On the topological implications of inhomogeneity

Boudewijn F. Roukema
Toruń Centre for Astronomy, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, ul. Gagarina 11, 87-100 Toruń, Poland and
Centre de Recherche Astrophysique de Lyon, UMR 5574, Univ. Lyon 1, 9 av. Charles André, 69230 St–Genis–Laval, France

Vincent Blanlœil
Département de Mathématiques, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg cedex, France

Jan J. Ostrowski
Toruń Centre for Astronomy, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, ul. Gagarina 11, 87-100 Toruń, Poland
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The approximate homogeneity of spatial sections of the Universe is well supported observationally, but the inhomogeneity of the spatial sections is even better supported. Here, we consider the implications of inhomogeneity in dust models for the connectedness of spatial sections at early times. We consider a non-global Lemaître-Tolman-Bondi (LTB) model designed to match observations, a more general, heuristic model motivated by the former, and two specific, global LTB models. We propose that the generic class of solutions of the Einstein equations projected back in time from the spatial section at the present epoch includes subclasses in which the spatial section evolves (with increasing time) smoothly (i) from being disconnected to being connected, or (ii) from being simply connected to being multiply connected, where the coordinate system is comoving and synchronous. We show that (i) and (ii) each contain at least one exact solution. These subclasses exist because the Einstein equations allow non-simultaneous big bang times. The two types of topology evolution occur over time slices that include significantly post-quantum epochs if the bang time varies by much more than a Planck time. In this sense, it is possible for cosmic topology evolution to be “mostly” classical.

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I. INTRODUCTION

The approximate homogeneity of spatial sections (hypersurfaces) of the Universe is well supported observationally. Both the assumption of homogeneity and fact of inhomogeneity play an important role in relativistic cosmological models. The Friedmann-Lemaître-Robertson-Walker (FLRW) models [1–5] are solutions of the Einstein equations in which the density is constant in any comoving spatial section. With the Concordance Model parameters of the metric [6], the FLRW models provide reasonably good fits to observational data (faint galaxy number counts [e.g. 7, 8], gravitational lensing [e.g. 9], supernovae type Ia magnitude-redshift relations [e.g. 10, 11]). However, there is no serious question of whether the Universe is inhomogeneous: the Earth, galaxies and galaxy clusters exist. The real question is whether the homogeneous, heuristic approach gives a sufficiently accurate approximation. The forcing of an FLRW model onto late-epoch observations requires a non-zero “dark energy” parameter \( \Omega_\Lambda \), suggesting that the latter is most simply interpreted as an artefact of forcing an oversimplified model onto the data [e.g. 12, 13].

The near-homogeneity is also a key element of the “Horizon Problem” for non-inflationary FLRW models: how was it possible for causally disconnected (but spatially connected) regions of the spatial section of the Universe to homogenise? In the context of dust models with comoving spatial sections, this question implicitly assumes that universe models with initially inhomogeneous spatial sections are relativistically valid and only have a problem with causal disconnectedness, not with comoving spatial disconnectedness. Is this assumption correct?

Although many families of inhomogeneous, exact, cosmological solutions of the Einstein equations are known (see the extensive compilation in [15] and a recent review in [16]), no generic model of exact solutions is known. The Lemaître-Tolman-Bondi (LTB) family of exact solutions [17, 19] is one well-known family of exact solutions. These solutions consist of exact solutions to the Einstein equations that are radially inhomogeneous and spherically symmetric with respect to an origin. In analogy with the way that the FLRW model is interpreted to apply to a 3-dimensionally averaged spatial section, an LTB solution can be interpreted to apply to a spatial section that has been averaged over every infinitesimally thin spherical shell, i.e. 2-dimensionally, prior to solving the Einstein equations. However, the Einstein equations do not imply their averaged equivalent: \( G(g) = 8\pi T(\rho) \neq G(\langle g \rangle) = 8\pi T(\langle \rho \rangle) [20] \) [e.g. 21]. That is, although homogeneity is often described in terms...
of the Cosmological “Principle”, the application of either an FLRW or an LTB solution to the real Universe is better seen as a heuristic calculational strategy rather than a physical principle, with the risk of averaging-related artefacts occurring in both cases. LTB solutions provide an intermediate step between the FLRW solutions that force full homogeneity and a more realistic, unknown family of generic solutions.

Thus, here we primarily consider LTB solutions. We first examine an LTB fit to recent observational data to see what it implies for the connectedness of spatial sections at early times (Sect. II). In Sect. III a less restricted situation is discussed by supposing that a solution with a Gaussian bang time function \( f(t, r) \) exists and considering its topology evolution. In Sect. IV we formally generalise to a wider class of inhomogeneous solutions that contains two subclasses of distinct types of topology evolution (Definition 1) and present a conjecture and a corollary regarding one of the subclasses. We give case examples in Sect. V A and Sect. V B using an LTB solution found previously \( [22] \), to show that the two subclasses given in Definition 1 are non-empty (Theorem 1). Interpretations are discussed in Sect. V and the second subclass is considered further in Sect. VI. Conclusions are given in Sect. VII. Unless otherwise stated, we only consider relativistic (non-quantum), comoving, dust solutions with a zero cosmological constant (the first fit of inhomogeneous exact cosmological solutions to supernovae type Ia data used radially inhomogeneous pressure solutions \( [23] \)).

