Computation of the “Enrichment” of a Value Functions of an Optimization Problem on Cumulated Transaction-Costs through a Generalized Lax-Hopf Formula

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Abstract

The Lax-Hopf formula simplifies the value function of an intertemporal optimization (infinite dimensional) problem associated with a convex transaction-cost function which depends only on the transactions (velocities) of a commodity evolution: it states that the value function is equal to the marginal function of a finite dimensional problem with respect to durations and average transactions, much simpler to solve. The average velocity of the value function on an investment temporal window is regarded as an enrichment, proportional to the profit and inversely proportional to the investment duration.

At optimum, the Lax-Hopf formula implies that the enrichment is equal to the cost of the average transaction on the investment temporal window.

In this study, we generalize the Lax-Hopf formula when the transaction-cost function depends also on time and commodity, for reducing the infinite dimensional problem to a finite dimensional problem. For that purpose, we introduce the moderated transaction-cost function which depends only on the duration and on a commodity.

Here again, the generalized Lax-Hopf formula reduces the computation of the value function to the marginal function of an optimization problem on durations and commodities involving the moderated transaction cost function. At optimum, the enrichment of the value function is still equal to the moderated transition cost-function of average transaction.

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Lax-Hopf formula, transaction of commodities, transaction-cost function, value-fonction of an intertemporal optimization problem, generalized Lax-Hopf formula, moderation of a transaction cost-function, terminal conditions

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1 Introduction

Given a value function \( t \mapsto V(t) \in \mathbb{R} \) (a cash-flow, for instance, or the value function of an intertemporal optimal control problem), we interpret its average velocity \( \frac{V(T) - V(T - \Omega)}{\Omega} \) on the temporal window \( T - \Omega, T \) as its enrichment\(^2\), where \( \Omega \geq 0 \) is the aperture\(^3\) (or duration) of the temporal window.

If we interpret \( V(T) - V(T - \Omega) \) as a profit, then the enrichment \( \frac{V(T) - V(T - \Omega)}{\Omega} \) is the ratio of the profit over the aperture of the temporal window. The larger the profit, the smaller the aperture, the larger the enrichment\(^4\).

Let us introduce \( X := \mathbb{R}^\ell \), regarded as a commodity space of commodities \( x := (x_h)_{1 \leq h \leq \ell} \) of amounts \( x_h \in \mathbb{R} \) of units \( e^h \) of goods or services labelled \( h = 1, \ldots, \ell \) (see an economic motivation based on a dynamical version of the Willingness to Pay issue in Section 2, p. 3, leading to such an intertemporal optimization problem, among many other economic examples and issues). The velocity \( x'(t) \) at time \( t \) of the evolution of a commodity \( x(\cdot) \) is regarded as a transaction (actually, an infinitesimal one) since it is the limit of average transactions \( \frac{x(T) - x(T - \Omega)}{\Omega} \) on the temporal window \( [T - \Omega, T] \) when the aperture converges to 0\(^5\).

In this study, we shall take for value function the one provided by an intertemporal optimal control problem of the form

\[
V(t, x) := \inf_{\Omega \geq 0} \inf_{x(\cdot)} \left( c(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^{T} I(x'(t)) dt \right)
\]  

(1)

Its (forward) interest rate is \( \frac{V(T) - V(T - \Omega)}{\Omega V(T)} \), its (backward) interest rate is \( \frac{V(T) - V(T - \Omega)}{\Omega V(T)} \) and its (symmetric) interest rate \( \frac{V'(t)}{V(t)} \) when \( \Omega \to 0^+ \) when \( V(\cdot) \) is differentiable from the left.

\(^2\)Its (forward) interest rate is \( \frac{V(T) - V(T - \Omega)}{\Omega V(T - \Omega)} \), its (backward) interest rate is \( \frac{V(T) - V(T - \Omega)}{\Omega V(T)} \) and its (symmetric) interest rate \( \frac{V'(t)}{V(t)} \) when \( \Omega \to 0^+ \) when \( V(\cdot) \) is differentiable from the left.

\(^3\)The inverse \( \frac{1}{\Omega} \) of the aperture \( \Omega \) of a temporal window can be regarded as a definition of the notion of "liquidity" (or "velocity", as is also called, although it does not mention the variable of which the inverse of aperture is the velocity in mathematical terminology). So, the enrichment is the product of the liquidity and the profit.

\(^4\)This ratio could be a basis for a "Shareholder Value Tax" inversely proportional to the investment duration \( \Omega \) and proportional to the profit (see Section 1.4, p. 18, of Time and Money. How Long and How Much Money is Needed to Regulate a Viable Economy, J. Aubin).

\(^5\)Derivatives from the left are used according to a suggestion of Efim Galperin, since derivatives from the right \( x'(t) = \lim_{h \to 0^+} \frac{x(t + h) - x(t)}{h} \) are "physically non-existent" since time \( t + h \) is not yet known, following the Jiri Buquoy, who in 1812, formulated the equation of motion of a body with variable mass, which retained only the attention of Poisson before being forgotten. This is also the reason why we use temporal windows \( [T - \Omega, T] \) where \( T \) is a flying present instead of using a present time \( T \geq 0 \) ranging over an unknown future, as it is currently done.
where the sum of the cost on the value \(x(T - \Omega)\) at the beginning of the temporal window and the cumulated cost of the transactions \(l(x'(t))\) on the temporal window are minimized with respect to both the aperture \(\Omega \geq 0\) and commodity evolutions defined below.

