BRANCHING RANDOM WALKS CONDITIONED ON RARELY SURVIVAL

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Abstract. In this paper, we show that a Galton-Watson tree conditioned to have a fixed number of particles in generation $n$ converges in distribution as $n \to \infty$, and with this tool we study the span and gap statistics of a branching random walk on such trees, which is the discrete version of Ramola, Majumdar and Schehr [13], generalized to arbitrary offspring and displacement distributions with moment constraints.

1. Introduction

Consider a Galton-Watson tree $T$ with offspring distribution $\mu$ and regularity conditions

$$\mu(0), \mu(1) > 0, \mu(0) + \mu(1) < 1,$$

(1.1)

$$m := \sum_{k=1}^{\infty} k\mu(k) \in (0, \infty), \sum_{k=1}^{\infty} k^2\mu(k) < \infty.$$

We denote by $Z_n$ the population of generation $n$, and by $\text{cut}_n(\text{pru}_n(T))$ the reduced tree formed by the family tree of nodes in generation $n$ up to their youngest common ancestor, as illustrated in Figure 1. Then our first result is that

Theorem 1.1. Fix $k \geq 1$. Under (1.1), one can construct an explicit probability measure $P_{st}^k$ (see Proposition 3.8) such that, as $n \to \infty$, for any set $B$ of finite trees,

$$P(\text{cut}_n(\text{pru}_n(T)) \in B \mid Z_n = k) \to P_{st}^k(B).$$

Figure 1. Essential structure $\text{cut}_n(\text{pru}_n(T))$ for $n = 4, k = 5$. 

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Moreover, consider the branching random walk \((V_u)_{u \in T}\) indexed by a Galton-Watson tree:

\[
V_u = \sum_{\emptyset < v \leq u} X_v, \quad V_\emptyset = 0,
\]

where \(X_v \sim \theta\) for all \(v \in T \setminus \{\emptyset\}\), and \(\theta\) is a distribution on \(\mathbb{R}\) with regularity conditions

\[
(1.2) \quad \mathbb{E}[X] = 0, \quad \text{Var}(X) = 1, \quad \mathbb{E}[\exp(tX)] < \infty, \quad \forall t \in \mathbb{R}, \quad \text{where} \quad X \sim \theta.
\]

![Figure 2. Spatial positions \((V_n^{(i)})\) illustrated with \(n = 4, k = 5\).](image)

We list the positions of nodes in generation \(n\) in increasing order,

\[
V_n^{(1)} \leq \cdots \leq V_n^{(Z_n)},
\]

as showed in Figure 2, then we study its span

\[
R_n := V_n^{(Z_n)} - V_n^{(1)}
\]

and its successive gaps

\[
g_n^i := V_n^{(i+1)} - V_n^{(i)}, \quad 1 \leq i \leq Z_n - 1.
\]

For simplicity, we write \(R\) and \((g^i)\) for the span and gaps of the last generation for a finite tree. Recall that \(m = \sum_{k=1}^{\infty} k \mu(k)\) is the expected number of children, and our second result is then

**Theorem 1.2.** Let \(k \geq 2\) and \(1 \leq i \leq k - 1\). If (1.1) and (1.2) are satisfied, then the law of \(R_n\) and \((g^i_n)\) under \(\mathbb{P}(\cdot | Z_n = k)\) converges in distribution to that of \(R\) and \((g^i)\) under \(\mathbb{P}^\text{st}_k(\cdot )\), and there are explicit constants \(C_1, C_2, C_3\) (see Lemma 4.1, Proposition 4.3) such that, as \(x \to \infty\),

\[
\mathbb{P}^\text{st}_k(R > x) = \begin{cases} 
(C_1 + o(1))x^{-2}, & m = 1, \\
\exp(-(C_2 + o(1))x), & m \neq 1,
\end{cases}
\]

\[
\mathbb{P}^\text{st}_k(g^i > x) = \begin{cases} 
(C_1 C_3 + o(1))x^{-2}, & m = 1, \\
\exp(-(C_2 + o(1))x), & m \neq 1.
\end{cases}
\]

**Remark 1.3.** We use the unified assumptions (1.1) and (1.2) for convenience. These assumptions can be further refined for specific cases:

- In the critical case \(m = 1\), \(\theta\) only need to have finite \((2 + \delta)\)-th moment (for any \(\delta > 0\)) instead of exponential moments.
• In the supercritical case $m > 1$, we only need the $L \log L$ condition,
  \[ \sum_{i=1}^{\infty} i \log i \mu(i) < \infty, \]
  instead of the finite variance condition in (1.1).

• In the subcritical case $m < 1$, we only need the $L \log L$ condition for
  Section 3 except in Part 2, Corollary 3.10.

This paper is mainly motivated by [12] and [13], where Ramola, Majumdar
and Schehr studied the span and gaps for the branching Brownian motion
via a PDE method. Their result corresponds to the continuous version of
Theorem 1.2 with geometric $\mu$ and Gaussian $\theta$. In particular, we show that
the asymptotic for gap statistics are no longer independent of $k$ and $i$ for the
critical case with non-geometric offspring distribution, see Remark 4.4.

The study of the reduced Galton-Watson tree $\text{pru}_n(T)$ at least dates
back to Fleischmann and Prehn [6], Fleischmann and Siegmund-Schultze [7],
where it is showed that the limit for the critical case $m = 1$ is the Yule tree.
See also Curien and Le Gall [5] for properties and applications of the Yule
tree. In particular, for the critical Galton Watson tree conditioned on non-
extinction, one has $Z_n = \Theta(n)$, and the conditioned case $\{0 < Z_n \leq \phi(n)\}$,
where $\phi(n) = O(n), \phi(n) \to \infty$ is recently studied in Liu and Vatutin [10].

The conditioned limiting behavior of the whole tree $T$ (in contrast to
the reduced tree) is known as the local limit. The general result (under
the condition $\{Z_n > 0\}$) for the local limit is the Kesten’s tree [9], see also
Geiger [8]. Further, there are detailed discussions for local limits conditioned
on rare events ($\{Z_n < \epsilon \mathbb{E}[Z_n | Z_n > 0]\}$ or $\{Z_n > \epsilon^{-1} \mathbb{E}[Z_n | Z_n > 0]\}$), see
Abraham and Delmas [2], [3], Abraham, Bouaziz and Delmas [1].

