TRACIAL ROKHLIN PROPERTY FOR FINITE GROUP ACTIONS ON NON-UNITAL SIMPLE C*-ALGEBRAS

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Abstract. We introduce the tracial Rokhlin property for finite group actions on simple not necessarily unital C*-algebras which coincides with Phillips’ definition in the unital case. We study its basic properties. Our main result is that if $\alpha : G \to \text{Aut}(A)$ is an action of a finite group $G$ on a simple (not necessarily unital) C*-algebra $A$ with tracial topological rank zero and $\alpha$ has the tracial Rokhlin property, then $A \rtimes_\alpha G$ and $A^\alpha$ have tracial topological rank zero. The main idea to show this is to prove that a simple non-unital C*-algebra has tracial topological rank zero if and only if it is Morita equivalent to a simple unital C*-algebra with tracial topological rank zero. Moreover, we show that all of the following classes of (not necessarily unital) simple C*-algebras are closed under taking crossed products and fixed point algebras with actions of finite groups with the tracial Rokhlin property: simple separable C*-algebras $A$ of real rank zero with $\text{TR}(A) \leq k$, simple separable C*-algebras of real rank zero, simple separable nuclear Z-stable C*-algebras, simple C*-algebras with Property (SP), and simple separable tracially Z-absorbing C*-algebras.

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1. Introduction

The Rokhlin property was studied in the classification of group actions on von Neumann algebras in the work of Connes [Con77], Jones [Jon80], and Ocneanu [Oc85]. Later, the Rokhlin property was investigated for actions on C*-algebras by Kishimoto [Ki96], Herman and Jones [HJ82], Herman and Ocneanu [HO84], and others. Izumi in [Iz04a] gave a modern definition of the Rokhlin property for finite group actions on unital C*-algebras and classified finite group actions on some classes of unital C*-algebras with the Rokhlin property in [Iz04b].

Actions with the Rokhlin property are rare and many C*-algebras do not admit any finite group action with the Rokhlin property. Indeed, the Rokhlin property imposes some $K$-theoretical obstructions. To deal with this problem, Phillips in [Ph11] defined a tracial analogue of the Rokhlin property for finite group actions on simple unital C*-algebras, and then the weak tracial Rokhlin property for actions on simple unital C*-algebras was considered in [HO13], [Ar08], and others.

Despite the fact that the Rokhlin property was extended to the case of actions on non-unital C*-algebras (see, e.g., [San15], [Naw16], and [GS16]), there have been no works on the extension of the tracial Rokhlin property to the simple non-unital context. However, there are several important simple non-unital C*-algebras like the Razak-Jacelon algebras, non-unital Kirchberg algebras, and non-unital simple purely infinite C*-algebras. Also, recently the study of stably projectionless C*-algebras (which are very far from being unital) turned out to be important in Elliott’s program for classification of C*-algebras (see [EGLN17]), and actions with the weak tracial Rokhlin property on these algebras may provide more classifiable C*-algebra crossed products.

In this paper we define a notion of (weak) tracial Rokhlin property for finite group actions on simple not necessary unital C*-algebras. The paper is devoted mostly to the general theory, while the study of special examples and some other non-unital related issues such as TAC algebras, tracially approximately representable actions, and $K$-theoretical results, are postponed to the subsequent papers. Our definition is the following.

**Definition 1.1.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$. We say that $\alpha$ has the *tracial Rokhlin property* if for every finite subset $F \subseteq A$, every $\varepsilon > 0$, and every positive elements $x, y \in A$ with $\|x\| = 1$, there exist orthogonal projections $(p_g)_{g \in G}$ in $A$ such that, with $p = \sum_{g \in G} p_g$, the following hold:

1. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;
(3) \((y^2 - ypy - \epsilon)_+ \lesssim x;\)
(4) \(\|pxp\| > 1 - \epsilon.\)

See Definition 3.1 for Phillips’ definition in the unital case. Condition (3) contains our main idea for a suitable notion of the tracial Rokhlin property in the non-unital case. This condition—which may be found strange at the first glance—says that the projection \(1 - p\) is small with respect to the Cuntz subequivalence relation. The rationale behind Condition (3) is that since \(y \in A_+\) is arbitrary, we can take it to be arbitrary large (i.e., close to 1) and so \(y^2 - ypy = y(1 - p)y\) is close to \(1 - p\). The \(\epsilon\) gap in (3) is a technical condition needed, e.g., when applying a key lemma in the Cuntz semigroup (Lemma 2.1 below).

Rokhlin property can be viewed as a noncommutative freeness, see [Ph09]. Following this point of view, we prove that an action with the (weak) tracial Rokhlin property is pointwise outer (Proposition 4.7). Hence, the resulting crossed product is simple. Moreover, we obtain some permanence properties like passing to invariant (unital) hereditary C*-subalgebras, direct limits, and tensor products.

Weaker versions of the tracial Rokhlin property for actions on simple unital C*-algebras, in which projections are replaced by positive contractions, have been considered by several authors, see [GHS17], [HO13], and [Gar17]. In the same spirit, we introduce the weak tracial Rokhlin property in the non-unital context and study some of its basic properties. In particular, we show that this weaker version of the tracial Rokhlin property in the non-unital case implies the pointwise outerness. Moreover, we compare this property when restricted to the unital case with other weaker versions of the tracial Rokhlin property studied in [GHS17], [HO13], and [Gar17] (see Proposition 4.6).

One of Phillips’ motivations to study a tracial version of the Rokhlin property was to study the crossed products of simple unital C*-algebras with tracial topological rank zero by finite group actions. Lin in [Lin01b] introduced the tracial topological rank for C*-algebras as a non-commutative analogue of the topological dimension. It is natural to ask how one can generalize Phillips’ result for simple non-unital C*-algebras of tracial topological rank zero. We recall that Lin in [Lin01b] first gave the definition of the tracial topological rank for unital C*-algebras (Definition 2.11), and then he defined the tracial topological rank of a non-unital C*-algebra to be the tracial topological rank of its minimal unitization. However, working with the unitization of C*-algebras is not always convenient. Moreover, the unitization of a simple non-unital C*-algebra is not a simple C*-algebra and so one can not use the techniques special to simple C*-algebras. Therefore, to study the crossed products of simple non-unital C*-algebras of tracial topological rank at most \(k\), we first
develop the approach of Property \((T_k)\) which unifies both unital and non-unital case in a single definition. (See Definition 2.7 for the definition of the class \(\mathcal{I}^{(k)}\).

**Definition 1.2.** Let \(A\) be a simple C*-algebra and let \(k\) be a non-negative integer. We say that \(A\) has Property \((T_k)\) if \(A\) has an approximate unit (not necessarily increasing) consisting of projections and for every positive elements \(x, y \in A\) with \(x \neq 0\), every finite set \(F \subseteq A\), and every \(\varepsilon > 0\), there is a C*-subalgebra \(E \subseteq A\) with \(E \in \mathcal{I}^{(k)}\) such that, with \(p = 1_E\), the following hold:

1. \(|pa - ap| < \varepsilon\) for all \(a \in F\);
2. \(pFp \subseteq E\);
3. \((y^2 - ypy - \varepsilon)_+ \preceq x\);
4. \(|pxp| > \|x\| - \varepsilon\).

We describe the relation between the two notions of tracial topological rank \(k\) and Property \((T_k)\) as follows:

**Theorem 1.3.** Let \(A\) be a simple C*-algebra.

1. \(A\) has tracial topological rank zero if and only if \(A\) has Property \((T_0)\).
2. Suppose that \(A\) has an approximate unit consisting of projections and let \(k\) be a non-negative integer. Then \(A\) has tracial topological rank at most \(k\) if and only if \(A\) has Property \((T_k)\).

Theorem 1.3 enables us to study tracial topological rank for a simple non-unital C*-algebra without considering its unitization. We prove permanence properties for Property \((T_k)\). Moreover, simple C*-algebras with Property \((T_0)\) have real rank zero and stable rank one. These results lead us to obtain a relation between simple non-unital C*-algebras of tracial topological rank zero and simple unital C*-algebras of tracial topological rank zero in terms of the Morita equivalence:

**Theorem 1.4.** Let \(A\) be a non-zero simple (not necessarily unital) C*-algebra. The following statement are equivalent:

1. \(\text{TR}(A) = 0\);
2. \(A\) is Morita equivalent to a simple C*-algebra \(B\) with \(\text{TR}(B) = 0\);
3. \(\text{TR}(pAp) = 0\) for some (any) non-zero projection \(p \in A\).

We use Theorem 1.4 to extend Phillips’ result in [Ph11] concerning the crossed product of simple unital C*-algebras of tracial topological rank zero to the non-unital case as follows:

**Theorem 1.5.** Let \(A\) be a simple C*-algebra with tracial topological rank zero and \(\alpha\) be an action of a finite group \(G\) on \(A\) with the tracial Rokhlin property.
Then the crossed product $A \rtimes_{\alpha} G$ and the fixed point algebra $A^\alpha$ are simple $C^*$-algebras with tracial topological rank zero.

We obtain a similar result for tracial rank at most $k$. The idea we use to prove Theorem 1.5 can be abstracted as follows:

**Proposition 1.6.** Let $G$ be a finite group and $\mathcal{C}$ be a class of simple (separable) $C^*$-algebras with the following properties:

1. If $A$ is a simple (separable) $C^*$-algebra and $p \in A$ is a non-zero projection, then $A \in \mathcal{C}$ if and only if $pAp \in \mathcal{C}$ (in particular, this is the case if $\mathcal{C}$ is closed under the Morita equivalence);
2. If $A \in \mathcal{C}$ is unital and $\alpha$ is an action of $G$ on $A$ with the tracial Rokhlin property then $A \rtimes_{\alpha} G \in \mathcal{C}$;
3. If $A \in \mathcal{C}$ and $B$ is a $C^*$-algebra with $A \cong B$, then $B \in \mathcal{C}$.

Then $\mathcal{C}$ is closed under crossed products of actions of $G$ with the tracial Rokhlin property (i.e., (2) above holds without the unital assumption).

It has been studied which properties of simple unital $C^*$-algebras pass to crossed products when the action has the tracial Rokhlin property, see [Ar11], [OT14], [For17], and [HO13]. Proposition 1.6 enables us to extend these results to the non-unital case. In particular, we prove the following theorem. (See Definition 6.6 for the definition of simple tracially $\mathcal{Z}$-absorbing $C^*$-algebras.)

**Theorem 1.7.** The following classes of simple not necessarily unital $C^*$-algebras are preserved under taking crossed products and fixed point algebras with actions of finite groups with the tracial Rokhlin property:

1. simple $C^*$-algebras of tracial topological rank zero;
2. simple separable $C^*$-algebras $A$ of real rank zero with $TR(A) \leq k$;
3. simple separable $C^*$-algebras of real rank zero;
4. simple separable $C^*$-algebras of stable rank one and real rank zero;
5. simple separable nuclear $\mathcal{Z}$-stable $C^*$-algebras;
6. simple separable tracially $\mathcal{Z}$-absorbing $C^*$-algebras;
7. simple $C^*$-algebras with Property (SP).

This paper is organized as follows. Section 2 is devoted to some basic results concerning the Cuntz subequivalence relation and to review briefly the tracial topological rank for $C^*$-algebras. In Section 3, we define the tracial Rokhlin property for finite group actions on simple (not necessarily unital) unital $C^*$-algebras and prove some basic results. We give a weaker notion of the tracial Rokhlin property in Section 4 and compare this notion with the Rokhlin property introduced in [San15]. In Section 5, we define Property $(T_k)$ and study its relation to the tracial topological rank. In Section 6, we prove that
the crossed product of a simple non-unital C*-algebra with tracial topological
rank zero by a finite group action with the tracial Rokhlin property is a simple
C*-algebra of tracial topological rank zero. Moreover, we study properties of
C*-algebras which are preserved under taking crossed products of finite group
actions with the tracial Rokhlin property in the case of simple non-unital
C*-algebras.

2. Preliminaries

In this section we recall some results and provide some lemmas which will be
used in the subsequent sections. We use the following (standard) notations.
For a C*-algebra $A$, $A^+$ denotes the positive cone of $A$. $A^+$ denotes the
proper minimal unitization of $A$. Also, $A^- = A$ if $A$ is unital and $A^- = A^+$
if $A$ is non-unital. The closed unit ball of $A$ is denoted by $A_1$. We write
$A_{+,1} = A_+ \cap A_1$. If $p, q \in A$ are projections then we write $p \sim_{\text{MvN}} q$ if $p$
is Murray-von Neumann equivalent to $q$. If $E, F$ are subsets of $A$ and $\varepsilon > 0$, we
write $E \subseteq_{\varepsilon} F$ if for every $a \in E$ there is $b \in F$ such that $\|a - b\| < \varepsilon$. We write
$K = K(\ell^2)$ and $M_n = M_n(\mathbb{C})$. We use $A \otimes B$ to denote the minimal tensor
product of C*-algebras $A$ and $B$.

Let $A$ be a C*-algebra. For $a, b \in A_+$, we write $a \preceq b$ if $a$ is Cuntz
subequivalent to $b$, i.e., there is a sequence $(v_n)$ in $A$ such that $\|a - v_n b v_n^*\| \to 0$.
We write $a \sim b$ if both $a \preceq b$ and $b \preceq a$. The following key lemma will be used
several times throughout the paper.

**Lemma 2.1 ([KR00], Lemma 2.2).** Let $A$ be a C*-algebra, $a, b$ be positive
elements in $A$, and $\varepsilon > 0$. If $\|a - b\| < \varepsilon$ then there is a contraction $d \in A$
such that $(a - \varepsilon)_+ = dbd^*$. In particular, $(a - \varepsilon)_+ \preceq b$.

**Proof.** We have

$$\|xbx^* - \varepsilon\| \leq \|x\|^2 \|b - \left(\frac{\varepsilon}{\|x\|^2}\right)_+\| \leq \|x\|^2 \frac{\varepsilon}{\|x\|^2} = \varepsilon.$$ 

Thus we get

$$(xbx^* - \varepsilon)_+ \preceq x\left(b - \frac{\varepsilon}{\|x\|^2}\right)_+ x^*.$$
If \( \|x\| \leq 1 \) then \( \frac{\varepsilon}{\|x\|} \leq \varepsilon \) and so
\[
(xb)^* - \varepsilon, \quad x(b - \varepsilon)^* x^* \preceq (b - \varepsilon)^*.
\]

Observe that if \( a \preceq b \) in \( A \) then, by definition, there is a sequence \( (v_n) \) in \( A \) such that \( \|a - v_n b v_n^*\| \to 0 \). But it is not the case that there exists always a bounded sequence with this property. However, we have the following lemma.

**Lemma 2.3** ([AGJP17]). Let \( A \) be a C*-algebra, \( a, b \in A_+ \), and \( \delta > 0 \). If \( a \preceq (b - \delta)_{+} \) then there exists a bounded sequence \( (v_n) \) in \( A \) such that \( \|a - v_n b v_n^*\| \to 0 \). We can take this sequence such that \( \|v_n\| \leq \|a\|^{-1} \delta^{-\frac{1}{2}} \) for every \( n \in \mathbb{N} \).

**Proof.** Let \( n \in \mathbb{N} \). Since \( a \preceq (b - \delta)_{+} \), there exists \( w_n \in A \) such that \( \|a - w_n (b - \delta)_{+} w_n^*\| < \frac{1}{n} \). By Lemma 2.1, there exists \( d_n \in A \) such that \( (a - \frac{1}{n})_{+} = d_n w_n (b - \delta)_{+} w_n^* d_n^* \). By [KR00, Lemma 2.4(i)], there exists \( v_n \in A \) such that \( (a - \frac{1}{n})_{+} = v_n b v_n^* \) and \( \|v_n\| \leq \|(a - \frac{1}{n})_{+}\|^{2} \delta^{-\frac{1}{2}} \). Therefore, \( v_n b v_n^* \to a \) and \( \|v_n\| \leq \|a\|^{2} \delta^{-\frac{1}{2}} \). \( \square \)

The following lemma should be part of the literature, however, we did not find a reference. We provide a proof for the convenience of the reader.

**Lemma 2.4.** Let \( A \) be a C*-algebra, \( a \in A_+ \), and \( p \in A \) be a projection. The following statements are equivalent:

1. \( p \preceq a \) in \( A \);
2. there exists \( v \in A \) such that \( p = vav^* \);
3. \( p \sim \text{MvN} q \) in \( A \) for some projection \( q \) in \( aAa \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( p \preceq a \). Choose \( 0 < \delta < 1 \). Then there exists \( w \in A \) such that \( \|p - waw^*\| < \delta \). Then by Lemma 2.1, \( (p - \delta)_{+} = d w a w^* d^* \) for some \( d \in A \). Note that \( (p - \delta)_{+} = (1 - \delta)p \). Put \( v = (1 - \delta)^{-\frac{1}{2}} w d \). Then \( p = vav^* \).

(2) \( \Rightarrow \) (3): Let \( p = vav^* \) for some \( v \in A \). Then \( q = a^{\frac{1}{2}} v^* v a^{\frac{1}{2}} \) is a projection in \( aAa \) and is Murray-von Neumann equivalent to \( p \).

(3) \( \Rightarrow \) (1): Let \( q \) be a projection in \( aAa \). Then \( p \sim q \preceq a \).

Alternatively, we give another argument (which avoids Lemma 2.1) to prove that (1) implies (3). Let \( p \preceq a \). Then there is \( z \in A \) such that \( \|p - zaz^*\| < \frac{1}{2} \). By [Lin01b, Lemma 2.5.4], there exists a projection \( r \) in \( C^*(zaz^*) \) such that \( \|p - r\| < 1 \). Thus \( p \sim \text{MvN} r \). We have

\[
r \in C^*(zaz^*) \subseteq zaz^*Azaz^* \subseteq zaAaz^*.
\]

By [PP15, Lemma 3.8], there is a projection \( q \) in \( az^*Az^*a \subseteq aAa \) such that \( q \sim \text{MvN} r \). Therefore, \( p \sim q \in aAa \). \( \square \)
In general, there is no upper bound for the norm of \( v \) in the previous lemma, unless there is a gap between \( p \) and \( a \), see the following lemma (which may be considered as a special case of [KR02, Lemma 2.4]).

**Lemma 2.5.** Let \( A \) be a C*-algebra, \( a \in A_+ \), \( \varepsilon > 0 \), and \( p \in A \) be a projection. Let \( p \precsim (a - \varepsilon)_+ \) in \( A \). Then there exists \( v \in A \) such that \( p = vav^* \) and \( \|v\| \leq \varepsilon^{-\frac{1}{2}} \).

**Proof.** By Lemma 2.4, there exists \( w \in A \) such that \( p = w(a - \varepsilon)_+ w^* \). Then [KR02, Lemma 2.4(i)] implies that there is \( v \in A \) such that \( p = vav^* \) and \( \|v\| \leq \varepsilon^{-\frac{1}{2}} \). \( \square \)

The following is a variant of [KR02, Lemma 2.4]. We shall use this lemma in the proof of Lemma 3.6.

**Lemma 2.6.** Let \( A \) be a C*-algebra, \( a, b \in A_+ \), and \( \varepsilon > 0 \).

(i) If \( a = x(b - \varepsilon)_+ \) for some \( x \in A \), then \( a = yb \) for some \( y \in A \) with \( \|y\| \leq \varepsilon^{-\frac{1}{2}}\|a\| \).

(ii) If \( a \in A(b - \varepsilon)_+ \) then there exists a sequence \( (v_n)_{n \in \mathbb{N}} \) in \( A \) such that \( \|a - v_n b\| \to 0 \) and \( \|v_n\| \leq \varepsilon^{-1}(\|a\| + \frac{1}{n}) \) for all \( n \in \mathbb{N} \).

