Instantons on a Non-commutative $T^4$
from
Twisted $(2,0)$ and Little String Theories

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We show that the moduli space of the $(2,0)$ and little-string theories compactified on $T^3$ with R-symmetry twists is equal to the moduli space of $U(1)$ instantons on a non-commutative $T^4$. The moduli space of $U(q)$ instantons on a non-commutative $T^4$ is obtained from little-string theories of NS5-branes at $A_{q-1}$ singularities with twists. A large class of gauge theories with $\mathcal{N} = 4$ SUSY in 2+1D and $\mathcal{N} = 2$ SUSY in 3+1D are limiting cases of these theories. Hence, the moduli spaces of these gauge theories can be read off from the moduli spaces of instantons on non-commutative tori. We study the phase transitions in these theories and the action of T-duality. On the purely mathematical side, we give a prediction for the moduli space of 2 $U(1)$ instantons on a non-commutative $T^4$.

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1. Introduction

In recent years, starting with the work of [1], the moduli-spaces of vacua have been found for a large class of gauge theories with 8 super-charges in 3+1D and in 2+1D. These solutions were derived from string dualities in [2] and the works that followed. String theory also suggested the existence of new theories in six dimensions [3,4] (see also [5-6]). Compactification of these theories to 3+1D reduces, in certain limits of the external parameter spaces, to ordinary gauge theories.

In this paper we will study compactifications of certain 6 dimensional theories down to 3 dimensions and examine their low energy behaviour. As we will see, all the previously solved gauge theories with \( \mathcal{N} = 2 \) supersymmetry and \( SU(N_1) \times \cdots \times SU(N_r) \) gauge groups [7] can be recovered at special limits of the external parameters of the compactification. We will start with the 6 dimensional theories that are the world-volume theories on \( k \) NS5-branes in type-IIA in the limit of vanishing string coupling keeping the string tension fixed [4]. We denote this theory \( S_A(k) \). It has \((2,0)\) supersymmetry. There is a similar theory coming from \( k \) NS5-branes in type-IIB in the limit of vanishing string coupling keeping the string tension fixed. We denote this theory \( S_B(k) \). It has \((1,1)\) supersymmetry. \( S_A(k) \) and \( S_B(k) \) are often referred to as the little-string theories. They both have an inherent scale, \( m_s \). In the limit \( m_s \to \infty \), \( S_A(k) \) becomes the theory on the world-volume of \( k \) M5-branes – the so called \((2,0)\) theory.

We will compactify these theories down to 3 dimensions. These theories have 16 super-charges, so if they are compactified on \( T^3 \) the resulting theories will have \( \mathcal{N} = 8 \) supersymmetry in three dimensions. The low energy behavior of \( \mathcal{N} = 8 \) theories is trivial. Instead we want to study theories with \( \mathcal{N} = 4 \) supersymmetry, i.e. 8 super-charges. So we have to compactify in a way that breaks half the supersymmetry. We will do that as in [8] by introducing holonomies of the R-symmetry around the three circles in \( T^3 \). To preserve half of the supersymmetries the holonomies were chosen inside a \( SU(2) \) subgroup of the \( Spin(4) \) R-symmetry group. The low energy behaviour of a \( \mathcal{N} = 4 \) theory in \( D = 3 \) is a sigma-model with the moduli-space of vacua as the target-space. So the low energy behaviour is given by the moduli-space of vacua and the its metric.

In [8] the case of \( k = 2 \) was studied in detail. The moduli space of vacua was identified explicitly. For general \( k \) it was conjectured that the moduli space of vacua was given as a moduli space of instantons on non-commutative \( T^4 \). The purpose of this paper will be to prove and generalize this and, at the same time, make the claim more precise.

Let us start by identifying all the parameters of the compactification. Consider \( S_A(k) \) compactified on \( T^3 \). The scale of \( S_A(k) \) is \( m_s \), the string mass. The \( T^3 \) is specified by a metric. For simplicity we will take it to be rectangular. It is easy to incorporate the more
general case. Furthermore there can be a flux of the 2-form \( B^{NS} \) field of type IIA through 2-cycles in the \( T^3 \). For simplicity we set \( B^{NS} = 0 \). It is again not hard to incorporate the more general case. Now we come to the most interesting parameters – the twists. The R-symmetry group of \( S_A(k) \) is \( \text{Spin}(4)_R \), corresponding to transverse rotations. The twists are taken inside

\[
U(1)_R \subset SU(2)_B \subset SU(2)_B \times SU(2)_U = \text{Spin}(4)_R
\]  

This preserves 8 of the 16 super-charges. There is a twist, \( \alpha_i \), along each of the 3 circles. The \( \alpha_i \)'s are periodic

\[
\alpha_i \rightarrow \alpha_i + 2\pi, \quad i = 1, 2, 3
\]  

The twists can be described in the following way. States that are charged under \( U(1)_R \) receive a phase shift in traversing a circle. In other words, momentum along the circle is shifted from \( \frac{n}{R} \) to \( \frac{n - \alpha}{2\pi} \). By performing T-duality along all circles of the \( T^3 \) we get \( S_B(k) \) on another \( T^3 \). Momentum has been exchanged with winding, so the T-dual of the twists has the following description. States that are charged under \( U(1)_R \) have fractional winding numbers; \( \frac{n - \alpha}{2\pi} \) instead of \( n \). We call this kind of twist an “\( \eta \)-twist.” By combining these two types of twists we learn that the most general twist around a circle shifts both momentum and winding. In other words the \( S_A(k) \) compactification on \( T^3 \) depends on 6 parameters

\[
\alpha, \eta_i, \quad i = 1, 2, 3
\]  

where \( \alpha \) shifts momentum and \( \eta_i \) shifts winding. The \( \alpha_i \)'s have a clear geometrical interpretation. In traversing the circle the transverse space is rotated. The \( \eta_i \)'s are harder to visualize. They are geometrical in the T-dual \( S_B(k) \).

We can actually generalize this system even more. Instead of \( k \) NS5-branes we can consider \( k \) NS5-branes on top of an \( A_{q-1} \) singularity. In other words the transverse space to the NS5-branes is \( \mathbb{R}^4/Z_q \), where \( Z_q \) is a subgroup of \( U(1)_R \). \( U(1)_R \) is still a symmetry of this space, so we can twist as before. These theories have 8 super-charges in 6 dimensions. The \( U(1)_R \) is a global symmetry which commutes with super-charges. The twists, therefore, do not break any more supersymmetry, so the compactified theory still has \( N = 4 \) in 3 dimensions.

Theories of branes on top of an ADE singularity have been studied in [9,10]. These 6 dimensional theories are, loosely speaking, quiver gauge theories coupled to tensor theories or vice versa, depending on whether it is in type-IIA or type-IIB.

The 3 dimensional theory, obtained after compactification with twists, has a low energy description as a sigma-model with a target-space, which is equal to the moduli-space of
vacua. In this paper we will prove that the moduli-space of vacua is equal to the moduli-space of $kU(q)$ instantons on a non-commutative $T^4$. The non-commutativity is set by the 6 parameters $\alpha_i$ and $\eta_i$. This generalizes the case of compactification without twists where the moduli-space of the theories turns out to be the moduli-space of ordinary instantons $[10,12]$.

This result implies similar results for all the theories which are special cases of this. This includes firstly the $(2,0)$ theory which can be obtained from $S_A(k)$ by $m_s \to \infty$. Secondly, it includes all three-dimensional $U(k)$ gauge theories with adjoint matter. By incorporating the $A_q$ singularity it also includes all gauge theories with group $U(k) \times \cdots \times U(k)$ and matter in $(k, \bar{k}, 1, \ldots, 1) + \text{permutations}$. By taking the gauge coupling to zero in some $U(k)$ we can get theories with the gauge group being $U(k) \times \cdots \times U(k)$ with fundamental and bi-fundamental matter in various combinations with generic masses.

Our results imply that all these 2+1D gauge theories have a moduli-space of vacua equal to the moduli-space of vacua of instantons on non-commutative $\mathbb{R}^3 \times S^1$. In the case of mass deformed $\mathcal{N} = 8$ this result was derived earlier in $[13]$.

By decompactifying one circle similar results hold for the moduli space of 4 dimensional gauge theories on $\mathbb{R}^3 \times S^1$.

We also find that for certain discrete values of the twists there are Higgs branches emanating from some locus of the Coulomb branch. We will identify these and calculate their dimensions. We will also calculate the existence of these branches from pure field theoretic arguments and find agreement in the structure of the Higgs branches. These branches generalize a branch found in $[8]$.

Moreover, combining our results with the formulas in $[8]$ for the special case of $q = 1$ and $k = 2$, we get a prediction for the moduli-space of two $U(1)$ instantons on a non-commutative $T^4$. This is a $K3$ (projecting out the center of mass) and the exact point in moduli-space was given in $[8]$ as a function of the twists.

The organization of the paper is as follows. In section (2) we present the proof that the moduli space is equal to the moduli space of instantons on non-commutative $T^4$. In section (3) we have a short review of non-commutative gauge theories. In section (4) we use this information about non-commutative gauge theory to make the claim about the moduli-space of non-commutative instantons precise and discuss some features of it. In section (5) we describe the decompactification limit to 3+1D (compactification of the 5+1D theories on $T^2$ with twists). In section (6) we present a more detailed geometrical formulation of $\alpha$-twists and especially $\eta$-twists. We conclude with a summary of the results and possible further direction.
2. The solution

In this section, we derive the solution to the moduli space of the twisted theory. To construct the solution we will start with type-IIA on a space

$$\mathbb{R}^{2,1} \times T^3 \times \mathbb{R}^4,$$

where $\times \mathbb{R}^4$ means that locally the space looks like $\mathbb{R}^{2,1} \times T^3 \times \mathbb{R}^4$ but as we go around a cycle of the $T^3$ we have to twist the transverse space $\mathbb{R}^4$ by the appropriate element of $Spin(4)$ corresponding to the twist. Now we take $k$ NS5-branes and let them stretch along $\mathbb{R}^{2,1} \times T^3$ and the origin of $\mathbb{R}^4$. The question what is the low-energy effective action for this system in the limit that the type-IIA string coupling constant $\lambda \to 0$.

