COHOMOLOGY AND EXTENSION PROBLEMS FOR SEMI $q$-CORONAE

ALBERTO SARACCO\textsuperscript{2},
GIUSEPPE TOMASSINI \textsuperscript{2}

ABSTRACT. We prove some extension theorems for analytic objects, in particular sections of a coherent sheaf, defined in semi $q$-coronae of a complex space. Semi $q$-coronae are domains whose boundary is the union of a Levi flat part, a $q$-pseudoconvex part and a $q$-pseudoconcave part. Such results are obtained mainly using cohomological techniques.

1. INTRODUCTION.

Let $X$ be a (connected and reduced) complex space. We recall that $X$ is said to be strongly $q$-pseudoconvex in the sense of Andreotti-Grauert [AG] if there exists a compact subset $K$ and a smooth function $\varphi : X \to \mathbb{R}$, $\varphi > 0$, which is strongly $q$-plurisubharmonic on $X \setminus K$ and such that:

a) $0 = \min_X \varphi < \min_K \varphi$;

b) for every $c > \max_K \varphi$ the subset

$$B_c = \{ x \in X : \varphi(x) < c \}$$

is relatively compact in $X$.

If $K = \emptyset$, $X$ is said to be $q$-complete. We remark that, for a space, being 1-complete is equivalent to being Stein.

Replacing the condition b) by

b’) for every $0 < \varepsilon < \min_K \varphi$ and $c > \max_K \varphi$ the subset

$$B_{\varepsilon, c} = \{ x \in X : \varepsilon < \varphi(x) < c \}$$

is relatively compact in $X$,

we obtain the notion of $q$-corona (see [AG], [AT]).

A $q$-corona is said to be complete whenever $K = \emptyset$.

The extension problem for analytic objects defined on $q$-coronae was studied by many authors (see e.g. [FG], [Se], [Si], [SiT], [T1]). In this
paper we deal with the larger class of the semi \( q \)-coronae which are defined as follows. Consider a strongly \( q \)-pseudoconvex space (or, more generally, a \( q \)-corona) \( X \), and a smooth function \( \phi : X \to \mathbb{R} \) displaying the \( q \)-pseudoconvexity of \( X \). Let \( B_{\varepsilon X} \) and let \( h : X \to \mathbb{R} \) be a pluriharmonic function (i.e. locally the real part of a holomorphic function) such that \( K \setminus \{h = 0\} \neq \emptyset \). A connected component of \( B_{\varepsilon X} \cap \{h = 0\} \) is, by definition, a semi \( q \)-corona.

Another type of semi \( q \)-corona is obtained by replacing the zero set of \( h \) with the intersection of \( X \) with a Levi flat hypersurface. More precisely, consider a closed strongly \( q \)-pseudoconvex subspace \( X \) of an open subset of \( \mathbb{C}^n \) and the \( q \)-corona \( C = B_{\varepsilon X} = B_c \setminus \overline{B}_\varepsilon \). Let \( H \) be a Levi flat hypersurface of a neighbourhood \( U \) of \( \overline{B}_c \) such that \( H \cap K = \emptyset \). The connected components \( C_m \) of \( C \cap H \) are called semi \( q \)-coronae.

In both cases the semi \( q \)-coronae are differences \( A_c \setminus \overline{A}_\varepsilon \) where \( A_c, A_\varepsilon \) are strongly \( q \)-pseudoconvex spaces. Indeed, the function \( \psi = \log h^2 \) (respectively \( \psi = \log \delta_H \zeta \), where \( \delta_H \zeta \) is the distance of \( z \) from \( H \)) is plurisubharmonic in \( W \cap \{h = 0\} \) (respectively \( W \cap H \)) where \( W \) is a neighbourhood of \( B_c \setminus \{h = 0\} \) (respectively \( B_c \setminus H \)). Let \( \chi : \mathbb{R} \to \mathbb{R} \) be an increasing convex function such that \( \chi \phi > \psi \) on a neighbourhood of \( B_c \setminus W \). The function \( \Phi = \sup(\chi \phi; \psi) + \phi \) is an exhaustion function for \( B_c \setminus \{h = 0\} \) (for \( B_c \setminus H \)) and it is strongly \( q \)-plurisubharmonic in \( B_c \setminus \{h = 0\} \) (in \( B_c \setminus H \)).

The interest for domains whose boundary contains a “Levi flat part” originated from an extension theorem for CR-functions proved in [LuT] (see also [La], [LaP], [St]).

Using cohomological techniques developed in [AG], [AT], [BS], [C] we prove that, under appropriate regularity conditions, holomorphic functions defined on a complete semi \( 1 \)-corona “fill in the holes” (Corollaries 4 and 6). Meanwhile we also obtain more general extension theorems for sections of coherent sheaves (Theorems 3 and 5). As an application, we finally obtain an extension theorem for divisors (Theorems 15 and 19) and for analytic sets of codimension one (Theorem 17).

We remark that this approach fails in the case when the objects to be extended are not sections of a sheaf defined on the whole \( B_c \). In particular, this applies for analytic sets of higher codimension. This is closely related with the general, definitely more difficult, problem of extending analytic objects assigned on some semi \( q \)-corona when the subsets \( B_c \) are not relatively compact in \( X \) i.e. when \( X \) is a genuine \( q \)-corona. It is worth noticing that a similar extension theorem for complex submanifold of higher codimension has been recently obtained in [DS] by different methods based on Harvey-Lawson’s theorem [HL].

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2. Cohomology and extension of sections.

2.1. **Closed q-corona.** Let $X$ be a strictly $q$-pseudoconvex space (respectively $X^n$ be a strictly $q$-pseudoconvex open set) and $H = \mathcal{F} h = 0\mathcal{G}$ (respectively $H$ Levi-flat), and $C = B_\mathcal{x} = B_\mathcal{c} \times \overline{B}_\mathcal{e}$ a q-corona.

