GROUND STATE AND NODAL SOLUTIONS FOR FRACTIONAL SCHröDINGER-MAXWELL-KIRCHHOFF SYSTEMS WITH PURE CRITICAL GROWTH NONLINEARITY

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(Communicated by Zhi-Qiang Wang)

ABSTRACT. In this paper, we consider the existence of a ground state nodal solution and a ground state solution, energy doubling property and asymptotic behavior of solutions of the following fractional critical problem

\[
\begin{aligned}
(a + b \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u|^2 \, dx)(-\Delta)^{\alpha}u + V(x)u + K(x)\phi u &= |u|^{2^*-2}u + \kappa f(x, u), \\
(-\Delta)^{\beta} \phi &= K(x)u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]

where \(a, b, \kappa\) are positive parameters, \(\alpha \in (\frac{3}{4}, 1), \beta \in (0, 1),\) and \(2^*_{\alpha} = \frac{6}{3-2\alpha},\) \((-\Delta)^{\alpha}\) stands for the fractional Laplacian. By the nodal Nehari manifold method, for each \(b > 0,\) we obtain a ground state nodal solution \(u_b\) and a ground-state solution \(v_b\) to this problem when \(\kappa \gg 1,\) where the nonlinear function \(f: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function. We also give an analysis on the behavior of \(u_b\) as the parameter \(b \to 0.\)

1. Introduction and main results. Our goal of this paper is to consider the existence of nodal solution and ground state solution of following fractional Schrödinger-Maxwell-Kirchhoff systems

\[
\begin{aligned}
(a + b \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u|^2 \, dx)(-\Delta)^{\alpha}u + V(x)u + K(x)\phi u &= f(x, u), \quad x \in \mathbb{R}^3, \\
(-\Delta)^{\beta} \phi &= K(x)u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]

where \(a, b > 0\) are parameters, \(2^*_{\alpha} = \frac{6}{3-2\alpha}\) is the Sobolev embedding exponent and \(V(x)\) is a continuous function. The fractional Laplacian operator \((-\Delta)^{\alpha}\) is defined by (see [15])

\[
(-\Delta)^{\alpha}u = C(\alpha) P.V. \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x-y|^{3+2\alpha}} \, dy, \quad u \in \mathcal{S}(\mathbb{R}^3),
\]

where P.V. stands for the Cauchy principal value of the integration. \(C(\alpha) = \left(\int_{\mathbb{R}^3} \frac{1 - \cos \xi_1}{|\xi|^{3+2\alpha}}\right)^{-1}\) is the normalized constant, \(\mathcal{S}(\mathbb{R}^3)\) is the Schwartz space of rapidly decreasing functions.

2020 Mathematics Subject Classification. Primary: 35J60, 35J50; Secondary: 35Q61.

Key words and phrases. Fractional Schrödinger-Maxwell-Kirchhoff systems, Nodal solution, ground state solution, Nehari manifold.

The first author is supported by NSF of China (11790271), Guangdong Basic and Applied basic Research Foundation(2020A1515011019), Innovation and Development Project of Guangzhou University.

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decaying functions. The fractional Laplacian operator $(-\Delta)^\alpha$ via the Fourier transform is presented as
\begin{equation}
(-\Delta)^\alpha u = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}(u)).
\end{equation}
When $a = 1, b = 0$, fractional Schrödinger-Maxwell-Kirchhoff systems reduced to the undermentioned fractional Schrödinger-Maxwell system
\begin{align*}
\begin{cases}
(-\Delta)^\alpha u + V(x)u + K(x)\phi u = f(x, u), & x \in \mathbb{R}^3, \\
(-\Delta)^\beta \phi = K(x)u^2, & x \in \mathbb{R}^3.
\end{cases}
\end{align*}
(1.3)
System (1.3) is derived from the standing wave of Schrödinger-Poisson systems. The nonlinearity $f(x, u)$ represents the particles interacting with each other. The term $K(x)\phi u$ represents the interaction with the electric field, which describes the interaction of quantum (non-relativistic) particles with the electromagnetic field generated by motion.

On the other hand, a great attention has recently been given to the so called fractional Kirchhoff equation (see [4, 5, 13] etc.)
\begin{equation}
(a + b \int_\Omega \frac{|(-\Delta)^{\alpha/2}u|^2}{2} dx)(-\Delta)^\alpha u = f(x, u)
\end{equation}
(1.4)
with $\Omega \subset \mathbb{R}^N$ being a bounded domain or $\Omega = \mathbb{R}^N$. Problem (1.4) is related to the stationary analogue of the fractional Kirchhoff equation
\begin{equation}
\frac{\partial^2 u}{\partial t^2} + (a + b \int_\Omega \frac{|(-\Delta)^{\alpha/2}u|^2}{2} dx)(-\Delta)^\alpha u = f(x, u).
\end{equation}
(1.5)

As a special significant case, the nonlocal aspect of the tension arises from nonlocal measurements of the fractional length of the string. The Kirchhoff’s model takes into account the length variation of the string produced by the transverse vibration, so the non-local term appears. For more mathematical and physical background on Schrödinger-Maxwell systems or Kirchhoff type problems, we refer the readers to [2, 3, 16] and the references therein.

In the remarkable work of Caffarelli and Silvestre [6], the authors express this nonlocal operator $(-\Delta)^\alpha$ as a Dirichlet-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half space. This technique is a valid tool to deal with the equations involving fractional operators in respect of regularity and variational methods. The appearance of nonlocal term not only makes it playing an important role in many physical applications, but also brings some difficulties and challenges in mathematical analysis. This fact makes the study of fractional Schrödinger-Maxwell-Kirchhoff systems and similar problems particularly interesting. A lot of interesting results on the existence of nonlocal problems were obtained recently in, for examples, [1, 5, 7, 9, 10, 11, 12, 13, 14, 17, 18, 19, 24, 27, 29] and the cited references.

In past few years, some researchers began to search for nodal solutions of Schrödinger type equations or similar problems and have got some interesting results. For example, Zhang [28] considered the following Schrödinger-Poisson system
\begin{align*}
\begin{cases}
-\Delta u + u + k(x)\phi u = a(x)|u|^{p-2}u + u^5, & x \in \mathbb{R}^3, \\
-\Delta \phi = k(x)u^2, & x \in \mathbb{R}^3,
\end{cases}
\end{align*}
(1.6)
where $p \in (4, 6)$, $k(x)$ and $a(x)$ are nonnegative functions. By using variational method, the existence of ground state solution and nodal solution was obtained.
Wang [21] studied the existence of a least energy sign-changing solution for the following Kirchhoff-type equation
\[
\begin{cases}
-(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = |u|^4 u + \lambda f(x, u), & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]
where \( \Omega \) is a bounded domain, \( \lambda, a, b > 0 \). \( f(x, \cdot) \) is continuously differentiable for a.e. \( x \in \Omega \). By using constraint variational method and degree theory, the existence of a least energy nodal solution to the Kirchhoff-type equation was obtained. Meanwhile, Wang, Zhang and Guan [22] studied the following Schrödinger-Poisson system with critical growth
\[
\begin{cases}
-\Delta u + V(x)u + b\phi u = |u|^4 u + \mu f(u), & x \in \mathbb{R}^3, \\
-\Delta \phi = u^2, & x \in \mathbb{R}^3,
\end{cases}
\]
where \( \mu, b > 0, f \in C^1(\mathbb{R}, \mathbb{R}) \). They got the existence and asymptotic behavior of least energy sign-changing solution to the above system.

However, as for fractional Schrödinger-Maxwell-Kirchhoff types systems, up to our knowledge, few results involved the existence and asymptotic behavior of ground state nodal solutions in case of critical growth except [26]. In [26], Ye and Teng considered the following Schrödinger-Poisson system with critical growth
\[
\begin{cases}
(-\Delta)^\alpha u + u + k(x)\phi u = |u|^{2^*_\alpha - 2} u + a(x)|u|^{p-2} u, & x \in \mathbb{R}^3, \\
(-\Delta)^\beta \phi = k(x)u^2, & x \in \mathbb{R}^3,
\end{cases}
\]
where \( k(x) \) and \( a(x) \) are nonnegative functions. Via the constraint variational method, they obtained the existence of ground state sign-changing solutions to system (1.8). However, if \( k(x) \equiv 1 \), their methods used in [28, 26] seems not valid because their result depends on the case \( k \in L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) for some \( p \in (1, 2^*_\alpha) \). Furthermore, they did not study associated energy character and the asymptotic behavior of sign-changing solutions.

