On the Time Derivative in an Obstacle Problem

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Abstract: We prove that the time derivative of the solution for the obstacle problem related to the Evolutionary $p$-Laplace Equation exists in Sobolev’s sense, provided that the given obstacle is smooth enough. We keep $p \geq 2$.

1 Introduction

The celebrated Evolutionary $p$-Laplace Equation is much studied and the regularity theory for the solutions is almost complete. We refer to the book [dB] about this fascinating equation. In general, the corresponding subsolutions and supersolutions do not possess that much regularity, they are semicontinuous. We are interested in a special kind of weak supersolutions of the Evolutionary $p$-Laplace Equation, namely the solutions of an obstacle problem. In the presence of a smooth obstacle the regularity improves a lot. Given a function $\psi = \psi(x,t)$ in a bounded domain $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$, we consider all functions $v$ such that

$$\frac{\partial v}{\partial t} \geq \nabla \cdot (|\nabla v|^{p-2} \nabla v) \quad \text{and} \quad v \geq \psi \quad \text{in} \quad \Omega_T.$$

The function $\psi$ acts as an obstacle. The smallest admissible $v$ is the solution of the obstacle problem. (This makes sense because a comparison principle is valid.) However, the above description was only formal. We will instead use Definition 1 below, which is more adequate since it comes with a variational inequality. — We will restrict ourselves to the case $p > 2$, the so-called slow diffusion case.
It is an established fact that if the obstacle $\psi$ is smooth enough, the solution to the obstacle problem inherits some regularity. Our objective is the time derivative $u_t$ of the solution $u$, which \textit{a priori} is only known to be a distribution. Our main result Theorem 2 states that, if $\psi$ has continuous second derivatives, then the time derivative $u_t$ exists in Sobolev’s sense and it belongs to the space $L^p_{loc}(\Omega_T)$. A formula is given for the derivative. The most laborious part of the proof is to show that $\Delta_p u = \nabla \cdot (|\nabla v|^{p-2} \nabla v)$ is a function so that the rule

$$\int_0^T \int_{\Omega} \varphi \Delta_p u \ dx \ dt = - \int_0^T \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \ dx \ dt$$

with test functions applies. The equation has first to be regularized, keeping the obstacle unaffected, and then difference quotients are used. The test functions in [L1] can be adjusted to work here.

An important feature, typical for obstacle problems, is that in the open set $\Upsilon = \{ u > \psi \}$ where the obstacle does not hinder, $u$ is, actually, a solution to the differential equation. Thus in $\Upsilon$ the equation $u_t = \Delta_p u$ holds in the weak sense. The boundary of the \textit{coincidence set} $\Xi = \{ u = \psi \}$ is crucial. This enables us to get an identity for the integral $\int u \varphi_t \ dx \ dt$, from which one can deduce the existence of the time derivative sought for. The special case with no obstacle present was treated in [L2]. —See also [BDM] for some general comments valid for “irregular” obstacles.

To this we may add a curious fact valid for $\psi \in C^2(\Omega_T)$. At all points in the coincidence set $\Xi$ the obstacle satisfies the inequality

$$\frac{\partial \psi}{\partial t} \geq \Delta_p \psi.$$ 

Thus a point at which $\frac{\partial \psi}{\partial t} < \Delta_p \psi$ cannot belong to the coincidence set. This piece of information follows from the characterization of continuous supersolutions as \textit{viscosity} supersolutions, cf. [JLM]. Then $\psi$ itself can do as a test function for the pointwise testing required in the theory of viscosity solutions. (The reader may consult [K] for some basic concepts.) —We will not need this observation.

It is likely that the time derivative belongs to the space $L^2_{loc}(\Omega_T)$, but an eventual proof of this improvement would require much stronger regularity considerations for $\nabla u$. We have kept $p > 2$, but one can expect a counterpart to Theorem 2 valid in the extended range $p > 2n/(n+2)$. The difficulty about further generalizations with $\Delta_p u$ replaced by some operator div $A_p$ is
the following. It is absolutely necessary that the solutions of the differential equation
\[
\frac{\partial u}{\partial t} = \text{div} \, A_p(x, t, u, \nabla u)
\]
enjoy the property of having a time derivative themselves, in order that the corresponding results could be extended to the related obstacle problem. This considerably restricts the possibilities.

## 2 Preliminaries

Let $\Omega$ be a bounded domain in the $n$-dimensional space $\mathbb{R}^n$ having a Lipschitz regular boundary. Suppose that a function $\psi = \psi(x, t)$ is given in the closure of the space-time cylinder $\Omega_T = \Omega \times (0, T)$. The function $\psi$ acts as an obstacle so that the admissible functions are forced to lie above $\psi$ in $\Omega_T$. We make the

Assumption: $\psi \in C(\overline{\Omega_T}) \cap W^{2,p}(\Omega_T)$.

