SEPARATRICES FOR REAL ANALYTIC VECTOR FIELDS IN THE PLANE

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Abstract. Let \(X\) be a germ of real analytic vector field at \((\mathbb{R}^2, 0)\) with an algebraically isolated singularity. We say that \(X\) is a topological generalized curve if there are no topological saddle-nodes in its reduction of singularities. In this case, we prove that if either the order \(\nu_0(X)\) or the Milnor number \(\mu_0(X)\) is even, then \(X\) has a formal separatrix, that is, a formal invariant curve at \(0 \in \mathbb{R}^2\). This result is optimal, in the sense that these hypotheses do not assure the existence of a convergent separatrix.

1. Introduction

A separatrix for a germ of real or complex analytic vector field, at \((\mathbb{R}^2, 0)\) or at \((\mathbb{C}^2, 0)\), is an invariant irreducible formal curve passing through the origin. The Separatrix theorem, by C. Camacho and P. Sad, asserts that, in the complex case, a convergent separatrix always exists (\cite{2}; see also \cite{4,15} for alternative proofs).

A comprehensive result of this sort is not true in the universe of real analytic vector fields. For instance, vector fields of center-focus type at \((\mathbb{R}^2, 0)\), such as \(X = y\partial/\partial x - x\partial/\partial y\), do not admit invariant curves through \(0 \in \mathbb{R}^2\), neither analytic nor formal. In this case, the complex separatrices of their complexifications to \((\mathbb{C}^2, 0)\), which exist by the Separatrix theorem, have trivial real traces. However, the search for real separatrices may have a successful outcome within specific families of vector fields, delimited, for instance, by conditions of algebraic nature imposed on the singularity. This is the approach of J.-J. Risler in \cite{14}, when proving that, in the family of vector fields at \((\mathbb{R}^2, 0)\) of real generalized curve type, a vector field \(X = X_\mathbb{R}\) with an algebraically isolated singularity has a convergent separatrix if either the algebraic multiplicity \(\nu_0(X_\mathbb{R})\) or the Milnor number \(\mu_0(X_\mathbb{R})\) is even. More generally, omitting the real generalized curve hypothesis, it is also proved in \cite{14} that the evenness of either \(\nu_0(X_\mathbb{R})\) or \(\mu_0(X_\mathbb{R})\) is sufficient to assure that \(X_\mathbb{R}\) has a characteristic orbit, that is, a trajectory accumulating to \(0 \in \mathbb{R}^2\) having a well defined tangent at this point. In other words, a center-focus vector field must have both \(\nu_0(X_\mathbb{R})\) and \(\mu_0(X_\mathbb{R})\) odd.

Let us explain some terms. We say that a germ of real analytic vector field \(X_\mathbb{R}\) has an algebraically isolated singularity if its coefficients vanish at \(0 \in \mathbb{R}^2\) and are relatively prime in the ring \(\mathbb{R}\{x, y\}\) of germs or real analytic functions at \((\mathbb{R}^2, 0)\). We say that \(X_\mathbb{R}\) — or the singular one-dimensional real foliation defined by it — is of real generalized curve type if
there are no saddle-nodes in its reduction of singularities. In this process, the final models are simple singularities — locally defined by vector fields having non-nilpotent linear part with real eigenvalues with ratio not in $\mathbb{Q}^+$. Saddle-nodes are simple singularities having one zero eigenvalue (see Section 2 for definitions).

A real saddle-node singularity, from the topological point of view, can be a saddle, a node or a saddle-node. We say that a germ of real analytic vector field $X_\mathbb{R}$ at $(\mathbb{R}^2,0)$ is a topological real generalized curve if there are no topological saddle-nodes in its reduction of singularities (Definition 2.1). For vector fields in this family we prove the following:

**Theorem.** Let $X_\mathbb{R}$ be a germ of real analytic vector field at $(\mathbb{R}^2,0)$, with an algebraically isolated singularity at $0 \in \mathbb{R}^2$, of topological real generalized curve type. If either its order $\nu_0(X_\mathbb{R})$ or its Milnor number $\mu_0(X_\mathbb{R})$ is even, then $X_\mathbb{R}$ has a (possibly formal) separatrix.

We remark that, with the hypotheses stated, we cannot ask for a convergent separatrix. Below, we reproduce an example from [13], a vector field which is a topological generalized curve having both $\nu_0$ and $\mu_0$ even, but admitting only one separatrix which is purely formal. Note that vector fields of generalized curve type or of center-focus type are, in particular, of topological generalized curve type. Hence, Risler’s results on separatrices and characteristic orbits mentioned above can be obtained as immediate corollaries of our theorem.

For proof purposes, our main theorem will be split in two, Theorems A and B, concerning, respectively, the situation of even $\nu_0$ and even $\mu_0$. Their proofs rely on a study of indices and of the reduction of singularities of the one-dimensional foliation induced by $X_\mathbb{R}$. At no moment the Separatrix theorem is invoked. As in [14], our strategy is to take the complexification $X_\mathbb{C}$ of $X_\mathbb{R}$ to $(\mathbb{C}^2,0)$ and use the fact that it induces a complex one-dimensional foliation that is invariant by the canonical anti-holomorphic involution $J : (x,y) \to (\bar{x},\bar{y})$. Thus, $J$ will preserve the separatrices of $X_\mathbb{C}$ and the fixed ones are precisely those whose real traces define separatrices for $X_\mathbb{R}$.

Two main ingredients are used in our proof. The first one is the tangency excess index (Definition 2.2), an invariant that computes the contributions of the orders of tangencies of saddle-node singularities along the divisor of the reduction of singularities. For a germ of complex analytic vector field, this is $C^\infty$-invariant [12]. In the real case, the fact that a vector field is a topological real generalized curve implies that its tangency excess index is even. The second ingredient is the notion of balanced divisor of separatrices. In the complex case, this is a tool that computes $\nu_0$ (see [7] and Proposition 3.1 below). For the complexification $X_\mathbb{C}$ of $X_\mathbb{R}$, the balanced divisor of separatrices can be taken symmetric with respect to the involution $J$. Hence, to prove our results, it is enough is to show that such a balanced divisor is supported in an odd number of separatrices. Theorem A is then obtained by a simple consideration of Proposition 3.1 (see Section 3).

In order to prove Theorem B, we use some tools from the polar theory of complex foliations, which are developed in Section 4. More specifically, we consider the polar intersection number — an invariant widely studied in [6, 3] — and prove Proposition 4.3, which shows a formula relating the order and the Milnor number of a complex foliation with polar intersection numbers and tangency excess indices. This result, interesting in itself, generalizes an inequality given in Proposition 2 of [3]. Finally, in Section 5 the application of this formula gives a proof to Theorem B.
2. Basic definitions and notation

We consider the field \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), of real or complex numbers. Along the text, we use the symbol \( \mathbb{K} \) as a prefix, replacing the adjectives “real” or “complex”, and as a subscript, indicating the field we are working with.

In \( \mathbb{K}^2 \) we take variables \((x, y)\). Whether they are real or complex will be made clear by the context. Let \( \mathbb{K}[[x, y]] \) denote the ring of formal power series in the variables \((x, y)\), and \( \mathbb{K}\{x, y\} \subset \mathbb{K}[[x, y]] \) denote the ring of convergent power series, the latter identified with the ring of germs of \( \mathbb{K} \)-analytic functions at \((\mathbb{K}^2, 0)\). Denote by \( \mathfrak{X}_\mathbb{K} \) and \( \Omega_\mathbb{K} \) the space of germs of \( \mathbb{K} \)-analytic vector fields and \( \mathbb{K} \)-analytic 1-forms at \((\mathbb{K}^2, 0)\).

The canonical complexification of \( \mathbb{R} \) is given by

\[
F_\mathbb{R}(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j \in \mathbb{R}[[x, y]] \quad \mapsto \quad F_\mathbb{C}(z, w) = \sum_{i,j \geq 0} a_{i,j} x^i y^j \in \mathbb{C}[[x, y]].
\]

It defines inclusions \( \mathbb{R}[[x, y]] \hookrightarrow \mathbb{C}[[x, y]] \) and \( \mathbb{R}\{x, y\} \hookrightarrow \mathbb{C}\{x, y\} \). This process is extended to \( \mathfrak{X}_\mathbb{K} \) or to \( \Omega_\mathbb{K} \), by simple substitution of coefficients in \( \mathbb{R}\{x, y\} \) by their complexifications in \( \mathbb{C}\{x, y\} \), so we also have inclusions \( \mathfrak{X}_\mathbb{R} \to \mathfrak{X}_\mathbb{C} \) and \( \Omega_\mathbb{R} \to \Omega_\mathbb{C} \).

