SPECTRAL PROPERTIES OF DYNAMICAL LOCALIZATION
FOR SCHRÖDINGER OPERATORS

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Abstract. We investigate the equivalence between dynamical localization and localization properties of eigenfunctions of Schrödinger Hamiltonians. We introduce three classes of equivalent properties and study the relationships between them. These relationships are shown to be optimal thanks to counter examples.

1. Introduction

In this note we investigate the equivalence between dynamical localization and localization properties of eigenfunctions for self-adjoint operators on a Hilbert spaces $\mathcal{H}$. Although the analysis holds in greater generality we are mainly interested in the particular cases where $\mathcal{H} = L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$. We shall nevertheless extend the discussion to operators on more general graphs. Our motivation comes from ergodic Schrödinger operators and more precisely from random and quasi-periodic Schrödinger operators, where dynamical localization has been proved, that is the non spreading of wave-packets under the time evolution coming from the Schrödinger equation [A, GDB, JJ, G, DS, GK2, GJ, BJ, GK3].

Although in the context of Anderson models, localization has been interpreted as pure point spectrum with exponentially localized eigenfunctions, it is by now well established that the latter is not sufficient to ensure dynamical localization, even with a uniform finite localization length, so that Del Rio, Jitomirskaya, Last and Simon raised the following natural question: what is localization? [DeRJLS1, DeRJLS2, GKT]. Actually even a single energy can be responsible for a nontrivial transport [JSS, DLS].

To go beyond Anderson localization, stronger forms of localization properties have been introduced in order to derive dynamical localization [DeRJLS1, DeRJLS2, G, GK2]. Note that if an eigenfunction $\phi$ decays as $|\phi(x)| \leq C_\phi e^{-\sigma|x-x_0|}$, then we can only conclude that $|\phi(x)| \leq \frac{1}{2}$ if $|x-x_0| \geq \frac{1}{\sigma} \log(2C_\phi)$, suggesting that not only the localization length $\frac{1}{\sigma}$ is of importance, but that the constant $C_\phi$ also matters. This trivial observation, combined to the fact that when dense point spectrum is observed a given non trivial wave-packet contains an infinite number of eigenfunctions, suggests that a better control on the exponential decay of eigenfunctions is required if we want to go beyond pure spectral results. In particular such a better control should ensure summability of the contributions of a (possible) infinite number of eigenfunctions. Two families of localization properties have been introduced in the literature which both solve this problem: semi-uniformly localized eigenfunctions (SULE) where the constant $C_\phi$ is explicit in the

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center \( x \), \cite{DeRJLS1,DeRJLS2} and \textit{semi-uniform decay of eigenfunction correlations} (SUDEC) where the two-sites eigenfunction correlation function is controlled in a summable way in energy \cite{G, GK2}. (SULE) and (SUDEC) properties are shown to be equivalent in great generality and to imply dynamical localization.

Besides the physical issue of controlling the time evolution of wave-packets, dynamical localization (DL) and the properties (SULE) and (SUDEC) have been shown to play a crucial role in the mathematical proof of several phenomenon of physical interest, like Mott formula \cite{KLM}, the quantum Hall effect \cite{H, Be} (including the quantification of the Hall conductance \cite{BES}, the existence of plateaux due to localized states \cite{AG, GKS1, GKS2}, the validity of the Kubo formula \cite{BoGKS}, the equality of the bulk / edge conductances when the Fermi level lies within a region of localization \cite{EGS, Ta}, the regularization of the edge conductance \cite{CG, EGS}). Also, these properties turn out to be a key ingredient in order to get relevant informations about the statistics of the eigenvalues of the Anderson model: finite multiplicity \cite{GK2, GK3}, simplicity of the spectrum \cite{KlM}, Poisson statistics \cite{M, GK11, GK12}, asymptotic eigenvalues ergodicity \cite{Klo}, level spacings statistics \cite{GK11, GK12}.

In this article, we come back to these properties that have been established and used in the mathematical physics literature over the past 20 years or so. We show that they can be gathered in three classes of equivalent properties and we study the relationships between them. The first class corresponds to dynamical localization, the second one to (SULE) and (SUDEC) for a basis of eigenvectors, and the third one to a stronger form of (SULE) and (SUDEC) where these properties hold for all vectors in the range of the eigenprojector (SULE+) and (SUDEC+). This last strong form of localization actually implies finite multiplicity of the eigenvalues. It was commonly believed that localization properties of eigenfunctions like (SULE) or (SUDEC) were stronger notions of localization than the non spreading of wave-packets (DL). However, the results available in \cite{DeRJLS2, T} indicate that this naive picture is not quite right and that detailed informations on the decay of eigenfunctions can be derived from the boundeness of moments of wave-packets. In this note, we extend the results of \cite{DeRJLS2, T} and present a clean picture of the situation. In particular, it solves an open question raised in \cite{DeRJLS2} about the equivalence between (DL) and (SULE), and the role payed by the multiplicity of the eigenvalues. We further extend the analysis to graphs or trees with moderate growth.

The paper is organized as follows. In section 2 we introduce the (DL), (SULE) and (SUDEC) properties and state our main results. In Section 3 we present the details of the proofs. In Section 4 we provide counter examples, showing that our results are optimal. Appendix A contains the proof of technical lemma, and in Appendix B we extend the first result of Section 2 to the random case.

2. Main results

We consider a self-adjoint operator \( H \) on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^d) \). The case \( \ell(\mathbb{Z}^d) \) is slightly simpler. At the end of this section we extend the results to graphs.

Given \( x \in \mathbb{R}^d \), we set \( |x| := \max\{|x_1|, |x_2|, \ldots, |x_d|\} \). We use \( |X_u| \) to denote the operator given by the multiplication by the function \( |x - u| \). By \( \Lambda_L(x) \) we denote the open box centered at \( x \in \mathbb{Z}^d \) with length side \( L > 0 \) and we write
\( \chi_{x,t} \) for its the characteristic function and set \( \chi_x := \chi_{x,1} \). Given an open interval \( I \subset \mathbb{R} \), we consider \( C_{c, +}^\infty (I) \) is the class of nonnegative real valued functions infinitely differentiable with compact support contained in \( I \). The notation \( \|A\|_2 \) corresponds to the Hilbert-Schmidt norm of the operator \( A \). We set \( P_E := \chi_E(H) \) the spectral projection associated to \( E \in \mathbb{R} \).

For a given \( \sigma > 0 \) and \( \zeta \in (0, 1] \), we introduce

\[
M_u(\sigma, \zeta, \mathcal{X}, t) := \left\| e^{\sigma |X_u|} e^{-itH} \mathcal{X}(H) \chi_u \right\|_2^2
\]

\[
= \text{tr} \{ \chi_u e^{itH} \mathcal{X}(H) e^{\sigma |X_u|} e^{-itH} \mathcal{X}(H) \chi_u \},
\]

the \((\sigma, \zeta)\)-subexponential moment at time \( t \) for the time evolution, initially localized near \( u \in \mathbb{Z}^d \) and localized in energy by the smooth function \( \mathcal{X} \in C_{c, +}(I) \).

The following theorem generalizes the main result of \([1]\).

Theorem 2.1. Let \( I \subset \sigma(H) \) be an interval and assume that \( H \) has pure point spectrum in \( I \). The following properties are equivalent.

(i) There exist \( \sigma > 0 \), \( \zeta \in (0, 1] \) so that for any \( \epsilon > 0 \), \( u \in \mathbb{Z}^d \) and \( \mathcal{X} \in C_{0,+}^\infty(I) \), there is a constant \( C_{\sigma, \zeta, \epsilon, \mathcal{X}} < \infty \), so that

\[
\sup_T M_u(\sigma, \zeta, \mathcal{X}, t) := \sup_T \frac{1}{T} \int_0^T M_u(\sigma, \zeta, \mathcal{X}, t) dt \leq C_{\sigma, \zeta, \epsilon, \mathcal{X}} \epsilon^{\|u\|_\infty}.
\]

(ii) There exist \( \sigma > 0 \), \( \zeta \in (0, 1] \) so that for any \( \epsilon > 0 \), \( u \in \mathbb{Z}^d \) and \( \mathcal{X} \in C_{0,+}^\infty(I) \), there is a constant \( C_{\sigma, \zeta, \epsilon, \mathcal{X}} < \infty \), such that

\[
\sup_T \frac{1}{T} \int_0^\infty e^{-\epsilon t} M_u(\sigma, \zeta, \mathcal{X}, t) dt \leq C_{\sigma, \zeta, \epsilon, \mathcal{X}} \epsilon^{\|u\|_\infty}.
\]

(iii) There exist \( \sigma > 0 \), \( \zeta \in (0, 1] \) so that for any \( \epsilon > 0 \), \( u \in \mathbb{Z}^d \) and \( \mathcal{X} \in C_{0,+}^\infty(I) \), there is a constant \( C_{\sigma, \zeta, \epsilon, \mathcal{X}} < \infty \), such that

\[
\sup_T M_u(\sigma, \zeta, \mathcal{X}, t) \leq C_{\sigma, \zeta, \epsilon, \mathcal{X}} \epsilon^{\|u\|_\infty}.
\]

(iv) There exist \( \zeta \in (0, 1] \), \( \sigma > 0 \) such that for all \( \epsilon > 0 \) and for any \( \mathcal{X} \in C_{0,+}^\infty(I) \), there is a constant \( C_{\zeta, \sigma, \epsilon, \mathcal{X}} < \infty \), so that

\[
\sup_T \| \chi_x e^{-itH} \mathcal{X}(H) \chi_u \|_2 \leq C_{\zeta, \sigma, \epsilon, \mathcal{X}} \epsilon^{\|u\|_\infty} e^{-\sigma |x-u| |\zeta|} \text{ for all } x, u \in \mathbb{Z}^d.
\]

(v) There exist \( \zeta \in (0, 1] \), \( \sigma > 0 \) such that for all \( \epsilon > 0 \) and for any \( \mathcal{X} \in C_{0,+}^\infty(I) \), there is a constant \( C_{\zeta, \sigma, \epsilon, \mathcal{X}} < \infty \), so that

\[
\sup_E \mathcal{X}(E) \| \chi_x P_E \chi_u \|_2 \leq C_{\zeta, \sigma, \epsilon, \mathcal{X}} \epsilon^{\|u\|_\infty} e^{-\sigma |x-u| |\zeta|} \text{ for all } x, u \in \mathbb{Z}^d.
\]

If \( H \) satisfies one of these properties, we say that \( H \) exhibits (subexponential) dynamical localization in \( I \). When \( \zeta = 1 \) we may talk about exponential dynamical localization.

