HOMOGENEOUS BESOV SPACES

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Abstract. This note is based on a series of lectures delivered in Kyoto University. This note surveys the homogeneous Besov space $\dot{B}^s_{pq}$ on $\mathbb{R}^n$ with $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ in a rather self-contained manner. Possible extensions of this type of function spaces are briefly discussed in the end of this article. In particular, the fundamental properties are stated for the spaces $\dot{B}^s_{pq}$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ as well as nonhomogeneous counterparts $B^s_{pq}$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ and $F^s_{pq}$ with $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$.

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1. Introduction

In this note we mean by a “function space” a linear function space made up of functions especially defined on $\mathbb{R}^n$. We envisage the following function spaces; $C^k$, $C^\infty$, $C^\infty_c$, $BC$ (the set of all bounded continuous functions), $O(\mathbb{C})$ (the set of all holomorphic functions on $\mathbb{C}$ under the canonical homeomorphism $\mathbb{C} \sim \mathbb{R}^2$) and the Sobolev space $W^{m,p}$ e.t.c.. What type of functions do these function spaces collect?

(1) Those which deal with size of functions: $L^p$, BC, $W^{m,p}$
(2) Those which deal with differentiability of functions: $C^k$, $C^\infty$, $C^\infty_c$, BC (0-times differentiability), $W^{m,p}$, $O(\mathbb{C})$.

The Sobolev spaces can handle both of them and are equipped with two parameters. We can say that the function spaces can describe many properties if they have many parameters. This in turn implies that the more properties they can describe, the more complicated their definition is.

1.1. Advantage of Besov spaces. Generally speaking, the function spaces are difficult to handle if their definition is complicated. However, although the definition of Besov spaces is complicated, Besov spaces are important in many senses.

1.1.1. Mathematical transforms. Besov spaces grasp many mathematical transforms nicely. Let us give examples without giving the precise definition of Besov spaces.

Theorem 1.1 (Riemann-Lebesgue). Define the Fourier transform $\mathcal{F}$ by

$$\mathcal{F}f(\xi) \equiv \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(x)e^{-ix\xi} \, dx \quad (\xi \in \mathbb{R}^n).$$

Then we have $\mathcal{F}$ maps $L^1$ continuously to $L^\infty$. 

Theorem 1.1 is known as the Riemann-Lebesgue theorem. However, in terms of the (nonhomogeneous) Besov spaces $B_{0\infty}^0$ and $B_{1\infty}^0$, we can refine Theorem 1.1 as follows:

**Theorem 1.2.**

1. $F$ maps $L^1$ continuously to $B_{0\infty}^0$.
2. $F$ maps $B_{01}^\infty$ continuously to $L^\infty$.

Since $B_{0\infty}^0$ is embedded into $L^\infty$ continuously and $B_{01}^\infty$ is embedded into $L^1$ continuously, Theorem 1.2 refines Theorem 1.1 in two different directions. We prove Theorem 1.2 in Section 3.5.

Let us give another example.

**Theorem 1.3.** Let $1 \leq p, q \leq \infty$ and $s > 0$. For $j = 1, 2, \ldots, n$, we define the $j$-th Riesz transform by:

$$R_j f(x) \equiv \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(x, \varepsilon)} \frac{(x_j - y_j)f(y)}{|x - y|^{n+1}} dy \quad (x \in \mathbb{R}^n).$$

Then $R_j$ maps $\dot{B}_s^{pq}$ continuously to itself.

This is a big achievement; $R_j$ does not map $L^1$ into $L^1$ continuously. Since $\dot{B}_{12}^0$ is almost the same as $L^1$, $\dot{B}_{12}^0$ nicely substitutes for $L^1$. We prove Theorem 1.3 in Section 5.1.

### 1.1.2. Special cases

Many important function spaces are realized as special cases of Besov spaces and Triebel-Lizorkin spaces, which are variants of Besov spaces. We refer to Sections 4.1 and 4.2 for the definition of the homogeneous Besov space $\dot{B}_s^{pq}$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ and the homogeneous Triebel-Lizorkin space $\dot{F}_s^{pq}$ with $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$, while we shall recall the definition of the nonhomogeneous Besov space $B_s^{pq}$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ and the nonhomogeneous Triebel-Lizorkin space $F_s^{pq}$ with $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$ in Sections 4.3 and 4.4, respectively.

**Theorem 1.4.** Denote by $\{e^{t\Delta}\}_{t>0}$ the heat semi-group.

1. For $1 < p < \infty$

   (1.1) $L^p \sim \dot{F}_{p2}^0 \sim \dot{F}_{p2}^0$.

   (1.2) $H^p \sim \dot{F}_{p2}^0$ for $0 < p < \infty$, where $H^p$ denotes the Hardy space of all distributions $f$ for which

   $$\|f\|_{H^p} \equiv \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_p < \infty \quad (f \in S').$$

   (3) $h^p \sim \dot{F}_{p2}^0$ for $0 < p < \infty$, where $h^p$ denotes the local Hardy space of all distributions $f$ for which

   (1.3) $\|f\|_{h^p} \equiv \left\| \sup_{t \in (0,1)} |e^{t\Delta} f| \right\|_p < \infty \quad (f \in S').$

2. Denote by BMO the space of bounded mean oscillations whose norm is given by:

   $$\|f\|_{\text{BMO}} \equiv \sup_{B: \text{balls}} \frac{1}{|B|} \int_B |f(x) - \frac{1}{|B|} \int_B f(y) dy| \ dx$$
for \( f \in L_{1,\text{loc}}^1 \). Then we have

\begin{align}
(1.4) \quad \text{BMO} & \sim F_{\infty}^0, \\
(5) \quad \text{Denote by bmo the space of local bounded mean oscillations whose norm is given by:}
\end{align}

\[
\|f\|_{\text{bmo}} = \sup_{B: \text{balls}, |B|=1} \int_B |f(x)| \, dx + \sup_{B: \text{balls}, |B| \leq 1} \frac{1}{|B|} \int_B |f(x) - \frac{1}{|B|} \int_B f(y) \, dy| \, dx.
\]

Then we have

\begin{align}
(1.5) \quad \text{bmo} & \sim F_{\infty}^0.
\end{align}

1.1.3. **Quantity and quality of functions.** We can easily grasp the meaning of the parameters \( p \) and \( s \) in Besov spaces and Triebel-Lizorkin spaces. As a result, we can deal with the quantity of functions and the quality of functions separately.

1.2. **Homogeneous and nonhomogeneous spaces.** Next, we move on to the homogeneous spaces and the nonhomogeneous spaces. Roughly speaking, homogeneous spaces are function spaces described by a set of partial derivatives of the same order; otherwise the space is nonhomogeneous.

**Example 1.5.** Let \( m \in \mathbb{N} \) and \( 1 \leq p \leq \infty \).

1. The **homogeneous Sobolev norm** \( \|f\|_{W_{m,p}} = \sum_{|\alpha|=m} \|\partial^\alpha f\|_p \) is homogeneous.

2. The **nonhomogeneous Sobolev norm** \( \|f\|_{\dot{W}_{m,p}} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p \) is nonhomogeneous.

Concerning the homogeneous norms, a couple of helpful remarks may be in order.

**Remark 1.6.** Since differentiation annihilates the polynomials or decreases the order of the polynomials, the homogenous norms do not have complete information of the function \( f \).

Despite this remark, we have the following good properties.

1. We can not use the nonhomogeneous norm to describe some properties of functions. For example, the dilation \( f \mapsto f(t \cdot) \) is a typical one.

2. Although the matters depends on the equations we consider, some invariant quantities can be realized as a sum of homogeneous norms. We need to handle each term elaborately.

1.3. **Notation.** We use the following notation in this note.

**Notation**.

1. The metric ball defined by \( \ell^2 \) is usually called a **ball**. We denote by \( B(x, r) \) the ball centered at \( x \) of radius \( r \). Given a ball \( B \), we denote by \( c(B) \) its center and by \( r(B) \) its radius. We write \( B(r) \) instead of \( B(0, r) \), where \( 0 \equiv (0,0,\ldots,0) \).

2. Let \( E \) be a measurable set. Then, we denote its **indicator function** by \( \chi_E \). If \( E \) has positive measure and \( E \) is integrable over \( f \), then denote by \( m_E(f) \) the **average of \( f \) over \( E \)**. The symbol \( |E| \) denotes the **volume of \( E \)**.
(3) Let \( A, B \geq 0 \). Then, \( A \lesssim B \) and \( B \gtrsim A \) mean that there exists a constant \( C > 0 \) such that \( A \leq CB \), where \( C \) depends only on the parameters of importance. The symbol \( A \sim B \) means that \( A \lesssim B \) and \( B \gtrsim A \) happen simultaneously. While \( A \simeq B \) means that there exists a constant \( C > 0 \) such that \( A = CB \).

(4) We define

\[
\mathbb{N} \equiv \{1, 2, \ldots\}, \quad Z \equiv \{0, \pm 1, \pm 2, \ldots\}, \quad \mathbb{N}_0 \equiv \{0, 1, \ldots\}.
\]

(5) For \( a \in \mathbb{R}^n \), we write \( \langle a \rangle \equiv \sqrt{1 + |a|^2} \).

(6) Suppose that \( \{f_j\}_{j=1}^\infty \) is a sequence of measurable functions. Then we write

\[
\|f_j\|_{L^p(\mathbb{R}^n)} \equiv \left( \int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty |f_j(x)|^q \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}}, \quad 0 < p, q \leq \infty
\]

and

\[
\|f_j\|_{\ell^p(L^q)} \equiv \left( \sum_{j=1}^\infty \left( \int_{\mathbb{R}^n} |f_j(x)|^p \, dx \right)^{\frac{1}{q}} \right)^{\frac{1}{p}}, \quad 0 < p, q \leq \infty
\]

(7) A distribution \( f \in \mathcal{S}' \) is said to belong to \( \mathcal{L}^1_{\text{loc}} \) if there exist constants \( C > 0 \) and \( N \in \mathbb{N} \) and \( F \in \mathcal{L}^1_{\text{loc}} \) such that

\[
|\langle f, \varphi \rangle| = \left| \int_{\mathbb{R}^n} F(x) \varphi(x) \, dx \right| \leq C \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \varphi(x)|
\]

for all \( \varphi \in C_c^\infty \). In this case one writes \( f \in \mathcal{L}^1_{\text{loc}} \cap \mathcal{S}' \), \( F \in \mathcal{L}^1_{\text{loc}} \cap \mathcal{S}' \), \( f \in \mathcal{S}' \cap \mathcal{L}^1_{\text{loc}} \) or \( F \in \mathcal{S}' \cap \mathcal{L}^1_{\text{loc}} \).

2. Schwartz spaces

2.1. Definitions—as sets and as topological spaces.

2.1.1. The Schwartz space \( \mathcal{S} \) and its dual \( \mathcal{S}' \). Here we recall the Schwartz space \( \mathcal{S} \) together with its topology.

**Definition 2.1.** The Schwartz space \( \mathcal{S} \) is the subspace of \( C^\infty \) given by

\[
\mathcal{S} = \bigcap_{N=1}^\infty \left\{ f \in C^\infty : \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} \langle x \rangle^N |\partial^\alpha f(x)| < \infty \right\}.
\]

The topology of \( \mathcal{S} \) is the weakest one for which the mapping \( f \in \mathcal{S} \mapsto p_N(f) \in \mathbb{R} \) is continuous for all \( N \in \mathbb{N} \), where

\[
p_N(f) \equiv \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} \langle x \rangle^N |\partial^\alpha f(x)|, \quad \langle x \rangle \equiv \sqrt{1 + |x|^2}.
\]

We admit the following fact:

**Theorem 2.2.** [44, p. 249, Théorème XII] The Fourier transform maps \( \mathcal{S} \) into \( \mathcal{S} \) isomorphically.

The proof is well known and omitted.

