Renormalization of Quantum Electrodynamics and Hopf Algebras

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Abstract

In 1999, A. Connes and D. Kreimer have discovered the Hopf algebra structure on the Feynman graphs of scalar field theory. They have found that the renormalization can be interpreted as a solving of some Riemann — Hilbert problem. In this work a generalization of their scheme to the case of quantum electrodynamics is proposed. The action of the gauge group on the Hopf algebra of diagrams are defined and the proof that this action is consistent with the Hopf algebra structure is given.

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1 Introduction

The mathematical theory of renormalization (R-operation) was given by N.N. Bogoliubov and O.S. Parasiuk [1]. K. Hepp has elaborated on their proofs [2].

In 1999, A. Connes and D. Kreimer [3, 4] have introduced the Hopf algebra structure on the Feynman graphs in scalar field theory with $\varphi^3$ interaction. The Hopf algebras play an important role in the theory of quantum groups and other noncommutative theories. (About noncommutative field theory and its relation to p-adic analysis see [5, 6].) The discovery of the Hopf algebra structure has clarified the complicated combinatorics of the renormalization theory. The Feynman amplitudes in this theory belongs to the group of characters. Let $U$ be a character corresponding to the nonrenormalized amplitudes, $R$ be a character corresponding to the renormalized amplitudes and $C$ be a character corresponding to the counterterms. The following equality holds.

$$R = C \star U.$$  \hspace{1cm} (1)

Here, the star means the group operation in the group of characters. Let $U_d$ be a character corresponding to the dimensionally regularized Feynman amplitudes. $U_d$ is holomorphic in a certain punctured neighbourhood of the point $d = 6$ ($d$ is a parameter of the dimensional regularization). We can consider $U_d$ as data of some Riemann — Hilbert problem on the group of characters [7]. A. Connes and D. Kreimer have proven that this problem has an unique solution, and the positive and negative parts of the Birkhoff de-
composition define renormalized amplitudes and counterterms respectively. (About future elaboration of this scheme see [8, 9, 10].)

In the present paper, we generalize this scheme to the case of quantum electrodynamics. In the theory of gauge fields, it is necessary to verify the gauge invariance of renormalized amplitudes in addition to the proof of the renormalizability by local counterterms. In the case of quantum electrodynamics the gauge invariance is expressed in terms of the Ward identities.

So we have the following two problems.

a) What is the action of the gauge group on the group of characters?

b) Is the action of the gauge group consistent with the Hopf algebra structure?

The theory of renormalization of nonabelian gauge fields was considered in [11]. The extension of the Hopf algebra formalism to the nonabelian field case requires additional investigation.

2 The Hopf algebra of Feynman diagrams

The Feynman rules for quantum electrodynamics are described in [11]. The Feynman graphs of quantum electrodynamics include the following elements.

a) The electron lines.

b) The photon lines.

c) The vertices of photon-electron interaction. These vertices have precisely two external electron lines and one external photon lines.

d) The vertices of photon self-energy renormalization. These vertices have
precisely two external photon lines.
e) The vertices of electron self-energy renormalization. These vertices have precisely two external electron lines.
f) The three-photon vertices,
g) The four-photon vertices.

Let us assign to each vertex $v$ a number

$$\Omega_v = 4 - 3/2n_E - n_F.$$  \hfill (2)

Here

$n_E$ is a number of electron lines coming into the vertex $v$.

$n_P$ is a number of photon lines coming into the vertex $v$.

By definition the space of all polynomials of degree $\Omega_v$ of momenta coming into the vertex $v$ is called the space of vertex operators at this point and denoted by $S_v$.

Let $\Phi$ be a Feynman graph, and $\varphi$ be a function which assigns to each vertex $v$ the element of $S_v$.

By definition the Feynman diagram is a pair $\Gamma = (\Phi, \varphi)$. We will write $\Phi_\Gamma, \varphi_\Gamma$ e.t.c. to point out the fact that these elements corresponds to the diagram $\Gamma$.

