Stability of analytical and numerical solutions of nonlinear stochastic delay differential equations

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Abstract
This paper deals with stability of analytical and numerical solutions of nonlinear stochastic delay differential equations (SDDEs). Sufficient conditions for stability in mean square, contractivity in mean square and asymptotic contractivity in mean square of the solutions of nonlinear SDDEs are given, which provide a unified theoretical treatment for constant delay and variable delay, bounded delay and unbounded delay. Stability in mean square, contractivity in mean square and asymptotic contractivity in mean square of the backward Euler method are investigated. It is shown that the backward Euler method can inherit the properties of the underlying system. The main theorems in this paper are different from Razumikhin-type theorems. It is not required to construct or find an appropriate Lyapunov functional when the theorems in this paper are applied.

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1 Introduction
Many real-life phenomena in physics, engineering, economics, etc can be modeled by stochastic differential equations (SDEs). The rate of change of such a system depends only on its present state and some noisy input. However, in many situations the rate of change of the state depends

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not only on the present but also on the past states of the system. Stochastic functional differential equations (SFDEs) give a mathematical formulation for such system. For more details on SFDEs, the reader may refer to [11, 12, 13] and the references therein.

SFDEs also can be regarded as a generalization of deterministic functional differential equations if stochastic effects are taken into account. For deterministic Volterra functional differential equations (VFDEs), Li [8] discussed stability, contractivity and asymptotic stability of the solutions in Banach spaces. The main techniques used in [8] are as follows: The author introduced a so-called $\frac{1}{n}$-perturbed problem and constructed an auxiliary function $Q(t)$. The former is beneficial to deal with a wide variety of delay arguments and the latter is the key to obtain main results in that paper. [8] provided a unified framework for stability analysis of solutions to nonlinear stiff problems in ordinary differential equations (ODEs), delay differential equations (DDEs), integro-differential equations (IDEs) and VFDEs of other type appeared in practice. In 2008, Wang [17] extended successfully the theory in [8] to nonlinear Volterra neutral functional differential equations (VNFDEs). Wang and Zhang [18] proved implicit Euler method can preserve the stability of VFDEs and VNFDEs.

It is natural to ask whether the solutions of SFDEs possess the properties similar to those presented in [8] and which methods can reproduce the properties. Due to features of the stochastic calculus the numerical analysis of SFDEs differs in some key areas from the already well-developed area of the numerical analysis of their deterministic counterparts. In the literature, much attention on numerical stability has been focused on a special class of SFDEs, that is, stochastic delay differential equations (SDDEs). The stability includes mean-square stability, asymptotic stability, exponential mean-square stability and so on. Some results on numerical stability can be found in [1, 10, 16, 19, 20, 21]. However, most of the existing numerical stability theory of SDDEs only deal with bounded lags. Far less is known for long-run behavior of SDDEs with unbounded lags. Very recently, Fan, Song and Liu [3] discussed the mean square stability of semi-implicit Euler methods for linear stochastic pantograph equation. Moreover, to the best knowledge of authors, there is no work on contractivity analysis of numerical methods for SDDEs. These motivate our work. This is the first paper to investigate stability and contractivity of general SDDEs with bounded and unbounded lags and study whether numerical methods can preserve the properties.

The main contributions of this paper could be summarized as follows:

(a) Sufficient conditions for stability in mean square, contractivity in mean square and asymptotic contractivity in mean square of the solutions of nonlinear SDDEs are given, which provide a unified theoretical treatment for constant and variable delay, bounded and unbounded delay. Applicability of the theory is illustrated by linear and nonlinear SDDEs with a wide variety of delay arguments such as constant delays, piecewise constant arguments, proportional delays and so on. The theorems established in this paper work for some SDDEs to which the existing results cannot be applied. The main results of analytic solution in this paper can be regarded as a generalization of those in [8] restricted in finite-dimensional Hilbert spaces and finitely many delays to the stochastic version.

(b) It is proved that the backward Euler method can reproduce stability in mean square, contractivity in mean square and asymptotic contractivity in mean square of the underlying system. In particularly, Theorem 4.2 and Theorem 4.4 show that the backward Euler method can inherit the contractivity and the asymptotic contractivity without any constraint on stepsize.

We point out that the main theorems in the present paper are different from the Razumikhin-type theorems in the literature [11]. It is not required to construct or find an appropriate
Lyapunov functional when the theorems in this paper are applied to prove the stability, in contrast to the Razumikhin-type theorems. In this sense, it is more convenient to use the theorems in this paper than the Razumikhin-type theorems.

The rest of the paper is arranged as follows. In section 2, we introduce some necessary notations and assumptions. In section 3, some criteria for the stability in mean square, contractivity in mean square and asymptotic contractivity in mean square of solutions of nonlinear SDDEs are established. The main results obtained in this section are applied to SDDEs with bounded and unbounded lags, respectively. In section 4, sufficient conditions on stability in mean square, contractivity in mean square and asymptotic contractivity in mean square of the backward Euler method are derived. Stability of analytic and numerical solutions of SDDEs with several delays is discussed in section 5.

2 Stochastic delay differential equations

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq a}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq a}\) satisfying the usual conditions (i.e., it is right continuous and \(\mathcal{F}_a\) contains all the \(\mathbb{P}\)-null sets). Let \(w(t) := (w_1(t), \ldots, w_m(t))^T\) be an \(m\)-dimensional Wiener process defined on the probability space. And \(|\cdot|\) is used to denote both the norm in complex Hilbert space \(\mathbb{C}^d\) and the trace norm (F-norm) in \(\mathbb{C}^{d \times m}\). Also, \(C([t_1, t_2]; \mathbb{C}^d)\) is used to represent the family of continuous mappings \(\psi\) from \([t_1, t_2]\) to \(\mathbb{C}^d\). Finally, \(L^p_{\mathcal{F}_t}([t_1, t_2]; \mathbb{C}^d)\) is used to denote a family of \(\mathcal{F}_t\)-measurable, \(C([t_1, t_2]; \mathbb{C}^d)\)-valued random variables \(\psi = \{\psi(u) : t_1 \leq u \leq t_2\}\) such that \(\|\psi\|_p^p := \sup_{t_1 \leq u \leq t_2} E|\psi(u)|^p < \infty\). \(\mathbb{E}\) denote mathematical expectation with respect to \(\mathbb{P}\).

Consider the following initial value problems of SDDEs in the sense of Itô

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(t, x(t), x(t - \tau(t)))dt + g(t, x(t), x(t - \tau(t)))dw(t), \quad t \in [a, b], \quad (2.1a) \\
x(t) &= \xi(t), \quad t \in [a - \tau, a], \quad (2.1b)
\end{align*}
\]

where \(a, b, \tau\) are constants, \(-\infty < a < b < +\infty, \tau \geq 0, \tau(t) \geq 0, \inf_{a \leq t \leq b} (t - \tau(t)) \geq a - \tau, \xi \in L^p_{\mathcal{F}_a}([a - \tau, a]; \mathbb{C}^d), p > 2, f : [a, b] \times \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^d, g : [a, b] \times \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^{d \times m}\) are given continuous mappings. We assume that \(f\) and \(g\) satisfy the following conditions.

For each \(R > 0\) there exists a constant \(C_R\), depending only on \(R\), such that

\[
|f(t, \bar{x}_1, \bar{y}) - f(t, \tilde{x}_2, \tilde{y})| \leq C_R|\bar{x}_1 - \tilde{x}_2|, \quad |\bar{x}_1| \vee |\tilde{x}_2| \vee |\bar{y}| \leq R, \quad (2.2)
\]

\[
\Re(x_1 - x_2, f(t, x_1, y) - f(t, x_2, y)) \leq \alpha(t)|x_1 - x_2|^2, \quad (2.3)
\]

\[
|f(t, x, y_1) - f(t, x, y_2)| \leq \beta(t)|y_1 - y_2|, \quad (2.4)
\]

\[
|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq \gamma_1(t)|x_1 - x_2| + \gamma_2(t)|y_1 - y_2|, \quad (2.5)
\]

for all \(t \in [a, b], x, x_1, x_2, y, y_1, y_2 \in \mathbb{C}^d\). Here \(\alpha(t), \beta(t), \gamma_1(t)\) and \(\gamma_2(t)\) are continuous real-valued functions. We introduce the following notations:

\[
\mu_1^{(0)} = \inf_{a \leq t \leq b} \tau(t) \geq 0, \quad \mu_2^{(0)}(t_1, t_2) = \inf_{t_1 \leq t \leq t_2} (t - \tau(t)) \geq a - \tau,
\]

\[
\forall t_1, t_2 : a \leq t_1 \leq t_2 \leq b.
\]

For convenience, we denote by \(SD(\alpha, \beta, \gamma_1, \gamma_2)\) the all problems (2.1) which satisfy the conditions (2.2)-(2.5). Examples of linear and nonlinear equations which satisfy the conditions (2.2)-(2.5) are given in the next section (see Example 3.16 and Example 3.20).
In order to deal with a wide variety of delay arguments in this paper, we introduced the so-called $\frac{1}{n}$-perturbed problem of (2.1) which first introduced by Li [8] for VFDEs. Let

$$f_n(t, x, \psi) = f(t, x, \psi^{(n,t)}), \quad g_n(t, x, \psi) = g(t, x, \psi^{(n,t)}),$$

for $t \in [a, b], x \in \mathbb{C}^d, \psi \in C([a - \tau, b]; \mathbb{C}^d)$, where $\psi^{(n,t)}$ is defined by

$$\psi^{(n,t)}(u) = \begin{cases} \psi(u), & u \in [a - \tau, t - \frac{1}{n}], \\ \psi(t - \frac{1}{n}), & u \in (t - \frac{1}{n}, b]. \end{cases}$$

Then the initial problem

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{dx(t)}{dt} = f_n(t, x(t), x(t - \tau(t))) dt + g_n(t, x(t), x(t - \tau(t))) dw(t), \quad t \in [a, b], \\
x(t) = \xi(t), \quad t \in [a - \tau, a]
\end{array} \right.
\end{align*}$$

is said to be the $\frac{1}{n}$-perturbed problem of the problem (2.1). Here the natural number $n > \frac{1}{\tau}$ can be arbitrarily given. Notice that for the case of $\tau = 0$, we can replace it by some positive number $\tilde{\tau}$ and define $\xi(u) = \xi(a)$ for $u \in [a - \tilde{\tau}, a]$. Therefore, without lose of generality, we always assume $\tau > 0$.