As will be clear later on, it is easier to solve the problem if we first replace the transverse $\mathbb{R}^4$ with another manifold $M_4$. In the limit that the curvature of $M_4$ is small at the position of the NS5-branes the switch from $\mathbb{R}^4$ to $M_4$ will not make a big difference. Moreover, we can argue that the quantum fluctuations in the transverse position of the NS5-brane are related to the fluctuations of the scalars of $S_A(k)$ as,

$$x \sim m_s^{-3}\lambda\Phi,$$

and for energy scales $m_s$, $\Phi$ is of the order of $m_s^2$. In the limit $\lambda \to 0$, the transverse fluctuations of the NS5-brane go to zero and if the point in $M_4$ is smooth, it would seem that the dynamics of the NS5-brane will be the same as on $\mathbb{R}^4$. This argument should be taken with caution since the actual solitonic solution of the NS5-brane has a cross-section of about $m_s$. In any case, we will not have to rely on this argument.

The manifold $M_4$ that we will use is the Taub-NUT space. The metric is,

$$ds^2 = \rho^2 U(dy - A_i dx^i)^2 + U^{-1} (d\vec{x})^2, \quad i = 1 \ldots 3, \quad 0 \leq y \leq 2\pi. \quad (2.1)$$

where,

$$U = \left(1 + \frac{\rho}{2|x|}\right)^{-1},$$

and $A_i$ is the gauge field of a monopole centered at the origin.

The Taub-NUT space has the following desirable properties (these properties were also used in [14]),

(1) If we excise the origin, what remains is a circle fibration over $\mathbb{R}^3 - \{0\}$. Eqn(2.1) is written such that $\vec{x}$ is the coordinate on this base $\mathbb{R}^3 - \{0\}$. For $|\vec{x}|$ restricted to a constant, the fibration is exactly the Hopf fibration of $S^3$ over $S^2$.  

4
The origin $\vec{x} = 0$ is a smooth point.

As $|\vec{x}| \to \infty$ the radius of the fiber becomes $\rho$.

The space has a $U(1)$ isometry group that preserves the origin $\vec{x} = 0$. An element $g(\theta) = e^{i\theta} \in U(1)$ acts by $y \to y + \theta$. It also acts on the tangent space $\mathbb{R}^4$ at the origin by embedding $e^{i\theta}$ inside

$$U(1) \to SU(2)_L \to (SU(2)_L \otimes SU(2)_R)/\mathbb{Z}_2 = SO(4).$$

Now that we have replaced the transverse $\mathbb{R}^4$ with a Taub-NUT space we have $k$ NS5-branes on the space,

$$\mathbb{R}^{2,1} \times T^3 \times_\alpha TN(\rho).$$

The $\alpha$-twists are incorporated as follows. As we go around a cycle of $T^3$ we have to act on the fiber $TN(\rho)$ with $g(\alpha_i)$ where $\alpha_i$ is the appropriate twist. In the limit $\rho \to \infty$, $TN(\rho)$ becomes $\mathbb{R}^4$ and the isometry $g(\alpha_i)$ becomes the element in $SO(4)$ that we have used for the twist. The virtue of working with $TN(\rho)$ instead of $\mathbb{R}^4$ is that at $\vec{x} = \infty$ the circle fiber becomes of finite size which will help in subsequent dualities.

To generalize the construction to the case of $k$ NS5-branes at an $A_{q-1}$ singularity, $\mathbb{R}^4/\mathbb{Z}_q$, we replace the transverse $\mathbb{R}^4/\mathbb{Z}_q$ with a $q$-centered Taub-NUT space, $TN_q(\rho)$ with radius $\rho \to \infty$. This space has similar properties,

(1') If we excise the origin, what remains is a circle fibration over $\mathbb{R}^3 - \{0\}$. For $|\vec{x}|$ restricted to a constant, the fibration is is a circle bundle over $S^2$ with first Chern-class $c_1 = q$.

(2') Near the origin $\vec{x} = 0$, $TN_q$ looks like $\mathbb{R}^4/\mathbb{Z}_q$.

(3') As $|\vec{x}| \to \infty$ the radius of the fiber becomes $\rho$.

(4') The space has a $U(1)$ isometry group that preserves the origin $\vec{x} = 0$. An element $g(\theta) = e^{i\theta} \in U(1)$ acts at $\vec{x} = \infty$ by $y \to y + \theta$. It also acts on the tangent space $\mathbb{R}^4/\mathbb{Z}_q$ at the origin by embedding $e^{i\theta}$ inside

$$U(1) \to SU(2)_L \to (SU(2)_L \otimes SU(2)_R)/\mathbb{Z}_2 = SO(4).$$

Note that the discrete $\mathbb{Z}_q$ by which we mod out is a subgroup of the same $U(1) \subset SU(2)_L$ as well.
2.1. Chains of Dualities

We have seen that the twisted compactified little-string theories can be realized as follows. Start with type-IIA on $\mathbb{R}^{2,1} \times T^3 \times T\mathbb{N}_q$, where the radii of $T^3$ are $R_i$ (of the order of $m_s$) and the radius of the fiber of the Taub-NUT space is taken to be $\rho$. Put $k$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$ and study the limit,

$$\lambda \to 0, \quad m_s \rho \to \infty.$$

In principle, we could probably settle on a constant $m_s \rho$ as well, since the transverse fluctuations of the NS5-brane are small. However, the transverse size of the NS5-brane, as a solitonic object, is of the order of $m_s^{-1}$. Therefore, to be on the safe side, we take $m_s \rho \to \infty$. The technique for solving theories with 8 supersymmetries is [2] to identify a parameter that decouples from the vector-multiplet and such that at one limit of this parameter the theory is described by gauge theory (or little-string theory, in our case) and in another limit a dual description becomes weakly coupled. In that second limit, the theory is no longer described by the gauge theory but the vacuum structure remains the same and is determined by the classical equations of motion. This method was also applied in [15,16,7].

In our case, to solve the problem we take the limit of strong coupling keeping the Taub-NUT radius large.

$$\lambda \to \infty, \quad m_s \rho \to \infty,$$

We will also require that $\lambda(m_s \rho)^{-3} \to \infty$. We can think of $\rho$ as being fixed but very large and $\lambda \to \infty$ much faster. We will not show that this corresponds to a parameter that is in a hyper-multiplet (and hence decouples from the vector-multiplets) but this is the basic assumption. Recall that in 2+1D hyper-multiplets and vector-multiplets can be distinguished with the help of the $U(1)_R \otimes SU(2)_U$ symmetry which is the unbroken subgroup of (1.1). The scalar fields of a vector-multiplet are invariant under $SU(2)_U$ while the scalar fields of a hyper-multiplet are in the 2 (see [17]). (The dilaton, which is a singlet, is a quadratic expression in these fields.) Similarly, the fermions of a hyper-multiplet are invariant under $SU(2)_U$ and the fermions of a vector-multiplet are in the 2.

The next step is to use string-dualities to convert the region (2.2) to a weakly coupled theory.

At this point we have $k$ NS5-branes in type-IIA on $\mathbb{R}^{2,1} \times T^3 \times T\mathbb{N}_q$ with string coupling $\lambda$, string scale $m_s$, $T^3$-radii $R_i$, and twists $\alpha_i$. For simplicity, we assume that $T^3$ is of the form $S^1 \times S^1 \times S^1$ with no NS-NS 2-form fluxes. Since $\lambda \to \infty$ we view this as $k$
M5-branes in M-theory on $\mathbb{R}^{2,1} \times T^3 \times S^1 \times T\mathbb{N}_q$. Let $M_p$ be the 11-dimensional Planck scale. The radius of $S^1$ is, $R$. They are related according to,

$$ R = \frac{\lambda}{m_s}, \quad M_p^3 = \frac{m_s^3}{\lambda}. $$

The radius of $T\mathbb{N}_q$ is, $\rho$.

**Step 1:** Since, in the limit (2.2),

$$ M_p \rho = m_s \lambda^{-1/3} \rho \rightarrow 0, $$

we should view the fiber of the Taub-NUT as the 11th small dimension and convert to type-IIA on $\mathbb{R}^{2,1} \times T^3 \times S^1 \times \mathbb{R}^3$. We also have $k$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$ and $T\mathbb{N}_q$ became $q$ D6-branes on $\mathbb{R}^{2,1} \times T^3 \times S^1$. The $\alpha$-twists became RR 1-form Wilson lines along the cycles of $T^3$. The string coupling constant is given by,

$$ \lambda' = \lambda^{-1/2} (m_s \rho^{3/2} \lambda^{-1/2} R_i \rightarrow 0). $$

The new string scale is,

$$ M'_s = m_s^{3/2} \rho^{1/2} \lambda^{-1/2}, $$

and the radii of $T^3$ satisfy,

$$ M'_s R_i = m_s^{3/2} \rho^{1/2} \lambda^{-1/2} R_i \rightarrow 0. $$

This means that we must perform T-duality on $T^3$.

**Step 2:** After T-duality on $T^3$ we obtain type-IIB on $\mathbb{R}^{2,1} \times \tilde{T}^3 \times S^1 \times \mathbb{R}^3$ with radii $\hat{R}_i$ which satisfy,

$$ M'_s \hat{R}_i = m_s^{-3/2} \rho^{-1/2} \lambda^{1/2} R_i^{-1} \rightarrow \infty. $$

There are now $k$ NS5-branes on $\mathbb{R}^{2,1} \times \tilde{T}^3$ and $q$ D3-branes on $\mathbb{R}^{2,1} \times S^1$. At this point the $\alpha$-twists became RR 2-form fluxes,

$$ \alpha_i \epsilon_{ijk} = \int_{C_{jk}} B^{RR}, \quad i, j, k = 1 \ldots 3, $$

where $C_{jk}$ is the 2-cycle made out of the $j^{th}$ and $k^{th}$ directions in $T^3$. The string coupling is now,

$$ \lambda^{(2)} = \frac{\lambda'}{m_s^{9/2} \rho^{3/2} \lambda^{-3/2} R_1 R_2 R_3} = \lambda m_s^{-3} R_1^{-1} R_2^{-1} R_3^{-1} \rightarrow \infty. $$
This means that we must do S-duality.

