Spinning Particles, Braid Groups and Solitons

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We develop general techniques for computing the fundamental group of the configuration space of $n$ identical particles, possessing a generic internal structure, moving on a manifold $M$. This group generalizes the $n$-string braid group of $M$ which is the relevant object for structureless particles. In particular, we compute these generalized braid groups for particles with an internal spin degree of freedom on an arbitrary $M$. A study of their unitary representations allows us to determine the available spectrum of spin and statistics on $M$ in a certain class of quantum theories. One interesting result is that half-integral spin quantizations are obtained on certain manifolds having an obstruction to an ordinary spin structure. We also compare our results to corresponding ones for topological solitons in $O(d+1)$-invariant nonlinear sigma models in $(d+1)$-dimensions, generalizing recent studies in two spatial dimensions. Finally, we prove that there exists a general scalar quantum theory yielding half-integral spin for particles (or $O(d+1)$ solitons) on a closed, orientable manifold $M$ if and only if $M$ possesses a spin$_c$ structure.

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1. Introduction

Consider a single particle moving on a smooth manifold $M$, $\dim M \geq 2$, and suppose that this particle has an internal structure. Denote by $Y$ the space parametrizing this structure. In the simplest case, the configuration space of this one-particle system is the cartesian product $M \times Y$. However, more generally, the particle’s internal coordinates may be correlated with its spatial position. Here, the configuration space can be the total space $E$ of a fiber bundle with base space $M$ and fiber $Y$. If we have $n$ such particles which are distinguishable (and noncoinciding) the configuration space becomes $E^n - \Delta_p$, where

$$\Delta_p = \{(e_1, \ldots, e_n) \in E^n \mid p(e_i) = p(e_j) \text{ for some } i \neq j\}$$

and $p : E \to M$ is the projection map of the above bundle. The corresponding configuration space for $n$ identical particles is the orbit space $Q^n_E(M) \equiv (E^n - \Delta_p)/S_n$, where the permutation group $S_n$ has the obvious action on $E^n - \Delta_p$. It is this space that will occupy our attention below.

The fundamental group $\pi_1(Q^n_E(M))$ plays an important role in describing various quantizations of the $n$ particle system. More precisely, to every irreducible unitary representation (IUR) $\rho$ of $\pi_1(Q^n_E(M))$ there exists a quantization of this system whose fixed-time state vectors are sections of an (irreducible) $\Phi^N$-bundle $B$ over $Q^n_E(M)$, $N = \dim \rho$. $B$ is equipped with a flat $U(N)$ connection whose holonomy realizes $\rho$. An alternative way of viewing these state vectors $\Psi$ is as multivalued functions from $Q^n_E(M)$ to $\Phi^N$, such that when the argument of $\Psi$ is brought around a loop in the homotopy class $[\ell] \in \pi_1(Q^n_E(M))$ we have $\Psi \to \rho([\ell])\Psi$. One may certainly consider more general quantum theories — for instance, those associated with nonflat complex vector bundles over $Q^n_E(M)$. However, until Section 8 we will restrict ourselves to those described above since many interesting features can already be seen at this level. For example, if the particles are structureless — that is, $Y$ is just a point — then $E = M$ and $\pi_1(Q^n_E(M))$ is the $n$-string braid group $B_n(M)$ of the manifold $M$. Given an IUR $\rho$ of $B_n(M)$, $n \geq 2$, one can determine the statistics of the $n$ identical particles in the corresponding quantum theory by restricting $\rho$ to those elements of $B_n(M)$ representing local permutations of the particles.

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1 Without loss of generality we assume that $E$, and therefore $Q^n_E(M)$, is path-connected. Hence, different choices of basepoint in $Q^n_E(M)$ lead to isomorphic fundamental groups and in what follows we will suppress this choice and simply write $\pi_1(Q^n_E(M))$.

2 For more on the notion of quantization used below, see [1][2] and references therein.
considers only scalar quantum theories (those associated with the one-dimensional IUR’s of $B_n(M)$), then it has been shown that the only allowed statistics for the particles are Bose and Fermi if $\text{dim } M \geq 3$ (both always being possible) [5]. By contrast, for open 2-manifolds (like $\mathbb{R}^2$) one obtains the full range of fractional (or $\theta$-) statistics for any $n \geq 2$ [7]. On the 2-sphere $S^2$ only a finite ($n$-dependent) subset of statistical angles $\theta$ are allowed [8], while on all other closed surfaces only Bose and Fermi statistics ($\theta = 0$ and $\pi$) are available [5]. However, by looking at nonscalar quantum theories (higher-dimensional IUR’s of $B_n(M)$), one can regain all the rational values of $\theta/\pi$ for $n$ identical particles on any closed, orientable 2-manifold [5] [10] [11]. Further, nonscalar quantum theories allow for the well-known parastatistics when $n \geq 3$ (and any $M$), as well as a complex generalization of them for three or more particles in two dimensions [5] [12] [13]. If $M$ is not simply connected, then there may be even more exotic possibilities such as ambistatistics where the superselection rule between bosons and fermions is effectively broken [5]. There is also a fractional version of ambistatistics on nonsimply connected 2-manifolds [10].

The groups $\pi_1(Q_n^E(M)) \equiv B_n^E(M)$, for a generic fiber bundle $E$ over $M$, generalize the braid groups $B_n(M)$. In this paper we will focus on the case where $Y$ parametrizes an internal spin degree of freedom for the particles. If $M$ is an orientable manifold, we choose a fixed orientation for $M$. Then the relevant $E$ is the bundle of oriented, orthonormal $d$-frames over $M$, $d = \text{dim } M$, whose fiber $Y$ is homeomorphic to the special orthogonal group $SO(d)$ [11] [14] [15]. It is the principal bundle associated with the tangent bundle of $M$. For nonorientable spaces, $E$ is the bundle of all $d$-frames over $M$ and $Y$ is the full orthogonal group $O(d)$. In both cases, we denote this bundle by $F(M)$. Just as above, we can determine the statistics of the particles associated with an IUR $\rho$ of $B_n^E(M)$ by restricting to the local permutations. But now we can also ask about their spin. We obtain this information by looking at how those elements of $B_n^E(M)$ which correspond to $2\pi$-rotations of the particles’ frames are represented by $\rho$. Note that there is no reason to expect a spin-statistics relation for these mechanical systems.

In what follows we will completely calculate the groups $B_n^E(M)$, for $M$ an orientable manifold of dimension $d \geq 3$, in terms of $\pi_1(M)$ and information about possible obstructions to a spin structure on $M$. One consequence of our results is that there exist theories where the particles have half-integral spin even though the ambient space $M$ is not a spin manifold. The nonspin manifolds for which this phenomenon occurs do, however, possess

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3 For a review of fractional statistics, including references to earlier literature, see [1].
generalized spin structures. We also comment on the nonorientable case, as well as the situation in two dimensions. Next, we discuss the relationship between the groups $B^E_n(M)$ and the fundamental group of the configuration space of the $O(d+1)$-invariant nonlinear sigma model with space manifold $M$. These models possess topological solitons, and the implications of our results for their spin and statistics are demonstrated. Finally, we return to the phenomenon of half-integral spin for particles and solitons on nonspin manifolds and study it from the point of view of more general quantum theories. The reader may find it useful to read the conclusions of this paper (Section 9) before proceeding to the main text. Although it uses some technical language which is only defined later, it may help to keep the organization and goals of the paper clearer as he or she proceeds.

We close this section with a word concerning references. In many places in the text we use reasonably well-known results or structures from the mathematical literature. Instead of providing a reference at each such occurrence (many of which are marked with italics), we have decided to give a general mathematical bibliography here. The results and definitions that we have utilized from various areas of mathematics can be found (and traced further) using the treatises cited below and references contained therein. For discrete group theory, see [16]; for algebraic topology, see [17] [18]; for bundle theory and characteristic classes, see [18] [19]; for spin structures on manifolds, see [20]. We have attempted to provide additional references at each occurrence in the text of a result which is less familiar.

2. Computing $B^E_n(M)$ - General Remarks

Before specializing to the case of interest, we make some remarks on the computation of $B^E_n(M)$ in general. A useful fact is that the fiber bundle $Y \xrightarrow{i} E \xrightarrow{p} M$ gives rise to a similar structure $Y^n \xrightarrow{i} Q^E_n(M) \xrightarrow{p_n} Q_n(M)$ for all $n \geq 1$. Here $Q_n(M) \equiv M^n - \Delta_{id_M}$ is the configuration space of $n$ identical structureless particles on $M$. The projection map $p_n : Q^E_n(M) \to Q_n(M)$ is given by $p_n([e_1, \ldots, e_n]) = [p(e_1), \ldots, p(e_n)]$, where $[x_1, \ldots, x_n]$ denotes an unordered $n$-tuple of points in the appropriate space. The long exact homotopy sequence of this fiber bundle yields\footnote{Note that $\pi_0(Y)$ does not, in general, possess a natural group structure. Thus, for the last two maps in this sequence we mean exact in the sense of pointed sets.}

$$\ldots \to \pi_2(Q_n(M)) \xrightarrow{\alpha_n} \pi_1(Y)^n \xrightarrow{i} B^E_n(M) \xrightarrow{(p_n)_*} B_n(M) \xrightarrow{\beta_n} \pi_0(Y)^n \to \{*\}.$$  (2.1)
In this sequence, the basepoint \( y_0 \in Y \) determines the basepoints \( i_*(y_0) \in Q_n^E(M) \) and \( p_n(i_*(y_0)) \in Q_n(M) \). Of course when \( n = 1 \), this reduces to the long exact homotopy sequence of the bundle \( Y \hookrightarrow E \twoheadrightarrow M \). The map \( \alpha_n \) in (2.1) is called the connecting homomorphism and there exists a simple relationship between \( \alpha_1 : \pi_2(M) \to \pi_1(Y) \) and \( \alpha_n, n \geq 2 \). More precisely, the inclusion of \( M^n - \Delta_{id_M} \) into \( M^n \) induces a homomorphism\( \mu_n : \pi_2(Q_n(M)) \to \pi_2(M)^n \), and the map \( \alpha_n \) is simply the composition \((\alpha_1)^n \circ \mu_n \).

More compactly, we may exhibit \( B_n^E(M) \) as an extension:

\[
\{e\} \to \pi_1(Y)^n/\text{Im } \alpha_n \xrightarrow{i} B_n^E(M) \xrightarrow{(p_n)_*} \text{Im } (p_n)_* \to \{e\}. \tag{2.2}
\]

The following information suffices to determine \( B_n^E(M) \) from the exact sequence (2.2).

First, we must know the groups \( G \equiv i_*(\pi_1(Y))^n/\text{Im } \alpha_n \) and \( H \equiv \text{Im } (p_n)_* \subset B_n(M) \).

Next, we need a single map \( \phi_s : H \to \text{Aut}(G) \), determined by choosing a transversal function \( s : H \to B_n(M) \) [that is, \((p_n)_* \circ s = id_H\)] and defining \( \phi_s(h), h \in H, \) to be the automorphism of the normal subgroup \( G \) of \( B_n^E(M) \) given by \( \phi_s(h)(g) = s(h)^{-1}gs(h), \ g \in G \). Note that neither \( \phi_s \) nor \( s \) need be homomorphisms, and different choices of \( s \) may lead to distinct maps \( \phi_s \). \( \phi_s \) is called the action of \( H \) on \( G \) associated with \( s \). It is true, however, that for any two transversal functions \( s_1 \) and \( s_2 \) the automorphisms \( \phi_{s_1}(h) \) and \( \phi_{s_2}(h), h \in H, \) differ only by an inner automorphism of \( G \). Thus, there is a unique map \( \phi : H \to \text{Out}(G) \) associated with the extension (2.2), where \( \text{Out}(G) = \text{Aut}(G)/\text{Inn}(G) \) is the outer automorphism group of \( G \). Moreover, \( \phi \) is a homomorphism and is called the coupling\( \sharp \) of (2.2).

If there exists a choice of \( s \) which is a homomorphism, then the extension (2.2) is said to be split and the above information completely determines \( B_n^E(M) \). In this case, \( \phi_s \) is a homomorphism and \( B_n^E(M) \) is isomorphic to the semidirect product of \( G \) by \( H \) corresponding to this action \( \phi_s \). If no such choice for \( s \) exists, then we also need information concerning the relevant obstructions. More precisely, suppose that a certain relation holds between various elements of \( H \). Without loss of generality we can take this relation to be of the form \( w = e \), where \( w \) is a product of elements in \( H \). Now for a given \( s \), the relation \( s(w) = e \) need not hold in \( B_n^E(M) \); all we know is that \( s(w) = g, \) for some \( g \in G \).

