Bounds for, and calculation of, the multipole moments of stationary spacetimes

Magnus Herberthson
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden.
E-mail: maher@mai.liu.se

Abstract. The multipole moments of stationary asymptotically flat spacetimes are considered. We demonstrate how the tensorial recursion of Geroch and Hansen can be replaced by a scalar recursion on $R^2$. We also give a bound on multipole moments. This confirms the “necessary part” of a long standing conjecture due to Geroch.

1. Introduction
The relativistic multipole moments of asymptotically flat spacetimes have been defined by Geroch [6] for static spacetimes and the generalized to the stationary case by Hansen [7]. The tensorial recursion which defines the multipole moments (1), is computationally rather complicated as it stands. In the axisymmetric (static or stationary) case, on the order hand, it was shown, [2], [3] that the recursion can be replaced by a scalar recursion on $R$, and that all the moments can be collected into one complex valued function $y$ on $R$, where the moments are given by the derivatives of $y$ at $0$. In this paper, we demonstrate that also in the general stationary case, the recursion (1) can be simplified to a scalar recursion, this time on $R^2$. This is accomplished by using normal coordinates, complex null geodesics and exploiting the conformal freedom of the conformal compactification.

Using this simplification we can partially confirm an extension of a long standing conjecture by Geroch [6]:

Given any set of multipole moments, subject to the appropriate convergence condition, there exists a static solution of Einstein’s equations having precisely these moments.

This conjecture has its natural extension to the stationary case. In this paper we will state the appropriate convergence condition in the general stationary case, which is necessary for the existence of a stationary solution to Einstein’s equations. In this presentation, the set-up and necessary theorems are provided with motivation. For proofs of the theorems, we refer to [4].

2. Multipole moments of stationary spacetimes
In this section we quote the definition of multipole moments given by Geroch and Hansen [6], [7]. Thus, consider a stationary spacetime $(M,g_{ab})$ with timelike Killing vector field $\xi^a$. We let $\lambda = -\xi^a\xi_a$ be the norm, and define the twist $\omega$ through $\nabla_a\omega = \epsilon_{abcd}\nabla^b\xi^c\xi^d$. If $V$ is the 3-manifold of trajectories, the metric $g_{ab}$ (with signature $(-,+,+,+)$) induces the positive definite
metric $h_{ab} = \lambda g_{ab} + \xi_a \xi_b$ on $V$. It is required that $V$ is asymptotically flat, i.e., there exists a 3-manifold $\hat{V}$ and a conformal factor $\Omega$ satisfying

(i) $\hat{V} = V \cup \Lambda$, where $\Lambda$ is a single point
(ii) $\hat{h}_{ab} = \Omega^2 h_{ab}$ is a smooth metric on $\hat{V}$
(iii) At $\Lambda, \Omega = 0, \hat{D}_a \Omega = 0, \hat{D}_a \hat{D}_b \Omega = 2 \hat{h}_{ab}$

where $\hat{D}_a$ is the derivative operator associated with $\hat{h}_{ab}$. On $M$, and/or $V$ one defines the scalar potential

$$\phi = \phi_M + i \phi_J, \quad \phi_M = \frac{\lambda^2 + \omega^2 - 1}{4 \lambda}, \quad \phi_J = \frac{\omega}{2 \lambda}.$$  

The multipole moments of $M$ are then defined on $\hat{V}$ as certain derivatives of the scalar potential $\hat{\phi} = \phi/\sqrt{\Omega}$ at $\Lambda$. Following [7], let $\hat{R}_{ab}$ denote the Ricci tensor of $\hat{V}$, and let $P = \hat{\phi}$. Define the sequence $P, P_{a1}, P_{a1a2}, \ldots$ of tensors recursively:

$$P_{a_1 \ldots a_n} = C[\hat{D}_{a_1} P_{a_2 \ldots a_n} - \frac{(n-1)(2n-3)}{2} \hat{R}_{a_1 a_2} P_{a_3 \ldots a_n}],$$  

where $C[ \cdot \cdot \cdot ]$ stands for taking the totally symmetric and trace-free part. The multipole moments of $M$ are then defined as the tensors $P_{a_1 \ldots a_n}$ at $\Lambda$. The requirement that all $P_{a_1 \ldots a_n}$ be totally symmetric and trace-free makes the actual calculations very cumbersome.

