Absolute continuity of the best Sobolev constant of a bounded domain

Grey Ercole ∗

Departamento de Matemática - ICEx, Universidade Federal de Minas Gerais,
Av. Antônio Carlos 6627, Caixa Postal 702, 30161-970, Belo Horizonte, MG, Brazil

May 22, 2014

Abstract

Let $\lambda_q := \inf \left\{ \|\nabla u\|_{L^p(\Omega)}^p / \|u\|_{L^q(\Omega)}^p : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}$, where $\Omega$ is a bounded and smooth domain of $\mathbb{R}^N$, $1 < p < N$ and $1 \leq q \leq p^* := \frac{Np}{N-p}$. We prove that the function $q \mapsto \lambda_q$ is absolutely continuous in the closed interval $[1, p^*]$.

2000 Mathematics Subject Classification. 46E35; 35J25; 35J70.

Keywords: Absolute continuity, Lipschitz continuity, $p$-Laplacian, Rayleigh quotient, Sobolev best constants.

1 Introduction.

Let $\Omega$ be a bounded and smooth domain of Euclidean space $\mathbb{R}^N$, $N \geq 2$, and let $1 < p < N$. For each $1 \leq q \leq p^* := \frac{Np}{N-p}$, let $R_q : W_0^{1,p}(\Omega) \setminus \{0\} \to \mathbb{R}$ be the Rayleigh quotient associated with the Sobolev immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. That is,

$$R_q(u) := \left( \int_{\Omega} |\nabla u|^p dx \right) \left( \int_{\Omega} |u|^q dx \right)^{-\frac{p}{q}} = \frac{\|\nabla u\|_p^p}{\|u\|_q^q}$$

where $\|\cdot\|_s := (\int_{\Omega} |\cdot|^s dx)^{\frac{1}{s}}$ denotes the usual norm of $L^s(\Omega)$.

It is well-known that the immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous if $1 \leq q \leq p^*$ and compact if $1 \leq q < p^*$. Hence, there exist

$$\lambda_q := \inf \left\{ R_q(u) : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}, \quad 1 \leq q \leq p^* \quad (1)$$

and $w_q \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$R_q(w_q) = \lambda_q, \quad 1 \leq q < p^*. \quad (2)$$

∗E-mail: grey@mat.ufmg.br. The author was supported by FAPEMIG and CNPq, Brazil.
Since $\mathcal{R}_q$ is homogeneous of degree zero the extremal function $w_q$ for the Rayleigh quotient can be chosen such that $\|w_q\|_q = 1$.

It is straightforward to verify that such a normalized extremal $w_q$ is a weak solution of the Dirichlet problem

$$
\begin{cases}
-\Delta_p u = \lambda_q |u|^{q-2}u & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(3)

for the $p$-Laplacian operator $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$. Hence, classical results imply that $w_q$ can still be chosen to be positive in $\Omega$ and that $w_q \in C^1(\overline{\Omega})$ for some $0 < \alpha < 1$.

In the case $q = p$, the constant $\lambda_p$ is the well-known first eigenvalue of the Dirichlet $p$-Laplacian and $w_p$ is the correspondent eigenfunction $L^p$-normalized.

If $q = 1$ the pair $(\lambda_1, w_1)$ is obtained from the Torsional Creep Problem:

$$
\begin{cases}
-\Delta_p u = 1 & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(4)

In fact, if $\phi_p$ is the torsion function of $\Omega$, that is, the solution of (4), then it easy to check that the only positive weak solution of (3) with $q = 1$ is $\lambda_1^{-\frac{1}{p-1}} \phi_p$. Thus, $w_1 = \lambda_1^{-\frac{1}{p-1}} \phi_p$ and since $\|w_1\|_1 = 1$ one has

$$
\lambda_1 = \frac{1}{\|\phi_p\|_{1}^{p-1}} \text{ and } w_1 = \frac{\phi_p}{\|\phi_p\|_1}.
$$

(5)

In the particular case where $\Omega = B_R(x_0)$, the ball of radius $R > 0$ centered at $x_0 \in \mathbb{R}^N$, the torsion function is explicitly given by $\phi_p(x) = \Phi_p(|x - x_0|)$ where

$$
\Phi_p(r) := \frac{p-1}{p} N r^{-\frac{1}{p-1}} \left( R^\frac{p}{p-1} - r^\frac{p}{p-1} \right), \quad 0 \leq r \leq R.
$$

Hence, for $\Omega = B_R(x_0)$ one obtains

$$
\lambda_1 = \left[ \frac{p + N(p-1)}{\omega_N (p-1)} \right]^{\frac{1}{p-1}} \frac{N}{R^{\frac{p}{p-1}}},
$$

(6)

and

$$
w_1(x) = \frac{p + N(p-1)}{p \omega_N R^N} \left( 1 - \frac{|x - x_0|}{R} \right)^{\frac{p}{p-1}}
$$

where $\omega_N$ is the $N$ dimensional Lebesgue volume of the unit ball $B_1(0)$. (More properties of the torsion function and some of its applications are given in [4] [7].)

