NON-ABELIAN Q-BALLS IN SUPERSYMMETRIC THEORIES

Minos Axenides\textsuperscript{1}, Emmanuel Floratos\textsuperscript{2}

\textit{Institute of Nuclear Physics, N.C.S.R. Demokritos}
\textit{153 10, Athens, Greece}

and

Alexandros Kehagias\textsuperscript{3}

\textit{Theory Division, CERN, CH-1211 Geneva 23, Switzerland}

ABSTRACT

We demonstrate the existence of non-abelian non-topological solitons such as Q-balls in the spectrum of Wess-Zumino models with non-abelian global symmetries. We conveniently name them Q-superballs and identify them for short as Q-sballs. More specifically, we show that in contrast to the non-supersymmetric case, they arise in renormalizable potentials with cubic self-interactions of only one dimensionful parameter and for the entire parameter space of the model available. We solve the field equations and present the explicit form of the Q-sball solution. We compute its main physical properties and observe that in the supersymmetrically invariant vacuum Q-sballs form domains of manifestly broken supersymmetry.
1 INTRODUCTION AND CONCLUSIONS

Non-topological solitons are extended finite energy configurations that arise in 3 + 1 dimensional field theories that contain scalar fields with unbroken global symmetries [1, 2, 3]. In the presence of attractive scalar self-interactions in their potential energy and a conserved global charge they admit non-dissipative time dependent solutions to their equation of motion with their total binding energy per unit charge $E/Q < \mu$ being smaller than the perturbative mass scale of free mesons. The total energy of such configurations scales with the conserved charge $Q$ as follows:

$$E \propto Q^s, \quad s \leq 1.$$  \hspace{1cm} (1)

They are spherically symmetric configurations of scalar fields which monotonically decrease to zero at spatial infinity. They receive both surface and volume contributions to their total energy for an arbitrary value of $Q$ with the latter dominating for sufficiently large values of $Q$. The resulting soliton is a homogeneous blob of matter possessing a very thin skin which can be neglected for all practical purposes. It has been called a Q ball [4]. More importantly it exists in the strict thermodynamic limit $V \to \infty, E/Q = \text{const}$. Such scalar field coherent configurations could also carry additional charges giving rise to long range gauge fields [2, 3, 4].

In the low energy world strong interactions alone respect strangeness and isospin. In the context of effective Lagrangians the possible existence of short lived Strange and Isospin balls as resonances in the low energy QCD spectrum has been discussed for some time now [3, 4].

In the minimal electroweak theory the pretty well respected baryon and lepton quantum numbers don't allow for such configurations. Recently, however, in supersymmetric extensions of the standard model such as the MSSM [5, 6] the existence of baryon and lepton balls was demonstrated. They are composed of charged squarks and sleptons along with Higgs scalar degrees of freedom. In what follows we will conveniently call a Q-ball which contains a supersymmetric scalar matter component such as sfermions a Q-superball denoting it for short Q-sball. Abelian Q-sballs typically arise through soft susy breaking interactions [8] such as non-renormalizable polynomial scalar self interactions that appear in the flat directions of the supersymmetric theories [10, 11]. They can also be generated from SUGRA
induced quantum logarithmic corrections [10]. It has been known from earlier work that the distinguishing characteristic of non-abelian Q-balls, such as $SO(3)$ or $SU(3)$, is that they can arise through solely renormalizable potentials that contain cubic interactions [2, 5, 12]. More precisely we consider a scalar model given by

$$ \mathcal{L} = \frac{1}{2} Tr(\partial_\mu \phi)^2 - Tr U(\phi), $$

and

$$ U(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{g \phi^3}{3!} + \frac{\lambda \phi^4}{4!}, $$

with $\mu^2 > 0$ and $\lambda > 0$. Here the field $\phi$ transforms according to the adjoint representation of $SU(3)$ $\phi \rightarrow u \phi u^\dagger$. Identically it can be taken to consist of five real valued scalar fields transforming according to the spinor representation of $SO(3)$ with the above potential being the most general $SO(3)$-invariant, renormalizable interaction.

