On Obtaining Simple Identities for Overshoots of Spectrally Negative Lévy Processes

R. L. Loeffen

First version: 3 November 2014
On obtaining simple identities for overshoots of spectrally negative Lévy processes

R. L. Loeffen*

October 20, 2014

Abstract

For a (killed) spectrally negative Lévy process we provide an analytic expression for the distribution of its overshoot over a fixed level in terms of the infinitesimal generator and the scale function of the process. Our identity involves an auxiliary function and the simplicity of the identity depends very much on the choice of this function. In particular, for specific choices one recovers various previous established formulas in the literature. We review several applications and also show that one can get in a similar way identities of overshoots for reflected and refracted spectrally negative Lévy processes.

AMS 2000 subject classifications. 60G51.
Key words and phrases. Spectrally negative Lévy processes, Fluctuation theory, Gerber-Shiu function, Reflected Lévy processes, Refracted Lévy processes.

1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process on the filtered probability space $(\Omega, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$, i.e. $X$ is a process with stationary and independent increments and no positive jumps. Here we exclude the case that $X$ is the negative of a subordinator, i.e. we exclude the case of $X$ having decreasing paths. The law of $X$ such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We work with the first passage times

$$\tau^+_a = \inf\{t > 0 : X_t > a\}, \quad \text{and} \quad \tau^-_a = \inf\{t > 0 : X_t < a\}.$$ 

In this paper we are interested in the following expectation concerning the overshoot over a level $a$, namely

$$\mathbb{E}_x \left[ e^{-q\tau^-_a} f(X_{\tau^-_a}) 1_{\{\tau^-_a < \tau^+_a\}} \right],$$

(1)

*School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom, e-mail: ronnie.loeffen@manchester.ac.uk
where $-\infty < a < b < \infty$, $q \geq 0$, $x \in [a, b]$ and $f : (-\infty, a) \rightarrow \mathbb{R}$ is a (penalty) function satisfying certain regularity conditions which we specify later. When $b \rightarrow \infty$, (1) is a specific case (namely, in which one does not consider the undershoot) of the so-called expected discounted penalty function introduced by Gerber and Shiu [3], which can be interpreted as a risk measure for an insurance company. Further, (1) appears in many applications of spectrally negative Lévy processes in which circumstances change when a given level is crossed downwards. Some examples are solving exit problems for refracted Lévy processes, see e.g. [7], determining Laplace transforms of occupation times, see e.g. [9–11, 15], or computing the value of multi-band strategies, see [1]. Analytic expressions for (1) in terms of the scale function and the Lévy triplet of the spectrally negative Lévy process are known, but for specific penalty functions it has been possible, usually after quite some effort, to come up with much simpler expressions; we will review some examples later on in Section 5. In this paper we shed some light on why this is possible and show how one can directly get such simple expressions. The novelty of our approach is that we introduce an extension $\tilde{f}$ to the interval $(-\infty, b]$ of the function $f$, which is free to choose (as long as it satisfies certain regularity conditions) and then provide an expression for (1) which depends on this extension. The existing expressions found for (1) turn out to correspond to particular choices of $\tilde{f}$. We will further see that given a penalty function, it is quite obvious how to choose the extension in such a way that the resulting formula is as simple as possible.

The rest of the paper is organised as follows. In the next section we briefly review some background on spectrally negative Lévy processes and state the main result whose proof is given in Section 3. The main idea of the proof also applies to certain modifications of spectrally negative Lévy processes and in Section 4 we give the analogues of (1) for reflected and refracted spectrally negative Lévy processes. Finally, in Section 5 we go over some examples to illustrate the usefulness of our result.

## 2 Main result

Before we state the main result, we briefly give some background information on spectrally negative Lévy processes and their scale functions; proofs can be found in for instance the book of Kyprianou [6]. A spectrally negative Lévy process is characterised in terms of a so-called Lévy triplet $(\gamma, \sigma, \Pi)$, where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ is called the Gaussian coefficient and $\Pi$, called the Lévy measure, is a measure on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge \theta^2) \Pi(d\theta) < \infty. \quad (2)$$

Note that for convenience we define the Lévy measure in such a way that it is a measure on the positive half line instead of the negative half line. As the Lévy process $X$ has no positive jumps, its Laplace transform exists and is given by

$$\mathbb{E}[e^{\lambda X_t}] = e^{\psi(\lambda)},$$

where $\psi(\lambda) = \gamma \lambda + \frac{1}{2} \sigma^2 \lambda^2$. 



2
for $\lambda, t \geq 0$, where

$$
\psi(\lambda) = \gamma \lambda + \frac{1}{2}\sigma^2 \lambda^2 + \int_0^\infty \left(e^{-\lambda \theta} - 1 + \lambda \theta \mathbf{1}_{\theta \leq 1}\right) \Pi(d\theta).
$$

The process $X$ has paths of bounded variation if and only if $\sigma = 0$ and $\int_0^1 \theta\Pi(d\theta) < \infty$.

