SMALL TIME ASYMPTOTICS FOR SPDES WITH LOCALLY MONOTONE COEFFICIENTS

SHIHU LI, WEI LIU* AND YINGCHAO XIE

School of Mathematics and Statistics
Jiangsu Normal University
Xuzhou 221116, China

(Communicated by Björn Schmalfuß)

ABSTRACT. This work aims to prove the small time large deviation principle (LDP) for a class of stochastic partial differential equations (SPDEs) with locally monotone coefficients in generalized variational framework. The main result could be applied to demonstrate the small time LDP for various quasilinear and semilinear SPDEs such as stochastic porous medium equations, stochastic p-Laplace equations, stochastic Burgers type equation, stochastic 2D Navier-Stokes equation, stochastic power law fluid equation and stochastic Ladyzhenskaya model. In particular, our small time LDP result seems to be new in the case of general quasilinear SPDEs with multiplicative noise.

1. Introduction. The small time LDP mainly studies the asymptotic behavior of the tails of a family of probability distributions at a given point in space when the time is very small. Specifically, we focus on the limiting behavior of the solution in time interval $[0, t]$ as $t$ goes to zero. The study of the small time asymptotics (large deviations) of finite dimensional diffusion processes was initiated by Varadhan in the influential work [57]. Due to its wide applications in extremal events arising in risk management, mathematical finance, statistical mechanics, quantum physics and many other areas, large deviation theory has become an important component of modern applied probability, see, e.g. [5, 6, 11, 14, 21, 22, 23, 25, 36, 52, 55, 57] and references therein.

Another main point being that the small time behaviour of a diffusion process can be characterized in terms of an energy/distance function on a Riemannian manifold, whose metric is induced from the inverse of the diffusion coefficient, that is, such small time asymptotics will be useful to get the following Varadhan identity

$$\lim_{t \to 0} 2t \log P(X(0) \in A_1, X(t) \in A_2) = -d^2(A_1, A_2),$$

where $d$ is an appropriate Riemann distance associated with the diffusion generated by $X$, see e.g. [1, 4, 13, 31, 56, 62] and references therein.

2020 Mathematics Subject Classification. Primary: 60H15, 60F10; Secondary: 76S05, 35J92, 35K57.

Key words and phrases. Small time asymptotics, large deviation principle, stochastic partial differential equations, locally monotone, porous medium equation.

The research of W. Liu is supported by NSFC (No. 11822106, 11831014, 11571147), the research of Y. Xie is supported by NSFC (No. 11771187, 11931004) and PAPD of Jiangsu Higher Education Institutions.

* Corresponding author: Wei Liu.
Apart from the above motivations, the small time asymptotic itself is also theoretically interesting, which has been studied a lot in the literatures. For instance, the small time asymptotics of infinite dimensional diffusion processes were studied in [2, 3, 20, 32, 62]. Subsequently, many authors have endeavored to derive the small time LDP for different types of SPDEs. An important development concerning small time LDP for stochastic 2D Navier-Stokes equation was established by Xu and Zhang [60]. In [50], Röckner and Zhang studied small time LDP for stochastic 3D tamed Navier-Stokes equation. Moreover, the second named author with Röckner and Zhu [44] also obtained the small time LDP for stochastic 2D quasi-geostrophic equations in the sub-critical case. The small time LDP of stochastic 3D primitive equations was investigated by Dong and Zhang [19]. Recently, the small time LDP of scalar stochastic conservation laws was also studied in [61]. The reader might refer to [12, 13, 33, 37] and references therein for further results on this subject.

However, most papers in the literature investigated small time asymptotics (LDP) only for semilinear type SPDEs. On the other hand, some very interesting quasi-linear SPDEs have been studied a lot recently, such as stochastic porous media equation and stochastic $p$-Laplace equation, see e.g. [9, 26, 27, 28, 29, 38, 39, 42, 43, 46, 49, 58]) and references therein. We would like to investigate whether small time asymptotics (LDP) results also hold for those SPDE models or not? This is one of the main motivations for us to study the small time LDP for a class of nonlinear SPDEs, where the coefficients satisfy local monotonicity condition under the (generalized) variational framework.

The variational framework has been used intensively for studying SPDE where the coefficients satisfying the classical monotonicity and coercivity conditions. It was first investigated in the seminal works of Pardoux [48] and Krylov and Rozovskii [34], where they adapted the monotonicity tricks to prove the existence and uniqueness of solutions for a class of SPDE. Recently, this framework has been substantially extended by the second named author and Röckner in [40, 41, 42, 43] for more general class of SPDE with coefficients satisfying the generalized coercivity and local monotonicity conditions. In recent years, various properties for SPDEs with monotone or locally monotone coefficients has been intensively investigated in the literature, such as small noise LDP [39, 45, 49, 59], random attractors [26, 27, 28, 29], Harnack inequality and applications [38], Wong-Zakai approximation and support theorem [46], ultra-exponential convergence [58], and existence of optimal controls [16].

The proof of the main result here mainly follows the idea in Zhang’s work [62] by using exponential equivalence arguments, which is a very powerful method used by many scholars to study the small time LDP for SPDEs, see, e.g. [12, 19, 37, 44, 50, 60, 61]. More precisely, consider a zero drift stochastic differential equation with the same initial data (see (4) below), where the small noise and small time asymptotics problems are equivalent. It is easy to see that the small noise LDP for the solution $Y^\varepsilon$ of zero drift stochastic differential equation holds, thus our task is to show that the law of $X^\varepsilon$ and $Y^\varepsilon$ are exponentially equivalent (see (5) below). Comparing with some related works on small time LDP for SPDEs, to deal with the stochastic differential equation with zero drift, one usually assumes that there exists another Hilbert space $H^1$ which is densely embedded in state space $H$. Working in the space of continuous $H^1$-valued trajectories, one is able to get the $H^1$-norm estimates by applying Itô’s formula to $\| \cdot \|^2_{H^1}$. However, in
the variational framework, we work with the Gelfand triple $V \subset H \subset V^*$, where $V$ is a reflexive Banach space such that $V \subset H$ is continuously and densely, and it is unavailable to get the $V$-norm estimates by applying Itô’s formula to $\| \cdot \|^2_V$ (e.g. in the quasilinear SPDE case). In order to overcome this difficulty, we use the concept of 2-smooth Banach space and get the $V$-norm estimates using the crucial BDG type inequality proved by Seidler [51] (cf. [64] for recent generalization) for stochastic integrals in the 2-smooth Banach space, where the sharp constant $p^{1/2}$ also plays an important role in our proof. This 2-smooth Banach space is introduced for establishing a theory of stochastic integration in Banach spaces and typical examples of such spaces are $L^p$ spaces with $p \geq 2$ and Sobolev spaces $W^{s,p}_0$ with $p \geq 2$ and $s \geq 1$. Thus, our main result is applicable to various types of SPDEs such as stochastic porous media equation, stochastic $p$-Laplace equation, stochastic Burgers type equation, stochastic 2D Navier-Stokes equation, stochastic power law fluid equation and stochastic Ladyzhenskaya model. In particular, by applying the abstract result to concrete models, our main result could cover the results in [60, 37], where the small time LDP for stochastic 2D Navier-Stokes equation and stochastic Ladyzhenskaya model was studied respectively. Moreover, to the best of our knowledge, our small time LDP results for general quasilinear SPDEs with multiplicative noise (such as porous media equation and $p$-Laplace equation) seem to be new in the literature.

The rest of the paper is organized as follows. In Section 2, we introduce the variational framework and formulate our main result. Section 3 is devoted to proving our main result. In Section 4, we apply the main result to various SPDE models as applications. Throughout the paper, $C$ and $C_p$ will denote positive constants which may change from line to line, here $C_p$ emphasize the dependence on parameter $p$.

2. Framework and main result. Let $(H, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space identified with its dual space $H^*$ by the Riesz isomorphism. Let $(V, \langle \cdot, \cdot \rangle_V)$ be a reflexive Banach space which is continuously and densely embedded into $H$. Then we have the following Gelfand triple:

\[ V \subset H \equiv H^* \subset V^*, \]

where $V^*$ is the dual space of $V$. Let $\langle \cdot, \cdot \rangle_V$ denote the dualization between $V$ and $V^*$, then it follows that

\[ V \cdot \langle u, v \rangle_V = \langle u, v \rangle, \quad u \in H, \quad v \in V. \]

Let $\{W(t)\}_{t \geq 0}$ be a cylindrical Wiener process in a separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$ on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Let $(L_2(U; H), \| \cdot \|_2)$ denote the space of all Hilbert-Schmidt operators from $U$ to $H$.

In this paper, we consider the following stochastic evolution equation:

\begin{equation}
\begin{cases}
    dX(t) = A(t, X(t))dt + B(X(t))dW(t), \\
    X(0) = x \in H,
\end{cases}
\end{equation}

where $A : [0, T] \times V \to V^*$ and $B : V \to L_2(U; H)$ are measurable.

Let us now state the precise conditions on the coefficients of (1).

Assumption 2.1. For fixed $\alpha > 1$, there exist constants $\beta \geq 0$, $\eta > 0$, $K$ and $C$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $t \in [0, T]$.

(A1) (Hemicontinuity) The map $s \mapsto \langle A(t, v_1 + sv_2), v \rangle_V$ is continuous on $\mathbb{R}$. 


(A2) (Local monotonicity)

\[2\nu \cdot \langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V \leq -\eta \|v_1 - v_2\|_V^2 + (K + \rho(v_2))\|v_1 - v_2\|_H^2,\]

where \(\rho: V \to [0, +\infty)\) is a measurable function and locally bounded in \(V\) such that

\[\rho(v) \leq C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\beta).\]

(A3) (Growth)

\[\|A(t, v)\|_{\gamma(V)}^\alpha \leq C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\beta).\]

In order to study the small time LDP, we also need to estimate the stochastic integrals in the Banach space \(V\). For a more specific example, consider the stochastic \(p\)-Laplace equation, it is common to take \(V = W_0^{1,p}\) for \(p \geq 2\) as in [42] and therefore we need to ensure the existence of the stochastic integral in (1) as an \(W_0^{1,p}\)-valued process. We recall that the Sobolev spaces \(W_0^{1,p}\) with \(p \geq 2\) belong to the class of 2-smooth Banach spaces since they are isomorphic to \(L^p(0, 1)\) according to [54, Remark 2 in Section 4.9] and hence they are well suited for the stochastic Itô integration (see e.g. Brzéniak et al. [7, 8] for the precise construction of the stochastic integral).