\(^5\) Since the codimension of \( \Delta_{id_M} \) in \( M^n \) is equal to the dimension \( d \) of \( M \), \( \mu_n \) is an isomorphism if \( d \geq 4 \) and an epimorphism if \( d = 3 \).

\(^6\) If \( G \) is an abelian group, then \( \text{Out}(G) = \text{Aut}(G) \) and for any choice of \( s, \phi_s \) is a homomorphism equal to \( \phi \).
(The element \( g \) depends on both \( s \) and \( w \).) So suppose \( R \) is a set of defining relations for \( H \), that is, a set of relations from which all others follow. Then in order to determine the extension \( B_n^E(M) \) we need, along with \( G \), \( H \), and a fixed \( \phi_s \), the set \( R(s) \) of relations in \( B_n^E(M) \) obtained by “lifting” those in \( R \) via \( s \). Again, if the extension is split, then \( s \) can be chosen such that \( s(w) = e \) if \( w = e \), and therefore the set \( R(s) \) contains no new information.

Often, the knowledge we possess about \( G \) and \( H \) is in terms of a presentation. We write \( G = < X_G \mid R_G > \) and \( H = < X_H \mid R_H > \) to denote that \( G \) (respectively, \( H \)) is generated by the set \( X_G \) (respectively, \( X_H \)) subject to the defining relations \( R_G \) (respectively, \( R_H \)). Given a transversal function \( s \), the above discussion then allows us to construct a presentation of \( B_n^E(M) \):

\[
B_n^E(M) = < X_G, s(X_H) \mid R_G, R_H(s), T_s >, \tag{2.3}
\]

where

\[
T_s = \{ s(y)^{-1} xs(y) = w^{(s)}_{x,y}, x \in X_G, y \in X_H \}. \tag{2.4}
\]

The object \( w^{(s)}_{x,y} \) is a representation of \( \phi_s(y)(x) \in G \) as a product of elements in \( X_G \cup X_G^{-1} \). The set of relations \( T_s \) provides the action of \( H \) on \( G \) associated with \( s \), while \( R_H(s) \) gives the required information on the possible obstructions to a splitting. This formula will prove very useful for us. Finally, we remark that the obstructions to a splitting homomorphism for (2.2) are related to certain obstructions to constructing a section for the map \( p_n \), that is, a continuous map \( \omega : Q_n(M) \rightarrow Q_n^E(M) \) with \( p_n \circ \omega = id_{Q_n(M)} \). One can try to construct \( \omega \) in stages — first by finding a section \( \omega^{(1)} \) over the 1-skeleton of the \( nd \)-dimensional manifold \( Q_n(M) \), then a section \( \omega^{(2)} \) over the 2-skeleton, etcetera. At each step there may be an obstruction. If \( \omega^{(1)} \) can be found, then the map \( \beta_n : B_n(M) \rightarrow \pi_0(Y)^n \) is trivial (\( Im (p_n)_* = B_n(M) \)). If \( \omega^{(2)} \) exists, there is a splitting homomorphism \( s = \omega^{(2)}(s) \) for (2.2).\(^7\) If \( \omega^{(3)} \) exists, then the homomorphism \( \alpha_n : \pi_2(Q_n(M)) \rightarrow \pi_1(Y)^n \) is also trivial. Moreover, it is straightforward to show that for \( n \geq 2 \) and fixed \( M \), a given partial section \( \omega^{(m)} \) for \( p_n \) exists if and only if the corresponding partial section exists for \( n = 1 \) (that is, for \( p \)).

\(^7\) Even if \( \omega^{(2)} \) does not exist, there may still be a splitting \( s \) for (2.2). However \( s \) will not be natural in the sense that it will have no topological origin.
As an alternative to (2.2), we may use the long exact homotopy sequence of the covering projection \( \chi_n : (E^n - \Delta_p) \to Q^E_n(M) \) to obtain another extension:

\[
\{e\} \to \pi_1(E^n - \Delta_p) \xrightarrow{(\chi_n)_*} B^E_n(M) \xrightarrow{\gamma_n} S_n \to \{e\}. \tag{2.5}
\]

For \( Y \) a point, this sequence has often been used to study \( B_n(M) \). There are two simplifications in (2.3) when \( d = \dim M \geq 3 \). First, one can show in general that the codimension of \( \Delta_p \) in \( E^n \) is \( d \). So when \( d \geq 3 \), this yields \( \pi_1(E^n - \Delta_p) = \pi_1(E)^n \). Moreover, using techniques similar to those developed for the structureless case [3], one can show that the extension (2.3) splits if \( d \geq 3 \). Thus, in this case, \( B^E_n(M) \) can be viewed as a semidirect product of \( \pi_1(E)^n \) by \( S_n \); the group \( S_n \) acts by permuting the \( n \) factors of \( \pi_1(E) \). This semidirect product is known as the \textit{wreath product} and denoted by \( \pi_1(E) \wr S_n \). To go further, one must look at the long exact sequence of \( Y \xrightarrow{i} E \xrightarrow{p} M \) to acquire information about \( \pi_1(E) \). It is simply a matter of taste whether one prefers to use the extension (2.2) or (2.5) to calculate \( B^E_n(M) \). Identical inputs are needed to take advantage of either one.

\section{B^E_n(M) for Spin Manifolds (d \geq 3)}

We now return to the situation where \( M \) is an orientable manifold of dimension \( d \geq 3 \), and \( E = F(M) \) is the frame bundle of \( M \). The fiber \( Y = SO(d) \) of \( F(M) \) is path-connected, so \( H = \text{Im } (p_n)_* = B_n(M) \). Moreover, \( \pi_1(SO(d \geq 3)) = \mathbb{Z}_2 \) and hence (2.2) becomes

\[
\{e\} \to \mathbb{Z}_2^n/\text{Im } \alpha_n \xrightarrow{i} B^F_n(M) \xrightarrow{(p_n)_*} B_n(M) \to \{e\}. \tag{3.1}
\]

We have \( \alpha_n = (\alpha_1)^n \circ \mu_n \) for all \( n \) (see Section 2), where \( \alpha_1 : \pi_2(M) \to \mathbb{Z}_2 \) can be described as follows. First, an element of \( \pi_2(M) \) may be thought of (up to homotopy) as a sequence of loops \( \ell_t \) in \( M \) (based at some point \( m \)), \( 0 \leq t \leq 1 \), with \( \ell_0 = \ell_1 = \) the constant loop. Now consider transporting a frame \( v \) at \( m \) along a given \( \ell_t \). After completing the loop, it will have been rotated by some element \( R_t \in SO(d) \). Since \( R_0 = R_1 \) = the identity, the sequence \( R_t \) defines a loop in \( SO(d) \). The map from \( \pi_2(M) \) to \( \pi_1(SO(d)) \) obtained in this way is the homomorphism \( \alpha_1 \).

As noted earlier, much is known about the groups \( B_n(M) \) appearing in (3.1). Indeed, using the discussion following (2.3) with \( Y \) a point, we can write \( B_n(M) = \pi_1(M) \wr S_n \). (This is not true for \( d = 2 \).) If \( \pi_1(M) = < X \mid R > \), then \( B_n(M) \) is generated by \( n \) copies of \( X \) along with certain elements \( \sigma_i, 1 \leq i \leq n - 1 \). These generators may be considered
to be (homotopy classes of) loops in the structureless particle configuration space \( Q_n(M) \).

Assume the particles are initially at the positions \( m_1, \ldots, m_n \), which can be taken to sit in a \( d \)-disk \( D \subset M \). Then an element \( x^{(i)} \) from the \( i \)th copy \( X^{(i)} \) of \( X \) represents a loop in \( Q_n(M) \) which takes the particle at \( m_i \) around a loop in \( M \) (avoiding all other particles) in the homotopy class \( x \in X \subseteq \pi_1(M) \). The element \( \sigma_i \) represents the local interchange in \( D \) of the particle at \( m_i \) with that at \( m_{i+1} \). The defining relations for \( B_n(M) \) will include \( n \) copies of \( R \), one for each \( X^{(i)} \), as well as

\[
\begin{align*}
x^{(i)} y^{(j)} &= y^{(j)} x^{(i)} & 1 \leq i, j \leq n; \ i \neq j, \\
x^{(i+1)} \sigma_i &= \sigma_i x^{(i)} & 1 \leq i \leq n - 1, \\
x^{(j)} \sigma_i &= \sigma_i x^{(j)} & 1 \leq i < n - 1; \ 1 \leq j \leq n; \ j \neq i, i + 1, \\
\sigma_i^2 &= e & 1 \leq i \leq n - 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \leq i \leq n - 2, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & 1 \leq i, j \leq n - 1; \ |i - j| \geq 2,
\end{align*}
\]

where \( x^{(i)}, y^{(i)} \in X^{(i)} \). A given \( X^{(i)} \) generates a subgroup isomorphic to \( \pi_1(M) \), and the \( \sigma_i \)'s generate an \( S_n \) subgroup. Clearly we have \( B_1(M) = \pi_1(M) \).

To compute the extension in (3.1), we still need to know the connecting homomorphism \( \alpha_n \) as well as how the relations in \( B_n(M) \) above lift to \( B_n^F(M) \). As mentioned previously, there exists a splitting homomorphism for (3.1) if there is a partial section \( \omega^{(2)} \) for \( \pi : F(M) \rightarrow M \). If \( \omega^{(2)} \) can be extended to \( \omega^{(3)} \), then \( \alpha_n \) is trivial. The obstructions to finding a section for \( p \) are related to various characteristic classes \( t_i \in H^i(M; \pi_{i-1}(SO(d))) \), \( i \geq 1 \). The map \( \omega^{(m)} \) exists for \( p \) if and only if \( t_i = 0 \) for all \( 1 \leq i \leq m \). We state a few basic facts about these classes. Since \( SO(d) \) is path-connected, \( t_1 = 0 \) for any \( M \). Next, the element \( t_2 \in H^2(M; \mathbb{Z}_2) \) is the second Stiefel-Whitney class of the tangent bundle \( \tau_M \) of \( M \), usually denoted by \( w_2 \). The class of orientable spaces which have \( w_2 = 0 \) are called spin manifolds. This is because they are precisely the manifolds for which the \( SO(d) \)-bundle \( F(M) \) can be extended to a principal \( Spin(d) \)-bundle over \( M \), where \( Spin(d) \) is the double cover of \( SO(d) \). Hence, spinors can be unambiguously defined on these spaces. Finally, \( t_3 = 0 \) since \( \pi_2(SO(d)) \) is trivial. Thus, we can construct the partial section \( \omega^{(3)} \) for \( p \) if and only if \( M \) is a spin manifold. So for these spaces \( M_{spin} \) we have a split extension

\[
\{e\} \rightarrow \mathbb{Z}_2^n \rightarrow B_n^F(M_{spin}) \xrightarrow{(p_n)_*} B_n(M_{spin}) \rightarrow \{e\}.
\]
All that remains here is to determine the coupling of (3.3). To this end, let us choose as our basepoint in $Q_n^F(M)$ the configuration where the $n$ identical spinning particles are located at the points $m_1, \ldots, m_n \in D \subset M_{\text{spin}}$. We must also pick a specific frame $v_i$ above each $m_i$. If we assume $\pi_1(M_{\text{spin}}) = \langle X \mid R \rangle$, then $B_n^F(M_{\text{spin}})$ in (3.3) is generated by $X^{(i)}, 1 \leq i \leq n$, and $\sigma_j, 1 \leq j \leq n-1, \text{as for } B_n(M_{\text{spin}})$, along with elements $r_k, 1 \leq k \leq n$. Here $r_k$ represents a $2\pi$-rotation of the frame $v_k$ above $m_k$; the $r_k$’s generate the subgroup $G = \pi_1(\mathbb{Z}^n_2)$. The defining relations for $B_n^F(M_{\text{spin}})$ consist of $n$ copies of $R$, the relations in (3.2), as well as

$$
\begin{align*}
  r_i^2 &= e \quad 1 \leq i \leq n, \\
  r_i r_j &= r_j r_i \quad 1 \leq i, j \leq n, \\
  x^{(i)} r_j &= r_j x^{(i)} \quad 1 \leq i, j \leq n, \\
  r_{i+1} \sigma_i &= \sigma_i r_i \quad 1 \leq i \leq n-1, \\
  r_j \sigma_i &= \sigma_i r_j \quad 1 \leq i \leq n-1; 1 \leq j \leq n; j \neq i, i + 1,
\end{align*}
$$

(3.4)

where $x^{(i)} \in X^{(i)}$. The first two sets of equations in (3.4) give relations which hold in $G$. It is not hard to convince oneself that the remaining three sets hold in $B_n^F(M_{\text{spin}})$, and that they provide the action of $H = B_n(M_{\text{spin}})$ on $G$. It is worth mentioning that the $\sigma_i$’s, along with the set $X^{(1)}$ and the element $r_1$, are all that’s needed to generate $B_n^F(M_{\text{spin}})$. However, eliminating the other generators from the above presentation leads to a more cumbersome set of relations. As a special case we have $B_1^F(M_{\text{spin}}) = \pi_1(F(M_{\text{spin}})) = \mathbb{Z}_2 \times \pi_1(M_{\text{spin}})$, and hence for any $n$ we can write $B_n^F(M_{\text{spin}}) = (\mathbb{Z}_2 \times \pi_1(M_{\text{spin}})) \wr S_n$. Our results for $B_n^F(M_{\text{spin}})$ apply, in particular, to closed, orientable 3-manifolds $M^{(3)}$ since they are all spin manifolds. It is interesting to note that if $M_1$ and $M_2$ are two spin manifolds (of dimension three or more) with the same fundamental group, then $B_n^F(M_1) = B_n^F(M_2)$.

