Power-law Distributions in the Kauffman Net

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Abstract

Kauffman net is a dynamical system of logical variables receiving two random inputs and each randomly assigned a boolean function. We show that the attractor and transient lengths exhibit scaleless behavior with power-law distributions over up to ten orders of magnitude. Our results provide evidence for the existence of the "edge of chaos" as a distinct phase between the ordered and chaotic regimes analogous to a critical point in statistical mechanics. The power-law distributions are robust to the changes in the composition of the transition rules and network dynamics.

87.10.+e, 05.50+9, 05.40j
Kauffman argues that in addition to Darwin’s natural selection and random mutation, self-organization in random complex systems is also responsible for the observed complexity in the biological world. A crucial question then is: what are the characteristics of complex dynamical systems that are best able to adapt and most suitable for evolving? It has been speculated that such optimal systems are poised on the boundary between the ordered and chaotic regimes [1–3].

Random Boolean network is a dynamical system of logical variables. Its dynamics, characterized by the period of its attractors, tends to be chaotic at large network connectivity $k$ and ordered at small $k$. The Kauffman net, with $k = 2$, is believed to be at the edge of chaos. In this Letter, we study the probability distributions of both the attractor and transient lengths of the Kauffman net. We find that the distributions obey power-laws over up to ten orders of magnitude. The exponent of the distribution depends on the system size, $N$, and approaches one at large $N$. This indicates that there is no typical attractor or transient length. The Kauffman net therefore possesses a wide range of behaviors, which supports the claim that it is most able to adapt [3,4].

Our finding also indicates the existence of a third phase between order and chaos. The transition from the ordered to the chaotic phase need not necessarily be continuous. In fact, some systems in statistical mechanics are known to have first order transitions from an ordered to a disordered phase. Our results provide concrete evidence for the existence of an intermediate third regime (popularly known as the ”edge of chaos”) as a distinct phase analogous to the critical point in the Ising model. Unlike in the Ising model, the power-law behavior in the Kauffman net does not need much special tuning. We have studied systems where the dynamics is made more chaotic or orderly by changing the composition of the rules
which determine its dynamics. The power-law distributions remain although the exponents are different.

The ability to study large networks of up to $N = 10^5$, which is much beyond previous works, is made possible by several algorithm innovations. We explored a special feature of the Kauffman net in which all the Boolean functions can be expressed in terms of logical functions that have hardware implementation in the modern computer. We used a method similar to “multiple spin coding” that packs bit variables into a machine word. Hardware implementation of these functions enabled us to update in parallel. At large system sizes, we used a new method to estimate the exponents of the power-law distribution from its median value.

Over the last twenty years, random Boolean networks have been related to the statistical properties of ontogeny and genetic regulatory systems [3,5]. A random Boolean network consists of a set of binary gates $S_i = \{0, 1\}$ (called spins) interacting with each other via certain logical rules and evolving discretely in time. The values of the spins represent the presence and absence of certain chemical species in a cell. They change over time through a web of complicated chemical reactions. The dynamics of this system of spins is as follows: the value of spin, $S_i(t + 1)$, at the next instant of time $t + 1$ is determined by the $k$ input spins, $S_{j^m}(t)(m = 1, 2, ..k)$, at time $t$ and the transition function $f_i$ associated with the site $i$:

$$S_i(t + 1) = f_i(S_{j_1^t}(t), S_{j_2^t}(t), ..., S_{j_k^t}(t))$$  \(1\)