II. OBSERVATIONAL ESTIMATE

There are many different fits of LTB models to observations—see Ref. \( [10] \) for a list of direct and inverse fits. Here, we consider a recent paper \( [12] \) that phenomenologically used the inverse method to find an LTB model. That is, the authors started with functions implied by the FLRW model with Concordance Model values of the metric parameters \( [6] \) and inferred LTB functions. A similar method and result are given in Ref. \( [24] \). By construction, the two fits found in Ref. \( [12] \) provide good fits to the observed supernovae type Ia angular-diameter-distance–redshift relation and to an observational estimate of Hubble parameter evolution with redshift, based on differential stellar ages of the oldest passings (Figs 3 and 11 of \( [12] \)). Let us extend the solution by assuming that \( t \) at the observer to \( \sim t_0 + 2 \text{ Gyr} \) on shells at an areal distance \( [20] \) of about 3.7 Gpc. Thus, as \( t - t_B(r) \to 0^+ \) at some given \( r \), the surface area of a spherical \((S^2)\) shell at \( r \) approaches zero.

In other words, the LTB family allows the age of the universe in a given universe model in a comoving spatial section to be a function \( t - t_B(r) \) that varies with the radial coordinate \( r \). Thus, since the authors deliberately aimed to avoid making arbitrary assumptions, Figs 3 and 12 of Ref. \( [12] \) show, unsurprisingly, that the \( t_B(r) \) solutions are not constant. In a comoving section at the present epoch \( t_0 \), the age of the universe increases from \( t_0 \) at the observer to \( \sim t_0 + 2 \text{ Gyr} \) on shells at an areal distance \( [20] \).

What are the topological properties of this solution? At times \( t > 0 \), let us assume that (i) the spatial section of the solution is simply connected. The spatial curvature is negative, since \( E(r) > 0 \) over the region of \( r > 0 \) studied (Figs 2, 11 of \( [12] \)). Let us extend the solution by assuming that (ii) \( E(r) > 0 \) \( \forall r > 0 \). Thus, spatial sections at \( t > 0 \) are the 3-manifold \( H^3 \), with non-constant curvature.

Figures 3 and 12 of Ref. \( [12] \) show that when \(-2 \text{ Gyr} \leq t < 0 \), a spatial section of the universe has a hole in the centre, where space has not yet emerged from the initial singularity. For example, consider a spatial section at \( t = -1 \text{ Gyr} \) in Fig. 3 of Ref. \( [12] \). In comoving coordinates, the closed 3-dimensional ball

\[
\mathcal{V} = \{ (r, \theta, \phi) : r \leq r_{\text{inf}}(t = -1 \text{ Gyr}) \}, \tag{8}
\]
and \( \partial \mathcal{V} \) is the physically defined spatial 3-manifold. The boundary \( \partial \mathcal{V} \) has zero metrical area \( 4\pi R^2 \) and corresponds to a spatial section through the initial singularity, i.e. \( \partial \mathcal{V} \subset \mathcal{V} \Rightarrow \partial \mathcal{V} \not\subset H^3 \setminus \mathcal{V} \).

The physically defined spatial section \( H^3 \setminus \mathcal{V} \) is shaded in grey up to an arbitrary cutoff radius.

where

\[
r_{\text{inf}}(t) := \inf\{ r : t - t_B(r) > 0 \} \tag{9}
\]

and \( r_{\text{inf}}(t = -1 \text{ Gyr}) \approx 1.7 \text{ Gpc} \), consists of the initial singularity \( \partial \mathcal{V} \) and a region of coordinate space beyond (earlier than) the singularity. The metric is only Lorentzian for \( t > t_B(r) \), i.e. \( r > r_{\text{inf}}(t) \), so the universe at \( t = -1 \) Gyr is \( H^3 \setminus \mathcal{V} \), i.e. a 3-manifold with a hole created by removing \( \mathcal{V} \) from \( H^3 \) (Fig. 1).

Thus, this universe model evolves from \( H^3 \setminus \mathcal{V} \) to \( H^3 \) at early times. What is the areal radius \( R(t, r) \) on the boundary \( \partial \mathcal{V} \)? This is given by (4), (7), and (2) [and (6) for \( r = 0 \)] of Ref. \cite{12}. As \( \eta(t, r) \rightarrow 0^+ \), we have \( \phi(t, r) \rightarrow 0^+ \) and \( \xi(t, r) \rightarrow 0^+ \), and thus \( R(t, r) \rightarrow 0^+ \), and \( t - t_B(r) \rightarrow 0^+ \), since \( E(r) > 0 \) and \( M(r) \) in Figs. 2, 4, 5, and 6 are non-zero (for \( r > 0 \)) functions of \( r \) only. Thus, the spatial volume of a shell at \( r \) shrinks to zero as \( t \rightarrow t_B(r) \) for fixed \( r \), or as \( r \rightarrow r_{\text{inf}}(t) \) at a fixed \( t \). Within the spatial section \( H^3 \setminus \mathcal{V} \) at \( t \), the boundary \( \partial \mathcal{V} \) appears metrically as a single missing point. In coordinate space imagined intuitively (Fig. 1) with, for example, a Euclidean metric, \( \partial \mathcal{V} \) would have an area of \( 4\pi R^2 \), but this is not physical; the metrical area of \( \partial \mathcal{V} \) is \( 4\pi R^2 = 0 \).

Relativistically, there is no problem with this solution.