**Step 3:** After S-duality we get type-IIB with $q$ D3-branes and $k$ D5-branes in the same geometry. The string coupling constant is now,

$$\lambda^{(3)} = \lambda^{-1} m_s^3 R_1 R_2 R_3 \to 0,$$

and the string scale is,

$$M_s^{(3)} = \lambda^{-1/2} m_s^3 (R_1 R_2 R_3)^{1/2}.$$

The radii satisfy,

$$M_s^{(3)} \hat{R}_i = \rho^{-1/2} (R_1 R_2 R_3)^{1/2} R_i^{-1} \to 0,$$

and the radius of $S^1$ satisfies,

$$M_s^{(3)} R = m_s^2 \rho^{1/2} (R_1 R_2 R_3)^{1/2} \to \infty.$$

At this point,

$$\alpha_i\varepsilon_{ijk} = \int_{C_{jk}} B^{NSNS}, \quad i,j,k = 1 \ldots 3.$$

Since $M_s^{(3)} \hat{R}_i \to 0$, we must perform another T-duality on $T^3$. However, because of the NS-NS 2-form fluxes, just as in [19], another T-duality will not help. Instead, let us do a T-duality on $S^1$ which brings us to the final setup of gauge theory on a non-commutative $T^4$.

**Step 4:** After T-duality along $S^1$ we get type-IIA with $k$ D6-branes and $q$ D2-branes. The string coupling is now,

$$\lambda^{(4)} = \frac{\lambda^{(3)}}{M_s^{(3)} R} = \lambda^{-1} m_s \rho^{-1/2} (R_1 R_2 R_3)^{1/2} \to 0,$$

and $\hat{M}_s = M_s^{(3)}$. The radii satisfy,

$$\hat{M}_s \hat{R}_i = \rho^{-1/2} (R_1 R_2 R_3)^{1/2} R_i^{-1} \to 0,$$

and the radius of the $S^1$ satisfies,

$$\hat{M}_s \hat{R} = m_s^{-2} \rho^{-1/2} (R_1 R_2 R_3)^{-1/2} \to 0.$$
At this point, the $\alpha$-twists are still NS-NS 2-form fluxes. We thus end up with a system of $k$ D6-branes on $T^4 \times \mathbb{R}^{2,1}$ and $q$ D2-branes which are points on $T^4$. The radii of $T^4$ are given, in terms of the 3 radii $R_i$ of the original $T^3$, as follows,
\begin{align}
\hat{R}_i &= \hat{M}_s^{-1} \rho^{-1/2}(R_1 R_2 R_3)^{1/2} R_i^{-1}, \\
\hat{R}_4 &= \hat{M}_s^{-1} m_s^{-2} \rho^{-1/2}(R_1 R_2 R_3)^{-1/2}.
\end{align}
(2.3)

Here $\hat{M}_s$ denotes the final type-IIA (with the D2-branes and D6-branes) string scale. The final string coupling constant is,
\[
\lambda = \lambda^{-1} m_s \rho^{-1/2}(R_1 R_2 R_3)^{1/2}.
\]

Similarly, we can start with $S_A(k)$ with 3 $\eta$-twists. By definition, this is $S_B(k)$ on the dual $T^3$ with 3 $\alpha$-twists. We realize this in type-IIB on the background $\mathbb{R}^{2,1} \times T^3 \times \text{Taub-NUT}_q$ and $k$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$. As before, the fiber of the Taub-NUT space is denoted by $\rho$. We first perform S-duality to replace the NS5-branes with $k$ D5-branes. At this point the $\eta$-twists are off-diagonal components of the metric $g_{i9}$ with $i$ in the direction of $T^3$ and 9 in the direction of the Taub-NUT fiber. Then, we perform T-duality on the direction of $\rho$ to obtain type-IIA on $\mathbb{R}^{2,1} \times T^3 \times S^1 \times \mathbb{R}^3$ with $q$ NS5-branes on $\mathbb{R}^{2,1} \times T^3$ and $k$ D6-branes on $\mathbb{R}^{2,1} \times T^3 \times S^1$. The $\eta$-twists became NS-NS 2-form fluxes $B_{i4}$ where 4 is the direction of $S^1$. Then, we do T-duality on the three directions of $T^3$. We obtain $k$ D3-branes on $\mathbb{R}^{2,1} \times S^1$ and $q$ NS5-branes. The $\eta$-twists are now off-diagonal components $g_{i4}$. We then do another S-duality to get $k$ D3-branes and $q$ D5-branes and, finally, another T-duality on $T^3$. At this point we are back with $k$ D6-branes and $q$ D2-branes. The $\eta$-fluxes are now NS-NS 2-form fluxes $B_{i4}$.

The moduli space is thus the same as the moduli space of $q$ D2-branes inside $k$ D6-branes on $T^4$ with NS-NS 2-form fluxes. In the case of $\alpha$-twists, these fluxes have both indices in the direction of $T^3 \subset T^4$. In the case of $\eta$-twists, the fluxes had one index in the direction of $T^3$ and the other index in the $4^{th}$ direction. In the generic case, we have both $\alpha$-twists and $\eta$-twists simultaneously. The result is that the NS-NS 2-form flux is nonzero for all 6 2-cycles of $T^4$. The string scale, string flux, string coupling, and the parameters of the $T^4$ are as calculated above. We could in principle follow the chain of dualities above with simultaneous $\alpha$-twists and $\eta$-twists but the intermediate steps would involve cumbersome non-linear expressions.

The moduli space of $q$ D2-branes inside $k$ D6-branes on $T^4$ with NS-NS 2-form fluxes, and in the limit that the size of the $T^4$ vanishes, was shown to be equivalent to the moduli space of $k$ instantons of $U(q)$ gauge theory on a non-commutative $T^4$ [18-21]. It is likely that this result is true even for $T^4$ of finite size, because the size decouples by arguments as above.

In the next sections we will review the non-commutative geometry and formulate a precise statement about the moduli space.
3. Review of Noncommutative Gauge Theory

In this section we will review the elements of non-commutative gauge theory which are relevant to our situation.

Non-commutative gauge theory first entered string theory in [19] where it was shown to provide a matrix model for M-theory on a torus with the $C^{(3)}$ field turned on along the light-like circle. Subsequently, a lot of interesting work on this topic was done [20,23]. What we need here is not the connection to matrix theory but just the study of D-branes with a $B^{NS}$ fields turned on.

Consider type-IIA on $\mathbb{R}^{1,9-d} \times T^d$ with $q$ D0-branes. The radii of $T^d$ are called $R_i, i = 1,..,d$, the string mass $m_s$ and the coupling $\lambda$. Furthermore let there be a constant $B^{NS}$ field along $T^d$. Let

$$ b_{ij} = \int_{ij} B^{NS}, \quad i, j = 1, \ldots, d \quad (3.1) $$

be the flux of $B^{NS}$ through the $T^2$ spanned by directions $i, j$. The $b_{ij}$ are periodic with period $2\pi$ due to the gauge invariance of $B^{NS}$.

In [27] this system was studied using the approach of [34]. The result is that the low energy physics is described by a $d+1$ dimensional $U(q)$ gauge theory on a dual torus, $	ilde{T}^d \times \mathbb{R}^{0,1}$ with radii

$$ \tilde{R}_i = \frac{1}{m_s^2 R_i} \quad (3.2) $$

and gauge coupling

$$ \frac{1}{g^2} = \frac{m_s^{2d-3} R_1 \ldots R_d}{\lambda}. \quad (3.3) $$

The effect of $b_{ij}$ is to change the action. Every time two fields are being multiplied, the multiplication is with the ∗-product defined as,

$$ (\phi^{(2)} \ast \phi^{(1)})(x) = e^{-\frac{b_{ij}}{2m_s^2 R_i R_j} (\partial_i^{(2)} \partial_j^{(1)} - \partial_j^{(2)} \partial_i^{(1)})} \phi^{(2)}(x_2)\phi^{(1)}(x_1) \bigg|_{x^{(2)} = x^{(1)} = x} $$

$$ \phi^{(a)}_i = \partial x^{(a)}_i, \quad a = 1, 2. \quad (3.4) $$

The action is the usual gauge theory action just with this modification.

If there had been no $B^{NS}$-field the resulting $d+1$ dimensional gauge theory could have been obtained by performing T-duality along $T^d$. The $q$ D0-branes would have turned into $q$ Dd-branes. The radii and gauge coupling of the $U(q)$ theory can be calculated in this way. The important point to remember is that the only change from having a $B^{NS}$-field
is to change the product into eq.(3.4). The radii and gauge coupling are independent of \(b_{ij}\). This result could not have been obtained by T-duality, since \(B^{NS}\)-fields change the formulas of T-duality and would have given other radii and gauge coupling.

There is another way of formulating this gauge theory. Instead of working with the \(*\)-product, eq.(3.4), one can say that the torus \(\tilde{T}^d\) is non-commutative. The algebra of functions on the torus is, \(A\), is generated by \(U_1, \ldots, U_d\) with relations

\[
U_i U_j = U_j U_i e^{ib_{ij}}
\]

(3.5)

The generalization of finite dimensional vector fields is finitely generated projective modules over \(A\). Let \(E\) be such a module. One can define connections, \(\nabla\), and curvature \(F_{ij}\) of this module \([19,21]\). One can define the Chern character of the module \(E\)

\[
\hat{\tau}(E) = \sum_{k=0} ch(E) \frac{\hat{\tau}(F^k)}{(2\pi i)^{k!k!}}
\]

(3.6)

\(\hat{\tau}\) is the trace on \(End_A(E)\). \(ch(E)\) can be regarded as an element in the cohomology, \(H^*(T^d, C)\), of \(T^d\), the original torus. \(ch(E)\) is not integral but there exists an integral cohomology class \(\mu(E) \in H^*(T^d, C)\) such that

\[
ch(E) = e^{i\pi \iota(b)} \mu(E)
\]

(3.7)

Here \(\iota(b)\) denotes contraction with \(b\) considered as an element of \(H_*(T^d, C)\) \([21]\).

