Refined inequalities on weighted logarithmic mean

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Abstract. Inspired by the recent work by R.Pal et al., we give further refined inequalities for a convex Riemann integrable function, applying the standard Hermite-Hadamard inequality. Our approach is different from their one in [9]. As corollaries, we give the refined inequalities on the weighted logarithmic mean and weighted identric mean. Some further extensions are also given.

Keywords: Weighted logarithmic mean, weighted identric mean, convex function, Hermite-Hadamard inequality and operator inequality

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1 Introduction

The inequalities on means attract many mathematicians for its developments. See [6] for example. Recently, in [9, Theorem 2.2], the weighted logarithmic mean was introduced properly and the inequalities among weighted means were shown as

\[ a\tilde{\ast}_v b \leq L_v(a, b) \leq a\nabla_v b, \tag{1} \]

where the weighted geometric mean \( a\tilde{\ast}_v b := a^{1-v}b^v \), the weighted arithmetic mean \( a\nabla_v b := (1-v)a + vb \) and the weighted logarithmic mean [9]:

\[ L_v(a, b) := \frac{1}{\log a - \log b} \left( \frac{1-v}{v}(a - a^{1-v}b^v) + \frac{v}{1-v}(a^{1-v}b^v - b) \right) \tag{2} \]

for \( a, b > 0 \) and \( v \in (0,1) \). We easily find that \( L_{1/2}(a, b) = \frac{a - b}{\log a - \log b} \), \( (a \neq b) \), with \( L_{1/2}(a, a) := a \). This is the so-called logarithmic mean. We also find that \( \lim_{v \to 0} L_v(a, b) = a \) and \( \lim_{v \to 1} L_v(a, b) = b \). Thus the inequalities given in [1] recover the well-known relations:

\[ \sqrt{ab} \leq \frac{a - b}{\log a - \log b} \leq \frac{a + b}{2}, \quad (a, b > 0). \]

R.Pal et al. obtained the inequalities given in [1] by their general result given in [9, Theorem 2.1] which can be regarded as the generalization of the famous Hermite-Hadamard inequality with weight \( v \in [0,1] \):

\[ f(a\nabla_v b) \leq C_{f,v}(a, b) \leq f(a)\nabla_v f(b) \tag{3} \]
where
\[ C_{f,v}(a, b) := \left( \int_0^1 f(a \nabla_v b) \, dt \right) \nabla_v \left( \int_0^1 f \left( (1 - v)(b - a)t + a \nabla_v b \right) \, dt \right) \]
(4)
for a convex Riemann integrable function, \( a, b > 0 \) and \( v \in [0, 1] \). By elementary calculations, we find that the inequalities given in (3) recover the standard Hermite-Hadamard inequalities:
\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2} \]
(5)

In this paper, we give a refinement of the inequalities given in (3) and as its consequence, we imply refined inequalities on the weighted logarithmic mean.

2 Main results

We firstly give the refined inequalities for (3) by repeating use of the standard Hermite-Hadamard inequalities given in (5).

**Theorem 2.1.** For every convex Riemann integrable function \( f : [a, b] \to \mathbb{R} \) and \( v \in [0, 1] \), we have
\[ f(a \nabla_v b) \leq R_{f,v}^{(1)}(a, b) \leq C_{f,v}(a, b) \leq R_{f,v}^{(2)}(a, b) \leq f(a) \nabla_v f(b), \]
(6)
where
\[ R_{f,v}^{(1)}(a, b) := f(a \nabla_v b) \nabla_v f(a \nabla_v \circ f) \]
(7)
and
\[ R_{f,v}^{(2)}(a, b) := (f(a) \nabla_v f(b)) \nabla_1/2 (f(a) \nabla_v b) \].
(8)

**Proof.** Applying the standard Hermite-Hadamard inequalities (5) on two intervals \([a, (1 - v)a + vb]\) and \([(1 - v)a + vb, b]\), we obtain respectively
\[ f \left( \frac{(2 - v)a + vb}{2} \right) \leq \frac{1}{v(b - a)} \int_a^{(1-v)a+vb} f(t) \, dt \leq \frac{f(a) + f((1 - v)a + vb)}{2} \]
(9)
and
\[ f \left( \frac{(1 - v)a + (1 + v)b}{2} \right) \leq \frac{1}{(1 - v)(b - a)} \int_{(1-v)a+vb}^b f(t) \, dt \leq \frac{f(b) + f((1 - v)a + vb)}{2}. \]
(10)

