Self–Dual Supersymmetry and Supergravity in Atiyah–Ward Space–Time

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ABSTRACT

We study supersymmetry and self-duality in a four-dimensional space-time with the signature (2, 2), that we call the Atiyah-Ward space-time. Dirac matrices and spinors, in particular Majorana-Weyl spinors, are investigated in detail. We formulate \( N \geq 1 \) supersymmetric self-dual Yang-Mills theories and self-dual supergravities. An \( N = 1 \) “self-dual” tensor multiplet is constructed and a possible ten-dimensional theory that gives rise to the four-dimensional self-dual supersymmetric theories is found. Instanton solutions are given as the zero modes in the \( N = 2 \) self-dual Yang-Mills theory. The \( N = 2 \) superstrings are conjectured to have no possible counter-terms at quantum level to all orders. These self-dual supersymmetric theories are to generate exactly soluble supersymmetric systems in lower dimensions.

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1Research supported by NSF grant \# PHY–91–19746
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1. Introduction

The relevance of the self-dual Yang-Mills (SDYM) theory in four dimensions [1] appears to be related with the fascinating conjecture (presumably, first raised by Atiyah and Ward [2]) that this theory is likely to be considered as the generating theory for all integrable or exactly soluble models in two and three dimensions, after some suitable compactifications or dimensional reductions. It has recently been extended to make a connection with string theory via the observation made by Ooguri and Vafa on the basis of string amplitude calculations [3] that the consistent backgrounds for $N = 2$ string propagation correspond to self-dual gravity configurations in the case of closed $N = 2$ strings, SDYM configurations in the case of open strings, and SDYM configurations coupled to gravity in the case of $N = 2$ heterotic strings, in four or lower dimensions. It has been known for a long time that the $N = 2$ strings, i.e. strings with $N = 2$ extended local superconformal symmetry on a world-sheet, live in 2 space-time dimensions [4]. Later on, it was emphasized [5] that these are in fact 2 complex dimensions, and the correct signature in real coordinates is $(2, 2)$. We call the four-dimensional space-time with this signature the Atiyah-Ward (AW) space-time in this paper. As will be shown below, the signature $(+, +, -, -)$ is crucial for introducing minimal space-time self-dual supersymmetry. The self-duality (SD) condition on real gauge fields makes sense in a space-time with the $(2, 2)$ or $(4, 0)$ signature, but not with the $(3, 1)$ or $(1, 3)$ one. The previous trials [6] based on the Minkowski or Euclidean signature are not appropriate for our purposes.

The aim of this paper is to extend the SDYM model and the self-dual gravity (SDG) to supersymmetric SDYM and self-dual supergravity (SDSG) models in the AW space-time. To formulate these theories, we need to supersymmetrize the SD condition on the YM and/or gravity field strengths, and the most appropriate way to do this is to use superspace. In this paper we formulate both SDYM and SDSG models with simple ($N = 1$) and extended ($N = 2$ and $N = 4$) supersymmetry by using superfield formulations of supersymmetric Yang-Mills (SYM) theories and supergravities (SG) in appropriate superspaces, revised to $2 + 2$ space-time dimensions. Our approach is based on reviewing the superspace constraints and the Bianchi identities for those theories in a search for a superfield SD condition. Since the SD condition puts the theory on-shell, we can avoid any complications related to the auxiliary field structure of supersymmetric YM and SG theories and restrict ourselves to their on-shell superspace formulations. Proceeding this, we also construct the $N = 1$ supersymmetric self-dual scalar and tensor multiplets, and notice a “no-go” barrier preventing an existence

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4This has been also reconfirmed by $\beta$-function calculation in the last paper in Ref. [3].

5Sometimes we use also the term $(2, 2)$ or $2 + 2$ space-time with the same meaning as $D = (2, 2)$. The expression $D = 4$ is also used, when the signature is not very important.
of the self-dual \( N = 2 \) hypermultiplet and the \( N = 4 \) SDYM.

This paper is organized as follows. In sect. 2 we start with the analysis of basic spinors and (extended) supersymmetry algebras existing in the AW space-time. Sect. 3 is devoted to formulations of the \( N = 1 \) supersymmetric SDYM model and the simple SDSG. Here various \( N = 1 \) supersymmetric self-dual multiplets are introduced. In sect. 4 we give formulations of the \( N = 2 \) supersymmetric SDYM and \( N = 2 \) SDSG theories in the \( N = 2 \) extended superspace. The construction of the \( N = 4 \) SDYM in the extended \( N = 4 \) superspace is presented in sect. 5. The “no-go” barrier is discussed in sect. 6. In sect. 7, we give the way to bypass the “no-go” barrier, introducing a propagating multiplier multiplet. In sect. 8, we give the instanton solutions for the SDYM models we have presented. We give our concluding remarks in sect. 9. Appendix A comprises our notation and conventions. Details about the Dirac gamma matrices in \( D = (2, 2) \) are collected in Appendix B. We briefly discuss the finiteness of the supersymmetric SDYM theories, SDSG and \( N = 2 \) superstrings in Appendix C. The alternative construction of SYM and SG theories in the AW space-time via dimensional reduction from higher dimensions is outlined in Appendix D.

2. Spinors and Supersymmetry in \( 2 + 2 \) Dimensions

To see what spinors can be introduced in \( D = (2, 2) \), it is a good idea to start from the Dirac equation in the AW space-time\footnote{We essentially follow here the construction in Ref. [7].}

\[
\left( i \Gamma^a \partial_a + m \right) \Psi = 0 ,
\]

where the \( 4 \times 4 \) Dirac gamma matrices \( \Gamma^a \) satisfy the Clifford algebra

\[
\{ \Gamma^a, \Gamma^b \} = 2 \eta_{(R)}^{ab} \equiv 2 \text{ diag } (+, +, -, -) .
\]

See the Appendix A for our notation and the Appendix B for details about the Dirac gamma matrices. One of the key issues in this matter is the charge conjugation matrix we are going to discuss in this section (see also Ref. [8] as for a general situation in \( s + t \) dimensions).

First, one notices that the \( \Gamma \)-matrices can be chosen in a way such that

\[
(\Gamma^1)^\dagger = +\Gamma^1, \quad (\Gamma^2)^\dagger = +\Gamma^2, \quad (\Gamma^3)^\dagger = -\Gamma^3, \quad (\Gamma^4)^\dagger = -\Gamma^4 .
\]

In particular, the two explicit representations given in the Appendix B do satisfy eq. (2.3). It follows that

\[
(\Gamma^a)^\dagger = -\Gamma_0 \Gamma^a \Gamma_0^{-1} ,
\]
where the \((2,2)\) analogue of the \((3+1)\)-dimensional \(\Gamma_0\)-matrix has been defined as

\[
\Gamma_0 \equiv \Gamma^1 \Gamma^2, \quad \Gamma_0^{-1} = \Gamma_0^\dagger = -\Gamma_0, \quad \Gamma_0^2 = -1 . \tag{2.5}
\]

To introduce the charge conjugation matrix \(C\) in the usual way, one notices that both \(±(\Gamma^a)^*\) and \(±(\Gamma^a)^T\) separately form equivalent representations of the Clifford algebra \((2.2)\). Therefore, there exist invertible matrices \(B\) and \(C\) for which

\[
(\Gamma^a)^* = \eta B \Gamma^a B^{-1}, \quad (\Gamma^a)^T = -\eta C \Gamma^a C^{-1}, \tag{2.6}
\]

where the sign \(\eta\), \(\eta = ±\), has been introduced. The correlation of the overall signs in these two equations appears to be needed for their consistency with eq. \((2.4)\). It also yields

\[
C = B^T \Gamma_0 , \tag{2.7}
\]

modulo a sign factor which is irrelevant here. It is not difficult to show that the matrix \(B\) is unitary and satisfies \([8]\)

\[
B^* B = 1 . \tag{2.8}
\]

The charge conjugation matrix \(C\) has the properties

\[
C^T = -C , \quad C^\dagger C = 1 , \tag{2.9}
\]

which follow just from the definitions above. It is optional to take \(C^* = -C\).

In the Majorana representation \((B.8)\) of the \(\Gamma\)-matrices, the matrix \(B\) can be chosen to be \(1\) while \(\eta = -1\). In that representation the “Lorentz” generators \(\Sigma^{ab} = (1/2) \left( \Gamma^a \Gamma^b - \Gamma^b \Gamma^a \right)\) are real. Dividing them into chiral pieces \(\Sigma^{ab}_± = (1/2) (1 ± \Gamma_5) \Sigma^{ab}\) yields two commuting \(sl(2,\mathbb{R})\) algebras generated by \(\Sigma^{ab}_+\) and \(\Sigma^{ab}_-\) respectively. This nicely illustrates the well-known isomorphism \([9]\)

\[
\overline{SO(2,2)} \cong SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R}) , \tag{2.10}
\]

where \(\overline{SO(2,2)}\) is the covering group of \(SO(2,2)\).

In the explicit representation \((B.10)\) we have

\[
(\Gamma^1)^* = -\Gamma^1, \quad (\Gamma^2)^* = +\Gamma^2, \quad (\Gamma^3)^* = +\Gamma^3, \quad (\Gamma^4)^* = -\Gamma^4 . \tag{2.11}
\]

It follows that

\[
\text{For } \eta = -1 : \quad B = \Gamma^3 \Gamma^2, \quad C = \Gamma^1 \Gamma^3 = \begin{pmatrix} +\tau_2 & 0 \\ 0 & +\tau_2 \end{pmatrix} , \tag{2.12}
\]

4
and

For $\eta = +1: \quad B = \Gamma^1 \Gamma^4$, \quad C = \Gamma^2 \Gamma^4 = \begin{pmatrix} +\tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix}.

(2.13)

In particular, $C^* = -C$ and $\tilde{\sigma} = \tau_2 (\sigma^2)^T \tau_2$ (at $\eta = -1$) in this representation. The $\tilde{\sigma}$ and $\sigma^2$ are in fact related by the charge conjugation matrix in any representation of the gamma matrices.

We are now in a position to discuss the simplest spinor representations of the lowest dimension in the AW space-time. The Dirac equation (2.1) defines the Dirac spinor $\Psi$ whose transformation properties follow from requiring the covariance of the Dirac equation. The Dirac spinor has 4 complex components and this representation is clearly reducible because of the existence of chiral projection operators. The chiral projectors $P_{\pm} = (1/2)(1 \pm \Gamma_5)$ can be used to define the chiral or Weyl spinors, $\psi_{\pm} = \pm \Gamma_5 \psi_{\pm}$.

In the convenient representation (B.10), we have

$$
\Psi_{\alpha} = \begin{pmatrix} \psi_{\alpha} \\ \tilde{\psi}_{\alpha} \end{pmatrix},
$$

(2.14)

where each Weyl spinor, $\psi$ or $\tilde{\psi}$, has 2 complex components, $\alpha = 1, 2; \quad \tilde{\alpha} = 1, \tilde{2}; \quad \alpha = (\alpha, \tilde{\alpha})$.

As in $D = (1,3)$, the antisymmetric tensors, $C^{\alpha \beta}$, $C_{\alpha \beta}$ and $C^{\alpha \beta}_{..}$, $C_{\alpha \beta}^{..}$, defined by chiral pieces of the charge conjugation matrix in eq. (2.13), can be used to raise and lower the chiral spinor indices $\alpha$ and $\tilde{\alpha}$. It is the specialty of the AW space-time that the projections $P_{\pm} \Gamma_{\alpha}$ separately transform under the $B$ conjugation of eq. (2.6).

The complex conjugation of the Dirac equation (2.1) yields

$$
\left[ i \Gamma^a \partial_a + (-\eta) m \right] \left( B^{-1} \Psi^* \right) = 0.
$$

(2.15)

Given $(-\eta)m = m$, this is the Dirac equation for $\Psi^c \equiv B^{-1} \Psi^*$. Since $B^* B = 1$, it is consistent to equate

$$
\Psi^* = B \Psi \quad \text{or} \quad \Psi^c = \Psi.
$$

(2.16)

The $\Psi^c$ is known as the Majorana-conjugated spinor, while eq. (2.16) is known as the Majorana condition. In the massive case, this is only possible if $\eta = -1$. In the massless case there is another option, $\eta = 1$, which allows to define the pseudo-Majorana spinors as those satisfying eq. (2.16). Introducing the $(2,2)$ analogue of the Dirac conjugation as

$$
\overline{\Psi} = \Psi^T \Gamma_0,
$$

(2.17)

we can rewrite the Majorana condition (2.16) to the form which is familiar from $D = (1,3)$:

$$
\Psi = \Psi^c = C \overline{\Psi}^T,
$$

(2.18)

\footnote{See eq. (B.11) for the definition of $\Gamma_5$.}
provided the representation of $\Gamma$-matrices, in which $C^* = -C$, is used. In the representation (B.10) the latter takes place, while the Majorana condition for the spinor (2.14) yields $\tau_1 \psi = \psi^*$ and $\tau_1 \tilde{\psi} = \tilde{\psi}^*$. Clearly, this distinguishes the $(2,2)$ case from its $(1,3)$ counterpart where we had $\psi^* = \tilde{\psi}$ instead.

It is now obvious that we can introduce \textit{Majorana-Weyl} (MW) spinors in $D = (2,2)$ without adding additional isospin indices. The chiral parts of the Majorana spinor just represent the MW spinors. In other words, the two constraints provided by the chirality and Majorana conditions can be simultaneously and consistently imposed on a spinor in the AW space-time! In the Majorana representation (B.8) with pure imaginary $\Gamma^a$, there exist real (Majorana) spinors, which satisfy the real Dirac equation. Since in the Majorana representation the $\Gamma_5$ is off-diagonal but real, the chirality condition makes perfect sense and leads to a 2-component real chiral spinor which is nothing but the MW spinor. In the representation (B.10), a four-component MW spinor $\Psi$ takes the form

$$\Psi_\alpha = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \quad \psi_\alpha = \begin{pmatrix} \psi \\ \psi^* \end{pmatrix}, \quad (2.19)$$

where the $\psi$ has just one complex component. Finally, from the viewpoint of the isomorphism (2.10), the MW spinor just realizes the fundamental (two-dimensional and real) representation of one of the $SL(2,\mathbb{R})$ factors on the r.h.s. of eq. (2.10).

It is worthy to mention here that the \textit{little group}$^8$ for the AW space-time is just $GL(1)$, but not $U(1)$. The $GL(1)$ is an abelian group whose irreducible representations (irreps) are one-dimensional. To have a non-trivial spin under the little group, the $\Pi$-irrep should be \textit{massive}. The \textit{massless} $\Pi$-irreps consist only of scalars. The meaning of the massless scalars in reference to the massless states with non-trivial spins had already been explained by Ooguri and Vafa [3]. Those scalars appear to be the potentials for the \textit{self-dual} fields which may have non-trivial spin but only one physical degree of freedom [3].

The \textit{supersymmetry} can be introduced in the usual way [10] as the “square root” of the AW space-time. The difference, however, is in the existence of \textit{two} different square roots in the AW space-time, related to either SD or \textit{anti-self-duality} (ASD). As explained in the Appendix B, the SD is associated with the choice $\mathbf{I}$ of the $\gamma$-matrices while the ASD picks up another choice $\mathbf{II}$. Therefore, one can introduce two sets of the (extended) supersymmetry charges $Q$ and $\bar{Q}$ which are linked to the choice $\mathbf{I}$ and $\mathbf{II}$ respectively, and they are completely independent on each other. The basic $N$-extended supersymmetry algebra is

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$^8$We define the little group in the AW space-time as the subgroup of the “Poincaré” group $\Pi$, $\Pi \equiv SO(2,2) \ltimes T^{2,2}$, which leaves invariant a non-zero “momentum” vector. Here the symbol $\ltimes$ means a semi-direct product, whereas the $T^{2,2}$ denotes the translation group.
described by
\[
\begin{align*}
\{Q^i_\alpha, Q^j_\beta\} &= \Sigma_{\alpha\beta} Z^{(ij)} + C_{\alpha\beta} Z^{[ij]}, \\
\{\bar{Q}^{\dagger}_{\bar{i}}, \bar{Q}^{\dagger}_{\bar{j}}\} &= 0, \\
\{Q_i^\alpha, \bar{Q}^{\dagger}_{\beta j}\} &= i\delta^i_j (\sigma^a_1)^{\alpha\beta} \partial_a, \\
\{\bar{Q}^{\dagger}_{\bar{i}}, \bar{Q}^{\dagger}_{\bar{j}}\} &= 0,
\end{align*}
\] (2.20)
\[
\begin{align*}
\{\tilde{Q}^{\dagger}_{\bar{i}}, \tilde{Q}^{\dagger}_{\bar{j}}\} &= \tilde{\Sigma}_{\alpha\beta} \tilde{Z}^{(ij)} + C_{\alpha\beta} \tilde{Z}^{[ij]}, \\
\{\tilde{Q}^{\dagger}_{\bar{i}}, \tilde{Q}^{\dagger}_{\bar{j}}\} &= i\delta^i_j (\sigma^a_1)^{\alpha\beta} \partial_a,
\end{align*}
\]
where \(i, j = 1, 2, ..., p; \bar{i}, \bar{j} = 1, 2, ..., q\), \(N = p + q\). The appearance of the two independent sets of supersymmetry generators resembles the two-dimensional case, where the \((p, q)\) supersymmetry algebras comprise \(p\) chiral and \(q\) anti-chiral Majorana spinor charges simultaneously [11]. In eq. (2.20) we have explicitly introduced the central charges \(Z^{(ij)}, Z^{[ij]}, \tilde{Z}^{(ij)}\) and \(\tilde{Z}^{[ij]}\) for more generality; the \(\tilde{\Sigma}\) represents the II-analogue of the symmetric matrix \(\Sigma\) (see the Appendix B). In this paper we will restrict ourselves to the case \(q = 0\) with all the central charges vanishing. However, in some cases, the central charges may really be important. In particular, it happens in the three-dimensional supersymmetric Chern-Simons theory where the central charges are active [12].

3. \(N = 1\) Supersymmetry and Self-Duality

In this section we consider \(N = 1\) supersymmetric scalar, tensor, vector and supergravity multiplets, and formulate supersymmetric SD conditions for them in the AW space-time. The \(N = 1\) supersymmetric SDYM theory, which is apparently relevant to integrable models and comprises all of the remarkable features mentioned above about spinors in \(D = (2, 2)\), will be in the heart of our discussion.

3.1. Self-Dual Scalar Multiplet

As an immediate corollary of the existence of MW spinors in \(D = (2, 2)\), we notice the existence of a real scalar \(N = 1\) multiplet (SM) in the AW space-time. The real scalar (or chiral) \(N = 1\) multiplet consists of a real scalar \(A\), a Majorana-Weyl spinor \(\tilde{\psi}\) and a real auxiliary scalar \(F\) (in total, \(2 + 2\) components off-shell and \(1 + 1\) components on-shell). The \(N = 1\) supersymmetry transformation laws (with the MW parameters \(\epsilon_\alpha\)
and $\bar{\epsilon}_{\alpha}$ read:

$$
\delta A = \bar{\epsilon}^\alpha \tilde{\psi}_\alpha^* , \\
\delta \tilde{\psi}_\alpha^* = i(\tilde{\sigma}^a)_\alpha^\beta \partial_a A \epsilon_\beta + F \bar{\epsilon} \alpha^* , \\
\delta F = -i \epsilon^\alpha (\sigma^a)_\alpha^\beta \partial_a \bar{\psi}_\beta^* .
$$

(3.1)

Hence, this $D = 4$ multiplet is very much like a $D = 2$ real SM! Given the Majorana representation for the $\gamma$-matrices (see the Appendix B), all the spinors in eq. (3.1) would be just real. The real (chiral) SM arises as the result of imposing the reality condition on a (chiral) SM, which is consistent in the AW space-time. An existence of the more fundamental real SM, compared to its $D = (1,3)$ complex scalar counterpart [13], is clearly a remarkable feature of supersymmetry in $D = (2,2)$. The real SM does not allow an action since there is no (undotted) spinor partner for the (dotted) MW spinor $\tilde{\psi}$ in order to form a fermionic kinetic term. Therefore, there seems to be a good reason to call the real chiral SM “self-dual scalar multiplet” (SDSM). We will find further evidence to support such identification, when considering the $N = 2$ supersymmetric SDYM multiplet from the viewpoint of the $N = 1$ multiplets it contains (see subsect. 4.2).

Another point of interest is the existence of $D = (2,2)$ models that mimic the structure of the chiral rings seen in conformal field theory [14]. To show the existence of such models, we first need to introduce a second “SDSM”. From the discussion above we have learned that $(A, \tilde{\psi}_\alpha^*, F)$ constitute a representation consistent with SD and supersymmetry. Now consider the multiplet $(B, \Omega_\alpha, U)$, where

$$
B \equiv F , \quad \Omega_\alpha \equiv -i(\sigma^a)_\alpha^\beta \partial_a \tilde{\psi}_\beta^* , \\
U \equiv \Box A ,
$$

(3.2)

and it is easy to see that the transformation laws of these are given by

$$
\delta B = \epsilon^\alpha \Omega_\alpha , \\
\delta \Omega_\alpha = -i(\sigma^a)_\alpha^\beta \tilde{\epsilon} (\partial_a B) + U \epsilon_\alpha , \\
\delta U = i \tilde{\epsilon} (\tilde{\sigma}^a)_\alpha^\beta (\partial_a \Omega_\beta) .
$$

(3.3)

Notice now a feature that appears to be unique to the SM. Even in the presence of the duality constraint, there still remain auxiliary fields. We can eliminate them by making specific choices. For example, we now regard $B$, $\Omega_\alpha$ and $U$ as independent functions with the transformation laws given by eq. (3.3). Introducing independent “superpotentials”
\( W'(B) \) and \( \tilde{W}'(A) \) in addition, we impose the constraints
\[
F \equiv W'(B) \quad \text{and} \quad U \equiv \tilde{W}'(A) .
\]
(3.4)

We can easily demonstrate an action for these ideas. As one might guess from \( D = (1,3) \) space, an appropriate lagrangian is given by
\[
\mathcal{L} = 2A \Box B + 2i \Omega^\alpha (\sigma^2)_\alpha^{\phantom{\alpha} \beta} \partial_\alpha \bar{\psi}_\beta + FU
- F\tilde{W}'(A) + \frac{1}{2} \bar{\psi}^{\tilde{\alpha}} \bar{\psi}_{\tilde{\alpha}} \tilde{W}''(A)
- UW'(B) + \frac{1}{2} \Omega^\alpha \Omega_\alpha W''(B) ,
\]
(3.5)

which clearly follows from the superfield expression
\[
\mathcal{L} = \int d^2 \theta d^2 \bar{\theta} \Phi \Psi - \int d^2 \theta \tilde{W}(\Phi) - \int d^2 \theta W(\Psi) .
\]
(3.6)

In this expression \( \Phi \) is a real anti-chiral superfield \( (D_\alpha \Phi = 0) \), while \( \Psi \) is an independent real chiral superfield \( (\bar{D}_{\tilde{\alpha}} \Psi = 0) \). The \( (A, \bar{\psi}_{\tilde{\alpha}}, F) \) components are contained in \( \Phi \) and the \( (B, \Omega_\alpha, U) \) components are contained in \( \Psi \). From the form of eq. (3.6) we can clearly see that as a final generalization a Kähler potential can be used
\[
\int d^2 \theta d^2 \bar{\theta} \Phi \Psi \rightarrow \int d^2 \theta d^2 \bar{\theta} K(\Phi, \Psi) .
\]
(3.7)

### 3.2 \( N = 1 \) Supersymmetric Self-Dual Yang-Mills Theory

We begin our construction of the \( N = 1 \) supersymmetric SDYM theory with component considerations, and then transfer the results to superspace. The superspace approach will be particularly useful when considering the extended supersymmetry and supergravity in the next sections.

