6.1). In particular, such a good basis for $X_r = A_r$ can be labelled by the standard tableaux (or, equivalently, by Gelfand-Tsetlin patterns [20]), so that the classical Littlewood-Richardson rule results. The modified Littlewood-Richardson rule (5.17) corresponds to

$$T^{(m)}_{\lambda, \mu} = \dim \left\{ V \in R(\mu; \nu - \lambda) \mid E(-\alpha_i)^{\nu_i+1}V = 0 \ \forall i \in \{0, 1, 2, \ldots, r\} \setminus m \right\}, \quad (6.2)$$

where we identify $E(-\alpha_0) := E(+\theta)$. A Weyl-transformed version of the good basis of [20] exists, and the rule involving the $m$-standard tableaux results. However, a natural generalization of the last two relations is

$$N^\nu_{\lambda, \mu} = \dim \left\{ V \in R(\mu; \nu - \lambda) \mid E(-\alpha_i)^{\nu_i+1}V = 0 \ \forall i \in \{0, 1, 2, \ldots, r\} \right\}. \quad (6.3)$$

This equation was conjectured in [21], and was motivated by the depth rule of [2]. But no basis of representations of $X_r$ exists that is good with respect to (3.3) [21], as can be verified explicitly for $X_r = A_r$ using the results of [21].

It seems then that a different approach is required, one relating fusions more directly to affine representations. This would allow the use of good bases for affine representations, or formulas for their characters involving the affine Bruhat order. In this context, we should mention the more recent work of Littelmann: in [14] the generalized Littlewood-Richardson rule of (5.17) was further generalized to a rule for tensor products of all symmetrizable Kac-Moody algebras, and the language of minimal defining chains was upgraded to one of certain paths.
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