Multiplying \((1 - v)\) and \(v\) to the both sides in (9) and (10) respectively and summing each side, we obtain
\[ R_{f,v}^{(1)}(a, b) \leq \frac{1 - v}{v(b - a)} \int_a^{(1-v)a+vb} f(t) \, dt + \frac{v}{(1 - v)(b - a)} \int_{(1-v)a+vb}^b f(t) \, dt \leq R_{f,v}^{(2)}(a, b), \]
(11)
which is equivalent to
\[ R_{f,v}^{(1)}(a, b) \leq C_{f,v}(a, b) \leq R_{f,v}^{(2)}(a, b), \]
(12)
by replacing the variables such as \( t := v(b - a)s + a \) in the first term and \( t := (1 - v)(b - a)u + (1 - v)a + vb \) in the second term of the integral parts in (11).

Finally we estimate \( R_{f,v}^{(1)}(a, b) \) and \( R_{f,v}^{(2)}(a, b) \). Since the function \( f \) is convex, we have
\[ R_{f,v}^{(1)}(a, b) \geq f \left( \left( \frac{(1 - v)(2 - v) + v(1 - v)}{2} \right) a + (v(1 - v) + v(1 + v)) b \right) = f(a \nabla_v b) \]
and
\[ R_{f,v}^{(2)}(a, b) \leq (f(a) \nabla_v f(b)) \nabla_1/2 (f(a) \nabla_v f(b)) = f(a) \nabla_v f(b). \]
Thus we completed the proof. \(\square\)
Corollary 2.2. For $a, b > 0$ and $v \in (0, 1)$, we have
\[
a^*_v b \leq \left( a^*_v b \right) \nabla_v \left( a^*_v b \right) \leq L_v(a, b) \leq (a \nabla_v b) \nabla_{1/2} (a^*_v b) \leq a \nabla_v b. \tag{13}
\]

Proof. Applying the convex function $f(t) := e^t$ in Theorem 2.1, we have for $b \geq a > 0$
\[
e^{(1-v)a+vb} \leq (1-v)e^{(2-v)a+vb} + ve^{(1-v)a+(1+v)b} \leq (1-v) \int_0^1 e^{v(b-a)t+a} \, dt
\]
\[+v \int_0^1 e^{(1-v)(b-a)t+(1-v)a+vb} \, dt \leq \frac{(1-v)e^a + ve^b + e^{(1-v)a+vb}}{2} \leq (1-v)e^a + ve^b.
\]

By elementary calculations, we have
\[
(1-v) \int_0^1 e^{v(b-a)t+a} \, dt + v \int_0^1 e^{v(b-a)t+(1-v)a+vb} \, dt
\]
\[= \frac{1-v}{v(b-a)} \left(e^{(1-v)a+vb} - e^a\right) + \frac{v}{(1-v)(b-a)} \left(e^b - e^{(1-v)a+vb}\right).
\]
Replacing $e^a$ and $e^b$ with $a$ and $b$ respectively, we obtain the inequalities (13) for $b \geq a > 0$ and $v \in (0, 1)$. Dividing $a$ in the both sides of the inequalities (13) and putting $\frac{b}{a} := t \geq 1$, we have
\[
t^v \leq (1-v)t^\frac{v}{t} + vt^\frac{v}{t} \leq L_v(1, t) \leq \frac{1}{2} ((1-v) + vt + t^v) \leq (1-v) + vt, \quad (t \geq 1, v \in (0, 1)). \tag{14}
\]

Putting $s := \frac{1}{t} \leq 1$ and $u := 1 - v$, and then multiplying $s > 0$ to both sides, we have
\[
s^u \leq (1-u)s^{-\frac{u}{s}} + us^\frac{1}{s} \leq L_u(1, s) \leq \frac{1}{2} ((1-u) + us + s^u) \leq (1-u) + us, \quad (0 < s \leq 1, u \in (0, 1)). \tag{15}
\]

by elementary calculations. Thus we have the inequalities:
\[
t^v \leq (1-v)t^\frac{v}{t} + vt^\frac{v}{t} \leq L_v(1, t) \leq \frac{1}{2} ((1-v) + vt + t^v) \leq (1-v) + vt, \quad (t > 0, v \in (0, 1)). \tag{16}
\]

Therefore we complete the proof by putting $t := \frac{b}{a}$ for any $a, b > 0$ in (16) and then multiplying $a > 0$ to both sides.

