Counting curves of any genus on $\mathbb{P}^2$  

Mendy Shoval  
Eugenii Shustin

Abstract

We compute Gromov-Witten invariants of any genus for del Pezzo surfaces of degree $\geq 2$. The genus zero invariants have been computed a long ago [3, 4], Gromov-Witten invariants of any genus for del Pezzo surfaces of degree $\geq 3$ have been found by Vakil [13]. We solve the problem in two steps: (1) we consider surfaces $\mathbb{P}^2_{a,1}$, the plane blown up at $a \geq 6$ points on a conic and one more point outside this conic, and, using techniques of tropical geometry, obtain a Caporaso-Harris type formula counting curves of any divisor class and genus subject to arbitrary tangency conditions with respect to the blown up conic, (2) then we express the Gromov-Witten invariants of $\mathbb{P}^2_7$ via enumerative invariants of $\mathbb{P}^2_{6,1}$, using Vakil’s version of Abramovich-Bertram formula.

1 Introduction

Del Pezzo surfaces (or Fano surfaces) are smooth rational surfaces which have an ample anticanonical class $-K$. Each of them is isomorphic either to the plane $\mathbb{P}^2$, or to the quadric $(\mathbb{P}^1)^2$, or to the plane blown up at $1 \leq k \leq 8$ generic points (below referred to as of type $\mathbb{P}^2_k$). The complex structure of a del Pezzo surface $\Sigma$ is generic in the space of almost complex structures, that particularly implies that the Gromov-Witten invariants $GW_g(\Sigma, D)$ are enumerative, i.e. they count irreducible complex curve of a given genus $g$ and of a given divisor class $D$ passing through $-KD + g - 1$ generic points on a surface: for the genus zero it was established in [3, 4], for higher genera observed in [13] Section 4.2. The celebrated Kontsevich formula [10] computes recursively all genus zero Gromov-Witten invariants of the plane. One can find there also similar formulas for the genus zero Gromov-Witten invariants of other Del Pezzo surfaces. All genus zero Gromov-Witten invariants of Del Pezzo surfaces have been computed in [3, 4]. In 1998 L. Caporaso and J. Harris [2] suggested another formula recursively computing the Gromov-Witten invariants of the projective plane for any genus. Vakil [13] extended the Caporaso-Harris approach further, producing a formula which
computes the number of curves of a given degree and genus in the plane or on a ruled surface and, in addition, has few fixed multiple points. In particular, Vakil computes the Gromov-Witten invariants of arbitrary genus for the plane blown up at \( q \leq 5 \) generic points and the numbers of curves in a given linear system and of a given genus in the plane blown up at 6 points on a conic. In the latter case, Vakil computes the invariants of \( \mathbb{P}^2_6 \) using results of Graber in Gromov-Witten theory [5] and deriving a suitable extension of the Abramovich-Bertram formula [1]. Since then there was no progress, and here we make the next step calculating the Gromov-Witten invariants of any genus for \( \mathbb{P}^2_7 \), leaving open the only case of \( \mathbb{P}^2_8 \).

The present work goes the way opened in [2] and developed further in [13]. We blow up the plane at \( a + 1 \geq 1 \) points, specializing \( a \) blown up points on a smooth conic and taking the \((a + 1)\)-st blown up point outside, and exhibit a recursive formula counting curves in any divisor class and of any genus of the obtained surface \( \mathbb{P}^2_{a,1} \). In the case \( a = 6 \), \( \mathbb{P}^2_{6,1} \) is almost Fano, and using [13, Theorem 4.2], we convert these enumerative numbers into genuine Gromov-Witten invariants of \( \mathbb{P}^2_7 \). Our recursive formula is very much similar to the Caporaso-Harris formula, counting plane curves of any degree and genus [2] and to Vakil’s formula [13, Theorem 6.8]. Its geometric core is a process of specialization of points in the constraint configuration (consisting of \(-DK + g - 1 \) points) to the divisor \( E \subset \mathbb{P}^2_{a,1} \), the strict transform of the conic through a blown up points. The counted curves degenerate either into irreducible curves of the same genus with certain tangency conditions with respect to \( E \), or they split off \( E \) and few other components. The comparison of possible degenerations and their deformations (as the constraint point leaves \( E \)) yields the recursion. We should like to mention that for \( \mathbb{P}^2_{a,1} \), one encounters a new phenomenon: the reducible degenerate curves can be non-reduced. This requires an additional deformation theory argument based on the local tropical geometry, and this is the main novelty of the present paper.

Our motivation included an application to the real enumerative geometry: computation of Welschinger invariants of real Del Pezzo surfaces of degree \( \geq 2 \) [8, 9], which is based on a conversion of the complex recursive formula into a formula counting real rational curves and on the real version of [13, Theorem 4.2]. Notice also that \( \mathbb{P}^2_{5,1} \) is just the plane blown up at 6 generic points, and our recursive formula directly computes the Gromov-Witten invariants of \( \mathbb{P}^2_6 \) (while in [13] it is done indirectly).

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## 2 Counting curves on $\mathbb{P}^2_{a,1}$

### 2.1 General setting

**Notation.** Denote by $\mathbb{Z}_+^\infty$ the direct sum of countably many additive semigroups $\mathbb{Z}_+ = \{m \in \mathbb{Z} \mid m \geq 0\}$, labeled by the naturals, and denote by $e_k \in \mathbb{Z}_+^\infty$ the generator of the $k$-th summand, $k \in \mathbb{N}$. For $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^\infty$, define

\[
\|\alpha\| = \sum_{k \geq 1} \alpha_k, \quad I\alpha = \sum_{k \geq 1} k\alpha_k, \quad I^n = \prod_{k \geq 1} k^{\alpha_k}, \quad \alpha! = \prod_{k \geq 1} \alpha_k!,
\]

and for $\alpha, \alpha^{(1)}, \ldots, \alpha^{(m)} \in \mathbb{Z}_+^\infty$ such that $\alpha \geq \alpha^{(1)} + \ldots + \alpha^{(m)}$ we define

\[
\binom{\alpha}{\alpha^{(1)}, \ldots, \alpha^{(m)}} = \frac{\alpha!}{\alpha^{(1)}! \ldots \alpha^{(m)}! (\alpha - \alpha^{(1)} - \ldots - \alpha^{(m)})!}.
\]

**Surfaces under consideration.** Let $\Sigma$ be a smooth rational surface which is a blow-up of $\mathbb{P}^2$, $E \subset \Sigma$ a smooth rational curve such that

- $-K_\Sigma$ is positive on each curve different from $E$,
- $-(K_\Sigma + E)$ is nef and effective.

Using these data and the genus formula, we immediately derive that

\[-(K_\Sigma + E)E = 2,
\]

that, for each smooth curve $\Lambda \neq E$,

\[\Lambda^2 \geq -1,
\]

and that, for each smooth rational $(-1)$-curve $\Lambda \neq E$,

\[-K_\Sigma \Lambda = 1, \quad E \Lambda \leq 1.
\]

Thus, the projection of $\Sigma$ onto $\mathbb{P}^2$ takes $E$ to a straight line or to a conic, and $\pi : \Sigma \to \mathbb{P}^2$ is a blow-up at several distinct points such that
• ("planar line-model") either some \(a \geq 0\) blown-up points lie on a straight line \(L_0\), \(E\) is a strict transform of \(L_0\), and \(b \leq 4\) points are blown-up outside \(L_0\), no three blown-up points lie on a line (except for those on \(L_0\)), no 6 blown-up points lie on a conic;

• ("planar conic-model") or some \(a \geq 0\) blown up points lie on a smooth conic \(C_0\), \(E\) is a strict transform of \(C_0\), and \(b \leq 1\) points are blown-up outside \(C_0\), no three blown-up points lie on a line.

We will assume that \(\Sigma\) is generic in the sense that, in the above models, the blown-up points are chosen generically subject to the condition that \(a\) of them lie on \(L_0\) or \(C_0\).

Without loss of generality in our considerations, we can blow up additional points. Notice that that a planar line-model with \(b \geq 3\) can be transformed by Cremona into a planar conic-model. Thus, in the sequel we will refer only to the planar conic-model with \(b = 1\) and denote it \(\mathbb{P}^2_{a,1}\). Without loss of generality, we suppose that \(a \geq 6\), in particular, \(E^2 \leq -2\).

When referring to the planar conic-model \(\pi: \mathbb{P}^2_{a,1} \to \mathbb{P}^2\), we denote by \(L\) the pull back of a general line, by \(E_1\) the exceptional divisor disjoint from \(E\), by \(E_2, ..., E_{a+1}\) the exceptional divisors crossing \(E\).

Notice that, in our setting, \((K_\Sigma + E)^2 = 0\), the linear system \(|-(K_\Sigma + E)|\) is one-dimensional, its generic element is a smooth rational curve, and it contains precisely two curves (denoted by \(L', L''\)) quadratically tangent to \(E\).

Introduce also the semigroups \(\text{Pic}_+(\Sigma, E)\) and \(\text{Pic}(\Sigma, E)\) generated by classes \(D \in \text{Pic}(\Sigma)\) of effective irreducible divisors such that \(DE > 0\) or \(DE \geq 0\), respectively. Under our assumptions \(E \not\in \text{Pic}(\Sigma, E)\).

**Families of curves.** For a given effective divisor class \(D \in \text{Pic}(\Sigma)\) and nonnegative integers \(g, n\), denote by \(\mathcal{M}_{g,n}(\Sigma, D)\) the moduli space of stable maps \(n: \hat{C} \to \Sigma\) of \(n\)-pointed, connected curves \(\hat{C}\) of genus \(g\) such that \(n_* \hat{C} \in |D|\) \([\mathbb{L}]\). This is an algebraic stack, whose open dense subset is formed by elements with a smooth curve \(\hat{C}\) of genus \(g\), and the other elements have connected \(\hat{C}\) of arithmetic genus \(g\) with at most nodes as singularities (cf. \([13, \text{Section 2}]\)).

Let a divisor class \(D \in \text{Pic}(\Sigma, E)\), an integer \(g\), and two elements \(\alpha, \beta \in \mathbb{Z}_+\) satisfy

\[
0 \leq g \leq g(\Sigma, D) = \frac{D^2 + DK_\Sigma}{2} + 1, \quad I\alpha + I\beta = DE. \tag{1}
\]

---

\(^1\)In what follows, by \(n_* \hat{C}\) we denote the \(n\)-image of \(\hat{C}\), whose components are taken with respective multiplicities, by \(n(\hat{C})\) we denote the reduced image.
Given a sequence \( p = (p_{ij})_{i \geq 1, 1 \leq j \leq \alpha_i} \) of \( \|\alpha\| \) distinct points of \( E \), define \( \mathcal{V}_\Sigma(D, g, \alpha, \beta, p) \subset \overline{M}_{g,\|\alpha\|}(\Sigma, D) \) to be the (stacky) closure of the set of elements \( \{n : \hat{C} \to \Sigma, \hat{p}\} \), \( \hat{p} = (\hat{p}_{ij})_{i \geq 1, 1 \leq j \leq \alpha_j} \) a sequence of distinct points of \( \hat{C} \), subject to the following restrictions:

- \( \hat{C} \) is smooth, \( n(\hat{p}_{ij}) = p_{ij} \) for all \( i \geq 1, 1 \leq j \leq \alpha_i \),

- \( n^*(n_*C \cap E) \) is the following divisor on \( \hat{C} \):
  \[
  n^*(n_*C \cap E) = \sum_{i \geq 1, 1 \leq j \leq \alpha_i} i \cdot \hat{p}_{ij} + \sum_{i \geq 1, 1 \leq j \leq \beta_i} i \cdot \hat{q}_{ij},
  \]

  where \( \{\hat{q}_{ij}\}_{i \geq 1, 1 \leq j \leq \beta_i} \) is a sequence of \( \|\beta\| \) distinct points of \( \hat{C} \setminus \hat{p} \).

Put
\[
R_\Sigma(D, g, \beta) = -D(K_\Sigma + E) + \|\beta\| + g - 1.
\]

At last, for a component \( V \) of \( \mathcal{V}_\Sigma(D, g, \alpha, \beta, p) \), define the intersection dimension \( \text{idim} \) to be the maximal number of a priori given generic distinct points of \( \Sigma \) lying in \( n_*\hat{C} \) for an element \( \{n : \hat{C} \to \Sigma, \hat{p}\} \in V \).

### 2.2 Dimension count and generic elements

**Proposition 2.1** In the notations of section 2.1, let \( D \in \text{Pic}(\Sigma, E) \), an integer \( g \geq 0 \), and vectors \( \alpha, \beta \in \mathbb{Z}_+^\infty \) satisfy (1), and let \( \mathcal{V}_\Sigma(D, g, \alpha, \beta, p) \neq \emptyset \). Then

1. \( R_\Sigma(D, g, \beta) \geq 0 \), and each component \( V \) of \( \mathcal{V}_\Sigma(D, g, \alpha, \beta, p) \) satisfies

\[
\text{idim} V \leq R_\Sigma(D, g, \beta);
\]

2. if \( D \neq sD_0 \) for any \( s \geq 2 \) and any divisor class \( D_0 \) such that \( -(K_\Sigma + E)D_0 = 0 \), and if \( V \) is a component of \( \mathcal{V}_\Sigma(D, g, \alpha, \beta, p) \) with \( \text{idim} V = R_\Sigma(D, g, \beta) \), then

\[
\text{idim} \mathcal{V}_{\Sigma,n}(D, g, \alpha, \beta, p) \leq -(K_\Sigma + E)D + g - 1 + n,
\]

(2i) a generic element \( \{n : \hat{C} \to \Sigma, \hat{p}\} \in V \) is an immersion, birational onto its image \( C = n(\hat{C}) \) which is nonsingular along \( E \); if, in addition \( E^2 \geq -3 \), then \( C \) is nodal.
(2ii) if \( R(\Sigma(D, g, \beta) > 0 \), then the family \( V \) has no base points outside \( p \), and the generic curve \( C = n(\hat{C}) \) crosses any a priori given curve \( C' \subset \Sigma \) transversally outside \( p \); furthermore, the image \( V_{\Sigma}(D, g, \alpha, \beta, \hat{p}) \subset |D| \) of the components of \( V_{\Sigma}(D, g, \alpha, \beta, \hat{p}) \) of intersection dimension \( R(\Sigma(D, g, \beta) \) by the projection

\[
pr : V_{\Sigma}(D, g, \alpha, \beta, \hat{p}) \rightarrow |D|, \quad \{ n : \hat{C} \rightarrow \Sigma, \hat{p} \} \mapsto n, \hat{C},
\]

is smooth at \( C \).

**Proof.** We divide our argument into several parts: in Steps 1-3 we prove statement (1), in Steps 4-9 we prove statement (2). We shall use the planar conic-model and the respective notations introduced in section 2.1.

**Step 1: Preliminaries.** Without loss of generality, assume that \( V_{\Sigma}(D, g, \alpha, \beta, \hat{p}) \) is irreducible.

From now on and till Step 8 below, we suppose that, for a generic element \( \{ n : \hat{C} \rightarrow \Sigma, \hat{p} \} \in V_{\Sigma}(D, g, \alpha, \beta, \hat{p}) \), the map \( n : \hat{C} \rightarrow C = n(\hat{C}) \) is birational, i.e. it is the normalization map.

Then inequalities (3) and (4) are evident if either \( D \) is a \((-1)\)-curve, or \( D = dL - d_1E_1 - ... - d_{a+1}E_{a+1} \), where \( d \leq 2 \) or \( d_i < 0 \) for some \( i \). So, we suppose that \( D = dL - d_1E_1 - ... - d_{a+1}E_{a+1} \) with \( d \geq 3 \), \( d_1, ..., d_{a+1} \geq 0 \).

Due to generic position of \( p \), one has

\[
\text{idim} \ V_{\Sigma}(D, g, \alpha, \beta, \hat{p}) = \text{idim} \ V_{\Sigma}(D, g, 0, \alpha + \beta, \emptyset) - \| \alpha \|,
\]

and, furthermore, if \( \{ n : \hat{C} \rightarrow \Sigma, \hat{p} \} \) is a generic element of \( V_{\Sigma}(D, g, \alpha, \beta, \hat{p}) \), then \( \{ n : \hat{C} \rightarrow \Sigma \} \) is generic for \( V(D, g, 0, \alpha + \beta, \emptyset) \). Hence we can let \( \alpha = 0 \) and \( p = \emptyset \). To shorten notation, we write (within the present proof) \( V(D, g, \beta) \) for \( V_{\Sigma}(D, g, 0, \beta, \emptyset) \).

Inequality (3) and its proof are completely analogous to [2, Propositions 2.1 and 2.2]: namely, the argument is developed for curves on any algebraic surface, and its applicability amounts to checking a number of sufficient numerical conditions, verified in [2] for the case of the plane. In the following we do not copy the reasoning of [2], but go through all numerical conditions and verify them in our setting.

**Step 2: Proof of (3).** The computation of \( \text{idim} \ V(D, g, \beta) \) literally goes along [2, Section 2.3], where one has to verify the following.

(D1) The conclusion of [2, Corollary 2.4] reads in our situation as

\[
\text{idim} \ V(D, g, (DE)e_1) \leq -DK_{\Sigma} + g - 1 .
\]
and it holds, since the hypothesis of [2, Corollary 2.4], equivalent to $DK_{\Sigma} < 0$ is true for any effective divisor class $D \in \text{Pic}(\Sigma, E)$.

(D2) The inequality $\deg(n^*O_{\Sigma}(-d)) \geq 0$ in [2, Page 363], where

$$d = \sum_{i \geq 1, 1 \leq j \leq \alpha_i} i \cdot \hat{p}_{ij} + \sum_{i \geq 1, 1 \leq j \leq \beta_i} (i - 1) \cdot \hat{q}_{ij}$$

(cf. [2]), reads in our situation as $\deg(n^*O_{\Sigma}(E)(-d)) \geq 0$, and it holds, since

$$\deg(n^*O_{\Sigma}(E)(-d)) = DE - \deg d \oplus \|\beta\| \geq 0.$$

(D3) The inequality

$$\deg(c_1(N_{\hat{C}}(-d) \otimes \omega_{\hat{C}}^{-1})) > 0$$

in [2, Page 363] ($N_{\hat{C}}$ is the normal sheaf on $\hat{C}$ and $\omega_{\hat{C}}$ is the dualizing bundle) reads in our setting as

$$-DK_{\Sigma} + 2g - 2 - \deg d + 2 - 2g = -D(K_{\Sigma} + E) + \|\beta\| = (d - d_1) + \|\beta\| > 0,$$

and it holds true since $d - d_1 \geq 1$ as $D$ is represented by a reduced, irreducible curve with $d \geq 3$.

After that, as in the end of the proof of [2, Proposition 2.1], we derive

$$\text{idim} V(D, g, \beta) \leq \deg(c_1(N_{\hat{C}}(-d)) - g + 1 = R_{\Sigma}(D, g, \beta),$$

which completes the proof of (3).

Step 3: Proof of (4). Again we can assume that $V_{\Sigma, n}(D, g, \alpha, \beta, p)$ is irreducible and that $\{n : \hat{C} \to \Sigma, \hat{p}\}$ is generic in $V_{\Sigma, n}(D, g, \alpha, \beta, p)$. Since $n$ is supposed to be birational on its image, the required statement is analogous to the inequality in [2, Corollary 2.7], and the proof literally coincides with the last paragraph of the proof of [2, Corollary 2.7].

Step 4: Immersion. The sufficient conditions for $n : \hat{C} \to \Sigma$ to be an immersion when $\{n : \hat{C} \to \Sigma, \hat{p}\} \in V(D, g, \beta)$ is generic, are

(I1) $\deg(c_1(N_{\hat{C}}(-d) \otimes \omega_{\hat{C}}^{-1}) \geq 2$, equivalent to $d - d_1 + \|\beta\| \geq 2$, for the immersion away from $n^{-1}(C \cap E)$ (see [2, First paragraph of the proof of Proposition 2.2])
Suppose that a generic $C = n(\hat{C})$ has a local singular branch at $z \in \Sigma \setminus E$. Then by Bézout’s bound for the intersection of $C$ with a line $l \in |L - E_1|$ passing through $z$, we get $d \geq d_1 + 2$, a contradiction by (I1).

