Landauer’s erasure, error correction and entanglement

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Landauer’s erasure, thermodynamics, classical and quantum error correction, entanglement. Classical and quantum error correction are presented in the form of Maxwell’s demon and their efficiency analyzed from the thermodynamic point of view. We explain how Landauer’s principle of information erasure applies to both cases. By then extending this principle to entanglement manipulations we rederive upper bounds on purification procedures thereby linking the "no local increase of entanglement" principle to the Second Law of thermodynamics.

I. INTRODUCTION

Landauer (1961) showed that any erasure of information is accompanied by an appropriate increase in entropy. This result was then used by Bennett (1982) to finally exorcise Maxwell’s demon in a Szilard-like set-up (Szilard 1929). His main conclusion was that the increase in entropy is not necessarily a consequence of observations made by the demon, but accompanies the resetting of the final state of the demon to be able to start a new cycle. In other words, information gained has to eventually be erased, which leads to an increase of entropy in the environment and prevents the Second Law of thermodynamics from being violated. In fact, the entropy increase in erasure has to be at least as large as the initial information gain. Bennett’s analysis was, however, completely classical. Soon after this, Zurek (1984) analyzed the demon quantum mechanically confirming Bennett’s results and since then there has been a number of other related works on this subject (e.g. Lubkin 1987 and Lloyd 1997). Here, however, we want to relate the notion of information erasure to the concept of quantum entanglement. Quantum theory tells us that a measurement process is, in fact, the creation of correlations (entanglement) between the system under observation and a measuring apparatus. Loosely speaking, the amount of entanglement tells us how much information is gained by the apparatus and therefore it seems natural to assume that Landauer’s principle has implication on entanglement manipulations. In this paper we show that this indeed is the case; we will argue that Landauer’s erasure entropy limits the increase in the amount of entanglement (more precisely, as will be seen later, it limits the average increase) between two quantum subsystems when each one is manipulated separately. This will then lead to an entirely new derivation of known entanglement measures such as the relative entropy of entanglement (Vedral & Plenio 1998) and the entanglement of creation (Bennett 1996b) (for a review see Plenio & Vedral 1998). However, in order to become more familiar with Landauer’s principle we first analyze classical error correction and compare it to quantum error correction using the concept of erasure of information.

The paper is organized as follows. In the next section we demonstrate formal analogies between classical error correction and a thermodynamical cycle in general. We show exactly how Landauer’s principle is manifested in such a protocol. In section 3 we repeat this analysis using quantum error correction and derive the most general statement for information erasure. In section 4 we apply Landauer’s principle to explain recent measures of entanglement, and link it to the principle of "no increase of entanglement by local means". This is then discussed from two different points of view, the individual and the ensemble, and a number of different questions is raised. Finally we discuss the implications of this work to other phenomena in quantum information theory and state other open problems implicated by our investigations.

II. CLASSICAL ERROR CORRECTION AND MAXWELL’S DEMON

We use a simple reversible cycle which is a slight modification of Bennett’s (1982) version of Maxwell’s demon to illustrate the process of error correction. In order to link information to thermodynamics we will use a box containing a single atom as a representation of a classical bit of information: the atom in the left hand half (LHH) will represent a 0, and the atom in the right hand half (RHH) will be a 1. Now, if the atom is already confined to one of these halves, and we expand it isothermally and reversibly to occupy the whole volume, then the entropy of the atom increases by
\[ \Delta S = k \log 2 \] and the free energy decreases by \[ \Delta F = -kT \log 2 \]. The atom does work \[ \Delta W = kT \log 2 \]. Suppose that initially we want to have our atom in one of the halves in order to be able to do some work as described. However, suppose also that there is a possibility of error, namely the atom has a chance of 1/2 to jump to the other half. Once this happens we cannot extract any work until we return the atom to the initial state. But this itself requires an amount of work of \( kT \log 2 \) in an isothermal compression. We would thus like to be able to correct this error and so we introduce another atom in a box to monitor the first one. This is represented in Fig. 1, and the whole error correction protocol goes through 5 stages.

1. Initially the atoms are in LHH and RHH of respective boxes.

2. Then an error happens to atom A so that it now has a 50/50 chance of being in LHH or RHH.

3. Atom B observes atom A and correlates itself to it, so that either both occupy LHHs or both occupy RHHs. We make no assumptions about how the observation is made.

4. Depending on the state of B we now compress A to one of the two halves; this involves no work, but the state of B is now not known—it has a 50/50 chance of being in LHH or RHH. Thus we have corrected A at the expense of randomizing B. It should be pointed out that by work, we always mean the work done by the atom (or on the atom) against the piston (or by the piston). As usual in thermodynamical idealizations of this kind, all other works are neglected (or assumed negligible). For example, the partition itself is assumed to be very light (in fact, with zero weight), so that there is no work in pushing it. Here there is no work done by the atom, since it is not contained in that part of the box which is compressed (this information about the position of A is recorded by B).

5. In order to be able to repeat the error correction we need to reset B to its initial state as in step 1. Thus we perform isothermal reversible compression of B.

