LOW DIMENSIONAL DISCRIMINANT LOCI AND SCROLLS

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ABSTRACT. Smooth complex polarized varieties \((X, L)\) with a vector subspace \(V \subseteq H^0(X, L)\) spanning \(L\) are classified under the assumption that the locus \(D(X, V)\) of singular elements of \(|V|\) has codimension equal to \(\dim(X) - i\), \(i = 3, 4, 5\), the last case under the additional assumption that \(X\) has Picard number one. In fact it is proven that this codimension cannot be \(\dim(X) - 4\) while it is \(\dim(X) - 3\) if and only if \((X, L)\) is a scroll over a smooth curve. When the codimension is \(\dim(X) - 5\) and the Picard number is one only the Plücker embedding of the Grassmannian of lines in \(\mathbb{P}^4\) or one of its hyperplane sections appear. One of the main ingredients is the computation of the top Chern class of the first jet bundle of scrolls and hyperquadric fibrations. Further consequences of these computations are also provided.

INTRODUCTION

Let \(X\) be an irreducible smooth complex projective variety of dimension \(n\). Given a line bundle \(L\) on \(X\) and a linear system \(|V|\) of dimension \(N\), defined by a vector subspace \(V \subseteq H^0(X, L)\), it is a classical problem to understand the (not necessarily irreducible) subvariety \(D(X, V)\) of \(|V|\) parameterizing the singular elements of \(|V|\). A first result on this respect is Bertini theorem showing that, under the assumption that \(V\) spans \(L\), the general element of \(|V|\) cannot be singular, so that \(D(X, V)\) has positive codimension in \(|V|\). The hypothesis that \(V\) spans \(L\) seems then natural just to have some control on the dimension of \(D(X, V)\) (at least to avoid the possibility of being the whole linear system). Moreover, when \(L\) is spanned by \(V\) a morphism \(\phi_V : X \to \phi_V(X) \subseteq \mathbb{P}^N\) appears and the related geometry enriches the picture. Imposing no other assumption on \(V\), in order to study the singular elements of \(|V|\), we need to deal, on the one hand, with the geometry of \(\phi_V(X) \subset \mathbb{P}^N\) and, on the other, with the geometry of the fibres of \(\phi_V\), see for example [LM1].

To overcome the problem represented by positive dimensional fibres it is natural to impose ampleness on \(L\), so that \(\phi_V\) is a finite morphism. According to [LPS1] these hypotheses, that is, \(L\) ample with \(V\) spanning \(L\), provide a good framework to study the object \(D(X, V)\), called the discriminant locus of the triplet \((X, L, V)\). In fact, in this setting, \(D(X, V)\) reflects a nice stratification of \(X\) determined by the rank of the differential \(d\phi_V\); in particular, a relevant role is played by the geometry of

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\( \phi_V(X) \subseteq \mathbb{P}^N \) and that of the ramification locus of \( \phi \) (for a more precise description see Section 0: (0.1) and (0.2)).

If furthermore \( L \) is very ample with \( \phi_V \) defining an embedding –from now on we refer to this context as the classical case– the discriminant variety is nothing else than the dual variety \( \phi_V(X) \vee \subseteq \mathbb{P}^{N \vee} \), a very classical object in projective geometry. For a survey on the topic of dual varieties see [T].

A series of papers, see [LPS1], [LPS2], [LM2], [LM3], is devoted to investigate to what extent results holding in the classical case are still true in the ample and spanned case. Mostly, these works study the basic invariants, say dimension and degree, of the discriminant locus. Let us focus on the dimension. The hypothesis of \( |V| \) being base point free allows us to write, by Bertini theorem, \( \dim(D(X, V)) = N - 1 - k \) where \( k \) is called, as usual, the (discriminant) defect of \((X, L, V)\). The problem of classifying positive defect triplets appears naturally. In the classical case a deep result by Beltrametti, Fania and Sommese, see [BFS, Thm. 1.2], states that a positive defect variety is always a fibration whose fibers are positive defect varieties with Picard number one and such that their defect is even bigger than that of the original variety. This shows that, when the defect is very big with respect to the dimension, only scrolls can appear, i.e., the fibers of the fibration are linear spaces. A concrete result, always in the classical case, is that if \( k \geq n - 3 > 0 \) then \( \phi_V(X) \) is either \( \mathbb{P}^N \) or a scroll over a smooth curve \( C \). To avoid any confusion in the terminology, we say that \((X, L)\) is a scroll over a smooth variety \( Y \) if there exists a vector bundle \( E \) over \( Y \) such that \( X = \mathbb{P}_{Y}(E) \) and \( L \) is the tautological line bundle.

In [LPS1] and [LM3] it is shown that, in the ample and spanned case, the information on dimension and degree of the discriminant variety is encoded in the Chern classes of the so-called first jet bundle \( J_1(L) \) of \( L \). In particular, positive defect is equivalent to the vanishing of the top Chern class of \( J_1(L) \). Let us consider the classification problem for triplets \((X, L, V)\), with \( L \) just ample and spanned by \( V \), whose defect is big with respect to \( n \). Since scrolls \( X = \mathbb{P}_{Y}(E) \) appear, at least in the classical case, and by some evidences saying that they seem to be the only examples, like [LPS1, Conj. 2.11], we need to compute the top Chern class of \( J_1(L) \) for scrolls. This is done in Section 1, in fact without any assumption on \( L \), and allows us to compute in Section 2, see Proposition 2.1, the defect of scrolls in the range \( n - 2 \dim(Y) \geq -1 \). Also irreducibility of the discriminant locus is shown when \( n - 2 \dim(Y) \geq 0 \) and a description of \( D \) when \((X, L)\) is the conormal variety of a smooth curve is provided. In Section 3, see Theorems 3.1 and 3.2, we prove that if the defect is bigger than or equal to \( n - 3 \) (and positive) then \((X, L)\) either is a scroll over a curve or \((X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\). Moreover, if the defect is equal to \( n-4 > 0 \) and the Picard number of \( X \) is one then \((X, L)\) is either \((\mathbb{G} (1,4), L)\), where \( \mathbb{G}(1,4) \) is the Grassmannian of lines in \( \mathbb{P}^4 \) and \( L \) defines the Plücker embedding \( \mathbb{G}(1,4) \subseteq \mathbb{P}^9 \), or there exists \( H \in |L| \) such that \((X, L) = (H, L_H)\), where \( L_H \) stands for the restriction of \( L \) to \( H \).

Our results of Section 3 rely on two basic facts. First one is the existence

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of linear systems $|W| \subset |V|$ not meeting the discriminant, that is, not containing any singular element. These systems produce exact sequences involving $J_1(L)$ that lead to the computation of its Chern classes. Second one is adjunction theory, more specifically, the classification of polarized varieties $(X, L)$ such that $K_X + (n-i)L$ is not ample for $i = 0, 1, 2$, see [I] and [F2]. For instance, the use of adjunction makes hyperquadric fibrations over a curve enter into the picture. This is the reason why we need to compute the top Chern class of $J_1(L)$ also for such polarized varieties. This computation is done in Section 1 and applied to exclude the possibility that defect is $n-3$ in Section 3. Here we embed the hyperquadric fibration into a scroll, fiberwise, as a divisor.

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0. Background material

(0.0) Let $(X, L, V)$ be a triplet where: $X$ is an irreducible smooth complex projective variety of dimension $n$, $L$ is an ample and spanned line bundle on $X$ and $V \subseteq H^0(X, L)$ spans $L$. Write $\dim(V) = N + 1$ and let $\phi_V : X \to \mathbb{P}^N$ (respectively $\phi_L$) be the morphism defined by $V$ (respectively by $H^0(X, L)$).

We use standard notation in algebraic geometry and the base field is always $\mathbb{C}$. In particular, $K_X$ will denote the canonical bundle of $X$.