It is easy to verify that, if problem (2.1) $\in \mathcal{SD}(\alpha, \beta, \gamma_1, \gamma_2)$, then its $\frac{1}{n}$-perturbed problem (2.8) $\in \mathcal{SD}(\alpha, \beta, \gamma_1, \gamma_2)$. Let $\tilde{\tau}(t) = \max\{\tau(t), \frac{1}{n}\}$, we have

$$\tilde{\mu}_1^{(0)} = \inf_{a \leq t \leq b} \tilde{\tau}(t) \geq \frac{1}{n}. \quad (2.9)$$

3 Stability analysis of SDDEs

We will discuss three types of stability of SDDEs in this section and here give definitions of them.

**Definition 3.1** The solution of problem (2.1) is said to be stable in mean square if

$$\mathbb{E}|x(t) - y(t)|^2 \leq C \sup_{a - \tau \leq \theta \leq a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2,$$

where $y(t)$ is a solution of any given perturbed problem

$$\begin{align*}
\left\{ \begin{array}{l}
dy(t) = f(t, y(t), y(t - \tau(t))) dt + g(t, y(t), y(t - \tau(t))) dw(t), \quad t \in [a, b], \\
y(t) = \eta(t), \quad t \in [a - \tau, a], \quad \eta \in L^p_{\mathcal{F}_a}([a - \tau, a]; \mathbb{C}^d).
\end{array} \right.
\end{align*}$$

**Definition 3.2** The solution of problem (2.1) is said to be contractive in mean square if (3.1) with $C \leq 1$ holds.

**Definition 3.3** The solution of problem (2.1) is said to be asymptotically contractive in mean square if

$$\lim_{t \to +\infty} \mathbb{E}|x(t) - y(t)|^2 = 0,$$

where $[a, b]$ replaced by $[a, +\infty)$. 

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Remark 3.4 In the strict sense, the solution of (2.7) should be called generalized contractive in mean square if (3.7) with \( C \leq 1 \) holds. For convenience, we call it contractive in mean square for brevity.

There exist well-known stability definitions in the literature which are closely related to those presented in this paper, but there are differences among them. The existing notions of stability include mean-square stability for stochastic differential equations, that is, \( \lim_{t \to +\infty} \mathbb{E}|x(t)|^2 = 0 \) (cf. [14]); exponential mean-square contraction of trajectories for stochastic differential equations with jumps (cf. [6]). The contractivity in mean square is weaker than that in [6].

The continuity of \( f \) and \( g \) implies that

\[
|f(t,0,0)| \leq C, \quad |g(t,0,0)| \leq C, \quad t \in [a,b],
\]

where \( C \) only depends on \( f, g \) and the interval \([a,b] \). We note that condition (2.5) implies that the diffusion coefficient \( g \) satisfies local Lipschitz condition

\[
|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq C_R (|x_1 - x_2| + |y_1 - y_2|), \quad t \in [a,b], x_1, x_2, y_1, y_2 \in \mathbb{C}^d, \quad |x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R.
\]

In fact, we can choose any \( C_R \) with \( C_R \geq \max \{ \max_{a \leq t \leq b} \gamma_1(t), \max_{a \leq t \leq b} \gamma_2(t) \} \). Using (2.3), (2.4) and (3.5), we have

\[
\Re \langle x, f(t,x,y) \rangle = \Re \left\langle x - 0, f(t,x,y) - f(t,0,y) + f(t,0,y) - f(t,0,0) + f(t,0,0) \right\rangle \\
\leq \alpha(t)|x|^2 + \beta(t)|y| + C|x| \leq C_1 \left( 1 + |x|^2 + |y|^2 \right),
\]

where \( C_1 \) only depends on \( C, \max_{a \leq t \leq b} \alpha(t) \) and \( \max_{a \leq t \leq b} \beta(t) \). By (2.5), we have

\[
|g(t,x,y)|^2 \leq 2|g(t,x,y) - g(t,0,0)|^2 + 2|g(t,0,0)|^2 \leq C_1 \left( 1 + |x|^2 + |y|^2 \right),
\]

where \( C_1 \) only depends on \( C, \max_{a \leq t \leq b} \gamma_1^2(t) \) and \( \max_{a \leq t \leq b} \gamma_2^2(t) \).

3.1 Finite interval

In order to prove the main theorems in this section, we prepare the following lemmas.

Lemma 3.5 Assume that problem (2.7) \( \in \mathcal{SD}(\alpha, \beta, \gamma_1, \gamma_2) \). Then for each \( p \geq 2 \) there is \( \bar{C} = \bar{C}(p,a,b,\alpha,\beta,\gamma_1,\gamma_2) \) such that

\[
\mathbb{E} \left( \sup_{a \leq t \leq b} |x(t)|^p \right) \leq \bar{C} \left( 1 + \mathbb{E} \left( \sup_{a-\tau \leq t \leq a} |\xi(t)|^p \right) \right) =: A.
\]

Proof. For every integer \( k \geq 1 \), define the stopping time

\[
\rho_k = \inf \{ t \in [a,b] : \sup_{a-\tau \leq \theta \leq t} |x(\theta)| \geq k \},
\]

where \( \tau > 0 \) is a fixed constant. Then for \( \tau \leq \rho_k \) we have

\[
\mathbb{E} \left( \sup_{a \leq t \leq b} |x(t)|^p \right) \leq \bar{C} \left( 1 + \mathbb{E} \left( \sup_{a-\tau \leq t \leq a} |\xi(t)|^p \right) \right) =: A.
\]
where we use the convention \( \inf \emptyset = b \). Clearly, \( \rho_k \uparrow b \) almost surely as \( k \to +\infty \). Let \( x^k(t) = x(t \wedge \rho_k) \). Using the Itô formula, we have for \( a \leq t \leq b \)

\[
\left( 1 + |x^k(t)|^2 \right)^{\frac{p}{2}} = \left( 1 + |\xi(a)|^2 \right)^{\frac{p}{2}}
+ p \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} \Re(x^k(s), f(s, x^k(s), x^k(s - \tau(s)))) I_{[a, \rho_k]} ds
+ \frac{p}{2} \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} |g(s, x^k(s), x^k(s - \tau(s)))|^2 I_{[a, \rho_k]} ds
+ \frac{p(p-2)}{2} \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} |(x^k(s))^T g(s, x^k(s), x^k(s - \tau(s))))|^2 I_{[a, \rho_k]} ds
+ p \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} \Re(x^k(s), g(s, x^k(s), x^k(s - \tau(s)))) I_{[a, \rho_k]} dw(s).
\]

It follows from (3.5), (3.6) and this expression that

\[
\left( 1 + |x^k(t)|^2 \right)^{\frac{p}{2}} \leq 2^{\frac{p-2}{2}} \left( 1 + |\xi(a)|^p \right) + 2pC_1 \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} \left( 1 + \sup_{s - \tau(s) \leq u \leq s} |x^k(u)|^2 \right) ds
+ pC_1 \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} \left( 1 + \sup_{s - \tau(s) \leq u \leq s} |x^k(u)|^2 \right) ds
+ \frac{p(p-2)}{2} \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} \Re(x^k(s), g(s, x^k(s), x^k(s - \tau(s)))) I_{[a, \rho_k]} ds
+ p \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} \Re(x^k(s), g(s, x^k(s), x^k(s - \tau(s)))) I_{[a, \rho_k]} dw(s)
\]

\[
\leq 2^{\frac{p-2}{2}} \left( 1 + |\xi(a)|^p \right) + p(p+1)C_1 \int_a^t \left( 1 + \sup_{s - \tau(s) \leq u \leq s} |x^k(u)|^2 \right)^{\frac{p}{2}} ds
+ p \int_a^t \left( 1 + |x^k(s)|^2 \right)^{\frac{p-2}{2}} \Re(x^k(s), g(s, x^k(s), x^k(s - \tau(s)))) I_{[a, \rho_k]} ds,
\]

which yields

\[
\mathbb{E} \sup_{a \leq s \leq t} \left( 1 + |x^k(s)|^2 \right)^{\frac{p}{2}} \leq 2^{\frac{p-2}{2}} \left( 1 + \mathbb{E} \sup_{a \leq s \leq a - \tau(s) \leq a} |\xi(s)|^p \right) + C_2 \mathbb{E} \int_a^t \left( 1 + \sup_{a - \tau(s) \leq u \leq a} |x^k(u)|^2 \right)^{\frac{p}{2}} ds
+ p \mathbb{E} \left( \sup_{a \leq s \leq t} \int_a^s \left( 1 + |x^k(u)|^2 \right)^{\frac{p-2}{2}} \Re(x^k(u), g(u, x^k(u), x^k(u - \tau(u)))) I_{[u, \rho_k]} dw(u) \right).
\]
Applying the Burkholder-Davis-Gundy inequality to the third term at the right hand side of this expression, we obtain the bound
\[
pE\left(\sup_{a \leq s \leq t} \int_a^t \left(1 + |x^k(u)|^2\right) \frac{d}{du} R(x^k(u), g(u, x^k(u), x^k(u - \tau(u)))) I_{[\alpha, \rho_t]}(u)dw(u)\right) \\
\leq C_\rho E\left(\int_a^t \left(1 + |x^k(u)|^2\right)^{p-2} |x^k(u)|^2 g(u, x^k(u), x^k(u - \tau(u)))^2 du\right)^{1/2} \\
\leq C_\rho E\left(\sup_{a \leq u \leq t} \left(1 + |x^k(u)|^2\right)^{\frac{p}{2}} \int_a^t \left(1 + |x^k(u)|^2\right)^{\frac{p-4}{2}} |x^k(u)|^2 g(u, x^k(u), x^k(u - \tau(u)))^2 du\right)^{1/2} \\
\leq \frac{1}{2} E\left(\sup_{a \leq u \leq t} \left(1 + |x^k(u)|^2\right)^{\frac{p}{2}} + \frac{C^2}{2} 2C\int_a^t \left(1 + \sup_{a-\tau \leq u \leq a} |x^k(s)|^2\right)^{\frac{p}{2}} du\right).
\]
Consequently,
\[
E\sup_{a \leq s \leq t} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}} \leq 2^{\frac{p}{2}} (1 + E\sup_{a-\tau \leq s \leq a} |\xi(s)|^p) + C_\beta E\int_a^t \left(1 + \sup_{a-\tau \leq s \leq u} |x^k(s)|^2\right)^{\frac{p}{2}} du \\
= 2^{\frac{p}{2}} (1 + E\sup_{a-\tau \leq s \leq a} |\xi(s)|^p) + C_\beta E\int_a^t \sup_{a-\tau \leq s \leq u} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}} du.
\]
Further, we notice that
\[
E\sup_{a-\tau \leq s \leq t} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}} \leq E\sup_{a-\tau \leq s \leq a} \left(1 + |\xi(s)|^p\right) + E\sup_{a \leq s \leq t} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}} \\
\leq 2^{\frac{p}{2}} (1 + E\sup_{a-\tau \leq s \leq a} |\xi(s)|^p) + E\sup_{a \leq s \leq t} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}}.
\]
Therefore,
\[
E\sup_{a-\tau \leq s \leq t} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}} \leq \frac{3}{2} 2^{\frac{p}{2}} (1 + E\sup_{a-\tau \leq s \leq a} |\xi(s)|^p) + C_\beta \int_a^t E\sup_{a-\tau \leq s \leq u} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}} du.
\]
Now the Gronwall’s inequality yields that
\[
E\left(\sup_{a-\tau \leq s \leq t} |x^k(s)|^p\right) \leq E\left(\sup_{a-\tau \leq s \leq t} \left(1 + |x^k(s)|^2\right)^{\frac{p}{2}}\right) \leq \frac{3}{2} 2^{\frac{p}{2}} (1 + E\sup_{a-\tau \leq s \leq a} |\xi(s)|^p) e^{C_\beta(t-a)}.
\]
Letting \(k \to +\infty\) and applying the Fatou’s lemma, we obtain the desired result.