For simplicity the obstacle $\psi$ also determines the values of the admissible functions on the parabolic boundary

$$\Gamma_T = \Omega \times \{0\} \cup \partial \Omega \times [0, T].$$

The class of admissible functions is

$$\mathcal{F}_\psi = \{v \in L^p(0, T; W^{1,p}(\Omega))| v \in C(\overline{\Omega_T}), v \geq \psi \text{ in } \Omega_T, v = \psi \text{ on } \Gamma_T\}.$$  

We keep $p \geq 2$.

**Definition 1** We say that the function $u \in \mathcal{F}_\psi$ is the solution to the obstacle problem, if the inequality

$$\int_0^T \int_\Omega \left( (|\nabla u|^{p-2}\nabla u, \nabla (\phi - u)) + (\phi - u) \frac{\partial \phi}{\partial t} \right) dx \, dt \geq \frac{1}{2} \int_\Omega (\phi(x, T) - u(x, T))^2 \, dx$$

(1)

holds for all smooth functions $\phi \in \mathcal{F}_\psi$. 

The solution exists and is unique, cf. [AL] and [C]. See also [KKS]. It is also a supersolution of the equation \( u_t \geq \Delta_p u \), i.e.,
\[
\int_0^T \int_\Omega \left( (|\nabla u|^{p-2} \nabla u, \nabla \varphi) - u \frac{\partial \varphi}{\partial t} \right) dx \, dt \geq 0
\]
for all non-negative \( \varphi \in C_0^\infty(\Omega_T) \).

Notice that nothing is assumed about the time derivative \( u_t \). Our main result is the theorem below.

**Theorem 2** The time derivative \( u_t \) of the solution \( u \) to the obstacle problem exists in the Sobolev sense and \( u_t \in L_{\text{loc}}^p(\Omega_T) \). It is the function
\[
u_t = \begin{cases} 
\psi_t & \text{in } \Xi \\
\Delta_p u & \text{in } \Omega_T \setminus \Xi
\end{cases}
\]
where \( \Xi = \{ u = \psi \} \) denotes the coincidence set.

In order to avoid the difficulty with the “forbidden” time derivative \( u_t \) in the proof, we have to regularize the equation, keeping the obstacle unchanged. We replace \( |\nabla u|^{p-2} \nabla u \) by
\[
\left( |\nabla u|^2 + \varepsilon^2 \right)^{p-2} \nabla u
\]
to obtain an equation which does not degenerate as \( \nabla u = 0 \).

**Lemma 3** There is a unique \( u^\varepsilon \in F_\psi \) such that
\[
\int_0^T \int_\Omega \left( (|\nabla u^\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \nabla u^\varepsilon, \nabla (\phi - u^\varepsilon) + (\phi - u^\varepsilon) \frac{\partial \phi}{\partial t} \right) dx \, dt 
\geq \frac{1}{2} \int_\Omega (\phi(x,T) - u^\varepsilon(x,T))^2 dx
\]
for all smooth functions \( \phi \) in the class \( F_\psi \). In the open set \( \{ u^\varepsilon > \psi \} \) the function \( u^\varepsilon \) is a solution of the equation
\[
\nabla \cdot \left( (|\nabla u^\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \nabla u^\varepsilon \right).
\]
In the case \( \varepsilon \neq 0 \) we have \( u^\varepsilon \in C^\infty(\Omega_T) \) and \( \frac{\partial u^\varepsilon}{\partial t} \in L^2(\Omega_T) \).
Proof: The existence can be extracted from the proof of [AL, Theorem 3.2]. The regularity for the nondegenerate case $\varepsilon \neq 0$ is according to the standard parabolic theory described in the celebrated book [LSU]. The proof of the Hölder continuity for the degenerate case $\varepsilon = 0$ is in [C].

When $\varepsilon \neq 0$, we can rewrite equation (3) in the more convenient form

$$
\int_0^T \int_\Omega \left( \left| \nabla u^\varepsilon \right|^2 + \varepsilon^2 \right)^{p-2} \left( \nabla u^\varepsilon, \nabla \eta \right) + \eta \left( \frac{\partial u^\varepsilon}{\partial t} \right) dx \, dt \geq 0 \tag{4}
$$

valid for all test functions $\eta$ such that $\eta \geq \psi - u^\varepsilon$ in $\Omega_T$ and $\eta = 0$ on $\Gamma_T$. We may even use any continuous $\eta \in L^p(0,T;W^{1,p}_0(\Omega))$ with $\eta(x,0) = 0$.