(0.1) The discriminant locus $D(X, V)$ of the triplet $(X, L, V)$ parameterizes the singular elements of $|V|$. More precisely, taking the incidence correspondence

$$\mathcal{Y} := \{(x, [s]) \in X \times |V| : j_1(s)(x) = 0\} \xrightarrow{p_1} X \xrightarrow{p_2} D(X, V) \subset \mathbb{P}^{N^\vee},$$

where $j_1(s)$ denotes the first jet of the section $s \in V$, $D(X, V)$ is the image of $\mathcal{Y}$ via the second projection of $X \times |V|$. Thus $D(X, V)$ is an algebraic variety in $|V| = \mathbb{P}^{N^\vee}$. By Bertini theorem, $\dim(D(X, V)) < N$. Hence, we can write $\dim(D(X, V)) = N - 1 - k$ where $\text{def}(X, V) := k \geq 0$ is called the (discriminant) defect of $(X, L, V)$. In particular $\text{def}(X, V)$ is the dimension of the largest linear projective space of $|V|$ not meeting $D(X, V)$.

If $\phi_V(X) \neq \mathbb{P}^N$ then the dual variety $\phi_V(X)^\vee \subset \mathbb{P}^{N^\vee}$ is a non-empty irreducible subvariety of $D(X, V)$. Furthermore, if $\phi_V$ is an immersion then $\phi_V(X)^\vee = D(X, V)$, see [LPS1, Rmk. 2.3.3]. Anyway, points in $D(X, V) \setminus \phi_V(X)^\vee$ are coming from points on $X$ where the differential of $\phi_V$ is not injective. In this context it is natural to define the jumping sets $J_i$ $(1 \leq i \leq n)$ as in [LPS1, (1.1)]:

$$J_i(X, V) = \{x \in X : \text{rk}(d\phi_V(x)) \leq n - i\}.$$
As in [LPS2, (0.3.1)] $X_i$ stands for $\mathcal{J}_i \setminus \mathcal{J}_{i+1}$, with the convention that $\mathcal{J}_0 = X$ and $\mathcal{J}_{n+1} = \emptyset$. This allows to define $\mathcal{D}_i(X, V) \subseteq \mathcal{D}(X, V)$ as $p_2 \circ p_1^{-1}(X_i)$, that is, the Zariski closure in $\mathbb{P}^N$ of the locus of elements of $|V|$ singular at points of $X_i$. This gives a sort of stratification of the discriminant locus, say

$$\mathcal{D}(X, V) = \bigcup_{i=0}^n \mathcal{D}_i(X, V),$$

and we can define $\text{def}_i(X, V) = N - 1 - \dim(\mathcal{D}_i(X, V))$.

Let us observe that, as a consequence of the Bertini theorem, for any choice of a vector subspace $W \subset V$ spanning $L$ we get $\text{def}(X, V) = \text{def}(X, W)$, see [LPS1, Thm. 2.7]. Hence, as suggested after (2.7.2) in [LPS1], we can write $\text{def}(X, L)$ instead of $\text{def}(X, V)$, this value being independent of the choice of $V$ spanning $L$. On the contrary the jumping sets are tightly related to the choice of the linear system $|V|$, think, for example, about projections of embedded projective varieties.

As shown in the definition of the discriminant locus and as will be shown in (0.4) the first jet bundle $J_1(L)$ is a very remarkable object in the study of the dimension and degree of the discriminant locus. For the sake of completeness let us recall briefly its construction, see [LPS1, (0.3)] and reference therein. Let $x \in X$ and let $m_x$ be the maximal ideal of the stalk $\mathcal{O}_{X,x}$ at $x$ of the structure sheaf of $X$. Then $m_x^2L_x$ is the $\mathcal{O}_{X,x}$ submodule of $L_x$ of the germs of sections of $L$ whose local expansion at $x$ has no terms of degree less than or equal to one. The quotient $J_1(L)_x = L_x/m_x^2L_X$ is a $\mathbb{C}$-vector space of dimension $n + 1$ and $J_1(L)$ is the collection $\cup_{x \in X} J_1(L)_x$ equipped with the unique holomorphic vector bundle structure inducing on $J_1(L)_x$ the preexistent $\mathbb{C}$-vector space structure. Locally near $x$ a section of $J_1(L)$ is given by a pair $(s, ds)$ where $s$ is a local section of $L$ and $ds$ is the differential of $s$ with respect to local coordinates of $X$ around $x$. The map sending $(s, ds)$ to $s$ gives rise to the following exact sequence, where $\Omega_X$ is the contangent bundle:

$$0 \to \Omega_X(L) \to J_1(L) \to L \to 0.$$

(0.4) The dimension and the degree of the discriminant locus can be computed by means of the Chern classes of the first jet bundle. In fact, let us recall that $\text{def}(X, L) \geq r$ if and only if $c_{n-r+1}(J_1(L)) = 0$, see [LPS1, Thm. 2.7]. In particular, for $(X, L)$ to be defective the top Chern class $c_n(J_1(L))$ must vanish. On the other hand, when $(X, L)$ is not defective, $c_n(J_1(L))$ can be interpreted in terms of the degree of the discriminant locus, usually called codegree and denoted by $\text{codeg}(X, L)$. For this interpretation see [LM3, Thm. 5.2]. Just recall here that if the general section in any maximal dimensional component of $\mathcal{D}(X, L)$ is singular at a single point and the singularity is ordinary quadratic of maximal rank then $c_n(J_1(L)) = \text{codeg}(X, L)$. When equality $c_n(J_1(L)) = \text{codeg}(X, L)$ holds, for example when $L$ is very ample, we say that $(X, L)$ has tame codegree, see [LM3, Def. 10.1].
1. **Top Chern classes of jet bundles**

As said in (0.4), to understand the dimension and degree of the discriminant locus of a triplet \((X, L, V)\) as in (0.0) we need to compute the top Chern class \(c_n(J_1(L))\) of the first jet bundle. This section is devoted to compute this Chern class for scrolls and for hyperquadric fibrations over curves. In the case of scrolls no assumption on the tautological line bundle is needed to do the computation. However, in Section 2 we will assume ampleness and spannedness to get nice results on the dimension of the discriminant locus of scrolls. In the case of hyperquadric fibrations we compute the top Chern class of the first jet bundle embedding our variety as a divisor into a scroll.

First we compute \(c_n(J_1(L))\) for scrolls. Recall the usual convention that \(\binom{m}{n} = 0\) if either \(n < 0\) or \(n > m\).

**Proposition 1.1.** Let \(Y\) be an irreducible smooth projective variety of dimension \(m\) and \(E\) a vector bundle of rank \(r\) on \(Y\). Consider the projective bundle \(\pi : \mathbb{P}_Y(E) \to Y\) and the corresponding tautological line bundle \(L\). Let \(n = m + r - 1\) be the dimension of \(X := \mathbb{P}_Y(E)\). In these conditions:

\[
c_n(J_1(L)) = \sum_{s_1, s_2 \geq 0, s_1 + s_2 \leq m} A_{s_1, s_2} L^{n-s_1-s_2} \pi^*(E^\vee) \pi^* c_{s_1}(T_Y)
\]

where

\[
A_{s_1, s_2} = \sum_{t=0}^{n} (-1)^t (n + 1 - t) \binom{r-s_1}{t-s_1-s_2}.
\]

**Proof.** By (0.3), see [BS, Lemma 1.6.4], we know that \(c_n(J_1(L)) = \sum_{t=0}^{n} (n + 1 - t)(-1)^t c_t(T_X) L^{n-t}\).

