Research Article

Hyers–Ulam–Mittag-Leffler Stability for a System of Fractional Neutral Differential Equations

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This article concerns with the existence and uniqueness for a new model of implicit coupled system of neutral fractional differential equations involving Caputo fractional derivatives with respect to the Chebyshev norm. In addition, we prove the Hyers–Ulam–Mittag-Leffler stability for the considered system through the Picard operator. For application of the theory, we add an example at the end. The obtained results can be extended for the Bielecki norm.

1. Introduction

Models of fractional differential equations (FDEs) have many applications in various fields of engineering and science such as mechanics, biology, chemistry, physics, and control and signal processing[1, 2]. For the different materials and phenomena, the heredity characteristics are well explained by FDEs, and as a result, many research papers and books have been published in this field [3–7]. FDEs in which the highest fractional derivatives (FDs) of unknown term appear both with and without delays are known as neutral FDEs. In the last few years, the study of neutral FDEs have developed dramatically. This is due the fact that the qualitative behavior of the aforesaid equations are quite different from those of nonneutral FDEs. Neutral FDEs also play a key role and has many advantages. For instance, they give more better description over population fluctuations. Also, neutral FDEs with delay appear in models of electrical networks containing lossless transmission lines, for more details see [8]. Different attempts have been made for the investigation of solutions of fractional and neutral FDEs [9–17].

Another imperative and more remarkable area of research is committed to the stability analysis of the solutions for the FDEs and ordinary orders. Many targets are achieved in this regard, for some recent work we refer the reader to see [18–35]. Niazi et al. [36] investigated the existence, uniqueness (EU), and Hyers–Ulam–Mittag–Leffler (HUML) stability for the following neutral FDEs:

\[
\begin{align*}
\mathcal{D}_0^\kappa u(\omega) &= g(\omega, u, \mathcal{D}_0^\sigma u), \quad \omega \in [0, 1], \\
\mathcal{D}_0^\sigma u(\omega) &= \phi_1(\omega), \quad \omega \in [-\xi, 0],
\end{align*}
\]

where \(1 < \kappa \leq 2\), \(0 < \sigma < 1\), \(\mathcal{D}_0^\kappa\) and \(\mathcal{D}_0^\sigma\) are the Caputo derivatives, and \(g: [0, 1] \times C_{\xi} \times C_{\xi} \rightarrow \mathbb{R}\) is continuous. Here, \(C_{\xi}\) denotes the Banach space of all continuous functions \(\phi_1: [-\xi, 1] \rightarrow \mathbb{R}\) with norm

\[
\|
\phi_1
\| = \max\{ |\phi_1(\omega)|: -\xi \leq \omega \leq 1 \}.
\]

If \(u_\omega \in C_{\xi}\), then is defined by

\[
u_\omega(\omega) = u(\omega + \tau), \quad \tau \in [-\xi, 1], \omega \in [0, 1].
\]

As far as we know, results on EU and HUML stability for a coupled system of neutral FDEs have not been investigated.
by the researchers. Many real-world problems required to be modeled into a coupled system of FDEs as they cannot be described into a single fractional differential equation, see [37] and references cited therein.

Motivated by the abovementioned work, in this article, we study the EU and HUMIL stability of the following implicit coupled neutral FDEs system involving Caputo FDS. The proposed coupled system (CS) is given by

\[
\begin{align*}
\frac{d^\mu u(t)}{dt^\mu} &= g(t, u(t), v(t)), & \mu \in J, \\
\frac{d^\sigma v(t)}{dt^\sigma} &= h(t, u(t), v(t)), & \sigma \in J, \\
u(t) &= \varphi_1(t), & \omega \in [\xi_1, 0], \\
v(t) &= \varphi_2(t), & \omega \in [\xi_1, 0],
\end{align*}
\]

where \( J = [0, 1], 1 < \kappa \leq 2 \) and \( \frac{d^\mu}{dt^\mu} \) and \( \frac{d^\sigma}{dt^\sigma} \) represent the Caputo FDS of orders \( \kappa \) and \( \sigma \), respectively. The set of all continuous functions \( \varphi_1: [-\xi_1, 1] \rightarrow \mathbb{R} \) with norms \( \|\varphi_1\| = \max\{|\varphi_1(\omega)|: -\xi_1 \leq \omega \leq 1\} \) and \( \|\varphi_2\| = \max\{|\varphi_2(\omega)|: -\xi_1 \leq \omega \leq 1\} \) is a Banach space denoted by \( C_0 \), and hence their product is also a Banach space. The functions \( g, h: J \times C_0 \times C_0 \rightarrow \mathbb{R} \) are continuous. If \( u, v: [-\xi_1, 1] \rightarrow \mathbb{R} \) are continuous, then \( u_\omega, v_\omega \in C_0 \) and the functions \( u_\omega, v_\omega \) are presented by \( u_\omega(\tau) = u(\omega + \tau), v_\omega(\tau) = v(\omega + \tau), \tau \in [-\xi_1, 1], -\xi_1 < 0. \)

2. Preliminaries

This section is concerned with some notions, definitions, and preliminary results used throughout the article.

**Definition 1** (see [38]). The \( \kappa > 0 \) order Riemann–Liouville integral for \( \theta^\kappa \) is

\[\Gamma_\kappa \theta^\kappa(\omega) = \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa-1} \theta^\kappa(s) ds,\]

where the integral is point-wise convergent.

**Definition 2** (see [38]). Suppose \( \theta^\kappa \) be a given function on \([a, b], \kappa \in (n - 1, n)\) order Caputo derivative of \( \theta^\kappa \) is stated by

\[\frac{d^n}{dt^n} \theta^\kappa(\omega) = \frac{1}{\Gamma(n - \kappa)} \int_a^\omega \left( \frac{d^n}{ds^n} \theta^\kappa(s) \right) ds,\]

with \([\kappa] = n - 1\). Furthermore, if domain of \( \theta^\kappa \) is \([a, b]\) and \( \kappa \in (0, 1)\), then

\[\frac{d^n}{dt^n} \theta^\kappa(\omega) = \frac{1}{\Gamma(1 - \kappa)} \int_a^\omega \left( \frac{d^n}{ds^n} \theta^\kappa(s) \right) ds,\]

where \( \theta^\kappa(s) = ((d^n\theta^\kappa)/ds^n)\).