In [1], [9] it was shown that (when the mass is non-zero) there exist a conformal factor $\omega$ and a chart, such that all components of the metric $\hat{h}_{ab}$ and the potential $\hat{\phi}$ are analytic in terms of the coordinates, in a neighbourhood of the infinity point. Expressed in these coordinates the exponential map becomes analytic. Therefore, we can use Riemannian normal coordinates, the exponential map becomes analytic. Therefore, we can use Riemannian normal coordinates and still have analyticity of the metric components and the potential. If the mass is zero, this analyticity condition will be assumed. Thus, henceforth we assume that $\Omega$ is chosen such that the (rescaled) metric and potential are analytic in a neighbourhood of $\Lambda$.

3. Multipole moments through a scalar recursion on $R^2$

Suppose that $(x^1, x^2, x^3) = (x, y, z)$ are normal coordinates (with respect to $\hat{h}_{ab}$) centered around $\Lambda$. This means that for any constants $a = a^1$, $b = a^2$, $c = a^3$ the curve $t \rightarrow (at, bt, ct)$ is a geodesic, i.e., in terms of coordinates that

$$\ddot{x}^i + \Gamma_{kl}^{i} \dot{x}^k \dot{x}^l = \Gamma_{kl}^{i} a^k a^l = 0$$

where the Christoffel symbols are evaluated at $(at, bt, ct)$ for appropriate $t$. Due to analyticity, this relation holds for complex values of $a, b, c$, i.e., we can consider geodesics in the complexification $\hat{V}_C$ of $\hat{V}$. Of particular interest is the one-parameter family of curves:

$$\gamma_\phi : t \rightarrow (t \cos \phi, t \sin \phi, it), \quad t \in [0, t_0], \quad \phi \in [0, 2\pi)$$

for some suitable $t_0$. The tangent vector $\eta^a = \eta^a_\phi(t) = \cos \phi \left( \frac{\partial}{\partial \phi} \right)^a + \sin \phi \left( \frac{\partial \phi}{\partial \phi} \right)^a + i \left( \frac{\partial \phi}{\partial \phi} \right)^a$ is seen to be a complex null vector along $\gamma_\phi$. Namely, from $\eta^a \hat{D}_a \eta^b = 0$, we infer that $\eta^a \hat{D}_a (\eta^b \eta_b) = 0$. The (constant) value of $\eta^b \eta_b$ is then found to be zero by evaluation at $t = 0$.

Next, consider the mapping $F : R^2 \rightarrow \hat{V}_C : (\xi, \zeta) \rightarrow (\xi, \zeta, i\sqrt{\xi^2 + \zeta^2})$. We let $S$ denote the 2-surface $F(U) \subset \hat{V}_C$, where $U \subset R^2$ is a suitable neighbourhood of $(\xi, \zeta) = (0, 0)$. $S$ is then a smooth surface, except at $\Lambda$ where it has a vertex point, closely resembling a null cone in a three dimensional Lorentzian space. The curves $\gamma_\phi$ are given by $\gamma_\phi(t) = F(t \cos \phi, t \sin \phi)$, and in particular $\eta^a$ lies along $S$. This suggests that we use the polar coordinates $\rho, \phi$ around $2$.
Λ on S defined via \( \xi = \rho \cos \varphi, \zeta = \rho \sin \varphi \). We now follow the approach from [3], where a useful vector field \( \eta^a \) on \( \hat{V} \) was introduced. In [3], where the spacetime was axisymmetric, \( \eta^a \) was explicitly expressed in terms of the metric cast in the Weyl-Papapetrou form [10], and was defined on the whole of \( \hat{V} \) except on the symmetry axis. In this paper, the axisymmetry condition is dropped, which makes the construction of a corresponding \( \eta^a \) more difficult. In addition, a general spacetime has \( 2n + 1 \) degrees of freedom for the multipole moment of order \( 2^n \), compared to one degree of freedom in the axisymmetric case [8]. Nevertheless, it will turn out to be sufficient to know the potential \( \hat{\phi} \) on S to determine all the moments. On S, \( \eta^a \) has the following properties

