Nd-M-fuzzy lattices as Nd-M-fuzzy relations

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Abstract
Tepavčević and Goran Trajakoviski introduced the concept of L-fuzzy lattices [1, 2] where a bounded lattice is fuzzified by using a complete lattice[3]. Also they showed the connection of L-fuzzy lattices and L-fuzzy relations. Now we introduced the idea of Nd-M-fuzzy lattices and Nd-M-fuzzy relations, where M being the structure of a multilattice[4, 9, 10]. In this paper we proved the equivalence of Nd-M-fuzzy lattices and Nd-M-fuzzy relations using Egli-Milner ordering of subsets.

Keywords
Nd-M fuzzy subset, Nd-M-fuzzy Relation, Nd-M-fuzzy lattice.

AMS Subject Classification
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1. Introduction
The problem of providing suitable fuzzification of crisp concepts[6] is an important topic which attracts the attention of a number of researchers. But in L-fuzzy sets introduced by Goguen[5], where L is a complete lattice[3], the membership function gives unique values in L for each element of its domain. Here we are fuzzifying a crisp concept through a membership function, where the member ship function gives a set of values to each element of its domain. Thus we introduced the non-deterministic M-fuzzy set in terms of nondeterministic membership functions[7, 8], where M has the structure of a multilattice[4, 9, 10].

Recently, Cordero[4] introduced an alternative definition of multilattices which is closely related to lattices. My investigation is how to use these kind of structures for the extention of fuzzy related concepts.

The L-fuzzy lattice was introduced by Tepavčević and Goran Trajakoviski[1, 2], where a bounded lattice is fuzzified by using a complete lattice. In this paper first we discuss Nd-M-fuzzy order relation. Then we extend the idea of L-fuzzy lattice[2] to the Nd-M-fuzzy lattice using Egli-Milner ordering of subsets. Here we fuzzified a bounded lattice by using a complete and consistent multilattice M.

The organisation of the paper is as follows. In section 2, the definition and preliminary (theoretical) results about multilattice, Nd-M-fuzzy subsets and L-fuzzy Lattice are introduced. Later in section 3 we define Nd-M-fuzzy lattices as Nd-m-fuzzy relations and finally in section 4 we show the equivalence of Nd-M-fuzzy lattices and Nd-M-fuzzy relations.

2. Preliminaries

If (M, ≤) is a partially ordered set and B⊆M then multisupremum of B is a minimal element of the set of upper bounds of B and Multisup(B) denote the multisuprema of B. Dually we define the multiminima.

Definition 2.1. [5, 6] A poset (M, ≤)is an ordered multilattice if and only if for any p, q, x with p ≤ x and q ≤ x, there exist
z ∈ Multisup {p, q} such that z ≤ x and its dual.

When comparing with lattices, we see that least upper bound (which is a unique element) is replaced by the non empty set of all minimal (instead of least) upper bounds and dually.

Definition 2.2. [9, 10] A multilattice M is distributive if for each p, q, r ∈ M, the conditions (p ∨ q) ∩ (p ∨ q) ≠ ∅ and...
\((p \land q) \cap (p \land r) \neq \emptyset \Rightarrow q = r\) (where \(\land\) - is the usual set intersection and \(\cup\) is the usual set union) similarly to lattice theory, if we define \(p \lor q\) = Multisup\((p, q)\) and \(p \land q\) = Multiinf\((p, q)\), then \((M, \land, \lor)\) be a algebraic multilattice and if we define \(p \leq q\) if and only if \(p \lor q = \{q\}\) and \(p \land q = \{p\}\) it is possible to obtain the order version of multilattice.

**Definition 2.3.** A complete multilattice is a partially ordered set \((M, \leq)\) such that every subset \(X \subseteq M\) the set of upper bounds of \(X\) has minimal (maximal) element, which are called multisuprema (multi infima), that is for any subset \(A\) of \(X\), multisup\((A)\) and multiSup\((A)\) exists and non empty.

**Definition 2.4.** A poset \((M, \leq)\) is said to be a multisemilattice if it satisfies that for all \(p, q, x \in M\) with \(p \leq x, q \leq x\), there exist \(z \in \text{multi sup}\{p, q\}\) such that \(z \leq x\) and dually.

