ANALYTIC ESTIMATE OF THE ORDER PARAMETER FOR MAGNETIC CHARGE CONDENSATION IN QCD

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The order parameter for monopole condensation is computed in terms of gauge invariant field strength correlators. Important properties emerge of the correlators in the confined phase, which could not be extracted by existing numerical determinations on the lattice.

Keywords: Confinement; Lattice QCD; Monopole condensation; Gauge invariant correlators.

1. INTRODUCTION

A possible mechanism for confinement of color is Dual Superconductivity of the Vacuum. The idea is that magnetic charges condense in the vacuum: the chromoelectric field acting between a $q - ar{q}$ pair is channeled by dual Meissner effect into flux tubes with energy proportional to the distance.

Flux tubes are indeed observed in lattice simulations, and the corresponding field configurations are consistent with expectations.

An order parameter $\langle \mu \rangle$ has been defined which is the vacuum expectation value of a magnetically charged operator $\mu$.

Numerical simulations show that $\langle \mu \rangle \neq 0$ in the confined phase, signaling Higgs breaking of magnetic $U(1)$ symmetry, $\langle \mu \rangle = 0$ in the deconfined phase. Deconfinement is an order-disorder phase transition.

For $SU(N)$ gauge group there exist $(N-1)$ magnetic charges and the corresponding operators $\mu^a$, $(a = 1, ..., N-1)$ can be written as

$$\mu^a(x,t) = e^{\frac{iq}{g}} \int d^3 y b_{\perp}(x - y) Tr(\Phi^a \bar{A}^0(y,t))$$ (1)

$q$ is an integer $\bar{b}_{\perp}$ is the field of a Dirac monopole with $\nabla \wedge \bar{b}_{\perp} = 0$ and $\nabla \cdot \bar{b}_{\perp} = \frac{q}{2\pi}$ Dirac string.

$$\Phi^a(x) \equiv U(x,y) \Phi^a U^\dagger(x,y)$$

with $U(x,y)$ an arbitrary gauge transformation which we shall take as a parallel transport to $x$ from a reference point $y$ along a path $C$, and

$$<\mu> = \langle \mu^a \rangle$$

for $a - a - > <(N-a)> = diag(\frac{N-a}{N}, ..., \frac{N-a}{N})$ (2)

$\langle \mu^a \rangle$ is gauge invariant. In the gauge in which

$$\Phi^a = \Phi^a_0$$ (Abelian Projection)

$$\mu^a(x,t) = e^{\frac{iq}{g}} \int d^3 y b_{\perp}(x - y) \bar{A}^0(y,t)$$ (3)

where $\bar{A}^0_{\perp}$ is the transverse chromoelectric field along the color direction $T^a$.

$$T^a = diag(0, ..., 0, 1, -1, 0, 0)$$ (4)

and, since $\bar{E}^a_{\perp}$ is the conjugate momentum to $\bar{A}^a_{\perp}$ one has

$$\mu^a(x,t) | \bar{A}^0_{\perp}(y,t) \rangle = | \bar{A}^0_{\perp}(y,t) + \frac{q}{2g} \bar{b}_{\perp}(x - y) \rangle$$ (5)

$\langle \mu^a \rangle$ is the ratio of two partition functions with the action proportional to $\beta \equiv \frac{2\pi}{g^2}$. At $\beta = 0 \mu^a = 1$. It proves convenient to use instead of $\langle \mu^a \rangle$ the susceptibility

$$\rho^a = \frac{\partial}{\partial \beta} \ln \langle \mu^a \rangle$$ (6)

in terms of which

$$\langle \mu^a \rangle = e^{\int_0^\beta d\beta \rho^a(\beta)}$$ (7)

$\rho^a$ is easily measured by lattice simulation, and this has been done for pure gauge theories compact $U(1)$, $SU(2)$, $SU(3)$, and
for \( N_f = 2 \) QCD \(^{11} \). In all cases the result is the following, in the thermodynamical limit 
\( V \equiv L^3 \to \infty \)

1) \( T < T_c \) (Confined )
\[ \rho^a \to \text{finite as } V \to \infty , \]
which by Eq(7) implies \( \langle \mu^a \rangle \neq 0 \)

2) \( T > T_c \) (Deconfined phase)
\[ \rho^a \approx -|c|L_s + c' \) or \( \langle \mu^a \rangle = 0 \)

3) \( T \approx T_c \) (Critical region)
\[ \rho^a \approx f(\tau L^3_s) \]
(finite size scaling)

Here \( \tau \equiv (1 - \frac{T}{T_c}) \), \( \nu \) is the critical index of the correlation length of the order parameter.

