Reconsideration of the multivariate moment problem and a new method for approximating multivariate integrals

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June 14, 2021

Abstract

Due to its intimate relation to Spectral Theory and Schrödinger operators, the multivariate moment problem has been a subject of many researches, so far without essential success (if one tries to compare with the one–dimensional case). In the present paper we reconsider a basic axiom of the standard approach - the positivity of the measure. We introduce the so–called pseudopositive measures instead. One of our main achievements is the solution of the moment problem in the class of the pseudopositive measures. A measure $\mu$ is called pseudopositive if its Laplace-Fourier coefficients $\mu_{k,l}(r)$, $r \geq 0$, in the expansion in spherical harmonics are non–negative. Another main profit of our approach is that for pseudopositive measures we may develop efficient "cubature formulas" by generalizing the classical procedure of Gauss–Jacobi: for every integer $p \geq 1$ we construct a new pseudopositive measure $\nu_p$ having "minimal support" and such that $\mu(h) = \nu_p(h)$ for every polynomial $h$ with $\Delta^p h = 0$. The proof of this result requires application of the famous theory of Chebyshev, Markov, Stieltjes, Krein for extremal properties of the Gauss-Jacobi measure, by employing the classical orthogonal polynomials $p_{k,l,j}$, $j \geq 0$, with respect to every measure $\mu_{k,l}$. As a byproduct we obtain a notion of multivariate orthogonality defined by the polynomials $p_{k,l,j}$. A major motivation for our investigation has been the further development of new models for the multivariate Schrödinger operators, which generalize the classical result of M. Stone saying that the one–dimensional orthogonal polynomials represent a model for the self–adjoint operators with simple spectrum.

1 Introduction

The univariate moment problem is one of the cornerstones of Mathematical Analysis where several areas of Pure and Applied Mathematics meet – continued fractions, quadrature formulas, orthogonal polynomials, analytic functions, finite differences, operator and spectral theory, scattering theory and inverse
problems, probability theory, and last but not least, control theory, see e.g. the
collection of surveys in [33] and the comprehensive recent account [13] on the
numerous applications of the moment problem to spectral theory. On the other
hand, the multivariate case is much more complicated, and we refer to [8], [15],
[37], [40], [41], [46] and the references given there for some recent developments.
However, the state of the art in the multivariate moment problem seems to be
well characterized by a remark in the versatile survey [20, p. 47], saying that
only comparatively little from the comprehensive theory of the classical moment
problem has been extended to dimension $d > 1$.

The main purpose of the present paper is to introduce a modified moment
problem for which the solutions are in general signed measures and belong to
the class of what we call pseudo–positive measures. The motivation for this new
notion is the possibility to generalize the univariate Gauß–Jacobi quadratures
to the multivariate setting, and thus to approximate multivariate integrals in a
new stable way. Let us emphasize that we do not claim to solve the multivariate
moment problem in its classical formulation.

In order to make our approach clear, let us first recall the usual formulation
of the multivariate moment problem: it asks for conditions on a sequence of real
numbers $c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ (here $\mathbb{N}_0$ denotes the set of all non–negative integers and
we use the multi–index notation $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$), such that there exists a
non-negative measure $\mu$ on $\mathbb{R}^d$ with

$$c_\alpha = \int_{\mathbb{R}^d} x^\alpha d\mu(x) \quad (1)$$

for all $\alpha \in \mathbb{N}_0^d$. Let us denote by $\mathbb{C}[x_1, x_2, \ldots, x_d]$ the space of all polynomials
in $d$ variables with complex coefficients. With the sequence $c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ we
associate by linear extension to $\mathbb{C}[x_1, x_2, \ldots, x_d]$ a functional $T_c$, by putting

$$T_c(x^\alpha) := c_\alpha \quad \text{for } \alpha \in \mathbb{N}_0^d. \quad (2)$$

By a theorem of Haviland, a necessary and sufficient condition for the existence
of a non-negative measure $\mu$ satisfying (1) is the positivity of the sequence $c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}$, i.e. $P \geq 0$ implies $T_c(P) \geq 0$ for all $P \in \mathbb{C}[x_1, x_2, \ldots, x_d]$ (here $P \geq 0$
means that $P(x) \geq 0$ for all $x \in \mathbb{R}^d$), cf. [21, p. 111]. As is well known,
the condition of positivity is difficult to apply, and in practice one uses the
weaker and easier to check algebraic condition that the sequence $c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}$
is positive definite; by definition the sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}$ is positive definite$^1$ if
and only if

$$T_c(P^*P) \geq 0 \quad \text{for all } P \in \mathbb{C}[x_1, x_2, \ldots, x_d];$$

here $P^*$ is the polynomial whose coefficients are the complex conjugates of
the coefficients of $P$. Let us remind that for $d = 1$, positivity and positive–
definiteness of $T_c$ are equivalent. However, for $d > 1$ this is not true, and this

$^1$Some authors [1] call such sequences "positive", while we are closer to the terminology of
say [2].
is a consequence of the fact, already known to D. Hilbert, that there exist non-negative polynomials which are not sums of squares of other polynomials, see e.g. [24], [8].

As mentioned above, we shall consider a different setting of the moment problem. Let us postpone at the moment the motivation for our approach, and let us concentrate on our new setting which needs some technical preparations from the theory of harmonic functions. We assume that for each $k = 0, 1, 2, ..., a_k$, the functions $Y_{k,l} : \mathbb{R}^d \to \mathbb{R}$, $l = 1, ..., a_k$, form a basis of the set of all harmonic homogeneous complex-valued polynomials$^2$ of degree $k \in \mathbb{N}_0$, and they are orthonormal with respect to the scalar product $\langle f, g \rangle_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(\theta) g(\theta) d\theta$, where $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere, and $r = |x| = \sqrt{x_1^2 + ... + x_d^2}$ is the euclidean norm, cf. [5] or [45]. For $x \in \mathbb{R}^d$ we will use further the representation $x = r\theta$ for $r \geq 0$, $\theta \in \mathbb{S}^{d-1}$.

The functions $Y_{k,l}$ are called solid harmonics and their restrictions to $\mathbb{S}^{d-1}$ spherical harmonics. An important property of the system $|x|^{2j} Y_{k,l}(x)$, $j, k \in \mathbb{N}_0$, $l = 1, ..., a_k$, is that it forms a basis for $\mathbb{C}[x_1, x_2, ..., x_d]$. This result follows from the Gauß decomposition of polynomials which says (cf. [5, Theorem 5.6, Theorem 5.21, p. 77 and p. 90], [14], or [31, Theorem 10.2]) that every polynomial $P$ may be expanded in the following way,

$$ P(x) = \sum_{j=0}^{\lfloor \deg(P)/2 \rfloor + 1} |x|^{2j} p_j(x), \quad (3) $$

where $p_j$ are harmonic polynomials, $\deg P$ denotes the degree of $P$ and $|x|$ is the integer part of a real number $x$. Since each $p_j$ is a linear combination of the solid harmonics $Y_{k,l}(x)$, $k \in \mathbb{N}_0$, $l = 1, ..., a_k$, it is clear that the system $|x|^{2j} Y_{k,l}(x)$, $j, k \in \mathbb{N}_0$, $l = 1, ..., a_k$, is a basis for $\mathbb{C}[x_1, x_2, ..., x_d]$. It is instructive to discuss the relationship between the Gauß decomposition and the Laplace-Fourier series: recall that for a sufficiently nice function $f : \mathbb{R}^n \to \mathbb{C}$ (e.g. continuous) the expansion

$$ f(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(r) Y_{k,l}(\theta), \quad (4) $$

is the Laplace–Fourier series with the Laplace–Fourier coefficients given by

$$ f_{k,l}(r) = \int_{\mathbb{S}^{d-1}} f(r\theta) Y_{k,l}(\theta) d\theta. \quad (5) $$

Suppose now that $f$ is a polynomial: then (4) implies that the Laplace-Fourier series (4) is a finite series, and the functions $f_{k,l}(r) r^{-k}$ are polynomials in the
variable $r^2$. Moreover from 3 directly follows that each $f_{k,l}(r) r^{-k}, k \in \mathbb{N}_0$, $l = 1, 2, ..., a_k$, is a polynomial of degree $\leq 2s - 2$ if and only if for all $x \in \mathbb{R}^n$

$$\Delta^s f(x) = 0.$$  (6)

Here $\Delta$ denotes the Laplace operator defined by $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$, and $\Delta^s$ is the $s$-th iterate of $\Delta$ for integers $s \geq 1$. In view of 3 let us recall that a function $f$ defined on an open subset $U$ in $\mathbb{R}^d$ is polyharmonic of order $s$ if $\Delta^s f(x) = 0$ for all $x \in U$, see 3.

Now we come to the cornerstone of our approach. Let $T_c : \mathbb{C} [x_1, x_2, ..., x_d] \rightarrow \mathbb{C}$ be a functional associated to a sequence of moments $c_\alpha, \alpha \in \mathbb{N}_0^n$. Using the basis $|x|^2 j Y_{k,l}(x) : j, k \in \mathbb{N}_0, l = 1, ..., a_k$ one can define a problem equivalent to the usual one 1, by means of the sequence $\{c(k,l)^{(j)}\}_{j \in \mathbb{N}_0}$ defined as follows,

$$c(k,l)^{(j)} := T_c \left( |x|^2 j Y_{k,l}(x) \right) \quad \text{for} \quad j = 0, 1, 2, ...$$  (7)

In order to distinguish them from the usual moments $c_\alpha$, the numbers $c(k,l)^{(j)}$ are sometimes called distributed moments, see 10, 11, 25, 26, 27, 29.

We say that the sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ or the associated functional $T_c$ is pseudo-positive definite if for every fixed pair of indices $(k, l)$ with $k \in \mathbb{N}_0$ and $l = 1, ..., a_k$ the sequences $\{c(k,l)^{(j)}\}_{j \in \mathbb{N}_0}$ and $\{c(k,l)^{(j+1)}\}_{j \in \mathbb{N}_0}$ are positive definite. Equivalently, for each solid harmonic $Y_{k,l}(x), k \in \mathbb{N}_0, l = 1, ..., a_k$, the component functional $T_{k,l} : \mathbb{C} [x_1] \rightarrow \mathbb{C}$ defined by

$$T_{k,l} (p) := T_c \left( p(|x|^2) Y_{k,l}(x) \right) \quad \text{for} \quad p \in \mathbb{C} [x_1]$$  (8)

has the property that $T_{k,l} (p^* (t) p (t)) \geq 0$ and $T_{k,l} (tp^* (t) p (t)) \geq 0$ for all $p \in \mathbb{C} [x_1]$.

Now we can formulate and successfully solve the following modified moment problem: Given a pseudo-positive definite sequence $c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ and its associated functional $T_c$, find the conditions for the existence of a signed measure $\mu$ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^n} P(x) \, d\mu = T_c (P) \quad \text{for all} \quad P \in \mathbb{C} [x_1, x_2, ..., x_d].$$  (9)

Two remarks are important: first, we allow $\mu$ to be a signed measure on $\mathbb{R}^n$, and this requirement is motivated by our constructive formulas for approximating integrals developed in later sections. Secondly, it follows from 3 that the measure $\mu$, considered as a functional on $\mathbb{C} [x_1, x_2, ..., x_d]$, is pseudo-positive definite. The remarkable thing which will be seen from our further development is that problem 3 has a solution $\mu$ which is pseudo-positive which means that the inequality

$$\int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x) \, d\mu (x) \geq 0$$  (10)

4In the last section we provide some remarks on the significance of the polyharmonic functions in approximation theory which have motivated also the present research.
holds for every non-negative continuous function \( h : [0, \infty) \to [0, \infty) \) with compact support and for all pairs of indices \((k,l)\) with \( k \in \mathbb{N}_0 \) and \( l = 1, 2, \ldots, a_k \).

As a first evidence that the pseudo-positivity is a reasonable generalization of the univariate positivity notion, we present in Section 2 the following solution to the modified moment problem (9), provided in two steps: 1. By a classical one-dimensional argument, for the component functionals \( T_{k,l} \) associated with the pseudo-positive definite functional \( T_c : \mathbb{C} [x_1, x_2, \ldots, x_d] \to \mathbb{C} \) there exist non-negative univariate representing measures \( \mu_{k,l} \) on \([0, \infty)\). 2. If they satisfy the summability assumption

\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty r^N r^{-k} d\mu_{k,l} (r) < \infty \quad \text{for all } N \in \mathbb{N}_0
\]  

then there exists a pseudo-positive signed measure \( \mu \) on \( \mathbb{R}^d \) representing \( T_c \), i.e. (9) holds. Further the following important identity

\[
\int_{\mathbb{R}^n} f(x) \, d\mu = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty f_{k,l}(r) r^{-k} d\mu_{k,l} (r)
\]  

holds for any continuous, polynomially bounded function \( f : \mathbb{R}^n \to \mathbb{C} \); here \( f_{k,l}(r) \) are the Laplace-Fourier coefficients. Equation (12) will be the key for defining polyharmonic Gauss–Jacobi cubatures as we shall show below.

Another strong supporting evidence for the nice properties of the notion of pseudo-positivity is the satisfactory solution of the question of determinacy in Section 3: we show that the representing measure \( \mu \) of a pseudo-positive definite functional \( T : \mathbb{C} [x_1, x_2, \ldots, x_d] \to \mathbb{C} \) is unique in the class of all pseudo-positive signed measures whenever each component functional \( T_{k,l} \) defined in (8) has a unique representing measure on \([0, \infty)\) in the sense of Stieltjes (for the precise definition see Section 3). And vice versa, if a pseudo-positive functional \( T \) is determinate in the class of all pseudo-positive signed measures and the summability condition (11) is satisfied, then each functional \( T_{k,l} \) is determinate in the sense of Stieltjes. The proof is essentially based on the properties of the Nevanlinna extremal measures.

Let us illustrate the notion of pseudo-positivity in the case where the signed measure \( \mu \) has a continuous density \( w(x) \) with respect to the Lebesgue measure \( dx \). We put \( d\mu (x) = w(x) \, dx \) in (10) and take into account that \( Y_{k,l} (x) = \left| x \right|^k Y_{k,l} (\theta) \), and \( dx = r^{d-1} d\theta dr \), (for detailed computations see Proposition 31). We see that the component functionals \( T_{k,l} \) defined in (9) are now given by

\[
T_{k,l} (p) = \int_0^\infty \left( r^2 \right)^{k+d-1} w_{k,l} (r) \, dr,
\]  

where \( w_{k,l} (r) \) are the Laplace-Fourier coefficients of the function \( w \) as defined in (9). From (13) it is obvious that the non-negativity of \( w_{k,l} \) implies that the measure \( \mu \) is pseudo-positive and the corresponding functional \( T \) defined by (9) is pseudo-positive definite. We regard now \( d\mu_{k,l} (r) = \left( r^{k+d-1} w_{k,l} (r) \right) dr \) as a univariate non-negative measure which represents the functional \( T_{k,l} \).
Now we are moving to our main theme, the construction of Gauß–Jacobi type cubatures for pseudo-positive measures. Let us first recall some terminology: By a cubature formula one usually means a linear functional of the form

\[ C(f) := \alpha_1f(x_1) + \cdots + \alpha_s f(x_s) = \int_{\mathbb{R}} \left( \sum_{j=1}^{s} \alpha_j \delta(x-x_j) \right) f(x) \, dx \]  

defined on the set \( C(\mathbb{R}^n) \), the set of all continuous complex-valued functions on \( \mathbb{R}^n \); here \( \delta \) is the Dirac delta function. The points \( x_1, \ldots, x_s \) are called nodes and the coefficients \( \alpha_1, \ldots, \alpha_s \in \mathbb{R} \) weights. A cubature formula \( C(\cdot) \) is exact on a subspace \( U \) of \( C(\mathbb{R}^n) \) with respect to a measure \( \nu \) if

\[ C(f) = \int f(x) \, d\nu \]  

holds for all \( f \in U \). If \( U_s \) is the set of all polynomials of degree \( \leq s \), and the cubature is exact on \( U_s \) but not on \( U_{s+1} \), we say that \( C \) has order \( s \).

It is common to call a cubature in the case \( d = 1 \) a quadrature.

In our construction we will use the Gauß-Jacobi quadrature, let us recall its definition: Let \( \nu \) be a non-negative measure on the interval \([0, R]\) and \( s \geq 1 \) be an integer. If the cardinality of the support of \( \nu \) is \( > s \) then there exist \( s \) different points \( t_j \) in the interval \((0, R)\) and \( s \) positive weights \( \alpha_j \) which define the classical Gauß–Jacobi measure

\[ d\nu^{(s)}(t) = \sum_{j=1}^{s} \alpha_j \delta(t-t_j) \, dt \]  

and the corresponding Gauß–Jacobi quadrature \( C(f) = \int_{0}^{R} f(t) \, d\nu^{(s)}(t) = \sum_{j=1}^{s} \alpha_j f(t_j) \) satisfies (15) for all polynomials of degree \( \leq 2s - 1 \). For formal reasons we put \( \nu^{(s)} \equiv \nu \) if the cardinality of \( \nu \) is \( \leq s \).  

