Coordinates on Schubert cells, Kostant’s harmonic forms, and the Bruhat-Poisson structure on $G/B$

Jiang-Hua Lu
Department of Mathematics, University of Arizona, Tucson, AZ 85721 USA
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1 Introduction and Notation

This work grew out of my attempts to understand the relations between results of Kostant [K] on the de Rham cohomology of a flag manifold and Poisson geometry.

Let $X = G/B$ be a flag manifold, where $G$ is a complex semi-simple Lie group and $B$ is a Borel subgroup of $G$. Let $K$ be a compact real form of $G$, so $X = G/B \cong K/T$, where $T = K \cap B$ is a maximal torus of $K$. In [K], Kostant constructs, for each element $w$ in the Weyl group $W$, an explicit $K$-invariant closed differential form $s^w$ on $X$ with $\deg(s^w) = 2l(w)$, where $l(w)$ is the length of $w$ (see Section 4). The cohomology classes of the $s^w$'s form a basis of $H(X, \mathbb{C})$ that, up to constant multiples, is dual to the basis of the homology of $X$ formed by the closures of the Bruhat (or Schubert) cells in $X$. These $K$-invariant forms on $X$ are $(d, \partial)$-harmonic, where $d$ is the de Rham differential operator and $\partial$ is a degree $-1$ operator introduced by Kostant.

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Our work was first motivated by wanting to understand the nature of the operator $\partial$ in terms of Poisson geometry.

The Poisson structure that is relevant to Kostant’s theorem is the so-called Bruhat-Poisson structure on $X$ \cite{E-W}. It has its origin in the theory of quantum groups. It has the special property that its symplectic leaves are precisely the Bruhat cells in $X$ (and hence the name). More properties of the Bruhat-Poisson structure are reviewed in Section 3.

The Bruhat-Poisson structure gives rise to the degree $-1$ operator

$$\partial_\pi = i_\pi d - di_\pi$$

on the space of differential forms on $X$ called the Koszul-Brylinski operator \cite{Ko} \cite{B}, where $\pi$ is the bi-vector field on $X$ defining this Poisson structure, and $i_\pi$ denotes the contraction operator of differential forms with $\pi$. It satisfies $\partial_\pi^2 = 0$. Its homology is called the Poisson homology of $\pi$. Poisson homology is closely related to cyclic homology of associative algebras, as is shown in \cite{B}.

It turns out that when restricted to $K$-invariant differential forms on $X$, Kostant’s operator $\partial$ and the Koszul-Brylinski operator $\partial_\pi$ differ by the contraction operator by a vector field $\theta_0$ called the modular vector field of $\pi$ (see \cite{W2} \cite{B-Z} \cite{E-L-W}). More explicitly, it is the infinitesimal generator of the $K$-action on $X$ in the direction of the element $2iH_\rho$ with $\rho$ being half of the sum of all positive roots. This result, together with others on the Poisson (co)homology of the Bruhat-Poisson structure, can be found in \cite{E-L}.

In this paper, we relate Kostant’s harmonic forms on $X$ with the Bruhat-Poisson structure.

More precisely, the Bruhat-Poisson structure on $X = K/T$ is $T$-invariant (but not $K$-invariant). Thus, each Bruhat cell $\Sigma_w$ inherits a $T$-invariant symplectic structure $\Omega_w$. Use $\phi_w : \Sigma_w \to t^*$ to denote the moment map and let $\mu_w$ be the Liouville volume form on $\Sigma_w$ defined by $\Omega_w$. Theorem 4.3 says that when restricted to the cell $\Sigma_w$, Kostant’s form $s^w$ is related to the Liouville volume form $\mu_w$ by

$$s^w|_{\Sigma_w} = e^{\langle \phi_w, 2iH_\rho \rangle} \mu_w.$$ 

Notice that the function $\langle \phi_w, 2iH_\rho \rangle$ is the Hamiltonian function for the modular vector field $\theta_0$ on $\Sigma_w$. (The vector field $\theta_0$ is not globally Hamiltonian (see Section 4), but it is on each cell). Theorem 4.3 thus expresses Kostant’s harmonic forms totally in terms of data coming from the Bruhat-Poisson structure. In particular, it shows that the integral $\int_{\Sigma_w} s^w$ is of the Duistermaat-Heckman type. Wanting to see this was another motivation for this work.
Theorem 4.3 is proved by writing everything down in some coordinates
\[ \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \]
on each Bruhat cell \( \Sigma_w \), where \( l = l(w) \). These coordinates are motivated by, but are independent of, the Bruhat-Poisson structure. Among the quantities that we write down explicitly in the coordinates are (see the later sections for the notation)

- (Theorem 3.4) the symplectic 2-form \( \Omega_w \) on \( \Sigma_w \) and thus the Liouville volume form \( \mu_w \):
  \[
  \Omega_w = \sum_{j=1}^{l} \frac{i}{\ll \alpha_j, \alpha_j \gg} \frac{1}{1 + |z_j|^2} dz_j \wedge d\bar{z}_j
  \]
  \[
  \mu_w = \prod_{j=1}^{l} \frac{i}{\ll \alpha_j, \alpha_j \gg} \frac{1}{1 + |z_j|^2} dz_j \wedge d\bar{z}_j
  \]

- (Theorem 3.4) the moment map for the \( T \)-action on \( (\Sigma_w, \Omega_w) \):
  \[
  \phi_w : \Sigma_w \rightarrow t^*: \quad \phi_w = \sum_{j=1}^{l} \left( -\frac{1}{2} \log(1 + |z_j|^2) \tilde{H}_{\alpha_j} \right)
  \]

- (Theorem 2.7) the Haar measure \( dn \) on the group \( N_w = N \cap wNw^{-1} \) that parametrizes \( \Sigma_w \):
  \[
  dn = \left( \prod_{j=1}^{l} \frac{i}{\pi \ll \beta_j, \beta_j \gg} \right) \prod_{j=1}^{l} \frac{2 \ll \rho, \beta_j \gg}{1 + |z_j|^2} \ll \beta_j, \beta_j \gg -1 dz_j \wedge d\bar{z}_j
  \]

- (Theorem 2.5) the \( A \)-component \( a_w(n) \) in the Iwasawa decomposition of the element \( \dot{w}^{-1}n\dot{w} \) for \( n \in N_w \) (where \( \dot{w} \) is any representative of \( w \) in \( K \)):
  \[
  a_w(n) = \prod_{j=1}^{l} \exp \left( \frac{1}{2} \log(1 + |z_j|^2) \tilde{H}_{\beta_j} \right)
  \]

- (Theorem 4.3) Kostant’s harmonic form \( s^w \) restricted to \( \Sigma_w \):
  \[
  s^w|_{\Sigma_w} = \prod_{j=1}^{l} \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2) \ll \alpha_j, \alpha_j \gg -1 dz_j \wedge d\bar{z}_j.
  \]

By comparing these explicit formulas, we immediately get (Corollaries 3.5 and 3.4)

\[
\phi_w = Ad_{\dot{w}} \log a_w(n)
\]
\[
\mu_w = \left( \prod_{j=1}^{l} \frac{\pi}{\ll \rho, \beta_j \gg} \right) a_w(n)^{-2\rho} dn.
\]
These relate the moment map $\phi_w$ and the Liouville volume form $\mu_w$ to the familiar map $a_w$ and the Haar measure $dn$ on $N_w$. This is desirable for understanding the geometry of the Bruhat-Poisson structure. The relation between the differential form $s^w|_{\Sigma_w}$ and the Liouville form $\mu_w$ also follows immediately from these formulas.

The coordinates $\{z_1, \bar{z}_1, z_2, \bar{z}_2, ..., z_l, \bar{z}_l\}$ are presented first in Section 2. Here we derive the formulas for the Haar measure $dn$ of $N_w$ and for the map $a_w : N \to A$. As one other application of these coordinates, we show that Harish-Chandra’s formula for the $c$-function (in the case of a complex group) follows easily as a product of 1-dimensional integrals. Our calculation here is easier because we have pushed the usual induction argument on the length of $w$ into the calculations for the explicit formulas for $a_w(n)$ and $dn$. The total amount of effort is probably the same.

In Section 3, we review some of the properties of the Bruhat-Poisson structure on $K/T$. We derive the formulas for the symplectic form $\Omega_w$, thus also the Liouville measure $\mu_w$, and the moment map for the $T$ action on each Bruhat cell in the $\{z_1, \bar{z}_1, z_2, \bar{z}_2, ..., z_l, \bar{z}_l\}$-coordinates. By using the formulas for $a_w(n)$ and $dn$ given in Section 2, we arrive at the (coordinate-free) interpretations for both the moment map $\phi_w$ and the Liouville measure $\mu_w$ in terms of $a_w(n)$ and $dn$ as given in Corollaries 3.5 and 3.6.

Kostant’s harmonic forms are reviewed in Section 4. Theorem 4.3 is given as an easy corollary of our formulas in the $z$-coordinates. Applications of Theorem 4.3 to the calculations of the Poisson (co)homology of the Bruhat-Poisson structure are given in [E-L].

In the Appendix, we discuss how our coordinates $\{z_1, \bar{z}_1, z_2, \bar{z}_2, ..., z_l, \bar{z}_l\}$ are related to the (complex) Bott-Samelson coordinates.

We now fix the notation.

Let $G$ be a finite dimensional complex semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let $H$ be a Cartan subgroup and $\mathfrak{h}$ its Lie subalgebra. Denote by $R$ the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $R^+$ be a choice of positive roots. We will also write $\alpha > 0$ for $\alpha \in R^+$. Let $\mathfrak{b} = \mathfrak{b}_+$ be the Borel subalgebra spanned by $\mathfrak{h}$ and all the positive root vectors, and let $B$ be the corresponding Borel subgroup.

Let $W$ be the Weyl group of $G$ relative to $H$. The Bruhat decomposition

\[ G = \bigcup_{w \in W} BwB, \]
where \( \dot{w} \) is a representative of \( w \) in the normalizer of \( H \) in \( G \), gives rise to the Bruhat decomposition

\[
G/B = \bigcup_{w \in W} B\dot{w}B/B
\]

of the flag manifold \( G/B \) into a disjoint union of cells. These cells, denoted by \( \Sigma_w = B\dot{w}B/B \), are called Bruhat or Schubert cells, and their closures, \( X_w = \overline{\Sigma_w} \), are called the Schubert varieties.