We now recall the definition of the $q$-scale function $W^{(q)}$. For $q \geq 0$, the $q$-scale function of the process $X$ is defined on $[0, \infty)$ as the continuous function with Laplace transform on $[0, \infty)$ given by

$$
\int_0^\infty e^{-\lambda y} W^{(q)}(y) dy = \frac{1}{\psi(\lambda) - q}, \quad \text{for } \lambda \text{ sufficiently large.} \tag{3}
$$

This function is unique, positive and strictly increasing for $x \geq 0$. We extend $W^{(q)}$ to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. Scale functions appear in various fluctuation identities and we will need the following one involving exiting the interval $[a, b]$ at the upper boundary,

$$
E_x \left[ e^{-q \tau_b^+} 1_{\{\tau_b^+ < \tau^a\}} \right] = \frac{W^{(q)}(x - a)}{W^{(q)}(b - a)} W^{(q)}(b - z) - W^{(q)}(x - z), \quad x \leq b, \tag{4}
$$

as well as the $q$-resolvent measure of $X$ killed upon exiting the interval $[a, b]$,

$$
\int_0^\infty e^{-qs} P_x(X_s \in dz, s < \tau_a^- \wedge \tau_b^+) ds = \left[ \frac{W^{(q)}(x - a)}{W^{(q)}(b - a)} W^{(q)}(b - z) - W^{(q)}(x - z) \right] dz, \tag{5}
$$

where $z \in [a, b]$ and $x \leq b$. We refer to Theorem 8.1 and Theorem 8.7 of [6] for the proof and origin of these two identities. When the Gaussian coefficient $\sigma = 0$, the process $X$ cannot creep downwards, i.e. $\mathbb{P}_x(\tau_a^- = a) = 0$ for all $x > a$. When $\sigma > 0$, $X$ does creep downwards and we have

$$
E_x \left[ e^{-q \tau_a^-} 1_{\{X_{\tau_a^- - a} = a, \tau_a^- < \tau_b^+\}} \right] = \frac{\sigma^2}{2} \left( \frac{W^{(q)\prime}(x - a)}{W^{(q)}(b - a)} - \frac{W^{(q)}(x - a)}{W^{(q)}(b - a)} W^{(q)\prime}(b - a) \right), \quad a < x \leq b. \tag{6}
$$

Note that (6) can be deduced from (5), see the proof of Corollary 2 in [14].

In order to state the main result, we need to introduce the following space of functions.

**Definition 1.** For $X$ a spectrally negative Lévy process with Lévy triplet $(\gamma, \sigma, \Pi)$ and $-\infty < a < b < \infty$, we define $\mathcal{H}(X; a, b)$ as the function space consisting of measurable, locally bounded functions $h : (-\infty, b] \to \mathbb{R}$ such that the following hold:

(i) $h$ is continuous on $(a, b]$,

(ii) there exists $\lambda > 0$ such that $x \mapsto \int_{\lambda}^\infty h(x - \theta) \Pi(d\theta)$ is bounded on $(a, b)$,

(iii) if $X$ has paths of unbounded variation, then $h$ is continuously differentiable on $(a, b)$ with the derivative being absolutely continuous on $(a, b)$ and having a density that is bounded on $(a, b)$. 

3
(iv) if $X$ has paths of bounded variation, then $h$ is absolutely continuous on $(a, b)$ with a density that is bounded and of bounded variation on $(a, b)$.

For $h \in \mathcal{H}(X; a, b)$ we define $\mathcal{A}h(x)$ by

$$\mathcal{A}h(x) = \gamma h'_-(x) + \frac{1}{2} \sigma^2 h''(x) + \int_0^\infty [h(x - \theta) - h(x) + h'_-(x)\theta \mathbf{1}_{\{\theta \leq 1\}}]\Pi(d\theta), \quad x \in (a, b),$$

where $h'_-$ denotes the left-derivative of $h$ and where if $\sigma = 0$, the term $\frac{1}{2} \sigma^2 h''(x)$ is understood to equal zero and if $\sigma > 0$, $h''$ denotes a version of the density of $h'$ (for all the results derived in this paper, it does not matter which version is chosen). Note that if $h \in \mathcal{H}(X; a, b)$ and $X$ has paths of bounded variation, then $h$ on $(a, b)$ can be written as the difference of two convex functions. This means that when $h \in \mathcal{H}(X; a, b)$, the right- and left-derivative $h'_+(x)$ and $h'_-(x)$ are always well-defined for $x \in (a, b)$. Further, note that the integral term in $\mathcal{A}h(x)$ is also well-defined when $h \in \mathcal{H}(X; a, b)$, see e.g. the proof of Lemma 4 below. It is well-known that the operator $\mathcal{A}$ coincides with the infinitesimal generator of $X$ on the space of twice continuously differentiable functions that vanish at infinity, together with the first two derivatives, cf. [16, Theorem 31.5].

We can now state the main theorem.