Suppose that a generic $C = n(\hat{C})$ has a singular local branch at $z \in E$. Then $\|\beta\| \geq 1$, and by the above Bézout’s bound we get $d - d_1 + \|\beta\| \geq 3$. Thus, by (I2), it only remains to analyze the case $d - d_1 + \|\beta\| = 3$, that means: $d_1 = d - 2$, $\|\beta\| = 1$ (equivalent to $(C \cdot E)_z = 2d - d_2 - ... - d_{a+1}$), and $z$ is a center of a unique local branch of multiplicity 2, i.e. a singularity $A_{2s}$, $s \geq 1$. Since $\delta(A_{2s}) = s$, by the genus formula we have

$$s \leq \frac{D^2 + DK_{\Sigma} + 2}{2} - g = d - 2 - \sum_{i=2}^{a+1} d_i(d_i - 1) - g .$$

(7)

We also recall, that after $s$ successive blow-ups, a curve germ of type $A_{2s}$ turns into a non-singular branch quadratically tangent to the last exceptional divisor, and the multiplicities of the exceptional divisors equal 2.

Suppose that

$$2d - d_2 - ... - d_{a+1} = (C \cdot E)_z = 2s + 1$$

(8)

(the maximal possible intersection number). We shall show that, in such a case, the family $V_1$ of curves of genus $g$ in the linear system $|D|$, having singularity $A_{2s}$ at $z$, has dimension $\leq g$. This will be a contradiction to the generality of $C$, since, allowing the point $z$ to move along $E$, we would get

$$\text{idim } V(D, g, \beta) \leq g + 1 = R_{\Sigma}(D, g, \beta) - 1 .$$

Indeed, consider the blow-up $\pi : \Sigma' \to \Sigma$ of $z$ and of $s - 1$ more infinitely near points of $C$ at $z$ (all the curves of the family $V_1$ have the same $s$ blown-up infinitely near points). Then the considered family of curves on $\Sigma$ goes to the family of immersed curves of genus $g$ in the linear system $|C^*|$ (the upper asterisk denotes the strict transform) quadratically tangent to the last exceptional divisor at the intersection point $z^*$ of $C^*$ with $E^*$. We have by (8)

$$\deg(c_1(\tilde{N}_{\hat{C}}(-2z^*))) - (2g - 2) = -K_{\Sigma'}C^* - 2 = 3d - d_1 - ... - d_{a+1} - 2s - 2 = 1 > 0 ,$$

(here $\tilde{N}_{\hat{C}}$ is the normal bundle of $\tilde{n} : \hat{C} \to C^* \hookrightarrow \Sigma'$) which ensures (cf. [2 Page 357]) that the considered family has dimension

$$\leq \deg(c_1(\tilde{N}_{\hat{C}}(-2z^*))) - g + 1 = g ,$$

(9)
and we are done, provided (8) holds.

Suppose that

\[ 2d - d_2 - \ldots - d_{a+1} = (C \cdot E)_z = 2k, \quad 1 \leq k \leq s. \quad (10) \]

Assume that \( k < s \). Since \( d_1 = d - 2 \), we derive from Bézout that \( d_2, \ldots, d_{a+1} \leq 2 \). Denote by \( r_1 \) and \( r_2 \) the number of \( d_i, \quad 2 \leq i \leq a + 1 \), equal to 1 and 2, respectively. Blowing down the divisors \( E_i \) with \( d_i = 1 \) and ignoring the condition to pass through the obtained \( r \) points on (the image of) \( E \), we obtain a germ of the family \( V_1 \) of curves of genus \( g \) having \( r_2 \) double points and a singularity \( A_{2s} \) on \( E \) so that the intersection number with \( E \) at this singular point equals \( 2k \). Fixing the position \( z \) of \( A_{2s} \) and additionally the position of \( s - 1 \) infinitely near points obtained in \( s - 1 \) successive blow-ups of this singularity, we obtain a family \( V_2 \) of curves of dimension \( \dim V_2 \geq \dim V_1 - 1 + (s - k) \). Performing \( s \) blow-ups of the singularity \( A_{2s} \), we transform the family \( V_2 \) into the family \( V_3 \) of curves on a surface \( \Sigma' \) which are quadratically tangent to the last exceptional divisor in a neighborhood of \( z^* \) (the tangency point of \( C^* \)). In this situation, taking into account the genus bound (7) which yields \( r_2 + s \leq d - 2 \), we have

\[
\deg(c_1(\tilde{N}_C(-z^*))) - (2g - 2) = -K_{\Sigma'}C^* - 1 = 3d - (d - 2) - 2r_2 - 2s - 1 > 0,
\]

and hence

\[
\dim V_1 \leq \dim V_2 + (1 + s - k) = \dim V_3 + (1 + s - k) = \deg(c_1(\tilde{N}_C(-z^*))) - g + 1 + (1 + s - k) = 2d - 2r_2 - s - k + 1 + g.
\]

Restoring the condition to pass through \( r_1 \) additional points on \( E \) and using the fact that these points are in general position, we derive that the original family of curves has dimension at most

\[
2d - 2r_2 - s - k + 1 + g - r_1 = (2d - 2r_2 - r_1 - 2k) - s + k + 1 + g \overset{(10)}{=} -s + k + 1 + g \\
\leq g + 1 = R_\Sigma(D, g, \beta) - 1.
\]

**Step 5: Nonsingularity along \( E \).** From now on we suppose that \( n : \hat{C} \to \Sigma \) is an immersion. In what follows, we argue by contradiction. Namely, assuming that \( C \) is singular at a point of \( C \cap E \), we derive that necessarily \( \dim V(D, g, \beta) < R_\Sigma(D, g, \beta) \).

Suppose that \( n \) takes \( s \geq 2 \) points of the divisor \( d \) to the same point \( z \in E \). Fixing the position of \( z \) in \( E \), we obtain a subvariety \( U \subset V(D, g, \beta) \) of dimension \( \dim U \geq \)
\( \text{idim} \mathcal{V}(D, g, \beta) - 1 \). On the other hand, the same argument as in the proof of [2, Proposition 2.1] gives
\[
\text{idim} \ U \leq h^0(\hat{C}, \mathcal{N}_C(-d - d')),
\]
where \( d' = \sum_{i,j \geq 1} \hat{q}_{ij} \). Assuming that \( c_1(\mathcal{N}_\hat{C}(-d - d')) \otimes \omega_{\hat{C}}^{-1} \) is positive on \( \hat{C} \) and applying [2, Observation 2.5], we get (cf. [2, Page 364])
\[
\text{idim} \mathcal{V}(D, g, \beta) \leq \text{idim} \mathcal{U} + 1 \leq h^0(\hat{C}, \mathcal{N}_\hat{C}(-d - d')) + 1
\]
where \( d' = \sum_{i,j \geq 1} \hat{q}_{ij} \).

The positivity required above is equivalent to
\[
-DK_\Sigma - \deg(d + d') = d - d_1 + \|\beta\| - s > 0,
\]
that holds due to \( \|\beta\| \geq s, d > d_1 \).

**Step 6: Nodality for \( E^2 \geq -3 \) (equivalently \( a \leq 7 \)).** Suppose that some distinct points \( w_1, w_2, w_3 \) of \( \hat{C} \) are mapped to the same point \( z \in \Sigma \setminus E \). Fixing the position of this point, we obtain a subvariety \( V \subset \mathcal{V}(D, g, \beta) \) of dimension \( \text{idim} V \geq \text{idim} \mathcal{V}(D, g, \beta) - 2 \). Thus, provided that \( c_1(\mathcal{N}_\hat{C}(-d - d')) \otimes \omega_{\hat{C}}^{-1} \) is positive on \( \hat{C} \), we get
\[
\text{idim} \mathcal{V}(D, g, \beta) \leq \text{idim} V + 2 \leq \text{idim} \mathcal{V}(D, g, \beta) - g + 3
\]
\[
-DK_\Sigma + 2g - 2 - \deg(d + d') - g + 2 = -D(K_\Sigma + E) + g + \|\beta\| - 1 = R_\Sigma(D, g, \beta),
\]
where \( d' = w_1 + w_2 + w_3 \). The required positivity amounts to \( -DK_\Sigma - \deg(d + d') = d - d_1 + \|\beta\| - 3 > 0 \). The only case, when the latter relation fails is \( \beta = 0 \) (that is \( 2d = d_2 + ... + d_{a+1} \) and \( d_1 = d - 3 \)). In such a case, the genus bound leads to
\[
\frac{(d - 1)(d - 2)}{2} \geq \frac{(d - 3)(d - 3 - 1)}{2} + \sum_{i=2}^{a+1} \frac{d_i(d_i - 1)}{2} + 3 + g
\]
\[
\implies d^2 - 3d + 2 \geq 2d \left( \frac{2d}{a} - 1 \right) + d^2 - 7a + 18
\]
\[
\implies 2d^2 - 2ad + 8a \leq 0,
\]
and the latter never holds as \( a \leq 7 \), since the discriminant is \( 9a^2 - 64 < 0 \).
Suppose that \( n^{-1}(z) = w_1 + w_2, w_1 \neq w_2 \in \hat{C} \) for some point \( z \in \mathbb{P}_{a,1}^2 \setminus E \), and the two local branches of \( C = n(\hat{C}) \) at \( z \) intersect with multiplicity \( s \geq 2 \) (singularity \( A_{2s-1} \)). In suitable coordinates in a neighborhood of \( z \), \( C \) is given by an equation \( y^2 + 2yx^s = 0 \). The tangent space to the local equisingular stratum in \( \mathcal{O}_{\Sigma, z} \) is the ideal \( I = \langle y + x^s, x^{s-1}y \rangle \), and it does not contain \( y \). If \( C \cap E \neq \emptyset \), we have the inequality \( d - d_1 + \|\beta\| > 2 \), which comes from \( \|\beta\| > 0 \) and the Bézout’s bound \( d \geq d_1 + 2 \) for the intersection of \( C \) with the line \( l \in |L - E_1| \) through \( z \), and this inequality can be rewritten as \( \deg(\mathcal{N}_{\hat{C}}(-d - w_1 - w_2)) > 2g - 2 \). Hence \( H^1(\hat{C}, \mathcal{N}_{\hat{C}}(-d - w_1 - w_2)) = 0 \). The latter relation immediately implies that there is a section of the bundle \( \mathcal{N}_{\hat{C}}(-d) \) which vanishes at \( w_1 \) and does not vanish at \( w_2 \). Moving \( n : \hat{C} \to \Sigma \) inside \( \mathcal{V}(D, g, \beta) \) along this section, we realize a (one-parameter) deformation of the local equation \( y^2 + 2yx^s = 0 \) of \( C \), tangent to the line spanned by the element \( y \in \mathcal{O}_{\mathbb{P}_{a,1}^2, z} \), and hence breaking the equisingularity. This contradiction proves that a generic \( C = n(\hat{C}) \) cannot have tangent local branches. The remaining option is \( C \cap E = \emptyset \). Then \( \|\beta\| = 0 \), and either \( d - d_1 > 2 \), which as above allows one to show that \( C \) cannot have tangent local branches outside \( E \), or \( d_1 = d - 2 \). In the latter case we have
\[
d_2 + \ldots + d_{a+1} = 2d, \quad d_2, \ldots, d_{a+1} \leq 2.
\]
Put \( r_j = \#\{i \in [2, a + 1] : d_i = j\}, \ j = 1, 2 \). Then \( r_1 + 2r_2 = 2d \). The genus bound
\[
\frac{(d - 1)(d - 2)}{2} \geq \frac{(d - 2)(d - 3)}{2} + r_2 + s
\]
yields
\[
r_1 \geq 4 + 2s.
\]
Blowing down the divisors \( E_i \) for \( d_i = 1 \) and ignoring the condition to pass through \( r_1 \) fixed points, we get to the case \( \|\beta''\| = r_1 > 0 \) in which we have derived that the family of curves with a singularity \( A_{2s-1} \) has dimension (here \( \Sigma' \) is the blown-up \( \Sigma \))
\[
< R_{\Sigma'}(D', g, \beta') = d - d_1 + g + \|\beta''\| - 1 = g + r_1 + 1.
\]
Restoring the condition to pass through \( r_1 \) generic points we obtain a family of dimension
\[
< g + 1 = d - d_1 + g - 1 = R_{\Sigma}(D, g, 0)
\]
which proves that a generic element of the original family cannot have singularity \( A_{2s-1} \), \( s \geq 2 \).
Step 7: Absence of base points. Assuming that \( R_\Sigma(D, g, \beta) > 0 \) and \( V_\Sigma(D, g, \alpha, \beta, p) \) has a base point \( z \in \Sigma \setminus p \), we obtain

\[
idim V_\Sigma(D, g, \alpha, \beta, p) \leq h^0(\hat{C}, N_\hat{C}(-d - w)),
\]

where \( n(w) = z \). To arrive to a contradiction, we use the inequality

\[
idim V_\Sigma(D, g, \alpha, \beta, p) \leq h^0(\hat{C}, N_\hat{C}(-d - d'))
\]

\[
= -DK_\Sigma + 2g - 2 - \deg(d + d') - g + 1
\]

\[
= -D(K_\Sigma + E) + g + \|\beta\| - 2 < R_\Sigma(D, g, \beta).
\]

This relation is based on the positivity of the bundle \( c_1(N_\hat{C}(-d - w)) \otimes \omega_{\hat{C}}^{-1} \) on \( \hat{C} \), which reduces to the inequality \( d - d_1 + \|\beta\| > 1 \). The only case when it does not hold is \( d_1 = d - 1, \|\beta\| = 0 \), but then \( g = 0 \) and \( R(D, 0, 0) = d - d_1 - 1 = 0 \), contrary to the assumptions made.

The same computation with \( w \in \hat{C} \) sent to a tangency point of \( C \setminus p \) and \( C' \) proves that the intersection \((C \setminus p) \cap C'\) is transversal.

Step 8: Smoothness of the family of curves. The smoothness of \( V_\Sigma(D, g, \alpha, \beta, p) \) at \( C \) is equivalent to

\[
H^1(\hat{C}, N_\hat{C}(-d)) = 0 \iff d - d_1 + \|\beta\| > 0,
\]

which clearly holds in the considered situation.

Step 9: Multiple covers. Assume that \( n : \hat{C} \to C = n(\hat{C}) \) is an \( s \)-fold covering, \( s \geq 2 \), for a generic element \( \{n : \hat{C} \to \Sigma, \hat{p}\} \in \mathcal{V} - \Sigma(D, g, \alpha, \beta, p) \). Then \( D = sD_0, 2 - 2g = s(2 - 2g') - r \), where \( g' \) is the geometric genus of \( C \), \( r \) is the total ramification multiplicity. Using (1) for the normalization map \( n' : C^n \to C \leftarrow \Sigma \), we get (cf. [2 Page 366] and [13 Page 62])

\[
idim V_{\Sigma, n}(D, g, \alpha, \beta, p) \leq -(K_\Sigma + E)D_0 + g' - 1 + \#(C \cap E \setminus p)
\]

\[
\leq -\frac{(K_\Sigma + E)D}{s} + \frac{g - 1 - r/2}{s} + n \leq -(K_\Sigma + E)D + g - 1 + n, \quad (13)
\]

the latter inequality coming from the nefness of \(- (K_\Sigma + E)\). Thus, we have proven both (3) and (4).

Now, the equality \( \text{idim } V_\Sigma(D, g, \alpha, \beta, p) = R(D, g, \beta) \), means the equality in (13) with \( n = \|\beta\| \), and this leaves the only possibility \( (K_\Sigma + E)D_0 = 0 \), equivalent to \( d = d_1 \), and hence either \( D_0 = E_i, 2 \leq i \leq a + 1 \), or \( D_0 = L - E_{a + 1} \). If \( D_0 = E_i, 2 \leq i \leq a + 1 \), then \( R_\Sigma(D, g, \beta) = 0 \). If \( D_0 = L - E_1 \) and \( R_\Sigma(D, g, \beta) > 0 \), then \( C = n(\hat{C}) \) is a generic line in
the one-dimensional linear system $L - E_i$, which, of course, crosses transversally any a priori given curve $C'$.

The proof of Proposition 2.1 is completed.

**Proposition 2.2** Let $D \in \text{Pic}_+(\Sigma, E)$, and let $V$ be a component of a nonempty family $\mathcal{V}_\Sigma(D, g, \alpha, \beta, p)$ such that, for a generic element $\{n : \hat{C} \to \Sigma, \hat{p}\} \in V$, $n : \hat{C} \to C = n(\hat{C})$ is an $s$-multiple cover, $s \geq 2$. Then

(i) either $C \subset \Sigma$ is a smooth rational $(-1)$-curve crossing $E$, $\alpha = 0$, $p = \emptyset$, $\beta = e_s$, $R_\Sigma(sC, 0, e_s) = 0$, and $n : \hat{C} \to C$ has at least two critical points, one of which is the preimage of $C \cap E$ and has ramification index $s$;

(ii) or $D = -s(K_\Sigma + E)$, $C \in |-(K_\Sigma + E)|$ is either $L'$, or $L''$, $g = 0$, $\alpha = 0$, $p = \emptyset$, $\beta = e_{2s}$, $R_\Sigma(D, 0, e_{2s}) = 0$, and $n : \hat{C} \to C$ has at least two critical points, one of which is the preimage of $C \cap E$ and has ramification index $s$;

(iii) or $D = -s(K_\Sigma + E)$, $C \in |-(K_\Sigma + E)|$ is a line crossing $E$ transversally at two points, $g = 0$, $\alpha = e_s$, $p$ is one of the points of $C \cap E$, $\beta = e_s$, $R_\Sigma(D, 0, e_s) = 0$, and $n : \hat{C} \to C$ has two critical points projected onto $C \cap E$;

(iv) or $D = -s(K_\Sigma + E)$, $C \in |-(K_\Sigma + E)|$ is a line crossing $E$ transversally at two points, $g = 0$, $\alpha = 0$, $p = \emptyset$, $\beta = 2e_s$, $R_\Sigma(D, 0, 2e_s) = 1$, and $n : \hat{C} \to C$ has two critical points of ramification multiplicity $s$ projected onto $C \cap E$.

**Proof.** The statement is straightforward from the consideration in Step 9 of the proof of Proposition 2.1.

**Proposition 2.3** Let $D \in \text{Pic}_+(\Sigma, E)$, $g \geq 0$, and $\alpha, \beta \in \mathbb{Z}_+^\infty$ satisfy (1), and let $p = (p_{ij})_{i \geq 1, 1 \leq j \leq a}$ be a sequence of generic distinct points of $E$. Assume that $R_\Sigma(D, g, \beta) = 0$, and that $\mathcal{V}_\Sigma(D, g, \alpha, \beta, p)$ contains an element $\{n : \hat{C} \to \Sigma, \hat{p}\}$ with $n$ birational onto its image. Then

(i) either $D$ is a class of a rational $(-1)$-curve $(E_i$ or $L - E_i - E_i$, $i = 2, \ldots, a + 1$, in the planar conic-model), $g = 0$, $\beta = e_1$, and $\# \mathcal{V}_\Sigma(D, 0, 0, e_1, \emptyset) = 1$;

(ii) or $D = -(K_\Sigma + E)$ and $g = 0$, $\beta = e_2$, and $\# \mathcal{V}(D, 0, 0, e_2, \emptyset) = 2$;
(iii) or $D = -(K_\Sigma + E)$, $g = 0$, $\alpha = \beta = e_1$, and $\#V(D, 0, e_1, e_1, p) = 1$;

(iv) or $-D(K_\Sigma + E) = 1$, $g = 0$, $I\alpha = DE > 0$, and $\#V(\Sigma, 0, \alpha, 0, p) = 1$.

Proof. Straightforward from the formula $R(D, g, \beta) = -D(K_\Sigma + E) + g + \|\beta\| - 1 = 0$ and the nefness of $-(K_\Sigma + E)$.

2.3 Enumerative numbers

Definition 2.4 Given a divisor class $D \in \text{Pic}(\Sigma, E)$, an integer $g \geq 0$, vectors $\alpha, \beta \in \mathbb{Z}^\infty_+$ satisfying (3), and a sequence $p = (p_{ij})_{i \geq 1, 1 \leq j \leq \alpha}$ of $\|\alpha\|$ generic points of $E$, we define

$$N_{\Sigma}(D, g, \alpha, \beta) = \begin{cases} 0, & \text{if } V_{\Sigma}(D, g, \alpha, \beta, p) = \emptyset, \\ \deg V_{\Sigma}(D, g, \alpha, \beta, p), & \text{if } V_{\Sigma}(D, g, \alpha, \beta, p) \neq \emptyset, \end{cases}$$

($V_{\Sigma}(D, g, \alpha, \beta, p)$ introduced in Proposition 2.1(2ii)).