The mathematical fact that the module \(E\) is characterized by integers is in exact agreement with our expectation from D-brane physics. Besides the \(q\) D0-branes on \(T^d\) there could be any number of D2-branes, D4-branes, etc. wrapped on \(T^d\). These numbers are exactly given by \(\mu(E)\). \(ch(E)\) measures the fact that D2-branes with \(B^{NS}\)-fields turned on have an effective D0-brane charge and the equivalent phenomena for other branes. Suppose for instance that only \(\mu_0\) and \(\mu_1\) are nonzero, then,

\[
ch_0 = \mu_0 + \frac{b_{12}}{2\pi} \mu_1, \quad ch_1 = \mu_1.
\]

(3.8)

This equation reflects the fact that the number of D2-branes is unchanged by the presence of the \(B^{NS}\)-field but the number of D0-branes is shifted by the product of the number of D2-branes and the \(B^{NS}\)-field along the D2-branes.
4. Noncommutative Instantons as the Moduli-space

Let us now go back to our system of $q$ D2-branes inside $k$ D6-branes given above. They have a common $\mathbb{R}^{1,2}$. This is the space-time in which the 3 dimensional theory is living. The 3 dimensional theory has a low energy description as a sigma model with the moduli space of vacua as target space. The moduli space of vacua is a Hyper-kähler manifold. The moduli space of vacua comes from the dynamics on the $T^4$, which is the same as the dynamics of $q$ D0-branes in $k$ D4-branes on $T^4$. The radii of the $T^4$, $\hat{R}_1, \hat{R}_2, \hat{R}_3, \hat{R}_4$, and the string coupling $\hat{\lambda}$ and string scale $\hat{M}_s$ are given in terms of the parameters of the $S_A(k)$ compactification in (2.3) which we repeat here,

$$\hat{R}_i = m_s^{-3}\lambda\rho^{-1}R_i^{-1}, \quad i = 1, 2, 3, \quad \hat{R}_4 = m_s^{-5}\lambda\rho^{-1}(R_1R_2R_3)^{-1},$$

$$\hat{M}_s = \lambda^{-1}m_s^3\rho^{1/2}(R_1R_2R_3)^{1/2}, \quad \hat{\lambda} = \lambda^{-1}m_s\rho^{-1/2}(R_1R_2R_3)^{1/2},$$

Furthermore there is a $B^{NS}$-field turned on along $T^4$,

$$\int_{12} B^{NS} = \alpha_3, \quad \int_{31} B^{NS} = \alpha_2,$$

$$\int_{23} B^{NS} = \alpha_1, \quad \int_{i4} B^{NS} = \eta_i, \quad i = 1, 2, 3. \quad (4.2)$$

but the vacuum structure of the vector-multiplets should be independent of $\rho$ in this limit.

According to the above review of non-commutative geometry, the moduli space is equal to the moduli space of $k$ instantons in $U(q)$ gauge theory on a non-commutative torus, $\tilde{T}^4$, with non-commutativity parameters equal to $\alpha_i, \eta_i$. As explained above the radii and gauge coupling of this gauge theory are the same as if $\alpha_i = \eta_i = 0$. Hence they can be found by T-duality on $T^4$. By this T-duality one obtains $k$ D2-branes in $q$ D6-branes on $\tilde{T}^4$ of radii,

$$\tilde{R}_1 = \frac{\lambda}{m_s^3R_2R_3}, \quad \tilde{R}_2 = \frac{\lambda}{m_s^3R_1R_3}, \quad \tilde{R}_3 = \frac{\lambda}{m_s^3R_1R_2}, \quad \tilde{R}_4 = \frac{\lambda}{m_s}, \quad (4.3)$$

and string mass, $\tilde{m}_s$, and coupling, $\tilde{\lambda}$,

$$\tilde{m}_s = \tilde{M}_s = \lambda^{-1}m_s^3\rho^{1/2}(R_1R_2R_3)^{1/2}, \quad \tilde{\lambda} = \lambda^{-1}m_s^3\rho^{3/2}(R_1R_2R_3)^{1/2}. \quad (4.4)$$

In the $U(q)$ theory, this gives a gauge coupling of,

$$\frac{1}{g^2} = \frac{\tilde{m}_s^{-3}}{\lambda} = \lambda^{-2}m_s^6R_1R_2R_3. \quad (4.5)$$
Observe that $\rho$ has dropped out of the radii and the gauge coupling.

What about the limit $\lambda \to \infty$ and $m_s$ fixed. To see that the moduli space of vacua is well defined in this limit we should remember that scalar fields in three dimensions have dimension $\frac{1}{2}$, if we want a standard kinetic term. We can either view the moduli space of vacua from the $U(q)$ gauge theory point of view or from the $U(k)$ theory on the D2-branes. From the last point of view the moduli space is the Higgs branch. The action of the $U(k)$ theory has a term,

$$\frac{1}{2} \frac{1}{\lambda \tilde{m}_s} \int d^3 x (\partial_\mu (\tilde{m}_s^{-2} X^i))^2$$  \hspace{1cm} (4.6)

We define $\Phi^i = \tilde{\lambda}^{-1/2} \tilde{m}_s^{-3/2} X^i$. This $\Phi$ has a standard kinetic term,

$$\frac{1}{2} \int d^3 x (\partial_\mu \Phi^i)^2$$  \hspace{1cm} (4.7)

The radii of the $\Phi^i$ are $R(\Phi^i) = \tilde{\lambda}^{-1/2} \tilde{m}_s^{-3/2} \tilde{R}^i$.

$$R(\Phi^1) = \sqrt{\frac{R_1}{R_2 R_3}}, \quad R(\Phi^2) = \sqrt{\frac{R_2}{R_1 R_3}}, \quad R(\Phi^3) = \sqrt{\frac{R_3}{R_1 R_2}} \quad R(\Phi^4) = m_s^2 \sqrt{R_1 R_2 R_3}.$$  \hspace{1cm} (4.8)

We see that the limit $\lambda \to \infty$ exists. This last discussion was really superfluous. Since $S_A(k)$ only depends on the combination $m_s^2$ and does not feel $\rho$, this had to be true. For finite $m_s \rho$, it could even be true for the full theory, not just the moduli space of vacua. The effect of the twists is just to deform the moduli space and so does not change the fact that the moduli space is independent of $\rho$ and has a limit when $\lambda \to \infty$, keeping $m_s$ fixed.

We can also see from (4.8) what happens in the limit of the $(2,0)$ theory. For this limit we take $m_s \to \infty$. We find that the $T^4$ degenerates to $T^3 \times \mathbb{R}$.

Let us now be more precise about the space of instantons on a non-commutative $T^4$. For this sake we will temporarily neglect the uncompactified directions and think of our system as $q$ D0-branes and $k$ D4-branes on $T^4$. According to the review of non-commutative geometry above, this is described by a gauge theory on the dual $\tilde{T}^4$ with non-commutativity parameters, $b_{ij}$, equal to the twists. By gauge theory we really mean a projective module, $E$, which is characterized by

$$\mu(E) = H^*(T^4, \mathbb{Z}).$$  \hspace{1cm} (4.9)

$\mu(E)$ has components in dimensions 0,2 and 4. $\mu_0 = q$ is the number of D0-branes on $T^4$. $(\mu_1)_{ij}$ is the number of D2-branes in the $T^2$ in direction $(i,j)$ with $i,j = 1,2,3,4$. $\mu_2 = k$ is the number of D4-branes. So far we have not specified the number of D2-branes. Since
we are interested in the low energy dynamics we should take the number of D2-branes to minimize the total energy in the D0,D2,D4 brane system. When $b_{ij} = 0$ this is done by setting $\mu_1 = 0$, i.e. no D2-branes. Let us turn on $b_{12}$, say. From the formula

$$ch(E) = e^{\frac{1}{2\pi} \epsilon(b) \mu(E)}$$  \hspace{1cm} (4.10)

we get

$$(ch_1)_{34} = (\mu_1)_{34} + \frac{b_{12}}{2\pi} \mu_2 = (\mu_1)_{34} + \frac{b_{12}}{2\pi} k.$$  \hspace{1cm} (4.11)

To minimize the energy, $(ch_1)_{34}$ should be minimized. We see that when $b_{12} > \frac{1}{2\pi} 2\pi k$ we can lower the energy by taking $(\mu_1)_{34} = -1$. This phenomena divides the space of $b_{ij}$ into “Brillouin” zones. Each zone is a six dimensional cube of length $\frac{2\pi k}{k}$ in each direction. Inside a zone the low energy physics is described by the gauge theory corresponding to a module with the $\mu(E)$ which minimizes the energy. In crossing the boundary between 2 zones, $\mu(E)$ jumps.

We also see another interesting phenomena. Whenever $\frac{b_{12} k}{2\pi}$ is an integer we have $(\mu_1)_{34} = -\frac{b_{12}}{2\pi}$ and hence $(ch_1)_{34} = 0$. This means that $ch(E)$ is nonzero only in dimensions 0 and 4 (We are keeping all other components of $b_{ij} = 0$. Only $b_{12} = n\frac{2\pi}{k}$). This is exactly like the pure D0,D4 system with no $B_{NS}$-field. This system has a phase where the D0-branes and D4-branes are separated. To reach this phase the system has to go through zero-size instantons. We thus conclude that whenever $b_{12} = n\frac{2\pi}{k}$, $n \in \mathbb{Z}$ there is another phase. Of course, there is nothing special about $b_{12}$. Similar statements could be made for the other 5 components of $b_{ij}$ and even for all of them simultaneously. The point is that for each center of the “Brillouin” zone there is another branch emanating from a locus on the Coulomb branch. It emanates from the points on the Coulomb branch where some instantons have shrunk to zero size. The other phase consists of the $k$ D4-branes with $-n$ D2-branes inside moving away from the $q$ D0-branes. Let us calculate the dimension of this branch. Suppose first $n = 1$, so there are $k$ D4-branes with $-1$ D2-brane inside (equivalently 1 anti D2-brane). This system has a bound state. It is not marginally bound. The system has an 8 dimensional moduli space. To see this we should really remember that it is really $k$ D6-branes with $-1$ D4-brane. 4 of the dimensions are $U(1)$ Wilson lines on the $T^4$. They are center of mass coordinates and are always present. We are not interested in these. The other 4 are 3 transverse positions and the dual photon in 3 dimensions. We conclude that the other phase is 4 dimensional. Furthermore it emanates from a point on the Coulomb branch, since all instantons have to shrink on top of each other. The only freedom is the point where they shrink, but that is a center of mass degree of freedom which we ignore.
Let us now take $n$ to be generic. Let $g = \gcd(n, k)$. The system of $n$ D2-branes inside $k$ D4-branes can split into $g$ separate systems. The dimension is thus $8g - 4$, subtracting the center of mass again. It emanates from the Coulomb branch on a locus of dimension $4g - 4$.

The special case of $q = 1, k = 2$ was studied in detail in [8]. Here it was found that there was another phase of dimension 4 for $\alpha = \pi$. We see that this agrees exactly with what was found here. However we get a much clearer picture of the other branch. In the next section we will understand these branches from a field theory point of view.

