The Hecke algebra of a reductive $p$-adic group: a geometric conjecture

Anne-Marie Aubert, Paul Baum and Roger Plymen

Abstract

Let $\mathcal{H}(G)$ be the Hecke algebra of a reductive $p$-adic group $G$. We formulate a conjecture for the ideals in the Bernstein decomposition of $\mathcal{H}(G)$. The conjecture says that each ideal is geometrically equivalent to an algebraic variety. Our conjecture is closely related to Lusztig’s conjecture on the asymptotic Hecke algebra. We prove our conjecture for $\text{SL}(2)$ and $\text{GL}(n)$. We also prove part (1) of the conjecture for the Iwahori ideals of the groups $\text{PGL}(n)$ and $\text{SO}(5)$.

1 Introduction

The reciprocity laws in number theory have a long development, starting from conjectures of Euler, and including contributions of Legendre, Gauss, Dirichlet, Jacobi, Eisenstein, Takagi and Artin. For the details of this development, see [28]. The local reciprocity law for a local field $F$, which concerns the finite Galois extensions $E/F$ such that $\text{Gal}(E/F)$ is commutative, is stated and proved in [42, p. 320]. This local reciprocity law was dramatically generalized by Langlands, see [5]. The local Langlands correspondence for $\text{GL}(n)$ is a noncommutative generalization of the reciprocity law of local class field theory [47]. The local Langlands conjectures, and the global Langlands conjectures, all involve, inter alia, the representations of reductive $p$-adic groups [5].

To each reductive $p$-adic group $G$ there is associated the Hecke algebra $\mathcal{H}(G)$, which we now define. Let $K$ be a compact open subgroup of $G$, and define $\mathcal{H}(G//K)$ as the convolution algebra of all complex-valued, compactly-supported functions on $G$ such that $f(k_1 x k_2) = f(x)$ for all $k_1, k_2$ in $K$. The Hecke algebra $\mathcal{H}(G)$ is then defined as

$$\mathcal{H}(G) := \bigcup_K \mathcal{H}(G//K).$$
The smooth representations of $G$ on a complex vector space $V$ correspond bijectively to the nondegenerate representations of $\mathcal{H}(G)$ on $V$, see [4, p.2].

In this article, we consider $\mathcal{H}(G)$ from the point of view of noncommutative (algebraic) geometry.

We recall that the coordinate rings of affine algebraic varieties are precisely the commutative, unital, finitely generated, reduced $\mathbb{C}$-algebras, see [16, II.1.1].

The Hecke algebra $\mathcal{H}(G)$ is a non-commutative, non-unital, non-finitely-generated, non-reduced $\mathbb{C}$-algebra, and so cannot be the coordinate ring of an affine algebraic variety.

The Hecke algebra $\mathcal{H}(G)$ is non-unital, but it admits local units, see [4, p.2].

The algebra $\mathcal{H}(G)$ admits a canonical decomposition into ideals, the Bernstein decomposition [4]:

$$\mathcal{H}(G) = \bigoplus_{s \in \mathfrak{B}(G)} \mathcal{H}^s(G).$$

Each ideal $\mathcal{H}^s(G)$ is a non-commutative, non-unital, non-finitely-generated, non-reduced $\mathbb{C}$-algebra, and so cannot be the coordinate ring of an affine algebraic variety.

In section 2, we define the extended centre $\tilde{Z}(G)$ of $G$. At a crucial point in the construction of the centre $Z(G)$ of the category of smooth representations of $G$, certain quotients are made: we replace each ordinary quotient by the extended quotient to create the extended centre.

In section 3 we define morita contexts, following [13].

In section 4 we prove that each ideal $\mathcal{H}^s(G)$ is Morita equivalent to a unital $k$-algebra of finite type, where $k$ is the coordinate ring of a complex affine algebraic variety. We think of the ideal $\mathcal{H}^s(G)$ as a noncommutative algebraic variety, and $\mathcal{H}(G)$ as a noncommutative scheme.

In section 5 we formulate our conjecture. We conjecture that each ideal $\mathcal{H}^s(G)$ is geometrically equivalent (in a sense which we make precise) to the coordinate ring of a complex affine algebraic variety $X^s$:

$$\mathcal{H}^s(G) \times \mathcal{O}(X^s) = \tilde{Z}^s(G).$$

The ring $\tilde{Z}^s(G)$ is the $s$-factor in the extended centre of $G$. The ideals $\mathcal{H}^s(G)$ therefore qualify as noncommutative algebraic varieties.

We have stripped away the homology and cohomology which play such a dominant role in [2], [8], leaving behind three crucial moves:
Morita equivalence, morphisms which are spectrum-preserving with respect to filtrations, and deformation of central character. These three moves generate the notion of geometric equivalence.

In section 6 we prove the conjecture for all generic points \( s \in \mathcal{B}(G) \).

In section 7 we prove our conjecture for \( \text{SL}(2) \).

In section 8 we discuss some general features used in proving the conjecture for certain examples.

In section 9 we review the asymptotic Hecke algebra of Lusztig.

The asymptotic Hecke algebra \( J \) plays a vital role in our conjecture, as we now proceed to explain. One of the Bernstein ideals in \( \mathcal{H}(G) \) corresponds to the point \( i \in \mathcal{B}(G) \), where \( i \) is the quotient variety \( \Psi(T)/W_f \). Here, \( T \) is a maximal torus in \( G \), \( \Psi(T) \) is the complex torus of unramified quasicharacters of \( T \), and \( W_f \) is the finite Weyl group of \( G \). Let \( I \) denote an Iwahori subgroup of \( G \), and define \( e \) as follows:

\[
e(x) = \begin{cases} 
\text{vol}(I)^{-1} & \text{if } x \in I, \\
0 & \text{otherwise.}
\end{cases}
\]

Then the **Iwahori ideal** is the two-sided ideal generated by \( e \):

\[
\mathcal{H}^i(G) := \mathcal{H}(G)e\mathcal{H}(G).
\]

There is a Morita equivalence \( \mathcal{H}(G)e\mathcal{H}(G) \sim e\mathcal{H}(G)e \) and in fact we have

\[
\mathcal{H}^i(G) := \mathcal{H}(G)e\mathcal{H}(G) \prec e\mathcal{H}(G)e \cong \mathcal{H}(G/\!/I) \cong \mathcal{H}(W,q_F) \times J
\]

where \( \mathcal{H}(W,q_F) \) is an (extended) affine Hecke algebra based on the (extended) Coxeter group \( W \), \( J \) is the asymptotic Hecke algebra, and \( \prec \) denotes geometric equivalence. Now \( J \) admits a decomposition into finitely many two-sided ideals

\[
J = \bigoplus_c J_c
\]

labelled by the two-sided cells \( c \) in \( W \). We therefore have

\[
\mathcal{H}^i(G) \cong \bigoplus J_c.
\]

This canonical decomposition of \( J \) is well-adapted to our conjecture.

In section 10 we prove that

\[
J_{c_0} \cong \mathcal{Z}^i(G)
\]

where \( c_0 \) is the lowest two-sided cell, for any connected \( F \)-split adjoint simple \( p \)-adic group \( G \). We note that \( \mathcal{Z}^i(G) \) is the ring of regular functions on the *ordinary quotient* \( \Psi(T)/W_f \).
In section 11 we prove the conjecture for $GL(n)$. We establish that for each point $s \in \mathcal{B}(G)$, we have

$$\mathcal{H}^s(GL(n)) \cong \tilde{s}(GL(n)).$$

In section 12 we prove part (1) of the conjecture for the Iwahori ideal in $\mathcal{H}(PGL(n))$.

In section 13 we prove part (1) of the conjecture for the Iwahori ideal in $\mathcal{H}(SO(5))$. Our proofs depend crucially on Xi’s affirmation, in certain special cases, of Lusztig’s conjecture on the asymptotic Hecke algebra $J$ (see [35, §10]).

In section 14 we discuss some consequences of the conjecture.

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2 The extended centre

Let $G$ be the set of rational points of a reductive group defined over a local nonarchimedean field $F$, and let $\mathcal{R}(G)$ denote the category of smooth $G$-modules. Let $(L, \sigma)$ denote a cuspidal pair: $L$ is a Levi subgroup of $G$ and $\sigma$ is an irreducible supercuspidal representation of $L$. The group $\Psi(L)$ of unramified quasicharacters of $L$ has the structure of a complex torus.

We write $[L, \sigma]_G$ for the equivalence class of $(L, \sigma)$ and $\mathcal{B}(G)$ for the set of equivalence classes, where the equivalence relation is defined by $(L, \sigma) \sim (L', \sigma')$ if $gLg^{-1} = L'$ and $g\sigma \simeq \nu'\sigma'$, for some $g \in G$ and some $\nu' \in \Psi(L')$. For $s = [L, \sigma]_G$, let $\mathcal{R}^s(G)$ denote the full subcategory of $\mathcal{R}(G)$ whose objects are the representations $\Pi$ such that each irreducible subquotient of $\Pi$ is a subquotient of a parabolically induced representation $i_P^G(\nu \sigma)$ where $P$ is a parabolic subgroup of $G$ with Levi subgroup $L$ and $\nu \in \Psi(L)$. The action (by conjugation) of $N_G(L)$ on $L$ induces an action of $W(L) = N_G(L)/L$ on $\mathcal{B}(L)$. Let $W_t$ denote the stabilizer of $t = [L, \sigma]_L$ in $W(L)$. Thus $W_t = N_t/L$ where

$$N_t = \{ n \in N_G(L) : n\sigma \simeq \nu \sigma, \text{ for some } \nu \in \Psi(L) \}.$$

It acts (via conjugation) on $\text{Irr}^t L$, the set of isomorphism classes of irreducible objects in $\mathcal{R}^t(L)$.

Let $\Omega(G)$ denote the set of $G$-conjugacy classes of cuspidal pairs $(L, \sigma)$. The groups $\Psi(L)$ create orbits in $\Omega(G)$. Each orbit is of the form $D_\sigma/W_t$ where $D_\sigma = \text{Irr}^t L$ is a complex torus.
We have
\[ \Omega(G) = \bigsqcup D_{\sigma}/W. \]

Let \( Z(G) \) denote the centre of the category \( \mathcal{R}(G) \). The centre of an abelian category (with a small skeleton) is the endomorphism ring of the identity functor. An element \( z \) of the centre assigns to each object \( A \) in \( \mathcal{R}(G) \) a morphism \( z(A) \) such that
\[ f \cdot z(A) = z(B) \cdot f \]
for each morphism \( f \in \text{Hom}(A,B) \).

According to Bernstein's theorem \(^\text{[4]}\) we have the explicit decomposition of \( \mathcal{R}(G) \):
\[ \mathcal{R}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{R}^s(G). \]

We also have
\[ Z(G) \cong \prod Z^s \]
where
\[ Z^s(G) = \mathcal{O}(D_{\sigma}/W) \]
is the centre of the category \( \mathcal{R}^s(G) \).

Let the finite group \( \Gamma \) act on the space \( X \). We define, as in \(^\text{[1]}\),
\[ \widetilde{X} := \{ (\gamma, x) : \gamma x = x \} \subset \Gamma \times X \]
and define the \( \Gamma \)-action on \( \widetilde{X} \) as follows:
\[ \gamma_1(\gamma, x) := (\gamma_1 \gamma \gamma_1^{-1}, \gamma_1 x). \]
The extended quotient of \( X \) by \( \Gamma \) is defined to be the ordinary quotient \( \widetilde{X}/\Gamma \). If \( \Gamma \) acts freely, then we have
\[ \widetilde{X} = \{ (1, x) : x \in X \} \cong X \]
and, in this case, \( \widetilde{X}/\Gamma = X/\Gamma \).

We will write
\[ \mathcal{O}(\widetilde{D}_{\sigma}/W) := \mathcal{O}(\widetilde{D}_{\sigma}/W). \]

We now form the extended centre
\[ \mathcal{O}(D_{\sigma}/W) = Z(G). \]
3 Morita contexts and the central character

A fundamental theorem of Morita says that the categories of modules over two rings with identity $R$ and $S$ are equivalent if and only if there exists a strict Morita context connecting $R$ and $S$.

A Morita context connecting $R$ and $S$ is a datum $(R, S_R M_{S,S} N_R, \phi, \psi)$ where $M$ is an $R$-$S$-bimodule, $N$ is an $S$-$R$-bimodule, $\phi : M \otimes_S N \rightarrow R$ is a morphism of $R$-$R$-bimodules and $\psi : N \otimes_R M \rightarrow S$ is a morphism of $S$-$S$-bimodules such that

\[
\phi(m \otimes n)m' = m\psi(n \otimes m')
\]

\[
\psi(n \otimes m)n' = n\psi(m \otimes n')
\]

for any $m, m' \in M, n, n' \in N$. A Morita context is strict if both maps $\phi, \psi$ are isomorphisms, see [13].

Let $R$ be a ring with local units, i.e. for any finite subset $X$ of $R$, there exists an idempotent element $e \in R$ such that $ex = xe = x$ for any $x \in X$. An $R$-module $M$ is unital if $RM = M$. Unital modules are called non-degenerate in [4].

Let $R$ and $S$ be two rings with local units, and $R - MOD$ and $S - MOD$ be the associated categories of unital modules. A Morita context for $R$ and $S$ is a datum $(R, S_R M_{S,S} N_R, \phi, \psi)$ with the condition that $M$ and $N$ are unital modules to the left and to the right. If the Morita context is strict, then we obtain by [13] Theorem 4.3 an equivalence of categories $R - MOD$ and $S - MOD$.

If $R$ and $S$ are rings with local units which are connected by a strict Morita context, then we will say that $R$ and $S$ are Morita equivalent and write

\[ R \sim_{\text{morita}} S. \]

We will use repeatedly the following elementary lemmas.

**Lemma 1.** Let $R$ be a ring with local units. Let $M_n(R)$ denote $n \times n$ matrices over $R$. Then we have

\[ M_n(R) \sim_{\text{morita}} R. \]

*Proof.* Let $M_{i \times j}(R)$ denote $i \times j$ matrices over $R$. Then we have a strict Morita context

\[ (R, M_{n \times n}(R), M_{1 \times n}(R), M_{n \times 1}(R), \phi, \psi) \]

where $\phi, \psi$ denote matrix multiplication. \qed
Lemma 2. Let \( R \) be a ring with local units. Let \( e \) be an idempotent in \( R \). Then we have
\[
ReR \sim_{\text{morita}} eRe.
\]

Proof. Given \( r \in R \), there is an idempotent \( e \in R \) such that \( r = er \in R^2 \) so that \( R \subset R^2 \subset R \). A ring \( R \) with local units is idempotent: \( R = R^2 \). This creates a strict Morita context
\[
(eRe, ReR, eR, Re, \phi, \psi)
\]
where \( \phi, \psi \) are the obvious multiplication maps in \( R \). We now check identities such as \((eR)(Re) = eRe, (Re)(eR) = ReR\).