This number, of course, does not depend on the choice of $p$ and it counts the curves in $V_{\Sigma}(D, g, \alpha, \beta, p)$ matching a generic configuration of $R_{\Sigma}(D, g, \beta)$ points in $\Sigma \setminus E$. In Theorem 2.1, section 2.6, we present a recursive formula for these numbers. The formula allows one to compute all these numbers starting with the following initial values:

Proposition 2.5 Let $D \in \text{Pic}_+(\Sigma, E)$, $g \geq 0$, and $\alpha, \beta \in \mathbb{Z}^\infty_+$ satisfy (3). Then

(1) $N(\Sigma, -s(K_\Sigma + E), 0, 0, 2e_s) = 1$ for all $s \geq 1$;

(2) If $R_{\Sigma}(D, g, \beta) = 0$, then $N_{\Sigma}(D, g, \alpha, \beta) = 0$ except for the following cases:

(i) $N_{\Sigma}(sD_0, 0, 0, e_s) = 1$ for any divisor class $D$ of a smooth $(-1)$-curve crossing $E$ and for any $s \geq 1$;

(ii) $N_{\Sigma}(-s(K_\Sigma + E), 0, 0, e_{2s}) = 2$ for all $s \geq 1$;

(iii) $N_{\Sigma}(-s(K_\Sigma + E), 0, e_s, e_s) = 1$ for all $s \geq 1$;

(iv) $N_{\Sigma}(D, 0, \alpha, 0) = 1$ for all divisor classes $D$ with $-D(K_\Sigma + E) = 1$, $I\alpha = DE$.

Proof. Straightforward from Propositions 2.2 and 2.3.
2.4 Degeneration

Let $D \in \text{Pic}(\Sigma, E)$, $g \geq 0$, and $\alpha, \beta \in \mathbb{Z}_+^\infty$ satisfy relations (\[1\]) and the inequality $R_\Sigma(D, g, \beta) > 0$. Let $p = (p_{ij})_{i \geq 1, 1 \leq j \leq \alpha_i}$ be a sequence of $\| \alpha \|$ generic points of $E$, and let $p$ be a generic point in $E \setminus p$. Assume that $V_\Sigma(D, g, \alpha, \beta, p) \neq \emptyset$ and introduce

$$V^p_\Sigma(D, g, \alpha, \beta, p) = \{ \{ n : \hat{\Sigma} \to \Sigma, \hat{p} \} \in V(D, g, \alpha, \beta, p) : p \in n(\hat{C}) \}.$$ 

Proposition 2.6 Let $V$ be a component of $V_\Sigma(D, g, \alpha, \beta, p)$ of dimension $R_\Sigma(D, g, \beta)$, and let $W$ be a component of $V \cap V^p_\Sigma(D, g, \alpha, \beta, p)$ of dimension $R_\Sigma(D, g, \beta) - 1$. Then a generic element $\{ n : \hat{C} \to \Sigma, \hat{p} \}$ of $W$ is as follows.

1. Assume that $n(\hat{C})$ does not contain $E$. Then $\hat{C}$ is smooth, and there is $k \geq 1$ such that $W$ is a component of $V_\Sigma(D, g, \alpha + e_k, \beta - e_k, p) \cup \{ p_{k, \alpha_k + 1} = p \}$.

2. Assume that $n(\hat{C})$ contains $E$. Then

(i) $\hat{C} = \hat{E} \cup X \cup Y \cup Z$, where $n : \hat{E} \to E$ is an isomorphism, $X$ is the union of components mapped by $n$ to curves positively intersecting with $E$ and not belonging to the linear system $| -(K_\Sigma + E)|$, $Y$ is the union of components mapped by $n$ to lines belonging to the linear system $| -(K_\Sigma + E)|$, $Z$ is the union of components contracted by $n$ to points;

(ii) $X = \bigcup_{1 \leq i \leq k} \hat{C}^{(i)}$, $Y = \bigcup_{k < i \leq m} \hat{C}^{(i)}$, where $0 \leq k \leq m$, and $\hat{C}^{(1)}, \ldots, \hat{C}^{(m)}$ are disjoint smooth curves;

(iii) for each $i = 1, \ldots, m$, the map $n : \hat{C}^{(i)} \to \Sigma$ represents a generic element in a component of some family $V_\Sigma(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}, \mathbf{p}^{(i)})$ of intersection dimension $R_\Sigma(D^{(i)}, g^{(i)}, \beta^{(i)})$ and such that

- $\sum_{i=1}^m D^{(i)} = D - E$,
- $\sum_{i=1}^m R_\Sigma(D^{(i)}, g^{(i)}, \beta^{(i)}) = R_\Sigma(D, g, \beta) - 1$,
- $\mathbf{p}^{(i)}$, $i = 1, \ldots, m$, are disjoint subsets of $\mathbf{p}$,
- $n : \hat{C}^{(i)} \to \Sigma$ is a birational immersion onto the image for each $i = 1, \ldots, k$; the images $n(\hat{C}^{(i)})$, $i = 1, \ldots, m$, are nonsingular along $E$, intersect each other transversally and only at their nonsingular points lying outside $E$;
- any quadruple $(D', g', 0, \beta')$ with $R_\Sigma(D', g', \beta') = 0$ appears in the list

$$(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}), \quad 1 \leq i \leq k,$$

at most once;
(iv) if, for some \( i = k + 1, \ldots, m \), \( n(\hat{C}^{(i)}) = L' \) (or \( L'' \)), then \( n : \hat{C}^{(i)} \to L' \) (resp. \( L'' \)) is an isomorphism.

Proof of Proposition 2.6(1,2i,2ii,2iii). We follow the lines of [2, Section 3], where a similar statement ([2, Theorem 1.2]) is proven for the planar case, and also use some arguments from the proof of [13, Theorem 5.1]. Again we do not copy all the details, but explain the main steps and perform all necessary computations, which, in fact, are consequences of the bounds (3) and (4), and of the relation \((K_\Sigma + E)E = -2\). The truly new claim, which we provide with a complete proof in the very end, is that, in the case (2), \( n_*\hat{C} \) does not contain multiple \((-1)\)-curves.

Consider a generic one-parameter family in the component \( V \) of \( V_\Sigma(D, g, \alpha, \beta, \hat{p}) \) with the central fibre belonging to \( W \). Using the construction of [2, Section 3], one can replace the given family with a family \( \{ n_t : \hat{C}_t \to \Sigma, \hat{p}_t \}_{t \in (\mathbb{C}, 0)} \) having the same generic fibres and a semistable central fibre such that (cf. conditions (b), (c), (e) in [2, Section 3.1]):

- the family is represented by a surface \( C \) with at most isolated singularities and two morphisms \( \pi_\Sigma : C \to \Sigma, \pi_\Sigma : C \to (\mathbb{C}, 0) \), so that for each \( t \neq 0 \), \( \pi_\Sigma : C_t \overset{\text{def}}{=} \pi_\Sigma^{-1}(t) \to \Sigma \) is isomorphic to \( n_t : \hat{C}_t \to \Sigma \),
- the fibre \( \hat{C}_0 = C_0 \) is a nodal curve,
- the family is minimal with respect to the above properties.

Notice that we have disjoint sections

\[ t \in (\mathbb{C}, 0) \setminus \{0\} \mapsto \hat{p}_{ij,t}, i \geq 1, 1 \leq j \leq \alpha_i, \quad t \in (\mathbb{C}, 0) \setminus \{0\} \mapsto \hat{q}_{ij,t}, i \geq 1, 1 \leq j \leq \beta_i, \quad (14) \]

defined by [2] for each fibre:

\[ n_t^*(E \cap n_t(\hat{C}_t)) = \sum_{i \geq 1, 1 \leq j \leq \alpha_i} i \cdot \hat{p}_{ij,t} + \sum_{i \geq 1, 1 \leq j \leq \beta_i} i \cdot \hat{q}_{ij,t}, \quad (15) \]

\[ n_t(\hat{p}_{ij,t}) = p_{ij} \in p, \ i \geq 1, 1 \leq j \leq \alpha_i. \]

They close up at \( t = 0 \) into some global sections, for which \( \hat{p}_{ij,0}, i,j \geq 1 \), remain disjoint.

In the case (1), if \( \hat{C}_0 \) splits into components \( \hat{C}^{(1)}, \ldots, \hat{C}^{(m)}, m \geq 1 \), mapped to curves and some components mapped to points, then the maps \( n_0 : \hat{C}^{(i)} \to \Sigma \) represent elements in some \( V_\Sigma(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}, \hat{p}^{(i)}) \), \( i = 1, \ldots, m \), for which we have

\[ \sum_{i=1}^{m} D^{(i)} = D, \quad \sum_{i=1}^{m} \| \beta^{(i)} \| \leq \| \beta \| - 1, \quad \sum_{i=1}^{m} (g^{(i)} - 1) \leq g - m. \]
(since some section $g_{kj,t}$ closes up at $p$, and at least $m - 1$ intersection points are smoothed out when deforming $\hat{C}_0$ into $\hat{C}_t$, $t \neq 0$). Hence (cf. [13, Page 66])

$$\sum_{i=1}^{m} \text{idim } \nu_{\Sigma}(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}, \hat{p}^{(i)}) = \sum_{i=1}^{m} (- (K_{\Sigma} + E) D^{(i)} + g^{(i)} + \|\beta^{(i)}\| - 1) \leq -(K_{\Sigma} + E) D + d + \|\beta\| - 1 - m = R_{\Sigma}(D, g, \beta) - m,$$

which implies $m = 1$ and thus, in view of Proposition 2.1, the statement (1).

In the case (2), assume that $\hat{C}_0 = \hat{E} \cup \hat{C}^{(1)} \cup \ldots \cup \hat{C}^{(m)} \cup Z$, where $n_0(\hat{E}) = E$, the components $\hat{C}^{(i)}$, $1 \leq i \leq m$, are mapped by $n_0$ to curves, and $n_0(Z)$ is finite. We have:

- $\hat{E}$ splits into components $\hat{E}_1, \ldots, \hat{E}_a$ such that $(n_0)_* \hat{E}_i = s_i E$, $s_i \geq 1$, $i = 1, \ldots, a$;
- $\{n_0 : \hat{C}^{(i)} \to \Sigma, \hat{p}^{(i)}\} \subset \nu_{\Sigma}(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}, \hat{p}^{(i)})$, where $\hat{p}^{(i)} = n_0(\hat{p}^{(i)}) \subset p$, and $\hat{p}^{(i)} = (\hat{p}_{kj,0} \in \hat{C}^{(i)})_{k,j \geq 1}$; furthermore, $\beta^{(i)} = \tilde{\gamma}^{(i)} + \gamma^{(i)}$, where $\gamma^{(i)}$ labels the multiplicities of the sections $\hat{g}_{kj,t}$ which land up on $\hat{C}^{(i)}$ for $t = 0$, $i = 1, \ldots, m$;
- denote also by $g'$ the sum of the geometric genera of all the algebraic components of $Z$, and by $n'$ the number of those connected components $Z'$ of $Z$, for which $\#Z' \cap \text{Sing}(\hat{C}_0)$ is greater than the number of algebraic components of $Z'$ plus $\#\{\hat{z} \in Z' \cap \bigcup_i \hat{C}^{(i)} : n_0(\hat{z}) \in E\}$.

By construction, the local branches $n_0 : (\hat{C}^{(i)}, \hat{p}_{kj,0}) \to \Sigma$ and $n_0 : (\hat{C}^{(i)}, \hat{q}_{kj,0}) \to \Sigma$ are deformed continuously into respective branches $n_t : (\hat{C}^{(i)}, \hat{p}_{kj,t}) \to \Sigma$ and $n_t : (\hat{C}^{(i)}, \hat{q}_{kj,t}) \to \Sigma$, $t \neq 0$. In turn, the other local branches $n_0 : (\hat{C}^{(i)}, \hat{z}) \to \Sigma$ with $n_0(\hat{z}) = z \in E$ are deformed so that they become disjoint from $E$ as $t \neq 0$; hence such a point $\hat{z} \in \hat{C}^{(i)}$ must be either an intersection point of $\hat{C}^{(i)}$ with $\hat{E}$, or an intersection point of $\hat{C}^{(i)}$ with a connected component of $Z$ which joins $\hat{C}^{(i)}$ with $\hat{E}$, and, in the deformation $(n_t : \hat{C}^{(i)} \to \Sigma)_{t \in (\Sigma, 0)}$, this intersection point is smoothed out. Moreover, all nodes of $\hat{C}_0$ are smoothed out in the deformation to $\hat{C}_t$, $t \neq 0$. Thus,

$$\sum_{i=1}^{a} (g(\hat{E}_i) - 1) + \sum_{i=1}^{m} (g^{(i)} - 1) + g' + n + n' + \sum_{i=1}^{m} \|\tilde{\gamma}_i\| \leq g - 1,$$

where $n$ is the number of nodes of $\hat{C}^{(1)}, \ldots, \hat{C}^{(m)}$. Applying (4), we get

$$\text{idim } W = R_{\Sigma}(D, g, \beta) - 1 = -(K_{\Sigma} + E) D + g - 1 + \|\beta\| - 1.$$
\[
\leq \sum_{i=1}^{m} \left( -(K_{\Sigma} + E)D^{(i)} + g^{(i)} - 1 + \#(n_0(\hat{C}^{(i)}) \cap E \setminus p) \right)
\]
\[
= -(K_{\Sigma} + E)(D - \sum_{i=1}^{a} s_iE) + \sum_{i=1}^{m} (g^{(i)} - 1) + \sum_{i=1}^{m} \|\beta^{(i)}\|
\]
\[
\leq -(K_{\Sigma} + E)D + g - 1 - \sum_{i=1}^{a} (2s_i - 1 + g(\hat{E}_i)) + \sum_{i=1}^{m} \|\gamma^{(i)}\| - n - n' - g'. \quad (16)
\]

Thus, we immediately obtain that

- \(a = 1, s_1 = 1\), and \(g(\hat{E}_1) = 0\), \text{i.e.} \(n_0 : \hat{E} \to E\) is an isomorphism;

- \(\sum_{i=1}^{m} \tilde{\gamma}^{(i)} = \beta\), \text{i.e.} all the points \(\hat{q}_{ij,0}\) belong to \(\hat{C}^{(1)} \cup \ldots \cup \hat{C}^{(m)}\);

- for each \(i = 1, \ldots, m\), \(\{n_0 : \hat{C}^{(i)} \to \Sigma, \hat{p}^{(i)}\}\) is a generic element of a component of \(\Sigma \times (\hat{C}^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}, \hat{p}^{(i)})\) of intersection dimension \(R_{\Sigma}(D^{(i)}, g^{(i)}, \beta^{(i)})\); in particular, all the sections \(t \in (\Sigma, 0) \mapsto \hat{p}_{kj,t}\) and \(t \in (\Sigma, 0) \mapsto \hat{q}_{kj,t}, k, j \geq 1\), are pairwise disjoint (cf. condition (d) in [2, Section 3.1]);

- \(n = 0\), \text{i.e.} \(\hat{C}^{(1)}, \ldots, \hat{C}^{(m)}\) are disjoint.

- \(\|\gamma^{(i)}\| > 0\) for each \(i = 1, \ldots, m\);

- \(g' = 0\), \text{i.e.} all the algebraic components of \(Z\) are rational, and \(n' = 0\), \text{i.e.} the connected components of \(Z\) are chains of rational curves, which either join the points \(\{\hat{z} \in \bigcup_{i} \hat{C}^{(i)} : n_0(\hat{z}) \in E, \hat{z} \neq \hat{p}_{kj,0}, \hat{q}_{kj,0}, k, j \geq 1\}\) with \(\hat{E}\) (cf. Assumption (b) in [2, Section 3.2]), or are attached by precisely one point to \(\hat{E}\) or to \(\bigcup_{i} \hat{C}^{(i)}\); however, the latter type connected components should not exist due to the minimality of \(\mathcal{C}\).

In view of Propositions \([2.1, 2.2]\) and \([2.3]\), it remains to prove that the points \(\hat{p}_{kj,0}\), which do not belong to \(\hat{C}^{(1)} \cup \ldots \cup \hat{C}^{(m)}\), lie on \(\hat{E}\). Indeed, if some point \(\hat{p}_{kj,0}\) were in \(Z \setminus (\hat{E} \cup \bigcup_{i} \hat{C}^{(i)})\), then some point \(\hat{z} \in \bigcup_{i} \hat{C}^{(i)} \setminus \hat{p}_0\) would have been mapped to \(p_{kj}\), which would have resulted in reducing 1 in the third and fourth line of bounds (16), thus, a contradiction.

The final step is to confirm that \(C = (n_0)_* \hat{C}_0\) does not contain multiple \((-1)\)-curves. Let, for instance, \(E_{a+1}\) have multiplicity \(s \geq 2\) in \(C\), and (in the above notations) let \(\hat{C}^{(k+1)}, \ldots, \hat{C}^{(m)}\) be all the components of \(\hat{C}_0\) mapped onto \(E_{a+1}\). Since \(R_{\Sigma}(D, g, \beta) > 0\), we have \(D = dL - d_1E_1 - \ldots - d_{a+1}E_{a+1}, d \geq 1, d_2, \ldots, d_{a+1} \geq 0\). Thus, \(C' = (n_0)_* (\hat{C}^{(1)} \cup \ldots \cup \hat{C}^{(k)})\) belongs to the linear system \(\lfloor (d-2)L-d_1E_1-(d_2-1)E_2-\ldots-(d_{a}-1)E_a-(d_{a+1}+s-1)E_{a+1} \rfloor\).
That is, $C'$ crosses $E_{a+1} \setminus E$ with multiplicity $d_{a+1} + s - 1$, and as explained above these intersection points persist in the deformation $(n_t : \hat{C}_t \to \Sigma)_{t \in (\mathbb{C}, 0)}$. Hence, $(n_t, \hat{C}_t)$ must cross $E_{a+1}$ with multiplicity $\geq d_{a+1} + s - 1 > d_{a+1}$, thus, a contradiction.

The proof of Proposition 2.6(2iv) will be given in part (6) of section 2.5. In turn the statement of Proposition 2.6(2iv) will not be used before that.

**Lemma 2.7** Pick a set $p$ of $n - 1$ generic points of $\Sigma \setminus E$. Let $\{ n : \hat{C} \to \Sigma, \hat{p} \} \in V(\Sigma, g, \alpha, \beta, p)$ and $C := n_* \hat{C}$ pass through $p$. Then $\{ n : \hat{C} \to \Sigma, \hat{p} \}$ satisfy the conditions of Proposition 2.6(2) and it can be uniquely restored from $C$. Furthermore, $C$ splits into distinct irreducible components in the following way:

\begin{equation}
C = E \cup \bigcup_{i=1}^{m} C(i) \cup s'L' \cup s''L'' \,,
\end{equation}

(i) $C^{(1)}, \ldots, C^{(m)}$ do not contain neither $L'$, nor $L''$, and, furthermore, each $C^{(i)}$ either is a reduced, irreducible curve, or is $kL(p_{kl})$, where $k \geq 2$, $p_{kl} \in p$, and the (uniquely defined) line $L(p_{kl}) \in |-(K_{\Sigma} + E)|$ contains $p_{kl}$, or is $kL(z)$, where $k \geq 2$, $z \in \overline{p}$, and the (uniquely defined) line $L(z) \in |-(K_{\Sigma} + E)|$ contains $z$,

(ii) the curve $C_{\text{red}}$ is nonsingular along $E$ and is immersed; furthermore, it is nodal as $E^2 \geq -3$.

In addition,

(iii) for each $i = 1, \ldots, m$, $C^{(i)}$ is a generic element in some $V(\Sigma, D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}, p^{(i)})$,

(iv) $\sum_{i=1}^{m} D^{(i)} = D - E + (s' + s'')(K_{\Sigma} + E)$,

(v) $\sum_{i=1}^{m} R(\Sigma, D^{(i)}, g^{(i)}, \beta^{(i)}) = R(\Sigma, D, g, \beta) - 1$,

(vi) $p^{(i)}$, $i = 1, \ldots, m$, are disjoint subsets of $p$;

(vii) $\overline{p}^{(i)} := \overline{p} \cap C^{(i)}$, $i = 1, \ldots, m$, form a partition of $\overline{p}$;

(viii) there is a sequence of vectors $\gamma^{(i)} \in \mathbb{Z}_{>0}$, $i = 1, \ldots, m$ such that

\begin{equation}
0 < \gamma^{(i)} \leq \beta^{(i)} \quad \text{and} \quad \beta = \sum_{i=1}^{m} (\beta^{(i)} - \gamma^{(i)}) \,. \tag{18}
\end{equation}

**Proof.** Straightforward from Proposition 2.6. $\square$

---

2The coefficients $s'$, $s''$ in the formula denote the multiplicities of $L', L''$ in $C$, respectively.
2.5 Deformation

2.5.1 Multiplicities

Having generic elements of $\mathcal{V}_G^p(D, g, \alpha, \beta, p)$ as described in Proposition 2.6, we intend to deform them into generic elements of $\mathcal{V}_G(D, g, \alpha, \beta, p)$ and compute the following multiplicities.

