K-theory Soergel bimodules

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Abstract
We initiate the study of $K$-theory Soergel bimodules, a
global and $K$-theoretic version of Soergel bimodules. We
show that morphisms of $K$-theory Soergel bimodules
can be described geometrically in terms of equivariant
$K$-theoretic correspondences between Bott–Samelson
varieties. We thereby obtain a natural categorification
of $K$-theory Soergel bimodules in terms of equivari-
ant coherent sheaves. We introduce a formalism of
stratified equivariant $K$-m motives on varieties with an
affine stratification, which is a $K$-theoretic analog of
the equivariant derived category of Bernstein–Lunts.
We show that Bruhat-stratified torus-equivariant $K$-
motives on flag varieties can be described in terms
of chain complexes of $K$-theory Soergel bimodules.
Moreover, we propose conjectures regarding an equiv-
ariant/monodromic Koszul duality for flag varieties and
the quantum $K$-theoretic Satake.

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1 | INTRODUCTION

Let $G \supset B \supset T$ be a connected split reductive group with a Borel subgroup $B$ and maximal torus $T$ such that the derived subgroup $G' \subset G$ is simply connected. Let $N_G(T)/T = \mathcal{W} \supset S$ be the Weyl group with set of simple reflections $S$, $X(T) = \text{Hom}_{\text{grp}}(T, \mathbb{G}_m)$ the character lattice and $X = G/B$ the flag variety. We fix some ring of coefficients $\Lambda$ and tacitly assume that everything is linear over $\Lambda$.

1.1 | Soergel bimodules

Soergel bimodules are graded bimodules over the $T$-equivariant ring of a point

$$S = H^*_T(pt) = H^*(BT) = \text{Sym}^*(X(T)\Lambda)$$

that arise as direct summands of the $T$-equivariant cohomologies of Bott–Samelson varieties

$$\text{BS}(s_1, \ldots, s_n) = P_{s_1} \times_B \ldots \times_B P_{s_n}/B,$$

which admit a simple description as iterated tensor products

$$H^*_T(\text{BS}(s_1, \ldots, s_n)) = S \otimes_{S^{s_1}} \ldots \otimes_{S^{s_n}} S,$$

see [34].

If $\Lambda$ is a field of characteristic 0, indecomposable Soergel bimodules yield the equivariant intersection cohomology of Schubert varieties in $X$. This is a consequence of the decomposition theorem for perverse sheaves, see [1], and Soergel’s Erweiterungssatz, see [33].

1.2 | $K$-theory Soergel bimodules

Our definition of $K$-theory Soergel bimodules follows the simple idea of replacing equivariant cohomology by equivariant $K$-theory: The ring $S$ is replaced by the representation ring of the torus $T$

$$R = R(T) = K^T_0(pt) = K_0(\text{Rep}(T)) = \Lambda[X(T)].$$

The $T$-equivariant $K$-theory of a Bott–Samelson variety is a bimodule over $R$ and can be computed as

$$K^T_0(\text{BS}(s_1, \ldots, s_n)) = R \otimes_{R^{s_1}} \ldots \otimes_{R^{s_n}} R,$$

see Subsection 4.4. Consequently, we define $K$-theory Soergel bimodules as direct sums and direct summands of these bimodules. We note that $K$-theory Soergel bimodules were also considered in [11, section 8].
1.3 | Atiyah–Segal completion theorem

If \( \Lambda \) is a field of characteristic 0, (cohomological) Soergel bimodules can be interpreted as a completed or infinitesimal version of \( K \)-theory Soergel bimodules.

Namely, the Atiyah–Segal completion theorem and the Chern character isomorphism exhibit the equivariant cohomology (= cohomology of the Borel construction) as the completion of the genuine equivariant \( K \)-theory (= \( K \)-theory of equivariant vector bundles) at the augmentation ideal \( I = \ker(K^0_T(pt) \to K_0(pt)) \). For a Bott–Samelson variety \( BS \), we obtain

\[
K^\wedge_0(BS)^{\wedge}_I \xrightarrow{\sim} K_0(ET \times_T BS) \xrightarrow{\sim} \prod_i H^i(ET \times_T BS) = \prod_i H^i_T(BS).
\]

Geometrically, cohomological and \( K \)-theory Soergel bimodules can be interpreted as coherent sheaves on the spaces

\[
\mathfrak{t}^\vee \times_{\mathfrak{t}^\vee/\mathcal{W}} \mathfrak{t}^\vee \text{ and } T^\vee \times_{T^\vee/\mathcal{W}} T^\vee,
\]

respectively, where \( T^\vee/\Lambda \) denotes the dual torus and \( \mathfrak{t}^\vee \) its Lie algebra. Hence, the former arise from the latter by passing to an infinitesimal neighborhood at the identity \( 1 \in T^\vee \).

1.4 | Correspondences and Erweiterungssatz

We will show that morphisms of \( K \)-theory Soergel bimodules admit a geometric description in terms of \( K \)-theoretic correspondences between Bott–Samelson varieties.

**Theorem** (Theorem 4.4). Let \( x, y \) be sequences of simple reflections. Then convolution induces an isomorphism

\[
\operatorname{act} : G^T_0(BS(x) \times_Y BS(y)) \sim \operatorname{Hom}_{R \otimes R}(K^T_0(BS(x)), K^T_0(BS(y))).
\]

Here \( G^T_0 \) denotes \( K_0(Coh_T(-)) \) which is the \( K \)-theoretic analog of Borel–Moore homology. The result is a \( K \)-theoretic analog of Soergel’s Erweiterungssatz, which implies a similar statement for equivariant Borel–Moore homology and cohomology. Correspondences can be composed via convolution. That allows to lift the result to an equivalence of categories between the Karoubi envelope of a category of \( K \)-theoretic correspondences and \( K \)-theory Soergel bimodules.

**Remark** 1.1. We note that Theorem 4.4 should also hold in the case of partial flag varieties and for Kac–Moody groups. This gives a geometric interpretation for homomorphism of singular \( K \)-theory Soergel bimodules associated to generalized Cartan matrices. For the cohomological case, see [6, 32].

1.5 | \( K \)-motives on flag varieties

Equivariant cohomology groups \( H^*_T \) can be interpreted as extensions in the equivariant derived category of constructible sheaves \( D_T \), see [3]. This yields another construction of (cohomological) Soergel bimodules in terms of equivariant sheaves on the flag variety \( X \).
In a modern formulation, this can be described via an equivalence of stable ∞-categories
\[ D_{mix}^{T,(B)}(X) \sim \text{Ch}^b(S\text{Bim}_S^X) \]
between a category of \( T \)-equivariant mixed sheaves, that are locally constant along Bruhat cells, and the category of chain complexes of graded Soergel bimodules.

Mixed sheaves \( D_{mix}(X) \) are a graded refinement of the category of constructible sheaves \( D^b(X) \) that can be constructed via mixed Hodge modules or mixed \( \ell \)-adic sheaves, see [2, 25], and, most satisfyingly, using mixed motives \( DM(X) \), see, for example, [15, 19, 37, 38].

We will prove a \( K \)-theoretic analog of this story that provides a third definition of \( K \)-theory Soergel bimodules.

Equivariant \( K \)-theory groups \( K^T_0 \) can be interpreted as morphisms in the category of \textit{equivariant} \( K \)-motives \( DK^T \). We will give a definition of this category based on the equivariant stable homotopy category \( SH^T \) constructed in [26]. We will see that \( DK^T \) comes equipped with a six functor formalism and behaves very similarly to \( D_{mix}^T \). In particular, we will discuss \textit{affine-stratified} \( K \)-motives in detail and discuss their formality using weight structures. We will then show:

**Theorem** (Corollary 5.3). Let \( \Lambda = \mathbb{Q} \). There is an equivalence of stable ∞-categories
\[ DK^T_{(B)}(X) \sim \text{Ch}^b(S\text{Bim}_R) \]
between the category of Bruhat-stratified \( T \)-equivariant \( K \)-motives on the flag variety and the category of chain complexes of \( K \)-theory Soergel bimodules over \( R \).