**Lemma 1** Suppose \( \hat{V} \) and \( S \) are defined as above. Then there exists a regularly direction dependent (at \( \Lambda \)) vector field \( \eta^a \) on S with the following properties:

a) \( \eta^a \hat{D}_a \eta^b \) is parallel to \( \eta^b \);

b) For all tensors \( T_{a_1 \ldots a_n}, \eta^{a_1} \ldots \eta^{a_n} T_{a_1 \ldots a_n} = \eta^{a_1} \ldots \eta^{a_n} C[T_{a_1 \ldots a_n}] \);

c) At \( \Lambda \), \( P_{a_1 \ldots a_n} \) (in \( \hat{V} \)) is determined by \( \eta^{a_1} \ldots \eta^{a_n} P_{a_1 \ldots a_n} \) (on S).

a) was demonstrated above, b) follows as in [3], while c) is proven in [4].

We can now replace the recursion (1) on \( \hat{V} \) with a scalar recursion on S. Again, we follow [3], and define

\[
\eta_{a_1 \ldots a_n} = \eta^{a_1} \ldots \eta^{a_n} P_{a_1 \ldots a_n}, \quad n = 0, 1, 2, \ldots
\] (2)

on S. In particular, \( f_0 = P = \hat{\phi} = \phi / \sqrt{\Omega} \). The moments \( P_{a_1 \ldots a_n} (\Lambda) \) will now be encoded in the trigonometric polynomial given by the direction dependent limit \( \lim_{\rho \to 0} f_n (\rho, \varphi) \), which takes the form

\[
\lim_{\rho \to 0} f_n (\rho, \varphi) = \sum_{j=0}^{n} a_j \cos^j \varphi \sin^{n-j} \varphi + i \sum_{j=0}^{n-1} b_j \cos^j \varphi \sin^{n-1-j} \varphi
\] (3)

Even if the moments are encoded in the coefficients \( a_n \) and \( b_n \), this encoding is dependent on the choice of normal coordinates, i.e., the orientation of the coordinate axes in \( T_\Lambda \hat{V}_\xi \). Also note that although \( P_{a_1 \ldots a_n} \) is analytic on \( \hat{V} \), \( f_n \) will not be analytic in terms of \( \xi, \zeta \) since \( \eta^a \) is direction dependent at \( \Lambda \in S \). In general we have the following lemmas, where we again refer to [4] for proofs:

**Lemma 2** Suppose \( f \) is an analytic function, on a ball of radius \( r_0 \) around \( \Lambda \) on \( \hat{V} \). Then the restriction of \( f \) to \( S \), \( f_L \), can be decomposed as \( f_L (\xi, \zeta) = f_1 (\xi, \zeta) + ip f_2 (\xi, \zeta) \), where \( f_1 \) and \( f_2 \) are analytic in terms of \( \xi \) and \( \zeta \) on the disk \( \xi^2 + \zeta^2 < r_0^2 / 2 \), and where \( \rho = \sqrt{\xi^2 + \zeta^2} \). Furthermore, if \( f \) is real-valued, then \( f_1 \) and \( f_2 \) are real-valued.

**Lemma 3** Suppose \( T_{a_1 \ldots a_n} \) is an analytic tensor field in a ball of radius \( r_0 \) around \( \Lambda \) on \( \hat{V} \). Then the scalar field \( f_L = \eta^{a_1} \ldots \eta^{a_n} T_{a_1 \ldots a_n} \) on \( S \) can be written as \( f_L (\xi, \zeta) = \rho^{-n} (f_1 (\xi, \zeta) + ip f_2 (\xi, \zeta)) \), where \( f_1 \) and \( f_2 \) are analytic (on the disk with radius \( r_0 / \sqrt{2} \) around the origin) in terms of \( \xi \) and \( \zeta \), and where \( \rho = \sqrt{\xi^2 + \zeta^2} \). Furthermore, if \( T_{a_1 \ldots a_n} \) is real-valued, then \( f_1 \) and \( f_2 \) are real-valued.

