Stability for acoustic wave motion with random force on the locally reacting boundary

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Abstract
This paper concerns about the stability for the acoustic wave motion with boundary frictions and random forces. We will show there exists a unique invariant measure for the stochastic evolution equation associated with this acoustic wave motion, and the invariant measure possesses the property of strong mixing. This result is new with respect to the literature on two accounts: (i) stochasticity is accounted for in the acoustic wave model; (ii) the controllability of the dynamical system modeling the acoustic wave motion implies the mixing.

Keywords: Stochastic wave equations, Acoustic boundary conditions, Observability inequality, Invariant measure, Strong mixing

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1. Introduction
Let $D \subset \mathbb{R}^3$ be an open and bounded domain that is full of some kind of idealized fluid. The equations of acoustic wave motion in $D$ will be considered below

$$\rho \frac{\partial u}{\partial t} = -\nabla p, \quad \kappa \frac{\partial p}{\partial t} = -\nabla \cdot u,$$

where the vector $u(t,x)$ is the fluid velocity, $p(t,x)$ the acoustic pressure, $\rho$ the uniform density of the fluid and $\kappa$ the adiabatic compressibility, see in [1]. When the acoustic wave is incident on the smooth boundary of $D$, denoted by $\partial D$, the acoustic pressure $p$ tends to make the boundary move, and forces some fluid into the pores of the boundary. As for any fluid motion that is normal to the boundary, there will be wave motion in the material forming the boundary. Then on the boundary we have the following physical relationship which is called the acoustic boundary condition.

$$\begin{cases}
-p = \text{Vibrating motion of the boundary,} \\
\mathbf{n} \cdot \mathbf{u} = \text{Velocity of the vibrating boundary.}
\end{cases}$$

Here, $\mathbf{n}$ is the unit exterior normal vector of the boundary. And if the motion of the boundary is perturbed by a small random fluctuation, the acoustic wave motion in $D$ will be also subject to the effect of the small random perturbation. It is interesting and important to consider the random fields governed by the stochastic acoustic wave equation driven by a small random perturbation.

Since $u$ can be obtained by taking the gradient of a scalar function $\phi(t,x)$, $u = -\nabla \phi$, then $\phi$ is always called the velocity potential. And it implies from [1] that $p = \rho \frac{\partial \phi}{\partial t}$ and $\phi$ satisfies the following wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \Delta \phi, \quad \text{in } \mathbb{R}^+ \times D.$$

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Here, \( c = \sqrt{\frac{1}{\kappa \rho}} \) is the speed of wave propagation. Assume that a portion \( \Gamma_0 \) of boundary \( \partial D \) has a very low acoustic impedance compared to the acoustic impedance of the idealized flow. In this case the portion of boundary is called sound soft [2]. Then we have

\[
\phi = 0, \quad \text{on } \mathbb{R}^+ \times \Gamma_0.
\]  

(B1)

As for the other portion \( \Gamma_1 = \partial D - \Gamma_0 \), the motion of the portion \( \Gamma_1 \) at one point will be related to motion at another point on \( \Gamma_1 \), and the relationship depends on the motion inside the material. Let \( \delta(t,x) \) be the normal displacement into the domain of a point \( x \in \Gamma_1 \) at time \( t \). Suppose that the material forming the boundary is elastic. If the motion on the different parts of \( \Gamma_1 \) does not influence each other directly, each point on \( \Gamma_1 \) acts like a resistive harmonic oscillator, which can deduces from [2] the locally reacting boundary condition as follows.

\[
\begin{aligned}
-\rho \frac{\partial \phi}{\partial t} &= m \delta_{tt} + d \delta_t + k \delta + \xi(t), \\
\frac{\partial \phi}{\partial n} &= \delta_t,
\end{aligned}
\]  

(B2)

Here, \( m \) is the mass per unit area of \( \Gamma_1 \), \( d \) is the resistivity, \( k \) is the stiffness, and \( \xi \) is a random perturbation.

The determined form of (B2) has been obtained in [3]. Without boundary random force, that is \( \xi(t) = 0 \), the mechanical energy associated with the system \( \{1,2,3,4,5\} \) is defined as

\[
\mathcal{E}(t) = \frac{1}{2} \int_D \rho |c^{-2} \phi|^2 + |\nabla \phi|^2 \, dx + \frac{1}{2} \int_{\Gamma_1} [k|\delta|^2 + m|\delta_t|^2] \, d\Gamma.
\]

The first part of energy is generated by the motion of acoustic wave, and the second part generated by the boundary vibration. Thus, we have

\[
\frac{d}{dt} \mathcal{E}(t) = -d \int_{\Gamma_1} |\delta_t|^2 \, d\Gamma \leq 0,
\]

(4)

from which we can see that the energy dissipated by the boundary friction. The author in [4] point out the energy \( \mathcal{E}(t) \) have no uniform decay rates. Afterwards, authors in [3] obtained polynomial decay of \( \mathcal{E}(t) \) under the following geometrical conditions

\[
\begin{aligned}
\Gamma_0 &= \{ x \in \Gamma : (x - x_0) \cdot n \leq 0 \}, \\
\Gamma_1 &= \{ x \in \Gamma : (x - x_0) \cdot n \geq c \}
\end{aligned}
\]

(5)

(6)

for some point \( x_0 \in \mathbb{R}^3 \) and some constant \( c > 0 \). We refer the reader to [3, 7, 8, 9] for deep studies on stability for the related acoustic systems. However, if the boundary random force \( \xi \neq 0 \), what is the propagation of acoustic wave? As we know, the friction tends to drive any system to a completely dead state, while the noise tends to keep the system alive. What is the balance between friction and noise? From the viewpoint of mathematics, we should concern about the long time behavior of solution to the acoustic system \( \{3,4,5\} \) under the interaction of the friction and random force both acting on the part of the boundary.