\( \rho^a \) is independent of the choice of the abelian projection\(^{12} 13 14 \).

From the definition of \( \langle \mu^a \rangle \) it follows

\[
\langle \mu^a \rangle = \sum_i^{\infty} \frac{\Delta_i}{2N} \frac{1}{n!} \int d^3y_1 \cdots d^3y_n \langle \vec{b}_{\perp}^a (\vec{x} - \vec{y}_i) \cdots \rangle
\]

\[ b_{\perp}^a (\vec{x} - \vec{y}_i) = \langle (\Phi^a \cdot \vec{E})_{i_1 \vec{y}_1} \cdots (\Phi^a \cdot \vec{E})_{i_n \vec{y}_n} \rangle \]

We have used the simplified notation \( \langle \Phi^a \cdot \vec{E} \rangle_{i_1 \vec{x}} \equiv Tr[\Phi^a(t, \vec{x}) \vec{E}(t, \vec{x})] \).

The \( \langle v \rangle \)'s in eq(8) are gauge invariant field strength correlators. We shall identify these correlators with those of the stochastic vacuum approach to QCD \(^{20} 21 22 \).

### 2. Cluster Expansion of \( \langle \mu^a \rangle \)

The idea of the stochastic vacuum approach is to approximate systematically the multiple gauge invariant field strength correlators by products of lowest order correlators in a systematic cluster expansion. The expansion is then truncated and only clusters up to order two are retained. Since the one point cluster \( \langle (\Phi^a \cdot \vec{E}) \rangle = 0 \), only two point clusters will survive the truncation, and hence only even terms in the expansion Eq(8) , which will be approximated as products of two point correlators \( \Phi^a_{i_1, i_2} (\vec{y}_1 - \vec{y}_2) = \langle (\Phi^a \cdot \vec{E})_{i_1 \vec{y}_1} (\Phi^a \cdot \vec{E})_{i_2 \vec{y}_2} \rangle \). The combinatorial factor of the term 2\( n \) is \((2n - 1)!!\) and Eq(8) becomes

\[
\langle \mu^a \rangle = e^{-\frac{q^2}{8\lambda^2}} \int d^3\gamma_1 \int d^3\gamma_2 \Phi^a_{i_1 \vec{y}_1} (\vec{y}_1) \Phi^a_{i_2 \vec{y}_2} (\vec{y}_2)
\]

or, since \( \beta = \frac{2N}{g} \) Eq(6) becomes

\[
\rho^a = -\frac{q^2}{16N} \partial \frac{\beta}{\rho} \int d^3\gamma_1 \int d^3\gamma_2 \Phi^a_{i_1 \vec{y}_1} (\vec{y}_1) \Phi^a_{i_2 \vec{y}_2} (\vec{y}_2) \]

We shall identify \( \Phi^a_{i_1 \vec{y}_1} \) with the two point correlators defined with a straight line parallel transport which are measured on the lattice \(^{17} 18 19 \), and are used as an input in the stochastic model of QCD. In fact for those correlators \( \langle E^a \cdot E^b \rangle = \delta^{ab} \) so that \( \rho^a \) is independent of \( a \), and this agrees with lattice determinations of \( \rho^a \) \(^9 \).

The validity of the cluster expansion is usually justified at large distances, and we are looking for infrared properties in the study of confinement.

A direct check can of be obtained by studying the dependence on the monopole charge \( q \). The truncated \( \rho \) is proportional to \( q^2 \) : higher correlators would introduce terms proportional to higher powers of \( q \). Old data \(^{8} 15 \) agree with proportionality to \( q^2 \) but a more systematic study of this issue is planned.

### 3. The Field Correlators

There exists a general parametrization of field strength correlators dictated by invariance arguments\(^{20} 21 \)

\[
\Phi^a_{i_1 \mu_1, i_2 \mu_2} (z_1 - z_2) = \frac{1}{N} (Tr F^a_{\mu_1 \mu_2} (z_1) V(z_1, z_2) F^a_{\mu_2 \mu_1} (z_2) V^\dagger(z_1, z_2))
\]

\[
\Phi^a_{i_1 \mu_1, i_2 \mu_2} (z_1 - z_2) = \bar{\delta}^{ab} \langle D(z_1 - z_2) \delta_{\mu_1 \mu_2} \delta_{\nu_1 \nu_2} - \delta_{\mu_1 \nu_2} \delta_{\mu_2 \nu_1} \rangle + \frac{1}{2} \partial \frac{\partial}{\partial z_{\mu_1}} \langle D(z_1 - z_2) \delta_{\mu_2 \nu_1} \rangle + \frac{1}{2} \partial \frac{\partial}{\partial z_{\nu_2}} \langle D(z_1 - z_2) \delta_{\mu_1 \mu_2} \rangle
\]

