A modification of the mixed joint universality theorem for a class of zeta-functions

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Abstract

The property of zeta-functions on mixed joint universality in the Voronin’s sense states that any two holomorphic functions can be approximated simultaneously with accuracy $\varepsilon > 0$ by suitable vertical shifts of the pair consisting from the Riemann zeta- and Hurwitz zeta-functions. In [1], it was shown rather general result, i.e., an approximating pair was composed of the Matsumoto zeta-functions’ class and the periodic Hurwitz zeta-function. In this paper, we prove that this set of shifts has a strict positive density for all but at most countably many $\varepsilon > 0$. Also, we give the concluding remarks on certain more general mixed tuple of zeta-functions.

Keywords: mixed joint universality, joint value distribution, periodic Hurwitz zeta-function, Matsumoto zeta-function, simultaneous approximation.

AMS classification: 11M06, 11M41, 11M36.

1 Introduction

Denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, non-negative integers, prime numbers, rational numbers, real numbers, and complex numbers, respectively, and by $s = \sigma + it$ a complex variable.

As it is well-known, the Riemann zeta-function $\zeta(s)$ is defined by the Dirichlet series and has a representation by the Euler product over primes, i.e.,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right), \quad \sigma > 1.$$

The Hurwitz zeta-function $\zeta(s, \alpha)$ with the real parameter $\alpha$, $0 < \alpha \leq 1$, is given by the series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1.$$

Both of the functions $\zeta(s)$ and $\zeta(s, \alpha)$ have an analytic continuation to the whole complex plane, except for a simple pole at $s = 1$ with residue 1. Note that the function $\zeta(s, \alpha)$ has an Euler product over primes only for the cases $\alpha = 1$ and $\alpha = \frac{1}{2}$, when $\zeta(s, 1) = \zeta(s)$ and $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$.

In the first decade of XXI century, the new type of universality was discovered by H. Mishou [2] and independently by J. Steuding and J. Sander [3]. They have opened so-called mixed joint universality in Voronin’s sense for the Riemann zeta- and the Hurwitz zeta-functions. More
precisely, they proved that a pair of analytic functions is simultaneously approximated by shifts of a pair \( (\zeta(s + i\tau), \zeta(s + i\tau, \alpha)) \) with transcendental \( \alpha \).

For the brevity, throughout this paper we use following notations and definitions. Let \( D(a, b) = \{ s \in \mathbb{C} : a < \sigma < b \} \) for every real numbers \( a < b \). Denote by \( \text{meas}\{A\} \) the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \), by \( B(S) \) the set of all Borel subsets of a topological space \( S \), and by \( H(D) \) the set of all holomorphic functions on \( D \). For any compact set \( K \subset \mathbb{C} \), denote by \( H^c(K) \) the set of all complex-valued continuous functions defined on \( K \) and holomorphic in the interior of \( K \), while by \( H_0^c(K) \) denote the subset of \( H^c(K) \), consisting of all elements which are non-vanishing on \( K \).

**Theorem 1** (see [2, Theorem 2]). Suppose that \( \alpha \) is a transcendental number. Let \( K_1 \) and \( K_2 \) be compact subsets of \( D(\frac{1}{2}, 1) \) with connected complements. Then, for any \( f_1(s) \in H_0^c(K_1), f_2(s) \in H^c(K_2) \) and every \( \varepsilon > 0 \), it holds that

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0.
\]  

(1)

In 2013, a statement of universality theorems in terms of density was proposed by J.-L. Mauclaire [4] and independently by A. Laurinčikas and L. Meška [5]. Such a statement of Theorem 1 was given in [6]. To be precise, let

\[
L(\alpha, \mathbb{P}) := \{(\log p : p \in \mathbb{P}), (\log(m + \alpha) : m \in \mathbb{N}_0)\}.
\]

Then the following modification of Theorem 1 was obtained.

**Theorem 2** (see [6, Theorem 2]). Suppose that the elements of the set \( L(\alpha, \mathbb{P}) \) are linearly independent over the field of rational numbers \( \mathbb{Q} \). Let \( K_1, K_2, f_1(s), f_2(s) \) be as in Theorem 1. Then the limit

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0
\]

exists for all but at most countably many \( \varepsilon > 0 \).

More general results of the same type can be found, for example, in [7].