Some common examples of spin manifolds are the Euclidean spaces $\mathbb{R}^d$ and the spheres $S^d$, as well as the real projective spaces $\mathbb{R}P^{4m+3}$ and the complex projective spaces $\mathbb{C}P^{2m+1}$. By the final remark of the preceding paragraph we have $B_n^F(\mathbb{R}^d) = B_n^F(S^d) = B_n^F(\mathbb{C}P^{2m+1}),$ for $d \geq 3$ and $m \geq 1$, since these spaces are all simply connected and have dimension at least 3 (recall that dim $\mathbb{C}P^m = 2m$). For these manifolds the set

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8 Moreover, these spaces are parallelizable (that is, $F(M^{(3)}) = SO(3) \times M^{(3)}$) because here $\omega^{(3)}$ is a full section for $p$, and sectioned principal bundles are homeomorphic to a product. Every closed, orientable 2-manifold is a spin manifold as well. However their treatment is somewhat different as we shall see in Section 5.
$X$ is empty and $B_n^F$ is just the wreath product $\mathbb{Z}_2 \wr S_n$ generated by the $\sigma_i$’s and the $r_j$’s. This group has $n \cdot 2^n$ elements. It is easy to check that the one-dimensional unitary representations of $\mathbb{Z}_2 \wr S_n$, $n \geq 2$, are given by the four possible combinations $\sigma_i = \pm 1$, $r_j = \pm 1$ ($1 \leq i \leq n-1$, $1 \leq j \leq n$). (If $n = 1$, then the group is a single $\mathbb{Z}_2$ generated by $r_1$ and there are only two IUR’s, namely, $r_1 = \pm 1$.) In the corresponding scalar quantum theories, the particles obey Bose (respectively, Fermi) statistics if $\sigma_i = 1$ (respectively, $-1$), and they have integral (respectively, half-integral) spin if $r_j = 1$ (respectively, $-1$). Note that there exist theories which violate the usual spin-statistics connection, since we can choose the $\sigma$’s and $r$’s to have opposite signs. Since $\mathbb{Z}_2 \wr S_n$ is nonabelian for $n \geq 2$, there will also be higher dimensional IUR’s in these cases and hence non scalar quantizations.

As an example, consider the case $n = 2$. The group $\mathbb{Z}_2 \wr \mathbb{Z}_2$ is isomorphic to the dihedral group of order 8. Along with the four IUR’s of dimension one described above, it has a two-dimensional IUR determined by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.5)$$

(The $2\pi$-rotation $r_1$ is given by $\sigma_1^{-1}r_2\sigma_1$.) In the associated quantum theory, the two particles obey a type of “half Bose - half Fermi” statistics which has been called ambistatistics. In the past, ambistatistics has been obtained for structureless particles either on nonsimply connected spaces [5][7][10], or on simply connected spaces in the presence of spectators which effectively create noncontractible loops [21][22]. Similar quantizations have also been found for extended objects such as identical geons in quantum gravity [23], topological solitons in certain nonlinear sigma models [24][25], and strings in mechanics [26] and certain gauge theories [22]. In (3.5), however, we have obtained ambistatistics for pointlike particles on a simply connected manifold with no spectators by introducing an internal spin degree of freedom. We also see that these particles possess what may be called ambispin.

For three or more particles we can construct IUR’s leading to parastatistics. These representations, when restricted to the $\sigma_i$’s, yield IUR’s of $S_n$ of dimension two or more. For instance, $\mathbb{Z}_2 \wr S_3$ has two IUR’s of dimension two given by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad r_3 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.6)$$

The matrices for $\sigma_1$ and $\sigma_2$ generate the two-dimensional IUR of $S_3$. So both of the representations in (3.6) give parastatistics for the three particles. If we choose the plus
sign for \( r_3 (= r_2 = r_1) \), then these paraparticles have integral spin. The minus sign yields half-integral spin. There are also four IUR’s of \( \mathbb{Z}_2 \wr S_3 \) of dimension three. They give rise to other types of exotic statistics. Similar results hold for \( n \geq 4 \).

For nonsimply connected spin manifolds \( M_{\text{spin}} \) \((d \geq 3)\), the groups \( B_n^F(M_{\text{spin}}) \) are “larger” than above, and their IUR’s correspondingly more complex. However, \( \mathbb{Z}_2 \wr S_n \) is always a homomorphic image of \( B_n^F(M_{\text{spin}}) \) (just set the generators in each \( X^{(i)} \) equal to \( e \)). As a consequence, every IUR of \( \mathbb{Z}_2 \wr S_n \) can be naturally viewed as an IUR of \( B_n^F(M_{\text{spin}}) \). If we let \( \xi \) denote the homomorphism from \( B_n^F(M_{\text{spin}}) \) onto \( \mathbb{Z}_2 \wr S_n \), then the IUR of \( B_n^F(M_{\text{spin}}) \) associated with the IUR \( \rho \) of \( \mathbb{Z}_2 \wr S_n \) is simply \( \rho \circ \xi \). Hence, the representations described in the simply connected case above, as well as their quantum mechanical interpretations, are still relevant for a general \( M_{\text{spin}} \). There may, of course, be further IUR’s. We conclude this section by noting one implication of the above discussion which will be of interest to us later. Namely, consider \( n \) identical particles moving on an arbitrary spin manifold of dimension three or more. Then, for any \( n \), there exist scalar quantizations of this system where the particles have half-integral spin.

4. \( B_n^F(M) \) for Orientable Nonspin Manifolds \((d \geq 3)\)

We now turn our attention to the situation where \( M \) is an orientable, nonspin manifold with \( d \geq 3 \). The exact sequence (3.1) is still valid here. One major difference from the spin manifold case is that the homomorphism \( \alpha_n \) is no longer trivial in general. We will see that if \( \alpha_n \) is not trivial, then it is onto and thus \( B_n^F(M) = B_n(M) \). But if it is trivial, another difference is that the extension (3.1) will not be split. Therefore we must understand exactly when \( \alpha_n \) is trivial, and if so how the relations in \( B_n(M) \) lift to \( B_n^F(M) \). The performance of this task requires studying in more detail the obstructions to defining a spin structure on \( M \).

As mentioned earlier, an orientable manifold \( M \) is spin if and only if the second Stiefel-Whitney class \( w_2 \in H^2(M; \mathbb{Z}_2) \) is trivial. However, there are two useful alternatives to this cohomological characterization of the obstruction to a spin structure. One involves the first two homotopy groups of \( M \), while the other uses the first two homology groups. To describe the homotopy-theoretic alternative, we exhibit \( \pi_1(F(M)) \) as an extension by using (3.1) with \( n = 1 \):

\[
\{e\} \to \mathbb{Z}_2/\text{Im} \alpha_1 \overset{i^*}{\to} \pi_1(F(M)) \overset{p_\ast}{\to} \pi_1(M) \to \{e\}.
\]
M is said to have a $\pi_1$-obstruction to a spin structure if (4.1) does not split. (Note that this implies that $\alpha_1$ is trivial.) We say that $M$ has a $\pi_2$-obstruction to a spin structure if $\alpha_1$ is nontrivial.

Every orientable, nonspin manifold has one or the other (but by definition not both) of these obstructions. Some well known examples are the complex projective spaces $\mathbb{C}P^{2m}$ and the real projective spaces $\mathbb{R}P^{4m+1}$, $m \geq 1$. The former spaces are simply connected, but possess a $\pi_2$-obstruction to a spin structure ($\pi_2(\mathbb{C}P^{2m}) = \mathbb{Z}$). The latter spaces have trivial $\pi_2$, but possess a $\pi_1$-obstruction (their fundamental group is $\mathbb{Z}_2$). Note that the product of a space with a $\pi_1$-obstruction and a space having a $\pi_2$-obstruction, like $\mathbb{R}P^5 \times \mathbb{C}P^2$, has a $\pi_2$-obstruction. In general, information about these two obstructions allows us to determine $B_n^F(M)$. For example, since (by definition) $\alpha_1$ is an epimorphism for spaces with a $\pi_2$-obstruction to a spin structure, we see that each $\alpha_n$, $n \geq 2$, is also onto for these manifolds. (Recall that $\alpha_n = (\alpha_1)^n \circ \mu_n$, and that $\mu_n$ is onto for $d \geq 3$.) Thus, we have:

If an orientable manifold $M$ has a $\pi_2$-obstruction to a spin structure, then $B_n^F(M) = B_n(M)$ for all $n \geq 1$.

As a consequence, in the quantum theories associated with IUR’s of $B_n^F(M)$ the $n$ identical particles must have integral spin (since the $2\pi$-rotations are homotopically trivial in $Q_n^F(M)$). There will still be, in general, a wide spectrum of statistics for the particles in these quantum theories. For instance, we are assured of at least one IUR yielding Fermi statistics. As an example, consider the spaces $\mathbb{C}P^{2m}$, $m \geq 1$, described above. It is easy to show that $B_n^F(\mathbb{C}P^{2m}) = S_n$ for any $m$. We thus obtain Bose, Fermi and parastatistical quantizations, but the identical particles in each of these theories have integral spin. In Section 8 we will see how half-integral spin may be obtained on $\mathbb{C}P^{2m}$ even though it does not allow ordinary spinors.

What about the case when $M$ has a $\pi_1$-obstruction? Here the map $\alpha_n$ is trivial and we have

$$\{e\} \to \mathbb{Z}_2 \xrightarrow{i} B_n^F(M) \xrightarrow{(p_n)_*} B_n(M) \to \{e\}.$$ (4.2)

In this case, $B_n^F(M)$ has the “same” set of generators as for $M$ a spin manifold. Further, (4.2) has the same coupling as (3.3) so that the relations in (3.4) hold here as well. The extension (4.2), however, does not split. So there are relations in $B_n(M)$ which get modified when lifted to $B_n^F(M)$. From the discussion in the last paragraph of Section 2, it can be deduced that the relations in (3.2) go over without change to $B_n^F(M)$. It is only the $n$
copies $R^{(i)}$ of the set $R$ of relations given in the presentation $\pi_1(M) = \langle X \mid R \rangle$ that must be modified. For each $1 \leq i \leq n$, the new set $\tilde{R}^{(i)}$ of lifted relations can be obtained as follows. First, choose a representative loop $\ell_x$ for each homotopy class $x \in X$. (We assume all loops are based at $m_i \in M$.) This allows us to find a representative for each word in these generators, that is, for each string of products of elements in $X \cup X^{-1}$. In particular, we can construct a representative $\ell_w$ for each word $w$ such that $w = e$ is a relation in $R$. We may then lift $\ell_w$ to a loop $\tilde{\ell}_w$ in $F(M)$ and deform it so that it lies completely in the $SO(d)$ fiber over $m_i$. If this procedure yields a loop which is homotopically trivial in $SO(d)$, then the relation $w = e$ carries over to the set $\tilde{R}^{(i)}$. On the other hand, if this deformation of $\tilde{\ell}_w$ lies in the class of the $2\pi$-rotation, then the new relation $w = r_i$ replaces $w = e$. This gives us $\tilde{R}^{(i)}$. To recap, let $M$ be an orientable manifold ($d \geq 3$) with a $\pi_1$-obstruction to a spin structure. Also let $\pi_1(M) = \langle X \mid R \rangle$. Then $B_n^F(M)$ is generated by the $n$ copies $X^{(i)}$ of the set $X$, along with the $n - 1$ exchanges $\sigma_i$ and the $n$ rotations $r_j$. The defining relations consist of the sets $\tilde{R}^{(i)}$ just described, as well as the relations in $[3,2]$ and $[3,4]$.