**Theorem 1** (see [39]). Let \( \kappa \in (n - 1, n) \), then \( \theta^\kappa \in C([a, b], \mathbb{R}) \), which implies \( (d^n/dt^n)\theta^\kappa(\omega) = 0 \) has the formula

\[\theta^\kappa(\omega) = \sum_{k=0}^{[\kappa]} c_k \omega^k,\]

where \( c_k \in \mathbb{R}, k = 1, 2, \ldots, [\kappa], [\kappa] + 1 = n.\)

**Definition 3** (see [40]). For a metric space \((Y_{\kappa, \sigma}, d)\), an operator \( A_{\kappa, \sigma}: Y_{\kappa, \sigma} \rightarrow Y_{\kappa, \sigma} \) is said to be Picard operator (PO) if there is \( x^* \in Y_{\kappa, \sigma} \) so that

(a) \( F_{A_{\kappa, \sigma}} = \{x^*\} \) where \( F_{A_{\kappa, \sigma}} = \{x \in Y_{\kappa, \sigma}: A_{\kappa, \sigma}(x) = x\} \).

(b) The sequence \((A_{\kappa, \sigma})^n(x_0)\) has the limit \( x^* \in Y_{\kappa, \sigma} \).

**Definition 4** (see [40]). For an ordered metric space \((Y_{\kappa, \sigma}, d, \leq)\), if \( A_{\kappa, \sigma}: Y_{\kappa, \sigma} \rightarrow Y_{\kappa, \sigma} \) is increasing PO with \( A_{\kappa, \sigma}(x^*) = x^* \), then \( \forall x \in Y_{\kappa, \sigma}, x \leq A_{\kappa, \sigma}(x) \) implies \( x \leq x^* \) and \( x \geq A_{\kappa, \sigma}(x) \) implies \( x \geq x^* \).

**Lemma 1** (see [41]). Let \( \theta_1, \theta_2 \in C(J, \mathbb{R}_+). \) If \( \kappa > 0 \), \( \theta_1 \) is increasing and there is \( m \geq 0 \) so that

\[\theta_1(\omega) = \theta_2(\omega) + m \int_0^\omega (\omega - s)^{n-1} \theta_1(s) ds, \omega \in J.\]

Then,

\[\theta_1(\omega) = \theta_2(\omega) + m \int_0^\omega \frac{\sum_{k=1}^{\infty} (m\Gamma(k\kappa))}{\Gamma(\kappa \times n \times k\kappa)} (\omega - s)^{n-1} \theta_1(s) ds, \omega \in J.\]

**Lemma 2** (see [42]). For \( \lambda > 0 \) and \( \kappa > 1 \), we have

\[\int_0^\omega (\omega - s)^{n-1} ds = \frac{\Gamma(\lambda)}{\Gamma(\kappa + 1)} \omega^{\lambda - 1}.\]

3. Main Results

In this section, we provide results regarding the EU and HUMIL stability for the solution of the considered system on the compact interval \( J = [0, 1], \) using the PO and Henry–Gronwall lemma [41]. Suppose \( g, h: J \times C_0 \times C_0 \rightarrow \mathbb{R} \) are continuous and \( C_0 \) is a Banach space of all the functions \( \varphi_1, \varphi_2: [-\xi_1, 1] \rightarrow \mathbb{R} \) which are continuous with norms \( \|\varphi_1\| = \max\{|\varphi_1(\omega)|: -\xi_1 \leq \omega \leq 1\} \) and \( \|\varphi_2\| = \max\{|\varphi_2(\omega)|: -\xi_1 \leq \omega \leq 1\} \). Consider the system

\[\begin{align*}
\frac{d^n}{dt^n} u(t) &= g(t, u(t), v(t)), & \omega \in J, \\
\frac{d^n}{dt^n} v(t) &= h(t, u(t), v(t)), & \omega \in J, \\
u(t) &= \varphi_1(t), & \omega \in [-\xi_1, 0], \\
v(t) &= \varphi_2(t), & \omega \in [-\xi_1, 0],
\end{align*}\]

where \( 1 < \kappa, \sigma \leq 2 \) and \( \varphi_1, \varphi_2 \in C([-\xi_1, 1], \mathbb{R}_+). \) For \( e_1, e_2, \)

we have the following inequalities:

\[\begin{align*}
\frac{d^n}{dt^n} u(t) - g(t, u(t), v(t)) \leq e_1 E_\kappa(\omega^\kappa), & \omega \in J, \\
\frac{d^n}{dt^n} v(t) - h(t, u(t), v(t)) \leq e_2 E_\kappa(\omega^\kappa), & \omega \in J,
\end{align*}\]

where \( E_\kappa(\cdot) \) and \( E_\kappa(\cdot) \) represent the Mittag-Leffler (ML) function defined by
\[ E_\kappa (\cdot) = \sum_{k=0}^{\infty} \frac{\cdot^k}{\Gamma (k\kappa + 1)} \]

\[ E_\sigma (\cdot) = \sum_{k=0}^{\infty} \frac{\cdot^k}{\Gamma (k\sigma + 1)} \]  

(\cdot) \in \mathbb{C}, \Re (\kappa), \Re (\sigma) > 0.