Let us mention that there have been relatively few analysis of the wave equation with stochastic boundary values. In 1993, Mao and Markus in [10] used the parallelogram identity to represent the solution explicitly. However, the 1-dimensionality of space is essential in their analysis. Afterwards, Kim in [11] combined the Galerkin method and the duality argument to establish the well-posedness for a 1-d wave equation with variable coefficients and white noise on the Neumann boundary. On the other hand, the parabolic equations with boundary white noises have been studied by some authors (e.g. [12, 13, 14]). Recently, Shirikyan in [15] developed a general framework for dealing with random perturbations acting though the boundary of the domain and applied this result to study the 2-d Navier-Stokes system driven by a random force on the Dirichlet boundary. To the best of my knowledge, the problem of mixing for the acoustic system with a random perturbation acting through the boundary has not been studied in earlier works.

The aim of this paper is devoted to the existence of uniqueness of invariant measures for solutions of the acoustic system \( \{3,4,5\} \) as well as to weak convergence of their transition probabilities. If we regard the boundary random force as a control function, the motivation is provided by comparisons with analogous problems of control theory for acoustic wave system. Then we will make effort to establish the observability
for the determined system associated with \(3-B1-B2\) which implies the controllability, to find a useful connection between the exact controllability, and finally to obtain strong mixing for the acoustic system. Note that the geometry of uncontrolled boundary \(\Gamma_0\) plays a decisive role in our proof. We also would like to state that our idea is stimulated by the significant papers \([11, 16, 17, 15]\).

This paper is organized as follows. In section 2, we give a well-posedness result by the theory of semigroups of linear operators. And we also obtain an observability inequality which is a key lemma (Lemma 2) to study the mixing problem for system \(3-B1-B2\). In section 3, we present our main theorem; we will apply the theory of exact controllability to prove the strong mixing of the invariant measure. Throughout this paper, \(c, C(\cdot)\) are as generic constants whose values may change from line to line.

2. Preliminary

Set \(H_0(\Omega) = \{v(x) \in H^1(\Omega) : v(x) = 0, \text{ on } \Gamma_0\}\). Define the finite energy space by

\[
\mathcal{H} = H_0(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1)
\]

where the inner product in \(\mathcal{H}\) is given by

\[
(f, g)_{\mathcal{H}} = \int_D \rho [c^{-2}f_2g_2 + \nabla f_1 \cdot \nabla g_1]dx + \int_{\Gamma_1} [k f_3 g_3 + m f_4 g_4]d\Gamma.
\]

The condition \(\phi|_{\Gamma_0} = 0\) and \(\frac{\partial \phi}{\partial n}|_{\Gamma_1} = \delta_t\) are interpreted in the weak sense

\[
\int_D [\Delta \phi v + \nabla \phi \cdot \nabla v]dx = \int_{\Gamma_1} \delta_t vd\Gamma, \quad \forall v \in H^1(\Omega),
\]

which can be deduced from integrating by parts.

Throughout the paper \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space with a right-continuous increasing family \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) of sub-\(\sigma\)-fields of \(\mathcal{F}_0\) each containing \(\mathbb{P}\)-null sets. \(\mathbb{E}(\cdot)\) stands for expectation with respect to the probability measure \(\mathbb{P}\). \(\mathcal{B}(\mathcal{H})\) denotes the Borel \(\sigma\)-algebra over \(\mathcal{H}\). \(\mathcal{P}(\mathcal{H})\) is the set of probability Borel measure on \(\mathcal{H}\). \(B_b(\mathcal{H})\) \(C(\mathcal{H})\), \(C_b(\mathcal{H})\), respectively is the set of bounded Borel functions (continuous, bounded and continuous functions, respectively) on \(\mathcal{H}\).

Let \(\{e_k\}\) be an arbitrary, completely, orthonormal basis in \(\mathcal{L}\). Here \(\mathcal{L} = L^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1)\). A \(\mathcal{L}\)-valued Wiener process \(W(t)\), \(t \geq 0\), defined on this probability space, can be defined as

\[
W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k, \quad t \geq 0,
\]

where \(\beta_k, k = 1, \cdots,\) are independent real processes. The random perturbation \(\xi(t)\) in the boundary condition \(\{B2\}\) is so-called white noise, which is the formal time derivative of \(W(t)\). From this viewpoint, system \(\{3-B1-B2\}\) is just deduced in a formal way to model the effect of the random perturbation on the acoustic wave motion.

Set \(X(t) = \{\phi, \phi_t, \delta, \delta_t\}\) and \(x := v(0) = \{\phi_0, \phi_1, \delta_0, \delta_1\}\) To be mathematical analysis, system \(\{3-B1-B2\}\) can be transformed into the following stochastic evolution equation on the Hilbert space \(\mathcal{H}\).

\[
\begin{cases}
  dX(t) = AX(t)dt + BDW(t), \\
  X(0) = x,
\end{cases}
\]

where

\[
A = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  c^2 \Delta & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & -\frac{d}{m} & -\frac{k}{m} & -\frac{d}{m}
\end{pmatrix}, \quad B = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -\frac{1}{m}
\end{pmatrix},
\]

and the domain of the operator \(A\) given by

\[
\text{Dom}(A) = \{f \in \mathcal{H} : \Delta f_1 \in L^2, f_2 \in H_{\Gamma_0}, \frac{\partial f_1}{\partial n}|_{\Gamma_1} = f_4\}.
\]
Lemma 1 (Well-posedness). If the initial data \( \mathbf{x} \) is \( \mathcal{F}_0 \)-measurable and belongs to \( \mathcal{H} \), there exists a unique mild solution, \( \mathbf{X}(t, \mathbf{x}) \in C([0, \infty), \mathcal{H}) \), to the system (7).

