A two-vertex theorem for normal tilings

GÁBOR DOMOKOS, ÁKOS G. HОРВАTH, and KRISZTINA REGŐS

Abstract. We regard a smooth, $d = 2$-dimensional manifold $M$ and its normal tiling $\mathcal{M}$, the cells of which may have non-smooth or smooth vertices (at the latter, two edges meet at 180 degrees.) We denote the average number (per cell) of non-smooth vertices by $\bar{v}^*$ and we prove that if $M$ is periodic then $\bar{v}^* \geq 2$. We show the same result for the monohedral case by an entirely different argument. Our theory also makes a closely related prediction for non-periodic tilings. In 3 dimensions we show a monohedral construction with $\bar{v}^* = 0$.

Mathematics Subject Classification. 52C20.

Keywords. Normal tiling, Sharp vertex.

1. Introduction

Both in nature and among man-made structures we often encounter cellular patterns where a domain is decomposed into mutually disjoint, non-overlapping cells. Such patterns, also called tessellations or tilings, arise in geology (as fracture patterns [1,3–5,14]), in engineering and architecture (as surface patterns or as urban structures [2,10]), in geography [6] or in biology (as cell tissues [8,12]).

In general, we will assume that the boundaries of cells are piecewise $C^1$-smooth, however, in sect. 3.1 we will require piecewise $C^2$-smoothness. We also assume that the tiling is normal [7], i.e. every cell is topologically equivalent to a disk, the intersection of any two cells is a connected set or the empty set, all cells are uniformly bounded, and the degree of the nodes is finite. In addition, for infinite tilings we will assume that they are also balanced [7].
i.e. the average number of vertices and edges exist independently from the
method of the counting, the detailed description of the latter process is given in Remark 2. Normal tilings have been described in various settings [9,11,13], however, the emphasis was on their combinatorial properties. Here we aim to describe a curios metric aspect of these patterns. Cells may have non-smooth or smooth vertices (at the latter, two edges meet at 180 degrees). We denote the average number (per cell) of the former by $\bar{v}^*$ and we prove that for periodic patterns $\bar{v}^* \geq 2$ (see Theorem 1 and Remark 1). In Theorem 1 we also provide a general formula for the lower bound on $\bar{v}^*$, including non-periodic patterns on arbitrary, smooth, 2D manifolds. In addition, we prove the $\bar{v}^* \geq 2$ statement for monohedral tilings (consisting of congruent cells) based on a different argument. Finally, we offer a 3-dimensional monohedral construction with $v^* = 0$.

1.1. A simple example and the main result

We start by giving an informal example, illustrating the basic concepts, based on which we can formulate our main result and we will provide more formal definitions later. We consider normal tilings of smooth, 2D manifolds. Fig. 1 illustrates three such patterns (illustrated in columns (A),(B),(C), respectively), all of which are periodic. In the bottom row we indicate some real-world examples which could be approximated by these patterns. We will characterize such patterns by the combinatorial nodal degree $\bar{n}$ and combinatorial cell degree $\bar{v}$ which count the average number of cells meeting at one node and the average number of nodes along the boundary of one cell. We could also interpret these degrees as counting the number of vertices meeting at a node and counting the number of vertices along the perimeter of a cell. In the case of periodic patterns, it is easy to compute these averages; for the particular patterns illustrated in Fig. 1 all nodes and cells are identical and we can see that the combinatorial degrees, illustrated and counted in the upper row, are identical for all three patterns.

Despite having identical combinatorial degrees, the three patterns look radically different. One way to capture these differences is to compute corner degrees: the nodal corner degree $\bar{n}^*$ and the cell corner degree $\bar{v}^*$ count only vertices where the angle of the half-tangents of the meeting arcs is not 180 degrees (and so, obviously, we have $\bar{n}^* \leq \bar{n}$, $\bar{v}^* \leq \bar{v}$). For the hexagonal honeycomb in column (A) of Fig. 1, the combinatorial and corner degrees coincide since none of the examined angles is 180 degrees. In the case of the brick wall pattern the corner degrees are smaller because we do have vertices with edge-angles of 180 degrees. The roof tile pattern has even smaller corner degrees.