The question is to compute the enrichment \(\frac{V(T) - V(T - \Omega)}{\Omega}\) in terms of the cost functions. When the transaction is convex and continuous\(^6\), the celebrated Lax-Hopf formula (see [Hopf, 16], [Lax, 18]) states that at optimal aperture and optimal evolution, the enrichment is provided by the formula

\[
\frac{V(T, x_*(T)) - V(T - \Omega_*, x_*(T - \Omega_*))}{\Omega_*} = \Omega_\star \left( \frac{x_*(T) - x_*(T - \Omega_*)}{\Omega_*} \right)
\]

stating that the cost of the average optimal transaction is the average velocity of the value function on the temporal window, or, in economic terms, that the enrichment is equal to the cost of the average transaction of the optimal evolution.

Once this formula recognized, we generalize this enrichment formula even when the transaction cost function depends on time and commodity by proving a generalization of the Lax-Hopf formula in Theorem 5.3, p. 10. We next pass from the Willingness to Pay example to the case of an economy involving the evolution of both commodities and their prices and taking for “potential” function the patrimonial value. Since its derivative involve not only the velocities of commodities (transactions) and the velocities of prices (price fluctuations, but also the values of the commodities and prices, the Lax-Hopf does not apply, but the generalized Lax-Hopf does.

**Organization of the Exposition:**

We begin by motivating the use of the Lax-Hopf by an example intertemporal optimal control problem derived from a dynamic version of the Willingness to Pay issue in Section 2, p. 3. Next, in Section 3, p. 6, we consider a version of an general optimal control problem posed on temporal windows \([T - \Omega, T]\) of unknown aperture \(\Omega \geq 0\) with terminal conditions instead of initial ones. When the transaction costs depend only on the transactions, we recall the Lax-Hopf formula in Section 4, p. 8, generalized in Section 5, p. 9 when the transaction costs depend also on time and commodities. In Section 6, p. 13, we take interest functions which are not fixed, but depend on time, commodities and transactions and extend to this case the generalized Lax-Hopf formula. We end this study in Section 7, p. 14 by applying the generalized Lax-Hopf formula to general economies involving commodities and prices.

## 2 An Economic Motivation: Willingness to Pay

“Willingness To Pay” (resp. Accept) is defined in the literature as the maximum amount a person would be willing to pay (in monetary units) of an exchange of an “economic state” to

\(^6\)Actually, lower semicontinuous.
receive (resp. accepting) the profit or avoid the sacrifice or something undesirable, such as pollution. (See [15, Hanemann], L’évaluation contingente : les valeurs ont-elles un prix ?, in Rendre possible, Jacques Weber, itinéraire d’un économiste passe-frontières, [5, Bouamrane et al] and Évaluation économique de la biodiversité [6, Brahic & Terreaux] de Brahic et J.-Ph. Terreaux and its bibliography, among an infinity of other publications on this topic).

Here, we follow the presentation of Chapter 6, p. 85, of Time and Money. How Long and How Much Money is Needed to Regulate a Viable Economy, [1, Aubin]).

A (static) economic perspective of this concept requires a Willingness To Pay Valuation valuation (function), where \( x \in \mathbb{R}^\ell \) is the “economic state” to evaluate and \( w \in \mathbb{R} \), the “wealth”. If \( x_0 \) and \( w_0 \) denote the original state and its value, and \( x \) and \( w \) another state and its value, the question arises to compute the value \( w \) of \( x \) such that as \( (x, w) \) has the same utility than \( (x_0, w_0) \), i.e., is a solution to

\[
  u(x, w) = u(x_0, w_0) \tag{3}
\]

Then \( \varpi(x; x_0, w_0) := w - w_0 \) is the transaction cost for obtaining \( x \) from \( x_0 \), the “Willingness To Pay” for exchanging \( x_0 \) with \( x \) defined implicitly as a solution to the equation

\[
  u(x, w_0 + \varpi(x; x_0, w_0)) = u(x_0, w_0) \tag{4}
\]

However, transactions defined as instantaneous exchange \( x'(t) \) of an evolving commodity \( x(t) \), involve some underlying evolutionary (dynamical) process for exchanging an initial commodity \( x_0 \) with a new one, which requires the introduction of

1. a time \( T \in \mathbb{R} \) (evolving present time);

2. a duration \( \Omega \geq 0 \);

defining the temporal window \([T - \Omega, T]\), the beginning of which is \( T - \Omega \) and the end of which is the present time \( T \), in which the current past time \( t \in [T - \Omega, T] \) evolves.

Instead of pairs of economic states \( x \in X := \mathbb{R}^\ell \) having the same “Willingness To Pay Valuation”, one, \( x(T) \), at ending time \( T \), the successor of another one, \( x(T - \Omega) \), at the beginning of the temporal window. In the example above, \( (x, w) \) is regarded as the successor of \( (x_0, w_0) \). In this evolutionary context, this would mean that the evolution between pairs \( (x_0, w_0) \) and \( (x, w) \) leaves constant the Willingness To Pay Valuation.

In this dynamical framework, instead of assuming that a Willingness To Pay Valuation (function) is given, as in the static case, we shall built it from the following data:

1. an evolutionary system governing a set \( \mathcal{A}_c(T - \Omega, T; x) \) of evolutions of the economic states \( x(\cdot) \) defined on the temporal window \([T - \Omega, T]\) and arriving at \( x \) at time \( T \) with velocities \( x'(t) \) bounded by \( c > 0 \): \( \|x'(t)\| \leq c \).
2. an “(instantaneous) cost function” \( x \mapsto c(T - \Omega, x) \) indexed by the beginning of the temporal window, interpreted as the cost of an investment at the beginning of the temporal window.

