The Dynamic Response of an Euler-Bernoulli Beam on an Elastic Foundation by Finite Element Analysis using the Exact Stiffness Matrix

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Abstract. In this study, finite element analysis of beam on elastic foundation, which received great attention of researchers due to its wide applications in engineering, is performed for estimating dynamic responses of shallow foundation using exact stiffness matrix. First, element stiffness matrix based on the closed solution of beam on elastic foundation is derived. Then, we performed static finite element analysis included exact stiffness matrix numerically, comparing results from the analysis with some exact analysis solutions well known for verification. Finally, dynamic finite element analysis is performed for a shallow foundation structure under rectangular pulse loading using trapezoidal method. The dynamic analysis results exist in the reasonable range comparing solution of single degree of freedom problem under a similar condition. The results show that finite element analysis using exact stiffness matrix is evaluated as a good tool of estimating the dynamic response of structures on elastic foundation.

1. Introduction

There are lots of methods for estimating the dynamic behaviour of structures. Especially because of simplicity and convenience, beam on elastic foundation analysis is used to analyse structures such as footings, piles, buried structures, slabs in mechanical and civil engineering. Beam on elastic foundations have been studied extensively over the years [1]. Many research of beam on foundation model, e.g. Winkler foundation, Pasternak foundation, have been combined with the finite element method for considering efficiently lots of loading conditions, shape, dynamic behaviours etc. However, there is not the approach of the finite element method using stiffness matrix, which is derived from the exact solution of beam on elastic foundation, in the dynamic problems.

The objective of this study is estimating the adaptability of exact stiffness matrix in the dynamic finite element analysis of beam on elastic foundation problems. The exact element stiffness of beam on foundation is formulated from the Winkler model [2], which is assumed that the foundation reaction is proportional to the deflection of the footing at the every point [3]. Then, simple structures resting on elastic foundation are analysed by static and dynamic finite element analysis using exact stiffness matrix. Though the number of elements is small and the length of elements is arbitrary, the finite element analysis results using exact stiffness matrix show a good agreement with theoretical or

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reference results. Thus, the use of elements based on exact shape function can be expected to reduce total elements in the analysis.

2. Dynamic FEM Formulation

2.1. Formulation of shape function matrix to the Winkler model
The solution of Euler-Bernoulli beam on Winkler foundation can be described the equation (1) in the form of matrix [2, 4].

\[ u(x) = \phi \mathbf{c} \]  \hspace{1cm} (1)

Where

\[ \phi = [e^{\beta x} \cos \beta x \ e^{\beta x} \sin \beta x \ e^{-\beta x} \cos \beta x \ e^{-\beta x} \sin \beta x], \quad \mathbf{c} = [c_1 \ c_2 \ c_3 \ c_4]^T \]

The equation (1) is expressed as the equation (2) in terms of a nodal displacement vector by applying element boundary conditions, \( u(0) = u_1, u'(0) = u'_1, u(L) = u_2, \) and \( u'(L) = u'_2. \)

\[ u(x) = \phi \mathbf{c} = \phi \mathbf{A}^{-1} \mathbf{d} = \mathbf{N} \mathbf{d} \]  \hspace{1cm} (2)

Where

\[ \mathbf{A} = \begin{bmatrix}
1 & 0 & 1 & 0 \\
\beta & 0 & -\beta & 0 \\
e^{\beta L} \cos \beta L & e^{\beta L} \sin \beta L & e^{-\beta L} \cos \beta L & e^{-\beta L} \sin \beta L \\
\beta e^{\beta L}(\cos \beta L - \sin \beta L) & \beta e^{\beta L}(\sin \beta L + \cos \beta L) & -\beta e^{-\beta L}(\cos \beta L + \sin \beta L) & \beta e^{-\beta L}(\sin \beta L - \cos \beta L)
\end{bmatrix} \]

\[ \mathbf{N} = \phi \mathbf{A}^{-1} \]  \hspace{1cm} (3)

2.2. Dynamic governing equation and exact stiffness matrix for finite element method
As the equation (3) is substituted into the weak form of governing equation of beam on elastic foundation [3, 5], the equation of motion can be derived as the (4) equation.

\[ M \ddot{\mathbf{d}} + C \dot{\mathbf{d}} + K \mathbf{d} = \mathbf{f}_{\text{ext}} \]  \hspace{1cm} (4)

