Performing $n$ steps of $\beta$-reduction to a given term in the $\lambda$-calculus can lead to an increase in the size of the resulting term that is exponential in $n$. The same is true for the possible depth increase of terms along a $\beta$-reduction sequence. We explain that the situation is different for the leftmost-outermost strategy for $\beta$-reduction: while exponential size increase is still possible, depth increase is bounded linearly in the number of steps. For every $\lambda$-term $M$ with depth $d$, in every step of a leftmost-outermost $\beta$-reduction rewrite sequence starting from $M$ the term depth increases by at most $d$. Hence the depth of the $n$-th reduct of $M$ in such a rewrite sequence is bounded by $d \cdot (n + 1)$.

We prove the lifting of this result to $\lambda$-term representations as orthogonal first-order term rewriting systems, which can be obtained by the lambda-lifting transformation. For the transfer to $\lambda$-calculus, we rely on correspondence statements via lambda-lifting. We argue that the linear-depth-increase property can be a stepping stone for an alternative proof of, and so can shed new light on, a result by Accattoli and Dal Lago (2015) that states: leftmost-outermost $\beta$-reduction rewrite sequences of length $n$ in the $\lambda$-calculus can be implemented on a reasonable machine with an overhead that is polynomial in $n$ and the size of the initial term.

Keywords: lambda calculus, beta reduction, leftmost-outermost strategy, complexity

1 Introduction

Accattoli and Dal Lago \cite{AC, AD} proved that the number of steps in a leftmost-outermost rewrite sequence to normal form provides an invariant cost model for the $\lambda$-calculus, in the following sense. There is an implementation $I$ on a reasonable machine (e.g., a Turing machine, or a random access machine) of the partial function that maps a $\lambda$-term to its normal form, whenever that exists, such that $I$ has the following property: there are integer polynomials $p(x, y)$ and $q(x)$ such that if a $\lambda$-term $N$ is the result of $n$ successive leftmost-outermost $\beta$-reduction steps performed to a $\lambda$-term $M$ of size $m$, then $I$ obtains a compact representation $C$ of $N$ from $M$ in time bounded by $p(n, m)$, and $N$ can be obtained from $C$ in time bounded by $q(\|N\|)$ where $\|N\|$ is the symbol size of the represented $\lambda$-term $N$.

To achieve this result, Accattoli and Dal Lago describe how to simulate leftmost-outermost $\beta$-reduction rewrite sequences in the $\lambda$-calculus by ‘leftmost-outermost useful’ rewrite sequences in the linear explicit substitution calculus. They employ substitution steps only insofar as those are needed to create the leftmost-outermost $\beta$-redex (representation), or to make it visible. In this way they work with shared representations of $\lambda$-terms in order to avoid size explosion. Then they show that ‘leftmost-outermost useful’ rewrite sequences in the linear explicit substitution calculus can be implemented on a reasonable machine with a polynomial overhead dependent on the length of the sequence, and the size of the initial term.

\* That the represented $\lambda$-term $N$ must be computable from its compact representation $C$ in time bounded by the size of $N$, which is implicit in the result of \cite{AC, AD}, is crucial to prevent ‘hiding’ of reduction work in the computation of ‘pretty printing’ $C$ as $N$. 

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\textit{Linear Depth Increase of Lambda Terms along Leftmost-Outermost Beta-Reduction}

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My goal is to connect this result with graph reduction techniques that are widely used for the compilation and runtime-evaluation of functional programs. In particular, I would like to obtain a graph rewriting implementation for leftmost-outermost $\beta$-reduction in the $\lambda$-calculus that demonstrates this result, but that is close in spirit to graph reduction as it is used in runtime evaluators for functional programming languages. My idea is to describe a port graph grammar [11] implementation that is based on TRS (term rewrite system) representations of $\lambda$-terms. These $\lambda$-term representations correspond closely to supercombinator systems that are obtained by lambda-lifting, as first described by Hughes [9].

That such an implementation is conceivable by employing subterm-sharing is suggested by a property of (plain, unshared) leftmost-outermost $\beta$-reduction rewrite sequences in the $\lambda$-calculus that we will show. The depth increase in each step of an arbitrarily long leftmost-outermost $\beta$-reduction rewrite sequence from a $\lambda$-term $M$ is uniformly bounded by $|M|$, the depth of $M$. As a consequence, for the depth of the $n$-th reduct $L_n$ of a $\lambda$-term $L_0$ in a leftmost-outermost $\beta$-reduction rewrite sequence $L_0 \rightarrow_{lo} L_1 \rightarrow_{lo} \cdots \rightarrow_{lo} L_n \rightarrow_{lo} \cdots$ it holds that: $|L_n| \leq |L_0| \cdot (n + 1)$, and hence $|L_n|/|L_0| \in O(n)$.

In the terminology of [1, 2] this property shows that leftmost-outermost rewrite sequences do not cause ‘depth explosion’ in $\lambda$-terms. This contrasts with general $\rightarrow_\beta$ rewrite sequences, among which the depth of terms may increase exponentially. The example below provides an illustration.

**Example 1** (‘depth-exploiting’ family under $\beta$-reduction, from Asperti and Lévy [3]). Consider the following families \{$M_i\}_{i \in \mathbb{N}}$ and \{$N_i\}_{i \in \mathbb{N}}$ of $\lambda$-terms:

\[
M_0 := xx, \quad N_0 := M_0 = xx,
\]

\[
M_{i+1} := \text{two}(\lambda x.M_i)x \quad (\text{for } n \in \mathbb{N}), \quad N_{i+1} := N_i[x := N_i] \quad (\text{for } n \in \mathbb{N}),
\]

where $\text{two} := \lambda x.\lambda y.x(xy)$ is the Church numeral for 2. By induction on $i$ it can be verified that it holds:

\[
|M_i| = \begin{cases} 1 & \text{if } i = 0 \\ 3(i + 1) & \text{if } i \geq 1 \end{cases} \in O(i) \quad M_i \rightarrow_\beta^i N_i \quad |N_i| = 2^i \in \Omega(2^i)
\]

and that the syntax tree of $N_i$ is the complete binary application tree with $2^i$ occurrences of $x$ at depth $|N_i| = 2^i$. The induction step for the statement on the rewrite sequence can be performed as follows:

\[
M_{i+1} = \text{two}(\lambda x.M_i)x \rightarrow_\beta^i \text{two}(\lambda x.N_i)x = (\lambda x.\lambda y.x(xy))(\lambda x.N_i)x \quad (\text{by the induction hypothesis})
\]

\[
\rightarrow_\beta (\lambda y.(\lambda x.N_i)((\lambda x.N_i)y))x
\]

\[
\rightarrow_\beta (\lambda x.N_i)((\lambda x.N_i)x) \rightarrow_\beta (\lambda x.N_i)N_i \rightarrow_\beta N_i[x := N_i] = N_{i+1}
\]

This $\rightarrow_\beta$ rewrite sequence is not leftmost-outermost, but it proceeds mainly in inside-out direction.

Let $i \geq 1$. Then for $n = 4i$ and $M := M_i$ it follows that $M_i$ reduces to its normal form $N_i$ in precisely $n$ $\beta$-reduction steps $M = M_i = L_0 \rightarrow_\beta L_1 \rightarrow_\beta \cdots \rightarrow_\beta L_n = N_i$, for with reducts $L_0, \ldots, L_n$, that the depth of the initial term is $|L_0| = |M| = |M_i| = 3(i + 1) \leq 4i = n$, and the depth of the final term is $|L_n| = |N_i| = |N_i| = 2^i = 2^{n/4}$. From this it follows $|L_n|/|L_0| \geq 2^{n/4}/n$.

This argument shows that for the relative depth increase of $\beta$-reduction rewrite sequences $L_0 \rightarrow_\beta^i L_n$ of length $n$ is exponential, because it holds in any case that $|L_n|/|L_0| \in \Omega(2^{(1/4) - \epsilon} n)$ for every $\epsilon > 0$.

Such an exponential depth increase with respect to general $\beta$-reduction contrasts sharply with the linear-depth-increase property of leftmost-outermost $\beta$-reduction that we will show here. We now lay out the basic insight that is at the basis of this result.

Underlying property, leading to the linear-depth-increase result. For every leftmost-outermost β-reduction rewrite sequence $L_0 \xrightarrow{\text{loβ}} L_1 \xrightarrow{\text{loβ}} \cdots \xrightarrow{\text{loβ}} L_n \xrightarrow{\text{loβ}} L_{n+1} (\xrightarrow{\text{loβ}} \cdots)$ in the λ-calculus, the following property can be shown: if $(\lambda z.P)Q$ is the leftmost-outermost β-redex in the $n$-th reduct $L_n$, its abstraction part has a representation $\lambda z.P \equiv (\lambda z.P_0)[z_1 := P_1, \ldots, z_k := P_k]$ with ‘scope part’ $\lambda z.P_0$ and ‘free subexpressions’ $P_0, P_1, \ldots, P_k$, where $z_1, \ldots, z_k \neq z$ are distinct variables that are free in $P_0$, such that an abstraction of the form $\lambda z.P' \equiv (\lambda z.P_0)[z_1 := P'_1, \ldots, z_k := P'_k]$ with the same scope part $\lambda z.P_0$, but possibly with different free subexpressions $P'_1, \ldots, P'_k$, occurs already in $L_0$ (perhaps as an α-conversion equivalent variant). This implies $|P_0| \leq |P| < |L_0|$ for the depth of $P_0$ in relation to the depth of the initial term $L_0$ of the sequence. Now if $L_n \equiv C[(\lambda z.P)Q]$ for some unary context $C$ with the leftmost-outermost β-redex highlighted, then the $n$-th step is of the form:

$$
L_n \equiv C[(\lambda z.P)Q] \equiv C[((\lambda z.P_0)[z_1 := P_1, \ldots, z_k := P_k])Q]
\equiv C[((\lambda z.P_0)[z_1 := P_1, \ldots, z_k := P_k])Q] \\
\xrightarrow{\text{loβ}} C[(P_0[z_1 := P_1, \ldots, z_k := P_k][z := Q]) \equiv C[P_0[z_1 := P_1, \ldots, z_k := P_k, z := Q]] \equiv L_{n+1}.
$$

In order to move the substitutions for $z_1, \ldots, z_k$ inside of the abstraction $\lambda z.P$, we have assumed here, for simplicity, that $z$ does not occur free in one of $P_1, \ldots, P_k$ (otherwise α-conversion would be needed to rename $z$ in $\lambda z.P$ first). This justifies taking up the substitution of $Q$ for $z$ into the simultaneous substitution expression after the $\xrightarrow{\text{loβ}}$ step. Now from the form of the step $L_n \xrightarrow{\text{loβ}} L_{n+1}$ we see that any depth increase can only stem from the substitution of $Q$ for one of the occurrences of $z$ in $P_0$. This can move the argument $Q$ of the β-redex deeper by at most $|P_0|$. So by using $|P_0| < |L_0|$, see above, we obtain $|L_{n+1}| < |L_n| + |L_0|$. In this way we recognize that the depth increase in the $n$-th leftmost-outermost β-reduction step is always bounded by the depth $|L_0|$ of the initial term $L_0$ of the sequence.

Concepts for showing the underlying property. For showing that scope parts of abstractions in leftmost-outermost redexes of leftmost-outermost β-reduction rewrite sequences trace back to the initial term of the sequence, we will use representations of λ-terms as orthogonal first-order term rewrite systems. We call these TRS representations λ-TRSs. They are closely connected to systems of supercombinators [4, 5], which are widely used for the compilation of functional programs. Supercombinator translations are obtained by ‘lambda-lifting’ [10]. This transformation rewrites higher-order terms with bindings (such as named abstractions in λ-terms) into applicative first-order terms, and a finite number of combinator definitions. For functional programs lambda-lifting is applied by construing them as generalized λ-terms with case and letrec constructs. A program is compiled into a finite number of combinator definitions of the form $C \ldots x_n = D[x_1, \ldots, x_n]$ where $D$ is an applicative combinator context.

Supercombinator representations are well-suited for the evaluation via leftmost-outermost evaluation. This is because evaluation can proceed by repeatedly applying a combinator definition to occurrences of combinators with their sufficient number of arguments. In the example as above these are occurrences of applicative terms of the form $C \ldots s_n$. In this way evaluation becomes a process of applying combinator definitions locally without having to carry out the substitutions of arguments for variable occurrences that are needed for β-reduction on λ-terms. Moreover, leftmost-outermost β-reduction can be simulated by evaluating combinator terms in a leftmost-outermost manner. In the λ-TRS formulation, supercombinator definitions are modeled by rewrite rules $@$($f(x_1 \ldots x_n), y) \rightarrow F[x_1, \ldots, x_n, y]$ where $f$ is a scope symbol, and $F$ an applicative context that may contain other scope symbols. λ-TRSs correspond to systems of supercombinators that are obtained by ‘fully-lazy lambda-lifting’ [4, 10].
This construction of first-order term representations of λ-terms guarantees that every redex of a term in the representing λ-TRS corresponds to a β-redex via the translation to λ-terms. Indeed, the leftmost-outermost redex on a λ-TRS term representation of a λ-term corresponds to the leftmost-outermost β-redex on the represented λ-term. But conversely, typically not all β-redexes in a λ-term will correspond directly to a redex on the λ-TRS-representation. Crucially, after a number of (typically leftmost-outermost) β-reduction steps $t_0 \to t_n$ have been simulated from a λ-TRS-term $t_0$ that represents a λ-term $M$, every redex \( \beta \) (\( f(t_1 \ldots t_n), u \)) in $t_n$ will involve a scope symbol $f$ that, under the translation to λ-calculus, represents the scope of $z$ in a a subterm $\lambda z. L$ that already occurred (modulo α-conversion) in $M$.

While the linear-depth-increase statement will be shown for rewrite sequences in a TRS for simulating leftmost-outermost β-reduction, its transfer to λ-terms via a lifting theorem along lambda-lifting will only be sketched. The lifting and projection statements needed for this part are similar to proofs for the correctness of fully-lazy lambda-lifting as described by Balabonski [3].

Notwithstanding the linear-depth-increase property for leftmost-outermost rewrite sequences that we will show here, it is important to realize that ‘size explosion’ (exponential size increase) can in fact take place. However, Accattoli and Dal Lago recognized that one cannot avoid exponential runtime cost, simply because the result of a λ-term will correspond to a λ-TRS corresponds to a λ-term.

Notwithstanding the linear-depth-increase property for leftmost-outermost rewrite sequences that we will show here, it is important to realize that ‘size explosion’ (exponential size increase) can in fact take place.

Example 2 (‘size exploding’ family under leftmost-outermost β-red., from Accattoli and Dal Lago [1, 2]). Consider the following two families \( \{M_n\}_{n \in \mathbb{N}} \) and \( \{N_n\}_{n \in \mathbb{N}} \) of λ-terms:

\[
M_0 := yxx , \\
N_0 := yxx , \\
M_{n+1} := (\lambda x.M_n)M_0 \quad \text{(for } n \in \mathbb{N}) , \\
N_{n+1} := yN_nN_n \quad \text{(for } n \in \mathbb{N}) .
\]

Every term $N_n$, for $n \in \mathbb{N}$ is a normal form, and it holds that:

\[
N_n[x := N_0] = N_{n+1} \quad \text{(for all } n \in \mathbb{N}) .
\]

This can be shown by induction. Furthermore the term $N_n$ is the normal form of $M_n$, for $n \in \mathbb{N}$, because there is a leftmost-outermost β-reduction rewrite sequence of length $n$ from $N_n$ to $M_n$:

\[
M_n \rightarrow^n_{\text{loβ}} N_n \quad \text{(for all } n \in \mathbb{N}) .
\]

The induction step in a proof of this statement can be verified as follows:

\[
M_{n+1} = (\lambda x.M_n)M_0 \rightarrow_{\text{loβ}} M_n[x := M_0] \rightarrow^n_{\text{loβ}} N_n[x := M_0] \quad \text{(by using that $M_0$ is normal form)}
\]

\[
= N_n[x := N_0] \quad \text{(by definition of $N_0$ and $M_0$ coincide)}
\]

Finally, the size of terms in \( \{M_n\}_n \) grows linearly, and the size of terms \( \{N_n\}_n \) exponentially:

\[
\|M_n\| = 5 + 8n \in O(n) , \quad \|N_n\| = 2^{n+4} \in \Omega(2^n) ,
\]

where by the size of the λ-term we understand the size of its syntax tree plus the number of symbols in variable occurrences.

Therefore naive implementations of leftmost-outermost β-reduction that operate directly on λ-terms cannot avoid exponential runtime cost, simply because the result of a leftmost-outermost β-reduction steps can be exponentially larger than the initial term. However, Accattoli and Dal Lago recognized that a leftmost-outermost β-reduction sequence can also be implemented in the linear substitution calculus by...
carrying out explicit-substitution steps of $\beta$-redex contractions in a lazy manner that only guarantees that the pattern of the next leftmost-outermost $\beta$-redex is always visibly created. They show that, in this way, the size of intermediate $\lambda$-term representations stays polynomially bounded by the length of the sequence.