Now we move on to the topological dual space \( \mathcal{S}' \).
**Definition 2.3.** One defines
\[ S' \equiv \{ f : S \to \mathbb{C} : f \text{ is linear and continuous} \} . \]
One equips \( S' \) with the weakest topology so that the mapping
\[ f \in S' \mapsto \langle f, \varphi \rangle \in \mathbb{C} \]
is continuous for all \( \varphi \in S \).

We recall the following fundamental characterization of \( S' \) for later considerations.

**Lemma 2.4.** For all \( f \in S' \) there exists \( N \in \mathbb{N} \) such that
\[ |\langle f, \varphi \rangle| \leq Np_N(\varphi) \]
for all \( \varphi \in S \).

**Proof.** Since \( f \) is continuous at 0,
\[ \{ \varphi \in S : |\langle f, \varphi \rangle| < 1 \} = f^{-1}(\{ z \in \mathbb{C} : |z| < 1 \}) \]
is an open set. Thus there exists \( N \in \mathbb{N} \) and \( \delta > 0 \) such that
\[ \{ \varphi \in S : p_N(\varphi) < \frac{1}{N} \} \subset \{ \varphi \in S : |\langle f, \varphi \rangle| < 1 \}, \]
implying \( |\langle f, \varphi \rangle| \leq Np_N(\varphi) \) for all \( \varphi \in S \). \( \square \)

**2.1.2. The Schwartz space \( S_\infty \) and its dual \( S'_\infty \).** The space \( S \) and its dual \( S' \) are the ingredients of the nonhomogeneous Besov space \( B^s_{p,q} \) for \( 0 < p,q \leq \infty \) and \( s \in \mathbb{R} \), while for the homogeneous Besov space \( B^s_{p,q} \) with \( 0 < p,q \leq \infty \) and \( s \in \mathbb{R} \) we need \( S_\infty \) and its dual \( S'_\infty \).

**Definition 2.5.** One defines \( S_\infty \equiv \bigcap_{\alpha \in \mathbb{N}_0^n} \left\{ \psi \in S : \int_{\mathbb{R}^n} x^\alpha \psi(x) \, dx = 0 \right\} \).

The main advantage of defining the class \( S_\infty \) is that when we are given \( \varphi \in S_\infty \) the function given by
\[ \psi = \mathcal{F}^{-1}[|\xi|^\alpha \cdot \mathcal{F}\varphi] \]
is in \( S_\infty \). In fact, for \( \varphi \in S \) \( \varphi \in S_\infty \) if and only if \( \mathcal{F}\varphi \) vanishes up to the arbitrary order at 0.

**Definition 2.6.**
(1) Equip \( S'_\infty \) with the weakest topology so that the evaluation
\[ f \in S' \mapsto \langle f, \varphi \rangle \in \mathbb{C} \]
is continuous for all \( \varphi \in S_\infty \).

(2) Denote by \( \mathcal{P} \) the set of all polynomials. Embed \( \mathcal{P} \) into \( S' \) canonically; for any \( \alpha \) the mononomial \( x^\alpha \) stands for the distribution
\[ \varphi \in S \mapsto \int_{\mathbb{R}^n} x^\alpha \varphi(x) \, dx \in \mathbb{C} . \]

Recall that we can endow the quotient \( X/\sim \) of a topological space \( (X, \mathcal{O}_X) \) and its equivalence relation \( \sim \) with a natural topology.

**Definition 2.7.** Let \( (X, \mathcal{O}_X) \) be a topological space. Denote by \( \sim \) the equivalence relation;
Let $p$ be a projection $X$ to $X/\sim$. The strongest topology of $X/\sim$ for which $p$ is continuous is called the quotient topology of $X/\sim$. Remark that $S'/\mathcal{P}$ is a quotient space; $[f] = [g]$ for $f, g \in S'$ if and only if $f - g \in \mathcal{P}$.

2.2. A fundamental theorem. We shall prove the following theorem:

Theorem 2.8.

1. For all $f \in S'_\infty$ there exists $F \in S'$ such that $F|S_\infty = f$.
2. The linear spaces $S'/\mathcal{P}$ and $S_\infty$ are isomorphic.
3. The mapping $R : F \in S' \mapsto F|S_\infty \in S'_\infty$ is continuous.
4. The mapping $R : F \in S' \mapsto F|S_\infty \in S'_\infty$ is open, that is, $R(U)$ is an open set in $S'_\infty$ for all open sets $U \subset S'$.

2.2.1. Remarks on Theorem 2.8 and its proof. Here we collect some facts for the proof of Theorem 2.8.

Remark 2.9.

1. A direct consequence of this theorem is that $[f] \in S'/\mathcal{P} \mapsto f|S_\infty \in S'_\infty$ is a topological isomorphism.
2. The continuity of $R$ is clear from the definition.
3. Since $S'$ is NOT metrizable, one cannot apply the Baire category theorem. It seems that there is no literature that allows us to apply a version of the Baire category theorem.
4. It seems that one cannot find the proof of the openness of $R$ in any literature before [36]. The proof of (1) and (2) is known in the book [54, 5.1.]. Yuan, Sickel and Yang proved (3) and (4) [64, Proposition 8.1]. However, there was a gap. They used the closed graph theorem. However, it seems unclear that one can use the closed graph theorem. The proof in this note is essentially to close the gap of (4) in the proof of [64, Proposition 8.1].
5. The idea for the proof is to extend many functionals we deal with to $V_N$ or $W_N$ carefully, where

\begin{align*}
V_N &\equiv \overline{S_\infty^p N} \subset C^N \\
W_N &\equiv \overline{S^p N} \subset C^N.
\end{align*}

2.2.2. ker $R$. We specify ker $R$ here. The following lemma is fundamental.

Theorem 2.10. Let $f \in S'$. Then the following are equivalent.

1. $\text{supp}(f) \subset \{0\}$.
2. $f$ is expressed as the following finite sum:

$$f = \sum_{\lambda \in \Lambda} c_\lambda \partial^\lambda \delta_0,$$

where $\Lambda \subset \mathbb{N}_0^n$ is a finite set and $\partial^\lambda \delta_0$ denotes the distribution defined by $\langle \partial^\lambda \delta_0, \varphi \rangle \equiv (-1)^{|\lambda|} \partial^\lambda \varphi(0)$ for $\varphi \in \mathcal{S}$.

Via the Fourier transform, we can prove:
Corollary 2.11. \( \ker(R) = \mathcal{P} \).

Proof of Theorem 2.10.

\( \circ \) Suppose \( \text{supp}(f) \subset \{0\} \). By Lemma 2.4, we can find \( N \in \mathbb{N} \) such that

\[
|\langle f, \varphi \rangle| \leq N \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (x)^N |\partial^\alpha \varphi(x)|
\]

for all \( \varphi \in \mathcal{S} \). Choose \( \{c_\alpha\}_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq 2N+1} \) such that

\[
\psi(x) = \varphi(x) - \sum_{|\alpha| \leq 2N+1} c_\alpha x^\alpha e^{-|x|^2} = O(|x|^{2N+2})
\]
as \( x \to 0 \). We claim

\[
\langle f, \psi \rangle = 0.
\]

Since each \( c_\alpha \) depends continuously on \( \varphi \), more precisely,

\[
c_\alpha = \sum_{\lambda \in \Lambda_\alpha} k_{\alpha, \lambda} \partial^\lambda \varphi(0)
\]

with \( \Lambda_\alpha \) a finite set in \( \mathbb{N}_0^n \), we have

\[
\langle f, \varphi \rangle = \sum_{|\alpha| \leq 2N+1} \langle f, x^\alpha e^{-|x|^2} \rangle c_\alpha.
\]

\( \circ \) Suppose \( f \) is expressed as the following finite sum as in (2). Then we have

\[
\langle \partial^\alpha \delta_0, \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(0) = 0.
\]

Thus \( \text{supp}(f) \subset \{0\} \) once we show (2.5).

Let us show (2.5). Recall that \( \mathcal{V}_N \) is defined by (2.1). Let \( \tau : \mathbb{R} \to \mathbb{R} \) be a function such that \( \chi_{(2, \infty)} \leq \tau \leq \chi_{(1, \infty)} \). Then

\[
\lim_{j \to \infty} p_N(\psi - \tau(j) \psi) = 0
\]

by the Leibniz rule, (2.3) and (2.4). Thus,

\[
\langle f, \psi \rangle = \lim_{j \to \infty} \langle f, \tau(j) \psi \rangle = 0
\]

since \( f \) is supported away from the origin. This proves (2.5).

\( \square \)

2.2.3. Surjectivity of \( R \). We aim here to show that \( R \) is surjective. To this end we choose \( f \in \mathcal{S}'_{\infty} \) arbitrarily. Then similar to Lemma 2.4 we can find \( N \in \mathbb{N} \) such that

\[
|\langle f, \varphi \rangle| \leq N p_N(\varphi)
\]

for all \( \varphi \in \mathcal{S}_{\infty} \).

Thanks to (2.6) \( f \) extends uniquely to a countinous linear functional \( g \) on the normed space \( \mathcal{V}_N \), i.e. \( f = g|_{\mathcal{S}_{\infty}} \). We use the Hahn-Banach theorem to extend \( g \) to \( \mathcal{W}_N \) to have a continuous linear functional \( G \) defined on \( \mathcal{W}_N \) such that \( G|\mathcal{V}_N = g \) and that \( |\langle G, \Phi \rangle| \leq N p_N(\Phi) \) for all \( \Phi \in \mathcal{V}_N \).
2.2.4. **Openness of** $R$. Everything hinges on the following observation:

**Theorem 2.12.** Let $\varphi_1, \varphi_2, \ldots, \varphi_N \in S_\infty$ and let $\psi_1, \psi_2, \ldots, \psi_K \in S$. Assume that the system $\{[\psi_1], [\psi_2], \ldots, [\psi_K]\}$ is linearly independent in $S/S_\infty$. Let

$$U \equiv \bigcap_{j=1}^N \{f \in S'_\infty : |\langle f, \varphi_j \rangle| < 1\}$$

be a neighborhood of $0 = 0 \in S'_\infty$ and

$$V \equiv \bigcap_{j=1}^N \{f \in S' : |\langle f, \varphi_j \rangle| < 1\} \cap \bigcap_{k=1}^K \{f \in S' : |\langle f, \psi_k \rangle| < 1\}$$

a neighborhood of $0 = 0 \in S'$. Then $R(V) = U$.

To prove Theorem 2.12, we need the following key lemma:

**Lemma 2.13.** Under the assumptions of Theorem 2.12, there exists $N \gg 1$ such that the system $\{[\psi_1], [\psi_2], \ldots, [\psi_K]\}$ is linearly independent in $W_N/V_N$.

**Proof.** Write

$$S^{2K-1} \equiv \left\{ (a_1, a_2, \ldots, a_K) \in \mathbb{C}^K : \sum_{k=1}^K |a_k|^2 = 1 \right\}.$$

We have only to show that

$$\left\{ \sum_{k=1}^K a_k \psi_k : (a_1, a_2, \ldots, a_K) \neq (0, 0, \ldots, 0) \in \mathbb{C}^K \right\} \cap V_N = \emptyset$$

or equivalently

$$\left\{ \sum_{k=1}^K a_k \psi_k : (a_1, a_2, \ldots, a_K) \in S^{2K-1} \right\} \cap V_N = \emptyset$$

for some large $N \gg n + 1$.

