HÖLDER AND LIPSCHITZ CONTINUITY
OF FUNCTIONS DEFINABLE OVER
HENSELIAN RANK ONE VALUED FIELDS

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ABSTRACT. Consider a Henselian rank one valued field \( K \) of equi-
characteristic zero with the three-sorted language \( \mathcal{L} \) of Denef–Pas.
Let \( f : A \to K \) be a continuous \( \mathcal{L} \)-definable (with parameters) func-
tion on a closed bounded subset \( A \subset K^n \). The main purpose is to
prove that then \( f \) is Hölder continuous with some exponent \( s \geq 0 \)
and constant \( c \geq 0 \); a fortiori, \( f \) is uniformly continuous. Further,
if \( f \) is locally Lipschitz continuous with a constant \( c \), then \( f \) is
(globally) Lipschitz continuous with possibly some larger constant \( d \).
Also stated are some problems concerning continuous and Lip-
schitz continuous functions definable over Henselian valued fields.

1. Introduction

Consider a Henselian rank one valued field \( K \) of equicharacteristic
zero along with the language \( \mathcal{L} \) of Denef–Pas, which consists of three
sorts: the valued field \( K \)-sort, the value group \( \Gamma \)-sort and the residue
field \( k \)-sort. The only symbols of \( \mathcal{L} \) connecting the sorts are the fol-
lowing two maps from the main \( K \)-sort to the auxiliary \( \Gamma \)-sort and
\( k \)-sort: the valuation map \( \nu \) and an angular component map \( \text{ac} \) which
is multiplicative, sends 0 to 0 and coincides with the residue map on
units of the valuation ring \( R \) of \( K \). The language of the \( K \)-sort is the
language of rings; that of the \( \Gamma \)-sort is any augmentation of the lan-
guage of ordered abelian groups (with \( \infty \)); finally, that of the \( k \)-sort
is any augmentation of the language of rings. Throughout the paper
the word ”definable” means ”definable with parameters” and \( K \)-sort
variables are \( x, y, z, \ldots \). We consider \( K^n \) with the product topology,
called the \( K \)-topology on \( K^n \), and adopt the following convention:

\[ |x| = |(x_1, \ldots, x_n)| := \max \{ |x_1|, \ldots, |x_n| \} \]

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and
\[ v(x) = v(x_1, \ldots, x_n) := \min \{ v(x_1), \ldots, v(x_n) \} \]
for \( x = (x_1, \ldots, x_n) \in K^n \).

The main purpose of this paper is to prove the following two theorems on Hölder and Lipschitz continuity.

**Proposition 1.1.** Let \( f : A \to K \) be a continuous \( \mathcal{L} \)-definable function \( f : A \to K \) on a closed bounded subset \( A \subset K^n \). Then \( f \) is Hölder continuous with some exponent \( r \geq 0 \) and constant \( c \geq 0 \), i.e.
\[ |f(x) - f(z)| \leq c|x - z|^r \]
for all \( x, z \in A \).

**Proposition 1.2.** Let \( f : A \to K \) be an \( \mathcal{L} \)-definable (with parameters) function \( f : A \to K \) on a closed bounded subset \( A \subset K^n \). Suppose \( f \) is locally Lipschitz continuous with a constant \( c \geq 0 \), i.e. each point \( a \in A \) has a neighbourhood \( U \) such that
\[ |f(x) - f(z)| \leq c|x - z| \]
for all \( x, z \in U \). Then \( f \) is (globally) Lipschitz continuous with possibly some larger constant \( d \).

An immediate consequence of Proposition 1.1 is the following

**Corollary 1.3.** Every continuous \( \mathcal{L} \)-definable function \( f : A \to K \) on a closed bounded subset \( A \subset K^n \) is uniformly continuous.

Proposition 1.1 follows immediately from a version of the Łojasiewicz inequality established in Section 2, which generalizes the version from our paper [11, Proposition 9.1]. Nevertheless, its formulation is more classical in comparison with the latter and its proof given in Section 2 follows similar arguments. Note that one of its basic ingredients is the closedness theorem from our paper [11, Theorem 3.1]. Proposition 1.2 relies directly on the closedness theorem. Section 3 contains the proofs of the two main theorems.