### 1.6 Further directions

This paper should be seen as a starting point to new possible \( K \)-theoretic approaches to geometric representation theory. We now discuss some of these further directions.

#### 1.6.1 Categorification of \( K \)-theory Soergel bimodules

The interpretation of \( K \)-theory Soergel bimodules and their morphisms in terms of \( K \)-theory of (fiber products of) Bott–Samelson varieties immediately yields a categorification
\[ R \otimes_{R^0} \cdots \otimes_{R^0} R = K^T_0(\text{BS}(x)) \xleftrightarrow{K^T_0(\text{BS}(y))} \text{D}^b(\text{Coh}_R(\text{BS}(x))) \]
\[ \text{Hom}_{R \otimes R}(K^T_0(\text{BS}(x)), K^T_0(\text{BS}(y))) \xleftrightarrow{K^T_0(\text{BS}(y))} \text{D}^b(\text{Coh}_R(\text{BS}(x) \times_x^y \text{BS}(y))) \]
in terms of the derived category of equivariant coherent sheaves on these spaces. Composition of morphisms is categorified with a convolution formula similar to Fourier–Mukai transformations. We will explore the implications in a future work.

#### 1.6.2 Diagrammatic calculus and algebraic properties

Cohomological Soergel bimodules admit a diagrammatic description that, roughly speaking, describes the relationship between the units and counits induced by the various Frobenius extensions \( S^s \subset S \) for \( s \in S \), see [14, 17, 20].
Very similarly, there are Frobenius extensions \( R^s = R(P_s) \subset R = R(T) \) for \( s \in S \) that arise from parabolic induction. They fulfill similar relationships and it is very imaginable that there is a diagrammatic calculus for \( K \)-theory Soergel bimodules. For example, there should be a nice diagrammatic basis for their homomorphisms corresponding to the affine strata of the fiber products \( BS(x) \times_X BS(y) \).

In this paper we completely ignore any algebraic questions such as a Krull–Schmidt property, uniqueness of indecomposable \( K \)-theory Soergel bimodules, and so on, which are probably best studied using diagrammatics.

1.6.3  |  Equivariant/monodromic duality

Koszul duality for flag varieties, see [2, 33], is an equivalence of categories between mixed sheaves on a flag variety \( X \) and its Langlands dual \( X^\vee \). Equivalently, Koszul duality provides an equivalence of the derived graded principal block of category \( \mathcal{O} \) of a complex reductive Lie algebra and its Langlands dual.

Remarkably, Koszul duality intertwines the Tate-twist and shift functor (1)[2] with the Tate twist (1). This motivated our construction of a nonmixed/ungraded Koszul duality for flag varieties, see [13],

\[
\text{DK}_{(B)}(X) \sim \text{D}_{(B)}(X^\vee),
\]

relating \( K \)-motives to constructible sheaves: \( K \)-motives admit a phenomenon called Bott periodicity which implies that (1)[2] is the identity functor, while the Tate twist (1) acts trivially on (nonmixed) constructible sheaves.

In the spirit of [6], this result should have a equivariant/monodromic lift:

**Conjecture 1.2** (Ungraded, uncompleted equivariant/monodromic Koszul duality). Let \( \Lambda = \mathbb{Q} \). There is an equivalence of categories

\[
\text{DK}^T_{(B)}(X) \sim \text{D}^{b,f g}_{B^\vee \times B^\vee -\text{mon}}(G^\vee(C)),
\]

between Bruhat-stratified \( T \)-equivariant \( K \)-motives on a flag variety and Bruhat-stratified \( B^\vee \times B^\vee \)-monodromic constructible sheaves on the Langlands dual group whose stalks are finitely generated under the fundamental group of \( B^\vee \times B^\vee \).

For each maximal ideal \( I \subset R \), this conjecture specializes to a Koszul duality between \( I \)-twisted equivariant sheaves and \( I \)-locally finite monodromic sheaves (see [22, 31]).

1.6.4  |  Quantum \( K \)-theoretic Satake

The approach to \( K \)-theoretic correspondences via \( K \)-motives developed here in the context of \( K \)-theory Soergel bimodules should shed new light on Cautis–Kamnitzer’s quantum \( K \)-theoretic Satake, see [9], which can be reformulated as the following:
Conjecture 1.3. There is an equivalence of categories

\[ \text{DK}_{r}^{G \times \mathbb{G}_m}(\text{Gr}) \sim D_{U_q(\mathfrak{g}^\vee)}^b(\mathcal{O}_q(G^\vee)), \]

between reduced $G \times \mathbb{G}_m$-equivariant $K$-motives on the affine Grassmannians and $U_q(\mathfrak{g}^\vee)$-equivariant $\mathcal{O}_q(G^\vee)$-modules.

Here reduced $K$-motives $\text{DK}_r$ should be constructed from $\text{DK}$ by removing the higher $K$-theory of the base point, as defined in the context of DM in [19]. In particular, the category $\text{DK}_{r}^{G \times \mathbb{G}_m}(\text{Gr})$ should have a combinatorial description in terms of singular $K$-theory Soergel bimodules. See also [16] for a different approach using a $q$-deformed Cartan matrix.

1.6.5 | Motivic springer theory

In the spirit of [12, 18], $K$-motives should be useful to construct categories of representations of $K$-theoretic convolution algebras, such as the affine Hecke algebra, geometrically. For example, we conjecture the following:

Conjecture 1.4. There is an equivalence of categories

\[ \text{DK}_{r}^{G \times \mathbb{G}_m, \text{Spr}}(\mathcal{N}) \sim D_{\text{perf}}(H_{aff}) \]

between the $G \times \mathbb{G}_m$-equivariant Springer $K$-motives on the nilpotent cone and the perfect derived category of the affine Hecke algebra.

1.7 | Structure of the paper

In Section 2, we introduce the formalism of $G$-equivariant $K$-motives $\text{DK}^G$ for diagonalizable groups $G$ and discuss the relation to equivariant $K$-theory and $G$-theory.

In Section 3, we consider $\mathcal{S}$-stratified $G$-equivariant $K$-motives $\text{DK}^G_{\mathcal{S}}(X)$ for varieties $X$ with an affine stratification $\mathcal{S}$. We construct a weight structure and discuss their formality.

In Section 4, we recall basic properties of equivariant $K$-theory of flag varieties and define $K$-theory Soergel bimodules. Moreover, we give a geometric construction of morphisms of $K$-theory Soergel bimodules in terms of $K$-theoretic correspondences. This can be read independently of the other sections and does not involve any $\infty$-categories.

In Section 5, we discuss the category $\text{DK}^T_{(\beta)}(X)$ of Bruhat-constructible $T$-equivariant $K$-motives on the flag variety and show that it can be described via chain complexes of $K$-theory Soergel bimodules.

2 | PRELIMINARIES ON EQUIVARIANT $K$-THEORY AND $K$-MOTIVES

In this section, we define a formalism of equivariant $K$-motives $\text{DK}^G(X)$ based on the equivariant stable motivic homotopy category introduced in [26]. Moreover, we discuss basic functorialities.
of \( K \)-theory and \( G \)-theory. Here, \( \Lambda \) is any ring of coefficients and \( k \) any base field. Moreover, by \( 1 \) we denote the tensor unit in any monoidal category.

### 2.1 Definition

Denote \( \text{pt} = \text{Spec}(k) \). Let \( G \) be an algebraic group over \( k \) of multiplicative type, for example, \( G \) is a finite product of groups of the form \( \mathbb{G}_m \) and \( \mu_n \). We use the term \( G \)-variety to denote a separated \( G \)-scheme \( X \) of finite type over \( k \) which is \( G \)-quasi-projective, that is, admits a \( G \)-equivariant immersion into \( \mathbb{P}(V) \) for a vector space \( V \) with linear \( G \)-action. In particular, if \( X \) is normal, quasi-projectivity implies \( G \)-quasi-projectivity. A morphism of \( G \)-varieties is a morphism of schemes that is \( G \)-equivariant.