The aim of this paper is to show an analogous result as of Theorem 2 for rather general classes of zeta-functions.

## 2 Statement of new result

A generalization of the Hurwitz zeta-function \( \zeta(s, \alpha) \) was introduced by A. Laurinčikas [8]. For the periodic sequence \( \mathcal{B} = \{b_m \in \mathbb{C} : m \in \mathbb{N}_0\} \) with a minimal period \( k \in \mathbb{N}_0 \) and a fixed real \( \alpha, 0 < \alpha \leq 1 \), the periodic Hurwitz zeta-function \( \zeta(s, \alpha; \mathcal{B}) \) is defined by the Dirichlet series

\[
\zeta(s, \alpha; \mathcal{B}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s}, \quad \sigma > 1.
\]

Since the sequence \( \mathcal{B} \) is periodic sequence, then

\[
\zeta(s, \alpha; \mathcal{B}) = \frac{1}{k^s} \sum_{m=0}^{k-1} b_m \zeta\left(s, \frac{m + \alpha}{k}\right).
\]
From this we deduce that the function $\zeta(s, \alpha; \mathfrak{B})$ has an analytic continuation to the whole complex plane except for a simple pole at the point $s = 1$ with residue $b = \frac{1}{k} \sum_{m=0}^{k-1} b_m$.

A class of the Matsumoto zeta-functions is a second class under our interest, particularly, since it covers a wide class of classical zeta-functions having the Euler product representation over primes. It was introduced by K. Matsumoto in [9]. For every $m \in \mathbb{N}$, let $g(m) \in \mathbb{N}$, and, for $j \in \mathbb{N}$ with $1 \leq j \leq g(m)$, $f(j, m) \in \mathbb{N}$. Denote by $p_m$ the $m$th prime number, and $a^{(j)}_m \in \mathbb{C}$. Assume that

$$g(m) \leq C_1 p_m^{\alpha_0} \quad \text{and} \quad |a^{(j)}_m| \leq p_m^{\beta_0}$$

with a positive constant $C_1$ and non-negative constants $\alpha_0$ and $\beta_0$. Define polynomials

$$A_m(X) = \prod_{j=1}^{g(m)} \left(1 - a^{(j)}_m X^{f(j, m)}\right)$$

of degree $f(1, m) + \cdots + f(g(m), m)$. The function

$$\bar{\varphi}(s) = \prod_{m=1}^{\infty} \left(A_m(p_m s)\right)^{-1}$$

is called the Matsumoto zeta-function. The product on right-hand-side of the equality (2) converges absolutely for $\sigma > \alpha_0 + \beta_0 + 1$. In this region, the function $\bar{\varphi}(s)$ has a Dirichlet series expansion as

$$\bar{\varphi}(s) = \sum_{k=1}^{\infty} \frac{\tilde{c}_k}{k^s}$$

with the coefficients satisfying an estimate $\tilde{c}_k = O(k^{\alpha_0 + \beta_0 + \varepsilon})$ for every $\varepsilon > 0$ if all prime factors of $k$ are large (for the details, see [10]). For brevity, we define its shifted version by

$$\varphi(s) := \bar{\varphi}(s + \alpha_0 + \beta_0) = \sum_{k=1}^{\infty} \frac{c_k}{k^s}$$

where $c_k = \tilde{c}_k k^{-\alpha_0 - \beta_0}$. Then it is easy to see that $\varphi(s)$ is absolutely convergent for $\sigma > 1$.

Also we assume that the function $\varphi(s)$ satisfies the following conditions:

(i) $\varphi(s)$ can be meromorphically continued to the region $\sigma \geq \sigma_0$, $\frac{1}{2} \leq \sigma_0 < 1$, and all poles in this region belong to a compact set which has no intersections with the line $\sigma = \sigma_0$;

(ii) $\varphi(\sigma + it) = O\left(|t|^{C_2}\right)$ for $\sigma \geq \sigma_0$ and a positive constant $C_2$;

(iii) it holds the mean-value estimate

$$\int_0^T |\varphi(\sigma_0 + it)|^2 \, dt = O(T).$$

All of functions satisfying above mentioned conditions construct the class of Matsumoto zeta-functions, and we denote the set of all such $\varphi(s)$ as $\mathcal{M}$.