In order to proceed to the limit under the integral sign in the forthcoming equations we need the convergence result below, where $u$ denotes the solution to the original obstacle problem, the one with $\varepsilon = 0$.

Lemma 4

$$
\lim_{k \to 0} \int_0^T \int_\Omega \left( |u^\varepsilon - u|^p + |\nabla u^\varepsilon - \nabla u|^p \right) dx \, dt = 0. \tag{5}
$$

Proof: It was established in [KL, Lemma 3.2] that

$$
\lim_{k \to 0} \int_0^T \int_\Omega |\nabla u^\varepsilon - \nabla u|^p dx \, dt = 0, \tag{6}
$$

but the strong convergence of the functions themselves requires, as it were, an extra compactness argument. Since $u^\varepsilon$ is a weak supersolution, there exists a Radon measure $\mu_\varepsilon$ such that

$$
\int_0^T \int_\Omega \left( \left| \nabla u^\varepsilon \right|^2 + \varepsilon^2 \right)^{p-2} \left( \nabla u^\varepsilon, \nabla \varphi \right) - u^\varepsilon \frac{\partial \varphi}{\partial t} \right) dx \, dt = \int_{\Omega_T} \varphi \, d\mu_\varepsilon
$$

for all functions $\varphi \in C_0^\infty(\Omega_T)$, whether positive or not. This is a consequence of Riesz’s Representation Theorem, cf. [EG, 1.8]. See [KLP] for details.

Given a regular open set (for example a polyhedron) $U \subset \subset \Omega_T$, we have to verify that

$$
\mu_\varepsilon(U) \leq M_U
$$
with a bound independent of $\varepsilon$, $0 < \varepsilon < 1$. Then the lemma follows as in [KLP, pp. 720-721]. (There [S] was used.) To this end, choose a test function $\phi \in C^\infty_0(\Omega_T)$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ in $U$. A rough estimation yields

$$\mu_\varepsilon(U) = \int_{\Omega_T} d\mu_\varepsilon = \int_{\Omega_T} \phi d\mu_\varepsilon$$

$$= \int_0^T \int_{\Omega} \left( \langle |\nabla u\varepsilon|^2 + \varepsilon^2 \frac{p-2}{2} \nabla u\varepsilon, \nabla \phi \rangle - u\varepsilon \frac{\partial \phi}{\partial t} \right) dx \, dt$$

$$\leq C_1 \left( \|\nabla u\varepsilon\|^p_{L^p(\Omega_T)} + \varepsilon \frac{p(p-2)}{p-1} \right) + C_2 \|u\varepsilon\|_{\infty}.$$

By the maximum principle $\|u\varepsilon\|_{\infty} \leq \|\psi\|_{\infty}$ and $\|\nabla u\varepsilon\|^p_{L^p(\Omega_T)}$ is uniformly bounded, since the gradients converge strongly. This yields the bound $M_U$. □

3 The gradient estimate

In order to prove that $\Delta_p u$ is a function, $u$ denoting the solution to the obstacle problem, we show that the function $F = |\nabla u|^{(p-2)/2} \nabla u$, where the usual power $p - 2$ has been replaced by $(p - 2)/2$, is in a suitable first order Sobolev $x$-space. This will immediately imply the desired result. At a first reading one had better to assume that the obstacle $\psi$ is as smooth as one pleases, say of class $C^2(\Omega_T)$. Actually, only the Sobolev derivatives $\psi_{x_i x_j}$ and $\psi_{x_t}$ are needed, while $\psi_{tt}$ does not appear at all. We recall our assumption $\psi \in C(\overline{\Omega_T}) \cap W^{2,p}(\Omega_T)$ and use the abbreviation

$$|D^2 \psi|^2 = \sum \psi_{x_i x_j}^2.$$

Under these assumptions about the obstacle $\psi = \psi(x, t)$ we have the following result.

**Theorem 5** For the solution $u$ to the obstacle problem, the derivative $DF$ of

$$F = |\nabla u|^{(p-2)/2} \nabla u$$
exists in Sobolev’s sense and belongs to $L_{loc}^{p/(p-1)}(\Omega_T)$. The estimate
\[
\int_0^T \int_\Omega \zeta^p |D^2 F|^2 \, dx \, dt \leq C \int_0^T \int_\Omega (\zeta^p + |\nabla \zeta|^p |\nabla u|^p) \, dx \, dt \\
+ C \int_0^T \int_\Omega \zeta^p |\nabla u|^2 \, dx \, dt + C \int_0^T \int_\Omega |\nabla \zeta|^p |\nabla \psi|^p \, dx \, dt \\
+ C \int_0^T \int_\Omega \zeta^p (|D^2 \psi|^p + |\nabla \psi|^2) \, dx \, dt + C \int_\Omega \zeta^p |\nabla \psi(x, T)|^2 \, dx
\]
holds for each non-negative test function $\zeta = \zeta(x)$ in $C_0^\infty(\Omega)$; and $C = C(p)$.