In this work, we assume \(V\) is a 2-smooth Banach space. Let us denote by \(\gamma(U, V)\) the space of the \(\gamma\)-radonifying operators from \(U\) to \(V\). We recall that \(\Psi \in \gamma(U, V)\) if the series

\[\sum_{k \geq 1} \gamma_k \Psi(u_k)\]

converges in \(L^2(\Omega, U)\) for any sequence \((\gamma_k)_{k \geq 0}\) of independent Gaussian real-valued random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and any orthonormal basis \((u_k)_{k \geq 0}\) of \(U\). Then, the space \(\gamma(U, V)\) is endowed with the norm

\[\|\Psi\|_{\gamma(U, V)} := \left( \mathbb{E} \left\| \sum_{k \geq 1} \gamma_k \Psi(u_k) \right\|_V^2 \right)^{1/2}\]

(which does not depend on \((\gamma_k)_{k \geq 0}\), nor on \((u_k)_{k \geq 0}\) and is a Banach space). In the following, we shall write \(\|\cdot\|_{\gamma}\) instead of \(\|\cdot\|_{\gamma(U, V)}\) for the simplicity of notations.

**Remark 2.1.** If \(V\) is a separable Hilbert space, clearly, \(V\) is 2-smooth. In this case, \(\gamma(U, V)\) consists of all Hilbert-Schmidt operators of mapping \(U\) into \(V\), and \(\|\cdot\|_{\gamma} = \|\cdot\|_2\) (see, e.g. [63, Example 2.8]). Typical examples of 2-smooth Banach space include every Hilbert space, \(L^p\) spaces with \(p \geq 2\) and Sobolev spaces \(W_0^{s,p}\) with \(p \geq 2\) and \(s \geq 1\).

We assume the following conditions on \(B\).

**Assumption 2.2.** There exists constant \(C\) such that the following conditions hold for all \(v, v_1, v_2 \in V\),

\[\|B(v)\|_V^2 \leq C(1 + \|v\|_V^2);\]

\[\|B(v_1) - B(v_2)\|_V^2 \leq C\|v_1 - v_2\|_H^2.\]

**Remark 2.2.** By Assumption 2.1 and Assumption 2.2, the coercivity of \(A\) and \(B\) is easily obtained as

\[\nu \cdot \langle A(t, v), v \rangle_V + \|B(v)\|_V^2 + \frac{\eta}{2}\|v\|_V^2 \leq C(1 + \|v\|_H^2).\]
Now, we recall the following definition.

**Definition 2.1.** A continuous $H$-valued ($\mathcal{F}_t$)-adapted process $\{X(t)\}_{t \in [0,T]}$ is called a strong solution of (1), if for its $dt \otimes \mathbb{P}$-equivalence class we have

$$X \in L^\alpha([0,T] \times \Omega; dt \otimes \mathbb{P}; V) \cap L^2([0,T] \times \Omega; dt \otimes \mathbb{P}; H)$$

and the following identity holds $\mathbb{P}$-a.s.

$$X(t) = x + \int_0^t A(s, X(s))ds + \int_0^t B(X(s))dW(s), \quad t \in [0,T].$$

The following well-posedness result is due to the second name author and Röckner [40, Theorem 1.1].

**Lemma 2.1.** Suppose that the conditions in Assumption 2.1 and Assumption 2.2 hold, then (1) has a unique solution $\{X(t)\}_{t \in [0,T]}$ such that for any $p \geq 2$

$$\mathbb{E} \left( \sup_{t \in [0,T]} \|X(t)\|_H^p + \int_0^T \|X(t)\|_V^\alpha dt \right) < \infty.$$

For $\varepsilon > 0$, in this paper, we aim to study the probabilistic asymptotic behavior for small time process $X(\varepsilon t)$ as $\varepsilon \to 0$.

Define a functional $I(g)$ on $C([0,T]; H)$ by

$$I(g) = \inf_{h \in \Gamma_g} \left\{ \frac{1}{2} \int_0^T |\dot{h}(s)|_V^2 ds \right\},$$

where

$$\Gamma_g = \left\{ h \in C([0,T]; H) : h(\cdot) \text{ is absolutely continuous and such that} \right\}$$

$$g(t) = x + \int_0^t B(g(s))\dot{h}(s)ds, \quad t \in [0,T].$$

Now we state the main result of this paper.

**Theorem 2.2.** Suppose that the conditions in Assumption 2.1 and Assumption 2.2 hold. Let $\mu^\varepsilon$ be the law of $X(\varepsilon \cdot)$ on $C([0,T]; H)$, then $\mu^\varepsilon$ satisfies the LDP with the rate function $I(\cdot)$ given by (2), i.e.,

\(\text{(i) For any closed subset } F \subset C([0,T]; H), \limsup_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(F) \leq -\inf_{g \in F} I(g), \)

\(\text{(ii) For any open subset } G \subset C([0,T]; H), \liminf_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(G) \geq -\inf_{g \in G} I(g). \)

**Remark 2.3.** (1) In section 4 below, we will apply Theorem 2.2 to concrete examples of SPDE models as applications. In particular, this covers the results in [60, 37], where the small time LDP for stochastic 2D Navier-Stokes equation and stochastic Ladyzhenskaya model was studied respectively.

(2) Furthermore, Theorem 2.2 can also be applied to study the small time LDP for many other SPDEs, such as stochastic Burgers type equation, stochastic porous medium equation, stochastic $p$-Laplace equation, stochastic 2D Boussinesq equations, stochastic 2D magneto-hydrodynamic equations, stochastic 2D magnetic Bénard problem, stochastic 3D Leray-$\alpha$ model, stochastic shell models of turbulence and stochastic power law fluid equation, which seem to not have been established in the literature before.
3. **Proof of main result.** In this section, we will give the proof of Theorem 2.2, which is mainly based on the exponential equivalence arguments.

More precisely, for $\varepsilon > 0$, by the scaling property of the Wiener process, it is easy to see that the small time process $X(\varepsilon \cdot)$ coincides in law with the solution of the following stochastic evolution equation:

$$X^\varepsilon(t) = x + \varepsilon \int_0^t A(\varepsilon s, X^\varepsilon(s))ds + \sqrt{\varepsilon} \int_0^t B(X^\varepsilon(s))dW(s).$$

Now, let $Y^\varepsilon(\cdot)$ be the solution of the following stochastic differential equation:

$$Y^\varepsilon(t) = x + \varepsilon \int_0^t B(Y^\varepsilon(s))dW(s)$$

and $\nu^\varepsilon$ be the law of $Y^\varepsilon(\cdot)$ on the $C([0,T]; H)$. Then, applying the weak convergence approach developed by Budhiraja and Dupuis [10], it easy to get that $\nu^\varepsilon$ satisfies the LDP with rate function $I(\cdot)$ given by (2) (see e.g. [39, 45]). Therefore, our task now is to show that the two families of probability measures $\mu^\varepsilon$ and $\nu^\varepsilon$ are exponentially equivalent, that is, for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \|X^\varepsilon(t) - Y^\varepsilon(t)\|_H^2 > \delta \right) = -\infty. \quad (5)$$

Then Theorem 2.2 follows from the fact that if one of the two exponentially equivalent families satisfies the LDP, so does the other (see e.g. [18, Theorem 4.2.13]).

We begin the proof with the following lemma which provides an estimate of the probability that the solution of (3) leaves an energy ball.

Set

$$(\|X^\varepsilon|_{H,V}(T))^p := \sup_{0 \leq t \leq T} \|X^\varepsilon(t)\|_H^p + \varepsilon \int_0^T \|X^\varepsilon(t)\|_H^{p-2} \|X^\varepsilon(t)\|_V^2 dt.$$

Then, we claim that

**Lemma 3.1.** For any $p \geq 2$,

$$\lim_{M \to \infty} \sup_{0 < \varepsilon \leq 1} \varepsilon \log \mathbb{P} ((\|X^\varepsilon|_{H,V}(T))^p > M) = -\infty. \quad (6)$$

**Proof.** According to Itô’s formula (cf. [42, Theorem 4.2.5]) and Remark 2.2, we have

$$\|X^\varepsilon(t)\|_H^p = \|x\|_H^p + \frac{p(p-2)}{2} \varepsilon \int_0^t \|X^\varepsilon(s)\|_H^{p-2} \|B(X^\varepsilon(s))^* X^\varepsilon(s)\|_V^2 ds$$

$$+ \frac{p}{2} \int_0^t \|X^\varepsilon(s)\|_H^{p-2} \{2V \cdot (A(\varepsilon s, X^\varepsilon(s)), X^\varepsilon(s))_V + \|B(X^\varepsilon(s))\|_2^2 ds$$

$$+ p \sqrt{\varepsilon} \int_0^t \|X^\varepsilon(s)\|_H^{p-2} \{X^\varepsilon(s), B(X^\varepsilon(s))dW(s)\}$$

$$\leq \|x\|_H^p - \frac{mp}{4} \varepsilon \int_0^t \|X^\varepsilon(s)\|_H^{p-2} \|X^\varepsilon(s)\|_V^2 ds + C \varepsilon \int_0^t \|X^\varepsilon(s)\|_H^p ds$$

$$+ p \sqrt{\varepsilon} \int_0^t \|X^\varepsilon(s)\|_H^{p-2} \{X^\varepsilon(s), B(X^\varepsilon(s))dW(s)\}.$$

Then it is easy to get

$$\|X^\varepsilon(t)\|_H^p + \varepsilon \int_0^t \|X^\varepsilon(s)\|_H^{p-2} \|X^\varepsilon(s)\|_V^2 ds$$
\[
\leq C_p (1 + \|x\|_H^p) + C_p \varepsilon \int_0^t \|X^\varepsilon(s)\|_H^p ds \\
+ C_p \varepsilon \int_0^t \|X^\varepsilon(s)\|_H^{p-2} (X^\varepsilon(s), B(X^\varepsilon(s))dW(s)),
\]
which implies that for any \( q \geq 2 \)
\[
(\mathbb{E}(\|X^\varepsilon\|_{H,V}(T))^p)^{1/q} \\
\leq C_p (1 + \|x\|_H^p) + C_p \varepsilon \left( \mathbb{E} \left( \int_0^T (\|X^\varepsilon\|_{H,V}(t))^p dt \right) \right)^{1/q} \\
+ C_p \varepsilon \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \|X^\varepsilon(s)\|_H^{p-2} (X^\varepsilon(s), B(X^\varepsilon(s))dW(s)) \right|^{q} \right)^{1/q}.
\]

To estimate the stochastic integral term, we will use the following martingale inequality from [17] that there exists a universal constant \( C \) such that, for any \( q \geq 2 \) and for any continuous martingale \( M_t \) with \( M_0 = 0 \), one has
\[
(\mathbb{E}(|M_t^*|^q))^{1/q} \leq C q^{1/2}(\mathbb{E}(M_t^{q/2}))^{1/q},
\]
where \( M_t^* = \sup_{0 \leq s \leq t} |M_s| \).

