AN ALTERNATIVE PROOF OF A RIGIDITY THEOREM
FOR THE SHARP SOBOLEV CONSTANT

STEFANO PIGOLA AND GIONA VERONELLI

Abstract. We provide a somewhat geometric proof of a rigidity theorem by M. Ledoux and C. Xia concerning complete manifolds with non-negative Ricci curvature supporting an Euclidean-type Sobolev inequality with (almost) best Sobolev constant. Using the same technique we also generalize Ledoux-Xia result to complete manifolds with asymptotically non-negative curvature.

1. Introduction

A Riemannian manifold \((M, \langle , \rangle)\) of dimension \(\dim M = m > p \geq 1\) is said to support an Euclidean-type Sobolev inequality if there exists a constant \(C_M > 0\) such that, for every \(u \in C^\infty_c(M)\),

\[
\left(\int_M |u|^{p^*} \, d\text{vol}\right)^{\frac{1}{p^*}} \leq C_M \left(\int_M |\nabla u|^p \, d\text{vol}\right)^{\frac{1}{p}},
\]

where

\[p^* = \frac{mp}{m-p}\]

and \(d\text{vol}\) denotes the Riemannian measure of \(M\). Clearly, (1) implies that there exists a continuous imbedding \(W^{1,p}(M) \hookrightarrow L^{p^*}(M)\), and can be expressed in the equivalent form

\[C^{-p}_M \leq \inf_{u \in \Lambda} \int_M |\nabla u|^p \, d\text{vol},\]

where

\[\Lambda = \left\{ u \in L^{p^*}(M) : |\nabla u| \in L^p \text{ and } \int_M |u|^{p^*} \, d\text{vol}_M = 1 \right\}.
\]

The validity of (1), as well as the best value of the Sobolev constant \(C_M\), have intriguing and deep connections with the geometry of the underlying manifold, many of which are discussed in the excellent lecture notes [5]. See also [8] for a survey in the more abstract perspective of Markov diffusion processes, and [9] for the relevance of (1) in the \(L^{p,q}\)-cohomology theory. For instance, we note that a complete manifold with non-negative Ricci curvature (but, in fact, a certain amount of negative curvature is allowed) and supporting an Euclidean-type Sobolev inequality is necessarily connected.
at infinity. This fact can be proved using (non-linear) potential theoretic arguments; see [10], [11].

It is known (see e.g. Proposition 4.2 in [5]) that

\[ C_M \geq K(m, p), \]

where \( K(m, p) \) is the best constant in the corresponding Sobolev inequality of \( \mathbb{R}^m \). It was discovered by M. Ledoux, [7], that for complete manifolds of non-negative Ricci curvature, the equality in (2) forces \( M \) to be isometric to \( \mathbb{R}^m \). This important rigidity result has been generalized by C. Xia, [12], by showing that, in case \( C_M \) is sufficiently close to \( K(m, p) \), then \( M \) is diffeomorphic to \( \mathbb{R}^m \). The first aim of this note is to provide a simple and somewhat geometric proof of the Ledoux-Xia rigidity result.

**Notation.** In what follows, having fixed a reference origin \( o \in M \), we set \( r(x) = \text{dist}_M(x, o) \) and we denote by \( B_t \) and \( \partial B_t \) the geodesic ball and sphere of radius \( t > 0 \) centered at \( o \). The corresponding balls and spheres in the \( m \)-dimensional Euclidean space are denoted by \( \mathbb{B}_t \) and \( \partial \mathbb{B}_t \). Finally, the symbols \( V(B_t) \) and \( A(\partial B_t) \) stand, respectively, for the Riemannian volume of \( B_t \) and the \((m-1)\)-dimensional Hausdorff measure of \( \partial B_t \).