We note that the third and forth inequalities have already given in [9, Lemma 2.3]. However the first and second inequalities are new results. In addition, our approaches are different from the authors in [9].

We give the inequalities on the weighted identric mean which was defined in [9] as
\[
I_v(a, b) := \frac{1}{e} (a \nabla_v b)^{(1-2v)/(1-v)(b-a)} \left( \frac{b^{vb}}{a^{vb}} \right)^{\frac{1}{1-v}}, \quad v \in (0, 1).
\tag{17}
\]
It is easy to check that $I_{1/2}(a, b)$ recovers the usual identric mean $I(a, b) := \frac{1}{e} \left( \frac{b^v}{a^v} \right)^{\frac{1}{1-v}}$, with $\lim_{v \to 0} I_v(a, b) = a$ and $\lim_{v \to 1} I_v(a, b) = b$.

Corollary 2.3. For $a, b > 0$ and $v \in (0, 1)$, we have
\[
a^*_v b \leq (a^*_v b)^{1/2} (a \nabla_v b) \leq I_v(a, b) \leq (a \nabla_v b)^{1/2} (a^*_v b) \leq a \nabla_v b. \tag{18}
\]
Proof. Applying the convex function $f(t) := -\log t$, ($t > 0$) in Theorem 2.1, we have for $b \geq a > 0$ with elementary calculations

$$\log a^{1-v}b^v \leq \log \left( a^{1-v}b^v \frac{(1-v)(a+vb)}{a} \right)$$

$$\leq \frac{1-v}{v(b-a)} \{(1-v)(a+vb)\log ((1-v)(a+vb) - (1-v)(a+vb) - a \log a + a\}$$

$$+ \frac{1-v}{(1-v)(b-a)} \{b \log b - b - ((1-v)(a+vb)\log ((1-v)(a+vb) + ((1-v)(a+vb)\}$$

$$\leq \log \left( \left( 1 - \frac{v}{2} \right) a + \frac{v}{2}b \right)^{1-v} \left( 1 + \frac{1+v}{2} b \right)^{v} \leq \log ((1-v)(a+vb).$$

We calculate the following

$$\frac{1-v}{(1-v)(b-a)} \{(1-v)(a+vb)\log ((1-v)(a+vb) - (1-v)(a+vb) - a \log a + a\}$$

$$+ \frac{1-v}{(1-v)(b-a)} \{b \log b - b - ((1-v)(a+vb)\log ((1-v)(a+vb) + ((1-v)(a+vb)\}$$

$$= \log \left( \left( 1 - \frac{v}{2} \right) a + \frac{v}{2}b \right) \left( \frac{(1-2v)(1-v)(a+vb)}{1-v(1-v)(b-a)} b \frac{1-v}{1-v(1-v)(b-a)} - 1 \right)$$

$$= \log 1 \{(1-v)(a+vb)\} \left( \frac{1-2v}{1-v(1-v)(b-a)} \frac{1-v}{1-v(1-v)(b-a)} - 1 \right).$$

Thus we complete the proof for any $a, b > 0$ by the similar way to the proof of Corollary 2.2. □

Our Corollary 2.3 clearly refines [8] Theorem 3.1.

According to the inequalities shown in [8] Theorem 3.3 for convex function $f$,

$$2 \min \{1-v, v\} \Delta_{f,1/2}(a, b) \leq \Delta_{f,v}(a, b) \leq 2 \max \{1-v, v\} \Delta_{f,1/2}(a, b)$$

(19)

where $v \in [0, 1]$ and

$$\Delta_{f,v}(a, b) := f(a) \nabla_v f(b) - f(a \nabla_v b) \geq 0,$$ we obtain the further refinements of Theorem 2.1

**Proposition 2.4.** Under the same assumption of Theorem 2.1, we have

$$f(a \nabla_v b) \leq Q_{f,v}^{(1)}(a, b) \leq R_{f,v}^{(1)}(a, b) \leq C_{f,v}(a, b) \leq R_{f,v}^{(2)}(a, b) \leq Q_{f,v}^{(2)}(a, b) \leq f(a) \nabla_v f(b),$$

(20)

where

$$Q_{f,v}^{(1)}(a, b) := f(a \nabla_v b) + 2 \min \{1 - v, v\} \Delta_{f,1/2} \left( a \nabla_\frac{1+v}{2} b, a \nabla_\frac{1-v}{2} b \right)$$

and

$$Q_{f,v}^{(2)}(a, b) := f(a) \nabla_v f(b) - \min \{1 - v, v\} \Delta_{f,1/2}(a, b).$$