Hence, we can get
\[
\sqrt{\varepsilon} \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \|X^\varepsilon(s)\|_H^{p-2} (X^\varepsilon(s), B(X^\varepsilon(s))dW(s)) \right|^q \right)^{1/q} \\
\leq C \sqrt{q \varepsilon} \left( \mathbb{E} \left( \int_0^T \|X^\varepsilon(t)\|_H^{2p-2} (1 + \|X^\varepsilon(t)\|_H^2) dt \right)^{q/2} \right)^{1/q} \\
\leq C \sqrt{q \varepsilon} \left( \int_0^T 1 + (\mathbb{E}\|X^\varepsilon(t)\|_H^{p})^{q/2} dt \right)^{1/2}.
\]

Therefore, combining the above estimates yields
\[
(\mathbb{E}(\|X^\varepsilon\|_{H,V}(T))^p)^{2/q} \\
\leq C_p (1 + \|x\|_H^p)^2 + C_p \varepsilon^2 \left( \mathbb{E} \left( \int_0^T (\|X^\varepsilon\|_{H,V}(t))^p dt \right) \right)^{2/q} \\
+ C_p q \varepsilon \left( \int_0^T 1 + (\mathbb{E}\|X^\varepsilon(t)\|_H^{p})^{2/q} dt \right) \\
\leq C_p (1 + \|x\|_H^p)^2 + C_p \varepsilon^2 \left( \mathbb{E}(\|X^\varepsilon\|_{H,V}(T))^{pq} \right)^{2/q} \\
+ C_p q \varepsilon T + C_p q \varepsilon \left( \int_0^T (\mathbb{E}(\|X^\varepsilon\|_{H,V}(t))^{pq})^{2/q} dt \right),
\]
where the second inequality is due to Minkowski’s inequality.

Applying Gronwall’s lemma we obtain that
\[
(\mathbb{E}(\|X^\varepsilon\|_{H,V}(T))^p)^{2/q} \leq \left( C_p (1 + \|x\|_H^p)^2 + C_p q \varepsilon T \right) \cdot \exp \left( C_p \varepsilon^2 + C_p q \varepsilon \right).
\]
Using Chebyshev’s inequality, for any $M > 0$, we have
\[
P \left( \left| X^\varepsilon \right|_{H,V}(T)^P > M \right) \leq \frac{\mathbb{E} \left( \left| X^\varepsilon \right|_{H,V}(T)^P \right)^q}{M^q}.
\]

Taking $q = 2/\varepsilon$, we get
\[
\varepsilon \log P \left( \left| X^\varepsilon \right|_{H,V}(T)^P > M \right) \leq -2 \log M + 2 C P + \log \left( C P (1 + \|x\|_H^P + 2 C T) \right).
\]

Then, it is easy to see
\[
\sup_{0 < \varepsilon \leq 1} \varepsilon \log P \left( \left| X^\varepsilon \right|_{H,V}(T)^P > M \right) \leq -2 \log M + \frac{2 C P}{\varepsilon} + \log \left( C P (1 + \|x\|_H^P + 2 C T) \right).
\]

Let $M \to \infty$ on both sides of (8), we complete the proof.

Since $V$ is dense in $H$, there exists a sequence $\{x_n\} \subset V$ such that
\[
\lim_{n \to +\infty} \|x_n - x\|_H = 0.
\]

Let $X^\varepsilon_n$ be the solution of (1) with the initial value $x_n$. From the proof of Lemma 3.1, it follows that
\[
\lim_{M \to \infty} \sup_{n \leq \varepsilon \leq 1} \varepsilon \log P \left( \sup_{0 \leq t \leq T} \left| Y^\varepsilon_n(t) \right|_{V}^2 > M \right) = -\infty.
\]

Let $Y^\varepsilon_n$ be the solution of (4) with the initial value $x_n$, i.e.
\[
Y^\varepsilon_n(t) = x_n + \sqrt{\varepsilon} \int_0^t B(Y^\varepsilon_n(s))dW(s).
\]

Then we can get the following estimate.

**Lemma 3.2.** For any $n \in \mathbb{Z}^+$,
\[
\lim_{M \to \infty} \sup_{0 < \varepsilon \leq 1} \varepsilon \log P \left( \sup_{0 \leq t \leq T} \left| Y^\varepsilon_n(t) \right|_{V}^2 > M \right) = -\infty.
\]

**Proof.** To estimate the stochastic integral term in the Banach space $V$, here we need to use the BDG type inequality for 2-smooth Banach space (cf. [51, Theorem 1.1]). Then for any $q > 1$, we can get
\[
\left( \mathbb{E} \left( \sup_{0 \leq t \leq T} \left| Y^\varepsilon_n(t) \right|_{V}^2 \right)^\frac{q}{2q} \right)^\frac{1}{q} \leq \|x_n\|_V + \sqrt{\varepsilon} \mathbb{E} \left( \sup_{0 \leq t \leq T} \left\| \int_0^t B(Y^\varepsilon_n(s))dW(s) \right\|_V^2 \right)^\frac{1}{2q} \frac{1}{q}.
\]
\[
\leq \|x_n\|_V + \sqrt{2q\varepsilon} C \left( \mathbb{E} \left( \int_0^T \|B(Y^\varepsilon_n(s))\|_V^2 ds \right)^q \right)^\frac{1}{q} \frac{1}{q}.
\]
\[
\leq \|x_n\|_V + \sqrt{2q\varepsilon} C \left( \mathbb{E} \left( \int_0^T (1 + \|Y^\varepsilon_n(s)\|_V^2 ds \right)^q ds \right)^\frac{1}{q} \frac{1}{q}.
\]
\[
\leq \|x_n\|_V + \sqrt{2q\varepsilon} C \left( \int_0^T (1 + \left( \mathbb{E} \left| Y^\varepsilon_n(s) \right|_{V}^{2q} \right)^{1/q}) ds \right)^\frac{1}{q}.
\]
where in the last inequality we use Minkowski’s inequality and the constant $C$ is independent of $q$ and $\varepsilon$.

Then, it is easy to get

$$
\left( E \left( \sup_{0 \leq t \leq T} ||Y_n^{\varepsilon}(t)||_V^{2q} \right) \right)^{1/2} \leq 2 ||x_n||_V^2 + 2q\varepsilon C + 2q\varepsilon C \int_0^T \left( E||Y_n^{\varepsilon}(s)||_V^{2q} \right)^{1/q} ds.
$$

Applying Gronwall’s Lemma yields

$$
\left( E \left( \sup_{0 \leq t \leq T} ||Y_n^{\varepsilon}(t)||_V^{2q} \right) \right)^{1/2} \leq \left( 2 ||x_n||_V^2 + 2q\varepsilon C \right) e^{2q\varepsilon C}.
$$

Fixing $M$ and taking $q = 1/\varepsilon$, we have

$$
\varepsilon \log P \left( \sup_{0 \leq t \leq T} ||Y_n^{\varepsilon}(t)||_V^2 > M \right) \leq \varepsilon \log \frac{E \left( \sup_{0 \leq t \leq T} ||Y_n^{\varepsilon}(t)||_V^{2q} \right)}{M^q}
$$

$$
\leq - \log M + \log \left( E \left( \sup_{0 \leq t \leq T} ||Y_n^{\varepsilon}(t)||_V^{2q} \right) \right)^{1/q}
$$

$$
\leq - \log M + \log \left( 2 ||x_n||_V^2 + 2C \right) + 2C. \quad (11)
$$

Let $M \to \infty$ on both sides of (11), we complete the proof.

Now, we establish the exponential convergence of $X_n^{\varepsilon} - X^{\varepsilon}$.

**Lemma 3.3.** For any $\delta > 0$, we have

$$
\lim_{n \to \infty} \sup_{0 < \varepsilon \leq 1} \varepsilon \log P \left( \sup_{0 \leq t \leq T} ||X_n^{\varepsilon}(t) - X^{\varepsilon}(t)||_H^2 > \delta \right) = -\infty.
$$

**Proof.** For $M > 0$, define stopping time

$$
\tau_M^{\varepsilon} = \inf \left\{ t : \varepsilon \int_0^t \|X^{\varepsilon}(s)||_H^{-2} \|X^{\varepsilon}(t)||_V^2 ds > M, \text{or } \|X^{\varepsilon}(t)||_H^2 > M \right\}.
$$