The high-\( r \) universe is born first, with the coordinate-space shell at \( r_{\text{inf}}(t) \), i.e. the 3-manifold boundary point at \( r_{\text{inf}}(t) \), representing the unfinished early big bang process. The flexibility of the areal radius \( R(t, r) \) in LTB solutions allows comoving space to continuously be born from this singularity, which moves to successively lower values of \( r_{\text{inf}}(t) \) as \( t \) increases up to \( t = 0 \). At \( t = 0 \), we have \( r_{\text{inf}}(0) = 0 \) and the singularity is replaced by ordinary spacetime points at \( (t > 0, r = 0) \). We can summarise these properties of \cite{12}'s solution:

\[
\begin{align*}
& r_{\text{inf}}(t) > 0, \forall t < 0 \\
& dr_{\text{inf}}(t)/dt < 0, \forall t < 0 \\
& \mathcal{V}(t) := \mathcal{V}[r \leq r_{\text{inf}}(t)], \forall t \leq 0 \\
& \int_{\partial \mathcal{V}(t)} d\Omega = \lim_{\tilde{r} \rightarrow r_{\text{inf}}} 4\pi R^2(t, \tilde{r}) = 0, \forall t \leq 0, \tag{10}
\end{align*}
\]

where the pre-big-bang universe \( \mathcal{V} \setminus \partial \mathcal{V} \) is considered to be non-physical, the metric is given in (1) of Ref. \cite{12}, and \( d\Omega \) is the metric area element. Comoving space is continuously born from the singularity \( \partial \mathcal{V} \) until \( t = 0 \) when the singularity disappears in the same way that it disappeared in parts of comoving space that were born earlier. Thus, this universe model evolves from \( H^3 \setminus \{0\} = S^2 \times \mathbb{R}^+ \) \cite{31} at \( t \leq 0 \) to \( H^3 \) at \( t > 0 \). This is a topology change, i.e. a change in \( \pi_2 \) homotopy classes for this comoving, synchronous spacetime foliation. Some 2-spheres cannot be continuously shrunk to a point at \( t \leq 0 \), but all 2-spheres can be continuously shrunk to a point at \( t > 0 \). Dropping the simplifying assumptions (i) and (ii) above does not make it possible to avoid the
topology change in this interpretation of \cite{12}'s solution, since it just replaces $H^3$ by a more generic 3-manifold $\mathcal{M}$.

### III. GAUSSIAN $t_B(r, \theta, \phi)$ DISTRIBUTION

The LTB solution presented in Ref. \cite{12} (and \cite{24}) is intended to demonstrate an example solution that fits key cosmological observations, but is not intended as a definitive replacement for the FLRW model with Concordance Model metric parameter values. Moreover, the LTB model is not a generic inhomogeneous model. An LTB solution constrained by more observational data could be expected to have a more complicated non-constant $t_B$ function (unless this is imposed by assumption). A more realistic solution using the inverse method would result from using the observational data to infer a more generic, inhomogeneous, dust solution. This could also reasonably be expected to have a non-constant $t_B$ function, as a function of three spatial variables rather than just one, i.e.

$$t_B = t_B(r, \theta, \phi)$$  \hspace{1cm} (11)

The solution \cite{12} has only one (continuous) comoving spatial region where $t_B < \max t_B$. This simplicity is unlikely to be a requirement of LTB models, or of more general cosmological (comoving dust) solutions of the Einstein equations expressed in comoving, synchronous coordinate systems.

In solution \cite{12}, lower density $\rho$ tends to correlate with older regions of the universe, i.e. more negative $t_B$ (cf. Figs 3 of \cite{12} and the solid curves in Fig. 10 of \cite{12}). A qualitative way to interpret this in terms of FLRW models is that for a fixed Hubble constant $H_0$, a lower matter density $\Omega_m$ universe is older than a higher matter density universe.\cite{32} This is only a qualitative guide to the LTB case, since both density and any typically defined equivalent of the Hubble parameter vary with $t$ and $r$ differently to the FLRW case. The same Figs 3 and 10 in \cite{12} show that this qualitative inference does not always hold: lower $\rho$ does not always correlate with more negative $t_B$.

In order to consider a more general solution than that of \cite{12}, let us suppose that a comoving dust solution to the Einstein equations expressed in synchronous, comoving, spherically symmetric coordinates has $t_B(r, \theta, \phi)$ drawn from a Gaussian distribution $G(0, \sigma)$, i.e. of mean zero and standard deviation $\sigma$ when smoothed on a length scale $\Delta x$. Gaussian density fluctuations on an FLRW background are a standard ingredient of modern cosmology, so even if it is unlikely that a given solution has a $t_B-\rho$ relation that is a function $t_B(\rho)$ (let alone a monotonic function), a Gaussian $t_B$ distribution is a heuristically reasonable hypothesis. Now consider an approximately flat, cubical, small region of side length $3\Delta x$ of which the central $(\Delta x)^3$ small cube contains a region with $t_B < -3\sigma$, i.e. born unusually early. The probability that this small cube is connected—in coordinate space—to another small cube with $t_B < -3\sigma$, i.e. that it is not isolated by iso-bang (constant $t_B$) contours, is the complement of the probability that the 26 small cubes around it all have $t_B \geq -3\sigma$, i.e. $P = 1 - (\frac{1}{2} [1 + \text{erf}(3/\sqrt{2})])^6 \approx 3\%$.\cite{33} The chances that the second $t_B < -3\sigma$ cube touches a third cube outside of the original $(3\Delta x)^3$ region, and that the $(n > 2)$-th cube touches another small cube yet further away for $n \geq 3$, rapidly decrease with increasing $n$.