**Definition 5.** Extending the definition of Hyers–Ulam (HU) stability for a CS [43], we say that system (12) is HUMLS stable with respect to \( E_\kappa (\omega^\sigma) \), \( E_\sigma (\omega^\kappa) \), if there is \( c_{\kappa, \sigma} = \max \{ c_\kappa, c_\sigma \} > 0 \) so that for every \( \epsilon = \max \{ \epsilon_1, \epsilon_2 \} > 0 \) and each solution \((\pi, \varphi) \in C_t \times C_t\) of the inequalities (13) there is a unique solution \((u, v) \in C_t \times C_t\) so that

\[ |(u, v) - (\pi, \varphi)| \leq c_{\kappa, \sigma} E_\kappa (\omega^\kappa), \quad \lambda = \max \{ \kappa, \sigma \}. \]  

**Remark 1.** Functions \( \pi, \varphi \in C^1 (J, \mathbb{R}) \) are, respectively, the solutions of the above inequalities \( \Leftrightarrow \) there is \( \theta_1, \theta_2 \in C^1 (J, \mathbb{R}) \) so that

1. \( |\theta_1 (\omega)| \leq \epsilon_1 E_\kappa (\omega^\sigma), |\theta_2 (\omega)| \leq \epsilon_2 E_\sigma (\omega^\kappa), \omega \in J; \)

2. \( \begin{align*}
\begin{cases}
\frac{d}{d\omega} D^\omega_0 u (\omega) = g (\omega, u, D^\omega_0 u, v, \omega, v), & \omega \in J, \\
\frac{d}{d\omega} D^\omega_0 v (\omega) = h (\omega, D^\omega_0 u, \omega, v, \omega), & \omega \in J,
\end{cases}
\end{align*} \]

\[ \varphi_1 (\omega), \quad \varphi_2 (\omega), \quad \omega \in [-z, 0]. \]  

**Theorem 2.** For \( 1 < \kappa, \sigma \leq 2, \) if \( \pi, \varphi \in C^1 (J, \mathbb{R}) \), are, respectively, the solutions of

\[ \begin{align*}
\begin{cases}
\frac{d}{d\omega} D^\omega_0 u (\omega) = g (\omega, u, D^\omega_0 u, v, \omega, v), & \omega \in J, \\
\frac{d}{d\omega} D^\omega_0 v (\omega) = h (\omega, D^\omega_0 u, v, \omega), & \omega \in J,
\end{cases}
\end{align*}
\]

Then, \((\pi, \varphi)\) satisfies the following integral inequalities:

\[ \begin{align*}
\begin{cases}
\Pi (\omega) - \Pi (0) - \Pi^* (0) \omega - \frac{1}{2} \Pi^* (0) \omega^2 \\
-\frac{1}{\Gamma (\kappa)} \int_0^\omega (\omega - s)^{\kappa - 2} g (s, \pi, D^\omega_0 \pi, \varphi, D^\omega_0 \varphi) \, ds \leq \epsilon_1 E_\kappa (\omega^\kappa),
\end{cases}
\end{align*} \]

\[ \begin{align*}
\begin{cases}
\Omega (\omega) - \Omega (0) - \Omega^* (0) \omega - \frac{1}{2} \Omega^* (0) \omega^2 \\
-\frac{1}{\Gamma (\sigma)} \int_0^\omega (\omega - s)^{\sigma - 2} h (s, D^\omega_0 \varphi, \pi, \varphi, D^\omega_0 \varphi) \, ds \leq \epsilon_2 E_\sigma (\omega^\sigma).
\end{cases}
\end{align*} \]

**Proof.** From Remark 1, we have

\[ \frac{d}{d\omega} D^\omega_0 \pi (\omega) = g (\omega, \pi, D^\omega_0 \pi, \varphi, \omega, v), \quad 1 < \kappa, \sigma \leq 2, \omega \in J. \]

Then,

\[ \begin{align*}
\Pi (\omega) - \Pi (0) - \Pi^* (0) \omega - \frac{1}{2} \Pi^* (0) \omega^2 \\
= \frac{1}{\Gamma (\kappa)} \int_0^\omega (\omega - s)^{\kappa - 2} g (s, \pi, D^\omega_0 \pi, \varphi, D^\omega_0 \varphi) \, ds
\end{align*} \]

\[ + \frac{1}{\Gamma (\kappa)} \int_0^\omega (\omega - s)^{\kappa - 2} \theta_1 (s) \, ds. \]  

So, we have

\[ \begin{align*}
\Pi (\omega) - \Pi (0) - \Pi^* (0) \omega - \frac{1}{2} \Pi^* (0) \omega^2 \\
- \frac{1}{\Gamma (\kappa)} \int_0^\omega (\omega - s)^{\kappa - 2} g (s, \pi, D^\omega_0 \pi, \varphi, D^\omega_0 \varphi) \, ds
\end{align*} \]

\[ \leq \frac{1}{\Gamma (\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} \theta_1 (s) \, ds \leq \frac{\epsilon_1}{\Gamma (\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} E_\kappa (\omega^\kappa). \]  

Using the same technique, we can get

\[ \Omega (\omega) - \Omega (0) - \Omega^* (0) \omega - \frac{1}{2} \Omega^* (0) \omega^2 \\
- \frac{1}{\Gamma (\sigma)} \int_0^\omega (\omega - s)^{\sigma - 2} h (s, D^\omega_0 \varphi, \pi, \varphi, D^\omega_0 \varphi) \, ds \leq \epsilon_2 E_\sigma (\omega^\sigma). \]

**Theorem 3.** Let the following assumptions hold:

\( (M_1) \) \( g, h \in C ([J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \theta_1, \theta_2, \xi, \xi \in C^1 ([J, \mathbb{R}]), \xi = D^\omega_0 \mu (\omega) \leq (1/\Gamma (2 - \kappa)) |u (\omega)|, \xi = D^\omega_0 \nu (\omega) \leq (1/\Gamma (2 - \sigma)) |\nu (\omega)|; \)

\( (M_2) \) There are \( \theta_1, \theta_2 > 0, \) so that \( \forall \omega \in J \) and \( u_j, v_j \in \mathbb{R}, j = 1, 2 \)

\[ |g (\omega, u_1, u_2) - g (\omega, v_1, v_2)| \leq \theta_1 \sum_{j=1}^2 |u_j - v_j|, \]

\[ |h (\omega, u_1, u_2) - h (\omega, v_1, v_2)| \leq \theta_2 \sum_{j=1}^2 |u_j - v_j|. \]
\[
\begin{align*}
\text{(M,)} \quad & \max_{\omega \in J} \left\{ \frac{\varrho_1}{\Gamma(\kappa - 1)} \left[ \frac{1}{\kappa(\kappa - 1)} + \frac{1}{\Gamma(3 - \sigma)} \right], \right. \\
& \quad \left. \frac{\varrho_2}{\Gamma(\sigma - 1)} \left[ \frac{1}{\sigma(\sigma - 1)} + \frac{1}{\Gamma(3 - \kappa)} \right] \right\} = K < 1.
\end{align*}
\]

Then,

(i) System (12) has a unique solution in \( C^2([-\xi, 1], \mathbb{R}) \cap C^2(1, \mathbb{R}) \);

(ii) problem (4) is HUML stable.

