Evading the theoretical no-go theorem for nonsingular bounces in Horndeski/Galileon cosmology

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Received 16 January 2019, revised 29 April 2019
Accepted for publication 29 May 2019
Published 18 June 2019

Abstract
We show that a nonsingular bounce, free of ghosts and gradient instabilities, can be realized in the framework of Horndeski or generalized Galileon cosmology. In particular, we first review that the theoretical \textit{no-go} theorem, which states that the above is impossible, is based on two very strong assumptions, namely that a particular quantity cannot be discontinuous during the bounce, and that there is only one bounce. However, as we show in the present work, the first assumption not only can be violated in a general Horndeski/Galileon scenario, but also it is necessarily violated at the bounce point within the subclass of Horndeski/Galileon gravity in which $K(\phi, X)$ becomes zero at $X = 0$. Additionally, concerning the second assumption, which is crucial in improved versions of the theorem which claim that even if a nonlinear free of pathologies can be realized it will lead to pathologies in the infinite past or infinite future, we show that if needed it can be evaded by considering cyclic cosmology, with an infinite sequence of nonsingular bounces free of pathologies, which forbids the universe to reach the ‘problematic’ regime at infinite past or infinite future. Finally, in order to make the analysis more transparent we provide explicit examples where nonsingular bounces without theoretical pathologies can be achieved.

Keywords: bounce cosmology, Horndeski theory, Galileon theory
1. Introduction

Nonsingular bouncing cosmologies may offer a potential solution to the problem of cosmological singularity [1]. In particular, although inflation is considered to be a crucial part of the history of our universe [2], it is still accompanied by the above problem, since such a big bang singularity is unavoidable if inflation is driven by a scalar field in the framework of general relativity [3]. Hence, alongside the efforts to alleviate the initial singularity through quantum gravity effects, a significant amount of research directs towards its solution through the bounce realization.

Bounce cosmology [4–9] can be realized by various modified gravity constructions [10–12], such as the pre-big-bang [13] and the ekpyrotic [14, 15] scenarios, higher-order gravity [16, 17], $f(R)$ gravity [18–20], $f(T)$ gravity [21], massive gravity [22], braneworld models [23, 24], non-relativistic gravity [25, 26], loop quantum cosmology [27–29], Lagrange modified gravity [30] etc. Alternatively, nonsingular bouncing cosmology may be studied through the application of effective field theory techniques, and the introduction of matter sectors that violate the null energy condition [31–34], or of non-conventional mixing terms [35, 36]. Such constructions can additionally provide an explanation for the scale invariant power spectrum [37, 38] and moderate non-Gaussianities [39, 40]. In summary, bouncing cosmology may be considered as a potential alternative to the big bang one.

A general class of gravitational modification are the so-called galileon theories [41–44], which are a re-discovery of the general scalar-tensor theory constructed by Horndeski under the requirement of maintaining the equations of motion second-ordered [45]. Application of the Horndeski/Galileon theory at a cosmological framework proves to be very interesting and thus it has been investigated in detail in the literature. In particular, one can study the late-time acceleration [46–50], inflation [51–53] and non-Gaussianities [54–56], cosmological perturbations [57–59], or use observational data to extract constraints on various sub-classes of the theory [60–62].

One interesting feature of Horndeski/Galileon theories is that they offer the framework for the realization of bouncing cosmology. In particular, one can obtain bouncing solutions in various sub-classes of the theory, describing both the background evolution as well as the generation of perturbations [63–74]. Despite the success of Horndeski/Galileon theories in generating nonsingular bouncing solutions, there is a discussion on whether these solutions are stable. In particular, in [75–78] the authors presented a theoretical no-go theorem stating that nonsingular models with flat spatial sections suffer in general from gradient instabilities or pathologies in the tensor sector. The proof of this theorem is based on two strong assumptions, namely that a specific non-observable quantity related to the tensor perturbation remains finite at the bounce point, and that there is only one bounce. However, this is not the general case, and indeed one can show that in successful and stable bounces the above assumption(s) are violated. Hence, the above theorem can be evaded and stable nonsingular bounces can be safely realized in the framework of Horndeski/Galileon cosmology. For instance, with the correspondence between the effective field theory (EFT) formalism and Horndeski/generalized Galileon theories made in [79], one may avoid this issue in bounce cosmology by modifying the dispersion relation for cosmological perturbations with the help of certain EFT operators [80–82].