Clearly properties (iii) and (iv) are equivalent, and (iii) \( \implies \) (ii) \( \implies \) (i). It will remain to show that (i) \( \implies \) (v) \( \implies \) (iv), which is the heart of Theorem 2.1. We point out that the underlying geometry of the Hilbert space only plays a role in the proof of (v) \( \implies \) (iv).
Remark 2.2. (i) Properties (iv) and (v) of Theorem 2.1 have been introduced in [DeRJLS2], and are respectively called by Semi-Uniform Dynamical Localization (SULD) and Semi-Uniform Localized Projections (SULP).

(ii) Theorem 2.1 actually shows that dynamical localization and time averaged dynamical localization are equivalent.

(iii) Theorem 2.1 generalizes the result of [1] in the sense that it provides the decay of the kernel of $e^{-itH}$.

(iv) By the RAGE theorem, the bound (2.3) implies that the spectrum of $H$ is pure point in $I$. If the multiplicity is finite, Theorem 2.1 will show that much more holds true.

(v) If one considers polynomial moments in (2.1) rather than (sub)exponential ones, then Theorem 2.1 still holds with polynomial decay in (iv) and (v).

We turn to the description of the decay properties of the eigenfunctions of $H$ and we start with some notations.

Let $\mathcal{E} \subset I$ be a collection of eigenvalues of $H$ that we assume to be nonempty ($\mathcal{E}$ may be infinite). Set $P_\mathcal{E} = \sum_{E \in \mathcal{E}} P_E$ and write $\mathcal{H}_E = P_E\mathcal{H}$ and $\mathcal{H}_E = P_E\mathcal{H}$.

We fix $\kappa > \frac{d}{2}$, and define $T$ as the operator on $\mathcal{H}$ given by multiplication by the function $T(x) = \langle x \rangle_\kappa$ for $x \in \mathbb{R}^d$, with $\langle x \rangle := \sqrt{1 + |x|^2}$. We set

$$\alpha_E := \text{tr} \{ T^{-1} P_E T^{-1} \} = \| T^{-1} P_E \|_2^2 \leq \text{tr} P_E. \quad (2.8)$$

Given a unit vector $\phi \in \mathcal{H}$, we denote by $P_\phi$ the rank one projection $P_\phi = |\phi\rangle \langle \phi|$, and let

$$\alpha_\phi := \text{tr} \{ T^{-1} P_\phi T^{-1} \} = \| T^{-1} P_\phi \|_2^2 = \| T^{-1} \phi \|_2^2 \leq 1. \quad (2.9)$$

If $\{ \phi_n \}_{n=1}^{\infty}$ is an orthonormal basis of $P_E\mathcal{H}$, with $N_E = \text{tr} P_E \leq \infty$, then $\sum_{n=1}^{N_E} \alpha_{\phi_n} = \alpha_E$. We assume the following finiteness condition

$$\alpha_{\mathcal{H},E} := \sum_{E \in \mathcal{E}} \alpha_E = \text{tr} \{ T^{-1} P_E T^{-1} \} < \infty. \quad (2.10)$$

If $\mathcal{E}$ is compact, condition (2.10) is known to hold for a large variety of Schrödinger and generalized Schrödinger operators [KKS, GK2].

Theorem 2.3. Let $\mathcal{G}_E = \{ \phi_n \}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}_E$, and assume (2.10). Then the following properties are equivalent:

(i) Summable Uniform Decays of Eigenfunction Correlations on $\mathcal{G}_E$ (SUDEC): there exist $\sigma > 0$, $\zeta \in (0,1]$ such that for all $\epsilon > 0$ and all $\phi_n \in \mathcal{G}_E$ and $x,u \in \mathbb{Z}^d$, we have

$$\| \chi_x \phi_n \chi_u \phi_n \|_2 = \| \chi_x \phi_n \| \| \chi_u \phi_n \| \leq C_{\zeta,\sigma,\epsilon} e^{\epsilon |u|^\zeta} e^{-\sigma |x-u|^\zeta}. \quad (2.11)$$

(i') There exist $\sigma > 0$, $\zeta \in (0,1]$ such that for all $\epsilon > 0$ and all $\phi_n \in \mathcal{G}_E$ and $x,u \in \mathbb{Z}^d$,

$$\| \chi_x \phi_n \| \| \chi_u \phi_n \| \leq C_{\zeta,\sigma,\epsilon} e^{\epsilon |u|^\zeta} e^{-\sigma |x-u|^\zeta}. \quad (2.12)$$

(ii) Semi Uniformly Localized Eigenfunctions on $\mathcal{G}_E$ (SULE): there exist $\sigma > 0$, $\zeta \in (0,1]$ such that for each $\phi_n \in \mathcal{G}_E$, we can find $x_{\phi_n} \in \mathbb{Z}^d$ so that for all $\epsilon > 0$ and $x \in \mathbb{Z}^d$, we have

$$\| \chi_x \phi_n \| \leq C_{\zeta,\sigma,\epsilon} e^{\epsilon |x_{\phi_n}|^\zeta} e^{-\sigma |x-x_{\phi_n}|^\zeta}. \quad (2.13)$$

Moreover, if (ii) holds, we may order the centers of localization $x_{\phi_n}$ in such a way that $|x_{\phi_n}| \geq Cn^{1/2\kappa}$. 

The (SULE) property has been introduced in [DeRJLS2], while the (SUDEC) property has been introduced in [G] and further developed in [GK2]. We single out (i') for it may look more natural to the reader. However, while (i) is shown to imply quite readily dynamical localization, using (i') would require a more involved analysis.

**Remark 2.4.** (i) Notice that if (2.11) and (2.13) are respectively replaced by
\[ \| x \phi_n \| \| x \phi_m \| \leq C_{\zeta', \sigma, \epsilon} \epsilon^{|u|} e^{-\sigma|x-u|^\zeta}, \] (2.14)
and
\[ \| x \phi_n \| \leq C_{\zeta', \zeta, \sigma, \epsilon} e^{|x_n|} e^{-\sigma|x-x_n|^\zeta}, \] (2.15)
with \( \zeta' > \zeta \) then the equivalence is lost. However (SUDEC), that is (2.14), is still strong enough to imply dynamical localization. This is not the case for (2.15), because of the lack of the quantity \( \alpha_{\phi_n} \). This situation is not as exotic as one may think! This is exactly what happens for random Schrödinger operators with singular measure (including Bernoulli), see [GK3].

Until now, the multiplicity of eigenvalues may be arbitrary. Now, we introduce a third class of properties with corresponds to stronger version of (SUDEC) and (SULE) and that will forces multiplicity to be finite. Our motivation comes from the theory of Anderson localization where eigenfunctions are shown to exhibit stronger localization properties than (SULE) or (SUDEC). We describe them in the following theorem.

**Theorem 2.5.** Assume (2.10). Let \( E \in \mathcal{E} \) be given and let \( \mathcal{G}_E = \{ \phi_n \} \) be an orthonormal basis of \( \mathcal{H}_E \). Then the following properties are equivalent:

(i) There exist \( \sigma > 0, \zeta \in [0,1] \) such that for any \( \epsilon > 0 \), for all \( \phi_n, \phi_m \in \mathcal{G}_E \) and for all \( x, u \in \mathbb{Z}^d \),
\[ \| x \phi_n \| \| x \phi_m \| \leq C_{\zeta, \sigma, \epsilon} \epsilon^{|u|} e^{-\sigma|x-u|^\zeta}. \] (2.16)

(ii) There exist \( \sigma > 0, \zeta \in [0,1] \) such that for any \( \epsilon > 0 \), for all \( \phi \in \text{Span} \mathcal{G}_E \) and for all \( x, u \in \mathbb{Z}^d \),
\[ \| x \phi \| \| x \phi \| \leq C_{\zeta, \sigma, \epsilon} \epsilon^{|u|} e^{-\sigma|x-u|^\zeta}. \] (2.17)

(iii) There exist \( \sigma > 0, \zeta \in [0,1] \) such that for any \( \epsilon > 0 \), for all \( \phi, \psi \in \text{Span} \mathcal{G}_E \) and for all \( x, u \in \mathbb{Z}^d \),
\[ \| x \phi \| \| x \psi \| \leq C_{\zeta, \sigma, \epsilon} \epsilon^{|u|} e^{-\sigma|x-u|^\zeta}. \] (2.18)

(iv) There exist \( \sigma > 0, \zeta \in [0,1] \) such that for any \( \epsilon > 0 \), for all \( x, u \in \mathbb{Z}^d \),
\[ \| x \phi \| \| x \phi \| \leq C_{\zeta, \sigma, \epsilon} \epsilon^{|u|} e^{-\sigma|x-u|^\zeta}. \] (2.19)

(iv') property (i), (ii), (iii) or (iv) holds with \( \alpha_{\bullet} = 1, \bullet = \phi, \psi \) normalized vectors or \( \bullet = E \), as in (2.12).