Proof. Due to \( \text{Dom}(A) \) is dense in \( \mathcal{H} \), then the operator \( A \) is a densely defined. Then we can deduce uniquely the duality operator \( A^* \) of \( A \)

\[
A^* = \begin{pmatrix}
0 & -1 & 0 & 0 \\
-c^2\Delta & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & \frac{\rho}{m} & \frac{k}{m} & -\frac{d}{m}
\end{pmatrix}
\]

and the domain of the operator \( A^* \) is given by

\[
\text{Dom}(A^*) = \{ \mathbf{g} \in \mathcal{H} : \Delta g_1 \in L^2, g_2 \in H_{\Gamma_0}, \frac{\partial g_1}{\partial n}|_{\Gamma_1} = g_4 \}.
\]

From the inequality (4), we have for any \( \mathbf{f} \in \text{Dom}(A) \)

\[
\Re(A\mathbf{f}, \mathbf{f})_{\mathcal{H}} = -d \int_{\Gamma_1} |f_4|^2 d\Gamma \leq 0,
\]

which implies that \( A \) is a dissipative operator. We want show that \( A \) is closable. Suppose it is not true. Then there is a sequence \( \mathbf{f}_n \in \text{Dom}(A) \) such that \( \mathbf{f}_n \to 0 \) and \( A\mathbf{f}_n \to \mathbf{h} \) with \( \|\mathbf{h}\|_{\mathcal{H}} = 1 \). Since \( A \) is dissipative, it follows that for every \( \lambda > 0 \) and \( \mathbf{f} \in \text{Dom}(A) \)

\[
\|\left(\frac{1}{\lambda} - A\right)(\mathbf{f} + \lambda^{-1}\mathbf{f}_n)\| \geq \frac{1}{\lambda} \|\mathbf{f} + \lambda^{-1}\mathbf{f}_n\|,
\]

that is,

\[
\|\left(\mathbf{f} + \lambda^{-1}\mathbf{f}_n\right) - (\lambda A\mathbf{f} + A\mathbf{f}_n)\| \geq \|\mathbf{f} + \lambda^{-1}\mathbf{f}_n\|.
\]

Letting \( n \to \infty \) and then \( \lambda \to 0 \) gives \( \|\mathbf{f} - \mathbf{h}\| \geq \|\mathbf{f}\| \), which is impossible obviously. Then \( A \) is closable. For any \( \mathbf{g} \in \text{Dom}(A^*) \), we have

\[
\Re(A^*\mathbf{g}, \mathbf{g}) = \int_D \rho [c^{-2}(-c^2\Delta g_1)\overline{g_2} + \nabla g_2 \nabla g_1] dx + \int_{\Gamma_1} [k(-g_3)\overline{g_4} + m\left(\frac{\rho}{m}g_2 + \frac{k}{m}g_3 - \frac{d}{m}g_4\right)\overline{g_4}] d\Gamma
\]

\[
= -d \int_{\Gamma_1} |g_4|^2 d\Gamma \leq 0.
\]

Hence \( A^* \) is also a dissipative operator. Thus, from Corollary 4.4 in [18, Chapter 1], \( A \) is the infinitesimal generator of a \( C_0 \) semigroup of contraction on \( \mathcal{H} \), denoted by \( S(t) \). It follows that \( \|S(t)\| \leq 1 \). Then we have

\[
\|S(t)f_1 - S(t)f_2\| \leq \|f_1 - f_2\|, \quad t > 0, \quad f_1, f_2 \in \mathcal{H}.
\]

Therefore, by Theorem 5.3.1 in [19], there exists an \( \mathcal{F}_t \)-adapted process \( \mathbf{X}(t) \), \( t \geq 0 \), satisfies the following integral equation

\[
\mathbf{X}(t) = S(t)\mathbf{x} + \int_0^t S(t-s)BdW(s), \quad t \in [0, T].
\]

which means \( \mathbf{X}(t) \) is a mild solution is a mild solution of (7). \( \square \)

To study the existence of invariant measure for the system (9), the geometric condition (5) is assumed on the boundary \( \Gamma_0 \). From Theorem A.4.1 in [20], [5] implies that there exist a vector field \( h(x) \) and a scalar function \( H(x) \)

\[
h(x) := \nabla H(x) \in [C^2(\bar{D})]^3
\]

such that \( \nabla H \cdot \mathbf{n} = 0 \) on \( \Gamma_0 \) and the Hessian matrix of \( H \) evaluated on \( \Gamma_0 \) is positive definite.
Lemma 2 (Observability inequality). There exists a constant $T_0 > 0$, depending only on $\Omega$, such that for $T > T_0$, there corresponds a positive constant $C_T$ satisfying

$$C_T \mathcal{E}(0) \leq \int_0^T \int_{\Gamma_1} \delta_t d\Gamma_1 dt.$$  

PROOF. Multiplying the first equation of (8) by $(h \cdot \nabla)\phi$, and integrating by parts, we have

$$0 = \int_{\tau_0}^{T_{\tau_0}} \int_D (h \cdot \nabla)\phi(\phi_t - c^2 \nabla \phi) dx dt$$

$$= \int_D (h \cdot \nabla)\phi(\phi_t - c^2 \nabla \phi) dx|_{\tau_0}^{T_{\tau_0}} - \int_{\tau_0}^{T_{\tau_0}} \int_D \nabla \cdot (\frac{h}{2} \phi_t^2) dx dt$$

$$- c^2 \int_{\tau_0}^{T_{\tau_0}} \int_{\Gamma_1} (h \cdot \nabla)\phi_t d\Gamma_1 dt + c^2 \int_{\tau_0}^{T_{\tau_0}} \int_D \nabla (h \cdot \nabla)\phi \cdot \nabla \phi dx dt$$

for some $\epsilon \in (0, \frac{T}{2})$. Since we have

$$\nabla (h \cdot \nabla)\phi \cdot \nabla \phi = \sum_{i,j} \frac{\partial h}{\partial x_i} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \sum_{i,j} h_j \frac{\partial}{\partial x_i} \left( \frac{\partial \phi}{\partial x_i} \right)^2$$

$$= \sum_{i,j} \frac{\partial h}{\partial x_i} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \nabla \cdot \left( \frac{h}{2} |\nabla \phi|^2 \right) - \frac{\nabla \cdot h}{2} |\nabla \phi|^2,$$

then from (10) and using the boundary condition, it deduces that

$$0 = \int_{\tau_0}^{T_{\tau_0}} \int_D \nabla \cdot \left( \frac{h}{2} |\nabla \phi|^2 \right) dx dt - \int_{\tau_0}^{T_{\tau_0}} \int_{\Gamma_1} \left\{ \frac{h}{2} |\nabla \phi|^2 - c^2 |\nabla \phi|^2 \right\} dx dt$$

$$- c^2 \int_{\tau_0}^{T_{\tau_0}} \int_{\Gamma_1} (h \cdot \nabla)\phi_t d\Gamma_1 dt + c^2 \int_{\tau_0}^{T_{\tau_0}} \int_D \sum_{i,j} \frac{\partial h}{\partial x_i} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx dt$$