Let \( J : (x, y) \in \mathbb{C}^2 \mapsto (\bar{x}, \bar{y}) \in \mathbb{C}^2 \) be the canonical anti-holomorphic involution. We can view \( \mathbb{R}^2 \) as the subset of \( \mathbb{C}^2 \) of fixed points of \( J \) and call it real trace. We establish, in \( \mathbb{C}[[x, y]] \) or in \( \mathbb{C}\{x, y\} \), the operator

\[
J^\vee : f_\mathbb{C} = \sum_{i,j \geq 0} a_{i,j} x^i y^j \mapsto f_\mathbb{C}^\vee = J^\vee(f_\mathbb{C}) = \sum_{i,j \geq 0} \bar{a}_{i,j} \bar{x}^i \bar{y}^j,
\]

defined in such a way that \( f_\mathbb{C}^\vee(J(x, y)) = \bar{f_\mathbb{C}}(x, y) \). Note that \( f_\mathbb{C} \in \mathbb{C}[[x, y]] \) is the complexification of \( f_\mathbb{R} \in \mathbb{R}[[x, y]] \) if and only if \( f_\mathbb{C}^\vee = f_\mathbb{C} \). By acting on the coefficients of a vector field or of a 1-form, \( J^\vee \) can be extended to \( \mathfrak{X}_\mathbb{C} \) or to \( \Omega_\mathbb{C} \). Following our notation, for \( X_\mathbb{C} \in \mathfrak{X}_\mathbb{C} \) and \( \omega_\mathbb{C} \in \Omega_\mathbb{C} \), we set \( J^\vee(X_\mathbb{C}) = X_\mathbb{C}^\vee \) and \( J^\vee(\omega_\mathbb{C}) = \omega_\mathbb{C}^\vee \). Elements of \( \mathfrak{X}_\mathbb{C} \) or of \( \Omega_\mathbb{C} \) in the image of the complexification map are precisely those that are fixed by \( J^\vee \).

Let \( X_\mathbb{K} \in \mathfrak{X}_\mathbb{K} \) and write \( X_\mathbb{K} = P_\mathbb{K} \partial_x + Q_\mathbb{K} \partial_y \), where \( P_\mathbb{K}, Q_\mathbb{K} \in \mathbb{K}\{x, y\} \) are assumed to be relatively prime. In a small neighborhood \( U_\mathbb{K} \) of \( 0 \in \mathbb{K}^2 \), the vector field \( X_\mathbb{K} \) induces a \( \mathbb{K} \)-analytic one-dimensional foliation, which is singular, with isolated singularity at the origin, if and only if \( P_\mathbb{K}(0, 0) = Q_\mathbb{K}(0, 0) = 0 \). We denote this foliation by \( \mathcal{F}_\mathbb{K} \), which is also defined by the equation \( \omega_\mathbb{K} = 0 \), where \( \omega_\mathbb{K} \in \Omega_\mathbb{K} \) is the dual 1-form \( \omega_\mathbb{K} = P_\mathbb{K} dy - Q_\mathbb{K} dx \). We will assume \( \mathcal{F}_\mathbb{C} \) to be the complexification of the real foliation \( \mathcal{F}_\mathbb{R} \), meaning that \( X_\mathbb{C} \in \mathfrak{X}_\mathbb{C} \) is a vector field, invariant by \( J^\vee \), that is the complexification of \( X_\mathbb{R} \in \mathfrak{X}_\mathbb{R} \). Thus, \( \mathcal{F}_\mathbb{C} \) will be \( J \)-invariant, meaning \( J \) sends leaves of \( \mathcal{F}_\mathbb{C} \) into leaves of \( \mathcal{F}_\mathbb{C} \). Besides, \( \mathcal{F}_\mathbb{R} = \mathcal{F}_\mathbb{C}|_{\mathbb{R}^2} \), that is, leaves of \( \mathcal{F}_\mathbb{R} \) are obtained by the restriction of those of \( \mathcal{F}_\mathbb{C} \) to the real trace \( \mathbb{R}^2 \). This convention will be abandoned only in Section 4 where the results presented apply to complex analytic vector fields in general.

A separatrix of a \( \mathbb{K} \)-analytic foliation \( \mathcal{F}_\mathbb{K} \) is an invariant irreducible \( \mathbb{K} \)-formal curve. That is, it is the object \( B_\mathbb{K} \) given by a reduced parametrization in one variable \( \gamma_\mathbb{K}(t) = (x_\mathbb{K}(t), y_\mathbb{K}(t)) \), where \( x_\mathbb{K}, y_\mathbb{K} \in t\mathbb{K}[[t]] \), defined up to right composition by formal diffeomorphisms in one variable, satisfying the formal relation

\[
(P_\mathbb{K} \circ \gamma_\mathbb{K}(t)) y_\mathbb{K}(t) - (Q_\mathbb{K} \circ \gamma_\mathbb{K}(t)) x_\mathbb{K}(t) = 0.
\]

When \( \gamma_\mathbb{K}(t) \) is \( \mathbb{K} \)-analytic, we have a \( \mathbb{K} \)-analytic separatrix. In this case, the parametrization defines a germ of geometric curve in a neighborhood of \( 0 \in \mathbb{K}^2 \), still denoted by \( B_\mathbb{K} \),
such that $B_\mathbb{K} \setminus \{0\}$ is a leaf (when $\mathbb{K} = \mathbb{C}$) or two leaves (when $\mathbb{K} = \mathbb{R}$) of $\mathcal{F}_\mathbb{K}$. We denote the family of all separatrices of $\mathcal{F}_\mathbb{K}$ by $\text{Sep}(\mathcal{F}_\mathbb{K})$.

The $J$-symmetry of $\mathcal{F}_\mathbb{C}$ implies that the involution $J$ takes separatrices into separatrices by the following correspondence:

$$B_\mathbb{C} : \gamma_\mathbb{C}(t) = (x_\mathbb{C}(t), y_\mathbb{C}(t)) \mapsto B_\mathbb{C}^\mathbb{C} : \gamma_\mathbb{C}^\mathbb{C}(t) = J(x_\mathbb{C}(\bar{t}), y_\mathbb{C}(\bar{t})).$$

Separatrices such that $B_\mathbb{C} = B_\mathbb{C}^\mathbb{C}$ are called real and we denote their family by $\text{Sep}^r(\mathcal{F}_\mathbb{C})$. They are in bijection with $\text{Sep}(\mathcal{F}_\mathbb{R})$. The other separatrices of $\mathcal{F}_\mathbb{C}$, those such that $B_\mathbb{C} \neq B_\mathbb{C}^\mathbb{C}$, are called purely complex and we denote their set by $\text{Sep}^c(\mathcal{F}_\mathbb{C})$. We then have a disjoint union $\text{Sep}(\mathcal{F}_\mathbb{C}) = \text{Sep}^r(\mathcal{F}_\mathbb{C}) \cup \text{Sep}^c(\mathcal{F}_\mathbb{C})$. Since $(B_\mathbb{C}^\mathbb{C})^\mathbb{C} = B_\mathbb{C}$, we have that separatrices in $\text{Sep}^r(\mathcal{F}_\mathbb{C})$ appear in pairs. This remark will be crucial in the development of our results.

For a germ of $\mathbb{C}$-analytic foliation $\mathcal{F}_\mathbb{C}$, induced by vector field $X_\mathbb{C}$, we consider the following invariants:

- $\nu_0(\mathcal{F}_\mathbb{C})$ is the order or the algebraic multiplicity of $\mathcal{F}_\mathbb{C}$ at $0 \in \mathbb{C}^2$, which is the minimum of the orders of the coefficients of $X_\mathbb{C}$.
- $\mu_0(\mathcal{F}_\mathbb{C})$, the Milnor number of $\mathcal{F}_\mathbb{C}$, defined as the dimension of the local algebra

$$\mu_0(\mathcal{F}_\mathbb{C}) = \dim_\mathbb{C} \frac{\mathbb{C}\{x, y\}}{(P_\mathbb{C}, Q_\mathbb{C})}.$$

For a germ of real foliation $\mathcal{F}_\mathbb{R}$ with complexification $\mathcal{F}_\mathbb{C}$, we set $\nu_0(\mathcal{F}_\mathbb{R}) = \nu_0(\mathcal{F}_\mathbb{C})$ and $\mu_0(\mathcal{F}_\mathbb{R}) = \mu_0(\mathcal{F}_\mathbb{C})$.

