To the memory of Leonid Avdeev

About higher order $\varepsilon$-expansion of some massive two- and three-loop master-integrals

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Abstract

For certain dimensionally-regulated massive two- and three-loop propagator-type diagrams the higher order $\varepsilon$-expansion is constructed.

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1 Introduction

In the light of the recent progress in the four-loop calculations [1, 2] one of the topical tasks is the construction of $\varepsilon$-expansion of one-, two-, and three-loop master-integrals of the existing packages like ONSHELL2 [3] or three-loop packages [4, 5, 6, 7] up to level sufficient for getting finite four-loop corrections in physical quantities. The all-order $\varepsilon$-expansion of the one-loop propagator diagram with arbitrary masses and external momentum, and the two-loop bubble diagram with arbitrary masses was constructed in [8, 9]. The results are expressible in terms of Nielsen polylogarithms [10] only. The construction of the $\varepsilon$-expansion for diagrams relating to QED/QCD problems was investigated in [11, 12]. The finite parts of the three-loop bubble integrals were presented in [13, 14, 15]. The higher order $\varepsilon$-expansion for some of the master-integrals from these packages was calculated in our previous papers [9, 16]. Independently of analytical calculations, the numerical approach to the evaluation of the single scale diagrams has been developed in recent years [17]. Based on this technique, the high-precision numerical values of higher order coefficients of the $\varepsilon$-expansion for four-loop bubble integrals [18] and three-loop propagator type integrals on mass shell [19] have been calculated recently.

The present paper is devoted to the analytical calculation of the higher order terms of the $\varepsilon$-expansion for the scalar integrals shown in Fig. 1. At $M = m$ these integrals enter in packages ONSHELL2 [3] ($V_{1111}$) and three-loop packages [4, 5] ($D_5$). The numerical values of higher order coefficients of the $\varepsilon$-expansion for diagram $V_{1111}$ and $D_5$ were calculated by Laporta in [17]. Another problem under consideration is further investigation and development of the “sixth root of unity” approach proposed by Broadhurst in [13] and developed in [14, 16, 9, 20, 18]. The main idea is that transcendental numbers occurring in the $\varepsilon$-expansion of single scale diagrams (diagrams with only one mass scale) are defined by the values of massive cuts. For the diagrams with zero-, one-, two-, and three- massive cuts, the set of transcendental numbers are related to generalized log-sine functions [21] or their generalization [9, 20] of special values of arguments $\{\pi, \pi/2, \pi/3, 2\pi/3\}$. For diagrams with four massive cuts new constants, associated with the elliptic function [18] appeared.

We work in the dimensional regularization [22] with space-time dimension $n = 4 - 2\varepsilon$. In our normalization each loop is divided by $(4\pi)^{2-\varepsilon}\Gamma(1+\varepsilon)$. We also use the following short notation for the auxiliary integrals appearing in our calculation:

$$J_{m_1m_2m_3} = \int \frac{d^n k_1 d^n k_2}{((k_1-p)^2 + m_1^2)(k_2^2 + m_2^2)((k_1-k_2)^2 + m_3^2)} |_{p^2 = -m_2},$$

$$V_{m_1m_2m_3} = \int \frac{d^n k_1 d^n k_2}{(k_1^2 + m_1^2)(k_2^2 + m_2^2)((k_1-k_2)^2 + m_3^2)},$$

$$B_0(m_1, m_2, m_3) = \int \frac{d^n k_1}{((k_1-p)^2 + m_1^2)(k_2^2 + m_2^2)} |_{p^2 = -m_3}.$$

\footnote{There is a typo in Eq. (4.10) of [9]: the coefficient before $\pi L_{s_4}(\frac{\pi}{4})$ should be $\frac{161}{54}$ instead of $\frac{161}{104}$. We are grateful to Y. Schröder and A. Vuorinen for correspondence.}
\[ A_0(m) = \int \frac{d^n k_1}{k_1^2 + m^2} \equiv \frac{4m^{n-2}}{(n-2)(n-4)}. \] (1.1)

2 \( V_{mmmM} \)

This integral enters in two-loop relation between pole and \( \overline{MS} \) masses of heavy particles like t-quark or Higgs boson within SM [23, 24]. The integrals with an arbitrary set of indices,

\[ V_{mmmM}(\alpha, \beta, \sigma, \lambda) = \int \frac{d^n k_1 d^n k_2}{((p - k_1)^2 + m^2)^\sigma((k_1 - k_2)^2 + m^2)^\alpha(k_2^2 + m^2)^\beta(k_1^2 + M^2)^\lambda}, \] (2.2)

where the external moment belongs to the mass shell, \( p^2 = -m^2 \), can be reduced to one master-integral with indices (1,1,1,1). For the calculation of the diagram, we use the differential equation technique [25]

\[
\frac{d}{dM^2} V_{mmmM}(1, 1, 1, 1) = V_{mmmM}(1, 1, 1, 1) \left[ \left( \frac{n}{2} - 2 \right) \frac{1}{M^2} + \frac{(n - 3)}{M^2 - 4m^2} \right] \\
- \left[ V_{mmm} + 2A_0(m)B_0(M, m, m) \right] \frac{1}{4m^2} \left( \frac{n}{2} - 1 \right) \left( \frac{1}{M^2} - \frac{1}{M^2 - 4m^2} \right) \\
+ J_{mmm} \frac{1}{m^2} \left( \frac{3n}{8} - 1 \right) \left( \frac{1}{M^2} - \frac{1}{M^2 - 4m^2} \right). \] (2.3)

The analytical solution of this equation up to part linear in \( \varepsilon \) was presented in [24]. Equation (2.3) has a very simple form in the framework of the geometrical approach [26]. Let us introduce a new variable

\[ \cos \theta = \frac{M}{2m}, \quad M \leq 2m. \] (2.4)
Taking into account all order $\varepsilon$-expansion of the one-loop propagator and two-loop bubble integrals \([27, 8]\) we rewrite the original differential equation (2.3)

\[
\frac{1}{\sin \theta \cos \theta} \frac{d}{d\theta} V_{mmmM} - 2 \left( \frac{\varepsilon}{\cos^2 \theta} + \frac{1 - 2\varepsilon}{\sin^2 \theta} \right) V_{mmmM} \]
\[- \frac{1}{2 \cos^2 \theta \sin^2 \theta (1 - 2\varepsilon)} \left\{ - \frac{1}{\varepsilon^2} \left[ 3 - 6 \cos^2 \theta + 2 \left( 4 \cos^2 \theta \right)^{1-\varepsilon} \right] \right. \]
\[+ \frac{2\theta}{\varepsilon} (4 \cos \theta \sin \theta)^{1-2\varepsilon} + (4 \cos \theta \sin \theta)^{1-2\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{(j + 1)!} \left[ 4L_{s,j+2} (2\theta) - 3L_{s,j+2} (4\theta) \right] \]
\[+ \frac{J_{mmm}}{m^2} \frac{2 - 3\varepsilon}{2 \sin^2 \theta \cos^2 \theta} = 0. \tag{2.5}
\]