Let $D \in \text{Pic}_+(\Sigma, E)$, a non-negative integer $g$, and vectors $\alpha, \beta \in \mathbb{Z}_\Sigma^+$ satisfy (1). Pick a sequence $p = \{p_{ij}\}_{i \geq 1, 1 \leq j \leq \alpha_i}$ of generic distinct points on $E$. Assume, in addition, that $n := R_\Sigma(D, g, \beta) > 0$ and $D \neq -s(K_\Sigma + E), s \geq 1$.

Then, by Proposition 2.2, the components of $\mathcal{V}_G^p(D, g, \alpha, \beta, p)$ of intersection dimension $R_\Sigma(D, g, \beta)$ have the genuine dimension $R_\Sigma(D, g, \beta)$, and their union birationally projects by (5) onto its image $\mathcal{V}_G(D, g, \alpha, \beta, p)$ in $|D|$. Pick a set $\mathbf{p}$ of $n - 1$ generic points of $\Sigma \setminus E$. Then

$$V_G(D, g, \alpha, \beta, p, \mathbf{p}) \overset{\text{def}}{=} \{C \in V_G(D, g, \alpha, \beta, p) : C \supset \mathbf{p}\}$$

is one-dimensional (or empty).

Let $p \in E \setminus \mathbf{p}$ be a generic point, and let $V$ be a component of $\mathcal{V}_G(D, g, \alpha, \beta, p)$ of (intersection) dimension $R_\Sigma(D, g, \beta)$ of $\mathbf{p}$ if $C \in V_G(D, g, \alpha, \beta, p, \mathbf{p})$. By construction, $\{n : C \in \mathcal{V}_G(D, g, \alpha, \beta, p, \mathbf{p})\}$ is a generic element of $V$; hence by Propositions 2.2 and 2.6, $C$ is nonsingular at $p$. Take a smooth curve germ $L_p$ transversally crossing $E$ and $C$ at $p$. Thus, we have a well defined map $\varphi_C : V_G(p, C) \rightarrow L_p$. The multiplicities we are interested in are the degrees $\deg \varphi_C$ (equal 0 if $V_G(p, C) = \emptyset$).

**Proposition 2.8** In the above notations and assumptions of Section 2.5, the following holds.

1. Let $\beta_k > 0$ and $C \in V_G(D, g, \alpha + e_k, \beta - e_k, p \cup \{p = p_k, \alpha_k + 1\})$. Then $\deg \varphi_C = k$.
2. Let $C \in |D|$ be given by (17) and satisfy the conditions of Lemma 2.7. Then

$$\deg \varphi_C = (s' + 1)(s'' + 1) \sum_{\gamma^{(i)}} \prod_{i \in S} I_i^{(o)} \prod_{i=1}^{m} \left(\frac{\beta^{(i)}}{\gamma^{(i)}}\right),$$

where the sum runs over all sequences $\gamma^{(i)} \in \mathbb{Z}_+^\infty, i = 1, \ldots, m$, subject to (18), and

$$S = \{i \in [1, m] : C^{(i)} \neq kL(p_{kl}), k \geq 2\}.$$
2.5.2 Deformation of an irreducible curve

Here we prove Proposition 2.8(1).

In view of (19), \( C \) possesses the properties listed in Proposition 2.1(2). Furthermore, the germ of \( V_\Sigma(D, g, \alpha + e_k, \beta - e_k, p \cup \{p = p_k, \alpha_k + 1\}) \) at \( C \) is smooth by Proposition 2.1(2ii). Excluding \( p \) from the set of fixed points, we deduce that \( V_\Sigma(D, g, \alpha, \beta, p) \) contains \( C \) and is smooth at \( C \) as well. In particular,

\[
h^0(\hat{C}, N_{\hat{C}}(-d - p)) = n - 1 \quad \text{and} \quad h^0(\hat{C}, N_{\hat{C}}(-d)) = n.
\]

In view of the generic choice of \( \bar{p} \), we obtain

\[
h^0(\hat{C}, N_{\hat{C}}(-d - p - \bar{p})) = 0 \quad \text{and} \quad h^0(\hat{C}, N_{\hat{C}}(-d - \bar{p})) = 1,
\]

which in its turn means that the germ \( V(\bar{p}, C) \) intersect transversally at \( C \) with the hyper-plane \( H_p = \{C' \in |D| : p \in C'\} \) in the linear system \( |D| \). The latter conclusion yields that \( V(\bar{p}, C) \) diffeomorphically projects onto the germ of \( E \) at \( p \) by sending a curve \( C' \in V(\bar{p}, C) \) to its intersection point with \( E \) in a neighborhood of \( p \). Thus, in suitable local coordinates \( x, y \) of \( \Sigma \) in a neighborhood of \( p \), we have \( p = (0, 0), E = \{y = 0\}, L_p = \{x = 0\} \), and a local parametrization

\[
C_t = \left\{ ay + b(x + t)^k + \sum_{i + kj \geq k} O(t) \cdot (x + t)^i y^j = 0 \right\}, \quad a, b \in \mathbb{C}^*, \ t \in (\mathbb{C}, 0),
\]

of \( V(\bar{p}, C) \). Hence, for any point \( p' = (0, \tau) \in L_p \), we obtain precisely \( k \) curves \( C_t \in V(\bar{p}, C) \) passing through \( p' \) and corresponding to the \( k \) values of \( t \)

\[
t = \left(-\frac{a}{b}\right)^{1/k} \tau^{1/k} + \text{h.o.t.},
\]

which completes the proof.

2.5.3 Deformation of a reducible curve

(1) Geometry of deformation.

**Lemma 2.9** Let \( C \) be as in Proposition 2.8(2), and let \( V(\bar{p}, C) \neq \emptyset \). Let \( \{C_t\}_{t \in (\mathbb{C}, 0)} \) be a parameterized branch of \( V(\bar{p}, C) \) centered at \( C = C_0 \), and let \( \{n_t : \hat{C}_t \to \Sigma, \hat{p}_t\} \), \( t \in (\mathbb{C}, 0) \) be its lift to \( V_\Sigma(D, g, \alpha, \beta, p) \). Then in the deformation \( C_0 \to C_t, t \neq 0 \), we have
(i) a point \( z \in \text{Sing}(C_{\text{red}}) \setminus E \) is either a node of some \( C^{(i)} \), \( 1 \leq i \leq m \), and then it deforms into a moving node of \( C_t \), \( t \neq 0 \), or \( z \) is a transverse intersection point of distinct irreducible components \( C', C'' \) of \( C_{\text{red}} \), both smooth at \( z \), and then the point \( z \) deforms into \( k'k'' \) nodal points of \( C_t \), \( t \neq 0 \), where \( k' \) and \( k'' \) are the multiplicities of \( C', C'' \) in \( C \), respectively;

(ii) if \( z = p_{kl} \in p \setminus \bigcup_{i=1}^{m} C^{(i)} \), then the germ of \( E \) at \( z \) turns into a smooth branch of \( C_t \) crossing \( E \) at \( z \) with multiplicity \( k \);

(iii) if \( z = p_{kl} \in p \cap C^{(i)} \) for some \( i = 1, \ldots, m \), then the germ of \( C^{(i)} \) at \( z \) deforms into a smooth branch of \( C_t \) centered at \( z \) and crossing \( E \) with multiplicity \( k \), and the germ of \( E \) at \( z \) deforms into a smooth branch of \( C_t \), \( t \neq 0 \), disjoint from \( E \) and crossing the former branch of \( C_t \) transversally at \( k \) points;

(iv) if \( z = q_{kl} \in C^{(i)} \cap E \setminus p \) for some \( i = 1, \ldots, m \), and \( q_{kl} = n(\hat{q}_{kl,0}) \), where \( (\hat{q}_{kl,t})_{t \in (C,0)} \) is defined by (14), (15), then, first, in case of \( C^{(i)} = sL(z) \), \( z \in p \), we have \( s = k \), and, second, the germ of \( C^{(i)} \) at \( z \) deforms into a smooth branch of \( C_t \), \( t \neq 0 \), crossing \( E \) at one point in a neighborhood of \( z \) with multiplicity \( k \), and the germ of \( E \) at \( z \) deforms into a smooth branch of \( C_t \), \( t \neq 0 \), disjoint from \( E \) and crossing the former branch of \( C_t \) transversally at \( k \) points;

(v) if a point \( z \in C^{(i)} \cap E \setminus p \) is not \( n(\hat{q}_{kl,0}) \) for any section \( (\hat{q}_{kl,t})_{t \in (C,0)} \) as in (iv), then the germ of \( E \cup C^{(i)} \) at \( z \) deforms into an immersed cylinder disjoint from \( E \);

(vi) if \( s > 0 \) (or \( s'' > 0 \)) then the union of \( s'L' \) (resp., \( s''L'' \)) with the germ of \( E \) at the point \( z = E \cap L' \) (resp., \( z = E \cap L'' \)) turns into an immersed disc disjoint from \( E \) and having \( s' \) (resp., \( s'' \)) nodes.

Proof. All the statements follow from Proposition 2.6(2). \( \square \)

(2) Strategy of the proof of Proposition 2.8(2). Let

\[(L_p, p) \rightarrow (\mathbb{C}, 0), \quad p' \in L_p \rightarrow \tau \in (C, 0), \]

be a parametrization of \( L_p \). Assume that \( V(\bar{p}, C) \neq \emptyset \), take one of its irreducible branches \( V = \{ C_t : t \in (\mathbb{C}, 0) \} \) with a parametrization normalized by a relation

\[p' = L_p \cap C_t \iff \tau = t^\mu \quad (24)\]
for some \( \mu \geq 1 \). Clearly, \( \deg \varphi_{C,\mu} = \mu \), and the curves \( C \in V \) passing through \( p' \) can be associated with different parameterizations of \( V \) obtained by substitutions \( t \mapsto \varepsilon t^\mu = 1 \).

Next, to a parameterized branch \( V \) we assign a collection of bivariate polynomials (called deformation patterns) which describe deformations of the curve \( C \) in neighborhood of isolated singularities of \( \bigcup_{i=1}^{m} C(i) \cap E \), in neighborhood of non-reduced components \( C(i) \), and in neighborhood of \( L', L'' \), when moving along the branch \( V \). We show that the constructed deformation patterns belong to an explicitly described finite set, and then prove that, given a collection of arbitrary deformation patterns from the above sets, there exists a unique parameterized branch \( V \) of \( V(p, C) \) which induces the given deformation patterns. Thus, the degree \( \deg \varphi_{C} \) appears to be the number of admissible collections of deformation patterns in the construction presented below.

We have chosen deformation patterns in view of their convenience for further real applications (see [8]), particularly, they suit well for the calculation of Welschinger signs of real nodal curves. Another advantage of deformation patterns is that we will be able to complete the classification of degenerations in Proposition 2.6, claim (2iv), by showing that the lines \( L', L'' \) cannot be multiply covered.

In parts (4) and (5) of the present section, we provide a detailed construction of deformation patterns for isolated singularities and for multiple lines \( L', L'' \), which are the most involved cases. In a similar way, in parts (7)-(9), we construct deformation patterns for multiply covered lines \( L(p_{kl}) \) and \( L(z) \), but omit routine details.

(3) Preliminary choice in the construction of deformation patterns. First, we fix a sequence \( \gamma(i) \in \mathbb{Z}_{+}^\infty, i = 1, \ldots, m \), satisfying (18).

Next, for each \( i = 1, \ldots, m \), take a subset of \( \|\gamma(i)\| \) points of \( C(i) \cap E \setminus p \) selected in such a way that, for precisely \( \gamma_k(i) \) of them, the intersection multiplicity of \( C(i) \) and \( E \) at the chosen point equals \( k \). We notice that the number of possible choices is \( \prod_{i=1}^{m} \left( \frac{\beta(i)}{\gamma(i)} \right) \), and we denote the chosen points by \( q'_{ki}, k \geq 1, 1 \leq j \leq \sum_{i} \gamma_k(i) \), assuming that \( (C(i) \cdot E)(q'_{ki}) = k \) as \( q'_{ki} \in C(i) \). In particular, for a component \( C(i) = kL(p_{kl}) \), the only point of \( L(p_{kl}) \cap E \setminus p \) is always chosen. Denote by \( z' \) the set of all the selected points.

In the next parts (4), (5), and (7)-(9), we suppose that \( V(p, C) \neq \emptyset \), and there exists a branch \( V \) of \( V(p, C) \) parameterized with normalization (24) and which lifts to a family \( \{ n_t : \hat{C}_t \to \Sigma, \hat{p}_t \} \in \mathcal{V}(D, g, \alpha, \beta, p), t \in (\mathbb{C}, 0) \), such that \( (n_0), \hat{C}_0 = C \), and the points \( n_0(q_{ij}t) \) do not belong to \( z' \) (here the sections \( \hat{q}_{ij}, t \in \hat{C}_t \) are uniquely defined by (14), (15)). Then put \( z = \{ n_0(q_{ij}t) : i, j \geq 1 \} \).
(4) Deformation patterns for isolated singularities. Pick a point $q_{kj} \in z' \cap \bigcup_{i=1}^m C^{(i)}$. Let $\Pi_E : \Sigma \to \mathbb{P}^2$ be the blow down of the (disjoint) $(-1)$-curves $E_1, ..., E_{a-2}, L - E_{a-1} - E_a, L - E_{a-1} - E_{a+1}, L - E_a - E_{a+1}$. In particular, it takes $E$ to the line through the points $\Pi_E(E_2), ..., \Pi_E(E_{a-2})$. Take an affine plane $\mathbb{C}^2 \subset \mathbb{P}^2$ with the coordinates $u, v$ such that $\Pi_E(E_2) \cap \mathbb{C}^2 = \{v = 0\}$. For the sake of notation, we write $u(z), v(z)$ for the coordinates of the point $\Pi(z)$ as $z$ is a point or a contracted divisor in $\Sigma$. For example, we write $p = (u(p), 0)$ with some $u(p) \neq 0$, and, furthermore, we suppose that $L_p = \{u = u(p)\}$.

Notice that this identification naturally embeds $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ into the space $P_{d_0}$ of polynomials in $u, v$ of degree $\leq d_0 = 2d - d_a - d_{a-1} - d_a + 1$ (where $D = dL - d_1E_1 - ... - d_aE_a + 1$).

The curve $C$ is then given by

$$F_C(u, v) := v \left( f_1(u) + v \sum_{k, l \geq 0} c_{kl} u^k v^l \right) = 0,$$

where

$$f_1(u) = \prod_{i=2}^{a-2} (u - u(E_i))^{d_i} \prod_{q_{ij} \in z'} (u - u(q_{ij}))^i \cdot \prod_{z \in \tilde{C} \cap \Sigma \setminus z'} (u - u(z))^{(\tilde{C} \cap \Sigma)(z)},$$

(26)

and $\tilde{C}$ is the union of the components of $C$ different from $E$.

A curve $C_t \in V$ is given by an equation

$$at^\lambda (f_2(u) + O(t)) + v \left( f_1(u) + O(t) + v \sum_{k, l \geq 0} (c_{kl} + O(t)) u^k v^l \right) = 0,$$

(27)

where

$$f_2(u) = \prod_{i=2}^{a-2} (u - u(E_i))^{d_i} \prod_{i \geq 1} \prod_{j=1}^{a_i} (u - u(p_{ij}))^i \cdot \prod_{i \geq 1} \prod_{j=1}^{\beta_i} (u - u(q_{ij}))^i$$

(28)

(here $p_{ij}$ runs over the points of $p$, and $q_{ij}$ are the projections of the central points $\hat{q}_{ij,0}$ of the sections $\hat{q}_{ij,t}$ defined by (14)). From (24), we immediately derive that

$$\lambda = \mu, \quad a = -\frac{f_1(u(p))}{f_2(u(p))}.$$

(29)

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In the coordinates \( x = u - u(q_{ij}) \), \( y = v \), where \( q_{ij} \in \mathcal{Z}' \) is the chosen point, the curve \( C \) is given by an equation
\[
c_i^{ij} y^2 + c_i^{ij} x \cdot y + \sum_{k+l>2i} c_{kl}^{ij} x^k y^l = 0, \quad c_{02}, c_{i1}^{ij} \neq 0,
\]
and the curve \( C_t \in V \ (t \neq 0) \) is given by an equation
\[
\Phi(x, y, t) := c_{02}^{ij} y^2 + c_i^{ij} x \cdot y + \sum_{k+l=2i} O(t) \cdot x^k y^l + \sum_{k+l>2i} (c_{kl}^{ij} + O(t)) x^k y^l
\]
\[
+ \sum_{k+l<2i} t^{n(k,l)} (c_{kl}^{ij} + O(t)) x^k y^l = 0,
\]
where the last sum does contain nonzero monomials, and for all of them \( n(k, l) > 0 \) and \( c_{kl}^{ij} \neq 0 \), in particular, by (27) and (29),
\[
n(0, 0) = \mu, \quad c_{00}^{ij} = -\frac{f_1(u(p)) f_2(u(q_{ij}^0))}{f_2(u(p))}.
\]
(30)

Making an additional coordinate change
\[
x \mapsto x - yt n(i^{-1,1}) \left( \frac{c_i^{ij}}{c_{02}^{ij}} + O(t) \right) (c_{i-1,1}^{ij} + O(t)),
\]
we can annihilate the monomial \( x^{i-1}y \) in \( \Phi(x, y, t) \).

Notice that
\[
n(k, 0) \geq n(0, 0) = \mu, \quad k > 0.
\]
(31)
Indeed, otherwise, we would have an intersection point of \( C_t \) and \( E \) converging to \( q_{ij}^0 \), a contradiction. Hence there exists
\[
\rho = \min_{k+i<2i} \frac{n(k, l)}{2i-k-il} > 0.
\]
(32)

Now we consider the equation \( \Psi(x, y, t) := t^{-2\rho} \Phi(x t^{\rho}, y t^{\rho}, t) = 0 \). We have
\[
\Psi(x, y, t) = c_{02}^{ij} y^2 + c_i^{ij} x \cdot y + \sum_{k+l=2i} O(t) \cdot x^k y^l + \sum_{k+l>2i} (c_{kl}^{ij} + O(t)) t^{\rho(k-il-2i)} x^k y^l
\]
\[
+ \sum_{k+l\leq2i} t^{n(k,l)-\rho(2i-k-il)} (c_{kl}^{ij} + O(t)) x^k y^l
\]
\[25]
The well-defined polynomial
\[
\Psi(x, y, 0) = c_{02}^{ij} y^2 + c_{11}^{ij} x^i y + \sum_{k+l<2i} b_{kl}^{ij} x^k y^l
\]
with the coefficients
\[
b_{kl}^{ij} = \begin{cases} 
    c_{kl}^{ij}, & \text{if } n(k, l) = \rho(2i - k - il), \\
    0, & \text{if } n(k, l) > \rho(2i - k - il). 
\end{cases}
\]
In view of (31), \(b_{k,0} = 0, k \geq 1\), and hence
\[
\Psi(x, y, 0) = c_{02}^{ij} y^2 + c_{11}^{ij} x^i y + \sum_{k=0}^{i-2} b_{k1}^{ij} x^k y + b_{00}^{ij},
\]
where at least one of the \(b_{k1}^{ij}\) or \(b_{00}^{ij}\) is nonzero. We claim that \(b_{00}^{ij} \neq 0\). Indeed, otherwise, \(\Psi(x, y, t) = 0\) would split into the line \(E = \{y = 0\}\) and a curve, crossing \(E\) in at least two points, which in turn would mean that, in the deformation of \(C\) into \(C_t\) in a neighborhood \(U_{ij}\) of \(q_{ij}'\), the germs of \(\tilde{C}\) and \(E\) at \(q_{ij}'\) would glue up with the appearance of at least two handles, thus breaking the claim of Lemma 2.9(v). Hence \(b_{00}^{ij} = c_{00}^{ij}\) given by (30), and as observed above the equation \(\Psi(x, y, 0) = 0\) defines an immersed affine rational curve. We call \(\Psi(x, y, 0)\) the deformation pattern for the point \(q_{ij}' \in z'\) associated with the parameterized branch \(V\) of \(V(\mathcal{P}, C)\). Its Newton polygon is depicted in Figure 1. We summarize the information on deformation patterns in the following statement:

**Lemma 2.10** (1) Given a curve \(C = E \cup \tilde{C}\) and a set \(z' \subset \tilde{C} \cap E\) as chosen in section 2.5.3(3), the coefficients \(c_{02}^{ij}, c_{11}^{ij},\) and \(b_{00}^{ij} = c_{00}^{ij}\) of a deformation pattern of the point \(q_{ij}' \in z'\) are determined uniquely up to a common factor.

