Floquet engineering to exotic topological phases in cold-atom systems

Hui Liu, 1 Tian-Shi Xiong, 2 Wei Zhang, 3, 4 and Jun-Hong An 1, *

1 School of Physical Science and Technology, Lanzhou University, Lanzhou 730000, China
2 Department of Physics, National University of Singapore, Singapore 117552
3 Department of Physics, Renmin University of China, Beijing 100872, China
4 Beijing Key Laboratory of Opto-Electronic Functional Materials and Micro-Nano Devices, Renmin University of China, Beijing 100872, China

Topological phases with large topological numbers have been widely studied as an example of exotic states of matter with potentially important applications. Although several models have been shown to support states with large topological numbers, they are difficult to realize in solid systems due to the complexity of various intervening factors. Inspired by the realization of synthetic spin-orbit coupling in cold-atom systems [Z. Wu, et al., Science 354, 83 (2016)], we propose a periodic quenching scheme to realize large-topological-number phases in optical lattices. Via introducing the periodic quenching to the Raman lattice, it is found that a large number of topological edge states can be induced in a controllable manner from the static topologically trivial system. Our result provides an experimentally accessible method to artificially synthesize and manipulate exotic topological states with large topological numbers.

Introduction.—Since the discovery of quantum Hall effect [1], exotic phases with topologically protected edge states have attracted extensive attention in the past decades. The study in this field enriches our understanding on topological nature of matters, and leads to the discovery of topological insulators [2, 3], topological superconductors [4], Weyl semimetals [5–8], and photonic topological insulators [9–12]. An intriguing direction in this field is to seek for novel phases with large topological numbers. Such phases can provide more edge channels which are expected to improve performance of certain devices by lowering the contact resistance in quantum anomalous Hall insulators [13, 14]. They may also be used to realize reflectionless waveguides, combiners, and one-way photonic circuits in photonic devices [15–17]. Although theoretical studies suggest several models that support such phases, an experimental realization is still lacking due to the complication of interplay among various types of degrees of freedom in solid state systems.

Another possible route toward the realization of topological phases is to periodically drive a traditional insulator to become a so-called Floquet topological insulators. The band structure of a given system can be drastically altered by periodic driving, such as oscillating electromagnetic fields [18–23] and periodic quenching [24–31]. Besides, periodic driving can also induce an equivalent long-range hopping which is crucial for certain exotic topological phases [32, 33]. However, the implementation of these schemes are usually difficult in veritable materials. Recently, the realization of synthetic spin-orbit coupling via Raman transition has paved the route toward quantum emulation of topological systems in ultracold atomic gases [34–36]. Owing to the high controllability therein, cold atoms confined in traps and optical lattices provide a promising platform to synthesize and study topological matters, as demonstrated in various examples [37–45].

In this work, we investigate the exotic phases with large topological numbers in an optical lattice with synthetic spin-orbit coupling implemented by a Raman lattice. By periodically driving the offset phase between the Raman lattice and the optical lattice, we find that a widely tunable number of edge states can be generated at ease in both the one- (1D) and two-dimensional (2D) cases. A criterion to determine the change of topological numbers when changing the driving parameters across the phase boundaries is established. As the experimental techniques of Raman lattice and periodic driving have been both successfully demonstrated in cold atoms, our proposal can be readily implemented.

Floquet topological phases.—We are interested in topological phase transitions in a time-periodic system $\hat{H}(t) = \hat{H}(t + T)$ with period $T$. The Floquet theorem promises a complete basis $|u_\alpha(t)\rangle$ from $\{\hat{H}(t) - i\hbar \partial_t\}|u_\alpha(t)\rangle = \epsilon_\alpha |u_\alpha(t)\rangle$. Here $|u_\alpha(t)\rangle$ and $\epsilon_\alpha$ play the...
same role as the stationary states and eigen energies in static systems. They are called quasistationary states and quasienergies [46, 47]. It is in the quasienergy spectrum that the topological properties of our periodic system is defined. The Floquet equation is equivalent to $U_T u_n(0) = e^{-i\epsilon_n T/\hbar} u_n(0)$, where $U_T = T e^{-i H(t)_{	ext{eff}}} H(t)_{	ext{eff}}$ is the evolution operator with $T$ being the time-ordering operator. Thus, $U_T$ defines an effective static system $H_{\text{eff}} \equiv \hbar^2 \ln U_T$ that shares the same (quasi)energies with the periodic system. Then one can use the well-developed tool of topological phase transition in static systems to study periodic systems via $H_{\text{eff}}$.

To facilitate the understanding on the underlying physics, we consider a Hamiltonian $H(k) = h(k) \cdot \sigma$ with the parameter $h$ periodically quenched between two choices $h_1$ and $h_2$ within the respective time duration $T_1$ and $T_2$ [48]. Applying the Floquet theorem, we obtain $H_{\text{eff}}(k) = h_{\text{eff}}(k) \cdot \sigma$ [33] with the Bloch vector $h_{\text{eff}}(k) = -\arccos(\epsilon/k) T$ and

$$
\epsilon = \cos \left[ T_1 h_1(k) \right] \cos \left[ T_2 h_2(k) \right] - h_1 \cdot h_2 \\
\times \sin \left[ T_1 h_1(k) \right] | \sin \left[ T_2 h_2(k) \right],
$$

$$
r = h_1 \times h_2 \sin \left[ T_1 h_1(k) \right] | \sin \left[ T_2 h_2(k) \right] - h_2 \cos \left[ T_1 h_1(k) \right] \\
\times \sin \left[ T_2 h_2(k) \right] - h_1 \cos \left[ T_2 h_2(k) \right] | \sin \left[ T_1 h_1(k) \right],
$$

where $T = T_1 + T_2$ and $\mathbf{v} \equiv |\mathbf{v}|$ is the unit vector of $\mathbf{v}$. The topological properties of the system are crucially dependent on the presence or absence of the chiral symmetry, which is tunable by the periodic quenching [49–51]. For example, the chiral symmetry is preserved if $U_T H(k) U_T^{-1} = -H(k)$, which is obviously satisfied as the $\alpha$-component of $h_{\text{eff}}$ vanishes and the unitary transformation is chosen as $U_T = \sigma_x$. Thus the chiral symmetry is present if the Bloch vector has only two components. Supplying a good way to recover the chiral symmetry by eliminating a component of $h_{\text{eff}}(k)$, the periodic quenching can be used to generate topological phases in different classes, e.g., multiple Majorana edge modes in Kitaev chains [52]. Note that if both of $H(k)$ have the identical chiral symmetry and $U_T$, a unitary transformation can convert $H_{\text{eff}}(k)$ into the one with the same chiral symmetry (see Supplemental Material [53]).