In 2015, R. Kačinskaitė and K. Matsumoto proved [1] a mixed joint universality theorem for wide class of Matsumoto zeta-functions and for the periodic Hurwitz zeta-function with transcendental parameter.

For the proof of mixed joint universality, the Bagchi method [11] can be used, but, in the case of the whole class $\mathcal{M}$ it is difficult to prove the denseness lemma. Therefore, we use one more restriction class, namely, the Steuding class $\tilde{S}$ (see [12]).

We say that $\varphi(s)$ belongs to the class $\tilde{S}$ if the following assumptions are fulfilled:
(a) \( \varphi(s) \) has a Dirichlet series expansion \( \varphi(s) = \sum_{m=1}^{\infty} a(m) m^{-s} \) with \( a_m = O(m^\epsilon) \) for every \( \epsilon > 0 \);

(b) there exists \( \sigma_\varphi < 1 \) such that \( \varphi(s) \) can be meromorphically continued to the region \( \sigma > \sigma_\varphi \), and is holomorphic there, except for a pole at \( s = 1 \);

(c) for any fixed \( \sigma > \sigma_\varphi \) and any \( \epsilon > 0 \), there exists a constant \( C_\varphi \geq 0 \) such that \( \varphi(\sigma + it) = O(|t|^{C_\varphi + \epsilon}) \);

(d) there exists the Euler product expansion

\[
\varphi(s) = \prod_{p \in \mathbb{P}} \prod_{j=1}^{l} \left( 1 - \frac{a_j(p)}{p^s} \right)^{-1} ;
\]

(e) there exists a constant \( \kappa > 0 \) such that

\[
\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a(p)|^2 = \kappa,
\]

where \( \pi(x) \) denotes the number of primes \( p \) up to \( x \).

Let \( \varphi(s) \in \tilde{S} \), and suppose that \( \sigma^* \) is an infimum of all \( \sigma_1 \) for which

\[
\frac{1}{2T} \int_{-T}^{T} |\varphi(\sigma + it)|^2 dt \sim \sum_{m=1}^{\infty} \frac{|a(m)|^2}{m^{2\sigma}}
\]

holds for any \( \sigma \geq \sigma_1 \). Then \( \frac{1}{2} \leq \sigma^* < 1 \), and we see that \( \tilde{S} \subset \mathcal{M} \).

In 2015, the first result on the mixed joint universality for the tuple \( (\varphi(s), \zeta(s, \alpha; \mathfrak{B})) \) was proved by R. Kačinskaitė and K. Matsumoto (see [1]). Later it was proved in a more general situation extending collection of periodic Hurwitz zeta-functions (see [10]).

**Theorem 3** ([1, Theorem 2.2]). Suppose that \( \varphi(s) \in \tilde{S} \), and \( \alpha \) is a transcendental number. Let \( K_1 \) be a compact subset of \( D(\sigma^*, 1) \), \( K_2 \) be a compact subset of \( D\left(\frac{1}{2}, 1\right) \) and both with connected complements. Then, for any \( f_1(s) \in H_0^\alpha(K_1) \), \( f_2(s) \in H^\alpha(K_2) \) and \( \epsilon > 0 \), it holds that

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \sup_{s \in K_1} |\varphi(s + i\tau) - f_1(s)| < \epsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{B}) - f_2(s)| < \epsilon\} > 0.
\]

The aim of the present paper is to prove the modification of Theorem 3 in terms of density and to give a further certain generalizations.

Now we state the main result of the present paper.

**Theorem 4.** Suppose that \( \varphi(s) \in \tilde{S} \) and \( \alpha \) is a transcendental number. Let \( K_1, K_2, f_1(s), f_2(s) \) be as in Theorem 3. Then, for all but at most countably many \( \epsilon > 0 \), it holds that

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas}\{\tau \in [0, T] : \sup_{s \in K_1} |\varphi(s + i\tau) - f_1(s)| < \epsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{B}) - f_2(s)| < \epsilon\} > 0.
\]

**Remark.** The transcendence of \( \alpha \) can be replaced by the assumption that the elements of the set \( L(\alpha, \mathbb{P}) \) are linearly independent over \( \mathbb{Q} \) as it is done in Theorem 2.
3 Two probabilistic results

For the proof of Theorem 4, the probabilistic approach is used. In this section, we present joint mixed limit theorems on weakly convergent probability measures in the space of analytic functions and a proposition on the support of probability measure.