**Definition.** The algebra $\mathcal{H}$ of Feynman diagrams is an unital commutative algebra generated by the pairs $(\Gamma, \sigma)$. Here $\Gamma$ is a Feynman diagram and $\sigma$ is a tensor-valued distribution with compact support on the space of momentas of external lines. The tensor structure of $\sigma$ corresponds to the types of particles coming into the diagram. $\sigma$ is called an external structure.
The generators \((\Gamma, \sigma)\) satisfy the following relations

\[
(\Gamma, \lambda \sigma_1 + \mu \sigma_2) = \lambda(\Gamma, \sigma_1) + \mu(\Gamma, \sigma_2),
\]

\[
\lambda(\Gamma_1, \sigma) + \mu(\Gamma_2, \sigma) = (\Gamma_3, \sigma).
\]

(3)

Here, \(\Gamma_1, \Gamma_2,\) and \(\Gamma_3\) are the same as graphs and for some fixed vertex \(v_0\) of \(\Phi_{\Gamma_1}\)

\[
\varphi_{\Gamma_1}(v) = \varphi_{\Gamma_2}(v) = \varphi_{\Gamma_3}(v), \text{ if } v \neq v_0
\]

\[
\varphi_{\Gamma_3}(v_0) = \lambda \varphi_{\Gamma_1}(v_0) + \mu \varphi_{\Gamma_2}(v_0).
\]

(4)

Let us introduce the following notation. Let \(\Omega_\Gamma = 4 - N_F - \frac{3}{2} N_E\), where \(n_F, n_E\) are the numbers of photon and electron lines coming into \(\Gamma\), respectively.

Let \(S_{\Gamma}\) be a space of all tensor-valued smooth functions on the space of momentas of external lines. The tensor structure of the elements of \(S_{\Gamma}\) corresponds to the types of particles coming into the diagram. \(S_{\Gamma}\) is a locally convex space with respect to the topology of uniform convergence of all derivatives on each compact. Let \(S_{\Gamma}^*\) be a dual space of the space \(S_{\Gamma}\). \(S_{\Gamma}^*\) is a space of all tensor-valued distributions with compact support.

Let \(S_{\Gamma}^{\Omega_\Gamma}\) be a subspace of \(S_{\Gamma}\) of all polynomials of degree less or equal to \(\Omega_\Gamma\). Let \(S_{\Gamma}^{\Omega_\Gamma*}\) be a dual space of \(S_{\Gamma}^{\Omega_\Gamma}\). Let \(\Pi_{\Omega_\Gamma} : S_{\Gamma} \rightarrow S_{\Gamma}^{\Omega_\Gamma}\) be a map which assigns to each function \(f\) its Taylor polynomial of degree \(\Omega_\Gamma\) with the center at zero. Let \(B_{\Gamma}^{\Omega_\Gamma} = \{l_{\Gamma}^\alpha\}, \alpha \in A_{\Gamma}\) be some basis of the space \(S_{\Gamma}^{\Omega_\Gamma}\). Let \(B_{\Gamma}^{\Omega_\Gamma*} = \{l_{\Gamma}^{\alpha*}\}\) be a dual basis of \(B_{\Gamma}^{\Omega_\Gamma}\). To each element \(l_{\Gamma}^{\alpha*} \in B_{\Gamma}^{\Omega_\Gamma*}\), assign the external structure \(\tau_{l_{\Gamma}^{\alpha*}} := l_{\Gamma}^{\alpha*} \circ \Pi_{\Omega_\Gamma}\).
Fix an one-particle irreducible diagram $\Gamma$. Let $V$ be a set of all vertices of diagram $\Gamma$. Let $V'$ be a subset of $V$. Let $R$ be a set of all lines of diagram $\Gamma$, $R_{in}'$ be a set of all internal lines of $\Gamma$ and $R_{ex}'$ be a set of all external lines of diagram $\Gamma$. Let $R_{in}'$ be a subset of all lines from $R_{in}$ such that: $\forall r \in R_{in}'$ booth of its endpoints are the elements of $V'$. Let $R_{ex}'$ be a set of all pairs $(r,v)$, where $r$ is a line (external or internal) and $v \in V'$ such that $r$ has only one endpoint from $V'$ and this endpoint is $v$.

Consider the diagram $\gamma$, such that the set of its vertices is $V'$, the set of its internal lines is $R_{in}'$, the set of its external lines is $R_{ex}'$, and the function $\varphi_{\gamma}$ is a restriction of $\varphi_{\Gamma}$ to $V'$. If $\gamma$ is one particle irreducible diagram than $\gamma$ is called an one-particle irreducible subdiagram.