**Lemma 3.6** Assume that problem (2.1) \(\in SD(\alpha, \beta, \gamma_1, \gamma_2)\). Then there exists a unique solution \(x(t)\) to equation (2.1).

**Proof.** Using (2.2), (2.4), (3.4), (3.5) and (3.6), we can complete the proof in a similar manner to those of Theorem 5.2.7 and Theorem 2.3.5 in [11].

**Lemma 3.7** Assume that problem (2.1) \(\in SD(\alpha, \beta, \gamma_1, \gamma_2)\). Then we have
\[
\lim_{t \to s \to 0} \sup_{a \leq s \leq t \leq b} E|x(t) - x(s)|^2 = 0. \tag{3.9}
\]
Proof. The proof uses the techniques employed in the proof of Theorem 2.2 in [5]. Integrating (2.1) gives for \( a \leq s < t \leq b \)

\[
x(t) - x(s) = \int_s^t f(u, x(u), x(u - \tau(u)))du + \int_s^t g(u, x(u), x(u - \tau(u)))dw(u).
\]

(3.10)

Let \( e(s, t) = x(t) - x(s) \),

\[
\rho_R = \inf\{t \in [a, b] : \sup_{a - \tau \leq \theta \leq t} |x(\theta)| \geq R\},
\]

where \( \inf \emptyset = b \). Using the Young inequality: for \( r^{-1} + q^{-1} = 1 \)

\[
ab \leq \frac{\delta}{r} a^r + \frac{1}{q^{\delta r}} b^q, \ \forall a, b, \delta > 0
\]

and letting \( r = \frac{p}{2}, q = \frac{p}{p-2} \), we thus have for any \( \delta > 0 \)

\[
\mathbb{E}\left(|e(s, t)|^2\right) = \mathbb{E}\left(|e(s, t)|^2 I_{\{\rho_R \geq b\}}\right) + \mathbb{E}\left(|e(s, t)|^2 I_{\{\rho_R < b\}}\right)
\]

\[
\leq \mathbb{E}\left(|e(s, t)|^2 I_{\{\rho_R \geq b\}}\right) + \frac{2\delta}{p} \mathbb{E}\left(|e(s, t)|^p\right) + \frac{1 - \frac{2}{p}}{\delta^{2/(p-2)}} \mathbb{P}(\rho_R < b),
\]

(3.11)

where \( p > 2 \). It follows from (3.7) that

\[
\mathbb{P}(\rho_R < b) = \mathbb{E}\left(I_{\{\rho_R < b\}} \frac{|x(\rho_R)|^p}{R^p}\right) \leq \frac{1}{R^p} \mathbb{E}\left(\sup_{a \leq t \leq b} |x(t)|^p\right) \leq \frac{A}{R^p}
\]

(3.12)

\[
\mathbb{E}(|e(s, t)|^p) \leq 2^{p-1} \mathbb{E}\left(\sup_{a \leq s \leq b} |x(s)|^p + \sup_{a \leq t \leq b} |x(t)|^p\right) \leq 2^p A.
\]

(3.13)

We then have

\[
\mathbb{E}(|e(s, t)|^2) \leq \mathbb{E}(|e(s, t)|^2 I_{\{\rho_R \geq b\}}) + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)A}{p\delta^{2/(p-2)}R^p}.
\]

(3.14)

Further, using the H"older’s inequality and the Itô isometry we obtain

\[
\mathbb{E}(|e(s, t)|^2 I_{\{\rho_R \geq b\}})
\]

\[
= \mathbb{E}\left(\left| \int_s^t f(u, x(u), x(u - \tau(u)))du + \int_s^t g(u, x(u), x(u - \tau(u)))dw(u) \right|^2 I_{\{\rho_R \geq b\}}\right)
\]

\[
\leq 2\mathbb{E}\left(\left| \int_s^t f(u, x(u), x(u - \tau(u)))du \right|^2 + \left| \int_s^t g(u, x(u), x(u - \tau(u)))dw(u) \right|^2 I_{\{\rho_R \geq b\}}\right)
\]

\[
\leq 2(t-s)\mathbb{E}(I_{\{\rho_R \geq b\}}) \int_s^t |f(u, x(u), x(u - \tau(u)))|^2 du + 2\mathbb{E} \int_s^t |g(u, x(u), x(u - \tau(u)))|^2 du
\]

\[
\leq 2(t-s)\mathbb{E}(I_{\{\rho_R \geq b\}}) \int_s^t |f(u, x(u), x(u - \tau(u))) - f(u, 0, 0) + f(u, 0, 0)|^2 du
\]

\[
+ 2\mathbb{E} \int_s^t |g(u, x(u), x(u - \tau(u)))|^2 du.
\]
By (2.2), (2.4), (3.3), (3.6) and Lemma 3.5, we have
\[
\mathbb{E}(|e(s,t)|^2 I_{\{s \geq b\}}) \\
\leq 8C_2^2(t - s) \int_s^t \left( \mathbb{E}|x(u)|^2 + \mathbb{E}|x(u - \tau(u))|^2 \right) du + 4(t - s)\mathbb{E} \int_s^t |f(u,0,0)|^2 du \\
+ 2C_1 \int_s^t \left(1 + \mathbb{E}|x(u)|^2 + \mathbb{E}|x(u - \tau(u))|^2\right) du \leq C_4(t - s),
\]
where $C_4$ is independent of $s$ and $t$. A combination of this expression and (3.14) leads to
\[
\mathbb{E}(|e(s,t)|^2) = C_4(t - s) + \frac{2^{p+1}\delta A}{p} + \frac{(p - 2)A}{p\delta^{2/(p-2)}R^p}.
\] (3.15)
Therefore, for any given $\varepsilon > 0$, we can choose $\delta$ and $R$ such that
\[
\frac{2^{p+1}\delta A}{p} \leq \frac{1}{3}\varepsilon, \quad \frac{(p - 2)A}{p\delta^{2/(p-2)}R^p} \leq \frac{1}{3}\varepsilon,
\] (3.16)
and then choose $t - s$ sufficiently small such that $C_4(t - s) < \frac{1}{3}\varepsilon$. Hence, we have
\[
\lim_{t - s \to 0, a \leq s < t \leq b} \sup_{a \leq s < t \leq b} \mathbb{E}(\|x(t) - x(s)\|^2) = \lim_{t - s \to 0, a \leq s < t \leq b} \sup_{a \leq s < t \leq b} \mathbb{E}(|e(s,t)|^2) = 0.
\]
The proof is complete.

**Lemma 3.8** Assume that problem (2.1) is SD($\alpha, \beta, \gamma_1, \gamma_2$). Then for any $t_1, t_2 : a \leq t_1 \leq t_2 \leq b$,
\[
\mathbb{E}|x(t_2) - y(t_2)|^2 \leq e^{\int_{t_1}^{t_2} (2\alpha(t) + \beta(t) + \gamma_1(t) + \gamma_2(t))dt} \mathbb{E}|x(t_1) - y(t_1)|^2 \\
+ \int_{t_1}^{t_2} (\beta(s) + \gamma_1(s) \gamma_2(s) + \gamma_2^2(s)) e^{\int_s^{t_2} (2\alpha(u) + \beta(u) + \gamma_1(u) \gamma_2(u) + \gamma_2^2(u))du} ds
\] (3.17)
\[
\cdot \sup_{\mu_1^{(0)}(t_1,t_2) \leq \theta \leq t_2 - \mu_1^{(0)}} \mathbb{E}|x(\theta) - y(\theta)|^2,
\]
where $y(t)$ is a solution of any given perturbed problem (3.3).

**Proof.** Let
\[
V(t, x(t)) = p(t)(\|x(t)\|^2 + \delta q(t)), \quad t_1 \leq t \leq t_2,
\] (3.18)
where
\[
p(t) = e^{\int_a^t (\alpha(u) + \beta(u) + \gamma_1(u) \gamma_2(u) - \gamma_2^2(u))du}, \\
q(t) = -(p(t))^{-1} \int_a^t \left(\beta(u) + \gamma_1(u) \gamma_2(u) + \gamma_2^2(u)\right)p(u)du,
\] (3.19)
$\delta$ is a constant to be determined. For convenience, from now on we use $\sigma(t)$ and $\varphi(t)$ to denote $2\alpha(t) + \beta(t) + \gamma_1(t) \gamma_2(t) + \gamma_2^2(t)$ and $\beta(t) + \gamma_1(t) \gamma_2(t) + \gamma_2^2(t)$, respectively; that is
\[
\sigma(t) = 2\alpha(t) + \beta(t) + \gamma_1(t) \gamma_2(t) + \gamma_2^2(t), \\
\varphi(t) = \beta(t) + \gamma_1(t) \gamma_2(t) + \gamma_2^2(t).
\] (3.20)
Then we have \( p'(t) = -\sigma(t)p(t), (p(t)q(t))' = -p(t)\varphi(t) \). By (3.18), (3.19) and the Itô formula, one can derive that, for \( a \leq t_1 \leq t_2 \leq b \),