We choose a compact real form \( \mathfrak{k} \) of \( \mathfrak{g} \) as follows: let \( \langle \cdot, \cdot \rangle \) be the Killing form of \( \mathfrak{g} \). For each positive root \( \alpha \), denote by \( H_\alpha \) the image of \( \alpha \) under the isomorphism \( \mathfrak{h}^* \to \mathfrak{h} \) via \( \langle \cdot, \cdot \rangle \), i.e., for any \( H \in \mathfrak{h} \),

\[
\langle H_\alpha, H \rangle = \alpha(H).
\]

Choose root vectors \( E_\alpha \) and \( E_{-\alpha} \) for \( \alpha \) and \( -\alpha \) respectively such that \( \langle E_\alpha, E_{-\alpha} \rangle = 1 \). Then \( [E_\alpha, E_{-\alpha}] = H_\alpha \). Set

\[
X_\alpha = E_\alpha - E_{-\alpha}, \quad Y_\alpha = i(E_\alpha + E_{-\alpha}).
\]

(1)

The real subspace

\[
\mathfrak{k} = \text{span}_\mathbb{R}\{iH_\alpha, X_\alpha, Y_\alpha : \alpha > 0\}
\]

is a compact real form of \( \mathfrak{g} \). Let \( K \) be the corresponding compact subgroup of \( G \). The intersection \( T = K \cap B \) is a maximal torus of \( K \), and its Lie algebra is

\[
t = \text{span}_\mathbb{R}\{iH_\alpha : \alpha > 0\}.
\]

Let \( \mathfrak{a} = i\mathfrak{t} \), and let \( \mathfrak{n} = \mathfrak{n}_+ \) be the subalgebra of \( \mathfrak{g} \) spanned by all the positive root vectors. Let \( A \) and \( N \) be the corresponding subgroups. Then

\[
G = KAN
\]

is the Iwasawa decomposition of \( G \) as a real semi-simple Lie group. It follows that the projection map from \( K \subset G \) to \( G/B \) induces an isomorphism from \( K/T \) to \( G/B \). From now on, we will identify \( G/B \) with \( K/T \) this way.

For \( g \in G \), let \( g = k_g a_g n_g \) be the Iwasawa decomposition of \( g \). The map

\[
G \times K \to K : (g, k) \mapsto k_gk
\]

defines a left action of \( G \) on \( K \). We will denote this action by \( (g, k) \to g \circ k \).

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2 The coordinates \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \) on \( N_w \)

Let \( w \in W \) be a Weyl group element with length \( l = l(w) \). Set

\[
R^+_w = \{ \alpha > 0 : w^{-1} \alpha < 0 \}
\]

and

\[
n_w = \text{span}_C \{ E_\alpha : \alpha \in R^+_w \},
\]

and let \( N_w \) be the subgroup of \( G \) with Lie algebra \( n_w \). Then \( N_w = N \cap wN_-w^{-1} \), where \( N_- \) is the “opposite” of \( N \), and the map

\[
j_w : N_w \rightarrow \Sigma_w : n \mapsto nw/B
\]

is a holomorphic diffeomorphism. Thus, any complex coordinate system on \( N_w \) will give one on \( \Sigma_w \). The coordinates \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \) that we will introduce in this section, however, will not be complex for a general \( w \) with \( l(w) > 1 \) (see Example 2.9). To obtain these coordinates, we make use of the compact real form \( K \) of \( G \) and the isomorphism of \( G/B \) and \( K/T \). They are motivated by the Bruhat-Poisson structure on \( K/T \).

Let \( \dot{w} \) be a representative of \( w \) in \( K \). For \( n \in N_w \), let \( a_w(n) \) be the \( A \)-component of the element \( \dot{w}^{-1}n\dot{w} \) for the Iwasawa decomposition, i.e.,

\[
\dot{w}^{-1}n\dot{w} = k_1a_w(n)m_1
\]

for some \( k_1 \in K \) and \( m_1 \in N \). Notice that the map \( a_w : N_w \rightarrow A \) depends only on \( w \) and not on the choice of \( \dot{w} \).

In this section, we will write down the map \( a_w : N_w \rightarrow A \) explicitly in the coordinates \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \) (Theorem 2.5). We will also write down the Haar measure of \( N_w \) in these coordinates (Theorem 2.7). As an application, we show how Harish-Chandra’s formula for the \( c \)-function also follows easily as a product of 1-dimensional integrals.

We now describe the coordinates \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \).

Let again \( \dot{w} \) be a representative of \( w \) in \( K \). Let

\[
C_{\dot{w}} = N_w \circ \dot{w}
\]

be the \( N_w \)-orbit in \( K \) through \( \dot{w} \) of the action \( (n, k) \rightarrow n \circ k \). Recall that \( n \circ k = k_1 \) if \( nk = k_1b \) is the Iwasawa decomposition of \( nk \) with \( k_1 \in K \) and \( b \in AN \). The map

\[
J_{\dot{w}} : N_w \rightarrow C_{\dot{w}} : n \mapsto n \circ \dot{w}
\]
is then a diffeomorphism. Moreover, $J_w$ followed by the projection map from $C_w \subset K \subset G$ to $G/B$ is just the map $j_w$ in (3). Thus $C_w$ is a lift of $\Sigma_w$ in $K$.

**The case of** $l(w) = 1$. When $w = \sigma_\gamma$ is the simple reflection corresponding to a simple root $\gamma$, we use, for notational simplicity, $N_\gamma$ to denote $N_{\sigma_\gamma}$ and $\hat{\sigma}_\gamma$. In this case, set

$$\hat{H}_\gamma = \frac{2}{\langle \gamma, \gamma \rangle} H_\gamma, \quad \hat{E}_\gamma = \sqrt{\frac{2}{\langle \gamma, \gamma \rangle}} E_\gamma, \quad \hat{E}_{-\gamma} = \sqrt{\frac{2}{\langle \gamma, \gamma \rangle}} E_{-\gamma}. $$

Then the map

$$\psi_\gamma : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g} : \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \mapsto \hat{H}_\gamma, \quad \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \mapsto \hat{E}_\gamma, \quad \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \mapsto \hat{E}_{-\gamma} \quad (5)$$

is a Lie algebra homomorphism. It induces a Lie group homomorphism

$$\Psi_\gamma : SL(2, \mathbb{C}) \rightarrow G,$$

and $\Psi_\gamma(SU(2)) \subset K$. For $z \in \mathbb{C}$, let

$$n_z = \Psi_\gamma \left( \begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) = \exp(z\hat{E}_\gamma) \in N_\gamma.$$ 

The map

$$\mathbb{C} \rightarrow N_\gamma : z \mapsto n_z$$

is clearly a parametrization of $N_\gamma$ by $\mathbb{C}$. This is a complex coordinate system on $N_\gamma$.

Notice that the element $\left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)$ in $SU(2)$ is a representative of the non-trivial element in the Weyl group of $SL(2, \mathbb{C})$. Let

$$\hat{\gamma} = \Psi_\gamma \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \in K.$$ 

It is a representative of $\sigma_\gamma$ in $K$. The following Iwasawa decomposition in $SL(2, \mathbb{C})$

$$\left( \begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) = \left( \begin{array}{cc} iz \sqrt{1 + |z|^2} & i \sqrt{1 + |z|^2} \\ i \sqrt{1 + |z|^2} & -i\bar{z} \sqrt{1 + |z|^2} \end{array} \right) \left( \begin{array}{cc} \sqrt{1 + |z|^2} & \bar{z} \sqrt{1 + |z|^2} \\ 0 & \frac{1}{\sqrt{1 + |z|^2}} \end{array} \right)$$

gives the Iwasawa decomposition of $n_z\hat{\gamma}$ in $G$ as

$$n_z\hat{\gamma} = \Psi_\gamma \left( \begin{array}{cc} iz \sqrt{1 + |z|^2} & i \sqrt{1 + |z|^2} \\ i \sqrt{1 + |z|^2} & -i\bar{z} \sqrt{1 + |z|^2} \end{array} \right) \Psi_\gamma \left( \begin{array}{cc} \sqrt{1 + |z|^2} & \bar{z} \sqrt{1 + |z|^2} \\ 0 & \frac{1}{\sqrt{1 + |z|^2}} \end{array} \right).$$
Thus
\[ n_z \circ \dot{\gamma} = \Psi_{\gamma} \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & i \\ \frac{i}{\sqrt{1+|z|^2}} & -\frac{iz}{\sqrt{1+|z|^2}} \end{pmatrix} \in K. \]

The map\[ \mathbb{C} \rightarrow C_{\gamma} : z \mapsto n_z \circ \dot{\gamma} = \Psi_{\gamma} \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & i \\ \frac{i}{\sqrt{1+|z|^2}} & -\frac{iz}{\sqrt{1+|z|^2}} \end{pmatrix} \]
is a parametrization of \( C_{\gamma} \) by \{ z, \bar{z} \}. We also see that
\[ a_{\sigma_{\gamma}}(n_z) = \exp\left(\frac{1}{2} \log(1 + |z|^2) H_{\gamma}\right). \]

**The general case.** For a general element \( w \in W \), let \( l = l(w) \) be the length of \( w \), and let
\[ w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l} \] (6)
be a reduced decomposition, where \( \gamma_1, \gamma_2, \ldots, \gamma_l \) are simple roots. Again for notational simplicity, we use \( N_{\gamma_j} \) to denote \( N_{\sigma_{\gamma_j}} \) for \( j = 1, \ldots, l \). We now have the Lie group homomorphism
\[ \Psi_{\gamma_j} : SL(2, \mathbb{C}) \rightarrow G \]
for each \( j \). Let again
\[ \dot{\gamma}_j = \Psi_{\gamma_j} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in K. \]
Write an element in \( N_{\gamma_j} \) as
\[ n_{z_j} = \exp(z_j \hat{E}_{\gamma_j}) \]
for \( z_j \in \mathbb{C} \). Set
\[ \dot{w} = \gamma_1 \gamma_2 \cdots \gamma_l \in K. \]

**Theorem 2.1.** There is a diffeomorphism (between real manifolds)
\[ F_w : N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l} \rightarrow N_w \]
characterized by
\[ F_w(n_1, n_2, \ldots, n_l) \circ \dot{w} = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l) \in K. \] (7)

The map
\[ \mathbb{C}^l \rightarrow N_w : (z_1, z_2, \ldots, z_l) \mapsto F_w(n_{z_1}, n_{z_2}, \ldots, n_{z_l}) \] (8)
gives coordinates \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \) on \( N_w \) (as a real manifold).

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Remark 2.2 This is the same as saying that the map
\[ C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_2} \times \cdots \times C_{\dot{\gamma}_l} \longrightarrow C_{\dot{w}} \]
given by multiplication in \( K \) is a diffeomorphism. This statement is given in [S]. The proof we give below contains a recursive formula for \( F_w \) that will be used later.

