CLOSEDNESS OF CONVEX SETS IN ORLICZ SPACES WITH APPLICATIONS TO DUAL REPRESENTATION OF RISK MEASURES

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Abstract. Let \((\Phi, \Psi)\) be a conjugate pair of Orlicz functions. A set in the Orlicz space \(L^\Phi\) is said to be order closed if it is closed with respect to dominated convergence of sequences of functions. A well known problem arising from the theory of risk measures in financial mathematics asks whether order closedness of a convex set in \(L^\Phi\) characterizes closedness with respect to the topology \(\sigma(L^\Phi, L^\Psi)\). (See [26, p. 3585].) In this paper, we show that for a norm bounded convex set in \(L^\Phi\), order closedness and \(\sigma(L^\Phi, L^\Psi)\)-closedness are indeed equivalent. In general, however, coincidence of order closedness and \(\sigma(L^\Phi, L^\Psi)\)-closedness of convex sets in \(L^\Phi\) is equivalent to the validity of the Krein-Smulian Theorem for the topology \(\sigma(L^\Phi, L^\Psi)\); that is, a convex set is \(\sigma(L^\Phi, L^\Psi)\)-closed if and only if it is closed with respect to the bounded-\(\sigma(L^\Phi, L^\Psi)\) topology. As a result, we show that order closedness and \(\sigma(L^\Phi, L^\Psi)\)-closedness of convex sets in \(L^\Phi\) are equivalent if and only if either \(\Phi\) or \(\Psi\) satisfies the \(\Delta_2\)-condition. Using this, we prove the surprising result that: If (and only if) \(\Phi\) and \(\Psi\) both fail the \(\Delta_2\)-condition, then there exists a coherent risk measure on \(L^\Phi\) that has the Fatou property but fails the Fenchel-Moreau dual representation with respect to the dual pair \((L^\Phi, L^\Psi)\). A similar analysis is carried out for the dual pair of Orlicz hearts \((H^\Phi, H^\Psi)\).

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1. Introduction

In the seminal paper [4], a theoretical foundation was laid for the problem of quantifying the risk of a financial position in terms of coherent risk measures. The theory of risk measures has since been an active and fruitful area of research in Mathematical Finance (cf. [2, 3, 5, 6, 8, 11, 14, 16, 23, 25]). It has also motivated new developments in Convex Analysis and Functional Analysis (cf. [9, 15, 21, 26]).

In the axiomatic treatment of risk measures, financial positions are modeled by a vector space $\mathcal{X}$, which includes constants, of random variables on a probability space $(\Omega, \Sigma, P)$. A coherent risk measure on $\mathcal{X}$ is a functional $\rho : \mathcal{X} \rightarrow (-\infty, \infty]$ that is proper (i.e., not identically $\infty$) and satisfies the following properties:

1. (Subadditive) $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$ for all $X_1, X_2 \in \mathcal{X}$.
2. (Monotone) $\rho(X_1) \leq \rho(X_2)$ if $X_1, X_2 \in \mathcal{X}$ and $X_1 \geq X_2$ a.s.
3. (Cash additive) $\rho(X + m\mathbb{1}) = \rho(X) - m$ for any $X \in \mathcal{X}$ and any $m \in \mathbb{R}$.
4. (Positively homogeneous) $\rho(\lambda X) = \lambda \rho(X)$ for any $X \in \mathcal{X}$ and any $0 < \lambda \in \mathbb{R}$.

Observe that a coherent risk measure is convex, i.e.,

$$\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2)$$

for all $X_1, X_2 \in \mathcal{X}$ and all $\lambda \in [0, 1]$.

An important topic in the theory of risk measures is to determine when a risk measure on $\mathcal{X}$ admits a representation with respect to some duality involving $\mathcal{X}$. The first major result in this direction was obtained by Delbaen [10], who used as model space $\mathcal{X}$ the space of all bounded random variables $L^\infty(P)$ and considered the duality $(L^\infty, L^1)$.

**Theorem 1.1** (Delbaen). The following are equivalent for every coherent risk measure $\rho$ on $L^\infty(P)$.

1. There is a set $\mathcal{Q}$ of nonnegative random variables with expectation 1 such that

$$\rho(X) = \sup_{Y \in \mathcal{Q}} \mathbb{E}[-XY] \text{ for any } X \in L^\infty,$$

2. $\rho$ satisfies the $(L^\infty)$-Fatou property:

$$\rho(X) \leq \liminf_n \rho(X_n) \text{ whenever } (X_n) \text{ is bounded in } L^\infty \text{ and } X_n \xrightarrow{a.s.} X.$$
In Delbaen’s theorem, the set $Q$ can be interpreted as a set of probability distributions (scenarios) and the risk measure of $X$ is obtained as the worst expected loss over the set of scenarios (stress tests). In general, such dual representations play an important role in optimization problems and portfolio selection.

The representation in (1) of Theorem 1.1 is connected with $\sigma(L^\infty, L^1)$-lower semicontinuity of $\rho$ via the Fenchel-Moreau Duality Theorem in convex analysis. Here, $\sigma(L^\infty, L^1)$-lower semicontinuity of $\rho$ refers to the property that the sets

$$\{\rho \leq \lambda\} = \{X \in L^\infty(\mathbb{P}) : \rho(X) \leq \lambda\}$$

are $\sigma(L^\infty, L^1)$-closed for any $\lambda \in \mathbb{R}$. On the other hand, condition (2) in Theorem 1.1 is equivalent to the fact that the sets $\{\rho \leq \lambda\}$ are closed with respect to dominated convergence of sequences.

In Theorem 1.1 as in other early framework for risk measures, the model space consists of bounded financial positions. The reader may refer to Föllmer and Schied [17] for a comprehensive treatment of the main results in this setting. More realistic models of financial positions may involve unbounded random variables; this motivates the study of risk measures on model spaces beyond $L^\infty$. The Orlicz spaces $L^\Phi$ and Orlicz hearts $H^\Phi$, being natural classes of Banach function spaces that generalize the $L^p$ spaces, have emerged as important model spaces of (unbounded) financial positions. Contributions to the study of risk measures on $L^\Phi$ and $H^\Phi$ may be found in Cheridito and Li [8], Biagini and Fritelli [5] and more recently in Krätschmer, Schied and Zähle [24] and Gao and Xanthos [19]. See also Gao et al. [18] and the references therein.

Let $(\Phi, \Psi)$ be a conjugate pair of Orlicz functions. (Refer to §2 for definitions and basic facts concerning Orlicz spaces.) In the paper [5], Biagini and Fritelli initiated the study of representation of risk measures on $L^\Phi$ with respect to the duality $(L^\Phi, L^\Psi)$. Observing that dominated convergence of a sequence of random variables in $L^\Phi$ implies $\sigma(L^\Phi, L^\Psi)$-convergence, they proposed the following version of the Fatou property on $L^\Phi$: A functional $\rho$ on $L^\Phi$ is said to satisfy the Fatou property if

$$\rho(X) \leq \liminf_n \rho(X_n) \text{ whenever } (X_n) \text{ is order bounded in } L^\Phi \text{ and } X_n \overset{a.s.}{\longrightarrow} X.$$
They then claimed a result similar to Theorem 1.1 for every conjugate pair \((L^\Phi, L^\Psi)\). Specifically, it was claimed that

for every coherent risk measure \(\rho : L^\Phi \to (-\infty, \infty]\),
\(\rho\) has the Fatou property if and only if
\((\star)\) there is a set \(Q\) of nonnegative random variables in \(L^\Psi\),
each having expectation 1, such that
\[\rho(X) = \sup_{Y \in Q} \mathbb{E}[-XY]\] for any \(X \in L^\Phi\).

As in Theorem 1.1, validity of \((\star)\) is closely linked to the equivalence of \(\sigma(L^\Phi, L^\Psi)\)-closedness and closedness under dominated convergence for convex sets. Their proof is based on an assertion that every Orlicz space enjoys a technical property (with regard to convex sets) which they called the \(C\)-property. Unfortunately, the validity of the \(C\)-property has been disproved in [19] when \(\Psi\) satisfies the \(\Delta_2\)-condition. Thus, the verity of the foregoing equivalence \((\star)\) for every conjugate pair \((L^\Phi, L^\Psi)\) remained an important outstanding problem in the theory of dual representation of risk measures. See, e.g., [26, p. 3585], where this problem is raised explicitly.

The main result of this paper is a comprehensive solution to this problem. It is shown that the equivalence \((\star)\) holds for a conjugate pair \((L^\Phi, L^\Psi)\) if and only if either \(\Phi\) or \(\Psi\) satisfies the \(\Delta_2\)-condition. First we relate the problem to the equivalence of order closedness and \(\sigma(L^\Phi, L^\Psi)\)-closedness for convex sets in \(L^\Phi\). It is shown that for a norm bounded convex set in \(L^\Phi\), order closedness and \(\sigma(L^\Phi, L^\Psi)\)-closedness are indeed equivalent. As a result, the validity of \((\star)\) for a dual pair \((L^\Phi, L^\Psi)\) is determined by whether the Krein-Smulian Theorem for the topology \(\sigma(L^\Phi, L^\Psi)\) holds. We complete the proof of the main result by showing that the Krein-Smulian Theorem for the topology \(\sigma(L^\Phi, L^\Psi)\) holds if and only if either \(\Phi\) or \(\Psi\) satisfies the \(\Delta_2\)-condition.