In addition, we also establish an extension of the Ratio theorem (cf. The-
om 1.7.4, [4]) as an byproduct. Indeed, let
\[ P_n(1,j) := P(Z_n = j | Z_0 = 1), \]

denote the transition probabilities for the population of $n$ generations of the
Galton-Watson tree, then we have that

**Proposition 1.4.** Fix $k \geq 2$. Under (1.1), we can construct constants $\gamma$ (see
Part 2, Proposition 2.1) and $C_4, C_5$ (see Remark 3.11) such that as $n \to \infty$,
\[ \frac{P_n(1,k)}{P_n(1,1)} - \frac{P_{n-1}(1,k)}{P_{n-1}(1,1)} = \begin{cases} 
(C_4 + o(1))n^{-2}, & m = 1, \\
(C_5 + o(1))\gamma^n, & m \neq 1, 
\end{cases} \]

The paper is organized as follows. In Section 2 we present systematically
the notations and concepts needed. In Section 3 we study the genealogical
properties of $\text{pru}_n(T)$ and $\text{cut}_n(\text{pru}_n(T))$, then we prove Theorem 1.1
and Proposition 1.4 in Proposition 3.8 and Remark 3.11. In Section 4 we
study the span and gaps, and prove Theorem 1.2 in Proposition 4.2 and
Proposition 4.3.

2. Preliminaries

2.1. Trees. A (locally finite, rooted) planar tree $T$ is a subset of integer-
valued words, $T \subset \cup_{n \geq 0} \mathbb{N}_+^n$, such that:

- The root $\varnothing \in T$, where by convention we denote $\mathbb{N}_+^0 = \{\varnothing\}$.
- If a node $u = (u_1, \cdots, u_n) \in T$, then its parent $\hat{u} := (u_1, \cdots, u_{n-1}) \in T$. 

For each node \( u = (u_1, \ldots, u_n) \in T \), there exists an integer \( k_u(T) \in \mathbb{N} \) called its number of children, such that for every \( j \in \mathbb{N}, (u_1, \ldots, u_n, j) \in T \) if and only if \( 1 \leq j \leq k_u(T) \).

We only consider locally finite trees, i.e. \( k_u(T) < \infty, \forall u \in T \). We give a few basic notations on trees:

- The set of all planar trees is denoted by \( T \).
- The generation/height of a node is its length as a word, i.e. if \( u = (u_1, \ldots, u_n) \), then \( |u| = n \). The height of a tree is then defined as \( H(T) := \max\{|u| : u \in T\} \in \mathbb{N} \cup \{\infty\} \).
- The population of generation \( n \) is defined as \( Z_n(T) := \#\{u \in T : |u| = n\} \). By construction, \( Z_0(T) = 1 \) for any tree \( T \).
- A node \( u = (u_1, \ldots, u_n) \in T \) is an ancestor of another one \( v = (v_1, \ldots, v_m) \in T \), denoted by \( u \prec v \), if \( n < m \) and \( u_i = v_i, 1 \leq i \leq n \). The (youngest) common ancestor of two nodes \( u, v \in T \), denoted by \( u \wedge v \), is then the node in \( T \) with maximum height, such that \( u \wedge v \preceq u, v \).
- For \( u \in T \), the subtree rooted at \( u \in u \) is defined as \( T[u] = \{v \in \mathbb{N}_n^u : uv \in T\} \), where \( uv \) stands for concatenation of words. It is not hard to check that this set is a tree. In particular, given that \( k_0(T) = r \), nodes in the first generations are labeled \( 1, 2, \ldots, r \) by construction, thus subtrees rooted at them are \( T[1], \ldots, T[r] \).

When there is no confusion for the tree under consideration, we omit the reliance on \( T \) and write, for instance, \( Z_n(T) \).

2.2. The prune and cut operation. To study the relative relations of nodes in generation \( n \) while omitting irrelevant information, we define the prune and cut operations on trees (recall the illustration in Figure [1]):

- For any \( T \in T \), we construct the pruned tree at height \( n \) by
  \[ \text{pru}_n(T) := \{u \in T : \exists v \in T, |v| = n, u \preceq v\} \].
  By convention, if \( Z_n(T) = 0 \), we take \( \text{pru}_n(T) = \{\emptyset\} \).
Moreover, we define the **cut operation** by
\[ \phi_n(T) = \bigwedge_{|u|=n, u \in T} u, \text{ and then } \text{cut}_n(T) = T[\phi_n(T)]. \]
where by convention, \( \phi_n(T) = \emptyset \) if \( Z_n(T) = 0 \).

In addition, we denote the **set of all pruned trees** at height \( n \) by \( T_n := \text{pru}_n(T) \), and the set of all pruned trees at height \( n \) with \( Z_n = k \) by \( T_{n,k} \).

In particular, \( T_{n,0} \) contains only one element \( \{\emptyset\} \). Since trees are assumed to be locally finite, we have that \( T_n = \bigcup_{k=0}^{\infty} T_{n,k} \). Since all these sets are countable, the problem of measurability is trivial.

These operations naturally extend to the branching random walk indexed by these trees, by translation such that the root is always pinned at 0. Then by construction,
\[ R_n(T) = R_{H(\text{cut}_n(T))}(\text{cut}_n(T)) = R_n(\text{pru}_n(T)), \]
and the same thing applies to the gaps.

### 2.3. Galton-Watson tree and Ratio theorem

Let \( \mu \) be a probability distribution on \( \mathbb{N} \). The law of a **Galton-Watson** tree with offspring distribution \( \mu \) is a probability measure \( P \) on the set of planar trees \( T \), such that for all nodes \( u \),
\[ k_u \text{i.i.d.} \sim \mu. \]

Clearly, the sequence \( (Z_n) \) is a Markov chain starting at \( Z_0 = 1 \) under \( P \), and one can then set its transition probabilities as
\[ P_n(i,j) := P(Z_{k+n} = j \mid Z_k = i), \]
where we take \( k \) such that \( P(Z_k = i) > 0 \). (Under the assumption \( \mu \leq 1 \), this is always possible by taking \( k \) large enough.) In particular,
\[ P_1(1,i) = \mu(i). \]

Moreover, we define the **generating function** of this process as
\[ f(s) := \mathbb{E}\left(s^{Z_1(T)}\right) = \sum_{i=0}^{\infty} P_1(1,i)s^i, \]
then its derivatives are
\[ f^{(r)}(s) = r! \sum_{\ell \geq r} \binom{\ell}{r} P_1(1,i)s^{\ell-r}, \]
and its iterations are
\[ f_n(s) := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}} = \mathbb{E}\left(s^{Z_n(T)}\right) = \sum_{\ell \geq 0} P_n(1,\ell)s^\ell. \]

We also define the **extinction probabilities** as
\[ q_n := P_n(1,0) = f_n(0), q := \lim_{n \to \infty} q_n. \]

Clearly, \( (q_n) \) is a bounded increasing sequence, which guarantees the existence of \( q \). Moreover, it is standard (Athreya and Ney [11, Theorem 1.5.1]) that \( q = 1 \) if \( m \leq 1 \) (except for the trivial case \( \mu = \delta_1 \)), and \( q < 1 \) if \( m > 1 \), where \( m := \sum_{i=1}^{\infty} i\mu(i) \) is the expected number of children.
We finish this section by citing some fundamental estimates that shall be used later,

**Proposition 2.1** (Athreya and Ney [4, Section 1.7-1.11]). Let $\mu$ be an offspring distribution such that

$$
\mu(0), \mu(1) > 0, \mu(0) + \mu(1) < 1, m < \infty.
$$

1. There exists a sequence $(\pi_j)$ such that for any $j \geq 1$,

$$
\lim_{n \to \infty} \frac{P_n(1,j)}{P_n(1,1)} \not\to \pi_j \in (0, \infty),
$$

where $\not\to$ means non-decreasing limit.

