Maximal entanglement concentration for \((n + 1)\)-qubit states

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Abstract

We propose two schemes for concentration of \((n + 1)\)-qubit entangled states that can be written in the form of \((\alpha|\varphi_0\rangle|0\rangle + \beta|\varphi_1\rangle|1\rangle)_{n+1}\) where \(|\varphi_0\rangle\) and \(|\varphi_1\rangle\) are mutually orthogonal \(n\)-qubit states. The importance of this general form is that the entangled states like Bell, cat, GHZ, GHZ-like, \(|\Omega\rangle\), \(|Q_s\rangle\), 4-qubit cluster states and specific states from the 9 SLOCC-nontwoblock families of 4-qubit entangled states can be expressed in this form. The proposed entanglement concentration protocol is based on the local operations and classical communications (LOCC). It is shown that the maximum success probability for ECP using quantum nondemolition (QND) technique is \(2\beta^2\) for \((n + 1)\)-qubit states of the prescribed form. It is shown that the proposed schemes can be implemented optically. Further it is also noted that the proposed schemes can be implemented using quantum dot and microcavity systems.

1 Introduction

Entanglement is one of the most important resources of quantum information processing. It plays a key role in quantum computation and communication. Specifically, maximally entangled states are often a prerequisite for quantum information processing. As entanglement between two or more distant parties cannot be created by local operations and classical communication (LOCC), it is required to be distributed. Usually a maximally entangled pure state is distributed among different parties by quantum channels and due to the presence of noise it transforms to either a mixed state or a less entangled pure state. Subsequently, maximally entangled state (MES) is extracted from an ensemble of mixed states or an ensemble of less entangled pure states. Entanglement concentration is a process in which one can extract maximally entangled state from nonmaximally (less entangled state) using LOCC with fidelity equal to 1. Similarly, the process of distilling a set of mixed states into a maximally entangled state is referred to as the entanglement purification.

Entanglement concentration protocol (ECP) was proposed by Bennett et al. \cite{1} in 1996. This pioneering protocol was based on the Schmidt projection method. Several ECPs were proposed thereafter using different methods. For example, entanglement-swapping \cite{2}, qubit-assisted \cite{3}, POVM \cite{4} and Bell-measurement \cite{5} based ECPs were proposed. Initial proposals for ECPs were restricted to the concentration of nonmaximally entangled Bell states. However, with the recent advances in applications of multipartite entangled states in quantum communication and quantum computation, it has become extremely interesting to design ECPs for multipartite entangled states. For example, in the recent past, ECPs for various multipartite entangled states like GHZ and cat state \cite{6}, GHZ-like state \cite{7}, cluster state \cite{8} \cite{9} \cite{10}, arbitrary W state \cite{11}, 4-qubit entangled state \cite{12}, etc. have been proposed. Nevertheless, apart from using different methods for designing ECPs, the possibility of implementation of ECPs is also studied for different technologies, like linear optics \cite{12} \cite{13} \cite{14} \cite{15} \cite{16}, quantum dot and microcavity systems \cite{17} \cite{18} \cite{19} \cite{20} and electronic technologies \cite{21} \cite{22}. Yamamoto et al. \cite{14} experimentally realized ECP from two copies of the state \(|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle\), where \(|\alpha|^2 + |\beta|^2 = 1\) using polarising beam splitters (PBSs), wave plates and photon detectors. The maximum probability achieved in these cases for single pair concentration is \(2|\alpha|^2|\beta|^2\). Another system which is used in ECP to obtain higher success probability is the cross-Kerr-linearity \cite{11} \cite{12}.