In the following we explicitly show how the theoretical no-go theorem on nonsingular bounces in Horndeski/Galileon cosmology can be evaded. We mention here that there is another no-go theorem from the observational perspective, which indicates that the parameter space for single-field nonsingular bounces is extremely limited due to the severe tension...
between tensor-to-scalar ratio and primordial non-Gaussianity [40, 83] (which in turn needs additional mechanisms to amplify the scalar perturbations [84]). In the present work we refer only to the theoretical no-go theorem, namely our goal is to show that there is not a ‘theoretical no-go theorem’, in the sense of a mathematically proven theorem of general validity, that forbids a non-singular bounce, and not to construct a bounce in perfect agreement with every observational requirement (which would require the thorough incorporation of background (SN1a, BAO, CMB shift parameter, $H_0$ measurements, etc) as well as perturbation ($f\sigma_8$) related data). Hence, even if a nonsingular bounce is difficult to be constructed from the observational point of view, it is not mathematically impossible.

The plan of the manuscript is as follows: in section 2 we review the theoretical no-go theorem, mentioning the assumptions on which it is based. In section 3 we show that the aforementioned theorem is based on two strong assumption which for general sub-classes of the theory can be violated, and thus offering a safe evading of the theorem. Additionally, we provide explicit examples where nonsingular bounces free of ghost and gradient instabilities can be realized in Horndeski/Galileon cosmology. Finally, in section 4 we summarize the obtained results.

2. The theoretical no-go theorem

In this section we review the theoretical no-go theorem which under specific assumptions states that nonsingular bounces in Horndeski/Galileon cosmology exhibit gradient instabilities or pathologies, following [75, 76].

We start by presenting Horndeski, or equivalently the generalized Galileon theory, and its cosmological application. The corresponding action is given by [44]

$$S = \int d^4x\sqrt{-g}\sum_{i=2}^{5}L_i,$$

with

$$L_2 = K(\phi, X),$$

$$L_3 = -G_3(\phi, X)\Box\phi,$$

$$L_4 = G_4(\phi, X)R + G_{4X}[\Box(\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)],$$

$$L_5 = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi)$$

$$-\frac{1}{6}G_{5X}[\Box(\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)$$

$$+2(\nabla_\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)],$$

with $R$ the Ricci scalar and $G_{\mu\nu}$ the Einstein tensor, and where we have set the Planck mass and the gravitational constant to $M_{pl}^2 \equiv 8\pi G = 1$ for simplicity. The functions $K$ and $G_i$ ($i = 3, 4, 5$) depend on the scalar field $\phi$ and its kinetic energy $X = -\partial^\mu\phi\partial_\mu\phi/2$, and moreover $G_{iX} \equiv \partial G_i/\partial X$ and $G_{i,\phi} \equiv \partial G_i/\partial \phi$.

Applying the above theory in a cosmological framework, namely imposing a flat Friedmann–Robertson–Walker (FRW) background geometry with metric
\[ \text{dx}^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \]  
with \( t \) the cosmic time, \( x^i \) the comoving spatial coordinates, and \( a(t) \) is the scale factor, one can extract the Friedmann equations as [44]

\[ \begin{align*}
2XK_X - K + 6\ddot{\phi}HG_{3,X} - 2XG_{3,\phi} - 6H^2G_4 \\
+ 24H^2X(G_{4,X} + XG_{4,XX}) - 12HX\dot{\phi}G_{4,\phi X} \\
- 6H\dot{\phi}G_{4,\phi} + 2H^2X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}) \\
- 6H^2X(3G_{5,\phi} + 2XG_{5,\phi X}) = 0,
\end{align*} \]  

(7)

\[ \begin{align*}
K - 2X(G_{3,\phi} + \ddot{\phi}G_{3,X}) + 2(3H^2 + 2\dot{H})G_4 \\
- 12H^2XG_{4,X} - 4HXG_{4,X} - 8HXG_{4,XX} \\
- 8HXXG_{4,XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} + 4XG_{4,\phi \phi} \\
+ 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X} + 4HX(X - HX)G_{5,\phi X} \\
- 2X(2H^3 + 2H\dot{H} + 3H^2\ddot{\phi})G_{5,X} \\
+ 2[2(HX + H\dot{X}) + 3H^2X]G_{5,\phi} \\
+ 4HX\dot{\phi}G_{5,\phi \phi} - 4H^2X^2\ddot{\phi}G_{5,XX} = 0,
\end{align*} \]  

