A new explicit formula for Kerov polynomials

P. Petrullo and D. Senato

Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata,
via dell’Ateneo Lucano 10, 85100 Potenza, Italia.
p.petrullo@gmail.com, domenico.senato@unibas.it

Abstract

We prove a formula expressing the Kerov polynomial $\Sigma_k$ as a weighted sum over the lattice of noncrossing partitions of the set $\{1, \ldots, k+1\}$. In particular, such a formula is related to a partial order $\leq_{irr}$ on the Lehner’s irreducible noncrossing partitions which can be described in terms of left-to-right minima and maxima, descents and excursions of permutations. This provides a translation of the formula in terms of the Cayley graph of the symmetric group $S_k$ and allows us to recover the coefficients of $\Sigma_k$ by means of the posets $P_k$ and $Q_k$ of pattern-avoiding permutations discovered by Bóna and Simion. We also obtain symmetric functions specializing in the coefficients of $\Sigma_k$.

keywords: symmetric group, symmetric functions, Cayley graph, Kerov polynomials, noncrossing partitions.

AMS subject classification: 05E10, 06A11, 05E05

1 Introduction

The $n$-th free cumulant $R_n$ can be thought as a function $R_n : \lambda \in \mathcal{Y} \to R_n(\lambda) \in \mathbb{Z}$, defined on the set of all Young diagrams $\mathcal{Y}$, which we identify with the corresponding integer partition, and taking integer values [2]. Indeed, after a suitable representation of a Young diagram $\lambda$ as a function in the plane $\mathbb{R}^2$, it is possible to determine the sequences of integers $x_0, \ldots, x_m$ and $y_1, \ldots, y_m$, consisting of the $x$-coordinates of the minima and maxima of $\lambda$, respectively. In this way, if we set

$$\mathcal{H}_\lambda(z) = \frac{\prod_{i=0}^m (z-x_i)}{\prod_{i=1}^m (z-y_i)},$$

then $R_n(\lambda)$ is the coefficient of $z^{n-1}$ in the formal Laurent series expansion of $\mathcal{K}_\lambda(z)$ such that

$$\mathcal{K}_\lambda(\mathcal{H}_\lambda(z)) = \mathcal{H}_\lambda(\mathcal{K}_\lambda(z)) = z.$$
It can be shown that $R_1(\lambda) = 0$ for all $\lambda$. So, the $k$-th Kerov polynomial is a polynomial $\Sigma_k(R_2, \ldots, R_{k+1})$ which satisfies the following identity,

$$\Sigma_k(R_2(\lambda), \ldots, R_{k+1}(\lambda)) = (n)_k \frac{\chi^\lambda(k, 1^{n-k})}{\chi^\lambda(1^n)},$$

where $\chi^\lambda(k, 1^{n-k})$ denotes the value of the irreducible character of the symmetric group $\mathfrak{S}_n$ indexed by the partition $\lambda$ on $k$-cycles. Two remarkably properties of $\Sigma_k$ have to be stressed. First, it is an “universal polynomial”, that is it does not depend on $\lambda$ nor on $n$. Second, its coefficients are nonnegative integers. A combinatorial proof of the positivity of $\Sigma_k$ is quite recent and is due to Féray [7]. Such a proof was then simplified by Doléga, Féray and Śniady [5]. Until now, several results on Kerov polynomials have been proved and conjectured, see for instance [3, 9, 11, 16] and [8] for a more detailed treatment.

Originally, free cumulants arise in the noncommutative context of free probability theory [14], and their applications in the asymptotic character theory of the symmetric group is due mainly to Biane. In 1992, Speicher [15] showed that the formulae connecting moments and free cumulants of a noncommutative random variable $X$ obey the M"obius inversion on the lattice of noncrossing partitions of a finite set. This result highlights the strong analogy between free cumulants and classical cumulants, which are related to the moments of a random variable variable $X$, defined on a classical probability space, via the M"obius inversion on the lattice of all partitions of a finite set. More recently, Di Nardo, Petrullo and Senato [6] have shown how the classical umbral calculus provides an alternative setting for the cumulant families which passes through a generalization of the Abel polynomials.

In 1997, it was again Biane [1] to show that the lattice $NC_n$ of noncrossing partitions of $\{1, \ldots, n\}$ can be embedded into the Cayley graph of the symmetric group $\mathfrak{S}_n$. So that, it seems reasonable that a not too complicated expression of the Kerov polynomials involving noncrossing partitions, or the Cayley graph of $\mathfrak{S}_n$, would exist. In particular, such a formula, conjectured in [2], appeared with a rather implicit description into the papers [5, 7].

In this paper, we state an explicit formula expressing $\Sigma_k$ as a weighted sum over the lattice $NC_{k+1}$. In particular, we introduce a partial order $\preceq_{irr}$ on the subset $NC_{irr}$ of $NC_n$ consisting of the noncrossing partitions having 1 and $k+1$ in the same block. Then, we prove that

$$\Sigma_k = \sum_{\tau \in NC_{irr, k+1}} \left[ \sum_{\pi: \tau \preceq_{irr} \pi} (-1)^{\ell(\pi)-1} W_\tau(\pi) \right] R_\tau,$$

where $\ell(\pi)$ is the number of blocks of $\pi$, $W_\tau(\pi)$ is a suitable weight depending on $\tau$ and $\pi$, and $R_\tau = \prod_B R_{|B|}$, $B$ ranging over the blocks of $\tau$ having at least 2 elements.

2
Since each \( \pi \in NC_{k+1} \) is obtained from a given \( \pi' \in NC_k \) simply by inserting \( k + 1 \) in the block containing 1, then the Biane embedding can be used to translate the formula in terms of the Cayley graph of the symmetric group \( \mathfrak{S}_k \).