In the critical case $q = p^*$ extremals for the Rayleigh quotient exist if the domain is the whole Euclidean space $\mathbb{R}^N$. In fact, in $\mathbb{R}^N$ one has the Sobolev Inequality

$$
S_{p,N} \|u\|_{L^{p^*}(\mathbb{R}^N)} \leq \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N)
$$

(7)

where (see [2]/[9]):

$$
S_{p,N} := \sqrt{\pi} N \frac{1}{p-1} \left( \frac{N-p}{p-1} \right)^{\frac{1}{p}} \left( \frac{\Gamma(N/p)}{\Gamma(1+N/N/p)} \right)^{\frac{1}{p}}
$$

(8)
This property of the critical case (7), that is: 

$$w(x) = a \left(1 + b |x - x_0|^p \right)^{-\frac{N-p}{p}}$$

(9)

for any \(a \neq 0, b > 0\) and \(x_0 \in \mathbb{R}^N\).

A remarkable fact is that for any domain \(\Omega\) (open, but non-necessarily bounded) the Sobolev constant \(S_{p,N}\) is still sharp with respect to the inequality (7), that is:

$$S^p_{p,N} = \lambda_{p^*} := \inf \left\{ \mathcal{R}_{p^*}(u) : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$  

(10)

This property of the critical case \(q = p^*\) may be easily verified by using a simple scaling argument. As a consequence, in this critical case, the only domain \(\Omega\) whose the Rayleigh quotient has an extremal is \(\mathbb{R}^N\). Indeed, if \(w \in W_0^{1,p}(\Omega) / \{0\}\) is an extremal for the Rayleigh quotient in \(\Omega\), then (by extending \(w\) to zero out of \(\Omega\)) \(w\) is also an extremal for the Rayleigh quotient in \(\mathbb{R}^N\). This implies that \(w\) must have an expression as in (7) and hence its support must be the whole space \(\mathbb{R}^N\), forcing thus the equality \(\Omega = \mathbb{R}^N\).

In this paper we are concerned with the behavior of \(\lambda_q\) with respect to \(q \in [1, p^*]\). Thus, we investigate the function \(q \mapsto \lambda_q\) defined by (7). We prove that this function is of bounded variation in \([1, p^*]\), Lipschitz continuous in any closed interval of the form \([1, p^* - \epsilon]\) for \(\epsilon > 0\), and left-continuous at \(q = p^*\). These combined results imply that \(\lambda_q\) is absolute continuous on \([1, p^*]\).

Up to our knowledge, the only result about the continuity of the function \(q \mapsto \lambda_q\) is given in [6] Thm 2.1, where the author proves the continuity of this function in the open interval \((1, p)\) and the lower semi-continuity in the open interval \((p, p^*)\).

Besides the theoretical aspects, our results are also important for the computational approach of the Sobolev constants \(\lambda_q\), since these constants or the correspondent extremals are not explicitly known in general, even for simple bounded domains. For recent numerical approaches related to Sobolev type constants we refer to [1, 5].

This paper is organized as follows. In Section 2 we derive a formula that describes the dependence of \(\mathcal{R}_q\) with respect to \(q\) and obtain, in consequence, the bounded variation of the function \(q \mapsto \lambda_q\) in the closed interval \([1, p^*]\) and also the left-continuity of this function at \(q = p^*\). Still in Section 2 we obtain a upper bound for \(S_{p,N}\) (see (12)) and we also show that for \(1 \leq q < p^*\) the Sobolev constant \(\lambda_q\) of bounded domains \(\Omega\) tends to zero when these domains tend to \(\mathbb{R}^N\).

By applying set level techniques, we deduce in Section 3 some estimates for \(w_q\) and in Section 4 we combine these estimates with the formula derived in Section 2 to prove the Lipschitz continuity of the function \(q \mapsto \lambda_q\) in each closed interval of the form \([1, p^* - \epsilon]\). Our results are in fact proved for the function \(q \mapsto |\Omega|^\frac{1}{p} \lambda_q\) where \(|\Omega|\) denotes the \(N\)-dimensional Lebesgue volume of \(\Omega\). But, of course, they are automatically transferred to the function \(q \mapsto \lambda_q\).
2 Bounded variation and left-continuity

We first describe the dependence of the Rayleigh quotient \( R_q(u) \) with respect to the parameter \( q \).

**Lemma 1** Let \( 0 \neq u \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega) \). Then, for each \( 1 \leq s_1 < s_2 \leq p^* \) one has

\[
|\Omega|^{-\frac{1}{p}} R_{s_2}(u) = |\Omega|^{-\frac{1}{p}} R_{s_1}(u) \exp \left( p \int_{s_1}^{s_2} K(t, u) \frac{dt}{t^2} \right) \tag{11}
\]

where

\[
K(t, u) := \frac{\int_{\Omega} |u|^t \ln |u|^t \, dx}{\|u\|^t_{\mathcal{L}}} + \ln \left( |\Omega| \|u\|^{-t} \right) \geq 0. \tag{12}
\]

Before proving Lemma 1 let us make a technical remark related to the assumptions of this lemma. If \( u \in W_0^{1,p}(\Omega) \) and \( 1 \leq t < p^* \), then

\[
\int_{\Omega} |u|^t \ln |u|^t \, dx = \int_{|u| < 1} |u|^t \ln |u|^t \, dx + t \int_{|u| \geq 1} |u|^t \ln |u| \, dx \leq \frac{|\Omega|}{e} + t \int_{|u| \geq 1} |u|^t \ln |u|^t \frac{|u|^t}{e(p^*-t)} \, dx \leq \frac{|\Omega|}{e} + \frac{p^* \|u\|^{p^*}}{e(p^*-t)} < \infty.
\]

However, we were not able to determine the finiteness of the integral \( \int_\Omega |u|^{p^*} \ln |u| \, dx \) without assuming that \( u \in L^{\infty}(\Omega) \). Fortunately, the assumption \( W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega) \) will be sufficient to our purposes in this paper.