The conserved charges of both models are assembled into a traceless hermitian matrix

$$ Q = i \int d^3 x \left[ \dot{\phi}, \phi \right]. $$

Both non-abelian models possess an identical set of defining equations provided that $u$ and $\phi$ are pure real and thus $Q$ is a pure imaginary. For a renormalizable potential with two independent dimensionful couplings $\mu$ and $g$ along with one dimensionless $\lambda$ Q-balls were shown to be present for a window in the space of free parameters given by

$$ 1 \leq \beta \equiv \frac{g^2}{\mu^2} \lambda \leq 9. $$

The upper bound guarantees that the state $\phi = 0$ is the unique vacuum of the model while the lower bound secures that the Q-balls does not decay into free mesons.

In the present work we explore the effect supersymmetry might have in the above picture by investigating the spectrum of global supersymmetric models that give rise to potentials of the same form as above. We establish

\footnote{As our arguments do not get modified in the presence of additional gauge fields and symmetries we restrict ourselves to scalar theories.}
the presence of Q-sballs in the spectrum of Wess-Zumino [13] supersymmetric
models of a single chiral superfield and non-abelian global symmetries. We
find that supersymmetry constrains the freedom between couplings in such
a way that the resulting potential is characterized by a single dimensionful
coupling \( m \) and a dimensionless one \( h \) and it possesses

\[
\beta = 3 .
\]  

(6)

In effect Q-sballs are present in its spectrum for every value of its free pa-
rameters. We solve analytically the equations of motion exhibiting explicitly
a Q-sball solution and compute its energy and charge densities. We observe
that in a supersymmetrically invariant vacuum the Q-sball interior breaks
supersymmetry explicitly. The supersymmetry breaking scale is set by the
internal rotational frequency of the superballs. Certainly in the presence of
additional soft susy breaking terms in the potential the above observation
does not hold as the vacuum breaks supersymmetry as well. It is in this very
respect that our present study makes apparent the distinctive characteristics
of non-abelian Q-sballs that arise through solely renormalizable interactions.

Our Q-sball solution is a minimal one as it has \( \det Q = 0 \). As such it is
an equally good solution for both \( SO(3) \) and \( SU(3) \) models. Non-minimal
Q-sball configurations with \( detQ \neq 0 \) are expected to exist for the \( SU(3) \)
model [12]. This is because the Q matrix can be unitarily rotated into a
diagonal form \( \text{diag}[q_1, q_2 - (q_1 + q_2)] \) with \( q_1 \cdot q_2 > 0 \). Consistently with Q
conservation

\[
\text{diag}(q_1, q_2, -(q_1 + q_2)) = \text{diag}(q_1, 0, -q_1) + (0, q_2, -q_2) ,
\]  

(7)

it has been shown that it is energetically favourable for a large Q-ball with
\( detQ \neq 0 \) to fission into two minimal ones with \( detQ = 0 \). The existence of a
minimal Q-sball automatically implies the presence of the same phenomenon
of fission in the \( SU(3) \) model as well. Our conclusions, as they apply to the
scalar sector of a more general theory, go through in the presence of additional
non-renormalizable interactions as well as gauge fields and charges that our
Q-sball configuration might carry. Our present study suggests the presence
of non-topological solitons in the spectrum of any supersymmetric gauge field
theory with residual non-abelian global symmetry and appropriate attractive
interactions.
2 MINIMAL AND MAXIMAL Q-SBALLS IN SU(3)

