Continuum limits of pluri-Lagrangian systems

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Abstract

A pluri-Lagrangian (or Lagrangian multiform) structure is an attribute of integrability that has mainly been studied in the context of multidimensionally consistent lattice equations. It unifies multidimensional consistency with the variational character of the equations. An analogous continuous structure exists for integrable hierarchies of differential equations. We present a continuum limit procedure for pluri-Lagrangian systems. In this procedure the lattice parameters are interpreted as Miwa variables, describing a particular embedding in continuous multi-time of the mesh on which the discrete system lives. Then we seek differential equations whose solutions interpolate the embedded discrete solutions. The continuous systems found this way are hierarchies of differential equations. We show that this continuum limit can also be applied to the corresponding pluri-Lagrangian structures. We apply our method to the discrete Toda lattice and to equations $H_1$ and $Q_{1,\delta=0}$ from the ABS list.

1. Introduction

A cornerstone of the theory of integrable systems is the idea that integrable equations come in families of compatible equations. In the continuous case these are hierarchies of differential equations with commuting flows. In the discrete case, in particular in the context of equations on quadrilateral graphs (quad equations) this property is known as multidimensional consistency. A classification of multidimensionally consistent quad equations was found by Adler, Bobenko, and Suris and is often referred to as the ABS list. Additionally, many integrable equations can be derived from a variational principle. The Lagrangian multiform or pluri-Lagrangian formalism, which grew out of a beautiful insight by Lobb and Nijhoff, combines these two aspects of integrability.
The discrete version of the pluri-Lagrangian theory is more developed than the continuous one, and arguably more fundamental. Hence, connecting both sides could lead to a better understanding of the continuous theory. It is known that the lattice parameters of a discrete pluri-Lagrangian system may play the role of independent variables in a corresponding continuous pluri-Lagrangian system of non-autonomous differential equations, see e.g. [14, 28]. This paper presents a different connection between discrete and continuous pluri-Lagrangian systems, where the continuous variables interpolate the discrete ones. The lattice parameters describe the size and shape of the mesh on which the discrete system lives, and thus they disappear in the continuum limit. The continuous systems found this way are hierarchies of autonomous differential equations. Pluri-Lagrangian structures for such hierarchies were studied independently of the discrete case in [25].

Some similar continuum limits can be found in the literature, for example in [16, 17, 18, 29] and in particular in [27], where the lattice potential KdV equation is shown to produce the potential KdV hierarchy in a suitable limit. On the level of the pluri-Lagrangian structure, the problem is essentially that of interpolation of discrete variational systems by continuous Lagrangian systems. This was studied in [26] because of its relevance in numerical analysis, in particular for backward error analysis of variational integrators. We will build on the ideas from that work to construct pluri-Lagrangian structures for hierarchies of differential equations that appear as continuum limits of lattice equations.

Section 2 contains a crash course on discrete and continuous pluri-Lagrangian systems. Section 3 provides an introduction to Miwa variables, which turn out to be a powerful tool for taking continuum limits. In fact on the level of equations, this is the only tool required to obtain a continuous system. However, that leaves the question whether the resulting differential equations are integrable. In Section 4 we look at the Lagrangian side of the continuum limit; first we review the method of [26] to take continuum limits of classical Lagrangian systems (regardless of their integrability), then we extend these ideas to pluri-Lagrangian systems. By recovering the pluri-Lagrangian structure on the continuous side, the question about integrability of the limit is settled in the affirmative. In Section 5 we study several examples in detail.

2. Pluri-Lagrangian systems

2.1. Discrete pluri-Lagrangian systems

Consider the lattice $\mathbb{Z}^N$ with basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$. To each lattice direction we associate a parameter $\lambda_i \in \mathbb{C}$. The equations we are interested in involve the values of a field $U : \mathbb{Z}^N \to \mathbb{C}$ on elementary squares in this lattice, or more generally, on $d$-dimensional plaquettes. Such a plaquette is a $2^d$-tuple of lattice points that form an elementary hypercube. We denote it by

$$\square_{i_1, \ldots, i_d}(\mathbf{n}) = \left\{ \mathbf{n} + \varepsilon_1 \mathbf{e}_{i_1} + \ldots + \varepsilon_d \mathbf{e}_{i_d} \mid \varepsilon_k \in \{0, 1\} \right\} \subset \mathbb{Z}^N,$$
Figure 1: Visualization of a discrete 2-surface in $\mathbb{Z}^3$.

where $\mathbf{n} = (n_1, \ldots, n_N)$. Plaquettes are considered to be oriented; an odd permutation of the directions $i_1, \ldots, i_d$ reverses the orientation of the plaquette. We will write $U(\square_{i_1, \ldots, i_d}(n))$ for the 2d-tuple

$$U(\square_{i_1, \ldots, i_d}(n)) = \left( U(n), U(n + \epsilon_{i_1}), \ldots, U(n + \epsilon_{i_1} + \epsilon_{i_d}) \right).$$

Occasionally we will also consider the corresponding “filled-in” hypercubes in $\mathbb{R}^N$,

$$\text{■}_{i_1, \ldots, i_d}(n) = \left\{ n + \alpha_1 \epsilon_{i_1} + \ldots + \alpha_d \epsilon_{i_d} \mid \alpha_k \in [0,1] \right\} \subset \mathbb{R}^N,$$

on which we consider the orientation defined by the volume form $dt_{i_1} \wedge \ldots \wedge dt_{i_d}$.

The role of a Lagrange function is played by a discrete d-form

$$L(U(\square_{i_1, \ldots, i_d}(n)), \lambda_{i_1}, \ldots, \lambda_{i_d}),$$

i.e. a function of the values of the field $U : \mathbb{Z}^N \to \mathbb{C}$ on a plaquette and of the corresponding lattice parameters, where

$$L(U(\square_{\sigma(i_1), \ldots, i_d}(n)), \lambda_{\sigma(i_1)}, \ldots, \lambda_{\sigma(i_d)}) = \text{sgn}(\sigma)L(U(\square_{i_1, \ldots, i_d}(n)), \lambda_{i_1}, \ldots, \lambda_{i_d})$$

for any permutation $\sigma$ of $i_1, \ldots, i_d$.

Consider a discrete d-surface $\Gamma = \{ \square_{\alpha} \}$ in the lattice, i.e. a set of d-dimensional plaquettes, such that the union of the corresponding filled-in plaquettes $\bigcup_{\alpha} \text{■}_{\alpha}$ is an oriented topological d-manifold (possibly with boundary). The action over $\Gamma$ is given by

$$S_\Gamma = \sum_{\square_{i_1, \ldots, i_d}(n) \in \Gamma} L(U(\square_{i_1, \ldots, i_d}(n)), \lambda_{i_1}, \ldots, \lambda_{i_d}).$$

The field $U$ is a solution to the pluri-Lagrangian problem if it is a critical point of $S_\Gamma$ (with respect to variations that are zero on the boundary of $\Gamma$) for all discrete d-surfaces $\Gamma$ simultaneously.

For $d = 1$ we have

$$S_\Gamma = \sum_{\{n, n + \epsilon_i\} \in \Gamma} L(U(n), U(n + \epsilon_i), \lambda_i).$$
The Euler-Lagrange equations at general elementary corners,
\[ \frac{\partial}{\partial U(n)} \left( L(U(n \pm e_i), U(n), \lambda_i) + L(U(n), U(n \pm e_j), \lambda_j) \right) = 0, \]
are sufficient conditions for \( U \) to be a solution to the pluri-Lagrangian problem.

For \( d = 2 \) we have
\[ S_\Gamma = \sum_{\{n, n+e_i, n+e_j, n+e_i+e_j\} \in \Gamma} L(U(n), U(n + e_i), U(n + e_j), \lambda_i, \lambda_j). \]

Since every surface can be constructed out of corners of cubes, it is sufficient to determine the Euler-Lagrange equations on these elementary building blocks. They are
\[ \frac{\partial}{\partial U} \left( L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) + L(U, U_k, U_{ik}, \lambda, \lambda_k) \right) = 0, \]
\[ \frac{\partial}{\partial U_i} \left( L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) - L(U, U_{ij}, U_{ik}, \lambda_j, \lambda_k) \right) = 0, \]
\[ \frac{\partial}{\partial U_{ij}} \left( L(U, U_i, U_{ij}, \lambda_i, \lambda_j) - L(U, U_{ij}, U_{ik}, \lambda_j, \lambda_k) \right) = 0, \]
\[ \frac{\partial}{\partial U_{ijk}} \left( -L(U, U_{ik}, U_{jk}, U_{ijk}, \lambda_i, \lambda_j) - L(U, U_{ij}, U_{ik}, U_{ijk}, \lambda_j, \lambda_k) \right) = 0. \]

These are sufficient conditions for \( U \) to be a solution to the pluri-Lagrangian problem. Often, \( L \) can be written in a three-leg form
\[ L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = A(U, U_i, \lambda_i) - A(U, U_j, \lambda_j) + B(U_i, U_j, \lambda_i - \lambda_j), \]
which renders the first and last corner equations trivial. In particular, this is the case for all equations from the ABS list.

For more details, we refer to [14], [7], [12, Chapter 12], and the references therein.

2.2. Continuous pluri-Lagrangian systems

In the continuous case, the lattice is replaced by a space \( \mathbb{R}^N \), which we refer to as multi-time. The Lagrangian in this context is a differential \( d \)-form
\[ \mathcal{L} = \sum_{i_1 < \ldots < i_d} \mathcal{L}_{i_1, \ldots, i_d}[u] \, dt_{i_1} \wedge \ldots \wedge dt_{i_d}, \]
where the square brackets denote dependence on the field $u : \mathbb{R}^N \to \mathbb{C}$ and an arbitrary number of its partial derivatives. We will always use lower case letters to denote continuous fields, as opposed to the upper case letters used for discrete fields. The field $u$ solves the \textit{pluri-Lagrangian problem} if for any $d$-dimensional submanifold $\Gamma$ of $\mathbb{R}^N$ it is a critical point of the action

$$S_\Gamma = \int_\Gamma \mathcal{L}$$

with respect to variations that are zero on the boundary of $\Gamma$.

The \textit{multi-time Euler-Lagrange equations}, which characterize solutions to the pluri-Lagrangian problem, were derived in [25]. The main idea of that derivation is to approximate any given smooth $d$-surface by a \textit{stepped surface}, a piecewise flat surface, the pieces of which are shifted sections of coordinate planes. Analogous to the discrete case, it is sufficient to look at the elementary building blocks of stepped surfaces.

In order to state the multi-time Euler-Lagrange equations we introduce a multi-index notation for partial derivatives. An \textit{N-index} $I$ is a $N$-tuple of non-negative integers. There is a natural bijection between $N$-indices and partial derivatives of $u : \mathbb{R}^N \to \mathbb{C}$. We denote by $u_I$ the mixed partial derivative of $u$, where the number of derivatives with respect to each $t_i$ is given by the entries of $I$. Note that if $I = (0, \ldots, 0)$, then $u_I = u$.

We will often denote a multi-index suggestively by a string of $t_i$-variables, but it should be noted that this representation is not always unique. For example,

$$t_1 = (1, 0, \ldots, 0), \quad t_N = (0, \ldots, 0, 1), \quad t_1 t_2 = t_2 t_1 = (1, 1, 0, \ldots, 0).$$

In this notation, we will also make use of exponents to compactify the expressions, for example

$$t_2^3 = t_2 t_2 t_2 = (0, 3, 0, \ldots, 0).$$

The notation $\Gamma t_j$ should be interpreted as concatenation in the string representation, hence it denotes the multi-index obtained from $I$ by increasing the $j$-th entry by one. Finally, if the $j$-th entry of $I$ is non-zero we say that $I$ contains $t_j$, and write $I \ni t_j$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{A stepped curve (left) and a stepped 2-surface (right) in $\mathbb{R}^3$}
\end{figure}
For $d = 1$ the multi-time Euler-Lagrange equations are

$$\frac{\delta L_i}{\delta u_I} = 0 \quad \forall I \not\ni t_i, \quad (1a)$$
$$\frac{\delta L_i}{\delta u_{It_j}} = \frac{\delta_j L_j}{\delta u_{It_j}} \quad \forall I, \quad (1b)$$

where $\frac{\delta}{\delta u_I}$ denotes a variational derivative in the $t_i$-direction,

$$\frac{\delta_i}{\delta u_I} = \sum_{k=0}^{\infty} (-1)^k D^k_{t_i} \frac{\partial}{\partial u_{It_k}} = \frac{\partial}{\partial u_I} - D_{t_i} \frac{\partial}{\partial u_{It_i}} + D^2_{t_i} \frac{\partial}{\partial u_{It_i,t_i}} - \ldots,$$

and $D_{t_i} = \frac{d}{dt_i}$. Equation (1a) is obtained from the straight parts of the stepped curve, Equation (1b) from the corners.