A set of one-particle irreducible nonintersecting subdiagrams of $\Gamma$ is called a subdiagram. An element of the subdiagram $\gamma$ is called a connected component of subdiagram. Let $M$ be a set of all connected components of $\gamma$. Let $\alpha$ be a map which to each element $i \in M$ assigns an element $\alpha(i)$ of $\mathcal{A}_\gamma$. $\alpha$ is called an multiindex.

If $\gamma'$ is one-particle irreducible subdiagram, we can replace it by corresponding vertex $v_0$. Now, we can essentially embed $S^{\gamma'}_{\gamma}$ into the space of vertex operators at the corresponding vertex $v_0$. To each subdiagram $\gamma \in \Gamma$ and multiindex $\alpha$, assign an element of the diagram algebra $\gamma_\alpha := \prod_{i \in M} (\gamma_i; \tau_{\gamma_i}(i)^\ast)$. To obtain the quotient diagram $\Gamma/\gamma_\alpha$, we need to replace each connected component $\gamma_i$ by the corresponding vertex $v_i$ and put $\varphi(v_i) = l_{\gamma_i}^{\alpha(i)}$. (Here, we use the previous identification.)

**Definition.** A coproduct $\Delta$ is a homomorphism $\mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ defined on
generators of $\mathcal{H}$ by the following formula:

$$\Delta((\Gamma, \sigma)) = (\Gamma, \sigma) \otimes 1 + 1 \otimes (\Gamma, \sigma) + \sum_{\emptyset \subset \gamma \subset \Gamma} \gamma_\alpha \otimes (\Gamma/\gamma_\alpha, \sigma). \quad (5)$$

**Remark.** In the previous formula, $\subset$ means strong inclusion. The expression $\gamma_\alpha \subset \Gamma$ means that $\gamma \subset \Gamma$. The sum is over all nonempty subdiagram and and multiindices $\alpha$.

Analogusly to [3, 4] we have the following theorems.

**Theorem 1.** The coproduct $\Delta$ is well defined and does not depend on the special choice of a basis $B^{\Omega, r}_\Gamma$ in the space $S^{\Omega, r}_\Gamma$.

**Theorem 2.** The homomorphism $\Delta$ is coassociative. Moreover we can find counit $\varepsilon$ and antipode $S$ and such that the set $(\mathcal{H}, \Delta, \varepsilon, S)$ is a Hopf algebra.

A character $\chi$ on the Hopf algebra $\mathcal{A}$ is a homomorphism $\chi : \mathcal{A} \to \mathbb{C}$. The set of all characters is a group with respect the convolution as a group operation.

### 3 Gauge transformation on the Hopf algebra of diagrams of quantum electrodynamics

To each one-particle irreducible Feynman diagram, we can assign the Feynman amplitude $\Sigma_\Gamma(...p,...,p',...,k,...)$. (Here, $...,p,...,p',...,k,...$ are the momentas, of external electron, positron, and photon lines respectively [11, 12].) If we regularize the infrared divergences by giving a nonzero mass to photon, than $\Sigma_\Gamma(...p...p'...k...) is a smooth function of external momentas for small...
nonzero $z := d - 4$. Here $d$ is a space-time dimensions. The Feynman amplitudes define a character on the Hopf algebra by the formula

$$U((\Gamma, \sigma)) = \int \Sigma\Gamma(..., p, ..., p', ..., k, ...)\sigma(..., p, ..., p', ..., k, ...)dp...dp'...dk...$$

(6)

**ξ-insertion.** ξ-insertion $[\Pi]$ can be made either into a vertex or into an electron line. Let us define the ξ-insertion into a vertex. Consider a vertex that has $n$ entering electron lines, $n$ entering positron lines and $m$ entering photon lines. Suppose that the vertex operator at this vertex has the form

$$P(p_1, ..., p_n|p'_1, ..., p'_n|k_1, ..., k_m).$$

(7)

Let $\Omega$ be a degree of $P(p_1, ..., p_n|p'_1, ..., p'_n|k_1, ..., k_m)$. By definition ξ-insertion assigns to this vertex operator the vertex operator of degree $\Omega - 1$

$$P_\xi^\mu(p_1, ..., p_n|p'_1, ..., p'_n|k_1, ..., k_m|k)$$

(8)

defined as

$$k^\mu P_\xi^\mu(p_1, ..., p_n|p'_1, ..., p'_n|k_1, ..., k_m|k) =$$

$$= \sum_{i=1}^n\{P(p_1, ..., p_i + k, ..., p_n|p'_1, ..., p'_n|k_1, ..., k_m) -$$

$$- P(p_1, ..., p_n|p'_1, ..., p'_i + k, ..., p'_n|k_1, ..., k_m)\}.$$

(9)

After ξ-insertion the vertex will have a new external photon appendix. The polynomial defined by this condition exists but it is defined ambiguously.