Where \( \mathbf{M}, \mathbf{C}, \mathbf{K}, \) and \( \mathbf{f}_{\text{ext}} \) is mass, damping, stiffness matrix, and nodal force vector, respectively. The stiffness matrix is composed with two parts, beam stiffness \( \mathbf{K}_b \) and ground stiffness \( \mathbf{K}_s \) as the equation (5).

\[ \mathbf{K} = \mathbf{K}_b + \mathbf{K}_s \]  \hspace{1cm} (5)

Where

\[ \mathbf{K}_b = \int_0^L \mathbf{B}^T \mathbf{E} \mathbf{B} \, dx, \quad \mathbf{K}_s = \int_0^L \mathbf{N}^T \mathbf{K}_n \mathbf{N} \, dx, \quad \mathbf{B} = \frac{\partial \mathbf{N}}{\partial x} \]

As substituting the shape function \( \mathbf{N} \) derived in section 2.1 into equation (4), not the 3rd polynomials which is used ordinarily as the shape function, we can obtain the stiffness matrix based on exact solution of the Winkler foundation.
3. Numerical examples

3.1. Problem statement
Dynamic finite element code using the formulation of exact stiffness matrix derived in section 2 is made in MATLAB language. To verify the code and analyze the characteristics of finite element method by using exact stiffness matrix, some numerical examples are analyzed. The reference values are obtained from theoretical solution of Winkler model and results of finite element analysis using 3rd order polynomial shape function [6]. All the examples are assumed the following conditions:

- The material properties of beam and ground are linear-elastic.
- The beam is slender and prismatic, not containing shear deformation.
- Except beams on elastic foundation, other members are frame elements. Flexural and Axial behaviour of the element is described by 3rd polynomials and linear function, respectively.
- The ground region which is not below the flexible beam has small flexural stiffness only.

3.2. Static finite element analysis
To verify our program codes, finite element models of infinite beam on foundation are created. The results are compared with the corresponding theoretical values [2]. The concentric load \( p = 1 \text{kN} \) or moment \( M = 10 \text{kN} \cdot \text{m} \) are loading on the center of infinite beam with the flexural stiffness \( EI = 132.533 \times 10^{-6} \text{m}^4 \), respectively. The beam is also supported by ground with \( k = 43 \text{MPa} \). The models compose 3 finite elements and use symmetric boundary conditions in the each case. Each result is shown in Figure 1. (a) and 1. (b), respectively.

![Figure 1. Comparison with FEM results of infinite beam on elastic foundation](image)

(a) Case 1: Concentric load  
(B) Case 2: Concentric moment

The results of using the exact stiffness matrix show that the variation of beam of deflection is not sensitive to the element length, ratio to flexural and ground stiffness than the case of the ordinary stiffness matrix. However, the results of two cases converge on the theoretical values as increasing the number of elements. Therefore, the number of elements and length is important than the shape function type. The variation of results according to the relative stiffness and element length are listed in the table 1 and 2.
Table 1. Comparisons with the midpoint deflection according to the relative stiffness (case 1)

| Type of Stiffness Element | Relative Stiffness ($\beta$) | Deflection ($\times 10^{-6}$, m) | Error $^b$ (%) | Relative Stiffness ($\beta$) | Deflection ($\times 10^{-6}$, m) | Error $^b$ (%) |
|---------------------------|-------------------------------|-----------------------------------|----------------|-------------------------------|-----------------------------------|----------------|
| Exact                     | 0.1688                        | 2.139                             | 9.0010         | 0.5337                        | 6.206                             | 0.0193         |
| Ordinary $^a$             |                               | 2.135                             | 8.8009         |                               | 5.477                             | 11.7325        |
| Exact                     | 0.3001                        | 3.546                             | 1.6047         |                               | 11.035                           | 0.0001         |
| Ordinary $^a$             |                               | 3.475                             | 0.4184         |                               | 7.247                             | 34.3271        |