**Theorem 2.** Let $-\infty < a < b < \infty$, $q \geq 0$ and $f : (-\infty, a] \to \mathbb{R}$ be a measurable, locally bounded function with the property that there exists $\lambda > b - a$ such that $x \mapsto \int_\lambda^\infty f(x - \theta)\Pi(d\theta)$ is bounded on $(a, b)$. Let $\tilde{f} : (-\infty, b] \to \mathbb{R}$ be an extension of $f$ that lies in $\mathcal{H}(X; a, b)$. Then for $a < x \leq b$,

$$\mathbb{E}_x \left[ e^{-q\tau_a^-} f(X_{\tau_a^-}) \mathbf{1}_{\{\tau_a^- < \tau_b^+\}} \right] = \tilde{f}(x) - \frac{W(q)(x - a)}{W(q)(b - a)} \tilde{f}(b) + \int_a^b (A - q) \tilde{f}(z) \frac{W(q)(x - a) - W(q)(b - z)}{W(q)(b - a)} \Pi(d\theta) \right] dz \quad \text{(7)}$$

where $\tilde{f}(a +) := \lim_{\epsilon \downarrow 0} \tilde{f}(a + \epsilon)$. Further, for the case $X_0 = a$, if $X$ has paths of bounded variation, then

$$\mathbb{E}_a \left[ e^{-q\tau_a^-} f(X_{\tau_a^-}) \mathbf{1}_{\{\tau_a^- < \tau_b^+\}} \right] = \tilde{f}(a +) - \frac{W(q)(0)}{W(q)(b - a)} \left[ \tilde{f}(b) - \int_a^b (A - q) \tilde{f}(z) W(q)(b - z) d\theta \right],$$

whereas if $X$ has paths of unbounded variation, $\mathbb{E}_a \left[ e^{-q\tau_a^-} f(X_{\tau_a^-}) \mathbf{1}_{\{\tau_a^- < \tau_b^+\}} \right] = f(a)$.

**Corollary 3.** Suppose the conditions in Theorem 2 hold and assume in addition that one of the following holds:
(i) \( \int_0^1 \theta \Pi(d\theta) < \infty \) and \( \sigma = 0 \),
(ii) \( \int_0^1 \theta \Pi(d\theta) < \infty \) and \( \tilde{f} \) is right-continuous at \( a \),
(iii) \( \int_0^1 \theta \Pi(d\theta) = \infty \) and \( \tilde{f} \) has a bounded density in a neighbourhood of \( a \).

Then for \( a < x \leq b \),
\[
E_x \left[ e^{-q\tau} f(X_{\tau^-}) 1_{\{\tau^- < \tau^+\}} \right] = \tilde{f}(x) - \int_a^x (A - q) \tilde{f}(z) W^{(q)}(x - z) dz
- \frac{W^{(q)}(x - a)}{W^{(q)}(b - a)} \left[ \tilde{f}(b) - \int_a^b (A - q) \tilde{f}(z) W^{(q)}(b - z) dz \right]. \tag{8}
\]

The identities (7) and (8) depend on the chosen extension \( \tilde{f} \). Note that one always has the option to choose \( \tilde{f} \) to be right-continuous at \( a \), with the result that the creeping term in (7) vanishes. Though (8) is more convenient to work with, it is not always possible to use it: in particular when \( \int_0^1 \theta \Pi(d\theta) = \infty \) and \( f \) is not left-continuous at \( a \), then the extra condition in Corollary 3 does not hold no matter which extension is chosen and so one then has to resort to (7). When we choose \( \tilde{f}(y) = 0 \) for all \( y \in (a, b] \), then (7) becomes,
\[
E_x \left[ e^{-q\tau} f(X_{\tau^-}) 1_{\{\tau^- < \tau^+\}} \right] = \int_a^b \int_{z-a}^\infty f(z - \theta) \Pi(d\theta) \left[ \frac{W^{(q)}(x - a)}{W^{(q)}(b - a)} W^{(q)}(b - z) - W^{(q)}(x - z) \right] dz \tag{9}
+ f(a) \frac{\sigma^2}{2} \left( \frac{W^{(q)}(x - a)}{W^{(q)}(b - a)} W^{(q)}(b - a) - W^{(q)}(x - a) \right) \frac{W^{(q)}(b - a)}{W^{(q)}(b - a)}.
\]

This identity is often used in the literature to deal with overshoots, see e.g. Equation (10.28) in [7], (5) in [11], (2.6) in [9] or in the case where \( b \to \infty \), (4) in [10]. It can be proved by using the compensation formula and has the advantage that one can easily incorporate the undershoot \( \lim_{t \to \tau^-} X_t - a \) as well, see e.g. [6, Equation (8.32)]. However, when one is only interested in the overshoot, we recommend to use (7) or (8) instead, since by choosing the extension \( \tilde{f} \) wisely, one can, in specific cases, get significantly simpler identities, which are not obvious to spot by using (9), see Section 5 for examples. Avram et al. [1] work with a different identity than (9), see Definition 5.2 and Proposition 5.5 in [1]. Their identity corresponds to the extension \( \tilde{f}(y) = f^-_1(a)y + f(a) \), \( y \in (a, b] \), assuming that the penalty function \( f \) admits a left-derivative at \( a \). As can be seen in [1], this choice is very convenient for computing the value function of multi-band strategies as in that case one deals with penalty functions which are affine in a neighbourhood of \( a \).