While it is clear that corner degrees may be smaller than combinatorial degrees, it is not obvious by how much these quantities may differ. In particular,
it is not clear what is the minimum for the cell corner degree which counts the average number of actual, non-smooth corners of a cell. Our goal is to prove

**Theorem 1.** Let $\mathcal{M}$ be a smooth, 2D manifold with Euler characteristic $\chi$ and let $M$ be a normal tiling of $\mathcal{M}$ with $E$ edges, combinatorial nodal degree $\bar{n}$, and cell corner degree $\bar{v}^*$. Then we have

$$\bar{v}^* \geq 2 - \frac{2\chi \bar{n}}{E(\bar{n} - 2) + \chi \bar{n}}.$$ 

As we can see, both in the $E \to \infty$ limit and for $\chi = 0$ Theorem 1 yields

$$\bar{v}^* \geq 2.$$  \hspace{1cm} (1)

**Remark 1.** As periodic patterns can be treated as infinite, equation (1) holds for such patterns.

**1.2. Structure of paper**

In Sect. 2 we first define the basic concepts and prove Theorem 1. In Sect. 3 we offer an alternative proof for the 2D monohedral case as well as a construction of a 3D monohedral tiling with no vertices.
2. Normal tiling of 2D manifolds

2.1. Combinatorial properties

Definition 1. Let $M$ be a normal [7] tiling of a smooth, $d = 2$-dimensional manifold $\mathcal{M}$ and let $M$ have $V$ vertices (0-dimensional cells or nodes), $E$ edges (1-dimensional cells) and $F$ faces (2-dimensional cells or simply cells). We call the number of edges meeting at the $i$th node ($i = 1, 2, \ldots V$) the combinatorial nodal degree and denote it by $n_i$. We call the number of edges on the boundary of the $j$th cell ($j = 1, 2, \ldots F$) the combinatorial cell degree and denote it by $v_j$. For finite $V, E, F$ we denote the average values of the combinatorial nodal and cell degrees by $\bar{n}, \bar{v}$, respectively. We also allow that $V, E, F$ to be infinite, however, in that case we also assume that $M$ is balanced [7], guaranteeing that a suitably chosen limit process on a sequence of finite tilings provides the corresponding averages $\bar{n}, \bar{v}$ for the nodal and cell degrees (for details see Remark 2). We refer to the $[\bar{n}, \bar{v}]$ plane as the (combinatorial) symbolic plane.

Remark 2. On compact manifolds and for finite tilings, counting averages is trivial. On infinite domains averages are again trivial if $M$ is periodic. For general normal tilings we compute the averages by counting the nodal and cell degrees in a finite ball $B(X, \rho)$ with center $X$ radius $\rho$ to obtain $\bar{n}(X, \rho), \bar{v}(X, \rho)$ associated with the finite normal tiling $M(X, \rho)$. Then we let the radius $\rho$ approach infinity. We say that the averages $\bar{n}, \bar{v}$ of the degrees exist if the limits $\lim_{\rho \to \infty} \bar{n}(X, \rho), \lim_{\rho \to \infty} \bar{v}(X, \rho)$ exist and are independent of $X$. Henceforth we will assume that the manifold $\mathcal{M}$ is either compact and $M$ is finite, or, $M$ is infinite and the averages converge in the sense described above and boundary effects decay.

Remark 3. Note that on the Euclidean plane the concepts of a tiling being “normal” and “balanced” are well-investigated (see Sect. 3.3 in [7]). Clearly, the definition of “normality” formally can be applied on arbitrary manifolds, but its precise connection with the property “balanced” is far from obvious, except on the Euclidean plane (see Theorem 3.3.2 of [7]. In this paper we rely on the strong conditions on averaging, given in Remark 2).

Remark 4. From the fact that the tiling $M$ is normal, it follows that each 2-cell is contractible, and the boundary of a 2-cell is a simple closed curve. We will later discuss $d$-dimensional manifolds and their tilings where we also use that $d$-dimensional cells are contractible and all $k < d$ dimensional faces are also contractible.