$^a$ 3rd polynomial

$^b$ referred to theoretical results

Table 2. Comparisons with the midpoint deflection according to the element length $^b$ (case 1)

| Element length Stiffness type | Displacement ($\times 10^{-6}$, m) | Node #1 | Node #2 | Node #3 | Node #4 | Node #5 | Node #6 |
|-------------------------------|------------------------------------|---------|---------|---------|---------|---------|---------|
| Uniform $^b$                  | Extra                              | 6.206   | 2.899   | 0.225   | 0.276   | 0.116   | 0.070   |
|                              | Ordinary                           | 6.174   | 2.877   | 0.219   | 0.275   | 0.114   | 0.070   |
|                              | Theoretical                        | 6.205   | 4.985   | 2.899   | 1.212   | 0.225   | 0.007   |
|                              | Extra                              | 6.206   | 4.984   | 2.897   | 1.231   | 0.281   | 0.116   |
| Non-uniform $^c$             | Ordinary                           | 6.192   | 4.973   | 2.897   | 1.231   | 0.228   | 0.007   |
|                              | Theoretical                        | 6.205   | 4.984   | 2.899   | 1.213   | 0.228   | 0.070   |
|                              | Extra                              | 6.206   | 0.276   | 0.214   | 0.116   | 0.020   | 0.070   |
| Non-uniform $^d$             | Ordinary                           | 5.104   | 0.679   | 0.347   | 0.099   | 0.073   | 0.216   |
|                              | Theoretical                        | 6.205   | 0.267   | 0.206   | 0.116   | 0.046   | 0.007   |

$^a$ Finite element models have 5 elements. All elements relative stiffness $\beta$ is 0.5337.

$^b$ element dimension : 2/2/2/2/2 (m) $^b$ element dimension : 1/1/1/1/6 (m) $^c$ element dimension : 6/1/1/1/1 (m)

3.3. Dynamic finite element analysis

We test our code by applying the static example and check the sensitivity of the element type in section 3.2. To examine the characteristic of finite element model using exact stiffness matrix, a simple frame structure resting on elastic foundation, as shown in the Figure 2, is considered.

![Figure 2. Simple frame structure supported elastic foundation](image-url)
The horizontal point load is 0.45kN rectangular pulse during 0.01 second. There are properties of the finite element model in the table 3. The time step is $\Delta t = 1 \times 10^{-6}$ second and the observation time is the same the duration of the pulse. To observe how the type of stiffness matrix influences the dynamic response, 3 cases numerical analysis are performed.

- Case 1: Single degree of freedom model with equivalent stiffness and mass (FEM_SDOF)
- Case 2: Finite element model with the exact stiffness matrix (FEM_exact)
- Case 3: Finite element model with the stiffness matrix derived $3^{rd}$ polynomial (FEM_polynomial)

| Elastic modulus (GPa) | Unit weight (kN/m$^3$) | Area (m$^2$) | Moment of inertia of Area (m$^4$) | Ground reaction coefficient (MPa) |
|----------------------|------------------------|-------------|----------------------------------|---------------------------------|
| 210                  | 76                     | 0.025       | 0.004019                         | 14                              |

Figure 3. Comparisons with the numerical results

Figure 4. Displacement according to the number of elements
In the Figure 3, the horizontal displacement of the node 4 at in the model obtained by using the exact stiffness matrix is plotted with the results of other cases. The case 1 can be considered as theoretical values because it is able to be determined analytically. Though it has a few elements in the model, the results of case 2 appear to exist in the reasonable range. Because the both stiffness of case 1 and 2 are derived from the solution of Winker model, this result is seemed to be natural. On the other hands, the results of case 3 have difference with the case 1. It can be considered because the ground stiffness is approximated by 3rd order polynomial shape function, in despite of theoretical vertical and flexural stiffness of beam on foundation is described as combination of exponential functions.

If the model is more discretized, the result of case 3 is close to the results of other cases. In the figure 4, the result of case 3 is improved with the increase of the number of elements of the model, 4, 6, and 12. However, the results of using element stiffness based on the exact solution are not sensitive to the number of elements of model and element length. It can be considered because the exact solution contain exponential function of $\beta x$, especially the relative stiffness ratio $\beta$ is small.

4. Conclusions

We perform the finite element analysis using the stiffness matrix derived from the exact solution of beam on elastic foundation. The stiffness and mass matrix are formulated to make the program code. The code is verified by performing the static finite element analysis of infinite beam on elastic foundation. The results show a good agreement with the theoretical solution of Winker model although our models have a few numbers of elements. The results obtained by using exact stiffness matrix are not sensitive to the number of elements and element length then the results of stiffness matrix using 3rd polynomial in the static and dynamic problems. Therefore, the application of exact stiffness matrix in the dynamic problem can be expected to reduce computational costs, time, and sensitivity of the number of element and size.

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