Let \( X \) be a complex affine algebraic variety, let \( \mathcal{O}(X) \) denote the algebra of regular functions on \( X \). Let
\[
k := \mathcal{O}(X).
\]
A \( k \)-algebra \( A \) is a \( \mathbb{C} \)-algebra which is also a \( k \)-module such that
\[
1_k \cdot a = a, \quad \omega(a_1 a_2) = (\omega a_1) a_2 = a_1 (\omega a_2), \quad \omega(\lambda a) = \lambda (\omega a) = (\lambda \omega) a
\]
for all \( a, a_1, a_2 \in A, \omega \in k, \lambda \in \mathbb{C} \).

If \( A \) has a unit then a \( k \)-algebra is a \( \mathbb{C} \)-algebra with a given unital homomorphism of \( \mathbb{C} \)-algebras from \( k \) to the centre of \( A \). If \( A \) does not have a unit, then a \( k \)-algebra is a \( \mathbb{C} \)-algebra with a given unital homomorphism of \( \mathbb{C} \)-algebras from \( k \) to the centre of the multiplier algebra \( M(A) \) of \( A \), see [4].

A \( k \)-algebra \( A \) is of \textit{finite type} if, as a \( k \)-module, \( A \) is finitely generated.

If \( A, B \) are \( k \)-algebras, then a morphism of \( k \)-algebras is a morphism of \( \mathbb{C} \)-algebras which is also a morphism of \( k \)-modules.

If \( A \) is a \( k \)-algebra and \( Y \) is a unital \( A \)-module then the action of \( A \) on \( Y \) extends uniquely to an action of \( M(A) \) on \( Y \) such that \( Y \) is a unital \( M(A) \)-module [4]. In this way the category \( A - \text{MOD} \) of unital \( A \)-modules is equivalent to the category \( M(A) - \text{MOD} \) of unital \( M(A) \)-modules. Since the \( k \)-algebra structure for \( A \) can be viewed as a unital homomorphism of \( \mathbb{C} \)-algebras from \( k \) to the centre of \( M(A) \), it now follows that given any unital \( A \)-module \( Y \), \( Y \) is canonically a \( k \)-module with the following compatibility between the \( A \)-action and the \( k \)-action:
\[
1_k \cdot y = y, \quad \omega(a y) = (\omega a) y = a (\omega y), \quad \omega(\lambda y) = \lambda (\omega y) = (\lambda \omega) y
\]
for all \( a \in A, y \in Y, \omega \in k, \lambda \in \mathbb{C} \).
If $A, B$ are $k$-algebras then a strict Morita context (in the sense of $k$-algebras) connecting $A$ and $B$ is a 6-tuple

$$(A, B, M, N, \phi, \psi)$$

which is a strict Morita context connecting the rings $A, B$. We require, in addition, that

$$\omega y = y\omega$$

for all $y \in M, \omega \in k$. Similarly for $N$. When this is satisfied, we will say that the $k$-algebras $A$ and $B$ are Morita equivalent and write

$$A \sim_{\text{morita}} B.$$ 

By central character or infinitesimal character we mean the following. Let $M$ be a simple $A$-module. Schur’s lemma implies that each $\theta \in k$ acts on $M$ via a complex number $\lambda(\theta)$. Then $\theta \mapsto \lambda(\theta)$ is a morphism of $\mathbb{C}$-algebras $k \to \mathbb{C}$ and is therefore given by evaluation at a $\mathbb{C}$-point of $X$. The map $\text{Irr}(A) \to X$ so obtained is the central character (or infinitesimal character). The notation for the infinitesimal character is as follows:

$$\text{inf.ch.} : \text{Irr}(A) \longrightarrow X.$$ 

If $A$ and $B$ are $k$-algebras connected by a strict Morita context then we have a commutative diagram in which the top horizontal arrow is bijective:

$$\text{Irr}(A) \longrightarrow \text{Irr}(B)$$

$$\downarrow \text{inf.ch.} \quad \downarrow \text{inf.ch.}$$

$$X \quad \longrightarrow \quad X$$

$$\text{id}$$

The algebras that occur in this paper have the property that

$$\text{Prim}(A) = \text{Irr}(A).$$

In particular, each Bernstein ideal $\mathcal{H}^s(G)$ has this property and any $k$-algebra of finite type has this property, see [2].

4 A Morita equivalence

Let $\mathcal{H} = \mathcal{H}(G)$ denote the Hecke algebra of $G$. Note that $\mathcal{H}(G)$ admits a set $E$ of local units. For let $K$ be a compact open subgroup of $G$ and define

$$e_K(x) = \begin{cases} 
\text{vol}(K)^{-1} & \text{if } x \in K, \\
0 & \text{otherwise}.
\end{cases}$$

Given a finite set $X \subset \mathcal{H}(G)$, we choose $K$ sufficiently small. Then we have $e_K x = x = xe_K$ for all $x \in X$. It follows that $\mathcal{H}$ is an idempotent algebra: $\mathcal{H} = \mathcal{H}^2$.

We have the Bernstein decomposition

$$\mathcal{H}(G) = \bigoplus_{s \in \mathcal{B}(G)} \mathcal{H}^s(G)$$

of the Hecke algebra $\mathcal{H}(G)$ into two-sided ideals.

**Lemma 3.** Each Bernstein ideal $\mathcal{H}^s(G)$ admits a set of local units.

**Proof.** Define

$$E^s := E \cap \mathcal{H}^s(G).$$

Then $E^s$ is a set of local units for $\mathcal{H}^s(G)$. \qed

We recall the notation from section 2:

$$s \in \mathcal{B}(G), \quad s = [L, \sigma]_G, \quad D_\sigma = \Psi(L)/\mathcal{G}.$$

**Theorem 1.** Let $s \in \mathcal{B}(G), k = \mathcal{O}(D_\sigma)$. The ideal $\mathcal{H}^s(G)$ is a $k$-algebra Morita equivalent to a unital $k$-algebra of finite type. If $W_i = \{1\}$ then $\mathcal{H}^s(G)$ is Morita equivalent to $k$.

**Proof.** Let $s = [L, \sigma]_G$. We will write

$$\mathcal{H}(G) = \mathcal{H} = \mathcal{H}^s \oplus \mathcal{H}'$$

where $\mathcal{H}'$ denotes the sum of all $\mathcal{H}^t$ with $t \in \mathcal{B}(G)$ and $t \neq s$. It follows from \cite[(3.7)]{4} that there is a compact open subgroup $K$ of $G$ with the property that $V^K \neq 0$ for every irreducible representation $(\pi, V)$ with $I(\pi) = s$. We will write $e_K = e + e'$ with $e \in \mathcal{H}^s, e' \in \mathcal{H}'$. Both $e$ and $e'$ are idempotent and we have $ee' = 0 = e'e$.

Given $h \in \mathcal{H}$, we will write $h = h^s + h'$ with $h^s \in \mathcal{H}^s, h' \in \mathcal{H}'$. By \cite[3.1, 3.4 – 3.6]{10}, we have $\mathcal{H}^s(G) \cong \mathcal{H}e\mathcal{H}$. We note that this follows from the general considerations in \cite[§3]{10}, and does not use the existence or construction of types. The ideal $\mathcal{H}^s$ is the idempotent two-sided ideal generated by $e$. By Lemma 2 we have

$$\mathcal{H}^s = \mathcal{H}e\mathcal{H} \sim_{\text{morita}} e\mathcal{H}e.$$

Since $eh' = he' = 0, e'h^s = h^se' = 0$ we also have

$$e\mathcal{H}e = e\mathcal{H}^s e = e_K \mathcal{H}^s e_K.$$

It follows that

$$\mathcal{H}^s \sim_{\text{morita}} e_K \mathcal{H}^s e_K.$$
Let $B$ be the unital algebra defined as follows:

$$B := e_K \mathcal{H}^s(G)e_K.$$ 

We will use the notation in [3, p.80]. Let $\Omega = D_\sigma/W_t$ and let $\Pi := \Pi(\Omega)^K, \Lambda := \text{End} \Pi$. There is a strict Morita context

$$(B, \Lambda, \Pi, \Pi^*, \phi, \psi).$$

We therefore have

$$\mathcal{H}^s(G) \sim_{\text{morita}} B \sim_{\text{morita}} \Lambda.$$ 

The intertwining operators $A_w$ generate $\Lambda$ over $\Lambda(D_\sigma)$, see [3, p.81]. We also have [3, p. 73]

$$\Lambda(D_\sigma) = \mathcal{O}(\Psi(L)) \rtimes \mathcal{G}$$

where the finite abelian group $\mathcal{G}$ is defined as follows:

$$\mathcal{G} := \{\psi \in \Psi(G) : \psi \sigma \cong \sigma\}.$$ 

Note that $\mathcal{G}$ acts freely on the complex torus $\Psi(L)$.

The $k$-algebras $\mathcal{O}(\Psi(L)) \rtimes \mathcal{G}$ and $\mathcal{O}(\Psi(L)/\mathcal{G})$ are connected by the following strict Morita context:

$$(\mathcal{O}(\Psi(L)) \rtimes \mathcal{G}, \mathcal{O}(\Psi(L)/\mathcal{G}), \mathcal{O}(\Psi(L)), \mathcal{O}(\Psi(L)), \phi, \psi).$$

We conclude that

$$\Lambda(D_\sigma) = \mathcal{O}(\Psi(L)) \rtimes \mathcal{G} \sim_{\text{morita}} \mathcal{O}(\Psi(L)/\mathcal{G}) = \mathcal{O}(D_\sigma) = k.$$ 

Therefore, $\Lambda$ is finitely generated as a $k$-module. Note that

$$\text{Centre} \Lambda(D_\sigma) \cong k.$$ 

If $W_t = \{1\}$ then there are no intertwining operators, so that $\Lambda = \Lambda(D_\sigma)$. In that case we have

$$\mathcal{H}^s(G) \sim_{\text{morita}} k.$$ 

\[\square\]
The conjecture

Let $k$ be the coordinate ring of a complex affine algebraic variety $X$, $k = \mathcal{O}(X)$. Let $A$ be an associative $\mathbb{C}$-algebra which is also a $k$-algebra. We work with the collection of all $k$-algebras $A$ which are countably generated. As a $\mathbb{C}$-vector space, $A$ admits a finite or countable basis.

We will define an equivalence relation, called geometric equivalence, on the collection of such algebras $A$. This equivalence relation will be denoted $\asymp$.

1. Morita equivalence of $k$-algebras with local units. Let $A$ and $B$ be $k$-algebras, each with a countable set of local units. If $A$ and $B$ are connected by a strict Morita context, then $A \asymp B$. Periodic cyclic homology is preserved, see [14, Theorem 1].

2. Spectrum preserving morphisms with respect to filtrations of $k$-algebras of finite type, as in [2]. Such filtrations are automatically finite, since $k$ is Noetherian. A morphism $\phi: A \to B$ of $k$-algebras of finite type is called

• spectrum preserving if, for each primitive ideal $q$ of $B$, there exists a unique primitive ideal $p$ of $A$ containing $\phi^{-1}(q)$, and the resulting map $q \mapsto p$ is a bijection from Prim($B$) onto Prim($A$);

• spectrum preserving with respect to filtrations if there exist increasing filtrations by ideals

$$(0) = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_{r-1} \subset I_r \subset A$$

$$(0) = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_{r-1} \subset J_r \subset B,$$

such that, for all $j$, we have $\phi(I_j) \subset J_j$ and the induced morphism

$$\phi_*: I_j/I_{j-1} \to J_j/J_{j-1}$$

is spectrum preserving. If $A, B$ are $k$-algebras of finite type such that there exists a morphism $\phi: A \to B$ of $k$-algebras which is spectrum preserving with respect to filtrations then $A \asymp B$.

3. Deformation of central character. Let $A$ be a unital algebra over the complex numbers. Form the algebra $A[t, t^{-1}]$ of Laurent polynomials with coefficients in $A$. If $q$ is a non-zero complex number, then we have the evaluation-at-$q$ map of algebras

$$A[t, t^{-1}] \longrightarrow A$$

which sends a Laurent polynomial $P(t)$ to $P(q)$. Suppose now that $A[t, t^{-1}]$ has been given the structure of a $k$-algebra i.e. we are given a unital map of algebras over the complex numbers from $k$ to the centre.
of $A[t, t^{-1}]$. We assume that for any non-zero complex number $q$ the composed map
\[ k \rightarrow A[t, t^{-1}] \rightarrow A \]
where the second arrow is the above evaluation-at-$q$ map makes $A$ into a $k$-algebra of finite type. For $q$ a non-zero complex number, denote the finite type $k$-algebra so obtained by $A(q)$. Then we decree that if $q_1$ and $q_2$ are any two non-zero complex numbers, $A(q_1)$ is equivalent to $A(q_2)$.

We fix $k$. The first two moves preserve the central character. This third move allows us to algebraically deform the central character.

Let $\cong$ be the equivalence relation generated by (1), (2), (3); we say that $A$ and $B$ are geometrically equivalent if $A \cong B$.

Since each move induces an isomorphism in periodic cyclic homology, we have
\[ A \cong B \implies HP_*(A) \cong HP_*(B). \]

In order to formulate our conjecture, we need to review certain results and definitions.

The primitive ideal space of $\tilde{\mathcal{Z}}^s(G)$ is the set of $\mathbb{C}$-points of the variety $\tilde{D}_\sigma/W_\iota$ in the Zariski topology.

We have an isomorphism
\[ HP_*(O(\tilde{D}_\sigma/W_\iota)) \cong H^*(\tilde{D}_\sigma/W_\iota; \mathbb{C}). \]

This is a special case of the Feigin-Tsygan theorem; for a proof of this theorem which proceeds by reduction to the case of smooth varieties, see [25].

Let $E_\sigma$ be the maximal compact subgroup of the complex torus $D_\sigma$, so that $E_\sigma$ is a compact torus.

Let $\text{Prim}^t \mathcal{H}^s(G)$ denote the set of primitive ideals attached to tempered, simple $\mathcal{H}^s(G)$-modules.