Since the chemical reactions are highly complex, as a first approximation, the logical rules $f_i$ are chosen at random from amongst all the $2^{2^k}$ possible Boolean rules of $k$ inputs
Each of the $j_i^m$ inputs is also chosen at random from amongst the $N$ sites in the network. This input-output connection structure and assignment of transition rules to each of the gates will be referred to as a realization and a specific combination of states of the spin as a configuration of the network. The realization is fixed in time. Since the state space is finite and the evolution is deterministic, any initial configuration of gates must after a finite number of time steps re-enter a configuration it had previously encountered. Thus, the maximum number of such time steps is $2^N$ and the minimum is one. Once it revisits this configuration, the network follows the same set of states again and does so forever. This repetitive set of states is called an attractor or cycle of the network and its size (called the cycle length or period) is the number of states comprising it. The number of states from the initial configuration required to enter an attractor is a measure of the transient time of that configuration. Since a network realization can have many attractors of different sizes, the state space is divided into these attractors by their basins of attraction which are the set of states flowing into the attractor. Hence the attractor is an interesting quantity to determine for the net.

In contrast to Hamiltonian systems like spin models, where the period is at most $2^N$, random boolean nets can have any period. Various levels of connectivity $k$ have been analyzed. For large $k$, since the gates are randomly connected, the system jumps randomly from one point to another among $2^N$ points in the state space. The period is found to scale approximately as $\sqrt{2^N}$. This is the chaotic regime with properties which are biologically impossible. When $k$ is reduced, the network becomes less chaotic making a transition to orderly behavior at some small $k$. In the annealed approximation, the transition from the chaotic to ordered phase is found to be at $k = 2$ for random Boolean function selection.
For \( k = 2 \), known as Kauffman net, the network flows into attractors of period much smaller than the volume of the state space \( (2^N) \). Kauffman found the median cycle length and number of cycles to scale approximately as \( \sqrt{N} \). He related the period to the cell replication time and the number of distinguishable attractors to the number of different cell types in an organism with an equivalent number of genes [5].

In order to calculate the attractor and transient lengths, we use Knuth’s algorithm [10] with our multispin coding implementation as described at the end of the Letter. A typical result for the period distribution is shown in Fig.1. Although the size of the network is only 64, periods are found over a wide range. The distribution is obtained by sampling over \( 5 \times 10^4 \) realizations. For each realization we sample 2000 initial configurations which are randomly chosen amongst the \( 2^{64} \) possible states. (We follow all initial configurations to its attractor and determine the periods). The frequency of each period is shown in Fig.1. Notice the distinct preponderance of the even over the odd periods. The even-odd oscillation is much bigger than the statistical noise. We believe the even-odd effect is an evidence for the existence of independent subnets. A subnet may have several periods. Since the period of a network composed of independent subnets is the least common multiple of the periods of the subnets, if the period of one of the subnets is even, then period of the entire network will be even. Therefore we are more likely to find even periods.

The second feature of the distribution is the power-law extending over ten orders of magnitude (Fig.2). The data in Fig.2 has been smoothened by averaging over even cycles in bins of logarithmically increasing sizes. The probability in a bin is obtained by summing over the probabilities of all the periods in the bin including those not found in the simulation. The straight lines in Fig.2 clearly show the power-law behavior with no sign of finite size
effect at large periods. The finite size deviation from the power-law presumably occurs at very large periods of order $\sqrt{2^N}$. Averaged distributions are shown for several network sizes. The exponents of these power-laws clearly depend on the network size.

The power-law comes mostly from averaging over realizations. Within each network realization, the distribution of attractors tend to be clustered, i.e. the periods tends to be either all large or all small.

The exponent $\alpha$ in the distribution $f(P) = AP^{-\alpha}$ of period $P$ is determined over the linear portion of the averaged distribution of the even periods in Fig. 2. The even periods are used because they give better statistics. The exponents measured with odd cycle lengths are close to the ones from the even cycles. Another method to determine the exponent is by computing the cumulative sum $F(Q) = \sum_{P=1}^{Q} f(P)$ which smooths out the noise. The power-law distribution for $f(P)$ implies that $(1 - F(Q)) \propto Q^{1-\alpha}$ when $\alpha > 1$. The error on the exponents is about 3%. The exponent for different system sizes are listed in Table 1.