For each $(a_1, a_2, \ldots, a_K) \in S^{2K-1}$, there exists $\alpha(a_1, a_2, \ldots, a_K) \in \mathbb{N}^n$ such that

$$\int_{\mathbb{R}^n} x^{\alpha(a_1, a_2, \ldots, a_K)} \sum_{j=1}^K a_j \psi_j(x) \, dx \neq 0.$$

Since the function

$$(b_1, b_2, \ldots, b_K) \in \mathbb{C}^K \mapsto \int_{\mathbb{R}^n} x^{\alpha(a_1, a_2, \ldots, a_K)} \sum_{j=1}^K b_j \psi_j(x) \, dx \in \mathbb{C}$$

is continuous, there exists an open neighborhood $U(a_1, a_2, \ldots, a_K) \subset \mathbb{C}^K$ of the point $(a_1, a_2, \ldots, a_K) \in S^{2K-1}$ such that

$$\int_{\mathbb{R}^n} x^{\alpha(a_1, a_2, \ldots, a_K)} \sum_{j=1}^K b_j \psi_j(x) \, dx \neq 0.$$
for all \((b_1, b_2, \ldots, b_K) \in U(a_1, a_2, \ldots, a_K)\). Since \(S^{2K-1}\) is compact, we can find a finite collection \((a'_1, a'_2, \ldots, a'_K)\) for \(l = 1, 2, \ldots, L\) such that

\[
S^{2K-1} \subset \bigcup_{l=1}^{L} U(a'_1, a'_2, \ldots, a'_K).
\]

Taking

\[
N \equiv n + 2 + \sum_{l=1}^{L} |\alpha(a'_1, a'_2, \ldots, a'_K)|,
\]

we obtain the desired result.

We prove Theorem 2.12. In fact, from Lemma 2.13, for any \(f \in U\), we can find \(G \in \mathcal{W}_N\) so that \(G|\mathcal{V}_N = g\) and that \(\langle G, \psi_k \rangle = 0\) for all \(k = 1, 2, \ldots, K\). In fact, we have a strictly increasing sequence of closed linear subspaces \(\{\text{Span}(\mathcal{V}_N \cup \{\psi_k\}_{k=1}^{K'})\}_{K'=1}\). Apply successively the Hahn Banach theorem to obtain the desired \(G\). Then observe that \(G|\mathcal{S} \in R\) and \(R(G|\mathcal{S}) = f\). Thus \(F = G|\mathcal{S}\) does the job.

**Remark 2.14.** Under the assumption of Theorem 2.12 set

\[
V_0 \equiv \bigcap_{j=1}^{N} \{f \in \mathcal{S}' : |\langle f, \varphi_j \rangle| < 1\} \cap \bigcap_{k=1}^{K} \{f \in \mathcal{S}' : \langle f, \psi_k \rangle = 0\}.
\]

Then \(R(V_0) = U\) as the above proof shows.

**Theorem 2.15.** Let \(\varphi_1, \varphi_2, \ldots, \varphi_N \in \mathcal{S}_{\infty}\) and let \(\psi_1, \psi_2, \ldots, \psi_K \in \mathcal{S}\). Assume that the system \(\{|\varphi_1|, |\varphi_2|, \ldots, |\varphi_K|\}\) is a maximal family of linearly independent elements in \(\{|\psi_1|, |\psi_2|, \ldots, |\psi_K|\}\) in \(\mathcal{S}/\mathcal{S}_{\infty}\), so that \(\tilde{K} \geq K\). More precisely,

\[
(2.7) \quad \psi_k = \tilde{\varphi}_k + \sum_{l=1}^{K} a_{kl}\psi_l
\]

for some \(\tilde{\varphi}_k \in \mathcal{S}_{\infty}\) and \(a_{kl} \in \mathbb{C}\), \(k = \tilde{K} + 1, \tilde{K} + 2, \ldots, K\) and \(l = 1, 2, \ldots, K\). Let

\[
U \equiv \bigcap_{j=1}^{N} \{f \in \mathcal{S}' : |\langle f, \varphi_j \rangle| < 1\} \cap \bigcap_{k=1}^{K} \{f \in \mathcal{S}' : |\langle f, \tilde{\varphi}_k \rangle| < 1\}
\]

be a neighborhood of \(0 = 0_{\mathcal{S}_{\infty}} \in \mathcal{S}'_{\infty}\) and

\[
V \equiv \bigcap_{j=1}^{N} \{f \in \mathcal{S}' : |\langle f, \varphi_j \rangle| < 1\} \cap \bigcap_{k=1}^{K} \{f \in \mathcal{S}' : |\langle f, \psi_k \rangle| < 1\}
\]

a neighborhood of \(0 = 0_{\mathcal{S}}' \in \mathcal{S}'\). Then \(R(U) \supset V\).

**Proof.** Let \(f \in V\). Then we can find a linear functional \(F \in \mathcal{S}'\) such that \(F|\mathcal{S}_{\infty} = f\) and that

\[
(2.8) \quad \langle F, \psi_k \rangle = 0
\]

for \(k = 1, 2, \ldots, \tilde{K}\) according to Remark 2.14. Observe that

\[
\langle F, \psi_k \rangle = \langle F, \tilde{\varphi}_k \rangle \in \{z \in \mathbb{C} : |z| < 1\}
\]

for \(k = \tilde{K} + 1, \tilde{K} + 2, \ldots, K\) from (2.7) and (2.8). Thus \(R(U) \supset V\).
3. Homogeneous Besov spaces

In this section we consider homogeneous Besov spaces. As we will see, justifying the definition is one of the hard tasks.

3.1. Definition. We define homogeneous Besov spaces and justify their definition.

Definition 3.1. Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Choose $\varphi \in \mathcal{S}$ so that $\chi_{B(8)} \setminus B(1) \leq \varphi \leq \chi_{B(4)} \setminus B(2)$. Define

$$
\dot{B}^s_{pq} = \left\{ f \in \mathcal{S}'/\mathcal{P} : \| f \|_{\dot{B}^s_{pq}} = \left( \sum_{j=-\infty}^{\infty} \langle 2^js \| \mathcal{F}^{-1} \varphi_j \cdot \mathcal{F} f \|_p \rangle^q \right)^{\frac{1}{q}} < \infty \right\}.
$$

The space $\dot{B}^s_{pq}$ is the set of all $f \in \mathcal{S}'/\mathcal{P}$ for which the quasi-norm $\| f \|_{\dot{B}^s_{pq}}$ is finite.

A couple of helpful remarks may be in order.

Remark 3.2.

(1) Since $\mathcal{S}'/\mathcal{P}$ is a quotient linear space, the expression $f \in \mathcal{S}'/\mathcal{P}$ is not appropriate. Instead, one should have written $[f] \in \mathcal{S}'/\mathcal{P}$, where $[f]$ denotes the class in $\mathcal{S}'/\mathcal{P}$ to which $f$ belongs. Nevertheless we write $f \in \mathcal{S}'/\mathcal{P}$ by habit.

(2) It does not make sense to consider $\| f \|_p$ for $f \in \mathcal{S}'$; $\mathcal{S}'$ is NOT included in $L^1_{\text{loc}}$. However, the distribution $\mathcal{F}^{-1} \varphi_j \cdot \mathcal{F} f$ is a $C^\infty$-function as is seen from the identity

$$
\mathcal{F}^{-1} \varphi_j \cdot \mathcal{F} f(x) = \frac{1}{\sqrt{(2\pi)^n}} \langle f, \mathcal{F}^{-1} \varphi_j(x - \cdot) \rangle \quad (x \in \mathbb{R}^n).
$$

(3) Note that $f \in \mathcal{P}$ if and only if $\mathcal{F}^{-1} \varphi_j \cdot \mathcal{F} f = 0$ for all $j \in \mathbb{Z}$. In fact when $f \in \mathcal{P}$, $\varphi_j \cdot \mathcal{F} f = 0$ for all $j \in \mathbb{Z}$, since $f \in \text{Span}\{\partial^\alpha \delta_0 : \alpha \in \mathbb{N}^n\}$. If $\mathcal{F}^{-1} \varphi_j \cdot \mathcal{F} f = 0$ for all $j \in \mathbb{Z}$, then $\varphi_j \cdot \mathcal{F} f = 0$. Choose a test function $\tau \in C_c^\infty$. Since $\tau$ is supported away from the origin and $\tau$ vanishes outside of a bounded set, one has

$$
\tau = \sum_{j=-\infty}^{\infty} \frac{\tau \varphi_j}{\Phi} \cdot \varphi_j,
$$

where

$$
\Phi = \sum_{j=-\infty}^{\infty} \varphi_j^2.
$$

It counts that the function $\varphi_j/\Phi$ makes sense as an element in $C_c^\infty$; compare the size of their support. Also, the expression (3.1) is essentially a finite sum, so that

$$
\langle \tau, \mathcal{F} f \rangle = \sum_{j=-\infty}^{\infty} \left\langle \frac{\tau \varphi_j}{\Phi}, \varphi_j \cdot \mathcal{F} f \right\rangle = 0.
$$

Thus, $\text{supp}(\mathcal{F} f) \subset \{0\}$, implying that $f \in \mathcal{P}$. 

The following observation justifies the definition $\dot{B}^s_{pq}$ as a linear space. In fact, we chose $\varphi$ so that the norm of $\dot{B}^s_{pq}$ depends on $\varphi$. But as the following theorem shows, $\dot{B}^s_{pq}$ is independent of $\varphi$ as a set, which justifies the definition of $\dot{B}^s_{pq}$.

**Theorem 3.3.** Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Let $\varphi, \tilde{\varphi} \in \mathcal{S}$ satisfy $\chi_{B(4)} \setminus B(2) \leq \varphi, \tilde{\varphi} \leq \chi_{B(8)} \setminus B(1)$. Set $\varphi_j := \varphi(2^{-j} \cdot)$ and $\tilde{\varphi}_j := \tilde{\varphi}(2^{-j} \cdot)$ for $j \in \mathbb{Z}$. Then we have

$$
\left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[\varphi_j \cdot F f] \|_p)^q \right)^{\frac{1}{q}} \sim \left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[\tilde{\varphi}_j \cdot F f] \|_p)^q \right)^{\frac{1}{q}}
$$

for all $f \in \mathcal{S}'$.

**Proof.** By symmetry, it suffices to show that

$$(3.2) \quad \left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[\varphi_j \cdot F f] \|_p)^q \right)^{\frac{1}{q}} \lesssim \left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[(\tilde{\varphi}_j - 1 + \tilde{\varphi}_j + \tilde{\varphi}_{j+1}) \cdot F f] \|_p)^q \right)^{\frac{1}{q}}.$$

Define $\psi \in C^\infty_c$ by

$$
\psi \equiv \frac{\varphi}{\varphi_{-1} + \varphi + \varphi_1}.
$$

Then we have $\varphi_j = \psi_j (\tilde{\varphi}_{j-1} + \tilde{\varphi}_j + \tilde{\varphi}_{j+1})$ for each $j \in \mathbb{Z}$. Inserting this relation into the right-hand side of (3.2), we obtain

$$
\left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[\varphi_j \cdot F f] \|_p)^q \right)^{\frac{1}{q}} = \left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[\psi_j \cdot (\tilde{\varphi}_{j-1} + \tilde{\varphi}_j + \tilde{\varphi}_{j+1}) \cdot F f] \|_p)^q \right)^{\frac{1}{q}}.
$$

By the relation between the Fourier transform, the convolution and the pointwise multiplication, we have

$$
\left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[\varphi_j \cdot F f] \|_p)^q \right)^{\frac{1}{q}} \approx \left( \sum_{j=-\infty}^{\infty} (2^j \| \mathcal{F}^{-1}[\psi_j \cdot (\tilde{\varphi}_{j-1} + \tilde{\varphi}_j + \tilde{\varphi}_{j+1}) \cdot F f] \|_p)^q \right)^{\frac{1}{q}}.
$$
By the Young inequality and the fact that \( \| F \phi_j \|_1 = \| F \phi_1 \|_1 \) for all \( j \in \mathbb{N} \), we obtain

\[
\left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [\phi_j \cdot F f] \|_p)^q \right)^\frac{1}{q} 
\leq \left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} \psi_j \|_1 \| F^{-1} [(\hat{\phi}_{j-1} + \hat{\phi}_j + \hat{\phi}_{j+1}) \cdot F f] \|_p)^q \right)^\frac{1}{q} 
\]
\[
\approx \left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [(\hat{\phi}_{j-1} + \hat{\phi}_j + \hat{\phi}_{j+1}) \cdot F f] \|_p)^q \right)^\frac{1}{q} .
\]

Finally, by the triangle inequality and the index shiftings, we have

\[
\left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [\phi_j \cdot F f] \|_p)^q \right)^\frac{1}{q} 
\leq \left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [\hat{\phi}_{j-1} F f] \|_p)^q \right)^\frac{1}{q} + \left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [\hat{\phi}_j \cdot F f] \|_p)^q \right)^\frac{1}{q} 
\]
\[
+ \left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [\hat{\phi}_{j+1} F f] \|_p)^q \right)^\frac{1}{q} 
\]
\[
\approx \left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [\hat{\phi}_j \cdot F f] \|_p)^q \right)^\frac{1}{q} ,
\]

which proves (3.2). \( \square \)

Before we conclude this section, we have a remark helpful till the end of this section.