Thus, $t_B < -3\sigma$ iso-bang contours in coordinate space will tend to form isolated 2-surfaces. That is, with a fixed smoothing scale $\Delta x$ and in a large enough region of comoving coordinate space, a Gaussian distribution in $t_B$ implies that there will tend to (statistically) exist a set of many regions $(3\text{-volumes}) \{W_i\}$ with $t_B < -3\sigma$ that are spatially isolated from one another in coordinate space, and thus also consist of isolated regions of the (metrically defined) 3-manifold. At $t \gg 0$, we label the latter $\mathcal{M}$. Let us assume that $\mathcal{M}$ is connected and that its volume is $\approx (\Delta x)^3$.

Now consider the coordinate-space spatial section at $t = -3\sigma$. The boundaries of the regions $\{W_i\}$ defined by $t_B < -3\sigma$, i.e. $\{\partial W_i\}$, are 2-spatial iso-bang contours. The regions $\{W_i\}$ have already emerged from the initial singularity, with $t - t_B = -3\sigma - t_B > 0$. Since the $\{W_i\}$ are isolated from one another, they constitute a set of disconnected 3-manifolds. Hence, the universe at $t = -3\sigma$ consists of the spatially disconnected 3-manifold $\mathcal{M} \setminus \bigcup \{W_i\}$, shown in Fig. 2.

The choice of $-3\sigma$ is for illustration only. Any reasonably high $x \gtrsim 3$ will (statistically) give a spatially disconnected universe at $t = -x\sigma$, given a large enough spatial volume and a Gaussian distribution of $t_B$ as stated above. At the same time $t$, the parts of “future” comoving space $\mathcal{M} \setminus \bigcup \{W_i\}$ have not yet emerged from the initial singularity and only exist in coordinate space. If we follow the spatial section back in time from $t = -3\sigma$, then the boundaries $\partial W_i$ correspond to $t = -x\sigma$ for increasing $x$, i.e. they shrink smoothly, possibly subdividing further, eventually vanishing into the singularity. For $t \leq \text{min} t_B(r, \theta, \phi)$, the global bang time, no more $W_i$ exist.

Moving forward in time, how do the $W_i$ merge together? The boundary $\partial W_i$ for the $i$-th disconnected region has zero 2-surface area, as in the case of $\partial (R(t, r_{\text{inf}})) = 0$ in the solution \cite{12}. That is, the boundary of $W_i$ is $S^2$ in coordinate space with zero 2-surface area, i.e. metrically it is a point-like singularity. Thus, $W_i$ can be thought of metrically as a 3-manifold with one point excluded. For intuitive purposes, it can be useful to think of an azimuthal equidistant projection of the Earth’s surface, centred at an arbitrary geographical location, with the antipode corresponding to the big bang initial singularity. The antipode can be thought of either as a large, coordinate-space, zero-circumference circle that bounds the 2-manifold from the “outside” in the projected map, or metrically as a single missing point “on” our usual
intuition of the Earth’s surface. Again, as in the solution [12], comoving space is born from this singularity, so that there is comoving growth of the spatial region \( \mathcal{W}_t \). As \( t \) increases, the \( t_B \) threshold for the iso-bang contours increases (\( dt > 0 \)), so that the \( \mathcal{W}_t \) eventually touch and pairs (or \( n \)-tuples) of \( \mathcal{W}_t \) merge together. Again writing \( t = -x\sigma \), as \( -x \) becomes more positive, \( t \) reaches a high enough \( -x\sigma \gg 0 \) such that the probability for an isolated \( t_B = -x\sigma \) region to exist becomes negligible and the universe becomes fully connected.

For a Gaussian \( t_B \) distribution, it could be expected that the zero 2-surface area of the boundary of an isolated region \( \mathcal{W}_t \) at \( t = -x\sigma \) will tend to topologically be \( S^2 \) in coordinate space, so that \( \mathcal{W}_t \) is \( S^3 \setminus \partial \mathcal{W}_t = S^3 \setminus \{0\} \) topologically. Other 2-manifolds for the coordinate space representation of \( \partial \mathcal{W}_t \) could also be possible.

Thus, we find that if a universe described by the Einstein equations is \( t_B \)-inhomogeneous, then, even with several simplifying assumptions (Gaussian distribution of \( t_B \) at a given smoothing length, \( \mathcal{M} \) connected and simply connected for \( t > 0 \)), there is a very high probability that it emerged from the (spacetime-smooth) mergers of comoving spatial sections that were spatially disconnected from each other prior to their mergers. The temporal sense of “merged” refers here to the comoving, synchronous spacetime coordinate system. Foliations of the same spacetime according to which there is no 3-spatial topology evolution are likely to exist, but are unlikely to provide a model as intuitively simple as the comoving, synchronous foliation. The early-epoch disconnectedness in the comoving, synchronous foliation is distinct from questions of causal connectivity. Interpretations of this topology evolution are discussed in Sect. IV after first proposing a generalisation and verifying that some examples of the proposed subclasses of solutions exist.

IV. RELATIVISTIC, POST–QUANTUM-EPOCH TOPOLOGY EVOLUTION

Let us formalise the meaning and existence of spacetimes that solve the Einstein equations and yet have “post–quantum epoch” spatial topology change.

**Definition 1** Let us define the generic class \( A \) where \( \{g^−\} \in A \) if \( g^− \) is a (regular) extension to \( t_B < t < t_0 \), using a synchronous coordinate system, of a dust (pressureless) metric \( g|_{t_0} \), i.e. \( g^− \) solves the Einstein equations over \( t_B < t < t_0 \), where \( g|_{t_0} \) on a comoving spatial section (3-manifold) at \( \sim t_0 \), which we call \( \mathcal{M} \),

1. solves the Einstein equations,
2. is regular, and
3. has an approximately homogeneous density \( \rho \).