The linear-depth-increase property along leftmost-outermost rewrite sequences suggests an alternative proof, which is based on graph rewriting, of the result by Accattoli and Dal Lago. The crucial idea is to use directed acyclic graph representations of terms in a $\lambda$-TRS with the property that the depth of a graph (which is defined due to acyclicity) corresponds closely to the depth of the represented term. Then the power of sharing is deployed to avoid size explosion of the graph representations. In Section 3 we sketch the basic idea for such a graph implementation, and estimate its complexity.

Overview. In Section 3 we introduce representations of $\lambda$-terms as first-order terms, and define a TRS that simulates the leftmost-outermost strategy (and a non-deterministic generalization) for $\beta$-reduction on $\lambda$-term representations. In Section 4 we define $\lambda$-TRSs, that is, representations of $\lambda$-terms as orthogonal term rewrite systems that are closely related to supercombinator representations. We also define the expansion of $\lambda$-TRS representations into first-order term representations of $\lambda$-terms. In Section 5 we adapt the leftmost-outermost $\beta$-reduction simulation TRS from Section 3 to $\lambda$-TRS representations of $\lambda$-terms. In Section 6 we show the linear-depth-increase result for simulated leftmost-outermost $\beta$-reduction sequences: we prove it for all rewrite sequences in the simulation TRS on $\lambda$-TRS representations. In Section 7 we sketch how the linear-depth-increase result can be transferred from $\lambda$-TRS representations to $\lambda$-terms. In Section 8 we briefly lay out our idea of using the linear-depth-increase result for developing an efficient graph rewriting system for simulating leftmost-outermost $\beta$-reduction on $\lambda$-TRS representations.

2 Preliminaries

By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the natural numbers including 0. For first-order term rewriting systems, terminology and notation from the standard text [12] will be used. Below we summarize the most important concepts and the notation that we will use.

First-order signatures, variables, and context holes. A (first-order) signature $\Sigma = (\Sigma, ar)$ is a set of function symbols that is equipped with an arity function $ar : \Sigma \rightarrow \mathbb{N}$. Such a signature may contain constants by which we mean function symbols of arity 0. When referring to signatures, we will mostly keep the arity function implicit, and write $\Sigma$ for $\Sigma$.

In addition to first-order signatures we will use countably infinite sets $\text{Var}$ of variables, and a countably infinite set $\square := \{\square_1, \square_2, \ldots\}$ of context hole symbols each of which carries an index. We will always tacitly assume that the set $\text{Var}$, the set $\square$, and the union of the set of function symbols in signature $\Sigma_1, \Sigma_2, \ldots$, under consideration are disjoint.

Terms and contexts over first-order signatures. By $\text{Ter}(\Sigma, \text{Var})$ we denote the set of terms over signature $\Sigma$ and set $\text{Var}$ of variables that are formed with function symbols in $\Sigma$ and variables in $\text{Var}$. By $\text{Ter}(\Sigma) := \text{Ter}(\Sigma, \varnothing)$ (with an empty set of variables) we define the set of ground terms over $\Sigma$, that is, the set of terms that are formed from only the function symbols in $\Sigma$. We use $\equiv$ to indicate syntactic equality of terms.

For $n \in \mathbb{N}$, $n > 0$, we denote by $\text{Ctx}_n(\Sigma, \text{Var})$ the set of contexts that are formed with function symbols in $\Sigma$ and variables in $\text{Var}$, and with $n$ kinds of holes $\square_1, \ldots, \square_n$. Note that an $n$-ary context may contain zero, one or more occurrences of each of the $n$ holes; so it does not need to have any hole occurrence at all,
in which case it is a term. As a consequence also $\text{Ter}(\Sigma, \text{Var}) \subseteq \text{Cxt}_n(\Sigma, \text{Var}) \subseteq \text{Cxt}_{n+1}(\Sigma, \text{Var})$ holds for all $n \in \mathbb{N}, n > 0$. We also use $\equiv$ to indicate syntactic equality of contexts. For $n \in \mathbb{N}, n > 0$, we define by $\text{Cxt}_n(\Sigma) := \text{Cxt}_n(\Sigma, \emptyset)$ the set of $n$-ary ground contexts over $\Sigma$, that is, the set of $n$-ary contexts that are formed from the function symbols in $\Sigma$. By $\text{Cxt}(\Sigma, \text{Var}) := \bigcup_{n\in\mathbb{N}, n>0} \text{Cxt}_n(\Sigma, \text{Var})$ we define the set of contexts over $\Sigma$ and $\text{Var}$ and with some of the holes in $\emptyset$. Note again that $\text{Ter}(\Sigma, \text{Var}) \nsubseteq \text{Cxt}(\Sigma, \text{Var})$ holds. By $\text{Cxt}(\Sigma)$ we denote the set of ground contexts over $\Sigma$.

For unary (1-ary) contexts $C \in \text{Cxt}_1(\Sigma, \text{Var})$ we permit to drop the subscript ‘1’ from the context hole $\square_1$, which then is the single context hole $\square_1$ that may occur in $C$, and thus we permit to write $\square$ for $\square_1$.

By $\text{Cxt}_{n,1}(\Sigma, \text{Var})$ we denote the subset of $\text{Cxt}_n(\Sigma, \text{Var})$ that is formed by the linear $n$-ary contexts in which every context hole $\square_i$, for $i \in \{1, \ldots, n\}$ is only permitted to occur once. By $\text{Cxt}_{n,1}(\Sigma)$ we denote the set of linear, $n$-ary, ground contexts over $\Sigma$.

Let $C \in \text{Cxt}_n(\Sigma, \text{Var})$ be an $n$-ary context. Then for terms $t_1, \ldots, t_n \in \text{Ter}(\Sigma, \text{Var})$ we denote by $C[t_1, \ldots, t_n]$ the term in $\text{Ter}(\Sigma, \text{Var})$ that results from $C$ by replacing each hole $\square_i$ in $C$ by $t_i$, for all $i \in \{1, \ldots, n\}$. Similarly, for contexts $C_1, \ldots, C_n \in \text{Cxt}_{n}(\Sigma, \text{Var})$ we denote by $C[C_1, \ldots, C_n]$ the context in $\text{Cxt}_{n}(\Sigma, \text{Var})$ that results from $C$ by replacing each hole $\square_i$ in $C$ by $C_i$, for all $i \in \{1, \ldots, n\}$.

**Depth and size of terms. Depth, hole depth, and size of contexts.** For a term $t$ we denote by $|t|$ the depth of $t$ by which we mean the length of the longest (cycle-free) path in the syntax tree of $t$ from the root to a leaf. For a context $C$ the depth $|C|$ of $C$ is defined analogously. By the size $||t||$ of a term $t$, and the size $||C||$ of a context $C$ we mean the size of the syntax tree of $t$, and $C$, respectively.

By the hole depth $|C|_{\square}$ of a context $C$ we mean the length of the longest (cycle-free) path in the syntax tree of $C$ from the root to a leaf at which some hole occurs. We will use the following two lemmas that express easy properties concerning the connection between depth and hole depth in filled contexts.

**Lemma 3.** $|C[s_1, \ldots, s_n, \square]|_{\square} \leq |C|$ for all terms $s_1, \ldots, s_n \in \text{Ter}(\Sigma)$, where $n \in \mathbb{N}$, and all contexts $C \in \text{Cxt}_{n+1}(\Sigma)$ in which there is at least one occurrence of $\square_{n+1}$.

**Lemma 4.** $|C[s_1, \ldots, s_n]| = \max \{|C|, |C|_{\square} + |s_i| \mid i \in \{1, \ldots, n\}\}$ for contexts $C \in \text{Cxt}_n(\Sigma)$, and terms $s_1, \ldots, s_n \in \text{Ter}(\Sigma)$.

**Term rewriting systems.** A term rewriting system (TRS) is a pair $(\Sigma, R)$ that consists of a signature $\Sigma$, and a set $R \subseteq \text{Ter}(\Sigma, \text{Var}) \times \text{Ter}(\Sigma, \text{Var})$ of pairs of terms over $\Sigma$ that are called rules. The rules are subject to two conditions: the left-hand side of a rule is not a variable, and the variables that occur on the right-hand side of a rule are a subset of the variables that occur on the left-hand side.

A term $s \in \text{Ter}(\Sigma, \text{Var})$ is a normal form of a TRS $R = (\Sigma, R)$ if no rule of $R$ is applicable to $s$.

**Notation for rewrite relations.** Let $R$ be a TRS with rewrite relation $\rightarrow$. Then we denote the many-step (zero, one or more step) rewrite relation of $R$ by $\rightarrow$, and the $n$ step rewrite relation of $R$ by $\rightarrow^n$, for $n \in \mathbb{N}$. By $\downarrow$ we mean the many-step rewrite relation of $R$ to a normal form. We will use the same notation convention for rewrite relations that are indexed by name abbreviations.

**$\lambda$-calculus.** Contrasting with terms in a TRS (first-order terms), $\lambda$-terms are viewed as $\alpha$-equivalence classes of pseudo-term representations with names for bound variables. For $\lambda$-terms, $\rightarrow_\beta$ denotes $\beta$-reduction, and $\rightarrow_{bo\beta}$ leftmost-outermost $\beta$-reduction.

A $\beta$-reduction redex in a $\lambda$-term $M$ is called leftmost-outermost if it is to the left, or outside of any other redex in $M$. The leftmost-outermost reduction strategy for the $\lambda$-calculus is a 1-step strategy that, for a given $\lambda$-term $M$ contracts the leftmost-outermost $\beta$-redex in $M$. 
Termination/strong normalization of rewrite relations. Let $\rightarrow$ be the rewrite relation (of a TRS or of $\lambda$-calculus), and let $t$ be a term. We say that $\rightarrow$ terminates from $s$, and also that $\rightarrow$ is strongly normalizing from $t$ if there is no infinite rewrite sequence from $s$ (and consequently all sufficiently long rewrite sequences from $t$ lead to a normal form with respect to $\rightarrow$). We say that $\rightarrow$ terminates, and also that is strongly normalizing, if $\rightarrow$ does not enable infinite rewrite sequences.

3 Simulation of leftmost-outermost rewrite sequences

We start with the formal definition of first-order representations of $\lambda$-terms, called $\lambda$-term representations, before describing a TRS for simulating leftmost-outermost $\beta$-reduction on $\lambda$-term representations.

Definition 5 ($\lambda$-term representations, denoted $\lambda$-terms). Let $\Sigma_\lambda := \{v_j \mid j \in \mathbb{N}\} \cup \{\cdot\} \cup \{\lambda v_j \mid j \in \mathbb{N}\}$ be the signature that consists of the variable symbols $v_j$, with $j \in \mathbb{N}$, which are constants (nullary function symbols), the binary application symbol $\cdot$, and the unary named abstraction symbols $(\lambda v_j)$, for $j \in \mathbb{N}$.

Now by a $\lambda$-term representation (a (first-order) representation of a $\lambda$-term) we mean a ground term in $\text{Ter}(\Sigma_\lambda)$. A $\lambda$-term representation $s$ denotes, by reading its symbols in the obvious way, and interpreting occurrences of variable symbols $v_j$ that are not bound, as the variable names $x_j$, a unique $\lambda$-term $[s]_\lambda$.

Example 6. $(\lambda v_0)(v_0), (\lambda v_1)(\lambda v_2)(v_1), \text{ and } (\lambda v_0)((\lambda v_1)((\lambda v_2)(\cdot((\cdot(n_0, v_1), v_1), v_1), v_2)))$ are $\lambda$-term representations that denote the $\lambda$-terms $I = \lambda x.x$, $K = \lambda xy.x$, and $S = \lambda xyz.xz(yz)$, respectively.

Below we formulate a TRS that facilitates the simulation, on $\lambda$-term representations, of the evaluation of $\lambda$-terms according to the leftmost-outermost strategy. We introduce this TRS as a motivation for a similar simulation TRS on supercombinator-based $\lambda$-term representations that is introduced later in Definition [3] and that will be crucial for obtaining the linear depth-increase result. While the TRS is designed to reason about leftmost-outermost rewrite sequences, it actually permits the simulation of generalizations of the leftmost-outermost rewrite sequences: $\beta$-redexes may also be contracted if they are leftmost-outermost in right subtrees immediately below stable parts of the term. This is because the search process for leftmost-outermost redexes will be initiated again in parallel positions just below stable spines. We will therefore use the abbreviation ‘lop’ in symbol names to hint at the non-deterministic evaluation strategy ‘leftmost-outermost, iterated in parallel positions below stable parts of the term’.

The idea behind the simulation TRS is as follows. The process is started on a term $\text{lop}(s)$, where $s$ is a $\lambda$-term representation that is to be evaluated. First $\text{lop}(s)$ is initialized to $\text{lop}_0(s)$ (via the rule $\text{(init)}$), where the index (which here is 0) will be used as a lower bound for yet unused variable indices. Then a term $s_0$ with an outermost applications in an expression $\text{lop}_n(s_0, t_1, \ldots, t_n)$ is uncurried into a representing expression with a stack of applications (by steps of the rule $\text{(desc}\_0)$) when descending over applications along the spine of the term until a variable or an abstraction is encountered (detected by one of the rules $\text{(desc}\_\lambda)$, $\text{(var}_0)$, or $\text{(var}_{n+1}$)). If an abstraction occurs, and the expression contains an argument for this abstraction, the representation of a leftmost-outermost $\beta$-redex has been detected, which is then contracted by a step corresponding to a $\beta$-contraction (applying the rule $\text{(contr)}$); the evaluation continues similarly from there on. If there is no argument for such an abstraction, then it is part of a head normal form context, and the evaluation descends into the abstraction (applying the rule $\text{(desc}\_1)$) to proceed recursively on the subterm. If a variable occurs on the left end of the spine (detected by one of the rules $\text{(var}_0)$ or $\text{(var}_{n+1}$)), then a head normal form context has been detected, which consists of a single variable (in case the applicable rule is $\text{(var}_0$)), or of the variable together with the recently uncurried applications (in case the applicable rule is $\text{(var}_{n+1}$)). In the first case evaluation stops in the present subterm, whereas
in the second case the simulating evaluation can continue (after applying \( \text{var}_{n+1} \)), possibly in parallel, from any immediate subterm of one of the recently uncurried applications. The rules:

\[
\begin{align*}
\text{lop}(x) & \rightarrow \text{lop}_0(x) & \text{(init)} \\
\text{lop}_n(\text{@}(x, y), y_1, \ldots, y_n) & \rightarrow \text{lop}_{n+1}(x, y, y_1, \ldots, y_n) & \text{(desc}_0) \\
\text{lop}_0((\lambda v_j)(x)) & \rightarrow (\lambda v_j)(\text{lop}_0(x)) & \text{(desc}_\lambda) \\
\text{lop}_{n+1}(\lambda v_j, y_1, y_2, \ldots, y_{n+1}) & \rightarrow \text{lop}_n(\text{subst}(x, v_j, y_1), y_2, \ldots, y_{n+1}) & \text{(contr}_{n+1}) \\
\text{lop}_0(v_j) & \rightarrow v_j & \text{(var}_0) \\
\text{lop}_{n+1}(v_j, y_1, \ldots, y_{n+1}) & \rightarrow \text{@}(\ldots\text{@}(v_j, \text{lop}_0(y_1))\ldots, \text{lop}_0(y_{n+1})) & \text{(var}_{n+1})
\end{align*}
\]

have to be extended with appropriate rules for \( \text{subst} \) that implement capture-avoiding substitution, which induce a rewrite relation \( \rightarrow_{\text{subst}} \). We do not provide those rules here, because the rewrite system above only serves us as a stepping stone for a similar rewrite system in Section 5 that operates on supercombinator representations of \( \lambda \)-terms (\( \lambda \)-TRSs) where substitution can be organized as context-filling.

Based on the simulation TRS, we denote by \( \rightarrow_{\text{contr}} \) the rewrite relation that is induced by the rule scheme \( \text{(contr}_{n+1} \) for \( n \in \mathbb{N} \). It defines steps that initiate the simulation of a \( \beta \)-reduction step which then proceeds with \( \rightarrow_{\text{subst}} \) steps that carry out the substitution in the contraction of the \( \beta \)-redex. By \( \rightarrow_{\text{init}}, \rightarrow_{\text{desc}_0}, \rightarrow_{\text{desc}_\lambda}, \text{ and } \rightarrow_{\text{var}} \) we designate the rewrite relations that are induced by the rules \( \text{(init)}, \text{(desc}_0), \text{(desc}_\lambda), \text{ and } \text{(var}_n \) for some \( n \in \mathbb{N} \), respectively. By \( \rightarrow_{\text{search}} \) we denote the union of \( \rightarrow_{\text{contr}}, \rightarrow_{\text{desc}_0}, \rightarrow_{\text{desc}_\lambda}, \text{ and } \rightarrow_{\text{var}}, \) because they organize the search for the next leftmost-outermost redex or of an outermost redex. Finally, we denote by \( \rightarrow_{\text{lop}} \) the rewrite relation that is induced by the entire TRS.