A qubit-assisted ECP for Bell state was initially proposed in \cite{3} using CNOT gate and Von Neumann measurement. There, it was shown that using iterative process maximal success probability probability up to \(2\beta^2\) can be obtained for Bell state. Recently, Sheng et al. \cite{12} have proposed a qubit-assisted concentration protocol for Bell state using PBS. They have shown that it is possible to obtain a higher success probability by iteratively using cross-Kerr-linearity. Various qubit-assisted entanglement concentration protocols \cite{23} \cite{24} \cite{25} \cite{26} \cite{27} \cite{28} \cite{29} \cite{30} for different entangled states using different systems have been reported thereafter, referring to the higher success probability mentioned in \cite{12}. However, upper bound on the success probability for ECP using quantum nondemolition (QND) technique is \(2\beta^2\).
Table 1: Entangled states and their corresponding less entangled counter parts expressed in the general form described by Eq. (1).

| $(n + 1)$-qubit state | Maximally entangled state | Non-maximally entangled state |
|------------------------|---------------------------|------------------------------|
| 2-qubit Bell           | $|00\rangle + |11\rangle$  | $(\alpha|0\rangle + \beta|1\rangle|1\rangle)_{(1+1)}$ |
| (2 + 1)-qubit GHZ      | $|000\rangle + |111\rangle$ | $(\alpha|00\rangle + \beta|11\rangle|1\rangle)_{(2+1)}$ |
| (n + 1)-qubit cat      | $|...00\rangle + |1...1\rangle$ | $(\alpha|00...0\rangle + \beta|1...1\rangle|1\rangle)_{(n+1)}$ |
| (2 + 1)-qubit GHZ-like | $|\psi^+\rangle + |\phi^+\rangle$ | $(\alpha|\psi^+\rangle + \beta|\phi^+\rangle|1\rangle)_{(2+1)}$ |
| $Q_5$ state            | $\frac{|0000\rangle + (|1011\rangle + |1101\rangle + |1110\rangle)}{\sqrt{2}}$ | $(\alpha|0000\rangle + |1111\rangle)0 + \beta(|101\rangle + |110\rangle)|1\rangle)_{(3+1)}$ |
| Cluster state          | $\frac{|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle}{\sqrt{2}}$ | $(\alpha|0000\rangle + |1100\rangle)0 + \beta(|001\rangle - |111\rangle)|1\rangle)_{(3+1)}$ |

probability was not discussed in these works. In what follows, we propose two ECPs for all $(n + 1)$-qubit entangled states that can be written in the form

$$|\alpha|\phi_0\rangle|0\rangle + |\beta|\phi_1\rangle|1\rangle)_{n+1},$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are mutually orthogonal $n$-qubit states and $\alpha^2 + \beta^2 = 1 : \alpha \neq \frac{1}{\sqrt{2}}$. Both the ECPs proposed here may be viewed as qubit-assisted ECPs as each of them is assisted by an ancillary qubit having the same Schmidt coefficients as that of the less entangled state to be concentrated. Further, we show that one of the proposed ECP which utilizes QND techniques achieves the maximum possible success probability (i.e., $2\beta^2$). To justify the importance of the states of the form (1), in Table 1 we have listed different entangled states which can be written in the general form described in Eq. (1). Further, it may be noted that the states of this particular form have vast applications in secure quantum communication, quantum secret sharing (QSS), bidirectional quantum teleportation and hierarchical quantum communication schemes \cite{31,32} e.g., hierarchical quantum information splitting (HQIS), probabilistic HQIS and hierarchical quantum secret sharing (HQSS). The general nature and applicability of quantum states of the form $(|\alpha|\phi_0\rangle|0\rangle + |\beta|\phi_1\rangle|1\rangle)_{n+1}$ have motivated us to construct ECPs for states of this form. Recently, we \cite{37} have proposed a Bell state-assisted ECP for the states of this particular form. Here we aim to propose two more ECPs for the same states and to improve the efficiency. The qubit-assisted ECP provides higher efficiency as shown in \cite{40} where efficiency of Bell state-assisted are compared with qubit-assisted ECP. Moreover, the latter requires less resource compared to the former. Further, if we use nonlinear resources (i.e., if we use QND technique or equivalently if we use cross-Kerr nonlinearity) then the efficiency approaches allowed upper bound.

Rest of the paper is organized as follows: In Section \ref{2} we have presented our scheme. In Section \ref{3} we have discussed that the entanglement is conserved during ECP and finally the paper is concluded in Section \ref{4}.

