On the mean curvature of submanifolds with nullity

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Abstract

In this paper, we investigate geometric conditions for isometric immersions with positive index of relative nullity to be cylinders. There is an abundance of noncylindrical $n$-dimensional minimal submanifolds with index of relative nullity $n - 2$, fully described by Dajczer and Florit [5] in terms of a certain class of elliptic surfaces. Opposed to this, we prove that nonminimal $n$-dimensional submanifolds in space forms of any codimension are locally cylinders provided that they carry a totally geodesic distribution of rank $n - 2 \geq 2$, which is contained in the relative nullity distribution, such that the length of the mean curvature vector field is constant along each leaf. The case of dimension $n = 3$ turns out to be special. We show that there exist elliptic three-dimensional submanifolds in spheres satisfying the above properties. In fact, we provide a parametrization of three-dimensional submanifolds as unit tangent bundles of minimal surfaces in the Euclidean space whose first curvature ellipse is nowhere a circle and its second one is everywhere a circle. Moreover, we provide several applications to submanifolds whose mean curvature vector field has constant length, a much weaker condition than being parallel.

1 Introduction

A fundamental concept in the theory of submanifolds is the index of relative nullity introduced by Chern and Kuiper [4]. At a point $x \in M^n$ the index of relative nullity $\nu(x)$ of an isometric immersion $f: M^n \to Q^m_c$ is the dimension of the relative nullity tangent subspace $\Delta_f(x)$ of $f$ at $x$, that is the kernel of the second fundamental form $\alpha^f$ at that point. Here, $Q^m_c$ is the simply connected space form with curvature $c$, that is, the Euclidean space $\mathbb{R}^m$, the sphere $S^m$ or the hyperbolic space $\mathbb{H}^m$, according to whether

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c = 0, c = 1 or c = −1, respectively. The kernels form an integrable distribution along any open subset where the index is constant and the images under f of the leaves of the foliation are totally geodesic submanifolds in the ambient space.

Cylinders are the simplest examples of submanifolds with positive index of relative nullity. An isometric immersion \( f : M^n \to \mathbb{R}^m \) is said to be a \( k \)-cylinder if the manifold \( M^n \) splits as a Riemannian product \( M^n = M^{n-k} \times \mathbb{R}^k \) and there is an isometric immersion \( g : M^{n-k} \to \mathbb{R}^{m-k} \) such that \( f = g \times \text{id}_{\mathbb{R}^k} \). A frequent theme in submanifold theory is to find geometric conditions for an isometric immersion with index of relative nullity \( \nu \geq k > 0 \) at any point to be a \( k \)-cylinder.

A fundamental result asserting that an isometric immersion \( f : M^n \to \mathbb{R}^m \) of a Riemannian manifold with positive index of relative nullity must be a cylinder is Hartman’s theorem \([21]\) that requires the Ricci curvature of \( M^n \) to be nonnegative. Even for hypersurfaces, the same conclusion does not hold if instead we assume that the Ricci curvature is nonpositive. Notice that the latter is always the case if \( f \) is a minimal immersion. Counterexamples easy to construct are the complete irreducible ruled hypersurfaces of any dimension discussed in \([7, p. 409]\).

The cylindricity of minimal submanifolds was studied in \([23, 8]\) under global assumptions. These results are truly global in nature since there are plenty of (noncomplete) examples of minimal submanifolds of any dimension \( n \) with constant index \( \nu = n - 2 \) that are not part of a cylinder on any open subset. They can be all locally parametrically described in terms of a certain class of elliptic surfaces (see \([3, \text{Th. 22}]\)). Some of the many papers containing characterizations of submanifolds as cylinders without the requirement of minimality are \([6, 20, 21, 26]\).

In this paper, we deal with nonminimal \( n \)-dimensional submanifolds of arbitrary codimension and index of relative nullity \( \nu \geq n - 2 \) at any point. Our aim is to provide geometric conditions, in terms of the mean curvature, for an isometric immersion to be a cylinder. The choice of the geometric condition is inspired by the observation that cylinders are endowed with a totally geodesic distribution contained in the relative nullity distribution, such that the mean curvature is constant along each leaf. Throughout the paper, the mean curvature of an isometric immersion \( f \) is defined as the length \( H = \|H\| \) of the mean curvature vector field given by \( H = \text{trace}(\alpha^f)/n \).

The following result provides a characterization of cylinders of dimension \( n \geq 4 \).

**Theorem 1.** Let \( f : M^n \to \mathbb{Q}^{n+p}, n \geq 4 \), be an isometric immersion such that \( M^n \) carries a totally geodesic distribution \( D \) of rank \( n - 2 \) satisfying \( D(x) \subseteq \Delta_f(x) \) for any \( x \in M^n \). If the mean curvature of \( f \) is constant along each leaf of \( D \), then \( f \) is minimal, or \( c = 0 \) and \( f \) is locally a \((n - 2)\)-cylinder over a surface on the open subset where the mean curvature is positive. Moreover, the submanifold is globally a cylinder if the leaves of \( D \) are complete.

It is interesting that the above theorem fails for substantial three-dimensional submanifolds of codimension \( p \geq 2 \). Being substantial means that the codimension can-
not be reduced. We show that besides cylinders, there exist elliptic three-dimensional submanifolds in spheres satisfying the properties assumed in Theorem 1. Thus the submanifolds being three-dimensional is special. The notion of elliptic submanifolds was introduced in [5]. In fact, the following result allows a parametrization of them in terms of minimal surfaces in the Euclidean space, the so-called bipolar parametrization, using the following construction.

Let \( g: L^2 \to \mathbb{R}^{n+1}, n \geq 5, \) be a minimal surface. The map \( \Phi_g: T^1L \to S^n \) defined on the unit tangent bundle of \( L^2 \) and given by

\[
\Phi_g(x, w) = g_x w
\]

parametrizes (outside singular points) an immersion with index of relative nullity at least one at any point.

**Theorem 2.** Let \( f: M^3 \to \mathbb{Q}^{3+p} \) be an isometric immersion such that \( M^3 \) carries a totally geodesic distribution \( D \) of rank one satisfying \( D(x) \subseteq \Delta_f(x) \) for any \( x \in M^3 \). If the mean curvature of \( f \) is constant along each integral curve of \( D \), then one of the following holds:

(i) The immersion \( f \) is minimal.

(ii) \( c = 0 \) and \( f \) is locally a cylinder over a surface.

(iii) \( c = 1 \) and the immersion \( f \) is elliptic and locally parametrized by (1), where \( g: L^2 \to \mathbb{R}^{n+1}, n \geq 5, \) is a minimal surface whose first curvature ellipse is nowhere a circle and the second curvature ellipse is everywhere a circle.

Minimal surfaces satisfying the conditions in part (iii) of the above theorem can be constructed using the Weierstrass representation by choosing appropriately the holomorphic data. It is worth noticing that minimal surfaces in the Euclidean space that satisfy the Ricci condition, or equivalently are locally isometric to a minimal surface in \( \mathbb{R}^3 \), fulfill these conditions (see Section 6 for details). These surfaces were classified by Lawson [25].

The above results allow us to provide applications to submanifolds with constant mean curvature and not necessarily constant but positive index of relative nullity.

Having constant mean curvature is a much weaker restriction on the mean curvature vector field than being parallel in the normal bundle. One can check that three-dimensional elliptic submanifolds described in Theorem 2 do not have parallel mean curvature vector field along the totally geodesic distribution. Combining this with Theorem 1 it follows that a submanifold is locally a cylinder provided that it carries a totally geodesic distribution of rank \( n - 2 \geq 1 \) that is contained in the relative nullity distribution, along which the mean curvature vector field is parallel in the normal connection.

Opposed to the fact that there is an abundance of noncylindrical \( n \)-dimensional minimal submanifolds with index of relative nullity \( n - 2 \) (see [5]), we prove the following result for submanifolds with constant and positive mean curvature.
Theorem 3. Let $f : M^n \to \mathbb{Q}^{n+p}_c, n \geq 3$, be a nonminimal isometric immersion with index of relative nullity $\nu \geq n - 2$ at any point. If the mean curvature of $f$ is constant and either $n \geq 4$ or $n = 3$ and $p = 1$, then $c = 0$. Moreover, there exists an open dense subset $V \subseteq M^n$ such that every point of which has a neighborhood $U \subseteq V$ so that $f(U)$ is an open subset of the image of a cylinder over a surface in $\mathbb{R}^{p+2}$, or over a curve in $\mathbb{R}^{p+1}$ with constant first Frenet curvature.

The following is an immediate consequence of the above result due to real analyticity of hypersurfaces with constant mean curvature.

Corollary 4. Let $f : M^n \to \mathbb{Q}^{n+1}_c, n \geq 3$, be a nonminimal isometric immersion with index of relative nullity $\nu \geq n - 2$. If the mean curvature of $f$ is constant, then $c = 0$ and $f(M)$ is an open subset of the image of a cylinder over a surface in $\mathbb{R}^{3}$ of constant mean curvature.

The next result extends Corollary 1 in [3] for hypersurfaces in every space form without any global assumption.

Corollary 5. Let $f : M^n \to \mathbb{Q}^{n+1}_c, n \geq 3$, be an isometric immersion with constant mean curvature. If $M^n$ has sectional curvature $K \leq c$, then either $f$ is minimal, or $c = 0$ and $f(M)$ is an open subset of the image of a cylinder over a surface in $\mathbb{R}^{3}$ of constant mean curvature. In the latter case, $f$ is a cylinder over a circle provided that $M^n$ is complete.

The following rigidity result that was proved in [6] for $c = 0$ is another consequence of our main results.

Corollary 6. Any nonminimal isometric immersion $f : M^n \to \mathbb{Q}^{n+1}_c, n \geq 3$, with constant mean curvature is rigid, unless $c = 0$ and $f(M)$ is an open subset of the image of a cylinder over a surface in $\mathbb{R}^{3}$ of constant mean curvature.

Our next result extends to any dimension a well-known theorem for constant mean curvature surfaces due to Klotz and Osserman [24] (see [2] for another extension).

Theorem 7. Let $f : M^n \to \mathbb{Q}^{n+1}_c, n \geq 3$, be an isometric immersion with constant mean curvature, where $c = 0$ or $c = 1$. If $M^n$ is complete and its extrinsic curvature does not change sign, then $f$ is either minimal, totally umbilical or a cylinder over a sphere of dimension $1 \leq k < n$.

For submanifolds with constant mean curvature of codimension two, we prove the following.
**Theorem 8.** Let \( f: M^n \to \mathbb{R}^{n+2}, n \geq 3, \) be a nonminimal isometric immersion with constant mean curvature. If the sectional curvature of \( M^n \) is nonpositive, then there exists an open dense subset \( V \subseteq M^n \) such that every point of which has a neighborhood \( U \subseteq V \) where one of the following holds:

(i) The neighborhood \( U \) splits as a Riemannian product \( U = M^2 \times W^{n-2} \) such that \( f|_U = g \times j \) is a product, where \( g: M^2 \to \mathbb{R}^4 \) is a surface with constant mean curvature and \( j: W^{n-2} \to \mathbb{R}^{n-2} \) is the inclusion.

(ii) The immersion on \( U \) is a composition \( f|_U = h \circ F \), where \( h = \gamma \times \text{id}_{\mathbb{R}^{n-1}}: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+2} \) is cylinder over a unit speed plane curve \( \gamma(s) \) with curvature \( k(s) \) and \( F: M^n \to \mathbb{R}^{n+1} \) is a hypersurface. Moreover, the mean curvature \( H_F \) of \( F \) is given by

\[
H_F^2 = H_j^2 \frac{1}{n^2} k^2 - F_a (1 - \langle \xi, a \rangle ^2)^2,
\]

where \( F_a \) and \( \langle \xi, a \rangle \) are the height functions, with respect to \( a = \partial/\partial s \), of \( F \) and its Gauss map \( \xi \), respectively.

(iii) The neighborhood \( U \) splits as a Riemannian product \( U = M_1^2 \times M_2^2 \times W^{n-4} \) such that \( f|_U = g_1 \times g_2 \times j \) is a product, where \( g_i: M_i^2 \to \mathbb{R}^3, i = 1, 2, \) are surfaces with constant mean curvature and \( j: W^{n-4} \to \mathbb{R}^{n-4} \) is the inclusion.