The labels for \( \rightarrow_{\text{contr}} \) and \( \rightarrow_{\text{search}} \) are motivated as follows: In a \( \rightarrow_{\text{contr}} \) step the representation of a leftmost-outermost redex is contracted, or the representation of a ‘stacked’ outermost redex that is leftmost-outermost below a stable part of the term (and that is bound to become a leftmost-outermost redex at some later stage, at least if the term has a normal form). And a \( \rightarrow_{\text{search}} \) step is part of the search in the term for the representation of the next leftmost-outermost redex or of an outermost redex that is bound to become a leftmost-outermost redex later.

**Example 7.** We consider the \( \lambda \)-term \( M = \lambda x.(\lambda y,y)((\lambda z.\lambda w. wz)x) \). Evaluating \( M \) with the leftmost-outermost rewrite strategy, symbolized by the rewrite relation \( \rightarrow_{\text{lop}} \), gives rise to the rewrite sequence:

\[
\lambda x.(\lambda y,y)((\lambda z.\lambda w. wz)x) \rightarrow_{\text{lop}} \lambda x.\lambda z.\lambda w. wz x \rightarrow_{\text{lop}} \lambda x.\lambda w. w x
\]

where the underlinings symbolize the \( \beta \)-redexes that are contracted in the next step. The term:

\[
s = ((\lambda v_1)((\lambda v_2)((\lambda v_3)((\lambda v_4)((\lambda v_5)(\text{@}(v_3, v_2)), v_0))))
\]

denotes \( M \), that is, \( [s]_\lambda = M \); other variable names are possible modulo ‘\( \alpha \)-conversion’. Simulating this leftmost-outermost rewrite sequence by means of the simulation TRS above:

\[
\text{lop}(s) \rightarrow_{\text{init}} \text{lop}_0((\lambda v_0)((\lambda v_1)((\lambda v_2)((\lambda v_3)((\lambda v_4)((\lambda v_5)(\text{@}(v_3, v_2)), v_0))))
\]

\[
\rightarrow_{\text{desc}_\lambda} (\lambda v_0)(\text{lop}_0((\lambda v_1)((\lambda v_2)((\lambda v_3)((\lambda v_4)((\lambda v_5)(\text{@}(v_3, v_2)), v_0))))
\]

\[
\rightarrow_{\text{desc}_0} (\lambda v_0)(\text{lop}_0(\text{subst}(v_1, v_1, ((\lambda v_2)((\lambda v_3)((\lambda v_4)((\lambda v_5)(\text{@}(v_3, v_2)), v_0))))
\]

\[
\rightarrow_{\text{contr}} (\lambda v_0)(\text{lop}_0(\text{subst}(v_1, v_1, ((\lambda v_2)((\lambda v_3)((\lambda v_4)((\lambda v_5)(\text{@}(v_3, v_2)), v_0))))
\]


Note that the $\rightarrow_{\text{contr}}$ steps indeed initiate, and the $\rightarrow_{\text{sub}}$ steps complete, the simulation of corresponding $\beta$-reduction steps in the $\rightarrow_{\text{lop}}$ rewrite sequence on $\lambda$-terms above, while the other steps organize the search for the next ($\lambda$-term representation of a) leftmost-outermost $\beta$-redex. The $\rightarrow_{\text{lop}}$ rewrite sequence $(\lambda x.\text{lo}p_0)\lambda v_0((\lambda v_3)((\lambda v_3)(\text{lo}p_0(v_3), v_0)))$ can be viewed as the projection of the $\rightarrow_{\text{lop}}$ rewrite sequence above under an extension of the denotation operation $\lambda -$ on $\lambda$-term representations yielding $\lambda$-terms (which works out substitutions, and interprets uncurried application expressions $\text{lo}p_n(s, t_1, \ldots, t_n)$ appropriately). Hereby $\rightarrow_{\text{contr}}$ steps project to $\rightarrow_{\text{lop}}$ steps, but all other steps vanish under the projection.

While the TRS above facilitates the faithful representation of leftmost-outermost rewrite sequences on $\lambda$-terms (which can be formulated formally analogous to Proposition 23, see page 23), it does not lend itself well to the purpose of proving the linear-depth-increase result. This is because it is not readily clear which invariant for reducts $t$ of a term $s$ in rewrite sequences $\sigma : s \rightarrow_{\text{lop}} t \rightarrow_{\text{lop}} u$ could make it possible to prove that the depth increase in the final step of $\sigma$ is bounded by a constant $d$ that only depends on the initial term $s$ of the sequence (but not on $t$). In the next section, however, we develop a concept that can overcome this problem. We define extensions of first-order $\lambda$-term representations in which the abstraction parts of representations of leftmost-outermost $\beta$-redexes are built up from contexts that trace back to contexts in the initial term of the rewrite sequence. This will guarantee that after a leftmost-outermost $\beta$-reduction rewrite sequence $M_0 \rightarrow_{\text{lop}}^n M_n$ a scope part of the abstraction part $\lambda z.L$ of the next leftmost-outermost $\beta$-redex $(\lambda z.L)P$ in $M_n$ does already occur in $M_0$.

## 4 $\lambda$-TRS representations of lambda terms

We now introduce $\lambda$-TRSs as orthogonal TRSs that are able to represent $\lambda$-terms. The basic idea is that, for a $\lambda$-term $M$, function symbols that are called ‘scope symbols’ are used to represent abstraction scopes. Hereby the scope of an abstraction $\lambda x.L$ in $M$ includes the abstraction $\lambda x$ and all occurrences of the bound variable $x$, but may leave room for subterms in $L$ without occurrences of $x$ bound by the abstraction. For example, the $\lambda$-term $\lambda x.zxyx$ may be denoted as the term $f(z, y)$ where the binary scope symbol $f$ represents the scope context $(\lambda x.\text{lo}p_1 x\text{lo}p_2 x)$. In our formalization of $\lambda$-term representations the free variables $z$ and $y$ will be replaced by variable constants, yielding for example the term $f(v_2, v_1)$. Furthermore, scopes are assumed to be strictly nested. Every scope symbol defines a rewrite rule that governs the behavior of the application of the scope to an argument. In the case of the $\lambda$-term $\lambda x.zxyx$ this leads to the first-order rewrite rule $\text{lo}p(f(z, y), x) \rightarrow \text{lo}p((\text{lo}p(f(z, x), y), x)$ for the scope symbol $f$ that corresponds to the $\lambda$-term scope context $(\lambda x.\text{lo}p_1 x\text{lo}p_2 x)$. Such a translation facilitates a correspondence between $\beta$-reduction steps in the $\lambda$-calculus, and first-order term rewriting steps on terms with adequately
defined scope symbols. In the example here the correspondence is between the steps:

\[(\lambda x. z y x) M \rightarrow_{\beta} z M y M\]  
\[\lambda (f, s) \rightarrow \lambda (\varnothing (\lambda (z, s), y), s)\]  

(\beta\text{-reduction in the \(\lambda\)-calculus),
(application of the corresponding \(\lambda\)\(\text{-TRS}\)-rule),
provided that the \(\lambda\)\(\text{-TRS}\)-term \(s\) represents the \(\lambda\)-term \(M\).

\(\lambda\)\(\text{-TRS}\)s are TRS-representations of systems of supercombinators that are obtained by the lambda-lifting transformation. I have been introduced to these \(\lambda\)-term representations by orthogonal TRSs by Vincent van Oostrom (personal communication, in the framework of the NWO-research project ‘Realising Optimal Sharing’, and our collaboration on ‘nested term graphs’ [F]). He strongly shaped my understanding of them, and pointed me to the studies of optimal reduction for weak \(\beta\)-reduction (\(\beta\)-reduction outside of abstractions or in ‘maximal free’ subexpressions) by Blanc, Lévy, and Maranget [5]. Also, he encouraged work by Ballabonski [F] on characterizations of optimal-sharing implementations for weak sharing, and our collaboration on ‘nested term graphs’ [7]). He strongly shaped my understanding of them, and pointed me to the studies of optimal reduction for weak \(\beta\)-reduction by term labelings. Later I discovered the direct connection with ‘fully-lazy lambda-lifting’, which was introduced in the early 1980-ies by Hughes [4, 5].

**Definition 8** (\(\lambda\)\(\text{-TRS}\)s). A \(\lambda\)\(\text{-TRS}\) is a pair \(L = (\Sigma, R)\), where \(\Sigma\) is a signature containing the binary application symbol \(\varnothing\), and the scope symbols in \(\Sigma^- := \Sigma \setminus \{\varnothing\}\), and where \(R = \{\rho_f \mid f \in \Sigma^-\}\) consists of the defining rules \(\rho_f\) for scope symbols \(f \in \Sigma^-\) with arity \(k\) that are of the form:

\[(\rho_f) \quad \varnothing (f(x_1, \ldots, x_k), y) \rightarrow F[x_1, \ldots, x_k, y]\]

with \(F\) a \((k + 1)\)-ary context of \(L\) that is called the scope context for \(f\). For scope symbols \(f, g \in \Sigma^-\) we say that \(f\) depends on the scope symbol \(g\), denoted by \(f \rightarrow g\), if \(g\) occurs in the scope context \(F\) for \(f\). We say that \(L\) is finitely nested if the converse relation of \(\rightarrow\), the nested-into relation \(\rightarrow\), is well-founded, or equivalently (using the axiom of dependent choice), if there is no infinite chain of the form \(f_0 \rightarrow f_1 \rightarrow f_2 \rightarrow \ldots\) on scope symbols \(f_0, f_1, f_2, \ldots \in \Sigma^-\).

**Example 9.** Let \(L = (\Sigma, R)\) be the \(\lambda\)\(\text{-TRS}\) with \(\Sigma^- = \{f, g, h, i\}\), where \(ar(f) = 2, ar(g) = ar(h) = 0,\) and \(ar(i) = 1\), and the following set \(R\) of rules:

\[(\rho_f) \quad \varnothing (f(x_1, x_2), x) \rightarrow \varnothing (x_1, \varnothing (x_2, x))\]

\[(\rho_h) \quad \varnothing (h, x) \rightarrow i(x)\]

\[(\rho_g) \quad \varnothing (g, x) \rightarrow x\]

\[(\rho_i) \quad \varnothing (i(x_1), x) \rightarrow \varnothing (x, x_1)\]

This finite \(\lambda\)\(\text{-TRS}\) is also finitely nested, as the depends-on relation consists only of a single link: \(h \rightarrow i\). It facilitates to denote the \(\lambda\)-term \(M\) in Example [5], see the expansion of \(f(g, h)\) in Example [5] below.

In order to explain how \(\lambda\)\(\text{-TRS}\) terms denote \(\lambda\)-term representations, we introduce, for every \(\lambda\)\(\text{-TRS}\) \(L\), an expansion TRS that makes use of the defining rules for the scope symbols in \(L\). Then ‘denoted \(\lambda\)-term representations’ will be defined as normal forms of terms in the expansion TRS. It uses function symbols \(\text{exp}_i\) with parameters \(i\) for expanding a \(\lambda\)\(\text{-TRS}\) term in a top–down manner. Thereby the indices \(i\) are used to guarantee that when an abstraction \(\lambda v_i\) is created the indexed variable name \(v_i\) is different from that of all abstractions \(\lambda x_j\) that have been created above it. In this way the arising \(\lambda\)-term representation will be uniquely named at vertical positions.

**Definition 10** (expansion TRS for a \(\lambda\)\(\text{-TRS}\)). Let \(L = (\Sigma, R)\) be a \(\lambda\)\(\text{-TRS}\). The expansion TRS \(E(L) = (\Sigma_{\text{exp}} \cup \text{Var}, R_{\text{exp}})\) for \(L\) has the signature \(\Sigma_{\text{exp}} := \Sigma \cup \Sigma_\lambda \cup \Sigma_{\text{exp}}\) with \(\Sigma_{\text{exp}} := \{\text{exp}_i \mid i \in \mathbb{N}\}\) where \(\text{exp}_i\) is unary for \(i \in \mathbb{N}\), and \(\Sigma^- \cap (\Sigma_\lambda \cup \Sigma_{\text{exp}}) = \emptyset\), and its set of rules \(R_{\text{exp}}\) consists of the rules:

\[\text{(\beta\text{-reduction in the } \lambda\text{-calculus),}}\]
\[\text{(application of the corresponding } \lambda\text{-TRS-rule),}}\]
exp_1(\emptyset(x_1, x_2)) \rightarrow @(@\prod(x_1), \exp_1(x_2))

exp_1(f(x_1, \ldots, x_k)) \rightarrow (\lambda v_i)(\exp_{i+1}(F[x_1, \ldots, x_k, v_i])) \quad \text{(where } F \text{ is the scope context for } f)

exp_1(\lambda v_j)(x) \rightarrow (\lambda v_j)(\exp_{\max\{i,j\}+1}(x))

exp_1(v_j) \rightarrow v_j

By \rightarrow_{\exp} we denote the rewrite relation of \mathcal{E}(\mathcal{L}).

**Lemma 11.** The expansion TRS \mathcal{E}(\mathcal{L}) of a \lambda-TRS \mathcal{L} = (\Sigma, R) is an orthogonal TRS. Hence its rewrite relation \rightarrow_{\exp} is confluent, and normal forms of terms, whenever they exist, are unique.

Since expansion TRSs are orthogonal TRSs, finite or infinite normal forms are unique. Furthermore they are constructor TRSs, i.e. they have rules whose right-hand sides are guarded by constructors. This can be used to show that all terms in an expansion TRS rewrite to a unique finite or infinite normal form.

**Definition 12** (\lambda-term representations denoted by \lambda-TRS-terms). Let \mathcal{L} = (\Sigma, R) be a \lambda-TRS. For a term \( s \in \text{Ter}(\Sigma) \) we denote by \[ s \]_{\exp} the finite or infinite \rightarrow_{\exp}-normal form of the term \exp_0(s) in \mathcal{E}(\mathcal{L}). If it is a \lambda-term representation, we say that \[ s \]_{\exp} is the denoted \lambda-term representation of \( s \), and write \[ s \]_{\exp, \lambda} for the \lambda-term \[ s \]_{\exp}.

**Example 13.** With the \lambda-TRS \mathcal{L} from Example 3, the \lambda-term \( M \) in Example 3 can be denoted as the term \( f(g, h) \) expands to a \lambda-term representation of \( M \) (the final \rightarrow_{\exp} step consists of two parallel \rightarrow_{\exp} steps):

\[
\exp_0(f(g, h)) \rightarrow_{\exp} (\lambda v_0)(\exp_1(\emptyset(g, \emptyset(h, v_0)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\exp_1(g, \exp_1(\emptyset(h, v_0)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(\exp_2(v_1)), \exp_1(\emptyset(h, v_0)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(\exp_1(h), \exp_2(v_0)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_1(h), \exp_2(v_0)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_1(h), v_0))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_2(v_1)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_2(v_1)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_2(v_1)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_2(v_1)))) \\
\rightarrow_{\exp} (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_2(v_1))))
\]

Hence \[ f(g, h) \]_{\exp, \lambda} = (\lambda v_0)(\emptyset(\lambda v_1)(v_1), \emptyset(\exp_2(v_1))))). This \lambda-term representation coincides with the term \( s \) in Example 3 (modulo \alpha-conversion), and furthermore, for the denoted \lambda-term it holds that \[ f(g, h) \]_{\exp, \lambda} = \lambda x. (\lambda y. g)((\lambda z. \lambda w. w) x) = M.

**Proposition 14.** Let \mathcal{L} be a finitely nested \lambda-TRS. Then for every ground term \( s \) of \mathcal{L}, \[ s \]_{\exp, \lambda} is a finite ground term over \Sigma_\lambda, hence a \lambda-term representation of the \lambda-term \[ s \]_{\exp, \lambda}.

For proving termination and finiteness of the expansion process for terms and contexts in finitely nested \lambda-TRSs, we now define two measures: the ‘nesting depth’ of scope symbols, and the ‘expansion size’ of terms and of contexts.

**Definition 15** (nesting depth of a scope symbol, maximal nesting depth of contexts and terms). Let \mathcal{L} = (\Sigma, R) be a finitely nested \lambda-TRS.
We define the nesting depth \( d_{\text{nest}}(f) \) of a scope symbol \( f \in \Sigma^- \) by means of well-founded induction on \( \sim \), the converse of the nested-into relation \( \circ \), as follows:

\[
d_{\text{nest}}(f) := \begin{cases} 
0 & \text{if } \not\exists g \in \Sigma^- \langle f \circ g \rangle , \\
1 + \max \{d_{\text{nest}}(g) \mid f \circ g, g \in \Sigma^- \} & \text{if } \exists g \in \Sigma^- \langle f \circ g \rangle .
\end{cases}
\]

Note that \( \sim \) is well-founded, since \( \mathcal{L} \) is finitely nested. Furthermore the maximum in this clause is always taken over a finite set, because \( f \circ g \) means that \( g \) occurs in the scope context \( F \) of \( f \), which is finite.

By the maximal nesting depth \( D_{\text{nest}}(C) \) of an \( n \)-ary context \( C \in \text{Ctx}_n(\Sigma \cup \Sigma_\lambda) \), we mean the maximal nesting depth of a scope symbol that occurs in \( C \). Similarly, by the maximal nesting depth \( D_{\text{nest}}(t) \) of a term \( t \in \text{Ter}(\Sigma \cup \Sigma_\lambda) \) we mean the maximal nesting depth of a scope symbol that occurs in \( t \).