(v) There is a common center of localization \( x_E \) for all \( \phi_n \in \mathcal{G}_E \) such that there are \( \sigma > 0, \zeta \in [0,1] \) so that for any \( \epsilon > 0 \), for all \( x \in \mathbb{Z}^d \),
\[ \| x \phi_n \| \leq C_{\zeta, \sigma, \epsilon} \epsilon^{|x_E|} e^{-\sigma|x-x_E|^\zeta}. \] (2.20)

(vi) There is a common center of localization \( x_E \) for all \( \phi \in \text{Span} \mathcal{G}_E \) such that there are \( \sigma > 0, \zeta \in [0,1] \) so that for any \( \epsilon > 0 \), for all \( x \in \mathbb{Z}^d \),
\[ \| x \phi \| \leq C_{\zeta, \sigma, \epsilon} \epsilon^{|x_E|} e^{-\sigma|x-x_E|^\zeta}. \] (2.21)
(vii) There exists \( x_E \in \mathbb{Z}^d \) such that there exist \( \sigma > 0, \zeta \in (0, 1] \) so that for any \( \epsilon > 0 \), for all \( x \in \mathbb{Z}^d \), we have
\[
\| \chi_x P_E \|_2 \leq C_{\zeta, \sigma, \epsilon} \sqrt{\alpha E} e^{\epsilon |x_E|} e^{-\sigma |x - x_E|}.
\]
(2.22)

We denote by (SUDEC+) any of properties (v), (vi), (vii), and by (SULE+) any of properties (v), (vi), (vii).

(vii’) property (v), (vi) or (vii) holds with \( \alpha_\bullet = 1, \bullet = \phi, \psi \) normalized vectors or \( \bullet = E \).

If one of the above properties holds then the eigenvalues have finite multiplicity and in addition,
\[
\text{tr} P_E \leq C_{\zeta, \sigma} \alpha_E \langle x_E \rangle^2 \kappa,
\]
(2.23)
and
\[
\tilde{N}_L := \# \{ E \in \mathcal{E}; |x_E| \leq L \} \leq C_{\zeta, \sigma} \alpha_E \text{H, E L}^2 \kappa \text{ for all } L \geq 1,
\]
(2.24)
where \( x_E \) is as in (vii).

Remark 2.6. (i) The bootstrap Multiscale Analysis of [GK1] yields (SULE+) and (SUDEC+). See also [GK2].

(ii) If (2.20) holds with \( \epsilon = 0 \), then the multiplicity is uniformly finite. This can be seen from Proposition 3.7, since (3.37) would hold with \( \delta = 0 \).

Next, notice that \( \| \chi_x P_E \chi_y \|_2 \leq \| \chi_x P_E \|_2 \| \chi_y P_E \|_2 \), so that (2.19) implies (2.7).

One way wonder whether fast decay of \( \| \chi_x P_E \chi_y \|_2 \) is equivalent to the one of \( \| \chi_x P_E \|_2 \| \chi_y P_E \|_2 \). Such a question was raised in [DeRJLS2]. In \( \ell^2(\mathbb{Z}^d) \), [DeRJLS2] proved the equivalence when the multiplicity is one (tr \( P_E = 1 \)), and [EGS] showed that if tr \( P_E < \infty \) and \( |\langle \delta_y, P_E \delta_x \rangle| \leq C e^{\epsilon(|x|+|y|)} e^{-\sigma|x-y|} \), then there exists a basis of \( \mathcal{H}_E \) with a (SUDEC) type property.

We summarize the relationships between the three classes of properties in the following optimal theorem. In particular it answers to [DeRJLS2]'s question about the equivalence between (DL) and (SULE), and the role played by the multiplicity (they were considering simple spectrum only).

**Theorem 2.7.** Assume (2.10). Then
(i) We have
\[
(SUDEC+) \Rightarrow (SULE) \Rightarrow (DL).
\]
(2.25)

(ii) Assume that tr \( P_E < \infty \) for any \( E \in \mathcal{E} \). Then
\[
(SUDEC) \iff (DL).
\]
(2.26)

(iii) There exist Schrödinger operators with eigenvalues of infinite multiplicity and for which (DL) holds but not (SULE/SUDEC). There exist Schrödinger operators for which (SULE/SUDEC) holds but not (SULE+/SUDEC+), as soon as eigenvalues are not simple. But (SULE/SUDEC) together with property (3.37) is equivalent to (SULE+/SUDEC+).

Remark 2.8. Of course, when the multiplicity is one, then (SULE/SUDEC) and (SULE+/SUDEC+) are the same. (SULE+/SUDEC+) provides a strong condition on the spatial repartition of the centers of localization described in Proposition 3.7 below. There is no reason for such a rigid condition on centers to hold in great generality, for eigenfunctions associated to a given eigenvalue may live far apart.
For instance, one may consider the Laplacian on a subgraph of $\mathbb{Z}^2$, for which there exist compactly supported eigenfunctions associated to the same eigenvalue and with disjoint supports.

However it is easy to see that (SULE/SUDEC) together with the property (3.37) implies (SULE+/SUDEC+).

As previously mentioned, these results remain valid in a general framework that we briefly outline. Let us consider an abstract separable Hilbert space $H$ equipped with a basis denoted by $\{e_n\}_{n \in \mathbb{N}}$ that we suppose to be orthonormal. Adopting notations of Section 2, we define the subexponential moment with parameters $\sigma$ and $\zeta$:

$$M_{e_u}(\sigma, \zeta, \mathcal{X}, t) := \sum_{n \geq 0} e^{\sigma n^\zeta} |\langle e^{-itH \mathcal{X}(H)} e_u, e_n \rangle_H|^2,$$

where $e_u \in H$ is an initial state and $\langle ., . \rangle_H$ denotes the inner product in $H$. Note that this corresponds to (2.1) with $\chi_u$ replaced by $\Pi e_u$ the rank one projection onto $e_u$. Theorem 2.1 is still valid in this context. Indeed, given $L > 0$ and $u \in \mathbb{N}$, we consider the ball $B_L(e_u) := \{e_n, |n - u| \leq L\}$. Notice that $\# B_L(e_u) \leq 2L + 1$ uniformly in $u$, so that Lemma 3.3 holds true with $d = 1$ in (3.13), which is the only place in the proof where the geometry plays a role.

However, we may object that we loose the physical interpretation of moments and of dynamical localization. From this point of view it is interesting to consider graphs as generalizations of the lattice $\mathbb{Z}^d$. Let $G$ be a graph with vertices $v \in V$, and set $H = \ell^2(V)$. Let $\{\delta_v\}_{v \in V}$ be the canonical basis of $\ell^2(V)$. We have a natural notion of distance $d$ in $V$: $d(u,v) = \inf \{\# p(u,v)\}$, where $p(u,v)$ is a path in $G$ joining $u$ and $v$ (if $G$ is a tree then there is only one such path, but $G$ may contain loops). We can thus define spheres $S_L(u) = \{v \in V; d(u,v) = L\}$ centered at $u \in G$ and of radius $L$.

We define $N_L(u) = \# S_L(u)$.

**Theorem 2.9.** Assume there exists $\beta \in [0, 1)$ such that

$$\sup_u N_L(u) \leq e^{L^\beta},$$

then Theorem 2.1 holds for $\zeta > \beta$.

The result thus still applies to graphs but with moderate growth. As example, rooted trees, as in [Br], satisfy to the growth condition (2.28). And random Schrödinger operators on such rooted trees are shown to exhibit dynamical localization [Br]. See also [Tau].

We turn to the (SUDEC) and (SULE) type properties. The geometry is further involved in the condition

$$\alpha_{H,E} = \sum_{u \in G} \langle \delta_u, T^{-1} P_e T^{-1} \delta_u \rangle < \infty,$$

where $T$ is now the operator given by the multiplication by $e^{\alpha |u|}$ for fixed $\alpha < \zeta$. We have

**Theorem 2.10.** Assume (2.28) for $\beta \in [0, 1)$ and (2.29) holds for $\alpha \in (0, 1)$, with $\beta < \alpha < \zeta$, then Theorem 2.3 and Theorem 2.5 hold.
3. Proofs

3.1. Dynamical localization. In this section, we prove Theorem 2.1 as a combination of the theorems below. Given \( u \in \mathbb{Z}^d \) we consider the function

\[
P_u(x, X) := \sup_k \mathcal{X}(E_k) \| \chi_x P_{E_k} \chi_u \|_2^2,
\]

(3.1)

and its corresponding moment

\[
L_u(\sigma, \zeta, X) := \sum_x e^{\sigma|x-u|^\zeta} P_u^2(x, X),
\]

(3.2)

for \( \sigma > 0, \zeta \in (0, 1] \) and where \( P_{E_k} \) denotes the eigenprojection associated to the eigenvalue \( E_k \). The role of the function \( P_u(x, X) \) above is to describe the decay of the eigenprojectors in terms of the subexponential moment (2.1), yielding directly (2.7).