We denote by $\pi_\mathbb{K} : (\tilde{\mathbb{K}}^2, D_\mathbb{K}) \to (\mathbb{K}^2, 0)$ the $\mathbb{K}$-blow-up at $0 \in \mathbb{K}^2$. The space $\tilde{\mathbb{K}}^2$ is a $\mathbb{K}$-analytic surface, obtained by replacing the origin by its set of tangent directions, parametrized by the projective line $D_\mathbb{K} \simeq \mathbb{P}_\mathbb{K}^1$. It is defined by coordinates $(x, u), (v, y) \in \mathbb{K}^2$ identified by the relations $y = ux$ and $x = vy$, in such a way that $\pi_\mathbb{K} : (x, u) \mapsto (x, ux), (v, y) \mapsto (vy, y)$. Restricted to $\tilde{\mathbb{K}}^2 \setminus D_\mathbb{K}$, the blow-up is a $\mathbb{K}$-analytic diffeomorphism onto its image.

When $\mathbb{K} = \mathbb{C}$, the involution $J : (x, y) \in \mathbb{C}^2 \mapsto (\bar{x}, \bar{y}) \in \mathbb{C}^2$ lifts to $\tilde{\mathbb{C}}^2$, defining a unique continuous involution $J_1 : \tilde{\mathbb{C}}^2 \to \tilde{\mathbb{C}}^2$ such that $\pi_\mathbb{C} \circ J_1 = J \circ \pi_\mathbb{C}$. We can canonically identify $\mathbb{R}^2$ with the fixed set of this lifting, that is $\mathbb{R}^2 = \mathbb{C}^2 \cap J_1(\mathbb{C}^2)$, so that $D_\mathbb{R} = D_\mathbb{C} \cap \mathbb{R}^2$ and $\pi_\mathbb{R} = \pi_\mathbb{C}|_{\mathbb{R}^2}$. We will say that $\mathbb{R}^2$ is the real trace of $\tilde{\mathbb{C}}^2$.

Given a germ of $\mathbb{K}$-analytic singular foliation $\mathcal{F}_\mathbb{K}$ at $(\mathbb{K}^2, 0)$, there is a unique $\mathbb{K}$-analytic singular foliation $\pi_\mathbb{K}^* \mathcal{F}_\mathbb{K}$, the strict transform of $\mathcal{F}_\mathbb{K}$, defined in $(\tilde{\mathbb{K}}^2, D_\mathbb{K})$ and having isolated singularities over $D_\mathbb{K}$, that corresponds diffeomorphically to $\mathcal{F}_\mathbb{K}$ over the points where $\pi_\mathbb{K}$ is a local diffeomorphism. If the line $D_\mathbb{K}$ is invariant by $\pi_\mathbb{K}^* \mathcal{F}_\mathbb{K}$, we say that the blow-up $\pi_\mathbb{K}$ or the component $D_\mathbb{K}$ is non-dicritical. It is dicritical otherwise.

We iterate the process of blowing up and consider a composition $\sigma_\mathbb{K} = \pi_\mathbb{K} \circ \cdots \circ \pi_\mathbb{K}$ of blow-ups $\pi_\mathbb{K}^j : (\tilde{M}_\mathbb{K}^j, D_\mathbb{K}^j) \to (\tilde{M}_\mathbb{K}^{j-1}, D_\mathbb{K}^{j-1})$, for $j = 1, \ldots, k$, where:

- $(\tilde{M}_\mathbb{K}^0, D_\mathbb{K}^0) \simeq (\mathbb{K}^2, 0)$, $(\tilde{M}_\mathbb{K}^1, D_\mathbb{K}^1) \simeq (\tilde{\mathbb{K}}^2, D_\mathbb{K})$ and $\pi_\mathbb{K}^1 : (\tilde{M}_\mathbb{K}^1, D_\mathbb{K}^1) \to (\tilde{M}_\mathbb{K}^0, D_\mathbb{K}^0)$ is the $\mathbb{K}$-blow-up at $0 \in \mathbb{K}^2$;
- each $\pi_\mathbb{K}^j$ is a blow-up at a point $p_{j-1} \in D_\mathbb{K}^{j-1}$;
- For $j = 1, \ldots, k$, $D_\mathbb{K}^j = (\pi_\mathbb{K}^\circ \cdots \circ \pi_\mathbb{K})^{-1}(0)$ is a normal crossings divisor, whose irreducible components are isomorphic to projective lines $\mathbb{P}_\mathbb{K}$.
Setting \((\tilde{M}_K, D_K) \simeq (\check{M}_K, D'_K)\), we denote this sequence of blow-ups by \(\sigma_K : (\check{M}_K, D_K) \to (\mathbb{K}^2, 0)\). The smooth points of \(D_K\) are called \textit{trace points}, while its singular points are called \textit{corners}. This iterated construction, applied to a germ of singular foliation \(F_K\), produces a strict transform foliation \(\check{\sigma}_K F_K \in (\check{M}_K, D_K)\). The irreducible components of \(D_K\) are classified as \textit{dicritical} —– non-invariant by \(\check{F}_K\) — or \textit{non-dicritical} — invariant by \(\check{F}_K\).

Suppose that \(0 \in \mathbb{K}^2\) is a singularity for \(F_K\). This singularity is said to be \textit{simple} if the linear part \(DX_K(0)\) of the vector field \(X_K\), that induces \(F_K\), has eigenvalues \(\lambda_1, \lambda_2 \in \mathbb{K}\) satisfying one of the two conditions:

i) \(\lambda_1, \lambda_2 \neq 0\) and \(\lambda_1/\lambda_2 \notin \mathbb{Q}^+\);

ii) \(\lambda_1 \neq 0, \lambda_2 = 0\).

In case (i) we have a \textit{non-degenerate} singularity and in case (ii) we have a \textit{saddle-node} singularity. In the real case, for reasons that will become clear later, we will sometimes refer to the second case as an \textit{algebraic saddle-node}. A simple singularity has exactly two smooth transversal separatrices (see [9]). In the non-degenerate case, both are convergent. In the saddle-node case, the one associated to the eigenspace of the non-zero eigenvalue is convergent and is called \textit{strong separatrix}. The other separatrix is, in principle, formal, and is called \textit{weak} of \textit{central separatrix}. A saddle-node singularity can be expressed, in formal coordinates, by a \(1\)–form of the type [10]:

\[
\omega_K = y(1 + \mu x^k)dx + x^{k+1}dy,
\]

where \(\mu \in \mathbb{K}\) and \(k \in \mathbb{Z}_+\) are formal invariants. In this writing, the strong separatrix corresponds to \(\{x = 0\}\) and the weak separatrix to \(\{y = 0\}\). The integer \(i_0^w(F_K) = k+1 > 1\) is called \textit{weak index} of the saddle-node.

When \(\mathbb{K} = \mathbb{R}\), an algebraic saddle-node singularity can be, from the topological point of view, a saddle, a node or a saddle-node. If \(i_0^w(F_K)\) is even, it is a topological saddle-node. If \(i_0^w(F_K)\) is odd, it can be either a saddle or a node (see [9, Th. 9.1]).

The foliation \(F_K\) admits a \textit{reduction of singularities} or \textit{desingularization} by Seidenberg’s theorem [16]. This means that there is a sequence of \(\mathbb{K}\)-blow-ups

\[
\sigma_K = (\check{M}_K, D_K) \to (\mathbb{K}^2, 0)
\]

such that strict transform foliation \(\check{F}_K = \sigma_K^* F\) has a finite number of singularities, all of them over \(D_K\), all of them simple. We can ask further \(\check{F}_K\) to be everywhere transversal to each dicritical component of \(D_K\) — we say in this case that \(\sigma_K\) desingularizes the set of separatrices \(\text{Sep}(F_K)\) (see [11]). Also, we require that the sequence of blow-ups \(\sigma_K\) is minimal with these properties, being thus unique up to isomorphism. Once and for all, we fix a minimal reduction of singularities \(\sigma_K\) for \(F_K\).