Motivated by the above references, in this paper, we study the existence of ground state and ground state nodal solutions of the following fractional Schrödinger-Maxwell-Kirchhoff systems with critical growth nonlinearity
\[
\begin{cases}
(a + b \int_{\mathbb{R}^3} \langle(-\Delta)^{\alpha/2} u \rangle dx) (-\Delta)^\alpha u + V(x)u + K(x)\phi u = |u|^{2^*_\alpha - 2} u + \kappa f(x, u), \\
(-\Delta)^\beta \phi = K(x)u^2, & x \in \mathbb{R}^3,
\end{cases}
\]
where \( a > 0, b \geq 0 \) are real numbers, \( \alpha \in \left(\frac{3}{4}, 1\right) \), so \( 2^*_\alpha \in (4, 6) \). Similarly to [23], we suppose that \( V \in C(\mathbb{R}^3, \mathbb{R}^+) \) being a function ensures that

(V): \( W_V \hookrightarrow L^p(\mathbb{R}^3) \) for \( 2 < p < 2^*_\alpha \), and \( W_V \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^3) \) is continuous, where \( W_V \) is a Hilbert space defined by (see [4])
\[
W_V = \begin{cases}
H^\alpha_r(\mathbb{R}^3) = \{ u \in H^\alpha(\mathbb{R}^3) : u(x) = u(|x|) \}, & \text{if } V(x) \text{ is a constant}, \\
\{ u \in D^{\alpha, 2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty \}, & \text{if } V(x) \text{ is not a constant},
\end{cases}
\]
with inner product
\[
\langle u, v \rangle = a \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} v dx + \int_{\mathbb{R}^3} V(x)uv dx, \forall u, v \in W_V
\]
and the norm \( \| \cdot \| \) defined by
\[
\| u \|^2 = a \int_{\mathbb{R}^3} \left( \left| (-\Delta)^{\alpha/2} u \right|^2 \right) dx + \int_{\mathbb{R}^3} V(x) u^2 dx.
\]
The fractional Sobolev space \( H^{\alpha}(\mathbb{R}^3) \) can be described by means of the Fourier transform as follows
\[
H^{\alpha}(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left( |\xi|^{2\alpha} + 1 \right) |\hat{u}|^2 d\xi < \infty \},
\]
with the equivalent norm \( \| u \|^2_{H^{\alpha}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left( \left| (-\Delta)^{\alpha/2} u \right|^2 + u^2 \right) dx \). The homogeneous fractional Sobolev space \( D^{\alpha,2}(\mathbb{R}^3) = \{ u \in L^{2\alpha}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left( |\xi|^{2\alpha} \right) |\hat{u}|^2 d\xi < \infty \} \) is a Hilbert space with inner product \((u,v)_\alpha = \int_{\mathbb{R}^3} \left( (-\Delta)^{\alpha/2} u \cdot (-\Delta)^{\alpha/2} v \right) dx\), and the Gagliardo seminorm \( \| u \|^2_\alpha = (u,u)_\alpha = \int_{\mathbb{R}^3} \left( \left| (-\Delta)^{\alpha/2} u \right|^2 \right) dx \). For more details, we refer to [15].

In this paper, we make the following assumptions on the potential \( K(x) \):

\( (K_1) \): There exists \( C > 0 \) and \( d > 0 \) such that \( 0 \leq K(x) \leq \frac{C}{(1+|x|)^d} \).

As for the function \( f \), we assume \( f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function and satisfies the following hypotheses:

\( (f_1) \): \( f(x,t) \cdot t > 0 \) for \( t \neq 0 \) and \( \lim_{t \to 0} \frac{f(x,t)}{t} = 0 \), for \( x \in \mathbb{R}^3 \) uniformly.

\( (f_2) \): There exists \( q \in (4,2^*_\alpha) \) such that \( \lim_{t \to \infty} \frac{f(x,t)}{t^q} = 0 \) for all \( t \in \mathbb{R} \setminus \{0\} \) and for \( x \in \mathbb{R}^3 \) uniformly;

\( (f_3) \): \( \frac{f(x,t)}{|t|^q} \) is a nondecreasing function with respect to \( t \) in \( (-\infty,0) \) and \( (0,\infty) \) for a.e. \( x \in \mathbb{R}^3 \).

**Remark 1.1.**

(i) We note that under the conditions \( (f_1)-(f_3) \), it is easy to see the function \( f(x,t) = t^3 \) is an example satisfying all conditions \( (f_1)-(f_3) \).

(ii) From assumption \( (f_3) \), we have
\[
f(x,t) t - 4F(x,t) \geq 0,
\]
and \( f(x,t) t - 4F(x,t) \) is nondecreasing with respect to \( t \) in \( (0,\infty) \) and non increasing with respect to \( t \) in \( (-\infty,0) \). The assumption \( (f_1) \) implies that
\[
F(x,t) \geq 0, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}.
\]

By the Lax-Milgram Theorem, for given \( u \in W_V \), there exists a unique \( \phi_u \in D^{\beta,2}(\mathbb{R}^3) \) such that \( (-\Delta)^\beta \phi = K(x) u^2 \) (see [26]). So the system (1.9) is merely a single equation on \( u \):
\[
(a + b \int_{\mathbb{R}^3} \left( \left| (-\Delta)^{\alpha/2} u \right|^2 \right) dx) (-\Delta)^\alpha u + V(x) u + K(x) \phi_u u = |u|^{2^*_\alpha - 2} u + \kappa f(x,u), \quad x \in \mathbb{R}^3.
\]

We define the functional associated with the system (1.9) as follows,
\[
J^b_\kappa(u) = \frac{1}{2} \| u \|^2 + \frac{b}{4} \| u \|^{4_\alpha} + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u u^2 dx - \kappa \int_{\mathbb{R}^3} F(x,u) dx - \frac{1}{2\alpha} \int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx
\]
for any \( u \in W_V \). By direct computations, \( J^b_\kappa(u) \in C^1(W_V, \mathbb{R}) \) (see [26]) and
\[
\langle (J^b_\kappa)'(u), v \rangle = \langle u, v \rangle + b \| u \|^{2_\alpha} \langle u, v \rangle + \int_{\mathbb{R}^3} K(x) \phi_u uv dx
\]
\[
- \kappa \int_{\mathbb{R}^3} f(x,u) v dx - \int_{\mathbb{R}^3} |u|^{2^*_\alpha - 2} uv dx, \quad \text{for any } u, v \in W_V.
\]
Theorem 1.1. Suppose that \((f_1)-(f_3)\) and \((K_1)\) are satisfied. Then, there exists a \(\kappa^* > 0\) such that for any \(\kappa \geq \kappa^*\), the systems (1.9) has a ground state nodal solution \(u_b\).

Remark 1.2. We denote \(u^+ = \max\{u(x), 0\}, \ u^- = \min\{u(x), 0\}\). By recalling that the nodal of a continuous function \(u : \mathbb{R}^3 \rightarrow \mathbb{R}\) is the surface \(u^{-1}(0)\), every connected component of \(\mathbb{R}^3 \setminus u^{-1}(0)\) is called a nodal domain. Talking about how many nodal domains of \(u_b\) is difficult for us now, we will continue to study this problem further.

Theorem 1.2. Suppose that \((f_1)-(f_3)\) and \((K_1)\) are satisfied. Then, there exists \(\kappa^{**} > 0\) such that for all \(\kappa \geq \kappa^{**}\), the \(\kappa\) is achieved by a ground state solution \(v_b\) of the systems (1.9) and \(J^{b}_{\kappa}(u_b) > 2c\), where \(c^{*} = \inf_{u \in \mathcal{N}^{b}_{\kappa}} J^{b}_{\kappa}(u), \mathcal{N}^{b}_{\kappa} = \{u \in W_V \setminus \{0\}|(J^{b}_{\kappa})'(u), u) = 0\}\) and \(u_b\) is the ground state nodal solution obtained in Theorem 1.1.

Theorem 1.3. Suppose that \((f_1)-(f_3)\) and \((K_1)\) are satisfied. Then, there exists \(\kappa^{***} > 0\) such that for any \(\kappa \geq \kappa^{***}\), for any ground state nodal solution sequence \(\{u_{b_n}\}\) with \(b_n \rightarrow 0\) as \(n \rightarrow \infty\), \(u_{b_n}\) converges to \(u_0\) weakly in \(W_V\) as \(n \rightarrow \infty\) (in subsequence sense), where \(u_0\) is a ground state nodal solution of the following problem

\[
\begin{cases}
a(\Delta)^{\alpha}u + V(x)u + K(x)\phi u = |u|^{2_{\alpha}^-}u + \kappa f(x, u), & x \in \mathbb{R}^3, \\
(\Delta)^{\beta} \phi = K(x)u^2, & x \in \mathbb{R}^3,
\end{cases}
\]  

(1.13)

Remark 1.3. As for the assumptions on potential \(V(x)\), in fact, we can also suppose that \(V(x) = V(|x|)\) is radical and indefinite in sign. More specifically, if we denote \(V^{-}(x) := \max\{-V(x), 0\}\) and \(\nu := \inf_{0 \neq u \in D^{\alpha,2}(\mathbb{R}^3)} \|u\|^2_{\alpha, \infty} / (f_{\alpha,2} |u|^{2_{\alpha}^-}dx)^{\frac{1}{2_{\alpha}}},\) we assume the following conditions holds.