We also define two slight different versions of the Foata bijection which give rise to a description of \( \leq \text{irr} \) in terms of left-to-right minima and maxima of permutations. Moreover, the maps \( \theta \) and \( f \), studied by Bóna and Simion \([4]\), allow us to compute \( \Sigma_k \) via the posets \( P_k \) and \( Q_k \) of pattern-avoiding permutations ordered by inclusion of descents sets and excedances sets respectively.

Finally, the special structure of the weight \( W_f(\pi) \) makes we able to determine symmetric functions \( g_\mu(x_0, \ldots, x_{k-1}) \) that specialized in \( x_i = i \) return the coefficient of \( \prod_{i \geq 2} R_i^m_i \) in \( \Sigma_k \), for every integer partition \( \mu \) of size \( k+1 \) having \( m_i \) parts equal to \( i \).

## 2 Kerov polynomials

Let \( n \) be a positive integer and let \( \lambda = (\lambda_1, \ldots, \lambda_l) \) be an integer partition of size \( n \), that is \( 1 \leq \lambda_1 \leq \cdots \leq \lambda_l \) and \( \sum \lambda_i = n \). As is well known, the Young diagram of \( \lambda \) (in the French convention) is an array of \( n \) left-aligned boxes, whose \( i \)-th row consists of \( \lambda_i \) boxes. Denote by \( \mathcal{Y}_n \) the set of all Young diagram of size \( n \), and set \( \mathcal{Y} = \bigcup \mathcal{Y}_n \). From now on, an integer partition and its Young diagram will be denoted by the same symbol \( \lambda \).

After a suitable representation of a Young diagram \( \lambda \) as a function in the plane \( \mathbb{R}^2 \), it is possible to determine the sequences of integers \( x_0, \ldots, x_m \) and \( y_1, \ldots, y_m \), consisting of the \( x \)-coordinates of its minima and maxima respectively. Then, by expanding the rational function

\[
\mathcal{H}_\lambda(z) = \frac{\prod_{i=0}^m (z - x_i)}{\prod_{i=1}^m (z - y_i)}
\]

as a formal power series in \( z^{-1} \) one has

\[
\mathcal{H}_\lambda(z) = z^{-1} + \sum_{n \geq 1} M_n(\lambda) z^{-(n+1)}.
\]

The integer \( M_n(\lambda) \) is said to be the \( n \)-th moment of \( \lambda \). Now, define \( \mathcal{K}_\lambda(z) = \mathcal{H}_\lambda^{-1}(z) \), that is \( \mathcal{K}_\lambda(\mathcal{H}_\lambda(z)) = \mathcal{H}_\lambda(\mathcal{K}_\lambda(z)) = z \), and consider its expansion as a formal Laurent series,

\[
\mathcal{K}_\lambda(z) = z^{-1} + \sum_{n \geq 1} R_n(\lambda) z^{n-1}.
\]

Then, the integer \( R_n(\lambda) \) is named the \( n \)-th free cumulant of \( \lambda \). It is not difficult to see that \( M_1(\lambda) = R_1(\lambda) = 0 \) for all \( \lambda \).

By setting

\[
\mathcal{M}_\lambda(z) = z^{-1}\mathcal{H}_\lambda(z^{-1}) \quad \text{and} \quad \mathcal{R}_\lambda(z) = z\mathcal{K}_\lambda(z),
\]

we obtain two formal power series in \( z \),

\[
\mathcal{M}_\lambda(z) = 1 + \sum_{n \geq 1} M_n(\lambda) z^n \quad \text{and} \quad \mathcal{R}_\lambda(z) = 1 + \sum_{n \geq 1} R_n(\lambda) z^n,
\]

3
such that
\[ \mathcal{M}_\lambda(z) = R_\lambda(z \mathcal{M}_\lambda(z)). \quad (2.1) \]

Let \( \lambda \) and \( \mu \) be two partitions of size \( n \), and denote by \( \chi^\lambda(\mu) \) the value of the irreducible character of \( S_n \) indexed by \( \lambda \) on the permutations of type \( \mu \). So that, if \( \mu = (k, 1^{n-k}) \), that is \( \mu_1 = k \) and \( \mu_2 = \cdots = \mu_{n-k+1} = 1 \), then the value of the normalized character \( \hat{\chi}^\lambda \) on the \( k \)-cycles of \( S_n \) is given by
\[ \hat{\chi}^\lambda(k, 1^{n-k}) = (n)_k \frac{\chi^\lambda(k, 1^{n-k})}{\chi^\lambda(1^n)}, \]
where \((n)_k = n(n-1) \cdots (n+k-1)\). The \( k \)-th Kerov polynomial is a polynomial \( \Sigma_k \), in \( k \) commuting variables, which satisfies the following identity,
\[ \Sigma_k(R_2(\lambda), \ldots, R_{k+1}(\lambda)) = \hat{\chi}^\lambda(k, 1^{n-k}). \]

If we think of \( R_n(\lambda) \) as the image of a map \( R_n : \lambda \in \mathcal{Y} \to R_n(\lambda) \in \mathbb{Z} \), then also Kerov polynomials become maps \( \Sigma_k = \Sigma_k(R_1, \ldots, R_{k+1}) \), which are polynomials in the \( R_n \)'s, such that \( \Sigma_k(\lambda) = \hat{\chi}^\lambda(k, 1^{n-k}) \).

Since the coefficients of \( \Sigma_k \) do not depend on \( \lambda \) nor on \( n \), but only on \( k \), such polynomials are said to be “universal”. A second remarkably property of Kerov polynomials is that all their coefficients are positive integers. This fact is known as the “Kerov conjecture” \([10]\). The first proof of the Kerov conjecture was given with combinatorial methods by Féray \([7]\). The same author with Doléga and Śniady \([5]\) have then simplified the proof. The following formula for \( \Sigma_k \) is due to Stanley \([16]\).