**Theorem 1.** Let \((M, \langle \cdot, \rangle)\) be a complete, \( m \)-dimensional Riemannian manifold, \( m > p > 1 \). Assume that \( ^M\text{Ric} \geq 0 \) and that the Euclidean-type Sobolev inequality (7) holds on \( M \). Then

\[ V(B_t) \geq V(\mathbb{B}_t) \geq \left( \frac{K(m, p)}{C_M} \right)^m V(B_t). \]

In particular, if \( C_M \) is sufficiently close to \( K(m, p) \) then \( M \) is diffeomorphic to \( \mathbb{R}^m \) and, in case \( C_M = K(m, p) \), \( M \) is isometric to \( \mathbb{R}^m \).

Actually, using the same technique, we shall prove that a lower control on the volume of geodesic balls from a fixed origin can be obtained even if we allow a certain amount of negative curvature. More precisely, we will prove the following

**Theorem 2.** Let \((M, \langle \cdot, \rangle)\) be a complete, \( m \)-dimensional Riemannian manifold, \( m \geq 3 \), with

\[ ^M\text{Ric}(y) \geq -(m - 1)G(r(y)) \text{ on } M \]

for some non-negative function \( G \in C^0([0, +\infty)) \). Assume that \( G \) satisfies the integrability condition

\[ \int_0^{+\infty} tG(t)dt = b < +\infty \]

and that the Euclidean-type Sobolev inequality (7) holds on \( M \), for some \( 1 < p < m \). Then

\[ e^{mb}V(\mathbb{B}_t) \geq V(B_t) \geq \hat{C}(m, p, C_M, b)V(\mathbb{B}_t), \]
where
\[
\hat{C}(m, p, C_M, b) \to \left(\frac{K(m, p)}{C_M}\right)^m \quad \text{as } b \to 0.
\]

Combining Theorem 2 with Theorem 3.1 in [13], see also [1], we immediately deduce the next rigidity result.

**Corollary 3.** Given \( m \geq 3 \), \( m > p \), there exist constants \( b_0 = b_0(m, p) > 0 \) and \( \varepsilon_0 = \varepsilon_0(m, p) > 0 \) such that, if \( M \) is an \( m \)-dimensional complete manifold supporting the Sobolev inequality \( (\frac{C_M}{K(m, p)}) \leq 1 - \varepsilon \) and such that

\[
M \operatorname{Sect} \geq -G(r) \quad \text{on } M,
\]

where \( G \) satisfies \((5)\) for some \( b \leq b_0 \), then \( M \) is diffeomorphic to \( \mathbb{R}^m \).

2. **Proof of the Ledoux-Xia theorem**

Recall that, in \( \mathbb{R}^m \), the equality in \( (1) \) with the best constant \( C_{\mathbb{R}^m} = K(m, p) \), is realized by the (radial) Bliss-Aubin-Talenti functions \( \phi_\lambda(x) = \varphi_\lambda(|x|) \) for every \( \lambda > 0 \), where \( |x| \) is the Euclidean norm of \( x \) and \( \varphi_\lambda(t) \) are the real-valued functions defined as

\[
\varphi_\lambda(t) = \beta(m, p) \left(\frac{\lambda + t}{\lambda + t} \right)^{\frac{m-p}{p-1}}.
\]

If we choose \( \beta(m, p) > 0 \) such that

\[
\int_{\mathbb{R}^m} \phi_\lambda^p(x) dx = 1
\]

then

\[
K(m, p)^{-p} = \int_{\mathbb{R}^m} |\varphi_\lambda'(|x|)|^p dx
\]

and, by the standard calculus of variations, the extremal functions \( \phi_\lambda \) obey the (nonlinear) Yamabe-type equation

\[
\Delta_p \phi_\lambda = -K(m, p)^{-p} \phi_\lambda^{p-1},
\]

where

\[
\Delta_p u = \operatorname{div} \left( |\nabla u|^{p-2} \nabla u \right)
\]

stands for the \( p \)-Laplacian of a given function \( u \).