The \( N = 1 \) SYM theory in four space-time dimensions is described off-shell in terms of the \( N = 1 \) non-Abelian real vector multiplet which contains (in the Wess-Zumino gauge) a gauge vector field \( A_\underline{a} \equiv A_\underline{a}^{I} t^{I} \), one Majorana spinor \( \Lambda \) and an auxiliary scalar \( D \), all in the adjoint representation of the gauge group [15]. The (anti-hermitian) generating matrices \( t^{I} \) of the gauge group act on the implicit gauge group indices of the above-mentioned fields. We omit the gauge indices, unless their absence may cause confusion.

\[\text{In our notation, a prime as a superscript always means a differentiation of a function with respect to a given argument.}\]
The supersymmetry transformation laws of the $N=1$ SYM multiplet take the usual form in the four-component notation for spinors:

$$
\delta A_\alpha = -i \bar{\epsilon} \Gamma_\alpha \Lambda ,
$$
$$
\delta \Lambda = \Sigma^{ab} F_{ab} \epsilon + D \epsilon ,
$$
$$
\delta D = -\frac{1}{4} i \bar{\epsilon} \Gamma_5 \Gamma^a \nabla_a \Lambda ,
$$

where the gauge-covariant YM derivatives $\nabla_a = \partial_a + A_\alpha$ and the non-abelian YM field strength $F_{ab}$ have been introduced.

The celebrated SD condition [1] on the YM field strength $F$ in four space-time dimensions with the $(2,2)$ signature is given by

$$
F_{ab} = \frac{1}{2} \epsilon_{abcd} F_{cd} ,
$$
in the real notation, where the totally antisymmetric Levi-Civita symbol $\epsilon^{abcd}$ with unit weight has been introduced. Given the complex notation (see the Appendix A for our notation and conventions), eq. (3.9) is equivalent to the two conditions [16,17]

$$
F_{ab} = \overline{F}_{ab} = 0 ,
$$
$$
\eta^{ab} F_{ab} = 0 .
$$

The first SD condition in eq. (3.10a) can be interpreted as an integrability condition for the existence of the holomorphic and anti-holomorphic spinors satisfying the equations $\nabla_a \Xi = 0$ and $\nabla_\alpha \Omega = 0$, respectively. The YM fields satisfying eq. (3.10a) can be referred to as the “hermitian” gauge configurations. Eq. (3.10a) can easily be solved in terms of the prepotential scalar fields $J$ and $\overline{J}$ as [18]

$$
A_a = J^{-1} \partial_a J , \quad A_\alpha = \overline{J}^{-1} \partial_\alpha \overline{J} .
$$

The second SD condition (3.10b) puts the YM theory on-shell since it implies the YM field equations of motion to be satisfied. Therefore, when discussing SD, it is always enough to consider an on-shell theory. As for the $N=1$ SYM theory, its equations of motion are given by

$$
\nabla^b F_{ab} = -i f^{IJK} \eta_{ac} \left[ \lambda^I \Gamma^{\beta a} (\sigma^a)^{\alpha}_{\dot{\alpha}} \lambda^J \lambda^K \right] ,
$$
$$
i (\sigma^a)^{\alpha}_{\dot{\alpha}} \nabla_\alpha \lambda^\dot{\alpha} = i (\bar{\sigma}^a_{\dot{\alpha}})^{\alpha}_{\dot{\alpha}} \nabla_\alpha \lambda^\dot{\alpha} = 0 ,
$$
$$
D = 0 ,
$$

where the gauge group structure constants $f^{IJK}$ have been introduced.
In the complex notation, the second line of eq. (3.8) can be rewritten on-shell as

$$
\delta \Lambda = \left( 2i\gamma_3 \epsilon^{ab} F_{ab} + 2i\gamma_5 \epsilon^{\alpha\beta} F_{\alpha\beta} + \frac{1}{2} \Sigma \eta^{\alpha\beta} F_{\alpha\beta} + \hat{\Sigma}^{\alpha\beta} F_{\alpha\beta} + D \right) \epsilon .
$$

(3.13)

Now we can separate the vanishing SDYM field strengths from the non-vanishing ones simply by taking the chiral projections of eq. (3.13),

$$
\delta \lambda_\alpha = \left( 2i\gamma_3 \epsilon^{ab} F_{ab} + 2i\gamma_5 \epsilon^{\alpha\beta} F_{\alpha\beta} + \frac{1}{2} \Sigma \eta^{ab} F_{ab} + D \right)_\alpha \beta \epsilon_\beta ,
$$

(3.14a)

$$
\delta \tilde{\lambda}_\alpha = F_{ab}(\Sigma^{ab})_{\alpha} \beta \epsilon_\beta ,
$$

(3.14b)

because of eqs. (B.27) and (B.28). This observation immediately gives rise to the $N = 1$ supersymmetric SDYM constraint to the Majorana spinor superpartner $\Lambda$ of the YM field $F$ in the simple form of the Majorana-Weyl condition:

$$
\frac{1}{2} (1 + \gamma_5) \Lambda_{SD} = 0 ,
$$

(3.15a)

while we still have

$$
D = 0 .
$$

(3.15b)

Therefore, the $N = 1$ supersymmetric SDYM multiplet consists of the SDYM field $F_{a\cdot b}$ and the MW spinor $\tilde{\lambda}_\alpha$ ($1 + 1$ components on-shell per one gauge index value).

In order to check the component super-SD conditions (3.10) and (3.15), first we notice the consistency of the SD conditions (3.10) on the YM fields with the SYM equations of motion (3.12), just because the gaugino source term on the r.h.s. of the YM field equations is now *vanishing*, subject to the constraint (3.15a). Second, we can also confirm that the gaugino field equations (3.12b) come out of the spinorial derivatives $\nabla_\alpha F_{ab} = 0$ and $\nabla_\alpha (\eta^{a\beta} F_{ab}) = 0$.

Given the second choice (II) in eq. (B.14) to represent the $\gamma$-matrices, the similar analysis would give rise to the *anti-self-dual* $N = 1$ SYM theory, characterized by the minus sign in front of the r.h.s. of eq. (3.9) and the constraint:

$$
\frac{1}{2} (1 - \gamma_5) \Lambda_{ASD} = 0 .
$$

(3.16)

We thus find the very interesting observation that the two different ways of defining a spin-structure in the AW space-time are co-related with the definition of self-dual or anti-self-dual supersymmetry.

In the $N = 1$ superspace associated with the AW space-time, the superfield formulation of the (non-self-dual) $N = 1$ SYM theory can be developed along the lines of the conventional $(3 + 1)$-dimensional case [19,20]. The SYM superfield potentials $A_A$
are used to define the gauge-covariant superspace derivatives $\nabla_A \equiv D_A + A_A$, where $A = (a, \alpha)$. These gauge-covariant derivatives define superfield strengths by

$$\{\nabla_A, \nabla_B\} = F_{AB} + T_{AB}^C \nabla_C,$$  \hspace{1cm} (3.17)

where the torsion $T_{AB}^C$ is the same one as in flat superspace, and some of the field strengths vanish [19,20]:

$$F_{\alpha\beta} = F_{\alpha}^{\dot{\beta}} = F_{\dot{\alpha}}^{\beta} = 0.$$  \hspace{1cm} (3.18)

Eq. (3.18) represents the whole set of the $N = 1$ SYM superspace constraints, whose only role is to remove as much independent components as possible for matching with the component SYM formulation, without getting a flat theory [20]. Given the constraints on some of the field strengths $F_{AB}$, the superspace Bianchi identities (BIds)

$$\nabla_{[A} F_{BC]} - T_{[AB]}^D F_{D|C} = 0$$  \hspace{1cm} (3.19)

are used to determine the implications of the constraints to all the other SYM field strengths.

We refer the reader to Refs. [19,21] for the details related to the analysis of the SYM Bianchi “identities” subject to the conventional SYM constraints for all $N \leq 4$. That analysis is still valid in $D = (2,2)$, before imposing any reality conditions. The outcome of such analysis is usually represented by some kind of the BIds “solution” that expresses all of the SYM field strengths in terms of a smaller number of the independent superfields, still satisfying certain superspace constraints. In particular, the off-shell $N = 1$ SYM theory is well-known to be described by a single Majorana spinor superfield $W_\alpha$ which comprises all of the SYM field components and has the $\Lambda$ as the leading one [20]. The MW chiral constituents $w_\alpha$ and $\bar{w}^{\dot{\alpha}}$ of the $W_\alpha$ are chiral and anti-chiral in the gauge-covariant superspace sense, respectively

$$\hat{\nabla}^{\dot{\alpha}} w_\beta = 0, \quad \nabla^\alpha \bar{w}^{\dot{\alpha}} = 0,$$  \hspace{1cm} (3.20)

and, in addition, satisfy the constraint [20,21]

$$\nabla^\alpha w_\alpha + \hat{\nabla}^{\dot{\alpha}} \bar{w}^{\dot{\alpha}} = 0.$$

(3.21)

The non-vanishing SYM field strengths take the form [20,21]

$$F^a_{\alpha} = -i (\sigma^a)^{\alpha\beta} w_\beta, \quad F^{\dot{a}}_{\dot{\alpha}} = -i (\bar{\sigma}^{\dot{a}})^{\dot{\alpha}\dot{\beta}} \bar{w}_{\dot{\beta}}$$  \hspace{1cm} (3.22a)

$$F_{ab} = \frac{1}{2} i \left[ \nabla^\alpha (\sigma_{ab})_{\alpha\beta} w_\beta + \hat{\nabla}^{\dot{\alpha}} (\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \bar{w}_{\dot{\beta}} \right].$$  \hspace{1cm} (3.22b)

We are now in a position to find out the SYM superfield SD condition easily. We simply notice that since the $w_\alpha$ superfield has $\lambda_\alpha$ as its $\theta = 0$-component, which is vanishing
in the SD case, the whole superfield \( w_\alpha \) should vanish, too. This gauge-chiral superfield \( w_\alpha \) just has the \( \lambda_\alpha, D \) and the anti-self-dual part \( F_{\alpha\beta} \) of the YM field strength as its independent components. The constraint (3.21), whose meaning was just the reality of the \( D \)-component, is now trivially satisfied because of eq. (3.15b), while the self-dual part \( F_{\alpha\beta} \) of the YM field strength remains in the decomposition (3.22b). The superspace equations of motion take formally the same form as in eq. (3.12b) after the substitution \( \Lambda \to W \), and they are obviously consistent with the SD condition. We have repeated the analysis of the BId (3.19) subject to the conventional SYM constraints (3.18) and the superspace SD condition in the most straightforward form \( F_{\alpha\beta} = 0 \), and reached the conclusion that all those constraints taken together, in order to be consistent, do imply that the Weyl part \( w \) of the Majorana spinor \( W \) should be simultaneously holomorphic and anti-holomorphic, \( \nabla_a w = \nabla_\pi w = 0 \), which means \( w = 0 \). In summary, we have found that the MW condition on the SYM spinor superfield strength \( W \),

\[
\frac{1}{2} (1 + \gamma_5) W_{SD} = 0 ,
\]

is the true \( N = 1 \) SYM SD constraint in superspace. It is usually the case in superspace that the origin of constraints on bosonic fields can be tracked back to constraints on some fermionic fields in lower-dimensional sector. We see that it indeed happens in our case when the first-order differential SD constraint on the YM field is replaced by the algebraic constraint on the spinor SYM superfield. The existence of the MW spinors was crucial in this respect.

3.3 \( N = 1 \) Self-Dual Supergravity

The \( N = 1 \) (simple) SDSG follows the pattern provided by the \( N = 1 \) supersymmetric SDYM theory above.

In the curved \( N = 1 \) superspace associated with the AW space-time, the covariant derivatives \( \nabla_A = E_A + \Omega_A \) are defined in terms of the (super)vielbein \( E_A = E_A^N D_N \) and the (super)connection \( \Omega_A = (\Omega_A)_\alpha^\beta M_\beta^\alpha + (\Omega_A)_\alpha^\beta \tilde{M}_\beta^\alpha \), where the \( M \)'s are “Lorentz” rotation generators. From the covariant derivatives one defines torsions and curvatures,

\[
[\nabla_A, \nabla_B ] = T_{AB}^C \nabla_C + R_{AB} \equiv T_{AB}^C \nabla_C + \frac{1}{2} R_{AB}{}^d M_d^C ,
\]

which satisfy the well-known \( N = 1 \) SG constraints [20]. By setting the auxiliary superfields equal to zero, the solution of the SG BId (subject to the SG constraints [20]) assumes the
form

\[
\{ \nabla_\alpha, \nabla_\beta \} = \left\{ \bar{\nabla}_\alpha, \bar{\nabla}_\beta \right\} = 0 , \quad \left\{ \nabla_\alpha, \bar{\nabla}_\beta \right\} = i \nabla_{\alpha \beta} ,
\]

\[
\left[ \bar{\nabla}_\alpha, i \nabla_\beta \right] = C_{\alpha \beta} W_{\beta \gamma}^\delta \mathcal{M}_\delta^\gamma , \quad \left[ \nabla_\alpha, i \nabla_\beta \right] = C_{\beta \alpha} \bar{W}^\delta_{\beta \gamma} \mathcal{M}_\delta^\gamma ,
\]

\[
\left[ i \nabla_{\alpha \beta}, i \nabla_\beta \right] = C_{\alpha \beta} \left[ \frac{1}{4!} \nabla (\alpha W_{\beta \gamma \delta}) \mathcal{M}_\delta^\gamma - W_{\alpha \beta \gamma} \nabla_\gamma \right] + C_{\beta \alpha} \left[ \frac{1}{4!} \bar{\nabla} (\alpha \bar{W}_{\beta \gamma \delta}^\bullet) \bar{\mathcal{M}}_\delta^\gamma - \bar{W}_{\alpha \beta \gamma} \bar{\nabla}_\gamma \right] ,
\]

where all of the torsion and curvature tensors have been expressed in terms of only two irreducible conformal superfield strengths \( W_{\alpha \beta \gamma} \) and \( \bar{W}^\bullet_{\alpha \beta \gamma} \), which are totally symmetric on their spinor indices. These tensors originate from the identification \[20\]

\[
W_{\alpha \beta \gamma} = \frac{1}{12} i R^\alpha_{(\alpha \beta \gamma)} = - \frac{1}{12} T_{(\alpha \beta \gamma)} ,
\]

and similarly for \( \bar{W}^\bullet_{\alpha \beta \gamma} \), and satisfy the relations \[20\]

\[
\bar{\nabla}_\alpha W_{\alpha \beta \gamma} = \nabla^\alpha W_{\alpha \beta \gamma} = 0 , \quad \nabla_\alpha \bar{W}_{\alpha \beta \gamma} = \bar{\nabla}^\alpha \bar{W}_{\alpha \beta \gamma} = 0 .
\]

The leading components of the \( W_{\alpha \beta \gamma} \) and \( \bar{W}^\bullet_{\alpha \beta \gamma} \) just represent the chiral parts of the on-shell Rarita-Schwinger field strength, while their symmetrized spinor derivatives \( W_{\alpha \beta \gamma \delta} \equiv (1/4!) \nabla (\alpha W_{\beta \gamma \delta}) \) and \( \bar{W}^\bullet^\bullet_{\alpha \beta \gamma \delta} \equiv (1/4!) \bar{\nabla} (\alpha \bar{W}_{\beta \gamma \delta}^\bullet) \) are the anti-self-dual and self-dual constituents of the Weyl curvature, respectively.

The SD condition is imposed on the Riemann curvature tensor, and in the real notation it takes the form

\[
R_{ab}^{\; cd} = \frac{1}{2} \epsilon_{ab}^{\; efh} R_{efh}^{\; cd} .
\]

Given the complex notation, eq. (3.28) is rewritten by

\[
R_{\alpha \beta}^{\; cd} = R_{\alpha \beta}^{\; cd} = 0 , \quad \eta^{\alpha \beta} R_{\alpha \beta}^{\; cd} = 0 ,
\]

which is quite similar to its SDYM counterpart in eq. (3.10). Eq. (3.29a) is known as the “Kähleriness” condition \[22\] which means that the base manifold (curved space-time) should be a hermitian manifold with the connection to be consistent with the complex structure. Eq. (3.29b) puts the theory on-shell and it is equivalent to the Ricci-flatness condition. As is well known in \( D = 4 \) \[23\], a Kählerian and Ricci-flat manifold has a self-dual curvature tensor and vice versa.

It now becomes clear that the \( N = 1 \) SDSG constraint in superspace is given by

\[
W_{\alpha \beta \gamma} = 0 .
\]

(3.30)
Eq. (3.30) means that the gravitino field becomes a Majorana-Weyl spinor (only $\tilde{W}_{\alpha\beta\gamma}^\cdot$ survives) while the anti-self-dual constituents of both Weyl and Riemann curvature tensors vanish. As a consistency check, we notice that both the SG contorsion tensor and the gravitino stress-energy tensor vanish as a consequence of eq. (3.30). The SD condition on the curvature remains unchanged in the $N = 1$ SDSG. Simultaneously, eq. (3.30) is consistent with the SG BIds and does not render the theory flat, just self-dual. The second-order differential constraint on the metric in eq. (3.28) is implied by the superspace constraint (3.30) which is the first-order differential equation on the gravitino field.

### 3.4 Self-Dual Tensor Multiplet

As a remarkable aspect of our AW space-time, we give here a completely new multiplet, different from any of the above mentioned multiplets, which satisfies a SD condition in a generalized sense. This multiplet is based on what is called the tensor multiplet (TM) or the linear multiplet, with the field content $(B_{\alpha\beta}, \Phi, \chi_{\alpha}, \tilde{\chi}_{\alpha}^\cdot)$. The superspace BIds are

\[
\frac{1}{6} \nabla_{(A} G_{B(CD)} - \frac{1}{4} T_{[AB]}^E G_{E(CD)} = 0 ,
\]

where $G_{ABC}$ is the superfield strength of $B_{AB}$. The constraints and constituency relations that solve these BIds are

\[
\begin{align*}
G_{\alpha\beta\gamma} &= -i(\sigma_2)_{\alpha\beta}^\cdot \Phi, \\
G_{\alpha\beta c} &= \frac{1}{\sqrt{2}}(\sigma_{bc})_{\beta}^\cdot \chi_{\alpha}^\cdot, \\
G_{\alpha\beta c}^\cdot &= \frac{1}{\sqrt{2}}(\tilde{\sigma}_{bc})_{\alpha}^\cdot \tilde{\chi}_{\beta}^\cdot, \\
\nabla_\alpha \Phi &= -\frac{1}{\sqrt{2}} \chi_{\alpha}^\cdot, \\
\tilde{\nabla}_\alpha \Phi &= -\frac{1}{\sqrt{2}} \tilde{\chi}_{\alpha}^\cdot, \\
\nabla_\alpha \tilde{\chi}_{\beta}^\cdot &= -i \frac{1}{\sqrt{2}} (\sigma^\epsilon)_{\alpha\beta}^\cdot \nabla_\epsilon \Phi + i \frac{1}{6 \sqrt{2}} \epsilon^{cdef}(\sigma_f)_{\alpha\beta}^\cdot G_{cde}, \\
\tilde{\nabla}_\alpha \chi_{\beta}^\cdot &= -i \frac{1}{\sqrt{2}} (\tilde{\sigma}^\epsilon)_{\alpha\beta}^\cdot \nabla_\epsilon \Phi - i \frac{1}{6 \sqrt{2}} \epsilon^{cdef}(\tilde{\sigma}_f)_{\alpha\beta}^\cdot G_{cde}, \\
\nabla_\alpha \chi_{\beta} &= 0, \\
\tilde{\nabla}_\alpha \tilde{\chi}_{\beta} &= 0, \\
T_{\alpha\beta c} &= 0, \\
T_{\alpha\beta c}^\cdot &= 0, \\
T_{\alpha\beta}^\cdot &= 0, \\
T_{\alpha\beta} &= 0.
\end{align*}
\]

The invariant lagrangian of this multiplet is

\[
\mathcal{L}_{TM}^{N=1} = \frac{1}{12} G^2_{mn} + \frac{1}{2} \eta^{mn}(\partial_\alpha \Phi)(\partial_\alpha \Phi) + i \chi^\alpha (\sigma^\varepsilon)_{\alpha}^\cdot \partial_\varepsilon \tilde{\chi}_{\beta}^\cdot.
\]

We turn to the question of SD in this system of TM. Based on our experience in other multiplets, it is natural to assume that such a SD is related to the Majorana-Weyl condition on the field $\chi$, namely the condition of vanishing of either $\chi_{\alpha}$ or $\tilde{\chi}_{\alpha}^\cdot$ would generate the SD. In our case, we find it convenient to choose

\[
\tilde{\chi}_{\alpha}^\cdot = 0 ,
\]
in order to have consistent couplings with other self-dual multiplets. Under eq. (3.34), we see that the bosonic condition

\[ G_{abc} = 2 \epsilon_{abc} \nabla_d \Phi \]

relates the two “field strengths” \( G_{abc} \) and \( \nabla_d \Phi \) in an unexpected way! Accordingly, the dilaton superfield \( \Phi \) is to be a chiral superfield to comply with the SD condition, because

\[ \nabla_a \Phi = -(1/\sqrt{2}) \tilde{\chi}_a = 0. \]

Actually, this multiplet plays an important role in the Wess-Zumino-Novikov-Witten term of the Green-Schwarz string action, which we have described in our previous paper [16]. Remarkably, the four \( N = 1 \) multiplets of SM, YM, SG and TM allow the SD conditions, and they are consistent with the Green-Schwarz string. The self-dual TM was also predicted by a previous work on the Neveu-Schwarz-Ramond \( N = (2, 0) \) heterotic \( \sigma \)-model [3]. In particular, it was noted that the only way the dilaton could be coupled to the string required that the dilaton must be chiral. Since the dilaton originates from the TM, it follows that it must satisfy some type of chirality condition. This chirality condition is precisely the vanishing of \( \tilde{\chi}_a \) (i.e. \( \nabla_a \Phi = 0 \)).