In the last section, we also investigate the dual representation problem on \(H^\Phi\) with respect to the dual pair \((H^\Phi, H^\Psi)\). This complements the results for the dual pair \((L^\Phi, L^\Psi)\) in Section 3 for the dual pair \((L^\Phi, H^\Psi)\) by Gao and Xanthos [19] and for the dual pair \((H^\Phi, L^\Psi)\) by Cheridito and Li [8].
In this section, we collect the basic facts regarding Orlicz spaces and set the notation. We adopt [1] and [13, 27] as standard references for unexplained terminology and facts on Banach lattices and Orlicz spaces, respectively. Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it is convex, increasing, and $\Phi(0) = 0$. Define the conjugate function of $\Phi$ by

$$\Psi(s) = \sup\{ts - \Phi(t) : t \geq 0\}$$

for all $s \geq 0$. If $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$ (equivalently, if $\Psi$ is finite-valued), then $\Psi$ is also an Orlicz function, and its conjugate is $\Phi$. Throughout this paper, $(\Phi, \Psi)$ stands for an Orlicz pair such that $\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty$ and that $\Phi(t) > 0$ for any $t > 0$.

The restrictions on $\Phi$ are minor as they only eliminate the case where $L_\Phi$ coincides with $L^1$ or $L^\infty$, in which cases our main results are either trivial or known.

Throughout this paper, $(\Omega, \Sigma, P)$ stands for a nonatomic probability space. The Orlicz space $L_\Phi := L_\Phi(\Omega, \Sigma, P)$ is the space of all real-valued random variables $X$ (modulo a.s. equality) such that

$$\|X\|_\Phi := \inf\{\lambda > 0 : E[\Phi\left(\frac{|X|}{\lambda}\right)] \leq 1\} < \infty.$$  

The norm $\|\cdot\|_\Phi$ on $L_\Phi$ is called the Luxemburg norm. The subspace of $L_\Phi$ consisting of all $X \in L_\Phi$ such that

$$E\left[\Phi\left(\frac{|X|}{\lambda}\right)\right] < \infty \quad \text{for all } \lambda > 0$$

is conventionally called the Orlicz heart of $L_\Phi$ and is denoted by $H_\Phi$. It is well known that $L^\infty \subset H_\Phi \subset L_\Phi \subset L^1$ and that $H_\Phi$ is a norm closed subspace of $L_\Phi$. We always endow the conjugate Orlicz space $L_\Psi$ and the conjugate Orlicz heart $H_\Psi$ with the Orlicz norm

$$\|Y\|_\Psi := \sup_{X \in L_\Phi, \|X\|_\Phi \leq 1} \left|E[XY]\right|$$

for all $Y \in L_\Psi$, which is equivalent to the Luxemburg norm on $L_\Psi$.

An Orlicz function $\Phi$ satisfies the $\Delta_2$-condition if there exist $t_0 \in (0, \infty)$ and $k \in \mathbb{R}$ such that $\Phi(2t) < k\Phi(t)$ for all $t \geq t_0$. It is well known that $L_\Psi = (H_\Phi)^*$ and that $L_\Phi$, being the order continuous dual of $L_\Phi$, is a lattice ideal in $(L_\Phi)^*$. Moreover, the following conditions are equivalent.

1. $L_\Psi = (L_\Phi)^*$, ...
(2) $L^\Phi = H^\Phi$.
(3) The Orlicz function $\Phi$ satisfies the $\Delta_2$-condition,
(4) $L^\Phi$ has order continuous norm, i.e.,
\[ X_\alpha \downarrow, \quad \inf_\alpha X_\alpha = 0 \text{ in } L^\Phi \implies \inf_\alpha \|X_\alpha\|_\Phi = 0. \]

A sequence $(X_n)$ in $L^\Phi$ is order bounded if there exists $X \in L^\Phi$ such that $|X_n| \leq X$ a.s. for all $n$. A sequence $(X_n)$ in $L^\Phi$ order converges to $X \in L^\Phi$, written as $X_n \overset{o}{\to} X$ in $L^\Phi$, if $X_n \overset{a.s.}{\to} X$ and $(X_n)$ is order bounded in $L^\Phi$. If $L^\Phi$ has order continuous norm, then
\[ X_n \overset{o}{\to} X \text{ in } L^\Phi \implies \|X_n - X\|_\Phi \to 0. \]

3. Dual representation with respect to the pair $(L^\Phi, L^\Psi)$

This section is the main part of the paper, where we show that the equivalence $(\ast)$ in the Introduction holds if and only if either $\Phi$ or $\Psi$ satisfies the $\Delta_2$-condition.

For a subset $C$ of $L^\Phi$, define its order closure in $L^\Phi$ to be the set
\[ C_o := \{X \in L^\Phi : X_n \overset{o}{\to} X \text{ for some sequence } (X_n) \text{ in } C\}. \]

We say that $C$ is order closed in $L^\Phi$ if $C = C_o$. In spite of the terminology, $C_o$ is not necessarily order closed. By Dominated Convergence Theorem,
\[ X_n \overset{o}{\to} X \text{ in } L^\Phi \implies \mathbb{E}[X_n Y] \to \mathbb{E}[XY], \quad \text{for any } Y \in L^\Psi. \]

Thus
\[ C_o \subset \overline{C}_{s(L^\Phi, L^\Psi)} \subset \overline{C}_s(L^\Phi, L^\Psi) \]
for any set $C \subset L^\Phi$, where $\overline{C}_{s(L^\Phi, L^\Psi)}$ denotes the $\sigma(L^\Phi, L^\Psi)$-sequential closure of $C$. In particular, every $\sigma(L^\Phi, L^\Psi)$-closed set is order closed.

We begin by examining the connection between $(\ast)$ and the equivalence of order closedness and $\sigma(L^\Phi, L^\Psi)$-closedness of convex sets in $L^\Phi$. The next two propositions are essentially known. We include the proofs for the sake of completeness.

**Proposition 3.1.** Let $\rho : L^\Phi \to (-\infty, \infty]$ be a proper convex functional. Then $\rho$ has the Fatou property if and only if the set $C_\lambda = \{X \in L^\Phi : \rho(X) \leq \lambda\}$ is order closed for any $\lambda \in \mathbb{R}$.

\[ \text{In the definition of order closures, we can equivalently use nets instead of sequences. Indeed, since } L^\Phi \text{ has the countable sup property, if } X_\alpha \overset{o}{\to} X \text{ in } L^\Phi, \text{ then there exists countably many } (\alpha_n) \text{ such that } X_{\alpha_n} \overset{o}{\to} X \text{ in } L^\Phi. \]
Proof. The fact that $C_\lambda$ is order closed if $\rho$ satisfies the Fatou property is immediate from the definitions. Conversely, suppose that $C_\lambda$ is order closed for any $\lambda \in \mathbb{R}$. Let $(X_n)$ be a sequence in $L^\Phi$ that order converges to $X \in L^\Phi$. If $\liminf_n \rho(X_n) = \infty$, then $\rho(X) \leq \liminf_n \rho(X_n)$ trivially. Otherwise, let $\lambda \in \mathbb{R}$ be such that $\lambda > \liminf_n \rho(X_n)$. Choose a subsequence $(X_{n_k})$ so that $\rho(X_{n_k}) < \lambda$ for all $k$. Then $X_{n_k} \in C_\lambda$ for all $k$ and $X_{n_k} \to X$. Thus $X \in C_\lambda$ and $\rho(X) \leq \lambda$. As this applies to any $\lambda > \liminf_n \rho(X_n)$, $\rho(X) \leq \liminf_n \rho(X_n)$. Therefore, $\rho$ has the Fatou property.

A functional $\rho : L^\Phi \to (-\infty, \infty]$ is said to be $\sigma(L^\Phi, L^\Psi)$-lower semicontinuous if the set $\{X \in L^\Phi : \rho(X) \leq \lambda\}$ is $\sigma(L^\Phi, L^\Psi)$-closed for all $\lambda \in \mathbb{R}$.

**Proposition 3.2.** The following are equivalent for a proper convex functional $\rho : L^\Phi \to (-\infty, \infty]$.

1. $\rho$ is lower $\sigma(L^\Phi, L^\Psi)$-semicontinuous.
2. Define $\rho^*(Y) = \sup_{X \in L^\Phi} (\mathbb{E}[XY] - \rho(X))$ for any $Y \in L^\Psi$. Then $\rho(X) = \sup_{Y \in L^\Psi} (\mathbb{E}[XY] - \rho^*(Y))$ for any $X \in L^\Phi$.

If $\rho$ is a coherent risk measure, then the conditions above are also equivalent to

3. There is a set $Q$ of nonnegative random variables in $L^\Psi$, each having expectation 1, such that $\rho(X) = \sup_{Y \in Q} \mathbb{E}[-XY]$ for any $X \in L^\Phi$.

Proof. (1) $\iff$ (2) is a consequence of the Fenchel-Moreau Duality Theorem ([7, Theorem 1.11]).