For constant sectional curvature submanifolds with constant mean curvature of codimension two, we prove the following theorem that extends results in [14, 11].

**Theorem 9.** Let \( f: M^n_c \to \mathbb{Q}^{n+2}_c, n \geq 3, \) be an isometric immersion of a Riemannian manifold of constant sectional curvature \( \tilde{c} \). If the mean curvature of \( f \) is constant and \( n \geq 4 \), or \( n = 3 \) and \( c = \tilde{c} \), then one of the following holds:

(i) \( f \) is totally geodesic or totally umbilical.

(ii) \( \tilde{c} = c = 0 \) and \( f = g \times j \), where \( g: M^2 \to \mathbb{R}^4 \) is a flat surface with constant mean curvature and \( j: W \to \mathbb{R}^{n-2} \) is an inclusion.

(iii) \( \tilde{c} = 0, c = -1 \) and \( f \) is a composition \( f = i \circ F \), where \( i: \mathbb{R}^{n+1} \to \mathbb{H}^{n+2} \) is the inclusion as a horosphere and \( F: M^n_c \to \mathbb{R}^{n+1} \) is cylinder over a circle.

Cylinder theorems for complete minimal Kähler submanifolds were proved in [9, 19]. For Kähler submanifolds with constant mean curvature, we prove the following results.

**Theorem 10.** Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 4, \) be an isometric immersion with constant mean curvature. If \( M^n \) is Kähler, then \( f \) is either minimal or \( f(M) \) is an open subset of the image of a cylinder over a surface in \( \mathbb{R}^3 \) with constant mean curvature.

**Theorem 11.** Let \( f: M^n \to \mathbb{R}^{n+2}, n \geq 4, \) be a nonminimal isometric immersion of a Kähler manifold \( M^n \) with constant mean curvature. If the Ricci curvature or the holomorphic curvature of \( M^n \) is nonpositive, then there exists an open dense subset \( V \subseteq M^n \) such that every point of which has a neighborhood \( U \subseteq V \) where \( f|_U \) is as in Theorem 8.
The paper is organized as follows: In Section 2, we recall well-known results about the relative nullity distribution, totally geodesic distributions that are contained in the relative nullity distribution, as well as results about their splitting tensor. In Section 3, we fix the notation, give some preliminaries and prove auxiliary results that will be used in the proofs of our main theorems. Section 4 is devoted to the proof of Theorem 1. In Section 5, we recall the notion of elliptic submanifolds, as well as the associated notion of higher curvature ellipses. We also discuss the polar and bipolar surfaces of elliptic submanifolds. In Section 6, we study the case of three-dimensional submanifolds. We provide a parametrization for these submanifolds in terms of certain elliptic surfaces, the so-called polar parametrization (see Theorem 21). Based on this, we give the proof of Theorem 2. We conclude this section by showing that minimal surfaces in the Euclidean space that are locally isometric to a minimal surface in $\mathbb{R}^3$ satisfy the conditions in part (iii) of Theorem 2. In Section 7, we prove Theorem 3 and the applications of our main results on submanifolds with constant mean curvature. In addition, we provide examples of submanifolds as in part (ii) of Theorems 8 and 9.

2 The relative nullity distribution

In this section, we recall some basic facts from the theory of isometric immersions that will be used throughout the paper.

Let $M^n, n \geq 3$, be a Riemannian manifold and let $f: M^n \rightarrow Q^m_c$ be an isometric immersion into a space form $Q^m_c$. The relative nullity subspace $\Delta_f(x)$ of $f$ at $x \in M^n$ is the kernel of its second fundamental form $\alpha_f: T(TM) \rightarrow N_fM$ with values in the normal bundle, that is,

$$\Delta_f(x) = \{X \in T_xM : \alpha_f(X, Y) = 0 \text{ for all } Y \in T_xM\}.$$  

The dimension $\nu(x)$ of $\Delta_f(x)$ is called the index of relative nullity of $f$ at $x \in M^n$.

A smooth distribution $D \subset TM$ on $M^n$ is totally geodesic if $\nabla_T S \in \Gamma(D)$ whenever $T, S \in \Gamma(D)$. Let $D$ be a smooth distribution on $M^n$ and $D^\perp$ denote the distribution on $M^n$ that assigns to each $x \in M^n$ the orthogonal complement of $D(x)$ in $T_xM$. We write $X = X^v + X^h$ according to the orthogonal splitting $TM = D \oplus D^\perp$ and denote $\nabla_X^h Y = (\nabla_X Y)^h$ for all $X, Y \in TM$, where $\nabla$ is the Levi-Civitá connection on $M^n$. The splitting tensor $C: D \times D^\perp \rightarrow D^\perp$ is given by

$$C(T, X) = -\nabla_X^h T$$

for any $T \in D$ and $X \in D^\perp$.

When $D$ is a totally geodesic distribution such that $D(x) \subseteq \Delta_f(x)$ for all $x \in M^n$, the following differential equation for the tensor $C_T = C(T, \cdot)$ is well-known to hold (cf. [12] or [7]):

$$\nabla_S^h C_T = C_T \circ C_S + C_{\nabla_S^h T} + c(S, T)I,$$

(2)
where $I$ is the identity endomorphism of $D^\perp$. Here $\nabla^h_S C_T \in \Gamma(\text{End}(D^\perp))$ is defined by

$$(\nabla^h_S C_T) X = \nabla^h_S C_T X - C_T \nabla^h_S X$$

for all $T, S \in D$ and $X \in D^\perp$. The Codazzi equation gives

$$\nabla_T A_\xi = A_\xi \circ C_T + A_\nabla T_\xi$$

for any $T \in D$, where the shape operator $A_\xi$ with respect to the normal direction $\xi$ is restricted to $D^\perp$ and $\nabla^\perp$ stands for the normal connection of $f$. In particular, the endomorphism $A_\xi \circ C_T$ of $D^\perp$ is symmetric, that is,

$$A_\xi \circ C_T = C^T_\xi \circ A_\xi.$$  

For later use, we recall the following well-known results (cf. [12]).

**Proposition 12.** Let $f : M^n \to \mathbb{Q}_c^n$ be an isometric immersion such that $M^n$ carries a smooth totally geodesic distribution $D$ of rank $0 < k < n$ satisfying $D(x) \subseteq \Delta_f(x)$ for all $x \in M^n$. If the splitting tensor $C$ vanishes, then $c = 0$ and $f$ is locally a $k$-cylinder.

**Proposition 13.** For an isometric immersion $f : M^n \to \mathbb{Q}_c^n$, the following assertions hold:

(i) The index of relative nullity $\nu$ is upper semicontinuous. In particular, the subset

$$M_0 = \{ x \in M^n : \nu(x) = \nu_0 \},$$

where $\nu$ attains its minimum value $\nu_0$ is open.

(ii) The relative nullity distribution $x \mapsto \Delta_f(x)$ is smooth on any subset of $M^n$ where $\nu$ is constant.

(iii) If $U \subseteq M^n$ is an open subset where $\nu$ is constant, then $\Delta_f$ is a totally geodesic (hence integrable) distribution on $U$ and the restriction of $f$ to each leaf is totally geodesic.

3 Auxiliary results

The aim of this section is to prove several lemmas that will be used in the proofs of our main results.

Throughout this section, we assume that $f : M^n \to \mathbb{Q}_c^{n+p}, n \geq 3$, is a nonminimal isometric immersion such that $M^n$ carries a smooth totally geodesic distribution $D$ of rank $n - 2$ satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^n$. We also suppose that the mean curvature of $f$ is constant along each leaf of $D$.

Hereafter we work on the open subset where the mean curvature is positive and we choose a local orthonormal frame $\xi_{n+1}, \ldots, \xi_{n+p}$ in the normal bundle $N_f M$, such that
\(\xi_{n+1}\) is collinear to the mean curvature vector field. We also choose a local orthonormal frame \(e_1, \ldots, e_n\) in the tangent bundle \(TM\) such that \(e_1, e_2\) span \(D^\perp\) and diagonalize \(A_{\xi_{n+1}}|_{D^\perp}\), where \(A_{\xi_{n+1}}\) denotes the shape operator of \(f\) with respect to \(\xi_{n+1}\). Then, we have \(A_{\xi_{n+1}}e_i = k_i e_i, \; i = 1, 2\), and consequently the mean curvature is given by \(nH = k_1 + k_2\), where \(k_1, k_2\) are the principal curvatures.

Since the mean curvature is positive, at least one of the principal curvatures \(k_1\) and \(k_2\) has to be different from zero. In the sequel, we assume without loss of generality, that \(k_1 \neq 0\) and we define the function

\[\rho = \frac{k_2}{k_1}.\]

On the open subset where the mean curvature is positive we have

\[k_1 = -\frac{nH}{\rho - 1} \quad \text{and} \quad k_2 = \frac{n\rho H}{\rho - 1}. \tag{5}\]

We use the above mentioned notation throughout the paper.

The following lemma gives the form of the splitting tensor.

**Lemma 14.** On the open subset where the mean curvature is positive, the splitting tensor is given by

\[C_T = \psi_1(T)L_1 + \psi_2(T)L_2\]

for any \(T \in \Gamma(D)\), where \(\psi_1, \psi_2\) are 1-forms dual to the vector fields \(\nabla_{e_2}e_2, \nabla_{e_1}e_2\), respectively, and \(L_1, L_2 \in \Gamma(\text{End}(D^\perp))\) are defined by \(L_1 e_1 = \rho e_1 = -L_2 e_2\) and \(L_1 e_2 = e_2 = L_2 e_1\). Moreover, the following hold:

\[T(k_1) = \rho k_1 \psi_1(T) + \sum_{\alpha = n+2}^{n+p} \langle \nabla_T^\perp \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_1 \rangle, \tag{6}\]

\[T(k_2) = k_2 \psi_1(T) - \sum_{\alpha = n+2}^{n+p} \langle \nabla_T^\perp \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_1 \rangle, \tag{7}\]

\[(k_1 - k_2)\omega(T) = k_2 \psi_2(T) + \sum_{\alpha = n+2}^{n+p} \langle \nabla_T^\perp \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_2 \rangle, \tag{8}\]

\[(k_1 - k_2)\omega(T) = -\rho k_1 \psi_2(T) + \sum_{\alpha = n+2}^{n+p} \langle \nabla_T^\perp \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_2 \rangle \tag{9}\]

for any \(T \in \Gamma(D)\), where \(\omega\) denotes the connection form given by \(\omega = \langle \nabla e_1, e_2 \rangle\).

**Proof:** From the Codazzi equation we have

\[
(\nabla_T A_{\xi_{n+1}}) e_i - (\nabla_{e_i} A_{\xi_{n+1}}) T = A_{\nabla_T^\perp \xi_{n+1}} e_i - A_{\nabla_{e_i}^\perp \xi_{n+1}} T
\]
for any $T \in \Gamma(D)$ and $i = 1, 2$. The above is equivalent to the following

$$T(k_1) = k_1 \langle \nabla e_1 e_1, T \rangle + \sum_{\alpha = n+2}^{n+p} \langle \nabla^\perp_{e_1} \xi_{n+1}, \xi_\alpha \rangle \langle A \xi_\alpha e_1, e_1 \rangle,$$

$$T(k_2) = k_2 \langle \nabla e_2 e_2, T \rangle - \sum_{\alpha = n+2}^{n+p} \langle \nabla^\perp_{e_2} \xi_{n+1}, \xi_\alpha \rangle \langle A \xi_\alpha e_1, e_1 \rangle,$$

$$(k_1 - k_2)\omega(T) = k_2 \langle \nabla e_1 e_2, T \rangle + \sum_{\alpha = n+2}^{n+p} \langle \nabla^\perp_{e_2} \xi_{n+1}, \xi_\alpha \rangle \langle A \xi_\alpha e_2, e_1 \rangle,$$

$$(k_1 - k_2)\omega(T) = k_1 \langle \nabla e_2 e_1, T \rangle + \sum_{\alpha = n+2}^{n+p} \langle \nabla^\perp_{e_1} \xi_{n+1}, \xi_\alpha \rangle \langle A \xi_\alpha e_1, e_2 \rangle.$$

Using the assumption that the mean curvature is constant along each leaf of the distribution $D$, the first two equations imply

$$\langle \nabla e_1 e_1, T \rangle = \rho \langle \nabla e_2 e_1, T \rangle$$

for any $T \in \Gamma(D)$. Additionally, the last two equations yield

$$\langle \nabla e_2 e_1, T \rangle = -\rho \langle \nabla e_1 e_2, T \rangle.$$

The above conclude the proof of the lemma.