Remark 5. Since all edges are assumed to be $C^1$-smooth, they have half-tangents at both endpoints. We also note that the angle of the two half-tangents can be calculated from the globally defined differential structure of the smooth manifold.
Definition 2. We call
\[ \bar{h} = \frac{\bar{v} \bar{n}}{\bar{v} + \bar{n}} \]
the harmonic degree of \( M \).

For finite \( M \), the Euler formula can be written as
\[ F - E + V = \chi. \quad (2) \]
According to Assumption (1) in Definition 1, and the fact that every edge has two vertices and belongs to two cells, the combinatorial nodal and cell degrees can be expressed as
\[ \bar{n} = \frac{2E}{V}, \quad \bar{v} = \frac{2E}{F}. \quad (3) \]
We can rewrite equation (2) as
\[ \frac{F}{2E} + \frac{V}{2E} = \frac{E + \chi}{2E}, \quad (4) \]
which, using equation (3) and Definition 2 translates into
\[ \bar{h} = \frac{2E}{E + \chi}. \quad (5) \]

In the infinite case we consider an averaging process analogous to the one given in Remark 2 where the number \( E(\rho) \) of edges (and the harmonic degree \( \bar{h}(\rho) \)) are defined as a function of the radius \( \rho \) of a finite ball. As \( \rho \to \infty \), we have
\[ \bar{h} := \lim_{\rho \to \infty} \bar{h}(\rho) = 2, \quad (6) \]
regardless of the Euler characteristic of \( M = \lim_{\rho \to \infty} M(\rho) \). Also, if \( \chi = 0 \) then we have \( \bar{h} = 2 \), one example is the torus \( T^2 \).

Remark 6. Observe, that the finiteness of the normal tiling \( M \) is equivalent to the compactness of the carrying manifold \( \mathcal{M} \).

2.2. Geometric properties: sharp corners

Definition 3. Let \( M \) be a normal and balanced tiling of a smooth, \( d = 2 \)-dimensional manifold \( \mathcal{M} \) and let \( M \) have \( V \) vertices (nodes), \( E \) edges and \( F \) faces (cells) and let \( n_i \) denote the combinatorial degree of the \( i \)th node, \( i = 1, 2, \ldots V \). We number the edges \( e_{i,j} \) \( (i = 1, 2 \ldots V, j = 1, 2, \ldots n_i) \) meeting at the \( i \)th node clockwise consecutively using their second subscript, so the pair \( e_{i,j}, e_{i,j+1} \) is not separated by any other edge. We also regard \( e_{i,n_i}, e_{i,1} \) as one pair, so we associate \( n_i \) pairs with the \( i \)th node. We call a pair degenerate if the two edges meet at 180 degrees, otherwise we call the pair generic. (The angle between two the edges of an edge-pair is identical to the alternate interior
angle of their respective half-tangents at the common vertex, thus we can distinguish between the angle zero (when the two half-tangents have opposite directions on their line) and the angle $\pi$ (when the two half-tangents have the same direction). We associate with the $i$th node the quantity $r_i \in \{0, 1, 2\}$, counting the number of degenerate pairs. We define the corner degree of the $i$th node as

$$n_i^* = n_i - r_i.$$ 

**Definition 4.** Let $M$ be a normal and balanced tiling of a smooth, $d = 2$-dimensional manifold $\mathcal{M}$ and let $M$ have $V$ vertices (nodes), $E$ edges and $F$ faces (cells) and let $v_i$ denote the combinatorial degree of the $i$th cell, ($i = 1, 2, \ldots F$). The edges $e_{i,j}$ ($i = 1, 2, \ldots F, j = 1, 2, \ldots n_i$) surrounding the $i$th cell are numbered clockwise consecutively, so the pair $e_{i,j}, e_{i,j+1}$ overlaps at a vertex. We also regard $e_{i,v_i}, e_{i,1}$ as one pair, so we associate $v_i$ pairs with the $i$th cell. We call a pair degenerate if the two edges meet at 180 degrees, otherwise we call the pair generic. We associate with the $i$th cell the quantity $q_i \in \{0, 1, 2, \ldots v_i\}$, counting the number of degenerate pairs. We define the corner degree of the $i$th cell as

$$v_i^* = v_i - q_i.$$ 

**Definition 5.** We call the $[\bar{n}^*, \bar{v}^*]$ plane the (geometric) symbolic plane.
Definition 6. Let

\[ \rho(M) = 1 - \frac{\sum_{i=1}^{V} r_i}{2V} \]

and we call \( M \) \( \rho \)-regular. In particular, we call \( M \) regular, semi-regular or irregular if \( \rho(M) = 1, 1/2, 0 \), respectively.