**Definition 2.5.** Let \((M, \leq)\) be a poset. The element \(p \in M\) is called a greatest element of \(M\) if there is no other element is smaller. That is \(p \geq x\) for every \(x \in M\). Similarly \(q \in M\) is said to be a smallest element of \(M\) if \(q \leq x\) for every \(x \in M\). If a multilattice has a greatest element and smallest element ,then \((M, \leq)\) is said to be bounded. Normally greatest element is taken as 1 and smallest element is taken as 0.

**Definition 2.6.** A multilattice \(M\) with 0 and 1 is called complemented if for each \(p \in M\), there is atleast one element \(q\) such that \(p \land q = \{0\}\) and \(p \lor q = \{1\}\).

Note that if \(M\) is complete distributive multilattice then every element in \(M\) has exactly one complement in \(M\).

Now we introduce three ordering among the subsets of posets which are the Hoare ordering, the Smyth ordering and the Egli -Milner ordering respectively.

**Definition 2.7.** [10] consider \(U, V \in 2^M\), then 
- \(U \subseteq_H V\) if and only if for all \(a \in U\) there exists \(b \in V\) such that \(a \leq b\)
- \(U \subseteq_S V\) if and only if for every \(b \in V\) there exists \(a \in U\) such that \(a \leq b\)
- \(U \subseteq_{EM} V\) if and only if \(U \subseteq_H V\) and \(U \subseteq_S V\).

**Definition 2.8.** [9] \((M, \land, \lor)\) - be a algebraic multilattice. Let \(p \in M\) and \(U, V\) be subsets of \(M\), then \(p \land U = \bigcup \{(p \land x) | x \in U\}\) \(p \lor U = \bigcup \{(p \lor x) | x \in U\}\) Also \(U \lor V = \bigcup \{(x \lor y) | x \in U, y \in V\}\) \(U \land V = \bigcup \{(x \land y) | x \in U, y \in V\}\)

**Definition 2.9.** [10] A multilattice \(M\) is said to be consistent if it satisfies the inequalities \(LB(U) \subseteq_{EM} \text{multiinf}(U)\) and \(\text{MultiSup}(U) \subseteq_{EM} UB(U)\) for all subset \(U\) of \(M\), Where \(LB(U)\) and \(UB(U)\) are the lower bound of \(U\) and upper bound of \(U\) respectively.

**Definition 2.10.** [7, 8] A Non deterministic M-Fuzzy subset of \(X\) is a function from \(X\) to \(2^M\), where \(M\) is a complete distributive multilattice. Then the collection of all the Nd-M-fuzzy subsets of \(X\) is called Nd-M-fuzzy space and is denoted by \((2^M)^X\).

**Definition 2.11.** A completely distributive multilattice \(M\) is called a \(F\)-multilattice if \(M\) has an order reversing involution \(\cdot : M \to M\).

Let \(X\) be a non empty ordinary set and \(M\) a \(F\) - multilattice. Let \(U \in (2^M)^X\). Then \(U'(a) = \{U(a)\}' = \{p' | p \in U(a)\}\)

If \(M\) is a completely distributive multilattice , then \(U'(a) = \{U(a)\}' = \{p' | p \in U(a)\}\) , then \(\cdot : (2^M)^X \to (2^M)^X\), the pseudo complementary operation on \((2^M)^X\), \(U'\) the pseudo complementary set of \(U\) in \((2^M)^X\).

**Definition 2.12.** [7, 8] Rules of set relations on \((2^M)^X\). Let \(U\) and \(V\) be two Nd-M-fuzzy subset of \(X\). Then 
1. \(U = V\) if \(U(a) = V(a)\), for every \(a \in X\)
2. \(U \leq V\) if \(U(a) \subseteq EM V(a)\)
3. \(C = U \lor V\) if \(C(a) = \text{ multisup}\{(U(a), V(a)) | for every \ a \in X\} = \bigcup \{p \lor q | p \in U(x) , q \in V(x)\}\) , for every \(a \in X\)
4. \(D = U \land V\) if \(D(a) = \text{ multiinf}\{(U(a), V(a)) | for every \ a \in X\} = \bigcup \{p \land q | p \in U(x) , q \in V(x)\}\) , for every \(a \in X\)
5. \(E = X - U\ then\ E(a) = \{p' | p \in U(a)\}\) for every \(a \in X\)