### 4.1. Phase Transitions from the Gauge Theory

With generic twists (non-commutativity parameters), the moduli-space that we obtain is smooth. However, for special values of the twists the moduli space has ADE-type singularities. We would now like to explain the origin of some of these singularities.

$S_B(k)$ is a gauge theory at low energies. Let us study it with an $\alpha$-twist along one circle and no twist along the other 2 circles. Since there is a circle without twist we can T-dualize on that direction to $S_A(k)$, so these remarks apply to $S_A(k)$ as well. We want to reproduce the existence of other branches of the moduli space. For a related discussion see [35].

The fields in 6 dimensions are a $U(k)$ vector-multiplet and an adjoint hypermultiplet. In 3 dimensions there is a tower of $U(k)$ vector-multiplets with masses $(\frac{n_1}{R_1}, \frac{n_2}{R_2}, \frac{n_3}{R_3})$, $n_i \in \mathbb{Z}$ and a tower of adjoint hypermultiplets with masses $(\frac{n_1-\alpha^2\pi}{R_1}, \frac{n_2}{R_2}, \frac{n_3}{R_3})$, $n_i \in \mathbb{Z}$. We remember that a mass in $\mathcal{N} = 4$ theories in 3 dimensions is specified by 3 numbers. The moduli space is $4k$-dimensional including the center of mass degrees of freedom. On the Coulomb branch the $U(k)$ is broken to $U(1)^k$. Each adjoint hypermultiplet splits into $k^2$ hypermultiplets of the following charges. There are hypermultiplets with charge $(0, \ldots, 0)$, and there are $k$ hypermultiplets with charges $(1, -1, \ldots, 0)$ plus permutations. There is a total of $k(k-1)$ of these. Some of these hypermultiplets can become massless on the Coulomb branch. For that to happen we have to turn on a Wilson line, $A_1$, along the first circle and set the other $3k$ moduli zero. $A_1$ has the form

$$A_1 = \begin{pmatrix}
a_1 & 0 & \ldots & 0 & 0 \\
0 & a_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & a_{k-1} & 0 \\
0 & 0 & \ldots & 0 & a_k
\end{pmatrix} \quad (4.12)$$

The tower of hypermultiplets is now as follows. There are $k$ of charge $(0, \ldots, 0)$ with mass $(\frac{n_1-\alpha^2\pi}{R_1}, \frac{n_2}{R_2}, \frac{n_3}{R_3})$ and for every $i \neq j$ there is a hypermultiplet with charge
\( (0, \ldots, 1, \ldots, -1, \ldots, 0) \) plus permutations with the 1 on the \( i^{th} \) place and the -1 on the \( j^{th} \) place. It has a mass \( \left( \frac{n_1 - \alpha + a_i}{2\pi R_1}, \frac{n_2 - a_j}{2\pi R_2}, \frac{n_3}{2\pi R_3} \right) \). The uncharged ones never become massless, as long as the twist is not a multiple of \( 2\pi \). The charged ones become massless if

\[
n_1 - \frac{\alpha}{2\pi} + \frac{a_i}{2\pi} - \frac{a_j}{2\pi} = n_2 = n_3 = 0. \tag{4.13}
\]

Now it is easy to make some of them massless by choosing \( A_1 \) appropriately. However to have a Higgs branch we need to have non trivial solutions to the D-flatness equations. For hypermultiplets charged under a \( U(1)^r \) group there should be at least \( r + 1 \) of them to have a non trivial solution. We thus need to find a number of massless hypermultiplets which is bigger than the number of \( U(1) \)'s under which they are charged. No hypermultiplets are charged under the diagonal \( U(1) \). Let us first find a situation of \( k \) massless hypermultiplets which are charged under \( U(1)^{k-1} \). The hypermultiplet of charge \( (1, -1, 0, \ldots, 0) \) is massless if,

\[
n_1 = 0, \quad a_1 - a_2 = \alpha. \tag{4.14}
\]

The one of charge \( (0, 1, -1, 0, \ldots, 0) \) is massless if,

\[
n_1 = 0, \quad a_2 - a_3 = \alpha \tag{4.15}
\]

and so on, up to the multiplet of charge \( (0, \ldots, 0, 1, -1) \) which is massless if,

\[
n_1 = 0, \quad a_{k-1} - a_k = \alpha \tag{4.16}
\]

This gives \( k - 1 \) massless hypermultiplets. To have one more we need \( (-1, 0, \ldots, 0, 1) \) to be massless. This is the case if,

\[
\frac{\alpha}{2\pi} = \frac{a_k - a_1}{2\pi} + n_1, \tag{4.17}
\]

for some integer \( n_1 \). Now,

\[
a_k - a_1 = (a_k - a_{k-1}) + \ldots + (a_2 - a_1) = -(k - 1)\alpha \tag{4.18}
\]

so we need \( \frac{k\alpha}{2\pi} \) to be an integer. So for \( \alpha = \frac{2\pi}{k} \) we have another phase of dimension 4. The dimension is 4 because there are \( k \) massless hypermultiplets each having 4 scalar fields and the D-flatness conditions remove \( 4(k - 1) \) dimensions leaving 4 real dimensions. This phase agrees exactly with the exact result from the previous section. We thus see that a naive field theory treatment, keeping all Kaluza-Klein modes, reproduces the result. This phase emanates from the Coulomb branch whenever \( a_i = a_{i-1} = \alpha \) as we saw.
above. This fixes the $a_i$ up to an overall shift. The overall shift is the $U(1)$ part which we discard anyway. This shows that the other phase emanates from one particular point on the Coulomb branch. Note that the field theory treatment is justified when $M_s R_i \gg 1$.

More generally, let us take $\alpha = n \frac{2\pi}{k}$ and $g = \gcd(n, k)$. Now we can play the same game as above but within $g$ blocks of the $U(k)$ matrix of size $\frac{k}{g}$. We thus get $g$ sets of $\frac{k}{g}$ massless fields. Each set is charged under a $U(1)^{\frac{k}{g} - 1}$ subgroup. This gives a $4g$ dimensional phase emanating from a locus on the Coulomb branch. This locus has dimension $4g - 4$. The $4g$ comes from the diagonal $U(1)$ in each of the $g$ blocks. The center of mass is subtracted again. This branch has a total dimension of $4g + 4g - 4 = 8g - 4$. We again find agreement with the exact result described previously.

The branches described above are the only ones coming from the naive field theory description besides the cases $\alpha = 2\pi n$, $n \in \mathbb{Z}$ which behave like $\alpha = 0$.

5. The 3+1D limit

In this section we will explain how to obtain the 3+1D Seiberg-Witten curves of the various theories compactified on $T^2$ with a twist. This time we only have two independent $\alpha$-twists corresponding to the two cycles of $T^2$. The way to obtain the 3+1D SW curves is to start with the moduli space of the theory compactified on $T^2 \times S^1$ where $S^1$ is of radius $R$ and take the limit $R \to \infty$. Let the 2+1D hyper-Kähler moduli space be of dimension $4n$. In the limit $R \to \infty$, it can be written as a fibration of $T^{2n}$ over a base of dimension $2n$. In the decompactification limit the fiber $T^{2n}$ shrinks to zero. We interpret it as the Jacobian variety of a Riemann surface of genus $n$ which varies over the base. This will then be the Seiberg-Witten curve (see [17]). Starting with the Blum-Intriligator little-string theories of $k$ NS5-branes at an $A_{q-1}$ singularity compactified on $T^2$ with twists we can get, in appropriate limits, a 3+1D gauge theory with,

$$SU(k)_1 \times \cdots \times SU(k)_q,$$

and massive adjoint hyper-multiplets in consecutive $(k, \bar{k})$ representations. The Seiberg-Witten curves for these models have been derived in [17]. As we will show below, we can reproduce these curves by taking the appropriate decompactification limit of the moduli space of $k$ $U(q)$ instantons on the non-commutative $T^4$.

To start, we will recall how the reduction of the untwisted compactified Blum-Intriligator theories works.
5.1. From instantons to quiver gauge theories

When we set all the $\alpha$-twists to zero we obtain the statement that the Coulomb-branch moduli space of the theories of $k$ NS5-branes on an $A_{q-1}$ singularity, compactified on $T^3$ is the same as the moduli space of $k$ ordinary instantons with a $U(q)$ gauge group on $T^4$. This result has already been established in [10,12]. Suppose we compactified on $T^3 = T^2 \times S^1$ and take the radius of $S^1$, $R \to \infty$. It can be checked (see (4.8)) that the auxiliary $T^4$ becomes a product $T^2_B \times T^2_F$. The complex structure of $T^2_F$ and $T^2_B$ are fixed as $R \to \infty$ while the area of $T^2_B$ is proportional to $R$ and the area of $T^2_F$ is proportional to $R^{-1}$. Now take a particular gauge configuration corresponding to an instanton of $U(q)$ with instanton number $k$. We can encode the information in the instanton as follows (see [36,37]). At a local point on the base, the gauge field reduces to two commuting $U(q)$ Wilson lines on the fiber. We can describe them uniquely as $q$ points on the dual $\tilde{T}^2$ of the fiber. These $q$ points vary over the base $B$. The instanton equations imply that they span a holomorphic curve $\Sigma_g$ of genus $g = qk + 1$. $\Sigma_g$ is called the “spectral curve”. To completely describe the instanton we also need to describe a line bundle over $\Sigma_g$ which corresponds to a point in the Jacobian of $\Sigma_g$ (recall that the Jacobian of a genus $g$ curve is $T^g$). The line bundle is called the “spectral-bundle”. Alternatively, we can represent the moduli space of $U(q)$ instantons at instanton number $k$ on $B \times F$ as the moduli space of $q$ D6-branes wrapped on $B \times F$ with $k$ D2-branes. The curves are obtained by T-duality along the two directions of $F$. We obtain a D4-brane wrapped on a curve $\Sigma_g$ of homology cycle $q[B] + k[F]$. The curve $\Sigma_g$ is the Seiberg-Witten curve of the point in the moduli space. It intersects a generic fiber $F$ in $q$ points and a zero section of the base $B$ at $k$ points. It is also easy to see that as the base $B$ decompactifies to $S^1 \times \mathbb{R}^1$ we reproduce exactly the curves from the brane construction of [7] for the quiver gauge theory.