As an example, consider the spaces $\mathbb{RP}^{4m+1}$, $m \geq 1$. For any $m$ we have $\pi_1(\mathbb{RP}^{4m+1}) = \langle x \mid x^2 = e \rangle = \mathbb{Z}_2$, and for each $1 \leq i \leq n$ the set $\tilde{R}^{(i)}$ contains the single relation $x^2 = r_i$. So the group $B_1^F(\mathbb{RP}^{4m+1}) = \pi_1(F(\mathbb{RP}^{4m+1}))$ is isomorphic to $\mathbb{Z}_4$, where the $2\pi$-rotation $r$ is the square of the generator $x$. There are four IUR’s of $\mathbb{Z}_4$ given by $x = \pm 1$ and $x = \pm i$. In the quantizations of the one-particle system corresponding to the first two IUR’s, the particle has integral spin ($x^2 = r = 1$). In the remaining two, the particle has half-integral spin ($x^2 = r = -1$). Of course for higher $n$ we have $B_n^F(\mathbb{RP}^{4m+1}) = \mathbb{Z}_4 \wr S_n$, and similar statements hold about the allowed spins for the $n$ identical particles. More generally, one can prove that there exists an IUR of $B_n^F(M)$ such that the corresponding quantum theory yields half-integral spin if and only if $M$ has no $\pi_2$-obstruction. It is curious that half-integral spin is allowed on spaces with a $\pi_1$-obstruction even though they do not admit ordinary spinors. In many cases, this is due to the existence of a generalized spin structure on the manifold called a $\text{spin}_c$ structure. This will be discussed in more detail in Section 8.

An alternative way of describing the possible obstructions to a spin structure for an orientable manifold $M$ utilizes the first two homology groups of $M$. More specifically, consider the following commutative diagram:

$$
\begin{array}{cccccc}
\pi_2(M) & \xrightarrow{\alpha_1} & \mathbb{Z}_2 & \xrightarrow{i} & \pi_1(F(M)) & \xrightarrow{p_*} & \pi_1(M) & \rightarrow & \{e\} \\
\downarrow & & \downarrow{id} & & \downarrow & & \downarrow & & \\
H_2(M) & \xrightarrow{\tau} & \mathbb{Z}_2 & \xrightarrow{i} & H_1(F(M)) & \xrightarrow{p_*} & H_1(M) & \rightarrow & \{e\}. \\
\end{array}
$$

(4.3)
The bottom row is the \textit{Serre exact homology sequence} of the frame bundle of \(M\), and the homomorphism \(\tau\) is known as the \textit{transgression}. Each of the vertical maps is the \textit{Hurewicz homomorphism}. In particular, the second one from the left, namely \(id\), is the identity map. We say that \(M\) has an \(H_1\)-\textit{obstruction} to a spin structure if the extension
\[
\{e\} \to \mathbb{Z}_2/\text{Im } \tau \xrightarrow{i^*} H_1(F(M)) \xrightarrow{p^*} H_1(M) \to \{e\}
\]
does not split. \(M\) is said to possess an \(H_2\)-\textit{obstruction} if \(\tau\) is nontrivial.

Again, every orientable, nonspin manifold has one or the other (but not both) of these obstructions. From the commutativity of \((4.3)\) we see that spaces with a \(\pi_2\)-obstruction necessarily have an \(H_2\)-obstruction. Thus, for each of the spaces \(\mathbb{C}\mathbb{P}^{2m}\) above has an \(H_2\)-obstruction to a spin structure. The spaces \(\mathbb{R}\mathbb{P}^{4m+1}\) discussed earlier have an \(H_1\)-obstruction since \(H_2\) of each of these spaces is trivial. However, there is no general connection between \(\pi_1\)- and \(H_1\)-obstructions. Indeed, there exist orientable manifolds which have a \(\pi_1\)- and an \(H_2\)-obstruction to a spin structure. The simplest example we know of is the 5-dimensional closed, flat Riemmanian manifold constructed in \([27]\). This nonspin manifold, which we denote by \(M_5\), has a nonabelian, torsion-free fundamental group and is \textit{aspherical} (that is, \(\pi_n(M_5)\) is trivial for \(n \geq 2\)). Thus, it must have a \(\pi_1\)-obstruction to a spin structure. However, \(M_5\) can be shown to possess an \(H_2\)-obstruction as well \([28]\).

Generalizing the methods of \([27]\), the author of \([30]\) constructed closed, flat Riemmanian manifolds of any odd dimension \(\geq 5\) which have \(w_2 \neq 0\). All of these spaces can be shown to have a \(\pi_1\)- and an \(H_2\)-obstruction to a spin structure. Note that if a manifold \(M\) has a \(\pi_1\)- and an \(H_2\)-obstruction, then the \(2\pi\)-rotation \(r \in \pi_1(F(M))\) is nontrivial and lies in the \textit{commutator subgroup}. Thus, half-integral spin quantizations for a particle on \(M\) exist, but they are necessarily nonscalar. (Similar results hold for \(n\) particles on \(M\).) In general, there exists a \textit{one-dimensional} IUR of \(B^F_n(M)\) yielding half-integral spin for \(n\) identical particles on a manifold \(M\) if and only if \(M\) has no \(H_2\)-obstruction to a spin structure.

We should point out that the notion of what we call here a \(\pi_2\)-obstruction to a spin structure has been discussed in the past (see, for example, \([31\ 32]\)). The authors of \([32]\) also discuss an obstruction to defining spinors having its origin in the fundamental group of the manifold \(M\) in question. This is similar to our \(\pi_1\)-obstruction. They further claim that if \(\pi_1(M)\) has no elements of even order, then there is no such obstruction. However the examples of \([27]\) and \([30]\) clearly show that noncontractible loops in \(M\) can be the source of an obstruction to a spin structure even if \(\pi_1(M)\) is torsion-free. Finally, as far as we can tell, the \(H_1\)- and \(H_2\)-obstructions defined above have not been discussed previously.

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\[9\] Indeed all closed, flat Riemmanian manifolds are aspherical, and all finite-dimensional aspherical spaces must have torsion-free fundamental groups \([28]\).
5. $B_n^F(M)$ for Orientable Surfaces

In the previous two sections we have calculated the groups $B_n^F(M)$, $n \geq 1$, for an arbitrary orientable manifold of dimension three or more. The computations for an orientable 2-manifold $M^{(2)}$ are more difficult. The additional complications have three main sources. First, for any bundle $E$ over $M^{(2)}$, the codimension of $\Delta_p$ in $E^n$ is only two and hence $\pi_1(E^n - \Delta_p) \neq \pi_1(E)^n$ in general. The low codimension of $\Delta_p$ further implies that the homomorphism $\mu_n : \pi_2(Q_n(M^{(2)})) \to \pi_2(M^{(2)})^n$ is not necessarily onto. Second, unlike in higher dimensions, the extension (2.5) generally does not split. In particular, the exchanges $\sigma_i$ may no longer square to the identity. This is the reason why fractional statistics can occur in two spatial dimensions. Finally, we have that $\pi_1(SO(2)) = \mathbb{Z}\mathbb{Z}$ and not $\mathbb{Z}_2$. This is why fractional spin is allowed in two dimensions (the $2\pi$-rotations $r_i$ need not square to the identity). The following exact sequence replaces (3.1):

$$\{e\} \to \mathbb{Z}^n/\text{Im } \alpha_n \xrightarrow{i} B_n^F(M^{(2)}) \xrightarrow{(p_n)^*} B_n(M^{(2)}) \to \{e\}. \quad (5.1)$$

If the characteristic class $t_2 \in H^2(M^{(2)}; \mathbb{Z})$ of the tangent bundle $\tau_{M^{(2)}}$ vanishes, then $p : F(M^{(2)}) \to M^{(2)}$ has a full section, implying that $\alpha_n$ is trivial and the extension (5.1) splits. (Note that $t_1$ is still zero.) But $t_2$ lives in the second integral cohomology group and so is no longer the second Stiefel-Whitney class; it is the Euler class. Hence, $t_2$ need not vanish even if $M^{(2)}$ is a spin manifold. It is worth mentioning here that if $\pi_1(M^{(2)}) = \langle X | R \rangle$, then, as for $d \geq 3$, $B_n^F(M^{(2)})$ can be generated by $n$ copies of $X$ along with the $\sigma_i$’s and $r_i$’s. The set of defining relations, however, is much more cumbersome.\footnote{When working in two spatial dimensions, it is important to make our orientation conventions for the $\sigma$’s and $r$’s explicit since they may not square to the identity. We always choose the $\sigma$’s to be “clockwise” exchanges, and the $r$’s similarly to be clockwise rotations.}

Things simplify for flat surfaces $M^{(2)}_{flat}$, since they are parallelizable. Examples are the plane $\mathbb{R}^2$, the torus $T^2$, and either of these with an arbitrary number of punctures (like the cylinder which is homeomorphic to $\mathbb{R}^2 - \{0\}$). For these spaces $t_2 = 0$, so that $B_n^F(M^{(2)}_{flat})$ is a semidirect product of $\mathbb{Z}^n$ by $B_n(M^{(2)}_{flat})$. The last four sets of equations in (3.4) hold in $B_n^F(M^{(2)}_{flat})$, with the last three giving the coupling of the above semidirect product. To these relations, one need only add those for $B_n(M^{(2)}_{flat})$ to obtain a complete presentation of $B_n^F$. For example, the last two sets of equations in (3.2) are a full set of defining relations.
for the classical braid group $B_n(\mathbb{R}^2)$, and using this we obtain $B_n^F(\mathbb{R}^2)$ \cite{33}. Its one-dimensional IUR’s are given by $\sigma_i = e^{i\theta}$, $r_j = e^{i\phi}$, for $0 \leq \theta, \phi < 2\pi$ and all $i$ and $j$. Thus we obtain the full spectrum of fractional spin and statistics for $n$ identical particles on $\mathbb{R}^2$. Higher-dimensional IUR’s of $B_n^F(\mathbb{R}^2)$ give rise to other types of exotic spin and statistics. The ordinary braid groups $B_n$ of the cylinder \cite{34} and the torus \cite{35} are also well known, and the above procedure can be used to determine $B_n^F$ for these spaces.

The groups $B_n$ of any closed, orientable surface are known as well (for $S^2$ see \cite{37}, for all higher genus surfaces see \cite{35} \cite{36}). But here, except for the torus, $t_2 \neq 0$ and much more work has to be done to calculate $B_n^F$. For a closed, orientable surface $M_g^{(2)}$ of genus $g$, we have $H^2(M_g^{(2)}; \mathbb{Z}) = \mathbb{Z}$ and we can choose $t_2 = 2(1 - g)$. If $g \geq 1$ then $\pi_2(M_g^{(2)}(1)) = \{e\}$, which has been shown to imply that $\pi_2(Q^F_n(M_g^{(2)})) = \{e\}$ for all $n \geq 1$ \cite{4}. This yields

$$\{e\} \rightarrow \mathbb{Z}^n \stackrel{i}{\rightarrow} B_{n}^F(M_g^{(2)}) \stackrel{(p_n)_*}{\rightarrow} B_{n}(M_g^{(2)}) \rightarrow \{e\}. \tag{5.2}$$

The group $B_{n}(M_g^{(2)}(1))$ is generated by the exchanges $\sigma_i$, $1 \leq i \leq n-1$, along with elements $\rho_l$, $\tau_l$, $1 \leq l \leq g$. The $\rho$’s (respectively, $\tau$’s) correspond to taking the particle at $m_1$ around the $g$ meridianal (respectively, longitudinal) homology cycles in $M_g^{(2)}$ in the manner described in \cite{36}. The defining relations among these generators are given by the last two sets of equations in (3.2) along with

$$\sigma_1 \ldots \sigma_{n-1}^2 \ldots \sigma_1 \tau_g \tau_g^{-1} \ldots \tau_1 (\rho_1^{-1} \tau_1^{-1} \rho_1) \ldots (\rho_g^{-1} \tau_g^{-1} \rho_g) = e, \tag{5.3}$$

and

$$\sigma_i \rho_l = \rho_l \sigma_i \quad 2 \leq i \leq n-1; \ 1 \leq l \leq g,$$

$$\sigma_i \tau_l = \tau_l \sigma_i \quad 2 \leq i \leq n-1; \ 1 \leq l \leq g,$$

$$\sigma_1 \rho_m \sigma_1 \rho_l = \rho_l \sigma_1 \rho_m \sigma_1 \quad m \geq l; \ 1 \leq l, m \leq g,$$

$$\sigma_1 \tau_m \sigma_1^{-1} \tau_l = \tau_l \sigma_1 \tau_m \sigma_1^{-1} \quad m \geq l; \ 1 \leq l, m \leq g,$$

$$\sigma_1 \tau_l \sigma_1 \tau_l = \tau_l \sigma_1 \tau_l \sigma_1 \quad 1 \leq l \leq g,$$

$$\sigma_1 \tau_m \sigma_1^{-1} \rho_l = \rho_l \sigma_1 \tau_m \sigma_1^{-1} \quad m \geq l; \ 1 \leq l, m \leq g,$$

$$\sigma_1 \rho_m \sigma_1 \tau_l = \tau_l \sigma_1 \rho_m \sigma_1 \quad m \geq l; \ 1 \leq l, m \leq g,$$

$$\sigma_1^{-1} \rho_l \sigma_1 \tau_l = \tau_l \sigma_1 \rho_l \sigma_1 \quad 1 \leq l \leq g.$$

\[11\] These local relations clearly hold in the braid group of any other 2-manifold, but there are, most often, numerous others as well.
The relation (5.3) gets modified when lifted to $B^F_n(M^{(2)}_{g \geq 1})$. The new relation reads

$$\sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1 \tau_g \tau_{g-1} \cdots \tau_1 (\rho_1^{-1} \tau_1^{-1} \rho_1) \cdots (\rho_g^{-1} \tau_g^{-1} \rho_g) = \tau_1^2(g-1).$$