It is not our intention to survey the numerous approaches to cubature formulas; one may consult the references in [38], [18], [47], [17] and in particular the monograph of S. L. Sobolev [44] where the minimization of the error functional of the formula in (15) is the main objective.

Our approach, which is rather different from the usual cubature formulas, is based on the identity (12). In order to be precise, let us begin with the assumptions: let \( \mu \) be a pseudo–positive measure and define the non-negative component measures \( \mu_{k,l} \) by the identity

\[ \int_{0}^{\infty} h(|x|) \, d\mu_{k,l} = \int_{\mathbb{R}^n} h(|x|) \, Y_{k,l}(x) \, d\mu \]  

valid for all continuous functions \( h : [0, \infty) \to \mathbb{C} \) with compact support. Let \( \psi : [0, \infty) \to [0, \infty) \) be the transformation \( \psi(t) = t^2 \) and let \( \mu_{k,l}^{\psi} \) be the image \( The points \( t_j \) are the zeros of the polynomial \( Q^s(t) \) which is the \( s \)-th orthogonal with respect to the measure \( \nu \) on \([0, R]\). It is important that \( t_j \in (0, R) \) if the support of \( \nu \) has cardinality \( > s \), cf. Chapter 1, Theorem 5.2 in [12], and Theorem 5.1 in Chapter 3.5 in [32].
measure of $\mu_{k,l}$ under $\psi$ (see (20) below). Then (12) becomes

$$\int_{\mathbb{R}} f(x) \, d\mu = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{0}^{\infty} f_{k,l} \left( \sqrt{t} \right) t^{-\frac{1}{2}k} \, d\mu_{k,l}(r).$$

(18)

The main idea is simple and consists in replacing in formula (15) the non-negative univariate measures $\mu_{k,l}^{\psi}$ by their univariate Gauss-Jacobi quadratures $\nu_{k,l}^{(s)}$ of order $2s - 1$. Let $\psi^{-1}$ be the inverse map of $\psi$ and put $\sigma_{k,l}^{(s)} = \left( \nu_{k,l}^{(s)} \right)^{\psi^{-1}}$. Then we obtain a pseudo-positive definite functional $T^{(s)}$ by setting

$$T^{(s)}(f) := \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{0}^{\infty} f_{k,l}(r) \, r^{-k} \, d\sigma_{k,l}^{(s)}(r)$$

(19)

$$= \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{0}^{\infty} f_{k,l} \left( \sqrt{t} \right) t^{-\frac{1}{2}k} \, d\nu_{k,l}^{(s)}(t).$$

As we have made it clear above, for $f \in \mathbb{C}[x_1, x_2, \ldots, x_d]$ we have $f_{k,l}(r) \, r^{-k} = p_{k,l}(r^2)$, where $p_{k,l}$ are polynomials. Hence, the functional $T^{(s)}$ is well-defined on $\mathbb{C}[x_1, x_2, \ldots, x_d]$, since then the series is finite. In fact, the most important thing is to find a condition on the measure $\mu$ which provides convergence of the series in (19) for the class of continuous, polynomially bounded functions $f$.

Since this is a very central result of our paper, let us give the main argument for proving the convergence of the series in (19) in the important case when all measures $\mu_{k,l}$ have their supports in the compact interval $[0, R]$. For the Laplace-Fourier coefficient, defined in (15), we have the simple estimate

$$|f_{k,l}(r)| \leq C \max_{|x| \leq R} |f(x)| \quad \text{for} \quad 0 \leq r \leq R,$$

based on the Cauchy inequality and the orthonormality of $\{Y_{k,l}(\theta)\}$. Hence,

$$\left| \int_{0}^{\infty} f_{k,l}(r) \, r^{-k} \, d\sigma_{k,l}^{(s)}(r) \right| \leq C \max_{|x| \leq R} |f(x)| \int_{0}^{\infty} r^{-k} \, d\sigma_{k,l}^{(s)}(r)$$

and

$$|T^{(s)}(f)| \leq C \max_{|x| \leq R} |f(x)| \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{0}^{\infty} r^{-k} \, d\nu_{k,l}^{(s)}(r).$$

(20)

Now here is the crux of the whole matter: for the convergence in (19) it would suffice to prove the inequality

$$\int_{0}^{\infty} r^{-k} \, d\nu_{k,l}^{(s)}(r) \leq \int_{0}^{\infty} r^{-k} \, d\mu_{k,l}(r).$$

(21)

The famous Chebyshev extremal property$^6$ of the Gauß-Jacobi quadrature provides us with a proof of (21). So we see that the convergence of the series in

$^5$The upper index $s$ will indicate the cardinality of the support.

$^6$This has been proved by A. Markov$^{36}$ and T. Stieltjes, cf.$^{32}$ Chapter 4 and$^{24}$ Chapter 3.
is a consequence of the summability condition with \( N = 0 \). Further
note that \( T^{(s)} \) is a continuous functional: by the Riesz representa-
tion theorem we infer the existence of a signed measure \( \sigma^{(s)} \) with support
in the closed ball \( B_R := \{ x \in \mathbb{R}^n : |x| \leq R \} \) such that
\[
T^{(s)}(f) = \int_{B_R} f(x) \, d\sigma^{(s)}(x)
\]
for all continuous functions \( f : B_R \to \mathbb{C} \). Moreover, the component measures of
the pseudo–positive measure \( \sigma^{(s)} \) are exactly the univariate measures \( \sigma^{(s)}_{k,l} \).

The exactness of the Gauß-Jacobi quadratures \( \nu_{k,l}^{(s)} \) for polynomials of degree
\( \leq 2s - 1 \) implies that \( T^{(s)} \) and \( \mu \) coincide on the set of all polynomials \( P \) such that
\( \Delta^{2s} P = 0 \). This is due to the fact that in the Laplace–Fourier expansion
the coefficients are given by \( f_{k,l}(r) = r^k p_{k,l}(r^2) \) where \( p_{k,l} \) are polynomials
of degree \( 2s - 1 \). For that reason we call the measure \( \sigma^{(s)} \) the 
\textit{polyharmonic Gauß–Jacobi measure} or the polyharmonic Gauß–Jacobi cubature of order \( s \).

In the following we want to discuss the properties of the \textit{polyharmonic Gauß–Jacobi cubature} and it is natural to compare them with those of the
univariate Gauß–Jacobi quadrature. Among the various existing quadratures (e.g. Newton-Cotes quadratures), the Gauß–Jacobi quadrature has the eminent
property that the weights are positive. This in turn is the key to prove the
convergence of the quadrature (see e.g. the discussion in [16, p. 353] based on
the theorems of Pólya and Steklov).

\textbf{Property 1} (Stieltjes) For every continuous function \( f \) the Gauß–Jacobi quadrature
\( \int_a^b f \, d\nu^{(s)} \) converges to \( \int_a^b f \, d\nu \), when \( s \) tends to infinity.

A second important property of the Gauß–Jacobi quadrature is the error
estimate due to A. Markov (see [16, p. 344])

\textbf{Property 2} (Markov) Let \( \nu \) be a non-negative measure on \( [a,b] \) whose support
has cardinality \( > s \). Then for any \( 2s \)-times continuously differentiable function
\( f : [a,b] \to \mathbb{R} \) there exists \( \xi \in (a,b) \) such that
\[
\int_a^b f(t) \, d\nu(t) - \int_a^b f(t) \, d\nu^{(s)}(t) = \frac{1}{(2s)!} f^{(2s)}(\xi) \int_a^b |Q_s(t)|^2
\]
where \( Q_s \) is the \( s \)-th orthogonal polynomial with respect to \( \nu \), with leading coefficient 1.

It is an amazing and non-trivial fact that properties 1 and 2 have analogs
for the polyharmonic Gauß–Jacobi cubature although the approximation measures \( \sigma^{(s)} \) are in general signed measures. In Theorem 22 we show that \( C_s(f) \)
converges to $\int f \, d\mu$ for every continuous function $f : \mathbb{R}^n \to \mathbb{C}$. This property implies the numerical stability of our cubature formula. In Section 5 we prove an estimate for the difference

$$\mu(f) - T^{(s)}(f)$$

for functions $f \in C^{2s}(\mathbb{R}^d)$ by their derivatives in the ball $B_R$ based on Markov’s error estimate.

Let us outline the structure of the paper: In Section 2 we introduce the notions of pseudo–positive definite functional and pseudo–positive measure, and we prove basic results about them. In Section 3 we consider the determinacy question. In Section 4 the polyharmonic Gauß–Jacobi cubature formula is presented in detail. Section 5 is devoted to a multivariate generalization of the Markov’s error estimate for the polyharmonic Gauß-Jacobi cubature.

Section 6 and 7 are devoted to definition of polyharmonic Gauß–Jacobi cubatures in other domains with symmetries as the annulus and the cylinder (periodic strip). In Section 6 we construct a Gauß-Jacobi cubature for pseudo-positive measures with support in a closed annulus $A_{\rho,R}$ which is exact on the space of all functions continuous on the closed annulus $A_{\rho,R}$ and polyharmonic of order $2s$ in the interior. While the case of the annulus is somewhat similar to that of the ball, we have to introduce a new notion of pseudo-positivity in the case of the cylinder (periodic strip) in Section 7 in order to obtain cubatures which preserve polyharmonic functions of order $2s$. Moreover it is not possible to use in the proof the usual univariate Gauß–Jacobi quadratures; instead we need the existence of quadratures of Gauß–Jacobi-type for Chebyshev systems (Theorem 33). The analog to the crucial inequality (21) follows from the Markov–Krein theory of extremal problems for the moment problem for Chebyshev systems.\(^7\)

In Section 8 we give explicit examples illustrating our results and provide miscellaneous properties of pseudo–positive measures. In the last Section 9 we discuss shortly aspects of numerical implementation and some background information about the polyharmonicity concept.

Finally, let us introduce some notations: the space of all continuous complex-valued functions on a topological space $X$ is denoted by $C(X)$. By $C_c(X)$ we denote the set of all $f \in C(X)$ having compact support. Further $C_{pol}(\mathbb{R}^d)$ is the space of all polynomially bounded, continuous functions, so for each $f \in C_{pol}(\mathbb{R}^d)$ there exists $N \in \mathbb{N}_0$ such that $|f(x)| \leq C_N (1 + |x|)^N$ for some constant $C_N$ (depending on $f$) for all $x \in \mathbb{R}^d$. Further, we define an useful space of test functions

$$C^\times(\mathbb{R}^d) := \left\{ \sum_{k=0}^N \sum_{l=1}^{a_k} f_{k,l}(|x|) Y_{k,l}(x) : N \in \mathbb{N}_0 \text{ and } f_{k,l} \in C[0,\infty) \right\}. \quad (23)$$

which can be rephrased as the set of all continuous functions with a finite Laplace-Fourier series. Moreover we set

$$C^\times_c(\mathbb{R}^d) := C^\times(\mathbb{R}^d) \cap C_c(\mathbb{R}^d). \quad (24)$$

\(^7\)We use the name “Markov–Krein theory” following [24, Chapter 3], while in [32] this is called “Chebyshev–Markov problem”.

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We need some terminology from measure theory: a signed measure on $\mathbb{R}^d$ is a set function on the Borel $\sigma$-algebra on $\mathbb{R}^d$ which takes real values and is $\sigma$-additive. For the standard terminology, as Radon measure, Borel $\sigma$-algebra, etc., we refer to [8]. By the Jordan decomposition [14, p. 125], a signed measure $\mu$ is the difference of two non-negative finite measures, say $\mu = \mu^+ - \mu^-$ with the property that there exist a Borel set $A$ such that $\mu^+(A) = 0$ and $\mu^-(\mathbb{R}^n \setminus A) = 0$. The variation of $\mu$ is defined as $|\mu| := \mu^+ + \mu^-$. The signed measure $\mu$ is called moment measure if all polynomials are integrable with respect to $\mu^+$ and $\mu^-$, which is equivalent to integrability with respect to the total variation. The support of a non-negative measure $\mu$ on $\mathbb{R}^d$ is defined as the complement of the largest open set $U$ such that $\mu(U) = 0$. In particular, the support of the zero measure is the empty set. The support of a signed measure $\sigma$ is defined as the support of the total variation $|\sigma| = \sigma^+ + \sigma^-$ (see [14, p. 226]). Recall that in general, the supports of $\sigma^+$ and $\sigma^-$ are not disjoint (cf. exercise 2 in [14, p. 231]). For a surjective measurable mapping $\varphi : X \to Y$ and a measure $\nu$ on $X$ the image measure $\nu\varphi$ on $Y$ is defined by
\[ \nu\varphi(B) := \nu(\varphi^{-1}B) \]
for all Borel subsets $B$ of $Y$. The equality $\int_X g(\varphi(x)) \, d\nu(x) = \int_Y g(y) \, d\nu\varphi(y)$ holds for all integrable functions $g$. We use the notation $\omega_{d-1}$ for the surface area of the unit sphere, so
\[ \omega_{d-1} := \int_{S^{d-1}} 1 \, d\theta. \]

By $B_R$ we denote the closed ball $\{ x \in \mathbb{R}^n : |x| \leq R \}$. For $0 \leq \rho < R \leq \infty$ we define the closed annulus by
\[ A_{\rho,R} := \{ x \in \mathbb{R}^d : \rho \leq |x| \leq R \}. \]
For a subset $B$ of $\mathbb{R}^n$ the interior is denoted by $B^\circ$. Further, we denote the closed and the open interval respectively by $[a,b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$ and $(a,b) = \{ x \in \mathbb{R} : a < x < b \}$.

2 The moment problem for pseudo-positive definite sequences

Let $T : \mathbb{C}[x_1, x_2, \ldots, x_d] \to \mathbb{C}$ be a linear functional. For any solid harmonic polynomial $Y_{k,l}(x)$ we define the component functional $T_{k,l}$ by
\[ T_{k,l}(p) := T_c \left( p(|x|^2)Y_{k,l}(x) \right) \text{ for every } p \in \mathbb{C}[x_1]. \]

Let us give the precise definition of pseudo-positive definiteness, which we already mentioned in the introduction:
Definition 3 A sequence \( c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \), or the associated functional \( T_c \), is pseudo-positive definite if for every \( k \in \mathbb{N}_0 \) and \( l = 1, \ldots, a_k \) the sequences \( \{c^{(k,l)}_j\}_{j \in \mathbb{N}_0} \) and \( \{c^{(k,l)}_{j+1}\}_{j \in \mathbb{N}_0} \) defined in (4) are positive definite. Clearly this is the same to say that \( T_{k,l}(p^*(t)p(t)) \geq 0 \) and \( T_{k,l}(t \cdot p^*(t)p(t)) \geq 0 \) for every \( p(t) \in \mathbb{C}[x_1] \).

First we recall the following result which may be found e.g. in [6] or [31].

Proposition 4 The Laplace-Fourier coefficient \( f_{k,l} \) of a polynomial \( f \) given by (3) is of the form

\[
f(x) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} p_{k,l}(|x|^2)Y_{k,l}(x).
\]

Equality (29) is a reformulation of the Gauß decomposition of a polynomial which we have provided in (3).

The next two Propositions characterize pseudo-positive definite sequences:

Proposition 5 Let \( c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) be a pseudo-positive definite sequence and \( T_c \) its associated functional. Then for each \( k \in \mathbb{N}_0, l = 1, \ldots, a_k \), there exist non-negative measures \( \sigma_{k,l} \) with support in \([0, \infty]\) such that

\[
T_c(f) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} f_{k,l}(r) r^{-k} d\sigma_{k,l}(r)
\]

holds for all \( f \in \mathbb{C}[x_1, x_2, \ldots, x_d] \) where \( f_{k,l}(r) \), \( k \in \mathbb{N}_0, l = 1, \ldots, a_k \), are the Laplace-Fourier coefficients of \( f \).