(2) The set \(\mathcal{P}(q_{ij}')\) of polynomials given by formula (33), having fixed coefficients \(c_{02}^{ij}, c_{11}^{ij},\) and \(b_{00}^{ij} = c_{00}^{ij}\) and defining a rational curve, consists of \(i\) elements. Moreover, their
coefficients $b_{ik}^{ij}$, where $k \equiv i + 1 \mod 2$, vanish, and, given a polynomial $\Psi(x,y) \in P(q_{ij}')$, the other members of $P(q_{ij}')$ can be obtained by the following transformations
\[
\Psi(x,y) \mapsto \Psi(x\varepsilon, y), \quad \text{where } \varepsilon^i = 1, \quad \text{if } i \equiv 1 \mod 2,
\]
\[
\Psi(x,y) \mapsto \Psi(x\varepsilon^i, y\varepsilon^i), \quad \text{where } \varepsilon^{2i} = 1, \quad \text{if } i \equiv 0 \mod 2.
\]

(3) The germ at $F \in P(q_{ij}')$ of the family of those polynomials with Newton triangle $\text{conv}\{(0,0),(0,2),(i,1)\}$, which define a rational curve, is smooth of codimension $(i - 1)$, and intersect transversally (at one point ${\{F}\}$) with the affine space of polynomials
\[
\left\{c_{02}^{ij}y^2 + c_{00}^{ij} + c_{11}^{ij}x^iy + \sum_{k=0}^{i-2} c_kx^ky : c_0, \ldots, c_{i-2} \in \mathbb{C}\right\}.
\]

**Proof.** The first statement comes from construction. In particular, $a_{00}^{ij}$ is determined by formula (30). The second statement follows from [12, Lemma 3.5]. At last, the second statement additionally implies that the family in the third statement is the (germ of the) orbit of $F$ by the action
\[
F(x,y) \mapsto F(\xi_1x + \xi_0, \xi_2y), \quad \xi, \xi_1, \xi_2 \in \mathbb{C}^*, \quad \xi_0 \in \mathbb{C},
\]
and the third claim follows. \qed

**Remark 2.11** (1) Since the coefficients $b_{ik}^{ij}$ in (33) vanish for $k \equiv i + 1 \mod 2$, and the other ones not, the values
\[
n(i - 2k, 1) = \rho(2i - (i - 2k) - i) = \frac{\mu}{2i} \cdot 2k = \frac{\mu}{i}, \quad 1 \leq k \leq \frac{i}{2},
\]
are integer; hence $\mu$ is divisible by $i$.

(2) The treatment of this section covers the cases of non-multiple components $C^{(i)} = L(p_{kl}), L(z), L', \text{or } L''$. In the next sections we consider nonreduced components $C^{(i)}$, however the case of multiplicity one is also covered there with the same (up to a suitable coordinate change) answer.

(5) **Deformation patterns for nonisolated singularities, I.** In this section we construct deformation patterns for a component $s'L'$ of $C$ with $s' \geq 1$.

**Step 1: Preparation.** To relax the notation, within this section we write $s$ for $s'$. 27
Consider the blow-down $\Pi : \Sigma \to \mathbb{P}^2$ which contracts $E_1, \ldots, E_{a+1}$. It takes $E$ to a conic $\Pi(E)$ passing through $a$ fixed points $\Pi(E_2), \ldots, \Pi(E_{a+1})$, and takes $L', L''$ to straight lines tangent to $\Pi(E)$ and passing through the fixed point $\Pi(E_1)$. The linear system $|D|$ on $\Sigma$ turns into the linear system $\Pi^* |D|$ of plane curves of degree $d$ with multiple points $\Pi(E_1), \ldots, \Pi(E_{a+1})$ of order $d_1, \ldots, d_{a+1}$, respectively. Take an affine plane $C^2 \subset \mathbb{P}^2$ with coordinates $x, y$ such that $\Pi(L') = \{y = 0\}$, $\Pi(E) = \{S(x, y) := y + xy + x^2 = 0\}$, $\Pi(E_1)$ is the infinite point of $\Pi(L')$, and the tangency point of $\Pi(L')$ and $\Pi(E)$ is the origin. The curves $\Pi(C_t)$, where $C_t \in V$, are then described by formula

$$F_t(x, y) = S(x, y)\tilde{G}_t(x, y) + t^\mu G_t(x, y),$$

(34)

which is a conversion of formula (27) (cf. also (29)) and in which we can suppose that $\tilde{G}_0(x, y) = y^s \tilde{G}'_0(x, y)$ with the polynomial $\tilde{G}'_0(x, y)$ defining the union of the components of $\Pi(C)$ different from $\Pi(L'), \Pi(L'')$ and such that $\tilde{G}'_0(0, 0) = 1$. Furthermore, the value $c = \tilde{G}_0(0, 0) \neq 0$ can be computed from relations (24) and (34) which yield

$$G_0(\Pi(p)) = -\frac{d}{d\tau} \left( S(x, y)\big|_{\Pi(L_p)} \right) \big|_{\Pi(p)} \cdot \tilde{G}'_0(\Pi(p)) .$$

(35)

In addition, moving all the terms with exponent of $t$ greater or equal to $\mu$ from the first summand in (34) to the second one, we can assume that

all exponents of $t$ occurring in $\tilde{G}_t(x, y)$ are strictly less than $\mu$.

(36)

**Step 2: Tropical limit.** Now we find the tropical limit of the family (34) in the sense of [12]. For the reader’s convenience, we shortly recall what is the tropical limit. Consider the family of the curves $K_t$ defined by polynomials (34) for $t \neq 0$ in the trivial family of the toric surfaces $C(\Delta)$, where $\Delta$ is the Newton polygon of a generic polynomial (34). Then we close up this family at $t = 0$ (all details can be found in [12]):

- to each monomial $x^i y^j$ we assign the point $(i, j, \nu_{ij}) \in \mathbb{Z}^3$, where $\nu_{ij}$ is the minimal exponent of $t$ in the coefficient of $x^i y^j$ in $F_t$, and then define a convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$, whose graph is the lower part of the convex hull of all the points $(i, j, \nu_{ij})$;

- the surface $C(\Delta)$ degenerates into the union of toric surfaces $C(\delta)$, where $\delta$ runs over the maximal linearity domains of $\nu$ (the subdivision of $\Delta$ into these linearity domains, which all are convex lattice polygons, we call $\nu$-subdivision),
then write \( F_t(x, y) = \sum_{(i,j) \in \Delta} (c_{ij} + O(t))t^{\nu(i,j)} \) and define the limit of the curves \( K_t \) at \( t = 0 \) as the union of the limit curves \( K_\delta \subset \mathbb{C}(\delta) \) given by limit polynomials
\[
f_\delta = \sum_{(i,j) \in \delta} c_{ij} x^i y^j
\]
for all pieces \( \delta \) of the \( \nu \)-subdivision of \( \Delta \).

Notice that
\[
\nu_{ij} \geq \nu(i, j) \quad \text{for all } i, j, \quad \text{and } \nu_{ij} = \nu(i, j) \quad \text{for the vertices of } \nu \text{-subdivision}.
\]

\[ (37) \]

\text{Step 3: Subdivision of the Newton polygon.} In Figure 2(a), we depicted the Newton polygon \( \Delta \) of \( F_t(x, y) \), whose part \( \delta_1 \) above the bold line is the Newton polygon of \( S(x, y)y^s\tilde{G}'_0(x, y) \), and it is the linearity domain of \( \nu \), where it vanishes. Below the bold line, \( \nu \) is positive, and hence this part \( \delta_2 \) is subdivided by other linearity domains of \( \nu \). We claim that \( \delta_2 \) is subdivided into two pieces as shown in Figure 2(b).

\[ (3A) \]
The truncation of the polynomial \( S(x, y)y^s\tilde{G}'_0(x, y) \) to the incline segment \( \sigma_1 \) of the bold line is \((y + x^2)y^s\), and the truncation to the horizontal segment \( \sigma_2 \) of the bold line is \( x^2y^s\tilde{G}'_0(x, 0) \). Observe that the roots of \( \tilde{G}'_0(x, 0) \) are the coordinates of the intersection points of the closure of \( \Pi(C \setminus (L' \cup E)) \) with \( \Pi(L') \) (counting multiplicities). By Lemma 2.9(i), the latter intersection points persist in the deformation \( C_t, t \in (\mathbb{C}, 0) \), which implies (cf. [12, Section 3.6]) that the subdivision of \( \delta_2 \) must contain one or few trapezes \( \delta \) with a pair of horizontal sides parallel to \( \sigma_2 \) of length at least \( \deg \tilde{G}'_0(x, 0) \), and of the total height \( s \) (like in Figure 2(c)), and the corresponding polynomials \( f_\delta \) are divisible by \( \tilde{G}'_0(x, 0) \).

\[ (3B) \]
Let us show that all such trapezes must be rectangles.

First, notice that \( \nu(0, 0) = \mu \), since \( F_t(0, 0) = t^\mu(cG_0(0, 0) + O(t)) \). The convexity of \( \nu \) and its constancy along the horizontal sides of the trapezes (cf. Figure 2(c)) yield that
\[
0 \leq \nu \leq \mu(s - l)/s \quad \text{along such a side on the height } 0 \leq l < s.
\]
Ordering the monomials of \( \tilde{G}'_t(x, y) \), first, by the growing power of \( x \) and then by the growing power of \( y \), and using (34), we inductively derive that the minimal exponents \( \mu_{il} \) in the coefficients of \( x^iy^l \) in \( \tilde{G}'_t(x, t) \) satisfy
\[
\mu_{il} \geq \nu(2, l) \quad \text{for all } i \geq 0, \; 0 \leq l \leq s.
\]
This excludes possible vertices \( v = (0, j) \) or \( (1, j) \) with \( 0 < j < s \) of the trapezes, since, otherwise, by formula (34) and relation (37), we would get \( \nu(v) = \nu_\nu \geq \min\{\mu_{0,j-1}, \mu_{1,j-1}\} \), contradicting (38) and the strict decrease of \( \nu \) with respect to the second variable.

Next we observe that the points \((0, 0)\) and \((1, 0)\) cannot serve as vertices of the lowest trapeze. Indeed, the values of \( \mu_{00} \) and \( \mu_{10} \) come from the second summand in (34), and hence are equal to \( \mu \). Thus, if the lowest trapeze is not a rectangle, its lower side has vertex
Figure 2: Deformation patterns for $sL'$
So, suppose that \((0, 0)\) is a vertex of the lower trapeze and bring this assumption to contradiction. The upper left vertex of that lowest trapeze must be \((2, j)\) with some \(1 \leq j \leq s\). Thus, we have

\[
0 \leq \nu_{2j} = \nu(2, j) \leq \mu \frac{s - j}{s},
\]

and by linearity of \(\nu\) in each trapeze,

\[
\nu(i, l) = \nu_{2j} \frac{l}{j} + \mu \frac{j - l}{j} \quad \text{for all } i \geq 2, \ 0 \leq l \leq j.
\]

Combining this bound with (38), we derive that

\[
\nu_{il} \geq \nu_{2j} \frac{l - 1}{j} + \mu \frac{j - l + 1}{l} > \nu_{2j} \frac{l}{j + 1} + \mu \frac{j + 1 - l}{j + 1}, \quad \text{for all } i = 0, 1, 1 \leq l \leq j.
\]

On the other hand, from (34), one obtains \(\nu(0, j + 1) = \mu_{0j} = \nu_{2j}\), and hence the preceding bound together with the convexity of \(\nu\) will imply that

\[
\nu_{il} > \nu(i, l) \quad \text{for all } i = 0, 1, 0 < l \leq j.
\]

In particular, we obtain that the subdivision of \(\delta_2\) contains the triangle

\[
\delta = \text{conv}\{(0, 0), (2, j), (0, j + 1)\} \quad \text{see Figure 2(d)},
\]

and, moreover, the limit polynomial \(f_\delta\) contains only three monomials corresponding to the vertices of \(\delta\). It is easy to see that then the limit curve \(K_\delta\) is of geometric genus \(\#(\text{Int}(\delta) \cap \mathbb{Z}^2) > 0\), which implies that the curve \(C_t\) has handles in a neighborhood of \(L'\) contradicting Lemma 2.9(vi).

(3C) Let us show that there is only one rectangle with horizontal/vertical edges in the subdivision of \(\delta_2\). Suppose that there are several rectangles like this (cf. Figure 2(e)). Let \((2, j), 0 < j < s\), be the left lower vertex of the upper rectangle. Arguing as in the preceding paragraph, one can derive that

\[
\nu_{il} > \nu(1, l) \quad \text{for all } 0 \leq l \leq s,
\]

\[
\nu_{0,j+1} = \nu(0, j + 1) = \nu_{2,j} = \nu(2, j) = \mu_{0,j},
\]

\[
\nu_{0l} \geq \nu(0, l) = \nu(2, l - 1) \quad \text{for all } j < l \leq s.
\]

From this we conclude that the parallelogram \(\delta_0 = \text{conv}\{(0, j + 1), (0, s + 1), (2, j), (2, s)\}\) is a part of the \(\nu\)-subdivision, and that the coefficients of \(f_{\delta_0}\) are nonzero only along the vertical sides of \(\delta_0\) and they all come from the coefficients of \(t^{\nu(0,l)}\) in the coefficients of \(y^{l-1}\) in \(\tilde{G}_t(x, y), j < l \leq s\). Particularly, \(f_{\delta_0} = (\sum_{i=j}^{s} c_i y^i)(y + x^2)\), and hence the curve \(K_{\delta_0}\) splits off the conic \(S_0 = \{y + x^2 = 0\}\) and \(s - j\) horizontal lines, each crossing \(S_0\) transversally at two
distinct points different from the origin. By a suitable variable change \((x, y) \mapsto (xt^\lambda_1, yt^\lambda_2)\) in \(F_t(x, y)\) and division by the minimal power of \(t\) in the obtained polynomial (still denoted by \(F_t\)), we can make the corresponding function \(\nu\) vanishing along \(\delta_0\), which means that \(F_t(x, t) = f_{\delta_0} + O(t)\). At the same time this operation takes the conic \(S_t = \{y + x^2 + xy \cdot O(t) = 0\}\). Notice that each line of \(K_{\delta_0}\) crosses \(S_t\) at two points convergent to its intersection points with \(S_0\). The statement of Lemma 2.9(vi) yields that the curves \(C_t, t \neq 0\), do not intersect \(E\) in a neighborhood of \(L'\). Hence, the intersection points of the lines of \(K_{\delta_0}\) with \(S_0\) must smooth out in the deformation \(\bigcup_{\delta} K_{\delta} \to K_t, t \neq 0\). However such a smoothing will develop at least one handle of \(C_t\) in a neighborhood of \(L'\), which is a contradiction to Lemma 2.9(vi).

\((3D)\) To complete the proof of the claim that the \(\nu\)-subdivision of \(\Delta\) is the one shown in Figure 2(b), it remains to exclude the subdivision depicted in Figure 2(f), and this can be done in the same manner as in the preceding paragraph, where we excluded from \(\nu\)-subdivision parallelograms with vertices \((0, j + 1), (0, s + 1), (2, j), (2, s)\).

**Step 4: Limit curves.** As we observed above, the limit curve corresponding to the rectangle splits into the union of vertical and horizontal lines. For the limit polynomial \(f_{\delta}\) of the trapeze \(\delta = \text{conv}\{(0, 0), (0, s + 1), (2, 0), (2, s)\}\), the consideration of part (3) in Step 3, notably, \((39)\), gives that \(f_{\delta}(x, y)\) does not contain monomials \(xy^l\), and the coefficients of the monomials \(y^{l+1}\) and \(x^2y^l\) equal the coefficient of \(t^{\nu(0,l+1)}\) standing at the monomial \(y^l\) in \(\tilde{G}_t(x, y)\) for all \(l = 0, ..., s - 1\). Then

\[
f_{\delta}(x, y) = c + (y + x^2)f(y),
\]

where \(c\) is defined by \((35)\), and \(f(y) = y^s + b_{s-1}y^{s-1} + ... + b_0\). We call this polynomial the **deformation pattern** for the component \(s'L'\) of \(C\) associated with the parameterized branch \(V\) of \(V(\mathcal{P}, C)\).

**Lemma 2.12**

1. Given a curve \(C = E \cup \tilde{C}\) and a set \(z' \subset \tilde{C} \cap E\) as chosen in section 2.5.3(3), the coefficient \(c\) is determined uniquely.

2. For any fixed \(c \neq 0\), the set \(\mathcal{P}(L')\) of the polynomials given by \((40)\) and defining a rational curve consists of \(s + 1\) elements, and they all can be found from the relation

\[
yf(y) + c = \frac{c}{2} \left( \text{cheb}_{s+1} \left( \frac{y}{(2s^{-1}c)^{1/(s+1)}} + y' \right) + 1 \right),
\]

where \(\text{cheb}_{s+1}(y) = \cos((s + 1)\text{arccos} y)\), and \(y'\) is the only positive simple root of \(\text{cheb}_{s+1}(y) - 1\).
(3) The germ at $f_δ \in P(L')$ of the set of those polynomials with Newton quadrangle conv\{(0, 0), (0, s + 1), (2, 0), (2, s)\} which define a rational curve, is smooth of codimension $s$, and intersects transversally (at one point \{f_δ\}) with the space of polynomials
$$\left\{(y + x^2) \left(y^s + \sum_{l=0}^{s-1} c_ly^l\right) : c_0, ..., c_{s-1} \in \mathbb{C}\right\}.$$

**Proof.** The first claim follows from the construction.

For the second claim, notice that $f_δ(-x, y) = f_δ(x, y)$ (cf. (40)); hence the curve $K_δ$ is the double cover of a nonsingular curve ramified at the intersection points with the toric divisors corresponding to the vertical sides of $δ$. It is easy to show that the double cover is rational (cf. Lemma 2.9(vi)), if and only if $f(y)$ has precisely $[s/2]$ double roots, and $yf(y) + c$ has precisely $[(s + 1)/2]$ double roots. Hence up to linear change in the source and in the target, all such polynomials $yf(y) + c$ must coincide with the Chebyshev polynomial $\text{cheb}_{s+1}(y)$, which is characterized by the following properties:

- it is real, has degree $s + 1$, is real, and its leading coefficient is $2^s$;
- for even $s$, it is odd, has $s/2$ maxima on the level $1$ and $s/2$ minima on the level $-1$.
- for odd $s$, it is even, has $(s - 1)/2$ maxima on the level $1$ and $(s + 1)/2$ minima on the level $-1$.

Thus, for $yf(y) + c$ we obtain precisely $s + 1$ possibilities given by formula (41). The smoothness and the dimension statement in the third claim follow, for instance, from [6, Theorem 6.1(iii)]. For the transversality statement, we notice that the tangent space to $M$ consists of polynomials vanishing at each of the $s$ nodes of $f_δ = 0$, whereas the tangent space to the affine space is $\{(y + x^2) \sum_{l=0}^{s-1} c_ly^l\}$, and its nontrivial elements cannot vanish at $s$ points outside $y + x^2 = 0$ with distinct $y$-coordinates.

**Remark 2.13** Similarly to Remark 2.11(1), we observe that the values of $ν$ must be integer at $(0, l)$, $0 \leq l \leq s + 1$, and hence $μ$ must be divisible by $s + 1$.