**Theorem 2.1.** Let \( R(z) = 1 + \sum_{n \geq 2} R_n z^n \). If
\[ F(z) = \frac{z}{R(z)} \quad \text{and} \quad G(z) = \frac{z}{F(z^{-1})(z^{-1})}, \]
then we have
\[ \Sigma_k = -\frac{1}{k} [z^{-1}]_\infty \prod_{j=0}^{k-1} G(z-j). \quad (2.2) \]

More precisely, if \([z^n]f(z)\) denotes the coefficient of \( z^n \) in the formal power series \( f(z) \), then \([z^{-1}]_\infty f(z) = [z] f(z^{-1}) \). This way, identity \((2.2)\) states that \( \Sigma_k \) is obtained by expressing the right-hand side in terms of the free cumulants \( R_n \)'s.

Moreover, if \( M(z) = 1 + \sum_{n \geq 1} M_n z^n \), then by virtue of \((2.1)\) we have \( z G(z)^{-1} = M(z^{-1}) \), and \((2.2)\) can be rewritten in the following equivalent form,
\[ \Sigma_k = -\frac{1}{k} [z^{k+1}]_\infty \prod_{j=0}^{k-1} \frac{1-jz}{M(1-jz)}, \quad (2.3) \]
3 Irreducible noncrossing partitions

A partition of a finite set $S$ is an unordered sequence $\pi = \{A_1, \ldots, A_t\}$ of its nonempty subsets, such that $A_i \cap A_j = \emptyset$, if $i \neq j$, and $\cup A_i = S$. We say that a partition $\tau$ refines a partition $\pi$, in symbols $\tau \leq \pi$, if and only if each block of $\pi$ is union of blocks of $\tau$. Moreover, if $T \subset S$, the restriction of a partition $\pi$ of $S$ to $T$ is the partition $\pi|_T$ obtained by removing from $\pi$ all the elements which do not belong to $T$.

There is a beautiful formula, due to Speicher [15], related to a special family of set partitions, which gives the expression of the moments $M_n$'s in terms of their respective free cumulants $R_n$'s. Let us recall it.

Denote by $[n]$ the set $\{1, \ldots, n\}$. A partition $\pi = \{A_1, \ldots, A_t\}$ of $[n]$ is said to be a noncrossing partition if and only if $a, c \in A_i$ and $b, d \in A_j$ implies $i = j$, whenever $1 \leq a < b < c < d \leq n$. The set of all the noncrossing partitions of $[n]$ is usually denote by $NC_n$. Its cardinality equals the $n$-th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$. Now, if for all $\pi = \{A_1, \ldots, A_t\} \in NC_n$ we set $R_{\pi} = R_{|A_1|} \cdots R_{|A_t|}$, then the formula of Speicher states that

$$M_n = \sum_{\pi \in NC_n} R_{\pi}.$$  

A noncrossing partition $\pi$ of $[n]$ is said to be irreducible if and only if 1 and $n$ lies in the same block of $\pi$. To the best of our knowledge, irreducible noncrossing partitions were introduced by Lehner [12]. According to Lehener’s notation, the set of all irreducible noncrossing partitions of $[n]$ will be denoted by $NC_{n}^{\irr}$.

By taking the sum of the monomials $R_{\pi}$’s, $\pi$ ranging in $NC_{n}^{\irr}$ instead of $NC_n$, one defines a quantity $B_n$ known as a boolean cumulant (see [12]),

$$B_n = \sum_{\pi \in NC_{n}^{\irr}} R_{\pi}.$$  

(3.1)

In particular, if $B(z) = \sum_{n \geq 1} B_n z^n$, then we have

$$\mathcal{M}(z) = \frac{1}{1 - B(z)}.$$  

(3.2)

Note that, a partition of $NC_{n+1}^{\irr}$ is obtained from a partition of $NC_n$ simply by inserting $n + 1$ in the block containing 1. This fixes a bijection between $NC_n$ and $NC_{n+1}^{\irr}$, which proves that $|NC_{n+1}^{\irr}| = |NC_n| = C_n$. If $\mu$ is an integer partition of size $n$, let $\ell(\mu)$ denote the number of its parts $\mu_i$’s, and define $NC_{\mu}^{\irr}$ to be the subset of $NC_{\mu}^{\irr}$ consisting of all the partitions of type $\mu$, namely the partitions $\pi = \{A_1, \ldots, A_t\}$ such that the sequence $(|A_1|, \ldots, |A_t|)$ is a rearrangement of $\mu$. It can be shown that, if exactly $m_i(\mu)$ parts of $\mu$ are equal to $i$, and if $m(\mu)! = m_1(\mu)! \cdots m_n(\mu)!$, then we have

$$|NC_{\mu}^{\irr}| = \frac{(n - 2)^{\ell(\mu) - 1}}{m(\mu)!}.$$  

(3.3)
The notion of noncrossing partition can be given for any totally ordered set \( S \). In particular, \( NC_S^{irr} \) will denote the set of all the noncrossing partitions of \( S \), such that the minimum and the maximum of \( S \) lies in the same block. Let us introduce a partial order on \( NC_S^{irr} \).

**Definition 3.1.** Let \( \tau, \pi \in NC_S^{irr} \). We set \( \tau \leq_{irr} \pi \) if and only if \( \tau \leq \pi \) and the restriction \( \tau|_A \) of \( \tau \) to each block \( A \) of \( \pi \), is in \( NC_A^{irr} \). In particular, we say that \( \pi \) covers \( \tau \) if and only if \( \tau \leq_{irr} \pi \) and \( \pi \) is obtained by joining two blocks of \( \tau \).