**Lemma 15.** Let $e_r, r \geq 3$, be an orthonormal frame of the distribution $D$. Then the functions $u_r := \psi_1(e_r)$ and $v_r := \psi_2(e_r)$ satisfy

$$2\rho(u_r u_s + v_r v_s) - c\delta_{rs} = \frac{\rho - 1}{nH} \sum_{\alpha = n+2}^{n+p} \langle \nabla^\perp_{e_r} \xi_{n+1}, \xi_\alpha \rangle \langle A \xi_\alpha e_1, e_1 \rangle - \langle A \xi_\alpha e_1, e_2 \rangle$$

(10)

for all $r, s \geq 3$, where $\delta_{rs}$ is the Kronecker delta.

**Proof:** Using Lemma 14, we have

$$(\nabla^h_{e_r} C_{e_s}) = e_r(u_s)L_1 + e_r(v_s)L_2 + u_s \nabla^h_{e_r} L_1 + v_s \nabla^h_{e_r} L_2$$

(11)

for any $r, s \geq 3$. A direct computation yields

$$(\nabla^h_{e_r} L_1)e_1 = -(\nabla^h_{e_r} L_2)e_2 = e_r(\rho) e_1 + (\rho - 1)\omega(e_r) e_2,$$

(12)

$$(\nabla^h_{e_r} L_1)e_2 = (\nabla^h_{e_r} L_2)e_1 = (\rho - 1)\omega(e_r) e_1.$$

(13)
Equations (6) and (7) imply that
\[
e_r(\rho) = -\rho(\rho - 1)u_r + \frac{(\rho - 1)^2}{nH} \sum_{\alpha=n+2}^{n+p} \langle \nabla_{e_r}^{\perp} \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_1 \rangle.
\] (14)

From equation (2) we know that the splitting tensor satisfies
\[
(\nabla^{h_i}_{e_r} C_{e_s}) e_i = C_{e_s} \circ C_{e_r} e_i + C_{e_r e_s} e_i + c \delta_{rs} e_i
\] (15)
for any \( r, s \geq 3 \) and \( i = 1, 2 \).

Let \( \omega_{rs} \) be the connection form given by \( \omega_{rs} = \langle \nabla e_r, e_s \rangle \) for all \( r, s \geq 3 \). Using equations (11)-(14), we find that (15) for \( i = 1 \) is equivalent to
\[
\rho e_r(u_s) = \rho(2\rho - 1)u_r u_s - \rho v_r v_s - (\rho - 1) v_s \omega(e_r)
- u_s(\rho - 1)^2 \frac{n + p}{nH} \sum_{\alpha=n+2}^{n+p} \langle \nabla_{e_r}^{\perp} \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_1 \rangle + \rho \sum_{t \geq 3} \omega_{st}(e_r) u_t + c \delta_{rs}
\] (16)
and
\[
e_r(v_s) = \rho u_r v_s + u_s v_r - (\rho - 1) u_s \omega(e_r) + \sum_{t \geq 3} \omega_{st}(e_r) v_t
\] (17)
for all \( r, s \geq 3 \). Moreover, equation (15) for \( i = 2 \) implies that
\[
e_r(u_s) = u_r u_s - \rho v_r v_s + (\rho - 1) v_s \omega(e_r) + \sum_{t \geq 3} \omega_{st}(e_r) u_t + c \delta_{rs}
\] (18)
for all \( r, s \geq 3 \).

Combining (16) and (17), we obtain
\[
2\rho u_r u_s + \rho v_r v_s - c \delta_{rs} - v_s(\rho + 1) \omega(e_r) = u_s \frac{\rho - 1}{nH} \sum_{\alpha=n+2}^{n+p} \langle \nabla_{e_r}^{\perp} \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_1 \rangle.
\]

Using (5), equation (19) is written as
\[
(\rho + 1) \omega(e_r) = -\rho v_r - \frac{\rho - 1}{nH} \sum_{\alpha=n+2}^{n+p} \langle \nabla_{e_r}^{\perp} \xi_{n+1}, \xi_\alpha \rangle \langle A_{\xi_\alpha} e_1, e_2 \rangle
\] (19)
and now the desired equation (10) follows directly from the above two equations.

We recall that the first normal space \( N^f_1(x) \) of the immersion \( f \) at a point \( x \in M^n \) is the subspace of its normal space \( N_f M(x) \) spanned by the image of its second fundamental form \( \alpha^f \) at \( x \), that is,
\[
N^f_1(x) = \text{span} \{ \alpha^f(X, Y) : X, Y \in T_x M \}.
\]
The rank condition and the symmetry of the second fundamental form imply that \( \dim N^f_1(x) \leq 3 \) for all \( x \in M^n \).

We consider the open subset

\[
M_3 = \left\{ x \in M^n : \dim N^f_1(x) = 3 \right\}.
\]

Lemma 16. If the open subset \( M^*_3 := M_3 \setminus \{ x \in M^n : H(x) = 0 \} \) is nonempty, then the splitting tensor vanishes on it.

Proof: On the subset \( M^*_3 \), we consider the orthogonal splitting \( N^f_1 = \hat{N}^f_1 \oplus \text{span}\{H\} \). We choose the local frame such that \( \xi_{n+1} \) is collinear to the mean curvature vector field \( H \), and \( \xi_{n+2}, \xi_{n+3} \) span the plane bundle \( \hat{N}^f_1 \). Then, we have

\[
\text{trace} A_{\xi_{n+2}} |_{D^\perp} = 0 = \text{trace} A_{\xi_{n+3}} |_{D^\perp}.
\]

Hence, we obtain

\[
A_{\xi_{n+2}} |_{D^\perp} \circ J = J^t \circ A_{\xi_{n+2}} |_{D^\perp} \quad \text{and} \quad A_{\xi_{n+3}} |_{D^\perp} \circ J = J^t \circ A_{\xi_{n+3}} |_{D^\perp},
\]

where \( J \) denotes the unique, up to a sign, almost complex structure acting on the plane bundle \( D^\perp \).

Equation (4) implies that for any \( T \in \Gamma(D) \) we have

\[
A_{\xi_{n+2}} |_{D^\perp} \circ C_T = C_T^t \circ A_{\xi_{n+2}} |_{D^\perp} \quad \text{and} \quad A_{\xi_{n+3}} |_{D^\perp} \circ C_T = C_T^t \circ A_{\xi_{n+3}} |_{D^\perp}.
\]

Since \( \hat{N}^f_1 \) is a plane bundle, the above imply that \( C_T \in \text{span}\{I, J\} \subseteq \text{End}(D^\perp) \). This, combined with Lemma 14, yields

\[
(\rho - 1)\psi_1(T) = 0 \quad \text{and} \quad (\rho - 1)\psi_2(T) = 0
\]

for any \( T \in \Gamma(D) \). Thus, the splitting tensor vanishes identically on \( M^*_3 \) and this concludes the proof of the lemma.

Hereafter, we assume that \( M_3 \) is not dense on \( M^n \) and we consider the open subset

\[
M_2 = \left\{ x \in M^n \setminus \overline{M}_3 : \dim N^f_1(x) = 2 \right\}.
\]

In the sequel, we assume that the open subset \( M^*_2 := M_2 \setminus \{ x \in M^n : H(x) = 0 \} \) is nonempty. We choose a local orthonormal frame such that \( \xi_{n+1} \) and \( \xi_{n+2} \) span the plane bundle \( \hat{N}^f_1 \) on this subset and \( \xi_{n+1} \) is collinear to the mean curvature vector field. Thus, there exist smooth functions \( \lambda, \mu \) such that

\[
A_{\xi_{n+2}} e_1 = \lambda e_1 + \mu e_2, \quad A_{\xi_{n+2}} e_2 = \mu e_1 - \lambda e_2 \quad \text{and} \quad \lambda^2 + \mu^2 > 0.
\]

We proceed with some auxiliary lemmas.
Lemma 17. The plane bundle $N^f_1$ is parallel in the normal connection along the distribution $D$ on the subset $M^*_2$. Moreover, the following hold:

\begin{align*}
\mu \psi_1(T) &= -\lambda \psi_2(T), \\
\mu \phi(T) &= -\frac{(\lambda^2 + \mu^2) - 1}{nH} \psi_2(T), \\
T(\mu) + 2\lambda \omega(T) + (\rho + 1) \lambda \psi_2(T) &= 0, \\
T(\lambda) - 2\mu \omega(T) - \mu \rho \psi_2(T) - \lambda \psi_1(T) &= \frac{n\rho H}{\rho - 1} \phi(T), \\
T(\lambda) - 2\mu \omega(T) - \mu \psi_2(T) - \lambda \rho \psi_1(T) &= \frac{nH}{\rho - 1} \phi(T)
\end{align*}

for any $T \in \Gamma(D)$, where $\phi$ is the normal connection form given by $\phi = \langle \nabla_{\perp} \xi_{n+1}, \xi_{n+2} \rangle$.

Proof: Equation (3) implies that 

$$\langle \nabla_{\perp} T \xi, \xi \rangle = 0$$

if $\alpha = n + 1, n + 2$ for any $T \in \Gamma(D)$ and any $\xi \in \Gamma(N^f_1)$.

Thus, the subbundle $N^f_1$ is parallel in the normal connection along the distribution $D$.

Moreover, from (3) we have

$$(\nabla_{\perp} T A)_{\xi_{n+2}} e_i = A_{\xi_{n+2}} \circ C_T e_i + A_{\nabla_{\perp} \xi_{n+2}} e_i, \quad i = 1, 2,$$

for any $T \in \Gamma(D)$. Bearing in mind the form of the splitting tensor given in Lemma 14, the above equations yield directly (23), (24) and the following

\begin{align*}
T(\mu) + 2\lambda \omega(T) + \lambda \rho \psi_2(T) - \mu \psi_1(T) &= 0, \\
T(\mu) + 2\lambda \omega(T) - \mu \rho \psi_1(T) + \lambda \psi_2(T) &= 0
\end{align*}

for any $T \in \Gamma(D)$. Subtracting the above equations, we obtain (20). Equation (21) follows by subtracting (23), (24) and using (20). Finally, plugging (20) into the first of the above equations, we have (22) and this completes the proof.

We now suppose that the subset $M_3 \cup M_2$ is not dense on $M^n$ and we consider the open subset

$$M_1 = \left\{ x \in M^n \setminus (M_3 \cup M_2) : \dim N^f_1(x) = 1 \right\}.$$

Lemma 18. If the subset $M^*_1 := M_1 \setminus \{ x \in M^n : H(x) = 0 \}$ is nonempty, then $c = 0$ and $f|_{M^*_1}$ is locally a cylinder over a surface in $\mathbb{R}^{p+2}$ or over a curve in $\mathbb{R}^{p+1}$.
Proof: On the subset $M^*_1$ we choose a local orthonormal frame $\xi_{n+1}, \ldots, \xi_{n+p}$ in the normal bundle such that $\xi_{n+1}$ is collinear to the mean curvature vector field. Then we have $A_{\xi_n} = 0$ for all $\alpha \geq n + 2$. The Codazzi equation yields

$$A_{\nabla^\perp \xi_n} e_r = A_{\nabla^\perp e_r} \xi_n$$

for all $\alpha \geq n + 2$, $i = 1, 2$, and $r \geq 3$. Thus, we obtain $\nabla^\perp \xi_{n+1} = 0$ and Lemma 15 gives

$$2\rho(u_r u_s + v_r v_s) = c\delta_{rs}$$

(25)

for all $r \geq 3$. Moreover, equation (14) becomes

$$e_r(\rho) = -\rho(\rho - 1)u_r.$$

Differentiating (25) with respect to $e_r$ and using the above along with equations (17) and (18), we obtain

$$\rho u_r(\rho - 3)(u_r^2 + v_r^2) - 2\rho v_r^2 + 2\rho \sum_{s \geq 3} \omega_{rs}(e_r)(u_s u_r + v_s v_r) = 0$$

for all $r \geq 3$. In view of (25), the above equation simplifies to the following

$$c(\rho + 1)u_r = 0.$$

Now we prove that $c = 0$. Arguing indirectly, we suppose that $c \neq 0$. Assume that the open set of points where $\rho \neq -1$ is nonempty. On this subset, we have $u_r = 0$ for all $r \geq 3$. Thus, equation (25) becomes $2\rho v_r^2 = c$ for all $r \geq 3$. Using (19), equation (18) yields $2\rho^2 v_r^2 = c(\rho + 1)$, which is a contradiction. Assume now that the set of points where $\rho = -1$ has nonempty interior. On this subset, (6) yields $u_r = 0$ and equation (8) implies that $v_r = 0$, which contradicts the assumption that $c \neq 0$.