**Proof of Lemma 1** We firstly note that

\[
\frac{d}{dq} \ln \left( \frac{|\Omega|^{\frac{1}{p}}}{\|u\|_{q}} \right) = \frac{d}{dq} \ln |\Omega| \frac{1}{q} \ln \left( \frac{\|u\|_{q}}{\|u\|_{\|u\|_q}} \right)
\]

\[
= - \frac{1}{q^2} \ln |\Omega| - \frac{1}{q^2} \left[ - \ln \|u\|^q_{\Omega} + \int_{\Omega} |u|^q \ln |u|^q \, dx \right] = - \frac{K(q, u)}{q^2}.
\]

Thus, integration on the interval \( [s_1, s_2] \) gives

\[
\frac{|\Omega|^{\frac{1}{p}}}{\|u\|_{s_2}} = \frac{|\Omega|^{\frac{1}{p}}}{\|u\|_{s_1}} \exp \left( - \int_{s_1}^{s_2} K(t, u) \frac{dt}{t^2} \right)
\]

from what (11) follows easily.

Since the continuous function \( h : [0, +\infty) \to \mathbb{R} \) defined by \( h(\xi) := \xi \ln \xi \), if \( \xi > 0 \), and \( h(0) = 0 \) is convex, it follows from Jensen’s inequality that

\[
h \left( |\Omega|^{-1} \int_{\Omega} |u|^t \, dx \right) \leq |\Omega|^{-1} \int_{\Omega} h(|u|^t) \, dx,
\]

thus yielding

\[
\|u\|_{s_2}^t \ln \left( |\Omega|^{-1} \|u\|_{s_1}^t \right) \leq \int_{\Omega} |u|^t \ln |u|^t \, dx,
\]

from what follows that \( K(t, u) \) defined in (12) is nonnegative. \( \square \)
Proposition 2 The function \( q \mapsto |\Omega|^{\frac{p}{p^*}} \lambda_q \) is strictly decreasing in \([1, p^*]\).

Proof. Let \( 1 \leq s_1 < s_2 \leq p^* \) and \( w_{s_1} \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega}) \) the positive and \( L^p \)-normalized extremal of the Rayleigh quotient \( R_{s_1} \). Note from the definition of \( w_{s_1} \) that

\[-\Delta_p w_{s_1} = \lambda_{s_1} w_{s_1}^{s_1-1} \text{ in } \Omega. \quad (13)\]

It follows from Lemma 1 that

\[|\Omega|^p \lambda_{s_1} = |\Omega|^p R_{s_2}(w_{s_1}) \exp \left( p \int_{s_1}^{s_2} K(t, w_{s_1}) \frac{dt}{t^2} \right) \geq |\Omega|^p R_{s_2}(w_{s_1}) > |\Omega|^p \lambda_{s_2}\]

since \( R_{s_1}(w_{s_1}) = \lambda_{s_1}, K(t, w_{s_1}) \geq 0 \) and \( R_{s_2}(w_{s_1}) > \lambda_{s_2} \). We need only to guarantee the strictness of the last inequality. Obviously, if \( s_2 = p^* \) the inequality is really strict because the Rayleigh quotient \( R_{p^*} \) does not reach a minimum value. Thus, let us suppose that \( \lambda_{s_2} = R_{s_2}(w_{s_1}) \) for \( s_2 < p^* \). Then

\[-\Delta_p (w_{s_1} / \|w_{s_1}\|_{s_2}) = \lambda_{s_2} (w_{s_1} / \|w_{s_1}\|_{s_2})^{s_2-1}\]

and hence the \((p-1)\)-homogeneity of the operator \( \Delta_p \) yields

\[-\Delta_p w_{s_1} = \lambda_{s_2} \|w_{s_1}\|_{s_2}^{p-s_2} w_{s_1}^{s_2-1} \text{ in } \Omega. \quad (14)\]

The combining of (13) with (14) produces

\[w_{s_1} = \left( \frac{\lambda_{s_1} \|w_{s_1}\|_{s_2}^{p-s_2}}{\lambda_{s_2}} \right)^{\frac{1}{s_2-1}} \text{ in } \Omega.\]

Since the only constant function in \( W_0^{1,p}(\Omega) \) is the null function we arrive at the contradiction \( 0 \equiv w_{s_1} > 0 \) in \( \Omega \).