For the rest of the proof, assume that $\rho$ is a coherent risk measure on $L^\Phi$. The implication (3) $\implies$ (1) is trivial. (2) $\implies$ (3). Assume that (2) holds. Choose $Z \in L^\Phi$ so that $\rho(Z) < \infty$. From positive homogeneity of $\rho$, one deduces easily that $\rho^*(\lambda W) = \lambda \rho^*(W)$ for any $W \in L^\Psi$ and any $0 < \lambda \in \mathbb{R}$. Hence $\rho^*(W) = 0$ or $\infty$ for any $W \in L^\Psi$. Suppose that $W \in L^\Psi$ and $\rho^*(W) = 0$. Let $A = \{\omega : W(\omega) > 0\}$ and set $X = 1_A$. Then $0 \leq X \in L^\Phi$ and $\mathbb{E}[XW] \geq 0$. For any $\lambda > 0$, by monotonicity of $\rho$,

$$\lambda \mathbb{E}[XW] + \mathbb{E}[ZW] = \mathbb{E}[(\lambda X + Z)W] \leq \rho(\lambda X + Z) + \rho^*(W) \leq \rho(Z) + \rho^*(W) < \infty.$$ 

Since $\lambda$ is arbitrary, $\mathbb{E}[W 1_{\{W > 0\}}] = \mathbb{E}[XW] = 0$ and hence $W \leq 0$ a.s. Define

$$Q := \{Y \in L^\Psi : \rho^*(-Y) = 0\}.$$ 

By the preceding argument, $Y \geq 0$ a.s. for any $Y \in Q$. Also, by (3) and the fact that $\rho^*(-W) = \infty$ for all $W \in L^\Psi \setminus Q$, we see that $\rho(X) = \sup_{Y \in Q} \mathbb{E}[-XY]$ for
any $X \in L^\Phi$. Finally, for any $m \in \mathbb{R}$,
\[
\rho(Z) - m = \rho(Z + m1) = \sup_{Y \in Q} (-\mathbb{E}[ZY] - m\mathbb{E}[Y]).
\]
Whence, for any $Y \in Q$ and any $m \in \mathbb{R}$,
\[
\rho(Z) \geq -\mathbb{E}[ZY] + m(1 - \mathbb{E}[Y]).
\]
Therefore, $\mathbb{E}[Y] = 1$ for any $Y \in Q$. This completes the proof of $(2) \implies (3)$. □

Let $\rho$ be a coherent risk measure on $L^\Phi$. The equivalence $(\ast)$ asserts that any coherent risk measure on $L^\Phi$ has the Fatou property if and only if it satisfies condition $(3)$ of Proposition 3.2. Since the set $C = \{X \in L^\Phi : \rho(X) \leq 0\}$ is convex, by the preceding propositions, the equivalence $(\ast)$ holds for $L^\Phi$ if for any convex set in $L^\Phi$, order closedness and $\sigma(L^\Phi, L^\Psi)$-closedness are equivalent. With this in mind, the following property was introduced in [5]. The topology $\sigma(L^\Phi, L^\Psi)$ is said to have the $C$-property if for given any net $(X_\alpha)$ in $L^\Phi$ that $\sigma(L^\Phi, L^\Psi)$-converges to $X \in L^\Phi$, there is a sequence $(Z_n)$ of convex combinations of $(X_\alpha)$ so that $Z_n \overset{\circ}{\to} X$.

Evidently, if $\sigma(L^\Phi, L^\Psi)$ has the $C$-property, then $C_{\sigma(L^\Phi, L^\Psi)} \subset C_\circ$ for any convex set $C$ in $L^\Phi$. Since the reverse inclusion always holds, it follows that order closedness and $\sigma(L^\Phi, L^\Psi)$-closedness would coincide for any convex set in $L^\Phi$. Consequently, the equivalence $(\ast)$ would hold for $L^\Phi$. Unfortunately, the next proposition shows that the $C$-property occurs rather sparsely.

**Proposition 3.3.** $C_\circ = C_{\sigma(L^\Phi, L^\Psi)}$ for every convex set $C$ in $L^\Phi$ if and only if $\Phi$ satisfies the $\Delta_2$-condition.

**Proof.** If $\Phi$ satisfies the $\Delta_2$-condition, then $L^\Phi = H^\Phi$ and thus $\sigma(L^\Phi, L^\Psi)$ is the weak topology on $L^\Phi$. By Mazur’s Theorem, $C_{\sigma(L^\Phi, L^\Psi)} \subset C_\circ$. Therefore, $\sigma(L^\Phi, L^\Psi)$-closedness would coincide for any convex set in $L^\Phi$. Conversely, suppose that $\Phi$ fails the $\Delta_2$-condition. By [27] (pp. 139, Theorem 5) (or [11] Theorem 4.51), there exist a sequence $(X_n)$ of pairwise disjoint random variables in $L^\Phi$ and a closed sublattice $X$ of $L^\Phi$ such that the map $T : \ell^\infty \to X$ defined by $T((a_n)_n) = \sum_n a_n X_n$ (pointwise sum) is a Banach lattice isomorphism. Denote by $e$ the identically 1 sequence in $\ell^\infty$. If $Y \in L^\Psi$, then
\[
\sum_n \mathbb{E}[X_n Y] \leq \sum_n \mathbb{E}[X_n] = \mathbb{E}[T e] < \infty.
\]
Hence $(\mathbb{E}[X_nY]) \in \ell^1$. Thus, if $w = (a_n) \in \ell^\infty$, then

\begin{equation}
\mathbb{E}[(Tw)Y] = \mathbb{E}\left(\sum a_n X_nY\right) = \sum a_n \mathbb{E}[X_nY] = \langle (\mathbb{E}[X_nY]), w \rangle.
\end{equation}

By Ostrovskii's Theorem (cf. [22, Theorem 2.34]), there exist a subspace $W$ of $\ell^\infty$ and $w \in \mathbb{W}^{(\ell^\infty, \ell^1)}$ such that $w$ is not the $\sigma(\ell^\infty, \ell^1)$-limit of any sequence in $W$. Let $C = T(W)$. Obviously, $C$ is a convex set in $L^\Phi$. Take a net $(w_\alpha) \subset W$ that $\sigma(\ell^\infty, \ell^1)$-converges to $w \in \ell^\infty$. Let $Y \in L^\Psi$. By (1),

$$
\mathbb{E}[(Tw_\alpha)Y] = \langle (\mathbb{E}[X_nY]), w_\alpha \rangle \rightarrow \langle (\mathbb{E}[X_nY]), w \rangle = \mathbb{E}[(Tw)Y].
$$

Thus $(Tw_\alpha) \sigma(L^\Phi, L^\Psi)$-converges to $Tw$ and $Tw \in \overline{C}^{\sigma(L^\Phi, L^\Psi)}$. Suppose, if possible, that $Tw \in \overline{C}^\circ$. Take a sequence $(w_k)$ in $W$ such that $Tw_k \xrightarrow{\sigma} Tw$ in $L^\Phi$. Clearly, $(Tw_k)$, being order bounded, is norm bounded in $L^\Phi$, so that $(w_k)$ is norm bounded in $\ell^\infty$. Write $w_k = (a_n^k) \in w = (a_n)$. Since the $X_n$'s are disjoint and $Tw_k = \sum_n a_n^k X_n \xrightarrow{a.s.} Tw = \sum_n a_n X_n$, $\lim_k a_n^k = a_n$ for each $n$. It follows that $(w_k) \sigma(\ell^\infty, \ell^1)$-converges to $w$, contrary to the choice of $w$.

However, the equality $\overline{C}^\circ = \overline{C}^{\sigma(L^\Phi, L^\Psi)}$ does hold in general for norm bounded convex sets $C \subset L^\Phi$.

**Theorem 3.4.** Let $C$ be any norm bounded convex set in $L^\Phi$. Then $\overline{C}^\circ = \overline{C}^{\sigma(L^\Phi, L^\Psi)}$.

In particular, $\overline{C}^\circ$ is order closed, and $C$ is order closed if and only if it is $\sigma(L^\Phi, L^\Psi)$-closed.

**Proof.** As has been observed, the inclusion $\overline{C}^\circ \subset \overline{C}^{\sigma(L^\Phi, L^\Psi)}$ always holds. To prove the reverse inclusion, it suffices to show that if $C$ is a convex subset of the unit ball of $L^\Phi$ such that $0 \in \overline{C}^{\sigma(L^\Phi, L^\Psi)}$, then $0 \in \overline{C}^\circ$. We divide the proof into three steps.