**Conjecture 1.** Let $s \in \mathcal{B}(G)$. Then

1. $\mathcal{H}^s(G)$ is geometrically equivalent to the commutative algebra $\tilde{\mathcal{Z}}^s(G)$:
   \[ \mathcal{H}^s(G) \cong \tilde{\mathcal{Z}}^s(G). \]

2. The resulting bijection of primitive ideal spaces
   \[ \text{Prim}^t \mathcal{H}^s(G) \leftrightarrow \tilde{D}_\sigma/W_\iota \]
   restricts to give a bijection
   \[ \text{Prim}^t \mathcal{H}^s(G) \leftrightarrow \tilde{E}_\sigma/W_\iota. \]
Remark 1. The geometric equivalence of part (1) induces an isomorphism

$$HP_*(H^s(G)) \cong H^*(\widetilde{D}_\sigma/W_1; \mathbb{C}).$$

Remark 2. The referee has posed a very interesting question: if $A, B$ are geometrically equivalent, what does this imply about the categories $A - \text{MOD}$ and $B - \text{MOD}$ of unital modules? It seems likely that for each ideal $H^s$ the category $H^s - \text{MOD}$ will have some resemblance to $\tilde{3}^s - \text{MOD}$. If the conjecture is true, then Prim $H^s$ is in bijection with $\tilde{D}_\sigma/W_1$. However, as the referee has indicated, there may be further resemblances between $H^s - \text{MOD}$ and $\tilde{3}^s - \text{MOD}$.

6 Generic points in the Bernstein spectrum

We begin with a definition.

Definition 1. The point $s \in \mathfrak{B}(G)$ is generic if $W_1 = \{1\}$.

For example, let $s = [G, \sigma]|_G$ with $\sigma$ an irreducible supercuspidal representation of $G$. Then $s$ is a generic point in $\mathfrak{B}(G)$. For a second example, let $s = [GL(2) \times GL(2), \sigma_1 \otimes \sigma_2]|_{\text{GL}(4)}$ with $\sigma_1$ not equivalent to $\sigma_2$ (after unramified twist). Then $s$ is a generic point in $\mathfrak{B}(\text{GL}(4))$.

Theorem 2. The conjecture is true if $s$ is a generic point in $\mathfrak{B}(G)$.

Proof. Part (1). This is immediate from Theorem 1. We conclude that

$$H^s(G) \times \mathcal{O}(D_\sigma) = \tilde{3}^s(G) = \tilde{3}^s(G).$$

Part (2). Let $\mathcal{C}(G)$ denote the Harish-Chandra Schwartz algebra of $G$, see [50]. We choose $e, e_K$ as in the proof of Theorem 1. Let $s \in \mathfrak{B}(G)$ and let $\mathcal{C}^s(G)$ be the corresponding Bernstein ideal in $\mathcal{C}(G)$. As in the proof of Theorem 1 $\mathcal{C}^s(G)$ is the two-sided ideal generated by $e$. We have $\mathcal{C}^s(G) = \mathcal{C}(G)e \mathcal{C}(G)$. By Lemma 2 we have

$$\mathcal{C}^s(G) = \mathcal{C}(G)e \mathcal{C}(G) \sim_{\text{morita}} e \mathcal{C}(G)e = e_K \mathcal{C}(G)e_K.$$

According to Mischenko’s theorem [39], the Fourier transform induces an isomorphism of unital Fréchet algebras:

$$e_K \mathcal{C}(G)e_K \cong C^\infty(E_\sigma, \text{End} E^K).$$

We also have

$$C^\infty(E_\sigma, \text{End} E^K) \cong M_n(C^\infty(E_\sigma)).$$
with $n = \dim_C(E^K)$. By Lemma 1, we have

$$\mathcal{C}^\sigma(G) \sim_{\text{morita}} \mathcal{C}^{\infty}(E_\sigma).$$

We now exploit the liminality of the reductive group $G$, and take the primitive ideal space of each side. We conclude that there is a bijection

$$\text{Prim}^t \mathcal{H}^\sigma(G) \longleftrightarrow E_\sigma.$$

\[\square\]

7 The Hecke algebra of $\text{SL}(2)$

Let $G = \text{SL}(2) = \text{SL}(2, F)$, a group not of adjoint type. Let $W$ be the Coxeter group with 2 generators:

$$W = \langle s_1, s_2 \rangle = \mathbb{Z} \rtimes W_f$$

where $W_f = \mathbb{Z}/2\mathbb{Z}$. Then $W$ is the infinite dihedral group. It has the property that

$$\mathcal{H}(W, q_F) = \mathcal{H}((\text{SL}(2))/I).$$

There is a unique isomorphism of $\mathbb{C}$-algebras between $\mathcal{H}(W, q_F)$ and $\mathbb{C}[W]$ such that

$$T_{s_1} \mapsto \frac{q_F + 1}{2} \cdot s_1 + \frac{q_F - 1}{2}, \quad T_{s_2} \mapsto \frac{q_F + 1}{2} \cdot s_2 + \frac{q_F - 1}{2},$$

where $T_{s_i}$ is the element of $\mathcal{H}(W, q_F)$ corresponding to $s_i$. We note also that

$$\mathbb{C}[\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}] \cong \mathbb{M} \rtimes \mathbb{Z}/2\mathbb{Z}$$

where

$$\mathbb{M} := \mathbb{C}[t, t^{-1}]$$

denotes the $\mathbb{Z}/2\mathbb{Z}$-graded algebra of Laurent polynomials in one indeterminate $t$. Let $\alpha$ denote the generator of $\mathbb{Z}/2\mathbb{Z}$. The group $\mathbb{Z}/2\mathbb{Z}$ acts as automorphism of $\mathbb{M}$, with $\alpha(t) = t^{-1}$. We define

$$\mathbb{L} := \{ P \in \mathbb{M} : \alpha(P) = P \}$$

as the algebra of balanced Laurent polynomials. We will write

$$\mathbb{L}^* := \{ P \in \mathbb{M} : \alpha(P) = -P \}.$$

Then $\mathbb{L}^*$ is a free of rank 1 module over $\mathbb{L}$, with generator $t - t^{-1}$. We will refer to the elements of $\mathbb{L}^*$ as \textit{anti-balanced} Laurent polynomials.
Lemma 4. We have

\[ M \rtimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{C}^2 \oplus L. \]

Proof. We will realize the crossed product as follows:

\[ M \rtimes \mathbb{Z}/2\mathbb{Z} = \{ f \in \mathcal{O}(\mathbb{C}^\times, M_2(\mathbb{C})) : f(z^{-1}) = a \cdot f(z) \cdot a^{-1} \} \]

where

\[ a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

We then have

\[ M \rtimes \mathbb{Z}/2\mathbb{Z} = \begin{pmatrix} L & L^* \\ L^* & L \end{pmatrix} \]

There is an algebra map

\[ \begin{pmatrix} L & L^* \\ L^* & L \end{pmatrix} \rightarrow \begin{pmatrix} L & L \\ L & L \end{pmatrix} \]

as follows:

\[ x_{11} \mapsto x_{11}, \ x_{22} \mapsto x_{22}, \ x_{12} \mapsto x_{12}(t - t^{-1}), \ x_{21} \mapsto x_{21}(t - t^{-1})^{-1}. \]

This map, combined with evaluation of \( x_{22} \) at \( t = 1, t = -1 \), creates an algebra map

\[ M \rtimes \mathbb{Z}/2\mathbb{Z} \rightarrow M_2(L) \oplus \mathbb{C} \oplus \mathbb{C}. \]

This map is spectrum-preserving with respect to the following filtrations:

\[ 0 \subset M_2(L) \subset M_2(L) \oplus \mathbb{C} \oplus \mathbb{C} \]

\[ 0 \subset I \subset M \rtimes \mathbb{Z}/2\mathbb{Z} \]

where \( I \) is the ideal of \( M \rtimes \mathbb{Z}/2\mathbb{Z} \) defined by the conditions

\[ x_{22}(1) = 0 = x_{22}(-1). \]

We therefore have

\[ M \rtimes \mathbb{Z}/2\mathbb{Z} \cong M_2(L) \oplus \mathbb{C}^2 \cong L \oplus \mathbb{C}^2. \]

It is worth noting that \( M \rtimes \mathbb{Z}/2\mathbb{Z} \) is not Morita equivalent to \( L \oplus \mathbb{C}^2 \). For the primitive ideal space \( \text{Prim}(M \rtimes \mathbb{Z}/2\mathbb{Z}) \) is not connected, whereas the primitive ideal space \( \text{Prim}(L \oplus \mathbb{C}^2) \) is disconnected (it has 3 connected components).

Theorem 3. The conjecture is true for \( \text{SL}(2, F) \).
Proof. For the Iwahori ideal \( \mathcal{H}^i(\text{SL}(2)) \) we have

\[
\mathcal{H}^i(G) \times \mathcal{H}(W, q_F) \times \mathbb{C}[W].
\]

Let \( \chi \) be a unitary character of \( F^\times \) of exact order two. Let \( G = \text{SL}(2, F) \) and let \( j = j(\lambda) = [T, \lambda]_G \in \mathfrak{B}(G) \) with \( \lambda \) is defined as follows:

\[
\lambda: \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mapsto \chi(x).
\]

Let \( c \) be the least integer \( n \geq 1 \) such that \( 1 + p_F^n \subset \ker(\chi) \). We set

\[
J_\chi := \left( \frac{\phi_F^x}{p_F^{[(c+1)/2]}}, \frac{p_F^{c/2}}{\phi_F^x} \right) \cap \text{SL}(2, F).
\]

Let \( \tau_\chi \) denote the restriction of \( \lambda \) to the compact torus

\[
\left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} : x \in \phi_F^x \right\}.
\]

Then \( (J_\chi, \tau_\chi) \) is an \( s \)-type in \( \text{SL}(2, F) \) and (for instance, as a special case of \cite[Theorem 11.1]{20}) the Hecke algebra \( \mathcal{H}(G, \tau_\chi) \) is isomorphic to \( \mathcal{H}(W, q_F) \). We have

\[
\mathcal{H}^i(G) \times \mathcal{H}(G, \tau_\chi) \cong \mathcal{H}(W, q_F) \times \mathbb{C}[W].
\]

To summarize: if \( s = i \) or \( j(\lambda) \) then we have

\[
\mathcal{H}^s(G) \times \mathbb{M} \cong \mathbb{Z}/2\mathbb{Z}.
\]

We have \( W_i = W_f = \mathbb{Z}/2\mathbb{Z} \). Let \( \Omega \) be the variety which corresponds to \( j \in \mathfrak{B}(G) \), so that \( \Omega = D/W_f \) with \( D \) a complex torus of dimension 1. Each unramified quasicharacter of the maximal torus \( T \subset \text{SL}(2) \) is given by

\[
\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mapsto s^{\text{val}(z)}
\]
with \( z \in \mathbb{C}^\times \). Let \( \mathcal{V}(zw - 1) \) denote the algebraic curve in \( \mathbb{C}^2 \) defined by the equation \( zw - 1 = 0 \). This algebraic curve is a hyperbola. The map \( \mathbb{C}^\times \to \mathcal{V}(zw - 1), z \mapsto (z, z^{-1}) \) defines the structure of algebraic curve on \( \mathbb{C}^\times \). The generator of \( W_f \) sends a point \( z \) on this curve to \( z^{-1} \) and there are two fixed points, 1 and \(-1\). The coordinate algebra of the quotient curve is given by

\[
\mathbb{C}[D/W_f] = L
\]
and the coordinate algebra of the extended quotient is given by
\[ C[\tilde{D}/W_f] = \mathbb{L} \oplus \mathbb{C} \oplus \mathbb{C}. \]

Then it follows from Lemma \[\text{[1]}\] that
\[ \mathcal{H}^s(\text{SL}(2)) \cong \tilde{\mathcal{Z}}^s(\text{SL}(2)) \]
if \( s = i \) or \( j(\lambda) \).

If \( s \neq i, j(\lambda) \) then \( s \) is a generic point in \( \mathcal{B}(\text{SL}(2)) \) and we apply
Theorem \[\text{[1]}\] Part (1) of the conjecture is proved.

Recall that a representation of \( G \) is called \textit{elliptic} if its character
is not identically zero on the elliptic set of \( G \).

It follows from \[\text{[19, Theorem 3.4]} \text{ (see also \text{[48, Prop. 1]}]} \] that the
(normalized) induced representation \( \text{Ind}^G_{TU}(\chi \otimes 1) \) has an elliptic consti-
tuent (and then the other constituent is also elliptic) if and only if
the character \( \chi \) is of order 2.

We then have
\[ \text{Ind}^G_{TU}(\chi \otimes 1) = \pi^+ \oplus \pi^-, \]
and we have the identity \( \theta^+ = \theta^- = 0 \), between the characters \( \theta^+, \theta^- \)
of \( \pi^+, \pi^- \).

Let \( s \in \mathbb{C}, |s| = 1 \). The corresponding (normalized) induced rep-
resentation will be denoted \( \pi(s) \):
\[ \pi(s) := \text{Ind}^G_{TU}(\chi s \lambda \otimes 1). \]

The representations \( \pi(1), \pi(-1) \) are reducible, and split into irre-
ducible components:
\[ \pi(1) = \pi^+(1) \oplus \pi^-(1) \]
\[ \pi(-1) = \pi^+(-1) \oplus \pi^-(-1) \]
These are \textit{elliptic} representations: their characters
\[ \theta^+(1), \theta^-(1), \theta^+(-1), \theta^-(-1) \]
are not identically zero on the elliptic set, although we do have the
identities
\[ \theta^+(1) + \theta^-(1) = 0 \]
\[ \theta^+(1) + \theta^-(1) = 0. \]

Concerning infinitesimal characters, we have
\[ \text{inf.ch.} \pi^+(1) = \text{inf.ch.} \pi^-(1) = \lambda \]
\[ \text{inf.ch.} \pi^+(1) = \text{inf.ch.} \pi^-(1) = (-1)^{\text{val}^F} \otimes \lambda. \]
Note that $E_{\sigma} = \mathbb{T}$, and recall that $\pi(s) \equiv \pi(s^{-1})$. The induced bijection

$$\text{Prim}^t \mathcal{H}^g(G) \to \tilde{E}_{\sigma}/W$$

is as follows. With $s \in \mathbb{C}, |s| = 1$:

$$\pi(s) \mapsto \{s, s^{-1}\}, \quad s^2 \neq 1$$

$$\pi^+(1) \mapsto 1, \quad \pi^-(1) \mapsto a$$

$$\pi^+(-1) \mapsto -1, \quad \pi^-(1) \mapsto b$$

where $a, b$ are the two isolated points in the compact extended quotient of $\mathbb{T}$ by $\mathbb{Z}/2\mathbb{Z}$. We note that the pair $\pi^+(1), \pi^-(1)$ are $L$-indistinguishable. The corresponding Langlands parameter fails to distinguish them from each other. The pair $\pi^+(-1), \pi^-(1)$ are also $L$-indistinguishable.