Here, \( t_0 \) is the age of the Universe at the location of our Galaxy, and \( t_B \) is a function of comoving spatial position defined by the initial big bang singularity. Two distinct subclasses of \( A \) are \( A^d \) (“disconnected”) and \( A^m \) (“multiply connected”) as follows, using coordinate time \( t \).

(i) \( A^d \), in which the universe is born from an initial singularity at \( t \to (\min t_B)^+ \) as two or more spatially disconnected regions (3-manifolds) \( \mathcal{W}_t \), each of which is bounded by at least one singularity (of zero spatial volume) from which comoving space emerges continuously. The \( \mathcal{W}_t \) successively merge together to form the connected 3-manifold \( \mathcal{M} \) at \( t \) where \( t > \max t_B > \min t_B \). The \( \mathcal{W}_t \) themselves are born, in general, at different times, and their enumeration changes as a function of \( t \), because of their mergers.

(ii) \( A^m \), in which the universe is born from an initial singularity at \( t \to (\min t_B)^+ \) as a connected, simply connected region \( \mathcal{W}_t \) bounded by at least two singularities during \( \min t_B < t < \min t_B + \delta t \) for some \( \delta t > 0 \). Comoving space is continuously born from the singularities, which join together smoothly in pairs (or \( n \)-tuples, with \( n > 1 \)), so that the spatial section at \( t > \max t_B \) is a connected, multiply connected 3-manifold \( \mathcal{M} \).

**Theorem 1** (i) The subclass \( A^d \) is non-empty. (ii) The subclass \( A^m \) is non-empty.

The heuristic Gaussian \( t_B \) discussion (Sect. III) suggests that (i) of Theorem 1 is correct, without establishing it rigorously in an exact solution of the Einstein equations. Neither (i) nor (ii) of Theorem 1 are relativistically problematic. However, both (i) and (ii), if they are correct, are contrary to common intuition, since, if the subclasses \( A^d \) and/or \( A^m \) are “common” according to a measure over the class of possible universes, then early, comoving, synchronous topology evolution is likely to occur at time slices that combine very early and very late universe ages \( t - t_B(r, \theta, \phi) \) within a single time slice at \( t \).

Examples of members of \( A^d \) and \( A^m \) are given in Sect. IV.A and Sect. IV.B proving Theorem 1. This provides the basis for hypothesising that solutions with non-constant \( t_B \) are more common than those with constant \( t_B \), in which case formal hypotheses about measure spaces are needed.

**Conjecture 1** For a measure \( \mu \) on \( A \) that is physically motivated at late epochs (and that does not contradict early disconnectedness), the measure of solutions that are not primordially disconnected (in terms of comoving, synchronous coordinate time \( t \)) is small, i.e. \( \mu(A\setminus A^d) \ll \mu(A) \).

**Corollary 1** If

(i) Conjecture 1 is correct, and

(ii) \( g^- \) for our real Universe is chosen randomly from \( A \), and
(iii) the standard deviation of the $t_B$ timescale is $\gg 10^{-60}$ times that estimated empirically in Ref. [12], then spatial disconnectedness occurred at early epochs $t$, in the sense that

$$1 \text{ s} \gg \max_{r,\theta,\phi} \{t - t_B(r, \theta, \phi)\},$$  \hspace{1cm} (12)$$

is satisfied on the spatial section at $t$, but that section also includes significantly post-quantum regions, i.e.

$$\max_{r,\theta,\phi} \{t - t_B(r, \theta, \phi)\} \gg 10^{-43} \text{ s}$$  \hspace{1cm} (13)$$
on the same spatial section, hereafter, a “mixed-epoch” spatial section or time slice.

A. Spatially disconnected sections that merge

We show that class $A^d$, i.e. (i) in Definition 1 is non-empty, using an explicit example of the “string of beads” LTB solution [22] (see also [34]). This is a positively curved solution. This class of solution requires the radial metric component $g_{rr}$ to be defined as a limit, because of behaviour at what (in the FLRW case) is the model’s equator [e.g. 28]. This particular example has a $t_B$ function with sinusoidal behaviour, with all the minima and maxima occurring at a single pair of values, $\min t_B$ and $\max t_B$, respectively. This is not a general requirement, it is just a characteristic of this particularly simple solution.

Using the LTB metric [1] and Equations (2)–(6) we consider an example of the

$$E(r) < 0 \forall r, t$$  \hspace{1cm} (14)$$

subcase. Following Section 8 of [22], define

$$E(r) := -\frac{1}{2}[1 - E_1 \sin^2(r)]$$

$$M(r) := M_0(1 + M_1 \cos r)$$

$$t_B(r) := \frac{-GM}{(-2E)^{3/2}} + GM_0(1 - M_1)$$

$$M_0 := \frac{\Omega_m}{2GH_0(\Omega_m - 1)^1.5}$$  \hspace{1cm} (15)$$

where the FLRW dimensionless matter density param-
FIG. 6. (colour online) As for Fig. 4 the Ricci scalar $R/6$ [17].

FIG. 7. (colour online) As for Fig. 4 the areal radius $R(t, r)$.

FIG. 8. (colour online) As for Fig. 4 the radial metric component $\sqrt{g_{rr}(t, r)}$.

FIG. 9. (colour online) As for Fig. 4 the radial proper length $d(t, r)$ [19] at several pre- and post-connection epochs $t$. The two thick curves match epochs shown in the previous figures.