**Step I:** For each $n \geq 1$ and each $0 \leq Y \in L^\Psi$, there exists a pair of disjoint random variables $Z_{Y,n}$ and $W_{Y,n}$ satisfying the following:

\begin{enumerate}
  \item $Z_{Y,n} \in L^\Phi$ and $W_{Y,n} \in H^\Phi$,
  \item $Z_{Y,n} + W_{Y,n} \in C$,
  \item $\mathbb{E}[\Phi(|Z_{Y,n}|)] \leq \frac{1}{2^r}$,
  \item $\mathbb{E}[|W_{Y,n}|Y] \leq 1$.
\end{enumerate}

Since $L^\Psi$ is a lattice ideal in $(L^\Phi)^*$, by [4] Theorem 3.50, the topological dual of $L^\Phi$ under $|\sigma|(L^\Phi, L^\Psi)$ is precisely $L^\Psi$. Thus, by Mazur’s Theorem,

$$0 \in \overline{C}^{\sigma(L^\Phi, L^\Psi)} = \overline{C}^{\sigma(L^\Phi, L^\Psi)},$$
so that there exists \( X \in \mathcal{C} \) such that \( \mathbb{E}[|X|Y] \leq 1 \). Since \( \mathcal{C} \) is in the unit ball, we have that \( \mathbb{E}[\Phi(|X|)] \leq 1 \). Now take \( k \geq 1 \) such that

\[
\mathbb{E}[\mathbb{1}_{\{|X| > k\}} \Phi(|X|)] \leq \frac{1}{2^n}.
\]

Set

\[
Z_{Y,n} = X \mathbb{1}_{\{|X| > k\}} \quad \text{and} \quad W_{Y,n} = X \mathbb{1}_{\{|X| \leq k\}}.
\]

Clearly, \( Z_{Y,n} \) and \( W_{Y,n} \) are disjoint. Conditions (1)-(3) are easily verified. Condition (4) holds because

\[
\mathbb{E}[|W_{Y,n}|Y] \leq \mathbb{E}[|X|Y] \leq 1.
\]

**Step II:** There exist sequences \((Z_n)\) and \((W_n)\) in \( L^\Phi \) such that for each \( n \geq 1 \),

1. \( X_n := Z_n + W_n \in \mathcal{C} \),
2. \( \mathbb{E}[\Phi(|Z_n|)] \leq \frac{1}{2^n} \),
3. \( \|W_n\|_\Phi \leq \frac{1}{2^n} \).

Keep the notation of Step I. For any \( n \geq 1 \), define \( A_n = \{W_{Y,n} : 0 \leq Y \in L^\Psi\} \subset H^\Phi \). Let \( Y_1, \ldots, Y_k \in L^\Psi \) and let \( \varepsilon > 0 \) be given. Set \( Y = \frac{1}{\varepsilon} \sum_{i=1}^k |Y_i| \in (L^\Psi)_+ \). By Step I,

\[
\|E[W_{Y,n}Y_i]\| \leq \|E[W_{Y,n}Y_i]\| \leq \varepsilon \|E[W_{Y,n}Y]\| \leq \varepsilon.
\]

This shows that \( 0 \) lies in the \( \sigma(H^\Phi, L^\Psi) \)-closed convex hull of \( A_n \). Since \( \sigma(H^\Phi, L^\Psi) \) is the weak topology on \( H^\Phi \), \( 0 \) lies in the norm closed convex hull of \( A_n \).

Now take \( W_{Y_i,n} \in A_n, 1 \leq i \leq k \), and a convex combination \( W_n = \sum_{i=1}^k c_i W_{Y_i,n} \) such that \( \|W_n\|_\Phi \leq \frac{1}{2^n} \). Put \( Z_n = \sum_{i=1}^k c_i Z_{Y_i,n} \). Then

\[
X_n := Z_n + W_n = \sum_{i=1}^k c_i (Z_{Y_i,n} + W_{Y_i,n}) \in \mathcal{C}
\]

by convexity of \( \mathcal{C} \). Moreover, since \( \Phi \) is a convex function,

\[
\mathbb{E}[\Phi(|Z_n|)] \leq \sum_{i=1}^k c_i \mathbb{E}[\Phi(|Z_{Y_i,n}|)] \leq \frac{1}{2^n}.
\]

**Step III.** In the notation of Step II, a subsequence of \((X_n)\) order converges to 0. Thus \( 0 \in \mathcal{C}' \).

From Step II, we know that \( \|W_n\|_\Phi \leq \frac{1}{2^n} \) for all \( n \) and hence \( \sum_1^\infty |W_n| \in L^\Phi \). Also, since \( \Phi \) is continuous and increasing,

\[
\mathbb{E}[\Phi(\sup_n |Z_n|)] = \mathbb{E}[\sup_n \Phi(|Z_n|)] \leq \sum_1^\infty \mathbb{E}[\Phi(|Z_n|)] \leq \sum_1^\infty \frac{1}{2^n} = 1,
\]
from which it follows that \( \sup_n |Z_n| \in L^\Phi \). Therefore,

\[
\tilde{X} := \sup_n |Z_n| + \sum_{1}^{\infty} |W_n| \in L^\Phi.
\]

Obviously, \( |X_n| < \tilde{X} \) for all \( n \geq 1 \). Thus \( (X_n) \) is an order bounded sequence in \( L^\Phi \). By Markov’s Inequality,

\[
\Phi(\varepsilon) P\{|Z_n| > \varepsilon\} \leq E[\Phi(|Z_n|)] \leq \frac{1}{2^n}
\]

for any \( \varepsilon > 0 \). It follows that \( (Z_n) \) converges to 0 in probability. Since \( \sum_{1}^{\infty} |W_n| \in L^\Phi \), \( (W_n) \) converges to 0 a.s. Therefore, a subsequence of \( (X_n) \) converges to 0 a.s., and thus in order, since the whole sequence \( (X_n) \) is order bounded.

Theorem 3.4 allows us to characterize general order closed convex sets in \( L^\Phi \) in terms of the topology \( \sigma(L^\Phi, L^\Psi) \).

**Corollary 3.5.** Denote by \( B \) the closed unit ball in \( L^\Phi \). For a convex set \( C \) in \( L^\Phi \), the following statements are equivalent:

1. \( C \) is order closed,
2. \( C \) is \( \sigma(L^\Phi, L^\Psi) \)-sequentially closed.
3. \( C \cap kB \) is \( \sigma(L^\Phi, L^\Psi) \)-closed for all \( k \geq 1 \).

**Proof.** The implication \((2) \implies (1)\) follows from the observation at the beginning of this section. Theorem 3.4 gives \((1) \implies (3)\). The implication \((3) \implies (2)\) follows from the fact that every \( \sigma(L^\Phi, L^\Psi) \)-convergent sequence is norm bounded. □

Let \( X \) be a Banach space with closed unit ball \( B \) and let \( \tau \) be a locally convex topology on \( X \). Say that \( \tau \) has the Krein-Smulian property if a convex set \( C \) in \( X \) is \( \tau \)-closed precisely when \( C \cap kB \) is \( \tau \)-closed for all \( k \geq 1 \). The well known Krein-Smulian Theorem says that for any Banach space \( X \), the weak* topology \( \sigma(X^*, X) \) on \( X^* \) has the Krein-Smulian property. Corollary 3.5 leads to the natural question of characterizing the pairs \( (\Phi, \Psi) \) so that \( \sigma(L^\Phi, L^\Psi) \) has the Krein-Smulian property. The next lemma is the key construction to solving this question. A set \( C \subseteq L^\Phi \) is said to be

1. **monotone** if \( X_1 \geq X_2 \in C \) implies \( X_1 \in C \),
2. **positively homogeneous** if \( \lambda C \subseteq C \) for any \( \lambda \geq 0 \), and
3. **additive** if \( C + C \subseteq C \).

A set that is positively homogeneous and additive is clearly convex.
Lemma 3.6. If $\Phi$ and $\Psi$ both fail the $\Delta_2$-condition, then $L^\Phi$ admits a monochromatic polynomial homogeneous and additive subset $C$ which is order closed but not $\sigma(L^\Phi, L^\Psi)$-closed. Furthermore, for any $X \in L^\Phi$, there exists $k \in \mathbb{R}$ so that $X - k1 \in C$.

Proof. Assume that both $\Phi$ and $\Psi$ fail the the $\Delta_2$-condition. We claim that there are a norm bounded set of disjoint positive random variables $\{X_n\}_{n \geq 1} \cup \{W_0\} \cup \{W_{ij}\}_{i,j \geq 1}$ in $L^\Phi$ and a norm bounded set of disjoint positive random variables $\{Y_n\}_{n \geq 1} \cup \{Z_0\} \cup \{Z_{ij}\}_{i,j \geq 1}$ in $L^\Psi$ such that

(a) $\text{supp } Y_n \subset \text{supp } X_n$, $\text{supp } W_0 \subset \text{supp } Z_0$ and $\text{supp } W_{ij} \subset \text{supp } Z_{ij}$ for all $n, i, j \geq 1$,

(b) The pointwise sums $\tilde{X} := \sum_n X_n$ and $\tilde{Z} := \sum_{i,j} Z_{ij}$ belong to $L^\Phi$ and $L^\Psi$ respectively,

(c) $E[X_n Y_n] = E[W_0 Z_0] = E[W_{ij} Z_{ij}] = 1$ for all $n, i, j \geq 1$.

Since $\mathcal{P}$ is nonatomic, there are three disjoint measurable subsets $\Omega_1, \Omega_2, \Omega_3$ of $\Omega$, each of which is atomless and has positive measure. Choose any $0 \leq W_0 \in L^\Phi(\Omega_2)$ and $0 \leq Z_0 \in L^\Psi(\Omega_2)$ such that $\text{supp } W_0 \subset \text{supp } Z_0$ and $E[W_0 Z_0] = 1$. Since $\Phi$ fails the $\Delta_2$-condition, we may apply [27, pp. 139, Theorem 5] to $L^\Phi(\Omega_1)$ to obtain a sequence $(X_n)$ of normalized disjoint positive random variables in $L^\Phi$ such that $\tilde{X} := \sum_n X_n \in L^\Phi$. Choose a norm bounded sequence $(Y_n) \subset L^\Psi$ so that $\text{supp } Y_n \subset \text{supp } X_n$ and that $E[X_n Y_n] = 1$ for all $n$. Similarly, since $\Psi$ fails the $\Delta_2$-condition, there is a normalized disjoint positive sequence $(Z_{ij})_{i,j \geq 1} \subset L^\Psi(\Omega_3)$ so that $\tilde{Z} := \sum_{i,j} Z_{ij} \in L^\Psi$. Then choose $(W_{ij}) \subset L^\Phi$ with the desired properties.