We say that \(B_K \in \text{Sep}(F_K)\) is a \textit{dicritical separatrix} if \(\check{B}_K = \sigma_K^* B_K\) touches \(D_K\) in a dicritical component. Otherwise, we say that \(B_K\) is an \textit{isolated} separatrix. This engenders a decomposition \(\text{Sep}(F_K) = \text{Iso}(F_K) \cup \text{Dic}(F_K)\), where notation is self-explanatory. Isolated separatrices are in one-to-one correspondence with trace singularities of \(\check{F}_K\). All purely formal separatrices of \(F_K\) are in \(\text{Iso}(F_K)\) and come from weak separatrices of saddle-node singularities of \(\check{F}_K\) that are not contained in \(D_K\).

When \(\mathbb{K} = \mathbb{C}\), the \(J\)-symmetry of \(F_C\) allows us to lift the involution \(J\) to a continuous involution \(\tilde{J} : \tilde{M}_C \to \tilde{M}_C\) such that \(\sigma_C \circ \tilde{J} = J \circ \sigma_C\). In this way \(\tilde{M}_R\) can be identified with
the fixed set of \( \tilde{J} \). The foliation \( \tilde{F}_C \) is evidently symmetric with respect to \( \tilde{J} \) and \( \tilde{F}_R \) is the restriction of \( \tilde{F}_C \) to the real trace \( \tilde{M}_R \). Also, \( \sigma_R \) is the restriction of \( \sigma_C \) to \( \tilde{M}_R \).

We have the following concept \([1, 14]\): a germ of \( \mathbb{K} \)-analytic singular foliation \( F_K \) at \((\mathbb{K}^2,0)\) is a \( \mathbb{K} \)-generalized curve (a \( \mathbb{K} \)GC foliation) if there are no saddle-nodes in its reduction of singularities. When \( \mathbb{K} = \mathbb{R} \), considering that algebraic saddle-nodes may differ in their topological picture, we delimit the following broader family of foliations:

**Definition 2.1.** A germ of \( \mathbb{R} \)-analytic singular foliation \( F_R \) at \((\mathbb{R}^2,0)\) is a topological \( \mathbb{R} \)-generalized curve if there are no topological saddle-nodes in its reduction of singularities.

The notion of generalized curve foliation can be weakened by the admission, in the reduction of singularities, of some saddle-node singularities with a good orientation. A saddle-node singularity of \( \tilde{F}_K \) is said to be tangent if its weak separatrix is contained in \( D_K \). For instance, a saddle-node in a corner of \( D_K \) is always tangent. We say that \( F_K \) is of \( \mathbb{K} \)-second type (a \( \mathbb{K} \)ST foliation) if there are no tangent saddle-nodes in its reduction of singularities \([1]\). The family of \( \mathbb{K} \)ST foliations evidently contains that of \( \mathbb{K} \)GC foliations.

To each \( D_K \subseteq D_K \) we associate a weight \( \rho(D_K) \in \mathbb{Z}_{>0} \), defined as the order at \( 0 \in \mathbb{K}^2 \) of a germ of \( \mathbb{K} \)-analytic curve whose strict transform by \( \sigma_K \) is transversal to \( D_K \). The weight also has a combinatorial description which can be seen in \([1]\). It appears in the definition of the following invariant, which is a measure of the existence of tangent saddle-nodes in the reduction of singularities of \( F_K \):

**Definition 2.2.** The \( \mathbb{K} \)-tangency excess of \( F_K \) at \( 0 \in \mathbb{K}^2 \) is the integer

\[
\tau_0(F_K) = \sum_{q \in SN(\tilde{F}_K)} \rho(D_{K,q})(i^w_q(\tilde{F}_K) - 1).
\]

In the formula above, \( SN(\tilde{F}_K) \) stands for the set of tangent saddle-nodes of \( \tilde{F}_K \) in \( D_K \), \( D_{K,q} \) is the component of \( D_K \) containing the weak separatrix of \( \tilde{F}_K \) at \( q \) and \( i^w_q(\tilde{F}_K) \) is the weak index. When \( F_K \) is has a simple singularity at \( 0 \in \mathbb{K}^2 \), the sum is understood as empty and \( \tau_0(F_K) = 0 \). Observe that \( \tau_0(F_K) \geq 0 \) and \( \tau_0(F_K) = 0 \) if and only if \( SN(\tilde{F}_K) = \emptyset \), i.e. if and only if \( F_K \) is a \( \mathbb{K} \)-second type foliation.

Our main theorems will be stated for the family of topological \( \mathbb{R} \)GC foliations. The following remark will be useful in their proofs:

**Remark.** When \( \mathbb{K} = \mathbb{R} \) and \( F_R \) is a topological \( \mathbb{R} \)GC foliation, then \( \tau_0(F_R) \) is even. Indeed, all tangent saddle-node are topological saddle or nodes, so their weak indices are odd and the contribution of each one to the tangency excess is even.

3. The algebraic multiplicity and separatrices

For \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{C} \), let \( X_\mathbb{K} \) denote germs of \( \mathbb{K} \)-analytic vector fields at \((\mathbb{K}^2,0)\) defining a foliation \( F_\mathbb{K} \). We keep in this section the convention that \( F_\mathbb{C} \) is the complexification of \( F_\mathbb{R} \). For them, we fix reductions of singularities as in \([2]\).

The valence \( Val(D_\mathbb{K}) \) of \( D_\mathbb{K} \subseteq D_K \) is the number of irreducible components of \( D_\mathbb{K} \) that intercept \( D_K \), other than \( D_K \) itself. Observe that, if \( D_\mathbb{C} = D_\mathbb{C}^0 \) and \( D_\mathbb{R} = D_\mathbb{C} \cap M_\mathbb{R} \), then \( Val(D_\mathbb{R}) \leq Val(D_\mathbb{C}) \), and the inequality can be strict. However, from the \( J \)-symmetry of \( \tilde{F}_C \), they have the same parity: \( Val(D_\mathbb{R}) \equiv Val(D_\mathbb{C}) \pmod{2} \). On the other hand, it is easy to see that \( \rho(D_\mathbb{R}) = \rho(D_\mathbb{C}) \).
A $K$-\textit{divisor of separatrices} for $\mathcal{F}_K$ is a formal sum

$$B_K = \sum_{B_K \in \text{Sep}(\mathcal{F}_K)} a_{B_K} \cdot B_K,$$

where the coefficients $a_{B_K} \in \mathbb{Z}$ are zero except for a finite number of $B_K \in \text{Sep}(\mathcal{F}_K)$. The \textit{order} of $B_K$ is calculated additively: $\nu_0(B_K) = \sum_{B_K \in \text{Sep}(\mathcal{F}_K)} a_{B_K} \nu_0(B_K)$. The support of $B_K$ is the set formed by all $B_K \in \text{Sep}(\mathcal{F}_K)$ such that $a_{B_K} \neq 0$. Whenever necessary, we can separate isolated and dicritical separatrices and decompose $B_K = B_{K,\text{iso}} + B_{K,\text{dic}}$.

We say that $B_K$ is a $K$-\textit{balanced divisor of separatrices} for $\mathcal{F}_K$ (see [7]) if its coefficients satisfy the following conditions:

- $a_{B_K} \in \{-1, 0, 1\}$;
- $a_{B_K} = 1$ for every $B_K \in \text{Iso}(\mathcal{F}_K)$;
- for a fixed dicritical component $D_K \subset D_K$, the following equality holds:

$$\sum_{B_K \in \text{Sep}(D_K)} a_{B_K} = 2 - \text{Val}(D_K),$$

where $\text{Sep}(D_K)$ is the set of separatrices associated to $D_K$.

Observe that $\nu_0(B_K)$ is the same for all balanced divisors associated to $\mathcal{F}_K$. Actually, in the complex case, it has the following relation with the order $\nu_0(\mathcal{F}_C)$:

\textbf{Proposition 3.1.} [7] Let $\mathcal{F}_C$ be a germ of singular foliation at $(C^2, 0)$ having $\mathcal{B}_C$ as a balanced divisor of separatrices. Then

$$\nu_0(\mathcal{F}_C) = \nu_0(\mathcal{B}_C) - 1 + \tau_0(\mathcal{F}_C).$$

Since $\tau_{C,0}(\mathcal{F}_C) \geq 0$, with equality if and only if $\mathcal{F}_C$ is a $C$-second type foliation, the proposition can be read in the following way:

$$\nu_0(\mathcal{B}_C) \leq \nu_0(\mathcal{F}_C) + 1,$$

with equality happening if and only if $\mathcal{F}_C$ is CST.