Since now, we are interested in the proof of a joint limit theorem for the tuple \( (\varphi(s), \zeta(s, \alpha; \mathfrak{B})) \), we deal with more specified regions than \( D(\sigma^*, 1) \) and \( D(1/2, 1) \) (for the arguments, we refer to [1] or [13]). As it is know, the function \( \varphi(s) \) has finitely many poles by condition (i) (describe them by \( \zeta \) functions \( H \)), the Cartesian product of the spaces \( \zeta_1 \) and pointwise multiplication, both tori \( \Omega_\gamma \) where \( \gamma = \{ s \in \mathbb{C} : \sigma > \sigma_0, \sigma \neq \Re s_j(\varphi), 1 \leq j \leq l \} \).

Since the function \( \zeta(s, \alpha; \mathfrak{B}) \) can be written as a linear combination of the Hurwitz zeta-functions \( \zeta(s, \alpha) \), it is entire or has at most a simple pole at \( s = 1 \). Let

\[
D_\zeta := \begin{cases} 
\{ s \in \mathbb{C} : \sigma > \frac{1}{2} \}, & \text{if } \zeta(s, \alpha; \mathfrak{B}) \text{ is entire}, \\
\{ s \in \mathbb{C} : \sigma > \frac{1}{2}, \sigma \neq 1 \}, & \text{if } s = 1 \text{ is a pole of } \zeta(s, \alpha; \mathfrak{B}),
\end{cases}
\]

and \( D_1 \) and \( D_2 \) be two open regions of \( D_\varphi \) and \( D_\zeta \), respectively. By \( H^2(D) \) we mean the Cartesian product of the spaces \( H(D_1) \) and \( H(D_2) \). Let \( T > 0 \), and for \( A \in B(H^2(D)) \), define

\[
P_T(A) = \frac{1}{T} \text{meas}\{ \tau \in [0, T] : Z(s + i\tau) \in A \}
\]

with \( s + i\tau = (s_1 + i\tau, s_2 + i\tau) \), \( s_1 \in D_1, s_2 \in D_2 \), and

\[
Z(s) = (\varphi(s_1), \zeta(s_2, \alpha; \mathfrak{B})).
\]

For the definition of limit measure, we need a certain probability space. Let \( \gamma = \{ s \in \mathbb{C} : |s| = 1 \} \). Define two tori

\[
\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=0}^{\infty} \gamma_m,
\]

where \( \gamma_p = \hat{\gamma} \) for all \( p \in \mathbb{P} \) and \( \gamma_m = \hat{\gamma} \) for all \( m \in \mathbb{N}_0 \), respectively. With the product topology and pointwise multiplication, both tori \( \Omega_1 \) and \( \Omega_2 \) become compact topological Abelian groups. Therefore, on (\( \Omega_1 \), \( \mathcal{B}(\Omega_1) \)) and (\( \Omega_2 \), \( \mathcal{B}(\Omega_2) \)), there exist the probability Haar measures \( m_{1H} \) and \( m_{2H} \) respectively. Thus, we get the probability spaces (\( \Omega_1 \), \( \mathcal{B}(\Omega_1), m_{1H} \)) and (\( \Omega_2 \), \( \mathcal{B}(\Omega_2), m_{2H} \)).

Denote by \( \omega_1(p) \) the projection of \( \omega_1 \in \Omega_1 \) to the coordinate space \( \gamma_p, p \in \mathbb{P} \), while, for \( m \in \mathbb{N}_0 \), let \( \omega_1(m) := \omega_1^{(m_1)}(p_1) \cdots \omega_1^{(m_p)}(p_p) \) according to the factorization of \( m \) into prime divisors \( m = p_1^{a_1} \cdots p_p^{a_p} \), \( \omega_2(m), m \in \mathbb{N}_0 \), the projection to the coordinate space \( \gamma_m \).

Now let \( \Omega = \Omega_1 \times \Omega_2 \), and denote the elements of \( \Omega \) by \( \omega = (\omega_1, \omega_2) \). Since \( \Omega \) is a compact topological Abelian group, we can define the probability Haar measure \( m_H := m_{1H} \times m_{2H} \) on (\( \Omega \), \( \mathcal{B}(\Omega) \)). This leads to a probability space (\( \Omega \), \( \mathcal{B}(\Omega) \), \( m_H \)).