**Remark 1.4.** Proposition 1.1 is a counterpart of a well known theorem that every continuous semi-algebraic or subanalytic function on a compact subset of \( \mathbb{R}^n \) is Hölder continuous. That theorem remains valid for continuous functions which are definable in polynomially bounded \( \sigma \)-minimal structures (cf. [7]). We thus see that, in a sense, also in algebraic geometry over Henselian rank one valued fields \( K \), closed bounded subsets of \( K^n \) correspond to compact subsets of \( \mathbb{R}^n \).
A different, more delicate problem is to examine locally Lipschitz continuous definable functions on subsets that are not closed. Then the conclusion of global Lipschitz continuity must be replaced by that of piecewise Lipschitz continuity. In the last Section 4, we refer to this problem over Henselian rank one valued fields and also state two extension problems: of Lipschitz continuous rational functions defined on algebraic subvarieties of $K^n$ as well as of continuous and Lipschitz continuous $\mathcal{L}$-definable functions defined on closed subsets of $K^n$.

Finally note that not all valued fields $K$ have an angular component map, but it exists if $K$ has a cross section, which happens whenever $K$ is $\aleph_1$-saturated (cf. [1], Chap. II). Moreover, a valued field $K$ has an angular component map whenever its residue field $k$ is $\aleph_1$-saturated (cf. [7], Corollary 1.6). In general, unlike for $p$-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. For both $p$-adic fields (Denef [5]) and Henselian equicharacteristic zero valued fields (Pas [16]), quantifier elimination was established by means of cell decomposition and a certain preparation theorem (for polynomials in one variable with definable coefficients) combined with each other. In the latter case, however, cells are no longer finite in number, but parametrized by residue field variables.

Nevertheless, all topological results about sets and functions definable in the language of rings augmented by the valuation map remain true even if an angular component does not exist. Indeed, since the $K$-topology induced by the valuation $v$ as well as closure and interior operations are $\mathcal{L}$-definable, the concept of continuity, Lipschitz continuity etc. are first order properties. Therefore elementary extensions can be used in the study of these properties. After replacing a given ground field with an $\aleph_1$-saturated elementary extension, one will thus have an angular component at hand.

2. A VERSION OF THE ŁOJASIEWICZ INEQUALITY

First we remind the reader that Henselian valued fields of equicharacteristic zero admit quantifier elimination and, more precisely, elimination of $K$-quantifiers in the language of Denef–Pas (Pas [16]). (In the case of non-algebraically closed fields, passing to the three sorts with additional two maps: the valuation $v$ and the residue map, is not sufficient.) Next note that every archimedean ordered group $\Gamma$ (which of course may be regarded as a subgroup of the additive group $\mathbb{R}$ of real numbers) admits quantifier elimination in the Presburger language $(<, +, -, 0, 1)$ with binary relation symbols $\equiv_n$ for congruences modulo $n > 1$, $n \in \mathbb{N}$, where 1 denotes the minimal positive element of...
\[ \text{if it exists or } 1 = 0 \text{ otherwise. Thus we can apply quantifier elimination in the } \Gamma\text{-sort whenever } K \text{ is a Henselian rank valued field of equicharacteristic zero.} \]

Here and in the next section, we shall still need the following easy consequence of the closedness theorem.

**Proposition 2.1.** Let \( f : A \to K \) be a continuous \( L \)-definable function on a closed bounded subset \( A \subset K^n \). Then \( f \) is a bounded function, i.e. there is an \( \alpha \in \Gamma \) such that \( v(f(x)) \geq \alpha \) for all \( x \in A \).

\[ \blacksquare \]

Now we can readily prove the following version of the Lojasiewicz inequality, which is a generalization of the one from \[11\] Proposition 9.1.