Consider the relative Euler and tangent exact sequences:

\[
0 \to \mathcal{O}_X \to \pi^*(E^\vee) \otimes L \to T_{X/Y} \to 0,
\]

\[
0 \to T_{X/Y} \to T_X \to \pi^* T_Y \to 0.
\]

By (1.1.1) and (1.1.2) we get:

\[
c_n(J_1(L)) = \sum_{t=0}^{n} (n + 1 - t)(-1)^t c_t(T_X) L^{n-t}.
\]

We end the proof computing the coefficient of \(L^{n-s_1-s_2} \pi^* c_{s_1}(E^\vee) \pi^* c_{s_2}(T_Y)\) in \(c_n(J_1(L))\) after substituting \(c_i(\pi^*(E^\vee) \otimes L)\) in (1.1.3) by

\[
c_i(\pi^*(E^\vee) \otimes L) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(\pi^*(E^\vee)) L^{i-j}.
\]

We can derive some consequences of Proposition 1.1 that will be useful in Section 2.
Proposition 1.2. In the conditions of (1.1) we get:

(1.2.1) if \( r \geq m + 2 \) then \( c_n(J_1(L)) = 0 \),

(1.2.2) if \( r = m + 1 \) then \( c_n(J_1(L)) = c_m(E) \),

(1.2.3) if \( r = m \) then \( c_n(J_1(L)) = c_{m-1}(E)(c_1(E) + K_Y) + mc_m(E) \).

Proof. Consider \( f : \mathbb{Z}^4 \to \mathbb{Z} \) defined as

\[
f(m, r, s_1, s_2) = \sum_{t=0}^{m+r-1} (-1)^t(m + r - t) \left( \frac{r - s_1}{t - s_1 - s_2} \right).
\]

We claim that:

(1.2.4) if \( m, r, s_1, s_2 \geq 0 \), \( s_1 + s_2 \leq m \) and \( r \geq m + 2 \) then \( f(m, r, s_1, s_2) = 0 \),

(1.2.5) if \( m, r, s_1, s_2 \geq 0 \), \( s_1 + s_2 \leq m \) and \( r = m + 1 \) then \( f(m, m+1, s_1, s_2) = 0 \) unless \( s_1 = m \) and \( s_2 = 0 \), in which case \( f(m, m+1, m, 0) = (-1)^m \),

(1.2.6) if \( m, r, s_1, s_2 \geq 0 \), \( s_1 + s_2 \leq m \) and \( r = m \) then \( f(m, m, s_1, s_2) = 0 \) unless \( (s_1, s_2) \in \{(m,0), (m-1,1), (m-1,0)\} \), in which cases \( f(m, m, m, 0) = (-1)^m m \), \( f(m, m, m-1, 1) = (-1)^m \), \( f(m, m, m-1, 0) = (-1)^{m+1} \).

Let us observe that (1.2.1) follows directly from (1.2.4) and (1.1). From (1.2.5) and (1.1) we get \( c_n(J_1(L)) = (-1)^m L^{m-n} c_m(\pi^*(E^\vee)) = c_m(E) \) so that (1.2.2) holds. From (1.2.6), (1.1) and basic properties of Chern classes we get:

\[
c_n(J_1(L)) = L^m \pi^* c_{m-1}(E) - L^{m-1} \pi^* c_{m-1}(E) \pi^* c_1(T_Y) + mL^{m-1} \pi^* c_m(E).
\]

Finally use the Chern–Wu relation \( \sum_{i=0}^{m} (-1)^i \pi^* c_i(E) L^{m-i} = 0 \) to substitute \( L^m \) in the previous formula and get (1.2.3).

Let us now prove (1.2.4), (1.2.5) and (1.2.6). The proof is based on two basic formulae, holding for \( m \geq 2 \):

(1.2.7) \[
\sum_{t=0}^{m-1} (-1)^t \binom{m - 1}{t} = \sum_{t=0}^{m} (-1)^t \binom{m}{t} = 0.
\]

First consider the decomposition of \( f(m, r, s_1, s_2) \) as a difference:

\[
\sum_{t=0}^{m+r-1} (-1)^t(m + r) \left( \frac{r - s_1}{t - s_1 - s_2} \right) - \sum_{t=0}^{m+r-1} (-1)^t \left( \frac{r - s_1}{t - s_1 - s_2} \right).
\]

Now make the change of variables \( z = t - s_1 - s_2 \). If \( (*) s_2 \neq m \) and \( (**) r - s_1 \geq 2 \) then by (1.2.7)

\[
0 = (-1)^{s_1 + s_2} f(m, r, s_1, s_2) = (m + r) \sum_{z=0}^{r-s_1} (-1)^z \binom{r - s_1}{z} - \sum_{z=0}^{r-s_1} (-1)^z (z + s_1 + s_2) \binom{r - s_1}{z}.
\]
If $r \geq m + 2$ then (**) holds and (*) holds unless $(s_1, s_2) = (0, m)$. Thus, to obtain (1.2.4) we only need to check that $f(m, r, 0, m) = 0$, which is an easy computation. If $r = m + 1$ then (**) holds unless $s_1 = m$. Thus, we obtain (1.2.5) just checking that $f(m, m + 1, m, 0) = (-1)^m$ and $f(m, m + 1, 0, m) = 0$. Finally, if $r = m$ then (**) holds unless $s_1 = m$ or $s_1 = m - 1$. We get (1.2.6) just checking that $f(m, m, m, 0) = (-1)^m m$, $f(m, m, m - 1, 1) = (-1)^m$, $f(m, m, m - 1, 0) = (-1)^{m+1}$ and $f(m, m, 0, m) = 0$. \(\square\)

Let us do a couple of remarks on this proposition. First we note that (1.2.3) is related with the invariant $v(Y, E)$ defined by Fukuma, see [Fu, Section 4]:

$$v(Y, E) = 1 + \frac{1}{2}((m - 2)c_m(E) + (K_Y + c_1(E))c_{m-1}(E)).$$

By (1.2.3) we get the following formula relating the top Chern classes of $J_1(L)$ and $E$ with the invariant $v(Y, E)$:

(1.2.8)  
$$c_n(J_1(L)) = 2v(Y, E) - 2 + 2c_m(E),$$

which will be of interest in (2.1.2).

Second we note that the computation of $c_n(J_1(L))$ can be done for $r < m$. However, formulae are more involved and difficult to control. For example, if $r = m - 1$ there are six non-zero coefficients in the expression of $c_n(J_1(L))$. Let us recall that $c_n(J_1(L)) \geq 0$, see [LPS1, Cor 2.6], for $L$ ample and spanned. Hence, these formulae provide non-negative expressions involving the Chern classes of $E$ and $T_Y$.

This concludes our computations on scrolls. Let us start the computation for hyperquadric fibrations over curves. We can use the following notion of hyperquadric fibration over a smooth curve, see [F1, §3].

**Definition.** Let $X$ be a smooth projective variety of dimension $n$ and $L$ an ample line bundle on $X$. The pair $(X, L)$ is a hyperquadric fibration over a smooth curve $B$ if there exists a morphism $f : X \to B$ such that any general fiber $F$ of $f$ is a smooth hyperquadric of $\mathbb{P}^n$ and $L$ induces the hyperplane bundle on $F$, i.e., $L_F = \mathcal{O}_F(1)$.

As in [F1, (3.2)] it is a natural construction to take the push-forward $E = f_* L$ which is a vector bundle of rank $n + 1$ onto $B$. This allows us to regard $X$ as a divisor in the projective bundle $\pi : \mathbb{P}_B(E) \to B$. In fact, denoting by $\xi$ the tautological line bundle on $\mathbb{P}_B(E)$, there exists a divisor $\beta \in \text{Pic}(B)$ of degree $b$ such that $X \in |2\xi - \pi^* \beta|$ and $L = \xi_X$. Set $e = c_1(E)$ and recall that $L^n = 2e - b$.