Using Itô’s formula, the local monotone condition (A2), we can get

$$
e^{-\varepsilon \int_0^{\tau_M^{\varepsilon}} (K + \rho(X^{\varepsilon}(s))) ds} \|X^{\varepsilon}(t) - X_n^{\varepsilon}(t)||_H^2
$$

$$
= \|x - x_n||_H^2
$$

$$
- \varepsilon \int_0^{\tau_M^{\varepsilon}} \left\{ e^{-\varepsilon \int_0^s (K + \rho(X^{\varepsilon}(r))) dr} \left[ (K + \rho(X^{\varepsilon}(s))) \|X^{\varepsilon}(s) - X_n^{\varepsilon}(s)||_H^2
$$

$$
- 2\nu \langle A(X^{\varepsilon}(s)) - A(X_n^{\varepsilon}(s)), X^{\varepsilon}(s) - X_n^{\varepsilon}(s) \rangle_N
$$

$$
+ \|B(X^{\varepsilon}(s)) - B(X_n^{\varepsilon}(s))\|^2_2 \right] ds
$$

$$
+ 2\sqrt{\varepsilon} \int_0^{\tau_M^{\varepsilon}} e^{-\varepsilon \int_0^s (K + \rho(X^{\varepsilon}(r))) dr} \langle X^{\varepsilon}(s) - X_n^{\varepsilon}(s),
$$

$$
(B(X^{\varepsilon}(s)) - B(X_n^{\varepsilon}(s)))dW(s) \rangle
$$

$$
\leq \|x - x_n||_H^2
$$

$$
+ 2\sqrt{\varepsilon} \int_0^{\tau_M^{\varepsilon}} e^{-\varepsilon \int_0^s (K + \rho(X^{\varepsilon}(r))) dr} \langle X^{\varepsilon}(s) - X_n^{\varepsilon}(s),
$$

$$
(B(X^{\varepsilon}(s)) - B(X_n^{\varepsilon}(s)))dW(s) \rangle.
$$
Then by the martingale inequality (7), we obtain
\[
\left( E\left[ \sup_{0 \leq t \leq \tau_M} e^{-\varepsilon \int_0^t (K + \rho(X^\varepsilon(r))) \, dr} \| X^\varepsilon(s) - X_n^\varepsilon(s) \|^2_H \right] \right)^{2/q} \leq 2 \| x - x_n \|_H^4 \\
+ C(q\varepsilon + t\varepsilon^2) \int_0^t \left( E\left[ \sup_{0 \leq s \leq t \wedge \tau_M} e^{-\varepsilon \int_s^t (K + \rho(X^\varepsilon(r))) \, dr} \| X^\varepsilon(s) - X_n^\varepsilon(s) \|^2_H \right] \right)^{2/q} \, ds.
\]
Applying Gronwall's lemma yields
\[
\left( E\left[ \sup_{0 \leq t \leq T \wedge \tau_M} e^{-\varepsilon \int_0^t (K + \rho(X^\varepsilon(r))) \, dr} \| X^\varepsilon(s) - X_n^\varepsilon(s) \|^2_H \right] \right)^{2/q} \leq 2 \| x - x_n \|_H^4 e^{CT(q\varepsilon + T\varepsilon^2)}.
\]
Hence, we have
\[
\left( E\left[ \sup_{0 \leq t \leq T \wedge \tau_M} \| X^\varepsilon(t) - X_n^\varepsilon(t) \|^2_H \right] \right)^{2/q} \leq 2 \| x - x_n \|_H^4 e^{CT(M + K + q\varepsilon + T\varepsilon^2)}.
\]
Fixing $M$ and taking $q = 2/\varepsilon$ we get
\[
\sup_{0 < \varepsilon \leq 1} \varepsilon \log P \left( \sup_{0 \leq t \leq T \wedge \tau_M} \| X^\varepsilon(t) - X_n^\varepsilon(t) \|^2_H > \delta \right) \leq \sup_{0 < \varepsilon \leq 1} \varepsilon \log \frac{E \sup_{0 \leq t \leq T \wedge \tau_M} \| X^\varepsilon(t) - X_n^\varepsilon(t) \|^2_H}{\delta^q} \leq \log 2 \| x - x_n \|_H^4 + C_{T,M,K} - 2 \log \delta \\
\rightarrow -\infty, \text{ as } n \rightarrow +\infty. \tag{12}
\]
By Lemma 3.1, for any $R > 0$, there exists a constant $M$ such that for every $\varepsilon \in (0,1]$ the following inequality holds:
\[
P \left( (\| X^\varepsilon \|_{H,V}(T))^p > M \right) \leq e^{-R/\varepsilon}. \tag{13}
\]
For such $M$, by (12) and the definition of stoping time $\tau_M$, there exists a positive integer $N$, such that for any $n \geq N$,
\[
\sup_{0 < \varepsilon \leq 1} \varepsilon \log P \left( \sup_{0 \leq t \leq T \wedge \tau_M} \| X^\varepsilon(t) - X_n^\varepsilon(t) \|^2_H > \delta, (\| X^\varepsilon \|_{H,V}(T))^p \leq M \right) \leq -R. \tag{14}
\]
Combining (12) and (14), we conclude that there exists a positive integer $N$ such that for any $n \geq N$ and $\varepsilon \in (0,1]$,
\[
P \left( \sup_{0 \leq t \leq T} \| X^\varepsilon(t) - X_n^\varepsilon(t) \|^2_H > \delta \right) \leq 2e^{-R/\varepsilon}.
\]
Since $R$ is arbitrary, the assertion of the lemma follows. \qed
Lemma 3.4. For any \( \delta > 0 \), we have
\[
\lim_{n \to \infty} \sup_{0 < \varepsilon \leq 1} \varepsilon \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \| Y_n^\varepsilon(t) - Y^\varepsilon(t) \|_H^2 > \delta \right) = -\infty. \tag{15}
\]

Proof. From (4) and (10), it is easy to see
\[
Y^\varepsilon(t) - Y_n^\varepsilon(t) = x - x_n + \sqrt{\varepsilon} \int_0^t (B(Y_n^\varepsilon(s)) - B(Y^\varepsilon(s))) dW(s).
\]
Applying Itô’s formula to \( \| Y_n^\varepsilon(t) - Y^\varepsilon(t) \|_H^2 \), we have
\[
\| Y^\varepsilon(t) - Y_n^\varepsilon(t) \|_H^2 = \| x - x_n \|_H^2 + \varepsilon \int_0^t \| B(Y^\varepsilon(s)) - B(Y_n^\varepsilon(s)) \|_2^2 ds
\]
\[
+ 2\sqrt{\varepsilon} \int_0^t (Y^\varepsilon(s) - Y_n^\varepsilon(s), (B(Y^\varepsilon(s)) - B(Y_n^\varepsilon(s))) dW(s).
\]
Then by the Assumption (2.2) and martingale inequality (7), we obtain
\[
\left( \mathbb{E} \left( \| Y^\varepsilon(t) - Y_n^\varepsilon(t) \|_H^{2q} \right) \right)^{\frac{1}{q}} \leq 2 \| x - x_n \|_H^{\frac{1}{q}} + (\varepsilon^2 C + \varepsilon C q) \int_0^t \left( \mathbb{E} \left( \| Y^\varepsilon(s) - Y_n^\varepsilon(s) \|_H^{2q} \right) \right)^{\frac{1}{q}} ds,
\]
where the constant \( C \) is independent of \( q \) and \( \varepsilon \).
Utilizing Gronwall’s lemma, we get
\[
\left( \mathbb{E} \left( \| Y^\varepsilon(t) - Y_n^\varepsilon(t) \|_H^{2q} \right) \right)^{\frac{1}{q}} \leq 2 \| x - x_n \|_H^{\frac{1}{q}} + \exp(\varepsilon^2 C + \varepsilon C q).
\]
Applying the same argument as the proof of (12) in Lemma 3.3, we complete the proof.

The following lemma says that for a fixed integer \( n \), the two families \( \{ X_n^\varepsilon, \varepsilon > 0 \} \) and \( \{ Y_n^\varepsilon, \varepsilon > 0 \} \) are exponentially equivalent.

Lemma 3.5. For any \( \delta > 0 \) and any positive integer \( n \), we have
\[
\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \| X_n^\varepsilon(t) - Y_n^\varepsilon(t) \|_H^2 > \delta \right) = -\infty.
\]

Proof. For \( M > 0 \), we define the following stopping times
\[
\tau_{1,n}^{M,\varepsilon} = \inf \left\{ t : \varepsilon \int_0^t \| X_n^\varepsilon(s) \|_{p-2} \| X_n^\varepsilon(s) \|_p ds > M, \text{ or } \| X_n^\varepsilon(t) \|_H^2 > M \right\},
\]
\[
\tau_{2,n}^{M,\varepsilon} = \inf \left\{ t : \| Y_n^\varepsilon(t) \|_V^2 > M \right\}.
\]
Setting \( \tau_{\varepsilon}^{M,n} = \tau_{1,n}^{M,\varepsilon} \wedge \tau_{2,n}^{M,\varepsilon} \), then we can get by applying Itô’s formula that
\[
\| X_n^\varepsilon(t \wedge \tau_{\varepsilon}^{M,n}) - Y_n^\varepsilon(t \wedge \tau_{\varepsilon}^{M,n}) \|_H^2
\]
\[
= \varepsilon \int_0^{t \wedge \tau_{\varepsilon}^{M,n}} 2\varepsilon \cdot \langle A(\varepsilon s, X_n^\varepsilon(s)) - A(\varepsilon s, Y_n^\varepsilon(s), X_n^\varepsilon(s) - Y_n^\varepsilon(s)) \rangle_V ds
\]
\[
+ \varepsilon \int_0^{t \wedge \tau_{\varepsilon}^{M,n}} (2\varepsilon \cdot \langle A(\varepsilon s, Y_n^\varepsilon(s)), X_n^\varepsilon(s) - Y_n^\varepsilon(s) \rangle_V + \| B(X_n^\varepsilon(s)) - B(Y_n^\varepsilon(s)) \|_2^2) ds.
\]
Applying Gronwall's lemma again, we obtain that

$$2 \sqrt{\varepsilon} \int_0^{t \wedge \tau_{n,M,\varepsilon}} \langle X_n^\varepsilon(s) - Y_n^\varepsilon(s), (B(X_n^\varepsilon(s)) - B(Y_n^\varepsilon(s)))dW(s) \rangle.$$ 

By condition (A3) and Young's inequality, we have

$$2 \sqrt{\varepsilon} \cdot \langle A(\varepsilon s, Y_n^\varepsilon(s)), X_n^\varepsilon(s) - Y_n^\varepsilon(s) \rangle \leq 2 \| A(\varepsilon s, Y_n^\varepsilon(s)) \| \| X_n^\varepsilon(s) - Y_n^\varepsilon(s) \|_V \leq C \| A(\varepsilon s, Y_n^\varepsilon(s)) \|_{L_2} + \theta \| X_n^\varepsilon(s) - Y_n^\varepsilon(s) \|_V$$

$$\leq C(1 + \| Y_n^\varepsilon(s) \|^\alpha_\varepsilon) + \theta \| X_n^\varepsilon(s) - Y_n^\varepsilon(s) \|_V^\alpha,$$

where \( \theta < \eta \) is a small positive constant. Then by condition (A2), we obtain that

$$\| X_n^\varepsilon(t \wedge \tau_{n,M,\varepsilon}) - Y_n^\varepsilon(t \wedge \tau_{n,M,\varepsilon}) \|_H^2$$

$$\leq \varepsilon \int_0^{t \wedge \tau_{n,M,\varepsilon}} (K + \rho(X_n^\varepsilon(s))) \| X_n^\varepsilon(s) - Y_n^\varepsilon(s) \|_H^2 ds + \varepsilon \int_0^{t \wedge \tau_{n,M,\varepsilon}} C(1 + \| Y_n^\varepsilon(s) \|^\alpha_\varepsilon) - (\eta - \theta) \| X_n^\varepsilon(s) - Y_n^\varepsilon(s) \|_V^\alpha ds$$

$$+ 2 \sqrt{\varepsilon} \int_0^{t \wedge \tau_{n,M,\varepsilon}} \langle X_n^\varepsilon(s) - Y_n^\varepsilon(s), (B(X_n^\varepsilon(s)) - B(Y_n^\varepsilon(s)))dW(s) \rangle.$$