**Proof of Proposition 2.6(2iv).** In the notation of Proposition 2.6 let $n : C^{(i)} \to L'$ is an $s$-multiple cover, $s \geq 2$. By Proposition 2.2(ii) and Proposition 2.6(2iii), $C^{(i)}$ is rational, and the cover $n : C^{(i)} \to L'$ has two ramification points, one of which $z \in L'$ differs from the tangency point of $L'$ and $E$. In particular, if we take a tubular neighborhood $U$ of $L'$ in $Σ$ and the projections $C_t \cap U \to L'$, $t \neq 0$, defined by a pencil of lines through a generic point outside $U$, then these projections will have ramification points converging to $z$. On
the other hand, it is easy to check that the critical points of the natural projection of the curve \( K_{\delta} \subset \mathbb{C}(\delta) \) given by \( f_{\delta}(x,y) = 0 \) with \( f_{\delta} \) as in Lemma 2.12 onto the toric divisor corresponding to the lower edge of \( \delta \), all lie in the big torus \((\mathbb{C}^*)^2 \subset \mathbb{C}(\delta)\), and hence the critical points of the projection \( C_t \cap U \to L' \) have coordinates \( ((\xi + O(t))t^\lambda, (\eta + O(t))t^{2\lambda}) \) with \( \lambda > 0, \xi, \eta \in \mathbb{C} \), thus, for \( t \to 0 \) they converge to the origin, which in our setting is the tangency point of \( E \) and \( L' \). Therefore, all ramification points of the projection converge to that tangency point, a contradiction.

(7) Deformation patterns for nonisolated singularities, II. In this section we construct deformation patterns for a component \( C^{(i)} = kL(p_{kj}) \) of \( C \) with \( k \geq 1 \), and we use the technique developed in part (5) of the present section.

Perform the blow-down \( \Pi : \Sigma \to \mathbb{P}^2 \), contracting \( E_1, \ldots, E_{a+1} \), and take an affine plane \( \mathbb{C}^2 \subset \mathbb{P}^2 \) with the coordinates \((x,y)\) such that \( \Pi(L(p_{kl})) = \{y = 0\} \), \( \Pi(E_1) \) is the infinitely far point of \( \Pi(L(p_{kl})) \), and the conic \( \Pi(E) \) crosses \( \Pi(L(p_{kl})) \) transversally at \( \Pi(p_{kl}) = (0,0) \) and at \((1,0)\), the latter point being smoothed out in the deformation of \( C \) along \( V \). There exists a unique transformation of \( \mathbb{C}^2 \)

\[
(x, y) \mapsto \left( x + \sum_{1 \leq i \leq k} c_i y^i, y \right),
\]

such that the equation of \( \Pi(E) \cap \mathbb{C}^2 \) becomes

\[
S(x, y) := y^k + x - x^2 + \sum_{j > k} c_{0j} y^j + \sum_{j > 0} (c_{1j} xy^j + c_{2j} x^2 y^j) = 0.
\]

Thus, the curve \( \Pi(C) \cap \mathbb{C}^2 \) is defined by a polynomial of the form \( F_0(x,y) = S(x,y)G_0(x,y) \) with Newton polygon \( \Delta \), whose lower part is the union of the segments \([(0,2k), (1,k)] \) and \([(1,k), (d-k,k)] \) (shown by fat line in Figure 3(a)). Here the root of the truncation of \( F_0 \) on the segment \([(0,2k), (1,k)] \) corresponds to the branch of \( \Pi(E) \) centered at the origin, and the roots of the truncation of \( F_0 \) on the segment \([(1,k), (d-k,k)] \) correspond to the intersection points of \( \Pi(E) \) with \( \Pi(L(p_{kl})) \) (equal to \((1,0)\)) and the intersection points of \( \Pi(L(p_{kl})) \) with the other components of \( \Pi(C) \). As in Step 1 of part (5), the curves \( \Pi(C_t) \cap \mathbb{C}^2 \) are defined by polynomials \( F_t(x,y) \) converging to \( F_0(x,y) \) as \( t \to 0 \) and given by \( [34] \) with the Newton polygon \( \Delta' \) being the convex hull of the union of \( \Delta \) with the points \((0,k), (1,0)\), and \((d-k,0) \) (see Figure 3(b)).

Then, using Lemma 2.9(i,iii,v) and proceeding as in Step 3 of part (5), we can derive that the tropical limit of the family \( F_t(x,y) \) is as follows:
Figure 3: Deformation patterns for $kL(p_{kl})$ and $kL(z)$
the area $\Delta' \setminus \Delta$ is subdivided as depicted in Figure 3(c),

the limit curve $K_{\delta_1}$ with the Newton parallelogram $\delta_1$ splits into two components, \emph{i.e.} its defining polynomial is $(x - b')(y^* - c'x)$ with some $b', c' \neq 0$,

the limit curve $K_{\delta_2}$ with the Newton trapeze $\delta_2$ is defined by the polynomial $((x - 1)y^* + c'x)\psi(x)$, where the roots of $\psi(x)$ correspond to the intersection points (counting multiplicities) of $\Pi(L(p_{kl}))$ with the other components of $\Pi(C)$.

The convex piece-wise linear function $\nu : \Delta' \to \mathbb{R}$ defining the above subdivision (cf. Step 2 of part (5)) is characterized by its values

\[\nu|_\Delta = 0, \quad \nu(0, k) = \xi > 0, \quad \nu(2, 0) = \eta > 0.\]

Particularly, this yields that $\nu(1, 0) = \xi + \eta$ and $\nu(u, 0) = \eta$ as $u \geq 2$. Combining this with (34) and the fact that the curves $C_t$, $t \neq 0$, cross $E$ at $p_{kl}$ and in a neighborhood of some intersection points of $E$ with the components of $C$ different from $L(p_{kl})$, we derive that $\xi = \eta = \mu$ with $\mu$ defined by (24), and that the limit curves are as follows:

(i) $\{y^k + x - b' = 0\}$ for the triangle $\text{conv}\{(0, k), (0, 2k), (1, k)\}$,

(ii) $\{(x - b')(y^k - c'x) = 0\}$ for the parallelogram $\delta_1 = \text{conv}\{(0, k), (1, 0), (2, 0), (1, k)\}$,

(iii) $\{((x - 1)y^k + c'x)\psi(x) = 0\}$ for the trapeze $\delta_2 = \text{conv}\{(2, 0), (1, k), (d - k, 0), (d - k, k)\}$.

The polynomial $(x - b')(y^k - c'x)$ (which uniquely determines the polynomials in (i) and (iii)) is called the deformation pattern for the component $kL(p_{kl})$ of $C$ associated with the parameterized branch $V$ of $V(p, C)$.

\textbf{Lemma 2.14} The parameters $b'$ and $c'$ in the polynomials (i)-(iii) are uniquely (up to a common factor) defined by the curve $C$, by the choice of the set $z'$ (cf. part (5)) and by relation (24).

\textbf{Proof.} Coming back to the coordinates $(u, v)$ introduces in part (4), we can see that equation (28) and relation (24) yield linear equations for $b'$ and $c'$, which can be uniquely resolved. \hfill $\square$

\textbf{(8) Deformation patterns for nonisolated singularities, III.} In this section we construct deformation patterns for a component $C^{(i)} = kL(z)$ of $C$ with $k \geq 1$ and $\gamma^{(i)} = e_k$, and again we apply the technique of parts (5) and (7).
Perform the blow down $\Pi : \Sigma \to \mathbb{P}^2$, contracting $E_1, ..., E_{a+1}$, and take an affine plane $\mathbb{C}^2 \subset \mathbb{P}^2$ with coordinates $x, y$ such that $\Pi(L(z)) \cap \mathbb{C}^2 = \{y = 0\}$, $\Pi(E_1)$ is the infinitely far point of $\Pi(L(z))$. Furthermore, we assume that the intersection point of $\Pi(E)$ and $\Pi(L(z))$ belonging to $\Pi(z')$ has coordinates $(0, 0)$, the other intersection point of $\Pi(E)$ and $\Pi(L(z))$ has coordinates $(1, 0)$, and the point $\Pi(z) \in \Pi(L_j)$ has coordinates $(x_0, 0)$, $x_0 \neq 0, 1$. At last $\Pi(E)$ is a conic, whose equation can be made $y^2 - x + x^2 = 0$ while keeping the preceding data.

The curve $\Pi(C) \cap \mathbb{C}^2$ is then defined by a polynomial of the form $F_0(x, y) = S(x, y)\tilde{G}_0(x, y)$ with Newton polygon $\Delta$, whose lower part is the union of the segments $[(0, 2 + k), (1, k)]$ and $[(1, k), (d - k, k)]$ (shown by fat line in Figure 3(d)). Here the root of the truncation of $F_0$ on the segment $[(0, 2 + k), (1, k)]$ corresponds to the branch of $\Pi(E)$ centered at the origin. In turn, the truncation of $F_0$ on the segment $[(1, k), (d - k, k)]$ equals $y^k x (x - 1) f(x)$, where the roots of $f(x)$ correspond to the intersection points of $\Pi(L(z))$ with the other components of $\Pi(C \setminus E)$. As in Step 1 of part (5), the curves $\Pi(C_t) \cap \mathbb{C}^2$ are defined by polynomials $F_t(x, y)$ converging to $F_0(x, y)$ as $t \to 0$ and given by (34) with the Newton polygon $\Delta' = \text{conv}(\Delta \cup \{(0, 0), (d - k, 0)\})$ (see Figure 3(e)).

Then, using Lemma 2.7(2i,iv,v) and proceeding as in Step 3 of part (5), we can derive that the tropical limit of the family $F_t(x, y)$ is as follows:

(i) the area $\Delta' \setminus \Delta$ is subdivided as depicted in Figure 3(f),

(ii) the limit curve $K_{\delta_1}$ with the Newton trapeze

$$\delta_1 = \text{conv}\{(0, 0), (d - k, 0), (1, k), (d - k, k)\}$$

is defined by a polynomial $F_{\delta_1}(x, y) = (x - 1)f(x)h^{(1)}(x, y)$ (with $f(x)$ introduced above), $h^{(1)}(x, y)$ has Newton triangle $\text{conv}\{(0, 0), (1, 0), (1, k)\}$, the curve $\{h^{(1)}(x, y) = 0\}$ crosses $\Pi(L(z))$ at $\Pi(z) = (x_0, 0)$ and crosses the line $x = 1$ at one point with multiplicity $k$;

(iii) the convex piece-wise linear function $\nu : \Delta \cup \delta_1 \cup \delta_2 \to \mathbb{R}$ whose graph projects to the above subdivision (cf. Step 2 of part (5)) is characterized by its values

$$\nu|_{\Delta} = 0, \quad \nu(u, 0) = \mu, \quad 0 \leq u \leq d - k .$$

Moreover, using (36), we can easily derive that
(iv) the limit curve \( K_{\delta_2} \) with the Newton triangle

\[
\delta_2 = \text{conv}\{(0,0), (0,2+k), (1,k)\}
\]  

is defined by a trinomial.

We call the polynomial \( h_1(x) \) in (ii) the deformation pattern for the component \( kL(z) \) of \( C \) associated with the parameterized branch \( V \) of \( V(\overline{p}, C) \).

**Lemma 2.15** (1) The limit curve \( K_{\delta_2} \), the polynomial \( f(x) \) and the coefficients \( c_{00}, c_{10}, c_{1k} \) of \( 1, x, \text{ and } xy_1y_k \) in \( h_1(x, y) \) are uniquely defined by the curve \( C \), by the choice of the set \( z' \), and by relation (43). Within these data, there are precisely \( k \) polynomials \( h_1(x, y) \) meeting conditions (ii) above.

(2) In the space of polynomials with Newton trapeze \( \delta_1 \) (cf. (42)), the germ at \( F_{\delta_1} \) of the family of polynomials, splitting into a product of binomials and a polynomial with Newton triangle \( \delta_2 \) (cf. (43)), which meets condition (ii) above, is smooth, and it intersect with the affine space of polynomials, having the same coefficients on the segments \([(1,k), (d-k,k)]\) and \([(0,0), (1,k)]\) as in \( F_{\delta_1} \) and vanishing at the point \((x_0,0)\), transversally at one element (equal to \( F_{\delta_1} \)).

**Proof.** Straightforward. \( \Box \)

We denote the set of polynomials \( h_1(x) \) as in Lemma 2.15 by \( P^{(1)}(L(z)) \).

**Remark 2.16** Observe that none of the coefficients of the polynomials \( h_1(x, y) \) in Lemma 2.15 vanishes, and hence the values \( \nu(1,i) = \mu(k-i)/k, \) \( 0 \leq i \leq k, \) must be integral, in particular, \( k \) divides \( \mu \).

(9) Deformation patterns for nonisolated singularities, IV. In this section, using the above approach, we construct deformation patterns for a component \( C^{(i)} = kL(z) \) of \( C \) with \( k \geq 1 \) and \( \gamma^{(i)} = 2e_k \).

Perform the blow down \( \Pi : \Sigma \to \mathbb{P}^2 \), contracting \( E_1, \ldots, E_{a+1} \), and take an affine plane \( \mathbb{C}^2 \subset \mathbb{P}^2 \) with coordinates \( x, y \) such that \( \Pi(L(z)) \cap \mathbb{C}^2 = \{ y = 0 \} \), \( \Pi(z) = (0,0) \), \( \Pi(E_1) \) is the infinitely far point of \( \Pi(L(z)) \), \( \Pi(L') \) is the infinitely far line, and its tangency point with \( \Pi(E) \) is the infinitely far point of the axis \( \{ x = 0 \} \). In addition to these assumptions, the equation of \( \Pi(E) \) can be brought to the form \( S(x,y) := y + P(x) = 0 \) with a nondegenerate quadratic polynomial \( P(x) \). The curve \( \Pi(C) \cap \mathbb{C}^2 \) is then defined by a polynomial of the
form $F_0(x, y) = S(x, y)\tilde{G}_0(x, y)$ with Newton polygon $\Delta$, whose lower part is the segment $[(0, k), (d - k, k)]$ (see Figure 3(g)). Here the root of the truncation of $F_0$ on the segment $[(0, k), (d - k, k)]$ equals $y^k P(x) f(x)$, where the roots of $f(x)$ correspond to the intersection points of $\Pi(L(z))$ with the other components of $\Pi(C \setminus E)$. As in Step 1 of part (5), the curves $\Pi(C_t) \cap C^2$ are defined by polynomials $F_t(x, y)$ converging to $F_0(x, y)$ as $t \to 0$ and given by (34) with the Newton polygon $\Delta' = \text{conv}(\Delta \cup \{(1, 0), (0, 1), (d - k, 0)\})$ (see Figure 3(h)). Using Lemma 2.9(i,v) and proceeding as in Step 3 of part (5), we can derive that the tropical limit of the family $F_t(x, y)$ is as follows:

(i) $\Delta'$ is subdivided as depicted in Figure 3(h);

(ii) the limit curve $K_{\delta'}$ with the Newton polygon $\delta' = \Delta' \setminus \Delta$ is defined by a polynomial $F_{\delta'}(x, y) = f(x)h_z^{(2)}(x, y)$ ($f(x)$ introduced above), $h_z^{(2)}(x, y)$ has Newton polygon $\text{conv}\{(0, 1), (1, 0), (2, 0), (2, k), (0, k)\}$;

(iii) the convex piece-wise linear function $\nu : \Delta' \to \mathbb{R}$, whose graph projects to the subdivision in Figure 3(h), is characterized by its values

$\nu|_{\Delta} = 0$, $\nu(u, 0) = \mu$, $1 \leq u \leq d - k$.

Using (36), we can easily derive that

$h_z^{(2)}(x, y) = y^k \varphi(y)P(x) + Q(x), \quad \varphi(y) = y^{k-1} + \sum_{j=0}^{k-2} c_j y^j. \quad (44)$

Finally, we observe that by Lemma 2.7(2v), $\{h_z^{(2)}(x, y) = 0\}$ is a rational curve. We call the polynomial $h_z^{(2)}$ the deformation pattern for the component $kL(z)$ of $C$ associated with the parameterized branch $V$ of $V(\mathcal{P}, C)$.

\textbf{Lemma 2.17} (1) The coefficients of the polynomial $Q(x) = b_1 x + b_2 x^2$ in (44) are uniquely defined by the curve $C$, by the choice of the set $z'$, and by relation (24). Within these data, there are precisely $k^2$ polynomials $h_z^{(2)}(x, y)$ satisfying (44) and defining a rational curve.

(2) In the space of polynomials with Newton polygon $\delta'$ (see Figure 3(h)), the germ at $F_{\delta'}$ of the family of polynomials, splitting into a product of binomials and a polynomial with Newton polygon $\delta$, which defines a rational curve, is smooth, and it intersects transversally at one element (equal to $F_{\delta'}$) with the affine space of polynomials $g(x, y)f(x)$, where $g(x, y)$ is given by formula (44) with free parameters $c_0, ..., c_{k-2}$. 

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Proof. We explain only the number $k^2$ of polynomials $h_z^{(2)}(x, y)$ in count. Computing singularities of \( \{ h_z^{(2)}(x, y) = 0 \} \) via relation (43) and equations

\[
h_z^{(2)}(x, y) = \frac{\partial}{\partial x} h_z^{(2)}(x, y) = \frac{\partial}{\partial y} h_z^{(2)}(x, y) = 0,
\]

we, first, obtain an equation $QP' - Q'P = 0$, which has two distinct roots $x_1, x_2$ (it follows from the fact that, by construction, the root of $Q(x)$ can be made generic with respect to the roots of $P(x)$), and, second, we derive that the polynomial $y \varphi(y)$ in (44) has precisely two critical levels (equal to $-Q(x_1)/P(x_1)$ and $-Q(x_2)/P(x_2)$), and hence (cf. the proof of Lemma 2.12), up to a uniquely defined affine modification in the source and in the target, and up to a shift, the polynomial $y \varphi(y)$ coincides with one of the $k$ Chebyshev polynomials of degree $k$. At last, the vanishing at $y = 0$ determines precisely $k$ possible shifts for each of the above modified Chebyshev polynomials.

We denote the set of polynomials $h_z^{(2)}$ as in Lemma 2.15 by $P^{(2)}(L(z))$.

Remark 2.18 Observe that none of the coefficients of the polynomials $h_z^{(2)}(x, y)$ in Lemma 2.15 vanishes, and hence the values $\nu(1, i) = \mu(k - i)/k$, $0 \leq i \leq k$, must be integral, in particular, $k$ divides $\mu$.

(10) From deformation patterns to deformation. Observe that formula (21) counts the number of possible collections of deformation patterns as described in Lemmas 2.10, 2.12, 2.14, 2.15, and 2.17 for all choices made in part (3) of the present section. Thus we complete the proof of Proposition 2.8(2) with the following

Lemma 2.19 In the notations and hypotheses of Proposition 2.8(2), let us be given the set $z'$, chosen as in part (3). Then

(1) for any collection of polynomials

\[
\begin{align*}
\Psi_{ij}(x, y) &\in \mathcal{P}(q'_{ij}) \quad \text{for all } q'_{ij} \in z', \text{ as in Lemma 2.10,} \\
\Psi'(x, y) &\in \mathcal{P}(L') \text{ and } \Psi''(x, y) \in \mathcal{P}(L''), \text{ as in Lemma 2.12,} \\
h_z^{(1)} &\in \mathcal{P}^{(1)}(L(z)) \quad \text{for all } C^{(i)} = kL(z) \text{ with } \gamma^{(i)} = \epsilon_k, \text{ as in Lemma 2.13,} \\
h_z^{(2)} &\in \mathcal{P}^{(2)}(L(z)) \quad \text{for all } C^{(i)} = kL(z) \text{ with } \gamma^{(i)} = 2\epsilon_k, \text{ as in Lemma 2.17,}
\end{align*}
\]

(45)

there exists a unique normally parameterized branch $V$ of $V(p, C)$ realizing the given polynomials as deformation patterns;

(2) there is one-to-one correspondence between the curves $C' \in V(p, C)$ passing through a given point $p' \in L_p$ and the collections of polynomials (43).
**Proof.** By technical reason, we shall work with $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ rather than with the linear system $|D|$, and the curves $C' \in |D|$ will be lifted to sections in $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ denoted by $F_{C'}$ and specified below.

Using the data of Lemma 2.19 and given deformation patterns (45), we will derive a provisional formula for the lift of $V$ to $H^0(\Sigma, \mathcal{O}_\Sigma(D))$. Then, using certain transversality conditions, we will show that the formula does define a unique parameterized branch $V$ of $V(p, C)$.