## 2 Maximal entanglement concentration

We have proposed the ECP for $(n + 1)$-qubit state of the form

$$|\psi\rangle = (|\alpha|\phi_0\rangle|0\rangle + |\beta|\phi_1\rangle|1\rangle)_{n+1}.$$ 

The protocol can be implemented on all states shown in Table 1. The protocol is described in Subsection \ref{2.1} and the corresponding quantum circuit is shown in Fig. 1. The purpose of presenting the quantum circuit is that the protocol can be realized in any technology. We propose concentration protocols using two independent methods: (i) post selection principle using linear optics and (ii) quantum nondemolition using cross-Kerr nonlinearity as described in Subsection \ref{2.2} and \ref{2.3} respectively. Many concentration protocols have been realized using linear optics as already stated in Section 1. Linear optics has two important features first is post selection and second is requirement of sophisticated single photon detectors. Post selection destroys the photons and they cannot be further used. The cross-Kerr nonlinearity is widely studied in the context of the CNOT gate \cite{33}, Bell state analysis \cite{34}, etc. It is a tool to construct quantum nondemolition detectors (QND) having the capability for conditioning the evolution of the system without necessarily destroying the state of the photon. It acts as parity check and single photon detector. Moreover, it is worthwhile to mention that the protocol here is implemented by linear optics, but can be used in other technologies as well for example in electronic systems like electronic polarization beam splitter and the charge detection \cite{10}, quantum dot spins \cite{35}, etc. The quantum nondemolition technique implemented using these electronic technologies have higher efficiency but the optimal value of success probability has not been attained in their works. In this work we show that by employing QND technique we can attain an optimal value concentration which is showed in Section \ref{2.3}.

### 2.1 Quantum circuit

Let Alice and Bob be two spatially separated parties. They share $N$ number of $n + 1$ pure entangled states such that Bob possesses first $n$ qubits and Alice possesses $(n + 1)^{th}$-qubit. The entangled state is of the form $|\psi\rangle = (|\alpha|\phi_0\rangle|0\rangle + |\beta|\phi_1\rangle|1\rangle)_{n+1}$
where $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are mutually orthogonal $n$-qubit states. The values of $\alpha$ and $\beta$ are real such that $\alpha^2 + \beta^2 = 1$ and $\alpha \neq \beta$. Alice also possess one more qubit $|\psi_s\rangle = \alpha|0\rangle + \beta|1\rangle$ which has the same Schmidt coefficients as that of the non-maximally entangled state shared by Alice and Bob. Therefore, Alice possesses two qubits which are $(n + 1)^{th}$ qubit and $(n + 2)^{th}$ qubit.

Input state of the circuit shown in Fig. 1 is

$$|\psi_1\rangle = (\alpha|\varphi_0\rangle|0\rangle + \beta|\varphi_1\rangle|1\rangle)_{1,2,\ldots,n+1} \otimes (\alpha|0\rangle + \beta|1\rangle)_{n+2}$$

(2)

Alice applies a CNOT gate on her qubits by using $(n + 1)^{th}$ qubit as the control qubit and the $(n + 2)^{th}$ qubit as the target qubit. As a consequence of this operation input state $|\psi_1\rangle$ transforms to

$$|\psi_2\rangle = (\alpha^2|\varphi_0\rangle|00\rangle + \alpha\beta|\varphi_1\rangle|11\rangle)_{1,2,\ldots,n+1,n+2} + \alpha\beta|\varphi_0\rangle|01\rangle + \beta^2|\varphi_1\rangle|10\rangle)_{1,2,\ldots,n+1,n+2}$$

(3)

Alice measures $(n + 2)^{th}$ qubit of $|\psi_2\rangle$ in the computational $\{|0\rangle,|1\rangle\}$ basis. From (3), we can easily observe that if Alice’s measurement yields $|1\rangle$ then the quantum state shared by Alice and Bob collapses to a normalized maximally entangled state $|\psi_{n+2}\rangle$. However, the circuit (equivalently an ECP represented by the circuit) fails when Alice’s measurement yields $|0\rangle$. Clearly, the success probability of the ECP is $2\alpha^2\beta^2$ and we are now left with a less entangled pure state with probability $1 - 2\alpha^2\beta^2$, i.e., $\alpha^4 + \beta^4$. We can repeat the ECP using the remaining less entangled pure states to maximize the number of maximally entangled states. The maximum probability that can be achieved by repeating this procedure is $2\beta^2$ which is provided later in Subsection 2.3.