Thus, we have concluded that \( |\Omega|^{\frac{p}{p^*}} \lambda_{s_1} > |\Omega|^{\frac{p}{p^*}} \lambda_{s_2} \) for \( 1 \leq s_1 < s_2 \leq p^* \). \( \square \)

The following corollary is immediate after writing \( \lambda_q \) as a product of two monotonic functions: \( \lambda_q = |\Omega|^{\frac{p}{p^*}} (|\Omega|^{\frac{q}{q^*}} \lambda_q) \).

Corollary 3 The function \( q \mapsto \lambda_q \) is of bounded variation in \([1, p^*]\).

Another consequence of Proposition 2 is that for each \( 1 \leq q < p^* \) the Sobolev constant \( \lambda_q \) of a bounded domain \( \Omega \) tends to zero as \( \Omega \nearrow \mathbb{R}^N \). In fact, this asymptotic behavior follows from the following corollary.

Corollary 4 Let \( B_R(x_0) \subset \mathbb{R}^N \) denote the ball centered at \( x_0 \) and with radius \( R \) and let

\[\lambda_q(R) := \min \left\{ \frac{\|\nabla u\|_{L^p(B_R(x_0))}^p}{\|u\|_{L^q(B_R(x_0))}^q} : u \in W_0^{1,p}(B_R(x_0))/\{0\} \right\}, \quad 1 \leq q < p^*.\]

Then

\[\lambda_q(R) \to 0 \quad \text{as} \quad R \to \infty. \quad (15)\]
Proof. It follows from Proposition 2 that
\[ \lambda_q(R) \leq \lambda_1(R)(\omega_N R^N)^{p(1 - \frac{1}{q})} \]
where, as before, \( \omega_N = |B_1(0)| \).
Now, replacing \( \lambda_1(R) \) by its expression (6) we obtain
\[ \lambda_q(R) \leq \left[ \frac{p + N(p - 1)}{\omega_N(p - 1)} \right]^{p-1}\frac{N(\omega_N)^{p(1 - \frac{1}{q})}}{R(N-p)(\frac{p}{q}-1)}, \tag{16} \]
yielding (15). \hfill \Box

Remark 5 Since \( \lambda_{p^*}(R) \equiv S_{N,p^*}^p, \omega_N = \pi N/2 / \Gamma(1 + N/2) \) and \( \frac{1}{p} - \frac{1}{p'} = \frac{1}{q} \), by making \( q = p^* \) in (16) we obtain the following upper bound for \( S_{N,p} \) with is quite comparable with the expression (8):
\[ S_{N,p} \leq \sqrt{\pi N\frac{1}{2}} \left( \frac{N - p}{p - 1} \right)^{\frac{p-1}{p}} \frac{(p - 1)^{\frac{p-1}{p}}}{\Gamma(1 + N/2)^{\frac{1}{2}}} \tag{17} \]

We now prove the left-continuity of the function \( q \mapsto \lambda_q \) in the interval \((1, p^*)\). Hence, as a particular case we obtain
\[ \lim_{q \to (p^*)^{-}} \lambda_q = \lambda_{p^*} \quad (= S_{p,N}^p). \tag{18} \]

Theorem 6 For each \( q \in (1, p^*) \) it holds \( \lim_{s \to q^-} \lambda_s = \lambda_q \).

Proof. Let us fix \( s < q \) and \( u \in C_{c}^\infty(\Omega) \setminus \{0\} \). If follows from Lemma 1 and Proposition 2 that
\[ |\Omega|^{\frac{p}{q}} \lambda_q < |\Omega|^{\frac{p}{q}} \lambda_s \leq |\Omega|^{\frac{p}{q}} \mathcal{R}_s(u) = |\Omega|^{\frac{p}{q}} \mathcal{R}_q(u) \exp \left( p \int_{s}^{q} \frac{K(t,u)}{t^2} \, dt \right). \tag{19} \]

For \( s \leq t \leq q \) Hölder’s inequality implies that
\[ |\Omega|^{-\frac{1}{q}} \|u\|_s \leq |\Omega|^{-\frac{1}{q}} \|u\|_t \leq |\Omega|^{-\frac{1}{q}} \|u\|_q. \]
Hence, since \( |\Omega|^{-\frac{1}{q}} \|u\|_s \to |\Omega|^{-\frac{1}{q}} \|u\|_q \) as \( s \to q \) we obtain
\[ \frac{|\Omega|^{-\frac{1}{q}} \|u\|_q}{2} \leq |\Omega|^{-\frac{1}{q}} \|u\|_s \leq |\Omega|^{-\frac{1}{q}} \|u\|_t \leq |\Omega|^{-\frac{1}{q}} \|u\|_q \]
for \( s \leq t \leq q \) with \( s \) sufficiently close to \( q \).