To define a ξ-insertion into the electron line, we need to insert a vertex of the photon-electron interaction into the electron line and put $P_\xi^\mu(p_1, p'_1|k) = \ldots$
\( \gamma^\mu \). After this \( \xi \)-insertion the vertex will have a new external appendix.

**Ward identities.** Let \( \Sigma_\Gamma \) be a Feynman amplitude corresponding to the one-particle irreducible diagram \( \Gamma \). Suppose that the diagram has \( n \) entering electron lines, \( n \) entering positron lines, and \( m \) entering photon lines. Then,

\[
\sum_{i=1}^{n} \{ \Sigma_\Gamma (p_1, \ldots, p_i + k, \ldots, p_n | p'_1, \ldots, p'_n | k_1, \ldots, k_m) - \\
- \Sigma_\Gamma (p_1, \ldots, p_n | p'_1, \ldots, p'_i + k, \ldots, p'_n | k_1, \ldots, k_m) \} = \\
\sum_{\xi} \{ k^\mu \Sigma_\Gamma^\mu_\xi (p_1, \ldots, p_n | p'_1, \ldots, p'_n | k_1, \ldots, k_m | k) \}. \tag{10}
\]

In the last line, the sum is over all \( \xi \)-insertion into the diagram \( \Gamma \). (The proof is similar to the proof in Bogoliubov and Shirkov’s monograph \( [1] \).)

**Gauge transformation on the Hopf algebra of Feynman diagrams.** Let us define now a gauge transformation on the Hopf algebra of diagrams \( \mathcal{H} \). For each distribution with compact support \( \alpha \in \mathcal{E}'(\mathbb{R}^4) \) define a map \( \delta_\alpha : \mathcal{H} \to \mathcal{H} \) by the following formulas:

\[
\delta_\alpha = \delta'_\alpha + \delta''_\alpha, \tag{11}
\]

\[
\delta'_\alpha((\Gamma, \sigma)) = (\Gamma, \sigma_\alpha), \quad \delta''_\alpha((\Gamma, \sigma)) = \sum_{\xi} (\Gamma_\xi, \xi_\alpha(\sigma)) \tag{12}
\]

\[
\sigma_\alpha (p_1, \ldots, p_n | p'_1, \ldots, p'_n | k_1, \ldots, k_m) = \\
\sum_{i=1}^{n} \int \alpha(k) \{ \sigma (\tilde{p}_i, \ldots, \tilde{p}_i - k, \ldots, \tilde{p}_n | \tilde{p}'_1, \ldots, \tilde{p}'_n | k_1, \ldots, k_m) - \\
- \sigma (\tilde{p}_1, \ldots, \tilde{p}_n | \tilde{p}'_1, \ldots, \tilde{p}'_i - k, \ldots, \tilde{p}_n | k_1, \ldots, k_m) \}, \tag{13}
\]

where

\[
\tilde{p}_i = p_i + \frac{k}{2n}, \tag{14}
\]
\[ \tilde{p}_i' = p_i' + \frac{k}{2n}. \] (15)

The sum in (12) is over all \( \xi \)-insertion into diagram \( \Gamma \) and

\[
\xi_\alpha(\sigma)(p_1, \ldots, p_n|p_1', \ldots, p_n'|k_1, \ldots, k_m) =
\]

\[
= \sigma(\tilde{p}_1, \ldots, \tilde{p}_n|\tilde{p}_1', \ldots, \tilde{p}_n'|k_1, \ldots, k_m|k)\alpha(k) \otimes k. \] (16)

Moment \( k \) corresponds to the new external line. \( \sigma \) is a tensor-valued distribution. We use the symbol \( \otimes \) in last formula in this sense. On the products of elements, we define \( \delta_\alpha \) by the Leibniz rule

\[ \delta_\alpha(ab) = \delta_\alpha(a)b + a\delta_\alpha(b). \]