Let us now analyze this process using entropy and free energy. In step 1 both of the atoms possess \( kT \log 2 \) of free energy. After A undergoes an error its free energy is decreased by \( \Delta F_A = -kT \log 2 \), and nothing happens to B. The total free energy is now \( \Delta F_{AB} = kT \log 2 \). In step 3 the total free energy is still \( \Delta F_{AB} = kT \log 2 \), but the atoms are correlated. This means that atom B has information about A (and vice versa). The amount of information is \( k \log 2 \). This enables the error correction step to take place in step 4. This does not change the total free energy, but the atoms are now decorrelated. In step 5 a work of \( kT \log 2 \) is invested into resetting the state of atom B so that the initial state in step 1 is reached. This completes the cycle which can now start again. What happens to entropy? The entropy of each atom is initially 0. Then error increases the entropy of A to \( \Delta S_A = k \log 2 \). In step 3 atoms get correlated so that they both have the same entropy, i.e. \( \Delta S_B = k \log 2 \). However, the crucial point is that the total entropy does not change from step 2 to step 3. This is the point of observation and the information gained by B about the state of A is \( S_A + S_B - S_{AB} = k \log 2 \). In step 4 \( \Delta S_A = -k \log 2 \) and there are no changes for atom B. In the resetting step \( \Delta S_B = -k \log 2 \), so that now both of the atoms have 0 entropy like at the beginning. Another change that took place, and this is the crux of Landauer’s principle, is that in the compression of atom B, work was invested and the entropy of the environment increased by \( k \log 2 \). This final entropy increase is necessary for resetting and is in this case equal to the amount of information gained in step 3. Landauer’s principle of erasure states that the entropy waste in resetting is at least as big as the information gain. If this were not so, we could use the above cycle to do work by extracting heat from the environment with no other changes and the Second Law of thermodynamics (Kelvin’s form) would be violated. Thus, here an error meant that atom A’s ability to do work has been destroyed and in order to correct this we needed another atom B to transfer its free energy to A. In this process atom B loses its ability to do work and, in order to regain it, an amount of \( k \log 2 \) of entropy has to be wasted (thus ”saving” the Second Law). To gain more familiarity with these kind of processes we will now analyze quantum error correction in general settings and then apply our reasoning to manipulations of entanglement.

III. QUANTUM ERROR CORRECTION AS MAXWELL’S DEMON

The aim of quantum error correction as presented in this section will be to preserve a given quantum state of a quantum mechanical system (Knill & Laflamme 1997), much as a refrigerator is meant to preserve the low temperature of food in a higher temperature environment (room). Some work is performed on the refrigerator which then reduces the entropy of food by increasing the entropy of the surroundings. In accord with the Second law, the entropy increase
in the environment is at least as large as the entropy decrease of the food. Analogously, when there is an error in the state of a quantum system, then the entropy usually increases (this is, however, not always the case as we will see later), and the error correction reduces it back to the original state thereby decreasing the entropy of the system, but increasing the entropy of the environment (or what we will call a garbage can). To quantify this precisely let us look at error correction process in detail (see also Nielsen et al. 1998).

Suppose we wish to protect a pure state \( |\psi\rangle = \sum_i c_i |a_i\rangle \), where \( \{|a_i\rangle\} \) form an orthonormal basis. This is usually done by introducing redundancy, i.e. encoding the state of a system in a larger Hilbert space according to some rule \( |a_i\rangle \rightarrow |C_i\rangle \) where \( \{|C_i\rangle\} \) are the so called code-words. Note that this step was omitted in the classical case. This is because the very existence of system \( B \) can be interpreted as encoding. The main difference between classical and quantum error correction is that errors in classical case can always be distinguished. In quantum mechanics these can lead to non-orthogonal states so that the errors cannot always be distinguished and corrected. So it might be said that the encoding in quantum mechanics makes errors orthogonal and hence distinguishable (a precise mathematical statement of this is given in Knill & Laflamme 1997). Of course, redundancy also exists in classical error correction (above we have another system, \( B \), to protect \( A \)), but the states are already orthogonal and distinguishable by the very nature of being classical. In the quantum case we will introduce an additional system, called the apparatus, in order to detect different errors; this will play the role that \( B \) plays in classical error correction. Now the error correction process can be viewed as a series of steps. First the initial state is

\[
|\psi_c\rangle |m\rangle |e\rangle
\]

where \( |\psi_c\rangle = \sum_i c_i |C_i\rangle \) is the encoded state, \( |m\rangle \) is the initial state of the measuring apparatus and \( |e\rangle \) is the initial state of the environment. Now, the second stage is the occurrence of errors, represented by the operators \( \{E_i\} \) which act on the state of the system only, after which we have

\[
\sum_i E_i |\psi_c\rangle |m\rangle |e_i\rangle
\]