The topological phase transition is associated with the closing and reopening of the quasienergy bands. We obtain from Eq. (2) that the bands close when

$$
h_1 = \pm h_2,
$$

$$
T_1 |h_1(k)| + T_2 |h_2(k)| = n \pi, \ n \in \mathbb{Z},
$$

with quasienergy being zero ($\pm \pi/T$) for even (odd) $n$, or

$$
T_j |h_j(k)| = n_j \pi, \ n_j \in \mathbb{Z}.
$$

As the sufficient condition for judging the phase transition, Eqs. (3–5) offer a guideline to design the quenching scheme to generate various topological phases at will.

Cold atom system.—Inspired by the experimental realization of synthetic spin-orbit coupling in cold-atom systems [34–36], we consider setups in $d$ dimensions $(d = 1, 2)$ as depicted in Fig. 1. The Hamiltonian is

$$
\hat{H} = p^2/2m + V_{\text{lat}}(r) + \mathbf{M}(r) \cdot \sigma + m_z \sigma_z,
$$

where $V_{\text{lat}}(r) = \sum_{j=1}^d V_j \cos^2 (k_0 r_j)$ is the optical lattice and $\mathbf{M}(r)$ is the Raman lattice. $V_{\text{lat}}(r)$ is formed by a standing wave $\mathbf{E} = \hat{a} E \cos(k_0 x)$ for $d = 1$, and by two standing waves $\mathbf{E}_1 = \hat{a} E_1 \cos(k_0 x)$ and $\mathbf{E}_2 = \hat{a} E_2 \cos(k_0 z)$ for $d = 2$ of frequency $\omega_1$. $\mathbf{M}(r)$ is formed by the combined actions of the aforementioned standing waves and additional running waves, which take the form $\mathbf{E}' = \hat{a} E' \cos((k_0 z + \varphi) r)$ for $d = 1$, and $\mathbf{E}'_1 = \hat{a} E'_1 \cos((k_0 z + \varphi) x)$ and $\mathbf{E}'_2 = \hat{a} E'_2 \cos((k_0 x + \varphi) z)$ for $d = 2$ of frequency $\omega_2$. Here, $\varphi$ is the initial phase and $\delta = L |\omega_2 - \omega_1|/c$ with $L$ being the optical path difference. The lattice potentials $V_{\text{lat}}(r)$ and $\mathbf{M}(r)$ together induce a two-photon Raman transition between the near-degenerate ground states $|g_{\uparrow} \downarrow\rangle$ mediated by the excited state $|e\rangle$ [54, 55]. By adiabatically eliminating $|e\rangle$, we have $\mathbf{M}(r) = (M \cos(k_0 x), 0, 0)$ with $M \propto EE'$ for $d = 1$, and $\mathbf{M}(r) = e^{i \varphi} (M_x \cos(k_0 x) \sin(k_0 z), M_z \cos(k_0 z) \sin(k_0 x), 0)$ for $d = 2$ with $M_x \propto E'_1$ and $M_z \propto E'_2$ when $\delta = \pi/2$.

Expanding Eq. (6) in the basis of $s$-band Wannier functions $\phi_{so}^{ij}(r)$, we obtain

$$
\hat{H} = -\sum_{i,j,\sigma} v_{ij}^{\uparrow \uparrow} \hat{c}_{i\sigma} \hat{c}_{j\sigma} + \sum_{i,j} [v_{ij}^{\uparrow \downarrow} \hat{c}_{i\sigma} \hat{c}_{j\uparrow} + H.c.] \\
+ \sum_i m_z \hat{c}_{i\uparrow} \hat{c}_{i\downarrow} - \hat{c}_{i\downarrow} \hat{c}_{i\uparrow},
$$

where $(i,j)$ denotes that the summation is subjected to nearest neighbors. $v_{ij}^{\uparrow \uparrow} = \int d^d r \phi_{so}^{ij}(r) |\mathbf{p}|^2/2m + V_{\text{lat}}(r) |\phi_{so}^{ij}(r)|^2$, and $v_{ij}^{\uparrow \downarrow} = \int d^d r \phi_{so}^{ij}(r) [\mathbf{M}(r) \cdot \sigma] \phi_{so}^{ij}(r)$ [35]. Owing to the periodicity of the potential, we have $v_{ij}^{\uparrow \uparrow} = v_{ij}^{\downarrow \downarrow}$ for both the 1D and 2D cases. Besides, one can verify that $v_{so}^{\uparrow \downarrow} = \pm (1) \pm (1) \pm (1) \pm (1)$ for $d = 1$, and $v_{so}^{\uparrow \downarrow} = \pm (1) \pm (1) \pm (1) \pm (1)$ for $d = 2$ [56, 57]. Defining $\hat{c}_{i\sigma} = e^{i \mathbf{k} \cdot \mathbf{r}} \hat{c}_{i\sigma}$ and making the Fourier transform $\hat{c}_{i\sigma} = \int_N \hat{c}_{k\sigma} e^{-i \mathbf{k} \cdot \mathbf{r}}$, with $N$ being the total site number, we obtain $\hat{H} = \sum_{\mathbf{k} \in \text{BZ}} C_{\mathbf{k}}^{\uparrow \downarrow} \mathbf{h}(\mathbf{k}) \cdot \sigma C_{\mathbf{k}}$ with $C_{\mathbf{k}}^{\uparrow \downarrow} = (c^{\uparrow \downarrow}_{\mathbf{k} \uparrow}, c^{\uparrow \downarrow}_{\mathbf{k} \downarrow})$ and the summation over the first Brillouin zone (BZ). The Bloch vectors $\mathbf{h}(\mathbf{k})$ read

$$
\mathbf{h}_{1D}(k) = (0, 2v_{so} \sin k, m_z - 2v_0 \cos k),
$$

$$
\mathbf{h}_{2D}(k) = (2v_{so} \sin k_z, 2v_{so} \sin k_x, \ m_z - 2v_0 (\cos k_x + \cos k_z)),
$$

where the lattice constant has been set to one. The particle-hole symmetry is naturally kept. Because $\mathbf{h}_{1D}(k)$ has two components, the static Hamiltonian possesses the chiral symmetry with $U_{\text{ch}} = \sigma_y$ and belongs to the symmetry class BDI [58, 59]. The topological properties are characterized by the winding number.
irrespective of $T$ and no phase transition can take place.

