ATTRACTORS AND ENTROPY BOUNDS FOR A NONLINEAR RDES WITH DISTRIBUTED DELAY IN UNBOUNDED DOMAINS

DALIBOR Pražák* AND JAKUB Slavík

Department of Mathematical Analysis, Charles University, Prague
Sokolovská 83, 186 75 Praha 8, Czech Republic

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Abstract. A nonlinear reaction-diffusion problem with a general, both spatially and delay distributed reaction term is considered in an unbounded domain $\mathbb{R}^N$. The existence of a unique weak solution is proved. A locally compact attractor together with entropy bound is also established.

1. Introduction. We are interested in the equation of the form
\[
\partial_t u - \text{div} \ a(\nabla u) + du = F(u^t),
\]
where $d, r > 0$ and $u^t(\theta) \equiv u(t + \theta)$ for $\theta \in [-r, 0]$ and
\[
F(u^t) = \int_{-r}^{0} \left( \int_{\Omega} b(u(t + \theta, y))f(x - y)e^{-|x-y|} dy \right) \xi(\theta, u(t), u^t) d\theta.
\]
The equation is posed in the unbounded spatial domain $x \in \Omega = \mathbb{R}^N$.

The equation can be seen as an abstract prototype of a nonlinear reaction diffusion system, which combines three nontrivial mathematical features: (i) nonlinear diffusion term $-\text{div} \ a(\nabla u)$, (ii) temporally and spatially distributed delay terms and (iii) the setting of unbounded domains. We will begin by discussing the difficulties related to these three issues, together with a selection of recent references.

Let us start with the last point (iii). It can be said that the dynamics in unbounded domains has attracted a growing attention of the PDE community during the last decade. The problem obviously has an inherent non-compactness or even non-separability. This calls for a careful rethinking of the proper choice for the functional setting, so that the results on the global attractor and its finite-dimensionality, which are typical in bounded domain setting, can find a proper generalized expression. A natural choice seems to be some space of uniformly locally integrable functions, see [4], [19], [2], [1]. In such a setting, the existence of locally compact attractor admitting natural entropy estimates is the expected result; see also [10] and [3].

Concerning the point (ii), we would remark that presence of temporally and spatially non-local reaction lower order terms arise naturally in describing both living

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* Corresponding author: Dalibor Pražák.
and non-living nature. We can mention the birth-death dynamics of maturing population or the spread of infection on the one hand, and the phenomena of yield or creep occurring in viscoelastic materials, or nonlocal interactions in phase transitions, on the other hand. The available mathematical techniques and results depend essentially on the complexity of non-local terms. In the case of linear delay of convolution type, linear techniques (theory of $C_0$-semigroups, linear stability results) can be used [7, 11, 5].

For a more general non-linear problems, perturbation and topological methods provide sufficient conditions for the existence of robust nontrivial structures like travelling waves, see [6], [17]. In the case of a bounded spatial domain, the existence of a global compact attractor was shown for the equation (1) in [13] with a linear diffusion $a(\nabla u) = \nabla u$; cf. also [12]. The existence of a global attractor for a similar linear equation in unbounded domain with $b(y, u(t+\theta, y))$ instead of $b(u(t+\theta, y))$ was proved in [9]. However, the authors in [9] study the equation in classical Sobolev spaces with $\xi \equiv 1$ and certain restrictions on $d$ and $r$ have to be met to obtain the existence of a compact attractor. Another similar linear equation with fixed delay and $N = 1$ was studied in [18] in the setting of bounded uniformly continuous functions. The existence of generalized attractors for delayed systems in unbounded domains was recently established in [16] and [15].

In the following we analyze the equation (1) in locally uniform spaces $L^2_b$ in the spirit of [19]. The main advantage of this setting, as compared to standard Lebesgue or weighted Lebesgue spaces, is the possibility to capture arbitrary spatial complexity of the dynamics, including (spatially) periodic patterns. The spatial uniformity of $L^2_b$ spaces, however, makes them similar to $L^\infty$ spaces and thus not a good choice as a target spaces for the underlying dynamical system. For example, one cannot in general expect that the solution will be continuous with values in $L^2_b$. Several auxiliary weaker spaces are thus necessary to be introduced in the course of the analysis. Here in particular, following [8], we employ a sort of parabolic version of locally uniform spaces $L^2_{b,1}(0; T; L^2)$ (see Section 2 below for definitions). The smoothing property of the dynamics can be easily proved in this parabolic setting, very much in the spirit of the so-called method of $\ell$-trajectories. This leads to the existence and entropy estimates of the global (locally uniform) attractor $\mathcal{A}$. No higher order regularity estimates and in particular, no restrictions on $d$ or $r$ other than $r, d > 0$ are needed. This low-cost (in terms of regularity) approach also enables us to work with a more general assumptions on the diffusion term (i): a general non-linear elliptic diffusion is possible, further generalizing the results common in the existing literature, where most often a linear dissipation (i.e. the Laplace operator) is considered.

The paper is organized as follows: the locally uniform spaces, corresponding duals and also their parabolic variants are briefly reviewed in Section 2. Existence and uniqueness of the weak solution are proven in Section 3. Locally compact attractor and its entropy estimates are established in Sections 4 and 5.

2. Function spaces and notation. Here we review the locally uniform spaces, following [19], [8].

Definition 2.1. Let $\bar{x} \in \mathbb{R}^N$ and $\varepsilon > 0$. The weighted Lebesgue space $L^2_{\bar{x},\varepsilon}(\Omega)$ is defined by the norm

$$
\|u\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 = \int_{\Omega} |u(x)|^2 e^{-\varepsilon|x-\bar{x}|} \, dx.
$$
Similarly the spaces $W^{1,2}_{x,\varepsilon}(\Omega)$ and $W^{-1,2}_{x,\varepsilon}(\Omega)$ are defined by the norms

$$
\|u\|^2_{W^{1,2}_{x,\varepsilon}(\Omega)} = \int_\Omega \left( |\nabla u(x)|^2 + |u(x)|^2 e^{-\varepsilon|x-\bar{x}|} \right) dx,
$$

$$
\|u\|^2_{W^{-1,2}_{x,\varepsilon}(\Omega)} = \sup_v \int_\Omega u(x)v(x)e^{-\varepsilon|x-\bar{x}|} dx,
$$

where the last supremum is taken over $v \in W^{1,2}_{x,\varepsilon}(\Omega)$ with unit norm.

We use the notation

$$
(u,v)_{\varepsilon,\bar{x}} \equiv \int_\Omega u(x)v(x)e^{-\varepsilon|x-\bar{x}|} dx.
$$

Similarly we can define the weighted locally uniform $L^2$ functions for a general $p \in [1, \infty)$ and the same actually holds for all the spaces defined in the rest of this section.

**Definition 2.2.** The space of locally uniform $L^2$ functions is defined by

$$
L^2_{loc}(\Omega) = \{ u \in L^2_{loc}(\Omega); \sup_{x_0 \in \Omega} \|u\|_{L^2(B(x_0,1))} < \infty \}.
$$

Here $B(x_0,1)$ stands for an r-ball centered in $x_0$. Let $x_k$, $k \in \mathbb{N}$, enumerate the points with half-integer coordinates, i.e. $(\mathbb{Z}/2)^N$, and let $C_k = C(x_k)$, $k \in \mathbb{N}$, be the unit cubes, centered in $x_k$. Then clearly the space $L^2_k(\Omega)$ has an equivalent norm

$$
\|u\|_{L^2_k(\Omega)} \equiv \|u\|_b = \sup_{k \in \mathbb{N}} \|u\|_{L^2(C_k)}.
$$

**Definition 2.3.** Let $\mu \geq 0$. An admissible weight function of growth rate $\mu$ is a measurable bounded function $\phi : \mathbb{R}^N \rightarrow (0, \infty)$ satisfying the inequalities

$$
C^{-1}e^{-\mu|x-y|} \leq \phi(x)/\phi(y) \leq Ce^{\mu|x-y|}, \quad |\nabla \phi(x)| \leq |\phi(x)|
$$

for some $C \geq 1$ and every $x, y \in \mathbb{R}^N$.

A typical example of an admissible weight function is the exponential $\phi(x) = e^{-q|x-\bar{x}|}$ with $\bar{x} \in \mathbb{R}^N$ and $q \in [0, 1]$. Trivially, $\phi(x) \equiv 1$ is and admissible weight function of growth rate $\mu = 0$. In fact we can define the locally uniform space in a more general manner and arrive at similar relations between weighted Lebesgue spaces and (weighted) locally uniform spaces. For more information see e.g. [2], Section 4.

**Definition 2.4.** Let $\phi$ be an admissible weight function. We define the space of weighted locally uniform $L^2$ functions $L^2_{b,\phi}(\Omega)$ by

$$
L^2_{b,\phi}(\Omega) = \{ u \in L^2_{loc}(\Omega); \sup_{x_0 \in \Omega} \phi(x_0)^{1/2} \|u\|_{L^2(B(x_0,1))} < \infty \}.
$$

For $\phi \equiv 1$, we simply write $L^2_b(\Omega)$.