(8)

with dots denoting derivatives with respect to \( t \), and where \( H \equiv \dot{a}/a \) is the Hubble function. Additionally, variation of (1) with respect to \( \phi(t) \) gives rise to its evolution equation

\[ \frac{1}{a^3} \frac{d}{dt}(a^3 J) = P_\phi, \]  

(9)

where

\[ J \equiv \ddot{\phi}K_X + 6HXG_{3,X} - 2\dot{\phi}G_{3,\phi} - 12HXG_{4,\phi X} \\
+ 6H^2\ddot{\phi}(G_{4,X} + 2XG_{4,XX}) \\
+ 2H^3X(3G_{5,X} + 2XG_{5,XX}) \\
- 6H^2\dot{\phi}(G_{5,\phi} + XG_{5,\phi X}), \]  

(10)

\[ P_\phi \equiv K_{,\phi} - 2X \left( G_{3,\phi \phi} + \ddot{\phi}G_{3,\phi X} \right) + 6(2H^2 + \dot{H})G_{4,\phi} \\
+ 6H(X + 2H\dot{X})G_{4,\phi X} \\
- 6H^2XG_{5,\phi \phi} + 2H^3X\ddot{\phi}G_{5,\phi X}. \]  

(11)

Note that in FRW geometry, \( \phi \) becomes a function of \( t \) only, and thus \( X(t) = \dot{\phi}^2(t)/2 \).

We proceed by examining the linear perturbations around the FRW background [44, 85, 86]. We work in the unitary gauge, i.e. \( \delta \phi = 0 \), and we perturb the spatial part of the metric as \( \gamma_{ij} = a^2(t)e^{2\zeta}(e^{h})_{ij} \), with \( \zeta \) the curvature perturbation and \( h_{ij} \) the tensor perturbation. We mention that the unitary gauge may lead to problems in the case where a particular quantity crosses zero at the bounce point (the \( \gamma \)-crossing of [73]), and thus one needs to apply the Newtonian gauge and show that the gauge variables remain non-singular, as it was done in [73]. However, in our work we use the unitary gauge because this gauge is used in [75, 76] where the no-go theorem was presented. The fact that the ‘proof’ of the no-go theorem may not be valid in the case of the \( \gamma \)-crossing, due to the use of the unitary gauge, could only serve as an additional argument against the mathematically proven universal validity of the no-go theorem.
Inserting these into (1) we extract the quadratic actions for tensor and scalar perturbations respectively as \[ S^{(2)}_h = \frac{1}{8} \int d^3 x a^3 \left[ G_T h_{ij}^2 - \frac{F_T}{a^2} (\partial h_{ij})^2 \right], \] (12)
and
\[ S^{(2)}_\zeta = \int d^3 x a^3 \left[ G_S \zeta^2 - \frac{F_S}{a^2} (\partial \zeta)^2 \right]. \] (13)

The coefficient functions are given by \[ F_T \equiv 2 \left[ G_4 - X \left( \phi G_{5,X} + G_{5,\phi} \right) \right], \] (14)
\[ G_T \equiv 2 \left[ G_4 - 2X G_{4,X} - X \left( H^2 G_{5,X} - G_{5,\phi} \right) \right], \] (15)

and
\[ F_S \equiv \frac{1}{a} \frac{d \xi}{d t} = F_T, \] (16)
\[ G_S \equiv \frac{\Sigma}{G_T^2} G_T^2 + 3 G_T, \] (17)

where
\[ \xi \equiv \frac{a G_T^2}{G_T}. \] (18)

