DIFFERENTIAL FORMS AND ODD SYMPLECTIC GEOMETRY

HOVHANNES M. KHUDAVERDIAN AND THEODORE TH. VORONOV

Abstract. We recall the main facts about the odd Laplacian acting on half-densities on an odd symplectic manifold and discuss a homological interpretation for it suggested recently by P. Ševera. We study the relationship of odd symplectic geometry with classical objects. We show that the Berezinian of a canonical transformation for an odd symplectic form is a polynomial in matrix entries and a complete square. This is a simple but fundamental fact, parallel to Liouville's theorem for an even symplectic structure. We draw attention to the fact that the de Rham complex on $M$ naturally admits an action of the supergroup of all canonical transformations of $\Pi\Pi^*M$. The infinitesimal generators of this action turn out to be the classical 'Lie derivatives of differential forms along multivector fields'.

Odd symplectic geometry (more generally, odd Poisson geometry) or the geometry of odd brackets is the mathematical basis of the Batalin–Vilkovisky method \cite{1, 2, 3} in quantum field theory.

Odd symplectic geometry possesses features connecting it with both classical ("even") symplectic geometry and Riemannian geometry. In particular, odd Laplace operators arise naturally on an odd symplectic manifold, i.e., the second order differential operators whose principal symbol is the odd quadratic form corresponding to the odd bracket \cite{4}. The key difference from the Riemannian case is that the definition of an odd Laplace operator, in general, requires an extra piece of data besides the "metric", namely, a choice of a volume form (even for a Laplacian acting on functions). This is due to the fundamental fact that on an odd symplectic manifold there is no invariant volume element \cite{4}.

However, as discovered by one of the authors, there is one isolated case where an odd Laplacian is defined canonically by the symplectic structure without any extra data \cite{6, 7, 8}. It is an operator acting on densities of weight $1/2$ (half-densities or semidensities). This fact is not obvious, and there is no simple explanation. A known proof is based on an analysis of the canonical transformations of the odd bracket. In works \cite{9, 11, 10} further phenomena related with odd Laplacians on odd Poisson manifolds were discovered, such as the existence of a natural 'master' groupoid acting on volume forms, its orbits corresponding to Laplacians on half-densities. The symplectic case is distinguished...
by the existence of a distinguished orbit, which gives the “canonical” operator.

In a very interesting recent paper [16], P. Ševera suggested a homological interpretation of the canonical odd Laplace operator on half-densities as one of the higher differentials in a certain natural spectral sequence associated with the odd symplectic structure.

In our paper we in particular discuss this interpretation and show that there is a simple but fundamental underlying fact from linear algebra, concerning the Berezinian of a canonical transformation for an odd symplectic bracket. It is the formula

$$\text{Ber} J = (\det J_{00})^2 \quad (1)$$

for $J$ in the odd symplectic supergroup, where $J_{00}$ is the even-even block. Hence the Berezinian is an entire rational function and, moreover, a complete square. There are many geometric facts related with formula (1), which can be found in the literature on odd brackets and the BV formalism. See, for example, [14, 15, 4, 5, 7]. We want to draw attention to it as a simple identity for matrices. In view of it, half-densities on an odd symplectic manifold are ‘tensor’ objects, i.e., transforming according to a polynomial representation. They can be seen as virtual differential forms on a Lagrangian surface. When such a surface is fixed, they become (isomorphic to) actual forms. More precisely, for a manifold or supermanifold $M$, we can identify (pseudo)integral forms on $M$, i.e., multivector densities, with half-densities on the odd symplectic manifold $\Pi T^* M$. (Pseudo)integral forms are related with (pseudo)differential forms by a sort of ‘Fourier transform’. Therefore we see that in the space of differential forms on an ordinary manifold, there is a natural representation of the supergroup of canonical transformations of the odd bracket. We give a clear description of this action in classical terms. The invariance of the de Rham differential under such a supergroup, which is absolutely transparent, is equivalent to the existence of the canonical odd Laplacian, but expressed in a different language.

1. Recollection of the canonical odd Laplacian

In this section we review the construction of the odd Laplacian on half-densities due to [6]. See also [14, 15, 4, 5, 7].

Let $M$ be a supermanifold endowed with an odd symplectic structure, given by an odd 2-form $\omega$. We shall refer to such supermanifolds as odd symplectic manifolds. (We always skip the prefix ‘super-’ unless required to avoid confusion.) Later we shall discuss the more general case of an odd Poisson manifold. A brief definition of the odd Laplacian acting on half-densities on $M$ follows.
Consider a cover of \( M \) by Darboux charts, in which the symplectic form takes the canonical expression \( \omega = dx^i d\xi_i \). Here \( x^i, \xi_i \) are canonically conjugate variables of opposite parity. We assume that the \( x^i \) are even; hence the \( \xi_i \), odd. Let \( Dy \), for any kind of variables \( y \), stand for the Berezin volume element. Then half-densities on \( M \) locally look like \( \sigma = s(x, \xi)(D(x, \xi))^{1/2} \). (Notice that we skip questions related with orientation.) We set

\[
\Delta \sigma := \frac{\partial^2 s}{\partial x^i \partial \xi_i} \left( D(x, \xi) \right)^{1/2},
\]

in Darboux coordinates, and call \( \Delta \), the canonical odd Laplacian on half-densities.