Taking into account again the $J$-symmetry of $\mathcal{F}_C$, we can easily see that if $D_C \subset D_C$ is a dicritical component, then $D_C^J = J(D_C)$ is also a dicritical component and $\text{Val}(D_C^J) = \text{Val}(D_C)$. It is also clear that if $D_C^J = D_C$, then $D_C = D_C \cap M_C$ is a dicritical component for the reduction of singularities $\sigma_M$. We can therefore produce a balanced divisor of separatrices $\mathcal{B}_C$ with the following two conditions:

i) $B_C$ and $B_C^J = J(B_C)$ have the same coefficient for every $B_C \in \text{Sep}(\mathcal{F}_C)$;

ii) if $D_C \subset D_C$ is a dicritical component such that $D_C = D_C^J$, then every $B_C \in \text{Sep}(D_C)$ with $a_{B_C} \neq 0$ is in $\text{Sep}^r(\mathcal{F}_C)$.

Keeping coherence with our terminology, we call such a balanced divisor $J$-\textit{symmetric}. We have a decomposition

(3)

$$B_C = B_C^r + B_C^c,$$

where $B_C^r$ comprises all separatrices in $\text{Sep}^r_C(\mathcal{F}_C)$ and $B_C^c$ those in $\text{Sep}^c_C(\mathcal{F}_C)$. By means of the identification $\text{Sep}^r_C(\mathcal{F}_C) \simeq \text{Sep}(\mathcal{F}_R)$, the divisor $B_C^r$ corresponds to a divisor of separatrices for $\mathcal{F}_R$. It is not, in principle, an $\mathbb{R}$-balanced divisor of separatrices for $\mathcal{F}_R$, since the real and complex valences of corresponding dicritical components may differ. However, given a $J$-symmetric balanced divisor $B_C$ for $\mathcal{F}_C$, we can produce, for $\mathcal{F}_R$, a
balanced divisor \( B_\mathbb{R} \) such that every non-zero coefficient of \( B_\mathbb{R} \in \text{Sep}(\mathcal{F}_\mathbb{R}) \) coincides with that of the corresponding \( B_\mathbb{C} \in \text{Sep}(\mathcal{F}_\mathbb{C}) \) in \( \mathcal{B}_\mathbb{C} \). Abusing terminology, we will say that \( B_\mathbb{R} \) is contained in \( \mathcal{B}_\mathbb{C} \).

**Proposition 3.2.** Let \( X_\mathbb{R} \) be a germ of real analytic vector field at \((\mathbb{R}^2, 0)\), \( X_\mathbb{C} \) be its complexification, \( \mathcal{F}_\mathbb{R} \) and \( \mathcal{F}_\mathbb{C} \) be the foliations associated to them. If \( B_\mathbb{R} \) and \( B_\mathbb{C} \) are balanced divisors of separatrices for \( \mathcal{F}_\mathbb{R} \) and \( \mathcal{F}_\mathbb{C} \), then

\[
\nu_0(B_\mathbb{R}) = \nu_0(B_\mathbb{C}) \quad \text{(mod 2)}.
\]

**Proof.** We can suppose that \( B_\mathbb{C} \) is \( J \)-symmetric and that \( B_\mathbb{R} \) is contained in \( B_\mathbb{C} \). Denoting \( \text{Iso}^r(\mathcal{F}_\mathbb{C}) = \text{Iso}(\mathcal{F}_\mathbb{C}) \cap \text{Sep}^r(\mathcal{F}_\mathbb{C}) \) and \( \text{Iso}^c(\mathcal{F}_\mathbb{C}) = \text{Iso}(\mathcal{F}_\mathbb{C}) \cap \text{Sep}^c(\mathcal{F}_\mathbb{C}) \), we have a splitting:

\[
\mathcal{B}_\mathbb{C,iso} = \sum_{B_C \in \text{Iso}^r(\mathcal{F}_\mathbb{C})} a_{B_C} \cdot B_C + \sum_{B_C \in \text{Iso}^c(\mathcal{F}_\mathbb{C})} a_{B_C} \cdot B_C = \mathcal{B}_\mathbb{C,iso}^r + \mathcal{B}_\mathbb{C,iso}^c.
\]

For symmetry reasons, \( \nu_0(\mathcal{B}_\mathbb{C,iso}^r) \) is even. On the other hand, separatrices in \( \text{Iso}^r(\mathcal{F}_\mathbb{C}) \) are in one to one correspondence with those in \( \text{Iso}(\mathcal{F}_\mathbb{R}) \). Therefore \( \nu_0(\mathcal{B}_\mathbb{C,iso}^r) = \nu_0(\mathcal{B}_\mathbb{R,iso}) \), allowing us to conclude that

\[
\nu_0(\mathcal{B}_\mathbb{C,iso}) = \nu_0(\mathcal{B}_\mathbb{R,iso}) \quad \text{(mod 2)}.
\]

Equations (4) and (5) prove the result. \( \square \)

We observe that, since \( \mathcal{F}_\mathbb{C} \) is \( J \)-symmetric, tangent saddle nodes outside the real trace appear in pairs and have the same weak indices. Thus,

\[
\tau_0(\mathcal{F}_\mathbb{C}) \equiv \tau_0(\mathcal{F}_\mathbb{R}) \quad \text{(mod 2)}.
\]

This is an ingredient for the next proposition, which gives sufficient conditions for the existence of separatrices for germs of real analytic vector fields:

**Proposition 3.3.** Let \( X_\mathbb{R} \) be a germ of real analytic vector field at \((\mathbb{R}^2, 0)\) inducing a foliation \( \mathcal{F}_\mathbb{R} \). Suppose that the algebraic multiplicity \( \nu_0(\mathcal{F}_\mathbb{R}) \) and the tangency excess \( \tau_0(\mathcal{F}_\mathbb{R}) \) have the same parity. Then \( \mathcal{F}_\mathbb{R} \) admits a formal separatrix.

**Proof.** We have, by (5), that \( \tau_0(\mathcal{F}_\mathbb{C}) \) and \( \tau_0(\mathcal{F}_\mathbb{R}) \) have the same parity. Since \( \nu_0(\mathcal{F}_\mathbb{C}) \) and \( \nu_0(\mathcal{F}_\mathbb{R}) \) are equal by definition, the proposition’s hypothesis gives that \( \nu_0(\mathcal{F}_\mathbb{C}) \) and \( \tau_0(\mathcal{F}_\mathbb{C}) \) have the same parity. By Proposition 3.2, we then have that \( \nu_0(B_\mathbb{C}) = 1 \) (mod 2). Finally, by Proposition 3.2, \( \nu_0(B_\mathbb{R}) = 1 \) (mod 2). This gives that \( B_\mathbb{R} \) has non-empty support, assuring the existence of a separatrix for \( \mathcal{F}_\mathbb{R} \). \( \square \)

The main result of this section comes as consequence of this proposition:
**Theorem A.** Let $X_\mathbb{R}$ be a germ of real analytic vector field at $(\mathbb{R}^2, 0)$ with associated foliation $\mathcal{F}_\mathbb{R}$. Suppose that $\mathcal{F}_\mathbb{R}$ is a real topological generalized curve foliation. If $\nu_0(\mathcal{F}_\mathbb{R})$ is even, then $\mathcal{F}_\mathbb{R}$ has a formal separatrix.

**Proof.** This is a consequence of the fact that $\tau_0(\mathcal{F}_\mathbb{R})$ is even for a real topological generalized curve foliation $\mathcal{F}_\mathbb{R}$. \qed

Observing that an $\mathbb{R}$GC foliation is also a topological $\mathbb{R}$GC foliation, and that all its separatrices are analytic, we can recover the following result from [14]:

**Corollary 3.4.** If $\mathcal{F}_\mathbb{R}$ is an $\mathbb{R}$-generalized curve foliation with $\nu_0(\mathcal{F}_\mathbb{R})$ is even, then it admits an analytic separatrix.

A germ of non-dicritical foliation $\mathcal{F}_\mathbb{R}$ at $(\mathbb{R}^2, 0)$ is of center-focus or of monodromic type if the following equivalent conditions happen (see, for instance, [9, Th. 9.13]):

- for every smooth $\mathbb{R}$-analytic semi-branch $\Gamma^+$ at $(\mathbb{R}^2, 0)$ there is a first return map $\rho: \Gamma^+ \to \Gamma^+$ (the monodromy map);
- there are no characteristic orbits, i.e., leaves that accumulate to $0 \in \mathbb{R}^2$ with well defined tangent at the origin;
- the reduced model $\tilde{\mathcal{F}}_\mathbb{R}$ has no trace singularities — and thus Sep($\mathcal{F}_\mathbb{R}$) is empty — and all corner singularities are topological saddles.