For instance, let \( \tau = \{\{1, 5\}, \{2, 3\}, \{4\}\}, \pi = \{\{1, 2, 3, 5\}, \{4\}\} \) and \( \pi' = \{\{1, 5\}, \{2, 3, 4\}\} \). Then \( \tau, \pi, \pi' \in NC_5^{irr} \) and \( \tau \) refines both \( \pi \) and \( \pi' \). However, \( \tau \leq_{irr} \pi \) and in particular \( \pi \) covers \( \tau \), while it is not true that \( \tau \leq_{irr} \pi' \), since \( \tau|_{\{2, 3, 4\}} = \{\{2, 3\}, \{4\}\} \) is not irreducible.

The singletons (i.e. blocks of type \( \{i\} \)) of the noncrossing partitions will play a special role. For all \( \tau \in NC_n \) we denote by \( U(\tau) \) the subset of \([n]\) consisting of all the integers \( i \) such that \( \{i\} \) is a block of \( \tau \), while \( \tilde{\tau} \) will be the partition obtained from \( \tau \) by removing the singletons. When \( \tau, \pi \in NC_n^{irr} \) and \( \tau \leq_{irr} \pi \), then \( \pi_\tau \) is the restriction of \( \pi \) to \( U(\tau) \). Note that \( \pi_\tau \in NC_{U(\tau)} \).

We define a tree-representation for the partitions of \( NC_n^{irr} \) in the following way. Assume \( \tau = \{A_1, \ldots, A_l\} \in NC_n^{irr} \) and \( \min A_i < \min A_{i+1} \). Construct a labeled rooted tree \( t_\tau \) by the following steps,

- choose \( A_1 \) as the root of \( t_\tau \),
- if \( 2 \leq i < j \leq l \) then draw an edge between \( A_i \) and \( A_j \) if and only if \( j \) is the lowest integer such that \( \min A_i < \min A_j < \max A_j < \max A_i \),
- label each edge \( \{A_i, A_j\} \) with \( \min A_j \).

For example, if \( \tau = \{\{1, 2, 7, 12\}, \{3, 5, 6\}, \{4\}, \{8, 9\}, \{10, 11\}\} \) then \( t_\tau \) is the following tree,

![Tree Representation](attachment:image.png)

Now, let \( E(\tau) \) be the set of labels of \( t_\tau \), and choose \( j \in E(\tau) \). We denote by \( t_{\tau,j} \) the tree obtained from \( t_\tau \) by deleting the edge labeled by \( j \) and joining its nodes (i.e. joining the blocks). In the following, we will say that \( t_{\tau,j} \) is the tree
obtained from \( t_\tau \) by “removing” \( j \). Hence, \( t_{\tau,3} \) is given by
\[
\{1, 2, 3, 5, 6, 7, 12\}
\]
\[
4 \quad 8 \quad 10
\]
\[
\{4\} \quad \{8, 9\} \quad \{10, 11\}
\]

Of course, \( t_{\tau,j} \) is the tree-representation of an irreducible noncrossing partition, here denoted by \( \tau_{(j)} \), whose blocks are the nodes of \( t_{\tau,j} \). By construction, we have \( \tau \leq_{\text{irr}} \tau_{(j)} \) and \( E(\tau_{(j)}) = E(\tau) - \{j\} \). More generally, given a subset \( S \subseteq E(\tau) \), we denote by \( \tau_S \) the only partition whose tree \( t_{\tau,S} \) is obtained from \( t_\tau \) by removing all labels in \( S \) successively. We remark that \( \tau_S \) depends only on the set \( S \) and not on the order in which labels are chosen. In the example, if \( S = \{3, 8\} \) then \( t_{\tau,S} \) is the tree below,
\[
\{1, 2, 3, 5, 6, 7, 8, 9, 12\}
\]
\[
4 \quad 10
\]
\[
\{4\} \quad \{10, 11\}
\]

This way, we have \( \tau_S = \{\{1, 2, 3, 5, 6, 7, 8, 9, 12\}, \{4\}, \{10, 11\}\} \). The following proposition is easy to prove.

**Proposition 3.1.** Let \( \tau, \pi \in \text{NC}_n^{\text{irr}} \). Then, we have \( \tau \leq_{\text{irr}} \pi \) if and only if \( \pi = \tau_S \) for some \( S \subseteq E(\tau) \). In particular, if \( \ell(\tau) \) is the number of blocks of \( \tau \), then we have
\[
|\{\pi \mid \tau \leq_{\text{irr}} \pi\}| = |2^{E(\tau)}| = 2^{\ell(\tau) - 1},
\]
\( 2^{E(\tau)} \) denoting the powerset of \( E(\tau) \), and
\[
|\{\pi \mid \pi \text{ covers } \tau\}| = |E(\tau)| = \ell(\tau) - 1.
\]

### 3.1 The Cayley graph of \( \mathfrak{S}_n \)

We start recalling some known results relating noncrossing partitions to the symmetric group in order to describe the partial order \( \leq_{\text{irr}} \) in terms of permutations.

The Cayley graph of \( \mathfrak{S}_n \) is the graph whose nodes are the elements of \( \mathfrak{S}_n \) and \( w, u \in \mathfrak{S}_n \) are connected by an edge if and only if there exists a transposition \( t \) such that \( u = wt \).

Denote by \( T_n \) the set of all transpositions of \( \mathfrak{S}_n \) and for all \( w \in \mathfrak{S}_n \) let \( \ell_T(w) \) denote the minimum number of transpositions in \( T_n \) whose product equals \( w \). If we set \( u \leq_T w \) if and only if \( \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w) \), then we obtain a partial order on \( \mathfrak{S}_n \), sometimes called the absolute order, whose Hasse diagram can be identified with the Cayley graph.