The simplicity of formula (2) is very deceptive. The expression \( \frac{\partial}{\partial x^i} \frac{\partial f}{\partial \xi_i} \) was originally suggested by Batalin and Vilkovisky, and is the famous ‘BV operator’. However, the trouble is that it is not well-defined on functions (actually, on any objects) unless we fix a volume form, which should therefore enter the definition. The geometrically invariant construction for functions, using a volume form, was first given in [4].

There is no canonical volume form on an odd symplectic manifold (unlike even symplectic manifolds, enjoying the Liouville form). In particular, the coordinate volume form \( D(x, \xi) \) for Darboux coordinates is not preserved by the (canonical) coordinate transformations (see later). Hence the invariance of the operator \( \Delta \) given by (2) is a deep geometric fact.

As we showed in [9], on any odd Poisson, in particular, odd symplectic, manifold there is a natural master groupoid of ‘changes of volume forms’ \( \rho \mapsto e^S \rho \) satisfying the master equation \( \Delta_\rho e^{S/2} = 0 \) (note \( 1/2 \) in the exponent; without it there would be no groupoid). Here \( \Delta_\rho \) is the odd Laplacian on functions with respect to the given volume form \( \rho \). It is defined by \( \Delta_\rho f := \text{div}_\rho X_f \), where \( X_f \) is the Hamiltonian vector field corresponding to \( f \). (See [4]; note also [12] for another approach.)

In a similar way one can define the odd Laplacian on any densities — again, depending on a chosen volume form. Now, half-densities are distinguished from densities of other weights precisely by the fact that for them the corresponding odd Laplacian would depend only on the orbit of a volume form with respect to the action of the above groupoid [9].

It turns out that on an odd symplectic manifold, all Darboux coordinate volume forms belong to the same orbit of the master groupoid. We can regard it as a ‘preferred orbit’; hence, in the absence of an invariant volume form, the odd Laplacian on half-densities defined by an arbitrary Darboux coordinate volume form is invariant. It is just (2).

2. Homological interpretation of the odd Laplacian

Now we are going to approach \( \Delta \) on half-densities from a very different angle.
Let \( \Omega(M) \) be the space of all pseudodifferential forms on \( M \), i.e., functions on \( \Pi T M \). (As usual, \( \Pi \) stands for the parity reversion functor on vector spaces, vector bundles, etc.) In coordinates such functions have the form \( s = s(x, \xi, dx, d\xi) \), where the differentials of coordinates are commuting variables of parity opposite to that of the respective coordinate. In our case \( dx^i \) are odd and \( d\xi_i \) are even. We do not assume that functions \( s(x, \xi, dx, d\xi) \) are polynomial in \( d\xi_i \). Of course they are (Grassmann) polynomial in \( dx_i \), because these variables are odd.

Consider the odd symplectic form \( \omega \). Since \( \omega^2 = 0 \), multiplication by \( \omega \) can be considered as a differential. Define the operator \( D = d + \omega \), where \( d \) is the de Rham differential. Since \( d\omega = 0 \), it follows that \( D^2 = 0 \) and we have a ‘double complex’ \( (\Omega(M), D = \omega + d) \). Warning: here a complex means just a \( \mathbb{Z}_2 \)-graded object.

The reader should bear in mind that since \( \omega = d\Theta \) for some even 1-form \( \Theta \), which is true globally, we have \( D = e^{-\Theta} \circ d \circ e^\Theta \) and the multiplication by the inhomogeneous differential form \( e^\Theta \) sets an isomorphism between the complexes \( (\Omega(M), D) \) and \( (\Omega(M), d) \). It follows that \( H(\Omega(M), D) \) is isomorphic to \( H(\Omega(M), d) \), which is just the de Rham cohomology of the underlying manifold \( M_0 \). (Note that the isomorphism \( e^\Theta \) preserves only parity, but not \( \mathbb{Z} \)-grading, even if we restrict it to differential forms on \( M \), i.e., polynomials in \( dx, d\xi \).)

The operator \( D = \omega + d \) was introduced in [16]. The idea was to consider the spectral sequence for \( (\Omega(M), D) \) regarded as a double complex. We shall follow it in a form best suiting our purposes and which is slightly different from [16]. (In particular, we do not assume grading in the space of forms.)

Although there is no \( \mathbb{Z} \)-grading present, single or double, one can still develop the machinery of spectral sequences as follows.