**Case III:** $h_{\alpha}(k) = h_{\bar{\alpha}}(k)$. Equation (3) reveals the bands touch at $k = 0$ or $\pi$. Using Eq. (4), we have

$$n_\alpha = T|m_z - 2e^{i\alpha}v_0|/\pi$$

(11)

for $n_\alpha \in \mathbb{Z}$ and $\alpha = 0, \pi$, and $h_{\alpha}(\alpha) = (0, 0, m_z - 2e^{i\alpha}v_0)$. The phase transition at zero quasienergy for even $n_\alpha$ requires $|m_z| < 2|v_0|$ like the static case, which causes the existence of a definite $k$ such that the bands keep closed according to **Case II**. It in turn rules out the phase transition at the zero quasienergy. Thus a phase transition only occurs for odd values of $n_\alpha$ satisfying Eq. (11).

Figures 2(a) and 2(b) show the quasienergy spectrum and the winding number with changing $T$. The parameters $|m_z| > 2|v_0|$ are chosen such that the static systems are topologically trivial. The nontrivial phases are induced when the periodic quenching is on [see Fig. 2(a) with $E_r = h^2k_0^2/2m$ being the recoil energy]. For small $T$, Eq. (5) is not fulfilled and a finite band gap exists at the zero quasienergy. With increasing $T$, the gap remains closed due to **Case I**. However, the gap at $\pm \pi/T$ is closed and reopened at $T\varepsilon_0/\pi = 1.25, 3.75, 5.0, 6.25$, and 8.75 accompanied by the corresponding change of $W$ [see Fig. 2(b)]. They correspond to Eq. (11) with $n_\alpha = 1, 3\pi, -1, 5\pi$, and $7\pi$, respectively. Figure 2(c) shows that the number of degeneracy of the formed bound modes exactly equals to $|W|$, as required by the bulk-edge correspondence [49]. More bound modes are achievable with further increasing $T$. As expected, all the bound modes are highly confined at the edges [see Fig. 2(d)].

To reveal how $W$ changes when $T$ crosses the phase boundaries, we check the change rate of $h_{\alpha}(k)$ across the quasienergy $\pm \pi/T$ at $\alpha$, i.e., $\lim_{\epsilon \to 0} \partial_k h_{\alpha}(k)|_{k=\alpha} = T = -\epsilon$. The Bloch vectors near $\alpha$ read $h_{\alpha}(\alpha + e^{i\theta}) = [0, (1/2)e^{i\theta}m_z - 2e^{i\alpha}v_0]$ with $\theta$ being an infinitesimal. Then we have $T, h_{\alpha}(\alpha + e^{i\theta}) = [4m_z - 2e^{i\alpha}v_0, 0, sgn(m_z - 2e^{i\alpha}v_0)]$ with $\epsilon' = e/m_z - 2e^{i\alpha}v_0$. Reminding that $n_\alpha$ is odd, we can easily find that the bands touch at $\sgn(m_z - 2e^{i\alpha}v_0)\pi/T$. The change rates of $h_{\alpha}(k)$ are

$$\lim_{\epsilon \to 0} T\partial_k h_{\alpha}(k)|_{k=\alpha} = (\frac{4\epsilon'v}{m_z - 2e^{i\alpha}v_0}, 0, 0),$$

(12)

with which the change rule of $W$ can be obtained.

For $m_z - 2e^{i\alpha}v_0 > 0$, the bands for both $\alpha = \pi$ and $0$ touch at $\pi/T$. Equation (12) reveals that $h_{\alpha}(k)$ crosses $\pi/T$ along the $-x$ (or $+x$) direction for $\alpha = \pi$ (or 0) with increasing $k$. This is confirmed by the dashed lines in Figs. 3(a) and 3(b). Since only the first Brillouin zone $[-\pi/T, \pi/T]$ of the quasienergy is meaningful, $h_{\alpha}(k)$ abruptly jumps from $\pi/T - \delta$ to $-\pi/T + \delta$ keeping the direction unchanged when $T$ crosses the phase boundary [see the solid lines in Figs. 3(a) and 3(b)]. Then a closed path with a clockwise wrapping to the origin is formed in Fig. 3(a) with $k$ running over $[-\pi, \pi]$. It causes $W$ changing from 0 to $-1$. Before the phase transition,
FIG. 3. Trajectories of $\mathbf{h}_{\text{eff}}(k)$ in the 1D case for $T$ (red dashed lines) and $T + \Delta T$ (blue solid lines) crossing the boundaries of $\alpha = 0$ marked by $\blacklozenge$ and $\pi$ by $\star$. The parameters satisfy $m_x - 2e^{i\alpha}v_0 > 0$ in (a, b) and $< 0$ in (c, d). We use $\Delta T = 0.002\pi/E_r$, $v_{so} = 0.05E_r$, and $(m_x/E_r, v_0/E_r, TE_r/\pi) = (0.5, 0.15, 1.249)$ in (a), $(0.6, 0.16, 3.571)$ in (b), $(-0.6, 0.05, 1.999)$ in (c), and $(-0.7, 0.25, 2.499)$ in (d).

$\mathbf{h}_{\text{eff}}(k)$ wraps the origin twice in the clockwise direction [see Fig. 3(b)] indicating $\mathcal{W} = -2$. After the phase transition, an anticlockwise path is formed and $\mathcal{W} = -1$. Thus $\mathcal{W}$ decreases (or increases) 1 with increasing $T$ across the phase boundary of $\alpha = \pi$ (or 0). This can be confirmed by $m_x - 2e^{i\alpha}v_0 < 0$, where the bands for both $\alpha = \pi$ and 0 touch at $-\pi/T$. Figures 3(c) and 3(d) demonstrate the cases that $\mathcal{W}$ changes from 1 (dashed line) to 0 (solid line) and from 1 (dashed line) to 2 (solid line), respectively.

FIG. 4. Phase diagram in the 1D case. The red solid and black dashed lines are phase boundaries from Eq. (11) with $n_\alpha$ labelled explicitly. We use $m_x = 0.5E_r$ and $v_{so} = 0.05E_r$.