Similarly as in the non-weighted case one may observe that the space $L^2_{b,\phi}(\Omega)$ has an equivalent norm

$$
\|u\|_{L^2_{b,\phi}(\Omega)} \equiv \|u\|_{b,\phi} = \sup_{k \in \mathbb{N}} \phi(x_k)^{1/2} \|u\|_{L^2(C_k)}.
$$

**Theorem 2.5** ([8], Theorem 2.1). Let $\phi$ be an admissible weight function with growth rate strictly smaller than $\varepsilon > 0$. The space $L^2_{b,\phi}(\Omega)$ admits an equivalent norm

$$
\|u\|^2_{b,\phi} = \sup_{x \in \Omega} \phi(x) \int_\Omega |u(x)|^2 e^{-\varepsilon|x-\bar{x}|} dx.
$$
Following the notation of [8], we define the $L^2_{b,\phi}$ seminorms corresponding to a subdomain $O \subseteq \Omega$. For $O \subseteq \Omega$ we define

$$\mathbb{I}(O) = \{ k \in \mathbb{N}; C_k \cap O \neq \emptyset \},$$

$$\|u\|_{L^2_{b,\phi}(O)} = \sup_{k \in \mathbb{I}(O)} \phi^{1/2}(x_k) \|u\|_{L^2(C_k)}.$$ 

We will need to use the so-called parabolic locally uniform spaces introduced in [8].

**Definition 2.6.** Let $\phi$ be an admissible weight function and $\varepsilon > 0$. We define the parabolic locally uniform spaces by their respective norms

$$\|u\|_{L^2_{b,\phi}(-r,\varepsilon; L^2(\Omega))} = \sup_{k \in \mathbb{N}} \phi(x_k)^{1/2} \|u\|_{L^2(-r,\varepsilon; L^2(C_k))},$$

$$\|u\|_{L^2_{b,\phi}(-r,\varepsilon; W^{1,2}(\Omega))} = \sup_{k \in \mathbb{N}} \phi(x_k)^{1/2} \|u\|_{L^2(-r,\varepsilon; W^{1,2}(C_k))},$$

$$\|u\|_{L^2_{b,\phi}(-r,\varepsilon; W^{-1,2}(\Omega))} = \sup_{k \in \mathbb{N}} \phi(x_k)^{1/2} \|u\|_{L^2(-r,\varepsilon; W^{-1,2}(C_k))}.$$ 

Once again, the symbol $\phi$ is dropped if $\phi \equiv 1$.

A simple variant of Theorem 2.4 from [8] implies that for $\phi$ of growth rate $\mu$ strictly smaller than $\varepsilon > 0$, the parabolic locally uniform spaces admit equivalent norms

$$\|u\|_{L^2_{b,\phi}(-r,\varepsilon; L^2(\Omega))} \approx \sup_{\bar{x} \in \Omega} \phi(\bar{x}) \int_{(-r,\varepsilon) \times \Omega} |u(t, x)|^2 e^{-\varepsilon|x-\bar{x}|} \, dx \, dt,$$  

(4)

$$\|u\|_{L^2_{b,\phi}(-r,\varepsilon; W^{1,2}(\Omega))} \approx \sup_{\bar{x} \in \Omega} \phi(\bar{x}) \int_{(-r,\varepsilon) \times \Omega} \left( |u(t, x)|^2 + |\nabla u(t, x)|^2 \right) e^{-\varepsilon|x-\bar{x}|} \, dx \, dt,$$  

(5)

$$\|u\|_{L^2_{b,\phi}(-r,\varepsilon; W^{-1,2}(\Omega))} \approx \sup_{v \in \Omega} \phi(\bar{x}) \int_{(-r,\varepsilon) \times \Omega} u(t, x)v(t, x) e^{-\varepsilon|x-\bar{x}|} \, dx \, dt,$$  

(6)

where the first supremum in the last equivalence is taken over the functions $v \in L^2_{b,\phi}(-r,\varepsilon; W^{1,2}(\Omega))$ with unit norm.

The parabolic uniformly bounded spaces and the Bochner spaces constructed over locally uniform spaces are related in the following way:

$$L^2(-r,\varepsilon; \Omega) \subseteq L^2_{b,\phi}(-r,\varepsilon; \Omega) \subseteq L^2_{t,\varepsilon}(\Omega) \subseteq L^2_{t,\varepsilon}(\Omega).$$ 

Recall that for $\varepsilon > 0$, a metric space $M$ and a precompact set $K \subseteq M$, the Kolmogorov $\varepsilon$-entropy is defined by

$$H_{\varepsilon}(K, M) = \log N_{\varepsilon}(K, M),$$

where $N_{\varepsilon}(K, M)$ is the smallest number of balls of radius $\varepsilon$ that cover the set $K$ in $M$.

**Lemma 2.7** ([8], Lemma 2.6). Let $\phi$ be an admissible weight function. Let $O \subseteq \Omega$ satisfy

$$\#I(O) \leq c_1 \text{vol}(O).$$  

(7)

Denote $Q = [-r,\ell] \times \Omega$ and define

$$\|\chi\|_{W_{b,\phi}(Q)} = \|\chi\|_{L^2_{b,\phi}(-r,\ell; \Omega)} + \|\partial_t \chi\|_{L^2_{b,\phi}(-r,\ell; \Omega)} + \|\partial_x \chi\|_{L^2_{b,\phi}(-r,\ell; \Omega)},$$  

(8)

Let $R > 0$ and $\theta \in (0,1)$. Then there exists $c_0 > 0$ such that

$$H_{\theta R}(B_R(\chi; W_{b,\phi}(Q)), L^2_{b,\phi}(-r,\ell; \Omega)) \leq c_0 \text{vol}(O),$$

where $\text{vol}(O)$ is the volume of the domain $O$. 


Lemma 2.11. Let $\kappa > 0$ for some $\kappa > 0$. Then the estimate
\[ \int_{\Omega \setminus B(0,R)} |u(x)|^2 e^{-\varepsilon |x-x_0|} \, dx < \delta \]
for every $u \in B$.

Lemma 2.9. Let $B \subseteq L^2_0(\Omega)$ be bounded and let $u_n, u \in B$. Then
\[ u_n \to u \text{ in } L^2_\varepsilon(\Omega) \Leftrightarrow u_n \to u \text{ in } L^2_{loc}(\Omega) \]
for every $\bar{x} \in \Omega, \varepsilon > 0$.

Lemma 2.10. Let $B \subseteq L^\infty(-r, \ell; L^2_\varepsilon(\Omega))$ be bounded and let $u_n, u \in B$. Then
\[ u_n \to u \text{ in } L^2(-r, \ell; L^2_{\varepsilon,x}(\Omega)) \Leftrightarrow u_n \to u \text{ in } L^2_{loc}((-r, \ell) \times \Omega) \]
for every $\bar{x} \in \Omega, \varepsilon > 0$.

Lemma 2.11. Let $p \in [1, \infty]$, $\bar{x} \in \mathbb{R}^N$, $u \in L^p_{x,\varepsilon}(\mathbb{R}^N)$ and let $G$ be a function such that $G_{\varepsilon/p} \in L^1(\mathbb{R}^N)$, where
\[ G_{\varepsilon/p}(y) = G(y)e^{\varepsilon |y|/p}. \]

Then the estimate
\[ \|u \ast G\|_{L^p_{x,\varepsilon}(\mathbb{R}^N)} \leq \|u\|_{L^p_{x,\varepsilon}(\mathbb{R}^N)} \|G_{\varepsilon/p}\|_{L^1(\mathbb{R}^N)} \]
holds true.

3. Well-posedness. We impose the following assumptions on the nonlinearities: Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function satisfying
\[ a(0) = 0, \quad (a(\zeta) - a(\eta)) \cdot (\zeta - \eta) \geq \kappa |\zeta - \eta|^2, \quad \forall \zeta, \eta \in \mathbb{R}^N, \quad (9) \]
\[ |a(\zeta) - a(\eta)| \leq \kappa \eta |\zeta - \eta|, \quad \forall \zeta, \eta \in \mathbb{R}^N, \quad (10) \]
\[ \zeta \to a(\zeta) \cdot \zeta \text{ is a convex function on } \mathbb{R}^N, \quad (11) \]
for some $\kappa > 0, \gamma \geq 1$.

Let $b : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz, i.e.
\[ b(0) = 0, \quad |b(r)| \leq C_b \text{ for every } r \in \mathbb{R}, \quad (12) \]
\[ |b(r) - b(s)| \leq C_b |r - s| \text{ for every } r, s \in \mathbb{R}. \quad (13) \]

Let $f : (\Omega - \Omega) \to \mathbb{R}$ be bounded, i.e.
\[ |f(x - y)| \leq C_f \text{ for every } x, y \in \Omega. \quad (14) \]

Next we choose $0 < \varepsilon < 1$ small enough, namely
\[ \varepsilon < \min \left( 2/\gamma, 2d/\kappa \gamma \right). \quad (15) \]
We remark that due to the unbounded domain setting, the absolute term $du$ is indespensable for dissipation as well as is some smallness condition on $\varepsilon$, which
corresponds to boundary condition at infinity. This particular choice of $\varepsilon$ does not affect the well-posedness of the problem, in fact the well-posedness can be proved for arbitrary $0 < \varepsilon < 1$ by the same argument as below.