**Proof:** The proof is based on the regularized obstacle problem and equation (4), where we abbreviate
\[
A_\varepsilon(x, t) = \left( \nabla u^\varepsilon \right)^2 + \varepsilon^2 \frac{\nabla u^\varepsilon}{p-2} \nabla u^\varepsilon.
\]
We denote its solution by $u$, suppressing the index $\varepsilon$. Thus $u$ means $u^\varepsilon$, to begin with. Given $\zeta$, the variable $x$ is given a small increment $h$ so that the test function
\[
\eta = \psi(x, t) - u(x, t) + \zeta(x)^p [u(x + h, t) - \psi(x + h, t)]
\]
\[
= \zeta(x)^p [u(x + h, t) - u(x, t)] - \zeta(x)^p [\psi(x + h, t) - \psi(x, t)]
\]
\[
- (1 - \zeta(x)^p) [u(x, t) - \psi(x, t)]
\]
is admissible in the regularized equation
\[
\int_0^T \int_\Omega \left( \langle A_\varepsilon(x, t), \nabla \eta \rangle + \eta \frac{\partial u}{\partial t} \right) \, dx \, dt \geq 0. \tag{7}
\]
Inserting the test function, we obtain
\[
\int_0^T \int_\Omega \left( \langle A_\varepsilon(x, t), \nabla (\zeta^p \Delta_h u) \rangle + \zeta^p \Delta_h u \frac{\partial u}{\partial t} \right) \, dx \, dt \\
- \int_0^T \int_\Omega \left( \langle A_\varepsilon(x, t), \nabla (\zeta^p \Delta_h \psi) \rangle + \zeta^p \Delta_h \psi \frac{\partial u}{\partial t} \right) \, dx \, dt \\
\geq \int_0^T \int_\Omega \left( \langle A_\varepsilon(x, t), \nabla ((1 - \zeta(x)^p) [u(x, t) - \psi(x, t)]) \rangle \\
+ (1 - \zeta(x)^p) [u(x, t) - \psi(x, t)] \frac{\partial u}{\partial t} \right) \, dx \, dt \\
\geq 0.
\]
The last integral is non-negative, because

\[(1 - \zeta(x)^p)[u(x, t) - \psi(x, t)]\]

will do as a test function in the equation (7). This observation is important here.

Aiming at difference quotients we give \(x\) the increment \(h\). The translated function \(u(x + h, t)\) solves the obstacle problem with the translated obstacle \(\psi(x + h, t)\), all this with respect to the shifted domain \(\Omega^h \times (0, T)\) where \(\Omega^h = \{ x \mid x + h \in \Omega \}\). For sufficiently small \(h\) we have

\[
\int_0^T \int_{\Omega^h} \left( \langle A_{\varepsilon}(x + h, t), \nabla \eta(x, t) \rangle + \eta(x, t) \frac{\partial u(x + h, t)}{\partial t} \right) \, dx \, dt \geq 0 \quad (8)
\]

whenever \(\eta(x, t) \geq \psi(x + h, t) - u(x + h, t)\) and \(\eta = 0\) on the parabolic boundary of \(\Omega^h \times (0, T)\). Here

\[
\eta = \psi(x + h, t) - u(x + h, t) + \zeta(x)^p[u(x, t) - \psi(x, t)]
\]

\[
= \zeta(x)^p[u(x + h, t) - u(x, t)] - \zeta(x)^p[\psi(x + h, t) - \psi(x, t)]
\]

\[
-(1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)]
\]

will do. We obtain

\[
- \int_0^T \int_{\Omega^h} \left( \langle A_{\varepsilon}(x + h, t), \nabla (\zeta^p \Delta_h u) \rangle + \zeta^p \Delta_h u \frac{\partial u(x + h, t)}{\partial t} \right) \, dx \, dt
\]

\[
+ \int_0^T \int_{\Omega^h} \left( \langle A_{\varepsilon}(x + h, t), \nabla (\zeta^p \Delta_h \psi) \rangle + \zeta^p \Delta_h \psi \frac{\partial u(x + h, t)}{\partial t} \right) \, dx \, dt
\]

\[
\geq \int_0^T \int_{\Omega^h} \left( \langle A_{\varepsilon}(x + h, t), \nabla (1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)] \rangle \right)
\]

\[
+ (1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)] \frac{\partial u(x + h, t)}{\partial t} \right) \, dx \, dt
\]

\[
\geq 0.
\]

The last integral is positive because

\[(1 - \zeta(x)^p)[u(x + h, t) - \psi(x + h, t)]\]

will do as a test function in the translated equation (8). This observation is essential here. The integrals in the left-hand member of the inequality
are, in fact, taken only over the support of the function \( \zeta(x) \). Hence we have an inequality with integrals taken only over \( \Omega_T \), provided that \( |h| < \text{dist}(\text{supp}\zeta, \partial\Omega) \). Thus \( \Omega^h \) is no longer directly involved.