Next we introduce the ‘expansion size’ of ground contexts (and thereby also of ground terms) over the signatures of an \( \lambda \)-TRS \( \mathcal{L} \), and the \( \lambda \)-term representations. We define it in such a way that the expansion size of a context \( C \) can later be recognized as the size of a normal form of \( \exp_0(C) \) in the expansion TRS \( \mathcal{E}(\mathcal{L}) \) when context holes, and remaining symbols \( \exp_1 \), are not counted. First, however, we will need this measure to show that every term \( \exp_0(C) \) has a normal form in \( \mathcal{E}(\mathcal{L}) \) at all.

**Definition 16** (expansion size). Let \( \mathcal{L} = (\Sigma, R) \) be a finitely nested \( \lambda \)-TRS. We define the expansion size \( \|C\|_{\exp} \) of contexts \( C \in \text{Ctx}(\Sigma \cup \Sigma_\lambda) \) by induction on the structure of \( C \), thereby distinguishing the five possible cases of outermost symbols:

\[
\|\Box(C_1, C_2)\|_{\exp} := 1 + \|C_1\|_{\exp} + \|C_2\|_{\exp},
\]

\[
\|f(C_1, \ldots, C_k)\|_{\exp} := 1 + \|F[\Box_1, \ldots, \Box_k, v_0]\|_{\exp} + \sum_{i=1}^k \|C_i\|_{\exp} \quad \text{(where } F \text{ is the scope context for } f \text{ in } \mathcal{L}),
\]

\[
\|(\lambda v_j)(C_0)\|_{\exp} := 1 + \|C_0\|_{\exp},
\]

\[
\|v_j\|_{\exp} := 1 \quad \text{(for all } j \in \mathbb{N}),
\]

\[
\|\Box_j\|_{\exp} := 0 \quad \text{(for all } j \in \mathbb{N}).
\]

In particular, we apply well-founded induction on \( \langle D_{\text{nest}}(C), \|C\| \rangle \) with respect to the lexicographic order on \( \mathbb{N} \times \mathbb{N} \), that is, induction on the nesting depth \( D_{\text{nest}}(C) \) of \( C \) with a subinduction on the size \( \|C\| \) of \( C \). Note that, in particular, \( \|f(C_1, \ldots, C_k)\|_{\exp} \) is well-defined: for \( \|F[\Box_1, \ldots, \Box_k, v_0]\|_{\exp} \) we can apply the induction hypothesis due to \( D_{\text{nest}}(F[\Box_1, \ldots, \Box_k, v_0]) < D_{\text{nest}}(f(C_1, \ldots, C_k)) \), since for all scope symbols \( g \) that occur in \( F \) it holds that \( g \sim f \); and \( \|C_i\|_{\exp} \) is well-defined for \( i \in \{1, \ldots, k\} \), because \( D_{\text{nest}}(C_i) \leq D_{\text{nest}}(f(C_1, \ldots, C_k)) \), and \( \|C_i\| < \|f(C_1, \ldots, C_k)\| \).

**Lemma 17.** \( \|C(C_1, \ldots, C_n)\|_{\exp} = \|C\|_{\exp} + \sum_{i=1}^n \|C_i\|_{\exp} \) holds for contexts \( C \in \text{Ctx}_n(\Sigma \cup \Sigma_\lambda) \) and \( C_1, \ldots, C_n \in \text{Ctx}_m(\Sigma \cup \Sigma_\lambda) \).

**Proof:** By a straightforward induction on the structure of the contexts \( C \in \text{Ctx}_n(\Sigma \cup \Sigma_\lambda) \).

On the basis of these preparations we can now show that the expansion rewrite relation with respect to a finitely nested \( \lambda \)-TRS always terminates (is strongly normalizing) on a term \( \exp_i(t) \) with \( i \in \mathbb{N} \), and with \( t \) a term over the signature of the \( \lambda \)-TRS and of \( \lambda \)-term representations.
Lemma 18 (termination of expansion in finitely nested \(\lambda\)-TRSs). Let \(\mathcal{L} = \langle \Sigma, R \rangle\) be a finitely nested \(\lambda\)-TRS. Then the following statements hold for \(\rightarrow_{\text{exp}}\) as defined in the expansion TRS \(E(\mathcal{L})\) of \(\mathcal{L}\):

(i) \(\rightarrow_{\text{exp}}\) terminates from \(\text{exp}_i(C)\) for every \(i \in \mathbb{N}\), and every context \(C \in \text{Cxt}(\Sigma \cup \Sigma_{\lambda})\).

(ii) \(\rightarrow_{\text{exp}}\) terminates from \(\text{exp}_i(t)\) for every \(i \in \mathbb{N}\), and every term \(t \in \text{Ter}(\Sigma \cup \Sigma_{\lambda})\).

Proof: By inspection of the four rules of the expansion TRS \(E(\mathcal{L})\) we find that in every step of the form \(\text{exp}_i(C) \rightarrow_{\text{exp}} C'\) where \(C \in \text{Cxt}(\Sigma \cup \Sigma_{\lambda})\), \(i \in \mathbb{N}\), and \(C' \in \text{Cxt}(\Sigma \cup \Sigma_{\lambda} \cup \Sigma_{\text{expand}})\) it holds for subexpressions \(\text{exp}_f(D)\) of \(C'\) that \(\|D\|_{\text{exp}} < \|C\|_{\text{exp}}\). This is immediate for the rules concerning application \(\odot\), named abstraction symbols \(\lambda v_i\), and variable symbols \(v_i\) for \(i \in \mathbb{N}\). For the rule concerning scope symbols \(f \in \Sigma^\ast\) this can be checked by using Lemma 17.

Now that we know that the expansion process for terms over the signatures of \(\lambda\)-TRSs, and of \(\lambda\)-term representations always terminate, and that \(\rightarrow_{\text{exp}}\) normal forms are always unique, we introduce notation and a name for these normal forms.

Definition 19 (expanded forms of contexts and terms). Let \(\mathcal{L} = \langle \Sigma, R \rangle\) be a finitely nested \(\lambda\)-TRS.

For every \(n\)-ary context \(C \in \text{Cxt}_n(\Sigma \cup \Sigma_{\lambda})\), where \(n \in \mathbb{N}\), and every term \(t \in \text{Ter}(\Sigma \cup \Sigma_{\lambda})\) we define the expanded form \(\langle C \rangle_{\text{exp}}^{(i)}\) of \(C\), and the expanded form \(\langle t \rangle_{\text{exp}}^{(i)}\) of \(t\) for all \(i \in \mathbb{N}\) by:

\[
\langle C \rangle_{\text{exp}}^{(i)} := \text{exp}_i(C)_{\text{exp}}, \quad \langle t \rangle_{\text{exp}}^{(i)} := \text{exp}_i(t)_{\text{exp}},
\]

where \(t_{\text{exp}}^{(i)}\) denotes the operation of taking the \(\rightarrow_{\text{exp}}\) normal form of \(t\). This normal form is well-defined, because \(\rightarrow_{\text{exp}}\) normal forms of terms \(\text{exp}_i(C)\) exist due to Lemma 18, and are unique due to Lemma 11.

For reasoning with expanded forms of terms and context later in Section 6 we will need representations of the expanded form of arbitrary contexts, and how this representation interacts with the context filling operation. The lemma below formulates the representation, and the subsequent lemma its property with respect to context filling.

Lemma 20 (context representation of expanded forms of contexts). Let \(\mathcal{L} = \langle \Sigma, R \rangle\) be a finitely nested \(\lambda\)-TRS. For every \(n\)-ary context \(C \in \text{Cxt}_n(\Sigma \cup \Sigma_{\lambda})\) (over the signature of \(\mathcal{L}\) and \(\lambda\)-term representations), and \(i \in \mathbb{N}\), the expanded form of \(C\) has a representation:

\[
\langle C \rangle_{\text{exp}}^{(i)} = D[\text{exp}_{i_1}(\langle j_1 \rangle_{\text{exp}}), \ldots, \text{exp}_{i_m}(\langle j_m \rangle_{\text{exp}})] = D[\langle j_1 \rangle_{\text{exp}}^{(i_1)}, \ldots, \langle j_m \rangle_{\text{exp}}^{(i_m)}]
\]

for some linear context \(D \in \text{Cxt}_m(\Sigma_{\lambda})\) (over the signature of \(\lambda\)-term representations), for \(m \in \mathbb{N}\), and \(i_1, \ldots, i_m \in \mathbb{N}\), and \(j_1, \ldots, j_m \in \{1, \ldots, n\}\) (therefore the context above on the right is \(n\)-ary just as \(C\)).

Proof: \(\langle C \rangle_{\text{exp}}^{(i)}\) was well-defined in Definition 19 on the basis of Lemma 18 and Lemma 11, as the unique \(\rightarrow_{\text{exp}}\) normal form \(C'\) in \(E(\mathcal{L})\) of \(\text{exp}_i(C)\). Since \(C'\) is a normal form with respect to \(\rightarrow_{\text{exp}}\), \(C'\)
does not contain subexpressions of the form \( \exp_j(\@ (E_1, E_2)) \), \( \exp_j(f(E_1, \ldots, E_k)) \), \( \exp_j((\lambda v_i)(E_0)) \), or \( \exp_j(v_i) \), with \( j, k, l \in \mathbb{N} \), and contexts \( E_0, E_1, E_2, \ldots, E_k \). The only possible occurrences of symbols \( \exp_j \) in \( C \) must therefore be of the ‘hole guarding’ form \( \exp_j \{ (\exp_j(p)) \} \) with \( j_1, \ldots, j_k, l \in \mathbb{N} \).

But proper stackings of symbols \( \exp \) in \( C' \) are not possible. We argue as follows. The form of the rules of \( \mathcal{E}(\mathcal{L}) \) guarantees that if \( \exp_1(C) \rightarrow^{\exp} E \) holds, then \( E \) does not contain symbols \( \exp_1 \) in nested positions. It follows that this also holds for the \( \rightarrow^{\exp} \) normal form \( C' \) of \( C \).

Therefore the only possible occurrences of symbols \( \exp_j \) in \( C \) are of the ‘linear hole guarding’ form \( \exp_j(p) \) with \( j, l \in \mathbb{N} \). As a consequence, \( C' \) can be written as of the form (3) for a linear context \( D \in \text{Cxt}_m(\Sigma) \), for \( m \in \mathbb{N} \), and \( i_1, \ldots, i_m \in \mathbb{N} \), and \( j_1, \ldots, j_m \in \{1, \ldots, n\} \).

\[ \langle \lambda \exp_1(C_1, \ldots, C_n) \rangle_{\exp} \equiv D[\langle C_{i_1} \rangle_{\exp}^{(i_1)}, \ldots, \langle C_{j_m} \rangle_{\exp}^{(j_m)}] \quad (4) \]

where \( D \) is a linear context \( D \in \text{Cxt}_m(\Sigma) \) (over the signature of \( \lambda \)-term representations), for \( m \in \mathbb{N} \), that describes the expanded form \( \langle \lambda \exp_1(C) \rangle_{\exp} \) of \( C \) via (3), for some \( i_1, \ldots, i_m \in \mathbb{N} \), and \( j_1, \ldots, j_m \in \{1, \ldots, n\} \).

**Proof:** By Lemma 20 there is a linear context \( D \in \text{Cxt}_m(\Sigma) \), for \( m \in \mathbb{N} \), that describes the expanded form \( \langle \lambda \exp_1(C) \rangle_{\exp} \) of \( C \) via (3), for some \( i_1, \ldots, i_m \in \mathbb{N} \), and \( j_1, \ldots, j_m \in \{1, \ldots, n\} \), and hence with:

\[ \exp_1(C) \rightarrow^{\exp} D[\exp_1(\Sigma_1), \ldots, \exp_1(\Sigma_m)]. \]

By filling \( C_1, \ldots, C_n \) in the holes \( \Box_1, \ldots, \Box_n \) of \( C \) in all contexts of this \( \rightarrow^{\exp} \) rewrite sequence we obtain another \( \rightarrow^{\exp} \) rewrite sequence that can furthermore be extended as follows:

\[ \exp_1(C[C_1, \ldots, C_n]) \rightarrow^{\exp} D[\exp_1(C_{i_1}), \ldots, \exp_1(C_{j_m})]. \]

by using the rewrite sequences \( \exp_1(C_{i_1}) \rightarrow^{\exp} \langle C_{i_1} \rangle_{\exp} \) for all \( l \in \{1, \ldots, m\} \), which exist by the definition of \( \langle C_{j_l} \rangle_{\exp} \). Since \( D \in \text{Cxt}(\Sigma) \), it does not contain any symbol \( \exp_1 \), for \( i \in \mathbb{N} \). Therefore the resulting context \( D[\langle C_{i_1} \rangle_{\exp}, \ldots, \langle C_{j_m} \rangle_{\exp}] \) is indeed a \( \rightarrow^{\exp} \) normal form. In this way we have justified the form (4) of \( \langle C[C_1, \ldots, C_n] \rangle_{\exp} \).

Finally, in Section 3 we will also need the following lemma. It states that all expanded forms of terms over the signature of an \( \lambda \)-TRS, and of the \( \lambda \)-term representations have the same unlabeled syntax tree, which entails that they have the same depth and size.

**Lemma 22.** Let \( \mathcal{L} = (\Sigma, R) \) be a finitely nested \( \lambda \)-TRS. Let \( C \in \text{Cxt}(\Sigma \cup \Sigma_\lambda) \) be a context, let \( t \in \text{Ter}(\Sigma \cup \Sigma_\lambda) \) be a term over the signatures of \( \mathcal{L} \), and \( \lambda \)-term representations. Let \( i_1, i_2 \in \mathbb{N} \) with \( i_1 \neq i_2 \).

Then \( \langle C \rangle_{\exp}^{(i_1)} \) and \( \langle C \rangle_{\exp}^{(i_2)} \) have the same unlabeled syntax tree, and at a position \( p \) they can only differ possibly in:

(i) a variable symbol \( v_{j_1} \) at \( p \) in \( \langle C \rangle_{\exp}^{(i_1)} \), and a variable symbol \( v_{j_2} \) at \( p \) in \( \langle C \rangle_{\exp}^{(i_2)} \).

(ii) an abstraction symbol \( (\lambda v_{j_1}) \) at \( p \) in \( \langle C \rangle_{\exp}^{(i_2)} \), and an abstraction symbol \( (\lambda v_{j_2}) \) at \( p \) in \( \langle C \rangle_{\exp}^{(i_2)} \).
(iii) a symbol \(\text{exp}_i\left(\square_j\right)\) at \(p\) in \(\langle C\rangle_{\exp}^{(i_1)}\), and a symbol \(\text{exp}_i\left(\square_j\right)\) at \(p\) in \(\langle C\rangle_{\exp}^{(i_2)}\), where \(j \in \{1, \ldots, n\}\).

Similarly, \(\langle t\rangle_{\exp}^{(i_1)}\) and \(\langle t\rangle_{\exp}^{(i_2)}\) have the same syntax tree, and at a position \(p\) they can only differ possibly as described in items (3) and (4).

**Proof:** From the rules of the expansion TRS \(E(\mathcal{L})\) we find that an expression \(\text{exp}_i(C)\) is a redex of \(E(\mathcal{L})\) irrespective of \(i \in \mathbb{N}\): indeed, it is a redex if and only if \(C\) is not a context hole. The role of the index \(i\) in a redex \(\text{exp}_i(C)\) is only used to determine the index \(j\) in variables \(v_j\) or abstractions \((\lambda v_j)\) that are possibly created by contracting this redex. Therefore for every rewrite sequence \(\text{exp}_i(C) \rightarrow_{\exp} C'_1\) has a ‘parallel’ rewrite sequence \(\text{exp}_i(C) \rightarrow_{\exp} C'_2\) where \(C'_1\) and \(C'_2\) have the same unlabeled syntax tree, and differ only possibly in the aspects (3), (4), and (5) of the lemma (with \(C'_1\) for \(\langle C\rangle_{\exp}^{(i_1)}\), and \(C'_2\) for \(\langle C\rangle_{\exp}^{(i_2)}\)). Then this fact holds clearly also for the expanded forms \(\langle C\rangle_{\exp}^{(i_1)}\) and \(\langle C\rangle_{\exp}^{(i_2)}\) of \(C\).

## 5 Simulation of leftmost-outermost \(\beta\)-reduction on \(\lambda\)-TRS-terms

We now adapt the TRS for the simulation of leftmost-outermost \(\beta\)-reduction rewrite sequences on \(\lambda\)-term representations, iterated in parallel positions (see page 8), to a ‘lopsim-TRS’ that facilitates such a simulation on terms of \(\lambda\)-TRSs. For every \(\lambda\)-TRS \(\mathcal{L}\), we introduce a lopsim-TRS with rules that are similar as before but differ for steps involving abstractions.