**Definition 2.13.** [1, 2] Let \(X\) be a lattice and \((L, \lor, \land)\) be a complete lattice with 0 and 1. Let \(R\) be a \(L\)-fuzzy set defined on \(X\). The \(p\) cut \((p \in L)\) of \(R\) is defined by \(\mu_p = \{a \in X : \mu(a) \geq p\}\). A fuzzy set \(\mu\) defined on \(L\) is a fuzzy sub lattice of \(L\), if
\[
(\mu(a \land b) \land \mu(a \lor b) \geq \min(\mu(a), \mu(b)) \land a, b \in X)
\[
(\mu(a \land b) \land \mu(a \lor b) \geq \mu(a) \land \mu(b))
\]

**Note 2.14.** \(\mu \in L^X\) is a \(L\)-fuzzy sub lattice of \(X\) if and only if \(\mu_p\) is a sub lattice of \(X\) for each \(p \in L\).

**Definition 2.15.** [7, 8] Let \((M, \land, \lor)\) be a complete and consistent multilattice with bottom element 0 and top element 1. Let \(X\) be a non-empty set. Then any mapping \(\bar{R} : X \times X \to 2^M\) is a Non deterministic \(M\) valued fuzzy relation on \(X\) called Nd-M-fuzzy relation on \(X\).

**Definition 2.16.** [7, 8] For \(\alpha \in 2^M\) an \(\alpha\) - level of \(\bar{R}\) is a mapping \(\bar{R}_\alpha : X \times X \to \{0, 1\}\), such that \(\bar{R}(x, y) = 1\) if and only if \(\alpha \subseteq EM \bar{R}(x, y)\). Then
\[
R_\alpha = \{(x, y) : \alpha \subseteq EM \bar{R}(x, y)\}
\] is the corresponding level set of \(\bar{R}\), which is a crisp relation on \(X\) called \(\alpha\) level of \(\bar{R}\).

**Definition 2.17.** [7, 8] An Nd-M fuzzy subset \(\mu \in (2^M)^L\) is a Nd-M-fuzzy sub lattice of \(L\) if \(\alpha \subseteq EM \mu(x) \land \mu(y)\) for every \(x, y \in \mu_\alpha\) and \(\mu_\alpha\) is a sub lattice of \(L\) for each \(\alpha \in 2^M\).
3. **nd-M-fuzzy lattices as nd-M-fuzzy relations**

In the previous section we defined *nd-M-fuzzy* lattices as *nd-M-fuzzy* algebraic structures. In this section, we introduce another approaches to *nd-M-fuzzy* lattices (via *nd-M-fuzzy* fuzzification of the order relation. Let \((M, \leq)\) is a complete multi lattice with bottom element \(0_M\) and top element \(1_M\) and let \(O\) be the one element lattice (which is also a multi lattice). Let \(M' = O \oplus M\). Clearly \((M', \leq)\) is a complete multi lattice with bottom element \(0_M\) and top element \(1_M\). Let \(R : L^2 \rightarrow 2^M\) be an *nd-M-fuzzy* relation. Let \(N_\alpha\) is the set defined by

\[
N_\alpha = \{ x \in L : \alpha \subseteq_{EM} R(x, x) \}
\]

Now we have the definition of a *nd-M-fuzzy* lattice (as an *nd-M-fuzzy* relation)

**Definition 3.1.** Let \(L\) be a non-empty set and \(M' = O \oplus M\) be a complete and consistent multilattice, then the pair \((L, \bar{R})\) where \(\bar{R} : L^2 \rightarrow 2^M\) is an *nd-M-fuzzy relation*, is called an *nd-M-valued fuzzy* lattice if \((L, R_0M)\) is a lattice and all the \(\alpha\)-levels of \(\bar{R}\), \(\alpha \in 2^M\), satisfies \(\alpha \subseteq_{EM} \bar{R}(x_1, y_1) \land_M \bar{R}(x_2, y_2)\), where \((x_1, y_1), (x_2, y_2) \in R_0M\) and also \(R_0M\) are sub lattice of it.

**Note 3.2.** We know that \(\{0_M\}\) is a level of \(R\) equal to \(L^2\) which is not an *nd-M-fuzzy* ordering relation and thus neither a *nd-M-fuzzy* lattice. Our aim is to find a *nd-M-fuzzy* sublattice of \(L\) that is why we introduce the artificial element \(\{0_L\}\).