To any \( G \)-variety \( X \), [26] associates the \( G \)-equivariant stable motivic homotopy category \( \text{SH}^G(X) \) that is a closed symmetric monoidal stable \( \infty \)-category. Moreover, there is a six functor formalism for \( \text{SH}^G(\_\_\_) \) that fulfills properties such as base change, localization sequences and projection formulae, see [26, Theorem 1.1].

In a next step, we pass from the stable homotopy category to \( K \)-motives. By [27], for each \( G \)-variety \( X \), there is a \( E_\infty \)-algebra \( KGL_X^G \in \text{SH}^G(X) \) representing homotopy invariant \( G \)-equivariant \( K \)-theory and we define the category of \( G \)-equivariant \( K \)-motives on \( X \) as

\[
\text{DK}^G(X) \overset{\text{def}}{=} \text{Mod}_{KGL_X^G}(\text{SH}^G(X))
\]

the closed symmetric monoidal stable \( \infty \)-category of \( KGL_X^G \)-modules in \( \text{SH}^G(X) \). The category of \( K \)-motives can be defined over any coefficient ring \( \Lambda \) via

\[
\text{DK}^T(X, \Lambda) \overset{\text{def}}{=} \text{DK}^T(X) \otimes_\mathbb{Z} \Lambda.
\]

We will mostly suppress the coefficients from the notation and work with \( \Lambda = \mathbb{Q} \) in Sections 3 and 5.

### 2.2 Six functors

By [27, Theorem 1.7], the collection of \( E_\infty \)-algebras \( KGL_X^G \) for all \( G \)-varieties \( X \) is a cocartesian section. That is, for a morphism \( f : X \to Y \) of \( G \)-varieties there is a natural equivalence \( f^* \ KGL_Y^G \cong KGL_X^G \) in \( \text{SH}^G(X) \). This implies that \( \text{DK}^G(X) \) inherits the six functor formalism from \( \text{SH}^G(X) \), see [7, Propositions 7.2.11 and 7.2.18]. We list some of the properties now, see [26, Theorem 1.1].

1. (Pullback and pushforward) For any morphism \( f : X \to Y \) of \( G \)-varieties, there are adjoint pullback and pushforward functors

\[
f_* : \text{DK}^G(Y) \rightleftarrows \text{DK}^G(X) : f_*.
\]

The functor \( f_* \) is monoidal.
(2) (Exceptional pullback and pushforward) For any morphism $f : X \to Y$ of $G$-varieties, there are adjoint exceptional pullback and pushforward functors

$$f_! : DK^G(X) \rightleftarrows DK^G(Y) : f^!.$$

(3) (Proper pushforward) If $f : X \to Y$ is a proper morphism of $G$-varieties, there is a canonical equivalence of functors

$$f_! \simeq f_* : DK^G(X) \to DK^G(Y).$$

(4) (Smooth pullback and Bott periodicity) If $f : X \to Y$ is a smooth morphism of $G$-varieties, there is a canonical equivalence of functors

$$f^! \simeq f^* : DK^G(Y) \to DK^G(X).$$

(5) (Base change) For a Cartesian square of morphism of $G$-varieties

$$\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
p \downarrow & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}$$

there are natural equivalences of functors

$$g^* f_! \simeq p_! q^*$$
and

$$g^! f_* \simeq p_* q^!.$$

(6) (Localization) Let $j : U \hookrightarrow X \hookrightarrow X / U : i$ be a $G$-equivariant open immersion and its closed complement. Then there are homotopy cofiber sequences of functors on $DK^G(X)$

$$j_! j^! \to \text{id} \to i_! i^!$$
and

$$i_! i^! \to \text{id} \to j_! j^*.$$

(7) (Projection formulae) For any morphism of $G$-varieties $f$, there are natural equivalences of functors

$$f_!( - \otimes f^*(-)) \simeq f_!( -) \otimes -, \quad \H om(f_!(-), -) \simeq f_* \H om(-, f^!),$$
and

$$f^! \H om(-, -) \simeq \H om(f^*-,-, f^!-).$$

(8) (Homotopy invariance) If $f : E \to X$ is a $G$-equivariant affine bundle over a $G$-variety $X$, then $f^* \simeq f^! : DK^G(X) \to DK^G(E)$ is fully faithful.

Remark 2.1. We note that the results of [26] work in greater generality. For example, one might work with linearly reductive groups $G$. Moreover, the $G$-quasi-projectivity assumption can be weakened for certain nice groups $G$, see [30].
Remark 2.2. A remarkable property of $K$-motives that is different from motivic sheaves or $\ell$-adic sheaves is Bott periodicity. Namely, the reduced $K$-motive of $\mathbb{P}^1$ is isomorphic to the unit object. This implies that the Tate-twist and shift $(1)[2]$ is isomorphic to the identity in $DK$. Bott periodicity is also reflected in the fact that $f^* \cong f^!$ for smooth maps $f$.

2.3 $K$-motives and $K$-theory

$K$-motives compute homotopy $K$-theory and $G$-theory. In particular, by [27, Proposition 4.6 and Remark 5.7] for a $G$-variety $f : X \to pt$ we get the following equivalences of spectra

$$\text{Maps}_{DK^G(pt)}(1, f_* f^* 1) \cong KH^G(X) \text{ for } X \text{ smooth and}$$

$$\text{Maps}_{DK^G(pt)}(1, f_* f^! 1) \cong G^G(X).$$

Here $KH^G(X) = \text{colim}_{n \in \Delta^o} \mathcal{K}(X \times \mathbb{A}^n)^G$ denotes Weibel’s homotopy $K$-theory spectrum, see [39, section IV.12], which is an $\mathbb{A}^1$-homotopy invariant version of the nonconnective $K$-theory spectrum $\mathcal{K}$ of the category of perfect $G$-equivariant complexes on $X$. Moreover, $G^G(X) = K(\text{Coh}_G(X))$ denotes the $G$-equivariant $G$-theory of $X$ that is the $K$-theory spectrum of the category of $G$-equivariant coherent sheaves on $X$.

In particular, passing to homotopy groups, there are isomorphisms

$$\text{Hom}_{DK^G(pt)}(1, f_* f^* 1(p)[q]) \cong KH^G_2^{2p-q}(X) \text{ for } X \text{ smooth and}$$

$$\text{Hom}_{DK^G(pt)}(1, f_* f^! 1(p)[q]) \cong G^G_2^{2p-q}(X).$$

We note that for regular $X$ the following natural maps are equivalences of spectra

$$K^G(X) \to KH^G(X) \to G^G(X)$$

(3)

where $K^G(X)$ denotes the $K$-theory spectrum of the category of $G$-equivariant perfect complexes.

The usual functorialities of $K$-theory and $G$-theory are induced by the appropriate unit and counit maps of the adjunctions $f^*$, $f_*$ and $f_!$, $f^!$ while making use of the fact that $f_! \cong f_*^!$ for $f$ proper and $f^! \cong f^*$ for $f$ smooth. So $KH^G$ admits arbitrary pullbacks and pushforwards along smooth and proper maps, while $G^G$ admits proper pushforwards and smooth pullbacks.

2.4 $K$-motives, correspondences and convolution

Let $S$ be a smooth $G$-variety, $p_X, p_Y, p_Z : X, Y, Z \to S$ $G$-quasi-projective proper $G$-morphisms such that $X, Y$ and $Z$ are smooth over $pt$. Then one can use base change to identify

$$\text{Hom}_{DK^G(S)}(p_X,1, p_Y,1) \cong G^G_0(X \times_S Y).$$

(4)

Moreover, this identification is compatible with convolution in the following way. Consider the maps

$$X \times_S X \times S Y \times Y \times_S Z \xleftarrow{\delta} X \times S X \times S Y \times S Z \xrightarrow{p} X \times S Z.$$
Then there is a convolution product defined via
\[ \star : G^G_0(X \times_S Y) \times G^G_0(Y \times_S Z) \to G^G_0(X \times_S Z), \alpha \star \beta = p_\delta \delta^*(\alpha \boxtimes \beta). \]

The obvious diagram comparing composition and convolution using the isomorphisms commutes. This is shown in the context of DM and Borel–Moore motivic cohomology in [21]. The same arguments apply to DK and G-theory.

