Algebraic Approach to Shape Invariance

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Abstract

The integrability condition called shape invariance is shown to have an underlying algebraic structure and the associated Lie algebras are identified. These shape-invariance algebras transform the parameters of the potentials such as strength and range. Shape-invariance algebras, in general, are shown to be infinite-dimensional. The conditions under which they become finite-dimensional are explored.

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I. INTRODUCTION

Supersymmetric quantum mechanics \[1\] and its connection to the factorization method \[2\] has been extensively investigated \[3\]. Since the ground state wavefunction, $\psi_0(x)$, for a bound system has no nodes it can be written as

$$\psi_0(x) = \exp\left(-\frac{\sqrt{2m}}{\hbar} \int W(x) dx\right).$$  \hspace{1cm} (1.1)

Introducing the operators

$$\hat{A} = W(x) + \frac{i}{\sqrt{2m}} \hat{p}$$

$$\hat{A}^\dagger = W(x) - \frac{i}{\sqrt{2m}} \hat{p},$$  \hspace{1cm} (1.2)

the Hamiltonian can be easily factorized

$$\hat{H} - E_0 = \hat{A}^\dagger \hat{A},$$  \hspace{1cm} (1.3)

where $E_0$ is the ground state energy. Since the ground state wavefunction satisfies the condition

$$\hat{A}|\psi_0\rangle = 0,$$  \hspace{1cm} (1.4)

the supersymmetric partner potentials

$$\hat{H}_1 = \hat{A}^\dagger \hat{A}$$

$$\hat{H}_2 = \hat{A} \hat{A}^\dagger$$  \hspace{1cm} (1.5)

have the same energy spectra except the ground state of $\hat{H}_1$ which has no corresponding state in the spectra of $\hat{H}_2$. The corresponding potentials are given by

$$V_1(x) = [W(x)]^2 - \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}$$

$$V_2(x) = [W(x)]^2 + \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}. $$  \hspace{1cm} (1.6)

It was shown that a subset of the potentials for which the Schrödinger equations are exactly solvable share an integrability condition called shape invariance \[4\]. The partner potentials of Eq. (1.6) are called shape invariant if they satisfy the condition

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1),$$  \hspace{1cm} (1.7)

where $a_{1,2}$ are a set of parameters that specify space-independent properties of the potentials (such as strength, range, diffuseness, etc.), $a_2$ is a function of $a_1$, and the remainder $R(a_1)$ is independent of $x$. One should emphasize that shape-invariance is not the most general integrability condition as not all exactly solvable potentials seem to be shape-invariant \[4\]. The purpose of this article is to show that shape invariance has an underlying algebraic structure and to identify the associated Lie algebras.
II. ALGEBRAIC PROPERTIES OF SHAPE INVARIANCE

The shape invariance condition of Eq. (1.7) can be rewritten in terms of the operators defined in Eq. (1.2)

\[ \hat{A}(a_1) \hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2) \hat{A}(a_2) + R(a_1), \]  \hspace{1cm} (2.1)

where \(a_2\) is a function of \(a_1\). We assume that replacing \(a_1\) with \(a_2\) in a given operator can be achieved with a similarity transformation:

\[ \hat{T}(a_1) \mathcal{O}(a_1) \hat{T}^{-1}(a_1) = \mathcal{O}(a_2). \]  \hspace{1cm} (2.2)

Such a transformation was used to construct coherent states for shape-invariant potentials \[6\]. So far two classes of shape-invariant potentials are found. In the first class the parameters \(a_1\) and \(a_2\) are related by a translation \[5,7\]:

\[ a_2 = a_1 + \eta, \]  \hspace{1cm} (2.3)

and in the second class they are related by a scaling \[8\]

\[ a_2 = qa_1. \]  \hspace{1cm} (2.4)

All textbook examples of exactly solvable potentials belong to the first class. In this article for definiteness we focus on the solutions of the shape-invariance condition involving translations of the parameters as shown in Eq. (2.3). For this class the operator \(\hat{T}(a_1)\) of Eq. (2.2) is simply given by

\[ \hat{T}(a_1) = \exp \left( \eta \frac{\partial}{\partial a_1} \right) \]
\[ \hat{T}^{-1}(a_1) = \hat{T}^\dagger(a_1) = \exp \left( -\eta \frac{\partial}{\partial a_1} \right). \]  \hspace{1cm} (2.5)

