Symplectic fixed points and Lagrangian intersections on weighted projective spaces

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Abstract
In this note we observe that Arnold conjecture for the Hamiltonian maps still holds on weighted projective spaces $\mathbb{C}P^n(q)$, and that Arnold conjecture for the Lagrange intersections for $(\mathbb{C}P^n(q),\mathbb{R}P^n(q))$ is also true if each weight $q_i \in q = \{q_1, \ldots, q_{n+1}\}$ is odd.

1 Introduction

A famous conjecture by Arnold [Ar] claimed that every exact symplectic diffeomorphism on a closed symplectic manifold $(P,\omega)$ has at least as many fixed points as the critical points of a smooth function on $P$. The homological form of it can be stated as: For a Hamiltonian map $\phi$, i.e., a time 1-map of a time-dependent Hamiltonian vector field $X_{h_t}$, $0 \leq t \leq 1$, the number of fixed points of $\phi$ satisfies the estimates

$$\begin{align*}
(AC_1) \left\{ \begin{array}{ll}
\sharp\text{Fix}(\phi) \geq CL(P) + 1, \\
\sharp\text{Fix}(\phi) \geq SB(P) & \text{if every point of Fix(\phi) is nondegenerate.}
\end{array} \right.
\end{align*}$$

Here $CL(P)$ is the cuplength of $P$ and $SB(P)$ is the sum of the Betti numbers of $P$. More generally, for a closed Lagrange submanifold $L$ in $(P,\omega)$ Arnold also conjectured:

$$\begin{align*}
(AC_2) \left\{ \begin{array}{ll}
\sharp(L \cap (\phi(L))) \geq CL(L) + 1, \\
\sharp(L \cap (\phi(L))) \geq SB(L) & \text{if } L \cap \phi(L).
\end{array} \right.
\end{align*}$$

After Conley and Zehnder [CoZe] first proved $(AC_1)$ for the standard symplectic torus $T^{2n}$, Fortune showed that $(AC_1)$ holds on $\mathbb{C}P^n$ with the standard structure. By generalizing the idea of Gromov [Gr], Floer [Fi1]-[Fi3] found a powerful approach to prove $(AC_1)$ and $(AC_2)$ for a large class of symplectic manifolds and their Lagrangian submanifolds; also see [HZ] for a different method in the case $\pi_2(P,L) = 0$. $(AC_2)$ was

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proved for $(CP^n, \mathbb{R}P^n)$ in [Ch] and [G]. Furthermore generalizations for $(AC_2)$ was made by Oh in [Oh1]-[Oh3]. Recently, by furthermore developing Floer’s method Fukaya-Ono [FuO] and Liu-Tian [LiuT] proved the second claim in $(AC_1)$ for all closed symplectic manifolds. The obstruction theory for Lagrangian intersection was developed in [FuOOO] very recently, and more general results for $(AC_2)$ was also obtained. For more complete history and references of the conjectures we refer to [FuOOO], [McSa] and [Se].

A symplectic orbifold is a natural generalization of a symplectic manifold. Recall that a symplectic orbifold is a pair $(M, \omega)$ consisting of an orbifold $M$ and a closed non-degenerate 2-form $\omega$ on it. That is, $\omega$ is a differential form which in each local representation is a closed nondegenerate 2-form. Many definitions on symplectic manifolds, e.g., Hamiltonian maps, symplectic group actions, moment maps and Hamiltonian actions can carry over verbatim to the category of symplectic orbifolds, cf., [Le]. Ones can, of course, raise the corresponding ones of the Arnold conjectures above on closed symplectic orbifolds.

Weighted projective spaces are typical symplectic orbifolds. Let $\mathbf{q} = (q_1, \cdots, q_{n+1})$ be a $(n + 1)$-tuple of positive integers. Recall that the weighted (twisted) projective space of type $\mathbf{q}$ is defined by

$$CP^n(\mathbf{q}) = (C^{n+1} \setminus \{0\})/C^*,$$

where $C^* = C \setminus \{0\}$ acts on $C^{n+1} \setminus \{0\}$ by

$$\alpha \cdot z = (\alpha^{q_1} z_1, \cdots, \alpha^{q_{n+1}} z_{n+1}) \quad (1.1)$$

for $z = (z_1, \cdots, z_{n+1}) \in C^{n+1}$ and $\alpha \in C^*$. Note that the above $C^*$-action is free iff $q_i = 1$ for every $i = 1, \cdots, n + 1$. If the largest common divisor $\text{gcd}(q_1, \cdots, q_{n+1}) = 1$, $CP^n(\mathbf{q})$ has only isolated orbifold singularities. Let $[z]_\mathbf{q}$ denote the orbit of $z \in C^{n+1} \setminus \{0\}$ under the above $C^*$-action, i.e., a point in $CP^n(\mathbf{q})$. Denote by $m(z)$ the largest common divisor of the set $\{q_j | z_j \neq 0\}$. The orbifold structure group $\Gamma_{[z]_\mathbf{q}}$ of $[z]_\mathbf{q}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. So $[z]_\mathbf{q}$ is a smooth point of $CP^n(\mathbf{q})$ if and only if $m(z) = 1$. Clearly, each point $[z]_\mathbf{q} \in CP^n(\mathbf{q})$ with all $z_j \neq 0$, is a smooth point. As on usual complex projective spaces ones can use symplectic reduction to describe the symplectic orbifold structure on $CP^n(\mathbf{q})$. Indeed, as showed in Proposition 2.8 of [Go] the action of $S^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$,

$$A^\mathbf{q}(z_1, \cdots, z_{n+1}) = (e^{iq_1 s} z_1, \cdots, e^{iq_{n+1} s} z_{n+1}) \forall s \in \mathbb{R}, \quad (1.2)$$

is a Hamiltonian circle action on $(C^{n+1}, \omega_0)$ with a moment map

$$K_\mathbf{q}(z_1, \cdots, z_{n+1}) = \frac{1}{2} \sum_{i=1}^{n+1} q_i |z_i|^2; \quad (1.3)$$

and each $t \neq 0$ is a regular value of $K_\mathbf{q}$. The circle action on $K^{-1}_\mathbf{q}(t)$ is locally free, and thus $CP^n(\mathbf{q}) \cong K^{-1}_\mathbf{q}(t)/S^1$ for each $t \neq 0$. Denote by $S^{2n+1}(\mathbf{q}) = K^{-1}_\mathbf{q}(\frac{1}{2})$, and by $\pi : S^{2n+1}(\mathbf{q}) \to S^{2n+1}(\mathbf{q})/S^1 = CP^n(\mathbf{q})$ be the natural projection. The reduction symplectic form $\omega^\mathbf{q}_S$ on $S^{2n+1}(\mathbf{q})/S^1$ is called standard orbifold symplectic
form on $\mathbb{C}P^n(\mathbf{q})$ without special statements (since $\omega^q_{FS} = \omega_{FS}$ has integral one on $\mathbb{C}P^1 \subset \mathbb{C}P^n$ if each component $q_i$ is equal to 1 in $\mathbf{q}$). It was computed in [Ka] that $H^i(\mathbb{C}P^n(\mathbf{q}); \mathbb{Z}) = \mathbb{Z}$ for $i = 2k$ and $0 \leq k \leq n$, and zero for other $i$. Let $\gamma_k$ denote the canonical generator of the group $H^{2k}(\mathbb{C}P^n(\mathbf{q}); \mathbb{Z})$ for $1 \leq k \leq n$. It was also proved in [Ka] that the multiplication is given by

$$\gamma_k \gamma_j = \frac{q_k q_j}{\gcd(q_k, q_j)} \gamma_{k+j} \text{ if } k + j \leq n, \quad \text{and } \gamma_k \gamma_j = 0 \text{ if } k + j \geq n.$$