Even though we skip all the details of the consistency check of the field equations, owing to the supersymmetry of the system it is sufficient to look into the consistency of our constraints with the SD condition at dimension \( d \leq 1 \) without even checking all the field equations at \( d \geq 3/2 \). The consistency of the constraints with our SD conditions under supersymmetry will guarantee the consistency of our SD conditions.

The curious reader may wonder if the couplings among all the self-dual multiplets are really consistent to all orders, especially when the SG multiplet is coupled. As an intuitive explanation of how this works, we note the following point: these multiplets are all re-derived from the corresponding multiplets in the usual \( D = (1, 3) \) space-time by appropriate replacements of the dotted spinors by tilded spinors, regarding them as independent quantities. By doing so, we see that in the canonical basis of all the fields, the energy-momentum tensors of all of them vanish, due to the involvements of at least one of the vanishing fields that appear in the SD conditions of those multiplets. The only subtlety is that the SD condition is to be chosen in the canonical basis, including the supergravity multiplet itself. This is because the SD conditions are not scale invariant conditions, so that some factors or other complications may arise. For example, after a Weyl rescaling, an unusual factor \( e^{\xi \Phi} \) will

\footnote{It is convenient to assign the dimensions to our bosonic superfields as \([e^{\xi \Phi}] = [\Phi] = [B_{mn}] = [A_{mn}] = 0 \) and for fermionic ones as \([\psi_{m\alpha}] = [\lambda_{a\alpha}] = 1/2 \). This is to be universal also for other bosonic and fermionic fundamental superfields, respectively. Thus, e.g., \([T_{mn}^{2k}] = 1/2, [T_{mn}] = 1, [F_{m\alpha L}] = 1/2, [F_{2L}] = 1, etc.\]}

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arise in front of the SD condition on the field strength $F_{ab}$,

$$F_{ab} = \frac{1}{\xi} e^{\xi \Phi} c_{ab} \Phi_{cd} F_{cd}$$, (3.36)

with some constant $\xi$. (See also Appendix C for quantum corrections.)

4. $N = 2$ Supersymmetry in AW Space-Time

Since we have established the $N = 1$ SDYM as well as the $N = 1$ SDSG, the next natural question is about extended supersymmetries. We will show in this section that the answer is actually in the affirmative, namely we construct the SDYM as well as the SDSG with $N = 2$ supersymmetry. However, it turns out to be impossible to formulate the self-dual $N = 2$ hypermultiplet, without the inclusion of additional propagating fields.

4.1 $N = 2$ Hypermultiplet in AW Space-Time

There exists an AW space-time analogue of the $N = 2$ hypermultiplet [24], which is straightforward to construct. In the $N = 2$ superspace, it is defined by the equations [25]

$$D_{\alpha i} \Phi_j + D_{\alpha j} \Phi_i = \tilde{D}_{\alpha i} \Phi_j + \tilde{D}_{\alpha j} \Phi_i = 0$$, (4.1)

in terms of a isoscalar (complex) $N = 2$ superfield $\Phi_i, \ i = 1, 2$. In the absence of central charges, eq. (4.1) does imply the equations of motion to be satisfied. Projecting the hypermultiplet into the $N = 1$ superspace yields two complex scalar $N = 1$ superfields $\phi_i$,

$$\phi_i = \Phi_i \mid$$, (4.2)

which are known to be chiral ($\phi_2$) and anti-chiral ($\phi_1$), respectively [24,26]. The on-shell hypermultiplet comprises a scalar isodoublet $\varphi_i(x)$ and a Dirac (complex) isoscalar spinor $\Psi_{\underline{a}} = (\psi_{\underline{a}}, \bar{\kappa}_{\underline{a}})$. The $N = 2$ supersymmetry transformation rules in the AW space-time are

$$\delta \varphi_i = \epsilon^a_i \psi_{\underline{a}} + \bar{\epsilon}^{\underline{a}}_i \bar{\kappa}_{\underline{a}}$$,

$$\delta \psi_{\underline{a}} = -2i (\sigma_{\underline{a}})_\alpha^\beta \bar{\epsilon}_{\underline{a}} \partial_{\beta} \varphi^\alpha$$, (4.3)

$$\delta \bar{\kappa}_{\underline{a}} = -2i (\bar{\sigma}_{\underline{a}})_\alpha^\beta \epsilon_{\beta \alpha} \partial_{\underline{a}} \varphi^\alpha$$.

The fields $\varphi_2$ and $\psi$ form the on-shell $N = 1$ chiral multiplet, while the fields $\varphi_1$ and $\bar{\kappa}$ enter into the on-shell $N = 1$ anti-chiral multiplet.

As we already know from the $N = 1$ considerations of sect. 3, the SD is associated with the real chiral objects of definite chirality and the certain choice of the $\gamma$-matrices. One
can use either of them, but not both simultaneously in one usual supermultiplet. Since the $N = 2$ hypermultiplet comprises the two $N = 1$ chiral multiplets of different chirality, one can not define the self-dual $N = 2$ hypermultiplet, even though the $N = 1$ self-dual and anti-self-dual SM do exist. One can, of course, introduce real chiral and real anti-chiral $N = 1$ scalar superfields to represent self-dual and anti-self-dual $N = 1$ SM, but one can not unify them into one object with $N = 2$ supersymmetry, since they correspond to the different supersymmetry algebras: one for the choice (I) and another for the choice (II) of the $\gamma$-matrices. (See Appendix B).

4.2 $N = 2$ Supersymmetric YM in AW Space-Time

The on-shell $N = 2$ SYM has the field content $(A_m^{I}, \lambda_{\alpha_i}^{I}, \tilde{\lambda}_{\dot{\alpha}_i}^{I}, S^I, T^I)$, where the indices $I, J, ...$ are for the adjoint representations, the indices $i, j, ... = 1, 2$ are for the two-dimensional representations of $Sp(1)$ group. The fields $S^I$ and $T^I$ are real scalars, while the $\lambda^I$ and $\tilde{\lambda}^I$ are Majorana-Weyl spinors of opposite chiralities, following the same notation of the $N = 1$ case. Our superspace BIds before imposing any SD condition are

$$\nabla_{[A}F_{BC]}^I - T_{[AB]}D_FD_{[C]}^I \equiv 0 \ .$$

These BIds at the dimensions $0 \leq d \leq 1$ are solved by the constraints

$$F_{\alpha i \beta j}^I = 2C_{\alpha \beta} \epsilon_{ij} T^I \ , \quad F_{\alpha i \beta j} = 2C_{\alpha \beta} \epsilon_{ij} S^I \ ,$$

$$F_{\alpha i \beta j} = -i(\sigma_{\alpha \beta})_{ijk} \tilde{\lambda}_{\dot{\alpha}_j}^I \ , \quad F_{\alpha i \beta j} = -i(\sigma_{\alpha \beta})_{\dot{\alpha}_j} \lambda_{\alpha_i}^I \ ,$$

$$F_{\alpha i \beta j}^I = 0 \ , \quad T_{\alpha i \beta j}^I = i(\sigma_{\alpha \beta})_{\alpha_i \beta_j} \delta_{ij} \ ,$$

$$\nabla_{\alpha_i} S^I = -\lambda_{\alpha_i}^I \ , \quad \tilde{\nabla}_{\alpha_i} S^I = 0 \ ,$$

$$\nabla_{\alpha_i} T^I = -\tilde{\lambda}_{\dot{\alpha}_j}^I \ , \quad \tilde{\nabla}_{\alpha_i} T^I = 0 \ ,$$

$$\nabla_{\alpha_i} \tilde{\lambda}_{\dot{\alpha}_j}^I = -i \epsilon_{ij} (\sigma_{\alpha \beta})_{\alpha \beta} \nabla_{\alpha_i} T^I \ ,$$

$$\tilde{\nabla}_{\alpha_i} \lambda_{\alpha_i}^I = +i \epsilon_{ij} (\sigma_{\dot{\alpha} \beta})_{\dot{\alpha} \beta} \nabla_{\dot{\alpha}_j} S^I \ ,$$

where $\epsilon_{ij} = -\epsilon_{ji}, \epsilon_{12} = +1$, stands for the invariant antisymmetric tensor for the $Sp(1)$ group. Needless to say, the raising and lowering of these $Sp(1)$ indices are by $\epsilon_{ij}$ and $\epsilon^{ij}$.

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The BIds of $3/2 \leq d \leq 2$ yield the superfield equations:

\[ i(\sigma^{\alpha})_{\alpha \beta} \nabla^{\beta} \lambda^{I} - 2 f^{IJK} \lambda^{J\alpha} T^{K} = 0 \ , \]

\[ i(\sigma^{\alpha})_{\beta \alpha} \nabla^{\beta} \lambda^{I} - 2 f^{IJK} \nabla^{\beta} \lambda^{I\alpha} S^{K} = 0 \ , \]

\[ \nabla_{\mathbf{h}} F^{abI} - 2 f^{IJK} (S^{J} T^{K} + T^{J} S^{K}) - 2 i f^{IJK} (\lambda^{i J} \sigma^{\alpha} \lambda^{I\alpha}) = 0 \ , \tag{4.6} \]

\[ \Box S^{I} + f^{IJK} (\lambda^{i J} \lambda^{I\alpha}) - 4 f^{IKJ} f^{JLM} S^{K} S^{L} T^{M} = 0 \ , \]

\[ \Box T^{I} - f^{IJK} (\lambda^{i J} \lambda^{I\alpha}) + 4 f^{IKJ} f^{JLM} S^{K} S^{L} T^{M} = 0 \ . \]

This system has an invariant lagrangian \[27\] (cf. Ref. \[28\])

\[ L_{N=2}^{\text{SYM}} = -\frac{1}{4} (F_{abI})^{2} - 2 i \lambda^{i I} \sigma^{\alpha} \nabla_{\mathbf{h}} \lambda^{I\alpha} - 2 (\nabla_{\mathbf{h}} T^{I})(\nabla_{\mathbf{h}} S^{I}) + 2 f^{IJK} (\lambda^{i J} \lambda^{I\alpha}) T^{K} - 2 f^{IJK} (\lambda^{i J} \lambda^{I\alpha}) S^{K} + 4 f^{IKJ} f^{JLM} S^{K} S^{L} T^{M} \tag{4.7} \]

Notice the peculiar kinetic term of $S^{I}$ and $T^{I}$, which is equivalent to two terms with the opposite signs in the diagonalized fields. This indicates the higher-dimensional origin of the system, such as $D = (3, 3)$, $N = 1$ SYM via simple dimensional reduction. In the Appendix D, we actually construct such a theory in $D = (5, 5)$.

We are so far free of any SD condition. Based on our $N = 1$ experience, we can easily postulate our SD condition to be

\[ S^{I} = 0 \ . \tag{4.8} \]

This condition implies through BIds other superfield equations such as

\[ \lambda_{\alpha}^{i} = 0 \ , \quad F_{abI} = \frac{1}{2} \epsilon_{abc} F_{cdI} \ , \tag{4.9} \]

as is desired in Ref. \[27\]. Being decomposed into the sub-multiplets with respect to the $N = 1$ supersymmetry, the $N = 2$ SDYM multiplet comprises the $N = 1$ SDYM multiplet and the $N = 1$ on-shell SDSM.

### 4.3 $N = 2$ Supergravity in AW Space-Time

An $N = 2$ SDSG in the AW space-time is also consistently constructed, based on the non-self-dual $N = 2$ SG multiplet first.

Our on-shell $N = 2$ SG multiplet has the fields $(e_{m}^{\alpha}, \psi_{m}^{\alpha i}, \bar{\psi}_{m}^{\alpha i}, A_{m}^{ij})$, where the gravitino is in the 2-representation of an $SO(2)$ group, and the graviphoton field $A_{m}^{ij}$ is the gauge field of the $SO(2)$. Our independent BIds before any SD condition are

\[ \nabla_{[A} T_{BC]} - T_{[AB]} (E_{E(C)} D - R_{[ABC]} D) \equiv 0 \ , \tag{4.10} \]

\[ \nabla_{[A} F_{BC]} - T_{[AB]} (D F_{D(C)} D) \equiv 0 \ . \]
These BIDs at \( 0 \leq d \leq 1 \) are solved by the constraints\(^\text{11}\):

\[
T_{\alpha i} j^k = i \delta_{ij}^k (\sigma^\alpha)^j = 0 , \quad T_{\alpha i} j^k = 0 ,
\]

\[
F_{\alpha i} j^k = \epsilon_{ij} C_{\alpha \beta} , \quad F_{\alpha i} j^k = \epsilon_{ij} C_{\alpha \beta} , \quad F_{\alpha i} j^k = 0 ,
\]

\[
T_{\alpha i} j^k = \frac{1}{2} \epsilon_{ij} (\sigma^\alpha)^k (F_{\alpha i} j^k + \tilde{F}_{\alpha i} j^k) , \quad T_{\alpha i} j^k = 0 ,
\]

\[
T_{\alpha i} j^k = \frac{1}{2} \epsilon_{ij} (\sigma_{\alpha \beta})_{\gamma_k} (F_{\alpha i} j^k - \tilde{F}_{\alpha i} j^k) , \quad T_{\alpha i} j^k = 0 ,
\]

\[
T_{\alpha i} j^k = 0 , \quad F_{\alpha i} j^k = 0 , \quad F_{\alpha i} j^k = 0 ,
\]

\[
R_{\alpha i} j^k = \epsilon_{ij} C_{\alpha \beta} (F_{\alpha i} j^k + \tilde{F}_{\alpha i} j^k) , \quad R_{\alpha i} j^k = \epsilon_{ij} C_{\alpha \beta} (F_{\alpha i} j^k - \tilde{F}_{\alpha i} j^k) ,
\]

\[
R_{\alpha i} j^k = \epsilon_{ij} C_{\alpha \beta} (\sigma^\alpha)^j \delta_{ij} \delta_{k} F_{\alpha i} j^k ,
\]

\[
R_{\alpha i} j^k = \epsilon_{ij} C_{\alpha \beta} (\sigma^\alpha)^j \delta_{ij} \delta_{k} F_{\alpha i} j^k ,
\]

\[
R_{\alpha i} j^k = 0 , \quad R_{\alpha i} j^k = 0 ,
\]

where the indices \( i, j, \ldots = 1, 2 \) are for the 2-representations of \( SO(2) \), while underlined spinorial indices \( \underline{\alpha}, \underline{i}, \ldots \) are for the pairs of indices \( (\alpha_i, \alpha_i), (\beta_i, \beta_i), \ldots \). The \( \tilde{F}_{\alpha i} j^k \equiv (1/2) \epsilon_{ab} F_{\alpha i} j^k \) is the dual of \( F_{\alpha i} j^k \).

The BIDs at \( d \geq 3/2 \) yield the superfield equations

\[
i (\sigma_{abc})_{\alpha i} T_{\alpha i} j^k = 0 , \quad i (\sigma_{abc})_{\alpha i} T_{\alpha i} j^k = 0 ,
\]

\[
\nabla_F^{ab} F_{\alpha i} j^k = 0 ,
\]

\[
R_{ab} + \frac{1}{2} \left( F_{ac} F_{bc}^{ab} - \tilde{F}_{ac} \tilde{F}_{bc}^{ab} \right)
\]

\[
= R_{ab} + \left[ F_{ac} F_{bc}^{ab} - \frac{1}{2} \eta_{ab} (F_{\gamma d} F_{ab})^2 \right] = 0 .
\]

Notice the similarity of this system to the \( N = 1 \) SG, we saw in sect. 3, which makes it easier to impose a SD condition. Define the superfields \( W_{\alpha i} \) and \( \tilde{W}_{\alpha i} \) by

\[
W_{\alpha i} \equiv (1/2) (\sigma_{abc})_{\alpha i} F_{\gamma d} , \quad \tilde{W}_{\alpha i} \equiv (1/2) (\sigma_{abc})_{\alpha i} F_{\gamma d} ,
\]

as the analogues of the \( w_{\alpha} \) and \( \tilde{w}_{\alpha} \) in the \( N = 1 \) SYM, or, equally, as the analogues of the \( W_{\alpha i} \) and \( \tilde{W}_{\alpha i} \) in the \( N = 1 \) SG. Then our SD condition is

\[
W_{\alpha i} = 0 .
\]

\(^{11}\)Here we are using \( SO(2) \) notation, so that some constraints look different from the \( Sp(1) \) ones. For example, all the raising and lowering of the \( SO(2) \) are done by the usual Kronecker’s delta, which does not cause any sign change, and effectively they do not matter, as opposed to the previous \( Sp(1) \) case in eq. (4.5).
This condition implies the superfield equations
\[ T_{ab}^{\gamma k} = 0, \quad \tilde{T}_{ab}^{\gamma k} = \frac{1}{2} \epsilon_{abc} \tilde{\epsilon}^{|d|e} \tilde{T}_{cd}^{\gamma k}, \]
\[ F_{ab} = \tilde{F}_{ab}, \]
\[ R_{ab} = 0, \quad R_{ab}^{cd} = \frac{1}{2} \epsilon_{abc} \tilde{\epsilon}^{|d|e} R_{cd}^{ef}, \]
via BIDs of \( d \geq 3/2 \). The consistency of our SD condition (4.14) with all the BIDs is also confirmed. In particular, we can verify that even the \( d = 5/2 \) BID is satisfied in a highly non-trivial way. First, we notice that under our SD condition the reduced BID\[ \nabla_{[a} T_{bc]}^{\gamma l} = 0 \]
holds, which helps us to confirm that
\[ \nabla_{a[i} [ R_{bc}^{de} - \frac{1}{2} \epsilon_{bc} \tilde{f}^g R_{fg}^{de} ] \varepsilon_{d]} = 0, \]
by the help of our superfield equations (4.15). Thus, there is no essential fundamental difference of \( N = 2 \) SDSG from the \( N = 1 \) SDSG, and everything is straightforward.

5. \( N = 4 \) Self-Dual Supergravity

The construction of \( N = 4 \) SDSG follows the same pattern we adopted for the \( N = 2 \) SDSG. We start with the non-self-dual case first as before, utilizing also the usual \( D = (1,3), N = 4 \) results. It has been known that there are three different superspace versions for the \( D = (1,3), N = 4 \) SG, namely the \( SO(4) \) theory [29], the \( SU(4) \) theory [30], and the \( SU(4) \) theory related to the heterotic string [31]. Among these it is easiest to utilize the \( SO(4) \) theory consistently truncated down to the \( N = 4 \) SDSG theory.

Our on-shell \( N = 4 \) SG with the global \( SO(4) \) symmetry before the SD condition has the field content \((e_m^i, \psi_m^{\alpha i}, \tilde{\psi}_m^{\dot{\alpha} i}, A_m^{ij}, A, B)\), where the indices \( i, j, \ldots = 1, \ldots, 4 \) are for the 4-representation of the \( SO(4) \) group. The central charge gauge fields \( A_m^{ij} \) are in the 6-representation. Since the \( D = (1,3), N = 4 \) superspace is similar to the \( D = (2,2), N = 4 \) superspace, we can rely on the results of Ref. [32]. The essential difference, however, is that the superfields \( W \) and \( \overline{W} \) in the former [32] are now replaced by the two completely independent real scalar superfields \( B \) and \( A \), respectively. In other words, \( A \) and \( B \) are not related to each other by complex conjugation in our \( D = (2,2) \) case.

Although the explicit construction to follow concentrates on the \( SO(4) \) theory, we emphasize that this is purely a matter of choice. In particular, the \( SU(4) \) theory [31] that is closely related to the \( D = (1,3), N = 4 \) heterotic string can also be made into a self-dual
theory. The key point is that this theory also possesses two superfield field strengths that permit the condition in (3.35) to be imposed. Upon the imposition of this constraint (3.35) on the $D = (1, 3)$, $N = 4$ supergravity theory, it too can be turned into a self-dual theory in AW space-time.