Hence, $c = 0$ and equation (25) becomes

$$\rho(u_r^2 + v_r^2) = 0$$

for all $r \geq 3$. If $\rho \neq 0$, then the splitting tensor vanishes and Proposition 12 implies that $f$ is locally a cylinder over a surface. If the subset of points where $\rho = 0$ has nonempty interior, then the Codazzi equation implies that the tangent bundle splits as an orthogonal sum of two parallel distributions one of which has rank $n - 1$. Thus, the manifold splits locally as a Riemannian product by the De Rham decomposition theorem. Since the second fundamental form is adapted to this splitting, the result follows from [12, Th. 8.4] and the proof is completed. \]
4 Submanifolds of dimension $n \geq 4$

We are now ready to give the proof of our first main result.

*Proof of Theorem 1:* If the open subset $M^*_3$ is nonempty, then Lemma 16 implies that the splitting tensor vanishes identically on it. Then, by Proposition 12 the immersion $f$ is locally a cylinder over a surface on $M^*_3$.

Now we assume that the subset $M^*_3$ is not dense on $M^*$ and suppose that $M^*_2$ is nonempty. Hereafter, we work on $M^*_2$. Due to the choice of the local orthonormal frame $\xi_{n+1}, \xi_{n+2}$ in the normal subbundle $N_1^f$, using (20) and (21), equation (10) of Lemma 15 takes the following form

$$v_r v_s \left( \lambda^2 + \mu^2 \right) \left( 2\rho - \left( \lambda^2 + \mu^2 \right) \frac{(\rho - 1)^2}{n^2 H^2} \right) = c\mu^2 \delta_{rs} \quad (26)$$

for any $r, s \geq 3$.

We claim that $v_r = 0$ for any $r \geq 3$. In fact, at points where

$$2\rho - \left( \lambda^2 + \mu^2 \right) \frac{(\rho - 1)^2}{n^2 H^2} \neq 0,$$

it follows from (26) that

$$v_r^2 = \frac{c\mu^2}{\left( \lambda^2 + \mu^2 \right) \left( 2\rho - \left( \lambda^2 + \mu^2 \right) \frac{(\rho - 1)^2}{n^2 H^2} \right)}$$

for any $r \geq 3$ and $v_r v_s = 0$ for $r \neq s \geq 3$. Thus, $v_r = 0$ for any $r \geq 3$ at those points.

It remains to prove that the same holds on the subset $U \subseteq M^*_2$ of points where

$$2\rho - \left( \lambda^2 + \mu^2 \right) \frac{(\rho - 1)^2}{n^2 H^2} = 0.$$

Notice that because of (5), the subset $U$ is the set of points where

$$\lambda^2 + \mu^2 = -2k_1 k_2. \quad (27)$$

In order to prove that $v_r = 0$ for any $r \geq 3$ on $U$, we assume that the interior of $U$ is nonempty. We suppose to the contrary that there exists $r_0 \geq 3$ such that $v_{r_0} \neq 0$ on an open subset of $U$. Differentiating (27) with respect to $e_{r_0}$ and using (22), (23), (6), (7) and (5), we obtain

$$\lambda^2 u_{r_0} - \lambda \mu v_{r_0} + (\rho + 1) k_1 k_2 u_{r_0} = \lambda (k_1 - 2k_2) \phi(e_{r_0}).$$

Multiplying by $\mu$ the above and using (21), we find that

$$\mu u_{r_0} \left( \lambda^2 + (\rho + 1) k_1 k_2 \right) = \lambda v_{r_0} \left( \mu^2 - \left( \lambda^2 + \mu^2 \right) (k_1 - 2k_2) \frac{\rho - 1}{nH} \right).$$
Taking into account (20), (5) and (27), the above yields
\[ \lambda v_r \rho + 1)(\lambda^2 + \mu^2) = 0. \]
Due to (27), we conclude that \( \lambda = 0 \) and consequently \( \mu \neq 0 \). Then, it follows from (20) that \( u_s = 0 \) for any \( s \geq 3 \). Equations (16) and (18) for \( s = r_0 \) imply that
\[ \rho v_r^2 + (\rho - 1)v_r \omega(e_{r_0}) - c = 0 \quad \text{and} \quad \rho v_r^2 - (\rho - 1)v_r \omega(e_{r_0}) - c = 0, \]
respectively. Hence \( \omega(e_{r_0}) = 0 \), and consequently (19) yields
\[ \rho v_r + \frac{\rho - 1}{nH} \mu \phi(e_{r_0}) = 0. \]
Using (21), (27) and (5) we find that \( \rho = 0 \), which contradicts (27). Thus we have proved the claim that \( v_r = 0 \) for any \( r \geq 3 \).

Now, we claim that \( u_r = 0 \) for any \( r \geq 3 \). Equation (20) implies that \( \mu u_r = 0 \) for any \( r \geq 3 \). Obviously, the function \( u_r \) vanishes at points where \( \mu \neq 0 \).

We assume that the set of points where \( \mu = 0 \) has nonempty interior and we argue on this subset. Since \( \lambda \neq 0 \) on this subset, it follows from (22) that \( \omega(e_r) = 0 \) for any \( r \geq 3 \), and consequently (23) and (24) yield
\[ \phi(e_r) = \frac{\rho - 1}{nH} \lambda u_r \quad \text{and} \quad e_r(\lambda) = (\rho + 1)\lambda u_r \]
for all \( r \geq 3 \). Using the first of the above equations, (10) is written equivalently as
\[ u_r^2 \left( 2 \rho - \frac{\lambda^2(\rho - 1)^2}{n^2H^2} \right) = c \]
for all \( r \geq 3 \).

Since we already proved that \( v_r = 0 \) for all \( r \geq 3 \), Lemma 14 implies that the image of the splitting tensor \( C: D \to \text{End}(D^\perp) \) satisfies \( \dim \text{Im} C \leq 1 \). Thus, \( \dim \ker C \geq n - 3 \).

We now suppose that \( \dim \ker C = n - 3 \). Then, there exists a unique \( r_0 \geq 3 \) such that \( u_{r_0} \neq 0 \) and \( u_s = 0 \) for any \( s \neq r_0 \). Thus, equation (29) implies that \( c = 0 \) and
\[ 2 \rho = \frac{\lambda^2(\rho - 1)^2}{n^2H^2}. \]
On account of (5), the above equation becomes \( \lambda^2 = -2k_1k_2 > 0 \). Differentiating this equation with respect to \( e_{r_0} \) and using the second of (28), (3), (7) and (5), we obtain \( 2\lambda^2 + k_1k_2 = 0 \), which contradicts the previous equation. Thus, the splitting tensor vanishes identically on the subset \( M_1^r \) and consequently, by Proposition 12, the immersion \( f \) is locally a cylinder over a surface.

If the open subset \( M_1^r \) is nonempty, then Lemma 18 implies that \( f \) is locally a cylinder over a surface or a curve and this completes the proof.
5 Elliptic submanifolds

In this section, we recall from [5] the notion of elliptic submanifolds of a space form as well as several of their basic properties.

Let \( f : M^n \to \mathbb{Q}^m \) be an isometric immersion. The \( \ell \)-th normal space \( N^f_\ell (x) \) of \( f \) at \( x \in M^n \) for \( \ell \geq 1 \) is defined as

\[
N^f_\ell (x) = \text{span} \left\{ \alpha^f_{\ell+1} (X_1, \ldots, X_{\ell+1}) : X_1, \ldots, X_{\ell+1} \in T_x M \right\}.
\]

Here \( \alpha^f_2 = \alpha^f \) and for \( s \geq 3 \) the so-called \( s \)-th fundamental form is the symmetric tensor \( \alpha^f_s : TM \times \cdots \times TM \to N^f_M \) defined inductively by

\[
\alpha^f_s (X_1, \ldots, X_s) = \pi^{s-1} \left( \nabla_{X_s} \cdots \nabla_{X_2} \alpha^f (X_2, X_1) \right),
\]

where \( \pi^k \) stands for the projection onto \( (N^f_1 \oplus \cdots \oplus N^f_{k-1})^\perp \).

An isometric immersion \( f : M^n \to \mathbb{Q}^m \) is called elliptic if \( M^n \) carries a totally geodesic distribution \( D \) of rank \( n - 2 \) satisfying \( D(x) \subseteq \Delta_f (x) \) for any \( x \in M^n \) and there exists an (necessary unique up to a sign) almost complex structure \( J : D^\perp \to D^\perp \) such that the second fundamental form satisfies

\[
\alpha^f (X, X) + \alpha^f (JX, JX) = 0
\]

for all \( X \in D^\perp \). Notice that \( J \) is orthogonal if and only \( f \) is minimal.

Assume that \( f : M^n \to \mathbb{Q}^m \) is substantial and elliptic. Assume also that \( f \) is nicely curved which means that for any \( \ell \geq 1 \) all subspaces \( N^f_\ell (x) \) have constant dimension and thus form subbundles of the normal bundle. Notice that any \( f \) is nicely curved along connected components of an open dense subset of \( M^n \). Then, along that subset the normal bundle splits orthogonally and smoothly as

\[
N_f M = N^f_1 \oplus \cdots \oplus N^f_{\tau_f},
\]

where all \( N^f_\ell \)'s have rank two, except possibly the last one that has rank one in case the codimension is odd. Thus, the induced bundle \( f^* T \mathbb{Q}^m \) splits as

\[
f^* T \mathbb{Q}^m = f_* D \oplus N^f_0 \oplus N^f_1 \oplus \cdots \oplus N^f_{\tau_f},
\]

where \( N^f_0 = f_* D^\perp \). Setting

\[
\tau^o_f = \begin{cases} 
\tau_f & \text{if } m - n \text{ is even} \\
\tau_f - 1 & \text{if } m - n \text{ is odd}
\end{cases}
\]

it turns out that the almost complex structure \( J \) on \( D^\perp \) induces an almost complex structure \( J_\ell \) on each \( N^f_\ell \), \( 0 \leq \ell \leq \tau^o_f \), defined by

\[
J_\ell \alpha^f_{\ell+1} (X_1, \ldots, X_\ell, X_{\ell+1}) = \alpha^f_{\ell+1} (X_1, \ldots, X_\ell, JX_{\ell+1}),
\]
where $\alpha^f_1 = f_*$.

The $\ell$th-order curvature ellipse $E^f_\ell(x) \subset N^f_\ell(x)$ of $f$ at $x \in M^n$ for $0 \leq \ell \leq \tau^o_f$ is

$$E^f_\ell(x) = \{\alpha^f_{\ell+1}(Z_\theta, \ldots, Z_\theta) : Z_\theta = \cos \theta Z + \sin \theta JZ \text{ and } \theta \in [0, \pi)\},$$

where $Z \in D^\perp(x)$ has unit length and satisfies $\langle Z, JZ \rangle = 0$. From ellipticity such a $Z$ always exists and $E^f_\ell(x)$ is indeed an ellipse.

We say that the curvature ellipse $E^f_\ell$ of an elliptic submanifold $f$ is a circle for some $0 \leq \ell \leq \tau^o_f$ if all ellipses $E^f_\ell(x)$ are circles. That the curvature ellipse $E^f_\ell$ is a circle is equivalent to the almost complex structure $J_\ell$ being orthogonal. Notice that $E^f_0$ is a circle if and only if $f$ is minimal.