Shape-invariant potentials are amenable to the treatment by the method of creation and annihilation operators originally developed for the harmonic oscillator. As such shape-invariant potentials are generalizations of the harmonic oscillator potential. One must, however, identify the creation and annihilation operators. The obvious choice of \(\hat{A}^\dagger\) and \(\hat{A}\) does not work as their commutator

\[ [\hat{A}, \hat{A}^\dagger] = 2 \frac{\hbar}{\sqrt{2m}} \frac{dW}{dx}, \]  \hspace{1cm} (2.6)

depends on the position. To establish the algebraic structure we first introduce the operators

\[ \hat{B}_+ = \hat{A}^\dagger(a_1) \hat{T}(a_1) \]
\[ \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1) \hat{A}(a_1), \]  \hspace{1cm} (2.7)

and rewrite the Hamiltonian of Eq. (1.3) as
\[ \hat{H} - E_0 = \hat{A}^\dagger \hat{A} = \hat{B}_+ \hat{B}_-. \]  

(2.8)

Using Eq. (2.1), one can easily prove the commutation relation:

\[ [\hat{B}_-, \hat{B}_+] = R(a_0), \]  

(2.9)

where we defined

\[ a_n = a_1 + (n - 1)\eta, \]  

(2.10)

and used the identity

\[ R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^\dagger(a_1), \]  

(2.11)

valid for any \( n \). Eq. (2.9) suggests that \( \hat{B}_- \) and \( \hat{B}_+ \) are the appropriate creation and annihilation operators provided that their non-commutativity with \( R(a_1) \) is taken into account. Indeed using the relations

\[ R(a_n)\hat{B}_+ = \hat{B}_+ R(a_{n-1}) \]
\[ R(a_n)\hat{B}_- = \hat{B}_+ R(a_{n+1}), \]  

(2.12)

which readily follow from Eqs. (2.7) and (2.11), one can write down the additional commutation relations

\[ [\hat{H}, \hat{B}_n^+] = (R(a_1) + R(a_2) + \cdots + R(a_n))\hat{B}_n^+, \]  

(2.13)

and

\[ [\hat{H}, \hat{B}_n^-] = -\hat{B}_n^-(R(a_1) + R(a_2) + \cdots + R(a_n)). \]  

(2.14)

Eqs. (2.13) and (2.14) are the generalization of the corresponding commutators for the harmonic oscillator creation and annihilation operators. Consequently the operators \( \hat{B}_+ \) and \( \hat{B}_- \) can be utilized as ladder operators for the spectra of the shape-invariant potentials. Using Eqs. (2.4) and (2.7) one can show that the ground state satisfies the condition

\[ \hat{B}_-|\psi_0\rangle = 0. \]  

(2.15)

In the following we set the energy scale so that the ground state energy, \( E_0 \), is zero. Using Eqs. (2.13) and (2.15) it follows that

\[ \hat{H} \left( \hat{B}_n^+|\psi_0\rangle \right) = (R(a_1) + R(a_2) + \cdots + R(a_n)) \left( \hat{B}_n^+|\psi_0\rangle \right), \]  

(2.16)

i.e., \( \hat{B}_n^+|\psi_0\rangle \) is an eigenstate of the Hamiltonian with the eigenvalue \( R(a_1) + R(a_2) + \cdots + R(a_n) \). Normalization should be carried out with some care as \( \hat{B}_n^+ \) in general does not commute with \( R(a_n) \). One can show that the normalized wavefunction is

\[ |\psi_n\rangle = \frac{1}{\sqrt{R(a_1)}} \hat{B}_+ \cdots \frac{1}{\sqrt{R(a_2)}} \hat{B}_+ \frac{1}{\sqrt{R(a_1)}} \hat{B}_+|\psi_0\rangle. \]  