**Proposition 1.3.** Let $(X, L)$ be a hyperquadric fibration $f : X \to B$ over a smooth curve $B$ of genus $g$. Let $n = \dim(X)$ and let $e$ and $b$ be as before. Then $c_n(J_1(L)) = 2Ae - Bb + 4C(1 - g)$, being

$$A = \sum_{t=0}^n (-1)^t(n + 1 - t)(\sum_{i=0}^t (-1)^i 2^i \binom{n}{t - i}),$$

which...
\begin{align*}
B &= \sum_{t=0}^{n} (-1)^t (n + 1 - t) \left( \sum_{i=0}^{t} (-1)^i 2^i \binom{n+1}{t-i} \right), \\
C &= \sum_{t=0}^{n} (-1)^t (n + 1 - t) \left( \sum_{i=0}^{t} (-1)^i 2^i \binom{n+1}{t-i-1} \right).
\end{align*}

**Proof.** For simplicity write \( P = \mathbb{P}_B(E) \) where \( E = f_*L \). The exact sequence associated to the embedding of \( X \) as a divisor in \( P \) is 0 \( \rightarrow T_X \rightarrow (T_P)_X \rightarrow \mathcal{O}_X(2\xi - \pi^*\beta) \rightarrow 0 \) and produces the following recursion law \( c_t(T_X) = c_t(T_P)_X - c_{t-1}(T_X)(2\xi - \pi^*\beta)_X \), for \( t = 1, \ldots, n \). This leads to the formula

\begin{equation}
(1.3.1) \quad c_t(T_X) = \sum_{i=0}^{t} (-1)^t c_{t-i}(T_P)_X (2\xi - \pi^*\beta)_X^i.
\end{equation}

Now observe that \( (2\xi - \pi^*\beta)_X = 2^i L^i - ib2^{i-1}L^{i-1}F \) and use the exact sequences on \( P \) analogous to (1.1.2) to get

\[ c_s(T_P)_X = (c_{s-1}(\pi^*(E^\vee) \otimes \xi)c_1(\pi^*(T_B)))_X + c_s(\pi^*(E^\vee) \otimes \xi)_X, \]

for \( s = 1, \ldots, n \). Since

\[ c_s(\pi^*(E^\vee) \otimes \xi)_X = \binom{n+1}{s} L^s - e \binom{n}{s-1} FL^{s-1} \]

we can substitute in \( c_s(T_P)_X \) to get

\[ c_s(T_P)_X = \binom{n+1}{s} L^s - e \binom{n}{s-1} FL^{s-1} + 2(1-g) \binom{n+1}{s-1} L^{s-1}F. \]

Then (1.3.1) gives:

\[ c_t(T_X) = \sum_{i=0}^{t} \left[ \binom{n+1}{t-i} 2^i L^t - \binom{n+1}{t-i} 2^{i-1} b L^{t-1}F \right] \]

\[ e \binom{n}{t-i-1} 2^i L^{t-1}F + 2^{i+1}(1-g) \binom{n+1}{t-i-1} L^{t-1}F. \]

Finally we use (1.1.1) together with \( L^{n-1}F = 2 \) and \( L^n = 2e - b \) to get the result. \( \square \)

As an example, let us write the formula for \( n = 4 \). We will use it in (3.2.3). Let \( (X, L) \) be as in (1.3). If \( \dim(X) = 4 \) then, with the same notation as there, \( A = 4, B = 16 \) and \( C = -1 \), so that \( c_4(J_1(L)) = 8e - 16b - 4(1-g) \). In fact we can rewrite it in the following equivalent way

\begin{equation}
(1.4) \quad c_4(J_1(L)) = (2e - b) + 3(2e - 5b) - 4(1-g),
\end{equation}

where \( 2e - b = L^4 > 0 \) by the ampleness of \( L \) and \( 2e - 5b \geq 0 \) because it corresponds to the number of singular fibers of \( f \), see [F1, (3.3)].
2. Discriminant loci of scrolls

Let us study the discriminant loci of scrolls. Recall the notation established in the introduction: a scroll \((X, L)\) is a projective bundle \(\pi : X \to Y\) polarized by its tautological line bundle, i.e., \(X = \mathbb{P}_Y(E)\) where \(E\) is an ample vector bundle of rank \(r\) on \(Y\) and \(L\) is the tautological line bundle. It is well known in the classical case (i.e., when \(L\) is very ample, see for example, [T, Thm. 7.21, p. 129]) that the defect of a scroll of dimension \(n\) over a \(m\)-dimensional base is greater than or equal to \(n - 2m\) and in fact equal when \(n - 2m \geq 0\). Using the computations of Proposition 1.2, since the vanishing of \(c_n(J_1(L))\) is needed to have positive defect, we give a proof of this fact which works in the ample and spanned setup. We can also deal with the case \(n - 2m = -1\) using the reformulation of (1.2.3) stated in (1.2.8) combined with a result of Wisniewski [Wi, Thm. 3.4] classifying ample and spanned vector bundles with \(c_m(E) = 1\). On the other hand, a result on the rank of this type of vector bundles (apparently implicit in Wisniewski’s result) can be proved by our previous results as we show in the following remark.

**Remark 2.0.** Let \(Y\) be an irreducible smooth projective variety of dimension \(m\) and \(E\) a rank \(r\) vector bundle on \(Y\). If \(E\) is ample and spanned and \(c_m(E) = 1\) then \(r = m\). In fact, if \(r < m\) then \(c_m(E) = 0\). If \(r > m+1\) then we can choose \(r - (m+1)\) general independent sections which give rise a homomorphism \(O_{\mathbb{P}_N}^{r-(m+1)} \to E\) of maximal rank in any fibre. This produces an exact sequence

\[
0 \to O_{\mathbb{P}_N}^{r-(m+1)} \to E \to Q \to 0,
\]

where \(Q\) is a vector bundle which is ample and spanned, so being \(E\). Consider the pair \((X = \mathbb{P}_Y(Q), L)\) where \(L\) is the tautological line bundle and write \(n = \dim(X)\). By (1.2.2) we get \(c_n(J_1(L)) = c_m(E) = 1\), that is, see [LM3, Thm. 5.2], codeg\((X, L) = 1\), contradicting [LM3, Thm. 5.13].

Let us introduce a bit of notation: Let \(Y\) be an irreducible smooth projective variety of dimension \(m\), \(E\) an ample vector bundle of rank \(r > 1\) on \(Y\) and \(X = \mathbb{P}_Y(E) \xrightarrow{\pi} Y\), \(\dim(X) = n\). Suppose that the tautological line bundle \(L\) is spanned by \(V \subseteq H^0(X, L)\), \(\dim(V) = N + 1\). Consider the following incidence variety \(I \subset |V| \times Y:\)

\[
I = \{(H, y) : \pi^{-1}(y) \subset H\} \xrightarrow{p_1} |V| = \mathbb{P}^N
\]

**Proposition 2.1.** With the notation of (2.1.0) we get that:

(2.1.1) If \(n - 2m \geq 0\) then \(\text{def}(X, L) = n - 2m\), \(p_1\) is generically finite and \(\mathcal{D}(X, V) = p_1(I)\).

(2.1.2) If \(n - 2m = -1\) and \(m \geq 3\) then \(\text{def}(X, L) = 0\) unless \(Y = \mathbb{P}^m\) and \(E = O_{\mathbb{P}_m}(1)^{\oplus m}\), in which case \((X, L) = (\mathbb{P}^m \times \mathbb{P}^{m-1}, O_{\mathbb{P}_m \times \mathbb{P}^{m-1}}(1, 1))\).
Proof. Let us first show the inequality \( \text{def}(X, L) \geq n - 2m \). Since \( \text{def}(X, L) \geq 0 \), we can suppose \( n > 2m \). By [LPS1, Thm. 2.7] it is enough to prove that \( c_{2m+1}(J_1(L)) = 0 \). Consider a general element \( X_1 \in |L| \) and denote \( L_1 = L|_{X_1} \). In [LM2, Lemma 1.13] it is shown that \( c_{2m+1}(J_1(L_1)) = c_{2m+1}(J_1(L))|_{X_1} \) if \( n > 2m + 1 \). Hence we can suppose \( n = 2m + 1 \), just taking general elements of \( |L| \). We thus conclude by (1.2.1).