**Proof.** Using the first inequality from relation (19) and replacing $a$ and $b$ by $a \nabla_\frac{1+v}{2} b$ and $a \nabla_\frac{1-v}{2} b$ respectively, we deduce

$$2 \min \{1 - v, v\} \Delta_{f,1/2}(a \nabla_\frac{1+v}{2} b, a \nabla_\frac{1-v}{2} b) \leq \Delta_{f,v}(a \nabla_\frac{1+v}{2} b, a \nabla_\frac{1-v}{2} b)$$

$$= R_{f,v}^{(1)}(a, b) - f \left( (a \nabla_\frac{1+v}{2} b) \nabla_\frac{1+v}{2} (a \nabla_\frac{1-v}{2} b) \right) = R_{f,v}^{(1)}(a, b) - f(a \nabla_v b).$$

Using the first inequality in (19) again, we have

$$R_{f,v}^{(2)}(a, b) = (f(a) \nabla_v f(b)) \nabla_1/2(f(a \nabla_v b)) = \frac{1}{2} \{f(a) \nabla_v f(b) + f(a \nabla_v b)\} \leq f(a) \nabla_v f(b) - \min \{1 - v, v\} \Delta_{f,1/2}(a, b) = Q_{f,v}^{(2)}(a, b) \leq f(a) \nabla_v f(b).$$

□

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Remark 2.5. (i) From the inequality $Q_{f,v}^{(2)}(a,b) \geq Q_{f,v}^{(1)}(a,b)$ in [20], we find that $$\Delta_{f,v}(a,b) \geq \min\{1-v,v\} \left( \Delta_{f,1/2}(a,b) + 2\Delta_{f,1/2} \left( a \nabla_{v} b, a \nabla_{1+v} b \right) \right) \geq 0.$$ 

(ii) From the second inequality of [19], we also find that $$R_{f,v}^{(1)}(a,b) \leq P_{f,v}^{(1)}(a,b), \quad P_{f,v}^{(2)}(a,b) \leq R_{f,v}^{(2)}(a,b)$$ where $$P_{f,v}^{(1)}(a,b) := f(a \nabla v b) + 2 \max\{1-v,v\} \Delta_{f,1/2} \left( a \nabla_{v} b, a \nabla_{1+v} b \right)$$ and $$P_{f,v}^{(2)}(a,b) := f(a) \nabla_v f(b) - \max\{1-v,v\} \Delta_{f,1/2} (a,b).$$ However there is no ordering between $P_{f,v}^{(1)}(a,b)$ and $P_{f,v}^{(2)}(a,b)$, since we have the following numerical examples. $P_{\exp,1/4}^{(1)}(4,1) - P_{\exp,1/4}^{(2)}(4,1) \simeq 4.35403$, $P_{\exp,1/4}^{(1)}(8,1) - P_{\exp,1/4}^{(2)}(8,1) \simeq -30.7996$.

3 Reverses and refinements by differentiable functions

We extend the above results for the differentiable functions. From [1], if $f : I \to \mathbb{R}$ is a differentiable function on $I^0$ (interior of $I$) and if $f' \in L[a,b]$ (the space of Riemann integrable function on $[a,b]$), where $a, b \in I$ with $a < b$, then the following equality holds for each $x \in [a,b]$:

$$f(x) - \frac{1}{b-a} \int_{a}^{b} f(t)dt = \frac{(x-a)^2}{b-a} \int_{0}^{1} v f'((1-v)a + vx)dv - \frac{(b-x)^2}{b-a} \int_{0}^{1} v f'((1-v)b + vx)dv.$$  (21)

If we choose $x = \frac{a+b}{2}$ in (21), then we have

$$f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(t)dt = \frac{b-a}{4} \left\{ \int_{0}^{1} v f'((1-v)a + v \frac{a+b}{2})dv - \int_{0}^{1} v f'((1-v)b + v \frac{a+b}{2})dv \right\}.$$  (22)

In [2] we found the following relation holds

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(t)dt = \frac{b-a}{2} \int_{0}^{1} (1-2v)f'(va + (1-v)b)dv.$$  (23)