For a modified function, $\tilde{V}(\theta)$ defined as

\[
V_{mmmM} = (\sin \theta)^{2-4\varepsilon} (\cos \theta)^{-2\varepsilon} (m^2)^{-2\varepsilon} \tilde{V}(\theta), \tag{2.6}
\]

the solution has the following form:

\[
2(1 - 2\varepsilon)^2 \tilde{V}(\theta) = -(2 - 3\varepsilon)(1 - 2\varepsilon) \frac{J_{mmm}}{(m^2)^{1-2\varepsilon}} \left[ (1 - 3\varepsilon)I_{Sc1,2} (\theta) - \frac{(\sin \theta)^{4\varepsilon} (\cos \theta)^{2\varepsilon}}{2 \sin^2 \theta} \right] \]
\[- \frac{1}{\varepsilon^2} \left[ 3(1 - \varepsilon)I_{Sc1,2} (\theta) + \frac{3}{2} \frac{(\sin \theta)^{4\varepsilon} (\cos \theta)^{2\varepsilon}}{\sin^2 \theta} - 4^{1-\varepsilon} (\sin \theta)^{4\varepsilon} \right] \]
\[+ 4^{1-2\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^{j-1}}{j!} \left\{ \frac{-\cos \theta}{(\sin \theta)^{1-2\varepsilon}} [4L_{s,j+1} (2\theta) - 3L_{s,j+1} (4\theta)] \right\} \]
\[-2^{-2\varepsilon} \sum_{k=0}^{\infty} \frac{(2\varepsilon)^k}{k!} \left[ 8 \ln^{k+j+1} (2 \sin \theta) \frac{k+j+1}{k+j+1} + 3\varepsilon L_{s,k+1} (2\theta) L_{s,j+1} (4\theta) \right] \]
\[-2^{2-2\varepsilon} \sum_{k=0}^{\infty} \frac{(2\varepsilon)^k}{k!} \int_{0}^{\theta} d\phi \left[ 2 \ln^k (2 \sin \phi) L_{s,j+1} (2\phi) + 3L_{s,k+1} (2\phi) \ln (2 \sin \phi) + \ln (2 \cos \phi) \right]^j \right] \]
\[+ 3 \cdot 4^{1-\varepsilon} \sum_{k=0}^{\infty} \frac{(2\varepsilon)^k}{k!} \int_{0}^{\theta} d\phi \frac{\cos \phi}{\sin \phi} \ln^k (2 \sin \phi) [\ln (2 \sin \phi) + \ln (2 \cos \phi)]^j \right\}, \tag{2.7}
\]

where $L_{s_j} (\theta)$ is a log-sine function \([21]\) and we have introduced a new function $I_{Sc_{a,b}} (\theta)$

\[
I_{Sc_{a,b}} (\theta) = \int_{0}^{\theta} d\phi \frac{\sin \phi}{\sin \phi} \frac{\phi^{ax} (\cos \phi)^{bx}}{\sin \phi \cos \phi}, \tag{2.8}
\]

where $a, b$ are integer numbers. The $\varepsilon$-expansion of this function is

\[
I_{Sc_{a,b}} (\theta) = \ln (\tan \theta) + \frac{1}{2} \sum_{i=1}^{\infty} \frac{\left( \frac{\varepsilon}{2} \right)^i}{(i + 1)!} \left( a^i \ln^{i+1} (\sin^2 \theta) - b^i \ln^{i+1} (\cos^2 \theta) \right) \]
\[- \frac{1}{2} \sum_{i=1}^{\infty} \left( -\frac{\varepsilon}{2} \right)^i \left( a^i S_{1,i} (\cos^2 \theta) - b^i S_{1,i} (\sin^2 \theta) \right) \]
can be parametrized by introducing a new class of functions $L_{sL_{sc}}^{k,i,j}$ where $S_{a,b}^{k,i,j}$ can be explicitly integrated.

Using the results of [11, 28] we get the following values for the first several coefficients: $V_{mmmM} = (\sin \theta)^{2-4\varepsilon} (\cos \theta)^{-2\varepsilon} (m^2)^{-2\varepsilon} \left\{ \tilde{V}(\theta) + \sum_{j=-1}^{\infty} \varepsilon^j v_j \right\}$, (2.11)

where the coefficients $v_j$ should be defined from the boundary condition. We choose a particular value of the diagram at $M = 0$ which corresponds to $\theta = \pi/2$. In this case the diagram reduces to $J_{mmm}$ and the product of one-loop vacuum diagrams

$V_{mmmM} \big|_{M=0} = \frac{3n - 8}{4(4 - n)} \frac{J_{mmm}}{m^2} - \frac{3(n - 2)^2}{8(n - 4)(n - 3)} \left[ \frac{A_0(m^2)}{m^2} \right]^2$.

Using the results of [11, 28] we get the following values for the first several coefficients:

$v_{-1} = 12 \ln 2$, $v_0 = 48 \ln 2 + 6\zeta_2 - 36 \ln^2 2$,

$v_1 = 144 \ln 2 + 24\zeta_2 - 5\zeta_3 - 12\zeta_2 \ln 2 - 144 \ln^2 2 + 72 \ln^3 2$,

$v_2 = 384 \ln 2 + 72\zeta_2 - 20\zeta_3 - 29\zeta_4 - 48\zeta_2 \ln 2 + 24\zeta_2 \ln^2 2 - 432 \ln^2 2 + 288 \ln^3 2 - 106 \ln^4 2 + 48 \text{Li}_4 \left( \frac{1}{2} \right)$,

$v_3 = 960 \ln 2 + 192\zeta_2 - 60\zeta_3 - 116\zeta_4 - \frac{633}{2} \zeta_5 - 144\zeta_2 \ln 2 + 96\zeta_2 \ln^2 2 - 48\zeta_2 \ln^3 2 + 306 \zeta_4 \ln 2 - 1152 \ln^2 2 + 864 \ln^3 2 - 424 \ln^4 2 + \frac{636}{5} \ln^5 2 + 192 \text{Li}_4 \left( \frac{1}{2} \right) + 288 \text{Li}_5 \left( \frac{1}{2} \right)$.

$v_4$ can also be calculated by using the results of [28].

From Eq. (2.7) it is evident, that the coefficients of the $\varepsilon$-expansion of the diagram $V_{1112}$ can be parametrized by introducing a new class of functions $L_{sL_{sc}}^{k,i,j}(\theta)$ defined as

$L_{sL_{sc}}^{k,i,j}(\theta) = \int_0^\theta d\phi L_{sL_{sc}}^{k+1}(\phi) \ln^{i-1} \left| 2 \sin \frac{\phi}{2} \right| \ln^{j-1} \left| 2 \cos \frac{\phi}{2} \right|$, (2.13)

where $k, i, j$ are integer numbers, $k \geq 0$ and $i, j \geq 1$. Some properties of these functions are collected in Appendix B.
Let us write the $\varepsilon$-expansion of the diagram in following form:

$$V_{\text{mmnM}}(\theta) = (m^2)^{-2\varepsilon} \left[ \frac{1}{2\varepsilon^2} F(\theta) - \frac{1}{\varepsilon} S(\theta) + F(\theta) + \varepsilon E(\theta) + \varepsilon^2 N(\theta) + \varepsilon^3 N_n(\theta) + \varepsilon^4 N_{nn}(\theta) + O(\varepsilon^5) \right].$$

(2.14)

The functions $S(\theta), F(\theta)$ and $E(\theta)$ were calculated in [24] (see Eq. (3.27)). The analytical results for $N(\theta)$ for arbitrary values of the angle $\theta$ read

$$N(\theta) = \frac{F(\theta) - E(\theta)}{\sin^2 \theta} - 4\theta \left[ 4L_3 (2\theta) - 3L_3 (4\theta) \right] - \frac{2\sin(2\theta)}{3} \frac{\sum_{j=0}^{3} (-1)^j \left[ 4L_{4-j} (2\theta) - 3L_{4-j} (4\theta) \right] l_{2\theta} (3)}{\sin^2 \theta} + \frac{1}{\sin^2 \theta} \left[ \frac{27}{2} + \frac{4}{3} \cos^2 \theta L_4^2 \right] + 24\zeta_2 \ln 2 - 14\zeta_3 - 12\zeta_2$$

$$- 24\Phi(2\theta) + 12L_4^{(1)} (2\theta) - 3L_4^{(1)} (4\theta) + 20 \left[ L_2 (2\theta) \right]^2 - 12L_2 (2\theta) L_2 (4\theta)$$

$$- 16\theta L_3 L_2 (2\theta) - 32\theta L_3 L_2 (2\theta) + 24\tau L_3 (\pi - 2\theta) \left[ L_2 + 2l_{2\theta} \right] - 12L_3 L_2 \left( \sin^2 \theta \right)$$

$$- 8\theta^2 L_3^2 - 32\theta^2 L_3 L_2 + 36\zeta_3 l_{2\theta} - 24L_3 L_2 + 12\zeta_2 L_2^2 + 48 \ln 2 \zeta_2 L_2$$

$$+ 42\zeta_3 \ln 2 - 31\zeta_4 + 2 \ln^4 2 - 36\zeta_2 \ln^2 2 + 48\tau \left( \frac{1}{2} \right),$$