A simulation starts on a term \(\text{lop}(s)\) where \(s\) is a \(\lambda\)-TRS ground term. Therefore initially all abstractions are represented by scope symbols. If under leftmost-outermost evaluation a scope symbol \(f\) is detected that does not have an argument, then the top of the \(\lambda\)-abstraction it represents is stable (that is, it is part of a head normal form context). Therefore it is expanded, giving rise to an abstraction representation \((\lambda v_i)\), and then leftmost-outermost evaluation continues immediately below. If, on the other hand, a scope symbol \(f\) is detected that has at least one applicative argument term \(s\), then it represents the \(\lambda\)-abstraction of a leftmost-outermost redex. In this case the \(\beta\)-reduction step for this leftmost-outermost redex is simulated by using the defining rule \(\rho\) of \(f\) in the \(\lambda\)-TRS, which involves filling the argument \(s\) into the scope context \(F\) of \(f\). The final term in an iterated simulation of a leftmost-outermost \(\rightarrow_{\beta}\) rewrite sequence to a \(\lambda\)-term normal form will be a normal form of the lopsim-TRS that is a \(\lambda\)-term representation with named abstraction symbols, but without any scope symbols.

The changes in the adapted simulation TRS concern \(\rightarrow_{\text{desc}_\lambda}\) steps that descend into an abstraction, and \(\rightarrow_{\text{cont}}\) steps that simulate the reduction of \(\beta\)-redexes. In both cases prior to the step the pertaining abstractions are represented by terms with a scope symbol at the root. Then in the steps the definition of the scope symbol in the underlying \(\lambda\)-TRS is used. Additional substitution rules are not necessary any more, because the substitution involved in the contraction of a (represented) \(\beta\)-redex can now be carried out by a single first-order rewrite step. This is because such a step includes the transportation of the argument of a redex into the scope context that defines the body of the abstraction. An additional parameter \(i\) of the operation symbols \(\text{lop}_{n,i}\) is used to prevent that any two nested abstractions refer to the same variable name, safeguarding that rewrite sequences denote meaningful reductions on \(\lambda\)-terms.

**Definition 23** (lopsim-TRS for \(\lambda\)-TRSs). Let \(\mathcal{L} = \langle \Sigma, R \rangle\) be a \(\lambda\)-TRS. The lopsim-TRS (leftmost-outermost-parallel \(\beta\)-reduction simulation TRS) \(\mathcal{L}_0(\mathcal{L}) = \langle \Sigma_{\text{lopsim}}, R_{\text{lopsim}} \rangle\) for \(\mathcal{L}\) has the signature \(\Sigma_{\text{lopsim}} := \Sigma \cup \Sigma_\lambda \cup \Sigma_{\text{lop}}\) with \(\Sigma_{\text{lop}} := \{\text{lop}\} \cup \{\text{lop}_{n,i} \mid n, i \in \mathbb{N}\}\), a signature of operation symbols (for simulating leftmost-outermost reduction) consisting of the unary symbol \(\text{lop}\), and the symbols \(\text{lop}_{n,i}\) with arity \(n + 1\).
for $n, i \in \mathbb{N}$; the rule set $R_{\text{lopsim}}$ of $\mathcal{LO}(\mathcal{L})$ consists of the following (schemes of) rewrite rules, which are indexed by scope symbols $f \in \Sigma^-$, and where $F$ is the scope context for scope symbol $f$:

\[
\begin{align*}
lop(x) & \to \lop_{0,0}(x) \quad \text{(init)} \\
lop_{n,i}(\oplus(x_1, x_2), y_1, \ldots, y_n) & \to \lop_{n+1,i}(x_1, x_2, y_1, \ldots, y_n) \quad \text{(desc$_\oplus$)$_{n,i}$} \\
lop_{0,i}(f(x_1, \ldots, x_k)) & \to (\lambda v_i)(\lop_{0,i+1}(F[x_1, \ldots, x_k, v_i])) \quad \text{(desc$_\lambda$)$_i$} \\
lop_{n+1,i}(f(x_1, \ldots, x_k), y_1, y_2, \ldots, y_{n+1}) & \to \lop_{n,i}(F[x_1, \ldots, x_k, y_1], y_2, \ldots, y_{n+1}) \quad \text{(contr)$_{n+1}$} \\
lop_{0,i}(v_j) & \to v_j \quad \text{(var)$_0$} \\
lop_{n+1,i}(v_j, y_1, \ldots, y_{n+1}) & \to \oplus(\ldots \oplus(v_j, \lop_{0,i}(y_1)), \ldots, \lop_{0,i}(y_{n+1})) \quad \text{(var)$_{n+1,i}$}
\end{align*}
\]

By $\to_{\text{lop}}$ we denote the rewrite relation of $\mathcal{LO}(\mathcal{L})$. By $\to_{\text{contr}}$ we denote the rewrite relation that is induced by the rule scheme (contr)$_f$, where $f \in \Sigma^-$ ranges over scope symbols of $\mathcal{L}$. By $\to_{\text{init}}$, $\to_{\text{desc$_\oplus$}}$, $\to_{\text{desc$_\lambda$}}$, and $\to_{\text{var}}$ we denote the rewrite relations that are induced by the rule schemes (init), (desc$_\oplus$)$_{n,i}$, (desc$_\lambda$)$_i$, and (var)$_n$, respectively, where the parameters range over $f \in \Sigma^-$, and $n, i \in \mathbb{N}$.

**Example 24.** For the $\lambda$-TRS $\mathcal{L}$ in Example 8, we reduce the term $f(g, h)$, which denotes the $\lambda$-term $M$ in Example 8, in the lopsim-TRS $\mathcal{LO}(\mathcal{L})$ for $\mathcal{L}$:

\[
\begin{align*}
\lop(f(g, h)) & \to_{\text{init}} \lop_{0,0}(f(g, h)) \\
& \to_{\text{desc$_\lambda$}} (\lambda v_0)(\lop_{0,1}(\oplus(g, \oplus(h, v_0)))) \\
& \to_{\text{desc$_\lambda$}} (\lambda v_0)(\lop_{1,1}(g, \oplus(h, v_0))) \\
& \to_{\text{contr}} (\lambda v_0)(\lop_{0,1}(\oplus(h, v_0))) \\
& \to_{\text{contr}} (\lambda v_0)(\lop_{0,1}(h, v_0)) \\
& \to_{\text{desc$_\lambda$}} (\lambda v_0)(\lop_{1,1}(h, v_0)) \\
& \to_{\text{contr}} (\lambda v_0)(\lop_{0,1}(i(v_0))) \\
& \to_{\text{desc$_\lambda$}} (\lambda v_0)(\lop_{0,1}(\oplus(v_1, v_0))) \\
& \to_{\text{desc$_\lambda$}} (\lambda v_0)(\lop_{1,2}(v_1, v_0)) \\
& \to_{\text{var$_1$}} (\lambda v_0)(\lop_{1,2}(v_1, \lop_{0,2}(v_0))) \\
& \to_{\text{var$_0$}} (\lambda v_0)(\lop_{0,2}(v_1, v_0))
\end{align*}
\]

We obtain an ‘$\alpha$-equivalent’ version of the $\lambda$-term representation at the end of the simulated leftmost-outmost reduction on $\lambda$-term representations in Example 8.

In order to define how terms in the lopsim-TRS denote $\lambda$-term representations we extend the expansion TRS from Definition 24 with rules that deal with operation and named- abstraction symbols. We want expansion to be an ‘X-ray picture’ of the current state of a term’s evaluation. Therefore operation symbols $lop$ and $lop_{n,i}$ will mainly be ignored. However, indices $i$ in operation symbols $lop_{n,i}$ will be taken into account to, ensure unique naming at comparable positions in the expanded $\lambda$-term representation.

**Definition 25** (expansion TRS for lopsim-TRS-terms, neglecting further evaluation). Let $\mathcal{L} = (\Sigma, R)$ be a $\lambda$-TRS. The expansion TRS $\mathcal{E}_{\text{lopsim}}(\mathcal{L}) = (\Sigma_{\text{lopsim}} \cup \Sigma_{\text{expand}}, R_{\text{exp}} \cup R_{\text{exp}'})$ for lopsim-TRS-terms, which neglects further evaluation according to $\to_{\text{lop}}$, has as its signature the union of the signature $\Sigma_{\text{lopsim}}$ of...
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$\mathcal{LO}(\mathcal{L})$ and the signature $\Sigma_{\text{expand}}$ of $\mathcal{E}(\mathcal{L})$, and as rules the rules $R_{\text{exp}}$ of $\mathcal{E}(\mathcal{L})$, see both in Definition 19, together with the set of rules $R_{\text{exp}}'$ that consists of:

\[
\begin{align*}
\exp_i(l\text{op}(x)) &\rightarrow \exp_i(x) \\
\exp_i(l\text{op}_{0,j}(x)) &\rightarrow \exp_{i'}(x) \quad \text{for } i' := \max \{i, j\} \\
\exp_i(l\text{op}_{n+1,j}(x, y_1, \ldots, y_{n+1})) &\rightarrow \exp_{i'}(\underbrace{\text{@}(\cdot \cdot \cdot \text{@$}(x, y_1, \ldots, y_{n+1})})_{\cdot} \quad \text{for } i' := \max \{i, j\}
\end{align*}
\]

The rewrite relation of $\mathcal{E}_{\text{lop}}(\mathcal{L})$ will again be denoted by $\rightarrow_{\text{exp}}$.

**Definition 26** (denoted $\lambda$-term (representation), extended to lopsim-TRS-terms). Let $\mathcal{L} = (\Sigma, R)$ be a $\lambda$-TRS. For terms $s \in \text{Ter}(\Sigma_{\text{lop}})$ in $\mathcal{LO}(\mathcal{L})$, we also denote by $\llbracket s \rrbracket^\Sigma$ the finite or infinite $\rightarrow_{\text{exp}}$-normal form of the term $\exp_0(s)$. If it is a $\lambda$-term representation, then we say that $\llbracket s \rrbracket^\Sigma$ is the denoted $\lambda$-term representation of $s$, and we again write $\llbracket s \rrbracket^\Sigma_\lambda$ for the $\lambda$-term $\llbracket \llbracket s \rrbracket^\Sigma \rrbracket_\lambda$.

6 Linear depth increase of leftmost-outermost $\beta$-red. simulation

In this section we establish that the depth increase of expanded terms along an arbitrary rewrite sequences in a lopsim-TRS is linear in the number of $\rightarrow_{\text{comp}}$ steps.

In order to reason directly on terms and contexts of the lopsim-TRS, we define the ‘expansion depth’ of terms and contexts as the depth of their expanded forms without counting expansion symbols $\exp_i$.

**Definition 27** (expansion depth of terms and contexts, expansion hole depth of contexts). Let $\mathcal{L} = (\Sigma, R)$ be a $\lambda$-TRS, and let $\mathcal{LO}(\mathcal{L}) = (\Sigma_{\text{lop}}, R_{\text{lop}})$ be the lopsim-TRS for $\mathcal{L}$.

For terms $t \in \text{Ter}(\Sigma_{\text{lop}}, \text{Var})$ and contexts $C \in \text{Ctx}_n(\Sigma_{\text{lop}}, \text{Var})$, where $n \in \mathbb{N}$, we define by:

\[
|t|_{\text{exp}} := |t|_{\text{exp}}(0)^{\text{exp}} \in \mathbb{N} \cup \{\infty\}, \quad |C|_{\text{exp}} := |C|_{\text{exp}}(0)^{\text{exp}} \in \mathbb{N} \cup \{\infty\}, \quad |C|_{\text{exp}, \square} := |C|_{\text{exp}, \square}(0)^{\text{exp}, \square} \in \mathbb{N} \cup \{\infty\},
\]

the expansion depth $|t|_{\text{exp}}$ of $t$, the expansion depth $|C|_{\text{exp}}$ of $C$, and the expansion hole depth $|C|_{\text{exp}, \square}$ of $C$, namely as the depth of the expanded form of $t$, the depth of the expanded form of $C$ while ignoring expansion symbols, and the hole depth of the expanded form of $C$ while ignoring expansion symbols, respectively. Here and below we denote by $|t|_{\text{exp}}^{\text{exp}}$ the operation that measures the depth of terms and contexts while ignoring symbols $\exp_i$ for $i \in \mathbb{N}$. So the expansion depth $|C|_{\exp}$, and the expansion hole depth $|C|_{\exp, \square}$ of a context $C$ ignore the symbols $\exp_{i_j}$ in guarded hole expressions $\exp_{i_j}(\square_{i_j})$ in the representation of the expanded forms of $C$ according to Lemma 20.

**Lemma 28.** Let $\mathcal{LO}(\mathcal{L}) = (\Sigma_{\text{lop}}, R_{\text{lop}})$ be the lopsim-TRSs $\mathcal{LO}(\mathcal{L})$ for finitely nested $\lambda$-TRSs $\mathcal{L}$.

Then for every term $t \in \text{Ter}(\Sigma_{\text{lop}})$ the expansion depth $|t|_{\text{exp}}$ of $t$ is finite, that is, a natural number. Also, for every context $C \in \text{Ctx}(\Sigma_{\text{lop}})$, the expansion depth $|C|_{\text{exp}}$ and the expansion hole depth $|C|_{\text{exp}, \square}$ of $C$ are finite.

**Proof:** We argued in Definition 19, the expanded form of terms in $\text{Ter}(\Sigma \cup \Sigma_\lambda)$ and of contexts in $\text{Ctx}(\Sigma \cup \Sigma_\lambda)$ are well-defined finite terms, and contexts, respectively. Now as the expansion depth of a term or context is defined as the depth of the expanded form of the term or context, it follows that that the expansion depth of the term or context in question is finite.

Since a $\lambda$-term representation $s$ and the $\lambda$-term $\llbracket s \rrbracket_\lambda$ denoted by it have the same depth, the expansion depth of a term $s$ that denotes a $\lambda$-term $M$ coincides with the depth of $M$. 

**Proposition 29.** Let $\mathcal{L} = (\Sigma, R)$ be a $\lambda$-TRS, and let $\mathcal{LO}(\mathcal{L})$ be the lopsim-TRS for $\mathcal{L}$. If for a term $s$ in $\mathcal{LO}(\mathcal{L})$ it holds that $[s]_\mathcal{L}^\mathcal{E} = M$ for a $\lambda$-term $M$, then $|s|_\exp = |[s]_\mathcal{L}^\mathcal{E}| = ||[s]_\mathcal{L}^\mathcal{E}|| = |M|$.

The following lemma formulates clauses for the expansion depth depending on the outermost symbol of a term in a lopsim-TRS. For finitely nested $\lambda$-TRSs, these clauses can be read as an inductive definition. They can be proved in a straightforward manner by making use of the definition via the expansion TRS of the $\lambda$-representations $[s]_\mathcal{E}$ for terms $s$ of the lopsim-TRS for a $\lambda$-TRS $\mathcal{L}$.

**Lemma 30** (inductive clauses for the expansion depth of terms and contexts). Let $\mathcal{L} = (\Sigma, R)$ be a finitely nested $\lambda$-TRS, and let $\mathcal{LO}(\mathcal{L}) = (\Sigma_{\text{lopsim}}, R_{\text{lopsim}})$ be the lopsim-TRS for $\mathcal{L}$.

The expansion depth $|C|_\exp$ of contexts $C \in \text{Cxt}(\Sigma_{\text{lopsim}}, \text{Var})$ satisfies the following clauses:

$$
|\text{x}|_\exp = 0 \quad (x \text{ variable in Var})
$$

$$
|\square_i|_\exp = 0 \quad (i \in \mathbb{N})
$$

$$
\mid@i(C_1, C_2)|_\exp = 1 + \max \{|C_1|_\exp, |C_2|_\exp\} \quad (i \in \mathbb{N})
$$

$$
|f(C_1, \ldots, C_k)|_\exp = 1 + |F|_{\mathcal{L}}[C_1, \ldots, C_k, \nu_0]|_\exp
$$

$$
|\nu_j|_\exp = 0 \quad (j \in \mathbb{N})
$$

$$
|\langle \lambda \nu_j \rangle (t)|_\exp = 1 + |t|_\exp
$$

$$
|\text{lap}(C)|_\exp = |C|_\exp
$$

$$
|\text{lap}_{n,i}(C_0, C_1, \ldots, C_n)|_\exp = |\text{@}(\cdots \text{@}(C_0, C_1) \ldots, C_n)|_\exp
$$

where $i, j, k, l, n \in \mathbb{N}$, $C_0, C_1, \ldots \in \text{Cxt}(\Sigma_{\text{lopsim}}, \text{Var})$ are contexts, and $f \in \Lambda^-$ with $\text{ar}(f) = k$ are scope symbols in $\mathcal{L}$ with appertaining scope contexts $F \in \mathcal{L}$. These clauses specialize to analogous clauses for terms in $\text{Ter}(\Sigma_{\text{lopsim}}, \text{Var})$, because terms can be viewed as contexts without hole occurrences.