**Theorem 3.1.** Fix \( \sigma > 0 \) and \( \zeta \in (0, 1] \). Then

\[
\liminf_{T \to \infty} M_u(\sigma, \zeta, X, T) \geq C_{\sigma, \zeta} L_u(\sigma, \zeta, X).
\]

(3.3)

for any \( X \in C_1^\infty(I) \) and all \( u \in \mathbb{Z}^d \). And thus

\[
P_u(x, X) \leq C_{\sigma, \zeta} \left( \liminf_{T \to \infty} M_u(\sigma, \zeta, X, T) \right)^{1/2} e^{-\frac{2}{\sigma} |x-u|^\zeta}.
\]

(3.4)

**Proof.** We first notice that

\[
M_u(\sigma, \zeta, X, T) \geq C_{\sigma, \zeta} \sum_{x \in \mathbb{Z}^d} e^{\sigma|x-u|^\zeta} \| \chi_x e^{-itH} \chi_u \|_2^2.
\]

(3.5)

For \( T > 0 \) and \( L \geq 1 \) we consider the finite volume time-averaged moment

\[
M_u^{LT}(\sigma, \zeta, X, T) := \frac{1}{T} \int_0^T \sum_{x \in \Lambda_L(u)} e^{\sigma|x-u|^\zeta} \| \chi_x e^{-itH} \chi_u \|_2^2 dt.
\]

The decomposition of the kernel over the eigenspaces allows us to write

\[
M_u^{LT}(\sigma, \zeta, X, T) = \sum_{k, k'} \mathcal{X}(E_k) \mathcal{X}(E_{k'}) \sum_{x \in \Lambda_L(u)} e^{\sigma|x-u|^\zeta} \operatorname{tr} \{ \chi_x P_{E_k} \chi_u P_{E_{k'}} \chi_x \} \left( \frac{1}{T} \int_0^T e^{-it(E_k - E_{k'})} dt \right),
\]

(3.6)

(3.7)

and a use of the dominated convergence theorem implies that

\[
\lim_{T \to \infty} M_u^{LT}(\sigma, \zeta, X, T) = \sum_k \sum_{x \in \Lambda_L(u)} \mathcal{X}^2(E_k) e^{\sigma|x-u|^\zeta} \| \chi_x P_{E_k} \chi_u \|_2^2.
\]

where we have used the fact that

\[
\frac{1}{T} \int_0^T e^{-it(E_k - E_{k'})} dt = \begin{cases} 1 & k = k' \\ \frac{e^{-iT(E_k - E_{k'})}}{-iT(E_k - E_{k'})} & k \neq k' \end{cases}.
\]
Since \( \liminf_{T \to \infty} \mathcal{M}_u(\sigma, \zeta, \mathcal{X}, T) \geq C_{\sigma, \zeta} \lim_{T \to \infty} \mathcal{M}_u^T(\sigma, \zeta, \mathcal{X}, T) \) and taking the limit when \( L \to \infty \), we deduce that
\[
\liminf_{T \to \infty} \mathcal{M}_u(\sigma, \zeta, \mathcal{X}, T) \geq C_{\sigma, \zeta} \sum_k \sum_{x \in \mathbb{Z}^d} \chi^2(E_k) e^{\sigma|x-u|} \| \chi x P_{E_k} \chi u \|_2^2 \geq C_{\sigma, \zeta} L_u(\sigma, \zeta, \mathcal{X}).
\]
(3.8)

As a consequence, (3.4) holds.

Theorem 3.2. Fix \( \sigma > 0 \) and \( \zeta \in (0, 1] \) and let \( \gamma \in (0, 1) \). Then
\[
\sup_t \| \chi x e^{-itH} \mathcal{X}(H) \chi u \|_2 \leq C_{\sigma, \zeta, d, \gamma, t} \mathcal{P}_u^{1-\gamma}(x, \mathcal{X}) \mathcal{L}_u^{\gamma/2}(\sigma, \zeta, \mathcal{X})
\]
for all \( x, u \in \mathbb{Z}^d \) and any function \( \mathcal{X} \in \mathcal{C}_{0, +}^2(I) \). In particular,
\[
\sup_t \| \chi x e^{-itH} \mathcal{X}(H) \chi u \|_2 \leq C_{\sigma, \zeta, d, \gamma, t} \liminf_{T \to \infty} \mathcal{M}_u(\sigma, \zeta, \mathcal{X}, T) \frac{1}{2} e^{-\frac{1}{(1-\gamma)(1-2\gamma)}} \| \sigma |x-u| \|_c.
\]
(3.10)

This relies partly on the following lemma which provides a bound on the number of elements contained in a box of size \( L \). Its proof is given in Appendix A.

Lemma 3.3. Fix \( \sigma > 0 \) and \( \zeta \in (0, 1] \). For \( k \in \mathbb{Z} \) and \( u \in \mathbb{Z}^d \), we set
\[
A_k(\sigma, \zeta, u) := \sum_x e^{\sigma|x-u|} \| \chi x P_{E_k} \chi u \|_2 \frac{\| \chi x P_{E_k} \chi u \|_2^2}{\| \chi u P_{E_k} \|_2^2}.
\]
(3.12)

Then
\[
N_{L, \zeta, \sigma, u} := \# \{ k \in \mathbb{Z}; E_k \in I, A_k(\sigma, \zeta, u) \leq L \} \leq C_{\sigma, \zeta, d} (\log L)^d \zeta \quad \text{for all} \quad L \in \mathbb{N},
\]
(3.13)

where \( C_{\sigma, \zeta, u} \) is a positive constant uniform in \( u \in \mathbb{Z}^d \).

In other terms, with new constant we can order \( A_k(\sigma, \zeta, u) \) increasingly so that
\[
A_k(\sigma, \zeta, u) \geq \exp(C_{\sigma, \zeta, d} k^{d/4}).
\]

Proof of Theorem 3.2. Write
\[
\sup_t \| \chi x e^{-itH} \mathcal{X}(H) \chi u \|_2 \leq \sum_{k; E_k \in I} \mathcal{X}(E_k) \| \chi x P_{E_k} \chi u \|_2 \\
\leq \mathcal{P}_u^{1-\gamma}(x, \mathcal{X}) \sum_{k; E_k \in I} \mathcal{X}(E_k) \| \chi x P_{E_k} \|_2^2 \| \chi u P_{E_k} \|_2^2.
\]
As in (11), we shall sacrifice some decay in space in order to recover the summability over \( k \). Given \( \sigma > 0 \), \( \zeta \in (0, 1] \), one has
\[
\mathcal{X}^2(E_k) \| \chi u P_{E_k} \|_2 A_k(\sigma, \zeta, u) = \sum_x e^{\sigma|x-u|} \mathcal{X}^2(E_k) \| \chi x P_{E_k} \chi u \|_2^2 \leq \mathcal{L}_u(\sigma, \zeta, \mathcal{X}).
\]
Thus
\[
\mathcal{X}(E_k) \| \chi u P_{E_k} \|_2 \leq A_k^{-1/2}(\sigma, \zeta, u) \mathcal{L}_u^{1/2}(\sigma, \zeta, \mathcal{X}),
\]
(3.14)

and
\[
\sup_t \| \chi x e^{-itH} \mathcal{X}(H) \chi u \|_2 \leq \mathcal{P}_u^{1-\gamma}(x, \mathcal{X}) \mathcal{L}_u^{\gamma/2}(\sigma, \zeta, \mathcal{X}) \sum_{k; E_k \in I} \| \chi x P_{E_k} \|_2^2 A_k^{-\gamma/2}(\sigma, \zeta, u).
\]
(3.15)
Hence

\[
\sum_{k \in \mathbb{E}_k \in I} \|\chi_x P_{E_k}\|_2^2 A_k^{-\gamma/2}(\sigma, \zeta, u) \leq \left( \sum_{k \in \mathbb{E}_k \in I} \|\chi_x P_{E_k}\|_2^2 \right)^{\gamma/2} \left( \sum_{k \in \mathbb{E}_k \in I} A_k^{-\gamma}(\sigma, \zeta, u) \right)^{(1-\gamma/2)}
\]


(3.16)

\[
= C_{\sigma, \zeta, d, \gamma, I} < \infty.
\]

Hence

\[
\sup_t \|\chi_x e^{-itH} \mathcal{X}(H)\chi_u\|_2 \leq C_{\sigma, \zeta, d, \gamma, I} P_1^{1-\gamma}(x, \mathcal{X}) \mathcal{E}_u^{\gamma/2}(\sigma, \zeta, \mathcal{X}).
\]

(3.17)

\[\square\]

**Proof of Theorem 2.1 (3.3)** shows that \((i) \Rightarrow (v)\), and \((3.10)\) that \((v) \Rightarrow (iv)\). \[\square\]

### 3.2. SULE, SUDEC

We now focus now on the second kind of criteria and we start with the proof of Theorem 2.3. It is a consequence of the theorem below which is the main technical result of this section. We may omit the index \(n\) and write \(\phi \in \mathcal{G}_E\) instead of \(\phi_n \in \mathcal{G}_E\). Similar to (2.13) and (2.11), we shall say that \(H\) verifies (SULE) if for any \(\mathcal{X}\) for some function \(f\) if these estimates are respectively replaced by

\[
\|\chi_x \phi\| \leq C_{\sigma, \zeta, f, f} e^{\|x\|\zeta} e^{-\sigma \|x\| \zeta}
\]

(3.18)

and

\[
\|\chi_x \phi\| \|\chi_u \phi\| \leq C_{\sigma, \zeta, f} e^{\|u\|\zeta} e^{-\sigma \|x-u\| \zeta}.
\]

(3.19)

**Theorem 3.4.** Let \(\mathcal{G}_E = \{\phi_n\}_{n \geq 1}\) be an orthonormal basis of \(\mathcal{H}_E\). Then the following properties are equivalent:

(i) there exists a nonnegative function such that for any \(\epsilon > 0\), \(f(s) \leq C_\epsilon e^{s^{-\gamma/2\epsilon}}\) for all \(0 < s \leq 1\) and for which \(H\) has (SUDEC) on \(\mathcal{G}_E\).