Remark 2.3 We use \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \) instead of \( \{ z_1, z_2, \ldots, z_l \} \) to denote the coordinates to emphasize the fact that the map in (8) is in general not a holomorphic diffeomorphism. See Example 2.9.

Remark 2.4 Even though we use \( F_w \) to denote the map in Theorem 2.1, it depends not only on \( w \) but also on the reduced decomposition \( w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l} \) for \( w \). Therefore, the coordinates \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \) depend on the choice of the reduced decomposition.

**Proof of Theorem 2.1.** We need to show that for every point \( (n_1, \ldots, n_l) \in N_{\gamma_1} \times \cdots \times N_{\gamma_l} \), there exists (a necessarily unique) \( F(n_1, \ldots, n_l) \in N_w \) such that (7) is satisfied. We also need to show that each \( n \in N_w \) arises this way. We prove this by induction on \( l(w) \). When \( l(w) = 1 \), the map \( F_w \) is the identity map. Now for \( w \) with \( l(w) > 1 \) and with the reduced decomposition given in (8), set
\[ w_1 = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_{l-1}} \]
so that \( w = w_1 \sigma_{\gamma_l} \). We wish to relate \( F_w \) and \( F_{w_1} \). To this end, we first recall that \( N_{w_1} \subset N_w \) [9]. In fact, the multiplication map in \( N \) gives a diffeomorphism
\[ N_{w_1} \times \dot{w}_1 N_{\gamma_1} \dot{w}_1^{-1} \longrightarrow N_w, \]
where
\[ \dot{w}_1 = \dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1}. \]
Now given \( (n_1, n_2, \ldots, n_l) \in N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l} \), let
\[ n' = F_{w_1}(n_1, n_2, \ldots, n_{l-1}) \in N_{w_1} \]
be such that
\[ n' \circ (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1}) = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) \in K. \]

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We search for $x \in w_1 N_{\gamma_l} w_1^{-1}$ such that $n = n' x \in N_w$ satisfies
\begin{equation}
 n \circ (\gamma_1 \gamma_2 \cdots \gamma_l) = (n_1 \circ \gamma_1)(n_2 \circ \gamma_2) \cdots (n_{l-1} \circ \gamma_{l-1}) \in K. \tag{9}
\end{equation}
Now
\begin{align*}
n \gamma_1 \cdots \gamma_i &= n' x \gamma_1 \cdots \gamma_{i-1} \gamma_i \\
&= (n' \gamma_1 \cdots \gamma_{i-1}) (w_1^{-1} x w_1) \gamma_i \\
&= (n' \gamma_1 \cdots \gamma_{i-1}) x' \gamma_i,
\end{align*}
where
\begin{equation}
x' = w_1^{-1} x w_1 \in N_{\gamma_l}.
\end{equation}
In the notation we have introduced so far, we have
\begin{equation}
n' \gamma_1 \cdots \gamma_{i-1} = (n_1 \circ \gamma_1)(n_2 \circ \gamma_2) \cdots (n_{i-1} \circ \gamma_{i-1}) a_{w_1} (n') m'
\end{equation}
for some $m' \in N$. Thus
\begin{equation}
n \gamma_1 \cdots \gamma_i = (n_1 \circ \gamma_1)(n_2 \circ \gamma_2) \cdots (n_{i-1} \circ \gamma_{i-1}) a_{w_1} (n') m' x' \gamma_i.
\end{equation}
Denote by $N_{\hat{\gamma}_i}$ the subgroup of $N$ with Lie algebra
\begin{equation}
n_{\hat{\gamma}_i} = \text{span}_C \{ E_\alpha : \alpha > 0, \alpha \neq \gamma_i \}.
\end{equation}
Then the multiplication map in $N$ induces a diffeomorphism $N_{\gamma_l} \times N_{\gamma_l} \to N$, so we have the decomposition $N = N_{\gamma_l} N_{\hat{\gamma}_l}$. Moreover, $\gamma_l^{-1} N_{\hat{\gamma}_l} \gamma_l = N_{\hat{\gamma}_l}$. Thus, if we take $x'$ to be the element $n' w_{l'} \in N_{\gamma_l}$ in the decomposition
\begin{equation}
(m')^{-1} a_{w_1} (n')^{-1} n \alpha a_{w_1} (n') = n n' \in N_{\gamma_l} N_{\gamma_l}, \tag{11}
\end{equation}
then
\begin{equation}
m' x' = a_{w_1} (n')^{-1} n \alpha a_{w_1} (n') (n' \hat{\gamma}_l)^{-1},
\end{equation}
and
\begin{align*}
n \gamma_1 \cdots \gamma_i &= (n_1 \circ \gamma_1)(n_2 \circ \gamma_2) \cdots (n_{i-1} \circ \gamma_{i-1}) n_i a_{w_1} (n') (n' \hat{\gamma}_l)^{-1} \gamma_i \\
&= (n_1 \circ \gamma_1)(n_2 \circ \gamma_2) \cdots (n_{i-1} \circ \gamma_{i-1}) n_i \gamma_i (\gamma_l^{-1} a_{w_1} (n') \gamma_l) (\gamma_l^{-1} (n' \hat{\gamma}_l)^{-1} \gamma_l).
\end{align*}
Set
\begin{equation}
m'' = \gamma_l^{-1} (n' \hat{\gamma}_l)^{-1} \gamma_l \in N_{\hat{\gamma}_l}
\end{equation}
and let
\[ n_i \gamma_i = (n_i \circ \gamma_i) a_i m_i \]
be the Iwasawa decomposition for \( n_i \gamma_i \), so \( a_i \in A \) and \( m_i \in N_{\gamma_i} \). Then
\[ n\gamma_1 \cdots \gamma_l = (n_1 \circ \gamma_1)(n_2 \circ \gamma_2) \cdots (n_l \circ \gamma_l) a_l m_l (\gamma_l^{-1} a_{w_1}(n') \gamma_l) m''', \]
or
\[ n\gamma_1 \cdots \gamma_l = (n_1 \circ \gamma_1)(n_2 \circ \gamma_2) \cdots (n_l \circ \gamma_l) a_l (\gamma_l^{-1} a_{w_1}(n') \gamma_l) m''', \]  
(12)
where
\[ m''' = (\gamma_l^{-1} a_{w_1}(n') \gamma_l)^{-1} m_l (\gamma_l^{-1} a_{w_1}(n') \gamma_l) m'' \in N_{\gamma_l} N_{\gamma_i} = N. \]

Therefore, with this choice of \( x' \in N_{\gamma_i} \), the element \( n = n' x = n' \dot{w}_1 x' \dot{w}_1^{-1} \in N_w \) satisfies (11). Notice that since \( N_{\gamma_i} \) normalizes \( N_{\gamma_i} \), we can first decompose \( (m')^{-1} \in N \) with respect to the decomposition \( N = N_{\gamma_i} N_{\gamma_i} \) to get
\[ (m')^{-1} = m''' m'' \]  
(13)
with \( m'' \in N_{\gamma_i} \) and \( m''' \in N_{\gamma_i} \). Then
\[ x' = n^\gamma_i = m'' a_{w_1}(n')^{-1} n_i a_{w_1}(n') \in N_{\gamma_i}. \]  
(14)

To summarize, we set
\[ f : N_{w_1} \times N_{\gamma_i} \longrightarrow N_w : (n', n_i) \longmapsto n' \dot{w}_1 (m'' a_{w_1}(n')^{-1} n_i a_{w_1}(n')) \dot{w}_1^{-1} \in N_w, \]  
(15)
and define
\[ F_w(n_1, \ldots, n_l) = n = n' \dot{w}_1 (m'' a_{w_1}(n')^{-1} n_i a_{w_1}(n')) \dot{w}_1^{-1} \in N_w. \]  
(16)

Then \( F_w : N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l} \to N_w \) is a well-defined map and
\[ F_w = f \circ (F_{w_1} \times \text{id}). \]

It is also clear now that \( F_w \) is a diffeomorphism.

Q.E.D.

Formula (12) also gives the following recursive formula for \( a_{w_1}(n) \):
\[ a_{w_1}(n) = a_l \gamma_l^{-1} a_{w_1}(n') \gamma_l, \]  
(17)
where \( n_t \dot{\gamma}_l = (n_t \circ \dot{\gamma}_l) \alpha_i m_i \) is the Iwasawa decomposition for \( n_t \dot{\gamma}_l \). Now for each \( j = 1, \ldots, l \), let

\[
n_j \dot{\gamma}_j = (n_j \circ \dot{\gamma}_j) \alpha_j m_j
\]

be the Iwasawa decomposition. We get from (17) that

\[
a_w(n) = a_l(\dot{\gamma}_l - 1) a_l^{-1} \dot{\gamma}_l (\dot{\gamma}_l - 1) a_l^{-2} \dot{\gamma}_l - 1 \cdot \cdot \cdot (\dot{\gamma}_l - 1) a_l^{-l} \dot{\gamma}_l - 1 \dot{\gamma}_l - 1 \dot{\gamma}_l - 1 \cdot \cdot \cdot (\dot{\gamma}_l - 1) a_l^{-1} \dot{\gamma}_l)
\]

or, since \( A \) is commutative,

\[
\dot{w}a_w(n)\dot{w}^{-1} = \prod_{j=1}^{l} \sigma_{\gamma_1} \sigma_{\gamma_2} \cdot \cdot \cdot \sigma_{\gamma_{j-1}} \sigma_{\gamma_j}(a_j).
\]

We know that in the \( \{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\} \) coordinates, \( a_j \) is given by

\[
a_j = \exp(\frac{1}{2}(1 + |z_j|^2)H_{\gamma_j})
\]

for \( j = 1, \ldots, l \). Thus

\[
\sigma_{\gamma_1} \sigma_{\gamma_2} \cdot \cdot \cdot \sigma_{\gamma_{j-1}} \sigma_{\gamma_j}(a_j) = \sigma_{\gamma_2} \cdot \cdot \cdot \sigma_{\gamma_{j-1}} \exp(-\frac{1}{2} \log(1 + |z_j|^2)H_{\gamma_j})
\]

\[
= \exp(-\frac{1}{2} \log(1 + |z_j|^2)H_{\alpha_j}),
\]

where, for each \( j = 1, \ldots, l \),

\[
\alpha_j = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdot \cdot \cdot \sigma_{\gamma_{j-1}}(\gamma_j).
\]

Recall that

\[
\{\alpha_1, \alpha_2, \ldots, \alpha_l\} = R_w^+ = \{\alpha > 0 : \alpha^{-1} \alpha < 0\}.
\]

Thus we have

\[
\dot{w}a_w(n)\dot{w}^{-1} = \prod_{j=1}^{l} \exp(-\frac{1}{2} \log(1 + |z_j|^2)H_{\alpha_j}).
\]

Let

\[
\beta_j = -\alpha_j = -\sigma_i \sigma_{i-1} \cdot \cdot \cdot \sigma_j(\gamma_j) = \sigma_i \sigma_{i-1} \cdot \cdot \cdot \sigma_{j-1}(\gamma_j),
\]

i.e.,

\[
\beta_1 = \sigma_{\gamma_1} \sigma_{\gamma_{i-1}} \cdot \cdot \cdot \sigma_{\gamma_2}(\gamma_1)
\]

\[
\beta_2 = \sigma_{\gamma_1} \sigma_{\gamma_{i-1}} \cdot \cdot \cdot \sigma_{\gamma_3}(\gamma_2)
\]

\[
\vdots
\]

\[
\beta_l = \gamma_l.
\]
We then know that
\[ \{\beta_1, \beta_2, \ldots, \beta_l\} = R^+_w - 1 = \{\beta > 0 : w\beta < 0\}. \]
We also know that \( \ll \beta_j, \beta \gg = \ll \alpha_j, \alpha \gg = \ll \gamma_j, \gamma \gg \) for each \( j = 1, \ldots, l \). This fact will be used later.