Remark: In \( f_L \), the subscript \( L \) stands for the ‘leading term’. We also note that the bound character of \( f_L (\xi, \zeta) = \rho^{-n} (f_1 (\xi, \zeta) + ip f_2 (\xi, \zeta)) \) when \( \rho \to 0 \) implies that both \( f_1 \) and \( f_2 \) have zeroes of sufficient order at \( (\xi, \zeta) = (0, 0) \). It also implies that \( f_L \) will be direction dependent there.

We can now contract (1) with \( \eta^a \) and get the following theorem

**Theorem 4** Let \( \hat{V} \) and \( S \) defined as in sections 2 and 3. Let \( \eta^a \) have the properties given by Lemma 1, and let \( f_n \) be defined by (2). Then the recursion (1) on \( \hat{V} \) takes the form
\[ f_n = \eta^a \tilde{D}_a f_{n-1} - \frac{1}{2} (n-1)(n-2) \eta^a \eta^b \tilde{R}_{ab} f_{n-2} \] (4)

on \( S \). The moments of order \( 2^n \) are captured in the direction dependent limit \( \lim_{\nu \to 0} f_n(\rho, \varphi) \).

**Proof.** That (1) takes the form (4) follows exactly as in [3] using that \( \eta^a \tilde{D}_a \eta^b = 0 \), although the recursion is defined on \( S \) rather than on \( \tilde{V} \). The last statement is the contents of Lemma 1c.

4. **Simplified calculation of the moments**

We will now demonstrate that it is possible to obtain the recursion (4) without the term involving the Ricci tensor. This will be accomplished by using the conformal freedom at hand, i.e., change \( \Omega \). The conformal freedom is \( \Omega \to \tilde{\Omega} / \alpha \) where \( \alpha \) is analytic near \( \Lambda \) with \( \alpha(\Lambda) = 1 \). \( \tilde{D}_a(1/\alpha) \) at \( \Lambda \) gives a shift of the moments which corresponds to a ‘translation’ of the physical space, \[ -6 \]. Hence we can assume that \( \tilde{D}_a(1/\alpha) = 0 \). It is to be noted that a change of \( \Omega \) changes the (rescaled) potential \( \phi / \sqrt{\tilde{\Omega}} \). It also changes the normal coordinates on \( \tilde{V} \) and hence all conclusions must be drawn with some care. In order to derive the simplified recursion (7), we will specify an \( \alpha \) through \( \alpha_L \), the restriction of \( \alpha \) to \( S \). However, in order to deduce that there exists a real-valued function \( \alpha \) with the prescribed values of \( \alpha_L \), we need to say more on the representation of \( \alpha_L \).

This result and a useful estimate is the contents of Lemma 5.

**Lemma 5** Let \( f_L(\xi, \zeta) = f_1(\xi, \zeta) + ipf_2(\xi, \zeta) \), where \( f_1 \) and \( f_2 \) are analytic on the ball \( U = \{|\xi|^2 + |\zeta|^2 < r_0^2\} \) and where \( \rho = \sqrt{\xi^2 + \zeta^2} \). We can then write

\[ f_L(\rho \cos \varphi, \rho \sin \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{l,m} e^{im\varphi} \rho^l \] (5)

where

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |c_{l,m}| e^{im\varphi} \rho^l < \infty, \quad \rho < r_0 \] (6)

Furthermore, the converse is true; if \( f_L \) is a function satisfying (5) and (6) then there are functions \( f_1 \) and \( f_2 \) analytic in \( U \) such that \( f_L = f_1 + ipf_2 \). The functions \( f_1 \) and \( f_2 \) are real-valued if and only if the coefficients \( c_{l,m} \) satisfy \( c_{l,m} = (-1)^{l-m} c_{l,-m} \).

This is proven by studying the functions \( f_1(\frac{x+y}{2}, \frac{x-y}{2}) \) and \( f_2(\frac{x+y}{2}, \frac{x-y}{2}) \), which are analytic in terms of \( x \) and \( y \).

