The compact support property for solutions to the stochastic partial differential equations with colored noise

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Abstract
We study the compact support property for nonnegative solutions of the following stochastic partial differential equations:

\[ \partial_t u = a^{ij} u_{x_i} x_j(t, x) + b^i u_x(t, x) + cu + h(t, x, u(t, x)) \hat{F}(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \]

where \( \hat{F} \) is a spatially homogeneous Gaussian noise that is white in time and colored in space, and \( h(t, x, u) \) satisfies \( K^{-1} u^\lambda \leq h(t, x, u) \leq K(1 + u) \) for \( \lambda \in (0, 1) \) and \( K \geq 1 \). We show that if the initial data \( u_0 \geq 0 \) has a compact support, then, under the reinforced Dalang’s condition on \( \hat{F} \) (which guarantees the existence and the Hölder continuity of a weak solution), all nonnegative weak solutions \( u(t, \cdot) \) have the compact support for all \( t > 0 \) with probability 1. Our results extend the works by Mueller-Perkins (21) and Krylov (15), in which they show the compact support property only for the one-dimensional SPDEs driven by space-time white noise on \((0, \infty) \times \mathbb{R}\).

Keywords: Stochastic partial differential equation, compact support property, colored noise, \( L_p \)-theory

MSC 2020 subject classification: 60H15, 35R60

1. Introduction

In this paper, we aim to study the compact support property of nonnegative solution to the following stochastic partial differential equations

\[ \partial_t u(\omega, t, x) = L u(\omega, t, x) + h(\omega, t, x, u(t, x)) \hat{F}(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d; \quad u(0, \cdot) = u_0, \]

where \( L \) is the random differential operator defined as

\[ Lu(\omega, t, x) = a^{ij}(\omega, t, x) u_{x_i x_j} + b^i(\omega, t, x) u_{x_i} + c(\omega, t, x) u \]

and \( h(\omega, t, x, u) \) is a function such that for some nonrandom constants \( K \geq 1 \) and \( \lambda \in (0, 1) \),

\[ K^{-1} u^\lambda \leq h(\omega, t, x, u) \leq K(1 + u) \quad \text{for all} \quad \omega \in \Omega, \quad t > 0, \quad x \in \mathbb{R}^d, \quad u \geq 0. \]

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We assume Einstein’s summation convention on $i$ and $j$, and $i$ and $j$ go from 1 to $d$. The detailed conditions on $a, b, c, \varphi$ and $h$ are specified below in Assumptions 2.3 and 2.4. The nonrandom initial data $u_0$ is a nonnegative Hölder continuous function with compact support. The noise $F$ is a centered generalized Gaussian random field whose covariance is given by

$$E[F(t, x)F(s, y)] = \delta_0(t - s)f(x - y),$$

where $\delta_0$ is the Dirac delta distribution and $f$ is a correlation function (or measure). In other words, $f$ is a nonnegative and nonnegative definite function/measure. We denote that the spectral measure $\mu$ of $f$ is a nonnegative tempered measure defined by

$$\mu(\xi) := F(f)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(dx)$$ (1.3)

which is the Fourier transform of $f$.

The compact support property (CSP) for (1.1) means that if the initial function $u_0$ has a compact support, then so does the solution $u(t, \cdot)$ for all $t > 0$ with probability 1, and this property has been studied extensively for one-dimensional stochastic heat equations driven by space-time white noise. More precisely, consider the following stochastic heat equations:

$$\partial_t u = u_{xx} + u^\lambda \dot{W}, \quad (t, x) \in (0, \infty) \times \mathbb{R}; \quad u(0, \cdot) = u_0,$$ (1.4)

where $\dot{W}$ is space-time white noise (i.e. $f = \delta_0$) and $\lambda > 0$ is a fixed constant.

When $\lambda = 1/2$ in (1.4), the solution to (1.4) can be regarded as a density of super-Brownian motion, and Iscoe (11) showed that CSP holds for (1.4) by using some singular elliptic boundary value problem related to properties of super-Brownian motion. Shiga (26) extended Iscoe’s technique to show CSP for (1.4) when $\lambda \in (0, 1/2)$. For $\lambda \in (1/2, 1)$, Mueller and Perkins (21) showed among other things that CSP holds for (1.4) by constructing and then applying a historical process that represents the ancestry of particles. In fact, the case where $\lambda \in (1/2, 1)$ is more delicate, as explained in [21], since this case can be regarded as a super-Brownian motion with birth and death rates slowed down (that is, the coefficient of the noise $u^\lambda$ is smaller than $u^{1/2}$ when $u$ is close to 0), but still the effect of noise is strong enough for (1.4) to have CSP. On the other hand, when $\lambda \geq 1$, Mueller (20) showed that CSP does not hold. In this case, when $u$ is close to 0, $u^\lambda$ for $\lambda \geq 1$ becomes quite small so that the positivity of the heat semigroup wins against the effect of noise. Hence, for (1.4), the critical exponent $\lambda$ that does not guarantee CSP is 1.

Regarding the one-dimensional SPDE (1.1) with the random second order differential operator $L$ and with space-time white noise, using completely different methods that are based on the weak solution to (1.1), Krylov (15) showed that CSP holds when $h(u) = u^\lambda$ for any $\lambda \in (0, 1)$ under the hypothesis of the existence of a suitable solution. Following the same argument by Krylov (15), Rippl (23) showed that CSP holds for (1.4) when $\dot{W}$ is replaced by $\dot{F}$ whose spatial covariance is of Riesz type, i.e., $f(x) = |x|^{-\alpha}$ for $\alpha \in (0, 1)$ and $x \in \mathbb{R}$. The proof therein strongly depends on the scaling property of the specific correlation function $|x|^{-\alpha}$. Moreover, both authors in [15] and [23] only considered the one-dimensional SPDEs, and the generalization of their techniques to higher dimensions is highly nontrivial, which is one of our main contributions in this paper.

When $d \geq 2$, in order to have a function-valued solution to (1.1), noise should be colored in space. In particular, when $L = \Delta$ and $h(u)$ is globally Lipschitz, Dalang (6) showed that if the spectral measure $\mu$ of $f$ satisfies

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty,$$ (1.5)
there exists a unique random field solution. However, this condition itself does not guarantee that the solution is continuous in $t$ and $x$ (see [3, 9]), and one sufficient condition that guarantees the Hölder continuity of the solution is the following:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^{1-\eta}} < +\infty \quad \text{for some } \eta \in (0, 1], \quad (1.6)$$

which is called the reinforced Dalang’s condition (see [2, 8, 23]). Under the condition (1.6), when $\mathcal{L} = \Delta$ and $h(u) \approx u$ in (1.4), following Mueller’s argument ([20]), Chen and Huang (2) showed that CSP does not hold. In addition, when $\mathcal{L} = \Delta$ and $h(u) = u^\lambda$ for $\lambda > 1$ in (1.1), following Mueller’s argument ([20]), one can show that CSP does not hold under the condition (1.6). Thus, a natural question arises whether CSP holds for (1.1) when $h$ satisfies (1.2) for $\lambda \in (0, 1)$. To the best of our knowledge, there is no literature on CSP for (1.1) even when $\mathcal{L} = \Delta$ on $\mathbb{R}^d$ for $d \geq 2$, and we provide an answer in the affirmative in this paper.

Our main result of this paper is that when $h$ satisfies (1.2) and the random second order differential operator $\mathcal{L}$ in (1.1) satisfies some natural condition (see Assumption 2.3), CSP holds for (1.1) only under the condition (1.6), which includes $f(x) = \delta_0(x)$, $f(x) = |x|^{-\alpha}$ for $\alpha \in (0, 2 \wedge d)$, $f \in C_c(\mathbb{R}^d)$, and $f(x) = 1$ for $x \in \mathbb{R}^d$. Note that $f \equiv 1$ means the noise is just white noise in time only, i.e., $F(t, x) = \hat{B}_t$, where $B_t$ is the standard one-dimensional Brownian motion. Therefore, our result highlights that CSP is essentially due to the sub-linearity $u^\lambda$ with $\lambda \in (0, 1)$, not to the choice of noise. In addition, it also suggests that, as for one-dimensional stochastic heat equations with space-time white noise, the critical exponent $\lambda$ that does not guarantee CSP for (1.1) is 1.

Let us now mention a few words about the proof. Since (1.1) includes the random second order differential operator $\mathcal{L}$ in (1.1) satisfies some natural condition (see Assumption 2.3), CSP holds for (1.1) only under the condition (1.6), which includes $f(x) = \delta_0(x)$, $f(x) = |x|^{-\alpha}$ for $\alpha \in (0, 2 \wedge d)$, $f \in C_c(\mathbb{R}^d)$, and $f(x) = 1$ for $x \in \mathbb{R}^d$. Note that $f \equiv 1$ means the noise is just white noise in time only, i.e., $F(t, x) = \hat{B}_t$, where $B_t$ is the standard one-dimensional Brownian motion. Therefore, our result highlights that CSP is essentially due to the sub-linearity $u^\lambda$ with $\lambda \in (0, 1)$, not to the choice of noise. In addition, it also suggests that, as for one-dimensional stochastic heat equations with space-time white noise, the critical exponent $\lambda$ that does not guarantee CSP for (1.1) is 1.

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on $a_1, a_2, \ldots, a_k$. The numeric value of $N$ can be changed line by line tacitly. For functions depending on $\omega, t,$ and $x$, the argument $\omega \in \Omega$ is omitted. Finally, for a function $\varphi(x)$, let $\hat{\varphi}(x)$ and $\hat{\varphi}(\xi)$ denote $\varphi(-x)$ and $\mathcal{F}(\varphi)(\xi)$, respectively.

2. Main Results

In this section, we first provide some basic definitions and assumptions related to \([1,11]\), and then state our main theorems (Theorems \([2,9]\) and \([2,11]\)).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, \({\mathcal{F}_t} : t \geq 0\) be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$ satisfying the usual conditions, and $\mathcal{P}$ be the predictable $\sigma$-field related to $\mathcal{F}_t$. Consider a mean-zero Gaussian process \({F(\phi) : \phi \in \mathcal{S}(\mathbb{R}^{d+1})}\) on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance functional given by

\[
\mathbb{E}[F(\phi)F(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} f(dx) (\phi(t, \cdot) * \hat{\psi}(t, \cdot))(x)
\]

\[
= \int_0^\infty dt \int_{\mathbb{R}^d} \mu(d\xi) \hat{\phi}(t, \xi) \overline{\hat{\psi}(t, \xi)},
\]

where $\hat{\psi}(t, x) := \psi(t, -x)$, $\hat{\phi}(t, \cdot) := \mathcal{F}(\phi(t, \cdot))(\xi)$ is the Fourier transform of $\phi$ with respect to $x$, $f$ is a nonnegative, and nonnegative definite tempered measure, and $\mu$ is the Fourier transform of $f$ introduced in \([1,3]\) so that it is again a nonnegative definite tempered measure. Following the Dalang-Walsh theory (see \([6,27]\)), the process $F$ can be extended to an $L^2(\Omega)$-valued martingale measure $F(ds, dx)$, where $F(ds, dx)$ is employed to formulate \([1,1]\) in the weak sense (see Definition \([2,7]\) and Remark \([2,8]\)). We impose the following assumption on $f$.

**Assumption 2.1.** The correlation measure $f$ satisfies the following: For some $\eta \in (0, 1]$,

\[
\begin{cases}
\int_{|x| < 1} |x|^{2 - 2\eta - d} f(dx) < +\infty & \text{if } 0 < 1 - \eta < \frac{d}{2}, \\
\int_{|x| < 1} \log \left( \frac{1}{|x|^d} \right) f(dx) < +\infty & \text{if } 1 - \eta = d/2, \\
\text{no conditions on } f & \text{if } 1 - \eta > d/2.
\end{cases}
\]

**Remark 2.2.** (i) \((2.2)\) is equivalent to the reinforced Dalang’s condition \((1.6)\) (see \([2,7]\) Proposition 5.3)]. As explained before, the reinforced Dalang’s condition guarantees that a solution exists and is Hölder continuous in $t$ and $x$ when $h$ is globally Lipschitz in $u$ in \([1,1]\) (see \([3]\) Theorem 6] or \([\text{5}\text{.3}]\)).

(ii) A large class of nonnegative, nonnegative definite tempered measures satisfies Assumption \((2.2)\).

For example, one may consider $f(dx) = f(x)dx$, where $f(x)$ can be the Riesz kernel $f(x) = |x|^{-\alpha}$ with $\alpha \in (0, 2 \land d)$, the Ornstein-Uhlenbeck-type kernel $f(x) = \exp(-|x|^\beta)$ with $\beta \in (0, 2]$, and the Brownian motion $f(x) \equiv 1$. Additionally, a continuous, nonnegative and nonnegative definite function $f$ with compact support satisfies \((2.2)\).

Below Assumptions \([2,3]\) and \([2,4]\) describe conditions on coefficients.

**Assumption 2.3.** (i) The coefficients $a_{ij}^t = a_{ij}(t,x)$, $b^i = b^i(t,x)$ and $c = c(t,x)$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) The functions $a_{ij}^t$, $a^i$, $b^i$, and $c$ are continuous in $x$. 

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Remark 2.5. (iii) There exists a finite constant $K \geq 1$ such that
\begin{equation}
K^{-1}|\xi|^2 \leq a^{ij} \xi^i \xi^j \leq K|\xi|^2 \quad \text{for all} \quad \omega \in \Omega, \ t > 0, \ x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d
\end{equation}
and
\begin{equation}
|a^{ij}(t, \cdot)|_{C^2(\mathbb{R}^d)} + |b^i(t, \cdot)|_{C^1(\mathbb{R}^d)} + |c(t, \cdot)|_{C(\mathbb{R}^d)} \leq K \quad \text{for all} \quad \omega \in \Omega, \ t > 0,
\end{equation}
where $C^k(\mathbb{R}^d) \ (k = 1, 2)$ is the set of all $k$ times continuously differentiable functions on $\mathbb{R}^d$ with finite norm
\[ \|u\|_{C^k(\mathbb{R}^d)} := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} |D^\alpha u(x)|. \]

**Assumption 2.4.** Let $\lambda \in (0, 1)$.

(i) The function $h(t, x, u)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R})$-measurable and $h(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for every $\omega \in \Omega, t > 0$ and $x \in \mathbb{R}^d$.

(ii) The function $h$ satisfies
\begin{equation}
h(t, x, u) = 0 \quad \text{for all} \quad \omega \in \Omega, \ t > 0, \ x \in \mathbb{R}^d, \ u \leq 0.
\end{equation}

(iii) There exists a finite constant $K \geq 1$ such that
\begin{equation}
K^{-1}u^\lambda \leq h(t, x, u) \leq K(1 + u) \quad \text{for all} \quad \omega \in \Omega, \ t > 0, \ x \in \mathbb{R}^d, \ u \geq 0.
\end{equation}

**Remark 2.5.** (i) The coefficients $a^{ij}, b^i, c,$ and $h$ are random functions.

(ii) Without loss of generality, we assume the same constant $K$ in Assumptions 2.3 and 2.4.

(iii) In (2.3), the upper bound of $h$ is used to obtain the existence and the Hölder regularity of a solution. On the other hand, the lower bound of $h$ is employed to calculate the probability estimate of the integral of the solution on circles $\{x : |x| = R\}$ and $\{x : |x| = R + r\}$ $(R, r > 0)$, which is essential to obtain the compact support property of a solution $u$; see Lemma 3.2 (ii).

Next we introduce the assumption on the initial data $u_0$.

**Assumption 2.6.** (i) The function $u_0(\cdot)$ is nonrandom, nonnegative, and $u_0(\cdot) \in C^n(\mathbb{R}^d)$, where $\eta$ is the constant introduced in (2.2).

(ii) The function $u_0(\cdot)$ has a compact support on $\mathbb{R}^d$. In other words, there exists $R_0 \in (0, \infty)$ such that support of $u_0$ is in $\{x \in \mathbb{R}^d : |x| < R_0\}$.

We now present the definition of a solution. First, we define
\[ C_{\text{tem}} := \left\{ u \in C(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |u(x)| e^{-a|x|} < \infty \text{ for any } a > 0 \right\}. \]

In the rest of this section, $\tau$ is any given bounded stopping time, i.e., $\tau \leq T$ for some nonrandom number $T > 0$ almost surely.