Note that at this stage the measurement has not yet been made so that the state of the apparatus is still disentangled from the rest. In general, the states of the environment \( |e_i\rangle \) need not be orthogonal (Vedral et al. 1997). If they are orthogonal this leads to a specific form of decoherence which we might call ”dephasing” and will be analysed later. However, the formalism we present here is completely general and applies to any form of errors. Now the measurement occurs and we obtain

\[
\sum_i E_i |\psi_c\rangle |m_i\rangle |e_i\rangle
\]

The error correction is seen as an application of \( E_i^{-1} \), conditional on the state \( m_i \) (this, of course, cannot always be performed, but the code-words have to satisfy conditions in Knill & Laflamme 1997 in order to be correctable. Here we need not worry about this, our aim is only to understand global features of error correction). After this the state becomes

\[
|\psi_c\rangle \sum_i |m_i\rangle |e_i\rangle
\]

and the state of the system returns to the initial encoded state; the error correction has worked. However, notice that the state of the apparatus and the environment is not equal to their initial state. This feature will be dealt with shortly. Before that let us note that the total state (system+apparatus+environment) is always pure. Consequently the von Neumann entropy is always zero. Therefore it is difficult to see how this process will be compared to refrigeration where entropy is kept low at the expense of the environment’s entropy increasing. However, in general, the environment is not accessible and we usually have no information about it (if we had this information we would not need error correction!). Thus the relevant entropies are those of the system and apparatus. This means that we can trace out the state of the environment in the above picture; this leads to dealing with mixed states and increasing and decreasing entropies. In addition, the initial state of the system might be pure or mixed (above we assumed a pure state) and these two cases we now analyze separately.
A. Pure states

We follow the above set of steps, but now the environment will be traced out of the picture after errors have occurred. Therefore in the first step the state is

1. after errors $\sum_i E_i |\psi_c\rangle |m\rangle |e_i\rangle$, where we assume the "perfect" decoherence, i.e. $\langle e_j|e_i\rangle = \delta_{ij}$, for simplicity.

2. Now the environment is traced out leading to $\sum_i E_i |\psi_c\rangle \langle \psi_c| E_i^\dagger \otimes |m\rangle |m\rangle$. Note that this is not a physical process, just a mathematical way of neglecting a part of the total state (we introduce the direct product sign just to indicate separation between the system and the apparatus; when there is no possibility of confusion we will omit it).

3. Then the system is observed, thus creating correlations between the apparatus and the system

$$\sum_i E_i |\psi_c\rangle \langle \psi_c| E_i^\dagger \otimes |m_i\rangle |m_i\rangle.$$ We assume that the observation is perfect so that $\langle m_j|m_i\rangle = \delta_{ij}$; we will deal with the imperfect observation in the following section. Note that we need different errors to lead to orthogonal states if we wish to be able to correct them. Here, also, if the observation is imperfect then error correction cannot be completely successful, since non-orthogonal states cannot be distinguished with perfect efficiency (see also the discussion at the end of this section).

4. The correction step happens and the system is decorrelated from the apparatus so that we have $|\psi_c\rangle \langle \psi_c| \otimes \sum_i |m_i\rangle \langle m_i|$. As we remarked before this is not equal to the initial state of the system and apparatus. If we imagine that we have to perform correction a number of times in succession, then this state of apparatus would not be helpful at all. We need to somehow reset it back to the original state $|m\rangle$.

5. This is done by introducing another system, called a garbage can (gc), which is in the right state $|m\rangle$, so that the total state is $|\psi_c\rangle \langle \psi_c| \otimes \sum_i |m_i\rangle \langle m_i| \otimes |m\rangle \langle m|$, and then swapping the state of the garbage can and the apparatus (this can be performed unitarily) so that we finally obtain $|\psi_c\rangle \langle \psi_c| \otimes |m\rangle \langle m| \otimes \sum_i |m_i\rangle \langle m_i|$. Only now the system and the apparatus are ready to undergo another cycle of error correction.

We can now apply the entropy analysis to this error correction cycle. In the first step the entropy of the system+apparatus has increased by $\Delta S_{S+A} = S(\rho)$, where $\rho = \sum_i E_i |\psi_c\rangle \langle \psi_c| E_i^\dagger$. Step 2 is not a physical operation and so there is no change in entropy. In the third step, there is also no change in entropy; it is only that the correlations between the system and the apparatus have been created. Step 4 is the same as the step 2, so no change in entropy on the whole. In step 5, the entropy of the system+apparatus is zero since they are in the total pure state. Thus, $\Delta S_{S+A} = -S(\rho)$, and now we see the formal analogy with the refrigeration process: the net change in entropy of the system+apparatus is zero, and the next error correction step can begin; however, the gc has at the end increased in entropy by $\Delta S_{gc} = S(\rho)$. This is now exactly the manifestation of Landauer’s erasure. The information gain in step 3 is equal to the mutual entropy between the system and the apparatus $I_{S+A} = S_S + S_A - S_{S+A} = S(\rho)$. The logic behind this formula is that before the observation the apparatus did not know anything about the system, therefore system’s state was uncertain by $S(\rho)$, whereas after the observations it is zero - the apparatus knows everything about the system. We note that this information is the Shannon mutual information which exists between the two (Schmidt) observables pertaining to the system and the apparatus. This needs to be erased at the end to start a new cycle and the entropy increase is exactly (in this case) equal to the information gained. So from the entropic point of view we have performed the error correction in the most efficient way, since, in general, the gc entropy increase is larger than the information gained (as we will see in subsection C).