The next theorem gives the necessary and sufficient conditions under which an *nd-M-fuzzy* relation is an *nd-M-fuzzy* lattice.

**Theorem 3.3.** Let \(L\) be a non-empty set and \(M\) complete and consistent multilattice. Then \(M' = O \oplus M\) is a complete and consistent multi lattice with the least element \(0\) and a unique atom \(0_M\). Then the mapping \(\bar{R} : L^2 \rightarrow 2^M\) is an *nd-M-fuzzy* lattice if and only if the following holds

1. \(\bar{R}\) is a weak *nd-M-fuzzy* ordering relation.
2. For all \(x, y \in L\) there exist \(S \in L\) such that for all \(\alpha \in \{0_M\} \cup \{\alpha \in 2^M / x, y \in N_\alpha\}\), the mapping \(\bar{R} : L^2 \rightarrow 2^M\) satisfies \(\alpha \subseteq_{EM} \bar{R}(x, x) \land_M \bar{R}(y, y)\).

**Proof.** Assume that \(\bar{R} : L^2 \rightarrow 2^M\) be an *nd-M-fuzzy* lattice. Let \(\alpha = \{0_M\}\). Then \((L, R_0M)\) is a lattice and for each \(\alpha \in 2^M, \bar{R}\) satisfies \(\alpha \subseteq_{EM} \bar{R}(x_1, y_1) \land_M \bar{R}(x_2, y_2)\), where \((x_1, y_1), (x_2, y_2) \in R_0M\) and also \(R_0M\) are sub lattice of it. This means that for any pair of elements \(x, y \in L\), \((x \land y)\) and \((x \lor y)\) exists. Let \(x \land y = S\) and \(x \lor y = I\), therefore the relations in 2 to 3 holds for \(\alpha = \{0_M\}\).

Suppose that \(\alpha \in 2^M, x, y \in N_\alpha\). Since \(R_0M\) is a sublattice of \(R_0M\), we have that supremum and infimum for elements \(x, y\) in lattices \((N_\alpha, R_\alpha)\) and \((L, R_0M)\) are the same.

Then \(\alpha \subseteq_{EM} \bar{R}(x, x)\) and \(\alpha \subseteq_{EM} \bar{R}(y, y)\), then for all \(s \in L\), the conditions in 2 and 3 holds.

Since \((L, R_0M)\) is a lattice and for each \(\alpha\) levels of \(\bar{R}\) satisfies \(\alpha \subseteq_{EM} \bar{R}(x_1, y_1) \land_M \bar{R}(x_2, y_2)\), where \((x_1, y_1), (x_2, y_2) \in R_0M\) and also \(R_0M\) are sub lattice of it they are ordering relations on subsets, that is all levels of \(\bar{R}\) is an *nd-M-fuzzy* ordering relations on \(L\), condition 1 is satisfied.

Conversely suppose that the mapping \(\bar{R} : L^2 \rightarrow 2^M\) , satisfies the conditions 1 to 3.

By weak reflexivity and condition 2, we have \(\bar{R}(x, y) \subseteq_{EM} \bar{R}(x, x)\) and \(\bar{R}(y, y) \subseteq_{EM} \bar{R}(x, y)\) for every \(x, y \in L\). Since \(\{0_M\} \subseteq_{EM} \bar{R}(x, x)\), we have that \(\bar{R}(0_M)\) for all \(x\).

This follows that \(\bar{R}(0_M)\) is an ordering relation and by condition 2 and \(3\) \((L, R_0M)\) is a lattice. Also from 2 and 3 , we have \(\alpha \subseteq_{EM} \bar{R}(x, y) \land_M \bar{R}(y, x)\) whenever \(\alpha \subseteq_{EM} \bar{R}(x_1, y_1)\) and \(\alpha \subseteq_{EM} \bar{R}(x_2, y_2)\). Also we see that \(\alpha\) level \(R_\alpha\) is an ordering relation on \(N_\alpha\). Thus \((N, R_\alpha)\) is a lattice and it is a sublattice of \((L, R_0M)\). 