The following theorem now follows immediately from (20).

**Theorem 2.5** In the \( \{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}\)-coordinates, the function \( a_w \) on \( N_w \) defined by (4) is explicitly given by
\[
a_w(n) = \prod_{j=1}^{l} \exp\left(\frac{1}{2} \log(1 + |z_j|^2) H_{\beta_j}\right),
\]
where the \( \beta_j \)'s are given by (21).

**Remark 2.6** This formula for \( a_w(n) \) also follows from a product formula found by Doug Pickrell in [P].

We now look at the left invariant Haar measure on \( N_w \).

**Theorem 2.7** In the \( \{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}\)-coordinates, a left invariant (and thus also bi-invariant) Haar measure on \( N_w \) is given as
\[
dn = \lambda_w \prod_{j=1}^{l} \frac{2 \ll \rho, \beta_j \gg}{\ll \beta_j, \beta_j \gg} (1 + |z_j|^2) \ll \beta_j, \beta_j \gg^{-1} dz_j \wedge d\bar{z}_j,
\]
where
\[
\lambda_w = \frac{1}{(-2\pi i)^l} \prod_{j=1}^{l} i \ll \rho, \beta_j \gg \ll \beta_j, \beta_j \gg = \prod_{j=1}^{l} \frac{i \ll \rho, \beta_j \gg}{\pi \ll \beta_j, \beta_j \gg}
\]
is such that
\[
\int_{N_w} a_w(n)^{-4\rho} dn = 1.
\]

**Proof.** Again we prove by induction on \( l(w) \). When \( l(w) = 1 \) so \( w = \sigma_\gamma \) for a simple root \( \gamma \), we have
\[
dn_\gamma = -\frac{1}{2\pi i} dz_1 \wedge d\bar{z}_1.
\]
Now for $w$ with $l(w) = l > 1$, we use the same notation as that in the proof of Theorem 2.1. Let

$$N_{\alpha_l} = \dot{w}_1 N_{\gamma_l} \dot{w}_1^{-1}$$

be the subgroup of $N$ with Lie algebra $\mathbb{C}E_{\alpha_l}$, where $\alpha_l = w_1(\gamma_l)$. Then the multiplication map

$$\mu : N_{w_1} \times N_{\alpha_l} \to N : (n', n_{\alpha_l}) \mapsto n'n_{\alpha_l}$$

is a diffeomorphism. Since $N$ is unipotent, we have, under the map $\mu$,

$$dn = \lambda \, dn' \, dn_{\alpha_l},$$

where $\lambda$ is a constant to be determined later, and we take

$$dn_{\alpha_l} = du \wedge d\bar{u}$$

if $N_{\alpha_l}$ is parametrized by $\{u, \bar{u}\}$ via $n_{\alpha_l} = \exp(u E_{\alpha_l})$.

Consider now the parametrization of $N_w$ by $C^l$ via $F_w$. Write $n = F_w(z_1, \bar{z}_1, ..., z_l, \bar{z}_l)$ if

$$n = F_w(n_1, ..., n_l) \quad \text{where} \quad n_j = \exp(z_j \bar{E}_{\gamma_j}) \quad \text{for each} \quad j.$$

Let again

$$n' = F_{w_1}(z_1, \bar{z}_1, ..., z_{l-1}, \bar{z}_{l-1}) \in N_{w_1}.$$

Recall that the element $m' \in N$ is given in (10) and that $m_{\gamma_l} \in N_{\gamma_l}$ is given in (13). Write

$$m_{\gamma_l} = \exp(m(z_1, \bar{z}_1, ..., z_{l-1}, \bar{z}_{l-1}) \bar{E}_{\gamma_l}).$$

Then we know from (16) that

$$n = n' \exp \left( (m(z_1, \bar{z}_1, ..., z_{l-1}, \bar{z}_{l-1}) + a_{w_1}(n')^{-\gamma_l} z_l) v_{\alpha_l} \right),$$

where $v_{\gamma_l} \in \mathbb{C}$ is such that $Ad_{\dot{w}_l} \bar{E}_{\gamma_l} = v_{l} E_{\gamma_l}$. Set

$$u = v_l(m(z_1, \bar{z}_1, ..., z_{l-1}, \bar{z}_{l-1}) + a_{w_1}(n')^{-\gamma_l} z_l).$$

Assume that $dn'$ is given as in the theorem for $w_1$. This means, noting the definition of the $\beta_j$'s, that

$$dn' = \lambda_{w_1} \prod_{j=1}^{l-1} \left( 1 + |z_j|^2 \right)^{2 \ll \rho} \ll \sigma_{\gamma_l} \beta_j \gg -1 dz_j \wedge d\bar{z}_j \quad \text{and}$$

$$= \lambda_{w_1} \prod_{j=1}^{l-1} \left( 1 + |z_j|^2 \right)^{2 \ll \gamma_l, \beta_j \gg (1 + |z_j|^2)^{2 \ll \beta_j \gg -1 dz_j \wedge d\bar{z}_j.}$$
Here we have just used the fact that $\rho - \sigma_{\gamma_l} = \gamma_l$. On the other hand, by Theorem 2.5, we have

$$a_{w_1}(n') = \prod_{j=1}^{l-1} \exp\left(\frac{1}{2} \log(1 + |z_j|^2) \hat{H}_{\sigma_{\gamma_l}(\beta_j)}\right),$$

so

$$a_{w_1}(n')^{-2\gamma_l} = \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\langle -2\gamma_l, \sigma_{\gamma_l} \beta_j \rangle} \langle \sigma_{\gamma_l} \beta_j, \sigma_{\gamma_l} \beta_j \rangle = \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\langle \beta_j, \beta_j \rangle}. $$

Therefore,

$$dn = \lambda \, dn' \, d_{\alpha_1} = \lambda' \, a_{w_1}(n')^{-2\gamma_l} \, dn' \, (dz_i \wedge d\bar{z}_i)$$

$$= \lambda' \lambda_{w_1} \left( \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\langle \beta_j, \beta_j \rangle} - 1 \right) \, dz_j \wedge d\bar{z}_j \wedge (dz_i \wedge d\bar{z}_i),$$

where $\lambda' = \lambda |v_l|^2$ is a new constant to be determined later. Since $\beta_l = \gamma_l$ is a simple root, we have

$$\langle 2 \ll \rho, \gamma_l \rangle = 1.$$

Thus

$$dn = \lambda' \lambda_{w_1} \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\langle \beta_j, \beta_j \rangle} - 1 \, dz_j \wedge d\bar{z}_j.$$

Using Theorem 2.5, we see that the integral

$$\int_N a_w^{-4\rho} \, dn$$

is now a product of 1-dimensional ones and is easily calculated. The constant $\lambda_w = \lambda' \lambda_{w_1}$ must be given by (24) for the above integral to be equal to 1.

Q.E.D.

**Example 2.8** We recall that for the complex group $G$ considered as a real Lie group, the $c$-function for a Weyl group element $w$ is defined to be (see [1], Chapter IV, §6)

$$c_w(\lambda) = \int_{N_{\bar{w}}} a(\bar{n})^{-i(\lambda + 2\rho)} \, d\bar{n},$$

where $N_{\bar{w}} \subset N_-$ is the “opposite” of $N_w$ and $a(\bar{n})$ is the $A$-component in the Iwasawa decomposition of $\bar{n} \in \bar{N}_w$. In our notation, we have

$$c_{w^{-1}}(\lambda) = \int_{N_w} a_w^{-i(\lambda + 2\rho)} \, dn.$$
Using our formulas for \( a_w(n) \) and for \( dn \) in the \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l \} \) coordinates, one immediately reduces the integral to a product of 1-dimensional ones and gets

\[
c_{w^{-1}}(\lambda) = \lambda_w \prod_{j=1}^{l(w)} \int_{\mathbb{R}^2} \frac{i\lambda + 2\rho, \beta_j}{\beta_j, \beta_j} + \frac{2}{\beta_j, \beta_j} - 1 \, dz_j \wedge d\bar{z}_j = \lambda_w \prod_{j=1}^{l(w)} \int_{\mathbb{R}^2} \frac{i\lambda, \beta_j}{\beta_j, \beta_j} - 1 (-2i)dx_j dy_j
\]

\[
= \lambda_w \prod_{j=1}^{l(w)} (-2\pi i) \int_{0}^{\infty} (1 + r_j^2) - \frac{i\lambda, \beta_j}{\beta_j, \beta_j} - 1 \, 2r_j dr_j
\]

\[
= (-2\pi i)^{l(w)} \lambda_w \prod_{j=1}^{l(w)} \frac{i\lambda, \beta_j}{\beta_j, \beta_j} \quad \text{if } \text{Re} \, i\lambda, \beta_j > 0 \quad \text{for each } j
\]

This is the well-known formula of Harish-Chandra (see Theorem 5.7 in [H]). As we have mentioned in the Introduction, our calculation here is easier because we have pushed the induction argument that is normally used in the calculations for the \( c \)-functions into the calculations for \( a_w(n) \) and \( dn \).