We add the two estimates, grouping the differences, and obtain

\[
+ \int_0^T \int_\Omega \langle A_\varepsilon(x + h, t) - A_\varepsilon(x, t), \nabla(\zeta^p \Delta_h u) \rangle \, dx \, dt \\
\leq \int_0^T \int_\Omega \langle A_\varepsilon(x + h, t) - A_\varepsilon(x, t), \nabla(\zeta^p \Delta_h \psi) \rangle \, dx \, dt \\
- \int_0^T \int_\Omega \zeta^p \Delta_h u \cdot \Delta_h \left( \frac{\partial u}{\partial t} \right) \, dx \, dt + \int_0^T \int_\Omega \zeta^p \Delta_h \psi \cdot \Delta_h \left( \frac{\partial u}{\partial t} \right) \, dx \, dt.
\]

The integrals with the time derivatives can be integrated by parts:

\[
- \int_0^T \int_\Omega \zeta^p \frac{\partial (\Delta_h u)^2}{2} \, dx \, dt + \int_0^T \int_\Omega \zeta^p \Delta_h \psi \cdot \Delta_h \left( \frac{\partial u}{\partial t} \right) \, dx \, dt \\
= - \int_\Omega \zeta^p(x) \frac{(\Delta_h u)^2}{2} \bigg|_0^T \, dx + \int_\Omega \zeta^p(x) \Delta_h \psi \cdot \Delta_h u \bigg|_0^T \, dx \\
- \int_0^T \int_\Omega \zeta^p \Delta_h u \cdot \Delta_h \left( \frac{\partial \psi}{\partial t} \right) \, dx \, dt.
\]

Since \( \Delta_h u = \Delta_h \psi \) when \( t = 0 \), the above expression is majorized by

\[
\frac{1}{2} \int_\Omega \zeta^p((\Delta_h \psi)^2)_T - (\Delta_h \psi)^2_0 \, dx + \frac{1}{2} \int_0^T \int_\Omega \zeta^p \left( (\Delta_h u)^2 + (\Delta_h \frac{\partial \psi}{\partial t})^2 \right) \, dx \, dt,
\]

where the inequality \( 2\Delta_h u \Delta_h \psi \leq (\Delta_h u)^2 + (\Delta_h \psi)^2 \) was used at time \( T \).

At this stage there are no “forbidden” time derivatives left and so we may safely let \( \varepsilon \) go to zero. By Lemma 3 we may pass to the limit under the integral sign and hence the estimate for the limit \( u \) (no longer \( u^\varepsilon \)) becomes

\[
\int_0^T \int_\Omega \langle (\Delta_h A, \nabla(\zeta^p \Delta_h u)) \rangle \, dx \, dt \\
\leq \int_0^T \int_\Omega \langle (\Delta_h A, \nabla(\zeta^p \Delta_h \psi)) \rangle \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega \zeta^p \left( (\Delta_h u)^2 + (\Delta_h \frac{\partial \psi}{\partial t})^2 \right) \, dx \, dt + \frac{1}{2} \int_\Omega \zeta^p(\Delta_h \psi)^2_0 \, dx.
\]
where
\[ \Delta_h A = A(x+h,t) - A(x,t). \]

We write this more conveniently as
\[
\int_0^T \int_\Omega \zeta^p \langle \langle \Delta_h A, \nabla \Delta_h u \rangle \rangle \, dx \, dt \\
\leq \int_0^T \int_\Omega p \zeta^{p-1} |\Delta_h A| |\Delta_h u| |\nabla \zeta| \, dx \, dt \\
+ \int_0^T \int_\Omega p \zeta^{p-1} |\Delta_h A| |\Delta_h \psi| |\nabla \zeta| \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega \zeta^p \left( (\Delta_h u)^2 (\Delta_h \frac{\partial \psi}{\partial t})^2 \right) \, dx \, dt + \frac{1}{2} \int_\Omega \zeta^p (\Delta_h \psi)^2 \, dx. \tag{9}
\]