We consider the Wess-Zumino Lagrangian \[13\]
\[
\mathcal{L} = i\partial_m \bar{\psi}_i \sigma^m \psi_i - \nabla_m \phi^i \nabla^m \phi^i - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi_i \psi_j - \frac{1}{2} \frac{\partial^2 W^*}{\partial \phi^{i*} \partial \phi^{j*}} \psi_i \psi_j - U(\phi, \phi^*),
\]
where $\phi$ and $\psi$ are the dynamical scalar and fermion components of an $N = 1$ chiral multiplet and $W$ is the superpotential. The potential is given in terms of the superpotential as
\[
U = \sum_i \left| \frac{\partial W}{\partial \phi^i} \right|^2.
\]
The above model obeys the following supersymmetry transformations
\[
\delta \xi \phi = \sqrt{2} \xi \psi,
\]
\[
\delta \xi \psi = i \sqrt{2} \sigma^m \xi \partial_m \phi.
\]
As the Lagrangian is explicitly supersymmetric it forces the fermions and scalar components to be degenerate in mass. In what follows we will focus in the scalar sector of the model. We will assume that the scalars belong to the adjoint representation of a global symmetry group $G$. For illustrative purposes, we will take $G$ to be $SU(N)$ in which case, $\phi$ is an $N \times N$ hermitian and traceless matrix. The most general superpotential which leads to renormalizable interactions is given by
\[
\mathcal{W} = m \text{Tr} \phi^2 + h \text{Tr} \phi^3 = m \phi_{ij} + h \phi_{ij} \phi_{jk} \phi_{kl}.
\]
As a result, the potential of the model model takes the form
\[
U = 4m^2 \text{Tr} \phi^2 + 12mh \text{Tr} \phi^3 + 9h^2 \text{Tr} \phi^4.
\]

We observe that in the general parametrization of the introduction, $\mu^2 = 8m^2$, $g = 72mh$ and $\lambda = 216h^2$
\[
\beta = 3.
\]
Remarkably the supersymmetric structure of the model while preserving the available number of renormalizable self interactions intact, it reduces the number of independent dimensionful parameters from two \( \mu \) and \( g \) in the non-susy case to one \( m \) in the present supersymmetric one. In effect it is associated to a theory which possesses a single value for the dimensionless ratio \( 1 < \beta = 3 < 9 \) for which previous analysis suggests that it admits Q-balls in its spectrum for all values of its parameters \( m \) and \( h \).

In what follows we will utilize the available formalism \[12\] to obtain the precise Q-sball configuration. We will focus in the \( SU(3) \) model although our formalism applies to the \( SU(N) \) case as well. We are interested in initial value data \( \phi \) and \( \Pi \equiv \dot{\phi} \) at a fixed time, which minimize the energy keeping \( Q \) fixed. Such data obey

\[
\delta [H + Tr \omega Q] = 0, \tag{14}
\]

where \( H \) is the energy written as a functional of \( \phi \) and \( \Pi \) at fixed time, and \( \Omega \) is a Lagrange multiplier, a traceless hermitian \( 3 \times 3 \) matrix. The above equation implies that

\[
\frac{\delta H}{\delta \phi} = -i [\Omega, \Pi], \tag{15}
\]

and

\[
\frac{\delta H}{\delta \Pi} = i [\Omega, \phi]. \tag{16}
\]

We can observe that \( \phi \) and \( \Pi \) are initial value data for a solution of the equation of motion that is in steady rotation in internal space

\[
\phi(x, t) = e^{i \Omega t} \phi(x, t)e^{-i \Omega t}. \tag{17}
\]

By restricting ourselves to large Q-balls, we consider a sphere of volume \( V \) in the interior of which \( \Pi \) and \( \phi \) are constant. In other words we neglect the contribution to \( Q \) and the energy from the surface region where \( \Pi \) and \( \phi \) go to zero. The energy is then given by

\[
E = V Tr \left( \frac{1}{2} \Pi^2 + U \right), \tag{18}
\]