The third of the above characterizations means, in particular, that a center-focus foliation is a topological $\mathbb{R}$GC foliation. The following fact is then an easy consequence of Theorem A (see [14, Cor. 3.5]):

**Corollary 3.5.** If $\mathcal{F}_\mathbb{R}$ is a center-focus foliation, then $\nu_0(\mathcal{F}_\mathbb{R})$ is odd.

**Proof.** If $\nu_0(\mathcal{F}_\mathbb{R})$ were even, there would be a formal separatrix, giving a contradiction. \qed

### 4. Local Polar Invariants of Complex Analytic Foliations

The main goal of this section is to prove Proposition 4.3. It permits the calculation of the sum of the algebraic multiplicity and of the Milnor number of a complex foliation in terms of an invariant of polar nature. Our result generalizes, for dicritical foliations, an inequality shown in [3, Prop. 2]. The results in this section apply to germs of $\mathcal{C}$-analytic foliations in general. All objects here are complex and we permit ourselves to lighten the notation by omitting the subscript \( \mathbb{C} \).

Let $X = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$ be a germ of complex analytic vector field at $(\mathbb{C}^2, 0)$ and $\omega = P(x, y)dy - Q(x, y)dx$ be its dual 1–form. Denote by $\mathcal{F}$ be the germ of foliation induced by them. The **polar curve** of $\mathcal{F}$ with respect to $(a : b) \in \mathbb{P}^1$, where $\mathbb{P}^1$ is the complex projective line, is the analytic curve $\mathcal{P}_{(a:b)}$ defined by the equation $aP - bQ = 0$. It consists of the sets of points where $X$ has inclination $a/b$ (when $b \neq 0$). For generic $(a : b) \in \mathbb{P}^1$, the curves $\mathcal{P}_{(a:b)}$ are reduced and equisingular. These curves are called **generic polar curves**. For instance, it is evident that an $\mathcal{F}$-invariant component of $\mathcal{P}_{(a:b)}$ must be a line through the origin of inclination $a/b$. Thus, except for the trivial radial foliation, the generic polar curve $\mathcal{P}_{(a:b)}$ is free from $\mathcal{F}$-invariant components. It is also clear that the equality $\nu_0(\mathcal{P}_{(a:b)}) = \nu_0(\mathcal{F})$ is true for a generic polar curve. We refer the reader to [6] for a more extensive discussion on polar curves and polar invariants of foliations.
Let $B$ be a germ of separatrix of $\mathcal{F}$ at $((C^2,0)$, with parametrization $\gamma_B(t)$. The polar intersection of $\mathcal{F}$ with respect to $B$ (see [8]) is defined as

$$(\mathcal{P}_{(a:b)}, B)_0 = \text{ord}_{t=0} ((aP - bQ) \circ \gamma_B(t)),$$

where $(\cdot, \cdot)_0$ stands for the intersection number of germs of complex formal curves at $(\mathbb{C},0)$ and $\mathcal{P}_{(a:b)}$ is a generic polar curve. We denote this number by $p_0(\mathcal{F},B)$. More generally, if $B = \sum_{B \in \text{Sep}(\mathcal{F})} a_B \cdot B$ is a divisor of separatrices of $\mathcal{F}$, we can define polar intersection of $\mathcal{F}$ with respect to $B$ in an additive way:

$$p_0(\mathcal{F},B) = \sum_{B \in \text{Sep}(\mathcal{F})} a_B p_0(\mathcal{F},B).$$

This definition will be used specifically in the case where $B$ is a balanced divisor of separatrices for $\mathcal{F}$ at $0 \in \mathbb{C}^2$.

Let $\Gamma$ be a formal branch at $((C^2,0)$, non-invariant by $\mathcal{F}$, with parametrization $\gamma(t)$. The tangency order of $\mathcal{F}$ with respect to $\Gamma$ at $0 \in \mathbb{C}^2$ is the integer

$$T_0(\mathcal{F},\Gamma) = \text{ord}_{t=0} \gamma^* \omega.$$ 

To a complex blow-up $\pi : (\mathbb{C}^2, D) \to (\mathbb{C}^2,0)$, we associate the following integer:

$$m = \begin{cases} 
\nu_0(\mathcal{F}) & \text{if } \pi \text{ is non-dicritical} \\
\nu_0(\mathcal{F}) + 1 & \text{if } \pi \text{ is dicritical.}
\end{cases}$$

Denote by $\tilde{\Gamma} = \pi^* \Gamma$ and by $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$ the strict transforms of $\Gamma$ and $\mathcal{F}$, and set $q = \tilde{\Gamma} \cap D$. The tangency orders $T_0(\mathcal{F},\Gamma)$ and $T_q(\tilde{\mathcal{F}},\tilde{\Gamma})$ are related in the following way:

**Proposition 4.1.** Let $\Gamma$ be a formal branch at $((C^2,0)$, non-invariant by the foliation $\mathcal{F}$. Then

$$T_0(\mathcal{F},\Gamma) = m \nu_0(\Gamma) + T_q(\tilde{\mathcal{F}},\tilde{\Gamma}),$$

where $m$, $\tilde{\mathcal{F}}$, $\tilde{\Gamma}$ and $q$ are defined as above.

**Proof.** Let us write $\gamma(t) = (x(t), y(t))$. By a linear change of coordinates, we can suppose that the $x$-axis is the tangent cone of $\Gamma$, so that $\nu_0(\Gamma) = \text{ord}_{x=0} x(t)$. Write, in coordinates, the blow-up map as $\pi(x,u) = (x, ux)$, so that $q = (0,0)$. Thus, we can obtain a parametrization $\tilde{\gamma}(t) = (x(t), u(t))$ of $\tilde{\Gamma}$ such that $\gamma(t) = \pi \circ \tilde{\gamma}(t)$, which is equivalent to $y(t) = u(t)x(t)$. If $\omega = P(x,y) dy - Q(x,y) dx$, then

$$\pi^* \omega = (-Q(x,ux) + uP(x,ux)) dx + xP(x,ux) du.$$

The foliation $\tilde{\mathcal{F}}$ is then defined by $\tilde{\omega} = (1/x^m) \pi^* \omega$, where $m$ is given by (7). We have $\gamma^* \omega = (\pi \circ \tilde{\gamma})^* \omega = \tilde{\gamma}^* \pi^* \omega$. Thus

$$T_0(\mathcal{F},\Gamma) = \text{ord}_{t=0} \{\gamma^* \omega\} = \text{ord}_{t=0} \{\tilde{\gamma}^* \pi^* \omega\} = \text{ord}_{t=0} \{(x(t))^m \tilde{\gamma}^* \tilde{\omega}\} = m \text{ord}_{t=0} x(t) + \text{ord}_{t=0} \{\tilde{\gamma}^* \tilde{\omega}\} = m \nu_0(\Gamma) + T_q(\tilde{\mathcal{F}},\tilde{\Gamma}).$$

$\square$
Consider a divisor of separatrices \( B = \sum_{B \in \text{Sep}(F)} a_B \cdot B \) and a blow-up \( \pi : (\mathbb{C}^2, D) \to (\mathbb{C}^2, 0) \). We define the strict transform of \( B \) by \( \pi \) in a standard manner:

\[
\tilde{B} = \pi^* B = \sum_{B \in \text{Sep}(F)} a_B \cdot \tilde{B},
\]

where \( \tilde{B} = \pi^* B \). We can germify this object at a fixed point \( q \in D \), which adds up to erasing all curves \( \tilde{B} \) not passing through \( q \). This is a divisor of separatrices for \( F \) at \( q \), that we denote by \( \tilde{B}_q \).