Proof. By the definition of pseudo-positive definiteness, \( T_{k,l}(p^*(t)p(t)) \geq 0 \) and \( T_{k,l}(t \cdot p^*(t)p(t)) \geq 0 \) for each univariate polynomial \( p(t) \) where the component functional \( T_{k,l} \) is defined in (28). By the solution of the Stieltjes moment problem there exists a non-negative measure \( \mu_{k,l} \) with support in \([0, \infty]\) representing the functional \( T_{k,l} \), i.e. satisfying

\[
T_{k,l}(p) = \int_0^\infty p(t) d\mu_{k,l}(t) \quad \text{for every } p \in \mathbb{C}[t].
\]

Let now \( \varphi : [0, \infty) \to [0, \infty) \) be defined by \( \varphi(t) = \sqrt{t} \). Then we put \( \sigma_{k,l} := \mu_{k,l}^{\varphi} \) where \( \mu_{k,l}^{\varphi} \) is the image measure defined in (28). We obtain

\[
\int_0^\infty h(t) d\mu_{k,l}(t) = \int_0^\infty h(r^2) d\mu_{k,l}^{\varphi}(r).
\]

Now use (28), the linearity of \( T \) and the definition of \( T_{k,l} \) in (28), and the equations (31) and (32) to obtain

\[
T_c(f) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} T_{k,l}(p_{k,l}) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} \int_0^\infty p_{k,l}(r^2) d\mu_{k,l}^{\varphi}(r).
\]
Since $p_{k,l}(r^2) = r^{-k}f_{k,l}(r)$ the claim follows from the last equation, which ends the proof. ■

The next result shows that the converse of Proposition is also true; not less important, it is a natural way of defining pseudo-positive definite sequences.

**Proposition 6** Let $\sigma_{k,l}, k \in \mathbb{N}_0, l = 1, ..., a_k$, be non-negative moment measures with support in $[0, \infty)$. Then the functional $T : \mathbb{C}[x_1, x_2, ..., x_d] \to \mathbb{C}$ defined by

$$T(f) := \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} \int_0^{\infty} f_{k,l}(r) r^{-k} d\sigma_{k,l}$$

is pseudo-positive definite, where $f_{k,l}(r), k \in \mathbb{N}_0, l = 1, ..., a_k$, are the Laplace-Fourier coefficients of $f$.

**Proof.** Let us compute $T_{k,l}(p)$ where $p$ is a univariate polynomial: by definition, $T_{k,l}(p) = T\left(p(|x|^2)Y_{k,l}(x)\right)$. The Laplace-Fourier series of the function $x \mapsto |x|^{2j}p(|x|^2)Y_{k,l}(x)$ is equal to $r^{2j}p(r^2) r^k Y_{k,l}(\theta)$, hence

$$T_{k,l}(t^j p(t)) = T\left(|x|^{2j}p(|x|^2)Y_{k,l}(x)\right) = \int_0^{\infty} r^{j}p(r^2) d\sigma_{k,l}$$

for every natural number $j$. Taking $j = 0$ and $j = 1$ one concludes that $T_{k,l}(p^* (t) p(t)) \geq 0$ and $T_{k,l}(tp^* (t) p(t)) \geq 0$ for all univariate polynomials $p$, hence $T$ is pseudo-positive definite. ■

The pseudo-positive definiteness is defined for a functional on $\mathbb{C}[x_1, x_2, ..., x_d]$. Now we introduce the concept of pseudo-positivity of a measure:

**Definition 7** A signed measure $\mu$ on $\mathbb{R}^n$ is called **pseudo-positive** if

$$\int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x) \, d\mu(x) \geq 0$$

holds for every non-negative continuous function $h : [0, \infty) \to [0, \infty)$ with compact support.

At first we need some basic properties of pseudo-positive measures.

**Proposition 8** Let $\mu$ be a pseudo-positive moment measure on $\mathbb{R}^d$. Then there exist unique moment measures $\mu_{k,l}$ defined on $[0, \infty)$, which we call component measures, such that

$$\int_0^{\infty} h(t) \, d\mu_{k,l}(t) = \int_{\mathbb{R}^d} h(|x|)Y_{k,l}(x) \, d\mu$$

holds for all $h \in C_{\text{pol}}[0, \infty)$. Further for each $f \in C_{\text{pol}}^{\infty}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} f(x) \, d\mu = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^{\infty} f_{k,l}(r) r^{-k} d\mu_{k,l}.$$
The last expression is finite since \( \mu \) applied to \( |f| \) is easy to see that the Laplace-Fourier coefficients \((42)\) below.

Note that statement note that each \( f \) on the other hand, it is obvious that convergence theorem for tone convergence theorem there exists a unique non-negative measure \( \mu_{k,l} \) such that \( M_{k,l} (h) = \int_0^\infty h(t) \, d\mu_{k,l} \) for all \( h \in C_e ([0, \infty)) \). We want to show that \((34)\) holds for all \( h \in C_{pol} [0, \infty) \). For this, let \( u_R : [0, \infty) \to [0, 1] \) be a cut-off function, so \( u_R \) is continuous and decreasing such that

\[
u_R (r) = 1 \text{ for all } 0 \leq r \leq R \text{ and } u_R (r) = 0 \text{ for all } r \geq R + 1. \tag{36}
\]

Let \( h \in C_{pol} [0, \infty) \). Then \( u_R h \in C_e ([0, \infty)) \) and

\[
\int_0^\infty u_R (t) \, h(t) \, d\mu_{k,l} = \int_{\mathbb{R}^d} u_R (|x|) \, h(|x|) \, Y_{k,l} (x) \, d\mu. \tag{37}
\]

Note that \( |u_R (t) \, h(t)| \leq |u_{R+1} (t) \, h(t)| \) for all \( t \in [0, \infty) \). Hence by the monotone convergence theorem

\[
\int_0^\infty |h(t)| \, d\mu_{k,l} = \lim_{R \to \infty} \int_0^\infty |u_R (t) \, h(t)| \, d\mu_{k,l}. \tag{38}
\]

On the other hand, it is obvious that

\[
\left| \int_{\mathbb{R}^d} u_R (|x|) \, h(|x|) \, Y_{k,l} (x) \, d\mu \right| \leq \int_{\mathbb{R}^d} |h(|x|) \, Y_{k,l} (x)| \, d|\mu|. \tag{39}
\]

The last expression is finite since \( \mu \) is a moment measure. From \((35)\), \((37)\) applied to \( |h| \) and \((39)\) it follows that \( |h| \) is integrable for \( \mu_{k,l} \). Using Lebesgue’s convergence theorem for \( \mu \) and \((37)\) it is easy to that \((36)\) holds. For the last statement note that each \( f \in C_{pol}^\infty (\mathbb{R}^d) \) has a finite Laplace-Fourier series, and it is easy to see that the Laplace-Fourier coefficients \( f_{k,l} \) are in \( C_{pol} [0, \infty) \), see \((40)\) below.

The next theorem is the main result of this section and it provides a simple sufficient condition for the pseudo-positive definite functional on \( \mathbb{C} [x_1, x_2, ..., x_d] \) defined in \((33)\) to possess a pseudo–positive representing measure. Let us note that not every pseudo-positive definite functional has a pseudo-positive representing measure, see Section \((8)\) for an example.

**Theorem 9** Let \( \sigma_{k,l}, k \in \mathbb{N}_0, l = 1, ..., a_k \), be non-negative measures with support in \( [0, \infty) \) such that for any \( N \in \mathbb{N}_0 \)

\[
C_N := \sum_{k=0}^\infty \sum_{l=1}^{a_k} \int_0^\infty r^{N} r^{-k} \, d\sigma_{k,l} < \infty. \tag{40}
\]

Then for the functional \( T : \mathbb{C} [x_1, x_2, ..., x_d] \to \mathbb{C} \) defined by \((33)\) there exists a pseudo-positive, signed moment measure \( \sigma \) such that

\[
T (f) = \int_{\mathbb{R}^n} f \, d\sigma \text{ for all } f \in \mathbb{C} [x_1, x_2, ..., x_d].
\]
Remark 10 1. If the measures $\sigma_{k,l}$ have supports in the compact interval $[\rho, R]$ for all $k \in \mathbb{N}_0$, $l = 1, \ldots, a_k$, then the measure $\sigma$ in Theorem 9 has support in the annulus $\{x \in \mathbb{R}^d : \rho \leq |x| \leq R\}$.

2. In the case of $R < \infty$, it obviously suffices to assume that $C_0 < \infty$ instead of $C_N < \infty$ for all $N \in \mathbb{N}_0$.

3. The proof of Theorem 9 shows that $\sigma_{k,l}$ is equal to the component measure induced by $\sigma$ with respect to the solid harmonic $Y_{k,l}(x)$.

Proof. 1. We show at first that $T$ can be extended to a linear functional $\tilde{T}$ defined on $C_{pol}(\mathbb{R}^d)$ by the formula

$$\tilde{T}(f) := \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty f_{k,l}(r) r^{-k} d\sigma_{k,l}$$  \hspace{1cm} (41)$$

for $f \in C_{pol}(\mathbb{R}^d)$, where $f_{k,l}(r)$ are the Laplace-Fourier coefficients of $f$. Indeed, since $f \in C_{pol}(\mathbb{R}^d)$ is of polynomial growth there exists $C > 0$ and $N \in \mathbb{N}$ such that $|f(x)| \leq C(1 + |x|^N)$. If follows from (5) that

$$|f_{k,l}(r)| \leq C (1 + r^N) \sqrt{\frac{\omega_{d-1}}{\omega_{d-1}}} \int_{S_{d-1}} |Y_{k,l}(\theta)|^2 d\theta = C (1 + r^N) \sqrt{\frac{\omega_{d-1}}{\omega_{d-1}}}, \hspace{1cm} (42)$$

where we used the Cauchy-Schwarz inequality and the fact that $Y_{k,l}$ is orthonormal. Hence,

$$\int_0^\infty |f_{k,l}(r)| r^{-k} d\sigma_{k,l} \leq \sqrt{\frac{\omega_{d-1}}{\omega_{d-1}}} \int_0^\infty (1 + r^N) r^{-k} d\sigma_{k,l}.$$ 

By assumption (40) the latter integral exists, so $f_{k,l}(r) r^{-k}$ is integrable with respect to $\sigma_{k,l}$. By summing over all $k, l$ we obtain by (40) that

$$\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty f_{k,l}(r) r^{-k} d\sigma_{k,l} < \infty,$$

which implies the convergence of the series in (41). It follows that $\tilde{T}$ is well-defined.

2. Let $T_0$ be the restriction of the functional $\tilde{T}$ to the space $C_c(\mathbb{R}^d)$. We will show that $T_0$ is continuous. Let $f \in C_c(\mathbb{R}^d)$ and suppose that $f$ has support in the annulus $\{x \in \mathbb{R}^d : \rho \leq |x| \leq R\}$ (for the case $\rho = 0$ this is a ball). Then by a similar technique as above above

$$|f_{k,l}(r)| \leq \sqrt{\frac{\omega_{d-1}}{\omega_{d-1}}} \max_{\rho \leq |x| \leq R} |f(x)|.$$ 

Using (41) one arrives at

$$|T_0(f)| \leq \max_{\rho \leq |x| \leq R} |f(x)| \sqrt{\frac{\omega_{d-1}}{\omega_{d-1}}} \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_\rho^R r^{-k} d\sigma_{k,l}. \hspace{1cm} (43)$$

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3. First consider the case that all measures $\sigma_{k,l}$ have supports in the interval $[\rho, R]$ with $R < \infty$ (cf. Remark 10). Then (43) and the Riesz representation theorem for compact spaces yield a representing measure with support in the annulus $\{x \in \mathbb{R}^d : \rho \leq |x| \leq R\}$. The pseudo-positivity of $\mu$ will be proved in item 5. below.

In the general case, we apply the Riesz representation theorem given in [8, p. 41, Theorem 2.5]: there exists a unique signed measure $\sigma$ such that

$$T_0(\sigma) = \int_{\mathbb{R}^d} g \, d\sigma \quad \text{for all } g \in C_c(\mathbb{R}^d).$$

4. Next we will show that the polynomials are integrable with respect to the variation of the representation measure $\sigma$. Let $\sigma = \sigma_+ - \sigma_-$ be the Jordan decomposition of $\sigma$. Following the techniques of Theorem 2.4 and Theorem 2.5 in [8, p. 42], we have the equality

$$\int_{\mathbb{R}^d} g(x) \, d\sigma_+ = \sup \left\{ T_0(h) : h \in C_c(\mathbb{R}^d) \text{ with } 0 \leq h \leq g \right\} \quad (44)$$

which holds for any non-negative function $g \in C_c(\mathbb{R}^d)$. Let $u_R$ be the cut-off function defined in (36). We want to estimate $\int_{\mathbb{R}^d} g(x) \, d\sigma_+$ for the function $g := |x|^N u_R(|x|^2)$. In view of (44), let $h \in C_c(\mathbb{R}^d)$ with $0 \leq h(x) \leq |x|^N u_R(|x|^2)$ for all $x \in \mathbb{R}^d$. Then for the Laplace-Fourier coefficient $h_{k,l}$ of $h$ we have the estimate

$$|h_{k,l}(r)| \leq \sqrt{\int_{S^{d-1}} |h(r\theta)|^2 \, d\theta} \sqrt{\int_{S^{d-1}} |Y_{k,l}(\theta)|^2 \, d\theta} \leq r^N u_R \left( r^2 \right) \sqrt{\omega_{d-1}}.$$ 

According to (41)

$$T_0(h) = |T_0(h)| \leq \sqrt{\omega_{d-1}} \sum_{k=0}^\infty \sum_{l=1}^{a_k} \int_0^\infty r^N r^{-k} \, d\sigma_{k,l} =: D_N.$$ 

It follows that $\int_{\mathbb{R}^d} |x|^N u_R(|x|^2) \, d\sigma_+ \leq D_N$ for all $R > 0$ (note that $D_N$ does not depend on $R$). By the monotone convergence theorem (note that $u_R(x) \leq u_{R+1}(x)$ for all $x \in \mathbb{R}^d$) we obtain

$$\int_{\mathbb{R}^d} |x|^N \, d\sigma_+ = \lim_{R \to \infty} \int_{\mathbb{R}^d} |x|^N u_R(|x|^2) \, d\sigma_+ \leq D_N.$$ 

Similarly one shows that $\int_{\mathbb{R}^d} |x|^N \, d\sigma_- < \infty$ by considering the functional $S = -T_0$. It follows that all polynomials are integrable with respect to $\sigma_+$ and $\sigma_-$. Using similar arguments it is not difficult to see that for all $g \in C^\infty(\mathbb{R}^d) \cap C_{\text{pol}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} g(x) \, d\sigma = \tilde{T}(g). \quad (45)$$
5. It remains to prove that $\sigma$ is pseudo-positive as given by Definition 7. Let $h \in C_c \left( (0, \infty) \right)$ be a non-negative function. The Laplace-Fourier coefficients $f_{k,l}$ of $f(x) := h(|x|) Y_k(x)$ are given by $f_{k,l}(r) = \delta_{kk'} \delta_{ll'} h(r) r^k$ and by it follows that

$$\int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x) \, d\sigma = \tilde{T}(f) = \int_0^\infty f_{k,l}(r) r^{-k} \, d\sigma_{k,l} = \int_0^\infty h(r) \, d\sigma_{k,l}.$$ 

Since $\sigma_{k,l}$ are non-negative measures, the last term is non-negative. According to definition (34), $\sigma$ is pseudo-positive. The proof is complete.

The following is a solution to the modified moment problem as explained in the introduction. It is an immediate consequence of Theorem 9.

**Corollary 11** Let $T : \mathbb{C} [x_1, x_2, \ldots, x_d] \to \mathbb{C}$ be a pseudo-positive definite functional. Let $\sigma_{k,l}, k \in \mathbb{N}_0, l = 1, \ldots, a_k$, be the non-negative measures with supports in $[0, \infty)$ representing the functional $T$ as obtained in Proposition 5. If for any $N \in \mathbb{N}_0$

$$\sum_{k=0}^\infty \sum_{l=1}^{a_k} \int_0^\infty r^N r^{-k} \, d\sigma_{k,l} < \infty,$$

then there exists a pseudo-positive, signed moment measure $\sigma$ such that

$$T(f) = \int f \, d\sigma \quad \text{for all } f \in \mathbb{C} [x_1, x_2, \ldots, x_d].$$

It would be interesting to see whether the summability condition (46) may be weakened, cf. also the discussion at the end of Section 8.

By the uniqueness of the representing measure in the Riesz representation theorem for compact spaces we conclude from Theorem 9:

**Corollary 12** Let $\mu$ be a signed measure with compact support. Then $\mu$ is pseudo-positive if and only if $\mu$ is pseudo-positive definite as a functional on $\mathbb{C} [x_1, x_2, \ldots, x_d]$.