**Proof:** The base cases of the inductive clauses can be verified as follows. For every $x \in \text{Var}$, we have $|\text{x}|_\exp = |\exp_0(\text{x})|_\mathcal{E} = |\text{x}| = 0$, and for every hole $\square_i \in \square$ we find $|\square_i|_\exp = |\exp_0(\square_i)|_\mathcal{E} = |\square_i| = 0$, for $i \in \mathbb{N}$. With the step $\exp_0(\nu_i) \to^{\exp} \nu_i$ we get $|\nu_i|_\exp = |\nu_i|_{\exp_0} = 0$.

Each of the other cases can be established by arguing with $\to^{\exp}$ steps. We provide two examples. For the first one we consider $C \equiv \text{@}(C_1, C_2)$. Then $\exp_0(\text{@}(C_1, C_2)) \to^{\exp} \text{@}(\exp_0(C_1), \exp_0(C_2))$ is an expansion step on $\exp_0(C)$, with which we can argue as follows:

$$
|\text{@}(C_1, C_2)|_\exp = \mid|\text{@}(C_1, C_2)|_{\exp_0} \mid_{\exp} = \mid|\exp_0(\text{@}(C_1, C_2))|_{\exp_0} \mid_{\exp} = \mid|\exp_0(\text{@}(C_1), \exp_0(C_2))|_{\exp_0} \mid_{\exp} = \mid|\exp_0(C_1), \exp_0(C_2)|_{\exp_0} \mid_{\exp} = 1 + \max \{|C_1|_{\exp_0}, |C_2|_{\exp_0} \}.
$$

As a second example, we consider a context $C \equiv f(C_1, \ldots, C_k)$. In this case there is an expansion step of the form $\exp_0(f(C_1, \ldots, C_k)) \to^{\exp} (\lambda \nu_0)(\exp_1((f[C_1, \ldots, C_k, \nu_0]))|_{\exp})$, with which we now argue as follows:

$$
|f(C_1, \ldots, C_k)|_\exp = |f(C_1, \ldots, C_k)|_{\exp_0} \mid_{\exp} = \mid\exp_0(f(C_1, \ldots, C_k))|_{\exp_0} \mid_{\exp} = \mid((\lambda \nu_0)(\exp_1((f[C_1, \ldots, C_k, \nu_0])))|_{\exp}) \mid_{\exp} \quad (\text{using the step here})
$$
Let \( f : \alpha \rightarrow \beta \) be a \( \lambda \)-TRS.

Let \( f \in \Sigma^- \) be a scope symbol in \( \mathcal{L} \) with arity \( k \). The expansion depth \(|f|_{\text{exp}}\) of \( f \in \Sigma^- \) is defined as

\[
|f(\Box_1, \ldots, \Box_k)|_{\text{exp}} = \max \{|f|_{\text{exp}} \mid f \in \Sigma^- \} \in \mathbb{N} \cup \{\infty\},
\]

that is, as the expansion depth of the \( k \)-ary context \( f(\Box_1, \ldots, \Box_k) \).

We also define by \(|\mathcal{L}|_{\text{exp}} := \max \{|f|_{\text{exp}} \mid f \in \Sigma^- \} \in \mathbb{N} \cup \{\infty\}\), the maximal expansion depth of a scope symbol in \( \mathcal{L} \).

Note that if a \( \lambda \)-TRS \( \mathcal{L} \) is finitely nested, then \(|f|_{\text{exp}} = |f(\Box_1, \ldots, \Box_k)|_{\text{exp}} \in \mathbb{N} \) due to Lemma \( \mathbb{B} \).

Furthermore, if in addition to being finitely nested \( \mathcal{L} \) is also finite, then \(|\mathcal{L}|_{\text{exp}} \in \mathbb{N} \).

**Lemma 32.** Let \( \mathcal{L} = (\Sigma, \mathcal{R}) \) be a \( \lambda \)-TRS, and let \( \mathcal{LO}(\mathcal{L}) \) be the lopsim-TRS for \( \mathcal{L} \). If for a term \( s \) in \( \mathcal{LO}(\mathcal{L}) \) it holds that \([s]_\mathcal{L} = M \) for a \( \lambda \)-term \( M \), then \(|\mathcal{L}|_{\text{exp}} \leq |M| \).

Next we establish expansion depth variants of the two easy context lemmas as formulated in Section \( \mathbb{E} \) of Lemma \( \mathbb{H} \) and Lemma \( \mathbb{J} \). Those lemmas are also crucial for the proof of the lemmas below.

**Lemma 33.** \(|C[s_1, \ldots, s_n, \Box]|_{\text{exp}} \leq |C|_{\text{exp}} \), holds, in a finitely nested \( \lambda \)-TRS \( (\Sigma, \mathcal{R}) \), for all terms \( s_1, \ldots, s_n \in \text{Ter}(\Sigma \cup \Sigma_\lambda) \), where \( n \in \mathbb{N} \), and all contexts \( C \in \text{Cxt}_{n+1}(\Sigma \cup \Sigma_\lambda) \) in which there is at least one occurrence of \( \Box_{n+1} \).

**Proof:** Let \( s_1, \ldots, s_n \in \text{Ter}(\Sigma \cup \Sigma_\lambda) \) be terms, and let \( C \in \text{Cxt}_{n+1}(\Sigma \cup \Sigma_\lambda) \) be a context in which \( \Box_{n+1} \) has an occurrence. Then due to Lemma \( \mathbb{D} \) and Lemma \( \mathbb{F} \) the expanded forms of \( C \) and of
By using (7) we obtain:

\[ C[s_1, \ldots, s_n, □] \text{ can be represented, with a linear context } D \in \text{Cxt}_{m,1}(\Sigma \cup \Sigma_λ) \text{ and } i_1, \ldots, i_m \in \mathbb{N}, \text{ and } j_1, \ldots, j_m \in \{1, \ldots, n+1\}, \text{ as follows:} \]

\[
\langle C \rangle^{(0)}_\text{exp} = D[\langle □, i_1 \rangle^\text{exp}, \ldots, \langle □, j_1 \rangle^\text{exp}],
\]

\[
\langle C[s_1, \ldots, s_n, □] \rangle^{(0)}_\text{exp} = D[\langle E_{j_1} \rangle^\text{exp}, \ldots, \langle E_{j_m} \rangle^\text{exp}],
\]

where \( E_i := \begin{cases} s_i & \text{if } i \in \{1, \ldots, n\} \\ □ & \text{if } i = n + 1 \end{cases} \in \text{Cxt}_1(\Sigma \cup \Sigma_λ), \text{ for } i \in \{1, \ldots, n+1\}. \]

Since □_{n+1} occurs in C, it follows that one of \( \langle E_{j_l} \rangle^\text{exp} \) for \( l \in \{1, \ldots, m\} \) is of the form \( \exp_{i_l}(□) \). We will use this in the application of Lemma 3 in the following argumentation that we now can perform on the basis of the preparation above:

\[
\langle C[s_1, \ldots, s_n, □] \rangle^{(0)}_\text{exp} = |\langle C[s_1, \ldots, s_n, □] \rangle^{(0)}_\text{exp}|_\text{exp, □} = D[\langle □, i_1 \rangle^\text{exp}, \ldots, \langle □, j_1 \rangle^\text{exp}] \]

(by def. of \( |\cdot|_\text{exp, □} \))

\[
\leq |D|^{\text{exp, □}} \quad \text{(by using a } |\cdot|^{\text{exp, □}}\text{-version of Lemma 3)}
\]

\[
= |D[\exp_{i_1}(□), \ldots, \exp_{i_m}(□)]|^{\text{exp, □}} \quad \text{(by def. of } |\cdot|^{\text{exp, □}}\text{)}
\]

\[
= |D(\langle □, i_1 \rangle^\text{exp}, \ldots, \langle □, j_1 \rangle^\text{exp})|^{\text{exp}} \quad \text{(by def. of } \langle □ \rangle^{(0)}_\text{exp}\text{)}
\]

\[
= |(C)^{(0)}_\text{exp}|^{\text{exp}} \quad \text{(by def. of } |\cdot|^{\text{exp, □}}\text{)}
\]

In this way we have established the inequality as stated by the lemma. 

**Lemma 34.** \( |C[s]|_\text{exp} = \max \{ |C|_\text{exp} | □ | s |_\text{exp} \} \) holds, in a finitely nested λTRS \( ⟨\Sigma, R⟩ \), for all contexts \( C \in \text{Cxt}_1(\Sigma \cup \Sigma_λ) \), and all terms \( s \in \text{Ter}(\Sigma \cup \Sigma_λ) \).

**Proof:** Let \( C \in \text{Cxt}_1(\Sigma \cup \Sigma_λ) \), and \( s \in \text{Ter}(\Sigma \cup \Sigma_λ) \). Due to Lemma 20 and Lemma 21 the expanded forms of \( C \) and of \( C[s] \) can be represented with a linear context \( D \in \text{Cxt}_{m,1}(\Sigma \cup \Sigma_λ) \) for \( m \in \mathbb{N} \), and \( i_1, \ldots, i_m \in \mathbb{N} \), and \( j_1, \ldots, j_m \in \{1, \ldots, n+1\} \) as follows:

\[
\langle C \rangle^{(0)}_\text{exp} = D[\langle □, i_1 \rangle^\text{exp}, \ldots, \langle □, j_1 \rangle^\text{exp}],
\]

\[
\langle C[s] \rangle^{(0)}_\text{exp} = D[\langle s \rangle^\text{exp}, \ldots, \langle s \rangle^\text{exp}].
\]

By using (7) we obtain:

\[
|C|_\text{exp} = |\langle C \rangle^{(0)}_\text{exp}|^{\text{exp}} = |D[\exp_{i_1}(□), \ldots, \exp_{i_m}(□)]|^{\text{exp}} = |D| = |D|^{\text{exp, □}},
\]

\[
|C[s]|_\text{exp} = |\langle C[s] \rangle^{(0)}_\text{exp}|^{\text{exp}} = |D[\exp_{i_1}(□), \ldots, \exp_{i_m}(□)]|^{\text{exp}} = |D| = |D|^{\text{exp, □}}.
\]

On the basis of these preparations we can now argue:

\[
|C[s]|_\text{exp} = |\langle C[s] \rangle^{(0)}_\text{exp}|^{\text{exp}} \quad \text{(by def. of } |\cdot|^{\text{exp}}\text{)}
\]

\[
= |D[\langle i_1 \rangle^\text{exp}, \ldots, \langle s \rangle^\text{exp}]|^{\text{exp}} \quad \text{(by } (8))
\]
Statement (12) follows by using (11) with \( j \in \{1, \ldots, n\} \). (by using a \(|\bullet|\) version of Lemma 34)

\[
\begin{align*}
\max \{ |D|_{\in}, D_{\in} + |(s)_j|_{\in} \} & = \max \{ |D|_{\in}, D_{\in} + |s|_{\in} \} \\
& = \max \{ |C|_{\in}, |C|_{\in} + |s|_{\in} \} \\
& = \max \{ |C|_{\in}, |C|_{\in} + |s|_{\in} \}
\end{align*}
\]

(by appeal to Lemma 23)

(by \( \beta \), and \( \alpha \)).

In this way we have shown the equation as stated by the lemma. □

For analyzing the depth increase of steps in lopsim-TRSs, the next two lemmas will be instrumental. They relate the expansion depth of contexts filled with terms to the expansion depths of occurring terms.

**Lemma 35.** Let \( \mathcal{L} = \langle \Sigma, R \rangle \) be a finitely nested \( \lambda \)-TRS. Then for all unary contexts \( C \in Cxt_1(\Sigma) \), terms \( s, t \in \text{Ter}(\Sigma) \), and \( d \in \mathbb{N} \) the following statements hold:

\[
|s|_{\exp} \leq |t|_{\exp} + d \implies |C[s]|_{\exp} \leq |C[t]|_{\exp} + d,
\]

\[ (11) \]

\[
|s|_{\exp} = |t|_{\exp} \implies |C[s]|_{\exp} = |C[t]|_{\exp},
\]

\[ (12) \]

**Proof:** Let \( C \in Cxt_1(\Sigma) \), \( s, t \in \text{Ter}(\Sigma) \), and \( d \in \mathbb{N} \). To verify (11) we assume that \( |s|_{\exp} \leq |t|_{\exp} + d \) holds, and show the inequality on the right-hand side in (11). For this we argue as follows:

\[
\begin{align*}
|C[s]|_{\exp} &= \max \{ |C|_{\exp}, |C|_{\exp, \Box} + |s|_{\exp} \} \\
& \leq \max \{ |C|_{\exp}, |C|_{\exp, \Box} + |t|_{\exp} + d \} \quad \text{(by Lemma 34)} \\
& \leq \max \{ |C|_{\exp} + d, |C|_{\exp, \Box} + |t|_{\exp} + d \} \quad \text{(using the assumption)} \\
& = \max \{ |C|_{\exp}, |C|_{\exp, \Box} + |t|_{\exp} + d \} \quad \text{(possibly increasing the maximum)} \\
& = |C[t]|_{\exp} + d \quad \text{(simplifying the maximum expression)} \\
& = |C[t]|_{\exp} + d \quad \text{(by Lemma 34)}. \\
\end{align*}
\]

Statement (12) follows by using (11), with \( d = 0 \) in both directions. □

**Lemma 36.** Let \( \mathcal{L} = \langle \Sigma, R \rangle \) be a finitely nested \( \lambda \)-TRS. Then for all contexts \( C \in Cxt_{k+1}(\Sigma) \), and terms \( s_1, \ldots, s_k, u \in \text{Ter}(\Sigma) \), with \( k \in \mathbb{N} \), the following statement holds:

\[
|C[s_1, \ldots, s_k, u]|_{\exp} \leq \max \{ |C[s_1, \ldots, s_k, \Box]|_{\exp}, |C|_{\exp} + |u|_{\exp} \}.
\]

\[ (13) \]

**Proof:** For contexts \( C \in Cxt_{k+1}(\Sigma) \), and terms \( s_1, \ldots, s_k, u \in \text{Ter}(\Sigma) \) with \( k \in \mathbb{N} \) we argue as follows. If \( \Box_{n+1} \) occurs in \( C \), then we can argue as follows:

\[
\begin{align*}
|C[s_1, \ldots, s_k, u]|_{\exp} &= \max \{ |C[s_1, \ldots, s_k, \Box]|_{\exp}, |C[s_1, \ldots, s_k, \Box]|_{\exp, \Box} + |u|_{\exp} \} \\
& \leq \max \{ |C[s_1, \ldots, s_k, \Box]|_{\exp}, |C|_{\exp} + |u|_{\exp} \} \quad \text{(by Lemma 34, using context } C[s_1, \ldots, s_k, \Box])
\end{align*}
\]

and have established the statement (13). If, on the other hand, \( \Box_{n+1} \) does not occurs in \( C \), then we argue:

\[
|C[s_1, \ldots, s_k, u]|_{\exp} = |C[s_1, \ldots, s_k, \Box]|_{\exp} \leq \max \{ |C[s_1, \ldots, s_k, \Box]|_{\exp}, |C|_{\exp} + |u|_{\exp} \}
\]

and have obtained (13) again. □
Now we can formulate, and prove, a crucial lemma (Lemma 37). Its central statement is that the depth increase in a →_contr step (with respect to a lopsim-TRS) at the root of a term is bounded by the depth of the scope context of the scope symbol that is involved in the step. See Figure 1 for an illustration of the underlying intuition for the analogous case of a step according to the defining rule of a scope symbol.

Lemma 37. Let \( \mathcal{L} = (\Sigma, R) \) be a finitely nested \( \lambda \)-TRS. Then for every scope symbol \( f \in \Sigma^- \) with arity \( k \) and scope context \( F \), and for all terms \( s_1, \ldots, s_k, u \in \text{Ter}(\Sigma) \), and all \( i \in \mathbb{N} \), it holds:

(i) \( |F[s_1, \ldots, s_k, u]|_{\text{exp}} \leq |@ (f(s_1, \ldots, s_k), u)|_{\text{exp}} + |f|_{\text{exp}} - 2 \).

(ii) \( |\text{lop}_{n,i}(F[s_1, \ldots, s_k, u_1], u_2, \ldots, u_{n+1})|_{\text{exp}} \leq |\text{lop}_{n+1,i}(f(s_1, \ldots, s_k), u_1, \ldots, u_{n+1})|_{\text{exp}} + |f|_{\text{exp}} - 2 \).

Proof: We let \( f, F, s_1, \ldots, s_k, t, \) and \( i \) be as assumed in the lemma. We establish statement (i) as follows:

\[
|F[s_1, \ldots, s_k, u]|_{\text{exp}} \\
\leq \max\{ |F[s_1, \ldots, s_k, \Box]|_{\text{exp}}, |F|_{\text{exp}} + |u|_{\text{exp}} \} \\
= \max\{ |f(s_1, \ldots, s_k)|_{\text{exp}} - 1, |f|_{\text{exp}} - 1 + |u|_{\text{exp}} \} \\
\leq \max\{ |f(s_1, \ldots, s_k)|_{\text{exp}} + |f|_{\text{exp}} - 1, |f|_{\text{exp}} - 1 + |u|_{\text{exp}} \}
\]

(by possibly increasing the maximum)

\[
= \left( \max\{ |f(s_1, \ldots, s_k)|_{\text{exp}}, |u|_{\text{exp}} \} \right) + |f|_{\text{exp}} - 1
\]

(simplification)

\[
= \left( 1 + \max\{ |f(s_1, \ldots, s_k)|_{\text{exp}}, |u|_{\text{exp}} \} \right) + |f|_{\text{exp}} - 2
\]

(rearrangement)

\[
= |@ (f(s_1, \ldots, s_k), u)|_{\text{exp}} + |f|_{\text{exp}} - 2
\]

(by Lemma 37).