Our independent BIDs are

$$\nabla_{[A T_{BC}]^D} - T_{[AB]} E_{E(C)}^D - R_{[ABC]}^D \equiv 0 \ ,$$

$$\nabla_{[A F_{BC}]^{ij}} - T_{[AB]}^D F_{D(C)}^{ij} \equiv 0 \ ,$$

with the superfield strengths $F_{AB}^{ij}$ for the the central charge gauge fields. Utilizing the similarities between the $D = (2, 2)$, $N = 4$ SG and the $D = (1, 3)$, $N = 4$ SG, we can solve these BIDs. In fact, we find that the appropriate supertorsion, supercurvature, and constituent constraints are as follows. First, the supertorsion constraints are

$$T_{a_1 a_2} = \frac{1}{4} A \Lambda (a_1 \delta_{\beta} a_2) \ , \quad T_{a_1 a_2} = C_{a_1 a_2} C_{a_1 a_2} \Lambda_{a_1 a_2} \ , \quad T_{a_1 a_2} = T_{a_1 a_2} = 0 \ ,$$

$$T_{a_1 a_2} = -\frac{1}{4} \delta_{a_1 a_2} \Lambda_{a_1 a_2} \ , \quad T_{a_1 a_2} = 2 \delta_{a_1 a_2} \Lambda_{a_1 a_2} \ , \quad T_{a_1 a_2} = - \delta_{a_1 a_2} \Lambda_{a_1 a_2} \ ,$$

$$T_{a_1 a_2} = \frac{1}{4} (A \tilde{D}_{a_1} B) \Lambda_{a_1 a_2} \delta_{a_1 a_2} + \frac{i}{2} C_{a_1 a_2} \Lambda_{a_1 a_2} \delta_{a_1 a_2} \Lambda_{a_1 a_2} \ ,$$

$$T_{a_1 a_2} = -C_{a_1 a_2} \Lambda_{a_1 a_2} \Lambda_{a_1 a_2} \Lambda_{a_1 a_2} \Lambda_{a_1 a_2} \ ,$$

We use the underlined indices $\underline{a}, \underline{b}, \ldots$ and $\underline{a}, \underline{b}, \ldots$ for $\alpha, \beta, \ldots$ and $\alpha, \beta, \ldots$, while $(A \tilde{D}_{a_1} B) \equiv A(D_{a_1} B) - (D_{a_1} A) B \ , \quad a = a(A, B) \equiv 1 - A^2 - B^2$. The new superfields on the r.h.s. of eq. (5.2) above and eq. (5.4) below are the ones, to which the whole set of the supertorsions and supercurvatures reduces. There exist also complementary constraints, obtained by exchanging $A \leftrightarrow B$, dotted $\leftrightarrow$ undotted indices, and tilded $\leftrightarrow$ untilded superfields. This is because $A$ and $B$ superfields are not related by complex conjugation in our case. For example,

$$T_{a_1 a_2} = -i C_{a_1 a_2} f_{a_1 a_2} \ ,$$

Their forms are thus parallel and straightforward to derive.
The supercurvatures are:
\[
R_{\omega^\gamma\delta} = 0, \quad R_{\omega^\gamma\delta} = 2C_{\alpha\beta}\bar{f}_{\gamma\delta ij},
\]
\[
R_{\omega^\gamma\delta} = -\frac{1}{2}C_{\alpha(\gamma[\Lambda_{\beta]i}]\bar{\Lambda}_{\beta}^j - \frac{1}{4}\delta_i^j\Lambda_{\beta]}\bar{\Lambda}_{\beta}^i],
\]
\[
R_{\omega^\gamma\delta} = i\frac{7}{16}C_{\alpha\beta}C_{ijkl}\bar{\Lambda}_{\beta}^j f_{\gamma\delta}^{kl} + i\frac{1}{32}C_{\beta(\gamma}C_{ijkl}\bar{\Lambda}_{\beta}^j f_{\delta)\alpha}^{kl} + \frac{3}{16}a^{-1}C_{(\alpha(D_{\beta)}A)\Lambda_{\beta]}i - \frac{5}{16}C_{\alpha\beta}(D_{(\gamma]A)\Lambda_{\delta)i},
\]
\[
R_{\omega^\gamma\delta} = -iC_{\alpha\beta}\bar{\Sigma}_{\beta\gamma\delta} - i\frac{1}{16}C_{\beta(\gamma}C_{ijkl}\bar{\Lambda}_{\beta}^j f_{\alpha\beta}^{kl} + \frac{1}{38}a^{-1}C_{\alpha\beta}C_{\beta(\gamma}(D_{\epsilon\delta)}A)\Lambda_{\epsilon}^i + \frac{3}{16}a^{-1}C_{\beta(\gamma}(D_{\alpha\beta)}A)\Lambda_{\beta]i},
\]
\[
R_{\omega^\gamma\delta} = -\frac{1}{2}C_{\alpha\beta}\left[V_{\alpha\beta\gamma\delta} + i\frac{1}{16}C_{\alpha(\gamma}\bar{\Lambda}_{\beta]}^i \left\{ D_{\beta}^i + \frac{3}{4}a^{-1} \left( \Lambda A_{\delta}^i \right) \right\} \Lambda_{\beta]}i
\]
\[
+ C_{\alpha(\gamma C_{\beta]} \left\{ \frac{1}{4}a^{-2}(D_{\epsilon\delta)}A)(D_{\epsilon\delta)}B) + i\frac{3}{16}a^{-1} \left( A_{\delta}^i \right) \Lambda_{\epsilon}^i,
\]
\[
+ \frac{15}{12\beta}i\bar{\Lambda}_{\beta}^i \Lambda_{\delta}^m \Lambda_{\epsilon}^i \Lambda_{\epsilon}^m \right\}
\]
\[
+ \frac{1}{2}C_{\alpha\beta} \left[ f_{\alpha\beta}^{ij} \bar{\Lambda}_{\beta}^i \Lambda_{\beta]}^j - \frac{1}{4}a^{-1}(D_{(\gamma]A)(D_{(\beta]}B)
\]
\[
- i\frac{1}{16}D_{(\gamma]A}(\Lambda_{\beta]}\bar{\Lambda}_{\beta]}^i \left( \Lambda_{\beta]}\bar{\Lambda}_{\beta]}^i \right)
\]
\[
- \frac{1}{64}i\bar{\Lambda}_{\beta}^i \Lambda_{\beta]}^m \Lambda_{\gamma]}^m \Lambda_{\delta]i}. \]

The central charge field strengths are:
\[
F_{\omega^\alpha}^{kl} = 2a^{-1} C_{\omega^\alpha} E_{ij}^{kl},
\]
\[
E_{ij}^{kl} = \frac{1}{2} \left[ \delta_i^{[k} \delta_j^{l]} + AC_{ij}^{kl} \right], \quad F_{\omega^\alpha}^{ij} = 0,
\]
\[
F_{\omega^\alpha}^{kl} = -i\frac{1}{2} a^{-1/2} C_{\omega^\alpha} \bar{\Lambda}_{\alpha}^i C_{ijmn} \bar{E}_{mkl}, \quad (5.5)
\]
\[
F_{\omega^\alpha}^{kl} = \frac{1}{2} a^{-1/2} \left[ C_{\omega^\alpha} f_{\omega^\alpha}^{mn} E_{mn}^{kl} + C_{\alpha\beta} \bar{f}_{\alpha\beta}^{mn} \bar{E}_{mkl} \right],
\]
\[
f_{\omega^\alpha}^{kl} = f_{\omega^\alpha}^{kl},
\]
where \( f_{\omega^\alpha}^{ij} \) and \( \bar{f}_{\omega^\alpha}^{ij} \) correspond to the component YM field strengths as their \( \theta = 0 \)-sectors.
Finally we need what is called the \textit{constituency relations} [32]:
\[
D_{\alpha i} B = (1 - A^2 - B^2) \Lambda_\alpha , \quad \bar{D}_{\alpha i} B = 0 ,
\]
\[
D_{\alpha i} \Lambda_{\beta j} = C_{ijkl} f_{\alpha \delta}^{kl} - \frac{3}{4} A A \alpha_{i} \Lambda_{\beta j} , \quad \bar{D}_{\alpha i} \Lambda_{\beta j} = 2i a^{-1} \delta_j^i (D_{\beta \alpha} B) + \frac{3}{4} B \Lambda_\beta \Lambda_{\alpha i} \Lambda_{\beta j} ,
\]
\[
\bar{D}_{\alpha i} f_{\beta \gamma}^{jk} = \frac{1}{2} B \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} + i \delta_m^i \left\{ D_{\beta \gamma} \Lambda_\alpha + \frac{i}{16} \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} + \frac{3}{4} a^{-1} \left( \Lambda D_{\beta \gamma} B \right) \right\} \Lambda_{\gamma n} ,
\]
\[
D_{\alpha i} f_{\beta \gamma}^{jk} = \delta_i \left[ \sum_{\alpha \beta} \gamma^k \right] - \frac{1}{2} A A \alpha_{i} f_{\beta \gamma}^{jk} + i \frac{1}{4} a^{-1} \delta_i \left[ \sum_{\alpha \beta} \gamma^k \right] C_{\alpha \beta} (D_{\gamma \delta} B) \Lambda_{\beta \delta} \Lambda_{\gamma m} + i \Lambda \alpha_{i} \Lambda_{\beta \gamma} \Lambda_{\gamma m} + \frac{3}{4} a^{-1} \Lambda D_{\beta \gamma} B \Lambda_{\gamma m} ,
\]
\[
\bar{D}_{\alpha i} \Sigma_{\beta \gamma}^{i} = \frac{1}{2} B \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} - \frac{1}{2} \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} - \frac{1}{2} a^{-2} (D_{\beta \gamma} A)(D_{\bar{\gamma} \delta} B) C_{\alpha \beta \delta} \Lambda_{\gamma m} + \frac{3}{4} \Lambda D_{\beta \gamma} B \Lambda_{\gamma m} ,
\]
\[
D_{\alpha i} \Sigma_{\beta \gamma}^{i} = \delta_i \Lambda_{\beta \gamma} \Lambda_{\gamma m} - \frac{1}{4} A A \alpha_{i} \Sigma_{\beta \gamma}^{i} - \frac{1}{2} a^{-2} (D_{\beta \gamma} A)(D_{\bar{\gamma} \delta} B) C_{\alpha \beta \delta} \Lambda_{\gamma m} + \frac{3}{4} \Lambda D_{\beta \gamma} B \Lambda_{\gamma m} ,
\]
\[
D_{\alpha i} V_{\beta \gamma} = \frac{1}{2} \left\{ D_{\beta \gamma} A - \frac{1}{4} a^{-1} \Lambda D_{\beta \gamma} B \right\} \Sigma_{\gamma \delta}^{i} + \left( \frac{3}{4} \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} - \frac{1}{2} \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} \right) \Lambda_{\gamma m} + \frac{3}{4} a^{-1} (D_{\beta \gamma} A) \Lambda_{\gamma m} ,
\]
\[
D_{\alpha i} V_{\beta \gamma} = \frac{1}{2} \left\{ D_{\bar{\gamma} \delta} A + \frac{1}{4} a^{-1} \Lambda D_{\beta \gamma} B \right\} \Sigma_{\gamma \delta}^{i} + \left( \frac{3}{4} \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} - \frac{1}{2} \Lambda_\alpha \Lambda_{\beta \gamma} \Lambda_{\gamma m} \right) \Lambda_{\gamma m} + \frac{3}{4} a^{-1} \Lambda_{\gamma m} \Lambda_{\gamma m} ,
\]
\[
B = 0 .
\]

We now concentrate on the question of SD condition. The slight difference of the present system from the $N = 1$ or $N = 2$ counterpart is that the scalar superfields $A$ and $B$ are the most fundamental superfields. For this reason, the natural SD condition is to put one of these scalars to be zero. The convenient choice is
\[
B = 0 .
\]

This generates other superfield equations, such as
\[
\Lambda_{\alpha i} = 0 ,
\]
\[
f_{\alpha \beta}^{ij} = 0 ,
\]
\[
\Sigma_{\alpha \beta}^{\gamma k} = 0 ,
\]
\[
V_{\alpha \beta \gamma} = 0 .
\]
through the BIDs of \( d \geq 3/2 \). Eqs. (5.7) and (5.8) are the complete set of field equations in the system. Eq. (5.8b) is nothing else than the SD condition on the central charge field strength, (5.8c) is the \( N = 4 \) analogue of the \( N = 1 \) condition \( W_{\alpha \beta \gamma} = 0 \) in eq. (3.30). Eq. (5.8d) implies the SD condition \( R_{\alpha \beta \gamma} = (1/2)\epsilon_{\alpha \beta \gamma}^{\delta \epsilon}R_{\delta \epsilon} \) for the Riemann tensor and the Ricci-flatness \( R_{\alpha \beta} = 0 \) because of the vanishing torsion \( T_{\alpha \beta \gamma} = 0 \) under eq. (5.8a). The consistency of the SD condition (5.7) with all the Bianchi identities can be easily confirmed by the inspection of the above constraints, especially of the constituency relations (5.6).

6. An Apparent \( N = 4 \) “No-Go” Barrier for SDYM

Before discussing the SD condition in the \( N = 4 \) SYM case, we need to formulate the non-self-dual version of this theory in \( D = (2, 2) \).

Our \( N = 4 \) SYM field content is \( (A^I, \lambda^I, \bar{\lambda}^I, S^I, T^I) \), where the indices \( i, j, \ldots \) are for the adjoint representation of a gauge group, the indices \( i, j, \ldots = 1, 2, 3, 4 \) are used for the \( SO(4) \) representations, while the indices \( i, j, \ldots = 1, 2, 3 \) are reserved for the 3-dimensional vector representations of the \( SU(2) \) factors in the \( SO(4) \approx SU(2) \otimes SU(2) \). The \( S^i \) and \( T^i \) can equivalently be represented by the self-dual: \( S_{ij} = (1/2)\epsilon_{ijkl}S_{kl} \), and anti-self-dual: \( T_{ij} = -(1/2)\epsilon_{ijkl}T_{kl} \) tensors of \( SO(4) \), respectively. The fields \( S \) and \( T \) are real scalars, while the \( \lambda \)’s and \( \bar{\lambda} \)’s are MW spinors, essentially in the same notation used for the \( N = 2 \) SYM case, considered in sect. 4.

Similarly to the familiar \( D = (1, 3), N = 4 \) SYM theory of Ref. [21], we impose the \( N = 4 \) superspace constraints on the \( N = 4 \) SYM field strengths as

\[
F_{\alpha \beta j}^I = 2C_{\alpha \beta}U_{ij}^I, \quad F_{\alpha \beta j}^I = 2C_{\alpha \beta}W_{ij}^I, \\
F_{\alpha \beta j}^I = 0,
\]

where the real scalar \( N = 4 \) superfields \( U \) and \( W \) have been introduced, \( U_{ij} = U_{ji} \) and \( W_{ij} = W_{ji} \).

Our goal is to obtain the irreducible component spectrum containing one YM field, so we impose the additional constraint beyond eq. (6.1), in order to get rid of redundant components, while maintaining the theory to be non-trivial, namely

\[
W_{ij} = \frac{1}{2}\epsilon_{ijkl}U_{kl} \equiv \bar{U}_{ij}.
\]

In fact, this additional constraint also goes along the lines of the \( N = 4 \) superspace formulation of the \( D = (1, 3), N = 4 \) SYM. (cf. Ref. [21]).
As the consequences of the \( N = 4 \) superspace BIDs, we have two equations for the \( U \) superfield to satisfy:
\[
\nabla_{\alpha i} U_{jk} + \nabla_{\alpha j} U_{ik} = 0 ,
\]
\[
\tilde{\nabla}_{\alpha i} \tilde{U}_{jk} + \tilde{\nabla}_{\alpha j} \tilde{U}_{ik} = 0 .
\]

(6.3)

It is important to notice that eq. (6.3) gives all the information about the \( U \) superfield which follows from the BIDs and the constraints (6.1) and (6.2). In particular, eq. (6.3) does imply the equations of motion for the \( N = 4 \) SYM theory to be satisfied.

It is now an easy exercise to check that we are left with the components
\[
U_{ij} , \ n_{\alpha i} U_{ij} , \ \tilde{\nabla}_{\alpha i} \tilde{U}_{ij} , \ F_{\underline{ab}} ,
\]

(6.4)
as the only independent ones. The self-dual and anti-self-dual parts of \( U \) represent the \( S \) and \( T \) above:
\[
U_{ij}^+ \equiv \frac{1}{2} (U_{ij} + \tilde{U}_{ij}) = S_{ij} ,
\]
\[
U_{ij}^- \equiv \frac{1}{2} (U_{ij} - \tilde{U}_{ij}) = T_{ij} ,
\]

(6.5)

the covariant spinor derivatives of \( U \) in eq. (6.4) are identified with the \( \lambda \)'s and \( \tilde{\lambda} \)'s,
\[
F_{\alpha i b} = -i (\sigma_\alpha)_{\alpha \beta} \lambda^\beta_{\ i} , \ F_{\alpha i b} = -i (\sigma_\alpha)_{\beta \alpha} \lambda^\beta_{\ i} ,
\]

(6.6)

whereas \( F_{ab} \) stands for the (non-self-dual) YM field strength:
\[
\nabla_{\alpha} i \lambda_{\beta j}^I = -\frac{1}{4} \delta_{ij} (\sigma_{cd})_{\alpha \beta} F_{cd}^I + \frac{1}{2} f^{IJK} C_{\alpha \beta \alpha} \epsilon^{ijkl} U_{kl}^I U_{pj}^K ,
\]
\[
\tilde{\nabla}_{\alpha} i \tilde{\lambda}_{\beta j}^I = \frac{1}{4} \delta_{ij} (\sigma_{cd})_{\alpha \beta} \tilde{F}_{cd}^I + \frac{1}{2} f^{IJK} C_{\alpha \beta \alpha} \epsilon^{ijkl} U_{kl}^I U_{pj}^K .
\]

(6.7)

Any higher number of covariant derivatives acting on \( U \) can be reduced, using the results above.

In the component formulation, the \( N = 4 \) supersymmetry transformation rules in the AW space-time take the form:
\[
\delta A_{ab}^I = -i \bar{\epsilon} \sigma_a \lambda^I - i \epsilon \sigma_a \tilde{\lambda}^I ,
\]
\[
\delta \lambda^I = -\frac{1}{4} \sigma^{ab} F_{ab}^I - \frac{1}{2} \alpha_i \sigma^{ab} \bar{\epsilon} \nabla_a S_i^I - \frac{1}{2} \beta_j \sigma^{ab} \bar{\epsilon} \nabla_a T_i^I + \frac{i}{4} \epsilon^{ijk} f^{IJK} \alpha_i \epsilon S_j^J S_k^K,
\]
\[
\delta \tilde{\lambda}^I = -\frac{1}{4} \bar{\sigma}^{ab} \tilde{F}_{ab}^I - \frac{1}{2} \alpha_i \bar{\sigma} \bar{\epsilon} \nabla_a S_i^I + \frac{1}{2} \beta_j \bar{\sigma} \bar{\epsilon} \nabla_a T_i^I + \frac{i}{4} \bar{\epsilon}^{ijk} f^{IJK} \alpha_i \bar{\epsilon} S_j^J S_k^K,
\]
\[
\delta S_{it} = i \epsilon \alpha_i \lambda^I + i \bar{\epsilon} \alpha_i \tilde{\lambda}^I , \ \ \ \delta T_{it} = i \epsilon \beta_i \lambda^I - i \bar{\epsilon} \beta_i \tilde{\lambda}^I ,
\]

(6.8)
where we have introduced the $4 \times 4$ $SO(4)$ gamma matrices $\alpha$ and $\beta$, which satisfy an algebra [33]
\[
\{\alpha^i, \alpha^j\} = 2\delta^{ij}, \quad [\alpha^i, \alpha^j] = 2i\epsilon^{ijk}\alpha^k,
\]
\[
\{\beta^i, \beta^j\} = 2\delta^{ij}, \quad [\beta^i, \beta^j] = 2i\epsilon^{ijk}\beta^k,
\]
\[
[\alpha^i, \beta^j] = 0.
\] (6.9)
All the $\alpha^i_{ij}$ and $\beta^i_{ij}$ matrices are supposed to be antisymmetric on their $SO(4)$ indices $i$ and $j$, where $i, j = 1, 2, 3, 4$ and $i, j = 1, 2, 3$. The $SO(4)$ indices as well as $\alpha, \beta$ are implicit in eq. (6.8). The $N = 4$ supersymmetry algebra reads in this case as
\[
[\delta_1, \delta_2] = \delta_t(\xi^g) + \delta_g(\Lambda_g),
\] (6.10)
where the space-time translation parameter $\xi$ and the gauge parameter $\Lambda_g$ are given by
\[
\xi^g = i \left[ (\bar{\epsilon}_1\bar{\alpha}\epsilon_2) - (\bar{\epsilon}_2\bar{\alpha}\epsilon_1) \right],
\]
\[
\Lambda_g^I = -i \left[ (\bar{\epsilon}_1\alpha^i\bar{\epsilon}_2) + (\epsilon_1\alpha^i\epsilon_2) \right] S^I_i - i \left[ (\bar{\epsilon}_1\beta^i\bar{\epsilon}_2) - (\epsilon_1\beta^i\epsilon_2) \right] T^I_i.
\] (6.11)
An invariant lagrangian and the equations of motion could also be written down here, which will eventually turn out to be unnecessary in what follows in this section.

We are now prepared enough to discuss the SD condition for this $N = 4$ theory. When trying to implement it, we encounter that the $N = 4$ SYM theory in $D = (2,2)$ does not follow the pattern developed for other supersymmetric theories in the previous sections. In superspace, the basic reason for this is that the relevant superfield $U$ is not chiral. Instead, both self-dual and anti-self-dual constituents of the YM field strength are contained in the one superfield, and there is no way to separate them consistently with the $N = 4$ supersymmetry. Any additional constraint, like the chirality or the SD of $U$, makes the theory trivial, as can be easily checked by the use of eq. (6.3). The result is always negative, in the trial of constructing the $N = 4$ SDYM.

To make it even more clear, let us consider the $D = (2,2), N = 4$ SYM theory by decomposing it into the $N = 2$ “sub-multiplets” it contains. The $N = 4$ non-self-dual SYM multiplet consists of the $N = 2$ non-self-dual SYM multiplet and the $N = 2$ non-self-dual hypermultiplet. But we already know from subsect. 4.1 that the self-dual hypermultiplet does not exist. In summary, the $N = 4$ SDYM does not exist because there is no real chiral (or self-dual) $N = 2$ hypermultiplet. The latter does not exist, simply because one needs both chiral and anti-chiral scalar $N = 1$ superfields to introduce the $N = 2$ hypermultiplet which is no longer represented by the chiral $N = 2$ superfield.

\[\text{However, we will give it in a different context in eq. (7.11).}\]
We finally give another reasoning for a non-existence of $N = 4$ supersymmetric SDYM, based on the transformation rules (6.8). Let us for simplicity think of the Abelian case in eq. (6.8), and try to put $\lambda^I = 0$. The obvious obstruction is that the r.h.s. of $\delta \lambda^I$ is to vanish, requiring that the $\alpha$ and $\beta$-matrix terms are to vanish. Since these two sorts of matrices are algebraically independent of each other, unless we kill both $S^{\hat{i}I}$ and $T^{\hat{i}I}$-fields, but we can not satisfy such a requirement, and thus the supersymmetry itself is truncated from $N = 4$.

The failure to construct the maximally extended $D = (2, 2), N = 4$ SDYM theory gives evidence that the $N$-extended SDSGs for $N \geq 4$ should not exist either in the same sense, namely without the inclusion of additional propagating fields. The basic reason for this feature seems to be the absence of appropriate chiral $N$-extended superfield strengths in their superspace on-shell formulations. The chirality of the relevant superfields turns out to be essential to separate the self-dual and anti-self-dual parts of the components.