Let us remark that Corollary 12 does not hold without the compactness assumption which follows from well known arguments in the univariate case: Indeed, let $\nu_1$ be a non-negative moment measure on $[0, \infty)$ which is not determined in the sense of Stieltjes; hence there exists a non-negative moment measure $\nu_2$ on $[0, \infty)$ such that $\nu_1(p) = \nu_2(p)$ for all univariate polynomials. Since $\nu_1 \neq \nu_2$ there exists a continuous function $h : [0, \infty) \to [0, \infty)$ with compact support that $\nu_1(h) \neq \nu_2(h)$. Without loss of generality assume that

$$\int_0^\infty h(r) \, d\nu_1 - \int_0^\infty h(r) \, d\nu_2 < 0.$$

For $i = 1, 2$ define $\mu_i = d\theta d\nu_i$, so for any $f \in C \left( \mathbb{R}^d \right)$ of polynomial growth

$$\int f \, d\mu_i = \int_0^\infty \int_{S^{d-1}} f(r\theta) \, d\theta d\nu_i.$$
For a polynomial $f$ let $f_0$ be the first Laplace–Fourier coefficient. Then $\int f d\mu_i = \int_0^\infty f_0(r) d\nu_i$ for $i = 1, 2$. Since $v_1(p) = v_2(p)$ for all univariate polynomials it follows that $\int f d\mu_1 = \int f d\mu_2$ for all polynomials. Then $\mu := \mu_1 - \mu_2$ is a signed measure which is pseudo-positive definite since $\mu(P) = 0$ for all polynomials $P$. It is not pseudo-positive since $\mu_0(h) = \int h(|x|) d\mu < 0$ by (47).

3 Determinacy for pseudo-positive definite functionals

Let $M^*(\mathbb{R}^d)$ be the set of all signed moment measures, and $M^+_*(\mathbb{R}^d)$ be the set of non–negative moment measures on $\mathbb{R}^d$. On $M^*(\mathbb{R}^d)$ we define an equivalence relation: we say that $\sigma \sim \mu$ for two elements $\sigma, \mu \in M^*(\mathbb{R}^d)$ if and only if $\int_{\mathbb{R}^d} f d\sigma = \int_{\mathbb{R}^d} f d\mu$ for all $f \in C_c[0, \infty)$. (47)

Definition 13 Let $\mu \in M^*(\mathbb{R}^d)$ be a pseudo-positive measure. We define

$$V_\mu = \{ \sigma \in M^*(\mathbb{R}^d) : \sigma \text{ is pseudo-positive and } \sigma \sim \mu \} .$$

We say that the measure $\mu \in M^*(\mathbb{R}^d)$ is determined in the class of pseudo-positive measures if $V_\mu$ has only one element, i.e. is equal to $\{ \mu \}$. (48)

Recall that a positive definite functional $\phi : \mathcal{P}_1 \rightarrow \mathbb{R}$ is determined in the sense of Stieltjes if the set

$$W_{\phi}^{Sti} := \left\{ \tau \in M^+_*([0, \infty)) : \int_0^\infty r^m d\tau = \phi(r^m) \text{ for all } m \in \mathbb{N}_0 \right\}$$

has exactly one element, cf. [9, p. 210]. According to Proposition 8, we can associate to a pseudo-positive measure $\mu$ the sequence of non-negative component measures $\mu_{k,l}, k \in \mathbb{N}_0, l = 1, \ldots, a_k$ with support in $[0, \infty)$. The measures $\mu_{k,l}$ contain all information about $\mu$. Indeed, we prove

Proposition 14 Let $\mu$ and $\sigma$ be pseudo-positive measures and let $\mu_{k,l}$ and $\sigma_{k,l}$ be as in Proposition 8. If $\mu_{k,l} = \sigma_{k,l}$ for all $k \in \mathbb{N}_0, l = 1, \ldots, a_k$ then $\mu = \sigma$.

Proof. Let $h \in C_c[0, \infty)$. Then, using the assumption $\mu_{k,l} = \sigma_{k,l}$, we obtain

$$\int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x) d\mu = \int_0^\infty h(t) d\mu_{k,l} = \int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x) d\sigma.$$ (49)

Since each $f \in C_c^\infty(\mathbb{R}^d)$ is a finite linear combination of functions of the type $h(|x|) Y_{k,l}(x)$, we obtain that $\int_{\mathbb{R}^d} f d\mu = \int_{\mathbb{R}^d} f d\sigma$ for all $f \in C_c^\infty(\mathbb{R}^d)$. We apply Proposition 15 to see that $\mu$ is equal to $\sigma$.

The following result is proved in [9, Proposition 3.1]:
**Proposition 15** Let $\mu$ and $\sigma$ be signed measures on $\mathbb{R}^d$. If $\int_{\mathbb{R}^d} f d\mu = \int_{\mathbb{R}^d} f d\sigma$ for all $f \in C_0^\infty(\mathbb{R}^d)$, then $\mu$ is equal to $\sigma$.

We can characterize $V_\mu$ in the case that only finitely many $\mu_{k,l}$ are nonzero.

**Theorem 16** Let $\mu$ be a pseudo-positive measure on $\mathbb{R}^n$ such that $\mu_{k,l} = 0$ for all $k > k_0, l = 1, \ldots, a_k$. Then $V_\mu$ is affinely isomorphic to the set

$$\bigoplus_{k=0}^{k_0} \bigoplus_{l=1}^{a_k} \{ \rho_{k,l} \in W_{\mu_{k,l}}^{Sti} : \int_0^\infty t^{-\frac{1}{2}k} d\rho_{k,l} < \infty \}$$  (49)

where the isomorphism is given by $\sigma \mapsto \sigma_{k,l}^{\psi} = (\sigma_{k,l}^{\psi})_{k=1, \ldots, k_0, l=1, \ldots, a_k}$ and the map $\psi : [0, \infty) \to [0, \infty)$ is defined by $\psi(t) = t^2$, cf. \[25\].

**Proof.** Let $\sigma$ be in $V_\mu$. Let $\sigma_{k,l}$ and $\mu_{k,l}$ be the unique moment measures obtained in Proposition 15. Then

$$\int_0^\infty h(t) d\sigma_{k,l}^{\psi} = \int_0^\infty h(t^2) d\sigma_{k,l} = \int_{\mathbb{R}^n} h(|x|^2) Y_{k,l}(x) d\sigma(x)$$

for all $h \in C_{pol}[0, \infty)$, and an analog equation is valid for $\mu_{k,l}$ and $\mu$. Taking polynomials $h(t)$ we see that $\sigma_{k,l} \in W_{\mu_{k,l}}^{Sti}$ using the assumption that $\mu \sim \sigma$.

Using a simple approximation argument it is easy to see from \[35\] that

$$\int_0^\infty t^{-\frac{1}{2}k} d\sigma_{k,l}^{\psi} = \int_{\mathbb{R}^n} Y_{k,l} \left( \frac{x}{|x|} \right) d\sigma(x).$$

Since $x \mapsto Y_{k,l} \left( \frac{x}{|x|} \right)$ is bounded on $\mathbb{R}^n$, say by $M$, we obtain the estimate

$$\left| \int_0^\infty t^{-\frac{1}{2}k} d\sigma_{k,l}^{\psi} \right| \leq M \int_{\mathbb{R}^n} 1 d|\sigma| < \infty.$$ 

It follows that $\sigma_{k,l}^{\psi} = (\sigma_{k,l}^{\psi})_{k=1, \ldots, k_0, l=1, \ldots, a_k}$ is contained in the set on the right hand side in \[49\].

Let now $\rho_{k,l} \in W_{\mu_{k,l}}^{Sti}$ be given such that $\int_0^\infty t^{-\frac{1}{2}k} d\rho_{k,l} < \infty$ for $k = 1, \ldots, k_0, l = 1, \ldots, a_k$. Define $\sigma_{k,l} = \rho_{k,l}^{\psi-1}$ and $\sigma_{k,l} = 0$ for $k > k_0$. Then by Theorem 16 there exists a measure $\tau \in V_\mu$ such that $\tau_{k,l} = \sigma_{k,l}$. This shows the surjectivity of the map. Let now $\sigma$ and $\tau$ be in $V_\mu$ with $\sigma_{k,l}^{\psi} = \tau_{k,l}^{\psi}$ for $k = 1, \ldots, k_0, l = 1, \ldots, a_k$. The property $\sigma \in V_\mu$ implies that $\sigma_{k,l}^{\psi} \in W_{\mu_{k,l}}^{Sti}$ for all $k \in \mathbb{N}_0, l = 1, \ldots, a_k$, hence $\sigma_{k,l}^{\psi} = 0$ for $k > k_0$, and similarly $\tau_{k,l}^{\psi} = 0$. Hence $\sigma_{k,l} = \tau_{k,l}$ for all $k \in \mathbb{N}_0, l = 1, \ldots, a_k$, and this implies that $\sigma = \tau$ by Proposition 16.

The following is a sufficient condition for a functional $T$ to be determined in the class of pseudo-positive measures.
Theorem 17 Let $T : \mathbb{C} [x_1, x_2, ..., x_d] \to \mathbb{R}$ be a pseudo-positive definite functional. If the functionals $T_{k,l} : \mathbb{C} [x_1] \to \mathbb{C}$ are determined in the sense of Stieltjes then there exists at most one pseudo-positive, signed moment measure $\mu$ on $\mathbb{R}^d$ with

$$T(f) = \int_{\mathbb{R}^d} f \, d\mu \quad \text{for all } f \in \mathbb{C} [x_1, x_2, ..., x_d]. \quad (50)$$

Proof. Let us suppose that $\mu$ and $\sigma$ are pseudo-positive, signed moment measures on $\mathbb{R}^d$ representing $T$. Taking $f = |x|^{2N} Y_{k,l}(x)$ we obtain from $(50)$ that

$$\int_{\mathbb{R}^d} |x|^{2N} Y_{k,l}(x) \, d\mu = T_{k,l}(t^N) = \int_{\mathbb{R}^d} |x|^{2N} Y_{k,l}(x) \, d\sigma.$$

for all $N \in \mathbb{N}_0$. Let $\mu_{k,l}$ and $\sigma_{k,l}$ as in Proposition 8 and consider $\psi : [0, \infty) \to [0, \infty]$ defined by $\psi(t) = t^2$. Then the image measures $\mu_{k,l}^\psi$ and $\sigma_{k,l}^\psi$ are non-negative measures with supports on $[0, \infty)$ such that

$$\int_0^\infty t^N \, d\mu_{k,l}^\psi = T_{k,l}(t^N) = \int_0^\infty t^N \, d\sigma_{k,l}^\psi.$$

Our assumption implies that $\mu_{k,l}^\psi = \sigma_{k,l}^\psi$, so $\mu_{k,l} = \sigma_{k,l}$. Proposition 17 implies that $\mu$ is equal to $\sigma$. \hfill \blacksquare

In the following we want to prove the converse of the last theorem, which is more subtle. We need now some special results about Nevanlinna extremal measures. Let us introduce the following notation: for a non-negative measure $\phi \in M_+^*(\mathbb{R})$ we put

$$[\phi] := \{\sigma \in M_+^*(\mathbb{R}) : \sigma \sim \phi\}.$$

Proposition 18 Let $\nu$ be a non-negative moment measure on $\mathbb{R}$ with support in $[0, \infty)$ which is not determined in the sense of Stieltjes, or applying the notation $W^\text{Sti}_\nu \neq \{\nu\}$. Then there exist uncountably many $\sigma \in W^\text{Sti}_\nu$ such that

$$\int_0^\infty u^{-k} \, d\sigma < \infty \quad \text{for all } k \in \mathbb{N}_0.$$

Proof. In the proof we will borrow some arguments about the Stieltjes problem as given in [13] or [32]. As in the proof of Proposition 4.1 in [32] let $\varphi : (-\infty, \infty) \to [0, \infty)$ be defined by $\varphi(x) = x^2$. If $\lambda$ is a measure on $\mathbb{R}$ define a measure $\lambda^\varphi$ by $\lambda^\varphi(A) := \lambda(-A)$ for each Borel set $A$ where $-A := \{-x : x \in A\}$. The measure is symmetric if $\lambda = \lambda^\varphi$. For each $\tau \in W^\text{Sti}_\nu$ define a measure $\tau^\varphi := \frac{1}{2} \left( \tau^\varphi + (\tau^\varphi)^	op \right)$ which is clearly symmetric, in particular $\nu$ is symmetric. As pointed out in [32], the map $\tau : W^\text{Sti}_\nu \to \nu$ is injective and the image is exactly the set of all symmetric measures in the set $[\nu]$. The inverse map $\tau$ defined on the image space is just the map $\sigma \to \sigma^\varphi$.

It follows that $\nu$ is not determined, so we can make use of the Nevanlinna theory for the indeterminate measure $\nu$, see p. 54 in [1]. We know by formula II.4.2 (9) and II.4.2 (10) in [1] that for every $t \in \mathbb{R}$ there exists a unique Nevanlinna–extremal measure $\sigma_t$ such that

$$\int_0^\infty \frac{d\sigma_t(u)}{u-z} = \frac{A(z) t - C(z)}{B(z) t - D(z)},$$

Here in order to avoid mixing of the notations, we retain the notation $[\phi]$ from the one-dimensional case in [1].
where \( A(z), B(z), C(z), D(z) \) are entire functions. Since the support of \( \sigma_t \) is the zero-set of the entire function \( B(z) t - D(z) \) it follows that the measure \( \sigma_t \) has no mass in 0 for \( t \neq 0 \), and now it is clear that \( \sigma_t([-\delta, \delta]) = 0 \) for \( t \neq 0 \) and suitable \( \delta > 0 \) (this fact is pointed out at least in the reference [9, p. 210]). It follows that

\[
\int_{-\infty}^{\infty} |u|^{-k} d\sigma_t < \infty
\]  

(51)

since the function \( u \mapsto |u|^{-k} \) is bounded on \( \mathbb{R} \setminus [-\delta, \delta] \) for each \( \delta > 0 \). Using the fact that the functions \( A(z) \) and \( B(z) \) of the Nevanlinna matrix are odd, while the functions \( B(z) \) and \( C(z) \) are even, one derives that the measure \( \rho_t := \frac{1}{2} \sigma_t + \frac{1}{2} \sigma_{-t} \) is symmetric. Further from the equation \( A(z) D(z) - B(z) C(z) = 1 \) it follows that \( \rho_t \neq \rho_s \) for positive numbers \( t \neq s \).

By the above we know that \( \rho_{\psi t} \neq \rho_{\psi s} \). This finishes the proof.

**Theorem 19** Let \( \mu \) be a pseudo-positive signed measure on \( \mathbb{R}^d \) such that the summability assumption (11) holds. Then \( V_\mu \) contains exactly one element if and only if each \( \mu_{k,l}^\psi \) is determined in the sense of Stieltjes.

**Proof.** Let \( \mu_{k,l} \) be the component measures as defined in Proposition 8. Assume that \( V_\mu = \{ \mu \} \) but that some \( \tau := \mu_{k_0,l_0}^\psi \) is not determined in the sense of Stieltjes where \( \psi(t) = t^2 \) for \( t \in [0, \infty) \). By Proposition 18 there exists a measure \( \sigma \in W_\tau^{Sti} \) such that \( \sigma \neq \tau \) and \( \int_0^\infty r^{-k} d\sigma < \infty \). By Theorem 9 there exists a pseudo-positive moment measure \( \tilde{\mu} \) representing the functional

\[
\tilde{T}(f) := \sum_{k=0, k \neq k_0}^{\infty} \sum_{l=1, l \neq l_0}^{a_k} \int_0^\infty f_{k,l}(r) r^{-k} d\mu_{k,l} + \int_0^\infty f_{k_0,l_0}(r) r^{-k} d\sigma^\psi r^{-1}.
\]

Then \( \tilde{\mu} \) is different from \( \mu \) since \( \sigma^\psi \neq \mu_{k_0,l_0} \) and \( \tilde{\mu} \in V_\mu \) since \( \sigma \in W_\tau^{Sti} \). This contradiction shows that \( \mu_{k_0,l_0}^\psi \) is determined in the sense of Stieltjes. The sufficiency follows from Theorem 17. The proof is complete.

**4 Polyharmonic Gauß–Jacobi cubatures**

In this section we will prove the main result of the paper, the existence of the polyharmonic Gauß–Jacobi cubature of order \( s \). The proof is based on application of the famous Chebyshev extremal property of the Gauß–Jacobi measure.

**Theorem 20** Let \( 0 \leq \rho < R \leq \infty \). Let \( \mu \) be a pseudo-positive signed measure with support in the closed annulus \( A_{\rho,R} \) such that

\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\mathbb{R}^d} Y_{k,l} \left( \frac{x}{|x|} \right) d\mu < \infty.
\]

Then for each natural number \( s \) there exists a unique pseudo-positive, signed measure \( \sigma^{(s)} \) with support in \( A_{\rho,R} \) such that

\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\mathbb{R}^d} Y_{k,l} \left( \frac{x}{|x|} \right) d\sigma^{(s)} < \infty.
\]
(i) The support of each component measure \( \sigma_{k,l}^{(s)} \) of \( \sigma^{(s)} \) (defined by (35)) has cardinality \( \leq s \).