2. For any $t \in \mathbb{Z}$, $i, j, k, l \geq 1$,

$$
\lim_{n \to \infty} \frac{P_n(i,t+j)}{P_n(k,l)} = q^{t-k}\frac{i\pi_j}{l}\pi_l,
$$

where $q$ is the extinction probability in (2.7), and $\gamma = f'(q)$.

3. If $m = 1$, $\sigma^2 := \sum_{i=1}^{\infty} i^2 \mu(i) < \infty$, then for any $i, j \geq 1$,

$$
\lim_{n \to \infty} n^2 P_n(i,j) = 2\pi_j q^{i-1}v_j,
$$

where $(v_j)$ is determined by $Q(s) = \sum_{j=0}^{\infty} vjs^j, 0 \leq s < 1$, with $Q$ the unique solution of

$$
Q(f(s)) = \gamma Q(s)(0 \leq s < 1), Q(q) = 0, \lim_{s \to q} Q'(s) = 1.
$$

2.4. **Branching random walk and Cramer’s theorem.** On a Galton-Watson tree $T$, we shall consider the branching random walk $(V_u)_{u \in T}$ by attaching i.i.d. spatial displacements

$$
X_u \overset{i.i.d.}{\sim} \theta, \forall u \in T\{\varnothing\},
$$

whose probability measure is still denoted by $P$ for simplicity. For other probability measures on trees that we shall construct, we also abuse the same notations for the corresponding spatial process.

Moreover, we cite a fundamental estimate for random walks,

**Lemma 2.2.** Let $\theta$ be a distribution on $\mathbb{R}$, and let $(X_i)$ be i.i.d. random variables distributed as $\theta$. Assume that $E[X_1] = 0$, $\text{Var}(X_1) = 1$.

1. [11] Theorem 5.7 If $E[|X_1|^{2+\delta}] < \infty$, then there exists $C_1, C_2, C_3 > 0$ such that for any $n \geq 1, x > 0$,

$$
P\left(\sum_{i=1}^{n} X_i > x\right) \leq \frac{C_1n}{x^3} + e^{-C_2x^2},
$$

$$
P\left(\sum_{i=1}^{n} X_i \leq x\right) - \Phi\left(\frac{x}{\sqrt{n}}\right) \leq C_3\frac{1}{\sqrt{n}},
$$

where $\Phi(x)$ is the cumulative distribution function of the standard Gaussian distribution $\mathcal{N}(0,1)$. 
(2) (Cramer’s theorem) If
\[
\Lambda(t) := \log E[\exp(tX_1)] < \infty, \forall t \in \mathbb{R},
\]
then
\[
\lim_{n \to \infty} \frac{1}{n} \log \left( P\left( \sum_{i=1}^{n} X_i \geq nx \right) \right) = -\sup_{t \in \mathbb{R}}(tx - \Lambda(t)).
\]

3. Galton-Watson trees conditioned on rarely survival

3.1. Pruned Galton-Watson trees. We first study the distribution of \( \text{pru}_n(T) \). Recall that \( \mathcal{T}_{n,k} \) stands for the set of all pruned trees with \( k \) nodes in generation \( n \).

Definition 3.1. For any pair of integers \((n,k)\) such that \( P_n(1,k) > 0 \), we denote by \( \text{P}^{\text{pru}}_{n,k} \) the law of \( \text{pru}_n(T) \), supported on \( \mathcal{T}_{n,k} \), where \( T \) is sampled under the law of \( P(\cdot \mid Z_n = k) \). In other words, for any \( A \subset \mathcal{T}_{n,k} \),

\[
(3.8) \quad \text{P}^{\text{pru}}_{n,k}(A) = \text{P}(\text{pru}_n(T) \in A \mid Z_n = k) = \frac{\text{P}(\text{pru}_n(T) \in A)}{P_n(1,k)}.
\]

Recall that \( T[i] \) denotes the \( i \)-th subtree of \( T \) rooted at the first generation, then

Proposition 3.2. Under (1.1), for any pair of integers \((n,k)\) such that \( P_n(1,k) > 0 \), let \( r \geq 1 \) and let \( k_1, \ldots, k_r \) be positive integers such that \( \sum_{i=1}^{r} k_i = k \). Then for any \( A_i \subset \mathcal{T}_{n-1,k_i}, 1 \leq i \leq r \),

\[
\text{P}^{\text{pru}}_{n,k}(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)
= \frac{1}{P_n(1,k)} \frac{f(r)(q_{n-1})}{r!} \prod_{i=1}^{r} P_{n-1}(1,k_i) \prod_{i=1}^{r} \text{P}^{\text{pru}}_{n-1,k_i}(A_i),
\]

where \( f \) and \( q_{n-1} \) are defined in (2.4), (2.7).

Proof. By (3.3),

\[
P_n(1,k) \text{P}^{\text{pru}}_{n,k}(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)
= \text{P}(Z_1(\text{pru}_n(T)) = r; \text{pru}_n(T)[i] \in A_i, 1 \leq i \leq r)
= \sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_\ell \leq \ell} \text{P}(Z_1(T) = \ell; \text{pru}_{n-1}(T[j_i]) \in A_i, i = 1, \ldots, r; Z_{n-1}(T[i]) = 0, i \notin \{j_1, \cdots, j_\ell\}).
\]

Let \( B_i = \{ T \in \mathcal{T} : \text{pru}_{n-1}T \in A_i \} \), by the self-similarity of Galton-Watson trees, the equation above can be further simplified as

\[
(3.9) \quad = \sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_\ell \leq \ell} P_1(1,\ell)q_{n-1}^{\ell-r} \prod_{i=1}^{r} \text{P}(B_i).
\]