\( \delta_\alpha \) is called a gauge transformation on the Hopf algebra of diagrams. Let \( G \) be the group of characters. Then \( \delta^*_\alpha : G \to TG, U \mapsto T_U G, \)

\[
(\delta^*_\alpha U)(X) := U(\delta_\alpha(X)), \] (17)

\( X \in H. \) \( \delta^* \) is called a gauge transformation on the group of characters. (Here, \( TG \) is the tangent bundle of \( G \) and \( T_U G \) is the tangent space at the point \( U. \))

**Definition.** A character \( U \) is called gauge-invariant if \( \delta^*(U) = 0. \)

**Remark.** Dimensionally regularized Feynman amplitudes satisfy the Ward identities; therefore the corresponding characters are gauge invariant.

**Definition.** A character \( U \) is called gauge invariant up to degree \( n \) if \( \delta^*(U)((\Gamma, \sigma)) = 0 \) for all diagrams \( \Gamma \) that have at most \( n \) vertices.
4 Main Results

Main Theorem. Let $C$ be a gauge invariant character up to degree $n$. Than, for all diagrams $\Gamma$ that have at most $n$ vertices the following relation holds:

$$\delta^*_\alpha(C \ast U)(\Gamma, \sigma) = C \ast (\delta^*_\alpha U)(\Gamma, \sigma).$$  \hspace{1cm} (18)

$\forall \alpha \in E'(\mathbb{R}^4)$

To prove the theorem, we need following lemma.

Lemma. Let $\Gamma$ be a one-particle irreducible diagram, $\gamma$ be a one-particle irreducible subdiagram of $\Gamma$, and $C$ be a gauge invariant character. Then,

$$\Gamma' := \sum_{\xi \in \gamma} \sum_{\alpha} C((\gamma \xi)\alpha)(\Gamma_{\xi}/(\gamma \xi)\alpha) = \sum_{\alpha} C(\gamma\alpha)(\Gamma/\gamma\alpha)\xi.$$  \hspace{1cm} (19)

Here, on the left-hand side, $\xi$-insertions are made into all vertexes and internal electron lines of $\gamma$, whereas on the right hand side, $\xi$-insertions are made into a vertex obtained by replacing $\gamma$ with a point.

Proof Suppose that there exists a smooth function $f_C^\gamma$ such that $C(\gamma, \sigma) = \int f_C^\gamma \sigma\ldots dp\ldots dk\ldots$. Denote by $v_0$ the vertex of the diagram $\Gamma_{\xi}/\gamma\xi$ obtained by replacing $\gamma\xi$ by point. Since $C$ is gauge invariant, we have

$$\sum_{i=1}^{n} \{ f_C^\gamma(p_1, \ldots, p_i + k, \ldots, p_n|p_{i+1}', \ldots, p_{n}|k_1, \ldots, k_m) - f_C^\gamma(p_1, \ldots, p_n|p_1', \ldots, p_{n}|k_1, \ldots, k_m) \} =$$

$$= \sum_{\xi} \{ k^\mu (f_C^{\gamma\xi})^\mu(p_1, \ldots, p_n|p_1', \ldots, p_{n}|k_1, \ldots, k_m|k) \}.$$  \hspace{1cm} (20)

Hence,

$$\Pi_{\Omega, \gamma} \{ \sum_{i=1}^{n} \{ f_C^\gamma(p_1, \ldots, p_i + k, \ldots, p_n|p_{i+1}', \ldots, p_{n}|k_1, \ldots, k_m) -$$
\[-f^\gamma_C(p_1, \ldots, p_n|p'_1, \ldots, p'_n + k, \ldots, p'_m|k_1, \ldots, k_m)\} =
= \Pi_{\Omega,\gamma}\{\sum_{\xi} k^\mu(f^\gamma_C)^\mu(p_1, \ldots, p_n|p'_1, \ldots, p'_n|k_1, \ldots, k_m|k)\}\} =
= k^\mu\Pi_{\Omega,\gamma}\{\sum_{\xi} ((f^\gamma_C)^\mu(p_1, \ldots, p_n|p'_1, \ldots, p'_n|k_1, \ldots, k_m|k)\}}. \quad (21)

But the last term in this formula is equal to \((\varphi_{\Gamma^v}(v_0))^\mu k^\mu\). Therefore, 
\((\varphi_{\Gamma^v}(v_0))^\mu\) is equal to the result of the \(\xi\)-insertion into \(v_0\). In order to prove 
this lemma in the general case, it is enough to note, that there exist smooth 
function \(f^\gamma_C\) such that 
\(C((\gamma, \Pi^*_{\Omega}(\sigma))) = \int f^\gamma_C \ldots \Pi^*_{\Omega}(\sigma) \ldots dp \ldots dk \ldots\)

The lemma is proved.