\[
\begin{align*}
& \mathbb{E}V(t_2, x(t_2) - y(t_2)) = \mathbb{E}V(t_1, x(t_1) - y(t_1)) \\
& + \int_{t_1}^{t_2} \left\{ -\sigma(t)p(t)\mathbb{E}|x(t) - y(t)|^2 - \delta p(t)\varphi(t) \\
& + 2p(t)\mathbb{E}\mathbb{R}\langle x(t) - y(t), f(t, x(t), x(t - \tau(t))) - f(t, y(t), y(t - \tau(t))) \rangle \\
& + p(t)\mathbb{E}|g(t, x(t), x(t - \tau(t))) - g(t, y(t), y(t - \tau(t)))|^2 \right\} dt \\
& \leq \mathbb{E}V(t_1, x(t_1) - y(t_1)) + \int_{t_1}^{t_2} \left\{ -\sigma(t)p(t)\mathbb{E}|x(t) - y(t)|^2 - \delta p(t)\varphi(t) \\
& + 2p(t)\alpha(t)\mathbb{E}|x(t) - y(t)|^2 + 2p(t)\beta(t)\mathbb{E}(|x(t) - y(t)||x(t - \tau(t)) - y(t - \tau(t))|) \\
& + p(t)\mathbb{E}\left( \gamma_1(t)|x(t) - y(t)| + \gamma_2(t)|x(t - \tau(t)) - y(t - \tau(t))| \right)^2 \right\} dt \\
& \leq \mathbb{E}V(t_1, x(t_1) - y(t_1)) + \int_{t_1}^{t_2} \left\{ -\sigma(t)p(t)\mathbb{E}|x(t) - y(t)|^2 - \delta p(t)\varphi(t) \\
& + 2p(t)\alpha(t)\mathbb{E}|x(t) - y(t)|^2 + p(t)\beta(t)\left( \mathbb{E}|x(t) - y(t)|^2 + \mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^2 \right) \\
& + p(t)\gamma_1^2(t)\mathbb{E}|x(t) - y(t)|^2 + p(t)\gamma_2^2(t)\mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^2 \\
& + 2p(t)\gamma_1(t)\gamma_2(t)\mathbb{E}(|x(t) - y(t)||x(t - \tau(t)) - y(t - \tau(t))|) \right\} dt \\
\end{align*}
\]

which yields

\[
\begin{align*}
& \mathbb{E}V(t_2, x(t_2) - y(t_2)) \\
& \leq \mathbb{E}V(t_1, x(t_1) - y(t_1)) + \int_{t_1}^{t_2} \left\{ p(t)\beta(t)\left( \mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^2 - \delta \right) dt \\
& + \int_{t_1}^{t_2} p(t)\gamma_2(t)\left( \mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^2 - \delta \right) dt \\
& + \int_{t_1}^{t_2} p(t)\gamma_1(t)\gamma_2(t)\left( \mathbb{E}|x(t - \tau(t)) - y(t - \tau(t))|^2 - \delta \right) dt. \\
\end{align*}
\]
Lemma 3.9 implies that $\mathbb{E} \sup_{a-\tau \leq t \leq b} |x(t)|^2 < +\infty$, $\mathbb{E} \sup_{a-\tau \leq t \leq b} |y(t)|^2 < +\infty$. Consequently, $\sup_{a-\tau \leq t \leq b} \mathbb{E}|x(t) - y(t)|^2 < +\infty$. Let $\delta = \sup_{\mu_2(t_1,t_2) \leq t \leq \mu_1(t_1,t_2)} \mathbb{E}(t_2, x(t_2) - y(t_2)) \leq \mathbb{E}(t_1, x(t_1) - y(t_1))$. It follows from (3.21) that

$$
\mathbb{E}(t_2, x(t_2) - y(t_2)) \leq \mathbb{E}(t_1, x(t_1) - y(t_1)).
$$

The required estimate (3.17) now follows from (3.22).

**Lemma 3.9** Assume that problem (2.7) $\in \mathcal{SD}(\alpha, \beta, \gamma_1, \gamma_2)$. Then

$$
\lim_{n \to \infty} \sup_{a-\tau \leq t \leq b} \mathbb{E}|x(t) - x_n(t)|^2 = 0,
$$

where $x_n(t)$ is a solution of the $\frac{1}{n}$-perturbed problem (2.8).

**Proof.** For any given natural number $n$, we can choose a natural number $q$ sufficiently large such that $\mu := (b - a)/q < \frac{1}{n}$. Let

$$
t_1 = a + (i - 1)\mu, \quad t_2 = a + i\mu, \quad i = 1, 2, \ldots, q,
$$

$$
a_0 = \max \left\{ \max_{a \leq t \leq b} \sigma(t), 0 \right\}, \quad \beta_0 = \max \left\{ \max_{a \leq t \leq b} \rho(t), 0 \right\}, \quad \gamma_0 = \max \left\{ \max_{a \leq t \leq b} (\sigma(t) + \rho(t)), 1 \right\}
$$

$$V(t, x(t)) = p(t)(|x(t)|^2 + \delta\rho(t)), \quad t_1 \leq t \leq t_2,$n

where

$$
\delta = \varepsilon_n + \sup_{a-\tau \leq \theta \leq t_1} \mathbb{E}|x(\theta) - x_n(\theta)|^2, \quad \varepsilon_n = \sup_{a \leq t \leq b} (\sup_{t - \frac{1}{n} \leq \theta \leq t} \mathbb{E}|x(\theta) - x(t - \frac{1}{n})|^2),
$$

$p(t)$ and $q(t)$ are defined by (3.19), $\sigma(t)$ and $\rho(t)$ are defined by (3.20). For $t_1 \leq t \leq t_2$, we can obtain the following estimate in a similar way to the proof of Lemma 3.8

$$
\mathbb{E}(t, x(t) - x_n(t)) \leq \mathbb{E}(t_1, x(t_1) - x_n(t_1)) + \int_{t_1}^t p(s)\beta(s)\left(\mathbb{E}|x(s - \tau(s)) - x_n(s - \tau(s))|^2 - \delta\right)ds + \int_{t_1}^t p(s)\gamma_2(s)\left(\mathbb{E}|x(s - \tau(s)) - x_n(s - \tau(s))|^2 - \delta\right)ds + \int_{t_1}^t p(s)\gamma_1(s)\gamma_2(s)\left(\mathbb{E}|x(s - \tau(s)) - x_n(s - \tau(s))|^2 - \delta\right)ds
$$

$$\leq \mathbb{E}(t_1, x(t_1) - x_n(t_1)) + \left(\sup_{a-\tau \leq u \leq t} \mathbb{E}|x(u) - x_n(u)|^2 - \delta\right)\int_{t_1}^t p(s)\rho(s)ds.
$$

Furthermore

$$
\sup_{a-\tau \leq u \leq t} \mathbb{E}|x(u) - x_n(u)|^2 \leq \sup_{a-\tau \leq u \leq t - \frac{1}{n}} \mathbb{E}|x(u) - x_n(u)|^2 + \sup_{t - \frac{1}{n} \leq u \leq t} \mathbb{E}|x(u) - x_n(t - \frac{1}{n})|^2
$$

$$\leq \sup_{a-\tau \leq u \leq t - \frac{1}{n}} \mathbb{E}|x(u) - x_n(u)|^2 + \sup_{a \leq t \leq b} \left(\sup_{t - \frac{1}{n} \leq u \leq t} \mathbb{E}|x(u) - x_n(t - \frac{1}{n})|^2\right) \leq \delta,
$$

where $\delta$ is defined by (3.21). It follows from (3.25) that

$$
\mathbb{E}(t, x(t) - x_n(t)) \leq \mathbb{E}(t_1, x(t_1) - x_n(t_1)),
$$

11
that is,
\[ E|x(t) - x_n(t)|^2 \leq e^{\int_0^t \sigma(u)du} E|x(t_1) - x_n(t_1)|^2 + \int_0^t \varphi(u) e^{\int_0^t \sigma(s)ds} du \left( \varepsilon_n + \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 \right) \]
\[ \leq \left( e^{\int_0^t \sigma(u)du} + \int_0^t \varphi(u) e^{\int_0^t \sigma(s)ds} du \right) \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 + \left( \int_0^t \varphi(u) e^{\int_0^t \sigma(s)ds} du \right) \varepsilon_n \]
\[ = \left( 1 + \int_0^t (\sigma(u) + \varphi(u)) e^{\int_0^t \sigma(s)ds} du \right) \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 + \left( \int_0^t \varphi(u) e^{\int_0^t \sigma(s)ds} du \right) \varepsilon_n \]
\[ \leq \left( 1 + \gamma_0 \mu e^{\alpha_0(b-a)} \right) \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 + \beta_0 \mu e^{\alpha_0(b-a)} \varepsilon_n \]
for all \( t \in [a + (i - 1)\mu, a + i\mu], \ i = 1, 2, \ldots, q \). Consequently,
\[ \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 = \max \left\{ \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2, \sup_{a + (i - 1)\mu \leq \theta \leq a + i\mu} E|x(\theta) - x_n(\theta)|^2 \right\} \]
\[ \leq \left( 1 + \gamma_0 \mu e^{\alpha_0(b-a)} \right) \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 + \beta_0 \mu e^{\alpha_0(b-a)} \varepsilon_n \]
for \( i = 1, 2, \ldots, q \). Therefore,
\[ \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 \leq \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 \]
\[ \leq C_\mu^q \sup_{a - \tau \leq \theta \leq b} E|x(\theta) - x_n(\theta)|^2 + \frac{C_\mu^q - 1}{\gamma_0 \mu e^{\alpha_0(b-a)} \beta_0 \mu e^{\alpha_0(b-a)}} \varepsilon_n \]  \( \quad (3.26) \)
\[ = \frac{\beta_0}{\gamma_0} (C_\mu^q - 1) \varepsilon_n, \]
where \( C_\mu = 1 + \gamma_0 \mu e^{\alpha_0(b-a)} \). By Lemma 3.7, we have \( \varepsilon_n = \sup_{a \leq \theta \leq b \frac{1}{n}} E|x(\theta) - x(t - \frac{1}{n})|^2 \to 0, \) as \( n \to \infty \). Let \( n \to \infty \) and take into account that
\[ \lim_{n \to \infty} C_\mu^q = \lim_{n \to \infty} \left( 1 + \frac{\gamma_0(b-a)e^{\alpha_0(b-a)}}{q} \right)^q = e^{\gamma_0(b-a)e^{\alpha_0(b-a)}}. \]
Then (3.26) leads to the relation (3.23). The proof is complete.