This final condition replaces the standard initial condition, since we are interested in time irreversible systems when only the past (described on the temporal window \([T - \Omega, \Omega]\)) is known and evolves with present time \( T \). The value \( x(T - \Omega) \) of the evolution \( x(\cdot) \) at the beginning of the temporal window is not prescribed in this study. One can define a “Willingness To Pay Valuation” \((T, \Omega; x) \mapsto V(T, \Omega; x)\) in the following way:

\[
W(T, \Omega; x) := \inf_{x(\cdot)\in A_c(T-\Omega;T;x)} c(T-\Omega, x(T-\Omega)) \tag{5}
\]

minimizing both the investment duration \( \Omega \geq 0 \) and the initial investment cost.

We observe that by taking the zero duration \( \Omega = 0 \), we derive from the construction of the Willingness To Pay valuation that the instantaneous boundary property (for zero duration)

\[
\forall T, \forall x \in X, \; W(T, 0; x) = c(T, x)
\]

holds true.

One of the required properties for a function to be regarded as a “Willingness To Pay valuation function” is that, as property (3), p. 4, in the static case,

\[
\forall t \in [T - \Omega, T], \; W(t, t - (T - \Omega), x(t)) := W(T, \Omega; x) \tag{6}
\]

(the dynamic programming property).

**Remark** — When the economic state is a pair \((x, w) \in X \times \mathbb{R}_+\) where \( x \in X \) is a commodity and \( w \) a wealth, consider the instantaneous data \( c(t, x, w) \). The derived Willingness To Pay feedback map \((t, d, x, w) \sim R(t, d, x, w)\) governs the Willingness To Pay evolutions according a differential inclusion

\[
\forall t \in [T - \Omega, T], \; (\overline{x}(t), \overline{w}(t)) \in R(t, t - (T - \Omega), x(t), w(t)) \tag{6}
\]

Therefore, for any \( t \in [T - \Omega, T], \; \overline{x}(T) - \overline{x}(t) = \int_t^T \overline{x}'(\tau)d\tau \) is the transaction between \( t \) and \( T \) and

\[
\forall t \in [T - \Omega, T], \; \overline{x}(x; x_t, w_t) := \int_{T-t}^T \overline{w}'(\tau)d\tau \tag{7}
\]

is its transaction cost, regarded as the Willingness To Pay in the static case which motivated this study. ☐

The “average deprivation” in the case of a process transforming \( x(T - \Omega) \) at the beginning of the temporal window to \( x(T) \) at the end of this window is equal to
\[
\frac{x(T) - x(T - \Omega)}{\Omega} = \frac{1}{\Omega} \int_{T-\Omega}^{T} x'(s) \, ds \text{ and involves the velocity of the evolution (regarded as a transaction in economic terms). If one wishes to integrate in the evaluation of deprivation a function } l : u \in X \mapsto l(u) \in \mathbb{R} \cup \{+\infty\} \text{ of the transactions, we can add the cumulated cost } \int_{T-\Omega}^{T} l(x'(s)) \, ds \text{ of transactions } x'(s) \text{ for defining a new “Willingness To Pay Valuation”.}
\]

Knowing both the evolutionary system \( A_c(T - \Omega, T; x) \), the cost function of the economic state at the beginning of the temporal window and the transaction cost function \( l \), one can define a “Willingness To Pay Valuation” \( (T, \Omega; x) \mapsto V(T, \Omega; x) \) in the following way:

\[
W(T, \Omega; x) := \inf_{x(\cdot) \in A_c(T - \Omega, T; x)} \left( c(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^{T} l(x'(t)) \, dt \right) \quad (8)
\]

Optimal duration \( \Omega^* \) and evolutions \( x^*(\cdot) \in A_c(T - \Omega, T; x) \) minimizing the Willingness To Pay Valuation valuation, if they exist, are regarded as willingness to pay investment durations and evolutions.

### 3 The Value Function of an Optimal Control Problem

Let \( X := \mathbb{R}^\ell \) be a vector space (the commodity space). We denote by \( x(\cdot) : t \mapsto x(t) \in X \) a commodity evolution (or “flow”). Its derivative \( x'(\cdot) : t \mapsto x'(t) \in X \) is regarded as a transaction evolution.

We introduce temporal window \([T - \Omega, T]\) where \( \Omega \geq 0 \) is its opening (and thus, \( T - \Omega \) is the departure time).

**Definition 3.1 [Departure and Arrival Map]** We consider two “cost functions” \( c \) and \( l \):

1. an instantaneous cost condition function \((t, x) \mapsto c(t, x) \in \mathbb{R} \cup \{+\infty\}\);

2. a Lagrangian \( l : (t, x, u) \mapsto l(t, x, u) \in \mathbb{R} \cup \{+\infty\} \), regarded as a transaction cost function \( u \mapsto l(t, x, u) \) depending on time and commodity

with which we associate

1. the departure tube \( C : \mathbb{R} \rightarrow X \) defined by

\[
C(t) := \{ x \in X \text{ such that } c(t, x) < +\infty \}
\]
2. the set-valued map $F : \mathbb{R} \times X \rightharpoonup X$ defined by

$$F(t, x) := \{u \in X \text{ such that } l(t, x, u) < +\infty\} \quad (9)$$

and the arrival map $A_l : \mathbb{R} \times X \rightharpoonup \mathcal{C}(-\infty, +\infty; X)$ associating with any final pair $(T, x)$ the set of evolutions $x(\cdot)$ governed by the differential inclusion

$$\forall t \in \mathbb{R}, \ x'(t) \in F(t, x(t)) \quad (10)$$

starting at some $s := x(T - \Omega) \in C(T - \Omega)$ and arriving at the prescribed terminal condition $x(T)x$.