We can suppose that $B_\mathcal{c} \times H$ has two connected components, $B_+ \text{ and } B_-$, and define $C_+ = B_+ \ \cup \ C, C = B_- \ \cup C$.

If $E \neq \text{Coh} (B_\mathcal{c})$, we define $p ((F \mathcal{E})) = \inf_{x \in B_\mathcal{c}} \text{ depth } (F \mathcal{E} x)$, the depth of $E$ on $B_\mathcal{c}$. If $E = 0$, the structure sheaf of $X$, we define $p (B_\mathcal{c}) = p (\mathcal{O})$.

**Theorem 1.** Let $E \neq \mathcal{Coh} (B_\mathcal{c})$. Then the image of the homomorphism

$$
H^r (F, B_+ ; \mathcal{E}) \rightarrow H^r (C_+ , \mathcal{E}) \rightarrow H^r (C, \mathcal{E})
$$

(all closures are taken in $B_\mathcal{c}$), defined by $(\xi, \eta) \in \mathcal{F}r + \mathcal{F}r_+$, $\eta_{\mathcal{F}r}$ has finite codimension provided that $q \geq 1$.

**Proof.** Consider the Mayer-Vietoris sequence applied to the closed sets $B_+$ and $C_-

$$
\delta (a \ b) = a_{\mathcal{F}r} \ b_{\mathcal{F}r}. B_+ (\mathcal{C}) = B_\mathcal{c} \times U \text{ where } U = B_\mathcal{c} \ \cup B_\mathcal{e}. U \text{ is q-complete, so the groups of compact support cohomology } H^r (U ; \mathcal{E}) \text{ are zero for } q \geq 1, p (\mathcal{E}) \ q.

From the exact sequence of compact support cohomology

$$
H^r (B_\mathcal{c} ; \mathcal{E}) \rightarrow H^r (B_\mathcal{c} \times U ; \mathcal{E}) \rightarrow H^r (U ; \mathcal{E})
$$

it follows that

$$
H^r (B_\mathcal{c} ; \mathcal{E}) \rightarrow H^r (B_\mathcal{c} \times U ; \mathcal{E}) \rightarrow H^r (U ; \mathcal{E})
$$

for $q \geq 1$. Since $B_\mathcal{c}$ is $q$-pseudoconvex,

$$
\dim C \ H^r (B_\mathcal{c} ; \mathcal{E}) < \infty
$$

for $q \geq \mathcal{D} \mathcal{C} \text{ Théorème 11}$, and so

$$
\dim C \ H^r (B_\mathcal{c} \times U ; \mathcal{E}) < \infty
$$

for $q \geq 1$. From (1) we see that $\dim C \ H^r (B_\mathcal{c} \times U ; \mathcal{E}) = \dim C \ H^r (\overline{B}_\mathcal{c} \times \mathcal{F}r ; \mathcal{E})$ is greater than or equal to the codimension of the homomorphism $\delta$. 2

**Corollary 2.** Under the same assumption of Theorem 1, if $K \ \setminus \ H = ?$, then

$$
\dim C \ H^r (\overline{B}_\mathcal{c} ; \mathcal{E}) < \infty
$$

for $q \geq 1, p (\mathcal{E}) \ q.$ 2.
Proof. Since $K \setminus H = \emptyset$, $\mathcal{B}_+$ is a $q$-pseudoconvex space, and by virtue of [AG Théorème 11] we have
\[ \dim_{\mathbb{C}} H^r(\mathcal{B}_+, \mathcal{F}) < \infty \]
for $r \neq q$. On the other hand, $\mathcal{C}$ is a $q$-corona, thus we obtain
\[ \dim_{\mathbb{C}} H^r(\mathcal{C}, \mathcal{F}) < \infty \]
for $q \neq r$ and $p(\mathcal{F}) = q$ 1 in view of [AT Theorem 3]. By Theorem [1] we then get that for $q \neq r$ and $p(\mathcal{F}) = q$ 1 the vector space $H^r(\mathcal{B}_+, \mathcal{F})$ has finite dimension and for $q = r \neq p(\mathcal{F}) = 2$ its image in $H^r(\mathcal{C}_+, \mathcal{F})$ has finite codimension. Thus $H^r(\mathcal{C}_+, \mathcal{F})$ has finite dimension for $q \neq r$ and $p(\mathcal{F}) = 2$.

Theorem 3. If $\mathcal{B}_+$ is a $q$-complete space, then
\[ H^r(\mathcal{C}, \mathcal{F}) \mid H^r(\mathcal{B}_+, \mathcal{F}) \]
for $q \neq r$ and the homomorphism
\[ H^q(\mathcal{B}_+, \mathcal{F}) \mapsto H^q(\mathcal{C}_+, \mathcal{F}) \]
is surjective for $p(\mathcal{F}) = 2q + 1$.

If $\mathcal{B}_+$ is a $1$-complete space and $p(\mathcal{F}) = 3$, the homomorphism
\[ H^0(\mathcal{B}_+, \mathcal{F}) \mid H^0(\mathcal{C}_+, \mathcal{F}) \]
is surjective.

Proof. Since by hypothesis $\mathcal{B}_+$ is a $q$-complete space, $H^r(\mathcal{B}_+, \mathcal{F}) = \mathcal{F}_0 g$ for $q \neq r$ [AG Théorème 5]. From (3) it follows that $H^r(\mathcal{B}_+, \mathcal{F}) = \mathcal{F}_0 g$ for $q \neq r$ and $p(\mathcal{F}) = 1$. Thus, the Mayer-Vietoris sequence (1) implies that $H^r(\mathcal{C}, \mathcal{F}) \mid H^r(\mathcal{B}_+, \mathcal{F})$ for $q \neq r$ and $p(\mathcal{F}) = 2q + 1$.