7. To Bypass No-Go Barrier

The SD condition, when considered as an equation of motion, presents a long unsolved problem. In particular, we would like to have an action whose variation yields the SD condition as an equation of motion. As a more stringent requirement, we might demand that such an action should contain no additional degrees of freedom. Under these restrictions there is no known action that is satisfactory, as we have seen in the previous sections. However, dropping the latter condition allows one to simply use a Lagrange multiplier. Variation of the gauge field yields an equation of propagation for the Lagrange multiplier. The type of action suggested by Parkes [34] and supersymmetrized by Siegel [35] is exactly of this nature. This can clearly be seen even at the level of superfields. In this section we give the superfield formulation for $N = 1$ and $N = 2$ SDYM and SDSG, and also $N = 4$ SDYM in component formulation to bypass the no-go barrier.

Let us start with the $N = 1$ SDYM theory, and consider the following action

$$I_{SDYM} = \int d^4x d^2\theta \, \Lambda^I \Lambda^I \, ,$$

(7.1)

where $W_\alpha$ is the usual field strength for a SYM multiplet. In this action $\Lambda_\alpha$ is a chiral ($\bar{\nabla}_\alpha \Lambda_\beta = 0$) lagrange multiplier. In order to prove that this is precisely the $N = 1$ Parkes-Siegel (PS) action, it is convenient to define the components contained in $\Lambda_\alpha$:

$$\Lambda_\alpha^I = \rho_\alpha^I \, , \quad \nabla_\alpha \Lambda_\beta^I = (a^{ab})_{\alpha\beta} G_{ab}^I + C_{\alpha\beta} \varphi^I \, ,$$

(7.2a)

$$\nabla^\gamma \nabla_\gamma \Lambda_\beta^I = \psi_\beta^I \, ,$$

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and the components of $W_\alpha$ have their usual form:

$$W_\alpha^I = \lambda_\alpha^I, \quad \nabla_\alpha W_\beta^I = (\sigma^{\underline{ab}})_{\alpha\beta} F_{\underline{ab}}^I(A) + C_{\alpha\beta}D^I,$$

$$\nabla^\gamma \nabla_\gamma W_\alpha^I = i(\sigma^a)_{\alpha\beta} \nabla_\alpha \lambda_\beta^I.$$

The superfield action above then reduces to

$$I_{SDYM} = \int d^4x [\frac{1}{2} C_{\underline{ab}}^I(F_{\underline{ab}}^I(A) - \frac{1}{2} C_{\underline{ab}}^I F_{\underline{cd}}^I(A)) + i \rho^{\alpha I}(\sigma^a)_{\alpha\beta} \nabla_\alpha \lambda_\beta^I + \varphi^I D^I + \psi^\alpha I \lambda_\alpha^I],$$

$$\tag{7.2b}$$

We thus clearly see the first two terms correspond to the "propagating" fields in the PS SDYM action. The latter two terms only yield trivial equations for auxiliary fields.

The existence of the self-dual $N = 1$ TM (3.31) - (3.35) as a solution to a set of superspace BIDs implies that the philosophy of the PS can be applied to that theory also. An $N = 1$ off-shell PS formulation is given by

$$I_{N=1}^{PS} = \int d^4x d^2\theta \bar{\Lambda}^{\alpha} \tilde{D} \Phi,$$

$$\tag{7.3}$$

where $\tilde{D} \Phi = 0$, and $D^2 \Phi = \tilde{D}^2 \Phi = 0$. It is a simple matter to show this action imposes eq. (3.35) as an equation of motion.

It is also obvious how to generalize the PS formulation to the case of SDSG. Namely we just make the replacements $\Lambda^\alpha \rightarrow \Lambda^{\alpha\beta\gamma}$ and $W_\alpha \rightarrow W_{\alpha\beta\gamma}$, where $\Lambda^{\alpha\beta\gamma}$ is a Lagrange multiplier and $W_{\alpha\beta\gamma}$ is the superfield conformal supergravity field strength. So the obvious action for $N = 1$ PS SDSG becomes

$$I_{SDSG}^{N=1} = \int d^4x d^2\theta \varphi^3 \Lambda^{\alpha\beta\gamma} W_{\alpha\beta\gamma},$$

$$\tag{7.5}$$

where $\varphi^3$ is the chiral density multiplet. The fact that SG fields appear only (besides the density factor) through $W_{\alpha\beta\gamma}$, the conformal field strength, implies that this action possesses space-time superconformal symmetry! This means that using a component tensor calculus approach, only superconformal $Q$-variations are required to prove the superinvariance of this action. Since only superconformal components enter eq. (7.5), the usual $S$ and $P$ auxiliary fields are absent from this action. Superconformal symmetry also tells us a lot more about eq. (7.5). For example, the component fields $\lambda_\alpha$, $F_{\underline{ab}}$, and $D$ are replaced respectively by $\psi_{\alpha\gamma}$, $W_{\underline{abcd}}$, $\theta_{[2]} A_{[2]}$ and $\phi_{\underline{a}} \gamma$. These latter quantities correspond to the gravitino field strength, the anti-self-dual part of the Weyl tensor, the curl of the SG axial vector auxiliary field, and finally the $S$-supersymmetry gauge field strength. (Of course, we are in a second order formulation so the spin-connection $\omega_{\underline{abc}}$ and $S$-supersymmetry gauge field $\phi_{\underline{a}} \gamma$ are

\footnote{It is interesting to note that $\Lambda_\alpha$ corresponds to a TM. Usually in a WZ gauge $\rho_\alpha = 0$ and $\varphi$ and $\psi_\alpha$ correspond to physical fields.}
expressed in terms of $e^m_a$ and $\psi^g_a$.) The component content of the Lagrange multiplier $\Lambda_{\alpha\beta\gamma}$ is also perfectly clear. The corresponding components to $\rho_\alpha$, $G_{ab}$, $\varphi$ and $\psi_\alpha$ are given by $\rho_{\alpha\beta\gamma}$, $\Lambda_{\alpha\beta\gamma}$, $t_{ab}$ and $\sigma_{\alpha\beta\gamma}$. The equations of motion for these respectively yield the vanishing gravitino field strength, the vanishing self-dual Weyl tensor, the vanishing self-dual auxiliary axial vector field strength and the vanishing $S$-supersymmetry gauge field strength tensor.

Once the $N = 1$ superfield formulation of PS-type constructions is clear, we can immediately generalize it to higher $N$. For example, the $N = 2$ SDYM PS action is just

$$I^{N=2}_{\text{SDYM}} = \int d^4x d^4\theta \Lambda^I S^I, \quad (7.6)$$

where $\Lambda^I$ is a chiral scalar Lagrange multiplier superfield and $S^I$ is the $N = 2$ SYM vector multiplet field strength. Similarly for $N = 2$ SG, we have

$$I^{N=2}_{\text{SDSG}} = \int d^4x d^4\theta \epsilon^{-1} \Lambda^{ab} W_{ab}, \quad (7.7)$$

with $\epsilon^{-1}$ denoting the $N = 2$ chiral density multiplet, $W_{ab}$ is the $N = 2$ conformal SG field strength tensor [36] and $\Lambda^{ab}$ is an $N = 2$ chiral Lagrange multiplier. Finally this construction works for the $N = 4$ conformal SG theory [37] as well:

$$I^{N=4}_{\text{SDSG}} = \int d^4x d^8\theta \epsilon^{-1} \Lambda W. \quad (7.8)$$

We now give the PS formulation of the $N = 4$ supersymmetric SDYM system in components. This is extremely interesting, because it is not known how to write the $N = 4$ analog of eq. (7.1) in superspace consistently. Nevertheless, by proceeding with a component formulation, we are able to write down such a theory. Due to the complication in the off-shell formulation for $N = 4$, we give the on-shell results in component fields. The field content is $(A^I_a, G^{ab}_I, \rho^I, \bar{\lambda}^I, S^I_\alpha, T^I_\beta)$, where $i, j, \ldots = 1, 2, 3$ are the indices for the $\alpha$ and $\beta$-matrices exactly as in sect. 6. We first give the invariant action $I^{N=4}_{\text{SDYM}} = \int d^4x \mathcal{L}^{N=4}_{\text{SDYM}}$, where

$$\mathcal{L}^{N=4}_{\text{SDYM}} = -\frac{1}{2} G^{ab}_I (F^{ab}_I - \frac{1}{2} \epsilon^{abc} F^{cI}_a) + \frac{1}{2} (\nabla_a S^I_\alpha)^2 - \frac{1}{2} (\nabla_a T^I_\beta)^2 + 2i (\rho^I \sigma^{ab} D_a \bar{\lambda}^I) - i f^{IJK} \left[ (\bar{\lambda}^I \alpha_i \lambda^J) S^K_\alpha + (\bar{\lambda}^I \beta_j \lambda^J) T^K_\beta \right]. \quad (7.9)$$

Because of the peculiar coupling between $G$ and $F$, the self-dual part of $G^{ab}_I$ is a “gauge” degree of freedom. This $G^{ab}_I$ plays a similar role to the anti-self-dual part of $F^{ab}_I$ in the
The supertranslation rule is

\[ \delta A^I_a = -i(\epsilon \sigma_a \tilde{\lambda}^I) , \]

\[ \delta G_{ab}^I = i(\tilde{\epsilon} \tilde{\sigma}_{[a} \nabla_{b]} \rho^I) - i \frac{1}{2} f^{IJK} \left[ (\epsilon \alpha_I \sigma_{ab} \rho^J) A^K_a - (\epsilon \beta_J \sigma_{ab} \rho^J) B^K_b \right] , \]

\[ \delta \rho^I = -\frac{1}{4} \sigma_{ab} \tilde{\epsilon} G_{ab}^I - \frac{1}{2} \alpha_I \sigma^a \tilde{\epsilon} \nabla_a S^I_i - \frac{1}{2} \beta_I \sigma^a \tilde{\epsilon} \nabla_a T^I_i \]

\[ + i \frac{1}{4} \epsilon \tilde{j}_k \alpha_I \epsilon \tilde{j}^I \kappa J S^I_{ik} S^K_j - i \frac{1}{4} \epsilon \tilde{j}_k \beta_I \epsilon \tilde{j}^I \kappa J T^I_{ik} T^K_j - \frac{1}{2} f^{IJK} \alpha_I \beta_J \epsilon \tilde{S}^I_{jk} T^K_j T^K_k \quad (7.10) \]

\[ \delta \tilde{\lambda}^I = -\frac{1}{4} \tilde{\sigma} a \tilde{\epsilon} F_{ab}^I - \frac{1}{2} \alpha_I \tilde{\tilde{\sigma}} \tilde{a} \kappa \nabla_a S^I_i + \frac{1}{2} \beta_I \tilde{\tilde{\sigma}} \tilde{a} \kappa \nabla_a T^I_i , \]

\[ \delta S^I_i = i(\epsilon \alpha_I \rho) + i(\tilde{\epsilon} \alpha_I \tilde{\lambda}^I) \quad \delta T^I_i = i(\epsilon \beta_I \rho) - i(\tilde{\epsilon} \beta_I \tilde{\lambda}^I) . \]

To compare the \( N = 4 \) SDYM (7.9) and (7.10) with the usual \( N = 4 \) non-self-dual SYM, we first give the lagrangian of the latter:

\[ \mathcal{L}^{N=4}_{\text{SYM}} = -\frac{1}{4} (F_{ab}^I)^2 + \frac{1}{2} (\nabla_a S^I_i)^2 - \frac{1}{2} (\nabla_a T^I_i)^2 + 2i(\lambda^I \sigma^a \nabla_a \tilde{\lambda}^I) \]

\[ - if^{IJK} \left[ (\lambda^I \alpha^a \lambda^J) + (\tilde{\lambda}^I \alpha^a \tilde{\lambda}^J) \right] S^K_a + if^{IJK} \left[ (\lambda^I \beta^a \lambda^J) - (\tilde{\lambda}^I \beta^a \tilde{\lambda}^J) \right] T^K_a \]

\[ - \frac{1}{12} (f^{IJK} S^I_a S^K_a)^2 - \frac{1}{4} (f^{IJK} T^I_a T^K_a)^2 + \frac{1}{2} (f^{IJK} T^I_a T^K_J T^K_K)^2 , \quad (7.11) \]

which is invariant under supersymmetry (6.8). The negative sign for the \( T \)-kinetic term in eq. (7.11) is peculiar to our \( D = (2, 2) \).

The most important property of the lagrangian (7.9) is the absence of quartic terms in scalar fields. Furthermore, the Yukawa interactions of the \( \rho \rho S \) or \( \rho \rho T \)-type are also absent there, while \( \lambda \lambda S \) or \( \lambda \lambda T \)-terms are present in eq. (7.11). The transformation law for \( G_{ab}^I \) in eq. (7.10) is very similar to that of the anti-self-dual part of \( F_{ab}^I \) derivable from eq. (6.8) except for the appearance of the bilinear terms. The \( A^I_a \) transformation law in eq. (7.10) lacks the \( \tilde{\epsilon} \sigma_{ab} \rho \)-term, which is expected from eq. (6.8). Another important difference is the absence of the bilinear bosonic terms for \( \delta \rho \) in eq. (7.10). The absence of some Yukawa terms and the quartic terms in eq. (7.9) are closely related to these properties. Since \( \rho^I_i \) is a superpartner of multiplier \( G_{ab}^I \), there is natural asymmetry between \( \rho \) and \( \lambda \) in eq. (7.10).

The on-shell closure of supersymmetry in eq. (7.10) is easily confirmed, especially by the use of the \( A^I_a \) and \( G -field equations. Even though such a check is parallel to the non-self-dual SYM case, there appear to be important differences, which result in the absent terms as well as the new ones in the transformation rule (7.10).

We point out the advantages as well as the drawbacks of the PS formulation for supersymmetric SDYM. The no-go barrier has been bypassed by the introduction of the additional propagating fields \( G_{ab}^I \) and \( \rho^I_i \). This has some advantage, especially when we are interested in the lower-dimensional integrable systems with such strong supersymmetry as \( N = 4 \).
However, the price to be paid is the inclusion of additional modes even for spin-one helicities out of $G_{\alpha\beta}^{\mu}$ increasing the field content in lower-dimensions after dimensional reductions. This may be a disadvantage, when the tight-field content under the SD condition plays an important role for integrability in lower-dimensions. Actually a similar situation is known in the context of $D = 10$ Type IIA and Type IIB superstrings, both of which give the same zero-mass spectrum in lower-dimensions after the simple dimensional reduction. The formulation in Ref. [35] seems similar to Type IIB superstring construction, due to its chiral features.

We should also point out that there is another more radical way which seems to hold the promise of overcoming the barrier. Although we will not discuss this other method in detail, it also holds the promise of providing very powerful new insights into the long unsolved problem of off-shell $D = 4, N = 4$ SYM theory. Let us return to the first assumptions (6.1) in deriving the no-go barrier. It is well known that for the $N = 4$ case (unlike any lower SYM theory) that eq. (6.1) actually implies the YM equation of motion for the gauge field in the multiplet. From this viewpoint, therefore, it is now suprising that the SD condition is inconsistent with the starting point in eq. (6.1). We may ask if it is possible to somehow change this starting point. The answer turns out to be in the affirmative. Instead we may write

\[
F_{\alpha i\beta j} = 2C_{\alpha\beta}^{I}U_{ij}^{I} + f_{(\alpha\beta)}^{(ij)}^{I}, \quad F_{\alpha i}^{\beta j I} = 2C_{\alpha\beta}^{I}W_{ij}^{I} + \tilde{f}_{(\alpha\beta)}^{(ij)}^{I}, \quad F_{\alpha i}^{\beta j} = 0.
\] (7.12)

where $f_{(\alpha\beta)}^{(ij)}^{I}$ and $\tilde{f}_{(\alpha\beta)}^{(ij)}^{I}$ are independent auxiliary superfields that have not been seen previously. When these superfields are present, the usual YM equations of motion are no longer imposed on the gauge field in the multiplet. A rather suggestive way to impose an asymmetry between $U_{ij}^{I}$ and $W_{ij}^{I}$ is to impose the vanishing of only one of the auxiliary superfields (say $\tilde{f}_{(\alpha\beta)}^{(ij)} = 0$). This might achieve the goal of making the $N = 4$ case more similar to the lower $N$ cases. Note for example that for lower $N$ values in the PS formulation, the Lagrange multiplier always comes from a Lagrange multiplier superfield that is different from the gauge supermultiplet upon which the SD condition is imposed. Note that this is contrary to the component level $N = 4$ theory presented above.

In closing, we note that it has been suggested by Siegel [35,38] that the space-time supersymmetric $N = 4$ versions of the theories above are closely linked to the $N = 2$ and/or $N = 4$ world-sheet supersymmetric string theories. In particular, in Ref. [35] even an $N = 8$ SDSG version was suggested in a way to be related to strings. Should this prove to be the case, we will be in the marvelous position of being able to find an $N = 8$ superconformal multiplet for the first time! Even in the case of the $N = 4$ theory, study of the open strings
should yield new insight into the structure of $N = 4$ SYM theory. This promises to be an area that begs further exploration.

8. Supersymmetric SDYM Zero Modes

The Euclidean formulation of a quantum field theory is well-known to be not just a matter of convenience but an appropriate framework to define the Green’s function generating functional of the theory. Another advantage of the Euclidean formulation is its relation with topology, which manifests itself in the existence of the *instantons* – classical solutions of the field equations with non-trivial topology.

When trying to connect the AW space-time formulation of the quantum field theory to the Euclidean one by a *double* Wick rotation, we immediately encounter the obstruction since, strictly speaking, this continuation can only be performed with the *complex* fields. The reason is that the reality conditions are sensitive to a choice of the space-time signature: there are both Majorana and MW spinors in $2 + 2$ dimensions, but there are no MW spinors in $1 + 3$ dimensions and there are no Majorana spinors at all in the Euclidean space-time. Restricting ourselves to the case of the $N$-extended supersymmetric SDYM theories for simplicity, we choose to consider the $N = 2$ supersymmetric SDYM model first. That theory can be defined in the AW space-time in terms of the complex chiral spinors, and they can be rotated to the Euclidean space-time easily.\[14]\n
Given the Euclidean version of the $N = 2$ SDYM theory, we can take advantage of the existing results about the YM instantons [1,40] and their quantum effects in supersymmetric gauge theories [41]. The well-known Euclidean BPSTH-instanton solution for the YM vector field,

$$A_2^I(x) = \frac{2}{g} \eta_{ab}^I (x - x_0)^b \sqrt{(x - x_0)^2 + \rho^2}, \quad (8.1)$$

and the associated SDYM field strength,

$$F_{ab}^I(x) = -\frac{4}{g} \eta_{ab}^I \frac{\rho^2}{[(x - x_0)^2 + \rho^2]^2}, \quad (8.2)$$

are parametrized by the center-of-instanton coordinate $(x_0)_a$ and the instanton size (the scale of the instanton) $\rho$. In eqs. (8.1) and (8.2) the $\eta_{ab}^I$ represents the constants known as the 't Hooft symbols [40]\[15\] whereas the $g$ is the gauge coupling constant. We restrict

---

\[14\] The relevance of the $N = 2$ supersymmetric YM theory in this context has been noticed in Ref. [39].

\[15\] In fact, one amusing point to note is that the $\alpha$ and $\beta$ matrices of Ref. [33] (see sect. 7) are exactly the same as the $\eta$ and $\bar{\eta}$-symbols of Ref. [40].
ourselves to the case of the \( SU(2) \) gauge group as usual, the extension to a general gauge group \( G \) could, in principle, be done in a Chevalley basis for \( G \).

The self-dual solution (8.1) can be interpreted as the vector zero mode, which is accompanied in the \( N = 2 \) supersymmetric SDYM case by the Dirac zero modes (see subsect. 4.2 for our notation):

\[
\begin{align*}
(\bar{\lambda}_{ss})^i_J(x) &= g (\bar{\sigma}_{ab})^{ij}_a F_{ab}^J(x) \bar{\alpha}^i, \\
(\bar{\lambda}_{sc})^i_J(x) &= g (\bar{\sigma}_{ab})^{ij}_a F_{ab}^J(x) i \left( \bar{\sigma}_c \right)_\beta \gamma(x - x_0) \bar{\beta}^\gamma_i.
\end{align*}
\] (8.3)

They do satisfy the Dirac equation, provided the YM field strength \( F \) is self-dual. The appearance of the \( \bar{\lambda}_{ss} \) is due to supersymmetry, while the existence of another zero mode \( \bar{\lambda}_{sc} \) is due to superconformal symmetry of the theory [41]. In eq. (8.3) the \( \bar{\alpha} \) and \( \beta \) represent anti-chiral and chiral Grassmannian constants respectively, which parametrize the solutions of the supersymmetric equations of motion. As was noted in Ref. [41], the \( (x_0, \bar{\alpha}) \) form the chiral superspace coordinates, while the collective coordinates \( (\rho, \beta) \) are united into a superfield parameter. Without adding new propagating fields, some kind of supersymmetry among the solutions survives by allowing the instanton size \( \rho \), as well as the \( \beta \), to transform [41]. The unusual (and incomplete) supersymmetry of Ref. [41] can be recast into the usual \( N = 2 \) supersymmetry if the theory is extended by the inclusion of a complex scalar, i.e. within the \( N = 2 \) supersymmetric YM theory constrained by the supersymmetric SD condition similar to that used in subsect. 4.2 but now in the Euclidean space. The last observation has first been made by Zumino [39], and we are not going to repeat here his presentation.

However, this is not the end of the story. Given the \( N = 2 \) SDYM spin-1 and spin-1/2 zero modes, there should be in addition, the spin-0 or scalar zero modes, which are supposed to satisfy the equation of motion presented in the last line of eq. (4.6) to be transformed to the Euclidean space. To this end, we are going to find the \( N = 2 \) SDYM scalar zero modes explicitly.