3 Proof

Lemma 4. Let \( -\infty < a < b < \infty \) and \( h \in H(X; a, b) \). Assume further that (i) \( \int_0^1 \theta \Pi(d\theta) = \infty \) and \( h \) has a bounded density in a neighbourhood of \( a \) or (ii) \( \int_0^1 \theta \Pi(d\theta) < \infty \). Then \( \int_a^b |Ah(x)| dx < \infty \).
**Proof.** We first consider the case where condition (i) holds. By Definition 1(iii), $h'$ has a bounded density on $(a, b)$ and we denote by $h''$ a version of this density. Then by the boundedness of $h''$ on $(a, b)$, $h$ and $h'$ are bounded on $(a, b)$. Hence $\int_a^b |\gamma h'(x) + \frac{1}{2}\sigma^2 h''(x)| dx < \infty$. To deal with the integral term, first note that by Taylor’s theorem we have for all $x \in (a, b)$,

$$
\int_0^{x-a} |h(x - \theta) - h(x) + h'(x)\cdot \mathbf{1}_{\{\theta \leq 1\}}\Pi(d\theta) | dx \leq \sup_{t \in (a, x)} |h''(t)| \int_0^{x-a} \frac{1}{2} \theta^2 \Pi(d\theta)
$$

where the last inequality is due to Definition 1(ii), inequality (2) and because by using Fubini,

Second, by Fubini,

$$
\int_a^b \int_{x-a}^{\infty} |h'(x)\cdot \mathbf{1}_{\{\theta \leq 1\}}\Pi(d\theta) | dx \leq \sup_{t \in (a, b)} |h'(t)| \int_a^b \int_{x-a}^{\infty} \mathbf{1}_{\{\theta \leq 1\}}\Pi(d\theta) dx
$$

Third, we have for any $\delta \in (0, b - a),

$$
\int_a^{a+\delta} \int_{x-a}^{\infty} |h(x - \theta) - h(x)|\Pi(d\theta) dx
$$

$$
\leq \int_a^{b+\delta} |h(x)|\Pi((x-a, \infty)) dx + \int_a^{b+\delta} \int_{x-a}^{\infty} |h(x - \theta)|\Pi(d\theta) dx
$$

$$
\leq \Pi(\delta, \infty) \int_a^{b+\delta} |h(x)| dx + \int_a^{b+\delta} \int_{x-a}^{b-a} |h(x - \theta)|\Pi(d\theta) dx + \int_a^{b+\delta} \int_{b-a}^{\infty} |h(x - \theta)|\Pi(d\theta) dx
$$

$$
< \infty,
$$

where the last inequality is due to Definition 1(ii), inequality (2) and because by using Fubini,

$$
\int_a^{b+\delta} \int_{x-a}^{b-a} |h(x - \theta)|\Pi(d\theta) dx \leq \sup_{t \in (2a+\delta-b,a)} |h(t)| \int_a^{b+\delta} \int_{x-a}^{b-a} \Pi(d\theta) dx
$$

$$
= \sup_{t \in (2a+\delta-b,a)} |h(t)| \int_{\delta}^{b-a} (\theta - \delta)\Pi(d\theta).
$$

Further,

$$
\int_a^{a+\delta} \int_{\delta}^{\infty} |h(x - \theta) - h(x)|\Pi(d\theta) dx \leq \Pi(\delta, \infty) \int_a^{a+\delta} |h(x)| dx + \int_a^{a+\delta} \int_{\delta}^{\infty} |h(x - \theta)|\Pi(d\theta) dx
$$

and the right hand side is finite by Definition 1(ii). Lastly, choosing $\delta > 0$ small enough such that $h$ has a bounded density in $(a - \delta, b)$, which we denote, with abuse of notation, by $h'$,
we get by Taylor’s theorem and Fubini,
\[
\int_a^{a+\delta} \int_{x-a}^{x} \left| h(x - \theta) - h(x) \right| \Pi(d\theta) dx \leq \int_a^{a+\delta} \sup_{t \in (x-\delta,x)} |h'(t)| \int_{x-a}^{x} \theta \Pi(d\theta) dx \\
\leq \sup_{t \in (a-\delta,b)} |h'(t)| \int_{a-\delta}^{a+\delta} \int_{x-a}^{x} \theta \Pi(d\theta) dx \\
= \sup_{t \in (a-\delta,b)} |h'(t)| \int_0^{\delta} \theta^2 \Pi(d\theta).
\]

Combining everything and recalling (2), gives us \(\int_a^b |Ah(x)| dx < \infty\).

We now assume that condition (ii) holds, i.e. \(\int_0^1 \theta \Pi(d\theta) < \infty\). Then by boundedness of \(h\) and \(h''\) (if \(\sigma > 0\)) on \((a,b)\), we have \(\int_a^b (\gamma + \int_0^1 \theta \Pi(d\theta)) h''(x) + \frac{1}{2} \sigma^2 h''(x) dx < \infty\). Further by Taylor’s theorem, we have for all \(x \in (a,b)\),
\[
\int_0^{x-a} \left| h(x - \theta) - h(x) \right| \Pi(d\theta) dx \leq \sup_{t \in (a,x)} |h'_-(t)| \int_0^{x-a} \theta \Pi(d\theta) \leq \sup_{t \in (a,b)} |h'_-(t)| \int_0^{b-a} \theta \Pi(d\theta).
\]
Moreover,
\[
\int_a^b \int_{x-a}^{x} \left| h(x - \theta) - h(x) \right| \Pi(d\theta) dx \\
\leq \sup_{t \in (a,b)} |h(t)| \int_a^b \int_{x-a}^{x} \Pi(d\theta) dx + \int_a^b \int_{x-a}^{x} \left| h(x - \theta) \right| \Pi(d\theta) dx \\
= \sup_{t \in (a,b)} \int_0^{\infty} \theta \wedge (b-a) \Pi(d\theta) + \sup_{t \in (2a-b, a)} \int_0^{b-a} \theta \Pi(d\theta) + \int_a^b \int_{b-a}^{\infty} \left| h(x - \theta) \right| \Pi(d\theta) dx
\]
and the right hand side is finite by Definition 1(ii). We conclude that also in this case \(\int_a^b |Ah(x)| dx < \infty\).