In particular, if $q = 1$ and $p(\mathcal{F}) = 3$ the homomorphism
\[ H^0(\mathcal{B}_+, \mathcal{F}) \mid H^0(\mathcal{C}_+, \mathcal{F}) \]
is surjective i.e. every section $\sigma$ of $H^0(\mathcal{C}_+, \mathcal{F})$ is a difference $\sigma_1 - \sigma_2$ of two sections $\sigma_1$ and $\sigma_2$. Since $\mathcal{B}_c$ is Stein, the cohomology group with compact supports $H^1_k(\mathcal{B}_c, \mathcal{F})$ is zero, and so the Mayer-Vietoris compact support cohomology sequence implies that the restriction homomorphism
\[ H^0(\mathcal{B}_c, \mathcal{F}) \mid H^0(\mathcal{B}_c \cap B, \mathcal{F}) = H^0(\mathcal{C}, \mathcal{F}) \]
is surjective, hence $\sigma_2 \in H^0(\mathcal{C}, \mathcal{F})$ is restriction of $\sigma_2 \in H^0(\mathcal{B}_c, \mathcal{F})$. So $\sigma$ is restriction to $\mathcal{C}_+$ of $\sigma_1 - \sigma_2$ and the restriction homomorphism is surjective.

Corollary 4. Let $\mathcal{B}_+$ be a $1$-complete space and $p(\mathcal{B}_c) = 3$. Then every holomorphic function on $\mathcal{C}_+$ extends holomorphically on $\mathcal{B}_+$. 

2.2. **Open \( q \)-coronae.** Most of the Theorems and Corollaries of the previous section still hold in the open case and their proofs are very similar. First we give the proof of the extension results using directly Theorem 3. We have to assume that \( H \) is the zero set of a pluriharmonic function \( h \) and we define \( B_{c}, C, B_{+}, B, C_{+} \) and \( C \) as we did before.

Let us suppose \( B_{+} \) is 1-complete and \( p (\mathcal{F}) \neq 3 \). Let \( s 2 H^{0} (\mathcal{C}_{+}; \mathcal{F}) \). For all \( \varepsilon > 0 \), we consider the closed semi 1-corona

\[
\overline{C}_{\varepsilon} = B_{\varepsilon+c} \setminus \{ \varepsilon \} \quad C_{+}
\]

Let \( \sigma_{\varepsilon} = \sigma_{c} \). By Theorem 5 (applied to \( \overline{C}_{\varepsilon}, H_{\varepsilon} = \{ \varepsilon \} \)), we obtain that \( \sigma_{\varepsilon} \) extends to a section \( \sigma_{\varepsilon} 2 H^{0} (\overline{B}_{\varepsilon}; \mathcal{F}) \), where \( \overline{B}_{\varepsilon} = \overline{B}_{+} \setminus \{ \varepsilon \} \). Since \( B_{+} = [\varepsilon \overline{B}_{\varepsilon} \), if for all \( \varepsilon_{2} > \varepsilon_{1} > 0 \),

\[
\sigma_{\varepsilon_{1}} \cap \overline{C}_{\varepsilon_{2}} = \sigma_{\varepsilon_{2}}
\]

the sections \( \sigma_{\varepsilon} \) can be glued together to a section \( \sigma 2 H^{0} (B_{+}; \mathcal{F}) \) extending \( s \).

Let \( \varepsilon_{1}, \varepsilon_{2} > \varepsilon_{1} > 0 \), be fixed. We have to show that (5) holds. By definition,

\[
\sigma_{\varepsilon_{1}} \cap \overline{C}_{\varepsilon_{2}} = \sigma_{\varepsilon_{2}}
\]

Thus, the support of \( \sigma_{\varepsilon_{1}} \cap \overline{C}_{\varepsilon_{2}} \) is an analytic set contained in \( \overline{B}_{\varepsilon_{2}} \cap C_{\varepsilon_{2}} \). Let us consider the family \( \phi_{\lambda} = \lambda (\varepsilon \phi_{\varepsilon_{2}} + (1 - \lambda) \phi_{\varepsilon_{1}} \varepsilon_{2}) \lambda_{2, \mathbb{R}} \{ \varepsilon \} \) of strictly plurisubharmonic functions. Let \( \lambda \) be the smallest value of \( \lambda \) for which \( \varepsilon \phi_{\lambda} = 0 \) \( \varepsilon \) \( S \) \( \varepsilon \). Then \( 0 \phi_{\lambda} \varepsilon \phi_{\lambda} = 0 \) \( \varepsilon \) \( S \) \( \varepsilon \). \( B_{+} \) is a Stein domain in which the analytic set \( S \) intersects the boundary; so the maximum principle for plurisubharmonic functions and the strict plurisubharmonicity of \( \phi_{\lambda} \) together imply that \( 0 \phi_{\lambda} = 0 \) \( \varepsilon \) \( S \) \( \varepsilon \) is a set of isolated points in \( S \). By repeating the argument, we show that \( S \) has no components of positive dimension. Hence \( \sigma_{\varepsilon_{1}} \cap \overline{C}_{\varepsilon_{2}} \) is zero outside a set of isolated points. Since \( p (\mathcal{F}) \neq 3 \), the only section of \( \mathcal{F} \) with compact support is the zero-section \( \mathcal{B}_{\varepsilon} \) Théorème 3.6 (a), p. 46), and so \( \sigma_{\varepsilon_{1}} \cap \overline{C}_{\varepsilon_{2}} \) is zero.

Hence, there exists a section \( \sigma 2 H^{0} (B_{+}; \mathcal{F}) \) such that \( \sigma_{C_{+}} = s \). Thus we have proved the following

**Theorem 5.** If a \( B_{+} \) is 1-complete space, \( \mathcal{F} \) a coherent sheaf on \( B_{+} \) with \( p (\mathcal{F}) \neq 3 \), the homomorphism

\[
H^{0} (B_{+}; \mathcal{F}) \rightarrow H^{0} (C_{+}; \mathcal{F})
\]

is surjective.