Here for $1 \leq k \leq n$,

$$l^{q_i q_j}_{k+j} = \text{lcm}\left(\frac{q_i \cdots q_{k+j}}{\gcd(q_i, \cdots, q_{k+j})} \mid 1 \leq i_1 < \cdots < i_{k+j} \leq n + 1\right),$$

and $l^{q_i q_j}_{k+j}/l^{q_i q_j}_{k+j}$ is always an integer. It follows that

$$\text{CL}(\mathbb{C}P^n(\mathbf{q}); \mathbb{Z}) + 1 = \text{SB}(\mathbb{C}P^n(\mathbf{q}); \mathbb{Z}) = n + 1.$$

As a generalization of Fortune’s theorem to $\mathbb{C}P^n(\mathbf{q})$, we have:

**Theorem 1.1** If $h : \mathbb{C}P^n(\mathbf{q}) \times [0, 1] \to \mathbb{R}$ is a $C^1$ time-dependent Hamiltonian on $\mathbb{C}P^n(\mathbf{q})$, then the time one map $\phi_1$ of $X_h$ has at least $n + 1$ fixed points. That is, the Arnold conjecture ($AC_1$) holds on $(\mathbb{C}P^n(\mathbf{q}), \omega^q_{FS})$.

No doubt the Oh’s main result in [Oh1] can be directly generalized to $T^{2n} \times \mathbb{C}P^k(\mathbf{q})$.

As a natural generalization $\mathbb{R}P^n \subset \mathbb{C}P^n$ we introduce a suborbifold $\mathbb{R}P^n(\mathbf{q}) \subset \mathbb{C}P^n(\mathbf{q})$ as follows:

$$\mathbb{R}P^n(\mathbf{q}) = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*,$$

called real projective space of weight $\mathbf{q}$. Here the action of $\mathbb{R}^*$ on $\mathbb{R}^{n+1} \setminus \{0\}$ is still defined by (1.1). The isotropy group at any point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ is given by $(\mathbb{R}^*)_x := \cap_{x \neq 0} G_{q_j} \cap \mathbb{R}$, where $G_{q_j} = \{e^{2\pi i k/q_j} \mid j = 0, \cdots, q_j - 1\}$ is the group of $q_j$-th roots of unity. Clearly, $(\mathbb{R}^*)_x$ is a subgroup of $\mathbb{Z}_2 = \{1, -1\}$, and thus $\mathbb{R}P^n(\mathbf{q})$ is an orbifold of dimension $n$. Clearly, we have an orbifold isomorphism

$$\mathbb{R}P^n(\mathbf{q}) \cong (\mathbb{R}^{n+1} \cap S^{2n+1}(\mathbf{q}))/\mathbb{Z}_2,$$

(1.4)

where the action of $\mathbb{Z}_2$ is induced by (1.1). From this it easily follows that $\mathbb{R}P^n(\mathbf{q})$ is a manifold if and only if all integers $q_1, \cdots, q_{n+1}$ are odd. In this case $\mathbb{R}P^n(\mathbf{q})$ is diffeomorphic to $\mathbb{R}P^n$. Hence $\text{CL}(\mathbb{R}P^n(\mathbf{q}); \mathbb{Z}_2) = n$ and $\text{SB}(\mathbb{R}P^n(\mathbf{q}); \mathbb{Z}_2) = n + 1$. If $\mathbb{C}P^n(\mathbf{q}) \equiv S^{2n+1}(\mathbf{q})/S^1$ is as above, and $[z]_q$ denotes the $S^1$-orbit of $z \in S^{2n+1}(\mathbf{q})$, then $\mathbb{R}P^n(\mathbf{q})$ is isomorphic to the Lagrangian suborbifold

$$L := \{[z]_q \mid \exists w \in [z]_q, w = u + iv, v = 0\}$$

in $(\mathbb{C}P^n(\mathbf{q}), \omega^q_{FS})$. As a generalization of a result due to Chang-Jiang [ChJi] and Givental [Gi] we have:

**Theorem 1.2** Let $q_1, \cdots, q_{n+1}$ be all odd. Then for any Hamiltonian map $\phi_1 : \mathbb{C}P^n(\mathbf{q}) \to \mathbb{C}P^n(\mathbf{q})$ as in Theorem 1.1 it holds that $\sharp(\phi_1(\mathbb{R}P^n(\mathbf{q}))) \cap \mathbb{R}P^n(\mathbf{q})) \geq n + 1$.

Namely, in this case $(AC_2)$ holds for $(\mathbb{C}P^n(\mathbf{q}), \mathbb{R}P^n(\mathbf{q}))$. 

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Let \( r(q) := \hat{z}(q_{i} | q_{i} \in 2\mathbb{Z} + 1) \). Note that \( \mathbb{R}^{n+1} \cap S^{2n+1}(q) \) is always diffeomorphic to \( S^{n} \). If \( r(q) = 0 \), by \( \mathbb{R}^{P^{n}(q)} = \mathbb{R}^{n+1} \cap S^{2n+1}(q) \) and hence \( SB(\mathbb{R}^{P^{n}(q)}) = 2 \) and \( CL(\mathbb{R}^{P^{n}(q)}) = 1 \). If \( 1 \leq r(q) \leq n \) then \( \mathbb{R}^{P^{n}(q)} \) is an orbifold, not a manifold. In fact, topologically \( \mathbb{R}^{P^{n}(q)} \) is a \( (n+1-r(q)) \)-fold unreduced suspension of \( \mathbb{R}^{P^{r(q)-1}} \), i.e., \( \mathbb{R}^{P^{n}(q)} = \Sigma^{(n+1-r(q))}(\mathbb{R}^{P^{r(q)-1}}) \). It follows that \( SB(\mathbb{R}^{P^{n}(q)}; \mathbb{Z}_{2}) = r(q) \) and

\[
CL(\mathbb{R}^{P^{n}(q)}; \mathbb{Z}_{2}) = 1 \text{ if } 2 \leq r(q) \leq n, \quad CL(\mathbb{R}^{P^{n}(q)}; \mathbb{Z}_{2}) = 0 \text{ if } r(q) = 1.
\]

As an example we take \( q = (2, 2, 3) \), then \( \mathbb{R}^{P^{3}}((2, 2, 3)) \) is homeomorphic to the unit disk \( D^{2} = \{ x_{1}^{2} + x_{2}^{2} \leq 1 \} \), and thus for any group \( G \) it holds that

\[
CL(\mathbb{R}^{P^{3}}((2, 2, 3)); G) = 0 \quad \text{and} \quad SB(\mathbb{R}^{P^{3}}((2, 2, 3)); G) = 1.
\]

Theorem \([11] \) and Theorem \([12] \) were observed when author lectured a course of Variational methods for graduates from March to June 2005. Though their proofs can be obtained by suitably changing ones in \([10] \) and \([13] \) respectively I am also to give main proof steps and necessary changes.

