Quantum Error-Correcting Codes over Mixed Alphabets

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Errors are inevitable during all kinds quantum informational tasks and quantum error-correcting codes (QECCs) are powerful tools to fight various quantum noises. For standard QECCs physical systems have the same number of energy levels. Here we shall propose QECCs over mixed alphabets, i.e., physical systems of different dimensions, and investigate their constructions as well as their quantum Singleton bound. We propose two kinds of constructions: a graphical construction based a graph-theoretical object composite coding clique and a projection-based construction. We illustrate our ideas using two alphabets by finding out some 1-error correcting or detecting codes over mixed alphabets, e.g., optimal ((6, 8, 3))14,121, ((6, 4, 3))14,122 and ((5, 16, 2))14,32 code and suboptimal ((5, 9, 2))4,32 code. Our methods also shed light to the constructions of standard QECCs, e.g., the construction of the optimal ((6, 16, 3))4 code as well as the optimal ((2n + 3, p2n+1, 2))p codes with p = 4k.

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Quantum error-correcting code (QECC) has been receiving much attention because it plays a vital role in many quantum information tasks such as fault-tolerant quantum computation, quantum key distribution and many quantum information tasks such as fault-tolerant quantum computation [5], quantum key distribution [5] and quantum computation [5]. In this Letter, we will study how to construct QECCs over mixed alphabets and their quantum Singleton bound [4]. Also some examples of optimal and suboptimal codes over mixed alphabets, e.g., the construction of the optimal ((6, 8, 3))14,121, ((6, 4, 3))14,122 and ((5, 16, 2))14,32 code and suboptimal ((5, 9, 2))4,32 code. Our methods also shed light to the constructions of standard QECCs, e.g., the construction of the optimal ((6, 16, 3))4 code as well as the optimal ((2n + 3, p2n+1, 2))p codes with p = 4k.

In what follows we shall illustrate our ideas by the construction of QECCs over 2 alphabets with an obvious generalization to more complicated cases. Here we denote a code over 2 alphabets, i.e., two kinds of physical systems of dimensions p and q, by ((n, K, d))p,q, which means that the system has n1 q-level particles (qubit) and n2 p-level particles (qupit) with n1 + n2 = n. When n1 = 0 or n2 = 0, it is reduced to a standard QECC. Depending on whether p and q are coprime or not we propose two different constructions.

In the case of reducible p and q, e.g., q = r · p for some integer r, a qubit can be regarded as the composite particle of a qubit and qudit. Denote the bit shift and phase shift operators of an l-level particle by Xi = \sum_{j \in \mathbb{Z}_l} |j + 1\rangle \langle j | and Zt = \sum_{j \in \mathbb{Z}_l} \omega_j |j\rangle \langle j | with \omega_l = e^{2\pi i / l} and \mathbb{Z}_l being the ring of addition modular l which satisfies ZlXl = \omega_lXlZl and Xl = Zl = I. It is easy to prove that the group \( \{X_p, Z_p \otimes \{X_r, Z_r\}\} \) forms an error basis of a qudit. Then the mixed-alphabet system, n1 qudits and n2 qudits, can be regarded as a composite system of an n-qudit and an n1-qubit subsystems, with a nice error basis given by

\[ \{E_p \otimes E_r := X_p^{s} Z_p^{t} \otimes X_r^{s'} Z_r^{t'} | s, t, s', t' \in \mathbb{Z}_p^{\otimes n1}\} \] (1)

Thus any less than d-bit error can be regarded as two errors on two subsystems respectively, i.e.,

\[ |E| = |E_p \cup E_r| = |\tilde{s} \cup \tilde{t} \cup s' \cup t'| < d, \] (2)

where \( \tilde{s} = \{i \in n | s_i \neq 0\} \) is the support of vector \( s \in \mathbb{Z}_p^{\otimes n1} \) and \( \|C\| \) indicates the number of elements in \( C \subset n \).