5.2. The rôle of the non-commutativity

Now let us repeat the same procedure but with two non-commutativity parameters $\alpha_1$ and $\alpha_2$. We can take $\alpha_1$ to be along the first cycle of the base $B = T^2$ and the first cycle of the fiber $F = T^2$ and we take $\alpha_2$ to be along the second cycle of the base $B$ and the first cycle of the fiber $F$. The $\eta$-twists will similarly correspond to non-commutativity along the second cycle of $B$ and one of the two cycles of $F$.

To translate this to the curve $\Sigma_g$ we take the system of $q$ D6-branes and $k$ D2-branes and put in NSNS 2-form fluxes according to the non-commutativity parameters. After T-duality along $F$ The NSNS fluxes become components of the metric $G_{IJ}$. 18
As a result, we obtain a tilted $T^4 \equiv \mathbb{R}^4/\Lambda$, where $\Lambda$ is a lattice spanned by the following vectors:

\[
\begin{align*}
\hat{e}_1 &= (1, 0, 0, 0), \\
\hat{e}_2 &= (\tau_1, \tau_2, 0, 0), \\
\hat{e}_3 &= (\alpha_1 + \eta_1 \tau_1, \eta_1 \tau_2, \chi, 0), \\
\hat{e}_4 &= (\alpha_2 + \eta_2 \tau_2, \eta_2 \tau_2, \chi \rho_1, \chi \rho_2).
\end{align*}
\] (5.1)

Here, $\tau \equiv \tau_1 + i\tau_2$ is the complex structure of $T^2_F$, $\rho \equiv \rho_1 + i\rho_2$ is the complex structure of $T^2_B$, and,

\[\chi = m_s(\tau_2 \rho_2)^{-1},\]

so that the overall volume of the unit cell will be $m_s^2$. We will denote the coordinates in $\mathbb{R}^4$ by $(x_1, x_2, x_3, x_4)$. The D2 and D6 branes became a single D4-brane in the homology class,

\[\left[\Sigma\right] = q[B'] + k[F'].\]

Here,

\[
\begin{align*}
F' &\equiv \{s\hat{e}_1 + t\hat{e}_2 \mid 0 \leq s, t \leq 2\pi\}, \\
B' &\equiv \{s\hat{e}_3 + t\hat{e}_4 \mid 0 \leq s, t \leq 2\pi\}. \tag{5.2}
\end{align*}
\]

are two faces of $T^4$. Similarly to [7] the D4-brane will find a minimal-area surface in this homology class. In the complex structure given by,

\[z = x_1 + ix_2, \quad w = x_3 + ix_4,\]

the cohomology class $\omega \in H^2(\mathbb{Z})$ which is Poincaré dual to $[\Sigma]$ will, generically, be a mixture of $(1, 1)$, $(0, 2)$ and $(2, 0)$ forms. However, it is always possible to find a complex structure (with respect to the flat metric) for which $\omega$ is entirely a $(1, 1)$ form. In this complex structure the $T^4$ is “algebraic” (see p315 of [38]). Given the complex structure, it is possible to write down the curve $\Sigma$ as the zero locus of a $\theta$-function on $T^4$. These $\theta$-functions are the sections of the line-bundle corresponding to $[\Sigma]$ and depend on $kq$ parameters which are the moduli (see [38] for further details).

It is easy to see that the “elliptic-models” of [7] are recovered in the special limit in which we get a gauge theory with massive hyper-multiplets. In this case $\tau \to \infty$ and there are no $\eta$-twists. The fiber $F'$ is replaced with a strip $S^1 \times \mathbb{R}^1$. The class $[\Sigma]$ is analytic (i.e. the class $\omega$ is a $(1, 1)$ 2-form) and the Seiberg-Witten curves of [7] are recovered.
6. Another Look at the $\eta$-twists

In this section, we write explicitly the solution for type-IIA (or type-IIB) theory, with both $\alpha$-twists and $\eta$-twists turned on. These solutions should be interpreted as string world-sheet $\sigma$-models with a $B$-field.

We will start with a Taub-NUT space without NS5-branes. It is straightforward to define the $\alpha$-twist. One starts with some given background, which is a principal $U(1)$ bundle cross a torus $T^d$. Locally, the $\alpha$-twist is just the change of coordinate in the $S^1$ fiber of the Taub-NUT space, of the form $y \rightarrow y + \sum \alpha_I \psi^I$. $y$ is the coordinate on the circle (see (2.1)) and $\psi^I$ is the coordinate on $T^3$ ($I = 1, 2, 3$). Since it is just the change of variables, the string theory equations of motion are trivially satisfied. But globally, this is not a valid coordinate transformation, since $\alpha_I \psi^I$ is not a periodic function on $T^3$ modulo $2\pi$. Therefore, we get a different background – we call it the $\alpha$-twisted background. As for $\eta$-twists, they are related to $\alpha$-twists by T-duality in $T^3$.

We will construct the background with both $\alpha$-twists and $\eta$-twists turned on in the following way. We first consider the background containing Taub-NUT space cross a three-torus, without any twists. We introduce $\alpha$-twists along the three-torus, with the parameters $\eta_I$. Then, we make a T-duality transformation, and get a background with $\eta$-twists. This new background is again a $U(1)$ bundle cross a (dual) torus, and we now $\alpha$-twist it. In this way, we get a background with both $\alpha$-twists and $\eta$-twists.

Let us do it explicitly. Start with $\mathbb{R}^{1,2} \times \text{TN}(\rho) \times T^3$. The metric is:

$$ds^2 = \rho^2 U_{[\rho]}(|\vec{r}|) \mathcal{A}^2 + U_{[\rho]}(|\vec{r}|)^{-1} (d\vec{r})^2$$

$$+ g_{IJ} d\psi^I d\psi^J - dx_0^2 + dx_1^2 + dx_2^2,$$

where we have denoted

$$U_{[\rho]}(|\vec{r}|) \equiv \left( 1 + \frac{\rho}{2|\vec{r}|} \right)^{-1}$$

and $\mathcal{A}$ is the connection one-form $\mathcal{A} = dy - \vec{A} \cdot d\vec{r}$. Also, we turn on the following $B$ field:

$$B = b_{IJ} d\psi^I \wedge d\psi^J$$

We wish to introduce $\alpha$-twists with the parameter $\eta_I$. As was explained above, this means just the change of variables $y \rightarrow y - \eta_I \psi^I$. This amounts to replacing $\mathcal{A}^2$ with $(\mathcal{A} - \eta_I d\psi^I)^2$ in (6.1).

Now we make three T-dualities. We do this by the standard technique of treating $V^{I'}_{\alpha} \equiv \partial_{\alpha} \psi^I$ (where $\alpha$ is a string world-sheet coordinate) as an independent variable and
inserting a Lagrange multiplier, $\tilde{\psi}_I$, for, $\partial_{[\alpha}V_{\beta]}^I$. We get the following metric:

$$
\begin{align*}
 ds^2 &= \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} (A - b^{IJ} \eta_I d\tilde{\psi}_J)^2 + U_{[\rho]}(|\vec{r}|)^{-1} (d\vec{r})^2 \\
 &\quad + l_s^4 \left( g^{IJ} - \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + \rho^2 (\eta, \eta) U_{[\rho]}(|\vec{r}|)} \eta^I \eta^J \right) d\tilde{\psi}_I d\tilde{\psi}_J - dx_0^2 + dx_1^2 + dx_2^2,
\end{align*}
$$

with the notation, $\eta^I = g^{IJ} \eta_J$, $(\eta, \eta) = \eta_I \eta^I$, and $g^{IJ} + b^{IJ}$ is the matrix inverse to $g_{IJ} + b_{IJ}$. Also, we have the following $B$ field:

$$
B = -\frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} \eta^I d\tilde{\psi}_I \wedge (A - b^{JK} \eta_J d\tilde{\psi}_K) + b^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J
$$

Notice that

$$
\frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} = \frac{\rho^2}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} U_{[\rho]} \left[ \frac{1}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} \right]
$$

If we start with a non-degenerate torus and a very small coupling constant, then T-duality gives us back a very small coupling constant.

Now we $\alpha$-twist this background. Again, $\alpha$-twisting is just a replacement,

$$
A \rightarrow A - \alpha^I d\tilde{\psi}_I,
$$

in all the formulas for the metric and the $B$ field. It is convenient to absorb $b^{IJ} \eta_I d\tilde{\psi}_J$ into $\alpha^I d\tilde{\psi}_I$. Then, the background fields are:

$$
\begin{align*}
 ds^2 &= R^2(|\vec{r}|)(A - \alpha^I d\tilde{\psi}_I)^2 + U_{[\rho]}(|\vec{r}|)(d\vec{r})^2 + (dx^\mu)^2 + l_s^4 G^{IJ}(|\vec{r}|) d\tilde{\psi}_I d\tilde{\psi}_J, \\
 B &= (A - \alpha^I d\tilde{\psi}_I) \wedge B^I d\tilde{\psi}_J + B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J
\end{align*}
$$

where

$$
\begin{align*}
 R^2(|\vec{r}|) &= \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} \\
 G^{IJ}(|\vec{r}|) &= g^{IJ} - \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} \eta^I \eta^J \\
 B^I(|\vec{r}|) &= \frac{\rho^2 U_{[\rho]}(|\vec{r}|)}{1 + (\eta, \eta)\rho^2 U_{[\rho]}(|\vec{r}|)} g^{IJ} \eta_J \\
 B^{IJ} &= b^{IJ}
\end{align*}
$$

Also, the dilaton is not constant. Let $\lambda$ be the string coupling at $|\vec{r}| \rightarrow \infty$. Then, the string coupling at finite $|\vec{r}|$ is:

$$
\lambda(|\vec{r}|) = \lambda \sqrt{\frac{1 + \eta^2 \rho^2}{1 + \eta^2 \rho^2 U_{[\rho]}(|\vec{r}|)}}
$$
The metric (6.7) is not, strictly speaking, Hyper-Kähler. Indeed, although it does have three complex structures, they are not covariantly constant with respect to the standard covariant derivative. But they must be covariantly constant, if we modify $\Gamma^{\rho}_{\mu\nu}$ with the torsion, proportional to $H = dB$.