(ii) there exist \(\sigma > 0\), \(\zeta \in (0, 1]\) such that for any \(\epsilon > 0\)

\[
\|\chi_x \phi\| \|\chi_u \phi\| \leq C_{\sigma, \zeta, \epsilon} e^{\|u\|\zeta} e^{-\sigma \|x-u\| \zeta},
\]

(3.20)

for all \(\phi \in \mathcal{G}_E\) and all \(x, u \in \mathbb{Z}^d\).

(iii) \(H\) exhibits (SUDEC) on \(\mathcal{G}_E\).

(iv) For any nonnegative function such that for any \(\epsilon > 0\), \(f(s) \geq C_\epsilon e^{-s^{-\gamma/2\epsilon}}\) for all \(0 < s \leq 1\), \(H\) has (SUDEC) on \(\mathcal{G}_E\).

Recall \(\alpha_\phi \leq 1\). Obviously, \((iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)\). It remains to prove that \((i) \Rightarrow (iv)\). This will be a consequence of the next two lemmas.

**Lemma 3.5.** Let \(f : \mathbb{R}^+ \to \mathbb{R}^+\) be a function. If there exist \(\zeta \in (0, 1]\) and \(\sigma > 0\) such that for all \(\epsilon > 0\),

\[
\|\chi_x \phi\| \|\chi_u \phi\| \leq C_{\zeta, \sigma, \epsilon} e^{\|u\|\zeta} e^{-\sigma \|x-u\| \zeta},
\]

(3.21)

for all \(x, u \in \mathbb{Z}^d\) and any \(\phi \in \mathcal{G}_E\) then there is a new constant \(C_{\zeta, \sigma, \epsilon}\) so that for all \(x \in \mathbb{Z}^d\), we have

\[
\|\chi_x \phi\| \leq C_{\zeta, \sigma, \epsilon} \frac{1}{\sqrt{\alpha_\phi}} e^{\|x\|\zeta} e^{-\sigma \|x-x_\phi\| \zeta},
\]

(3.22)

where \(x_\phi\) maximizes \(x \mapsto \|\chi_x \phi\|\).
In particular, taking \( f(s) = s \) says that if (SUDEC) holds on \( \mathcal{G}_E \) then (SULE) holds on \( \mathcal{G}_E \) and with the same parameters \( \zeta \) and \( \sigma \).

This lemma tells us that if (SUDEC) \( f \) holds for a given function \( f \) then (SULE) \( g \) occurs where \( g : s \mapsto \frac{f(s)}{\sqrt{s}} \).

Proof. We set \( \tilde{\phi} = \frac{1}{\sqrt{\alpha_\phi}} \phi = \phi / \|T^{-1}\phi\| \) and we pick \( x_\phi \in \mathbb{Z}^d \) (not unique) such that

\[
\|\chi_{x_\phi} \tilde{\phi}\| = \max_{u \in \mathbb{Z}^d} \|\chi_u \tilde{\phi}\|. \tag{3.23}
\]

Since

\[
1 = \|T^{-1}\tilde{\phi}\|^2 = \sum_{u \in \mathbb{Z}^d} \|\chi_u T^{-1} \tilde{\phi}\|^2 \leq \|\chi_{x_\phi} \tilde{\phi}\|^2 \sum_{u \in \mathbb{Z}^d} \|\chi_u T^{-1}\tilde{\phi}\|^2 \leq C_d \|\chi_{x_\phi} \tilde{\phi}\|^2, \tag{3.24}
\]

we get

\[
\|\chi_{x_\phi} \tilde{\phi}\| \geq C_d^{-1/2}. \tag{3.25}
\]

It follows now from (3.24) that

\[
\|\chi_x \phi\| \leq C_d^{1/2} \frac{1}{\sqrt{\alpha_\phi}} \|\chi_x \phi\| \|\chi_{x_\phi} \tilde{\phi}\| \leq C_{d,\zeta,\sigma,\epsilon} e^{\epsilon |x_\phi|^\zeta} \epsilon^{-\sigma} |x-x_\phi|^\zeta, \tag{3.26}
\]

for all \( x \in \mathbb{Z}^d \).

Furthermore, we establish a control on \( \alpha_\phi \) in term of the center of localization \( x_\phi \) according to:

**Lemma 3.6.** Suppose that (SULE) \( f \) holds with some function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for all \( \epsilon > 0 \),

\[
f(s) \leq C_\epsilon e^{\epsilon s^{-\zeta/2\kappa}} \text{ for all } s \in [0,1]. \tag{3.27}
\]

Then there exists a constant \( C > 0 \) (independant of \( \mathcal{G}_E \)), so that

\[
\alpha_\phi \geq C (x_\phi)^{-2\kappa} \text{ for all } \phi \in \mathcal{G}_E. \tag{3.28}
\]

Proof. We note that from (3.18) we get

\[
\|\chi_{|x-x_\phi| \geq R \phi}\|^2 \leq C_{2,\zeta,\sigma,\epsilon} f^2(\alpha_\phi) \sum_{|x-x_\phi| \geq R} e^{2|\sigma|^\zeta} e^{-2\sigma |x-x_\phi|^\zeta} \leq \frac{1}{9}, \tag{3.29}
\]

if we take

\[
R \geq R_\phi := \left( \frac{\epsilon}{\sigma} \right)^{1/\zeta} |x_\phi| + \left( \frac{1}{\sigma} \log f(\alpha_\phi) + \frac{1}{\alpha_\phi} \log(3 C_{\zeta,\sigma,\epsilon}) \right)^{1/\zeta}. \tag{3.30}
\]

Since \( |x-x_\phi| \leq R_\phi \) implies that \( |x| \leq |x_\phi| + R_\phi \) and using (3.29) and (3.30), we have

\[
\alpha_\phi = \|T^{-1}\phi\|^2 \geq \sum_{x \in \Lambda_{R_\phi}(x_\phi)} \|\chi_x T^{-1}\phi\|^2 \geq \langle |x_\phi| + R_\phi \rangle^{-2\kappa} \left( \|\chi_{\Lambda_{R_\phi}(x_\phi)} \tilde{\phi}\|^2 \right)^2
\]

\[
\geq \frac{8}{9} \left\{ (1 + \left( \frac{\epsilon}{\sigma} \right)^{1/\zeta}) |x_\phi| + \left( \frac{\epsilon}{\sigma} \right)^{1/\zeta} \alpha_\phi^{-1/2\kappa} + C'_{\zeta,\sigma,\epsilon} \right\}^{-2\kappa},
\]

for any \( \phi \in \mathcal{G}_E \). Thus, choosing \( \epsilon \) small enough, yields (3.28).

We complete the proof of Theorem 3.4.
Proof of Theorem 3.4. As mentioned above, it is enough to prove that (i) implies (iv). If there exists a function $f$ such that for any $\epsilon > 0$, we have

$$f(s) \leq C_\epsilon e^{\epsilon s - \epsilon^2 \langle 2 \epsilon \rangle}$$

for all $s \in [0, 1]$, and (3.19) holds, then the (SULE) property (3.18) will occur with a factor $\frac{f(\alpha s)}{\sqrt{\alpha s}}$ in view of Lemma 3.5. Proceeding now as in [G, Proof of Proposition A.1] and making use of (3.18), we get

$$\left\| \chi_{x \phi} \right\| \| \chi_{u \phi} \| \leq \frac{1}{\alpha \phi} \frac{f(\alpha \phi)^2}{\alpha \phi} \| \xi_{x \phi} \| \| \xi_{u \phi} \| \leq \frac{1}{\alpha \phi} \frac{f(\alpha \phi)^2}{\alpha \phi} \| \xi_{x \phi} \| \| \xi_{u \phi} \|$$

for all $x, u \in \mathbb{Z}^d$ and with $\epsilon' < \sigma$. We note that it follows from (3.28) that

$$\frac{1}{\alpha \phi} \frac{f(\alpha \phi)^2}{\alpha \phi} \| \xi_{x \phi} \| \| \xi_{u \phi} \| \leq \frac{1}{\alpha \phi} \frac{f(\alpha \phi)^2}{\alpha \phi} \| \xi_{x \phi} \| \| \xi_{u \phi} \|$$

(3.33)

for some postive and finite constants $C_1, C_2$. Taking $\epsilon' > (C_1 + 2)\epsilon$, we conclude that (3.19) follows for any function $f \geq 0$ such that for any $\epsilon > 0$, $f(s) \geq C_\epsilon e^{-\epsilon s - \epsilon^2 \langle 2 \epsilon \rangle}$ for all $0 < s \leq 1$. □

Proof of Theorem 2.5. The “equivalence” part of the proof is currently provided by Theorem 3.4 and Lemma 3.5. It remains to show that the centers of localization $\{x_{n \phi}\}_n$ can be reordered in such a way that $|x_{n \phi}|$ increases with $n$. We proceed as in [DeRJLS2].