(2.15)

where

$$L_\theta = \ln (2 \cos \theta), \quad l_{\theta} = \ln (2 \sin \theta), \quad l_{m\theta} = \ln (2 \sin m\theta),$$

(2.16)

and

$$\Phi(\theta) \equiv \int_0^\theta d\phi L_2 (\phi) \ln \left( \frac{2 \cos \frac{\phi}{2}}{2} \right) = \text{LsLsc}_{1,2} (\theta),$$

is the function defined in Eq. (2.41) of [29]. The results for $N_n(\theta)$ and $N_{nn}(\theta)$ are sufficiently lengthy to be published here. It should be mentioned that in the order of $\varepsilon^3$ the following combination appears:

$$2L_3 \text{LsLsc}_{1,2} (2\theta) + L_3 \text{LsLsc}_{0,3,2} (2\theta) + 2L_3 \text{LsLsc}_{0,3,2} (\pi - 2\theta) + L_3 \text{LsLsc}_{1,1,3} (2\theta) - L_3 \text{LsLsc}_{1,1,3} (\pi - 2\theta)$$

(2.17)

To write results valid in other regions of the variable ($M > 2m$), a proper analytical continuation of all expressions should be constructed. For generalized log-sine functions it is described in detail in [9, 31]. In terms of the variable

$$y \equiv e^{i\sigma \theta}, \quad \ln (-y - i\sigma 0) = \ln y - i\sigma \pi,$$

(2.18)

the analytical continuation of all generalized log-sine can be expressed in terms of Nielsen polylogarithms, whereas for the function $\Phi(\theta)$ the result is expressible in terms of harmonic polylogarithms (see Eq. (3.4) in [29]) introduced by Remiddi and Vermaseren [32]. In terms of conformal variable (2.18), the analytical continuation of the LsLsc functions can be written.
in terms of harmonic polylogarithms (see the discussion in [9, 29, 33]). In particular, an analytical continuation of the \( L_{sLsc1,1,3}(\theta) \)-function produces integral \( f H_{-1,0,1}(z) \frac{dz}{z+2} \). The result is relatively lengthy to be published here. The analytically continued results of the order \( \varepsilon^2 \) checked by heavy mass expansion [34] with the help of the packages described in [35].

For a particular case \( M = m \) the integral \( V_{mm,mM} \) converts into the master-integral \( V_{1111} \) from the package \textbf{ONSHELL2} [3, 30]. Its \( \varepsilon \)-expansion is (we present only new coefficients of the expansion)

\[
N \left( \frac{\pi}{3} \right) \equiv V_{1111}[\varepsilon^2] = \frac{211}{2} - 32 \frac{\pi}{\sqrt{3}} + 12 \frac{\pi}{\sqrt{3}} \ln 3 - 2 \frac{\pi}{\sqrt{3}} \ln^2 3 + \frac{1}{6} \frac{\pi}{\sqrt{3}} \ln^3 3 \\
-34 \zeta_2 + 16 \zeta_2 \ln 3 - 4 \zeta_2 \ln^2 3 - 32 \zeta_3 + \frac{39}{2} \zeta_3 \ln 3 - 18 \zeta_2 \frac{\pi}{\sqrt{3}} + \frac{9}{2} \zeta_2 \frac{\pi}{\sqrt{3}} \ln 3 + 9 \zeta_3 \frac{\pi}{\sqrt{3}} \\
-84 \frac{L_{s2}(\frac{\pi}{3})}{\sqrt{3}} + 28 \frac{L_{s4}(\frac{\pi}{3})}{\sqrt{3}} \ln 3 + \frac{16}{3} \pi L_{s2}(\frac{\pi}{3}) - 42 \frac{L_{s4}(\frac{2\pi}{3})}{\sqrt{3}} \\
-\frac{7}{2} \frac{L_{s4}(\frac{\pi}{3})}{\sqrt{3}} \ln^2 3 - \frac{8}{3} \pi L_{s2}(\frac{\pi}{3}) \ln 3 + \frac{21}{2} \frac{L_{s3}(\frac{2\pi}{3})}{\sqrt{3}} \ln 3 - 7 \pi L_{s3}(\frac{2\pi}{3}) \\
-\frac{219}{4} \zeta_4 - 7 \frac{L_{s4}(\frac{2\pi}{3})}{\sqrt{3}} + \frac{27}{2} L_{s4}^{(1)}(\frac{2\pi}{3}) + \frac{14}{3} \left[ L_{s2}(\frac{\pi}{3}) \right]^2 \\
+24 \zeta_2 \ln 2 + 18 \zeta_2 \ln^2 2 - 18 \zeta_2 \ln 2 \ln 3 + 9 \zeta_2 \pi L_{i2} \left( \frac{\pi}{3} \right), \tag{2.19}
\]