(ii) \( \int P d\mu = \int P d\sigma^{(s)} \) for all polynomials \( P \) with \( \Delta^{2s} P = 0 \).

**Proof.** By Proposition 8 the following identity holds

\[
\int_{\mathbb{R}^d} f(x) d\mu(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\rho}^{R} f_{k,l}(r) r^{-k} d\mu_{k,l}(r)
\]

for any \( f \in C_{\text{pol}}^{\infty}(\mathbb{R}^d) \) where \( \mu_{k,l}(h) = \int h(|x|) Y_{k,l}(x) d\mu(x) \). It is clear that \( \mu_{k,l} \) has support in the interval \([\rho, R]\). If the cardinality of the support of \( \mu_{k,l} \) is \( \leq s \) we define \( \sigma_{k,l}^{(s)} := \mu_{k,l} \). If the cardinality is strictly larger than \( s \) we define \( \sigma_{k,l}^{(s)} \) as the non-negative measure such that

\[
\int_{\mathbb{R}} r^{2j} d\sigma_{k,l}^{(s)}(r) = \int_{\mathbb{R}} r^{2j} d\mu_{k,l}(r)
\]

for all \( j = 0, \ldots, 2s-1 \). The existence of \( \sigma_{k,l}^{(s)} \) is proved as follows: Let \( \psi : [\rho, R] \to [\rho^2, R^2] \) be the map \( \psi(t) = t^2 \). Then the image measure \( \mu_{k,l}^{\psi} \) is a measure on \([\rho^2, R^2]\) and its support has clearly cardinality \( > s \). Let \( \nu_{k,l}^{(s)} \) be the Gauß–Jacobi quadrature of \( \mu_{k,l}^{\psi} \). From the Gauß–Jacobi quadrature formula (16) (see also the footnote after it) follows that \( \nu_{k,l}^{(s)} \) has support in the open interval \((\rho^2, R^2)\) and

\[
\int_{\rho^2} r^{2j} d\nu_{k,l}^{(s)}(r) = \int_{\rho^2} r^{2j} d\mu_{k,l}(r)
\]

for \( j = 0, \ldots, 2s-1 \). Now it is easily seen that \( \sigma_{k,l}^{(s)} := \left( \nu_{k,l}^{(s)} \right)^{\psi^{-1}} \) satisfies (54).

We now define a functional \( T : \mathbb{C} [x_1, x_2, \ldots, x_d] \to \mathbb{C} \) by putting

\[
T^{(s)}(f) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} \int_{\rho}^{R} f_{k,l}(r) r^{-k} d\sigma_{k,l}^{(s)}.
\]

Let us show that

\[
\int_{\mathbb{R}^d} P(x) d\mu(x) = T^{(s)}(P)
\]

for all polynomials \( P \) with \( \Delta^{2s} P(x) = 0 \). Indeed, according to (29) the Laplace-Fourier series of a polyharmonic polynomial of order \( 2s \) can be written as

\[
P(x) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} p_{k,l}(r^2) r^{k} Y_{k,l}(\theta)
\]

and the univariate polynomials \( p_{k,l}(t) \) have degree \( \leq 2s - 1 \) (see e.g. Theorem 10.42, Remark 10.43]). Combining (57) with (56), (54) and (53) gives (56).
Now we want to prove that $T(s)$ can be represented by a signed measure. We claim that
\[ \int_{\rho}^{R} r^{-k} d\sigma(s)(r) \leq \int_{\rho}^{R} r^{-k} d\mu_{k,l}(r) < \infty \] (58)
for all $k \in \mathbb{N}_0, l = 1, \ldots, a_k$. If $\sigma(s) = \mu_{k,l}$ there is nothing to prove, so we can assume that the cardinality of the support of $\mu_{k,l}$ is bigger than $s$. After taking the image measures under the map $\psi$ we see that we have to prove
\[ \int_{\rho^2}^{R^2} t^{-\frac{1}{2}k} d\nu(s)(t) \leq \int_{\rho^2}^{R^2} t^{-\frac{1}{2}k} d\mu_{k,l}(t). \] (59)
For $\rho > 0$ this follows from the Chebyshev extremal property of the Gauß–Jacobi measure (see e.g. [32, Chapter 4, Theorem 1.1]) applied to the function $f(t) := t^{-\frac{1}{2}k}$. The same result works in the case $\rho = 0$ but due to the singularity of $f$ we have to use essentially the fact that all points of $\nu(s)$ are in the open interval $(0, R^2)$ and to apply Remark 1.2 in Chapter 4 of [32].

By our assumption (52) and by (58) we can apply Theorem 11 and (i) is proved. Property (ii) follows from (56) which we have proved above.

Let us prove the uniqueness of $\sigma(s)$. Assume that $\tau$ is a signed pseudo-positive measure with compact support, and with properties (i) and (ii). Since $\tau$ is pseudo-positive there exists by Proposition 8 univariate measure $\tau_{k,l}$ such that
\[ \int h(t) d\tau_{k,l} = \int h(|x|) Y_{k,l}(x) d\tau(x) \] (61)
for any polynomially bounded continuous function $h$. Since $\Delta^{2s}(|x|^{2j} Y_{k,l}(x)) = 0$ for $j = 0, \ldots, 2s - 2$, we infer that
\[ \int |x|^{2j} Y_{k,l}(x) d\tau(x) = \int |x|^{2j} Y_{k,l}(x) d\mu(x). \]
Hence $\int t^{2j} d\tau_{k,l} = \int t^{2j} d\mu_{k,l}$ for $j = 0, \ldots, 2s - 2$, so
\[ \int t^{j} d\tau_{k,l} = \int t^{j} d\mu_{k,l}. \] (62)

It is curious that Stieltjes proved that the Gauß–Jacobi quadrature measure solves a three-dimensional spherically symmetric extremal problem with a singular function $f(t) = \frac{1}{\sqrt{t}}$, see the complete description in [32] Chapter 4.2, formula (2.6).
By property (i) the support $\tau_{k,l}$ has cardinality $\leq s$, hence $\tau_{k,l}^\psi$ has cardinality $\leq s$. The uniqueness of the Gauß-Jacobi quadrature shows that $\tau_{k,l}^\psi$ is equal to $\sigma_{k,l}^\psi$, which means that $\tau_{k,l} = \sigma_{k,l}$. If the support of $\mu_{k,l}$ has less than $s$ points then $\sigma_{k,l}^\psi$ in our construction is defined to be $\mu_{k,l}$.

Proposition 14 yields $\tau = \sigma^{(s)}$.

**Definition 21** The measure $\sigma^{(s)}$ constructed in Theorem 20 will be called the **polyharmonic Gauß-Jacobi measure of order $s$** for the measure $\mu$.

The following is an analog to the theorem of Stieltjes about the convergence of the univariate Gauß–Jacobi quadrature formulas.

**Theorem 22** Let $0 < R < \infty$ and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order $s$ for the measure $\mu$, obtained in Theorem 20. Then

$$\int f(x) \, d\sigma^{(s)} \to \int f(x) \, d\mu$$

holds for every function $f \in C(B_R)$.

**Proof.** Item (ii) of Theorem 20 implies that for any polynomial $P$ the convergence $T^{(s)}(P) \to P$ holds for $s \to \infty$. Theorem 14.4.4 in [16] shows that the convergence $T^{(s)}(f) \to f$ carries over to all continuous functions $f : B_R \to \mathbb{C}$ provided there exists a constant $C > 0$ such that for all natural numbers $s$ and all $f \in C(B_R)$

$$|T^{(s)}(f)| \leq C \max_{|x| \leq R} |f(x)|.$$

But that is just estimate (43) in the proof of Theorem 11. The proof is complete.

Using the same techniques as in Theorem 20 we may prove a generalization of the Chebyshev extremal property of the Gauß–Jacobi quadrature:

**Theorem 23** Let $0 \leq \rho < R < \infty$ and let $\mu$ be a pseudo-positive signed measure with support in $A_{\rho,R}$ satisfying the summability condition (52) and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order $s$. Let $f \in C^{2s}(\mathbb{R}^d)$ be such that for all $t \in (\rho^2, R^2)$ holds

$$\frac{d^{2s}}{dt^{2s}} \left[ f_{k,l} \left( \sqrt{t} \right) t^{-\frac{k}{2}} \right] \geq 0,$$

for all $k \in \mathbb{N}_0$, $l = 1, 2, \ldots, a_k$. Then the following inequality

$$\int f(x) \, d\sigma^{(s)} \leq \int f(x) \, d\mu$$

holds.
Although the measures $\sigma_{k,l}^{(s)}$ are based on point evaluations, it is clear that our approximation measures $\sigma^{(s)}$ are not point evaluations. The following gives a description of the support of the polyharmonic Gauß–Jacobi measure $\sigma^{(s)}$ when only finitely many measures $\mu_{k,l}$ are non-zero:

**Proposition 24** Suppose that $\sigma_{k,l}$, $k = 1, \ldots, k_0$, $l = 1, \ldots, a_k$, are non-negative measures with finite support in the open interval $(0, \infty)$ and suppose that $\sigma_{k,l} = 0$ for all $k > k_0$. Then the support of the representing measure of the functional $T : C [x_1, x_2, \ldots, x_d] \rightarrow \mathbb{C}$ defined by (33) is contained in the union of finitely many spheres with positive radius and with center 0.

**Proof.** Let $S_{k,l}$ be the finite support of $\sigma_{k,l}$ and let $S$ be the union of all sets $S_{k,l}$ with $k = 1, \ldots, k_0$, $l = 1, \ldots, a_k$. Let $\overline{S} = \{ x \in \mathbb{R}^d : |x| \in S \}$. We show that the support of $\sigma$ is contained in $\overline{S}$. Indeed, one can estimate as in (33)

$$|T(P)| \leq \max_{x \in \overline{S}} |P(x)| \sqrt{2d-1} \sum_{k=0}^{k_0} \sum_{l=1}^{a_k} \int_0^R r^{-k} d\sigma_{k,l}(t).$$

By the Riesz representation theorem, $T$ can be represented by a signed measure $\sigma$ which has support in the compact set $\overline{S}$. □

**Remark 25** An interesting characteristic feature of the classical Gauß–Jacobi quadrature measure is the minimality of its support among all non-negative measures which are exact of the same degree $2s - 1$. One might see above some analogy with this phenomenon if one considers $\mu_{k,l}(r)$ and $\sigma_{k,l}^{(s)}(r)$ as measures defined on the space $K = \{ (k, l, r) : k \in \mathbb{N}_0, l = 1, 2, \ldots, a_k, r \in [0, \infty) \}$.

## 5 Markov type error estimates

In this section we want to give an error estimate for our cubature formula. The proof is based on the Markov estimate for the Gauß-Jacobi measure provided in (22).

Let $s \in \mathbb{N} \cup \{ \infty \}$. For an open subset $U$ of $\mathbb{R}^d$ we denote by $C^s(U)$ the space of all $f \in C(U)$ which are continuously differentiable in $U$ up to the order $s$.

**Theorem 26** Let $0 \leq \rho < R < \infty$ and let $\psi : [0, \infty) \rightarrow [0, \infty)$ be defined by $\psi(t) = t^2$. Let $\mu$ be a pseudo-positive signed measure with support in $A_{p,R}$ satisfying the summability condition (22), and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order $s$. Define for every $f \in C(A_{p,R})$ the error functional

$$E_s(f) := \int f(x) d\mu(x) - \int f(x) d\sigma^{(s)}(x).$$

If $f \in C^{2s}(A_{p,R}) \cap C(A_{p,R})$ then $E_s(f)$ is lower equal to

$$\frac{1}{(2s)!} \sum_{k=0}^{\infty} \sum_{l=0}^{a_k} \sup_{\rho^2 < \xi < R^2} \left| \frac{d^{2s}}{dt^{2s}} \left[ f_{k,l} \left( \sqrt{t} \right) t^{-\frac{1}{2}k} \right](\xi) \right| \int_{R^2} |Q_{k,l}(t)|^2 d\mu^\psi.$$
Here \( Q_{k,l}^s(t) \) is the orthogonal polynomial of degree \( s \) with respect to the measure \( \mu_{k,l}^\psi \), normalized so that the leading coefficient is equal to 1; if the support of \( \mu_{k,l} \) has less than \( s \) points, \( Q_{k,l}^s \) is defined to be 0.

**Proof.** Since \( f \in C^{2s} \left( A_{p,R}^\psi \right) \cap C \left( A_{p,R} \right) \) it is easy to see that the Laplace-Fourier coefficients \( f_{k,l} \in C^{2s} \left( \rho, R \right) \cap C \left( \rho, R \right) \). Let \( \mu_{k,l} \) and \( \sigma_{k,l} \), \( k \in \mathbb{N}_0 \), \( l = 1, \ldots, a_k \), and \( \sigma(s) \) be as in Theorem \( 20 \). From the definitions it follows

\[
E_s(f) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\rho}^{R} f_{k,l}(r) r^{-k} d\mu_{k,l} - \int_{\rho}^{R} f_{k,l}(r) r^{-k} d\sigma_{k,l}^{(s)}.
\]

Further \( f_{k,l}(r) r^{-k} \) is integrable with respect to \( \mu_{k,l} \) since \( f_{k,l} \) is continuous on \( \left[ \rho, R \right] \) and condition \( \left[ 22 \right] \) holds. Let us fix the pair of indices \( (k, l) \). If the support of \( \mu_{k,l} \) has less than \( s \) points we know that \( \mu_{k,l} = \sigma_{k,l}^{(s)} \). So assume that the support of \( \mu_{k,l} \) has at least \( s \) points. Then the support of \( \mu_{k,l} \) has at least \( s \) points and in our construction \( \nu_{k,l}(s) \) is the Gauß-Jacobi measure of \( \mu_{k,l}^\psi \). Consequently

\[
e(f_{k,l}) := \int_{\rho}^{R} f_{k,l}(r) r^{-k} d\mu_{k,l}(r) - \int_{\rho}^{R} f_{k,l}(r) r^{-k} d\sigma_{k,l}^{(s)}(r)
\]

\[
= \int_{R^2} f_{k,l} \left( \sqrt{t} \right) t^{-\frac{1}{2} k} d\mu_{k,l}(t) - \int_{R^2} f_{k,l} \left( \sqrt{t} \right) t^{-\frac{1}{2} k} d\sigma_{k,l}^{(s)}(t)
\]

By the proof of Markov’s error estimate \( \left[ 22 \right] \) given in \( \left[ 16 \right] \) one easily obtains with \( g_{k,l}(t) := f_{k,l} \left( \sqrt{t} \right) t^{-\frac{1}{2} k} \) the inequality

\[
e(f_{k,l}) \leq \frac{1}{(2s)!} \sup_{\rho^2 < \xi < R^2} \left| g_{k,l}^{(2s)}(\xi) \right| \int_{\rho^2}^{R^2} \left| Q_{k,l}^s(\xi) \right|^2 d\mu_{k,l}(\xi)
\]

The proof is complete. \( \blacksquare \)

In the following we want to give a Markov type error estimates for holomorphic functions \( f \). We will need the following property which was observed in \( \left[ 9 \right] \):

**Lemma 27** Let \( f \in C^\infty \left( B_\tau^2 \right) \). Then \( f_{k,l} \in C^\infty \left( [0,R] \right) \) and \( \frac{d^m}{dr^m} f_{k,l}(0) = 0 \) for \( m = 0, \ldots, k - 1 \).