Let $f : M^n \rightarrow \mathbb{Q}_c^{m-c}, c \in \{0, 1\}$, be a substantial nicely curved elliptic submanifold. Assume that $M^n$ is the saturation of a fixed cross section $L^2 \subset M^n$ to the foliation of the distribution $D$. The subbundles in the orthogonal splitting (30) are parallel in the normal connection (and thus in $\mathbb{Q}_c^{m-c}$) along $D$. Hence each $N^f_\ell$ can be seen as a vector bundle along the surface $L^2$.

A polar surface to $f$ is an immersion $h$ of $L^2$ defined as follows:

(a) If $m - n - c$ is odd, then the polar surface $h : L^2 \rightarrow \mathbb{S}^{m-1}$ is the spherical image of the unit normal field spanning $N^f_{\tau^o_f}$.

(b) If $m - n - c$ is even, then the polar surface $h : L^2 \rightarrow \mathbb{R}^m$ is any surface such that $h_\ast T_xL = N^f_{\tau^o_f}(x)$ up to parallel identification in $\mathbb{R}^m$.

Polar surfaces always exist since in case (b) any elliptic submanifold admits locally many polar surfaces.

The almost complex structure $J$ on $D^\perp$ induces an almost complex structure $\tilde{J}$ on $TL$ defined by $P \circ \tilde{J} = J \circ P$, where $P : TL \rightarrow D^\perp$ is the orthogonal projection. It turns out that a polar surface to an elliptic submanifold is necessarily elliptic. Moreover, if the elliptic submanifold has a circular curvature ellipse then its polar surface has the same property at the “corresponding” normal bundle. As a matter of fact, up to parallel identification it holds that

$$N^h_s = N^f_{\tau^o_{\tau^o_f-s}} \text{ and } J^h_s = (J^f_{\tau^o_{\tau^o_f-s}})^t, \quad 0 \leq s \leq \tau^o_f.$$  \hspace{1cm} (31)

In particular, the polar surface is nicely curved.

A bipolar surface to $f$ is any polar surface to a polar surface to $f$. In particular, if we are in case $f : M^3 \rightarrow \mathbb{S}^{m-1}$, then a bipolar surface to $f$ is a nicely curved elliptic surface $g : L^2 \rightarrow \mathbb{R}^m$. 

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6 Three-dimensional submanifolds

In this section, we study the case of three-dimensional submanifolds and we provide the proof of Theorem 2. To this purpose, we need the following results.

**Proposition 19.** Let \( f : M^3 \to \mathbb{Q}^{3+p} \) be an isometric immersion such that \( M^3 \) carries a totally geodesic distribution \( D \) of rank one satisfying \( D(x) \subseteq \Delta_f(x) \) for any \( x \in M^3 \). If the mean curvature of \( f \) is constant along each integral curve of \( D \) and the normal bundle of \( f \) is flat, then \( f \) is minimal, or \( c = 0 \) and \( f \) is locally a cylinder.

**Proof:** We assume that \( f \) is nonminimal. If the open subset \( M^3_* \) is nonempty, then Lemma 16 and Proposition 12 imply that the immersion \( f \) is locally a cylinder over a surface.

We now assume that the open subset \( M^2_* \) is nonempty and we argue on it. Having flat normal bundle implies that \( \mu = 0 \) and according to (20) we obtain \( v_3 = 0 \). Consequently, (18) is written as

\[
e_3(u_3) = u_3^2 + c.
\]

Comparing equations (23) and (24), we obtain

\[
\phi(e_3) = \frac{\rho - 1}{nH} \lambda u_3.
\]

Thus,

\[
e_3(\rho) = (\rho + 1)\lambda u_3
\]

and consequently equation (14) becomes

\[
e_3(\rho) = u_3(\rho - 1)(\tau - \rho),
\]

where \( \tau \) is the function given by

\[
\tau = \frac{\lambda^2(\rho - 1)^2}{n^2 H^2}.
\]

Moreover, equation (10) is written as \( u_3^2(2\rho - \tau) = c \). Differentiating with respect to \( e_3 \) and using equations (32), (33) and (34), we derive that

\[
u_3^2(\rho + 1)(\rho - \tau) = 0.
\]

Now we claim that \( u_3 = 0 \). Arguing indirectly, we suppose that \( u_3 \neq 0 \) on an open subset. We observe that \( \rho \neq -1 \) due to our assumption and equation (4). Hence \( \rho = \tau \), or equivalently \( \rho n^2 H^2 = \lambda^2(\rho - 1)^2 \) and \( e_3(\rho) = 0 \) by (34). Thus \( e_3(\rho) = 0 \), which contradicts (33) since \( \lambda \neq 0 \). This proves the claim that \( u_3 = 0 \) and consequently the splitting tensor vanishes. That the immersion \( f \) is locally a cylinder on \( M^3_* \) follows from Proposition 12.

If the open subset \( M^1_* \) is nonempty, then Lemma 18 implies that the immersion \( f \) is locally a cylinder over a surface or over a curve. Thus the proof is completed. \( \blacksquare \)
Proposition 20. Let \( f : M^3 \to \mathbb{Q}^{3+p} \) be a nonminimal isometric immersion such that \( M^3 \) carries a totally geodesic distribution \( D \) of rank one satisfying \( D(x) \subseteq \Delta_f(x) \) for any \( x \in M^3 \). If the mean curvature of \( f \) is constant along each integral curve of \( D \) and \( f \) is not locally a cylinder, then the splitting tensor of \( f \) is an almost complex structure on \( D^\perp \). Moreover, \( f \) is a spherical elliptic submanifold with respect to this almost complex structure and its first curvature ellipse is a circle.

Proof: Since by assumption the immersion \( f \) is not a cylinder on any open subset, it follows from Lemma 16, Proposition 12 and Lemma 18 that the open subsets \( M^*_3 \) and \( M^*_1 \) are both empty.

Proposition 19, implies that the immersion \( f \) has nonflat normal bundle on \( M^*_2 \). Thus, we have \( \mu \neq 0 \) and \( \rho \neq -1 \). Using (20) and (21), equations (19), (14), (22), (23), (17) and (10) are written as

\[
\omega(e_3) = -\frac{\rho - \tau}{\rho + 1} v_3, \\
e_3(\rho) = \frac{\lambda}{\mu}(\rho - 1)(\rho - \tau)v_3, \\
e_3(\mu) = -\frac{\lambda}{\rho + 1} (2\tau + \rho^2 + 1) v_3, \\
e_3(\lambda) = \left(\frac{2\mu}{\rho + 1} \tau - \frac{2\mu \rho}{\rho + 1} - \frac{\lambda^2}{\mu}(\rho + 1)\right)v_3, \\
e_3(v_3) = \frac{\lambda}{\mu(\rho + 1)} ((\rho - 1)\tau - (2\rho^2 + \rho + 1))v_3^2, \\
(\lambda^2 + \mu^2)(2\rho - \tau)v_3^2 = c\mu^2, \\
(\lambda^2 + \mu^2)(\rho - 1)^2 = 19
\]

respectively, where \( \tau \) is the function given by

\[
\tau = (\lambda^2 + \mu^2)\frac{(\rho - 1)^2}{n^2 H^2}.
\]

By differentiating equation (37) and using all the above equations, we obtain

\[
\lambda(\lambda^2 + \mu^2)\left(\rho(5\rho^2 + 6\rho + 5) - (4\rho^2 + 2\rho + 4)\tau - 2\tau^2\right)v_3^3 = c\lambda \mu^2 v_3.
\]

We claim that \( \lambda v_3 \neq 0 \). Arguing indirectly, we assume that the open subset where \( \lambda v_3 \neq 0 \) is nonempty. Thus, comparing the above equation with equation (37), we derive that \( \tau = \rho \). This along with (35) imply that \( e_3(\tau) = e_3(\rho) = 0 \). By the definition of \( \tau \), it follows that \( e_3(\lambda^2 + \mu^2) = 0 \). Using the above equations it is easy to see that

\[
e_3(\lambda^2 + \mu^2) = -2\frac{\lambda}{\mu}(\lambda^2 + \mu^2)(\rho + 1)v_3,
\]
which is a contradiction and this proves our claim.

Now we claim that \( v_3 \) cannot vanish on any open subset. Arguing indirectly, we suppose that \( v_3 \neq 0 \) on an open subset. Then equation (20) implies that \( u_3 = 0 \). By Lemma 14, the splitting tensor vanishes and consequently the immersion \( f \) would be a cylinder by Proposition 12. This contradicts our assumption.

Since we already proved that \( \lambda v_3 = 0 \), we obtain \( \lambda = 0 \) and equation (20) implies that \( u_3 = 0 \). It follows from equation (36) that

\[
\mu^2 = \frac{\rho n^2 H^2}{(\rho - 1)^2}.
\]

In particular, we have \( \rho > 0 \). This, along with equation (37) yield

\[
\rho v_3^2 = c.
\]

Hence, \( c = 1 \). We now observe that the splitting tensor satisfies \( C_3^2 = -I \), where \( I \) is the identity endomorphism of \( D^\perp \), that is, \( C_3 \) is an almost complex structure \( J: D^\perp \to D^\perp \). Using equation (39) and the fact that the shape operator \( A_{\xi_j} \) satisfies \( A_{\xi_j} e_i = \mu e_j \) for \( i \neq j = 1, 2 \), we easily verify that the second fundamental form of \( f \) satisfies \( \alpha^J(e_1, e_2) = \alpha^J(e_1, Je_2) \). This is equivalent to the ellipticity of the immersion \( f \).

In order to prove that the first curvature ellipse of \( f \) is a circle, it is sufficient to prove that the vector fields \( \alpha^J(e_1, e_1) \) and \( \alpha^J(e_1, Je_1) \) are of the same length and perpendicular. Obviously, they are perpendicular since

\[
\alpha^J(e_1, e_1) = k_1 \xi_4 \quad \text{and} \quad \alpha^J(e_1, Je_1) = \mu v_3 \xi_5.
\]

Using equations (5) and (38), we obtain

\[
\frac{\|\alpha^J(e_1, Je_1)\|^2}{\|\alpha^J(e_1, e_1)\|^2} = \rho v_3^2.
\]

Bearing in mind equation (39), we conclude that the first curvature ellipse is a circle and this completes the proof. 

The following result parametrizes all three-dimensional submanifolds in spheres that carry a totally geodesic distribution of rank one, contained in the relative nullity distribution, such that the mean curvature is constant along each integral curve. This parametrization, given in terms of their polar surfaces, was introduced in \([5]\) as the polar parametrization.

**Theorem 21.** Let \( h: L^2 \to Q_c^{N+1}, c \in \{0, 1\}, N \geq 5 \), be a nicely curved elliptic surface of substantial even codimension, such that the curvature ellipses \( \mathcal{E}^{h}_{\tau_{h} - 2}, \mathcal{E}^{h}_{\tau_{h}} \) are circles and \( \mathcal{E}^{h}_{\tau_{h} - 1} \) is nowhere a circle. Then, the map \( \Psi_h: M^3 \to S^{N+c} \) defined on the circle bundle \( \tilde{M}^3 = U N^h \tau_h \) by \( \Psi_h(x, w) = w \) is a nonminimal elliptic isometric immersion with polar
surface $h$. Moreover, $M^3$ carries a totally geodesic distribution $D$ of rank one satisfying $D(p) \subseteq \Delta_{\Psi_h}(p)$ for any $p \in M^3$ such that the mean curvature of $\Psi_h$ is constant along each integral curve of $D$.

Conversely, let $f: M^3 \to \mathbb{S}^{3+p}, p \geq 2$, be a substantial nonminimal isometric immersion such that $M^3$ carries a totally geodesic distribution $D$ of rank one satisfying $D(x) \subseteq \Delta_f(x)$ for any $x \in M^3$. If the mean curvature of $f$ is constant along each integral curve of $D$, then $f$ is elliptic and there exists an open dense subset of $M^3$ such that for each point there exist a neighborhood $U$, and a local isometry $F: U \to UN^h$ such that $f = \Psi_h \circ F$, where $h$ is a polar surface to $f$ with curvature ellipses as above.

**Proof:** Let $h: L^2 \to \mathbb{Q}^{N^1}, c \in \{0, 1\}$, be a substantial elliptic surface, where $N = 2m + 3, m \geq 1$. We choose a local orthonormal frame $e_1, e_2$ in the tangent bundle of $L^2$ such that the almost complex structure $J$ of the elliptic surface is given by

$$Je_1 = be_2 \quad \text{and} \quad Je_2 = -\frac{1}{b}e_1,$$

where $b$ is a positive smooth function.