**Remark 3.4.** Let \( \zeta \in C_c^\infty (\mathbb{R}^n \setminus \{0\}) \). The above proof shows that we have many possibilities of \( \zeta \) in the definition of \( \dot{B}^{s}_{pq} \):

\[
\| f \|_{B^{s}_{pq}} \equiv \left( \sum_{j=-\infty}^{\infty} (2^j \| F^{-1} [\zeta_j \cdot F f] \|_p)^q \right)^\frac{1}{q} .
\]

A particular choice of \( \zeta \); \( \zeta = \psi - \psi(2\cdot) \), where \( \chi_{B(1)} \leq \psi \leq \chi_{B(2)} \), is useful for later considerations.

### 3.2. Hölder-Zygmund spaces and Besov spaces.

So far, we justified the definition of the homogeneous Besov spaces \( \dot{B}^{s}_{pq} \) with \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \) as a linear space (or a normed space). However, it is rather hard to show that Besov space \( B^{s}_{pq} \) is complete. As is often the case, it is a burden to construct the limit point when we are given a Cauchy sequence in metric spaces.

The main aim of this section is twofold; one is to create a tool to obtain a candidate of the limit point when we are given a Cauchy sequence in the space \( B^{s}_{pq} \).
The other is to exhibit an example showing that the homogeneous Besov space \( \dot{B}^{s}_{\infty\infty} \) is useful. In fact, as a special case the homogeneous Hölder-Zygmund spaces are realized with the scale \( \dot{B} \).

3.2.1. **The difference operator.** To define homogeneous Hölder-Zygmund spaces, we need the notion of difference of higher order. We start with the following elementary identity:

**Lemma 3.5.** \[
\sum_{l=1}^{m} (-1)^l m! C_l = (-1)^m m! \text{ for all } m \in \mathbb{N}.
\]

**Proof.** Compare the coefficient of \( t^m \) of the function \( t^m + \cdots = (e^t - 1)^m = \sum_{l=0}^{m} e^{lt} m! C_l (-1)^{m-l} \).

Then we have
\[
1 = \sum_{l=1}^{m} (-1)^l m! C_l,
\]
as was to be shown. \( \square \)

Let \( y \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R} \) be a mapping. Define inductively \( \Delta_y^m f \) by
\[
\Delta_y^1 f \equiv f(\cdot + y) - f, \quad \Delta_y^{m+1} f \equiv \Delta_y^m(\Delta_y^1 f).
\]

Based on Lemma 3.5, we shall obtain a formula to connect the difference with the operator \( f \in \mathcal{S}' \mapsto \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F} f] \). Set
\[
\Phi \equiv \sum_{r=1}^{m} \sum_{l=1}^{m} \frac{r^n}{(rl)^n} m! C_r \cdot m! C_l (-1)^{r+l+m+1} \Psi \left( \frac{\cdot}{rl} \right)
\]

**Lemma 3.6.** Let \( f \in L^1_{\text{loc}} \) and \( \Psi \in C^\infty_c \). Then
\[
2^j \Phi(2^j \cdot) f(x) - m! \int_{\mathbb{R}^n} \Psi(y) dy \times f(x) = \sum_{r=1}^{m} (-1)^{r+m+1} \int_{\mathbb{R}^n} \Psi(y) \Delta_y^m(\Delta_y^{2-j} f(x)) dy.
\]

**Proof.** First we observe that
\[
\int_{\mathbb{R}^n} \Phi(y) dy = \sum_{r=1}^{m} \sum_{l=1}^{m} r^n (-1)^{r+l+1+m} m! C_r \cdot m! C_l \int_{\mathbb{R}^n} \Psi(y) dy
\]
\[
= (-1)^{m+1} \sum_{r=1}^{m} r^n (-1)^{r} m! C_r \times \sum_{l=1}^{m} (-1)^l m! C_l \times \int_{\mathbb{R}^n} \Psi(y) dy
\]
\[
= m! \int_{\mathbb{R}^n} \Psi(y) dy
\]
using Lemma 3.5. Thus, it follows that
\[
2^{jn} \Phi(2^j \cdot) * f(x) - m! \int_{\mathbb{R}^n} \Psi(y) dy \times f(x)
\]
\[
= \sum_{r=1}^{m} \sum_{l=1}^{m} r^m m! C_r \cdot m C_l (-1)^{r+l+m+1} \int_{\mathbb{R}^n} \Psi(y) f(x - 2^{-j} r l y) dy
\]
\[
- m! \int_{\mathbb{R}^n} \Psi(y) f(x) dy
\]
\[
= \sum_{r=1}^{m} \sum_{l=0}^{m} r^m m! C_r \cdot m C_l (-1)^{r+l+m+1} \int_{\mathbb{R}^n} \Psi(y) f(x - 2^{-j} r l y) dy
\]
\[
= \sum_{r=1}^{m} r^m m C_r (-1)^{r+m+1} \int_{\mathbb{R}^n} \Psi(y) \Delta_{2^{-j} r l} f(x - 2^{-j} r l y) dy,
\]
as was to be shown. \(\square\)

The equivalent expression in the next lemma is useful when we consider the difference operator.

**Lemma 3.7.** Suppose that \(\Psi \in S\) satisfies
\[
\chi_{B(4) \setminus B(2)} \leq \mathcal{F} \Psi \leq \chi_{B(8) \setminus B(1)}.
\]

Define \(\Phi\) by:
\[
\Phi \equiv \sum_{r=1}^{m} \sum_{l=1}^{m} \frac{r^m m! C_r \cdot m C_l (-1)^{r+l+m+1}}{(rl)^n} \Psi \left( \frac{\cdot}{rl} \right).
\]

1. The function \(\mathcal{F} \Phi\) is constant in a neighborhood of the origin.
2. Let \(1 \leq p, q \leq \infty\) and \(s \in \mathbb{R}\). Then by setting
\[
\varphi_j \equiv \mathcal{F} \Phi(2^{-j-1} \cdot) - \mathcal{F} \Phi(2^{-j} \cdot) \quad (j \in \mathbb{Z})
\]
and
\[
\|f\|_{\dot{B}_{pq}^s}^p \equiv \left( \sum_{j=-\infty}^{\infty} 2^{js} \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F} f]\|_p \right)^{\frac{p}{q}},
\]
we obtain a norm equivalent to \(\dot{B}_{pq}^s\).

**Proof.**

1. Just observe
\[
\mathcal{F} \Phi \equiv \sum_{r=1}^{m} \sum_{l=1}^{m} r^m m C_r \cdot m C_l (-1)^{r+l+m+1} \mathcal{F} \Psi (rl \cdot).
\]

2. This assertion follows from Remark 3.4 \(\square\)
3.2.2. The space $C_s$. Denote by $P_r$ the set of all polynomials $f \in P$ having degree at most $r$.

**Definition 3.8.** The Hölder space $\dot{C}_s$ is the set of all continuous functions $f$ on $\mathbb{R}^n$ for which the semi-norm

$$
\|f\|_{\dot{C}_s} \equiv \sup \left\{ \frac{\|\Delta_y^{[s+1]} f\|_{L^\infty}}{|y|^s} : y \in \mathbb{R}^n \right\} < \infty.
$$

Sometimes one considers $\dot{C}_s$ modulo $P_s$.

**Examples 3.9.** Denote by $C^{\mathbb{R}^n}$ the set of all complex-valued functions defined on $\mathbb{R}^n$ and by $C$ the set of all continuous functions defined on $\mathbb{R}^n$.

1. Let $0 < s < 1$. Then

$$
\dot{C}_s = \left\{ f \in C : |f(x + y) - f(x)| \lesssim |y|^s \text{ for all } x, y \in \mathbb{R}^n \right\}
$$

2. Let $s = 1$. Then

$$
\dot{C}_s = \left\{ f \in C^{\mathbb{R}^n} : |f(x + y) - 2f(x) + f(x - y)| \lesssim |y| \text{ for all } x, y \in \mathbb{R}^n \right\}.
$$

Remark that this set is NOT equal to $\dot{C}_s = \left\{ f \in C^{\mathbb{R}^n} : |f(x + y) - 2f(x) + f(x - y)| \lesssim |y|^s \text{ for all } x, y \in \mathbb{R}^n \right\}$, as one can show using the Hamel basis, the basis of the linear space $\mathbb{R}^n$ over $\mathbb{Q}$. [34, Proposition A.1].

3.2.3. The space $\dot{B}^s_{\infty\infty}$. Let $s > 0$ and $f \in \dot{B}^s_{\infty\infty}$. Let us now choose $\psi \in S$ so that

$$
(3.3) \quad \chi_{B(1)} \leq \psi \leq \chi_{B(2)}.
$$

Define

$$
(3.4) \quad \varphi_j \equiv \psi(2^{-j}) - \psi(2^{-j+1})
$$

for $j \in \mathbb{Z}$. Set

$$
(3.5) \quad H \equiv \sum_{j=1}^{\infty} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f].
$$

It is easy to see that $H$ is a continuous function since

$$
\sum_{j=1}^{\infty} \|\varphi_j(D)f\|_\infty \leq \sum_{j=1}^{\infty} 2^{-js} \|f\|_{\dot{B}^s_{\infty\infty}} \sim \|f\|_{\dot{B}^s_{\infty\infty}}.
$$

The function $H$ is called the high frequency part of $f$.

Meanwhile, the function

$$
\tilde{G} \equiv \sum_{j=1}^{\infty} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f].
$$

is called the low frequency part of $f$. The trouble is that there is no guarantee that the right-hand side defining $\tilde{G}$ is convergent in some suitable topology. To circumvent this problem, we consider its derivative as follows:
Lemma 3.10. Let \( f \in \dot{B}^s_{\infty, \infty} \) with \( s > 0 \). Let \( \psi \in \mathcal{S} \) satisfy (3.3). Define \( \varphi_j \) by (3.3) for \( j \in \mathbb{Z} \). If \( \alpha \geq [s + 1] \), then

\[
G_{\alpha} \equiv \sum_{j=-\infty}^{0} \partial^\alpha \left[ \mathcal{F}^{-1} [\varphi_j \cdot Ff] \right]
\]

is convergent in \( L^\infty \) and hence this function is smooth.