**Proof of Theorem 2.** Let \(x \in [a,b]\). In order to deal with the possibility that \(\tilde{f}\) is not right-continuous at \(a\), we introduce the function \(g : (-\infty, b] \rightarrow \mathbb{R}\) defined by
\[
g(x) = \begin{cases} 
\tilde{f}(x) & x \neq a, \\
\tilde{f}(a^+) & x = a.
\end{cases}
\]

For notational convenience, let \(T = \tau_a^+ \wedge \tau_b^-\). Then by the regularity assumptions, \(g\) is smooth enough on \([a, b]\) in order to use the Meyer-Itô formula. In particular, using Theorem 70 of [12] in combination with the fact that the continuous part of the quadratic variation of the Lévy process is given by \([X, X]_c = \sigma^2 t\) (cf. [4, Theorem I.4.52, Definition II.2.6 and Corollary II.4.19]) and (i) Corollary 1 on p.216 of [12] in the case where \(X\) has paths of
unbounded variation or (ii) Corollary 3 on p.225 of [12] in the case where $X$ has paths of bounded variation, we get under $\mathbb{P}_x$:

$$e^{-q(t\wedge T)}g(X_{t\wedge T}) = g(X_0) + \int_{0+}^{t\wedge T} e^{-qs} dg(X_s) - \int_{0+}^{t\wedge T} qe^{-qs} g(X_s) ds$$

$$= g(X_0) + \int_{0+}^{t\wedge T} e^{-qs} \left( \frac{g^2}{2} g''(X_s) - qg(X_s) \right) ds + \int_{0+}^{t\wedge T} e^{-qs} g'_-(X_s) dX_s$$

$$+ \sum_{0<s\leq t\wedge T} e^{-qs} [\Delta g(X_s) - g'_-(X_s-1) \Delta X_s]$$

$$= g(X_0) + \int_{0+}^{t\wedge T} e^{-qs} (A - q) g(X_s) ds$$

$$+ \left\{ \int_{0+}^{t\wedge T} e^{-qs} g'_-(X_s) d\left( X_s - \gamma s - \sum_{0<s\leq s} \Delta X_s 1_{\{|\Delta X_s| > 1\}} \right) \right\}$$

$$+ \left\{ \sum_{0<s\leq t\wedge T} e^{-qs} (\Delta g(X_s) + \Delta X_s) - g'_-(X_s-1) \Delta X_s 1_{\{|\Delta X_s| \leq 1\}} \right\}$$

$$- \int_{0+}^{t\wedge T} \int_{0+}^{\infty} e^{-qs} (g(X_s - \theta) - g(X_s)) + g'_-(X_s) \theta 1_{\{0<\theta \leq 1\}} \Pi(d\theta) ds \right\}.$$

Here we have used the following notation: $X_s^- = \lim_{u \downarrow s} X_u$, $\Delta X_s = X_s - X_{s-}$ and $\Delta g(X_s) = g(X_s) - g(X_{s-})$. By the Lévy-Itô decomposition (cf. [6, Section 2.1]) the expression between the first pair of curly brackets is a zero-mean martingale and by the compensation formula (cf. [6, Corollary 4.6]) the expression between the second pair of curly brackets is also a zero-mean martingale. Hence taking expectations under $\mathbb{P}_x$ and letting $t \to \infty$, we have with the aid of the dominated convergence theorem and the regularity properties of $g$ on $[a, b]$,

$$\mathbb{E}_x \left[ e^{-q(\tau_a^- \wedge \tau_b^+)} g(X_{\tau_a^- \wedge \tau_b^+}) \right] = g(x) + \mathbb{E}_x \left[ \int_0^{\tau_a^- \wedge \tau_b^+} e^{-qs} (A - q) g(X_s) ds \right]$$

$$= g(x) + \int_a^b (A - q) g(z) \int_0^\infty e^{-qs} \mathbb{P}_x(X_s \in dz, s < \tau_a^- \wedge \tau_b^+) ds. \quad (10)$$

Since by the lack of upward jumps,

$$\mathbb{E}_x \left[ e^{-q(\tau_a^- \wedge \tau_b)} g(X_{\tau_a^- \wedge \tau_b}) \right] = \mathbb{E}_x \left[ e^{-q\tau_a^-} g(X_{\tau_a^-}) 1_{\{\tau_a^- < \tau_b^+\}} \right] + g(b) \mathbb{E}_x \left[ e^{-q\tau_b} 1_{\{\tau_a^- > \tau_b^+\}} \right], \quad (11)$$