and
\[ \Sigma \equiv X K_X + 2X^2 K_{XX} + 12H \phi X G_{3,X} \\
+ 6H \phi X^2 G_{3,XX} - 2X G_{4,\phi} - 2X^2 G_{3,\phi X} - 6H^2 G_4 \\
+ 6 \left[ H^2 \left( 7X G_{4,X} + 16X^2 G_{4,XX} + 4X^3 G_{4,XXX} \right) \\
- H \phi \left( G_{4,\phi} + 5X G_{4,\phi X} + 2X^2 G_{4,\phi XX} \right) \right] \\
+ 30H^3 \phi X G_{5,X} + 26H^3 \phi X^2 G_{5,XX} \\
- 6H^2 X \left( 6G_{5,\phi} + 9X G_{5,\phi X} + 2X^2 G_{5,\phi XX} \right) \\
+ 4H^3 \phi X^3 G_{5,XXX}. \] (19)
\[ \Theta \equiv - \phi X G_{1,X} + 2HG_{4} - 8HX G_{4,X} \\
- 8HX^2 G_{4,XX} + \phi G_{4,\phi} + 2X \phi G_{4,\phi X} \\
- H^2 \phi \left( 5X G_{5,X} + 2X^2 G_{5,XX} \right) \\
+ 2HX \left( 3G_{5,\phi} + 2 X G_{5,\phi X} \right). \] (20)

In summary, from (12) and (13) we deduce that in order for the theory to be free of ghost and gradient instabilities we must have
\[ F_S > 0; \ G_S > 0; \ F_T > 0; \ G_T > 0. \] (21)
There are two crucial assumptions for the proof of the theoretical no-go theorem [75]. The first is that $\Theta$ in (20) can never cross zero, which implies that $\xi$ in (18) cannot be discontinuous, which finally implies that $F_S$ is a smooth function everywhere. The second (although not clearly stated but definitely used) is that there is only one bounce, namely that the universe is always contracting before the bounce, and always expanding after it. Under these assumptions the proof is the following.

From the definition of $F_S$ in (16) we deduce that the condition for gradient instabilities absence, namely $F_T > 0$, can be rewritten as
\[ \frac{d\xi}{dt} > aF_T > 0, \]  
which after integration from $t_i$ to $t_f$ becomes
\[ \xi_f - \xi_i > \int_{t_i}^{t_f} aF_T dt. \]  
If the universe evolution is not singular one has $a(t) > \text{const} > 0$ for all times. Now, the integral in (23) for $t_f \to \infty$ and $t_i \to -\infty$, can be convergent or not, depending on the asymptotic behavior of $F_T$. In the case where it is non-convergent relation (23) implies $-\xi_f < -\xi_i - \int_{t_i}^{t_f} aF_T dt$, and since the integral is a positive and increasing function of $t_f$ ($F_T > 0$ according to (21)), for sufficiently large $t_f$ the right hand side will become negative. This means that $\xi_f > 0$. On the other hand writing (23) as $-\xi_i > -\xi_f + \int_{t_i}^{t_f} aF_T dt$ we see that for $t_i \to -\infty$ the right hand side will become positive and thus $\xi_i < 0$. Hence, since $\xi_f > 0$ and $\xi_i < 0$ one could deduce that $\xi$ crosses zero. However, according to (18), if $\xi$ is not discontinuous then it can never cross zero for a nonsingular bounce, namely for $a(t) > \text{const} > 0$ (note that $G_2^T > 0$ for every theory that has general relativity as a particular limit, since in general relativity $G_4 = 1$). Hence, in [75] it is concluded that the nonsingular condition $a(t) > \text{const} > 0$ must be relaxed if we desire not to have instabilities (i.e. if $F_T > 0$), and thus $a(t)$ should be zero at a specific time. Finally, the proof is completed by considering the case where the integral in (23) is convergent, which requires $F_T \to 0$ sufficiently fast either in the asymptotic past or future. However, as $F_T \to 0$ the normalization of vacuum quantum fluctuations implies that they diverge (strong-gravity problem), and thus tensor perturbations will asymptotically exhibit pathologies.

In summary, under the assumption that $\Theta$ in (20) can never cross zero, i.e. that $\xi$ in (18) cannot be discontinuous, and that there is only one bounce, in [75] it was shown that the condition for instabilities absence in the tensor sector, namely $F_T > 0$, implies that $a(t)$ should be zero at a specific time, and hence a nonsingular bounce is impossible in the framework of Horndeski/Galileon cosmology. Finally, one can extend the above arguments and proof in the case where there are more degrees of freedom in the scalar perturbations [75], as well as in the case of multi-galileon theory [76].