**Definition 2.7.** A continuous $C_{\text{tem}}$-valued function $u = u(t, \cdot)$ defined on $\Omega \times [0, \tau]$ is said to be a solution to (1.1) on $[0, \tau]$ if for any $\phi \in \mathcal{S}$
(i) the process \( \int_{\mathbb{R}^d} \phi(x)u(t \wedge \tau, x)dx \) is well-defined, \( \mathcal{F}_t \)-adapted, and continuous;

(ii) the process

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(t \wedge \tau, y, u(t \wedge \tau, y))\phi(y)h(t \wedge \tau, y + x, u(t \wedge \tau, y + x))\phi(y + x)f(dx)dy
\]

is well-defined, \( \mathcal{F}_t \)-adapted, and continuous;

(iii) the process

\[
\int_{0}^{\tau} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(t, y, u(t, y))\phi(y)h(t, y + x, u(t, y + x))\phi(y + x)f(dx)dydt < \infty
\]

Almost surely;

(iii) the equation

\[
(u(t, \cdot), \phi) = (u_0, \phi) + \int_{0}^{t} (u(s, \cdot), (\mathcal{L}^* \phi)(s, \cdot))ds + \int_{0}^{t} \int_{\mathbb{R}^d} h(s, x, u(s, x))\phi(x)F(ds, dx)
\]

holds for all \( t \leq \tau \) almost surely, where

\[
(\mathcal{L}^* \phi)(s, x) = (a_{ij}^x x_i x_j - b_i^x x_i + c) \phi + (2a_{ij}^x - b_i^x) \phi x_i + a_{ij}^x \phi x_i x_j.
\]

Remark 2.8. (i) In (2.7), the Dalang-Walsh theory is employed to define a stochastic integral with respect to \( F(ds, dx) \) and we call it Walsh’s stochastic integral; see [6, 7]. As in [5, 8], Walsh’s stochastic integral can be written as an infinite summation of Itô’s stochastic integral; for example, if \( X(t, \cdot) \) is a predictable process such that

\[
X(t, x) = \zeta(x)1_{[\tau_1, \tau_2]}(t),
\]

where \( \tau_1, \tau_2 \) are bounded stopping times, \( (\tau_1, \tau_2) := \{ (\omega, t) : \tau_1(\omega) < t \leq \tau_2(\omega) \} \), and \( \zeta \in C_{c}^{\infty} \), then we have

\[
\int_{0}^{t} \int_{\mathbb{R}^d} X(s, x)F(ds, dx) = \sum_{k=1}^{\infty} \int_{0}^{t} \int_{\mathbb{R}^d} X(s, x) (f * e_k)(x) dxdw_k^s,
\]

where \( \{ w_k^s, k \in \mathbb{N} \} \) is a collection of one-dimensional independent Wiener processes and \( \{ e_k, k \in \mathbb{N} \} \subseteq \mathcal{S} \) is a complete orthonormal system of a Hilbert space \( \mathcal{H} \) induced by \( f \); see [5, 8]. The construction and properties of \( \mathcal{H} \) and \( \{ e_k, k \in \mathbb{N} \} \) are described in [4, Remark 2.6].

(ii) Thanks to (2.6) and (2.9), the stochastic integral term in (2.7) can be written as

\[
\int_{0}^{t} \int_{\mathbb{R}^d} h(s, x, u(s, x))\phi(x)F(ds, dx) = \sum_{k=1}^{\infty} \int_{0}^{t} \int_{\mathbb{R}^d} h(s, x, u(s, x))\phi(x) (f * e_k)(x) dxdw_k^s.
\]

Throughout this paper, we always assume that Assumptions 2.1, 2.3, 2.4, and 2.6 hold. Now we show the existence and Hölder regularity of a solution. Since the proof of the following theorem is rather classical, we include the proof at the end (see Section A).
Theorem 2.9 (Weak existence and Hölder regularity of the solution). There exists a stochastically weak solution \( u \in C([0, \tau]; C_{tem}(\mathbb{R}^d)) \) to \((1.1)\) satisfying \( u \geq 0 \). In addition, for any solution \( u \in C([0, \tau]; C_{tem}(\mathbb{R}^d)) \) to \((1.1)\) and any \( \gamma \in (0, \eta/2) \), we have that for every \( a > 0 \),
\[
\|\Psi_a u\|_{C^\gamma([0, \tau] \times \mathbb{R}^d)} < \infty \quad \text{almost surely},
\]
where \( \eta \) is the constant introduced in \((2.2)\) and \( \Psi_a(x) = \Psi_a(|x|) = \frac{1}{\cosh(a|x|)} \).

Remark 2.10. When \( \mathcal{L} = \Delta \) in \((1.1)\), Mytnik-Perkins-Sturm \((22)\) showed the existence of a stochastically weak solution by using a compactness argument. Indeed, they also showed the pathwise uniqueness of a solution under certain conditions on \( h \) and \( f \). Here, we also use a compactness argument but in a different way. That is, we use the \( L_p \)-theory to get tightness conditions (see Section A). However, since we have the random second order differential operator in \((1.1)\), it is not easy to show the pathwise uniqueness by applying the arguments in \((22)\). Thus, we do not assume the solution is unique, but this does not affect our proof since we only care about the compact support property of any solution.

Now we introduce the main result of this paper, which states the compact support property holds for \((1.1)\).

Theorem 2.11 (Compact support property). Suppose \( u \) is a nonnegative solution to \((1.1)\) and \( R_0 \) is the constant introduced in Assumption 2.6. Then, for almost every \( \omega \), there exists \( R = R(\omega) \in (R_0, \infty) \) such that \( u(t, x) = 0 \) for all \( t \leq \tau \) and \( x \in \{ x \in \mathbb{R}^d : |x| > R \} \).

3. Proof of Theorem 2.11

In this section, we assume the results of Theorem 2.9 and provide the proof of Theorem 2.11. Our proof is based on the approach of Krylov \((15)\). We first present two lemmas (Lemmas 3.1 and 3.2), which are essential tools for the proof of Theorem 2.11 and whose proofs are given in Sections 5 and 4 respectively. Throughout this section, \( u \) is referred to any solution to \((1.1)\) that is nonnegative and continuous almost surely. We also use the following notations through the remaining part of the paper: For \( a > 0 \), define
\[
\Psi_a(x) := \Psi_a(|x|) := \frac{1}{\cosh(a|x|)}.
\]
For \( r > 0 \), set
\[
Q_r := \{ x \in \mathbb{R}^d : |x| > r \}.
\]
The measure \( d\sigma_R \) denotes surface measure on \( \partial Q_R \). Let \( T > 0 \) be an arbitrary fixed real constant and \( R_0 > 0 \) be the constant introduced in Assumption 2.6 i.e., \( \text{supp}(u_0) \subseteq Q_{R_0} \). The following lemma shows the limit behavior of \( u \) on \( \partial Q_R \) as \( R \to \infty \).

Lemma 3.1. Let \( \tau \leq T \) be a bounded stopping time. Suppose that there exist \( a, H \in (0, \infty) \) such that
\[
\sup_{t \leq \tau} \sup_{x \in \mathbb{R}^d} \Psi_a(x)u(t, x) \leq H \quad \text{almost surely},
\]
where \( \Psi_a \) is the function introduced in \((3.1)\). Then, for every \( \delta > 0 \), we have
\[
\limsup_{R \to \infty} E \left[ \int_0^\tau e^{\delta R} \int_{\partial Q_R} u(t, \sigma) d\sigma_R dt \right] = 0.
\]
In Lemma 3.2 below, a version of the maximum principle for \( u \) is introduced. In other words, we show that if \( u \) is zero on \( \partial Q_R \) almost surely, \( u \) is also zero on \( Q_R \). The last part of the lemma, which has a crucial role in the proof of Theorem 2.11, shows a probability estimate of the integral value of the solution on circles.

**Lemma 3.2.** Let \( \tau \leq T \) be a bounded stopping time and \( K \) be the constant introduced in Assumptions 2.3 and 2.4. Set \( l := \frac{2\lambda+a}{\gamma+d} \in (0,1) \), where \( \gamma \) is the constant introduced in Theorem 2.4. Suppose that there exist \( a, H \in (0, \infty) \) such that

\[
\|\Psi_a u\|_{C^2([0, \tau] \times R^d)} \leq H \quad \text{almost surely,}
\]  

where \( \Psi_a \) is the function introduced in (3.1). Let \( R_0 \) be the constant introduced in Assumption 2.6.

Then, we have the following:

(i) For \( R > R_0 \) ∨ 1,

\[
P \left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds = 0 \right) \leq P \left( u(s, x) = 0 \quad \text{for all } s \in [0, \tau] \text{ and } x \in Q_R \right).
\]

(ii) For \( R > R_0 \) ∨ 1, \( p, q > 0 \),

\[
0 < r < \frac{R}{2} \land \left[ 2^{-\frac{(\gamma+1)}{\gamma}} H^{-\frac{d}{\gamma}} R^{d-1} \left( \frac{d\pi^{d/2}}{1(d/2 + 1)} \right) \right] ^{\frac{r}{\gamma} + l} \land 1,
\]  

and \( \alpha \in (0, 1) \), there exists a point \( r \in (r, 2r) \) such that

\[
P \left( \int_0^\tau \left( \int_{\partial Q_{R+r}} u(s, \sigma) d\sigma_{R+r} \right)^l \, ds \geq p^l \right)
\leq \frac{1}{p} \left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \geq q^l \right) \]
\[+ NR^{\alpha \left( \frac{d}{\gamma} + \frac{d(d-1)}{\gamma^2} + \frac{L(d-1)}{\gamma} \right)} e^{\alpha R(l-\lambda + \frac{\theta}{2})} \left( \frac{q^l}{p} \right)^{\alpha l},
\]

where \( N = N(\alpha, \gamma, \lambda, a, d, f, H, K, T) > 0 \) and \( L := \frac{\gamma+1}{\gamma-1} = \frac{2(\gamma+2)+d}{\gamma(\gamma+2)+d} > 1 \).

**Remark 3.3.** Using Lemma 3.2 (i), we have that for all \( R > R_0 \) ∨ 1 and \( \theta > \theta' > 0 \),

\[
P \left( \int_0^\tau \left( \int_{\partial Q_{R+\theta}} u(s, \sigma) d\sigma_{R+\theta} \right)^l \, ds > 0 \right) \leq P \left( \int_0^\tau \left( \int_{\partial Q_{R+\theta'}} u(s, \sigma) d\sigma_{R+\theta'} \right)^l \, ds > 0 \right). 
\]  

Indeed, Lemma 3.3 (i) implies that

\[
P \left( \int_0^\tau \left( \int_{\partial Q_{R+\theta'}} u(s, \sigma) d\sigma_{R+\theta'} \right)^l \, ds = 0 \right) \leq P \left( u(s, x) = 0 \quad \text{for all } s \in [0, \tau] \text{ and } x \in Q_{R+\theta'} \right)
\leq P \left( \int_0^\tau \left( \int_{\partial Q_{R+\theta}} u(s, \sigma) d\sigma_{R+\theta} \right)^l \, ds = 0 \right),
\]

which proves (3.7).
We now provide the proof of Theorem 2.11.

**Proof of Theorem 2.11.** Let \( \tau \leq T \) be the given bounded stopping time and \( u \) be the nonnegative solution introduced in Definition 2.7. For \( R > 0 \), we set

\[
\Omega_R := \{ \omega : u(\omega, t, x) = 0 \quad \text{for all} \quad t \leq \tau, \quad x \in Q \}.
\]

We show that \( P(\bigcup_R \Omega_R) = 1 \). Since \( \Omega_R \subset \Omega_{R_2} \) for \( R_1 \leq R_2 \), it suffices to show

\[
\lim_{R \to \infty} P(\Omega_R) = 1.
\]

By Lemma 3.2 (i), we have

\[
P\left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) ds = 0 \right) \leq P(\Omega_R).
\]

Therefore, it suffices to show that

\[
\limsup_{R \to \infty} P\left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) ds > 0 \right) = 0. \tag{3.8}
\]

Now, let \( \gamma \in (0, \eta/2) \) be a constant where \( \eta \) is the constant in Assumption 2.1 and \( a > 0 \) be fixed. For \( n \in \mathbb{N} \), define a bounded stopping time \( \tau_n \) such that

\[
\tau_n := n \wedge \tau \wedge \inf \{ t \geq 0 : \| \Psi_a u \|_{C^\gamma([0,t] \times \mathbb{R}^d)} \geq n \},
\]

where \( \Psi_a \) is the function introduced in (3.1). Note that stopping \( \tau_n \) is well-defined due to Theorem 2.9. Additionally, notice that \( \tau_n \to \tau \) almost surely due to (2.10). Observe that

\[
P\left( \int_0^{\tau_n} \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) ds > 0 \right) \leq P\left( \int_0^{\tau_n} \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) ds > 0 \right) + P(\tau_n < \tau).
\]

Since \( \tau_n \to \tau \) almost surely, it is enough to prove that

\[
\limsup_{R \to \infty} P\left( \int_0^{\tau_n} \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) ds > 0 \right) = 0. \tag{3.9}
\]

For proof of (3.9), let us denote \( \tau \) instead of \( \tau_n \) for simplicity. In addition, we may assume \( H > 0 \) satisfies

\[
\| \Psi_a u \|_{C^\gamma([0,\tau] \times \mathbb{R}^d)} \leq H. \tag{3.10}
\]

Let \( \xi \in (0, 1) \) be a constant that will be specified later. Choose \( \varepsilon > 0 \) and set \( r_k := \varepsilon(k + 1)^{-2} \), \( p_k := \xi e^{-k} \) for \( k = 0, 1, 2, \ldots \) such that \( r_0 \) satisfies the condition on \( r \) in (3.5). Let \( \alpha \in (0, 1) \). By
Lemma 3.2 \[^{[11]}\] for all \( R > R_0 \lor 1 \), there exists \( \theta_1 \in (r_0, 2r_0) \) such that

\[
P\left( \int_0^\tau \left( \int_{\partial Q_{R+\theta}} u(s, \sigma)d\sigma_{R+\theta} \right)^t ds \geq p_1^t \right)
\leq P\left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma)d\sigma_R \right)^t ds \geq p_0^t \right) + NR^a(\frac{2}{\gamma} + \frac{d(\alpha - 1)}{\gamma}) e^{\alpha R(1 - \lambda + \frac{2}{\gamma})} \xi^t \left( \frac{p_L}{p_k} \right)^\alpha,
\]

where \( N = N(\alpha, \gamma, \lambda, a, d, f, H, K, T) \) and \( L := \frac{\gamma + 1}{\gamma + 1} = \frac{\gamma(\gamma + d) + \gamma}{\gamma(\gamma + d) + \gamma + d} > 1 \). Now, we proceed the iteration as follows: For \( k = 2, 3, \ldots \), we take \( R + \theta_k - 1 \) instead of \( R \) and again apply Lemma 3.2 \[^{[11]}\] to find \( \theta_k \) such that \( \delta_k - \theta_k - 1 \in (r_{k-1}, 2r_{k-1}) \) and

\[
P\left( \int_0^\tau \left( \int_{\partial Q_{R+\theta_k}} u(s, \sigma)d\sigma_{R+\theta_k} \right)^t ds \geq p_k^t \right)
\leq P\left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma)d\sigma_{R+\theta_k-1} \right)^t ds \geq p_{k-1}^t \right) + NR^a(\frac{2}{\gamma} + \frac{d(\alpha - 1)}{\gamma}) e^{\alpha R(1 - \lambda + \frac{2}{\gamma})} r_{k-1}^{-\alpha(1 + \frac{2}{\gamma})} \left( \frac{p_L}{p_k} \right)^\alpha,
\]

where \( N = N(\alpha, \gamma, \lambda, a, d, f, H, K, T) \). Here, we used the fact that

\[
0 < R + \theta_k \leq R + 2 \sum_{j=0}^k \theta_j \leq R + \frac{\pi^2 \xi}{3} < 5R
\]

for all \( k \geq 1 \). Due to \( |\theta_k - \theta_{k-1}| \leq 2r_{k-1} = 2\xi k^{-2} \) for all \( k = 1, 2, \ldots \), there exists \( \theta \) such that \( \theta_k \to \theta \) as \( k \to \infty \). From (3.11), by summing over \( k = 1, 2, \ldots \), we have

\[
P\left( \int_0^\tau \left( \int_{\partial Q_{R+\theta}} u(s, \sigma)d\sigma_{R+\theta} \right)^t ds > 0 \right)
\leq P\left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma)d\sigma_R \right)^t ds \geq \xi^t \right) + N_1 R^a(\frac{2}{\gamma} + \frac{d(\alpha - 1)}{\gamma}) e^{\alpha R(1 - \lambda + \frac{2}{\gamma})} \xi^t \left( \frac{p_L}{p_k} \right)^\alpha
\]

where

\[
N_1 := N \sum_{k=1}^\infty r_{k-1}^{-\alpha(1 + \frac{2}{\gamma})} e^{\alpha(k - (k-1)L)} < \infty.
\]

The last inequality holds thanks to \( L > 1 \). Choose \( \delta > \frac{a(1 - \lambda + \frac{2}{\gamma})}{(lL - 1)} \) and set \( \xi := e^{-\delta R} \), where \( K \) is the
constant introduced in Assumptions 2.3 and 2.4. Then, by Chebyshev’s and Jensen’s inequalities,
\[ P \left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \geq \xi^l \right) \leq \xi^{-l} E \left[ \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \right] \leq \xi^{-l} N \left( \int_0^\tau \int_{\partial Q_R} u(s, \sigma) d\sigma_R ds \right)^l \leq N \left( E \left[ \int_0^\tau e^{\delta R} \int_{\partial Q_R} u(s, \sigma) d\sigma_R ds \right] \right)^l, \]
where \( N = N(\gamma, \lambda, d, T) \). Thus, by Lemma 3.1 (see (3.3)),
\[ \limsup_{R \to \infty} P \left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \geq \xi^l \right) = 0. \] (3.14)
In addition, since \( L > 1 \), we have
\[ \limsup_{R \to \infty} R^\alpha \left( \frac{d(d-1)}{2^d} + \frac{L(d-1)}{5} \right) e^{\alpha R(l-\gamma+\frac{\theta}{\lambda})} \xi^{(L-1)\alpha l} \]
\[ = \limsup_{R \to \infty} R^\alpha \left( \frac{d(d-1)}{2^d} + \frac{L(d-1)}{5} \right) e^{\alpha R(l-\gamma+\frac{\theta}{\lambda})-\delta \alpha (L-1) R} \] (3.15)
Therefore, by applying (3.14) and (3.15) to (3.13), we have
\[ \limsup_{R \to \infty} P \left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds > 0 \right) = 0, \]
which completes the proof.