Proof. Note that

\[
\mathcal{F}^{-1}[\varphi_j \cdot Ff] = \mathcal{F}^{-1}[(\varphi_{j-1} + \varphi_j + \varphi_{j+1}) \cdot \varphi_j \cdot Ff] 
\]

\[
\simeq_n \mathcal{F}^{-1}[\varphi_{j-1} + \varphi_j + \varphi_{j+1}] \ast \mathcal{F}^{-1}[\varphi_j \cdot Ff] 
\]

Write \( \tau \equiv F^{-1}\varphi_j \). Then we have

\[
\mathcal{F}^{-1}[\varphi_j \cdot Ff] \simeq (2^{(j-1)n} \tau(2^{j-1})) + 2^{jn} \tau(2^j) + 2^{(j+1)n} \tau(2^{j+1}) \ast \mathcal{F}^{-1}[\varphi_j \cdot Ff].
\]

By the Hölder inequality, we have

\[
\| \partial^\alpha \left[ \mathcal{F}^{-1}[\varphi_j \cdot Ff] \right] \|_\infty 
\]

\[
= \| (2^{(j-1)\alpha(n+n)} \partial^\alpha \tau(2^{j-1})) + 2^{j\alpha(n+n)} \partial^\alpha \tau(2^j) + 2^{(j+1)\alpha(n+n)} \partial^\alpha \tau(2^{j+1})) \|_1 
\]

\[
\times \| \mathcal{F}^{-1}[\varphi_j \cdot Ff] \|_\infty 
\]

\[
\lesssim 2^{j\alpha} \| \mathcal{F}^{-1}[\varphi_j \cdot Ff] \|_\infty.
\]

As a result we have

\[
\sum_{j=-\infty}^{0} \| \partial^\alpha \left[ \mathcal{F}^{-1}[\varphi_j \cdot Ff] \right] \|_\infty \lesssim \sum_{j=-\infty}^{0} 2^{j\alpha} \| \mathcal{F}^{-1}[\varphi_j \cdot Ff] \|_\infty 
\]

\[
\leq \sum_{j=-\infty}^{0} 2^{j\alpha - js} \| f \|_{\dot{B}^s_{\infty, \infty}} 
\]

\[
= \sum_{j=-\infty}^{0} 2^{j(s+1)-s} \| f \|_{\dot{B}^s_{\infty, \infty}} 
\]

\[
\simeq \| f \|_{\dot{B}^s_{\infty, \infty}}.
\]

To construct a substitute of the lower part, we need to depend on a geometric property of \( \mathbb{R}^n ; H^1_{\text{DR}}(\mathbb{R}^n) = 0 \). Applying this fact, we can prove the following:

Lemma 3.11. Let \( N \in \mathbb{N} \). Suppose that we have \( \{ f_\alpha \}_{\alpha \in \mathbb{N}_0^n \setminus \{ \alpha \} = N} \subset C^\infty \) satisfying

\[
\partial^\alpha f_\alpha = \partial^\beta f_\beta
\]

for all \( \alpha, \alpha', \beta, \beta' \in \mathbb{N}_0^n \) with \( |\alpha| = |\alpha'| = N \) and \( \alpha + \beta = \alpha' + \beta' \). Then there exists \( f \in C^\infty \) such that \( \partial^\alpha f = f_\alpha \).

The proof is in Appendix; see Section 5.2. By using Lemma 3.11, we have the following control of the low frequency part:

Corollary 3.12. Let \( f \in \dot{B}^s_{\infty, \infty} \) with \( s > 0 \). Let \( \psi \in \mathcal{S} \) satisfy (3.3). Define \( \varphi_j \) by (3.3) for \( j \in \mathbb{Z} \). There exists a function \( G \in C^\infty \) such that

\[
\partial^\alpha G = \sum_{j=-\infty}^{0} \partial^\alpha \left[ \mathcal{F}^{-1}[\varphi_j \cdot Ff] \right]
\]
for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \geq [s + 1]$.

We further investigate the property of $G$ in Corollary 3.12.

**Proposition 3.13.** Let $s > 0$ and $f \in B^s_{\infty \infty}$. Choose $\psi \in \mathcal{S}$ so that $\chi_{B(1)} \leq \psi \leq \chi_{B(2)}$. Define $\varphi_j \equiv \psi(2^{-j} \cdot) - \psi(2^{-j+1} \cdot)$ for $j \in \mathbb{Z}$.

1. Define $G$ and $H$ by (3.7) and (3.8), respectively. Then $f - (G + H) \in \mathcal{P}$.

2. $G + H \in \mathcal{C}^\infty$.

3. If $G'$ is such that

$$\partial^\alpha G' = \sum_{j=\infty}^0 \partial^\alpha [F^{-1}[\varphi_j \cdot Ff]]$$

for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \geq [s + 1]$, then $G' - G$ is a polynomial of order less than or equal to $|s|$.

**Proof.**

1. Observe that $\partial^\alpha [F^{-1}[\varphi_j \cdot F(f - G - H)]] = 0$ for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \geq [s + 1]$ and for all $j \in \mathbb{Z}$. Thus $F^{-1}[\varphi_j \cdot F(f - G - H)] = 0$ for all $j \in \mathbb{Z}$. Thus $f - G - H \in \mathcal{P}$.

2. Write $F = G + H$. We need to show that

$$|\Delta^{[s+1]} G(x) | \lesssim |y|^s \|f\|_{B^s_{\infty \infty}}.$$

By the triangle inequality we have

$$|\Delta^{[s+1]} G_j(x) | \lesssim |y|^s \|F^{-1}[\varphi_j \cdot Ff]\|_{\infty}.$$

Meanwhile, by the mean value theorem we have

$$|\Delta^{[s+1]} F^{-1}[\varphi_j \cdot Ff](x) | \lesssim 2^{j[s+1]} |y|^{[s+1]} \|F^{-1}[\varphi_j \cdot Ff]\|_{\infty}.$$

Let $j_0 \in \mathbb{Z}$ be chosen so that

$$1 \leq 2^{j_0} |y| < 2.$$

We define

$$H_{j_0} = \sum_{j=j_0}^{\infty} F^{-1}[\varphi_j \cdot Ff], \quad G_{j_0} = G + H - H_{j_0}.$$

Once we prove

$$|\Delta^{[s+1]} G_{j_0}(x) | \lesssim |y|^s \|f\|_{B^s_{\infty \infty}}$$

and

$$|\Delta^{[s+1]} H_{j_0}(x) | \lesssim |y|^s \|f\|_{B^s_{\infty \infty}},$$

then we have (3.8).

To prove (3.12), we use the mean-value theorem and (3.10) to have

$$|\Delta^{[s+1]} G_{j_0}(x) | \lesssim |y|^{[s+1]} \|\nabla^{[s+1]} G_{j_0}\|_{\infty}$$

$$= |y|^{[s+1]} \left\| \sum_{j=-\infty}^{j_0-1} \nabla^{[s+1]} [F^{-1}[\varphi_j \cdot Ff]] \right\|_{\infty}$$

$$\leq \sum_{j=-\infty}^{j_0-1} |y|^{[s+1]} \left\| \nabla^{[s+1]} [F^{-1}[\varphi_j \cdot Ff]] \right\|_{\infty}.$$
Arguing as we did in Lemma 3.10 we have
\[ |\Delta_y^{[s+1]}G_{j_0}(x)| \lesssim \sum_{j_0}^{j_0-1} 2^{[s+1]}|y|^{[s+1]}\|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_\infty \]
\[ \leq \sum_{j=-\infty}^{j_0-1} 2^{j([s+1]-\alpha)}|y|^{[s+1]}\|f\|_{B_{\infty}^{s+1}} \]
\[ \sim 2^{j_0([s+1]-\alpha)}|y|^{[s+1]}\|f\|_{B_{\infty}^{s+1}} \]
\[ \sim |y|^s\|f\|_{B_{\infty}^{s+1}} \]
using (3.11). To prove (3.13) we use (3.9) to have
\[ |\Delta_y^{[s+1]}G_{j_0}(x)| \leq 2^{[s+1]} \sum_{j=j_0}^{\infty} \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_\infty \lesssim \sum_{j=j_0}^{\infty} 2^{-j} s\|f\|_{B_{\infty}^{s+1}} \sim |y|^s\|f\|_{B_{\infty}^{s+1}}, \]
as was to be shown.

(3) As we did in (1), we \( \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}(G'-G)] = 0 \). Thus, \( G-G' \) is a polynomial. Since \( \partial^\alpha G' = \partial^\alpha G \) for all \( \alpha \in \mathbb{N}_0 \) with \( |\alpha| \geq [s+1] \), \( G-G' \) is a polynomial of order less than or equal to \( [s] \).

\[ \Box \]

3.2.4. The isomorphism \( \mathcal{C}^s \sim \hat{B}_{\infty}^{s+1} \). We can summarize the above observations as follows:

Theorem 3.14. Let \( s > 0 \). Choose an auxiliary \( \psi \in \mathcal{S} \) so that \( \chi_{B(1)} \leq \psi \leq \chi_{B(2)} \). Set \( \varphi \equiv \psi(2^{-1}) - \psi \).

(1) \( \hat{C}^s \rightarrow \mathcal{S}_\infty^s \), where \( \hat{C}^s \) is the quotient linear space modulo \( \mathcal{P}_{[s]} \).

(2) For \( F \in \hat{C}^s \), define
\[ f : \rho \in \mathcal{S}_\infty \mapsto \int_{\mathbb{R}^n} F(x)\rho(x) \, dx \in \mathbb{C}. \]
Then \( f \in \hat{B}_{\infty}^{s+1} \).

(3) Let \( f \in \hat{B}_{\infty}^{s+1} \). Then there exists a continuous function \( F \in \hat{C}^s \) such that \( f-F \in \mathcal{P} \). More precisely, \( F \) can be taken as a sum of continuous functions \( G \) and \( H \) so that
\[ \partial^\alpha G = \sum_{j=-\infty}^{0} \partial^\alpha[\varphi_j(D)f], \quad H = \sum_{j=1}^{\infty} \varphi_j(D)f. \]

Proof.

(1) Suppose that \( f \in \hat{C}^s \), where we do NOT consider \( \hat{C}^s \) as the quotient space. Write \( m \equiv [s+1] \). For \( x \in \mathbb{R} \), let us set
\[ g_k(y) = \begin{cases} \sum_{l=0}^{m} (-1)^{m-l} m C_l \cdot f(ky + ly) & (k \geq 0), \\ \sum_{l=-k}^{m} (-1)^{m-l} m C_l \cdot f(ky + ly) & (-m \leq k < 0), \\ 0 & (k < -m) \end{cases} \]
for \( y \in \mathbb{R}^n \).
Then we have
\[ f(ky) = \sum_{l_m=-m}^{k} \sum_{l_{m-1}=-m}^{l_m} \cdots \sum_{l_1=-m}^{l_2} g_k(y). \]

For each \( x \) with \( |x| > 1 \), choose \( k \in \mathbb{N} \) so that \( k < |x| \leq k + 1 \). Set \( x = ky \).

Then we have \( |y| \geq k^{-1} |x| > 1 \) and
\[ |f(x)| \lesssim (1 + |x|)^m \]
as was to be shown.

(2) According to Lemma 3.7 and the analogue of Lemma 3.6, we have
\[ \mathcal{F}^{-1}\varphi_j * f(x) = \sum_{r=1}^{m} (-1)^{r+m+1} r^m \int_{\mathbb{R}^n} \Psi(y)(\Delta_{2^{-j}ry}^m F(x) - \Delta_{2^{-j}ry}^m F(x)) \, dy. \]

Thus
\[ |\mathcal{F}^{-1}\varphi_j * f(x)| \leq \sum_{r=1}^{m} \int_{\mathbb{R}^n} |\Psi(y)|(|\Delta_{2^{-j}ry}^m F(x)| + |\Delta_{2^{-j}ry}^m F(x)|) \, dy \]
\[ \leq \|F\|_{\dot{C}^s} \sum_{r=1}^{m} \int_{\mathbb{R}^n} |\Psi(y)| \cdot |2^{-js}y|^s \, dy \]
\[ \lesssim 2^{-js} \|F\|_{\dot{C}^s}, \]
and hence \( 2^{js} |\mathcal{F}^{-1}\varphi_j * f(x)| \lesssim \|F\|_{\dot{C}^s} \) for \( x \in \mathbb{R}^n \), as was to be shown.

(3) This is included in Proposition 3.13.

\[ \Box \]

### 3.3. Fundamental properties

Here we investigate fundamental properties of homogeneous Besov spaces. Let \( \psi \in \mathcal{S} \) satisfy (3.3). Define \( \varphi_j \) by (3.4) for \( j \in \mathbb{Z} \).

We set
\[ \|f\|_{\dot{B}^s_{pq}} = \left( \sum_{j=-\infty}^{\infty} (2^{js} \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F} f]\|_p)^q \right)^{\frac{1}{q}} \]
for \( f \in \mathcal{S}'/\mathcal{P} \) as before.