$$+ \left| \int_D (h \cdot \nabla)\phi_t dx \right|_{\tau_0}^{T_{\tau_0}},$$

which implies

$$c^2 h_0 \int_{\tau_0}^{T_{\tau_0}} \int_D |\nabla \phi|^2 dx dt$$

$$\leq c^2 \int_{\tau_0}^{T_{\tau_0}} \int_D \sum_{i,j} \frac{\partial h}{\partial x_i} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx dt$$

$$= \int_{\tau_0}^{T_{\tau_0}} \int_D \nabla \cdot \left( \frac{h}{2} |\nabla \phi|^2 \right) dx dt - \int_{\tau_0}^{T_{\tau_0}} \int_{\Gamma_1} \left\{ \frac{h}{2} |\nabla \phi|^2 - c^2 |\nabla \phi|^2 \right\} dx dt$$

$$- c^2 \int_{\tau_0}^{T_{\tau_0}} \int_{\Gamma_1} (h \cdot \nabla)\phi_t d\Gamma_1 dt + \left| \int_D (h \cdot \nabla)\phi_t dx \right|_{\tau_0}^{T_{\tau_0}}$$

$$\leq C_h \int_{\tau_0}^{T_{\tau_0}} \int_{\Gamma_1} \left| \phi_t \right|^2 + |\nabla \phi|^2 d\Gamma_1 dt + \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt + C_h \mathcal{E}(0)$$

$$+ \left| \int_{\tau_0}^{T_{\tau_0}} \int_D \nabla \cdot \left( \frac{h}{2} |\nabla \phi|^2 \right) dx dt \right|.$$
Here, $\nabla_\| \|$ is the tangential derivative on the boundary and $C_h$ is a positive constant depending on $h$. Multiplying the first equation of (8) by $\phi(\nabla \cdot h)$, and integrating by parts, we have

\[
0 = \int_{\tau_0}^{T-\tau_0} \int_D \phi(\nabla \cdot h)(\partial_t \phi - c^2 \Delta \phi) \, dx \, dt
\]
\[
= \left[ \int_D \phi(\nabla \cdot h) \phi_1 \, dx \right]_{\tau_0}^{T-\tau_0} - \int_{\tau_0}^{T-\tau_0} \int_D (\nabla \cdot h) \phi_1^2 - c^2 \nabla (\phi \nabla \cdot h) \cdot \nabla \phi \, dx \, dt
\]
\[
- c^2 \int_{\tau_0}^{T-\tau_0} \int_{\Gamma_1} \phi(\nabla \cdot h) \frac{\partial \phi}{\partial n} \, d\Gamma dt
\]
\[
= \left[ \int_D \phi(\nabla \cdot h) \phi_1 \, dx \right]_{\tau_0}^{T-\tau_0} - \int_{\tau_0}^{T-\tau_0} \int_D (\nabla \cdot h) \phi_1^2 - c^2 |\nabla \phi|^2 \, dx \, dt
\]
\[
+ c^2 \int_{\tau_0}^{T-\tau_0} \int_D \phi \nabla \phi \cdot \nabla (\nabla \cdot h) \, dx \, dt - c^2 \int_{\tau_0}^{T-\tau_0} \int_{\Gamma_1} \phi(\nabla \cdot h) \delta_t d\Gamma dt,
\]

which deduce that

\[
\int_{\tau_0}^{T-\tau_0} \int_D (\nabla \cdot h) \phi_1^2 - c^2 |\nabla \phi|^2 \, dx \, dt
\]
\[
= \langle \phi(\nabla \cdot h), \phi_1 \rangle_{\tau_0}^{T-\tau_0} + c^2 \int_{\tau_0}^{T-\tau_0} \int_D \phi \nabla \phi \cdot \nabla (\nabla \cdot h) \, dx \, dt - c^2 \int_{\tau_0}^{T-\tau_0} \int_{\Gamma_1} \phi(\nabla \cdot h) \delta_t d\Gamma dt. \tag{12}
\]

Here, $\langle \cdot, \cdot \rangle$ means the the product in $H^{-\eta}(D) \times H^{\eta}(D)$ for $\eta > 0$. We introduce the standard denotation for the terms which are below the level of energy, that is,

\[\text{LOT}(\Phi, \Psi) : = \| (\Phi, \Psi) \|^2_{C([0,T]; M)},\]

in which $\Phi = \{ \phi, \phi_1 \}$, $\Psi = \{ \delta, \delta_1 \}$ and $M = H^{1-\eta}(D) \times H^{-\eta}(D) \times H^{-\eta}(\Gamma_1) \times H^{-\eta}(\Gamma_1)$. Then by (12) and using the Young’s inequality, we have

\[
\int_{\tau_0}^{T-\tau_0} \int_D (\nabla \cdot h) \phi_1^2 - c^2 |\nabla \phi|^2 \, dx \, dt \leq \tau_1 \int_{\tau_0}^{T-\tau_0} \int_D |\nabla \phi|^2 \, dx \, dt + C(\tau_1) \int_0^T \int_{\Gamma_1} \delta_t^2 \, d\Gamma \, dt + \text{LOT}(\Phi, \Psi). \tag{13}
\]

Combining (11) and (13), it deduces that

\[
\int_{\tau_0}^{T-\tau_0} \int_D \rho |c^{-1} \phi_1|^2 + |\nabla \phi|^2 \, dx \, dt
\]
\[
\leq C(\tau_1, h) \{ \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma dt + \int_{\tau_0}^{T-\tau_0} \int_{\Gamma_1} \phi_1^2 + |\nabla ||\phi|^2 | d\Gamma dt \} + C(h)\mathcal{E}(0) + \text{LOT}(\Phi, \Psi). \tag{14}
\]

Let $\kappa(t) \in C_0^\infty(\mathbb{R})$ be cut-off function given by

\[
\kappa(t) = \begin{cases} \begin{array}{ll} 1, & t \in [\epsilon_0, T - \epsilon_0], \\ 0, & t \in (0, \epsilon_0) \cup (T - \epsilon_0, T), \\ a C^\infty \text{ function with range in } (0, 1), & t \in (-\infty, 0) \cup (T, \infty). \end{array} \end{cases}
\]