In particular,

**Corollary 6.** If \( B_{+} \) is a 1-complete space and \( p (\mathcal{B}_{c}) \neq 3 \), every holomorphic function on \( C_{+} \) can be holomorphically extended on \( B_{+} \).
Theorem 7. Let \( \text{Sing}(B_c) = \emptyset \). Let \( F \subset \text{Coh}(B_c) \). Then the image of the homomorphism

\[
H^r(B_c; \mathfrak{i}F) \rightarrow H^r(C; \mathfrak{i}F) \rightarrow H^r(C_+; \mathfrak{i}F)
\]

defined by \( (\xi; \eta) \mapsto \xi \in \mathcal{C} \), \( \eta \in \mathcal{C}_+ \) has finite codimension for \( q \leq r \) \( p \geq 1 \) \( q < 2 \). For \( q = 1 \) the thesis holds true also dropping the assumption \( \text{Sing}(B_c) = \emptyset \).

Proof. Consider the Mayer–Vietoris sequence applied to the open sets \( B_+ \) and \( C \)

\[
\begin{align*}
\delta (a, b) &= a \mathfrak{i}_{B_+} \cap b \mathfrak{i}_C, \\
B_+ \cap C &= B_c \cap K_0 \text{ where } K_0 = \overline{B} \setminus \overline{B}_c. \quad K_0 \text{ has a } q\text{-complete neighbourhoods system and so the local cohomology groups } H^{r+1}_K(B_c; \mathfrak{i}F) \text{ are zero for } q < r < p \text{ (in the general case for } q = 1, \text{ see [BS, Lemme 2.3, p. 29]})}.
\end{align*}
\]

Then, from the local cohomology exact sequence

\[
\begin{align*}
\delta \left( H^r(B_c; \mathfrak{i}F) \right) &\rightarrow H^r(B_c; \mathfrak{i}F), \\
H^r(K_0; \mathfrak{i}F) &\rightarrow H^r(B_c; \mathfrak{i}F)
\end{align*}
\]

follows that

\[
H^r(B_c; \mathfrak{i}F) \rightarrow H^r(B_c; \mathfrak{i}F)
\]

for \( q < r < p \leq 1 \).

Since \( B_c \) is \( q\)-pseudoconvex,

\[
\dim C \cdot H^r(C; \mathfrak{i}F) < \infty
\]

for \( q < r \leq 1 \). [AG Théorème 11], and so

\[
\dim C \cdot H^r(B_c; \mathfrak{i}F) < \infty
\]

for \( q < r < p \leq 1 \).

From (6) we see that \( \dim C \cdot H^r(B_c; K_0; \mathfrak{i}F) = \dim C \cdot H^r(B_+; \mathfrak{i}F) \) is greater than or equal to the codimension of the homomorphism \( \delta \). 2

Corollary 8. Under the same assumption of Theorem 7 if \( K \setminus H = \emptyset \),

\[
\dim C \cdot H^r(C_+; \mathfrak{i}F) < \infty
\]

for \( q < r < p \leq 2 \).

Proof. The proof is similar to that of Corollary 2. 2

Theorem 9. Suppose that \( \text{Sing}(B_c) = \emptyset \) and \( B_+ \) is a \( q\)-complete space, then

\[
H^r(C; \mathfrak{i}F) \rightarrow H^r(C_+; \mathfrak{i}F)
\]
for \( q \neq r \) \( p \in \mathbb{F} \) \( q \neq 2 \) and the homomorphism

\[
H^q \left( B_+ ; \mathbb{F} \right) \rightarrow H^q \left( C ; \mathbb{F} \right) \rightarrow H^q \left( C_+ ; \mathbb{F} \right)
\]

is surjective if \( p \in \mathbb{F} \) \( 2q + 1 \). If \( q = 1 \), both results hold true for an arbitrary complex space \( B_+ \).

**Proof.** The proof is similar to that of Theorem 1.2

### 2.3. Corollaries of the extension theorems.

From now on, unless otherwise stated, by \( B, B+, B_\varepsilon, C \) and \( C_+ \) we denote both the open sets and their closures, and we suppose that \( H = \varepsilon h = 0 \), \( h \) pluriharmonic.

#### 2.3.1. Let \( f \in H^0 \left( B_+ ; \mathbb{O} \right) \). In the hypothesis of Corollaries 4 and 6 both \( f \) and \( 1 = f \) extend holomorphically on \( B_+ \). Hence:

**Corollary 10.** If \( B_+ \) is a 1-complete space and \( p \left( B_+ \right) \), then the restriction homomorphism

\[
H^0 \left( B_+ ; \mathbb{O} \right) \rightarrow H^0 \left( C_+ ; \mathbb{O} \right)
\]

is surjective.

#### 2.3.2. In Theorems 3 and 5 we have established the isomorphism

\[
H^r \left( C ; \mathbb{F} \right) \rightarrow H^r \left( C_+ ; \mathbb{F} \right)
\]

In some special cases this leads to vanishing-cohomology theorems for \( C_+ \).

An example is provided by a \( q \)-corona \( C \) which is contained in an affine variety. In such a situation, we have that \( H^r \left( C ; \mathbb{F} \right) = f0g \), for \( q = r \), \( p \in \mathbb{F} \) \( q \neq 2 \) [AT], and consequently \( H^r \left( C_+ ; \mathbb{F} \right) = f0g \) in the same range of \( r \).

#### 2.3.3. Let \( X \) be a Stein space. Let \( H = \varepsilon h = 0 \) \( X \) be the zero set of a pluriharmonic function, and let \( S \) be a real hypersurface of \( X \) with boundary, such that \( S \setminus H = bS = bA \), where \( A \) is an open set in \( H \). Let \( D \in X \) be the relatively compact domain bounded by \( S \setminus A \). In [LuT] it is proved that, for \( X = \mathbb{C}^n \), \( C_{2n} \)-functions on \( S \) extend holomorphically to \( D \). As a corollary of the previous theorems, we can obtain a similar result for section of a coherent sheaf on an arbitrary Stein space \( X \).