Next we consider correction of mixed states. This might at first appear useless, because we might think that a mixed state is one that has already undergone an error. This is, however, not necessarily so, and this situation occurs when we are, for example, protecting a part of an entangled bipartite system (see Vedral et al. 1997). It might be thought that an analogous case does not exist in classical error correction. There an error was represented by a free expansion of the atom $A$ from step 1 to step 2. However, we could have equally well started from $A$ occupying the whole volume and treating an error as a “spontaneous” compression of the atom to one of the halves. If $A$ was correlated to some other atom $C$ (so that they both occupied LHH or RHH), then this compression would result in decorrelation which really is an error. Thus classical and quantum error correction are in fact very closely related which is also shown by their formal analogy to Maxwell’s demon.
Now suppose that systems $A$ and $B$ are entangled and that we are only performing error correction on the system $A$. Then this is in our case the same as protecting a mixed state. We are not saying that protecting a mixed state is in general the same as protecting entanglement. For example, if a state $|00\rangle + |11\rangle$ flips to $|01\rangle + |10\rangle$ with probability $1/2$, then the entanglement is destroyed, but the reduced states of each subsystem are still preserved. What we mean is that quantum error correction is here developed to protect any pure state of a given system. In that case, any mixed state is also protected, and also any entanglement that it might have with some other systems. This also means that, using the standard quantum error correction (Knill & Laflamme 1997), an entangled pair can be preserved just by protecting each of the subsystems separately. Now, for simplicity say that we have a mixture of two orthogonal states $|\psi\rangle, |\phi\rangle$. The initial state is then without normalization (and without the system $B$)

$$
(|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) \otimes |e\rangle \otimes |m\rangle \langle m|
$$

And now we can go through all the above stages.

1. error: $\sum_i E_i (|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) E_i^\dagger \otimes |e_i\rangle \langle e_i| \otimes |m\rangle \langle m|;

2. tracing out the environment: $\sum_i E_i (|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) E_i^\dagger \otimes |m\rangle \langle m|;

3. observation: $\sum_i E_i (|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) E_i^\dagger \otimes |m_i\rangle \langle m_i|;

4. correction: $(|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) \otimes \sum_i |m_i\rangle \langle m_i|;

5. resetting: $(|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) \otimes \sum_i |m_i\rangle \langle m_i| \otimes |m\rangle \langle m| \\
\rightarrow (|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) \otimes |m\rangle \otimes \sum_i |m_i\rangle \langle m_i|).

The entropy analysis is now as follows. In step 1, $\Delta S_{S+A} = S(\rho_f) - S(\rho_i)$, where $\rho_f = \sum_i E_i (|\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|) E_i^\dagger$ and $\rho_i = |\psi\rangle \langle \psi| + |\phi\rangle \langle \phi|$. In steps 2 and 3 there is no change of entropy, although in step 3 an amount of $I = S(\rho_f)$ information was gained if the correlations between the system and the apparatus are perfect (i.e. $\langle m_i | m_i \rangle = \delta_{ij}$). In step 4, $\Delta S_{S+A} = S(\rho_i)$ as the system and the apparatus become decorrelated. In step 5, $\Delta S_{S+A} = -S(\rho_f)$, but the entropy of the gc increases by $S(\rho_f)$. Thus altogether $\Delta S_{S+A} = 0$, and the entropy of the gc has increased by exactly the same amount as the information gained in step 3 thus confirming Landauer’s principle again.

Now we want to analyze what happens if the observation in step 3 is imperfect. Suppose for simplicity that we only have two errors $E_1$ and $E_2$. Then there would be only two states of the apparatus $|m_1\rangle$ and $|m_2\rangle$; an imperfect observation would imply that $|m_1\rangle |m_2\rangle = a > 0$. Now the entropy of information erasure is $S(|m_1\rangle \langle m_1| + |m_2\rangle \langle m_2|)$ which is smaller than when $|m_1\rangle$ and $|m_2\rangle$ are orthogonal. This implies via Landauer’s principle that the information gained in step 3 would be smaller than when the apparatus states are orthogonal and this in turn leads to imperfect error correction. Thus, doing perfect error correction without perfect information gain is forbidden by the Second law of thermodynamics via Landauer’s principle. This is analogous to von Neumann’s (1952) proof that being able to distinguish perfectly between two non-orthogonal states would lead directly to violation of the Second Law of thermodynamics.

C. General erasure

Previously we described erasure as a swap operation between the gc and the system. Now we will describe a more general way of erasing information, but which will be central to our understanding of entanglement manipulations in section 4. We follow Lubkin’s (1987) analysis in somewhat more general settings.