Define \( \tilde{\phi}_\lambda : M \to \mathbb{R} \) as \( \tilde{\phi}_\lambda(x) := \varphi_\lambda(r(x)) \). The idea of our proof is simply to apply Karp version of Stokes theorem, \( [6] \), to the vector field \( X_\lambda := \tilde{\phi}_\lambda |\nabla \tilde{\phi}_\lambda|^{p-2} \nabla \tilde{\phi}_\lambda \), once we have observed that, by \((5)\) and the Laplacian comparison theorem, each function \( \tilde{\phi}_\lambda \) on \( M \) satisfies

\[
\Delta_p \tilde{\phi}_\lambda \geq -K(m, p)^{-p} \tilde{\phi}_\lambda^{p-1}.
\]

This leads directly to inequality \((2.2)\) in \([12]\) and the argument can be completed essentially as explained by Xia.
Proof of Theorem 7. We claim that
\begin{equation}
\begin{aligned}
(i) & \quad \int_M \hat{\phi}^p \, d\text{vol} \leq 1, \\
(ii) & \quad |\nabla \hat{\phi}| \in L^p(M),
\end{aligned}
\end{equation}
Indeed, since $^M\text{Ric} \geq 0$, according to the Bishop-Gromov comparison theorem, [3, 10], $A(\partial B_t) / A(\partial B_t)$ is a decreasing function of $t > 0$ and, therefore,
\begin{equation}
A(\partial B_t) \leq A(\partial B_t), \quad V(B_t) \leq V(B_t).
\end{equation}
The validity of (9) follows from the co-area formula. Furthermore, since $^M\text{Ric} \geq 0$, by Laplacian comparison, [10], $\Delta r \leq (m - 1)/r$ pointwise on $M \setminus \text{cut(o)}$ and weakly on all of $M$. This means that
\begin{equation}
- \int_M \langle \nabla r, \nabla \eta \rangle \, d\text{vol} \leq \int_M \eta \frac{m - 1}{r} \, d\text{vol}
\end{equation}
for all $0 \leq \eta \in W^{1,2}_c(M)$. Let $0 \leq \xi \in C^\infty_c(M)$ to be chosen later and apply (11) with
\begin{equation}
\eta = -\left(\xi \hat{\phi}_\lambda\right) |\varphi'_\lambda(r)|^{p-2} \varphi'_\lambda(r),
\end{equation}
thus obtaining
\begin{equation}
\int_M \varphi'_\lambda(r) |\varphi'_\lambda(r(y))|^{p-2} \langle \nabla r, \nabla \left(\xi \hat{\phi}_\lambda\right) \rangle \, d\text{vol}
\leq -\int_M |\varphi'_\lambda(r(y))|^{p-2} \left[(p - 1) \varphi''_\lambda(r) + \frac{(m - 1)}{r} \varphi'_\lambda(r)\right] \left(\xi \hat{\phi}_\lambda\right) \, d\text{vol}.
\end{equation}
On the other hand, according to [5],
\begin{equation}
|\varphi'_\lambda(t)|^{p-2} \left\{(p - 1) \varphi''_\lambda(t) + \frac{m - 1}{t} \varphi'_\lambda(t)\right\} = -K(m, p)^{-p} \varphi''_\lambda^{-1}(t)
\end{equation}
for all $t > 0$, and inserting into (13) gives
\begin{equation}
- \int_M K(m, p)^{-p} \xi \hat{\phi}^p \, d\text{vol}
\leq -\int_M |\varphi'_\lambda(r(y))|^{p-2} \varphi'_\lambda(r) \langle \nabla r, \hat{\phi}_\lambda \nabla \xi + \xi \varphi'_\lambda(r) \nabla r \rangle \, d\text{vol}
\leq -\int_M \xi |\nabla \hat{\phi}_\lambda| \, d\text{vol} - \int_M \hat{\phi}_\lambda \varphi'_\lambda(r)|\varphi'_\lambda(r(y))|^{p-2} \langle \nabla r, \nabla \xi \rangle \, d\text{vol}.
\end{equation}
Now choose $\xi = \xi_R$ such that $\xi \equiv 1$ on $B_R$, $\xi \equiv 0$ on $M \setminus B_{2R}$ and $|\nabla \xi| < 2/R$. By Cauchy-Schwarz inequality and volume comparison we get
\begin{align*}
& \left| \int_M \hat{\phi}_\lambda \varphi'_\lambda(r)|\varphi'_\lambda(r(y))|^{p-2} \langle \nabla r, \nabla \xi \rangle \, d\text{vol} \right|
\leq \frac{2}{R} \int_R^{2R} \varphi_\lambda(t)(-\varphi'_\lambda(t))^{p-1} A(\partial B_t) dt \to 0
\end{align*}
as $R \to \infty$, for every $m > p$. Then, taking the limits as $R \to \infty$ in (15) we obtain
\[ K(m, p)^{-p} \int_M \hat{\phi}_\lambda^{p^*} d\text{vol} \geq \int_M |\nabla \hat{\phi}_\lambda|^p d\text{vol}, \]
proving that
\begin{equation} \label{eq:16} \frac{\int_M |\nabla \hat{\phi}_\lambda|^p}{\int_M \hat{\phi}_\lambda^{p^*}} \leq K(m, p)^{-p}. \end{equation}