Let us consider the connected component \( Y \) of \( X \setminus H \) containing \( D \), the closure \( \overline{D} \) of \( D \) in \( Y \), and let be \( F = Y \setminus D \) and \( S_Y = S \setminus Y \). For every coherent sheaf \( \mathbb{F} \) on \( X \), with \( p \in \mathbb{F} \) \( 3 \) we have the Mayer-Vietoris exact sequence

\[
0 \rightarrow \overline{D} \left( \mathbb{F} \right) \rightarrow H^0 \left( F ; \mathbb{F} \right) \rightarrow H^0 \left( S_Y ; \mathbb{F} \right) \rightarrow H^1 \left( \mathbb{F} \right)
\]

Since \( Y \) is Stein, \( H^1 \left( Y ; \mathbb{F} \right) \) is zero, and every section \( \sigma \) on \( S_Y \) is a difference \( s_1 - s_2 \), where \( s_1 \in H^0 \left( \overline{D} ; \mathbb{F} \right) \) and \( s_2 \in H^0 \left( F ; \mathbb{F} \right) \). By choosing \( \varepsilon \) big enough so that \( \sigma \) is contained in the ball \( B_\varepsilon \) \( (x_0) \) of radius \( \varepsilon \) of \( X \) centered in \( x_0 \), we can apply Theorem 5 to the semi 1-corona \( C_+ = Y \setminus \left( B_\varepsilon \setminus Y \right) \), to extend \( s_2 \) \( x_+ \) to a section \( \tilde{s}_2 \) defined on \( Y \). In order to conclude that \( s_1 = \tilde{s}_2 \) extends the section \( \sigma \), we have to prove that \( s_2 \) \( \tilde{s}_2 = 0 \).
As before, we consider the set $\Sigma = s_{2,j} \tilde{s}_{2,j} \in 0 \ B_\epsilon \setminus Y$ and conclude that $\Sigma$ is a set of isolated points. Since $p(\mathcal{F}) = 3$, $\mathcal{F}$ has no non zero section with compact support [BS, Théorème 3.6 (a), p. 46]. Thus $\Sigma = ?$ and we have obtained the following:

**Corollary 11.** Let $X$ be a Stein space. Let $H = \mathcal{F} = 0 \mathcal{G}$ be the zero set of a pluriharmonic function, and $S$ be a real hypersurface of $X$ with boundary, such that $S \setminus H = bS = bA$, where $A$ is an open set in $H$. Let $D \ X$ be the relatively compact domain bounded by $S \setminus A$ and $F$ be a coherent sheaf with $p(F) = 3$. All sections of $F$ on $S$ extend (uniquely) to $D$.

We can go further:

**Corollary 12.** Let $X$ be a Stein manifold, $\mathcal{F}$ a coherent sheaf on $X$ such that $p(\mathcal{F}) = 3$ and $D$ be a bounded domain and $K$ a compact subset of $bD$ such that $bD \setminus K$ is smooth. Assume that $K$ is $\mathcal{O}(D)$-convex, i.e. $K = \{ z \in D : \sup_{z \in K} |f(z)| \leq M \}$ for some $M > 0$.

Then every section of $\mathcal{F}$ on $bD \setminus K$ extends to $D$.

**Proof.** We recall that since $U$ is an open subset of a Stein manifold there exists an envelope of holomorphy $\Theta$ of $U$ (cfr. [DG]) $\Theta$ is a Stein domain $\pi_U : \Theta \to X$ over $X$ and there exists and open embedding $j : U \to \Theta$ such that $\pi_U \circ j = id_U$ and $J : \Theta \to \Theta$ is an isomorphism. In particular $\pi_U^{-1}(\mathcal{F})$ is a coherent sheaf with the same depth as $\mathcal{F}$, which extends $\mathcal{F}$. Let $\theta$ be the holomorphic extension of $\theta$ to $\Theta$. The hypersurface

$$H = \{ z \in \Theta : \sup_{z \in K} |\theta(z)| \leq M \}$$

is the zero-set of a pluriharmonic function and, by construction,

$$x 2 \Theta_+ = \{ z \in \Theta : \sup_{z \in K} |\theta(z)| \leq M \}$$

Now we are in the situation of Corollary [1] so $\sigma$ extends to a section on $\Theta_+$. Since $x 2 \Theta_+$, this ends the proof. 2

3. **Extension of divisors and analytic sets of codimension one.**

First of all, we give an example in dimension $n = 2$ of a regular complex curve of $C_+$ which does not extend on $B_+$. Hence, not every divisor on $C_+$
Example. Using the same notation as before, let $B_c$ be the ball $\frac{x_1}{2} + \frac{x_2}{2} < c$, $c > 2$, in $\mathbb{C}^2$, and $H$ be the hyperplane $\mathbb{R} = 0 (z_j = x_j + iy_j)$. Let $2 < \varepsilon < c, C = B_c \cap B_e, B_+ = B_c \setminus \mathbb{R} > 0 g, C_+ = C \setminus \mathbb{R} > 0 g.$

Consider the connected irreducible analytic set of codimension one

$$A = f (z_1; z_2) 2 B_+ : z_1 z_2 = 1 g$$

and its restriction $A_C$ to $C_+$. If $A_C$ has two connected components, $A_1$ and $A_2$, we try to extend $A_1$ (analytic set of codimension one on $C_+$) to $B_+$, its restriction to $C_+$ will contain also $A_2$. So $A_1$ is an analytic set of codimension one on $C_+$ that does not extend on $B_+$.