Here, we have the equality:

$$\int_{0}^{1} (1-2v)f'(va + (1-v)b)dv = \int_{0}^{1} (2v-1)f'((1-v)a + vb)dv = \frac{2}{(b-a)^2} \int_{a}^{b} \left( t - \frac{a+b}{2} \right) f'(t)dt.$$  

Thus we have the following equality from [23] with this equality

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(t)dt = \frac{1}{b-a} \int_{a}^{b} \left( t - \frac{a+b}{2} \right) f'(t)dt.$$  (24)
Theorem 3.1. For every convex differentiable function \( f : [a, b] \rightarrow \mathbb{R} \) with \( f' \in L[a, b] \) and \( |f'(x)| \leq K \), we have

\[
C_{f,v}(a, b) - R^{(1)}_{f,v}(a, b) \leq \frac{v(1-v)K(b-a)}{2}
\]

and

\[
R^{(2)}_{f,v}(a, b) - C_{f,v}(a, b) \leq \frac{v(1-v)K(b-a)}{2}.
\]

Proof. If \( |f'(x)| \leq K \), then from (22) we deduce

\[
\frac{1}{b-a} \int_a^b f(t)dt - f \left( \frac{a + b}{2} \right) \leq \frac{K}{4}(b-a),
\]

and from (24) we obtain

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{K}{b-a} \int_a^b \left| t - \frac{a + b}{2} \right| dt = \frac{K}{4}(b-a).
\]

We obtain (25) by applying the inequalities (27) on two intervals \([a, (1-v)a + vb] \) and \([(1-v)a + vb, b] \), and then multiplying \((1-v)\) and \(v\) to them and summing them. By the same way with (28), we obtain (26).

Corollary 3.2. For \( b \geq a > 0 \) and \( v \in (0,1) \), we have

\[
L_{v}(a,b) \leq \left( a^\#_v b \right) \nabla_v \left( a^\#_v \frac{b}{2} \right) + \frac{v(1-v)b}{2} \log \frac{b}{a}
\]

and

\[
(a \nabla_v b) \nabla_{1/2} \left( a^\#_v b \right) \leq L_{v}(a,b) + \frac{v(1-v)b}{2} \log \frac{b}{a}.
\]

Proof. Applying the convex function \( f(t) := e^t \) in Theorem 3.1, we have the relations of the statement, since we have

\[
C_{\exp,v}(a,b) = L_{v}(e^a, e^b),
\]

\[
R^{(1)}_{\exp,v}(a,b) = \left( e^{a^\#_v e^b} \right) \nabla_v \left( e^{a^\#_v \frac{b}{2} e^b} \right),
\]

\[
R^{(2)}_{\exp,v}(a,b) = \left( e^{a \nabla_v e^b} \right) \nabla_{1/2} \left( e^{a^\#_v e^b} \right)
\]

and we can take \( K = e^b \) for \( t \in [a, b] \). Finally we replace \( e^a \) and \( e^b \) by \( a \) and \( b \), respectively.

The inequalities (29) and (30) give (difference type) reverses for the 2nd and 3rd inequalities in (13), respectively.

Corollary 3.3. For \( b \geq a > 0 \) and \( v \in (0,1) \), we have

\[
\left( a \nabla_{1/2} b \right) \sharp_v \left( a \nabla_{1/2} b \right) \leq e^{\frac{v(1-v)(b-a)}{2a}} I_{v}(a,b)
\]

and

\[
I_{v}(a,b) \leq e^{\frac{v(1-v)(b-a)}{2a}} \left( a^\#_v b \right) \sharp_{1/2} (a \nabla_v b).
\]
Proof. Applying the convex function \( f(t) := -\log t, \( t > 0 \) in Theorem 3.1, we have the relations of the statement, since we have
\[
C_{\log,v}(a,b) = -\log I_v(a,b),
\]
\[
R^{(1)}_{\log,v}(a,b) = -\log \left(a \nabla_{\frac{v}{2}} b \right) \left(a \nabla \frac{1+v}{2} b \right),
\]
\[
R^{(2)}_{\log,v}(a,b) = -\log \left(a^* \nabla_{\frac{v}{2}} b \right) \left(a \nabla_v b \right)
\]
and we can take \( K = \frac{1}{a} \) for \( t \in [a,b] \).

The inequalities (31) and (32) give (ratio type) reverses for the 3rd and 2nd inequalities in (18), respectively.