If \( f \in \mathcal{S}_\infty \), then the notation \( [f] \in \mathcal{S}'/\mathcal{P} \) makes sense. Here we can say more about this and we can present some fundamental examples of elements in \( \dot{B}^s_{pq} \).

**Theorem 3.15.** Let \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \). Then \( \mathcal{S}_\infty \hookrightarrow \dot{B}^s_{pq} \).

We use the following lemma:

**Lemma 3.16.** Let \( x \in \mathbb{R}^n \) and \( j, k \in \mathbb{Z} \). Then we have
\[ \int_{\mathbb{R}^n} \frac{2^{(j+k)n} dy}{(4^j|y|^2 + 1)^{n+1}} \sim \frac{2^{\min(j,k)n}}{(4^{\min(j,k)}|x|^2 + 1)^{n+1}}. \]
Proof. The proof is simple; by the Fourier transform and the Plancherel theorem, we have
\[
\int_{\mathbb{R}^n} \frac{2^{(j+k)n}}{(4|y|^2 + 1)^{\frac{n+1}{2}}} \text{dy} \sim \int_{\mathbb{R}^n} \exp(-\xi \cdot xi - (2^{-j} + 2^{-k})|\xi|) \text{d}\xi \\
\sim \frac{(2^{-j} + 2^{-k})^n}{((2^{-j} + 2^{-k})^2|x|^2 + 1)^{\frac{n+1}{2}}} \sim \frac{2^{\min(j,k)n}}{(4^{\min(j,k)}|x|^2 + 1)^{\frac{n+1}{2}}}.
\]
\]
\[\square\]

We now prove Theorem 3.15.

Proof of Theorem 3.15. Let \(\theta \in S_\infty\). We seek to show
\[
\|\theta\|_{\dot{B}^{s}_{p,q}} = \left( \sum_{j=-\infty}^{\infty} (2^j \|F^{-1}[\varphi_j \cdot F\theta]\|_p)^q \right)^{\frac{1}{q}} < \infty.
\]

Let \(x \in \mathbb{R}^n\) be fixed. It matters that \(\tilde{\theta} \equiv F^{-1}(|\xi|^{-2L}F\theta)\) belongs to \(S_\infty\) for all \(L \in \mathbb{N}\) and that \(\Delta \tilde{\theta} = (-1)^L\theta\). Then we have
\[
F^{-1}\varphi_j \ast \theta(x) = 2^{jn} \int_{\mathbb{R}^n} F^{-1}\varphi(2^j y)\theta(x-y) \text{dy} \\
= (-1)^L 2^{jn} \int_{\mathbb{R}^n} F^{-1}\varphi(2^j y)\Delta^L \theta(x-y) \text{dy} \\
= (-1)^L 2^{jn+2jL} \int_{\mathbb{R}^n} (\Delta^L F^{-1}\varphi)(2^j y)\theta(x-y) \text{dy}.
\]
From Lemma 3.16, we obtain
\[
|F^{-1}\varphi_j \ast \theta(x)| \lesssim \frac{2^{2jL}}{(|x| + 1)^{n+1}}.
\]
Likewise by letting \(\tilde{\varphi}(\xi) \equiv |\xi|^{-2L}\varphi(\xi)\), we obtain
\[
|F^{-1}\varphi_j \ast \theta(x)| \lesssim \frac{2^{-2jL}}{(2^j|x| + 1)^{n+1}} \leq \frac{2^{-(n+1)j-2jL}}{(|x| + 1)^{n+1}}.
\]
Let \(L \gg 1\). As a result,
\[
\|\theta\|_{\dot{B}^{s}_{p,q}} \lesssim \left( \sum_{j=-\infty}^{\infty} (2^j |x|^{(n+1) - 2jL})q \right)^{\frac{1}{q}} < \infty,
\]
as was to be shown. \[\square\]

Next we aim to consider the role of the parameter \(s\). Let \(f \in S'_\infty\). Then
\(f = \sum_{j=-\infty}^{\infty} F^{-1}[\varphi_j \cdot Ff]\) in \(S'_\infty\) since \(\Phi = \sum_{j=-\infty}^{\infty} F^{-1}[\varphi_j \cdot F\Phi]\) in \(S_\infty\) for all \(\Phi \in S_\infty\). Observe that
\[
(-\Delta)^{\frac{s}{2}} \Phi \equiv \sum_{j=-\infty}^{\infty} F^{-1}[|\xi|^{(n+1)} \varphi_j \cdot F\Phi],
\]
where the convergence takes place in $S_\infty$. Therefore, we can define \((-\Delta)^{\frac{\alpha}{2}} f\) by
\[
(-\Delta)^{\frac{\alpha}{2}} f \equiv \sum_{j=\infty}^{\infty} F^{-1}[|\xi|^\alpha \cdot \varphi_j \cdot \hat{f}]
\]
where the convergence takes place in $S'_\infty$. It is not so hard to show that the definition of \((-\Delta)^{\frac{\alpha}{2}} f\) does not depend on $\varphi$ satisfying (4.3).

Theorem 3.17. Let $1 \leq p, q \leq \infty$ and $s, \alpha \in \mathbb{R}$. Then \((-\Delta)^{\frac{\alpha}{2}} : \dot{B}^s_{pq} \to \dot{B}^{s-\alpha}_{pq}\) is an isomorphism.

Proof. We have only to show that \((-\Delta)^{\frac{\alpha}{2}} : \dot{B}^s_{pq} \to \dot{B}^{s-\alpha}_{pq}\) is continuous. Indeed, once this is proved, then we have \((-\Delta)^{\frac{\alpha}{2}} : \dot{B}^{s-\alpha}_{pq} \to \dot{B}^s_{pq}\) is continuous, which is the inverse of \((-\Delta)^{\frac{\alpha}{2}}\).

Let $f \in \dot{B}^s_{pq}$. Then we have
\[
\|(-\Delta)^{\frac{\alpha}{2}} f\|_{\dot{B}^{s-\alpha}_{pq}} = \left( \sum_{j=\infty}^{\infty} (2^{js-\alpha}) \|F^{-1}[|\xi|^\alpha \cdot \varphi_j \cdot \hat{f}]\|_p^q \right)^{\frac{1}{q}} \]
\[
= \left( \sum_{j=\infty}^{\infty} (2^{js-\alpha}) \|F^{-1}[(\varphi_{j-1} + \varphi_j + \varphi_{j+1}) \cdot |\xi|^\alpha \cdot \varphi_j \cdot \hat{f}]\|_p^q \right)^{\frac{1}{q}} \]
\[
\approx \left( \sum_{j=\infty}^{\infty} (2^{js-\alpha}) \|F^{-1}[(\varphi_{j-1} + \varphi_j + \varphi_{j+1}) \cdot |\xi|^\alpha] * F^{-1}[\varphi_j \cdot \hat{f}]\|_p^q \right)^{\frac{1}{q}}.
\]

If we use the Young inequality, then we have
\[
\|(-\Delta)^{\frac{\alpha}{2}} f\|_{\dot{B}^{s-\alpha}_{pq}} \leq \left( \sum_{j=\infty}^{\infty} (2^{js-\alpha}) \|F^{-1}[(\varphi_{j-1} + \varphi_j + \varphi_{j+1}) \cdot |\xi|^\alpha] \|_1 \|F^{-1}[\varphi_j \cdot \hat{f}]\|_p^q \right)^{\frac{1}{q}} \]
\[
\approx \left( \sum_{j=\infty}^{\infty} (2^{js}) \|F^{-1}[\varphi_j \cdot \hat{f}]\|_p^q \right)^{\frac{1}{q}}
\]
\[
= \|f\|_{\dot{B}^s_{pq}},
\]
as was to be shown.

\[\square\]

Theorem 3.18. Let $1 \leq p, q, r \leq \infty$ and $s \in \mathbb{R}$.

(1) If $r \geq q$, then $\dot{B}^s_{pq} \hookrightarrow \dot{B}^s_{pr}$.

(2) In the sense of continuous embedding
\[
\dot{B}^s_{pq} \hookrightarrow \dot{B}^{s-n/p}_{\infty q}.
\]

Proof.

(1) This is clear because $\ell^q(\mathbb{Z}) \hookrightarrow \ell^r(\mathbb{Z})$. 

(2) Let \( f \in \dot{B}^{s}_{pq} \). Then we have

\[
\|f\|_{\dot{B}^{s-n/p}_{\infty q}}
\]

\[
= \left( \sum_{j=-\infty}^{\infty} (2^{j(s-n/p)} \|F^{-1}[\varphi_j \cdot F f]\|_{\infty})^q \right)^{\frac{1}{q}}
\]

\[
= \left( \sum_{j=-\infty}^{\infty} (2^{j(s-n/p)} \|F^{-1}[\varphi_{j-1} + \varphi_j + \varphi_{j+1}] \cdot \varphi_j \cdot F f]\|_{\infty})^q \right)^{\frac{1}{q}}
\]

\[
\approx \left( \sum_{j=-\infty}^{\infty} (2^{j(s-n/p)} \|F^{-1}[\varphi_{j-1} + \varphi_j + \varphi_{j+1}] \ast F^{-1}[\varphi_j \cdot F f]\|_{\infty})^q \right)^{\frac{1}{q}}
\]

By the Hölder inequality, we have

\[
\|f\|_{\dot{B}^{s-n/p}_{\infty q}} \lesssim \left( \sum_{j=-\infty}^{\infty} (2^{j(s-n/p)} \|F^{-1}[\varphi_{j-1} + \varphi_j + \varphi_{j+1}] \ast F^{-1}[\varphi_j \cdot F f]\|_{\infty})^q \right)^{\frac{1}{q}}
\]

\[
= \|f\|_{\dot{B}^{s}_{pq}}.
\]

Thus we have the desired result.

\[
\square
\]

**Theorem 3.19.** Let \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \).

1. \( \dot{B}^{s}_{pq} \hookrightarrow S'_{\infty} \).
2. \( \dot{B}^{s}_{pq} \) is complete.

**Proof.**

1. According to Theorems 3.14 and 3.18

\[
(3.15) \quad \dot{B}^{s}_{pq} \hookrightarrow \dot{B}^{s-n/p}_{\infty q} \hookrightarrow \dot{B}^{s-n/p}_{\infty} \sim \dot{C}^{s-n/p} \hookrightarrow S'_{\infty}
\]

as long as \( s > n/p \). Thus, when \( s > n/p \), \( \dot{B}^{s}_{pq} \hookrightarrow S'_{\infty} \). According to Theorem 3.17 we still have \( \dot{B}^{s}_{pq} \hookrightarrow S'_{\infty} \) when \( s \leq n/p \).

2. According to Theorem 3.17 we may assume \( s > n/p \). Let \( \{f_j\}_{j=1}^{\infty} \) be a Cauchy sequence in \( \dot{B}^{s}_{pq} \). Then, according to (3.15) we have \( \dot{B}^{s}_{pq} \hookrightarrow \dot{C}^{s-n/p} \). Thus \( \{f_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( \dot{C}^{s-n/p} \). Since \( \dot{C}^{s-n/p} \) is complete, \( \{f_j\}_{j=1}^{\infty} \) is convergent to \( f \in \dot{C}^{s-n/p} \). Since \( \dot{C}^{s-n/p} \hookrightarrow S'_{\infty} \) and \( F^{-1}[\varphi_j \cdot F f](x) \approx \langle F^{-1}[\varphi_j \cdot F f](x) \rangle \), we have

\[
\lim_{l \to \infty} F^{-1}[\varphi_j \cdot F f_l](x) = F^{-1}[\varphi_j \cdot F f](x).
\]
By the Fatou theorem we have
\[
\|f - f_k\|_{\dot{B}^s_{pq}} = \left( \sum_{j=-\infty}^{\infty} (2^j s \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}[f - f_k]]\|_p)^q \right)^{\frac{1}{q}} \\
\leq \left( \sum_{j=-\infty}^{\infty} \liminf_{l \to \infty} (2^j s \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}[f_l - f_k]]\|_p)^q \right)^{\frac{1}{q}} \\
\leq \liminf_{l \to \infty} \left( \sum_{j=-\infty}^{\infty} (2^j s \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}[f_l - f_k]]\|_p)^q \right)^{\frac{1}{q}}.
\]

Thus, by letting \(k \to \infty\), we learn \(f_k \to f\) in \(\dot{B}^s_{pq}\).

\[\square\]

3.4. Realization. When we consider the partial differential equation, it is not comfortable to work on the quotient space. One of the reasons is that the quotient space does not give us any information of the value of functions. Therefore, at least we want to go back to the subspace of \(S'\). Although the evaluation mapping does not make sense in \(S'\), we feel that the situation becomes better in \(S'\) than in \(S'_\infty \simeq S'/\mathcal{P}\). Such a situation is available when \(s\) is small enough.