**Example 2.9** Consider the example of \( g = sl(3, \mathbb{C}) \). We take \( w \) be to the longest Weyl group element \( w_0 = (1, 2)(2, 3)(1, 2) \). In this case, parametrize \( N_{w_0} = N \) by complex coordinates

\[
(u_1, u_2, u_3) \mapsto n = \begin{pmatrix} 1 & u_1 & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We have

\[
\hat{\gamma}_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{\gamma}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{\gamma}_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

so

\[
\hat{w} = \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]
The Iwasawa decomposition of $n\dot{w}$ in $SL(3, \mathbb{C})$ is

$$n\dot{w} = \begin{pmatrix} -u_3 & -u_1 & -1 \\ -u_2 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-u_3}{\Delta_3} & \frac{-u_1(1 + |u_2|^2)}{\Delta_2\Delta_3} - \frac{1}{\Delta_2} & \frac{-u_3}{\Delta_3} \\ \frac{-u_2}{\Delta_3} & \frac{1 + |u_2|^2 - u_1u_2\bar{u}_3}{\Delta_2\Delta_3} & \frac{-\bar{u}_1}{\Delta_2} \\ \frac{-1}{\Delta_3} & \frac{\bar{u}_2 + u_1\bar{u}_3}{\Delta_2\Delta_3} & \frac{\bar{u}_3 - \bar{u}_1\bar{u}_2}{\Delta_2} \end{pmatrix} \begin{pmatrix} \Delta_3 & \bar{u}_2 + u_1\bar{u}_3 & \bar{u}_3 \\ \bar{u}_1 & \frac{\Delta_3}{\Delta_2} & \frac{\bar{u}_1(1 + |u_2|^2) - u_2\bar{u}_3}{\Delta_2\Delta_3} \\ 0 & \Delta_2 & \frac{\bar{u}_1(1 + |u_2|^2) - u_2\bar{u}_3}{\Delta_2\Delta_3} \end{pmatrix},$$

where

$$\Delta_1 = \sqrt{1 + |u_1|^2}, \quad \Delta_2 = \sqrt{1 + |u_1|^2 + |u_1u_2 - u_3|^2}, \quad \Delta_3 = \sqrt{1 + |u_2|^2 + |u_3|^2}.$$

On the other hand, for $z_1, z_2$ and $z_3$ in $\mathbb{C}$, let

$$n_1 = \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 1 & z_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$(n_1 \circ \gamma_1)(n_2 \circ \gamma_2)(n_3 \circ \gamma_3) = \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3} \begin{pmatrix} iz_1 & i & 0 \\ i & -i\bar{z}_1 & 0 \\ 0 & 0 & \varepsilon_1 \end{pmatrix} \begin{pmatrix} \varepsilon_2 & 0 & 0 \\ 0 & iz_2 & i \\ 0 & i & -i\bar{z}_2 \end{pmatrix} \begin{pmatrix} iz_3 & i & 0 \\ i & -i\bar{z}_3 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

$$= \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3} \begin{pmatrix} -\varepsilon_2z_1z_3 - iz_2 & -\varepsilon_2z_1 + iz_2\bar{z}_3 & -\varepsilon_3 \\ -\varepsilon_2\bar{z}_1z_2 + i\bar{z}_2 & -\varepsilon_2 - i\bar{z}_1z_2\bar{z}_3 & \bar{z}_1\varepsilon_3 \\ -\varepsilon_1 & \varepsilon_1\bar{z}_3 & -i\varepsilon_1\bar{z}_2\bar{z}_3 \end{pmatrix},$$

where $\varepsilon_j = \sqrt{1 + |z_j|^2}$ for $j = 1, 2, 3$. By setting

$$n \circ \dot{w} = (n_1 \circ \gamma_1)(n_2 \circ \gamma_2)(n_3 \circ \gamma_3),$$

we get the following coordinate change between our coordinates $\{(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3)\}$ and the $u$'s:

$$u_1 = z_1, \quad u_2 = \frac{\varepsilon_2z_3 - i\bar{z}_1z_2}{\varepsilon_1}, \quad u_3 = \frac{\varepsilon_2z_1z_3 + iz_2}{\varepsilon_1},$$

or

$$z_1 = u_1, \quad z_2 = \frac{u_1u_2 - u_3}{\Delta_1}, \quad z_3 = \frac{\bar{u}_1u_3 + u_2}{\Delta_2}.$$
Notice that this is not a holomorphic change of coordinates. Thus the \( \{ z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3 \} \) coordinates are not complex. Now in the \( u \)-coordinates, we have

\[
\begin{pmatrix}
\Delta_3 & 0 & 0 \\
0 & \Delta_2 & 0 \\
0 & 0 & \frac{1}{\Delta_2}
\end{pmatrix}
\]

Under the coordinate change, we have

\[
\Delta_1 = \varepsilon_1, \quad \Delta_2 = \varepsilon_1 \varepsilon_2, \quad \Delta_3 = \varepsilon_2 \varepsilon_3.
\]

Thus we get \( a_w(n) \) in the \( \{(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3) \} \) coordinates as:

\[
\begin{pmatrix}
\varepsilon_2 \varepsilon_3 & 0 & 0 \\
0 & \varepsilon_1 & 0 \\
0 & 0 & \frac{1}{\varepsilon_1 \varepsilon_2}
\end{pmatrix}
\]

This is the same as what one would obtain from Theorem 2.5. Similarly, the left invariant Haar measure \( dn \) is, up to a constant multiple, given in the \( u \)-coordinates by

\[
dn = du_1 \wedge d\bar{u}_1 \wedge du_2 \wedge d\bar{u}_2 \wedge du_3 \wedge d\bar{u}_3.
\]

After the change of coordinates, we get

\[
dn = \varepsilon_2^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3.
\]

Again, this is the same as what one would obtain from Theorem 2.7.

The following proposition will be used in Section 4.

**Proposition 2.10** We have

\[
F_w(e, ..., e, n_j, e, ..., e) = (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1}) n_j (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{j-1})^{-1} \in N_{\alpha_j}
\]

for \( j = 1, ..., l \) and \( n_j \in N_j \), and

\[
(F_w)_*(0) \left( \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} \wedge \cdots \wedge \frac{\partial}{\partial z_l} \wedge \frac{\partial}{\partial \bar{z}_l} \right) = \left( \prod_{j=1}^{l} i_{\alpha_j} \right) E_{\alpha_1} \wedge i E_{\alpha_1} \wedge \cdots \wedge E_{\alpha_l} \wedge i E_{\alpha_l},
\]

where \((F_w)_*(0)\) is the differential of \( F_w \) at 0.
Proof. Let $n = (\gamma_1 \gamma_2 \cdots \gamma_{j-1}) n_j (\gamma_1 \gamma_2 \cdots \gamma_{j-1})^{-1}$. Let

\[ n_j \gamma_j = (n_j \circ \gamma_j) a_j m_j \]

be the Iwasawa decomposition of $n_j \gamma_j$, where $m_j \in N_{\gamma_j}$. Then

\[ n_j \gamma_j \gamma_j + \cdots \gamma_i = (n_j \circ \gamma_j)(\gamma_j + \cdots \gamma_i)^{-1} a_j (\gamma_j + \cdots \gamma_i)^{-1} m_j (\gamma_j + \cdots \gamma_i) \]

Set

\[ a'_j = (\gamma_j + \cdots \gamma_i)^{-1} a_j (\gamma_j + \cdots \gamma_i) \in A \]

\[ m'_j = (\gamma_j + \cdots \gamma_i)^{-1} m_j (\gamma_j + \cdots \gamma_i) \in N. \]

Then

\[ n \gamma_1 \cdots \gamma_i = \gamma_1 \cdots \gamma_{j-1} (n_j \circ \gamma_j) (\gamma_j + \cdots \gamma_i) a'_j m'_j. \]

Thus

\[ n \circ (\gamma_1 \cdots \gamma_i) = \gamma_1 \cdots \gamma_{j-1} (n_j \circ \gamma_j) \gamma_j + \cdots \gamma_i \]

Hence

\[ F_w(e, ..., e, n_j, e, ..., e) = n. \]

Write $n_j = \exp(z_j \bar{E}_{\gamma_j})$. Then

\[ (\gamma_1 \gamma_2 \cdots \gamma_{j-1}) n_j (\gamma_1 \gamma_2 \cdots \gamma_{j-1})^{-1} = \exp(z_j \text{Ad}_{\gamma_1 \gamma_2 \cdots \gamma_{j-1}}(\bar{E}_{\gamma_j})) \]

\[ = \exp(\sqrt{\frac{2}{\langle \gamma_j, \gamma_j \rangle}} z_j \text{Ad}_{\gamma_1 \gamma_2 \cdots \gamma_{j-1}}(E_{\gamma_j})). \]

Since $\sigma_1 \sigma_2 \cdots \sigma_{j-1}(\gamma_j) = \alpha_j$ by definition, we have

\[ \text{Ad}_{\gamma_1 \gamma_2 \cdots \gamma_{j-1}}(E_{\gamma_j}) = c_j E_{\alpha_j} \]

for some complex number $c_j$. Write $z_j = x_j + iy_j$ and $c_j = u_j + iv_j$. Using the fact that $\langle \gamma_j, \gamma_j \rangle = \langle \alpha_j, \alpha_j \rangle$, we get

\[ (F_w)_*(0)(\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j}) = \frac{i}{2} (F_w)_*(0)(\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j}) \]

\[ = \frac{i}{\langle \alpha_j, \alpha_j \rangle} (u_j E_{\alpha_j} + v_j (i E_{\alpha_j})) \wedge (-v_j E_{\alpha_j} + u_j (i E_{\alpha_j})) \]

\[ = \frac{i}{\langle \alpha_j, \alpha_j \rangle} |c_j|^2 E_{\alpha_j} \wedge i E_{\alpha_j}. \]

Since both root vectors $E_{\gamma_j}$ and $E_{\alpha_j}$ have length 1 with respect to the $K$-invariant Hermitian form on $g$ induced by $\mathfrak{k}$, we have $|c_j|^2 = 1$. The statement about $(F_w)_*(0)$ now follows immediately from this.
We now look at the $T$-action on $N_w$ by conjugations:

$$T \times N_w \rightarrow N_w : (t, n) \mapsto t n t^{-1}.$$  

For a given $t \in T$, set

$$t_1 = t$$
$$t_2 = \tilde{\gamma}_1 t \tilde{\gamma}_1^{-1}$$
$$t_3 = \tilde{\gamma}_2 \tilde{\gamma}_1 t (\tilde{\gamma}_2 \tilde{\gamma}_1)^{-1}$$
$$\ldots$$
$$t_i = (\tilde{\gamma}_{i-1} \tilde{\gamma}_{i-2} \cdots \tilde{\gamma}_1) t (\tilde{\gamma}_{i-1} \tilde{\gamma}_{i-2} \cdots \tilde{\gamma}_1)^{-1}.$$  

Equip $N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l}$ with the $T$-action given by

$$t \cdot (n_1, n_2, \ldots, n_l) = (t_1n_1t_1^{-1}, t_2n_2t_2^{-1}, \ldots, t_ln_lt_l^{-1}). \quad (25)$$

**Proposition 2.11** 1) With respect to the $T$-actions on $N_w$ by conjugation and on $N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l}$ as given by (25), the map $F_w$ is $T$-equivariant;

2) In the $\{z_1, \tilde{z}_1, z_2, \tilde{z}_2, \ldots, z_l, \tilde{z}_l\}$ coordinates, the $T$-action on $N_w$ is given by

$$t \cdot (z_1, \tilde{z}_1, \ldots, z_l, \tilde{z}_l) = (t^{\alpha_1}z_1, t^{-\alpha_1}\tilde{z}_1, \ldots, t^{\alpha_l}z_l, t^{-\alpha_l}\tilde{z}_l), \quad (26)$$

where, recall, $\alpha_j = \sigma_{\gamma_1}\sigma_{\gamma_2} \cdots \sigma_{\gamma_{j-1}}(\tilde{\gamma}_j)$ for $1 \leq j \leq l$.