FIG. 5. (a) Phase diagram in the 2D case. The black (red) lines are from Eq. (13) with $\alpha = \beta = 0$ (0) and $n$ labelled explicitly. The white dashed line depicts the case of $(\alpha, \beta) = (0, \pi)$ or $(\pi, 0)$. (b) Quasienergy spectrum and (c) site distribution of the edge states with $z$ direction opened when $(v_0/E_r, TE_r/\pi) = (0.3, 2.6)$. The blue solid (red dashed) lines marked by $\Delta$, $\square$ and $\circ$ are the zero $(\pi/T)$ modes. We use $m_x = 0.5E_r$, $v_{so} = 0.28E_r$, and $(T_1, T_2) = (0.7, 0.3)T$.

The change of $\mathcal{W}$ can be verified by the phase diagram in Fig. 4. The solid and dashed lines depict the phase boundaries analytically evaluated from Eq. (11). With increasing $T$, $\mathcal{W}$ increases 1 through a solid line and decreases 1 through a dashed line. Note that the phases with large $\mathcal{W}$ and multiple edge states can be obtained at large $T$. The physics behind this originates from the ability of periodic driving in effectively engineering the long-range hopping [52]. The phase diagram gives a map for experimentally designing the parameters to engineer exotic topological phases. We emphasize that the findings above are qualitatively valid in the general case with $T_1 \neq T_2$ (see Supplementary Material [53]).

Periodic quenching in 2D case.—Using the protocol Eq. (10), arbitrary number of edge states can also be generated in 2D case. Without loss of generality, we choose $T_1 \neq T_2$. It can be proved that both of Eq. (3) with “−” and Eq. (5) do not support phase transition (see Supplementary Material [53]). Equation (3) with “+” reveals the band touching points $k = (\alpha, \beta)$ with $\alpha, \beta = 0$ or $\pi$, such that Eq. (4) reads

$$T|m_x - 2(e^{i\alpha} + e^{i\beta})v_0| = n\pi.$$

Thus, the bands touch at $\pi/T$ (0) for odd (even) $n$.

As depicted in the phase diagram Fig. 5(a), a widely tunable $C$ ranging from $-2$ to 2 can be formed. The boundaries match well with Eq. (13). In the same mechanism, when $T$ increases across the phase boundaries, $\mathbf{h}_{\text{eff}}(k)$ has an abrupt jump in sign at $k = (\alpha, \beta)$. It causes the change of the wrapping time of $\mathbf{h}_{\text{eff}}(k)$ to the origin of the BZ. The chirality $\text{Ch}(k)$ for $k = (0, 0)$ and $(\pi, \pi)$ is $-1$ (see Supplementary Material [53]). Thus when $T$ increases across the phase boundary contributed
by these two points, $\Delta C = \text{sgn}[m_z - 2v_0(e^{i\alpha} + e^{i\beta})]$ for odd $n$ and $-\text{sgn}[m_z - 2v_0(e^{i\alpha} + e^{i\beta})]$ for even $n$, respectively. This is verified by the black and red lines in Fig. 5(a). The chirality $\text{Ch}(k)$ for $k = (0, \pi)$ and $(\pi, 0)$ is $+1$. Both of the points contribute to $\Delta C$ being $2\text{sgn}[-(-1)^nm_z]$, as verified by the white dashed line in Fig. 5(a).

The quasienergy spectrum in Fig. 5(b) reveals that although $C = -2$, there are six independent edge states. The bulk-edge correspondence can be recovered by considering the respective contribution of $\pi/T$- and zero-mode edge states to $C$ [62, 63]. Figure 5(c) shows the site distribution of the six edge states with a positive group velocity $\partial_{k_x} \varepsilon$, where the number of the zero- and $\pi/T$-modes are both three. Two of the zero-modes locate at the left edge and one at the right, which gives $C_0 = -1$. Two of the $\pi/T$-modes reside at the right and one at the left, which gives $C = 1$ [see Supplementary Material [53]]. The total Chern number is $C = C_\pi + C_0 = -2$.

**Conclusion.**—In summary, we have proposed a periodic quenching scheme to generate exotic phases with large topological numbers and multiple edge states both in one- and two-dimensional cold-atom systems. Resorting to the periodic switching of the phase of the Raman lattice between $0$ and $\pi$, our scheme can be readily implemented in experiments with existing techniques of synthesizing spin-orbit coupling [34–36], and supplies an avenue to controllably design topological devices in an experimentally friendly way.

**Acknowledgments.**—This work is supported by the National Natural Science Foundation (Grant Nos. 11875150, 11434011, 11522436, and 11774425), the National Key R&D Program of China (Grant No. 2018YFA0306501), the Beijing Natural Science Foundation (Grant No. Z180013), and the Fundamental Research Funds for the Central Universities of China.

---

* anjhong@lzu.edu.cn

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).

[2] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).

[3] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).

[4] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).

[5] A. A. Burkov and L. Balents, Phys. Rev. Lett. 107, 127205 (2011).

[6] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).

[7] B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, Z. Fang, X. Dai, T. Qian, and H. Ding, Phys. Rev. X 5, 031013 (2015).

[8] A. A. Burkov, M. D. Hook, and L. Balents, Phys. Rev. B 84, 235126 (2011).

[9] A. B. Khanikaev, S. Hossein Mousavi, W.-K. Tse, M. Kargarian, A. H. MacDonald, and G. Shvets, Nature Materials 12, 233 (2012).

[10] O. Zilberberg, S. Huang, J. Guglielmon, M. Wang, K. P. Chen, Y. E. Kraus, and M. C. Rechtsman, Nature 553, 59 (2018).

[11] A. Slobozhanyuk, S. H. Mousavi, X. Ni, D. Smirnova, Y. S. Kivshar, and A. B. Khanikaev, Nature Photonics 11, 130 (2016).

[12] L. J. Maczewsky, J. M. Zeuner, S. Nolte, and A. Szameit, Nature Communications 8, 13756 (2017).

[13] J. Wang, B. Lian, H. Zhang, Y. Xu, and S.-C. Zhang, Phys. Rev. Lett. 111, 136801 (2013).

[14] C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. Lett. 112, 046801 (2014).

[15] F. D. M. Haldane and S. Raghu, Phys. Rev. Lett. 100, 013904 (2008).

[16] S. A. Skirlo, L. Lu, and M. Soljačić, Phys. Rev. Lett. 113, 113904 (2014).

[17] S. A. Skirlo, L. Lu, Y. Igarashi, Q. Yan, J. Joannopoulos, and M. Soljačić, Phys. Rev. Lett. 115, 253901 (2015).

[18] W. Yao, A. H. MacDonald, and Q. Niu, Phys. Rev. Lett. 99, 047401 (2007).

[19] T. Oka and H. Aoki, Phys. Rev. B 79, 081406 (2009).