**Proof.** Define the continuous functions \( f_\varepsilon, g_\varepsilon : [0, \infty) \to [0, \infty) \) for \( \varepsilon > 0 \) as in [KR02, Lemma 2.4], that is,

\[
f_\varepsilon(t) = \begin{cases} \sqrt{\frac{t - \varepsilon}{t}} & t \geq \varepsilon, \\ 0 & t < \varepsilon, \end{cases} \quad \text{and} \quad g_\varepsilon(t) = \begin{cases} \frac{1}{t} & t \geq \varepsilon, \\ \varepsilon^{-2}t & t < \varepsilon. \end{cases}
\]

Then \( tf_\varepsilon(t)^2 = (t - \varepsilon)_+ \) and \( f_\varepsilon(t)^2 = (t - \varepsilon)_+ g_\varepsilon(t) \). Thus \( bf_\varepsilon(b)^2 = (b - \varepsilon)_+ \) and \( f_\varepsilon(b^2) = (b - \varepsilon)_+ g_\varepsilon(b) \). Note that \( \|g_\varepsilon(b)\| \leq \varepsilon^{-1} \).

To prove (i), put \( y = xf_\varepsilon(b)^2 \). Then \( yb = xf_\varepsilon(b)^2 b = x(b - \varepsilon)_+ = a \). Also,

\[
yy^* = xf_\varepsilon(b)^4 x^* = (b - \varepsilon)^2 g_\varepsilon(b)^2 x^* \leq \|g_\varepsilon(b)^2\|\|x\|\|b - \varepsilon\|^2 x^* \leq \varepsilon^{-2}a^*.
\]

Thus \( \|y\| \leq \varepsilon^{-1}\|a\| \).

For (ii), let \( a \in A(b - \varepsilon)_+ \) and fix \( n \in \mathbb{N} \). Then there is \( w_n \in A \) such that \( \|a - w_n(b - \varepsilon)_+\| < \frac{1}{n} \). Put \( a_n = w_n(b - \varepsilon)_+ \). Thus \( \|a_n\| \leq \|a\| + \frac{1}{n} \).

By (i) there is \( v_n \in A \) such that \( a_n = v_n b \) and \( \|v_n\| \leq \varepsilon^{-1}(\|a\| + \frac{1}{n}) \). Then \( \|a - v_n b\| = \|a - a_n\| < \frac{1}{n} \). \( \square \)

In the rest of this section we collect some basic definitions and results on the tracial topological rank of C*-algebras. We refer the reader to [Lin01b] for more details.

**Definition 2.7** ([Lin01b], Definition 2.7). We denote by \( \mathcal{I}^{(0)} \) the class of all finite dimensional C*-algebras, and denote by \( \mathcal{I}^{(k)} \) the class of all unital
C*-algebras which are (isomorphic to) unital hereditary C*-subalgebras of C*-algebras of the form $C(X) \otimes F$, where $X$ is a $k$-dimensional finite CW complex and $F \in \mathcal{I}^{(0)}$.

**Remark 2.8.** Observe that $A \in \mathcal{I}^{(k)}$ if and only if $A \cong \bigoplus_{i=1}^{m} p_i M_{n_i}(C(X_i)) p_i$ where each $X_i$ is a $k$-dimensional finite CW complex and each $p_i$ is a projection in $M_{n_i}(C(X_i))$ (because one can take $X$ to be the disjoint union of the $X_i$). Also, $\mathcal{I}^{(k)} \subseteq \mathcal{I}^{(k+1)}$, for all $k \geq 0$, and $\mathcal{I}^{(k)}$ is closed under finite direct sums.

**Definition 2.9** ([Lin01b], Definition 2.3). Let $0 < \sigma_1 < \sigma_2 \leq 1$ be two positive numbers. Define a continuous function $f_{\sigma_1}^{\sigma_2} : [0, \infty) \to [0, 1]$ by

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1 & t \geq \sigma_2, \\ \text{linear} & \sigma_1 \leq t < \sigma_2, \\ 0 & 0 \leq t < \sigma_1. \end{cases}$$

**Definition 2.10.** Let $a$ and $b$ be two positive elements in a C*-algebra $A$. We write $[a] \leq [b]$ if there exists $x \in A$ such that $x^* x = a$ and $x x^* \in b A b$ (see [Lin01b, Definition 2.2] for another equivalent definition). Let $n$ be a positive integer. We write $n[a] \leq [b]$ if there are $n$ mutually orthogonal positive elements $b_1, \ldots, b_n \in b A b$ such that $[a] \leq [b_i]$ for all $i = 1, \ldots, n$.

**Definition 2.11** ([Lin01b], Definition 3.1). A unital C*-algebra $A$ is said to have tracial topological rank at most $k$ if for any $\varepsilon > 0$, any finite subset $F$ containing a non-zero element $b \geq 0$, any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, and any integer $n > 0$, there exist a non-zero projection $p \in A$ and a C*-subalgebra $B \in \mathcal{I}^{(k)}$ with $1_B = p$ such that

1. $\|p a - a p\| < \varepsilon$ for all $a \in F$;
2. $p F p \subseteq B$;
3. $n[1 - p] \leq [p]$ and $n[f_{\sigma_1}^{\sigma_2}(1 - p) b (1 - p)] \leq [f_{\sigma_3}^{\sigma_4}(p b p)]$.

If $A$ has tracial topological rank at most $k$, we will write TR$(A) \leq k$. If furthermore, TR$(A) \not< k - 1$ then we say TR$(A) = k$. A non-unital C*-algebra $A$ is said to have TR$(A) = k$ if TR$(A^\sim) = k$.

Lin also introduced a weaker version of the tracial topological rank in [Lin01b] as follows.

**Definition 2.12** ([Lin01b], Definition 3.4). Let $A$ be a unital C*-algebra. We write TR$_w(A) \leq k$ if for any $\varepsilon > 0$, any finite subset $F$ containing a non-zero element $b \geq 0$, any integer $n > 0$, and any full element $x \in A_+$, there exist a non-zero projection $p \in A$ and a C*-subalgebra $B \in \mathcal{I}^{(k)}$ with $1_B = p$ such that

1. $\|p a - a p\| < \varepsilon$ for all $a \in F$;
(2) $pFp \subseteq \varepsilon B$ and $\|pfp\| \geq \|b\| - \varepsilon$;
(3) $n[1 - p] \leq [p]$ and $[1 - p] \leq [x]$.

We write $\text{TR}_w(A) = k$ if $\text{TR}_w(A) \leq k$ and the statement $\text{TR}_w(A) \leq k - 1$ is false.

If $A$ is non-unital we define $\text{TR}_w(A) = \text{TR}_w(A^\sim)$. Observe that for any $C^*$-algebra $A$, $\text{TR}_w(A) = 0$ if and only if $A$ is TAF in the sense of $[\text{Lin01c}]$.

**Remark 2.13.** Observe that in Definition 2.12, we may omit the assumption that $b$ is positive. In fact, if $b$ is not positive we may assume that $\|b\| = 1$ and then use $b^*b$ instead of $b$.

The following theorem follows from Theorem 6.13 and Remark 6.12 of $[\text{Lin01b}]$.

**Theorem 2.14** ($[\text{Lin01b}]$). Let $A$ be a simple unital $C^*$-algebra and $k$ be a non-negative integer. Then the following are equivalent:

(i) $\text{TR}(A) \leq k$;
(ii) $\text{TR}_w(A) \leq k$;
(iii) for every finite set $F \subseteq A$, every $\varepsilon > 0$, and every non-zero positive element $x \in A$, there is a non-zero $C^*$-subalgebra $B \subseteq A$ with $B \in \mathcal{I}(k)$ such that, with $p = 1_B$, the following hold:

1. $\|pa - ap\| < \varepsilon$ for all $a \in F$;
2. $pFp \subseteq \varepsilon B$;
3. $1 - p \preceq x$.

Moreover, $\text{TR}_w(A) = k$ if and only if $\text{TR}(A) = k$.

**Remark 2.15.** Observe that Theorem 2.14 is not true for simple non-unital $C^*$-algebras, that is, Conditions (i) and (ii) are not equivalent in general. For example, let $C$ be a simple unital $C^*$-algebra with properties stated in (the proof of) $[\text{Lin01b}, \text{Example 4.7}]$, that is, $\text{TR}_w(C) = \text{TR}(C) = 1$ and $\text{TR}_w(B) = 0$ where $B = C \otimes K$. Then $\text{TR}_w(B) \neq \text{TR}(B)$. In fact, suppose that $\text{TR}(B) = 0$. Let $q \in K$ be a rank one projection and put $p = 1_C \otimes q \in B$. By $[\text{Lin01b}, \text{Theorem 5.3}]$ we have $\text{TR}(pBp) = 0$. On the other hand, $pBp \cong C$ and so $\text{TR}(C) = 0$ which is a contradiction.

### 3. Tracial Rokhlin Property

In this section we introduce the tracial Rokhlin property for finite group actions on simple not necessarily unital $C^*$-algebras (Definition 3.2) and study some basic properties. First we recall Phillips’ definition of the tracial Rokhlin property for finite group actions on simple unital $C^*$-algebras.
Definition 3.1 ([Ph11], Definition 2.1). Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on an infinite dimensional simple separable unital C*-algebra $A$. We say that $\alpha$ has the tracial Rokhlin property if for every finite subset $F \subseteq A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there exist orthogonal projections $p_g \in A$ for $g \in G$ such that, with $p = \sum_{g \in G} p_g$, the following hold:

1. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. the projection $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$, or equivalently, $1 - p \precsim x$ (by Lemma 2.4);
4. $\|pxp\| > 1 - \varepsilon$.

We will use the above definition even if $A$ is not separable.

We now give a definition of the tracial Rokhlin property for finite group actions on simple not necessarily unital C*-algebras.

Definition 3.2. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$. We say that $\alpha$ has the tracial Rokhlin property if for every finite subset $F \subseteq A$, every $\varepsilon > 0$, and every positive elements $x, y \in A$ with $\|x\| = 1$, there exist orthogonal projections $(p_g)_{g \in G}$ in $A$ such that, with $p = \sum_{g \in G} p_g$, the following hold:

1. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $(y^2 - ypy - \varepsilon) + \precsim x$;
4. $\|pxp\| > 1 - \varepsilon$.

Condition (3) is the main difference between Definitions 3.1 and 3.2. It says that the projection $1 - p$ is small with respect to the Cuntz subequivalence relation. The rationale behind Condition (3) in Definition 3.2 is that since $y \in A_+$ is arbitrary, we can take it to be arbitrary large (i.e., close to 1) and so $y^2 - ypy = y(1 - p)y$ is close to $1 - p$. Moreover, the $\varepsilon$ gap in (3) is a technical condition we will need when applying Lemma 2.1.

Lemma 3.3. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$. Let $x \in A_+$ and $\|x\| = 1$. Suppose that a positive element $y \in A$ has the following property. For every $\varepsilon > 0$ and every finite subset $F \subseteq A$ there exist orthogonal projections $(p_g)_{g \in G}$ in $A$ such that, with $p = \sum_{g \in G} p_g$, the following hold:

1. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $(y^2 - ypy - \varepsilon) + \precsim x$;
4. $\|pxp\| > 1 - \varepsilon$. 
Then every positive element \( z \in \overline{A_y} \) also has the above property of \( y \).

\[ \begin{aligned}
\text{Proof.} & \quad \text{Let} \ z \in \overline{A_y} \ \text{be a positive element and let we are given a finite set} \ F \subseteq A, \\
& \quad \text{\varepsilon > 0, and a non-zero element} \ x \in A_+. \ \text{Let} \ \delta \ \text{be such that} \\
& \quad 0 < \delta < \min \left\{ 1, \frac{\varepsilon}{4(2\|z\| + 1)} \right\}.
\end{aligned} \]

Since \( z \in \overline{A_y} \), there exists a non-zero element \( w \in A \) with \( \|z - wy\| < \delta \). Choose \( 0 < \eta < \min \{\varepsilon, \frac{\varepsilon}{2\|w\|} \} \). By assumption, there exists mutually orthogonal projections \( (p_g)_{g \in G} \) in \( A \) such that, with \( p = \sum_{g \in G} p_g \), the following hold:

\[ \begin{aligned}
(1) & \quad \|p_g a - ap_g\| < \eta \ \text{for all} \ a \in F \ \text{and all} \ g \in G; \\
(2) & \quad \|\sigma_g(p_h) - p_{gh}\| < \eta \ \text{for all} \ g,h \in G; \\
(3) & \quad (y^2 - ypy - \eta)_+ \preceq x; \\
(4) & \quad \|pxp\| > 1 - \eta.
\end{aligned} \]

Since \( \eta < \varepsilon \), (1), (2), and (4) above also hold for \( \varepsilon \) in place of \( \eta \). It remains to show that \( (z^2 - zpz - \varepsilon)_+ \preceq x \). To see this, first by Lemma 2.2 we have

\[ \begin{aligned}
& \quad \left( wy^2w^* - wypyw^* - \frac{\varepsilon}{2} \right)_+ \preceq w \left( y^2 - ypy - \frac{\varepsilon}{2\|w\|^2} \right)_+ w^* \\
& \quad \preceq \left( y^2 - ypy - \frac{\varepsilon}{2\|w\|^2} \right)_+ \\
& \quad \preceq (y^2 - ypy - \eta)_+ \\
& \quad \preceq x.
\end{aligned} \]

On the other hand, we have

\[ \begin{aligned}
& \quad \|z^2 - zpz - \left( wy^2w^* - wypyw^* - \frac{\varepsilon}{2} \right)_+ \| \\
& \quad \leq \|z^2 - zpz - (wy^2w^* - wypyw^*)\| + \frac{\varepsilon}{2} \\
& \quad \leq \|z^2 - wyz\| + \|wyz - wy^2w^*\| + \|wyz - wypz\| \\
& \quad + \|wypz - wypyw^*\| + \frac{\varepsilon}{2} \\
& \quad \leq \delta(2\|z\| + 2\|wyz\|) + \frac{\varepsilon}{2} \\
& \quad \leq \delta(4\|z\| + 2\delta) + \frac{\varepsilon}{2} \\
& \quad \leq \delta(4\|z\| + 2) + \frac{\varepsilon}{2} \\
& \quad < \varepsilon.
\end{aligned} \]

Therefore, by Lemma 2.1, \( (z^2 - zpz - \varepsilon)_+ \preceq \left( wy^2w^* - wypyw^* - \frac{\varepsilon}{2} \right)_+ \preceq x \), as desired. This finishes the proof. \( \square \)

Remark 3.4. Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple C*-algebra \( A \).
(i) If the property stated in Definition 3.2 holds for some $y \in A_+$ then it also holds for every positive element $z \in \overline{Ay}$ (by Lemma 3.3).

(ii) For a unital C*-algebra $A$, Definition 3.2 is equivalent to Definition 3.1. In fact, that Definition 3.2 implies Definition 3.1 follows easily from Lemma 2.4 by putting $y = 1$ in Definition 3.2 and using the fact that $(1 - p - \varepsilon)_+ = (1 - \varepsilon)(1 - p) \sim (1 - p)$. The converse follows from Lemma 2.4 and item (i).

(iii) If $A$ is $\sigma$-unital then $\alpha$ has the tracial Rokhlin property if some strictly positive element $y$ in $A$ has the property stated in Definition 3.2. This follows from (i) and that $A = yAy = Ay$.

(iv) In Definition 3.2, if moreover $A$ is purely infinite then Condition (3) is unnecessary.

Remark 3.5. Observe that in Definition 3.2, it is enough to take $y$ in a norm dense subset of $A_+$. Moreover, if $(e_i)_{i \in I}$ is a approximate unit for $A$, it is enough to take $y$ from the set $\{e_i \mid i \in I\}$. This follows from Lemma 3.3 and the fact that the set $\{y \in A_+ \mid y \in \overline{Ae_i} \text{ for some } i \in I\}$ is dense in $A_+$.

In Definition 3.2, if moreover $A$ has Property (SP) then we may assume that $x$ is a non-zero projection, thanks to the following lemma.

Lemma 3.6. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$ with Property (SP). Then $\alpha$ has the tracial Rokhlin property if and only if for every positive element $y \in A$, every $\varepsilon > 0$, every finite subset $F \subseteq A$, and every non-zero projection $q \in A$, there exist orthogonal projections $(p_g)_{g \in G}$ in $A$ such that, with $p = \sum_{g \in G} p_g$, the following hold:

1. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $(y^2 - ypy - \varepsilon)_+ \preceq q$;
4. $\|pqp\| > 1 - \varepsilon$.

Proof. If $\alpha$ has the tracial Rokhlin property (Definition 3.2) then it is obvious that the condition stated in the statement holds. For the converse let $y$, $F$, $\varepsilon > 0$, and $x$ be as in Definition 3.2. Choose $0 < \delta < 1$ such that

$$\left(\frac{\delta}{2 - \delta}(1 - \frac{\varepsilon}{2})\right)^2 > 1 - \varepsilon.$$

Note that this is possible since $(1 - \frac{\varepsilon}{2})^2 > 1 - \varepsilon$. Put $z = (x^{\frac{1}{2}} - \delta)_+.

Then $(x^{\frac{1}{2}} - \delta)_+ \neq 0$ since $\|x\| = 1$ and $\delta < 1$. Since $A$ has Property (SP), there is a non-zero projection $q \in \overline{Az}$. Applying the condition stated in the statement to $y$, $F$, $\frac{\varepsilon}{2}$, and $q$, there exist orthogonal projections $(p_g)_{g \in G}$.
in $A$ such that (1) and (2) above hold with $\frac{\epsilon}{2}$ in place of $\epsilon$. Moreover, with $p = \sum_{g \in G} p_g$ we have $(y^2 - ypy - \epsilon)_+ \preceq q$ and $\|pqp\| > 1 - \frac{\epsilon}{2}$. Thus (1) and (2) in Definition 3.2 hold. As $q \in zAz$, we have

$$(y^2 - ypy - \epsilon)_+ \preceq z = (x^\frac{1}{2} - \delta)_+ \preceq x.$$  

Therefore (3) in Definition 3.2 holds. It remains to show (4) in that definition.

We have $q \in zAz \subseteq Az = A(x^{\frac{1}{2}} - \delta)_+$. Thus by Lemma 2.6 there exists a sequence $(v_n)_{n \in \mathbb{N}}$ in $A$ such that $\|v_n x^{\frac{1}{2}} - q\| \to 0$ and $\|v_n\| \leq (\|q\| + \frac{1}{n})\delta^{-1} = (1 + \frac{\epsilon}{2})\delta^{-1}$. Then $\|pv_n x^{\frac{1}{2}}p - pqp\| \to 0$. Since $\|pqp\| > 1 - \frac{\epsilon}{2}$ and $\delta < 1$, there is $n \in \mathbb{N}$ such that $\|pv_n x^{\frac{1}{2}}p\| > 1 - \frac{\epsilon}{2}$ and $\frac{1}{n} < 1 - \delta$. Hence,

$$1 - \frac{\epsilon}{2} < \|pv_n x^{\frac{1}{2}}p\| \leq \|x^{\frac{1}{2}}p||v_n\| \leq \|x^{\frac{1}{2}}p\|(1 + \frac{1}{n})\delta^{-1} \leq \|x^{\frac{1}{2}}p\|(2 - \delta)\delta^{-1}.$$  

Thus,

$$\|pxp\| = \|x^{\frac{1}{2}}p\|^2 > \left(\frac{\delta}{2 - \delta}(1 - \frac{\epsilon}{2})\right)^2 > 1 - \epsilon.$$  

Therefore, $\alpha$ has the tracial Rokhlin property.  

In the following lemma, we give a (seemingly) stronger equivalent definition of the tracial Rokhlin property for finite group actions on simple C*-algebras.

**Lemma 3.7.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$. Then $\alpha$ has the tracial Rokhlin property if and only if the following holds. For every $\epsilon > 0$, every finite subset $F \subseteq A$, and every positive elements $x, y, z \in A$ with $x \neq 0$ and $\|z\| = 1$, there exist orthogonal projections $(p_g)_{g \in G}$ in $A$ such that, with $p = \sum_{g \in G} p_g$, we have $p \in A^\alpha$ and the following hold:

1. $\|p_g a - ap_g\| < \epsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(p_h) - p_{gh}\| < \epsilon$ for all $g, h \in G$;
3. $(y^2 - ypy - \epsilon)_+ \preceq x$;
4. $\|pzp\| > 1 - \epsilon$.