and

\[
N_n \left( \frac{\pi}{3} \right) \equiv V_{1111}[\varepsilon^3] = \frac{665}{2} - 80 \frac{\pi}{\sqrt{3}} + 32 \frac{\pi}{\sqrt{3}} \ln 3 - 6 \frac{\pi}{\sqrt{3}} \ln^2 3 + \frac{2}{3} \frac{\pi}{\sqrt{3}} \ln^3 3 - \frac{1}{24} \frac{\pi}{\sqrt{3}} \ln^4 3 \\
-54 \zeta_2 \frac{\pi}{\sqrt{3}} + 18 \zeta_2 \frac{\pi}{\sqrt{3}} \ln 3 - \frac{9}{4} \zeta_2 \frac{\pi}{\sqrt{3}} \ln^2 3 - 148 \zeta_2 - 152 \zeta_3 - 157 \zeta_4 + \frac{1081}{24} \zeta_2 \zeta_3 + \frac{3503}{12} \zeta_5 \\
+48 \zeta_2 \ln 3 - 16 \zeta_2 \ln^2 3 + \frac{8}{3} \pi \zeta_2 \ln^3 3 + 36 \zeta_3 \frac{\pi}{\sqrt{3}} - 9 \zeta_3 \frac{\pi}{\sqrt{3}} \ln 3 + 78 \zeta_3 \ln 3 - \frac{30}{3} \zeta_3 \ln^2 3 \\
+63 \zeta_4 \ln 3 - \frac{171}{8} \zeta_4 \frac{\pi}{\sqrt{3}} - 224 \frac{L_{s3}(\frac{\pi}{3})}{\sqrt{3}} + 84 \frac{L_{s3}(\frac{\pi}{3})}{\sqrt{3}} \ln 3 - 14 \frac{L_{s3}(\frac{\pi}{3})}{\sqrt{3}} \ln^2 3 + \frac{7}{6} \frac{L_{s4}(\frac{\pi}{3})}{\sqrt{3}} \ln^3 3 \\
+16 \pi L_{s2}(\frac{\pi}{3}) - \frac{32}{3} \pi L_{s2}(\frac{\pi}{3}) \ln 3 + \frac{8}{3} \pi L_{s2}(\frac{\pi}{3}) \ln^2 3 + 6 \pi \zeta_2 L_{s2}(\frac{\pi}{3}) \\
-126 \frac{L_{s3}(\frac{2\pi}{3})}{\sqrt{3}} + 42 \frac{L_{s3}(\frac{2\pi}{3})}{\sqrt{3}} \ln 3 - \frac{21}{4} \frac{L_{s3}(\frac{2\pi}{3})}{\sqrt{3}} \ln^2 3 - 14 \pi L_{s3}(\frac{2\pi}{3}) \left[ 2 - \ln 3 \right] \\
+\frac{23}{18} \pi L_{s4}(\frac{\pi}{3}) - \frac{33}{3} \pi L_{s4}(\frac{2\pi}{3}) - \frac{11}{3} \frac{L_{s4}(\frac{2\pi}{3})}{\sqrt{3}} \ln 3 + 27 L_{s4}^{(1)}(\frac{2\pi}{3}) \left[ 2 - \ln 3 \right] \\
-\frac{7}{2} \frac{L_{s5}(\frac{2\pi}{3})}{\sqrt{3}} + \frac{25}{3} \left[ L_{s2}(\frac{\pi}{3}) \right]^2 \left[ 2 - \ln 3 \right] - 4 L_{s2}(\frac{\pi}{3}) \pi L_{s3}(\frac{2\pi}{3}) - \frac{25}{6} \pi \zeta_2 L_{s2}(\frac{\pi}{3}) \\
+9 L_{s4}^{(1)}(\frac{2\pi}{3}) - \frac{8}{3} \chi_5 + 36 L_{sLsc1,1,3}(\frac{\pi}{3}) \ln 2 \\
-4 \ln^4 2 + 24 \zeta_2 \ln^2 2 - 96 \pi L_{i4}(\frac{\pi}{3}) - \frac{45}{3} \ln^5 2 - 63 \zeta_3 \ln^2 2 - 288 \left[ L_{i4}(\frac{1}{2}) \ln 2 + L_{i5}(\frac{1}{2}) \right] \\
+L_{i2}(\frac{1}{4}) \left[ 36 \zeta_2 + \frac{63}{2} \zeta_3 - 18 \zeta_2 \ln 2 - 18 \zeta_2 \ln 3 \right] - 9 \zeta_2 L_{i3}(\frac{1}{4}) - 18 \zeta_2 S_{1,2}(\frac{1}{4}) \\
+72 L_{i4}(\frac{1}{2}) \ln 3 + 3 \ln 3 \ln^4 2 - 72 \zeta_2 \ln 2 \ln 3 + 18 \zeta_2 \ln 2 \ln^2 3. \tag{2.20}
\]

Numerical values of the coefficients, \( V_{1111}[\varepsilon^2] \) and \( V_{1111}[\varepsilon^3] \) are in full agreement with the proper results of [17].
3 \( D_{mmmmM} \)

The differential equation for this integral is

\[
-M^2 \frac{d}{dM^2} D_{mmmmM} = \left( 5 + 4(3 - n) \frac{m^2}{M^2 - 4m^2} - \frac{3n}{2} \right) D_{mmmmM}
+ B_N \left( \frac{3}{2} n - 4 \right) \frac{1}{M^2 - 4m^2} + V_{MmmM} A_0(m) \frac{4 - 2n}{M^2 - 4m^2},
\]

(3.21)

where \( B_N \equiv B_N(0, 0, 1, 1, 1, 1) \) was defined by Broadhurst in \([11, 12]\). Using again the angle variable defined in (2.4), we rewrite equation (3.21) in the following form:

\[
\cos \theta \frac{d}{d\theta} D_{mmmmM} - \left( -1 + 3\varepsilon + \frac{1 - 2\varepsilon}{\sin^2 \theta} \right) D_{mmmmM}
= -B_N \left( 2 - 3\varepsilon \right) \frac{1}{m^2} \frac{1 - 2\varepsilon}{4 \sin^2 \theta} \frac{1}{(1 - \varepsilon)(1 - 2\varepsilon)} \left\{ -\frac{1}{\varepsilon^2} \left[ 1 - 2\cos^2 \theta + \left( 4\cos^2 \theta \right)^{1-\varepsilon} \right] \right\}
+(4 \cos \theta \sin \theta)^{1-2\varepsilon} \sum_{j=0}^{\infty} \frac{(-\varepsilon)^j}{(j+1)!} \left[ 2L_{s,j+2} (2\theta) - L_{s,j+2} (4\theta) \right],
\]

(3.22)

For auxiliary function \( \tilde{D}(\theta) \) defined as

\[
D_{mmmmM} = (\sin \theta)^{2-4\varepsilon} (\cos \theta)^{-2\varepsilon} (m^2)^{1-3\varepsilon} \tilde{D}(\theta)
\]

the solution is

\[
(1 - \varepsilon)(1 - 2\varepsilon)^2 \tilde{D}(\theta) =
-\frac{(1 - \varepsilon)(2 - 3\varepsilon)(1 - 2\varepsilon)}{2} B_N \left( \frac{1 - 3\varepsilon}{(m^2)^{2-3\varepsilon}} \left[ (1 - 3\varepsilon)\text{Isc}_{4,2} (\theta) - \frac{(\sin \theta)^{4\varepsilon} (\cos \theta)^{2\varepsilon}}{2 \sin^2 \theta} \right] \right)
+ \frac{2}{\varepsilon^2} \left[ (1 - \varepsilon)\text{Isc}_{4,2} (\theta) + \frac{(\sin \theta)^{4\varepsilon} (\cos \theta)^{2\varepsilon}}{2 \sin^2 \theta} - 2^{1-2\varepsilon} (\sin \theta)^{4\varepsilon} \right]
-2^{3-4\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{(j+1)!} \left\{ \frac{\cos \theta}{(\sin \theta)^{1-2\varepsilon}} \left[ 2L_{s,j+2} (2\theta) - L_{s,j+2} (4\theta) \right] \right\}
-2^{2-2\varepsilon} \sum_{k=0}^{\infty} \frac{(2\varepsilon)^k}{k!} \left[ \frac{\ln^{k+j+2} (2\sin \phi)}{k+j+2} + \varepsilon L_{s,k+1} (2\theta) L_{s,j+2} (4\theta) \right]
-2^{2-2\varepsilon} \sum_{k=0}^{\infty} \frac{(2\varepsilon)^k}{k!} \left\{ \frac{\cos \phi}{\sin \phi} \ln^k (\sin \theta) \left[ \ln (2\sin \phi) + \ln (2\cos \phi) \right]^{j+1} + \ln^k (\sin \theta) L_{s,j+2} (2\phi) \right\}
+2^{2-2\varepsilon} \sum_{k=0}^{\infty} \frac{(2\varepsilon)^k}{k!} \left\{ \frac{\cos \phi}{\sin \phi} \ln^k (2\sin \phi) \left[ \ln (2\sin \phi) + \ln (2\cos \phi) \right]^{j+1} \right\},
\]

(3.23)

where the last integral can be explicitly integrated with the help of Eq. (2.10) and \( \text{Isc}_{a,b} (\theta) \) is defined in (2.9). The result for the diagram \( D_{mmmmM} \) is

\[
D_{mmmmM} = (\sin \theta)^{2-4\varepsilon} (\cos \theta)^{-2\varepsilon} (m^2)^{1-3\varepsilon} \left\{ \tilde{D}(\theta) + \sum_{j=-3}^{\infty} \varepsilon^j d_j \right\},
\]

(3.24)
where the coefficients \( d_j \) are defined from the value of the original diagrams at \( M = 0 \), which corresponds to \( \theta = \pi/2 \):

\[
m^2 D_{mmmmM} \bigg|_{M=0} = V_{0mm} A_0(m) \frac{n-2}{n-4} - B_N \frac{3n-8}{4(n-4)}.
\]