Then applying Gronwall's lemma yields that

$$\| X_n^\varepsilon(t \wedge \tau_{n,M,\varepsilon}) - Y_n^\varepsilon(t \wedge \tau_{n,M,\varepsilon}) \|_H^2$$

$$\leq \left[ C \varepsilon \int_0^{t \wedge \tau_{n,M,\varepsilon}} (1 + \| Y_n^\varepsilon(s) \|^\alpha_\varepsilon) ds \right. + \left. 2 \sqrt{\varepsilon} \int_0^{t \wedge \tau_{n,M,\varepsilon}} \langle X_n^\varepsilon(s) - Y_n^\varepsilon(s), (B(X_n^\varepsilon(s)) - B(Y_n^\varepsilon(s)))dW(s) \rangle \right]$$

$$\cdot e^{\varepsilon \int_0^{t \wedge \tau_{n,M,\varepsilon}} (K + \rho(X_n^\varepsilon(s))) ds}.$$

Using (7), we obtain by the definition of the stopping time \( \tau_{n,M,\varepsilon} \) that

$$\left( \mathbb{E} \left[ \sup_{0 \leq s \leq t \wedge \tau_{n,M,\varepsilon}} \| X_n^\varepsilon(s) - Y_n^\varepsilon(s) \|_H^2 \right] \right)^{\frac{q}{2}} \leq C e^{M+\alpha_\varepsilon t \varepsilon} (C(t \varepsilon + M \frac{\alpha_\varepsilon}{2} t \varepsilon)^2 + q \varepsilon$$

$$\cdot \int_0^t \left( \mathbb{E} \left[ \sup_{0 \leq r \leq s \wedge \tau_{n,M,\varepsilon}} \| X_n^\varepsilon(r) - Y_n^\varepsilon(r) \|_H^2 \right] \right)^{\frac{q}{2}} ds \right).$$

Applying Gronwall’s lemma again, we obtain that

$$\left( \mathbb{E} \left[ \sup_{0 \leq s \leq T \wedge \tau_{n,M,\varepsilon}} \| X_n^\varepsilon(s) - Y_n^\varepsilon(s) \|_H^2 \right] \right)^{\frac{q}{2}} \leq C e^{M+\alpha_\varepsilon T \varepsilon} \exp \left( C_q T \varepsilon e^{M+\alpha_\varepsilon T \varepsilon} \right).$$

Fixing \( M \) and taking \( q = 2/\varepsilon \) we have

$$\varepsilon \log \mathbb{P} \left( \sup_{0 \leq s \leq T \wedge \tau_{n,M,\varepsilon}} \| X_n^\varepsilon(t) - Y_n^\varepsilon(t) \|_H^2 > \delta \right)$$
\[ \begin{align*}
&\leq \varepsilon \log \frac{\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_{M,\varepsilon}} \left\| X_n^\varepsilon(t) - Y_n^\varepsilon(t) \right\|^2_H \right]}{\delta^q} \\
&\leq \log C(T\varepsilon + M^{\alpha+\beta}T\varepsilon)^2 + CTe^{M+(M+C)T\varepsilon} + M + (M + C)T\varepsilon - 2 \log \delta \\
&\to -\infty, \text{ as } \varepsilon \to 0.
\end{align*} \]

By (9) and Lemma 3.2, for any \( R > 0 \), there exists a constant \( M \) such that the following inequalities hold:

\[ \sup_{0 < \varepsilon \leq 1} \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| X_n^\varepsilon(t) \right\|^2_H > \delta \right) \leq e^{-R/\varepsilon}, \tag{17} \]

\[ \sup_{0 < \varepsilon \leq 1} \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| Y_n^\varepsilon(t) \right\|^2_V > M \right) \leq e^{-R/\varepsilon}. \tag{18} \]

For such \( M \), by (16) and the definition of stopping time \( \tau_{M,\varepsilon} \), there exists \( \varepsilon_0 \), such that for every \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_0 \),

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| X_n^\varepsilon(t) - Y_n^\varepsilon(t) \right\|^2_H > \delta, \left( |X_n^\varepsilon(t)|_{H,V}(T) \right)^p \leq M, \sup_{0 \leq t \leq T} \left\| Y_n^\varepsilon(t) \right\|^2_V \leq M \right) \]

\[ \leq \mathbb{P} \left( \sup_{0 \leq t \leq T \wedge \tau_{M,\varepsilon}} \left\| X_n^\varepsilon(t) - Y_n^\varepsilon(t) \right\|^2_H > \delta \right) \leq e^{-R/\varepsilon}. \tag{19} \]

Combining (17), (18) and (19), we conclude that there exists \( \varepsilon_0 \), such that for every \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_0 \),

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| X_n^\varepsilon(t) - Y_n^\varepsilon(t) \right\|^2_H > \delta \right) \leq 3e^{-R/\varepsilon}. \]

Since \( R \) is arbitrary, the assertion of the lemma follows. \( \Box \)

We can now complete the proof of our main result.

**Proof of Theorem 2.2:** Due to Lemmas 3.3 and 3.4, for any \( R > 0 \), there exists a \( N_0 \) satisfying

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| X_{N_0}^\varepsilon(t) - X(t) \right\|^2_H > \frac{\delta}{3} \right) \leq e^{-R/\varepsilon}, \text{ for any } 0 < \varepsilon \leq 1; \tag{20} \]

and

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| Y_{N_0}^\varepsilon(t) - Y(t) \right\|^2_H > \frac{\delta}{3} \right) \leq e^{-R/\varepsilon}, \text{ for any } 0 < \varepsilon \leq 1. \tag{21} \]

For such \( N_0 \), according to Lemma 3.5, there exists \( \varepsilon_0 \), such that for every \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_0 \),

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| X_{N_0}^\varepsilon(t) - Y_{N_0}^\varepsilon(t) \right\|^2_H > \frac{\delta}{3} \right) \leq e^{-R/\varepsilon}. \tag{22} \]

Combining (20)-(22), for any \( 0 < \varepsilon \leq \varepsilon_0 \), we have

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| X^\varepsilon(t) - Y^\varepsilon(t) \right\|^2_H > \delta \right) \leq 3e^{-R/\varepsilon}. \]

Since \( R \) is arbitrary, we obtain

\[ \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left( \sup_{0 \leq t \leq T} \left\| X^\varepsilon(t) - Y^\varepsilon(t) \right\|^2_H > \delta \right) = -\infty, \]

\[ \text{SMALL TIME ASYMPTOTICS FOR SPDES 4813} \]
i.e. (5) holds. Hence the conclusion of Theorem 2.2 holds by using the exponential equivalence result of LDP, see e.g. [18, Theorem 4.2.13]. \[\square\]

4. Application to examples. The main result of this paper is applicable to a large class of SPDE with local monotone coefficients, and we illustrate the applicability of our main result to the following concrete examples of SPDE models.

In this section we use \( \Lambda \subseteq \mathbb{R}^d \) to denote an open bounded domain with a smooth boundary and \( C_0^\infty(\Lambda, \mathbb{R}^d) \) denote the set of all smooth functions from \( \Lambda \) to \( \mathbb{R}^d \) with compact support. For \( p \geq 1 \), let \( L^p(\Lambda, \mathbb{R}^d), \| \cdot \|_{L^p} \) be the vector valued \( L^p \)-space. For any integer \( m > 0 \), let \( W_0^{m,p}(\Lambda, \mathbb{R}^d) \) denote the standard Sobolev space on \( \Lambda \) with values in \( \mathbb{R}^d \), i.e. the closure of \( C_0^\infty(\Lambda, \mathbb{R}^d) \) with respect to the following norm:

\[
\|u\|_{W^{m,p}} = \left( \sum_{0 \leq |\alpha| \leq m} \int_\Lambda |D^\alpha u|^p\,dx \right)^{\frac{1}{p}}.
\]

For the reader’s convenience, we recall the following Gagliardo-Nirenberg interpolation inequality (cf. e.g. [53, Theorem 2.1.5]).

If \( m, n \in \mathbb{N} \) and \( q \in [1, \infty] \) such that

\[
\frac{1}{q} = \frac{1}{2} + \frac{n}{d} - \frac{m\theta}{d}, \quad \frac{n}{m} \leq \theta \leq 1,
\]

then there exists a constant \( C > 0 \) such that

\[
\|u\|_{W^{n,q}} \leq C\|u\|_{W^{m,2}}^{\frac{\theta}{m}}\|u\|_{L^2}^{1-\theta}, \quad u \in W^{m,2}(\Lambda, \mathbb{R}^d).
\] (23)

4.1. Stochastic multidimensional Burgers type equation. The first example is stochastic multidimensional Burgers type equation. Consider the Gelfand triple \( V := W_0^{1,2} \subset H := L^2(\Lambda) \subset (W_0^{1,2})^* = V^* \) and the following semilinear stochastic partial differential equation

\[
\begin{cases}
    dX(t) = (\Delta X + (f(X), \nabla X) + g(X(t)))dt + B(X(t))dW(t), \\
    X(0) = x \in H,
\end{cases}
\] (24)

where \( f = (f_1, \cdots, f_d) : \mathbb{R} \to \mathbb{R}^d \) is a Lipschitz functions and \( (\ , \ ) \) denotes the inner product in \( \mathbb{R}^d \), \( W \) is a cylindrical Wiener process in \( U \) defined on a probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \). Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function with \( g(0) = 0 \) such that for some constants \( C, r, s \in [0, \infty[\]

\[
|g(x)| \leq C(|x|^r + 1), x \in \mathbb{R};
\] (25)

\[
(g(x) - g(y))(x - y) \leq C(1 + |y|^s)(x - y)^2, x, y \in \mathbb{R}.
\] (26)

Now, for \( \varepsilon > 0 \), we consider the small time process \( X(\varepsilon t) \). Let \( \mu^\varepsilon \) be the law of \( X(\varepsilon \cdot) \) on \( C([0, T], H) \), we have the small time LDP for (24).