3. Evading the theoretical no-go theorem

In the previous section we reviewed the theoretical no-go theorem presented in [75], stating that a nonsingular bounce cannot be realized in Horndeski/Galileon cosmology if we desire not to have ghost and gradient instabilities. As we mentioned, the proof is based on two very strong assumptions, namely that $\Theta$ in (20) can never cross zero and hence that $\xi$ in (18) cannot be discontinuous, and that there is only one bounce. However, as we will show in this section,
not only these assumptions can be violated in usual bouncing scenarios, but on the contrary for general sub-classes of the theory it is impossible not to violate them.

The main condition of the bounce realization is that the Hubble function must be zero at the bounce point. Thus, as one can see, the majority of terms in $\Theta$ definition in (20) become zero at a general bounce. Now, observing the first Friedmann equation of Horndeski/Galileon cosmology, namely equation (7), we can see that if the function $K(\phi, X)$ becomes zero at $X = 0$, then the above main bounce condition is realized if $\dot{\phi}$, i.e. $\dot{\phi}$, becomes zero at the bounce point. But $\dot{\phi} = 0$ implies that $\Theta$ in (20) crosses zero at the bounce point, or equivalently $\xi$ in (18) becomes discontinuous. Hence, we conclude that the assumption on which the theoretical no-go theorem is based is always violated in a nonsingular bounce if $K(\phi, 0) = 0$. Note that $K(\phi, 0) = 0$ (which for instance is satisfied in the ‘kinetic’ choices where $K$ is a polynomial of $X$ [64]) is a sufficient condition, not a necessary one, since $\Theta$ can become zero at the bounce point for other suitable choices of $K(\phi, X)$ too. However, the above sub-case ensures the successful evading of the above theoretical no-go theorem.

Let us provide a specific example where the theoretical no-go theorem is evaded as we described, and a nonsingular bounce free from ghost and gradient instabilities can be realized in the framework of Horndeski/Galileon cosmology. We will follow the method presented in [70], in which one inserts the desired scale factor, as well as the ansatizes of some of the involved functions, and reconstructs the rest of them in order to obtain self-consistency. We first consider a specific nonsingular bounce scale factor of the form

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{The evolution of the functions $\Theta(t)$ (upper graph) and $\xi(t)$ (lower graph), for the nonsingular bounce (24) with $a_0 = 0.2$, $B = 10^{-5}$, under the choice $K = X^2$, $G_4 = 1 + X^2$, $G_5 = 0$. All quantities are measured in units where $M_{pl} = 1$, and the vertical line at $t = 0$ is drawn for convenience.}
\end{figure}
\[ a(t) = a_b (1 + B t^2)^{1/3}, \]  
with \( a_b \) the scale factor value at the bounce and \( B \) a positive parameter, i.e. time varies between \(-\infty\) and \( +\infty \) and the bounce is realized at \( t = 0 \). Additionally, we consider a shift-symmetric Horndeski/Galileon model with \( K = X^2, \ G_4 = 1 + X^2, \ G_5 = 0. \)

Thus, inserting these into the Friedmann equations and assuming that \( G_3(\phi, X) = G_3(X) \) one can numerically extract the solution for \( \phi(t) \) and reconstruct the \( G_3(X) \) form that generates the above bounce realization [70]. Finally, knowing the behaviour of all background quantities, we can numerically calculate the perturbation quantities \( F_S, G_S, F_T, G_T \) and examine whether they are positive, i.e. satisfying the conditions for absence of ghost and gradient instabilities (21).

In figure 1 we depict the behavior of \( \Theta(t) \) and \( \xi(t) \) for the nonsingular bounce (24). As we can see, the basic assumption of the theoretical no-go theorem is evaded, namely \( \Theta(t) \) crosses zero at the bounce point, and thus \( \xi(t) \) becomes discontinuous and transits from positive to negative values without crossing zero and being always an increasing function. Additionally, in figure 2 we present the corresponding behavior of the quantities \( F_S \) and \( G_S \) that are related to scalar perturbations, while in figure 3 we show the corresponding behavior of \( F_T \) and \( G_T \) that are related to tensor perturbations. As we observe all of them are positive and thus the conditions (21) for the absence of ghost and gradient instabilities are satisfied.
In summary, with the general justification we presented in the beginning of this section, we showed that a nonsingular bounce free from ghost and gradient instabilities can indeed be realized in the framework of Horndeski/Galileon cosmology, and without loss of generality we verified it with the specific example given above.