**Proof.** For each $j = 1, 2, \ldots, l$, we have

$$(t_jn_jt_j^{-1}) \circ \tilde{\gamma}_j = t_j(n_j \circ \tilde{\gamma}_j)(\tilde{\gamma}_j t_j^{-1} \tilde{\gamma}_j^{-1}) = t_j(n_j \circ \tilde{\gamma}_j)t_{j+1}^{-1},$$

where $t_{i+1}$ is defined to be equal to $(\tilde{\gamma}_i \tilde{\gamma}_{i-1} \cdots \tilde{\gamma}_1) t (\tilde{\gamma}_i \tilde{\gamma}_{i-1} \cdots \tilde{\gamma}_1)^{-1} = \tilde{w}^{-1}t\tilde{w}$. Thus,

$$\prod_{j=1}^{l} (t_jn_jt_j^{-1}) \circ \tilde{\gamma}_j = t \left( \prod_{j=1}^{l} (n_j \circ \tilde{\gamma}_j) \right) \tilde{w}^{-1}t\tilde{w}.$$  

On the other hand, let $n = F_w(n_1, \ldots, n_l) \in N_w$. We have

$$(t n t^{-1}) \circ \tilde{w} = t(n \circ \tilde{w})(\tilde{w}^{-1}t\tilde{w}).$$
It now follows from the definition of $F_w$ that

$$F_w(t_1n_1t_1^{-1}, \cdots t_in_i^{-1}) = t n t^{-1}.$$  

Write $t = \exp(iH) \in T$ for $H \in \mathfrak{a}$ and $n_j = \exp(z_j \tilde{E}_{\gamma_j}) \in N_{\gamma_j}$. Then (26) follows from the following calculation:

$$t_j n_j t_j^{-1} = \exp(z_j \ Ad_{t_j} \tilde{E}_{\gamma_j})$$
$$= \exp(z_j e^{i\sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_{j-1}}(\gamma_j)(H)} \tilde{E}_{\gamma_j})$$
$$= \exp(t^\alpha z_j \ E_{\gamma_j}).$$

Q.E.D.

3 The Bruhat-Poisson structure

Recall that a Poisson structure on a manifold $M$ ([W1]) is a bivector field $\pi$ on $M$ such that the bracket operation on the algebra $C^\infty(M)$ of smooth functions on $M$ defined by

$$\{\phi, \varphi\} = \pi(d\phi, d\varphi), \quad \phi, \varphi \in C^\infty(M)$$

satisfies Jacobi’s identity. The condition on $\pi$ is that $[\pi, \pi] = 0$, where $[ , ]$ denotes the Schouten bracket on the space of multivector fields on $M$ ([Ko]).

The bivector field $\pi$ can also be regarded as the bundle map

$$\tilde{\pi} : T^*M \longrightarrow TM : (\tilde{\pi}(\alpha), \beta) = \pi(\alpha, \beta).$$

When $\pi$ is of maximal rank ($M$ is then necessarily even dimensional), the bundle map $\tilde{\pi}$ is invertible, and the 2-form $\omega$ on $M$ defined by $\omega(x, y) = \pi(\tilde{\pi}^{-1}(x), \tilde{\pi}^{-1}(y))$ is closed and non-degenerate and is thus a symplectic 2-form. In general, the image of $\pi$ defines a (generally singular) involutive distribution on $M$. It has integrable submanifolds which inherit symplectic structures [W1]. They are called the symplectic leaves of $\pi$ in $P$. Therefore, symplectic manifolds are special cases of Poisson manifolds and every Poisson manifold is a disjoint union of symplectic manifolds.

The Bruhat-Poisson structure on $K/T$ comes from a Poisson structure $\pi$ on $K$ defined by

$$\pi = \Lambda^r - \Lambda^l,$$
where
\[ \Lambda = \frac{1}{2} \sum_{\alpha > 0} X_\alpha \wedge Y_\alpha \in \mathfrak{k} \wedge \mathfrak{k} \]
with \( X_\alpha \) and \( Y_\alpha \) given by (1), and \( \Lambda^r \) (resp. \( \Lambda^l \)) is the right (resp. left) invariant bi-vector field on \( K \) with value \( \Lambda \) at the identity element \( e \). We summarize some properties of \( \pi \) in the next theorem. For details, see [STS] [S] and [L-W].

**Theorem 3.1**

a) The bi-vector field \( \pi \) defines a Poisson structure on \( K \);

b) Equip \( K \times K \) with the product Poisson structure \( \pi \oplus \pi \). Then the multiplication map \( K \times K \to K : (k_1, k_2) \mapsto k_1k_2 \) is a Poisson map, making \((K, \pi)\) into a Poisson Lie group;

c) The symplectic leaves of \( \pi \) in \( K \) are precisely the orbits of AN in \( K \) for the action given by (2). These are the Bruhat cells in \( K \): for each Weyl group element \( w \in W \) with a fixed representative \( \dot{w} \) of \( w \) in \( K \), the symplectic leaf through \( \dot{w} \) is the \( N_w \)-orbit \( C_{\dot{w}} = N_w \cdot \dot{w} \) introduced in Section 2. For each \( t \in T \), the subset \( C_{\dot{w}}t \) is the symplectic leaf through the point \( \dot{w}t \). As \( w \) runs over \( W \) and \( t \) over \( T \), these are all the symplectic leaves in \( K \);

d) Both left and right translations by elements in \( T \) leave \( \pi \) invariant;

e) The image of \( \pi \) under the projection map \( K \to K/T \) is a well defined bi-vector field on \( K/T \) which we still denote by \( \pi \). It defines a Poisson structure on \( K/T \) called the Bruhat-Poisson structure;

f) Symplectic leaves of the Bruhat-Poisson structure on \( K/T \) are precisely the Bruhat cells \( \Sigma_w \), for \( w \in W \), in \( K/T \) (and thus the name);

g) With respect to the left translations by elements in \( K \), the Bruhat-Poisson structure is \( T \)-invariant but not \( K \)-invariant; The action map
\[ K \times K/T \to K/T : (k_1, k_2/T) \mapsto k_1k_2/T \]
is a Poisson map, making \((K/T, \pi)\) into a Poisson homogeneous \((K, \pi)\)-space.

Therefore, for each \( w \in W \) with a fixed representative \( \dot{w} \) in \( K \), both \( C_{\dot{w}} \) and \( \Sigma_w \) inherit symplectic structures as symplectic leaves in \( K \) and \( K/T \) respectively, and the projection from \( C_{\dot{w}} \) to \( \Sigma_w \) is a symplectic diffeomorphism. Moreover, the symplectic structure on \( \Sigma_w \) is invariant under the action of \( T \) by left translations. The goal of this section is to write down both the
symplectic structure - thus also the Liouville measure - on $\Sigma_w$ and the moment map for the $T$-action in the $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}$ coordinates. Here, we regard $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}$ as coordinates on $\Sigma_w$ via the parametrization of $\Sigma_w$ by $N_w$.

Property b) of $\pi$ on $K$ is called the multiplicativity of $\pi$. As we will see shortly, it is exactly this property that enables us to decompose, as symplectic manifolds, the higher dimensional $C_{\dot{w}}$’s into products of 2-dimensional ones. In fact, this is also the motivation for introducing the coordinates $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}$.

As in Section 2, let $l = l(w)$ and fix a reduced decomposition

$$w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l}.$$

For each $j = 1, \ldots, l$, let $\dot{\gamma}_j$ be a representative of $\sigma_{\gamma_j}$ in $K$, so $\dot{w} = \dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_l$ is a representative of $w$ in $K$. Property c) of $\pi$ on $K$ says that the symplectic leaf through $\dot{\gamma}_j$ is the 2-dimensional cell $C_{\dot{\gamma}_j}$. Theorem [2.1] (see Remark [2.2]) says that the map

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_1} \times \cdots \times C_{\dot{\gamma}_l} \rightarrow C_{\dot{w}}. \quad (27)$$

is a diffeomorphism.

**Proposition 3.2** The map in (27) is a symplectic diffeomorphism, where the left hand side has the product symplectic structure.

**Proof.** (Notice how the multiplicativity of $\pi$ is used in the proof.) The inclusion map

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_1} \times \cdots \times C_{\dot{\gamma}_l} \hookrightarrow K \times K \times \cdots \times K \quad (l\text{-copies})$$

is a Poisson map because each $C_{\dot{\gamma}_j}$ is a symplectic leaf and thus a Poisson submanifold of $K$. The multiplicativity of $\pi$ says that the multiplication map

$$K \times K \times \cdots \times K \rightarrow K : (k_1, k_2, \cdots, k_l) \mapsto k_1k_2\cdots k_l$$

is a Poisson map. Thus, composing the two, we see that

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_1} \times \cdots \times C_{\dot{\gamma}_l} \rightarrow K : (k_1, k_2, \cdots, k_l) \mapsto k_1k_2\cdots k_l$$

is a Poisson map. But it has its image in $C_{\dot{w}}$, which is a Poisson submanifold of $K$. Thus, regarded as a map to $C_{\dot{w}}$, the above is a Poisson map, and thus a Poisson and therefore a symplectic diffeomorphism.
It now remains to determine the symplectic structure on the 2-dimensional leaves. Recall that for each simple root $\gamma$, we have a Lie group homomorphism

$$\Phi_\gamma : SL(2, \mathbb{C}) \rightarrow G$$

which maps $SU(2)$ to $K$.