For showing statement (ii) we proceed by lifting the inequality in statement (i) into a context by means of Lemma 35. More precisely, we argue as follows by means of the inductive clauses in Lemma 30, and by appealing to Lemma 33 for the context \( C := @ (\cdots @ (\Box, u_1) \cdots, u_{n+1}) \):

\[
|\text{lop}_{n,i}(F[s_1, \ldots, s_k, u_1], u_2, \ldots, u_{n+1})|_{\text{exp}} \\
= |@ (\cdots @ (F[s_1, \ldots, s_k, u_1], u_2) \cdots, u_{n+1})|_{\text{exp}} \\
\leq |@ (\cdots @ (@ (f(s_1, \ldots, s_k), u_1), u_2) \cdots, u_{n+1})|_{\text{exp}} + |f|_{\text{exp}} - 2 \\
= |\text{lop}_{n+1,i}(f(s_1, \ldots, s_k), u_1, \ldots, u_{n+1})|_{\text{exp}} + |f|_{\text{exp}} - 2 .
\]

In this way we have now also justified the inequality in statement (ii). □

Lemma 38. Let \( \mathcal{L}_0(\mathcal{L}) = (\Sigma_{\text{lopsim}}, R_{\text{lopim}}) \) be the lopim-TRS for a finitely nested \( \lambda \)-TRS \( \mathcal{L} = (\Sigma, R) \).

Then every →_search step in \( \mathcal{L}_0(\mathcal{L}) \) preserves the expansion depth, and every →_contr step increases the expansion depth by less than the expansion depth of the scope symbol \( f \) involved in the contraction. More precisely, the following statements hold for all \( t_1, t_2 \in \text{Ter}(\Sigma_{\text{lopsim}}) \):

(i) If \( t_1 \to_{\text{search}} t_2 \), then \( |t_1|_{\text{exp}} = |t_2|_{\text{exp}} \).

(ii) If \( t_1 \to_{\text{contr}} t_2 \), then \( |t_1|_{\text{exp}} \geq |t_2|_{\text{exp}} \).
(ii) If $t_1 \rightarrow_{\text{contr}} t_2$, then $|t_2|_{\exp} \leq |t_1|_{\exp} + |f|_{\exp} - 2$, where $f$ is the scope symbol involved in the step.

**Proof:** We first reduce the proof obligation for both items of the lemma to statements that pertain to rewrite steps that take place at the root of the term $t_1$. This is because for non-root $\rightarrow_{\text{search}}$ and $\rightarrow_{\text{contr}}$ steps the corresponding property can be lifted into a rewriting context by using Lemma 35. For instance, consider a step $t_1 \rightarrow_{\text{contr}} t_2$ that does not take place at the root of $t_1$. As such it is of the form $t_1 \equiv C[t_{10}] \rightarrow_{\text{contr}} C[t_{20}] \equiv t_2$ for some non-trivial unary context $C \neq \Box$ and subterms $t_{10}$ and $t_{20}$ of $t_1$ and $t_2$, respectively, such that $t_{10} \rightarrow_{\text{contr}} t_{20}$ is a root step. Now under the assumption that (ii) holds for root $\rightarrow_{\text{contr}}$ steps, we have $|t_{20}|_{\exp} \leq |t_{10}|_{\exp} + |f|_{\exp} - 2$. Then by using equation (10) in Lemma 35 we obtain the desired inequality as follows:

$$|t_2|_{\exp} = |C[t_{20}]|_{\exp} \leq |C[t_{10}]|_{\exp} + |f|_{\exp} - 2 = |t_1|_{\exp} + |f|_{\exp} - 2,$$

For non-root $\rightarrow_{\text{search}}$ steps, preservation of expansion depth can be argued analogously by using equation (12) in Lemma 35 under the assumption that root $\rightarrow_{\text{search}}$ steps preserve expansion depth.

It remains to show that the statements in (ii) and (ii) hold for root steps. We start with showing this for item (i), by inspecting the rules of the lopsim-TRS, and by using the clauses of expansion depth in Lemma 30. The case of a root $\rightarrow_{\text{init}}$ step is straightforward. Now we consider the case of a root $\rightarrow_{\text{desc}}$ step, which is of the form:

$$t_1 \equiv \text{lop}_{n,i}(\@ (s_1, s_2), u_1, \ldots, u_n) \rightarrow_{\text{desc}} \text{lop}_{n+1,i}(s_1, s_2, u_1, \ldots, u_n) \equiv t_2,$$

for some $n, i \in \mathbb{N}$. Here we easily conclude with the clauses for the expansion depth in Lemma 35:

$$|t_1|_{\exp} = \left|\text{lop}_{n,i}(\@ (s_1, s_2), u_1, \ldots, u_n)\right|_{\exp} = \left|\@ (\ldots \@ (\@ (s_1, s_2), u_1) \ldots, u_n)\right|_{\exp} = \left|\text{lop}_{n+1,i}(s_1, s_2, u_1, \ldots, u_n+1)\right|_{\exp} = |t_2|_{\exp}.$$

Next we consider a root $\rightarrow_{\text{desc}}$ step. With some $i \in \mathbb{N}$ it is of the form:

$$t_1 \equiv \text{lop}_{0,i}(f(s_1, \ldots, s_k)) \rightarrow_{\text{desc}} (\lambda v_i)(\text{lop}_{0,i+1}(F[s_1, \ldots, s_k, v_i])) \equiv t_2.$$

Here we argue as follows by using clauses for the expansion depth in Lemma 35:

$$|t_2|_{\exp} = \left|(\lambda v_i)(\text{lop}_{0,i+1}(F[s_1, \ldots, s_k, v_i]))\right|_{\exp} = 1 + \left|\text{lop}_{0,i+1}(F[s_1, \ldots, s_k, v_i])\right|_{\exp} = 1 + \left|F[s_1, \ldots, s_k, v_i]\right|_{\exp} = \left|(\lambda v_i)(F[s_1, \ldots, s_k, v_i])\right|_{\exp} = \left|f(s_1, \ldots, s_k)\right|_{\exp} = \left|\text{lop}_{0,i}(f(s_1, \ldots, s_k))\right|_{\exp} = |t_1|_{\exp}.$$

The case of a root $\rightarrow_{\text{var}}$ step is again easy, both according to the rule $(\text{var})_{0,i}$, also according to the rule $(\text{var})_{n+1,i}$, by using the clauses for $\text{lop}_{n,i}(x)$, and for $\text{lop}_{n+1,i}(x, y_1, \ldots, y_{n+1})$, respectively.

For showing the restriction of item (ii) to root steps, we consider a $\rightarrow_{\text{contr}}$ steps at the root. Such a step is of the form:

$$t_1 \equiv \text{lop}_{n+1,i}(f(s_1, \ldots, s_k), u_1, \ldots, u_{n+1}) \rightarrow_{\text{contr}} \text{lop}_{n,i}(F[s_1, \ldots, s_k, u_1], u_2, \ldots, u_{n+1}) \equiv t_2.$$
Then the desired expansion depth inequality $|t_2|_\exp \leq |t_1|_\exp + |f|_\exp - 2$ follows from Lemma 37 ($\blacksquare$).

By a direct application of this lemma we obtain our main result concerning the depth increase of terms in $\rightarrow_{\text{lop}}$ rewrite sequences.

**Theorem 39.** Let $\mathcal{L} = \langle \Sigma, R \rangle$ be a finite, and finitely nested $\lambda$-TRS, and let $D := |\mathcal{L}|_\exp$. Let $\sigma$ be a finite or infinite $\rightarrow_{\text{lop}}$ rewrite sequence $\sigma$ with initial term $s$. Then $\sigma$ can be construed as a sequence of $\rightarrow_{\text{search}}$ and $\rightarrow_{\text{contr}}$ steps:

$$
\sigma : s = u_0 \rightarrow_{\text{search}} u_0' \rightarrow_{\text{contr}} u_1 \rightarrow_{\text{search}} u_1' \rightarrow_{\text{contr}} \cdots \rightarrow_{\text{contr}} u_n \rightarrow_{\text{search}} u_n' \rightarrow_{\text{contr}} (\cdots \rightarrow_{\text{contr}} u_n+1 \rightarrow_{\text{search}} \cdots),
$$

and then the following statements hold for all $n \in \mathbb{N}$ with $n \leq l$ where $l \in \mathbb{N} \cup \{\infty\}$ is the length of $\sigma$:

(i) $|u_n|_\exp = |u_n'|_\exp$, and $|u_{n+1}|_\exp \leq |u_n|_\exp + (D - 2)$ if $n + 1 \leq l$, that is more verbally, the expansion depth remains the same in the $\rightarrow_{\text{search}}$ steps, and it increases by at most $D - 2$ in the $\rightarrow_{\text{contr}}$ steps.

(ii) $|u_n|_\exp, |u_n'|_\exp \leq |s|_\exp + (D - 2) \cdot n$, that is, the increase of the expansion depth along $\sigma$ is linear in the number of $\rightarrow_{\text{contr}}$ steps performed, with $(D - 2)$ as multiplicative constant.

**Proof:** Statement (i) follows directly from Lemma 38, (i), and (ii). Statement (ii) follows by adding up the uniform bound $D$ on the expansion depth increase in the $n \rightarrow_{\text{contr}}$ steps of the rewrite sequence $\sigma$. ($\blacksquare$)

### 7 Transfer to leftmost-outermost $\beta$-reduction in the $\lambda$-calculus

In this section we sketch how the linear-depth-increase result can be transferred from simulating rewrite sequences on terms of the lopsim-TRS $\mathcal{L} \mathcal{O}(\mathcal{L})$ for a $\lambda$-TRS $\mathcal{L}$ to leftmost-outermost $\beta$-reduction rewrite sequences on terms of the $\lambda$-calculus. We formulate correspondence statements via projection and lifting. In particular, we formulate statements about the projections of $\rightarrow_{\text{lop}}$ steps to $\beta$-reduction steps on $\lambda$-terms, where the projection takes place via expansion to expanded-form $\lambda$-term representations, and about the lifting of leftmost-outermost $\beta$-reduction rewrite sequences to leftmost-outermost rewrite sequences in lopsim-TRSs, where the lifting has to be defined via fully-lazy lambda-lifting. We do not prove these statements here, but we illustrate them by means of our running example. On the basis of such correspondences between rewrite sequences, the linear-depth-increase result for leftmost-outermost $\beta$-reduction in the $\lambda$-calculus follows from the linear-depth-increase result for lopsim-TRS in Section 6.

The first correspondence statement concerns the projection of $\rightarrow_{\text{lop}}$ steps to $\rightarrow_{\beta}$ steps or empty steps on $\lambda$-terms with the property that leftmost-outermost $\rightarrow_{\text{contr}}$ steps project to leftmost-outermost $\rightarrow_{\beta}$ steps.

**Proposition 40 (Projection of $\rightarrow_{\text{lop}}$ steps via $[[\cdot]]_\xi$).** Let $\mathcal{L} \mathcal{O}(\mathcal{L}) = \langle \Sigma_{\text{lopsim}}, R_{\text{lopsim}} \rangle$ be the lopsim-TRS for a $\lambda$-TRS $\mathcal{L} = \langle \Sigma, R \rangle$. Let $s \in \text{Ter}(\Sigma_{\text{lopsim}})$ be a term in $\mathcal{L} \mathcal{O}(\mathcal{L})$ such that $[[s]]_\xi = M$ for a $\lambda$-term $M$.

Then the following statements hold concerning the projection of $\rightarrow_{\text{lop}}$ steps via $[[\cdot]]_\xi$ to steps on $\lambda$-terms, for all $s, s_1 \in \text{Ter}(\Sigma_{\text{lopsim}})$:

(i) If $s \rightarrow_{\text{search}} s_1$, then $[[s]]_\lambda = [[s_1]]_\lambda$. That is, the projection of a $\rightarrow_{\text{search}}$ step via $[[\cdot]]_\xi$ is a trivial step.

(ii) If $s \rightarrow_{\text{contr}} s_1$, then $[[s]]_\lambda \rightarrow_{\beta} [[s_1]]_\lambda$. That is, the projection of a $\rightarrow_{\text{contr}}$ step via $[[\cdot]]_\xi$ is a $\rightarrow_{\beta}$ step.

(iii) If $s \rightarrow_{\text{contr}} s_1$ is a leftmost-outermost step, then $[[s]]_\lambda \rightarrow_{\text{lo}\beta} [[s_1]]_\lambda$ holds. That is, the projection of a leftmost-outermost $\rightarrow_{\text{contr}}$ step via $[[\cdot]]_\xi$ is $\rightarrow_{\text{lo}\beta}$ steps.
Linear Depth Increase of Lambda Terms along Leftmost-Outermost Beta-Reduction

A proof of this statement can be obtained by defining the projection via the expansion rewrite relation \( \rightarrow_{\exp} \), and in particular, via the reduction \( \downarrow_{\exp} \) to expanded forms, which yields \( \lambda \)-term representations. Then it can be shown that \( \rightarrow_{\text{search}} \) steps do not change the expanded form, and that \( \rightarrow_{\text{contr}} \) steps correspond to the contraction of \( \beta \)-redexes on the represented \( \lambda \)-terms.

Example 41. We illustrate the projection of \( \rightarrow_{\text{lop}} \) rewrite sequences in a lopsim-TRS to \( \rightarrow_{\text{loβ}} \) sequences in the \( \lambda \)-calculus at our standard example. For this, we consider the \( \lambda \)-term \( M := \lambda x. (\lambda y. y)((\lambda z. \lambda w. w)z)x \) from Example [8], and the \( \lambda \)-TRS \( \mathcal{L} = \langle \Sigma, R \rangle \) with \( \Sigma^- = \{ f, g, h, i \} \) as defined in Example 3, for which \( \llbracket f(g, h) \rrbracket_\lambda = M \) holds, that is, the \( \lambda \)-TRS-term \( f(g, h) \) represents the \( \lambda \)-term \( M \).

Then the leftmost (and leftmost-outermost) \( \rightarrow_{\text{lop}} \) rewrite sequence in \( \mathcal{LO}(\mathcal{L}) \) from Example 24 projects to the leftmost-outermost \( \rightarrow_{\text{loβ}} \) rewrite sequence in the \( \lambda \)-calculus from Example 7 as follows, where we indicate the projection by writing the denoted \( \lambda \)-terms beneath the corresponding \( \lambda \)-term representations:

\[
\begin{align*}
lo(f(g, h)) & \rightarrow_{\text{search}} lop_{0,0}(f(g, h)) & \rightarrow_{\text{search}} (\lambda v_0)(lo_{0,1}(\hat{\langle g, \hat{\langle h, v_0 \rangle} \rangle))) \\
M & \equiv \lambda x. (\lambda y. y)((\lambda z. \lambda w. w)z)x & \equiv \lambda x. (\lambda y. y)((\lambda z. \lambda w. w)z)x \\
& \rightarrow_{\text{search}} (\lambda v_0)(lo_{1,1}(\hat{\langle g, \hat{\langle h, v_0 \rangle} \rangle})) & \rightarrow_{\text{contr}} (\lambda v_0)(lo_{0,1}(\hat{\langle h, v_0 \rangle}))) \\
& \equiv \lambda x. (\lambda y. y)((\lambda z. \lambda w. w)z)x & \equiv \lambda x. (\lambda z. \lambda w. w)z \\
& \rightarrow_{\text{search}} (\lambda v_0)(lo_{0,1}(\hat{\langle h, v_0 \rangle}))) & \rightarrow_{\text{contr}} (\lambda v_0)(lo_{0,1}(\hat{\langle h, v_0 \rangle}))) \\
& \equiv \lambda x. (\lambda z. \lambda w. w)(\langle v_1, lo_{0,2}(v_0) \rangle) & \equiv \lambda x. \lambda w. w \\
& \rightarrow_{\text{search}} (\lambda v_1)(lo_{0,1}(\hat{\langle v_1, \hat{\langle v_0 \rangle} \rangle})) & \rightarrow_{\text{search}} (\lambda v_1)(lo_{0,1}(\hat{\langle v_1, \hat{\langle v_0 \rangle} \rangle})) \\
& \equiv \lambda x. (\lambda z. \lambda w. w)z & \equiv \lambda x. \lambda w. w \\
& \equiv \lambda x. (\lambda z. \lambda w. w)z & \equiv \lambda x. \lambda w. w \\
& \equiv \lambda x. (\lambda z. \lambda w. w)z & \equiv \lambda x. \lambda w. w \\
& \equiv \lambda x. (\lambda z. \lambda w. w)z & \equiv \lambda x. \lambda w. w
\end{align*}
\]

As in Example 7 we have underlined redexes that are contracted in \( \rightarrow_{\text{loβ}} \) steps. This parallelization of steps can help to recognize, for the latter ones quite directly, that projection takes place by taking the expanded form of the lopsim-TRS term, and interpreting that as a \( \lambda \)-term (modulo \( \alpha \)-conversion).