The integrand on left-hand side is \( \langle \Delta_h A, \nabla \Delta_h u \rangle \) =
\[ \langle |\nabla u(x+h,t)|^{\frac{p-2}{2}} \nabla u(x+h,t) - |\nabla u(x,t)|^{\frac{p-2}{2}} \nabla u(x,t), \nabla u(x+h,t) - \nabla u(x,t) \rangle \geq \frac{4}{p^2} |F(x+h,t) - F(x,t)|^2 = \frac{4}{p^2} |\Delta_h F|^2, \tag{10}\]

where the elementary inequality
\[ \frac{4}{p^2} \left( b^{\frac{p-2}{2}} b - a^{\frac{p-2}{2}} a \right)^2 \leq \langle |b|^{p-2} b - |a|^{p-2} a, b - a \rangle \]
for vectors was used. We aim at an estimate for the integral of \( \zeta^p |\Delta_h F| \).

We divide the \( \Delta_h \)-terms by \( |h| \) so that the desired difference quotients appear. The estimate
\[ \left| \frac{\Delta_h A}{h} \right| \leq (p-1) \left| \frac{\Delta_h F}{h} \right| \left( |\nabla u(x+h,t)|^{\frac{p-2}{2}} + |\nabla u(x,t)|^{\frac{p-2}{2}} \right), \]
coming from the elementary vector inequality
\[ |b|^{p-2} b - |a|^{p-2} a \leq (p-1) \left( |b|^{\frac{p-2}{2}} + |a|^{\frac{p-2}{2}} \right) \left| b^{\frac{p-2}{2}} b - |a|^{\frac{p-2}{2}} a \right|, \]
is used in the integrands of I, II, and III. In I we split the factors so that
\[
p\zeta^p \left| \frac{\Delta_h A}{h} \right| \left| \frac{\Delta_h u}{h} \right| \left| \nabla \zeta \right|
\]
\[
\leq p(p-1) \left[ \zeta^\frac{p}{2} \left| \frac{\Delta_h F}{h} \right| \right] \left[ \frac{\Delta_h u}{h} \right] \left| \nabla \zeta \right| \left[ \zeta^\frac{p-2}{2} \left( |\nabla u(x,t)|^{\frac{p-2}{2}} + |\nabla u(x+h,t)|^{\frac{p-2}{2}} \right) \right]
\]
and use Young’s inequality
\[
abc \leq \frac{\varepsilon^2 a^2}{2} + \frac{\varepsilon^{-p} b^p}{p} + \frac{(p-2)c^{2p-2}}{2p}
\]
to get the bound
\[
\frac{I}{|h|^2} \leq \frac{p(p-1)\varepsilon^2}{2} \int_0^T \int_\Omega \zeta^p \left| \frac{\Delta_h F}{h} \right|^2 \, dx \, dt
\]
\[
+ (p-1)\varepsilon^{-p} \int_0^T \int_\Omega \left| \frac{\Delta_h u}{h} \right|^p \left| \nabla \zeta \right| \, dx \, dt
\]
\[
+ c_p \int_0^T \int_\Omega \zeta^p \left( |\nabla u(x,t)|^p + |\nabla u(x+h,t)|^p \right) \, dx \, dt
\]
The integral $\frac{II}{|h|^2}$ has a similar majorant, the only difference being that $\Delta_h u$ be replaced by $\Delta_h \psi$. The integrand of III is estimated in a similar way:
\[
p\zeta^p \left| \frac{\Delta_h A}{h} \right| \left| \frac{\Delta_h \psi}{h} \right|
\]
\[
\leq p(p-1) \left[ \zeta^\frac{p}{2} \left| \frac{\Delta_h F}{h} \right| \right] \left[ \frac{\Delta_h u}{h} \right] \left| \nabla \left( \frac{\Delta_h \psi}{h} \right) \right| \left[ \zeta^\frac{p-2}{2} \left( |\nabla u(x,t)|^{\frac{p-2}{2}} + |\nabla u(x+h,t)|^{\frac{p-2}{2}} \right) \right]
\]
\[
\leq \frac{p(p-1)\varepsilon^2}{2} \zeta^p \left| \frac{\Delta_h F}{h} \right|^2 + (p-1)\varepsilon^{-p} \zeta^p \left| \nabla \left( \frac{\Delta_h \psi}{h} \right) \right|^p
\]
\[
+ c_p \zeta^p \left( |\nabla u(x,t)|^p + |\nabla u(x+h,t)|^p \right).
\]
Adding up the three integrated estimates, we arrive at

\[
\frac{1 + II + III}{|h|^2} \leq \\
3 \frac{p(p - 1)\varepsilon^2}{2} \int_0^T \int_\Omega \left| \frac{\Delta_h F}{h} \right|^2 dx dt \\
+ (p - 1)\varepsilon^p \int_0^T \int_\Omega \left( \left| \frac{\Delta_h u}{h} \right|^p |\nabla \xi|^p + \left| \frac{\Delta_h \psi}{h} \right|^p |\nabla \xi|^p + \varepsilon^p \left| \nabla \frac{\Delta_h \psi}{h} \right|^p \right) dx dt \\
+ 3c_2 \int_0^T \int_\Omega \xi^p \left( |\nabla u(x, t)|^p + |\nabla u(x + h, t)|^p \right) dx dt.
\]