(5.5)

By contrast, the relations in (5.4) remain valid when lifted. To complete the presentation of $B^F_n(M^{(2)}_{g \geq 1})$ we add to (5.4) and (5.3) the following:

$$\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \leq i \leq n-2, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & 1 \leq i, j \leq n-1; |i-j| \geq 2, \\
r_i r_j &= r_j r_i & 1 \leq i, j \leq n, \\
r_{i+1} \sigma_i &= \sigma_i r_i & 1 \leq i \leq n-1, \\
r_j \sigma_i &= \sigma_i r_j & 1 \leq i \leq n-1; 1 \leq j \leq n; j \neq i, i+1, \\
\rho_l r_j &= r_j \rho_l & 1 \leq l \leq g; 1 \leq j \leq n, \\
\tau_l r_j &= r_j \tau_l & 1 \leq l \leq g; 1 \leq j \leq n,
\end{align*}$$

(5.6)

which are the relevant portions of (3.2) and (3.4). For $n = 1$, set each $\sigma_i = e$ in (5.3). (To obtain $B^F_n$ for the connected sum $\mathbb{R}^2 \# M^{(2)}_{g \geq 1}$, or equivalently $M^{(2)}_{g \geq 1}$ with a single point removed, simply ignore (5.5) in the presentation of $B^F_n(M^{(2)}_{g \geq 1})$). For a partial treatment of the representation theory of these groups, and the attendant consequences for the fractional spin and statistics of identical particles on these spaces, see [10][11]. These systems will also possess quantizations in which the particles obey fractional ambistatistics and have fractional ambispin.

The case $M^{(2)} = S^2$ (that is, $g = 0$) is a bit different. $F(S^2)$ is homeomorphic to $\mathbb{RP}^3$ and thus $B^F(S^2) = \pi_1(F(S^2)) = \mathbb{Z}_2$. This group is generated by the single $2\pi$-rotation, which here does square to the identity. So fractional spin is not allowed for the particle. For any $n \geq 2$, the following genus zero version of (5.3) holds in $B^F_n(S^2)$ [11]:

$$\sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1 = r_1^{-2}.$$  

(5.7)

As $S^2$ is simply connected, there is no analog of the relations in (5.4). To complete the presentation of $B^F_n(S^2)$, we simply add the first five sets of relations in (5.6). This can be proven using, among other things, the results in [38]. In particular, we note that for $n \geq 3$ we have $\pi_2(Q_n(S^2)) = \{e\}$, while $\pi_2(Q_2(S^2)) = \mathbb{Z}$ and the homomorphism $\alpha_2$ is nontrivial.
For any open orientable surface $M_{\text{open}}^{(2)}$, it is straightforward to show that $\pi_2(M_{\text{open}}^{(2)}) = \{e\}$. Again, this implies that $\pi_2(Q_n(M_{\text{open}}^{(2)})) = \{e\}$ for all $n \geq 1$. Hence, the homomorphisms $\alpha_n$ are trivial for these spaces and there is an exact sequence for $B^n_F(M_{\text{open}}^{(2)})$ analogous to (5.2) for $M_{g \geq 1}^{(2)}$. If $M_{\text{open}}^{(2)}$ is flat, we have already given a characterization of this extension in terms of $B_n(M_{\text{open}}^{(2)})$. All other cases must be handled individually, since there are no simple classification theorems for open 2-manifolds.

6. $B^n_F(M)$ for Nonorientable Spaces

If $M_{\text{NO}}$ is a nonorientable manifold of dimension $d \geq 2$, then the fiber of $F(M_{\text{NO}})$ is the full orthogonal group $O(d)$. For any $d$ we have $\pi_0(O(d)) = \mathbb{Z}_2$, and the map $\beta_n : B_n(M_{\text{NO}}) \to \mathbb{Z}_2^n$ in (2.1) is onto. (The partial section $\omega^{(1)}$ does not exist because $t_1 \in H^1(M_{\text{NO}}; \pi_0(O(d)))$, which is the first Stiefel-Whitney class $w_1$, is nontrivial.) Thus, unlike the orientable case, the group $H = \text{Im}(p_n)_*$ is a proper subgroup of $B_n(M_{\text{NO}})$; it is the subgroup generated by local exchanges of the identical particles, and single particle excursions around orientation preserving loops in $M_{\text{NO}}$. However for $n = 1$ we can always reduce the calculation back to the orientable case by noting that $F(M_{\text{NO}})$ is homeomorphic to $F(\tilde{M})$, where $\tilde{M}$ is the orientable double cover of $M_{\text{NO}}$. Hence, $B^F_1(M_{\text{NO}}) = \pi_1(F(M_{\text{NO}}))$ is isomorphic to $B^F_1(\tilde{M}) = \pi_1(F(\tilde{M}))$. This equality persists to larger values of $n$ if $d \geq 3$, as can be seen from the discussion following (2.5). That is, we have:

If $M_{\text{NO}}$ is a nonorientable manifold of three or more dimensions, and $\tilde{M}$ is its orientable double cover, then $B^n_F(M_{\text{NO}}) = B^n_F(\tilde{M})$ for all $n \geq 1$.

Thus, in this case, we may compute $B^n_F(M_{\text{NO}})$ using $\tilde{M}$ and the methods described in sections 3 and 4. The interpretations of the generators of $B^n_F(\tilde{M})$ given in these sections remain valid in $B^n_F(M_{\text{NO}})$. They now represent particle rotations, exchanges and loops on $M_{\text{NO}}$. It is important to note that the above result is not generally valid for $d = 2$. For example, it can be shown that $B^F_2(\mathbb{R}P^2)$ is not isomorphic to $B^F_2(S^2)$.

For any closed, nonorientable surface $M_{\text{NO}}^{(2)}$ it has been shown that $\pi_2(Q_n(M_{\text{NO}}^{(2)})) = \{e\}$ (for $\mathbb{R}P^2$ see [38], for all higher genuses see [4]). Thus, (2.2) becomes (compare to (5.2))

$$\{e\} \to \mathbb{Z}^n \xrightarrow{i_*} B^n_F(M_{\text{NO}}^{(2)}) \xrightarrow{(p_n)_*} H \to \{e\},$$

(6.1)
where $H$ is the subgroup of $B_n(M_{NO}^{(2)})$ described above. The ordinary braid groups $B_n$ of closed, nonorientable surfaces are all known (for $\mathbb{RP}^2$ see [39], for higher genuses see [40]), and in principle the subgroup $H$ can be determined from these presentations. (For example, it can be shown from [39] that $H \subset B_2(\mathbb{RP}^2)$ is isomorphic to $\mathbb{Z}_4$.) However, in all but the simplest cases this is a formidable task. If $H$ can be determined the next step is to compute the extension in (6.1), which then can be used to discuss the spectrum of spin and statistics for identical particles on $M_{NO}^{(2)}$. We leave the completion of this program for future investigations.

7. Relations to the $O(d+1)$-Invariant Sigma Model in $(d+1)$-Dimensions

There is a class of nonlinear field theories whose topological properties are intimately related to those of the above quantum mechanical systems. These are the $O(d+1)$-invariant sigma models in $(d+1)$-dimensions. At any fixed time, such a system is described (classically) by a map from the space manifold $M$ (which we assume to be closed and orientable) to the target space $S^d$, $d = \dim M$. Thus, the configuration space is the set $X(M) \equiv \text{Map}(M, S^d)$ of all such maps (with the compact-open topology). This model has topological solitons labelled by the degree $n$ of the map from $M$ to $S^d$ ($\pi_0(X(M)) = \mathbb{Z}$). Thus, we may write $X(M)$ as $\bigcup_{n=-\infty}^{\infty} X_n(M)$, where the component $X_n(M)$ contains only the maps of degree $n$. A degree $n$ soliton configuration $\phi_n : M \to S^d$ can be constructed by choosing a $d$-disk $D$ in $M$ and letting $\phi_n$ be constant everywhere except in the interior of $D$. The disk $D$ with its boundary identified to a point can be thought of as a $d$-sphere and $\phi_n$ is chosen to map this sphere $n$ times around the target $S^d$. All other maps in $X_n$ are homotopic to this configuration. In particular, $\phi_n$ is homotopic to a map consisting of isolated solitons of degree one which resembles the $n$ identical particle configurations on $M$ encountered in the quantum mechanical examples discussed earlier. Indeed, for each $n \geq 1$ there is a continuous map $f : Q_n^F(M) \to X_n(M)$ obtained by choosing $n$ nonoverlapping disks centered at the positions of the framed particles in each configuration $q \in Q_n^F(M)$, and then inserting a soliton of degree one on each of them. The radius of these disks (in a fixed metric on $M$) can be taken as, say, one-fourth of the distance between the two closest particles in $q$. The frame on a given particle is used to determine the orientation of the corresponding soliton. Since $f$ induces a homomorphism $f_* : \pi_1(Q_n^F(M)) \to \pi_1(X_n(M))$, each of the elements of the braid group $B_n^F(M) = \pi_1(Q_n^F(M))$ has a counterpart in $\pi_1(X_n(M))$. 18
We now compute the groups $\pi_1(X_n(M))$, whose representations provide information about the spectrum of spin and statistics for the above solitons.

We begin by reviewing some known results \cite{14} \cite{15} \cite{41}. Corresponding to the exchanges $\sigma_i$, the $2\pi$-rotations $r_j$ and the one-particle loops $x^{(k)}$ in $B_n^F(M)$, there will be the analogous operations $f_*(\sigma_i)$, $f_*(r_j)$ and $f_*(x^{(k)})$ for the $n$ degree one solitons. These operations can be shown to generate $\pi_1(X_n(M))$. However, due to the possibility of creation and annihilation of soliton-antisoliton pairs, the $f_*(\sigma_i)$'s are all homotopic to each other. We denote their common homotopy class by $\sigma$. A similar result holds for the $f_*(r_j)$'s and we denote the associated class by $r$. Moreover, there is a topological spin-statistics relation which states that $\sigma = r$. Finally, for $d \geq 3$ it can be shown that any two loops $f_*(x^{(k)})$ and $f_*(y^{(l)})$ commute, even if $k = l$. We may thus obtain a partial presentation of $\pi_1(X_n(M))$ by taking the image under $f_*$ of the generators and relations for $B_n^F(M)$, and adding the new relations\footnote{For $n \geq 2$, the relations in (7.2) are a consequence of those in $B_n^F(M)$ and (7.1). For $n = 1$ (7.1) is vacuous, while (7.2) in general provides new relations.}

\begin{align*}
    f_*(\sigma_i) &= f_*(r_j) \equiv r & 1 \leq i \leq n - 1, \ 1 \leq j \leq n \quad (7.1) \\
    f_*(x^{(k)}) f_*(y^{(k)}) &= f_*(y^{(k)}) f_*(x^{(k)}) \quad 1 \leq k \leq n, \ d \geq 3. \quad (7.2)
\end{align*}

Although there could, in principle, be additional relations that are needed, we will demonstrate that this is not the case; the above procedure defines a complete presentation of $\pi_1(X_n(M))$. In other words, if we denote by $G_n(M)$ the group obtained from $B_n^F(M)$ by adding the relations in (7.1) and (7.2), then $\pi_1(X_n(M)) = G_n(M)$.