**Proposition 2.2.** Let \( f, g_1, \ldots, g_m : A \to K \) be continuous \( L \)-definable functions on a closed (in the \( K \)-topology) bounded subset \( A \subset K^n \). If \( \{ x \in A : g_1(x) = \ldots = g_m(x) = 0 \} \subset \{ x \in A : f(x) = 0 \} \), then there exist a positive integer \( s \) and a constant \( c \geq 0 \) such that

\[ |f(x)|^s \leq c \cdot |(g_1(x), \ldots, g_m(x))| \]

for all \( x \in A \).

**Proof.** Put \( g = (g_1, \ldots, g_m) \). It is easy to check that the set

\[ A_\gamma := \{ x \in A : v(f(x)) = \gamma \} \]

is a closed \( L \)-definable subset of \( A \) for every \( \gamma \in \Gamma \). By the hypothesis and the closedness theorem \[11\] Theorem 3.1, the set \( g(A_\gamma) \) is a closed \( L \)-definable subset of \( K^m \setminus \{0\} \), \( \gamma \in \Gamma \). The set \( v(g(A_\gamma)) \) is thus bounded from above, i.e.

\[ v(g(A_\gamma)) \leq \alpha(\gamma) \]

for some \( \alpha(\gamma) \in \Gamma \). By elimination of \( K \)-quantifiers, the set

\[ \Lambda := \{(v(f(x)), v(g(x))) \in \Gamma^2 : x \in A \} \subset \{ (\gamma, \delta) \in \Gamma^2 : \delta \leq \alpha(\gamma) \} \]

is a definable subset of \( \Gamma^2 \) in the Presburger language, and thus it is described by a finite number of linear inequalities and congruences. Hence

\[ \Lambda \cap \{ (\gamma, \delta) \in \Gamma^2 : \gamma > \gamma_0 \} \subset \{ (\gamma, \delta) \in \Gamma^2 : \delta \leq s \cdot \gamma \} \]

for a positive integer \( s \) and some \( \gamma_0 \in \Gamma \). We thus get

\[ v(g(x)) \leq s \cdot v(f(x)) \text{ if } x \in A, v(f(x)) > \gamma_0. \]

Again, by the hypothesis, we have

\[ g(\{ x \in A : v(f(x)) \leq \gamma_0 \}) \subset K^m \setminus \{0\}. \]
Therefore it follows from the closedness theorem that the set
\[
\{ v(g(x)) \in \Gamma : v(f(x)) \leq \gamma_0 \}
\]
is bounded from above, say, by a constant \( \delta \). Hence and by Proposition 2.1, we get
\[
s \cdot v(f(x)) - v(g(x)) \geq \beta := \min \{ 0, s \cdot \alpha - \delta \}
\]
for all \( x \in A \). This is the desired conclusion formulated in terms of valuation with constant \( c := \exp(-\beta) \). \( \Box \)

3. Proofs of the main results

Proof of Proposition 1.1. Apply Proposition 2.2 to the functions
\[
f(x) - f(y) \quad \text{and} \quad g_i(x, y) = x_i - y_i, \ i = 1, \ldots, n.
\]
\( \Box \)

Proof of Proposition 1.2. Let \( \mathbb{P}^1(K) \) stand for the projective line over \( K \). Define an \( L \)-definable subset \( E \) of \( A \times A \times \mathbb{P}^1(K) \) by putting \((x, z, u) \in E \) iff
\[
u(u) = f(x) - f(z)\]
for some \( i = 1, \ldots, n \) such that \( x_i \neq z_i \) and, moreover, the value \( v(u) \) is the largest from among the values \( v(w_j) \) of the fractions
\[
\frac{f(x) - f(z)}{x_j - z_j}, \ j = 1, \ldots, n,
\]
with \( x_j \neq z_j \). Let \( \overline{E} \) be the closure of \( E \) in \( A \times A \times \mathbb{P}^1(K) \). Denote by
\[
\phi : K^n \times K^n \times \mathbb{P}^1(K) \to \mathbb{P}^1(K)
\]
the canonical projection. Again, by the closedness theorem, the image \( \phi(\overline{E}) \) is a closed subset of \( \mathbb{P}^1(K) \). But by Proposition 2.1, the function \( f \) is bounded. Therefore, since \( f \) is locally Lipschitz continuous with a fixed constant \( c \), it is not difficult to deduce that \( \overline{E} \) is actually a subset of \( A \times A \times K \). Thus the image \( \phi(\overline{E}) \) is a subset of \( K \) and a closed subset of \( \mathbb{P}^1(K) \). Hence \( \phi(\overline{E}) \) is a bounded subset of \( K \), i.e. \( v(\phi(\overline{E})) \geq \delta \) for some \( \delta \in \Gamma \). Then \( d := \exp(-\delta) > 0 \) is the Lipschitz constant we are looking for. \( \Box \)
4. SOME PROBLEMS ON CONTINUOUS AND LIPSCHITZ CONTINUOUS FUNCTIONS DEFINABLE OVER HENSELIAN VALUED FIELDS