\((V_{0})\) \(V^{-} \in L^{3/2\alpha}(\mathbb{R}^3)\) and \(\int_{\mathbb{R}^3} |V^{-}(x)|^{3/2\alpha} dx < \nu^{3/2\alpha}\),

\((V_{1})\) there exist \(\gamma > 0\) and \(C_V > 0\) such that

\(V(x) \leq V_{\infty} - C_V e^{-\gamma|x|}, \) for a.e. \(x \in \mathbb{R}^3,\)

where \(0 < V_{\infty} := \lim_{x \rightarrow +\infty} V(x).\)

If we set

\(E_V = \{u \in H^{\alpha}_{rad}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(|x|)u^2 dx < \infty\},\)

where \(H^{\alpha}_{rad}(\mathbb{R}^3) = \{u \in H^{\alpha}(\mathbb{R}^3), u(x) = u(|x|)\}\). According to the Lemma 2.1 in [8], the quadratic form \(u \rightarrow (a \int_{\mathbb{R}^3}((-\Delta)^{\alpha/2}u^2) dx + V^{-}(x)u^2)\) defines a norm in \(E_V\), which is equivalent to the usual norm of \(D^{\alpha,2}(\mathbb{R}^3)\). Moreover, we can use Hölder’s inequality and Sobolev’s inequality to obtain

\[
\int_{\mathbb{R}^3} V^{-}(x)u^2 dx \leq \|V^{-}\|_{3/2\alpha} \|u\|^2_{2_{\alpha}} \leq \nu^{-1}\|V^{-}\|^{\frac{1}{3/2\alpha}}_{3/2\alpha} \int_{\mathbb{R}^3} ((-\Delta)^{\alpha/2}u^2) dx.
\]

Therefore, we conclude that the norm \(\|\cdot\|\)

\[
\|u\|^2 = a \int_{\mathbb{R}^3} ((-\Delta)^{\alpha/2}u^2) dx + \int_{\mathbb{R}^3} V(|x|)u^2 dx
\]
is well defined on $E_V$ and is equivalent to $\|u\|_\alpha$. In this case, we can show that the above three results are still true for the following systems
\[
\begin{cases}
(a + b \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 dx)(-\Delta)^{\alpha} u + V(|x|)u + K(|x|)\phi u = f(|x|, u), & x \in \mathbb{R}^3, \\
(-\Delta)^{\beta} \phi = K(|x|)u^2, & x \in \mathbb{R}^3,
\end{cases}
\]
where $|x| = \sum_{i=1}^3 x_i^2$ is the radial radius. The proofs are similar to the processes in Section 3 and Section 4.

It’s worth noting that, the Brouwer degree method used in [24, 22] strictly depends on the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$, so we have to find new tricks to solve our modeling where we only allow $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ being a Carathéodory function. On the other hand, in our modeling, both of the nonlocal pseudo-differential operator $(-\Delta)^{\alpha}$ and nonlocal terms $\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 dx$ appear, we need to solve the difficulties caused by the nonlocal terms under a suitable variational framework.

The remainder of this paper is organized as follows. In Section 2, we give some useful preliminaries. In Section 3 we study the existence of a nodal solution of (1.9) and give a proof of Theorem 1.1. In Section 4, we prove Theorem 1.2 and Theorem 1.3.

2. Some technical lemmas. To fix some notations, the letter $C$, $C_i$ will be repeatedly used to denote various positive constants whose exact values are irrelevant.

We first list some properties of the fractional Laplacian operator $(-\Delta)^{\alpha}$ for further use.

(i) $(-\Delta)^{\alpha}$ is a self-adjoint operator for any $\alpha \in (0, 1)$, i.e.
\[
\int_{\mathbb{R}^3} (-\Delta)^{\alpha} u \cdot v dx = \int_{\mathbb{R}^3} (-\Delta)^{\alpha} v \cdot u dx = \frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{[u(x) - u(y)] \cdot [v(x) - v(y)]}{|x - y|^{3 + 2\alpha}} dx dy.
\]

(ii) For any $\alpha \in (0, 1)$, there holds $(-\Delta)^{\alpha/2} \cdot (-\Delta)^{\alpha/2} = (-\Delta)^\alpha$.

(iii) For any $\alpha \in (0, 1)$, there holds
\[
\int_{\mathbb{R}^3} (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} v dx = \frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{[u(x) - u(y)] \cdot [v(x) - v(y)]}{|x - y|^{3 + 2\alpha}} dx dy.
\]

The property (i) comes from (1.1). More general than the property (ii), for any $s, t \in (0, 1)$ with $s + t \in (0, 1)$, from (1.2) we have
\[
(-\Delta)^s \cdot (-\Delta)^t u = \mathcal{F}^{-1}(|\xi|^{2s} |\xi|^{2t} \mathcal{F} u(\xi)) = \mathcal{F}^{-1}(|\xi|^{2s+t} \mathcal{F} u(\xi)) = (-\Delta)^{s+t} u.
\]
Then property (iii) is a consequence of the properties (i) and (ii).

We now list some properties of $\phi_u$, which can be founded in [17, 20].

Proposition 2.1. For any $u \in W_V$, we have

(i) there exists $C > 0$ such that
\[
\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leq C\|u\|^4, \quad \forall u \in W_V;
\]

(ii) $\phi_u \geq 0, \forall u \in W_V$;

(iii) $\phi_{tu} = t^2 \phi_u, \forall t > 0$ and $u \in W_V$;

(iv) if $u_n \rightharpoonup u$ in $W_V$, then $K(x)\phi_{u_n} \rightharpoonup K(x)\phi_u$ in $D^{3,2}(\mathbb{R}^3)$ and
\[
\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx \to \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx.
\]
Proposition 2.2. If \{||u_n^±||\} is bounded and \(u_n^± \to u^±\) in \(W_V\), then up to a subsequence, there hold \(\lim_{n \to \infty} ||u_n^±||^2 = \lim_{n \to \infty} ||u_n^± - u^±||^2 + ||u^±||^2\), and further more \(\lim_{n \to \infty} ||u_n^±||^2 = \lim_{n \to \infty} ||u_n^± - u^±||^2 + ||u^±||^2\).

Proof. Because \(u_n^± \to u^±\) in \(W_V\), \(W_V\) is a Hilbert space, we can deduce
\[
\|u_n^±\|^2 - \|u_n^± - u^±\|^2 = 2\langle u_n^±, u^± \rangle - \|u^±\|^2.
\]
We can assume that the sequence \{\{||u_n^±||\}\} is convergent, so we have
\[
\lim_{n \to \infty} ||u_n^±||^2 = \lim_{n \to \infty} ||u_n^± - u^±||^2 + ||u^±||^2.
\]
Also we can assume that the sequence \{\{||u_n^±||\}_n\} is convergent, so we have
\[
\lim_{n \to \infty} ||u_n^±||^2 = \lim_{n \to \infty} ||u_n^± - u^±||^2 + ||u^±||^2.
\]

For fixed \(u \in W_V\) with \(u^± \neq 0\), we let
\[
Q_4(s)u^± = s^2\|u^±\|^2 + b\|su^+ + tu^-\|_2^2 (su^+ + tu^-)_{±}
\]
\[
+ s^4 \int K(x)\phi_{u^+}|u^+|^2\,dx + s^2 t^2 \int K(x)\phi_{u^-}|u^-|^2\,dx
\]
\[
- aC(\alpha)st \int \int K(x)\phi_{u^+}u^+(y)u^+(x)\,dxdy,
\]
\[
Q_4(t)u^± = t^2\|u^±\|^2 + b\|su^+ + tu^-\|_2^2 (su^+ + tu^-)
\]
\[
+ tu^-, tu^- \rangle_{±} + t^4 \int K(x)\phi_{u^-}|u^-|^2\,dx + s^2 t^2 \int K(x)\phi_{u^+}|u^+|^2\,dx
\]
\[
- aC(\alpha)st \int \int K(x)\phi_{u^-}u^-(y)u^+(x)\,dxdy,
\]
\[
Q_{2^*}(s)u^± = s^{2^*} \int |u^±|^{2^*}\,dx + \kappa \int f(x, su^+)su^+\,dx,
\]
\[
Q_{2^*}(t)u^± = t^{2^*} \int |u^±|^{2^*}\,dx + \kappa \int f(x, tu^-)tu^-\,dx,
\]
\[
\varphi_u(s, t) = J_b^\kappa (su^+ + tu^-)
\]
and the nodal Nehari manifold is defined by
\[
\mathcal{M}_\kappa^b = \{u \in W_V, u^± \neq 0 and \langle (J_b^\kappa)'(u), u^± \rangle = \langle (J_b^\kappa)'(u), u^- \rangle = 0\}. \tag{2.1}
\]

Remark 2.1. For any \(u \in \mathcal{M}_\kappa^b\), we have \(\langle (J_b^\kappa)'(u), u \rangle = 0\).

Lemma 2.1. Assume that \((f_1)-(f_3)\) are satisfied, if \(u \in W_V\) with \(u^± \neq 0\), then there is a unique pair \((s_u, t_u) \in (0, \infty) \times (0, \infty)\) such that \(s_u u^+ + t_u u^- \in \mathcal{M}_\kappa^b\), which is also the unique maximum point of \(\varphi_u\) on \([0, \infty) \times [0, \infty)\).

Proof. We follow the ideas of [22] and [23]. It is easy to see that the pair \((s, t)\) is a critical point of \(\psi_u\) with \(s, t > 0\) iff \(su^+ + tu^- \in \mathcal{M}_\kappa^b\).