The unramified unitary principal series is defined as follows:

$$\omega(s) := \text{Ind}_{TU}^G (\chi_s \otimes 1).$$

Recall that $\omega(s) = \omega(s^{-1})$. The representation $\omega(-1)$ is reducible:

$$\omega(-1) = \omega^+(-1) \oplus \omega^-(1).$$

For that part of the tempered spectrum which admits non-zero Iwahori-fixed vectors, the induced bijection

$$\text{Prim}^t \mathcal{H}^g(G) \to \tilde{E}_{\sigma}/W$$

is as follows. With $s \in \mathbb{C}, |s| = 1$:

$$\omega(s) \mapsto \{s, s^{-1}\}, \quad s \neq -1$$

$$\omega^+(-1) \mapsto -1, \quad \omega^-(1) \mapsto c$$

$$\text{St}(2) \mapsto d$$

where $c, d$ are the two isolated points in the compact extended quotient of $\mathbb{T}$ by $\mathbb{Z}/2\mathbb{Z}$, and $\text{St}(2)$ denotes the Steinberg representation of $\text{SL}(2)$.

These maps are induced by the geometric equivalences and are therefore not quite canonical, because the geometric equivalences are not canonical.

If $s \neq i, i(\lambda)$ then $s$ is a generic point in $\mathfrak{B}(\text{SL}(2))$ and the induced bijection is as follows:

$$\text{Prim}^t \mathcal{H}^g(G) \to \tilde{E}_{\sigma}/W$$

takes the form of the identity map $\mathbb{T} \to \mathbb{T}$ or the identity map $pt \to pt$. \qed

There are 2 non-generic points in $\mathfrak{B}(\text{SL}(\mathbb{Q}_p))$ with $p > 3$; there are 4 non-generic points in $\mathfrak{B}(\text{SL}(2, \mathbb{Q}_2))$. 

18
8 Iwahori-Hecke algebras

The proof of Theorem \( \mathcal{H}(G) \) shows that \( \mathcal{H}(G) \) is Morita equivalent to a unital \( k \)-algebra which we will denote by \( A_s \). The next step in proving the conjecture will be to relate this algebra \( A_s \) to a generalized Iwahori-Hecke algebra, as defined below.

Let \( W' \) be a Coxeter group with generators \( (s)_{s \in S} \) and relations

\[
(ss')^{m_{s,s'}} = 1, \quad \text{for any } s, s' \in S \text{ such that } m_{s,s'} < +\infty,
\]

and let \( L \) be a weight function on \( W' \), that is, a map \( L : W' \to \mathbb{Z} \) such that \( L(w w') = L(w) + L(w') \) for any \( w, w' \in W' \) such that \( \ell(w w') = \ell(w) + \ell(w') \), where \( \ell \) is the usual length function on \( W' \). Clearly, the function \( \ell \) is itself a weight function.

Let \( \Omega \) be a finite group acting on the Coxeter system \((W', S)\). The group \( W := W' \rtimes \Omega \) will be called an extended Coxeter group. We extend \( L \) to \( W \) by setting \( L(w \omega) := L(w) \), for \( w \in W', \omega \in \Omega \).

Let \( A := \mathbb{Z}[v, v^{-1}] \) where \( v \) is an indeterminate. We set \( u := v^2 \) and \( v_s := v^{L(s)} \) for any \( s \in S \). Let \( : A \to A \) be the ring involution which takes \( v^n \) to \( v^{-n} \) for any \( n \in \mathbb{Z} \).

Let \( \mathcal{H}(W, u) = \mathcal{H}(W, L, u) \) denote the \( A \)-algebra defined by the generators \( (T_s)_{s \in S} \) and the relations

\[
(T_s - v_s)(T_s + v_s^{-1}) = 0 \quad \text{for } s \in S,
\]

\[
\underbrace{T_s T_{s'} T_s \cdots}_{m_{s,s'} \text{ factors}} = \underbrace{T_{s'} T_s T_{s'} \cdots}_{m_{s',s} \text{ factors}}, \quad \text{for any } s \neq s' \in S \text{ such that } m_{s,s'} < +\infty.
\]

For \( w \in W \), we define \( T_w \in \mathcal{H}(W, u) \) by \( T_w = T_{s_1} T_{s_2} \cdots T_{s_m} \), where \( w = s_1 s_2 \cdots s_m \) is a reduced expression in \( W \). We have \( T_1 = 1 \), the unit element of \( \mathcal{H}(W, u) \), and \( (T_w)_{w \in W} \) is an \( A \)-basis of \( \mathcal{H}(W, u) \). The \( v_s \) are called the parameters of \( \mathcal{H}(W, u) \).

Let \( \mathcal{H}(W', u) \) be the \( A \)-subspace of \( \mathcal{H}(W, u) \) spanned by all \( T_w \) with \( w \in W' \). For each \( q \in \mathbb{C}^\times \), we set \( \mathcal{H}(W, q) := \mathcal{H}(W, u) \otimes_A \mathbb{C} \), where \( \mathbb{C} \) is regarded as an \( A \)-algebra with \( u \) acting as scalar multiplication by \( q \). The algebras of the form \( \mathcal{H}(W, q) \) where \( W \) is an extended Coxeter group and \( q \in \mathbb{C} \) will be called extended Iwahori-Hecke algebras. In the case when the Coxeter group \( W' \) is an affine Weyl group, we will say that \( \mathcal{H}(W, q) \) is an extended affine Iwahori-Hecke algebra.

We now observe that \( W_t \) is a (finite) extended Coxeter group. Indeed, there exists a root system \( \Phi_t \) with associate Weyl group denoted \( W'_t \) and a subset \( \Phi_t^+ \) of positive roots in \( \Phi_t \), such that, setting

\[
C_t := \{ w \in W_t : w(\Phi_t^+) \subset \Phi_t^+ \},
\]
we have
\[ W_t = W'_t \times C_t. \]
This follows from \([21\text{ Prop. 4.2]}\) and \([23\text{ Lem. 2}].\)

It is expected, and proved, using the theory of types of \([10]\), for level-zero representations in \([10], [11]\), for principal series representations of split groups in \([16]\), for the group \(\text{GL}(n, F)\) in \([9], [11]\), for the group \(\text{SL}(n, F)\) in \([20]\), for the group \(\text{Sp}(4)\) in \([6]\), and for a large class of representations of classical groups in \([26], [27]\), that there exists always an extended affine Iwahori-Hecke algebra \(H'_s\) such that the following holds:

1. there exists a (finite) Iwahori-Hecke algebra \(H'_s\) with corresponding Coxeter group \(W'_t\) and a Laurent polynomial algebra \(B_t\) satisfying \(H'_s = H'_s \otimes_{\mathbb{C}} B_t\);
2. there exists a two-cocycle \(\mu: C_t \times C_t \to \mathbb{C}^\times\) and an injective homomorphism of groups \(\iota: C_t \to \text{Aut}_{\mathbb{C}-\text{alg}} H'_s\) such that \(A_s\) is Morita equivalent to the twisted tensor product algebra \(H'_s \otimes_{\mathbb{C}} \mathbb{C}[C_t]_{\mu}\).

In the case of \(\text{GL}(n, F)\) (see \([9]\)), and in the case of principal series representations of split groups with connected centre (see \([16]\)), we always have \(C_t = \{1\}\). The references quoted above give examples in which \(C_t \neq \{1\}\). The results in \([20]\) also show that the algebra \(H'_s \otimes_{\mathbb{C}} \mathbb{C}[C_t]_{\mu}\) is not always isomorphic to an extended Iwahori-Hecke algebra.

There are no known example in which the cocycle \(\mu\) is non-trivial. In the case of unipotent level zero representations \([37], [32]\), of principal series representations \([16]\), and of the group \(\text{Sp}(4)\) \([6]\), it has been proved that \(\mu\) is trivial.

From now we restrict attention to the case where \(C_t = \{1\}\), so that \(A_s\) is expected to be Morita equivalent to a generalized affine Iwahori-Hecke algebra \(H'_s\).

In particular, if \(L\) is a torus and \(s = [T, 1]_G\), then \(A_s\) is isomorphic to the commuting algebra \(\mathcal{H}(G//I)\) in \(G\) of the induced representation from the trivial representation of an Iwahori subgroup \(I\) of \(G\). We have
\[ \mathcal{H}(G//I) \simeq \mathcal{H}(W, q_F), \]
where \(q_F\) is the order of the residue field of \(F\) and \(W\) is defined as follows (see for instance \([12, \S 3.2\text{ and } 3.5]\)). Here the weight function is taken to be equal to the length function. In particular, we are in the equal parameters case.

Let \(T\) be a maximal split torus in \(G\), and let \(X^*(T), X_s(T)\) denote its groups of characters and cocharacters, respectively. Let \(\Phi(G, T) \subset\)
$X^*(T), \Phi^\vee(G, T) \subset X_*(T)$ be the corresponding root and coroot systems, and $W_f$ the associated (finite) Weyl group. Then

$$W = X_*(T) \rtimes W_f,$$

Now let $X'_*(T)$ denote the subgroup of $X_*(T)$ generated by $\Phi^\vee(G, T)$. Then $W' := X'_*(T) \rtimes W_f$ is a Coxeter group (an affine Weyl group) and $W = W' \rtimes \Omega$, where $\Omega$ is the group of elements in $W$ of length zero.

Let $L G^0$ be the Langlands dual group of $G$, and let $L T^0$ denote the Langlands dual of $T$, a maximal torus of $L G^0$. By Langlands duality, we have

$$W' = X'_*(T) \rtimes W_f = X^*(L T^0) \rtimes W_f.$$

The isomorphism

$$L T^0 \cong \Psi(T), \quad t \mapsto \chi_t$$

is fixed by the relation

$$\chi_t(\phi(\varpi_F)) = \phi(t)$$

for $t \in L T^0, \phi \in X_*(T) = X^*(L T^0)$, and $\varpi_F$ a uniformizer in $F$. This isomorphism commutes with the $W_f$-action, see \cite{[13], Section I.2.3].

The group $W_f$ acts on $L T^0$, and we form the quotient variety $L T^0/W_f$.

Let $i \in \mathcal{B}(G)$ be determined by the cuspidal pair $(T, 1)$. We have

$$Z^i = \mathbb{C}[L T^0/W_f],$$

$$\bar{Z}^i = \mathbb{C}[\bar{L} T^0/W_f].$$

9 The asymptotic Hecke algebra

There is a unique algebra involution $h \mapsto h^!$ of $\mathcal{H}(W', u)$ such that $T_s^! = -T_s^{-1}$ for any $s \in S$, and a unique endomorphism $h \mapsto \bar{h}$ of $\mathcal{H}(W', u)$ which is $A$-semilinear with respect to $\cdot: A \to A$ and satisfies $\bar{T}_s = T_s^{-1}$ for any $s \in S$. Let

$$A_{<0} := \bigoplus_{m \leq 0} \mathbb{Z} v^m = \mathbb{Z}[v^{-1}], \quad A_{\leq 0} := \bigoplus_{m < 0} \mathbb{Z} v^m,$$

$$\mathcal{H}(W', u)_{\leq 0} := \bigoplus_{w \in W'} A_{\leq 0} T_w, \quad \mathcal{H}(W', u)_{<0} := \bigoplus_{w \in W'} A_{<0} T_w.$$

Let $z \in W'$. There is a unique $c_z \in \mathcal{H}(W', u)_{\leq 0}$ such that $\bar{c}_z = c_z$ and $c_z = T_z \mod \mathcal{H}(W', u)_{<0}$, \cite{[8], Theorem 5.2 (a)]. We write $c_z = $
\[ \sum_{y \in W} p_{y,z} T_y, \] where \( p_{y,z} \in A_{\leq 0} \). For \( y \in W' \), \( \omega, \omega' \in \Omega \), we define \( p_{\omega, \omega'} \) as \( p_{y,z} \) if \( \omega = \omega' \) and as 0 otherwise. For \( w \in W \), we set \( c_w := \sum_{y \in W} p_{y,w} T_y \). Then it follows from [38, Theorem 5.2 (b)] that \((c_w)_{w \in W}\) is an \( A \)-basis of \( \mathcal{H}(W, u) \).

For \( x, y, z \) in \( W \), we define \( f_{x,y,z} \in A \) by

\[ T_x T_y = \sum_{z \in W} f_{x,y,z} T_z. \]

From now on, we assume that \( W' \) is a bounded weighted Coxeter group, that is, that there exists an integer \( N \in \mathbb{N} \) such that \( v^{-N} f_{x,y,z} \in A_{\leq 0} \) for all \( x, y, z \) in \( W \).

For \( x, y, z \) in \( W \), we define \( h_{x,y,z} \in A \) by

\[ c_x \cdot c_y = \sum_{z \in W} h_{x,y,z} c_z. \]

It follows from [38, §13.6] that, for any \( z \in W \), there exists an integer \( a(z) \in [0, N] \) such that

\[ h_{x,y,z} \in v^{a(z)} \mathbb{Z}[v^{-1}] \quad \text{for all} \quad x, y \in W, \]

\[ h_{x,y,z} \not\in v^{a(z)-1} \mathbb{Z}[v^{-1}] \quad \text{for some} \quad x, y \in W. \]

Let \( \gamma_{x,y,z} \) be the coefficient of \( v^{a(z)} \) in \( h_{x,y,z} \).

Let \( \Delta(z) \geq 0 \) be the integer defined by

\[ p_{1,z} = n_z v^{-\Delta(z)} + \text{strictly smaller powers of } v, \quad n_z \in \mathbb{Z} - \{0\}, \]

and let \( \mathcal{D} \) denote the following (finite) subset of \( W \):

\[ \mathcal{D} := \{ z \in W : a(z) = \Delta(z) \}. \]

In [38, chap. 14.2] Lusztig stated a list of 15 conjectures \( P_1, \ldots, P_{15} \) and proved them in several cases [38, chap. 15, 16, 17]. Assuming the validity of the conjectures, Lusztig was able to define in [38] partitions of \( W \) into left cells, right cells and two-sided cells, which extend the theory of Kazhdan-Lusztig from the case of equal parameters (that is, \( v_s = v_{s'} \) for any \( (s, s') \in S^2 \)) to the general case. In the case of equal parameters the conjectures mentioned above are known to be true.