Finally, concerning the form of the distributed delay, we impose the following conditions on the function $\xi : (-r, 0) \times H \rightarrow \mathbb{R}$, where

$$H = L^2_0(\Omega) \times L^2_0(-r, 0; L^2(\Omega))$$

is the space of initial conditions:

(i) For every $M > 0$ there exists $L = L(M) > 0$ such that for every $\bar{x} \in \Omega$ and $(v^i, \psi^i) \in H$ satisfying

$$\|v^i\|_{L^2_{\bar{x},\varepsilon}}^2 + \int_{-r}^0 \|\psi^i(\xi)\|_{L^2_{\bar{x},\varepsilon}}^2 \, ds \leq M^2, \quad i = 1, 2,$$

the following holds:

$$\int_{-r}^0 |\xi(\theta, v^1, \psi^1) - \xi(\theta, v^2, \psi^2)| \, d\theta \leq L \left(\|v^1 - v^2\|_{L^2_{\bar{x},\varepsilon}}^2 + \int_{-r}^0 \|\psi^1(\theta) - \psi^2(\theta)\|_{L^2_{\bar{x},\varepsilon}}^2 \, d\theta \right)^{1/2}.$$  \hspace{1cm} (16)

(ii) There exists $C_\xi > 0$ such that for every $(v, \psi) \in H$ we have

$$\|\xi(\cdot, v, \psi)\|_{L^2(-r, 0)} \leq C_\xi.$$  \hspace{1cm} (17)

Note that the condition (ii) implies that $F(u^i) \in L^2_{\bar{x},\varepsilon}(\Omega)$ uniformly for every $u : (t - r, t) \times \Omega \rightarrow \mathbb{R}$ with $(u(t), u^i) \in H$.

Function $u : [0, T] \times \Omega \rightarrow \mathbb{R}$ will be called a (weak solution), if for every $\bar{x} \in \Omega$

$$u \in C([0, T]; L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2(-r, 0; L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\Omega)),
$$

$$\partial_t u \in L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\Omega)),$$

and $u$ satisfies the variational formulation

$$-\int_0^T (u(t), \partial_t \psi(t)) \, dt + \int_0^T (a(\nabla u(t)), \nabla \psi(t)) \, dt + d \int_0^T (u(t), \psi(t)) \, dt = \int_0^T (F(u^i), \psi(t)) \, dt$$  \hspace{1cm} (18)

for every $\psi \in \mathcal{D}((0, T) \times \Omega)$ and the initial conditions

$$u(0) = u_0 \in L^2_0(\Omega), \quad u_{(-r, 0)} = \varphi \in L^2_0(-r, 0; L^2(\Omega))$$  \hspace{1cm} (19)

hold true. For arbitrary $\bar{x} \in \Omega$ we may use a standard density argument and arrive to the duality with respect to $L^2_{\bar{x},\varepsilon}(\Omega)$:

$$(u(T), \psi(T))_{\varepsilon, \bar{x}} - \int_0^T (u(t), \partial_t \psi(t))_{\varepsilon, \bar{x}} \, dt + \int_0^T \left( a(\nabla u(t)), \nabla \psi(t) - \varepsilon \frac{x - \bar{x}}{|x - \bar{x}|} \psi(t) \right)_{\varepsilon, \bar{x}} \, dt + d \int_0^T (u(t), \psi(t))_{\varepsilon, \bar{x}} \, dt = \int_0^T (F(u^i), \psi(t))_{\varepsilon, \bar{x}} \, dt + (u(0), \psi(0))_{\varepsilon, \bar{x}}$$  \hspace{1cm} (20)
for any \( \psi \in L^2(0,T;W^{1,2}_{x,e}(\Omega)) \cap W^{1,2}(0,T;L^2_{x,e}(\Omega)) \). Indeed, one can replace \( \psi \) in (18) by \( \psi\chi_ne^{-\varepsilon|x-x^\ast|} \), where \( \chi_n \) is some sequence of cut-off functions such that \( \chi_n \to 1 \), \( \nabla\chi_n \to 0 \) and \( |\chi_n| + |\nabla\chi_n| \leq c \) a.e. It is clear that (20) in turn implies (18).

**Theorem 3.1.** Let (9) - (15) hold and let \( \xi \) satisfy the conditions (i),(ii). Then for every \( \bar{\omega} \) in (18) by then pass to the limit. Let \( \Omega \)

\[
\text{Proof.}\ \text{The proof is a variant of the original proof for the linear case in a bounded domain (see [13], Theorem 1). We need to handle the limit of nonlinear diffusion term (cf. [8], Theorem 3.2); otherwise, standard techniques for unbounded domains are used ([19]).}

We approximate the problem (18) by a sequence of problems solvable on bounded domains and then pass to the limit. Let \( \Omega_n = B_n(0) \subseteq \mathbb{R}^N \) and let \( \psi_n \in C^\infty(\Omega, [0,1]) \) satisfy \( \psi_n \equiv 1 \) on \( \bar{\Omega}_{n-1} \), \( \text{supp} \psi_n \subseteq \Omega_n \), and define \( u_{0,n} = u_0\psi_n \) and \( \varphi_n(\theta) = \varphi(\theta)\psi_n \) for \( \theta \in [-r,0] \). Using Theorem 2.5 and the Lebesgue’s dominated convergence theorem we immediately obtain

\[
u_{0,n} \to u_0 \text{ in } L^2_{x,e}(\Omega), \quad \varphi_n \to \varphi \text{ in } L^2(-r,0; L^2_{x,e}(\Omega)) \quad (21)
\]

for every \( \bar{x} \in \Omega \). Next we define the operator

\[
A_n : W^{1,2}_0(\Omega_n) \to W^{-1,2}(\Omega_n), \quad \langle A_n v, z \rangle = \int_{\Omega_n} a(\nabla v(x)) \cdot \nabla z(x) \, dx
\]

and the approximate problem

\[
\partial_t u_n + A_n u_n + d u_n = \int_{-r}^0 \left( \int_{\Omega_n} b(u_n(t+\theta,y))f(x-y)e^{-|x-y|} \, dy \right) \xi(\theta, u_n(t), u_n') \, d\theta.
\]

with the initial condition

\[
u_n(0) = u_{0,n}, \quad u_n|_{(-r,0)} = \varphi_n. \quad (22)
\]

A nonlinear variant of Theorem 1 from [13] implies that the equation (22) with the initial equation (23) has a solution

\[
u_n \in C([0,T], L^2(\Omega_n)) \cap L^2(-r,T; L^2(\Omega_n)) \cap L^2(0,T; W^{1,2}_0(\Omega_n)).
\]

First we aim to show that

\[
u_n \xrightarrow{*} u \text{ in } L^\infty(0,T,L^2_{x,e}(\Omega)) \cap L^2(0,T; W^{1,2}_{x,e}(\Omega)) \quad (24)
\]

for some \( u \in L^\infty(0,T,L^2_{x,e}(\Omega)) \cap L^2(0,T; W^{1,2}_{x,e}(\Omega)) \). Let us extend \( u_n \) by zero outside of \( \Omega_n \) (note that then \( u_n \in L^\infty(0,T; L^2_{x,e}(\Omega)) \cap L^2(0,T; W^{1,2}_{x,e}(\Omega)) \) and thus \( \xi(\theta, u_n(t), u_n') \) makes good sense) and test (22) by \( u_n(t,x)e^{-\varepsilon|x-x^\ast|} \) to get

\[
th{2} \int_{\Omega} |u_n(t,x)|^2 e^{-\varepsilon|x-x^\ast|} \, dx + \int_{\Omega} \left( \nabla u_n(t,x) - \varepsilon \frac{x-x^\ast}{|x-x^\ast|} u_n(t,x) \right) e^{-\varepsilon|x-x^\ast|} \, dx
\]

\[
= \int_{\Omega} F(u_n(t,x)) u_n(t,x) e^{-\varepsilon|x-x^\ast|} \, dx.
\]