**Theorem 3.10** Assume that problem (2.7) \( \in SD(\alpha, \beta, \gamma_1, \gamma_2) \). Let \( c = \max_{a \leq t \leq b} (2\alpha(t) + 2\beta(t) + \gamma_1(t) + 2\gamma_1(t)\gamma_2(t) + \gamma_2(t)) \). Then \( \forall t \in [a, b] \),
\[ E|x(t) - y(t)|^2 \leq e^{c(t-a)} \sup_{a - \tau \leq \theta \leq a} E|\xi(\theta) - \eta(\theta)|^2, \]  \( \quad (3.27) \)
\[ E|x(t) - y(t)|^2 \leq \sup_{a - \tau \leq \theta \leq a} E|\xi(\theta) - \eta(\theta)|^2, \]  \( \quad (3.28) \)
where \( y(t) \) is a solution of any given perturbed problem (3.3).
\((3.27)\) and \((3.28)\) mean that problem \((2.1)\) is stable in mean square and contractive in mean square, respectively.

**Proof.** We divide the proof into two cases: \(\mu_1(0) > 0\) and \(\mu_1(0) = 0\).

**Case A:** \(\mu_1(0) > 0\). In this case, we can obtain the desired result in a similar manner as in the proof of Theorem 2.1 in [8]. In fact, replacing \(\alpha(t)\), \(\beta(t)\), \(\|y(t) - z(t)\|\) in [8] with \(2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)\), \(\beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)\), \(\mathbb{E}|x(t) - y(t)|^2\), respectively, using Lemma 3.8 and following the proof of Theorem 2.1 in [8], we can obtain either \((3.27)\) or \((3.28)\) immediately.

**Case B:** \(\mu_1(0) = 0\). Note that

\[
\begin{align*}
\mathbb{E}|x(t) - y(t)|^2 &\leq \mathbb{E}(|x(t) - x_n(t)| + |x_n(t) - y_n(t)| + |y(t) - y_n(t)|)^2 \\
&= \mathbb{E}|x(t) - x_n(t)|^2 + \mathbb{E}|x_n(t) - y_n(t)|^2 + \mathbb{E}|y(t) - y_n(t)|^2 \\
&\quad + 2\mathbb{E}(|x(t) - x_n(t)||x_n(t) - y_n(t)|) + \mathbb{E}(|x(t) - x_n(t)||y(t) - y_n(t)|) \\
&\quad + \mathbb{E}(|x_n(t) - y_n(t)||y(t) - y_n(t)|) \\
&\leq \mathbb{E}|x(t) - x_n(t)|^2 + \mathbb{E}|x_n(t) - y_n(t)|^2 + \mathbb{E}|y(t) - y_n(t)|^2 \\
&\quad + 2\left(\mathbb{E}|x(t) - x_n(t)|^2\right)^{1/2}\left(\mathbb{E}|x_n(t) - y_n(t)|^2\right)^{1/2} \\
&\quad + \left(\mathbb{E}|x(t) - x_n(t)|^2\right)^{1/2}\left(\mathbb{E}|y(t) - y_n(t)|^2\right)^{1/2} \\
&\quad + \left(\mathbb{E}|x_n(t) - y_n(t)|^2\right)^{1/2}\left(\mathbb{E}|y(t) - y_n(t)|^2\right)^{1/2},
\end{align*}
\]

where \(y_n(t)\) is the solution of the \(\frac{1}{n}\)-perturbed problem

\[
\left\{ \begin{array}{l}
dy(t) = f_n(t, y(t), y(\cdot))dt + g_n(t, y(t), y(\cdot))dw(t), \ t \in [a, b], \\
y(t) = \eta(t), \ t \in [a - \tau, a],
\end{array} \right.
\]

(3.29)

of the problem \((3.2)\). \(f_n, g_n\) are defined by \((2.6)\). It is known that, problem \((2.1)\) in \(SD(\alpha, \beta, \gamma_1, \gamma_2)\) implies that problem \((2.8)\) in \(SD(\alpha, \beta, \gamma_1, \gamma_2)\). It follows from \((2.3)\) that \(\mu_1(0) \geq \frac{1}{n} > 0\). Therefore, by case A, for \(\mathbb{E}|x_n(t) - y_n(t)|^2\), either \((3.27)\) holds if \(c > 0\) or \((3.28)\) holds if \(c \leq 0\). Letting \(n \to +\infty\) and using Lemma 3.9 we can obtain the desired estimate of \(\mathbb{E}|x(t) - y(t)|^2\) in this case.

**Corollary 3.11** Under the assumptions of Theorem 3.10. Suppose \(f(t, 0, 0) = 0\) and \(g(t, 0, 0) = 0\), then \(\forall t \in [a, b]\),

\[
\begin{align*}
\mathbb{E}|x(t)|^2 &\leq e^{c(t-a)}\sup_{a-\tau \leq \theta \leq a}\mathbb{E}|\xi(\theta)|^2, \quad \text{if } c > 0, \\
\mathbb{E}|x(t)|^2 &\leq \sup_{a-\tau \leq \theta \leq a}\mathbb{E}|\xi(\theta)|^2, \quad \text{if } c \leq 0.
\end{align*}
\]

**Lemma 3.12** Suppose problem \((2.7)\) in \(SD(\alpha, \beta, \gamma_1, \gamma_2)\), and that

\[
\frac{2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)}{\beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)} \leq \nu < 1, \quad \forall t \in [a, b],
\]

(3.30)
where $a_0$ and $\nu$ are constants. Then for any given $c_1, c_2, c_3 : a \leq c_1 < c_2 < c_3 \leq b$, we have

$$\mathbb{E}|x(t) - y(t)|^2 \leq \left( \nu + (1 - \nu)e^{a_0(c_2 - c_1)} \right) \sup_{\mu_2^{(0)}(c_1, c_2) \leq \theta \leq c_2} \mathbb{E}|x(\theta) - y(\theta)|^2,$$

$$\forall t \in [c_2, c_3]. \quad (3.31)$$

**Proof.** We divide the proof into two cases: $\mu_1^{(0)} > 0$ and $\mu_1^{(0)} = 0$.

**Case A:** $\mu_1^{(0)} > 0$. In this case, replacing $\alpha(t), \beta(t), \|y(t) - z(t)\|$ in $[8]$ with $2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t), \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t), \mathbb{E}|x(t) - y(t)|^2$, respectively, using Lemma $3.8$ and following the proof of Case B of Theorem $3.10$, we can obtain (3.31) immediately.

**Case B:** $\mu_1^{(0)} = 0$. In this case, we can obtain the estimate (3.31) in a similar manner as in the proof of Case B of Theorem $3.10$.

### 3.2 Infinite interval

Let us now proceed to discuss the equation (2.1) which satisfies conditions (2.2)-(2.5) but the integration interval $[a, b]$ replaced by $[a, +\infty)$. Accordingly, interval $[a - \tau, b]$ is replaced by $[a, +\infty)$. We introduce the symbol $\mathbb{S}^{\infty}(\alpha, \beta, \gamma_1, \gamma_2)$ to denote this class of problems.

**Theorem 3.13** Assume that problem (2.1) is in $\mathbb{S}^{\infty}(\alpha, \beta, \gamma_1, \gamma_2)$, and

$$\lim_{t \to +\infty} (t - \tau(t)) = +\infty, \sup_{a \leq t < +\infty} \left( 2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t) \right) = a_0 < 0,$$

$$\sup_{a \leq t < +\infty} \frac{\beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)}{2\alpha(t) + \beta(t) + \gamma_1(t)\gamma_2(t) + \gamma_1^2(t)} = \nu < 1.$$

Then, for any given constant $\mu > 0$, there exists a strictly increased sequence $\{t_k\}$ which diverges to $+\infty$ as $k \to +\infty$, where $t_0 = a$, such that

$$\sup_{t_k \leq t \leq t_{k+1}} \mathbb{E}|x(t) - y(t)|^2 \leq C^{k+1}_\mu \sup_{a - \tau \leq t \leq a} \mathbb{E}|\xi(t) - \eta(t)|^2, \quad k = 0, 1, 2, \ldots \quad (3.32)$$

where $C_\mu = \nu + (1 - \nu)e^{a_0\mu} \in (0, 1)$. Further,

$$\lim_{t \to +\infty} \mathbb{E}|x(t) - y(t)|^2 = 0. \quad (3.33)$$

**Proof.** It is obvious that (3.32) implies (3.33). So, only the proof of (3.32) is required. First we construct a sequence $\{t_k\}$ by induction. Let $t_0 = a$. Suppose that $t_k$ is chosen appropriately, where $k \geq 0$. Because $\lim_{t \to +\infty} (t - \tau(t)) = +\infty$, there exists a $M$ such that for all $t \geq M$, we have $t - \tau(t) \geq t_k$ and therefore $\mu_2^{(0)}(M, +\infty) \geq t_k$. So we can choose $t_{k+1} = M + \mu$ and have the relation

$$t_k \leq \mu_2^{(0)}(t_{k+1} - \mu, +\infty) \leq t_{k+1} - \mu < t_{k+1}. \quad (3.34)$$

Using (3.34) and Lemma $3.12$ we get

$$\sup_{t_k \leq t \leq t_{k+1}} \mathbb{E}|x(t) - y(t)|^2 \leq \left( \nu + (1 - \nu)e^{a_0\mu} \right) \sup_{\mu_2^{(0)}(t_{k+1} - \mu, t_{k+1}) \leq t \leq t_k} \mathbb{E}|x(t) - y(t)|^2$$

$$\leq C_\mu \sup_{t_{k-1} \leq t \leq t_k} \mathbb{E}|x(t) - y(t)|^2 \leq \ldots \leq C^{k+1}_\mu \sup_{a - \tau \leq t \leq a} \mathbb{E}|\xi(t) - \eta(t)|^2.$$

The proof is complete.
Corollary 3.14 Under the same conditions as Theorem 3.13. Furthermore, suppose that \( f(t,0,0) = 0, g(t,0,0) = 0 \), then
\[
\sup_{t_k \leq t \leq t_{k+1}} \mathbb{E}|x(t)|^2 \leq C_{\mu}^{k+1} \sup_{a-\tau \leq t \leq a} \mathbb{E}|\xi(t)|^2, \quad k = 0, 1, 2, \ldots, \tag{3.35}
\]
\[
\lim_{t \to +\infty} \mathbb{E}|x(t)|^2 = 0. \tag{3.36}
\]

Remark 3.15 Li [8] discussed the stability of nonlinear stiff Volterra functional differential equations in Banach spaces. Theorem 3.10 and Theorem 3.13 can be regarded generalizations of Theorem 2.1 and Theorem 2.2 of [8] restricted in finite-dimensional Hilbert spaces \( \mathbb{C}^d \) and finitely many delays to the stochastic version, respectively. It should be pointed out that function \( f \) is required to be locally Lipschitz continuous in this paper whereas the corresponding condition is not required in [8]. It is known that local Lipschitz continuity is not a strong restriction.