We continue the investigation examining some ‘improvements’ of the theoretical no-go theorem that have appeared in the literature. In [76] it was argued that the no-go theorem could also be proved in the case where \( \Theta(t) \) crosses zero, i.e. \( \xi(t) \) becomes discontinuous, at the bounce point, however again under the crucial assumption that this happens only one time (which due to the fact that \( \xi(t) \) must be monotonous according to (22) allows one to deduce that \( \lim_{t \to \pm \infty} \xi = \text{const.} \)). Note that similar arguments under the single-bounce assumption are also made in [87], where \( \Theta \) is denoted by \( \gamma \), and thus the \( \Theta \)-crossing is called \( \gamma \)-crossing (nevertheless even assuming a single bounce these authors do not exclude the evading of the no-go theorem in the case where \( \Theta \) (i.e. \( \gamma \)) and \( G_T \), namely the denominator and numerator in (18), vanish at the same time).

As we mentioned, the assumption of a single bounce remains crucial in the updated versions of the theoretical no-go theorem [75–78] (see also [87–89]), since the proof does admit that the nonsingular bounce itself can indeed be free of any pathologies, however suitably far from the bounce, either in the infinite past or in the infinite future, even if \( F_S, G_S, F_T, G_T \) remain non-negative we will have \( F_T \to 0 \) (or \( F_S \to 0 \)) which leads to pathologies and the onset of strong coupling. Although the principle that in order to study a local bounce

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**Figure 3.** The evolution of the quantities \( F_T(t) \) (upper graph) and \( G_T(t) \) (lower graph) related to tensor perturbations, for the nonsingular bounce (24) with \( a_0 = 0.2, B = 10^{-5} \), under the choice \( K = X^2, G_4 = 1 + X^2, G_5 = 0 \). All quantities are measured in units where \( M_{\text{pl}} = 1 \), and the vertical line at \( t = 0 \) is drawn for convenience.
one should examine the global behavior of the universe is a bit uncomfortable\(^6\) (we mention that for instance in the specific example we presented above the time scale of the evolution depends on the parameter \(B\), and thus choosing it arbitrarily small could push the \(F_T \to 0, F_S \to 0\) regimes arbitrarily far), still the assumption that the universe expands forever before or after the bounce is a very strong one.

Indeed, it is known that many modified gravities may lead to cyclic cosmology [91, 92], namely to an infinite series of bounces and turnarounds, and Horndeski/Galileon theory is one

\(^6\)This issue, namely whether a pathologies-free bounce that may be accompanied by pathologies in the phase far before or far after the bounce is acceptable or not, has led to a debate in the literature [71, 72, 90].
of them [50, 70, 74]. Hence, one can clearly see that in a multiple realization of the pathologies-free nonsingular bounce, which the proof of the theoretical no-go theorem does admit that it can exist, the universe never reaches the regime $\mathcal{F}_T \to 0$ and/or $\mathcal{F}_S \to 0$, since there is not infinite past and infinite future regime before and after any bounce respectively.

**Figure 5.** The evolution of the quantities $\mathcal{F}_T(t)$ (upper graph) and $\mathcal{G}_T(t)$ (lower graph) related to tensor perturbations, for the cyclic scale factor (26) with $a_i = 0.01, A = 10^{-3}$, $\omega = 0.5$, under the choice $K = X + V(\phi), G_3 = X, G_4 = 1 + X^2, G_5 = 0$. All quantities are measured in units where $M_{pl} = 1$, and the vertical line at $t = 0$ is drawn for convenience.
In order to again give a specific example of such a possibility we follow the method of [70] and we impose the nonsingular oscillating scale factor\(^2\)

\[ a(t) = A \sin(\omega t) + a_c, \tag{26} \]

where \(a_c - A > 0\) is the scale factor value at the bounce, with \(A + a_c\) the scale factor value at the turnaround. Note that this is not the most general cyclic scale factor, since its minima and maxima happen at the same values, however it is adequate for the subsequent discussion. We moreover consider \(K = X + V(\phi), G_3 = X, G_4 = 1 + X^2, G_5 = 0\), while we must also include the matter sector in order to be consistent with the whole universe history (the matter sector does not interfere with the discussion on the bounce stability and the no-go theorem). Inserting these into the Friedmann equations we can numerically extract the solution for \(\phi(t)\) and reconstruct the \(V(\phi)\) form that generates the above cyclic scale factor [70]. Finally, knowing the behaviour of all background quantities, we can numerically calculate \(F_S, G_S, F_T, G_T\). In figure 4 we present the evolution of \(F_S\) and \(G_S\) that are related to scalar perturbations, while in figure 5 we show the corresponding behavior of \(F_T\) and \(G_T\) that are related to tensor perturbations. As we observe all of them are positive and thus the conditions (21) for the absence of ghost and gradient instabilities are satisfied. Furthermore, the regimes \(F_T \to 0\) and/or \(F_S \to 0\) are never reached since after any bounce the universe cannot expand forever since it is followed by a turnaround and a next bounce.