Fix a Henselian rank one valued field $K$ of equicharacteristic zero. It is more delicate to examine Lipschitz continuous definable functions whose domains are non-closed subsets of $K^n$. Then the conclusion of global Lipschitz continuity must be replaced by that of piecewise Lipschitz continuity. One of the most fundamental questions is just the following problem concerning the latter property.

**Problem 1.** Consider a semi-algebraic (or $\mathcal{L}$-definable or, more generally, definable in a suitable language) function $f : A \rightarrow K$ with $A \subset K^n$, and suppose that $f$ is locally Lipschitz continuous with a Lipschitz constant $c$. Then is $f$ piecewise Lipschitz continuous with possibly some larger constant $d$? In other words, does there exist a finite semi-algebraic (or $\mathcal{L}$-definable etc.) partition $\{A_1, \ldots, A_s\}$ of $A$ such that the restriction of $f$ to each subset $A_i$ is Lipschitz continuous with some constant $d$? Further, what can one say about a new constant $d$ in comparison to $d$? Is it possible to take the same constant $d = c$?

**Remark 4.1.** Here a suitable language may be, in the first place, one linked with so called topological systems in the sense of van den Dries [6]. That notion seems to be very useful in relating definable topologies and model theoretical treatment. In particular, the language of Denef–Pas can be translated into such a language of topological system on a given Henselian valued field $K$ with the topology induced by the valuation (by adding the inverse images under the valuation and angular component map of relations on the value group and residue field).

Over the real number field $\mathbb{R}$, the affirmative answer to Problem 1 for semi-algebraic and (globally) subanalytic functions was given by Kurdyka [11] and Parusiński [13, 14]; an o-minimal version over real closed fields was presented by Pawlucki [15]. Moreover, one can take $d = Mc$ where the constant $M > 1$ depends only on the dimension $n$ of the ambient affine space. Finally, let us mention that one of Kurdyka’s results, namely [11, Corollary C], was inspired by a question of Professor Łojasiewicz.

Over the $p$-adic number fields $\mathbb{Q}_p$ (or their finite extensions), the affirmative answer for semi-algebraic and subanalytic functions was given by Cluckers–Comte–Loeser [2, Theorem 2.1]. Their proof relies on a certain compatible cell decomposition for the function $f$. It takes into account some comparison of distances from the centers of cells in the domain and image of $f$, which is made by means of a Jacobian property for definable functions. This result is crucial for the theory
of local density and local metric properties of \( p \)-adic definable sets, developed in the paper \([3]\). But it would be rather difficult to directly deduce from that paper itself how the constant \( d \) depends on \( c \).

In the \( p \)-adic case, however, it is possible to take the same Lipschitz constant \( d = c \), which is no longer true over the field of real numbers. This was established by Cluckers–Halupczok \([4]\, \text{Theorem 1}\). Their approach combines the compatible cell decomposition mentioned above and a kind of simultaneous piecewise approximation of \( f \) and its derivative by a "monomial with fractional exponent" and its derivative.

Problem 1 seems to be open in the case where \( K \) is an arbitrary Henselian rank one valued field. Observe that in this case, too, the sets definable in the language of Denef–Pas admit decomposition into a finite number of cells "combed" by finitely many congruences, as demonstrated in our paper \([11]\, \text{Corollary 2.7}\] and recalled below. We begin with the concept of a cell. Consider an \( \mathcal{L} \)-definable subset \( B \) of \( K^n \times \mathbb{k}^m \), a positive integer \( \nu \) and three \( \mathcal{L} \)-definable functions \( a(x, \xi), b(x, \xi), c(x, \xi) : B \to K \).