[20] N. H. Lindner, G. Refael, and V. Galitski, Nature Physics 7, 490 (2011).

[21] T. Kitagawa, T. Oka, A. Brataas, L. Fu, and E. Demler, Phys. Rev. B 84, 235108 (2011).

[22] B. Dóra, J. Caysols, F. Simon, and R. Moessner, Phys. Rev. Lett. 108, 056602 (2012).

[23] J.-i. Inoue and A. Tanaka, Phys. Rev. Lett. 105, 017401 (2010).

[24] M. S. Foster, M. Dzero, V. Gurarie, and E. A. Yuzbashyan, Phys. Rev. B 88, 104511 (2013).

[25] P. D. Sacramento, Phys. Rev. E 90, 032138 (2014).

[26] U. Bhattacharya, J. Hutchinson, and A. Dutta, Phys. Rev. B 95, 144304 (2017).

[27] I. C. Fulga and M. Maksymenko, Phys. Rev. B 93, 075405 (2016).

[28] M. D. Caio, N. R. Cooper, and M. J. Bhaseen, Phys. Rev. Lett. 115, 236403 (2015).

[29] M. S. Foster, V. Gurarie, M. Dzero, and E. A. Yuzbashyan, Phys. Rev. Lett. 113, 076403 (2014).

[30] L. C. Wang, X. P. Li, and C. F. Li, Phys. Rev. B 95, 104308 (2017).

[31] L. Zhou and J. Gong, Phys. Rev. B 97, 245430 (2018).

[32] T. Mikami, S. Kitamura, K. Yasuda, N. Tsuji, T. Oka, and H. Aoki, Phys. Rev. B 93, 144307 (2016).

[33] T.-S. Xiong, J. Gong, and J.-H. An, Phys. Rev. B 93, 184306 (2016).

[34] L. Huang, Z. Meng, P. Wang, P. Peng, S.-L. Zhang, L. Chen, D. Li, Q. Zhou, and J. Zhang, Nature Physics 12, 540 (2016).

[35] Z. Wu, L. Zhang, W. Sun, X.-T. Xu, B.-Z. Wang, S.-C. Ji, Y. Deng, S. Chen, X.-J. Liu, and J.-W. Pan, Science 354, 83 (2016).

[36] H.-R. Chen, K.-Y. Lin, P.-K. Chen, N.-C. Chiu, J.-B. Wang, C.-A. Chen, P.-P. Huang, S.-K. Yip, Y. Kawaguchi, and Y.-J. Lin, Phys. Rev. Lett. 121, 113204 (2018).

[37] A. Eckardt, Rev. Mod. Phys. 89, 011004 (2017).

[38] J. Jüennemann, A. Piga, S.-J. Ran, M. Lewenstein, M. Rizzi, and A. Bermudez, Phys. Rev. X 7, 031057 (2017).
[39] D. T. Tran, A. Dauphin, A. G. Grushin, P. Zoller, and N. Goldman, Science Advances 3, e1701207 (2017).
[40] C. Gross and I. Bloch, Science 357, 995 (2017).
[41] I.-D. Potirniche, A. C. Potter, M. Schleier-Smith, A. Vishwanath, and N. Y. Yao, Phys. Rev. Lett. 119, 123601 (2017).
[42] S. Wang, J.-S. Pan, X. Cui, W. Zhang, and W. Yi, Phys. Rev. A 95, 043634 (2017).
[43] X.-Y. Mai, D.-W. Zhang, Z. Li, and S.-L. Zhu, Phys. Rev. A 95, 063616 (2017).
[44] H. M. Price, T. Ozawa, and N. Goldman, Phys. Rev. A 95, 023607 (2017).
[45] L.-L. Wang, Q. Sun, W.-M. Liu, G. Juzeliūnas, and A.-C. Ji, Phys. Rev. A 95, 053628 (2017).
[46] H. Sambe, Phys. Rev. A 7, 2203 (1973).
[47] C. Chen, C.-J. Yang, and J.-H. An, Phys. Rev. A 93, 062122 (2016).
[48] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. 106, 220402 (2011).
[49] J. K. Asbóth, B. Tarasinski, and P. Delplace, Phys. Rev. B 90, 125143 (2014).
[50] M. Rodríguez-Vega and B. Seradjeh, Phys. Rev. Lett. 121, 036402 (2018).
[51] X.-Y. Xu, Q.-Q. Wang, W.-W. Pan, K. Sun, J.-S. Xu, G. Chen, J.-S. Tang, M. Gong, Y.-J. Han, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. 120, 260501 (2018).
[52] Q.-J. Tong, J.-H. An, J. Gong, H.-G. Luo, and C. H. Oh, Phys. Rev. B 87, 201109 (2013).
[53] The Supplementary Material includes the derivation of the recovered chiral symmetry when $T_1 \neq T_2$ in the 1D case, the phase transition condition and the change rule of the topological number in the 2D case.
[54] X.-J. Liu, M. F. Borunda, X. Liu, and J. Sinova, Phys. Rev. Lett. 102, 046402 (2009).
[55] X.-J. Liu, K. T. Law, and T. K. Ng, Phys. Rev. Lett. 112, 086401 (2014).
[56] X.-J. Liu, Z.-X. Liu, and M. Cheng, Phys. Rev. Lett. 110, 076401 (2013).
[57] J.-S. Pan, X.-J. Liu, W. Zhang, W. Yi, and G.-C. Guo, Phys. Rev. Lett. 115, 045303 (2015).
[58] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016).
[59] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B 82, 235114 (2010).
[60] L. Li, Z. Xu, and S. Chen, Phys. Rev. B 89, 085111 (2014).
[61] D. Sticlet, F. Piéchon, J.-N. Fuchs, P. Kalugin, and P. Simon, Phys. Rev. B 85, 165456 (2012).
[62] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, Phys. Rev. X 3, 031005 (2013).
[63] P. M. Perez-Piskunow, G. Usaj, C. A. Balseiro, and L. E. F. F. Torres, Phys. Rev. B 89, 121401 (2014).
Supplemental material for “Floquet engineering to exotic topological phases in cold-atom systems”

Hui Liu, Tian-Shi Xiong, Wei Zhang, and Jun-Hong An

1 School of Physical Science and Technology, Lanzhou University, Lanzhou 730000, China
2 Department of Physics, National University of Singapore, Singapore 117542
3 Department of Physics, Renmin University of China, Beijing 100872, China
4 Beijing Key Laboratory of Opto-Electronic Functional Materials and Micro-Nano Devices, Renmin University of China, Beijing 100872, China