**Proposition 3.3** (See also [3]) For each simple root $\gamma$, equip $SU(2)$ with the Poisson structure given by

$$\pi_\gamma = \Lambda^r_\gamma - \Lambda^l_\gamma,$$

where

$$\Lambda_\gamma = \frac{1}{4} \langle \gamma, \gamma \rangle \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \wedge \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \in su(2) \wedge su(2),$$

and $\Lambda^r_\gamma$ (resp. $\Lambda^l_\gamma$) denotes the right (resp. left) invariant bi-vector field on $SU(2)$ with value $\Lambda_\gamma$ at the identity element. Then 1) $\Psi_\gamma : (SU(2), \pi_\gamma) \rightarrow (K, \pi)$ is a Poisson map, and 2) the symplectic leaf of $\pi_\gamma$ in $SU(2)$ through the point $\left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)$ is

$$C_0 = \left\{ \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \in C \right\}.$$

Using $(z, \bar{z})$ as coordinates on $C_0$, the induced symplectic structure is given by

$$\Omega = \frac{i}{\langle \gamma, \gamma \rangle} \frac{1}{1 + |z|^2} dz \wedge d\bar{z}.$$

**Proof.** 1) Think of $a + n$ as a real Lie algebra and identify $a + n$ with $\mathfrak{k}^*$ using the imaginary part of the Killing form as the pairing. The fact that $\Psi_\gamma$ is a Poisson map then follows from the fact that the subspace $\Psi_\gamma(su(2))^\perp$ of $a + n$ consisting of all elements that annihilate $\Psi_\gamma(su(2))$ is an ideal of $a + n$. See [L-W].

2) As a special case of Theorem 3.1 (up to a constant multiple), the symplectic leaves of $\pi_\gamma$ in $SU(2)$ are either 2 or 0-dimensional. We know from Section 2 that the set $C_0$ is the symplectic leaf through the point $\left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)$.

Write an element of $SU(2)$ as

$$u = \left( \begin{array}{cc} \xi & \eta \\ -\bar{\eta} & \bar{\xi} \end{array} \right).$$
The Poisson brackets defined by $\pi_\gamma$ are

$$\{\xi, \bar{\xi}\} = -i \ll \gamma, \gamma \gg |\eta|^2 \quad \{\xi, \eta\} = \frac{1}{2} i \ll \gamma, \gamma \gg \xi \eta$$

$$\{\xi, \bar{\eta}\} = \frac{1}{2} i \ll \gamma, \gamma \gg \xi \bar{\eta} \quad \{\eta, \bar{\eta}\} = 0.$$

Set

$$\xi = \frac{iz}{\sqrt{1 + |z|^2}}, \quad \eta = \frac{i}{\sqrt{1 + |z|^2}},$$

so

$$z = \xi/\eta, \quad \bar{z} = \bar{\xi}/\bar{\eta}.$$

Using the Leibniz rule for the Poisson bracket, we get

$$\{z, \bar{z}\} = \{\xi/\eta, \xi/\bar{\eta}\} = \frac{1}{|\eta|^2} \{\xi, \bar{\xi}\} - \frac{\bar{\xi}}{|\eta|^2} \{\xi, \bar{\eta}\} = -i \ll \gamma, \gamma \gg (1 + \frac{|\xi|^2}{|\eta|^2})$$

$$= -i \ll \gamma, \gamma \gg (1 + |z|^2).$$

Thus the symplectic 2-form $\Omega$ on $C_0$ is given as stated.

**Q.E.D.**

**Theorem 3.4** In the $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}$ coordinates, the induced symplectic structure on $\Sigma_w$ (as a symplectic leaf of the Bruhat-Poisson structure) and the Liouville volume form $\mu_w$ associated to $\Omega_w$ are given by

$$\Omega_w = \sum_{j=1}^l \ll \alpha_j, \alpha_j \gg \frac{i}{1 + |z_j|^2} dz_j \wedge d\bar{z}_j$$

$$\mu_w := \frac{1}{l!} (\Omega_w)^l = \prod_{j=1}^l \ll \alpha_j, \alpha_j \gg \frac{1}{1 + |z_j|^2} dz_j \wedge d\bar{z}_j.$$

The moment map for the $T$-action on $\Sigma_w$ by left translations satisfying $\phi_w(0) = 0$ is given by

$$\phi_w : \Sigma_w \to t^* \cong a : \phi_w = \sum_{j=1}^l (-\frac{1}{2} \log(1 + |z_j|^2) \bar{H}_{\alpha_j}).$$

Here we are identifying $t^*$ with $a$ by using the imaginary part of the Killing form as the pairing.
Proof. The formula for $\Omega_w$ follows immediately from Propositions 3.2 and 3.3 and the fact that $\ll \gamma_j, \gamma_j \gg = \ll \alpha_j, \alpha_j \gg$.

The formula for the moment $\phi_w$ follows from the explicit formula for the $T$-action on $\Sigma_w$ in the $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}$ coordinates as given in Proposition 2.11; let $iH \in t$, where $H \in a$. By Proposition 2.11, the generating vector field of the $T$ action on $\Sigma_w$ in the direction of $iH$ is given by

$$V_{iH} = \sum_{j=1}^l \alpha_j(H)(-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}).$$

Set

$$V_{iH} \Omega_w = d \langle \phi_w, iH \rangle.$$ 

We see that the only solution of $\phi_w$ that satisfies this equation for all $H \in a$ and the condition that $\phi_w(0) = 0$ is given by (30).

Q.E.D.

The following two corollaries follow immediately by comparing the formulas in Theorems 2.5, 2.7 and 3.4 (see also identity (20)).

Corollary 3.5 Think of the moment map $\phi_w$ as a map from $N_w$ to $a$ via the parametrization of $\Sigma_w$ by $N_w$. Then

$$\phi_w = Ad_{\dot{w}} \log a_w(n).$$

Corollary 3.6 Think of the Haar measure $dn$ of $N_w$ as a volume form on $\Sigma_w$ via the parametrization of $\Sigma_w$ by $N_w$. It is related to the Liouville measure $\mu_w$ by

$$\mu_w = \left( \prod_{j=1}^l \ll \rho, \beta_j \gg \right) a_w(n)^{-2\rho} dn.$$ 

We have thus connected the moment map $\phi_w$ and the Liouville measure $\mu_w$ with the familiar map $a_w : N_w \to A$ and the Haar measure $dn$ on $N_w$. Such a connection is desirable for understanding the geometry of the Bruhat-Poisson structure. We have arrived at this by comparing their formulas in coordinates. This is why we wanted to write down the formulas for $a_w$ and $dn$ in the $\{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_l, \bar{z}_l\}$ coordinates in Section 2.
4 Kostant’s theorem on $H(G/B)$

In [K], Kostant constructs, for each element $w$ in the Weyl group $W$, an explicit $K$-invariant closed differential form $s^w$ on $X$ with $\deg(s^w) = 2l(w)$, such that the cohomology classes of the $s^w$’s form a basis of $H(X, \mathbb{C})$ that, up to scalar multiples, is dual to the basis of the homology of $X$ formed by the closures of the Bruhat cells in $X$. We now recall the definition of these forms in more details.

Let $C = \oplus C^\bullet$ be the space of all $K$-invariant differential forms on $G/B \cong K/T$. By identifying $g$ and $g^*$ via the Killing form $\langle\ ,\ \rangle$ so that $(g/h)^* \cong n_- + n$, we can identify $C^\bullet \cong \wedge^\bullet(n_- + n)^T$.

Introduce the operators $E$ and $L_0$ on $C^\bullet$ by

$$E = 2 \sum_{\alpha > 0} \text{ad}_{E^{-\alpha}} \otimes \text{ad}_{E^{\alpha}}$$

and

$$L_0(E^{-\alpha_1} \wedge E_{\alpha_2} \wedge \cdots \wedge E^{-\alpha_p} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_q}) =
\begin{cases}
0 & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 = 0 \\
\frac{1}{\|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2} & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 \neq 0
\end{cases}$$

Set

$$R = -L_0E.$$

It is a nilpotent operator.

Now let $w \in W$ be a Weyl group element. As before, use $l$ to denote $l(w)$. Let

$$\{\alpha_1, \alpha_2, \ldots, \alpha_l\} = \{\alpha > 0 : w^{-1}\alpha < 0\},$$

and let

$$\beta_j = -w^{-1}\alpha_j$$

so that

$$\{\beta_1, \beta_2, \ldots, \beta_l\} = \{\beta > 0 : w\beta < 0\}.$$
Set
\[ h^{w-1} = \left( \frac{i}{2} \right)^l E_{-\beta_1} \wedge E_{-\beta_2} \wedge \cdots \wedge E_{-\beta_l} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_l}, \]
and
\[ s^w = (1 - R)^{-1} h^{w-1} = h^{w-1} + Rh^{w-1} + R^2 h^{w-1} + \cdots. \] (31)

**Theorem 4.1 (Kostant [K])**

1. The forms \( s^w \) for \( w \in W \) are closed, and their cohomology classes form a basis of the de Rham cohomology of \( G/B \cong K/T \) (with complex coefficients);

2. Let \( j_w : N_w \to \Sigma_w : n \mapsto nw/T \) be the parametrization map. Then
\[ j_w^*(s^w|\Sigma_w) = \begin{cases} 0 & \text{if } l(w_1) = l(w) \text{ but } w_1 \neq w \\ a_w(n)^{-2(\rho - w^{-1}\rho)}(dn)_1 & \text{if } w_1 = w, \end{cases} \]
where \( s^w|\Sigma_w = i_{w_1}^* s^w \) if \( i_{w_1} \) is the inclusion map of \( \Sigma_w \) into \( K/T \), the map \( a_w : N_w \to A \) is, as before, defined by (4), and \( (dn)_1 \) is the left invariant Haar measure on \( N_w \) normalized by the condition
\[ ((dn)_1, E_{\alpha_1} \wedge iE_{\alpha_1} \wedge E_{\alpha_2} \wedge iE_{\alpha_2} \wedge \cdots \wedge E_{\alpha_l} \wedge iE_{\alpha_l}) = 1. \]

3.
\[ \int_{\Sigma_w} s^w = \prod_{j=1}^l \pi \left\langle \rho, \alpha_j \right\rangle. \]

The purpose of this section is to relate the form \( s^w \) with the Liouville volume form \( \mu_w \) on \( \Sigma_w \) induced by the Bruhat-Poisson structure. This is now easy due to Corollaries 3.5 and 3.6. We will also give a simple proof of 3).