Biane \([\Pi]\) has shown that the lattice \( (\text{NC}_n, \leq) \) can be embedded into the Cayley graph of \( \mathfrak{S}_n \) through a map, here denoted by \( \beta \), such that \( \tau \leq \pi \) if
and only if $\beta(\tau) \leq_T \beta(\pi)$. This embedding has a quite simple description. If $A = \{i_1, \ldots, i_h\} \subseteq [n]$ and $1 < \cdots < i_h$, let $\beta(A) = (i_1 \cdots i_h) \in S_n$. Then, set $\beta(\tau) = \beta(A_1) \cdots \beta(A_t)$ whenever $\tau = \{A_1, \ldots, A_t\}$. This way, if $NC(S_n) = \{\beta(\tau) \mid \tau \in NC_n\}$, then $NC(S_n)$ is the interval $[id_n, c_n] = \{w \in S_n \mid \underline{id}_n \leq_T w \leq_T \underline{c}_n\}$ where $\underline{id}_n = (1) \cdots (n)$ and $\underline{c}_n = (1 \cdots n)$. Moreover, it is easy to see that if $NC_{irr}(S_n) = \{\beta(\tau) \mid \tau \in NC_{irr}\}$, then $NC_{irr}(S_{n+1}) = \{(1n+1w \mid w \in NC(S_n))\}$.

Incidentally, the Biane map $\beta$ allows us to obtain a further enumerative result. Indeed, consider the expression of a permutation $w$ as a product of its disjoint cycles, $w = (i_1,1 \cdots i_{1,n_1}) \cdots (i_{l,1} \cdots i_{l,n_l})$. For $1 \leq h \leq l$ define

$$T_{(i_1,1 \cdots i_{h,n_h})} = \{(i j) \in T_n \mid i_1 \leq j < i_{h,n_h}\},$$

and then set

$$T_w = \bigcup_{1 \leq h \leq l} T_{(i_1,1 \cdots i_{h,n_h})}.$$

Now, the following proposition is easy to prove.

**Proposition 3.2.** Let $\tau, \pi \in NC_{irr}^\ast$, $u = \beta(\tau)$ and $w = \beta(\pi)$. Then, $\pi$ covers $\tau$ if and only if there exists $t \in T_w$ such that $u = wt$. In particular, we have

$$|\{\tau \in NC_{irr}^\ast \mid \pi \text{ covers } \tau\}| = |T_w| = \sum_{A \in \tau} \binom{|A| - 1}{2}.$$

### 3.2 Left-to-right minima and maxima

Let $w = w_1 \cdots w_n$ be a permutation of $S_n$ written as a word, that is $w_i = w(i)$. A left-to-right maximum of $w$ is an integer $w_i$ such that $w_j < w_i$ for all $j < i$. Analogously, a left-to-right minimum of $w$ is an integer $w_i$ such that $w_j > w_i$ for all $j < i$. Of course, every $w \in S_n$ has a trivial left-to-right minimum in 1, and a trivial left-to-right maximum in $n$. Denote by Max($w$) and Min($w$) the sets of all the nontrivial left-to-right maxima and left-to-right minima of $w$, respectively.

There is a well known bijection, named the Foata bijection, showing that the number of permutations in $S_n$ with $k$ cycles equals the number of permutations in $S_n$ with $k$ left-to-right maxima. In this paper we use two slight different versions of the Foata bijection which we are going to describe.

Let $w = (i_{1,1} \cdots i_{1,n_1}) \cdots (i_{l,1} \cdots i_{l,n_l}) \in NC_{irr}(S_n)$. Arrange the cycles of $w$ in decreasing order of their minima from left to right, and define $\tilde{w}$ to be the permutation (in the word notation) obtained by removing the parenthesis. Clearly, the minimum of each cycle in $w$ is a left-to-right minimum of $\tilde{w}$. Moreover, it is easy to see that the map $w \to \tilde{w}$ is a bijection.

Now, consider the same permutation $w$. First, arrange the cycles in decreasing order of their maxima from left to right. If $w'$ is the word obtained by
removing the parenthesis, then let $\hat{w}$ denote the reflection of $w$ with respect to its middle-point, that is $\hat{w}_1 = w'_{n-i+1}$. This way, the maximum of each cycle of $w$ is a left-to-right-maximum of $\hat{w}$, and the map $w \to \hat{w}$ is a bijection too. Note also that, if $\beta(\tau) = w$, then we have $\min(\hat{w}) \cap \max(\hat{w}) = U(\tau)$. In fact, $\min(\hat{w}) \cap \max(\hat{w})$ consists of the fixed points of $w$, that is the singletons of $\tau$.

For example, consider $w = (1 \ 2 \ 10) \ (4) \ (5 \ 6 \ 7) \ (3) \ (8 \ 9) \in NC_n^{irr}(\mathfrak{S}_{10})$. By arranging the cycles in decreasing order of their minima we have $w = (8 \ 9) \ (5 \ 6 \ 7) \ (4) \ (3) \ (1 \ 2 \ 10)$, then $\hat{w} = 89567431210$ and $\min(\hat{w}) = \{3,4,5,8\}$.

Therefore, if we arrange the cycles in decreasing order of their maxima we obtain $w = (1 \ 2 \ 10) \ (8 \ 9) \ (5 \ 6 \ 7) \ (4) \ (3)$, so that $w' = 12108956743$ and finally $\hat{w} = 34765981021$. This way, $\max(\hat{w}) = \{3,4,7,9\}$.

**Proposition 3.3.** Let $\tau, \pi \in NC_n^{irr}$, $w = \beta(\tau)$ and $u = \beta(\pi)$. If $\tau \leq \pi$ then $\min(\hat{u}) \subseteq \min(\hat{w})$ and $\max(\hat{u}) \subseteq \max(\hat{w})$. Moreover, once fixed $\tau$, the maps $\pi \in \{\pi \mid \tau \leq \pi\} \to \min(\hat{u})$ and $\pi \in \{\pi \mid \tau \leq \pi\} \to \max(\hat{u})$ are bijections.