From now on we shall assume the validity of these conjectures. Let us recall some of them below:

**P1.** For any \( z \in W \) we have \( a(z) \leq \Delta(z). \)

**P2.** If \( d \in \mathcal{D} \) and \( x, y \in W \) satisfy \( \gamma_{x,y,d} \neq 0 \), then \( x = y^{-1}. \)

**P3.** If \( y \in W \), there exists a unique \( d \in \mathcal{D} \) such that \( \gamma_{y,y^{-1},d} \neq 0. \)
P4. If $z' \leq_{LR} z$ then $a(z') \geq a(z)$. Hence, if $z' \sim_{LR} z$, the $a(z') = a(z)$.

P5. If $d \in D$, $y \in W$, $\gamma_{y-1, y, d} \neq 0$, then $\gamma_{y-1, y, d} = n_d = \pm 1$.

P6. If $d \in D$, then $d^2 = 1$.

P7. For any $x, y, z$ in $W$ we have $\gamma_{x, y, z} = \gamma_{y, z, x}$.

P8. Let $x, y, z$ in $W$ be such that $\gamma_{x, y, z} \neq 0$. Then $x \sim_{L} y^{-1}$, $y \sim_{L} z^{-1}$, $z \sim_{L} x^{-1}$.

For any $z \in W$, we set $\hat{n}_z := n_d$ where $d$ is the unique element of $D$ such that $d \sim_{L} z^{-1}$.

Let $J$ denote the free Abelian group with basis $(t_w)_{w \in W}$. We set $t_x \cdot t_y := \sum_{z \in W} \gamma_{x, y, z} t_z$.

(This is a finite sum.) This defines an associative ring structure on $J$. The ring $J$ is called the based ring of $W$. It has a unit element $\sum_{d \in D} t_d$ (see [38, §18.3]).

The C-algebra $J(W) := J \otimes_{\mathbb{Z}} \mathbb{C}$ is called the asymptotic Hecke algebra of $W$.

According to property (P8), for each two-sided cell $c$ in $W$, the subspace $J_c$ spanned by the $t_w$, $w \in c$, is a two-sided ideal of $J$. The ideal $J_c$ is in fact an associative ring with unit $\sum_{d \in D \cap c} T_d$, which is called the based ring of the two-sided cell $c$, [38, §18.3], and

$$J = \bigoplus_c J_c$$

is a direct sum decomposition of $J$ as a ring.

Let $J(W, u) := A \otimes_{\mathbb{Z}} J$. We recall from [38, Theorem 18.9] (which extends [33, 2.4]) that the $A$-linear map $\phi : \mathcal{H}(W, u) \to J(W, u)$ given by

$$\phi(c_w^\uparrow) := \sum_{x \in W, d \in D, a(d) = a(z)} h_{x, d, z} \hat{n}_z t_z \quad (x \in W)$$

is a homomorphism of $A$-algebra with unit (note that the conjecture $(P_{15})$ is used here).

Let

$$\phi_q : \mathcal{H}(W, q) \to J \otimes_{\mathbb{Z}} \mathbb{C}$$

be the C-algebra homomorphism induced by $\phi$.

Let $\mathcal{H}(W, q)^{\geq i}$ be the $C$-subspace of $\mathcal{H}(W, q)$ spanned by all the $c_w^\uparrow$ with $w \in W$ and $a(w) \geq i$. This a two-sided ideal of $\mathcal{H}(W, q)$, because of [38, §13.1] and (P7). Let

$$\mathcal{H}(W, q)^i := \mathcal{H}(W, q)^{\geq i} / \mathcal{H}(W, q)^{\geq i+1};$$

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this is an \( \mathcal{H}(W,q) \)-bimodule. It has as \( \mathbb{C} \)-basis the images \( [c_w^+] \) of the \( c_w^+ \in \mathcal{H}(W,q)^\mathfrak{z} \) such that \( a(w) = i \).

We may regard \( \mathcal{H}(W,q)^i \) as a \( J \)-bimodule with multiplication defined by the rule:

\[
\begin{align*}
t_x * [c_w^+] &= \sum_{z \in W, a(z) = i} \gamma_{x,w,z} \hat{n}_w \hat{n}_z c_z^+, \\
[c_w^+] * t_x &= \sum_{z \in W, a(z) = i} \gamma_{w,x,z} \hat{n}_w \hat{n}_z c_z^+, \quad (w, x \in W, a(w) = i).
\end{align*}
\]

We have (see \[38\, 18.10\]):

\[
hf = \phi_q(h) * f, \text{ for all } f \in \mathcal{H}(W,q)^i, h \in \mathcal{H}(W,q). \tag{4}
\]

On the other side:

\[
(j * f)h = j * (fh), \text{ for all } f \in \mathcal{H}(W,q)^i, h \in \mathcal{H}(W,q), j \in J. \tag{5}
\]

Let

\[
f_i := \sum_{d \in D, a(d) = i} [c_d^+] \in \mathcal{H}(W,q)^i.
\]

**Lemma 5.** We have

\[
t_x * f_i = f_i * t_x = \hat{n}_x [c_x^+].
\]

**Proof.** By definition of *, we have

\[
t_x * f_i = \sum_{d \in D, a(d) = i} \sum_{z \in W, a(z) = i} \gamma_{x,d,z} \hat{n}_d \hat{n}_z c_z^+.
\]

Since (because of P7) \( \gamma_{x,d,z} = \gamma_{d,z} = \gamma_{z,x} \), it follows from (P2) that if \( \gamma_{x,d,z} \neq 0 \) then \( z = x \). Then, using (P3) and (P5), we obtain \( t_x * f_i = \hat{n}_x c_x^+ \). The proof for \( f_i * t_x \) is similar. \( \square \)

Let \( k \) denote the centre of \( \mathcal{H}(W,q) \). Lusztig proved the following result in \[34\, Proposition 1.6 (i)\] in the case of equal parameters. Our proof will follow the same lines.

**Proposition 1.** The centre of \( J \otimes_{\mathbb{Z}} \mathbb{C} \) contains \( \phi_q(k) \).

**Proof.** It is enough to show that \( \phi_q(z) * t_x = t_x * \phi_q(z) \) for any \( z \in k, x \in W \). Assume that \( a(x) = i \). Let \( z \in k \). Using Lemma 5, we obtain

\[
(\phi_q(z) t_x) * f_i = \phi_q(z) * t_x * f_i = \hat{n}_x \phi_q(z) * [c_x^+] = \hat{n}_x \phi_q(z) * [c_x^+].
\]
On the other side, using equation (4), we get

\((t_x \phi_q(z)) \ast f_i = t_x \ast (\phi_q(z) \ast f_i) = t_x \ast (zf_i)\).

Since \(zf_i = f_iz\), it gives

\((t_x \phi_q(z)) \ast f_i = t_x \ast (f_iz),\)

and then, using equation (5) and again Lemma 5, we obtain

\((t_x \phi_q(z)) \ast f_i = (t_x \ast f_i)z = \hat{n}_x[c^1_x]z.\)

Now, since \(z \in k\), we have \(z[c^1_x] = [c^1_x]z\). Hence

\((\phi_q(z)t_x) \ast f_i = (t_x \phi_q(z)) \ast f_i.\) (6)

It follows from the combination of (P4) and (P8) that \(\gamma_{x,y,z} \neq 0\) implies \(a(x) = a(y) = a(z)\). Hence we have

\[\phi_q(z)t_x = \sum_{x' \in W \atop a(x') = i} \alpha_{x'} t_{x'},\]

\[t_x \phi_q(z) = \sum_{x' \in W \atop a(x') = i} \beta_{x'} t_{x'},\]

with \(\alpha_{x'}, \beta_{x'} \in \mathbb{C}\). Then (4) implies that

\[\sum_{x' \in W \atop a(x') = i} \alpha_{x'} [c^1_{x'}] = \sum_{x' \in W \atop a(x') = i} \beta_{x'} [c^1_{x'}].\]

Hence \(\alpha_{x'} = \beta_{x'}\) for all \(x' \in W\) such that \(a(x') = i\). It gives \(\phi_q(z)t_x = t_x \phi_q(z)\), as required.

\[\square\]

**Remark 3.** The above proposition provides \(J \otimes_{\mathbb{Z}} \mathbb{C}\) (and also each \(J_\ell\)) with a structure of \(k\)-algebra. This \(k\)-algebra structure is not canonical: it depends on \(q\). Our move (3) precisely allows us to pass from one \(k\)-algebra structure, depending on \(q_1\), to another \(k\)-algebra structure, depending on \(q_2\).

From now on we will assume that the weight function is equal to the length function \(\ell\). We will assume that \(q = 1\), in which case \(H(W,q)\) is the group algebra of \(W\); or \(q\) is not a root of unity, in which case we can take for \(q\) the order \(q_F\) of the residue field of \(F\).

Let \(E\) be a simple \(\mathcal{H}(W,q)\)-module (resp. \(J \otimes_{\mathbb{Z}} \mathbb{C}\)-module). We attach to \(E\) an integer \(a_E\) by the following two requirements:
1. \(c_wE = 0\) (resp. \(t_wE = 0\)) for any \(w\) with \(a(w) > a_E\);

2. \(c_wE \neq 0\) (resp. \(t_wE \neq 0\)) for some \(w\) such \(a(w) = a_E\).

Then Lusztig proved in [34, Cor. 3.6] (see also [36, Th. 8.1]) that there is a unique bijection \(E \mapsto E'\) between the set of isomorphism classes of simple \(H(W, q)\)-modules and the set of isomorphism classes of simple \(J \otimes \mathbb{C}\)-modules such that \(a_E' = a_E\) and such that the restriction of \(E'\) to \(H(W, q)\) via \(\phi_q\) is an \(H(W, q)\)-module with exactly one composition factor isomorphic to \(E\) and all other composition factors of the form \(\bar{E}\) with \(a_{\bar{E}} < a_E\).

As shown in [2, Th. 9], it follows that \(\phi_q\) is spectrum preserving with respect to filtrations. Hence

\[
H(W, q) \times J \otimes \mathbb{C}.
\]  

(7)

Let \(G\) be a connected \(F\)-split adjoint simple \(p\)-adic group. By Langlands duality we have

\[
W := X_*(T) \times W_f = X^*(L(T^0(\mathbb{C}))) \times W_f.
\]  

(8)

Lusztig proved in [35, Theorem 4.8] that the unipotent conjugacy classes in \(L^G\) are in bijection with the two-sided cells in \(W\).

Let \(J\) be the based ring attached to \(W\). We fix a two-sided cell \(c\) in \(W\). Let \(\mathcal{O}_c\) be the unipotent conjugacy class in \(L^G\) corresponding to \(c\). Let \(\varphi: \text{SL}(2)(\mathbb{C}) \to L^G\) be a homomorphism of algebraic groups such that \(u = \varphi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) belongs to \(\mathcal{O}_c\) and let \(F_c\) be a maximal reductive algebraic subgroup of the centralizer \(C_{L^G}(u)\). The reductive group \(F_c\) may be disconnected: the identity component of \(F_c\) will be denoted \(F^0_c\).

Let \(Y\) be a finite \(F_c\)-set (that is, a set with an algebraic action of \(F_c\); thus, \(F^0_c\) acts trivially). An \(F_c\)-vector bundle on \(Y\) is a collection of finite dimensional \(\mathbb{C}\)-vector spaces \(V_y\) (\(y \in Y\)) with a given algebraic representation of \(F_c\) on \(\bigoplus_{y \in Y} V_y\) such that \(G \cdot V_y = V_{gy}\) for all \(g \in F_c\), \(y \in Y\). We now consider the finite \(F_c\)-set \(Y \times Y\) with diagonal action of \(F_c\) and denote by \(K_{F_c}(Y \times Y)\) the Grothendieck group of the category of \(F_c\)-vector bundles on \(Y \times Y\). One can define an associative ring structure on \(K_{F_c}(Y \times Y)\) (see [35, §10.2]).

Then the conjecture of Lusztig in [35, §10.5] states in particular that there should exist a finite \(F_c\)-set \(Y\) and a bijection \(\pi\) from \(c\) onto the set of irreducible \(F_c\)-vector bundles on \(Y \times Y\) (up to isomorphism) such the \(\mathbb{C}\)-linear map \(J_c \to K_{F_c}(Y \times Y) \otimes \mathbb{C}\) sending \(t_w\) to \(\pi(w)\) is an algebra isomorphism (preserving the unit element).

Let \(|Y|\) denote the cardinality of \(Y\). This number is expected to be the number of left cells contained in \(c\). When \(F_c\) is connected,
\( K_{F_c}(Y \times Y) \) is isomorphic to the \(|Y| \times |Y|\) matrix algebra \( M_{|Y|}(R_C(F_c)) \) over the (complexified) rational representation ring \( R_C(F_c) \) of \( F_c \). It is important to note: when \( F_c \) is connected, the Lusztig conjecture asserts that \( J_c \) is Morita equivalent to a commutative algebra.

The Lusztig conjecture has been proved by Xi for any two-sided cell \( c \) when \( G \) is one of the following groups \( \text{GL}(n) \), \( \text{PGL}(n) \), \( \text{SL}(2) \), \( \text{SO}(5) \) and \( G_2 \), and for the lowest two-sided cell \( c_0 \) (see next section) when \( G \) is any connected \( F \)-split adjoint simple \( p \)-adic group.

10 The ideal \( J_{c_0} \) in \( J \)

As above we assume that \( G \) is a connected \( F \)-split adjoint simple \( p \)-adic group. Let \( J \) be the based ring attached to \( W \), with \( W \) as in [5]. The centralizer of 1 is of course \( L G^0 \). Under the bijection cited above, the unipotent class 1 corresponds to the lowest two-sided cell \( c_0 \), that is the subset of all the elements \( w \) in \( W \) such that \( a(w) \) equals the number of positive roots in the root system of \( W_f \).

Xi proved the Lusztig conjecture for this ideal \( J_{c_0} \) in [5, Theorem 1.10]. According to his result, we have a ring isomorphism

\[
J_{c_0} \cong M_{|W_f|}(R_{C}(L G^0)).
\]

The character map \( Ch \) creates an isomorphism

\[
R_{C}(L G^0) \cong (R_{C}(L T^0))^{W_f}.
\]

The \( W_f \)-invariant subring of the (complexified) representation ring of \( L T^0 \) is precisely the coordinate ring of the quotient torus \( L T^0 / W_f \).

Since

\[
\Psi(T) = L T^0
\]

we have a Morita equivalence

\[
J_{c_0} \sim Z_i(G)
\]

where \( i \) is the quotient variety \( \Psi(T)/W_f \). Therefore, we obtain the following result.