For any $X \in L^\Phi$,

$$\sum_{i,j} |E[X Z_{ij}]| \leq E[|X \tilde{Z}|] \leq \|X\|_\Phi \|\tilde{Z}\|_\Psi.$$ 

Thus the map $T$ defined by

$$TX = \left( E[X Y_n] \right)_n \oplus E[X Z_0] \oplus \left( E[X Z_{ij}] \right)_{ij}$$ 

is a bounded linear operator from $L^\Phi$ into $\ell^\infty \oplus \mathbb{R} \oplus \ell^1(\mathbb{N} \times \mathbb{N})$. Clearly, $T$ is a positive operator. Define the summing operator $S : \ell^1 \rightarrow \ell^\infty$ by

$$S((a_j)_j) = \left( \sum_{j=1}^n a_j \right)_n.$$
For any $y = (y(i, j))_{i,j\geq 1} \in \ell^1(\mathbb{N} \times \mathbb{N})$, put $y_i = (y(i, j))_{j \in \ell^1(\mathbb{N} \times \mathbb{N})}$ for any $i \geq 1$. Let $\mathcal{C}$ be the subset of $L^\Phi$ consisting of all functions $X \in L^\Phi$ for which, if we write $TX = u \oplus a \oplus v$, there are $\lambda \in \mathbb{R}$ and $y \in \ell^1(\mathbb{N} \times \mathbb{N})$ such that

$$\lambda \geq 0, y \geq 0, \text{ and } \sum_i 2^i \|y_i\|_1 = 1,$$

$$a \geq -\lambda, \quad v \geq \lambda y, \quad \text{and} \quad u \geq \lambda \sum_{i=1}^l 4^i S y_i \text{ for all } l \geq 1.$$ 

If the above occurs, we write $X \sim (\lambda, y)$.

**Claim I**: $\mathcal{C}$ is monotone, positively homogeneous and additive.

Indeed, if $X' \geq X \in \mathcal{C}$ and $X \sim (\lambda, y)$, then clearly $X' \sim (\lambda, y)$ and $\mu X \sim (\mu \lambda, y)$ for any $\mu \geq 0$, so that $X', \mu X \in \mathcal{C}$. Now suppose that $X \sim (\lambda, y)$ and $X' \sim (\lambda', y')$.

Since $y, y' \geq 0$, it follows that

$$\sum_i 2^i \|\lambda y_i + \lambda'y'_i\|_1 = \lambda \sum_i 2^i \|y_i\|_1 + \lambda' \sum_i 2^i \|y'_i\|_1 = \lambda + \lambda'.$$

Thus we can find $0 \leq y'' \in \ell^1(\mathbb{N} \times \mathbb{N})$ such that

$$(\lambda + \lambda') y'' = \lambda y + \lambda'y'.$$

Let us show that $X + X' \sim (\lambda + \lambda', y'')$. Indeed, write $TX = u \oplus a \oplus v$ and $TX' = u' \oplus a' \oplus v'$, then

$$T(X + X') = (u + u') \oplus (a + a') \oplus (v + v').$$

Now

$$a + a' \geq (-\lambda) + (-\lambda') = -(\lambda + \lambda'),$$

$$v + v' \geq \lambda y + \lambda'y' = (\lambda + \lambda') y'',$$

and

$$u + u' \geq \lambda \sum_{i=1}^l 4^i S y_i + \lambda' \sum_{i=1}^l 4^i S y'_i = \sum_{i=1}^l 4^i S (\lambda y_i + \lambda'y'_i)$$

$$= (\lambda + \lambda') \sum_{i=1}^l 4^i S y''_i, \quad \text{for all } l \geq 1.$$ 

This proves that $X + X' \sim (\lambda + \lambda', y'')$, so that $X + X' \in \mathcal{C}$, as desired.

Since order intervals in $\ell^1$ are norm compact, for any norm convergent positive sequence $(v_p)$ in $\ell^1$, the set $\cup_p [0, v_p]$ is relatively norm compact in $\ell^1$. 


Claim II: $\mathcal{C}$ is order closed in $L^\Phi$.

Let $(U_p)$ be a sequence in $\mathcal{C}$ that order converges to some $U \in L^\Phi$. We want to show that $U \in \mathcal{C}$. Write
\[
\mathcal{T} U_p = u_p \oplus a_p \oplus v_p \quad \text{and} \quad \mathcal{T} U = u \oplus a \oplus v.
\]
For each $n \geq 1$, denote by $x(n)$ the $n$-th coordinate of a vector $x$ in $\ell^\infty$. By Dominated Convergence Theorem, for any $n \geq 1$,
\[
\lim_{p} u_p(n) = \lim_{p} \mathbb{E}[U_p Y_n] = \mathbb{E}[UY_n] = u(n).
\]
Moreover, since $(U_p)$ is order bounded, and therefore, norm bounded, in $L^\Phi$, it is easy to see that $(u_p)$ is norm bounded in $\ell^\infty$. It follows that $(u_p) \sigma(\ell^\infty, \ell^1)$-converges to $u$. Similarly, $(a_p)$ converges to $a$. For any $(b_{ij}) \in \ell^\infty(\mathbb{N} \times \mathbb{N})$,
\[
\left| \sum_{i,j} b_{ij} \mathbb{E}[(U_p - U)Z_{ij}] \right| \leq \mathbb{E}\left[|U_p - U| \sum_{i,j} |b_{ij}| Z_{ij} \right] \\
\leq \sup_{i,j} |b_{ij}| \cdot \mathbb{E}[|U_p - U|] \to 0.
\]
Hence $(v_p)$ converges to $v$ with respect to the topology $\sigma(\ell^1(\mathbb{N} \times \mathbb{N}), \ell^\infty(\mathbb{N} \times \mathbb{N}))$. Since $\ell^1(\mathbb{N} \times \mathbb{N})$ has the Schur property, $(v_p)$ norm converges to $v$.

For each $p$, suppose that $U_p \sim (\lambda_p, y_p)$ and write $y_{pi} = (y_p(i, j))_{j=1}^\infty$ for each $i$. Choose $M$ so that $\|u_p\|_{\infty} \leq M$ for all $p$. If $l \geq 1$, then
\[
(2) \quad M \geq u_p(n) \geq \lambda_p \sum_{i=1}^{l} 4^i S y_{pi}(n) \to \lambda_p \sum_{i=1}^{l} 4^i \|y_{pi}\|_1 \quad \text{as} \quad n \to \infty.
\]
In particular, $M \geq \lambda_p \sum_{i=1}^{l} 4^i \|y_{pi}\|_1 = \lambda_p \geq 0$, so that $(\lambda_p)$ is a bounded sequence.

Take a subsequence if necessary to assume that $(\lambda_p)$ converges to some $\lambda \geq 0$. If $\lambda = 0$, put $y$ to be any positive element in $\ell^1(\mathbb{N} \times \mathbb{N})$ such that $\sum_i 2^i \|y_i\|_1 = 1$, where $y_i = (y(i, j))_{j=1}^\infty$. Then it is easy to see that $a \geq -\lambda$, $v \geq \lambda y$ and $u \geq \lambda \sum_{i=1}^{l} 4^i S y_i$ for all $l$. Hence, $U \sim (\lambda, y)$, and $U \in \mathcal{C}$.

For the rest of the proof, assume that $\lambda > 0$. Since $v_p \geq \lambda_p y_p \geq 0$ for all $p$ and $(v_p)$ is norm convergent in $\ell^1(\mathbb{N} \times \mathbb{N})$, it follows that $(\lambda_p y_p)$ is relatively norm compact in $\ell^1(\mathbb{N} \times \mathbb{N})$. Passing to a subsequence again, we may assume that $(\lambda_p y_p)_p$ converges in norm to some $z$ in $\ell^1(\mathbb{N} \times \mathbb{N})$. Set $y = \frac{z}{\lambda}$. Then $y \geq 0$, and $(y_p)$ converges to $y$ in norm. To complete the proof, we will verify that $U \sim (\lambda, y)$. Clearly, for any $i \geq 1$, we have $2^i \|y_{pi}\|_1 \to 2^i \|y_i\|_1$ as $p \to \infty$. Choose $p_0$ such that
\[ \lambda_p \geq \frac{1}{2} \text{ for all } p \geq p_0. \] By (2), if \( p \geq p_0 \), then \( 0 \leq 2^i \| y_{pi} \|_1 \leq \frac{M}{\lambda^{i-1}} \) for any \( i \geq 1 \). It follows from Dominated Convergence Theorem that

\[ \sum_i 2^i \| y_i \|_1 = \lim_p \sum_i 2^i \| y_{pi} \|_1 = 1. \]

Furthermore,

\[ a = \lim_p a_p \geq -\lim \lambda_p = -\lambda, \]

and

\[ v = \lim_p v_p \geq \lim \lambda_p y_p = \lambda y. \]

Finally, for each \( n \) and each \( i \), \( S y_{pi}(n) \to S y_i(n) \) as \( p \to \infty \). Thus, for any \( l \),

\[ u(n) = \lim_p u_p(n) \geq \lim_p \lambda_p \sum_i 4^i S y_{pi}(n) = \lambda \sum_i 4^i S y_i(n). \]

This proves that

\[ u \geq \lambda \sum_i 4^i S y_i \text{ for any } l. \]

Clearly, (3)-(6) show that \( U \sim (\lambda, y) \), as desired.