2.3. Trivial lower bounds on the cell and nodal degrees for large cell decompositions

Here we consider the \( E \to \infty \) limit of a balanced normal tiling. If \( M \) is regular (\( \rho = 1 \)) then we have

\[ \bar{v} = \bar{v}^*, \quad \bar{n} = \bar{n}^*. \]  \hfill (7)

Equation (6) implies

\[ \bar{n}, \bar{v} \geq 2, \]  \hfill (8)

which, via (7) is equivalent to

\[ \bar{n}^*, \bar{v}^* \geq 2. \]  \hfill (9)

If, in addition to being regular, \( M \) is also combinatorially equivalent to a convex mosaic then we have

\[ \bar{n}^*, \bar{v}^* \geq 3. \]  \hfill (10)

All the listed bounds are trivial. Also, because of the inverse functional relationship (6) between \( \bar{n} \) and \( \bar{v} \), the lower bound of one of the combinatorial averages is tied to the upper bound of the other combinatorial average.

If \( M \) is not regular (\( \rho < 1 \)) then there is no general functional relationship between \( \bar{n} \) and \( \bar{v} \). Based on Definitions 3 and 4 for non-regular tilings we have

\[ \bar{v} > \bar{v}^*, \quad \bar{n} > \bar{n}^* \]  \hfill (11)

and we can also see that the corner degrees attain their respective minima if \( M \) is irregular, i.e. we have \( \rho = 0 \). Our next goal is to find the lower bound for \( \bar{v}^* \) for irregular normal tilings.

2.4. Proof of theorem 1

Proof. Let \( M \) be a regular normal tiling with \( F \) faces (cells), \( V \) vertices (nodes) and \( E \) edges and respective cell and nodal combinatorial degrees given in equation (3). Let us deregularize \( M \) by transforming generic edge-pairs into degenerate ones. We perform this by local, continuous transformation in the vicinity of the nodes. Assume that by such local transformations we create \( 2V(1 - \rho) \) degenerate pairs to obtain the \( \rho \)-regular tiling \( M(\rho) \). While the
combinatorial degrees remain constant, the corner degrees can be computed as a function of $\rho$:

$$\bar{v}^\star(\rho) = \frac{2E - 2(1 - \rho)V}{F} = \bar{v}
\left(\frac{\bar{n} - 2(1 - \rho)}{\bar{n}}\right),$$

$$\bar{n}^\star(\rho) = \frac{2E - 2(1 - \rho)V}{V} = \bar{n} - 2(1 - \rho).$$

(12)

We can immediately see that

$$\frac{\bar{v}}{\bar{n}} = \frac{\bar{v}^\star(\rho)}{\bar{n}^\star(\rho)},$$

(13)

so the evolution path of topological deregularization on the $[\bar{n}, \bar{v}]$ symbolic plane is a straight line passing through the origin. By subsituting $\rho = 0$ into (12) we obtain

$$\bar{n}^\star \geq \frac{2E - 2V}{V} = \bar{n} - 2,$$

(14)

$$\bar{v}^\star \geq \frac{2E - 2V}{F} = \bar{v}
\left(\frac{\bar{n} - 2}{\bar{n}}\right).$$

(15)

Equation (5) is equivalent to

$$\bar{v} = \frac{2E\bar{n}}{\bar{n}(E + \chi) - 2E}. (16)$$

Substituting (16) into (15) yields the formula of Theorem 1.

\[ \square \]

3. Related issues and discussion

3.1. The monohedral case in $d = 2$ dimensions

Here we discuss the special case where the tiling of the Euclidean plane consists of congruent cells $C$. In this case each cell has the same corner degree $v^\star$, hence $\bar{v}^\star = v^\star$. Our next argument requires that at every smooth point of the boundary of $C$ the signed curvature $\kappa$ exists. To guarantee this property we assume here that the smooth part of the boundary is the finite union of two-times continuously differentiable arcs and for brevity, we call such tilings $C^2$-tilings. Theorem 1 implies

**Corollary 1.** In the case of monohedral, normal $C^2$-tilings of the Euclidean plane, for the corner degree of the cell we have $\bar{v}^\star \geq 2$.