The early epoch curve in Fig. 7, i.e. for $R(0.42 h^{-1} \text{Gyr}, r)$ shows numerically what can be seen in (4), (5), and (6): provided that the factors that include $E$ and $M$ are well-behaved, the one-sided limit $\eta \to 0^+ \Leftrightarrow \xi \to 0^+$ corresponds to $R(t, r) \to 0^+$, and $t \to t_B(r) \to 0^+$. Thus, as for the solution [12], zero-surface area 2-spheres, i.e. point-like singularities, bound the post-big-bang parts of the universe model.

In coordinate space, it is clear that the spatial sections of the universe are disconnected at $t < 0$. What happens at and near the coordinate points $(t, r) = (0, (2n +
Let us, w.l.o.g., consider \( (t, r) = (0, \pi) \). Since \( R = 0 \) at this point, the metric (1) has a non-Lorentzian signature: this is the initial big bang singularity from which the comoving spatial point \((t > 0, \pi)\) is born. Thus, at \( t = 0 \), the two parts of the spatial section \((0, -\pi < r < \pi)\) and \((0, \pi < r < 2\pi)\) are disconnected from one another by the point \((0, \pi)\) in coordinate space. While disconnection by a single missing point might seem trivial, since mathematically, adding a point to a manifold can remove a singularity, the physical significance would be non-trivial. The addition of a single point “at infinity” to infinite Euclidean 3-space \( \mathbb{R}^3 \) is enough to transform the latter into \( S^3 \), although physically, this would be absurd.

How does the radial component of the metric behave near the connection points \((0, (2n + 1)\pi)\)? For \( t > 0 \), Figs.7 and 8 show the anisotropic way in which the metric evolves. As \( t \to 0^+ \), \( R \) decreases (Fig. 7) but \( g_{rr} \) increases (Fig. 8). The latter increases without bound as \( t \to 0^+ \) at \((t, (2n + 1)\pi)\) and remains infinite at the \( r \) boundaries of the disconnected sections. However, the integrated proper length

\[
d(t, r) := \int_0^r \sqrt{g_{rr}(t, \hat{r})} d\hat{r}
\]

from the centre of an initially disconnected section to its boundary, i.e. to the big-bang singularity, remains finite (bottom-right).

Unless a pre-big-bang scenario is introduced, the comoving spatial sections of the universe during \( \min t_B < t < 0 \) consist of the disjoint union \( \bigcup_{i \in \mathbb{Z}} S^2 \times (0, 1) \). This is not an issue of particle horizons within acausal spatial sections; the spatial sections are disconnected. With the parameters chosen, the delay before these grow and merge with the “rest” of the future-to-be-created spatial section is more than \( 100 \times 10^{-5} \) Myr, i.e. long after nucleosynthesis.

Thus, \( A^d \) of Definition 1 is a non-empty set. The age of the universe \( t_0 \) used in this example is that for an FLRW model with \( \Omega_m = 1.013, \Omega_L = 0 \); a \( t_0 = 10^{-5} \) Gyr model can be calculated trivially by modifying \( M_0 \).

B. A connected, simply connected section that becomes multiply connected

Taking the solution (16), (17), apply the holonomy

\[
\gamma : (t, r, \theta, \phi) \mapsto (t, r + 2\pi, \theta, \phi).
\]

The spatial sections for \( t > 0 \) are multiply connected, i.e. \( S^2 \times S^1 \). This is an exact, non-vacuum solution of the Einstein equations with a multiply connected spatial section, similar for \( t > 0 \) to the \( S^2 \times S^1 \) solution published earlier [36][37].

But at \( t < 0 \), the spatial section is \( S^2 \times (0, 1) \) i.e. it is a single, connected, simply connected 3-manifold. Hence, a simply connected universe can smoothly become multiply connected at early (post-quantum) epochs: the class of solutions \( A^0 \) of Definition 1 is non-empty, establishing Theorem 1. Figure 10 shows the universal covering space of this solution.

V. DISCUSSION

A. Does relativistic, post–quantum-epoch topology evolution require teleology?

Section IV establishes that universe models that evolve from being disconnected to being connected, and from being simply connected to being multiply connected, exist as classical, relativistic spacetimes. If Conjecture 1 is correct, i.e. if disconnected solutions are common, then Corollary 1 implies that the inverse method of using extragalactic, astronomical observations to extrapolate back towards the initial singularity, for example, numerically using the \((3+1)\)-formalism [e.g. 38], would be likely to yield evidence of spatial disconnectedness in post–quantum-epoch time slices, that merge together as coordinate time \( t \) increases. This is intuitively surprising.

Is this a problem of teleology? [39] How is it possible for the singularities in spatially disconnected regions to “know” where other regions and their singularities are “located” in order for the singularities to join together by the “creation” of new comoving space? A coordinate system such as that used for LTB models is convenient to work with, but if there are spatial islands in the spatial part of the coordinate system, then the remaining “sea” of space consists of a purely fictional construct—useful for coordinate-based intuition—until comoving space is born there, converting it from fictive, coordinate space
to physical (metric) space. If we only have a relativistic spacetime (with a Lorentzian metric everywhere), then individual \( W_i \) cannot “know” that they must be embedded in a future coordinate system that will unite them. The transition from simple to multiple connectedness is conceptually simpler, since the singularities exist in the same, connected initial manifold, but still appears to require spacelike physical interaction.