We now state that we can choose \( \alpha \) such that the Ricci term in (4) vanishes.

**Lemma 6** There exists a real-valued real analytic function \( \alpha \) on \( \tilde{V} \) such that the Ricci tensor \( \tilde{R}_{ab} \) of the new metric \( \tilde{h}_{ab} = \alpha^{-2} h_{ab} \) satisfies \( \eta^a \eta^b \tilde{R}_{ab} = 0 \) on \( \tilde{S} \), where both \( \tilde{\eta}^a \), \( \tilde{S} \) and the mapping \( F \) are defined in terms of coordinates which are normal with respect to the new metric \( \tilde{h}_{ab} \).

The proof of this uses the conformal properties of the Ricci tensor [10], together with well known theory for ODEs, and as a first step one shows that it is possible to make \( \eta^a \eta^b \tilde{R}_{ab} = 0 \) on \( S \). However, it then also follows that \( \tilde{\eta}^a \tilde{\eta}^b \tilde{R}_{ab} = 0 \) on \( \tilde{S} \). For, from \( \alpha \) we get \( \tilde{\Omega} = \Omega \alpha \), and the corresponding new metric \( \tilde{h}_{ab} = \alpha^{-2} h_{ab} \). Note that in (4), where now \( \eta^a \eta^b \tilde{R}_{ab} = 0 \), the recursion is stated in terms of \( \tilde{D}_a \), i.e., it is expressed in terms of \( \tilde{h}_{ab} \) instead of \( h_{ab} \). From \( \tilde{h}_{ab} \) we get new normal coordinates, i.e., \( (x, y, z) \) in \( T_{\Lambda} \tilde{V}_C \), are mapped into \( V_C \) using the exponential map belonging to using the exponential map belonging to \( h_{ab} \). We then construct \( \tilde{\eta}^a \) and the mapping \( F \) with respect to these coordinates.

\[ 1 \quad \text{A more natural condition is the equivalent statement } \Omega \to \tilde{\Omega} = \Omega \alpha. \text{ However, this formulation gives slightly neater calculations.} \]
Now, null geodesics, of which $S$ consists, are conformally invariant, although they become non-affinely parametrised. This means that $\tilde{\eta}^a \propto \eta^a$, and points in $S$ are mapped into points in $\tilde{S}$. Thus $\tilde{\eta}^a \tilde{\eta}^b \tilde{R}_{ab} \propto \eta^a \eta^b \tilde{R}_{ab} = 0$ on $\tilde{S}$ (or $S$).

Henceforth we denote all entities defined via $\tilde{h}_{ab}$ instead of $h_{ab}$ with a tilde. In particular, $\tilde{D}_a$ will denote the derivative operator associated with $\tilde{h}_{ab}$. Applying Lemma 6 to Theorem 4 we immediately get the following theorem.

**Theorem 7** Let $\tilde{V}$ and $\tilde{S}$ defined as in sections 2 and 3, where $\tilde{S}$ is defined in terms of the normal coordinates associated to $\tilde{h}_{ab}$. Let $\tilde{\eta}^a$ have the properties given by Lemma 1 with respect to $\tilde{h}_{ab}$, and let $\tilde{f}_n$ be defined by (2) with $\tilde{\eta}^a$ in the place of $\eta^a$. Then the recursion (1) on $V$ takes the form

$$\tilde{f}_n = \tilde{\eta}^a \tilde{D}_a \tilde{f}_{n-1} = (\tilde{\eta}^a \tilde{D}_a)^n \tilde{f}_0 = \frac{\partial^n}{\partial \tilde{\rho}^n} \tilde{f}_0$$

(7)

on $\tilde{S}$. The moments of order $2^n$ are captured in the direction dependent limit $\lim_{\tilde{\rho} \to 0} \tilde{f}_n(\tilde{\rho}, \varphi)$.