For $d = 2$ the multi-time Euler-Lagrange equations are

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \quad \forall I \not\ni t_i, t_j, \quad (2a)$$
$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \quad \forall I \not\ni t_i, \quad (2b)$$
$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i,t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j,t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k,t_i}} = 0 \quad \forall I, \quad (2c)$$

where

$$\frac{\delta_{ij}}{\delta u_I} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^{k+\ell} D^k_{t_i} D^\ell_{t_j} \frac{\partial}{\partial u_{It_k,t_{i}t_{j}}}.$$
3. Miwa variables

To motivate our approach to the continuum limit, we start by considering the opposite direction.\footnote{The author is grateful to Yuri Suris for suggesting the motivation presented here.} The problem of integrable discretization has been studied at impressive length in the monograph [24]. Let us briefly summarize the “recipe” for discretizing Toda-type systems from Section 2.9 of that work. It starts from an integrable ODE with a Lax representation of the form

\[ L_t = [L, \pi_+(f(L))] \] (3)

in a Lie algebra \( g = g_+ \oplus g_- \), where \( \pi_+ \) denotes projection onto \( g_+ \). Here \( L \) denotes the Lax operator and is not to be confused with a Lagrangian. Such an equation is part of an integrable hierarchy, given by

\[ L_{tk} = [L, \pi_+(f(L)^k)] \] (4)

A related integrable difference equation can be formulated in the corresponding Lie group \( G \), with subgroups \( G_+ \) and \( G_- \) having Lie algebras \( g_+ \) and \( g_- \) respectively. Any element \( x \in G \) close to the unit \( \text{Id} \in G \) can be factorized as \( x = \Pi_+(x)\Pi_-(x) \), where \( \Pi_{\pm}(x) \in G_{\pm} \).

The difference equation is given by

\[ \widetilde{L} = \Pi_+(F(L)^{-1} L \Pi_+(F(L)), \] (5)

where the tilde \( \sim \) denotes a discrete time step and

\[ F(L) = \text{Id} + \lambda f(L) \]

for some small parameter \( \lambda \).

Solutions of the differential equation (3) are given by

\[ L(t) = \Pi_+(e^{tf(L_0)})^{-1} L_0 \Pi_+(e^{tf(L_0)}) \].

A simultaneous solution to the whole hierarchy (4) takes the form

\[ L(t_1, t_2, \ldots) = \Pi_+(e^{t_1 f(L_0)+t_2 f(L_0)^2+\ldots})^{-1} L_0 \Pi_+(e^{t_1 f(L_0)+t_2 f(L_0)^2+\ldots}) \]. (6)

A solution of the discretization (5) is given by

\[ L(n) = \Pi_+(F^n(L_0))^{-1} L_0 \Pi_+(F^n(L_0)) \]

\[ = \Pi_+(e^{n \log(1+\lambda f(L_0))})^{-1} L_0 \Pi_+(e^{n \log(1+\lambda f(L_0))}) \]

\[ = \Pi_+(e^{n \lambda f(L_0)-\frac{\lambda^2}{2} f(L_0)^2+\ldots})^{-1} L_0 \Pi_+(e^{n \lambda f(L_0)-\frac{\lambda^2}{2} f(L_0)^2+\ldots}) \]. (7)
Comparing equations (6) and (7), it is natural to identify a discrete step \( n \mapsto n + 1 \) with a time shift
\[
(t_1, t_2, \ldots, t_i, \ldots) \mapsto \left( t_1 + \lambda, t_2 - \frac{\lambda^2}{2}, \ldots, t_i + (-1)^{i+1} \frac{\lambda_i}{i}, \ldots \right).
\]
This gives us a map from the discrete space \( \mathbb{Z}^N(n_1, \ldots, n_N) \) into the continuous multi-time \( \mathbb{R}^N(t_1, \ldots, t_N) \). We associate a parameter \( \lambda_i \) with each lattice direction and set
\[
t_i = (-1)^{i+1} \left( n_1 \frac{\lambda_1}{i} + \ldots + n_N \frac{\lambda_N}{i} \right).
\]
Note that a single step in the lattice (changing one \( n_j \)) affects all the times \( t_i \), hence we are dealing with a very skew embedding of the lattice. We will also consider a slightly more general correspondence,
\[
t_i = (-1)^{i+1} \left( n_1 \frac{c\lambda_1}{i} + \ldots + n_N \frac{c\lambda_N}{i} \right) + \tau_i,
\]
for constants \( c, \tau_1, \ldots, \tau_N \) describing a scaling and a shift of the lattice. The variables \( n_j \) and \( \lambda_j \) are known in the literature as Miwa variables and have their origin in [15]. In the present work we will call the \( n_j \) discrete coordinates, the \( \lambda_j \) lattice parameters and the \( t_i \) continuous coordinates or times. We will call Equation (8) the Miwa correspondence. Let \( \lambda = (\lambda_1, \ldots, \lambda_N) \) and consider the \( N \times N \) matrix
\[
M_\lambda = \left( (-1)^{i+1} \frac{\lambda_j}{i} \right)_{i,j=1}^N.
\]
Then we can write the Miwa correspondence as
\[
t = cM_\lambda n + \tau,
\]
where \( t = (t_1, \ldots, t_N)^T, n = (n_1, \ldots, n_N)^T, \) and \( \tau = (\tau_1, \ldots, \tau_N)^T \). In other words, we consider the mesh \( \mathbb{Z}^N \) under the affine transformation
\[
A_{c,\lambda,\tau} : \mathbb{R}^N \rightarrow \mathbb{R}^N : t \mapsto cM_\lambda t + \tau.
\]
We will use the Miwa correspondence (8) even if the discrete system is not generated by the recipe described above. In many cases one can justify this in a similar way by considering plane wave factors, solutions of the linearized system. For more on this perspective, see e.g. [19, 20, 27] and [12, Chapter 5].

For a completely different motivation for Miwa variables, note that for \( N \) distinct parameter values \( \lambda_1, \ldots, \lambda_N \) the corresponding vectors
\[
\nu(\lambda) = \left( c\lambda, -\frac{c\lambda^2}{2}, \ldots, -(-1)^N \frac{c\lambda_N}{N} \right)
\]
are linearly independent. Up to projective transformations, \( \nu \) is the only curve with that property. It is known as the rational normal curve \[11\].

To perform the continuum limit of a difference equation involving \( U : \mathbb{Z}^N \to \mathbb{C} \), we associate to it a function \( u : \mathbb{R}^N \to \mathbb{C} \) that interpolates it:

\[
U(n) = u(A_{c, \lambda, \tau}(n)) \quad \forall n \in \mathbb{Z}^N.
\]

We denote the shift of \( U \) in the \( i \)-th lattice direction by \( U_i \). If \( U(n) = u(t_1, \ldots, t_N) \), it is given by

\[
U_i = U(n + e_i) = u\left(t_1 + c\lambda_i, t_2 - \frac{c\lambda_i^2}{2}, \ldots, t_n - (-1)^N \frac{c\lambda_i^N}{N}\right),
\]

which we can expand as a power series in \( \lambda_i \). The difference equation thus turns into a power series in the lattice parameters. If all goes well, its coefficients will define differential equations that form an integrable hierarchy. Examples can be found in Section 5.

Note that such a procedure is strictly speaking not a continuum limit; sending \( \lambda_i \to 0 \) would only leave the leading order term of the power series. A more precise formulation is that the continuous \( u \) interpolates the discrete \( U \) for sufficiently small values of \( \lambda_i \), where \( U \) is defined on a mesh that is embedded in \( \mathbb{R}^N \) using the Miwa correspondence. Since \( \lambda_i \) is still assumed to be small, it makes sense to think of the outcome as a limit, but it is important to keep in mind that higher order terms should not be disregarded.

# 4. Continuum limits of Lagrangian forms

## 4.1. Modified Lagrangians in the classical variational problem

In \[26\] we performed a continuum limit on Lagrangian systems in the context of variational integrators for ODEs. Given a discrete Lagrangian, we constructed a continuous modified Lagrangian whose critical curves interpolate solutions of the discrete problem. A similar approach can be used in the context of pluri-Lagrangian systems, but first we present the relevant ideas in the context of the classical variational formulation of a P\( \Delta \)E. Here we use parameters \( h_j \) representing the mesh size of the lattice. In Section 4.2 we will consider the pluri-Lagrangian problem and reinterpret the parameters as Miwa variables.

In the classical discrete variational principle we consider elementary plaquettes of full dimension, so it is sufficient to label them only by position, leaving out the subscripts denoting the direction. We consider Lagrangians \( L_{\text{disc}}(\square(n), h_1, \ldots, h_d) \) depending on the values of the field \( U : \mathbb{Z}^d \to \mathbb{C} \) on a plaquette \( \square(n) \) and on the mesh sizes \( h_1, \ldots, h_d \). As before, we denote lattice shifts by subscripts:

\[
U = U(n), \quad U_i = U(n + e_i), \quad U_{-i} = U(n - e_i), \quad U_{ij} = U(n + e_i + e_j), \quad \ldots.
\]

We identify points of a discrete solution with mesh size \((h_1, \ldots, h_d)\) with evaluations of an interpolating field \( u : \mathbb{R}^d \to \mathbb{C} \),

\[
U(n_1, \ldots, n_d) = u(h_1n_1, \ldots, h_dn_d).
\]
Using a Taylor expansion we can write the discrete Lagrangian $L_{\text{disc}}(\square(n), h_1, \ldots, h_d)$ as a function of the interpolating field $u$ and its derivatives,

$$L_{\text{disc}}([u], h_1, \ldots, h_d) = L_{\text{disc}}\left(\left\{u + \sum_{k=1}^{d} \varepsilon_k h_k u_{t_k} + \frac{1}{2} \sum_{k=1}^{d} \sum_{\ell=1}^{d} \varepsilon_k \varepsilon_{\ell} h_k h_{\ell} u_{t_k t_{\ell}} + \ldots \biggm| \varepsilon_k \in \{0, 1\}\right\}, h_1, \ldots, h_d\right),$$

where the square brackets denote dependence on $u$ and any number of its partial derivatives.

So far we have only written the discrete Lagrangian as a function of the continuous field. The corresponding action is still a sum:

$$S(U, h_1, \ldots h_d) = \sum_{n \in \mathbb{Z}^d} L_{\text{disc}}(U(\square(n)), h_1, \ldots, h_d) = \sum_{n \in \mathbb{Z}^d} L_{\text{disc}}([u(n)], h_1, \ldots, h_d).$$

We want to write the action as an integral. This can be done using the Euler-Maclaurin formula, which relates sums to integrals \[1,\; \text{Eq. 23.1.30}\]:

$$\sum_{k=0}^{m-1} F(a + kh) = \frac{1}{h} \int_a^{a+mh} F(t) \, dt + \sum_{i=1}^{\infty} h^{i-1} \frac{B_i}{i!} \left(F(i-1)(a + mh) - F(i-1)(a)\right)$$

$$= \frac{1}{h} \int_a^{a+mh} \left(\sum_{i=0}^{\infty} h^i \frac{B_i}{i!} F^{(i)}(t)\right) \, dt,$$

where $B_i$ denote the Bernoulli numbers $1, -\frac{1}{2}, 0, -\frac{1}{6}, 0, \ldots$. Applying this to $L_{\text{disc}}$ in each of the lattice directions, we obtain the meshed modified Lagrangian

$$L_{\text{mesh}}([u], h_1, \ldots h_d) = \sum_{i_1, \ldots, i_d=0}^{\infty} \frac{B_{i_1} \cdots B_{i_d}}{i_1! \cdots i_d!} D_{i_1}^{i_1} \cdots D_{i_d}^{i_d} L_{\text{disc}}([u], h_1, \ldots h_d).$$

The power series in the Euler-Maclaurin Formula generally does not converge. The same is true for the series defining $L_{\text{mesh}}$. Formally, it satisfies

$$S(U, h_1, \ldots h_d) = \int_{\mathbb{R}^d} L_{\text{mesh}}([u(t)], h_1, \ldots h_d) \, dt,$$

where $dt = dt_1 \wedge \ldots \wedge dt_d$. This property also holds locally,

$$L_{\text{disc}}(U(\square(n)), h_1, \ldots, h_d) = \int_{\square(n)} L_{\text{mesh}}([u(t)], h_1, \ldots h_d) \, dt. \quad (9)$$
The word \textit{meshed} refers to the fact that the discrete system provides additional structure for the continuous variational problem. In the \textit{meshed variational problem}, non-differentiable fields are admissible as long as their singular points are consistent with the mesh, i.e. if they only occur on the boundaries of mesh cells. This imposes additional conditions on critical curves, related to the natural boundary conditions and to the Weierstrass-Erdmann corner conditions (see e.g. [10, Sec. 6 and 13] for these two concepts). In [26] these conditions were used to turn the meshed modified Lagrangian into a true modified Lagrangian which does not depend on higher derivatives. We will not discuss this method here. Instead we will find that the pluri-Lagrangian structure provides us with simpler tools to eliminate unwanted derivatives.