Let \( \Gamma \) be a formal branch at \((\mathbb{C}^2, 0)\), non-invariant by \( F \). Denote by \( I(\Gamma) \) its set of infinitely near points. We define the tangency excess of \( F \) along \( \Gamma \) as the integer

\[
(8) \quad \tau_0(F, \Gamma) = \sum_{q \in I(\Gamma)} \tau_q(F)\nu_q(\tilde{\Gamma}),
\]

where \( \tau_q \) and \( \nu_q \) are calculated for the strict transforms of \( F \) and \( \Gamma \) by the sequence of blow-ups that produces \( q \). Note that the non-invariance of \( \Gamma \) implies that the sum has a finite number of non-zero terms. If \( \Gamma \) is a reduced curve, without \( F \)-invariant components, we write its decomposition in irreducible components \( \Gamma = \bigcup_{i=1}^k C_i \) and define additively

\[
\tau_0(F, \Gamma) = \sum_{i=1}^k \tau_0(F, C_i).
\]

The following result is a generalization of [3, Cor. 1]:

**Lemma 4.2.** Let \( F \) be a germ of complex analytic foliation at \((\mathbb{C}^2, 0)\). Let \( B \) be a balanced divisor of separatrices for \( F \) and \( \Gamma \) be a non-invariant formal branch. Then

\[
(B, \Gamma)_0 = T_0(F, \Gamma) + 1 - \tau_0(F, \Gamma).
\]

**Proof.** Let us define

\[
\kappa_0(F, \Gamma) = T_0(F, \Gamma) + 1 - (B, \Gamma)_0.
\]

We will first see how \( \kappa_0(F, \Gamma) \) is modified by a blow-up \( \pi : (\mathbb{C}^2, D) \to (\mathbb{C}^2, 0) \) at the origin. As before, denote by \( \bar{F} = \pi^* F, \bar{\Gamma} = \pi^* \Gamma \) and \( \bar{B} = \pi^* B \) the strict transforms of \( F, \Gamma \) and \( B \in \text{Sep}(F) \). If \( q = \bar{\Gamma} \cap D \), let \( \bar{B}_q = (\pi^* B)_q \) denote the germ at \( q \) of the strict transform of the balanced divisor \( B \). The germ \( D_q \) of \( D \) at \( q \) is a separatrix of \( \bar{F} \) at \( q \) if and only if \( \pi \) is a non-dicritical blow-up. Only in this case it should be included in a balanced divisor for \( \bar{F} \):

\[
B_q = \bar{B}_q + \epsilon D_q,
\]

where

\[
\epsilon = \begin{cases} 
1 & \text{if } \pi \text{ is non-dicritical} \\
0 & \text{if } \pi \text{ is dicritical}
\end{cases}
\]

is a balanced divisor of separatrices for \( \bar{F} \) at \( q \) [8, Lem. 3.11].

Noether’s formula (see, for instance, [5, Lem. 3.3.4]), applied to \( B \in \text{Sep}(F) \), gives

\[
(B, \Gamma)_0 = \nu_0(B)\nu_0(\Gamma) + (\bar{B}, \bar{\Gamma})_q.
\]

By linearity, this formula can be extended to the balanced divisor \( B \):

\[
(B, \Gamma)_0 = \nu_0(B)\nu_0(\Gamma) + (\bar{B}_q, \bar{\Gamma})_q.
\]
Proposition 3.1 applied to this expression, gives

\[ (B, \Gamma)_0 = (\nu_0(F) + 1 - \tau_0(F))\nu_0(\Gamma) + (\tilde{B}_q, \tilde{\Gamma})_q. \]

On the other hand,

\[ \epsilon \nu_0(\Gamma) + (\tilde{B}_q, \tilde{\Gamma})_q = \epsilon(D_q, \tilde{\Gamma})_q + (\tilde{B}_q, \tilde{\Gamma})_q = (\tilde{B}_q + \epsilon D_q, \tilde{\Gamma})_q = (B_q, \tilde{\Gamma})_q. \]

Inserting this in (9), we find

\[ (B, \Gamma)_0 = (\nu_0(F) + 1 - \epsilon - \tau_0(F))\nu_0(\Gamma) + (B_q, \tilde{\Gamma})_q = (m - \tau_0(F))\nu_0(\Gamma) + (B_q, \tilde{\Gamma})_q, \]

where \( m \) is as (7). From Proposition 4.1 and (10), we have

\[ \kappa_0(F, \Gamma) - \kappa_q(\tilde{F}, \tilde{\Gamma}) = (T_0(F, \Gamma) + 1 - (B, \Gamma)_0) - (T_q(\tilde{F}, \tilde{\Gamma}) + 1 - (B_q, \tilde{\Gamma})_q) \]

\[ = (T_0(F, \Gamma) - T_q(\tilde{F}, \tilde{\Gamma})) - ((B, \Gamma)_0 - (B_q, \tilde{\Gamma})_q) \]

\[ = m\nu_0(\Gamma) - (m - \tau_0(F))\nu_0(\Gamma) = \tau_0(F)\nu_0(\Gamma) \]

Let \( q_0, q_1, \ldots, q_n \) be successive points in \( I(\Gamma) \), where \( q_0 = 0 \in \mathbb{C}^2 \). Let us denote by \( \tilde{F}_j \) and by \( \tilde{\Gamma}_j \) the strict transforms of \( F \) and \( \Gamma \) at \( q_j \), with the convention that \( F = \tilde{F}_0 \) and \( \Gamma = \tilde{\Gamma}_0 \). We can obtain \( n \) such that, at \( q_n \), both \( \tilde{\Gamma}_n \) and \( \tilde{F}_n \) are regular and transversal, implying that \( \kappa_{q_n}(\tilde{F}_n, \tilde{\Gamma}_n) = 0 \). Indeed, we can fix local coordinates \((x, y)\) at \( q_n \) such that \( \tilde{F}_n \) is defined by \( \omega = dx \) and \( \tilde{\Gamma}_n \) is the \( x \)-axis. Thus, \( B_{q_n} = \{ x = 0 \} \) is the only separatrix of \( \tilde{F}_n \), so that the balanced divisor is \( B_{q_n} = B_{q_n} \). It is then straightforward that \( T_q(\tilde{F}, \tilde{\Gamma}) = 0 \) and that \( (B_q, \tilde{\Gamma})_q = (B, \Gamma)_q = 1 \).

Now, we have

\[ T_0(F, \Gamma) + 1 - (B, \Gamma)_0 = \kappa_0(F, \Gamma) \]

\[ = \kappa_{q_0}(\tilde{F}_0, \tilde{\Gamma}_0) - \kappa_{q_n}(\tilde{F}_n, \tilde{\Gamma}_n) \]

\[ = \sum_{j=0}^{n-1} \left( \kappa_{q_j}(\tilde{F}_j, \tilde{\Gamma}_j) - \kappa_{q_{j+1}}(\tilde{F}_{j+1}, \tilde{\Gamma}_{j+1}) \right) \]

\[ = \sum_{j=0}^{n-1} \tau_j(\tilde{F}_j)\nu_j(\tilde{\Gamma}_j) = \tau_0(F, \Gamma) \quad (\text{by } (11) \text{ and } (8)), \]

proving the Lemma. \( \square \)

Now we can prove the main result of this section:

**Proposition 4.3.** Let \( F \) be a germ of singular complex analytic foliation at \((\mathbb{C}^2, 0)\) and \( B \) be a balanced divisor of separatrices for \( F \). Then

\[ p_0(F, B) = \mu_0(F) + \nu_0(F) - \tau_0(F, \Gamma), \]

where \( \Gamma \) is a generic polar curve.

**Proof.** Let \( \Gamma = \mathcal{P}_{(a:b)} \) be a generic polar curve for \( F \). We can suppose that \( a = 1 \). Write \( \Gamma = \bigcup_{i=1}^{k} C_i \) the decomposition of \( \Gamma \) in irreducible components. As a consequence of Lemma
For each $C_i$, we have

\begin{equation}
(12) \quad p_0(\mathcal{F}, \mathcal{B}) = (\Gamma, \mathcal{B})_0 = \sum_{i=1}^{k} (C_i, \mathcal{B})_0 = \sum_{i=1}^{k} (T_0(\mathcal{F}, C_i) + 1 - \tau_0(\mathcal{F}, C_i)) = \sum_{i=1}^{k} (T_0(\mathcal{F}, C_i) + 1) - \tau_0(\mathcal{F}, \Gamma).
\end{equation}

Thus, from the equation $P - bQ = 0$ of $\Gamma$, we find that $P(\gamma_i(t)) = bQ(\gamma_i(t))$ for all $t$. We have

\begin{equation}
(13) \quad T_0(\mathcal{F}, C_i) + 1 = \text{ord}_{t=0} (\gamma_i^* \omega) + 1 = \text{ord}_{t=0} \left( Q(\gamma_i(t)) \left( by_i(t) - x_i(t) \right) \right) + 1 = \text{ord}_{t=0} Q(\gamma_i(t)) + \text{ord}_{t=0} (by_i(t) - x_i(t)) + 1 = \text{ord}_{t=0} Q(\gamma_i(t)) + \text{ord}_{t=0} (by_i(t) - x_i(t)) = (Q, C_i)_0 + \nu_0(C_i).
\end{equation}

The proof is then completed by putting (13) in (12), observing that

$$
\sum_{i=1}^{k} (Q, C_i)_0 = (Q, \Gamma)_0 = (Q, P - bQ)_0 = (Q, P)_0 = \dim \mathbb{C} \mathbb{C}(x, y)/(P, Q) = \mu_0(\mathcal{F})
$$

and that $\sum_{i=1}^{k} \nu_0(C_i) = \nu_0(\Gamma)$. 

\[\square\]

5. The Milnor number and separatrices

In this section, we resume the study of the pair $\mathbb{R}$-analytic vector field and its complexification, $X_\mathbb{R}$ and $X_\mathbb{C}$, along with their associated foliations, $\mathcal{F}_\mathbb{R}$ and $\mathcal{F}_\mathbb{C}$. Hence, the subscripts $\mathbb{R}$ or $\mathbb{C}$ will be reincorporated to our notation.