Using Lemma 7.2 in [21], we have

\[
\int_{\tau_0}^{T-\tau_0} \int_{\Gamma_1} |\nabla ||\phi| |^2 d\Gamma dt
\]
\[
\leq \int_0^T \int_{\Gamma_1} |\nabla (\kappa \phi)|^2 d\Gamma dt
\]
\[
\leq C(T, \epsilon_0) \{ \int_0^T \int_{\Gamma_1} |\partial_t \phi|^2 d\Gamma dt + \int_0^T \int_{\Gamma_1} |\partial_t (\kappa \phi)|^2 d\Gamma dt \} + \text{LOT}(\Phi, \Psi)
\]
\[
\leq C(T, \epsilon_0) \{ \int_0^T \int_{\Gamma_1} |\delta_t|^2 d\Gamma dt + \int_0^T \int_{\Gamma_1} \kappa^2 |\phi_t|^2 d\Gamma dt \} + \text{LOT}(\Phi, \Psi).
\]
We estimate \( \int_0^{T-\epsilon_0} \int_{\Gamma_1} \phi_i^2 d\Gamma_1 dt \) and \( \int_0^T \int_{\Gamma_1} \kappa^2 |\phi_i|^2 d\Gamma_1 dt \). The boundary condition on \( \Gamma_1 \) shows that

\[
\int_0^{T-\epsilon_0} \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt \leq \int_0^T \int_{\Gamma_1} \kappa^2 |\phi_i|^2 d\Gamma_1 dt \leq C(\rho, m, d, k) \int_0^T \int_{\Gamma_1} (|\delta_{tt}^2 + \delta_t^2 + \delta^2| d\Gamma_1 dt
\]

Then from \( (14) \) it deduces that

\[
\int_0^{T-\epsilon_0} \mathcal{E}(t) dt \leq C(\tau_1, h, T, \epsilon_0, \rho, m, d, k) \int_0^T \int_{\Gamma_1} (|\delta_{tt}^2 + \delta_t^2 + \delta^2| d\Gamma_1 dt + C(h)\mathcal{E}(0) + \text{LOT}(\Phi, \Psi).
\]

From \( \mathcal{E}(T) = \mathcal{E}(0) - \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt \), it follows that

\[
(T - 2\epsilon_0)|\mathcal{E}(0) - \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt|
\leq (T - 2\epsilon_0)E(T)
\leq \int_0^{T-\epsilon_0} \mathcal{E}(t) dt
\leq C(\tau_1, h, T, \epsilon_0, \rho, m, d, k) \int_0^T \int_{\Gamma_1} (|\delta_{tt}^2 + \delta_t^2 + \delta^2| d\Gamma_1 dt + C(h)\mathcal{E}(0) + \text{LOT}(\Phi, \Psi),
\]

which implies

\[
(T - 2\epsilon_0 - C(h))\mathcal{E}(0) \leq C(\tau_1, h, T, \epsilon_0, \rho, m, d, k) \int_0^T \int_{\Gamma_1} (|\delta_{tt}^2 + \delta_t^2 + \delta^2| d\Gamma_1 dt + \text{LOT}(\Phi, \Psi),
\]

Differentiating the boundary equation in time, multiplying \( \delta_t \) and integrating by parts, we have

\[
0 = \int_0^T \int_{\Gamma_1} \delta_t (\rho \phi_t + m \delta_{tt} + d \delta_t + k \delta_t) d\Gamma_1 dt
\]

\[
= \rho \int_{\Gamma_1} \delta_t \phi_t d\Gamma_1 \bigg|_0^T - \rho \int_0^T \int_{\Gamma_1} \delta_t \phi_t d\Gamma_1 dt + [m \int_{\Gamma_1} \delta_t \delta_{tt} d\Gamma_1] \bigg|_0^T - m \int_0^T \int_{\Gamma_1} \delta_{tt}^2 d\Gamma_1 dt
\]

\[
+ \frac{d}{2} \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt + k \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt,
\]

which implies

\[
\int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt \leq C(\rho, m, d)\mathcal{E}(0) + \frac{k}{m} \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt + \frac{\rho}{2m} \int_0^T \int_{\Gamma_1} (\delta_{tt} + \phi_t)^2 d\Gamma_1 dt
\]

\[
\leq C(\rho, m, d)\mathcal{E}(0) + \frac{k}{m} \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt + \frac{\rho}{2m} \int_0^T \int_{\Gamma_1} (\delta_t + \delta)^2 d\Gamma_1 dt
\]

\[
\leq C(\rho, m, d)\mathcal{E}(0) + \frac{k + \rho}{m} \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt + \frac{\rho}{m} \int_0^T \int_{\Gamma_1} \delta^2 d\Gamma_1 dt.
\]

Multiplying \( \delta_t \) and integrating by parts, we have

\[
0 = \int_0^T \int_{\Gamma_1} \delta_t (\rho \phi_t + m \delta_{tt} + d \delta_t + k \delta_t) d\Gamma_1 dt
\]

\[
= \frac{\rho}{m} \int_{\Gamma_1} \delta_t \phi_t d\Gamma_1 \bigg|_0^T - m \int_0^T \int_{\Gamma_1} \delta_t \phi_t d\Gamma_1 dt + [m \int_{\Gamma_1} \delta_t \delta_{tt} d\Gamma_1] \bigg|_0^T - m \int_0^T \int_{\Gamma_1} \delta_{tt}^2 d\Gamma_1 dt
\]

\[
+ \frac{d}{2} \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt + k \int_0^T \int_{\Gamma_1} \delta_t^2 d\Gamma_1 dt,
\]
which implies
\[
\int_0^T \int_{\Gamma_1} \delta^2 d\Gamma_1 dt \leq C(\rho, m, d, k)\mathcal{E}(0) + \tau_2 \int_0^T \int_{\Gamma_1} \phi^2 d\Gamma_1 dt + \left( C(\tau_2, \rho) + m \right) \int_0^T \int_{\Gamma_1} \delta^2 d\Gamma_1 dt.
\]