I. PERIODIC QUenchING PROTOCOL

In our scheme, the periodic driving is realized by periodically quenching the system parameters within the respective time duration $T_1$ and $T_2$. Applying the Floquet theorem to the two-band Hamiltonian $\mathcal{H}_j(k) = \mathbf{h}_j(k) \cdot \sigma$ ($j = 1, 2$), we obtain

$$\hat{U}_T = \varepsilon \hat{U}_{2x2} + i \mathbf{r} \cdot \sigma$$

and

$$\mathcal{H}_{\text{eff}}(k) = \mathbf{h}_{\text{eff}}(k) \cdot \sigma$$

with $\mathbf{h}_{\text{eff}}(k) = -\arccos(\varepsilon) \mathbf{r} / T$ and

$$\varepsilon = \cos |T_1| h_1(k) | \cos |T_2| h_2(k) | - h_1 \cdot h_2 \sin |T_1| h_1(k) | \sin |T_2| h_2(k) |,$$

$$\mathbf{r} = h_1 \times h_2 \sin |T_1| h_1(k) | \sin |T_2| h_2(k) | - h_2 \cos |T_1| h_1(k) | \sin |T_2| h_2(k) | - h_1 \cos |T_2| h_2(k) | \sin |T_1| h_1(k) |.$$

(S1)

(S2)

Here, $T = T_1 + T_2$ and $\mathbf{v} \equiv \mathbf{v} / |\mathbf{v}|$ is the unit vector of $\mathbf{v}$ [1]. The topological phase transition is associated with the closing and reopening of the quasienergy bands. We obtain from Eqs. (S2) that the bands close when

$$h_1 = \pm h_2,$$

$$T_1 |h_1(k)| \pm T_2 |h_2(k)| = n\pi, \ n \in \mathbb{Z},$$

at quasienergy zero ($\pm \pi / T$) if $n$ is even (odd), or

$$T_j |h_j(k)| = n_j \pi, \ n_j \in \mathbb{Z}.$$  

(S3)

(S4)

(S5)

The equations above give the sufficient conditions for the occurrence of topological phase transition. For the configurations discussed in the main text, the static Hamiltonians for the 1D and 2D cases can be written explicitly as

$$\mathbf{h}_{1D}(k) = (0, 2v_{so} \sin k, m_z - 2v_0 \cos k),$$

$$\mathbf{h}_{2D}(k) = [2v_{so} \sin k_z, 2v_{so} \sin k_x, m_z - 2v_0 (\cos k_x + \cos k_z)].$$

(S6)

(S7)

The periodic quenching protocol under consideration reads

$$v_{so}(t) = \begin{cases} -v_{so}, & t \in [mT, mT+T_1) \\ v_{so}, & t \in [mT+T_1, (m+1)T), \ m \in \mathbb{Z}. \end{cases}$$

(S8)

II. RECOVERED CHIRAL SYMMETRY IN THE 1D CASE WHEN $T_1 \neq T_2$

In this section, we show that the chiral symmetry in the 1D case when $T_1 \neq T_2$ can be recovered by a unitary transformation. This supplies another subtle way to engineer the large topological number in systems belonging to symmetry class D with $Z_2$ topological invariant.

For a periodically quenched two-band system, the evolution operator $\hat{U}_T$ can be written as

$$\hat{U}_T = e^{-i\hat{H}_1 T_1} e^{-i\hat{H}_2 T_2},$$

where $\hat{H}_j = \mathbf{h}_j(k) \cdot \sigma$ ($j = 1, 2$). Applying a unitary transformation $\hat{U} = e^{i\hat{T}_1 T_1 / 2}$ to $\hat{U}_T$, we obtain

$$\hat{U}_T = \hat{U}_j' \hat{U}_j'$$

with $\hat{U}_j' = e^{-i\hat{H}_1 T_1 / 2} e^{-i\hat{H}_2 T_2 / 2}$ and $\hat{U}_j' = e^{-i\hat{H}_2 T_2 / 2} e^{-i\hat{H}_1 T_1 / 2}$. According to Eq. (S2), we have

$$\hat{U}_j' = e^{i\mathbf{r}' \cdot \sigma}$$

and

$$\mathbf{r}' = (-1)^j \mathbf{a} \mathbf{h}_1 - b \mathbf{h}_2 - c \mathbf{h}_1.$$  

(S9)
FIG. S1. (a) Quasienergy spectrum, (b) Majorana number, and (c) winding number with the change of \( T_2 \) for the 1D case. The green \( \Diamond \), blue +, red \( \bullet \), brown \( * \), and pink \( \Box \) symbols denote the cases of one, two, three, four, and six degenerate edge modes, respectively. (d) Trajectories of \( h'_{\text{eff}}(k) \) for the duration of \( T \) (red dashed lines) and \( T + \Delta T \) (blue solid lines) by crossing the phase boundaries of \( k = \pi \) by \( \star \) for \( m_z - 2 e^{i\alpha} v_0 > 0 \) with \( (T, T + \Delta T) E_r/\pi = (8.14, 8.16) \). Other parameters are \( m_z = 0.5 E_r \), \( v_0 = 0.15 E_r \), \( v_{\text{so}} = 0.25 E_r \), and \( T_1 = 0.6 \pi / E_r \).

where \( a = \sin[T_1 h_{1}/2] \sin[T_2 h_{2}/2] \), \( b = \cos[T_1 h_{1}/2] \sin[T_2 h_{2}/2] \), and \( c = \cos[T_2 h_{2}/2] \sin[T_1 h_{1}/2] \). Then we can obtain \( \hat{U}_T' = \epsilon' I_{2 \times 2} + i r' \cdot \sigma \), where \( \epsilon' = \langle \epsilon \rangle^2 - r_1' \cdot r_2' \) and \( r_1' = 2h_1(-\epsilon_1' c + ac h_1 \cdot h_2 + ab) - 2h_2(\epsilon_1' b + ac + abh_1 \cdot h_2) \). \( \langle \epsilon \rangle \) is determined by \( \hat{h} \). From Eq. (S10), one can find that the symmetry of \( \hat{U}_T' \) is determined by \( h_1(k) \) and \( h_2(k) \). If \( h_1(k) \) and \( h_2(k) \) have the same symmetry with the same symmetry operator, \( \hat{U}_T' \) will inherit their symmetry.