## 2 Proofs of Theorems

### 2.1 Proof of Theorem \([1.1] \)

Later when talking the \( S^{1} \)-action on \( C^{n+1} \) or \( S^{2n+1}(q) \) we always mean one given by \([12] \) without special statements. Denote by \( B^{2n+2}(q) = \{ z \in C^{n+1} | K_{q}(z) \leq 1/2 \} \). It is a compact convex set containing the origin as an interior point and has boundary \( S^{2n+1}(q) \). Since each nonzero \( z \in C^{n+1} \) can be uniquely expressed as \( z = r_{x}z' \), \( r_{x} > 0 \) and \( z' \in S^{2n+1}(q) \), for a smooth family of functions \( h_{t} : C^{P^{n}(q)} \to \mathbb{R}, 0 \leq t \leq 1 \), we can uniquely define a smooth family of functions \( h_{t} : C^{n+1} \to \mathbb{R} \) by

\[
H_{t}(z) = r_{x}^{2}h_{t}(z') = r_{x}^{2}h_{t}(\Pi(z')) \quad \text{and} \quad H_{t}(0) = 0.
\]

Clearly, each \( H_{t} \) is invariant under the action in \([12] \), and positive homogeneous of degree two and restricts to \( h_{t} \circ \Pi \) on \( S^{2n+1}(q) \). By the standard symplectic reduction theory, cf., \([14] \) and \([15] \), for any constant \( \lambda \) it is easily checked that \( \Pi_{e}(X_{H_{t} + \lambda X_{K_{q}}}) = X_{H_{t}} \). If \( z : [0, 1] \to C^{P^{n}(q)} \) satisfies \( \hat{z}(t) = X_{H_{t}}(z(t)) \) and \( z(0) = z(1) \) then there must exist a \( \bar{z} : [0, 1] \to S^{2n+1}(q) \) to satisfy \( \bar{z}(t) = X_{H_{t}}(\bar{z}(t)) \) and \( z(t) = \Pi(\bar{z}(t)) \) for any 0 \( \leq t \leq 1 \), and hence there is some \( s \in [0, 2\pi] \) such that \( \bar{z}(0) = A_{\frac{\pi}{2} + 2k\pi}^{q} \bar{z}(1) \) for any \( k \in \mathbb{Z} \). Define \( \bar{u}_{k}(t) = A_{\frac{\pi}{2} + 2k\pi}^{q} \bar{z}(t) \) for \( t \in [0, 1] \). Then \( \bar{u}_{k}(0) = \bar{u}_{k}(1) \) and \( \bar{u}_{k}(t) = X_{H_{t} + \lambda X_{K_{q}}}(\bar{u}_{k}(t)) \) for any \( t \in [0, 1] \) and \( s + 2\pi \mathbb{Z} \); see the proof of Proposition 1 on page 144 in \([10] \). Conversely, if \( \bar{z} : [0, 1] \to C^{n+1} \) satisfies \( \bar{z}(0) = \bar{z}(1) \in S^{2n+1}(q) \) and \( \bar{z}(t) = X_{H_{t} + \lambda X_{K_{q}}}(\bar{z}(t)) \) for some \( \lambda \in \mathbb{R} \) and any \( t \in [0, 1] \), then \( \bar{z}(0), \bar{z}(1) \in S^{2n+1}(q) \) and \( z = \Pi \circ \bar{z} : [0, 1] \to C^{P^{n}(q)} \) satisfies \( z(t) = X_{H_{t}}(z(t)) \) and \( z(0) = z(1) \); moreover for two such pairs \( (z_{1}, A_{1}) \) and \( (z_{2}, A_{2}), z_{1} = \Pi \circ z_{1} = \Pi \circ z_{2} = 2z \) implies \( A_{1} - A_{2} \in 2\pi \mathbb{Z} \). Hence each closed integral curve \( z \) of \( X_{H_{t}} \) on \( C^{P^{n}(q)} \) corresponds to a family

\[
\Omega_{z} := \left\{ (u_{k}, s + 2k\pi) \mid \bar{u}_{k}(t) = X_{H_{t} + s + 2k\pi X_{K_{q}}}(u_{k}(t)), u_{k}(0) = u_{k}(1) \right\}
\]

and \( \Pi \circ u_{k} = \Pi \circ u_{0} \forall k \in \mathbb{Z} \).
Clearly, the family $\Omega_z$ is diffeomorphic to $S^1 \times (2\pi \mathbb{Z})$, and different families correspond to different fixed points of $\phi_1$. So it suffice to prove:

There are always at least $(n + 1)$ distinct families as $\Omega_z$. 

In order to transfer it into a variational problem let $Z := L^2(\mathbb{R}/\mathbb{Z}, C^{n+1})$ and

$$X = \left\{ u = \sum_{k \in \mathbb{Z}} u_k \exp(2\pi ik t) \in Z \mid |u_0|^2 + \sum_{k \in \mathbb{Z}} |k||u_k|^2 < \infty \right\}.$$ 

Both carry respectively complete Hermitian inner products

$$(u, v) = \sum_{k \in \mathbb{Z}} (u_k, v_k)_{C^{n+1}}$$ and $$(u, v) = (u_0, v_0)_{C^{n+1}} + \sum_{k \in \mathbb{Z}} |k|(u_k, v_k)_{C^{n+1}},$$

where $(\cdot, \cdot)_{C^{n+1}}$ is the standard Hermitian inner-product on $C^{n+1}$. Let $|u| = (u, u)^{\frac{1}{2}}$ and $\|u\| = (u, u)^{\frac{1}{2}}$ be the corresponding norms. Denote by $X^+ = \{ u \in X \mid u_k = 0 \forall k < 0 \}, X^- = \{ u \in X \mid u_k = 0 \forall k \geq 0 \}$ and $X^0 = \{ u \in X \mid u_k = 0 \forall k \neq 0 \}$. Then the natural splitting of $X$, $X = X^+ \oplus X^0 \oplus X^-$, is an orthogonal decomposition of $X$ for the scalar products $(\cdot, \cdot)$ and $(\cdot, \cdot)_{C^{n+1}}$. Let $P_+, P_0$ and $P_-$ be the corresponding orthogonal projections. Consider the densely defined self-adjoint linear operator $L : Z \supset D(L) \to Z$ given by $Lu = -i\dot{u}$. Then $\sigma(L) = 2\pi\mathbb{Z}$ and each $2\pi k$ has multiplicity $n + 1$; moreover, Ker($L$) $\cong C^{n+1}$ and normalized eigenvectors corresponding to $2\pi k \in 2\pi\mathbb{Z}$ are $\phi_{k,j} = e^{2\pi ik t} \varepsilon_j$, where $\varepsilon_1, \ldots, \varepsilon_{n+1}$ are the canonical basis in $C^{n+1}$. Define a compact self-adjoint linear operator $L : X \to X$ by

$$L(u) = 2\pi(u^+ - u^-) = 2\pi \sum_{k \in \mathbb{Z}} ku_k, \quad (2.3)$$

Clearly, it is an extension of $L$ to $X$ since $X$ can be compactly embedded in $Z$. Consider a $C^1$ functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2} (Lu, u) = \pi \|u^+\|^2 - \|u^-\|^2.$$ 