Because the power series defining $L_{\text{mesh}}$ usually does not converge, we introduce the following concept of criticality.

\textbf{Definition 1.} A field $u : \mathbb{R}^d \rightarrow \mathbb{C}$ is $k$-critical for the action

$$ \int L([u], h_1, \ldots, h_d) \, dt $$

if for any variation $\delta u$ there holds

$$ \delta \int L([u], h_1, \ldots, h_d) \, dt = O(h_1^{k+1} + \ldots + h_d^{k+1}). $$

In the discrete case the definition is analogous, with integrals replaced by sums.

Note that in contrast to [26] we do not consider parameter-dependent families of fields. This is because we do not want the lattice parameters to survive in the continuum limit; $u$ should not depend on the lattice parameters. A welcome consequence of this restriction is that it allows us to avoid much of the cumbersome analysis of [26]. In the current setting the following property is quite obvious.

\textbf{Proposition 2.} A field $u : \mathbb{R}^d \rightarrow \mathbb{C}$ is $k$-critical for the action

$$ \int L([u(t)], h_1, \ldots, h_d) \, dt $$

if and only if it satisfies the Euler-Lagrange equations with a defect of order $O(h_1^{k+1} + \ldots + h_d^{k+1})$,

$$ \frac{\delta L([u], h_1, \ldots, h_d)}{\delta u} = O(h_1^{k+1} + \ldots + h_d^{k+1}). $$

\textbf{4.2. From discrete to continuous pluri-Lagrangian structures}

In the pluri-Lagrangian context we consider a discrete Lagrangian $d$-form in a higher dimensional lattice $\mathbb{Z}^N$, $N > d$. Furthermore, from now on the lattice parameters are interpreted as Miwa variables, hence they will not have the immediate interpretation of mesh
size. Through the Miwa correspondence \( \mathcal{M} \) they still determine a lattice embedded in the continuous space \( \mathbb{R}^N \), albeit a very skew one.

Consider \( N \) pairwise distinct lattice parameters \( \lambda_1, \ldots, \lambda_N \) and denote by \( \mathbf{e}_1, \ldots, \mathbf{e}_N \) the unit vectors in the lattice \( \mathbb{Z}^N \). The differential of the Miwa correspondence maps them to linearly independent vectors in \( \mathbb{R}^N \):

\[
\mathbf{e}_i \mapsto \mathbf{v}_i = \left( c \lambda_i, -\frac{c \lambda^2_i}{2}, \ldots, (-1)^{N+1} \frac{c \lambda_i^N}{N!} \right).
\]

We calculate the modified Lagrangian in the transformed coordinate system. The evaluations \( u(t), u(t + \mathbf{v}_i), u(t + \mathbf{v}_i + \mathbf{v}_j), \ldots \) of a continuous field correspond to a discrete field evaluated on the plaquette located at \( A_{c, \lambda, \tau}(t) \). Hence

\[
\mathcal{L}_{\text{disc}}([u \circ A_{c, \lambda, \tau}^{-1}], \lambda_1, \ldots, \lambda_d) = \mathcal{L}_{\text{mesh}}([u \circ A_{c, \lambda, \tau}^{-1}], \lambda_1, \ldots, \lambda_d) = \sum_{i_1, \ldots, i_d} \frac{B_{i_1} \cdots B_{i_d}}{i_1! \cdots i_d!} \partial_{i_1} \cdots \partial_{i_d} \mathcal{L}_{\text{disc}}([u \circ A_{c, \lambda, \tau}^{-1}], \lambda_1, \ldots, \lambda_d).
\]

Note that \( \mathcal{L}_{\text{Miwa}} \) depends on a field parameterized in Miwa variables, whereas \( \mathcal{L}_{\text{disc}} \) and \( \mathcal{L}_{\text{mesh}} \) are inherited form the previous subsection and thus depend on a field described in orthonormal coordinates.

**Lemma 3.** Consider a filled-in plaquette of the embedded lattice, \( A_{c, \lambda, \tau}(\square_{i_1, \ldots, i_d}(n)) \), and let \( \eta_k \) be the 1-forms dual to the Miwa shifts,

\[
\eta_k = (A_{c, \lambda, \tau}^{-1})^* dt_k,
\]

where \( * \) denotes the pullback. Then \( \mathcal{L}_{\text{Miwa}} \) satisfies

\[
\int_{A_{c, \lambda, \tau}(\square_{i_1, \ldots, i_d}(n))} \mathcal{L}_{\text{Miwa}}([u], \lambda_{i_1}, \ldots, \lambda_{i_d}) \eta_{i_1} \wedge \ldots \wedge \eta_{i_d} = \mathcal{L}_{\text{disc}}(\square_{i_1, \ldots, i_d}(n), \lambda_{i_1}, \ldots, \lambda_{i_d}).
\]
Proof. In Equation (9) we have the corresponding result for $\mathcal{L}_{\text{mesh}}$, so the proof is a simple change of variables:

$$
\int_{A_{\gamma,\tau}(\mathbf{n})} \mathcal{L}_{\text{Miwa}}([u], \lambda_1, \ldots, \lambda_{id}) \eta_{i_1} \wedge \ldots \wedge \eta_{id} = 
\int_{A_{\gamma,\tau}(\mathbf{n})} \mathcal{L}_{\text{mesh}}([u \circ A^{-1}_{c,\lambda,\tau}], \lambda_1, \ldots, \lambda_{id}) (A^{-1}_{c,\lambda,\tau})^*(dt_1 \wedge \ldots \wedge dt_{id}) = 
\int_{\square_{i_1,\ldots,i_{id}}(\mathbf{n})} \mathcal{L}_{\text{mesh}}([u], \lambda_1, \ldots, \lambda_{id}) dt_1 \wedge \ldots \wedge dt_{id} = 
\mathcal{L}_{\text{disc}}(\square_{i_1,\ldots,i_{id}}(\mathbf{n}), \lambda_1, \ldots, \lambda_{id}).
$$

We want to use this result for plaquettes in arbitrary directions. This suggests the Lagrangian $d$-form

$$
\sum_{1 \leq i_1 < \ldots < i_{id} \leq N} \mathcal{L}_{\text{Miwa}}([u], \lambda_1, \ldots, \lambda_{id}) \eta_{i_1} \wedge \ldots \wedge \eta_{id}.
$$

Up to a truncation error, this $d$-form can be written in a much more convenient way. Let $T_N$ denote truncation of a power series after degree $N$ in each variable,

$$
T_N \left( \sum_{i_1,\ldots,i_{id}=1}^{\infty} \lambda_1^{i_1} \ldots \lambda_{id}^{i_{id}} f_{i_1,\ldots,i_{id}} \right) = \sum_{i_1,\ldots,i_{id}=1}^{N} \lambda_1^{i_1} \ldots \lambda_{id}^{i_{id}} f_{i_1,\ldots,i_{id}}.
$$

Lemma 4. Assume that every term in the power series $\mathcal{L}_{\text{Miwa}}$ is of strictly positive degree in each $\lambda_i$,

$$
\mathcal{L}_{\text{Miwa}}([u], \lambda_1, \ldots, \lambda_{d}) = \sum_{i_1,\ldots,i_{d}=1}^{\infty} (-1)^{i_1+\ldots+i_{d}} c^{d}_{i_1 i_2} \ldots \frac{\lambda_{d}^{i_{id}}}{i_{id}} \mathcal{L}_{i_1,\ldots,i_{id}}[u],
$$

then

$$
\sum_{1 \leq j_1 < \ldots < j_{id} \leq N} T_N(\mathcal{L}_{\text{Miwa}}([u], \lambda_{j_1}, \ldots, \lambda_{j_{id}})) \eta_{j_1} \wedge \ldots \wedge \eta_{j_{id}} = \sum_{1 \leq i_1 < \ldots < i_{id} \leq N} \mathcal{L}_{i_1,\ldots,i_{id}}[u] dt_{i_1} \wedge \ldots \wedge dt_{id}.
$$

Note in Equation (10) that the factors $(-1)^{i_1+\ldots+i_{id}} c^{d}_{i_1 i_2} \ldots \frac{\lambda_{d}^{i_{id}}}{i_{id}}$ are terms of $(d \times d)$-minors of the transformation matrix $c M_{\lambda}$.

Proof of Lemma 4. First observe that, just like the discrete Lagrangian, the Lagrangian $\mathcal{L}_{\text{Miwa}}([u], \lambda_1, \ldots, \lambda_{id})$ is skew-symmetric as a function of $(\lambda_1, \ldots, \lambda_{id})$. Therefore, the coefficients $\mathcal{L}_{i_1,\ldots,i_{id}}[u]$ are skew-symmetric as a function of $(i_1, \ldots, i_{id})$. 

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We pair the form

\[
\mathcal{L} = \sum_{1 \leq i_1 < \ldots < i_d \leq N} \mathcal{L}_{i_1, \ldots, i_d}[u] \, dt_{i_1} \wedge \ldots \wedge dt_{i_d}
\]

with a \(d\)-tuple of vectors \((\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_d}) = (cM_\lambda \mathbf{e}_{j_1}, \ldots, cM_\lambda \mathbf{e}_{j_d})\):

\[
\langle \mathcal{L}, (\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_d}) \rangle = \frac{1}{d!} \sum_{i_1, \ldots, i_d=1}^N \left( \mathcal{L}_{i_1, \ldots, i_d}[u] \sum_{\sigma \in S_d} \left( \text{sgn}(\sigma) \prod_{k=1}^d \langle dt_{i_k(\sigma)}, \mathbf{v}_{j_k} \rangle \right) \right)
\]

Due to the skew-symmetry of \(\mathcal{L}_{i_1, \ldots, i_d}[u]\), this can be written as

\[
\langle \mathcal{L}, (\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_d}) \rangle = \frac{1}{d!} \sum_{i_1, \ldots, i_d=1}^N \sum_{\sigma \in S_d} \mathcal{L}_{i_{\sigma(1)}, \ldots, i_{\sigma(d)}}[u] \prod_{k=1}^d \langle dt_{i_k(\sigma)}, \mathbf{v}_{j_k} \rangle
\]

Since the first sum is over all \(d\)-tuples \((i_1, \ldots, i_d)\) with strictly positive integer entries, permuting \((i_1, \ldots, i_d)\) yields a different term of this sum. Hence the additional summation over permutations \(\sigma \in S_d\) amounts to multiplication by \(d!\). We find

\[
\langle \mathcal{L}, (\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_d}) \rangle = \sum_{i_1, \ldots, i_d=1}^N \mathcal{L}_{i_1, \ldots, i_d}[u] \prod_{k=1}^d \langle dt_{i_k}, \mathbf{v}_{j_k} \rangle
\]

\[
= \sum_{i_1, \ldots, i_d=1}^N \mathcal{L}_{i_1, \ldots, i_d}[u] \prod_{k=1}^d (-1)^{i_k} \frac{\lambda_{i_k}^{j_k}}{i_k}
\]

\[
= T_N(\mathcal{L}_\text{Miwa}([u], \lambda_{j_1}, \ldots, \lambda_{j_d})). \quad \blacksquare
\]