4. Proof of Lemma 3.2

In this section, we provide the proof of Lemma 3.2. To prove Lemma 3.2, we assume the results of Theorem 2.9 and Lemma 3.1, and introduce three auxiliary lemmas: Lemmas 4.1, 4.2, and 4.3. Lemma 4.1 shows that the quadratic variation of the stochastic part in (2.7) on \( Q_R \) is controlled by the integration of the solution \( u \) on \( \partial Q_R \).

Throughout this section, we assume that there exist \( a, H \in (0, \infty) \) satisfying (3.4). For \( R > 1 \), we define
\[ M(x) := M_R(x) := (|x| - R)_+ \] (4.1)
and choose $K_1$ and $K_2$ such that
\[ K_1 > 2K^2 \vee a \quad \text{and} \quad K_2 \geq K + 4KK_1 + K_1^2K =: N_1. \quad (4.2) \]
where $K$ is the constant in Assumption 2.8.

**Lemma 4.1.** Let $\tau \leq T$ be a bounded stopping time and $u$ be a nonnegative solution to (1.1) introduced in Definition 2.7. Suppose $R > R_0 \vee 1$ is a constant, where $R_0$ is the constant introduced in Assumption 2.6. Then, for every $\alpha \in (0, 1)$, we have
\[ E \left[ \left( \int_0^\tau \int_{R^d} e^{-2Kz-M(M+y)+M(y)} M(x+y)(u(s, x+y)u(s, y))^{\alpha} dy ds \right)^{\alpha/2} \right] \leq N E \left( \int_0^\tau \int_{\partial Q_R} e^{-Kz} u(s, \sigma) d\sigma ds \right)^\alpha, \quad (4.3) \]
where $N := N(\alpha, d, K)$.

**Proof.** To consider a weak solution $u$ of (1.1) on $Q_R$, we need to choose appropriate test functions. Let $\psi \in C^\infty_0(\mathbb{R})$ be a nonnegative symmetric function satisfying $\psi(z) = 1$ on $|z| < 1$ and $\psi(z) = 0$ on $|z| \geq 2$ and set $\psi_n(x) := \psi(|x|/n) := \psi(|x|)/n$ for $n \in \mathbb{N}$. Take a nonnegative function $\zeta \in C^\infty_0(\mathbb{R})$ satisfying $\int \zeta(z) dz = 1$, symmetric, and $\zeta = 0$ on $|z| \geq 1$. Define for $m \in \mathbb{N}$,
\[ \phi_m(x) := m \int R \mathbf{1}_{|z| > R + \frac{a}{m}}(z) \zeta \left( m \left( |z| - a \right) \right) \, dz = \int R \mathbf{1}_{|z| > R + \frac{a}{m}}(z) \zeta(z) \, dz. \quad (4.4) \]
It should be noted that $\phi_m(x) = 0$ on $|x| \leq R$ and $\phi_m(x) \to 1_{|x| > R}(x)$ as $m \to \infty$. Additionally, observe that
\[ \phi_m(x) = 1 \quad \text{on} \quad |x| > R + \frac{2}{m}. \quad (4.5) \]
Indeed, for $|x| > R + \frac{2}{m}$ and $|z| < 1$, we have $|x| - \frac{a}{m} > |z| - \frac{a}{m} > R + \frac{2}{m} - \frac{|z|}{m} > R + \frac{1}{m}$. Therefore, by the definition of $\phi_m$, (4.4) implies (4.5). Moreover, (4.4) ensures that
\[ \phi_m(x) = 0 \quad \text{for} \quad |x| > R + \frac{2}{m}. \quad (4.6) \]
Furthermore, if $|x| - \frac{a}{m} > R + \frac{1}{m}$ and $|z| < 1$, then we have $R < |x| - \frac{a}{m} - \frac{1}{m} \leq |x| + \frac{|z|}{m} - \frac{1}{m} \leq |x|$. Thus,
\[ \phi_m(x) \leq \mathbf{1}_{|x| > R}. \quad (4.7) \]
For $g \in L_{1,\text{loc}}(\mathbb{R}^d)$ and $\varepsilon > 0$, set
\[ g^{(\varepsilon)}(x) := \varepsilon^{-d} \int_{\mathbb{R}^d} g(y) \zeta(\varepsilon^{-1}(x-y)) \, dy \quad (4.8) \]
Then, for $m, n \in \mathbb{N}, \varepsilon > 0$, and $\varepsilon_1 > 0$, Itô’s formula yields
\[ \int_{\mathbb{R}^d} e^{-Kz} u(x, t) M^{(\varepsilon)}(x) \psi_n(x) \phi_m(x) e^{-K_1 M^{(\varepsilon_1)}(x)} \, dx \]
\[ = \int_0^\tau \int_{\mathbb{R}^d} e^{-Kz} u(s, x) L^* \left( M^{(\varepsilon)}(x) \psi_n(x) \phi_m(x) e^{-K_1 M^{(\varepsilon_1)}(x)} \right) \, dx ds \]
\[ - K_2 \int_0^\tau \int_{\mathbb{R}^d} e^{-Kz} u(s, x) M^{(\varepsilon)}(x) \psi_n(x) \phi_m(x) e^{-K_1 M^{(\varepsilon_1)}(x)} \, dx ds + m^{m,n,\varepsilon_1}, \quad (4.9) \]
where
\[ m_{\tau}^{m,n,\varepsilon,\varepsilon_1} := \int_0^\tau \int_{\mathbb{R}^d} e^{-K_2 s h(s, u(s, x))} M^{(\varepsilon)}(x) \psi_m(x) \phi_m(x) e^{-K_1 M^{(\varepsilon)}(x)} F(ds, dx). \]

For the properties of \( m_{\tau}^{m,n,\varepsilon,\varepsilon_1} \), see Remark 2.8 and 2.14. Note that the quadratic variation of \( m_{\tau}^{m,n,\varepsilon,\varepsilon_1} \) is related to the LHS of (4.3). We will get the inequality (4.3) by taking various limits in (4.9) and using some martingale theory. The remaining part of the proof is separated into four steps. From Step 1 to Step 3, the limits are taken in \( \varepsilon_1 \to 0, \varepsilon \to 0 \), and then \( m \to \infty \). In Step 4, we take the limit \( n \to \infty \) (4.9) and obtain (4.3).

(Step 1) We consider the case \( \varepsilon_1 \downarrow 0 \). For convenience, set
\[ \Phi^{m,n,\varepsilon}(x) := M^{(\varepsilon)}(x) \psi_m(x) \phi_m(x). \] (4.10)

Then, from (4.9) and (2.8), we have
\[
\mathcal{L}^* \left( \Phi^{m,n,\varepsilon}(x) e^{-K_1 M^{(\varepsilon)}(x)} \right)
\begin{align*}
&= \left( \partial_{ij}^2 \Phi^{m,n,\varepsilon}(x) e^{-K_1 M^{(\varepsilon)}(x)} + \left( 2\partial_{ij} - b_i^j \right) \Phi^{m,n,\varepsilon}(x) e^{-K_1 M^{(\varepsilon)}(x)} \right)_{x^i} \\
&= \left[ 2\partial_{ij}^2 - b_i^j - 2K_1 a^{ij} M^{(\varepsilon)}(x) \right] (\Phi^{m,n,\varepsilon}(x))_{x^i} e^{-K_1 M^{(\varepsilon)}(x)} \\
&\quad + \Phi^{m,n,\varepsilon}(x)_{x^i} e^{-K_1 M^{(\varepsilon)}(x)} + D^{(s)}(s, x) \Phi^{m,n,\varepsilon}(x) e^{-K_1 M^{(\varepsilon)}(x)},
\end{align*}
\] (4.11)
where
\[
D^{(s)}(s, x) = \partial_{ij}^2 - b_i^j + c - K_1 \left( 2\partial_{ij} - b_i^j \right) M^{(\varepsilon)}(x) - K_1 a^{ij} M^{(\varepsilon)}(x) + K_1^2 a^{ij} M^{(\varepsilon)}(x) M^{(\varepsilon)}(x).
\] (4.12)

To deal with \( D^{(s)}(s, x) \), notice that if we let \( \varepsilon_1 \downarrow 0 \), then on \( |x| > R \),
\[
M^{(\varepsilon)}(x) \to M(x), \quad M^{(\varepsilon)}(x) \to M(x), \quad M^{(\varepsilon)}(x) \to M(x),
\] (4.13)
where
\[
M(x) = \frac{x^i}{|x|}, \quad M_{x^i}(x) = \begin{cases} \frac{x^i}{|x|} - \frac{x^i x^i}{|x|^2} & \text{if } d \geq 2, \\
0 & \text{if } d = 1,
\end{cases}
\] (4.14)
and \( \delta^{ij} \) is the Kronecker delta. Thus, by applying (4.13) and (4.14) to (4.12), on \( |x| > R \), we have
\[
\limsup_{\varepsilon_1 \to 0} D^{(s)}(s, x)
\begin{align*}
&= \partial_{ij}^2 - b_i^j + c - K_1 \left( 2\partial_{ij} - b_i^j \right) M_{x^i}(x) - K_1 a^{ij} M_{x^i}(x) + K_1^2 a^{ij} M_{x^i}(x) M_{x^i} \leq N_1,
\end{align*}
\] (4.15)
where \( N_1 \) is the constant introduced in (4.2). Therefore, if we take the limit in probability as \( \varepsilon_1 \to 0 \) in (4.9), (4.15) yields
\[
\int_{\mathbb{R}^d} e^{-K_2 \tau u(t, x) M^{(\varepsilon)}(x)} \psi_m(x) \phi_m(x) e^{-K_1 M(x)} dx \\
\leq \int_0^\tau e^{-K_2 s} (A(m, n, \varepsilon) + B(m, n, \varepsilon)) ds + m_{\tau}^{m,n,\varepsilon}
\] (4.16)
where
\[ A(m, n, \varepsilon) := \int_{\mathbb{R}^d} \left[ 2a_{ij}^e x^i - b^i - 2K_1 a_{ij}^{(e)} M_{x^j}(x) \right] u(s, x) (\Phi_{m, n, \varepsilon}(x))_{x^i} e^{-K_1 M(x)} d\nu, \tag{4.17} \]
\[ B(m, n, \varepsilon) := \int_{\mathbb{R}^d} u(s, x) \left( a_{ij}^{(e)} (\Phi_{m, n, \varepsilon}(x))_{x^i} - (K_2 - N_1) \Phi_{m, n, \varepsilon}(x) \right) e^{-K_1 M(x)} d\nu, \tag{4.18} \]
\[ m_{m, n, \varepsilon} := \int_{0}^{\infty} e^{-K_2 s} h(s, u(s, x)) M^{(\varepsilon)}(x) \psi_n(x) \phi_m(x) e^{-K_1 M(x)} F(ds, dx). \tag{4.19} \]

Note that one can show that \( m_{m, n, \varepsilon} \) converges in probability to \( m_{m, n, \varepsilon}^{\varepsilon_1} \) as \( \varepsilon_1 \downarrow 0 \) by the Burkholder-Davis-Gundy inequality, the bounded convergence theorem, and the fact that \( \psi_n \in C^\infty_c(\mathbb{R}^d) \).

(Step 2) In this step, our aim is to show that
\[
\lim_{m \to \infty} \limsup_{\varepsilon \to 0} A(m, n, \varepsilon) \leq -K^{-1} K_1 \int_{Q_R} u(s, x) \psi_n(x) e^{-K_1 M(x)} d\nu + N \sum_i \int_{Q_R} u(s, x) M(x) \psi_{n^i}(x) e^{-K_1 M(x)} d\nu, \tag{4.20} \]
where \( A(m, n, \varepsilon) \) is in \( \{1.17\} \) and \( N = N(K) \). We separate \( A(m, n, \varepsilon) \) into three parts:
\[
A(m, n, \varepsilon) = A_1(m, n, \varepsilon) + A_2(m, n, \varepsilon) + A_3(m, n, \varepsilon),
\]
where
\[
A_1(m, n, \varepsilon) := \int_{\mathbb{R}^d} \left( 2a_{ij}^{(e)} x^i - b^i - 2K_1 a_{ij}^{(e)} M_{x^j}(x) \right) u(s, x) M^{(\varepsilon)}(x) \psi_n(x) \phi_m(x) e^{-K_1 M(x)} d\nu,
\]
\[
A_2(m, n, \varepsilon) := \int_{\mathbb{R}^d} \left( 2a_{ij}^{(e)} x^i - b^i - 2K_1 a_{ij}^{(e)} M_{x^j}(x) \right) u(s, x) M^{(\varepsilon)}(x) \psi_{n^i}(x) \phi_m(x) e^{-K_1 M(x)} d\nu,
\]
\[
A_3(m, n, \varepsilon) := \int_{\mathbb{R}^d} \left( 2a_{ij}^{(e)} x^i - b^i - 2K_1 a_{ij}^{(e)} M_{x^j}(x) \right) u(s, x) M^{(\varepsilon)}(x) \psi_n(x) \phi_{m^i}(x) e^{-K_1 M(x)} d\nu.
\]
We use \( \{1.14\} \), Assumption \( \{2.3\} \) and \( \{4.2\} \) to get that
\[
\lim_{m \to \infty} \limsup_{\varepsilon \to 0} A_1(m, n, \varepsilon)
\]
\[
= \limsup_{m \to \infty} \int_{Q_R} u(s, x) \left[ 2a_{ij}^{(e)} \frac{x^i}{|x|} - b^i \frac{x^i}{|x|} - 2K_1 a_{ij}^{(e)} \frac{x^i x^j}{|x|^2} \right] \psi_n(x) \phi_m(x) e^{-K_1 M(x)} d\nu
\]
\[
\leq \left[ 2K - 2K^{-1} K_1 \right] \limsup_{m \to \infty} \int_{Q_R} u(s, x) \psi_n(x) \phi_m(x) e^{-K_1 M(x)} d\nu
\]
\[
\leq -K^{-1} K_1 \int_{Q_R} u(s, x) \psi_n(x) e^{-K_1 M(x)} d\nu. \tag{4.21} \]
In addition, since $|M_{x'}| \leq 1$ (see (4.14)), we have

$$\limsup_{m \to \infty} \limsup_{\varepsilon \to 0} A_2(m, n, \varepsilon)$$
\[
= \limsup_{m \to \infty} \int_{\mathbb{R}^d} \left[ 2n_{x'}^{ij} b^i - 2K_1 a^{ij} M_{x'}(x) \right] u(s, x)M(x)\psi_{nx'}(x)\phi_n(x)e^{-K_1 M(x)} \, dx \\
\leq [2K + 2KK_1] \limsup_{m \to \infty} \int_{\mathbb{R}^d} u(s, x)M(x) |\psi_{nx'}(x)| \phi_n(x)e^{-K_1 M(x)} \, dx \\
\leq N \int_{Q_R} u(s, x)M(x) |\psi_{nx'}(x)| e^{-K_1 M(x)} \, dx,
\] (4.22)

where $N = N(K)$. To control $A_3(m, n, \varepsilon)$, observe that (4.6) and (4.7) imply

$$\limsup_{\varepsilon \to 0} |A_3(m, n, \varepsilon)|$$
\[
\leq [2K + 2KK_1] m \int_{|x| < R + \frac{2}{K}} u(s, x)M(x)\psi_n(x) \left| \frac{x}{|x|} \right| \int_{\mathbb{R}} \frac{\varepsilon}{|x| - \frac{1}{M}} \frac{z}{M} \phi_{nx'}(z) e^{-K_1 M(x)} \, dx \\
\leq Nm \int_{|x| < R + \frac{2}{K}} u(s, x)M(x)\psi_n(x) \varepsilon M e^{-K_1 M(x)} \, dx \\
= Nm^{-1} \int_{\partial Q_{R+m^{-1}K}} u(s, \sigma) \rho \psi_n(R + m^{-1} \rho) e^{-m^{-1}K_1 \rho} d\sigma_{R+m^{-1}K} d\rho,
\] (4.23)

where $N = N(d, K)$. Since all terms in the last integral are bounded and continuous almost surely, we have

$$\limsup_{m \to \infty} \limsup_{\varepsilon \to 0} |A_3(m, n, \varepsilon)| = 0. \quad (4.24)$$

By combining (4.21), (4.22), and (4.24), we have (4.20).