For a time-dependent Hamiltonian $G_t$ on $C^{n+1}$ we also define $\mathcal{G} : X \to \mathbb{R}$ by

$$\mathcal{G}(u) = \int_0^1 G_t(u(t)) dt.$$ 

Set $\Phi_G = \Phi - \mathcal{G}$. As in Proposition 2.1 in [Fo] ones can easily prove that 1-periodic orbits of $X_G$ correspond to critical points of $\Phi_G$ in a one-to-one way. The $S^1$-action in $\mathbb{R}^{2n+1}$ induces an orthogonal $S^1$-representation $\{ T_s \}_{s \in S^1}$ on $X$ as follows:

$$T_s(u) = \sum_{k \in \mathbb{Z}} (A^s q_k u_k) \exp(2\pi k t) \quad (2.4)$$

if $u = \sum_{k \in \mathbb{Z}} u_k \exp(2\pi k t)$. (When saying $S^1$-action on $X$ below we always mean this $S^1$-action without special statements). The representation also preserve the orthogonal splitting $X = X^+ \oplus X^0 \oplus X^-$. So if each $G_t$ is $S^1$-invariant then the functional $\mathcal{G}$ and thus $\Phi_G$ is invariant with respect to the $S^1$-representation $\{ T_s \}_{s \in S^1}$. 

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Taking $H_t = G_t + K_q$ we can write
\[ \Phi_{H+\lambda K_q}(u) = \Phi_H - \lambda K_q(u), \quad \text{where} \quad K_q(u) = \int_0^1 K_q(u(t))dt. \]

They are all $C^1$-smooth on $X$, and $S(q) := K^{-1}_q(1)$ is a Hilbert manifold. Since both $H_t$ and $K_q$ are positive homogeneous of degree two, by Lagrange multiplier theorem the critical points of $\Phi_{H+\lambda K_q}$ are in a one-to-one correspondence with critical points of $\Phi_H$ constrained to $S(q)$. Precisely speaking, if $u$ is a critical point of $\Phi_H|_{S(q)}$, then it is a critical point of $\Phi_{H+\lambda K_q}$ in $X$ with $\lambda = \Phi_H(u)$; conversely, if $u$ is a critical point of $\Phi_{H+\lambda K_q}$ in $X$ and also sits in $S(q)$, then $u$ is a critical point of $\Phi_H|_{S(q)}$ with critical value $\lambda$. Let $u_1, u_2$ be two critical points $\Phi_H|_{S(q)}$ with corresponding critical values $\lambda_1 = \Phi_H(u_1)$ and $\lambda_2 = \Phi_H(u_2)$. If $\lambda_1 - \lambda_2 \notin 2\pi\mathbb{Z}$ then $u_1$ and $u_2$ correspond to two geometrically different 1-periodic orbits of $X_{\bar{h}_t}$ on $\mathbb{C}P^n(q)$. Therefore $\Phi_{H+\lambda K_q}$ is reduced to prove:

**Theorem 2.1** There are always at least $(n+1)$ critical values of $\Phi_H|_{S(q)}$, $\lambda_i, i = 1, \cdots, n+1$, such that $\lambda_i - \lambda_j \notin 2\pi\mathbb{Z}$ for any $i \neq j$.

Clearly, we can always assume $h_t \geq 0$ and thus $H_t \geq 0$. By (2.1) ones have
\[ H_t(z) \leq 2 \max_{(x,t) \in \mathbb{C}P^n(q) \times I} h_t(x)K_q(z) \leq 2MK_q(z) \]
for any $z \in \mathbb{C}^{n+1}$ and $t \in \mathbb{R}$. Here $M := \sup_{(x,t) \in \mathbb{C}P^n(q) \times I} h_t(x)$ and $I = [0,1]$. It immediately gives the first claim in the following lemma.

**Lemma 2.2** The functional $\mathcal{H} : X \to \mathbb{R}$ defined by $\mathcal{H}(u) = \int_0^1 H_t(u(t))dt$, satisfies:
(i) $0 \leq \mathcal{H}(u) \leq M = 2 \max_{(x,t) \in \mathbb{C}P^n(q) \times I} h_t(x)$ for any $u \in K_q^{-1}(1)$;
(ii) $\nabla \mathcal{H} : X \to X$ is compact and equivariant, i.e., $\nabla \mathcal{H}(T_su) = T_s \nabla \mathcal{H}(u)$ for any $u \in X$ and $s \in \mathbb{R}/2\pi\mathbb{Z}$.

The second properties is Proposition 2.3 in [Fo].

**Lemma 2.3** The operator $X \to X, u \mapsto \nabla K_q(u)$ is linear and bounded, and also respects the splitting of $X = X^+ \oplus X^0 \oplus X^-$. 

**Proof.** It only need to check the final claim. Let $u(t) = (u^{(1)}(t), \cdots, u^{(n+1)}(t))$ and $u_k = (u_k^{(1)}, \cdots, u_k^{(n+1)})$ for $k \in \mathbb{Z}$. Since
\[
\langle \nabla K_q(u), v \rangle = dK_q(u)(v) = \frac{1}{2} \int_0^1 \sum_{j=1}^{n+1} q_j [(u^{(j)}(t), v^{(j)}(t)) + (v^{(j)}(t), u^{(j)}(t))]dt
\]
\[= \frac{1}{2} \sum_{j=1}^{n+1} \sum_{k \in \mathbb{Z}} q_j (u_k^{(j)}v_k^{(j)} + v_k^{(j)}u_k^{(j)}) \]
for any $u, v \in X$, by the definitions of $X^+, X^0$ and $X^-$, it easily follows that $\nabla K_q(X^+) \subset X^+, \nabla K_q(X^0) \subset X^0$ and $\nabla K_q(X^-) \subset X^-$. \qed
Note that for any $u \in X$, \[
\frac{1}{2} |u|^2 \leq \frac{\min_i q_i |u|^2}{2} \leq K_q(u) \leq \frac{\max_i q_i |u|^2}{2} \leq \frac{q_i}{2} |u|^2.
\]

Carefully checking the proof of Proposition 3.2 in [Fo] ones can easily use Lemma 2.2 and Lemma 2.3 to obtain:

**Lemma 2.4** $\Phi_H|_{S(q)}$ satisfies the Palais-Smale condition.

Note that $\Phi_H = \Phi - \mathcal{H}$. For $c \in \mathbb{R}$ and $\delta > 0$ let

- $K_c := \{u \in S(q) | \Phi_H(u) = c, \ d(\Phi_H|_{S(q)})(u) = 0\}$,
- $N_\delta(K_c) := \{u \in S(q) | d(u, K_c) = \inf_{v \in K_c} \|u - v\| < \delta\}$,
- $(\Phi_H|_{S(q)})^c := \{u \in S(q) | \Phi_H(u) \leq c\}$.