From \((f_1)\) and \((f_2)\), for any \(\varepsilon > 0\), there is \(C_\varepsilon > 0\) satisfying
\[
|f(x, t)| \leq \varepsilon |t| + C_\varepsilon |t|^q-1, \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.2}
\]
Following the above equality, by using the Sobolev’s embedding theorem, we can get
\[
H(s, t) := ((J_b^\kappa)'(su^+ + tu^-), su^+) = Q_4(s)u^+ - Q_{2^*}(s)u^+
\]
\[
\geq (1 - k\varepsilon C_2)s^2\|u^+\|^2 - C_1s^{2^*}\|u^+\|^{2^*} - kC_3s^q\|u^+\|^q. \tag{2.3}
\]
Next, we will consider $(s, t)$ in range of $(s, t) \in (0, \infty) \times (0, \infty)$. According to the arbitrariness of $\varepsilon > 0$, we let $(1 - \kappa \varepsilon C_2) > 0$ such that $H(s, t) > 0$ for $0 < s \ll 1$. Similarly, there holds

$$G(s, t) := (\partial_k^\alpha (s u^+ + tu^-), tu^-) = Q_1(t)u^- - Q_2 s(t)u^- > 0, \quad \forall 0 < t \ll 1. \quad (2.4)$$

In the above two estimations, we have used the fact

$$D(u) := (u^+, u^-)_\alpha = -\frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x - y|^{3+2\alpha}} \, dx \, dy \geq 0.$$

Hence, there exists $\xi_1 > 0$ such that $H(\xi_1, t) > 0, \quad G(s, \xi_1) > 0$ for all $(s, t) \in (0, +\infty) \times (0, +\infty)$.

By (1.11), choosing $s = \xi_2 > \xi_1$, for $t \in [\xi_1, \xi_2]$ and $\xi_2' > 1$, we have

$$H(\xi_2', t) \leq P_4(\xi_2')u^+ - P_{2s}(\xi_2')u^+ \leq 0,$$

where $P_4(\xi_2')u^+ = (\xi_2')^2 \|u^+\|^2 + b\|u^+ + tu^-\|^2 (\xi_2')u^+ + tu^-(\xi_2')u^+\alpha + \xi_2'\int_{\mathbb{R}^3} K(x)\phi_u |u^+|^2 dx + (\xi_2')^4 \int_{\mathbb{R}^3} K(x)\phi_u |u^-|^2 dx + a(\xi_2')^2 D(u), \quad P_{2s}(\xi_2')u^+ = (\xi_2')^2 \int_{\mathbb{R}^3} |u^+|^{2s} dx.$

In the same way, we have

$$G(s, t) \leq P_4(t)u^- - P_{2s}(t)u^-,$$

in which

$$P_4(t)u^- = t^2 \|u^-\|^2 + b\|su^+ + tu^-\|^2 (su^+ + tu^-)_\alpha + t^4 \int_{\mathbb{R}^3} K(x)\phi_u |_u^-|^2 dx + 2t^2 \int_{\mathbb{R}^3} K(x)\phi_u |_u^-|^2 dx + ast D(u),$$

and

$$P_{2s}(t)u^- = t^{2s} \int_{\mathbb{R}^3} |u^-|^{2s} dx.$$

By taking $\xi_2 > \xi_2' > 1$, we obtain $H(\xi_2, t) < 0, \quad G(s, \xi_2) < 0$ for all $s, t \in [\xi_1, \xi_2]$.

By using Miranda’s Theorem (see Lemma 2.4 in [11]), there is $(s_u, t_u) \in (0, \infty) \times (0, \infty)$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_\kappa$.

Now, we turn to prove the uniqueness of $(s_u, t_u)$. In case of $u \in \mathcal{M}_\kappa$, we have

$$\|u^+\|^2 + aD(u) + b\|u^+\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^+|u^+|^2 dx + \int_{\mathbb{R}^3} K(x)\phi_u u^-|u^-|^2 dx$$

$$= \int_{\mathbb{R}^3} |u^+|^{2s} dx + \kappa \int_{\mathbb{R}^3} f(x, u^+) s_0 u^+ dx. \quad (2.5)$$

We will show that the pair $(s_u, t_u) = (1, 1)$ is the unique pair such that $s_u u^+ + t_u u^- \in \mathcal{M}_\kappa$. Let $(s_0, t_0)$ satisfies $s_0 u^+ + t_0 u^- \in \mathcal{M}_\kappa$ with $0 < s_0 \leq t_0$. We have

$$s_0^2 \|u^+\|^2 + b\|s_0 u^+ + t_0 u^-\|^2 (s_0 u^+ + t_0 u^-, s_0 u^+)_\alpha + s_0^4 \int_{\mathbb{R}^3} K(x)\phi_u u^+|u^+|^2 dx$$

$$+ s_0^2 t_0^2 \int_{\mathbb{R}^3} K(x)\phi_u u^-|u^-|^2 dx + ast_0 D(u)$$

$$= s_0^2 \int_{\mathbb{R}^3} |u^+|^{2s} dx + \kappa \int_{\mathbb{R}^3} f(x, s_0 u^+) s_0 u^+ dx. \quad (2.6)$$

and

$$t_0^2 \|u^-\|^2 + b\|s_0 u^+ + t_0 u^-\|^2 (s_0 u^+ + t_0 u^-, t_0 u^-)_\alpha$$

$$+ t_0^4 \int_{\mathbb{R}^3} K(x)\phi_u u^-|u^-|^2 dx + s_0^2 t_0^2 \int_{\mathbb{R}^3} K(x)\phi_u u^+|u^+|^2 dx + ast_0 D(u)$$
\[
= t_0^{2s} \int_{\mathbb{R}^3} |u^+|^{2s} \, dx + \kappa \int_{\mathbb{R}^3} f(x, t_0u^-) t_0u^- \, dx.
\] (2.7)

By using \(0 < s_0 \leq t_0\) and \(\langle u^+, u^- \rangle \geq 0\), from (2.7), we deduce
\[
\frac{\|u^+\|^2 + AD(u)}{t_0^2} \geq \frac{\|u^-\|^2 + b\|u\|^2_{2\alpha} (u, u^-)_{\alpha} + \int_{\mathbb{R}^3} K(x) \phi_u^- |u^-|^2 \, dx + \int_{\mathbb{R}^3} K(x) \phi_u^+ |u^-|^2 \, dx}{t_0^2}
\geq \int_{\mathbb{R}^3} |u^-|^{2s} \, dx + \kappa \int_{\mathbb{R}^3} \left[ \frac{f(x, t_0u^-)}{(t_0u^-)^{3\beta}} \right] (u^-)^4 \, dx.
\] (2.8)

Combining (2.5) with (2.8), one has that
\[
\left( \frac{1}{t_0^2} - 1 \right) (\|u^-\|^2 + AD(u)) \geq \int_{\mathbb{R}^3} |u^-|^{2s} \, dx + \kappa \int_{\mathbb{R}^3} \left[ \frac{f(x, t_0u^-)}{(t_0u^-)^{3\beta}} \right] (u^-)^4 \, dx.
\]

By using the assumption \((f_3)\), we get \(t_0 \leq 1\). Analogously, from (2.5), (2.6) and \(0 < s_0 \leq t_0\), we have
\[
\left( \frac{1}{s_0^2} - 1 \right) (\|u^+\|^2 + AD(u)) \leq (s_0^{2s} - 1) \int_{\mathbb{R}^3} |u^+|^{2s} \, dx + \kappa \int_{\mathbb{R}^3} \left[ \frac{f(x, s_0u^+)}{(s_0u^+)^{3\beta}} \right] (u^+)^4 \, dx,
\]

it implies \(s_0 \geq 1\). Consequently, \(s_0 = t_0 = 1\).

In the case \(u \notin \mathcal{M}^b_\kappa\), similar to the proof of Lemma 2.1 in [22] to transform the case to the previous case, we can complete the proof of this claim.

In the following computations, we denote \(P_1(s, t)\) a nonzero polynomial with nonnegative coefficients at most i-degree (i=3,4). By direct computations, we deduce
\[
\varphi_u(s, t) = P_4(s, t) - \frac{t_0^{2s}}{2\alpha} \int_{\mathbb{R}^3} |u^-|^{2s} \, dx - \frac{s_0^{2s}}{2\alpha} \int_{\mathbb{R}^3} |u^+|^{2s} \, dx
- \kappa \int_{\mathbb{R}^3} F(x, su^+) \, dx - \kappa \int_{\mathbb{R}^3} F(x, tu^-) \, dx.
\]

Since \(2s^* > 4\), it is obviously \(\lim_{t \to \infty} \varphi_u(s, t) = -\infty\).

By direct computation, it follows that
\[
(\varphi_u)'_s(s, t) = P_3(s, t) - s_0^{2s - 1} \int_{\mathbb{R}^3} |u^+|^{2s} \, dx
- \kappa \int_{\mathbb{R}^3} f(x, su^+) su^+ \, dx > 0, \quad \forall 0 < s \ll 1.
\]

It implies that the boundary point \((0, t)\) is not a maximum point of \(\varphi_u\) with \(t \geq 0\). Similarly, \((s, 0)\) is also not a maximum point of \(\varphi_u\) with \(s \geq 0\). Thus the maximum point of \(\varphi_u\) in \([0, +\infty) \times [0, +\infty)\) should be a critical point of \(\varphi_u\). \(\square\)

**Lemma 2.1.** There exist \(\kappa^* > 0\) such that for all \(\kappa \geq \kappa^*\), \(c^*_\kappa = \inf_{u \in \mathcal{M}^b_\kappa} J^b_\kappa(u)\) is achieved by an element \(u_b \in \mathcal{M}^b_\kappa\).