(12)
At $T \neq 0$ the electric field correlators are not equal to those of the magnetic field, so one has four form factors, $D_E, D_{1E}, D_H, D_{1H}$. For the electric field correlators $\mu_1 = \mu_2 = 0, \nu_1 = i_1, \nu_2 = i_2$ and

$$\Phi^{ab}_{i_1i_2} = \delta^{ab}\delta_{i_1i_2}(D_E + \frac{1}{2}D_{1E}) + \frac{\partial}{\partial z_{i_1}}$$

(13)

In the convolution with $\delta_{i_1i_2}$ the derivative terms give 0. For the same reason the replacement can be performed

$$\delta_{i_1i_2} \rightarrow \delta_{i_1i_2} - \frac{k_{i_1}k_{i_2}}{k^2}$$

Going to the Fourier transform Eq(10) becomes

$$\rho^a = -\frac{q^2}{16} \frac{\partial}{\partial \beta} \left[ \beta \int \frac{d^3k}{(2\pi)^3} b_1(k) b_2(-k) \right]$$

$$\bar{D}_E(k^2) \frac{1}{k^2} [k^2 \delta_{k_{i_1}k_{i_2}} - k_{i_1}k_{i_2}]$$

(14)

where $\bar{D}_E(k^2)$ is the Fourier transform of $(D_E + \frac{1}{2}D_{1E})$. The identity

$$(k^2 \delta_{i_1i_2} - k_{i_1}k_{i_2}) b_1(k) b_2(-k) = |\tilde{H}(k)|^2$$

(15)

, together with the explicit form of $\tilde{H}(k)$

$$\tilde{H}(k) = \vec{k} \wedge \vec{\beta}_{i_1}(\vec{k})$$

(16)

with $\vec{n}$ the direction of the Dirac string (we shall call it $z$), gives

$$|\tilde{H}(k)|^2 = -\frac{1}{k^2} + \frac{1}{k_z^2}$$

(17)

and for $\rho^a$

$$\rho^a = \frac{q^2}{16} \frac{\partial}{\partial \beta} \left[ \beta \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{k^2} - \frac{1}{k_z^2} \right) f(k^2) \right]$$

(18)

Here $f(k^2) = \frac{1}{2}D_E(k^2)$. Again we notice that identifying our correlators with those of the stochastic model implies that $\rho^a$ is independent both on $a$ and on the abelian projection.

At large $\beta$ (deconfined phase) $f(k^2)$ can be approximated by first order perturbation theory

$$f(k^2) = \frac{1}{2Nk}$$

(19)

the only dependence on $\beta$ is the explicit factor in Eq(18) so that

$$\rho^a = \frac{q^2}{16N} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} - \frac{1}{k_z^2}$$

(20)

The integral is easily evaluated with UV cut-off $\frac{1}{a}$ (a the lattice spacing) and IR cut-off $\frac{1}{L_{na}}$ ($L_{na}$ the spacial lattice size). The result is

$$\rho^a = \frac{q^2}{16N} \frac{1}{(2\pi)^2} \left[-\sqrt{2}L_{a} + 2ln(L_{a}) + const\right]$$

(21)

which, as explained in Sect. 1 means $\langle \mu^a \rangle = 0$ in the thermodynamical limit $L_{a} \rightarrow \infty$.

The two point correlators have been measured on the lattice both at $T = 0$ and at finite $T$. They are well represented in the range of distances $0.1fm \leq x \leq 1fm$ by a form

$$D_E = Ae^{-\frac{\pi x}{a\lambda_{a}}} + \frac{b}{x^4}e^{-\frac{\pi x}{a\lambda_{b}}}$$

(22)

with $\lambda_{a} \approx 2\lambda_{b}$ and $\lambda_{b} \approx 3fm$. $A$ and $b$ are independent on $\beta$ from $T = 0$ up to $T \approx .95T_c$ Approaching further $T_c$ $A$ decreases rapidly to zero.