**(Step 3)** Now we control $B(m, n, \varepsilon)$. In this step, we show that for fixed $n > R$,

$$\limsup_{m \to \infty} \limsup_{\varepsilon \to 0} B(m, n, \varepsilon)$$
\[
\leq N \int_{Q_R} u(s, x) (|M_{x'}(x)| \psi_{nx'}(x) + M(x)\psi_{nx',x'}(x)) e^{-K_1 M(x)} \, dx \\
+ 2K \int_{Q_R} u(s, x)\psi_n(x)e^{-K_1 M(x)} \, dx + K \int_{\partial Q_R} u(s, \sigma) d\sigma_R,
\] (4.25)

where $N := N(d, K)$ and $B(m, n, \varepsilon)$ are introduced in (4.18). We decompose $B(m, n, \varepsilon)$ as

$$B(m, n, \varepsilon) = B_1(m, n, \varepsilon) + B_2(m, n, \varepsilon) + B_3(m, n, \varepsilon) + B_4(m, n, \varepsilon) + B_5(m, n, \varepsilon) + B_6(m, n, \varepsilon),$$
(4.26)
where

\[ B_1(m, n, \varepsilon) := (N_1 - K_2) \int_{\mathbb{R}^d} u(s, x) M^{(\varepsilon)}(x) \phi_n(x) \phi_m(x) e^{-K_1 M(x)} dx, \]

\[ B_2(m, n, \varepsilon) := \int_{\mathbb{R}^d} u(s, x) a^{ij}(s, x) M^{(\varepsilon)}(x) \phi_n(x) \phi_m(x) e^{-K_1 M(x)} dx, \]

\[ B_3(m, n, \varepsilon) := 2 \int_{\mathbb{R}^d} u(s, x) a^{ij}(s, x) M^{(\varepsilon)}(x) \phi_n(x) \phi_{mx^i} e^{-K_1 M(x)} dx, \]

\[ B_4(m, n, \varepsilon) := 2 \int_{\mathbb{R}^d} u(s, x) a^{ij}(s, x) M^{(\varepsilon)}(x) \phi_{nx^i} \phi_{mx^j} e^{-K_1 M(x)} dx, \]

\[ B_5(m, n, \varepsilon) := \int_{\mathbb{R}^d} u(s, x) a^{ij}(s, x) M^{(\varepsilon)}(x) \phi_n(x) \phi_{mx^i} e^{-K_1 M(x)} dx, \]

\[ B_6(m, n, \varepsilon) := \int_{\mathbb{R}^d} u(s, x) a^{ij}(s, x) \left( 2 M^{(\varepsilon)}(x) \phi_{nx^i}(x) + M^{(\varepsilon)}(x) \phi_{nx^i x^j}(x) \right) \phi_m(x) e^{-K_1 M(x)} dx. \]

In the case of \( B_1(m, n, \varepsilon) \), by the choice of \( K_2 \) (see (4.12)), we have

\[ \lim_{m \to \infty} \lim_{\varepsilon \to 0} B_1(m, n, \varepsilon) \leq 0. \]  \hspace{1cm} (4.27)

In the case of \( B_2(m, n, \varepsilon) \) (4.13) yield

\[ \lim_{m \to \infty} \lim_{\varepsilon \to 0} |B_2(m, n, \varepsilon)| = \lim_{m \to \infty} \sup_{\varepsilon \to 0} \int_{\mathbb{R}^d} u(s, x) a^{ij}(s, x) \left( \frac{\delta^{ij}}{|x|} - \frac{x^i x^j}{|x|^3} \right) \psi_n(x) \phi_m(x) e^{-K_1 M(x)} dx \]

\[ \leq 2K \int_{Q_R} u(s, x) \psi_n(x) e^{-K_1 M(x)} dx. \]  \hspace{1cm} (4.28)

To bound \( B_3(m, n, \varepsilon) \), we mimic the estimation for \( A_3(m, n, \varepsilon) \); see (4.23). Then, we have

\[ \lim_{\varepsilon \to 0} B_3(m, n, \varepsilon) \]

\[ = 2m \int_{|x| < R + \frac{1}{m}} u(s, x) a^{ij}(s, x) \frac{x^i x^j}{|x|^2} \psi_n(x) \int_{\mathbb{R}} \frac{1}{|x| - \frac{1}{m} - \frac{\zeta}{R + \frac{1}{m}}} e^{-K_1 M(x)} dx \]

\[ = 2m \int_{\partial Q_R} u(s, \rho) a^{ij}(s, \rho) \frac{\sigma^i \sigma^j}{\rho^2} \psi_n(\rho) \int_{\mathbb{R}} \frac{1}{|\rho - \frac{1}{m} - \frac{\zeta}{R + \frac{1}{m}}} e^{-K_1 (\rho - \rho^2) \sigma R + \frac{1}{m}} d\sigma \rho \]

\[ = 2 \int_{\partial Q_R} u(s, \rho) a^{ij}(s, \rho) \frac{\sigma^i \sigma^j}{(R + \rho m^{-1})^2} d\sigma R + \frac{1}{m} \zeta \rho (\rho - 1) e^{-K_1 \rho/m} d\rho. \]

By taking the limit \( m \to \infty \), we have

\[ \lim_{m \to \infty} \lim_{\varepsilon \to 0} B_3(m, n, \varepsilon) = 2R^{-2} \psi_n(R) \int_{\partial Q_R} u(s, \rho) a^{ij}(s, \rho) \sigma^i \sigma^j d\sigma R \]

\[ = 2R^{-2} \int_{\partial Q_R} u(s, \rho) a^{ij}(s, \rho) \sigma^i \sigma^j d\sigma R, \]  \hspace{1cm} (4.29)
since $\psi_n(R) = 1$ for $n > R$. In the case of $B_4(m, n, \varepsilon)$, note that

$$\limsup_{\varepsilon \to 0} |B_4(m, n, \varepsilon)|$$

\[ \leq N \int_{|x| < R + \frac{1}{m}} u(s, x) \left| a^{ij}(s, x) \right| M(x) |\psi_{nx}(x)| |\phi_{mx}(x)| e^{-K_1 M(x)} dx \]

\[ \leq N m \int_{|x| < R + \frac{1}{m}} u(s, x) M(x) |\psi_{nx}(x)| 1_{|x| > R} e^{-K_1 M(x)} dx \]

\[ \leq N m^{-1} \int_0^2 \int_{\partial Q_{R + m^{-1} \rho}} (s, \sigma) d\sigma \left| \psi_{nx}(R + m^{-1} \rho) \right| |\rho e^{-K_1 m^{-1} \rho}| d\rho, \]

where $N := N(K)$. By taking the limit $m \to \infty$, we have

$$\limsup_{m \to \infty} \limsup_{\varepsilon \to 0} |B_4(m, n, \varepsilon)| = 0. \quad (4.30)$$

To control $B_5(m, n, \varepsilon)$, recall that $\phi_{mx+ij}(x) = 0$ for $|x| > R + \frac{2}{m}$ (see (4.10) and (4.11)). Thus, similar to the estimation for $B_4(m, n, \varepsilon)$, we get

$$\limsup_{\varepsilon \to 0} B_5(m, n, \varepsilon)$$

\[ \begin{align*}
&= m^2 \int_{|x| < R + \frac{1}{m}} u(s, x) a^{ij}(s, x) \frac{x^i x^j}{|x|^2} M(x) \psi_{nx}(x) \int_{\mathbb{R}} 1_{|x| - \frac{1}{m} < x < R + \frac{1}{m}} \zeta''(z) dz e^{-K_1 M(x)} dx \\
&\quad + m \int_{|x| < R + \frac{1}{m}} u(s, x) a^{ij}(s, x) M(x) \psi_{nx}(x) \int_{\mathbb{R}} 1_{|x| - \frac{1}{m} < x < R + \frac{1}{m}} \zeta''(z) dz e^{-K_1 M(x)} dx \\
&= m^2 \int_{|x| < R + \frac{1}{m}} u(s, x) a^{ij}(s, x) \frac{x^i x^j}{|x|^2} M(x) \psi_{nx}(x) \int_{\mathbb{R}} 1_{|x| - \frac{1}{m} < x < R + \frac{1}{m}} \zeta''(z) dz e^{-K_1 M(x)} dx \\
&\quad + m \int_{|x| < R + \frac{1}{m}} u(s, x) a^{ij}(s, x) M(x) \psi_{nx}(x) \int_{\mathbb{R}} 1_{|x| - \frac{1}{m} < x < R + \frac{1}{m}} \zeta''(z) dz e^{-K_1 M(x)} dx \\
&= \int_{\partial Q_{R + m^{-1} \rho}} u(s, \sigma) a^{ij}(s, \sigma) \sigma^i \sigma^j d\rho \psi_n(R + m^{-1} \rho) \zeta(\rho) \frac{\rho}{(R + m^{-1} \rho)^2} e^{-K_1 M(R + m^{-1} \rho)} d\rho \\
&\quad + m^{-1} \int_{\partial Q_{R + m^{-1} \rho}} u(s, \sigma) a^{ij}(s, \sigma) \frac{\sigma^i \sigma^j}{R + m^{-1} \rho} d\rho \psi_n(R + m^{-1} \rho) \zeta(\rho - 1) \rho e^{-K_1 M(R + m^{-1} \rho)} d\rho.
\end{align*} \]

The bounded convergence theorem implies that

$$\lim_{m \to \infty} \limsup_{\varepsilon \to 0} B_5(m, n, \varepsilon) = -R^{-2} \psi_n(R) \int_{\partial Q_R} u(s, \sigma) a^{ij}(s, \sigma) \sigma^i \sigma^j d\sigma_R$$

$$= -R^{-2} \int_{\partial Q_R} u(s, \sigma) a^{ij}(s, \sigma) \sigma^i \sigma^j d\sigma_R. \quad (4.31)$$

In the case of $B_6(m, n, \varepsilon)$, by (4.13), we have

$$\limsup_{m \to \infty} \limsup_{\varepsilon \to 0} |B_6(m, n, \varepsilon)|$$

\[ \begin{align*}
&\leq N \int_{Q_R} u(s, x) (2 |M(x)\psi_{nx}(x)| + M(x) |\psi_{nx+ij}(x)|) e^{-K_1 M(x)} dx,
\end{align*} \]

(4.32)
Again recall that $A$ surely. These show that $N$ has exponential decay as $\varepsilon \downarrow 0$ and $m \to \infty$ due to the same reason as in the end of (Step 1).

By the choice of $K_1$, we get

$$(-K^{-1}K_1 + 2K) \int_{Q_R} u(s, x) \psi_n(x) e^{-K_1 M(x)} dx \leq 0;$$

see (4.2). Then, by applying (4.20) and (4.25) to (4.16), we have

$$0 \leq \int_{Q_R} e^{-K_2 \tau} u(\tau, x) M(x) \psi_n(x) e^{-K_1 M(x)} dx \leq {\mathfrak A}^n_\tau + {\mathfrak B}_\tau + m^n_\tau,$$

where

$${\mathfrak A}^n_\tau := N^{-1} \int_0^\tau \int_{Q_R} e^{-K_2 s} u(s, x) \left( |\psi_{x'}(x/n)| M(x) + (|M_{x'}(x)| \psi_{x'}(x/n) + M(x) |\psi_{x,x'}(x/n)| ) \right) e^{-K_1 M(x)} dx ds$$

and

$${\mathfrak B}_\tau := N(d, K) \int_0^\tau \int_{Q_R} u(s, \sigma) d\sigma R ds.$$

Recall that $\psi_n(x) = \psi(|x|/n)$. Since $\psi_{x'}, \psi_{x'x'},$ and $\sup_{s \leq \tau, x \in R^d} \Psi_a(x) u(s, x)$ are bounded, and $e^{-(K_1-a)M(x)}$ has exponential decay as $|x| \to \infty$, the integral (4.35) is finite. Note that for any arbitrary stopping time $\kappa \leq \tau$, we have $(m^n_\kappa)_- \leq \mathfrak A^n_\kappa + \mathfrak B_\kappa$ from (4.33). Since $m^n_\kappa$ is a local martingale, for any fixed $n$, one can choose a sequence of stopping times $\{\tau_i\}_i$ such that $\tau_i \to \infty$ and $m^n_{\tau_i \wedge T}$ is a martingale on $[0, T]$ for each $i \in N$. Therefore, $m^n_{\tau_i \wedge T}$ is a martingale by the optional sampling theorem, and thus implies that $E[m^n_{\tau_i \wedge T}] = 0$ for any $i \leq T$. This ensures that $E[(m^n_{\tau_i \wedge T})_-] = E[(m^n_{\tau_i \wedge T})_+]$. Thus, we have

$$E[|m^n_{\tau_i \wedge T}|] \leq 2E[|A^n_\kappa + B_\kappa|],$$

recalling that $A^n_\tau + B_\tau$ is nondecreasing in $t$. Therefore, Fatou’s lemma shows that

$$E[|m^n_\kappa|] \leq 2E[|A^n_\kappa + B_\kappa|].$$

Note that the above inequality holds for an arbitrary stopping time $\kappa \leq \tau$. Since $A^n_\kappa$ is nonnegative, by Itô’s inequality (e.g. [12] Theorem III.6.8)), for $\alpha \in (0, 1)$, we have

$$E \left[ \sup_{t \leq \tau} |m^n_t|^{\alpha} \right] \leq 2^{\alpha} \frac{2-\alpha}{1-\alpha} E \left[ \sup_{t \leq \tau} |A^n_t + B_t|^{\alpha} \right].$$

Again recall that $A^n_\tau$ and $B_\tau$ are nondecreasing in $t$, and $\sup_{t \leq \tau, x \in R^d} \Psi_a(x) u(t, x)$ is bounded almost surely. These show that

$$\lim_{n \to \infty} \sup_{t \leq \tau} E \left[ \sup_{t \leq \tau} |A^n_t + B_t|^{\alpha} \right] \leq E[|B_\tau|^{\alpha}].$$

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On the other hand, by the Burkholder-Davis-Gundy inequality (e.g. [13, Theorem IV.4.1]) with (2.5) and passing to the limit as \( n \to \infty \), we can see that

\[
E \left[ \int_0^\tau \int_{Q_n} e^{-2K_2u - K_1(M(x+y) + M(y))} M(x + y) M(y) (u(s, x + y) u(s, y))^\lambda dy f(dx) ds \right]^{\alpha/2} \]

\[
\leq N \cdot \limsup_{n \to \infty} E \left[ \sup_{t \leq \tau} |m_t^n|^\alpha \right] \leq NE \left[ |\mathfrak{M}_\tau|^\alpha \right],
\]

where \( N = N(\alpha, d, K) \). This completes the proof of (4.3).

Next we present integral inequalities. In the proof of Lemma 3.2, these inequalities are used to adjust the exponent of integrals.

**Lemma 4.2.** Let \( \gamma, \lambda \in (0, 1) \), \( 0 \leq a < b < \infty \), and \( H > 1 \).

(i) Suppose \( g : \mathbb{R}^d \to \mathbb{R} \) is a bounded, nonnegative, and continuous function such that

\[
g(x) \leq H \quad \text{and} \quad |g(x) - g(y)| \leq H |x - y|^{\gamma} \quad \text{for} \quad x, y \in \mathbb{R}^d.
\]

Then, for \( R > 1 \) and \( r > 0 \) satisfying

\[
0 < r < \frac{R}{b} \wedge \left[ \frac{1}{b - a} \left( 2^{-\frac{(\gamma + 1)}{\gamma}} H^{-\frac{\gamma}{\gamma}} R^{d-1} \left( \frac{d\pi^{d/2}}{\Gamma(d/2 + 1)} \right)^{\frac{1}{\gamma + d}} \right) \right],
\]

we have

\[
\left( \int_{R + ar < |x| < R + br} g(x) dx \right)^\frac{\lambda d + d}{\gamma + d} \leq NR \frac{d(d-1)}{\gamma + d}(r(b - a))^{-d(\gamma + d - 1)\gamma + d} \int_{R + ar < |x| < R + br} (g(x))^{\lambda} dx,
\]

where \( N = N(\gamma, \lambda, H) \).

(ii) Let \( T \in (0, \infty) \). Suppose \( g : \mathbb{R}_+ \to \mathbb{R} \) is a nonnegative and continuous function on \( \mathbb{R} \) such that

\[
|g(t) - g(s)| \leq H |t - s|^{\gamma} \quad \text{for} \quad t, s \in (0, \infty).
\]

If \( g(0) = 0 \), we have

\[
\left( \int_0^T g(t) dt \right)^\frac{\gamma}{\gamma + 1} \leq NH^1/\gamma \int_0^T (g(t))^{\lambda} dt,
\]

where \( N = N(\gamma, \lambda) \).

**Proof.** The idea of proof follows from [1, Theorem 4] and [23, Lemma 10.4.2]. To prove Lemma 4.2 (i), we separate the proof into two steps.

**Step 1** In this step, we consider the case

\[
\int_{R + ar < |x| < R + br} g(x) dx = 1.
\]
First we assume \( \sup_{R + ar < |x| < R + br} g(x) < 1 \). Notice that \( |g(x)| < |g(x)|^\lambda \) for \( x \in \{ x : R + ar < |x| < R + br \} \). Therefore, (4.42) yields

\[
\int_{R + ar < |x| < R + br} (g(x))^\lambda \, dx \geq 1. \tag{4.43}
\]

Now we consider the case where \( \sup_{R + ar < |x| < R + br} g(x) \geq 1 \). Due to the continuity of \( g \), there exists \( x_0 \in \{ x : R + ar < |x| < R + br \} \) such that \( g(x_0) > 1/2 \). Then, since \( g \) satisfies (4.38), for any \( x \in \{ x : R + ar < |x| < R + br \} \) with \( |x - x_0| \leq (4H)^{-1/\gamma} \), we have

\[
g(x) \geq g(x_0) - |g(x) - g(x_0)| \geq \frac{1}{2} - H|x - x_0|^\gamma \geq \frac{1}{2} - H(4H)^{-1} = 1/4.
\]

Therefore, we have

\[
\int_{R + ar < |x| < R + br} (g(x))^\lambda \, dx \geq 4^{-\lambda} \int_{R + ar < |x| < R + br} (g(x))^\lambda \mathbb{1}_{\{ |x-x_0| < (4H)^{-1/\gamma} \}} \, dx
\]

\[
\geq 4^{-\lambda} \prod_i \int_{R + ar < x_i < R + br} \mathbb{1}_{\{ |x_i-x_i^0| < (4H)^{-1/\gamma} \gamma \}} \, dx_i \geq \frac{N(\lambda, d) \left( (4H)^{-1/\gamma} \wedge (b - a)r \right)^d}{\gamma}.
\]

The last inequality follows from [23] Lemma 10.4.1. Combining (4.43) and (4.44), we get

\[
\int_{R + ar < |x| < R + br} (g(x))^\lambda \, dx \geq N \left( (4H)^{-1/\gamma} \wedge (b - a)r \right)^d, \tag{4.45}
\]

where \( N = N(\lambda, d) \) is a positive constant.