**Theorem 4.1.** (stochastic multidimensional Burgers type equation) Assume \( g \) satisfies the above conditions, \( B \) satisfies the assumption 2.2. If \( d = 1, r = 3, s = 2, \) or \( d = 2, r < 3, s = 2, \) or \( d = 3, r = \frac{7}{3}, s = \frac{4}{3}, \) then (24) has a unique solution \( X(t) \) and \( \mu^\varepsilon \) satisfies the LDP with the rate function \( I(\cdot) \) given by (2).\[\square\]

**Proof.** According to [42, Example 5.1.8], we know the coefficients in (24) satisfies the hemicontinuity, local monotonicity and growth properties (A1)-(A3). Therefore, the assertion follows by Lemma 2.1 and Theorem 2.2.\[\square\]
Remark 4.1. If $d = 1$, $f(x) = x$ and $g = 0$, Theorem 4.1 can be applied to the classical stochastic Burgers equation. Here, we also allow a polynomial perturbation term $g$ in the drift of (24). For example, one can take $g(x) = -x^3 + c_1x^2 + c_2x$ ($c_1, c_2 \in \mathbb{R}$) and show that (25)-(26) hold. Hence (24) also covers some stochastic reaction-diffusion type equations.

Besides from the example of semilinear SPDE above, we can also apply the main result to the following quasilinear SPDEs such as stochastic $p$-Laplace equation and stochastic porous medium equation, which have been studied a lot in recent years (see e.g. [26, 27, 28, 38, 39, 42, 43, 46, 49, 58] and references therein).

4.2. Stochastic $p$-Laplace equation. We consider the triple

$$V := W^{1,p}_0 \subset H := L^2(\Omega) \subset (W^{1,p}_0)^* = V^*$$

and the following stochastic $p$-Laplace equation

$$dX(t) = [\text{div}(\nabla X(t))|\nabla X(t)|^{p-2}\nabla X(t)] - c|X(t)|^{p-2}X(t)dt + B(X(t))dW(t),$$

$$X(0) = x \in H,$$

where $2 \leq p < \infty$, $1 \leq \tilde{p} \leq p$, $c$ is positive constant and $W(t)$ is a cylindrical Wiener process in $H$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

It is well known that the $p$-Laplace operator satisfies the hemicontinuity, monotonicity and growth properties (A1)-(A3) (see, e.g. [39, Example 5.5]).

Consider the small time process $X(\varepsilon t)$ and let $\mu^\varepsilon$ be the law of $X(\varepsilon \cdot)$ on $C([0, T], H)$, by applying our main result, we formulate the small time LDP for Eq. (27).

**Theorem 4.2.** (stochastic $p$-Laplace equation) Assume that $B$ satisfies the assumption 2.2, then (27) has a unique solution $X(t)$ and $\mu^\varepsilon$ satisfies the LDP with the rate function $I(\cdot)$ given by (2).

4.3. Stochastic porous medium equation. The main result in this work can also be applied to stochastic porous medium equation. Let $(E, \mathcal{M}, \mathbf{m})$ be a separable probability space and $(L, \mathcal{D}(L))$ a negative definite self-adjoint linear operator on $(L^2(\mathbf{m}), \langle \cdot, \cdot \rangle)$ with spectrum contained in $(-\infty, -\lambda_0]$ for some $\lambda_0 > 0$. Then the embedding

$$H^1 := \mathcal{D}(\sqrt{-L}) \subseteq L^2(\mathbf{m})$$

is dense and continuous. Define $H$ is the dual Hilbert space of $H^1$ realized through this embedding. Assume $L^{-1}$ is continuous on $L^{r+1}(\mathbf{m})$.

For fixed $r > 1$, we consider the following Gelfand triple

$$V := L^{r+1}(\mathbf{m}) \subset H := H \subset V^*$$

and the stochastic porous medium equation

$$dX(t) = [L\Psi(t, X(t)) + \Phi(t, X(t))]dt + B(X(t))dW(t),$$

$$X(0) = x \in H,$$

where $\Psi, \Phi : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable and continuous in the second variable, $W(t)$ is a cylindrical Wiener process in $H$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Suppose that there exist two constants $\delta > 0$ and $K$ such that

$$|\Psi(t, x)| + |\Phi(t, x)| \leq K(1 + |x|^r), \quad t \in [0, T], x \in \mathbb{R};$$

$$-\langle \Psi(t, u) - \Psi(t, v), u - v \rangle - \langle \Phi(t, u) - \Phi(t, v), L^{-1}(u - v) \rangle$$

$$\leq -\delta\|u - v\|_{L^{r+1}}^2 + K\|u - v\|_H^2, \quad t \in [0, T], u, v \in V.$$

\[\tag{30}\]
It is easy to see that the drift part of Eq. (28) satisfies the conditions (A1)-(A3) (cf. [39, Example 5.3]). Let $\mu^\varepsilon$ be the law of $X(\varepsilon \cdot)$ on $C([0,T], H)$, by applying our main result, we formulate the small time LDP for Eq. (28).

**Theorem 4.3.** (stochastic porous medium equation) Assume that $\Psi, \Phi$ satisfy the above conditions (29)-(30) and $B$ satisfies the assumption 2.2, then (28) has a unique solution $X(t)$ and $\mu^\varepsilon$ satisfies the LDP with the rate function $I(\cdot)$ given by (2).

**Remark 4.2.** If we take $L = \Delta$, the Laplace operator on a smooth bounded domain in a complete Riemannian manifold with Dirichlet boundary condition. A simple example for $\Psi$ and $\Phi$ satisfy the above conditions (29)-(30) is given by

$\Psi(t, x) = f(t) |x|^{r-1} x$, $\Phi(t, x) = g(t) x$

for some strictly positive continuous function $f$ and bounded function $g$ on $[0,T]$.

In the following, we will show that the main result is also applicable to many stochastic hydrodynamical systems.

### 4.4. Stochastic 2D Navier-Stokes equation.

Our next example is the stochastic 2D Navier-Stokes equation. The classical Navier-Stokes equation is a very important model in fluid mechanics to describe the time evolution of incompressible fluids, it can be formulated as follows (2D case):

$$\partial_t u = \nu \Delta u - (u \cdot \nabla) u - \nabla p + f, \text{ div}(u) = 0,$$

where $u = (u_1(x,t), u_2(x,t))$ is the velocity of a fluid, $p$ is the pressure, $\nu$ is the Kinematic viscosity, and $f$ denote the external force of the fluid, and $u \cdot \nabla = \sum_{j=1}^{d} u_j \partial_j$.

Let $\Lambda \subset \mathbb{R}^2$ be an open bounded domain with smooth boundary. Define

$$V := \{ v \in W_0^{1,2}(\Lambda, \mathbb{R}^2) : \text{ div}(v) = 0 \}, \| v \|_V := \left( \int_{\Lambda} |\nabla v|^2 \, dx \right)^{1/2},$$

and $H$ is the closure of $V$ in the following norm

$$\| v \|_H := \left( \int_{\Lambda} |v|^2 \, dx \right)^{1/2}.$$  

We define the stokes operator $A$ by

$$Au = P_H \Delta u, \forall u \in W^{2,2}(\Lambda, \mathbb{R}^2) \cap V,$$

where $P_H$ (Helmholtz-Leray projection) is the projection operator from $L^2(\Lambda, \mathbb{R}^2)$ to $H$, and the nonlinear operator

$$F(u, v) = -P_H( (u \cdot \nabla)v ), \quad F(u) = F(u, u).$$

Then (31) can then be written in form:

$$\partial_t u = \nu Au + F(u) + f(x), \quad u(0) = u_0.$$  

Now, we study the following stochastic 2D Navier-Stokes equation

$$
\begin{cases}
    dX(t) = (\nu AX(t) + F(X(t)) + f(x))\,dt + B(X(t))\,dW(t), \\
    X(0) = x \in H,
\end{cases}
$$

(32)
where \( W(t) \) is a cylindrical Wiener process in \( U \) defined on a probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\).

It is well known that stochastic 2D Navier-Stokes equation satisfies the conditions (A1)-(A3) (see, e.g. [40, Example 3.3]).

Let \( \mu^{\varepsilon} \) be the law of \( X(\cdot) \) on \( C([0,T], H) \). By applying our main result, we have the small time LDP for stochastic 2D Navier-Stokes equation (32).

**Theorem 4.4.** (stochastic 2D Navier-Stokes equation) Assume that \( B \) satisfies the assumption 2.2, then (32) has a unique solution \( X(t) \) and \( \mu^\varepsilon \) satisfies the LDP with the rate function \( I(\cdot) \) given by (2).

**Remark 4.3.**
(1) The small time LDP for stochastic 2D Navier-Stokes equation have been established by Xu and Zhang [60].

(2) Beside the stochastic 2D Navier-Stokes equation, many other hydrodynamical systems also satisfy the local monotonicity condition (A2) and growth condition (A3). For example, Chueshov and Millet [15] have studied the well-posedness and small noise LDP for an abstract stochastic evolution equations, covering a wide class of fluid dynamical models such as stochastic 2D Boussinesq equations, stochastic 2D magneto-hydrodynamic equations, stochastic 2D magnetic Bénard problem, stochastic 3D Leray-\( \alpha \) model and also shell models of turbulence. We refer the reader to [15] (and the references therein) for the details of these models. Note that the assumptions in [15] imply the conditions (A1)-(A3) (cf. [46, section 3.1] for a detail proof).

(3) Furthermore, below we will show that the main result in this work is also applicable to stochastic power law fluid equation and stochastic Ladyzhenskaya model.

### 4.5. Stochastic power law fluid equation.

As one of the important models in hydrodynamical, stochastic power law fluid equation can be used to characterize the dynamic properties of various incompressible non-Newtonian fluids. We can refer to [24, 47] for the study of this type of equation.

Let \( \Lambda \) be the open bounded domain with smooth boundary on \( \mathbb{R}^d(d \geq 2) \), \( u : \Lambda \to \mathbb{R}^d \) be a vector field. Define

\[
e(u) : \Lambda \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad e_{i,j}(u) = \frac{\partial_i u_j + \partial_j u_i}{2}, \quad i, j = 1, \ldots, d.
\]

\[
\tau(u) : \Lambda \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad \tau(u) = 2\nu(1 + |e(u)|)^{p-2}e(u),
\]

where \( \nu > 0 \) is the viscosity coefficient of the fluid, \( p > 1 \) is a constant.