**Proof.**

(i) Consider

\[
C^\kappa_0 u(\omega) = g(\omega, u_0, C^\kappa_0 v_0), \quad 1 < \kappa, \sigma \leq 2, \omega \in J.
\]

The solution is equivalent to

\[
u(\omega) - u(0) - u'(0)\omega - \frac{1}{2} u''(0)\omega^2 = \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, C^\kappa_0 v_s) ds, \]

\[
u(\omega) = u(0) + u'(0)\omega + \frac{1}{2} u''(0)\omega^2 + \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, C^\kappa_0 v_s) ds.
\]

Thus,

\[
u(\omega) = \left\{ \begin{array}{ll}
u(0) + u'(0)\omega + \frac{1}{2} u''(0)\omega^2 \\
+ \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, C^\kappa_0 v_s) ds, & \omega \in J, \\
p_1(\omega), & \omega \in [-\xi, 1].
\end{array} \right.
\]

Similarly, on considering

\[C^\kappa_0 v(\omega) = h(\omega, C^\kappa_0 u_\omega, v(\omega)), \quad 1 < \kappa, \sigma \leq 2, \omega \in J,\]

we can obtain

\[
\nu(\omega) = \left\{ \begin{array}{ll}
u(0) + v'(0)\omega + \frac{1}{2} v''(0)\omega^2 \\
+ \frac{1}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma - 1} h(s, C^\kappa_0 u_s, v_s) ds, & \omega \in J, \\
p_2(\omega), & \omega \in [-\xi, 1].
\end{array} \right.
\]

Let \( Y_\xi = \{ u \in C([-\xi, 1], \mathbb{R}) \cap C^2([-\xi, 1], \mathbb{R}) \} \) and define norm on \( Y_\xi \) by \( \| u \| = \max_{\omega \in J} | u(\omega) | + \max_{\omega \in J} | C^\kappa_0 u(\omega) | \) and \( Y_\xi = \{ v \in C([-\xi, 1], \mathbb{R}) \cap C^2([-\xi, 1], \mathbb{R}) \} \). Let \( \nu \in C([-\xi, 1], \mathbb{R}) \) and \( \nu \in C^2([-\xi, 1], \mathbb{R}) \) respectively represent the class of continuous and continuously differentiable functions from \([-\xi, 1]\) to \( \mathbb{R} \). The norm is defined in such a way that the norm of each term depends on the derivatives of the fractional order of the other terms of \( Y_\xi \). The product \( Y_\xi \times Y_\xi \) is also a Banach space with norm \( \| (u, v) \| = \| u \| + \| v \| \).

Define operators \( k = (H, G) : Y_\xi \times Y_\xi \to Y_\xi \times Y_\xi \) by

\[
H(u, v)(\omega) = \left\{ \begin{array}{ll}
u(0)u'(0)\omega + \frac{1}{2} v''(0)\omega^2 \\
+ \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, C^\kappa_0 v_s) ds, & \omega \in J, \\
p_1(\omega), & \omega \in [-\xi, 0],
\end{array} \right.
\]

\[
G(u, v)(\omega) = \left\{ \begin{array}{ll}
u(0) + v'(0)\omega + \frac{1}{2} v''(0)\omega^2 \\
+ \frac{1}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma - 1} h(s, C^\kappa_0 u_s, v_s) ds, & \omega \in J, \\
p_2(\omega), & \omega \in [-\xi, 0].
\end{array} \right.
\]

First, we show that \( k \) is a contraction mapping. It is clear that

\[
\| k(u, v)(\omega) - k(\bar{u}, \bar{v})(\omega) \| = \| H(u, v)(\omega) - H(\bar{u}, \bar{v})(\omega) \| + \| G(u, v)(\omega) - G(\bar{u}, \bar{v})(\omega) \|
\]

\[
\leq | H(u, v)(\omega) - H(\bar{u}, \bar{v})(\omega) | + | G(u, v)(\omega) - G(\bar{u}, \bar{v})(\omega) |
\]

\[
= | \varphi_1(\omega) - \varphi_1(\omega) | + | \varphi_2(\omega) - \varphi_2(\omega) | = 0,
\]

\( \forall \omega \in [-\xi, 0]. \)

For \( \omega \in J \), we have

\[
\max_{0 \leq s \leq \omega} | u_s(\omega) - \bar{u}_s(\omega) | \]

\[
\leq \max_{0 \leq s \leq \omega} | u_s - \bar{u}_s | + \max_{0 \leq s \leq \omega} | v_s - \bar{v}_s |,
\]

\[
\leq \max_{0 \leq s \leq \omega} | u(s + \tau) - \bar{u}(s + \tau) | + \max_{0 \leq s \leq \omega} | v(s + \tau) - \bar{v}(s + \tau) |
\]

\[
= \max_{0 \leq s \leq \omega} | u(s + \tau) - \bar{u}(s + \tau) | + \max_{0 \leq s \leq \omega} | v(s + \tau) - \bar{v}(s + \tau) |
\]

\[
= | u(s + \tau) - \bar{v}(s + \tau) | + | v(s + \tau) - \bar{v}(s + \tau) | = \| (u, v) - (\bar{u}, \bar{v}) \|.
\]