We argue for the case where $m \geq 2$. The case where $m = 1$ can be handled in a similar manner. We know from (30) that the normal bundle splits orthogonally as

$$N_hL = N^h_1 \oplus \cdots \oplus N^h_{m-1} \oplus N^h_m \oplus N^h_{m+1}.$$ 

Let $\zeta_1, \ldots, \zeta_{2m+4}$ be an orthonormal frame in the normal bundle, defined on an open subset $V \subseteq L^2$, such that $\zeta_{2s+1}, \zeta_{2s+2}$ span the plane subbundle $N^h_s$ for any $1 \leq s \leq m+1$. The corresponding normal connection forms $\omega_{\alpha\beta}$ are given by $\omega_{\alpha\beta} = \langle \nabla^1 \zeta_\alpha, \zeta_\beta \rangle, \alpha, \beta = 3, \ldots, 2m+4$.

Due to our hypothesis, we may choose the frame such that

$$\alpha^h_m(e_1, \ldots, e_1) = \kappa_{m-1}\zeta_{2m-1}, \quad \alpha^h_m(e_1, \ldots, e_1, e_2) = \frac{\kappa_{m-1}}{b}\zeta_{2m}$$

and

$$\alpha^h_{m+2}(e_1, \ldots, e_1) = \kappa_{m+1}\zeta_{2m+3}, \quad \alpha^h_{m+2}(e_1, \ldots, e_1, e_2) = \frac{\kappa_{m+1}}{b}\zeta_{2m+4},$$

where $\kappa_{m-1}, \kappa_{m+1}$ denote the radii of the circular curvature ellipses $\mathcal{E}^h_{m-1}, \mathcal{E}^h_{m+1}$, respectively. Since the curvature ellipse $\mathcal{E}^h_m$ is nowhere a circle, we may choose $\zeta_{2m+1}, \zeta_{2m+2}$ to be collinear to the major and minor axes of this ellipse, respectively. Thus, we may write

$$\alpha^h_{m+1}(e_1, \ldots, e_1) = v_{11}\zeta_{2m+1} + v_{12}\zeta_{2m+2} \quad \text{and} \quad \alpha^h_{m+1}(e_1, \ldots, e_1, e_2) = v_{21}\zeta_{2m+1} + v_{22}\zeta_{2m+2},$$

where $v_{ij}$ are smooth functions such that

$$b^2v_{21}v_{22} + v_{11}v_{12} = 0, \quad \kappa_m = \left(v_{11}^2 + b^2v_{21}^2\right)^{1/2}, \quad \mu_m = \left(v_{12}^2 + b^2v_{22}^2\right)^{1/2} \quad (40)$$

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and $\kappa_m, \mu_m$ denote the lengths of the semi-axes of the curvature ellipse $E^h_m$.

Bearing in mind the definition of the higher fundamental forms, their symmetry and the ellipticity of the surface $h$, we have

$$\alpha^h_{s+1}(e_1, \ldots, e_1, e_2) = (\nabla^\perp_{e_2} \alpha^h_s(e_1, \ldots, e_1))^N_h = (\nabla^\perp_{e_1} \alpha^h_s(e_1, \ldots, e_1))^N_h,$$

for $s = m, m + 1$. From these we obtain

$$\omega_{2m-1,2m+1}(e_1) = \frac{v_{11}}{\kappa_{m-1}}, \quad \omega_{2m-1,2m+2}(e_1) = \frac{v_{12}}{\kappa_{m-1}},$$

$$\omega_{2m-1,2m+1}(e_2) = \frac{v_{21}}{\kappa_{m-1}}, \quad \omega_{2m-1,2m+2}(e_2) = \frac{v_{22}}{\kappa_{m-1}},$$

$$\omega_{2m,2m+1}(e_1) = \frac{bv_{21}}{\kappa_{m-1}}, \quad \omega_{2m,2m+2}(e_1) = \frac{bv_{22}}{\kappa_{m-1}},$$

$$\omega_{2m,2m+1}(e_2) = -\frac{v_{11}}{b\kappa_{m-1}}, \quad \omega_{2m,2m+2}(e_2) = -\frac{v_{12}}{b\kappa_{m-1}},$$

$$\omega_{2m,1,2m+3}(e_1) = \frac{b\kappa_{m-1}}{\kappa_m \mu_m} v_{22}, \quad \omega_{2m+1,2m+3}(e_2) = \frac{\kappa_{m+1}}{b\kappa_m \mu_m} v_{12},$$

$$\omega_{2m+1,2m+4}(e_1) = -\frac{\kappa_{m+1}}{\kappa_m \mu_m} v_{12}, \quad \omega_{2m+1,2m+4}(e_2) = -\frac{\kappa_{m+1}}{\kappa_m \mu_m} v_{12},$$

$$\omega_{2m+2,2m+3}(e_1) = -\frac{b\kappa_{m+1}}{\kappa_m \mu_m} v_{21}, \quad \omega_{2m+2,2m+3}(e_2) = -\frac{b\kappa_{m+1}}{\kappa_m \mu_m} v_{21},$$

$$\omega_{2m+2,2m+4}(e_1) = \frac{\kappa_{m+1}}{\kappa_m \mu_m} v_{11}, \quad \omega_{2m+2,2m+4}(e_2) = -\frac{\kappa_{m+1}}{\kappa_m \mu_m} v_{11}.$$

Let $\Pi: M^3 \to L^2$ the natural projection of the circle bundle

$$M^3 = UN^h_{m-1} = \{(x, \delta) \in N^h_{m+1}: \|\delta\| = 1, x \in L^2\}.$$

We parametrize $\Pi^{-1}(V)$ by $V \times \mathbb{R}$ via the map

$$(x, \theta) \mapsto (x, \cos \theta \zeta_{2m+3}(x) + \sin \theta \zeta_{2m+4}(x))$$

and consequently, we have

$$\Psi_h(x, \theta) = \cos \theta \zeta_{2m+3} + \sin \theta \zeta_{2m+4}.$$

We notice that $\nabla^\perp N^h_{m+1} \subseteq N^h_m \oplus N^h_{m+1}$. Then we easily find that

$$\Psi_h E_i = (\cos \theta \omega_{2m+3,2m+1}(e_i) + \sin \theta \omega_{2m+4,2m+1}(e_i)) \zeta_{2m+1} + (\cos \theta \omega_{2m+3,2m+2}(e_i) + \sin \theta \omega_{2m+4,2m+2}(e_i)) \zeta_{2m+2},$$

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where the vector fields \( E_i \in TM, i = 1, 2 \), are given by

\[
E_i = e_i - \omega_{2m+3,2m+4}(e_i) \frac{\partial}{\partial \theta}.
\]

Using equations (45), (46), (47) and (48), we obtain

\[
\Psi_h E_1 = \frac{\kappa_{m+1}}{\kappa_m \mu_m} \left( (-bv_{22} \cos \theta + v_{12} \sin \theta) \zeta_{2m+1} + (bv_{21} \cos \theta - v_{11} \sin \theta) \zeta_{2m+2} \right) \tag{49}
\]

and

\[
\Psi_h E_2 = \frac{\kappa_{m+1}}{\kappa_m \mu_m} \left( -\left( \frac{v_{12}}{b} \cos \theta + v_{22} \sin \theta \right) \zeta_{2m+1} + \left( \frac{v_{11}}{b} \cos \theta + v_{21} \sin \theta \right) \zeta_{2m+2} \right) \tag{50}
\]

Additionally, we have

\[
\Psi_h (\partial/\partial \theta) = -\sin \theta \zeta_{2m+3} + \cos \theta \zeta_{2m+4}. \tag{51}
\]

It follows that the normal bundle of the isometric immersion \( \Psi_h \) is given by

\[
N_{\Psi_h} M = c \text{ span}\{h\} \oplus N^h_1 \oplus \cdots \oplus N^h_{m-2} \oplus N^h_{m-1}.
\]

It is easy to see that the first normal bundle of \( \Psi_h \) is \( N^{\Psi_h}_1 = N^h_{m-1} \). Moreover, it follows easily that the distribution \( D = \text{span}\{\partial/\partial \theta\} \) is contained in the nullity distribution \( \Delta_{\Psi_h} \) of \( \Psi_h \). In particular, from equation (51) and the Gauss formula we derive that \( \nabla_{\partial/\partial \theta} \partial/\partial \theta = 0 \). This implies that the distribution \( D \) is totally geodesic.

It remains to show that the mean curvature of the immersion \( \Psi_h \) is constant along each integral curve of \( D \). The shape operator \( A_{\zeta_{2m-j}} \) of \( \Psi_h \) with respect to the normal direction \( \zeta_{2m-j}, j = 0, 1 \), is given by the Weingarten formula as

\[
-\Psi_h (A_{\zeta_{2m-j}} E_i) = \nabla_{\xi_i} \zeta_{2m-j} - \left( \nabla_{\xi_i} \zeta_{2m-j} \right)^{N^h_{m-2} \oplus N^h_{m-1}} = \left( \nabla_{\xi_i} \zeta_{2m-j} \right)^{N^h_m}, \quad i = 1, 2, \tag{52}
\]

since \( \zeta_{2m-1}, \zeta_{2m} \in N^h_{m-1} \). Here, \( \nabla \) stands for the induced connection of the induced bundle \( h^*TQ^N_{\xi_c} \). Using (41)-(44), equations (52) yield

\[
\Psi_h (A_{\zeta_{2m-1}} E_1) = -\frac{1}{\kappa_{m-1}} (v_{11} \zeta_{2m+1} + v_{12} \zeta_{2m+2}), \tag{53}
\]

\[
\Psi_h (A_{\zeta_{2m-1}} E_2) = -\frac{1}{\kappa_{m-1}} (v_{21} \zeta_{2m+1} + v_{22} \zeta_{2m+2}), \tag{54}
\]

\[
\Psi_h (A_{\zeta_{2m}} E_1) = -\frac{b}{\kappa_{m-1}} (v_{21} \zeta_{2m+1} + v_{22} \zeta_{2m+2}), \tag{55}
\]