For the p-level subsystem, consider a \( \mathbb{Z}_p \)-weighted graph \( G_p = (V, \Gamma_p) \) composed of a set V of n vertices and a set of weighted edges specified by the adjacency matrix \( \Gamma_p \) which is an n × n matrix with zero diagonal entries and the matrix element \( \Gamma_{ab} \in \mathbb{Z}_p \) indicating the weight of the edge connecting vertices a and b. The graph state on \( G_p \) reads \( |\Gamma_p\rangle = \prod_{a,b \in V} (U_{ab})^{\Gamma_{ab}} |\theta_0\rangle^V \), where \( |\theta_0\rangle^V = \left( \frac{1}{\sqrt{p}} \sum_{j \in \mathbb{Z}_p} |j\rangle \right)^\otimes n \) is the joint +1 eigenstate of all \( \{X_p^{s}| s \in \mathbb{Z}_p^{\otimes n}\} \) and \( U_{ab} = \sum_{i,j \in \mathbb{Z}_p} \omega_j^{ab} |i\rangle |a \otimes |j\rangle \langle j|_b \)
is the non-binary controlled phase gate between qubits $a$ and $b$. Thus $|\Gamma_p\rangle$ is the joint $+1$ eigenstate of a stabilizer group $\{g_p := X_p^s Z_p^{s'}\Gamma_p | s, s' \in \mathbb{Z}_2^n \}$. Given a $Z_r$-weighted graph $G_r = (V_1, \Gamma_r)$ for the $r$-level subsystem with $V_1 \subset V$ indicating the first $n_1$ vertices of $V$ as well, then

$$\{Z_p^r |\Gamma_p\rangle \otimes Z_r^c |\Gamma_r\rangle | c \in \mathbb{Z}_2^{n_1}, c' \in \mathbb{Z}_r^c \} \quad (3)$$

defines a basis of the mixed-alphabet system. Since graph states own a good feature that any bit shift error can be replaced by a phases shift error, i.e., $X^s Z^{s'} |\Gamma\rangle \propto Z^{t-s} |\Gamma\rangle$, it allows us to introduce a composite coding clique for the mixed-alphabet system in below.

**Definition** Given graphs $G_p = (V, \Gamma_p)$ and $G_r = (V_1, \Gamma_r)$ for an $n$-qubit and an $n_1$-qubit subsystems respectively, we define the $d$-uncoverable set as

$$D_d = \{t \in \mathbb{Z}_2^n \otimes \mathbb{Z}_r^c \mid 0 < |\hat{s} \cdot \hat{r} \cdot \hat{s}' \cdot \hat{r}'| < d \} \quad (4)$$

and the $d$-purity set as

$$S_d = \{s \otimes s' \in \mathbb{Z}_2^n \otimes \mathbb{Z}_r^c \mid |\hat{s} \cdot \hat{s}' \cdot \hat{r} \cdot \hat{r}'| < d \}. \quad (5)$$

A composite coding clique $C^K_d$ is a collection of $K$ different vectors $\{c_i \otimes c_i' | i = 1, \cdots, K\}$ in $\mathbb{Z}_2^n \otimes \mathbb{Z}_r^c$ that satisfy:

(i) $0 \in C^K_d$;

(ii) $\omega^c \cdot c_i \otimes c_i'$ is for all $s \otimes s' \in S_d$ and $c \otimes c' \in C^K_d$;

(iii) $(c_i - c_j) \otimes (c_i' - c_j') \in D_d$ for all $c_i \otimes c_i', c_j \otimes c_j'$ in $C^K_d$.

With this definition we have the following.

**Theorem 1** Given a composite coding clique $C^K_d$ on two graphs $G_p = (V, \Gamma_p)$ and $G_r = (V_1, \Gamma_r)$, the subspace spanned by the basis

$$\{Z_p^c |\Gamma_p\rangle \otimes Z_r^c |\Gamma_r\rangle | c \otimes c' \in C^K_d \} \quad (6)$$

defines a mixed-alphabet code $((n, K, d), q^{n_1 + n_2})$ with $q = rp$.