**Step 2** For general \( g \), we set

\[
G := \left( \int_{R + ar < |x| < R + br} g(x) \, dx \right)^{1/d} \quad \text{and} \quad G(x) := G^{-\gamma} g(Gx).
\]

Then, by the change of variable, we have

\[
\int_{R + ar < |x| < R + br} G(x) \, dx = G^{-(\gamma + d)} \int_{R + ar < |x| < R + br} g(x) \, dx = 1.
\]

Furthermore, since

\[
|G(x) - G(y)| \leq G^{-\gamma} |g(Gx) - g(Gy)| \leq HG^{-\gamma} |Gx - Gy|^\gamma = H|x - y|^\gamma,
\]

we have

\[
N(\lambda, d) \left( (4H)^{-1/\gamma} \wedge \frac{(b - a)r}{G} \right)^d \leq \int_{R + ar < |x| < R + br} (G(x))^\lambda \, dx
\]

\[
= G^{-\gamma \lambda - d} \int_{R + ar < |x| < R + br} (g(x))^\lambda \, dx.
\]

(4.46) yields

\[
N(\lambda, d) \left( (4H)^{-1/\gamma} \wedge \frac{(b - a)r}{G} \right)^d \leq \int_{R + ar < |x| < R + br} (G(x))^\lambda \, dx
\]

\[
= G^{-\gamma \lambda - d} \int_{R + ar < |x| < R + br} (g(x))^\lambda \, dx.
\]
Observe that
\[
\frac{G}{r(b - a)} = \frac{1}{r(b - a)} \left( \int_{R + ar < |x| < R + br} g(x) dx \right)^{\frac{\lambda + d}{\gamma + d}}
\]
\[
\leq \frac{1}{r(b - a)} \left( \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \right)^{\frac{\lambda + d}{\gamma + d}} ((R + br)^d - (R + ar)^d)^{\frac{\lambda + d}{\gamma + d}} H^{\frac{\lambda + d}{\gamma + d}} \tag{4.48}
\]
\[
\leq \left( \frac{d H 2^{d-1} \pi^{d/2}}{\Gamma(d/2 + 1)} \right)^{\frac{\lambda + d}{\gamma + d}} R^{\frac{d-1}{\gamma + d}} (r(b - a))^{-\frac{\gamma + d-1}{\gamma + d}}.
\]

Also, note that for any \( a, b > 0 \), if \( a \leq b \leq N_0 \) for some \( N_0 \in (0, \infty) \), then
\[
1 \leq N_0 \left( \frac{1}{a} \vee \frac{1}{b} \right) \leq N_0 \left( \frac{1}{a} + \frac{1}{b} \right). \tag{4.49}
\]
Thus, since \( R, H > 1 \) and \( r \) satisfies \((4.39)\), if we multiply \( G^{\gamma + d} \) both sides of \((4.47), (4.48) \) and \((4.49)\) imply
\[
\left( \int_{R + ar < |x| < R + br} g(x) dx \right)^{\frac{\lambda + d}{\gamma + d}} \leq N \left( (4H)^{1/\gamma} \vee \frac{G}{r(b - a)} \right)^d \int_{R + ar < |x| < R + br} (g(x))^\lambda dx
\]
\[
\leq N \left( (4H)^{1/\gamma} + R^{\frac{d-1}{\gamma + d}} (r(b - a))^{-\frac{\gamma + d-1}{\gamma + d}} \left( \frac{d H 2^{d-1} \pi^{d/2}}{\Gamma(d/2 + 1)} \right)^{\frac{\lambda + d}{\gamma + d}} \right)^d \int_{R + ar < |x| < R + br} (g(x))^\lambda dx
\]
\[
\leq N R^{\frac{d(d-1)}{\gamma + d}} (r(b - a))^{-\frac{d(\gamma + d-1)}{\gamma + d}} \int_{R + ar < |x| < R + br} (g(x))^\lambda dx,
\]
where \( N = N(\gamma, \lambda, d, H) \). Thus, Lemma 4.2 (i) is proved.

Finally, we prove Lemma 4.2 (ii). Since the proof is similar to the one of Lemma 4.2 (i) with \( d = 1 \), we only point out the differences. Instead of \((4.50)\), we have
\[
\left( \int_0^T g(t) dt \right)^{\frac{\lambda + 1}{\gamma + 1}} \leq N \left( (4H)^{1/\gamma} \vee \frac{G}{T} \right) \int_0^T (g(t))^\lambda dt, \tag{4.51}
\]
where \( G := \left( \int_0^T g(t) dt \right)^{1/(\gamma + 1)} \). Since we assume \( g(0) = 0 \), we have \( g(t) = g(t) - g(0) \leq H |t - 0|^\gamma \leq HT^\gamma \) for all \( t \in (0, T) \), and thus \( G \leq H^{1/(\gamma + 1)} T \). Therefore, from \((4.51)\) and \((4.49)\),
\[
\left( \int_0^T g(t) dt \right)^{\frac{\lambda + 1}{\gamma + 1}} \leq N \left( (8H)^{\frac{\gamma}{\gamma + 1}} + H^{\frac{\gamma}{\gamma + 1}} T^{\frac{\gamma}{\gamma + 1}} \right) \int_0^T (g(t))^\lambda dt,
\]
which completes the proof.
\[\square\]

Now we introduce an integral inequality that produces a lower bound of correlation measure \( f \) in \((2.1)\). The motivation of the proof follows from \((2.4)\).
Lemma 4.3. For the given correlation measure \( f \), there exists a nonnegative function \( \varphi \in C_c^\infty(\mathbb{R}^d) \) with \( \int_{\mathbb{R}^d} \varphi(x)dx > 0 \) such that for any bounded nonnegative measurable function \( g \) with compact support on \( \mathbb{R}^d \), we have
\[
\int_{\mathbb{R}^d} (g \ast \tilde{g})(x) f(dx) \geq \int_{\mathbb{R}^d} \|(g \ast \varphi)(x)\|^2 dx,
\]
where \( \tilde{g}(x) = g(-x) \).

Proof. Let us first construct \( \varphi \) by approximating \( f \) with a continuous nonnegative function. Define
\[
\psi(x) := 4^{-d} \prod_{i=1}^d (2 - |x^i|)_+^2|_{|x^i| \leq 2}(x).
\]
Observe that \( \psi \) is a continuous, nonnegative and nonnegative definite function on \( \mathbb{R}^d \). In addition, \( \psi \) has a compact support and \( \int_{\mathbb{R}^d} \psi(x)dx = 1 \). For \( \varepsilon > 0 \), define \( \psi_\varepsilon(x) := e^{-d\varepsilon \psi(x/\varepsilon)} \). Then, we have
\[
|\hat{\psi}_\varepsilon(\xi)|^2 = |\hat{\psi}(\xi/\varepsilon)|^2 = \int_{\mathbb{R}^d} e^{-i\varepsilon \xi \cdot x} \psi(x)dx \leq \int_{\mathbb{R}^d} \psi(x)dx \leq 1,
\]
where \( \hat{\psi}(\xi) := \mathcal{F}(\psi)(\xi) \). With given \( \psi_\varepsilon \), set \( f_\varepsilon(x) := (\psi_\varepsilon \ast \psi_\varepsilon \ast f)(x) \). Then, \( f_\varepsilon \) is a nonnegative and nonnegative definite \( C^2(\mathbb{R}^d) \) function that is not identically 0. Thus, \( f_\varepsilon(0) > 0 \) since \( f_\varepsilon \) achieves its maximum at the origin (thanks to the fact that \( f_\varepsilon \) is a nonnegative definite function). Thus, there exists a constant \( r > 0 \) satisfying \( \inf_{|x| \leq r} f_\varepsilon(x) \geq c > 0 \) for some \( c > 0 \), which implies that there exists \( \varphi \in C_c^\infty(\mathbb{R}^d) \) such that \( \varphi \geq 0 \), \( \text{supp}(\varphi) \subseteq \{ |x| \leq r \} \), and \( 0 < \sup_{x \in \mathbb{R}^d} (\varphi \ast \hat{\varphi})(x) = \| \varphi \|_{L^2} \leq c/2 \).

Next, we approximate \( g \) by \( C_c(\mathbb{R}^d) \) functions to apply (2.1). Choose \( \zeta \in C_c^\infty(\mathbb{R}^d) \) such that \( \int_{\mathbb{R}^d} \zeta(x)dx = 1 \) and set \( \zeta_\delta(x) := \delta^{-d}\zeta(x/\delta) \) with \( \delta > 0 \). Recall \( g^{(\delta)}(x) = (g \ast \zeta_\delta)(x) \) in (4.8). Since \( g^{(\delta)}(\cdot) \in C_c(\mathbb{R}^d) \), by Fatou’s lemma and (2.1), we have
\[
\int_{\mathbb{R}^d} (g \ast \varphi)(x)^2 dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x)g(y)(\varphi \ast \hat{\varphi})(x - y)dxdy
\leq \liminf_{\delta \downarrow 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x)g(y)f_\varepsilon(x - y)dxdy
\leq \liminf_{\delta \downarrow 0} \int_{\mathbb{R}^d} \left| g^{(\delta)}(x) \right|^2 \tilde{f}_\varepsilon(x)dx
= \liminf_{\delta \downarrow 0} \int_{\mathbb{R}^d} \left| g^{(\delta)}(\xi) \right|^2 \hat{\tilde{f}}(d\xi).
\]

The last equality follows from \( f_\varepsilon := \psi_\varepsilon \ast \psi_\varepsilon \ast f \). By applying (4.53) to (4.54), we have
\[
\int_{\mathbb{R}^d} (g \ast \varphi)(x)^2 dx \leq \liminf_{\delta \downarrow 0} \int_{\mathbb{R}^d} \left| g^{(\delta)}(\xi) \right|^2 \hat{f}(d\xi)
= \liminf_{\delta \downarrow 0} \int_{\mathbb{R}^d} \left( g^{(\delta)} \ast \hat{g}^{(\delta)} \right)(x)f(dx)
= \int_{\mathbb{R}^d} (g \ast \hat{g})(x)f(dx),
\]

where the last equality is justified by the bounded convergence theorem. This completes the proof.

We are now ready to prove Lemma 3.2

**Proof of Lemma 3.2.** First we prove (i). Let \( p, q > 0, \alpha \in (0, 1) \), and recall \( \ell = \frac{\alpha \lambda + d}{\gamma + d} \in (0, 1) \), where \( \gamma \) is the constant introduced in Theorem 2.9. Define

\[
\kappa := \tau \land \inf \left\{ t \geq 0 : \int_0^t \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) l \, ds \geq q^l \right\},
\]

and

\[
U_R(s, x) := e^{-K_2 s} e^{-K_1 M(x)} M(x)(u(s, x))^\lambda,
\]

where \( M(x) \) is the function introduced in (4.1). By Chebyshev’s inequality, (4.3), and the definition of \( \kappa \), we have

\[
P \left( \int_0^\tau \int_{R^d} (U_R(s, \cdot) \ast \tilde{U}_R(s, \cdot))(x)f(dx)ds \geq p^l \right)
\]

\[
\leq P(\kappa < \tau) + P \left( \int_0^\tau \int_{R^d} (U_R(s, \cdot) \ast \tilde{U}_R(s, \cdot))(x)f(dx)ds \geq p^l \right)
\]

\[
\leq P \left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \geq q^l \right) + p^{-\alpha/2} 2N \mathbb{E} \left[ \left( \int_0^\kappa \int_{\partial Q_R} u(s, \sigma) d\sigma_R ds \right)^\alpha \right],
\]

where \( N = N(\alpha, d, K) \). Now we control the last term of (4.57). Due to (3.1), we have

\[
\left| \int_{\partial Q_R} u(t, \sigma) d\sigma_R - \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right| \leq \int_{\partial Q_R} |u(t, \sigma) - u(s, \sigma)| d\sigma_R \leq N(d) R^{d-1} e^{aR} |t - s|^{\gamma}
\]

for \( t, s \in (0, T) \). Additionally, note that \( \int_{\partial Q_R} u(0, \sigma) d\sigma_R = \int_{\partial Q_R} u_0(\sigma) d\sigma_R = 0 \). Therefore, Lemma 4.2 (ii) implies

\[
\int_0^\kappa \int_{\partial Q_R} u(s, \sigma) d\sigma_R ds \leq N \left( \int_0^\kappa \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \right)^{\frac{\gamma + 1}{\gamma + \alpha}},
\]

(4.58)

where \( N = N(\gamma, d, R, H) \). Thus, by definition of \( \kappa \) and applying (4.58) to the last term of (4.57), we have

\[
\mathbb{E} \left[ \left( \int_0^\kappa \int_{\partial Q_R} u(s, \sigma) d\sigma_R ds \right)^\alpha \right] \leq N \mathbb{E} \left[ \left( \int_0^\kappa \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \right)^{\frac{\gamma + 1}{\gamma + \alpha}} \right] \leq N q^{\frac{\gamma + 1}{\gamma + \alpha}},
\]

(4.59)

where \( N = N(\alpha, \gamma, a, d, H, R) \). Therefore, by applying (4.59) to the last term of (4.57),

\[
P \left( \int_0^\tau \int_{R^d} (U_R(s, \cdot) \ast \tilde{U}_R(s, \cdot))(x)f(dx)ds \geq p^l \right)
\]

\[
\leq P \left( \int_0^\tau \left( \int_{\partial Q_R} u(s, \sigma) d\sigma_R \right)^l \, ds \geq q^l \right) + N \left( \frac{q^{\frac{\gamma + 1}{\gamma + \alpha}}}{p^{1/\alpha}} \right)^{1/\alpha},
\]

(4.60)
where \( N = N(\alpha, \gamma, \lambda, a, d, R, H, K) \). By letting \( q \downarrow 0 \) and \( p \downarrow 0 \) in order, (4.60) implies

\[
P \left( \int_0^T \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) \frac{i}{l} ds = 0 \right) \leq P \left( \int_0^T \int_{\mathbb{R}^d} (U_R(s, \cdot) \ast \tilde{U}_R(s, \cdot))(x) f(dx) ds = 0 \right). \tag{4.61}
\]

Next, we claim that

\[
\left\{ \omega \in \Omega : \int_0^T \int_{\mathbb{R}^d} (U_R(s, \cdot) \ast \tilde{U}_R(s, \cdot))(x) f(dx) ds = 0 \right\} \subset \left\{ \omega \in \Omega : u(s, x) = 0 \text{ for all } s \in [0, \tau] \text{ and } x \in Q_R \right\}. \tag{4.62}
\]

Indeed, for \( S > R \), let us set

\[
g_{R,S}(s, x) := M(x) e^{-K_1 M(x)} (u(s, x))^\lambda 1_{Q_R \setminus Q_S}(x)
\]

and choose \( \varphi \in C^\infty_c(\mathbb{R}^d) \) introduced in Lemma 4.3. Since \( g_{R,S} \) is a bounded nonnegative measurable function with compact support on \( \mathbb{R}^d \), we have

\[
\int_{\mathbb{R}^d} \left( g_{R,S}(s, \cdot) \ast \tilde{g}_{R,S}(s, \cdot) \right)(x) f(dx) \geq \int_{\mathbb{R}^d} |(g_{R,S}(s, \cdot) \ast \varphi)(x)|^2 dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_{R,S}(s, x) g_{R,S}(s, y) (\varphi \ast \tilde{\varphi})(x-y) dy dx. \tag{4.63}
\]

Therefore, if we take \( \omega \in \Omega \) satisfying \( \int_0^T \int_{\mathbb{R}^d} (U_R(s, \cdot) \ast \tilde{U}_R(s, \cdot))(x) f(dx) ds = 0 \), then \( U_R \geq g_{R,S} \), nonnegativity of \( f \), and (4.63) yield

\[
\int_0^T \int_{Q_R \setminus Q_S} \int_{Q_R \setminus Q_S} e^{-2K_2 x - K_1 (M(x) + M(y))} M(x) M(y) (u(t, x))^\lambda (u(t, y))^\lambda (\varphi \ast \tilde{\varphi})(x-y) dy dx ds = 0.
\]

Thus,

\[
(u(s, x))^\lambda (u(s, y))^\lambda (\varphi \ast \tilde{\varphi})(x-y) = 0
\]

almost every \((s, x, y) \in (0, \tau) \times (Q_R \setminus Q_S) \times (Q_R \setminus Q_S)\). Since \((\varphi \ast \tilde{\varphi})(0) > 0\), we have \((u(t, x))^2 \lambda = 0\) for almost every \((t, x) \in (0, \tau) \times (Q_R \setminus Q_S)\). Since \( S > R \) is arbitrary, the continuity of \( u \) implies (4.62). Then, by (4.61) and (4.62), we have (i).