**Step 1.** In the linear system $|D|$, consider the germ at $C$ of the family of curves, which

- pass through $p \setminus \{ z : L(z) \subset C \}$,
- in neighborhood of the points of
  $$p \cup (\text{Sing}(C_{\text{red}}) \setminus E) \cup (z \setminus \{ q_{kl} \in z : q_{kl} \in L(z), L(z) \subset C \})$$
  realize local deformations as described in Lemma 2.7(ii,iii,iv),
- in neighborhood of the points $\{ q_{kl} \in z : q_{kl} \in L(z), L(z) \subset C \}$, realize deformations in which the local branches of $E$ do not glue up with local branches of the lines $L(z)$ (weaker that that in Lemma 2.9(iv), where we additionally require a special tangency condition).

We will show that this family lifts to a smooth variety germ $M \subset H^0(\Sigma, \mathcal{O}_\Sigma(D))$. Namely, we will describe $M$ as an intersection of certain smooth germs and then verify the transversality of the intersection. In the following computations we use the model of part (4) with affine coordinates $u, v$.

In a neighborhood of a point $p_{kl} \in p \setminus \bigcup_i C^{(i)}$, in the coordinates $x = u - u(p_{kl})$, $y = v$, we have

$$F_C(x, y) = y(\bar{c} + \psi(x, y)), \quad \bar{c} \neq 0, \quad \psi(0,0) = 0.$$  
By Lemma 2.7(ii), the local deformation of $C$ along $V(p, C)$ can be described as

$$y(\bar{c} + \psi(x, y)) + \sum_{kk' + \ell \geq k} c_{k'\ell} x^{k'} y^{\ell'}, \quad c_{k'\ell} \in (\mathbb{C},0),$$
which defines a germ of a linear subvariety $M_{p_{kl}} \subset H^0(\Sigma, \mathcal{O}_\Sigma(D))$ which embeds into $\mathcal{O}_{\Sigma,p_{kl}}$ as the intersection of the ideal $I_{p_{kl}} = \langle y, x^k \rangle$ with the image of $H^0(\Sigma, \mathcal{O}_\Sigma(D))$.

\[3\]There is no canonical map $H^0(\Sigma, \mathcal{O}_\Sigma(D)) \to \mathcal{O}_{\Sigma,p_{kl}}$, but the image of $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ in $\mathcal{O}_{\Sigma,p_{kl}}$ is defined correctly.
In a neighborhood of a point $p_{kl} \in \mathbf{p} \cap C^{(i)}$, where $C^{(i)}$ is reduced, in the coordinates $x = u - u(p_{kl})$, $y = v$, we have

$$F_C(x, y) = y(cy + c'x^k + \psi(x, y)), \quad \overline{cc'} \neq 0, \quad \psi(x, y) = \sum_{k' + k'' > 2k} b_{k' k''} x^{k'} y^{k''}. \quad (46)$$

By Lemma 2.7(ii), the local deformation of $C$ along $V(\mathbf{p}, C)$ can be described as

$$\left( y + \sum_{k', l' \geq 0} c_{k' l'} x^{k'} y^{l'} \right) \left( cy + c'x^k + \psi(x, y) + \sum_{k' + k'' > 2k} c'_{k' l'} x^{k'} y^{k''} \right),$$

$c_{k' l'}, c'_{k' l'} \in (\mathbb{C}, 0)$. It is immediate that this formula defines the germ of a smooth subvariety $M_{p_{kl}} \subset H^0(\Sigma, \mathcal{O}_D)$, whose tangent space embeds into $\mathcal{O}_{\Sigma, p_{kl}}$ as the intersection of the ideal $I_{p_{kl}} = \langle cy + c'x^k + \psi(x, y), y^2, x^k y \rangle$ with the image of $H^0(\Sigma, \mathcal{O}_D)$. In the same way, for a point $p_{kl} \in \mathbf{p} \cap C^{(i)}$, where $C^{(i)} = kL(p_{kl})$, the considered deformation as in Lemma 2.7(iii) is realized in a smooth variety germ $M_{p_{kl}} \subset H^0(\Sigma, \mathcal{O}_D)$, whose tangent space embeds into $\mathcal{O}_{\Sigma, p_{kl}}$ as the intersection of the ideal $I_{p_{kl}} = \langle x^k + \psi(x, y), y^2, x^k y \rangle$ with the image of $H^0(\Sigma, \mathcal{O}_D)$.

In a neighborhood of a point $q_{kl} \in \mathbf{z} \cap C^{(i)}$, where $C^{(i)}$ is reduced, in the coordinates $x = u - u(q_{kl})$, $y = v$, we again have (46), and by Lemma 2.7(iv), the local deformation of $C$ along $V(\mathbf{p}, C)$ can be described as

$$\left( y + \sum_{k', l' \geq 0} c_{k' l'} x^{k'} y^{l'} \right) \left( cy + d(x + c')x^k + \psi(x, y) + \sum_{k' + k'' > 2k} c'_{k' l'} x^{k'} y^{k''} \right),$$

c_{k' l'}, c'_{k' l'} \in (\mathbb{C}, 0)$. This formula also defines the germ of a smooth subvariety $M_{q_{kl}} \subset H^0(\Sigma, \mathcal{O}_D)$, whose tangent space embeds into $\mathcal{O}_{\Sigma, q_{kl}}$ as the intersection of the ideal $I_{q_{kl}} = \langle cy + d(x + c')x^k + \psi(x, y), y^2, x^{k-1} y \rangle$ with the image of $H^0(\Sigma, \mathcal{O}_D)$.

For a point $q_{kl} \in \mathbf{z} \cap C^{(i)}$, where $C^{(i)} = kL(z)$, in the local coordinates $x = u - u(q_{kl})$, $y = v$, where, in addition, $L(z) = \{x = 0\}$, the considered deformation can be described as

$$\left( y + \sum_{k', l' \geq 0} c_{k' l'} x^{k'} y^{l'} \right) \left( x^k + \sum_{k' + l' \geq 0} c'_{k' l'} x^{k'} y^{l'} \right),$$

c_{k' l'}, c'_{k' l'} \in (\mathbb{C}, 0)$. This formula again defines the germ of a smooth subvariety $M_{q_{kl}} \subset H^0(\Sigma, \mathcal{O}_D)$, whose tangent space embeds into $\mathcal{O}_{\Sigma, q_{kl}}$ as the intersection of the ideal $I_{q_{kl}} = \langle y, x^k \rangle$ with the image of $H^0(\Sigma, \mathcal{O}_D)$.

At last, we notice that, if $z \in \Sigma \setminus E$ is a center of $l \geq 2$ distinct smooth local branches $P_1, ..., P_l$ of $C$ of multiplicities $r_1, ..., r_l$, respectively, and, in some local coordinates $x, y$, we
have $z = (0, 0)$, $P_i = \{f_i(x, y) = 0\}$, $i = 1, \ldots, l$, then (cf. [12 Section 5.2]) the closure $M_z \subset H^0(\Sigma, \mathcal{O}_\Sigma(D))$ of the set of sections defining in a neighborhood of $z$ curves with $\sum_{1 \leq i < j \leq 1} r_i r_j (P_i \cdot P_j)$ nodes, is smooth at $F_C$ and its tangent space embeds into $\mathcal{O}_{\Sigma, z}$ as the intersection of the ideal $I_z = \{\prod_{i \neq j} f_i^r : j = 1, \ldots, l\}$ with the image of $H^0(\Sigma, \mathcal{O}_\Sigma(D))$.

The branch $V$ we are looking for must lie inside the intersection of the above germs $M_{pkl}, M_{qkl}, M_z$ with the space of sections vanishing at $\mathfrak{p} \setminus \{z : L(z) \subset C\}$. We claim that this intersection is transverse, or, equivalently, that $H^1(\Sigma, \mathcal{J}_{Z_1/\Sigma}(D)) = 0$, where $\mathcal{J}_{Z_1/\Sigma} \subset \mathcal{O}_\Sigma$ is the ideal sheaf of the scheme $Z_1 \subset \Sigma$ supported at $p \cup z \cup (\text{Sing}(\widetilde{C}_{\text{red}}) \setminus E)$, where it is defined by the above local ideals $I_{pkl}, I_{qkl}, I_z$, respectively, and supported at $\mathfrak{p} \setminus \{z : L(z) \subset C\}$, where it is defined by the maximal ideals. From the exact sequence of sheaves (cf. [7])

$$0 \to \mathcal{J}_{Z_1:E/\Sigma}(D - E) \to \mathcal{J}_{Z_1/\Sigma}(D) \to \mathcal{J}_{Z_1 \cap E/E}(DE) \to 0$$

we obtain the cohomology exact sequence

$$H^1(\Sigma, \mathcal{J}_{Z_1:E/\Sigma}(D - E)) \to H^1(\Sigma, \mathcal{J}_{Z_1/\Sigma}(D)) \to H^1(E, \mathcal{J}_{Z_1 \cap E/E}(DE)) \, .$$

(47)

Here $H^1(E, \mathcal{J}_{Z_1 \cap E/E}(D)) = 0$ in view of Riemann-Roch, since $\deg(Z_1 \cap E) = DE$. Thus, it remains to prove

$$H^1(\Sigma, \mathcal{J}_{Z_1:E/\Sigma}(D - E)) = 0 \, ,$$

(48)

Notice that the scheme $Z_1 : E$ coincides with $Z_1$ in $\Sigma \setminus E$, and it is defined at the points $p_{kl} \in p \cap C^{(i)}$ by the ideals $J_{p_{kl}} = \langle y, x^k \rangle \subset \mathcal{O}_{\Sigma, p_{kl}}$, at the points $q_{kl} \in z \setminus \{q_{kl} : q_{kl} \in L(z), L(z) \subset C\}$ by the ideals $J_{q_{kl}} = \langle y, x^{k-1} \rangle \subset \mathcal{O}_{\Sigma, q_{kl}}$. From the definition of the ideals $I_z, z \in \text{Sing}(\widetilde{C}_{\text{red}}) \setminus E$, and the Noether fundamental theorem, we have a canonical decomposition

$$H^0(\Sigma, \mathcal{J}_{Z_1:E/\Sigma}(D - E)) \simeq \bigoplus_{C^{(i)} \neq s', s'' : s'' L'' \neq \emptyset} \prod_{s' \neq i} F_{C^{(i)}} (F_{L'})^{s'} (F_{L''})^{s''} \cdot H^0(\Sigma, \mathcal{J}_{Z_1(C^{(i)})/\Sigma(C^{(i)})})$$

$$\oplus \prod_{s' \neq i} F_{C^{(i)}} (F_{L'})^{s'} \cdot H^0(\Sigma, \mathcal{O}_{\Sigma}(-s'(K_{\Sigma} + E)))$$

$$\oplus \prod_{s' \neq i} F_{C^{(i)}} (F_{L'})^{s'} \cdot H^0(\Sigma, \mathcal{O}_{\Sigma}(-s''(K_{\Sigma} + E))) \, ,$$

(49)

where $Z_1(C^{(i)}) \subset C^{(i)}$ is the part of $Z_1 : E$ supported along $C^{(i)}$ and disjoint from the other components of $C$ different from $E$ (in particular, $Z_1(kL(z)) = \emptyset$). It follows that (48) turns into a sequence of equalities

$$H^1(\Sigma, \mathcal{J}_{Z_1(C^{(i)})/\Sigma(C^{(i)})}) = 0, \quad i = 1, \ldots, m \, .$$

(50)

We intend to prove even a stronger result:
• for a reduced $C^{(i)} \not\subseteq \Sigma$, we replace (50) with
\[
H^1(\Sigma, \mathcal{J}_{Z_2(C^{(i)})}/\Sigma(C^{(i)})) = 0 ,
\]
where the zero-dimensional scheme $Z_2(C^{(i)})$ is obtained from $Z_1(C^{(i)})$ by adding the (simple) points $\bar{p} \cap C^{(i)}$ and the scheme supported at $z \cap C^{(i)}$ and defined in $q_{kl}' \in z \cap C^{(i)}$ by the ideal $J_{q_{kl}'} = \langle y, x^{k-1} \rangle \subset \mathcal{O}_{\Sigma, q_{kl}'}$;

• for $C^{(i)} = kL(z)$ such that $\gamma^{(i)} = e_k$, we replace (50), which in view of $Z_1(kL(z)) = \emptyset$ reads as $H^1(\Sigma, \mathcal{O}_\Sigma(kL(z))) = 0$, with
\[
H^1(\Sigma, \mathcal{J}_{Z_2(kL(z))/\Sigma(kL(z))}) = 0 ,
\]
where the zero-dimensional scheme $Z_2(kL(z))$ is supported at the point $q_{kl} \in L(z) \cap z$ and is defined there by the ideal $\langle y, x^k \rangle$;

• for $C^{(i)} = kL(z)$ such that $\gamma^{(i)} = 2e_k$, we replace (50) with
\[
H^1(\Sigma, \mathcal{J}_{Z_2(kL(z))/\Sigma(kL(z))}) = 0 ,
\]
where the zero-dimensional scheme $Z_2(kL(z))$ is supported at the point $z \in L(z) \cap \bar{p}$ and is defined there by the ideal $\langle x, y^k \rangle$ in the coordinates $x, y$ introduced in part (9).

To derive (51), we notice that $H^1(\Sigma, \mathcal{J}_{Z_1(C^{(i)})\setminus\bar{p}/\Sigma(C^{(i)}))} = 0$ is equivalent to
\[
H^1(\hat{C}^{(i)}, \mathcal{N}_{\hat{C}^{(i)}}(-d^{(i)})) = 0 ,
\]
where $\hat{C}^{(i)}$ is the normalization of $C^{(i)}$, and
\[
d^{(i)} = \sum_{p_{kl} \in \mathcal{P}
C^{(i)}} k \cdot \hat{p}_{kl} + \sum_{q_{kl} \in \Sigma \cap C^{(i)}} (k - 1) \cdot \hat{q}_{kl} + \sum_{q_{kl}' \in z \cap C^{(i)}} (k - 1) \cdot \hat{q}_{kl}'
\]
($\hat{q}_{kl}'$ being the lift of $q_{kl}'$ to $\hat{C}^{(i)}$). In its turn, (52) comes from (9). This yields, in particular, that (details of computation left to the reader)
\[
h^0(\Sigma, \mathcal{J}_{Z_1(C^{(i)})\setminus\bar{p}/\Sigma(C^{(i))}) = h^0(\Sigma, \mathcal{O}_\Sigma(C^{(i)})) - \deg(Z_1(C^{(i)}) \setminus \bar{p}) = \#(\bar{p} \cap C^{(i)}) + 1 .
\]
Due to the generic choice of the configuration $\bar{p}$, we conclude that
\[
h^0(\Sigma, \mathcal{J}_{Z_2(C^{(i)})\setminus\Sigma(C^{(i))}) = h^0(\Sigma, \mathcal{J}_{Z_1(C^{(i)})\setminus\bar{p}/\Sigma(C^{(i))}) - \#(\bar{p} \cap C^{(i)}) = 1 ,
\]
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which immediately implies (51). Similarly, relations (50) for $C^{(i)} = kL(p_{kl})$ and relations (52), (53) for $C^{(i)} = kL(z)$ lift to the $k$-sheeted coverings $n : \hat{L}(p_{kl}) \to L(p_{kl})$, $n : \hat{L}(z) \to L(z)$ (cf. Proposition 2.2(iii,iv)) in the form

$$H^1(\hat{L}(p_{kl}), \mathcal{N}_{\hat{L}(p_{kl})}(-n^*(p_{kl}))) = H^1(\hat{L}(z), \mathcal{N}_{\hat{L}(z)}(-n^*(q_{kl}))) = H^1(\hat{L}(z), \mathcal{N}_{\hat{L}(z)}(-n^*(z))) = 0,$$

which all come from Riemann-Roch due to

$$\deg n^*(p_{kl}) = \deg n^*(q_{kl}) = \deg n^*(z) = k.$$

Thus, we conclude that the considered family $M$ is smooth.

**Step 2.** Now we construct a certain parametrization of $M$. For, choose a special basis for its tangent space $H^0(\Sigma, \mathcal{J}_{Z_1/\Sigma}(D))$.

**2(a)** In view of (18), we have

$$H^0(\Sigma, \mathcal{J}_{Z_1/\Sigma}(D)) \simeq F_E \cdot H^0(\Sigma, \mathcal{J}_{Z_1:E/\Sigma}(D-E)) \oplus H^0(E, \mathcal{J}_{Z_1 \cap E/E}(DE)),$$

and hence there is a section $F \in H^0(\Sigma, \mathcal{J}_{Z_1: \Sigma}(D))$, which in the coordinates $u, v$ of part (4) can be written in the form

$$f_2(u) + v f_2(u, v),$$

where $f_2$ is given by (26).

**2(b)** Using (51) for a reduced $C^{(i)} \notin -(K_{\Sigma} + E)$, we get $H^0(\Sigma, \mathcal{J}_{Z_2/(C^{(i)})/\Sigma}(C^{(i)})) = F_{C^{(i)}} \cdot \mathbb{C}$, where $F_{C^{(i)}}$ is specified so that, in the coordinates $x, y$ in a neighborhood of each point $q_{kl} \in z' \cap C^{(i)}$ (see part (4)), the coefficient of $y^2$ in $F_{C^{(i)}}(x, y)$ equals $c_{02}^{kl}$ as defined in part (4). Next, from the definition of the scheme $Z_2(C^{(i)})$, we get that $H^0(\Sigma, \mathcal{J}_{Z_2/(C^{(i)})/\Sigma}(C^{(i)}))$, $1 \leq l \leq m$, admits a basis consisting of $F_{C^{(i)}}$ and the sections $F_{i,j,k,l}$ labeled by the points $q_{kl} \in z' \cap C^{(i)}$ and the numbers $0 \leq j < k - 1$, such that

- jet$_{k-2,q_{kl}}(F_{i,j,k,l}(x,0)) = x^j$, $0 \leq j < k - 1$, in the coordinates $x = u - u(q_{kl})$, $y = v$;
- jet$_{k-2,q'_{kl}}(F_{i,j,k,l}(x,0)) = 0$ in the coordinates $x = u - u(q'_{kl})$, $y = v$, for all $i, j, k, l$ and $q'_{kl} \in z' \cap C^{(i)}$, $(k', l') \neq (k, l)$.

**2(c)** The scheme $Z_1(kL(p_{kl}))$ is determined by the condition to cross $E$ at the point $p_{kl}$ with multiplicity $k$. Hence $H^0(\Sigma, \mathcal{J}_{Z_1(kL(p_{kl}))}/\Sigma(kL(p_{kl}))) = (F_{L(p_{kl})})^k \cdot \mathbb{C}$.

**2(d)** For the $k$-dimensional space $H^0(\Sigma, \mathcal{O}_\Sigma(C^{(i)}))$, where $C^{(i)} = kL(z)$, we choose the basis as follows:

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• if \( \gamma = e_k \), we take \((F_{L(z)})^j(F_{L(z)})^{k-j}, 0 \leq j \leq k\), where, in the coordinates \(x, y\) of part (8), \(L(z) \in |-(K_{\Sigma} + E)|\) is the infinitely far line (respectively, \(F_{L(z)}(x, y) = y, F_{\tilde{L}(z)}(x, y) = 1\));

• if \( \gamma = 2e_k \), we take \((F_{L(z)})^j(F_{L(z)})^{k-j}, 0 \leq j \leq k\), where, in the coordinates \(x, y\) of part (9), \(\tilde{L}(z) \in |-(K_{\Sigma} + E)|\) is the infinitely far line (respectively, \(F_{L(z)}(x, y) = y, F_{\tilde{L}(z)}(x, y) = 1\)).

\((2e)\) Finally, for the summands \(H^0(\Sigma, O_{\Sigma}(-s'(K_{\Sigma} + E)))\) and \(H^0(\Sigma, O_{\Sigma}(-s''(K_{\Sigma} + E)))\) in (49), choose the following bases

\[
F_L^{\nu}, F_L^{\nu} - j, 0 \leq j < s', \quad \text{and} \quad F_L^{\nu}, F_L^{\nu} - j, 0 \leq j < s'',
\]

where \(F_L = y, F_{\tilde{L}}(x, y) = 1\) in the coordinates \(x, y\) introduced in Step 1 of part (5), and \(F_L^{\nu}, F_{\tilde{L}}^{\nu}\) are defined similarly.

Thus, our basis for \(H^0(\Sigma, \mathcal{J}_{\Sigma}/\Sigma(D))\) contains the section \(F\) (see (2a)), the sections built in (2b)-(2e) (multiplied with extra factors along (49), (55)), and certain unspecified completion.