Moreover, by (3.8), we have that

\[
\text{P}(B_i) = P_{n-1}(1,k_i) \text{P}^{\text{pru}}_{n-1,k_i}(A_i),
\]
put it in \([3.9]\), and it suffices to show that
\[
\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_r \leq \ell} P_1(1, \ell) q_{n-1}^{\ell-r} = \frac{f^{(r)}(q_{n-1})}{r!}.
\]
Indeed, we have
\[
\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_r \leq \ell} P_1(1, \ell) q_{n-1}^{\ell-r} = \sum_{\ell \geq r} \binom{\ell}{r} P_1(1, \ell) q_{n-1}^{\ell-r} = \frac{f^{(r)}(q_{n-1})}{r!},
\]
where the second line follows from \([2.5]\). \(\square\)

Moreover, we give two more properties of \(P_{n,k}^{pru}\):

**Corollary 3.3.** Under \([1.1]\), for any \(1 \leq u \leq n\), any \(A \subseteq T_{n,k}\),
\[
P_{n,k}^{pru}(Z_1(T) = \cdots = Z_{n-u}(T) = 1, T[\underbrace{1 \cdots 1}_{n-u \text{ times}}] \in A)
= \frac{P_n(1, 1) P_n(1, k)}{P_n(1, k) P_n(1, 1)} P_{n,k}^{pru}(A),
\]
where \(\underbrace{1 \cdots 1}_{n-u \text{ times}}\) means the first node (and also the only node, under the condition \(Z_1(T) = \cdots = Z_{n-u}(T) = 1\)) in generation \(n-i\).

**Proof.** Take \(r = 1\) in Proposition 3.2 we have that
\[
P_{n,k}^{pru}(Z_1(T) = 1, T[1] \in B) = \frac{P_{n-1}(1, k)}{P_n(1, k)} f'(q_{n-1}) P_{n,k}^{pru}(B)
= \frac{P_{n-1}(1, k)}{P_n(1, k)} \frac{P_n(1, 1)}{P_n(1, 1)} P_{n,k}^{pru}(1, k) (B),
\]
for any \(B \subseteq T_{n-1,k}\), where we use \([2.6]\) and \([2.7]\) to deduce that \(f'(q_{n-1}) = \frac{f_r'(0)}{f_{n-1}'(0)} = \frac{P_n(1, 1)}{P_n(1, 1)}\). The result follows by using this relation \(n-u\) times inductively. \(\square\)

**Corollary 3.4.** Under \([1.1]\), for any \(r, k \geq 2\), if \(f^{(r)}(q) < \infty\), then
\[
\lim_{n \to \infty} \frac{P_{n,k}^{pru}(Z_1(T) = r)}{P_n(1, 1)^{r-1}} = \gamma^{-r} f^{(r)}(q) \frac{1}{r!} \sum_{k_1, k_2, \ldots, k_r \geq 1, k_1 + \cdots + k_r = k} \frac{\pi_{k_1} \cdots \pi_{k_r}}{\pi_k},
\]
\[
\lim_{n \to \infty} \frac{P_{n,k}^{pru}(Z_1(T) \geq r)}{P_n(1, 1)^{r-1}} = \gamma^{-r} f^{(r)}(q) \frac{1}{r!} \sum_{k_1, k_2, \ldots, k_r \geq 1, k_1 + \cdots + k_r = k} \frac{\pi_{k_1} \cdots \pi_{k_r}}{\pi_k},
\]
where \(f\) is defined in \([2.4]\), \(q, \gamma\) and \((\pi_k)\) are defined in Proposition 2.1.

**Proof.** For the first equation, by Proposition 3.2 we take \(A_i = T_{n-1,k_i}\) and sum over all choices of \((k_i)\), then
\[
P_{n,k}^{pru}(Z_1(T) = r) = \frac{1}{P_n(1, k)} \frac{f^{(r)}(q_{n-1})}{r!} \sum_{k_1, k_2, \ldots, k_r \geq 1, k_1 + \cdots + k_r = k} \prod_{i=1}^{r} P_{n-1}(1, k_i).
\]
Divide both sides by $P_n(1,1)^{r-1}$, we have that
\[ \frac{P_{n,k}^{\text{ru}}(Z_1(T) = r)}{P_n(1,1)^{r-1}} = \frac{P_n(1,1)}{P_n(1,k)} \frac{f(r)(q_{n-1})}{r!} \sum_{k_1,k_2,\ldots,k_r \geq 1 \atop k_1 + \cdots + k_r = k} \prod_{i=1}^r P_{n-1}(1,k_i) P_n(1,1). \]

For fixed $r$ and $k$, there are only finitely many terms on the right hand side, thus we can take the limit separately for each fraction by Proposition 2.1 and the result follows.

To deal with $P_{n,k}^{\text{ru}}(Z_1(T) \geq r)$, we sum over
\[ P_{n,k}^{\text{ru}}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1) \]
for all $\sum_{i=1}^r k_i = k$. In other words, we consider the first $r-1$ subtrees to give $k_i$ offspring each, while the rest subtrees give $k_r$ offspring in total. By the proof of Proposition 2.1 we have that
\[ P_n(1,k)P_{n,k}^{\text{ru}}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1) = \sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_{\ell-1} < \ell} P(\ell; Z_{n-1}(T[j_i]) = k_i, i = 1, \ldots, r-1; \sum_{i \notin \{j_1, \cdots, j_{\ell-1}\}} Z_{n-1}(T[i]) = 0; \sum_{i > j_{\ell-1}} Z_{n-1}(T[i]) = k_r) \]
\[ = \sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_{\ell-1} < \ell} P(1,\ell) \prod_{i=1}^{r-1} P_{n-1}(1,k_i) \cdot q_{n-1}^{j_i-1-r+1} P_{n-1}(\ell - j_{\ell-1}, k_r). \]

Write $j_{\ell-1} = j$ for short, and we can simplify this term into
\[ P_n(1,k)P_{n,k}^{\text{ru}}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1) = P(1,\ell) \prod_{i=1}^{r-1} P_{n-1}(1,k_i) \sum_{\ell = r}^{\infty} \sum_{j = r-1}^{\ell-1} \left( \frac{j-1}{r-2} \right) q_{n-1}^{j-r+1} P_{n-1}(\ell - j, k_r). \]

Divide by $(P_n(1,1))^r$, then
\[ \frac{P_n(1,k)}{(P_n(1,1))^r} P_{n,k}^{\text{ru}}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1) = P(1,\ell) \prod_{i=1}^{r-1} P_{n-1}(1,k_i) \sum_{\ell = r}^{\infty} \sum_{j = r-1}^{\ell-1} \left( \frac{j-1}{r-2} \right) q_{n-1}^{j-r+1} \frac{P_{n-1}(\ell - j, k_r)}{P_n(1,1)}. \]