The explicit values of the first several coefficients are

\[
\begin{align*}
  d_{-3} &= \frac{4}{3}, &
  d_{-2} &= \frac{20}{3} - 8 \ln 2, &
  d_{-1} &= \frac{68}{3} - \frac{16}{3} \zeta_2 - 40 \ln 2 - 8 \ln^2 2, \\
  d_0 &= \frac{196}{3} + \frac{20}{3} \zeta_3 - \frac{80}{3} \zeta_2 - 136 \ln 2 - 40 \ln^2 2 + \frac{176}{3} \ln^3 2, \\
  d_1 &= -\frac{400}{3} \ln^4 2 + \frac{860}{3} \ln^3 2 + 16 \zeta_2 \ln^2 2 - 136 \ln^2 2 - 392 \ln 2 \\
  &\quad - \frac{27}{3} \zeta_2 + \frac{100}{3} \zeta_3 + \frac{116}{3} \zeta_4 - 64 \text{Li}_4 \left(\frac{1}{2}\right) + 172, \\
  d_2 &= 428 + \frac{2012}{15} \ln^5 2 - \frac{2000}{3} \ln^4 2 + \frac{2002}{3} \ln^3 2 - 392 \ln^2 2 - 1032 \ln 2 \\
  &\quad - 32 \zeta_2 \ln^3 2 + 80 \zeta_2 \ln^2 2 - 408 \text{Li}_4 \ln 2 - \frac{784}{3} \zeta_2 + \frac{340}{3} \zeta_3 + \frac{580}{3} \zeta_4 \\
  &\quad + \frac{16}{3} \zeta_2 \zeta_3 + 422 \zeta_5 - 320 \text{Li}_4 \left(\frac{1}{2}\right) - 384 \text{Li}_5 \left(\frac{1}{2}\right),
\end{align*}
\]

The next coefficient \( d_3 \) contains the \( U_{5,1} \) constant [12] which could be rewritten in terms of the generalized log-sine function of the argument \( \pi/2 \) [9]. Collecting all the results we get the first several coefficients of the \( \varepsilon \)-expansion of the diagram

\[
(m^2)^{-1+3\varepsilon} (1-\varepsilon)(1-2\varepsilon)^2 D_{mmmmM}(\theta) = -\frac{2}{3\varepsilon^3} \left\{ 1 + 2 \cos^2 \theta \right\} - \frac{2}{3\varepsilon^2} \left\{ 1 - 12 \cos^2 \theta L_\theta \right\}
\]

\[
-\frac{2}{\varepsilon} \left\{ 1 + 4 \cos^2 \theta \theta^2 - 2 \sin(2\theta) \left[ 4 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta) \right] \right\}
\]

\[
- \frac{6}{\varepsilon} \left[ 16 \cos^2 \theta \theta^3 - 4 \sin^2 \theta \left[ 4 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta) \right] - \frac{2}{3} \zeta_3 \right]\n\]

\[
+ 4 \sin(2\theta) \left[ 4 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta) \right] - 8 \sin(2\theta) l_{2\theta} \left[ 4 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta) \right]
\]

\[
+ \varepsilon \left\{ -18 + \frac{56}{3} \zeta_3 + 8 \sin^2 \theta \left[ 2 L_\theta + L_\theta \right] \left[ 4 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta) \right] \right\}
\]

\[
+ 4 \sin^2 \theta \left[ 8 \text{F}(2\theta) - 4 \text{Li}_4^{(1)}(2\theta) + \text{Li}_4^{(1)}(4\theta) \right]
\]

\[
+ 16 \sin^2 \theta \left[ \theta [2 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta)] - \text{L}_2(2\theta) [2 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta)] + \zeta_3 (2 L_\theta - 3 l_{2\theta}) \right]
\]

\[
+ 4 \sin^2 \theta \left[ 17 \zeta_4 - 16 \text{Li}_4 \left(\frac{1}{2}\right) + 4 \zeta_2 \ln^2 2 - 14 \zeta_2 \ln 2 - \frac{2}{3} \ln^4 2 \right] - \frac{8}{3} \cos^2 \theta L_\theta^4
\]

\[
+ 8 \sin(2\theta) l_{2\theta}^2 \left[ 2 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta) \right] - 8 \sin(2\theta) l_{2\theta} [2 \text{Cl}_3(2\theta) - \text{Cl}_3(4\theta)]
\]

\[
+ \frac{8}{3} \sin(2\theta) [2 \text{Li}_4(2\theta) - \text{Li}_4(4\theta)] \right\} + \mathcal{O}(\varepsilon^2),
\]

where \( l_{m\theta} \) and \( L_\theta \) are defined in (2.16). The analytical results for the next two coefficients are expressible in terms of the \( \text{LsLsc} \)-functions. In the order \( \varepsilon^2 \) the previous combination (2.17) of \( \text{LsLsc} \) functions is included. The divergent parts of this integral were calculated previously in [36]. In the regions of the variable \( M > 2m \) the proper analytical continuation of all
expressions can be constructed. It completely coincides with the previous case $V_{mmmM}$. The result is relatively lengthy and therefore will not be presented here.

For the case of equal masses $M = m$ (the master-integral $D_5$) we get

$$(m^2)^{-1+3\varepsilon}(1 - \varepsilon)(1 - 2\varepsilon)^2D_5 = -\frac{1}{\varepsilon^3} - \frac{2}{3\varepsilon^2} - \frac{2}{\varepsilon}\left\{1 - 6\frac{L_{S_2}(\pi/3)}{\sqrt{3}}\right\}$$

$$-6 + 6\zeta_2\frac{\pi}{\sqrt{3}} + 6\zeta_3 + 18\frac{L_{S_3}(2\pi/3)}{\sqrt{3}} - 12\frac{L_{S_4}(\pi/3)}{\sqrt{3}}\ln 3$$

$$-\varepsilon\left\{18 + 12\zeta_3\frac{\pi}{\sqrt{3}} + 6\zeta_2\frac{\pi}{\sqrt{3}}\ln 3 - \frac{56}{3}\zeta_3 + 26\zeta_3\ln 3 - 63\zeta_4 - 6\frac{L_{S_2}(\pi/3)}{\sqrt{3}}\ln^2 3ight. $$

$$+ 18\frac{L_{S_3}(2\pi/3)}{\sqrt{3}}\ln 3 - 12\pi L_{S_3}\left(\frac{2\pi}{3}\right) + 18L_{S_4}^{(1)}\left(\frac{2\pi}{3}\right) + 8\left[L_{S_2}(\pi/3)\right]^2 - 12\frac{L_{S_4}(2\pi/3)}{\sqrt{3}}\right\}$$

$$+ \varepsilon^2\left\{-54 + 3\zeta_2\frac{\pi}{\sqrt{3}}\ln^2 3 + 12\zeta_3\frac{\pi}{\sqrt{3}}\ln 3 + 56\zeta_3 - 136\zeta_4 - \frac{793}{18}\zeta_2\zeta_3 - \frac{3593}{9}\chi_5ight.$$

$$+ 26\zeta_3\ln^2 3 + \frac{57}{2}\zeta_2\frac{\pi}{\sqrt{3}} - 24\zeta_4\ln 3 - 2L_{S_2}(\pi/3)\ln^3 3 + 9\frac{L_{S_3}(2\pi/3)}{\sqrt{3}}\ln^2 3 - 12\frac{L_{S_4}(2\pi/3)}{\sqrt{3}}\ln 3$$

$$+ 16\left[L_{S_2}(\pi/3)\right]^2\ln 3 + 6\frac{L_{S_3}(2\pi/3)}{\sqrt{3}} + \frac{38}{9}\pi\zeta_2L_{S_2}(\pi/3) - 12L_{S_4}^{(1)}\left(\frac{2\pi}{3}\right) + \frac{3}{2}\chi_5$$

