On the Poincaré-Lindstedt perturbation method for a Non-Linear Rayleigh Oscillator with periodic damping coefficient

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Abstract. In this paper the Poincaré-Lindstedt perturbation method will be applied to analyse an oscillator with Rayleigh type with Periodic Damping Coefficient. The mathematical model of the oscillator describes flow-induced vibrations in a uniform wind field. The horizontal cylinder supported by springs as a model can be designed vibrate in vertical direction. It will be studied a solution approximation of the oscillator by using Poincaré-Lindstedt perturbation method. The basic idea of the Poincaré-Lindstedt perturbation is that from the simple harmonic oscillator, the period of oscillation depends on the amplitude of the motion. The Lindstedt perturbation expansion allows the frequency to adapt to the nonlinearity by defining the “stretched time variable” The periodic solution of Limit cycle will be studied in this paper.

1. Introduction
The mathematical model of the oscillator describes flow-induced vibrations in a uniform wind field. The aerodynamic instability can be accured vibration with relatively large amplitudes because of rain in vibrations with relatively large amplitudes. This condition can be called as a galloping phenomenon. In the experiment, the criterion can cause galloping phenomenon by considering several assumptions. This assumption usually called Den Hortog’s criteria (see [1-3]). The time periodic damping coefficient in cable stayed bridges model have been studied by Van Horssen, et.al [4] and Waluya [5]. Vibration mechanisms and controls of long-span bridges also have been studied by Fujino and Siringoringo [6]. Torino [7] have been studied Rain-Wind Induced Vibrations in Stay Cables. Many researchers have studied of many type oscillators. El- Naggar and Ismail [8] have studied strongly nonlinear Duffing oscillator analytically. Conservative oscillator with strong odd nonlinearity have been studied by Guo [9] by using iterative homotopy harmonic balancing approach. Chen, [10] have studied an efficient homotopy-based poicare-lindstedt for nonlinear autonomous oscillators. The various perturbation techniques have been studied for analyzing nonlinear oscillators, such as a homotopy harmonic balancing [9], Floquet Theory [11], Multiple-Scales Lindstedt-Poincaré Method [12], approximation of first integral perturbation [13], Averaging perturbation method [14], [15], and many others. An oscillator with Rayleigh type perturbation will be studied in this paper.

In the next section will be discussed mathematical model, the analysis of the solution, and conclusion.

2. Mathematical Model
The mathematical model describing vibration of an oscillator in one degree of freedom will be derived. The equation of vibration describing flow induced vibration in wind field arrive second order
differential equation with Rayleigh type and periodic damping coefficient. The mathematical model equation for non-linear with Rayleigh type can be seen in [5], [7], and [16]. The equation can be given by
\[ \ddot{y} + b e \sin(2\tau) \dot{y} + k \gamma y = e \left( 1 - \frac{1}{3} \dot{y}^2 \right) \dot{y}, \]
where \( b \) is the periodic damping coefficient, \( e > 0 \), and \( k \gamma > 0 \) is the constant of spring. \( \dot{y} \) is velocity. Without loss of generality let \( b = \frac{1}{2}, k \gamma = 1. \)

3. Solution of The Model
In this section will be studied how the Poincar’e- Lindstedt perturbation can be applied to approximate the solution of (1). Furthermore, the Limit Cycle will be shown from the solution of equation (1). Let the solution of (1) can be written in the form of
\[ y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \cdots \]
with
\[ \tau = \omega t, \]
where \( \omega \) is frequency of the response. The frequency \( \omega \) is expanded in power series in the \( \varepsilon \), that is
\[ \omega = 1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \omega_3 \varepsilon^3 + \cdots \]
By changing variable (3), the equation (1) can be rewrite in the form of
\[ \omega^2 y'' + \frac{1}{2} \varepsilon \sin \left( \frac{2\tau}{\omega} \right) \omega y' + y = e \left( 1 - \frac{1}{3} \omega^2 \dot{y}' \right) \omega y' \]
By substituting solution (2) and expansion of \( \omega \) (4) into equation (4), and collected in the power series in the \( \varepsilon \) then will be obtained problems in order \( O(1), O(\varepsilon), O(\varepsilon^2), \cdots \)
The order \( O(1) \) is
\[ y_0'' + y_0 = 0, \]
Solving equation (5) with initial condition \( y_0(0) = A_0, y_0'(0) = 0 \), will be obtained
\[ y_0 = A_0 \cos \tau \]
The problem of order \( O(\varepsilon) \) is
\[ y_1'' + y_1 + 2\omega_1 y_0^2 + \left( 1 - \frac{1}{3} \dot{y}_0^2 + \frac{1}{2} \sin 2\tau \right) y_0' = 0 \]
Substituting solution (6) into equation (7) and remove secular term, that is \( \omega_1 = -\frac{1}{8}, A_0 = 2 \), then will be obtained
\[ y_1'' + y_1 + \frac{1}{2} \cos(3\tau) - \frac{2}{3} \sin(3\tau) = 0. \]
The solution of equation (8) with initial condition \( y_1(0) = A_1, y_1'(0) = 0 \), is
\[ y_1 = \frac{1}{4} \sin(\tau) + \left( A_1 - \frac{1}{16} \right) \cos(\tau) + \frac{1}{16} \cos(3\tau) - \frac{1}{12} \sin(3\tau) \]
Substituting the equation (6) and (9) into problem in order \( O(\varepsilon^2) \), remove the secular term the will be obtained \( \omega_2 = -\frac{7}{128}, A_1 = \frac{19}{96} \). The frequency up to order \( O(\varepsilon^2) \) is
\[ \omega = 1 - \frac{1}{8} \varepsilon - \frac{7}{128} \varepsilon^2. \]
The approximation solution is
\[ y = 2 \cos(\omega t) + e \left( \frac{1}{4} \sin(\omega t) + \frac{13}{96} \cos(\omega t) + \frac{1}{16} \cos(3\omega t) - \frac{1}{12} \sin(3\omega t) \right) + O(\varepsilon^2). \]
Plot of the solution approximation (Limit Cycle) for \( \varepsilon = 0.5 \) can be given in Figure (1). Phase portrait by using Runge-Kutta method for \( \varepsilon = 0.5 \) can be given in Figure (2).
Figure 1. Plot phase portrait (Limit Cycle) for $\epsilon = 0.5$.

From Figure (1) it can be shown that the limit cycle equation (10) with the period $T = 2\pi + \frac{1}{4}\epsilon \pi + \frac{9}{64}\pi \epsilon^2$.

Figure 2. Phase portrait by using Runge-Kutta method for $\epsilon = 0.5$.

From Figure (2) Limit Cycle exist. The trajectory with near zero tend to that Limit Cycle and the trajectory from outside tend to the Limit Cycle.

4. Conclusion

It has been discussed an oscillator model describing wind induced vibration with periodic damping coefficient in Rayleigh type. The approximation solution of the model can be given by $y = 2\cos(\omega t) + \epsilon \left(\frac{1}{4}\sin(\omega t) + \frac{13}{96}\cos(\omega t) + \frac{1}{16}\cos(3\omega t) - \frac{1}{12}\sin(3\omega t)\right) + O(\epsilon^2)$, where $\omega = 1 - \frac{1}{8}\epsilon - \frac{7}{128}\epsilon^2$. This solution is a Limit Cycle.
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