(32)
where $s^* = s + \tau$ and $-\xi \leq \tau < 0$. Thus,

\[
\|\mathcal{K}(u, v) - \mathcal{K}(\bar{u}, \bar{v})\| = \|\mathcal{H}(u, v) - \mathcal{H}(\bar{u}, \bar{v})\| + \|\mathcal{G}(u, v) - \mathcal{G}(\bar{u}, \bar{v})\|
\]

\[
\leq \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} |g(s, u, s\mathcal{D}_0^\mu v) - g(s, \bar{u}, s\mathcal{D}_0^\mu \bar{v})| ds
\]

\[
+ \frac{1}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma - 1} |h(s, s\mathcal{D}_0^\mu \bar{u}, v) - h(s, \bar{u}, s\mathcal{D}_0^\mu \bar{v})| ds
\]

\[
\leq \frac{\beta_1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} \max_{0 \leq s \leq \omega} |u(s + \tau) - \bar{u}(s + \tau)| + \max_{0 \leq s \leq \omega} |s\mathcal{D}_0^\mu v(s + \tau) - s\mathcal{D}_0^\mu \bar{v}(s + \tau)| ds
\]

\[
+ \frac{\beta_2}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma - 1} \max_{0 \leq s \leq \omega} |s\mathcal{D}_0^\mu u(s + \tau) - s\mathcal{D}_0^\mu \bar{u}(s + \tau)| + \max_{0 \leq s \leq \omega} |v(s + \tau) - \bar{v}(s + \tau)| ds
\]

\[
\leq \frac{\beta_1}{\Gamma(\kappa)} \max_{-\xi \leq s^* \leq 1} |u(s^*) - \bar{u}(s^*)| + \max_{-\xi \leq s^* \leq 1} |s\mathcal{D}_0^\mu v(s^*) - s\mathcal{D}_0^\mu \bar{v}(s^*)| \int_0^\omega (\omega - s)^{\kappa - 1} ds
\]

\[
+ \frac{\beta_2}{\Gamma(\sigma)} \max_{-\xi \leq s^* \leq 1} |v(s^*) - \bar{v}(s^*)| + \max_{-\xi \leq s^* \leq 1} |s\mathcal{D}_0^\mu u(s^*) - s\mathcal{D}_0^\mu \bar{u}(s^*)| \int_0^\omega (\omega - s)^{\sigma - 1} ds
\]

\[
\leq \frac{\beta_1}{\Gamma(\kappa + 1)} \|u - \bar{u}\| + \frac{\beta_2}{\Gamma(\sigma + 1)} \|v - \bar{v}\|
\]

Also,

\[
\|s\mathcal{D}_0^\mu, \mathcal{D}_0^\mu \mathcal{K}(u, v) - s\mathcal{D}_0^\mu, \mathcal{D}_0^\mu \mathcal{K}(\bar{u}, \bar{v})\|
\]

\[
= \max_{w \in \mathcal{J}} \left|s\mathcal{D}_0^\mu \mathcal{H}(u, v), s\mathcal{D}_0^\mu \mathcal{G}(u, v) - s\mathcal{D}_0^\mu \mathcal{H}(\bar{u}, \bar{v}), s\mathcal{D}_0^\mu \mathcal{G}(\bar{u}, \bar{v})\right|
\]

\[
\leq \max_{w \in \mathcal{J}} \left|s\mathcal{D}_0^\mu \mathcal{H}(u, v) - s\mathcal{D}_0^\mu \mathcal{H}(u, v)\right| + \max_{w \in \mathcal{J}} \left|s\mathcal{D}_0^\mu \mathcal{G}(u, v) - s\mathcal{D}_0^\mu \mathcal{G}(\bar{u}, \bar{v})\right|
\]

\[
\leq \frac{1}{\Gamma(2 - \sigma)} \max_{w \in \mathcal{J}} \left|\int_0^\omega (\omega - s)^{1-\sigma} \left(\frac{(\kappa - 1)(\kappa - 2)}{\Gamma(\kappa)}\right) \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u, s\mathcal{D}_0^\mu v) ds\right|
\]

\[
+ \frac{1}{\Gamma(2 - \kappa)} \max_{w \in \mathcal{J}} \left|\int_0^\omega (\omega - s)^{\kappa - \sigma} \frac{(\kappa - 1)(\kappa - 2)}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, \bar{u}, s\mathcal{D}_0^\mu \bar{v}) ds\right|
\]

\[
\leq \frac{\beta_1}{\Gamma(2 - \sigma)\Gamma(\kappa)} \|u - \bar{u}\| \int_0^\omega (\omega - s)^{1-\sigma} ds + \frac{\beta_2}{\Gamma(2 - \kappa)\Gamma(\sigma)} \|v - \bar{v}\| \int_0^\omega (\omega - s)^{1-\sigma} ds
\]

\[
\leq \frac{\beta_1}{\Gamma(3 - \sigma)\Gamma(\kappa)} \|u - \bar{u}\| + \frac{\beta_2}{\Gamma(3 - \kappa)\Gamma(\sigma)} \|v - \bar{v}\|
\]
So, we obtain
\[
\|\mathcal{K}(u, v) - \mathcal{K}(\overline{u}, \overline{v})\|
\leq \frac{\mathcal{G}_1}{\Gamma(\kappa + 1)} \|u - \overline{u}\| + \frac{\mathcal{G}_2}{\Gamma(\sigma + 1)} \|v - \overline{v}\| + \frac{\mathcal{G}_1}{\Gamma(\kappa - 1)} \|u - \overline{u}\|
\]
\[
+ \frac{\mathcal{G}_2}{\Gamma(\sigma - 1)} \left( \frac{1}{\kappa(\kappa - 1)} + \frac{1}{\Gamma(\kappa - 1)} \right) \|u - \overline{u}\| + \frac{1}{\Gamma(\sigma)} \left( \frac{1}{\sigma(\sigma - 1)} + \frac{1}{\Gamma(\sigma - 1)} \right) \|v - \overline{v}\|
\]
\[
\leq \mathcal{K}(\|u, v\| - \|\overline{u}, \overline{v}\|).
\]

(35)

This shows that $\mathcal{K}$ is a contraction operator.