**Theorem 4.** Let \( G \) be a connected \( F \)-split adjoint simple \( p \)-adic group. There is a Morita equivalence between \( J_{c_0} \) and the coordinate ring of the Bernstein variety \( \Psi(T)/W_f \).

According to our conjecture, the other ideals \( J_c \) account (up to geometric equivalence) for the rest of the extended quotient of \( \Psi(T) \) by \( W_f \).
The classical Satake isomorphism is an isomorphism between the spherical Hecke algebra $\mathcal{H}(G//K)$ and the ring $R_\mathbb{C}(L^0)$. Further, a theorem of Bernstein (see e.g. [30, Proposition 8.6]) asserts that the centre $Z(\mathcal{H}(G//I))$ of the Iwahori-Hecke algebra $\mathcal{H}(G//I)$ is also isomorphic to $R_\mathbb{C}(L^0)$.

At this point, we need the map $\phi_{qF,c_0}$ defined in section 1.7 of Xi’s paper [53]. This map is the composition of $\phi_{qF}$ and of the projection of $J$ onto $J_{c_0}$.

Xi has proved in [53, Theorem 3.6] that the image $\phi_{qF,c_0}(Z(H(G//I)))$ is the centre $Z(J_{c_0})$ of the algebra $J_{c_0}$. This creates the following diagram:

\[
\begin{array}{ccc}
\mathcal{H}(G//K) & \longrightarrow & R_\mathbb{C}(L^0) \\
\downarrow & & \downarrow \\
\mathcal{H}(G//I) & \phi_{qF,c_0} & \longrightarrow & J_{c_0}
\end{array}
\]

in which the top horizontal map is the Satake isomorphism, the left vertical map is induced by the inclusion $K \subset I$, the right vertical map sends $R_\mathbb{C}(L^0)$ onto the centre of $J_{c_0}$ and the bottom horizontal map is Xi’s map $\phi_{qF,c_0}$. The vertical maps are injective. We expect that this diagram is commutative.

11 The Hecke algebra of $GL(n)$

Theorem 5. The conjecture is true for $GL(n)$.

Proof. In this proof, we follow [3] rather closely; we have refined the proof at certain points. The occurrence of an extended quotient in the smooth dual of $GL(n)$ was first recorded in [22], in the context of Deligne-Langlands parameters.

Let $G := GL(n)$, $s = [L,\sigma]_G \in \mathfrak{B}(G)$ and $t = [L,\sigma]_L \in \mathfrak{B}(L)$. We can think of $t$ as a vector of irreducible supercuspidal representations of smaller general linear groups. If the vector is

$$(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_t, \ldots, \sigma_t)$$

with $\sigma_i$ repeated $e_i$ times, $1 \leq i \leq t$, and $\sigma_1, \ldots, \sigma_t$ pairwise distinct (after unramified twist) then we say that $t$ has exponents $e_1, \ldots, e_t$.

Each representation $\sigma_i$ of $G_i := GL(m_i)$ has a torsion number: the order of the cyclic group of all those unramified characters $\eta$ for which $\sigma_i \otimes \eta \cong \sigma_i$. The torsion number of $\sigma_i$ will be denoted $r_i$.

Hence

$L \simeq \prod_{i=1}^{t} G_i^{e_i}$ and $\sigma \simeq \bigotimes_{i=1}^{t} \sigma_i^{\otimes e_i}$,
Each $\sigma_i$ contains a maximal simple type $(K_i, \lambda_i)$ in $G_i$ \cite{9}. Let

$$K_L := \prod_{i=1}^{t} K_i^{e_i} \quad \text{and} \quad \tau_L := \bigotimes_{i=1}^{t} \lambda_i^{\otimes e_i}.$$  

Then $(K_L, \tau_L)$ is a $t$-type in $L$. We have

$$W_t \simeq \prod_{i=1}^{t} S_{e_i}.$$  

Let $W_{e_i}$ denote the extended affine Weyl group associated to $\text{GL}(e_i, \mathbb{C})$.

Let $(K, \tau)$ be a semisimple $s$-type, see \cite{9, 10, 11}. It is worth pointing out that we do not need the type explicitly. Instead, we need certain items attached to the type: the idempotent $e_{\tau}$ and the endomorphism-valued Hecke algebra $\mathcal{H}(G, \tau)$. Let $e_{\tau}$ be the idempotent attached to the type $(K, \tau)$ as in \cite{10, Definition 2.9}:

$$e_{\tau}(x) = \begin{cases} (\text{vol } K)^{-1} (\text{dim } \tau) \text{tr}(\tau(x^{-1})) & \text{if } x \in K, \\ 0 & \text{if } x \in G, x \notin K. \end{cases}$$  

The idempotent $e_{\tau}$ is then a special idempotent in the Hecke algebra $\mathcal{H}(G)$ according to \cite{10, Definition 3.11}. Let $\mathcal{H} = \mathcal{H}(G)$. It follows from \cite{10, §3} that

$$\mathcal{H}^e(G) = \mathcal{H} * e_{\tau} * \mathcal{H}.$$  

We then have a Morita equivalence

$$\mathcal{H} * e_{\tau} * \mathcal{H} \sim_{\text{morita}} e_{\tau} * \mathcal{H} * e_{\tau}.$$  

Now let $\mathcal{H}(K, \tau)$ be the endomorphism-valued Hecke algebra attached to the semisimple type $(K, \tau)$. By \cite{10, 2.12} we have a canonical isomorphism of unital $\mathbb{C}$-algebras:

$$\mathcal{H}(G, \tau) \circledast \mathbb{C} \text{End}_\mathbb{C}W \cong e_{\tau} * \mathcal{H}(G) * e_{\tau}$$  

so that the algebra $e_{\tau} * \mathcal{H}(G) * e_{\tau}$ is Morita equivalent to the algebra $\mathcal{H}(G, \tau)$. Now we quote the main theorem for semisimple types in $\text{GL}(n)$ \cite{11, 1.5}: there is an isomorphism of unital $\mathbb{C}$-algebras

$$\mathcal{H}(G, \tau) \cong \bigotimes_{i=1}^{t} \mathcal{H}(W_{e_i}, q_{l_i}^{t_i}).$$  

The factors $\mathcal{H}(W_{e_i}, q_{l_i}^{t_i})$ are (extended) affine Hecke algebras whose structure is given explicitly in \cite{9, 5.4.6, 5.6.6}. 

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We conclude that
\[ H^s(G) \simeq \bigotimes_{i=1}^t H(W_{e_i}, q_F^{r_i}). \]

On the other hand, from (7), we have
\[ \bigotimes_{i=1}^t H(W_{e_i}, q_F^{r_i}) \simeq \bigotimes_{i=1}^t J(W_{e_i}). \]

Finally we will prove that that
\[ \bigotimes_{i=1}^t J(W_{e_i}) \simeq \bar{3}^s. \]

Let \( LT^0 \) be the maximal standard torus of \( LG^0 = GL(n, \mathbb{C}) \) and let \( W \) be the extended affine Weyl group associated to \( GL(n, \mathbb{C}) \). We have \( W := X^*(LT^0) \rtimes S_n = W_n \). For each two-sided cell \( c \) of \( W \) we have a corresponding partition \( \lambda \) of \( n \). Let \( \mu \) be the dual partition of \( \lambda \). Let \( u \) be a unipotent element in \( GL(n, \mathbb{C}) \) whose Jordan blocks are determined by the partition \( \mu \). Let the distinct parts of the dual partition \( \mu \) be \( \mu_1, \ldots, \mu_p \) with \( \mu_r \) repeated \( n_r \) times, \( 1 \leq r \leq p \).

Let \( C_G(u) \) be the centralizer of \( u \) in \( G = GL(n, \mathbb{C}) \). Then the maximal reductive subgroup \( F_c \) of \( C_G(u) \) is isomorphic to \( GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C}) \times \cdots \times GL(n_p, \mathbb{C}) \). For the non-trivial combinatorics which underlies this statement, see [17, §2.6].

Let \( J \) be the based ring of \( W \). For each two-sided cell in \( W \), let \( |Y| \) be the number of left cells contained in \( c \). The Lusztig conjecture says that there is a ring isomorphism

\[ J_c \simeq M_{|Y|}(R_{F_c}), \quad t_w \mapsto \pi(w) \]

where \( R_{F_c} \) is the rational representation ring of \( F_c \). This conjecture for \( GL(n, \mathbb{C}) \) has been proved by Xi [51, 1.5, 4.1, 8.2].

Since \( F_c \) is isomorphic to a direct product of the general linear groups \( GL(n_i, \mathbb{C}) \) \((1 \leq i \leq p)\) we see that \( R_{F_c} \) is isomorphic to the tensor product over \( \mathbb{Z} \) of the representation rings \( R_{GL(n_i, \mathbb{C})} \), \( 1 \leq i \leq p \). For the ring \( R(GL(n, \mathbb{C})) \) we have

\[ R(GL(n, \mathbb{C})) = \mathbb{Z}[X^*(T(\mathbb{C}))]^{S_n} \]

where \( T(\mathbb{C}) \) is the standard maximal torus in \( GL(n, \mathbb{C}) \), and \( X^*(T(\mathbb{C})) \) is the set of rational characters of \( T(\mathbb{C}) \), by [11, Chapter VIII]. Therefore we have
\[ R_F c \otimes \mathbb{Z} C \simeq \mathbb{C}[\text{Sym}^{n_1} C^x \times \cdots \times \text{Sym}^{n_p} C^x]. \]

Let \( \gamma \in S_n \) have cycle type \( \mu \), let \( X = (C^x)^n \). Then

\[
\begin{align*}
X^\gamma &\simeq (C^x)^{n_1} \times \cdots \times (C^x)^{n_p} \\
Z(\gamma) &\simeq (\mathbb{Z}/\mu_1 \mathbb{Z}) \wr S_{n_1} \times \cdots \times (\mathbb{Z}/\mu_p \mathbb{Z}) \wr S_{n_p} \\
X^\gamma / Z(\gamma) &\simeq \text{Sym}^{n_1} C^x \times \cdots \times \text{Sym}^{n_p} C^x
\end{align*}
\]

and so

\[ R_F c \otimes \mathbb{Z} C \simeq \mathbb{C}[X^\gamma / Z(\gamma)]. \]

Then, using (1), we obtain

\[ J \otimes \mathbb{Z} C \simeq \bigoplus_e (J_c \otimes \mathbb{Z} C) \sim \bigoplus_e (R_F c \otimes \mathbb{Z} C) \simeq \mathbb{C}[\widetilde{X}/S_n]. \]

The algebra \( J \otimes \mathbb{Z} C \) is Morita equivalent to a reduced, finitely generated, commutative unital \( \mathbb{C} \)-algebra, namely the coordinate ring of the extended quotient \( \widetilde{X}/S_n \). This finishes the proof of part (1) of the conjecture.

Part (2) of the conjecture for GL(\( n \)) is a consequence of [43, Theorem 5.1].

\[ \square \]

12 The Iwahori ideal in \( \mathcal{H}(\text{PGL}(n)) \)

Let \( G = \text{PGL}(n) \), let \( T \) be its standard maximal torus. Let \( W := X_*(T) \times W_f \). Then \( \mathcal{L}G^0 = \text{SL}(n, \mathbb{C}) \) is the Langlands dual group. Its maximal torus will be denoted \( \mathcal{L}T^0 \).

The discrete group \( W \) is an extended Coxeter group:

\[ W = \langle s_1, s_2, \ldots, s_n \rangle \rtimes \mathbb{Z}/n\mathbb{Z} \]

where \( \mathbb{Z}/n\mathbb{Z} \) permutes cyclically the generators \( s_1, \ldots, s_n \). We have

\[ \mathcal{H}(W, q_F) = \mathcal{H}(G//I). \]

The symmetric group \( W_f = S_n \) acts on \( \mathcal{L}T^0 \) by permuting coordinates, and we form the quotient variety \( \mathcal{L}T^0 / S_n \).

Let \( i \in \mathfrak{B}(G) \) be determined by the cuspidal pair \((T, 1)\). We have

\[ \mathfrak{Z}^i = \mathbb{C}[\mathcal{L}T^0 / S_n], \]

\[ \mathfrak{Z}^\tilde{i} = \mathbb{C}[\widetilde{\mathcal{L}T^0} / S_n]. \]
Theorem 6. Let $\mathcal{H}^i(G)$ denote the Iwahori ideal in $\mathcal{H}(G)$. Then $\mathcal{H}^i(G)$ is geometrically equivalent to the extended quotient of $L^0T^0$ by the symmetric group $S_n$:

$$\mathcal{H}^i(G) \cong \mathbb{C}[L^0T^0/S_n].$$

Proof. The non-unital algebra $\mathcal{H}^i(G)$ is Morita equivalent to the unital affine Hecke algebra $\mathcal{H}(W, q_F)$:

$$\mathcal{H}^i(G) = \mathcal{H}e\mathcal{H} \sim_{\text{morita}} e\mathcal{H}e \cong \mathcal{H}(W, q_F).$$

From (7), we have

$$\mathcal{H}(W, q_F) \sim J \otimes \mathbb{Z} \mathbb{C}.$$

The Langlands dual of $\text{PGL}(n, F)$ is $\text{SL}(n, \mathbb{C})$. For each two-sided cell $c$ of $W$ we have a corresponding partition $\lambda$ of $n$. Let $\mu$ be the dual partition of $\lambda$. Let $u$ be a unipotent element in $\text{SL}(n, \mathbb{C})$ whose Jordan blocks are determined by the partition $\mu$. Let the distinct parts of the dual partition $\mu$ be $\mu_1 < \cdots < \mu_p$ with $\mu_r$ repeated $n_r$ times, $1 \leq r \leq p$.

Let $C_G(u)$ be the centralizer of $u$ in $G = \text{SL}(n, \mathbb{C})$. Then the maximal reductive subgroup $F'_\lambda$ of $C_G(u)$ is isomorphic to $(\text{GL}(n_1, \mathbb{C}) \times \text{GL}(n_2, \mathbb{C}) \times \cdots \times \text{GL}(n_p, \mathbb{C})) \cap \text{SL}(n, \mathbb{C})$. For details of the injective map

$$(\text{GL}(n_1, \mathbb{C}) \times \text{GL}(n_2, \mathbb{C}) \times \cdots \times \text{GL}(n_p, \mathbb{C})) \cap \text{SL}(n, \mathbb{C}) \rightarrow \text{SL}(n, \mathbb{C})$$

see [17].