$$- 24\pi L_{S_3}\left(\frac{2\pi}{3}\right)\ln 3 - \frac{46}{27}\pi L_{S_4}\left(\frac{2\pi}{3}\right) + 12\pi L_{S_4}\left(\frac{\pi}{3}\right) + 36L_{S_4}^{(1)}\left(\frac{2\pi}{3}\right)\ln 3 - 48L_{S_{Lsc}}^{1,1,3}\left(\frac{\pi}{3}\right)$$

$$- 32\zeta_2\ln^2 2 + \frac{16}{3}\ln^4 2 + 128\text{Li}_4\left(\frac{1}{2}\right)$$

$$- 64\zeta_2\ln^3 2 + 84\zeta_3\ln^2 2 + \frac{64}{7}\ln^5 2 + 384\left[\ln 2\text{Li}_4\left(\frac{1}{2}\right)\ln 2 + \text{Li}_5\left(\frac{1}{2}\right)\right]$$

$$+ 24\zeta_2\ln^2 2\ln 3 - 42\zeta_3\text{Li}_2\left(\frac{1}{4}\right) - 96\text{Li}_4\left(\frac{1}{2}\right)\ln 3 - 4\ln^4 2\ln 3\right\}.$$

As a non-trivial check of these results we established full agreement between the numerical values for the coefficients of $\varepsilon$-expansion of the $D_5$-integral with the proper Laporta results [17].

4 Conclusion

In this paper, the higher order $\varepsilon$-expansion of the diagrams shown in Fig. (1) have been constructed (2.7), (3.23). The coefficients of the $\varepsilon$-expansion are parametrized in terms of generalized log-sin functions and LsLsc-functions described in Appendix B. At $M = m$ these integrals enter in FORM [38] based packages for calculation of two-loop on-shell self-energy diagrams [3] ($V_{1111}$) and three-loop vacuum integrals [4, 5] ($D_5$). The numerical values of the calculated coefficients (2.19),(2.20) and (3.28) coincide with the results presented in [17].

We shown that the basis of transcendental numbers for single scale diagrams with two- and three-massive cuts contains new elements in addition to the odd/even basis of weight 5 constructed in [14, 9]. Some of these elements are the product of the lowest weight elements
of the "sixth root of unity" basis:

\[
\zeta_2 \times \left\{ \ln 2 \ln 3, \ln 2 \ln^2 3, \text{Li}_3 \left( \frac{1}{4} \right), S_{1,2} \left( \frac{1}{4} \right) \right\}, \quad \text{Li}_2 \left( \frac{1}{4} \right) \times \{ \zeta_2, \zeta_3, \zeta_2 \ln 2, \zeta_2 \ln 3 \}, \\
\text{Li}_4 \left( \frac{1}{4} \right) \times \{ \ln 2, \ln 3 \}, \quad \ln 3 \ln^4 2.
\]

(4.29)

Only one extra term should be added to the weight 5 basis. This new constant can be related to a \( \text{LsLsc}_{1,1,3} \)-function of the argument \( \frac{\pi}{3} \). The results of analytical calculation of three-loop master-integrals, evaluated in [9] and in the present paper are available on

http://theor.jinr.ru/~kalmykov/three-loop/master.uu

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A Auxiliary integrals

Below we present the values of some integrals appearing in Sec. 2.

\[
\int_0^\theta d\phi \frac{\cos \phi}{(\sin \phi)^{3-\varepsilon}} = -\frac{1}{(2-a\varepsilon)} \frac{(\sin \theta)^{a\varepsilon}}{\sin^2 \theta}, \quad (A.1)
\]

\[
\left( 1 - \frac{a\varepsilon}{2} \right) \int_0^\theta d\phi \frac{\cos \phi}{(\sin \phi)^{a\varepsilon}} \frac{(\cos \phi)^{b\varepsilon}}{\sin^3 \phi} = \frac{1}{2} (\sin \theta)^{a\varepsilon} (\cos \theta)^{b\varepsilon} - \frac{b\varepsilon}{2} \text{Isc}_{a,b} (\theta), \quad (A.2)
\]

\[
\left( 1 - \frac{a\varepsilon}{2} \right) \int_0^\theta d\phi \frac{(\sin \phi)^{a\varepsilon} (\cos \phi)^{b\varepsilon}}{\sin^3 \phi \cos \phi} = \left[ 1 - \frac{(a+b)\varepsilon}{2} \right] \text{Isc}_{a,b} (\theta) - \frac{(\sin \theta)^{a\varepsilon} (\cos \theta)^{b\varepsilon}}{2 \sin^2 \theta}, \quad (A.3)
\]

\[
(1-a\varepsilon) \int_0^\theta d\phi \frac{1}{\sin^2 \phi} (\sin \phi)^{a\varepsilon} \text{Ls}_{j+1} (2\phi) = -\frac{\cos \theta}{(\sin \theta)^{1-a\varepsilon}} \text{Ls}_{j+1} (2\theta)
\]

\[
-2^{-a\varepsilon} \sum_{k=0}^{\infty} \frac{(a\varepsilon)^k}{k!} \left[ \frac{\ln^{k+j+1} (2 \sin \theta)}{k+j+1} + a\varepsilon \int_0^\theta d\phi \ln^k (2 \sin \phi) \text{Ls}_{j+1} (2\phi) \right], \quad (A.4)
\]

\[
(1-a\varepsilon) \int_0^\theta d\phi \frac{1}{\sin^2 \phi} (\sin \phi)^{a\varepsilon} \text{Ls}_{j+1} (4\phi) = -\frac{\cos \theta}{(\sin \theta)^{1-a\varepsilon}} \text{Ls}_{j+1} (4\theta)
\]

\[
+2^{-1-a\varepsilon} a\varepsilon \sum_{k=0}^{\infty} \frac{(a\varepsilon)^k}{k!} \text{Ls}_{k+1} (2\theta) \text{Ls}_{j+1} (4\theta)
\]

\[
+2^{1-a\varepsilon} a\varepsilon \sum_{k=0}^{\infty} \frac{(a\varepsilon)^k}{k!} \int_0^\theta d\phi \text{Ls}_{k+1} (2\phi) \left[ \ln (2 \sin \phi) + \ln (2 \cos \phi) \right]^j
\]
\[-2^{2-\varepsilon} \sum_{k=0}^{\infty} \frac{(a\varepsilon)^k}{k!} \int_{\theta}^0 d\phi \cos \phi \sin \phi \ln^k (2 \sin \phi) \ln (2 \sin \phi) + \ln (2 \cos \phi)]^j, \quad (A.5)\]

where \( \text{Isc}_{a,b} (\theta) \) is defined in (2.8).

### B Auxiliary function

Let us describe here some properties of the new function \( \text{LsLsc}_{k,i,j} (\theta) \) defined as

\[
\text{LsLsc}_{k,i,j} (\theta) = \int_{0}^{\theta} d\phi L_{k+1}(\phi) \ln^{i-1} \left| \frac{2 \sin \frac{\phi}{2}}{2} \right| \ln^{j-1} \left| \frac{2 \cos \frac{\phi}{2}}{2} \right|, \quad (B.1)
\]

where \( k, i, j \) are integer numbers, \( k \geq 0 \) and \( i, j \geq 1 \), and \( \theta \) is an arbitrary real number.