(ii) Next, let $(\overline{u}, \overline{v}) \in C^2(J, \mathbb{R})$ be the approximate solution of (13) and $(u, v) \in C^2(J, \mathbb{R})$ be the unique solutions of system (12), that is,
\[
\begin{align*}
\mathcal{C}D_0^u u(\omega) &= g(\omega, u_\omega, \mathcal{C}D_0^u v_\omega), & \omega \in J, 1 < \kappa, \sigma \leq 2, \\
\mathcal{C}D_0^v v(\omega) &= h(\omega, \mathcal{C}D_0^u u_\omega, v_\omega), & \omega \in J, 1 < \kappa, \sigma \leq 2, \\
u(\omega) &= \overline{u}(\omega), v(\omega) = \overline{v}(\omega), & \omega \in [-\xi, 0],
\end{align*}
\]

(36)

then
\[
\begin{align*}
u(\omega) &= \nu(0) + \nu'(0)\omega + \frac{1}{2} \nu''(0)\omega^2 \\
&\quad + \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, \mathcal{C}D_0^u v_s) ds
\end{align*}
\]

\[
\begin{align*}
u(\omega) &= \nu(0) + \nu'(0)\omega + \frac{1}{2} \nu''(0)\omega^2 \\
&\quad + \frac{1}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma - 1} h(s, \mathcal{C}D_0^u u_s, v_s) ds.
\end{align*}
\]

(37)

For $\omega \in [-\xi, 0]$, we have
\[
\|u(\omega) - \nu(\omega)\| \leq |\overline{u}(\omega) - \nu(\omega)| + |\overline{v}(\omega) - \nu(\omega)| = 0.
\]

(38)

Also, for $\omega \in J$, we have
\[
|(u, v)(\omega) - (\overline{u}, \overline{v})(\omega)| \leq |u(\omega) - \overline{u}(\omega)| + |v(\omega) - \overline{v}(\omega)|.
\]

(39)

Now,
\[
\begin{align*}
|\nu(\omega) - \overline{v}(\omega)|
\leq |\nu(\omega) - \nu(0)| + |\nu(0) - \overline{v}(0)| - \frac{1}{2} u''(0)\omega^2 + \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, \mathcal{C}D_0^u v_s) ds
\]
\[
\leq |\nu(\omega) - \nu(0)| + |\nu(0) - \overline{v}(0)| - \frac{1}{2} u''(0)\omega^2 + \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, \mathcal{C}D_0^u v_s) ds
\]
\[
+ \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, \mathcal{C}D_0^u v_s) ds
\]
\[
\leq |\nu(\omega) - \nu(0)| + |\nu(0) - \overline{v}(0)| - \frac{1}{2} u''(0)\omega^2 + \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, u_s, \mathcal{C}D_0^u v_s) ds
\]
\[
\leq \epsilon_1 E_\omega(\omega^\nu) + \frac{g_1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} [g(s, u_s, \mathcal{C}D_0^u v_s) - g(s, \overline{u}_s, \mathcal{C}D_0^u \overline{v}_s)] |ds
\]
\[
\leq \epsilon_1 E_\omega(\omega^\nu) + \frac{g_1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} |u_s - \overline{u}_s| + |\mathcal{C}D_0^u v_s - \mathcal{C}D_0^u \overline{v}_s| |ds
\]
\[
\leq \epsilon_1 E_\omega(\omega^\nu) + \frac{g_1}{\Gamma(\kappa)} \left( \int_0^\omega (\omega - s)^{\kappa - 1} |u_s - \overline{u}_s| + \int_0^\omega (\omega - s)^{\kappa - 1} |\mathcal{C}D_0^u v_s - \mathcal{C}D_0^u \overline{v}_s| \right) ds.
\]

(40)

Similarly, we can get
\[
\begin{align*}
|\nu(\omega) - \overline{v}(\omega)|
\leq |\nu(\omega) - \nu(0)| + |\nu(0) - \overline{v}(0)| - \frac{1}{2} u''(0)\omega^2 + \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} g(s, \mathcal{C}D_0^u u_s, v_s) ds
\]
\[
+ \frac{1}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma - 1} h(s, \mathcal{C}D_0^u u_s, v_s) ds.
\]

(41)
Let \( \chi, \theta \in C([-\xi, 1], \mathbb{R}_+) \) and in view of inequalities (40) and (41) take \( A_{k, \sigma} = (A_k, A_\sigma) \), where \( A_k, A_\sigma : C([-\xi, 1] \times [-\xi, 1], \mathbb{R}_+) \rightarrow C([-\xi, 1], \mathbb{R}_+) \) are defined by

\[
A_k(\chi, \theta)(\omega) = \begin{cases} 
\varepsilon_1 E_\chi(\omega^\kappa) + \frac{\Theta_1}{\Gamma(\kappa)} \left( \int_0^\omega (\omega - s)^{\kappa-1} \chi_s ds + \int_0^\omega (\omega - s)^{\kappa-1} D_0^\sigma \theta_s ds \right), & \omega \in J, \\
0, & \omega \in [-\xi, 0],
\end{cases}
\]

\[
A_\sigma(\chi, \theta)(\omega) = \begin{cases} 
\varepsilon_2 E_\sigma(\omega^\kappa) + \frac{\Theta_2}{\Gamma(\sigma)} \left( \int_0^\omega (\omega - s)^{\sigma-1} \theta_s ds + \int_0^\omega (\omega - s)^{\sigma-1} D_0^\kappa \chi_s ds \right), & \omega \in J, \\
0, & \omega \in [-\xi, 0].
\end{cases}
\]

We will show that \( A_{k, \sigma} \) is PO and consequently need to show that \( A_{k, \sigma} \) is a contraction. Consider

\[
|A_{k, \sigma}(\chi, \theta)(\omega) - A_{k, \sigma}(\chi^*, \theta^*)(\omega)|
\]

\[
\leq |A_k(\chi, \theta)(\omega) - A_k(\chi^*, \theta^*)(\omega)| + |A_\sigma(\chi, \theta)(\omega) - A_\sigma(\chi^*, \theta^*)(\omega)|
\]

\[
\leq \frac{\Theta_1}{\Gamma(\kappa)} \left( \int_0^\omega (\omega - s)^{\kappa-1} |\chi_s - \chi_s^*| ds + \int_0^\omega (\omega - s)^{\kappa-1} |D_0^\sigma \theta_s - D_0^\sigma \theta_s^*| ds \right)
\]

\[
+ \frac{\Theta_2}{\Gamma(\sigma)} \left( \int_0^\omega (\omega - s)^{\sigma-1} |\theta_s - \theta_s^*| ds + \int_0^\omega (\omega - s)^{\sigma-1} |D_0^\kappa \chi_s - D_0^\kappa \chi_s^*| ds \right)
\]

\[
\leq \frac{\Theta_1}{\Gamma(\kappa)} \left( \max_{0 \leq s \leq \omega} |\chi_s - \chi_s^*| + \max_{0 \leq s \leq \omega} |D_0^\sigma \theta_s - D_0^\sigma \theta_s^*| \right) \int_0^\omega (\omega - s)^{\kappa-1} ds
\]

\[
+ \frac{\Theta_2}{\Gamma(\sigma)} \left( \max_{0 \leq s \leq \omega} |\theta_s - \theta_s^*| + \max_{0 \leq s \leq \omega} |D_0^\kappa \chi_s - D_0^\kappa \chi_s^*| \right) \int_0^\omega (\omega - s)^{\sigma-1} ds
\]

\[
\leq \frac{\Theta_1}{\Gamma(\kappa + 1)} \|\chi - \chi^*\| + \frac{\Theta_2}{\Gamma(\sigma + 1)} \|\theta - \theta^*\|.
\]