Below we give a proof of Corollary 1 which is independent of the proof of Theorem 1.
Let us parametrize the boundary $C$ of the cell $C$ by the arclength $s$ on the unit interval $I \equiv [0, 1]$, so we have $s \in [0, 1]$. We assumed that $C$ is piecewise $C^2$-smooth. We will consider $I$ as the union of the (open) set of smooth points $I_S$ and the (discrete) set of combinatorial nodes $N_i$, $i = 1, 2, \ldots v$:

$$I = I_S \cup \{N_i\}. \quad (17)$$

We observe that the discrete angle increments $\Delta \alpha_i$ differ from zero only at corners. To simplify our computation, we re-assign indices $i \in [1, 2, \ldots v^*]$ to these non-zero relative angles and so, instead of (18) we may write:

$$K = \int_{I_S} \kappa(s) ds + \sum_{i=1}^{v^*} \Delta \alpha_i = 2\pi. \quad (19)$$
Since the tiling is monohedral, the neighbor cells are congruent to the investigated cell $C$. Now we observe that for every smooth part $\gamma$ of the boundary there is at least one orientation preserving congruence $f_\gamma : \mathcal{M} \to \mathcal{M}$ which sends the cell $C_\gamma$, overlapping with $C$ along the arc $\gamma$, to $C$. Let $\gamma'$ be the image of $\gamma$ by $f_\gamma$. If $\gamma' = \gamma$ then the value of the integral (19) on $\gamma$ is zero. If this is not the case then the value of the integral on the union $\gamma \cup \gamma'$ is zero. Since $f_\gamma(C)$ is the neighbour of $C$ along the arc $\gamma'$, if there exists another smooth part $\gamma^*$ for which $f_{\gamma^*}(\gamma^*) = \gamma'$ then we have $(f_{\gamma}^{-1} \circ f_{\gamma^*})(C) = C$ and so $f_{\gamma}^{-1} \circ f_{\gamma^*}$ is an orientation preserving symmetry transformation of $C$. Thus $\gamma'' := f_{\gamma}^{-1} \circ f_{\gamma^*}(\gamma')$ is congruent to $\gamma'$. If $\gamma'' \neq \gamma'$ then on the union of $\gamma$, $\gamma^*$, $\gamma'$ and $\gamma''$ the value of the integral is zero. Let us consider the arc $\gamma'$ with endpoints $P$ and $Q$. If $\gamma'' = \gamma'$ then we have two cases: either the endpoints $P$, $Q$ of the arc $\gamma'$ remain fixed or they are interchanged. In both cases, the orientation preserving congruence $f_{\gamma}^{-1} \circ f_{\gamma^*}$ fixes the segment $PQ$ and either $f_{\gamma}^{-1} \circ f_{\gamma^*}$ is the identity (which is a contradiction) or it is a reflection at the center of $PQ$. This latter possibility means that the value of the integral is zero in $\gamma'$ and also in $\gamma$. It is easy to see by an inductive argument that on the whole smooth part the integral is zero and so

$$K = \int_{I_\gamma} \kappa(s) ds + \sum_{i=1}^{v^*} \Delta \alpha_i = \sum_{i=1}^{v^*} \Delta \alpha_i = 2\pi. \quad (20)$$

As $\Delta \alpha_i \leq \pi$, this implies that for the number $v^*$ of corners in $C$ we have $v^* \geq 2$ and thus it proves the Corollary.

**Remark 7.** It is easy to see that the proof of Corollary 1 also implies that for normal, periodic tilings of the Euclidean plane we have $\bar{v}^* \geq 2$.