We want to study the moduli space of the theory on the NS5-brane, sitting at $\vec{r} = 0$ in this background. As we remarked in section (2), the NS5-brane has a size of $l_s$ and, although it is very heavy, it could affect the metric. We will explore this later in this section. For now, we will assume that it is safe to forget about the NS5-brane. To study the moduli space, we perform the chain of dualities. It is most convenient to think of these dualities as acting on the asymptotic ($|\vec{r}| \rightarrow \infty$) values of the fields. Therefore, we would like to discuss how the background fields near the position of the NS5-brane ($|\vec{r}| \rightarrow 0$) are related to the asymptotic values of the fields at $|\vec{r}| \rightarrow \infty$.

Let us look first at the geometry near the origin in $\mathbb{R}^3$. From (6.4) and (6.6) we see that the geometry becomes flat when the following two conditions are satisfied:

$$|\vec{r}| \ll \rho \quad \text{and} \quad |\vec{r}| \ll \frac{\rho}{1 + (\eta, \eta) \rho^2} \quad (6.10)$$

In this limit, we have just $\mathbb{R}^{1,6} \times \mathbb{T}^3$ with the metric

$$ds^2 = (dx^\mu)^2 + |d(e^{i\alpha^J \bar{\psi}_J z_1})|^2 + |d(e^{-i\alpha^J \bar{\psi}_J z_2})|^2 + g^{IJ} d\bar{\psi}_I d\bar{\psi}_J \quad (6.11)$$

The $B$ field becomes:

$$B = -\eta^I d\bar{\psi}_I \wedge \text{Im}(z_1^* dz_1 + z_2^* dz_2) + b^{IJ} d\bar{\psi}_I \wedge d\bar{\psi}_J \quad (6.12)$$

We wish to study the moduli space for the NS five-brane sitting at $\vec{r} = 0$. Notice that the transversal fluctuations of this five-brane at energy scale $\simeq m_s^2$ have the characteristic size $\Delta X_\perp \simeq \lambda l_s$. If we take $\rho \simeq l_s$ and general $\eta$, then both of the inequalities (6.10) are satisfied for $|\vec{r}| \equiv \Delta X_\perp$. This suggests that the parameter $\rho \simeq l_s$ actually does not affect the moduli space. The reason why it might be not true is that the transversal size of the NS5-brane is, actually, of the order $l_s$. Therefore the curvature of the background should, presumably, affect the physics even in the limit $\lambda \rightarrow 0$. The answer we will get shows that the moduli space does not really depend on $\rho$.

Now let us look at the fields at infinity. They are given by the formulae (6.7) and (6.8) with $|\vec{r}| = \infty$. We will denote the limits of $R^2(|\vec{r}|)$, $G^{IJ}(|\vec{r}|)$ and $B^I(|\vec{r}|)$ as $|\vec{r}| \rightarrow \infty$ by $R^2$, $G^{IJ}$ and $B^I$. It is convenient to have a dictionary relating the fields at $|\vec{r}| = \infty$ with
the fields at $|\vec{r}| = 0$. Let us first summarize our notations. We have already introduced the matrices $g_{IJ}, b_{IJ}, g^{IJ}$ and $b^{IJ}$ satisfying:

$$(g^{IJ} + m_s^2 b^{IJ})(g_{JK} + i_s^2 b_{JK}) = \delta^K_I$$

We have also introduced $G^{IJ}$ and $B^{IJ}$ in (5.4). Now, we define $G_{IJ}, B_{IJ}, g^{-1}_{IJ}$ and $G^{-1}_{IJ}$ in the following way:

$$(G_{IJ} + B_{IJ})(G^{JK} + B^{JK}) = \delta^K_I, \quad g^{-1}_{IJ} g^{JK} = \delta^K_I, \quad G^{-1}_{IJ} G^{JK} = \delta^K_I \quad (6.13)$$

Then, we have the following dictionary, relating asymptotic background to the local background:

$$\rho^2 = R^2 + (B, B), \quad R^{-2} = \rho^{-2} + (\eta, \eta),$$

$$g^{IJ} = G^{IJ} + R^{-2} B^I B^J, \quad G^{-1}_{IJ} = g^{-1}_{IJ} + \rho^2 \eta_I \eta_J,$$

$$\eta_I = \frac{R^{-2} G^{-1}_{IJ} B^J}{1 + R^{-2} (B, B)}, \quad B^I = \frac{\rho^2}{1 + \rho^2 (\eta, \eta)} g^{IJ} \eta_J,$$

$$B^{IJ} = b^{IJ}. \quad (6.14)$$

The local value, $\lambda_0$, of the string coupling is related to the asymptotic value $\lambda$ by the formula which follows from (5.9):

$$\lambda_0^2 = (1 + (\eta, \eta) \rho^2) \lambda^2 \quad (6.15)$$

### 6.1. The chain of dualities.

We start by replacing the Taub-NUT circle with the M-theory circle. We get a D6-brane wrapped on $\mathbf{T}^4$, with the NS5-brane on top of it.

At this point it is useful that we remember how the fields of type-IIA theory are related to the fields of M-theory. M-theory on a $\mathfrak{U}(1)$ bundle is type-IIA on the base of this bundle. Suppose that the action of $\mathfrak{U}(1)$ is associated to the vector field $v$. The M-theory three-form $C_M$ splits as follows:

$$C_M = \pi^* A^{(3)} + A \wedge \pi^* B \quad (6.16)$$

Also, we choose some local trivialization, and define the connection one-form $A^{(1)}$ on the base, $dA^{(1)} = \mathcal{F}$ ($\mathcal{F}$ is the curvature two-form on the base, $dA = \pi^* \mathcal{F}$). It should be identified with the RR one-form $C^{(1)}$ of type-IIA. Also, $B$ should be identified with the $B$ field of type-IIA (this follows from its coupling to the fundamental string). What is the relation between $A^{(3)}$ and the Ramond-Ramond three-form $C^{(3)}$ of type-IIA? Let
us remember the general formula for the couplings of the Ramond-Ramond fields to the D-brane [18]:

\[ S_{RR} = \int \mu_p C \wedge \text{tr} F - B \] (6.17)

For example, for the D2 brane we get:

\[ S_{RR} = \mu_2 \int C^{(3)} - C^{(1)} \wedge (B - F) \] (6.18)

Here \( C^{(1)} \) should be identified with the connection one-form, \( A = d\phi + C^{(1)} \). We have to keep in mind that various forms participating in this formula are, in general, subject to gauge transformations. For example, under the gauge transformation \( C^{(1)} \rightarrow C^{(1)} - d\psi \) we should have \( C^{(3)} \rightarrow C^{(3)} - d\psi \wedge B \) (this is needed for the coupling (6.18) to be correctly defined). This suggests that

\[ C^{(3)} = A^{(3)} + C^{(1)} \wedge B \] (6.19)

(that is, \( C_M = \pi^* C^{(3)} + d\phi \wedge \pi^* B \)). We may derive how Ramond-Ramond fields transform under T duality from their coupling to D branes. It follows that \( Ce^{-B} \) transforms as a spinor of \( O(d, d, \mathbb{Z}) \).

Let us return to our dualities. We assume that the M Theory circle in our original configuration has radius \( S = \lambda l_s \), where \( l_s \) is the string scale in the configuration we start with, and \( \lambda \) is the original coupling constant (which has to be very small, if we want to get Little String Theory on NS5 brane). The three-form of M Theory is read from (6.7):

\[ C_M = (\mathcal{A} - \alpha^I d\tilde{\psi}_I) \wedge B^J d\tilde{\psi}_J \wedge d\theta + B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J \wedge d\theta \] (6.21)

If we now treat the Taub-NUT circle as the M-theory circle, we get (6.16) with

\[ A^{(3)} = B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J \wedge d\theta, \quad B = B^I d\tilde{\psi}_I \wedge d\theta. \]

(Notice that \( \mathcal{A} - \alpha^I d\tilde{\psi}_I \) is just the connection 1-form after \( \alpha \)-twist.)

In the new type-IIA theory, obtained by compactifying M Theory on the Taub-NUT circle, we have the following asymptotic values of the background fields:

\[
\begin{align*}
    ds^2 &= S^2 d\theta^2 + l_s^4 G^{IJ} d\tilde{\psi}_I d\tilde{\psi}_J + dr^2 + (dx^\mu)^2, \\
    B &= B^I d\tilde{\psi}_I \wedge d\theta, \\
    Ce^{-B} &= \alpha^I d\tilde{\psi}_I + d\theta \wedge B^{IJ} d\tilde{\psi}_I \wedge d\tilde{\psi}_J.
\end{align*}
\] (6.22)
We have used (6.20) to find $Ce^{-B}$ in type-IIA. The new string length is:

$$l_1^2 = \frac{S^2}{R_s^2} = \frac{\lambda_0^3}{\rho}$$

and the new string coupling constant is:

$$\lambda_1 = \left(\frac{R}{l_s}\right)^{3/2} \frac{1}{\sqrt{\lambda}}$$

(6.24)

Making three $T$ duality transformations along $T^3$, we get:

$$ds^2 = \frac{l_s^2 \lambda_0^2}{\rho^2} \left[ \rho^2 d\theta^2 + G_{IJ}^{-1} d\psi^I d\psi^J + 2G_{IJ}^{-1} B^I d\psi^J d\theta \right] + dr^2 + (dx^\mu)^2,$$

$$B^{RR} = \alpha^I \epsilon_{IJK} d\psi^J \wedge d\psi^K + d\theta \wedge \epsilon_{IJK} B^{IJK} d\psi^K,$$

$$B^{NS} = 0,$$

with the string coupling constant,

$$\lambda_2 = \frac{\lambda}{l_s^3 \sqrt{\det G}}.$$  

(6.26)

The NS5-brane remains an NS5-brane, wrapped on $T^3$, and D6-brane becomes D3-brane. It shares with NS5 the directions of $R^{1,2}$.