On (\( \Omega \), \( \mathcal{B}(\Omega), m_H \)), define \( H^2(D) \)-valued random element \( Z(s, \omega) \) by the formula

\[
Z(s, \omega) = (\varphi(s_1, \omega_1), \zeta(s_2, \alpha, \omega_2; \mathfrak{B})).
\]

Here \( s = (s_1, s_2) \in D_1 \times D_2 \),

\[
\varphi(s_1, \omega_1) = \sum_{m=1}^{\infty} \frac{c_m \omega_1(m)}{m} \quad \text{and} \quad \zeta(s_2, \alpha, \omega_2; \mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^{s_2}}
\]

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are $H(D_1)$-valued and $H(D_2)$-valued random elements defined on $(\Omega_1, \mathcal{B}(\Omega_1), m_{1H})$ and $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, respectively. Denote by $P_Z$ the distribution of the random element $Z(\underline{s}, \omega)$, i.e.,

$$P_Z(A) = m_H\{\omega \in \Omega : Z(\underline{s}, \omega) \in A\}, \quad A \in \mathcal{B}\left(H^2(D)\right).$$

Now we are in position to state a mixed joint limit theorem for the tuple of the class of zeta-functions.

**Theorem 5.** Suppose that $\varphi(s) \in \mathcal{M}$, and $\alpha$ is a transcendental number. Then the measure $P_T(A)$ converges weakly to $P_Z(A)$ as $T \to \infty$.

**Proof.** The proof of this theorem is given in [1, Section 3, Theorem 2.1]. We only note that the transcendence of $\alpha$ plays an essential role in the proof. \qed

The second probabilistic result used in the proof of Theorem 4 is that we need to construct an explicit form of the support of the measure $P_Z$. For an obtaining of the mentioned result, we use the positive density method. Therefore, it is necessary to assume that the function $\varphi(s)\in \tilde{S}$, particularly, the condition (e) must to be satisfied (for the details, see [1, Section 4, Remark 4.4]).

Let $\varphi(s), K_1, K_2, f_1(s)$ and $f_2(s)$ be as in Theorem 3. Then there exists a real number $\sigma_0, \sigma^* < \sigma_0 < 1$ and a sufficiently large positive number $M$ such that $K_1$ belongs to

$$D_M = \{s \in \mathbb{C} : \sigma_0 < \sigma < 1, |t| < M\}.$$  

Since $\varphi(s) \in \tilde{S}$, it has only one pole at $s = 1$, then put $D_{\varphi} = \{s \in \mathbb{C} : \sigma > \sigma_0, \sigma \neq 1\}$. Therefore, $D_M \subset D_{\varphi}$. Analogously we can find a sufficiently large positive number $N$ such that $K_2$ belongs to

$$D_N = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < N\}.$$  

Now, if in Theorem 5 take $D_1 = D_M$ and $D_2 = D_N$, we get an explicit form of the $P_Z$’s support.

**Theorem 6.** The support of the measure $P_Z$ is the set $S := S_{\varphi} \times H(D_N)$, where $S_{\varphi} := \{f_1(s) \in H(D_M) : f_1(s) \neq 0 \text{ for all } s \in D_M, \text{ or } f_1(s) \equiv 0\}$.

**Proof.** The proof of the theorem can be found in [1, Lemma 4.3]. \qed

### 4 Proof of Theorem 4

First we remind two propositions used in the proof of the main result of the paper.

Recall that a set $A \in \mathcal{B}(S)$ is said to be a continuity set of the probability measure $P$ if $P(\partial A) = 0$, where $\partial A$ is the boundary of $A$. Note that the set $\partial A$ is closed, therefore, it belongs to the class $\mathcal{B}(S)$. We are interested on the property of probability measures defined in terms of continuity sets, which is equivalent to weak convergence. Therefore we use a following fact.

**Theorem 7.** Let $P_n$ and $P$ be probability measures on $(S, \mathcal{B}(S))$. Then the following assertions are equivalent:

1) $P_n$ converges weakly to $P$ as $n \to \infty$;

2) $\lim_{n \to \infty} P_n(A) = P(A)$ for all continuity sets $A$ of $P$.  