**Claim III:** \( -W_0 \in \overline{C}^{\sigma(L^\Phi, L^\Psi)} \setminus C \) and thus \( C \) is not \( \sigma(L^\Phi, L^\Psi) \)-closed.

Clearly

\[ \mathcal{T}(-W_0) = 0 \oplus -1 \oplus 0. \]

If \( -W_0 \in C \), then there would exist \( \lambda \geq 0 \) and \( 0 \leq y \in \ell^1(\mathbb{N} \times \mathbb{N}) \) such that \( -1 \geq -\lambda, 0 \geq \lambda y \), and \( \sum_i 2^i \| y_i \|_1 = 1 \), where \( y_i = (y(i, j))_j \). It follows that \( \lambda \geq 1 \), forcing \( y = 0 \), which is impossible. This proves that \( -W_0 \not\in C \).

Next, we show that \( -W_0 \in \overline{C}^{\sigma(L^\Phi, L^\Psi)} \). Let \( V_1, \ldots, V_l \in L^\Psi \) and \( \varepsilon > 0 \) be given. Set \( V = \frac{1}{\varepsilon} \sum_{l=1}^l |V_l| \). Since \( \sup_{ij} \mathbb{E}[W_{ij} V] \leq (\sup_{ij} \| W_{ij} \| \Phi) \| V \| \Psi < \infty \), there exists \( s \geq 1 \) large enough so that

\[ \mathbb{E}[W_{ij} V] < 2^{s-1} \]

for all \( i, j \). Since \( \sum_n \mathbb{E}[X_n V] \leq \mathbb{E}[\bar{X} V] < \infty \), there exists \( r \geq 1 \) such that

\[ \sum_{n=r}^{\infty} \mathbb{E}[X_n V] < \frac{1}{2^{s+1}}. \]
Let \( y \in \ell^1(\mathbb{N} \times \mathbb{N}) \) be defined by 
\[
y(i, j) = \begin{cases} 
\frac{1}{2^s} & \text{if } (i, j) = (s, r) \\
0 & \text{otherwise}
\end{cases}
\]
and 0 otherwise. Simple computations show that if 
\[
X := 2^s \sum_{n=r}^{\infty} X_n - W_0 + \frac{W_{sr}}{2^s}
\]
then \( X \sim (1, y) \), so that \( X \in \mathcal{C} \). Moreover, if \( 1 \leq t \leq l \), then
\[
\left| \mathbb{E}[X_V] - \mathbb{E}[(W_0)_V] \right| \leq \varepsilon \mathbb{E}[|X + W_0|V] 
\leq \varepsilon 2^s \sum_{n=r}^{\infty} \mathbb{E}[X_n V] + \frac{\varepsilon}{2^s} \mathbb{E}[W_{sr} V] < \varepsilon.
\]
This proves that \( -W_0 \in \mathcal{C} \).

Finally, suppose that there exists \( X \in L^\Phi \) so that \( X - k \mathbb{1} \in \mathcal{C} \) for all \( k \in \mathbb{R} \). Let \( u \) be the first component of \( TX \). Then the first component of \( T(X - k \mathbb{1}) \) is 
\[
u - k(\mathbb{E}[Y_n])_n.
\]
Since \( X - k \mathbb{1} \in \mathcal{C} \), \( u - k(\mathbb{E}[Y_n])_n \geq 0 \). Thus 
\[
(\mathbb{E}[Y_n])_n \leq \frac{u}{k} \text{ for all } k \geq 1,
\]
from which it follows that \( \mathbb{E}[Y_n] = 0 \) for all \( n \), contrary to the choice of \( Y_n \). \( \square \)

We are now ready to present the main result of this section.

**Theorem 3.7.** The following statements are equivalent for an Orlicz space \( L^\Phi \) defined on a nonatomic probability space.

1. Let \( \rho : L^\Phi \rightarrow (-\infty, \infty] \) be a coherent risk measure. Then \( \rho \) has the Fatou property if and only if there is a set \( \mathcal{Q} \) of nonnegative random variables in \( L^\Psi \), each having expectation 1, such that 
\[
\rho(X) = \sup_{Y \in \mathcal{Q}} \mathbb{E}[-XY] \text{ for any } X \in L^\Phi.
\]
2. Let \( \rho : L^\Phi \rightarrow (-\infty, \infty] \) be a proper convex functional. Define \( \rho^*(Y) = \sup_{X \in L^\Phi}(\mathbb{E}[XY] - \rho(X)) \) for any \( Y \in L^\Psi \). Then \( \rho \) has the Fatou property if and only if 
\[
\rho(X) = \sup_{Y \in L^\Psi}(\mathbb{E}[XY] - \rho^*(Y)) \text{ for any } X \in L^\Phi.
\]
3. Every order closed convex set in \( L^\Phi \) is \( \sigma(L^\Phi, L^\Psi) \)-closed.
4. Every \( \sigma(L^\Phi, L^\Psi) \)-sequentially closed convex set in \( L^\Phi \) is \( \sigma(L^\Phi, L^\Psi) \)-closed.
5. \( \sigma(L^\Phi, L^\Psi) \) has the Krein-Smulian property.
Either $\Phi$ or $\Psi$ satisfies the $\Delta_2$-condition.

Proof. (3) $\implies$ (2) $\implies$ (1) follows from Propositions 3.1 and 3.2 and the observation that every $\sigma(L^\Phi, L^\Psi)$-closed set is order closed. By Corollary 3.3, we have (3) $\iff$ (4) $\iff$ (5). If $\Phi$ satisfies the $\Delta_2$-condition, then $\sigma(L^\Phi, L^\Psi)$ is the weak topology, which has the Krein-Smulian property. If $\Psi$ satisfies the $\Delta_2$-condition, then $L^\Phi$ is the norm dual of $L^\Psi$, and $\sigma(L^\Phi, L^\Psi)$ is the weak* topology, which has the Krein-Smulian property by the Krein-Smulian Theorem. This shows that (6) $\implies$ (5).

Finally, suppose that $\Phi$ and $\Psi$ both fail the $\Delta_2$-condition. Let $C$ be the set obtained by applying Lemma 3.6. Define $\rho : L^\Phi \to (-\infty, \infty]$ by

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m1 \in C\}.$$ 

Using the properties of the set $C$, it is easy to check that $\rho$ is a coherent risk measure. Clearly, $C \subset \{\rho \leq 0\}$ and $\{\rho < 0\} \subset C$ by monotonicity of $C$. It follows from the order closedness of $C$ that $X \in C$ if $\rho(X) = 0$. Therefore, $\{\rho \leq 0\} = C$, so that

$$\{\rho \leq m\} = C - m1 \quad \text{for any } m \in \mathbb{R}.$$ 

Thus $\{\rho \leq m\}$ is order closed for all $m$. By Proposition 3.1, $\rho$ has the Fatou property. If condition (1) holds, then from the representation

$$\rho(X) = \sup_{Y \in \mathcal{Q}} \mathbb{E}[-XY] \quad \text{for any } X \in L^\Phi,$$

it is clear that $C = \{\rho \leq 0\}$ is $\sigma(L^\Phi, L^\Psi)$-closed, contrary to the choice of $C$. This proves that (1) $\implies$ (6). \qed

Remark 3.8. In the special case where $\Psi$ satisfies the $\Delta_2$-condition, Theorem 3.4 was announced by Delbaen and Owari at the Vienna Congress on Mathematical Finance, September 12-14, 2016. From it they obtained the dual representation result Theorem 3.7 (2) when $\Psi$ satisfies the $\Delta_2$-condition. Their paper, which appeared in the ArXiv in November 2016, was preceded by the first version of the present paper (ArXiv October 2016), which already contained the complete results Theorems 3.4 and 3.7.

4. Dual representation with respect to the pair $(H^\Phi, H^\Psi)$

Besides the duality $(L^\Phi, L^\Psi)$ considered in §3, duality theory for risk measures on $H^\Phi$ with respect to the pair $(H^\Phi, L^\Psi)$ was studied in [8], based on the fact that $L^\Psi = \ldots$
We are motivated to study dual representations of risk measures on \( H^\Phi \) using the smaller and more manageable space \( H^\Psi \) as the dual. Despite Theorem 3.7 asserts that coherent risk measures on \( L^\Phi \) with the Fatou property may fail a dual representation with respect to the pair \((L^\Phi, L^\Psi)\), the paper [19] established a duality theory for risk measures on \( L^\Phi \) with respect to the pair \((L^\Phi, H^\Psi)\) whenever \( L^\Phi \neq L^1 \). In this context, \( H^\Psi \) consists precisely of all random variables \( Y \) such that

\[(X_n) \text{ is norm bounded in } L^\Phi, \quad X_n \xrightarrow{a.s.} X \implies \mathbb{E}[X_nY] \rightarrow \mathbb{E}[XY].\]

This observation motivated the introduction of a stronger version of the Fatou property. A functional \( \rho : H^\Phi \rightarrow (-\infty, \infty] \) satisfies the strong Fatou property if

\[(X_n) \text{ is norm bounded in } H^\Phi, \quad X_n \xrightarrow{a.s.} X \implies \rho(X) \leq \liminf_n \rho(X_n).\]

In this section, to complement and complete the three cases mentioned above, we investigate the connection between the strong Fatou property of a coherent risk measure or a proper convex functional on \( H^\Phi \) and its dual representation with respect to the duality \((H^\Phi, H^\Psi)\). Say that a set \( C \subset H^\Phi \) is boundedly a.s. closed if for any norm bounded sequence \((X_n)\) in \( C \) that a.s. converges to some \( X \in H^\Phi \), \( X \in C \).