8
we have by (10), (11) and the definition of $g$,
\[
\mathbb{E}_x \left[ e^{-q\tau_a} f(X_{\tau_a}) \mathbf{1}_{\{\tau_a < \tau_a^+\}} \right] = g(x) - \mathbb{E}_x \left[ e^{-q\tau_b} \mathbf{1}_{\{\tau_b > \tau_b^+\}} \right] \tilde{f}(b)
\]
\[
+ \int_a^b (A - q) \tilde{f}(z) \int_0^\infty e^{-qs} \mathbb{P}_x(X_s \in dz, s < \tau_a^- \land \tau_b^+ \lor a) ds \quad (12)
\]
\[
+ (f(a) - \tilde{f}(a+)) \mathbb{E}_x \left[ e^{-q\tau_a} \mathbf{1}_{\{X_{\tau_a} = a, \tau_a^- < \tau_a^+\}} \right].
\]
Now the identities of the theorem follow by plugging (4), (5) and (6) into the above equation, while noting that if $x = X_0 = \alpha$ and $X$ has paths of bounded variation, then $X_{\tau_a^-} = \alpha$ is an event which has probability 0, whereas if $x = X_0 = \alpha$ and $X$ has paths of unbounded variation, then $\tau_a^- = 0$ and $X_{\tau_a^-} = \alpha$ almost surely, which implies in addition that $W^{(q)}(0) = 0$, cf. (4).

**Proof of Corollary 3.** The corollary follows easily, since by the extra condition assumed, the last term of (7) vanishes and by Lemma 4 we are allowed to split the integral into two terms.

**4 Overshoot identities for reflected and refracted Lévy processes**

Following the proof of Theorem 2, one can easily establish identities involving the overshoot of reflected or refracted spectrally negative Lévy processes as well. To this end, let $Z = \{Z_t : t \geq 0\}$ be the process $X$ reflected at level $b \in \mathbb{R}$, i.e.
\[
Z_t = X_t - \xi_t, \quad \text{where} \quad \xi_t = \left( \sup_{0 \leq s \leq t} (X_s - b) \lor 0 \right)
\]
and define the stopping time
\[
T_a^- = \inf\{t > 0 : Z_t < a\}.
\]
Further, let $U = \{U_t : t \geq 0\}$ be the process $X$ refracted at level $c \in \mathbb{R}$, i.e. $U$ is the strong solution to the stochastic differential equation,
\[
dU_t = dX_t - \delta \mathbf{1}_{\{U_t > c\}} dt, \quad U_0 = X_0, \quad (13)
\]
where $0 < \delta < \gamma + \int_0^1 \theta \Pi(d\theta)$. By Theorem 1 of [7], the process $U$ is well-defined. We denote the first passage times of $U$ above and below a level by
\[
\kappa_a^+ = \inf\{t > 0 : U_t > a\}, \quad \text{and} \quad \kappa_a^- = \inf\{t > 0 : U_t < a\}.
\]
Then under the conditions of Theorem 2, we have for \( a < x \leq b \),

\[
\mathbb{E}_x \left[ e^{-qT_x} f(Z_{T_x}) \right] = \tilde{f}(x) - \mathbb{E}_x \left[ \int_{0}^{T_x} e^{-qs} d\xi_s \right] \tilde{f}'(b) \\
+ \int_a^b (A - q) \tilde{f}(z) \left[ \int_{z}^{\infty} e^{-qs} \mathbb{P}_x(Z_s \in dz, s < T_x) ds \right] dz \\
+ (f(a) - \tilde{f}(a+)) \mathbb{E}_x \left[ e^{-qT_x} 1_{\{X_{T_x} = a\}} \right]
\]

and for \( c \in (a, b) \),

\[
\mathbb{E}_x \left[ e^{-q\tau_x} f(U_{\kappa_x}) 1_{\{\kappa_x^- < \kappa_x^+\}} \right] = \tilde{f}(x) - \mathbb{E}_x \left[ e^{-q\tau_x} 1_{\{\kappa_x^+ < \kappa_x^-\}} \right] \tilde{f}(b) \\
+ \int_a^b ((A - q) \tilde{f}(z) - \delta 1_{\{z>c\}} \tilde{f}'(z)) \left[ \int_{0}^{\infty} e^{-qs} \mathbb{P}_x(U_s \in dz, s < \kappa_x^- \land \kappa_x^+) ds \right] dz \\
+ (f(a) - \tilde{f}(a+)) \mathbb{E}_x \left[ e^{-q\tau_x} 1_{\{U_{\kappa_x} = a, \kappa_x^- < \kappa_x^+\}} \right].
\]

The proof of these two identities is almost the same as the proof of (12) given in Section 3 and we leave the details to the reader. Note that all the expectations and resolvent measures on the right hand sides of (14) and (15) admit analytic expressions in terms of scale functions. In particular, see Theorem 10.3 in [6], respectively Theorem 4(i) in [7], for the first expectation on the right hand side of (14), respectively (15). Further, see Theorem 1.(ii) in [13] and Theorem 6(i) in [7] for the two \( q \)-resolvent measures and note that the two expectations involving the event of creeping are non-zero if and only if the Gaussian coefficient is non-zero and in that case, these expectations can be derived from the corresponding resolvent measure, see the proof of Corollary 2 in [14]. Note also that (14) for the special case \( \tilde{f}(y) = f'_-(a)y + f(a), \ y \in (a, b] \) is given in Proposition 5.5 of [1].