**Theorem 5.** Let \(\mathcal{L}_\text{disc}\) be a discrete Lagrangian \(d\)-form, such that every term in the corresponding power series \(\mathcal{L}_\text{Miwa}\) is of strictly positive degree in each \(\lambda_i\), i.e. such that \(\mathcal{L}_\text{Miwa}\) is of the form (10). Consider the differential \(d\)-form

\[
\mathcal{L} = \sum_{1 \leq i_1 < \ldots < i_d \leq N} \mathcal{L}_{i_1, \ldots, i_d}[u] \, dt_{i_1} \wedge \ldots \wedge dt_{i_d},
\]

built out of the coefficients of \(\mathcal{L}_\text{Miwa}\). Then a field \(u : \mathbb{R}^N \to \mathbb{C}\) is a solution to the continuous pluri-Lagrangian problem for \(\mathcal{L}\) if and only if the corresponding discrete fields

\[
U_{\tau} : \mathbb{Z}^N \to \mathbb{C} : \mathbf{n} \mapsto u(A_{c, \lambda, \tau}(\mathbf{n})), \quad \tau \in \mathbb{R}^N,
\]

are \(N\)-critical for the discrete pluri-Lagrangian problem for \(\mathcal{L}_\text{disc}\).
**Proof.** Consider the \((d+1)\)-dimensional cube
\[
C = \left\{ \sum_{k=1}^{d+1} \varepsilon_k e_j \mid \varepsilon_k \in \{0,1\} \right\} \quad (1 \leq j_1 < \ldots < j_{d+1} \leq N)
\]
in the lattice \(\mathbb{Z}^N\). It corresponds to a \((d+1)\)-dimensional parallelotope
\[
P_\tau = \left\{ \tau + \sum_{k=1}^{d+1} \alpha_k v_{j_k} \mid \alpha_i \in [0,1] \right\} \quad (1 \leq j_1 < \ldots < j_{d+1} \leq N)
\]
in \(\mathbb{R}^N\). Combining Lemma 3 and Lemma 4 we find that
\[
\sum_{\text{facets } \square \text{ of } C} L_{\text{disc}}(\square) = \int_{\partial P_\tau} \sum_{1 \leq i_1 < \ldots < i_d \leq N} L_{\text{Miwa}}([u], \lambda_{i_1}, \ldots, \lambda_{i_d}) \eta_{i_1} \wedge \ldots \wedge \eta_{i_d}
\]
\[
= \int_{\partial P_\tau} L[u] + O(\lambda_1^{N+1} + \ldots + \lambda_N^{N+1}).
\]
Note that the integral \(\int_{\partial P_\tau} L[u]\) still depends on the \(\lambda_i\) because the parallelotope \(P_\tau\) depends on them. From this relation it follows that if \(u\) is a solution to the pluri-Lagrangian problem for \(L\), then the discrete action is \(N\)-critical.

On the other hand, if the discrete fields \(U_\tau\) are \(N\)-critical for the pluri-Lagrangian problem for every \(\tau \in \mathbb{R}^N\), then the continuous action for the \(d\)-form \(L[u]\) is \(N\)-critical on every parallelootope \(P_\tau\). Therefore, the continuous action is \(N\)-critical on any corner of such a parallelootope (see Figure 3). In [25] such elementary corners were used as building blocks for stepped surfaces and shown to be a sufficiently large set of \(d\)-surfaces to derive the multi-time Euler-Lagrange equations. The skewness of the Miwa coordinates does not affect the argument. Hence the \(N\)-criticality on corners implies that the multi-time Euler-Lagrange equations are satisfied with an \(O(\lambda_1^{N+1} + \ldots + \lambda_N^{N+1})\)-defect. Since both the field \(u\) and the Euler-Lagrange equations for \(L\) are independent of the parameters \(\lambda_i\), the \(O(\lambda_1^{N+1} + \ldots + \lambda_N^{N+1})\)-defect must be exactly zero. \(\square\)

### 4.3. Eliminating alien derivatives

Suppose a pluri-Lagrangian \(d\)-form in \(\mathbb{R}^N\) produces multi-time Euler-Lagrange equations of evolutionary type,
\[
u_{t_k} = f_k[u] \quad \text{for } k \in \{d, d+1, \ldots, N\}.
\]
Unlike in the classical Lagrangian framework, Euler-Lagrange equations in the pluri-Lagrangian context are often evolutionary. The multi-time Euler-Lagrange equations and their differential consequences can be written as
\[
u_I = f_I[u] \quad \text{with } I \ni t_k \text{ for some } k \in \{d, d+1, \ldots, N\} \quad (11)
\]
Figure 3: Two elementary corners (solid), and the parallelotopes they belong to (dotted), in Miwa coordinates \( v_i = (c\lambda_i, -c\frac{\lambda_i^2}{2}) \) in \( \mathbb{R}^2 \).

In this context it is natural to consider the first \( d - 1 \) coordinates \( t_1, \ldots, t_{d-1} \) as space coordinates and the others as time coordinates.

**Definition 6.** A function \( f[u] \) is called

(a) \{\( i_1, \ldots, i_d \}\}-native if it only depends on \( u \) and its derivatives with respect to \( t_{i_1}, \ldots, t_{i_d} \) and with respect to the space coordinates \( t_1, \ldots, t_{d-1} \),

(b) \{\( i_1, \ldots, i_d \}\}-alien if it is not \{\( i_1, \ldots, i_d \}\}-native, i.e. if it depends on a \( t_k \)-derivative with \( k \not\in \{1, \ldots, d - 1, i_1, \ldots, i_d\} \).

A multi-index \( I \) is said to be native or alien if the corresponding derivative \( u_I \) is of that type.

We would like the coefficient \( L_{i_1 \ldots i_d} \) to be \{\( i_1, \ldots, i_d \}\}-native. A naive approach would be to use the multi-time Euler-Lagrange equations (11) to eliminate all alien derivatives. Let \( R_{i_1 \ldots i_d} \) denote the operator that replaces all \{\( i_1, \ldots, i_d \}\}-alien derivatives using (11). We denote the native version of the pluri-Lagrangian coefficients by

\[
\overline{L}_{i_1 \ldots i_d} = R_{i_1 \ldots i_d}(L_{i_1 \ldots i_d})
\]

and the \( d \)-form with these coefficients by \( \overline{L} \). A priori there is no reason to believe that the \( d \)-form \( \overline{L} \) will be equivalent to the original pluri-Lagrangian \( d \)-form \( L \). For example, the 1-dimensional Lagrangian \( L(u, u_t, u_{tt}) = \frac{1}{2} uu_{tt} \) leads to the Euler-Lagrange equation \( u_{tt} = 0 \), but any curve is critical for the Lagrangian \( \overline{L}(u, u_t, u_{tt}) = 0 \). However, in many cases the pluri-Lagrangian structure guarantees that \( L \) and \( \overline{L} \) have the same critical fields.

**Theorem 7.** If either

- \( d = 1 \) and \( L_1[u] \) does not depend on any alien derivatives, or
- \( d = 2 \) and for all \( j \) the coefficient \( L_{1j}[u] \) does not contain any alien derivatives,

then every critical field \( u \) for the pluri-Lagrangian \( d \)-form \( L \) is also critical for \( \overline{L} \).
The condition for \( d = 2 \) might seem restrictive, but given a Lagrangian 2-form, we can often find an equivalent one with coefficients \( L_{ij}[u] \) that satisfy this condition by inspection.

**Proof of Theorem 7** In this proof we consider the variation operator \( \delta \) as the vertical exterior derivative in the variational bicomplex. A short introduction to the variational bicomplex is given in Appendix A.

First we consider the case \( d = 1 \). Let

\[
F_{i,J}[u] = R_i(u_J),
\]

i.e. \( F_{i,J} = u_J \) if \( J \) is \( \{i\} \)-native and \( F_{i,J} \) is the native replacement for \( u_J \) otherwise. Note that \( D_{t_i} \) and \( R_i \) commute, hence \( D_{t_i} F_{i,J} = F_{i,J} \). We have

\[
\delta \mathcal{L} = \sum_{1 \leq i \leq N} \sum_{J \neq t_i} R_i \left( \frac{\partial L_i}{\partial u_J} \right) \delta F_{i,J} \wedge dt_i
\]

\[
= \sum_{1 \leq i \leq N} \sum_{J \neq t_i} R_i \left( \frac{\delta_i L_i}{\delta u_J} + D_{t_i} \frac{\delta_i L_i}{\delta u_{J t_i}} \right) \delta F_{i,J} \wedge dt_i
\]

\[
= \sum_{1 \leq i \leq N} \left( \sum_{J \neq t_i} R_i \left( \frac{\delta_i L_i}{\delta u_J} \right) \delta F_{i,J} + \sum_{J} D_{t_i} \left( R_i \left( \frac{\delta_i L_i}{\delta u_{J t_i}} \right) \delta F_{i,J} \right) \right) \wedge dt_i.
\]

Hence on solutions of the pluri-Lagrangian problem for \( \mathcal{L} \) there holds that

\[
\delta \mathcal{L} = \sum_{1 \leq i \leq N} \left( D_{t_i} \sum_{J} \frac{\delta_i L_1}{\delta u_{J t_i}} \delta F_{i,J} \right) \wedge dt_i.
\]

Using the assumption that no alien derivatives occur in \( L_1 \), we can simplify this to

\[
\delta \mathcal{L} = \sum_{1 \leq i \leq N} D_{t_i} \left( \sum_{\alpha=0}^{\infty} \frac{\partial L_1}{\partial u_{t_i}^{\alpha+1}} \delta u_{t_i}^{\alpha} \right) \wedge dt_i = d \left( -\sum_{\alpha=0}^{\infty} \frac{\partial L_1}{\partial u_{t_i}^{\alpha+1}} \delta u_{t_i}^{\alpha} \right).
\]

The fact that \( \delta \mathcal{L} \) is exact with respect to \( d \) implies that \( \delta \int_\Gamma \mathcal{L} = 0 \) for all curves \( \Gamma \) and all variations that are zero on the endpoints of \( \Gamma \). Hence \( u \) is a solution to the pluri-Lagrangian problem for \( \mathcal{L} \).

Now we consider the case \( d = 2 \). Let

\[
F_{ij,J} = R_{ij}(u_J).
\]
Note that $R_{ij}$ commutes with both $D_{t_i}$ and $D_{t_j}$. We have

$$\delta \mathcal{L} = \sum_{1 \leq i < j \leq N} \sum_J R_{ij} \left( \frac{\partial L_{ij}}{\partial u_J} \right) \delta F_{ij,J} \wedge dt_i \wedge dt_j$$

$$= \sum_{1 \leq i < j \leq N} \left( \sum_{J \neq t_i, t_j} R_{ij} \left( \frac{\partial L_{ij}}{\partial u_J} \right) \delta F_{ij,J} + \sum_{J \neq t_i} D_{t_i} \left( R_{ij} \left( \frac{\partial L_{ij}}{\partial u_{Jt_i}} \right) \delta F_{ij,J} \right) \right) \wedge dt_i \wedge dt_j$$

On solutions of the pluri-Lagrangian problem for $\mathcal{L}$ there holds that

$$\delta \mathcal{L} = \sum_{1 \leq i < j \leq N} \left( \sum_{J \neq t_i} D_{t_i} \left( \frac{\partial L_{ij}}{\partial u_{Jt_i}} \delta F_{ij,J} \right) - \sum_{J \neq t_i} D_{t_j} \left( \frac{\partial L_{ij}}{\partial u_{Jt_j}} \delta F_{ij,J} \right) \right) \wedge dt_i \wedge dt_j,$$

where we have left out the $R_{ij}$ because the $L_{ij}$ do not contain any alien derivatives. For the same reason, only terms where $J = \{i, j\}$-native can be nonzero, so in all nonvanishing terms we find $F_{ij,J} = u_J$. Therefore,

$$\delta \mathcal{L} = \sum_{1 \leq i < j \leq N} \left( D_{t_i} \left( \sum_{J \neq t_j} \frac{\partial L_{ij}}{\partial u_{Jt_i}} \delta u_J + \sum_J D_{t_j} \left( \frac{\partial L_{ij}}{\partial u_{Jt_j}} \delta u_J \right) \right) \right) \wedge dt_i \wedge dt_j$$

$$- D_{t_j} \left( \sum_{J \neq t_i} \frac{\partial L_{ij}}{\partial u_{Jt_i}} \delta u_J + \sum_J D_{t_i} \left( \frac{\partial L_{ij}}{\partial u_{Jt_i}} \delta u_J \right) \right) \wedge dt_j,$$

This implies that $\delta \int_{\Gamma} \mathcal{L} = 0$ for all surfaces $\Gamma$ and all variations that are zero on the boundary of $\Gamma$. Hence $u$ is a solution to the pluri-Lagrangian problem for $\mathcal{L}$. $\square$

5. Examples

The plan for this section is as follows. We begin with the 1-form case and discuss the continuum limit of the discrete Toda lattice. After that we present three examples for the
2-form case. The first one is a linear quad equation. This will help us understand how to proceed for the two non-linear quad equations that follow, \( H_1 \) and \( Q_1 \). From the ABS list, in each of the examples we first perform the continuum limit on the level of equations and then discuss the pluri-Lagrangian structure.