We extend the above results for the twice differentiable functions. From (3), (4) and (5), assume that \( f : I \to \mathbb{R} \) is a continuous on \( I \), twice differentiable on \( I^o \) and there exist \( m = \inf_{x \in I^o} f''(x) \) and \( M = \sup_{x \in I^o} f''(x) \), \( a, b \in I \) with \( a < b \), then the following inequalities hold:
\[
\frac{m}{3} \left( \frac{b-a}{2} \right)^2 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{M}{3} \left( \frac{b-a}{2} \right)^2
\]  \hspace{1cm} (33)
and
\[
\frac{m}{6} \left( \frac{b-a}{2} \right)^2 \leq \frac{1}{b-a} \int_a^b f(t) dt - f \left( \frac{a+b}{2} \right) \leq \frac{M}{6} \left( \frac{b-a}{2} \right)^2.
\]  \hspace{1cm} (34)

Theorem 3.4. Assume that \( f : I \to \mathbb{R} \) is a continuous on \( I \), twice differentiable on \( I^o \) and there exist \( m = \inf_{x \in I^o} f''(x) \) and \( M = \sup_{x \in I^o} f''(x) \), \( a, b \in I \) with \( a < b \), we have
\[
\frac{v(1-v)m}{6} \left( \frac{b-a}{2} \right)^2 \leq C_{f,v}(a,b) - R^{(1)}_{f,v}(a,b) \leq \frac{v(1-v)M}{6} \left( \frac{b-a}{2} \right)^2
\]  \hspace{1cm} (35)
and
\[
\frac{v(1-v)m}{3} \left( \frac{b-a}{2} \right)^2 \leq R^{(2)}_{f,v}(a,b) - C_{f,v}(a,b) \leq \frac{v(1-v)M}{3} \left( \frac{b-a}{2} \right)^2.
\]  \hspace{1cm} (36)

Proof. Applying the inequality (33) on two intervals \([a, (1-v)a + vb]\) and \([(1-v)a + vb, b]\), we obtain
\[
\frac{m}{6} \left( \frac{v(b-a)}{2} \right)^2 \leq \frac{1}{v(b-a)} \int_a^b f(t) dt - f \left( a \nabla_{\frac{v}{2}} b \right) \leq \frac{M}{6} \left( \frac{v(b-a)}{2} \right)^2
\]  \hspace{1cm} (37)
and
\[
\frac{m}{6} \left( \frac{(1-v)(b-a)}{2} \right)^2 \leq \frac{1}{(1-v)(b-a)} \int_a^b f(t) dt - f \left( a \nabla \frac{1+v}{2} b \right) \leq \frac{M}{6} \left( \frac{(1-v)(b-a)}{2} \right)^2.
\]  \hspace{1cm} (38)

Multiplying \((1-v)\) and \(v\) to the both sides in (37) and (38) respectively and summing each side, we obtain the relations of the statement. Similar, applying the inequality (34), we deduce the inequality (33).

Corollary 3.5. For \( b \geq a > 0 \) and \( v \in (0,1) \), we have
\[
\frac{v(1-v)a}{24} \log^2 \frac{b}{a} \leq L_v(a,b) - \left(a^* \nabla_{\frac{v}{2}} b \right) \left(a^* \nabla \frac{1+v}{2} b \right) \leq \frac{v(1-v)b}{24} \log^2 \frac{b}{a}
\]  \hspace{1cm} (39)
and
\[
\frac{v(1-v)a}{12} \log^2 \frac{b}{a} \leq (a \nabla_v b) \nabla \frac{1}{2} (a^* \nabla_{\frac{v}{2}} b) - L_v(a,b) \leq \frac{v(1-v)b}{12} \log^2 \frac{b}{a}.
\]  \hspace{1cm} (40)
Proof. Applying the convex function \( f(t) := e^t \) in Theorem 3.4, we have the relations of the statement, since \( m = e^a \) and \( M = e^b \). Finally we replace \( e^a \) and \( e^b \) by \( a \) and \( b \), respectively. \( \square \)

The inequalities (39) and (40) give a better (difference type) refinement for the 2nd and 3rd inequality in (13), respectively.