On the other hand, because of (9), we can use $\hat{\phi}_\lambda$ into (1) and get
\begin{equation} \label{eq:17} \frac{\int_M |\nabla \hat{\phi}_\lambda|^p}{\int_M \hat{\phi}_\lambda^{p^*}} \geq \frac{\int_M |\nabla \hat{\phi}_\lambda|^p}{(\int_M \hat{\phi}_\lambda^{p^*})^{\frac{p}{p^*}}} \geq C^{-p}_M. \end{equation}

Combining (16) and (17) we obtain
\[ 1 \geq \int_M \hat{\phi}_\lambda^{p^*} \geq \left( \frac{K(m, p)}{C_M} \right)^m. \]

From this latter, using (7), the co-area formula and integrating by parts, it follows that
\begin{equation} \label{eq:18} 0 \leq \int_M \left( \frac{C_M}{K(m, p)} \right)^m \hat{\phi}_\lambda^{p^*} d\text{vol} - \int_{\mathbb{R}^m} \hat{\phi}_\lambda^{p^*} (x) dx \\
= \int_0^\infty v_M(t) V(B_t) \frac{d}{dt} \left( -\varphi_\lambda^{p^*} (t) \right) dt \end{equation}
where, by Bishop-Gromov, the function
\[ v_M(t) := \left[ \left( \frac{C_M}{K(m, p)} \right)^m \frac{V(B_t)}{V(B_t)} - 1 \right] \]
is non-increasing. In order to prove (8), it is enough to show that
\[ \lim_{t \to \infty} v_M(t) \geq 0. \]