We define linear relations (see [13]) on \( \Omega(M) \):

\[
\partial_0 := \left\{ (\alpha, \beta) \in \Omega(M) \times \Omega(M) \mid \omega \alpha = \beta \right\},
\]

and

\[
\partial_r := \left\{ (\alpha, \beta) \in \Omega(M) \times \Omega(M) \mid \exists \alpha_1, \ldots, \alpha_r \in \Omega(M) : \omega \alpha = 0,
\right. \\
\left. d\alpha + \omega \alpha_1 = 0, \ldots, d\alpha_{r-2} + \omega \alpha_{r-1} = 0, d\alpha_{r-1} + \omega \alpha_r = \beta \right\}
\]

for all \( r = 1, 2, 3, \ldots \). We also set \( \partial_{-1} := \{(\alpha, 0)\} \). We have subspaces \( \text{Ker} \partial_r \), \( \text{Def} \partial_r \) (the domain of definition), \( \text{Ind} \partial_r \) (the indeterminancy), and \( \text{Im} \partial_r \) in \( \Omega(M) \), and by a direct check

\[
\text{Im} \partial_r \subset \text{Ker} \partial_r ,
\]

\[
\text{Def} \partial_r = \text{Ker} \partial_{r-1} ,
\]

\[
\text{Ind} \partial_r = \text{Im} \partial_{r-1} .
\]
That is, we have a sequence of differential relations on $\Omega(M)$, defining a spectral sequence $(E_r, d_r)$ where

$$E_r := \frac{\text{Ker } \partial_{r-1}}{\text{Im } \partial_{r-1}} = \frac{\text{Def } \partial_r}{\text{Ind } \partial_r}$$

and the homomorphism $d_r : E_r \to E_r$ is induced by $\partial_r$ in the obvious way. (In fact, differential relations like this is the shortest way of defining spectral sequences, see [13, p. 340].)

Clearly $E_0 = \Omega(M)$. The relation $\partial_0$ is simply the graph of the linear map $d_0 : \Omega(M) \to \Omega(M)$, $d_0 \alpha = \omega \alpha$. What is $E_1$?

**Theorem 1.** The space $E_1$ can be naturally identified with the space of half-densities on $M$.

A proof consists of two independent steps. First, we find the cohomology of $d_0$ using algebra. Second, we identify the result with a geometrical object. The first part goes as follows.

The operator $d_0 = \omega$ is a Koszul type differential, since in an arbitrary Darboux chart $\omega = dx^i d\xi_i$. Introduce a $\mathbb{Z}$-grading by the degree in the odd variables $dx^i$. The operator $d_0$ increases the degree by one. (This grading is not preserved by changes of coordinates.) From general theory it follows that the cohomology should be concentrated in the “maximal degree”. Indeed, suppose that dim $M = n$ and consider the linear operator $H$ on pseudodifferential forms defined as follows. For $\sigma = \sigma(x, \xi, dx, d\xi)$,

$$H\sigma(x, \xi, dx, d\xi) := \int_0^1 dt t^{n-1} \frac{\partial^2 \sigma}{\partial dx^i \partial d\xi_i}(x, \xi, t^{-1} dx, t d\xi),$$

— notice the similarity with the $\Delta$-operator. The operator $H$ is well defined on all forms of degree less than $n$ in $dx^i$ and on forms of ‘top’ degree if they vanish at $d\xi_i = 0$. (In both cases there will be no problem with division by $t$.) For forms on which $H$ makes sense one can check that

$$(Hd_0 + d_0 H)\sigma = \sigma.$$ 

In particular, if a form $\sigma$ is $d_0$-closed and of degree less than $n$ in $dx^i$, then $\sigma = d_0 H \sigma$. The same applies for a top degree form taking a non-zero value at $d\xi_i = 0$. Hence the $d_0$-cohomology “sits on” pseudodifferential forms of degree $n$ in $dx^i$ that do not depend on $d\xi_i$:

$$\sigma = s(x, \xi) dx^1 \ldots dx^n.$$ 

No non-zero form of this appearance can be cohomologous to zero: indeed, any $d_0$-exact form, $d_0 \tau = \omega \tau$, vanishes at $d\xi_i = 0$.

Hence, each $d_0$-cohomology class has a unique representative in a given Darboux coordinate system $x^i, \xi_i$. It is obtained by taking an arbitrary form from the class, extracting its component of degree $n$ in $dx^i$ and evaluating at $d\xi_i = 0$. By applying this to the class of $dx^1 \ldots dx^n$, we immediately arrive at
Lemma 2.1. Elements of the cohomology space $E_1 = H(\Omega(M), \omega)$ are represented in Darboux coordinates as classes
\[
\sigma = s(x, \xi) [dx^1 \ldots dx^n],
\]
where under a change of Darboux coordinates
\[
x^i = x^i(x', \xi'),
\]
\[
\xi_i = \xi_i(x', \xi')
\]
the class $[dx^1 \ldots dx^n]$ transforms as follows:
\[
[dx^1 \ldots dx^n] = \det J_{00} \cdot [dx'^1 \ldots dx'^n].
\]
Here $J_{00} = \frac{\partial x}{\partial x'}$ is the even-even block of the Jacobi matrix $J = \frac{\partial (x, \xi)}{\partial (x', \xi')}$. To better appreciate the statement, notice that $dx^i = dx^i' \frac{\partial x^i}{\partial x'^i} + d\xi^i' \frac{\partial x^i}{\partial \xi'^i}$. Hence
\[
dx^1 \ldots dx^n = dx'^1 \ldots dx'^n \cdot \det \left( \frac{\partial x^i}{\partial x'^i} \right) + \text{terms containing } d\xi^i'.
\]
Passing to cohomology is equivalent to discarding these lower order terms.