**Proof.** The backward implication is obvious. For the forward implication, let $\alpha$ has the tracial Rokhlin property and $\epsilon, F, x, y, z$ be as in the statement. We may assume that $F$ is contained in the closed unit ball of $A$. Let $n = |G|$ and put $\epsilon_0 = \min\{\frac{\epsilon}{2}, \frac{1}{2n}, \frac{\epsilon}{2\|z\|}\}$. By [Lin01a, Lemmas 2.5.1 and 2.5.4], there is $\delta_0 > 0$ satisfying the following property: If $b$ is a self-adjoint element in a C*-algebra $B$ and $\epsilon$ is a projection in $B$ such that $\|b - \epsilon\| < n\delta_0$, then there is a projection $p$ in $C^*(b)$ and a unitary $u \in B^\sim$ satisfying $upeu^* = p$ and $\|u - 1\| < \epsilon_0$ where $1$ denotes the unit of $B^\sim$. We may assume that $\delta_0 < \epsilon_0$. Choose $0 < \delta < 1$ such that

$$\left(\frac{\delta}{2 - \delta}(1 - \frac{\delta_0}{2})\right)^2 > 1 - \delta.$$  


Put $z_1 = (z^2 - \delta)$. Since $A$ is simple, [Ph14, Lemma 2.6] implies that there is a positive element $d \in z_1 A z_1$ such that $d \leq x$ and $\|d\| = 1$. Applying Definition 3.2 with $y$ and $F$ as given, with $\frac{\delta}{2}$ in place of $\varepsilon$, and with $d$ in place of $x$, there exist orthogonal projections $(e_g)_{g \in G}$ in $A$ such that, with $e = \sum_{g \in G} e_g$, the following hold:

(i) $\|e_g a - a e_g\| < \frac{\delta}{2}$ for all $a \in F$ and all $g \in G$;
(ii) $\|\alpha_g(e_h) - e_{gh}\| < \frac{\delta}{2}$ for all $g, h \in G$;
(iii) $(y^2 - y e y - \frac{\delta}{2}) \geq d$;
(iv) $\|e e\| > 1 - \frac{\delta}{2}$.

By (iv), similar to the proof of Lemma 3.6 (with $\delta_0$ in place of $\varepsilon$, with $z$ in place of $x$, and with $z_1$ in place of $z$) we get

(v) $\|e e\| > 1 - \delta_0$.

Now we give an argument analogous to the proof of [Ph11, Lemma 3.17] to obtain the desired $p$ in $A^\alpha$. Put $b = \frac{1}{n} \sum_{g \in G} \alpha_g(e)$. Then by (ii), $\|b - e\| < n\delta_0$. Note that $b \in A^\alpha$. By the choice of $\delta_0$, there is a projection $p$ in $A^\alpha$ and a unitary $u \in A^\sim$ such that $u e u^* = p$ and $\|u - 1\| < \varepsilon_0$ where $1$ denotes the unit of $A^\sim$. Put $p_g = u e_g u^*$, for all $g \in G$. Thus $(p_g)_{g \in G}$ are orthogonal projections in $A$ and $p = \sum_{g \in G} p_g$. We have

(vi) $\|p_g - e_g\| < 2\varepsilon_0$, for all $g \in G$, and $\|p - e\| < 2\varepsilon_0$.

Now we prove (1)–(4) in the statement. For (1), by (i) and (vi) we have

$\|p_g a - ap_g\| \leq \|p_g a - e_g a\| + \|e_g a - a e_g\| + \|a e_g - a p_g\| < 5\varepsilon_0 < \varepsilon$.

For (2), Conditions (ii) and (vi) yield

$\|\alpha_g(p_h) - p_{gh}\| \leq \|\alpha_g(p_h) - \alpha_g(e_h)\| + \|\alpha_g(e_h) - e_{gh}\| + \|e_{gh} - p_{gh}\| < 5\varepsilon_0 < \varepsilon$.

To see (3), first by (vi) we get

$\| (y^2 - y_{pp}) - (y^2 - y e y - \frac{\delta}{2})_+ \| \leq \frac{\delta}{2} + \| y_{pp} - y e y \| < \varepsilon_0 + 2\varepsilon_0 \|y\| \leq \varepsilon$.

Hence by (iii) and Lemma 2.1, $(y^2 - y_{pp} - \varepsilon)_+ \preceq (y^2 - y e y - \frac{\delta}{2})_+ \preceq q \preceq x$.

To prove (4), by (v) and (vi) we can compute

$\|p z p\| = \|e e + (p - e) z p + e z (p - e)\|$

$\geq \|e e\| - 2\|p - e\|$

$\geq 1 - \delta_0 - 4\varepsilon_0$

$\geq 1 - \varepsilon$.

Pointwise outerness can be regarded as a form of freeness for actions on noncommutative $C^*$-algebras (see [Ph09]). An action $\alpha : G \to \text{Aut}(A)$ is called pointwise outer if for any $g \in G \setminus \{1\}$, the automorphism $\alpha_g$ is outer, i.e., is not of the form $\text{Ad} u$ for any unitary $u$ in the multiplier algebra of $A$. 
The main use of pointwise outerness in the literature (at least so far) has been in the proof of the simplicity of crossed products. By Lemma 1.5 of [Ph11], any action of a finite group on a unital simple C*-algebra with the tracial Rokhlin property is pointwise outer. We show that this result also holds in our non-unital case.

**Proposition 3.8.** Let $\alpha$ be an action of a finite group $G$ on a non-zero simple C*-algebra $A$. If $\alpha$ has the tracial Rokhlin property then $\alpha_g$ is outer for any $g \in G \setminus \{1\}$.

**Proof.** We will show in Proposition 4.7 that a more general result holds, that is, every action with the weak tracial Rokhlin property is pointwise outer. □

**Remark 3.9.** Let $\alpha$ be an action of a finite group $G$ on a non-zero simple C*-algebra $A$ with the tracial Rokhlin property. If $|G| > 1$ then $A$ is non-elementary. In fact, by Proposition 3.8, $\alpha_g$ is outer for any $g \in G \setminus \{1\}$, but every automorphism of a simple elementary C*-algebra (i.e., $K(H)$ for some Hilbert space $H$) is inner.

**Corollary 3.10.** Let $\alpha$ be an action of a finite group $G$ on a simple C*-algebra $A$. If $\alpha$ has the tracial Rokhlin property then $A \rtimes \alpha G$ is simple.

**Proof.** This follows from [Ki81, Theorem 3.1] and Proposition 3.8. □

**Corollary 3.11.** Let $\alpha$ be an action of a finite group $G$ on a simple C*-algebra $A$. If $\alpha$ has the tracial Rokhlin property then the fixed point algebra $A^\alpha$ is Morita equivalent to $A \rtimes \alpha G$.

**Proof.** By [Ros79], there exists a projection $p$ in multiplier algebra of $A \rtimes \alpha G$ such that $A^\alpha \cong p(A \rtimes \alpha G)p$. Note that $A \rtimes \alpha G$ is simple by Corollary 3.10, and so $A^\alpha$ is isomorphic to a full corner of $A \rtimes \alpha G$. This shows that $A^\alpha$ is Morita equivalent to $A \rtimes \alpha G$. □

The following is a non-unital analogue of [Ph11, Proposition 1.12]. We also do not assume that $G$ is finite.

**Lemma 3.12.** Let $A$ be a simple C*-algebra with Property (SP) and let $\alpha$ be a pointwise outer action of a discrete group $G$ on $A$. Then every non-zero hereditary C*-subalgebra of the reduced crossed product $A \rtimes_{\alpha, r} G$ has a non-zero projection which is Murray-von Neumann equivalent (in $A \rtimes_{\alpha, r} G$) to a projection of $A$. In particular, $A \rtimes_{\alpha, r} G$ has Property (SP).

**Proof.** Assume that $B$ is a hereditary C*-subalgebra of $A \rtimes_{\alpha, r} G$. Since $A$ is simple, by Proposition 1.4 of [PP15] the action $\alpha$ is pointwise spectrally non-trivial. Thus, Lemma 3.2 and Proposition 3.9 of [PP15] imply that there exist a non-zero hereditary C*-subalgebra $E$ of $A$ and an injective *-homomorphism
\( \phi : E \to B \) such that \( \phi(x) \sim x \) in \( A \rtimes_{\alpha,r} G \), for all \( x \in E_+ \). Since \( A \) has Property (SP), there is a non-zero projection \( p \) in \( E \). Hence, \( \phi(p) \) is a projection in \( B \) and \( p \preceq \phi(p) \). So there is a projection \( q \in A \rtimes_{\alpha,r} G \) such that \( q \sim_M p \) and \( q \leq \phi(p) \). Since \( \phi(p) \in B \) and \( B \) is a hereditary C*-subalgebra of \( A \rtimes_{\alpha,r} G \), we have \( q \in B \). This completes the proof.

Alternatively, the statement follows from [JO98, Theorem 4.2] by taking \( N = \{1\} \). Note that the assumption that \( A \) is unital is not used in the proof of [JO98, Theorem 4.2].

3.1. Permanence Properties. In this subsection we show that the definition of the tracial Rokhlin property for finite group actions on not necessarily unital C*-algebras works well with restrictions (to subgroups or certain subalgebras), direct limits, and tensor products of actions.

**Proposition 3.13.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple C*-algebra \( A \) with the tracial Rokhlin property. If \( H \) is a subgroup of \( G \), then the restriction of \( \alpha \) to \( H \) has the tracial Rokhlin property.

**Proof.** Following the proof of Lemma 5.6 of [ELPW10], put \( m = |G|/|H| \) and let \( T \) be a set of right coset representations for \( H \) in \( G \), so \( |T| = m \). Let we are given a finite subset \( F \subseteq A \), \( \varepsilon > 0 \), and positive elements \( x, y \in A \) with \( \|x\| = 1 \). Put \( \varepsilon_0 = \frac{\varepsilon}{m} \). Applying Definition 3.2 to the action \( \alpha : G \to \text{Aut}(A) \) with \( F, x, y \) as given and with \( \varepsilon_0 \) in place of \( \varepsilon \), we get mutually orthogonal projections \( (p_g)_{g \in G} \) such that

1. \( \|p_ga - ap_g\| < \varepsilon_0 \) for all \( a \in F \) and all \( g \in G \);
2. \( \|a_g(p_h) - p_{gh}\| < \varepsilon_0 \) for all \( g, h \in G \);
3. \( (y^2 - ypy - \varepsilon) + \preceq x \);
4. \( \|pxp\| > 1 - \varepsilon_0 \).

Define \( \varepsilon_h = \sum_{t \in T} p_{ht} \) for any \( h \in H \) to obtain a family of mutually orthogonal projections. Then we have \( \|e_ha - a\varepsilon_h\| \leq \sum_{t \in T} \|p_{ht}a - ap_{ht}\| < m\varepsilon_0 = \varepsilon \) for all \( a \in F \) and all \( h \in H \). Also, for all \( k, h \in H \) we have

\[
\|a_h(e_k) - e_h\varepsilon_h\| \leq \sum_{t \in T} \|a_h(p_{kt}) - p_{hkt}\| < m\varepsilon_0 = \varepsilon.
\]

Put \( \varepsilon = \sum_{h \in H} e_h \). Then \( \|e\varepsilon e\| > 1 - \varepsilon \) as \( e = \sum_{g \in G} p_g = p_\varepsilon \). It only remains to show that \( (y^2 - y\varepsilon - \varepsilon)_+ \preceq x \). But this follows from (3) above by noting that \( \varepsilon_0 \leq \varepsilon \) and \( e = p \). \( \square \)

The following lemma is known. It follows from Lemmas 2.5.5 and 2.5.12 of [Lin01a]. We will need this lemma in the proof of Proposition 3.15.

**Lemma 3.14.** For every \( \varepsilon > 0 \) and every \( n \in \mathbb{N} \) there exists \( \delta = \delta(\varepsilon, n) > 0 \) with the following property. If \( A \) is a C*-algebra and \( a_1, \ldots, a_n \) are positive
contractions in $A$ such that $\|a_i^2 - a_i\| < \delta$ and $\|a_ia_j\| < \delta$ for all $i, j = 1, \ldots, n$ with $i \neq j$, then there are mutually orthogonal projections $p_1, \ldots, p_n \in A$ such that $\|a_i - p_i\| < \varepsilon$ for all $i = 1, \ldots, n$.

The following is a non-unital analogue of [Ph11, Lemma 3.7].

**Proposition 3.15** (cf. Proposition 4.13). Let $A$ be a simple $C^*$-algebra and $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ with the tracial Rokhlin property. Let $B$ be a unital $\alpha$-invariant hereditary $C^*$-subalgebra of $A$ and $\beta : G \to \text{Aut}(B)$ be the restriction of $\alpha$ to $B$. Then $\beta$ has the tracial Rokhlin property.

**Proof.** Let $p$ be the unit of $B$. Thus $p$ is a projection in $A^\alpha$, the fixed point algebra. (Note that $pAp$ is $\alpha$-invariant if and only if $p \in A^\alpha$.) Let $\beta : G \to \text{Aut}(B)$ be the restriction of $\alpha$ to $B$. Let $F \subseteq B$ be a finite set, $0 < \varepsilon < 1$, and $x \in B_+$ with $\|x\| = 1$. We have to find orthogonal projections $(q_g)_{g \in G}$ in $B$ such that with $q = \sum_{g \in G} q_g$ the following hold:

1. $\|q_ga - aq_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(q_h) - q_{gh}\| < \varepsilon$ for all $g, h \in G$;
3. $p - q \preceq x$;
4. $\|qxq\| > 1 - \varepsilon$.

We may assume that $p \in F$ and $F \subseteq B_1$ where $B_1$ denotes the closed unit ball of $B$. Set $n = |G|$ and $\varepsilon_1 = \frac{\varepsilon}{5n+1}$. Choose $\delta = \delta(\varepsilon_1, n)$ as in Lemma 3.14. We may assume that $\delta < \varepsilon_1$. Since by assumption $\alpha$ has the tracial Rokhlin property, applying Definition 3.2 with $p$ in place of $y$, with $\delta$ in place of $\varepsilon$, and with $F$ and $x$ as given, there exist orthogonal projections $(e_g)_{g \in G}$ in $A$ such that with $e = \sum_{g \in G} e_g$ the following hold:

1. $\|e_ga - ae_g\| < \delta$ for all $a \in F$ and all $g \in G$;
2. $\|\alpha_g(e_h) - e_{gh}\| < \delta$ for all $g, h \in G$;
3. $(p - epe - \delta) \preceq x$;
4. $\|exe\| > 1 - \delta$.

Put $f_g = pe_gp$, for all $g \in G$. By (i) we have

$$\|f_g^2 - f_g\| = \|pe_gpe_gp - pe_gp\| \leq \|pe_g - e_gp\| < \delta.$$  

Also, by (i), for all $g, h \in G$ with $g \neq h$ we have

$$\|f_gf_h\| = \|pe_gpe_hp\| \leq \|e_gpe_hp\| = \|e_g(pe_h - e_hp)p\| < \delta.$$  

Thus by Lemma 3.14, there are mutually orthogonal projections $(q_g)_{g \in G}$ in $B$ such that

1. $\|q_g - f_g\| < \varepsilon_1$, for all $g \in G$.
Put $q = \sum_{g \in G} q_g$. We show that (1)–(4) above hold. To prove (1), by (i) and (v) for any $a \in F$ and $g \in G$ we have
\[
\|q_g a - a q_g\| \leq \|q_g a - f_g a\| + \|f_g a - a f_g\| + \|a f_g - a q_g\| < 2\varepsilon_1 + \|pe_g p a - ape_g p\| = 2\varepsilon_1 + \|pe_g a p - pa e_g p\| \leq 2\varepsilon_1 + \|e_g a - a e_g\| < 2\varepsilon_1 + \delta < 3\varepsilon_1 < \varepsilon.
\]

To see (2), by (ii) and (v) we have
\[
\|\alpha_g(q_h) - q_{gh}\| \leq \|\alpha_g(q_h) - \alpha_g(f_h)\| + \|\alpha_g(f_h) - f_{gh}\| + \|f_{gh} - q_{gh}\| < 2\varepsilon_1 + \|\alpha_g(pe_h p) - pe_{gh} p\| \leq 2\varepsilon_1 + \|\alpha_g(e_h) - e_{gh}\| < 2\varepsilon_1 + \delta < 3\varepsilon_1 < \varepsilon.
\]

For (3), first by (v) we have
\[
\|(p - q) - (p - pep - \delta)\|_+ \leq \delta + \|q - pep\| < \delta + n\varepsilon_1 < (n + 1)\varepsilon_1 < \varepsilon.
\]
Thus, by Lemma 2.1 and (iii) we have
\[
p - q \sim (1 - \varepsilon)(p - q) = (p - q - \varepsilon)_+ \precsim (p - pep - \delta)_+ \precsim x.
\]

To prove (4), first by (i) we have
\[
\|pexep - exe\| \leq \|pexep - epxep\| + \|exep - exe\| \leq 2\|pe - ep\| < 2n\delta.
\]

Put $f = \sum_{g \in G} f_g$. By (v) we have $\|f - q\| < n\varepsilon_1$. Hence $\|f\| < n\varepsilon_1 + 1 \leq 2$. Using these three last inequalities and (iv) we have
\[
\|qxq\| \geq \|fxf\| - \|(f - q)xf\| - \|qx(f - q)\| \geq \|pepxep\| - n\varepsilon_1\|f\| - n\varepsilon_1 > \|pexep\| - 3n\varepsilon_1 > \|exep\| - 2n\delta - 3n\varepsilon_1 > 1 - \delta - 2n\delta - 3n\varepsilon_1 > 1 - (5n + 1)\varepsilon_1 \geq 1 - \varepsilon.
\]

Therefore, $\beta$ has the tracial Rokhlin property. □

The following lemma follows immediately from Definition 3.2.
Lemma 3.16. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$. Suppose that for every finite set $F \subseteq A$ and every $\varepsilon > 0$ there is an $\alpha$-invariant simple C*-subalgebra $B$ of $A$ such that $F \subseteq_{\varepsilon} B$ and the restriction of $\alpha$ to $B$ has the tracial Rokhlin property. Then $\alpha : G \to \text{Aut}(A)$ has the tracial Rokhlin property.

The following corollary follows immediately from Lemma 3.16. See [Ph17, Proposition 3.24] for the definition of the direct limit of actions.

Corollary 3.17. Let $G$ be a finite group. Let $((G, A_i, \alpha^{(i)})_{i \in I}, (\varphi_{j,i})_{i \leq j})$ be a direct system of simple $G$-algebras. Let $A$ be the direct limit of the $A_i$ and let $\alpha : G \to \text{Aut}(A)$ be the direct limit of the $\alpha^{(i)}$. If each $\alpha^{(i)}$ has the the tracial Rokhlin property then so does $\alpha$.

The following result states a criterion for the non-unital tracial Rokhlin property in terms of the unital tracial Rokhlin property.

Proposition 3.18. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$. Suppose that $A$ has an approximate unit (not necessarily increasing) $(p_i)_{i \in I}$ consisting of projections such that each $p_i$ is in $A^{\alpha}$. Then $\alpha : G \to \text{Aut}(A)$ has the tracial Rokhlin property if and only if the restriction of $\alpha$ to $p_i A p_i$ has the tracial Rokhlin property for every $i \in I$.

Proof. The “if” part follows from Lemma 3.16 and the “only if” part follows from Proposition 3.15.

The next theorem, in particular, enables us to construct examples of finite group actions with the tracial Rokhlin property on simple not necessarily unital C*-algebras. This theorem extends [Wan13, Proposition 2.4.6] to the non-unital case and at the same time removes the assumption of Property (SP).