Proof. For the proof, see [14, Theorem 2.1].

We also recall the Mergelyan theorem on the approximation of analytic functions by polynomials.

**Theorem 8.** Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on $K$ analytic inside $K$. Then, for any $\varepsilon > 0$, there exists a polynomial $p(s)$ such that

$$
\sup_{s \in K} |f(s) - p(s)| < \varepsilon.
$$

**Proof.** The proof of the theorem can be found in [15].

Proof of Theorem 4. Since $f_1(s) \neq 0$ on $K_1$, by the Mergelyan theorem, there exist polynomials $p_1(s)$ and $p_2(s)$ such that, for every $\varepsilon > 0$,

$$
\sup_{s \in K_1} |f_1(s) - \exp(p_1(s))| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{s \in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}.
$$

(3)

In view of Theorem 6, an element $(\exp(p_1(s)), p_2(s))$ belongs to the set $S$, i.e., to the support of the measure $P_Z$.

Consider the set

$$
G = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - \exp(p_1(s))| < \frac{\varepsilon}{2}, \sup_{s \in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2} \right\}.
$$

This set is an open subset in $H^2(D)$ and, by Theorem 6, an open neighbourhood of an element $(\exp(p_1(s)), p_2(s))$. Therefore, by Theorems 5 and 7, the inequality $P_Z(G) > 0$ holds.

Now, for $f_1(s)$ and $f_2(s)$ fulfilling the conditions of Theorem 4, define the set $G_\varepsilon$ by

$$
G_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| < \frac{\varepsilon}{2}, \sup_{s \in K_1} |g_2(s) - f_2(s)| < \frac{\varepsilon}{2} \right\}
$$

with the boundary

$$
\partial G_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| = \varepsilon \right\}
\cup \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| < \varepsilon \right\}
\cup \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| = \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| = \varepsilon \right\}.
$$

Easy to see that with different $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ the boundaries $\partial G_{\varepsilon_1}$ and $\partial G_{\varepsilon_2}$ are disjoint. Therefore, only countable many sets $\partial G_\varepsilon$ can have positive measure $P_Z$.

Hence, $P_Z(\partial G_\varepsilon) = 0$ for at most countable set of values $\varepsilon > 0$, i.e., $G_\varepsilon$ is a continuity set of $P_Z$ for all but at most countable many $\varepsilon > 0$.

Moreover, in view of (3), $G \subset G_\varepsilon$. Therefore, by Theorem 5, we have

$$
\lim_{T \to \infty} P_T(G_\varepsilon) = P_Z(G_\varepsilon) > 0
$$

for all but at most countable many $\varepsilon > 0$. This together with the definitions of $P_T$ and $G_\varepsilon$ prove the theorem.


5 Concluding remarks

Theorem 4 can be generalised in a following direction.

Suppose that \( \alpha_j \) is a real number such that \( 0 < \alpha_j < 1 \), and \( l(j) \) is a positive integer, \( j = 1, \ldots, r \). Let \( \lambda = l(1) + \cdots + l(r) \). For each \( j \) and \( l \), \( 1 \leq j \leq r \), \( 1 \leq l \leq l(j) \), let \( \mathcal{B}_{jl} = \{ b_{mjl} : m \in \mathbb{N}_0 \} \) be a periodic sequence of complex numbers \( b_{mjl} \) with minimal period \( k_{jl} \), and let \( \zeta(s, \alpha_j; \mathcal{B}_{jl}) \) be the corresponding periodic Hurwitz zeta-function. Denote by \( k_j \) the least common multiple of periods \( k_{jl}, \ldots, k_{jl(j)} \). Let \( B_j \) be a matrix consisting of elements \( b_{mjl} \) from the periodic sequences \( \mathcal{B}_{jl}, j = 1, \ldots, r, l = 1, \ldots, l(j), \) i.e.,

\[
B_j := \begin{pmatrix}
    b_{1j1} & \cdots & b_{1jl(j)} \\
    \vdots & \ddots & \vdots \\
    b_{kj1} & \cdots & b_{kjl(j)}
\end{pmatrix}, \quad j = 1, \ldots, r.
\]