### 2.2 Entanglement concentration using linear optics

In optical implementation of an ECP, we consider that qubits are realized using polarization states of photon where horizontal (H) and vertical (V) photon represent the logical bits 0 and 1. Thus, the quantum state to be concentrated can be expressed as $\alpha|\varphi_0H_a\rangle + \beta|\varphi_1V_a\rangle$. A linear optics based scheme for entanglement concentration of this state is shown in Fig. 2 and the action of the same is described below. Initial state of the system can be expressed as

$$|\psi_3\rangle = |\psi\rangle \otimes |\psi_s\rangle = [\alpha|\varphi_0H_a\rangle + \beta|\varphi_1V_a\rangle \otimes (\alpha|H_b\rangle + \beta|V_b\rangle)]_{n+2}$$

$$= \alpha^2|\varphi_0H_aH_b\rangle + \beta^2|\varphi_1V_aV_b\rangle + \alpha\beta|\varphi_0H_aV_b\rangle + \alpha\beta|\varphi_1V_aH_b\rangle$$

(4)

Here, subscripts $a$ and $b$ represent the modes of Alice’s photon. To be precise, $a$ corresponds to the $(n+1)^{th}$ qubit of the shared entangled state and $b$ corresponds to the additional single photon state $(n+2)^{th}$ qubit prepared by Alice to implement the ECP. Now, Alice uses a half wave plate HWP$_{90}$ to rotate the polarization of photon in mode $b$ by $90$ degrees to transform $|\psi_3\rangle$ to

$$|\psi_4\rangle = \alpha^2|\varphi_0H_aV_b\rangle + \beta^2|\varphi_1V_aH_b\rangle + \alpha\beta|\varphi_0H_aH_b\rangle + \alpha\beta|\varphi_1V_aV_b\rangle$$

(5)

Photons in mode $a$ and $b$ enter polarized beam splitter (PBS). As the PBS transmits the horizontal polarization component and reflects the vertical polarization component after the PBS the state of the system transforms to
Finally, if the detector detects and similarly if the detector
the Hamiltonian of the nonlinear medium or interaction between the signal beam and probe beam can be written as
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where

Note that in the first two terms in (6), the last two photons are in the same spatial mode, whereas in the last two terms
one photon is present in each output mode. Alice selects the case where each mode contains one photon. Therefore, after this
choice Alice and Bob obtain a shared state \( \frac{1}{\sqrt{2}} (|\varphi_0\rangle|H_0\rangle + |\varphi_1\rangle|V_0\rangle) \) with probability \( 2\alpha^2\beta^2 \). Now, Alice allows the photon in mode \( b' \) to pass through \( R_{45} \) which rotates the polarization by 45 degrees. As a consequence the state of the system transforms to

Finally, if the detector detects \( (n + 2) \) photon in the horizontal mode (i.e., if \( D_H \) clicks) then Alice and Bob obtain the state

and similarly if the detector \( D_V \) clicks, then they obtain the state

It is easy to observe that on application of the unitary operation in both the cases we obtain a maximally entangled state
\( \frac{1}{\sqrt{2}} (|\varphi_0\rangle|H\rangle + |\varphi_1\rangle|V\rangle) \). Therefore, the probability \( P_1 \) to obtain the maximally entangled state \( \frac{1}{\sqrt{2}} (|\varphi_0\rangle|H\rangle + |\varphi_1\rangle|V\rangle) \) or
\( \frac{1}{\sqrt{2}} (|\varphi_0\rangle|H\rangle - |\varphi_1\rangle|V\rangle) \) is \( 2(\alpha\beta)^2 \).