**Proof of the main theorem.** We have

\[
\delta^\alpha_*(C \ast U)((\Gamma, \sigma)) = (C \ast U)\{\delta^\alpha((\Gamma, \sigma))\} =
= (C \ast U)\{\delta'_\alpha((\Gamma, \sigma)) + \delta''^\alpha((\Gamma, \sigma))\}, \quad (22)
\]

\[
(C \ast U)(\delta'_\alpha((\Gamma, \sigma))) = (C \ast U)(((\Gamma, \sigma_\alpha)) =
= C((\Gamma, \sigma_\alpha)) + \sum_{\emptyset \leq \gamma_\beta \subset \Gamma} C(\gamma_\beta)U((\Gamma/\gamma_\beta, \sigma_\alpha)), \quad (23)
\]

\[
(C \ast U)(\delta''^\alpha((\Gamma, \sigma))) = \sum_{\xi} (C \ast U)((\Gamma_\xi, \xi_\alpha(\sigma))) =
= (\delta''_{\alpha}^\ast C)((\Gamma, \sigma)) + (\delta''_{\alpha}^\ast U)((\Gamma, \sigma)) + \sum_{\emptyset \subset \gamma_\beta \subset \Gamma_\xi} C(\gamma_\beta)U((\Gamma_\xi/\gamma_\beta, \xi_\alpha(\sigma))). \quad (24)
\]

To each subdiagram \(\gamma\) of \(\Gamma\) and \(\xi\)-insertion into \(\Gamma\), assign a subdiagram \(\gamma'\) of 
a diagram \(\Gamma_\xi\) in the following way: if the \(\xi\)-insertion is not a \(\xi\)-insertion into 
\(\gamma\), then \(\gamma' = \gamma\). If the \(\xi\)-insertion is a \(\xi\)-insertion into \(\gamma\), than \(\gamma' = \gamma_\xi\). The
subdiagram $\gamma' \subseteq \Gamma_\xi$ is proper, if and only if $\gamma$ is a proper subdiagram of $\Gamma$.

We have the following expression for the last term in (24):

$$
\sum_{\emptyset \subset \gamma \subset \Gamma} \sum_{\xi \in \gamma} C(\gamma_\beta) U((\Gamma/\gamma_\beta)_\xi, \xi_\alpha(\sigma)) + \sum_{\emptyset \subset \gamma \subset \Gamma, \xi \not\in \gamma} C(\gamma_\beta) U(((\Gamma/\gamma_\beta)_\xi, \xi_\alpha(\sigma))).
$$

(25)

The first term of this expression is equal to

$$
\sum_{\emptyset \subset \gamma \subset \Gamma} \sum_{\xi} C(\gamma_\beta) U(((\Gamma/\gamma_\beta)_\xi, \xi_\alpha(\sigma))).
$$

(26)

Here, the sum is over all $\xi$-insertion into the vertices of the diagram $\Gamma/\gamma_\beta$ obtained by replacing all connected components of $\gamma$ by points. We have

$$
(\delta'^{\prime \prime}_{\alpha}(C \ast U))((\Gamma, \sigma)) = (\delta'^{\prime \prime}_{\alpha} C)((\Gamma, \sigma)) + (\delta'_\alpha U)((\Gamma, \sigma)) + \sum_{\emptyset \subset \gamma_\beta \subset \Gamma} C(\gamma_\beta) U(((\Gamma/\gamma_\beta)_\xi, \xi_\alpha(\sigma))) = (\delta'^{\prime \prime}_{\alpha} C)((\Gamma, \sigma)) + (\delta'^{\prime \prime}_{\alpha} U)((\Gamma, \sigma)) + \sum_{\emptyset \subset \gamma_\beta \subset \Gamma} C(\gamma_\beta) (\delta'^{\prime \prime}_{\alpha} U)((\Gamma/\gamma_\beta, \sigma)).
$$

(27)

It follows from (23) and (27) that

$$
\delta'_\alpha (C \ast U)((\Gamma, \sigma)) = C \ast (\delta'_\alpha U)((\Gamma, \sigma)).
$$

(28)

The theorem is completely proved.