By contradiction, suppose there exist positive constants $\epsilon$ and $T$ such that $v_M(t) \leq -\epsilon$ for all $t \geq T$. Define $T_0 = \sup\{t < T : v_M(t) \geq 0\}$. Then $0 < T_0 < T$ and
\begin{equation} \label{eq:19} \int_0^\infty v_M(t) V(B_t) \frac{d}{dt} \left( -\varphi_\lambda^{p^*} (t) \right) dt \leq v_M(0) \int_0^{T_0} V(B_t) \frac{d}{dt} \left( -\varphi_\lambda^{p^*} (t) \right) dt \\
- \epsilon \int_T^\infty V(B_t) \frac{d}{dt} \left( -\varphi_\lambda^{p^*} (t) \right) dt. \end{equation}
Observe that the 1-parameter family of functions
\[
V(B_t) \frac{d}{dt} \left( -\varphi^p_\lambda(t) \right) = \omega_m \frac{mp}{p-1} \beta(m, p)^p \lambda^p \frac{t^{\frac{1}{p-1} + m}}{(\lambda + t^{\frac{1}{p-1}})^{m+1}}
\]
is decreasing in \(\lambda\), provided \(\lambda >> 1\). Then we can apply the dominated convergence theorem to deduce
\[
\lim_{\lambda \to +\infty} \int_0^T V(B_t) \frac{d}{dt} \left( -\varphi^p_\lambda(t) \right) dt = 0. \quad (20)
\]
On the other hand, using the co-area formula once again,
\[
\int_0^\infty V(B_t) \frac{d}{dt} \left( -\varphi^p_\lambda(t) \right) dt = \int_0^\infty A(\partial B_t) \varphi^p_\lambda(t) dt = \int_{\mathbb{R}^m} \partial^p_\lambda(x) dx = 1,
\]
for all \(\lambda > 0\). Therefore, by (20),
\[
\lim_{\lambda \to +\infty} \int_T^\infty V(B_t) \frac{d}{dt} \left( -\varphi^p_\lambda(t) \right) dt = 1. \quad (21)
\]
Using (20) and (21) into (19) we conclude that, up to choosing \(\lambda > 0\) large enough,
\[
\int_0^\infty v_M(t) V(B_t) \frac{d}{dt} \left( -\varphi^p_\lambda(t) \right) dt < 0,
\]
which contradicts (18). We have thus proven the validity of (3).

Remark 4. It should be noted that, in order to prove the isometry with \(\mathbb{R}^m\), we can merely use the existence of a single minimizing function \(\varphi_\lambda\). Indeed, assume \(C_M = K(m, p)\). Since \(v_M(t) \geq 0\) and the selected function \(-\varphi^p_\lambda(t)\) is increasing, from (18) we immediately deduce that \(v_M(t)\) vanishes identically. The presence of an entire family \(\varphi_\lambda\) of minimizers well behaving with respect to the parameter \(\lambda\) is actually needed to reach the general volume estimate and the consequent diffeomorphism with \(\mathbb{R}^m\). On the other hand, as Ledoux pointed out, the original argument is oriented towards the conjecture that a sharp Euclidean Sobolev inequality, without any curvature assumption, implies a sharp Euclidean volume lower bound.