In this study, we shall look for both

1. an opening $\Omega \geq 0$ of the temporal window;

2. an evolution $x(\cdot) : [T - \Omega, T] \mapsto X$ belonging to $A_l(T, x)$ (regulated by the differential inclusion $F$ and arriving at $x$ at time $T$)

satisfying the following optimality criterion:

**Definition 3.2 /Value Function/** The value function of the associated intertemporal optimization problem with respect to the aperture $\Omega \geq 0$ and $x(\cdot) \in A_l(T, x)$ is defined by

$$V(T, x) := \inf_{\Omega \geq 0} \inf_{x(\cdot) \in A_l(T, x)} \left( c(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^{T} l(t, x(t), x'(t))dt \right) \quad (11)$$
This optimization problem minimizes

1. the aperture $\Omega$ of the temporal window $[T - \Omega, T]$;
2. the sum of the initial condition at departure time $T - \Omega$ and the cumulated sum of the transaction costs on the temporal window $[T - \Omega, T]$.

We refer to *Time and Money. How Long and How Much Money is Needed to Regulate a Viable Economy*, [1] Aubin, for the theorem stating the existence of optimal evolutions to this infinite dimensional problem as well as to Chapter 14, p. 563, of *Viability Theory. New Directions*, [3] Aubin, Bayen & Saint-Pierre for more examples.

The purpose of this study is to adapt to the case of general transaction-cost functions $l : (t, x, u) \mapsto l(t, x, u)$ the Lax-Hopf formula proved for convex transaction-cost functions $l : u \mapsto l(u)$ independent of $t$ and $x$.

### 4 The Lax-Hopf formula

When the transaction cost function $u \mapsto l(u)$ is convex and lower semicontinuous (i.e., when its epigraph $\mathcal{E}_p(l)$ is convex and closed) and *depend neither on time nor on commodity*, the celebrated Lax-Hopf formula[7] states that the value function $V$ can be drastically simplified:

$$
V(T, x) := \inf_{\Omega \geq 0} \inf_{\Upsilon \in \text{Dom}(l)} \left( c(T - \Omega, x(T - \Omega)) + \Omega l(\Upsilon) \right)
$$

where $\Upsilon \in \text{Dom}(l) \subset X$ range over the domain[8] of the transaction cost function $l$.

Indeed, the infinite dimensional optimization problem (11), p. 7 is reduced to a finite dimensional optimization (12), p. 8 on $\mathbb{R}_+ \times X$.

It allows us to solve analytically the optimization problem and to simplify its numerical calculation.

At optimal aperture $\Omega_*$ and transaction $\Upsilon_* \in \text{Dom}(l)$, evolutions $x_*(\cdot) \in A_l(T, x)$ achieve the minimum of the value function:

$$
\frac{V(T, x_*(T)) - V(T - \Omega_*, x_*(T - \Omega_*))}{\Omega_*} = 1 \left( \frac{x_*(T) - x_*(T - \Omega_*)}{\Omega_*} \right)
$$

which states that *the enrichment of the optimal value function is equal to the transaction cost of the average transaction* on the temporal window $[T - \Omega_*, T]$.

Furthermore, the dynamic optimality property

[7] See Partial Differential Equations, [14] Evans, *Semicontinuous Functions, Hamilton-Jacobi Equations, and Optimal Control*, [7] Cannarsa & Sinestrari, [2] Aubin, Bayen & Saint-Pierre, [3]  [Claudel & Bayen], [13] Désilles and Section 11.5, p. 465, of *Viability Theory. New Directions*, [3] Aubin, Bayen & Saint-Pierre.

[8] The domain $\text{Dom}(l)$ is the subset of $\Upsilon \in X$ such that the transaction cost $l(\Upsilon) < +\infty$ is finite.
∀ t ∈ [T − Ω, T], V(t) = V(t, x(t)) := c(T − Ω, x(T − Ω)) + \int_{T−Ω}^{t} l(x′(τ))dτ \quad (14)
holds true and satisfies the boundary conditions \( V(T) = V(T, x) = V(T, x(T)) \) and \( V(T − Ω, x(T − Ω)) = c(T − Ω, x(T − Ω)) \).

5 The Generalized Lax-Hopf Formula

The purpose of this study is to extend this Lax-Hopf formula to the case when the transaction cost function depends on the time and/or the commodities. In this case, we introduce the concept of moderation of a transition cost function:

Definition 5.1 [Moderation of a Transition Cost Function] The moderated transition cost function \((T, x, Ω, Υ) \rightarrow Λ_l(T, x, Ω, Υ)\) of a the transaction cost function \((t, x, u) \rightarrow l(t, x, u)\) is the value function of the intertemporal problem

\[
Λ_l(T, x, Ω) := \inf_{\{x(\cdot) \in \mathcal{A}_l(T, x) | \frac{1}{\Omega} \int_{T−Ω}^{T} x′(s)ds = Υ\}} \frac{1}{\Omega} \int_{T−Ω}^{T} l(t, x(t), x′(t))dt
\]

which depends on

1. the aperture \(Ω ≥ 0\) of the temporal window;
2. the (average) transactions \(Υ ∈ X\).