Slightly changing the proof of Theorem 3.1 in [Fo] we can get:

**Lemma 2.5** For any given $\delta > 0$ and $c \in \mathbb{R}$ there exists an $\varepsilon > 0$ and an equivariant homeomorphism $\eta : X \setminus \{0\} \to X \setminus \{0\}$ such that:

(i) $\eta|_{S(q)}$ is an equivariant homeomorphism to $S(q)$,

(ii) $\eta(S(q) \cap (\Phi_H|_{S(q)})\{c+\varepsilon\} \setminus N_\delta(K_c)) \subset S(q) \cap (\Phi_H|_{S(q)})\{c-\varepsilon\}$,

(iii) $\eta(u) = Bu + K(u)$, where $B : X \to X$ is an equivariant linear isomorphism of the form $exp(-tL)$ for some $t > 0$ and $K : X \setminus \{0\} \to X$ is compact.

Note that the fixed point set of the $S^1$-action defined by $\{T_s\}_{s \in S^1}$,

$$\text{Fix}(\{T_s\}_{s \in S^1}) := \{u \in X | T_s(u) = u \forall s \in \mathbb{R}/2\pi\mathbb{Z}\} = \{0\}.$$ 

Let $A$ be a family of all closed and $S^1$-invariant subset $S \subset X \setminus \{0\}$, and

$$\Lambda = \{h \in C^0(X, X) | h \circ T_s = T_s \circ h \forall s \in \mathbb{R}/2\pi\mathbb{Z}\}.$$ 

Benci’s index [Be] is a map $\tau : A \to \mathbb{N} \cup \{0, +\infty\}$ defined by

$$\tau(S) = \left\{ \begin{array}{ll}
\min \{m \in \mathbb{N} | \exists \phi \in C^0(S, C^m \setminus \{0\}) \exists k \in \mathbb{N} : \\
\phi(T_su) = e^{2\pi ikx} \phi(u) \forall (u, s) \in S \times \mathbb{R}/2\pi\mathbb{Z}\}, & \text{if } \{\ldots\} \neq \emptyset, \\
+\infty, & \text{if } \{\ldots\} = \emptyset,
\end{array} \right.$$

and $\tau(\emptyset) = 0$. For properties of the index $\tau$ see Proposition 2.9 in [Be].

Let $\{R_s\}_{s \in \mathbb{R}/2\pi\mathbb{Z}}$ be an $S^1$-representation on $\mathbb{C}^k$ with 0 as the only fixed point, and

$$E^+ = X^+,$$

$$E^- = X^- \oplus X^0,$$

$$P_{E^+} = P^+, \quad P_{E^-} = P^- \oplus P_0.$$ 

Then we get a $S^1$-representation $\{(T \oplus R)_s = T_s \oplus R_s\}_{s \in \mathbb{R}/2\pi\mathbb{Z}}$ by

$$(T \oplus R)_s(u \oplus x) = (T_s u) \oplus (R_s x) \quad (2.5)$$

for any $u \oplus x \in E^- \oplus \mathbb{C}^k$ and $s \in \mathbb{R}/2\pi\mathbb{Z}$, which has 0 as the only fixed point. In [BLMR] a relative index (relative to $E^+$)

$$(2.6)$$
was defined as the minimum \( m \in \mathbb{N} \) for which there is an \( S^1 \)-representation \( \{ R_s \}_{s \in \mathbb{R} / 2\pi \mathbb{Z}} \) on \( \mathbb{C}^m \) with the fixed point set \( \{ 0 \} \), and an equivariant continuous map \( \phi : S \to (E^+ \oplus \mathbb{C}^m) \setminus \{ (0, 0) \} \) with respect to \( \mathbb{S}^1 \)-representations \( \{ R_s \}_{s \in \mathbb{R} / 2\pi \mathbb{Z}} \) on \( \mathbb{C}^m \) and \( \{(T \oplus R)_s = T_s \oplus R_s\}_{s \in \mathbb{R} / 2\pi \mathbb{Z}} \) on \( E^- \oplus \mathbb{C}^m \) such that \( P_{E^-} \circ \phi^- = P_{E^-} + K \) for some compact map \( K : S \to E^- \) (i.e., a continuous map which maps bounded subsets in \( S \) into relatively compact subsets in \( E^- \)). Here \( \phi^- \) is the \( E^- \)-component of \( \phi \). As before, if no such \( m \) exist \( \gamma(S) \) is defined as \( +\infty \). Moreover, \( \gamma(0) = 0 \).

**Lemma 2.6** The relative index \( \gamma \) satisfies:

(i) \( \gamma(S \cup R) \leq \gamma(S) + \gamma(R) \) for any \( S, R \in \mathcal{A} \).

(ii) If an equivariant isomorphism \( h : X \to X \) leaves \( E^- \) invariant, then for any \( S \in \mathcal{A} \) it holds that \( h(S) \in \mathcal{A} \) and \( \gamma(h(S)) = \gamma(S) \).

(iii) If \( \gamma(S) \geq m \) and \( E^+ = F_1 \oplus F_2 \), where \( F_i, i = 1, 2 \) are \( S^1 \)-invariant and \( \dim_{\mathbb{C}} F_i < m \), then \( S \cap F_2 \neq \emptyset \).

(iv) If \( F \subset E^+ \) is a complex \( k \)-dimensional invariant subspace and \( S(F, r) = \{ u \in E^- \oplus F \mid K_q(u) = r \} \), then \( \gamma(S(F, r)) = k \).

(v) For \( S, R \in \mathcal{A} \), if there exists a continuous bounded map \( \varphi : S \to R \) such that \( P_{E^-} \circ \varphi = P_{E^-} + K \) for some compact map \( K : S \to E^- \), then \( \gamma(S) \leq \gamma(R) \).

(vi) For \( S, R \in \mathcal{A} \), if \( \gamma(R) < \infty \) then \( \mathcal{S}(\mathcal{R}) \in \mathcal{A} \) and \( \gamma(\mathcal{S}(\mathcal{R})) \geq \gamma(S) - \tau(R) \).