We first need to relate the left Haar measures \( (dn)_1 \) and \( dn \) as given in Theorem 2.7.

**Lemma 4.2** We have
\[ (dn)_1 = \prod_{j=1}^l \pi \left\langle \rho, \beta_j \right\rangle dn. \]

**Proof.** This follows from Proposition 2.10 and the fact that \( \left\langle \alpha_j, \alpha_j \right\rangle = \left\langle \beta_j, \beta_j \right\rangle \) for each \( j = 1, \ldots, l \).

Q.E.D.

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Theorem 4.3  When restricted to the Schubert cell $\Sigma_w$, Kostant’s harmonic form $s^w$ is related to the Liouville volume form $\mu_w$ on $\Sigma$ by

$$s^w|_{\Sigma_w} = (a_w)^{2w-1} \mu_w = e^{<\phi_w, 2iH_\rho>} \mu_w. \tag{32}$$

Proof. This is a direct consequence of Corollaries 3.5 and 3.6. Explicitly, we have

$$s^w|_{\Sigma_w} = \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2) - \frac{2}{\ll \alpha_j, \alpha_j \gg} - \frac{1}{\ll \alpha_j, \alpha_j \gg} dz_j \wedge d\bar{z}_j \tag{33}$$

and

$$<\phi_w, 2iH_\rho> = -\sum_{j=1}^l \frac{2}{\ll \alpha_j, \alpha_j \gg} \log(1 + |z_j|^2) \tag{34}$$

and

$$\mu_w = \prod_{j=1}^l \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2) - 1 dz_j \wedge d\bar{z}_j.$$

Thus we have (32).

Q.E.D.

From the explicit formula for $s^w|_{\Sigma_w}$, we immediately get

$$\int_{\Sigma_w} s^w = \prod_{j=1}^l \int_{\mathbb{R}^2} \frac{i}{\ll \alpha_j, \alpha_j \gg} (1 + |z_j|^2) - \frac{2}{\ll \alpha_j, \alpha_j \gg} - \frac{1}{\ll \alpha_j, \alpha_j \gg} dz_j \wedge d\bar{z}_j$$

$$= \prod_{j=1}^l \frac{\pi}{\ll \rho, \alpha_j \gg}.$$

This integral was first calculated in [K-K] using induction on $l(w)$. Again, as in the case of the c-functions, our simple proof is due to the fact that the induction argument has been pushed to the calculations for $a_w(n)$ and $dn$. This is in fact a special case of the c-function with $i\lambda = -2w^{-1}\rho$.

Remark 4.4 When $w = w_0$ is the longest Weyl group element, the form $s^{w_0}$ is a $K$-invariant volume form so it coincides with the Haar measure on $K/T$. When restricted to the biggest cell $\Sigma_{w_0}$, the Liouville volume form $\mu_{w_0}$, the form $s^{w_0}$, and the Haar measure $dn$ for $N_{w_0} = N$ are related by

$$s^{w_0}|_{\Sigma_{w_0}} = (a_{w_0})^{-2\rho} \mu_{w_0} = \left(\prod_{\alpha > 0} \frac{\pi}{\ll \rho, \alpha \gg}\right) a_{w_0}(n)^{-4\rho} dn.$$
Remark 4.5 The function \( \langle \phi_w, 2iH_0 \rangle \) on \( \Sigma_w \) is the Hamiltonian function for the generating vector field \( \theta_0 \) of the \( T \)-action in the direction of \( 2iH_0 \). This vector field is intrinsic to the Bruhat-Poisson structure in the sense that it is its modular vector field. As \( |z_j| \to +\infty \), the function \( \langle \phi_w, 2iH_0 \rangle \to -\infty \). Thus the modular vector field \( \theta_0 \) is not globally Hamiltonian on \( K/T \). We say that the Bruhat-Poisson structure is not unimodular. See [W2] [B-Z] [E-L-W].

Remark 4.6 The second identity in (32) expresses the form \( s^w|_{\Sigma_w} \) totally in terms of data coming from the Bruhat-Poisson structure. In particular, it says that the integral \( \int_{\Sigma_w} s^w \) is of the Duistermaat-Heckman type. Wanting to see this was another motivation for this work. Theorem 4.3 can be used to describe generators of the so-called Poisson - de Rham cohomology of the Bruhat-Poisson structure. We do this in [E-I].

5 Appendix: Relation to the Bott-Samelson coordinates

As before, let \( w \in W \) be a Weyl group element with a fixed reduced decomposition:

\[
w = \sigma_{\gamma_1} \sigma_{\gamma_2} \cdots \sigma_{\gamma_l}
\]

where \( l = l(w) \). Then we have the Lie group homomorphism \( \Psi_{\gamma_j} : SL(2, \mathbb{C}) \to G \) and the element

\[
\dot{\gamma}_j = \Psi_{\gamma_j} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in K
\]

for each \( j = 1, ..., l \). Set again \( \dot{\gamma}_w = \dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_l \in K \). Then the map

\[
F'_w : N_{\gamma_1} \times \cdots \times N_{\gamma_l} \to N_w :
\]

\[
(n_1, n_2, ... , n_l) \mapsto n_1 \dot{\gamma}_1 n_2 \dot{\gamma}_2 \cdots n_l \dot{\gamma}_l \dot{\gamma}_w^{-1} = n_1 (\dot{\gamma}_1 n_2 \dot{\gamma}_1^{-1}) \cdots (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1}) n_l (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_{l-1})^{-1}
\]

is a holomorphic diffeomorphism. Thus, if we parametrize \( N_{\gamma_j} \) by \( \mathbb{C} \) via

\[
z'_j \mapsto n(z'_j) := \Psi_{\gamma_j} \begin{pmatrix} 1 & z'_j \\ 0 & 1 \end{pmatrix} = \exp(z'_j E_{\gamma_j}) \in N_{\gamma_j}
\]

for each \( j \), we get a parametrization of \( N_w \) by \( \mathbb{C}^l \):

\[
\mathbb{C}^l \to N_w : (z'_1, z'_2, ... , z'_l) \mapsto F'_w(n(z'_1), n(z'_2), ... , n(z'_l)).
\]

We call it the Bott-Samelson parametrization of \( N_w \) and call \( \{z'_1, z'_2, ... , z'_l\} \) the Bott-Samelson coordinates on \( N_w \) because of the close relation to the Bott-Samelson desingularization of the Schubert variety \( X_w \) [I].
The change of coordinates between \( \{z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_i, \bar{z}_i\} \) and the Bott-Samelson coordinates \( \{z'_1, z'_2, \ldots, z'_i\} \) can be described as follows: by composing \( F_w \) (in Theorem 2.7) and \( F'_w \) with the map
\[
j_w : N_w \to \Sigma_w : n \mapsto nw/B,
\]
we can think of both \( F_w \) and \( F'_w \) as mapping \( N_{\gamma_1} \times \cdots \times N_{\gamma_1} \) diffeomorphically to \( \Sigma_w \). The map \( F'_w \) first sends \((n_1, n_2, \ldots, n_i)\) to the product of \( n_1\gamma_1 n_2\gamma_2 \cdots n_i\gamma_i \) in \( G \) and then projects the product to \( G/B \). But for \( F_w \), we first pick up the \( K \)-component \( k_j \) in the Iwasawa decomposition of \( n_j\gamma_j \) for each \( j \), multiply the \( k_j \)'s inside \( K \) and then project the product to \( K/T \cong G/B \).

Consider now the change of coordinates
\[
(z'_1, z'_2, \ldots, z'_i) = I_w(z_1, z_2, \ldots, z_i).
\]
We can get a recursive formula for \( I_w \) from the proof of Theorem 2.7. Clearly, when \( w \) is a simple reflection, the map \( I_w \) is the identity map. For a general \( w \) with the reduced decomposition \( w = \sigma_{\gamma_1}\sigma_{\gamma_2} \cdots \sigma_{\gamma_1} \), let again \( \omega_1 = \sigma_{\gamma_1}\sigma_{\gamma_2} \cdots \sigma_{\gamma_{I-1}} \) so that \( w = \omega_1\sigma_{\gamma_1} \). Keeping the same notation as in the proof of Theorem 2.7, we know that \( I_w \) is given by
\[
\begin{align*}
(z'_1, z'_2, \ldots, z'_{I-1}) &= I_{\omega_1}(z_1, z_2, \ldots, z_{I-1}) \\
z'_{I} &= m(z_1, z_2, \ldots, z_{I-1}) + a_{\omega_1}(n)^{-\gamma_i}z_i,
\end{align*}
\]
where, recall, \( m(z_1, z_2, \ldots, z_{I-1}) \in \mathbb{C} \) is such that \( \exp(m(z_1, z_2, \ldots, z_{I-1})E_{\gamma_i}) \) is the \( N_{\gamma_i} \)-component of \( (m')^{-1} \in N \) with respect to the decomposition \( N = N_{\gamma_i}N_{\gamma_i} \), and \( m' \in N \) is the \( N \)-component in the Iwasawa decomposition of \( nw_1 \). The change of coordinates is in general not complex because the function \( m(z_1, z_2, \ldots, z_{I-1}) \) is in general not holomorphic. This is also seen from the following example.

**Example 5.1** For \( G = SL(3, \mathbb{C}) \) and \( w = (1, 2)(2, 3)(1, 2) \), the Bott-Samelson parametrization of \( N_w \) is by
\[
(z'_1, z'_2, z'_3) \mapsto \begin{pmatrix}
1 & z'_1 & z'_1 z'_3 + iz'_2 \\
0 & 1 & z'_3 \\
0 & 0 & 1
\end{pmatrix}.
\]
The change of coordinates between \((z_1, z_2, z_3)\) and \(\{z'_1, z'_2, z'_3\}\) are (see Example 2.9)
\[
z'_1 = z_1, \quad z'_2 = \varepsilon_1 z_2 \quad z'_3 = \frac{\varepsilon_2 z_3 - i\bar{z}_1 z_2}{\varepsilon_1},
\]
or
\[
z_1 = z'_1, \quad z_2 = \frac{z'_2}{\eta_1} \quad z_3 = \frac{\eta_2 z'_3 + i\bar{z}_1 z'_2}{\eta_2},
\]
where \( \varepsilon_j = \sqrt{1 + |z_j|^2} \) for \( j = 1, 2 \) and \( \eta_1 = \sqrt{1 + |z'_1|^2} \), \( \eta_2 = \sqrt{1 + |z'_1|^2 + |z'_2|^2} \).
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