Hence, as we showed in detail in this section, a nonsingular bounce free from ghost and gradient instabilities can indeed be realized in the framework of Horndeski/Galileon cosmology. The reason behind the evading of the theoretical no-go theorem is that \(\Theta(t)\) crosses zero at the bounce point, and thus \(\xi(t)\) becomes discontinuous and transits from positive to negative values without crossing zero and being always an increasing function. Additionally, even if one ‘improves’ the no-go theorem by claiming that although a nonsingular bounce free of pathologies can be realized at some point, it will lead to strong-gravity-related pathologies at infinite past or infinite future, this can also be evaded by considering cyclic cosmology, namely an infinite sequence of nonsingular bounces free of pathologies, which forbids the universe to reach the ‘problematic’ regime at infinite past or infinite future. Lastly, note also the interesting possibility that a nonsingular bounce free of pathologies is accompanied by a singular bounce free of pathologies, in which case all the arguments of the theoretical no-go theorem of [75–78, 87, 88] collapse, and the nonsingular bounce free of pathologies can clearly exist.

4. Conclusions

In this work we showed that a nonsingular bounce, free of ghosts and gradient instabilities, can be realized in the framework of Horndeski or generalized Galileon cosmology. This result was known through specific models [64, 65, 68–74], however in this work we proved why the theoretical no-go theorem which states that such a realization is impossible [75, 76] can be evaded.

In particular, we first reviewed that this theoretical no-go theorem is based on two very strong assumptions, namely that a particular quantity, \(\xi\) in (18), cannot be discontinuous, and that there is only one bounce. Concerning the first assumption we showed that not only can be violated in a general Horndeski/Galileon scenario, but that it is necessarily violated at the bounce point in the subclass of Horndeski/Galileon gravity in which \(K(\phi, 0) = 0\) (as for instance in the kinetic choices where \(K\) is a polynomial of \(X\)). In order to make the analysis \(^2\)Cyclic cosmology may exhibit the old entropy problem (although the works of Frampton et al may offer ways to evade it, see e.g. [93]), nevertheless this is a completely different issue from the mathematical ‘no-go theorem’.

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more transparent, and without loss of generality, we provided an explicit example where a nonsingular bounce is realized, with all stability conditions being satisfied. Concerning the second assumption, which is also crucial in improved versions of the theoretical no-go theorem which claim that even if a nonsingular bounce free of pathologies can be realized it will lead to pathologies in the infinite past or infinite future, we showed that it can be evaded by considering cyclic cosmology, with an infinite sequence of nonsingular bounces free of pathologies, which forbids the universe to reach to the ‘problematic’ regime at infinite past or infinite future. In this case we also provided a specific example with the above behavior, with all stability conditions being satisfied eternally.

In conclusion, stable nonsingular bounce realizations are not mathematically impossible in Horndeski/Galileon cosmology, which may serve as an additional advantage for this class of gravitational modification.

Acknowledgments

We are grateful to J Barrow, Y Cai, D Easson, X Gao, T Kobayashi, S Mironov, T Qiu, A Vikman, D G Wang, P Zhang and M Zhu for stimulating discussions. This article is based upon work from COST Action ‘Cosmology and Astrophysics Network for Theoretical Advances and Training Actions’, supported by COST (European Cooperation in Science and Technology). The work of YFC is supported in part by the National Youth Thousand Talents Program of China, by the NSFC (Nos. 11722327, 11653002, 11421303, J1310021), by the CAST Young Elite Scientists Sponsorship (2016QNRC001), and by the Fundamental Research Funds for Central Universities. The work of ENS is partly supported by the International Visiting Professorship at USTC. Part of numerics are operated on the computer cluster LINDA in the particle cosmology group at USTC.

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