Let $\Gamma_\mathbb{R}$ be a germ of real analytic curve, non-invariant by $\mathcal{F}_\mathbb{R}$. We can define, in an obvious manner, a real version of tangency excess of a foliation along $\Gamma_\mathbb{R}$. If $\Gamma_\mathbb{R}$ is a branch, $\tau_0(\mathcal{F}, \Gamma_\mathbb{R})$ is the integer obtained, in formula (8), by simply replacing $\tau_0(\mathcal{F}_\mathbb{C})$ by its real equivalent $\tau_q(\mathcal{F}_\mathbb{R})$. If $\Gamma_\mathbb{R}$ is a union of branches, $\tau_0(\mathcal{F}, \Gamma_\mathbb{R})$ is defined additively, as we did before. Thus, if $\Gamma_\mathbb{R}$ is a germ of $\mathbb{R}$-analytic curve, without $\mathcal{F}_\mathbb{R}$-invariant branches, and $\Gamma_\mathbb{C}$ is its complexification, we have, by (6),

\begin{equation}
(14) \quad \tau_0(\mathcal{F}_\mathbb{R}, \Gamma_\mathbb{R}) \equiv \tau_0(\mathcal{F}_\mathbb{C}, \Gamma_\mathbb{C}) \pmod{2}.
\end{equation}

We chose, for $\mathcal{F}_\mathbb{C}$, a generic polar curve $\Gamma_\mathbb{C} = \mathcal{P}_{(a, b)}$ with $a, b \in \mathbb{R}$. Its equation $aP_\mathbb{C} - bQ_\mathbb{C} = 0$ has real coefficients and its decomposition in irreducible components is such that the components which are not $J$-symmetric appear in pairs. Let $\Gamma_\mathbb{R}$ be the real curve corresponding to the union of $J$-symmetric components of $\Gamma_\mathbb{C}$. We will call it a \textit{generic real polar curve} for $\mathcal{F}_\mathbb{R}$. We set

\begin{equation}
p_0(\mathcal{F}_\mathbb{R}, \mathcal{B}_\mathbb{R}) = (\Gamma_\mathbb{R}, \mathcal{B}_\mathbb{R})_0,
\end{equation}
where $B_R$ is balanced divisor of separatrices for $F_R$ and the intersection number should be understood as that of complexified curves. Again, it is a consequence of the $J$-symmetry of $F_C$ that

$$p_0(F_R, B_R) \equiv p_0(F_C, B_C) \pmod{2}.$$  \hspace{1cm} (15)

Considering these definitions and equations (14) and (15), we can state the following real version for Proposition 4.3:

**Proposition 5.1.** Let $F_R$ be a germ of real analytic foliation at $(R^2, 0)$. If $B_R$ is a balanced divisor of separatrices, then

$$p_0(F_R, B_R) = \mu_0(F) + \nu_0(F) - \tau_0(F, \Gamma_R) \pmod{2},$$

where $\Gamma_R$ is a generic real polar curve.

We can assure the existence of a separatrix for $F_R$ by showing that $p_0(F_R, B_R)$ is non-zero. Indeed, this implies that $B_R$ has non-empty support. In this way, we can use Proposition 5.1 to prove the following result:

**Theorem B.** Let $X_R$ be a germ of real analytic vector field at $(R^2, 0)$ with associated foliation $F_R$. Suppose that $F_R$ is a real topological generalized curve foliation. If $\mu_0(F_R)$ is even, then $F_R$ has a formal separatrix.

**Proof.** If $\nu_0(F_R)$ is even, there is a separatrix by Theorem A. Thus, we can assume that $\nu_0(F_R)$ is odd. The foliation $F_R$ is a topological RGC, so there are no topological saddle-nodes in its reduction of singularities. Thus, all tangency excess indices $\tau_q(F_R)$ for the infinitely near points of the real polar curve $\Gamma_R$ are even, giving that $\tau_0(F, \Gamma_R)$ is also even. Now, we find by Proposition 5.1 that $p_0(F_R, B_R) = 1 \pmod{2}$, which implies the existence of a separatrix. \hfill \Box

With arguments similar to those of Corollaries 3.4 and 3.5, we obtain the following results of [14]:

**Corollary 5.2.** If $F_R$ is an $R$-generalized curve foliation and $\mu_0(F_R)$ is even, then it admits an analytic separatrix.

**Corollary 5.3.** If $F_R$ is a center-focus foliation, then $\mu_0(F_R)$ is odd.

The following example shows that, with the hypothesis stated, we cannot ask for analytic separatrices in Theorems A and B:

**Example 5.4.** Let $a(x) \in R\{x\}$ be a convergent series in one variable such that $a(0) = a'(0) = 0$, and consider the $R$-analytic vector field

$$X_R = (y^2 + x^4) \frac{\partial}{\partial x} + (-xy + x^3a(x) + \frac{a(x)}{x}y^2) \frac{\partial}{\partial y}.$$  

It is proved in [13] Prop. 10] that $X_R$ has a unique separatrix which is, for a generic choice of $a(x)$, purely formal. By a first blow-up $\pi_1 : (x, y) \mapsto (x, y = xy_1)$, the strict transform foliation has a unique singularity over $D_1 = \pi_1^{-1}(0)$, placed at $(x, y_1) = (0, 0)$, which is a saddle-node with strong separatrix contained in the $D_1$. Another blow-up
\[ \pi_1 : (x, y_2) \mapsto (x, y_1 = xy_2) \] produces a strict transform foliation induced at the origin by a vector field of the form
\[ x^3 \frac{\partial}{\partial x} + (-y_2(1 + b(x, y_2)) + a(x)) \frac{\partial}{\partial y_2}, \]
where \( b(x, y_2) \) contains terms of order at least two. The weak index of this saddle-node is \( \iota^w = 3 \) and it is not a topological saddle-node (actually, it is a topological saddle, that can be converted into a topological node by changing \( y \) to \(-y\) in the coefficients of \( X_\mathbb{R} \)). The vector field \( X_\mathbb{R} \) is thus a topological real generalized curve whose only separatrix (for a generic choice of \( a(x) \)) is purely formal. It is easy to see that \( \nu_0(X_\mathbb{R}) = 2 \) and \( \mu_0(X_\mathbb{R}) = 6 \), showing that the statements of Theorems A and B cannot be improved by asking a convergent separatrix.

Our main results could also have been stated in terms of the more general family of \textit{topological second type foliations}: real foliations without tangent topological saddle nodes. We finish by mentioning this fact, placing it as a consequence of Theorems A and B:

**Corollary 5.5.** Let \( \mathcal{F}_\mathbb{R} \) be a germ of topological \( \mathbb{R} \)-second type foliation at \((\mathbb{R}^2, 0)\). If either \( \nu_0(\mathcal{F}_\mathbb{R}) \) or \( \mu_0(\mathcal{F}_\mathbb{R}) \) is even, then \( \mathcal{F}_\mathbb{R} \) has a formal separatrix.

**Proof.** It there were no separatrices, the reduced foliation \( \tilde{\mathcal{F}}_\mathbb{R} \) would have no trace singularities and, so, \( \mathcal{F}_\mathbb{R} \) would be a topological \( \mathbb{R} \)-generalized curve foliation. This would contradict Theorems A and B. \( \Box \)

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