Moreover,

\[
|\hat{D}_0^\sigma \hat{D}_0^\kappa A_{k, \sigma}(\chi, \theta) - (\hat{D}_0^\sigma \hat{D}_0^\kappa) A_{k, \sigma}(\chi^*, \theta^*)|
\]

\[
= |\hat{D}_0^\sigma \hat{D}_0^\kappa A_k(\chi, \theta) - (\hat{D}_0^\sigma \hat{D}_0^\kappa) A_k(\chi^*, \theta^*)| + |\hat{D}_0^\sigma \hat{D}_0^\kappa A_\sigma(\chi, \theta) - (\hat{D}_0^\sigma \hat{D}_0^\kappa) A_\sigma(\chi^*, \theta^*)|
\]

\[
\leq \frac{1}{\Gamma(2 - \sigma)} \int_0^\omega (\omega - s)^{\kappa-3} \left( \frac{(k - 1)(k - 2)}{\Gamma(k)} \int_0^\omega (\omega - s)^{\kappa-3} \max_{0 \leq t \leq \omega} |\chi_t - \chi_t^*| ds \right)
\]

\[
+ \frac{(k - 1)(k - 2)}{\Gamma(k)} \int_0^\omega (\omega - s)^{\kappa-3} \max_{0 \leq t \leq \omega} |D_0^\sigma \theta_t - D_0^\sigma \theta_t^*| ds
\]

\[
+ \frac{1}{\Gamma(2 - \sigma)} \int_0^\omega (\omega - s)^{\sigma-3} \left( \frac{(\sigma - 1)(\sigma - 2)}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-3} \max_{0 \leq t \leq \omega} |\theta_t - \theta_t^*| ds \right)
\]

\[
+ \frac{(\sigma - 1)(\sigma - 2)}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-3} \max_{0 \leq t \leq \omega} |D_0^\kappa \chi_t - D_0^\kappa \chi_t^*| ds
\]

\[
\leq \frac{\Theta_1}{\Gamma(3 - \kappa) \Gamma(\kappa - 1)} \|\chi - \chi^*\| + \frac{\Theta_2}{\Gamma(3 - \kappa) \Gamma(\sigma - 1)} \|\theta - \theta^*\|.
\]
Thus, we get
\[
\| A_{\kappa}(\chi, \theta)(\omega) - A_{\kappa}(\chi^*, \theta^*)(\omega) \|
\leq \frac{\bar{q}_1}{\Gamma(\kappa - 1)} \left[ \frac{1}{\kappa(\kappa - 1)} + \frac{1}{\Gamma(3-\sigma)} \right] \| \chi - \chi^* \| + \frac{\bar{q}_2}{\Gamma(\sigma - 1)} \left[ \frac{1}{\Gamma(3-\kappa)} + \frac{1}{\Gamma(\kappa - 1)} \right] \| \theta - \theta^* \|.
\] (45)

Therefore, \( \| A_{\kappa}(\chi, \theta)(\omega) - A_{\kappa}(\chi^*, \theta^*)(\omega) \| \leq K \| (\chi, \theta) - (\chi^*, \theta^*) \| \forall \chi, \theta, \chi^*, \theta^* \in C([-\xi, \xi], R_+), \) showing that \( A_{\kappa, \sigma} \) is a contraction operator. By Definition 3, \( A_{\kappa, \sigma} \) is a PO and \( F_{A_{\kappa, \sigma}} = \{(x^*, y^*)\}, \) then

\[
x^*(h_2) - x^*(h_1) \leq \epsilon_1 (E_k(h_2^*) - E_k(h_1^*)) + \frac{\bar{q}_1}{\Gamma(\kappa)} \int_0^{h_1} [(h_2 - s)^{\kappa - 1} - (h_1 - s)^{\kappa - 1}] [x_s^* + D_0^\sigma y_s^*] ds
+ \frac{\bar{q}_1}{\Gamma(\kappa)} \int_0^{h_1} (h_2 - s)^{\kappa - 1} [x_s^* + D_0^\sigma y_s^*] ds
\geq \epsilon_1 (E_k(h_2^*) - E_k(h_1^*)) + \frac{\bar{q}_1}{\Gamma(\kappa)} \int_0^{h_1} [(h_2 - s)^{\kappa - 1} - (h_1 - s)^{\kappa - 1}] ds
+ \frac{\bar{q}_1}{\Gamma(\kappa)} \int_0^{h_1} (h_2 - s)^{\kappa - 1} ds
\geq \epsilon_1 (E_k(h_2^*) - E_k(h_1^*)) + \frac{\bar{q}_1}{\Gamma(\kappa)} (h_2^* - h_1^*) > 0.
\] (47)

Similarly, we can get
\[
y^*(\omega) \geq \epsilon_1 (E_\sigma(h_2^*) - E_\sigma(h_1^*)) + \frac{\bar{q}_1}{\Gamma(\sigma + 1)} (h_2^* - h_1^*) > 0.
\] (48)