As a special case, let the two-sided cell $c$ correspond to the partition $\lambda = (1, 1, 1, \ldots, 1)$ of $n$. Then the dual partition $\mu = (n)$. The unipotent matrix $u$ has one Jordan block, and its centralizer $C_G(u) = Z$ the centre of $\text{SL}(n, \mathbb{C})$. The maximal reductive subgroup $F'_\lambda$ of $C_G(u)$ is the finite group $Z$. This is the case $p = 1$, $\mu_1 = n$, $n_1 = 1$.

By the theorem of Xi [51, 8.4] we have

$$J_c \otimes \mathbb{Z} \mathbb{C} \sim_{\text{morita}} R_{\mathbb{C}}(F'_\lambda) = R_{\mathbb{C}}(Z) = \mathbb{C}^n.$$

Let $\gamma$ have cycle type $(n)$. Then the fixed set $(L^0T^0)^\gamma$ comprises the $n$ fixed points

$$\text{diag}(\omega^j, \ldots, \omega^j) \in L^0T^0$$

where $\omega = \exp(2\pi i/n)$ and $0 \leq j \leq n - 1$. These $n$ fixed points correspond to the $n$ generators in the commutative ring $\mathbb{C}^n$. 

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We expect that the corresponding points in Irr(PGL(N)) arise as follows. The unramified unitary twist
\[ z^{\text{valordet}} \otimes \text{St}(n) \]
of the Steinberg representation of GL(n) has trivial central character if and only if \( z \) is an \( n \)th root of unity. For these values \( 1, \omega, \omega^2, \ldots, \omega^{n-1} \) of \( z \), we obtain \( n \) irreducible smooth representations of PGL(n).

From now on, we will assume that \( \lambda \neq (1,1,1,\ldots,1) \). Then \( F_\lambda' \) is a connected Lie group.

We will write \( T_\lambda(C) \) for the standard maximal torus of \( F_\lambda' \). The Weyl group is then \( W(\lambda) = S_{n_1} \times \cdots \times S_{n_p} \).

According to Bourbaki [7, Chapter 8], the map \( Ch \), sending each (virtual) representation to its (virtual) character, creates an isomorphism:
\[ Ch : R(F_\lambda') \cong \mathbb{Z}[X^*(T_\lambda(C))]^{W(\lambda)}. \]

Note that a complex linear combination of rational characters of \( T_\lambda(C) \) is precisely a regular function on \( T_\lambda(C) \).

For each two-sided cell \( c \) of \( W \) the \( \mathbb{Z} \)-submodule \( J_c \) of \( J \), spanned by all \( t_w \), \( w \in c \), is a two-sided ideal of \( J \). The ring \( J_c \) is the based ring of the two-sided cell \( c \). Now apply the theorem of Xi [51, 8.4]. We get
\[ J_c \otimes \mathbb{C} \sim_{\text{Morita}} R_C(F_\lambda') \cong \mathbb{C}[T_\lambda(C)]^{W(\lambda)} \cong \mathbb{C}[T_\lambda(C)/W(\lambda)]. \]

Let \( \gamma \in S_n \) have cycle type \( \mu \). Then the \( \gamma \)-centralizer is a direct product of wreath products:
\[ Z(\gamma) \cong (\mathbb{Z}/\mu_1 \mathbb{Z}) \wr S_{n_1} \times \cdots \times (\mathbb{Z}/\mu_p \mathbb{Z}) \wr S_{n_p}. \]

The image of \( T_\lambda(C) \) in the inclusion \( T_\lambda(C) \rightarrow L T^0 \) is precisely the subtorus of \( L T^0 \) fixed by \( \mathbb{Z}/\mu_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/\mu_p \mathbb{Z} \). We therefore have
\[ (L T^0)^\gamma / Z(\gamma) \cong T_\lambda(C)/W(\lambda). \]

We conclude that
\[ R(F_\lambda') \otimes \mathbb{C} \cong \mathbb{C}[(L T^0)^\gamma / Z(\gamma)] \]
Then
\[ J \otimes \mathbb{C} = \oplus_c (J_c \otimes \mathbb{C}) \sim \oplus_c (R(F_\lambda') \otimes \mathbb{C}) \cong \mathbb{C}[L T^0 / S_n] \]
The algebra \( J \otimes \mathbb{C} \) is Morita equivalent to a reduced, finitely generated, commutative unital \( \mathbb{C} \)-algebra, namely the coordinate ring of the extended quotient \( L T^0 / S_n \). \( \square \)
13 The Iwahori ideal in $\mathcal{H}(\text{SO}(5))$

Let $G$ denote the special orthogonal group $\text{SO}(5,F)$. We view it as the group of elements of determinant 1 which stabilise the symmetric bilinear form
\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

Let $T$ be the group of diagonal matrices
\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \lambda_2^{-1}
\end{pmatrix}, \quad \lambda_1, \lambda_2 \in F^\times.
\]

The extended affine Weyl group $W = X_*(T) \rtimes W_f$ is of type $\tilde{B}_2$, with $W_f \simeq S_2 \ltimes (\mathbb{Z}/2)^2$ a finite Weyl group of type $B_2$.

Here $L^G_0 = \text{Sp}(4,\mathbb{C})$, and $L^T_0$ is the group of diagonal matrices
\[
d(t_1, t_2) := \begin{pmatrix}
t_1 & 0 & 0 & 0 \\
0 & t_2 & 0 & 0 \\
0 & 0 & t_2^{-1} & 0 \\
0 & 0 & 0 & t_1^{-1}
\end{pmatrix}, \quad t_1, t_2 \in \mathbb{C}^\times.
\]

We have $N_{L^G_0}(L^T_0)/L^T_0 \simeq W_f$. The group $L^G_0 = \text{Sp}(4,\mathbb{C})$ is simply connected. In [52 §11.1], Xi has proved Lusztig’s conjecture for the group $W$.

The extended Coxeter group $W = W' \rtimes \Omega$ has four two-sided cells $c_e, c_1, c_2$ and $c_0$ (see [52 §11.1]):
\[
c_e = \{w \in W : a(w) = 0\} = \{e, \omega\} = \Omega,
\]
\[
c_1 = \{w \in W : a(w) = 1\},
\]
\[
c_2 = \{w \in W : a(w) = 2\},
\]
\[
c_0 = \{w \in W : a(w) = 4\} \quad \text{(the lowest two-sided cell)}.
\]

We have
\[
J = J_{c_e} \oplus J_{c_1} \oplus J_{c_2} \oplus J_{c_0}.
\]

Let $i \in \mathfrak{B}(G)$ be determined by the cuspidal pair $(T, 1)$. We have
\[
\mathfrak{Z}^i = \mathbb{C}[L^T_0/W_f], \quad \mathfrak{Z}_i = \mathbb{C}[\tilde{L}^T_0/W_f].
\]
Lemma 6. Let

\[ \mathbb{L} := \mathbb{C}[t, t^{-1}]^\mathbb{Z} / 2\mathbb{Z} \]

denote the balanced Laurent polynomials in one indeterminate \( t \), where the generator \( \alpha \) of \( \mathbb{Z} / 2\mathbb{Z} \) acts as follows: \( \alpha(t) = t^{-1} \). Then the coordinate algebra of the extended quotient of \( L^T \) by \( W_f \) is the \( \mathbb{C} \)-algebra

\[ \mathbb{C}^5 \oplus \mathbb{L}^3 \oplus \mathbb{C}[L^T / W_f] \].

Proof. Let \( X := L^T \). The 8 elements \( \gamma_1, \ldots, \gamma_8 \) of \( W_f \) can be described as follows:

\[
\begin{align*}
\gamma_1(d(t_1, t_2)) &= (d(t_1, t_2)) & \gamma_2(d(t_1, t_2)) &= d(t_2, t_1) \\
\gamma_3(d(t_1, t_2)) &= d(t_1^{-1}, t_2) & \gamma_4(d(t_1, t_2)) &= d(t_1, t_2^{-1}) \\
\gamma_5(d(t_1, t_2)) &= d(t_2, t_1^{-1}) & \gamma_6(d(t_1, t_2)) &= d(t_1^{-1}, t_2^{-1}) \\
\gamma_7(d(t_1, t_2)) &= d(t_2^{-1}, t_1) & \gamma_8(d(t_1, t_2)) &= d(t_1, t_2^{-1})
\end{align*}
\]

We have \( \gamma_5 = \gamma_2 \gamma_3 = \gamma_4 \gamma_7 = \gamma_3 \gamma_4 = \gamma_4 \gamma_3 \). The elements \( \gamma_2, \gamma_3, \gamma_6, \gamma_7 \) and \( \gamma_8 \) are of order 2, the elements \( \gamma_5 \) and \( \gamma_7 \) are of order 4. We obtain

\[
X^{\gamma_1} = X, \quad X^{\gamma_2} = \{d(t, t) : t \in \mathbb{C}^x\},
\]

\[
X^{\gamma_3} = X^{\gamma_4} = \{d(1, t_2), d(-1, t_2) : t_2 \in \mathbb{C}^x\},
\]

\[
X^{\gamma_5} = X^{\gamma_7} = \{d(1, 1), d(-1, -1)\},
\]

\[
X^{\gamma_6} = \{d(1, 1), d(1, -1), d(-1, 1), d(-1, -1)\}.
\]

The elements \( \gamma_1, \gamma_6 \) are central, and we have

\[
Z(\gamma_2) = \{\gamma_1, \gamma_2, \gamma_6, \gamma_8\}, \quad Z(\gamma_3) = \{\gamma_1, \gamma_3, \gamma_4, \gamma_6\}, \quad Z(\gamma_7) = \{\gamma_1, \gamma_5, \gamma_6, \gamma_7\}.
\]

There are five \( W_f \)-conjugacy classes:

\[
\{\gamma_1\}, \{\gamma_6\}, \{\gamma_2, \gamma_8\}, \{\gamma_3, \gamma_4\}, \{\gamma_5, \gamma_7\}.
\]

As representatives, we will take \( \gamma_1, \gamma_2, \gamma_3, \gamma_5, \gamma_6 \).

- We have

\[
\mathbb{C}[X^{\gamma_6} / Z(\gamma_6)] = \mathbb{C}[X^{\gamma_6} / W_f] = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}
\]

since there are three \( W_f \)-orbits in \( X^{\gamma_6} \), namely

\[
\{d(1, 1)\}, \{d(1, -1), d(-1, 1)\}, \{d(-1, -1)\}
\]

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• We have
\[ \mathbb{C}[X^3/Z(\gamma_3)] = \mathbb{C}[X^3] = \mathbb{C} \oplus \mathbb{C} = R_\mathbb{C}(Z) = \mathbb{C}[\Omega] = J_{c_1} \]
since \( Z(\gamma_3) \) acts trivially on \( X^3 \).

• We have
\[ X^3 = \{d(1,t) : t \in \mathbb{C}^x\} \cup \{d(-1,t) : t \in \mathbb{C}^x\} \]
The \( Z(\gamma_3) \)-orbit of \( d(1,t) \) is the unordered pair \( \{d(1,t), d(1,t^{-1})\} \)
and the \( Z(\gamma_3) \)-orbit of \( d(-1,t) \) is the unordered pair \( \{d(-1,t), d(-1,t^{-1})\} \).
Therefore we have
\[ \mathbb{C}[X^3/Z(\gamma_3)] = \mathbb{C} \oplus \mathbb{C}. \]

• We have
\[ X^2 = \{d(t,t) : t \in \mathbb{C}^x\} \cong \{t : t \in \mathbb{C}^x\} \]
The \( Z(\gamma_2) \)-orbit of the point \( t \) is the unordered pair \( \{t,t^{-1}\} \). So we have
\[ X^2/Z(\gamma_2) \cong \mathbb{C}^x/Z/2 \]
and \( \mathbb{C}[X^2/Z(\gamma_2)] = \mathbb{C}. \)

• We have \( \mathbb{C}[X^1/Z(\gamma_1)] = \mathbb{C}[L T^0/W_f] \).

\[ \square \]

The reductive group \( F_{c_1} \) is the center of \( LG^0 \), \( F_{c_1} = (\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{C}^x \)
where \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{C}^x \) by \( z \mapsto z^{-1} \), and \( F_{c_2} = (\mathbb{Z}/2\mathbb{Z}) \ltimes SL(2,\mathbb{C}) \)
and there is a ring isomorphism \( J_{c_2} \cong M_4(R_{F_{c_2}}) \), where \( R_{F_{c_2}} \) is the rational representation ring of \( F_{c_2} \). \[ [32] \text{Theorem 11.2}. \]

We have \( F_{c_1} = \langle \alpha \rangle \ltimes \mathbb{C}^x \) where \( \alpha \) generates \( \mathbb{Z}/2\mathbb{Z} \). Note the crucial relation
\[ \alpha z = z^{-1} \alpha \]
with \( z \in \mathbb{C}^x \). The (semisimple) conjugacy classes in \( F_{c_1} \) are:
\[ \{1\}, \quad \{-1\}, \quad \{\{z, z^{-1} \} : z \in \mathbb{C}^x, z^2 \neq 1\}, \quad \alpha \cdot \mathbb{C}^x \]

**Lemma 7.** Let \( M := \mathbb{C}[t, t^{-1}] \) denote the Laurent polynomials in one indeterminate \( t \). We have
\[ J_{c_1} \cong \mathbb{C} \oplus (M \rtimes \mathbb{Z}/2\mathbb{Z}) \]
where the generator \( \alpha \) of \( \mathbb{Z}/2\mathbb{Z} \) acts as follows: \( \alpha(t) = t^{-1} \).
Proof. Let $F = F_{e_1}$, $F = \langle \alpha \rangle \ltimes \mathbb{C}^\times$. We have to construct the simple $J_{e_1}$-modules explicitly, following Xi [52, p. 51, 107]. We will use Xi’s explicit proof of the Lusztig conjecture for $B_2$. Let $Y = \{1, 2, 3, 4\}$ be the $F$-set such that as $F$-sets we have $\{1\} \cong \{2\} \cong F/F$ and $\{3, 4\} \cong F/F^0$. The simple $J_{e_1}$-modules are given by

$$E_{s, \rho} := \text{Hom}_{A(s)}(\rho, H_s(Y^s))$$

where $A(s)$ denotes the component group $C_F(s)/C_F(s)\Delta$ and $\rho$ is a simple $A(s)$-module which appears in the homology group $H_s(Y^s)$. The set $Y^s$ denotes the $s$-fixed set: $Y^s = \{y \in Y : sy = y\}$. The pair $(s, \rho)$ is chosen up to $F$-conjugacy.