**Proof.** The proofs of these properties can be found in [BLMR] and [Be]. Ones only need to note that (iv) can be proved by slightly changing the proof of Proposition 2.10 in [BLMR]. In the present case we shall obtain a map from the elliptic sphere in \( \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^j \) into \( \mathbb{C}^n \times \mathbb{C}^j \times \mathbb{C}^j \) with \( j < k \) which is equivariant (with respect to our \( S^1 \)-action as the above \( \{(T \oplus R)_s\}_{s \in \mathbb{R} / 2\pi \mathbb{Z}} \) and leaves \( \mathbb{C}^j \), the fixed-point set, invariant. It is not hard to check that the \( S^1 \)-version of the Borsuk-Ulam theorem due to [FHR] can be still used to get the desired result. \( \square \)

For each \( m \in \mathbb{N} \) let \( \Gamma_m(S(q)) = \{ S \in \mathcal{A} \mid S \subset S(q), \gamma(S) \geq m \} \) and

\[
c_m = \inf_{S \in \Gamma_m(S(q))} \sup_{u \in S} \Phi_H(u).
\]

For \( j = 1, \cdots, n + 1 \), let \( \mathbb{C}^{n+1,j} \subset \mathbb{C}^{n+1} \) be the complex 1-dimensional subspace consisting of \( w \in \mathbb{C}^{n+1} \) whose \( k \)-components are zero for \( k \neq j \). Then for each \( m \in \mathbb{N} \),

\[
F_{m,j} = \oplus_{k=1}^m \mathbb{C}^{n+1,j} \exp(2\pi ikt) \subset E^+
\]

is a complex \( m \)-dimensional invariant subspace. By Lemma 2.6 (iv), \( S(F_{m,j}, r) \in \Gamma_m(S(q)) \) and \( P_{E^+}(S(F_{m,j}, r)) \) is also compact. So it follows from Lemma 2.2 (i) that

\[
c_m \leq \sup_{u \in S(F_{m,j}, 1)} \left[ \pi(||u^+||^2 - ||u^-||^2) - H(u) \right] \\
\leq \sup_{u \in S(F_{m,j}, 1)} \left[ \pi(||u^+||^2 - ||u^-||^2) + M \right] \\
\leq \sup_{u \in S(F_{m,j}, 1)} \pi ||u^+||^2 + M \\
\leq \sup_{u \in P_{E^+}(S(F_{m,j}, 1))} \pi ||u||^2 + M < +\infty
\]
since \( P_{E^+}(S(F_{m,j}, 1)) \) is compact. Using Lemmas 2.4-2.6 the standard minimax arguments lead to:

**Theorem 2.7** \( c_1 \leq c_2 \leq \cdots < +\infty \). If \( c_m > -\infty \) then it is a critical value of \( \Phi_H|_{S(q)} \). Moreover, if \( c_m = c_{m+1} = \cdots = c_{m+k} = c \) is finite then \( \gamma(K_c) \geq k \).

Since \( \Phi = \Phi_H + \mathcal{H} \), if we set \( d_m = \inf_{S \in \Gamma_m(S(q))} \sup_{u \in S} \Phi(u) \), it follows from the proof of Proposition 3.5 in [Fo] that

\[
d_m - M \leq c_m \leq d_m \quad \forall m \in \mathbb{N}. \tag{2.7}
\]

Here \( M \) is defined as in Lemma 2.2(i). Clearly, the \( d_m \) have the same properties as the \( c_m \) in Theorem 2.7. In particular, if \( d_m \) is finite, it is a critical value of \( \Phi|_{S(q)} \), and thus is an eigenvalue of the linear eigenvalue problem

\[
Lu = \mu \nabla K_q(u) \quad \text{on } S(q). \tag{2.8}
\]

Without loss of generality we now assume that

\[
q_1 \geq q_2 \geq \cdots \geq q_{n+1} \quad \text{and} \quad q_1 \geq 2. \tag{2.9}
\]

**Lemma 2.8** Under the assumption \( (2.1), (2.3) \) has eigenvalues

\[
\lambda_{1} = \pm \frac{2\pi}{q_1}, \quad \lambda_{2} = \pm \frac{2\pi}{q_{n+1}}, \quad \lambda_{3} = \pm \frac{2\pi(2k+1)}{q_1}, \quad \lambda_{4} = \pm \frac{2\pi(q_{n+1})}{q_{n+1}}, \quad \ldots \quad \lambda_{(k+1)(n+1)} = \frac{2\pi(2k+1)}{q_{n+1}},
\]

\[
\lambda_0 = 0 \quad \text{with multiplicity } n + 1, k = 1, 2, \ldots.
\]

Moreover, all \( \phi_{k,j} = e^{2\pi i k t} \xi_j, k \in \mathbb{Z} \) and \( j = 1, \ldots, n + 1 \), are still the corresponding eigenvectors.

**Proof.** Assume that \( Lu = \mu \nabla K_q(u) \) for some \( u \in S(q) \) and \( \mu \in \mathbb{R} \). Let \( u(t) = \sum_{k \in \mathbb{Z}} u_k \exp(2\pi i k t), \quad u_k = (u(1)_k, \ldots, u_{(n+1)}_k) \), and \( v(t) = \sum_{k \in \mathbb{Z}} v_k \exp(2\pi i k t), \quad v_k = (v(1)_k, \ldots, v_{(n+1)}_k) \). As in the proof of Lemma 2.8 \( \langle Lu, v \rangle = \mu \langle \nabla K_q(u), v \rangle = \mu d K_q(u)(v) \) if and only if

\[
2\pi \sum_{k \neq 0} \sum_{j=1}^{n+1} q_j k u_k^{(j)} v_k^{(j)} = \frac{\mu}{2} \sum_{j=1}^{n+1} \sum_{k \in \mathbb{Z}} q_j (u_k^{(j)} v_k^{(j)} + v_k^{(j)} u_k^{(j)}).
\]

The desired conclusions are easily derived from it. \( \square \)

Denote by

\[
\cdots \leq \mu_{-2} \leq \mu_{-1} = 0 < \mu_1 \leq \mu_2 \leq \cdots \tag{2.10}
\]

the sequence of eigenvalues of the eigenvalue problem \( (2.8) \), each repeated according multiplicity. Let \( \hat{\phi}_k \) be the eigenfunction corresponding to \( \mu_k \) for \( k \in \mathbb{Z} \). By Lemma 2.8 we can normalize the \( \hat{\phi}_k \) so that \( \langle \nabla K_q(\hat{\phi}_k), \hat{\phi}_l \rangle = \delta_{kl}, k, l \in \mathbb{Z} \). Note that each \( \hat{\phi}_k \) is the normalization of some \( \phi_{k,j} \), and that \( \mu_k > 0 \) if and only if \( i > 0 \). Moreover, it is clear that \( \{ \hat{\phi}_k \mid k \in \mathbb{Z} \} \) form a complete orthogonal system in \( X \). So for each \( u \in X \) it holds that \( u = \sum_{k \in \mathbb{Z}} (u, \hat{\phi}_k) \hat{\phi}_k \). Especially, \( u = \sum_{k \in \mathbb{Z}} (u, \hat{\phi}_k) \hat{\phi}_k \in S(q) \) if and only if