5.1. Toda lattice

5.1.1. Equation

The Toda lattice consists of a number of particles on a line with an exponential nearest-neighbor force. If we denote the positions of the particles with

\[ q(t) = (q^{[0]}(t), q^{[1]}(t), \ldots, q^{[N]}(t)), \]

then their motion is described by the equation

\[
\frac{d^2 q^{[k]}}{dt^2} = \exp(q^{[k+1]} - q^{[k]}) - \exp(q^{[k]} - q^{[k-1]}).
\]

There are two common conventions regarding boundary conditions: periodic (formally \( q^{[N+1]} \equiv q^{[1]} \)) and open-end (formally \( q^{[0]} \equiv +\infty \) and \( q^{[N+1]} \equiv -\infty \)). An integrable discretization of the Toda lattice is given by [24, Chapter 5]

\[
\frac{1}{\lambda_i} (\exp(Q_i^{[k]} - Q_i^{[k]}) - \exp(Q_i^{[k]} - Q_{-i}^{[k]})) \\
+ \lambda_i (\exp(Q_i^{[k]} - Q_{i}^{[k-1]}) - \exp(Q_{-i}^{[k+1]} - Q_{i}^{[k]})) = 0,
\]

where the subscripts \( i \) and \(-i\) denote forward and backward shifts respectively and \( \lambda_i \) is a lattice parameter.

We use the Miwa correspondence \( \mathcal{S} \) with \( c = 1 \) to identify discrete steps with continuous time shifts

\[
Q_i^{[k]} = q_i^{[k]}(t_1, t_2, t_3, \ldots), \\
Q_i^{[k]} = q_i^{[k]}(t_1 + \lambda_i, t_2 - \frac{\lambda_i^2}{2}, t_3 + \frac{\lambda_i^3}{3}, \ldots) , \\
Q_{-i}^{[k]} = q_i^{[k]}(t_1 - \lambda_i, t_2 + \frac{\lambda_i^2}{2}, t_3 - \frac{\lambda_i^3}{3}, \ldots) .
\]

We plug these identifications into Equation (12) and perform a Taylor expansion in \( \lambda \):

\[
\begin{align*}
&(-\exp(q^{[k+1]} - q^{[k]}) + \exp(q^{[k]} - q^{[k-1]}) + q_{1i}^{[k]} \lambda) \\
&+ (\exp(q^{[k+1]} - q^{[k]}) q_1^{[k+1]} - \exp(q^{[k]} - q^{[k-1]}) q_1^{[k-1]} + q_1^{[k]} q_{1i}^{[k]} - q_{12}^{[k]} \lambda^2 = \mathcal{O}(\lambda^3),
\end{align*}
\]
where the subscripts \( i \) are a shorthand for \( t_i \) and denote partial derivatives. As long as one remembers that discrete fields are printed in upper case and continuous fields in lower case, there should be no confusion between partial derivatives and lattice shifts. In the leading order term we recognize the first Toda equation

\[
q^{[k]}_{11} = \exp(q^{[k+1]}[1] - q^{[k]}[1]) - \exp(q^{[k]}[1] - q^{[k-1]}[1]) .
\]  

(13)

Using this equation, we find that the coefficient of \( \lambda^2 \) is

\[
\begin{align*}
\exp(q^{[k+1]}[1] - q^{[k]}[1]) q^{[k+1]}_{11} - \exp(q^{[k]}[1] - q^{[k-1]}[1]) & q^{[k]}_{11} + q^{[k]}[1] q^{[k]}_{11} - q^{[k]}_{12} \\
= \exp(q^{[k+1]}[1] - q^{[k]}[1]) (q^{[k+1]}_{11} - q^{[k]}_{11}) - \exp(q^{[k]}[1] - q^{[k-1]}[1]) (q^{[k]}_{11} - q^{[k]}_{11}) + 2q^{[k]}[1] q^{[k]}_{11} - q^{[k]}_{12} \\
= D_{11} \left( \exp(q^{[k+1]}[1] - q^{[k]}[1]) + \exp(q^{[k]}[1] - q^{[k-1]}[1]) + (q^{[k]}_{11})^2 - q^{[k]}_{12} \right).
\end{align*}
\]

Under the differentiation one can recognize the second Toda equation

\[
q^{[k]}_{2} = (q^{[k]}_{11})^2 + \exp(q^{[k+1]}[1] - q^{[k]}[1]) + \exp(q^{[k]}[1] - q^{[k-1]}[1]) .
\]  

(14)

Similarly, the higher order terms correspond to the subsequent equations of the Toda hierarchy.

5.1.2. Pluri-Lagrangian structure

A pluri-Lagrangian structure for the discrete Toda equation was studied in [6]. The Lagrangian is given by

\[
L(Q, Q^i, \lambda^i) = \frac{1}{\lambda^i} \sum_k \left( \exp(Q^i_k - Q^k) - 1 - (Q^i_k - Q^k) \right) \\
- \lambda^i \sum_k \exp(Q^k - Q^{k-1}) .
\]  

(15)

Performing a Taylor expansion and applying the Euler-Maclaurin formula as in Section 4.2 we obtain

\[
\mathcal{L}_{\text{Miwa}}(q, \lambda) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\lambda^j}{j} \mathcal{L}_j[q]
\]
with coefficients

\[ L_1 = \sum_k \left( \frac{1}{2} (q_1^{[k]})^2 - \exp(q^{[k]} - q^{[k-1]}) \right) , \]

\[ L_2 = \sum_k \left( q_1^{[k]} q_2^{[k]} - \frac{1}{3} (q_1^{[k]})^3 - (q_1^{[k]} + q_1^{[k-1]}) \exp(q^{[k]} - q^{[k-1]}) \right) , \]

\[ L_3 = \sum_k \left( -\frac{1}{4} \left( (q_1^{[k+1]})^2 + 4q_1^{[k+1]} q_1^{[k]} + (q_1^{[k]})^2 + q_1^{[k+1]} \right) \exp(q^{[k+1]} - q^{[k]}) \right. \]
\[ \left. + \frac{1}{4} \left( -q_1^{[k+1]} + q_1^{[k]} - 3q_2^{[k]} - 3q_2^{[k+1]} \right) \exp(q^{[k+1]} - q^{[k]}) \right) \]
\[ + \frac{1}{8} (q_1^{[k]})^4 - \frac{3}{4} (q_1^{[k]})^2 q_2^{[k]} - \frac{1}{8} (q_1^{[k+1]})^2 + \frac{3}{8} (q_2^{[k]})^2 + q_1^{[k]} q_3^{[k]} \right) , \]

By Theorem 5, these are the coefficients of a pluri-Lagrangian 1-form \( \mathcal{L} = \sum_i \mathcal{L}_i \, dt_i \) for the Toda hierarchy \((13), (14), \ldots \).

Note that \( \mathcal{L}_3 \) contains derivatives with respect to \( t_2 \). We replace these using the second Toda equation and find

\[ \mathcal{L}_3 = \sum_k \left( -\frac{1}{4} (q_1^{[k]})^4 - \left( (q_1^{[k+1]})^2 + q_1^{[k+1]} q_1^{[k]} + (q_1^{[k]})^2 \right) \exp(q^{[k+1]} - q^{[k]}) \right. \]
\[ \left. + q_1^{[k]} q_3^{[k]} - \exp(q^{[k+2]} - q^{[k]}) - \frac{1}{2} \exp(2(q^{[k+1]} - q^{[k]})) \right) , \]

Similarly one can obtain \( \mathcal{L}_i \) for \( i \geq 4 \). By Theorem 6, the corresponding 1-form \( \mathcal{L} \) is equivalent to \( \mathcal{L} \). The Lagrangian 2-form \( \mathcal{L} \) is identical to the one that was found in [22] using the variational symmetries of the Toda lattice.

5.2. A linear quad equation

5.2.1. Equation

Consider the linear quad equation

\[ (\alpha_1 - \alpha_2)(U - U_{12}) = (\alpha_1 + \alpha_2)(U_1 - U_2) . \] (16)

It is a discrete analogue of the Cauchy-Riemann equations \([5]\) and also the linearization of the lattice potential KdV equation, which will be discussed in Section 5.3. Therefore all the results in this section are consequences of those in Section 5.3. Nevertheless, this simple quad equation is a good subject to illustrate some of the subtleties of the continuum limit procedure.
To get meaningful equations in the continuum limit, we need to write the quad equation in a suitable form. Since in the Miwa correspondence the parameter enters linearly in the $t_1$-coordinate and with higher powers in the other coordinates, the leading order of the expansion of the shifts of $U$ will only contain derivatives with respect to $t_1$. Other derivatives enter at higher orders. Since we want to obtain PDEs in the continuum limit, not ODEs, we must require that the leading order of the expansion yields a trivial equation.

Written in terms of difference quotients, Equation (16) reads

$$\frac{U_1 - U_2}{\alpha_1 - \alpha_2} = \frac{U - U_{12}}{\alpha_1 + \alpha_2},$$

but setting $U = u(t_1, \ldots)$, $U_i = u(t_1 + \alpha_i, \ldots)$, etc., this would yield $u_{t_1} = -u_{t_2}$ in the leading order of the expansion. In order to avoid this, we introduce new parameters $\lambda_i = \alpha_i^{-1}$. Then Equation (16) reads

$$\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)(U - U_{12}) - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)(U_1 - U_2) = 0. \tag{17}$$

or, equivalently,

$$\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 \lambda_2} \left(\frac{U_1 - U_2}{\lambda_1 - \lambda_2} - \frac{U_{12} - U}{\lambda_1 + \lambda_2}\right) = 0.$$

Inside the brackets we find $u_{t_1} = u_{t_2}$ in the leading order if we set $U = u(t_1, \ldots)$, $U_i = u(t_1 + \lambda_i, \ldots)$, etc., which is trivial as desired.

We use the Miwa correspondence (8) with $c = -2$. This choice will give us a nice normalization of the resulting differential equations. We apply the Miwa correspondence to Equation (17) and expand to find a double power series in $\lambda_1$ and $\lambda_2$,

$$\sum_{i,j} \frac{4(-1)^{i+j}}{i!j!} F_{ij}[u] \lambda_1^i \lambda_2^j = 0,$$

where $F_{ji} = -F_{ij}$. The factor $(-1)^{i+j}\frac{4}{i!j!}$ is chosen to normalize the $F_{0j}$, but does not influence the final result. The first few of these coefficients are

$$F_{01} = u_{t_2},$$
$$F_{02} = -u_{t_1 t_1 t_1} + \frac{3}{2} u_{t_1 t_2} + u_{t_3},$$
$$F_{03} = -\frac{4}{3} u_{t_1 t_1 t_1 t_1} + \frac{4}{3} u_{t_1 t_3} + u_{t_2 t_2} + u_{t_4},$$
$$F_{04} = -u_{t_1 t_1 t_1 t_1 t_1} - \frac{5}{3} u_{t_1 t_1 t_1 t_2} + \frac{5}{4} u_{t_1 t_2 t_2} + \frac{5}{4} u_{t_1 t_4} + \frac{5}{3} u_{t_2 t_3} + u_{t_5},$$
$$\vdots$$
We see that the flows corresponding to even times are trivial. In the odd orders we find a hierarchy of linear equations,

\[ u_{t_2} = 0, \quad u_{t_3} = u_{t_1 t_1}, \quad u_{t_4} = 0, \quad u_{t_5} = u_{t_1 t_1 t_1 t_1}, \quad \cdots \]

For \( i \geq 1 \), the equations \( F_{ij} = 0 \) are consequences of these equations.