Like the attractors, we also determined the transient distributions. The distribution has a tail of rare long transients (Fig. 3). The exponent of the power-law distribution, $\beta$, is larger than in the case of attractors for the same $N$ and depends on $N$ significantly as shown in Table 1.

One interesting consequence of a power-law distribution is that the mean of the distribution diverges for $\alpha < 2$. We see from Fig. 4 that the exponent $\alpha$ decreases monotonically with increasing size $N$. The exponent $\alpha$ becomes smaller than 2 for $N$ larger than about 120. The average of the attractor and transient lengths determines the simulation time and thus it becomes increasingly difficult to simulate for large $N$. The amount of computing time is dominated by very long cycles for large $N$ and is of the order $(2^{N/2})^{2-\alpha}$ for $\alpha < 2$. 
This exponential growth with $N$ limits analysis to small $N$.

In order to study the power-law at large system size, we developed a method that estimates the exponent of the power-law (shown in Fig.4 for large $N$) from an approximate calculation of median cycle length $P_m$. The median cycle length is derived from the data of small periods up to $P_m$. This is very efficient since for large $N$, $\alpha < 2$ and most of the computing time is spent on tracking large attractors. A configuration is evolved up to a cutoff time $M_c$. The algorithm guarantees to find the transient $T$ and period $P$ in at most $2(T + P)$ steps and we find the period of all the configurations whose $(T + P)$ is smaller than $M_c/2$. With a cutoff, the period and transient can easily be found for a large percentage(%) of initial configurations (more than 50%). We calculate an estimated median $\bar{P}$ from the configurations whose periods have been determined by assuming that the rest of the configurations have periods larger than $\bar{P}$. The medians thus calculated are upper bounds to the true median because an undetermined period may be smaller than $\bar{P}$. When the cutoff $M_c$ is increased, $P$ and $T$ for a larger percentage of nets are found, thus providing a tighter upper bound. With higher cutoffs, we find that $\bar{P}$ versus %s tends to saturate at $(60 - 65)$% for most nets. Also, the median period is found to scale approximately as $\sqrt{N}$ up to about 20000 after which it grows faster. More details will be available from another publication [11]. We also obtain the probability $f(\bar{P})$ at the median. Assuming $\bar{P}$ and $f(\bar{P})$ are very close to the true values and also assuming that the power-law extends to the median, we can compute the exponent from the definition of the median: $\int_{P_m}^{\infty} f(P_m)(\frac{P}{P_m})^{-\alpha}dP = \frac{1}{2}$. We obtain $\alpha - 1 = 2P_m f(P_m)$. The exponent $\alpha$ determined by this method are shown as triangles in Fig.4. Although the data at large $N$ is noisy due to the difficulty associated with determining $f(P_m)$, the exponent agrees quite well with the direct measurement at
intermediate $N$. Our data suggest that $\alpha - 1 = CN^{-\gamma}$ where $C = 5.24$, $\gamma = 0.35 \pm 0.03$.

The power-law is robust to changes in the dynamics of the net. Updating sequentially produced similar power-laws with different exponents and fewer attractors of smaller periods. We also tried to use different combinations of transition rules. The rules by themselves produce different dynamics. Some of the rules are chaotic (XOR and EQUIVALENCE) in the sense that the dynamics exclusively due to them produce enormous periods while two others (CONTRADICTION and TAUTOLOGY) produce a constant output of either 1 or 0. We formed nets consisting of all except the two chaotic or the two constant-output rules. Deleting the two chaotic rules produces a more ordered net. Similarly, deleting the two constant rules produces large cycles. The distribution of cycle and transients lengths in both cases still obey power-law with different exponents. Our results indicate the important contribution of the constant output rules in reducing the lengths of the cycles as well as the number of cycles of a net. The constant rules produce islands of gates fixed in the evolution. Consequently, any cycle consisting of these islands of frozen gates would have fewer distinct states that can be produced by the remaining active gates, thereby reducing the size of the cycle. Therefore, deleting the constant rules should produce large cycle lengths.