## 5. Bounds on the moments

All multipole moments are encoded in $\tilde{f}_0$, and we note that the recursion (7) is identical to the recursion emanating from a scalar function in $\mathbb{R}^3$ (after inversion). It is clear that many different functions on $\mathbb{R}^3$ will produce the same moments, but if we also require that the function, $g$ say, is harmonic, $g$ is uniquely determined by the moments.

Thus, provided that we can connect a function which is harmonic in a neighbourhood of $0 \in \mathbb{R}^3$ to each $\tilde{f}_0$, we have the following theorem:

**Theorem 8** Suppose that $(M, g_{ab})$ is a stationary asymptotically flat spacetime, admitting analytic (rescaled) potential and an analytic chart on the conformally compactified manifold of timelike Killing trajectories, around the infinity point $\Lambda$. Then there exists a (flat-)harmonic function $g$ in a neighbourhood of $0 \in T_\Lambda \tilde{V} \cong \mathbb{R}^3$, such that all multiple moments of $M$ are given by

$$P_{a_1 \cdots a_n}(\Lambda) = (\nabla a_1 \cdots \nabla a_n g)(0),$$

(8)

where $\nabla a_i$ in the LHS of (8) is the flat derivative operator in $\mathbb{R}^3$. This puts a bound on the multipole moments, since the Taylor expansion $\sum_{|\alpha| \geq 0} \frac{\alpha^n}{\alpha!} (\partial_\alpha g)(0)$ of $g$ converges in a neighbourhood of the origin in $\mathbb{R}^3$.

The proof, which again is found in ([4]), starts by taking the function $\tilde{f}_L$ in the form (5), which fully determines the moments. Next, one considers

$$g(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m+l} a_{l,m} Y^m_l(\theta, \varphi) r^l,$$

(9)

where

$$a_{l,m} = c_{l,m} i^{-m-l} 2^{l+1} \pi \frac{\sqrt{(l+m)! (l-m)!}}{\Gamma(l + \frac{1}{2}) \sqrt{2l + 1}}$$

It is then possible to show that the RHS of (9) converges for $r < r_0$, and that (9) defines a function on $\mathbb{R}^3$ which is analytic in a neighbourhood of the origin. Finally, it is shown that $g$ produces the same moments as $\tilde{f}_L$.

2 As discussed before, the analyticity has been proved for the case of non-zero mass.
6. Discussion
In this paper we have studied the multipole moments of stationary asymptotically flat spacetimes. By using normal coordinates, and by exploiting the conformal freedom, we could demonstrate that the tensorial recursion (1) could be replaced by the scalar recursion (7). This recursion is a direction dependent recursion on $\mathbb{R}^2$, where the moments are encoded in the direction dependent limits at $\Lambda$.

Using this setup, we could also show that the multipole moments cannot grow too fast. In essence, the rescaled potential behaves (locally) in the manner of a harmonic function on $\mathbb{R}^3$. The bounds on the moments given in theorem 8 gives the necessary part in a conjecture due to Geroch [6], and it is of course tempting to conjecture that this condition on the moments also will be sufficient (as long as the monopole is real-valued).

Whether this can be proved using the techniques presented here is still an open question.

We also remark that similar questions concerning the convergence of of asymptotic expansions in the static case are currently being studied by Friedrich, using a different technique, [3].

References
[1] Beig R and Simon W 1981 Proc. R. Soc. London A 376 333
[2] Bäckdahl T and Herberthson M 2005 Class. Quantum Grav. 22 1607
[3] Bäckdahl T and Herberthson M 2005 Class. Quantum Grav. 22 3585
[4] Bäckdahl T and Herberthson M 2006 Class. Quantum Grav. 23 5997
[5] Friedrich H 2005 On the convergence of certain expansions at space-like infinity of asymptotically flat, static vacuum solutions http://www.newton.cam.ac.uk/webseminars/pg+ws/2005gmr/1202/friedrich/
[6] Geroch R 1970 J. Math. Phys. 11 2380
[7] Hansen R O 1974 J. Math. Phys. 15 46
[8] Herberthson M 2004 Class. Quantum Grav. 21 5121
[9] Kundu P 1981 J. Math. Phys. 22 2006
[10] Wald R M 1984 General Relativity (Chicago: The University of Chicago Press)