**Remark 2.3.** We briefly discuss why the convolution product \( \star \) is well-defined. For \( \delta^* \) to be well-defined, we need that \( \delta \) is of finite Tor-dimension. As \( Y \) is smooth over \( \text{pt} \) by assumption, the diagonal map \( \Delta : Y \to Y \times Y \) is a regular immersion and hence of finite Tor-dimension. The property is preserved under base change, so the same holds true for \( \delta \). Now \( p_* \) is well-defined because \( p \) is proper. The exterior product \( \boxtimes \) is well-defined unconditionally.

## 3 Preliminaries on Stratified Equivariant \( K \)-Motives

We introduce \( \delta \)-stratified \( G \)-equivariant \( K \)-motives on varieties with \( G \)-equivariant affine stratifications and discuss basis properties, such as the existence of weight structures. In this section, we work with rational coefficients \( \Lambda = \mathbb{Q} \) everywhere, the base field \( k = \mathbb{F}_q \) or \( k = \overline{\mathbb{F}}_q \) and let \( \text{pt} = \text{Spec}(k) \).

### 3.1 Constant Equivariant \( K \)-Motives

For an algebraic group \( G \) over \( k \) we denote by \( R(G) = K_0(\text{Rep}_k(G)) = K^G_0(\text{pt}) \) the representation ring. Let \( G \) be an algebraic group over \( k \) of multiplicative type. For a \( G \)-variety \( X \) we consider the category of constant equivariant \( K \)-motives

\[ DK^G(X) \subset DK^G(X) \]

as generated by the tensor unit \( 1 \) by finite colimits and retracts.

In some cases, \( DK^G(X) \) admits an explicit description in terms of modules over the representation ring

\[ R(G) \overset{\text{def}}{=} \text{End}_{DK^G(\text{pt})}(1) = K^G_0(\text{pt}) = K_0(\text{Rep}(G)). \]

**Proposition 3.1.** Let \( G \) be a diagonalizable algebraic group and \( V \in \text{Rep}(G) \) Then

\[ \text{Hom}_{DK^G(V)}(1[n], 1) = \begin{cases} R(G) & \text{if } n = 0 \text{ and } \\ 0 & \text{else.} \end{cases} \]

**Proof.** By homotopy invariance for \( DK^G \), we can assume that \( V = \text{pt} \) with the trivial \( G \) action. In this case,

\[ \text{Hom}_{DK^G(\text{pt})}(1[n], 1) = K^G_n(\text{pt}) = K^G_0(\text{pt}) \otimes \mathbb{Q} K_n(\text{pt}), \]
where the first equality follows from (1) and (3) and the second from [28, Theorem 1.1(b)]. By assumption, \( pt = \text{Spec}(\mathbb{F}_q) \) or \( pt = \text{Spec}(\mathbb{F}_p) \) and we use rational coefficients. Hence, \( K_n(pt) = 0 \) for \( n \neq 0 \) and the statement follows.

The vanishing of \( \text{Hom}_{\text{DKT}^G(V)}(1[n], 1) \) for \( n < 0 \) allows to define the following weight structure (for an overview over weight structures and weight complex functors for \( \infty \)-categories, see [19, section 2.1.3]) on \( \text{DKT}^G(V) \), which exists by [5, Proposition 1.2.3(6)].

**Definition 3.2.** Let \( G \) be a diagonalizable algebraic group and \( V \in \text{Rep}(G) \). The standard weight structure \( w \) on \( \text{DKT}^G(V) \) is defined as the unique weight structure on \( \text{DKT}^G(V) \) with heart \( \text{DKT}^G(V)^{w=0} \) generated by \( 1 \) by finite direct sums and retracts.

The vanishing of \( \text{Hom}_{\text{DKT}^G(V)}(1[n], 1) \) for \( n > 0 \) implies that the weight complex functor

\[
\text{DKT}^G(V) \to \text{Ch}^b(\text{Ho}(\text{DKT}^G(V)^{w=0}))
\]

To the category of chain complexes of the homotopy category of the heart of the weight structure is an equivalence of categories. The category \( \text{Ho}(\text{DKT}^G(V)^{w=0}) \) is equivalent to the category of finitely generated projective \( R(G) \)-modules and hence we obtain:

**Proposition 3.3.** Let \( G \) be a diagonalizable algebraic group and \( V \in \text{Rep}(G) \). There is an equivalence of categories between constant \( G \)-equivariant \( K \)-motives and the perfect derived category of the representation ring \( R(G) \)

\[
\text{DKT}^G(V) \sim \text{D perf}(R(G)).
\]

The description is compatible with pullback/pushforward along surjective \( G \)-equivariant maps using the homotopy invariance of \( \text{DK}^T \).

**Proposition 3.4.** Let \( G \) be a diagonalizable algebraic group and \( f : V \to W \) be a surjective map in \( \text{Rep}(G) \). Then

\[
f^*1 \cong f^!1 \cong 1 \in \text{DK}^G(V) \text{ and } f_*1 \cong f_!1 \cong 1 \in \text{DK}^G(W).
\]

**Proof.** As \( f \) is smooth \( f^* \cong f^! \) which implies the first chain of isomorphisms. The homotopy invariance of \( \text{DK}^G \) implies the second.

**Corollary 3.5.** In the notation of Proposition 3.4, the functors \( f_? \), \( f^? \) are weight exact and there are homotopy commutative diagrams

\[
\begin{array}{ccc}
\text{DKT}^G(V) & \xrightarrow{\alpha} & \text{D perf}(R(G)) \\
\downarrow f_* & & \downarrow \text{id} \\
\text{DKT}^G(W) & \xrightarrow{\alpha} & \text{D perf}(R(G))
\end{array}
\]

for \( ? = *, ! \) where the horizontal maps are induced from the weight complex functor.
Proof. Follows from Proposition 3.4 and the fact that the weight complex functor commutes with weight exact functors, see [35].

3.2 Affine-stratified varieties

In this section, we consider $K$-motives for $G$-varieties with $G$-equivariant affine stratifications, that is, $G$-varieties that are stratified by $G$-representations.

Definition 3.6. Let $G$ be an algebraic group and $X$ a $G$-variety. A $G$-equivariant affine stratification $\mathcal{S}$ is a decomposition

$$X = \bigsqcup_{s \in \mathcal{S}} X_s$$

of $X$ into $G$-invariant locally closed subsets, called *strata*, such that for each $s \in \mathcal{S}$ the closure $\overline{X_s}$ is a union of strata and there is a $G$-equivariant isomorphism $X_s \cong V$ for some $V \in \text{Rep}(G)$. We denote the inclusion of a stratum by $i_s : X_s \hookrightarrow X$.

We need a notion of morphism between $G$-varieties with $G$-equivariant affine stratification, that is built from surjective linear maps of $G$-representations.

Definition 3.7. Let $(X, \mathcal{S})$ and $(Y, \mathcal{S}')$ be $G$-varieties with $G$-equivariant affine stratifications. A $G$-equivariant affine stratified morphism is a $G$-equivariant morphism $f : X \to Y$ such that

1. for each $s \in \mathcal{S}'$, the preimage $f^{-1}(Y_s)$ is a union of strata;
2. for each $X_s$ mapping into $Y_{s'}$, there is a commutative diagram

$$
\begin{array}{ccc}
X_s & \longrightarrow & Y_{s'} \\
\downarrow & & \downarrow \\
V & \longrightarrow & W,
\end{array}
$$

where $V \to W$ is a surjective map in $\text{Rep}(G)$.

We now define $K$-motives that are constant along the strata of a stratification.