Before going further, we wish to point out that although the map $f$ above is only defined for $n$ positive, it is still possible to speak of the spin and statistics (as well as other properties) of the degree one solitons in $X_n(M)$ for \textit{any} $n \in \mathbb{Z}$ — even $X_0(M)$ which has no analog in our mechanical systems. Degree one soliton rotation and loop operations in an arbitrary $X_n(M)$ can be described (for example) by starting with a configuration where the field is constant everywhere except on a disk $D$ in $M$. Next, create a degree one soliton anti-soliton pair in the vacuum outside of $D$, and then perform the appropriate operation on the soliton before finally annihilating the pair. To define the soliton exchange operation we must create \textit{two} degree one soliton anti-soliton pairs, exchange the two solitons, and then annihilate. The group $\pi_1(X_n(M))$ for any $n$ is generated by (the homotopy classes of)
these processes. (Indeed, for \( n \geq 1 \) this is the same set of generators as that described above using \( f_* \).) We also note that there is a homomorphism \( \psi_{n,m} : \pi_1(X_n(M)) \to \pi_1(X_m(M)) \), for any \( n, m \in \mathbb{Z} \), which sends each such degree one process in \( X_n(M) \) to the corresponding one in \( X_m(M) \). This shows, among other things, that the spin-statistics connection holds in every \( X_n(M) \).

The starting point for our demonstration of the equality \( \pi_1(X_n(M)) = G_n(M) \) is the exact sequence \([42]\)

\[
H^{d-2}(M; \mathbb{Z}) \xrightarrow{\theta} H^d(M; \pi_{d+1}(S^d)) \xrightarrow{\lambda} \pi_1(X_n(M)) \xrightarrow{p} H^{d-1}(M; \mathbb{Z}) \to \{e\},
\]

which may be obtained using a Postnikov decomposition of the target sphere \( S^d \). For \( d = 2 \) we have \( \pi_3(S^2) = \mathbb{Z} \) and the map \( \theta \) in \((7.3)\) is given by \( \theta(x) = y^{2|n|} \), where \( x \in H^0(M; \mathbb{Z}) = \mathbb{Z} \) and \( y \in H^2(M; \mathbb{Z}) = \mathbb{Z} \) are suitably chosen generators. For \( d \geq 3 \) we have \( \pi_{d+1}(S^d) = \mathbb{Z}_2 \) and \( \theta \) is equal to the composite \( \text{Sq}^2 \circ \rho \), where \( \rho : H^{d-2}(M; \mathbb{Z}) \to H^{d-2}(M; \mathbb{Z}_2) \) is mod 2 reduction and \( \text{Sq}^2 : H^{d-2}(M; \mathbb{Z}_2) \to H^d(M; \mathbb{Z}_2) \) is the Steenrod square operation. Since \( M \) is closed and orientable we may write (for any \( d \geq 2 \)) \( H^d(M; \pi_{d+1}(S^d)) = \pi_{d+1}(S^d) \), and by Poincaré duality \( H^{d-1}(M; \mathbb{Z}) = H_1(M) \). Thus, \((7.3)\) becomes

\[
\{e\} \to \pi_{d+1}(S^d)/\text{Im} \theta \xrightarrow{\lambda} \pi_1(X_n(M)) \xrightarrow{p} H_1(M) \to \{e\}.
\]

The image under \( \lambda \) of the generator of \( \pi_{d+1}(S^d) \) is the 2\( \pi \)-rotation \( r \). Under the homomorphism \( p \), the operation of taking a degree one soliton around a loop \( \ell \) in \( M \) goes to the homology class of \( \ell \).

We now assume \( d \geq 3 \). Then, using the techniques described (for example) in \([14]\), it can be shown that each of the homomorphisms \( \psi_{n,m} : \pi_1(X_n(M)) \to \pi_1(X_m(M)) \) above is an isomorphism. (The isomorphism class of \( G_n(M) \) is similarly \( n \)-independent and we will simply write \( \pi_1(X(M)) \) and \( G(M) \) in what follows.) The exact sequence \((7.4)\) becomes

\[
\{e\} \to \mathbb{Z}_2/\text{Im} \theta \xrightarrow{\lambda} \pi_1(X(M)) \xrightarrow{p} H_1(M) \to \{e\},
\]

where \( \theta = \text{Sq}^2 \circ \rho \). It is also known that \( \text{Sq}^2(x) \) is equal to the cup product \( x \cup U_2 \), where \( x \in H^{d-2}(M; \mathbb{Z}_2) \) and the Wu class \( U_2 \in H^2(M; \mathbb{Z}_2) \) is equal to \( w_2 + w_1 \cup w_1 \). Here \( w_1 \) and \( w_2 \) are the first and second Stiefel-Whitney classes of \( \tau_M \) respectively. Since \( M \) is orientable we have \( w_1 = 0 \). Hence \( \text{Sq}^2(x) = x \cup w_2 \). We will now show that:

The homomorphism \( \theta \) is trivial if and only if \( M \) has no \( H_2 \)-obstruction to a spin structure.
To prove this we use Lemma 1 of [45] which implies that \( \theta \) is trivial if and only if \( w_2 \) is the mod 2 reduction of an integral class \( z_2 \in H^2(M; \mathbb{Z}) \) which has finite order. Thus, all we need to show is that there exists such an element \( z_2 \) if and only if \( M \) has no \( H_2 \)-obstruction.

We first tackle the “only if” implication. Manifolds with no \( H_2 \)-obstruction fall into two classes. First the spin manifolds which can be handled easily since they have \( w_2 = 0 \). (Simply take \( z_2 = 0 \).) The remaining spaces are those with an \( H_1 \)-obstruction. If we write by the Universal Coefficient Theorem

\[
\{ e \} \to \text{Ext}(H_1(M), \mathbb{Z}_2) \xrightarrow{\alpha} H^2(M; \mathbb{Z}_2) \xrightarrow{\beta} \text{Hom}(H_2(M), \mathbb{Z}_2) \to \{ e \}, \tag{7.6}
\]

then the Stiefel-Whitney class \( w_2 \) for such a space lives completely in the Ext subgroup. As such, it can be seen as the mod 2 reduction of an element \( z_2 \in \text{Ext}(H_1(M), \mathbb{Z}) \subseteq H^2(M; \mathbb{Z}) \) which clearly has finite order. Finally, to deal with the “if” implication, we must show that if \( M \) does possess an \( H_2 \)-obstruction, then no such element \( z_2 \) of finite order exists. This follows from the fact that \( \beta(w_2) \in \text{Hom}(H_2(M), \mathbb{Z}_2) \) is the transgression \( \tau \) (see Section 4), which is nontrivial for these spaces. So if \( w_2 \) is the reduction of an integral class \( z_2 \), then \( z_2 \) must map nontrivially into \( \text{Hom}(H_2(M), \mathbb{Z}) \) which is torsion-free. This completes the proof.

As a result of the above theorem, we see that for manifolds having no \( H_2 \)-obstruction to a spin structure we have

\[
\{ e \} \to \mathbb{Z}_2 \xrightarrow{\lambda} \pi_1(X(M)) \xrightarrow{\rho} H_1(M) \to \{ e \}. \tag{7.7}
\]

We may write down a similar sequence for the group \( G(M) \) in this case. That is, it is straightforward to show (using the results of Sections 3 and 4) that \( G(M)/\mathbb{Z}_2 = H_1(M) \), where the normal subgroup \( \mathbb{Z}_2 \) is generated by the \( 2\pi \)-rotation \( r \). Since adjoining any additional (nonredundant) relations to \( G(M) \) would ruin this property, we must have \( \pi_1(X(M)) = G(M) \) here. For spin manifolds \( M_{\text{spin}} \), one can use the results of Section 3 to further demonstrate that \( \pi_1(X(M_{\text{spin}})) = G(M_{\text{spin}}) = \mathbb{Z}_2 \times H_1(M_{\text{spin}}) \). By contrast, the extension \( \{ 7.7 \} \) does not split if \( M \) has an \( H_1 \)-obstruction. However \( \pi_1(X(M)) = G(M) \) is still abelian here — for instance, \( \pi_1(X(\mathbb{R}P^{4m+1})) = \mathbb{Z}_4 \) for \( m \geq 1 \). In the case of manifolds with an \( H_2 \)-obstruction, the map \( \theta \) is onto. Hence the rotation \( r \) is trivial and from \( \{ 7.5 \} \) we have \( \pi_1(X(M)) = H_1(M) \) in this situation. Using the results of Section 4 we see that \( G(M) = H_1(M) \) as well. We have thus shown that \( \pi_1(X(M)) = G(M) \) for all closed, orientable manifolds of three or more dimensions. Note that unlike the case of
identical particles on $M$, there are only scalar quantizations of the above nonlinear sigma models (for $d \geq 3$) since $\pi_1(X(M))$ is always abelian. However, as in the particle case, there exists a one-dimensional IUR of $\pi_1(X(M))$ yielding half-integral spin for the degree one solitons if and only if $M$ has no $H_2$-obstruction.

For a closed, orientable surface $M^{(2)}$ of genus $g$, the extension (7.4) becomes

$$\{e\} \to \mathbb{Z}_{2|n|} \xrightarrow{\lambda} \pi_1(X_n(M^{(2)})) \xrightarrow{\rho} H_1(M^{(2)}) \to \{e\}. \quad (7.8)$$

Two major differences from the situation in higher dimensions are: (1) $\pi_1(X_n(M^{(2)}))$ depends on $n$ (actually, only on $|n|$); and (2) these groups are nonabelian for $g \geq 1$. Clearly we have $\pi_1(X_n(S^2)) = \mathbb{Z}_{2|n|}$. The extension (7.8) for the groups $\pi_1(X_n(M^{(2)}_{g \geq 1}))$ has been computed in [42]. They are generated by degree one soliton loops $\rho_l$ and $\tau_l$, $1 \leq l \leq g$ (analogous to those defined in Section 5) along with the degree one soliton rotation $r$. The defining relations are $r^{2n} = e$ along with

$$\begin{align*}
\rho_l r &= r \rho_l \quad 1 \leq l \leq g, \\
\tau_l r &= r \tau_l \quad 1 \leq l \leq g, \\
\rho_l \rho_k &= \rho_k \rho_l \quad 1 \leq l, k \leq g, \\
\tau_l \tau_k &= \tau_k \tau_l \quad 1 \leq l, k \leq g, \\
\rho_l \tau_k &= \tau_k \rho_l \quad 1 \leq l \neq k \leq g, \\
\rho_l \tau_l &= r^2 \tau_l \rho_l \quad 1 \leq l \leq g.
\end{align*}$$

(7.9)

Using the results of Section 5 it is easy to prove that $\pi_1(X_n(M^{(2)})) = G_n(M^{(2)}_{g \geq 1})$ for $g \geq 0$ and $n \geq 1$ (see also [10][15]). Thus, the equality of $\pi_1(X_n(M))$ and $G_n(M)$ holds for all closed, orientable manifolds of dimension two or more.

The above results can be extended to an (infinite volume) open manifold $\hat{M} = \mathbb{R}^d \# M$ of dimension $d \geq 2$, whose one-point compactification is the closed, orientable manifold $M$. Here the fields of the $O(d+1)$-invariant sigma model must go to a constant at spatial infinity in order to have finite energy. Thus, the space manifold $\hat{M}$ is effectively compactified to $M$ and the appropriate configuration space is the set $X^*(M) \equiv \text{Map}_s(M, S^d)$ of all basepoint preserving maps from $M$ to $S^d$. That is, for every $\phi_s \in X^*(M)$ we have $\phi_s(m_0) = s_0$, where $m_0$ is the “point at infinity” on $M$ and $s_0$ is a fixed element of $S^d$. The model still possesses topological solitons ($\pi_0(X^*(M)) = \mathbb{Z}$) and we may again compute the group $\pi_1(X^*_n(M))$ for the degree $n$ sector $X^*_n(M)$, $n \in \mathbb{Z}$. There is a simple fibration relating
\[ X^*(M) \text{ and the space } X(M) \text{ considered previously. More precisely, for any } n \text{ there is a fibering } \mu : X_n(M) \to S^d \text{ given by } \mu(\phi) = \phi(m_0) \text{ for all } \phi \in X(M). \text{ The fiber above a given point } s_0 \in S^d \text{ is just the space } X_n^*(M). \text{ The long exact homotopy sequence of } \mu \text{ yields}
\[
\cdots \to \pi_2(S^d) \to \pi_1(X_n^*(M)) \to \pi_1(X_n(M)) \to \pi_1(S^d) \to \cdots.
\]
(7.10)

If \( d \geq 3 \), we see that \( \pi_1(X_n^*(M)) = \pi_1(X_n(M)) \) and hence the results of this section apply. (Note that \( B_n^F(M) = B_n^F(M) \) in this case as well.) If \( d = 2 \), all we know from (7.10) is that \( \pi_1(X_n(M_{g}^{(2)})) \) is a homomorphic image of \( \pi_1(X_n^*(M_{g}^{(2)})) \). These latter groups have been calculated in [10]; all one has to do is remove the relation \( \iota^{2n} = e \) from the above presentation of \( \pi_1(X_n(M_{g}^{(2)})) \). In other words \( \pi_1(X_n^*(S^2)) = \mathbb{Z} \), while the groups \( \pi_1(X_n^*(M_{g_{\geq 2}}^{(2)})) \) are generated by \( r, \rho_1 \) and \( \tau_1 \) subject only to the relations in (7.9). Note that unlike the situation for \( X_n(M_{g_{\geq 2}}^{(2)}) \), these groups are independent of \( n \) and (like the case of particles on \( \hat{M}_{g_{\geq 2}}^{(2)} \)) the full range of fractional spin and statistics can be obtained at the level of scalar quantizations. Finally, it can be shown that \( \pi_1(X^*(M_{g}^{(2)})) = G(M_{g}^{(2)}) \) for all \( g \geq 0 \) [13].