\[
1 = \langle \nabla K_q(\sum_{k \in \mathbb{Z}} (u, \hat{\phi}_k) \hat{\phi}_k), \sum_{k \in \mathbb{Z}} (u, \hat{\phi}_k) \hat{\phi}_k \rangle = \sum_{k \in \mathbb{Z}} |(u, \hat{\phi}_k)|^2. \tag{2.11}
\]
Furthermore, assume that \( u \in S(\mathbf{q}) \cap \text{span}\{\hat{\phi}_l | l \leq k\} \); then we have
\[
\Phi(u) = \frac{1}{2} \langle Lu, u \rangle = \frac{1}{2} \left( \sum_{l \leq k} \left\langle u, \hat{\phi}_l \right\rangle \hat{\phi}_l, \sum_{l \leq k} \left\langle u, \hat{\phi}_l \right\rangle \hat{\phi}_l \right) = \frac{1}{2} \sum_{l \leq k} \sum_{j \leq k} \left\langle u, \hat{\phi}_l \right\rangle \left\langle u, \hat{\phi}_j \right\rangle (L \hat{\phi}_l, \hat{\phi}_j) = \frac{1}{2} \sum_{l \leq k} \sum_{j \leq k} \left\langle u, \hat{\phi}_l \right\rangle \left\langle u, \hat{\phi}_j \right\rangle \mu_l \langle \nabla K_{\mathbf{q}}(\hat{\phi}_l), \hat{\phi}_j \rangle = \frac{1}{2} \sum_{l \leq k} \mu_l |\langle u, \hat{\phi}_l \rangle|^2 \leq \mu_k.
\]

The final step is because of \( \Delta \). Hence we get
\[
d_k \leq \mu_k \quad \forall k \geq 1. \tag{2.12}
\]

On the other hand, since \( \phi_j \in E^+ \) for any \( j > 0 \), for any \( S \in \Gamma_k(S(\mathbf{q})) \) with \( k \geq 2 \) it follows from Lemma 2.6(iii) that the intersection \( S \cap \text{span}\{\hat{\phi}_j | j \geq k\} \) must be nonempty since \( E^+ = \text{span}\{\hat{\phi}_j | 1 \leq j \leq k - 1\} + \text{span}\{\hat{\phi}_j | j \geq k\} \) is an orthogonal decomposition of invariant subspaces. Let \( v \in S \cap \text{span}\{\hat{\phi}_j | j \geq k\} \). Then
\[
\sup_{u \in S} \Phi(u) \geq \Phi(v) = \frac{1}{2} \sum_{l \geq k} \mu_l |\langle u, \hat{\phi}_l \rangle|^2 \geq \mu_k.
\]

This and \( \Delta \) together yield:

**Proposition 2.9** ([Prop.3.6, BLMR]) \( d_m = \mu_m \) for any \( m > 1 \).

**Remark 2.10** In Proposition 3.6 of [BLMR] it was also claimed that \( d_1 = \mu_1 \). However the arguments in the second step of proof therein seem not to be complete for \( k = 1 \). Precisely, for a set \( B \in \Gamma_1(G_1) \) with \( \gamma_r(pB) = 1 \), I do not know how their Corollary 2.9 is used to derive \( B \cap \text{span}\{\phi_i | i \geq 1\} \neq \emptyset \). From the proof of their Proposition 2.8 it is impossible to improve their condition “\( \dim F_1 < k \)” to “\( \dim F_1 \leq k \)”.

Now \( \Delta \), Lemma 2.8 and Proposition 2.9 together yield
\[
\frac{2\pi}{q_1} \leq \mu_2 \leq \frac{2\pi}{q_2} \quad \text{and} \quad \mu_m - M \leq c_m \leq \mu_m \quad \forall m \geq 2. \tag{2.13}
\]

**Proof of Theorem 2.1** Let \( t_0 \geq 1 \) be the smallest integer such that \( M \leq 2t_0\pi \). By \( \Delta \), \( \max_i q_i = q_1 \geq 2 \). Then \( 4\pi/q_1 \leq 2\pi \) and
\[
2\pi/q_2 \leq \cdots \leq 2\pi/q_{n+1} \leq 2\pi.
\]

These imply that there are at least \( (n + 1)'s \mu_k \) (counting multiplicity) in the interval \( (2\pi/q_1, 2\pi] \). Using Lemma 2.8 it is easily seen that for each integer \( s > t_0 \) the interval \( (2(1 + t_0)\pi, 2s\pi] \) contains at least \( (s - t_0 - 1)(n + 1)'s \mu_k \). Let them be
\[
\mu_{i+1} \leq \mu_{i+2} \leq \cdots \leq \mu_{i+(s-t_0-1)(n+1)}, \quad l \geq 1.
\]
Then by (2.13), corresponding with them we have
\[ c_{l+1} \leq c_{l+2} \leq \cdots \leq c_{l+(s-t_0-1)(n+1)}, \quad l \geq 1. \]  
(2.14)
By (2.13) ones easily derive that these are in the interval \( I_s = (2\pi, 2s\pi] \), and thus that they are critical values of \( \Phi_H|_{S(q)} \).
If there are \( c_k, c_{k'} \in I_s, k, k' \geq l+1, k \neq k' \) such that \( c_k = c_{k'} \), the conclusion is obvious. So we can assume:
\[ c_k \neq c_{k'}, \quad k, k' \geq l+1 \]
Then (2.14) shows that
\[ \sharp(\{c_k \mid k \geq l+1\} \cap I_s) \geq (s-t_0-1)(n+1). \]  
(2.15)
Two elements \( c \) and \( c' \) in \( \{c_k \mid k \geq l+1\} \cap I_s \) is said to be equivalent if \( c - c' \) is an integer multiple of \( 2\pi \). Denote by \( N_s \) the number of the equivalent classes. Without loss of generality we can assume that \( N_s \) is finite. Then
\[ \sharp(\{c_k \mid k \geq l+1\} \cap I_s) \leq N_s(s-1). \]  
(2.16)
Take \( s > 1 \) so large that \( t_0(n+1)/(s-1) < 1 \). Then (2.15) and (2.16) give
\[ N_s \geq (n+1) - \frac{t_0(n+1)}{s-1} \]
and thus \( N_s \geq n + 1 \). The desired result is proved. 
\[ \square \]

2.2 Proof of Theorem 1.2

The original problem is reduced to estimate the number of distinct solutions of the following boundary value problem:
\[ \dot{z}(t) = X_{h_t}(z(t)), \quad z(0) = p_0 \in L, \quad z(1) = p_1 \in L. \]  
(2.17)
Let \( H_t \) be defined by (2.1). As in [ChJi], modify \( H_t \) outside some open neighborhood of \( B^{2n+2}(q) \) so that \( H_t \) is \( C^1 \)-bounded, and then consider a boundary value problem:
\[ \left\{ \begin{array}{l}
\dot{z}(t) = X_{H_t}(z(t)) + \lambda X_{K_{q}}(z(t)), \\
z(j) \in \mathbb{R}^{n+1} \cap S^{2n+1}(q), \quad j = 0, 1.
\end{array} \right. \]  
(2.18)
Since \( \Pi_s(X_{H_t} + \lambda X_{K_{q}}) = X_{h_t} \), with the similar arguments to Lemma 2.1 and Lemma 2.2 in [ChJi] we easily get:

**Lemma 2.11** Each solution \( z \) of (2.14) sits in \( S^{2n+1}(q) \), and \( u(t) = \Pi(A_{\lambda,t}^q z(t)) \) solves (2.17). Moreover, if \( (z^1, \lambda_1) \) and \( (z^2, \lambda_2) \) are two solutions of (2.18), then
\[ \Pi(A_{\lambda,t}^q z^1(t)) = \Pi(A_{\lambda,t}^q z^2(t)) \forall t \implies \lambda_1 = \lambda_2 \pmod{\pi}. \]
Consider an operator $A : L^2([0, 1], \mathbb{C}^{n+1}) \cong D(A) \to L^2([0, 1], \mathbb{C}^{n+1})$ given by $Au = -i \hat{u}$, where $D(A) = \{ z \in H^1([0, 1], \mathbb{C}^{n+1}) \mid z(0), z(1) \in \mathbb{R}^{n+1} \}$. It is self-adjoint, and $\sigma(A) = \pi \mathbb{Z}$. Moreover, each eigenvalue $k \pi$ has multiplicity $n + 1$, and corresponding eigenspace is spanned by $\varphi_{k,j} = e^{\pi i k t} \varepsilon_j$, $j = 1, \cdots, n + 1$. According to the spectral decomposition

$$L^2([0, 1], \mathbb{C}^{n+1}) = \oplus_{k \in \mathbb{Z}} \text{span}\{ \varphi_{k,j} \mid j = 1, \cdots, n + 1 \},$$

the operator $A$ can be decomposed into the positive, zero and negative parts: $A = A^+ + A^0 - A^-$. Let us denote $D(|A|^{1/2})$ by

$$X = \left\{ u = \sum_{k \in \mathbb{Z}} u_k \exp(\pi i k t) \in L^2([0, 1], \mathbb{C}^{n+1}) \mid |u_0|^2 + \sum_{k \in \mathbb{Z}} |k||u_k|^2 < \infty \right\}.$$

It is a Hilbert space with inner product $(u, v)_X = \sum_{k \in \mathbb{Z}} (1 + |k\pi|)(u_k, v_k)_{\mathbb{C}^{n+1}}$ and the corresponding norm $\|u\|_X = (u, u)_X^{1/2}$. Let $\mathcal{K}_q : X \to \mathbb{R}$ be defined by $\mathcal{K}_q(u) = \int_0^1 K_q(u(t))dt$. Introduce the manifold $S(q) = \{ u \in X \mid \mathcal{K}_q(u) = 1 \}$ and define a functional $J_H : X \to \mathbb{R}$ by

$$J_H(u) = \frac{1}{2} (Bu, u)_X - \int_0^1 H_t(u(t))dt,$$

where $B : X \to X$ is defined by $B(u) = \pi \sum_{k \in \mathbb{Z}} ku_k$. Slightly changing the proof of Lemma 2.4 in [ChJi] one can get:

**Lemma 2.12** If $z_0 \in S(q)$ is a critical point of $J_H|_{S(q)}$ and $\lambda_0$ is the corresponding Lagrange multiplier, then $(z_0, \lambda_0)$ solves (2.15) and $J_H(z_0) = \lambda_0$.

There is an obvious $\mathbb{Z}_2$-action induced by (1.12),

$$g \cdot u = (g^{q_1} u_1, \cdots, g^{q_{n+1}} u_{n+1}) \forall g \in \mathbb{Z}_2 = \{ 1, -1 \},$$

under which $J_H$ is invariant. Thus $J_H$ can be viewed as a functional on the quotient $P(q) := S(q)/\mathbb{Z}_2$. **Since all** $q_1, \cdots, q_{n+1}$ **are odd**, the action in (2.14) is free on $X \setminus \{ 0 \}$, and hence $P(q)$ is a Hilbert manifold. By (2.14), $-u = (-1) \cdot u$ for any $u \in X$, and thus $J_H(-z) = J_H(z)$ for any $z \in X$. Note that for a given $z \in \mathbb{C}^{n+1}$, $z \varphi_{k,j} \in X$ if and only if $z \in \mathbb{R}^{n+1}$. We set

$$X_m = \oplus_{|k| \leq m} \oplus_{j=1}^{n+1} (\mathbb{R}\varphi_{k,j}), \quad P(q)_m = (S(q) \cap X_m)/\mathbb{Z}_2,$$

$$X_m^+ = \oplus_{k=1}^m \oplus_{j=1}^{n+1} (\mathbb{R}\varphi_{k,j}), \quad P(q)_m^+ = (S(q) \cap X_m^+)/\mathbb{Z}_2,$$

and $J_m = J_H|_{P_m(q)}$. Then $C\ell(\cup_{m=0}^{\infty} X_m) = X$. By the proof of Lemma 3.1 in [ChH] ones can easily get: $J_H$ satisfies $(PS)^*$ with respect to the smooth Hilbert filtration of finite dimension $P_1(q) \subset P_2(q) \subset \cdots \subset P_m(q) \subset \cdots$ of $P(q)$; that is, for any sequence $u^{(m)} \in P_m(q)$, $m = 1, 2, \cdots$, if $\{ J_m(u^{(m)}) \}$ is bounded and $\lim_{m \to \infty} dJ_m(u^{(m)}) = 0$, then $\{ u^{(m)} \}$ has a convergent subsequence.

The key is that $P_m(q)$ is diffeomorphic to $\mathbb{R}^{(2m+1)(n+1)-1}$. Almost repeating the arguments in [ChH] ones can get the desired result. □
2.3 Open questions and concluding remarks

(i) If $1 \leq r(q) \leq n$, the fixed point set of the action in (2.19) is given by

$$\text{Fix}_{Z_2} = \{ u = (u_1, \cdots, u_{n+1}) \in X \mid u_i = 0 \text{ if } q_i \notin 2\mathbb{Z} \}.$$ 

Both $\text{Fix}_{Z_2}$ and $X \setminus \text{Fix}_{Z_2}$ are infinite dimension subspaces. In this case the above methods fail. Will $(AC_2)$ hold in the cases $0 \leq r(q) \leq n$?

(ii) Hofer's method in [Ho] seem to be able to prove the following result:

Let $(M, \omega)$ be a closed symplectic orbifold and $M^{sm}$ be its smooth locus. If $L \subset (M^{sm}, \omega)$ is a compact Lagrange submanifold without boundary satisfying $\pi_2(M, L) = 0$, then for any Hamiltonian map $\phi : M \to M$ it holds that $\sharp(L \cap \phi(L)) \geq \text{CL}(L; \mathbb{Z}_2) + 1$. Here $\text{CL}(L; \mathbb{Z}_2)$ denotes the $\mathbb{Z}_2$-cuplength of $L$.

Even if $\pi_2(M, L) \neq 0$, but $L$ is monotone and its minimal Maslov number $N_L \geq 2$ it is also possible to generalize some results in [Oh2] to the case that $(M, \omega)$ is a closed symplectic orbifold and $L \subset M^{sm}$.

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