The scalar equation of motion in the \( N = 2 \) SDYM theory with the \( SU(2) \) gauge group takes the form

\[
\begin{align*}
\partial^a \partial_a T^I + g \epsilon^{IJK} \left( \partial^a A^J_a \right) T^K + 2 g \epsilon^{IJK} A^a J_a T^K + 2 g^2 \epsilon^{IJKLM} T^{MP} (A^L_a A^N_a) T^K = \epsilon^{IJK} (\bar{\lambda}^i J \bar{\lambda}^K_i),
\end{align*}
\] (8.4)

where \( \epsilon^{IJK} \) are the \( SU(2) \) structure constants; \( i, j, k = 1, 2, 3 \). Using the explicit solutions for the fields \( A^I_a(x) \) and \( \bar{\lambda}^i_J(x) \) given above in eqs. (8.1), (8.2) and (8.3), and the
identities for the $\eta$-symbols:

$$
\eta_{ab}^I = \frac{1}{2} \epsilon_{abcd} \eta_{cd}^I ,
$$

$$
\eta_{ab}^I \eta_{cd} = \delta_{IJ} \eta_{bc} + \epsilon^{IJK} \eta_{bc}^K ,
$$

$$
\epsilon^{IJK} \eta_{ab}^I \eta_{cd}^K = \eta_{ac} \eta_{bd}^I - \eta_{ad} \eta_{bc}^I - \eta_{bc} \eta_{ad}^I + \eta_{bd} \eta_{ac}^I ,
$$

(8.5)

and for the $\sigma$-matrices:

$$
\tilde{\sigma}_{ab} = \frac{1}{2} \epsilon_{abcd} \tilde{\sigma}_{cd} ,
$$

$$
\tilde{\sigma}_{ab} \tilde{\sigma}_{cd} = - \epsilon_{abcd} + \frac{1}{2} ( \eta_{ad} \tilde{\sigma}_{bc} + \eta_{bd} \tilde{\sigma}_{ac} + \eta_{ac} \tilde{\sigma}_{bd} + \eta_{bc} \tilde{\sigma}_{ad} ) ,
$$

$$
\sigma \tilde{\sigma} \eta_{ab} \eta_{cd} = \frac{1}{2} ( \eta_{bc} \sigma_{ad} - \eta_{bd} \sigma_{ac} + \epsilon_{abcd} \sigma^d ) ,
$$

(8.6)

allows us to considerably simplify eq. (8.4) to the form

$$
\partial^2 \partial_x T^I (x) - \frac{8(x - x_0)^2}{((x - x_0)^2 + \rho^2)^2} T^I (x) + \frac{4\rho^4}{((x - x_0)^2 + \rho^2)^2} q^I (x) = 0 ,
$$

(8.7)

where we have introduced the notation

$$
q_{ss}^I (x) = 16 \eta_{ab}^I ( \tilde{\alpha}^{i} \tilde{\sigma}_{ab} \tilde{\alpha}^{i} ) \equiv q^I ,
$$

$$
q_{sc}^I (x) = -16 \eta_{ab}^I ( \beta^{i} \sigma_{cd} \tilde{\sigma}_{ab} \tilde{\sigma}_{cd} \beta^{i} ) (x - x_0)^2 (x - x_0)^2
$$

$$
\equiv p_{(cd)}^I (x - x_0)^2 (x - x_0)^2 .
$$

(8.8)

In eq. (8.8) the $q^I$ and $p_{(cd)}^I$ are the covariant nilpotent constants. In particular, the $p_{ab}^I$ is traceless,

$$
p_{aa}^I = 0 ,
$$

(8.9)

because of the identity

$$
\sigma \tilde{\sigma} \eta_{ab} \eta_{cd} = 0 .
$$

(8.10)

We now substitute

$$
T_{ss}^I (x) = t_{ss} (y) q^I , \quad y = (x - x_0)^2 ,
$$

(8.11)

in the case of the supersymmetric scalar zero mode $T_{ss} (x)$, which results in the ordinary differential equation

$$
y t_{ss}'' + 2 t_{ss}' - \frac{2y}{(y + \rho^2)^2} t_{ss} + \frac{\rho^4}{(y + \rho^2)^2} = 0 .
$$

(8.12)

After the substitution

$$
t_{ss} (y) = \frac{\rho^4 f_{ss} (y)}{(y + \rho^2)^2}
$$

(8.13)

\footnote{In our notation here, prime always means a differentiation, e.g. $t' (y) \equiv dt/dy , \quad t'' (y) \equiv d^2 t/dy^2 , \quad$ etc.}
eq. (8.12) takes the form

$$y(y + \rho^2)^2 f''_{ss} + 2(y + \rho^2)(\rho^2 - y)f'_{ss} - 4\rho^2 f_{ss} + 1 = 0.$$  \hspace{1cm} (8.14)

This equation has a constant particular solution,

$$f_{ss} = \frac{1}{4\rho^2},$$  \hspace{1cm} (8.15)

while the two independent solutions of the associated homogeneous equation \((q = 0)\) take the form

$$f_1(y) = \frac{1}{3\rho^2}y^3 + \frac{4}{3\rho^2}y^2 + \frac{2}{\rho^2}y + 1,$$  \hspace{1cm} (8.16)

$$f_2(y) = \frac{\rho^2}{y} + 1.$$

Both solutions in eq. (8.16) seem to be irrelevant to the instantons, since the scalar field \(T^I(x)\) defined by eqs. (8.11) and (8.13) either goes to infinity when \(x \to \infty\) in the case of the \(f_1(y)\) to be added to the solution (8.15), or has a singularity at \(y = 0\) in the case of the added \(f_2(y)\).

Therefore, we conclude that the correct scalar zero mode corresponding by supersymmetry to \(A^I_a(x)\) and \(\tilde{\lambda}^{iI}_{ss}(x)\), takes the form

$$T^{I}_{ss}(x) = \frac{4\rho^2}{[(x - x_0)^2 + \rho^2]^{2} \eta^{I}_{ab} (\tilde{\alpha}_i \tilde{\sigma}^{ab} \tilde{\alpha}_i) = (\tilde{\alpha}_i \tilde{\lambda}^{iI}_{ss})}. \hspace{1cm} (8.17)$$

This solution is to be expected since in the second equality of eq. (8.17) we just get the \(N = 2\) supersymmetry transformation law of the \(N = 2\) SDYM theory.

Let us turn now to the superconformal zero mode \(T_{sc}(x)\) and substitute

$$T_{sc}^{I}(x) = \frac{f_{sc}(z)}{\rho^2 z(z - 1)^3 \lambda^{I}_{ab} (x - x_0)^{\mu}(x - x_0)^{\nu}} \hspace{1cm} (8.18)$$

where we have introduced the new argument, \(z\),

$$z = \rho^{-2}y + 1 \equiv \frac{(x - x_0)^2}{\rho^2} + 1 \hspace{1cm} (8.19)$$

The substitution (8.18) reduces the partial differential equation (8.7) to a second-order ordinary differential equation, because of the use of eq. (8.9). It leads to the equation of the Fuchsian type [42] and, ultimately, to the hypergeometric equation of Gauss.\(^{17}\) The equation for the \(f_{sc}(z)\) takes the form

$$f_{sc}^{n} - 2 \left( \frac{1}{z - 1} + \frac{1}{z} \right) f_{sc} + \frac{2}{z(z - 1)}f_{sc} + \frac{(z - 1)^2}{z^3}f_{sc} + \frac{(z - 1)^2}{z^3} = 0.$$  \hspace{1cm} (8.20)

\(^{17}\)We appreciate discussions with G. Gilbert, who pointed out this fact.
The general solution of eq. (8.20) reads

\[ f_{\text{sc}}(z) = -F(z) \int^z \frac{z_1^2(z_1 - 1)^2}{F^2(z_1)} \, dz_1 \int^{z_1} \frac{F(z_2)}{z_2^6} \, dz_2 \]  

(8.21)

in terms of the solution \( F(z) \) to the associated homogeneous equation, which is just the standard hypergeometric function \( F(\alpha, \beta, \gamma; z) \) [43] with the parameters \( \alpha, \beta \) and \( \gamma \) to be determined from the coefficients of eq. (8.20) according to the rule [43]

\[ \alpha + \beta = -5, \quad \alpha \beta = 2, \quad \gamma = -2. \]  

(8.22)

Since the \( \gamma \) is a negative integer, the solution to the homogeneous hypergeometric equation is in fact [43]

\[ F(z) = \frac{F(\alpha, \beta, -2; z)}{\Gamma(-2)} = \frac{1}{6} \alpha(\alpha + 1)(\alpha + 2)\beta(\beta + 1)(\beta + 2)z^3F(\alpha + 3, \beta + 3, 4; z), \]  

(8.23)

where

\[ \alpha = \frac{-5 - \sqrt{17}}{2}, \quad \beta = \frac{-5 + \sqrt{17}}{2}. \]  

(8.24)

The behaviour of the solution \( T_{\text{sc}}(x) \) at the infinity \( x \to \infty \) is given by

\[ T_{\text{sc}}^I(x) \to \frac{\rho^4}{2(x^2)^3} p_{ab}^{I} x^a x^b. \]  

(8.25)

Near the instanton center, where \( x \to x_0 \), we have

\[ T_{\text{sc}}^I(x) \to -\frac{(x - x_0)^2}{4\rho^4} p_{ab}^{I}(x - x_0) x^a x^b. \]  

(8.26)

Eq. (8.25) guarantees the finiteness of the Euclidean action for the scalar superconformal zero modes, whereas the vanishing of the scalar solution at \( x_0 = 0 \) in eq. (8.26) clearly matches the similar property of the corresponding Dirac zero mode in the second line of eq. (8.3).\[\text{The vanishing of the solutions at } x_0 = 0 \text{ is due to the fact that we are in fact talking about the orbital part of the superconformal transformations in question. The spin part of them is zero for the scalars. The superconformal symmetry among the solutions is trivially restored at } x \to \infty.\]
to the supersymmetric scalar zero mode in eq. (8.17), the superconformal scalar zero mode can not be represented as the superconformal transformation of the superconformal Dirac zero mode because, though the $N = 2$ SDYM equations of motion are superconformally invariant, their instanton solutions are not.\footnote{The superconformal invariance may be recovered in the quantum theory after integrating over $\rho$ and $\beta$.\[41\].}

There seem to be no obstructions in a continuation of the Euclidean solutions to the AW space-time.\footnote{Of course, the relation with topology becomes much more obscure, if any, in the AW space-time compared to the Euclidean space, and there is no well-defined value for the scalar action in $2 + 2$ dimensions any more.}

In many respects, there are analogues between various objects in $D = (4, 0)$ and $D = (2, 2)$ dimensions due to a similarity of the “Lorentz” group structure in both cases:

$$SO(4) \cong SU(2) \otimes SU(2) ,$$

$$SO(2, 2) \cong SU(1, 1) \otimes SU(1, 1) \cong Sp(2) \otimes Sp(2) . \quad (8.27)$$

In particular, the ’t Hooft’s $\eta$-symbols in both cases are nothing but the generators of the self-dual subgroups $SU(2)$ or $Sp(2)$, respectively,

$$J^I_{SD} = \frac{1}{4} \eta^I_{ab} M^{ab} , \quad J^I_{ASD} = \frac{1}{4} \eta^I_{ab} M^{ab} , \quad (8.28)$$

where we have introduced the “Lorentz” generators $M^{ab}$ and the generators of the self-dual ($J^I_{SD}$) and anti-self-dual ($J^I_{ASD}$) subgroups.

Finally, it is worthwhile to take a look at the $N = 4$ supersymmetric SDYM theory which exists in the PS formulation. The Euclidean version of that theory could easily be read off from the results of sect. 7. First, let us look at the equations of motion which follow from the lagrangian (7.9) for the scalars $S^I_i(x)$ and $T^I_i(x)$, and the spinors $\rho^I_j(x)$ and $\tilde{\lambda}^I_j(x)$.

The search for their solutions associated with the YM instanton leads to the same equations as were considered above in this section, after taking into account rather obvious redefinitions of the covariant nilpotent constants $q^I$ and $p^I_{ab}$. The appearance of the additional internal symmetry indices does not play any essential role here, and the dynamical factors for the scalar and spinor instanton solutions uncovered in this section remain intact. The only novel feature is due to the presence of the additional propagating field $G_{ab}(x)$ whose equation of motion resembles the YM case:

$$D^{ab} G^{-}_{ab} \equiv \frac{1}{2} D^{ab} \left( G_{ab} - \frac{1}{2} \epsilon_{abcd} G^{cd} \right) = 0 . \quad (8.29)$$

Eq. (8.29) simply states that the solution for the $G^{-}_{ab}$ should be represented by the anti-self-dual field:

$$\left( G^{-} \right)^I_{ab} = \text{const} \cdot \eta^I_{ab} \frac{\rho^2}{\left[ (x - x_0)^2 + \rho^2 \right]^2} , \quad (8.30)$$
where the symbols $\eta_{ab}^f$ have also been introduced by 't Hooft [40].

9. Concluding Remarks

The relevance of a $D = 4$ space-time with the $(2, 2)$ signature (the AW space-time) appears to be based on the assumption that the underlying “master” theory of all exactly solvable (integrable) models generated by the SDYM theory in $D = (2, 2)$ might be just the $N = 2$ heterotic string. The search for the underlying theory of integrable systems is conceptually justified by noticing that it could explain the origin of hidden symmetries in a variety of physically interesting examples of such systems. Space-time supersymmetry by itself is a good motivation to introduce the supersymmetric SDYM and SDSG theories, which have a good chance to be the generating theories for all \textit{supersymmetric} integrable models in lower dimensions. Just another motivation comes from the anticipated space-time supersymmetry in the $N = 2$ superstring theory [38]. The explicit superspace constructions given above prove the consistency between SD and \textit{(extended)} supersymmetry in $D = (2, 2)$. The indefinite $D = (2, 2)$ space-time signature and the existence of MW spinors in this space-time were crucial in all of the constructions. The real difference between the superspace formulations of various supersymmetric gauge theories and supergravities with extended supersymmetry in $D = (1, 3)$ and $D = (2, 2)$ stems from the reality conditions imposed on the spinor derivatives, the superspace torsion and curvature components with spinor indices. The maximally extended $N = 4$ SYM and the $N > 4$ SG appear to be incompatible with the SD condition without the inclusion of new propagating fields. The reason for this is the apparent importance of the MW property for the relevant superfield strengths, which is only possible for $N \leq 1$ in the case of SM, $N \leq 2$ in the case of SYM and $N \leq 4$ in the case of SG.

Both SDYM and SDG equations of motion are non-linear partial differential equations which allow an infinite number of conservation laws. The supersymmetry does not violate the basic reason of integrability of the SDYM equations, namely their interpretation as the zero curvature conditions in the Hamiltonian formulation, which makes it possible to apply the inverse scattering method for their integration [45]. This is simply because the SD condition on the YM field strength is \textit{not} modified in \textit{(extended)} supersymmetry, but just accompanied by...

\footnote{The $N = 2$ \textit{twisted} SYM theory in the \textit{Euclidean} space-time is relevant for the $D = 4$ Donaldson theory [44] describing topology of low-dimensional manifolds, as advocated by Witten [16]. However, there seem to be no reason to link our construction of the $N = 2$ SDYM theory directly with the Witten’s $N = 2$ SYM theory, since they are very different. Nevertheless, there might be a possibility to exploit the \textit{topological} information provided by the $N = 2$ SDYM theory to get some insights into the Witten’s topological $N = 2$ SYM theory.}
by the associated equations on the other components of the self-dual supermultiplet under consideration. This is particularly amusing in extended SDSGs, where both scalars and vectors are added to the graviton field, and the SDG condition implies the SDYM condition. As was argued by Mason [46] (see also Ref. [47]), the dimensional reduction of the SDYM equations with two null vectors and the gauge group $SL(\infty)$ describes the SD Einstein equations in the so-called Plebanski [48] form re-discovered by Penrose [49] in the context of general relativity, by Park [50] in the context of $W$-algebras, and by Ooguri and Vafa [3] in the context of strings. Therefore, from the viewpoint of integrability, the SDG equations may not be considered as independent.

Another understanding of integrability of SDYM and SDG equations in $D = 4$ is based on their equivalence to the two-dimensional non-linear sigma-models with a pure Wess-Zumino term in their actions [50]. This observation allows to connect four-dimensional self-dual theories with two-dimensional conformal field theories. In particular [50], the twistor construction of the $D = 4$ gravitational instantons given by Penrose [49], can be identified with the inverse scattering approach to the relevant $D = 2$ non-linear sigma-model. It has been known for a while [51], that the moduli space of $D = 4$ YM instantons and that of $D = 2$ instantons valued in the loop group, are equivalent. The important issue in this respect is the relevant gauge symmetry. As was shown in Ref. [50], in the case of SDG the symplectic diffeomorphisms of a $D = 2$ surface act as a gauge group, while in the case of SDYM one has a specific infinite-dimensional gauge group too, which generalizes the Kac-Moody algebra as well as the universal $W_\infty$-algebra. This conclusion agrees with the statement made by Mason [46], that the transformation group $SL(\infty)$ among the solutions of the SD equation is a loop group of area-preserving diffeomorphisms of a $D = 2$ surface. It indicates an existence of some general relation between the SD and the $W_\infty$-symmetries which can be interpreted as the special area-preserving diffeomorphisms [52]. Our construction of the $N = 4$ SDSG gives the first evidence for the existence of a $N = 4$ extended $W_\infty$-algebra.

The obvious application of our construction is the reduction of the supersymmetric self-dual systems from $D = 4$ to $D = 2$, the work of which is in progress now. The reduction means imposing an appropriate ansatz for the gauge connection after introducing two Killing symmetries. As for the SDYM is concerned, the use of the compact Lie group and Euclidean signature yields the $D = 2$ Toda field theory [53], while the use of non-compact $SL(2)$ group and the $(2,2)$ signature gives rise to the KdV equation [54] (see also Ref. [55] as for the further generalizations of this construction). Therefore, one now has a natural way to connect the $D = 2$ super-Toda field equations with the $D = (2,2)$ supersymmetric

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22The Plebanski equation is nothing but the constancy condition on the metric determinant, which ensures the Ricci-flat condition in the case of Kähler geometry. It is the theorem in $D = 4$ [23] that a Ricci-flat 4-manifold is a hyper-Kähler one when the curvature is self-dual.
SDYM, and supersymmetrize the KdV equation as well as the well-known KdV flows [56]. It has recently been conjectured [55], that SDG in the AW space-time provides a universal integrable system of equations, which yields all the KdV-flows by truncation, so that it is just enough to supersymmetrize the reduction of the $D = 4$ SDSG in order to get the super-KdV flows.

In Appendix C we argue that the $N = 2$ superstring is finite to all orders in string loops. Though there is now some evidence [35,38] that the $N = 2$ as well as $N = 4$ superstrings do possess space-time supersymmetry, a precise connection between the $N = 2$ superstrings and the space-time supersymmetry is yet to be elaborated. It is well-known that the Green-Schwarz superstrings, when coupled to SYM and/or SG backgrounds, do require them to satisfy their equations of motion [57]. Technically, it originates from requiring the Siegel’s invariance of the coupled Green-Schwarz action, which is crucial for the consistency of the theory. Taking the supersymmetric space-time background on-shell is just enough for the consistency of the theory, but it is the non-trivial and important observation that the SD condition in addition does not violate that consistency [17].

In Appendix D, we indicate an interesting possibility that a SG plus SYM theory in $D = (5,5)$ may be even a more fundamental theory, that generates the $D = (2,2)$ SDSG and SDYM, via simple dimensional reductions. If this is indeed the case, the SDYM and SDSG in $D = (2,2)$ themselves are some effective theories of the more fundamental theory. Since the $D = 10$ is supposed to be the highest dimension, where SG and SYM theories are possible notwithstanding particular signatures [8], it is not unlikely that the $D = (5,5)$ SG + SYM theory is singled out in a significant way, as in the case of $N = 1$ superstring model.

**Acknowledgements**

We are grateful for R. Brooks, D. Depireux, T. Hübsch, G. Gilbert, A. Schwarz, W. Siegel, C. Vafa and E. Witten for stimulating discussions.
Appendix A: General Notation and Conventions

For the flat space-time with four dimensions and signature \((2, 2)\), the metric form in \textit{real} coordinates \(y^a = (y^1, y^2, y^3, y^4)\) reads

\[
ds^2 = \eta^{(R)}_{ab} dy^a dy^b \equiv (dy^1)^2 + (dy^2)^2 - (dy^3)^2 - (dy^4)^2 .
\] (A.1)

The very form of the flat \((2, 2)\) metric suggests to use \textit{complex} coordinates

\[
x^1 = \frac{1}{\sqrt{2}} \left( y^1 + iy^2 \right) , \quad x^2 = \frac{1}{\sqrt{2}} \left( y^3 + iy^4 \right)
\] (A.2)

and their conjugates, \(x^a = (x^1, x^2, \bar{x}^1, \bar{x}^2)\), instead of the real ones. The complex notation is natural in the \((2, 2)\) space-time, because of the classical isomorphism \([9]\) \(SO(2, 2; \mathbb{R}) \cong SU(1, 1; \mathbb{C}) \otimes SU(1, 1; \mathbb{C})\). In other words, there is an obvious complex structure in this space-time which gives the simplest example of a \(D = 4\) hermitian manifold. In the complex notation, the metric takes the form

\[
ds^2 = \eta_{ab} dx^a dx^b \equiv 2 \left( dx^1 dx^1 - dx^2 d\bar{x}^2 \right) ,
\] (A.3)

where we have introduced the notation

\[
x^a = (x^1, x^2) , \quad x^\bar{a} = \bar{x}^a = (\bar{x}^1, \bar{x}^2) , \quad (a = (a, \bar{a}) ; \ a = 1, 2; \ \bar{a} = 1, 2) .
\] (A.4)

The metric tensor \(\eta_{ab}\) in this notation is

\[
\eta_{ab} = \begin{pmatrix} 0 & \eta_{\bar{a}b} \\ \eta_{\bar{a}b} & 0 \end{pmatrix} , \quad \eta_{a\bar{b}} = \eta_{\bar{b}a} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\] (A.5)

The metric form can now be rewritten as

\[
ds^2 = 2\eta_{\bar{a}b} dx^a d\bar{x}^\bar{b} .
\] (A.6)

The inner product of \((2, 2)\) vectors \(V\) and \(U\) reads: \(V^a U_a \equiv V^a U_a + V^\bar{a} U_{\bar{a}} = V^a \eta_{ab} U^b + V^\bar{a} \eta_{\bar{a}b} U^b\).