3.3 Examples
System (2.1) includes the following three classes of SDDEs as special cases

- SDDEs with constant delays: \( \tau(t) \equiv \tau \).
- Stochastic pantograph equations: \( t - \tau(t) = qt \), where \( 0 < q < 1 \) is a constant.
- SDDEs with piecewise constant arguments: \( t - \tau(t) = [t-i] \), where \( [t] \) denotes the largest integer number less than or equal to \( t \), \( i \) is a nonnegative integer.

Therefore, Theorem 3.10, Theorem 3.13, Corollary 3.11 and Corollary 3.14 are valid for the three classes of SDDEs mentioned above.

Example 3.16 Consider the following linear SDDEs
\[
dx(t) = \left( A_1(t)x(t) + A_2(t)x(t - \tau(t)) + F(t) \right) dt \\
+ \left( B_1(t)x(t) + B_2(t)x(t - \tau(t)) + G(t) \right) dw(t), \tag{3.37}
\]
where \( A_1(t), A_2(t), B_1(t), B_2(t) \in \mathbb{C}^{d \times d}, F(t), G(t) \in \mathbb{C}^d \) are continuous with respect to \( t \), \( w(t) \) is an 1-dimensional Wiener process. For the problems (3.37), it is easy to verify the conditions (2.2)-(2.7) with
\[
\alpha(t) = \frac{\lambda_{\text{max}}^{A_1(t)} + A_1(t)}{2}, \quad \beta(t) = |A_2(t)|, \quad \gamma_1(t) = |B_1(t)|, \quad \gamma_2(t) = |B_2(t)|,
\]
where \( \lambda_{\text{max}}^{A_1(t)} \) denotes the largest eigenvalue of the Hermite matrix \( \frac{A_1(t)+A_1(t)}{2} \).

Applying Theorem 3.10 and Corollary 3.11 to (3.37), we have

Corollary 3.17 The solutions of (3.37) satisfy
\[
\mathbb{E}|x(t) - y(t)|^2 \leq e^{c(t-a)} \sup_{a-\tau \leq \theta \leq a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2, \quad t \in [a,b], \text{ if } c > 0,
\]
\[
\mathbb{E}|x(t) - y(t)|^2 \leq \sup_{a-\tau \leq \theta \leq a} \mathbb{E}|\xi(\theta) - \eta(\theta)|^2, \quad t \in [a,b], \text{ if } c \leq 0,
\]

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where \( x(t), y(t) \) are the solutions of (3.37) corresponding to the initial functions \( \xi(t) \) and \( \eta(t) \), respectively,

\[
c = \max_{a \leq t \leq b} \left( 2\lambda_{\max}^{-1} + 2|A_2(t)| \left| (|B_1(t)| + |B_2(t)|)^2 \right. \right).
\]

Furthermore, if \( F(t) = 0 \) and \( G(t) = 0 \), then the solutions of (3.37) satisfy

\[
E|x(t)|^2 \leq e^{c(t-a)} \sup_{a-\tau \leq \theta \leq a} E|\xi(\theta)|^2, \quad t \in [a, b], \text{ if } c > 0,
\]

\[
E|x(t)|^2 \leq \sup_{a-\tau \leq \theta \leq a} E|\xi(\theta)|^2, \quad t \in [a, b], \text{ if } c \leq 0.
\]

Applying Theorem 3.13 and Corollary 3.14 to (3.37), we have

**Corollary 3.18** If \( \lim_{t \to +\infty} (t - \tau(t)) = +\infty \),

\[
\sup_{a \leq t < +\infty} \left( 2\lambda_{\max}^{-1} + |A_2(t)| + |B_1(t)| |B_2(t)| + |B_1(t)|^2 \right) < 0,
\]

\[
\sup_{a \leq t < +\infty} \left( 2\lambda_{\max}^{-1} + |A_2(t)| + |B_1(t)| |B_2(t)| + |B_1(t)|^2 \right) < 1,
\]

then the solutions of (3.37) satisfy \( \lim_{t \to +\infty} E|x(t) - y(t)|^2 = 0 \). Furthermore, if \( F(t) = 0 \) and \( G(t) = 0 \), then the solutions of (3.37) satisfy \( \lim_{t \to +\infty} E|x(t)|^2 = 0 \).

In particularly, if \( d = 1 \), \( A_1(t), A_2(t), B_1(t), B_2(t) \) are constants, that is, \( A_1(t) = A_1, A_2(t) = A_2, B_1(t) = B_1, B_2(t) = B_2 \), then the problems (3.37) are mean-square stable if

\[
\Re A_1 + |A_2| + \frac{1}{2} \left( |B_1| + |B_2| \right)^2 < 0. \tag{3.38}
\]

**Remark 3.19** For linear scalar SDDE with constant delay and linear scalar stochastic pantograph equation, [11] [10] and [3] obtain the condition (3.38), respectively. Therefore, specializing Corollary 3.14 to the case of linear scalar SDDE with constant delay and linear scalar stochastic pantograph equation, respectively, the corresponding result is accordant with that in the literature.

**Example 3.20** Consider the following nonlinear equation

\[
dx(t) = \left( A_1(t)x + A_2(t)x^3 + A_3(t)\sqrt{x^2(t - \tau(t))} + 1 + F(t) \right) dt
\]

\[
+ \left( B_1(t) \sin x(t) + B_2(t) \arctan x(t - \tau(t)) + G(t) \right) dw(t), \tag{3.39}
\]

where \( A_1(t), A_2(t), A_3(t), B_1(t), B_2(t), F(t), G(t) \) are continuous real-valued functions in \( t \) and \( A_2(t) < 0 \). It is easy to verify that (3.39) satisfies the conditions (2.3), (2.5) with

\[
\alpha(t) = A_1(t), \quad \beta(t) = |A_3(t)|, \quad \gamma_1(t) = |B_1(t)|, \quad \gamma_2(t) = |B_2(t)|.
\]

Applying Theorem 3.10, Corollary 3.11, Theorem 3.13 and Corollary 3.14 to (3.39), we can obtain properties of solutions of (3.39) which are similar to those of Corollary 3.17 and Corollary 3.18. For the sake of brevity, we do not present the corresponding results for (3.39).

**Remark 3.21** The drift coefficient \( f(t, x, y) \) of (3.39) satisfies local Lipschitz condition and one-sided Lipschitz condition with respect to \( x \) but global Lipschitz condition. The stability analysis in this paper is based on the local Lipschitz condition and the one-sided Lipschitz condition instead of a more restrictive global Lipschitz condition.
4 Stability of backward Euler method

In this section, we investigate whether numerical methods can reproduce the contractivity in mean square. For the deterministic differential equations, it is known that the contractivity of numerical methods is too strong [2, 7]. The existing theory [2, 15] shows that only backward Euler method and two-stage Lobatto IIIC Runge-Kutta method can preserve the contractivity of nonlinear delay differential equations. Therefore, in the stochastic setting, we only focus on the backward Euler method instead of other methods.

For simplicity, from now on, we assume that
\[ \alpha(t) \equiv \alpha, \ \beta(t) \equiv \beta, \ \gamma_1(t) \equiv \gamma_1, \ \gamma_2(t) \equiv \gamma_2, \ t \in [a, b]. \]

On a finite time interval \([a, b]\), a uniformly partition is defined by
\[ t_i = a + ih, \ i = 0, 1, \ldots, h = \frac{b - a}{N}. \]

The backward Euler (BE) method applied to (2.1) yields
\[
\begin{align*}
X_{n+1} &= X_n + hf(t_{n+1}, X_{n+1}, X^h(t_{n+1} - \tau(t_{n+1}))) \\
& \quad + g(t_n, X_n, X^h(t_n - \tau(t_n)))\Delta w_n, \quad n = 0, 1, \ldots, N - 1, \\
X^h(t) &= \pi^h(t, \xi, X_1, X_2, \ldots, X_n), \ a - \tau \leq t \leq t_n, 
\end{align*}
\]

(4.1a)

where \(\pi^h\) is an appropriate interpolation operator which approximates to the exact solution \(x(t)\) on the interval \([a - \tau, b]\). \(X_n\) is an approximation to the exact solution \(x(t_n)\), \(\Delta w_n = w(t_{n+1}) - w(t_n)\). It is well known that the backward Euler method is convergent with strong order only 1/2 for stochastic differential equations. We can use the following piecewise linear interpolation
\[
X^h(t) = \begin{cases} \frac{1}{h}[(t_{i+1} - t)X_i + (t - t_i)X_{i+1}], & t_i \leq t \leq t_{i+1}, i = 0, 1, 2, \ldots, N - 1, \\ \xi(t), & a - \tau \leq t \leq a. \end{cases}
\]

(4.2)

Applying BE to the perturbed problem (3.2) we can obtain the corresponding scheme
\[
\begin{align*}
Y_{n+1} &= Y_n + hf(t_{n+1}, Y_{n+1}, Y^h(t_{n+1} - \tau(t_{n+1}))) \\
& \quad + g(t_n, Y_n, Y^h(t_n - \tau(t_n)))\Delta w_n, \quad n = 0, 1, \ldots, N - 1, \\
Y^h(t) &= \pi^h(t, \eta, Y_1, Y_2, \ldots, Y_n), \ a - \tau \leq t \leq t_n. 
\end{align*}
\]

(4.3)

For simplicity, for any given nonnegative integer \(n\), we write
\[
P_n = X_n - Y_n, \ Q_n = \max_{1 \leq i \leq n} \max |P_i|^2, \ \sup_{a - \tau \leq t \leq a} \mathbb{E}|\xi(t) - \eta(t)|^2, \ n \geq 1, \\
Q_0 = \sup_{a - \tau \leq t \leq a} \mathbb{E}|\xi(t) - \eta(t)|^2.
\]

(4.4)

Moreover, for convenience, we introduce notations to denote the values of drift and diffusion coefficients at specific points.
\[
\begin{align*}
f_{xx}(n + 1) &= f(t_{n+1}, X_{n+1}, X^h(t_{n+1} - \tau(t_{n+1}))), \\
f_{yy}(n + 1) &= f(t_{n+1}, Y_{n+1}, Y^h(t_{n+1} - \tau(t_{n+1}))), \\
f_{xy}(n + 1) &= f(t_{n+1}, Y_{n+1}, Y^h(t_{n+1} - \tau(t_{n+1}))), \\
g_{xx}(n) &= g(t_n, X_n, X^h(t_n - \tau(t_n))), \ g_{yy}(n) = g(t_n, Y_n, Y^h(t_n - \tau(t_n))).
\end{align*}
\]

(4.5)
Lemma 4.1 Under the conditions (2.3) and (2.4), if \((\alpha + \beta)h < 1\), the implicit equation (4.1a) admits a unique solution.