\[
\Psi_h (A_{\zeta_{2m}} E_2) = \frac{1}{b \kappa_{m-1}} (v_{11} \zeta_{2m+1} + v_{12} \zeta_{2m+2}). \tag{56}
\]
We may set
\[ A_{\zeta_{2m-1}} E_i = \lambda_{i1} E_1 + \lambda_{i2} E_2 \quad \text{and} \quad A_{\zeta_{2m}} E_i = \gamma_{i1} E_1 + \gamma_{i2} E_2, \quad i = 1, 2, \tag{57} \]
where \( \lambda_{ij} \) and \( \gamma_{ij} \) are smooth functions on the manifold \( M^3 \). From equations (49), (53), (54) and the first one of (57), we obtain
\[
\lambda_{11} = \frac{1}{\kappa_{m-1}\kappa_{m+1}} \left( (v_{11}^2 + v_{12}^2) \cos \theta + b (v_{11} v_{21} + v_{12} v_{22}) \sin \theta \right)
\]
and
\[
\lambda_{22} = \frac{1}{\kappa_{m-1}\kappa_{m+1}} \left( -b^2 (v_{21}^2 + v_{22}^2) \cos \theta + b (v_{11} v_{21} + v_{12} v_{22}) \sin \theta \right)
\]
Hence
\[
\text{trace} A_{\zeta_{2m-1}} = \frac{1}{\kappa_{m-1}\kappa_{m+1}} \left( (v_{11}^2 + v_{12}^2 - b^2 v_{21}^2 - b^2 v_{22}^2) \cos \theta + 2b (v_{11} v_{21} + v_{12} v_{22}) \sin \theta \right)
\]
Similarly, from equations (50), (55), (56) and the second of (57), we find that
\[
\gamma_{11} = \frac{1}{\kappa_{m-1}\kappa_{m+1}} \left( b (v_{11} v_{21} + v_{12} v_{22}) \cos \theta + b^2 (v_{21}^2 + v_{22}^2) \sin \theta \right)
\]
and
\[
\gamma_{22} = \frac{1}{\kappa_{m-1}\kappa_{m+1}} \left( b (v_{11} v_{21} + v_{12} v_{22}) \cos \theta - (v_{11}^2 + v_{12}^2) \sin \theta \right)
\]
Then, it follows that
\[
\text{trace} A_{\zeta_{2m}} = \frac{1}{\kappa_{m-1}\kappa_{m+1}} \left( 2b (v_{11} v_{21} + v_{12} v_{22}) \cos \theta - (v_{11}^2 + v_{12}^2 - b^2 v_{21}^2 - b^2 v_{22}^2) \sin \theta \right)
\]
Thus, the mean curvature of the isometric immersion \( \Psi_h \) is given by
\[
\| \mathcal{H}_{\Psi_h} \|^2 = \frac{1}{(3\kappa_{m-1}\kappa_{m+1})^2} \left( (v_{11}^2 + v_{12}^2 + b^2 v_{21}^2 + b^2 v_{22}^2)^2 - 4(v_{11}^2 v_{21}^2 + b^2 v_{21}^2 v_{22}^2)^2 \right)
\]
Using equations (40), the above equation becomes
\[
\| \mathcal{H}_{\Psi_h} \| = \frac{|\kappa_m^2 - \mu_m^2|}{3\kappa_{m-1}\kappa_{m+1}}.
\]
It is obvious that the mean curvature of the isometric immersion \( \Psi_h \) is constant along each integral curve of the distribution \( D \). This completes the proof of the direct statement of the theorem for \( m \geq 2 \). The case \( m = 1 \) can be treated in a similar manner. In this case, the mean curvature of \( \Psi_h \) is given by
\[
\| \mathcal{H}_{\Psi_h} \| = \frac{|\kappa_1 - \mu_1^2|}{3\kappa_2^2}.
\]
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Conversely, let \( f: M^3 \to S^{3+p} \) be a nonminimal isometric immersion. Suppose that \( M^3 \) carries a totally geodesic distribution \( D \) of rank one satisfying \( D(x) \subseteq \Delta_f(x) \) for any \( x \in M^3 \) such that the mean curvature is constant along each integral curve of \( D \). From Proposition 20, we know that \( f \) is an elliptic submanifold and its first curvature ellipse is a circle. Hereafter, we work on a connected component of an open dense subset where \( f \) is nicely curved.

We consider a polar surface \( h: L^2 \to \mathbb{Q}_{c}\mathbb{Q}^{p-c+4} \) to the immersion \( f \), where \( c = 0 \) if \( p \) is even and \( c = 1 \) if \( p \) is odd. We notice that \( \tau^0_f = \tau_h - 1 \). Using equations \( (31) \), we conclude that the curvature ellipse \( \mathcal{E}^h_{\tau_h - 2} \) of the surface \( h \) is a circle and the curvature ellipse \( \mathcal{E}^h_{\tau_h - 1} \) is nowhere a circle.

We claim that the last curvature ellipse \( \mathcal{E}^h_{\tau_h} \) is a circle. We notice that \( N^h_{\tau_0} = \text{span}\{\xi, \eta\} \), where the sections \( \xi, \eta \) of the normal bundle \( N_hL \) are given by \( \xi = f \circ \pi \) and \( \eta = f_*e_3 \circ \pi \). Here \( \pi \) denotes the natural projection \( \pi: M^3 \to L^2 \) onto the fixed cross section \( L^2 \subset M^3 \) to the foliation generated by the distribution \( D \).

Let \( X_1, \ldots, X_{\tau_h} \in TL \) be arbitrary vector fields. By \( (31) \) we have \( N^h_{\tau_h - 1} = N^f_{\tau_0} = f_*D^\perp \). Thus, there exists \( X \in \Gamma(D^\perp) \) such that

\[
\alpha^h_{\tau_h}(X_1, \ldots, X_{\tau_h}) = f_*X.
\]

For every vector field \( Y \in TL \) there exists a vector field \( Z \in \Gamma(D^\perp) \) such that \( Y = \pi_*Z \). Then we have

\[
\alpha^h_{\tau_h+1}(X_1, \ldots, X_{\tau_h}, Y) = (\nabla^h_Y \alpha^h_{\tau_h}(X_1, \ldots, X_{\tau_h}))^N_{\tau_h} = -\langle f_*X, f_*Z \rangle \xi - \langle f_*X, \nabla_Z f_*e_3 \rangle \eta.
\]

Using the Gauss formula and the definition of the splitting tensor, the above equation becomes

\[
\alpha^h_{\tau_h+1}(X_1, \ldots, X_{\tau_h}, Y) = -\langle X, Z \rangle \xi + \langle X, C_3 Z \rangle \eta.
\]

From Proposition 20 we know that the splitting tensor in the direction of \( e_3 \) is the almost complex structure \( J^f_0: D^\perp \to D^\perp \) of \( f \). Hence, we obtain

\[
\alpha^h_{\tau_h+1}(X_1, \ldots, X_{\tau_h}, Y) = -\langle X, Z \rangle \xi + \langle X, J^f_0 Z \rangle \eta.
\]

On account of \( \pi_* \circ J^f_0 = J^0 \circ \pi_* \), we have \( J^0_h Y = \pi_*J^f_0 Z \). Thus, it follows that

\[
\alpha^h_{\tau_h+1}(X_1, \ldots, X_{\tau_h}, J^0_h Y) = -\langle X, J^0_h Z \rangle \xi - \langle X, Z \rangle \eta.
\]

Since \( \xi, \eta \) is an orthonormal frame of the subbundle \( N^h_{\tau_h} \), it is now obvious that the normal vector fields \( \alpha^h_{\tau_h+1}(X_1, \ldots, X_{\tau_h+1}, Y) \) and \( \alpha^h_{\tau_h+1}(X_1, \ldots, X_{\tau_h}, J^0_h Y) \) are of the same length and perpendicular. Hence, the last curvature ellipse of the polar surface \( h \) is a circle.

We observe that the isometric immersion \( f \) is written as the composition \( f = \Psi_h \circ F \), where \( F: U \to UN^h_{\tau_h} \) is the local isometry given by \( F(x) = (\pi(x), f(x)) \), \( x \in U \), and \( U \) is the saturation of the cross section \( L^2 \subset M^3 \). This completes the proof.
Remark 22. It follows from the computation of the mean curvature of the submanifold \( \Psi_h \) in the proof of Theorem 21 that the mean curvature is constant by properly choosing the elliptic surface \( h \). Ejiri [13] proved that tubes in the direction of the second normal bundle of a pseudoholomorphic curve in the nearly Kähler sphere \( S^6 \) have constant mean curvature. Opposed to our case, the index of relative nullity of these tubes is zero.

Proof of Theorem 2: Assume that the isometric immersion \( f \) is neither minimal nor locally a cylinder. Proposition 20 implies that \( f \) is spherical. Thus, from Theorem 21 we know that for each point on an open dense subset there exist an elliptic surface \( h \): \( L^2 \to Q_p^{c+4} \), where \( c = 0 \) if \( p \) is even and \( c = 1 \) if \( p \) is odd, a neighborhood \( U \) and a local isometry \( F: U \to UN_{\tau h}^h \) such that \( f = \Psi_h \circ F \). In fact, the elliptic surface \( h \) is a polar to \( f \). Moreover, we know that the curvature ellipses \( E_{\tau h}^h \) and \( E_{\tau h}^h \) are circles, while the curvature ellipse \( E_{\tau h}^h \) is nowhere a circle.

Now, we consider a bipolar surface \( g \) to \( f \), that is, a polar surface to the elliptic surface \( h \). Then it follows from equations (31) that the curvature ellipse \( E_0^g \) of \( g \) is a circle. This means that the bipolar surface is minimal. Furthermore, its first curvature ellipse is nowhere a circle and the second one is a circle. That the isometric immersion \( f \) is locally parametrized by (1) follows from the fact that \( f = \Psi_h \circ F \) and \( N_0^g = N_{\tau h}^h \). This completes the proof.

6.1 Minimal surfaces

The following proposition provides a way of constructing minimal surfaces in \( \mathbb{R}^6 \) that satisfy the properties that are required in Theorem 2(iii).

Proposition 23. Let \( \hat{g}: M^2 \to \mathbb{R}^6 \) be the minimal surface defined by
\[
\hat{g} = \cos \varphi g_\theta \oplus \sin \varphi g_{\theta + \pi/2};
\]
where \( g_\theta, \theta \in [0, \pi) \), is the associated family of a simply connected minimal surface \( g: M^2 \to \mathbb{R}^3 \) with negative Gaussian curvature, and \( \oplus \) denotes the orthogonal sum with respect to an orthogonal decomposition of \( \mathbb{R}^6 \). If \( \varphi \neq \pi/4 \), then its first curvature ellipse is nowhere a circle and its second curvature ellipse is a circle.

Let \( g: M \to \mathbb{R}^n \) be an oriented minimal surface. The complexified tangent bundle \( TM \otimes \mathbb{C} \) is decomposed into the eigenspaces \( T'M \) and \( T''M \) of the complex structure \( J \), corresponding to the eigenvalues \( i \) and \( -i \). The \( r \)-th fundamental form \( \alpha^g_r \), which takes values in the normal subbundle \( N^g_{r-1} \), can be complex linearly extended to \( TM \otimes \mathbb{C} \) with values in the complexified vector bundle \( N^g_{r-1} \otimes \mathbb{C} \) and then decomposed into its \((p, q)\)-components, \( p + q = r \), which are tensor products of \( p \) differential 1-forms vanishing on \( T''M \) and \( q \) differential 1-forms vanishing on \( T'M \). The minimality of \( g \) is equivalent to the vanishing of the \((1, 1)\)-component of the second fundamental form. Hence, the \((p, q)\)-components of \( \alpha^g_r \) vanish unless \( p = r \) or \( p = 0 \).
It is known (see [30, Lem. 3.1]) that the curvature ellipse of order \( r - 1 \) is a circle if and only if the \((r,0)\)-component of \( \alpha \) is isotropic, that is
\[
\langle \alpha_r(X, \ldots, X), \alpha_r(X, \ldots, X) \rangle = 0
\]
for any \( X \in T'M \), where \( \langle \cdot, \cdot \rangle \) denotes the bilinear extension over the complex numbers of the Euclidean metric.

**Proof of Proposition 23:** We choose a local tangent orthonormal frame \( e_1, e_2 \) such that the shape operator \( A \) of \( g \) satisfies \( AE = k\bar{E} \), where \( E = e_1 + ie_2 \) and \( k \) is a positive smooth function. The associated family satisfies \( g_{\theta} = dg \circ J_{\theta} \), where \( J_{\theta} = \cos \theta I + \sin \theta J \) and \( I \) is the identity endomorphism of the tangent bundle. Then we have
\[
\hat{g}_*E = e^{-i\theta} (\cos \varphi g_*E, -i \sin \varphi g_*E). \tag{58}
\]

Using the Gauss formula and the fact that the shape operator \( A_{\theta} \) of \( g_{\theta} \) is given by \( A_{\theta} = A \circ J_{\theta} \), we find that the second fundamental form \( \hat{\alpha} \) of \( \hat{g} \) satisfies
\[
\hat{\alpha}(E, E) = 2ke^{-i\theta} (\cos \varphi N, -i \sin \varphi N), \tag{59}
\]
where \( N \) is the Gauss map of \( g \). It is obvious that \( \hat{\alpha}(E, E) \) is not isotropic if \( \varphi \neq \pi/4 \), which implies that the first curvature ellipse of \( \hat{g} \) is nowhere a circle.