Observe that the integration in the previous equation is over \( \Omega \) instead of \( \Omega_n \). This is possible since \( u_n \equiv 0 \) outside of \( \Omega_n \) and \( b(0) = 0, a(0) = 0 \). Using the boundedness
of the functions $b$ and $f$, from (ii) it follows that

$$
(F(v^t), z) \leq C\|z\|_{\bar{x},\varepsilon}.
$$

The previous estimate, (25), (9), (10) and Young’s inequality immediately give

$$
\frac{d}{dt}\|u_n(t)\|_{\bar{x},\varepsilon}^2 + \sigma (\|\nabla u_n(t)\|_{\bar{x},\varepsilon}^2 + \|u_n(t)\|_{\bar{x},\varepsilon}^2) \leq C_1\|u_n(t)\|_{\bar{x},\varepsilon}^2 + C_2
$$

for some $\sigma > 0$. Gronwall’s inequality applied to

$$
\chi_n(t) = \|u_n(t)\|_{\bar{x},\varepsilon}^2 + \sigma \left(\int_0^t \|\nabla u_n(s)\|_{\bar{x},\varepsilon}^2 ds + \int_0^t \|u_n(s)\|_{\bar{x},\varepsilon}^2 ds \right) + C_2
$$

gives

$$
\chi_n(t) \leq (\|u_n(0)\|_{\bar{x},\varepsilon}^2 + C_2)e^{C_1t},
$$
in other words

$$
\|u_n(t)\|_{\bar{x},\varepsilon}^2 + \int_0^t \|\nabla u_n(s)\|_{\bar{x},\varepsilon}^2 ds + \int_0^t \|u_n(s)\|_{\bar{x},\varepsilon}^2 ds \leq (\|u_n(0)\|_{\bar{x},\varepsilon}^2 + C_2)e^{C_1t} - C_2. \quad (26)
$$

From (21) it follows that the sequence $\{u_n(t)\}_{n=1}^\infty$ is bounded in $L^\infty(0, T; L^2_{\bar{x},\varepsilon}(\Omega))$, $L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\Omega))$ and using a standard argument we finally have (24). Observe that if we take the supremum of (26) over $x \in \Omega$, using (3) and (5) we obtain

$$
u \in L^\infty(0, T; L^2_{\bar{x}}(\Omega)) \cap L^2(0, T, W^{1,2}_{\bar{x}}(\Omega)). \quad (27)
$$

Following a similar argument we can show

$$
\partial_t u_n \rightarrow \partial_t u \text{ in } L^2(0, T; W^{-1,2}_{\bar{x}}(\Omega))
$$

and therefore we immediately have

$$
u \in C([0, T], L^2_{\bar{x},\varepsilon}(\Omega)).
$$

Also note that since the norms $L^2_{\bar{x},\varepsilon}(\Omega)$ are equivalent for different $\bar{x} \in \Omega$, the function $u$ is independent of $\bar{x}$. Next we show

$$
u_n \rightarrow u \text{ in } L^2(0, T; L^2_{\bar{x},\varepsilon}(\Omega)), \quad (28)
$$

The first step is to establish the convergence

$$
u_n \rightarrow u \text{ in } L^2(0, T; L^2(\Omega_m)) \quad (29)
$$

for $m \in \mathbb{N}$ fixed. We proceed similarly as in the proof of Theorem 3.2 from [8]. The weak convergence (24) implies

$$
u_n \rightharpoonup u \text{ in } L^\infty(0, T, L^2(\Omega_m)) \cap L^2(0, T; W^{1,2}(\Omega_m)). \quad (30)
$$

Then we choose $n > m$ and test the equation (22) with $v \in L^2(0, T; W^{1,2}(\Omega_m))$ extended by zero outside of $\Omega_m$ to get

$$
\partial_t \nu_n \rightarrow \partial_t u \text{ in } L^2(0, T; W^{-1,2}(\Omega_m)). \quad (31)
$$

The desired convergence (29) then follows from (30), (31) and the Aubin-Lions lemma.

Using the continuity of $\nu_n$ and $u$ and the estimate (27) we can find a set $B \subseteq L^2_{\bar{x}}(\Omega)$ such that

$$
u_n(t) - u(t) \in B \text{ for a.e. } t \in [0, T]
$$

and $B$ is bounded in $L^2_{\bar{x}}(\Omega)$. Choose $\delta > 0$ and use Lemma 2.8 to find $n_0 \in \mathbb{N}$ such that

$$
\int_{\Omega \setminus \Omega_{n_0}} |u_n(t, x) - u(t, x)|^2 e^{-\varepsilon_0|x-x|} \, dx < \delta
$$
for $n \geq n_0$. Since this estimate is uniform in $n \geq n_0$ and $t \in [0,T]$, from the strong convergence in $\Omega_{n_0}$ (29) we thus obtain the desired convergence (28). We emphasize the convergence (28) holds for all $\bar{x} \in \mathbb{R}^N$ and $0 < \varepsilon < 1$.

Now we will prove $F(u'_n) \to F(u')$ in $L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))$. Let $v \in L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))$ be fixed. We split the estimate into two separate integrals:

\[
\int_0^T (F(u'_n) - F(u'), v(t))_{\bar{x},\varepsilon} \, dt = \int_0^T \left( \int_{\Omega_r} \left( \int_{-r}^0 \left( \int_{\Omega} b(u_n(t + \theta, y)) f(x - y)e^{-|x-y|} \, dy \right) \xi(\theta, u_n(t), u'_n) \, d\theta \right) v(t,x)e^{-\varepsilon |x|} \, dx \right) \, dt \\
+ \int_0^T \left( \int_{\Omega_r} \left( \int_{-r}^0 \left( \int_{\Omega} b(u(t + \theta, y)) f(x - y)e^{-|x-y|} \, dy \right) \xi(\theta, u_n(t), u'_n) \right) \, d\theta \right) v(t,x)e^{-\varepsilon |x|} \, dx \right) \, dt \equiv I_1 + I_2.
\]

The integral $I_1$ can be estimated using the boundedness of $f$ and $\xi$ (17), Fubini’s theorem, the Lipschitz continuity of $b$ (13), Hölder’s inequality and Lemma 2.11 with $G(y) = e^{-|x-y|}$ in the following way (for convenience we denote $\omega(t,x) \equiv u_n(t,x) - u(t,x)$):

\[
|I_1| \leq C_3 \int_0^T \int_{-r}^0 \left( \int_{\Omega} |\omega(t + \theta)|_{L^1_{\bar{x},\varepsilon}(\Omega)} \, v(t,x)e^{-\varepsilon |x|} \, dx \right) |\xi(\theta, u_n(t), u'_n)| \, d\theta \, dt \\
\leq C_3 \int_0^T \int_{-r}^0 \left( \int_{\Omega} |\omega(t + \theta)|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 e^{-\varepsilon |x|} \, dx \right)^{1/2} |\xi(\theta, u_n(t), u'_n)| \, d\theta \, dt \\
\leq C_4 \int_0^T \int_{-r}^0 \left( \int_{\Omega} |\omega(t + \theta)|_{L^2_{\bar{x},\varepsilon}(\Omega)} \, d\theta \right)^{1/2} \left[ \int_{-r}^0 |\xi(\theta, u_n(t), u'_n)|^2 \, d\theta \right]^{1/2} \, dt \\
\leq C_5 \left( \int_0^T |\omega(t)|_{L^2_{\bar{x},\varepsilon}}^2 \, dt \right)^{1/2} \left( \int_0^T \int_{-r}^0 \left| \omega(t + \theta) \right|_{L^2_{\bar{x},\varepsilon}}^2 \, d\theta \, dt \right)^{1/2} \\
\leq C_6 \left( \|\varphi_n - \varphi\|_{L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))} + \|u_n - u\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))} \right) \|v\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))}. \tag{32}
\]

The integral $I_2$ can be treated in a similar manner. Since $b$ and $f$ are bounded, the condition (16) implies

\[
|I_2| \leq C_7 \int_0^T \left( \int_{\Omega} \left( \int_{-r}^0 |\xi(\theta, u_n(t), u'_n) - \xi(\theta, u(t), u')| \, d\theta \right) v(t,x)e^{-\varepsilon |x|} \, dx \right) \, dt \\
\leq C_8 \int_0^T \left( \|\omega(t)|_{L^2_{\bar{x},\varepsilon}}^2 + \int_{-r}^0 |\omega(t + \theta)|_{L^2_{\bar{x},\varepsilon}}^2 \, d\theta \right)^{1/2} \|v(t)|_{L^2_{\bar{x},\varepsilon}(\Omega)} \, dt \\
\leq C_9 \left( \|\varphi_n - \varphi\|_{L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))} + \|\omega\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))} \right) \|v\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))}. \tag{33}
\]

Using the obvious continuous inclusion $L^2_{\bar{x},\varepsilon}(\Omega) \to L^1_{\bar{x},\varepsilon}(\Omega)$, the estimates (32) and (33) imply
\[ \|F(u_n') - F(u')\|_{L^2(0,T;L^2_{x,r}((\Omega)))} \]
\[ \leq C_{10} \left( \|\varphi_n - \varphi\|_{L^2(-r,0;L^2_{x,r}((\Omega)))} + \|u_n - u\|_{L^2(0,T;L^2_{x,r}((\Omega)))} \right). \]

Also note that the argument leading to (32) and (33) shows the Lipschitz continuity of \( F \)
\[ \|F(v') - F(w')\|_{L^2_{x,r}((\Omega))} \leq C\|v - w\|_{L^2(t-r,T;L^2_{x,r}((\Omega)))} \quad (34) \]
for \( v, w \in L^2(t-r,T;L^2_{x,r}((\Omega))) \).