Remark — The function \(Λ_l\) does not depend upon the instantaneous cost function \(c\) and can be computed off-line for each pair \((Ω, Υ)\). If it exists, the optimal evolutions \(x(Ω, Υ)(\cdot) \in \mathcal{A}_l(T, x)\) satisfying

\[
\frac{1}{Ω} \int_{T−Ω}^{T} x′(Ω, Υ)(s)ds = Υ \quad \text{and} \quad Λ_l(T, x, Ω, Υ) := \frac{1}{Ω} \int_{T−Ω}^{T} l(t, x(Ω, Υ)(t), x′(Ω, Υ)(t))dt
\]
are computed once and for all.

The moderation of a convex lower semicontinuous transition cost function coincides it when it depends only on transitions:

Lemma 5.2 [Moderation of a Lower Semicontinuous Convex Function Depending Only on Transitions] If the transition cost function \(l : u \rightarrow l(u)\) in independent of time and commodity and convex and lower semicontinuous, then it coincides with its moderation \(Λ_l\):

\[
∀ Υ ∈ \text{Dom}(l), \quad Λ_l(T, x, Ω, Υ) := l(Υ)
\]

9
Proof — Since the evolution defined by \( x_T(t) := x - \Upsilon(T - t) \) belongs to \( A_l(T, x) \), then

\[
A_l(T, x, \Omega, \Upsilon) \leq \frac{1}{\Omega} \int_{T-\Omega}^{T} l(x'_T(t))dt = l(\Upsilon)
\]  

(18)

The opposite inequality follows from the Jensen inequality (see [17, Jensen]) stating that whenever \( l : u \rightarrow l(u) \) is lower semicontinuous and convex, then

\[
l(\Upsilon) = l \left( \frac{1}{\Omega} \int_{T-\Omega}^{T} x'(s)ds \right) \leq \frac{1}{\Omega} \int_{T-\Omega}^{T} l(x'(s))ds
\]

(19)

implying that \( l(\Upsilon) \leq A_l(T, x, \Omega, \Upsilon) \), so that \( l(\Upsilon) = A_l(T, x, \Omega, \Upsilon) \).

\[\blacksquare\]

Theorem 5.3 [The Generalized Lax-Hopf Formula for Commodity Dependent Transaction Costs] The generalized Lax-Hopf formula states that the value function \( V \) defined by

\[
V(T, x) := \inf_{\Omega \geq 0} \inf_{x(\cdot)} \left( c(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^{T} l(x'(t))dt \right)
\]

is equal to

\[
V(T, x) := \inf_{\Omega \geq 0} \inf_{\Upsilon} \left( c(T - \Omega, x(T - \Omega)) + \Omega A_l(T, x, \Omega, \Upsilon) \right)
\]

(20)

which is a finite dimensional minimization problem on \( \mathbb{R}_+ \times X \).

This is the generalization of the Lax-Hopf formula [13], p. 8 when the transition cost function the average transition cost \( u \sim l(u) \) is convex, lower semicontinuous and not depends only on transactions since \( A_l(T, x; \Omega, \Upsilon) = l(\Upsilon) \) in this case.

Proof — The proof is as simple as the proof of Lax-Hopf formula.

1. We can write the value function \( V \) in the form

\[
\left\{ V(T, x) := \inf_{\Omega \geq 0} \inf_{x(\cdot)} \inf_{\{x(\cdot) \in A_l(T, x) \text{ such that } f_{T-\Omega}^{T} x'(t) = \Omega \Upsilon \}} \left( c(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^{T} l(t, x(t), x'(t))dt \right) \right\}
\]

(21)

which is equal to

\[
\left\{ V(T, x) := \inf_{\Omega \geq 0} \inf_{\Upsilon} \left( c(T - \Omega, x(T - \Omega)) + \right. \right.
\]

\[
\left. + \left( \inf_{\{x(\cdot) \in A_l(T, x) \text{ such that } f_{T-\Omega}^{T} x'(t) = \Omega \Upsilon \}} \int_{T-\Omega}^{T} l(t, x(t), x'(t))dt \right) \right\}
\]

(22)

This is the formula which we were looking for;
2. Lax-Hopf formula (13), p. 8 follows from Lemma 5.2, p. 9 and Theorem 5.3, p. 10.

We recall that if we assume that the cost function \( l \) is Marchaud, the optimization problem has a solution.

**Definition 5.4 [Marchaud Transition Cost Functions]** We shall say that a transaction cost function \( (t, x, u) \mapsto l(t, x; u) \in \mathbb{R} \cup \{+\infty\} \) is Marchaud if it is a lower semicontinuous function convex with respect to \( u \) and if there exists a finite positive constant \( c > 0 \) such that

\[
\text{Dom}(l(t, x; \cdot)) \subset c(\|x\| + \|d\| + 1)B \quad \text{and is closed}
\]

\[
\forall u \in \text{Dom}(l(t, x; \cdot)), \quad 0 \leq l(d, x; u) \leq c(\|x\| + \|d\| + 1)
\]  

Under this condition, the value function inherits the properties of optimization problems: see Theorem 13.4.2, p. 533, 13.5.1, p. 538 and 13.5.2, p. 539 of Viability Theory. New Directions, [3, Aubin, Bayen & Saint-Pierre] that we summarize in the next statement.