while

\[
Q = iV [\Pi, \phi]. \tag{19}
\]
In order to firstly look for minimal Q-balls with $detQ = 0$, i.e. those for which one eigenvalue of $Q$ vanishes it is convenient to make a unitary transformation so that $\phi$ is diagonal
\[
\phi = \text{diag}(\phi_1, \phi_2, \phi_3),
\]
(20)
with the eigenvalues taking the following order
\[
\phi_1 \geq \phi_2 \geq \phi_3.
\]
(21)
When $\phi$ is diagonal, $\Pi = i[\omega, \phi]$ and $Q = i[\phi, \dot{\phi}]$ take a pure off-diagonal form
\[
\Pi_{ij} = i\Omega_{ij}(\phi_j - \phi_i),
\]
(22)
and
\[
Q_{ij} = -V\Omega_{ij}(\phi_j - \phi_i)^2.
\]
(23)
Thus we can write $E$ in terms of $Q$ and $\phi$ as
\[
E = \sum_{i>j} |Q_{ij}|^2 V(\phi_i - \phi_j)^2 + VTrU.
\]
(24)
By varying $V$ in the energy expression by keeping $\phi$ and $Q$ fixed we find
\[
Tr\frac{1}{2}\Pi^2 = TrU.
\]
(25)
Using the equations of motion
\[
\ddot{\phi} = [\omega, [\Omega, \phi]],
\]
(26)
it can be easily shown that
\[
Tr(\phi \frac{\partial U}{\partial \phi}) = Tr(\ddot{\phi} \phi) = Tr\Pi^2.
\]
(27)
Hence by eliminating $Tr\Pi^2$ from eqs(25,27) and we get
\[
Tr(\phi \frac{\partial U}{\partial \phi}) = 2TrU.
\]
(28)
For the general renormalizable potential given before, the above relation takes the form
\[
Tr(2g\phi^3 + \lambda\phi^4) = 0.
\]
(29)
We now proceed to determine the Q-sball configuration in detail by the method utilized for a general potential [12].

We may choose to minimize the energy by taking \( Q_{13} = Q^*_{31} = iq \) a positive imaginary and the only nonzero matrix element of \( Q \). Since it is a constant of motion it must commute with \( \omega \). With no loss of generality we choose the \( \Omega \) matrix in complete similarity with the \( Q \) matrix to have only one nonzero off diagonal element

\[
\Omega_{13} = \Omega^*_{31} = -i\omega. \tag{30}
\]

Consequently,

\[
\omega = \frac{q}{V(\phi_1 - \phi_3)^2}. \tag{31}
\]

We see that \( \phi \) is real at all times. In other words every minimal \( SU(3) \) Q-ball is unitarily equivalent to an \( SO(3) \) one as we asserted before. Rewriting the equations of motion as before for a renormalizable potential \( U \) and the chosen \( \omega \) and \( \phi \) we get

\[
\mu^2 \phi + \frac{1}{2} g \phi^2 + \frac{1}{6} \lambda \phi^3 - \frac{1}{3} \text{Tr}(\frac{1}{2} g \phi^2 + \frac{1}{6} \lambda \phi^3)I = [\Omega, [\Omega, \phi]]
\]

\[
= 2\omega^2 (\phi_1 - \phi_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{32}
\]

We note that the (22) element of the above equation contains no reference to \( \omega \). Along with eq.(29) we can solve them to obtain the two independent eigenvalues of \( \phi \). Once we have these two, we can use them in the remaining of eq.(32) to determine \( \omega \). Finally once we have \( \omega \) we can determine \( V \) from \( Q \) in eq.(31). To this end we adopt the parametrization of [12] for the eigenvalues of \( \phi \) in terms of \( \phi_2 \) and a dimensionless variable \( y \) as follows:

\[
\phi_1 = -\frac{1}{2} \phi_2 (1 + y), \quad \phi_3 = -\frac{1}{2} \phi_2 (1 - y). \tag{33}
\]