So, let us prove that $A_C$ has indeed two connected components. A point of $A$ (of $A_C$) can be written as $z_1 = \rho e^{j\theta}, z_2 = \frac{1}{\rho} e^{i\theta}$, with $\rho > 2 \mathbb{R}^+$ and $\theta > \frac{\pi}{2} i \frac{\pi}{2}$. Hence, points in $A_C$ satisfy

$$2 < \varepsilon < \rho^2 + \frac{1}{\rho^2} < c \quad 2 < \rho^2 + \frac{1}{\rho^2} < c$$

Since $f (\rho) = \rho + 1 = \rho$ is monotone decreasing up to $\rho = 1$ (where $f (1) = 2$), and then monotone increasing, there exist $a$ and $b$ such that the inequalities are satisfied when $a < \rho < b < 1$, or when $1 < 1 = b < \rho < 1 = a$. $A_C$ is thus the union of the two disjoint open sets

$$A_1 = \rho e^{j\theta} \frac{1}{\rho} e^{i\theta} 2 \mathbb{C}^2 \ a < \rho < b; \quad \frac{\pi}{2} < \theta < \frac{\pi}{2};$$

$$A_2 = \rho e^{i\theta} \frac{1}{\rho} e^{i\theta} 2 \mathbb{C}^2 \ a < \frac{1}{\rho} < b; \quad \frac{\pi}{2} < \theta < \frac{\pi}{2}.$$ 

The aim of this section is to prove an extension theorem for divisors, i.e. to prove that, under certain hypothesis, the homomorphism

$$(10) \quad H^0 (B_+; \mathcal{T}) \rightarrow H^0 (C_+; \mathcal{T})$$

is surjective.

In order to get this result, we observe that from the exact sequence

$$(11) \quad 0 \rightarrow M \rightarrow D \rightarrow 0$$

we get the commutative diagram (horizontal lines are exact)

$$\begin{array}{cccccccc}
H^0 (B_+; \mathcal{M}) & \rightarrow & H^0 (B_+; \mathcal{T}) & \rightarrow & H^1 (B_+; \mathcal{O}) & \rightarrow & H^1 (B_+; \mathcal{M}) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & \\
H^0 (C_+; \mathcal{M}) & \rightarrow & H^0 (C_+; \mathcal{T}) & \rightarrow & H^1 (C_+; \mathcal{O}) & \rightarrow & H^1 (C_+; \mathcal{M})
\end{array}$$
Thus, in view of the “five lemma”, in order to conclude that $\beta$ is surjective it is sufficient to show that $\alpha$ and $\gamma$ are surjective, and $\delta$ is injective.

**Lemma 13.** If $\text{Sing} (B_+) = \emptyset$, $B_c$ is $1$-complete and $p (B_+)$ 3, then $\alpha$ is surjective.

**Proof.** Let $f$ be a meromorphic invertible function on $C_+$. Since $C_+$ is an open set of the Stein manifold $B_+$, $f = f_1 f_2 ^{-1}$, $f_1 f_2 2 H^0 (C_+ ; \mathcal{O})$. By Corollary 4 (10), $f_1$ and $f_2$ extend to holomorphic functions on $B_+$ and consequently $f$ extends on $B_+$ as well. 2

**Lemma 14.** Assume that the restriction $H^2 (B_+ ; \mathcal{O}) \neq H^2 (C_+ ; \mathcal{O})$ is surjective. If $B_c$ is $1$-complete and $p (B_+)$ 4, then $\gamma$ is surjective.

We remark that if $H^2 (C_+ ; \mathcal{O}) = \emptyset$ the first condition is satisfied.

**Proof.** From the exact sequence

\[
\begin{array}{ccccccccc}
0 & ! & Z & ! & \mathcal{O} & ! & \mathcal{O} & ! & 0 \\
\end{array}
\]

we get the commutative diagram (horizontal lines are exact)

\[
\begin{array}{ccccccccc}
H^1 (B_+ ; \mathcal{O}) & \longrightarrow & H^1 (B_+ ; \mathcal{O}) & \longrightarrow & H^2 (B_+ ; \mathcal{O}) & \longrightarrow & H^2 (B_+ ; \mathcal{O}) \\
| f_2 | & & | \gamma | & & | f_4 | & & | f_5 | \\
H^1 (C_+ ; \mathcal{O}) & \longrightarrow & H^1 (C_+ ; \mathcal{O}) & \longrightarrow & H^2 (C_+ ; \mathcal{O}) & \longrightarrow & H^2 (C_+ ; \mathcal{O})
\end{array}
\]

where $H^1 (B_+ ; \mathcal{O}) = H^2 (B_+ ; \mathcal{O}) = \emptyset$ because $B_+$ is Stein, and $f_4$ is surjective by hypothesis. Thus in order to prove that $\gamma$ is surjective by the “five lemma” it is sufficient to show that $f_2$ is surjective, i.e. that $H^1 (C_+ ; \mathcal{O}) = \emptyset$.

Since $p (B_+) = 4$, by Theorem 3 (9) it follows that

\[
H^1 (C_+ ; \mathcal{O}) \neq H^1 (C_+ ; \mathcal{O}) ;
\]

Consider the local, respectively compact support, cohomology exact sequence

\[
\begin{array}{ccccccccc}
H^1_{\mathcal{B}_k} (B_c ; \mathcal{O}) & \longrightarrow & H^1 (B_c ; \mathcal{O}) & \longrightarrow & H^1 (C_+ ; \mathcal{O}) & \longrightarrow & H^2_{\mathcal{B}_k} (B_c ; \mathcal{O}) \\
| \ | & & | \ | & & | \ | & & | \ |
\end{array}
\]

\[
\begin{array}{ccccccccc}
H^1 (B_+ ; \mathcal{O}) & \longrightarrow & H^1 (B_+ ; \mathcal{O}) & \longrightarrow & H^1 (C_+ ; \mathcal{O}) & \longrightarrow & H^2 (B_+ ; \mathcal{O}) \\
\end{array}
\]

Since $B_c$ is Stein, $H^1 (B_+ ; \mathcal{O}) = \emptyset$ and $H^r (B_+ ; \mathcal{O}) = H^r_{\mathcal{B}_k} (B_+ ; \mathcal{O}) = \emptyset$ for

\[
1 \quad p (B_+) \quad 1 [\mathbb{C}].
\]