Proof. Let \(\tilde{f}(z) := f(\cdot, z, z^h(\cdot))\), then implicit equation (4.1a) can be rewritten as

\[
z = h\tilde{f}(z) + b = hf(\cdot, z, z^h(\cdot)) + b,
\]

where \(z\) is unknown whereas \(b\) and \(h\) are known. Inserting the interpolation operator (4.2) into (4.6), we have

\[
z = h\tilde{f}(z) + b = hf(\cdot, z, lz + b_0) + b,
\]

where \(0 \leq l \leq 1, l\) and \(b_0\) are also known. It follows from (2.3), (2.4) and (4.7) that

\[
\Re(\langle z_1 - z_2, \tilde{f}(z_1) - \tilde{f}(z_2) \rangle) = \Re(\langle z_1 - z_2, f(\cdot, z_1, lz_1 + b_0) - f(\cdot, z_2, lz_2 + b_0) \rangle)
\]

\[
+ \Re(\langle z_1 - z_2, f(\cdot, z_2, lz_1 + b_0) - f(\cdot, z_2, lz_2 + b_0) \rangle) \leq \alpha|z_1 - z_2|^2 + \beta|z_1 - z_2|^2.
\]

The assertion follows immediately from [4] Theorem 5.6.1.

Theorem 4.2 Assume that problem (2.1) \(\in SD(\alpha, \beta, \gamma_1, \gamma_2)\). Let \(\{X_n\}\) and \(\{Y_n\}\) be two sequences of numerical solutions obtained by the backward Euler schemes (4.1) and (4.3), respectively. Write \(c = 2\alpha + 2\beta + \gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2\).

(i) If \(c > 0\), for any given \(c_0 \in (0, 1)\), then we have for \(hc \leq c_0\)

\[
\mathbb{E}|X_n - Y_n|^2 \leq e^{\tilde{c}(t_n - a)} \sup_{a - \tau \leq t \leq a} \|\xi(t) - \eta(t)\|^2, \quad n = 1, 2, \cdots, N,
\]

where \(\tilde{c} = \frac{c_1}{h}, c_1 = \max\left\{\frac{1 + b\gamma_2^2 + 2b\gamma_2 + b\gamma_2^2}{1 - 2\alpha - 2\beta}, \frac{1 + b\gamma_2 + b\gamma_2^2 + 2b\gamma_2 + b\gamma_2^2}{1 - 2\alpha - h\beta}\right\} > 1\).

(ii) If \(c \leq 0\), then we have for any \(h > 0\)

\[
\mathbb{E}|X_n - Y_n|^2 \leq \max_{a - \tau \leq t \leq a} \mathbb{E}(|\xi(t) - \eta(t)|^2), \quad n = 1, 2, \cdots, N.
\]

Note that the stability in mean square (4.8) and the contractivity in mean square (4.9) can be regarded as numerical analogs of (3.27) and (3.28) for the analytical solution of the problem (2.1), respectively.

Proof. (i) By (4.1) and (4.3), we have

\[
P_{n+1} - h(f^{xx}(n+1) - f^{yy}(n+1)) = P_n + (g^{xx}(n) - g^{yy}(n))\Delta w_n,
\]

which yields

\[
|P_{n+1}|^2 - 2h\Re(P_{n+1}, f^{xx}(n+1) - f^{yy}(n+1)) + h^2|f^{xx}(n+1) - f^{yy}(n+1)|^2
\]

\[
= |P_n|^2 + 2\Re(P_n, (g^{xx}(n) - g^{yy}(n))\Delta w_n) + |(g^{xx}(n) - g^{yy}(n))\Delta w_n|^2.
\]
Taking expectation and using \((2.3)-(2.5)\) and \((4.2)\), we get

\[
E|P_{n+1}|^2 \leq E|P_n|^2 + 2h\bE\langle P_{n+1}, f_{tx}(n+1) - f_{ty}(n+1) + h\b E|g_{tx}(n) - g_{ty}(n)|^2
\]

\[
\leq E|P_n|^2 + 2h\b E\langle P_{n+1}, f_{tx}(n+1) - f_{ty}(n+1) + h\b E|g_{tx}(n) - g_{ty}(n)|^2
\]

\[
+ 2h\b E\langle P_{n+1}, f_{tx}(n+1) - f_{ty}(n+1) + h\b E|g_{tx}(n) - g_{ty}(n)|^2
\]

\[
\leq E|P_n|^2 + 2h\a E|P_{n+1}|^2 + 2h\b E|P_{n+1}|^2
\]

\[
+ h\b E\langle P_{n+1}|X^h(t_{n+1} - \tau(t_n)) - Y^h(t_{n+1} - \tau(t_n))\rangle + h\b E(\gamma_1|P_n| + \gamma_2)X^h(t_n - \tau(t_n)) - Y^h(t_n - \tau(t_n))^2
\]

\[
\leq E|P_n|^2 + 2h\a E|P_{n+1}|^2 + h\b E|P_{n+1}|^2
\]

\[
+ h\b \max\{ \max_{1 \leq i \leq n+1} E|P_i|^2, \sup_{a-\tau \leq t \leq a} E(\xi(t) - \eta(t))^2 \}
\]

\[
+ h(\gamma_1^2 + \gamma_1\gamma_2)E|P_n|^2 + h(\gamma_1^2 + \gamma_1\gamma_2)Q_n,
\]

where we used the piecewise linear interpolation \((4.2)\) and the following inequality

\[
E|(1 - \delta)P_1 + \delta P_{i+1}|^2 \leq \max\{ E|P_i|^2, E|P_{i+1}|^2 \}, \quad 0 \leq \delta \leq 1.
\]

It is clear from \((4.10)\) that

\[
(1 - 2h\a - h\b)E|P_{n+1}|^2 \leq (1 + h\gamma_1^2 + h\gamma_1\gamma_2)E|P_n|^2
\]

\[
+ h\b \max\{ \max_{1 \leq i \leq n+1} E|P_i|^2, \sup_{a-\tau \leq t \leq a} E(\xi(t) - \eta(t))^2 \} + h(\gamma_1^2 + \gamma_1\gamma_2)Q_n,
\]

We now consider two cases:

\begin{align*}
(\text{a}) & \quad \max\{ \max_{1 \leq i \leq n+1} E|P_i|^2, \sup_{a-\tau \leq t \leq a} E(\xi(t) - \eta(t))^2 \} = E|P_{n+1}|^2, \\
(\text{b}) & \quad \max\{ \max_{1 \leq i \leq n+1} E|P_i|^2, \sup_{a-\tau \leq t \leq a} E(\xi(t) - \eta(t))^2 \} \neq E|P_{n+1}|^2.
\end{align*}

In the case of (a), it follows from \((4.12)\) that

\[
(1 - 2h\a - 2h\b)E|P_{n+1}|^2 \leq (1 + h\gamma_1^2 + 2h\gamma_1\gamma_2 + h\gamma_2^2)Q_n,
\]

which yields

\[
E|P_{n+1}|^2 \leq \frac{1 + h\gamma_1^2 + 2h\gamma_1\gamma_2 + h\gamma_2^2}{1 - 2h\a - 2h\b}Q_n \leq c_1 Q_n.
\]

In the case of (b), \((4.12)\) implies that

\[
(1 - 2h\a - h\b)E|P_{n+1}|^2 \leq (1 + h\gamma_1^2 + h\gamma_1\gamma_2)E|P_n|^2 + h\b Q_n + h(\gamma_1^2 + \gamma_1\gamma_2)Q_n
\]

\[
\leq (1 + h\b + h\gamma_1^2 + 2h\gamma_1\gamma_2 + h\gamma_2^2)Q_n,
\]

which yields

\[
E|P_{n+1}|^2 \leq \frac{1 + h\b + h\gamma_1^2 + 2h\gamma_1\gamma_2 + h\gamma_2^2}{1 - 2h\a - h\b}Q_n \leq c_1 Q_n.
\]
To summarize, both in the cases we have shown that $E|P_{n+1}|^2 \leq c_1 Q_n$, which yields

$$Q_n \leq Q_{n-1} + E|P_n|^2 \leq (1 + c_1)Q_{n-1}.$$  \hfill (4.16)

By induction, we further obtain

$$E|X_n - Y_n|^2 = E|P_n|^2 \leq Q_n \leq (1 + c_1)Q_{n-1} \leq \cdots \leq (1 + c_1)^n Q_0 \leq e^{c_1 n} Q_0 = e^{c(t_n-a)} \sup_{a-t\leq t\leq a} E|\xi(t) - \eta(t)|^2.$$  

(ii) When $c \leq 0$, noting that (4.14), (4.15) and

$$\frac{1 + h(\gamma_1 + \gamma_2 + \gamma_0^2)}{1 - 2h\alpha - 2h\beta} \leq 1, \quad \frac{1 + h\gamma + \gamma_1^2 + 2h\gamma_1\gamma_2 + \gamma_2^2}{1 - 2h\alpha - h\beta} \leq 1,$$

we have for any $h > 0$

$$E|X_n - Y_n|^2 \leq Q_{n-1} \leq Q_{n-2} \leq \cdots \leq Q_0 = \sup_{a-t\leq t\leq a} E|\xi(t) - \eta(t)|^2.$$  \hfill (4.17)

Therefore we have completed the proof of the theorem.