By Proposition 2.1 and dominated convergence, this term converges to
\[ P(1,\ell) \gamma^{-r} \prod_{i=1}^{r-1} \pi_{k_i} \sum_{\ell = r}^{\infty} \sum_{j = r-1}^{\ell-1} \left( \frac{j-1}{r-2} \right) q^{j-r+1+\ell-j-1}(\ell - j) \pi_{k_r} \]
\[ = P(1,\ell) \gamma^{-r} \prod_{i=1}^{r} \pi_{k_i} \sum_{\ell = r}^{\infty} \left( \frac{\ell}{r} \right) q^{\ell - r} \]
\[ = \gamma^{-r} \frac{f(r)(q)}{r!} \prod_{i=1}^{r} \pi_{k_i}, \]
by the elementary identities

\[
\sum_{i=r-1}^{\ell-1} (\ell - i) \binom{i - 1}{r - 2} = \ell \binom{\ell}{r}, \quad f^{(r)}(s) = \sum_{\ell \geq r} r! \binom{\ell}{r} P_1(1, \ell) s^{\ell - r}.
\]

Moreover, by Proposition 2.1 again, we have that

\[
\frac{P_n(1, k)}{(P_n(1, 1))^r} = (1 + o(1)) \frac{\pi_k}{(P_n(1, 1))^{r-1}},
\]

thus (3.10) gives

\[
\lim_{n \to \infty} \frac{\pi_k}{(P_n(1, 1))^{r-1}} P^\text{pru}_{n,k}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r - 1)
\]

\[
= \gamma^{-r} f^{(r)}(q) \frac{1}{r!} \pi_{k_1} \cdots \pi_{k_r},
\]

and the result follows by summing over all choices of \((k_i)\).

Remark 3.5. The condition \(f^{(r)}(s) < \infty\) is always true if \(s < 1\), while \(q < 1\) if and only if \(m \leq 1\). Thus the condition \(f^{(r)}(q) < \infty\) in Corollary 3.4 is trivially satisfied if \(m > 1\). If \(m \leq 1\), one can verify that this condition is equivalent to the moment condition \(\sum_{i=1}^{\infty} i^r \mu(i) < \infty\).

3.2. The small tree measure. Now we study the composition of the cut and prune operation.

Definition 3.6. For any pair of integers \((n, k)\) such that \(P_n(1, k) > 0\) and \(k \geq 2\), we denote by \(P^\text{st}_{n,k}\) the law of \(\text{cut}_n(T)\), where \(T\) is sampled under the law of \(P^\text{pru}_{n,k}\). In other words, for any \(A \subseteq T\),

\[
P^\text{st}_{n,k}(A) = P^\text{pru}_{n,k}(\text{cut}_n(T) \in A).
\]

We remark that \(k\) is assumed to be at least 2, since for \(k = 1\), the cut operation will always give the trivial tree \(\{\emptyset\}\). This measure \(P^\text{st}_{n,k}\) is developed in the following lemma:

Lemma 3.7. Under [1,1], for any pair of integers \((n, k)\) such that \(P_n(1, k) > 0\) and \(k \geq 2\), \(P^\text{st}_{n,k}\) is supported on \(\cup_{u=1}^{n} T_{u,k} \cap \{T : Z_1 \geq 2\}\). Moreover, for any \(A \subseteq \cup_{u=1}^{n} T_{u,k} \cap \{T : Z_1 \geq 2\}\),

\[
P^\text{st}_{n,k}(A) = \frac{P_n(1, 1)}{P_n(1, k)} \sum_{u=1}^{n} \frac{P_u(1, k)}{P_u(1, 1)} P^\text{pru}_{u,k}(A \cap T_{u,k}).
\]

Proof. For the first assertion, by construction, \(\text{cut}_n(T)\) has at least 2 nodes in its first generation. Moreover, the trees on which we perform the cut operation are those in \(T_{n,k}\), so \(\text{cut}_n(T)\) has height at most \(n\), with all its leaves in the last generation. Thus it is an element in \(\cup_{u=1}^{n} T_{u,k} \cap \{T : Z_1(T) \geq 2\}\). The converse is trivial.
Then it suffices to prove the second assertion for \( A \subseteq \mathcal{T}_{u,k} \cap \{ T : Z_1(T) \geq 2 \} \). Indeed, by Definition 3.6 and Corollary 3.3
\[
\mathbb{P}^{st}_{n,k}(A) = \mathbb{P}^{pru}_{n,k}(\text{cut}(T) \in A)
= \mathbb{P}^{pru}_{n,k}(Z_1(T) = \cdots = Z_{n-u}(T) = 1, T[\underbrace{1 \cdots 1}_n-u \text{ times}] \in A)
= \frac{P_u(1,1)}{P_n(1,1)} P_u(1,k) \mathbb{P}^{pru}_{u,k}(A).
\]

Then the conclusion follows by partitioning a general set \( A \subseteq \bigcup_{u=1}^n \mathcal{T}_{u,k} \cap \{ T : Z_1(T) \geq 2 \} \) into \( \bigcup_{u=1}^n (A \cap \mathcal{T}_{u,k}) \), and use the above equation on each part \( A \cap \mathcal{T}_{u,k} \).

In fact, we notice that \( n \) no longer plays a major role in Lemma 3.7 and we are motivated to take the limit \( n \to \infty \). Moreover, this limit measure still has Galton-Watson-type branching properties:

**Proposition 3.8.** Under (1.1), fix \( k \geq 2 \), let \( n \to \infty \), then the measures \( \{ \mathbb{P}^{st}_{n,k} \}_{n} \) converge to a measure \( \mathbb{P}^{st}_k \) supported on \( \bigcup_{u=1}^\infty \mathcal{T}_{u,k} \cap \{ T : Z_1(T) \geq 2 \} \), defined by

\[
\mathbb{P}^{st}_k(A) = \frac{1}{\pi_k} \sum_{u=1}^\infty \frac{P_u(1,k)}{P_u(1,1)} \mathbb{P}^{pru}_{u,k}(A \cap \mathcal{T}_{u,k}),
\]

for any \( A \subseteq \bigcup_{u=1}^\infty \mathcal{T}_{u,k} \cap \{ T : Z_1(T) \geq 2 \} \). Moreover, fix any \( u > 0 \) such that \( P_u(1,k) > 0 \), let \( r \geq 2 \) and \( k_1, \ldots, k_r \geq 1 \) such that \( \sum_{i=1}^r k_i = k \). Then for any \( A_i \subseteq \mathcal{T}_{u-1,k_i} \),

\[
\mathbb{P}^{st}_k(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)
= \frac{\pi_{k_1} \cdots \pi_{k_r}}{\pi_k} \frac{P_{u-1}(1,1)}{P_u(1,1)} \frac{f(r)(q_{u-1})}{r!} \prod_{i=1}^r \mathbb{P}^{st}_{k_i}(\text{cut}_{u-1}(A_i)).
\]