If the strict inequality \( \text{def}(X, L) > n - 2m \geq 0 \) holds then \( c_{2m}(J_1(L)) = 0 \), again by [LPS1, Thm. 2.7]. Exactly as in the previous paragraph we can suppose \( n = 2m \) (just taking general elements of \( |L| \)), i.e., \( r = m + 1 \). By (1.2.2) we get \( 0 = c_{2m}(J_1(L)) = c_m(E) \), which contradicts the ampleness of \( E \), see [BG, Cor. 1.2]. This shows the first part of (2.1.1).

Now we observe that the incidence variety \( I \) of (2.1.0) is a projective bundle over \( Y \) whose fiber is a linear space of dimension \( N - 1 - (n - m) \), in particular it is irreducible and of dimension \( N - 1 - (n - 2m) \). Hence \( p_1(I) \) is irreducible and \( \dim(p_1(I)) \leq N - 1 - (n - 2m) \). Furthermore, since \( r > 1 \), \( \mathcal{D}(X, V) \subseteq p_1(I) \) so that \( \dim(\mathcal{D}(X, V)) \leq N - 1 - (n - 2m) \) or, equivalently, \( \text{def}(X, L) \geq n - 2m \). Moreover, the equality \( \text{def}(X, L) = n - 2m \) implies that all inequalities \( \dim(\mathcal{D}(X, V)) \leq \dim(p_1(I)) \leq \dim(I) = N - 1 - (2n - m) \) are equalities so that \( p_1 \) is generically finite and \( \mathcal{D}(X, V) = p_1(I) \). This completes the proof of (2.1.1).

If \( n - 2m = -1 \) then \( r = m \). Hence, by (1.2.8), we get \( c_n(J_1(L)) = 2v(Y, E) - 2 + 2c_m(E) \). Since \( m \geq 3 \), \( v(Y, E) \geq 0 \), see [Fu, Thm. 4.1], and \( c_m(E) > 0 \) by [BG, Cor. 1.2]. Hence \( c_n(J_1(L)) = 0 \) gives \( v(Y, E) = 0 \) and \( c_m(E) = 1 \). Then (2.1.2) follows from [Wi, Thm. 3.4]. \( \square \)

Remark 2.2. Let us point out that the inclusion \( \mathcal{D}(X, V) \subseteq p_1(I) \) provides a geometrical proof of the inequality \( \text{def}(X, L) \geq n - 2m \). On the other hand, since \( p_1(I) \) is irreducible, in the conditions of (2.1.1) the discriminant \( \mathcal{D}(X, V) \) is irreducible. In particular, \( \mathcal{I}_n(X, V) = \emptyset \).

Let us observe that if \( n - 2m < -1 \) then \( \dim(I) = N - 1 - (n - 2m) > N \). Hence \( p_1 \) cannot be generically finite. In the following example \( n - 2m = -1 \) and \( p_1(I) \subsetneq \mathcal{D}(X, V) \), showing the necessity of the hypothesis \( n - 2m \geq 0 \) in (2.1.1).

Example 2.3. Consider the Segre embedding of the product \( \mathbb{P}^1 \times \mathbb{P}^2 \) in \( \mathbb{P}^5 \) as a scroll over \( \mathbb{P}^2 \). Here \( X = \mathbb{P}_2(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) \) and the tautological line bundle \( L \) is very ample. Then \( \dim(I) = 5 \) and \( X^V = \mathcal{D}(X, L) \subsetneq p_1(I) = (\mathbb{P}^5)^V \).

The hypothesis \( r > 1 \) on the rank of \( E \) in Proposition 2.1 is crucial to get the inclusion \( \mathcal{D}(X, V) \subseteq p_1(I) \). The following construction will allow us to produce examples (see Example 2.4.3) in which \( r = 1 \) and \( \mathcal{D}(X, V) \) is not contained in \( p_1(I) \), being in particular reducible. Let us also point out that the following construction is of interest by its own. In fact it relates the second order infinitesimal information of a variety \( Y \subset \mathbb{P}^N \) with the discriminant locus of a special scroll over \( Y \), namely its conormal variety, see [R].
Consider an irreducible non-degenerate smooth projective variety \( Y \subset \mathbb{P}^N \) with \( \dim(Y) = m \), such that the twist of the normal bundle \( \mathcal{N}_{Y/\mathbb{P}^N}(-1) \) is ample. For instance, this assumption is satisfied when \( m = 1 \); or when \( m = 2 \) and \( \text{Pic}(Y) \cong \mathbb{Z}H \) (\( H \) being the hyperplane section); or when \( Y \subset \mathbb{P}^N \) is a complete intersection. As in [LM2, Example 3.2] consider the conormal variety \( X = \mathbb{P}(\mathcal{N}_{Y/\mathbb{P}^N}(-1)) \subset \mathbb{P}^N \times \mathbb{P}^{N^\vee} \) (note that \( n := \dim(X) = N - 1 \)) and the corresponding projections \( \pi \) and \( \pi_2 \):

\[
\mathbb{P}^N \times \mathbb{P}^{N^\vee} \supset \mathbb{P}(\mathcal{N}_{Y/\mathbb{P}^N}(-1)) = \begin{array}{c}
X \\
\downarrow \pi_2
\end{array} \quad Y \subset \mathbb{P}^N \quad \mathbb{P}^{N^\vee} \supset Y^\vee
\]

where, by definition, \( \pi_2(X) = Y^\vee \). The following triplet is as in (0.0):

\[ (X, L = \pi_2^* \mathcal{O}_{\mathbb{P}^{N^\vee}}(1), V = \pi_2^* H^0(\mathbb{P}^{N^\vee}, \mathcal{O}_{\mathbb{P}^{N^\vee}}(1))). \]

**Remark 2.4.1.** For a triplet as in (2.4) we have \( \mathcal{D}_0(X, V) = (Y^\vee)^\vee = Y \subset \mathcal{D}(X, V) \), so that \( \text{def}_0(X, V) = N - 1 - \dim(\mathcal{D}_0) = N - 1 - m = n - m \). Moreover, by (2.1.1), if \( N - 1 \geq 2m \) then \( \text{def}(X, L) = n - 2m \). Since \( n - m > n - 2m \), we conclude that when \( N \geq 2m + 1 \) there is a strict inclusion \( \mathcal{D}_0(X, V) \subset \mathcal{D}(X, V) \).

The following lemma relates the jumping sets \( \mathcal{J}_i(X, V) \) of a triplet as in (2.4) with the hyperplane sections of \( Y \subset \mathbb{P}^N \) whose singularities are not general, i.e., not ordinary quadratic of maximal rank. This shows the interaction between the second order infinitesimal information of \( Y \subset \mathbb{P}^N \) and the discriminant locus of its conormal variety.

**Lemma 2.4.2.** For \( (X, L, V) \) as in (2.4) it follows that \( \mathcal{J}_i = \{(y, H) \in X : \text{the quadratic part of the defining equation of } Y \cap H \text{ locally at } y \text{ has rank } \leq m - i \} \).

**Proof.** Recall that, by definition, \( \mathcal{J}_i = \{(y, H) \in X : \text{rank}(d\pi_2(y, H)) \leq \dim(X) - i \} \). Then the lemma is a consequence of the local expression of \( \pi_2 \) at \( (y, H) \), see for example [T, pp. 58–59] or [R, Sect. 2]. \( \square \)

Here is the promised example.

**Example 2.4.3.** Take \( Y \subset \mathbb{P}^2 \) a plane curve and the triplet \( (X, L, V) \) as in (2.4). Using the description of \( \mathcal{J}_1 \) of Lemma 2.4.2 we get that \( \mathcal{J}_1 = \{(y, H) \in X : \text{the quadratic part of the defining equation of } Y \cap H \text{ locally at } y \text{ vanishes} \} \). Then, the description of the discriminant locus of (0.2) is the following: \( \mathcal{D}(X, V) = \mathcal{D}_0(X, V) \cup \mathcal{D}_1(X, V) \) where \( \mathcal{D}_0(X, V) = Y \) (see Remark 2.4.1) and \( \mathcal{D}_1(X, V) \) is the union of all inflectional tangent lines to \( Y \). Any of these lines corresponds to the pencil of lines through the cusp of \( Y^\vee \) dualizing a flex of \( Y \). In particular \( \mathcal{D}(X, V) \) is reducible.