Given $L > 0$, let $R_L := \delta L + C\delta$ as in (3.30) for some $\delta > 0$ (that depends on $\zeta$ and $\sigma$) and where we have taken $f = 1$, it follows from (3.20) that

$$\left\| \chi_{x_{n \phi}, R_L \phi n} \right\|^2 \geq 9$$

whenever $|x_{n \phi}| \leq L$, (3.35)

and if $N_L$ is the cardinal of the set $\{n, \phi \in \mathcal{G}_\zeta; |x_{n \phi}| \leq L\}$ then we conclude that

$$\frac{1}{9} N_L \leq \sum_{n, |x_{n \phi}| \leq L} \|\chi_{x_{n \phi}, R_L \phi n}\|^2 \leq \|\chi_{0, L + R_L P_x \phi}\|^2_2$$

$$\leq C L^{2\kappa} \alpha \zeta,$$

(3.36)

for some finite constant $C$ that depending $\zeta$ and $\sigma$. Since $N_L < \infty$ for all $L > 0$ by, (2.10), we may reorder the centers of localization in increasing order in $n$, which yields $|x_{n \phi}| \geq C_{\zeta, \sigma, \zeta} n^{\frac{1}{2\kappa}}$. □

We now turn to Theorem 2.5 and the strong forms of (SUDEC) and (SULE).

Proposition 3.7. Assume (SULE)/(SUDEC) for vectors in the range of $P_E$. For any $\delta > 0$ there is a constant $C_\delta$ such that, for any $\phi, \psi \in \text{ran } P_E$ and $E \in \mathcal{E}$, their localization centers $x_{\phi}, x_{\psi}$ satisfy

$$|x_{\phi} - x_{\psi}| \leq \delta |x_{\phi}| + C_\delta.$$

(3.7)
Next, we show that (3.30) where we take $f \equiv 1$ that yields that for any $\delta > 0$,
\[ \|x|_x \leq R_\phi \| \leq \frac{1}{3} \quad \text{for } R_\phi = \delta|x|_x + C_\phi. \]

If $|x_\phi - x_\psi| \leq 2(R_\phi + R_\psi)$, then (3.37) follows from the definition of $R_\phi, R_\psi$. Assume $|x_\phi - x_\psi| \geq 2(R_\phi + R_\psi)$ and set $\varphi = \frac{1}{\sqrt{2}}(\phi + \psi) \in \text{Ran}P_E$. As a consequence,
\[ \|x|_x \leq R_\varphi \| \geq \frac{1}{\sqrt{2}} \left\| x|_x \leq R_\phi \varphi - \frac{1}{\sqrt{2}} \right\| x|_x \leq R_\psi \| \right\| \geq \frac{2}{3} \sqrt{2} - \frac{1}{\sqrt{2}} \left\| x|_x \leq R_\phi \| \right\| \geq \frac{2}{3} \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{3} \right\| \right\} (3.38) \]

In the same manner, we have $\|x|_x \leq R_\varphi \| \geq \frac{1}{3} \sqrt{2}$. Having in mind that we assumed $|x_\phi - x_\psi| - (R_\phi + R_\psi) \geq \frac{1}{2}|x_\phi - x_\psi|$ and applying (SUDEC) to $\varphi$, we get
\[ \frac{1}{18} \leq \|x|_x \leq R_\varphi \| \left\| x|_x \leq R_\varphi \| \right\| \leq C_\xi, \sigma, \epsilon e^{C_\xi, \sigma, \epsilon} \left\| x|_x \leq R_\varphi \| \right\| \left\| x|_x \leq R_\varphi \| \right\| \leq C_\xi, \sigma, \epsilon e^{C_\xi, \sigma, \epsilon} e^{-\sigma(\frac{1}{2}|x_\phi - x_\psi|)}. \right\| (3.39) \]

The result follows.

\begin{remark}
Notice that (3.37) asserts that if (SULE) holds for all vectors in the span of $G_E$ then the multiplicity has to be finite, since a ball of given radius can only contain a finite number of centers of localization by Theorem 2.3.
\end{remark}

\begin{proof}[Proof of Theorem 2.4] Since $\|x_\phi\| \leq \|x_\phi P_E\|_2$ for any $\phi \in \text{Ran}P_E$, we immediately get (iv) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (ii), (i), and (vii) $\Rightarrow$ (v) $\Rightarrow$ (v). Next, we have (iv) $\Leftrightarrow$ (vii) using the same strategy as in the proof of Theorem 2.3.

To see that (i) $\Rightarrow$ (iv), let $(\phi_n)_{n \geq 1}$ be an orthonormalized basis of $\text{Ran} P_E$ verifying (2.10). Then
\[ \|x_\phi P_E\|_2 \|x_\phi P_E\|_2 = \sum_{n, m} \|x_\phi \phi_n\|^2 \|x_\phi \phi_m\|^2 \leq \left( \sum_{n} \alpha_{\phi_n} \right)^2 C_\xi, \sigma, \epsilon e^{2g(|x|^{\xi} + u^{\xi})} e^{-2(\sigma - \alpha)|x - u|^{\xi}} \right. \leq C_\xi, \sigma, \epsilon e^{2g(|x|^{\xi} + u^{\xi})} e^{-2(\sigma - \alpha)|x - u|^{\xi}}. \right\} (3.40) \]

Finite multiplicity follows. Indeed, there exists $u \in \mathbb{Z}^d$ such that $\|x_\phi P_E\|_2 \neq 0$ (otherwise tr $P_E = 0$), hence for all $E \in \mathcal{E}$, tr $P_E = \sum_{x \in \mathbb{Z}^d} \|x_\phi P_E\|_2 < \infty$ by (3.40).

Next, we show that (v) $\Rightarrow$ (i). We write
\[ \|x_\phi \phi_n\| \|x_\phi \phi_m\| \leq C_\xi, \sigma, \epsilon e^{2g|x_E|^{\xi}} e^{-\sigma(|x|x_E|^{\xi} + u-x_E|^{\xi})} \leq C_\xi, \sigma, \epsilon e^{-2g|x_E|^{\xi}} e^{2g(|x|^{\xi} + u^{\xi})} e^{-2(\sigma - \alpha)|x - u|^{\xi}}, \]
with $\epsilon < \sigma/2$. Then (i) follows since $e^{-2g|x_E|^{\xi}} \leq C(x_E)^{-2g} \leq \sqrt{\alpha_{\phi_n} \alpha_{\phi_m}}$ by (3.38).
We thus have \((iv) \Rightarrow (vii) \Rightarrow (v) \Rightarrow (i) \Rightarrow (iv)\), and the equivalence is proved \(((iv') \land (vii')\) can be deduced from Theorem 2.3 and Lemma 3.4. At last, we show that \((ii) \Rightarrow (vi)\). We have to show that we can get (SULE) with a common center of localization. By Lemma 3.3 we get a (SULE) bound for all \(\phi \in \mathcal{G}_E\), with centers of localization \(x_\phi\). Let \(x_\phi\) be one of them, but given. By Proposition \ref{prop:SULE} 

\[|x_\phi - x_v| \leq \delta|x_v| + C_\delta, \tag{3.41}\]

with \(\epsilon' = \epsilon(1 + \delta)^\zeta + \sigma\delta^\zeta\).

The bound \((2.23)\) is given by an argument similar to the proof of Lemma 3.6. Indeed, there are \(\zeta \in (0, 1], \sigma > 0\) such that for any \(\epsilon > 0\) there is a finite constant \(C_{\zeta, \sigma, \epsilon}'\) for which

\[\|\chi_{x-x_E}|_{|E|} \leq \frac{1}{2}, \quad \text{where} \quad R_E = \left(\frac{\epsilon}{\sigma}\right)^{1/\zeta} \|x_E \| + C_{\zeta, \sigma, \epsilon}' P_E. \tag{3.44}\]

Since

\[\|\chi_{|x-x_E|} \leq R_E P_E\| \leq \|\chi_{|x|} \leq (1 + \frac{1}{\delta})^{1/\zeta} \|x_E \| + C_{\zeta, \sigma, \epsilon}' P_E\| \leq \|\chi_{|x-x_E|} \leq R_E P_E\|^{2}, \tag{3.45}\]

and with \(\epsilon\) small enough one gets

\[\|\chi_{|x-x_E|} \leq R_E P_E\| \leq C_{\zeta, \sigma}(x_E)^{2\epsilon} \alpha_E, \tag{3.46}\]

and thus \(\text{tr} P_E = \|P_E\|_1 = \|P_E\|_2 \leq \|\chi_{|x-x_E|}\| \leq \frac{1}{2} + C_{\zeta, \sigma}(x_E)^{2\epsilon} \alpha_E\). Finally, the last bound \((2.24)\) could be deduced from the equation \((3.44)\) and in proceeding analogously to \((3.36)\).

\[\square\]

Proof of Theorem 2.7. The first claim follows immediately from \((2.10)\) applied to the case \(n = m\) and from \((2.19)\) that we combine with \(\|x_{P} \chi_{x_E} \|_2 \leq \|x_{P} \chi_{P} \|_2 \|x_{u} P_E \|_2\). For the second part, notice that the implications from the left to the right are still valid. The novelty here is that under the hypothesis of finite multiplicity, all these properties become equivalent.