**Proof.** Convergence and (3.11) follows directly from Lemma 3.7 and Part 1 of Proposition 2.1.

Then for (3.12), by (3.11) and Proposition 3.2, we have that

\[
\mathbb{P}^{st}_k(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)
= \frac{1}{\pi_k} \frac{P_u(1,k)}{P_u(1,1)} \mathbb{P}^{pru}_{u,k}(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)
= \frac{1}{\pi_k} \frac{1}{P_u(1,1)} \frac{f(r)(q_{u-1})}{r!} \prod_{i=1}^r P_{u-1}(1,k_i) \prod_{i=1}^r \mathbb{P}^{pru}_{u-1,k_i}(A_i).
\]

Decompose \( A_i \) by the height of trees after the cut operation,

\[
A_i^{(x)} := \{ T \in A_i : H(\text{cut}_{u-1}(T)) = x \},
\]

then by Corollary 3.3

\[
\mathbb{P}^{pru}_{u-1,k_i}(A_i) = \frac{P_{u-1}(1,1)}{P_{u-1}(1,k_i)} \sum_{x=1}^{u-1} P_x(1,k_i) \frac{P_{u-1}(1,k_i)}{P_x(1,1)} \mathbb{P}^{pru}_{x,k_i}(\text{cut}_{u-1}(A_i^{(x)})).
\]

Since \( \text{cut}_{u-1} \) is injective on \( A_i \), we have

\[
\text{cut}_{u-1}(A_i^{(x)}) = \text{cut}_{u-1}(A_i) \cap \mathcal{T}_{x,k_i},
\]
thus by (3.11) again,
\[ \mathbf{P}^\text{pru}_{u-1,k}(A_i) = \frac{P_{u-1}(1,1)}{P_{u-1}(1,k_i)} \cdot \pi_k \mathbf{P}^\text{st}_{k_i}(\text{cut}_{u-1}(A_i)). \]

Put this back into (3.13), and we get (3.12). \[ \square \]

This enables us to give further descriptions of \( \mathbf{P}^\text{st}_k \). Recall that \( H(T) \) is the height of a tree, then

**Corollary 3.9.** Under (1.1), for any \( r, k \geq 2 \), if \( f^{(r)}(q) < \infty \), then
\[
\lim_{u \to \infty} \frac{\mathbf{P}^\text{st}_k(Z_1(T) = r \mid H(T) = u)}{P_u(1,1)^{r-2}} = \lim_{u \to \infty} \frac{\mathbf{P}^\text{st}_k(Z_1(T) \geq r \mid H(T) = u)}{P_u(1,1)^{r-2}} = \frac{2\gamma^{2-r} f^{(r)}(q)}{r! \sum_{k_1, k_2 \geq 1} \pi_{k_1} \cdot \pi_{k_2} \cdot \sum_{k_1 + k_2 = k} \pi_{k_1} \cdot \pi_{k_2}}.
\]

**Proof.** By (3.11),
\[ \mathbf{P}^\text{st}_k(Z_1(T) = r \mid H(T) = u) = \frac{\mathbf{P}^{\text{pru}}_{u,k}(Z_1(T) = r)}{\mathbf{P}^{\text{pru}}_{u,k}(Z_1(T) \geq 2)}, \]
and we apply Corollary \( 3.4 \). Changing \( Z_1(T) = r \) to \( Z_1(T) \geq r \) is idem. \[ \square \]

**Corollary 3.10.** Under (1.1), fix any \( k \geq 2 \).

1. For any \( u \geq 1 \),
\[ \mathbf{P}^\text{st}_k(H(T) = u) = \frac{1}{\pi_k} \left( \frac{P_u(1,k)}{P_u(1,1)} - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)} \right). \]

2. \[
\lim_{u \to \infty} \frac{\mathbf{P}^\text{st}_k(H(T) = u)}{P_u(1,1)} = \gamma^{-2} \frac{f''(q)}{2} \sum_{1 \leq i \leq k-1} \pi_i \pi_{k-i}.
\]

**Proof.**

1. Take \( A = T_{u,k} \cap \{ T : Z_1(T) \geq 2 \} \) in Proposition 3.8 we have that
\[ \mathbf{P}^\text{st}_k(H(T) = u) = \mathbf{P}^\text{st}_k(T_{u,k} \cap \{ T : Z_1(T) \geq 2 \}) = \frac{1}{\pi_k} \frac{P_u(1,k)}{P_u(1,1)} \mathbf{P}^{\text{pru}}_{u,k}(Z_1(T) \geq 2) = \frac{1}{\pi_k} \frac{P_u(1,k)}{P_u(1,1)} \left( 1 - \mathbf{P}^{\text{pru}}_{u,k}(Z_1(T) = 1) \right), \]

then we use Corollary 3.3 to conclude that
\[ \frac{1}{\pi_k} \frac{P_u(1,k)}{P_u(1,1)} \left( 1 - \mathbf{P}^{\text{pru}}_{u,k}(Z_1(T) = 1) \right) = \frac{1}{\pi_k} \frac{P_u(1,k)}{P_u(1,1)} \left( 1 - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)} \frac{P_u(1,1)}{P_{u-1}(1,1)} \right) = \frac{1}{\pi_k} \left( \frac{P_u(1,k)}{P_u(1,1)} - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)} \right). \]
(2) In the proof of Part 1, we deduced that

$$P_k^{st}(H(T) = u) = \frac{1}{\pi_k} P_{u}(1,k) P_{u,k}^{\text{pru}}(Z_1(T) \geq 2),$$

and the conclusion follows from Corollary 3.4 with \( r = 2 \).

\[ \square \]

**Remark 3.11.** As a byproduct of Corollary 3.10, we have that

$$\frac{1}{\pi_k} \left( \frac{P_u(1,k)}{P_u(1,1)} - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)} \right) = (1 + o(1)) \gamma^{-2} f''(q) \sum_{1 \leq i < k-1} \pi_i \pi_{k-i} P_u(1,1).$$

Together with the asymptotic of \( P_u(1,1) \) in Proposition 2.1, we deduce that

$$\frac{P_u(1,k)}{P_u(1,1)} - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)} = \begin{cases} (C_4 + o(1)) u^{-2}, & m = 1, \\ (C_5 + o(1)) \gamma^n, & m \neq 1, \end{cases}$$

where

$$C_4 = \gamma^{-2} f''(q) \sum_{1 \leq i < k-1} \pi_i \pi_{k-i}, \quad C_5 = \frac{1}{2} \gamma^{-2} v_1 f''(q) \sum_{1 \leq i \leq k-1} \pi_i \pi_{k-i},$$

with \( \sigma^2, v_1 \) defined in Proposition 2.1.