It follows from these estimates that
\[ K(t,u) = \frac{t \int_{\Omega} |u|^t \ln |u| \, dx}{\|u\|^t} + t \ln \left( \frac{\|u\|^t}{\|u\|_t} \right) \leq \frac{t \ln \|u\|_{\infty}}{\|u\|^t} \int_{\Omega} |u|^t \, dx \quad + \quad t \ln \left( \frac{2 |\Omega|^{\frac{1}{q}}}{\|u\|^t} \right) = t M_q(u). \]
Therefore,
\[
\exp\left( p \int_s^q K(t, u) \frac{dt}{t^2} \right) \leq \exp\left( p M_q(u) \ln\left( \frac{q}{s} \right) \right) = \left( \frac{q}{s} \right)^{p M_q(u)}
\]
and (19) yields
\[
|\Omega|^{\frac{s}{p}} \lambda_q \leq |\Omega|^{\frac{q}{p}} \lambda_s \leq |\Omega|^{\frac{q}{p}} R_q(u) \left( \frac{q}{s} \right)^{p M_q(u)}.
\]

By making \( s \to q^- \) we conclude that
\[
\lambda_q \leq \liminf_{s \to q^-} \lambda_s \leq \limsup_{s \to q^-} \lambda_s \leq R_q(u)
\]
for each \( u \in C^\infty_c(\Omega) \setminus \{0\} \). Since \( C^\infty_c(\Omega) \) is dense in \( W^{1, p}_0(\Omega) \) this clearly implies that
\[
\lambda_q \leq \liminf_{s \to q^-} \lambda_s \leq \limsup_{s \to q^-} \lambda_s \leq R_q(u) \quad \text{for all } u \in W^{1, p}_0(\Omega) \setminus \{0\}.
\]

Therefore,
\[
\lambda_q \leq \liminf_{s \to q^-} \lambda_s \leq \limsup_{s \to q^-} \lambda_s \leq \lambda_q
\]
from what follows that \( \lim_{s \to q^-} \lambda_s = \lambda_q \). \(\square\)

## 3 Bounds for \( w_q \)

In this section we deduce some bounds for the extremal \( w_q \) defined by (2). Our results are based on level set techniques and inspired by [3] and [8].

**Proposition 7** Let \( 1 \leq q < p^* \) and \( \sigma \geq 1 \). Then, it holds
\[
2^{\frac{N(p-1)+p}{p}} C_q \|w_q\|_\infty^{\frac{N(p-1)+p}{p}} \leq \|w_q\|_C^p
\]
where
\[
C_q := \left( \frac{p}{p + N(p-1)} \right)^{N+1} \left( \frac{S^p_{N,p}}{\lambda_q} \right)^{\frac{p}{p}}.
\]

**Proof.** Since \( w_q \) is a positive weak solution of (3) we have that
\[
\int_{\Omega} |\nabla w_q|^{q-2} \nabla w_q \cdot \nabla \phi dx = \lambda_q \int_{\Omega} w_q^{\lambda_q - 1} \phi dx
\]
for all test function \( \phi \in W^{1,p}_0(\Omega) \).

For each \( 0 < t < \|w_q\|_\infty \) define \( A_t = \{ x \in \Omega : w_q > t \} \). Since \( w_q \in C^{1,a}(\Omega) \) for some \( 0 < a < 1 \) it follows that \( A_t \) is open and \( \nabla(w_q - t)^+ = \nabla w_q \) in \( A_t \).
Thus, the function

$$(w_q - l)^+ = \max \{w_q - l, 0\} = \begin{cases} w_q - l, & \text{if } w_q > l \\ 0, & \text{if } w_q \leq l \end{cases}$$

belongs to $W^{1,p}_0(\Omega)$ and by using it as a test function in (22) we obtain

$$\int_{A_t} |\nabla w_q|^p \, dx = \lambda_q \int_{A_t} w_q^{-1} (w_q - t) \, dx \leq \lambda_q \|w_q\|_{\infty}^{q-1} (\|w_q\|_{\infty} - t) |A_t|.$$  \hspace{1cm} (23)

Now, we estimate $\int_{A_t} |\nabla w_q|^p \, dx$ from below. Applying Hölder and Sobolev inequalities we obtain

$$\left( \int_{A_t} (w_q - t) \, dx \right)^p \leq \left( \int_{A_t} (w_q - t)^{p^*} \, dx \right)^{\frac{p}{p^*}} |A_t|^{\frac{p}{p^*} - \frac{p}{p}} \leq S_{N,p}^p |A_t|^{\frac{p}{p^*} - \frac{p}{p}} \int_{A_t} |\nabla w_q|^p \, dx$$

and thus,

$$S_{N,p}^p |A_t|^{\frac{p}{p^*} - \frac{p}{p}} \left( \int_{A_t} (w_q - t) \, dx \right)^p \leq \int_{A_t} |\nabla w_q|^p \, dx.$$  \hspace{1cm} (24)

By combining this inequality with (23) we obtain

$$S_{N,p}^p |A_t|^{\frac{p}{p^*} - \frac{p}{p}} \left( \int_{A_t} (w_q - t) \, dx \right)^p \leq \lambda_q \|w_q\|_{\infty}^{q-1} (\|w_q\|_{\infty} - t) |A_t|.$$  \hspace{1cm} (25)

Since $\frac{1}{p^*} + \frac{1}{N} = \frac{1}{p}$ the previous inequality can be rewritten as

$$\left( \int_{A_t} (w_q - t) \, dx \right)^\frac{N}{p} \leq \left[ \lambda_q S_{N,p}^{-p} \|w_q\|_{\infty}^{q-1} (\|w_q\|_{\infty} - t) \right]^{\frac{N}{p}} |A_t|.$$  \hspace{1cm} (26)