Definition 3.8. Let $G$ be a diagonalizable algebraic group and $(X, \mathcal{S})$ a $G$-variety with a $G$-equivariant affine stratification. The category of $\mathcal{S}$-stratified $G$-equivariant $K$-motives on $X$ is the full subcategory

$$\text{DK}_G^{\mathcal{S}}(X) = \{ M \in \text{DK}^G(X) \mid i_s^? \in \text{DKT}^G(X_s) \text{ for } s \in \mathcal{S}, ? = *, ! \}.$$ 

Next, we study well-behaved stratifications.

Definition 3.9. In the notation of Definition 3.8, the stratification is called *Whitney–Tate* if $i_{s,1} \in \text{DK}_G^{\mathcal{S}}(X)$ for all $s \in \mathcal{S}$ and $? = *, !$. 
In the case of a Whitney–Tate stratification, the category $\text{DK}^G_\mathcal{S}(X)$ is generated by the objects $i_{s,1}^*$ (or $i_{s,1}$) under finite colimits and retracts. For example, the Whitney–Tate condition is fulfilled if there are $G$-equivariant affine-stratified resolutions of stratum closures:

**Definition 3.10.** A $G$-variety $(X, \mathcal{S})$ with a $G$-equivariant affine stratification affords $G$-equivariant affine-stratified resolutions if for all $s \in \mathcal{S}$ there is a $G$-equivariant map $p_s : \overline{X}_s \to \overline{X}$, such that

1. $\overline{X}_s$ is smooth projective and has a $G$-equivariant affine stratification,
2. $p_s$ is $G$-equivariant affine-stratified morphism and an isomorphism over $X_s$.

There is a weight structure on constructible equivariant $K$-motives by gluing the standard weight structures on the strata, see Definition 3.2.

**Proposition 3.11.** Let $G$ be a diagonalizable algebraic group and $(X, \mathcal{S})$ a $G$-variety with a Whitney–Tate $G$-equivariant affine stratification. Setting

\[
\text{DK}^G_\mathcal{S}(X)^{w \leq 0} = \{ M \in \text{DK}^G_\mathcal{S}(X) | i_{s,1}^* M \in \text{DKT}^G(X_s)^{w \leq 0} \text{ for all } s \in \mathcal{S} \}
\]

\[
\text{DK}^G_\mathcal{S}(X)^{w \geq 0} = \{ M \in \text{DK}^G_\mathcal{S}(X) | i_{s}^* M \in \text{DKT}^G(X_s)^{w \geq 0} \text{ for all } s \in \mathcal{S} \}
\]

defines a weight structure on $\text{DK}^G_\mathcal{S}(X)$ that we call standard weight structure.

**Proof.** The existence follows from an iterative application of [4, Theorem 8.2.3].

Stratified equivariant $K$-motives and their weight structure are compatible with affine-stratified equivariant maps in the following way.

**Proposition 3.12.** Let $G$ be a diagonalizable algebraic group, $(X, \mathcal{S}), (Y, \mathcal{S}^{'})$ $G$-varieties with Whitney–Tate $G$-equivariant affine stratification and $f : X \to Y$ a $G$-equivariant affine-stratified morphism. Then the following holds.

1. The functors $f^*$, $f_*$, $f_!$, $\otimes$ and $\hom$ preserve $\text{DK}^G_\mathcal{S}$.
2. The functors $f_!, f^*$ preserve nonnegative weights.
3. The functors $f_!, f^*$ preserve nonpositive weights.

**Proof.** Follows as in [15, Propositions 3.8 and 3.12].

The heart of the weight structure can be described in terms of the $K$-motives of resolutions of the closures of the strata.

**Proposition 3.13.** Let $G$ be a diagonalizable algebraic group, $(X, \mathcal{S})$ a $G$-variety with a $G$-equivariant affine stratification that affords $G$-equivariant affine-stratified resolutions $p_s : \overline{X}_s \to \overline{X}$, Then the heart of the weight structure $\text{DK}^G_\mathcal{S}(X)^{w \geq 0}$ is equal to the thick subcategory of $\text{DK}^G_\mathcal{S}(X)$ generated by the objects $p_{s,1}^*$ for $s \in \mathcal{S}$ by finite direct sums and retracts.

**Proof.** By an induction on the number of strata one shows that the objects $p_{s,1}^*$ generate the category $\text{DK}^G_\mathcal{S}(X)$ with respect to finite colimits. By Proposition the objects $p_{s,1}^*$ are contained in
The statement follows from the uniqueness of generated weight structures, see [5, Proposition 1.2.3(6)].

### 3.3 Pointwise purity and weight complex functor

With an additional pointwise purity assumption, stratified equivariant $K$-motives can be described in terms of their weight zero part.

**Definition 3.14.** Let $G$ be a diagonalizable algebraic group, $(X, \mathcal{S})$ be a $G$-variety with Whitney–Tate $G$-equivariant affine stratification. Let $? \in \{*,!\}$. An object $M \in \text{DK}^G_{\mathcal{S}}(X)$ is called $?$-pointwise pure if $i^?_s M \in \text{DK}^G(X)^{w=0}$ for all $s \in \mathcal{S}$. The object is called pointwise pure if it is $?$-pointwise pure for both $? = * ,!$.

**Proposition 3.15.** In the notation of Definition 3.14, let $M, N \in \text{DK}^G(X)$ be $*$- and $!$-pointwise pure, respectively, then $\text{Hom}_{\text{DK}^G(X)}(M, N[n]) = 0$ for all $n \neq 0$.

**Proof.** Follows by an induction on the number of strata and Proposition 3.1, see [15, Lemma 3.16].

**Theorem 3.16.** In the notation of Definition 3.14, assume that all objects in $\text{DK}^G_{\mathcal{S}}(X)^{w=0}$ are pointwise pure. Then the weight complex functor is an equivalence of categories

$$\text{DK}^G_{\mathcal{S}}(X) \to \text{Ch}^b(\text{Ho} \text{DK}^G_{\mathcal{S}}(X)^{w=0}).$$

The assumptions of Theorem 3.16 are, for example, fulfilled if there are $G$-equivariant stratified resolutions of stratum closures.

**Proposition 3.17.** Under the assumptions of 3.13, all objects in $\text{DK}^G_{\mathcal{S}}(X)^{w=0}$ are pointwise pure.

**Proof.** The generators $p_{s,!*1}$ of $\text{DK}^G_{\mathcal{S}}(X)^{w=0}$ are pointwise pure by base change and $p_{s,!} = p_{s,*}$, see [15, Proposition 3.15].

### 4 $K$-THEORY SOERGEL BIMODULES

The goal of this section is to define $K$-theory Soergel bimodules. Similarly to usual, cohomological, Soergel bimodules, they arise from the equivariant $K$-theory of Bott–Samelson resolutions of Schubert varieties. We start the section with basic notations and results on representation rings and the equivariant $K$-theory of flag varieties and Bott–Samelson varieties. Here, $\Lambda$ is any ring of coefficients and $k$ any base field.

#### 4.1 Flag varieties and Bott–Samelson varieties

Let $G \supset B \supset T$ be a split reductive connected group over $k$ that has a simply connected derived group with a Borel subgroup $B$ and maximal torus $T$. Denote by $\text{Hom}(T, G_m) = X(T) \supset \Phi$ the
Denote by $\mathcal{W} = N_G(T)/T$ the Weyl group, $S \subset \mathcal{W}$ the set of simple reflections with respect to $B$ and by $w_0 \in \mathcal{W}$ the longest element. Let $U \subset B$ the unipotent radical, $U^- = U^{w_0}$ its opposite and $U_w = U \cap wU^{-w^{-1}}$ for $w \in \mathcal{W}$.

Let $\Phi^+ \subset \Phi$ be the set of roots that appear in the tangent space of $G/B$, so the roots that appear in $\text{Lie}(U^-)$. We make this nonstandard choice of positive roots to obtain a nice Weyl character formula.