5 Examples

In order to illustrate why (7) and (8) are useful with a simple example, let us take \( f(y) = 1 \) for \( y \leq a \) in (1). It is well known, see e.g [6, Theorem 8.1] or combine (4) and (5), that for \( x \in [a, b] \),

\[
\mathbb{E}_x \left[ e^{-q\tau_x} 1_{\{\tau_x^- < \tau_x^+\}} \right] = Z(q)(x - a) - \frac{W^{(q)}(x - a)}{W^{(q)}(b - a)} Z(q)(b - a),
\]

(16)
where \( Z^{(q)}(y) = 1 + q \int_0^y W^{(q)}(z) \, dz, \ y \in \mathbb{R} \). Using (9), which corresponds to choosing the extension \( \tilde{f}(y) = 0 \) for \( a < y \leq b \), we get for \( a < x \leq b \),

\[
\mathbb{E}_x \left[ e^{-\eta \tau_z} 1_{\{\tau_z < \tau_b^+\}} \right] = \int_a^b \Pi(z-a, \infty) \left[ \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} W^{(q)}(b-z) - W^{(q)}(x-z) \right] \, dz + \frac{\sigma^2}{2} \left( W^{(q)}(x-a) - \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} W^{(q)}(b-a) \right).
\]

If we did not know the identity (16), it would not at all be obvious that the right hand side of (17) actually simplifies to the right hand side of (16). If we instead use the extension \( \tilde{f}(y) = 1 \) for \( a < y \leq b \), then Corollary 3 gives us directly (16), since for that choice \( (A - q) \tilde{f}(z) = -q \), \( a < z < b \).

We now consider a more interesting example. Let \( U = \{ U_t : t \geq 0 \} \) be the refracted Lévy process defined by the stochastic differential equation

\[
dU_t = dX_t - \delta 1_{\{U_t > b\}} dt, \quad U_0 = X_0,
\]

where \( 0 < \delta < \gamma + \int_0^1 \theta \Pi(d\theta) \). Note that compared to (13) we have now denoted the refraction level by \( b \) instead of \( c \), so that we are consistent with the notation used in e.g. [7, 9–11, 15]. We further let \( Y = \{ Y_t : t \geq 0 \} \) be the spectrally negative Lévy process defined by \( Y_t = X_t - \delta t \) and denote by \( W^{(q)}(x) \) the \( q \)-scale function of \( Y \). With

\[
\nu_a^+ = \inf\{ t > 0 : Y_t > a \}, \quad \text{and} \quad \nu_a^- = \inf\{ t > 0 : U_t < a \},
\]

we look at the following special case of (1),

\[
\mathbb{E}_x \left[ e^{-\eta \nu_a^-} W^{(q)}(Y_{\nu_a^-}) 1_{\{\nu_a^- < \nu_b^+\}} \right],
\]

where \( p, q \geq 0 \), \( b \leq x \leq c \) and recall \( W^{(q)}(x) \) is the scale function of \( X \). This particular case has appeared in the study of refracted Lévy processes and occupation times of (refracted) spectrally negative Lévy processes, see [7, 9–11, 15]. We want to obtain an as simple as possible expression for (18). For this we use (8) with \( \tilde{f}(y) = W^{(q)}(y) \) being the obvious choice for the extension. However, in general it is unknown whether this extension satisfies the required smoothness conditions of Theorem 2; in particular it is unknown if \( A W^{(q)}(x) \) is well-defined. We therefore restrict ourselves to the case were \( X \) (equivalently \( Y \)) has paths of bounded variation or \( \sigma > 0 \). In that case \( W^{(q)}(x) \) (restricted to \( (-\infty, b] \)) lies in \( \mathcal{H}(X; a, b) = \mathcal{H}(Y; a, b) \), cf. Equations (4)-(6) and Theorem 1 in [2]. Further, when \( X \) has paths of bounded variation, one can easily show by taking Laplace transforms and using (3) and [6, Lemma 8.6] that \( (A - q) W^{(q)}(x) = 0 \) for almost every \( x > 0 \). If \( \sigma > 0 \), then
\((A - q)W^{(q)}(x) = 0\) for every \(x > 0\), see e.g. p.694 of [2]. Combining everything and denoting \(A h(x) = A h(x) - \delta h''(x)\) for \(h \in \mathcal{H}(X; a, b)\), we conclude that

\[
\mathbb{E}_x \left[ e^{-p \nu_\delta} W^{(q)}(Y_{\nu_\delta}^-) \mathbf{1}_{\{\nu_\delta^- < \nu_\delta^+\}} \right]
\]

\[
= W^{(q)}(x) - \int_b^c (A - p)W^{(q)}(z) \mathbb{W}^{(p)}(x - z) dz
\]

\[
- \frac{\mathbb{W}^{(p)}(x - b)}{\mathbb{W}^{(p)}(c - b)} \left[ W^{(q)}(c) - \int_b^c (A - p)W^{(q)}(z) \mathbb{W}^{(p)}(b - z) dz \right]
\]

\[
= W^{(q)}(x) - \int_b^c \left( (q - p)W^{(q)}(z) - \delta W^{(q)}(z)^\prime \right) \mathbb{W}^{(p)}(x - z) dz
\]

\[
- \frac{\mathbb{W}^{(p)}(x - b)}{\mathbb{W}^{(p)}(c - b)} \left[ W^{(q)}(c) - \int_b^c \left( (q - p)W^{(q)}(z) - \delta W^{(q)}(z)^\prime \right) \mathbb{W}^{(p)}(b - z) dz \right].
\]