Next we prove (ii). Let \( \alpha \in (0, 1), p, q > 0, \) and \( r > 0 \) satisfy (3.5). Again, we set

\[
\kappa := \tau \wedge \inf \left\{ t \geq 0 : \int_0^t \left( \int_{\partial Q_R} u(s, \sigma) d\sigma \right) \frac{i}{l} ds \geq q \right\}.
\]
Then, by Chebyshev’s inequality and Jensen’s inequality, we have
\[
\begin{align*}
\tau^{-1} \int_{-\tau}^{\tau} \mathbb{P} \left( \int_{0}^{\tau} \int_{\partial Q_{R+\tau}} u(s, \sigma) d\sigma R+z \right)^l ds \geq p_l \right) dz \\
\leq \mathbb{P}(\kappa < \tau) + \tau^{-1} \int_{-\tau}^{\tau} \mathbb{P} \left( \int_{0}^{\tau} \int_{\partial Q_{R+\tau}} u(s, \sigma) d\sigma R+z \right)^l ds \geq p_l \right) dz \\
\leq \mathbb{P} \left( \int_{0}^{\tau} \int_{\partial Q_{R}} u(s, \sigma) d\sigma R \right)^l ds \geq q^l \\
& \quad + \tau^{-1} p^{-\alpha l} \int_{-\tau}^{\tau} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{\partial Q_{R+\tau}} u(s, \sigma) d\sigma R+z \right)^l ds \right]^\alpha dz.
\end{align*}
\]
(4.64)

To bound the last term of (4.64), we use Jensen’s inequality to get
\[
\begin{align*}
\int_{-\tau}^{\tau} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{\partial Q_{R+\tau}} u(s, \sigma) d\sigma R+z \right)^l ds \right]^\alpha dz \\
& \leq \tau^{1-\alpha l} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{\partial Q_{R+\tau}} u(s, \sigma) d\sigma R+z \right)^l ds \right]^\alpha \\
& = \tau^{1-\alpha l} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{R+\tau < |x| < R+2\tau} u(s, x) ds \right)^l ds \right]^\alpha.
\end{align*}
\]
(4.65)

Thanks to (3.4), we can use Lemma 4.2 (i) to see that the last term of (4.65) is dominated by
\[
\begin{align*}
\mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{R+\tau < |x| < R+2\tau} u(s, x) ds \right)^l ds \right]^\alpha \\
& \leq N e^{\alpha l a R} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{R+\tau < |x| < R+2\tau} \Psi(a) u(s, x) ds \right)^l ds \right]^\alpha \\
& \leq N R^{\frac{a(d-1)}{\gamma+1} - \frac{a(l-1)}{\gamma+1}} e^{\alpha(l-\lambda) a R} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{R+\tau < |x| < R+2\tau} (u(s, x))^\lambda ds \right)^{\frac{\alpha}{l}} ds \right],
\end{align*}
\]
(4.66)

where \( l = \frac{2\lambda + d}{\gamma + d} \) and \( N = N(\alpha, \gamma, \lambda, a, d, H) \). Since the difference between \( Q_{R+\tau} \setminus Q_{R+2\tau} \) and \( \{ x : R+\tau < |x| < R+2\tau \} \) has measure zero, by combining (4.65) and (4.66), we have
\[
\begin{align*}
\tau^{-1} \int_{-\tau}^{\tau} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{\partial Q_{R+\tau}} u(s, \sigma) d\sigma R+z \right)^l ds \right]^\alpha dz \\
& \leq N R^{\frac{a(d-1)}{\gamma+1} - \frac{a(l-1)}{\gamma+1}} e^{\alpha(l-\lambda) a R} \mathbb{E} \left[ \left( \int_{0}^{\tau} \int_{Q_{R+\tau} \setminus Q_{R+2\tau}} (u(s, x))^\lambda ds \right)^{\frac{\alpha}{l}} ds \right].
\end{align*}
\]
(4.67)
where \( N = N(\alpha, \gamma, \lambda, a, d, H, K) \).

To proceed further, we set

\[
g(s, x) := M(x) e^{-K_1 M(x)} (u(s, x))^\lambda e^\frac{R}{2} Q_{R+\delta/2} e^\frac{R}{2} (s, x).
\]

and let \( \varphi \in C_c(\mathbb{R}^d) \) be the function introduced in Lemma 4.3. Without loss of generality, we may assume that \( \text{supp}(\varphi) \subseteq Q_R^c \). Then, note that

\[
\left( \int_{\mathbb{R}^d} g(s, x) dx \right)^2 \leq N \left( \int_{|x|<2(R+2\tau)} (g(s, \cdot) * \varphi)(x) dx \right)^2
\]

where \( N = N(d, f) \). The equality in the second line holds since \( \text{supp}(g(s, \cdot)) + \text{supp}(\varphi) \subseteq \{ |x| < 2(R+2\tau) \} \). Thus, by applying Lemma 4.3 to (4.68), we have

\[
\left( \int_{\mathbb{R}^d} g(s, x) dx \right)^2 \leq NR^d \int_{|x|<2(R+2\tau)} |(g(s, \cdot) * \varphi)(x)|^2 dx,
\]

where \( N = N(d, f) \). We now use (4.69) and Jensen’s inequality to get that

\[
R^{-cd/2} e^{-2K_1 R^d} \left( \int_0^T \int_{Q_{R+\delta/2}} (u(s, x))^\lambda ds \right)^\alpha \leq R^{-cd/2} \left( \int_0^T \int_{Q_{R+\delta/2}} g(s, x) ds \right)^\alpha \leq N \left( \int_0^T \int_{\mathbb{R}^d} e^{-2K_1 R^d} (g(s, \cdot) * \varphi)(x) f(dx) ds \right)^{\alpha/2},
\]

where \( N = N(\alpha, d, f, K, T) \). Then, by combining (4.70) and Lemma 4.1 since \( \tau \leq 1 \), we have

\[
E \left[ \left( \int_0^T \int_{Q_{R+\delta/2}} (u(s, x))^\lambda ds \right)^\alpha \right] \leq NR^{-cd/2} e^{-2K_1 R^d} \left( \int_0^\infty \int_{\partial Q_R} e^{-K_2 s} u(s, \sigma) d\sigma ds \right)^\alpha,
\]

where \( N = N(\alpha, \lambda, d, f, H, K, T) \). On the other hand, (3.4) implies

\[
\left| \int_{\partial Q_R} u(t, \sigma) d\sigma - \int_{\partial Q_R} u(s, \sigma) d\sigma \right| \leq N(d) R^{d-1} e^{aR} H |t - s|^\gamma.
\]
where $L := \frac{\gamma + d}{\gamma \lambda + d} : \frac{\gamma}{(\gamma \lambda + d)} + \gamma + d \in (0, \gamma, \lambda, d, H)$ and $N = N(\alpha, \gamma, \lambda, a, d, H)$. Therefore, by combining (4.64), (4.67), (4.71), and (4.72), we have

$$r^{-1} \int_{\tau}^{2\tau} P \left( \int_{0}^{r} \left( \int_{\partial Q_{R+z}} u(s, \sigma) d\sigma_{R+z} \right)^l ds \geq p^l \right) dz$$

$$\leq P \left( \int_{0}^{r} \left( \int_{\partial Q_{R}} u(s, \sigma) d\sigma_{R} \right)^l ds \geq q^l \right)$$

$$+ NR^{\alpha(\frac{d}{2} + d(\frac{1-\gamma}{\gamma}) + \frac{d(1-\gamma)}{\gamma} + \frac{d(1-\gamma)}{\gamma})},$$

where $L := \frac{\gamma + d}{\gamma \lambda + d} : \frac{\gamma}{(\gamma \lambda + d)} + \gamma + d \in (0, \gamma, \lambda, a, d, H, K, T) > 0$. This implies (5.6) and we complete the proof.

5. Proof of Lemma 3.1

In this section, we provide the proof of Lemma 3.1. Throughout this section, we assume that there exist $a, H \in (0, \infty)$ satisfying (3.2). To prove Lemma 3.1, we need an $L_1$ bound of the solution.

Lemma 5.1. Let $\tau \leq T$ be a bounded stopping time. Then, for every $R > (R_0 \vee 1) + 1$ and $\delta > 0$, we have

$$E \left[ \int_{Q_R} |x| e^{\delta |x|} u(\tau, x) dx \right] \leq N$$

(5.1)

where $N = N(\delta, a, d, m, H, K, R_0, T)$ and $R_0$ is the constant introduced in Assumption 2.6. In particular,

$$\lim_{R \to \infty} E \left[ \int_{Q_R} |x| e^{\delta |x|} u(\tau, x) dx \right] = 0.$$  

(5.2)

Proof. For convenience, set

$$q(x) := |x| e^{\delta |x|}$$

on $\mathbb{R}^d$. Choose $R_1 \in (R_0 \vee 1, (R_0 \vee 1) + 1)$. Let $\psi \in C^\infty_c(\mathbb{R})$ be a nonnegative symmetric function satisfying $\psi(z) = 1$ on $|z| < 1$ and $\psi(z) = 0$ on $|z| \geq 2$ and define $\psi_n(x) := \psi(|x|/n)$ for $n \in \mathbb{N}$. Take a nonnegative function $\zeta \in C^\infty_c(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \zeta(z) dz = 1$, symmetric, and $\zeta(z) = 0$ on $|z| \geq 1$. For $m \in \mathbb{N}$, define

$$\varphi_m(x) := m \int_{\mathbb{R}} \varphi_{\psi_n}(x - \frac{m}{n} \zeta(m(|x| - z)) dz = \int_{\mathbb{R}} \varphi_{\psi_n}(x - \frac{m}{n} \zeta)|x| > R_1 - \frac{m}{n} \zeta(z) \zeta(z) dz.$$
Additionally, for $g \in L_{1,\text{loc}}(\mathbb{R}^d)$ and $\varepsilon > 0$, set $g^{(\varepsilon)}(x)$ as in (5.8). Choose $K_0 > 0$ such that

$$K_0 > (5 + 6\delta + \delta^2)K$$

(5.3)

where $K$ is the constant introduced in Assumption 2.3. Then, for any $t \leq T$ and large $m \in \mathbb{N}$, since $\text{supp}(u_0) \subset \{ ||x|| \leq R_0 \}$ and $R_0 < R_1$, we have

$$E \left[ \int_{Q_{R_1}} e^{-K_0(t \wedge \tau)} u(t \wedge \tau, x) q^{(\varepsilon)}(x) \psi_n(x) dx \right]$$

$$\leq E \left[ \int_{R^d} e^{-K_0(t \wedge \tau)} u(t \wedge \tau, x) q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) dx \right]$$

$$= E \left[ \int_0^{t \wedge \tau} \int_{R^d} e^{-K_0 s} u(s, x) \left[ \mathcal{L}^* \left( q^{(\varepsilon)} \psi_n \varrho_m \right) (s, x) - K_0 q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) \right] dx ds \right].$$

Note that

$$\left( q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) \right)_{x^{i}}$$

$$= q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) + q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) + q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x),$$

$$\left( q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) \right)_{x^{i}x^{j}}$$

$$= q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) + q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) + q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x)$$

$$+ 2q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) + 2q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x) + 2q^{(\varepsilon)}(x) \psi_n(x) \varrho_m(x).$$

Thus, if we apply (5.6) to the last term of (5.4), we have

$$\mathcal{L}^* \left( q^{(\varepsilon)} \psi_n \varrho_m \right) (s, x) - K_0 \left( q^{(\varepsilon)} \psi_n \varrho_m \right) (x)$$

$$= a^{ij}_{x^{i}x^{j}} - b^i_{x^{i}} + c - K_0 \left( q^{(\varepsilon)} \psi_n \varrho_m \right) (x) + \left( 2a^{ij}_{x^{i}x^{j}} - b^i \right) \left( q^{(\varepsilon)} \psi_n \varrho_m \right)_{x^{i}} (x)$$

$$+ a^{ij} \left( q^{(\varepsilon)} \psi_n \varrho_m \right)_{x^{i}x^{j}} (x)$$

(5.6)

$$\leq \left( K - K_0 \right) \left( q^{(\varepsilon)} \psi_n \right) (x) + \left( 2a^{ij}_{x^{i}x^{j}} - b^i \right) \left( q^{(\varepsilon)} \psi_n \right)_{x^{i}} (x) + a^{ij} \left( q^{(\varepsilon)} \psi_n \right)_{x^{i}x^{j}} (x) \varrho_m (x)$$

$$+ \left( 2a^{ij}_{x^{i}x^{j}} - b^i \right) \left( q^{(\varepsilon)} \psi_n \right) (x) + 2a^{ij} \left( q^{(\varepsilon)} \psi_n \right)_{x^{i}} (x) + a^{ij} \left( q^{(\varepsilon)} \psi_n \varrho_m \right)_{x^{i}x^{j}} (x).$$

Observe that on $||x|| > R_1 > 1$, by letting $\varepsilon \downarrow 0$,

$$q^{(\varepsilon)}(x) \rightarrow q(x), \quad q^{(\varepsilon)}_{x^{i}}(x) \rightarrow q_{x^{i}}(x) = (||x||^{-1} + \delta) \frac{x^{i}}{||x||} q(x), \quad |q_{x^{i}}(x)| \leq (1 + \delta)q(x),$$

(5.7)

and

$$q^{(\varepsilon)}_{x^{i}x^{j}}(x) \rightarrow q_{x^{i}x^{j}}(x) = \left[ \left( 1 \frac{1}{||x||^2} + \frac{\delta}{||x||} \right) \frac{\delta^{ij}}{||x||^2} - \frac{x^{i}x^{j}}{||x||^2} \right] q(x) + \left( \frac{2\delta}{||x||} + \frac{\delta^2}{||x||^2} \right) \frac{x^{i}x^{j}}{||x||^2} q(x),$$

(5.8)

$$|q_{x^{i}x^{j}}(x)| \leq (2 + 4\delta + \delta^2)q(x),$$

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where $\delta^{ij}$ is the Kronecker delta. Thus, if we let $n$ be large enough and $\varepsilon \downarrow 0$ in the product rule, by (5.7), and (5.3), we have that for $|x| > R_1$

$$(K - K_0)q(x)\psi_n(x) + \left(2a_{xj}^{ij} - b^i\right) \left(q(x)\psi_n(x)\right)_{x^i} + a^{ij} \left(q(x)\psi_n(x)\right)_{x^ix^j}$$

$$= (K - K_0)q(x)\psi_n(x) + \left(2a_{xj}^{ij} - b^i\right) \left(q(x)\psi_n(x) + q(x)\psi_{nx^i}\right)$$

$$+ a^{ij} q_{x^i} x^j(x) \psi_n(x) + 2a^{ij} q_{x^i} x^j(x) \psi_{nx^i}(x) + a^{ij} q(x) \psi_{nx^i x^j}(x)$$

$$\leq \left[(5 + 6\delta + \delta^2) K + N(\delta, K) n^{-1} - K_0\right] q(x)\psi_{2n}(x).$$

To obtain the last inequality, we employ

$$|\psi(|x|/n)| + |\psi'(|x|/n)| + |\psi''(|x|/n)| \leq N \psi \left(\frac{|x|}{2n}\right),$$

where $N$ is independent of $n$. Thus, if $n$ large enough,

$$(K - K_0) q(x) \psi_n(x) + \left(2a_{xj}^{ij} - b^i\right) \left(q(x)\psi_n(x)\right)_{x^i} + a^{ij} \left(q(x)\psi_n(x)\right)_{x^ix^j} \leq 0. \quad (5.10)$$

In addition, observe that

$$\limsup_{n \to \infty, \varepsilon \downarrow 0} \left[\left(2a_{xj}^{ij} - b^i\right) \left(q(x)\psi_n(x)\right)_{x^i} + a^{ij} \left(q(x)\psi_n(x)\right)_{x^ix^j}\right]$$

$$\leq \left(2a_{xj}^{ij} - b^i\right) q(x) + 2a^{ij} q_{x^i} x^j(x) \left[|\varrho_{mx^j}(x)| + a^{ij} q(x)|\varrho_{nx^ix^j}(x)|\right]$$

$$\leq N(d, K) q(x) \left(|\varrho_{mx^j}(x)| + |\varrho_{nx^ix^j}(x)|\right). \quad (5.11)$$

Thus, by employing (5.6), (5.10), and (5.11) to (5.4), and letting $\varepsilon \downarrow 0$ and $n \to \infty$ in order, we have

$$\mathbb{E} \left[\int_{Q_{R_1}} e^{-K_0(t \wedge \tau)} u(t \wedge \tau, x) q(x) \, dx\right]$$

$$\leq N(d, K) \mathbb{E} \left[\int_{0}^{t \wedge \tau} \int_{|x| < R_1} e^{-K_0 s} u(s, x) q(x) \left(|\varrho_{mx^j}(x)| + |\varrho_{nx^ix^j}(x)|\right) \, dx \, ds\right] \quad (5.12)$$

$$\leq N(\delta, d, m, K, R_0) \mathbb{E} \left[\int_{0}^{t \wedge \tau} \int_{|x| < R_1} e^{-K_0 s} u(s, x) \, dx \, ds\right].$$

Therefore, by (5.2)

$$\mathbb{E} \left[\int_{Q_{R_1}} q(x) u(t, x) \, dx\right] \leq N \mathbb{E} \left[\int_{0}^{t \wedge \tau} \int_{|x| < R_1} e^{-K_0 s} u(s, x) \, dx \, ds\right]$$

$$\leq N \int_{|x| < R_1} e^{a|x|} \, dx,$$

$$\leq N,$$

where $N = N(\delta, a, d, m, H, K, R_0, T)$. Since $Q_R \subset Q_{R_1}$ for any $R > (R_0 \vee 1) + 1$, we have (5.1). To obtain (5.2), we employ the dominated convergence theorem with the fact that for $R > R_1$,

$$1_{Q_{R_1}} q(x) u(t, x) \leq 1_{Q_{R_1}} q(x) u(t, x).$$

The lemma is proved. \qed
Proof of Lemma 3.1. Since the method of proof is similar to the proof of Lemma 4.1 we only sketch the proof.