In the sequel we use twice the following Fubini’s theorem: if $u \geq 0$ is measurable, $\sigma \geq 1$, and $E_\sigma = \{x : u(x) > \sigma\}$, then

$$\int_{\Omega} u(x)^\sigma \, dx = \sigma \int_0^\infty \tau^{\sigma-1} |E_\tau| \, d\tau.$$  \hspace{1cm} (27)

Let us define $g(t) := \int_{A_t} (w_q - t) \, dx$. It follows from (27) that

$$g(t) = \int_0^t |\{w_q > t\}| \, d\tau = \int_t^\infty |\{w_q > s\}| \, ds = \int_t^\infty |A_s| \, ds = \int_t^{\|w_q\|_{\infty}} |A_s| \, ds$$

and therefore $g'(t) = -|A_t| \leq 0$. Thus, (24) can be written as

$$\left[ \lambda_q S_{N,p}^{-p} \|w_q\|_{\infty}^{q-1} (\|w_q\|_{\infty} - t) \right]^{\frac{N}{p(N+1)}} \leq -g(t) \frac{N}{p(N+1)} g'(t)$$

and integration over the interval $[t, \|w_q\|_{\infty}]$ produces

$$C_q \|w_q\|_{\infty}^{\frac{N(q-1)}{p}} (\|w_q\|_{\infty} - t) \frac{N(q-1)}{p} \leq g(t)$$  \hspace{1cm} (28)

where $C_q$ is given by (21).
By using the fact that $g(t) \leq (||w_q||_{\infty} - t) |A_1|$ we obtain from (25) that

$$C_q \parallel w_q \parallel_{\infty} \frac{N(q-1)}{p} \left( ||w_q||_{\infty} - t \right) \frac{N(p-1)}{p} \leq |A_1|.$$ 

If $\sigma \geq 1$, multiplying the previous inequality by $\sigma t^{\sigma-1}$ and integrating over the interval $[0, ||w_q||_{\infty}]$, we get

$$C_q \parallel w_q \parallel_{\infty} ^{\frac{N(q-1)}{p}} \sigma \int_0 ^{\parallel w_q \parallel_{\infty}} \left( ||w_q||_{\infty} - t \right) ^{\frac{N(p-1)}{p}} t^{\sigma-1} dt \leq \parallel w_q \parallel_{\sigma} ^{\sigma} \quad (27)$$

since (25) gives

$$\parallel w_q \parallel_{\sigma} ^{\sigma} = \int_\Omega w_q ^{\sigma} dx = \sigma \int_0 ^{\parallel w_q \parallel_{\infty}} t^{\sigma-1} |A_1| dt.$$

The change of variable $t = \tau \parallel w_q \parallel_{\infty}$ produces

$$\int_0 ^{\parallel w_q \parallel_{\infty}} (||w_q||_{\infty} - t) ^{\frac{N(p-1)}{p}} t^{\sigma-1} dt = \parallel w_q \parallel_{\infty} ^{\frac{N(q-1)}{p}} \int_0 ^{1} (1 - \tau) ^{\frac{N(p-1)}{p}} \tau^{\sigma-1} d\tau. \quad (28)$$

Since we have

$$\int_0 ^{1} (1 - \tau) ^{\frac{N(p-1)}{p}} \tau^{\sigma-1} d\tau \geq (1/2) \int_0 ^{\frac{1}{2}} \tau^{\sigma-1} d\tau = \frac{2^{\frac{N(p-1)+p}{p}}}{\sigma},$$

combining this with (27) and (28) produces (20). \hfill \Box

We remark from Theorem 6 that

$$\lim_{q \rightarrow p^*} C_q = \left( \frac{p}{p + N(p-1)} \right) ^{N+1} < 1. \quad (29)$$

Thus it follows from the monotonicity of the function $q \mapsto |\Omega|^\frac{q}{p} \lambda_q$ that both $C_q$ and $(C_q)^{-1}$ are bounded.

**Corollary 8** If $1 \leq q \leq p^* - \epsilon$, then

$$|\Omega|^\frac{q}{p} \leq \parallel w_q \parallel_{\infty} \leq C_\epsilon \quad (30)$$

where $C_\epsilon$ is a positive constant that depends on $\epsilon$ but not on $q$.

**Proof.** The first inequality is trivial, since $\parallel w_q \parallel_q = 1$. Let us suppose that $1 \leq q \leq p$. It follows from (20) with $\epsilon = 1$ that

$$2^{\frac{N(p-1)+p}{p}} C_q \parallel w_q \parallel_{\infty} ^{\frac{N(q-1)+p}{p}} \leq \parallel w_q \parallel_1 \leq \frac{|\Omega|^\frac{q-1}{p}}{\lambda_q} \parallel w_q \parallel_q = |\Omega|^\frac{q-1}{p}.$$

Thus,

$$\parallel w_q \parallel_{\infty} \leq \bar{A} := \max_{1 \leq q \leq p} \left( 2^{\frac{N(p-1)+p}{p}} |\Omega|^\frac{q-1}{p} \right).$$
Now, let us consider \( p \leq q \leq p^* - \epsilon \). Then, by making \( \sigma = q \) in (20) we obtain
\[
2^{-\frac{N(p-1)+qp}{p}} C_q \|w_q\|_\infty^{\frac{N(p-q)+qp}{p}} \leq \|w_q\|_q^q = 1.
\]
that is,
\[
\|w_q\|_\infty \leq \tilde{B}_e := \max_{p \leq q \leq p^* - \epsilon} \left( \frac{2^{-\frac{N(p-1)+qp}{p}}}{C_q} \right)^{\frac{N(p-q)+qp}{p}},
\]
since \( N(p-q)+qp = (N-p)(p^*-q) \).