**Proof.** Note that, the labels of the tree $t_\tau$ are exactly the nontrivial minima of the cycles of $w$, that is $E(\tau) = \min(\hat{w})$. Moreover, if we define a new label on $t_\tau$ by replacing $\min A_j$ with $\max A_j$, then Proposition 3.1 is again true. So, the proof follows by means of Proposition 3.1.

\[ \square \]

### 3.3 Descents, excedances and pattern-avoiding permutations

An integer $i \in [n-1]$ is a descent for a permutation $w = w_1 \ldots w_n \in \mathfrak{S}_n$ if $w_i > w_{i+1}$, while it is called an excedance of $w$ if $w_i > i$. We denote by $\text{Des}(w)$ the set of all the descents of $w$, and by $\text{Exc}(w)$ the set of all its excedances.

Consider the poset $NC_n$ under the refinement order. Following Bóna and Simion [4], let $P_n$ denote the set of all 132-avoiding permutations of $\mathfrak{S}_n$, and let $Q_n$ denote the set of all 321-avoiding permutations of $\mathfrak{S}_n$. A partial order can be introduced on $P_n$ and $Q_n$ by assuming $u \leq w$ in $P_n$ (resp. in $Q_n$) if and only if $\text{Des}(u) \subseteq \text{Des}(w)$ (resp. $\text{Exc}(u) \subseteq \text{Exc}(w)$). Then, there are two order-preserving bijections $f : NC_n \to P_n$ and $\theta : NC_n \to Q_n$ with the following properties:

- $i \geq 1$ is a descent of $f(\tau)$ if and only if $i+1$ is the minimum of its block in $\tau$,
- $i \geq 1$ is an excedance of $\theta(\tau)$ if and only if $i+1$ is the minimum of its block in $\tau$.

We also observe that, if $\tau \in NC_n^{irr}$ and if $w = f(\tau)$, then $w_{n+1} = n + 1$. Analogously, if $w = \theta(\tau)$ then $w_{n+1} = n + 1$. Finally, this says that the image of $NC_n^{irr}$ under $f$ (resp. $\theta$) can be identified with $P_n$ (resp. $Q_n$).
4 Kerov polynomial formula

By means of the results of Section 2 and Section 3, we are able to give a new formula for the Kerov polynomial $\Sigma_k$. In particular, such a formula is related to the order $\preceq_{\text{irr}}$ on the irreducible noncrossing partitions of the set $[k+1]$. Furthermore, the map $\beta$ makes we able to compute Kerov polynomials via the Cayley graph of $\mathcal{S}_k$.

Let $j$ be a nonnegative integer and denote by $\lambda \boxplus j$ the image of the diagram $\lambda$ under the translation of the plane given by $x \rightarrow x + j$. The $i$-th minimum and maximum of $\lambda \boxplus j$ are $x_i + j$ and $y_i + j$ respectively, so that

$$H_{\lambda \boxplus j}(z) = \prod_{m=0}^{n} z - (x_i + j) \prod_{i=1}^{m} z - (y_i + j)$$

and $M_{\lambda \boxplus j}(z) = \frac{1}{1 - jz} \frac{z}{1 - jz}$.

In this way we may rewrite (2.3) as follows,

$$\Sigma_k(R_2(\lambda), \ldots, R_{k+1}(\lambda)) = -\frac{1}{k} [z^{k+1}] \prod_{j=0}^{k-1} \frac{1}{M_{\lambda \boxplus j}(z)}. \quad (4.1)$$

Now, let $R_n(\lambda \boxplus j)$ denote the $n$-th free cumulant of $\lambda \boxplus j$, that is the coefficient of $z^n$ in the formal power series $R_{\lambda \boxplus j}(z)$ such that $M_{\lambda \boxplus j}(z) = R_{\lambda \boxplus j}(z)$. Hence, it is immediate to verify that $R_{\lambda \boxplus j}(z) = jz + R_{\lambda}(z)$, or equivalently

$$R_n(\lambda \boxplus j) = R_n(\lambda) + j\delta_{1,n}, \quad (4.2)$$

where $\delta_{1,n}$ is the Kronecker delta.

Theorem 4.1 (The formula for Kerov polynomials). We have

$$\Sigma_k(R_2(\lambda), \ldots, R_{k+1}(\lambda)) = \sum_{\tau \in NC^{\text{irr}}_{k+1}} \sum_{\pi : \tau \preceq_{\text{irr}} \pi} (-1)^{\ell(\pi) - 1} W_\tau(\pi) R_\tau, \quad (4.3)$$

where

$$W_\tau(\pi) = \frac{1}{k!} \sum_{w \in \mathcal{S}_k} (w(1) - 1)^{|A_1|} \cdots (w(k) - 1)^{|A_k|},$$

if $\pi = \{A_1, \ldots, A_l\}$ and $A_i = \emptyset$ for $i > l$.

Proof. Let $B_n(j)$ denote the $n$-th boolean cumulant $B_n(\lambda \boxplus j)$ of $\lambda \boxplus j$. Since $R_1(\lambda) = 0$, then from (3.1) and (4.2) we deduce

$$B_n(j) = \sum_{\pi \in NC_n^{\text{irr}}} j^{u(\pi)} R_\pi(\lambda), \quad (4.4)$$

where $u(\pi) = |U(\pi)|$. Via (3.2) we have $[z^n] (M_{\lambda \boxplus j}(z))^{-1} = -B_n(j)$, then the right-hand side in (4.1) is equal to

$$\sum_{\mu} (-1)^{\ell(\mu) - 1} m(\mu)! k^{\ell(\mu)} \sum_{w \in \mathcal{S}_k} \prod_{i=1}^{k} B_{\mu_i}(w(i) - 1).$$
Here $\mu = (\mu_1, \ldots, \mu_l)$ ranges over all the integer partitions of size $k + 1$ with at most $k$ parts, and $\mu_i = 0$ if $i > \ell(\mu)$. However, by taking into account \[\ref{eq:4.3}\] we may rewrite it in the following form,