Theorem 3.19. Let $\alpha : G \to \text{Aut}(A)$ and $\beta : G \to \text{Aut}(B)$ be actions of a finite group $G$ on simple C*-algebras $A$ and $B$. Let $\alpha$ have the tracial Rokhlin property and let $B^{\beta}$ have an approximate unit (not necessarily increasing) consisting of projections. Then the action $\alpha \otimes \beta : G \to \text{Aut}(A \otimes B)$ has the tracial Rokhlin property.

Proof. Let we are given a finite set $F \subseteq A \otimes B$, $\varepsilon > 0$, and $x, y \in (A \otimes B)_+$ with $\|x\| = 1$. We find orthogonal projections $(e_g)_{g \in G}$ in $A \otimes B$ such that, with $e = \sum e_g G$, the following hold:

(i) $\|e_g f - f e_g\| < \varepsilon$ for all $f \in F$ and all $g \in G$;
(ii) $\|e_g (e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$;
(iii) $(y^2 - y e y - \varepsilon)_+ \precsim x$;
(iv) $\|e x e\| > 1 - \varepsilon$. 

□
We may assume that there exist $c_1, \ldots, c_m$ in $A_1$ and $d_1, \ldots, d_m$ in $B_1$ such that $F = \{c_i \otimes d_i \mid 1 \leq i \leq m\}$. By Remark 3.5, we may assume that $y = y_1 \otimes y_2$ for some $y_1 \in A_{+1}$ and $y_2 \in B_{+1}$. Choose $0 < \delta < \frac{\varepsilon}{3}$ such that $((1 - \delta)^2 - 4\delta - \delta^2) > 1 - \varepsilon$. Choose $\frac{1}{2} < \delta_1 < 1$ such that

$$((1 - \delta)^2 - 4\delta - \delta^2) \delta_1^2 > 1 - \varepsilon.$$  

Put $z = (x - \delta_1)_+$. It follows from Kirchberg’s Slice Lemma (Lemma 4.1.9 of [Ro02]) that there are elements $x_1 \in A_+$ and $x_2 \in B_+$ such that $x_1 \otimes x_2 \preceq z$ in $A \otimes B$ and that $\|x_1\| = \|x_2\| = 1$. By Lemma 2.3, there exists $w \in A \otimes B$ such that $\|wxw^* - x_1 \otimes x_2\| < \delta^2$ and $\|w\| \leq \delta_1^{-\frac{1}{2}} < \delta_1^{-1}$. Thus there is $v \in A \otimes B$ (the algebraic tensor product) such that the following holds:

(v) $\|vxv^* - x_1 \otimes x_2\| < \delta^2$ and $\|v\| < \delta_1^{-1}$.

Write $v = \sum_{i=1}^{k} v_i \otimes w_i$ where $v_i \in A$ and $w_i \in B$ for all $i = 1, \ldots, k$. Put $E = \{c_i \mid 1 \leq i \leq m\} \cup \{v_i \mid 1 \leq i \leq k\}$. Proposition 2.7(v) of [KR00] implies that there is $n \in \mathbb{N}$ such that

(vi) $\left(\frac{y_2}{y_2} - \delta_1\right) \preceq x_2 \otimes 1_n$ in $M_\infty(B)$.

By Remark 3.9, $A$ is not elementary and so it follows from Corollary IV.1.2.6 of [B06] that $A$ is not of Type I. Now, Lemma 2.4 of [Ph14] implies that there is a non-zero element $x_0 \in A_+$ such that

(vii) $x_0 \otimes 1_n \preceq x_1$ in $M_\infty(A)$.

Put $M = 1 + \sum_{i=1}^{k} \|v_i\| + \sum_{i=1}^{k} \|w_i\|$. Choose $\eta > 0$ such that $\eta < \frac{\delta}{2Mk|G|}$. Applying Lemma 3.7 to the action $\alpha$ with $E$ in place of $F$, with $\eta$ in place of $\varepsilon$, with $x_0$ in place of $x$, with $y_1$ in place of $y$, and with $x_1$ in place of $z$, there are orthogonal projections $(p_g)_{g \in G}$ in $A$ such that, with $p = \sum_{g \in G} p_g$, the following hold:

(viii) $\|p_g c - cp_g\| < \eta$ for all $c \in E$ and all $g \in G$;
(ix) $\|\alpha_g(p_h) - p_{gh}\| < \eta$ for all $g, h \in G$;
(x) $(y_1^2 - y_1py_1 - \eta)_+ \preceq x_0$ in $A$;
(xi) $\|px_1p\| > 1 - \eta$.

On the other hand, since $B^\beta$ has an approximate unit consisting of projections, there is a projection $q \in B^\beta$ such that the following holds:

(xii) $\|y_2qy_2 - y_2\| < \eta$, $\|[q, d_i]\| < \eta$ for all $1 \leq i \leq m$, $\|[q, w_j]\| < \eta$ for all $1 \leq j \leq k$, and $\|qx_2q\| > 1 - \eta$.

Put $e_g = p_g \otimes q$ for all $g \in G$, and put $e = \sum_{g \in G} e_g$. Then $(e_g)_{g \in G}$ is a family of mutually orthogonal projections in $A \otimes B$. We show that (i)–(iv) hold. For
(i), let $1 \leq i \leq m$. Then by (viii) and (xii) we have

\[
\| [e_g, c_i \otimes d_i] \| = \| [p_g \otimes q, c_i \otimes d_i] \|
\]
\[
= \| (p_g c_i) \otimes (q d_i) - (c_i p_g) \otimes (d_i q) \|
\]
\[
\leq \| [p_g, c_i] \otimes (q d_i) \| + \| (c_i p_g) [q, d_i] \|
\]
\[
< \eta + \eta < 2\delta < \varepsilon.
\]

Part (ii) follows from (ix). To prove (iii), first using (xii) in the third place we have

\[
\| (y^2 - y e y) - (y^2 - y_1 p y_1 - \eta) \| \otimes (y^2 - \delta) \|
\]
\[
\leq \| (y^2 \otimes y^2) - (y_1 p y_1) \otimes (y_2 q y_2) - (y^2 - y_1 p y_1) \| + \eta + \delta
\]
\[
= \| (y_1 p y_1) \otimes (y_2 q y_2 - y^2) \| + 2\delta
\]
\[
< \delta + 2\delta < \varepsilon.
\]

Then by Lemma 2.1 in the first step, by (vi) and (x) in the second step, and by (vii) in the forth step, we get

\[
(y^2 - y e y - \varepsilon) \leq (y^2 - y_1 p y_1 - \eta) \otimes (y^2 - \delta)
\]
\[
\leq x_0 \otimes (x_2 \otimes 1_n)
\]
\[
\sim (x_0 \otimes 1_n) \otimes x_2
\]
\[
\leq x_1 \otimes x_2
\]
\[
\leq z
\]
\[
\leq x.
\]

To prove (iv), first by (viii) and (xii) we have

\[
\| ev - ve \| = \left\| \sum_{i=1}^{k} (p v_i) \otimes (q w_i) - \sum_{i=1}^{k} (v_i p) \otimes (w_i q) \right\|
\]
\[
\leq \sum_{i=1}^{k} \| (p v_i - v_i p) \otimes (q w_i) \| + \sum_{i=1}^{k} \| (v_i p) \otimes (q w_i - w_i q) \|
\]
\[
\leq M \sum_{i=1}^{k} \sum_{g \in G} \| p_g v_i - v_i p_g \| + M \sum_{i=1}^{k} \| q w_i - w_i q \|
\]
\[
< M k |G| \eta + M k \eta
\]
\[
\leq 2 M k |G| \eta
\]
\[
< \delta.
\]
Also, by (xi) and (xii) we get
\[
\|e(x_1 \otimes x_2)e\| = \|(px_1p) \otimes (qx_2q)\|
\]
\[
= \|px_1p\| \cdot \|qx_2q\|
\]
\[
> (1 - \delta)(1 - \eta)
\]
\[
> (1 - \delta)^2.
\]

Then by using these two latter inequalities and (v) we calculate
\[
(1 - \delta)^2 < \|e(x_1 \otimes x_2)e\|
\]
\[
\leq \|e(vxv^* - x_1 \otimes x_2)e\|
\]
\[
< \|vexv^*e\| + \|e(vxv^* - x_1 \otimes x_2)e\|
\]
\[
\leq \|vexv^*e\| + \|ex(v^*e - ev^*)\| + \delta\|v\| + \delta^2
\]
\[
\leq \|v\|^2\|exe\| + \delta\|v\| + \delta\|v\| + \delta^2
\]
\[
\leq \delta_1^{-2}\|exe\| + 2\delta\delta_1^{-1} + \delta^2
\]
\[
< \delta_1^{-2}\|exe\| + 4\delta + \delta^2.
\]

Therefore, by the choice of \(\delta_1\) we get
\[
\|exe\| > (1 - \delta)^2 - 4\delta - \delta^2 > 1 - \varepsilon.
\]

This completes the proof. \(\square\)

Note that the proof of Theorem 3.19 works for the trivial group if we further assume that \(A\) is non-elementary.

The following corollary follows from Theorem 3.19 by taking \(B = M_n\) and \(\beta\) the trivial action.

**Corollary 3.20.** Let \(\alpha: G \to \text{Aut}(A)\) be an action of a finite non-trivial group \(G\) on a simple C*-algebra \(A\). Then the induced inflation action of \(G\) on \(M_n(A)\) has the tracial Rokhlin property for any \(n \in \mathbb{N}\).

Theorem 3.19 can be used to construct examples of actions with the tracial Rokhlin property on non-unital C*-algebras (see Section 3).

**4. Weak Tracial Rokhlin Property**

A C*-algebra should have enough projections to admit actions with the tracial Rokhlin property, this imposes a strong restriction. To tackle this obstruction in the unital case, a weaker version of the tracial Rokhlin property, in which projections are replaced by positive contractions, has been considered, see, e.g., [HO13] and [GHS17]. In this section we define the weak tracial Rokhlin property for finite group actions on simple not necessarily unital C*-algebras similarly to the non-unital tracial Rokhlin property in Section 3. First we
recall the definition of the unital case. There are several similar definitions for the unital case (not all of them are equivalent in general). We discuss the relation between them.

**Definition 4.1** ([GBS17], Definition 2.2). Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple unital C*-algebra \( A \). Then \( \alpha \) has the weak tracial Rokhlin property if for every \( \epsilon > 0 \), every finite set \( F \subseteq A \), and every positive element \( x \in A \) with \( \|x\| = 1 \), there exist positive contractions \( (f_g)_{g \in G} \) in \( A \) such that, with \( f = \sum_{g \in G} f_g \), the following hold:

1. \( \|f_g a - af_g\| < \epsilon \) for all \( a \in F \) and all \( g \in G \);
2. \( \|\alpha_g(f_h) - f_{gh}\| < \epsilon \) for all \( g,h \in G \);
3. \( 1 - f \preceq x \);
4. \( \|fxf\| > 1 - \epsilon \);
5. \( f_g f_h \leq \epsilon \) for all \( g,h \in G \) with \( g \neq h \).

**Definition 4.2** ([Gar17], Definition 3.11). Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple unital C*-algebra \( A \). Then \( \alpha \) has the weak tracial Rokhlin property if for every \( \epsilon > 0 \), every finite set \( F \subseteq A \), and every positive element \( x \in A \) with \( \|x\| = 1 \), there exist orthogonal positive contractions \( (f_g)_{g \in G} \) in \( A \) such that, with \( f = \sum_{g \in G} f_g \), the following hold:

1. \( \|f_g a - af_g\| < \epsilon \) for all \( a \in F \) and all \( g \in G \);
2. \( \|\alpha_g(f_h) - f_{gh}\| < \epsilon \) for all \( g,h \in G \);
3. \( 1 - f \preceq x \);
4. \( \|fxf\| > 1 - \epsilon \).

**Definition 4.3** ([HO13], Definition 5.2). Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple unital C*-algebra \( A \). Then \( \alpha \) has the generalized tracial Rokhlin property if for every \( \epsilon > 0 \), every finite set \( F \subseteq A \), and every non-zero positive element \( x \in A \), there exist normalized positive contractions \( (f_g)_{g \in G} \) in \( A \) such that, with \( f = \sum_{g \in G} f_g \), the following hold:

1. \( \|f_g a - af_g\| < \epsilon \) for all \( a \in F \) and all \( g \in G \);
2. \( \|\alpha_g(f_h) - f_{gh}\| < \epsilon \) for all \( g,h \in G \);
3. \( 1 - f \preceq x \);
4. \( f_g f_h = 0 \) for all \( g,h \in G \) with \( g \neq h \).

Our non-unital weak tracial Rokhlin property is based on Definition 4.2, however, we show that Definitions 4.1 and 4.2 are equivalent and imply Definition 4.3. First we define a non-unital weak tracial Rokhlin property.

**Definition 4.4.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple C*-algebra \( A \). We say that \( \alpha \) has the weak tracial Rokhlin property if for every \( \epsilon > 0 \), every finite set \( F \subseteq A \), and every positive elements \( x,y \in A \)
with \( \|x\| = 1 \), there exist orthogonal positive contractions \((f_g)_{g \in G}\) in \( A \) such that, with \( f = \sum_{g \in G} f_g \), the following hold:

1. \( \| f_g a - a f_g \| < \varepsilon \) for all \( a \in F \) and all \( g \in G \);
2. \( \| \alpha_g(f_h) - f_{gh} \| < \varepsilon \) for all \( g, h \in G \);
3. \( (y^2 - y f y - \varepsilon)_+ \preceq x \);
4. \( \| f x f \| > 1 - \varepsilon \).

**Remark 4.5.** A result similar to Lemma 3.3 holds for the weak tracial Rokhlin property (Definition 4.4). More precisely, Lemma 3.3 holds if we replace “orthogonal projections” by “orthogonal positive contractions.” In particular, if Definition 4.4 holds for some \( y \in A_+ \) then it also holds for every positive element \( z \in \mathcal{A}_{\bar{y}} \). Therefore, in Definition 4.4 if moreover \( A \) is unital then \( \alpha \) has the weak tracial Rokhlin property (in the sense of Definition 4.4) if and only if the condition of Definition 4.4 holds only for \( y = 1 \) (and every \( \varepsilon, F, x \) as in that definition).

The following result shows, in particular, that in the unital case, Definition 4.4 is equivalent to Definition 4.2. Thus our non-unital definition extends the unital definition. Some parts of the following proposition should be known (especially, the equivalence of parts (a) and (c)). However, we provide a proof for the completeness.

**Proposition 4.6.** Let \( \alpha : G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple unital C*-algebra \( A \). The following statements are equivalent:

(a) \( \alpha \) has the weak tracial Rokhlin property in the sense of Definition 4.2;
(b) \( \alpha \) has the weak tracial Rokhlin property in the sense of Definition 4.4;
(c) \( \alpha \) has the weak tracial Rokhlin property in the sense of Definition 4.1.

Moreover, in this case the action \( \alpha \) has the generalized tracial Rokhlin property in the sense of Definition 4.3.

**Proof.** The implication (a) \( \Rightarrow \) (c) is clear. To prove (c) \( \Rightarrow \) (b), by Remark 4.5, it is enough to take \( y = 1 \) in Definition 4.4. Let \( \varepsilon > 0 \), \( F \subseteq A \) be a finite set, and \( x \in A_+ \) with \( \|x\| = 1 \). We may assume that \( F \subseteq A_1 \). Set \( n = |G| \). By [Lin01a, Lemma 2.5.12], there exists \( \delta > 0 \) such that if \((f_g)_{g \in G}\) are positive contractions in \( A \) satisfying \( \|f_g f_h\| < \delta \) for all \( g, h \in G \) with \( g \neq h \), then there are orthogonal positive contractions \((a_g)_{g \in G}\) in \( A \) such that \( \|f_g - a_g\| < \frac{\delta}{4n} \).

We may assume that \( \delta < \frac{\varepsilon}{3} \). Applying Definition 4.1 with \( F, x \) as given and with \( \delta \) in place of \( \varepsilon \), we obtain positive contractions \((f_g)_{g \in G}\) in \( A \) that satisfy Conditions (1)–(5) of Definition 4.1. By the choice of \( \delta \), there exist orthogonal positive contractions \((a_g)_{g \in G}\) in \( A \) such that \( \|f_g - a_g\| < \frac{\varepsilon}{4n} \). Then it is not difficult to see that \((a_g)_{g \in G}\) satisfy Definition 4.4.
To show (b) ⇒ (a), let $F$, $x$, $\varepsilon$ be as in Definition 4.2. We will find orthogonal positive contractions $(f_g)_{g \in G}$ in $A$ satisfying (1)-(4) of Definition 4.2. We may assume that $F \subseteq A_1$. Choose $0 < \delta < \frac{\varepsilon}{2n+1}$. Applying Definition 4.4 with $\delta$ in place of $\varepsilon$, with $y = 1$, and with $x, F$ as given, there are orthogonal positive contractions $(e_g)_{g \in G}$ in $A$ such that with $e = \sum_{g \in G} e_g$ the following hold:

(i) $\|e_g a - ae_g\| < \delta$ for all $a \in F$ and all $g \in G$;
(ii) $\|e_g (e_h) - e_{gh}\| < \delta$ for all $g, h \in G$;
(iii) $(1 - e - \delta)_+ \lesssim x$;
(iv) $\|exe\| > 1 - \delta$.

Write $G = \{g_1, \ldots, g_n\}$. Define a c.p.c. order zero map $\phi : \mathbb{C}^n \to A$ by

$$\phi(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i e_{g_i}.$$ 

Then $\phi(1) = e$. Let $\eta : [0, 1] \to [0, 1]$ be the continuous function defined by

$$\eta(\lambda) = \begin{cases} 
(1 - \delta)^{-1}\lambda & 0 \leq \lambda \leq 1 - \delta, \\
1 & 1 - \delta < \lambda \leq 1. 
\end{cases}$$

Using the functional calculus for c.p.c. order zero maps ([WZ08, Corollary 4.2]), define $\psi = \eta(\phi)$. Thus $\psi : \mathbb{C}^n \to A$ is a c.p.c. order zero map. Similar to the argument given in the proof of [ABP17, Lemma 2.8], we conclude that $\|\psi(z) - \phi(z)\| \leq \delta\|z\|$ for all $z \in \mathbb{C}^n$ with $\|z\| = 1$, and that

$$1 - \psi(1) = \frac{1}{1 - \delta} (1 - \phi(1) - \delta)_+ \sim (1 - \phi(1) - \delta)_+ \lesssim x.$$ 

Put $f_{g_i} = \psi(E_i)$, where $E_i$ is the element of $\mathbb{C}^n$ having 1 in the $i$-th coordinate and 1 elsewhere. Thus $(f_g)_{g \in G}$ are orthogonal positive contractions in $A$ and we have

(v) $\|f_g - e_g\| \leq \delta$ for all $g \in G$.

Moreover, with $f = \sum_{g \in G} f_g$ we have $1 - f = 1 - \psi(1) \lesssim x$. Thus Condition (3) in Definition 4.2 holds. It is easy to see that (i) and (v) imply Condition (1), and (ii) together with (v) yield Condition (2) in Definition 4.2. Finally, Condition (4) in Definition 4.2 follows from (iv) and (v). This completes the proof of (b) ⇒ (a).

The last part of the statement is well known. □

Actions on simple unital C*-algebras with the weak tracial Rokhlin property are pointwise outer (by [HO13, Proposition 5.3] and Proposition 4.6). We prove that this result also holds in our non-unital context, however the proof is more technical.
Proposition 4.7. Let $\alpha$ be an action of a finite group $G$ on a simple $C^*$-algebra $A$. If $\alpha$ has the weak tracial Rokhlin property then $\alpha_g$ is outer for any $g \in G \setminus \{1\}$.