In particular, since $p (B_+) = 4$, it follows that

\[
\emptyset = H^1 (B_c ; \mathcal{O}) \neq H^1 (C_+ ; \mathcal{O}) ;
\]

(13) and (14) give

\[
\emptyset = H^1 (B_c ; \mathcal{O}) \neq H^1 (C_+ ; \mathcal{O}) ;
\]
and this proves the lemma. 2

In the case $H^2(C_+;\mathbb{Z}) = f0g$ we remark that from the proof of Lemma 14 it follows that the sequence

$$f0g \to H^1(C_+;\mathcal{O}) \to f0g$$

is exact, that is $H^1(C_+;\mathcal{O}) = f0g$. Hence, the commutative diagram relative to (1) becomes (horizontal lines are exact)

$$
\begin{array}{cccccc}
H^0(B_+;\mathcal{M}) & \to & H^0(B_+;\mathcal{D}) & \to & H^1(B_+;\mathcal{O}) \\
\alpha & \downarrow & \beta & \downarrow & \gamma \\
H^0(C_+;\mathcal{M}) & \to & H^0(C_+;\mathcal{D}) & \to & f0g
\end{array}
$$

and it is then easy to see that a divisor on $C_+$ can be extended to a divisor on $B_+$.

Thus we have proved the following:

**Theorem 15.** Let $B_c$ be 1-complete, $p(B_c) = 4$, and $C_+$ satisfy the topological condition $H^2(C_+;\mathbb{Z}) = f0g$. Then, if $\text{Sing}(B_+) = \emptyset$, all divisors on $C_+$ extend (uniquely) to divisors on $B_+$.

**Corollary 16.** Let $B_c$ be 1-complete, $p(B_c) = 4$, $\text{Sing}(B_+) = \emptyset$, and $\xi$ be a divisor on $C_+$ with zero Chern class in $H^2(C_+;\mathbb{Z})$. Then $\xi$ extends (uniquely) to a divisor on $B_+$.

**Proof.** Use diagram (15). 2

**Theorem 17.** Assume that $H^2(C_+;\mathbb{Q}) = f0g$. If $\text{Sing}(B_+) = \emptyset$, $B_+$ is 1-complete and $p(B_c) = 4$, then all analytic sets of codimension 1 on $C_+$ extend to analytic sets on $B_+$.

**Proof.** Let $A$ be an analytic set of codimension 1 on $C_+$. Since $B_+$ is a Stein manifold, $C_+$ is locally factorial, and so there exists a divisor $\xi$ on $C_+$ with support $A$. Since $H^2(C_+;\mathcal{O}) = f0g$, there exists $n \in \mathbb{N}$ such that $nc_2(\xi) = 0$ in $H^2(C_+;\mathbb{Z})$. Hence $n\xi$ has zero Chern class in $H^2(C_+;\mathbb{Z})$, and so, by Corollary 14, $n\xi$ can be extended to a divisor $n\xi$ on $B_+$. The support of $n\xi$ is an analytic set $A$ which extends to $B_+$ the support $A$ of $n\xi$. 2

In Theorem 15 the condition $H^2(C_+;\mathbb{Z}) = f0g$ can be relaxed and replaced by the weaker one: the restriction map $H^2(B_+;\mathbb{Z}) \to H^2(C_+;\mathbb{Z})$ is surjective. We need the following

**Lemma 18.** $\delta$ is injective.

**Proof.** First we prove lemma for $C_+$ closed. Let $\xi_2 H^1(B_+;\mathcal{M})$ be such that $\xi_2 \xi_+ = 0$. Consider the set

$$A = f\eta \cap [0;\varepsilon] : \xi_2 \xi_+ \xi_\eta = 0.$$
If we prove that $0 < A$, we are done, because $0 = \xi_{\mathcal{B}_+} \circ \mathcal{B}_0 = \xi_{\mathcal{B}_+} = \xi$. Obviously $\eta_0 2 A$ implies $6\eta > \eta_0$, $\eta 2 A$.

$A \not\emptyset ?$. Since $C_+ = \mathcal{B}_+ \circ \mathcal{B}_0$ and $\xi_{\mathcal{B}_+} = 0, e 2 A$.

$A$ is closed. If $\eta_0 2 A$, for all $n$, and $\eta_n \not\emptyset$ $\eta_{\infty}$, $\mathcal{B}_+ \circ \mathcal{B}_{\eta_n} = \mathcal{B}_+ \circ \mathcal{B}_{\eta_{\infty}} = 0$ for all $n$ implies $\xi_{\mathcal{B}_+} \circ \mathcal{B}_{\eta_{\infty}} = 0$, i.e. $\eta_{\infty} 2 A$.

$A$ is open. Suppose $0 < \eta_0 2 A$. We denote $C_{\eta_0} = \mathcal{B}_+ \circ \mathcal{B}_{\eta_0}$. Let $A$ be the family of open covering $\mathcal{E}U_{i,j} \circ \mathcal{B}_{\eta_0}$ of $\mathcal{B}_+$ such that:

$\alpha)$ $U_i$ is isomorphically equivalent to an holomorphy domain in $C^n$;

$\beta)$ If $U_i \setminus \mathcal{B}_{\eta_0} \not\emptyset$, the restriction homomorphism

$$H^0 (U_i \circ \mathcal{O}) \cap H^0 (U_i \setminus C_{\eta_0} \circ \mathcal{O})$$

is bijective;

$\gamma)$ $U_i \setminus U_j$ is simply connected.