**Theorem 3.20.** Let \(1 \leq p, q \leq \infty\). Assume
\[s < \frac{n}{p}\]
or
\[s = \frac{n}{p}, \quad q = 1.\]
Choose \(\psi \in S\) so that \(\chi_{B(1)} \leq \psi \leq \chi_{B(2)}\). Set \(\varphi_j \equiv \psi(2^{-j} \cdot ) - \psi(2^{-j+1} \cdot )\). Then for all \(f \in \dot{B}_{pq}^{s}\), \(\sum_{j=-\infty}^{0} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\) is convergent in \(L^\infty\) and \(\sum_{j=1}^{\infty} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\) is convergent in \(S'\).

**Proof.** We use \(\|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_\infty \lesssim 2^{jn/p}\|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_p\) to show that \(\sum_{j=-\infty}^{0} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\) is convergent in \(L^\infty\) as before. The fact that \(\sum_{j=1}^{\infty} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\) is convergent in \(S'\) is a general fact. Or to show this, we can use the lift operator to have \(\sum_{j=1}^{\infty} (-\Delta)^{-L} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\) is convergent in \(S'\) for any \(L \in \mathbb{N}\). In fact, we can check that
\[
\sum_{j=1}^{\infty} \|(-\Delta)^{-L} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_\infty \lesssim \sum_{j=1}^{\infty} 2^{j(n/p - 2L)} \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_p < \infty
\]
as long as \(2L > \frac{n}{p}\). \[\square\]
3.5. **Refinement of the Fourier transform.** We supplement and prove Theorem 1.3.

**Theorem 3.21.**

1. The Fourier transform $\mathcal{F}$ maps $L^1$ into $\dot{B}^{0}_{\infty 1}$.
2. $\dot{B}^{0}_{\infty 1} \hookrightarrow BC$, where $\dot{B}^{0}_{\infty 1}$ can be regarded as the subset of $S'$ by way of Theorem 3.20.

**Proof.** We assume that $\varphi \in S$ satisfies

$$\text{supp}(\varphi) \subset B(4) \setminus B(1), \quad \sum_{j=-\infty}^{\infty} \varphi_j = \chi_{\mathbb{R}^n \setminus \{0\}},$$

where $\varphi_j(\xi) \equiv \varphi(2^{-j}\xi)$ for $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$.

1. $\|\mathcal{F}f\|_{\dot{B}^{0}_{\infty 1}} = \sum_{j=-\infty}^{\infty} \|\mathcal{F}^{-1}[\varphi_j \cdot f(\cdot)]\|_\infty \leq \sum_{j=-\infty}^{\infty} \|\varphi_j \cdot f(\cdot)\|_1 = \|f\|_1$ where we used the original Hausdorff-Young theorem to obtain the inequality.

(2) The assumption $f \in \dot{B}^{0}_{\infty 1}$ is equivalent to

$$\sum_{j=-\infty}^{\infty} \|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_\infty < \infty.$$

Thus, the series

$$f = \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]$$

converges uniformly. Since $\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f] \in BC$, it follows that $f \in BC$. □

From the remark below we see that Theorem 3.21 improves the classical Hausdorff-Young theorem.

**Remark 3.22.** Remark that BUC is a closed subspace of $L^\infty$, which is of course equipped with the $L^\infty$-norm.

This implies that BUC $\neq \dot{B}^{0}_{\infty 1}$. In fact, the above proof shows that BUC $\supset \dot{B}^{0}_{\infty 1}$. Assume that equality holds. Then the function

$$f_j(x) \equiv \max(1, \min(-1, jx_1)) \in \text{BUC}$$

forms a bounded set. This implies that $\{f_j\}_{j=1}^{\infty} \subset \dot{B}^{0}_{\infty 1}$ is a bounded set by virtue of the closed graph theorem. Since $f_j \to 2\chi_{\{x_1 > 0\}} - 1$ in $S'$, $2\chi_{\{x_1 > 0\}} - 1$ would be a member in $\dot{B}^{0}_{\infty 1}$. This is a contradiction to Theorem 3.21(2).

4. **More about function spaces**

4.1. **The homogeneous Besov space $\dot{B}^{s}_{pq}$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.** We extend $\dot{B}^{s}_{pq}$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

**Definition 4.1.** Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Choose $\varphi \in S$ so that $\chi_{B(4) \setminus B(1)} \leq \varphi \leq \chi_{B(8) \setminus B(1)}$. Define

$$\dot{B}^{s}_{pq} \equiv \left\{ f \in S'/\mathcal{P} : \|f\|_{\dot{B}^{s}_{pq}} \equiv \left( \sum_{j=-\infty}^{\infty} (2^{js}\|\mathcal{F}^{-1}[\varphi_j \cdot \mathcal{F}f]\|_p)^q \right)^{1/q} < \infty \right\}.$$
The space $\dot{B}_{pq}^s$ is the set of all $f \in S'/\mathcal{P}$ for which the quasi-norm $\|f\|_{\dot{B}_{pq}^s}$ is finite.

Let $0 < \eta < \infty$. We define the powered Hardy-Littlewood maximal operator $M^{(\eta)}$ by

$$M^{(\eta)}f(x) \equiv \sup_{R > 0} \left( \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|^{\eta} \, dy \right)^{1/\eta}. \tag{4.1}$$

The powered Hardy-Littlewood maximal operator $M^{(\eta)}$ comes about in the context of the Plancherel-Polya-Nikolskii inequality:

$$\sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{(1+Ry)^{\eta/\eta}} \lesssim M^{(\eta)}(x)$$

for all $R > 0$ and $f \in S'$ such that supp$(f) \subset B(R)$.

The thrust of considering the spaces $\dot{B}_{pq}^s$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ is the pointwise product $f \cdot g$, which corresponds to the fact that $L^p \cdot L^q = L^r$ for any $p, q, r > 0$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

### 4.2. The homogeneous Triebel-Lizorkin space $\dot{F}_{pq}^s$ for $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$

We let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ instead of letting $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

**Definition 4.2.** Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Choose $\varphi \in S$ so that $\chi_{B(4) \setminus B(2)} \leq \varphi \leq \chi_{B(8) \setminus B(1)}$. Define

$$\dot{F}_{pq}^s = \left\{ f \in S'/\mathcal{P} : \|f\|_{\dot{F}_{pq}^s} \equiv \left( \sum_{j=-\infty}^{\infty} \|2^j \mathcal{F}^{-1}[\varphi f \cdot \mathcal{F}f]|^q \|_p \right)^{1/q} \right\}. \tag{4.2}$$

The space $\dot{F}_{pq}^s$ is the set of all $f \in S'/\mathcal{P}$ for which the quasi-norm $\|f\|_{\dot{F}_{pq}^s}$ is finite.

To handle the convolution, we use the following theorem:

**Theorem 4.3.** Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If $0 < \eta < \infty$ and $\{f_j\}_{j=1}^{\infty}$ is a sequence of measurable functions, then

$$\left\| \left( \sum_{j=1}^{\infty} Mf_j \right)^{\eta/q} \right\|_p \lesssim_{p,q,\eta} \left\| \left( \sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_p. \tag{4.3}$$

### 4.3. The nonhomogeneous Besov space $B_{pq}^s$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$

Choose $\psi \in S$ so that

$$\chi_{B(2)} \leq \psi \leq \chi_{B(4)}. \tag{4.4}$$

Define

$$\varphi_j \equiv \psi(2^{-j}) - \psi(2^{-j+1}) \tag{4.5}$$

for $j \in \mathbb{N}$.

Using (4.2) and (4.3), we define the nonhomogeneous Besov space $B_{pq}^s$ as follows:
Definition 4.4. Let \(0 < p, q \leq \infty\) and \(s \in \mathbb{R}\). Define
\[
B_{pq}^s = \left\{ f \in S' : \|f\|_{B_{pq}^s} \equiv \|F^{-1}[\psi \cdot Ff]\|_p + \left( \sum_{j=1}^{\infty} (2^j s \|F^{-1}[\varphi_j \cdot Ff]\|_p)^q \right)^{\frac{1}{q}} < \infty \right\}.
\]

The nonhomogeneous Besov space \(B_{pq}^s\) is the set of all \(f \in S'\) for which the quasi-norm \(\|f\|_{B_{pq}^s}\) is finite.

Here we collect some fundamental properties of Besov spaces, which we prove as we did in this note.

Remark 4.5.

1. The definition of \(B_{pq}^s\) does not depend on \(\psi\) satisfying (4.2).
2. \(B_{pq}^s\) is a complete space, that is, \(B_{pq}^s\) is a quasi-Banach space when \(0 < p, q \leq \infty\) and \(B_{pq}^s\) is a Banach space when \(1 \leq p, q \leq \infty\).
3. \(S \hookrightarrow B_{pq}^s \hookrightarrow S'\) in the sense of continuous embedding.
4. \(B_{pq}^s = L^2\) with norm equivalence.
5. Define \(C^s \equiv \dot{C}^s \cap L^\infty\), where \(\dot{C}^s\) is NOT a linear space modulo \(\mathcal{P}_{[s]}\). Then, \(C^s\) is a Banach space and \(B_{pq}^s = C^s\) for \(s > 0\) with norm equivalence.
6. Let \(0 < p_0 < p_1 < \infty\), \(0 < q \leq \infty\) and \(-\infty < s_0 < s_1 < \infty\). Assume
\[
s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.
\]

Then \(B_{p_0q}^{s_0} \hookrightarrow B_{p_1q}^{s_1}\).

4.4. The nonhomogeneous Triebel-Lizorkin space \(F_{pq}^s\) for \(0 < p < \infty\), \(0 < q \leq \infty\) and \(s \in \mathbb{R}\). Let \(\psi\) and \(\varphi\) satisfy (4.2) and (4.3), respectively.

Definition 4.6. Let \(0 < p, q \leq \infty\) and \(s \in \mathbb{R}\). Define
\[
F_{pq}^s = \left\{ f \in S' : \|f\|_{F_{pq}^s} \equiv \|F^{-1}[\psi \cdot Ff]\|_p + \left( \sum_{j=1}^{\infty} (2^j s \|F^{-1}[\varphi_j \cdot Ff]\|_p)^q \right)^{\frac{1}{q}} < \infty \right\}.
\]

The nonhomogeneous Triebel-Lizorkin space \(F_{pq}^s\) is the set of all \(f \in S'\) for which the quasi-norm \(\|f\|_{F_{pq}^s}\) is finite.

Here we collect some fundamental properties of Triebel-Lizorkin spaces, whose proof overlaps largely the ones in this note. Here we comment what else idea we need if necessary.

Remark 4.7.