**Proposition 4.1.** Let \( C \) be a convex set in \( H^\Phi \) and let \( X \in H^\Phi \). If \( X \in \mathcal{C}^{\sigma(H^\Phi, H^\Psi)} \), then \( X \) is the a.s.-limit of a sequence in \( C \). The converse holds if \( C \) is norm bounded. Therefore, a norm bounded convex set \( C \) in \( H^\Phi \) is \( \sigma(H^\Phi, H^\Psi) \)-closed if and only if it is boundedly a.s. closed.

**Proof.** Let \( C \) be a convex set in \( H^\Phi \) and let \( X \in \mathcal{C}^{\sigma(H^\Phi, H^\Psi)} \). Since \( H^\Psi \) is a lattice ideal of \( L^\Psi = (H^\Phi)^* \), by [10, Theorem 3.50], the topological dual of \( H^\Phi \) under \( |\sigma|(H^\Phi, H^\Psi) \) is precisely \( H^\Psi \). Thus, by Mazur’s Theorem,

\[X \in \mathcal{C}^{\sigma(H^\Phi, H^\Psi)} = \mathcal{C}^{\sigma|(H^\Phi, H^\Psi)} \].

Consequently, since \( 1 \in H^\Psi \), there is a sequence \((X_n)\) in \( C \) such that

\[\mathbb{E}[|X_n - X|] \leq \frac{1}{2^n}\]

for all \( n \geq 1 \). Since

\[\sup_{n \leq m \leq k} (|X_m - X| \wedge 1) \leq \sup_{m \geq n} (|X_m - X| \wedge 1),\]

for all \( n \geq 1 \).
it follows that
\[ E \left[ \sup_{m \geq n} (|X_m - X| \wedge 1) \right] = \lim_k E \left[ \sup_{n \leq m \leq k} (|X_m - X| \wedge 1) \right] \leq \lim_k E \left[ \sum_{n \leq m \leq k} |X_m - X| \right] \leq \frac{1}{2^{n-1}}. \]

Therefore,
\[ E \left[ \inf_{n \geq m \geq n} (|X_m - X| \wedge 1) \right] = 0, \]
so that \( \inf_{n} \sup_{m \geq n} (|X_m - X| \wedge 1) = 0 \). It follows that \( X_n \xrightarrow{a.s.} X \).

Suppose that \( C \) is a norm bounded convex set in \( H^\Phi \) and that \( (X_n) \) is a sequence in \( C \) that converges a.s. to some \( X \in H^\Phi \). Since \( L^\Phi = (H^\Psi)^* \) and \( H^\Psi \) has order continuous norm, by \([1\text{, Theorem 4.18}]\), for any \( Y \in H^\Psi \) and any \( \varepsilon > 0 \), there exists \( X_0 \in L^\Phi \) such that
\[ E\left[ (|X_n - X| - X_0)^+|Y| \right] < \varepsilon. \]

Therefore,
\[ |E[(X_n - X)Y]| \leq E[|X_n - X||Y|] = E[(|X_n - X| \wedge X_0)|Y|] + E[(|X_n - X| - X_0)^+|Y|] \leq E[(|X_n - X| \wedge X_0)|Y|] + \varepsilon. \]

By Dominated Convergence Theorem, \( E[(|X_n - X| \wedge X_0)|Y|] \to 0 \), and thus \( \limsup_n E[|(X_n - X)|Y|] \leq \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, we conclude that \( (X_n) \sigma(H^\Phi, H^\Psi) \)-converges to \( X \).

The next corollary is the counterpart of Corollary 3.5 and can be proved similarly.

\textbf{Corollary 4.2.} Denote by \( B \) the closed unit ball of \( H^\Phi \). For a convex set \( C \) in \( H^\Phi \), the following are equivalent:

1. \( C \) is boundedly a.s. closed.
2. \( C \) is \( \sigma(H^\Phi, H^\Psi) \)-sequentially closed,
3. \( C \cap kB \) is \( \sigma(H^\Phi, H^\Psi) \)-closed for all \( k \geq 1 \).

\textbf{Theorem 4.3.} The following statements are equivalent for an Orlicz heart \( H^\Phi \) defined on a nonatomic probability space.
(1) Let \( \rho : H^\Phi \to (-\infty, \infty] \) be a coherent risk measure. Then \( \rho \) has the strong Fatou property if and only if there is a set \( Q \) of nonnegative random variables in \( H^\Psi \), each having expectation 1, such that
\[
\rho(X) = \sup_{Y \in Q} \mathbb{E}[-XY] \text{ for any } X \in H^\Phi.
\]

(2) Let \( \rho : H^\Phi \to (-\infty, \infty] \) be a proper convex functional. Define \( \rho^*(Y) = \sup_{X \in H^\Phi} (\mathbb{E}[XY] - \rho(X)) \) for any \( Y \in H^\Psi \). Then \( \rho \) has the strong Fatou property if and only if
\[
\rho(X) = \sup_{Y \in H^\Psi} (\mathbb{E}[XY] - \rho^*(Y)) \text{ for any } X \in H^\Phi.
\]

(3) Every boundedly a.s. closed convex set in \( H^\Phi \) is \( \sigma(H^\Phi, H^\Psi) \)-closed.

(4) Every \( \sigma(H^\Phi, H^\Psi) \)-sequentially closed convex set in \( H^\Phi \) is \( \sigma(H^\Phi, H^\Psi) \)-closed.

(5) \( \sigma(H^\Phi, H^\Psi) \) has the Krein-Smulian property.

(6) Either \( \Phi \) or \( \Psi \) satisfies the \( \Delta_2 \)-condition.

**Sketch of proof.** We omit the proofs of the implications (5) \( \implies \) (4) \( \implies \) (3) \( \implies \) (2) \( \implies \) (1) as they are similar to the proofs of the corresponding implications in Theorem 3.7.

If \( \Phi \) satisfies the \( \Delta_2 \)-condition, then \( H^\Phi = L^\Phi \) is the dual space of \( H^\Psi \), so that \( \sigma(H^\Phi, H^\Psi) \) is a weak* topology, which has the Krein-Smulian property by the Krein-Smulian Theorem. If \( \Psi \) satisfies the \( \Delta_2 \)-condition, then \( H^\Psi = L^\Psi \) is the norm dual of \( H^\Phi \), so that \( \sigma(H^\Phi, H^\Psi) \) is a weak topology and also has the Krein-Smulian property. This proves (6) \( \implies \) (5).

Once again, the proof that (1) \( \implies \) (6) relies on a construction, whose verification we postpone for the moment. If \( \Phi \) and \( \Psi \) both fail the \( \Delta_2 \)-condition, let \( C \) be a subset of \( H^\Phi \) as given by Lemma 4.4. Then the functional \( \rho \) defined on \( H^\Phi \) by
\[
\rho(X) = \inf\{m \in \mathbb{R} : X + m1 \in C\}
\]
is a coherent risk measure on \( H^\Phi \) that has the strong Fatou property (due to the bounded a.s. closedness of \( C \)) but is not \( \sigma(H^\Phi, H^\Psi) \)-lower semicontinuous. Thus \( \rho \) cannot be represented as in condition (1). \( \square \)

**Lemma 4.4.** Assume that \( \Phi \) and \( \Psi \) both fail the \( \Delta_2 \)-condition. There is a monotone, positively homogeneous and additive subset \( C \) of \( H^\Phi \) which is boundedly a.s. closed but fails to be \( \sigma(H^\Phi, H^\Psi) \)-closed.
Sketch of proof. Observe that, for any positive random variable $X$ in $L^\Phi$ (respectively, $L^\Psi$), the truncation $X1_{\{X \leq k\}}$ lies in $H^\Phi$ (respectively, $H^\Psi$). Moreover, $\|X1_{\{X \leq k\}}\| \uparrow_k \|X\|$. As in the proof of Lemma 3.6 and applying suitable truncations, we find a norm bounded set of disjoint, positive functions $\{X_n\}_{n \geq 1} \cup \{W_0\} \cup \{W_i\}_{i \geq 1}$ in $H^\Phi$ and a norm bounded set of disjoint, positive functions $\{Y_n\}_{n \geq 1} \cup \{Z_0\} \cup \{Z_i\}_{i \geq 1}$ in $H^\Psi$ such that

(a) $\text{supp } Y_n \subset \text{supp } X_n$, $\text{supp } W_0 \subset \text{supp } Z_0$ and $\text{supp } W_i \subset \text{supp } Z_i$ for all $n, i \geq 1$,

(b) The pointwise sums $\tilde{X} = \sum_n X_n \in L^\Phi$ and $\tilde{Z} = \sum_i Z_i \in L^\Psi$,

(c) $\mathbb{E}[X_n Y_n] = \mathbb{E}[W_0 Z_0] = \mathbb{E}[W_i Z_i] = 1$ for all $n, i \geq 1$.