4. **Application to branching random walks**

As we shall deal with trees without fixed heights in this section, we abbreviate \( R(T) = R_{H(T)}(T) \) and \( g'(T) = g'_{H(T)}(T) \). By (2.3) and Proposition 3.8, for the span we have that

$$\lim_{n \to \infty} \mathbb{P}(R_n > x \mid Z_n = k)$$

$$= \lim_{n \to \infty} \mathbb{P}(R(\text{cut}_n(\text{pru}_n T)) > x \mid Z_n = k)$$

$$= \lim_{n \to \infty} \mathbb{P}_{n,k}^{st}(R(T) > x) = \mathbb{P}_k^{st}(R(T) > x),$$

and it is idem for the gaps. In other words, \( R_n \) and \( (g_n') \) under \( \mathbb{P}(\cdot \mid Z_n = k) \) converge to \( R(T) \) and \( (g'(T)) \) under \( \mathbb{P}_k^{st}(\cdot) \) as \( n \to \infty \). Thus to prove Theorem 1.2 it suffices to study the span and gaps under \( \mathbb{P}_k^{st} \).

4.1. **The span.** Take any tree \( T \in \mathcal{T}_{n,k} \cap \{ Z_1(T) \geq 2 \} \) under \( \mathbb{P}_k^{st} \), we divide the span \( R_n(T) \) into two parts: the span of the first (in lexicographical order) node in the last layer of each subtree is denoted by

$$S_n(T) \ := \ \text{the span of } \{ 1 \leq i \leq Z_1(T) : V_{i1} \cdots V_{in}^{\text{length } n} (T) \},$$

and the maximum span among each subtree is denoted by

$$G_n(T) \ := \ \max_{1 \leq i \leq Z_1(T)} \{ R_{n-1}(T[i]) \}.$$
For simplicity, since trees under $P_{st}^k$ do not have a fixed height, we write $S(T)$, $G(T)$, $R(T)$ for $S_{H(T)}(T)$, $G_{H(T)}(T)$, $R_{H(T)}(T)$.

**Lemma 4.1.** Fix $k \geq 2$. Under the conditions (1.1) and (1.2), as $x \to \infty$,

$$P_{st}^k(S(T) > x) = \begin{cases} (C_1 + o(1))x^{-2}, & m = 1, \\ \exp(-(C_2 + o(1))x), & m \neq 1, \end{cases}$$

where

$$C_1 = \frac{f''(q)}{\gamma^2 \sigma^2} \sum_{1 \leq i \leq k-1} \pi_i \pi_{k-i},$$

$$C_2 = \inf_{s \in (0, \infty)} \left( -s \log \gamma + \sup_{t \in \mathbb{R}} (t - s\Lambda(t)) \right),$$

and all parameters appearing in the $C_1, C_2$ are those in Proposition 2.1 and Lemma 2.2.

**Proof.** Denote by $X^{(j)}_i$ independent random variables distributed as $\theta$ for all $i, j \in \mathbb{N}$. Denote

$$F(m, u, x) := P \left( \max_{1 \leq a, b \leq m} \left\{ \sum_{i=1}^{u} X^{(a)}_i - \sum_{j=1}^{u} X^{(b)}_j \right\} > x \right),$$

then

$$P_{st}^k(S(T) > x) = E_{st}^k(F(Z_1(T), H(T), x)) = \sum_{u \geq 1} P_{st}^k(H(T) = u)E_{st}^k(F(Z_1(T), u, x) \mid H(T) = u).$$

Moreover, by the union bound,

$$F(m, u, x) \leq m^2 F(2, u, x),$$

thus

$$E_{st}^k(F(Z_1(T), H(T), x)) - \sum_{u \geq 1} P_{st}^k(H(T) = u)F(2, u, x)$$

(4.16)

$$\leq \sum_{u \geq 1} P_{st}^k(H(T) = u) \cdot k^2 F(2, u, x)P_{st}^k(Z_1(T) \geq 3 \mid H(T) = u).$$

Then by Corollary 3.9, the error term in (4.16) is negligible, so

$$P_{st}^k(S(T) > x) = E_{st}^k(F(Z_1(T), H(T), x)) = (1+o(1)) \sum_{u \geq 1} P_{st}^k(H(T) = u)F(2, u, x).$$
Thus it suffices to show that
\[
\sum_{u \geq 1} P_k^u(H(T) = u)F(2, u, x) = \begin{cases} (C_1 + o(1))x^{-2}, & m = 1, \\ \exp(-(C_2 + o(1))x), & m \neq 1. \end{cases}
\]

If \( m = 1 \), by Part 2 of Corollary \ref{corollary:2} Part 3 of Proposition \ref{proposition:2} and Part 1 of Lemma \ref{lemma:2}, we have that
\[
\sum_{u \geq 1} P_k^u(H(T) = u)F(2, u, x)
\]
\[= \sum_{u > x^{3/2}} C_1 + o(1) \left[ 2 \left(1 - \Phi\left(\frac{x}{\sqrt{2u}}\right)\right) + O\left(\frac{1}{\sqrt{u}}\right)\right] + \sum_{u \leq x^{3/2}} o\left(\frac{1}{u^2} \cdot \frac{u}{x^2 \log x}\right)
\]
\[= (2C_1 + o(1)) \int_{x^{3/2}}^{\infty} \frac{1-\Phi(x/\sqrt{2y})}{y^{2}} dy + o(x^{-2})
\]
\[= \frac{2C_1 + o(1)}{x^2} \int_{0}^{\infty} \frac{1-\Phi(1/\sqrt{2z})}{z^{2}} dz = \frac{2C_1 + o(1)}{x^2}.
\]

If \( m \neq 1 \), similarly, we can choose suitable constants \( C, s_1, s_2 \) such that
\[
\sum_{u \geq 1} P_k^u(H(T) = u)F(2, u, x)
\]
\[= \sum_{s_1 x < u < s_2 x} (C + o(1)) \gamma u^{-(1+o(1))} \sup_{t \in \mathbb{R}} \left( (tx-u\Lambda(t)) + O(R(2, s_1 x, x) + \gamma^s x) \right)
\]
\[= e^{-(C_2 + o(1))x}.
\]

We remark that since
\[
\lim_{s \to +\infty} \left(-s \log \gamma + \sup_{t \in \mathbb{R}} \left( -s \log \gamma + (0 - s\Lambda(0)) \right) \right) \geq \lim_{s \to +\infty} \left(-s \log \gamma + (-s\log \gamma) \right) = +\infty,
\]
and
\[
\lim_{s \to 0+} \left(-s \log \gamma + \sup_{t \in \mathbb{R}} \left( -s \log \gamma + \sup_{t \in \mathbb{R}} (t - s\Lambda(t)) \right) \right) = \lim_{s \to 0+} \sup_{t \in \mathbb{R}} (t - s\Lambda(t)) = +\infty,
\]
the infimum over \((0, \infty)\) in \( C_2 \) is equivalent to the infimum among a bounded interval \([\epsilon, \epsilon^{-1}]\).