Proof. Suppose to the contrary that there is $g_0 \in G \setminus \{1\}$ and $u \in U(M(A))$ such that $\alpha_{g_0} = \text{Ad} \: u$. Choose $0 < \varepsilon < 1$ such that

$$
\frac{\sqrt{1 - \varepsilon - n\varepsilon}}{n} > 0 \text{ and } \left(\frac{\sqrt{1 - \varepsilon - n\varepsilon}}{n}\right)^2 - 4\varepsilon > 0
$$

where $n = |G|$. By [Lin01a, Lemma 2.5.11] (with $f(t) = t^{1/2}$ there), there is $\delta > 0$ such that if $x, y \in A$ are positive contractions with $\|xy - yx\| < \delta$ then $\|x^{1/2}y - yx^{1/2}\| < \varepsilon$. We may assume that $\delta < \varepsilon$. Choose a positive element $b \in A^\alpha$ with $\|b\| = 1$. Applying Definition 4.4 with $F = \{b, bu^*\}$, $\delta$ in place of $\varepsilon$, $x = b$, and $y = 0$, there are positive contractions $\{f_g\}_{g \in G}$ in $A$ such that

1. $\|f_gb - bf_g\| < \delta$ and $\|f_gb u^* - bu^*f_g\| < \delta$ for all $g \in G$;
2. $\|\alpha_g(f_h) - f_{gh}\| < \delta$ for all $g, h \in G$;
3. with $f = \sum_{g \in G} f_g$ we have $\|bf\| > 1 - \delta$.

Using (1) above we have ($e$ denotes the neutral element of $G$):

$$
\|fb^{1/2} - \sum_{g \in G} \alpha_g(eb^{1/2})\| = \|\sum_{g \in G} f_gb^{1/2} - \sum_{g \in G} \alpha_g(f_e)b^{1/2}\|
$$

$$
\leq \sum_{g \in G} \|f_g - \alpha_g(f_e)\|
$$

$$
< n\delta.
$$

Thus, using (3) above we have

$$
n\delta > \|fb^{1/2}\| - \sum_{g \in G} \|\alpha_g(f_e b^{1/2})\| \geq \|bf\|^{1/2} - n\|f_e b^{1/2}\| > \sqrt{1 - \delta - n\|f_e b^{1/2}\|}.
$$

Hence,

$$
\|f_e b^{1/2}\| > \frac{\sqrt{1 - \delta - n\delta}}{n} > \frac{\sqrt{1 - \varepsilon - n\varepsilon}}{n}.
$$

By (1), $\|f_e b - bf_e\| < \delta$ and so $\|f_e^{1/2} b - bf_e^{1/2}\| < \varepsilon$. Thus,

$$
\|f_e b - f_e^{1/2} bf_e^{1/2}\| \leq \|f_e^{1/2} b - bf_e^{1/2}\| < \varepsilon.
$$

Similarly, since $\|fgb - bf_g\| < \delta$ we have

$$
\|fg_0 b - f_{g_0}^{1/2} bf_{g_0}^{1/2}\| < \varepsilon.
$$

Note that $f_{g_0}^{1/2} bf_{g_0}^{1/2} = f_e^{1/2} bf_e^{1/2}$ and thus by (4.1) we have

$$
\|f_{g_0}^{1/2} bf_{g_0}^{1/2} - f_e^{1/2} bf_e^{1/2}\| \geq \|f_{g_0}^{1/2} bf_{g_0}^{1/2}\| = \|f_e^{1/2} bf_e^{1/2}\|^2 \geq \|f_e b^{1/2}\|^2 > \left(\frac{\sqrt{1 - \varepsilon - n\varepsilon}}{n}\right)^2.
$$
Moreover, using (1) we have
\[\|uf_ebu^* - bf_e\| = \|uf_ebu^* - \alpha_{g_0}(b)f_e\|\]
\[= \|uf_ebu^* - ubu^*f_e\|\]
\[= \|f_ebu^* - bu^*f_e\| < \delta.\] (4.4)

Finally, using (2), (4.2), (4.3), and (4.4) we have
\[\|\alpha_{g_0}(f_e b) - uf_ebu^*\| \geq \|\frac{1}{n}f_{g_0}^2bf_{g_0} - \frac{1}{n}f_e^2bf_e\| \geq \|\frac{1}{n}f_{g_0}^2bf_{g_0} - f_{g_0}b\|
- \|f_{g_0}b - \alpha_{g_0}(f_e b)\| - \|uf_ebu^* - bf_e\|
- \|bf_e - f_{g_0}^2bf_{g_0}\|
> \left(\frac{\sqrt{1 - \varepsilon - n\varepsilon}}{n}\right)^2 - \varepsilon - 2\delta - \varepsilon
> \left(\frac{\sqrt{1 - \varepsilon - n\varepsilon}}{n}\right)^2 - 4\varepsilon > 0\] which is a contradiction. \(\square\)

The proofs of the following two corollaries are similar to the proofs of Corollaries 3.10 and 3.11 (except that we need Proposition 4.7 instead of Proposition 3.8).

**Corollary 4.8.** Let \(\alpha\) be an action of a finite group \(G\) on a simple C*-algebra \(A\). If \(\alpha\) has the weak tracial Rokhlin property, then \(A \rtimes_\alpha G\) is simple.

**Corollary 4.9.** Let \(\alpha\) be an action of a finite group \(G\) on a simple C*-algebra \(A\). If \(\alpha\) has the weak tracial Rokhlin property, then the fixed point algebra \(A^\alpha\) is Morita equivalent to \(A \rtimes_\alpha G\).

Nawata introduced a notion of Rokhlin property for finite group actions on \(\sigma\)-unital C*-algebras using projections in the central sequence algebra of the given C*-algebra [Naw16]. This definition coincides with Izumi’s definition for unital separable C*-algebras. Later, Santiago in [San15] defined a notion of Rokhlin property for actions of finite groups on non-unital C*-algebras which generalizes Nawata’s definition. We show that an action of a finite group on a simple C*-algebra with the Rokhlin property has the weak tracial Rokhlin property. First we recall the Santiago’s definition.

**Definition 4.10** ([San15], Definition 3.2). Let \(A\) be a (not necessarily unital) C*-algebra and \(\alpha : G \to \text{Aut}(A)\) be an action of a finite group \(G\) on \(A\). We say that \(\alpha\) has the **Rokhlin property** if for any \(\varepsilon > 0\) and any finite set \(F \subseteq A\) there exist mutually orthogonal positive contractions \((r_g)_{g \in G}\) such that

(i) \(\|\alpha_g(r_h) - r_{gh}\| < \varepsilon\) for all \(g, h \in G\);

(ii) \(\|r_ga - ar_g\| < \varepsilon\) for all \(g \in G\) and all \(a \in F\);
(iii) \(\|(\sum_{g \in G} r_g)a - a\| < \varepsilon\) for all \(a \in F\).

**Proposition 4.11.** Let \(A\) be a simple \(C^*\)-algebra and \(\alpha : G \to \text{Aut}(A)\) be an action of a finite group \(G\) on \(A\). If \(\alpha\) has the Rokhlin property (Definition 4.10) then it has the weak tracial Rokhlin property (Definition 4.4).

**Proof.** Let \(\alpha\) has the Rokhlin property (Definition 4.10). Let \(x, y \in A_+\) with \(\|x\| = 1\), \(F \subseteq A\) be a finite set, and \(\varepsilon > 0\). We may assume that \(y \neq 0\) and \(x, y \in F\). Also, by Lemma 3.3 we may further assume that \(\|y\| = 1\). By Definition 4.10, there exist mutually orthogonal positive contractions \((f_g)_{g \in G}\) such that, with \(f = \sum_{g \in G} f_g\), we have:

(i) \(\|\alpha_g(f_h) - f_{gh}\| < \frac{\varepsilon}{2}\) for all \(g, h \in G\);

(ii) \(\|f_ga - af_g\| < \frac{\varepsilon}{2}\) for all \(g \in G\) and all \(a \in F\);

(iii) \(\|fa - a\| < \frac{\varepsilon}{2}\) for all \(a \in F\).

Thus (1) and (2) in Definition 4.4 are satisfied. Since \(y \in F\), by (iii) we have

\[\|y^2 - yfy\| \leq \|y - yf\| < \frac{\varepsilon}{2} < \varepsilon.\]

Thus \((y^2 - yfy - \varepsilon)_+ = 0 \leq x\). Hence (3) in Definition 4.4 is also satisfied. To prove (4), by (iii) and that \(x \in F\) we have

\[\|fxf - x\| \leq \|fxf - x\| + \|xf - x\| < \|fx - x\| + \|xf - x\| < \varepsilon.\]

Thus \(\|fxf\| > \|x\| - \varepsilon = 1 - \varepsilon\). Therefore, \(\alpha\) has the weak tracial Rokhlin property. \(\square\)

**Remark 4.12.** Many results which are proved for the tracial Rokhlin property have analogues for the weak tracial Rokhlin property in which one considers “orthogonal positive contractions” instead of “orthogonal projections.” More precisely, we mean Lemma 3.3, Remark 3.5, Lemma 3.6, Lemma 3.7 (without the requirement that \(p \in A^0\)), Remark 3.9 (except that the action of the trivial group on any simple \(C^*\)-algebra has the weak tracial Rokhlin property), Proposition 3.13, Lemma 3.16, Corollary 3.17, Proposition 3.18, and Corollary 3.20 (except that it holds also for the trivial group). Some other results with a different statement are addressed in the sequel.

**Proposition 4.13** (cf. Proposition 3.15). Let \(A\) be a simple \(C^*\)-algebra and \(\alpha : G \to \text{Aut}(A)\) be an action of a finite group \(G\) with the weak tracial Rokhlin property. Let \(B\) be an \(\alpha\)-invariant hereditary \(C^*\)-subalgebra of \(A\) and \(\beta : G \to \text{Aut}(B)\) be the restriction of \(\alpha\) to \(B\). Then \(\beta\) has the weak tracial Rokhlin property.

**Proof.** The proof is similar to the proof of Proposition 3.15 except that one needs [Lin01a, Lemma 2.5.12] instead of Lemma 3.14. More precisely, first choose an approximate unit \((e_i)_{i \in I}\) for \(B\) such that each \(e_i\) is in \(B^\beta\). Let us
are given a finite set $F \subseteq B$, $\varepsilon > 0$, and $x, y \in B_+$ with $\|x\| = 1$. Choose $i \in I$ such that $e_i a$, $ae_i$, and $e_i ae_i$ are all sufficiently close to $a$, for each $a \in F$. Choose a sufficiently small $\delta$ according to [Lin01a, Lemma 2.5.12]. Applying Definition 4.4 to $\alpha$ with $x, y$ as given, with $F \cup \{e_i\}$ in place of $F$, and with $\delta$ in place of $\varepsilon$, we get orthogonal positive contractions $(e_g)_{g \in G}$. Put $f_g = e_i e_g e_i$, for all $g \in G$. Then $(f_g)_{g \in G}$ is a family of almost orthogonal positive contractions in $B$ satisfying the condition of Definition 4.4 (with a smaller $\varepsilon$). Applying [Lin01a, Lemma 2.5.12], we obtain a family of orthogonal positive contractions $(r_g)_{g \in G}$ in $B$ such that each $r_g$ is close to $f_g$. Now it is easy to see (similar to the proof of Proposition 3.15) that the family $(r_g)_{g \in G}$ has the desired properties.

**Proposition 4.14.** Let $\alpha: G \to \text{Aut}(A)$ and $\beta: G \to \text{Aut}(B)$ be actions of a finite group $G$ on simple C*-algebras $A$ and $B$. If $\alpha$ has the weak tracial Rokhlin property then so does $\alpha \otimes \beta: G \to \text{Aut}(A \otimes B)$.

**Proof.** First observe that $B$ has an approximate unit $(u_i)$ such that each $u_i$ is in $B^\beta$. Indeed, if $(v_i)$ is an approximate unit for $A$ then we can take $u_i = \frac{1}{|C|} \sum_{g \in G} \beta_g(v_i)$. Using this observation, the statement can be concluded by a similar argument given in proof of Theorem 3.19. $\square$

5. **C*-algebras with Property (T$_k$)**

In this section, we define the notion of Property (T$_k$) for simple C*-algebras (Definition 5.1), where $k$ is a non-negative integer, and we study some of its properties. The main aim of this section is to make preparations for Section 6 in which we show that tracial topological rank zero is preserved under taking crossed products by actions of finite groups with the tracial Rokhlin property. We also consider the tracial topological rank at most $k$. We show that a simple C*-algebra $A$ has Property (T$_0$) if and only if $\text{TR}(A) = 0$. Also, $A$ has Property (T$_k$) if and only if $\text{TR}(A) \leq k$ and $A$ has an approximate unit consisting (not necessarily increasing) of projections.

We recall that Lin in [Lin01b] first gave the definition of tracial topological rank for unital C*-algebras (Definition 2.11), and then he defined the tracial topological rank of a non-unital C*-algebra to be the tracial topological rank of its minimal unitization. However, working with the unitization of C*-algebras is not always convenient. Moreover, the unitization of a simple non-unital C*-algebra is not a simple C*-algebra and so one can not use the techniques special to simple C*-algebras. Therefore, to study the crossed products of simple non-unital C*-algebras of tracial topological rank at most $k$, we first develop the approach of Property (T$_k$) which unifies both unital and non-unital case in a single definition (Definition 5.1).
The main result we need to use in Section 6 is that tracial topological rank zero is preserved under the Morita equivalence in the class of simple (not necessarily unital nor separable) C*-algebras (and a similar result for tracial topological rank at most $k$). This result may be proved using generalizations of some results of [Lin01b] to the non-unital simple case (such as the direct limit), and a result from [BP91] which says that real rank zero is preserved under Morita equivalence. However, in this section we use Property (T$_k$) in a systematic way to prove this result.

**Definition 5.1.** Let $A$ be a simple C*-algebra and let $k$ be a non-negative integer. We say that $A$ has Property (T$_k$) if $A$ has an approximate unit (not necessarily increasing) consisting of projections and for every positive elements $x, y \in A$ with $x \neq 0$, every finite set $F \subseteq A$, and every $\varepsilon > 0$, there is a C*-subalgebra $E \subseteq A$ with $E \in \mathcal{I}(k)$ such that, with $p = 1_E$, the following hold:

1. $\|pa - ap\| < \varepsilon$ for all $a \in F$;
2. $pFp \subseteq \varepsilon E$;
3. $(y^2 - ypy - \varepsilon)_+ \lesssim x$;
4. $\|pxp\| > \|x\| - \varepsilon$.

We need the following lemma to compare Property (T$_k$) with the tracial topological rank of simple unital C*-algebras. The proof is very similar to that of Lemma 3.3 and so it is omitted.

**Lemma 5.2.** Let $A$ be a C*-algebra, $x \in A_+$ be non-zero, and $k$ be a non-negative integer. Let $y \in A_+$ have the following property. For any finite set $F \subseteq A$ and any $\varepsilon > 0$ there exist a projection $p \in A$ and a C*-subalgebra $E \in \mathcal{I}(k)$ of $A$ with unit $p$ such that the following hold:

1. $\|pa - ap\| < \varepsilon$ for all $a \in F$;
2. $pFp \subseteq \varepsilon E$;
3. $(y^2 - ypy - \varepsilon)_+ \lesssim x$;
4. $\|pxp\| > \|x\| - \varepsilon$.

Then every positive element $z \in \overline{Ay}$ also has the above property of $y$.

The following proposition states the relation between Property (T$_k$) and tracial topological rank at most $k$ for simple unital C*-algebras.

**Proposition 5.3.** Let $A$ be a simple unital C*-algebra and let $k$ be a non-negative integer. The following statements are equivalent:

1. $A$ is has Property (T$_k$);
2. $\text{TR}(A) \leq k$;
3. $\text{TR}_w(A) \leq k$;
4. for any $x, y, \varepsilon, F$ as in Definition 5.1 there is a non-zero C*-subalgebra $E \subseteq A$ with $E \in \mathcal{I}(k)$ such that (1), (2), and (3) in Definition 5.1 hold.
\textbf{Proof.} By Theorem 2.14 we have (ii) ⇔ (iii). The implication (i) ⇒ (iv) is obvious. Moreover, (iv) ⇒ (iii) follows from Theorem 2.14 and by applying (iv) with \( y = 1 \) and using the fact that for \( 0 < \varepsilon < 1 \),
\begin{equation}
(1 - p - \varepsilon)_+ = (1 - \varepsilon)(1 - p) \sim (1 - p).
\end{equation}

To see (iii) ⇒ (i), note that (iii) together with (5.1) imply that Definition 5.1 is satisfied for \( y = 1 \). Now by Lemma 5.2, Definition 5.1 is satisfied for every \( y \in A_+ \). Therefore, (i) holds. \( \Box \)

We need the following lemma in the proof of Proposition 5.5.

\textbf{Lemma 5.4.} Let \( A \) be a simple \( C^* \)-algebra with Property \((T_k)\). Then every unital hereditary \( C^* \)-subalgebra of \( A \) has Property \((T_k)\).

\textbf{Proof.} The proof is similar to the proof of [Lin01a, Lemma 3.6.5] with some differences. Let \( B = qAq \) be a unital hereditary \( C^* \)-subalgebra of \( A \) where \( q \) is a projection of \( A \). Let \( F \subseteq B \) be a finite subset, \( x, y \in B_+ \) with \( x \neq 0 \), and \( \varepsilon > 0 \). We may assume that \( F \cup \{ x, y \} \subseteq B_1 \), the closed unit ball of \( B \). Put \( G = F \cup \{ q \} \). Choose \( \delta > 0 \) with \( \delta < \min \{ \frac{1}{6}, \frac{\varepsilon}{43} \} \). Since \( A \) has Property \((T_k)\), there is a subalgebra \( E \in \mathcal{I}(k) \) such that, with \( p = 1_E \), the following hold:
\begin{enumerate}
\item \( \|pa - ap\| < \delta \) for all \( a \in F \);
\item \( pGp \subseteq E \);
\item \( (y^2 - ypy - \delta)_+ \preceq x \);
\item \( \|pxp\| > \|x\| - \delta \).
\end{enumerate}

Then by (i) we have
\[
\| (qpq)^2 - qpq \| = \| qpq - qpq \| \leq \| qpq - pq \| < \delta.
\]

Thus by [Lin01a, Lemma 2.5.5] (note that the assumption \( \|a\| \geq \frac{1}{2} \) is unnecessary in the statement of [Lin01a, Lemma 2.5.5]), there is a projection \( q_1 \in B \) such that:
\begin{enumerate}
\item \( \|q_1 - qpq\| < 2\delta \).
\end{enumerate}

By (ii) there is \( e \in E \) such that \( \|qpq - e\| < \delta \). Then \( \|q_1 - e\| < 3\delta \). Thus by [Lin01a, Lemma 2.5.4] (note that the assumption that \( a \) is self-adjoint is unnecessary in the statement of [Lin01a, Lemma 2.5.4]), there is a projection \( e \in E \) such that:
\begin{enumerate}
\item \( \|q_1 - e\| < 6\delta \).
\end{enumerate}

Hence by [Lin01a, Lemma 2.5.1], there is a unitary \( u \in A^\sim \) such that:
\begin{enumerate}
\item \( u^*eu = q_1 \) and \( \|u - 1_{A^\sim}\| < 12\delta \).
\end{enumerate}

Put \( D = u^*eEeu \). Then \( D \) is in \( \mathcal{I}(k) \) which is a subalgebra of \( B \) since \( D = q_1u^*Eeuq_1 \subseteq qAq = B \), and \( 1_D = u^*eu = q_1 \). We show that:
\begin{enumerate}
\item \( \|q_1a - aq_1\| < \varepsilon \) for all \( a \in F \);
\end{enumerate}
(2) \( q_1 F q_1 \subseteq \varepsilon D; \)
(3) \( (y^2 - yq_1 y - \varepsilon)_+ \preceq x; \)
(4) \( \|q_1 x q_1\| > \|x\| - \varepsilon. \)

By (i) and (v) we have \( \|q_1 - pq\| \leq \|q_1 - qpq\| + \|qpq - pq\| < 3\delta. \) Thus,
(viii) \( \|q_1 - pq\| = \|q_1 - qp\| < 3\delta. \)
To see (1), by (i) and (viii) for all \( a \in F \) we have
\[
\|q_1 a - a q_1\| \leq \|q_1 a - pqa\| + \|pa - ap\| + \|aqp - aq_1\| < 7\delta < \varepsilon.
\]
To show (2) let \( a \in F. \) By (ii) there is \( b \in E \) such that \( \|pab - b\| < \delta. \) Put \( d = u^* e beu \in D. \) Then by (vi), (vii), and (viii) we have:
\[
\|q_1 a q_1 - d\| \leq \|q_1 a q_1 - eq_1 a q_1 e\| + \|eq_1 a q_1 e - epqaqpe\|
+ \|epape - ebe\| + \|ebe - d\|
< 12\delta + 6\delta + \delta + 24\delta = 43\delta < \varepsilon.
\]
To see (4), by (iv) and (viii) we have
\[
\|q_1 x q_1\| \geq \|pq x q p\| - 6\delta = \|px p\| - 6\delta > \|x\| - 7\delta > \|x\| - \varepsilon.
\]
To prove (3) first we have
\[
\|(y^2 - ypy - \delta)_+ - (y^2 - yq_1 y)\| \leq \delta + \|(y^2 - ypy) - (y^2 - yq_1 y)\|
= \delta + \|ypq y - yq_1 y\|
< 4\delta < \varepsilon.
\]
Therefore, \( (y^2 - yq_1 y - \varepsilon)_+ \preceq (y^2 - ypy - \delta)_+ \preceq x. \)

To compare Property \((T_k)\) with the tracial topological rank at most \( k \) for simple not necessarily unital \( C^*\)-algebras, we need the following result.