4. Relation between two types of nd-M-fuzzy lattices

**Theorem 4.1.** Let \((L, \land, \lor, \land_M, \lor_M)\) be a lattice and \((M, \land_M, \lor_M)\) be a complete multilattice with \(0_M\) and \(1_M\). Then \(M' = O \oplus M\) be a complete multi lattice. Let \(\bar{L} : L \rightarrow 2^M\) be an *nd-M-fuzzy* lattice satisfying \(\alpha \subseteq_{EM} \bar{L}(x) \land_M \bar{L}(y)\), for every \(x, y \in L\).

Then the mapping \(\bar{R} : L^2 \rightarrow 2^M\) is defined by

\[
\bar{R}(x, y) = \bar{L}(x) \land_M \bar{L}(y) \text{ if } x \leq y
\]

\[
= \{0_M\} \text{ otherwise}
\]

is an *nd-M-fuzzy* lattice (as an *nd-M-fuzzy* relation).

Moreover, \((L_\alpha, N_\alpha, \bar{R}_\alpha)\) for \(\alpha \in 2^M\) are the same sub lattice of \(M\).

**Proof.** Assume that \(\bar{L} : L \rightarrow 2^M\) be an *nd-M-fuzzy* lattice, then for each \(\alpha \in 2^M, \bar{L}\) satisfies \(\alpha \subseteq_{EM} \bar{L}(x) \land_M \bar{L}(y)\) for all \(x, y \in L\). If \(\alpha = \{0_M\}\), then \(\{0_M\} \subseteq_{EM} \bar{L}(x)\) for every \(x \in L\).

Hence \(0_M \subseteq_{EM} \bar{L}(x) \land \bar{L}(y)\) and so \(0_M \subseteq_{EM} \bar{R}(0_M)\). That is \(\bar{R}(0_M) = 1\) for all \(x \leq y\). If \(\bar{R}(0_M)(x, y) = 0\), we have that \((L, R_0M)\) is the same lattice as \((L, \land, \lor, \land_M, \lor_M)\).

Now let \(\alpha \in 2^M\). If \(x \in L_\alpha\) if and only if \(\alpha \subseteq_{EM} \bar{L}(x)\) if
and only if $\alpha \subseteq EM \bar{R}(x,x)$ if and only if $x \in N_\alpha$. Hence for all $\alpha \in 2^M$, the sets $\bar{L}_\alpha$ and $N_\alpha$ are equal. Now let $x,y \in \bar{L}_\alpha x \leq y$, then $\alpha \subseteq EM \bar{L}(x)$ and $\alpha \subseteq EM \bar{L}(y)$, this implies $\alpha \subseteq EM \bar{R}(x,y)$. That is $(x,y) \in R_\alpha$.

If $(x,y) \in R_\alpha$, then $\alpha \subseteq EM \bar{R}(x,y)$, hence $\bar{R}(x,y) \neq \emptyset$ and $\alpha \subseteq EM \bar{L}(x) \land \bar{L}(y)$ and $x \leq y$.

Then we have to prove that the relations $\bar{R}_\alpha$ on $L_\alpha$ and $\leq$ on $L_\alpha$ are same. Since $(L_\alpha, \leq)$ is a lattice, and also it is a sub lattice of $(L, \leq)$. This means that $(N_\alpha, \bar{R}_\alpha)$ is a sub lattice of $(L, \bar{R}\{0M\})$. Therefore the mapping $\bar{R}$ is an nd-M-fuzzy Lattice(nd-M-fuzzy relation).

Example 4.2. Consider the example 3.2, the corresponding nd-M-fuzzy lattice (as a nd-M-fuzzy relation) is mapping $\bar{R}$: $L^2 \rightarrow M'$. Given in the table below, where $L = \{0_L, a, b, c, d, f, g, h, 1_L\}$ and $M'$ is the figure 1

| $R(x,y)$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $1_L$ |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|------|
| $0_L$    | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $a$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $b$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $c$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $d$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $e$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $f$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $g$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $h$      | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |
| $1_L$    | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ | $1_L$ |

5. Conclusion

We have showed the equivalence of Nd-M-fuzzy relations and Nd-M-fuzzy Lattices. Also we observed that in a L-fuzzy lattice the membership value is replaced by a set of values in a multilattice in M, then the extension of L-fuzzy lattice to Nd-M-fuzzy lattice is valid with respect to the Egli Milner ordering of subsets.

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