Then from (16) and choosing \( T_0 := \frac{2\alpha + C(h) + C(\rho, m, d, k)}{1 - \tau_2} \), combining (17) and (18) gives
\[
(T - T_0)\mathcal{E}(0) \leq C(\tau_1, \tau_2, h, T, \epsilon, \rho, m, d, k) \int_0^T \int_{\Gamma_1} \delta^2 d\Gamma_1 dt + \text{LOT}(\Phi, \Psi).
\]

Claim that there exists a constant \( C_T > 0 \) such that the solution of (8) satisfies the inequality
\[
\text{LOT}(\Phi, \Psi) \leq C_T \|\delta\|_{L^2([0, T] \times \Gamma_1)}^2.
\]

We use the method of contradiction. Suppose that this claim is false. Then there exists a sequence of solutions \( \{\Phi^{(n)}(t), \Psi^{(n)}(t)\} \) to (8) with the initial data \( \{\Phi^{(n)}(0), \Psi^{(n)}(0)\} \subset \mathcal{H} \) such that
\[
\|\{\Phi^{(n)}(0), \Psi^{(n)}(0)\}\|_{\mathcal{H}} \text{ is bounded. Then there is a subsequence, still denoted by}
\]
\[
\|\{\Phi^{(n)}(t), \Psi^{(n)}(t)\}\|_{C([0, T]; \mathcal{H})} \text{ is bounded.}
\]
\[
\{\Phi^{(n)}(t), \Psi^{(n)}(t)\} \text{ weakly } \rightarrow \{\hat{\Phi}(t), \hat{\Psi}(t)\}, \text{ in } \mathcal{H}
\]
as \( n \to \infty \). Let \( \{\hat{\Phi}(t), \hat{\Psi}(t)\} \) be the solution of (8) subject to the initial data \( \{\hat{\Phi}(0), \hat{\Psi}(0)\} \). By (4), we see that \(\|\{\Phi^{(n)}(t), \Psi^{(n)}(t)\}\|_{C([0, T]; \mathcal{H})} \) is bounded, and then
\[
\{\Phi^{(n)}(t), \Psi^{(n)}(t)\} \text{ weak star } \rightarrow \{\hat{\Phi}(t), \hat{\Psi}(t)\}, \text{ in } L^\infty([0, T]; \mathcal{H}).
\]

Let \( \mathcal{N} = H^{-\eta}(D) \times (H^1(D))' \times H^{-\eta}(\Gamma_1) \times H^{-\eta}(\Gamma_1) \). For any \( \omega \in H^1 \) and \( t \in (0, T) \), we have the following equation holds true
\[
\langle \phi_{tt}^{(n)}, \omega \rangle_{H^{-1} \times H^1} = \int_D \Delta \phi^{(n)} \omega dx - \int_{\Gamma_1} \delta^{(n)} \omega d\Gamma_1 - \int_D \nabla \phi^{(n)} \cdot \nabla \omega dx,
\]
which implies that \( \|\phi_{tt}^{(n)}\|_{H^{-1} \times H^1} \leq C(\|\delta^{(n)}\|_{L^2}, \|\nabla \phi^{(n)}\|_{L^2}) \|\omega\|_{H^1}. \) Hence \( \phi_{tt}^{(n)} \in L^\infty([0, T], (H^1(D))') \). Thus, it follows that \(\|\{\Phi^{(n)}(t), \Psi^{(n)}(t)\}\|_{L^\infty([0, T]; \mathcal{N})} \) is bounded uniformly. Due to \( \mathcal{H} \subset \subset \mathcal{M} \subset \mathcal{N} \), we deduce from the Aubin’s theorem (22) that
\[
\{\Phi^{(n)}(t), \Psi^{(n)}(t)\} \text{ strongly } \rightarrow \{\hat{\Phi}(t), \hat{\Psi}(t)\}, \text{ in } L^\infty([0, T]; \mathcal{M}),
\]
which gives
\[
\|\{\hat{\Phi}(t), \hat{\Psi}(t)\}\|_{C([0, T]; \mathcal{M})} = \text{LOT}(\Phi^{(n)}, \Psi^{(n)}) = 1.
\]

By (21) and (22), we have \( \delta_t = 0 \). Let \( P = \hat{\phi}_{tt} \). Then we have
\[
\begin{cases}
P_{tt} = c^2 \Delta P, \quad \text{in } \mathbb{R}^+ \times D, \\
\frac{\partial P}{\partial n} = 0, \quad \text{on } \mathbb{R}^+ \times \Gamma_1 \\
P = 0, \quad \text{on } \mathbb{R}^+ \times \Gamma_0.
\end{cases}
\]

By Holmgren’s Uniqueness Theorem, choosing \( T > 2\text{diam}(D) \), it follows that \( P = \hat{\phi}_{tt} = 0 \). And then we obtain
\[
\begin{cases}
\Delta \hat{\phi} = 0, \quad \text{in } D, \\
\frac{\partial \hat{\phi}}{\partial n} = 0, \quad \text{on } \Gamma_1 \\
\hat{\phi} = 0, \quad \text{on } \Gamma_0,
\end{cases}
\]
which implies that \( \hat{\phi} = 0 \). The boundary condition shows that \( \hat{\delta} = 0 \). Then we obtain \( \{\hat{\Phi}(t), \hat{\Psi}(t)\} = 0 \) which contradicts (23). Therefore, we prove the inequality (20). \( \square \)
3. Main Result

The following theorem is the main result of this paper.

**Theorem 1 (Mixing).** Assume that \( \mathbf{x} \in \mathcal{H} \) is \( \mathcal{F}_0 \)-measurable and \( \mathbb{E}\|\mathbf{x}\|_\mathcal{H}^2 \leq \infty \). Let \( X(t, \mathbf{x}) \) be the mild solution of system \( \mathbf{7} \) with the initial data \( \mathbf{x} \). There exists a unique invariant measure \( \mu \) for the system \( \mathbf{7} \) which is strong mixing, that is,

\[
\lim_{t \to \infty} \mathbb{E} f(X(t, \mathbf{x})) = \int_{\mathcal{H}} f(z) \mu(dz), \quad \text{strongly in } L^2(\mathcal{H}, \mu)
\]

for any \( f \in C_b(\mathcal{H}) \).