1. The definition of \(F_{pq}^s\) does not depend on \(\psi\) satisfying (4.2).
2. \(F_{pq}^s\) is a complete space, that is, \(F_{pq}^s\) is a quasi-Banach space when \(0 < p, q \leq \infty\) and \(F_{pq}^s\) is a Banach space when \(1 \leq p, q \leq \infty\).
3. \(S \hookrightarrow B_{p_{\min}(p,q)}^s \hookrightarrow F_{pq}^s \hookrightarrow B_{p_{\max}(p,q)}^s \hookrightarrow S'\) in the sense of continuous embedding.
4. Let \(1 < p < \infty\). Then \(F_{p2}^0 = L^p\) with norm equivalence. This is proved by using the Rademacher sequence.
(5) Let $0 < p_0 < p_1 < \infty$, $0 < q \leq \infty$ and $-\infty < s_1 < s_0 < \infty$. Assume

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$  

Then $F^{s_0}_{p_0,\infty} \hookrightarrow F^{s_1}_{p_1, q}$.

(6) Let $0 < p_0 < p_1 < \infty$, $0 < q \leq \infty$ and $-\infty < s_1 < s_0 < \infty$. Assume

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$  

Then $B^{s_0}_{p_0, p_1} \hookrightarrow F^{s_1}_{p_1, q}$ and $F^{s_0}_{p_0, \infty} \hookrightarrow B^{s_1}_{p_1, p_0}$.  

### 5. Appendix

#### 5.1. Proof of Theorem 1.3

Let $\psi \in \mathcal{S}$ satisfy (3.3). Define $\varphi_j$ by (3.4) for $j \in \mathbb{Z}$.

We adopt the following definition of the Besov norm:

$$\|f\|_{B^{s}_{p,q}} \equiv \left( \sum_{j=-\infty}^{\infty} (2^{js} \|F^{-1}[\varphi_j \cdot Ff]\|_p)^q \right)^{\frac{1}{q}}$$

for $f \in \mathcal{S}'/\mathcal{P}$. Set

$$\psi_{j,k}(\xi) \equiv \begin{cases} \xi_k |\xi|^{-1} (\varphi_{j-1}(\xi) + \varphi_j(\xi) + \varphi_{j+1}(\xi)) & (\xi \notin \mathbb{R}^n \setminus \{0\}), \\ 0 & (\xi = 0). \end{cases}$$

Then arguing as before, we have

$$\|R_k f\|_{B^{s}_{p,q}} = \left( \sum_{j=-\infty}^{\infty} (2^{js} \|F^{-1}[\psi_{j,k} \varphi_j \cdot Ff]\|_p)^q \right)^{\frac{1}{q}}$$

$$= \left( \sum_{j=-\infty}^{\infty} (2^{js} \|F^{-1}[\psi_{j,k} \cdot Ff]\|_p)^q \right)^{\frac{1}{q}}$$

$$\preceq \left( \sum_{j=-\infty}^{\infty} (2^{js} \|F^{-1}[\psi_{j,k} \cdot Ff]\|_p)^q \right)^{\frac{1}{q}}$$

$$\preceq \|f\|_{B^{s}_{p,q}},$$

as was to be shown.

#### 5.2. Proof of Lemma 3.11

We prove this lemma by induction on $N$. If $N = 1$, then the result is immediate from the Poincaré lemma, or equivalently, $H^1_{\text{DR}}(\mathbb{R}^n) = 0$. Let $N \geq 2$. Define $g_{j\gamma} \equiv f_{e_j + \gamma}$ if $j = 1,2,\ldots,n$ and $|\gamma| = N - 1$. Let $e_j \equiv (0,\ldots,0,1,0,\ldots,0)$, where 1 is in the $j$-th lot. Then we have

$$\partial^\delta g_{j\gamma} = \partial^\delta f_{e_j + \gamma} = \partial^\delta f_{e_j} + \gamma = \partial^\delta g_j$$

for all $j,j' = 1,2,\ldots,n$, $\gamma, \delta, \delta' \in \mathbb{N}_0^n$ with $|\gamma| = N - 1$ and $e_j + \delta = e_{j'} + \delta'$. Thus we are in the position of applying the Poincaré lemma to have $g_j \in C^\infty$ satisfying $g_{j\gamma} = \partial^\gamma g_j$ for all $\gamma$ with $|\gamma| = N - 1$ and $j = 1,2,\ldots,n$.  

### 5.3. Appendix
Let $\alpha, \alpha', \gamma, \gamma' \in \mathbb{N}$ satisfy $|\gamma| = |\gamma'| = N-1$ and $|\alpha| = |\alpha| > 0$ and $\alpha + \gamma = \alpha' + \gamma'$. Suppose $\alpha - e_j, \alpha' - e_j' \geq 0$. Then we have
\[
\partial^\alpha g_\gamma = \partial^{\alpha-e_j} \partial^j g_\gamma = \partial^{\alpha-e_j} g_{\gamma_j} = \partial^{\alpha-e_j} f_{e_j+\gamma}
\]
and
\[
\partial^{\alpha'} g_{\gamma'} = \partial^{\alpha'-e_j'} \partial^j g_{\gamma'} = \partial^{\alpha'-e_j'} g_{j\gamma'} = \partial^{\alpha'-e_j'} f_{e_j'+\gamma'}.
\]
Since $(\alpha - e_j) + (e_j + \gamma) = (\alpha' - e_j') + (e_j' + \gamma')$, we obtain $\partial^\alpha g_\gamma = \partial^{\alpha'} g_{\gamma'}$.

Thus by the induction assumption to have a function $f \in C^\infty$ such that $g_\gamma = \partial^\gamma f$. Consequently, if $|\alpha| = N$, then by choosing $j$ so that $e_j \leq \alpha$ we have $f_\alpha = \partial_j g_{\alpha-e_j} = \partial_j \partial^{\alpha-e_j} f = \partial^\alpha f$.

6. Historical notes

6.1. Function spaces. Let us go back to the study of Sobolev spaces initiated in \cite{1, 2}. Zygmund found out that the difference of the second order is useful in $\mathbb{R}$. The space $W^s_p$ with $s \in (0, \infty) \setminus \mathbb{N}$ and $1 \leq p \leq \infty$ dates back to \cite{3, 4, 5, 6}. The space $\Lambda^s_p$ is defined by Besov in \cite{8, 9}. Let $f \in L^p$ with $1 \leq p < \infty$. When $0 < s < 1$, the norm $\|f\|_{\Lambda^s_p}$ is given by
\[
\|f\|_{\Lambda^s_p} \equiv \|f\|_{L^p} + \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-s} \|f(\cdot + h) - 2f + f(\cdot - h)\|_p.
\]

When $s = 1$, the norm $\|f\|_{\Lambda^s_p}$ is given by
\[
\|f\|_{\Lambda^s_p} \equiv \|f\|_{W^1_p} + \sup_{h \in \mathbb{R}^n \setminus \{0\}} \|\nabla[f(\cdot + h) - 2f + f(\cdot - h)]\|_p.
\]

When $1 < s < 2$, the norm $\|f\|_{\Lambda^s_p}$ is given by
\[
\|f\|_{\Lambda^s_p} \equiv \|f\|_{W^1_p} + \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-s+1} \|\nabla[f(\cdot + h) - 2f + f(\cdot - h)]\|_p.
\]

Let $1 \leq p \leq \infty$ and $s > 0$. The space $H^s_p$ was introduced by Aronszajn and Smith in \cite{10} and by Calderón in \cite{11}.

6.2. Theorems \cite{1,2} and \cite{3,21}. Taibleson, Rivière and Sagher proved that $F$ maps $B^1_{1,1}$ to $L^\infty$; see \cite{32, 51}. See also \cite{11} p. 116 (3]) for the fact that the Fourier transform maps $B^1_{1,1}$, the homogeneous Besov space, into $L^\infty$. See \cite{11} Chapter 6) for various extensions of this boundedness. Gabisoniya proved $F$ maps $B^{n/2}_{21}$ to $L^1$ assuming $f \in L^2$ \cite{20}. On the torus Bernstein considered a counterpart to the Fourier series. Golovkin and Solonnikov proved the results assuming that both sides are finite in \cite{24} Section 3, Theorem 7]. Madych gave a sharper version of this fact in \cite{33}.

6.3. Theorem \cite{1,3}. The boundedness of the Riesz transform on $\dot{B}^s_{pq}$ is an immediate consequence of \cite{54, 5, 2, 22}.

6.4. Theorem \cite{1,4}. The equivalence \cite{1,1} can be found in \cite{30, 91, 32} and it is the beginning of the Littlewood and Paley theory. The equivalence \cite{1,2}, which asserts $H^p \approx F^0_{pq}$, dates back to Peetre \cite{39, 40}. Meanwhile, the equivalence \cite{1,3}, which asserts $H^p \approx F^0_{pq}$, is due to \cite{10}. This equivalence motivates the definition of the space $F^0_{pq}$.

\cite{1,2} and \cite{1,3} date back to Goldberg \cite{22, 23}. See also \cite{54} p. 93, Remark] and \cite{54} p. 92, Theorem] for the proof of \cite{1,3} and \cite{1,2}, respectively. We can find \cite{1,4}
in [54, Remark 3]. In [22][23], Goldberg investigated local bmo space as well as local Hardy spaces. (1.3) is due to [22, p. 36].

6.5. **Theorem 2.2.** Theorem 2.2 is due to Schwartz [44, p. 105, Théorème XII].

6.6. **Theorem 2.8.** We can find a detailed proof of Theorem 2.8 (1)-(3) in [64, Proposition 8.1]. We refer to [36, Section 6] for the proof of (4).

6.7. **Theorem 2.10.** The author was not sure when Theorem 2.10 initially appeared. To the best knowledge of the author, we can find it in the textbook [4] without proof.

6.8. **Hölder-Zygmund spaces.** See [65] for Hölder-Zygmund spaces.

6.9. **Theorems 2.12 and 2.15.** We refer to [36, Section 6] for the proof.

6.10. **Theorem 3.3.** See the textbooks [4][41] for the definition of the Besov space $B^s_{pq}$ with $1 \leq p,q \leq \infty$ and $s \in \mathbb{R}^n$ as in this book. We can find the definition of the inhomogeneous space $B^s_{pq}$ in [41, Definition 6.2.2] and [41, p. 48, Definition 1]. For the definition of homogeneous Besov spaces see [4, 6.3].

6.11. **Theorem 3.14.** Theorem 3.14 is essentially due to Taibleson [51, Theorem 4]. See also [54, p. 90, Theorem]. For $s > 2$, $C^s$ is a function space applied to elliptic differential equations in the paper [11]. See the textbook of Miranda [35] for the detailed background. We refer to [14, 20, 53] for further details.

6.12. **Theorems 3.15 and 3.19.** We can find Theorems 3.15 and 3.19 in [54, p. 240, Theorem].

6.13. **Theorem 3.17.** We can find Theorem 3.17 in [54, 5.2.3].

6.14. **Theorem 3.18.** The embedding (3.14) dates back to Triebel [52] when $1 < p,q < \infty$. See also [4, Theorem 6.5.1]. For the general case (3.14) is due to Jawerth [29]. See also [54, p. 129, Theorem].

6.15. **Theorem 3.20.** We refer to the paper [7] for more about the case for general $s$.

6.16. **Theorem 4.3.** We refer to [16] for Theorem 4.3. See also [19][50].

6.17. **Definition of function spaces.** There is a long history in the definition of the function spaces. So the proof is a little simpler. Triebel used the Fourier multiplier very systematically in [52, Theorem 3.5] to define $B^s_{pq}$ with $1 < p,q < \infty$ and $s \in \mathbb{R}$. We refer to [41, p. 225–227] and [39] for the motivation of function spaces $B^s_{pq}$ with $0 < p < 1$. For the definition of $B^s_{pq}$ with $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$ with $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$ we refer to [41, p. 48, Definition 1] and [41, p. 232, Definition 1]. In his 1976 book [41] Peetre showed that the definition of $B^s_{pq}$ is independent of the choice of $\varphi$ whose proof in that book is similar to the one in this book.

6.18. **Textbooks on Besov spaces.** We list [9][15][14][18][25][27][28][43][46][55][56][57][58][59][60][61][62] as textbooks of function spaces.
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