We conclude this section with a brief discussion of the \( O(d+1) \)-invariant sigma model with a closed, nonorientable space manifold \( M_{NO} \) of dimension \( d \). One major difference from the orientable case is that \( \pi_0(X(M_{NO})) = \mathbb{Z}_2 \). Thus there exist solitons, but they can be continuously deformed into their antisolitons. (More precisely, solitons are turned into antisolitons by bringing them around an orientation reversing loop in \( M_{NO} \).) Another way of saying this is that the degree of a map from \( M_{NO} \) to \( S^d \) is only defined mod 2. Hence, we need only consider \( X_0 \) and \( X_1 \). The exact sequence (7.3) is still valid for \( M_{NO} \) (where \( n \) is to be taken mod 2), only now \( H^d(M_{NO}; \mathbb{Z}) = \mathbb{Z}_2 \) for all \( d \) and we cannot use Poincaré duality on \( H^{d-1}(M_{NO}; \mathbb{Z}) \). In place of (7.4) we therefore have
\[
\{e\} \to \mathbb{Z}_2/\text{Im } \theta \xrightarrow{\lambda} \pi_1(X_n(M_{NO})) \xrightarrow{p} H^{d-1}(M_{NO}; \mathbb{Z}) \to \{e\},
\]
(7.11)
where the map \( \theta \) is still \( \text{Sq}^2 \circ \rho \) for \( d \geq 3 \), but is now trivial for \( d = 2 \). The spin-statistics connection remains valid, and the image under \( \lambda \) of the generator of \( \mathbb{Z}_2 \) is the homotopy class of the \( 2\pi \)-rotation of a degree one soliton. It has also been shown that \( \pi_1(X_n(M_{NO})) \) is abelian (and independent of \( n \)) for all \( d \), and further that (7.11) splits for \( d = 2 \) [12]. Note that there are two major differences from the situation for identical particles on \( M_{NO} \) (see Section 6). First, for any closed, nonorientable surface \( M_{NO}^{(2)} \) the above \( 2\pi \)-rotation squares to the identity in \( \pi_1(X_n(M_{NO}^{(2)})) = \mathbb{Z}_2 \times H^1(M_{NO}^{(2)}) \), while the
analogous operation for particles has infinite order (see (6.1)). Second, if $d \geq 3$ we have seen that $B^F_n(M_{NO}) = B^F_n(\tilde{M})$, where $\tilde{M}$ is the orientable double cover of $M$. However, in general, the groups $\pi_1(X_n(M_{NO}))$ and $\pi_1(X_n(\tilde{M}))$ are different for any $d$ — for example, $\pi_1(X_n(S^1 \times \mathbb{R}P^2)) \neq \pi_1(X_n(S^1 \times S^2))$. We leave details concerning the spin and statistics of $O(d+1)$ solitons on nonorientable manifolds for future work.

8. Half-Integral Spin on Nonspin Manifolds - A General Treatment

The purpose of this section is to obtain a better understanding of why we have encountered scalar quantum theories possessing half-integral spin objects (in both mechanics and field theory) on manifolds having an $H_1$-obstruction to a spin structure. We will also see how to get similar results on certain spaces having an $H_2$-obstruction by considering more general quantum theories than those based on IUR’s of the fundamental group of the relevant configuration space $Q$. Although we restrict ourselves to scalar quantizations, we will consider state vectors that are sections of an arbitrary complex line bundle $L$ over $Q$. Such bundles can be classified by the elements of $H^2(Q; \mathbb{Z})$. By the Universal Coefficient Theorem this group can be written as the direct product of the free part of $H_2(Q)$ and the torsion part of $H_1(Q)$. The flat bundles are associated with these latter torsion elements. In [17], a procedure is given which (for $d \geq 3$) allows us to determine the spin of the above particle-like objects in the quantum theory associated with a bundle $L$. First, we choose a configuration $q_0 \in Q$ which contains such an object and construct the map $g : SO(d) \rightarrow Q$ given by $g(R) = Rq_0$. Here $R$ is a rotation in $SO(d)$ and $Rq_0$ denotes the configuration in which the particle has been appropriately rotated. The map $g$ induces a homomorphism $g^* : H^2(Q; \mathbb{Z}) \rightarrow H^2(SO(d); \mathbb{Z})$. If the line bundle $L$ corresponds to the element $z \in H^2(Q; \mathbb{Z})$, then the particle has half-integral spin in the corresponding quantum theory if and only if $g(z)$ is the nontrivial element of $H^2(SO(d); \mathbb{Z}) = \mathbb{Z}_2$. There is always at least one integral spin quantization since we may choose $L$ to be the trivial line bundle which corresponds to the identity element of $H^2(Q; \mathbb{Z})$. If the particle-like objects live on a manifold $M$ (which for the remainder of this section is assumed to be closed, orientable and of dimension $d \geq 3$), then one may have guessed that there exist half-integral spin quantizations if and only if $M$ is a spin manifold. However in previous

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13 One source of this difference is the fact that $H^{d-1}(M_{NO}; \mathbb{Z}) \neq H_1(\tilde{M})$ in general.
14 Flat line bundles associated with distinct one-dimensional representations of $\pi_1(Q)$ may correspond to the same element of $H^2(Q; \mathbb{Z})$. 

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sections we have seen that this is not the case. More precisely, such quantizations indeed exist if $M$ is a spin manifold, but they arise in certain other cases as well. A study of these situations leads us to the following conjecture:

*In the above mechanical and field-theoretic models, there exist half-integral spin scalar quantizations if and only if the underlying space manifold $M$ possesses a spin$_c$ structure.*

A manifold $M$ has a spin$_c$ structure if the principal $SO(d)$-bundle $F(M)$ can be extended to a principal $Spin_c(d)$-bundle over $M$, where $Spin_c(d) = Spin(d) \times U(1)/\mathbb{Z}_2$. (The $\mathbb{Z}_2$ action sends $(S,u) \in Spin(d) \times U(1)$ to $(-S,-u)$.) Such a structure exists on $M$ if and only if the second Stiefel-Whitney class $w_2 \in H^2(M;\mathbb{Z}_2)$ is the mod 2 reduction of an element of $H^2(M;\mathbb{Z})$. This is equivalent to the statement $b(w_2) = 0$, where $b: H^2(M;\mathbb{Z}_2) \to H^3(M;\mathbb{Z})$ is the Bockstein homomorphism. Clearly all spin manifolds have a spin$_c$ structure. Moreover, by the discussion following (7.6), all manifolds with an $H_1$-obstruction to a spin structure possess spin$_c$ structures. This is also true of some, but not all, manifolds having an $H_2$-obstruction. For example the spaces $\mathbb{C}P^{2m}$, $m \geq 1$, are spin$_c$ manifolds with an $H_2$-obstruction. The same is true of the examples in [27] and [30] (see [29]). On the other hand, consider the five-dimensional coset space $K = SU(3)/SO(3)$. We have $\pi_1(K) = \{e\}$, $\pi_2(K) = \mathbb{Z}_2$ and $H^2(K;\mathbb{Z}_2) = \mathbb{Z}_2$. The generator of this last group can be shown to be $w_2$. Thus, $K$ has a $\pi_2$- (and hence an $H_2$-) obstruction to a spin structure. However because $H^2(K;\mathbb{Z})$ is trivial, $K$ is not a spin$_c$ manifold [20]. Since it is known that all closed, orientable manifolds of dimension less than five are spin$_c$, this example is in some sense minimal. There are also examples of closed, flat Riemmanian manifolds (in any even dimension $\geq 6$) which have a $\pi_1$- and an $H_2$-obstruction to a spin structure and are not spin$_c$ [29].

We will first prove the above conjecture for the simple case of a single particle on $M$. Here $Q = F(M)$ and the map $g: SO(d) \to F(M)$ is just the inclusion map of an $SO(d)$ fiber into $F(M)$. The induced map $g^*$ can be determined from the *Serre exact cohomology sequence* for the frame bundle:

$$\{e\} \to H^2(M;\mathbb{Z}) \xrightarrow{p^*} H^2(F(M);\mathbb{Z}) \xrightarrow{g^*} H^2(SO(d);\mathbb{Z}) \xrightarrow{\beta} H^3(M;\mathbb{Z}).$$

(8.1)

It can be shown that $\beta$ maps the generator of $H^2(SO(d);\mathbb{Z}) = \mathbb{Z}_2$ to $b(w_2) \in H^3(M;\mathbb{Z})$. Thus, $g^*$ is onto (showing the existence of a half-integral spin scalar quantum theory) if
and only if $M$ possesses a spin$_c$ structure. This result can be extended to the case of $n$ distinguishable particles on $M$, $n \geq 1$. The configuration space is $\hat{Q}_n^F(M) \equiv F^n(M) - \Delta_p$, and to find the spin of the $i$th particle we use the map $g_i : SO(d) \to \hat{Q}_n^F(M)$ which rotates this particle in a given configuration. We may also relate this situation to the one-particle case through the commutative diagram

$$SO(d) \xrightarrow{g_i} \hat{Q}_n^F(M) \xrightarrow{\hat{h}_i} \hat{h}_i \downarrow \quad g \downarrow F(M),$$

(8.2)

where the map $\hat{h}_i$ simply reads off the coordinates and frame of the $i$th particle. (8.2) yields the following diagram of cohomology groups:

$$H^2(SO(d); \mathbb{Z}) \xleftarrow{g_i^*} H^2(\hat{Q}_n^F(M); \mathbb{Z}) \xrightarrow{\hat{h}_i^*} H^2(F(M); \mathbb{Z}).$$

(8.3)

The commutativity of (8.3) tells us that $g_i^*$ is onto if $M$ is spin$_c$ (since $g^*$ is onto in this case). That is, for each $i$, there exists at least one scalar quantum theory in which particle $i$ has half-integral spin. Conversely, if $M$ is not a spin$_c$ manifold then $g^*$ is trivial. However this does not imply that $g_i^*$ is trivial, since it is still possible that $g_i^*$ is onto and $\text{Im} \; \hat{h}_i^* \subseteq \text{Ker} \; g_i^*$. To show that this does not occur, we will need to compute $H^2(\hat{Q}_n^F(M); \mathbb{Z})$. Without loss of generality we may assume that $d \geq 5$, since all closed manifolds of dimension less than five are spin$_c$. Since $\Delta_p$ has codimension $d$ in $F(M)^n$, the inclusion of $\hat{Q}_n^F(M)$ into $F(M)^n$ induces an isomorphism on $H^2$ in this case. So we can write $H^2(\hat{Q}_n^F(M); \mathbb{Z}) = H^2(F(M)^n; \mathbb{Z}) = H^2(F(M); \mathbb{Z})^n \times A$, where $A$ is free abelian and we have used the Kunneth formula to obtain the last equality. (This result holds whether or not $M$ is spin$_c$.) The map $\hat{h}_i^*$ in (8.3) is then an isomorphism onto the $i$th factor of $H^2(F(M); \mathbb{Z})$ in $H^2(\hat{Q}_n^F(M); \mathbb{Z})$, and the commutativity of the diagram implies that $g_i^*$ is trivial for all $i \geq 1$. Therefore, there are no half-integral spin scalar quantizations for any particle. This completes the proof of our conjecture for $n$ distinguishable particles on $M$.

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15 Note that in order to construct half-integral spin scalar quantum theories on spin$_c$ manifolds having an $H_2$-obstruction to an ordinary spin structure, such as $\mathbb{C}P^{2m}$, we must use nonflat bundles.