A more general way of conducting erasure (resetting) of the apparatus is to assume that there is a reservoir which is in thermal equilibrium in a Gibbs state at certain temperature $T$. To erase the state of the apparatus we just throw it into the reservoir and bring in another pure state. The entropy increase of the operation now consists of two parts: the apparatus reaches the state of the reservoir and this entropy is now added to the reservoir entropy, and also
the rest of the reservoir changes its entropy due to this interaction which is the difference in the apparatus internal energy before and after the resetting (no work is done in this process). This quantum approach to equilibrium was also studied by Partovi (1989). A good model is obtained by imagining that the reservoir consists of a great number of systems (of the same "size" as the apparatus) all in the same quantum equilibrium state $\omega$. Then the apparatus, which is in some state $\rho$, interacts with these reservoir systems one at a time. Each time there is an interaction, the state of the apparatus approaches more closely the state of the reservoir, while that single reservoir system also changes its state away from the equilibrium. However, the systems in the bath are numerous so that after a certain number of collisions the apparatus state will approach the state of the reservoir, while the reservoir will not change much since it is very large (this is equivalent to the so called Born-Markov approximation that leads to irreversible dynamics of the apparatus described here).

Bearing all this in mind, we now reset the apparatus by plunging it into a reservoir in a thermal equilibrium at temperature $T$. Let the state of the reservoir be

$$\omega = \frac{e^{-\beta H}}{Z} = \sum_j q_j |\varepsilon_j\rangle \langle \varepsilon_j|$$

where $H = \sum_i \varepsilon_i |\varepsilon_i\rangle \langle \varepsilon_i|$ is the Hamiltonian of the reservoir, $Z = \text{tr}(e^{-\beta H})$ is the partition function and $\beta^{-1} = kT$, where $k$ is the Boltzmann constant. Now suppose that due to the measurement the entropy of the apparatus is $S(\rho)$ (and an amount $S$ of information has been gained), where $\rho = \sum_i r_i |r_i\rangle \langle r_i|$ is the eigen expansion of the apparatus state. Now the total entropy increase in the erasure is (there are two parts as we argued above: 1. change in the entropy of the apparatus and 2. change in the entropy of the reservoir)

$$\Delta S_{er} = \Delta S_{app} + \Delta S_{res}$$

We immediately know that $\Delta S_{app} = S(\omega)$, since the state of apparatus (no matter what state it was before) is now erased to be the same as that of the reservoir. On the other hand, the entropy change in the reservoir is the average over all states $|r_i\rangle$ of heat received by the reservoir divided by the temperature. This is minus the heat received by the apparatus divided by the temperature; the heat received by the apparatus is the internal energy after the resetting minus initial internal energy $\langle r_i \mid H \mid r_i \rangle$. Thus,

$$\Delta S_{res} = -\sum_k r_k \frac{\text{tr}(\omega H) - \langle r_k \mid H \mid r_k \rangle}{T}$$

$$= \sum_k (r_k \sum_j |\langle r_k \mid \varepsilon_j\rangle|^2 - q_k)(-\log q_k - \log Z)$$

$$= -\text{tr}(\rho - \omega)(\log \omega - \log Z)$$

$$= \text{tr}(\omega - \rho) \log \omega$$

Altogether we have that

$$\Delta S_{er} = -\text{tr}(\rho \log \omega)$$

(This result generalizes Lubkin’s (1987) result which applies only when $|\rho, \omega| = 0$). In general, however, the information gain is equal to $S(\rho)$, the entropy increase in the apparatus. Thus, we see that

$$\Delta S_{er} = -\text{tr}(\rho \log \omega) \geq S(\rho) = I$$

and Landauer’s principle is confirmed (the inequality follows from the fact that the quantum relative entropy $S(\rho||\omega) = -\text{tr}(\rho \log \omega) - S(\rho)$ is non-negative). So the erasure is least wasteful when $\omega = \rho$, in which case the entropy of erasure is equal to $S(\rho)$, the information gain. This is when the reservoir is in the same state as the state of the apparatus we are trying to erase. In this case we just have a state swap between the new pure state of the apparatus which is used to replace our old state $\rho$. This, in fact, was the case in all our examples of error correction above. However, sometimes it is impossible to meet this condition and it is this case we turn to next.
IV. ENTANGLEMENT PURIFICATION

A. General Considerations

Entanglement purification is a procedure whereby an ensemble of bipartite quantum systems all in a state \( \rho \) is converted to a subensemble of pure maximally entangled states by local operation on the systems separately and with the aid of classical communications (Bennett et al. 1996a). The rest of the pairs end up completely separable (i.e. disentangled) and can be taken to be in a pure state of the form \( |\psi\rangle |\phi\rangle \). For the sake of simplicity let us first assume that the initial ensemble is in a pure, but not maximally entangled state. To link this with our previous analysis let us see how this situation might arise. Let a system \( S \) be in the state

\[
|\psi_S\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |s_i\rangle
\]

where \( \{ |s_i\rangle \} \) is an orthonormal basis. Let now the apparatus observe this superposition after which the state is