Now we study a hydrodynamic equation with a power law property:

\[
\partial_t u = \text{div}(\tau(u)) - (u \cdot \nabla) u - \nabla p + f, \quad \text{div}(u) = 0,
\]

where \( u = u(t, x) = (u_i(t, x))_{i=1}^d \) denote the velocity field of the fluid, \( p \) is pressure, \( f \) denote the external force of the fluid,

\[
\text{div}(\tau(u)) = \left( \sum_{j=1}^d \partial_j \tau_{i,j}(u) \right)_{i=1}^d.
\]

The power law fluid equation defined above is the classical Navier-Stokes equation if \( p = 2 \).

Now we consider the following Gelfand triple

\[
V \subseteq H \subseteq V^*,
\]
where
\[ V = \{ u \in W^{1,p}_0(\Lambda; \mathbb{R}^d) : \text{div}(u) = 0 \}; \quad H = \{ u \in L^2(\Lambda; \mathbb{R}^d) : \text{div}(u) = 0 \}. \]

Let \( P_H \) be the projection operator on \( L^2(\Lambda; \mathbb{R}^d) \to H \). Then we can extend the operator
\[
\mathcal{A} : W^{2,p}(\Lambda; \mathbb{R}^d) \cap V \to H, \quad \mathcal{A}(u) = P_H[\text{div}(\tau(u))];
\]
\[
F : (W^{2,p}(\Lambda; \mathbb{R}^d) \cap V) \times (W^{2,p}(\Lambda; \mathbb{R}^d) \cap V) \to H;
\]
\[
F(u, v) = -P_H[(u \cdot \nabla)v], F(u) := F(u, u)
\]
to the map (see [41]):
\[
\mathcal{A} : V \to V^*; \quad F : V \times V \to V^*.
\]

In particular, we have
\[
\langle \mathcal{A}(u), v \rangle_V = -\int_\Lambda \sum_{i,j=1}^d \tau_{i,j}(u)e_{i,j}(v) dx, \quad u, v \in V;
\]
\[
V \cdot \langle F(u, v), w \rangle_V = -V \cdot \langle F(u, w), v \rangle_V, \quad V \cdot \langle F(u, v), v \rangle_V = 0, \quad u, v, w \in V.
\]

Then the power law fluid equation defined above can be written in variational form:
\[
\partial_t u = \mathcal{A}u(t) + F(u(t)) + f(t), \quad u(0) = u_0.
\]

Now study the following stochastic power law fluid equation
\[
\begin{cases}
\d X(t) = (\nu \mathcal{A}X(t) + F(X(t)) + f(x)) dt + B(X(t))dW(t), \\
X(0) = x \in H,
\end{cases}
\tag{33}
\]

where \( W(t) \) is a cylindrical Wiener process in \( U \) defined on a probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\).

Let \( \mu^\varepsilon \) be the law of \( X(\varepsilon \cdot) \) on \( C([0, T], H) \). We will show the small time LDP and its proof by applying our main result.

**Theorem 4.5.** (stochastic power law fluid equation) Let \( p \geq \frac{d+2}{2} \) and \( B \) satisfy the assumption 2.2. Then (33) has a unique solution \( X(t) \) and \( \mu^\varepsilon \) satisfies the LDP with the rate function \( I(\cdot) \) given by (2).

**Proof.** Assume without loss of generality that viscosity coefficient \( \nu = 1 \). By [47, Lemma 1.19], we have
\[
\int_\Lambda |e(u)|^p dx \geq C_p \|u\|_{W^{1,p}_0}, \quad u \in W^{1,p}_0(\Lambda; \mathbb{R}^d);
\]
\[
\sum_{i,j=1}^d \tau_{i,j}(u)e_{i,j}(u) \geq C(|e(u)|^p - 1);
\]
\[
\sum_{i,j=1}^d (\tau_{i,j}(u) - \tau_{i,j}(v))(e_{i,j}(u) - e_{i,j}(v)) \geq C(|e(u) - e(v)|^2 + |e(u) - e(v)|^p);
\]
\[
|\tau_{i,j}(u)| \leq C(1 + |e(u)|)^{p-1}, \quad i, j = 1, \ldots, d.
\]

According to the inequality above, for any \( u, v \in V \), we have
\[
V \cdot \langle F(u) - F(v), u - v \rangle_V = -V \cdot \langle F(u - v), v \rangle_V
\]
\[
= V \cdot \langle F(u - v), u - v \rangle_V
\]
\[
\leq C\|v\|_V \|u - v\|_{L^{2p}}^{2p}.
\]
Similar to Theorems 4.4 and 4.5, the operators order variant of the power law fluid where the stress tensor has the form
\[ (23) \]
we have
\[ \text{Stochastic Ladyzhenskaya model.} \]

When
\[ p > \frac{d}{2} \]
and shear thinning when
\[ 1 < p < \frac{d}{2}. \]

Let
\[ q = \frac{dp}{d-p}, \gamma = \frac{d}{(d+1)2-p-2d}, \]
by the Gagliardo-Nirenberg interpolation inequality (23) we have
\[ \|v\|_{L^{\frac{2p}{d-p}}} \leq \|v\|_{L^{\frac{2p}{2}}}^{\frac{1}{2}} \|v\|_{L^{2}}^{\frac{1}{2}} \leq C \|v\|_{V}^{\frac{1}{2}} \|v\|_{H}^{\frac{1}{2}}. \]

Since
\[ p \geq \frac{d+2}{2}, \]
the condition (A3) holds.

Therefore, the assertion follows by Lemma 2.1 and Theorem 2.2.

4.6. **Stochastic Ladyzhenskaya model.** The Ladyzhenskaya model is a higher order variant of the power law fluid where the stress tensor has the form
\[ \tilde{\tau}(u): \Lambda \to \mathbb{R}^{d} \otimes \mathbb{R}^{d}, \quad \tilde{\tau}(u) = 2\mu_{0}(1 + |e(u)|^{2})^{{\frac{p-2}{2}}}e(u) - 2\mu_{1}\Delta e(u). \]

This model was pioneered by Ladyzhenskaya [35] and further analyzed by various authors (see [29, 30] and the references therein). Compared to the power law fluids considered above, there is an additional fourth order term \( \text{div}(-2\mu_{1}\Delta e(u)) \) present in the equation. The fluids are shear thinning when \( 1 < p < 2 \) and shear thickening when \( p > 2 \).

Martingale and stationary solutions for this model was established by Guo et al. in [30]. Recently, the small time LDP for this model has been studied for \( d = 2, p \in (1, \frac{5}{2}) \) by Lin and Sun in [37].

Consider the Gelfand triple \( V \subset H \subset V^{*} \), where
\[ V = \left\{ u \in W^{2,2}_{0}(\Lambda; \mathbb{R}^{d}) : \text{div}(u) = 0 \text{ in } \Lambda \right\}; \]
\[ H = \left\{ u \in L^{2}(\Lambda; \mathbb{R}^{d}) : \text{div}(u) = 0 \text{ in } \Lambda, \ u \cdot n = 0 \text{ on } \partial \Lambda \right\}. \]

Let \( P_{H} \) be the orthogonal (Helmholtz-Leray) projection from \( L^{2}(\Lambda, \mathbb{R}^{d}) \) to \( H \).

Similar to Theorems 4.4 and 4.5, the operators
\[ \tilde{A} : C_{c}^{\infty}(\Lambda; \mathbb{R}^{d}) \cap V \to H, \quad N(u) := P_{H} \left[ \text{div}(\tilde{\tau}(u)) \right]; \]
\[ F : (C_{c}^{\infty}(\Lambda; \mathbb{R}^{d}) \cap V) \times (C_{c}^{\infty}(\Lambda; \mathbb{R}^{d}) \cap V) \to H; \]
\[ F(u, v) := -P_H [ (u \cdot \nabla) v], \quad F(u) := F(u, u); \]
can be extended to the well defined operators:
\[ \tilde{\mathcal{A}} : V \to V^*; \quad F : V \times V \to V^*. \]

With these preparations, we can write our model in the abstract form
\[ \begin{aligned}
\frac{dX(t)}{dt} &= (\tilde{\mathcal{A}}(X(t)) + F(X(t)) + f(x))dt + B(X(t))dW(t), \\
X(0) &= x \in H,
\end{aligned} \tag{34} \]
where \( W(t) \) is a cylindrical Wiener process in \( U \) defined on a probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \).

Let \( \mu^\varepsilon \) be the law of \( X(\varepsilon \cdot) \) on \( C([0, T], H) \). We then have the small time LDP by applying our main result, which covers the result in [37]. Since the proof is similar with power law fluid in Theorem 4.5, we omit it here, the reader might refer to [37] for some further detailed calculations.

**Theorem 4.6.** (Stochastic Ladyzhenskaya model) Let \( d = 2, p \in (1, \frac{5}{2}] \) and \( B \) satisfy the assumption 2.2. Then Eq. (34) has a unique solution \( X(t) \) and \( \mu^\varepsilon \) satisfies the LDP with the rate function \( I(\cdot) \) given by (2).

**Remark 4.4.** The restriction on parameter \( p \) allows us to understand the nonlinear term as a perturbation of the linear term. In fact, for general \( d \geq 2 \), using the Gagliardo-Nirenberg inequality, it is possible to find a “maximal” range \((1, p_d]\) of \( p \) to which the locally monotone variational framework could apply (cf. [29]).

**REFERENCES**

[1] H. Abdallah, A Varadhan type estimate on manifolds with time-dependent metrics and constant volume, *J. Math. Pures Appl.*, 99 (2013), 409–418.

[2] S. Aida and H. Kawabi, Short time asymptotics of a certain infinite dimensional diffusion process, *Stochastic Analysis and Related Topics, VII (Kusadasi, 1998)*, Progr. Probab., Birkhäuser Boston, Boston, MA, 48 (1998), 77–124.

[3] S. Aida and T. S. Zhang, On the small time asymptotics of diffusion processes on path groups, *Potential Anal.*, 16 (2002), 67–78.

[4] T. Ariyoshi and M. Hino, Small-time asymptotic estimates in local Dirichlet spaces, *Electron. J. Probab.*, 10 (2005), 1236–1259.