In connection with Example 2.4.3, let us include a remark about the degree of the discriminant. Since in this example \( X \) is isomorphic to \( Y \) then \( L = (d - 1)H \), where \( d = \deg(Y) \) and \( H \) defines the embedding \( Y \subset \mathbb{P}^2 \). Hence \( c_1(J_1(L)) = \)

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As an application of Proposition 2.1 we can completely describe the discriminant locus for a triplet as in (2.4) in the range where 2.1 is meaningful.

**Proposition 2.4.4.** Let \((X, L, V)\) be a triplet as in (2.4). If \(N \geq 2m + 1\) then the discriminant locus \(D(X, V)\) can be identified with the tangent developable \(TY \subset \mathbb{P}^N\).

**Proof.** We use the identification between \(D(X, V)\) and \(p_1(I)\) of (2.1.1). For \(y \in Y\) we have that \(\pi^{-1}(y) = \{H \in \mathbb{P}^{N \vee} : T_{Y,y} \subset H\}\) by definition of conormal variety. Under the natural identification between \(\mathbb{P}^{N \vee} \) and \(\mathbb{P}^N = |V|\) the set \(\{\pi^{-1}(y) \subset H\} \subset |V| \times Y\) is sent onto \(T_{Y,y}\) by \(p_1\). This shows that \(TY = D(X, V)\). \(\square\)

**3 Characterization of \(\text{def}(X, L) > n - 4\)**

In the classical context, varieties with big defect with respect to their dimension tend to be scrolls. To be precise, if \(L\) is very ample and \(\phi_V\) is an embedding, then \(\text{def}(X, L) \geq n - 3 > 0\) if and only if \((X, L)\) is a scroll over a curve or \((X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\), see [LS, Cor. 3.4], [E, Thm. 3.2] and Landman’s parity Theorem [L], [K3, II (22)]. Moreover, when \(\text{def}(X, L) \geq n - 4 > 0\) we have three possibilities, see [M, Prop. 2.5]: either \((X, L)\) is a scroll over a surface, or \(X\) embedded by \(|L|\) is \(G(1, 4) \subset \mathbb{P}^9\), i.e., the Plücker embedding of the Grassmannian of lines in \(\mathbb{P}^4\), or a smooth hyperplane section of it. This shows that in this range examples different from scrolls are very few.

We are going to use the computations of Section 1 to prove results of this type in the ample and spanned case. In fact we also give new proofs of the classical results when \(\text{def}(X, L) \geq n - 3\). To be concrete: let \((X, L)\) be as in (0,0), then \(\text{def}(X, L) \leq n\) with equality if and only if \((X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\), [LPS1, Thm. 2.8]. Furthermore, the next case \(\text{def}(X, L) \neq n - 1\) cannot occur, see [LPS1, Thm. 2.8]. Here we deal with the next-to-maximal cases \(\text{def}(X, L) = n - i, i = 2, 3, 4\). We characterize scrolls over curves by the equality \(\text{def}(X, L) = n - 2\), we exclude the possibility for the defect to be \(n - 3\) and we classify the pairs \((X, L)\) in case \(\text{def}(X, L) = n - 4\) under the assumption that the Picard number of \(X\) is one.

The proofs in the classical setup use adjunction theory and either topology or the linearity of the general contact locus. In our proofs for the ample and spanned case we only use the adjunction theoretic results on the ampleness of \(K_X + (\dim(X) - i)L\) for \(i = 0, 1, 2\) and \(L\) ample and spanned together with the
exact sequence of \[LPS1, (2.8.2)\].

**Lemma 3.0.** Let \((X, L)\) be as in (0.0) and suppose that \((X, L) \neq (\mathbb{P}^n, O_{\mathbb{P}^n}(1))\). If \(\text{def}(X, L) > 0\) then \(K_X + (\text{def}(X, L) + 2)L\) is not ample.

**Proof.** Suppose by contradiction that \(\text{def}(X, L) = n - s > 0\) with \(s\) an integer \(n - 1 \geq s \geq 0\) and \(K_X + (n - s + 2)L\) is ample. By [LM2, Lemma 1.13] and adjunction formula, just restricting iteratively to general elements of \(|L|\), we can suppose \(n = s + 1\), \(\text{def}(X, L) \geq 1\) and \(K_X + 3L\) ample. For any integer \(r, 1 \leq r \leq n + 1\) consider the filtration of \(\mathcal{O}_X\) associated to (0.3) (see [H, pp. 127–128]):

\[
\land^r J_1(L) = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_r \supseteq F_{r+1} = 0,
\]

where \(F_i/F_{i+1} \cong \land^i (\Omega_X \otimes L) \otimes \land^{r-i} L\), that is, \(\Omega^i_X(rL)\) for \(i = r - 1, r\) and 0 otherwise. This provides the exact sequence:

\[
0 \to \Omega^r_X(rL) \to \land^r J_1(L) \to \Omega^{r-1}_X(rL) \to 0.
\]

Tensor this sequence by \(M = -(K_X + \alpha L)\) to get

\[
0 \to \Omega^r_X(M + rL) \to \land^r J_1(M) \to \Omega^{r-1}_X(M + rL) \to 0.
\]

The line bundle \(-(M + rL) = K_X + (\alpha - r)L\) is ample by hypothesis if \(\alpha - r \geq 3\). Then we can use the Nakano vanishing theorem, [La, Thm. 4.2.3, p. 250], to get \(h^j(X, \Omega^r_X(M + rL)) = 0\) when \(j + r < n\) and \(h^j(X, \Omega^{r-1}_X(M + rL)) = 0\) when \(j + r - 1 < n\). This leads to this vanishing:

\[
(3.0.1) \quad h^j(X, \land^r J_1(-K_X + \alpha L)) = 0, \quad \alpha - r \geq 3, \quad j + r < n.
\]

Since \(\text{def}(X, L) \geq 1\) we can choose a general line in \(|L|\) not meeting \(\mathcal{D}(X, L)\). This allows us construct the following exact sequence, exactly as in \([LPS1, (2.8.2)]\):

\[
(3.0.2) \quad 0 \to \mathcal{O}_X^{\oplus 2} \to J_1(L) \to Q \to 0,
\]

where \(Q\) is a rank \(n - 1\) vector bundle on \(X\). Now we claim that:

\[
(3.0.3) \quad h^j(X, \land^i Q(-(K_X + (n + 1)L))) = 0, \quad 0 \leq i \leq n - 2, \quad 0 \leq j \leq n - i - 1.
\]

Before proving the claim let us show how it leads to a contradiction. Consider the exact sequence dual of (0.3):

\[
0 \to \mathcal{O}_X(-L) \to J_1(L)^\vee \to T_X(-L) \to 0.
\]

Since \((X, L) \neq (\mathbb{P}^n, O_{\mathbb{P}^n}(1))\) then \(h^0(X, T_X(-L)) = 0\), see [W] or [BS, Thm. 5.4.5]. Moreover, \(h^0(X, -L) = 0\), whence \(h^0(X, J_1(L)^\vee) = 0\). Taking the dual exact sequence of (3.0.2) we get \(h^1(X, Q^\vee) > 0\). Let us recall that \(Q^\vee = \land^{n-2} Q(-(K_X + (n + 1)L))\) so that \(h^1(X, \land^{n-2} Q(-(K_X + (n + 1)L))) = h^1(X, Q^\vee) > 0\), contradicting the claim (3.0.3).
Now we can prove the claim by induction on \( i \). For \( i = 0 \), it follows from the Kodaira vanishing theorem that \( h^j(X, \mathcal{O}_X(-(K_X + (n+1)L))) = 0 \) for \( 0 \leq j \leq n-1 \), since \( n+1 \geq 3 \).

