New Lower Bounds for 18 Classical 2-Color Ramsey Numbers

Geoffrey Exoo
Department of Mathematics and Computer Science
Indiana State University
Terre Haute, IN 47809
g@cs.indstate.edu

Milos Tatarevic
Alameda, CA 94501
milos.tatarevic@gmail.com

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Abstract

New lower bounds for 18 classical 2-color Ramsey numbers are presented. Several of these constructions are derived from two well-known colorings: the Paley coloring on 101 vertices and the cubic coloring on 127 vertices.

1 Introduction

The classical Ramsey number \( R(s,t) \) is the smallest integer \( n \) such that in any two-coloring of the edges of the complete graph \( K_n \) there is a monochromatic copy of \( K_s \) in the first color or a monochromatic copy of \( K_t \) in the second color. In this note, we describe some new variations on computer construction methods that were used to improve 18 new lower bounds for
The reader is referred to Radziszowski’s survey on Small Ramsey Numbers \([6]\) for basic terminology related to the problem. The survey contains a comprehensive summary of the current state of the art.

All of the colorings described in this paper are available online at the following location.

\[
\text{http://cs.indstate.edu/ge/RAMSEY/ExTa}
\]

## 2 New Lower Bounds

The table below lists the new bounds obtained by the methods outlined below. Some of the new bounds were obtained by efficient implementations of old methods (circle colorings) and others by some new variations. Several of our new colorings are derived from two well known colorings; the Paley (or quadratic) coloring of \(K_{101}\) and the cubic coloring of \(K_{127}\). The former coloring was used to establish the current lower bound for \(R(6, 6)\) \([4]\), while the latter was used to establish lower bounds for both \(R(4, 12)\) \([7]\) and \(R(4, 4, 4)\) \([3]\). In several cases, we used one of these colorings as a starting point, completing the coloring using various local search procedures. Brief summaries of our methods are given below. When discussing a procedure for improving the lower bound on \(R(s, t)\) we use the term *bad subgraphs* to mean monochromatic complete graph of order \(s\) in color 1 or monochromatic complete subgraphs of order \(t\) in color 2.
| Ramsey Number | Old Bound | New Bound | Method       |
|---------------|-----------|-----------|--------------|
| R(4, 8)       | 58        | 59        | circulant blocks |
| R(4, 11)      | 98        | 102       | circulant blocks |
| R(4, 13)      | 133       | 136       | cubic(127)   |
| R(4, 14)      | 141       | 146       | cubic(127)   |
| R(4, 15)      | 153       | 155       | cubic(127)   |
| R(4, 16)      | 164       | 166       | cubic(127)   |
| R(5, 10)      | 144       | 149       | circulant-plus |
| R(5, 11)      | 171       | 174       | circulant    |
| R(5, 12)      | 191       | 194       | circulant    |
| R(5, 13)      | 213       | 218       | circulant    |
| R(5, 14)      | 239       | 242       | circulant    |
| R(5, 15)      | 265       | 267       | circulant    |
| R(5, 16)      | 289       | 290       | circulant    |
| R(6, 7)       | 113       | 115       | Paley(101)   |
| R(6, 8)       | 132       | 134       | Paley(101)   |
| R(6, 9)       | 169       | 175       | circulant    |
| R(6, 10)      | 179       | 185       | circulant    |
| R(7, 9)       | 241       | 242       | circulant    |

### 3 Methods

#### 3.1 Circulants

The colorings in the table labeled *circulant* were produced by a search method that combines steepest descent with tabu search [1]. The goal is to produce a good coloring on the complete graph of order $n$. The procedure begins with a random circulant coloring matrix (adjacency matrix) and maintains a record of the last $M$ colorings. During each iteration, the procedure considers recoloring the edges for each of the $\lfloor n/2 \rfloor$ chord lengths. If such a recoloring produces a coloring that was previous visited, it is eliminated from consideration for that iteration. Among the remaining chord lengths, one whose recoloring produces the minimum number of monochromatic subgraphs is chosen, and recolored.