We can write then a parametrization of \(M\) in the form

\[
F_E \cdot \prod_{C^{(i)} \neq kL(p_k), kL(z)} \left( F_{C^{(i)}} + \sum_{j,k} \xi_{j,k,l} F_{i,j,k,l} \right) \cdot \prod_{C^{(i)} = kL(p_k)} (F_{L(p_k)})^k \\
\times \prod_{C^{(i)} = kL(z)} \left( (F_{L(z)})^k + \sum_{j} \xi_{j,z} (F_{L(z)})^{k-j} (F_{\tilde{L}(z)})^j \right) \\
\times \left( F_L^{s' - 1} + \sum_{j=0}^{s'-1} \xi'_{j} F_L F_L^{s'-j} \right) \left( F_L^{s'' - 1} + \sum_{j=0}^{s''-1} \xi''_{j} F_L^{2} F_L^{s''-j} \right) \\
+ \xi_0 F + O(\xi^2), \quad (56)
\]

where \(\xi\) is the sequence of

\[
N := \sum_{C^{(i)} \neq kL(z)} (I_{\gamma^{(i)}} - |\gamma^{(i)}|) + \sum_{C^{(i)} = kL(z)} k + s' + s'' + 1 \quad (57)
\]

free parameters

\[
\xi_0, \xi_{j,s}, \xi_{ijkl}, \xi'_{s}, \xi''_{s} \in (\mathbb{C}, 0).
\]
Step 3. We are ready to construct the desired parameterized branch $V$ of $V(\overline{p}, C)$ realizing deformation patterns \[(45)\]. Namely, imposing additional conditions to the parameters $\xi$ in \[(56)\], we will expose a formula for a lift of the required branch $V$ to $H^0(\Sigma, O(D))$.

Put
\[
\mu = \operatorname{lcm}\left\{ k : q_{kl} \in z' \right\} \cup \left\{ s' + 1, s'' + 1 \right\} \cup \left\{ k : kL(z) \in \{C^{(i)}\}_{i=1,...,m} \right\}
\] \[(58)\]
(cf. Remarks 2.11, 2.13, 2.15, and 2.18) and let $\xi_0 = t^\mu c(t)$ with $t \in (\mathbb{C}, 0)$ and $c(0) = c$ given by formula \[(29)\].

(3a) For a point $\xi_{kl} \in z \cap C^{(i)}$ such that $C^{(i)} \not\subset \{ - (K_\Sigma + E) \}$ is reduced, choose a deformation pattern
\[
\Psi_{kl}(x, y) = c_{02}^k y^2 + c_{21}^k x^k y + \sum_{j < k, j \equiv k \mod 2} b_{j1}^k x^j y + b_{00}^k
\]
(cf. (33), (45), and Lemma 2.10(2)). Let $F_{kl}(x, y)$ be the section \[(56)\] written in the coordinates $x = u - u(q_{kl}), y = v$. The coefficient $c_{k-1,1}^k(\xi)$ of $x^{k-1}y$ in $F_{kl}$ vanishes at $\xi = 0$. Since the coefficient of $x^k y$ in $F_{kl}$ turns into $c_{k1}^k \neq 0$ as $\xi = 0$, there is a function $\tau_{kl}(\xi)$ vanishing at the origin such that the coefficient of $x^{k-1}y$ in $F_{kl}(x - \tau(\xi), y)$ equals $0$, and hence the coefficient of $x^j y$, $j < k - 1$, in $F_{kl}(x - \tau(\xi), y)$ equals $\xi_{i,j,kl} + \lambda_{i,j,kl}(\xi)$ with a function $\lambda_{i,j,kl} \in O(\xi^2)$. Thus, due to the smoothness and transversality statement in Lemma 2.10(3), the conditions to fit the given deformation pattern $\Psi_{ij}$ and to realize a local deformation as described in Lemma 2.7(v) amounts to a system of relations
\[
\xi_{i,j,kl} + \lambda_{i,j,kl}(\xi) = \begin{cases} 
\mu^{(k-j)/(2k)}(b_{j1}^k + O(t)), & j \equiv k \mod 2, \\
O(\mu^{(k-j)/(2k)}+1), & j \not\equiv k \mod 2,
\end{cases} \quad 0 \leq j < k - 1.
\] \[(59)\]

(3b) Let $C^{(i)} = kL(z)$ with $\gamma^{(i)} = e_k$, and let, in the coordinates $x, y$ of part (8),
\[
h_z^{(1)}(x, y) = x(y^k + \sum_{j=0}^{k-1} b_{z,j} y^j) - x_0 b_{z,0} \in \mathcal{P}^{(1)}(L(z)).
\] In these coordinates, expression \[(56)\] reads as
\[
(y^2 - x + x^2) \left(y^k + \sum_{j=0}^{k-1} \xi_{z,j} y^j\right) \left(1 + O(x, y, \bar{\xi}) \right) + t^\mu(x_0 b_{z,0} + O(t, \bar{\xi}))(x - 1 + O(x^2, y, \bar{\xi})).
\]

In view of the smoothness and transversality statement in Lemma 2.15(2), the conditions to fit the deformation pattern $h_z^{(1)}(x, y)$, to realize a local deformation as described in Lemma 2.7(2iv), and to hit the point $z \in L(z) \cap \overline{p}$ altogether amount to a system of equations
\[
\xi_{z,j} + O(\bar{\xi}) = t^\mu(k-j)/k(b_{z,j} + O(t, \bar{\xi})), \quad 0 \leq j < k.
\] \[(60)\]
Similarly, let \( C^{(i)} = kL(z) \) with \( \gamma^{(i)} = 2\epsilon_k \), and let, in the coordinates \( x, y \) from part (9), \( h^{(2)}_x(x, y) = P(x)(y^k + \sum_{j=1}^{k-1} c_{x, j} y^j) + Q(x) \in \mathcal{P}^{(2)}(L(z)) \) be a deformation pattern (cf. (44)). In these coordinates, expression (56) reads as

\[
(y^2 + P(x)) \left( y^k + \sum_{j=0}^{k-1} \xi_{x, j} y^j \right) \left( 1 + O(x, y, \xi) \right) + t^\mu(F|_z + O(t, x, y, \xi)).
\]

Again the smoothness and transversality statement in Lemma 2.17(2) convert the condition to fit the deformation pattern \( h^{(2)}_x(x, y) \) and to realize a deformation of the union of \( kL(z) \) with the germs of \( E \) at the points \( E \cap L(z) \) into an immersed cylinder, into a system of equations

\[
\xi_{x, j} + O(\xi^2) = t^\mu(k-j)/k(b_{x, j} + O(t, \xi)), \quad 1 \leq j < k, \quad \xi_{x, 0} = t^\mu(-F|_z + O(t, \xi)).
\]  

(3c) Let \( \Psi'(x, y) \in \mathcal{P}(L') \) be a deformation pattern for the component \((L')^{s'}\) of \( C \). In agreement with (40) we can write (slightly modifying notations)

\[
\Psi'(x, y) = b' + (y + x^2) f'(y), \quad f'(y) = y^{s'} + b'_{s'-1} y^{s'-1} + ... + b'_0,
\]

with \( f'(y) \) defined as in Lemma 2.12. On the other hand, in the coordinates \( x, y \) introduced in Step 1 of part (5), expression (56) with \( \xi_0 = t^\mu(c + O(t)) \) reads as

\[
(y + xy + x^2) \left( y^{s'} + \sum_{k=0}^{s'-1} \xi_k y^k + O(\xi^2) \right) \left( 1 + O(x, y, \xi) \right) + t^\mu(b' + O(t, \xi)).
\]

Again in view of the smoothness and transversality statement in Lemma 2.17(3), the conditions to fit the given deformation pattern \( \Psi' \) together with the demand to realize a deformation in a neighborhood of the point \( L' \cap E \) as described in Lemma 2.7(2vi), turn into a system of equations

\[
\xi_k + O(\xi^2) = t^\mu(s'+1-k)/(s'+1) (b'_k + O(t, \xi)), \quad k = 0, ..., s' - 1.
\]  

Similarly, for the component \((L'')^{s''}\) of \( C \) with a given deformation pattern \( \Psi'' \), we obtain equations

\[
\xi_k + O(\xi^2) = t^\mu(s''+1-k)/(s''+1) (b''_k + O(t, \xi)), \quad k = 0, ..., s'' - 1.
\]  

(3d) In the right-hand side of \( N - 1 \) \( C^{(i)} \neq kL(z) \) \( (I_{\gamma^{(i)}} - |\gamma^{(i)}|) + \sum_{C^{(i)} = kL(z)} k + s' + s'' \) equations (59), (60), (62), and (63), we have \( N - 1 \) unknown functions. Observe that the
linearization of this unified system has a nondegenerate diagonal form. Hence it is uniquely soluble as far as we take \( \xi_0 = t^\mu (a + O(t)) \). This provides a uniquely defined parameterized branch \( V \) of \( V(\mathcal{P}, C) \) matching all the given deformation patterns.

Had we taken \( \xi_0 = t^{r\mu} (c + O(t)) \) with some \( r > 1 \) in the presented construction, we were coming (due to the uniqueness claim) to the branch of \( V(\mathcal{P}, C) \), geometrically coinciding with \( V \), with the parametrization obtained from that of \( V \) via the parameter change \( t \mapsto t^r \).

The proof of Lemma 2.19 and Proposition 2.8(2) is completed. \( \square \)

### 2.6 Recursive formula

The set \( \text{Pic}(\Sigma, E) \times \mathbb{Z} \times \mathbb{Z}_+^\infty \times \mathbb{Z}_+^\infty \) is a semigroup with respect to the operation

\[
(D', g', \alpha', \beta') + (D'', g'', \alpha'', \beta'') = (D' + D'', g' + g'' - 1, \alpha' + \alpha'', \beta' + \beta'') .
\]

Denote by \( A(\Sigma, E) \subset \text{Pic}(\Sigma, E) \times \mathbb{Z} \times \mathbb{Z}_+^\infty \times \mathbb{Z}_+^\infty \) the subsemigroup generated by

- all the quadruples \( (D, g, \alpha, \beta) \) with reduced, irreducible \( D, g \geq 0 \), and \( \alpha, \beta \) satisfying

\[ \bigcup \]

- the quadruples \((−s(K_\Sigma + E), 0, e_s, e_s)\) and \((−s(K_\Sigma + E), 0, 0, 2e_s)\) for all \( s \geq 2 \).

Notice that one may have \( g < 0 \) for elements \((D, g, \alpha, \beta) \in A(\Sigma, E)\); in such a case we always assume \( N_{\Sigma}(D, g, \alpha, \beta) = 0 \).

**Theorem 2.1** Given an element \((D, g, \alpha, \beta) \in A(\Sigma, E)\) such that

\[ R_\Sigma(D, g, \beta) > 0 \quad \text{and} \quad (D, g, \alpha, \beta) \neq (-s(K_\Sigma + E), 0, 0, 2e_s), s \geq 1 , \]

we have

\[
N_{\Sigma}(D, g, \alpha, \beta) = \sum_{j \geq 1, \beta_j > 0} j \cdot N_{\Sigma}(D, g, \alpha + e_j, \beta - e_j)
\]

\[
+ \sum \left( \alpha^{(1)}, \ldots, \alpha^{(m)} \right) \frac{(n - 1)!}{n_1! \cdots n_m!} \binom{k + 3}{3} \prod_{i \in S} \left( \frac{\beta^{(i)}}{\gamma^{(i)}} \right)^{\Gamma^{(i)}} N_{\Sigma}(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}) ,
\]

where

\[ n = R_\Sigma(D, g, \beta), \quad n_i = R_\Sigma(D^{(i)}, g^{(i)}, \beta^{(i)}), \quad i = 1, \ldots, m , \]

\[ S = \{ i \in [1, m] : (D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}) \neq (-s(K_\Sigma + E), 0, e_s, e_s), s \geq 1 \} , \]

and the second sum is taken

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over all integer $k \geq 0$ and all splittings in $A(\Sigma, E)$

$$(D - E + k(K_\Sigma + E), g', \alpha', \beta') = \sum_{i=1}^{m} (D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}) , \quad (65)$$

such that

$g^{(i)} \geq 0$ and $(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)}) \neq (-s(K_\Sigma + E), 0, 0, s\alpha_2), \ s \geq 1, \ i = 1, \ldots, m$,

of all possible quadruples $(D - E + k(K_\Sigma + E), g', \alpha', \beta') \in A(\Sigma, E)$ with $k \geq 0$, satisfying

(a) $\alpha' \leq \alpha, \ \beta' \geq \beta, \ g - g' = \|\beta' - \beta\| + 1$,

(b) each summand $(D^{(i)}, g^{(i)}, \alpha^{(i)}, \beta^{(i)})$ with $n_i = 0$ and $\alpha^{(i)} = 0$ appears in (65) at most once,

over all splittings

$$\beta' = \beta + \sum_{i=1}^{m} \gamma^{(i)}, \ \|\gamma^{(i)}\| > 0, \ 1 \leq i \leq m \quad (66)$$

satisfying the restrictions $\beta^{(i)} \geq \gamma^{(i)}$ for all $1 \leq i \leq m$,

and, finally, the second sum in (64) is factorized by simultaneous permutations in both splittings (65) and (66).

Proof. Immediately follows from from Propositions 2.6 and 2.8. So, the first summand corresponds to the degeneration described in Proposition 2.6(1), and its multiplicity $j$ was computed in Proposition 2.8(1). The second sum corresponds to the degeneration described in Proposition 2.6(2), and its multiplicity was computed in Proposition 2.8(2). Notice only that the subtraction of $-k(K_\Sigma + E)$ in (65) corresponds to all possible combinations $s'L' \cup s''L'', \ s' + s'' = k$, which, by Proposition 2.8(2), altogether contribute to the multiplicity $(k + 1) + k \cdot 2 + (k - 1) \cdot 3 + \ldots + (k + 1) = \binom{k+3}{k}$.

Note that the second sum in the right hand side of (64) becomes empty if $D - E$ is not effective.

Theorem 2.2 All the numbers $N_{\Sigma}(D, g, \alpha, \beta)$ with $(D, g, \alpha, \beta) \in A(\Sigma, E)$ and $R_{\Sigma}(D, g, \beta) > 0$ are determined recursively by formula (64) and the initial values $N_{\Sigma}(D, g, \alpha, \beta)$ indicated in Proposition 2.5.
**Proof.** Straightforward. □

In the important case of genus zero, all components of degenerate curves in Proposition 2.6 are rational, and the general formula (64) reduces to the count of only genus zero terms. Furthermore, it can be simplified so that the terms corresponding to multiple covers will not explicitly appear.

Let us write for short $N_{\Sigma}(D,\alpha,\beta) := N_{\Sigma}(D,0,\alpha,\beta)$ and introduce the semigroup $A_0(\Sigma,E) \subset \text{Pic}(\Sigma,E) \times (\mathbb{Z}_+^\infty)^2$, generated by the triples $(D,\alpha,\beta)$ with reduced, irreducible $D$ and $I(\alpha + \beta) = DE$.

**Corollary 2.20** Given an element $(D,\alpha,\beta) \in A_0(\Sigma,E)$ such that $R_{\Sigma}(D,0,\beta) > 0$, we have

$$N_{\Sigma}(D,\alpha,\beta) = \sum_{j \geq 1, \beta_j > 0} j \cdot N_{\Sigma}(D,\alpha + e_j,\beta - e_j)$$

$$+ \sum_{k \geq 0, \beta(0) !} 2^{\|\beta(0)\|} \binom{k+3}{3} \binom{n-1}{n_1,\ldots,n_m} \prod_{i=1}^m \left( \binom{\beta(i)}{\gamma(i)} N_{\Sigma}(D^{(i)},\alpha^{(i)},\beta^{(i)}) \right),$$

where

$$n = R_{\Sigma}(D,0,\beta), \quad n_i = R_{\Sigma}(D^{(i)},0,\beta^{(i)}), \quad i = 1,\ldots,m,$$

and the second sum is taken

- over all integers $k \geq 0$ and vectors $\alpha^{(0)},\beta^{(0)} \in \mathbb{Z}_+^\infty$ such that $\alpha^{(0)} \leq \alpha$ and $\beta^{(0)} \leq \beta$;

- over all sequences

$$(D^{(i)},\alpha^{(i)},\beta^{(i)}) \in A_0(\Sigma,E), \quad 1 \leq i \leq m,$$

such that

(i) $D^{(i)} \neq -(K_\Sigma + E)$ and $R(D^{(i)},0,\beta^{(i)}) \geq 0$ for all $i = 1,\ldots,m$,

(ii) $D - E = \sum_{i=1}^m D^{(i)} - (k + I\alpha^{(0)} + I\beta^{(0)})(K_\Sigma + E)$,

(iii) $\sum_{i=0}^m \alpha^{(i)} \leq \alpha, \sum_{i=0}^m \beta^{(i)} \geq \beta$,

(iv) each triple $(D^{(i)},0,\beta^{(i)})$ with $n_i = 0$ appears in (68) at most once,
• over all sequences
\[ \gamma(i) \in \mathbb{Z}_+^\infty, \quad \|\gamma(i)\| = 1, \quad i = 1, \ldots, m, \quad (69) \]
satisfying
\[ \beta(i) \geq \gamma(i), \quad i = 1, \ldots, m, \quad \text{and} \quad \sum_{i=1}^{m} (\beta(i) - \gamma(i)) = \beta - \beta(0), \]
and the second sum in \((67)\) is factorized by simultaneous permutations in the sequences \((68)\) and \((69)\).

3 Counting curves on \(\mathbb{P}^2_7\)

Since \(\mathbb{P}^2_{6,1}\) is almost Fano in the sense of [13, Section 4.1], i.e. \(K_{\mathbb{P}^2_{6,1}}\) is negative on all curves except for \(E\). Hence [13, Theorem 4.2] yields the following relation between \(GW_g(\mathbb{P}^2_7, D)\) and enumerative invariants of \(\mathbb{P}^2_{6,1}\) (abusing notations, we use the same symbol for divisor classes corresponding via a natural isomorphism \(\text{Pic}(\mathbb{P}^2_7) \cong \text{Pic}(\mathbb{P}^2_{6,1})\)).

**Theorem 3.1** For any effective divisor class \(D\) on \(\mathbb{P}^2_7\), the Gromov-Witten invariant \(GW_g(\mathbb{P}^2_7, D)\) is given by the formula
\[ GW_g(\mathbb{P}^2_7, D) = \sum_{i \geq 0} \binom{DE + 2i}{i} N_{\mathbb{P}^2_{6,1}}(D - iE, g, 0, (DE + 2i)e_1). \quad (70) \]

**Proof.** We have to make only one remark. The maps \(n : C' \cup C'' \rightarrow \mathbb{P}^2_{6,1}\) which are counted in [13, Theorem 2], where the components of \(C'\) are isomorphically mapped onto \(E\), and the components of \(C''\) are not mapped onto \(E\), the part \(C''\) is connected. Thus, by Proposition [2.1(2)] it is smooth of genus \(g\), and its image in \(\mathbb{P}^2_{6,1}\) crosses \(E\) transversally. The binomial coefficient reflects the choice of intersection points of \(C'\) and \(C''\) among the points of \(n^{-1}(n_*(C'') \cap E)\). \(\square\)

**Example 3.1** For the popular case of \(D = -2K_6 = 6L - 2(E_1 + \ldots + E_6)\) (cf. [13, Page 119, Table II], [4, Page 88], [13, Section 9.2]), we compute Gromov-Witten invariants for all genera in two ways, once using Theorem 2.1 for \(\mathbb{P}^2_{6,1} = \mathbb{P}^2_6\), and another time using Theorem 2.1 for \(\mathbb{P}^2_{6,1}\) with \(D = 6L - 2(E_2 - \ldots - E_7)\) and then applying Theorem 3.1 (cf. [13, Section 9.2]).
For the case of $D = -2K_{\mathbb{P}^2} = 6L - 2(E_1 + \ldots + E_7)$ (cf. [4, Page 88]), our formula gives the following values for the Gromov-Witten invariants:

| $g$ | 0  | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|----|
| $GW$ | 576 | 236 | 26 | 1  |

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Institute of Mathematics,
Hebrew University of Jerusalem,
Givat Ram, Jerusalem, 91904, Israel
E-mail: mendy.shoval@mail.huji.ac.il

School of Mathematical Sciences,
Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel
E-mail: shustin@post.tau.ac.il