### 5.2.2. Pluri-Lagrangian structure

The linear quad equation (16) possesses a pluri-Lagrangian structure \[5, 13\],

\[
L(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = U(U_i - U_i) - \frac{1}{2} \frac{\alpha_i + \alpha_j}{\alpha_i - \alpha_j}(U_i - U_j)^2.
\]

The following Lemma will help us put this Lagrangian in a more convenient form.

**Lemma 8.** \( L_0(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = (U + U_{ij})(U_i - U_j) \) is a null Lagrangian (i.e. its multi-time Euler-Lagrange equations are trivially satisfied)

**Proof.** Consider the discrete 1-form given by \( \eta(U, U_i) = UU_i \) and \( \eta(U_i, U) = -UU_i \). Its discrete exterior derivative is

\[
\Delta \eta(U, U_i, U_{ij}, U_j) = UU_i + U_i U_{ij} - U_{ij} U_j - U_j U = L_0.
\]

Just like in the continuous case, this means that the action of \( L_0 \) over any discrete surface only depends on values of \( U \) at the boundary of the surface. Hence all fields are critical with respect to variations in the interior. \( \square \)

Using Lemma 8 we see that the Lagrangian (18) is equivalent to (denoted with = by abuse of notation)

\[
L(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = \frac{1}{2}(U_i - U_j)(U - U_{ij}) - \frac{1}{2} \frac{\alpha_i + \alpha_j}{\alpha_i - \alpha_j}(U_i - U_j)^2,
\]

or, in terms of the parameters \( \lambda_k \),

\[
L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = \frac{1}{2}(U_i - U_j)(U - U_{ij}) + \frac{1}{2} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}(U_i - U_j)^2.
\]

Since the Taylor expansion of \((U_i - U_j)^2\) contains a factor \( \lambda_i - \lambda_j \), the expansion of the Lagrangian does not contain any negative order terms. In fact all zeroth order terms vanish as well, so Theorem 5 applies: the coefficients of the power series

\[
\mathcal{L}_{\text{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} 4(-1)^{i+j} \frac{1}{ij} L_{ij}[u] \lambda_1^i \lambda_2^j
\]
define a pluri-Lagrangian 2-form

$$\mathcal{L} = \sum_{1 \leq i < j \leq N} \mathcal{L}_{ij} dt_i \wedge dt_j.$$ 

We find

$$\mathcal{L}_{12} = ut_1 ut_2,$$

$$\mathcal{L}_{13} = -ut_1 ut_1 t_1 + \frac{3}{4} u_{t_2}^2 + ut_1 ut_3,$$

$$\mathcal{L}_{23} = -ut_1 ut_1 t_2 + ut_1 t_1 t_1 t_2 - 2u_{t_1} u_{t_1} u_{t_2} - 3u_{t_1} u_{t_2} u_{t_2} - 3u_1 u_{t_2} u_{t_2} + u_{t_2} u_{t_3},$$

: 

We will not study this example in more detail. Instead we move on to one of its non-linear cousins.

5.3. Lattice potential KdV (H1)

5.3.1. Equation

Consider equation H1 from the ABS list \[\text{[2]},\] also known as the lattice potential Korteweg-de Vries (lpKdV) equation,

$$(V_{12} - V)(V_2 - V_1) = \alpha_1 - \alpha_2.$$ \tag{19}

We would like write Equation (19) in terms of difference quotients. To achieve this, we identify $\alpha_1 = -\lambda_1^{-2}$ and $\alpha_2 = -\lambda_2^{-2}$. Then Equation (19) is equivalent to

$$\frac{V_{12} - V}{\lambda_1 + \lambda_2} \frac{V_2 - V_1}{\lambda_2 - \lambda_1} = \frac{1}{\lambda_1^2 \lambda_2^2}.$$ 

The left hand side is now a product of meaningful difference quotients, but the right hand side explodes as the parameters tend to zero. (Setting $\alpha_i = -\lambda_i^{-2}$ instead would cause the same problem as in the first attempt of Section 5.2.) To avoid this we make a non-autonomous change of variables

$$V(n_1, \ldots, n_N) = U(n_1, \ldots, n_N) + \frac{n_1}{\lambda_1} + \ldots + \frac{n_N}{\lambda_N}.$$ 

Then the lpKdV equation takes the form

$$\left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{12} - U \right) \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1 \right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}. \tag{20}$$
This is the form in which the lpKdV equation was originally found and studied, usually with parameters $p = \lambda_1^{-1}$ and $q = \lambda_2^{-1}$, see [19] for an overview. In terms of difference quotients, the equation reads

\[
\frac{U_{12} - U}{\lambda_1 + \lambda_2} - \frac{U_2 - U_1}{\lambda_2 - \lambda_1} - \lambda_1 \lambda_2 \frac{U_{12} - U}{\lambda_1 + \lambda_2} \frac{U_2 - U_1}{\lambda_2 - \lambda_1} = 0.
\]

If we identify $U = u(t_1, \ldots)$, $U_i = u(t_1 + \lambda_i, \ldots)$, etc., then the negative powers of the parameters cancel. In the leading we find the tautological equation $u_{t_1} - u_{t_1} = 0$. Therefore, this form of the difference equation is a suitable candidate for the continuum limit.

Again we use the Miwa correspondence [8] with $c = -2$. From Equation (20) we find a double power series in $\lambda_1$ and $\lambda_2$,

\[
\sum_{i,j} 4(-1)^{i+j} F_{ij}[u] \lambda_i^j = 0,
\]

where $F_{ji} = -F_{ij}$. The first few of these coefficients are

\[
F_{01} = u_2,
\]
\[
F_{02} = -3u_1^2 - u_{111} + \frac{3}{2} u_{12} + u_3,
\]
\[
F_{03} = -8u_1 u_{11} + 4u_1 u_2 - \frac{4}{3} u_{1111} + \frac{4}{3} u_{13} + u_{22} + u_4,
\]
\[
F_{04} = -5u_1^2 - \frac{20}{3} u_1 u_{111} - 10u_1 u_{12} - 5u_{11} u_2 - \frac{5}{4} u_2^2 + \frac{10}{3} u_1 u_3 - u_{11111}
\]
\[
- \frac{5}{3} u_{1112} + \frac{5}{4} u_{122} + \frac{5}{4} u_{14} + \frac{5}{3} u_{23} + u_5,
\]

\[
\vdots
\]

where once again we use the subscript $i$ rather than $t_i$ to denote partial derivatives of $u$. We see that the flows corresponding to even times are trivial. In the odd orders we find the pKdV equations,

\[
u_2 = 0,
\]
\[
u_3 = 3u_1^2 + u_{111},
\]
\[
u_4 = 0,
\]
\[
u_5 = 10u_1^3 + 5u_{11}^2 + 10u_1 u_{111} + u_{11111},
\]

\[
\vdots
\]

For $i \geq 1$, the equations $F_{ij} = 0$ are consequences of these equations.
5.3.2. Pluri-Lagrangian structure

A pluri-Lagrangian description of Equation (19) was found in [14], the Lagrange function itself goes back to [8]. It reads

\[ L(V, V_i, V_j, V_{ij}, \alpha_i, \alpha_j) = V(V_i - V_j) - (\alpha_i - \alpha_j) \log(V_i - V_j). \]

Using Lemma 8, we see that this Lagrangian is equivalent to (denoted with \( = \) by abuse of notation)

\[ L(V, V_i, V_j, V_{ij}, \alpha_i, \alpha_j) = \frac{1}{2}(V - V_{ij})(V_i - V_j) + (\alpha_i - \alpha_j) \log(V_i - V_j). \]

In terms of \( U \) and \( \lambda \) it is (up to a constant)

\[ L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = \frac{1}{2} \left( U - U_{ij} - \lambda_i^{-1} - \lambda_j^{-1} \right) \left( U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1} \right) + \left( \lambda_i^{-2} - \lambda_j^{-2} \right) \log \left( 1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}} \right). \]

**Lemma 9.** \( L_0(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = (\lambda_i^{-1} + \lambda_j^{-1})(U_i - U_j) + (\lambda_i^{-1} - \lambda_j^{-1})(U - U_{ij}) \) is a null Lagrangian.

**Proof.** Consider the discrete 1-form \( \eta \) defined by \( \eta(U, U_i, \lambda_i) = \lambda_i^{-1}(U + U_i) \) and, dictated by symmetry, \( \eta(U_i, U, \lambda_i) = -\lambda_i^{-1}(U + U_i) \). Its discrete exterior derivative is

\[ \Delta \eta(U, U_i, U_{ij}, U_j, \lambda_i, \lambda_j) = \frac{U + U_i}{\lambda_i} + \frac{U_i + U_{ij}}{\lambda_j} - \frac{U_{ij} + U_j}{\lambda_i} - \frac{U_j + U}{\lambda_j} = L_0. \]

Lemma 9 implies that \( L \) is equivalent to

\[ L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = \frac{1}{2} \left( U - U_{ij} - 2\lambda_i^{-1} - 2\lambda_j^{-1} \right) \left( U_i - U_j \right) + \left( \lambda_i^{-2} - \lambda_j^{-2} \right) \log \left( 1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}} \right). \]  

\[ (21) \]

To see why this Lagrangian is preferable, do a first order Taylor expansion of the logarithm and admire the cancellation. Thanks to this cancellation we avoid terms of non-positive order in the series expansion.

Applying the Miwa correspondence [6] with \( c = -2 \), a Taylor expansion, and the Euler-Maclaurin formula to the Lagrangian (21), we obtain a power series

\[ \mathcal{L}_{\text{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{ij=1}^{\infty} 4(-1)^{i+j} \frac{i+j}{ij} \mathcal{L}_{ij}[u]\lambda_i^j\lambda_2, \]
whose coefficients define a continuous pluri-Lagrangian 2-form for the KdV hierarchy. The first row of coefficients reads:

\[
\begin{align*}
L_{12} &= u_1 u_2 \\
L_{13} &= -2u_1^3 - u_1 u_{111} + \frac{3}{4} u_2^2 + u_1 u_3 \\
L_{14} &= -4u_2^3 u_2 - \frac{4}{3} u_1 u_{112} - \frac{2}{3} u_{11} u_{12} - \frac{2}{3} u_{111} u_2 + \frac{4}{3} u_2 u_3 + u_1 u_4 \\
L_{15} &= \frac{10}{3} u_1 u_{11}^2 - \frac{5}{2} u_1 u_2^2 - \frac{10}{3} u_1^2 u_3 + \frac{5}{9} u_{11} u_{111} + \frac{1}{9} u_1 u_{1111} - \frac{10}{9} u_1 u_{113} - \frac{5}{6} u_2^2 \\
&\quad - \frac{5}{12} u_1 u_{122} - \frac{5}{9} u_{11} u_{13} - \frac{5}{6} u_{112} u_2 - \frac{5}{12} u_{11} u_{22} - \frac{5}{9} u_1 u_{111} + \frac{5}{9} u_3 + \frac{5}{3} u_2 + \frac{5}{2} u_4 + u_1 u_5 \\
\vdots
\end{align*}
\]

Note that we can get rid of the alien derivatives in each \( L_{1j} \) by adding a total derivative \( D_{t_1} c_j \) and discarding terms that have a double zero on solutions. To make sure we get an equivalent Lagrangian 2-form, we also add \( D_{t_i} c_j \) to the coefficients \( L_{ij} \), which amounts to adding the closed form \( d \left( \sum c_j dt_j \right) \) to \( L \). In this particular example we take

\[
\sum c_j dt_j = \left( \frac{4}{3} u_1 u_{12} - \frac{2}{3} u_{11} u_2 \right) dt_4 \\
+ \left( \frac{10}{3} u_1^2 u_{11} - \frac{4}{9} u_{11} u_{111} - \frac{1}{9} u_1 u_{1111} + \frac{10}{9} u_1 u_{13} + \frac{5}{12} u_1 u_{22} - \frac{5}{9} u_1 u_3 \right) dt_5 \\
+ \ldots.
\]

Now that we have disposed of the alien derivatives in the \( L_{1j} \), we can use Theorem 7 to eliminate the remaining alien derivatives in all other \( L_{ij} \). For \( i < j \leq 5 \), the coefficients obtained this way are displayed in Table 1.