Both the two static Hamiltonians \( h_1(k) \) and \( h_2(k) \) in one dimension possess the chiral symmetry. Choosing \( T_1 \neq T_2 \), the chiral symmetry of the effective system \( h_{\text{eff}}(k) \) determined by \( \hat{U}_T' \) would be broken. The corresponding topological properties can only be characterized by the topological invariant Majorana number \( M = \text{sgn}[h_{\text{eff}}(\pi) \cdot h_{\text{eff}}(\pi)] z \) \([2, 3]\). One can readily find that

\[
M = \text{sgn}[\sin(T(m_z + 2v_0)) \sin(T(m_z - 2v_0))].
\]

Following the discussion above, the unitary transformation \( \hat{U} \) can recover the chiral symmetry. The resulting effective \( h'_{\text{eff}}(k) \) determined by \( \hat{U}_T' \) preserves the chiral symmetry of \( h_1(k) \) and \( h_2(k) \). Thus, its topological properties can be characterized by the winding number \( W \). This gives us another way to realize large topological numbers.

We plot in Figs. S1(a)-S1(c) the quasienergy spectrum, the Majorana number \( M \) determined by \( \hat{U}_T \), and the winding number \( W \) determined by \( \hat{U}_T' \) with the change of \( T_2 \), respectively. With increasing \( T_2 \), the gap is closed and reopened at \( T_2 E_r/\pi = 0.65, 3.15, 4.4, 5.65, \) and \( 8.15 \) for the quasienergy \( \pi / T \), and \( T_2 E_r/\pi = 1.9, 4.4, 6.9, \) and 9.4 for the quasienergy \( 0 \). They are clearly reflected by both of \( M \) and \( W \). However, \( M \) only characterizes the parity of the numbers of the formed edge modes (\( M = -1 \) for odd pairs and \( M = 1 \) for even pairs), while \( 2|W| \) equals exactly to the numbers of the edge modes. Therefore, via the unitary transform, we have perfectly recovered the bulk-edge correspondence in the \( Z_2 \). In Fig. S1(d), we show the trajectory of \( h'_{\text{eff}}(k) \) of \( \hat{U}_T' \) crossing the quasienergy \( \pi / T \) for the band touching point \( k = \pi \). According to Eq. (12) in the main text, \( h'_{\text{eff}}(k) \) for \( m_z + 2v_0 > 0 \) crosses \( \pi / T \) along the \( -y \) direction. The corresponding \( W \) changes from \(-5 \) to \(-6 \). This verifies again our result on the changing rule of the topological number.

III. PHASE TRANSITION CONDITION AND BULK-EDGE CORRESPONDENCE IN THE 2D CASE

In this section, we give the derivation of phase transition condition and the change rule of the topological number in the periodically quenched 2D cold atom system.

Equations (S3) and (S4) or (S5) give the quasienergy band touching points of \( \mathcal{H}_{\text{eff}}(k) \), from which the Chern number can be calculated according to

\[
C = \frac{1}{2} \sum_{k \in \mathbb{D}_z} \text{sgn}[h_{\text{eff}}(k)]_z \cdot \text{Ch}(k),
\]

where \( \text{Ch}(k) = \text{sgn}[\partial_k h_{\text{eff}}(k) \times \partial_k h_{\text{eff}}(k)]_z \) is the chirality and \( \mathbb{D}_z \) is the set of band touching points for \( h_{\text{eff}}(k) \) excluding the \( z \) component \([4]\).
According to Eqs. (S3)-(S5), the boundaries of the quasienergy band closing are determined as follows.

**Case I**: $T_{j}|h_{1}(k)| = n_{j} \pi$. In the neighbourhood of the band touching point satisfying this condition, i.e., $T_{j}|h_{1}(k)| = n_{j} \pi - \epsilon_{j}$ with $\epsilon_{j}$ an infinitesimal, we obtain $h_{\text{eff}}(k) = (-1)^{n_{1}+n_{2}+1}[\epsilon_{2}h_{2}(k) + \epsilon_{1}h_{1}(k)]/T_{1,2}$. Then one can obtain
\[
\lim_{\epsilon_{1,2} \to 0} \partial_{k} h_{\text{eff}}(k) = \lim_{\epsilon_{1,2} \to 0} \frac{(-1)^{n_{1}+n_{2}+1} \epsilon_{2} \partial_{k} h_{2}(k) + \epsilon_{1} \partial_{k} h_{1}(k)}{T_{1,2}} = 0,
\]
where $p = k_{x}$ or $k_{y}$. The expression above indicates that at the direction of phase transition, the chirality of system is zero. Thus, this case makes no contribution to the Chern number or the phase transition.

**Case II**: $h_{1}(k) = -h_{2}(k)$. Equation (S3) determines that the bands touch at $k_{0} = (k_{x}, 0, k_{z})$ satisfying $(\cos k_{x} + \cos k_{z}) = \frac{m}{2v_{so}}$. Substituting this condition into Eq. (S4), we obtain $(T_{1} - T_{2})2v_{so}\sin^{2} k_{x} + \sin^{2} k_{z} = n\pi$. In the neighbourhood of $k_{0}$, i.e., $(T_{1} - T_{2})2v_{so}\sin^{2} k_{x} + \sin^{2} k_{z} = n\pi - \epsilon$, we have
\[
\frac{\partial_{k} h_{\text{eff}}(k)}{T_{1,2}} \bigg|_{k \to k_{0}} = 8v_{so} \sin \frac{|T_{1}h_{1}(k)| \sin |T_{2}h_{1}(k)| \sin k_{z}}{T_{1}h_{1}(k)^{2}} [\sin k_{x}, -\sin k_{z}, \frac{h_{1}(k) |\sin |T_{1}h_{1}(k)| |\sin |T_{2}h_{1}(k)|}{4v_{so} \sin |T_{1}h_{1}(k)| \sin |T_{2}h_{1}(k)|}],
\]
which lead to $\partial_{k_{x}} h_{\text{eff}}(k) \times \partial_{k_{y}} h_{\text{eff}}(k) = 0$. Thus this case cannot induce a topological phase transition either.

**Case III**: $h_{1}(k) = h_{2}(k)$. Equation (S3) requires $k = (\alpha, \beta)$ with $\alpha, \beta = 0$ and $\pi$. Substituting this into Eq. (S4), we have
\[
n = T|m_{z} - (2e^{i\alpha} + 2e^{i\beta})v_{0}|/\pi,
\]
which determines the phase boundaries.