### 3.2. Construction of a monohedral tiling with no vertices in 3 dimensions

In 3 dimensions the deregularization algorithm, presented in the proof of Theorem 1, would work in an analogous manner as in 2 dimensions, since in any dimension $d$, at any point, the number of hypercells (of dimension $d-1$) tangent to each other is exactly 2. This algorithm reduces the nodal degree by 2, and thus we expect (15) to remain valid. However, unlike in 2 dimensions, in the 3D case this equation will not provide the actual lower bound on $\bar{v}^*$: we have to consider that in 3 dimensions one can use a different deregularization algorithm as well, where not vertices but edges are being smoothed. The general formulae for the second algorithm appear to be less trivial as the smoothing of edges is not a local operation. Nevertheless, it is not difficult to establish a sharp lower bound in 3 dimensions:

**Lemma 1.** In dimension $d = 3 \bar{v}^* \geq 0$. 

Figure 4. Construction of a $v^* = 0$ monohedral tiling in $d = 3$ dimensions. The rectangular prism has its vertical edges at $(x, y) = \pm 1$ and the saddle-like surface at the top and the bottom are given as $z = \sqrt{1 - x^2} - \sqrt{1 - y^2} \pm 1$

Proof. We prove Lemma 1 by showing a monohedral tiling in 3D with $v^* = 0$, illustrated in Fig. 4. □

Funding Open access funding provided by Budapest University of Technology and Economics.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[1] Adler, P.M., Thovert, J.-F.: Fractures and fracture networks. Springer, Dordrecht (1999)
[2] Chang, W.: Application of tessellation in architectural geometry design. E3S Web of Conferences, ICEMEE 2018 38(03015), (2018). https://doi.org/10.1051/e3sconf/20183803015
[3] Domokos, Gábor., Jerolmack, Douglas J., Kun, Ferenc, Török, János.: Plato’s cube and the natural geometry of fragmentation. Proc. of the National Academy of Sciences 117(31), 18178–18185 (2020)
[4] Goehring, L., Morris, S.W.: Scaling of columnar joints in basalt. Journal of Geophysical Research: Solid Earth 113 issue B10, (2008)
[5] Goehring, Lucas: Evolving fracture patterns: columnar joints, mud cracks and polygonal terrain. Philosophical Trans. of the Royal Soc. A: Math. Phys. and Eng. Sciences 371(2004), 20120353 (2013)
[6] Gold, C.: Tessellations in gis: Part ii-making changes. Geo-spatial Inf. Science 19, 157–167 (2016)
[7] Grünbaum, B., Shepard, G.C.: Tilings and Patterns. Freeman and Co., New York (1987)
[8] Heller, E., Fuchs, E.: Tissue patterning and cellular mechanics. J. of Cell Biology 211, 219–231 (2015)
[9] Horne, C.E.: Geometric Symmetry in Patterns and Tilings. Woodhead Publishing, Published in North and South America by CRC Press LLC (2000)
[10] Jiang, B.: A topological pattern of urban street networks: Universality and peculiarity. Phys. A 384, 647–655 (2007)
[11] Kazanci, D., Vince, A.: A property of normal tilings. Am. Math. Monthly 111, 813–816 (2004)
[12] Laruelle, E., Spassky, N., Genovesio, A.: Unraveling spatial cellular pattern by computational tissue shuffling. Communication Biology 3, 605 (2020)
[13] Schattschneider, D., Senechal, M.: Tilings. chapter 3 in: Discrete and computational Geometry. CRC Press, New York (2004)
[14] Steacy, S., Sammis, C.: An automaton for fractal patterns of fragmentation. Nat. 353, 250–252 (1991)

Gábor Domokos
MTA-BME Morphodynamics Research Group and Department of Mechanics, Materials and Structures
Budapest University of Technology and Economics
Műegyetem rakpart 1-3
Budapest 1111
Hungary
e-mail: domokos@iit.bme.hu

Ákos G. Horváth
Department of Geometry
Budapest University of Technology and Economics
Műegyetem rakpart 3
Budapest H-1111
Hungary
e-mail: ghorvath@math.bme.hu