By changing the sign of \( y \) the values of \( \phi_1 \) and \( \phi_3 \) get exchanged with no effect in our equations. With no loss of generality we can thus take \( y \) to be positive. In terms of these variables eq.(29) takes the form

\[
\frac{3g}{2} \phi^3_2 (1 - y^2) + \frac{1}{8} \lambda \phi^4_2 (3 + y^2)^2 = 0. \tag{34}
\]
The (22) element of eq.(32) is identified to be

$$\mu^2 \phi_2 + \frac{1}{12} g \phi_2^2 (3 - y^2) + \frac{1}{24} \lambda \phi_2^3 (3 + y^2) = 0. \quad (35)$$

Substituting into it the expression for $\phi_2$

$$\phi_2 = \frac{12g(y^2 - 1)}{\lambda(3 + y^2)^2}, \quad (36)$$

we obtain a cubic equation for $z = y^2$ which if expressed in terms of the parameters $m, h$ takes the form

$$z^3 - 15z^2 - 9z - 9 = 0. \quad (37)$$

There exist two imaginary roots and one real one which is relevant for us and it is

$$z \approx 15.613. \quad (38)$$

In terms of $z$, $\phi_2$ is given by

$$\phi_2 = 4 \frac{m}{h} \frac{z - 1}{(3 + z)^2} \approx 0.169 \frac{m}{h}. \quad (39)$$

It automatically fixes the other two components $\phi_1$ and $\phi_2$ through eq.(33) giving

$$\phi_1 = -0.418 \frac{m}{h}, \quad \phi_3 = 0.249 \frac{m}{h}. \quad (40)$$

Thus a minimal Q-sball configuration with $\det Q = 0$ is fully determined in terms of $m$ and $h$. We can now proceed to specify its physical properties. The rotational frequency of the Q-sball is determined to be

$$\omega^2 = \frac{g}{48} \phi_2 (z - 9) = \frac{6(z - 1)(z - 9)}{(3 + z)^2} m^2 \approx 1.67m^2, \quad (41)$$

We observe that $\omega^2 < \mu^2 = 8m^2$, hence the binding meson energy is smaller than the corresponding free meson mass which is the statement of existence of a coherent scalar bound state in the spectrum of the model. The charge density is given by

$$\frac{q}{V} = \omega \phi_2^2 \sqrt{z} = 154.85 \frac{(z - 1)^{5/2}(z - 9)^{1/2}}{(3 + z)^5} m^3 h^2 \approx 0.145 \frac{m^3}{h^2}. \quad (42)$$
Non-abelian minimal Q-sballs are thus generically present in the scalar spectrum of a Wess-Zumino model with a non-abelian global symmetry. We argue that in the presence of degenerate in mass fermionic fields $\psi$ which are components of the chiral superfield our scalar Q-ball is stable. Indeed the global charge is not favourable to be radiated away from the surface of our configuration by the fermions as long as supersymmetry in the vacuum is manifest. The evaporation of the Q-sball\[^{14}\] will only take place in the presence of fermions with mass smaller than the characteristic energy per unit charge of our soliton given by

$$\frac{E}{q} = \omega = 1.292m. \quad (43)$$

This can conceivably occur in the above model through the introduction of additional non-renormalizable soft supersymmetry breaking terms bringing down the fermion mass gap which is given by $\mu \approx 2.83m$.

An interesting property of the Q-sball solution is that it breaks explicitly supersymmetry. Indeed the solution given above violates supersymmetry since

$$\delta \xi \psi \propto \sigma^0 \bar{\xi} \dot{\phi} \propto \sigma^0 [\Omega, \phi] \neq 0. \quad (44)$$

In the supersymmetrically invariant theory Q-sballs are domains where supersymmetry is explicitly broken. Fermions and bosons traversing such configurations are not degenerate in mass in their interior. Our discussion and demonstration of the existence of minimal non-abelian Q-sballs in Wess-Zumino type of models do not get modified in the presence also of gauge fields[^5, ^7] and additional gauge symmetries in the Lagrangian.