Since $Y_n$‘s are disjoint, $Y_n \xrightarrow{a.s.} 0$, so that $\lim_n \mathbb{E}[XY_n] = 0$ for any $X \in H^\Phi$. Thus, $(\mathbb{E}[XY_n])_n \in c_0$. Also, since

$\sum_i |\mathbb{E}[XZ_i]| \leq \mathbb{E}[||X\tilde{Z}||] < \infty,$

$(\mathbb{E}[XZ_i])_i \in \ell^1$. Therefore, the map

$T : X \mapsto (\mathbb{E}[XY_n])_n \oplus \mathbb{E}[XZ_0] \oplus (\mathbb{E}[XZ_i])_i$

is a bounded positive linear operator from $H^\Phi$ into $c_0 \oplus \mathbb{R} \oplus \ell^1$. For any $y = (y(i, j))_{i, j \geq 1} \in \ell^1(\mathbb{N} \times \mathbb{N})$, put $y_i = (y(i, j))_j \in \ell^1$ for any $i \geq 1$. Let $(s_j)_j$ be the summing basis for $c_0$, i.e.,

$s_j = (1, \ldots, 1, 0, \ldots),$

with 1 occurring in the first $j$ coordinates, and let $(w_j)$ be the standard basis for $\ell^1$, i.e.,

$w_j = (0, \ldots, 0, 1, 0, \ldots),$

with 1 occurring in the $j$-th coordinate. Let $C$ be the subset of $H^\Phi$ consisting of all random variables $X \in H^\Phi$ for which, if we write $TX = u \oplus a \oplus v$, there are $\lambda \in \mathbb{R}$ and $y \in \ell^1(\mathbb{N} \times \mathbb{N})$ such that

$\lambda \geq 0$, $y \geq 0$, $\sum_i 2^i \|y_i\|_1 = 1$, and $\sum_i 4^i \|y_i\|_1 < \infty$.

$a \geq -\lambda$, $v \geq \lambda \sum_j \left( \sum_i 4^i y(i, j) \right) w_j$, $u \geq \lambda \sum_j \left( \sum_i y(i, j) \right) s_j$.

If the above occurs, we write $X \sim (\lambda, y)$. 
We omit the verifications that $C$ is monotone, positively homogeneous and additive. For $j \geq 1$, define the projection $P_j$ on $\ell^1$ by

$$P_j(b_1, b_2, \ldots) = (0, \ldots, 0, b_j, b_{j+1}, \ldots).$$

Then the condition on $u$ is equivalent to

$$(7) \quad u \geq \lambda \left( \sum_i \|P_1 y_i\|_1, \sum_i \|P_2 y_i\|_1, \ldots \right).$$

**Claim I**: $U \in C$ whenever there exists a norm bounded sequence $(U_p)_p$ in $C$ such that $(U_p)_p$ converges a.s. to $U \in H^\Phi$.

Suppose that $TU_p = u_p \oplus a_p \oplus v_p$, and $TU = u \oplus a \oplus v$.

Write $u_p = (u_p(j))_j$ and $u = (u(j))_j$. Then

$$u_p(j) = \mathbb{E}[U_p Y_j] \to \mathbb{E}[UY_j] = u(j)$$

for each $j$. Moreover, since $(U_p)$ is norm bounded in $H^\Phi$, $(u_p)$ is norm bounded in $c_0$. Thus, $u_p \xrightarrow{\sigma(c_0, \ell^1)} u$. There is a sequence of convex combinations, $(\sum_{j=p_{n-1}}^{p_n} c_j u_j)$, $0 = p_0 < p_1 < p_2 < \cdots$, which converge to $u$ in the norm of $c_0$. By replacing $U_p$’s with the corresponding convex combinations (which also converge a.s. to $U$), we may assume that $(u_p)$ converges to $u$ in $c_0$-norm. Similarly, $v_p \to v$ coordinatewise and

$$\|v_p\|_1 \leq \sum_i \mathbb{E}[|U_p|Z_i] = \mathbb{E}[\|U_p\|\tilde{Z}] \leq \|U_p\|\Phi \|\tilde{Z}\|_\Psi.$$ 

Therefore, $v_p \xrightarrow{\sigma(\ell^1, c_0)} v$. Clearly, $a_p \to a$.

For each $p$, let $U_p \sim (\lambda_p, y_p)$. Write $y_{pi} = (y_p(i,j))_{j=1}^\infty$ for each $i$. Then there exists some $M > 0$ such that

$$(8) \quad M \geq \|v_p\|_1 \geq \lambda_p \sum_i 4^i y_p(i,j) = \lambda_p \sum_i 4^i \|y_{pi}\|_1.$$ 

In particular, $M \geq \lambda_p \sum_i 2^i \|y_{pi}\|_1 = \lambda_p \geq 0$. Thus $(\lambda_p)$ is a bounded sequence. Passing to a subsequence, we may assume that $(\lambda_p)$ converges to some $\lambda \geq 0$. If $\lambda = 0$, set $y$ to be any positive element in $\ell^1(\mathbb{N} \times \mathbb{N})$ such that $\sum_i 2^i \|y_i\|_1 = 1$ and
\[ \sum_i 4^i \|y_i\|_1 < \infty. \] It is easily checked that \( U \sim (0, y) \) and hence \( U \in C \). Assume that \( \lambda > 0 \). By (8), for all sufficiently large \( p \),

\[ \frac{\lambda}{2} \|P_j(2^i y_{pi})\|_1 \leq \lambda p \|2^i y_{pi}\|_1 \leq \frac{M}{2}, \]

for all \( i, j \geq 1 \). It follows that, for each \( j \), the sequence \((\|P_j(2^i y_{pi})\|_1)_{i \geq 1}\), being contained in an interval of \( \ell^1 \), is relatively norm compact in \( \ell^1 \). By passing to a subsequence, we may assume that there exists \((b_{ij})_{i \geq 1}\) such that

\[ \lim_p (\|P_j(2^i y_{pi})\|_1) = (b_{ij}), \text{ in } \ell^1 \text{-norm}, \]

for each \( j \geq 1 \). In particular,

\[ b_{i,j+1} \leq b_{ij} \text{ and } \sum_i b_{i1} = \lim_p \sum_i \|2^i y_{pi}\|_1 = 1. \]

Set

\[ y(i, j) = \frac{b_{ij} - b_{i,j+1}}{2^i} \text{ for all } i, j \geq 1, \]

and

\[ y := (y(i, j)) \geq 0. \]

We claim that \( U \sim (\lambda, y) \) and hence \( U \in C \). Note that

\[ b_{ij} - b_{i,j+1} = \lim_p (\|P_j(2^i y_{pi})\|_1 - \|P_{j+1}(2^i y_{pi})\|_1) = \lim_p 2^i y_{pi}(i, j). \]

Thus \( y_p(i, j) \to y(i, j) \) for any \( i, j \geq 1 \). It follows from Fatou’s Lemma and (8) that \( \sum_i 4^i \|y_i\|_1 < \infty \). Since \((u_p)\) converges to \( u \) in \( c_0 \), \( \lim_j u_p(j) = 0 \) uniformly in \( p \). By condition (7) for \( u_p \), \( \lim_j \sum_i \|P_j y_{pi}\|_1 = 0 \) uniformly in \( p \). In particular, for each \( i \), \( \lim_j \|P_j y_{pi}\|_1 = 0 \) uniformly in \( p \), so that \( \lim_j b_{ij} = 0 \) for each \( i \). Therefore,

\[ \|2^i y_i\|_1 = \sum_j (b_{ij} - b_{i,j+1}) = b_{i1} \text{ for all } i, \]

and

\[ \sum_i \|2^i y_i\|_1 = \sum b_{i1} = 1. \]

Fatou’s Lemma also implies that

\[ v(j) = \lim_p v_p(j) \geq \lim \inf_p \lambda p \sum_i 4^i y_p(i, j) \geq \lambda \sum_i 4^i y(i, j), \]

and that (using (7) for \( u_p \))

\[ u(j) = \lim_p u_p(j) \geq \lim \inf_p \lambda_p \sum_i \|P_j y_{pi}\|_1 \geq \lambda \sum_i \|P_j y_i\|_1. \]

This completes the proof that \( U \sim (\lambda, y) \).
Claim II: \( C \) is not \( \sigma(H^\Phi, H^\Psi) \)-closed. Precisely, \( -W_0 \notin \sigma(H^\Phi, H^\Psi) \setminus C \).

Since \( T(-W_0) = 0 \oplus -1 \oplus 0 \), it is easy to see that \( -W_0 \notin C \). On the other hand, observe that, for any \( s, r \geq 1 \),

\[
X_{sr} := \frac{1}{2^s} \sum_{n=1}^{r} X_n - W_0 + 2^s W_r \sim (1, y),
\]

where \( y(i, j) = \frac{1}{2^s} \) if \( (i, j) = (s, r) \) and 0 otherwise, so that \( X_{sr} \in C \). As in the proof of Lemma 3.6, one can show that \( -W_0 \) lies in the \( \sigma(H^\Phi, H^\Psi) \)-closure of the double sequence \( (X_{sr})_{s,r \geq 1} \).

□

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