These functions have appeared in higher-order epsilon expansion of two- and three-loop Feynman diagrams investigated in this paper (see Eqs.(2.7) and (3.23)). \( \text{LsLsc} \)-functions are also related to the \( \varepsilon \)-expansion of hypergeometric functions. As was shown in [20] the multiple inverse binomial sums \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{x^n}{n^2} S_1(n-1) \) and \( \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \frac{x^n}{n^2} S_1(2n-1) \), where \( S_n(n) \) is the harmonic sum, are expressible in terms of \( \text{LsLsc}_{0,i,j} (\theta) \), \( \text{LsLsc}_{1,i,1} (\theta) \) and \( \text{LsLsc}_{1,1,j} (\theta) \).

Moreover, the elements \( \chi_5 \) and \( \tilde{\chi}_5 \) contributing to odd/even basis of a set of transcendental numbers [9] are related to \( \text{LsLsc}_{1,3,1} \left( \frac{2\pi}{3} \right) \) and \( \text{LsLsc}_{1,3,1} \left( \frac{\pi}{2} \right) \), respectively (see Eq. (15) in [20]). \( \text{LsLsc}_{1,3,1} (\theta) \) also appears in higher order \( \varepsilon \)-expansion of the one-loop triangle diagram (see Eq. (4.5) in [29]). For lower values of the parameters, function (B.1) reduces to the generalized log-sine functions. In particular,

- for \( k = 0 \) and \( i = 1 \) or \( j = 1 \)
  \[
  \text{LsLsc}_{0,i,1} (\theta) = L_{s+i+1}(\theta), \quad (B.2)
  \text{LsLsc}_{0,1,j} (\theta) = L_{s+j+1}(\pi - \theta) - L_{s+j+1}(\pi) - \pi \left[ L_{s+j}(\pi - \theta) - L_{s+j}(\pi) \right]. \quad (B.3)
  \]

- for \( k = 0, i = j = 1 \)
  \[
  \text{LsLsc}_{0,2,2} (\theta) = \frac{1}{8} L_{s+1}(2\theta) - \frac{1}{2} L_{s+1}(\theta) - \frac{1}{2} \left[ L_{s+1}(\pi - \theta) - L_{s+1}(\pi) \right] + \frac{1}{2} \pi \left[ L_{s+3}(\pi - \theta) - L_{s+3}(\pi) \right]. \quad (B.4)
  \]

- for an arbitrary \( k \) and \( i = j = 1 \)
  \[
  \text{LsLsc}_{k,i,1} (\theta) = \theta L_{s+k+1}(\theta) - L_{s+k+2}(\theta). \quad (B.5)
  \]

- for \( k = 1, i = 2 \) and \( j = 1 \)
  \[
  \text{LsLsc}_{1,2,1} (\theta) = -\frac{1}{2} \left[ L_{s+2}(\theta) \right]^2. \quad (B.6)
  \]
For \( k = 1, \ i = 1 \) and \( j = 2 \) we get a function, which was denoted as \( \Phi(\theta) \) in [29] (see Eq.(2.41) in [29]),

\[
L_{sL_{sc}1,1,2}(\theta) = \Phi(\theta) . \tag{B.7}
\]

The relation between the functions of the opposite arguments is

\[
L_{sL_{sc}k,i,j}(-\theta) = L_{sL_{sc}k,i,j}(\theta) , \tag{B.8}
\]

The symmetry relation for the new functions

\[
L_{sL_{sc}k,i,j}(\theta) - L_{sL_{sc}k,i,j}(2\pi - \theta) + L_{sL_{sc}k,i,j}(2\pi) = 2L_{sL_{sc}k+1}(\pi) [L_{sc}_{i,j}(2\pi - \theta) - L_{sc}_{i,j}(2\pi)] \tag{B.9}
\]

includes the generalized log-sine-cosine integral introduced in [9] (see Appendix A.2). Its definition is

\[
L_{sc}_{i,j}(\theta) = -\int_0^\theta d\phi \ln^{i-1}\left|\frac{\sin\phi}{2}\right|\ln^{j-1}\left|\frac{2\cos\phi}{2}\right|. \tag{B.10}
\]

In particular, the following relation is valid:

\[
\int_0^\theta d\phi L_{sc}_{i,j}(\phi) = \theta L_{sc}_{i,j}(\theta) - L_{sL_{sc}0,i,j}(\theta) . \tag{B.11}
\]

At the same time, the relation (B.9) can be considered as definition of the generalized log-sine-cosine functions (\( L_{sL_{sc}k}(\pi) \) is a normalization constant). For \( k = 0 \) there is an extra symmetry relation,

\[
L_{sL_{sc}0,i,j}(\theta) - L_{sL_{sc}0,j,i}(\pi - \theta) + L_{sL_{sc}0,j,i}(\pi) = -\pi [L_{sc}_{j,i}(\pi - \theta) - L_{sc}_{j,i}(\pi)] . \tag{B.12}
\]

When \( i \) or \( j \) are equal to unity, the following relations are valid:

\[
\begin{align*}
L_{sL_{sc}k,1,1}(\theta) + L_{sL_{sc}k-1,k+1,1}(\theta) &= -L_{sL_{sc}k+1}(\theta) L_s(\theta) , \tag{B.13} \\
L_{sL_{sc}k,1,j}(\theta) + L_{sL_{sc}k-1,k+1,j}(\pi - \theta) - L_{sL_{sc}k-1,k+1,j+1}(\pi) &= L_{sL_{sc}k+1}(\theta) L_{sL_{sc}j}(\pi - \theta) . \tag{B.14}
\end{align*}
\]

The values of the function \( L_{sL_{sc}k,i,j}(\theta) \) of the argument \( \theta = 2\pi \) can be extracted from relation (B.9). Using the symmetric properties of the \( L_{sc}_{i,j}(\theta) \)-function,

\[
\begin{align*}
L_{sc}_{i,j}(-\theta) &= -L_{sc}_{i,j}(\theta) , \\
L_{sc}_{i,j}(2\pi - \theta) &= L_{sc}_{i,j}(2\pi) - L_{sc}_{i,j}(\theta) , \\
L_{sc}_{i,j}(2\pi) &= 2L_{sc}_{i,j}(\pi) , \\
L_{sc}_{i,j}(\pi) &= L_{sc}_{j,i}(\pi) , \tag{B.15}
\end{align*}
\]

it is easy to get

\[
L_{sL_{sc}k,i,j}(2\pi) = -2L_{sL_{sc}k+1}(\pi) L_{sc}_{i,j}(\pi) . \tag{B.16}
\]
where the values of $Ls_{k+1}(\pi)$ and $Lsc_{i,j}(\pi)$ can be founded in Lewin’s book [21]. Taking into account that $Ls_2(\pi) = 0$ we get for $k = 1$

$$LsLs_{1,i,j}(2\pi) = 0.$$  \hspace{1cm} (B.17)

Using relations (B.16), Eq. (B.9) can be rewritten as

$$LsLsc_{k,i,j}(\theta) - LsLsc_{k,i,j}(2\pi - \theta) = -2Ls_{k+1}(\pi)[Lsc_{i,j}(\theta) - Lsc_{i,j}(\pi)].$$  \hspace{1cm} (B.18)

From relation (B.12) we get for $\theta = \pi$

$$LsLsc_{0,i,j}(\pi) + LsLsc_{0,j,i}(\pi) = \pi Lsc_{i,j}(\pi).$$  \hspace{1cm} (B.19)