**Lemma 28** Let \( f \) be a holomorphic function on the open ball \( B_\tau^2 := \{ w \in \mathbb{C}^d : \sum_{j=1}^{d} |w_j|^2 < \tau^2 \} \) for \( \tau > 0 \). Let \( f_{k,l} \) be the Laplace-Fourier coefficient of \( f \) given by \( \left[ 4 \right] \) and let \( p_{k,l}(t) \) be defined by the equation \( f_{k,l}(r) = p_{k,l}(r^2) r^k \) for \( 0 < r < \tau \). Let \( \rho \) and \( t \) satisfy \( 0 \leq t^2 < \rho < \tau \). Then

\[
\left| \frac{d^s}{dr^s} p_{k,l}(t) \right| \leq \sqrt{\omega_d} \max_{u \in [0,2\pi], \theta \in \mathbb{R}^{d-1}} \left| f \left( e^{iu} \rho \theta \right) \right| \frac{\rho^{2-k} s!}{(\rho^2 - t)^{s+1}}
\]

(63)

hold for all \( s = 0, 1, 2, \ldots \)
Proof. Let $\theta \in S^{d-1}$. The map $\varphi_\theta : \{ z \in \mathbb{C} : |z| < \tau \} \to B^C_\tau (0)$ defined by $\varphi_\theta (z) = z \theta$ is clearly holomorphic. Hence $f_\theta$ defined by $f_\theta (z) = f (z \theta) = f \circ \varphi_\theta (z)$ is holomorphic. It follows that $f_{k,l} (z)$ defined by

$$f_{k,l} (z) = \int_{S^{d-1}} f (z \theta) Y_{k,l} (\theta) \, d\theta \quad (64)$$

is a holomorphic extension of $f_{k,l}$ to $\{ z \in \mathbb{C} : |z| < \tau \}$. Cauchy’s inequality shows that for $\rho = |z|$

$$|f_{k,l} (z)|^2 \leq \int_{S^{d-1}} |f (z \theta)|^2 \, d\theta \cdot \int_{S^{d-1}} |Y_{k,l} (\theta)|^2 \, d\theta \leq \omega_d \max_{u \in [0, 2\pi], \theta \in S^{d-1}} |f (e^{i\theta} \rho)|^2.$$

The Cauchy estimates $|g^{(k)} (0) | \leq \frac{k!}{\rho^k} \max_{|z| = \rho} |g (z)|$ for a holomorphic function $g$ and the last estimate imply for $0 < \rho < \tau$

$$\left| \frac{d^{m+k}}{dz^{m+k}} f_{k,l} (0) \right| \leq \sqrt{\omega_d} \max_{u \in [0, 2\pi], \theta \in S^{d-1}} |f (e^{i\theta} \rho)| \cdot \frac{(k+m)!}{\rho^{m+k}} \quad (65)$$

Let us write $f_{k,l} (z) = \sum_{m=k}^{\infty} \frac{1}{m!} \frac{d^{m}}{dr^{m}} f_{k,l} (0) \cdot z^m$ for $|z| < \tau$. It is known that $r^{-k} f_{k,l} (r)$ is an even function, hence we can write

$$p_{k,l} (r^2) = r^{-k} f_{k,l} (r) = \sum_{m=0}^{\infty} \frac{1}{(k+2m)! (m-s)!} \frac{d^{2m+k}}{dr^{2m+k}} f_{k,l} (0) \cdot r^{2m},$$

Then for $t = r^2$ we have

$$\frac{ds}{dt} p_{k,l} (t) = \sum_{m=s}^{\infty} \frac{1}{(k+2m)! (m-s)!} \frac{d^{2m+k}}{dr^{2m+k}} f_{k,l} (0) \cdot t^{m-s}.$$ 

Now (65) implies

$$\left| \frac{ds}{dt} p_{k,l} (t) \right| \leq \sqrt{\omega_d} \max_{u \in [0, 2\pi], \theta \in S^{d-1}} |f (e^{i\theta} \rho)| \cdot \frac{1}{\rho^{k+2s}} \sum_{m=s}^{\infty} \frac{m!}{(m-s)!} \left( \frac{t}{\rho^2} \right)^{m-s}.$$ 

Since for $|t| < 1$ we have

$$\sum_{m=s}^{\infty} \frac{m!}{(m-s)!} t^{m-s} = \frac{d^{s}}{dt^{s}} \sum_{m=0}^{\infty} m^s = \frac{d^{s}}{dt^{s}} \frac{1}{1-t} = s! (1-t)^{-s-1}$$

a straightforward computation yields (64). $\blacksquare$

Now combining Theorem 26 and the last Lemma, we obtain the final estimate.

**Theorem 29** Let $0 < R < \infty$ and let $\mu$ be a pseudo-positive signed measure with support in $B_R$ satisfying the summability condition (52) and let $\sigma^{(s)}$ be the polyharmonic Gauß-Jacobi measure of order $s$. Then the following error estimate

$$E (f) \leq \frac{\sqrt{\omega_d \rho^2}}{(\rho^2 - \tau^2)^{2s+1}} \max_{u \in \mathbb{C}^n, |w| \leq \rho} |f (w)| \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \frac{1}{\rho^k} \int_0^R |Q_{k,l} (t)|^2 \, d\mu_{k,l}^\psi (t)$$

holds for all functions $f : B_R \to \mathbb{C}$ which possess a holomorphic extension to the complex ball $B^C_\tau$ for $\tau > R$ and for any $\rho$ with $R < \rho < \tau$. 

26
6 Polyharmonic Gauß–Jacobi cubature in the annulus

Let us imagine a function \( f \) which is say holomorphic in the ball \( B_R \) with some singularities in the smaller ball \( B^\circ_\rho \), and one needs to find the integral of the function on the annulus \( A_\rho,R \) with an estimate for the error of approximation. This example is a motivation to consider polyharmonic Gauß–Jacobi cubatures in the annulus \( A_\rho,R \) which generalize the construction of Section 4.

By the results in Section 4 the polyharmonic Gauß-Jacobi measure \( \sigma(s) \) of order \( s \) has support in \( A_\rho,R \) and it is exact on the space of all polynomials \( f \) such that \( \Delta^2s f = 0 \).

In the present section we will seek polyharmonic Gauß–Jacobi cubatures which are exact on the larger space

\[
PH^{2s} (A_\rho,R) = \left\{ f \in C (A_\rho,R) \cap C^{2s} (A^\circ_\rho,R) : \Delta^{2s} f (x) = 0 \text{ for all } x \in A^\circ_\rho,R \right\}
\]

where \( A^\circ_\rho,R \) is the interior of \( A_\rho,R \). Exactness with respect to \( PH^{2s} (A_\rho,R) \) is related to the expectation that the integrals of functions with singularities in the inner open ball \( \{|x| < \rho\} \) will be better approximated.

It turns out that the problem can be solved in a way very similar to Theorem 20. The proof is so far based on the “generalized Gauß–Jacobi quadratures for Chebyshev systems” which have been developed mainly by A. Markov, [32, Chapter 4]. We restrict our discussion to the case of compact annulus which is less technical. Let us now formulate the result precisely:

**Theorem 30** Let \( 0 < \rho < R < \infty \). Let \( \mu \) be a pseudo-positive signed measure with support in \( A_\rho,R \) such that

\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\rho}^{R} r^{-k} d\mu_{k,l} (r) < \infty.
\]

(66)

Let \( s \) be a natural number. Then there exists unique pseudo-positive measure \( \tau^{(2s)} \) with support in \( A_\rho,R \) such that

(i) The cardinality of the support of each component measure \( \tau^{(2s)}_{k,l} \), \( k \in \mathbb{N}_0 \), \( l = 1, 2, ..., a_k \), is \( \leq 2s \).

(ii) \( \int f (x) d\tau^{(2s)} (x) = \int f (x) d\mu (x) \) for all \( f \in PH^{2s} (A_\rho,R) \).

Let us compare the result with Theorem 20. In the latter case we obtained a measure \( \sigma^{(s)} \) with support in \( A_\rho,R \) which is exact for all polynomials \( P \) with \( \Delta^{2s} P = 0 \). The support of the component measure \( \sigma^{(s)}_{k,l} \) has at most \( s \) points. In contrast, the component measure \( \tau^{(2s)}_{k,l} \) of the solution \( \tau^{(2s)} \) has a support of cardinality \( \leq 2s \) which is twice bigger. This is caused by the fact that \( \tau^{(2s)} \) is exact on the larger subspace \( PH^{2s} (A_\rho,R) \).

For the proof we will first need the representation of a polyharmonic function in the annulus which is somewhat more sophisticated than that of a polynomial as given in formula (29). Let us introduce the operators

\[
L(k) := \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{k(n+k-2)}{r^2}
\]

(67)
which may be written as
\[ L_{(k)}f(r) = \frac{1}{r^{n+k-1}} \frac{d}{dr} \left[ r^{n+2k-1} \frac{d}{dr} \left( \frac{1}{r^k f(r)} \right) \right], \quad (68) \]

see e.g. (10.18) in [31]. Let \( L_{(k)}^{2s} \) denote the 2s-th iterate of \( L_{(k)} \). These operators are the radial part of the polyharmonic operators \( \Delta^{2s} \). A set of the 4s linearly independent solutions of the equation
\[ L_{(k)}^{2s}f(r) = 0 \quad \text{for } r > 0, \quad (69) \]
can be explicitly constructed; e.g. for \( k \geq 4s \) a set of solutions is given by
\[ r^{2j-k} \quad j = 0, 1, \ldots, s-1; \quad r^{2j+k} \quad j = 0, 1, \ldots, 2s-1. \quad (70) \]

For \( k < 4s \) one has to be careful with multiplicities, and we refer to [31] for an explicit description.

We have the following result (see e.g. Theorem 10.39 in [31]):

**Proposition 31** Let \( h \in PH^{2s}(A_{\rho,R}) \). Then the Laplace–Fourier series
\[ h(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} h_{k,l}(r) Y_{k,l}(\theta) \] (71)
converges absolutely and uniformly on compact subsets of \( A_{\rho,R} \). The Laplace–Fourier coefficients \( h_{k,l} \) are solutions of (69).

In the following we will mimic the proof of Theorem [20]. The measures \( \sigma_{k,l}^{(s)} \) in the proof of Theorem [20] had the feature that they were exact on the solutions 1, \( r^2, \ldots, r^{4s-1} \). Proposition [31] shows that we need now quadratures which are exact on the solutions of (69). This motivates to recall the theory of A. Markov and M. Krein on quadratures for Chebyshev systems.

**Definition 32** Let \( u_0, \ldots, u_N \) be continuous functions on \([a,b] \). We say that they form a Chebyshev system of order \( N+1 \) on \([a,b] \) if every non–trivial linear combination \( \sum_{j=0}^{N} \gamma_j u_j(t) \) has at most \( N \) zeros on \([a,b] \), i.e. the determinant
\[ \det (u_j(t_i))_{i,j=0}^{N} \] (72)
is not zero on \([a,b] \). The system \( u_0, \ldots, u_N \) is \( T_+ \) on \([a,b] \) if \( \det (u_j(t_i))_{i,j=0}^{N} > 0 \) holds for every choice of \( t_j \in [a,b] \) with \( t_0 < t_1 < \ldots < t_N \).

Note that the definition of a \( T_+ \)-system depends on the order of the functions \( u_0, \ldots, u_N \).

The following theorem is the generalization of the Gauß-Jacobi quadratures for Chebyshev systems (see Theorem 1.1 and Remark 1.2 in Chapter 4, and Theorem 1.4 in Chapter 2 in [32]).
Theorem 33 Let $N = 2s - 1$ for a natural number $s$ and let $\sigma$ be a non-negative measure on $[a, b]$ with cardinality of the support $> s$. Let the continuous functions $u_0, \ldots, u_N$ be a Chebyshev system on the interval $[a, b]$ and assume that $u_{N+1} := \Omega$ is a continuous function on $[a, b]$. If $u_0, \ldots, u_{N+1}$ is a $T_+-$system on $[a, b]$ then there exists a unique measure $\sigma^{(s)}$ with support of cardinality $s$ such that
\[
\int_a^b u_j(t) \, d\sigma(t) = \int_a^b u_j(t) \, d\sigma^{(s)}(t) \quad \text{for } j = 0, 1, \ldots, N. \quad (73)
\]
The support of $\sigma^{(s)}$ is contained in the open interval $(a, b)$.

The measure $\sigma^{(s)}$ is called in [32] the "lower chief representation" which is also very natural to be called Gauß–Jacobi–Markov quadrature, and we will use this name further.

A second major result in the Krein-Markov theory is the extremal property of the truncated moment problem due to Chebyshev, Markov and Stieltjes.

Theorem 34 With the notations and assumptions of Theorem 33 let $\sigma^{(s)}$ be the Gauß–Jacobi–Markov quadrature of $\sigma$. The measure $\sigma^{(s)}$ attains the minimum in the problem
\[
\min_{\nu} \int_a^b \Omega(t) \, d\nu(t) \quad (74)
\]
where $\nu$ ranges over all non-negative measures $\nu$ such that
\[
\int_a^b u_j(t) \, d\nu(t) = \int_a^b u_j(t) \, d\sigma(t) \quad \text{for } j = 0, 1, \ldots, N.
\]

We return now to our case of polyharmonic cubatures on annuli. At first we note

Proposition 35 Any linear independent system $R_{k,j}^s, j = 1, 2, \ldots, 4s, a/s$ of solutions of (69) is a Chebyshev system of order $2s$ on every interval $[a, b]$ with $a > 0$. For $k \geq 2s$ the system of solutions in (70) is a $T_+-$system on $[a, b]$ with $a > 0$.

The proof follows from the results in Section II.5 and Theorem II.5.2 in [32] and uses the representation (68). In view of the Krein-Markov theory we need the following stronger result:

Proposition 36 Let $k \in \mathbb{N}_0$, $l = 1, 2, \ldots, a_k$ be fixed, with $k \geq 4s$, and define $u_{4s} = \Omega \equiv 1$. If we denote by $u_0, \ldots, u_{4s-1}$ the system of solutions in (70) then $u_0, \ldots, u_{4s}$ is a $T_+-$system on $[a, b]$ for $a > 0$.

Proof. By the example in [32] Chapter II, Section 2.1(c)] the system
\[
\left\{ e^{t(2j-k)}, \quad j = 0, 1, \ldots, 2s - 1; \quad e^0; \quad e^{t(2j+k)}, \quad j = 0, 1, \ldots, 2s - 1 \right\}
\]

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is a $T_\pm$-system since the numbers $2j - k$ and $2j + k$ are all different due to $k \geq 4s$. Then the reordered system
\[
\begin{align*}
\{e^{i(2j-k)}, & \quad j = 0, 1, \ldots, 2s - 1; \\
\{e^{i(2j+k)}, & \quad j = 0, 1, \ldots, 2s - 1; \}
\end{align*}
\]
has the same determinant sign of (72) as the above. By a change of the variable $r = e^t$, one concludes that the system $u_0, \ldots, u_{4s}$ is a $T_\pm$-system.

Now we are prepared to make the proof.

**Proof of Theorem 30.** Fix a pair of indices $(k,l)$ with $k \in \mathbb{N}_0$, $l = 1, 2, \ldots, a_k$, and let $\mu_{k,l}$ be the component measure. If the support of $\mu_{k,l}$ has less than $2s$ points, put $\tau_{k,l}(2s) = \mu_{k,l}$. Assume now it has more than $2s$ points. Let $u_0, \ldots, u_{4s-1}$ be the system (70). By Markov–Krein’s Theorem 33 applied to $\sigma := \mu_{k,l}$ there exists a Gauß–Jacobi–Markov measure $\tau_{k,l}(2s)$ with support in $(\rho,R)$, and its support has cardinality $2s$. An essential point is to prove that (at least) for sufficiently large $k$ one has
\[
\int_{\rho}^{R} 1d\tau_{k,l}(2s) (r) \leq \int_{\rho}^{R} 1d\mu_{k,l} (r) .
\]
(75)

For $k \geq 4s$ this follows immediately from Markov–Krein’s Theorem 34 by means of Proposition 36.

Further we proceed as in the proof of Theorem 20. We want to define a functional $T^{(2s)}$ on $C(A_\rho,R)$ by putting
\[
T^{(2s)} (f) := \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\rho}^{R} f_{k,l} (r) d\tau_{k,l}(2s) (r)
\]
(76)
for $f \in C(A_\rho,R)$ where $f_{k,l}$ are its Laplace–Fourier coefficients. Indeed, by using the standard estimate for the Laplace-Fourier coefficients the inequality
\[
\left| T^{(2s)} (f) \right| \leq C \max_{\rho \leq |x| \leq R} |f(x)| \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\rho}^{R} d\tau_{k,l}(2s) (r)
\]
is easily established for all $f \in C(A_\rho,R)$. Now with (75) and our assumption (66) it follows that $T^{(2s)}$ is well-defined. We may apply the Riesz representation theorem and obtain a representing measure, denoted by $\tau^{(2s)}$, with support in $A_\rho,R$. Since the constant function is in $C(A_\rho,R)$ it is clear that $\tau^{(2s)}$ is a finite measure. Let us remark that due to our assumption (66) the identity
\[
\int_{A_\rho,R} f (x) d\mu = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_{\rho}^{R} f_{k,l} (r) d\mu_{k,l} (r)
\]
holds for all $f \in C(A_\rho,R)$, since the right hand side defines a continuous functional on $C(A_\rho,R)$ which agrees with $\mu$ on the dense subspace $C^\infty(A_\rho,R)$. Due to the exactness property of all measures $\tau_{k,l}(2s)$ and the representation (70), it follows that $\tau^{(2s)}$ satisfies ii) of Theorem 30 for all $f \in PH^{2s}(A_\rho,R)$. 

The pseudo-positivity of $\tau^{(2s)}$ follows from Corollary 12 since $T^{(2s)}$ is clearly pseudo-positive definite. The uniqueness of $\tau^{(2s)}$ follows from the uniqueness of the Gauß–Jacobi–Markov measure $\tau^{(2s)}_{k,l}$ as in the proof of Theorem 20.