16 This is even true for $d = 4$. For $d = 3$, the induced map on $H^2$ is an epimorphism.
At this point it would be natural to treat the case of \( n \) identical particles on \( M \). However, we find it easier to first consider identical \( O(d+1) \) solitons on \( M \) and then return to the particle situation. We therefore wish to consider the groups \( H^2(X_n(M); \mathbb{Z}) \) for \( M \) a closed, orientable manifold of dimension \( d \geq 3 \). We will first treat the case \( n = 1 \), and then use this result to deal with all other values of \( n \). We can relate the \( n = 1 \) case to our previous results for a single particle by using the following commutative diagram:

\[
\begin{array}{ccc}
SO(d) & \xrightarrow{g} & F(M) \\
\downarrow & & \downarrow f \\
\downarrow j & & X_1(M).
\end{array}
\]  

(8.4)

Here \( j \) is the rotation map for a configuration containing a single degree one soliton, and \( f \) is the map of the previous section. This yields the following diagram of cohomology groups:

\[
\begin{array}{ccc}
H^2(SO(d); \mathbb{Z}) & \xleftarrow{g^*} & H^2(F(M); \mathbb{Z}) \\
\downarrow & & \downarrow f^* \\
\downarrow j^* & & H^2(X_1(M); \mathbb{Z}).
\end{array}
\]  

(8.5)

Without any explicit information about \( f^* \), the commutativity of (8.5) along with our results for the single particle case already show that there are no half-integral spin scalar quantizations for degree one solitons in \( X_1(M) \) when \( M \) is not \( \text{spin}_c \). That is, in this case \( g^* \) — and hence \( j^* = g^* \circ f^* \) — is trivial. To show that such quantizations exist (or equivalently, \( j^* \) is onto) if \( M \) is \( \text{spin}_c \), we must compute \( H^2(X_1(M); \mathbb{Z}) \). The torsion part of this group can be computed using the results of the previous section on \( \pi_1(X_1(M)) \). The torsion-free part can be found using standard techniques in the theory of localization of homotopy types (in particular rational homotopy theory) \[48\]. Putting these results together, it is straightforward to show that \( H^2(X_1(M); \mathbb{Z}) \) has a direct factor isomorphic to \( H^2(F(M); \mathbb{Z}) \). The map \( f^* \) in (8.5) then sends this factor onto the target \( H^2(F(M); \mathbb{Z}) \). The commutativity of (8.4), along with the single particle result, then implies that \( j^* \) is onto if \( M \) is \( \text{spin}_c \). This takes care of the \( n = 1 \) case. For an arbitrary \( n \) consider a configuration in \( X_n(M) \) which contains (among other things) an isolated degree one soliton, and let \( j : SO(d) \to X_n(M) \) be the rotation map for this soliton. We must show that \( j^* : H^2(X_n(M); \mathbb{Z}) \to H^2(SO(d); \mathbb{Z}) \) is onto if and only if \( M \) is \( \text{spin}_c \). One can demonstrate (by an extension of the arguments used to show the \( n \)-independence of \( \pi_1(X_n(M)) \)) that for
any integers \( n \) and \( m \) there is an isomorphism \( \Psi_{n,m} : H^2(X_n(M); \mathbb{Z}) \to H^2(X_m(M); \mathbb{Z}) \) which makes

\[
\begin{align*}
H^2(SO(d); \mathbb{Z}) & \xleftarrow{j^*} H^2(X_m(M); \mathbb{Z}) \\
\downarrow & \quad \uparrow \Psi_{n,m} \\
H^2(X_n(M); \mathbb{Z}) & \xleftarrow{j^*}
\end{align*}
\]

(commute). By choosing \( m = 1 \) and using our results for \( X_1(M) \), we see that for any \( n \in \mathbb{Z} \) the map \( j^* \) is onto if and only if \( M \) is spin\(_c\). Thus, we have proven the conjecture for degree one solitons on \( M \) in any \( X_n(M) \).\(^{[17]}\) Of course the same result holds for solitons of any odd degree, while even degree solitons necessarily have integral spin.

Finally, we return to the case of \( n \) identical particles on \( M \). We will use the commutative diagram \((n \geq 1)\)

\[
\begin{array}{cccc}
g_i & \xrightarrow{\hat{Q}_n^F(M)} & \chi_n \\
\uparrow & \downarrow & \downarrow \\
SO(d) & \xrightarrow{\hat{g}_i} & Q_n^F(M) & \xrightarrow{f} X_n(M),
\end{array}
\]

where \( \hat{g}_i \) (respectively, \( j_i \)) is the rotation map of the particle (respectively, degree one soliton) in the \( i \)th position in an \( n \) identical particle (respectively, soliton) configuration, and \( \chi_n \) is the covering projection defined in Section 2. Again, we have the diagram of cohomology groups

\[
\begin{align*}
\begin{array}{cccc}
g_i^* & \xleftarrow{\hat{g}_i^*} & H^2(\hat{Q}_n^F(M); \mathbb{Z}) & \uparrow \chi_n^* \\
\downarrow & \downarrow & \uparrow f^* \\
H^2(SO(d); \mathbb{Z}) & \xleftarrow{j_i^*} & H^2(Q_n^F(M); \mathbb{Z}) & \xrightarrow{H^2(X_n(M); \mathbb{Z})}
\end{array}
\end{align*}
\]

From the commutativity of the upper triangle in \((8.8)\) and the result for distinguishable particles, we see that there are no half-integral spin scalar quantizations for the identical particles if \( M \) is not a spin\(_c\) manifold. From the commutativity of the lower triangle and

\(^{[17]}\) We also note that the spin-statistics relation still holds in the above general scalar quantizations of the \( O(d + 1) \)-invariant sigma model \([47]\).
the result for solitons we have that there exist half-integral spin scalar quantizations if $M$ is a spin$_c$ manifold. This completes the proof of the full conjecture.

To summarize, we have considered the existence of half-integral spin scalar quantum theories on a closed, orientable manifold $M$ of dimension $d \geq 3$. We have shown that such quantizations exist for particles (identical or not) or $O(d + 1)$ solitons if and only if $M$ is a spin$_c$ manifold. Recall that if $M$ has a $\pi_1$- and an $H_2$-obstruction to a spin structure, then we have shown (see Section 4) that there exist half-integral spin nonscalar quantizations for $n$ identical particles on $M$ even if $M$ is not spin$_c$ (as in the examples constructed in [29]). Presumably these quantizations utilize structures on $M$ more general than spin$_c$ in order to obtain half-integral spin.

9. Comments and Conclusions

The purpose of this paper has been fourfold. First, to define a generalization of the braid groups $B_n(M)$ which apply to identical particles on $M$ having an internal structure described by a fiber bundle $Y \hookrightarrow E \to M$, and to develop general techniques for the computation of these new groups $B^n_E(M)$.[18] Second, to apply these techniques to the case of identical particles possessing an internal spin degree of freedom — that is, $E$ is the frame bundle $F(M)$ of the manifold $M$. Also, to construct the IUR’s of the spin braid groups $B^n_F(M)$ in order to discuss the spectrum of spin and statistics for the particles. Third, to relate the groups $B^n_F(M)$ to the fundamental group of the configuration space of the $O(d + 1)$-invariant nonlinear sigma model with space manifold $M$, $d = \dim M$. These models contain topological solitons and the above relationship sheds light on their spin and statistics. Finally, to discuss necessary and sufficient conditions on $M$ for the existence of a (general) scalar quantum theory which yields half-integral spin for either particles or solitons.

Our main results may be stated as follows. In Section 2 we displayed $B^n_E(M)$ as an extension of a quotient group of $\pi_1(Y)^n$ by a subgroup of $B_n(M)$. We also discussed[19] The result for $O(d + 1)$ solitons can be extended to include manifolds of the form $\mathbb{R}^d \# M$, with $M$ closed and orientable, while that for distinguishable particles is actually true for any orientable manifold. Although we have no explicit proof, we believe that the result for identical particles can also be extended to any orientable manifold.

[18] For applications to particles carrying an internal $U(1)$ charge in the presence of flux tubes, see [49].
the information about the bundle $E$ needed to compute this extension, as well as some simplifications when $d \geq 3$. In Sections 3 and 4 we were able to completely determine the groups $B^F_n(M)$ for an arbitrary orientable manifold $M$ of dimension three or more in terms of $\pi_1(M)$ and information about the possible obstructions to a spin structure on $M$. More precisely, we showed that $B^F_n(M) = (\mathbb{Z}_2 \times \pi_1(M)) \rtimes S_n$ if $M$ is a spin manifold (Section 3), and $B^F_n(M) = B_n(M) = \pi_1(M) \rtimes S_n$ if $M$ has a $\pi_2$-obstruction to a spin structure (Section 4). If $M$ has a $\pi_1$-obstruction then $B^F_n(M)$ is an extension of $\mathbb{Z}_2$ by $\pi_1(M) \rtimes S_n$ which is also described in Section 4. We gave a presentation of these groups in terms of $2\pi$-rotations of the particle’s frames, local particle interchanges and single particle loops, which we used to obtain information on the available spectrum of spin and statistics. In particular, we were able to prove that there exists an IUR of $B^F_n(M)$ whose corresponding quantum theory provides half-integral spin for the identical particles if and only if $M$ does not have a $\pi_2$-obstruction to a spin structure. Moreover, a one-dimensional IUR yielding half-integral spin exists if and only if $M$ does not have an $H_2$-obstruction. In Section 5 we discussed various general properties of the groups $B^F_n(M)$ in two dimensions, and displayed these groups when the orientable surface $M$ is closed. We further determined $B^F_n(M)$ in terms of $B_n(M)$ when $M$ is flat. Section 6 contains a brief discussion of nonorientable spaces. The main result is that the spin braid groups of a nonorientable manifold and its orientable double cover are isomorphic in three or more dimensions.

In Section 7 we turned our attention to the $O(d+1)$-invariant nonlinear sigma model in $(d+1)$-dimensions with closed, orientable space manifold $M$. We defined a homomorphism $f_*$ from $B^F_n(M)$ to the fundamental group $\pi_1(X_n(M))$ of the degree $n$ sector $X_n(M)$ of the sigma model configuration space, which sent particle exchanges, rotations and loops to the corresponding operations for degree one solitons. We then demonstrated that $f_*$ was onto for all $d \geq 2$. All one has to do is add the new relations (7.1) and (7.2) to our presentation of $B^F_n(M)$ in order to obtain $\pi_1(X_n(M))$ — each of these relations being a consequence of soliton creation and annihilation processes. (In particular, there is a spin-statistics relation for solitons.) For $d \geq 3$, the above result implies that the groups $\pi_1(X_n(M))$ are always abelian, and hence all quantizations are scalar. Moreover, as for particles, there is a scalar quantization yielding half-integral spin for degree one solitons in $X_n(M)$ if and only if $M$ has no $H_2$-obstruction to a spin structure. We closed the section with a treatment of solitons on the open manifolds $\mathbb{R}^d \# M$, proving results analogous to the closed case, as well as a brief discussion of solitons on nonorientable spaces. Finally, in Section 8 we set out to better understand the half-integral spin quantizations we had encountered on certain
nonspin manifolds, as well as to show how to obtain such quantizations on other nonspin manifolds, by considering quantum theories built from arbitrary complex line bundles over the appropriate configuration space. We proved that there exists such a general scalar quantum theory yielding half-integral spin for particles or $O(d+1)$ solitons if and only if the closed, orientable space manifold $M$ possesses a spin$_c$ structure.

The results of this paper may be extended in several directions. For example, in mechanics we may consider systems with $k$ distinct species of particles, there being $n_i$ particles of species $i$. Working along similar lines we will obtain the braid groups $B^{E}_{n_1,n_2,...,n_k}(M)$ which are generalizations of the “partially colored” braid groups $B_{n_1,n_2,...,n_k}(M)$ in the structureless case [21,22]. We can further include the possibility of pair creation and annihilation in these mechanical systems [14,15,50]. In field theory, we can consider much broader classes of nonlinear sigma models. The existence and properties of solitons in these systems will depend intricately on the nature of both the space manifold and the target space (which will no longer be simply $S^d$). For some general results and specific examples along this line, see [4,24,25,51]. (For instance, certain models with closed space manifolds and nonsimply connected target spaces contain ambi-statistical solitons [24,25].) It would also be interesting to further investigate the nature of half-integral spin quantizations on nonorientable manifolds from the point of view of pin and pin$_c$ structures [20], as well as to look at the possibility of half-integral spin on nonspin$_c$ manifolds by considering generalized nonscalar quantum theories — that is, quantum theories whose state vectors are sections of an arbitrary complex vector bundle over the appropriate configuration space. These latter theories may utilize structures on the space manifold more general than spin$_c$ (such as those in [24]) in order to obtain half-integral spin. Finally, we note that a discussion of the rather peculiar topic of particle spin and statistics on the circle $S^1$ can be found in [4,23], while a reasonably complete treatment of the spin and statistical properties of topological solitons on $S^1$ in a sigma model with an arbitrary target space is given in [24].

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