If $M$ is large relative to $n$, it is possible that during this procedure there will be no edges which can be recolored without visiting a previous position. When this (rare) event occurs, we restart with a new random coloring. For
graphs of order \(n\), where \(150 < n < 250\) (which applies to several of the cases considered here), a value of \(M\) of approximately 1000 seemed to produce good results.

In the case of \(R(5,10)\) (labeled \textit{circulant-minus} in the Table), we began with a coloring of \(K_{149}\) that was nearly a good coloring. This particular coloring had 149 monochromatic \(K_5\)'s and no monochromatic \(K_{10}\)'s. So one vertex was deleted, and a local search for a good coloring succeeded.

### 3.2 Circulant Blocks

For a graph of order \(n\), let \(m\) be a positive integer such that \(m \mid n\). We can partition the coloring matrix into \(m \times m\) square blocks, and constrain the blocks to be circulant matrices. This method is a good alternative when the cycle colorings can not improve the lower bound, as for example for \(n < 102\) \cite{[2]}. But perhaps due to the larger search space, this method has not produced many of the current lower bounds. However, some of the colorings listed here were obtained by extending this method.

First, note that the diagonal blocks must be symmetric circulant matrices. For the off-diagonal matrices, one could allow the blocks to be asymmetric or also require them to be symmetric. The latter case significantly reduces the size of the search space but does not always provide better results. For small \(n\), we search for all available patterns for the off-diagonal blocks.

A search restricted to coloring matrices with symmetric circulant blocks may rarely produce good colorings, but often will produce colorings with relatively few bad subgraphs. So we typically selected a threshold value for the number of bad subgraphs, and saved any coloring that we found that was as good or better than an empirically determined threshold. For such colorings, we ran a simulated annealing \cite{[5]} procedure to eliminate the remaining bad subgraphs, and in some cases succeeded. In other cases, we deleted problem vertices and repeated the simulated annealing.

In our search to improve the lower bound for \(R(4,8)\), we examined coloring for \(K_n\) for \(60 \leq n \leq 64\), for all \(2 \leq m \leq 10\), such that \(m \mid n\). We set the threshold to 250 bad subgraphs. We manually tested all the graphs we could found with the less 250 bad subgraphs. For \(m = 4\) and \(m = 5\), we found graphs on 60 vertices that we used to extract a \((4,8)\)-coloring on 58 vertices. In the first case, the block construction contained 60 monochromatic \(K_4\)'s and 45 monochromatic \(K_8\)'s. In the second, it contained 63 monochromatic \(K_4\)'s and no monochromatic \(K_8\)'s. In both cases, the best coloring
was reached after removing two vertices. We note that these graphs on 60 vertices were not those with the minimum possible number of bad subgraphs (among block circulant colorings). Many colorings were found with less than 100 bad subgraphs, and a few with less than 50, but we were not be able to modify those to obtain good colorings.

Similarly, for $R(4, 11)$, we searched for good candidates on 105 vertices, using $m = 7$. To simplify our search, we started from a block construction for $n = 60$ and $m = 4$, the same one we used to reach the new lower bound of $R(4, 8)$. We extended this graph by adding circulant blocks without changing the initial coloring on 60 vertices. First we found a promising $(4, 9)$-coloring on 75 vertices. The procedure was repeated for $(4, 10)$-colorings with $n = 90$ and $m = 6$ until reaching a local minimum. It was then applied to a search for a $(4, 11)$-coloring on 105 vertices. Once we reached a local minimum there, and were convinced it could not be improved with circulant blocks, we performed the individual edge recoloring procedure, and finally removed selected vertices to obtain a good coloring on 101 vertices and thus establish the new lower bound.

### 3.3 Using Special Colorings

In the table, several of the colorings are labeled *Paley* or *cubic*. For these cases, we used the Paley coloring of order 101, or the cubic coloring of order 127, as the starting blocks in a block circulant coloring. Each block would be a copy of one of these two colorings. (Note: we did not use both of these special colorings on the same problem.) In cases where the desired coloring was not an integer multiple of 101 or 127 we deleted rows and columns until a matrix of the desired size resulted. In some cases, the resulting graph had very few bad subgraphs. A local improvement procedure was then applied to produce a good coloring. In the case of the colorings based on the cubic residue graph, it was surprising how quickly we were able to obtain good colorings.

### References

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