The next lemma states that every leftmost-outermost \( \beta \)-reduction step \( M \rightarrow_{\text{loβ}} M_1 \) can be lifted to a sequence \( s \rightarrow_{\text{search}} \cdot \rightarrow_{\text{contr}} s_1 \) of leftmost steps in a lopsim-TRS, provided that \( s \) denotes \( M \), and \( s \) has been obtained by the simulation of a \( \rightarrow_{\text{loβ}} \) rewrite sequence.

Lemma 42 (Lifting of \( \rightarrow_{\text{loβ}} \) steps to \( \rightarrow_{\text{search}} \cdot \rightarrow_{\text{contr}} \) steps w.r.t. \( \llbracket \cdot \rrbracket_\lambda \)). Let \( \mathcal{L} = \langle \Sigma, R \rangle \) be a \( \lambda \)-TRS. Let \( s \in \text{Ter}(\Sigma) \) be a ground term such that \( \llbracket s \rrbracket_\lambda = M_0 \) for a \( \lambda \)-term \( M_0 \). Furthermore let \( u \in \text{Ter}(\Sigma_{\text{lopsim}}) \) with \( \llbracket u \rrbracket_\lambda = M \) for a \( \lambda \)-term \( M \) be the final term of a leftmost-outermost rewrite sequence \( lop(s) \rightarrow_{\text{lop}} u \).

Then for a \( \rightarrow_{\text{loβ}} \) step \( \phi : [u]_\lambda = M \rightarrow_{\text{loβ}} M_1 \) with \( \lambda \)-term \( M_1 \) as target there are terms \( u', u_1 \in \text{Ter}(\Sigma_{\text{lopsim}}) \) and a leftmost-outermost \( \rightarrow_{\text{lop}} \) rewrite sequence \( \hat{\phi} : u \rightarrow_{\text{search}} u' \rightarrow_{\text{contr}} u_1 \) whose projection via \( \llbracket \cdot \rrbracket_\lambda \) amounts to the step \( \phi \), and hence, \( \llbracket u' \rrbracket_\lambda = M \), and \( \llbracket u_1 \rrbracket_\lambda = M_1 \).

We note that in the lemma ‘leftmost-outermost’ in ‘leftmost-outermost rewrite sequence \( lop(s) \rightarrow_{\text{lop}} u \)’ and ‘leftmost-outermost \( \rightarrow_{\text{lop}} \) rewrite sequence \( \hat{\phi} : u \rightarrow_{\text{search}} u' \rightarrow_{\text{contr}} u_1 \)’ could both be replaced by ‘leftmost’. The reason is as follows. In a lopsim-TRS \( \mathcal{LO}(\mathcal{L}) \) for a \( \lambda \)-TRS \( \mathcal{L} \) it holds for all rewrite sequences \( lop(t) \rightarrow_{\text{lop}} u \) for a ground term \( t \) over the signature of \( \mathcal{L} \) and of \( \lambda \)-term representations that
and hence that all leftmost-outermost steps from \( u \) identically or parallel positions). From this it follows that all redexes of the lopsim-TRS in \( u \) do not have occurrences of operation symbols \( \Sigma_{\text{lop}} \) in nested positions (but only at identical or parallel positions). We expect that Lemma 42 can be proved in close analogy to the correctness statement for fully-lazy lambda-lifting. In particular, it is possible to use the correspondence between weak \( \beta \)-reduction steps on \( \lambda \)-terms and combinator reduction steps on supercombinator representations obtained by fully-lazy lambda-lifting. The latter result was formulated and proved by Balabonski in [4].

Now by using Lemma 42 in a proof by induction on the length of a \( \rightarrow_{\text{lop}} \) rewrite sequence the theorem below can be obtained. It justifies the use of lopsim-TRSs for the simulation of \( \lambda \)-terms and combinator reduction steps on supercombinator representations obtained by fully-lazy lambda-lifting.

**Proposition 43** (Lifting of \( \rightarrow_{\text{lop}} \) to leftmost-outermost \( \rightarrow_{\text{lop}} \) rewrite sequences). Let \( L = (\Sigma, R) \) be a \( \lambda \)-TRS. Let \( s \in \text{Ter}(\Sigma) \) be a ground term with \( \llbracket s \rrbracket^L = M \) for a \( \lambda \)-term \( M \). Then every \( \rightarrow_{\text{lop}} \) rewrite sequence:

\[
\sigma : M = L_0 \rightarrow_{\text{lop}} L_1 \rightarrow_{\text{lop}} \cdots \rightarrow_{\text{lop}} L_k \rightarrow_{\text{lop}} \cdots
\]

of finite or infinite length \( l \in \mathbb{N} \cup \{\infty\} \) lifts via \( \llbracket \cdot \rrbracket^L \) to a leftmost-outermost \( \rightarrow_{\text{lop}} \) rewrite sequence:

\[
\hat{\sigma} : \text{lop}(s) = u_0 \rightarrow_{\text{search}} \cdots \rightarrow_{\text{contr}} u_1 \rightarrow_{\text{search}} \cdots
\]

\[
\cdots \rightarrow_{\text{contr}} u_k \rightarrow_{\text{search}} \cdots
\]

with precisely \( l \rightarrow_{\text{contr}} \) steps such that furthermore \( \llbracket u_i \rrbracket^L = L_i \) holds for all \( i \in \{0, 1, \ldots, l\} \).

For the same reason as argued above for Lemma 42, the formulation ‘leftmost-outermost \( \rightarrow_{\text{lop}} \) rewrite sequence’ in this proposition could be replaced by ‘leftmost \( \rightarrow_{\text{lo}} \) rewrite sequence’.

Now by using the lifting of \( \rightarrow_{\text{lop}} \) rewrite sequences to \( \rightarrow_{\text{lop}} \) rewrite sequences (Proposition 43), that \( \lambda \)-term and \( \lambda \)-term representation depths coincide (Proposition 29), and that the depth of an \( \lambda \)-TRS that can represent a \( \lambda \)-term \( M \) is bounded by the depth of \( M \) (Lemma 32), the theorem above entails our main theorem, the linear-depth-increase result for leftmost-outermost \( \beta \)-reduction rewrite sequences.

**Theorem 44** (Linear depth increase in \( \rightarrow_{\text{lop}} \)-rewrite sequences). Let \( M \) be a \( \lambda \)-term. Then for every finite or infinite leftmost-outermost rewrite sequence \( \sigma : M = L_0 \rightarrow_{\text{lop}} L_1 \rightarrow_{\text{lop}} \cdots \rightarrow_{\text{lop}} L_k \rightarrow_{\text{lop}} \cdots \) from \( M \) with length \( l \in \mathbb{N} \cup \{\infty\} \) it holds:

(i) \( |L_{n+1}| \leq |L_n| + |M| \) for all \( n \in \mathbb{N} \) with \( n + 1 \leq l \), that is, the depth increase in each step of \( \sigma \) is uniformly bounded by \( |M| \).

(ii) \( |L_n| \leq |M| + n \cdot |M| = (n + 1) \cdot |M| \), and hence \( |L_n| = |M| \in O(n) \), for all \( n \in \mathbb{N} \) with \( n \leq l \), that is, the depth increase along \( \sigma \) to the \( n \)-th reduct is linear in \( n \), with \( |M| \) as multiplicative constant.

8 Idea for a graph rewriting implementation

The linear-depth-increase result suggests a directed-acyclic-graph implementation of leftmost-outermost \( \beta \)-reduction that is based on the following idea. It keeps subterms shared as much as possible, particularly in the search for the representation of the next leftmost-outermost redex. Steps that are used in the search for the next leftmost-outermost redex do not perform any unsharing, but only use markers to organize the search, and to keep track of its progress. All search steps together increase the size of the graph only by at most a constant multiple. Then the number of search steps that are necessary for finding the next leftmost-outermost redex is linear in the size of the current graph. Unsharing of the graph only takes place
once the next (representation of the) leftmost-outermost redex is found: then the part of the graph between this redex and the root is unshared (copied), and subsequently the (represented) redex is contracted.

The idea is to develop a graph rewriting calculus $G(\Sigma)$ such that its rewrite relation $\Rightarrow$ implements leftmost-outermost $\rightarrow_{\text{search}} \cdot \rightarrow_{\text{contr}}$ rewrite sequences in the corresponding lopsim-TRS $LO(\Sigma)$. We know from Proposition 44 that those leftmost-outermost $\rightarrow_{\text{search}} \cdot \rightarrow_{\text{contr}}$ rewrite sequences in turn implement $\rightarrow_{\text{lo}}$ rewrite sequences in the $\lambda$-calculus. Starting from a $\lambda$-term $M_0$, a leftmost-outermost $\beta$-reduction rewrite sequence from $M_0$ is thus first lifted to a $\rightarrow_{\text{search}} \cdot \rightarrow_{\text{contr}}$ rewrite sequence from a $\lambda$-term representation $t_0$ of $M$ in an lopsim-TRS $LO(\Sigma)$ for a $\lambda$-TRS $\mathcal{L} = (\Sigma, R)$ with $[t_0]_\lambda = M_0$, and then to a $\Rightarrow$ rewrite sequence from a directed-acyclic graph $G_0$ that represents $t_0$:

$$
M_0 \rightarrow_{\text{lo}} M_1 \rightarrow_{\text{lo}} M_2 \rightarrow_{\text{lo}} \ldots \rightarrow_{\text{lo}} M_{n-1} \rightarrow_{\text{lo}} M_n \ (\lambda)
$$

$$
t_0 \rightarrow_c t_1 \rightarrow_c \ldots \rightarrow_c t_{n-1} \rightarrow_c t_n \ (LO(\Sigma))
$$

$$
G_0 \Rightarrow G_1 \Rightarrow G_2 \Rightarrow \ldots \Rightarrow G_{n-1} \Rightarrow G_n \ (G(\Sigma))
$$

(here we have shortened the subscripts in $\rightarrow_{\text{search}} \cdot \rightarrow_{\text{contr}}$ steps) where it holds for all $i \in \{1, \ldots, n\}$:

$$
[t_i]_\lambda = M_i, \ G_i \text{ represents } t_i, \ |M_i| \leq (i + 1) \cdot |M_0|, \ |t_i| \leq |t_i|_{\text{exp}} = |M_i| \leq (i + 1) \cdot |t_0| = (i + 1) \cdot |M_0|.
$$

Here we have used the linear-depth-increase results Theorem 44 for $\lambda$-terms, and Theorem 39 for $\lambda$-term representations. Now it seems feasible to develop the graph rewrite calculus $G(\Sigma)$ in such a way that the depth $|G_i|$ of the (acyclic) graph representations $G_i$ of the lopsim-TRS terms $t_i$ are bounded by a constant $c$ multiplied with the depth of $t_i$, and consequently also bounded by $c$ multiplied with the $\lambda$-depth of $t_i$, or the depth of $M_i$:

$$
|G_i| \leq c \cdot |t_i| \leq c \cdot |t_i|_{\text{exp}} = c \cdot |M_i|, \ \text{hence: } |G_i| \leq c \cdot (i + 1) \cdot |M_0| \text{ (for all } i \in \{0, 1, \ldots, n\}).
$$

The reason for the possible depth increase in the graph representations consists in the use of additional controle nodes for keeping track of the progress of leftmost-outermost evaluation: links will be used in order to indicate positions to which the leftmost-outermost evaluation needs to backtrack after having reduced a subexpression to a normal form, or having detected that a subexpression is a normal form. The depth of the graphs $G_i$ are well-defined because they are acyclic.

The idea for simulating a step $t_i \rightarrow_{\text{search}} \rightarrow_{\text{contr}} t_{i+1}$ consists in unsharing the graph representation $G_i$ of $t_i$ only between the graph’s root and the representation of the $\beta$-redex in the $\rightarrow_{\text{contr}}$ step, and then carrying out the representation of the $\rightarrow_{\text{contr}}$ step that involves replacing the symbol $f$ by a graph version of its scope context $F$, together with adapting links accordingly. We can expect the size increase in the graph rewrite step $G_i \Rightarrow G_{i+1}$ to be bounded linearly in the depth $|G_i|$ of $G_i$, for the first part, and to be bounded by linearly the size $|F|$ for the second part, and hence the size $|M_0|$ of $M_0$ for the contraction part. That is, we want to guarantee that for all $i \in \{0, 1, \ldots, n-1\}$ it holds:

$$
\|G_{i+1}\| \leq \|G_i\| + d \cdot |G_n| + c \cdot |M_0|,
$$

hence:

$$
\|G_{i+1}\| - \|G_i\| \leq d \cdot |G_i| + c \cdot |M_0| \leq c \cdot d \cdot (i + 1) \cdot |M_0| + c \cdot |M_0|.
$$

We may also assume that $d \in \mathbb{N}$ is at the same time a multiplicative constant for bounding the size of $G_0$ by the sizes of $t_0$ and $M_0$:

$$
\|G_0\| \leq d \cdot \max \{\|t_0\|, \|M_0\}\.
On the basis of these assumptions a bound for the size of the \( n \)-th graph \( G_n \) of the graph rewrite sequence can be calculated as follows:

\[
\|G_n\| = \left( \sum_{i=0}^{n-1} (\|G_{i+1}\| - \|G_i\|) \right) + \|G_0\|
\]

\[
= \|G_0\| + \sum_{i=0}^{n-1} (c \cdot d \cdot (i + 1) \cdot \|M_0\| + c \cdot \|M_0\|)
\]

\[
= \|G_0\| + c \cdot d \cdot \|M_0\| \cdot \sum_{i=1}^{n} i + c \cdot n \cdot \|M_0\|
\]

\[
\leq d \cdot \|M_0\| + \frac{1}{2} \cdot cd \cdot \|M_0\| \cdot n(n + 1) + c \cdot n \cdot \|M_0\|
\]

\[
\leq d \cdot \|M_0\| + 0.5 \cdot cd \cdot \|M_0\| \cdot n(n + 1) + c \cdot n \cdot \|M_0\| \in O(\|M_0\| \cdot n^2)
\]

Now the time for computing the \( i \)-th rewrite step \( G_i \Rightarrow G_{i+1} \) will consist of two parts: the time \( \text{Time}_{\text{search}}(G_i) \) for searching the occurrence of the leftmost-outermost redex in \( G_i \), and the time \( \text{Time}_{\text{contr}}(G_i) \) for performing the graph representation of the \( \Rightarrow_{\text{contr}} \) step. The search part \( \text{Time}_{\text{search}}(G_i) \) can be organized as a graph traversal of \( G_i \), and therefore can be expected to be performed in time that depends linearly on the size of \( G_i \). The contraction part \( \text{Time}_{\text{contr}}(G_i) \) consists of the necessary unsharing of the \( G_i \) between its root and the represented leftmost-outermost redex, and by performing the \( \Rightarrow_{\text{contr}} \) step on the shared representation \( G_i \). The first subpart necessitates copying work of time that is linearly dependent on the depth \( |G_i| \) of \( G_i \). The second subpart involves the addition of a graph context from \( G_0 \) that corresponds to the scope context \( F \) of the scope symbol \( f \) that is part of the \( \Rightarrow_{\text{contr}} \) redex that is contracted; it therefore requires copying \( F \), and since \( F \) occurs already in \( G_0 \), this can be expected to be work that depends linearly on the size of \( G_0 \). Together we obtain that for some \( e, f \in \mathbb{N} \) it holds:

\[
\text{Time}_{\text{search}}(G_i) + \text{Time}_{\text{contr}}(G_i) \\
\leq e \cdot \|G_i\| + f \cdot (\|G_i\| + \|G_0\|) \\
\leq (e + 2f) \cdot \|G_i\| \in O(\|M_0\| \cdot i^2)
\]

From this we now obtain the following rough estimate of the time needed to implement the leftmost-outermost rewrite sequence \( M_0 \xrightarrow{\beta^{\ast}} M_n \) by the graph rewrite sequence \( G_0 \Rightarrow G_n \), for some \( g \in \mathbb{N} \):

\[
\text{Time}_{\text{search}}(G_i) + \text{Time}_{\text{contr}}(G_i) \\
= \sum_{i=1}^{n-1} \text{Time}(G_i \Rightarrow G_{i+1}) \\
\leq \sum_{i=0}^{n-1} g \cdot \|M_0\| \cdot i^2 = g \cdot \|M_0\| \cdot \sum_{i=0}^{n-1} i^2 \in O(\|M_0\| \cdot n^3)
\]

This can yield a polynomial cost function for the work that is needed to faithfully implement a leftmost-outermost \( \beta \)-reduction rewrite sequence of length \( n \) by ‘atomic’ graph manipulation steps.

An implementation of such graph rewriting representations of leftmost-outermost \( \beta \)-reduction sequences, broken down into the atomic steps of a port graph rewrite system [1], on a reasonable machine could lead to an alternative proof of the invariance result of Accattoli and Dal Lago.

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