So that the solution \((x^*(\omega), y^*(\omega))\) is increasing and \(D_0^\sigma x^*, D_0^\sigma y^* > 0\). Also
\[
x^*(\omega) \leq \epsilon_1 E_k(\omega^*) + \frac{\bar{q}_1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa - 1} x_s^* (s) ds
+ \frac{\bar{q}_1}{\Gamma(\kappa)\Gamma(2-\sigma)} \int_0^\omega (\omega - s)^{\kappa - 1} x_s^* (s) ds
\leq \epsilon_1 E_k(\omega^*) + \frac{\bar{q}_1}{\Gamma(\kappa)} \left( 1 + \frac{1}{\Gamma(2-\sigma)} \right) \int_0^\omega (\omega - s)^{\kappa - 1} x_s^* (s) ds.
\] (49)

Thus, by using Gronwall’s lemma we have
\[
x^*(\omega) \leq c_{E_k} \epsilon_1 E_k(\omega^*),
\] (50)

where \( c_{E_k} = E_k)((\bar{q}_1/\Gamma(\kappa))(1 + (1/\Gamma(2-\sigma)))) \). In the same way
\[
y^*(\omega) \leq c_E \epsilon_2 E_\sigma(\omega^*),
\] (51)

where \( c_E = E_\sigma((\bar{q}_2/\Gamma(\sigma))(1 + (1/\Gamma(2-\sigma)))) \). Particularly, if \( x = |u - v| \) and \( y = |v - \bar{v}| \), then by (40) and (41), we have \( x(\omega) \leq A_k \) and \( y(\omega) \leq A_\sigma \); therefore, by Gronwall’s lemma we obtain
\[
|u(\omega) - \bar{u}(\omega)| \leq c_E \epsilon_1 E_k(\omega^*),
\] (52)

\[
|v(\omega) - \bar{v}(\omega)| \leq c_E \epsilon_2 E_\sigma(\omega^*).
\] (53)

Thus, from the above we have
\[
\|(u,v) - (\bar{u}, \bar{v})\| \leq \|u - \bar{u}\| + \|v - \bar{v}\|
\leq c_E \epsilon_1 E_k(\omega^*) + c_E \epsilon_2 E_\sigma(\omega^*) \leq 2c_E \epsilon_4 E_4(\omega^*).
\] (53)

Hence, it follows that (4) is ML stable. □

4. Illustrative Example

As an application of our results, we consider the following example.

Example 1. Consider the problem
\begin{equation}
\begin{aligned}
\mathcal{D}^{3/2}_0 u(\omega) &= \frac{1}{4e^\omega |u(\omega - 1)| + |\mathcal{D}^{3/2}_0 v(\omega - 1)|}, \\
\omega &\in J = [0, 1], \\
\mathcal{D}^{5/3}_0 v(\omega) &= \frac{1}{10e^{2\omega} + (1 + |\mathcal{D}^{3/2}_0 u(\omega - 1)| + |v(\omega - 1)|)}, \\
\omega &\in J = [0, 1], \\
u(\omega) = 0, v(\omega) = 0, \omega \in [-1, 0].
\end{aligned}
\tag{54}
\end{equation}

Set the functions as
\begin{equation}
\begin{aligned}
g_{3/2}(u, u(\omega), v(\omega)) &= \frac{1}{4e^\omega (1 + |u| + |v|)}, \\
h_{5/3}(u, u(\omega), v(\omega)) &= \frac{1}{10e^{2\omega} + (1 + |u| + |v|)}.
\end{aligned}
\end{equation}

and then for every \( u, v, \mathfrak{u}, \mathfrak{v} \) from \( \mathbb{R} \), \( \omega \in [0, 1] \) we have
\begin{equation}
\begin{aligned}
\|g_{3/2}(u, u(\omega), v(\omega)) - g_{3/2}(u, \mathfrak{u}, \mathfrak{v}(\omega))\| \\
&\leq \frac{1}{4} |u - \mathfrak{u}| + \frac{1}{4} |v - \mathfrak{v}|, \\
\|h_{5/3}(u, u(\omega), v(\omega)) - h_{5/3}(u, \mathfrak{u}, \mathfrak{v}(\omega))\| \\
&\leq \frac{1}{10} |u - \mathfrak{u}| + \frac{1}{10} |v - \mathfrak{v}|.
\end{aligned}
\tag{56}
\end{equation}

Therefore, \((M_2)\) is satisfied with \( \kappa = 3/2 \) and \( \sigma = 5/3 \), and from the inequalities in (56), \( \tilde{\epsilon}_1 = 1/4 \) and \( \tilde{\epsilon}_2 = 1/10 \). Thus, we have \( \mathbf{K} = 0.34601 < 1 \). Therefore, by Theorem 2, (54) has a unique solution. After calculations, we obtain
\begin{equation}
\|u, v) - (\mathfrak{u}, \mathfrak{v})\| \leq c_{5/3}\mathbb{E}_{3/2}(\omega^{3/2}), \quad \omega \in [0, 1].
\tag{57}
\end{equation}

Using Theorem 3, the solution of (54) is Hyers–Ulam–ML stable.

**Remark 2.** If we consider another space \( B_{u,v} = [u, v; u, v \in C[-\xi, 1], \mathcal{D}^{\alpha}_0 u, \mathcal{D}^{\alpha}_0 v \in C[-\xi, 1]] \) with modified Bielecki’s norms defined by
\begin{equation}
\begin{aligned}
\|u\| &= \max_{\omega \in J} (|u(\omega)|e^{-\lambda\omega} + |\mathcal{D}^\alpha_0 u(\omega)|e^{-\rho\omega}), \\
\|v\| &= \max_{\omega \in J} (|v(\omega)|e^{-\lambda\omega} + |\mathcal{D}^\alpha_0 v(\omega)|e^{-\rho\omega}),
\end{aligned}
\tag{58}
\end{equation}

where \( 0 < \rho < \lambda \), \( J = [0, 1] \), then the results similar to Theorem 3 can be obtained for the solution of (4). 

5. **Conclusion**

We gave sufficient conditions for the EU of the solutions to the nonlinear implicit CS of neutral FDEs. Our main tool was the Banach contraction principle. Likewise under specific conditions, we have found the HUML stability results for the solution of the CS given in (4).

**Data Availability**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this article.

**Authors’ Contributions**

The main idea of the paper was given by Zada. Results were proved by Ahmad and Zada. Ali helped in the example. Fu and Xu drafted the paper, and Jiang helped in the revision and gave financial support for publishing.

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