Equation (19) has been derived earlier, see [7, Theorem 16] in combination with (9) for the case \(p = q\) and \(X\) having paths of bounded variation, see [11, Section 2] for the case \(\delta = 0\) and see [15, Lemma 3.1] in combination with [11, Equation (6)] for the general case. Whereas in these references it took quite some effort to get to (19), with Theorem 2 or Corollary 3 it is obvious how to come up with this relatively simple expression. We remark that getting as simple as possible, analytic expressions for (18) (and in general (1)) is not just useful for evaluating this expectation. It also allows one to tackle more complicated cases, compare e.g. the main results of [10] and [11]. Further in the context of refracted Lévy processes, obtaining (19) (for the case \(p = q\)) was a crucial step in [7] for ultimately showing the existence of these processes in the case where \(X\) has paths of unbounded variation and also for solving the related optimal control problem with bounded dividend rates, cf. [8].

Remark 5. The expressions for the overshoot derived in this paper are all given in terms of scale functions for which in general only the Laplace transform is known. However, there are plenty of examples of spectrally negative Lévy processes for which an explicit formula (though the degree of explicitness can vary case by case) exists for the scale function \(W^{(q)}(x)\), cf. Chapter 9 of [6]. We refer to Example 2 in [7] and Example 1.1 in [11] for examples where (a subcase of) (19) is worked out in the case where the Lévy measure is given by a mixture of exponentials. On the other hand, in cases where a closed-form expression for the scale function is not available, there are good numerical methods for dealing with Laplace inversion of the scale function, see Section 5 of [5].
References

[1] F. Avram, Z. Palmowski, and M.R. Pistorius, *On Gerber-Shiu functions and optimal dividend distribution for a Lévy risk-process in the presence of a penalty function - a probabilistic approach*, 2014. arXiv:1110.4965v4 [math.PR].

[2] T. Chan, A. E. Kyprianou, and M. Savov, *Smoothness of scale functions for spectrally negative Lévy processes*, Probab. Theory Related Fields 150 (2011), no. 3-4, 691–708.

[3] Hans U. Gerber and Elias S. W. Shiu, *On the time value of ruin*, N. Am. Actuar. J. 2 (1998), no. 1, 48–78. With discussion and a reply by the authors.

[4] Jean Jacod and Albert N. Shiryaev, *Limit theorems for stochastic processes*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003.

[5] A. Kuznetsov, A.E. Kyprianou, and V. Rivero, *The theory of scale functions for spectrally negative Lévy processes*, Lévy matters II, Lecture Notes in Mathematics, vol. 2061, Springer, Heidelberg, 2012, pp. 97–186.

[6] Andreas E. Kyprianou, *Fluctuations of Lévy processes with applications*, 2nd ed., Universitext, Springer, Heidelberg, 2014. Introductory lectures.

[7] A. E. Kyprianou and R. L. Loeffen, *Refracted Lévy processes*, Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 1, 24–44 (English, with English and French summaries).

[8] Andreas E. Kyprianou, Ronnie Loeffen, and José-Luis Pérez, *Optimal control with absolutely continuous strategies for spectrally negative Lévy processes*, J. Appl. Probab. 49 (2012), no. 1, 150–166.

[9] A.E. Kyprianou, J.C. Pardo, and J.L. Pérez, *Occupation times of refracted Lévy processes*, 2013. To appear in J. Theor. Probab.

[10] D. Landriault, J.-F. Renaud, and X. Zhou, *Occupation times of spectrally negative Lévy processes with applications*, Stochastic Process. Appl. 121 (2011), no. 11, 2629–2641.

[11] Ronnie L. Loeffen, Jean-François Renaud, and Xiaowen Zhou, *Occupation times of intervals until first passage times for spectrally negative Lévy processes*, Stochastic Process. Appl. 124 (2014), no. 3, 1408–1435.

[12] Philip E. Protter, *Stochastic integration and differential equations*, 2nd ed., Applications of Mathematics (New York), vol. 21, Springer-Verlag, Berlin, 2004. Stochastic Modelling and Applied Probability.

[13] M. R. Pistorius, *On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum*, J. Theoret. Probab. 17 (2004), no. 1, 183–220.

[14] Martijn R. Pistorius, *A potential-theoretical review of some exit problems of spectrally negative Lévy processes*, Séminaire de Probabilités XXXVIII, Lecture Notes in Math., vol. 1857, Springer, Berlin, 2005, pp. 30–41.

[15] J.-F. Renaud, *On the time spent in the red by a refracted Lévy risk process*, 2014. To appear in J. Appl. Probab.

[16] Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original; Revised by the author.