For \( m, n \in \mathbb{N} \), choose \( \psi_n \) and \( \phi_m \) as in the proof of Lemma 4.1. Fix \( m, n \in \mathbb{N}, \varepsilon > 0 \), and \( \varepsilon_1 > 0 \). Recall that \( K_1, K_2 \) and \( N_1 \) in (4.2). Then, by applying Itô’s formula and taking expectation, we have

\[
E \left[ \int_{\mathbb{R}^d} e^{K_2 \tau} u(\tau, x) M(\varepsilon(x)) \psi_n(x) \phi_m(x) e^{K_1 M(\varepsilon(x))} \, dx \right] = E \left[ \int_0^\tau \int_{\mathbb{R}^d} e^{K_2 \tau} u(s, x) \mathcal{L}^* \left( M(\varepsilon(x)) \psi_n(x) \phi_m(x) e^{K_1 M(\varepsilon(x))} \right) \, dx \, ds \right] + E \left[ K_2 \int_0^\tau \int_{\mathbb{R}^d} e^{K_2 \tau} u(s, x) M(\varepsilon(x)) \psi_n(x) \phi_m(x) e^{K_1 M(\varepsilon(x))} \, dx \, ds \right].
\]

(5.14)

We note that \( K_1 \) and \( K_2 \) are plugged in the exponent of exponential functions instead of \(-K_1\) and \(-K_2\). Then, by combining (5.15), (5.18), and (5.19) with taking \( \hat{a}_{ij}^j \) of the proof of Lemma 4.1, we obtain

\[
\hat{a}_{ij}^j - b_i^j + c + K_1 \left( 2a_{ij} - b^i \right) M_{x^i} + K_1 a_{ij}^j M_{x^x} + K_2 a_{ij} M_{x^2} \geq -N_1,
\]

we have

\[
E \left[ \int_{\mathbb{R}^d} e^{K_2 \tau} u(\tau, x) M(\varepsilon(x)) \psi_n(x) \phi_m(x) e^{K_1 M(x)} \, dx \right] \geq E \left[ \int_0^\tau e^{K_2 \tau} \left( \hat{A}(m, n, \varepsilon) + \hat{B}(m, n, \varepsilon) \right) \, ds \right],
\]

(5.15)

where

\[
\hat{A}(m, n, \varepsilon) := \int_{\mathbb{R}^d} \left[ 2a_{ij} - b^i + 2K_1 a_{ij}^j M_{x^j} \right] u(s, x) \left( \Phi^{m, n, \varepsilon}(x) \right)_{x^j} e^{K_1 M(x)} \, dx,
\]

(5.16)

\[
\hat{B}(m, n, \varepsilon) := \int_{\mathbb{R}^d} u(s, x) \left( a_{ij}^j \left( \Phi^{m, n, \varepsilon}(x) \right)_{x^j} \right) e^{K_1 M(x)} \, dx.
\]

(5.17)

The function \( \Phi^{m, n, \varepsilon}(x) \) is introduced in (4.10) and the constant \( N_1 \) is introduced in (4.2).

Next, similar to (Step 2) and (Step 3) of the proof of Lemma 4.1 we obtain

\[
\lim_{n \to \infty} \lim_{m \to \infty} \lim_{\varepsilon \to 0} \hat{A}(m, n, \varepsilon) \geq K^{-1} K_1 \int_{\mathbb{R}} u(s, x) e^{K_1 M(x)} \, dx,
\]

(5.18)

\[
\lim_{n \to \infty} \lim_{m \to \infty} \lim_{\varepsilon \to 0} \hat{B}(m, n, \varepsilon) \leq -2K \int_{\mathbb{R}} u(s, x) e^{K_1 M(x)} \, dx + K^{-1} \int_{\partial \mathbb{R}} u(s, \sigma) \, d\sigma.R.
\]

(5.19)

The only difference is to consider \(-K\) as a lower bound of the coefficients \( a_{ij}^j, b^i, \) and \( c \).

Then, by combining (5.15), (5.18), and (5.19) with taking \( K_1 \geq 2K^2 \), we have

\[
E \left[ \int_0^\tau \int_{\partial \mathbb{R}} u(s, \sigma) \, d\sigma.R \, ds \right] \leq K E \left[ \int_{\mathbb{R}} e^{K_2 \tau} u(\tau, x) M(x) e^{K_1 M(x)} \, dx \right] \leq K e^{-K_1 R} E \left[ \int_{\mathbb{R}} e^{K_2 \tau} u(\tau, x) |x| e^{K_1 |x|} \, dx \right],
\]

and thus

\[
e^{K_1 R} E \left[ \int_0^\tau \int_{\partial \mathbb{R}} u(s, \sigma) \, d\sigma.R \, ds \right] \leq N(K, T) E \left[ \int_{\mathbb{R}} u(\tau, x) |x| e^{K_1 |x|} \, dx \right].
\]

Thus, by (5.2), we have (3.3) for all \( \delta = K_1 \geq 2K^2 \) (see (4.2)). The case of \( \delta \in (0, 2K^2) \) also follows immediately from the above display. The lemma is proved. □
A. Proof of Theorem 2.9

This section is devoted to proving Theorem 2.9. The proof is based on the $L_p$-theory for stochastic partial differential equations; see [14][16]. The $L_p$-theory enables us to show the existence of a solution to (1.1) in some stochastic Banach spaces and also Hölder continuity of the solution through the Sobolev embedding theorem. We briefly introduce the definitions of stochastic Banach spaces below. For more information, see [10][16].

**Definition A.1** (Bessel potential space). Let $p > 1$ and $\gamma \in \mathbb{R}$. The space $H_\gamma^p = H_\gamma^p(\mathbb{R}^d)$ is the set of all tempered distributions $u$ on $\mathbb{R}^d$ satisfying

$$
\|u\|_{H_\gamma^p} := \left\| (1 - \Delta)^{\gamma/2} u \right\|_{L_p} = \left\| \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{\gamma/2} \mathcal{F}(u)(\xi) \right] \right\|_{L_p} < \infty.
$$

Similarly, $H_\gamma^p(\ell_2) = H_\gamma^p(\mathbb{R}^d; \ell_2)$ is the space of $\ell_2$-valued functions $g = (g^1, g^2, \cdots)$ satisfying

$$
\|g\|_{H_\gamma^p(\ell_2)} := \left\| (1 - \Delta)^{\gamma/2} g \right\|_{L_p} = \left\| \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{\gamma/2} \mathcal{F}(g)(\xi) \right] \right\|_{\ell_2} < \infty.
$$

For $\gamma = 0$, we set $H_0^p := L_p$ and $H_0^p(\ell_2) := L_p(\ell_2)$.

Below lemma is well-know result for the H"older embedding theorem for the Bessel potential spaces.

**Lemma A.2.** Let $p > 1$ and $\gamma \in \mathbb{R}$. If $\gamma - d/p = \nu$ for some $\nu \in (0, 1)$, then we have

$$
[u]_{C(\mathbb{R}^d)} + [u]_{C^\nu(\mathbb{R}^d)} \leq N \|u\|_{H_\gamma^p}, \tag{A.1}
$$

where $N = N(\gamma, d, p)$.

**Proof.** For example, see [18] Theorem 13.8.1. \qed

**Remark A.3.** It is well-known that for $\gamma \in (0, \infty)$ and $u \in S$, we have

$$(1 - \Delta)^{-\gamma/2} u(x) = \int_{\mathbb{R}^d} R_\gamma(x - y) u(y) dy,$$

where

$$
|R_\gamma(x)| \leq N(\gamma, d) \left( e^{-|x|/2} 1_{|x| \geq 2} + A_\gamma(x) 1_{|x| < 2} \right)
$$

and

$$
A_\gamma(x) := \begin{cases} 
|x|^{\gamma-d} + 1 + O(|x|^{\gamma-d+2}) & \text{if } 0 < \gamma < d, \\
\log(2/|x|) + 1 + O(|x|^2) & \text{if } \gamma = d, \\
1 + O(|x|^{\gamma-d}) & \text{if } \gamma > d.
\end{cases}
$$

Also, we have

$$(R_\gamma \ast R_\gamma)(x) = R_{2\gamma}(x).$$

For more information, see [10] Proposition 1.2.5.]

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions and $\mathbb{P}$ is the predictable $\sigma$-field related to $\mathcal{F}_t$. Next we introduce definitions and properties of stochastic Banach spaces.
Definition A.4 (Stochastic Banach spaces). For a bounded stopping time \( \tau \leq T \), let us denote \( (0, \tau] := \{(\omega, t) : 0 < t \leq \tau(\omega)\} \).

(i) For \( \tau \leq T \),

\[
H^\gamma_p(\tau) := L_p((0, \tau], \mathcal{P}, d\mathbb{P} \times dt; H^\gamma_p),
\]

\[
H^\gamma_p(\tau, \ell_2) := L_p((0, \tau], \mathcal{P}, d\mathbb{P} \times dt; H^\gamma_p(\ell_2)),
\]

\[
U^\gamma_p := L_p(\Omega, \mathcal{F}_0, d\mathbb{P}; H^{\gamma - 2/p}_p).
\]

For convenience, we write \( L_p(\tau) := H^0_p(\tau) \) and \( L_p(\tau, \ell_2) := H^0_p(\tau, \ell_2) \).

(ii) The norm of each space is defined in the natural way. For example,

\[
\|u\|_{H^\gamma_p(\tau)} := \mathbb{E} \left[ \int_0^T \|u(t)\|_{H^\gamma_p}^2 dt \right].
\] (A.2)

Definition A.5. Let \( \tau \leq T \) be a bounded stopping time and \( u \in H^\gamma_p(\tau) \).

(i) We write \( u \in H^\gamma_p(\tau) \) if \( u_0 \in U^\gamma_p \) and there exists \( (f, g) \in H^{-2}_p(\tau) \times H^{-1}_p(\tau, \ell_2) \) such that

\[
du = f dt + \sum_{k=1}^{\infty} g^k dw_k^k, \quad t \in (0, \tau]; \quad u(0, \cdot) = u_0
\]

in the sense of distributions, i.e., for any \( \phi \in C_c^\infty \), the equality

\[
(u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s, \cdot), \phi) dw_k^k
\] (A.3)

holds for all \( t \in [0, \tau] \) almost surely. In this case, we write

\[
Du := f, \quad Su := g.
\]

(ii) The norm of \( H^\gamma_p(\tau) \) is defined by

\[
\|u\|_{H^\gamma_p(\tau)} := \|u\|_{H^\gamma_p(\tau)} + \|Du\|_{H^{-2}_p(\tau)} + \|Su\|_{H^{-1}_p(\tau,\ell_2)} + \|u(0, \cdot)\|_{U^\gamma_p}.
\]

Next, we introduce the H"older embedding theorem for \( H^\gamma_p(\tau) \).

Theorem A.6. (i) If \( \gamma \in \mathbb{R}, p > 2, 1/2 > \beta > \alpha > 1/p \), then for any \( u \in H^\gamma_p(\tau) \), we have \( u \in C^{\alpha - 1/p}([0, \tau]; H^{\gamma - 2\beta}_p) \) almost surely and

\[
\mathbb{E} \|u\|_{C^{\alpha - 1/p}([0, \tau]; H^{\gamma - 2\beta}_p)}^p \leq N(\alpha, \beta, d, p, T) \|u\|_{H^\gamma_p(\tau)}^p.
\]

(ii) If \( \gamma \in (0, 1), p > 2, \alpha, \beta \in (0, \infty) \) satisfy

\[
\frac{1}{p} < \alpha < \beta < \frac{1}{2} \left( \gamma - \frac{d}{p} \right),
\]

then for any \( u \in H^\gamma_p(\tau) \), we have \( u \in C^{\alpha - 1/p}([0, \tau]; C^{\gamma - 2\beta - d/p}) \) almost surely and

\[
\mathbb{E} \|u\|_{C^{\alpha - 1/p}([0, \tau]; C^{\gamma - 2\beta - d/p})}^p \leq N(\alpha, \beta, d, p, T) \|u\|_{H^\gamma_p(\tau)}^p.
\]
Proof. For (i), see Theorem 7.2 of [16]. For (ii), combine (A.1) and (i). In other words,
\[ E\|u\|_{C^{a-1/p}([0,\tau];C^{n-2\beta-d/p}(\mathbb{R}^d))}^p \leq N E\|u\|_{C^{a-1/p}([0,\tau];H^a_{\gamma})}^p \leq N\|u\|_{\mathcal{H}_{\gamma}(\tau)}. \]

Note that if \( h(t, x, u) \) is globally Lipschitz in \( u \), the \( L_p \)-theory says that there is a unique solution \( u \in \mathcal{H}_{\gamma}(\tau) \) for each \( \gamma > 0 \) to (1.1) (see [8] Theorem 6 or [9] Theorem 3.5). However, since \( h(t, x, u) \) is not globally Lipschitz in \( u \) (see Assumption 2.4), we approximate \( h \) by a sequence of Lipschitz functions, and then show that the sequence is tight and the limit would be a solution to (1.1).

Let us take a symmetric function \( \psi \in C_c^\infty(\mathbb{R}) \) such that \( 0 \leq \psi \leq 1 \), \( \psi(z) = 1 \) on \( |z| \leq 1 \), \( \psi(z) = 0 \) on \( |z| \geq 2 \), and \( \sup_{z \in \mathbb{R}} |\psi'(z)| \leq 1 \). Let \( n \geq 1 \) and set \( \psi_n(z) := \psi(z/n) \). Choose \( \zeta \in C_c^\infty(\mathbb{R}) \) such that \( \int_\mathbb{R} \zeta dz = 1 \) and \( \zeta(z) = 0 \) if \( z < 0 \) or \( z > 1 \). Define, for all \( n \geq 1 \),
\[ h_n(t, x, u) := n \int_\mathbb{R} h(t, x, z) \zeta(n(u - z)) \psi_n(u) = \int_0^1 h(t, x, u - z/n) \zeta(z) \psi_n(u). \] (A.4)

By Assumption 2.4 it is easy to see that
\[ h_n(t, x, 0) = 0, \] (A.5)
\[ |h_n(t, x, u)| \leq N(1 + u), \]
\[ |h_n(t, x, u) - h_n(t, x, v)| \leq N_n|u - v|, \] (A.6)
and
\[ |h_n(t, x, u) - h(t, x, u)| \to 0 \]
uniformly on compacts as \( n \to \infty \). The constant \( N_n \) introduced in (A.6) is a positive constant only depending on \( n \), and the other constant \( N \) does not depend on \( n \).

The following lemma shows the existence of a nonnegative solution to (1.1) with \( h_n \) instead of \( h \) and provides a uniform estimate of the moments of \( u_n \) in some sense (see (A.8) and (A.9)).

Lemma A.7. Let \( \tau \leq T \) be a bounded stopping time. Suppose that for each \( n \in \mathbb{N} \), \( h_n \) is the function defined in (A.4) and \( \eta \in (0, 1] \) is the constant introduced in Assumption 2.4. Then, for any \( p \geq 2 \), there exists \( u_n \in \mathcal{H}_{\gamma}^p(\tau) \) such that \( u_n \geq 0 \), and for any \( \phi \in \mathcal{S} \), the equality
\[ (u_n(t, \cdot), \phi) = (u_0, \phi) + \int_0^t (u_n(s, \cdot), (\mathcal{L}^\ast \phi)(s, \cdot)) ds + \int_0^t \int_{\mathbb{R}^d} h_n(s, x, u_n(s, x)) \phi(x) F(ds, dx) \] (A.7)
holds for all \( t \leq \tau \) almost surely, where \( \mathcal{L}^\ast \) is given in (2.8). Furthermore, if \( \alpha, \beta > 0 \) and \( p > 2 \) satisfy
\[ \frac{1}{p} < \alpha < \beta < \frac{\eta}{2} - \frac{d}{2p}, \]
then \( u_n \in C^{a-1/p}([0, \tau]; C^{n-2\beta-d/p}(\mathbb{R}^d)) \) almost surely. In addition, for \( a > 0 \), we have
\[ \sup_{n \geq 1} E\|u_n \Psi_{a/p}\|_{C^{a-1/p}([0, \tau]; C^{n-2\beta-d/p}(\mathbb{R}^d))}^p \leq N + N\|u_0 \Psi_{a/p}\|_{L_p}^p, \] (A.8)
where \( \Psi_{a/p}(x) := \frac{1}{\cosh(a|x|/p)} \) and \( N = N(\alpha, \beta, \eta, \lambda, a, d, p, K, T) \). In particular,
\[ \sup_{n \geq 1} \left[ \sup_{t \leq \tau, x \in \mathbb{R}^d} (u_n(t, x))^{p} e^{-a|x|} \right] \leq N, \] (A.9)
where \( N = N(\eta, \lambda, a, d, p, K, T, u_0) \).
Proof. We first note that the stochastic integral part in (A.7) can be written as

\[ \int_0^t \int_{\mathbb{R}^d} h_n(s, x, u_n(s, x)) \phi(x) F(ds, dx) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} h_n(s, x, u_n(s, x)) \phi(x)(f * e_k) dx dw_k^k, \]

where \( e_k \) and \( w_k^k \) are introduced in Remark 2.8. This allows us to use the \( L^p \)-theory of SPDEs (14)(16). Indeed, since \( h_n(s, x, u) \) is Lipschitz in \( u \), [8, Theorem 6] (or [5, Theorem 3.5]) yields that there exists a unique solution \( u_n \in H^p_0(\tau) \) satisfying (A.7). Furthermore, by Theorem [A.6] [17], we have \( u_n \in C^{\alpha-1/p}([0, \tau]; C^{\beta-d/p}(\mathbb{R}^d)) \) almost surely.