This complements (9). Recall (10). The next step is to absorb the first integral above in the right-hand member into the minorant in (10) by fixing \(\varepsilon\) small enough, say

\[
3 \frac{p(p - 1)\varepsilon^2}{2} = \frac{2}{p^2}.
\]

The resulting estimate, written out without abbreviations, is

\[
\int_0^T \int_\Omega \xi^p \left| \frac{F(x + h, t) - F(x, t)}{h} \right|^2 dx dt \\
\leq a_p \int_0^T \int_\Omega \left| \frac{u(x + h, t) - u(x, t)}{h} \right|^p |\nabla \xi|^p dx dt \\
+ a_p \int_0^T \int_\Omega \left| \frac{\psi(x + h, t) - \psi(x, t)}{h} \right|^p |\nabla \xi|^p dx dt \\
+ a_p \int_0^T \int_\Omega \xi^p \left| \frac{\nabla \psi(x + h, t) - \nabla \psi(x, t)}{h} \right|^p dx dt \\
+ b_p \int_0^T \int_\Omega \xi^p \left( |\nabla u(x, t)|^p + |\nabla u(x + h, t)|^p \right) dx dt \\
+ c_p \int_0^T \int_\Omega \xi^p \left| \frac{u(x + h, t) - u(x, t)}{h} \right|^2 dx dt \\
+ c_p \int_0^T \int_\Omega \xi^p \left| \frac{\psi(x + h, t) - \psi(x, t)}{h} \right|^2 dx dt \\
+ c_p \int_\Omega \xi^p \left| \frac{\psi(x + h, T) - \psi(x, T)}{h} \right|^2 dx,
\]
where the constants depend only on $p$. Finally, letting the increment $h \to 0$ in any desired direction, we arrive at the estimate in the theorem. Here we use the characterization of Sobolev spaces in terms of integrated differential quotients, cf. [G, Chapter 8.1]. This concludes our proof of Theorem 5.

**Corollary 6** If $u$ is the solution to the obstacle problem with the obstacle $\psi$, then $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ belongs to the space $L^{\frac{p}{p-1}}_{\text{loc}}(\Omega_T)$ and

$$\int_0^T \int_{\Omega} \varphi \Delta_p u \, dx \, dt = \int_0^T \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \, dx \, dt$$

for all test functions $\varphi$ in $C^\infty_0(\Omega_T)$.

**Proof:** Since $F$ is in Sobolev’s space and $p \geq 2$, we can differentiate $|\nabla u|^{p-2} \nabla u = |F|^{\frac{p-2}{p}} F$ and hence

$$\left| \frac{\partial}{\partial x_j} (|\nabla u|^{p-2} \nabla u) \right| \leq 2 \left( 1 - \frac{1}{p} \right) |F|^{\frac{p-2}{p}} \left| \frac{\partial F}{\partial x_j} \right|.$$

By Hölder’s inequality

$$\frac{\partial}{\partial x_j} (|\nabla u|^{p-2} \nabla u) \in L^{\frac{p}{p-1}}_{\text{loc}}(\Omega_T),$$

since $F \in L^2(\Omega_T)$ and $DF \in L^2(\Omega_T)$. □

### 4 The Time Derivative

For the proof of the Theorem we notice that the contact set $\Xi = \{ u = \psi \}$ is a closed subset of $\overline{\Omega_T}$ and that its complement $\Upsilon = \Omega_T \setminus \Xi$ is open. In the set $\Upsilon$, where the obstacle does not hinder, $u$ is a solution to the Evolutionary $p$-Laplace Equation $u_t = \Delta_p u$. In other words, whenever $\phi \in C^\infty(\Upsilon)$,

$$\int_0^T \int_{\Omega} u \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^T \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, dx \, dt = - \int_0^T \int_{\Omega} \phi \Delta_p u \, dx \, dt,$

the actual set of integration being $\Upsilon$. Here Corollary was used. Thus $u_t$ is available, but only in $\Upsilon$ to begin with. (See also [L2].) Let $\phi$ denote an
arbitrary test function in \( C_0^\infty(\Omega_T) \). We need a specific test function with compact support in \( \Upsilon \). To construct it, define

\[
\theta_k = \min\{1, k(u - \psi)\}, \quad k = 1, 2, \ldots.
\]

Then \( 1 - \theta_k = 1 \) in \( \Xi \) and pointwise the monotone convergence \( 1 - \theta_k \to \chi_\Xi \) holds. Moreover, the support of \( \theta_k \) is compact in \( \Upsilon \). The time derivative of \( \theta_k \) is available!