For the \(D = 4\) vectorial indices we use the \textit{underlined} letters \(\underline{a} \equiv (a, \bar{a})\), \(\underline{b} \equiv (b, \bar{b})\), ..., which are equivalent to the pairs of two-dimensional complex coordinate indices \(a, b, \ldots = 1, 2\) and their conjugates \(\pi, \bar{\pi}, \ldots = 1, 2\). As for the spinorial indices, we use the \textit{undotted} and \textit{dotted} letters \(\alpha, \beta, \ldots = 1, 2\) and \(\alpha^*, \beta^*, \ldots = 1, 2\) for Weyl spinors of each chirality. We also use the \textit{underlined} spinorial indices \(\underline{a} \equiv (a, \alpha)\), \(\underline{b} \equiv (b, \beta)\), ..., denoting the pairs of \textit{dotted} and \textit{undotted} indices. The chiral spinor dotted and undotted indices can be raised and lowered by the use of the antisymmetric charge conjugation matrix \(C_{\alpha\beta}, C^{\alpha\beta}\) and \(C_{\alpha^*\beta^*}, C^{\alpha^*\beta^*}\). (See Appendix B for details).

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As a rule for \( N = 1 \) supersymmetry, we use capital Greek letters to represent any Majorana spinor field in the four-component notation for spinors, while \textit{the same} lower case Greek letter is used to identify the chiral (Weyl) constituents of the same spinor, e.g.

\[
\Psi_\alpha = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix}.
\]

(A.7)

The same rule is sometimes used for the diagonal products of gamma matrices, e.g. the spinor generators \( \Sigma^{ab} \) (see the Appendix B) are decomposable as

\[
\Sigma^{ab} = \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \bar{\sigma}^{ab} \end{pmatrix}.
\]

(A.8)

A real \((2,2)\)-dimensional vector \( \mathbf{V}_a = (V_1, V_2, V_3, V_4) \) can be represented by a complex pair \( \mathbf{V}_a = (V_1, V_2) \) in the complex notation, where \( V_1 = V_1 + iV_2 \) and \( V_2 = V_3 + iV_4 \). Then we have \( \mathbf{V}_a = (V_1, V_2) = (\tilde{V}_1, \tilde{V}_2) \), respectively.

The \( \sigma \)-matrices of eq. (B.16) below can be used to convert a vector index into a pair of spinor indices,

\[
V^{\alpha\dot{\alpha}} = V_a (\sigma^a)^{\alpha\dot{\alpha}}, \quad V_{\dot{a}} = \frac{1}{2} (\bar{\sigma}^a)^{\dot{a}}_{\alpha} V^{\alpha\dot{\alpha}}.
\]

(A.9)
Appendix B: Dirac Matrices

The *Klein-Gordon* equation for a scalar \( \Phi \) in the real coordinates,

\[
\left( \Box - m^2 \right) \Phi(y) \equiv \left( \eta^{ab} \partial_a \partial_b - m^2 \right) \Phi(y) = 0 ,
\]

is rewritten in the complex coordinates as

\[
\left( \Box - m^2 \right) \Phi(x) \equiv \left( 2\eta^{ab} \partial_a \partial_b - m^2 \right) \Phi(x) = 0 .
\]

The *Dirac equation* results from the factorization procedure of the Klein-Gordon kinetic operator in eq. (B.1), and in the real notation it takes the form

\[
\left( i\Gamma^a \partial_a + m \right) \Psi(y) = 0 ,
\]

where the Dirac gamma matrices \( \Gamma^a \) satisfy the Clifford algebra

\[
\{ \Gamma^a, \Gamma^b \} = 2\eta^{ab} ,
\]

Eq. (B.2) can also be factorized, and it leads to the Dirac equation in the form

\[
\left( i\gamma^a \partial_a + m \right) \Psi(x) \equiv \left( i\gamma^a \partial_a + i\gamma^\beta \partial_\beta + m \right) \Psi(x) = 0 ,
\]

where we have introduced the corresponding Dirac matrices \( \gamma^a = (\gamma^a, \gamma^\beta) \). The latter satisfy an algebra

\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} ,
\]

\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 0 ,
\]

\[
\gamma^\beta \gamma^\beta + \gamma^\beta \gamma^\beta = 0 .
\]

In particular, all of those \( \gamma^a \)-matrices are nilpotent: \( (\gamma^a)^2 = 0 \) (no sum).

The representation theory of the Clifford algebra (B.4) can be developed along the lines of the familiar \( D = (1, 3) \) case. There exists only one non-trivial \( 4 \times 4 \) matrix representation of eq. (B.4), and its (equivalent) explicit forms can easily be constructed by the use of \( 2 \times 2 \) Pauli matrices,

\[
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\]

The *Majorana* representation of the \( \Gamma \)-matrices takes the form

\[
\Gamma^1 = \begin{pmatrix} 0 & -i \tau_3 \\ i \tau_3 & 0 \end{pmatrix} , \quad \Gamma^2 = \begin{pmatrix} 0 & -i \tau_1 \\ i \tau_1 & 0 \end{pmatrix} ,
\]

\[
\Gamma^3 = \begin{pmatrix} 0 & \tau_2 \\ -\tau_2 & 0 \end{pmatrix} , \quad \Gamma^4 = \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix} ,
\]
where 0 and \( I_2 \) are the \( 2 \times 2 \) zero and identity matrices, respectively. In this representation all of the components of its \( \Gamma \)-matrices are pure imaginary, i.e. all the entries of the \( i\Gamma^\alpha \) in the Dirac equation are purely real.

Another explicit representation, which is particularly useful in supersymmetry, has the diagonal \( \Gamma_5 \)-matrix. It provides a preferred basis for introducing the two-component notation for spinors in four space-time dimensions, and generically has the form

\[
\Gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ \bar{\sigma}^\alpha & 0 \end{pmatrix}
\]

(B.9)

with some \( 2 \times 2 \)-dimensional entries \( \sigma^\alpha \) and \( \bar{\sigma}^\alpha \). The appropriate explicit representation is given by

\[
\Gamma^1 = \begin{pmatrix} 0 & -i\tau_1 \\ +i\tau_1 & 0 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & -i\tau_2 \\ +i\tau_2 & 0 \end{pmatrix},
\]

(B.10)

\[
\Gamma^3 = \begin{pmatrix} 0 & +\tau_3 \\ -\tau_3 & 0 \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} 0 & +iI_2 \\ +iI_2 & 0 \end{pmatrix},
\]

since in this representation we have

\[
\Gamma_5 \equiv \Gamma^1\Gamma^2\Gamma^3\Gamma^4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.
\]

(B.11)

Given an explicit representation of the \( \Gamma \)-matrices, it is easy to construct an explicit representation of the \( \gamma \)-matrices (B.6) by forming the linear combinations

\[
V_\pm \equiv \frac{1}{\sqrt{2}} \left( \Gamma^1 \pm i\Gamma^2 \right), \quad W_\pm = \frac{1}{\sqrt{2}} \left( \Gamma^3 \pm i\Gamma^4 \right).
\]

(B.12)

It follows that

\[
V_\pm^2 = W_\pm^2 = 0, \quad \{V_+, V_-\} = -\{W_+, W_-\} = 2, \quad VW + VW = 0,
\]

(B.13)

the latter being true for any assignment of the subscripts \( \pm \). Therefore, each choice,

(I) : \( \gamma^a = (V_+, W_+), \quad \gamma^\beta = (V_-, W_-) \), \hspace{1cm} (II) : \( \gamma^a = (V_+, W_-), \quad \gamma^\beta = (V_-, -W_+) \)

(B.14)

yields the corresponding representation for the \( \gamma \)-matrices. The two possibilities are just related to either the SD (I) or the ASD (II). For simplicity, we restrict ourselves to the SD here and choose (I) for the rest of the Appendix.
Explicitly, we find

\[
\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ +i\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \\ -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

(B.15)

\[
\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & +i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & +\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & +\sqrt{2} & 0 & 0 \end{pmatrix},
\]

This representation has the structure (B.9) which allows to introduce the \( 2 \times 2 \) \( \sigma \)-matrices as

\[
\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \tilde{\sigma}^a & 0 \end{pmatrix}, \quad \gamma^\pi = \begin{pmatrix} 0 & \sigma^\pi \\ \tilde{\sigma}^\pi & 0 \end{pmatrix}.
\]

(B.16)

The \( \sigma \)-matrices satisfy an algebra

\[
\sigma^a \tilde{\sigma}^b + \sigma^b \tilde{\sigma}^a = 2\eta^{ab},
\]

(B.17a)

\[
\sigma^a \tilde{\sigma}^b + \sigma^b \tilde{\sigma}^a = 0,
\]

(B.17b)

\[
\sigma^\pi \tilde{\sigma}^\pi + \tilde{\sigma}^\pi \sigma^\pi = 0,
\]

(B.17c)

which follows from eqs. (B.6) and (B.16).

We find it convenient to define a basis in the space of \( 2 \times 2 \) matrices by introducing the following set

\[
P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

(B.18)

Comparing eqs. (B.15), (B.16) and (B.18), we find

\[
\sigma^a = (-i\sqrt{2}\tau_-, -\sqrt{2}P_-), \quad \tilde{\sigma}^a = (+i\sqrt{2}\tau_-, -\sqrt{2}P_+),
\]

\[
\sigma^\pi = (-i\sqrt{2}\tau_+, +\sqrt{2}P_+), \quad \tilde{\sigma}^\pi = (+i\sqrt{2}\tau_+, +\sqrt{2}P_-).
\]

(B.19)

Linearly independent covariant products of \( \gamma \)-matrices form a basis in the space of all \( 4 \times 4 \) matrices, and our choice of the basis is
\[ I_4 ; \]
\[ \gamma^a, \gamma^\bar{a} ; \]
\[ \gamma_3, \Sigma^{ab}, \gamma_\mp ; \]
\[ \gamma^a \gamma_\mp, \gamma_3 \gamma_\mp ; \]
\[ \gamma_5 , \] (B.20)

where \( \gamma_5 \equiv \Gamma_5 \), while \( \gamma_3 \) and \( \gamma_\mp \) represent the covariant products
\[ \gamma^a \gamma^b = 2i \epsilon^{ab} \gamma_3 , \quad \gamma^\bar{a} \gamma^\bar{b} = 2i \epsilon^{\bar{a}\bar{b}} \gamma_\mp . \] (B.21)

In eq. (B.21) the Levi-Civita symbols have been introduced, \( \epsilon^{ab} = - \epsilon^{ba} = \epsilon_{ab} \), \( \epsilon_{12} = -1 \); \( \epsilon^{\bar{a}\bar{b}} = - \epsilon^{\bar{b}\bar{a}} = \epsilon_{\bar{a}\bar{b}} \), \( \epsilon_{12} = 1 \). The \( \Sigma^{ab} \) in eq. (B.20) are similar to their \((3 + 1)\)-dimensional counterparts and take the form
\[ \Sigma^{ab} = - \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a) . \] (B.22)

Those four matrices are covariantly reducible as
\[ \Sigma^{ab} = \frac{1}{2} \eta^{ab} \Sigma + \hat{\Sigma}^{ab} , \] (B.23)
where we have introduced the irreducible pieces of \( \Sigma^{ab} \) as
\[ \Sigma \equiv \eta_{ab} \Sigma^{ab} = \gamma_\mp_3 - \gamma_3 \gamma_\mp , \quad \eta_{ab} \hat{\Sigma}^{ab} = 0 . \] (B.24)

The commutators involving the \( \gamma_3 \), \( \gamma_\mp \) and \( \Sigma^{ab} \) take the form
\[ \left[ \gamma_3, \Sigma^{ab} \right] = - \eta^{ab} \Sigma \gamma_3 = - \Sigma , \quad \left[ \gamma_3, \Sigma^{\bar{a}\bar{b}} \right] = \eta^{\bar{a}\bar{b}} \gamma_3 , \quad \left[ \gamma_\mp, \Sigma^{ab} \right] = - \eta^{ab} \gamma_\mp , \]
\[ \left[ \Sigma^{ab}, \Sigma^{\bar{c}\bar{d}} \right] = \eta^{\bar{a}\bar{d}} \Sigma^{\bar{b}\bar{c}} - \eta^{\bar{b}\bar{d}} \Sigma^{\bar{a}\bar{c}} . \] (B.25)

This set of matrices forms the Lie algebra which is in fact isomorphic to \( so(2, 2) \), as it should. The \( \left\{ \Sigma^{ab} \right\} \) form the subalgebra which is isomorphic to \( u(2) \). Clearly, the \( \left\{ \gamma_3, \Sigma^{ab}, \gamma_\mp \right\} \) are nothing but the \( SO(2, 2) \) generators in the spinor representation, while \( \left\{ \Sigma^{ab} \right\} \) represent the generators of the \( U(2) \) subgroup in the same representation. Eq. (B.23) reflects the relevant Lie algebra decomposition \( u(2) = su(2) \oplus u(1) \).

In the explicit representation (B.20), the \( \gamma_3 \), \( \Sigma \) and \( \gamma_\mp \) take the form
\[ \gamma_3 = \begin{pmatrix} \tau_- & 0 \\ 0 & \tau_+ \end{pmatrix} , \quad \Sigma = \begin{pmatrix} \tau_3 & 0 \\ 0 & 0 \end{pmatrix} , \quad \gamma_\mp = \begin{pmatrix} \tau_- & 0 \\ 0 & \tau_+ \end{pmatrix} , \] (B.26)
and therefore, they separately form an su(2) subalgebra.

The $\gamma_3$ and $\gamma_\pi$ are nilpotent and hermitian-conjugated to each other, $\gamma_3^\dagger = \gamma_\pi$. All matrices in eq. (B.26) obviously commute with the $\gamma_5$ and satisfy the funny property:

$$\gamma_3 = \gamma_5 \gamma_3 , \quad \gamma_\pi = \gamma_5 \gamma_\pi , \quad \Sigma = \gamma_5 \Sigma . \quad (B.27)$$

The $\hat{\Sigma}^{ab}$ is also commuting with the $\gamma_5$, but has the property

$$\gamma_5 \hat{\Sigma}^{ab} = - \hat{\Sigma}^{ab} . \quad (B.28)$$

In addition, we find the relations:

$$\gamma^a \gamma_3 = \gamma_3 \gamma^a = \gamma_5 \gamma_3 = 0 ,$$

$$[\gamma^a, \gamma_\pi] = -i \epsilon^{ab} \eta_{ac} \gamma^c , \quad [\gamma_\pi, \gamma_3] = -i \epsilon^{ac} \eta_{ac} \gamma^c . \quad (B.29)$$

and some useful identities:

$$\eta_{ab} \gamma^a \gamma^b = 2 - 2 \Sigma , \quad \Sigma \gamma^a = -i \epsilon^{ac} \eta_{ac} \gamma^c . \quad (B.30)$$

The chiral projection operators can be rewritten as

$$\frac{1}{2} (1 + \gamma_5) = (\gamma_3 + \gamma_\pi)^2 = \gamma_3 \gamma_3 + \gamma_\pi \gamma_\pi = 2 \gamma_3 \gamma_\pi + \Sigma , \quad (B.31a)$$

$$\frac{1}{2} (1 - \gamma_5) = 1 - \gamma_3 \gamma_3 - \gamma_\pi \gamma_\pi = 1 - \Sigma - 2 \gamma_3 \gamma_\pi . \quad (B.31b)$$

It is interesting that there are more hermitian projectors for spinors in $D = (2, 2)$. Defining

$$\gamma_{33} \equiv \gamma_3 \gamma_\pi , \quad \gamma_{\pi 3} \equiv \gamma_\pi \gamma_3 , \quad (B.32)$$

we find

$$\gamma_{3\pi}^2 = \gamma_3^\dagger = \gamma_{33} , \quad \gamma_{\pi 3}^2 = \gamma_\pi^\dagger = \gamma_{\pi 3} ,$$

$$(\gamma_3 \gamma_\pi)^2 = \gamma_3 \gamma_\pi , \quad (\gamma_\pi \gamma_3)^2 = \gamma_\pi \gamma_3 . \quad (B.33)$$

In the explicit representation (B.15) or (B.19), we have

$$\gamma_{33} = \gamma_3 \gamma_3 = \left( \begin{array}{cc} P_+ & 0 \\ 0 & 0 \end{array} \right) , \quad \gamma_{\pi 3} = \gamma_\pi \gamma_3 = \left( \begin{array}{cc} P_- & 0 \\ 0 & 0 \end{array} \right) . \quad (B.34)$$

Clearly, they are commuting with the $\gamma_5$. In addition, we find

$$[\gamma^a, \gamma_{3\pi}] = -i \epsilon^{ac} \eta_{ac} \gamma_3 \gamma^b , \quad [\gamma^a, \gamma_{\pi 3}] = -i \epsilon^{ac} \eta_{ac} \gamma_\pi \gamma^b , \quad (B.35)$$

$$[\gamma_\pi, \gamma_{3\pi}] = -i \epsilon^{ac} \eta_{ac} \gamma^b \gamma_3 , \quad [\gamma_\pi, \gamma_{\pi 3}] = -i \epsilon^{ac} \eta_{ac} \gamma^b .$$
Appendix C: Finiteness of $N = 2$ Superstrings

Within the standard background field approach [58] to the quantum four-dimensional SYM theory, it is not difficult to extend the non-renormalization theorem (NRT) [59] from the flat to the super-instanton background [41]. In other words, none of the possible counter-terms survives, when the background fields are restricted to be self-dual. In terms of the super-instanton solution, any correction is supposed to be invariant under the shifts of the $\theta$-coordinate (NRT!). But there is no way to cancel it because of the absence of an anti-chiral instanton collective coordinate (the instanton superfield is chiral!) [41]. In terms of the relevant superfield strengths, there is simply no way to write down any counter-term for the supersymmetric SDYM systems, since the super-SD is always dictated by the conditions that chiral or anti-chiral superfields are to vanish. The physical significance of vanishing quantum corrections is the integrability of the initial classical system combined with supersymmetry.

Unlike the SYM theories, the SGs are non-renormalizable for general backgrounds. Nevertheless, as was shown in Ref. [60], the SD condition eliminates all the counter-terms to all orders, except the topological ones, the latter being quadratic in the curvature. The SDSG finiteness follows as a consequence of both supersymmetry and fermionic chiral invariance [60]. The main assumption used in Ref. [60] is just the validity of supersymmetry or the existence of the supersymmetrical regularization.

In parallel with the supersymmetric SDYM instanton solutions, the SDSG instantons could also be developed. The solutions to eq. (3.28) in the Euclidean space are known as the gravitational instantons. They are Ricci-flat and may be used to describe tunneling between different gravitational (or string) vacua, as well as “topological” fluctuations of the gravitational field (the Hawking’s foam) [61]. According to Ref. [61], all asymptotically locally flat non-singular solutions of the Einstein equations are self-dual (or anti-self-dual). Therefore, it seems to be reasonable to take on the only known compact solution – the so-called Eguchi-Hanson (EH) instanton [62] – as a basis. In the AW space-time it is described by the metric

$$ds^2 = \frac{dr^2}{1-1/r^4} - \frac{1}{4}r^2 \left[ d\vartheta^2 + \sinh^2 \vartheta d\phi^2 \right]$$

$$+ \frac{1}{4}r^2 \left( 1 - 1/r^4 \right) \left[ d\psi^2 + 2 \cosh \vartheta d\psi d\phi + \cosh^2 \vartheta d\phi^2 \right],$$

where we have introduced the parametrization

$$x^1 = r \cosh \left( \frac{1}{4} \vartheta \right) \exp \left[ \frac{1}{2} i (\psi + \phi) \right],$$

$$x^2 = r \sinh \left( \frac{1}{4} \vartheta \right) \exp \left[ \frac{1}{2} i (\psi - \phi) \right].$$

The AW space-time complex coordinates $(x^1, x^2)$ are defined in the Appendix A, see eqs. (A.1)–(A.4).
The EH-instanton solution allows a supersymmetrization within the \( N \)-extended SDSG \((N \leq 4)\) to a full \( N \)-extended chiral superfield. The chirality implies the UV-finiteness through the NRT. Again, the extended SG is needed to move freely between the AW and Euclidean space-times in quantum theory. The typical example is the possible three-loop correction to the \( N = 1 \) quantum SG [63]

\[
I_{N=1}^{(3)} = \int d^4x \, d^2\theta \, d^2\bar{\theta} \, W_{\alpha\beta\gamma} W_{\alpha'\beta'\gamma'} W_{\alpha\beta'\gamma} W_{\alpha'\beta\gamma'},
\]

which comprises the supersymmetrization of the Bel-Robinson tensor squared [63],

\[
T_{abcd}^2 = \left[ R_{a\alpha}^c R_{b\beta}^d + \tilde{R}_{a\alpha}^c \tilde{R}_{b\beta}^d \right]^2,
\]

where the \( \tilde{R}_{abcd} \) is the dual of \( R_{abcd} \). Obviously, if we restrict the SG background to be self-dual by setting \( W_{\alpha\beta\gamma} = 0 \), this integral is to vanish, since it contains both chiral and anti-chiral superfields \( W_{\alpha\beta\gamma} \) and \( \tilde{W}_{\alpha\beta\gamma} \) multiplicatively. This property persists to all orders in the quantum perturbations [60]. In the usual \( D = (1, 3) \) space-time, we could not impose \( W_{\alpha\beta\gamma} = 0 \), while keeping \( \tilde{W}_{\alpha\beta\gamma} \neq 0 \), which is now possible in our \( D = (2, 2) \) case, since \( \tilde{W}_{\alpha\beta\gamma} \) is no longer the complex conjugate of \( W_{\alpha\beta\gamma} \).

The above considerations can equally be applied to the (heterotic) \( N = 2 \) superstring, whose backgrounds are just supersymmetric SDYM and SDSG, to prove its finiteness. If all the counter-terms are to be local at a given string-loop order, as they are expected to be from the experience with the \( N = 1 \) superstring, then any quantum corrections at any genus of the world-sheet are to possess the same features as the tree level counter-terms. Combined with the arguments above, it means the total finiteness of the \( N = 2 \) superstring theory.