It shows that the limit of the average of \( f(X(t, \mathbf{x})) \) is a quantity that does not depend on the initial point. Next, we will give the proof of this main result.

**Proof.** The subsequent proof consists of several steps.

1. **Step 1.** Let \( P_t \) stand for the transition function of the family defined as the law of \( X(t, \mathbf{x}) \) under the probability measure \( \mathbb{P} \)

\[
P_t(\mathbf{x}, \Lambda) = \mathbb{P}(X(t, \mathbf{x}) \in \Lambda), \quad \Lambda \in \mathcal{B}(\mathcal{H}), \quad t > 0.
\]

The corresponding Markov semigroups are given by

\[
M_t : C(\mathcal{H}) \to C(\mathcal{H}), \quad M_t f(\mathbf{x}) = \int_{\mathcal{H}} P_t(\mathbf{x}, dz) f(z),
\]

\[
M_t^* : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}), \quad M_t^* \lambda(\Lambda) = \int_{\mathcal{H}} P_t(\mathbf{x}, \Lambda) \lambda(d\mathbf{x}).
\]

And these two semigroups satisfy the duality relation \( (M_t f, \lambda) = (f, M_t^* \lambda) \) for any \( f \in C(\mathcal{H}), \lambda \in \mathcal{P}(\mathcal{H}) \).

Due to \( \frac{1}{2}\|X(t, \mathbf{x})\|_{\mathcal{H}}^2 = E(t) \), then from Young’s inequality it follows that

\[
\|X(t, \mathbf{x})\|_{\mathcal{H}}^2 = \|\mathbf{x}\|_{\mathcal{H}}^2 - 2d \int_0^t \int_{\Gamma_1} \delta_1^2 d\Gamma_1 dt - 2 \int_0^t \int_0^t \delta_1 dW(s) d\Gamma_1
\]

\[
\leq \|\mathbf{x}\|_{\mathcal{H}}^2 - 2d \int_0^t \int_{\Gamma_1} \delta_1^2 d\Gamma_1 dt + 2d \int_{\Gamma_1} (\int_0^t \delta_1 dW(s))^2 d\Gamma_1 + \frac{\|\Gamma_1\|}{4d}.
\]

(24)

By the Itô isometry \( \int_{\Gamma_1} E\left(\int_0^t \delta_1 dW(s)^2\right)^2 d\Gamma_1 = \int_{\Gamma_1} E\left(\int_0^t \delta_1 ds\right)^2 d\Gamma_1 \), we have from (24)

\[
E\|X(t, \mathbf{x})\|_{\mathcal{H}}^2 \leq E\|\mathbf{x}\|_{\mathcal{H}}^2 + \frac{\|\Gamma_1\|}{4d}.
\]

(25)

Define the linear operator

\[
Q_t \mathbf{x} = \int_0^t S(s) B B^* S^*(s) \mathbf{x} ds, \quad \mathbf{x} \in \mathcal{H}.
\]

Due to \( \sup_{t \geq 0} E\|X(t, 0)\|_{\mathcal{H}}^2 = \sup_{t \geq 0} \text{Tr} Q_t \), from (23) we get \( \sup_{t \geq 0} \text{Tr} Q_t \leq \infty \). Hence by Theorem 6.2.1 in [19] there exists an invariant measure \( \mu \) for system \( \mathbf{7} \), that is, \( M_t^* \mu = \mu \).

2. **Step 2.** Note that \( X(t, \mathbf{x}) \) is a Gaussian random variable with mean \( S(t) \mathbf{x} \) and covariance \( Q_t \). From the Doob’s theorem [23], if the transition function \( P_t(\mathbf{x}, \cdot) \) defined on \( \mathcal{B}(\mathcal{H}) \) is \( t_0 \)-regular for some \( t_0 > 0 \), then \( \mu \) is the unique invariant measure for \( P_t \), which is also strong mixing. Since \( X(t) \) is Gaussian, the regularity of \( P_t \) is a consequence of the strong Feller property.

Assume that the range of \( S(t) \) is a subset of the range of \( Q_t^+ \), that is, \( \text{Ran} S(t) \subset \text{Ran} Q_t^+, \ t > 0 \). Then we can denote a linear bounded operator \( R(t) = Q_t^+ \circ S(t) \). By the Cameron-Martin formula, we have for any \( \mathbf{x} \in \mathcal{H} \)

\[
\frac{dP_t(\mathbf{x}, \cdot)}{dP_t(0, \cdot)}(\mathbf{\bar{x}}) = F(t, \mathbf{x}, \mathbf{\bar{x}})
\]

(26)
with
\[ F(t, x, \tilde{x}) = \exp\left(\left(\frac{Q_t}{2} \circ S(t)x, Q_t^{-\frac{1}{2}} \tilde{x}\right)_{\mathcal{H}} - \frac{1}{2} \|Q_t^{-\frac{1}{2}} \circ S(t)x\|_{\mathcal{H}}^2\right)\].

Let \( \{e_k(t)\} \) be a complete orthonormal basis on \( \mathcal{H} \) which diagonalizes \( Q_t \), and \( \{\eta_k(t)\} \) the corresponding set of eigenvalues of \( Q_t \). Set
\[ G_N(t, x, \tilde{x}) = \exp\left[\sum_{k=1}^{N} \left[\frac{(S(t)x, e_k(t))(\tilde{x}, e_k(t))}{\eta_k(t)} - \frac{1}{2} \left(\frac{(S(t)x, e_k(t))}{\eta_k(t)}\right)^2\right]\right].\]

It is easy to check that for any \( \tilde{x} \in \mathcal{H} \) we have \( \lim_{N \to \infty} G_N(t, x, \tilde{x}) = F(t, x, \tilde{x}) \) almost surely in \( L^2(\mathcal{H}, \mathcal{B}(\mathcal{H}), P_t(0, \cdot)) \).