Using

\[
\int_0^T \int_\Omega u \frac{\partial}{\partial t} (\theta_k \phi) \, dx \, dt = \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla (\theta_k \phi) \rangle \, dx \, dt,
\]

we write

\[
\int_0^T \int_\Omega \phi \Delta_p u \, dx \, dt = - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, dx \, dt
\]

\[
= - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla (\theta_k \phi + (1 - \theta_k) \phi) \rangle \, dx \, dt
\]

\[
= - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla (\theta_k \phi) \rangle \, dx \, dt - \int_0^T \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla ((1 - \theta_k) \phi) \rangle \, dx \, dt
\]

\[
= - \int_0^T \int_\Omega \frac{\partial}{\partial t} (\theta_k \phi) \, dx \, dt + \int_0^T \int_\Omega (1 - \theta_k) \phi \, dx \, dt.
\]

The last integral has the limit

\[
\lim_{k \to 0} \int_0^T \int_\Omega (1 - \theta_k) \phi \, dx \, dt = \iint_\Xi \phi \, \Delta_p \psi \, dx \, dt.
\]

In the integral with the time derivative we write

\[
-u \frac{\partial}{\partial t} (\theta_k \phi) = -u \frac{\partial \phi}{\partial t} + (u - \psi) \frac{\partial}{\partial t} (1 - \theta_k) \phi + \psi \frac{\partial}{\partial t} (1 - \theta_k) \phi
\]

and obtain

\[
- \int_0^T \int_\Omega u \frac{\partial}{\partial t} (\theta_k \phi) \, dx \, dt = \int_0^T \int_\Omega -u \frac{\partial \phi}{\partial t} \, dx \, dt
\]

\[
+ \int_0^T \int_\Omega (u - \psi) \frac{\partial}{\partial t} ((1 - \theta_k) \phi) \, dx \, dt - \int_0^T \int_\Omega (1 - \theta_k) \phi \frac{\partial \psi}{\partial t} \, dx \, dt.
\]
where an integration by parts has produced the last integral. It has the evident limit
\[
\lim_{k \to 0} \int_0^T \int_\Omega (1 - \theta_k) \phi \frac{\partial \psi}{\partial t} \, dx \, dt = \int_\Xi \phi \frac{\partial \psi}{\partial t} \, dx \, dt.
\]

The middle integral vanishes as \( k \to 0 \):
\[
\begin{align*}
&\int_0^T \int_\Omega (u - \psi)(1 - \theta_k) \frac{\partial \phi}{\partial t} \, dx \, dt \\
= &\int_0^T \int_\Omega (u - \psi)(1 - \theta_k) \frac{\partial \phi}{\partial t} \, dx \, dt - \int_0^T \int_\Omega \phi (u - \psi) \frac{\partial \theta_k}{\partial t} \, dx \, dt \\
= &\int_0^T \int_\Omega (u - \psi)(1 - \theta_k) \frac{\partial \phi}{\partial t} \, dx \, dt - \frac{1}{2k} \int_0^T \int_\Omega \phi \frac{\partial \theta_k^2}{\partial t} \, dx \, dt \\
= &\int_0^T \int_\Omega (u - \psi)(1 - \theta_k) \frac{\partial \phi}{\partial t} \, dx \, dt + \frac{1}{2k} \int_0^T \int_\Omega \theta_k^2 \frac{\partial \phi}{\partial t} \, dx \, dt \quad \xrightarrow{k \to \infty} \quad 0 + 0.
\end{align*}
\]

Collecting results,
\[
\int_0^T \int_\Omega \phi \Delta_p u \, dx \, dt = - \int_0^T \int_\Omega \phi_t \, dx \, dt - \int_\Xi (\psi_t - \Delta_p \psi) \phi \, dx \, dt.
\]

In other words, the final formula
\[
- \int_0^T \int_\Omega u \phi_t \, dx \, dt = \int_0^T \int_\Omega \phi [\Delta_p u + (\psi_t - \Delta_p \psi) \chi_\Xi] \, dx \, dt
\]
holds for every \( \phi \) in \( C_0^\infty(\Omega_T) \). Therefore
\[
u_t = \Delta_p u + (\psi_t - \Delta_p \psi) \chi_\Xi
\]
and this is a function belonging to \( L_{loc}^{p/(p-1)}(\Omega_T) \). This concludes the proof of Theorem 2. \( \Box \)

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