Assuming that \(H\) exhibits \((2.7)\) in \(\mathcal{G}_E\), we construct a family \(\mathcal{G}_E\) of orthonormalized eigenfunctions that verifies \((2.13)\), namely (SULE) property. For any given \(E \in \mathcal{E}\), since \(\sum_{x \in Z^d} \|x_{P} \chi_{x_E} \|_2^2 = \text{tr} P_E = N < \infty\) there exists \(x_E \in Z^d\) which maximizes \(\|x_{P} \chi_{x_E} \|_2\). Note that \(\|P_E \chi_{x_E} \|_2 \neq 0\), otherwise we would have \(\|P_E \chi_{x_E} \|_2 = 0\) for all \(x\) which is not possible since \(\text{tr} P_E \neq 0\). Now, we pick a unit vector \(\eta \in \mathcal{H}\) such that \(\|\eta\| = 1\) and \(\|P_E \chi_{x_E} \eta\| \geq \frac{1}{2} \|P_E \chi_{x_E} \|\), and set

\[\phi_1 = \frac{P_E \chi_{x_E} \eta}{\|P_E \chi_{x_E} \eta\|} \in P_E \mathcal{H} = \mathcal{H}_E. \tag{3.47}\]
We have
\[
\alpha_1 := \text{tr}(T^{-1} P_{\varphi_1} T^{-1}) = \|T^{-1} \varphi_1\|^2
\]
(3.48)\]
\[
= \sum_{x \in \mathbb{Z}^d} \|\chi_x T^{-1} \varphi_1\|^2
\]
\[
\leq \sum_{x \in \mathbb{Z}^d} \|\chi_x T^{-1}\|^2 \frac{\|\chi_x P_E \chi_{x,E}\eta\|^2}{\|P_E \chi_{x,E}\eta\|^2} \leq \sum_{x \in \mathbb{Z}^d} \|\chi_x T^{-1}\|^2 \|\chi_x P_E\|^2
\]
\[
\leq C_d \|P_E \chi_{x,E}\|^2 \leq 4C_d \|P_E \chi_{x,E}\|^2.
\]
(3.49)
As
\[
\|\chi_x \varphi_1\| \leq \frac{\|\chi_x P_E \chi_{x,E}\|}{\|P_E \chi_{x,E}\eta\|},
\]
we get from (2.7) and (3.49), that
\[
\|\chi_x \varphi_1\| \leq \tilde{C}_{\zeta,\sigma,\epsilon} \frac{1}{\alpha_1} e^{\epsilon |x_E|^\zeta} e^{-\sigma |x-x_E|^\zeta}.
\]
(3.50)
We repeat this procedure with \(P_{E,1} := P_E - P_{\varphi_1}\), and so on with \(P_{E,n+1} := P_{E,n} - P_{\varphi_{n+1}}\), until the rank is zero. The finiteness of the rank of \(P_E\), denoting by \(N\), ensures that the process will stop. Notice that the projectors \(P_{E,n}\) exhibit (2.7), and so on with \(P_{E,n}\). For instance \(\|\chi_x P_{E,1} \chi_u\|_2\) is a sum of two decaying quantities
\[
\|\chi_x P_{E,1} \chi_u\|_2 \leq \|\chi_x P_E \chi_u\|_2 + \|\chi_x \varphi_1\| \|\chi_u \varphi_1\|
\]
(3.51)
\[
\leq \tilde{C}_{\zeta,\sigma,\epsilon} e^{\epsilon |x_u|^\zeta} e^{-\sigma |x-u|^\zeta},
\]
where the decay of the second term in the r.h.s of (3.51) results from (3.50). Therefore, by induction we get \(N\) orthonormalized functions \(\varphi_n\) satisfying the (SULE)-like estimate in the sense that
\[
\|\chi_x \varphi_n\| \leq \tilde{C}_{\zeta,\sigma,\epsilon} \frac{1}{\alpha_n} e^{\epsilon |x_{E,n}|^\zeta} e^{-\sigma |x-x_{E,n}|^\zeta},
\]
(3.52)
for any \(n \in \{1, \ldots, N\}\). We can get rid of \(\alpha_n^{-1/2}\) from the proof of Theorem 3.4 in which case only an arbitrary small fraction of the mass \(\sigma\) is lost. Alternatively, at each step, one can follow [EGS]. Proof of Lemma 4 and bound \(\|\chi_x \varphi_n\|\) by the geometric mean of (2.7) and \(\|\chi_x \varphi_n\| \leq \|\chi_{E,n} P_{E,n}\|\). In this latter case, the final \(\sigma\) is divided by 2 at each step.

\[\square\]

We turn to the proof of Theorems 2.9 and 2.10. Theorem 2.9 follows immediately from the proof of Theorem 2.1. The main point is to notice that the technical Lemma 3.3 is still valid in the case of subexponential growth, where the r.h.s of (3.13) becomes \(e^{C_{\alpha,\zeta,\epsilon}(\log L)^{\beta/\zeta}}\).

In view of the proof Theorem 2.9 the Theorem 2.10 can be deduced by adapting the different steps which involve the geometry of the space. In particular, the technical result in Lemma 3.6 and Theorem 3.4 remain true if we take \(f(s) \leq C_\epsilon e^{(-\epsilon \log s)^{\zeta/\alpha}}\) in (i) and \(f(s) \geq C_\epsilon e^{(-\epsilon \log s)^{\zeta/\alpha}}\) in (iv) for \(s \in (0, 1]\) in which case \(\alpha_\varphi \geq C_\epsilon e^{-|x_b|^\alpha}\).
4. COUNTEREXAMPLES

The first model is the free Landau Hamiltonian $H_B := (-i\nabla - A)^2$ on $L^2(\mathbb{R}^2)$ where $A$ is the vector potential $A = B R (-x_2, x_1)$ and $B > 0$ is the strength of the constant magnetic field. It is well known that the Landau levels are infinitely degenerated and that it exhibits the property (2.7) and thus dynamical localization. We claim that (SUDEC) does not occur for $H_B$. In fact, consider for instance the eigenfunctions associated to the first Landau level and whose expression is given by

$$\varphi_n(z) = \left(\frac{B^n}{2\pi 2^n n!}\right)^{1/2} z^n e^{-\frac{B}{4}|z|^2}. \quad (4.1)$$

For $n$ integer, we define the radial function $f_n(r) = r^{2n} e^{-\frac{2B}{r^2}}$ for which the maximum is achieved for the radius $r_{\text{max}} = \left(\frac{2nB}{B}\right)^{1/2}$. Let $z_1$ and $z_2$ to be affixes of two opposite points on this maximal circle. A simple computation yields

$$|\varphi_n(z_1)\varphi_n(z_2)| = \frac{n^n}{2\pi n!} e^{-n}. \quad (4.2)$$

Together with the Stirling’s formula, it gives that there are no positive constants $c_1$ and $c_2$ ($c_2$ depends on $B$) such that $\frac{1}{\sqrt{n}} \leq c_1 e^{-c_2\sqrt{n}}$ for all $n$.

Remark 4.1. Another way to see that $H_B$ does not has (SUDEC) can be derived from the theory of the quantum Hall effect. Indeed, if (SUDEC) would occur for a basis of eigenvectors then the Hall conductance $\sigma_H$ would be constant at Landau levels by [GKS1], while $\sigma_H$ is known to have jumps.

Next, let us consider the discrete Laplacian $-\Delta$ on subgraphs of $\mathbb{Z}^2$. It is enough to consider a subgraph given by $J \geq 2$ disjoint copies $C_j$ of a given finite cluster $C_1$ and we set $H_j := -\Delta|_{C_j}$. The operators $H_j$, $j = 1, \cdots, J$, have the same discrete spectrum with compactly supported eigenfunctions. The operator $-\Delta|_{\cup_j C_j} = \oplus_j H_j$ for $1 \leq j \leq J$ for $1 \leq j \leq J$, has (SULE) since we obtain a basis of compactly supported eigenfunctions. But (SULE+) and (SUDEC+) does not hold as soon as copies $C_i$ and $C_j$ for $i \neq j$, are far enough so that Proposition 3.7 is violated. We mention that such finite clusters appear in a natural way in percolation theory. We refer to [KiM] for more details.

5. APPENDIX A

In this section, we shall order the moments (3.12) given in Section 3, eq (3.12), uniformly on the space.

Proof of Lemma 3.3. Set

$$a_{kx}(u) := \frac{\|\chi_x P_k \chi_u\|_2^2}{\|\chi_u P_k\|_2^2},$$

which verify

$$\sum_x a_{kx}(u) = \frac{\|\chi_u P_k\|_2^2}{\|\chi_u P_k\|_2^2} = 1 \quad \text{for all } u \in \mathbb{Z}^d \text{ and all } k \in \mathbb{Z}, \quad (5.1)$$

and

$$\sum_{k, E_k \in I} a_{kx}(u) \leq \sum_{k, E_k \in I} \|\chi_x P_k\|_2^2 = \text{tr}(\chi_x P_1 \chi_x) \leq 1 \quad \forall x, u \in \mathbb{Z}^d, \quad (5.2)$$
where $P_I$ denotes the projection on the interval $I$.

For $L \in \mathbb{N}$, define the following set

$$J_u(L) := \{ k \in \mathbb{Z}, E_k \in I; \sum_{x \notin \Lambda_L(u)} a_{kx}(u) \leq 1/2 \},$$

and consider the sum

$$S_u(L) := \sum_{k \in J_u(L)} \sum_{x \in \Lambda_L(u)} a_{kx}(u).$$

We will estimate the cardinal of $J_u(L)$ in terms of the volume of the box $\Lambda_L(u)$.