**Proof.** We need to prove that for any error that $0 < |E| < d$, the encoding space satisfies the Knill-Laflamme condition $\langle i | E | j \rangle = f(\hat{E}) \delta_{ij}$. Firstly if the error is accidentally proportional to a stabilizer of state $|\Gamma_p\rangle \otimes |\Gamma_r\rangle$, i.e., $E = f(\hat{E}) \cdot g_p^s \otimes g_r^s$, with $f(\hat{E})$ being phase factor, we have

$$\langle i | E | j \rangle = f(\hat{E}) \langle \Gamma_p | Z^{-c_i} g_p^s Z^c_i | \Gamma_p \rangle \langle \Gamma_r | Z^{-c_j} g_r^s Z^c_j | \Gamma_r \rangle = f(\hat{E}) \omega_p^c g_p^s g_r^s f(\hat{E}) \delta_{ij}, \quad (7)$$

where we have used the fact $s \otimes s' \in S_d$. Secondly if the error $E \propto X_p^s Z_p^{s'} \otimes X_r^t Z_r^{t'}$ is not a stabilizer of state $|\Gamma_p\rangle \otimes |\Gamma_r\rangle$, then

$$\langle \Gamma_p | Z_p^{-c_i} (X_p^s Z_p^{s'} |\Gamma_p\rangle |\Gamma_r \rangle | Z_r^{-c_j} (X_r^t Z_r^{t'} |\Gamma_r\rangle \rangle \propto \langle \Gamma_p | Z_p^{-c_i} Z_r^{-t'} s^{-1} |\Gamma_p\rangle |\Gamma_r \rangle | Z_r^{-c_j} Z_r^{t'} s^{-1} |\Gamma_r\rangle \rangle \quad (8)$$

which vanishes because condition (iii) of $C^K_d$ makes at least one of $c_i - c_j + s - t - s - t - r - r \neq 0$ and $c_j - c_i + t - t' + s' - s' - r' \neq 0$ holds. Thus we have proved that the encoding space as in Eq. (4) is an $((n, K, d), q^{n_1 + n_2})$ code.

In the case of coprime $p$ and $q$ ($p < q$), the method of composite coding clique does not work since the mixed-alphabet system cannot be divided into some subsystems. Here we introduce an ancillary system with all the $n$ particles being of $q$ dimensions. Denote the projector of a qudit onto a qudit by $P_i$ which satisfies $P_i | P_i = P_i$.

**Theorem 2** Given an $n$-qubit ancillary system, an $q^{n_1 + n_2}$ mixed-alphabet system and the corresponding projector $P$, for any error on the mixed-alphabet system that $|E| < d$, if a $K$-dimensional encoding space of the ancillary system with basis $\{|l\rangle | l = 1, \cdots, K\}$ can correct the corresponding error $P E P$, then the subspace spanned by

$$\{|l'\rangle := \frac{P}{\sqrt{d}} |l\rangle | l = 1, \cdots, K\} \quad (9)$$

defines a mixed-alphabet code $((n, K, d), q^{n_1 + n_2})$.

**Proof.** Firstly for any error $|E| < d$, we have

$$\langle i' | E | j' \rangle = \frac{1}{d} \frac{1}{\langle i | P | j \rangle} \langle i | P | E | P | j \rangle = f(\hat{E}) \delta_{ij}. \quad (10)$$

Secondly any two basis $|i'\rangle$ and $|j'\rangle$ satisfy that

$$\langle i' | j' \rangle = \frac{1}{\langle i | P | j \rangle} \langle i | P | P | j \rangle = 0, \quad (i \neq j) \quad (11)$$

which means the dimension of the encoding space of the mixed-alphabet system is still $K$. Thus this is a $((n, K, d), q^{n_1 + n_2})$ code.