**Proposition 5.5.** Let \( A \) be a simple non-unital \( C^*\)-algebra and let \( k \) be a non-negative integer. If \( A \) has Property \((T_k)\) then the following holds. For every \( \varepsilon > 0, \) every \( n \in \mathbb{N}, \) every non-zero positive element \( x \in A^\sim, \) every finite subset \( F \subseteq A^\sim \) which contains a non-zero positive element \( x_1, \) and every \( \sigma_i, 1 \leq i \leq 4, \) with \( 0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1, \) there is a \( C^*\)-subalgebra \( E \subseteq A \) with \( E \in \mathcal{I}(k) \) such that, with \( p = 1_E, \) the following hold:

1. \( \|pa - ap\| < \varepsilon \) for all \( a \in F; \)
2. \( p F p \subseteq \varepsilon E \) and \( \|px_1 p\| \geq \|x_1\| - \varepsilon; \)
3. \( 1 - p \preceq x \) and \( n[1 - p] \leq [p]; \)
4. \( n[f^{\sigma_1}_{\sigma_4}(1 - p)x_1(1 - p))] \leq [f^{\sigma_4}_{\sigma_1}(px_1 p)]. \)

The converse holds for \( k = 0. \)
Proof. To prove the first part of the statement let $A$ be a simple non-unital C*-algebra with Property (T) $k$. By Definition 5.1, there is a net $(p_i)_{i \in I}$ of projections in $A$ which is an approximate unit for $A$. For any $y \in A^\sim$ we have

\[ ||y|| = \lim_{i \to \infty} ||p_iyp_i||. \tag{5.2} \]

In fact, write $y = \lambda + a$ where $\lambda \in \mathbb{C}$ and $a \in A$. Since $A$ is not unital we have $||y|| = \sup\{||yb|| : b \in A_1\}$ where $A_1$ denotes the closed unit ball of $A$. Let $\delta > 0$. Then there is $b \in A_1$ such that $||yb|| > ||y|| - \delta$. Note that $p_iyp_ib = \lambda p_i b + p_iap_i b$ which tends to $\lambda b + ab = yb$. Thus there is $j \in I$ such that $||p_iyp_ib|| > ||y|| - \delta$ for all $i \geq j$. Then for every $i \geq j$ we have

\[ ||y|| - \delta < ||p_iyp_ib|| \leq ||p_iyp_i|| \leq ||y|| \]

and so (5.2) holds.

Now let $\varepsilon, n, F, x_1,$ and $x$ be as in the statement. Write $F = \{x_1, \ldots, x_m\}$ and $x_j = \lambda_j + a_j$ where $\lambda_j \in \mathbb{C}$ and $a_j \in A$ for all $1 \leq j \leq m$. Choose $d_3, d_4$ with $\sigma_4 < d_3 < d_4 < \sigma_1$. By [Lin01b, Lemma 2.6], there exists $\eta > 0$ such that if $a, b \in A^\sim$ are positive elements with $||a||, ||b|| \leq ||x_1||$ and $||a - b|| < \eta$ then $[f^{d_4}_{d_3}(a)] \leq [f^{d_4}_{d_3}(b)]$. (Note that in Lemma 2.6 of [Lin01b] it is assumed that $||a||, ||b|| \leq 1$ but the proof of this lemma works for any $M > 0$ instead of 1.) Choose $\delta$ with $0 < \delta < \min\{\frac{\varepsilon}{4}, \frac{\eta}{3}\}$. By the previous remark and that $(p_i)_{i \in I}$ is an approximate unit for $A$, there is $i \in I$ such that, with $a_j' = p_ia_jp_i$, the following hold:

(i) $||p_ix_jp_i|| > ||x_j|| - \delta$;

(ii) $||a_jp_i - a_j|| < \delta$ and $||p_ia_j - a_j|| < \delta$ for all $1 \leq j \leq m$;

(iii) $||a_j' - a_j|| < \delta$ for all $1 \leq j \leq m$;

(iv) $p_ix_jp_i \neq 0$.

Set $G = \{a_j' \mid 1 \leq j \leq m\}$. By Lemma 5.4, $B = p_iAp_i$ has Property (T) $k$ and hence $\text{TR}(B) \leq k$ by Proposition 5.3. Then, by [Lin01b, Theorem 5.6], there is a C*-subalgebra $D \subseteq B$ with $D \in \mathcal{I}^{(k)}(k)$ such that, with $q = 1_D$, the following hold:

(v) $||qa_j' - a_j'q|| < \delta$ for all $1 \leq j \leq m$;

(vi) $qGq \subseteq D$ and $||qb_1q|| \geq ||b_1|| - \delta$ where $b_1 = p_ix_1p_i$;

(vii) $p_i - q \preceq p_ixp_i$ and $[p_i - q] \leq [q]$.

(viii) $n[f^{d_4}_{d_3}(p_i - q) b_1(p_i - q))] \leq [f^{d_4}_{d_3}(qb_1q)]$.

Put $E = C(1 - p_i) + D$ and $p = 1 - p_i + q$ which is the unit of $E$ (here 1 denotes the unit of $A^\sim$). Then $E \in \mathcal{I}^{(k)}$ (see Remark 2.8). We show that (1)–(4) in the statement hold. To see (1), by (iii) and (v), for all $1 \leq j \leq m$ we have

\[ ||px_j - x_jp|| = ||pa_j - a_jp|| \leq ||pa_j - pa_j'|| + ||pa_j' - a_jp|| + ||a_j'p - a_jp|| < 2\delta + ||qa_j' - a_j'q|| < 3\delta < \varepsilon. \]
To see (2), fix $1 \leq j \leq m$. By (vi) there is $d \in D$ such that $\|qa_j'q - d\| < \delta$. Put $e = \lambda_j p + d \in E$. Then by (ii) we have

$$\|px_j p - e\| = \|pa_j p - d\| \leq \|qa_j'q - d\| + \|pa_j p - qa_j'q\|
< \delta + \|(1 - p_i)a_j(1 - p_i) + (1 - p_i)aq + qa_j(1 - p_i)\|
< \delta + 2\|a_j - p_i a_j\| + \|a_j - a_i p_i\|
< 4\delta < \epsilon.$$

For the second part of (2), by (i), (ii), and (vi) we have

\[
\|px_1 p\| \geq \|(1 - p_i)x_1(1 - p_i) + qx_1 q\|
- \|(1 - p_i)x_1q\| - \|qx_1(1 - p_i)\|
= \max \{\|(1 - p_i)x_1(1 - p_i)\|, \|qx_1 q\|\}
- \|(1 - p_i)a_1 q\| - \|qa_1(1 - p_i)\|
> \|qx_1 q\| - 2\delta = \|xp_1 p p_1 q\| - 2\delta
\geq \|p_i x_1 p_1\| - 3\delta
\geq \|x_1\| - 4\delta
> \|x_1\| - \epsilon.
\]

To prove (3), note that $1 - p = p_i - q$. Thus by (vii), $n[1 - p] \leq [q] \leq [p]$.

Also, we have $1 - p = p_i - q \preceq p_i x p_i \preceq x$.

To see (4), first note that

\[(p_i - q)b_1(p_i - q) = (p_i - q)x_1(p_i - q) = (1 - p)x_1(1 - p).\]

Thus by (viii), $n[f_{d_1}(1 - p)x_1(1 - p)] \leq [f_{d_3}(q b_1 q)] = [f_{d_3}(qx_1 q)]$. Thus to prove (4) it is enough to show that

\[
(5.3) \quad \quad [f_{d_3}(qx_1 q)] \leq [f_{d_3}(px_1 p)]
\]

For this, first by (ii) we have (recall that $x_1 = \lambda_1 + a_1$):

\[
\|px_1 p - (qx_1 q + \lambda_1(1 - p_i))\| = \|(1 - p_i)x_1 q + qx_1(1 - p_i)
+ (1 - p_i)x_1(1 - p_i) - \lambda_1(1 - p_i)\|
= \|(1 - p_i)a_1 q + qa_1(1 - p_i) + (1 - p_i)a_1(1 - p_i)\|
< 3\delta < \eta.
\]

On the other hand, we have $\|px_1 p\| \leq \|x_1\|$ and

\[
\|qx_1 q + \lambda_1(1 - p_i)\| = \max \{\|qx_1 q\|, \|\lambda_1(1 - p_i)\|\} \leq \|x_1\|.
\]

Thus by the choice of $\eta$ we have

\[
(5.4) \quad \quad [f_{d_3}(qx_1 q + \lambda_1(1 - p_i))] \leq [f_{d_3}(px_1 p)].
\]
Also, since \( qx_1q \perp \lambda_1(1 - p_i) \) and \( f_{d_3}^{d_4}(0) = 0 \) we have
\[
f_{d_3}^{d_4}(qx_1q + \lambda_1(1 - p_i)) = f_{d_3}^{d_4}(qx_1q) + f_{d_3}^{d_4}(\lambda_1(1 - p_i)),
\]
and hence,
\[
(5.5) \quad [f_{d_3}^{d_4}(qx_1q)] \leq [f_{d_3}^{d_4}(qx_1q + \lambda_1(1 - p_i))].
\]
Combining (5.4) and (5.5), we get (5.3). Therefore, the proof of the first part of the statement is complete at this point.

Now we prove the second part of the statement. Let the condition of the statement holds for \( k = 0 \) (we do not use (4) in the proof). We show that \( A \) has Property (T0). Note that this condition is stronger than the definition of that \( \text{TR}_w(A^\sim) = 0 \) (i.e., \( A^\sim \) is TAF), since it is not assumed that \( x \in (A^\sim)_+ \) is full.

Observe that in [Lin01c, Proposition 2.7] the assumption that \( a \in (A^\sim)_+ \) is full is not used in the proof of both parts. Now let \( \varepsilon > 0, x, y \in A_+, \) and \( F \subseteq A \) be as in Definition 5.1. By Lemma 5.2 we may assume that \( \|y\| \leq 1. \) Then by (the proof of) [Lin01c, Proposition 2.7] with \( x \) in place of \( x_1, \) \( F = F \cup \{x, y\}, \) \( 2 \varepsilon \) in place of \( \varepsilon, \) and \( x \) in place of \( a, \) there exist two orthogonal projections \( p_1, p_2 \in A \) and a finite dimensional \( C^* \)-subalgebra \( E \subseteq A \) such that \( p_1 = 1_E \) and the following hold:

\begin{enumerate}
  
  \item[(a)] \( \|p_1a - ap_1\| < \frac{\varepsilon}{3} \) for all \( a \in F \) and \( i = 1, 2; \)
  
  \item[(b)] \( p_1Fp_1 \subseteq \frac{\varepsilon}{3} \) \( E, \) \( \|p_1xp_1\| \geq \|x\| - \frac{\varepsilon}{3}, \) and \( \|(p_1 + p_2)a - a\| < \frac{\varepsilon}{3} \) for all \( a \in F; \)
  
  \item[(c)] \( p_2 \precsim x. \)
\end{enumerate}

By (a) and (b), the finite dimensional \( C^* \)-subalgebra \( E \subseteq A \) with unit \( p_1 \) satisfies (1), (2), and (4) in Definition 5.1. To see (3), by (a) and (b) we have:
\[
\|y^2 - y_{p_1}y - y_{p_2}y\| \leq \|y^2 - y^2p_1 - y^2p_2\| + \|y^2p_1 - y_{p_1}y\| + \|y^2p_2 - y_{p_2}y\|
\leq \|y\|\|(y - y(p_1 + p_2))\| + \|y_{p_1} - y_{p_1}y\| + \|y_{p_2} - y_{p_2}y\|
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
Thus by (c) and Lemma 2.1 we have \( (y^2 - y_{p_1}y - \varepsilon)_+ \precsim y_{p_2}y \succeq p_2 \succeq x. \) Therefore, \( A \) has Property (T0). (Note that (b) above implies that \( A \) has an approximate unit consisting of projections.)

\[\square\]

Remark 5.6. Observe that Proposition 5.5 holds also for a simple unital \( C^* \)-algebra \( A \) and its converse holds for any \( k \geq 0 \) in this case (note that in this case \( A^\sim = A \) according to our convention). This follows from Proposition 5.3 and [Lin01b, Theorem 5.6].

Remark 5.7. Let \( A \) be a simple non-unital \( C^* \)-algebra. Then \( A \) has Property (T0) (equivalently, \( \text{TR}(A) = 0 \) by Theorem 5.19) if and only if Conditions (1)–(3) in Proposition 5.5 hold for \( k = 0 \) (because Condition (4) is not
Then there are non-zero projection \( \|F\| = 1 \). We have

\[ \text{TR}(\cdot) = 1 \]

Let \( b \equiv 1 \). In fact, by (ii) there is \( d \) such that \( \|pa_1p - d\| < \delta \). If \( \|d\| < 1 \), put \( f = d \). If \( \|d\| > 1 \) put \( f = d/\|d\| \). Then \( \|pa_1p - f\| < 2\delta \) since if \( \|d\| > 1 \) we have

\[ \|pa_1^p p - f\| \leq \|pa_1^p p - d\| + \|d - f\| < 2\delta. \]

Put \( b_1 = ff^* \). Then \( 0 \leq b_1 \leq p \) and we have

\[ \|pa_1p - b_1\| \leq \|pa_1p - pa_1^p pa_1 p\| + \|pa_1^p p - fpa_1^p p\| + \|fpa_1^p p - f f^*\| < 5\delta. \]

Similarly, there is \( b_2 \in B \) such that \( 0 \leq b_2 \leq p \) and \( \|pa_2p - b_2\| < 5\delta \). We have

\[ \|b_1b_2 - b_2\| \leq \|b_1b_2 - pa_1pb_2\| + \|pa_1pb_2 - pa_1pa_2p\| \]

\[ + \|pa_1pa_2p - pa_1a_2p\| + \|pa_2p - b_2\| \]

\[ < 16\delta. \]

The following proposition implies that in Definition 5.1, even without the assumption of having an approximate unit of projections, the algebra \( A \) has Property (SP).

**Proposition 5.9.** Let \( A \) be a simple \( C^* \)-algebra which satisfies Conditions (1), (2), and (4) of Definition 5.1 for \( k = 0 \). Then \( A \) has Property (SP).

**Proof.** Let \( D \) be a non-zero hereditary \( C^* \)-subalgebra of \( A \). We will find a non-zero projection \( q \) in \( D \). We may assume that \( D \) is infinite dimensional. Then there are \( a_1, a_2 \in D_+ \) such that \( \|a_1\| = \|a_2\| = 1 \) and \( a_1a_2 = a_2 \). Put \( F = \{a_i, a_i^* \mid i = 1, 2\} \). Let \( 0 < \epsilon < \frac{1}{\sqrt{2}} \) and \( 0 < \delta < \epsilon \) (\( \delta \) to be determined later). By assumption there is a finite dimensional \( C^* \)-subalgebra \( B \) of \( A \) such that, with \( p = 1_B \), the following hold:

1. \( \|pa - ap\| < \delta \) for all \( a \in F \);
2. \( pFp \subseteq B \);
3. \( \|pa_2p\| > 1 - \frac{\epsilon}{2} \).

Then there are \( b_1, b_2 \in B \) such that \( 0 \leq b_1, b_2 \leq p \) and \( \|pa_i p - b_i\| < 5\delta \) for \( i = 1, 2 \). In fact, by (ii) there is \( d \in B \) such that \( \|pa_1^p p - d\| < \delta \). If \( \|d\| < 1 \), put \( f = d \). If \( \|d\| > 1 \) put \( f = d/\|d\| \). Then \( \|pa_1^p p - f\| < 2\delta \) since if \( \|d\| > 1 \) we have

\[ \|pa_1^p p - f\| \leq \|pa_1^p p - d\| + \|d - f\| < 2\delta. \]

Put \( b_1 = ff^* \). Then \( 0 \leq b_1 \leq p \) and we have

\[ \|pa_1p - b_1\| \leq \|pa_1p - pa_1^p pa_1 p\| + \|pa_1^p p - fpa_1^p p\| + \|fpa_1^p p - f f^*\| < 5\delta. \]

Similarly, there is \( b_2 \in B \) such that \( 0 \leq b_2 \leq p \) and \( \|pa_2p - b_2\| < 5\delta \). We have

\[ \|b_1b_2 - b_2\| \leq \|b_1b_2 - pa_1pb_2\| + \|pa_1pb_2 - pa_1pa_2p\| \]

\[ + \|pa_1pa_2p - pa_1a_2p\| + \|pa_2p - b_2\| \]

\[ < 16\delta. \]
Thus, \( \| (p - b_1)b_2 \| < 16\delta \). By choosing \( \delta \) small enough (according to [Lin01a, Lemma 2.5.12]), there are positive elements \( c'_1, c_2 \) in the closed unit ball of \( B \) such that
\[
\| c'_1 - (p - b_1) \| < \varepsilon, \quad \| c_2 - b_2 \| < \varepsilon, \quad c'_1 c_2 = 0.
\]
Note that \( c_2 \neq 0 \) since we have
\[
\| c_2 \| > \| b_2 \| - \varepsilon > \| pa_2 p \| - \delta - \varepsilon > 1 - 2\delta - \varepsilon > 1 - 3\varepsilon > 0.
\]
Put \( c_1 = p - c'_1 \). Then \( c_1 c_2 = c_2 \) and \( \| c_1 - b_1 \| < \varepsilon \). Let \( e \) be any non-zero projection in \( C^*(c_2) \subseteq B \). Then \( c_1 e = e = e c_1 \). By (i) we have
\[
\| a_1^\frac{1}{2} p a_1^\frac{1}{2} - c_1 \| \leq \| a_1^\frac{1}{2} p a_1^\frac{1}{2} - a_1 p \| + \| a_1 p - p a_1 p \| + \| p a_1 p - b_1 \| + \| b_1 - c_1 \| < 8\varepsilon.
\]
Put \( b = a_1^\frac{1}{2} p a_1^\frac{1}{2} e a_1^\frac{1}{2} p a_1^\frac{1}{2} \). Since \( D \) is hereditary, \( b \in D \). Using the above inequality and that \( c_1 e = e = e c_1 \), we have
\[
\| b - e \| \leq \| b - a_1^\frac{1}{2} p a_1^\frac{1}{2} e c_1 \| + \| a_1^\frac{1}{2} p a_1^\frac{1}{2} e - c_1 e \| < 16\varepsilon < \frac{1}{2}.
\]
Hence, by [Lin01a, Lemma 2.5.4], there exists a projection \( q \in D \) such that \( \| q - e \| < 1 \). Since \( e \neq 0 \), we have \( q \neq 0 \). Therefore, \( A \) has Property (SP). \( \square \)

5.1. **Permanence Properties.** In this subsection we study some properties of Property \( (T_k) \). We begin with the following proposition which shows that if a simple \( C^* \)-algebra has the local Property \( (T_k) \) then it has Property \( (T_k) \).