$$
\frac{1}{k!} \sum_{\pi} (-1)^{\ell(\pi)} \sum_{w \in \mathcal{G}_k} \prod_{i=1}^{k} B_{|A_i|}(w(i) - 1),
$$

where $\pi = \{A_1, \ldots, A_l\}$ ranges over all the irreducible noncrossing partitions of $[k + 1]$ (which in fact have at most $k$ blocks), and $A_i = \emptyset$ if $i > \ell(\pi)$. The second sum in the expression above equals, via identity \[\ref{eq:4.4}\], the following quantity,

$$
\sum_{\tau_1, \ldots, \tau_k \in \mathcal{G}_k} \sum_{w \in \mathcal{G}_k} (w(1) - 1)^{u(\tau_1)} \cdots (w(k) - 1)^{u(\tau_k)} R_{\tau_1}(\lambda) \cdots R_{\tau_k}(\lambda),
$$

where $\tau_i$ ranges over all $NC_{A_i}^{irr}$, with $NC_{\emptyset}^{irr} = \emptyset$. Now, if we set $\tau = \tau_1 \cup \cdots \cup \tau_k$, then $\tau \in NC_{\emptyset}^{irr} \subseteq \pi$ and $R_{\tau}(\lambda) = R_{\tau_1}(\lambda) \cdots R_{\tau_k}(\lambda)$. Finally, $u(\tau_i)$ is the number of singletons in $\tau_i = \tau_{i,1}$, that is the cardinality of the set $A_i \cap \mathcal{U}(\tau)$, which if nonempty is a block of $\pi_\tau$. This completes the proof. \[\Box\]

So, for all integer partitions $\mu$ of size $k + 1$, if $\hat{\mu}$ is obtained from $\mu$ by removing all parts equal to 1, then the monomial $R_{\hat{\mu}} = R_{\mu_1} \cdots R_{\mu_l}$ occurs in $\Sigma_k$ with a nonnegative coefficient. Thanks to \[\ref{eq:4.3}\] and Proposition 3.1 we known that such a coefficient is

$$
\sum_{\tau \in NC_{\hat{\mu}}^{irr}} \sum_{\pi: \tau \subseteq \pi \subseteq \mu} (-1)^{\ell(\pi)} W_\tau(\pi) = \sum_{\tau \in NC_{\hat{\mu}}^{irr}} \sum_{S \subseteq E(\tau)} (-1)^{|E(\tau)| - |S|} W_\tau(S),
$$

where $W_\tau(S) = W_\tau(\pi)$ if $\pi = \pi_S$.

However, via the map $\beta$ the same coefficient can be recovered on the Cayley graph of $\mathcal{G}_k$. Indeed, if $u^* = (1 + k + 1)u$ for all $u \in NC(\mathcal{G}_k)$ then we may rewrite it in the following form,

$$
\sum_{u \in NC(\mathcal{G}_k)} \sum_{u^* \in NC_{\hat{\mu}}^{irr}(S_{\hat{\mu}})} (-1)^{\ell(u)} W_u(u),
$$

with $\ell(u)$ and $W_u(u)$ defined in the suitable way.

Proposition 3.3 provides connections between Kerov polynomials and left-to-right minima and maxima. While, the maps $f$ and $\theta$ give relations between $\Sigma_k$ and descents and excedances, and allow us to recover $\Sigma_k$ from the posets $P_k$ and $Q_k$ of Bóna and Simion.

Now, let $\{x_0, \ldots, x_{k-1}\}$ be a set of commuting variables and consider the polynomial $\Omega_k(x_0, \ldots, x_{k-1})$ defined by

$$
\Omega_k(x_0, \ldots, x_{k-1}) = \frac{1}{k[k+1]} \prod_{j=0}^{k-1} \frac{1 - x_j z}{M(1-x_j z)},
$$
Of course, $\Omega_k$ is symmetric with respect to the $x_i$’s. Moreover, by virtue of (2.3) we obtain $\Omega_k(0, 1, \ldots, k - 1) = \Sigma_k$. A formula for $\Omega_k(x_0, \ldots, x_{k-1})$ is obtained simply by replacing $j$ with $x_j$ in (4.3). More precisely, if $\mu$ is an integer partition of size $k + 1$, then the coefficient of $R^\mu$ in the polynomial $\Omega_k(x_0, \ldots, x_{k-1})$ is given by

$$
\sum_{\tau \in NC^\mu_{(n)}} \sum_{S \subseteq E(\tau)} (-1)^{|E(\tau)| - |S|} W_\tau(S; x_0, \ldots, x_{k-1}),
$$

where $W_\tau(S; x_0, \ldots, x_{k-1})$ is simply obtained by replacing $j$ with $x_j$ in the definition of $W_\tau(S)$. Let $\lambda_\tau(S)$ denote the integer partition corresponding to the type of $\pi_\tau$, with $\pi = \tau S$. Then, it is not difficult to see that the weight (4.5) satisfies

$$
k! W_\tau(S; x_0, \ldots, x_{k-1}) = m(\lambda_\tau(S))! (k - \ell(\lambda_\tau(S)))! m_{\lambda_\tau(S)}(x_0, \ldots, x_{k-1}),$$

$m_{\lambda_\tau(S)}(x_0, \ldots, x_{k-1})$ being the monomial symmetric function indexed by $\lambda_\tau(S)$ [13]. So that the coefficient of $R^\mu$ in $\Omega_k(x_0, \ldots, x_{k-1})$ is a symmetric function of degree $m_1(\mu)$. Denote it by $g^\mu(x_0, \ldots, x_{k-1})$ and assume

$$
g^\mu(x_0, \ldots, x_{k-1}) = \sum_{\lambda} g^{\mu, \lambda} m(\lambda_\tau(S))(x_0, \ldots, x_{k-1}).$$