To finish the proof of existence, we need to deal with the nonlinear diffusion. The process is standard. We follow [8, Theorem 3.2]. First we observe that the convergence (24) and the assumption (10) implies
\[ a(\nabla u_n) \to \alpha \text{ in } L^2(0,T;L^2_{x,r}((\Omega))) \quad (35) \]
for fixed \( \bar{x} \in \Omega \). Note that at this stage \( \alpha \) might depend on the choice of \( \bar{x} \). Test the equation (22) by a fixed \( v \in L^2(0,T;W^{1,2}_{x,r}((\Omega))) \) and take the limit with respect to \( n \) to obtain
\[ \int_{\Omega} \partial_t u(t,x)v(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx + \int_{\Omega} \alpha(t,x) \left( \nabla v(t,x) - \varepsilon \frac{x - \bar{x}}{|x - \bar{x}|} v(t,x) \right) e^{-\varepsilon|x-\bar{x}|} \, dx \]
\[ + d \int_{\Omega} u(t,x)v(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx = \int_{\Omega} F(u(t,x))v(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx \quad (36) \]
for a.e. \( t \in (0,T) \). Let us go back to (25), integrate over \( (0,T) \) and take the limit superior with respect to \( n \) to get
\[ \limsup_{n \to \infty} \int_0^T \int_{\Omega} a(\nabla u_n(t,x)) \cdot \nabla u_n(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx \]
\[ \leq -\frac{1}{2} \liminf_{n \to \infty} \int_{\Omega} |u_n(T,x)|^2 e^{-\varepsilon|x-\bar{x}|} \, dx + \frac{1}{2} \limsup_{n \to \infty} \int_{\Omega} |u_n(0,x)|^2 e^{-\varepsilon|x-\bar{x}|} \, dx \]
\[ - \liminf_{n \to \infty} d \int_0^T \int_{\Omega} |u_n(t,x)|^2 e^{-\varepsilon|x-\bar{x}|} \, dx \, dt \]
\[ + \limsup_{n \to \infty} \int_0^T \int_{\Omega} F(u_n')(t,x)u_n(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx \, dt \]
\[ + \limsup_{n \to \infty} \varepsilon \int_0^T \int_{\Omega} a(\nabla u_n(t,x)) \frac{x - \bar{x}}{|x - \bar{x}|} u_n(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx \, dt \quad (37) \]

Using (21), (24) and the lower semicontinuity of norms we immediately see that the first three terms have a well defined limit. Also using (28) and (35) we have
\[ \lim_{n \to \infty} \varepsilon \int_0^T \int_{\Omega} a(\nabla u_n(t,x)) \frac{x - \bar{x}}{|x - \bar{x}|} u_n(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx \, dt \]
\[ = \varepsilon \int_0^T \int_{\Omega} \alpha(t,x) \frac{x - \bar{x}}{|x - \bar{x}|} u(t,x)e^{-\varepsilon|x-\bar{x}|} \, dx \, dt \quad (38) \]
Finally, the strong convergence (28) and Lipschitz continuity of \( F \) (34) imply
\[ \left( F(u_n'), u_n \right)_{\bar{x},\varepsilon} = \left( F(u^r_n), u_n - u \right)_{\bar{x},\varepsilon} + \left( F(u^r_n) - F(u'), u \right)_{\bar{x},\varepsilon} + \left( F(u'), u \right)_{\bar{x},\varepsilon} \]
\[ \to \left( F(u'), u \right)_{\bar{x},\varepsilon} . \]
Now we are ready to compare (37) and (36) (with \( v = u \)) integrated over time. We arrive to

\[
\limsup_{n \to \infty} \int_0^T \int_\Omega a (\nabla u_n(t, x)) \cdot \nabla u_n(t, x) e^{-\varepsilon|x-x_i|} \, dx \, dt
\leq \int_0^T \int_\Omega a(t, x)\nabla u(t, x) e^{-\varepsilon|x-x_i|} \, dx \, dt.
\]

The assumptions (9)-(11) assure that \( a \) induces a strictly monotone operator from \( L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\Omega)) \) to \( L^2(0, T; W^{-1,2}_{\bar{x},\varepsilon}(\Omega)) \), and a standard argument leads us to the equality

\[
\alpha(t, x) = a (\nabla u(t, x)) \quad (e^{-\varepsilon|x-x_i|} \, dx) \text{-a.e. in } \Omega \text{ and a.e. in } (0, T).
\]  

(39)

Clearly the equality (39) holds also a.e. with respect to the standard Lebesgue measure in \( (0, T) \times \Omega \) and therefore \( \alpha \) is independent of the choice of \( \bar{x} \). Therefore we may use (39) to substitute in (36), which finishes the proof of existence.

The proof of the uniqueness is analogous to the proof for the bounded domain. Let \( u \) and \( v \) be two solutions with the respective initial conditions \((u_0, \varphi), (v_0, \psi) \in H \) and denote \( w(t) = u(t) - v(t) \). Test the equations for \( u \) and \( v \) by \( w \). Subtracting these and using a similar argument as in the derivation of the inequalities (32) and (33) we obtain

\[
\frac{d}{dt} \|w(t)\|^2_{\bar{x},\varepsilon} + \sigma \|w(t)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)} \leq C_{10} \|w(t)\|^2_{\bar{x},\varepsilon} + C_{11} \int_{-r}^0 \|w(t + \theta)\|^2_{\bar{x},\varepsilon} \, d\theta
\]

\leq C_{12} \left( \|w(t)\|^2_{\bar{x},\varepsilon} + \sigma \int_0^t \|w(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)} \, ds \right) + C_{13} \int_{-r}^0 \|w(\theta)\|^2_{\bar{x},\varepsilon} \, d\theta
\]

for some \( \sigma > 0 \). The Gronwall’s lemma applied to the function

\[
Y(t) = \|w(t)\|^2_{\bar{x},\varepsilon} + \sigma \int_0^t \|w(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)} \, ds
\]

implies

\[
Y(t) \leq \left( Y(0) + \int_{-r}^0 \|w(\theta)\|^2_{\bar{x},\varepsilon} \, d\theta \right) C(t).
\]

The estimate can be rewritten in the form

\[
\sup_{t \in [0, T]} \|u(t) - v(t)\|^2_{\bar{x},\varepsilon} + \sigma \int_0^t \|u(s) - v(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)} \, ds
\]

\leq C(T) \left( \|u_0 - v_0\|^2_{\bar{x},\varepsilon} + \int_{-r}^0 \|\varphi(\theta) - \psi(\theta)\|^2_{\bar{x},\varepsilon} \, d\theta \right),
\]

which gives the uniqueness of the solutions.

\[ \square \]

**Definition 3.2.** We define the solution operator \( S(t) : H \to H \) by

\[
S(t)(u_0, \varphi) = (u(t), u^t),
\]

where \( u(t) \) is the solution from Theorem 3.1.

As is usual in locally uniform spaces, we cannot generally expect the solution \( u(t) \) to be continuous in the space \( L^2_{\bar{x}}(\Omega) \). Continuity in \( L^2_{\bar{x}}(\Omega) \) can be achieved for more regular initial data; cf. [18]. In the following we will compensate for the lack of additional regularity by working in weighted Lebesgue spaces and by using the method of \( \ell \)-trajectories.
Corollary 1. The operator
\[ S(t) : L^2_{\bar{x},\varepsilon}(\Omega) \times L^2(-r,0; L^2_{\bar{x},\varepsilon}(\Omega)) \to L^2_{\bar{x},\varepsilon}(\Omega) \times L^2(-r,0; L^2_{\bar{x},\varepsilon}(\Omega)) \]
is Lipschitz continuous on \( H \) uniformly with respect to \( t \in [0,T] \) for every \( \bar{x} \in \Omega \) and \( 0 < \varepsilon < 1 \). The solution operator \( S(t) \) paired with \( H \) as its phase space form a dynamical system.

**Proof.** The continuity in time follows from the continuity of the solution and the locally uniform Lipschitz continuity from (41). The semigroup property is obvious by uniqueness. The joint continuity w.r.t. \((t,\chi)\), where \( \chi \in H \), follows by time continuity of solutions and locally uniform continuity w.r.t. \( \chi \). \( \Box \)

4. Locally compact attractor.

**Theorem 4.1.** Let the assumptions of Theorem 3.1 hold and let \( \phi \) be an admissible weight function with growth rate smaller than \( \varepsilon \). Then for \( t \geq r \) we have the estimate
\[ \|u(t)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 + C_1 \|u\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 e^{-\sigma(t-r)} + C_2 \leq \|u_0\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 + C_2, \] (42)
where \( \sigma, C_1, C_2 > 0 \) are dependent of the data of the equation and independent of the initial data \( u_0, \varphi \).