What kind of geometrical object is this?

Lemma 2.2. Objects of the form $\sigma = s(x, \xi) [dx^1 \ldots dx^n]$, in Darboux coordinates, with the transformation law given in Lemma 2.1 can be identified with half-densities on $M$.

This is the crucial claim. There is a simple but fundamental fact from linear algebra behind Lemma 2.2 which will be proved in the next section.

The transformation law for $[dx^1 \ldots dx^n]$ can be obtained from the formal "law" $[dx^i] = [dx'^i] \frac{\partial x^i}{\partial x'^i}$. Unfortunately, it does not define a geometric object, because it does not obey the cocycle condition. In a way, it is only a ‘virtual’ transformation law, which will make sense only if an extra structure is imposed on $M$.

Now as we have the space $E_1$, let us check the differential $d_1$ on it. It is induced by the differential relation $\partial_1$ on $\Omega(M)$. Take an element $\sigma = s(x, \xi) [dx^1 \ldots dx^n] \in E_1$, take its representative $\alpha = s(x, \xi) dx^1 \ldots dx^n$ and consider $\beta \in \Omega(M)$ such that $d\alpha + \omega \alpha_1 = \beta$, for $\alpha_1 \in \Omega(M)$. We will have $[\beta] = d_1 \sigma$ for the class $[\beta]$ in $E_1$. Notice that $d\alpha = d\xi^i \frac{\partial s}{\partial \xi^i} dx^1 \ldots dx^n$ and it will vanish at $d\xi_i = 0$, therefore it is an $\omega$-exact form, according to our previous analysis. Thus $d_1 = 0$ identically and $E_2 = E_1$.

Consider $d_2$ on $E_1 = E_2 = H(\Omega(M), \omega)$. By definition, $d_2$ maps the class $\sigma = s(x, \xi) [dx^1 \ldots dx^n]$, with a local representative $\alpha =
\[ s(x, \xi) \, dx^1 \ldots dx^n, \] to the class of \( \beta \in \Omega(M) \) such that \( d\alpha + \omega \alpha_1 = 0, \ d\alpha_1 + \omega \alpha_2 = \beta \), for some \( \alpha_1 \) and \( \alpha_2 \). We may set \( \alpha_1 := -Hd\alpha \), where \( H \) is the homotopy operator defined above, and \( \beta := d\alpha_1 = -dHd\alpha \).

Directly:

\[
Hd\alpha = H \left( \frac{\partial s}{\partial \xi_i} \, dx^1 \ldots dx^n \right) = \sum (-1)^{i+\hat{s}} \frac{\partial s}{\partial \xi_i} \, \hat{dx}^1 \ldots \hat{dx}^i \ldots dx^n
\]

and

\[
\beta = -dHd\alpha = -d \sum (-1)^{i-1+\hat{s}} \frac{\partial s}{\partial \xi_i} \, dx^1 \ldots \hat{dx}^i \ldots dx^n =
- \frac{\partial^2 s}{\partial x^i \partial \xi_i} \, dx^1 \ldots dx^n + \text{lower order terms in } dx.
\]

Hence in \( E_1 \) we get:

\[
d_2\sigma = d_2 \left( s(x, \xi) \, [dx^1 \ldots dx^n] \right) = -\frac{\partial^2 s}{\partial x^i \partial \xi_i} \, [dx^1 \ldots dx^n] = -\Delta \sigma,
\]
which is quite remarkable. What about the space \( E_3 \) and the differential \( d_3 \), and so on?

It is not hard to notice that the cohomology of the \( \Delta \)-operator on half-densities on \( M \) is isomorphic to the de Rham cohomology of the underlying ordinary manifold \( M_0 \) (we shall say more about this later). Locally the cohomology vanishes except for constants: \( \sigma = \text{const} \cdot [dx^1 \ldots dx^n] \). Thus \( d_3 = 0 \), and \( E_4 = E_3 \); the same continues for \( d_4 = 0 \), \( E_5 = E_4 = E_3 \), and so on. We arrive at the following statement (which was the main result of [16]):

**Theorem 2.** With the identification of the space \( E_1 = H(\Omega(M), \omega) \) with half-densities on \( M \), the differential \( d_1 \) vanishes and the next differential \( d_2 \) coincides up to a sign with the canonical odd Laplacian. The spectral sequence \( (E_r, d_r) \) degenerates at the term \( E_3 \), which is the cohomology of the operator \( \Delta \).

The importance of Theorem 2 is in the fact that it gives an alternative proof of the invariance of the odd Laplacian on half-densities \( \Delta \), by identifying it with an operator in a spectral sequence invariantly associated with the odd symplectic structure.