Now we do S-duality, so that $B^{RR}$ becomes $B^{NS}$, and NS5 becomes D5. Also, we get the new string coupling and the new string length:

$$\lambda_3 = \frac{l_s^3 \sqrt{\det G}}{\lambda}, \quad l_3 = \lambda_0 \sqrt{\frac{(\det g^{-1})^{1/2}}{\rho}}$$

(6.27)

Then, doing T-duality along the circle parameterized by $\theta$. We have now D6 brane wrapped on the four-torus, and the D2 brane inside it, orthogonal to the torus. We end up with the following string coupling and string length,

$$\lambda_4 = \frac{l_s^2}{\lambda_0 \sqrt{\rho (\det g^{-1})^{1/2}}}, \quad l_4 = \lambda_0 \sqrt{\frac{(\det g^{-1})^{1/2}}{\rho}}$$

(6.28)

and the following metric and $B$ field,

$$ds_4^2 = \frac{l_s^2}{\rho^2} \lambda_0^2 \left[ l_s^{-4} (\det g^{-1})(d\tilde{\theta} - \epsilon_{IJK} b^{IJK} d\psi^K)^2 + g^{-1}_{IJ} d\psi^I d\psi^J \right],$$

$$B = \alpha^I \epsilon_{IJK} d\psi^J \wedge d\psi^K + \eta_I d\theta \wedge d\psi^I.$$  

(6.29)

Let us summarize. We have started with $k$ NS5-branes sitting at the center of the Taub-NUT space, string coupling $\lambda_0$ and string length $l_s$. The background fields are given by the equations (6.11) and (6.12), they correspond to both $\alpha$-twists and $\eta$-twists present. By the chain of dualities, we have mapped this configuration to $k$ D6 branes wrapped on $T^4$, and one D2 brane, the metric and the $B$ field given by (6.28) and (6.29). Notice that the volume of $T^4$ is $l_s^4$. In the limit we are interested in ($\lambda_0 \to 0$) it remains finite in the string units (specified by $l_4$). The shape of the torus does not depend on $\rho$. 

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6.2. World-sheet T-duality in the limit $\rho \to \infty$

Let us now see what happens in the limit $\rho \to \infty$. The strategy will be to start with type-IIA string-theory on the purely geometrical background which realizes the $\alpha$-twist. We will then perform world-sheet T-duality on $S^1$ to obtain a nonlinear world-sheet $\sigma$-model. Finally, we will insert the NS5-branes back.

To describe the geometrical background we choose,

$$X_6, \ldots, X_9,$$

as the transverse coordinates (on which the R-symmetry $SO(4)$ acts). These replace the coordinates $y$ and $\vec{r}$ of $TN(\rho)$. We will denote,

$$Z_1 = X_6 + iX_7, \quad Z_2 = X_8 - iX_9.$$

The other coordinates will be denoted,

$$X_0, \ldots, X_5,$$

where $X_5$ is periodic with period $2\pi$. They are the world-sheet fields corresponding to $x_0, x_1, x_2, \psi_1, \psi_2, \psi_3$ from the previous section. The bosonic part of the world-sheet action is,

$$L_0 = \sum_{\mu, \nu=0}^{4} \eta^{\mu\nu} \partial_\alpha X_\mu \partial^\alpha X_\nu + R^2 \partial_\alpha X_5 \partial^\alpha X_5 + \sum_{i=1,2} \partial_\alpha Z_i \partial^\alpha Z_i.$$

Let us, for simplicity, twist only along $X_5$ ($= \psi_3$). The twist implies that $Z_i$ are not single-valued but rather,

$$W_i = Z_i e^{-i \frac{\alpha}{2\pi} X_5}, \quad i = 1, 2$$

are single-valued. The world-sheet Lagrangian now reads,

$$L_0 = \sum_{\mu, \nu=0}^{4} \eta^{\mu\nu} \partial_\alpha X_\mu \partial^\alpha X_\nu + R^2 \partial_\alpha X_5 \partial^\alpha X_5 + \sum_{j=1,2} |\partial_\alpha W_j + \frac{i\alpha}{2\pi} W_j \partial_\alpha X_5|^2.$$

Next we perform T-duality by the standard technique of treating $V_\alpha \equiv \partial_\alpha X_5$ as an independent field and inserting a Lagrange multiplier $Y$ for $\partial_\alpha V_\beta$.

The result is a world-sheet action corresponding to the metric and $B$-field,

$$ds^2 = \sum_{\mu, \nu=0}^{4} \eta^{\mu\nu} dx_\mu dx_\nu + |dW_1|^2 + |dW_2|^2 + \frac{dY^2 + \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)^2}{R^2 + \frac{\alpha^2}{4\pi^2} (|W_1|^2 + |W_2|^2)},$$

$$B_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{dY \wedge \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)}{R^2 + \frac{\alpha^2}{4\pi^2} (|W_1|^2 + |W_2|^2)}.$$
6.3. Adding in the NS5-brane

Now we repeat the same exercise with the NS5-brane metric. In string units, the metric is,

\[ L_0 = \sum_{\mu, \nu=0}^{4} \eta^{\mu \nu} \partial_\alpha X_\mu \partial^\alpha X_\nu + R^2 \partial_\alpha X_5 \partial^\alpha X_5 + \frac{1}{|Z_1|^2 + |Z_2|^2} \sum_{i=1,2} \partial_\alpha Z_i \partial^\alpha Z_i. \]

The dilaton is given by,

\[ g_s^2 = \frac{1}{|Z_1|^2 + |Z_2|^2}, \]

and the solution is to be trusted when \( g_s \ll 1 \). (See discussion in [39].) After T-duality we obtain,

\[ ds^2 = \sum_{\mu, \nu=0}^{4} \eta^{\mu \nu} dX_\mu dX_\nu + \frac{|dW_1|^2 + |dW_2|^2}{|W_1|^2 + |W_2|^2} dY^2 + \frac{1}{\|W\|^2} \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)^2 \]

\[ B_{\mu \nu} dx^\mu \wedge dx^\nu = \frac{dY \wedge \sum_j (iW_j d\overline{W}_j - i\overline{W}_j dW_j)}{(R^2 + \frac{\alpha^2}{4\pi\alpha})\|W\|^2}. \]

This is to be trusted when,

\[ \|W\|^2 \equiv |W_1|^2 + |W_2|^2 \gg 1. \]

We see that as \( R \to 0 \), the \( Y \)-direction stays of finite size \( \frac{2\pi}{\alpha} \).

6.4. Large radius limit

An interesting question is what is the low-energy description of \( S_B(k) \) compactified on \( S^1 \) of radius \( R \) with a fixed \( \eta \)-twist in the limit \( R \to \infty \). Naively, one can argue as follows. To perform an \( \eta \)-twist we have to go over the “fundamental” degrees of freedom of \( S_B(k) \) (whatever they are!) and separate them according to their charge \( Q \) under the \( U(1) \) subgroup of the R-symmetry and according to their momenta \( n \) and winding \( w \) along \( S^1 \). We then add \( \eta QR \) to the mass of this field. In the limit \( R \to \infty \) and for generic \( \eta \), this will push all the \( Q \)-charged fields to high energy and we will be left with only the \( Q \)-neutral sector. Thus, if we start with \( \mathcal{N} = (1,1) \ U(k) \) SYM in 5+1D, as the effective low-energy description, the conclusion would be that we are left with \( \mathcal{N} = (1,0) \ U(k) \) SYM. This conclusion cannot be correct since the gluinos of the \( \mathcal{N} = (1,0) \) vector-multiplet are chiral and the theory has a local gauge anomaly.
One possibility is that there is no 5+1D limit. For this to be true we must show that there are no BPS states corresponding to light KK states. On the type-IIA side we must show that there are no states made by strings wrapped on the T-dual \( S^1 \) which would become light. Perhaps, when the circle is small enough, they do not form bound states any more?

7. Conclusion

Let us summarize the results:

1. The moduli space of the little-string theories of \( k \) NS5-branes compactified on \( T^3 \) with \( \text{Spin}(4) \) R-symmetry \( \alpha \)-twists is equal to the moduli space of \( k \) \( U(1) \) instantons on a non-commutative \( T^4 \). The shape of the \( T^4 \) is determined by the shape and size of the physical \( T^3 \) and by the NSNS 2-form fluxes along it. The non-commutativity parameters are determined from the values of the twists.

2. In principle, there are 6 non-commutativity parameters on \( T^4 \). They are determined from the 3 geometrical \( \alpha \)-twists and the 3 non-geometrical \( \eta \)-twists. The moduli space depends only on the 3 self-dual combinations of the non-commutativity parameters and hence only on the sum of the \( \eta \)-twists and \( \alpha \)-twists.

3. Combining the result for \( k = 2 \) with the result of [8], we obtain a concrete prediction for the moduli space of 2 \( U(1) \) instantons on a non-commutative \( T^4 \). This 8-dimensional moduli space is a resolution of \( (T^4 \times T^4)/\mathbb{Z}_2 \) by blowing up the singular locus. It can also be described as a \( T^4 \) fibration over a \( \mathbb{Z}_4^2 \) quotient of a particular \( K3 \). The fiber corresponds to the “center-of-mass” of the NS5-branes and the structure group is \( \mathbb{Z}_4^2 \) acting as translations of the fiber. The particular point in the moduli space of hyper-Kähler metrics on the \( K3 \) was constructed in [8] as a function of the \( \alpha \)-twists, i.e. the non-commutativity parameters. This \( K3 \) turns out to have a \( \mathbb{Z}_4^2 \) isometry. The \( K3 \) can be described by blowing up \( T^4/\mathbb{Z}_2 \) and the \( \mathbb{Z}_4^2 \) acts by permuting the exceptional divisors of the blow-up. Note that this \( \mathbb{Z}_4^2 \) does not act freely.

4. Similarly, the moduli space of the little-string theories of \([10]\) of \( k \) NS5-branes at an \( A_{q-1} \) singularity, compactified on \( T^3 \) with \( \alpha \)-twists (twists in the global \( U(1) \)), is equal to the moduli space of \( k \) \( U(q) \) instantons on a non-commutative \( T^4 \).

5. We studied the phase transitions which occur at singular points of the moduli space.

\[1\] In [8], the global “center-of-mass” of the NS5-brane and the \( \mathbb{Z}_4^2 \) were ignored, and only the \( K3 \) was studied.
6. If instead of the little-string theories we start with the (2,0) theory (or the SCFT theory of \cite{9} in item (4) above), we obtain the moduli spaces of instantons on a non-commutative $T^3 \times \mathbb{R}$. The non-commutativity parameters are only along $T^3$, which is in accord with the fact that there are no $\eta$-twists for this problem.

Let us conclude with 3 open problems:

a. Generalize to other gauge groups, in particular to D-type and E-type little-string theories.

b. Generalize to NS5-branes at D-type or E-type singularities.

c. Study the $\eta$-twists, in particular how they are described at large compactification radii.

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