2.3 Entanglement concentration using QND detectors (cross-Kerr-nonlinearity)

Quantum nondemolition detector (QND) is generally based on cross-Kerr-nonlinearity. After their introduction by Nemoto
and Munro many works precisely in entanglement concentration protocols of pure entangled states were reported. The Hamiltonian of the nonlinear medium or interaction between the signal beam and probe beam can be written as

\[
\hat{H} = \hbar \chi \hat{a}_s \hat{a}_p
\]

where \( \hbar \chi \) is the coupling strength of the cross-Kerr material, \( \hat{a}_s \) and \( \hat{a}_p \) are the number operators of signal mode \( s \) and probe mode \( p \), respectively. Let the signal mode be in input state \( |\varphi\rangle = a|H\rangle + b|V\rangle \) and the probe mode be in a coherent state \( |\alpha\rangle \). The cross-Kerr nonlinear medium causes the combined input state of \( |\varphi\rangle \) and coherent state \( |\alpha\rangle \) to evolve as

\[
U_{cross-Kerr}|\varphi\rangle|\alpha\rangle = e^{i\theta t/\hbar} (a|0\rangle + b|1\rangle)|\alpha\rangle
\]

where \( \theta = \chi t \) and \( t \) is the interaction time. In Eq. 8, \( |n\rangle : n \in 0, 1 \) represents the Fock state contains \( n \) photons. Therefore, we observe that \( |\varphi\rangle \) remains same and the probe state picks up a phase shift and it can be measured by homodyne detection. The phase shift is directly proportional to the number of photons. In this way, we can nondestructively determinate the state of the signal mode by measuring the probe mode. Thus, QND can condition the evolution of the system by not destroying

Figure 2: Proposed linear optics circuit for ECP for non-maximally entangled \( (n + 1) \)-qubit state \( \alpha|\varphi_0\rangle|0\rangle + \beta|\varphi_1\rangle|1\rangle \).

\[
|\psi_5\rangle = \alpha^2|\varphi_0\rangle|H_0\rangle + \beta^2|\varphi_1\rangle|V_0\rangle + \alpha\beta|\varphi_0\rangle|H_0\rangle + \alpha\beta|\varphi_1\rangle|V_0\rangle.
\]
Therefore, if detector process. The probability of success of second iteration is given by

\[ P = P_1 + (1 - P_1) [P_2 + (1 - P_2) [P_3 + \cdots + (1 - P_{N-1}) P_N]]. \]
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3 Conservation of entanglement

It is well known that there exists a large number of measures of entanglement, each having its merit and demerit. In the investigations related to entanglement concentration and purification, a specific measure known as entanglement of single pair purification (ESPP) is often used [2, 3]. This measure was introduced by Bose et al. [2] as the maximum probability with which two parties sharing a non-maximally entangled pure Bell-type state can transform it to a Bell state by using LOCC (local operation and classical communication). Historically, Bose et al., defined ESPP in context of Bell states as in their original work they described an ECP for Bell state and showed that ESPP is such a measure of entanglement which remains conserved in the ECP described by them. Later on Bandyopadhyay [3] showed that ESPP remains conserved in a qubit-assisted ECP for Bell state. Motivated by this observation, here we plan to show that ESPP is also conserved in the QND-based ECP proposed here. To do so, first we need to generalize the definition of ESPP for states other than Bell as our ECP is valid for all states of the form described in Eq. (1). The generalized definition of ESPP is as follows: ESPP is the maximum probability with which we can obtain a MES from the corresponding less entangled state for

where $x = \frac{\beta}{\alpha}$ and $0 < x < 1$. For large $N$, i.e., $N \to \infty$, above series (the series present within square bracket of the expression of $P$) becomes an infinite series and that converges to 1 for $x \in (0, 1)$. Therefore, $P = 2 \beta^2 (\alpha^2 + \beta^2) = 2 \beta^2$. Thus, the maximum probability with which we can obtain a MES from the corresponding less entangled state for

The generalized definition of ESPP is as follows: ESPP is the maximum probability with which two parties sharing a non-maximally entangled pure entangled state can transform it to the corresponding maximally entangled state.

where $\alpha |0\rangle + \beta |1\rangle$ and $\alpha |0\rangle + \beta |1\rangle$ are the nonmaximally entangled pure state of the form provided in Eq. (1) and $|\psi\rangle = (\alpha |\varphi_0\rangle |0\rangle + \beta |\varphi_1\rangle |1\rangle)_{n+1}$ is $2 \beta^2$. It is a straightforward exercise to show that the same bound on the maximum success probability is also applicable to the recent proposals of Sheng et. al. [11, 12].

Figure 4: Schematic of ECP with QND.