For particular values of the $LsLsc$-functions of weight 5 at $\theta = \pi, \pi/3$ and $\theta = 2\pi/3$ the PSLQ analysis [37] yields

\[
\begin{align*}
LsLsc_{0,3,2}(\pi) &= \frac{2}{5} \zeta_2 \ln^3 2 - \frac{2}{15} \ln^5 2 - \frac{7}{4} \zeta_3 \ln^2 2 - 4 \ln 2 L_{i4} \left( \frac{1}{2} \right) + \frac{155}{12} \zeta_5 - \frac{1}{5} \zeta_2 \zeta_3 - 4 L_{i5} \left( \frac{1}{2} \right) \\
&= 1.13461087559610383910839668272 \ldots,
\end{align*}
\]

\[
\begin{align*}
LsLsc_{1,1,3}(\pi) &= \frac{2}{5} \ln^5 2 - 2 \zeta_2 \ln^3 2 + \frac{2}{15} \ln^3 2 + \frac{1}{3} \ln^2 2 + 12 \ln 2 L_{i4} \left( \frac{1}{2} \right) - \frac{7}{8} \zeta_2 \zeta_3 - \frac{155}{16} \zeta_5 + 12 L_{i5} \left( \frac{1}{2} \right) \\
&= 0.6301340120387385042438188494654 \ldots,
\end{align*}
\]

\[
\begin{align*}
LsLsc_{1,3,1}(\pi) &= -\frac{2}{15} \ln^5 2 + \frac{7}{8} \zeta_2 \ln^3 2 - \frac{1}{4} \zeta_3 \ln^2 2 - 4 \ln 2 L_{i4} \left( \frac{1}{2} \right) + \frac{155}{32} \zeta_5 - 4 L_{i5} \left( \frac{1}{2} \right) \\
&= 0.8874478318089418243337609880871 \ldots,
\end{align*}
\]

\[
\begin{align*}
LsLsc_{1,2,2}(\pi) &= \frac{2}{5} \ln^5 2 - \frac{7}{8} \zeta_2 \ln^3 2 + \frac{1}{3} \zeta_3 \ln^2 2 + 4 \ln 2 L_{i4} \left( \frac{1}{2} \right) - \frac{79}{64} \zeta_5 + 4 L_{i5} \left( \frac{1}{2} \right) \\
&= -0.3851859504113720162670058305844 \ldots,
\end{align*}
\]

\[
\begin{align*}
LsLsc_{1,3,1}(\frac{\pi}{3}) &= \frac{11}{18} \zeta_5 = 0.6336780725876149549802789098028 \ldots,
\end{align*}
\]

\[
\begin{align*}
LsLsc_{1,3,1}(2\frac{2}{3}) &= -\frac{55}{286} \pi \zeta_2 L_{i2} \left( \frac{2}{3} \right) - \frac{1225}{1296} \zeta_2 \zeta_3 + \frac{621}{1296} \zeta_5 + \frac{23}{972} \pi L_{s4} \left( \frac{2}{3} \right) \\
&\quad - \frac{5}{18} \pi L_{s4} \left( \frac{2}{3} \right) - \frac{1}{8} L_{i2} \left( \frac{2}{3} \right) L_{s2} \left( \frac{2}{3} \right) L_{s3} \left( \frac{2}{3} \right) + \frac{1}{3} L_{s5} \left( \frac{2}{3} \right) \left( \frac{2}{3} \right) - \frac{1}{3} \chi_5 \\
&= 0.744148498381945153773320072924 \ldots,
\end{align*}
\]

\[
\begin{align*}
LsLsc_{1,2,2}(\frac{2}{3}) &= -\frac{55}{286} \pi \zeta_2 L_{i2} \left( \frac{2}{3} \right) - \frac{1}{4} L_{i2} \left( \frac{2}{3} \right) L_{s2} \left( \frac{2}{3} \right) L_{s3} \left( \frac{2}{3} \right) - \frac{23}{1296} \pi L_{s4} \left( \frac{2}{3} \right) \\
&\quad - \frac{5}{24} \pi L_{s4} \left( \frac{2}{3} \right) - \frac{1225}{1296} \zeta_2 \zeta_3 - \frac{1379}{576} \zeta_5 - \frac{1}{4} L_{s5} \left( \frac{2}{3} \right) + \frac{1}{6} \chi_5 \\
&= -0.3438441037234077110928896695989 \ldots,
\end{align*}
\]

\[
\begin{align*}
LsLsc_{0,3,2}(\frac{2}{3}) &= -LsLsc_{1,1,3}(\frac{2}{3}) + \frac{55}{216} \pi \zeta_2 L_{i2} \left( \frac{2}{3} \right) + \frac{1}{2} L_{i2} \left( \frac{2}{3} \right) L_{s2} \left( \frac{2}{3} \right) L_{s3} \left( \frac{2}{3} \right) + \frac{233}{864} \zeta_2 \zeta_3 \\
&\quad - \frac{793}{864} \chi_5 - \frac{29}{648} \pi L_{s4} \left( \frac{2}{3} \right) - \frac{1}{8} \pi L_{s4} \left( \frac{2}{3} \right) + \frac{1}{4} L_{s5} \left( \frac{2}{3} \right) + \frac{1}{32} \chi_5 + 4 L_{i5} \left( \frac{1}{2} \right) + 4 \ln 2 L_{i4} \left( \frac{1}{2} \right) - \frac{2}{3} \zeta_2 \ln^3 2 + \frac{7}{4} \zeta_3 \ln^2 2 + \frac{1}{15} \ln^5 2 \\
&= -0.1728468405935728535297398287784 \ldots
\end{align*}
\]
\[
\text{LsLsc}_{0,3,2} \left( \frac{2\pi}{3} \right) = \text{LsLsc}_{1,1,3} \left( \frac{\pi}{3} \right) - \frac{55}{216} \pi \zeta_2 Ls_2 \left( \frac{\pi}{3} \right) - \frac{1}{2} Ls_3 \left( \frac{2\pi}{3} \right) Ls \left( \frac{2\pi}{3} \right) + \frac{1565}{864} \zeta_2 \zeta_3 \\
+ \frac{7183}{864} \zeta_5 + \frac{55}{648} \pi Ls_4 \left( \frac{\pi}{3} \right) - \frac{1}{12} \pi Ls_4 \left( \frac{2\pi}{3} \right) - \frac{1}{2} Ls_5^{(1)} \left( \frac{2\pi}{3} \right) - \frac{31}{32} \chi_5 \\
- 12 Li_5 \left( \frac{1}{2} \right) - 12 \ln 2 Li_4 \left( \frac{1}{2} \right) + 2 \zeta_2 \ln^3 2 - \frac{21}{4} \zeta_3 \ln^2 2 - \frac{3}{8} \ln^5 2 \\
= -0.2193285689662125608977695929575\ldots
\]

\[
\text{LsLsc}_{1,1,3} \left( \frac{2\pi}{3} \right) = -\text{LsLsc}_{1,1,3} \left( \frac{\pi}{3} \right) + \frac{55}{216} \pi \zeta_2 Ls_2 \left( \frac{\pi}{3} \right) + \frac{1}{2} Ls_2 \left( \frac{\pi}{3} \right) Ls_3 \left( \frac{2\pi}{3} \right) - \frac{685}{864} \zeta_2 \zeta_3 \\
- \frac{102335}{864} \zeta_5 - \frac{23}{648} \pi Ls_4 \left( \frac{\pi}{3} \right) - \frac{1}{12} \pi Ls_4 \left( \frac{2\pi}{3} \right) + \frac{1}{2} Ls_5^{(1)} \left( \frac{2\pi}{3} \right) + \frac{1}{2} \chi_5 \\
+ 16 Li_5 \left( \frac{1}{2} \right) + 16 \ln 2 Li_4 \left( \frac{1}{2} \right) - \frac{8}{3} \zeta_2 \ln^3 2 + 7 \zeta_3 \ln^2 2 + \frac{8}{15} \ln^5 2 \\
= 0.4544220738685365169841248301889\ldots
\]  

\begin{equation}
(\text{B.20})
\end{equation}

where the high-precision numerical evaluation of the generalized log-sine functions was performed with the lsjk-program [39] and

\[
\text{LsLsc}_{1,1,3} \left( \frac{\pi}{3} \right) = 0.32403774485559909073259711644693958993\ldots
\]

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