3. **Berezinian of a canonical transformation**

Consider a vector space \( V = V_0 \oplus V_1 \) with an odd symplectic structure, i.e., an odd non-degenerate antisymmetric bilinear form. (A choice of ‘antisymmetric’ or ‘symmetric’ does not make any difference.) Necessarily \( \text{dim } V = n|n \). We call matrices preserving this form, *symplectic*. This should not cause problems; when comparing them
with ordinary symplectic matrices corresponding to an even symplectic structure, we shall make the reference to the parity of the bilinear form explicit.

**Theorem 3.** Suppose that $J$ is a symplectic matrix for an odd symplectic space. Let

$$ J = \begin{pmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{pmatrix} $$

be its standard block decomposition. Then

$$ \text{Ber} \, J = (\det J_{00})^2. \quad (3) $$

**Proof.** We can write the matrix of our symplectic form as

$$ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

The relation for $J$ is $JBJ^T = B$, where the operation of matrix transpose takes into account the parities of the blocks:

$$ \begin{pmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{pmatrix}^T = \begin{pmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -J_{01}^T & J_{10}^T \\ J_{11}^T & J_{00}^T \end{pmatrix}. $$

Hence we obtain

$$ J_{00}J_{01}^T = (J_{00}J_{01}^T)^T, \quad (4) $$
$$ J_{11}J_{10}^T = -(J_{11}J_{10}^T)^T, \quad (5) $$
$$ J_{00}J_{11}^T + J_{01}J_{10}^T = 1. \quad (6) $$

From (6) we may express

$$ J_{11} = J_{00}^{-1} + J_{10}J_{01}^{-1}J_{00}^{-1} = J_{00}^{-1} + J_{10}J_{00}^{-1}J_{01}, \quad \text{taking into account (4)}.$$

We arrive at the identity

$$ J_{11} - J_{10}J_{00}^{-1}J_{01} = J_{00}^{-1}. \quad (7) $$

Therefore

$$ \text{Ber} \, J = \frac{\det J_{00}}{\det(J_{11} - J_{10}J_{00}^{-1}J_{01})} = \frac{\det J_{00}}{\det J_{00}^{-1}} = (\det J_{00})^2. $$

Notice that the steps we followed in the proof are the same as may be used for proving the classical Liouville theorem, i.e., that $\det J = 1$ for an ordinary symplectic matrix $J \in \text{Sp}(2n)$. The decomposition of $V$ in that case will be the decomposition into the sum of Lagrangian subspaces and an identity similar to (7) will be valid. The difference will arise only when calculating the determinant: instead of the ratio of the determinants of the blocks, there will be the product, which will give 1 instead of $\det J_{00}^2$. 
It is easy to generalize. Let $V$ be a vector (super)space with a symplectic structure, even or odd. Consider its decomposition into the sum of two Lagrangian subspaces. In the even case they will have the same dimensions; in the odd case, the opposite, i.e., $p|n-p$ and $n-p|p$. Denote the chosen decomposition by $V = V_0 \oplus V_1$. Here the indices have nothing to do with parity. By picking ‘canonically conjugate’ bases in $V_0$ and $V_1$ we arrive at a picture formally the same as above. The Berezinian can be calculated using the corresponding block decompositions. It will be either

$$\text{Ber} J = \text{Ber} J_{00} \cdot \text{Ber} (J_{11} - J_{10} J_{00}^{-1} J_{01})$$

or

$$\text{Ber} J = \frac{\text{Ber} J_{00}}{\text{Ber} (J_{11} - J_{10} J_{00}^{-1} J_{01})}$$

depending on the parity of the symplectic form (in the odd case the ‘formats’ of the matrix blocks will be the opposite, hence division). Then the analog of the identity (7) should be applied. For even symplectic structure we thus obtain the analog of Liouville’s theorem, and for odd, we arrive at

**Theorem 4.** Let $J$ be a symplectic transformation of an odd symplectic space $V$. Then for an arbitrary decomposition into the sum of Lagrangian subspaces, $V = V_0 \oplus V_1$ (indices not indicating parity), the identity

$$\text{Ber} J = (\text{Ber} J_{00})^2$$

holds.

**Remark 3.1.** Theorem 3 gives, in particular, that the Berezinian of a symplectic matrix is a polynomial in the matrix entries and, moreover, a complete square. This is somewhat masked in the more general Theorem 4.

There is an ‘abstract’ argument parallel to the calculation above. Consider a decomposition into Lagrangian subspaces $V = V_0 \oplus V_1$, for an even or odd symplectic form. (This works in the same way for symmetric forms.) Consider the dual space $V_1^*$. We have $V_1^* \cong \text{Ann} V_0 \subset V^*$ and, using the form, can replace the annihilator by the orthogonal complement:

$$V_1^* \cong V_0^\perp \subset V$$

for the even form or

$$V_1^* \cong \Pi V_0^\perp \subset \Pi V$$

for the odd form. Recalling that $V_0$ and $V_1$ are Lagrangian subspaces, we get $V_1^* \cong V_0$ or $V_1^* \cong \Pi V_0$. Hence

$$\text{Ber} V = \text{Ber} (V_0 \oplus V_1) = \text{Ber} V_0 \otimes \text{Ber} V_1$$

$$= \text{Ber} V_0 \otimes \text{Ber} V_0^* = 1$$
(even symplectic form) or

\[ \text{Ber} V_0 \otimes \text{Ber} \Pi V_0^* = (\text{Ber} V_0)^{\otimes 2} \]

(odd symplectic form). Equalities here mean natural isomorphisms.