\[
|\psi_{S+A}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |s_i\rangle |a_i\rangle
\]

but let the observation be imperfect so that \( \{ |a_i\rangle \} \) is NOT an orthogonal set (which means that \( S \) and \( A \) are not maximally entangled). However, suppose that by acting on the apparatus we can transform the whole state \( |\psi_{S+A}\rangle \) into the maximally entangled state

\[
|\phi_{S+A}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |s_i\rangle |b_i\rangle
\]

where \( \{ |b_i\rangle \} \) is an orthogonal set. This does not increase the information between the apparatus and the system since we are not interacting with the system at all. The crucial question is: what is the probability with which we can do this? Let

\[
\text{tr}_S(|\psi_{S+A}\rangle \langle \psi_{S+A}|) = \rho_A
\]

be the state of the apparatus after the perfect measurement. Then Landauer’s principle says that the entropy of erasure is \( S(\rho_A) \) and this has to be greater than or equal to the information gain. But let us look at what the information gain is after we purified the state to \( |\phi_{S+A}\rangle \) with a probability \( p \). First of all, we gained \( p \log N \) information about the system since we have maximal correlations now. Secondly the rest of the state contains no information (we assume it is completely disentangled) and this is with probability \( (1-p) \). Thus writing Landauer’s principle leads to

\[
S(\rho_A) \geq p \log N
\]

The upper bound to purification efficiency is therefore

\[
p \leq \frac{S(\rho_A)}{\log N}
\]

That the upper bound is achievable was shown by Bennett et al (1996a). We stress that in this reasoning we used that the entropy erasure is greater than or equal to the information gain before purification (by Landauer), and this in turn is greater than or equal to the information after purification (since, in purification, the apparatus does not interact with the system).

Let us now analyze when the initial state of the system and the apparatus is mixed in a state \( \rho \). Then by Lubkin’s method the entropy of erasure is

\[
\Delta S_{er} = -\text{tr}(\rho \log \omega)
\]

The information gain after the purification is

\[
I = S(\rho) + p \log N
\]
where $S(\rho)$ comes from the fact that all the states after purification are pure (being either maximally entangled or disentangled), so that the total information gain is the uncertainty before (i.e. $S(\rho)$) minus the uncertainty after (i.e. zero). We can view this in yet another way. The fact that $\rho$ is mixed means that it is entangled to another system, which we call ancilla. The total state of system+apparatus+ancilla can always be chosen to be pure. Now, system+apparatus are correlated to the ancilla, since the system+apparatus itself is in a mixed state (the ancilla is also mixed and has the same eigen-spectrum as system+apparatus which means that it also has the same von Neumann entropy). After the purification this correlation disappears, and the resulting information is $S(\rho)$. Using Landauer’s principle now gives

$$-\text{tr}(\rho \log \omega) \geq S(\rho) + p \log N$$

or

$$p \leq S(\rho|\omega)/\log N$$

where $S(\rho|\omega) = -\text{tr}(\rho \log \omega) - S(\rho)$ is quantum relative entropy which we met in the previous section. There a tight upper bound on purification is found when $S(\rho|\omega)$ is minimal. However, remembering that we now acted on the system and the apparatus separately (i.e. all the operations were local), the state of the reservoir for resetting will also have to be local (i.e. separable or disentangled as in Vedral & Plenio 1998). This means that the system is reset by plunging it into its own reservoir and the apparatus is reset by plunging it into its own, but separate, reservoir. By reservoir we will always mean two of these reservoirs together unless stated otherwise. We can always assume that they are in a separable, but classically correlated state as will be shown later in this section. Let us call the set of all separable states of the system and the apparatus $D$. Then,

$$p \leq \min_{\omega \in D} S(\rho|\omega)/\log N$$

The quantity $E_{RE}(\rho) = \min_{\omega \in D} S(\rho|\omega)$ is known as the relative entropy of entanglement and has been recently argued by Vedral and Plenio (1998) and separately Rains (1998) to be an upper bound on the efficiency of purification procedures. For $\rho$ pure this reduces to our previous result. These results were originally derived from the principle that entanglement, being a non-local property, cannot increase under local operations and classical communications. This implied finding a mathematical form of local operations and then finding a measure which is non-increasing under them (Vedral & Plenio 1998). Now, we have derived the same result in a simpler and more physical way, but from at first sight completely different direction, by taking Landauer’s principle. We have thus related the "no local increase of entanglement" principle to the Second Law of thermodynamics. Let us briefly summarize this link. Local increase of entanglement is in this context equivalent to perfectly distinguishing between non-orthogonal states. If we have this ability we can then violate the Second law with Maxwell’s demon that Bennett (1982) used in his analysis (also von Neumann (1952) showed this by a different argument). Therefore, local increase of entanglement is seen to be prohibited by the Second Law. The additional principle that we used is that information about some system can only be gained if we interact with it. We emphasize that this principle is not necessarily related only to entanglement. We saw in the previous section that, in error correction, the apparatus correlates itself with the system in order to correct errors and these correlations are purely classical in nature. It is here, however, that the link between “no local increase of entanglement” and Landauer’s principle is most clearly seen: entanglement between two subsystems cannot be increased unless one subsystem gains more information about the other one, but this cannot be done locally without interaction.