Differentiating (59) with respect to \( E \) and using the Weingarten formula, we obtain
\[
\nabla_E \hat{\alpha}(E, E) = 2e^{-i\theta} E(k) (\cos \varphi N, -i \sin \varphi N) - 2k^2 e^{-i\theta} (\cos \varphi g_*\bar{E}, -i \sin \varphi g_*\bar{E}),
\]
where \( \nabla \) is the connection of the induced bundle of \( \hat{g} \). Since \( \hat{g}_*E \) and \( \hat{g}_*\bar{E} \) span \( N^g_0 \otimes \mathbb{C} \), the above equation along with (58) yield
\[
(\nabla_E \hat{\alpha}(E, E))^N^g_0 \otimes \mathbb{C} = -2k^2 e^{-2i\theta} \cos 2\varphi \hat{g}_*\bar{E}.
\]
Equation (59) implies that \( N^g_1 \otimes \mathbb{C} = \text{span}_\mathbb{C}\{\xi, \eta\} \), where \( \xi = (N, 0) \) and \( \eta = (0, iN) \). It follows directly that
\[
(\nabla_E \hat{\alpha}(E, E))^N^g_1 \otimes \mathbb{C} = 2e^{-i\theta} E(k) (\cos \varphi N, -i \sin \varphi N).
\]
Using the above and since the \((3,0)\)-component of the third fundamental form of \( \hat{g} \) is given by
\[
\hat{\alpha}_3(E, E, E) = (\nabla_E \hat{\alpha}(E, E))^{(N^g_0 \otimes \mathbb{C} \oplus N^g_1 \otimes \mathbb{C})^\perp},
\]
we obtain
\[
\hat{\alpha}_3(E, E, E) = k^2 e^{-i\theta} \sin 2\varphi (-\sin \varphi g_*\bar{E}, i \cos \varphi g_*\bar{E}).
\]
Thus the \((3,0)\)-component of the third fundamental form of \( \hat{g} \) is isotropic, and the proof is completed. \( \blacksquare \)
7 Submanifolds with constant mean curvature

In this section, we provide the proofs of the applications of our main results to submanifolds with constant mean curvature.

Proof of Theorem 3: The manifold $M^n$ is written as the disjoint union of the subsets

$$M_{n-i} = \{ x \in M^n : \nu(x) = n - i \}, \quad i = 1, 2.$$

Assume that the subset $M_{n-2}$ is nonempty. Then, using Proposition 13 it follows from Theorem 1 for $n \geq 4$, or Theorem 2 for $n = 3$ and $p = 1$, that the isometric immersion $f$ is locally a cylinder over a surface on $M_{n-2}$.

Suppose that the interior $\text{int}(M_{n-1})$ of the subset $M_{n-1}$ is nonempty. It follows from the Codazzi equation that the relative nullity distribution is parallel in the tangent bundle along $\text{int}(M_{n-1})$. Thus, the tangent bundle splits as an orthogonal sum of two parallel orthogonal distributions of rank one and $n - 1$ on $\text{int}(M_{n-1})$. By the De Rham decomposition theorem, $\text{int}(M_{n-1})$ splits locally as a Riemannian product of two manifolds of dimension one and $n - 1$. Then, the Gauss equation yields $c = 0$. Since the second fundamental form is adapted to the orthogonal decomposition of the tangent bundle, it follows that $f$ is a cylinder over a curve in $\mathbb{R}^{p+1}$ with constant first Frenet curvature (see [12, Th. 8.4]).

We observe that the open subset $V = \text{int}(M_{n-1}) \cup M_{n-2}$ is dense on $M^n$, and this completes the proof.

In order to proceed to the proofs of the applications of our main results, we need to recall Florit’s estimate of the index of relative nullity for isometric immersions with nonpositive extrinsic curvature. The extrinsic curvature of an isometric immersion $f : M^n \to \tilde{M}^{n+p}$ for any point $x \in M^n$ and any plane $\sigma \in T_x M$ is given by

$$K_f(\sigma) = K_M(\sigma) - K_{\tilde{M}}(f_\star \sigma),$$

where $K_M$ and $K_{\tilde{M}}$ are the sectional curvatures of $M^n$ and $\tilde{M}^{n+p}$, respectively. Florit [15] proved that the index of relative nullity satisfies $\nu \geq n - 2p$ at points where the extrinsic curvature of $f$ is nonpositive.

Proof of Corollary 5: We have that the index of relative nullity of $f$ satisfies $\nu \geq n - 2$. Theorem 3 implies that $c = 0$ and, on an open dense subset, $f$ splits locally as a cylinder over a surface in $\mathbb{R}^3$ of constant mean curvature. By real analyticity, the splitting is global. If $M^n$ is complete, then the surface is also complete with nonnegative Gaussian curvature. That the surface is a cylinder over a circle follows from [24].

Proof of Corollary 6: Assume that the hypersurface is nonrigid. Then, the well-known Beez-Killing Theorem (see [12]) implies that the index of relative nullity satisfies $\nu \geq n - 2$. The result follows from Corollary 4.

Proof of Theorem 7: We suppose that the hypersurface is nonminimal.
We first assume that the extrinsic curvature is nonnegative. If $c = 0$, a result of Hartman \cite{22} asserts that $f(M^n) = S^k_R \times \mathbb{R}^{n-k}$, where $1 \leq k \leq n$. If $c = 1$, then $M^n$ is compact by the Bonnet-Myers theorem. According to \cite[Th. 2]{28}, $f$ is totally umbilical.

In the case of nonpositive extrinsic curvature, the result follows from Corollary 5.

**Proof of Theorem 8**

According to the aforementioned result due to Florit \cite{15}, we have $\nu \geq n - 4$. Clearly the manifold $M^n$ is written as the disjoint union of the subsets

$$M_{n-i} = \{ x \in M^n : \nu(x) = n - i \}, \quad i = 1, 2, 3.$$  

We distinguish the following cases.

**Case I**: We suppose that the subset $M_{n-4}$ is nonempty. According to Proposition 13, this subset is open. Using \cite[Th. 1]{16}, we have that on an open dense subset of $M_{n-4}$ the immersion $f$ is locally a product $f = f_1 \times f_2$ of two hypersurfaces $f_i : M^{n_i} \rightarrow \mathbb{R}^{n_i+1}$, $i = 1, 2$, of nonpositive sectional curvature. The assumption that $f$ has constant mean curvature implies that both hypersurfaces have constant mean curvature as well. Each hypersurface $f_i, i = 1, 2$, has index of relative nullity $n_i - 2$. Then it follows from Corollary 4 that the submanifold is locally as in part (iii) of the theorem.

**Case II**: Assume that the interior of the subset $M_{n-3}$ is nonempty. Due to \cite[Th. 1]{17}, on an open dense subset of int($M_{n-3}$), $f$ is written locally as a composition $f = h \circ F$, where $h = \gamma \times id_{\mathbb{R}^{n-1}} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$ is cylinder over a unit speed plane curve $\gamma(s)$ with nonvanishing curvature $k(s)$ and $F : M^n \rightarrow \mathbb{R}^{n+1}$ is a hypersurface. The second fundamental form of $f$ is given by

$$\alpha^f(X,Y) = h_* \alpha^F(X,Y) + \alpha^h(F_*X, F_*Y), \quad X,Y \in TM.$$  

From this we obtain $k\langle F_*T, \partial/\partial s \rangle^2 = 0$ for any $T \in \Delta_f$. This implies that the height function $F_a = \langle F, \partial/\partial s \rangle$ is constant along the leaves of $\Delta_f$. Then, the mean curvature vector field of $f$ is given by

$$nH_f = nH_fh_*\xi + k \circ F_a \|\text{grad}F_a\|^2\eta,$$

where $\xi, \eta$ stand for the Gauss maps of $F$ and $h$, respectively. Using that

$$\|\text{grad}F_a\|^2 = 1 - \langle \xi, a \rangle^2,$$

it follows that the mean curvature of $F$ is given as in part (ii) of the theorem.

**Case III**: Suppose that the subset $M_{n-2} \cup M_{n-1}$ has nonempty interior. Then Theorem 3 implies that the submanifold is locally as in part (i) of the theorem, and this completes the proof.  

**Proof of Theorem 9**

It follows from \cite[Th. 5.1]{12} that $\bar{c} \geq c$ if $n \geq 4$. We distinguish the following cases.
Case I: We assume that $\tilde{c} > c$. From [29, Lem. 8] or [10, Prop. 9], we have that the second fundamental form splits orthogonally and smoothly as

$$\alpha^f(\cdot, \cdot) = \beta(\cdot, \cdot) + \sqrt{\tilde{c} - c} \langle \cdot, \cdot \rangle \eta,$$

where $\eta$ is a unit normal vector field and $\beta$ is a flat bilinear form. Thus, the shape operator $A_\xi$, associated to a unit normal vector field $\xi$ perpendicular to $\eta$, has rank $A_\xi \leq 1$. The mean curvature $H$ of $f$ is given by

$$H^2 = \frac{k^2}{n^2} + \frac{\tilde{c} - c}{n},$$

where $k = \text{trace} A_\xi$. Obviously, the function $k$ is constant. If $k = 0$, then $f$ is totally umbilical.

Assume now that $k \neq 0$. Let $X$ be a unit vector field such that $A_\xi X = kX$. The Codazzi equation

$$(\nabla_X A_\eta)T - (\nabla_T A_\eta)X = A_{T^\perp \xi} T - A_{\xi^\perp T} X$$

implies that

$$\nabla^\perp_T \xi = \nabla^\perp_T \eta = 0$$

for any $T \in \ker A_\xi$. Moreover, from the Codazzi equation

$$(\nabla_X A_\xi)T - (\nabla_T A_\xi)X = A_{T^\perp \eta} T - A_{\xi^\perp T} X$$

it follows that

$$\nabla_T X = 0 \quad \text{and} \quad \langle \nabla_X T, X \rangle = \langle \nabla^\perp_T \xi, \eta \rangle$$

for any $T \in \ker A_\xi$. Hence the orthogonal distributions $D^1 = \text{span}\{X\}$ and $D^{n-1} = \ker A_\xi$ are parallel. By the De Rham decomposition theorem, the manifold splits locally as a Riemannian product $M^n_\tilde{c} = M^1 \times M^{n-1}$. Consequently, we have $\tilde{c} = 0$ and $c = -1$. Clearly $M^{n-1}$ is flat and the second fundamental form is adapted to this decomposition. Then it follows that $f$ is a composition $f = i \circ F$, where $i: \mathbb{R}^{n+1} \to \mathbb{H}^{n+2}$ is the inclusion as a horosphere and $F: M^n_\tilde{c} \to \mathbb{R}^{n+1}$ is the cylinder over a circle (see [12, Th. 8.4]).

Case II: We suppose that $c = \tilde{c}$. It is known that $\nu \geq n - 2$ (see Example 1 and Corollary 1 in [27]). Then, the result follows from Theorem 3.

If $n = 3$, then Theorem 2 implies that either $c = 0$ and $f(M)$ is an open subset of a cylinder over a flat surface $g: M^2 \to \mathbb{R}^4$ of constant mean curvature, or $c = 1$ and $f$ is parametrized by (1). In the latter case, by Proposition 20 we have that $f$ is either totally geodesic or elliptic. However, the ellipticity of $f$ implies that the sectional curvature cannot be equal to one. This completes the proof.

Proof of Theorem 10: Assume that $f$ is nonminimal. According to Abe [1], the index of relative nullity satisfies $\nu \geq n - 2$. Thus, Corollary 4 and Proposition 12 conclude the proof.
Proof of Theorem 11: Using [18], it follows that $\nu \geq n - 4$. The rest of the proof is omitted since it is similar to the proof of Theorem 8.

The following example produces submanifolds satisfying the conditions in part (ii) of Theorems 8 or 11.

Example 24. Let $F = g \times id_{\mathbb{R}^{n-2}} : U \times \mathbb{R}^{n-2} \rightarrow \mathbb{R}^{n+1}$ be a cylinder over a rotational surface $g(x, \theta) = (x \cos \theta, x \cos \theta, \varphi(x)), (x, \theta) \in U$, where $\varphi(x)$ is a smooth function. We consider a cylinder $h = \gamma \times id_{\mathbb{R}^n}$ in $\mathbb{R}^{n+2}$ over a unit speed plane curve $\gamma$ with curvature $k$. Then the isometric immersion $f = h \circ F$ satisfies the conditions in part (ii) of Theorems 8 and 11 with constant constant curvature $H$ and $a = (1, 0, \ldots, 0)$, if the function $\varphi(x)$ solves the ordinary differential equation

$$\varphi'' - 1 - \varphi'^2 = \pm \varphi \sqrt{(1 + \varphi'^2)(n^2H^2(1 + \varphi'^2)^2 - k^2)}.$$

In particular, $g$ can be chosen as a Delaunay surface and $\gamma$ as the curve with curvature $k = c_0(1 + \varphi'^2)$ for a constant $c_0$ such that $0 < |c_0| < n|H|$.

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