A different abstract argument based on the well-known interpretation of the space \( \text{Ber} V \) as the cohomology of a Koszul complex and justifying the equality \( \text{Ber} V = (\text{Ber} V_0)^{\otimes 2} \) for an odd symplectic space was given in [16].

A weak point of abstract arguments is that they do not really give information about matrices, which is necessary in applications such as a proof of Lemma 2.2.

4. “Supersymmetries” of differential forms

In this section we change viewpoint. We would like to phrase the previous constructions entirely in the language of ‘classical’ differential-geometric objects. In this way we shall see how the canonical odd Laplacian on half-densities on an odd symplectic manifold can be seen as a ‘classical’ object equipped with extra symmetries.

Let \( M \) now stand for an arbitrary manifold or supermanifold. Previously we worked with odd symplectic manifolds. It is known that any such odd symplectic manifold can be non-canonically identified with \( \Pi T^* M \) considered with the natural odd bracket, for some \( M \). A change of identifying symplectomorphism is equivalent to a symplectomorphism (or canonical transformation) of the space \( \Pi T^* M \). Therefore, we can restrict ourselves to objects on \( \Pi T^* M \), but should analyze them from the viewpoint of the larger supergroup of all canonical transformations of \( \Pi T^* M \), not just that of diffeomorphisms of \( M \).

We shall consider multivector fields (and multivector densities) and differential forms on \( M \). When \( M \) is a supermanifold, we actually speak of pseudodifferential forms.

Multivector fields on \( M \) are identified with functions on \( \Pi T^* M \). In local coordinates, we have \( X = X(x, x^*), \) where \( x^a \) are coordinates on \( M \) and \( x^*_a \) are the corresponding coordinates on the fibers, of the opposite parity, transforming as \( x^*_a = \frac{\partial x^d}{\partial x^a} x^*_d \). There is no problem with canonical transformations of \( \Pi T^* M \) acting on multivector fields on \( M \)—it is just the pull-back of functions.

Multivector densities on \( M \) have the form \( \sigma = s(x, x^*) Dx \) and at first glance it is not obvious how a transformation mixing \( x \) and \( x^* \) can be applied to them. However, in view of Theorems 3 and 4, for a canonical transformation \( F: \Pi T^* M \to \Pi T^* M \) one can set

\[ F^* \sigma := s(y(x, x^*), y^*(x, x^*)) \text{Ber} \left( \frac{\partial y}{\partial x} \right) Dx \]

if in coordinates \( F: (x, x^*) \mapsto (y = y(x, x^*), y^* = y^*(x, x^*)) \). This is a well-defined action. In other words, we identify multivector densities on
M with half-densities on $\Pi T^*M$ and apply the natural action, taking into account identity (8). In integration theory, multivector densities are known as integral forms (more precisely, pseudointegral, if we insist on differentiating between arbitrary smooth functions and polynomials). Therefore we can make a remark: integral forms on an arbitrary supermanifold $M$ are the same as half-densities on the odd symplectic manifold $\Pi T^*M$. In this language we see that integral forms have more symmetries than those obvious ones given by diffeomorphisms of $M$.

Consider now pseudodifferential forms on $M$, i.e., functions on $\Pi T^*M$. They are related with (pseudo)integral forms, i.e., multivector densities on $\Pi T^*M$ by the Fourier transform:

$$\omega(x, dx) = \int_{\Pi T^*_y M} Dx^* e^{i \partial_a x^*_a} s(x, x^*)$$

and conversely

$$s(x, x^*) = \text{const} \int_{\Pi T_x M} D(dx) e^{-i \partial_a x^*_a} \omega(x, dx).$$

From here we obtain the action of the canonical transformations of the odd symplectic manifold $\Pi T^*M$ on forms on $M$ as follows:

$$(F^*\omega)(x, dx) = \text{const} \int_{\Pi T^*_y M \times \Pi T_x M} Dx^* D(dy) e^{i (\partial_a x^*_a - dy^a y^*_a)(y(x, x^*))} \omega(y(x, x^*), dy) \text{Ber} \frac{\partial y}{\partial x}(x, x^*),$$

where, as above, $F: (x, x^*) \mapsto (y = y(x, x^*), y^* = y^*(x, x^*))$. In general, this action is non-local. We shall consider the representation of the infinitesimal canonical transformations of $\Pi T^*M$ on forms and multivector densities on $M$. As it turns out, the description in both cases will be very simple.

For the odd symplectic manifold $\Pi T^*M$, the canonical odd Laplacian on half-densities $\Delta$ on $\Pi T^*M$ is just the familiar divergence of multivector densities $\delta$ on $M$. Indeed,

$$\delta \sigma = \frac{\partial^2 s}{\partial x^a \partial x^*_a}(x, x^*) Dx,$$

if $\sigma = s(x, x^*) Dx$. Consider the infinitesimal canonical transformation of $\Pi T^*M$ generated by a function ("Hamiltonian") $H = H(x, x^*)$. Denote by $L_H$ the corresponding Lie derivative. Notice that from the viewpoint of $M$, the function $H$ is a multivector field.