For \( i = 1 \) the claim follows from (3.0.1), the Kodaira vanishing theorem and the exact sequence obtained by twisting (3.0.2) by \( M = -(K_X + (n+1)L) \):

\[
0 \to \mathcal{O}_X^\oplus 2(M) \to J_1(L)(M) \to Q(M) \to 0.
\]

For \( i \geq 2 \), using the filtration of \( \wedge^i J_1(L) \) associated to (3.0.2), we get the following exact sequences:

\[
0 \to F_1 \to \wedge^i J_1(L) \to \wedge^i Q \to 0,
\]

\[
0 \to \wedge^{i-2} Q \to F_1 \to (\wedge^{i-1} Q)^\oplus 2 \to 0,
\]

where \( F_1 \) is a vector bundle on \( X \). Tensoring the second exact sequence by \( -(K_X + (n+1)L) \), by induction hypothesis, we get \( h^j(X, F_1(-(K_X + (n+1)L))) = 0 \) for \( 0 \leq j \leq n-i \). Tensoring the first one by the same line bundle and using (3.0.1) we get the claim. \( \square \)

**Theorem 3.1.** Let \( (X, L) \) be as in (0.0) with \( n \geq 3 \). The following are equivalent:

(3.1.1) \( X = \mathbb{P}_C(E) \) where \( C \) is a smooth curve, \( E \) is a vector bundle on \( C \) and \( L = \mathcal{O}_X(1) \) is the tautological bundle;

(3.1.2) \( \text{def}(X, L) = n - 2 \).

**Proof.** If (3.1.1) holds then, since \( n \geq 3 \), \( \text{def}(X, L) = n - 2 \) by (2.1.1).

Let us now prove that (3.1.2) implies (3.1.1) by induction on \( n \). Let us start with \( n = 3 \), so that \( \text{def}(X, L) = 1 \). Since by Lemma 3.0 \( K_X + 3L \) is not ample then we conclude by [I, (1.3)].

If \( n > 3 \), consider a general element \( X_1 \in |V| \), set \( L_1 = L|_{X_1} \) and denote by \( V_1 \) the image of \( V \) via the restriction map \( r : H^0(X, L) \to H^0(X_1, L_1) \). Then the triplet \( (X_1, L_1, V_1) \) is as in (0.0) and \( \text{def}(X_1, L_1) \geq \dim(X_1) - 2 \) by [LM2, Lemma 1.13]. It cannot be \( (X_1, L_1) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1)) \), otherwise it would be \( \text{def}(X, L) = n \). Hence, by induction, \( X_1 \) is a scroll over a smooth curve, \( \dim(X_1) \geq 3 \) and thus we conclude by [BS, Thm. 5.5.2]. \( \square \)

**Remarks 3.1.6.** Let us point out that Theorem 3.1 improves [LM2, Corollary 4.6]. Observe also that in (2.1.2) we impose \( m \geq 3 \) to use the results of Fukuma. If \( m = 2 \) then \( n = 3 \) and \( X \) is a scroll \( \pi : X = \mathbb{P}_Y(E) \to Y \) over a smooth surface \( Y \), \( \text{rank}(E) = 2 \). If \( \text{def}(X, L) > 0 \) then \( \text{def}(X, L) = \dim(X) - 2 \) so that, by Theorem 3.1, \( X \) must also be a scroll over a smooth curve \( C \), i.e., \( \pi' : X = \mathbb{P}_C(E') \to C \). These two structures of scroll on \( X \) lead easily to \( Y = \mathbb{P}^2 \) and \( E = \mathcal{O}_{\mathbb{P}^2}(1)^\oplus 2 \). Then (2.1.2) is also true when \( m = 2 \). Moreover, it is also true when \( m = 1 \), even though the hypothesis \( r > 1 \) does not hold.
In the classical case Landman’s parity theorem says that def\((X, L)\) and dim\((X)\) have the same parity, see reference above or [T, Thm. 7.4] and references inside. In particular def\((X, L)\) \(\neq n - 3\). We are going to prove this last result in the ample and spanned case. Together with [LPS1, Thm. 2.8] this seems to be a new evidence towards Landman’s parity theorem in the ample and spanned case.

Let us first recall the following definition, see [I, (0.11)] or [BS]. Let \((X, L)\) be a polarized pair of dimension \(n\), i.e., \(X\) is a smooth projective variety and \(L\) is an ample line bundle on \(X\). An effective divisor \(E \subset X\) is called a \((-1)\)-hyperplane if \(E \simeq \mathbb{P}^{n-1}\), \(\mathcal{O}_X(L) \otimes \mathcal{O}_E = \mathcal{O}_E(1)\) and \(\mathcal{O}_X(E) \otimes \mathcal{O}_E = \mathcal{O}_E(-1)\). A smooth polarized variety \((X', L')\) is called a reduction of \((X, L)\) if it is obtained contracting all the \((-1)\)-hyperplanes of \((X, L)\), that is, \(\sigma : X \to X'\) is the blowing up of a smooth variety \(X'\) at \(s\) distinct points \(p_1, \ldots, p_s \in X'\), \(E_i = \sigma^{-1}(p_i)\) and \(L = \sigma^*L' - E\) where \(E = E_1 + \cdots + E_s\) the total exceptional divisor. For \(n \geq 3\) the contraction \(\sigma\) is uniquely determined by \((X, L)\). Let us point out that \(|L'|\) can have base points. Anyway, the discriminant locus \(\mathcal{D}(X, L') \subseteq |L'|\) makes sense, though in principle it could be that \(\mathcal{D}(X', L') = |L'|\). Let us write as an abuse of notation \(\text{def}(X', L') = \dim(|L'|) - 1 - \dim(\mathcal{D}(X', L'))\). In particular it could be that \(\text{def}(X', L') = -1\). However this cannot happen. In fact, we can prove the following lemma.

**Lemma 3.2.** Let \((X, L)\) be as in (0.0), \(n \geq 3\). We get \(\text{def}(X, L) \leq \text{def}(X', L')\) for \((X', L')\) a reduction of \((X, L)\).

**Proof.** By definition \(X\) is the blowing up \(\sigma\) of \(X'\) at \(s\) distinct points \(p_1, \ldots, p_s\) and \(L = \sigma^*L' - E\) being \(E = E_1 + \cdots + E_s\) the total exceptional divisor. If \(H \in |L|\) is any smooth divisor then it does not contain any exceptional divisor and its image \(H'\) in \(X'\) is also smooth. In fact \(\text{mult}_{p_i}(H') = \deg(C_{p_i}H') = \deg(H|_{E_i}) = 1\) for every \(i\), where \(C_{p_i}H'\) denotes the tangent cone of \(H'\) at \(p_i\). Indeed this implies that any linear subspace of \(|L|\) containing only smooth divisors maps to a linear subspace of \(|L'|\) of the same dimension containing only smooth divisors. Just by definition, see (0.1), the defect cannot decrease and the lemma follows. \(\Box\)

**Theorem 3.2.** For \((X, L)\) as in (0.0) with \(n \geq 4\) we have \(\text{def}(X, L) \neq n - 3\).

**Proof.** Suppose by contradiction that \(\text{def}(X, L) = n - 3\). Let us observe that for a general \(X_1 \in |L|\), denoting \(L_1 = L|_{X_1}\) we have \(\text{def}(X_1, L_1) \geq \text{def}(X, L) - 1\). By [LPS1, Thm. 2.8] and Theorem 3.1 the strict inequality cannot hold. Hence, by induction we can suppose \(n = 4\) and \(\text{def}(X, L) = 1\). Thus, by Theorem 3.1, \((X, L)\) is not a scroll over a smooth curve so that \(K_X + 4L\) is ample. By Lemma 3.0 we know that \(K_X + 3L\) is not ample. Recall that \(L\) is ample and base point free. Hence one of the following holds, see [I, (1.6)]:

1. \((3.2.2)\) \((X, L)\) is a Del Pezzo variety, i.e., \(-K_X = 3L\),
2. \((3.2.3)\) \((X, L)\) is a hyperquadric fibration over a smooth curve,
3. \((3.2.4)\) \((X, L)\) is a scroll over a smooth surface,
(3.2.5) there exists a reduction \((X', L')\) such that \(K_{X'} + 3L'\) is ample.