It should be noted that the reservoirs in general have to be classically correlated (in order to obtain a tight upper bound on purification; otherwise, the bound still holds, but is in general too high). This poses a question as to how this could be achieved in practice. A way to do that is to remember that this result is applicable in the asymptotic case, meaning that we can average over a large number of different, but uncorrelated reservoir states, which are certainly natural to consider. So, if $\omega = \sum_i p_i \omega^i_\beta \otimes \omega^j_A$ is the state of the reservoir achieving the minimum (where $S$ refers to the reservoir into which the system is immersed and $A$ to the reservoir into which the apparatus is immersed to erase their mutual information), then this implies that we would be using reservoirs of the form $\omega^i_\beta \otimes \omega^j_A$ with frequencies $p_i$ and this would on average produce the state $\omega$. Therefore if we consider at $n$ initial mixed states of system+apparatus, then to delete these correlations, we use $p_1 n$ times the uncorrelated reservoir state $\omega^i_\beta \otimes \omega^j_A$, $p_2 n$ times the uncorrelated state $\omega^2_\beta \otimes \omega^2_A$ and so on. Note that while each of these reservoirs is in a thermal state with a well defined temperature, the total (classically correlated) state does not have a well defined temperature. In general, however, the bound is universal and holds no matter how this purification is performed as long as it is local in character. Also note that if we are allowed to have a common reservoir for the system and the apparatus, then the erasure can be different to the above, i.e. it can be smaller than the above local erase. However, if we allow nonlocal operations, then we can also create additional entanglement and the above analysis would not apply. An
alternative way to view this local erasure is to allow a common reservoir for the system and the apparatus, but to restrict its thermal states to separable states only. Then we are sure that no additional entanglement is created to the already existing between the system and the apparatus and all our results remain true.

This now leads us to state another, equivalent way of interpreting the relative entropy of entanglement. Suppose that we again have a disentangled bipartite system+apparatus, but this time in a thermal state \( \omega = e^{-\beta H} / Z \), where \( \beta^{-1} = kT \), \( k \) is the Boltzmann constant, \( T \) the temperature, \( H \) is the Hamiltonian and \( Z \) the partition function (note that this is now the state of system+apparatus which is the same as the state of the reservoir used for resetting. Here, however, there are no further locality requirements and we do have a well defined temperature). The question that we wish to ask now is how much free energy would this system gain by going to the entangled state \( \rho \) (see Donald 1987 for applications of quantum relative entropy to statistical mechanics, and Partovi 1989 for the quantum basis of thermodynamics)? Note that here we allow any operations, and are not restricted by locality requirements. First of all, the free energy is given by

\[
F = U - TS
\]

so that

\[
F(\rho) = \text{tr}(\rho H) - \beta^{-1} S(\rho) \\
F(\omega) = -\beta^{-1} \log Z
\]

Therefore the difference in free energies of \( \rho \) and \( \omega \) is

\[
F(\rho) - F(\omega) = \beta^{-1} S(\rho | \omega)
\]

In the light of this, entanglement as given by the relative entropy would be proportional to the amount of free energy lost in deleting all the quantum correlations and creating solely classical correlations by e.g. plunging the system+apparatus into a reservoir whose thermal state is given by that classically correlated state (now we do allow a common reservoir with a well defined thermal state and temperature). The constant of proportionality is the temperature times the Boltzmann constant \( k \).

We stress that entanglement of creation, another measure of quantum correlations introduced by Bennett et al (1996b), can also be interpreted using above methods. It is also an upper bound to the efficiency of purification procedures although it is not tight as it is larger than the relative entropy of entanglement (Vedral & Plenio 1998). It arises from applying Landauer’s principle to each pure state in the decomposition of \( \rho = \sum_i \rho_i |\psi_i\rangle \langle \psi_i| \) (note that this is not necessarily the eigen-decomposition). In fact, we can define the entanglement of creation via the free energy as the average decrease in free energy due to resetting each of \( |\psi_i\rangle = \sum_j c_{ij} |a_j\rangle \langle b_j| \) to a completely separable state \( \omega_i = \sum_j |c_{ij}|^2 |a_j\rangle \langle b_j| \langle a_j| \langle b_j| \) (strictly speaking, the entanglement of creation is actually the minimum of this quantity over all possible decompositions of \( \rho \)). Mathematically this can be expressed as

\[
E_C = \frac{\min_{\text{decomp of } \rho} \sum_i \Delta F_i}{T}
\]

where \( \Delta F_i = \beta^{-1} S(\rho_i | \omega_i) \) (which is equal to the von Neumann reduced entropy of \( |\psi_i\rangle \) Vedral & Plenio 1998).