**Proof.** According to (2.5), there exists a bounded minimum sequence \(\{u_n\}\) of \(c^*_\kappa\) in \(\mathcal{M}^b_\kappa \subset W_V\). So, in the subsequence sense, there exists \(u = u^+ + u^- \in W_V\) such that \(u_n \rightharpoonup u\). Since the embedding \(W_V \hookrightarrow L^p(\mathbb{R}^3)\) is compact for \(p \in (2, 2s^*)\), we deduce
\[
u^+_n \rightharpoonup u^+ \text{ in } W_V, \quad u^-_n \rightharpoonup u^- \text{ in } L^p(\mathbb{R}^3), \quad u^+_n(x) \rightarrow u^+(x) \text{ a.e. } x \in \mathbb{R}^3.
\]
On the other hand, for any \( u \in \mathcal{M}_k^b \), by (2.5) and \( D(u) \geq 0 \), we deduce
\[
(1 - \kappa \varepsilon C_1)\|u^\pm\|^2 \leq \kappa C_2\|u^\pm\|^q + C_3\|u^\pm\|^{2\alpha}.
\]
Similarly as the proof in Lemma 2.1, we can find \( \|u^\pm\| \) is distant from zero, i.e.
\[
\|u^\pm\| \geq \rho_k > 0, \quad \forall u \in \mathcal{M}_k^b.
\]
Hence, by using \( f(x,t) - 4F(x,t) \geq 0 \) which induced from (f3), one gets
\[
J_k^b(u) = J_k^b(u) - \frac{1}{4}\langle (J_k^b)'(u), u \rangle \geq \frac{1}{4}\|u\|^2, \quad \forall u \in \mathcal{M}_k^b.
\]
From the above discussions, we can see that \( \beta_k^c \geq 0 \) is well-defined.

Let \( u \in W_k^1 \) with \( u^\pm \neq 0 \) be fixed. There exists \( s_k, t_k > 0 \) such that \( s_k u^+ + t_k u^- \in \mathcal{M}_k^b \). Hence, by (1.11), the Sobolev’s embedding theorem and Proposition 2.1, we have
\[
\begin{align*}
0 \leq & c_k^b \leq J_k^b(s_k u^+ + t_k u^-) \\
\leq & s_k^2\|u^+\|^2 + t_k^2\|u^-\|^2 + 2bs_k^4\|u^+\|^4 + 2bt_k^4\|u^-\|^4 + 2Cs_k^4\|u^+\|^4 + 2Ct_k^4\|u^-\|^4
\end{align*}
\]
for some constant \( C > 0 \). Hence, we have that
\[
\begin{align*}
& \leq s_k^2\|u^+\|^2 + t_k^2\|u^-\|^2 + 2bs_k^4\|u^+\|^4 + 4bt_k^4\|u^-\|^4 + 8Cs_k^4\|u^+\|^4 + 8Ct_k^4\|u^-\|^4.
\end{align*}
\]
It follows from \( 2s_k^* > 4 \) that \( (s_k, t_k) \) is bounded, so there exists \( s_0 \) and \( t_0 \) such that \( (s_0, t_0) \) as \( k \to \infty \) (in subsequence sense). Similar to the proof processes of Lemma 2.1, by making a contradiction, we get \( s_0 = t_0 = 0 \), that is \( \lim_{k \to \infty} c_k^b = 0 \).

Denote \( \tilde{\beta} := \frac{2}{2}(S)^{\frac{1}{2\alpha}} > 0 \), where
\[
S := \inf_{u \in W_k^1 \setminus \{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3}|u|^{2\alpha}dx}.
\]
From the above discussions, we know that there exists \( \kappa^* > 0 \) such that \( c_k^\beta < \tilde{\beta} \) for all \( \kappa \geq \kappa^* \). By using (2.2), we have \( \int_{\mathbb{R}^3} F(x,su^+_n)dx \to \int_{\mathbb{R}^3} F(x,su^\pm)dx \). Thus, we get
\[
\begin{align*}
& \lim_{n \to \infty} \inf \frac{s_k^2}{2} \lim_{n \to \infty} \|u^n_+ - u^-\|^2 + \|u^n_+\|^2 \\
& + \frac{bs_k^4}{2} \lim_{n \to \infty} \|u^n_+ - u^-\|^2 + \|u^n_+\|^2 + \frac{bt_k^4}{4} \lim_{n \to \infty} \|u^n_+ - u^-\|^2 + \|u^n_+\|^2 \\
& + ast \lim_{n \to \infty} D(u^n) + bs_k^2 t_k^2 \lim_{n \to \infty} D^2(u^n) + \frac{bs_k^4 t_k^4}{2} \lim_{n \to \infty} \inf_D |u^n_+|^2 \|u^n_+\|^2 \\
& + bs_k^3 t_k \lim_{n \to \infty} D(u^n)|u^n_+|^2 + bs_k^3 t_k \lim_{n \to \infty} D(u^n)|u^n_+|^2 \\
& + \frac{s_k^2}{2} \lim_{n \to \infty} \int_{\mathbb{R}^3} K(x)\phi_{u^n_+} u^n_+^2 dx + \frac{t_k^4}{4} \lim_{n \to \infty} \int_{\mathbb{R}^3} K(x)\phi_{u^n_-} u^n_-^2 dx \\
& - \frac{s_k^2}{2\alpha} \lim_{n \to \infty} (|u^n_+|^2 + |u^n_+|^2) + \frac{s_k^2}{2\alpha} \lim_{n \to \infty} (|u^n_-|^2 + |u^n_-|^2) \\
& - \kappa \int_{\mathbb{R}^3} F(x,su^+_n)dx - \kappa \int_{\mathbb{R}^3} F(x,su^\pm)dx.
\end{align*}
\]
\[ + \frac{s^2 t^2}{4} \lim_{n \to \infty} \int_{\mathbb{R}^3} K(x) \phi_{u_n^+} |u_n^+|^2 \, dx + \frac{s^2 t^2}{4} \lim_{n \to \infty} \int_{\mathbb{R}^3} K(x) \phi_{u_n^-} |u_n^-|^2 \, dx. \]

In view of \(-\frac{u_n^+(x) u_n^-(y) + u_n^+(y) u_n^-(x))}{|x-y|^{|\alpha|+2}}\) \(\geq 0\), by using Fatou's Lemma, there holds

\[
\liminf_{n \to \infty} J_n^b(su_n^+ + tu_n^-) \geq J_n^b(su^+ + tu^-) + \frac{s^2}{2} \lim_{n \to \infty} \|u_n^+ - u^+\|^2
\]

\[+ \frac{t^2}{2} \lim_{n \to \infty} \|u_n^- - u^-\|^2 - \frac{s^2}{2} \lim_{n \to \infty} |u_n^+ - u^+|^2 - \frac{t^2}{2} \lim_{n \to \infty} |u_n^- - u^-|^2
\]

\[+ \frac{b s^4}{2} \lim_{n \to \infty} \|u_n^+ - u^+\|^2 + \frac{b s^4}{4} (\lim_{n \to \infty} |u_n^+ - u^+|^2)^2
\]

\[+ \frac{b t^4}{2} \lim_{n \to \infty} \|u_n^- - u^-\|^2 \leq \frac{b t^4}{4} A_4 + \frac{b s^4}{4} A_3^2 + \frac{b s^4}{4} A_2 + \frac{b t^4}{4} A_1^2,
\]

where

\[A_1 = \lim_{n \to \infty} \|u_n^+ - u^+\|^2, \quad A_2 = \lim_{n \to \infty} \|u_n^- - u^-\|^2,
\]

\[B_1 = \lim_{n \to \infty} |u_n^+ - u^+|^2, \quad B_2 = \lim_{n \to \infty} |u_n^- - u^-|^2,
\]

\[A_3 = \lim_{n \to \infty} \|u_n^+ - u^+\|^2, \quad A_4 = \lim_{n \to \infty} \|u_n^- - u^-\|^2.
\]

Hence, we have

\[J_n^b(su^+ + tu^-) + \frac{s^2}{2} A_1 - \frac{s^2}{2} B_1 + \frac{t^2}{2} A_2 - \frac{t^2}{2} B_2
\]

\[+ \frac{b s^4}{2} A_3 + \frac{b s^4}{4} A_3^2 + \frac{b t^4}{2} A_4 + \frac{b t^4}{4} A_1^2 \leq c_6^b,
\]

(2.10) for all \((s, t) \in (0, \infty) \times (0, \infty)\). After proving the following four claims, the proof of this lemma is complete.

**Claim 1.** \(u^+ \neq 0\). In fact, by contradiction, if \(u^+ = 0\), we divide it into two cases.

**Case 1:** \(B_1 = 0\). When \(A_1 = 0\), by (2.9), we have \(|u|^2 > 0\). If \(A_1 > 0\), by taking \(t = 0\) in (2.10), we get \(\frac{s^2}{2} A_1 + \frac{b s^4}{2} A_3^2 \leq c_6^b\) \(\forall s > 0\). It is impossible either.

**Case 2:** \(B_1 > 0\). In this case, by the definition of \(S\), we deduce

\[\bar{\beta} = \frac{\alpha}{3} S^{\frac{1}{2\alpha}} \leq \frac{\alpha}{3} \left( \frac{A_1}{(B_1)^{\frac{1}{2\alpha}}} \right)^{\frac{1}{2\alpha}}.
\]

On the other hand, we get

\[\frac{\alpha}{3} \left( \frac{A_1}{(B_1)^{\frac{1}{2\alpha}}} \right)^{\frac{1}{2\alpha}} \leq \max_{s \geq 0} \left\{ \frac{s^2}{2} A_1 - \frac{s^2}{2} B_1 \right\}
\]

\[\leq \max_{s \geq 0} \left\{ \frac{s^2}{2} A_1 - \frac{s^2}{2} B_1 + \frac{b s^4}{2} A_3 + \frac{b s^4}{4} A_2 \right\}.
\]
Since $c^*_p < \hat{\beta}$, we have that
\[
\hat{\beta} \leq \max_{s \geq 0} \left\{ \frac{s^2}{2} A_1 - \frac{s^2}{2} s^2 \alpha B_1 + \frac{bs^4}{2} A_3 \| u^+ \|^2_\alpha + \frac{bs^4}{4} A_3^2 \right\} \leq c^*_p < \hat{\beta}.
\]
It is also a contradiction. The proof of $u^- \neq 0$ is similar.