**Proposition 5.10.** Let \( A \) be a simple \( C^* \)-algebra with the following property: for every \( \varepsilon > 0 \) and every finite subset \( F \subseteq A \) there exists a simple \( C^* \)-subalgebra \( B \) of \( A \) with Property \( (T_k) \) such that \( F \subseteq \varepsilon B \). Then \( A \) has Property \( (T_k) \).

**Proof.** Let \( A \) be a simple \( C^* \)-algebra with the property in the statement. Observe that \( A \) has an approximate unit consisting of projections. Let \( x, y, \varepsilon, \) and \( F \) be as in Definition 5.1. We may assume that \( \varepsilon < 1 \) and \( \| x \| = 1 \). Also, by Lemma 5.2 we may assume that \( \| y \| < \frac{1}{2} \). Write \( F = \{ f_1, \ldots, f_m \} \). Choose \( 0 < \delta < \frac{\varepsilon}{4} \) such that \( (2 + \delta)\delta < \frac{\varepsilon^2}{16} \). Set \( \tilde{F} = F \cup \{ x^\frac{1}{2}, y^\frac{1}{2} \} \). By assumption there is a simple \( C^* \)-subalgebra \( B \) of \( A \) with Property \( (T_k) \) such that \( \tilde{F} \subseteq \delta B \).

Thus there exists \( b \in B \) such that \( \| x^\frac{1}{2} - b \| < \delta \). Then
\[
\| b^* b - x \| \leq \| b^* b - b^* x^\frac{1}{2} \| + \| b^* x^\frac{1}{2} - x \|
\leq \| b \| \delta + \| x^\frac{1}{2} \| \delta
\leq (\| x^\frac{1}{2} \| + \delta + \| x^\frac{1}{2} \|) \delta < \frac{\varepsilon^2}{16}.
\]
Also there exists \( c \in B \) such that \( \| y^\frac{1}{2} - c \| < \delta \). Similarly, we have
\[
\| c^* c - y \| < \frac{\varepsilon^2}{16}.
\]
Set \( w = c^*c \). So \( \|w\| < 1 \). Also set \( d = (b^*b - \frac{\varepsilon}{12})_+ \). Note that \( d \neq 0 \) since \( \|b^*b\| > 1 - \frac{\varepsilon}{12} > \frac{1}{2} \) and \( \frac{\varepsilon}{12} < \frac{1}{2} \). Moreover, there exist \( b_1, \ldots, b_m \in B \) such that \( \|b_i - f_i\| < \delta \) for every \( i = 1, \ldots, m \). Put \( D = \{b_1, \ldots, b_m\} \). Since \( B \) has Property \( (T_k) \), by Definition 5.1 there is a subalgebra \( E \subset A \) in \( I^{(k)} \) such that, with \( p = 1_E \), the following hold:

(i) \( \|pb_i - b_i p\| < \delta \) for all \( i = 1, \ldots, m \);

(ii) \( pDp \subseteq \delta E \);

(iii) \( (w^2 - wpw - \delta)_+ \preceq d \);

(iv) \( \|pd p\| > \|d\| - \delta \).

Now we show that (1)–(4) in Definition 5.1 hold. For (1) we have

\[
\|pf_i - f_i p\| \leq \|pf_i - pb_i\| + \|pb_i - b_i p\| + \|b_i p - f_i p\| < 3\delta < \varepsilon.
\]

To see (2) fix \( 1 \leq i \leq m \). By (ii) above there is \( e \in E \) such that \( \|pb_i p - e\| < \delta \).

Then we have

\[
\|pf_i p - e\| \leq \|pf_i p - pb_i p\| + \|pb_i p - e\| < 2\delta < \varepsilon.
\]

To show (4) in Definition 5.1, note that by (iv) we can compute as follows

\[
\|xp\| \geq \|pd p\| - \|p(x - d)p\|
\]

\[
> \|d\| - \delta - \frac{\varepsilon}{12}
\]

\[
> \|b^*b\| - \frac{\varepsilon}{12} - \frac{\varepsilon}{4} - \frac{\varepsilon}{12}
\]

\[
> \|x\| - \frac{\varepsilon}{12} - \frac{\varepsilon}{4}
\]

\[
> \|x\| - \varepsilon.
\]

To prove (3) in Definition 5.1 first we have

\[
\|(w^2 - wpw - \delta)_+ - (y^2 - ypy)\|
\leq \|(w^2 - wpw - \delta)_+ - (w^2 - wpw)\|
\]

\[
+ \|(w^2 - wpw) - (y^2 - ypy)\|
\leq \delta + \|w^2 - y^2\| + \|wpw - wpw\| + \|wpw - ypy\|
\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{6} + \frac{\varepsilon}{12} + \frac{\varepsilon}{12} < \varepsilon.
\]

Therefore, by (iii), \( (y^2 - ypy - \varepsilon)_+ \preceq (w^2 - wpw - \delta)_+ \preceq d \preceq x \). ☐

The following is an immediate consequence of Proposition 5.10.

**Corollary 5.11.** Let \( A \) be a C*-algebra which is the inductive limit of a family of simple C*-algebras with Property \( (T_k) \). Then \( A \) has Property \( (T_k) \).

The following characterization of Property \( (T_k) \) is essential in the following,
**Proposition 5.12.** Let $A$ be a simple C*-algebra. Then $A$ has Property $(T_k)$ if and only if there exists an approximate unit (not necessarily increasing) consisting of projections $(p_i)_{i \in I}$ for $A$ such that $\text{TR}(p_iAp_i) \leq k$ for all $i \in I$.

*Proof.* The forward implication follows from Definition 5.1, Lemma 5.4, and Proposition 5.8. For the backward implication, let $(p_i)_{i \in I}$ be as in the statement. Proposition 5.3 implies that each $p_iAp_i$ is a simple C*-algebra with Property $(T_k)$. Let $F \subseteq A$ be a finite subset and $\varepsilon > 0$. Since $(p_i)_{i \in I}$ is an approximate unit, there exists $i \in I$ such that $F \subseteq_{\varepsilon} p_iAp_i$. Now Proposition 5.10 yields that $A$ has Property $(T_k)$. □

With the above characterization of Property $(T_k)$, we can obtain more properties of simple C*-algebras with Property $(T_k)$, in particular when $k = 0$.

**Theorem 5.13.** Let $A$ be a simple C*-algebra with Property $(T_0)$. Then $A$ has real rank zero and stable rank one.

*Proof.* Let $A$ be a non-zero simple C*-algebra with Property $(T_0)$. Proposition 5.12 yields the existence of a non-zero projection $p \in A$ such that $\text{TR}(pAp) = 0$. Thus, by [Lin01b, Theorem 7.1], $pAp$ has real rank zero. Since $A$ is simple, $pAp$ is a full corner of $A$ and so $pAp$ is Morita equivalent to $A$. Then by [BP91, Theorem 3.8], $A$ has also real rank zero. To see that $A$ has stable rank one, first note that [Lin01b, Theorem 6.9] and [Lin01b, Theorem 6.13] imply that $\text{tsr}(pAp) = 1$. Moreover, by [Bl04, Corollary 4.6], $\text{tsr}(A) \leq \text{tsr}(pAp)$. Hence, $A$ has stable rank one. □

**Proposition 5.14.** Let $A$ be a simple C*-algebra with Property $(T_k)$ and let $B$ be a hereditary C*-subalgebra of $A$. Then $B$ has Property $(T_k)$ if and only if it has an approximate unit consisting of (not necessarily increasing) projections.

*Proof.* The forward implication follows from Definition 5.1. For the backward implication let $A$ be a simple C*-algebra with Property $(T_k)$ and let $B$ be a hereditary C*-subalgebra of $A$ which contains an approximate unit consisting of projections $(p_i)_{i \in I}$. For each $i \in I$ we have $p_iBp_i = p_iAp_i$ which has Property $(T_k)$ by Lemma 5.4. Therefore, Propositions 5.3 and 5.12 imply that $B$ has Property $(T_k)$. □

**Theorem 5.15.** Let $A$ be a simple C*-algebra with real rank zero and let $k$ be a non-negative integer. The following statements are equivalent:

1. $A$ has Property $(T_k)$;
2. $x^*Ax$ has Property $(T_k)$ for all $x \in A_+$;
3. $\text{TR}(pAp) \leq k$ for all projections $p \in A$. 


Proof. (1) $\Rightarrow$ (2): This follows from Proposition 5.14 and the fact that $xAx$ has real rank zero. (2) $\Rightarrow$ (3): This follows from Proposition 5.8. Finally, the implication (3) $\Rightarrow$ (1) follows from Proposition 5.12. □

Corollary 5.16. Let $A$ be a simple $C^*$-algebra. The following are equivalent:

1. $A$ has Property $(T_0)$;
2. $xAx$ has Property $(T_0)$ for all $x \in A_+$;
3. $A$ has real rank zero and $\operatorname{TR}(pAp) = 0$ for all projections $p \in A$.

Proof. The statement follows from Theorem 5.15 and noting that each of parts (1), (2), and (3) implies that $A$ has real rank zero (by Theorem 5.13). □

Lemma 5.17. Let $A$ be a simple $C^*$-algebra with Property $(T_k)$ where $k$ is a non-negative integer. Then $M_n(A)$ has Property $(T_k)$ for all $n \in \mathbb{N}$.

Proof. Let $(p_i)_{i \in I}$ be an approximate unit consisting of projections for $A$. Put $q_i = \operatorname{diag}(p_i, \ldots, p_i) \in M_n(A)$. Then $(q_i)_{i \in I}$ is an approximate unit consisting of projections for $M_n(A)$. Lemma 5.4 and Proposition 5.3 yield that $\operatorname{TR}(p_iAp_i) \leq k$. Thus $q_iM_n(A)q_i = M_n(p_iAp_i)$ has tracial topological rank at most $k$, by [Lin01b, Theorem 5.8]. Therefore, Proposition 5.12 implies that $M_n(A)$ has Property $(T_k)$. □

Proposition 5.12 enables us to describe relation between tracial topological rank at most $k$ and having Property $(T_k)$ as follows.

Theorem 5.18. Let $A$ be a simple $C^*$-algebra and $k$ be a non-negative integer. Then $A$ has Property $(T_k)$ if and only if $\operatorname{TR}(A) \leq k$ and $A$ has an approximate unit consisting of (not necessarily increasing) projections.

Proof. The forward implication follows from Proposition 5.8. For the other direction, let $\operatorname{TR}(A) \leq k$ and let $(p_i)_{i \in I}$ be an approximate unit consisting of projections for $A$. Then Theorem 5.3 of [?] implies that $\operatorname{TR}(p_iAp_i) \leq k$ for all $i \in I$. Now, Proposition 5.12 implies that $A$ has Property $(T_k)$. □

Theorem 5.19. Let $A$ be a simple (not necessarily unital) $C^*$-algebra. Then $A$ has Property $(T_0)$ if and only if $\operatorname{TR}(A) = 0$.

Proof. The forward implication follows from Proposition 5.8. For other direction, let $A$ be a simple $C^*$-algebra with $\operatorname{TR}(A) = 0$. We may assume that $A$ is non-unital since the unital case follows from Proposition 5.3. As $\operatorname{TR}(A) = 0$, [Lin01b, Corollary 5.7] implies that $\operatorname{TR}_w(A) = 0$ (recall that, by definition, $\operatorname{TR}(A) = \operatorname{TR}(A^\sim)$ and $\operatorname{TR}_w(A) = \operatorname{TR}_w(A^\sim)$). Thus, $A$ is TAF in the sense of [Lin01c]. Now by [Lin01c, Corollary 2.8], $A$ has an approximate unit (not necessarily increasing) consisting of projections. (Note that the separability assumption is unnecessary in [Lin01c, Corollary 2.8].) Thus, by Theorem 5.18, $A$ has Property $(T_0)$. □
Theorem 5.19 has this advantage that we do not need to consider the unitization of simple non-unital C*-algebras to study the tracial rank zero.

We end this subsection with some applications of Theorem 5.19.

Corollary 5.20 (cf. [Lin01b], Proposition 4.8). Let \( A \) be a simple C*-algebra which is an inductive limit of simple C*-algebras of tracial topological rank zero. Then \( \text{TR}(A) = 0 \).

**Proof.** This follows from Theorem 5.19 and Corollary 5.11. \( \square \)

The following corollary was proved by Lin in the unital case. More precisely, part (1) in the simple unital case follows from [Lin01b, Theorem 7.1] and [Lin01a, Theorem 3.6.11]. Part (2) is proved in [Lin01b, Theorem 5.8] in the unital not necessarily simple case. Part (3) in the case of unital hereditary follows from [Lin01b, Theorem 5.3].

**Corollary 5.21.** Let \( A \) be a simple (not necessarily unital) C*-algebra with \( \text{TR}(A) = 0 \). Then the following hold:

1. \( A \) has real rank zero and stable rank one;
2. \( \text{TR}(M_n(A)) = 0 \) for all \( n \in \mathbb{N} \);
3. if \( B \) is a hereditary C*-subalgebra of \( A \) then \( \text{TR}(B) = 0 \).

**Proof.** Part (1) follows from Theorems 5.13 and 5.19. Part (2) follows from Lemma 5.17 and Theorem 5.19. Finally, part (3) follows from Part (1), Proposition 5.14, and Theorem 5.19. \( \square \)

5.2. Morita Equivalence. In this subsection we show that a (not necessarily unital) C*-algebra is Morita equivalent to a simple unital C*-algebra with tracial topological rank zero if and only if it has tracial topological rank zero. In particular, we prove that the class of simple C*-algebras with tracial topological rank zero is closed under the Morita equivalence. We also consider the tracial topological rank at most \( k \).

We begin by investigating the class of C*-algebras which are stably isomorphic to simple unital C*-algebras with tracial topological rank at most \( k \).

**Proposition 5.22.** Let \( A \) be a simple C*-algebra and \( k \) be a non-negative integer. If \( A \) has Property \((T_k)\) then so does \( A \otimes K \). The converse holds if \( A \) has real rank zero.

**Proof.** The first part of the statement follows from Lemma 5.17, Corollary 5.11, and the fact that \( A \otimes K \) is isomorphic to an inductive limit \( \lim_{\to} M_n(A) \). For the second part, suppose that \( A \otimes K \) has Property \((T_k)\). Let \( x \in A_+ \) and \( p \) be
a rank one projection in $\mathcal{K}$. We have $\overline{xAx} \cong (x \otimes p)(A \otimes \mathcal{K})(x \otimes p)$. In fact, the map
\[
\varphi : \overline{xAx} \to (x \otimes p)(A \otimes \mathcal{K})(x \otimes p)
\]
defined by $\varphi(b) = b \otimes p$, $b \in \overline{xAx}$, is a surjective $\ast$-isomorphism. Since $A \otimes \mathcal{K}$ has Property $(T_k)$, so does $(x \otimes p)(A \otimes \mathcal{K})(x \otimes p)$, by Theorem 5.15. Thus $\overline{xAx}$ has property $(T_k)$. Since $x \in A_\infty$ was arbitrary, by Theorem 5.15, $A$ has Property $(T_k)$.

**Lemma 5.23.** Let $A$ be a simple $C^*$-algebra with real rank zero and let $k$ be a non-negative integer. The following are equivalent:

1. $A$ is a $\sigma$-unital $C^*$-algebra with Property $(T_k)$;
2. $A$ is stably isomorphic to a simple unital $C^*$-algebra $B$ with $\text{TR}(B) \leq k$.

**Proof.** $(1) \Rightarrow (2)$: Let $A$ be a simple $\sigma$-unital $C^*$-algebra with Property $(T_k)$. We may assume that $A$ is non-zero. Thus there is a non-zero projection $p \in A$. Put $B = pAp$. Since $A$ is simple and $\sigma$-unital, $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, by [Br77]. Note that by Lemma 5.4 and Proposition 5.3, $B$ is a simple unital $C^*$-algebra with $\text{TR}(B) \leq k$.

$(2) \Rightarrow (1)$: Let $A$ be stably isomorphic to a simple unital $C^*$-algebra $B$ with $\text{TR}(B) \leq k$. Then $B$ has Property $(T_k)$, by Proposition 5.3. Thus by Propositions 5.22, $A$ has Property $(T_k)$. It remains to show that $A$ is $\sigma$-unital. In general, if $A$ and $B$ are stably isomorphic $C^*$-algebras and $B$ is $\sigma$-unital then so is $A$. This should be part of the literature. We prove this for the convenience of the reader. Since $B$ is $\sigma$-unital, $B \otimes \mathcal{K}$ is also $\sigma$-unital.

Write $D = A \otimes \mathcal{K}$. Thus there is an approximate unit $(e_n)_{n \in \mathbb{N}}$ for $D$. We may write $D$ as the inductive limit $D = \lim \to M_n(A)$ where the connecting maps $\varphi_n : M_n(A) \to M_{n+1}(A)$ are given by $\varphi_n(x) = \text{diag}(x, 0)$, $x \in M_n(A)$. Then there is a subsequence $(e_{n_k})_{k \in \mathbb{N}}$ and a sequence $(f_{n_k})_{k \in \mathbb{N}}$ in $D$ such that $\|e_{n_k} - f_{n_k}\| < \frac{1}{k}$, $0 \leq f_{n_k} \leq 1$, and $f_{n_k} \in M_n(A)$ for all $k \in \mathbb{N}$. Thus $(f_{n_k})_{k \in \mathbb{N}}$ is an approximate unit (not necessarily increasing) for $D$. Put $a_k = (f_{n_k})_{1,1}$, the $(1,1)$-th entry of $f_{n_k}$. Then $(a_k)_{k \in \mathbb{N}} \in A_{+,1}$ where $A_{+,1}$ denotes the set of positive elements in the closed unit ball of $A$. It follows that $(a_k)_{k \in \mathbb{N}}$ is an approximate unit (not necessarily increasing) for $A$. In fact, let $a \in A$. Then we have
\[
\|aa_k - a\| = \|(af_{n_k} - a)_{1,1}\| \leq \|af_{n_k} - a\| \to 0.
\]
Put $x = \sum_{k \in \mathbb{N}} \frac{a_k}{2^k}$. Then it is easily seen that $x$ is a strictly positive element for $A$ and so $A$ is $\sigma$-unital. □

**Theorem 5.24.** Let $A$ be a simple $C^*$-algebra with real rank zero and $k$ be a non-negative integer. The following are equivalent:
(1) \( A \) has Property \( (T_k) \);
(2) \( \text{TR}(A) \leq k \);
(3) \( \text{TR}(pAp) \leq k \) for some (any) non-zero projection \( p \in A \);
(4) \( A \) is Morita equivalent to a simple C*-algebra \( B \) with \( \text{TR}(B) \leq k \).

**Proof.** Let \( A \) be a non-zero simple C*-algebra with real rank zero. By Theorem 5.18, parts (1) and (2) are equivalent. The implication \( (1) \Rightarrow (3) \) follows from Lemma 5.4 and Proposition 5.3. For \( (3) \Rightarrow (4) \), put \( B = pAp \) for some non-zero projection \( p \in A \). Such a projection exists since \( \text{RR}(A) = 0 \). It remains to show that \( (4) \Rightarrow (1) \). Let \( A \) be Morita equivalent to a simple C*-algebra \( B \) with \( \text{TR}(B) \leq k \). We may assume that \( B \) is unital, since \( \text{RR}(B) = 0 \) (by [BP91, Theorem 3.8]) and so there is a non-zero projection \( q \in B \). Thus we can consider \( qBq \) instead of \( B \). Since we assume that \( A \) has real rank zero, \( A \) has an approximate unit (not necessarily increasing) consisting of projections \( (p_i)_{i \in I} \). For each \( i \in I \), the simple unital C*-algebra \( p_iAp_i \) is Morita equivalent to \( A \), and so it is Morita equivalent to \( B \). Since both \( p_iAp_i \) and \( B \) are unital, they are stably isomorphic (by [BGR77]). Thus by Lemma 5.23, \( p_iAp_i \) has Property \( (T_k) \). Therefore, Propositions 5.3 and 5.12 imply that \( A \) has Property \( (T_k) \). \( \square \)

**Remark 5.25.** Theorem 5.24 implies that if \( A \) and \( B \) are simple C*-algebras of real rank zero which are Morita equivalent, then \( \text{TR}(A) = \text{TR}(B) \).