(2.17)
In addition to the oscillator like commutation relations of Eqs. (2.13) and (2.14) one gets new commutation relations

\[
\left[ \hat{B}_+, R(a_0) \right] = (R(a_1) - R(a_0)) \hat{B}_+,
\]

\[
\left[ \hat{B}_+, (R(a_1) - R(a_0)) \hat{B}_+ \right] = \{ (R(a_2) - R(a_1)) - (R(a_1) - R(a_0)) \} \hat{B}_+,
\]

and so on. In general there is an infinite number of these commutation relations. These commutation relations and their complex conjugates along with Eq. (2.9) form an infinite-dimensional Lie algebra, realized here in a unitary representation.

To classify algebras associated with the shape-invariant potentials one can utilize the fact that for confining potentials the \( n \)th eigenvalue \( E_n \) for large \( n \) obeys the constraint \[ E_n \leq \text{constant} \times n^2. \] (2.20)

Here we will show that those potentials where \( E_n \) is given by

\[ E_n = \beta n^2 + \delta n + \gamma, \] (2.21)

lead to a finite shape-invariance algebra. Using Eq. (2.16) one can then show that

\[ R(a_n) = E_n - E_{n-1} = 2\beta n + \delta - \beta, \] (2.22)

which gives

\[ R(a_n) - R(a_{n-1}) = 2\beta. \] (2.23)

Consequently for systems that satisfy Eq. (2.21) the resulting Lie algebra is finite:

\[
\left[ \hat{B}_-, \hat{B}_+ \right] = R(a_0), \quad \left[ \hat{B}_+, R(a_0) \right] = 2\beta \hat{B}_+.
\] (2.24)

If \( \beta \) is nonzero, depending on its sign, this algebra is either \( SU(2) \) or \( SU(1,1) \). If \( \beta \) is zero it is the Heisenberg-Weyl algebra. For those potentials the eigenvalues of which which do not satisfy Eq. (2.21) the shape-invariance algebras remain infinite-dimensional.

### III. EXAMPLES

To illustrate the discussion in the previous section we first explicitly work out two cases where one gets a finite Lie algebra and then give the most general conditions on the superpotentials for the shape-invariance Lie algebras to be finite.
A. Morse Potential

For the Morse potential, \( V(x) = V_0(e^{-2\lambda x} - 2be^{-\lambda x}) \), the superpotential is

\[
W(x; a_n) = \sqrt{V_0}(a_n - e^{-\lambda x}).
\]  

(3.1)

The remainder in Eq. (1.7) is given by

\[
R(a_n) = 2\lambda\hbar\sqrt{\frac{V_0}{2m}} \left(a_n - \frac{1}{2} \frac{\lambda\hbar}{\sqrt{2mV_0}}\right),
\]

(3.2)

where

\[
a_n = b - \frac{\lambda\hbar}{\sqrt{2mV_0}}(n - \frac{1}{2}).
\]

(3.3)

Introducing the dimensionless operators

\[
K_0 = \frac{m}{\hbar^2\lambda^2} R(a_0)
\]

and

\[
K_{\pm} = \frac{\sqrt{m}}{\hbar \lambda} \hat{B}_{\pm},
\]

(3.5)

one finds that the shape-invariance algebra for the Morse potential is \( SU(1, 1) \):

\[
[K_+, K_-] = -2K_0, \quad [K_0, K_{\pm}] = \pm K_{\pm}.
\]

(3.6)

We note that this algebra is distinct from the \( SU(2) \) algebra used in the usual group-theoretical approach to the problem of finding bound-state solutions of the one-dimensional Morse potential [10,11]. As

\[
K_0 = \frac{\sqrt{2mV_0}}{\hbar \lambda}
\]

the shape-invariance algebra relates a series of Morse potentials with different depths.