7 Polyharmonic Gauß–Jacobi cubature in the cylinder (periodic strip)

The concept of pseudo-positivity which we have studied so far depends on the expansion of the polyharmonic functions in Laplace-Fourier series which uses the rotational symmetry of the ball and the annulus. The polyharmonic Gauß-Jacobi cubature in the ball was defined respectively by the application of the Gauß-Jacobi quadrature to the Laplace-Fourier coefficients. It is natural to extend this concept on expansions available in other domains with symmetries, and we will do so for the case of the cylinder (which may be considered also as a periodic strip), where the Fourier series is the natural expansion.

Let now $-\infty \leq a < b \leq \infty$. We consider functions $f : [a, b] \times \mathbb{R}^{d-1} \rightarrow \mathbb{C}$ depending on $x = (t, y)$ in the strip $[a, b] \times \mathbb{R}^{d-1}$ which are $2\pi$-periodic with respect to the variable $y$, i.e. satisfy the equality

$$ f(t, y + 2\pi k) = f(t, y) \quad \text{for all } t \in \mathbb{R}, \ y \in \mathbb{R}^{d-1}, \ k \in \mathbb{Z}^{d-1}. $$

Let us introduce the cylinder

$$ X = [a, b] \times \mathbb{T}^{d-1} $$

where $\mathbb{T}^{d-1} = \mathbb{S}^{d-1}$ is the $d-1$ dimensional torus. One may interpret the space $X$ also as a periodic strip. The space of $2\pi$-periodic in $y$ functions coincides with the space $C(X)$. By $PH^s ([a, b] \times \mathbb{T}^{d-1})$ we denote the space of functions $f : X \rightarrow \mathbb{C}$ which are polyharmonic of order $s$ on $X$, i.e. the space of functions $f \in X$ such that the corresponding $2\pi$-periodic in $y$ function $f \in C ([a, b] \times \mathbb{R}^{d-1})$ is polyharmonic of order $s$ in $[a, b] \times \mathbb{R}^{d-1}$ of the variables $x$.

If we fix $t \in [a, b]$, the equality

$$ f(t, y) = \sum_{k \in \mathbb{Z}^{d-1}} f_k(t) e^{i(k,y)} \quad (77) $$

is the Fourier series expansion of $f \in C(X)$ where the Fourier coefficients $f_k(t)$, $k \in \mathbb{Z}^{d-1}$, of $f(t, \cdot)$ are given by

$$ f_k(t) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{T}^{d-1}} f(t, y) e^{-i(k,y)} dy = \frac{1}{(2\pi)^{d-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} f(t, y) e^{-i(k,y)} dy. \quad (78) $$

---

10 The case of non–periodic functions on the strip $[a, b] \times \mathbb{R}^{d-1}$ is similar but needs more care.
In the following we want to construct polyharmonic Gauß-Jacobi cubatures for measures \( \mu \) defined on \( \mathbb{R} \times \mathbb{T}^{d-1} \), or equivalently, defined on \( \mathbb{R}^d \) and \( 2\pi \)-periodic with respect to the variable \( y \).

Next we introduce pseudo-positivity in this setting:

**Definition 37** A measure \( \mu \) on \( \mathbb{R} \times \mathbb{T}^{d-1} \) with support in the cylinder \( X \) is **pseudo-positive** (in order to avoid mixing with the notion of pseudo-positivity introduced in Section \( \S \) we will say sometimes pseudo-positive on the cylinder \( X \)), if for each \( k \in \mathbb{Z}^{d-1} \) and for each non-negative continuous function \( h : \mathbb{R} \to [0, \infty) \) with compact support the inequality

\[
\int_X h(t) e^{i\langle k, y \rangle} d\mu(t, y) \geq 0
\]

holds.

For every \( k \in \mathbb{Z}^{d-1} \), one may apply the Riesz representation theorem to prove the existence of a unique non-negative measure \( \mu_k \) on \( \mathbb{R} \) such that

\[
\int_{-\infty}^{\infty} h(t) d\mu_k(t) = \int_X h(t) e^{i\langle k, y \rangle} d\mu(t, y)
\]

(79) holds for all \( h \in C_c(\mathbb{R}) \). If we assume that \( \mu \) has support in \( X = [a, b] \times \mathbb{T}^{d-1} \) with \( -\infty < a < b < \infty \) then it is clear that (79) holds for all \( h \in C(\mathbb{R}) \).

Now we are going to prove the existence of polyharmonic Gauß-Jacobi cubature for the case of the cylinder.

**Theorem 38** Let \( -\infty < a < b < \infty \). Let \( \mu \) be a finite, signed measure with support in the cylinder \( X = [a, b] \times \mathbb{T}^{d-1} \) which is pseudo-positive on \( X \). Suppose that

\[
C := \sum_{k \in \mathbb{Z}^{d-1}} \int_X e^{i\langle k, y \rangle} d\mu(t, y) < \infty.
\]

Then for each natural number \( s \) there exists a unique finite signed measure \( \sigma^{(2s)} \) with support in \( X \) such that

(i) The support of each component measure \( \sigma_{k}^{(2s)} \), \( k \in \mathbb{Z}^{d-1} \) is in \( [a, b] \) and has cardinality \( \leq 2s \).

(ii) \( \int f d\mu = \int f d\sigma^{(2s)} \) for all functions \( f \in PH^{2s}(X) \).

(iii) \( \sigma^{(2s)} \) is pseudo-positive on the cylinder \( X \).

**Proof.** Let us give a characterization of the space \( PH^{2s}(X) \). If \( f \in PH^{2s}(X) \) then \( f \) is representable in the Fourier series (77) where for every \( k \in \mathbb{Z}^{n-1} \) the function \( f_k(t) \) is a \( C^\infty \)-solution to the equation

\[
\left( \frac{d^2}{dt^2} - k^2 \right)^{2s} g(t) = 0,
\]
cf. Theorem 9.3 in [31]. All solutions of the latter equation are linear combinations of the following functions

\[ u_j(t) = t^j e^{-|k|t} \quad \text{for } j = 0, 1, \ldots, 2s - 1, \]
\[ u_{2s+j}(t) = t^j e^{|k|t} \quad \text{for } j = 0, 1, \ldots, 2s - 1. \]

Further define the function \( u_{4s} \equiv 1 \). Then the system of functions \( u_0, \ldots, u_{4s} \) is a \( T_+ \) system – this follows from example c) in Chapter 2.2 of [32].

Let \( \mu_k \) be the non-negative measure defined by (79). If the support of \( \mu_k \) has less or equal than \( 2s \) points we define \( \sigma_{k}^{(2s)}(2s) := \mu_k \). If the support of \( \mu_k \) has more than \( 2s \) points, there exists according to Theorem 33 a non-negative measure \( \sigma_{k}^{(2s)} \) such that

\[ \int_a^b u_j(t) d\sigma_{k}^{(2s)}(t) = \int_a^b u_j(t) d\mu_k(t) \quad \text{for } j = 0, \ldots, 4s - 1, \]

and the support of \( \sigma_{k}^{(2s)} \) has \( \leq 2s \) points and lies in \([a,b] \). From the Krein–Markov Theorem 34 it follows

\[ \int_\mathbb{R} 1 d\sigma_{k}(t) \leq \int_\mathbb{R} 1 d\mu_k(t). \quad (80) \]

Following the usual scheme, we want to define the functional \( T^{(2s)} \) on \( C(X) \) by putting

\[ T^{(2s)}(f) := \frac{1}{(2\pi)^{n-1}} \sum_{k \in \mathbb{Z}^{n-1}} \int_\mathbb{R} f_k(t) d\sigma_{k}^{(2s)}(t), \]

We have to show that the functional \( T^{(2s)} \) is well-defined. Note that

\[ \left| T^{(2s)}(f) \right| \leq \frac{1}{(2\pi)^{n-1}} \sum_{k \in \mathbb{Z}^{n-1}} \max_{t \in [a,b]} |f_k(t)| \int_\mathbb{R} 1 d\sigma_{k}(t). \quad (81) \]

Moreover it is clear that

\[ \max_{t \in [a,b]} |f_k(t)| \leq \max_{t \in [a,b]} \max_{y \in \mathbb{T}^{n-1}} |f(t,y)|. \]

By (80) we obtain

\[ \left| T^{(2s)}(f) \right| \leq \max_{t \in [a,b]} \max_{y \in \mathbb{T}^{n-1}} |f(t,y)| \sum_{k \in \mathbb{Z}^{n-1}} \int_\mathbb{R} 1 d\mu_k(t), \]

Thus the functional \( T^{(2s)}(f) \) is well-defined on \( C(X) \), and by the Riesz representation theorem there exists a signed representing measure with support in \( X \), denoted by \( \sigma^{(2s)} \). Arguments similar to those presented at the end of the proof of Theorem 30 show that \( \sigma^{(2s)} \) is pseudo-positive, unique and that the exactness property (ii) is satisfied. The details are omitted. \[ \square \]
8 Examples and miscellaneous results

In this section we provide some examples and results on pseudo-positive measures which throw more light on these new notions.

8.1 The univariate case

It is instructive to consider the univariate case of our theory: then \( d = 1 \), \( S^0 = \{-1, 1\} \), and the normalized measure is \( \omega_0 (\theta) = \frac{1}{2} \) for all \( \theta \in S^0 \). The harmonic polynomials are the linear functions, their basis are the two functions defined by \( Y_0 (x) = 1 \) and \( Y_1 (x) = x \) for all \( x \in \mathbb{R} \). The following is now immediate from the definitions:

**Proposition 39** Let \( d = 1 \). A functional \( T : \mathbb{C}[x] \to \mathbb{C} \) is pseudo-positive definite if and only if \( T \left( p^* \left( x^2 \right) p \left( x^2 \right) \right) \geq 0 \) and \( T \left( xp^* \left( x^2 \right) p \left( x^2 \right) \right) \geq 0 \) for all \( p \in \mathbb{C}[x] \).

It follows from the last proposition that a Stieltjes moment sequence is always pseudo-positive definite; by definition the functional \( T : \mathbb{C}[x] \to \mathbb{C} \) has the stronger property that \( T \left( q^* \left( x \right) q \left( x \right) \right) \geq 0 \) and \( T \left( xq^* \left( x \right) q \left( x \right) \right) \geq 0 \) for all \( q \in \mathbb{C}[x] \). Below we give an example of a pseudo-positive definite functional which is not positive definite, in particular it does not define a Stieltjes moment sequence.

As pointed out in [45 Chapter 4.1], the Laplace–Fourier expansion of \( f \) is given by

\[
 f \left( r \theta \right) = f_0 \left( r \right) Y_0 \left( \theta \right) + f_1 \left( r \right) Y_1 \left( \theta \right)
\]

for \( x = r \theta \) with \( r = |x| \) and \( \theta \in S^0 \), where

\[
 f_0 \left( r \right) = \int_{S^0} f \left( r \theta \right) Y_0 \left( \theta \right) d \omega_0 \left( \theta \right) = \frac{f \left( r \right) + f \left( -r \right)}{2}
\]

\[
 f_1 \left( r \right) = \int_{S^0} f \left( r \theta \right) Y_1 \left( \theta \right) d \omega_0 \left( \theta \right) = \frac{f \left( r \right) - f \left( -r \right)}{2}
\]

are the usual even and odd functions.

**Example 40** Let \( \sigma \) be a non-negative finite measure on the interval \([a, b]\) with \( a > 0 \). Then the functional \( T : \mathbb{C}[x] \to \mathbb{C} \) defined by

\[
 T \left( f \right) = \int_a^b f \left( x \right) d \sigma - \int_a^b f \left( -x \right) d \sigma
\]

is pseudo-positive definite but not positive definite.

**Proof.** Let \( f_1 \) be as above. Then \( T \left( f \right) = 2 \int_a^b f_1 \left( r \right) d \sigma \), so \( T \) is pseudo-positive definite by Proposition 6. Since \( T \left( 1 \right) = 0 \) and \( T \neq 0 \) it is clear that \( T \) is not positive definite. \( \Box \)
8.2 A criterion for pseudo-positivity

The following is a simple criterion for pseudo-positivity:

**Proposition 41** Let $\mu$ be a signed moment measure on $\mathbb{R}^d$. Assume that $\mu$ has a density $w(x)$ with respect to the Lebesgue measure $dx$ such that $\theta \mapsto w(r\theta)$ is in $L^2(S^{d-1})$ for each $r > 0$. If the Laplace-Fourier coefficients of $w$,

$$w_{k,l}(r) := \int_{S^{d-1}} w(r\theta) Y_{k,l}(\theta) d\theta$$

are non-negative then $\mu$ is pseudo-positive and

$$d_{\mu_{k,l}}(r) = r^{k+d-1}w_{k,l}(r), \quad (82)$$

$$\int_0^\infty r^{-k}d_{\mu_{k,l}}(r) = \int_0^\infty w_{k,l}(r) \cdot r^{d-1} dr \quad (83)$$

if the last integral exists. The measures $\mu_{k,l}$ are defined by means of equality (35).

**Proof.** Since $\mu$ has a density $w(x)$ we can use polar coordinates to obtain for $f \in C_{pol}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} f d\mu = \int_{\mathbb{R}^d} f(x) w(x) dx = \int_0^\infty \int_{S^{d-1}} f(r\theta) w(r\theta) r^{d-1} d\theta dr. \quad (84)$$

For any $h \in C_{pol}(0, \infty)$ we put $f(x) = h(|x|) Y_{k,l}(x)$, then we obtain

$$\int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x) d\mu = \int_0^\infty \int_{S^{d-1}} h(r) r^{k+d-1} Y_{k,l}(\theta) w(r\theta) d\theta dr. \quad (85)$$

Since $\theta \mapsto w(r\theta)$ is in $L^2(S^{d-1})$, we know that $w_{k,l}(r) = \int_{S^{d-1}} w(r\theta) Y_{k,l}(\theta) d\theta$. Hence, by the definition of $\mu_{k,l}$, we obtain

$$\int_0^\infty h(r) d_{\mu_{k,l}} := \int_{\mathbb{R}^d} h(|x|) Y_{k,l}(x) d\mu = \int_0^\infty h(r) w_{k,l}(r) r^{k+d-1} dr. \quad (86)$$

Thus the measure $\mu$ is pseudo-positive, and (82) follows. Let us prove (83): we define the cut-off functions $h_m \in C_{pol}(0, \infty)$ such that $h_m(t) = t^{-k}$ for $t \geq 1/m$ and such that $h_m \leq h_{m+1}$. Now use (83) and the monotone convergence theorem to obtain (83). 

The next example addresses the question of whether there is a relationship among the supports of the Gauß-Jacobi quadratures $\sigma_{k,l}$ in Theorem 20.

**Proposition 42** Let $s$ be a natural number. Then there exists a pseudo-positive measure $\mu$ with support in the unit ball such that the component measures $\sigma^{(s)}_{k,l}$ of the polyharmonic Gauß-Jacobi cubature in Theorem 20 of order $s$ have identical supports.
Proof. Let $0 < a < b$ and consider the density
\[ w(re^{it}) = \sum_{k=1}^{\infty} 1_{[a,b]}(r) \frac{r^k}{k^2} P_k(\theta) \]
where $1_{[a,b]}$ is the indicator function of $[a,b]$. Then $w$ induces a pseudo-positive measure and $\int pd\mu_{k,1} = \frac{2^{k+d-1}}{k} \int_a^b p(t) dt$ according to (86). It follows that for all $k \in \mathbb{N}$ the orthogonal polynomials of degree $s$ associated with $\mu_{k,1}$ are identical up to a factor. Hence the supports of the measures $\sigma_{k,1}^{(2s)}$ are identical for all $k \in \mathbb{N}$. 

8.3 The two–dimensional case

Let us consider the case $d = 2$, and take the usual orthonormal basis of solid harmonics, defined by $Y_0(e^{it}) = \frac{1}{\sqrt{\pi}}$ and
\[ Y_{k,1}(re^{it}) = \frac{1}{\sqrt{\pi}} r^k \cos kt \quad \text{and} \quad Y_{k,2}(re^{it}) = \frac{1}{\sqrt{\pi}} r^k \sin kt \quad \text{for} \quad k \in \mathbb{N}. \quad (87) \]

We define a density $w^{(\alpha)} : \mathbb{R}^n \to [0, \infty)$, depending on parameter $\alpha > 0$, by
\[ w^{(\alpha)}(re^{it}) := (1 - r^\alpha) P(re^{it}) \quad \text{for} \quad 0 \leq r < 1 \]
\[ w^{(\alpha)}(re^{it}) = 0 \quad \text{for} \quad r \geq 1; \]
here the function $P(re^{it})$ is the Poisson kernel for $0 \leq r < 1$ given by (see e.g. 5.1.16 in [2, p. 243])
\[ P(re^{it}) := \frac{1 - r^2}{1 - 2r \cos t + r^2} = 1 + \sum_{k=1}^{\infty} 2r^k \cos kt. \quad (88) \]

By Proposition 43 the measure $d\mu^{\alpha} := w^{(\alpha)}(x) dx$ is pseudo-positive. For $k > 0$, by (88) and (87) we obtain
\[ \int r^{-k} d\mu_{k,1}^{(\alpha)} = 2\sqrt{\pi} \int_0^1 r^{k+1} (1 - r^\alpha) dr = \frac{2\sqrt{\pi}^\alpha}{(k+1)(\alpha+k+2)}. \]
It follows that $w^{(\alpha)}(x) dx$ satisfies the summability condition (11), so we can apply our cubature formula to this kind of measures.