To reveal how $C$ changes with $T$ by crossing the phase boundaries, we examine the signs of the changing rate of $h_{\text{eff}}(k)$ across the quasienergy 0 or $\pm \pi/T$ at $(\alpha, \beta)$ both in the $k_{x}$- and $k_{z}$-directions, i.e., $\lim_{\epsilon \to 0} \partial_{k_{x}} h_{\text{eff}}(k)|_{k \to (\alpha, \beta)}$ and $\lim_{\epsilon \to 0} \partial_{k_{z}} h_{\text{eff}}(k)|_{k \to (\alpha, \beta)}$ at $T_{j} = T - \epsilon$. Using $h_{j}(\alpha + e^{i\alpha} \delta, \beta) = [0, (-1)^{j}2v_{so}q, \delta], h_{j}(\alpha, \beta + e^{i\beta} \delta) = [(-1)^{j}2v_{so}\delta, 0, q]$ with $\delta$ being an infinitesimal, and Eq. (S2), we have
\[
h_{\text{eff}}(\alpha + e^{i\alpha} \delta, \beta) = \frac{1}{T_{j}} \frac{4v_{so}q}{q} \sin |T_{1}q| \sin |T_{2}q|, \frac{2v_{so}q}{q} \sin (T_{2} - T_{1})|q|, sgn(q) \sin |T_{\epsilon}q|,
\]
\[
h_{\text{eff}}(\alpha, \beta + e^{i\beta} \delta) = \frac{1}{T_{j}} \frac{2v_{so}q}{q} \sin (T_{2} - T_{1})|q|, -\frac{4v_{so}q}{q} \sin |T_{1}q| \sin |T_{2}q|, sgn(q) \sin |T_{\epsilon}q|,
\]
where $q = m_z - 2v_0(e^{i\alpha} + e^{i\beta})$. Remembering $n \in \mathbb{Z}$, we conclude that the band touching occurs at the quasienergy $\text{sgn}(q)\pi/T$ for odd $n$ and $0 - \text{sgn}(q)/T$ for even $n$. The changing rates of $h_{\text{eff}}(k)$ at $(\alpha, \beta)$ are

$$
\lim_{\epsilon \to 0} \partial_{k_z} h_{\text{eff}}(k)|_{k \to (\alpha, \beta)} = \frac{4v_{so}e^{i\alpha}}{q^T}\sin(T_1q)\sin(T_2q) - \frac{\sin(T_2 - T_1)q}{2}, \quad (S19)
$$

$$
\lim_{\epsilon \to 0} \partial_{k_z} h_{\text{eff}}(k)|_{k \to (\alpha, \beta)} = \frac{4v_{so}e^{i\beta}}{q^T}\sin(T_2 - T_1)q - \frac{\sin(T_1q)\sin(T_2q)}{2}, \quad (S20)
$$

Then $\partial_{k_z} h_{\text{eff}}(k) \times \partial_{k_z} h_{\text{eff}}(k)|_{k \to (\alpha, \beta)} = -\frac{16\pi^2}{q^T}e^{i(\alpha + \beta)}[0, 0, \sin^2 T_1q\sin^2 T_2q + \frac{\sin^2(T_2 - T_1)q}{4}]$. The chirality of band touching points can be calculated $\text{Ch}[k = (0, 0)] = \text{Ch}[k = (\pi, \pi)] = -1$ and $\text{Ch}[k = (0, \pi)] = \text{Ch}[k = (\pi, 0)] = +1$. According to Eq. (S12), the change of Chern number $\Delta C$ can be evaluated with the obtained $\text{Ch}(k)$ and the sign of the $z$-component of Eqs. (S17) and (S18), we have

$$
\text{sgn}[h_{\text{eff}}(\alpha, \beta)|_{T+\epsilon}] = \text{sgn}(q)\text{sgn}[\sin |q(T + \epsilon)|] = \text{sgn}(q)\text{sgn}([-1]^n\epsilon'); \quad (S21)
$$

$$
\text{sgn}[h_{\text{eff}}(\alpha, \beta)|_{T-\epsilon}] = \text{sgn}(q)\text{sgn}[\sin |q(T - \epsilon)|] = \text{sgn}(q)\text{sgn}([-1]^{n+1}\epsilon'), \quad (S22)
$$

where Eq. (S16) has been used and $\epsilon' = |q|\epsilon$. According to $\Delta C[k = (\alpha, \beta)] = \frac{\text{Ch}[k = (\alpha, \beta)]}{2}[\text{sgn}[h_{\text{eff}}]|_{T+\epsilon} - \text{sgn}[h_{\text{eff}}]|_{T-\epsilon}]$, we obtain

$$
\Delta C[k = (0, 0)] = \text{sgn}([-1]^n(m_z - 4v_0)], \quad (S23)
$$

$$
\Delta C[k = (\pi, \pi)] = \text{sgn}([-1]^{n+1}(m_z + 4v_0)], \quad (S24)
$$

$$
\Delta C[k = (0, \pi)] = \Delta C[k = (\pi, 0)] = \text{sgn}([-1]^n m_z). \quad (S25)
$$

The bulk-edge correspondence in the 2D case can be revealed from the energy spectra and site distribution of edge states with positive group velocity $\partial_{k_z} \epsilon$ as shown in Fig. S2. Figure S2(a1) only has one pair of $\pi/T$-mode edge states. The corresponding $C = 1$ is uniquely contributed by one of the pair with positive group velocity, which residing on the left edge [Fig. S2(a2)]. Thus the $\pi/T$-mode left-edge state contribute 1 to $C$. Similarly, Figs. S2(b1) and S2(b2) demonstrate that the $\pi/T$-mode right-edge state contributes $-1$ to $C$, while Figs. S2(c1) and S2(c2) show that the 0-mode right-edge state contributes 1 to $C$. Figure S2(d1) has one pair of 0-mode and one pair of $\pi$-mode edge states. The $\pi$-mode state resides on the right edge and thus contributes $C_{\pi} = -1$. The 0-mode state resides on the left edge and gives $C_0 = -1$. Then the total Chern number $C = C_0 + C_{\pi} = -2$ can be justified.

[1] T.-S. Xiong, J. Gong, and J.-H. An, Phys. Rev. B 93, 184306 (2016).
[2] J. K. Asbóth, B. Tarasinski, and P. Delplace, Phys. Rev. B 90, 125143 (2014).
[3] L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. 106, 220402 (2011).
[4] D. Sticlet, F. Piéchon, J.-N. Fuchs, P. Kalugin, and P. Simon, Phys. Rev. B 85, 165456 (2012).