**Theorem 5.5 [Existence and Properties of Optimal Evolutions]** Let us assume that the transaction-cost function \( l \) is a Marchaud function and that the initial cost function \( c \) is lower semicontinuous. Then there exist an optimal aperture and an optimal evolution.

At optimal aperture \( \Omega_* \) and optimal average transaction \( \Upsilon_* \), optimal evolutions \( x_*(\cdot) \) satisfy

\[
\Upsilon_* = \frac{1}{\Omega_*} \int_{T-\Omega_*}^{T} x'_*(t)dt \quad \text{and} \quad \Lambda_1(T, x, \Omega_*, \Upsilon_*) = \frac{1}{\Omega_*} \int_{T-\Omega_*}^{T} l(t, x_*(t), x'_*(t))dt
\]  

(24)

and the Isaacs-Bellman dynamic optimal property stating that the function \( V : [T - \Omega, T] \mapsto V(t) \) defined by

\[
V(t) := V(t, x_*(t)) := c(T - \Omega_*, x_*(T - \Omega_*)) + \int_{T-\Omega_*}^{t} l(t, x_*(t), x'_*(t))dt
\]  

(25)

is still the optimal value function at \( (t, x_*(t)) \) satisfying \( V(T) = V(T, x) = V(T, x_*(T)) \) and \( V(T - \Omega_*) = V(T - \Omega_*, x_*(T - \Omega_*)) = c(T - \Omega_*, x_*(T - \Omega_*)). \)

The generalized Lax-Hopf condition states that

\[
\Lambda_1 \left( T, x, \Omega_*, \frac{x_*(T) - x_*(T - \Omega_*)}{\Omega_*} \right) = \frac{V(T, x_*(T)) - V(T - \Omega_*, x_*(T - \Omega_*))}{\Omega_*}
\]  

(26)

so that the enrichment \( \frac{V(T) - V(T - \Omega_*)}{\Omega_*} \) of the optimal value function \( t \mapsto V(t) := V(t, x_*(t)) \) on the temporal window \( [T - \Omega_*, T] \) is equal the optimal moderated transaction cost \( \Lambda_1(T, x, \Omega_*, \Upsilon_*) \).
Example — In the case when \(c(t, x) = 0\) when \(t = 0\) and \(x = 0\) and \(c(t, x) = +\infty\) otherwise, the intertemporal optimal value function boils down to

\[
V(T, x) := \inf_{\Omega \geq 0} \inf_{\{x(t) \in A(T, x)\}} \int_{T-\Omega}^{T} l(t, x(t), x'(t)) dt
\]

(27)

The above formula (26), p. 11 boils down to

\[
\begin{cases}
\Upsilon_* = \frac{x_*(T)}{\Omega_*} \\
\Lambda_1(T, x; \Omega_*, \Upsilon_*) = \frac{V(T, x_*)}{\Omega_*}
\end{cases}
\]

(28)

Remark — We recall that the value function, when it is differentiable, is a solution to the Hamilton-Jacobi equation. This was for solving Hamilton-Jacobi equations that the Lax-Hopf formula was derived. Nowadays, we can associate with any transaction function \(l: (t, x, u) \rightarrow l^*(t, x, p) \times \mathbb{R} \cup \{+\infty\}\) its conjugate function\(^9\) (also called the Legendre-Fenchel transform) \(l^*: (t, x, p) \rightarrow l^*(t, x, p) \times \mathbb{R} \cup \{+\infty\}\) defined by

\[
l^*(t, x, p) := \sup_{u \in X} \langle p, u \rangle - l(t, x, u)
\]

(29)

The Fenchel theorem states that whenever \(u \sim l(t, x, u)\) is convex and lower semicontinuous, then its conjugate \(p \sim l^*(t, x, p)\) is also convex and lower semicontinuous, and, furthermore, that the conjugate \(l^* = l\) of \(l^*\) is equal to \(l\).

Recall also that by definition, \(p \in \partial l(t, x, u)\) belongs to the subdifferential of \(l\) if and only if \(\langle p, u \rangle = l(t, x, u) + l^*(t, x, p)\), so that both are equivalent to \(u \in \partial l(t, x, p)\). The Hamilton-Jacobi equation associated with \(l^*\) is

\[
\frac{\partial V(t, x)}{\partial t} := l^* \left( t, x, \frac{\partial V(t, x)}{\partial x} \right)
\]

(30)

The condition associated with the cost function \(c\) is written

\[
\forall (t, x), \quad V(t, x) \leq c(t, x)
\]

(31)

When \(l\) is Marchaud, the value function, when it is differentiable, is a solution to the Hamilton-Jacobi equation. Otherwise, when it is not differentiable, but only lower semicontinuous, we can give a meaning to a solution as a solution in the Barron-Jensen/Frankoska sense, using for that purpose subdifferential of lower semicontinuous functions defined in non-smooth analysis (Set-valued analysis, [4, Aubin & Frankowska], [2, 3, Aubin, Bayen,

\[^9\]In physics, when \(l\) is interpreted as a Lagrangian, its conjugate function is called an Hamiltonian.
So, under this assumption, the value function if and only if it is a generalized solution of the Hamilton-Jacobi equation and if and only if it is a “viability” solution. So we can prove the Lax-Hopf formula for Hamilton-Jacobi equation in two steps, the first one uses the fact that the solution is the value function, the second one, by using the generalized Lax-Hopf formula for the value function. The adaptation of the results of Chapters 13 and 17 of *Viability Theory. New Directions*, [3, Aubin, Bayen & Saint-Pierre] is straightforward.