**Theorem 9.** Suppose that \( \alpha_1, \ldots, \alpha_r \) are algebraically independent over \( \mathbb{Q} \), \( \text{rank}(B_j) = l(j) \), \( 1 \leq j \leq r \), and \( \varphi(s) \) belongs to the class \( \hat{S} \). Let \( K_1 \) be a compact subset of \( D(\sigma^*, 1) \) and \( K_{2j} \) be a compact subset of \( D(\frac{1}{2}, 1) \), all of them with connected complements. Suppose that \( f_1(s) \in H_0^\lambda(K_1) \) and \( f_{2j}(s) \in H_0^\lambda(K_{2j}) \). Then, for all but at most countably many \( \varepsilon > 0 \), it holds that

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K_1} |\varphi(s + i\tau) - f_1(s)| < \varepsilon, \right. \\
\left. \quad \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l(j)} \sup_{s \in K_{2j}} |\zeta(s + i\tau, \alpha_j; \mathcal{B}_{jl}) - f_{2j}(s)| < \varepsilon \right\} > 0. \tag{4}
\]

**Proof.** In [10], the joint mixed universality it was proved under the same conditions as in the theorem instead \( \lim \) studying \( \liminf \) for every \( \varepsilon > 0 \). Therefore, arguing in similar way as in the proof of Theorem 4, we can show the universality inequality (4).

However, we give some highlights. Let \( H^{\lambda+1}(D) := H(D_1) \times H(D_2) \times H(D_2) \), and let \( p_{2j}(s) \) be a polynomials satisfying the second inequality of (3) for each \( j = 1, \ldots, r, l = 1, \ldots, l(j) \).

Instead of the set \( \mathcal{G} \) in the proof of Theorem 4, we consider the set

\[
\mathcal{G}_* = \left\{ (g_1, g_211, \ldots, g_2l(1), \ldots, g_{2r1}, \ldots, g_{2rl(r)}) \in H^{\lambda+1}(D) : \\
\sup_{s \in K_1} |g_1(s) - \exp(p_1(s))| < \frac{\varepsilon}{2}, \quad \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l(j)} \sup_{s \in K_{2j}} |g_{2j}(s) - p_{2j}(s)| < \frac{\varepsilon}{2} \right\},
\]

and show that \( P^\lambda_{\mathcal{G}}(\mathcal{G}_*) > 0 \), where \( P^\lambda_\mathcal{G} \) is a distribution of the \( H^{\lambda+1}(D) \)-random element constructed for the collections of zeta-functions in the theorem. For details, we refer to [10].

Next we define the set

\[
\mathcal{G}_* = \left\{ (g_1, g_211, \ldots, g_2l(1), \ldots, g_{2r1}, \ldots, g_{2rl(r)}) \in H^{\lambda+1}(D) : \\
\sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon, \quad \sup_{1 \leq j \leq r} \sup_{1 \leq l \leq l(j)} \sup_{s \in K_{2j}} |g_{2j}(s) - f_{2j}(s)| < \varepsilon \right\}
\]

and obtain that it is a continuity set of the measure \( P^\lambda_\mathcal{G} \) for all but at most countably many \( \varepsilon > 0 \). Again, arguing as for \( \mathcal{G} \) and \( \mathcal{G}_* \), we get that \( \mathcal{G} \subset \mathcal{G}_* \). Therefore, for all but at most countably many \( \varepsilon > 0 \), \( \lim_{T \to \infty} P^\lambda_T(\mathcal{G}_*) = P^\lambda_\mathcal{G}(\mathcal{G}_*) > 0 \). In view of the similarity of \( P^\lambda_T \)’s construction to \( P_T \) extending a collection of the periodic Hurwitz zeta-functions (for exact definition of \( P_T \), see p. 195 in [10]), this and the definition of \( \mathcal{G}_* \) complete the proof. \qed
Finally, we would like to mention that Theorem 9 can be shown under different conditions than the algebraic independence over $\mathbb{Q}$ of the parameters the parameters $\alpha_1, \ldots, \alpha_r$. Particularly, we can prove that the universality inequality (4) holds if the elements of the set \{(log $p$ : $p \in \mathbb{P}$), (log($m + \alpha_j$) : $m \in \mathbb{N}_0$, $j = 1, \ldots, r$)\} are linearly independent over $\mathbb{Q}$.

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