We know that the embedding $W_2 \hookrightarrow L^{2s}_2 (\mathbb{R}^3)$ is only continuous but may not be compact. The following claim says that we have partial compactness for the embedding at least in the sequence \{ $u^\ast_p$ \}. The proof of the following claim is a key trick to overcome the difficulties bringing from the critical index case.

**Claim 2.** $B_1 = B_2 = 0$. We suppose that $B_1 > 0$ indirectly.

**Case 1: $B_2 > 0$.** Assume $s_a$ and $t_b$ are the maximum points, i.e.
\[
\frac{s^2}{2} A_1 - \frac{s^2}{2} s^2 \alpha B_1 + \frac{bs^4}{2} A_3 \| u^+ \|^2_\alpha + \frac{bs^4}{4} A_3^2 \]
\[
= \max_{s \geq 0} \left\{ \frac{s^2}{2} A_1 - \frac{s^2}{2} s^2 \alpha B_1 + \frac{bs^4}{2} A_3 \| u^+ \|^2_\alpha + \frac{bs^4}{4} A_3^2 \right\},
\]
\[
\frac{t_b^2}{2} A_2 - \frac{t_b^2}{2} s^2 \alpha B_2 + \frac{bt_b^4}{2} \| u^- \|^2_\alpha + \frac{bt_b^4}{4} A_4^2 \]
\[
= \max_{t \geq 0} \left\{ \frac{t^2}{2} A_2 - \frac{t^2}{2} s^2 \alpha B_2 + \frac{bt^4}{2} \| u^- \|^2_\alpha + \frac{bt^4}{4} A_4^2 \right\}.
\]
Since $\varphi_u$ is continuous, we have that $(s_a, t_u) \in [0, s_a] \times [0, t_b]$ satisfying
\[
\varphi_u(s_u, t_u) = \max_{(s, t) \in [0, s_a] \times [0, t_b]} \varphi_u(s, t).
\]

Similar to the proof of Lemma 2.4 (Claim 2) in [21], we get $(s_a, t_u) \notin [0, s_a] \times \{0\}$ and $(s_a, t_u) \notin \{0\} \times [0, t_b]$. By direct computation, we get
\[
\frac{s^2}{2} A_1 - \frac{s^2}{2} s^2 \alpha B_1 + \frac{bs^4}{2} A_3 \| u^+ \|^2_\alpha + \frac{bs^4}{4} A_3^2 > 0,
\]
\[
\frac{t_b^2}{2} A_2 - \frac{t_b^2}{2} s^2 \alpha B_2 + \frac{bt_b^4}{2} \| u^- \|^2_\alpha + \frac{bt_b^4}{4} A_4^2 > 0 \tag{2.11}
\]
for all $(s, t) \in (0, s_a) \times (0, t_b)$, moreover, there holds
\[
\hat{\beta} \leq \frac{s_a^2}{2} A_1 - \frac{s_a^2}{2} s^2 \alpha B_1 + \frac{bs_a^4}{2} A_3 \| u^+ \|^2_\alpha + \frac{bs_a^4}{4} A_3^2 + \frac{t^2}{2} A_2 - \frac{t^2}{2} s^2 \alpha B_2 + \frac{bt^4}{2} \| u^- \|^2_\alpha + \frac{bt^4}{4} A_4^2,
\]
\[
\hat{\beta} \leq \frac{t_b^2}{2} A_2 - \frac{t_b^2}{2} s^2 \alpha B_2 + \frac{bt_b^4}{2} \| u^- \|^2_\alpha + \frac{bt_b^4}{4} A_4^2.
\]
In view of (2.10), it follows that for all $(s, t) \in (0, s_a) \times (0, t_b)$, there holds
\[
\varphi_u(s, t_b) \leq 0, \quad \varphi_u(s_a, t) \leq 0.
\]

Hence, we can deduce that $(s_a, t_u) \in (0, s_a) \times (0, t_b)$. By Lemma 2.1, one get $s_u u^+ + t_u u^- \in M^b_{\alpha}$. By (2.10), (2.11), we deduce
\[
c^*_p \geq J^b_p(s_u u^+ + t_u u^-) + \frac{s_a^2}{2} A_1 - \frac{s_a^2}{2} s^2 \alpha B_1 + \frac{t_b^2}{2} A_2 - \frac{t_b^2}{2} s^2 \alpha B_2 + \frac{bt_b^4}{2} \| u^- \|^2_\alpha + \frac{bt_b^4}{4} A_4^2
\]
\[
> J^b_p(s_u u^+ + t_u u^-) \geq c^*_p.
\]
It is a contradiction.

Case 2: $B_2 = 0$. We note that

$$\tilde{\beta} \leq \frac{s_u^2}{2} (A_1 - \frac{s_u^2}{2\alpha} B_1 + \frac{b^2_4}{2} A_3 \|u^+\|_\alpha^2 + \frac{b_4}{2} A_2 - \frac{l_2^2}{2\alpha} B_2 + \frac{b^4}{4} A_4 \|u^+\|_\alpha^2 + \frac{b^4}{4} A_4^2),$$

for all $t \in [0, \infty)$. Similarly as the process of Case 1, we know $(s_u, t_u)$ is an inner maximizer of $\varphi_u$ in $[0, s_u] \times [0, \infty)$. We can also make a contradiction of $c_b^\alpha > J_b^\alpha(s_u u^+ + t_u u^-) \geq c_b^\alpha$ if $B_2 > 0$. So the Claim 2 is true.

Claim 3. If $\langle (J_b^\alpha)'(u), u^\pm \rangle \leq 0$, then $0 < s_u, t_u \leq 1$. Suppose $s_u \geq t_u > 0$. One has

$$s_u^2 \|u^+\|^2 + 2s_u b u D(u) + b(s_u^2 \|u^+\|_\alpha^2 + s_u^2 \|u^-\|_\alpha^2 + 2s_u^2 D(u))(s_u^2 \|u^+\|_\alpha^2 + s_u^2 D(u))$$

$$+ s_u^4 \int_{\mathbb{R}^3} K(x)\phi_{u^+}|u^+|^2dx + s_u^4 \int_{\mathbb{R}^3} K(x)\phi_{u^-}|u^+|^2dx$$

$$\geq s_u^2 \int_{\mathbb{R}^3} |u^+|^2dx + \kappa \int_{\mathbb{R}^3} f(x, s_u u^+)s_u u^+ dx. \quad (2.12)$$

On the other hand, we deduce

$$\|u^+\|^2 + 2D(u) + b(\|u^+\|_\alpha^2 + \|u^-\|_\alpha^2 + 2D(u))(\|u^+\|_\alpha^2 + D(u))$$

$$+ \int_{\mathbb{R}^3} K(x)\phi_{u^+}|u^+|^2dx + \int_{\mathbb{R}^3} K(x)\phi_{u^-}|u^+|^2dx$$

$$\leq \int_{\mathbb{R}^3} |u^+|^2dx + \kappa \int_{\mathbb{R}^3} f(x, u^+)u^+ dx. \quad (2.13)$$

By (2.12) - (2.13), noticing that $D(u) \geq 0$, we get

$$\left(\frac{1}{s_u^2} - 1\right)(\|u^+\|^2 + aD(u))$$

$$\geq (s_u^2 a^2 - 1) \int_{\mathbb{R}^3} |u^+|^2dx + \kappa \int_{\mathbb{R}^3} f(x, s_u u^+) - f(x, u^+) \frac{u^+}{(u^+)^3} \frac{4}{(u^+)^4}dx.$$

It implies $s_u \leq 1$. So the Claim 3 is true.