**Proposition 4.2.** Fix \( k \geq 2 \). Under the conditions \((1.1)\) and \((1.2)\), as \( x \to \infty \),
\[
P_k^u(R(T) > x) = \begin{cases} (C_1 + o(1))x^{-2}, & m = 1, \\ \exp(-(C_2 + o(1))x), & m \neq 1, \end{cases}
\]
where \( C_1, C_2 \) are those in Lemma \ref{lemma:4}.1

**Proof.** By \((4.15)\) and Lemma \ref{lemma:4.1} it suffices to show that
\[
(4.17) \quad P_k^u(G(T) > x^{1-\epsilon}) = o(P_k^u(S(T) > x))
\]
for some \( \epsilon > 0 \).

In fact, \( G(T) \) is determined by the structures of \( \text{cut}(T[i]) \). If we denote by
\[
\tilde{H}(T) := \max_{1 \leq i \leq Z_1(T)} H(\text{cut}(T[i])),
\]
then by the union bound, $G(T)$ is determined by at most $k^2$ pairs of nodes within the same subtrees, in other words,

$$
\mathbf{P}_k^{st}(G(T) > x^{1-\epsilon}) \leq k^2 \mathbf{P}_k^{st}\left[ \max_{1 \leq u \leq \bar{H}(T)} F(2, u, x^{1-\epsilon}) \right] \\
\asymp \sum_{u \geq 1} \mathbf{P}_k^{st}(\bar{H}(T) = u) F(2, u, x^{1-\epsilon}).
$$

Sum over all possible cases by (3.12), notice that integer values at most $r$ and $(k_i)$ can only take

to sum up,

$$
\mathbf{P}_k^{st}(G(T) > x^{1-\epsilon}) \lesssim \sum_{u \geq 1} \mathbf{P}_u(1, 1) \mathbf{P}_k^{st}(H(T) = u). \\
$$

To sum up,

$$
\mathbf{P}_k^{st}(G(T) > x^{1-\epsilon}) \lesssim \sum_{u \geq 1} \mathbf{P}_u(1, 1) \mathbf{P}_k^{st}(H(T) = u) F(2, u, x^{1-\epsilon}),
$$

while

$$
\mathbf{P}_k^{st}(S(T) > x) \asymp \sum_{u \geq 1} \mathbf{P}_k^{st}(H(T) = u) F(2, u, x).
$$

Therefore, by the proof of Lemma 4.1, we have the desired dominance in (4.17).

4.2. The gaps.

**Proposition 4.3.** Let $k \geq 2$ and $1 \leq i \leq k - 1$. Under 1.1 and 1.2, as $x \to \infty$,

$$
\mathbf{P}_k^{st}(g^i(T) > x) = (C_3 + o(1)) \mathbf{P}_k^{st}(R(T) > x) \\
= \begin{cases} 
(C_1 C_3 + o(1)) x^{-2}, & m = 1, \\
\exp(-C_2 + o(1)) x, & m \neq 1,
\end{cases}
$$

where $C_3 = \frac{\tau_1 \tau_{k-1}}{\sum_{i=1}^{k-1} 3^{k-1}}$.

**Proof.** Recall that trees under $\mathbf{P}_k^{st}$ have $k$ nodes in the last generation, and we write their positions in increasing order,

$$
V^{(1)}(T) \leq \cdots \leq V^{(k)}(T).
$$

In the previous section, we showed that

$$
\mathbf{P}_k^{st}(Z_1(T) \geq 3, R(T) > x) = o(\mathbf{P}_k^{st}(R(T) > x)), \\
\mathbf{P}_k^{st}(G(T) > x^{1-\epsilon}) = o(\mathbf{P}_k^{st}(R(T) > x)),
$$

thus it suffices to consider the case where

$$
\{V^{(1)}(T), \cdots, V^{(i)}(T)\} \text{ and } \{V^{(i+1)}(T), \cdots, V^{(k)}(T)\}
$$

are exactly positions of nodes in the last generation of the two subtrees $T[1], T[2]$. In other words,

$$
\mathbf{P}_k^{st}(g^i(T) > x) \\
= \frac{1}{2} \mathbf{P}_k^{st}(S(T) > x, \#T[1] = i, \#T[2] = k - i) \\
+ \frac{1}{2} \mathbf{P}_k^{st}(S(T) > x, \#T[1] = k - i, \#T[2] = i) + o(\mathbf{P}_k^{st}(R(T) > x)),
$$

where
where the factors $\frac{1}{2}$ are to distinguish the symmetric cases $g^i(T) > x$ and $g^{k-i}(T) > x$.

Moreover, by Proposition 3.8 we have that
\[
\begin{align*}
\mathbb{P}_k^x(S(T) > x, \#T[1] = i, \#T[2] = k - i) \\
= \sum_{u \geq 1} F(2, u, x) \mathbb{P}_k^x(H(T) = u, \#T[1] = i, \#T[2] = k - i) \\
= (C_3 + o(1)) \sum_{u \geq 1} F(2, u, x) \mathbb{P}_k^x(H(T) = u),
\end{align*}
\]
and the conclusion follows from the proof of Lemma 4.1. □

**Remark 4.4.** As an example, consider the canonical case where the offspring distribution $\mu$ is the geometric distribution with parameter $\frac{1}{2}$, i.e. $\mu(k) = 2^{-k-1}$, then one can explicitly show (cf. e.g. [4, Section 1.4]) that its generating function satisfies
\[
f_n(s) = \sum_{i=0}^{\infty} P(Z_n = i) s^i = 1 - \frac{1}{n + \frac{1}{1-s}},
\]
and its transition probabilities are
\[
P_n(1, k) = \frac{1}{k!} \left. \frac{d^k f_n(s)}{ds^k} \right|_{s=0} = \frac{n^{k-1}}{(n + 1)^{k+1}}.
\]

Therefore,
\[
\pi_k = \lim_{n \to \infty} P_n(1, k) = 1, \ \forall k \in \mathbb{N}_+.
\]

Thus in this case, the constant
\[
C_1C_3 = \frac{f''(q)}{\gamma^2 \sigma^2} \frac{\pi_1 \pi_{k-1}}{\pi_k \sum_{i=1}^{\infty} \pi_i (\mu(0))^i} = \frac{f''(q)}{\gamma^2 \sigma^2} \frac{1}{\sum_{i=1}^{\infty} (\mu(0))^i}
\]
in Proposition 4.3 no longer depends on the choice of $i$ or $k$, as is showed in [13].

Finally, we formally conclude that

**Proof of Theorem 1.3** The conclusion follows from (4.14) (with its counterpart for the gaps), Proposition 4.2 and Proposition 4.3. □

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