6 Lax-Hopf Formula with Transaction-Dependent Interest Rates

We next introduce time, commodity and transaction dependent interest rate \( m(t, x, u) \). Such interest rates can be prescribed constants \( m \), as it usually assumed, or prescribed time dependent rates \( m(t) \), or, more interestingly, \( m(t, x(t), x'(t)) \) dependent also on the evolution of the commodity and the transaction.

For every evolution \( x(\cdot) \in A_{(l, m)}(T, x) \), we set

\[
M_{x(\cdot)}(t) := \int_0^t m(\tau, x(\tau), x'(\tau))d\tau
\]

and define the value function with interest rates by

\[
\begin{align*}
V_{(l, m)}(T, x) := & \inf_{\Omega \geq 0} \inf_{x(\cdot) \in A_{(l, m)}(t, x)} \\
& \left( e^{M_{x(\cdot)}(\Omega)} c(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^T e^{M_{x(\cdot)(T-\tau)}} (l(\tau, x(\tau), x'(\tau)))d\tau \right) \\
\end{align*}
\]

We introduce the value function

\[
\Lambda_{(l, m)}(T, x, \Omega, \Upsilon) := \inf_{\{x(\cdot) \in A_{(l, m)}(T, x)| \int_{T-\Omega}^T x'(s)ds = \Omega \Upsilon\}} \frac{1}{\Omega} \int_{T-\Omega}^T e^{M_{x(\cdot)(T-\tau)}} l(t, x(t), x'(t))dt
\]

Therefore,

**Theorem 6.1** *The Lax-Hopf Formula for Commodity Dependent Transaction Costs with Interest Rates* The value function \( V \) defined by (33), p. 13 is equal to

\[
V_{(l, m)}(T, x) = \inf_{(\Omega, \Upsilon) \in \mathbb{R}_+ \times X} \left( e^{M_{x(\cdot)}(\Omega)} c(T - \Omega, x - \Omega \Upsilon) + \Omega \Lambda_{(l, m)}(T, x, \Omega, \Upsilon) \right)
\]
which is a finite dimensional minimization problem on $\mathbb{R}_+ \times X$ as the Lax-Hopf formula with time, commodity and transaction interest rates and costs. At optimal aperture $\Omega_*$ and optimal average $\Upsilon_*$, optimal evolutions satisfy

$$\Lambda_{(1,m)}(T, x; \Omega_*, \Upsilon_*) := \frac{1}{\Omega_*} \int_{T-\Omega_*}^T e^{M_{(1,m)}(\tau,\Upsilon_*)} 1(t, x(\Omega_*, \Upsilon_*)(t), x'(\Omega_*, \Upsilon_*))(t)dt$$

and the Isaacs-Bellman dynamic optimal property stating that the function $V : [T - \Omega, T] \mapsto V(t)$ defined by

$$V(t) = e^{M_{(1,m)}(\Omega_*)} c(T - \Omega_*, x_*(T - \Omega_*)) + \int_{T - \Omega_*}^t c(M_{(1,m)}(\tau, \Upsilon_*) = 1) 1(t, x_*(\tau), x'_*(\tau))d\tau$$

is still the optimal actualized (at the end of the temporal window value) function $V(t) := V_{(1,m)}(t, x_*(t))$ at $(t, x_*(t))$. The generalized Lax-Hopf formula states that at optimum,

$$\Lambda_{(1,m)}(T, x; \Omega_*, \Upsilon_*) := \frac{1}{\Omega_*} \int_{T-\Omega_*}^T e^{M_{(1,m)}(\tau,\Upsilon_*)} 1(t, x(\Omega_*, \Upsilon_*)(t), x'(\Omega_*, \Upsilon_*))(t)dt$$

is the ratio between the optimal actualized profit and the optimal aperture of the temporal window, and thus, the actualized enrichment (at terminal time $T$) of the actualized value function $V$.

7 Generalized Lax-Hopf Formula for a Dynamic Economy

The dual $X^* := \mathbb{R}^\ell^*$ of the commodity space is the space of prices $p = (p^h)_{1 \leq h \leq \ell} : x \mapsto \langle p, x \rangle \in \mathbb{R}$ associating with any commodity $x \in X$ its value $\langle p, x \rangle := \sum_{h=1}^\ell p^h x_h$.

The velocity $p'(t)$ at time $t$ of the evolution of a price $p(\cdot)$ is regarded as the price fluctuations. The impact of price fluctuation $\langle p'(t), x(t) \rangle$ on a commodity $x(t)$ is related to the concept of inflation\(^\text{10}\).

We consider a set of $n$ (economic) agents. We denote by $X^n$ the set of allocations $x := (x_i)_{i=1,...,n}$ of commodities $x_i \in X$ among $n$ agents. The patrimonial value\(^\text{11}\) $t \mapsto U(x(t), p(t))$

\(^{10}\)Iflation is measured as the impact of price fluctuations $\langle p'(t), b \rangle$ on a numéraire or a consumer price indexes $b \in X$: $\langle p'(t), x(t) \rangle = \left( \frac{\langle p'(t), x(t) \rangle}{\langle p'(t), b \rangle} \right) \langle p'(t), b \rangle$.