Claim 4. $c_b^\alpha$ is achieved by the weak limit of $\{u_n\}$ defined above. Since $u^\pm \neq 0$, by Lemma 2.1, there are $s_u, t_u > 0$ such that $\bar{u} := s_u u^+ + t_u u^- \in M_b^\alpha$. On the other hand, $u_n \to u$ in $W_V$, then we get limitinf$_{n \to \infty} \|u_n\|_\alpha \geq \|u\|_\alpha$. On the other hand, by using (2.2) and Proposition 2.1, we get

$$\langle (J_b^\alpha)'(u), u^\pm \rangle \leq \liminf_{n \to \infty} \|u_n^\pm\|^2 + b \liminf_{n \to \infty} \|u_n\|_\alpha^2 (u_n, u_n^\pm)_\alpha + \liminf_{n \to \infty} aD(u_n)$$

$$+ \liminf_{n \to \infty} \int_{\mathbb{R}^3} K(x)\phi_{u_n^+}|u_n^+|^2dx - \liminf_{n \to \infty} \int_{\mathbb{R}^3} f(x, u_n^+)u_n^+ dx - \liminf_{n \to \infty} \int_{\mathbb{R}^3} |u_n^+|^2dx \leq 0.$$

So by Claim 3, we have $s_u, t_u \in (0, 1]$. We deduce by using Claim 2 that

$$c_b^\alpha \leq \frac{1}{4}(\|s_u u^+\|^2 + 2as_u t_u D(u_n) + \|t_u u^-\|^2 + \frac{1}{4} - \frac{1}{2\alpha})(\|s_u u^+\|_\alpha^2 + \|t_u u^-\|_\alpha^2)$$

$$+ \frac{\kappa}{4} \int_{\mathbb{R}^3} f(x, s_u u^+)|s_u u^+|^2dx - 4F(x, s_u u^+) dx$$

$$+ \frac{\kappa}{4} \int_{\mathbb{R}^3} f(x, t_u u^-)(t_u u^-)^2 - 4F(x, t_u u^-) dx.$$

By Remark 1.1, we can get

$$c_b^\alpha \leq J_b^\alpha(u) - \frac{1}{4} \langle (J_b^\alpha)'(u), u \rangle \leq \liminf_{n \to \infty} [J_b^\alpha(u_n) - \frac{1}{4} \langle (J_b^\alpha)'(u_n), u_n \rangle] = c_b^\alpha.$$
Hence, we get the Claim 4 proved. Now we can take \( u_b = u \) in Lemma 2.1 and complete the proof. \( \square \)

3. Existence of ground state Nodal solution. In this section, we will give a proof of Theorem 1.1.

**Proof.** We should show that the minimizer \( u_b \) obtained in Lemma 2.1 is a critical point of the functional \( J_b^k \) in \( W_V \). Indirectly, if \( (J_b^k)'(u_b) \neq 0 \), then there exists \( \delta > 0 \) and \( \theta > 0 \) such that

\[
\|(J_b^k)'(v)\| \geq \theta, \quad \text{for all } \|v - u_b\| \leq 3\delta.
\]

Let \( \varepsilon := \min\{c^\kappa_b - r^\kappa_b/4, \theta\delta/8\} \) and \( S_\delta := B(u_b, \delta) \), according to Lemma 2.3 of [25], there exists a deformation \( \eta \in C([0, 1] \times W_V, W_V) \) satisfying

(a) \( \eta(t, v) = v \) if \( t = 0 \), or \( v \notin (J_b^k)^{-1}[(c^\kappa_b - 2\varepsilon, c^\kappa_b + 2\varepsilon)] \cap S_\delta; \)

(b) \( \eta(1, (J_b^k)^{\kappa_b + \varepsilon} \cap S_\delta) \subset (J_b^k)^{\kappa_b - \varepsilon}; \)

(c) \( J_b^k(\eta(1, v)) \leq J_b^k(v) \) for all \( v \in W_V; \)

(d) \( J_b^k(\eta(\cdot, v)) \) is non-increasing for every \( v \in W_V. \)

Let \( Q := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2}) \), \( g(s, t) = su_b^+ + tu_b^- \) with \( (s, t) \in Q. \) Similarly to the proof of Theorem 1.1 in [14], we know

\[
\max_{(s, t) \in Q} J_b^k(\eta(1, g(s, t))) < c^\kappa_b. \tag{3.1}
\]

In fact, it follows from Lemma 2.1 that \( g(s, t) \in (J_b^k)^{\kappa_b + \varepsilon}. \) On the other hand, from (a) and (d), we get

\[
J_b^k(\eta(1, v)) \leq J_b^k(\eta(0, v)) = J_b^k(v), \quad \forall v \in W_V. \tag{3.2}
\]

For \( (s, t) \in Q \), when \( s \neq 1 \) or \( t \neq 1 \), according to (3.2), we get

\[
J_b^k(\eta(1, g(s, t))) \leq J_b^k(g(s, t)) < c^\kappa_b.
\]

If \( s = 1 \) and \( t = 1 \), that is, \( g(1, 1) = u_b \), so that it holds \( g(1, 1) \in (J_b^k)^{\kappa_b + \varepsilon} \cap S_\delta \), then by the property (b), we have

\[
J_b^k(\eta(1, g(1, 1))) \leq c^\kappa_b - \varepsilon < c^\kappa_b.
\]

Thus, (3.1) holds.

In the following, we prove that \( \eta(1, g(Q)) \cap M_b^\kappa \neq \emptyset. \) Let \( \varphi(s, t) := \eta(1, g(s, t)). \)

We define

\[
\Psi(s, t) := \frac{1}{s}((J_b^k)'(\varphi(s, t)), (\varphi(s, t)^\kappa_b)^\circ), \frac{1}{t}((J_b^k)'(\varphi(s, t)), (\varphi(s, t)^\kappa_b)^\circ)).
\]

We should show that there exists \( (s_0, t_0) \in Q \) such that \( \Psi(s_0, t_0) = (0, 0). \) Since

\[
\|g(s, t) - u_b\|^2 = \|(s - 1)u_b^+ + (t - 1)u_b^-\|^2 \geq |s - 1|^2\|u_b^+\|^2 \geq |s - 1|^2(6\delta)^2,
\]

and \( |s - 1|^2(6\delta)^2 > 4\delta^2 \Leftrightarrow s < 2/3 \) or \( s > 4/3 \), using (i) and the range of \( s, \)

for \( s = \frac{1}{2} \) and for every \( t \in [\frac{1}{2}, \frac{3}{2}] \) we have \( g(\frac{1}{2}, t) \notin S_\delta, \) so from (a), we have

\[
\varphi(\frac{1}{2}, t) = g(\frac{1}{2}, t). \]

Thus

\[
\Psi(\frac{1}{2}, t) = \left(2((J_b^k)'(\frac{1}{2}u_b^+ + tu_b^-), \frac{1}{2}u_b^+), \frac{1}{t}((J_b^k)'(\frac{1}{2}u_b^+ + tu_b^-), tu_b^-)\right)
\]

\[
= \left(2H(\frac{1}{2}, t), \frac{1}{t}G(\frac{1}{2}, t)\right).
\]
We remind that the functions \( H(s,t) \) and \( G(s,t) \) are defined in (2.3) and (2.4) respectively. From (2.3) and \( u_b \in \mathcal{M}_b^k \), we have that

\[
((J_b^{\kappa})(tu_b), tu_b^+) = t^2(1 - t^2)(\|u_b^+\|^2 + aD(u_b)) + (t^4 - t^{2\kappa}) \int_{\mathbb{R}^3} |u_b^+|^2^\kappa dx + k t^4 \int_{\mathbb{R}^3} \left( f(x, u_b^+) - \frac{f(x, tu_b^+)}{t^3} \right) u_b^+ dx.
\]

Then, according to (f3) and \( 2\alpha > 4 \), we have that

\[
H(\frac{1}{2}, t) &= \|\frac{3}{2} u_b^+\|^2 + \frac{3ta}{2} D(u_b) + b(\frac{9}{4} \|u_b^+\|^2 + t^2 \|u_b^-\|^2 + 3taD(u_b)) \left( \frac{9}{4} \|u_b^+\|^2 + \frac{3ta}{2} D(u_b) \right) \\
&+ \int_{\mathbb{R}^3} K(x) \phi_{\frac{3}{2} u_b^+ + tu_b^-} \left( \frac{3}{2} u_b^+ \right)^2 dx - \frac{1}{2^{2\kappa}} \int_{\mathbb{R}^3} |u_b^+|^{2\kappa} dx - \kappa \int_{\mathbb{R}^3} f\left( \frac{3}{2} u_b^+ \right) \frac{3}{2} u_b^+ dx \\
&\geq (\langle J_b^{\kappa} \rangle(\frac{1}{2} u_b^+), \frac{1}{2} u_b^+) = \frac{3}{16} (\|u_b^+\|^2 + aD(u_b)) \\
&+ (\frac{1}{16} - \frac{1}{2^{2\kappa}}) \int_{\mathbb{R}^3} |u_b^+|^2 dx + \frac{k}{16} \int_{\mathbb{R}^3} \left( f(x, u_b^+) - 8f\left( \frac{1}{2} u_b^+ \right) \right) u_b^+ dx > 0
\]

for every \( t \in [\frac{1}{2}, \frac{3}{2}] \).

Similarly, for \( s = \frac{3}{2} \) and for every \( t \in [\frac{1}{2}, \frac{3}{2}] \), we have \( \varphi(\frac{3}{2}, t) = g(\frac{3}{2}, t) \), so that

\[
H(\frac{3}{2}, t) = \|\frac{3}{2} u_b^+\|^2 + \frac{3ta}{2} D(u_b) + b\left( \frac{9}{4} \|u_b^+\|^2 + t^2 \|u_b^-\|^2 + 3taD(u_b) \right) \left( \frac{9}{4} \|u_b^+\|^2 + \frac{3ta}{2} D(u_b) \right) \\
&+ \int_{\mathbb{R}^3} K(x) \phi_{\frac{3}{2} u_b^+ + tu_b^-} \left( \frac{3}{2} u_b^+ \right)^2 dx - (\frac{3}{2})^{2\kappa} \int_{\mathbb{R}^3} |u_b^+|^{2\kappa} dx - \kappa \int_{\mathbb{R}^3} f\left( \frac{3}{2} u_b^+ \right) \frac{3}{2} u_b^+ dx \\
&\leq (\langle J_b^{\kappa} \rangle(\frac{3}{2} u_b^-), \frac{3}{2} u_b^+) < 0
\]

for every \( t \in [\frac{1}{2}, \frac{3}{2}] \).