For the block universe interpretation of a Lorentzian spacetime [e.g. 40], there is no problem of teleology: all of spacetime in the Lorentzian 4-manifold “just is”. Lorentzian causality concerns the past and future time cones of a given spacetime event, not the time coordinate of a given spacetime foliation. The topology of the 4-dimensional spacetime viewed in terms of the comoving, synchronous representation of the metric is a property of the choice of foliation. A foliation defined by a time coordinate which makes the universe age constant within any given spatial section could be defined for the same spacetime [e.g. Sect. II.A.41. For this refoliation of the model of Sect. [V.A] there would be no topology evolution until epochs shortly before the big crunch (when the universe would become disconnected and the disconnected sections would end “simultaneously” in disconnected, individual big crunches). However, the coordinates would be non-comoving or asynchronous or both. The question of interpretation would then be: would it be reasonable to have initial conditions in a constant-\( t_B \) foliation whose later evolution (with constant 3-spatial topology) describes a 4-dimensional spacetime that can equivalently be described with a simpler expression for the metric, i.e. in comoving, synchronous coordinates, but with spatial topology evolution? The evolution from a complicated metric expression to a simpler one could be seen as teleological.

Similarly, for the multiplied connected model of Sect. [V.B] a constant-\( t_B \) refoliation would imply an interpretation that the universe is born multiply connected, with a non-comoving and/or asynchronous representation of the metric, and becomes simply connected when the big crunch appears as two individual spatial singularities into which all of comoving space disappears. Is this simpler than a universe which is born simply connected, but has a metric representation that is comoving and synchronous at all times?

In both cases, there is a conflict in terms of Occam’s Razor and avoidance of teleology. What is the preferred model: a metric that can be expressed in a simple way with an evolving topology, or a simple (trivial) early topology evolution with a complicated metric expression? To help consider the former possibility, we speculate on the minimal properties that an extension of general relativity could require in Sect. [V.B]

B. Evolution of a connected 3-manifold

Let us consider a more conservative hypothesis than Conjecture \( \text{[I]} \) i.e. a hypothesis that rejects primordial disconnectedness as unlikely, but does not force \( t_B \) to be constant.

Conjecture 2 For a measure \( \mu \) on \( A \) that is physically motivated (and does not contradict the mergers of early epoch singularities),

(i) the measure of solutions that are disconnected is zero, i.e. \( \mu(A^0) = 0 \), and

(ii) the measure of the class of solutions with constant \( t_B \) on comoving (always connected) spatial sections is small, i.e. \( \mu(A \setminus A^{1B}) \ll \mu(A^{1B}) \), where \( A^{1B} \) is the class of solutions with non-constant \( t_B \) and spatial sections that are always connected over \( \min t_B < t < t_0 \).

By the definition of \( A^m \) (Definition [I]), \( A^m \subset A^{1B} \).

If Conjecture 2 is correct, then a universe is most likely to be born connected at \( \min t_B \), with at least one singularity that disappears later (as in Sect. [I]). If the universe is born with many singularities, then some may disappear individually (as in Sect. [I]), some may disappear in pairs (as in Sect. [V.B]), and others could, in principle, disappear in \( n \)-tuples with \( n > 2 \), even though it may be hard to find exact metrics as examples of regular mergers of \( n > 2 \) primordial singularities. Thus, if Conjecture 2 is correct and if the universe is born with many (\( N \gg 1 \)) singularities, then an example of a minimal extension of general relativity that would describe the evolution of the universe would be a definition \( \forall i, j \in \mathbb{Z} : i, j \leq N \) of

\[
\begin{align*}
(\text{i}) & \quad P_1(g(t), t), \text{ the probability that singularity } i \text{ at time } t \\
& \text{i} \text{ disappears at time } t \text{ in a way such that } g \text{ is regular } \forall t' < t + \delta \text{ for some } \delta > 0 \text{ over the whole spatial section}, \\
(\text{ii}) & \quad \forall n : 2 \leq n \leq N, P_n(g(t), t), \text{ the probability that the singularities } t_1, t_2, \ldots, t_n \text{ at time } t \text{ merge together at time } t \text{ in a way such that } g \text{ is regular } \forall t' < t + \delta \text{ for some } \delta > 0 \text{ over the whole spatial section}.
\end{align*}
\]

Given the existence of the numerical solution in Sect. [I] and the analytical solution in Sect. [V.B] and the requirement that in the latter case, the two pre-merger metrics must be post-merger compatible, it would seem reasonable that \( P_1 \gg P_2 \) and \( i < j \Rightarrow P_i \gg P_j \), although this is only speculation. Two obvious classes of models would be those that define the probabilities \( P_1 \) and \( P_n, n \geq 2 \) to be independent of the (comoving) spatial locations of neighbourhoods of the singularities, and those that define the probabilities to be dependent on the spatial locations or on global properties (e.g. mean 3-Ricci scalar, topology) of the spatial section. The \( P_1 \) and \( P_n, n \geq 2 \) could
also depend on the comoving spatial number density of the singularities.

A model of the functions $P_n, n \geq 1$ would provide a minimal extension of general relativity which could be used to calculate the probabilities that a universe evolved from a connected, simply connected spatial section to a connected, multiply connected spatial section, and which topologies would be more likely to remain at $t > \max t_B$. If the $P_2$ and the number of spatial singularities are high enough, then evolution to a multiply connected spatial section would become more likely than to a simply connected spatial section. If the standard deviation of the $t_B$ timescale is $\gg 10^{-60}$ times that estimated empirically in Ref. [12], then this model would apply at significantly post-quantum epochs. Nevertheless, the requirement of discreteness (since the singularities are discrete within comoving spatial sections) and the suggested probabilistic nature of the model suggest a quantum model.