Note that the equations \( u_{t_{2i}} = 0 \) restrict the dynamics to a space of half the dimension. We can also restrict the pluri-Lagrangian formulation to this space:

\[
L = \sum_{i,j} L_{2i+1,2j+1} dt_{2i+1} \wedge dt_{2j+1}
\]

is a pluri Lagrangian 2-form for the hierarchy of non-trivial pKdV equations,

\[
\begin{align*}
u_3 &= 3u_1^2 + u_{111}, \\
u_5 &= 10u_1^3 + 5u_{11}^2 + 10u_1 u_{111} + u_{11111}, \\
\vdots
\end{align*}
\]
\[ \mathcal{L}_{12} = u_1 u_2 \]
\[ \mathcal{L}_{13} = -2u_1^3 - u_1 u_{111} + u_1 u_3 \]
\[ \mathcal{L}_{14} = u_1 u_4 \]
\[ \mathcal{L}_{15} = -5u_1^4 + 10u_1 u_{11}^2 - u_{111}^2 + u_1 u_5 \]
\[ \mathcal{L}_{23} = -3u_1^2 u_2 - u_1 u_{112} + u_{11} u_{12} - u_{111} u_2 \]
\[ \mathcal{L}_{24} = 0 \]
\[ \mathcal{L}_{25} = -10u_1^3 u_2 + 20u_1 u_{11} u_{12} - 5u_{11}^2 u_2 - 10u_1 u_{111} u_{12} - 2u_{111} u_{112} + 2u_{111} u_{12} - u_{1111} u_2 \]
\[ \mathcal{L}_{34} = 3u_1^2 u_4 + u_1 u_{114} - u_{11} u_{14} + u_{111} u_4 \]
\[ \mathcal{L}_{35} = 6u_1^5 - 15u_1^2 u_{11}^2 + 20u_1^3 u_{111} - 10u_1^4 u_3 + 7u_1^2 u_{111} + 6u_1 u_{11}^2 u_{111} - 12u_1 u_{111} u_{1111} + 3u_1^2 u_{1111} + 20u_1 u_{11} u_{113} - 5u_{11}^2 u_3 - 10u_1 u_{111} u_3 + 3u_1^2 u_5 - u_{1111}^2 + u_{111} u_{111} - 2u_{1111} u_{113} + u_{11} u_{115} + 2u_{1111} u_{13} - u_{111} u_{15} - u_{11111} u_{13} + u_{11111} u_5 \]
\[ \mathcal{L}_{45} = -10u_1^3 u_4 + 20u_1 u_{11} u_{14} - 5u_{11}^2 u_4 - 10u_1 u_{111} u_{14} - 2u_{111} u_{114} + 2u_{111} u_{14} - u_{11111} u_4 \]

\textbf{Table 1:} Coefficients $\mathcal{L}_{ij}$ for $H1$, after eliminating alien derivatives.

On the level of equations we could have restricted to the odd-numbered coordinates $t_1, t_3, \ldots$ from the beginning. However, on the level of Lagrangians we need to consider the even-numbered coordinates as well, at least in the theoretical arguments, because otherwise there is no interpretation for the (generally non-zero) coefficients of $\lambda_1^{2i} \lambda_2^{2j}$ in the power series $\mathcal{L}_{Miwa}$

5.3.3. The double continuum limit of Wiersma and Capel

In [27] Wiersma and Capel presented a continuum limit of the lpKdV equation

\[ (\mu_1 + \mu_2 + U_{12} - U)(\mu_1 - \mu_2 + U_1 - U_2) = \mu_1^2 - \mu_2^2, \tag{22} \]

which is equivalent to equation (20) under the transformation $\mu_i = \lambda_i^{-1}$. Their procedure consists of two steps. First they obtain a hierarchy of differential-difference equations. A second continuum limit, applied to any single equation of this hierarchy, then yields the potential KdV hierarchy. Some ideas concerning this limit procedure were already developed in [21, 23]. Here we will summarize both limits in one step.

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The limit procedure from [27] uses the lattice parameters $\nu = \mu_1 - \mu_2$ and \mu_1 itself, and skew lattice coordinates:
\[ V(n, m) = U(n - m, m). \]

Consider an interpolating function $u$. If
\[ V(n, m) = U(n - m, m) = u(t_1, t_3, t_5, \ldots), \]
then after the double limit of [27], lattice shifts correspond to multi-time shifts as follows:
\[ V_1 = U_1 = u\left(t_1 - \frac{2}{\mu_1}, t_3 - \frac{2}{3\mu_1^3}, t_5 - \frac{2}{5\mu_1^5}, \ldots\right) \]
and
\[ V_2 = U_{-1,2} = u\left(t_1 + \frac{2}{\mu_1^2} - \frac{\nu^2}{\mu_1^3} + \frac{\nu^3}{3\mu_1^4} - \ldots, t_3 + \frac{2}{\mu_1^1} - \frac{\nu^2}{2\mu_1^2} + \frac{\nu^3}{3\mu_1^3} \ldots, t_5 + \frac{2}{\mu_1^1} - \frac{\nu^3}{6\mu_1^2} + \frac{\nu^4}{3\mu_1^3} \ldots \right). \]

The series occurring here can be recognized as Taylor expansions:
\[ V_2 = u\left(t_1 - \frac{2}{\mu_1 + \nu} - \frac{2}{\mu_1}, t_3 - \frac{1}{3} \left(\frac{2}{(\mu_1 + \nu)^3} - \frac{2}{\mu_1^3}\right), t_5 - \frac{1}{5} \left(\frac{2}{(\mu_1 + \nu)^5} - \frac{2}{\mu_1^5}\right), \ldots\right). \]

Going back to the straight lattice coordinates and the original lattice parameters $\mu_1$ and $\mu_2 = \mu_1 + \nu$, we find
\[ U_2 = V_{12} = u\left(t_1 - \frac{2}{\mu_2}, t_3 - \frac{2}{3\mu_2^3}, t_5 - \frac{2}{5\mu_2^5}, \ldots\right), \]
\[ U_1 = V_1 = u\left(t_1 - \frac{2}{\mu_1}, t_3 - \frac{2}{3\mu_1^3}, t_5 - \frac{2}{5\mu_1^5}, \ldots\right). \]

Hence the end result of the double limit of Wiersma and Capelli is the same as the limit we obtain using the odd-numbered Miwa variables only.

5.4. Cross-ratio equation (Q1$_{\delta=0}$)

5.4.1. Equation

Consider equation Q1 from the ABS list [2], with parameter $\delta = 0$,
\[ \alpha_1(V_2 - V)(V_{12} - V_1) - \alpha_2(V_1 - V)(V_{12} - V_2) = 0. \]
It is also known as the cross-ratio equation \[4, 19\] and as the lattice Schwarzian KdV equation \[12, \text{Chapter } 3\]. As before, we would like to view Equation (24) as a consistent numerical discretization of some differential equation. To achieve this, we identify \(\alpha_1 = \lambda_1^2\) and \(\alpha_2 = \lambda_2^2\). Then Equation (19) is equivalent to

\[
\frac{V_1 - V_12 - V_2}{\lambda_1} - \frac{V_2 - V_12 - V_1}{\lambda_2} = 0. \tag{25}
\]

If we identify \(V = v(t_1, \ldots), V_i = v(t_1 + \lambda_i, \ldots),\) etc., then the leading order expansion yields \(v_{t_1}^2 - v_{t_1}^2 = 0\). This is a tautological equation, as desired. Hence in this case there is no need for an additional change of variables.

Once more we use the Miwa correspondence \(8\) with \(c = -2\). A Taylor expansion of (25) yields

\[
\sum_{i,j} 4(-1)^{i+j} \mathcal{F}_{ij}[v] \lambda_1^i \lambda_2^j = 0
\]

with

\[
\begin{align*}
\mathcal{F}_{01} &= v_1 v_2, \\
\mathcal{F}_{02} &= \frac{3}{2} v_{11}^2 - v_1 v_{111} + \frac{3}{2} v_1 v_{12} + \frac{3}{2} v_{11} v_2 + \frac{3}{8} v_2^2 + v_1 v_3, \\
\mathcal{F}_{03} &= \frac{8}{3} v_{11} v_{111} - \frac{4}{3} v_1 v_{1111} + 4 v_{11} v_{12} + \frac{4}{3} v_1 v_{13} + \frac{4}{3} v_{11} v_2 + 2 v_{12} v_2 + v_1 v_{22} + \frac{4}{3} v_{11} v_3 \\
&\quad + \frac{2}{3} v_2 v_3 + v_1 v_4, \\
\mathcal{F}_{04} &= -\frac{10}{9} v_{111}^2 - \frac{5}{3} v_{11} v_{1111} + v_1 v_{11111} + \frac{5}{3} v_1 v_{1112} - 5 v_{11} v_{112} - 10 \frac{5}{3} v_{111} v_{12} - 5 \frac{2}{2} v_{12}^2 \\
&\quad - \frac{5}{4} v_1 v_{122} - \frac{10}{3} v_{11} v_{13} - \frac{5}{4} v_1 v_{14} - \frac{5}{6} v_{1111} v_2 - \frac{5}{2} v_{11} v_{12} v_2 - 5 \frac{1}{3} v_1 v_2 v_2 - \frac{5}{4} v_{11} v_{22} \\
&\quad - \frac{5}{8} v_{22} v_{22} - 5 \frac{5}{3} v_1 v_{23} - 10 \frac{9}{9} v_{111} v_3 - 5 \frac{3}{3} v_{12} v_3 - \frac{5}{18} v_2^2 - \frac{5}{4} v_{11} v_4 - \frac{5}{8} v_2 v_4 - v_1 v_5, \\
\vdots
\end{align*}
\]

We assume that \(v_1 \neq 0\). Then we see that the flows corresponding to even times are trivial. In the odd orders we find the hierarchy of Schwarzian KdV equations,

\[
\begin{align*}
v_2 &= 0, \\
\frac{v_3}{v_1} &= -\frac{3 v_{11}^2}{2 v_1^2} + \frac{v_{111}}{v_1}, \\
v_4 &= 0, \\
\frac{v_5}{v_1} &= -\frac{45 v_{11}^4}{8 v_1^4} + \frac{25 v_{11}^2 v_{111}}{2 v_1^3} - \frac{5 v_{1111}^2}{2 v_1^2} - \frac{5 v_{111} v_{1111}}{v_1^2} + \frac{v_{11111}}{v_1}, \\
\vdots
\end{align*}
\]
For $i \geq 1$, the equations $F_{ij} = 0$ are differential consequences of these equations.

5.4.2. Pluri-Lagrangian structure

A Pluri-Lagrangian description of Equation (24) was found in [14]

$$L = \alpha_i \log(V - V_i) - \alpha_j \log(V - V_j) - (\alpha_i - \alpha_j) \log(V_i - V_j),$$

which is equivalent to

$$L = \lambda_i^2 \log \left( \frac{V - V_i}{\lambda_i} \right) - \lambda_j^2 \log \left( \frac{V - V_j}{\lambda_j} \right) - (\lambda_i^2 - \lambda_j^2) \log \left( \frac{V_i - V_j}{\lambda_i - \lambda_j} \right).$$

Each term of the series $L_{\text{Miwa}}$ constructed from this discrete Lagrangian contains strictly positive powers of both $\lambda_i$ and $\lambda_j$. Thus by Theorem 5 we can identify the coefficients of this power series with the coefficients of a pluri-Lagrangian 2-form. Some of these coefficients, after eliminating alien derivatives, are given in Table 2.

Again we can restrict the pluri-Lagrangian formulation to a space of half the dimension and consider

$$L = \sum_{i,j} L_{2i+1,2j+1} dt_{2i+1} \wedge dt_{2j+1}$$

as a pluri Lagrangian 2-form for the non-trivial equations of the SKdV hierarchy.

5.4.3. The generating PDE of Nijhoff, Hone, and Joshi

Nijhoff, Hone, and Joshi [21] introduced a non-autonomous PDE for a function $z_{n,m}(t, s)$ depending on a pair of continuous variables $(s, t)$, and a pair of parameters $(m, n)$. They noted that the flow of this PDE in continuous $(s, t)$-coordinates commutes with the difference equations

$$\frac{(z_{n,m} - z_{n+1,m})(z_{n,m+1} - z_{n+1,m+1})}{(z_{n,m} - z_{n,m+1})(z_{n+1,m} - z_{n+1,m+1})} = \frac{s}{t}. \quad (28)$$

Equation (28) is nothing but equation $Q_1$ of $\delta = 0$. Hence it is possible to switch between the continuous and discrete picture by reversing the roles of parameters and independent variables.