Similarly, we have,

\[
G(\frac{1}{2}, t) > 0, \text{ for every } s \in [\frac{1}{2}, \frac{3}{2}], \quad G(\frac{3}{2}, t) < 0, \text{ for every } s \in [\frac{1}{2}, \frac{3}{2}].
\]

Since \( \Psi \) is continuous on \( Q \), still by Miranda’s theorem mentioned above, we have \( \Psi(s_0, t_0) = (0, 0) \) for some \( (s_0, t_0) \in Q \), so \( \eta(1, g(s_0, t_0)) = \varphi(s_0, t_0) \in \mathcal{M}_b^\kappa \). By comparing to (3.1), we have a contradiction with the definition of \( c_b^* \). 

\[\Box\]

4. Ground state solution and convergence properties.

4.1. The proof of Theorem 1.2.

**Proof.** Similar to the proof of Lemma 2.1, there exists \( \kappa^*_1 > 0 \), when \( \kappa \geq \kappa^*_1 \), we have \( v_b \in \mathcal{N}_b^\kappa \) such that \( J_b^{\kappa}(v_b) = c^*_\kappa = \inf_{u \in \mathcal{N}_b^\kappa} J_b^{\kappa}(u) \) and there is \( \kappa^* > 0 \) such that \( c_b^* < \beta \) for all \( \kappa \geq \kappa^* \). Analogically, there exists a minimum sequence \( v_n \in \mathcal{N}_b^\kappa \) such that \( v_n \to v_b \). Therefore,

\[
c_b^* = \lim_{n \to \infty} J_b^{\kappa}(v_n) = \lim_{n \to \infty} J_b^{\kappa}(tv_n) \geq J_b^{\kappa}(tv_b) + \frac{t^2}{2} A - \frac{t^{2\kappa}}{2\kappa} B + \frac{bt^4}{4} (C^2 + 2C\|v_b\|^2),
\]

where \( A = \lim_{n \to \infty} \|v_n - v_b\|^2, B = \lim_{n \to \infty} |v_n - v_b|^{2\kappa}, \) and \( C = \|v_n - v_b\|^2 \).

One can deduce \( v_b \neq 0 \) analogically to the proof of Lemma 2.1.
We further claim that $B = 0$. In fact, we can maximize $\varphi_{v_b}(t) = J^b_k(t v_b)$ in $[0, \infty)$. Indeed, there exists $t_0 \in [0, \infty)$ such that $J^b_k(t v_b) \leq 0$ for all $t \in [t_0, \infty)$. Let $t_v$ be an inner maximizer of $\varphi_{v_b}$ in $[0, \infty)$, then $t_v v_b \in \mathcal{N}^b_k$. Recalling that
\[
\hat{\beta} = \frac{\alpha}{3} S^\frac{1}{2} \leq \frac{\alpha}{3} \left( \frac{A}{B} \right)^{\frac{1}{2}} = \frac{t_v^2}{2} A - \frac{t_v^n}{2^n B} = \max_{t \geq 0} \left\{ \frac{t_v^2}{2} A - \frac{t_v^n}{2^n B} \right\}.
\]
If $B > 0$, $J^b_k(\hat{t} v_b) + \hat{\beta} \leq J^b_k(\hat{t} v_b) + \frac{t^2}{2} A - \frac{t^n}{2^n B} + \frac{b t^4}{4} (C^2 + 2C \| v_b \|)^2 \leq \epsilon^* < \hat{\beta}$ implies that $J^b_k(\hat{t} v_b) < 0$. So $t_v \leq \hat{t}$ and $\frac{t_v^2}{2} A - \frac{t_v^n}{2^n B} + \frac{b t^4}{4} (C^2 + 2C \| v_b \|)^2 > 0$. Thus,
\[
c^*_v \leq J^b_k(t_v v_b) \leq J^b_k(t v_b) + \frac{t_v^2}{2} A - \frac{t_v^n}{2^n B} + \frac{b t^4}{4} (C^2 + 2C \| v_b \|)^2 \leq c^*_v.
\]
It is a contradiction.

Lastly, similar to the proof of the Claim 4 in Lemma 2.2, we get $c^*$ is achieved by $v_b \in \mathcal{N}^b_k$. Furthermore, if $u_b = u^+ + u^- \in \mathcal{M}^b_k$ is obtained in Theorem 1.1, there exists $s^\pm \in (0, 1)$ such that $s^\pm u^\pm \in \mathcal{N}^b_k$. Hence
\[
2c^* \leq J^b_k(s^+ u^+) + J^b_k(s^- u^-) \leq J^b_k(s^+ u^+ + s^- u^-) < J^b_k(u^+ + u^-) = c^*_b. \quad \Box
\]

4.2. Proof of Theorem 1.3.

Proof. We recall that $u_{b_n}$ is a ground state nodal solution of system (1.9) with $b = b_n$.

Claim (a). $\{u_{b_n}\}$ is bounded in $W_V$. For any $b \in (0, 1)$, there exists a positive pair $(\theta, \theta')$, such that
\[
\langle (J^b_k)'(\theta \eta^+ + \theta' \eta^-), \theta \eta^+ \rangle < 0, \quad \langle (J^b_k)'(\theta \eta^+ + \theta' \eta^-), \theta' \eta^- \rangle < 0, \quad \forall \eta \in C^\infty_c(\mathbb{R}^3), \quad \eta^\pm \neq 0.
\]
Thus, according to Lemma 2.1, for any $b \in [0, 1]$, there is a unique pair $s_b, t_b \in (0, 1] \times (0, 1]$ such that $\tilde{s}_b := s_b \theta^+ + t_b \theta' \eta^- \in \mathcal{M}^b_k$. Hence by using (2.2), we get
\[
J_k^b(u_b) \leq \frac{1}{4} (k_1^2 \eta^+ ||^2 + k_2^2 \eta^- ||^2) + \frac{k}{4} \int_{\mathbb{R}^3} (C_3 \theta^2 \eta^+ \eta^+ - C_3 \theta^2 \eta^- \eta^-) \eta^+ dx \\
+ \frac{1}{4} \int_{\mathbb{R}^3} (C_3 \theta^2 \eta^- \eta^- - C_3 \theta^2 \eta^- \eta^-) \eta^- dx + \frac{1}{2} \int_{\mathbb{R}^3} \eta^+ \eta^- dx.
\]
Thus, we deduce $C^* + 1 \geq J_k^b(u_{b_n}) \geq \frac{1}{4} ||u_{b_n}||^2$.

Claim (b). Existence of a ground state nodal solution $u_0$ of system (1.13).

According to Claim (a), in subsequence sense, there exists $u_0 \in W_V$ such that $u_{b_n} \rightarrow u_0$ in $W_V$. Thanks to $\{u_{b_n}\}$ is a ground state nodal solution of system (1.9) with $b = b_n$, we have that
\[
\langle u_{b_n}, v \rangle + b_n ||u_{b_n}||^2 \langle u_{b_n}, v \rangle + \int_{\mathbb{R}^3} K(x) \phi_{u_{b_n}} u_{b_n} v dx \\
- \kappa \int_{\mathbb{R}^3} f(x, u_{b_n}) v dx - \int_{\mathbb{R}^3} |u_{b_n}|^{2^*_b - 2} u_{b_n} v dx = 0, \quad \forall v \in C^\infty_c(\mathbb{R}^3). \quad (4.1)
\]
By direct computations, we know that
\[
\langle u_0, v \rangle + \int_{\mathbb{R}^3} K(x) \phi_{u_0} u_0 v dx - \kappa \int_{\mathbb{R}^3} f(x, u_0) v dx - \int_{\mathbb{R}^3} |u_0|^{2^*_b - 2} u_0 v dx = 0.
\]
Since $u_{b_n} \in M_{b_n}^\rho$, in view of Claim (a) and Sobolev’s embedding theorem, we obtain
\[ \rho \leq \|u_{b_n}^+\|^2 \leq \varepsilon \|u_{b_n}^+\|^2 + C\|u_{b_n}^+\|^2 + \kappa C_\varepsilon \|u_{b_n}^+\|^q. \]
It implies $u_{b_n}^+ \neq 0$.

By standard processes, there exists $\kappa_2^* > 0$ such that, for all $\kappa \geq \kappa_2^*$, the problem (1.13) possesses a ground state nodal solution $v_0$. Let $\kappa^{***} = \max\{\kappa^*, \kappa_2^*\}$, according to Lemma 2.1, there exists a positive pair $(s_{b_n}, t_{b_n})$ such that $s_{b_n} v_{b_n}^0 + t_{b_n} v_{b_n}^0 \in M_{b_n}^\rho$.

By $(f_3)$, we can deduce $s_{b_n} \to s_0$ and $t_{b_n} \to t_0$ as $b_n \to 0$ (in subsequence sense). Similar to the proof of Theorem 1.3 in [22], we get $(s_0, t_0) = (1, 1)$ and
\[ J_k^0(v_0) \leq J_k^0(u_0) \leq J_k^0(v_0). \]
So $u_0$ is a ground state nodal solution of problem (1.13).

Acknowledgments. We would like to thank you for following the instructions above very closely in advance. It will definitely save us lot of time and expedite the process of your paper’s publication.

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Received July 2020; revised October 2020.

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