3. The case of manifolds with asymptotically non-negative curvature

This section is devoted to a proof of Theorem 2. To this end, we follow exactly the strategy we used to prove Theorem 1. Clearly, this time we have to take into account the (small) perturbations of (8) introduced by the negative curvature.
Proof of Theorem 1. Let $h \in C^2([0, +\infty))$ be the solution of the problem
\[
\begin{align*}
 h''(t) - G(t)h(t) &= 0, \\
 h(0) &= 0, \\
 h'(0) &= 1,
\end{align*}
\]
and consider the $m$-dimensional model manifold $M_h$ defined as $M_h := (\mathbb{R} \times S^{m-1}, ds^2 + h^2(s)d\theta^2)$, where $d\theta^2$ is the standard metric on $S^{m-1}$. We shall use an index $'h$ to denote objects and quantities referred to $M_h$. Thus, we denote by $B^h_t$ and $\partial B^h_t$ the geodesic ball and sphere of radius $t > 0$ in $M_h$. Moreover we introduce the family of functions $\phi_{\lambda,h} : M_h \to \mathbb{R}$ defined by $\phi_{\lambda,h}((s,\theta)) := \varphi_{\lambda}(s)$. For later purposes, we recall that, [13], [10],
\[
\begin{align*}
 V(B^h_t) &\geq V(B_t), \\
 t &\geq 0.
\end{align*}
\]
Furthermore, we observe that, according to the Bishop-Gromov comparison theorem and its generalizations, [3], [10], $A(\partial B_t) / A(\partial B^h_t)$ is a decreasing function of $t > 0$ and the following relations hold
\[
\begin{align*}
 A(\partial B_t) &\leq A(\partial B^h_t) \leq e^{b(m-1)} A(\partial B^h_t), \\
 V(B_t) &\leq V(B^h_t) \leq e^{bn} V(B^h_t).
\end{align*}
\]
By the co-area formula, these imply
\[
\begin{align*}
 \int_M \hat{\phi}_{\lambda,h}^{*} d\text{vol} &\leq \int_M \phi_{\lambda,h}^{*} d\text{vol}_h \leq e^{b(m-1)}, \\
 |\nabla \hat{\phi}_{\lambda}| &\in L^p(M), \\
 r(x)^{-1}\hat{\phi}_{\lambda}|\nabla \hat{\phi}_{\lambda}|^{p-1} &\in L^1(M), \\
 \int_{B_R} \hat{\phi}_{\lambda}|\nabla \hat{\phi}_{\lambda}|^{p-1} d\text{vol} &\to 0, \text{ as } R \to +\infty.
\end{align*}
\]
Here $d\text{vol}_h$ stands for the Riemannian measure on $M_h$. We need also to recall from [2] that the validity of (4) implies that there exists a (small) constant $\gamma = \gamma(m,p,C_M) > 0$ (depending continuously on $C_M$) such that
\[
V(B_t) \geq \gamma V(B^h_t).
\]
Now, by Laplacian comparison, assumption (4) yields
\[
\Delta r \leq \frac{(m-1)e^b}{r}
\]
pointwise on $M \setminus \text{cut}(o)$ and weakly on all of $M$. This means that
\[
- \int_M \langle \nabla r, \nabla \eta \rangle d\text{vol} \leq \int_M \eta \frac{(m-1)e^b}{r} d\text{vol},
\]
for all $0 \leq \eta \in W^{1,2}_c(M)$. Let $0 \leq \xi \in C^\infty_c(M)$ to be chosen later and apply (26) with $\eta$ defined in (42) thus obtaining
\[
\begin{align*}
 \int_M \phi_{\lambda}(r) - \phi_{\lambda}^{*}(r)|^{p-2} \left\langle \nabla r, \nabla \left( \xi \hat{\phi}_{\lambda} \right) \right\rangle d\text{vol} &\leq - \int_M |\phi_{\lambda}(r)|^{p-2} \left[ (p-1)\phi_{\lambda}''(r) + \frac{(m-1)e^b}{r} \phi_{\lambda}'(r) \right] \left( \xi \hat{\phi}_{\lambda} \right) d\text{vol}.
\end{align*}
\]
Whence, inserting (14) gives

\[(28)\]
\[-\int_M K(m, p)^{-p} \xi \hat{\phi}^p \lambda d\text{vol} + \int_M \lambda \hat{\phi} \lambda' \hat{\phi} \lambda' \hat{\phi}^p (r) |\hat{\phi} \lambda' \hat{\phi} \lambda' (r) |^p - 2 (m - 1) (e^b - 1) \xi d\text{vol}\]
\[\leq -\int_M \xi \left| \nabla \hat{\phi} \lambda \right|^p d\text{vol} - \int_M \hat{\phi} \lambda \hat{\phi} \lambda' \hat{\phi} \lambda' \hat{\phi}^p (r) |\hat{\phi} \lambda' \hat{\phi} \lambda' (r) |^p - 2 \langle \nabla r, \nabla \xi \rangle d\text{vol}\]
\[\leq -\int_M \xi \left| \nabla \hat{\phi} \lambda \right|^p d\text{vol} - \int_M \hat{\phi} \lambda \hat{\phi} \lambda' \hat{\phi} \lambda' \hat{\phi}^p (r) |\hat{\phi} \lambda' \hat{\phi} \lambda' (r) |^p - 2 |\nabla \xi| d\text{vol}.\]