If \( h \in \mathcal{H} \), we have
\[ \left(\frac{d}{dx} G_N(t, x, \tilde{x}), h\right) = G_N(t, x, \tilde{x}) \sum_{k=1}^{N} \left[\frac{(S(t)h, e_k(t))(\tilde{x}, e_k(t))}{\eta_k(t)} - \frac{(S(t)x, e_k(t))(S(t)h, e_k(t))}{\eta_k(t)}\right],\]

which implies
\[ \lim_{N \to \infty} \left(\frac{d}{dx} G_N(t, x, \tilde{x}), h\right) = F(t, x, \tilde{x})\left\{\left(\frac{Q_t^{-\frac{1}{2}} \circ S(t)h_1, Q_t^{-\frac{1}{2}} \tilde{x}}{\eta_k(t)}\right) - \left(\frac{Q_t^{-\frac{1}{2}} \circ S(t)x, Q_t^{-\frac{1}{2}} \circ S(t)h}{\eta_k(t)}\right)\right\}\]
\[ = F(t, x, \tilde{x})(R(t)h, Q_t^{-\frac{1}{2}} \tilde{x} - R(t)x). \]

For any \( f(x) \in B_b(\mathcal{H}) \), combining (27) and (28) gives
\[ \left(\frac{d}{dx} M_t f(x), h\right) = \left(\frac{d}{dx} \int_{\mathcal{H}} P_t(x, dz)f(z), h\right) \]
\[ = \frac{d}{dx} \int_{\mathcal{H}} F(t, x, z)P_t(0,dz)f(z), h\]
\[ = \int_{\mathcal{H}} (R(t)h, Q_t^{-\frac{1}{2}} z - R(t)x)F(t, x, \tilde{x})P_t(0, dz)f(z) \]
\[ = \int_{\mathcal{H}} (R(t)h, Q_t^{-\frac{1}{2}} z - R(t)x)P_t(x, dz)f(z) \]
\[ = \int_{\mathcal{H}} (R(t)h, Q_t^{-\frac{1}{2}} z)P_t(0, dz)f(S(t)x + z).\]

Then it deduce from (28) that for any \( x_1, x_2 \in \mathcal{H} \) and \( \tau \in [0, 1] \)
\[ \|M_tf(x_1) - M_tf(x_2)\| = \left(\frac{d}{dx} M_t f(x_1 + \tau(x_2 - x_1), x_2 - x_1)\right) \]
\[ = \int_{\mathcal{H}} (R(t)(x_2 - x_1), Q_t^{-\frac{1}{2}} z)P_t(0, dz)f(S(t)(x_1 + \tau(x_2 - x_1)) + z) \]
\[ \leq \sup_{x \in \mathcal{H}} |f(x)| \left[\int_{\mathcal{H}} \left(\frac{(R(t)(x_2 - x_1), Q_t^{-\frac{1}{2}} z)\right|\|P_t(0, dz)\|_{\mathcal{H}}^\frac{1}{2} \right] \]
\[ \leq \sup_{x \in \mathcal{H}} |f(x)| \|R(t)\| \|x_2 - x_1\|_{\mathcal{H}}. \]

Since \( f(x) \in B_b(\mathcal{H}) \) and \( R(t) \) is a bounded operator, we conclude from (28) that \( P_t(x, \cdot) \) is a strong Feller semigroup.

**Step 3.** In this step, we will prove \( \text{Ran} S(t) \subset \text{Ran} Q_t^{\frac{1}{2}} \). Considering the deterministic system
\[ \begin{cases} dY(t) = A(Y(t))dt + Bv, \\ Y(0) = y_0. \end{cases} \]
Let \( T > 0 \). The solution of (30) can be written as
\[ Y(t) = S(t)y_0 + \int_0^t S(t-s)Bvds, \quad t \in [0, T]. \]
Define the following operator

\[ F_T : L^2([0,T];\mathcal{L}) \to \mathcal{H}, \quad F_T \mathbf{v} = \int_0^T S(T-s)B\mathbf{v}ds. \]

Note that \( \text{Ran}F_T \) consists of all states reachable in time \( T \) from zero. Let \( Z_T \in \text{Dom}(A^*) \). Let us also recall that \( \text{Dom}(A^*) \) is dense in \( \mathcal{H} \). Then for \( \mathbf{v} \in L^2([0,T];\mathcal{L}) \) we have

\[ \langle \mathbf{v}, F_T^*(Z_T) \rangle_{L^2([0,T];\mathcal{L})} = \langle F_T(\mathbf{v}), Z_T \rangle_{\mathcal{H}} = \int_0^T \langle \mathbf{v}, B^*S^*(T-s)Z_T \rangle_{\mathcal{H}}ds, \]

which implies that

\[ \langle F_T^*\mathbf{x}(s) = B^*S^*(T-s)\mathbf{x}, \quad \mathbf{x} \in \mathcal{H}, s \in [0,T]. \]

By the definition of \( Q_T^\perp \), we see that \( Q_T = F_TF_T^* \), and \( \text{Ran}Q_T^\perp = \text{Ran}F_T \).

Due to \( Y(T) = S(t)y_0 + F_T\mathbf{v} \), system (30) is null controllable in time \( T \), which means that given any \( y_0 \in \mathcal{H} \), there exists a control function \( \mathbf{v} \in L^1([0,T];\mathcal{L}) \) such that \( Y(T) = 0 \), if and only if \( \text{Ran}S(t) \subset \text{Ran}F_T \). Therefore, to prove \( \text{Ran}S(t) \subset \text{Ran}Q_T^\perp \), it remains to show that system (30) is null controllable in time \( T \).

From [24, Theorem 2.44], the system (30) is null controllable in time \( T \) if and only if there exists a positive constant \( C_{\text{obs}} > 0 \) such that

\[ \int_0^T \| B^*S^*(t)Z \|^2_{\mathcal{L}}ds \geq C_{\text{obs}}\| S^*(T)Z \|^2_{\mathcal{H}}, \quad \forall Z \in \text{Dom}(A^*). \]  

(31)

Let \( Z(t) \) be the solution of the adjoint system

\[ \begin{cases} 
    dZ(t) = A^*(Z(t))dt, & t \in [0,T] \\
    Z(T) = Z_T \in \text{Dom}(A^*). 
\end{cases} \]

Then the inequality (31) is equivalent to

\[ \int_0^T \| B^*Z(t) \|^2_{\mathcal{L}}ds \geq C_{\text{obs}}\| Z(0) \|^2_{\mathcal{H}}, \]

which is a consequence of Lemma 2.

This concludes the proof of our main theorem. \( \square \)

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