We consider the Bruhat stratification of the flag variety $X = G/B$

$$X = \bigsqcup_{w \in \mathcal{W}} X_w$$

into $B$-orbits $X_w = BwB/B$ called Bruhat cells. For $w \in \mathcal{W}$ there is a $T$-equivariant isomorphism $U_w \to X_w, u \mapsto uwB/B$ where $T$ acts on $U_w$ by conjugation and on $X_w$ by left multiplication. There is an isomorphism $U_w = \mathfrak{a}_T^{w(e(w))}$ and the action of $T$ on $U_w$ is linear with set of characters $\Phi^- \cap w(\Phi^+)$. For a simple reflection $s \in S$, let $P_s = B \cup BsB \subset G$ denote the associated parabolic subgroup. For a sequence of simple reflections $x = (s_1, \ldots, s_n) \in S^n$ denote the associated Bott–Samelson variety and map to the flag variety by

$$p_x : \text{BS}(x) = P_{s_1} \times^B \cdots \times^B P_{s_n} / B \to X, [p_1, \ldots, p_n] \mapsto p_1 \cdot \cdots \cdot p_n B/B.$$ 

The variety $\text{BS}(x)$ is smooth projective and arises as quotient of $P_{s_1} \times \cdots \times P_{s_n}$ by the action of $B^n$ via

$$(b_1, \ldots, b_n) \cdot (p_1, \ldots, p_n) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \ldots, b_{n-1} p_n b_n^{-1}).$$

The torus $T$ acts on $\text{BS}(x)$ from the left and there is a $T$-equivariant affine stratification on $\text{BS}(x)$ indexed by the $2^n$ subsequences of $x$. Moreover, the map $p_x$ is a $T$-equivariant affine-stratified map in the sense of Definition 3.7, see [23, Proposition 2.1] and [24, Proposition 3.0.2].

## 4.2 Representation rings and Frobenius extensions

We now discuss various representation rings and their relation to each other. The discussion mostly follows [29] and [8]. We denote by

$$R = R(T) = K_T^0(\text{pt}) = \Lambda[X(T)]$$

the representation ring of $T$. For a character $\lambda \in X(T)$, we write $e^\lambda$ for the corresponding element in $R$. This way, $e^\lambda = [k_\lambda]$ denotes the class of the one-dimensional representation $k_\lambda$ on which $T$ acts via $\lambda$. The ring $R(T)$ is isomorphic to a Laurent polynomial ring in $\text{rank}(T)$ many variables. Moreover, there is a natural action of $\mathcal{W}$ on $R(T)$.

The representation ring $R(G) = K_G^0(\text{pt})$ is related to $R = R(T)$ via two natural maps

$$\text{Ind}_T^G : R(T) \hookrightarrow R(G), \quad \text{Res}_T^G : R(G) \twoheadrightarrow R(T).$$

We will describe these maps in detail and see that they form a Frobenius extension.

The map $\text{Res}_T^G$ is an injective algebra homomorphism defined by restricting a $G$-representation to $T$. The image of $\text{Res}_T^G$ are exactly the $\mathcal{W}$-invariants and we hence identify $R(G) = R^\mathcal{W} \subset R$. 

The map $\text{Ind}_T^G$ is obtained by inducing representations from $T$ to $G$. It maps the class $[V] \in R(T)$ of a representation $V$ of $T$ to the alternating sum of the cohomology groups of the $G$-equivariant vector bundle $G \times^B V$ on the flag variety $X$, where $B$ acts on $V$ via the quotient map $B \to T$. Namely, $\text{Ind}_T^G$ is the composition of maps

$$\begin{align*}
R &= R(T) \xrightarrow{\sim} K_0^G(X) \xrightarrow{\pi_*} K_0^G(\text{pt}) = R(G) = R^W \\
[V] &\mapsto [G \times^B V] \mapsto \sum_i (-1)^i [H^i(X, G \times^B V)]
\end{align*}$$

where the first isomorphism comes from the induction equivalence

$$R(T) = K_0^T(\text{pt}) \cong K_0^B(\text{pt}) \cong K_0^G(G \times^B \text{pt}) = K_0^G(X)$$

and $\pi_*$ is pushforward along the projection map $\pi : X \to \text{pt}$.

The map $\text{Ind}_T^G$ is an $R(G)$-module homomorphism. Namely, let $[V'] \in R(G)$ be the class of a representation of $G$. Then there is a $G$-equivariant trivialization

$$G \times^B V' \sim G/B \times V', [g, v] \mapsto (gB, gv),$$

which shows that $\text{Ind}_T^G(\text{Res}_G^T[V']) = [V']$. Similarly, if $[V] \in R(T)$ is the class of a representation $V$ of $T$, then $G \times^B (V' \otimes V)$ is the tensor product of the vector bundles $G \times^B V' \cong G/B \times V'$ and $G \times^B V$. It follows that $\text{Ind}_T^G(\text{Res}_G^T([V'])[V]) = [V'] \text{Ind}_T^G([V])$.

The Weyl character formula allows to explicitly compute $\text{Ind}_T^G$ as

$$\text{Ind}_T^G(e^\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}),$$

where $w \cdot \lambda = w(\lambda + \rho) - \rho$ denotes the dot-action of $W$ and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ is the half-sum of all positive roots.

The map $\text{Ind}_T^G$ induces a pairing

$$\langle , \rangle : R(T) \otimes R(T) \to R(G), \langle [V], [V'] \rangle = \text{Ind}_T^G([V][V']) = \pi_*(G \times^B (V \otimes V')).$$

There is an $R(G)$-basis $\{ e_w \}_{w \in W}$ of $R(T)$ constructed in [36] such that

$$\det(\langle e_w, e_{w'} \rangle)_{w,w'} = 1,$$

see [29, Proposition 1.6]. This implies that there is a dual basis $\{ e_w^* \}_{w \in W}$ such that $\langle e_w, e_{w'}^* \rangle = \delta_{w,w'}$. Hence, $\text{Res}_G^T$ and $\text{Ind}_T^G$ form a Frobenius extension. It follows that the functors

$$R(T) \otimes_{R(G)} - : \text{Mod}_{R(G)} \cong \text{Mod}_{R(T)} : \text{Hom}_{R(T)}(R(T), -)$$

are adjoint in both ways.

We remark that the discussion also applies to standard parabolic subgroups $B \subset P \subset G$ by taking $G = L = P / \text{Rad}_u(P)$ as Levi factor of $P$ and using that $R(P) \cong R(L)$ and $P/B \cong L/(B \cap L)$.
4.3  The rank two case

The previous discussion specializes to the following formulas for minimal parabolics \( P_s = B \cup BsB \subset G \) for simple reflections \( s \in S \). Namely, one can identify \( R(P_s) = R^s \) and there is a Frobenius extension

\[
\text{Ind}_{P_s}^T : R(T) \leftrightarrow R(P) : \text{Res}_{P_s}^T,
\]

where \( \text{Res}_{P_s}^T \) corresponds to the inclusion \( R^s \subset R \) and \( \text{Ind}_{P_s}^T \) is given by

\[
\text{Ind}_{P_s}^T(e^\lambda) = \frac{e^\lambda - e^{s \cdot \lambda}}{1 - e^{-\alpha_s}} = \frac{e^{\lambda + \alpha_s/2} - e^{s(\lambda) - \alpha_s/2}}{e^{\alpha_s/2} - e^{-\alpha_s/2}},
\]

where \( \alpha_s \in \Phi^+ \) is the simple root corresponding to \( s \). Hence, the \( \Delta_s = \text{Ind}_{P_s}^T \text{Res}_{P_s}^T \) for \( s \in S \) are the Demazure operators on \( R = R(T) \), see [10].

4.4  Equivariant \( K \)-theory of flag and Bott–Samelson varieties

We study the \( T \)-equivariant \( K \)-theory of the flag variety, Bruhat cells and Bott–Samelson varieties. There are isomorphisms

\[
K_0^T(G/B) \cong K_0^{T \times T}(G) \cong R \otimes R \omega R \quad (5)
\]

where the second isomorphism is induced from the pullback \( R \otimes R = K^{T \times T}(pt) \to K^{T \times T}(G) \). In particular, we can interpret modules over \( K_0^T(G/B) \cong R \otimes R \omega R \) as \( R \)-bimodules.