Corollary 3.6. For \( b \geq a > 0 \) and \( v \in (0,1) \), we have
\[
e^{-v(1-v)(b-a)^2/24a^2} \left( a \nabla_v^2 b \right)^{1/2} \left( a \nabla_v^{1+v} b \right) \leq I_v(a,b) \leq e^{-v(1-v)(b-a)^2/24b^2} \left( a \nabla_v^{1/v} b \right)^{1/2} \left( a \nabla_v^{1+v} b \right) \tag{41}
\]
and
\[
e^{v(1-v)(b-a)^2/12b^2} \left( a \nabla_v^2 b \right)^{1/2} \left( a \nabla_v^{1/v} b \right) \leq I_v(a,b) \leq e^{v(1-v)(b-a)^2/12a^2} \left( a \nabla_v^2 b \right)^{1/2} \left( a \nabla_v^{1/v} b \right). \tag{42}
\]

Proof. Applying the convex function \( f(t) := -\log t, \ (t>0) \) in Theorem 3.4 we have the relations of the statement, since \( m = \frac{1}{t^2} \) and \( M = \frac{1}{t^2} \).

The inequalities (41) and (42) give a better (ratio type) refinement for the 3rd and 2nd inequality in (18), respectively.

4 Concluding remarks

Our obtained results in this paper can be extended to the operator inequalities. We give operator inequalities corresponding to Corollary 2.2. We omit the other cases. For strictly positive operators \( A \) and \( B \), the weighted geometric operator mean and arithmetic operator mean are defined as
\[
A^\#_v B := A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^v A^{1/2}, \quad A \nabla_v B := (1-v)A + vB.
\]

It is known that an operator mean \( M(A,B) \) is associated with the representing function \( f(t) = m(1,t) \) with a mean \( m(a,b) \) for positive numbers \( a, b \), in the following
\[
M(A,B) = A^{1/2} f \left( A^{-1/2} B A^{-1/2} \right) A^{1/2}
\]
in the general operator mean theory by Kubo-Ando [7]. Thus it is understood that the weighted logarithmic operator mean \( A\ell_v B \) is defined by through the representing function \( L_v(1,t) \) for \( v \in (0,1) \).

From Corollary 2.2 and Kubo-Ando theory (or standard functional calculus), we can obtain the following operator inequalities. However, we state an alternative proof for the scalar inequalities on the representing functions.

Theorem 4.1. For any \( v \in (0,1) \) and strictly positive operators \( A \) and \( B \), we have
\[
A^\#_v B \leq (1-v)A^\#_{\frac{v}{2}} B + vA^\#_{1+v} B \leq A\ell_v B \leq \frac{1}{2} \left( A^\#_v B + A \nabla_v B \right) \leq A \nabla_v B.
\]

Proof. It is sufficient to prove the following scalar inequalities:
\[
t^v \leq (1-v)t^{v/2} + vt^{(1+v)/2} \leq L_v(1,t) \leq \frac{1}{2} \left( t^v + (1-v) + vt \right) \leq (1-v) + vt \tag{43}
\]
where
\[
L_v(1,t) := \frac{1}{\log t} \left( \frac{1-v}{v} (t^v - 1) + \frac{v}{1-v} (t - t^v) \right), \ (t > 0, \ v \in (0,1)).
\]

\[\]
The 4th inequality in (43) is trivial and 3rd one in (43) was proven in [9, Lemma 2.3]. The 1st inequality in (43) can be proven by the fact that the arithmetic mean is greater or equal to the geometric mean as 

\[(1 - v)\frac{t^{v/2} + vt^{(1+v)/2}}{2} \geq \frac{t^{(1-v)/2}t^{(1+v)/2}}{2} = t^{v}.
\]

The 2nd inequality in (43) can be proven by the use of the following inequality:

\[\frac{x^2 - 1}{\log x^2} \geq x, \quad (x > 0).\]  \hspace{1cm} (44)

Putting \(x := t^{v/2}\) and \(x := t^{(v-1)/2}\) in (44), we have respectively

\[t^{v/2} \leq \frac{t^v - 1}{v \log t} \quad \text{and} \quad t^{(v-1)/2} \leq \frac{t^{v-1} - 1}{(v - 1) \log t} \iff t^{(1+v)/2} \leq \frac{t - t^v}{(1 - v) \log t}.
\]

Multiplying \((1 - v)\) and \(v\) to the 1st and 2nd inequality in the above and then summing them, we obtain the 2nd inequality in (43). Finally, replacing \(t\) by \(A^{-1/2}BA^{-1/2}\) in the inequalities (43) and then multiplying \(A^{1/2}\) from both sides, we complete the proof.  

The upper bound of \(A\ell_vB\) has already given in [9, Theorem 2.4]. But the lower bound of \(A\ell_vB\) is a new result in Theorem [4].

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