**Proof.** The proof follows the proof of Theorem 3.2 in [8]. Let \( u \) be the solution of (18), (19). Then using the Cauchy-Schwartz and Young’s inequalities, (9) and the initial choice of \( \varepsilon \) in (15) we get
\[ \frac{d}{dt} \|u(t)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 + \sigma (\|\nabla u(t)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 + \|u(t)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2) \leq C_1 \] (43)
for some \( C_1, \sigma > 0 \). The Gronwall’s lemma implies
\[ \|u(t)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 \leq \|u(0)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 e^{-\sigma t} + C_2 (1 - e^{-\sigma t}) \leq \|u(0)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 e^{-\sigma t} + C_3 \]
and integrating (43) from \( t - r \) to \( t \) leads to
\[ \|u(t)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 + \sigma \int_{t-r}^t (\|\nabla u(s)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 + \|u(s)\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2) \, ds \leq \|u_0\|_{L^2_{\bar{x},\varepsilon}(\Omega)}^2 e^{-\sigma(t-r)} + C_4. \]
Finally we multiply the previous estimate by \( \phi(\bar{x}) \) and take the supremum over \( \bar{x} \in \Omega \). From the definition of parabolic locally uniform spaces we obtain (42). \( \Box \)

**Corollary 2.** The solution operator
\[ S(t) : L^2_{\bar{x},\varepsilon}(\Omega) \times L^2(-r,0; L^2_{\bar{x},\varepsilon}(\Omega)) \to L^2_{\bar{x},\varepsilon}(\Omega) \times L^2(-r,0; L^2_{\bar{x},\varepsilon}(\Omega)) \]
adopts a positively invariant bounded absorbing set \( W \subseteq H \). Moreover, \( W \) absorbs not only bounded subsets of \( H \), but also the sets of the form \( B \times L^2_{\bar{x},\varepsilon}(\Omega) \), where \( B \subseteq L^2_{\bar{x},\varepsilon}(\Omega) \) is bounded.

**Proof.** Using the notation of Theorem 4.1 (cf. (42)), let \( \tilde{W} = B(0,C_2+1) \subseteq H \). The set \( \tilde{W} \) is clearly absorbing. Then we find \( t_0 > 0 \) such that \( S(t)\tilde{W} \subseteq \tilde{W} \) for every \( t \geq t_0 \) and set
\[ W = \bigcup_{t \geq t_0} S(t)\tilde{W}. \]
The fact that the set \( W \) absorbs even the sets of the form \( B \times L^2_{\bar{x},\varepsilon}(\Omega) \) for \( B \subseteq L^2_{\bar{x},\varepsilon}(\Omega) \) bounded follows immediately from the form of the estimate (42). \( \Box \)
Since all the solutions from Theorem 3.1 are continuous for \( t \geq 0 \) and we are interested in the asymptotic dynamics, from now on we may consider only \( S(t) : X \to X \), where

\[
X = \{ u \in C([-r, 0], L^2_{\bar{x}, \varepsilon}(\Omega)) ; \text{ } u \text{ is a (weak) solution from Theorem 3.1} \}
\]

is equipped with \( L^2(-r, 0; L^2_{\bar{x}, \varepsilon}(\Omega)) \) topology for fixed \( \bar{x} \in \Omega \). The choice of particular \( \bar{x} \) does not play any role in the analysis as the solution \( u(t) \) satisfies (27).

Corollary 2 implies that the dynamical system \((X, S(t))\) admits a bounded absorbing set

\[
\mathcal{B} \subseteq L^2(-r, 0; L^2_{\bar{x}, \varepsilon}(\Omega)) \cap L^2_0(-r, 0; L^2(\Omega)).
\]

Moreover, the continuity of the solutions allows us to assume

\[
\mathcal{B} \subseteq C([-r, 0]; L^2_{\bar{x}, \varepsilon}(\Omega)).
\]

Clearly all the long time dynamics will take place in the absorbing set \( \mathcal{B} \). Now we may define the space of short trajectories similarly as in [12].

**Definition 4.2.** The space of short trajectories is given by

\[
\mathcal{X} = \{ \chi \in C([0, \ell], L^2_{\bar{x}, \varepsilon}(\Omega)) ; \text{ } \chi \text{ is a solution of (1) in } [0, \ell] \text{ with } \chi|_{[-r, 0]} \in \mathcal{B} \},
\]

together with the \( L^2(-r, \ell; L^2_{\bar{x}, \varepsilon}(\Omega)) \) topology. The evolution operator \( L(t) : \mathcal{X} \to \mathcal{X} \) is defined by

\[
(L(t)\chi)(s) = u(t + s), \quad s \in [-r, \ell],
\]

where \( u \) is the solution from Theorem 3.1 satisfying \( u|_{(-r, 0)} = \chi \).

Finally, the operator \( e : \mathcal{X} \to \mathcal{X} \) is defined by

\[
e(\chi) = (L(r + \ell)\chi)|_{[-r, 0]}.
\]

We note that \( \mathcal{X} \) in general is not complete with respect to its metric; we will see however in Lemma 4.6 that \( L(t) \) is asymptotically compact on \( \mathcal{X} \).

**Theorem 4.3.** The operator \( L(t) : \mathcal{X} \to \mathcal{X} \) is Lipschitz continuous uniformly with respect to \( t \in [0, T] \). Moreover, the pair \((\mathcal{X}, L(t))\) forms a dynamical system.

**Proof.** Let \( \chi_1, \chi_2 \in \mathcal{X} \) and let \( u_1 \) and \( u_2 \) be the respective solutions from Theorem 3.1. Define \( w(t) = u_1(t) - u_2(t) \). We start from the estimate (40), use the condition (10) and integrate over \( \tau \in [s, t] \), where \( s \in (0, \ell), t \in (\ell, \ell + T) \), to obtain

\[
\|w(t)\|_{W_{2, \varepsilon}^1(\Omega)}^2 + \sigma \int_0^t \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau \leq \|w(s)\|_{W_{2, \varepsilon}^1(\Omega)}^2 + C_3 \int_0^t \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau.
\]

Then we integrate over \( s \in (0, \ell) \) and get

\[
\|w(t)\|_{W_{2, \varepsilon}^1(\Omega)}^2 + \sigma \int_0^\ell \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau \leq C_4 \left( \int_0^\ell \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau + \int_0^t \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau \right).
\]

Again we apply the Gronwall’s lemma to the function

\[
Y(s) = \|w(s)\|_{W_{2, \varepsilon}^1(\Omega)}^2 + \int_0^s \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau
\]

and obtain the estimate

\[
\sup_{t \in [0, T]} \|w(t)\|_{W_{2, \varepsilon}^1(\Omega)}^2 + \int_0^{\ell + T} \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau \leq C(T) \int_{-r}^\ell \|w(\tau)\|_{W_{2, \varepsilon}^1(\Omega)}^2 \, d\tau,
\]
which gives
\[ \sup_{t \in [0, T]} \| L(t) \chi_1 - L(t) \chi_2 \|_X^2 \leq C(T) \| \chi_1 - \chi_2 \|_X^2. \]

It remains to prove that \((\mathcal{X}, L(t))\) is a dynamical system. The continuity of the operator \(L(t)\) in time follows from the continuity of solutions and the uniform continuity of \(L(t)\) in \(\mathcal{X}\) follows from the previous part of the proof. \(\square\)

**Definition 4.4.** We define the space \(\mathcal{W}\) as the set \(\mathcal{X}\) with the norm
\[
\| \chi \|_{\mathcal{W}}^2 = \int_{-r}^{\ell} \| \chi(s) \|_{W_{x,\varepsilon}^{1,2}(\Omega)}^2 ds + \int_{-r}^{\ell} \| \partial_t \chi(s) \|_{W_{x,\varepsilon}^{-1,2}(\Omega)}^2 ds.
\]

**Lemma 4.5.** The mapping \(L(r + \ell) : \mathcal{X} \rightarrow \mathcal{W}\) is Lipschitz continuous.

**Proof.** Using the notation of the proof of Theorem 4.3, choosing \(t = r + \ell\) in the estimate (45) immediately gives
\[
\int_{\ell}^{2\ell+r} \| w(s) \|_{W_{x,\varepsilon}^{1,2}(\Omega)}^2 ds \leq C \int_{-r}^{\ell} \| w(s) \|_{W_{x,\varepsilon}^{1,2}(\Omega)}^2 ds. \tag{46}
\]

From the equation we have
\[
\int_{\ell}^{2\ell+r} \langle \partial_t w(s), \varphi(s) \rangle_{\bar{x},\varepsilon} ds \tag{47}
\]
\[
= - \int_{\ell}^{2\ell+r} \left( a(\nabla u) - a(\nabla v), \nabla \varphi(s) - \hat{\varepsilon} \frac{x - \bar{x}}{|x - \bar{x}|} \varphi(s) \right)_{\bar{x},\varepsilon} ds \tag{48}
\]
\[
- d \int_{\ell}^{2\ell+r} (w(s), \varphi(s))_{\bar{x},\varepsilon} ds + \int_{\ell}^{2\ell+r} (F(u^s) - F(v^s), \varphi(s))_{\bar{x},\varepsilon} ds, \tag{49}
\]
where \(\langle \cdot, \cdot \rangle_{\bar{x},\varepsilon}\) denotes the duality on \(\left(W_{x,\varepsilon}^{-1,2}(\Omega), W_{x,\varepsilon}^{1,2}(\Omega)\right)\).