For a stratum \( X_w = BwB/B \subset X \), one has

\[
K_0^T(X_w) \cong K_0^{T \times T}(wT) \cong R_w, \quad (6)
\]

where \( R_w \) is isomorphic to \( R \) as a ring but has a twisted \( R \)-bimodule structure, given by \( r \cdot m = rm \) and \( m \cdot r = mw(r) \) for \( r \in R, m \in R_w \).

Next, we compute the \( T \)-equivariant \( K \)-theory of Bott–Samelson varieties. For this, we make use of the following statement:

**Lemma 4.1.** Let \( B \subset P \subset G \) be a standard parabolic. Let \( Y \) be a \( B \)-variety. Then there is a natural isomorphism

\[
K_0^T(P \times^B Y) \cong R \otimes_{R \omega P} K_0^T(Y).
\]

**Proof.** There is the following chain of isomorphisms

\[
K_0^T(P \times^B Y) \cong R \otimes_{R \omega P} K_0^P(P \times^B Y) \cong R \otimes_{R \omega P} K_0^B(Y) \cong R \otimes_{R \omega P} K_0^T(Y).
\]

The first isomorphism is [8, Theorem 6.1.22], the second the induction equivalence and the last the reduction property.

\[\Box\]
Let \( x = (s_1, \ldots, s_n) \in S^n \) be a sequence of simple reflections. By applying Lemma 4.1 inductively, one obtains that

\[
K_T^T(BS(x)) = K_T^T(P_{s_1} \times_B \cdots \times_B P_{s_n}/B) \cong R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_n}} R
\]

as an \( R \)-bimodule.

### 4.5 K-theory Soergel bimodules

Soergel bimodules arise from (direct summands of) the \( T \)-equivariant cohomology of Bott–Samelson varieties, interpreted as bimodules over the \( T \)-equivariant cohomology ring of a point \( H^*_T(pt) = H^*(BT) \). It is hence natural to define \( K \)-theory Soergel bimodules in the same way, replacing equivariant cohomology by equivariant \( K \)-theory.

**Definition 4.2.** The category of \( K \)-theory Soergel bimodules \( SBim_R \) is the full thick subcategory of the category of \( R \)-bimodules generated by the bimodules

\[
K_T^T(BS(x)) = R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_n}} R
\]

for all sequences \( x = (s_1, \ldots, s_n) \in S^n \) of simple reflections by finite direct sums and retracts.

**Remark 4.3.** In fact, it will turn out that (with rational coefficients) the category is \( SBim_R \) is already generated by the collection of bimodules \( K_T^T(BS(w)) \) for any fixed choice of reduced expressions \( w \) for the elements \( w \in W \). This follows from the geometric description in terms of weight zero \( K \)-motives, see Corollary 5.2, and Proposition 3.2.

### 4.6 K-theory Soergel bimodules via convolution

We will now show how homomorphisms between \( K \)-theory Soergel bimodules can be described via a convolution product. This yields an equivalent definition of the category \( K \)-theory Soergel bimodules via correspondences.

Namely, for two sequences of simple reflections \( x \in S^n \) and \( y \in S^m \) convolution defines a natural map

\[
\operatorname{act} : G_T^T(BS(x) \times_X BS(y)) \to \operatorname{Hom}_{K_T^T(X)}(K_T^T(BS(x)), K_T^T(BS(y))), \beta \mapsto (\alpha \mapsto \alpha \ast \beta),
\]

where \( \alpha \ast \beta = p_\ast \delta^*(\alpha \boxtimes \beta) \) is the convolution of \( \alpha \) and \( \beta \) and the maps \( \delta^* \) and \( p_\ast \) are induced by the diagonal and projection maps

\[
BS(x) \times BS(x) \times_X BS(y) \xleftarrow{\delta} BS(x) \times_X BS(y) \xrightarrow{p} BS(y),
\]

see Subsection 2.4.

**Theorem 4.4.** The map \( \operatorname{act} \) is an isomorphism.
Proof. Step 1: We reduce the statement to the case when $\mathbf{x}$ is the empty sequence and hence $BS(\mathbf{x}) = B/B = X_e$.

For this, let $s \in S$ be a simple reflection and write $sx \in S^{n+1}$ for the concatenation. Then $BS(sx) = P_s \times^B BS(\mathbf{x})$. We abbreviate $P = P_s, M = BS(\mathbf{x})$ and $N = BS(\mathbf{y})$. Our goal is to construct a commutative diagram

$$
\begin{array}{c}
G^B_0((P \times^B M) \times_X N) \xrightarrow{\sim} G^B_0(M \times_X (P \times^B N)) \\
\downarrow{\text{act}} \quad \downarrow{\text{act}}
\end{array}
$$

$$
\begin{array}{c}
\text{Hom}_{R \otimes R}(K^B_0(P \times^B M), K^B_0(N)) \xrightarrow{\sim} \text{Hom}_{R \otimes R}(K^B_0(M), K^B_0(P \times^B N)).
\end{array}
$$

The upper horizontal isomorphism can be constructed as follows. The isomorphism

$$
\phi : (P \times M) \times_X N \to M \times_X (P \times N), (p, m, n) \mapsto (m, p^{-1}, n)
$$

is equivariant with respect to the actions by $B \times B$ on $(P \times M) \times_X N$ and $M \times_X (P \times N)$ given by $(b_1, b_2)(p, m, n) = (b_1 pb_2^{-1}, b_2 m, b_1 n)$ and $(b_1, b_2)(m, p, n) = (b_2 m, b_2 pb_1^{-1}, b_1 n)$, respectively. Hence, $\phi$ induces via the induction equivalence an isomorphism

$$
G^B_0((P \times^B M) \times_X N) = G^B_0((P \times M) \times_X N)
$$

$$
\sim G^B_0(M \times_X (P \times N)) = G^B_0(M \times_X (P \times^B N)).
$$

The lower horizontal isomorphism in the commutative diagram comes from the following chain of isomorphisms

$$
\begin{aligned}
\text{Hom}_{R \otimes R}(K^B_0(P \times^B M), K^B_0(N)) &\sim \text{Hom}_{R \otimes R}(R \otimes_R K^B_0(M), K^B_0(N)) \\
&\sim \text{Hom}_{R \otimes R}(K^B_0(M), K^B_0(N)) \\
&\sim \text{Hom}_{R \otimes R}(K^B_0(M), R \otimes_R K^B_0(N)) \\
&\sim \text{Hom}_{R \otimes R}(K^B_0(M), K^B_0(P \times^B N)).
\end{aligned}
$$

The first and last isomorphisms are given by Lemma 4.1. The second isomorphism is the Hom-tensor adjunction and sends a map $f$ to $(x \mapsto f(1 \otimes x))$. The third isomorphism comes from the Frobenius extension $R^s \subset R$ with trace map $\Delta_s : R \to R_s$ and is given by pushforward along the map $r \otimes y \mapsto \Delta_s(r)y$. In total, the composition of the first two isomorphisms is induced by the pullback along the map $P \times M \to M$. Dually, the composition of the last two isomorphism is induced by the pushforward along the map $P \times N \to N$.

The commutativity of the above square boils down to the commutativity of the diagram

$$
\begin{array}{c}
G^B_0((P \times M) \times_X N) \xrightarrow{\phi^*} G^B_0(M \times_X (P \times N)) \\
\downarrow{\text{act'}} \quad \downarrow{\text{act''}}
\end{array}
$$

$$
\begin{array}{c}
\text{Hom}_{R' \otimes R}(K^B_0(M), K^B_0(N))
\end{array}
$$
where \( \text{act}' \) and \( \text{act}'' \) are defined via the exterior tensor product as well as pullback and pushforward along the diagrams
\[
M \times (P \times M) \times_X N \xleftarrow{\xi} (P \times M) \times_X (P \times M) \times_X N \xleftarrow{\delta} (P \times M) \times_X N \xrightarrow{\rho} N
\]
and
\[
M \times M \times_X (P \times N) \xleftarrow{\delta} M \times_X (P \times N) \xrightarrow{\rho} P \times N \xrightarrow{\pi} N,
\]
respectively. We remark that the pushforwards are well-defined, as they can be represented by pushforwards along proper maps when taking the quotient by the appropriate free \( B \)-action. The two actions clearly agree with respect to the map \( \phi \).