For each \( \xi \in \mathbb{k}^m \) set
\[
C(\xi) := \{(x, y) \in K^n_x \times K^n_y : (x, \xi) \in B, \quad v(a(x, \xi)) <_1 v((y - c(x, \xi))^\nu) <_2 v(b(x, \xi)), \quad \overline{ac}(y - c(x, \xi)) = \xi_1\},
\]
where \( <_1, <_2 \) stand for \( <, \leq \) or no condition in any occurrence. If the sets \( C(\xi), \xi \in \mathbb{k}^m \), are pairwise disjoint, the union
\[
C := \bigcup_{\xi \in \mathbb{k}^m} C(\xi)
\]
is called a cell in \( K^n \times K \) with parameters \( \xi \) and center \( c(x, \xi) \); \( C(\xi) \) is called a fiber of the cell \( C \).

**Proposition 4.2.** Every \( \mathcal{L} \)-definable subset \( A \) of \( K^n \times K \) is a finite disjoint union of sets each of which is a subset
\[
F := \bigcup_{\xi \in \mathbb{k}^m} F(\xi)
\]
of a cell \( C \) with center \( c(x, \xi) \):
\[
C := \bigcup_{\xi \in \mathbb{k}^m} C(\xi)
\]
determined by finitely many congruences:
\[
F(\xi) = \{(x, y) \in C(\xi) : v(f_i(x, \xi)(y - c(x, \xi))^{k_i}) \equiv_M 0, \quad i = 1, \ldots, s\},
\]
where \( f_1, \ldots, f_s \) are \( \mathcal{L} \)-definable functions and \( k_1, \ldots, k_s, M \in \mathbb{N} \).
A subset $F \subseteq K^n$ of the form as above will be called a *combed cell* in $K^n \times K$. The conclusion of Proposition 4.2 may thus be rephrased as follows:

*Every $\mathcal{L}$-definable subset $A$ of $K^n \times K$ is a finite disjoint union of combed cells.*

We now turn to the problem of extending continuous rational functions from an algebraic subvariety to a continuous rational function on the ambient variety, which was solved in the papers [9, 11]. The former paper deals with real and $p$-adic varieties and the latter with varieties over an arbitrary Henselian rank one valued field of equicharacteristic zero.

Below we state the extension problem for Lipschitz continuous rational functions, which is open as yet. It may be connected with the open problem on extending rational functions of class $\mathcal{C}^p$, $p \in \mathbb{N}$, posed at the end of [11, Section 13].

**Problem 2.** Let $f : V \to K$ be a rational function on an algebraic subvariety $V$ of $K^n$. Suppose that $f$ is Lipschitz continuous with a constant $c$. Does $f$ extend to a rational function $F : K^n \to K$ that is Lipschitz continuous with some constant $d$? Further, what can one say about a new constant $d$ in comparison to $d$? Is it possible to take the same constant $d = c$?

Finally, we pose two problems concerning a Henselian analogue of the Tietze–Urysohn extension theorem.

**Problem 3.** Let $f : A \to K$ be an $\mathcal{L}$-definable function on a closed subset $A$ of $K^n$.

1) If $f$ is continuous, does there exist a continuous $\mathcal{L}$-definable extension $F : K^n \to K$?

2) If $f$ is Lipschitz continuous with a constant $c$, does there exist a Lipschitz continuous $\mathcal{L}$-definable extension $F : K^n \to K$ with a constant $d$? Can one take the Lipschitz constant $d = c$?

The affirmative answer to the first question is given in our paper [12] being in preparation. We also expect an affirmative answer to the second question, although we are currently able to construct only a Lipschitz continuous extension $F$ with the same constant $c$, but without ensuring its definability. Note that in the realm of pure topology, a non-archimedean analogue on extending continuous functions from an ultraparacompact space into a complete metric space was established by Ellis [4].
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