On the other hand, there exist pseudo-positive measures which do not satisfy the summability condition (11):

**Proposition 43** Let $w(re^{it}) := P(re^{it})$ for $0 \leq r < 1$ and $w(re^{it}) := 0$ for $r \geq 1$ where $P(x)$ is given by (88). Then $d\mu := w(x) dx$ is a pseudo-positive, non-negative moment measure which does not satisfy the summability condition (11).
\textbf{Proof.} It follows from (85) for \( k \geq 1 \)
\[
\int r^{-k} d\mu_{k,1} = \int_0^\infty w_{k,1}(r) \cdot r^{d-1} dr = 2\sqrt{\pi} \int_0^1 r^{k+1} dr = \frac{2\sqrt{\pi}}{(k+2)},
\]
so we see that the summability condition (11) is not fulfilled. \( \blacksquare \)

Next we compute explicitly the error in Section 5 for the function \( w^{(\alpha)}(x) \) with \( \alpha = 2 \).

\textbf{Theorem 44} Let \( d\mu := w^{(2)}(x) dx \), and let \( \sigma^{(s)} \) be the polyharmonic Gauß-Jacobi measure of order \( s \). Then for every \( f \in C^{2s}(\mathbb{R}^d) \) the error \( E_s(f) = \int f(x) d\mu - \int f(x) d\sigma^{(s)} \) can be estimated by
\[
\sum_{k=0}^\infty \sup_{0 < \xi < 1} \left| \frac{d^{2s}}{dt^{2s}} \left[ f_{k,l} \left( \sqrt{t} \right) t^{-\frac{k}{2}} \right](\xi) \right| \frac{s! \Gamma(s+k+1) \Gamma(s+1) \Gamma(s+k)}{(2s+k+1)! (2s+k+2)!}.
\]

\textbf{Proof.} For \( k \geq 0 \) we obtain by (86) and (88) the equality
\[
\int_0^1 p(t) d\mu_{k,1} = 2\sqrt{\pi} \int_0^1 p(r) (1-r^2) r^{2k+1} dr,
\]
and clearly \( \mu_{k,2} = 0 \). Let \( \psi \) be the usual one defined by \( \psi(t) = t^2 \). Then \( \mu_{k,1}^\psi \) can be computed by
\[
\int_0^1 p(t) d\mu_{k,1}^\psi = \int_0^1 p(t^2) d\mu_{k,1} = 2\sqrt{\pi} \int_0^1 p(r^2) (1-r^2) r^{2k+1} dr. \quad (89)
\]

According to Theorem 26 we have only to compute \( \int_0^{|R^2|} \left| Q_{k,l}^s(t) \right|^2 d\mu_{kl}^\psi \), where \( Q_{k,l}^s(t) \) is the \( s \)-th orthogonal polynomial with leading coefficient 1. The substitution \( t = r^2 \) in the integral (88) yields \( \int_0^1 p(t) d\mu_{k,1}^\psi = \sqrt{\pi} \int_0^1 p(t) (1-t) t^k dt \). The substitution \( t = \frac{1}{2} (x+1) \) and \( dt = \frac{1}{2} dx \) shows that
\[
\int_0^1 p(t) d\mu_{k,1}^\psi = \frac{\sqrt{\pi}}{2^{k+2}} \int_{-1}^1 p \left( \frac{1}{2} (x+1) \right) (1-x) \cdot (1+x)^k dx.
\]
Let \( P^{(\alpha,\beta)}_s(x) \) be the Jacobi polynomial of degree \( s \) (see [21] p. 30]) normalized with \( \int_{-1}^1 P^{(\alpha,\beta)}_s(t) dt = \frac{2^{2s+\alpha+\beta+1} \Gamma(s+\alpha+1) \Gamma(s+\beta+1)}{s! \Gamma(s+\alpha+\beta+1)} \). Further
\[
h_s := \int_{-1}^1 \left| P^{(\alpha,\beta)}_s(x) \right|^2 dw_{\alpha,\beta} = \frac{2^{2s+\alpha+\beta+1} \Gamma(s+\alpha+1) \Gamma(s+\beta+1)}{2s+\alpha+\beta+1 \Gamma(s+\alpha+\beta+1)}.
\]
Define \( P^{(\alpha,\beta)}_s(t) := P^{(\alpha,\beta)}_s(2t-1) \) for \( 0 \leq t \leq 1 \). Then \( \bar{P}^{(1,k)}_S \) are orthogonal polynomials for \( \mu_{k,1}^\psi \) since
\[
\int \bar{P}^{(1,k)}_s(x) \bar{P}^{(1,k)}_m(x) dx = \frac{\sqrt{\pi}}{2^{k+2}} \int_{-1}^1 P^{(1,k)}_s(x) P^{(1,k)}_m(x) (1-x) \cdot (1+x)^k dx. \quad (90)
\]
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Now choose a sequence of polynomials \( p_{s,k} \) such that \( \|p_{s,k}\|_\infty \leq 1 \). Thus \( Q_{k,1}^* (t) \) and we obtain from (33)

\[
\int Q_{k,1}^* (t)^2 \, d\mu_{k,1} = \frac{1}{k_s} \int_{-1}^{1} \left| p_{s,k}(x) \right|^2 (1-x)(1+x)^k \, dx.
\]

Now this is equal to

\[
\left( \frac{2s+k+1}{s} \right)^{-1} \frac{\sqrt{s}}{2^{k+2}} \frac{\Gamma(s+2)\Gamma(s+k+1)}{2s+k+2} \frac{s!\Gamma(s+k+2)}{s!\Gamma(s+k+2)}
\]

giving

\[
\int Q_{k,1}^* (t)^2 \, d\mu_{k,1} = \frac{s!(s+k+1)!(s+1)!(s+k)!}{(2s+k+1)! (2s+k+2)!} \sim \frac{1}{k^{2s+2}}
\]

for large \( k \). \( \blacksquare \)

### 8.4 The summability condition

We are now turning back to the general situation. The next result shows that the spectrum of the component measures \( \sigma_{k,l} \) is contained in the spectrum of the representation measure \( \mu \).

**Theorem 45** Let \( \sigma_{k,l} \) be non-negative measures on \([0, \infty)\). If the functional \( T: \mathbb{C} [x_1, x_2, \ldots, x_d] \to \mathbb{C} \) defined by (33) possesses a representing moment measure \( \mu \) with compact support then

\[
\sigma_{k,l}\left( \{ |x|^2 \} \right) \leq \max_{\theta \in S^{d-1}} |Y_{k,l}(\theta)| \cdot |x|^k \cdot |\mu| \left( |x|^2 S^{d-1} \right)
\]

for any \( x \in \mathbb{R}^d \) where \( |\mu| \) is the total variation and \( |x|^2 S^{d-1} = \{ |x|^2 \theta : \theta \in S^{d-1} \} \).

**Proof.** Let the support of \( \mu \) be contained in \( B_R \). Let \( x_0 \in \mathbb{R}^d \) be given. For every univariate polynomial \( p(t) \) with \( p\left( |x_0|^2 \right) = 1 \) we have

\[
\sigma_{k,l}\left( \{ |x_0|^2 \} \right) \leq \int_0^\infty p(r^2) \, d\sigma_{k,l} \leq \int_{\mathbb{R}^d} |p(|x|^2)Y_{k,l}(x)| \, d|\mu|
\]

\[
\leq \max_{\theta \in S^{d-1}} |Y_{k,l}(\theta)| \int_{\mathbb{R}^d} |p(|x|^2)| \, |x|^k \, d|\mu|.
\]

Now choose a sequence of polynomials \( p_m \) with \( p_m\left( |x_0|^2 \right) = 1 \) which converges on \([0, R]\) to the function \( f \) defined by \( f\left( |x_0|^2 \right) = 1 \) and \( f(t) = 0 \) for \( t \neq |x_0|^2 \).

Since \( |\mu| \) has support in \( B_R \) Lebesgue’s convergence theorem shows that

\[
\sigma_{k,l}\left( \{ |x_0|^2 \} \right) \leq \max_{\theta \in S^{d-1}} |Y_{k,l}(\theta)| \int_{\mathbb{R}^d} |f(x)| \, |x|^k \, d|\mu|.
\]

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The last implies our statement.

The following result shows that the summability condition is sometimes equivalent to the existence of a pseudo-positive representing measure:

**Corollary 46** Let \( d = 2 \). Let \( \sigma_{k,l} \) be non-negative measures on \([0, \infty)\) and assume that they have disjoint and at most countable supports. If the functional \( T : \mathbb{C}[x_1, x_2] \to \mathbb{C} \) defined by (33) possesses a representing moment measure with compact support then

\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty r^{-k} d\sigma_{k,l}(r) < \infty.
\]

**Proof.** Let \( \Sigma_{k,l} \) be the support set of \( \sigma_{k,l} \). The last theorem shows that \( \sigma_{k,l}(\{0\}) = 0 \), hence \( 0 \notin \Sigma_{k,l} \). Moreover it tells us that

\[
\int_0^\infty r^{-k} d\sigma_{k,l}(r) \leq \max_{\theta \in S^{d-1}} |Y_{k,l}(\theta)| \cdot \sum_{r \in \Sigma_{k,l}} |\mu|(r S^{d-1}).
\]

Since \( d = 2 \) we know that \( \max_{\theta \in S^{d-1}} |Y_{k,l}(\theta)| \leq 1 \). Hence

\[
\sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \int_0^\infty r^{-k} d\sigma_{k,l}(r) \leq \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{r \in \Sigma_{k,l}} |\mu|(r S^{d-1}) \leq |\mu|(\mathbb{R}^d)
\]

where the last inequality follows from the fact that \( \Sigma_{k,l} \) are pairwise disjoint.

Recall that the converse of the last theorem holds under the additional assumption that the supports of all \( \sigma_{k,l} \) are contained in some interval \([0, R]\).

**Theorem 47** There exists a functional \( T : \mathbb{C}[x_1, x_2, \ldots, x_d] \to \mathbb{C} \) which is pseudo-positive definite but does not possess a pseudo-positive representing measure.

**Proof.** Let \( \sigma \) be a non-negative measure over \([0, R]\). Let \( f \in \mathbb{C}[x_1, \ldots, x_d] \) and let \( f_{k,l} \) be the Laplace-Fourier coefficients of \( f \). By Proposition 2 it is clear that

\[
T(f) := \int_0^R f_{1,1}(r) r^{-1} d\sigma(r)
\]

is pseudo-positive definite. We take now for \( \sigma \) the Dirac functional at \( r = 0 \). Suppose that \( T \) has a signed representing measure \( \mu \) which is pseudo-positive. Then the component measure \( \mu_{11} \) is non-negative, and it is defined by the equation

\[
\int_0^\infty h(r) d\mu_{11}(r) := \int_{\mathbb{R}^n} h(|x|) Y_{11}(x) d\mu
\]

for any continuous function \( h : [0, \infty) \to \mathbb{C} \) with compact support. Take now \( h(r) = r^2 \). Then by Proposition 7

\[
\int_0^\infty r^2 d\mu_{11}(r) = \int_{\mathbb{R}^n} |x|^2 Y_{11}(x) d\mu = T\left(|x|^2 Y_{11}(x)\right) = 0.
\]
It follows that \( \mu_{11} \) has support \( \{0\} \). On the other hand, if we take a sequence of functions \( h_m \in C_c([0, \infty)) \) such that \( h_m \to 1_{\{0\}} \), then we obtain

\[
\mu_{11}(\{0\}) = \lim_{m \to \infty} \int_{\mathbb{R}^n} h_m(|x|) Y_{11}(x) \, d\mu.
\]

But \( h_m(|x|) Y_{11}(x) \) converges to the zero-function, and Lebesgue’s theorem shows that \( \mu_{11}(\{0\}) = 0 \), so \( \mu_{11} = 0 \). This is a contradiction since

\[
\int_0^\infty 1 \, d\mu_{11}(r) = \int_{\mathbb{R}^n} Y_{11}(x) \, d\mu = T(Y_{11}) = \int_0^R 1 \, d\sigma(r) = 1.
\]

The proof is complete.

\section{Concluding Remarks}

One important feature of the polyharmonic Gauß–Jacobi cubature which deserves to be discussed is its numerical significance. At first glance, one may object that the cubature needs the knowledge of the Laplace-Fourier coefficients of the function \( f : \mathbb{R}^n \to \mathbb{C} \), and these are based on integrals as well. However, if one works with polynomials, the Gauß decomposition \( \mathfrak{3} \) can be constructed by an efficient differentiation algorithm, see \( \mathfrak{4} \). Decomposing the harmonic polynomials \( h_j \) according to our fixed orthonormal basis \( Y_{k,l}(x) \) one obtains from \( \mathfrak{3} \) the expansion

\[
f(x) = \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} p_{k,l}(|x|^2) Y_{k,l}(x),
\]

where \( p_{k,l} \) are uniquely determined univariate polynomials. For the Gauß–Jacobi quadrature there exists suitable software to compute the weights \( \alpha_{k,l}^{1}, \ldots, \alpha_{k,l}^{s} \) and the nodes \( r_1^{(k,l)}, \ldots, r_s^{(k,l)} \) of the measures \( \sigma_{k,l}^{(s)} \). Then

\[
T^{(s)}(f) := \sum_{k=0}^{\deg f} \sum_{l=1}^{a_k} \sum_{i=1}^{s} \alpha_{k,l}^{i} p_{k,l} \left((r_i^{(k,l)})^2\right).
\]

In practice one also has to bound the number of spherical harmonics \( Y_{k,l}(x) \) in the formula. Our main results Theorem \( \mathfrak{20} \) and Theorem \( \mathfrak{22} \) show that by increasing the number \( k \) and the number \( s \) the algorithm remains stable.

What concerns the polyharmonic Gauß–Jacobi cubature in the annulus and the strip, we note that recently efficient algorithms for finding Gauß–Jacobi–Markov quadratures for Chebyshev systems have been studied in \( \mathfrak{35} \).

A second point to be made clear, is the motivation why we choose the space of polyharmonic functions of order \( s \) as the exactness space for the multivariate generalization. One main reason is the fact that recently polyharmonic functions have shown to be an efficient tool in approximation theory and more generally, in

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mathematical analysis, see e.g. [3], [28], [30], [31], and [34]. Another motivation stems from potential theory: two non-negative measures $\mu$ and $\nu$ with compact supports are *gravitationally equivalent* if

$$\int_{\mathbb{R}^n} h(x)\,d\mu(x) = \int_{\mathbb{R}^n} h(x)\,d\nu(x)$$

(91)

for all harmonic functions $h$ defined on a neighborhood of the supports of the measures. If $\mu$ and $\nu$ are gravitationally equivalent then they produce the same potential outside of their support. Graviequivalent measures are very important in inverse problems in Geophysics and Geodesy, and a new mathematical area has grown extensively during the last two decades or so under the title Quadrature Domains, see the comprehensive survey and references in [23], as well as [49].

In analogy to (91) one could define two (generally speaking, signed) measures $\mu$ and $\nu$ as polyharmonically equivalent of order $s$ if (91) holds for all polyharmonic functions of order $s$ in a neighborhood of their support. Similar notions of equivalence have been developed by L. Ehrenpreis [19] in the form of a generalized balayage. To our knowledge the equivalence of two measures (and more generally, distributions) with respect to the solutions of an elliptic operator has been for the first time rigorously formulated and studied in the case of non–negativity in [42].

Let us remark that the polyharmonic Gauß–Jacobi measure $\nu^{(s)}$ which we have introduced in the present paper is related to the concept of "mother body" (a non–negative measure $\nu$ satisfying (91) for a given $\mu$, and having minimal support) in the theory of Quadrature Domains, cf. [23], and Remark 25 as well as [27] and [29].

ACKNOWLEDGMENT. Both authors acknowledge the support of the Institutes Partnership project with the Alexander von Humboldt Foundation.

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