**Theorem 5.** On multivector densities (= integral forms) on $M$, the Lie derivative w.r.t. the infinitesimal canonical transformation of $\Pi T^*M$ generated by $H$ is given by the formula:

$$L_H = [\delta, H],$$
where at the r.h.s. stands the commutator of the divergence operator $\delta$ and multiplication by the multivector field $H$.

A proof can be given by a direct computation. It fact, the statement mimics a similar and more general statement concerning odd Laplace operators acting on densities of various weights, see [9].

**Corollary 4.1.** The operator $\delta$ on multivector densities on $M$ is invariant under all canonical transformations of the odd symplectic manifold $\Pi T^*M$. (At least those given by a Hamiltonian.)

*Proof.* We need to show that $\delta$ commutes with all Lie derivatives $L_H$. Indeed, $[\delta, L_H] = [\delta, [\delta, H]] = [\delta^2, H] = 0$, since $\delta^2 = 0$. $\square$

We can adopt the following viewpoint. Suppose we do not know anything about the operator $\Delta$ on half-densities in odd symplectic geometry. Instead we concentrate on a familiar object, the operator $\delta$ on multivector densities on a manifold $M$. The operator $\delta$, as shown, is invariant under much larger group of transformations than just diffeomorphisms of $M$. It is invariant under symplectomorphisms of $\Pi T^*M$. We can then take $\delta$ as the definition of $\Delta$ for $\Pi T^*M$. Since any odd symplectic manifold $N$ is symplectomorphic to some $\Pi T^*M$, we can use this to define $\Delta$ on $N$. The invariance of $\Delta = \delta$ for $\Pi T^*M$ under symplectomorphisms of $\Pi T^*M$ shows that $\Delta$ on $N$ is well-defined, i.e., its action on half-densities on $N$ does not depend on an arbitrary choice of the identifying symplectomorphism $N \simeq \Pi T^*M$.

It may seem that there is a gap in such an argument as the invariance was proved only infinitesimally or, equivalently, for transformations that can be included into a Hamiltonian flow. In fact, there is no gap. Consider the supergroup $\text{Can} \Pi T^*M$ of all canonical transformations. If $M$ is an ordinary manifold, the structure of this supergroup was described in [7]. It is the product of the three subgroups:

1. Transformations induced by diffeomorphisms of $M$;
2. Shifts in the fibers of $\Pi T^*M$ of the form
   \[ F^*x^a = x^a, \quad F^*x^a_\ast = x^a_\ast + \frac{\partial \Phi}{\partial x^a}, \]
   where $\Phi = \Phi(x)$ is an odd function on $M$;
3. Transformations identical on the submanifold $M \subset \Pi T^*M$.

Since $\delta$ is invariant under diffeomorphisms of $M$, all that remains is to study transformations of types 2 and 3. Canonical transformations of types 2 and 3 can be included into Hamiltonian flows. Indeed, for type 2 one can take the flow with the Hamiltonian $\Phi$. For type 3 there is also a Hamiltonian flow, with the Hamiltonian of the form $\Psi = \Psi^{ab}(x, x^\ast)x^a_\ast x^b_\ast$, as shown in [7]. Therefore, transformations of types 2 and 3 are covered by the argument in the proof of Corollary 4.1 and this completes the proof.
Now let us turn our attention to (pseudo)differential forms on $M$.

Under the Fourier transform (9),(10), the divergence operator $\delta$ becomes the exterior differential $d$, up to a multiple of $i$. The multiplication by a multivector field $H = H(x, x^*)$ becomes the ‘convolution’ (or ‘cap product’):

$$(H \ast \omega)(x, dx) = \text{const} \int \int Dx^* D(dx) e^{i(dx^a - \overline{dx})^a x^*_a} H(x, x^*) \omega(x, \overline{dx})$$

$$= \int D(dx) \tilde{H}(x, dx - \overline{dx}) \omega(x, \overline{dx}).$$

where $\tilde{H} = \tilde{H}(x, dx)$ is the inverse Fourier transform of $H$. In other words, if we denote

$$i_H \omega := H \ast \omega,$$

we have

$$i_H = H \left( x, -i \frac{\partial}{\partial dx} \right)$$

the differential operator, w.r.t. the variables $dx^a$, with the symbol $H$. It is clear that up to $i$’s, it is just the classical internal product of a form by a multivector field, if we deal with ordinary differential forms and multivectors on an ordinary manifold.

We immediately get

**Theorem 6.** On (pseudo)differential forms on $M$, the Lie derivative w.r.t. the infinitesimal canonical transformation of $\Pi T^*M$ generated by $H$ is given by the ‘Cartan like formula’:

$$iL_H = [d, i_H],$$

where at the r.h.s. stands the commutator of the de Rham differential and the interior product by the multivector field $H$ as defined by (11).