The left-hand side of (4.5) assures us that, for every $\lambda$ of size $m_1(\mu)$ we have

$$
g^{\mu, \lambda} = \frac{1}{k!} \sum_{\tau \in NC^\mu_{(n)}} \sum_{S \subseteq E(\tau)} (-1)^{|E(\tau)| - |S|} m(\lambda)! (k - \ell(\lambda))!,$$

hence the $g^{\mu, \lambda}$’s are rational numbers. Moreover, since $g^\mu(0, \ldots, k - 1)$ is the coefficient of $R^\mu$ in $\Omega_k$, then the Kerov conjecture implies it is a nonnegative integer. Hence, we may check if the $g^{\mu, \lambda}$’s are nonnegative integers too. This is not true. In fact, we have

$$
g_{(3,1,1,1)} = \frac{14}{5} m_{(1,1,1,1)} - \frac{4}{5} m_{(1,2)} + \frac{4}{5} m_{(3)}.$$

More generally, the expansion of $g_{(3,1,1,1)}(x_0, \ldots, x_{k-1})$ in terms of all the classical basis of the ring of symmetric functions, namely elementary functions $e_\lambda$, complete homogeneous functions $h_\lambda$, power sum functions $p_\lambda$ and Schur functions $s_\lambda$, have rational coefficients which are not positive integers. In fact we have,

$$
g_{(3,1,1,1)} = \frac{14}{5} h_{(1,1,1,1)} - 7 h_{(1,2)} + 5 h_{(3)} = \frac{4}{5} e_{(1,1,1,1)} - 3 e_{(1,2)} + 5 e_{(3)} = \frac{5}{3} p_{(1,1,1,1)} - p_{(1,2)} + \frac{2}{15} p_{(3)} = \frac{4}{5}s_{(1,1,1,1)} - \frac{7}{5} s_{(1,2)} + \frac{14}{5} s_{(3)}.$$
We conclude this paper by stating a second formula expressing $\Sigma_k$ as a weighted sum over the whole $NC_{k+1}$. To this aim, let us introduce the notion of an irreducible component of a noncrossing partition.

Given $\tau \in NC_n$, let $j_1$ be the greatest integer lying in the same block of 1. Set $\tau_1 = \tau_{|j_1}$ so that $\tau_1$ is an irreducible noncrossing partition of $[j_1]$. Now, let $j_2$ be the greatest integer lying in the same block of $j_1 + 1$ and set $\tau_2 = \tau_{|j_1+1,j_2}$. By iterating this process, we determine the sequence of irreducible noncrossing partitions $\tau_1, \ldots, \tau_d$, which we name the irreducible components of $\tau$, such that $\tau = \tau_1 \cup \cdots \cup \tau_d$. For all $\tau \in NC_n$, we denote by $d(\tau)$ the number of its irreducible components. Note that, $d(\tau) = 1$ if and only if $\tau$ is an irreducible noncrossing partition. The proof of the following theorem is omitted.

**Theorem 4.2.** We have

$$\Sigma_k = \sum_{\tau \in NC_{k+1}} \left[ (-1)^{d(\tau)-1} V_\tau \right] R_\tau,$$

where

$$V_\tau = \frac{1}{k} \sum_{i_1 < \cdots < i_d \leq k-1} i_{u(\tau_1)}(i_1) \cdots i_{u(\tau_d)}(i_d),$$

if $d = d(\tau)$.

**References**

[1] P. Biane, *Some properties of crossings and partitions*, Discrete Math. 175 (1997), 41-53.

[2] P. Biane, *Characters of the symmetric group and free cumulants*, Lecture Notes in Math. 1815 (2003), Springer, Berlin, 185-200.

[3] P. Biane, *On the formula of Goulden and Rattan for Kerov Polynomials*, Sém. Lothar. Combin. 55 (2006).

[4] M. Bóna, R. Simion, *A self-dual poset on objects counted by the Catalan numbers and a type-B analogue*, Discrete Math. 220 (2000), 35-49.

[5] M. Dolega, V. Féray and P. Śniady, *Explicit combinatorial interpretation of Kerov character polynomials as number of permutation factorizations*, arXiv: 0810.3209v2 (2008).

[6] E. Di Nardo, P. Petrullo and D. Senato, *Cumulants and convolutions via Abel polynomials*, preprint.

[7] V. Féray, *Combinatorial interpretation and positivity of Kerov’s character polynomials*, J. Algebraic Combin. 29 (2009), 473-507.

[8] V. Féray, *Ph.D. Thesis* (2009), available at [http://feray.fr/valentin/soutenance](http://feray.fr/valentin/soutenance).
[9] I.P. Goulden, A. Rattan, *An explicit form for Kerov’s character polynomials*, Trans. Amer. Math. Soc. 359 (2007), 3669-3685.

[10] S.V. Kerov, talk at IHP Conference (2000).

[11] M. Lassalle, *Two positive conjectures for Kerov polynomials*, Adv. in Appl. Math. 41 (2008), 407-422.

[12] F. Lehner, *Free cumulants and enumeration of connected partitions*, Europ. J. Combin. 22 (2002), 1025-1031.

[13] I.G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford University Press, Oxford (1995).

[14] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge University Press (2006).

[15] R. Speicher, *Multiplicative functions on the lattice on no-crossing partitions and free convolution*, Math. Ann., 298 (1994), 611-628.

[16] R.P. Stanley, *Kerov’s character polynomial and irreducible symmetric group characters of rectangular shape*, Transparencies from a talk at CMS meeting (2002), Quebec City.