**Step 2:** By the first step, it suffices to show that the map
\[
\text{act} : G^T_0(X_e \times_X \text{BS}(y)) \to \text{Hom}_{K^T_0(X)}(K^T_0(X_e), K^T_0(\text{BS}(y)))
\]
is an isomorphism. To see this, we abbreviate \( N = \text{BS}(y) \) and \( N_w = X_w \times_X N \). Denote by \( i : N_e \to N \leftarrow N \setminus N_e : u \) the inclusions. We identify \( K^T_0(X_e) = R \). Then \( act(\alpha)(1) = i_* \alpha \). Each space \( N_w \) admits a stratification such that the strata are affine bundles over \( X_w \). By the cellular fibration lemma, see [8, Lemma 5.5.1], it follows that \( K^T_0(N_w) = G^T_0(N_w) \) admits a filtration with subquotients of the form \( K^T_0(X_w) = R_w \). Moreover, there is a short exact sequence
\[
0 \to K^T_0(N_e) = G^T_0(N_e) \xrightarrow{i_*} G^T_0(N) \xrightarrow{u^*} G^T_0(N \setminus N_e) \to 0
\]
where \( G^T_0(N \setminus N_e) \) is a successive extension of modules of the form \( R_w \) for \( w \neq e \). In the associated exact sequence
\[
0 \to \text{Hom}(R, G^T_0(N_e)) \to \text{Hom}(R, G^T_0(N)) \to \text{Hom}(R, G^T_0(N \setminus N_e)),
\]
where we abbreviate \( \text{Hom} = \text{Hom}_{K^T_0(X)} \), the right-hand term vanishes because
\[
\text{Hom}_{K^T_0(X)}(R, R_w) = \text{Hom}_{R \otimes R}(R, R_w) = 0
\]
for \( w \neq e \). This implies that \( \text{act} \) is an isomorphism. \( \Box \)

**Remark 4.5.**

(1) The isomorphism \( \text{act} \) is compatible with composition in the following sense. If \( z \in S^k \) is a third sequence of simple reflections, one can define the convolution product
\[
\star : G^T_0(\text{BS}(x) \times_X \text{BS}(y)) \times G^T_0(\text{BS}(y) \times_X \text{BS}(z)) \to G^T_0(\text{BS}(x) \times_X \text{BS}(z))
\]
see Subsection 2.4. By associativity of convolution \( \text{act}(\beta) \circ \text{act}(\alpha) = \text{act}(\alpha \star \beta) \).

(2) The above discussion yields the following equivalent construction of the category \( \text{SBim}_R \). Namely, consider the category of \( K \)-theoretic correspondences of Bott–Samelson resolutions \( \text{KCorr} \) with objects sequences of simple reflections \( x \in S^n \) and morphisms \( \text{Hom}_{\text{KCorr}}(x, y) = K^T_0(\text{BS}(y) \times_X \text{BS}(x)) \). Composition is given by convolution \( \star \). Then the maps \( \text{act} \) define a functor \( \text{KCorr} \to \text{SBim}_R \) that is fully faithful by Theorem 4.4. In fact, the induced functor from the Karoubian envelope \( \text{Kar}(\text{KCorr}) \to \text{SBim}_R \) yields an equivalence of categories.
(3) The category Kar(KCorr) has a more conceptual construction. Namely, there is an equivalence \( \text{DK}^T_K(B)(X)^{w=0} \sim \text{Kar}(K\text{Corr}) \) with the category of weight zero objects in the category of Bruhat-stratified \( T \)-equivariant \( K \)-motives on the flag variety. In this context, Theorem 4.4 can be seen as a \( K \)-theoretic analogue of Soergel’s Erweiterungssatz. It is equivalent to the statement that the functor \( \kappa : \text{DK}^T_K(B)(X)^{w=0} \to \text{SBim}_R \), which sends a \( K \)-motive to its \( K \)-theory, see Remark 5.4, is fully faithful. In fact, our proof of Theorem 4.4 closely follows the proof of the Erweiterungssatz in the context of equivariant motives, see [37, Proposition III.6.11].

5  || **\( K \)-THEORY SOERGEL BIMODULES VIA \( K \)-MOTIVES ON FLAG VARIETIES**

We now combine the results from Section 3 and 4 to obtain a combinatorial description of Bruhat-stratified torus-equivariant \( K \)-motives on flag varieties in terms of (complexes of) \( K \)-theory Soergel bimodules. In this section, our ring of coefficients is \( \mathbb{Q} \) and \( k = \mathbb{F}_q \) or \( \mathbb{F}_p \).

5.1  || **Bruhat-stratified \( K \)-motives**

We continue in the notation of Subsection 4.1. We consider the flag variety \( X = G/B \) with its action by the maximal torus \( T \). By the discussion there, the Bruhat stratification is a \( T \)-equivariant affine stratification of \( X \) in the sense of Definition 3.6 and we denote it by \( (B) \). It hence makes sense to consider the category \( \text{DK}^T_K(B)(X) \) of Bruhat-stratified \( T \)-equivariant \( K \)-motives on the flag variety.

Moreover, for a reduced expression \( w = (s_1, ..., s_n) \) of an element \( w \in \mathcal{W} \), the map \( p_w : \text{BS}(w) \to X \) provides a resolution of singularities of the Schubert variety \( \overline{X}_w \) and hence \( X \) affords \( T \)-equivariant affine-stratified resolutions in the sense of Definition 3.10. This shows that the Bruhat-stratification is Whitney–Tate and that there is a weight structure on \( \text{DK}^T_K(B)(X)^{w=0} \) such that the objects in the heart \( \text{DK}^T_K(B)(X)^{w=0} \) are pointwise pure by Proposition 3.17. Hence, Theorem 3.16 implies the following:

**Theorem 5.1.** The weight complex functor induces an equivalence of categories

\[
\text{DK}^T_K(B)(X) \to \text{Ch}^b(\text{Ho} \text{DK}^T_K(B)(X)^{w=0}).
\]

5.2  || **A combinatorial description**

Let \( x \in S^n \), \( y \in S^m \) be sequences of simple reflections and denote by \( p_x : \text{BS}(x) \to X \) and \( p_y : \text{BS}(y) \to X \) the Bott–Samelson resolutions. Then combining the discussion in Subsection 2.4 and Theorem 4.4, we obtain isomorphisms

\[
\text{Hom}_{\text{DK}^T_K(X)}(p_x, 1, p_y, 1) \sim C^T_0(\text{BS}(x) \times X \text{BS}(y))
\]

\[
\sim \text{Hom}_{K^T_0(X)}(K^T_0(\text{BS}(x)), K^T_0(\text{BS}(y)))
\]
compatible with composition. As the categories $\text{Ho} \text{DK}_T^T(X)^{w=0}$ and $\text{SBim}_R$ are generated by direct sums and direct summands of the objects $p_{x,1}$ and $K_T^T(\text{BS}(x))$, respectively, we obtain the following statement.

**Corollary 5.2.** There is an equivalence of categories

$$\text{Ho} \text{DK}_T^T(X)^{w=0} \sim \text{SBim}_R.$$ 

Together with Theorem 5.1, this yields:

**Corollary 5.3.** There is an equivalence of categories

$$\text{DK}_T^T(X) \sim \text{Ch}^h(\text{SBim}_R).$$

**Remark 5.4.** The equivalence $\text{Ho} \text{DK}_T^T(X)^{w=0} \sim \text{SBim}_R$ can also be constructed via the functor $\mathbb{K} : \text{HoDK}_T(X) \to \text{Mod}_{K_T(X)}, M \mapsto \text{Hom}_{K_T(X)}(1, M)$.

Hence, Corollary 5.2 can be seen as a $K$-theoretic analog of Soergel’s Erweiterungssatz, see also Remark 4.5.

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