**Corollary 4.3** Assume that problem (4.1) $\in SD(\alpha, \beta, \gamma_1, \gamma_2)$. Let $\{X_n\}$ be a sequence of numerical solutions obtained by the backward Euler method (4.1). Furthermore, if $f(t), g(t, 0, 0) = 0, g(t, 0, 0) = 0, t \geq 0$, then

(i) for the case of $c > 0$, for any given $c_0 \in (0, 1)$, we have for $hc \leq c_0$

$$E|X_n|^2 \leq e^{c(t_n-a)} \max_{a-t\leq t\leq a} E|\xi(t)|^2, \quad n = 1, 2, \cdots, N;$$  \hfill (4.18)

(ii) for the case of $c \leq 0$, we have for any $h > 0$

$$E|X_n|^2 \leq \max_{a-t\leq t\leq a} E|\xi(t)|^2, \quad n = 1, 2, \cdots, N.$$  \hfill (4.19)

**Theorem 4.4** Assume that problem (4.1) $\in SD(\alpha, \beta, \gamma_1, \gamma_2)$, and

$$\lim_{t \to +\infty} (t - \tau(t)) = +\infty, \quad c = 2\alpha + 2\beta + \gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2 < 0.$$  \hfill (4.20)

Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of numerical solutions obtained by the backward Euler schemes (4.7) and (4.3). Then,

(i) there exists a strictly increased positive integer sequence $\{n_k\}$ which diverges to $+\infty$ as $k \to +\infty$, where $n_0 = 0$, such that for any given $h > 0$,

$$\max_{n_k + 1 \leq n \leq n_k + 1} E|X_n - Y_n|^2 \leq c_2^{k+1} \sup_{a-t\leq t\leq a} E|\xi(t) - \eta(t)|^2, \quad k = 0, 1, 2, \ldots, \hfill (4.21)$$

where $c_2 = \max\left\{\frac{1 + h\gamma_1^2 + 2h\gamma_1\gamma_2 + \gamma_2^2}{1 - 2h\alpha - 2h\beta}, \frac{1 + h\beta + h\gamma_1^2 + 2h\gamma_1\gamma_2 + \gamma_2^2}{1 - 2h\alpha - h\beta}\right\} < 1$;

(ii) for any given $h > 0$,

$$\lim_{n \to +\infty} E|X_n - Y_n|^2 = 0.$$  \hfill (4.22)
Note that the contractivity in mean square \((4.21)\) and the asymptotic contractivity in mean square \((4.22)\) can be regarded as numerical analogs of \((3.32)\) and \((3.33)\) for the analytical solution of the problem \((2.1)\), respectively.

**Proof.** It is obvious that \((4.21)\) implies \((4.22)\), and we only need to prove \((4.21)\). By \((4.20)\), we have \(2\alpha + \beta + \gamma_1\gamma_2 + \gamma_1^2 < 0, (\beta + \gamma_1\gamma_2 + \gamma_1^2)/2\alpha + \beta + \gamma_1\gamma_2 + \gamma_1^2 | < 1\) and \(c_2 < 1\).

First, as done in \([9, 18]\), we can construct a strictly increased sequence of integers \(\{n_k\}\) which diverges to \(+\infty\) as \(k \to +\infty\), such that

\[
t - \tau(t) > t_{n_k+1}, \quad \forall t \geq t_{n_k+1},
\]

where \(n_0 = 0\). In fact, suppose that \(n_k(k \geq 0)\) has been chosen appropriately. Then there exists a constant \(M > t_{n_k}\) such that for all \(t \geq M\) we have \(t - \tau(t) > t_{n_k} + h\) since \(\lim_{t \to +\infty}(t - \tau(t)) = +\infty\). If \(M\) is a node, let \(t_{n_k+1} = M\), otherwise there exists natural number \(j\) such that \(t_j < M < t_{j+1}\), then we let \(n_{k+1} = j + 1\) and \(t_{n_k+1} = t_{j+1}\). Thus we obtain the required sequence \(\{n_k\}\) which satisfies

\[
t_0 < t_1 < \cdots < t_{n_1} < \cdots < t_{n_2} \cdots < t_{n_k} < \cdots.
\]

For \(n_k < n + 1 \leq n_{k+1}\), by the second inequality of \((4.10)\) and conditions \((2.3)\) and \((2.5)\), we have

\[
\mathbb{E}|P_{n+1}|^2 \leq \mathbb{E}|P_n|^2 + 2h\mathbb{E}\langle P_{n+1}, f^{xx}(n + 1) - f^{yy}(n + 1) \rangle
\]

\[
+ 2h\mathbb{E}\langle P_{n+1}, f^{xx}(n + 1) - f^{yy}(n + 1) \rangle + h\mathbb{E}|g^{xx}(n) - g^{yy}(n)|^2
\]

\[
\leq \mathbb{E}|P_n|^2 + 2h\alpha\mathbb{E}|P_{n+1}|^2 + 2h\beta\mathbb{E}(\|P_{n+1}\|X^h(t_{n+1} - \tau(t_{n+1})) - Y^h(t_{n+1} - \tau(t_{n+1})))
\]

\[
+ h\mathbb{E}(\gamma_1|P_n| + \gamma_2|X^h(t_n - \tau(t_n)) - Y^h(t_n - \tau(t_n)))^2
\]

\[
\leq \mathbb{E}|P_n|^2 + 2h\alpha\mathbb{E}|P_{n+1}|^2 + h\beta\mathbb{E}|X^h(t_{n+1} - \tau(t_{n+1})) - Y^h(t_{n+1} - \tau(t_{n+1})))^2
\]

\[
+ h(\gamma_1^2 + \gamma_1\gamma_2)\mathbb{E}|P_n|^2 + h(\gamma_2^2 + \gamma_1\gamma_2)\mathbb{E}|X^h(t_n - \tau(t_n)) - Y^h(t_n - \tau(t_n)))^2,
\]

which yields

\[
(1 - 2\alpha - h\beta)\mathbb{E}|P_{n+1}|^2
\]

\[
\leq (1 + \gamma_1^2 + \gamma_1\gamma_2)\mathbb{E}|P_n|^2 + h\beta\mathbb{E}|X^h(t_{n+1} - \tau(t_{n+1})) - Y^h(t_{n+1} - \tau(t_{n+1})))^2
\]

\[
+ h(\gamma_2^2 + \gamma_1\gamma_2)\mathbb{E}|X^h(t_n - \tau(t_n)) - Y^h(t_n - \tau(t_n)))^2
\]

\[
\leq (1 + \gamma_1^2 + \gamma_1\gamma_2)\mathbb{E}|P_n|^2 + h\beta \max_{n_k-1 < i \leq n+1} \mathbb{E}|P_i|^2 + h(\gamma_2^2 + \gamma_1\gamma_2) \max_{n_k-1 < i \leq n} \mathbb{E}|P_i|^2,
\]

where we used the piecewise linear interpolation operator \((4.2)\) and the inequality \((4.11)\). We now consider the following two cases.

If \(\max_{n_k-1 < i \leq n+1} \mathbb{E}|P_i|^2 = \mathbb{E}|P_{n+1}|^2\), we have

\[
\mathbb{E}|P_{n+1}|^2 \leq \frac{1 + \gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2}{1 - 2\alpha - 2h\beta} \max_{n_k-1 < i \leq n} \mathbb{E}|P_i|^2 \leq c_2 \max_{n_k-1 < i \leq n} \mathbb{E}|P_i|^2.
\]

If \(\max_{n_k-1 < i \leq n+1} \mathbb{E}|P_i|^2 \neq \mathbb{E}|P_{n+1}|^2\), we have

\[
\mathbb{E}|P_{n+1}|^2 \leq \frac{1 + \gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2}{1 - 2\alpha - h\beta} \max_{n_k-1 < i \leq n} \mathbb{E}|P_i|^2 \leq c_2 \max_{n_k-1 < i \leq n} \mathbb{E}|P_i|^2.
\]
In both cases, we have obtained the inequality
\[ \mathbb{E}|P_{n+1}|^2 \leq c_2 \max_{n_k-1 < i \leq n_k} \mathbb{E}|P_i|^2, \quad n_k < n + 1 \leq n_{k+1}. \] (4.23)

with \( n = n_k \) reduces to \( \mathbb{E}|P_{n_k+1}|^2 \leq c_2 \max_{n_k-1 < i \leq n_k} \mathbb{E}|P_i|^2 \). By induction, we have
\[
\max_{n_k < i \leq n_{k+1}} \mathbb{E}|X_i - Y_i|^2 = \max_{n_k < i \leq n_{k+1}} \mathbb{E}|P_i|^2 \leq c_2 \max_{n_k-1 < i \leq n_k} \mathbb{E}|P_i|^2 \\
\leq \cdots \leq c_2^{k+1} \max_{a - \tau \leq t \leq a} \mathbb{E}|\xi(t) - \eta(t)|^2.
\]
The proof is complete.

**Corollary 4.5** Under the same assumptions of Theorem 4.4. Let \( \{X_n\} \) be a sequence of numerical solution obtained by the backward Euler method (4.1). Furthermore, if \( f(t, 0, 0) = 0 \) and \( g(t, 0, 0) = 0 \), then,

(i) there exists a strictly increased positive integer sequence \( \{n_k\} \) which diverges to \( +\infty \) as \( k \to +\infty \), where \( n_0 = 0 \), such that for any given \( h > 0 \),
\[
\max_{n_k < i \leq n_{k+1}} \mathbb{E}|X_i|^2 \leq c_2^{k+1} \sup_{a - \tau \leq t \leq a} \mathbb{E}|\xi(t)|^2, \quad k = 0, 1, 2, \ldots.
\]

(ii) for any given \( h > 0 \), \( \lim_{n \to +\infty} \mathbb{E}|X_n|^2 = 0 \).

5 **SDDEs with several delays**

Consider the following SDDEs with several delays
\[
\begin{cases}
    dx(t) = f(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_r(t)))dt + g(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_r(t)))dw(t), & t \geq a, \\
    x(t) = \xi(t), & t \in [a - \tau, a],
\end{cases}
\] (5.1)

where \( \tau_i(t) \geq 0, i = 1, 2, \ldots, r \) and \( \max_{1 \leq i \leq r, t \geq a} \inf_{t \leq i \leq r} \geq a - \tau \). All results given in this paper can be extended easily to the case of several delays. For the sake of brevity, we do not present the corresponding results for (5.1).

6 **Conclusions and future work**

In this paper, we investigate stability of analytical and numerical solutions of nonlinear SDDEs. Sufficient conditions for stability in mean square, contractivity in mean square and asymptotic contractivity in mean square to the solutions of nonlinear SDDEs are given, which provide a unified theoretical treatment for stochastic differential equations, stochastic delay differential equations with constant and variable delays, bounded and unbounded delays. Furthermore, it is proved that the backward Euler method can preserve the properties of the underlying system. The main results of analytic solution in this paper can be regarded as generalization of those in [8] restricted in finite-dimensional Hilbert spaces and finitely many delays to the stochastic version. We have encountered problems when we tried to obtain a unified framework for general
stochastic functional differential equations. It is worth noting that whether the results in [8] can be extended to general stochastic functional differential equations or not. One area for the future work is to give a positive or negative answer for the question. Neutral stochastic delay differential equation (NSDDE) is more general than stochastic delay differential equation. It is interesting to investigate whether the theory of this paper can be extended to NSDDEs and corresponding numerical methods. It will also be our future work.

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