Among all kinds of quantum bounds, the Singleton bound (qSB) and the Hamming bound (qHB) are two most important ones. Comparatively the qSB is stronger for short codes and weaker for long codes than the qHB. Since the qHB can easily be generalized to the case of mixed alphabets, here we focus on the generalization of the qSB. Consider a mixed-alphabet code with parameters $((n, K, d), q^{n_1 + n_2})$ where $p_i$ indicates the dimensions of the $i$th particle. Considering an arbitrary partition of
quantum Singleton bound. These two codes are both optimal since they saturate the $Z$ with the composite coding clique containing 2 generators 000110 100100 110000 001100 010010 011010 101001 100110 110000, which means \{Z^{14}Z^{3/4}6', Z^{25}Z^{3/4}6', Z^{3/4}6'Z^{1/2}6\}. Then the subspace spanned by basis \{Z^cL_6 \otimes Z^c' L_6' | c \otimes c' \in \mathbb{C}^{16}_3\} forms the optimal ((6, 16, 3)) code. The standard stabilizer code [[6, 2, 3]]_4 has been constructed in [22] and [24]. While our (6, 16, 3)_4 code gives another stabilizer construction whose stabilizer has 8 generators with addition modular 2 that takes the following form:

$$\begin{align*}
\begin{bmatrix}
XZZZZX & XZIIIIZ \\
ZXXZZX & ZXZIII \\
ZZXXZX & IHIHII \\
IHHIII & ZXXZZX \\
XZIIIZ & \IIZXZI \\
ZXLXI & ZIIIIZX \\
IZXXZI & YYZIIIZ \\
YXYYZI & IZXZII
\end{bmatrix}
\end{align*}$$

Example 2 Given two 2-weighted loop graph $L_6$ and $L'_6$ with the corresponding vertices paired up as shown in Fig.1(B), we find a ((6, 8, 3))_4 code whose composite coding clique contains 3 generators as \{Z^1 Z^2 3'4', Z^{345}Z^1, Z^{56}Z^2 4\}. Similarly via loop graphs $L_6$ and $L'_4$ paired up as Fig.1(C), we find a ((6, 4, 3))_4 code with the composite coding clique containing 2 generators as \{Z^{1235}Z^2 4', Z^{346}Z^3 1\}. Inequality (13) shows that these two codes are both optimal since they saturate the quantum Singleton bound.

Example 3 For $d = 2$, via two 2-weighted loop graphs paired up as Fig.2(A), the optimal ((3, 4, 2))_4 code can be constructed by 2 generators \{Z^1 Z^2, Z^2 Z^3\}. Similarly if three loop graphs are combined as shown in Fig.2(B), we can construct the optimal ((3, 8, 2))_8 code with the composite coding clique containing 3 generators as \{Z^1 Z^2, Z^2 Z^3, Z^3 Z^2\}. It is known that the direct product of the encoding space of two codes ((n, K, d)_p and ((n, K', d)_q constructs an ((n, KK', d)_pq code [21]. Thus all ((3, 2^n, 2))_2 codes can be constructed via ((3, 4, 2))_4 and ((3, 8, 2))_8. Since all ((3, p, 2))_p codes for odd $p$ are known, the optimal ((3, p, 2))_p codes with $p = 4k$ for any integer $k$ can be constructed.

Example 4 Method of stabilizer pasting [23, 26] can be used to construct longer mixed-alphabet QECCs. Past together the ((3, 4, 2))_4 code constructed above and a trivial ((2, 1, 2))_2 code can construct a ((5, 16, 2))_4 code.
code with the stabilizer taking the following form

\[
\begin{bmatrix}
Z_1^2 Z_2^2 X_3^4 X_4^3 Z_5^5 \otimes I^4 I^4 I^3 \\
I^1 I^2 I^3 I^5 \otimes X^1 Z_2^3 Z_3^3 \\
X_1^1 Z_2^2 Z_3^3 X_4^3 \otimes Z_1^4 X_2^2 Z_3^3 \\
Z_1^2 X_2^2 Z_2^3 I^4 I^5 \otimes Z_2^2 Z_2^3 Z_3^3
\end{bmatrix}
\]