Note that it follows from (5.1) that for $k \in J_u(L)$, we have

$$\sum_{x \in \Lambda_L(u)} a_{kx}(u) = \sum_{x} a_{kx}(u) - \sum_{x \notin \Lambda_L(u)} a_{kx}(u) \geq 1/2.$$

Thus $S_u(L) \geq \frac{1}{2} \#(J_u(L))$. Moreover, the bound (5.2) yields

$$S_u(L) \leq \sum_{k, E_k \in I} \sum_{x \in \Lambda_L(u)} a_{kx}(u) \leq \sum_{x \in \Lambda_L(u)} 1 \leq C_d L^d,$$

and hence

$$\#(J_u(L)) \leq C_d L^d. \tag{5.3}$$

Now given $\sigma > 0$ and $\zeta \in (0, 1]$, we set

$$I_u(L, \sigma, \zeta) = \{ k \in \mathbb{Z}, E_k \in I; A_k(\sigma, \zeta, u) \leq \frac{1}{2} e^{\sigma L^\zeta} \},$$

and notice that

$$A_k(\sigma, \zeta, u) \geq e^{\sigma L^\zeta} \sum_{x \notin \Lambda_L(u)} a_{kx}(u),$$

which shows that $I_u(L, \sigma, \zeta) \subset J_u(L)$. Taking the exponential rescaling $l = \frac{e^{\sigma L^\zeta}}{2}$ and using (5.3), we obtain

$$N(l) := \# \{ k \in \mathbb{Z}, E_k \in I; A_k(\sigma, \zeta, u) \leq l \} \leq C_{\sigma, \zeta, d} (\log l)^{d/\zeta},$$

and thus the finiteness of the set $\{ k \in \mathbb{Z}, A_k(\sigma, \zeta, u) \leq l \}$ follows.

For any $u \in \mathbb{Z}^d$, there exists a new order $j_u : k \mapsto j_u(k)$ for $k \in \mathbb{Z}$ in such a way that $A_{j_u(k)}(\sigma, \zeta, u)$ increases. So $N(A_{j_u(k)}) = |j_u(k)|$ and with $A_{j_u(k)}(\sigma, \zeta, u) = l$, one gets

$$|j_u(k)| \leq C_{\sigma, \zeta, d} \left( \log(A_{j_u(k)}(\sigma, \zeta, u)) \right)^{d/\zeta}.$$

We conclude that $A_k(\sigma, \zeta, u)$ may be ordered so that the increase with $k$ in the sense

$$A_k(\sigma, \zeta, u) \geq e^{C_{\sigma, \zeta, d} k^{\phi/d}},$$

for a positive constant $C_{\sigma, \zeta, d}$ which is uniform in $u \in \mathbb{Z}^d$. □
6. Appendix B

In this part we review the first result of the Section 2 in the case of random Hamiltonians. More precisely, we consider a \( \mathbb{Z}^d \)-ergodic operator \( H_\omega \). We adapt the notations and the quantities used previously. We consider the random \((\sigma, \zeta)\)-subexponential moment

\[
M_{u, \omega}(\sigma, \zeta, \mathcal{X}, t) := \left\| e^{\frac{\sigma}{2}|X-u|\zeta} e^{-itH_\omega} \mathcal{X}(H_\omega) \chi_u \right\|^2. \tag{6.1}
\]

We establish a similar version in expectation of Theorem 2.1 that we formulate as Theorem 6.1.

**Theorem 6.1.** Let \( I \subset \sigma(H) \) be an interval and assume that \( H \) has pure point spectrum in \( I \). The following properties are equivalent.

(i) There exist \( \sigma > 0, \zeta \in (0, 1] \) so that for any \( \mathcal{X} \in C^\infty_{0, +}(I) \),

\[
\sup_T M_{u, \omega}(\sigma, \zeta, \mathcal{X}, T) := \sup_T \frac{1}{T} \int_0^T \mathbb{E}\{M_{u, \omega}(\sigma, \zeta, \mathcal{X}, t)\} dt < \infty. \tag{6.2}
\]

(ii) There exist \( \sigma > 0, \zeta \in (0, 1] \) so that for any \( \mathcal{X} \in C^\infty_{0, +}(I) \),

\[
\sup_T \frac{1}{T} \int_0^\infty e^{-t/T} \mathbb{E}\{M_{u, \omega}(\sigma, \zeta, \mathcal{X}, t)\} dt < \infty. \tag{6.3}
\]

(iii) There exist \( \sigma > 0, \zeta \in (0, 1] \) any \( \mathcal{X} \in C^\infty_{0, +}(I) \),

\[
\mathbb{E}(\sup_T M_{u, \omega}(\sigma, \zeta, \mathcal{X}, t)) < \infty. \tag{6.4}
\]

(iv) There exist \( \zeta \in (0, 1], \sigma > 0 \) such that for for any \( \mathcal{X} \in C^\infty_{0, +}(I) \), there is a constant \( C_{\zeta, \sigma, \mathcal{X}} < \infty \), so that

\[
\mathbb{E}(\sup_T \| \chi_x e^{-itH\mathcal{X}(H)\chi_u} \|_2) \leq C_{\zeta, \sigma, \mathcal{X}} e^{-\sigma|x-u|\zeta} \text{ for all } x, u \in \mathbb{Z}^d. \tag{6.5}
\]

(v) There exist \( \zeta \in (0, 1], \sigma > 0 \) such that for any \( \mathcal{X} \in C^\infty_{0, +}(I) \), there is a constant \( C_{\zeta, \sigma, \mathcal{X}} < \infty \), so that

\[
\mathbb{E}(\sup_k \mathcal{X}(E_{k, \omega}) \| \chi_x P_{k, \omega} \chi_u \|_2) \leq C_{\zeta, \sigma, \mathcal{X}} e^{-\sigma|x-u|\zeta} \text{ for all } x, u \in \mathbb{Z}^d. \tag{6.6}
\]

If \( H \) satisfies one of these properties, we say that \( H \) exhibits strong dynamical localization in \( I \).

The proof is similar to that of Theorem 2.1 and notice that the ergodicity allows us to study the dynamics just from the origin \((u = 0)\) through the moments. Furthermore, we should take the randomness in account and add it in all other quantities that we have introduced.

**Proof.** Once again, the points that we should prove are (i) \(\Rightarrow\) (v) \(\Rightarrow\) (iv). As in (3.1) and (3.2), we introduce

\[
P_{\omega}(x, \mathcal{X}) := \sup_k \mathcal{X}(E_{k, \omega}) \| \chi_x P_{k, \omega} \chi_0 \|_2, \tag{6.7}
\]

\[
\mathcal{L}_{\omega}(\sigma, \zeta, \mathcal{X}) := \sum_{x \in \mathbb{Z}^d} e^{\sigma|x|\zeta} P^2_{\omega}(x, \mathcal{X}), \tag{6.8}
\]
and
\[ \mathcal{L}(\sigma, \zeta, \mathcal{X}) := \sum_{x \in \mathbb{Z}^d} e^{\sigma|x|^\xi} E \left( P_\omega^2(x, \mathcal{X}) \right). \] (6.9)

Using the same strategies, we have
\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T M_{0, \omega}(\sigma, \zeta, \mathcal{X}, T) \geq C_{\sigma, \zeta} \sum_k \sum_{x \in \mathbb{Z}^d} \mathcal{X}^2(E_{\omega, k}) e^{\sigma|x|^\xi} \| \chi_x P_{\omega, k} \chi_0 \|_2^2 \\
\geq C_{\sigma, \zeta} \sum_{x \in \mathbb{Z}^d} e^{\sigma|x|^\xi} \left( \sup_k \mathcal{X}^2(E_{\omega, k}) \| \chi_x P_{\omega, k} \chi_0 \|_2^2 \right). 
\]

Taking the expectation, we obtain
\[
E \left( \liminf_{T \to \infty} \frac{1}{T} \int_0^T M_{0, \omega}(\sigma, \zeta, \mathcal{X}, T) \right) \geq C_{\sigma, \zeta} \sum_{x \in \mathbb{Z}^d} e^{\sigma|x|^\xi} E \left( \sup_k \mathcal{X}^2(E_{\omega, k}) \| \chi_x P_{\omega, k} \chi_0 \|_2^2 \right)^2, 
\]

and the Fatou lemma yields
\[
\liminf_{T \to \infty} M_{0}(\sigma, \zeta, \mathcal{X}, T) \geq C_{\sigma, \zeta} \mathcal{L}(\sigma, \zeta, \mathcal{X}). \] (6.10)

Consequently, we get a similar result to (3.4). For the last point, we go back to Theorem 3.2 and Lemma 3.3 that we restore for \( \omega \) fixed. Then for any \( \gamma \in (0,1) \), there exists a constant \( C_{\sigma, \zeta, d, \gamma} \) which is uniform in \( \omega \) such that
\[
\sup_t \| \chi_x e^{-itH_{\omega}} \mathcal{X}(H_{\omega}) \chi_0 \|_2 \leq C_{\sigma, \zeta, d, \gamma} P_{\omega}^{1-\gamma}(x, \mathcal{X}) \mathcal{L}_{\omega}^{\gamma/2}(\sigma, \zeta, \mathcal{X}),
\]

and hence
\[
E \left( \sup_t \| \chi_x e^{-itH_{\omega}} \mathcal{X}(H_{\omega}) \chi_0 \|_2 \right) \leq C_{\sigma, \zeta, d, \gamma} E \left( P_{\omega}(x, \mathcal{X}) \right)^{1-\gamma} E \left( \mathcal{L}(\sigma, \zeta, \mathcal{X}) \right)^{\gamma/2},
\]

thanks to the H"older inequality that we appley with conjugate exponents \( p = \frac{1}{1-\gamma} \) and \( p' = 1/\gamma \) and to Jensen’s inequality.

\[\Box\]

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