We now show non-negativity of \( u_n \). Define \( \bar{h}_n(t, x, u) := h_n(t, x, u) \mathbb{1}_{u \geq 0} = h_n(t, x, u_+) \) and consider the following SPDE

\[ \partial_t \bar{u}_n(\omega, t, x) = \mathcal{L}u_n(\omega, t, x) + \sum_{k=1}^{\infty} \bar{h}_n(\omega, t, x, \bar{u}_n(t, x))(f * e_k)(t, x) dx dw_k^k \quad (t, x) \in (0, \infty) \times \mathbb{R}^d \tag{A.10} \]

with initial data \( \bar{u}_n(0, \cdot) = u_0(\cdot) \geq 0. \) Since \( \bar{h}_n \) is also Lipschitz in \( u \), there exists a unique solution \( \bar{u}_n \in H^p_0(\tau) \) to (A.10). Here, if we show \( \bar{u}_n \geq 0 \), then \( \bar{h}_n(t, x, \bar{u}_n) = h_n(t, x, \bar{u}_n) = h_n(t, x, \bar{u}_n) \). Thus, by the uniqueness, \( \bar{u}_n = u_n \), which results in the non-negativity of \( u_n \). Now, for the non-negativity of \( \bar{u}_n \), it just follows from the proof of [5, Appendix A] (see also [17, Theorem 1.1]); the main idea is to consider the finite sum \( \sum_{m=1}^{\infty} \) in the noise part in (A.10) whose solutions (say \( u_{n,m} \)) are nonnegative, thanks to [17, Theorem 1.1], and then show the convergence \( u_{n,m} \rightarrow u_n \) as \( m \rightarrow \infty \).

To obtain (A.8) and (A.9), for \( a, p > 0 \), define \( \Psi(x) = \Psi_{a/p}(x) = \frac{1}{\cosh(a|x|/p)}. \) As in [5, Lemma 5.5], we have

\[ \Psi_{a/p}(x) \leq \frac{a}{p} \Psi(x) \quad \text{and} \quad \Psi_{a/p}(x) \leq \frac{a^2}{p^2} \Psi(x). \tag{A.11} \]

Set \( v_n(t, x) := u_n(t, x) \Psi(x). \) Then, for any \( \phi \in \mathcal{S}, v_n(t, x) \) satisfies

\[ (v_n(t, \cdot), \phi) = (\Psi u_0, \phi) + \int_0^t (v_n(s, \cdot), (\mathcal{L}^* \phi)(s, \cdot)) + (g(s, \cdot), \phi) ds \]

\[ + \int_0^t \int_{\mathbb{R}^d} h_n(s, x, u_n(s, x)) \phi(x)(f * e_k) dx dw_k^k, \tag{A.12} \]

where \( \mathcal{L}^* \) is in (2.3), and

\[ g := u_n(a^{ij} \Psi_{x^i} + 2a^{ij} \Psi_{x^i x^j} - b^i \Psi_{x^i}) - (2a_n a^{ij} \Psi_{x^i x^j})_{x^j}. \tag{A.13} \]

Since \( u_n \in H^p_0(\tau) \subset H^p(\tau), \) we have

\[ a^{ij} v_{nx^i x^j} + b^i v_{nx^i} + c v_n + g \in H^{p-2}(\tau) \quad \text{and} \quad \Psi u_0 \in L^p(\Omega, \mathcal{F}_0, H^{p-2/p}_p). \tag{A.14} \]

In addition, if we set \( \nu \equiv (f * e_1, f * e_2, \ldots), \) then we have

\[ h_n(u_n) \Psi \nu \in H^{p-1}(\tau, \ell). \tag{A.15} \]
Indeed, to see (A.15), observe that

\[
\left|(1 - \Delta)^{-\frac{1}{2}} h_n(u_n) \Psi \mu \right|^2_{L^2} \\
= \sum_{k=1}^\infty \left|(1 - \Delta)^{-\frac{1}{2}} h_n(u_n) \Psi (f * e_k) \right|^2 \\
= \sum_{k=1}^\infty |R_{1-n} * (h_n(u_n) \Psi (f * e_k))|^2 \\
= \sum_{k=1}^\infty |(R_{1-n}(x - \cdot) h_n(u_n) \Psi, f * e_k)|^2 \\
= \int_{\mathbb{R}^d \times \mathbb{R}^d} R_{1-n}(x - (y - z)) R_{1-n}(x + z) h_n(u_n) \Psi(y - z) h_n(u_n) \Psi(-z) dz f(dy).
\]

Thus, by Minkowski’s inequality and Hölder’s inequality,

\[
\|h_n(u_n) \Psi \mu\|_{H^p_{\ell^2}(\ell_2)}^p \\
= \int_{\mathbb{R}^d} \left|(1 - \Delta)^{-\frac{1}{2}} h_n(u_n) \Psi \mu \right|^2_{L^2} dx \\
= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} R_{1-n}(x - (y - z)) R_{1-n}(x + z) h_n(u_n) \Psi(y - z) h_n(u_n) \Psi(-z) dz f(dy) \right|^{p/2} dx \\
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |R_{1-n}(y - z) R_{1-n}(z)(h_n(u_n) \Psi(x + y - z) h_n(u_n) \Psi(x - z) dz f(dy)|^{p/2} dx \right)^{2/p} dz f(dy) \\
\leq \left( \int_{\mathbb{R}^d} R_{1-n}(y - z) R_{1-n}(z) \left( \int_{\mathbb{R}^d} (h_n(u_n) \Psi(x + y - z) h_n(u_n) \Psi(x - z))^{p/2} dx \right)^{2/p} dz f(dy) \right)^{p/2} \\
\leq \|h_n(u_n) \Psi\|_{L^p}^p \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} R_{1-n}(y - z) R_{1-n}(z) dz f(dy) \right)^{p/2} \\
= \|h_n(u_n) \Psi\|_{L^p}^p \left( \int_{\mathbb{R}^d} R_{2-n}(y) f(dy) \right)^{p/2}.
\]

Note that

\[
\int_{\mathbb{R}^d} R_{2-2n}(y) f(dy) < \infty
\]

by (2.2) and Remark A.3. Therefore,

\[
\|h_n(u_n) \Psi \mu\|_{H^p_{\ell^2}(\ell_2, \tau)}^p \leq NE \int_0^\tau \int_{\mathbb{R}^d} |h_n(u_n) \Psi|^p dx dt \leq NE \int_0^\tau \int_{\mathbb{R}^d} |\Psi|^p + |v_n|^p dx dt < \infty,
\]

which implies (A.15). Combining this with (A.14), we have \(v_n \in H^p_{\ell^2}(\tau)\). Besides, by applying [10] Theorem 5.1, for any \(t > 0\), we have

\[
\|v_n\|_{H^p_{\ell^2}(\tau, \ell_2)}^p \leq N \left( \|\Psi u_0\|_{L^p}^p + \|g\|_{H^p_{\ell^2}(\tau, \ell_1)}^p ds + \|h_n(u_n) \Psi \mu\|_{H^p_{\ell^2}(\tau, \ell_2)}^p \right),
\]

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where \( N = N(\eta, d, p, K, T) \) and \( g \) is given in (A.13). Due to (A.11), for any \( s \leq \tau \wedge t \), we have
\[
\|g(s, \cdot)\|_{H^p_{p-2}}^p \leq \|u_n(s, \cdot)(a^{ij}\Psi_x + 2a^{ij}\Psi_x \cdot b^i\Psi_x)\|_{H^p_{p-2}}^p + \|(2u_n(s, \cdot)a^{ij}\Psi_x)\|_{H^p_{p-2}}^p \leq N\|v_n(s, \cdot)\|_{L^p},
\]
where \( N = N(\eta, a, d, p, K) \). In addition, by (A.18),
\[
\|h_n(s, \cdot, u_n(s, \cdot))\|_{H^p_{p-1}(\ell^2)}^p \leq N\|h_n(s, \cdot, u_n(s, \cdot))\|_{L^p}^p \leq N + N\|v_n(s, \cdot)\|_{L^p},
\]
where \( N = N(\eta, a, d, p, K) \) is independent of \( n \). We now apply (A.20), (A.21) to (A.19), and use Theorem A.6 [4], we have
\[
\|v_n\|_{H^p_{p}(\tau \wedge t)}^p \leq N + N\|\Psi u_0\|_{L^p}^p + N\|v_n\|_{L^p(\tau \wedge t)}^p \leq N + N\|\Psi u_0\|_{L^p}^p + N\int_0^t E\|v_n\|_{C([0,\tau \wedge s])}^p ds \leq N + N\|\Psi u_0\|_{L^p}^p + N\int_0^t \|v_n\|_{H^p_{p}(\tau \wedge s)}^p ds,
\]
where \( N = N(\eta, \lambda, a, d, p, K, T) \). Thus, by the Gronwall’s inequality, we have
\[
\|v_n\|_{H^p_{p}(\tau \wedge T)}^p \leq N + N\|\Psi u_0\|_{L^p}^p,
\]
where \( N = N(\eta, \lambda, a, d, p, K, T) \). Therefore, by Theorem A.6 [4], we have (A.8). In addition, (A.9) follows from
\[
E \left[ \sup_{t \leq \tau, x \in \mathbb{R}^d} (u_n(t, x))^p e^{-a|x|^2} \right] = E \left[ \sup_{t \leq \tau, x \in \mathbb{R}^d} (u_n(t, x))^p \frac{1}{\cosh(a|x|)} \right] \leq E \left[ \sup_{t \leq \tau, x \in \mathbb{R}^d} |u_n(t, x)|^p \right].
\]
The lemma is proved. \( \square \)

We now provide the proof of Theorem 2.9.

**Proof of Theorem 2.9** We first show that there is a nonnegative solution \( u \) to (1.1) and then show that any solution to (1.1) is Hölder continuous almost surely.

**Steps**

**Step 1** (Existence) Since we use the standard compactness argument for the existence of a stochastically weak solution (see e.g. [22, Theorem 1.2] and [26, Theorem 2.6]), we only outline the proof.

From the bounds (A.8) and (A.9), the Kolmogorov type tightness criterion (see [26, Lemma 6.3]) implies that \( \{u_n\}_{n \in \mathbb{N}} \) is tight in \( C(\mathbb{R}^+, C_{tem}) \). Then, by Prokhorov’s theorem combined with Skorokhod’s representation theorem, we have an appropriate probability space and the sequence of solutions \( \tilde{u}_n \) on it, which is identical in law to \( \{u_n\} \) and converges to some nonnegative \( u \) almost surely in \( C(\mathbb{R}^+, C_{tem}) \).

**Step 2** (Hölder regularity) We assume \( u \) is a solution to (1.1) in the sense of Definition 2.1. Let \( p > \frac{d+2}{\eta} \). For \( n \in \mathbb{N} \) and \( a > 0 \), set
\[
\tau_n := \tau \wedge n \wedge \inf \left\{ t \geq 0 : \sup_{x \in \mathbb{R}^d} u(t, x)e^{-a|x|} \geq n \right\}.
\]
Fix \( t \leq \tau_n \) and take \( \Psi(x) = \Psi_{2a}(x) = \frac{1}{\cosh(2a|x|)} \). Then, since we assume \( 2.5 \), as in \([A.21]\), we have
\[
\| h(u)\Psi u \|_{H^p_{\tau_n}((t_2, \tau_n))} \leq N \| h(u)\Psi \|_{L^p(\tau_n)} \leq N \int_{\mathbb{R}^d} \left( 1 + e^{P_0|x|} \right) |\Psi(x)|^p dx < \infty,
\]
In addition, if we define
\[
g := u \left( a^{ij} \Psi_{x^{i}x^{j}} + 2a^i_{,x^i} \Psi_{x^i} - b^i \Psi_{x^i} \right) - (2a^{ij} \Psi_{x^i}(x)u)_{,x^j},
\]
then, similar to \([A.20]\), we have
\[
\| g \|_{H^p_{\tau_n}((t_2, \tau_n))} < \infty. \quad \text{Since } \Psi u_0 \in L^p(\Omega, F_0, H^{\eta-2}/p), \quad \text{[16] Theorem 5.1} \text{ yields that there exists a unique } v \in H^\eta_{\tau_n} \text{ such that}
\]
\[
\partial_t v = a^{ij} v_{x^{i}x^{j}} + b^i v_{x^i} + cv + g + h(u)\Psi \hat{F}, \quad t \leq \tau; \quad v(0, \cdot) = \Psi u_0,
\]
in the sense of Definition \([2.7]\) (see also \([16] \text{ Remark 5.3}\)). Note that \( h(u)\Psi \) is used in the stochastic integral part, not \( h(v)\Psi \). Observe that for fixed \( \omega \in \Omega \), \( \tilde{u} := v - \Psi u \) satisfies
\[
\partial_t \tilde{u} = a^{ij} \tilde{u}_{x^{i}x^{j}} + b^i \tilde{u}_{x^i} + cv, \quad 0 < t < \tau_n; \quad \tilde{u}(0, \cdot) = 0.
\]
Since \( \Psi u, v \in L^p((0, \tau_n) \times \mathbb{R}^d) \), we have \( \tilde{u} \in L^p((0, \tau_n) \times \mathbb{R}^d) \). Thus, by the deterministic version of \( L^p \) theory (e.g. \([19]\)), we have \( \tilde{u}(t, \cdot) = 0 \) in \( L^p(\mathbb{R}^d) \) for all \( t \leq \tau_n \) almost every \( \omega \). Thus, \( \Psi u = v \in H^\eta_{\tau_n} \). Since \( p > \frac{d+2}{\eta} \), by Theorem \([A.6] (i) \), for any \( \gamma < \eta/2 \), we have
\[
\Psi u \in C^{\gamma,\gamma}_{t,x}([0, \tau_n] \times \mathbb{R}^d),
\]
almost surely. The theorem is proved. \( \square \)

References

[1] Krzysztof Burdzy, Carl Mueller, and Edwin Perkins. Nonuniqueness for nonnegative solutions of parabolic stochastic partial differential equations. Illinois Journal of Mathematics, 54(4):1481–1507, 2010.

[2] Le Chen and Jingyu Huang. Comparison principle for stochastic heat equation on \( \mathbb{R}^d \). Annals of Probability, 47(2):989–1035, 2019.

[3] Le Chen, Jingyu Huang, Davar Khoshnevisan, and Kunwoo Kim. Dense blowup for parabolic spdes. Electronic Journal of Probability, 24:1–33, 2019.

[4] Le Chen and Kunwoo Kim. Stochastic comparisons for stochastic heat equation. Electronic Journal of Probability, 25:1–38, 2020.

[5] Jae-Hwan Choi and Beom-Seok Han. A regularity theory for stochastic partial differential equations with a super-linear diffusion coefficient and a spatially homogeneous colored noise. Stochastic Processes and their Applications, 135:1–30, 2021.

[6] Robert Dalang et al. Extending the martingale measure stochastic integral with applications to spatially homogeneous SPDE’s. Electronic Journal of Probability, 4, 1999.
[7] Robert C Dalang and Nicholas E Frangos. The stochastic wave equation in two spatial dimensions. *Annals of Probability*, pages 187–212, 1998.

[8] Marco Ferrante and Marta Sanz-Solé. Spdes with coloured noise: analytic and stochastic approaches. *ESAIM: Probability and Statistics*, 10:380–405, 2006.

[9] Mohammud Foondun and Davar Khoshnevisan. On the stochastic heat equation with spatially-colored random forcing. *Transactions of the American Mathematical Society*, 365(1):409–458, 2013.

[10] Loukas Grafakos. Modern Fourier analysis. 250, Springer, 2009.

[11] Ian Iscoe. On the supports of measure-valued critical branching Brownian motion. *Annals of Probability*, pages 200–221, 1988.

[12] Davar Khoshnevisan. Analysis of stochastic partial differential equations. American Mathematical Society, Vol.119, 2014.

[13] Nicolai Krylov. Introduction to the theory of diffusion processes. Providence, 1995.

[14] Nicolai Krylov. On $L_p$-theory of stochastic partial differential equations in the whole space. *SIAM Journal on Mathematical Analysis*, 27(2):313–340, 1996.

[15] Nicolai Krylov. On a result of C. Mueller and E. Perkins. *Probability Theory and Related Fields*, 108(4):543–557, 1997.

[16] Nicolai Krylov. An analytic approach to SPDEs. *Stochastic partial differential equations: six perspectives, in Mathematical Surveys and Monographs*, 64:185–242, 1999.

[17] Nicolai Krylov. Maximum principle for SPDEs and its applications. *Stochastic Differential Equations: Theory And Applications: A Volume in Honor of Professor Boris L Rozovskii*, pages 311–338. World Scientific, 2007.

[18] Nicolai Krylov. Lectures on elliptic and parabolic equations in Sobolev spaces. American Mathematical Society, 2008.

[19] Olga Ladyženskaja, Vsevolod Solonnikov, and Nina Ural’ceva. Linear and quasi-linear equations of parabolic type, volume 23. American Mathematical Soc., 1988.

[20] Carl Mueller. On the support of solutions to the heat equation with noise. *Stochastics*, 37(4):225–245, 1991.

[21] Carl Mueller and Edwin Perkins. The compact support property for solutions to the heat equation with noise. *Probability Theory and Related Fields*, 93(3):325–358, 1992.

[22] Leonid Mytnik, Edwin Perkins, and Anja Sturm. On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients. *Annals of Probability*, 34(5):1910–1959, 2006.

[23] Thomas Rippl. Pathwise uniqueness of the stochastic heat equation with Hölder continuous diffusion coefficient and colored noise. Doctoral dissertation, Georg-August-Universität Göttingen 2013.

[24] Marta Sanz-Solé and Monica Sarrà. Path properties of a class of gaussian processes with applications to SPDE’s. In *Stochastic processes, physics and geometry: new interplays, I*, pages 303–316. Amer. Math. Soc., 2000.
[25] Marta Sanz-Solé and Monica Sarrà. Hölder continuity for the stochastic heat equation with spatially correlated noise. In Seminar on Stochastic Analysis, Random Fields and Applications III, pages 259–268. Springer, 2002.

[26] Tokuzo Shiga. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. Canadian Journal of Mathematics, 46(2):415–437, 1994.

[27] John B Walsh. An introduction to stochastic partial differential equations. In École d’Été de Probabilités de Saint Flour XIV-1984, pages 265–439. Springer, 1986.

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