The case (3.2.2) can be easily excluded looking at the classification of Del Pezzo manifolds, see for example [F3, Thm. 8.11, p. 72]. The case \(L^4 = 1\) does not occur since \(|L|\) is base point free. If \(L^4 = 2\) then \(X\) is a double cover of \(\mathbb{P}^4\) ramified along a smooth quartic so that \(\mathcal{D}_1\) is a hypersurface. If \(L^4 \geq 3\) then \(L\) is very ample. Here, in order to avoid the computations of \(c_4(J_1(L))\), we can use the results in the classical case to exclude this possibility.

The cases (3.2.3) and (3.2.4) come from the computation of the top Chern class that we have done in Section 1. If \((X, L)\) is a hyperquadric fibration then, by (1.4), we get \(c_4(J_1(L)) = (2e - b) + 3(2e - 5b) - 4(1 - g) > -4(1 - g)\). In order to have positive defect we need that \(c_4(J_1(L)) = 0\). Hence \(g = 0\). Then \(c_4(J_1(L)) = 0\) implies \((2e - b) + 3(2e - 5b) = 4\). Since \(2e - b > 0\) and \(2e - 5b \geq 0\), it is straightforward to check that the vanishing of \(c_4(J_1(L))\) cannot occur. If \((X, L)\) is a scroll over a smooth surface then \(c_4(J_1(L)) = c_2(E) > 0\) by (1.2.2) and the ampleness of \(E\), see [K1, Thm. 3].

Finally let us consider (3.2.5). By Lemma 3.2.0 we can choose a line \(T' \subset |L'|\) such that any element in \(T'\) is smooth. Consider two distinct points in \(T'\) and their corresponding sections \(s_0, s_1 \in |L'|\). They give rise to an exact sequence:

\[ 0 \to \mathcal{O}_{X'}^\oplus 2 \to J_1(L') \to Q' \to 0, \]

where \(Q'\) is a rank 3 vector bundle on \(X'\). Let us show the injectivity of the map \(f : \mathcal{O}_{X'}^\oplus 2 \to J_1(L')\) in the vector space fibre. The arguments of [LPS1, (2.6)] work at points outside the base locus of \(|L'|\). So we can confine to the base points. Since \(\text{Bs}|L'| \subset \{p_1, \ldots, p_s\}\) we can suppose that \(\text{Bs}|L'| = \{p_1, \ldots, p_t\}\), where \(t \leq s\). Consider the point \(p_1\) (the same arguments works for \(p_i, i \leq t\)). Locally around \(p_1\) the morphism \(f : \mathcal{O}_{X'}^\oplus 2 \to J_1(L')\) is given by

\[
\begin{align*}
    f_{p_1}(\lambda_0, \lambda_1) &= \lambda_0(s_0(p_1), ds_0(p_1)) + \lambda_1(s_1(p_1), ds_1(p_1)) = \\
    &= \lambda_0(0, ds_0(p_1)) + \lambda_1(0, ds_1(p_1)) = (0, \lambda_0 ds_0(p_1) + \lambda_1 ds_1(p_1)).
\end{align*}
\]

Since the line \(T'\) does not meet the vector subspace of sections of \(L'\) singular at \(p_1\) we get that \(f_{p_1}\) is injective. Now we can reproduce the arguments of Lemma 3.0 based on the non vanishing of \(h^1(Q^\vee)\) to contradict the ampleness of \(K_{X'} + 3L'\). \(\square\)

Now we study the next step \(\text{def}(X, L) = n - 4 > 0\). By Lemma 3.0 we get \(K_X + (n - 2)L\) is not ample. In the case \(\text{def}(X, L) = n - 3\), when \(K_X + (n - 1)L\) is not ample we know that \(K_{X'} + (n - 1)L'\) is ample on the reduction \((X', L')\) and we can do a complete classification using Lemma 3.2.0. For the case \(\text{def}(X, L) = n - 4\) we cannot reproduce the same argument because if \(K_X + (n - 2)L\) is not ample then we can only get the semi-ampleness (that is, global generation of a power) of the adjoint bundle \(K_{X'} + (n - 2)L'\) on the reduction. However, we can prove the following under the assumption that the Picard number is one.
Proposition 3.3. Let \((X, L)\) be as in \((0,0)\) with \(\text{def}(X, L) = n - 4 > 0\). If the Picard number of \(X\) is equal to one then either \(X = \mathbb{G}(1, 4)\) is the Grassmannian of lines in \(\mathbb{P}^4\) and \(L\) defines the Plücker embedding in \(\mathbb{P}^9\) or \(X\) is a smooth hyperplane section \(Y\) of \(\mathbb{G}(1, 4) \subset \mathbb{P}^9\) and \(L = \mathcal{O}_{\mathbb{P}^9}(1) \otimes \mathcal{O}_Y\).

Proof. Recall that \(K_X + (n - 2)L\) is not ample by Lemma 3.0. Since \(\text{Pic}(X) = \mathbb{Z}H\) with \(H\) an ample line bundle, we easily see that \(H = L\). Moreover, either \((X, L)\) is a Del Pezzo manifold or \(K_X + (n - 2)L\) is semi-ample, \((X, L)\) coinciding with its reduction, by [I, (1.7)]. In the former case we can use the classification of Del Pezzo manifolds, [F3, Thm. 8.11, p. 72], to conclude that \((X, L)\) is as in the statement.

In the latter case \(-K_X = (n - 2)L\), that is, \((X, L)\) is a Mukai variety. Recall that \(|L|\) is base point free by hypothesis. As a consequence of the study of the anticanonical linear system for Fano threefolds of [Is] we get the following, see [Mu, Prop. 1]: either \(L\) is very ample, or \(|L|\) defines a degree two morphism onto \(\mathbb{P}^n\), or onto a quadric \(\mathbb{Q}^n \subset \mathbb{P}^{n+1}\). By [M, Prop. 2.5] \(L\) cannot be very ample. If we have a degree two map onto \(\mathbb{P}^n\), \(\phi_L : X \to \mathbb{P}^n\), then the branch locus of \(\phi_L\) is a smooth hypersurface (of degree \(> 1\)) and its dual variety is contained in \(\mathcal{D}(X, L)\). This contradicts \(\text{def}(X, L) > 0\). If we have a degree two map \(\phi_L : X \to \mathbb{Q}^n \subset \mathbb{P}^{n+1}\) onto a quadric then \((X, L)\) has \(\Delta\)-genus 2 and fits into [F3, (10.8.2), p. 89], so that \(\mathbb{Q}^n \subset \mathbb{P}^{n+1}\) is smooth, being \(n \geq 3\). But then \(\mathbb{Q}^{n+1} \simeq \mathbb{Q}^n \subseteq \mathcal{D}(X, L)\) contradicting \(\text{def}(X, L) > 0\). \(\square\)

The following table summarizes the results on the classification of varieties with high defect with respect to their dimension in comparison with the classical case.

| \(\text{def}(X, L) > 0\) | \(n\) | \(n-1\) | \(n-2\) | \(n-3\) | \(n-4\) |
|---------------------|-----|-----|-----|-----|-----|
| \(L\) ample and spanned by \(V\) | \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\) | Not possible | Scroll over a curve | Not possible | \(\mathbb{G}(1, 4) \subset \mathbb{P}^9\) if \(\text{Pic}(X) = \mathbb{Z}\) |
| \(\phi_V\) an embedding | \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))\) | Not possible | Scroll over a curve | Not possible | \(\mathbb{G}(1, 4) \subset \mathbb{P}^9\) |

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