Assuming that the \(L^2(\ell, 2\ell + r; W_{x,\varepsilon}^{1,2}(\Omega))\)-norm of \(\varphi\) is one, we can use (10) to obtain the estimates
\[
\int_{\ell}^{2\ell+r} \left( a(\nabla u) - a(\nabla v), \nabla \varphi(s) - \hat{\varepsilon} \frac{x - \bar{x}}{|x - \bar{x}|} \varphi(s) \right)_{\bar{x},\varepsilon} ds \leq C \| w \|_{L^2(\ell, 2\ell + r; W_{x,\varepsilon}^{1,2}(\Omega))}, \tag{50}
\]
\[
\int_{\ell}^{2\ell+r} (w(s), \varphi(s))_{\bar{x},\varepsilon} ds \leq C \| w \|_{L^2(\ell, 2\ell + r; W_{x,\varepsilon}^{1,2}(\Omega))}, \tag{51}
\]
and similarly as in the proof of Theorem 3.1 we obtain
\[
\int_{\ell}^{2\ell+r} (F(u^s) - F(v^s), \varphi(s))_{\bar{x},\varepsilon} ds \leq C \| w \|_{L^2(\ell, 2\ell + r; L_{x,\varepsilon}^2(\Omega))}, \tag{52}
\]
which gives
\[
\| \partial_t L(r + \ell) \chi_1 - \partial_t L(r + \ell) \chi_2 \|_{L^2(-r, \ell; W_{x,\varepsilon}^{-1,2}(\Omega))} \leq C \| \chi_1 - \chi_2 \|_X. \tag{53}
\]
Combining (46) - (53) gives the desired continuity
\[
\| L(r + \ell) \chi_1 - L(r + \ell) \chi_2 \|_{\mathcal{W}} \leq C \| \chi_1 - \chi_2 \|_X. \]
The previous lemma also implies that the mapping \( e : \mathcal{X} \to X \) is Lipschitz continuous. This follows immediately from (44), since we have
\[
\|e(\chi_1) - e(\chi_2)\|_X = \|(L(r + \ell)\xi_1 - L(r + \ell)\xi_2)\|_{[-r,0]}\|_X
\leq \|L(\ell + r)\chi_1 - L(\ell + r)\chi_2\|_W \leq C\|L(\ell + r)\chi_1 - L(\ell + r)\chi_2\|_X.
\]  

(54)

**Lemma 4.6.** The dynamical system \((\mathcal{X}, L(t))\) is asymptotically compact.

**Proof.** Let \( \chi_n \in \mathcal{X} \) be a bounded sequence and \( t_n \to \infty \). We aim to show
\[
L(t_n)\chi_n \to \chi \text{ in } L^2(-r, \ell, L^2_\mathcal{X}(\Omega))
\]  

(55)

where \( \chi \in \mathcal{X} \), up to a subsequence.

From Theorem 4.1 and Lemma 4.5 we see that \( L(t_n)\chi_n \) is bounded in the norms \( L^2(-r, \ell; W^{1,2}(B)), W^{1,2}(-r, \ell; W^{-1,2}(B)) \), where \( B \subseteq \Omega \) is an arbitrary compact set. The Aubin-Lions lemma implies that
\[
L(t_n)\chi_n \to \chi \text{ in } L^2(-r, \ell; L^2(B))
\]  

and therefore \( L(t_n)\chi_n \to \chi \) in \( L^2_{loc}((-r, \ell) \times \Omega) \). Since the sequence is also bounded in \( L^\infty((-r, \ell; L^2_\mathcal{X}(\Omega)) \), Lemma 2.10 immediately gives us the strong convergence (55).

Theorem 4.1 and Lemma 4.5 also imply
\[
L(t_n)\chi_n \to \chi \text{ in } L^2(-r, \ell; W^{1,2}_{\mathcal{X},\mathcal{X}}(\Omega)), \quad \partial_t L(t_n)\chi_n \to \partial_t \chi \text{ in } L^2(-r, \ell; W^{-1,2}_{\mathcal{X},\mathcal{X}}(\Omega)),
\]  

which together with the strong convergence (55), (10) and the Lipschitz continuity of \( F \) (34) justifies taking a limit in the equation in a similar manner as in the proof of existence (see Theorem 3.1) and thus \( \chi \in \mathcal{X} \).

\( \square \)

**Theorem 4.7.** The dynamical system \((X, S(t))\) has a locally compact attractor, more precisely a \( (L^2_\mathcal{X}([-r,0); L^2_\mathcal{X}(\Omega)), L^2_{loc}((-r,0) \times \Omega)) \)-attractor.

**Proof.** First we observe that the dynamical system \((\mathcal{X}, L(t))\) has a global attractor \( \mathcal{A}_t \). By Theorem 4.1 it has a bounded absorbing set; the asymptotic compactness was proved in Lemma 4.6 and we apply a standard result (see e.g. [14, Theorem 23.12]). Define
\[
\mathcal{A} = e(\mathcal{A}_t).
\]  

(56)

It remains to check that \( \mathcal{A} \) is the desired attractor.

Observe that
\[
S(t)e(\chi) = e(L(t)\chi),
\]
therefore \( \mathcal{A} \) is invariant under \( S(t) \). The compactness in \( L^2_{loc}((-r,0) \times \Omega) \) follows from the compactness of \( \mathcal{A}_t \), the continuity of \( e : \mathcal{X} \to X \) and Lemma 2.10. To show that \( \mathcal{A} \) attracts bounded sets of \( X \), we observe that
\[
S(r + \ell)B = e(B_t)
\]  

for \( B \subseteq X \), where
\[
B_t = \{ \chi \in C([-r,0]; L^2_{\mathcal{X},\mathcal{X}}(\Omega)); \chi \text{ is a solution from Theorem 3.1 with the initial condition } (\varphi(0), \varphi) \text{ for } \varphi \in B \}.
\]

By Theorem 4.1, the set \( B_t \) is bounded in \( \mathcal{X} \) for \( B \) bounded in \( X \). Then we have the estimate
\[
\text{dist}_X(S(t + r + \ell)B, \mathcal{A}) = \text{dist}_X(S(t)S(r + \ell)B, \mathcal{A}) = C_1 \text{dist}_X(S(t)e(B_t), \mathcal{A}) = C_1 \text{dist}_X(e(L(t)B_t), \mathcal{A}) \leq C_2 \text{dist}_X(L(t)B_t, \mathcal{A}),
\]
where the last estimate uses the Lipschitz continuity of \( e \), cf. (54).  

\( \square \)
5. Entropy estimates. We estimate the entropy of the attractor constructed in Theorem 4.7 using the general method presented in [19], that has been adapted to the setting of $t$-trajectories and parabolic uniform spaces in [8]. Actually, the rest of the proof follows the latter article quite closely.

We need some preliminary results. First we formulate the Lipschitz continuity of the operators $L(t), e$ and the smoothing property in the context of parabolic uniformly bounded spaces.

Corollary 3. Let $\psi$ be an admissible weight function of growth rate smaller than $\varepsilon$. Then

1. $L(t): L^2_{b,\psi}(-r, t; L^2(\Omega)) \rightarrow L^2_{b,\psi}(-r, t; L^2(\Omega))$ is Lipschitz continuous uniformly with respect to $t \in [0, T]$, 
2. $e: L^2_{b,\psi}(-r, t; L^2(\Omega)) \rightarrow L^2_{b,\psi}(-r, 0; L^2(\Omega))$ is Lipschitz continuous,
3. the mapping $L(\ell + r): L^2_{b,\psi}(-r, t; L^2(\Omega)) \rightarrow W_{b,\psi}(Q)$, where $Q = [-r, \ell] \times \Omega$ and $W_{b,\psi}(Q)$ is defined in (8), is Lipschitz.

The Lipschitz constants only depend on $C$ and $\mu$ in (2) and not on the particular form of the weight function $\psi$.

Proof. Multiply (45) by $\psi(x)$ and take supremum over $x \in \Omega$ to obtain

$$
\sup_{x \in \Omega} \psi(x) \int_{(t-r, t+\varepsilon) \times \Omega} |w(s, x)|^2 e^{-\varepsilon|x-x_0|} \, dx \, ds
\leq C(T) \sup_{x \in \Omega} \psi(x) \int_{(t-r, t) \times \Omega} |w(s, x)|^2 e^{-\varepsilon|x-x_0|} \, dx \, ds.
$$

The first assertion then follows from the equivalence of the norms (4).

The remaining assertions can be proved in a similar manner from a variant of (54), the equivalence of norms (4)-(6) and from Lemma 4.5.

We will also need an auxiliary admissible weight function (cf. [19]). Let $x_0 \in \Omega$ and $R \geq 1$. Then we define

$$
\psi(x_0, R) \equiv \psi(x_0, R)(x) = \begin{cases} 
1 & \text{if } |x - x_0| \leq R + \sqrt{d}, \\
\frac{1}{e^{c(R + \sqrt{d} - |x - x_0|)/2}} & \text{otherwise.}
\end{cases}
$$

Observe that $\psi(x_0, R)$ satisfies (2) with $\mu = \varepsilon/2$ and some $C > 0$ independent of $x_0 \in \mathbb{R}^N$ and $R \geq 1$.