Therefore, \( \|w_q\|_\infty \leq \max \{ A, \tilde{B}_e \} := C_e. \)

Note from (29) that \( \tilde{B}_e \to \infty \) as \( \epsilon \to 0^+ \).

4 Absolute continuity

In this section we prove our main result: the absolute continuity of the function \( q \mapsto \lambda_q \) in the closed interval \([1, p^*]\). For this we first prove the Lipschitz continuity of the function \( q \mapsto \|\Omega\|^{\frac{q}{p}} \lambda_q \) in each close interval of the form \([1, p^* - \epsilon]\). Obviously, this is equivalent to the Lipschitz continuity of the function \( q \mapsto \lambda_q \) in same interval.

**Theorem 9** For each \( \epsilon > 0 \), there exists a positive constant \( \mathcal{L}_e \) such that
\[
\|\Omega\|^{\frac{p}{q}} \lambda_s - \|\Omega\|^{\frac{p}{q}} \lambda_q \leq \mathcal{L}_e |s - q|
\]
for all \( s, q \in [1, p^* - \epsilon] \).

**Proof.** Without loss of generality let us suppose that \( s < q \). Thus, the monotonicity of the function \( \tau \mapsto \|\Omega\|^{\frac{p}{\tau}} \lambda_\tau \) implies
\[
\|\Omega\|^{\frac{p}{q}} \lambda_s - \|\Omega\|^{\frac{p}{q}} \lambda_q \leq \|\Omega\|^{\frac{p}{q}} \lambda_s - \|\Omega\|^{\frac{p}{q}} \lambda_q.
\]

Take \( t \in \mathbb{R} \) so that \( s \leq t \leq q \). It follows from (20) with \( \sigma = 1 \) that
\[
2^{-\frac{N(p-1)+p}{p}} C_q \|w_q\|_\infty^{1+\frac{N(p-q)}{p}} \leq \|w_q\|_1 \leq \|\Omega\|^{1-\frac{1}{p}} \|w_q\|_1
\]
and therefore
\[
\|\Omega\|^{\frac{1}{p}} \|w_q\|_\infty \leq \frac{2^{-\frac{N(p-1)+p}{p}} \|\Omega\|}{C_q \|w_q\|_\infty^{\frac{N(p-q)}{p}}}
\]
Hence, for \( 1 \leq q \leq p \) the first inequality in (30) gives
\[
\|\Omega\|^{\frac{1}{p}} \|w_q\|_\infty \leq \frac{2^{-\frac{N(p-1)+p}{p}} \|\Omega\|}{C_q \|w_q\|_\infty^{\frac{N(p-q)}{p}}} \leq A := \frac{2^{\frac{N(p-1)+p}{p}}}{C_q} \max_{1 \leq q \leq p} \|\Omega\|^{1+\frac{N(p-q)}{p}}
\]
while for $p \leq q \leq p^* - \epsilon$ the second inequality in (30) gives

$$\frac{\|\Omega^\dagger \|_\infty \|w_q\|_{\ell}}{\|w_q\|_1} \leq 2 \frac{\|\Omega\|_{\ell}^{N(p - 1) + p}}{C_q} \|w_q\|_{\ell}^{N(\lambda p)} \leq B_c := 2 \frac{\|\Omega\|_{\ell}^{N(p - 1) + p}}{p \leq q \leq p^* - \epsilon} \max_{\|w_q\|_1} \frac{C_e}{C_q}.$$

Therefore,

$$\frac{\|\Omega^\dagger \|_\infty \|w_q\|_{\ell}}{\|w_q\|_1} \leq D_c := \max \{A, B_c\}. \quad (31)$$

Thus,

$$K(t, w_q) = t \int \frac{| \omega_q |^t \ln | \omega_q |}{\| \omega_q \|_1} dx + t \ln \left( \frac{\|\Omega^\dagger \|_\infty \|w_q\|_{\ell}}{\|w_q\|_1} \right) \leq t \left( \int \frac{| \omega_q |^t \ln | \omega_q |}{\| \omega_q \|_1} dx \right) + t \ln \left( \frac{\|\Omega^\dagger \|_\infty \|w_q\|_{\ell}}{\|w_q\|_1} \right) = t \ln \left( \frac{\|\Omega^\dagger \|_\infty \|w_q\|_{\ell}}{\|w_q\|_1} \right) \leq t D_c$$

and we obtain

$$\exp \left( p \int_s^q \frac{K(t, w_q)}{t^2} dt \right) \leq \exp \left( p D_c \int_s^q \frac{dt}{t} \right) = \left( \frac{q}{s} \right)^{p D_c}.$$