\(^{11}\)This is the simplest example of an “economic potential” chosen for the sake of simplicity. In physics, the gradient of a potential function $U : x \mapsto U(x)$ is interpreted as a force: along an evolution $t \mapsto x(t)$, $\frac{d}{dt} U(x(t)) = \left( \frac{\partial U(x(t))}{\partial x} \right) x(t)$ is a power. In economics, the variable $x$ is replaced by the allocation-price pair $(p, x)$ and the impetus plays the role of the mechanical power.
along an evolution \( t \mapsto (x(t), p(t)) \) is defined by

\[
U(x(t), p(t)) := \sum_{i=1}^{n} \langle p_i(t), x_i(t) \rangle \quad (39)
\]

and its “impetus” \( t \mapsto E(x(t), p(t)) \) by Definition 7.1.1, p. 107, of Time and Money. How Long and How Much Money is Needed to Regulate a Viable Economy, [1, Aubin]:

\[
E(x(t), p(t)) := \frac{d}{dt}U(x(t), p(t)) = \sum_{i=1}^{n} (\langle p_i(t), x'_i(t) \rangle + \langle p'_i(t), x_i(t) \rangle) \quad (40)
\]

An instantaneous cost function \((T, x, p) \mapsto c(T, x, p)\) of the state \((x, p) \in X^n \times X^*\) at instant \( T := [T, T] \) (of zero aperture) is a function which associates the cost of \((x, p)\) at instant \( T\), regarded as a temporal window with zero aperture.

We shall assume once and for all that this instantaneous cost function is lower semicontinuous.

**Definition 7.1** [*Impetus Cost Function*] The impetus cost function described by a priori

1. a convex lower semicontinuous function \( l : E \in \mathbb{R} \mapsto l(E) \in \mathbb{R}_+ \) with which we associate the impetus cost function \( t \mapsto l(E(x(t), p(t))) \);

2. dynamical behaviors described by bounds \( \gamma(\cdot) := (\gamma_0(\cdot), \gamma_i(\cdot))_{i=1,\ldots,n} \) where:
   
   (a) bound \( 0 \leq \gamma_0(t) < +\infty \) on the norm of the price fluctuations;
   
   (b) bound \( 0 \leq \gamma_i(t) < +\infty \) on the norm of the commodity transactions of the agents \( i, i=1,\ldots,n \).

We introduce the (dynamical) impetus cost function defined by

\[
l'_i(E(x(t), p(t))) :=
\begin{cases}
  l(E(x(t), p(t))) & \text{if } \max(\max_i(\|x'_i(t)\| - c_i(t)), \|p'(t)\| - c_0(t)) \leq 0 \\
  +\infty & \text{if not}
\end{cases}
\quad (41)
\]

We denote by \( A(T, x, p) := A_1(T, x, p) \) the set of evolutions \( (x(\cdot), p(\cdot)) \) of evolutions \( t \mapsto x(t) := (x_i(t))_{i=1,\ldots,n} \) and \( t \mapsto p(t) \) with bounded velocities arriving at \((x, p)\) at terminal time \( T \).

We modify the concept of “endowment function” introduced in Chapter 7, p. 105, of Time and Money. How Long and How Much Money is Needed to Regulate a Viable Economy [1, Aubin], as suggested in Footnote 3, p.109 by introducing impetus cost functions.
Definition 7.2 [The Economic Value Function] The instantaneous cost $c$, the bounds of the canonical dynamic system being given, the endowment function $W : (T, x, p) \mapsto (T, x, p)$ is defined by

\[
W(T, x, p) := \inf_{\Omega \geq 0} \inf_{(x(\cdot), p(\cdot)) \in \mathcal{A}_\gamma(T, x, p)} \left( c(T - \Omega, x(T - \Omega), p(T - \Omega)) + \int_{T-\Omega}^T \mathbf{1}_\gamma(E(x(t), p(t))) dt \right)
\]

which is the infimum over the set of evolutions $(x(\cdot), p(\cdot))$ of their cumulated impetus cost.

Since the impetus cost function depends upon the time, the commodity and the price, we have to introduce the moderated impetus cost function $\Lambda_l(T, x, t, \Omega, \Upsilon_x, \Upsilon_p)$ defined by

\[
\Lambda_l(T, x, t, \Omega, \Upsilon_x, \Upsilon_p) := \inf \left\{ \frac{1}{\Omega} \int_{T-\Omega}^T x'(t) = \Upsilon_x(t) \mathbf{1}_T \right\} \inf \left\{ \frac{1}{\Omega} \int_{T-\Omega}^T p'(t) = \Upsilon_p(t) \mathbf{1}_T \right\} \frac{1}{\Omega} \int_{T-\Omega}^T \mathbf{1}_\gamma(E(x(t), p(t))) dt
\]

Therefore,

\[
W(T, x, p) := \inf_{\Omega \geq 0} \inf_{\Upsilon_x, \Upsilon_p} (c(T - \Omega, x - \Omega \Upsilon_x, p - \Omega \Upsilon_p) + \Lambda_l(T, x, t, \Omega, \Upsilon_x, \Upsilon_p))
\]

Then, at optimum, the enrichment of the economic value function is equal to the moderated impetus function at average optimal transaction and price fluctuation on the temporal window:

\[
\left\{ \begin{array}{l}
W(T, x_*(T), p_*(T)) - W(T - \Omega_*, x_*(T - \Omega_*), p_*(T - \Omega_*)) \\
:= \Lambda_l \left( T, x, p, \frac{x_*(T) - x_*(T - \Omega_*)}{\Omega_*}, \frac{p_*(T) - p_*(T - \Omega_*)}{\Omega_*} \right)
\end{array} \right.
\]
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