(A). $s = 1$, $\rho = 1$. Then $Y^s = \{1, 2, 3, 4\}$. Also $H_s(Y^s)$ is the free $\mathbb{C}$-vector space $V := \mathbb{C}^4$ on $\{1, 2, 3, 4\}$: we will denote its basis by $\{e_1, e_2, e_3, e_4\}$. $A(s) = \mathbb{Z}/2\mathbb{Z}$. The generator of $A(s)$ permutes $e_3, e_4$. Let $V_1$ denote the span of $e_1, e_2, e_3 + e_4$, let $V_2$ denote the span of $e_3 - e_4$. Then

$$E_{1,1} := \text{Hom}_{A(s)}(\rho, V) = V_1 = \mathbb{C}^3.$$ 

(B). $s = 1$, $\rho = \epsilon$ where $\epsilon$ is the sign representation of $\mathbb{Z}/2\mathbb{Z}$. We have

$$E_{1,\epsilon} := \text{Hom}_{A(s)}(\rho, V) = V_2 = \mathbb{C}.$$

Note that we have

$$E_{1,1} \oplus E_{1,\epsilon} = V = \mathbb{C}^4$$

as $A(s)$-modules.

(C). $s = -1$, $\rho = 1$. We have $Y^s = Y$, $A(s) = \mathbb{Z}/2\mathbb{Z}$ and

$$E_{-1,1} := \text{Hom}_{A(s)}(\rho, V) = V_1 = \mathbb{C}^3.$$ 

(D). $s = -1$, $\rho = \epsilon$. We have $Y^s = Y$, $A(s) = \mathbb{Z}/2\mathbb{Z}$ and

$$E_{-1,\epsilon} := \text{Hom}_{A(s)}(\rho, V) = V_2 = \mathbb{C}$$

Note that we have

$$E_{-1,1} \oplus E_{-1,\epsilon} = V = \mathbb{C}^4$$

as $A(s)$-modules.

(E) $s = z$, $\rho = 1$ where $z \in \mathbb{C}^\times$, $z^2 \neq 1$. We have $Y^s = Y$, $A(s) = \{1\}$ and

$$E_{z,1} := \text{Hom}(\mathbb{C}, V) = V = \mathbb{C}^4.$$ 

(F). $s = \alpha$, $\rho = 1$. We have $Y^s = \{1, 2\}$, $H_s(Y^s) = \mathbb{C}^2$, $A(s) = \{1, -1, \alpha, -\alpha\} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have

$$E_{\alpha,1} := \text{Hom}_{A(s)}(\mathbb{C}, \mathbb{C}^2) = \mathbb{C}^2.$$
This concludes the list of simple $J_{c_1}$-modules. We now turn to the Lusztig-Xi isomorphism of unital algebras:

$$J_{c_1} \cong K_F(Y \times Y) \otimes \mathbb{Z} \mathbb{C}.$$ 

The equivariant $K$-theory $K_F(Y \times Y) \otimes \mathbb{Z} \mathbb{C}$ is equipped with the convolution product. The action of $F$ on the set $Y$ leads to the following description. We identify $K_F(Y \times Y) \otimes \mathbb{Z} \mathbb{C}$ with the $C$-algebra of $4 \times 4$ matrices $(a_{ij})$ where $a_{11}, a_{12}, a_{21}, a_{22} \in R(F) \otimes \mathbb{Z} \mathbb{C}$ and all other entries are in $R(\mathbb{C}^\times) \otimes \mathbb{Z} \mathbb{C}$, subject to the following conditions:

$$a_{14} = a_{13}, \quad a_{24} = a_{23}, \quad a_{41} = a_{31}, \quad a_{42} = a_{32}, \quad a_{44} = a_{33}, \quad a_{43} = a_{34}$$

where, for all $z \in \mathbb{C}^\times$,

$$a_{ij}(z) = a_{ij}(z^{-1}).$$

Let

$$\mathbb{M} := \mathbb{C}[t, t^{-1}]$$

denote the Laurent polynomials in one indeterminate $t$. We have an injective homomorphism of unital $\mathbb{C}$-algebras:

$$\psi : \mathbb{C} \oplus (\mathbb{M} \times \mathbb{Z}/2\mathbb{Z}) \longrightarrow K_F(Y \times Y) \otimes \mathbb{Z} \mathbb{C}$$

$$(\lambda, p + \alpha \cdot q) \mapsto \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & p & \eta \\ 0 & 0 & q & \bar{\eta} \end{pmatrix}$$

We claim that this map is a spectrum-preserving morphism of unital finite-type $k$-algebras. We view each entry in $K_F(Y \times Y)$ as a virtual character of $F$ or $\mathbb{C}^\times$. We can then evaluate each matrix entry at $z \in \mathbb{C}^\times$. Evaluation at $z \in \mathbb{C}^\times$ determines an algebra homomorphism

$$K_F(Y \times Y) \otimes \mathbb{Z} \mathbb{C} \longrightarrow M_4(\mathbb{C}).$$

This homomorphism gives $\mathbb{C}^4$ the structure of a $J_{c_1}$-module.

- When $z^2 \neq 1$, this homomorphism gives $\mathbb{C}^4$ the structure of a simple $J_{c_1}$-module, namely $E_{z,1}$.
- When $z = 1$, the module $\mathbb{C}^4$ splits into simple $J_{c_1}$-modules of dimensions 3 and 1, namely $E_{1,1}$ and $E_{1,e}$.
- When $z = -1$, the module $\mathbb{C}^4$ splits into simple $J_{c_1}$-modules of dimensions 3 and 1, namely $E_{-1,1}$ and $E_{-1,e}$. 

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When restricted to the lower right $2 \times 2$ block of the $4 \times 4$ matrix algebra $(a_{ij})$, each simple 4-dimensional module splits into the direct sum of a 2-dimensional 0-module and a 2-dimensional simple module over the crossed product $\mathbb{M} \rtimes \mathbb{Z}/2\mathbb{Z}$. At 1 and $-1$, the module $\mathbb{C}^4$ restricts to a 2-dimensional 0-module and two 1-dimensional simple modules. It now follows that $\text{Prim}(J_{c_1})$ with one deleted point is in bijection (via our morphism of algebras) with $\text{Prim}(\mathbb{M} \rtimes \mathbb{Z}/2\mathbb{Z})$. There is one remaining point in $\text{Prim}(J_{c_1})$, namely $E_{\alpha,1}$. This point in $\text{Prim}(J_{c_1})$ maps, via our morphism of algebras, to the one remaining primitive ideal $0 \oplus (\mathbb{M} \rtimes \mathbb{Z}/2\mathbb{Z})$.

We conclude from this that

$$J_{c_1} \simeq \mathbb{C} \oplus (\mathbb{M} \rtimes \mathbb{Z}/2\mathbb{Z}).$$

\[\square\]

**Theorem 7.** Let $\mathcal{H}^i(\text{SO}(5))$ denote the Iwahori ideal in $\text{SO}(5)$. We have

$$\mathcal{H}^i(\text{SO}(5)) \simeq \tilde{3}^i(\text{SO}(5)).$$

**Proof.** We note that

- $J_{c_\varepsilon} = \mathbb{C}[\Omega] = \mathbb{C}^2$
- $J_{c_2} \simeq R_{\mathbb{C}}(F_{c_2}) = \mathbb{L}^2$ since

$$R_{\mathbb{C}}(F_{c_2}) = R_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z} \rtimes \text{SL}(2, \mathbb{C})) = R_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z}) \otimes R_{\mathbb{C}}(\text{SL}(2, \mathbb{C})) = \mathbb{L} \oplus \mathbb{L}.$$

By Lemmas 4 and 7, we have

$$J = J_{c_\varepsilon} \oplus J_{c_1} \oplus J_{c_2} \oplus J_{c_0} \simeq \mathbb{C}^2 \oplus (\mathbb{C}^3 \oplus \mathbb{L}) \oplus \mathbb{L}^2 \oplus J_{c_0}.$$

By Lemma 6 we have

$$\mathbb{C}[\widetilde{L^T 0}/W_f] = \mathbb{C}^5 \oplus \mathbb{L}^3 \oplus \mathbb{C}[\widetilde{L^T 0}/W_f].$$

We conclude, by Theorem 3, that

$$\mathcal{H}^i(\text{SO}(5)) \simeq J \simeq \mathbb{C}[\widetilde{X}/W_f] = \tilde{3}^i(\text{SO}(5))$$

as required. This confirms part (1) of our conjecture for the Iwahori ideal of $\mathcal{H}(\text{SO}(5))$.

It is worth noting that, in this example, there are 4 two sided cells and 5 $W_f$-conjugacy classes.
14 Consequences of the conjecture

Parametrization of the smooth dual. In this section we will suppose that the conjecture is true for the Iwahori ideal \( \mathcal{H}_i(G) \). We then have

\[
\mathcal{H}_i(G) \cong \mathcal{O}(L^0/W_f).
\]

We now take the primitive ideal space of each side. We obtain a bijection

\[
\text{Prim } \mathcal{H}_i(G) \leftrightarrow L^0/W_f.
\]

Now \( \text{Prim } \mathcal{H}_i(G) \) may be identified with the subset \( \text{Irr}_I(G) \) of the smooth dual \( \text{Irr}(G) \) which admits nonzero Iwahori-fixed vectors. This leads to a parametrization of \( \text{Irr}_I(G) \):

\[
\text{Irr}_I(G) \leftrightarrow L^0/W_f.
\]

This is a parametrization of \( \text{Irr}_I(G) \) by the \( \mathbb{C} \)-points in a complex affine algebraic variety (with several components). This parametrization is not quite canonical: it depends on the specific finite sequence of elementary steps (and filtrations) connecting \( \mathcal{H}_i(G) \) to \( \mathcal{O}(L^0/W_f) \).

This parametrization will assign to each \( \omega \in \text{Irr}_I(G) \) a pair

\[
(s, \gamma)
\]

with \( s \in L^0, \gamma \) a \( W_f \)-conjugacy class.

More generally, the conjecture leads to a parametrization of \( \text{Irr}(G) \) by the \( \mathbb{C} \)-points in a complex affine locally algebraic variety (with countably many components). The dimensions of the components are less than or equal to the rank of \( G \).

Langlands parameters. We wish to make a comparison with Langlands parameters. We first recall some background material.

Let \( \mathcal{W}_F \) denote the Weil group of \( F \), and let \( \mathcal{I}_F \subset \mathcal{W}_F \) be the inertia subgroup so that \( \mathcal{W}_F/\mathcal{I}_F \simeq \mathbb{Z} \). Denote the Frobenius generator by \( \text{Frob} \) in order that \( \mathcal{W}_F = \mathcal{I}_F \ltimes \langle \text{Frob} \rangle \).

By Langlands parameter we mean a continuous homomorphism

\[
\varphi: \mathcal{W}_F \times \text{SL}(2, \mathbb{C}) \to L^0 G^0
\]

which is rational on \( \text{SL}(2, \mathbb{C}) \) and such that \( \varphi(\text{Frob}) \) is semisimple.

Call \( \varphi \) unramified if it has trivial restriction to \( \mathcal{I}_F \). The unramified Langlands parameters are parameterized by pairs \( (s, u) \) with \( s \in L^0 G^0 \) semisimple, \( u \in L^0 G^0 \) unipotent with \( su s^{-1} = u^{qF} \).

To each Langlands parameter \( \varphi \) should correspond a finite packet \( \Pi_\varphi \) of irreducible smooth representations of \( G \). According to the Langlands philosophy, the unramified Langlands parameters should
be those for which the representations in $\Pi_\varphi$ are unipotent in the sense of [37]. This has been proved by Lusztig [37] when $G$ is the group of $F$-points of a split adjoint simple algebraic group.

More precisely, let $Z$ be the centre of $L^G$, let $C(g)$ denote the centralizer in $L^G$ of $g \in L^G$, and $C(s, u) := C(s) \cap C(u)$. We denote by $C(s, u)^0$ the connected component of $C(s, u)$ and set $A(s, u) := C(s, u)/Z \cdot C(s, u)^0$. It is proved in [37] that the isomorphism classes of unipotent representations of $G$ are naturally in one-to-one correspondence with the set of triples $(s, u, \rho)$ (modulo the natural action of $L^G$) where $s, u$ as above and $\rho$ an irreducible representation (up to isomorphism) of $A(s, u)$.

In the case when we restrict ourselves to representations in $\text{Irr}_I(G)$ the above classification specializes to the Kazhdan-Lusztig classification, which provides a proof of a refined form of a conjecture of Deligne and Langlands: if $G$ is the group of $F$-points of any split reductive group over $F$, the set $\text{Irr}_I(G)$ is naturally in bijection with the set of triples $(s, u, \rho)$ as above (modulo the natural action of $L^G$) such that $\rho$ appears in the homology of the variety $B_{s, u}$ of the Borel subgroups of $L^G$ containing both $s$ and $u$ (see [24], [44]).

For $\text{GL}(n)$, the simultaneous centralizer $C(s, u)$ is connected, and so each Deligne-Langlands parameter is a pair $(s, u)$. There is a bijection between unipotent classes in $\text{GL}(n, \mathbb{C})$ and partitions of $n$ (via Jordan canonical form). There is a second bijection between partitions of $n$ and conjugacy classes in $W_f = S_n$ the symmetric group. In this way we obtain a perfect match

$$(s, u) \leftrightarrow (s, \gamma).$$

The Deligne-Langlands parameters can be arranged to form an extended quotient of the complex torus $\mathbb{C}^\times$ by the symmetric group $S_n$. The details of this correspondence were recorded in [22].

**Irreducibility of Induced Representations.** If $W_i$ acts freely then $D_\sigma/W_i \cong D_\sigma/W_1$ and the conjecture in this case predicts irreducibility of the induced representations $i^G_F(\sigma \otimes \chi)$. This situation is discussed in [45], with any maximal Levi subgroup of $G$, and also with the following (non maximal) Levi subgroup

$$L = (\text{GL}(2) \times \text{GL}(2) \times \text{GL}(4)) \cap \text{SL}(8)$$

of $\text{SL}(8)$ and $\sigma$ defined as in [45] p.127.

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Anne-Marie Aubert, Institut de Mathématiques de Jussieu, U.M.R. 7586 du C.N.R.S., Paris, France
Email: aubert@math.jussieu.fr

Paul Baum, Pennsylvania State University, Mathematics Department, University Park, PA 16802, USA
Email: baum@math.psu.edu

Roger Plymen, School of Mathematics, University of Manchester, Manchester M13 9PL, England
Email: plymen@manchester.ac.uk