Now, we are in a position to conclude that the class of simple C*-algebras with tracial topological rank zero is closed under Morita equivalence.

**Corollary 5.26.** Let \( A \) be a non-zero simple (not necessarily unital) C*-algebra. The following statement are equivalent:

(1) \( \text{TR}(A) = 0 \);
(2) \( A \) is Morita equivalent to a simple C*-algebra \( B \) with \( \text{TR}(B) = 0 \);
(3) \( \text{TR}(pAp) = 0 \) for some (any) non-zero projection \( p \in A \).

**Proof.** The implication \( (1) \Rightarrow (3) \) follows from part (3) of Corollary 5.21. Also, \( (3) \Rightarrow (2) \) is obvious. For \( (2) \Rightarrow (1) \), let \( B \) be a simple C*-algebra with \( \text{TR}(B) = 0 \) such that \( B \) is Morita equivalent to \( A \). By part (1) of Corollary 5.21 we have \( \text{RR}(B) = 0 \). By [BP91, Theorem 3.8], having real rank zero is preserved under the Morita equivalence, hence we get \( \text{RR}(A) = 0 \). Now, Theorem 5.24 implies that \( \text{TR}(A) = 0 \). \( \square \)

We will use Corollary 5.26 in the next section to study the crossed products of simple non-unital C*-algebras with tracial topological rank zero.
6. Crossed Products

This section is devoted to study the crossed products of simple (not necessarily unital) C*-algebras by finite group actions with the tracial Rokhlin property. Our main result is a non-unital analogue of [Ph11, Theorem 2.6] (see Theorem 6.2). We show that all of the following classes of C*-algebras are closed under taking crossed products and fixed point algebras with actions of finite groups with the tracial Rokhlin property: simple C*-algebras of tracial topological rank zero, simple separable C*-algebras $A$ of real rank zero with $\text{TR}(A) \leq k$, simple separable C*-algebras of real rank zero, simple separable C*-algebras of stable rank one and real rank zero, simple separable nuclear Z-stable C*-algebras, simple C*-algebras with Property (SP), and simple separable tracially Z-absorbing C*-algebras (see Theorem 6.7).

The following proposition reduces to the unital case. However, there is another approach to obtain more results which does not use the previous unital results and works for both unital and non-unital cases. This is based on a suitable notion of non-unital TAC algebras. We will work on this approach in subsequent paper.

**Proposition 6.1.** Let $G$ be a finite group and $\mathcal{C}$ be a class of simple (separable) C*-algebras with the following properties:

1. if $A$ is a simple (separable) C*-algebra and $p \in A$ is a non-zero projection, then $A \in \mathcal{C}$ if and only if $pAp \in \mathcal{C}$ (in particular, this is the case if $\mathcal{C}$ is closed under the Morita equivalence);
2. if $A \in \mathcal{C}$ is unital and $\alpha$ is an action of $G$ on $A$ with the tracial Rokhlin property then $A \rtimes_{\alpha} G \in \mathcal{C}$;
3. if $A \in \mathcal{C}$ and $B$ is a C*-algebra with $A \cong B$, then $B \in \mathcal{C}$.

Then $\mathcal{C}$ is closed under crossed products of actions of $G$ with the tracial Rokhlin property (i.e., (2) above holds without the unital assumption).

**Proof.** Let $\mathcal{C}$ be a class of simple C*-algebras as in the statement. Let $A \in \mathcal{C}$ and $\alpha : G \to \text{Aut}(A)$ be an action with the tracial Rokhlin property. We show that $A \rtimes_{\alpha} G \in \mathcal{C}$. We may assume that $A$ is non-zero. First we prove that there exists a non-zero projection $q$ in $A^\alpha$. Choose $x \in A_+$ with $\|x\| = 1$. Since $\alpha$ has the tracial Rokhlin property, applying Definition 3.2 with $\varepsilon = \frac{1}{3n}$ (where $n = |G|$) and with $x$ as given (we do not need Conditions (1) and (3) in Definition 3.2 here and so we do not specify $F$ and $y$), there exist orthogonal projections $(p_g)_{g \in G}$ in $A$ such that, with $p = \sum_{g \in G} p_g$, the following hold:

1. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$;
2. $\|xp\| > 1 - \varepsilon$. 


Put $b = \frac{1}{n} \sum_{g \in G} \alpha_g(p)$. Then $b$ is a positive contraction in $A^\alpha$. By (i) we have
\[
\|b - p\| = \frac{1}{n} \left\| \sum_{g \in G} (\alpha_g(p) - p) \right\| \leq \frac{1}{n} \sum_{g \in G} \|\alpha_g(p) - p\|
\]
\[
\leq \frac{1}{n} \sum_{g \in G} \sum_{h \in G} \|\alpha_g(wh) - pwh\|
\]
\[
< n\varepsilon < \frac{1}{2}.
\]
Thus by [Lin01a, Lemma 2.5.4], there exists a projection $q \in C^* \subseteq A^\alpha$ such that $\|q - p\| < 1$. By (ii), $p \neq 0$. Thus $q \neq 0$.

Let $\beta : G \to \text{Aut}(qAq)$ be the restriction of $\alpha$ to $qAq$. Now, Proposition 3.15 implies that $\beta$ has the tracial Rokhlin property. By Condition (1), $qAq \in C$. Thus, by Condition (2), $qAq \rtimes G \in C$. Observe that $qAq \rtimes G \cong q(A \rtimes G)q$. In fact, the map $\varphi : qAq \rtimes G \to q(A \rtimes G)q$ defined by $\varphi(\sum_{g \in G} b_g u_g) = \sum_{g \in G} b_g \delta_g$ where $b_g \in qAq$ for all $g \in G$, is easily seen to be a surjective $*$-isomorphism. Thus by Condition (3), $q(A \rtimes G)q \in C$. Now Condition (2) implies that $A \rtimes G \in C$.

If we assume that $C$ is a class of separable C*-algebras satisfying Conditions (1) and (3), and satisfying the separable version of Condition (2), then the above argument again works by noting that $A \rtimes G$ is separable if $A$ is separable and $G$ is finite.

In the following theorem we extend Phillips’ result [Ph11, Theorem 2.6] to the non-unital case. We also consider the fixed point algebra.

**Theorem 6.2.** Let $A$ be a simple C*-algebra with tracial topological rank zero and $\alpha$ be an action of a finite group $G$ on $A$ with the tracial Rokhlin property. Then the crossed product $A \rtimes G$ and the fixed point algebra $A^\alpha$ are simple C*-algebras with tracial topological rank zero.

**Proof.** Let $C$ denote the class of simple C*-algebras with tracial topological rank zero. By Corollary 5.26, $C$ satisfies Condition (1) in Proposition 6.1. Also, by [Ph11, Theorem 2.6], $C$ satisfies Condition (2) in Proposition 6.1. (Note that the assumption of separability is unnecessary in [Ph11, Theorem 2.6].) Clearly, $C$ satisfies Condition (3) in Proposition 6.1. Thus, Proposition 6.1 implies the first part of the statement about the crossed product.

The second part of the statement about the fixed point algebra follows from Corollary 3.11 which says that $A^\alpha$ is Morita equivalent to $A \rtimes G$, and Corollary 5.26 which implies that the tracial topological rank zero is preserved under the Morita equivalence in the class of simple C*-algebras. □

Let $A$ be a simple unital separable C*-algebra with $\text{TR}(A) \leq k$ for some non-negative integer $k$, and let $\alpha$ be an action of a finite group $G$ on $A$ with the tracial Rokhlin property. Then it follows from Theorem 3.9 of [OT14]...
that $\text{TR}(A \rtimes_\alpha G) \leq k$. We give a similar result by removing the assumption of being unital and adding the assumption that $\text{RR}(A) = 0$.

**Theorem 6.3.** Let $A$ be a simple separable $C^*$-algebra of real rank zero with $\text{TR}(A) \leq k$ for some non-negative integer $k$. Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$ with the tracial Rokhlin property. Then we have $\text{TR}(A \rtimes_\alpha G) = \text{TR}(A^\alpha) \leq k$.

**Proof.** Let $k$ be a non-negative integer. Denote by $\mathcal{C}$ the class of simple separable $C^*$-algebras $A$ of real rank zero with $\text{TR}(A) \leq k$. Then by Lemma 5.4, Theorem 5.24, and that the real rank zero is preserved under the Morita equivalence ([BP91, Theorem 3.8]), $\mathcal{C}$ satisfies Condition (1) in Proposition 6.1. Also by [OT14, Corollary 3.6 and Theorem 3.9], $\mathcal{C}$ satisfies Condition (2) in Proposition 6.1. Thus, by Proposition 6.1, $\mathcal{C}$ is closed under crossed products by finite group actions with the tracial Rokhlin property. Thus, if $\alpha: G \to \text{Aut}(A)$ is as in the statement then we have $\text{TR}(A \rtimes_\alpha G) \leq k$. By Corollary 3.11, $A^\alpha$ and $A \rtimes_\alpha G$ are Morita equivalent. Thus $\text{RR}(A^\alpha) = \text{RR}(A \rtimes_\alpha G) = 0$. Therefore, $\text{TR}(A \rtimes_\alpha G) = \text{TR}(A^\alpha)$, by Remark 5.25. □

**Proposition 6.4.** Let $A$ be a simple separable $C^*$-algebra and let $\alpha$ be an action of a finite group $G$ on $A$ with the tracial Rokhlin property.

(i) If $A$ has real rank zero then $A \rtimes_\alpha G$ and $A^\alpha$ have real rank zero;

(ii) If $A$ has stable rank one and has real rank zero, then $A \rtimes_\alpha G$ and $A^\alpha$ have stable rank one.

**Proof.** To see (i), let $\mathcal{C}$ denote the class of simple separable $C^*$-algebras with real rank zero. Then [BP91, Theorem 3.8] implies that $\mathcal{C}$ satisfies (1) in Proposition 6.1. Also, [OT14, Corollary 3.6] implies that $\mathcal{C}$ satisfies (2) in Proposition 6.1. Therefore, $\mathcal{C}$ is closed under crossed products by finite group actions with the tracial Rokhlin property. Thus, if $\alpha: G \to \text{Aut}(A)$ is as in the statement then $\text{RR}(A \rtimes_\alpha G) = 0$. Since real rank zero is preserved under the Morita equivalence, Corollary 3.11 implies that $\text{RR}(A^\alpha) = 0$. Thus, (i) holds.

For (ii), let $\mathcal{D}$ denote the class of simple separable $C^*$-algebras with stable rank one and real rank zero. Then by [Bl04, Theorem 4.6] and [BP91, Theorem 3.8], $\mathcal{D}$ satisfies (1) in Proposition 6.1. Also, by [OT14, Theorem 3.4 and Corollary 3.6], $\mathcal{D}$ satisfies (2) in Proposition 6.1. Thus, $\mathcal{D}$ is closed under crossed products by finite group actions with the tracial Rokhlin property. Thus, if $\alpha: G \to \text{Aut}(A)$ is as in the statement then $\text{tsr}(A \rtimes_\alpha G) = 1$. Now we show that $\text{tsr}(A^\alpha) = 1$. By (i) we have $\text{RR}(A^\alpha) = 0$. We may obviously assume that $A^\alpha \neq 0$. Then there is a non-zero projection $p \in A^\alpha$. Corollary 3.11 implies that $A \rtimes_\alpha G$ and $pA^\alpha p$ are Morita equivalent. Since both these algebras are separable, we have $(A \rtimes_\alpha G) \otimes \mathcal{K} \cong (pA^\alpha p) \otimes \mathcal{K}$. Since having stable rank...
one passes to matrix algebras and inductive limits (by [Lin01a, Theorem 3.1.9 and Proposition 3.2.1]), $(A \rtimes_\alpha G) \otimes K$ has stable rank one. Thus, $(pA^\alpha p) \otimes K$ has stable rank one. Note that $pA^\alpha p$ is isomorphic to a unital hereditary C*-subalgebra of $(pA^\alpha p) \otimes K$. Then by [Lin01a, Theorem 3.1.8], $\text{tsr}(pA^\alpha p) = 1$. Therefore, [Bl04, Theorem 4.6] implies that $\text{tsr}(A^\alpha) = 1$. □

Proposition 6.5. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple nuclear separable C*-algebra $A$ with the tracial Rokhlin property. If $A$ is $\mathcal{Z}$-stable then so are $A \rtimes_\alpha G$ and $A^\alpha$.

Proof. Let $\mathcal{C}$ denote the class of simple separable nuclear $\mathcal{Z}$-stable C*-algebras. By Corollary 3.2 of [TW07], $\mathcal{Z}$-stability is preserved under the Morita equivalence in the class of separable C*-algebras. Moreover, by Theorem 3.15 of [HRW07], nuclearity is preserved under the Morita equivalence. Thus the class $\mathcal{C}$ satisfies Condition (1) in Proposition 6.1. On the other hand, Corollary 5.7 of [HO13] implies that the class $\mathcal{C}$ also satisfies Condition (2) in Proposition 6.1. Therefore, by Proposition 6.1 the class $\mathcal{C}$ is preserved under taking crossed products with action of finite groups with the tracial Rokhlin property. The corresponding result holds for $A^\alpha$ by noting that $A \rtimes_\alpha G$ and $A^\alpha$ are Morita equivalent (and separable).

Alternatively, the statement follows from part (vi) of Theorem 6.7. More precisely, let $A$ and $\alpha : G \to \text{Aut}(A)$ be as in the statement. It is shown in [AGJP17] that every simple $\mathcal{Z}$-stable (not necessarily unital) C*-algebra is tracially $\mathcal{Z}$-absorbing. Thus $A$ is simple, separable, and tracially $\mathcal{Z}$-absorbing. Now, part (vi) of Theorem 6.7 implies that $A \rtimes_\alpha G$ and $A^\alpha$ are simple and tracially $\mathcal{Z}$-absorbing. Note that these are also nuclear. Since $\alpha$ has the tracial Rokhlin property, Lemma 3.7 implies that $A^\alpha$ has a non-zero projection, and so does $A \rtimes_\alpha G$ (we may assume that $A \neq 0$). On the other hand, it is shown in [AGJP17] that every simple separable nuclear C*-algebra which is not stably projectionless, is $\mathcal{Z}$-stable. Therefore, $A \rtimes_\alpha G$ and $A^\alpha$ are $\mathcal{Z}$-stable. □

We quote the definition of a simple (not necessarily unital) tracially $\mathcal{Z}$-absorbing C*-algebra from [AGJP17]. We need this notion in Theorem 6.7.

Definition 6.6 ([AGJP17]). Let $A$ be a simple C*-algebra. Then $A$ is called tracially $\mathcal{Z}$-absorbing if $A \ncong \mathcal{C}$ and for every $x, a \in A_+$ with $a \neq 0$, every finite set $F \subseteq A$, every $\epsilon > 0$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi : M_n \to A$ such that

1. $(x^2 - x\varphi(1)x - \epsilon)_+ \lesssim a$;
2. $\|[[\varphi(z), b]] < \epsilon$ for any normalized element $z \in M_n$ and any $b \in F$. 

In the following theorem we list some properties preserved under taking crossed products and fixed point algebras with actions of finite groups with the tracial Rokhlin property.

**Theorem 6.7.** The following classes of simple not necessarily unital C*-algebras are preserved under taking crossed products and fixed point algebras with actions of finite groups with the tracial Rokhlin property:

(i) simple C*-algebras of tracial topological rank zero;
(ii) simple separable C*-algebras A of real rank zero with TR(A) ≤ k;
(iii) simple separable C*-algebras of real rank zero;
(iv) simple separable C*-algebras of stable rank one and real rank zero;
(v) simple separable nuclear Z-stable C*-algebras;
(vi) simple separable tracially Z-absorbing C*-algebras;
(vii) simple C*-algebras with Property (SP).

*Proof.* Parts (i) and (ii) follow from Theorems 6.2 and 6.3, respectively. Also, parts (iii) and (iv) follow from Proposition 6.4, and part (v) follows from Proposition 6.5. To see (vi), let \( \mathcal{C} \) denote the class of simple separable tracially Z-absorbing C*-algebras. By [HO13, Theorem 5.6] and Proposition 4.6, \( \mathcal{C} \) satisfies (2) in Proposition 6.1. It is shown in [AGJP17] that the tracial Z-absorption is preserved under Morita equivalence in the class of simple separable C*-algebras. Hence, \( \mathcal{C} \) satisfies (1) in Proposition 6.1. Therefore, (vi) follows from Proposition 6.1.

To see (vii), let \( \alpha: G \to \text{Aut}(A) \) be an action of a finite group \( G \) on a simple C*-algebra \( A \). Let \( \alpha \) has the tracial Rokhlin property and let \( A \) has Property (SP). By Lemma 3.12, \( A \rtimes_{\alpha} G \) has Property (SP). Also, by [Ros79], \( A^\alpha \) is isomorphic to a hereditary C*-subalgebra of \( A \rtimes_{\alpha} G \). Thus \( A^\alpha \) has Property (SP). (Alternatively, (vii) follows from [KK97, Remark 8].) \( \square \)

7. Examples

As we mentioned in the introduction, this paper is devoted mainly to the general theory, and working on important examples is postponed to the subsequent paper. In particular, we are investigating actions on the Razak-Jacelon algebras and non-unital Kirchebrg algebras. However, in this section we provide some (rather trivial) examples to show that there are many examples of actions on non-unital simple algebras with the (weak) tracial Rokhlin property.

It follows from Theorem 3.19 (Proposition 4.14) that if \( \alpha: G \to \text{Aut}(A) \) is an action of a finite group \( G \) on a simple unital C*-algebra \( A \) and \( \alpha \) has the (weak) tracial Rokhlin property, then \( \alpha \otimes \text{id}: G \to \text{Aut}(A \otimes K) \) has the (weak) tracial Rokhlin property.
Recall that a simple C*-algebra $A$ is called purely infinite if $A \not\cong C$ and $a \sim b$ for any $a, b \in A_+ \setminus \{0\}$ (cf. [KR00, Definition 4.1]).

**Proposition 7.1.** Let $A = \bigotimes_{k=1}^{\infty} M_3$ be the UHF algebra of type $3^\infty$ and let $B$ be a (not necessarily unital) purely infinite simple C*-algebra. Suppose that $\alpha: \mathbb{Z}_2 \to \text{Aut}(A)$ is defined by

$$\alpha = \bigotimes_{k=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

Then the action $\alpha \otimes \text{id} : \mathbb{Z}_2 \to \text{Aut}(A \otimes B)$ has the tracial Rokhlin property.

**Proof.** By [Ph17, Example 13.23 and Remark 14.9], $\alpha$ has the tracial Rokhlin property and it does not have the Rokhlin property. On the other hand, we have $\text{RR}(B) = 0$. We show this for the convenience of the reader. Let $B \neq 0$. By [KR00, Proposition 5.4], there exists a non-zero (infinite) projection $p \in B$. Then $pBp$ is a simple unital purely infinite C*-algebra (by [KR00, Proposition 4.17]). Now, we have $\text{RR}(pBp) = 0$ (e.g., by [Lin01a, Theorem 3.5.15]). Since having real rank zero is preserved under the Morita equivalence ([BP91, Theorem 3.8]), we get $\text{RR}(B) = 0$. Thus $B$ has an approximate unit consisting of projections. Therefore, by Theorem 3.19, $\alpha \otimes \text{id} : \mathbb{Z}_2 \to \text{Aut}(A \otimes B)$ has the tracial Rokhlin property. \hfill \Box

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