This is very remarkable. Suppose $M$ is an ordinary manifold. The operation $i_H$, up to the imaginary unit, is the familiar interior product with a multivector field, generalizing the interior product with a vector field. For Lie derivatives along vector fields one proves the Cartan formula $L_X = [d, i_X]$. For multivector fields, as opposed to vector fields, this equation is taken as the definition of a ‘Lie derivative of a differential form along a multivector field’. In the classical picture it is not seen how these derivatives corresponds to actual transformations. Now we see that they are generators of odd canonical transformations acting on differential forms.

Notice that in general $L_H$ is not a derivation of the algebra $\Omega(M)$. Of course,

$$L_{[H,G]} = [L_H, L_G]$$

where at the l.h.s. stands the Schouten bracket of multivector fields. Equation (12) implies that the de Rham differential on $M$ is invariant under the canonical transformations of $\Pi T^*M$. Again, one can see the
\( \Delta \) operator as the de Rham differential considered together with these extra symmetries.

Some of the arguments of this section were implicit in our earlier works \([4, 6, 7, 8]\).

\section*{References}

\begin{itemize}
  \item [1] I. A. Batalin and G. A. Vilkovisky. Gauge algebra and quantization. \textit{Phys. Lett.}, 102B:77–81, 1981.
  \item [2] I. A. Batalin and G. A. Vilkovisky. Quantization of gauge theories with linearly dependent generators. \textit{Phys. Rev.}, D28:2567–2582, 1983.
  \item [3] I. A. Batalin and G. A. Vilkovisky. Closure of the gauge algebra, generalized Lie equations and Feynman rules. \textit{Nucl. Phys.}, B234:106–124, 1984.
  \item [4] O. M. Khudaverdian \(^1\). Geometry of superspace with even and odd brackets. Preprint of the Geneva University, UGVA-DPT 1989/05-613, 1989. Published in: \textit{J. Math. Phys.} 32 (1991), 1934–1937.
  \item [5] O. M. Khudaverdian. Batalin-Vilkovisky formalism and odd symplectic geometry. In P. N. Pyatov and S. N. Solodukhin, eds., \textit{Proceedings of the Workshop “Geometry and Integrable Models”, Dubna, Russia, 4-8 October 1994}. World Scientific Publ., 1995, \texttt{hep-th 9508174}.
  \item [6] O. M. Khudaverdian. \( \Delta \)-operator on semidensities and integral invariants in the Batalin–Vilkovisky geometry. Preprint 1999/135, Max-Planck-Institut für Mathematik Bonn, 19 p, 1999, \texttt{arXiv:math.DG/9909117}.
  \item [7] H. M. Khudaverdian. Semidensities on odd symplectic supermanifolds. \textit{Comm. Math. Phys.}, 247(2):353–390, 2004, \texttt{arXiv:math.DG/0012256}.
  \item [8] H. M. Khudaverdian. Laplacians in odd symplectic geometry. In Th. Voronov, ed., \textit{Quantization, Poisson Brackets and Beyond}, volume 315 of \textit{Contemp. Math.}, pages 199–212. Amer. Math. Soc., Providence, RI, 2002, \texttt{arXiv:math.DG/0212354}.
  \item [9] H. M. Khudaverdian and Th. Th. Voronov. On odd Laplace operators. \textit{Lett. Math. Phys.}, 62:127–142, 2002, \texttt{arXiv:math.DG/0205202}.
  \item [10] H. M. Khudaverdian and Th. Th. Voronov. Geometry of differential operators, and odd Laplace operators. \textit{Russian Math. Surveys}, 58:197–198, 2003, \texttt{arXiv:math.DG/0301236}.
  \item [11] H. M. Khudaverdian and Th. Th. Voronov. On odd Laplace operators. II. In V. M. Buchstaber and I. M. Krichever, eds., \textit{Geometry, Topology and Mathematical Physics. S. P. Novikov's seminar: 2002–2003}, volume 212 of \textit{Amer. Math. Soc. Transl. (2)}, pages 179–205. Amer. Math. Soc., Providence, RI, 2004, \texttt{arXiv:math.DG/0212311}.
  \item [12] Y. Kosmann-Schwarzbach and J. Monterde. Divergence operators and odd Poisson brackets. \textit{Ann. Inst. Fourier}, 52:419–456, 2002, \texttt{arXiv:math.QA/0002209}.
  \item [13] S. Mac Lane. \textit{Homology}. Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press Inc., Publishers, New York, 1963.
  \item [14] A. S. Schwarz. Geometry of Batalin-Vilkovisky quantization. \textit{Comm. Math. Phys.}, 155(2):249–260, 1993.
  \item [15] A. S. Schwarz. Symmetry transformations in Batalin-Vilkovisky formalism. \textit{Lett. Math. Phys.}, 31(4):299–301, 1994.
  \item [16] P. Severa. On the origin of the BV operator on odd symplectic supermanifolds, \texttt{arXiv:math.DG/0506331}.
\end{itemize

\footnote{O. M. Khudaverdian = H. M. Khudaverdian.}