**Remark.** Actually, we have proved the following proposition. if $C$ is gauge invariant character up to degree $n - 1$ and $U$ is a character, than

$$
\delta'_\alpha \{U(\bullet) + \sum_{\emptyset \subset \gamma_\alpha \subset \bullet} C(\gamma_\alpha) U(\bullet/\gamma_\alpha)\} = \delta'_\alpha U(\bullet) + \sum_{\emptyset \subset \gamma_\alpha \subset \bullet} C(\gamma_\alpha) (\delta'_\alpha U)(\bullet/\gamma_\alpha).
$$

(29)
The following theorem holds.

**Theorem 3.** The set of all gauge-invariant characters is a subgroup of the group of characters.

**Proof.** The fact that a product of two gauge invariant characters is a gauge invariant character follows from the main theorem. The counit is obviously gauge invariant. We can prove the gauge invariance of $U^{-1}$ using the remark to the main theorem and the following relation:

$$ U^{-1}(\Gamma) = -U(\Gamma) - \sum_{\emptyset \subset \gamma_a \subset \Gamma} U^{-1}(\gamma_a)U(\Gamma/\gamma_a). $$

(30)

**The Riemann—Hilbert problem.** Let $U_z$ be a character holomorphic in $z$ in a small punctured neighbourhood $O \setminus \{0\}$ of zero. To solve the Riemann-Hilbert problem, we must to find the Birkhoff decomposition of $U_z$, i.e., to find characters $R_z$ and $C_z$ such that $R_z$ holomorphic in $\{0\}$, $C_z$ holomorphic in $C \setminus \{0\}$, the following gluing conditions holds

$$ R_z = C_z \star U_z $$

(31)

in $O \setminus \{0\}$, and $C_z \to \varepsilon$ as $z \to \infty$. This problem has an unique solution. This fact follows from the Liouville theorem.

**Remark.** We say that the character $U_z$ is continuous, holomorphic etc. with in $z$ if $U_z(X)$ is continuous, holomorphic etc. with in $z$ for all $X \in H$.

The following theorem generalize the result of Connes and Kreimer for the scalar model to the case of quantum electrodynamics.

**Theorem 4.** A solution of the Riemann-Hilbert problem exists.
Proof. We follow the Connes-Kreimer proof for the scalar field. A solution of the problem is given by the explicit formulas

\[ C_z(\Gamma) = -T(U_z(\Gamma) + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} C_z(\gamma_\alpha)U_z(\Gamma/\gamma_\alpha)), \quad (32) \]

\[ R_z(\Gamma) = (1 - T)(U_z(\Gamma) + \sum_{\emptyset \subset \gamma_\alpha \subset \Gamma} C_z(\gamma_\alpha)U_z(\Gamma/\gamma_\alpha)). \quad (33) \]

Here by definition the operator \( T \) assigns to each function \( f \) holomorphic in \( O \setminus \{0\} \) its pole part.

Remark. Let \( U_z \) be a character corresponding to the dimensionally regularized Feynman amplitudes. In this case \( C_z \) corresponds to counterterms, and \( R_z \) corresponds to the set of renormalized amplitude.

Theorem 5. If the data of the Riemann-Hilbert problem is gauge-invariant, than the solution of this problem is gauge-invariant too.

Proof. This fact follows from the remark to the main theorem and the obvious fact that the gauge transformation commutes with \( T \).

5 Conclusion

In this paper, we have given a generalization of Connes-Kreimer method in renormalization theory to the case of quantum electrodynamics. We have introduced the Hopf algebra of diagrams, that generalize the corresponding construction of Connes and Kreimer. We have obtained two main results. The first one is that the set of gauge-invariant characters is a group. The
second one is that the Riemann — Hilbert problem has a gauge-invariant solution if the data of this problem is gauge-invariant. It is interesting to generalize these results to the case of nonabelian gauge fields.

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