[5] M. Avellaneda, D. Boyer-Olson, J. Busca and P. Friz, Application of large deviation methods to the pricing of index options in finance, *C. R. Math. Acad. Sci. Paris*, 336 (2003), 263–266.

[6] H. Berestycki, J. Busca and I. Florent, Computing the implied volatility in stochastic volatility models, *Comm. Pure Appl. Math.*, 57 (2004), 1352–1373.

[7] Z. Brzéziak, On stochastic convolution in Banach spaces and applications, *Stochastics Stochastics Rep.*, 61 (1997), 245–295.

[8] Z. Brzéziak and S. Peszat, Space-time continuous solutions to SPDE’s driven by a homogeneous Wiener process, *Studia Math.*, 137 (1999), 261–299.

[9] Z. Brzéziak, W. Liu and J. H. Zhu, Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise, *Nonlinear Anal. Real World Appl.*, 17 (2014), 283–310.

[10] A. Budhiraja and P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, *Probab. Math. Statist.*, 20 (2000), 39–61.

[11] A. Budhiraja, P. Dupuis and V. Maroulas, Large deviations for infinite dimensional stochastic dynamical systems, *Ann. Probab.*, 36 (2008), 1390–1420.

[12] Y. Chen, H. J. Gao and L. L. Fan, Well-posedness and the small time large deviations of the stochastic integrable equation governing short-waves in a long-wave model, *Nonlinear Anal. Real World Appl.*, 29 (2016), 38–57.

[13] Z.-Q. Chen, S. Z. Fang and T. S. Zhang, Small time asymptotics for Brownian motion with singular drift, *Proc. Amer. Math. Soc.*, 147 (2019), 3567–3578.

[14] P. L. Chow, Large deviation problem for some parabolic Itô equations, *Comm. Pure Appl. Math.*, 45 (1992), 97–120.
[15] I. Chueshov and A. Millet, *Stochastic 2D hydrodynamical type systems: Well posedness and large deviations*, Appl. Math. Optim., 61 (2010), 379–420.

[16] E. A. Coayla-Teran, P. M. Dias de Magalhães and J. Ferreira, *Existence of optimal controls for SPDE with locally monotone coefficients*, International J. Control, (2018).

[17] B. Davis, *On the L^p-norms of stochastic integrals and other martingales*, Duke Math. J., 43 (1976), 697–704.

[18] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Jones and Bartlett, Boston, MA, 1993.

[19] Z. Dong and R. Zhang, *On the small-time asymptotics of 3D stochastic primitive equations*, Math. Methods Appl. Sci., 41 (2018), 6336–6357.

[20] S. Fang and T. S. Zhang, *On the small-time behavior of Ornstein-Uhlenbeck processes with unbounded linear drifts*, Probab. Theory Related Fields, 114 (1999), 487–504.

[21] J. Feng, J.-P. Fouque and R. Kumar, *Small-time asymptotics for fast mean-reverting stochastic volatility models*, Ann. Appl. Probab., 22 (2012), 1541–1575.

[22] M. Forde and A. Jacquier, *Small-time asymptotics for an uncorrelated local-stochastic volatility model*, Appl. Math. Finance, 18 (2011), 517–535.

[23] M. Forde, A. Jacquier and R. Lee, *The small-time smile and term structure of implied volatility under the Heston model*, SIAM J. Financial Math., 3 (2012), 690–708.

[24] J. Frehse and M. Růžička, *Non-homogeneous generalized Newtonian fluids*, Math. Z., 260 (2008), 355–375.

[25] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Grundlehren der Mathematischen Wissenschaften, 260. Springer-Verlag, New York, 1984.

[26] B. Gess, *Random attractors for singular stochastic evolution equations*, J. Differential Equations, 255 (2013), 524–559.

[27] B. Gess, *Random attractors for degenerate stochastic partial differential equations*, J. Dynam. Differential Equations, 25 (2013), 121–157.

[28] B. Gess, W. Liu and M. Röckner, *Random attractors for a class of stochastic partial differential equations driven by general additive noise*, J. Differential Equations, 251 (2011), 1225–1253.

[29] B. Gess, W. Liu and A. Schenke, *Random attractors for locally monotone stochastic partial differential equations*, J. Differential Equations, 268 (2020), In press.

[30] B. L. Guo, C. X. Guo and J. J. Zhang, *Martingale and stationary solutions for stochastic non-Newtonian fluids*, Differential Integral Equations, 23 (2010), 303–326.

[31] M. Hino and K. Matsunura, *An integrated version of Varadhan’s asymptotics for lower-order perturbations of strong local Dirichlet forms*, Potential Anal., 48 (2018), 257–300.

[32] M. Hino and J. A. Ramirez, *Small-time Gaussian behaviour of symmetric diffusion semigroup*, Ann. Probab., 31 (2003), 1254–1295.

[33] T. Jegaraj, *Small time asymptotics for stochastic evolution equations*, J. Theoret. Probab., 24 (2011), 756–788.

[34] N. V. Krylov and B. L. Rozovskii, *Stochastic evolution equations*, Stochastic Differential Equations: Theory and Applications, 1–69, Interdiscip. Math. Sci., Vol. 2, World Sci. Publ., Hackensack, NJ, 2007.

[35] O. A. Ladyzhenskaya, *New equations for the description of the viscous incompressible fluids and solvability in large of the boundary value problems for them*, Volume V of Boundary Value Problems of Mathematical Physics, Amer. Math. Soc., Providence, 1970.

[36] S. H. Li, W. Liu and Y. C. Xie, *Large deviations for stochastic 3D Leray-α model with fractional dissipation*, Commun. Pure Appl. Anal., 18 (2019), 2491–2510.

[37] H. Liu and C. F. Sun, *On the small time asymptotics of stochastic non-Newtonian fluids*, Math. Methods Appl. Sci., 40 (2017), 1139–1152.

[38] W. Liu, *Harnack inequality and applications for stochastic evolution equations with monotone drifts*, J. Evol. Equ., 9 (2009), 747–770.

[39] W. Liu, *Large deviations for stochastic evolution equations with small multiplicative noise*, Appl. Math. Optim., 61 (2010), 27–56.

[40] W. Liu and M. Röckner, *SPDE in Hilbert space with locally monotone coefficients*, J. Funct. Anal., 259 (2010), 2902–2922.

[41] W. Liu and M. Röckner, *Local and global well-posedness of SPDE with generalized coercivity conditions*, J. Differential Equations, 254 (2013), 725–755.

[42] W. Liu and M. Röckner, *Stochastic Partial Differential Equations: An Introduction*, Universitext, Springer, Cham, 2015.
[43] W. Liu, M. Röckner and J. L. da Silva, Quasi-linear (stochastic) partial differential equations with time-fractional derivatives, SIAM J. Math. Anal., 50 (2018), 2588–2607.

[44] W. Liu, M. Röckner and X.-C. Zhu, Large deviation principles for the stochastic quasi-geostrophic equations, Stochastic Process. Appl., 123 (2013), 3299–3327.

[45] W. Liu, C. Tao and J. Zhu, Large deviation principle for a class of SPDE with locally monotone coefficients, Sci. China Math., (2020), In press.

[46] T. Ma and R.-C. Zhu, Wong-Zakai approximation and support theorem for SPDEs with locally monotone coefficients, J. Math. Anal. Appl., 469 (2019), 623–660.

[47] J. Málek, J. Nečas, M. Rokyta and M. Růžička, Weak and Measure-Valued Solutions to Evolutionary PDEs, Applied Mathematics and Mathematical Computation, 13. Chapman & Hall, London, 1996.

[48] E. Pardoux, Équations aux Dérivées Partielles Stochastiques non Linéaires Monotones, Thèse de Doctorat, Université Paris-Sud, 1975.

[49] J. G. Ren and X. C. Zhang, Freidlin-wentzell’s large deviations for stochastic evolution equations, J. Funct. Anal., 254 (2008), 3148–3172.

[50] M. Röckner and T. S. Zhang, Stochastic 3D tamed Navier-Stokes equations: Existence, uniqueness and small time large deviation principles, J. Differential Equations, 252 (2012), 716–744.

[51] J. Seidler, Exponential estimates for stochastic convolutions in 2-smooth Banach spaces, Electron. J. Probab., 15 (2010), 1556–1573.

[52] D. W. Stroock, An Introduction to the Theory of Large Deviations, Universitext. Springer-Verlag, New York, 1984.

[53] K. Taira, Analytic Semigroups and Semilinear Initial Boundary Value Problems, London Mathematical Society Lecture Note Series, 223. Cambridge University Press, Cambridge, 1995.

[54] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, Second edition, Johann Ambrosius Barth, Heidelberg, 1995, 532 pp.

[55] S. R. S. Varadhan, Asymptotic probabilities and differential equations, Comm. Pure Appl. Math., 19 (1966), 261–280.

[56] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math., 20 (1967), 431–455.

[57] S. R. S. Varadhan, Diffusion processes in a small time interval, Comm. Pure Appl. Math., 20 (1967), 659–685.

[58] F.-Y. Wang, Exponential convergence of non-linear monotone SPDEs, Discrete Contin. Dyn. Syst., 35 (2015), 5239–5253.

[59] J. Xiong and J. L. Zhai, Large deviations for locally monotone stochastic partial differential equations driven by Lévy noise, Bernoulli, 24 (2018), 2842–2874.

[60] T. G. Xu and T. S. Zhang, On the small time asymptotics of the two-dimensional stochastic Navier-Stokes equations, Ann. Inst. Henri Poincaré Probab. Stat., 45 (2009), 1002–1019.

[61] R. Zhang, On the small time asymptotics of scalar stochastic conservation laws, arXiv:1907.03397.

[62] T. S. Zhang, On the small time asymptotics of diffusion processes on Hilbert spaces, Ann. Probab., 28 (2000), 537–557.

[63] X. C. Zhang, Stochastic Volterra equations in Banach spaces and stochastic partial differential equation, J. Funct. Anal., 258 (2010), 1361–1425.

[64] J. H. Zhu, Z. Brzeźniak and W. Liu, Maximal inequalities and exponential estimates for stochastic convolutions driven by Lévy-type processes in Banach spaces with application to stochastic quasi-geostrophic equations, SIAM J. Math. Anal., 51 (2019), 2121–2167.

Received for publication August 2019.

E-mail address: shihuli@jsnu.edu.cn
E-mail address: weiliu@jsnu.edu.cn
E-mail address: ycxie@jsnu.edu.cn