Another finiteness argument comes from the anticipated equivalence between the \( N = 2 \) and \( N = 4 \) superstrings [38]. In the \( N = 4 \) formulation the \( SO(2, 2) \) (or “Lorentz”) covariance is manifest, which makes it possible to clearly distinguish between the Neveu-Schwarz and Ramond sectors of the theory and prove a presence of space-time supersymmetry in it. For the \( N = 4 \) superstring, the \( \sigma \)-model field equations are implied classically, rather than by quantum \( \beta \)-function calculations. In other words, the \( N = 4 \) superstring action remains unrenormalized against quantum corrections because of the \( N = 4 \) supersymmetry on the world-sheet [64]. The \( N = 4 \) supersymmetry is associated with the hyper-Kähler geometry [64] which is nothing but the self-dual geometry for the four-dimensional target [23]. Hence, it is the \( N = 4 \) world-sheet supersymmetry (which is quite obscure in the \( N = 2 \) formulation) that is responsible for the \( N = 2 \) superstring finiteness. The string-loop corrections may even vanish for the \( N = 2 \) superstring, which would then constitute a topological field theory. The last viewpoint has recently been advocated by Siegel [35].

It is amazing that the SD condition associated with the chiral superfields plays such an important role for the \( N = 2 \) superstrings. In the \( N = 1 \) superstring theory, there is no such
a strong condition as the SD condition on the backgrounds, so that the $N = 1$ superstring finiteness at the string-loop level is rather obscure.
Appendix D: $D = (5,5), N = 1$ SG + SYM and Dimensional Reduction

In this appendix we give a $D = 10$ theory in the space-time signature $(5,5)$ with $N = 1$ local supersymmetry, which produces our $D = (2,2)$ SYM and SG with various values $N$ of supersymmetry, after dimensional reduction and compactifications. Due to the no-go barrier described in sect. 6, only some of these resultant models can be consistently truncated to generate the SDSG and supersymmetric SDYM theories in $D = (2,2)$. We give one example of a $D = (2,2), N = 1$ system after such dimensional reduction/compactification of our $D = (5,5), N = 1$ initial theory.

The interesting case related to the SDSG and SD SYM is the $D = (2,2), N = 1$ theory based on a technique of dimensional reduction, similar to what we call dimensional reduction (DR) à la Witten [66]. This technique has been developed in component formulation, based on the Calabi-Yau (CY) compactification [67] in the usual $D = (1,9), N = 1$ heterotic superstring theory. The non-compactness [68] of the “extra” $E = (3,3)$ dimensional CY space with the indefinite signature $(3,3)$ poses no problem, as we will discuss later.

We first establish our initial $D = (5,5), N = 1$ SG + SYM system. To this end, we have to clarify the spinorial properties of the space-time. There are good similarities as well as important differences among the $D = (5,5), D = (1,9)$ and $D = (2,2)$ cases, which we can utilize to simplify the calculation. First of all, there exist Majorana, Weyl as well as Majorana-Weyl spinors in the $D = (5,5)$ space-time, as can be easily noticed by the help of Ref. [8]. The main difference of the $D = (5,5)$ case from the $D = (1,9)$ case is about the anti-chiral Weyl spinors, namely the chirality-conjugates are not complex conjugates of the undotted Weyl spinors similarly to the $D = (2,2)$ case. This is due to the reason very similar to the $D = (2,2)$ case, i.e. the matrix $\Gamma^{11} \equiv \Gamma_1 \Gamma_2 \cdots \Gamma_{10}$ [8] gives $(\Gamma^{11})^2 = 1$, so that $\Gamma^{11}$ can be purely real. Therefore, the chiral projectors $P_{\pm} \equiv (1/2)(1 \pm \Gamma^{11})$ are not complex conjugate to each other, just like the $D = (2,2)$ case. It is to noted also that the bars in our $D = (5,5)$ case denote just the spinorial scalar products, exactly like the $D = (1,9)$ case considered in Ref. [69].

After this arrangement of notation, we can construct our initial $D = (5,5), N = 1$ SG + SYM theory with the field content $(e^m_a, \psi^\alpha_m, B_{mn}, \chi_\alpha, \Phi; A^I_m, \lambda^{\alpha I})$, where instead of the dots for spinorial indices, the superscripts and subscripts of spinorial indices denote respectively the chiral and anti-chiral spinors in $D = 10$. Therefore, $\psi^\alpha_m$ and $\lambda^{\alpha I}$ are all chiral Majorana-Weyl spinors, while $\chi_\alpha$ is anti-chiral Majorana-Weyl spinor. This system

---

23We mention that the $D = (2,2), N = 4$ theory with global $SU(4)$ is easily obtained by what is called the simple dimensional reduction, described in Ref. [65]. Since this theory does not yield the SD theory due to the global symmetry $SU(4)$ as mentioned in sect. 5, we do not give the details of the derivation or the results.
is described by the BIDs:

\[
\nabla_{[A}T_{BC]}^D - T_{[AB]}^E T_{E(C)}^D - R_{[ABC]}^D \equiv 0 ,
\]

\[
\frac{1}{6} \nabla_{[A}G_{BCD]} - \frac{1}{4} T_{[AB]}^E G_{E(CD)} \equiv \frac{1}{7} F_{[AB} T_{FCD]}^I \equiv X_{ABCD}^4 , \quad (D.1)
\]

\[
\nabla_{[A}F_{BC]}^I - T_{[AB]}^D F_{D(C)}^I \equiv 0 .
\]

These BIDs are solved by the following set of constraints:

\[T_{\alpha\beta}^c = i(\Gamma^c)_{\alpha\beta} , \quad G_{\alpha\beta\gamma} = G_{abc} = 0 , \quad G_{\alpha\beta c} = i\frac{1}{2}(\Gamma_c)_{\alpha\beta} , \quad T_{ab}^c = 0 , \]

\[T_{\alpha\beta}^\gamma = i\frac{1}{16}(\Gamma_b)_{\alpha\beta}^\gamma (\Gamma^e)_{\gamma} , \quad T_{ab}^\gamma = -\sqrt{2}[\delta_{\alpha\beta} \lambda^\gamma + (\Gamma^\gamma)_{\alpha\beta} (\Gamma_{e\alpha\beta})] , \]

\[T_{\alpha\beta}^c = -2G_{ab}^c , \quad \nabla_\alpha \Phi = -\frac{1}{\sqrt{2}} \chi_\alpha , \]

\[F_{ab}^I = i\frac{1}{\sqrt{2}} (\Gamma_b)_{\alpha} , \quad (D.2)
\]

\[\nabla_\alpha \chi_\beta = -i\frac{1}{\sqrt{2}} (\Gamma^c)_{\alpha} \nabla_\beta \Phi + i\frac{1}{12\sqrt{2}} (\Gamma^{ced})_{\alpha} (G_{ced} - i\frac{1}{4} \chi_\gamma \Gamma_{ced} \chi + i\frac{1}{8} \Gamma^i \Gamma_{ced} \chi^i) , \]

\[\nabla_\alpha \lambda^{\beta I} = -\frac{1}{2\sqrt{2}} (\Gamma^{ced})_{\alpha} \lambda^{\beta} F_{cd}^I - \frac{1}{\sqrt{2}} (\Gamma^{ced})_{\alpha} (\chi_\gamma T_{cd} \lambda^i) - \sqrt{2} \chi_\alpha \lambda^{\beta I} - \frac{1}{2} \delta_{\alpha} (\chi \lambda^i) , \]

\[R_{\alpha\beta\gamma} = -2i(\Gamma^e)_{\alpha\beta} G_{cede} - \frac{1}{16} (\Gamma^{efg})_{\alpha\beta} (\chi_\gamma T_{efg} \lambda^i) - \frac{1}{8} (\Gamma^{efg})_{\alpha\beta} (\chi_\gamma T_{efg} \lambda^i) , \]

\[\nabla_\alpha G_{bcd} = i\frac{1}{4} (\Gamma_{[a} T_{cd]} \alpha) + i\frac{1}{\sqrt{2}} (\Gamma_{[b]} \lambda^i)_{\alpha} F_{[cd]}^I , \]

\[R_{abcd} = -i(\Gamma_{[c} T_{d]} \alpha) + i\frac{1}{\sqrt{2}} (\Gamma_{[b]} \lambda^i)_{\alpha} F_{[cd]}^I . \]

Here the \( G_{mnp} \) contains the Chern-Simons term of the YM:

\[G_{mnp} \equiv \frac{1}{2} \partial_{[m} B_{np]} + X_{mnp}^3 , \quad (D.3)\]

where \( X_{ABC}^3 \) satisfies the relation

\[\frac{1}{6} \nabla_{[A}X_{BCD]}^3 - \frac{1}{4} T_{[AB]}^E X_{E(CD)}^3 \equiv X_{ABCD}^4 \equiv 0 . \quad (D.4)\]

This set of constraints is a \( D = (5,5) \) analogue of the constraints developed for the \( D = (1,9) \) case in Ref. [70] in order to simplify the \( \beta \)-function calculations in Green-Schwarz \( \sigma \)-models, owing to the vanishing exponential functions of the dilaton field \( \Phi \), as well as the simplification of superfield equations, whose explicit forms we skip here.

We next look into the issue of compactification to the \( D = (2,2) \), \( N = 1 \) theory, based on the DR à la Witten on a CY manifold [66]. Due to the similarity between \( D = (1,9) \), \( N = 1 \) and \( D = (5,5) \), \( N = 1 \) theories, we can utilize the main results in the compactification of \( D = (1,9) \), \( N = 1 \) heterotic string, applying them to the special non-compact manifold [71] with the signature \( (3,3) \). However, as has been well-known, the Kaluza-Klein (KK)-type compactification on a CY manifold, keeping all the curved background, is very hard in
practice. Fortunately the DR à la Witten provides us a convenient way [66] to handle this problem relying on a simple principle in a component field formulation. The main principle is that we keep only the $SU(3)$ singlet quantities among all the zero mass fields upon the compactification, where $SU(3)$ is the holonomy group of the CY manifold. By this prescription, the component field computation is drastically simplified, because all the fields we maintain are invariant under the $SU(3)$ group. Our task is to apply this method to our external non-compact space-time with the signature $(3,3)$.

Our next question is the validity of this prescription in superspace. Fortunately, a superspace version of this prescription has also been developed in Ref. [72] for the purely SG sector. Due to another similarity between the $D=(2,2), N=1$ and $D=(1,3), N=1$ SG systems, we can borrow the results in Ref. [72] for our purposes rather easily.

Keeping these points in mind, we establish our notation first. The indices we use are $a, b, \ldots$ and $\alpha, \beta, \ldots$ for respectively vector and spinor indices in $D=(5,5), N=1$ superspace, while $\underline{a}, \underline{b}, \ldots$ and $\alpha, \beta, \ldots$ (or $\underline{\alpha}, \underline{\beta}, \ldots$) for $D=(2,2), N=1$ vector and spinor (or antispinor) indices, respectively. (The $\alpha, \beta, \ldots$ indices are used both in $D=(5,5)$ and $D=(2,2)$, as long as they are not confusing from the context.) For $E=(3,3)$ we use $\underline{a}, \underline{b}, \ldots$ as their vectorial indices, while we use no indices for spinors, because they are not needed in our calculation. For example, our $\Gamma$-matrices in $D=(5,5)$ are decomposed as

$$ (\Gamma^a)_{\alpha\beta} \rightarrow \left\{ \begin{array}{c} (\sigma^a)_{\alpha\beta} \\ \Gamma^a_{\alpha\beta} \end{array} \right\}. \tag{D.5} $$

Accordingly, we have $\Gamma_{11} = \Gamma_7 \Gamma_5$, where $\Gamma_7$ characterizes the chiralities in $E=(3,3)$.

We can now study the detailed significance of the $E=(3,3)$ space, which is a non-compact analog of the $E=(0,6)$ CY space. There are similarities as well as differences between our “non-compact CY” (NCCY) and the usual CY space [67]. One of the differences is that the holonomy group of our NCCY is no longer $SU(3)$, but it is a subgroup of $SL(2, C)$, which can be understood by noticing the isomorphism $SO(3,3) \approx SL(4, R)$. If we exclude the torsion and the dilaton condensate: $G_{abc} = 0, \Phi = 0$ in the purely bosonic background, the gravitino transformation rule obtained from (D.2) implies that the Ricci-flatness solves its Killing spinor condition. Accordingly, the covariant constant spinor enables us to show the covariant constancy of an almost complex structure, as in the usual CY case [67]. Therefore the extra space-time can be Ricci-flat and Kähler. Relevantly, there must be a Killing spinor, which is a singlet under a little group of $SO(3,3) \approx SL(4, R)$. A typical example is $SL(4, R) \rightarrow SO(2) \otimes SO(3) \otimes SO(3)$ under which the original spinor $4$ of $SL(4, R)$ goes to $4 \rightarrow (2,1) + (1,2)$. Since we need at least a singlet, we have to

\footnote{Due to the indefinite signature, there may be other solutions, but this Ricci-flat solution gives the Killing spinor condition.}
go further down. A simplest choice is \( SL(4, R) \rightarrow SO(2) \otimes SO(2) \otimes SO(3) \), whereupon \( 4 \rightarrow 1 + 1 + 2 \), regarding \( SO(2) \otimes SO(3) \) as the holonomy group of the NCCY. Eventually, the original supersymmetry \( \epsilon^\alpha \) yields the \( N = 1 \) supersymmetry in \( D = (2, 2) \). Now the only remaining Killing spinor equation is from the gaugino:

\[
\delta \lambda^\alpha I = \frac{1}{\sqrt{2}} \epsilon^\alpha \epsilon^\beta F_{\hat{c}\hat{d}} = 0 ,
\]

implying the vanishing first Chern class. Thus we see that our NCCY shares the similar properties: the Ricci-flatness, Kählerness, and vanishing first Chern class, with the usual CY manifold \([67]\).\(^{25}\)

This can be also understood, considering the spinors in \( E = (3, 3) \). The \( D = (5, 5) \) chirality operator \( \Gamma_{11} \) is a product of those for \( E = (3, 3) \) and \( D = (2, 2) \): \( \Gamma_{11} = \Gamma_7 \Gamma_5 \). This implies that the Majorana-Weyl spinor with the positive chirality in \( D = (5, 5) \) has either \((+, +)\) or \((-,-)\) chiralities in \( E = (3, 3) \) and \( D = (2, 2) \), respectively. We can also impose the Majorana condition additionally on the Weyl spinor in \( E = (3, 3) \), and the resulting \( D = (2, 2) \) theory will have the \( N = 2 \) supersymmetry. For these extended supersymmetries, we can keep the holonomy group \( SO(3,3) \) for \( E = (3,3) \), while to get the \( N = 1 \) supersymmetry, we need to reduce the holonomy group into \( SO(2) \otimes SU(2) \), as above.

We now review the DR à la Witten for the usual CY. Its principle was that we truncate all the quantities which are non-invariant under the holonomy group \( SU(3) \) of the CY extra space. Under this principle, any quantity with indices \( \hat{a}, \hat{b}, \ldots \) put to zero, except for the antisymmetric tensor, which can be proportional to the \( SU(3) \) invariant tensor \([67]\) \( \epsilon_{\hat{a}\hat{b}} \). As for spinors, their indices are singlets under the \( SU(3) \) group. The NCCY analog of this prescription is to truncate all the quantities, which carry non-singlet indices with respect to the holonomy group \( SU(2) \), as we described. Hence all the fields carrying \( E = (3,3) \) vectorial indices are truncated, and all the \( D = (5,5) \) spinorial indices are reduced to be singlets under the \( SU(2) \) holonomy group of our NCCY.\(^{26}\)

We next show how our DR works for our constraints (D.2). As an example, we consider

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\(^{25}\)We acknowledge T. Hübisch for explaining this point.

\(^{26}\)Eventually what we will get after our DR is formally independent of the details of the holonomy group for our NCCY. This is due to the fact that we are reducing one of the two \( 2 \)'s in \( 4 \rightarrow 2 + 2 \) under \( SL(4, R) \rightarrow SO(2) \otimes SL(2, C) \) into singlets, reducing \( N = 2 \) into \( N = 1 \) in \( D = (2, 2) \), as described above.
the case of $\nabla_{a}\chi_{\beta}$ in (D.2):

$$
\nabla_{a} \chi_{\beta} \rightarrow \left\{ \begin{array}{l}
\nabla_{a} \bar{\chi}_{\beta} = -i \frac{\sqrt{2}}{2} (\sigma^{a}_{\alpha \beta} \chi_{\alpha} - \sigma^{a}_{\beta \alpha} \alpha \beta) \nabla_{a} \Phi \\
+ i \frac{1}{2} \sigma^{a}_{\alpha \beta} (G_{\alpha \beta} - i \frac{1}{2} \chi_{\alpha} \chi_{\beta}^{'}) + i \frac{1}{4} \lambda^{I} \sigma_{\alpha \beta} \lambda^{I} ,
\end{array} \right.
$$

(D.7)

Needless to say, we can make use of the Fierz identities, such as

$$(1/6) (\sigma^{a \beta}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'}) = (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'}) = 2 C_{\alpha \gamma} \chi_{\alpha} \chi_{\beta}^{'},$$

in $D = (2,2)$. (See Ref. [72] for other details.) As a result of this rule, we get the following $D = (2,2)$, $N = 1$ constraints:

$$
T_{\alpha \beta} = i (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}'), \quad \nabla_{a} \Phi = -\frac{1}{\sqrt{2}} \chi_{\alpha} ,
$$

$$
G_{\beta \gamma} = \frac{1}{2} (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}'), \quad G_{\alpha \beta} = 0 , \quad G_{\alpha \beta} = 0 , \quad G_{\alpha \gamma} = 0 ,
$$

$$
T_{\alpha \beta} = 0 , \quad T_{\alpha \beta} = \sqrt{2} \delta_{\alpha} \gamma \chi_{\beta}^{'}, \quad T_{\alpha \beta} = -\sqrt{2} \delta_{\alpha} \gamma \chi_{\beta}^{'},
$$

$$
\nabla_{a} \bar{\chi}_{\beta} = -i \frac{1}{\sqrt{2}} (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'}) + i \frac{1}{2} \sigma^{a}_{\alpha \beta} (G_{\alpha \beta} - i \frac{1}{2} \chi_{\alpha} \chi_{\beta}^{'}) (2 \chi_{\alpha} \chi_{\beta}^{'}) - \lambda^{I} \lambda^{I} ,
$$

$$
T_{\alpha \beta} = -i \frac{1}{\sqrt{2}} (\sigma_{\gamma \alpha} \chi_{\beta}^{'}) + (\lambda^{I} \lambda_{\beta}^{'}) \chi_{\alpha} \chi_{\beta}^{'}, \quad T_{\alpha \beta} = 0 , \quad T_{\alpha \beta} = -2 G_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'},
$$

$$
R_{\alpha \beta} = -2 i (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}'),
$$

$$
F_{\alpha \beta} = i \frac{1}{\sqrt{2}} (\sigma_{\gamma \alpha} \chi_{\beta}^{'}) \chi_{\alpha} \chi_{\beta}^{'},
$$

$$
\nabla_{a} \chi^{\alpha} = -i \frac{1}{\sqrt{2}} (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'}) F_{\alpha \beta}, \quad \nabla_{a} \chi^{\alpha} = -i \frac{1}{\sqrt{2}} (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'}) F_{\alpha \beta} - i \frac{1}{2} \sigma^{a}_{\alpha \beta} (\epsilon^{c} \chi^{d} \lambda^{I}) - \sqrt{2} \chi_{\alpha} \lambda^{I} - \frac{1}{\sqrt{2}} \delta_{\alpha} \beta (\lambda^{I} + \chi^{'}) ,
$$

$$
\nabla_{a} \chi^{\alpha} = -i \frac{1}{\sqrt{2}} (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'}) F_{\alpha \beta} - i \frac{1}{\sqrt{2}} (\sigma^{a}_{\alpha \beta} \chi_{\alpha} \chi_{\beta}^{'}) F_{\alpha \beta} - i \frac{1}{2} \sigma^{a}_{\alpha \beta} (\epsilon^{c} \chi^{d} \lambda^{I}) - \sqrt{2} \chi_{\alpha} \lambda^{I} - \frac{1}{\sqrt{2}} \delta_{\alpha} \beta (\lambda^{I} + \chi^{'}) ,
$$

(D.8)

As is easily seen, this contains a SG multiplet $(e_{m}^{\alpha}, \psi_{m}^{\alpha}, \bar{\psi}_{m}^{\alpha})$, a TM $(B_{mn}, \chi_{\alpha}, \bar{\chi}_{\alpha}, \Phi)$ and a YM multiplet $(A_{m}^{I}, \lambda_{\alpha}^{I}, \bar{\lambda}^{I}_{\alpha})$. By putting the YM and TM to zero, we can re-obtain the purely SG constraints in eqs. (3.25) and (3.27), after appropriate field rescalings. At this stage, the truncation into the SDSG or the supersymmetric SDYM is straightforward.

What we have performed so far, have been the practically simplified DR rules to get the theories in $D = (2,2)$ (the AW space-time), which do not contain any effect of KK modes in $E = (3,3)$. Our DR rule fortunately bypasses the problem associated with the non-compactness of the extra space, because we do not keep the KK massive modes. In the usual field theory, the unitarity of the initial theory in higher-dimensions is rigorously required, and, therefore, the indefinite signature $(5,5)$ is not acceptable for such purposes. In our case, however, this does not concern us, since the resultant theory of $D = (2,2)$ itself has the indefinite metric.

We have considered only the $E = (3,3)$-type CY extra space, but we can also perform the usual KK compactification now on the compact $E = (0,6)$ CY internal manifold, which
yields the *usual* \( D = (1, 3), N = 1 \) theory! However, in such a case the resulting theory will perhaps have some problems with unitarity, caused by the indefinite metric in the original \((5, 5)\) theory itself. We also note that the result for such a DR is *formally* the same as the above case. This is also why we could utilize the result of the DR on the usual CY \([72]\).

Even though we have considered only the case of \( D = (5, 5) \) as the initial higher-dimensional theory, we could formulate other supersymmetric theories equally as well, such as a \( D = (3, 3) \) theory in six-dimensions, which is similar to a \( D = (1, 5), N = 2 \) theory constructed in Ref. \([73]\). Actually, we could perform similar DR as above, to get the \( D = (2, 2), N = 1 \) SDSG and supersymmetric SDYM after subsequent truncation, the details of which we skip in this paper.

We finally mention the possibility that such a \( D = (5, 5) \) theory with the indefinite metric is related to some superstring theory, that we may have overlooked in the past. Since even the \( N = 2 \) superstring has the non-compact target space such as \( D = (2, 2) \) with *two* time directions, it is not totally senseless to introduce more time coordinates in a target space-time for some superstring theories.
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