B. Scarf Potential

For the Scarf potential, \( V(x) = -V_0/\cosh^2 \lambda x \), the superpotential is

\[
W(x; a_n) = \frac{\hbar \lambda}{\sqrt{2m}} a_n \tanh \lambda x,
\]

(3.7)

with
\[
R(a_n) = \frac{\hbar^2 \lambda^2}{2m} (2a_n - 1) \tag{3.8}
\]

where
\[
a_n = \frac{1}{2} \left( \sqrt{\frac{8mV_0}{\hbar^2 \lambda^2} + 1} - 2n + 1 \right). \tag{3.9}
\]

Again introducing the dimensionless operators
\[
K_\pm = \sqrt{\frac{2m}{\hbar \lambda}} \hat{B}_\pm, \tag{3.10}
\]

and
\[
K_0 = \frac{m}{\hbar^2 \lambda^2} R(a_0) \tag{3.11}
\]

we again obtain the shape-invariance algebra to be an \(SU(1,1)\) algebra:
\[
[K_+, K_-] = -2K_0, \ [K_0, K_\pm] = \pm K_\pm. \tag{3.12}
\]

Once again the shape-invariance algebra relates a series of potentials with different depths as \(K_0\) is given as
\[
K_0 = \left( \frac{2mV_0}{\hbar^2 \lambda^2} + \frac{1}{4} \right)^{1/2}. \tag{3.13}
\]

C. General conditions for finite shape-invariance algebras

Eqs. (2.10) and (2.22) imply that when the shape-invariance algebra is finite \(R(a_n)\) is linear in \(a_n\), which in turn requires the superpotential to be of the form
\[
W(x; a_n) = f(x) a_n + g(x). \tag{3.14}
\]

For \(R(a_n)\) to be independent of \(x\), the functions \(f(x)\) and \(g(x)\) must satisfy the equations
\[
\eta \frac{\hbar}{\sqrt{2m}} \frac{df}{dx} - \eta^2 f^2 = \beta, \tag{3.15}
\]

and
\[
\frac{\hbar}{\sqrt{2m}} \frac{dg}{dx} - \eta f(x) g(x) = \frac{\delta}{2} + \frac{\beta a_1}{\eta}. \tag{3.16}
\]

The resulting \(R(a_n)\) is of the form
\[
R(a_n) = 2\beta a_n - \frac{\beta}{\eta} a_1 - \beta + \delta. \tag{3.17}
\]

One can then catalog those potentials for which the shape-invariance algebra is finite-dimensional. Eq. (3.13) indicates that \(f(x)\) is a superpotential yielding to a reflectionless potential. Thus it is either constant or proportional to \(\tanh x\). Eq. (3.16) then implies that \(g(x)\) is either constant, proportional to \(x\) or \(\exp(x)\). This indicates that constant potential, harmonic oscillator potential, Morse potential, and Scarf potential provide the complete list of potentials for which the shape-invariance algebra is finite-dimensional.
IV. CONCLUSIONS

Shape invariance is shown to have an underlying algebraic structure and the associated Lie algebras are identified. These Lie algebras transform the parameters of the potentials such as strength or range. In general shape-invariance algebras are infinite-dimensional. The conditions under which they become finite-dimensional are elaborated.

Shape-invariance was originally introduced in the context of one dimensional quantum mechanics via the definition given in Eq. (1.7). However, it is possible to define shape-invariance in terms of operators only, as given in Eq. (2.1), without any explicit reference to a potential function. An alternative approach to the study of quantum systems is to introduce algebraic Hamiltonians and exploit dynamical symmetries associated with such Hamiltonians [12]. The method introduced in this article can easily be extended to explore shape-invariance properties of algebraic Hamiltonians. One such application is to utilize shape-invariance algebras in many-body problems where pairing plays an important role. Indeed solutions for the generalized pairing Hamiltonian for spherical nuclei has been derived by introducing an infinite-dimensional algebra [13]. It may be possible to extend this result to deformed nuclei using the techniques described here.

It has been shown that harmonic oscillators with spin-orbit couplings naturally lead to the superalgebras both as dynamical symmetry algebras and spectrum-generating algebras [14,15]. It would be interesting to see if those results can be generalized to all shape-invariant potentials.

Finally, in this paper we omitted those shape-invariant potentials where parameters are related by scaling. One expects that shape-invariance algebras for such potentials can be related to q-algebras. A detailed study of this connection is deferred to a later publication.

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