At the end of this section we emphasize again why local operations were important when we wanted to derive a bound on purification of entanglement. First of all, it is believed that entanglement does not increase under local operations and classical communication. Here, however, we did not use this fact to derive bounds on the efficiency of purification procedures. We only used the fact that local operations on one subsystem (e.g. apparatus) do not tell us anything more about some other subsystem than is already known; in other words, if the operations were not local the above analysis would be wrong. In fact, the upper bounds we derived on entanglement purification can now be restated: the most successful purification is the one which wastes no information, i.e. the free energy needed to reset the state of the ensemble before purification should be equal to the free energy needed to reset the state of the ensemble after purification (i.e. Landauer’s erasure bound is saturated). At the end of this section we showed an equivalent way of interpreting the relative entropy of entanglement where the operations are not restricted to the local ones only, but the reservoir state still has to be disentangled.
B. Ensemble versus single trial view

All the results we considered above are actually appropriate from the ensemble point of view. To explain what we mean by this consider the following problem. Suppose that Alice and Bob share a single pair of entangled systems in a state

\[ |\psi\rangle = a |0\rangle |0\rangle + b |1\rangle |1\rangle \]

Then it can be proven (Lo & Popescu 1997) that the best efficiency (i.e., the highest probability) with which they can purify this state by acting locally on their own systems is \( 2b^2 \) if \( b < a \). However, \( 2b^2 < -a^2 \log a^2 - b^2 \log b^2 \) (the equality is achieved only when \( b^2 = 1/2 \), and the state is already maximally entangled) which is the efficiency we derived above from Landauer’s erasure principle. Thus Landauer’s efficiency limit is only reached asymptotically when Alice and Bob share and infinite ensemble of entangled systems and they operate on all of their particles at the same time. So, although Landauer’s erasure holds true even if we operate on single pairs (a “single shot” measurement), it gives an overestimate of the efficiency of this process. We note that, strictly speaking, both of these problems involve ensembles, but in the single shot view we are allowed to act on only one pair at a time, whereas otherwise we can act on all of them simultaneously. It is therefore no surprise that the former method, which is a special case of the latter method, is less efficient. This is the reason why the bound we presented in this paper is too high for the single shot purification. On the whole, however, by performing erasure the way we imagined, where we use thermal reservoirs to delete information, we have derived a universal upper bound no matter how the purification is performed. Thus an open question would be whether it is more appropriate to use a different measure for erasure in the single shot case since the amount of entanglement as measured by the purification efficiency is different for a single pair measurement and for an ensemble measurement. This would be important to consider, since most of the practical manipulations at present involve only a few entangled particles and, as we said, the entropic measures overestimate various efficiencies of entanglement processing. So, in our bounds on entanglement purification, \( \rho \) should actually stand for \( \rho^n = \rho \otimes \rho \ldots \otimes \rho \) (\( n \) times), where each \( \rho \) now refers to a single pair of quantum systems (also \( \log N \) would become \( n \log N \). This brings us to the question of whether \( \min_{\omega \in D} S(\rho^n | | \omega) = n \min_{\omega \in D} S(\rho | | \omega) \), i.e., whether the relative entropy of entanglement is additive, which is still open (although see Rains 1998). A reasonable conjecture is that in all the quantum information manipulations that involve large ensembles the above reasoning will be suitable. Examples of this are quantum data compression (Schumacher 1995) and capacity of a quantum channel (see, for example, Feynman’s (1996) derivation of Shannon’s coding for classical binary symmetric channels using Landauer’s principle). In quantum data compression, for instance, the free energy lost in deleting before compression is \( n \beta^{-1} S(\rho) \) and after the compression is \( m \beta^{-1} \log N \). These two free energies should be equal if no information is lost (i.e., if we wish to have maximum efficiency) in compressing and therefore \( m/n = S(\rho)/ \log N \) as shown by Schumacher 1995.

V. CONCLUSIONS

We have analyzed general classical and quantum error correcting procedure from the entropic perspective. We have shown that the amount of information gained in the observation step needed to perform correction is then turned into an equal amount of wasted entropy in a gc. This gc is needed to reset the apparatus to its initial state so that the next cycle of error correction can be performed. This fact is equally true both in the classical the quantum case and is known as Landauer’s principle of information erasure. We then analyzed purification procedures using the same principle. Surprisingly, Landauer’s principle when applied appropriately yields the correct upper bounds to the efficiency of purification procedures. Whether these bounds can be achieved in general remains an open question. Landauer’s principle therefore provides a physical basis for several entanglement measures, notably relative entropy of entanglement and the entanglement of creation. In addition this provides a link between the principle of “no local increase of entanglement” and the Second Law of thermodynamics. Further open questions are the implications of Landauer’s erasure to other forms of quantum information manipulations such as quantum cloning.

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Fig 1. Classical error correction as a Maxwell’s demon. Steps are detailed in the text and their significance explained:
(1) states of atoms A and B are initially uncorrelated; (2) atom A undergoes an error; (3) atom B observes the atom A, and the atoms thereby become correlated; (4) atom A is corrected to its initial state and the atoms are now uncorrelated; (5) atom B is returned to its initial state and the whole cycle can start again.