To determine the attractors and transients, we use an algorithm that utilizes minimum storage space. Two identical nets ($S(0)$ and $S'(0)$) are evolved, one at twice the speed of the other till their Hamming distance becomes zero after say $T_0$ evolutions. Then the nets $S(T_0)$ and $S(0)$ are evolved and they become identical after the transient length $T$. Finally $S(T)$ is evolved and it revisits itself after $P$ steps equal to the period. We have been able to optimize our simulation using a technique similar to multi-spin coding in Monte Carlo simulations of Ising model. All the sixteen Boolean functions used was expressed
as combinations of logical functions OR, AND, EXCLUSIVE OR and NEGATIVE, all of
which have bitwise hardware implementation on a modern workstation. Since the hardware
performs logical calculation on all the bits of a word in a CPU cycle, we can sample $N_{\text{word}}$
random initial configurations in parallel for a realization, where $N_{\text{word}}$ is the size of the
machine word. For each configuration assigned to a bit, we have an individual time counter
for that bit which independently measures the time evolution along with the other bits.
Whenever the period and transient are found for a bit, a new random initial configuration
is assigned to that bit so that all the bits are always busy.

In summary, we find the probability distributions of the attractor and transient lengths
of the Kauffman net to be power-laws with the exponent strongly dependent on the net size.
The effects of different rule composition and dynamics of the net produce different statistics
but still display power-laws. Assuming that the Kauffman net describes adequately the
statistical properties of the genetic regulatory network [3], our result implies that the cell
replication time of organisms with the same gene size obeys a power-law. It has been
speculated that in the vicinity of phase transitions, physical systems can behave like a
computer [2]. Information processing arises spontaneously and is optimal in systems at the
critical state. Recently, self-organized criticality was proposed as an underlying mechanism
of certain systems to achieve a critical state by itself (without the necessity of tuning any
external parameter) where it possesses a wide range of behaviors described by power-law
distributions [15]. Therefore, the power-laws [16] are concrete evidence that the Kauffman
net is poised in a critical state where it is most versatile.

We would like to thank Elihu Abrahams, Wentian Li, and Leo Kadanoff for useful discus-
sions. We are particularly indebted to Jayanth Banavar for his constant encouragement and
contributions. The work was supported in part by the ONR Grant No. N00014-92-J-1340 and by a grant from the ARL at Penn State University.
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| Net Size N | $\alpha$ | $\beta$ | Configurations | Realizations |
|------------|----------|----------|----------------|--------------|
| 32         | 2.55     | 3.13     | 2000           | 50000        |
| 64         | 2.24     | 2.60     | 2000           | 50000        |
| 128        | 1.98     | 2.01     | 2000           | 50000        |
| 256        | 1.67     | 1.89     | 2000           | 3500         |
| 960        | 1.49     | 2.43     | 2000           | 1000         |

TABLE I. Attractor and transient length distribution exponents for net size $N$ and the number of realizations and initial configurations simulated. The error bar on $\alpha$ and $\beta$ is within 3%.
FIGURES

FIG. 1. Distribution of attractors (in Log-Log scale) for $N = 64$. The initial points are joined to clearly show the even-odd effect with a distinct preponderance of even over the odd cycles.

FIG. 2. Distribution of attractors (in Log-Log scale) for $N = 32, 128$ and $960$. The distribution is averaged over many random realizations and configurations as noted in Table 1. Averaging over all the periods in bins along the x axis gives the power-law in a Log-Log plot.

FIG. 3. Distribution of the transients (in Log-Log scale) for $N = 32, 64, 128$. No averaging has been performed on the data.

FIG. 4. $(\alpha - 1)$ versus $N$ for the attractor distribution, where $\alpha$ is the attractor length exponent. Exponents for large $N$ derived from medians are shown as triangles. A least square fit to the data points gives slope of $-0.35 \pm 0.03$. 