$A$ is not empty and it is cofinal in the set of open coverings of $\mathcal{B}_+$ [AG] Lemma 2, p. 222]. Let $U = \mathcal{E}U_{i,j} \circ \mathcal{B}_{\eta_0}$, and $f_{ij} \in Z^1 (U, \mathcal{O}_U, \mathcal{B}_{\eta_0}$ be a representative of $\xi$. Let $W_i = U_i \setminus C_{\eta_0}$. Since $\eta_0 2 A$, if $W_i \setminus W_j \not\emptyset$, $f_{ij} \in W_i \setminus W_j = f_i f_j^{-1} (f_{ij} \in H^0 (W_i \circ \mathcal{O}_U, \mathcal{B}_{\eta_0}$). By $\alpha)$, $f_{ij} = p_{ij} q_{ij}^{-1}$, $p_{ij} q_{ij} \in H^0 (W_i \circ \mathcal{O}_U, \mathcal{B}_{\eta_0}$).

By $\beta)$, both $p_{ij}$ and $q_{ij}$ can be holomorphically extended on $U_{ij}$, with $p_{ij}$ and $q_{ij}$. Hence we have $f_{ij} = p_i p_j^{-1} (p_i p_j^{-1} \in 1)$ on $U_i \setminus U_j$ (which is simply connected, so that there is no polidromy). So $\xi = 0$ in an open neighborhood $U$ of $C_{\eta_0}$ and, by compactness, there exists $e > 0$ such that $C_{\eta_0} \circ e 0 \subset U$. So $\eta_0 \circ e 2 A$ and consequently $A$ is open.

Thus $A = \emptyset \circ e 1$ and the lemma is proved if $C_+$ is closed.

If $C_+$ is open, we consider $C_+$ as a union of the closed semi 1-coronae

$$C_+ \cap \mathcal{E}h > e \mathcal{G}_C \subset C_+:$$

Let $\xi 2 H^1 (\mathcal{B}_+ \circ \mathcal{M} )$ be such that $\xi_{\mathcal{B}_+} = 0$. Then $\xi_{\mathcal{B}_+} = 0$, for all $e > 0$. Consequently from what we have already proved $\xi_{\mathcal{B}_+} = 0$, where $\mathcal{B}_+ = \mathcal{B}_+ \cap \mathcal{E}h > e \mathcal{G}_C$. Since $\{ \xi_{\mathcal{B}_+} = 0, \xi = 0$ and the lemma is proved. 2

Lemma 13, Lemma 14 and Lemma 18 lead to the following generalization of Theorem 13

**Theorem 19.** Assume that the restriction $H^2 (\mathcal{B}_+ \circ \mathcal{M}) \cap H^2 (\mathcal{C}_+ \circ \mathcal{M})$ is surjective. If $\mathcal{S}ing (\mathcal{B}_+ ) = \emptyset$, $B_c$ is 1-complete and $p (B_c)$ 4, then all divisors on $C_+$ extend to divisors on $B_+$.

**REFERENCES**

[AG] Andreotti, A., Grauert, H.: Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France 90, 193-259 (1962).

[AT] Andreotti, A., Tomassini, G.: A remark on the vanishing of certain cohomology groups, Comp. Math. 21, 417-430 (1969).

[BS] Bănică, C., Stănășilă, O.: Méthodes algébriques dans la théorie globale des espaces complexes, Editura Academiei et Gauthier-Villars, Bordas Paris (1977).
COHOMOLOGY AND EXTENSION PROBLEMS FOR SEMI-$q$-CORONAE

[C] Cartan, H.: Faisceaux analytiques cohérents. In: corso C.I.M.E. “Funzioni e varietà complesse”, Varenna (1963).

[DS] Della Sala, G., Saracco, A.: Extension of non compact maximally complex submanifolds, (2005). [arXiv: math.CV/0503430]

[DG] Docquier, F., Grauert, H.: Levi's Problem and Runge's Satz für Teilgebiete Steinscher Mannigfaltigkeiten, Math. Ann. 140, 94-123 (1960).

[FG] Frisch, J., Guenot, J.: Prolongement de faisceaux analytiques cohérents, Invent. Math. 7, 321-343 (1969).

[HL] Harvey, F. R., Lawson, H. B.: On boundaries of complex analytic varieties I, Ann. of Math. 102, 223-290 (1975).

[J] Jöricke, B.: Boundaries of singularity sets, removable singularities, and CR-invariant subsets of CR-manifolds, J. Geom. Anal. 9, 257-300 (1999).

[La] Laurent-Thiébaut, C.: Sur l’extension des fonctions CR dans une variété de Stein, Ann. Mat. Pura Appl. 150, 141-151 (1988).

[LaP] Laurent-Thiébaut, C., Porten, E.: Analytic extension from non-pseudoconvex boundaries and $A(D)$-convexity, Ann. Inst. Fourier (Grenoble) 53, 847-857 (2003).

[Lu] Lupacciolu, G.: A theorem on holomorphic extension of CR-functions, Pac. J. Math. 124, 177-191 (1986).

[LuT] Lupacciolu, G., Tomassini, G.: Un teorema di estensione per le CR-funzioni, Ann. Mat. Pura Appl. 137, 257-263 (1984).

[P] Perotti, A.: Extension of CR-forms and related problems, Rend. Sem. Mat. Univ. Padova 77, 37-55 (1987).

[Se] Serre, J. P.: Prolongement de faisceaux analytiques cohérents, Ann. Inst. Fourier (Grenoble) 16, 363-374 (1966).

[Si] Siu, Y. T.: Extension Problem in Several Complex Variables. In: Proc. Int. Congr. IMU, Helsinki, 669-673 (1978).

[SiT] Siu, Y. T., Trautmann, G.: Gap-sheaves and extension of coherent analytic subsheaves. Lecture Notes in Mathematics 172 Springer-Verlag, Berlin-New York (1971).

[St] Stout, E. L.: Removable singularities for the boundary values of holomorphic functions of several complex variables. In: Proc. Mittag-Leffler special year in complex variables, Stockholm 1987-1988, Princeton Univ. Press, Princeton, 600-629 (1993).

[T1] Tomassini, G.: Inviluppo d’olomorfia e spazi pseudoconcavi, Ann. Mat. Pura Appl. 87, 59-86 (1970).

[T2] Tomassini, G.: Extension d’objèts CR, Math. Z. 194, 471-486 (1987).