Now choose \(\xi = \xi_R\) such that \(\xi \equiv 1\) on \(B_R\), \(\xi \equiv 0\) on \(M \setminus B_2R\) and \(|\nabla \xi| < 2/R\). Then, taking the limits as \(R \to +\infty\) in (28) and recalling (24) we obtain

\[\int_M \left| \nabla \hat{\phi} \lambda \right|^p d\text{vol} - K(m, p)^{-p} \int_M \hat{\phi}^p \lambda d\text{vol}\]
\[\leq (m - 1) \left( \frac{m - p}{p - 1} \right)^{p - 1} \beta^p (e^b - 1) \lambda^{\frac{m - p}{p - 1}} \int_M (\lambda + r^{\frac{p}{p - 1}})^{-(m - 1)} d\text{vol},\]

proving that

\[(29)\]
\[\frac{\int_M \left| \nabla \hat{\phi} \lambda \right|^p}{\int_M \hat{\phi}^p \lambda} \leq K(m, p)^{-p} + C_1,\]

where we have set

\[(30)\]
\[C_1(m, p, \lambda, b) := (m - 1) \left( \frac{m - p}{p - 1} \right)^{p - 1} \beta^p (e^b - 1) \lambda^{\frac{m - p}{p - 1}} \int_M (\lambda + r^{\frac{p}{p - 1}})^{-(m - 1)} d\text{vol} \lambda \int_M (\lambda + r^{\frac{p}{p - 1}})^{-(m - 1)} d\text{vol}.\]

By (23) and computing explicitly the integrals on \(\mathbb{R}^m\), we get

\[\int_M (\lambda + r^{\frac{p}{p - 1}})^{-(m - 1)} d\text{vol} \leq \int_0^\infty \frac{A(\partial \mathbb{B}_1) e^{b(m - 1)}}{(\lambda + t^{\frac{p}{p - 1}})^{m - 1}} dt = e^{b(m - 1)} A(\partial \mathbb{B}_1) \lambda^{\frac{m - p}{p}} \Gamma(m - \frac{m}{p}) \Gamma(m - 1),\]

where \(\Gamma\) denotes the Euler Gamma function. On the other hand,

\[(31)\]
\[\frac{V(B_t)}{(\lambda + t^{\frac{p}{p - 1}})^m} \leq \frac{V(\mathbb{B}_t^1)}{(\lambda + t^{\frac{p}{p - 1}})^m} \leq e^{bm} \frac{V(\mathbb{B}_t^1)}{(\lambda + t^{\frac{p}{p - 1}})^m} \to 0,\]

as \(t \to \infty\). Therefore, we can integrate by parts using the co-area formula, apply (25), integrate by parts again and compute explicitly the integrals on
\[ \int_{M} (\lambda + r^\frac{p}{m-1})^{-m} d\vol = \int_{0}^{\infty} V(B_t) \left( -\frac{d}{dt} (\lambda + t^\frac{p}{m-1})^{-m} \right) dt \]
\[ \geq \gamma \int_{0}^{\infty} V(B_t) \left( -\frac{d}{dt} (\lambda + t^\frac{p}{m-1})^{-m} \right) dt \]
\[ = \gamma \int_{0}^{\infty} A(\partial B_t) \frac{d}{dt} (\lambda + t^\frac{p}{m-1})^{-m} dt \]
\[ = \gamma A(\partial B_1) \lambda^{-\frac{m}{p}} \frac{\Gamma(m - \frac{m}{p}) \Gamma(\frac{m}{p})}{\Gamma(m)}. \]

Inserting into (30) and (29), it follows that
\[ \int_{M} \left| \nabla \hat{\phi}_{\lambda} \right|^p \leq K(m, p)^{-p} + C_2, \]
where
\[ C_2(m, p, b, C_M) := \frac{(m-1)^2}{m-p} \left( \frac{m-p}{p-1} \right)^{p-1} \beta^{-\frac{m^2}{m-p}} (e^b - 1) \frac{e^{b(m-1)}}{\gamma}. \]

On the other hand, because of (24), we can use \( \hat{\phi}_{\lambda} \) into (11) and get
\[ \int_{M} \left| \nabla \hat{\phi}_{\lambda} \right|^p \left( \int_{M} \hat{\phi}_{\lambda} \frac{\partial}{\partial \lambda} \right)^{\frac{p}{p}} \geq C^{-\frac{p}{p}}. \]