This can be written as

$$P = 2 \alpha^2 \beta^2 + \frac{2 \alpha^2 + 2 \beta^2}{(\alpha^2 + \beta^2)^2} + \frac{2 \alpha^2 + 2 \beta^2}{(\alpha^2 + \beta^2)^2} \cdots + \frac{2 \alpha^2 + 2 \beta^2}{(\alpha^2 + \beta^2)^2} \cdots$$

$$+ \frac{2 \alpha^2 + 2 \beta^2}{(\alpha^2 + \beta^2)^2} \cdots (\alpha^2 + \beta^2)^2$$

$$= 2 \alpha^2 \beta^2 + 2 \beta^4 \left[\frac{1}{1+2x^2} + \frac{1}{1+2x^2} \cdots \right]$$

Thus, the maximum probability with which we can obtain a MES from the corresponding less entangled state for

$$(\alpha |\varphi_0\rangle |0\rangle + \beta |\varphi_1\rangle |1\rangle)_{n+1}$$

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3 Conservation of entanglement

It is well known that there exists a large number of measures of entanglement, each having its merit and demerit. In the investigations related to entanglement concentration and purification, a specific measure known as entanglement of single pair purification (ESPP) is often used [2, 3]. This measure was introduced by Bose et al. [2] as the maximum probability with which two parties sharing a non-maximally entangled pure Bell-type state can transform it to a Bell state by using LOCC (local operation and classical communication). Historically, Bose et al., defined ESPP in context of Bell states as in their original work they described an ECP for Bell state and showed that ESPP is such a measure of entanglement which remains conserved in the ECP described by them. Later on Bandyopadhyay [3] showed that ESPP remains conserved in a qubit-assisted ECP for Bell state. Motivated by this observation, here we plan to show that ESPP is also conserved in the QND-based ECP proposed here. To do so, first we need to generalize the definition of ESPP for states other than Bell as our ECP is valid for all states of the form described in Eq. (1). The generalized definition of ESPP is as follows: ESPP is the maximum probability with which two parties sharing a non-maximally entangled pure entangled state can transform it to the corresponding maximally entangled state by using LOCC. Following Lo and Popescu [20] we may note that the maximum probability with which a nonmaximally entangled pure state of the form provided in Eq. (1) can be concentrated is twice the modulus square of the Schmidt coefficient of the smaller magnitude. Therefore, before concentration the average entanglement shared between Alice and Bob is $E_{before} = 2 \beta^2$ where $\alpha > \beta$. In the proposed QND based protocol, the maximum probability for $(n+1)$-qubit state after the concentration is $E_{after} = 2 \alpha^2 \beta^2 + 2 \beta^4 = 2 \beta^2$. This implies that the average ESPP is conserved, the maximum probability to concentrate $(n+1)$-qubit state is $2 \beta^2$ and that the remaining will be totally disentangled if the protocol is continued for sufficiently long time. Here we have followed the works of Bose et al. [2] and Bandyopadhyay [3] to establish the conservation of entanglement in the similar sense. However, it may be noted that this conclusion (i.e., conservation of entanglement) is entanglement measure specific and it may and may not hold for another measure of entanglement.
4 Conclusion

We have proposed two practical schemes for entanglement concentration for all entangled states that can be expressed in the form $(\alpha |\varphi_0\rangle |0\rangle + \beta |\varphi_1\rangle |1\rangle)_{n+1}$. Specifically, we have proposed an ECP using linear optics i.e., using PBSs and single photon detectors. Subsequently, we have also proposed an ECP using cross-Kerr-nonlinearity and have obtained the maximum possible success probability $2\beta^2$. The proposed ECP uses less resources compared to the earlier proposals. We have presented a quantum circuit for the proposed ECP so that the protocol can be implemented using other technologies like atomic system, NMR, etc. The proposed ECP can be applied to many entangled states like Bell and cat states, GHZ, GHZ-like, $|\Omega\rangle$, $|Q_5\rangle$, 4-qubit cluster states and specific states from the 9 SLOCC-nonequivalent families of 4-qubit entangled states and other states which can be expressed as $(\alpha |\varphi_0\rangle |0\rangle + \beta |\varphi_1\rangle |1\rangle)_{n+1}$.

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