We also define

$$
\Omega_{x_0, R} = \Omega \cap B(x_0, R) \subseteq \mathbb{R}^N.
$$

Lemma 5.1 ([8], Lemma 5.4). Let $\lambda_0 > 0$. Then there exists $c_1 > 0$ such that for every $x_0 \in \Omega$, $R \geq 1$, $\lambda \in (0, \lambda_0)$ and $\chi_1, \chi_2 \in \mathcal{X}$ we have

$$
\|\chi_1 - \chi_2\|^2_{L^2_{b,\psi}(\Omega_{x_0, R})(-r, t; L^2(\Omega))} \leq \max\{\|\chi_1 - \chi_2\|^2_{L^2_{b,\psi}(\chi_1, R)(-r, t; L^2(\Omega_{x_0, R}))}, \lambda\},
$$

where

$$
R(\lambda) = R + c_1 \left(1 + \log \frac{1}{\lambda}\right).
$$

We are now ready to prove the entropy estimate.

Theorem 5.2. Let $x_0 \in \Omega$ and $R > 0$. Then there exist $c_0, c_1, \lambda_0 > 0$ such that for every $x_0 \in \Omega$, $R \geq 1$ and $\lambda \in (0, \lambda_0)$ the entropy estimate

$$
H_{\lambda} \left(A, L^2_{b}(-r, 0; L^2(\Omega_{x_0, R}))\right) \leq c_0 \left(R + c_1 \log \frac{1}{\lambda}\right)^N \log \frac{1}{\lambda}
$$

(57)
holds.

Proof. The proof follows the proof of Theorem 5.1 in [8] almost word by word. First we observe that it suffices to prove a similar estimate for the global attractor of \((A_\ell, \mathcal{X})\), namely

\[
H_\lambda \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) \leq c_0 \left( R + c_1 \log \frac{1}{\lambda} \right)^N \log \frac{1}{\lambda}. \tag{58}
\]

The estimate (57) then follows immediately using the relation (56), the Lipschitz continuity of \(e\) leading to the estimate

\[
H_\lambda \left( A, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) \leq H_{\lambda/\kappa} \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right),
\]

where \(\kappa > 0\) is the Lipschitz constant of the mapping \(e\) from Corollary 3, and the obvious estimate

\[
H_\lambda \left( A, L^2_{b}(-\ell, 0; L^2(\Omega)) \right) \leq H_\lambda \left( A, L^2_{b,\psi(x_0,R)}(-\ell, 0; L^2(\Omega)) \right).
\]

The proof of the estimate (58) relies on the recurrent estimate

\[
H_{\alpha/2} \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) \leq H_\alpha \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) + c_0 \left( R + c_1 \log \frac{1}{\alpha} \right)^N. \tag{59}
\]

Then we may choose \(\lambda_0 > 0\) large enough so that

\[
H_{\lambda_0} \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) = 0
\]

and for \(\lambda \in (0, \lambda_0)\) we find \(k \in \mathbb{N}\) such that \(2^{-k}\lambda_0 \leq \lambda \leq 2^{-k+1}\lambda_0\). The estimate (58) then follows from the recurrent estimate (59) and the inequality \(k \leq c \log 1/\lambda\) holding for all \(\lambda\) sufficiently small:

\[
H_\lambda \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) \\
\leq H_{2^{-k}\lambda_0} \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) - H_{\lambda_0} \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) \\
\leq \sum_{i=1}^{k} \left\{ H_{2^{-i}\lambda_0} \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) - H_{2^{-i+1}\lambda_0} \left( A_\ell, L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega)) \right) \right\} \\
\leq \sum_{i=1}^{k} c_0 \left( R + c_1 \log \frac{2^{i-1}}{\lambda_0} \right)^N \leq c_0 \left( R + c_1 \log \frac{1}{\lambda} \right)^N \log \frac{1}{\lambda}.
\]

It remains to prove the recurrent estimate (59). Recall that the short trajectory attractor \(A_\ell\) is compact in \(L^2(-\ell, \ell; L^2_{\ell,\psi}(\Omega))\), it is also compact in the space \(L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega))\), therefore for every \(\alpha > 0\) we may find \(m \in \mathbb{N}\) such that

\[
A_\ell \subseteq \bigcup_{i=1}^{m} B_\alpha(\chi_i; L^2_{b,\psi(x_0,R)}(-\ell, \ell; L^2(\Omega))).
\]

Using Corollary 3 and the invariance of \(A_\ell\) we have

\[
A_\ell \subseteq \bigcup_{i=1}^{m} B_{\alpha \ell}(\hat{\chi}_i; W_{b,\psi(x_0,R)}(Q)).
\]
for some $\kappa > 0$. Lemma 2.7 now implies that
\[
H_{\alpha/2} \left( B_{\kappa\alpha}(\tilde{\chi}; W_{b,\psi}(x_0, R) (Q)) , L^2_{b,\psi}(x_0, R) (-r, \ell; L^2(\Omega_{x_0,R}(\alpha/2))) \right) \\
\leq \tilde{c}_0 \left( R + \tilde{c}_1 (1 + \log \frac{2}{\alpha}) \right)^N \leq c_0 \left( R + c_1 \log \frac{1}{\alpha} \right)^N.
\]
From Lemma 5.1 it follows that an $\alpha/2$-covering in $L^2_{b,\psi}(x_0, R) (-r, \ell; L^2(\Omega))$ is also an $\alpha/2$-covering in $L^2_{b,\psi}(x_0, R) (-r, \ell; L^2(\Omega))$, which finishes the proof.

REFERENCES

[1] J. M. Arrieta, J. W. Cholewa, T. Dlotko and A. Rodríguez-Bernal, Dissipative parabolic equations in locally uniform spaces, Math. Nachr., 280 (2007), 1643–1663.
[2] J. M. Arrieta, A. Rodríguez-Bernal, J. W. Cholewa and T. Dlotko, Linear parabolic equations in locally uniform spaces, Math. Models Methods Appl. Sci., 14 (2004), 253–293.
[3] M. Efendiev, Finite and Infinite Dimensional Attractors for Evolution Equations of Mathematical Physics, Gakkōtosho Co., Ltd., Tokyo, 2010.
[4] M. A. Efendiev and S. V. Zelik, The attractor for a nonlinear reaction-diffusion system in an unbounded domain, Comm. Pure Appl. Math., 54 (2001), 625–688.
[5] M. Grasselli and V. Pata, Uniform attractors of nonautonomous dynamical systems with memory, in Evolution equations, semigroups and functional analysis (eds. A. Lorenzi and B. Ruf), Birkhäuser, Basel, 50 (2002), 155–178.
[6] M. Grasselli, D. Pražák and G. Schimperna, Attractors for nonlinear reaction-diffusion systems in unbounded domains via the method of short trajectories, J. Differential Equations, 249 (2010), 2287–2315.
[7] X. Li and Z. X. Li, The Global Attractor of a Non-Local PDE Model with Delay for Population Dynamics in $\mathbb{R}^n$, Acta Math. Sin. (Engl. Ser.), 27 (2011), 1121–1136.
[8] A. Miranville and S. Zelik, Attractors for dissipative partial differential equations in bounded and unbounded domains, in Handbook of differential equations: Evolutionary equations, Vol. IV (eds. C. M. Dafermos and M. Pokorný), Elsevier/North-Holland, Amsterdam, 2008, 103–200.
[9] V. Pata, Stability and exponential stability in linear viscoelasticity, Milan J. Math., 77 (2009), 333–360.
[10] D. Pražák, Exponential attractors for abstract parabolic systems with bounded delay, Bull. Austral. Math. Soc., 76 (2007), 285–295.
[11] A. V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays, J. Math. Anal. Appl., 326 (2007), 1031–1045.
[12] G. R. Sell and Y. You, Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.
[13] Y. Wang and P. E. Kloeden, The uniform attractor of a multi-valued process generated by reaction-diffusion delay equations on an unbounded domain, Discrete Continuous Dynam. Systems - A, 34 (2014), 4343–4370.
[14] Y. Wang, L. Wang and W. Zhao, Pullback attractors for nonautonomous reaction-diffusion equations in unbounded domains, J. Math. Anal. Appl., 336 (2007), 330–347.
[15] Z. Wang, W. Li and S. Ruan, Travelling wave fronts in reaction-diffusion systems with spatio-temporal delays, J. Differential Equations, 222 (2006), 185–232.
[16] T. Yi, Y. Chen and J. Wu, Global dynamics of delayed reaction-diffusion equations in unbounded domains, Z. Angew. Math. Phys., 63 (2012), 793–812.
[19] S. V. Zelik, Attractors of reaction-diffusion systems in unbounded domains and their spatial complexity, Comm. Pure Appl. Math., 56 (2003), 584–637.

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E-mail address: prazak@karlin.mff.cuni.cz
E-mail address: slavikj@karlin.mff.cuni.cz