Combining (32) and (33) we obtain
\[ \int_{M} \hat{\phi}_{\lambda}^{\frac{p}{p}} \geq C_3^{-1}, \]
with
\[ C_3(m, p, b, C_M) := \left[ \left( \frac{C_M}{K(m, p)} \right)^p + C_M C_2 \right]^{m/p}. \]

From this latter, using (24), (31), the co-area formula and integrating by parts, it follows that
\[ 0 \leq C_3 e^{b(m-1)} \int_{M} \hat{\phi}_{\lambda}^{\frac{p}{p}} d\vol - \int_{M} \hat{\phi}_{\lambda,h}^{\frac{p}{p}} d\vol_h \]
\[ = \int_{0}^{\infty} v_{M,h}(t) V(B_t) \frac{d}{dt} \left( -\hat{\phi}_{\lambda}^{\frac{p}{p}} (t) \right) dt \]
where, by Bishop-Gromov, the function
\[ v_{M,h}(t) := \left[ C_3 e^{b(m-1)} \frac{V(B_t)}{V(B_{\hat{B}_t})} - 1 \right] \]
is non-increasing. In view of (22), in order to prove (6), it’s enough to show that \( \lim_{t \to \infty} v_{M,h}(t) \geq 0 \). By contradiction, suppose there exist positive
constants $\epsilon$ and $T$ such that $v_{M,h}(t) \leq -\epsilon$ for all $t \geq T$. In this assumption, $T_0 := \sup \{ t < T : v_{M,h}(t) \geq 0 \}$ is well defined and $0 < T_0 < T$. Then

\begin{equation}
\int_0^\infty v_{M,h}(t) V(\mathbb{B}_t^h) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt \\
\leq v_{M,h}(0) \int_0^{T_0} V(\mathbb{B}_t^h) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt \\
- \epsilon \int_T^\infty V(\mathbb{B}_t^h) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt.
\end{equation}

Observe that

\begin{equation}
\lim_{\lambda \to +\infty} \int_0^{T_0} V(\mathbb{B}_t^h) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt \\
\leq e^{b_{m}} \lim_{\lambda \to +\infty} \int_0^{T_0} V(\mathbb{B}_t) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt = 0.
\end{equation}

On the other hand, using (22) and the co-area formula, we have

\begin{equation}
\int_0^\infty V(\mathbb{B}_t^h) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt \\
\geq \int_0^\infty V(\mathbb{B}_t) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt \\
\geq \int_0^\infty A(\partial \mathbb{B}_t) \varphi^p_{s\lambda}(t) dt \\
= \int_{\mathbb{R}^m} \varphi^p_{s\lambda}(x) dx = 1,
\end{equation}

for all $\lambda > 0$. Therefore, by (36),

\begin{equation}
\lim_{\lambda \to +\infty} \int_T^\infty V(\mathbb{B}_t^h) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt \geq 1.
\end{equation}

Inserting (36) and (37) into (35) we conclude that, up to choosing $\lambda > 0$ large enough,

\begin{equation}
\int_0^\infty v_{M,h}(t) V(\mathbb{B}_t^h) \frac{d}{dt} \left( -\varphi^p_{s\lambda}(t) \right) dt < 0,
\end{equation}

which contradicts (34). Setting $\hat{C}(m,p,C_M,b) := C_{3}^{-1} e^{-b(m-1)}$, we have thus proven the validity of (6). □

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Dipartimento di Fisica e Matematica, Università dell’Insubria - Como, via Valleggio 11, I-22100 Como, Italy.

E-mail address: stefano.pigola@uninsubria.it

Dipartimento di Matematica, Università degli Studi di Milano, via Saldini 50, I-20133 Milano, Italy.

E-mail address: giona.veronelli@unimi.it