The main feature of the PDE in question is that it generates the SKdV hierarchy through the identification

$$z_{n,m}(t, s) = v \left( x_1 + \frac{2n}{t}, x_3 + \frac{2n}{3t}, \ldots, x_{2j+1} + \frac{2n}{(2j + 1)(t^{2j+1})}, \ldots \right).$$

Note that there is an error in the second SKdV equation as stated in [21]: the Lagrangian is missing the term $-z_{i,j}'x_i z_i^2$ at the corresponding order and in the equation itself the factor 2 of the first term should be removed.
\[
L_{12} = \frac{v_{11}}{2v_1} - \frac{v_2}{4v_1} \\
L_{13} = \frac{v_{11}}{4v_1} - \frac{v_3}{4v_1} \\
L_{14} = \frac{v_{11}^3}{3v_1^3} - \frac{v_{11}v_{111}}{3v_1^2} - \frac{v_4}{4v_1} \\
L_{15} = -\frac{5v_4^4}{16v_1^4} + \frac{5v_{11}^2v_{11}}{18v_1^3} - \frac{5v_{111}}{36v_1^2} + \frac{5v_{11}v_{1111}}{36v_1} - \frac{v_{1111}}{16v_1} - \frac{v_5}{4v_1} \\
L_{23} = \frac{v_{11}v_{12}}{4v_1} + \frac{v_{11}v_{12}}{4v_1^2} - \frac{v_{13}}{2v_1} + \frac{v_{11}v_{12}}{4v_1^2} - \frac{v_{111}v_{12}}{4v_1^2} \\
L_{24} = -\frac{v_{11}v_{12}}{3v_1^3} + \frac{v_{11}v_{12}}{3v_1^3} - \frac{v_{14}}{2v_1} \\
L_{25} = \frac{v_{11}v_{112}}{36v_1} + \frac{v_{11}v_{112}}{9v_1^2} + \frac{v_{11}v_{112}}{2v_1^3} - \frac{7v_{111}v_{112}}{18v_1^2} + \frac{v_{11}v_{12}}{4v_1} - \frac{11v_{11}v_{111}v_{12}}{9v_1^3} \\
+ \frac{19v_{11}v_{111}v_{12}}{36v_1^2} - \frac{v_{15}}{2v_1} + \frac{27v_{11}v_{12}}{32v_1^2} - \frac{17v_{11}v_{111}v_{12}}{8v_1^2} + \frac{7v_{11}v_{12}}{8v_1^2} + \frac{3v_{11}v_{111}v_{12}}{4v_1^2} \\
- \frac{v_{111}v_{12}}{4v_1^2} \\
L_{34} = -\frac{v_{11}v_{113}}{3v_1^3} - \frac{v_{114}}{4v_1} + \frac{v_{11}v_{13}}{3v_1^3} - \frac{v_{11}v_{14}}{4v_1^2} - \frac{v_{11}v_{14}}{8v_1^3} + \frac{v_{11}v_{14}}{4v_1^2} \\
L_{35} = \frac{45v_{11}^6}{64v_1^6} - \frac{57v_{11}v_{111}}{32v_1^5} + \frac{19v_{11}^2v_{111}}{16v_1^4} - \frac{7v_{111}^3}{8v_1^3} + \frac{3v_{11}v_{111}v_{1111}}{8v_1^2} + \frac{3v_{11}v_{111}v_{1111}}{4v_1^2} - \frac{v_{1111}}{4v_1^2} \\
- \frac{3v_{11}v_{1111}}{8v_1^3} + \frac{v_{11}v_{1111}}{4v_1^2} - \frac{v_{11}v_{1111}}{36v_1} + \frac{v_{11}v_{1111}}{9v_1^2} + \frac{v_{11}v_{113}}{2v_1^3} - \frac{18v_{111}}{4v_1} \\
+ \frac{v_{11}v_{113}}{4v_1^2} - \frac{11v_{11}v_{111}v_{113}}{9v_1^3} + \frac{19v_{11}v_{111}v_{113}}{36v_1^2} - \frac{v_{11}v_{15}}{36v_1} + \frac{27v_{11}v_{13}}{32v_1^2} - \frac{17v_{11}v_{111}v_{13}}{8v_1^2} \\
- \frac{v_{11}v_{13}}{4v_1^2} - \frac{v_{11}v_{13}}{8v_1^3} + \frac{v_{11}v_{13}}{4v_1^2} + \frac{7v_{11}v_{111}v_{13}}{8v_1^3} + \frac{3v_{11}v_{111}v_{13}}{4v_1^2} \\
L_{45} = -\frac{v_{11}v_{114}}{36v_1} + \frac{v_{11}v_{114}}{9v_1^2} + \frac{v_{11}v_{114}}{2v_1^3} - \frac{7v_{111}v_{114}}{18v_1^2} + \frac{v_{11}v_{115}}{3v_1^2} + \frac{v_{11}v_{14}}{4v_1^4} - \frac{11v_{11}v_{111}v_{14}}{9v_1^3} \\
+ \frac{19v_{11}v_{111}v_{14}}{36v_1^2} - \frac{v_{11}v_{15}}{3v_1^2} + \frac{27v_{11}v_{14}}{32v_1^2} - \frac{17v_{11}v_{111}v_{14}}{8v_1^2} + \frac{7v_{11}v_{14}}{8v_1^2} + \frac{3v_{11}v_{111}v_{14}}{4v_1^2} \\
- \frac{v_{11}v_{114}}{4v_1^2} \\

Table 2: Coefficients $L_{ij}$ for Q1, after eliminating alien derivatives.
Because of this it has become known as the generating PDE \cite{14,28} for the SKdV hierarchy. Renaming the parameters $t = \lambda^2_1$ and $s = \lambda^2_2$ we obtain once again the odd order Miwa shifts

$$z_{n,m} = v \left( x_1 + \frac{2n}{\lambda_1} + \frac{2m}{\lambda_2}, x_3 + \frac{2n}{3\lambda_1^2} + \frac{2m}{3\lambda_2^2}, \ldots,\right.$$

$$x_{2j+1} + \frac{2n}{(2j+1)\lambda_1^{2j+1}} + \frac{2m}{(2j+1)\lambda_2^{2j+1}}, \ldots \big).$$

Hence our continuum limit of $Q_{1,\delta=0}$ is implicitly present in \cite{21}. The relation between the (non-autonomous) generating PDE, the quad equation, and the hierarchy of (autonomous) PDEs is illustrated in Figure 4.

### 6. Conclusion

We have presented a method to perform continuum limits of discrete pluri-Lagrangian systems. In this approach, a single (parameter-dependent) discrete equation produces a full hierarchy of differential equations, and the pluri-Lagrangian structure is carried over from the discrete system to the continuous one.

Although the method can be stated in a general way, it can only be executed if we can find a form of the discrete equation and its Lagrangian that allows a suitable Taylor expansion in the parameters. Finding such a form is a non-trivial task. We solved this on a case-by-case basis for a few important examples. In a future publication we will discuss many more examples, including all ABS equations of type Q.

### A. The variational bicomplex

This is a minimal introduction to the variational bicomplex. Much more on this topic can be found for example in \cite[Chapter 19]{9}.

Our starting point is the idea that the exterior derivative can be split into a vertical part $\delta$ and a horizontal part $d$. An $(a,b)$-form is a differential $(a + b)$-form structured as

$$\omega^{a,b} = \sum f_{I_1,\ldots,I_a}([u]) \delta u_{I_1} \wedge \ldots \wedge \delta u_{I_a} \wedge dt_{j_1} \ldots \wedge dt_{j_b}.$$
Some authors use contact forms instead of the $\delta u_I$, see for example \[3\]. We denote the space of $(a, b)$-forms by $\Omega^{(a,b)} \subset \Omega^{a+b}$. These spaces are related to each other by $d$ and $\delta$ as in the following diagram:

\[
\begin{array}{cccccccc}
\Omega^{(2,0)} & d & \to & \Omega^{(2,1)} & d & \to & \cdots & d & \to & \Omega^{(2,n-1)} & d & \to & \Omega^{(2,n)} \\
\delta & & \uparrow & & \delta & & \cdots & & \delta & & \cdots & & \delta \\
\Omega^{(1,0)} & d & \to & \Omega^{(1,1)} & d & \to & \cdots & d & \to & \Omega^{(1,n-1)} & d & \to & \Omega^{(1,n)} \\
\delta & & \uparrow & & \delta & & \cdots & & \delta & & \cdots & & \delta \\
\Omega^{(0,0)} & d & \to & \Omega^{(0,1)} & d & \to & \cdots & d & \to & \Omega^{(0,n-1)} & d & \to & \Omega^{(0,n)}
\end{array}
\]

The horizontal and vertical exterior derivatives are characterized by the anti-derivation property,

\[
\begin{align*}
\delta (\omega_1^{p_1,q_1} \wedge \omega_2^{p_2,q_2}) &= \delta \omega_1^{p_1,q_1} \wedge \omega_2^{p_2,q_2} + (-1)^{p_1+q_1} \omega_1^{p_1,q_1} \wedge \delta \omega_2^{p_2,q_2}, \\
\delta^2 &= d^2 = \delta d + d \delta = 0.
\end{align*}
\]

and by the way they act on $(0, 0)$-forms, and basic $(1, 0)$ and $(0, 1)$-forms:

\[
\begin{align*}
df &= \sum_j (D_t^j f) dt_j \\
\delta f &= \sum_I \frac{\partial f}{\partial u_I} \delta u_I \\
d(\delta u_I) &= -\sum_j \delta u_{Ij} \wedge dt_j \\
\delta(\delta u_I) &= 0 \\
d(dt_j) &= 0 \\
\delta(dt_j) &= 0
\end{align*}
\]

One can verify that $d + \delta : \Omega^a \to \Omega^{a+1}$ is the usual exterior derivative and that

\[
\delta^2 = d^2 = \delta d + d \delta = 0.
\]

Furthermore, for any (generalized) vector field $V$, there holds

\[
\iota_V d + d \iota_V = 0.
\]

The derivative $D_t^i$ acts on elementary $(1, 0)$ forms as $D_t^i \delta u_I = \delta u_{I_t^i}$ and satisfies the Leibniz rule with respect to $\wedge$.

We assume that the vertical forms form a trivial bundle. Then we can consider the integral of an $(a, b)$-form over a $b$-dimensional manifold to be a vertical $(a, 0)$-tensor

\[
\sum_{I_1, \ldots, I_a} c_{I_1, \ldots, I_a} \delta u_{I_1} \wedge \cdots \wedge \delta u_{I_a}.
\]
The variational principle \( \delta \int_\Gamma L = 0 \), for an \((0,d)\)-form \( L = \sum L_1 \cdots d t_1 \wedge \cdots \wedge d t_d \), can be understood as follows. For every vertical vector field \( V = v \frac{\partial}{\partial u} \) that vanishes on the boundary \( \partial \Gamma \), its prolongation

\[
\text{pr} V = \sum_I (D_I v) \frac{\partial}{\partial u_I}
\]

must satisfy

\[
\int_\Gamma \iota_{\text{pr} V} \delta L = \int_\Gamma \sum_I \iota_{\text{pr} V} \delta L_1 \wedge \cdots \wedge dt_1 \wedge \cdots \wedge dt_d = 0.
\]

An important property of classical Lagrangian systems is that changing the Lagrangian by a full derivative (or divergence) does not affect the Euler-Lagrange equations. The following Proposition is the pluri-Lagrangian generalization of this property.

**Proposition 10.** The field \( u \) is a critical point of the action \( \int_\Gamma L[u] \) for all \( \Gamma \) if there exists a \((1, d-1)\)-form \( \Theta[u] \) such that \( \delta L = d\Theta \).

**Proof.** Since the horizontal exterior derivative \( d \) anti-commutes with the interior product operator \( \iota_{\text{pr} V} \) for the prolongation of a vertical vector field \( V \), it follows that for any variation \( V \) of the field \( u \) that is zero on the boundary of a manifold \( \Gamma \):

\[
\int_\Gamma \iota_{\text{pr} V} \delta L = -\int_\Gamma d (\iota_{\text{pr} V} \Theta) = -\int_{\partial \Gamma} \iota_{\text{pr} V} \Theta = 0. \quad \square
\]

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