C. Inferences from recent time cone observations

Let us now reconsider inferences from observations. Suppose that a given function $t_B$ estimated from observations has a standard deviation $\sigma(t_B)$ over the three spatial coordinates that is $\sim 10^{20}$ times lower than $\sigma(t_B)$ for the empirically derived $t_B$ in the solution [12], i.e. a much more conservative estimate by many orders of magnitude. The study of LTB models with what are called “decaying modes” (in a perturbed FLRW context) in comparison with observations [11, 12] indicates that $\sigma(t_B)$ is likely to be several orders of magnitude lower than that estimated for illustrative purposes by [13]. Let us also suppose that a comoving synchronous metric accurately describes the evolution of the observed recent Universe backwards towards the initial singularity, and that (for simplicity) we ignore the need to consider the change to a radiation-dominated epoch.

In this case, we would infer topology evolution of spatial sections that are significantly post-Planck but early, i.e. combining [12] and [13] as

$$1 \, s \gg \max \{t - t_B(r, \theta, \phi)\} \gg 10^{-43} \, s.$$  

(VI. CONCLUSION)

We have examined the early epoch topology evolution that corresponds to non-simultaneous big bang times in non-empty, inhomogeneous dust models of the Universe using a recent empirical estimate and an older analytical exact solution. Even if $\sigma(t_B)$ estimated empirically is overestimated by several tens of orders of magnitude (in their introduction, the authors suggest that a more realistic time scale would be $\sim 100 \, \text{yr}$, i.e. about $10^7$ times shorter than their empirical solution [12]), it is still post-quantum unless $t_B$ is constant to within the Planck time scale, i.e. $\sim 10^{-43} \, s$. Other estimates of $\sigma(t_B)$ vary from $\sim 20 \, \text{Gyr}$ over about $4 \, \text{Gpc}$, suggesting $\sigma(t_B)$ of about the same order of magnitude. The solution in [12] has $\max t_B - \min t_B \gtrsim 2 \, \text{Gyr}$ over about $4 \, \text{Gpc}$, suggesting $\sigma(t_B)$ of about the same order of magnitude. Thus, the temporal evolution implied by $t_B$-inhomogeneous models (ignoring the need to enter the radiation-dominated epoch) may imply 3-spatial topology evolution for a comoving, synchronous representation of the metric, either from disconnected spatial sections to connected spatial sections (Sect. IV.A), or from multiply connected to simply connected spatial sections (Sect. IV.B).

This surprising implication could be avoided by imposing $t_B$ = constant as an assumption in cosmological modelling. One problem in assuming constant $t_B$ is that for flat LTB solutions, generalisations beyond the FLRW model are rejected. That is, the combination of $E(r) = 0$ and $t_B(r) = 0$ leaves no freedom to adjust the third “arbitrary” function $M(r)$; see VIII (63a), XIV.B in Ref. [35]. Section XIV.B of Ref. [35] also discusses the restrictions on LTB models implied by imposing $t_B(r) = 0 \forall r$ in the positive and negative $E(r)$ cases. More importantly from a physical point of view, allowing $t_B$ spatial dependence to be a result of comparison between models and observations rather than an assumption could potentially lead to evidence for early universe spatial sections that in comoving, synchronous coordinates undergo topology evolution. This evidence would be artificially suppressed if $t_B = 0 \forall r$ were forced.

We have formalised some of the possible properties of subclasses of solutions of this type and of possible implications in Definition [4], Theorem [4], Conjecture [4], Corollary [4] and Conjecture [2]. Conjecture [2] opens the way to calculations of the probabilities of a simply connected initial spatial section smoothly evolving into a multiply connected spatial section, based on a choice of functions $P_n(\mathbf{g}(t), t), \forall n : 2 \leq n \leq N, P_n(t, r, \theta, \phi)$ as defined in the requirements of a physical theory suggested above (Sect. IV.B). Understanding how a multiply connected spatial section arises would have considerable observational interest [e.g. [52]], since it would offer an alternative to the topological acceleration effect [53] for theoretical understanding of the topology of the present-day (i.e. recent time-cone) Universe (see also [54, 55]).

How was it possible that post-quantum-epoch topology change without causality problems was overlooked in cosmic topology literature? It has generally been thought that if the spatial sections of the Universe are compact,
then the topology of spatial sections of the Universe cannot have evolved at post-quantum epochs, because this would imply the existence of closed timelike curves or a discontinuity in the choice of the forward light cone, as a consequence of Geroch’s Theorem [58, Sect. 9.4.1.59, 60], and both are generally considered unphysical. Singularities make spacelike sections non-compact, so that the theorem no longer applies, but it is not immediately obvious that an astrophysically realistic black (let alone white) hole could change the large-scale, global topology of spatial sections in a way that leads to approximate homogeneity in the late-time Universe. What was overlooked was the fact that a non-constant $\Omega$ provides (in general) an arbitrary number of singularities in early, post-quantum comoving spatial sections, which in spacetime constitute just one singularity—the initial big bang singularity which is generally accepted as physical in relativistic, non-quantum cosmology. Moreover, as illustrated above, the density and curvature inhomogeneities near vanished singularities/connection points can become much weaker, i.e. enter a “decaying mode” in a perturbed FLRW context. The LTB models provide a useful tool for studying examples of characteristics that are counter-intuitive for FLRW-like models.

Although in this work we only consider topology change implied by non-constant $t_B$ in classical relativity, see Ref. [60] for a quantum gravity approach using sums of histories and Morse theory.

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