(15)

Pasting the \((3,4,2)_{4}\) code with \(n\) copies of the trivial \((2,1,2)_{4}\) code can construct the \((2n + 3, 4^{2n+1}, 2)_{4}\) code. Similarly the \((2n + 3, 8^{2n+1}, 2)_{8}\) code can be constructed via pasting the \((3,8,2)_{8}\) code with \(n\) copies of the trivial \((2,1,2)_{8}\) code. Since all the \((2n + 3, p^{2n+1}, 2)_{p}\) codes for odd \(p\) are known, the \((2n + 3, p^{2n+1}, 2)_{p}\) codes with \(p = 4k\) for any integer \(k\) can be constructed.

Example 5 To construct the \((5, 5, 2)_{3}+(2)_{1}\) code, from \(3\) and \(2\) are coprime, a 5-qutrit ancillary system is needed.

For a qutrit, \(X = |1\rangle|0\rangle|2\rangle|1\rangle+|0\rangle|0\rangle + \omega|1\rangle|1\rangle + \omega^2|2\rangle|2\rangle\) with \(\omega = e^{i2\pi/3}\). Here we choose the projector as \(P_1 = |0\rangle\langle 0| + |1\rangle\langle 1|\), then

\[P = I \otimes I \otimes I \otimes P_3 = 2I + (1 + \omega^2)Z^5 + (1 + \omega)(Z^2)^2\]

(16)

up to some phase factor. For any 1-bit error \(E\) on the first four particles, considering that \([E, P] = 0\), we have \(P_1 \mathcal{E}P = 2\mathcal{E} + (1 + \omega^2)Z^5 + (1 + \omega)(Z^2)^2\). To the last particle, the bit flip and phase flip errors are \(X' = |1\rangle|0\rangle + |0\rangle|1\rangle\) and \(Z' = |0\rangle|0\rangle - |1\rangle|1\rangle\) respectively. Thus \(P_1 Z^5 P = 1 - \omega^5 Z^5 + (1 - \omega)(Z^2)^2\) and \(P_1 X' Z^5 P = X^5 + X^5 Z^5 + X^5 (Z^2)^2 + (X^5)^2 + \omega^2 (X^5)^2 Z^5 + \omega (X^5 Z^5)^2\) up to some phase factors. Then the problem is reduced to that of finding a code on the 5-qutrit ancillary system which can detect all 1-bit errors as well as all \(\{Z_5 \mathcal{E}\} \) and \(\{Z_5^2 \mathcal{E}\}\) 2-bit errors. Given a 3-weighted loop graph \(L_3\) with all edges weighted 1, we find a 9-dimension subspace for the ancillary system with the coding clique as

\[
\begin{bmatrix}
00000, 01020, 02110, 11010, 10222 \\
12200, 20210, 21102, 22120
\end{bmatrix}
\]

(17)

that can detect such errors. Thus a \((5, 5, 2)_{3}+(2)_{1}\) code is constructed. However, this is a suboptimal code according to the Singleton bound.

In this Letter, we extend the range of quantum codes to the mix-alphabet codes which have not been studied before. And the generalized quantum Singleton bound is also given. The composite coding clique is quite powerful for both mix-alphabet and standard QECCs. Many families of codes that saturate the Singleton bound can be easily constructed by this approach. However, on the other hand, the clique searching problem is intrinsically an NP-complete problem. Thus other methods need to be involved in constructing longer codes over mixed alphabets. We have used the method of stabilizer pasting to build longer codes of distance 2. And it is still an opening question to generate it to the case of \(d \geq 3\). To the projection-based construction, the \((5,9,2)_{3}+(2)_{1}\) code is suboptimal since the method of coding clique is not the best choice for this projector. And the problem of finding perfect method for different projectors needs to be further explored.

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