In fact the above parametrization cannot be valid at shorter distances, where the operator product expansion and the non-existence of condensates of dimension less than 4 require that

$$D_E \approx \frac{b}{2}\left[\frac{1}{(x+ie)^4} + \frac{1}{(x-ie)^4}\right] + c + dx^2$$

(23)

The prescription on the singularity is dictated by the match to perturbation theory. At larger distances a stronger infrared cut-off at some distance $\Lambda$ is expected since colored particles cannot propagate at infinite distance. This feature needs a further numerical investigation on the lattice. Up to $T \approx .95T_c$ the only dependence on $\beta$ in Eq(14) is again the explicit factor so that the result is the same as Eq(21) with the lattice size $L_{a}$ replaced by the infrared cut-off $\Lambda$

$$\rho^a = \frac{q^2}{16} \frac{1}{(2\pi)^2} \left[-\sqrt{2}\Lambda + 2ln(\Lambda) + const\right]$$

(24)
This expression gives a finite value of $\rho^a$ for $T < T_c$ and hence $\langle \mu^a \rangle \neq 0$ which means dual superconductivity for any finite value of the UV cut-off $a$. However in the continuum limit $a \to 0$ $\rho^a$ diverges so that $\langle \mu^a \rangle$ needs a renormalization. This is similar to what happens for the Polyakov line in the quenched theory. Existing Lattice data support this statement [See Fig(2) of ref. 8], but a more systematic investigation is planned. When the critical temperature is approached both the IR cut-off $\Lambda$ and the coefficient $A$ in Eq(22) strongly depend on $\beta : A$ diverges and $A$ tends to zero. A more detailed calculation, which will be reported elsewhere gives

$$
\rho^a = \frac{q^2}{16N} \frac{\partial}{\partial \beta} \left( \frac{1}{2} \ln \left( \frac{\sqrt{2}}{a} \right) + \frac{\Lambda}{\pi a} \right)
+ \text{const} + 2N\lambda_b^a A_E \left(1 + \frac{\Lambda}{\pi \lambda_b} \right)
$$

(25)

Numerical determinations of the temperature dependence of $A_E$ around $T_c$ exist. Not much is known about the behavior of $\Lambda$. Further study is needed to understand how the scaling law of $\rho^a$ described in Sect 1 works in Eq(25), and this is already on the way. In conclusion an interesting interrelation has been found between the large distance behavior of correlators and confinement, which deserves further study.

References

1. G. ’tHooft: High Energy Physics EPS International Conference, Palermo 1975, A. Zichichi ed.
2. S. Mandelstam: Phys. Rept23C (1976) 245
3. G. ’tHooft, Nucl. Phys. B190 (1981) 455
4. H.B. Nielsen, P. Olesen, Nucl. Phys. B57 (1973) 367
5. A. Di Giacomo, Acta Phys. Polon. B25 (1994) 215
6. L. Del Debbio, A. Di Giacomo, G. Paffuti, P. Pieri: Phys. Lett. B355 (1995) 513
7. A. Di Giacomo, G. Paffuti: Phys. Rev. D56 (1997) 6816
8. A. Di Giacomo, B. Lucini, L. Montesi and G. Paffuti: Phys. Rev. D 61 (2000) 034503
9. A. Di Giacomo, B. Lucini, L. Montesi and G. Paffuti: Phys. Rev. D 61 (2000) 034504
10. L. Del Debbio, A. Di Giacomo, B. Lucini, G. Paffuti: Abelian projection in SU(N) gauge theories, e-Print Archive, hep-lat/0203023
11. M. D’Elia, A. Di Giacomo, B. Lucini, G. Paffuti, C. Picc: Phys. Rev. D71 (2005) 114502
12. A. Di Giacomo: “Independence on the Abelian projection of monopole condensation in QCD,” arXiv:hep-lat/0206018.
13. J. M. Carmona, M. D’Elia, A. Di Giacomo, B. Lucini and G. Paffuti: Phys. Rev. D 64 (2001) 114507
14. See also A. Di Giacomo, G. Paffuti: Nucl. Phys. Proc. Suppl 129 (2004) 647.
15. M. D’Elia, A. Di Giacomo, B. Lucini: Phys. Rev. D 69 (2004) 077504
16. J. M. Carmona, M. D’Elia, L. Del Debbio, A. Di Giacomo, B. Lucini and G. Paffuti: Phys. Rev. D 66 (2002) 011503
17. A. Di Giacomo, H. Panagopoulos: Phys. Lett B3285 (1992) 133
18. M. D’Elia, A. Di Giacomo, E. Meggiolaro: Phys. Lett B408 (1997) 315
19. M. D’Elia, A. Di Giacomo, E. Meggiolaro: Phys. Rev. D67 (2003) 114504
20. H. G. Dosch: Phys. Lett. B190 (1987) 177
21. Yu. A. Simonov: Nucl. Phys. B307 (1988) 512
22. For a review see A. Di Giacomo, H. G. Dosch, V. I. Shevchenko, Yu. A. Simonov: Phys. Rep C372 (2002) 320
23. F. Zantow, O. Kaczmarek, F. Karsch, P. Petreczky: Phys. Lett. B543 (2002) 41
24. A. Di Giacomo, E. Meggiolaro, H. Panagopoulos: Nucl. Phys. B483 (1997) 315