We complete our demonstration of existence of non-abelian Q-sballs by discussing non-minimal Q-sballs in the model considered. They arise through the minimization of the energy functional by keeping $det Q \neq 0$ fixed. We adapt the arguments of previous work[^12] in a straightforward manner. For convenience we will focus in the $SU(3)$ model identifying the corresponding non minimal non-dissipative solutions as maximal. To that end we take the meson charge to be diagonal of the form

$$Q = \begin{pmatrix} N & 0 & 0 \\ 0 & n - N & 0 \\ 0 & 0 & -n \end{pmatrix}. \quad (45)$$
The minimum energy of such a meson will be given by
\[ E \sim \mu(n + N). \] (46)
However as it was argued before [12] the configuration can attain a lower energy by fissioning into two minimal Q-sball solutions. The charge decomposition consistently with global charge conservation is the following
\[ Q = \begin{pmatrix} N - n & 0 & 0 \\ 0 & n - N & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -n \end{pmatrix}. \] (47)
The energy of the configuration of two widely separated minimal Q-sballs with the above charge assignments is reduced to
\[ E \sim \mu N. \] (48)
By adapting the argument to our case we expect that if Q-sballs with \( \det Q \not= 0 \) exist they are energetically favorable to fission into two minimal Q-sballs with \( \det Q = 0 \).
By introducing the variable \( y = n/N \) we can denote departure from Q charge minimality. One can numerically demonstrate that for \( 0 < y < 0.06 \), i.e. for not large values of \( \det Q \), and in a renormalizable meson theory such as is our supersymmetric model at hand, maximal Q-sball configurations exist which are favorable to fission into two minimal ones with charge of the form given above. For more details of the algebra involved one can consult previous work done [12] on the issue.

3 Acknowledgements
One of us A.K. thanks Dr Kusenko for discussing his work with him.

References
[1] J. Rosen, J. Math. Phys. 9 (1998)996; \textit{ibid} 9 (1968)999.
[2] R. Friedberg, T.D. Lee and R. Sirlin, Phys. Rev. D13 (1976)2739; Nucl. Phys. B115 (1976)1, 32.
[3] T.D. Lee and Y. Pang, Phys. Rep. 221 (1992) and refs therein.
[4] S. Coleman, Nucl. Phys. B262 (1985)263.
[5] M. Axenides, Harvard Univ. 1986 Ph.D. Thesis unpublished; M. Axenides, Int. J. Mod. Phys. A7 (1992)7169.
[6] J. Distler, B. Hill, and D. Spector, Phys. Letts. B182 (1986)71. A.P. Balachandran, B. Rai and A.M. Srivastava, Phys. Rev. Lett. 59 (1987)853. A.P. Balachandran, B. Rai, G. Sparano and A.M. Srivastava, Int. J. Mod. Phys. A3, (1988)2621.
[7] A. Kusenko, M. Shaposhnikov and P.G. Tinyakov, JETP Lett. 67 (1998)247, hep-th/980104.
[8] A. Kusenko, Phys. Lett. B 406 (1997)26, hep-ph/9705361; Nucl. Phys. Proc. Suppl. 62A/C (1998)248, hep-ph/9707306.
[9] A. Kusenko and M. Shaposhnikov, Phys. Lett. B418 (1998)46, hep-ph/9709492. G. Dvali, A. Kusenko and M. Shaposhnikov, Phys. Lett. B417 (1998)99, hep-ph/9707423.
[10] K. Enqvist and J. McDonald, Phys. Lett. B425 (1998)309, hep-ph/9711514, hep-ph/9806213.
[11] M. Dine, L. Randall and S. Thomas, Nucl. Phys. B458 (1996)291.
[12] A. Safian, S. Coleman and M. Axenides, Nucl. Phys. B297 (1988)498; A. Safian Nucl. Phys. B304 (1988)403.
[13] J. Wess and J. Bagger in Introduction to Supersymmetry and Supergravity, Princeton University Press.
[14] A. Cohen, S. Coleman, H. Georgi and A. Manohar, Nucl. Phys. B272 (1986)301.