But

$$|\Omega|^\frac{\ell}{\lambda_q} \leq |\Omega|^\frac{\ell}{\lambda_q} \mathcal{R}(w_q)$$

$$= |\Omega|^\frac{\ell}{\lambda_q} \mathcal{R}(w_q) \exp \left( p \int_s^q \frac{K(t, w_q)}{t^2} dt \right) = |\Omega|^\frac{\ell}{\lambda_q} \lambda_q \exp \left( p \int_s^q \frac{K(t, w_q)}{t^2} dt \right)$$

yields

$$|\Omega|^\frac{\ell}{\lambda_q} \lambda_q - |\Omega|^\frac{\ell}{\lambda_q} \lambda_q \leq |\Omega|^p \lambda_1 \left[ \exp \left( p \int_s^q \frac{K(t, w_q)}{t^2} dt \right) - 1 \right]$$

$$\leq |\Omega|^p \lambda_1 \left[ \exp \left( p \int_s^q \frac{K(t, w_q)}{t^2} dt \right) - 1 \right] \leq |\Omega|^p \lambda_1 \left( \frac{q}{s} \right)^{p D_c} - 1.$$

Therefore,

$$0 \leq \frac{|\Omega|^\frac{\ell}{\lambda_q} \lambda_q - |\Omega|^\frac{\ell}{\lambda_q} \lambda_q}{q - s} \leq \frac{|\Omega|^p \lambda_1 \left( \frac{q}{s} \right)^{p D_c} - 1}{q - s} \leq |\Omega|^p \lambda_1 H\left( \frac{q}{s} \right)$$

where

$$H(\xi) = \frac{\xi^{p D_c} - 1}{\xi - 1}; \quad 1 \leq \xi \leq p^* - \epsilon.$$

Since

$$\lim_{\xi \to 1^+} H(\xi) = p D_c,$$

we conclude that $H$ is bounded in $[1, p^* - \epsilon]$ and thus

$$\exp \left( p \int_s^q \frac{K(t, w_q)}{t^2} dt \right) - 1 \leq \mathcal{L}_c := \max_{1 \leq \xi \leq p^* - \epsilon} H(\xi).$$

Hence, for $1 \leq s < q \leq p^* - \epsilon$ we have

$$|\Omega|^\frac{\ell}{\lambda_q} \lambda_q - |\Omega|^\frac{\ell}{\lambda_q} \lambda_q \leq |\Omega|^\frac{\ell}{\lambda_q} \lambda_q - |\Omega|^\frac{\ell}{\lambda_q} \lambda_q \leq \mathcal{L}_c (q - s) = \mathcal{L}_c |s - q|. \quad \Box$$
Theorem 10 The function $q \mapsto \lambda_q$ is absolutely continuous in $[1, p^*]$.

Proof. According to Corollary 3 the function $q \mapsto \lambda_q$ is of bounded variation. Therefore, its derivative $(\lambda_q)'$ exists almost everywhere in $[1, p^*]$ and it is Lebesgue integrable in this interval. Thus, Lebesgue’s dominated convergence theorem implies that

$$\lim_{q \to p^*} \int_1^q (\lambda_s)' ds = \int_1^{p^*} (\lambda_s)' ds. \quad (32)$$

On the other hand, since Lipschitz continuity implies absolute continuity it follows from Theorem 9 that $\lambda_q$ is absolutely continuous in each interval of the form $[1, p^* - \epsilon]$. Therefore,

$$\lambda_q = \lambda_1 + \int_1^q (\lambda_s)' ds, \quad \text{for } 1 \leq q < p^*. \quad (33)$$

Hence, the left-continuity (18) combined with (32) imply that (33) is also valid for $q = p^*$. We have concluded that $\lambda_q$ is the indefinite integral of a Lebesgue integrable (its derivative) function what guarantees that $\lambda_q$ is absolutely continuous. \(\square\)

References

[1] P. F. Antonietti, A. Pratelli, Finite element approximation of the Sobolev constant, Numer. Math. 117 (2011) 37–64.

[2] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differ. Geom. 11 (1976), 573–598.

[3] C. Bandle, Rayleigh-Faber-Krahn inequalities and quasilinear elliptic boundary value problems, Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday. Vol. 1,2, Kluwer Acad. Publ., Dordrecht, 2003, pp 227–240.

[4] H. Bueno and G. Ercole, Solutions of the Cheeger problem via torsion functions, J. Math. Anal. Appl. 381 (2011) 263–279.

[5] A. Caboussat, R. Glowinski and A. Leornar, Looking for the best constant in a Sobolev inequality: a numerical approach, Calcolo 47 (2010) 211–238.

[6] Y. X. Huang, A note on the asymptotic behavior of positive solutions for some elliptic equation, Nonlinear Analysis TMA 29 (1997), 533–537.

[7] B. Kawohl, On a family of torsional creep problems, J. reine angew. Math. 410 (1990), 1–22.

[8] O. Ladyzhenskaya and N. Uraltseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York-London, 1968.

[9] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976), 353–372.