Research article

Subclass of analytic functions defined by $q$-derivative operator associated with Pascal distribution series

B. A. Frasin$^{1,*}$ and M. Darus$^2$

$^1$ Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan
$^2$ Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia

* Correspondence: Email: bafrasin@yahoo.com.

Abstract: The purpose of the present paper is to find the necessary and sufficient condition and inclusion relation for Pascal distribution series to be in the subclass $TC_q(\lambda, \alpha)$ of analytic functions defined by $q$-derivative operator. Further, we consider an integral operator related to Pascal distribution series, and several corollaries and consequences of the main results are also considered.

Keywords: analytic functions; Hadamard product; $q$-starlike functions; $q$-convex functions; Pascal distribution series

Mathematics Subject Classification: 30C45

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

(1.1)

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{U}.$$

(1.2)

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A, B), \tau \in \mathbb{C}\setminus\{0\}, -1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U}.$$
This class was introduced by Dixit and Pal [13].

The theory of $q$-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, $q$-difference and $q$-integral equations, as well as geometric function theory of complex analysis. The application of $q$-calculus was initiated by Jackson [23]. Recently, many researchers studied $q$-calculus such as Srivastava et al. [52], Muhammad and Darus [31], Kanas and Răducanu [28], Aldweby and Darus [2–4] and Muhammad and Sokol [30]. For details on $q$-calculus one can refer [1, 5–7, 9, 20, 23, 25, 38, 39, 43, 44, 46, 48–51] and also the reference cited therein.

For $0 < q < 1$ the Jackson's $q$-derivative of a function $f \in \mathcal{A}$ is, by definition, given as follows [23]

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases}$$  \hspace{1cm} (1.3)$$

and

$$D_q^2 f(z) = D_q(D_q f(z)).$$

From (1.3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$  \hspace{1cm} (1.4)

where

$$[n]_q = \frac{1 - q^n}{1 - q},$$  \hspace{1cm} (1.5)

is sometimes called the basic number $n$. If $q \rightarrow 1-$, $[n]_q \rightarrow n$.

For a function $h(z) = z^n$, we obtain

$$D_q h(z) = D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1},$$

and

$$\lim_{q \rightarrow 1-} D_q h(z) = \lim_{q \rightarrow 1-} ([n]_q z^{n-1}) = nz^{n-1} = h'(z),$$

where $h'$ is the ordinary derivative.

Using the above defined $q$-calculus, several subclasses belonging to the class $\mathcal{A}$ have already been investigated in geometric function theory. Ismail et al. [26] were the first who used the $q$-derivative operator $D_q$ to study the $q$-calculus analogous of the class $S^*$ of starlike functions in $\mathbb{U}$ (see Definition 1.1 below). However, a firm footing of the $q$-calculus in the context of geometric function theory was presented mainly and basic (or $q$-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [45], p.347 et seq.); see also [46]).

For $0 < q < 1$, we define the class $S^*_q(\alpha)$ of $q$- starlike functions and the class $C_q(\alpha)$ of $q$- convex functions of order $\alpha (0 \leq \alpha < 1)$ (see, [26, 40, 41]), as below:

**Definition 1.1.** A function $f \in \mathcal{A}$ is said to be in the class $S^*_q(\alpha)$ if it satisfies

$$\Re\left(\frac{zD_q f(z)}{f(z)}\right) > \alpha, \hspace{0.5cm} (z \in \mathbb{U}).$$
**Definition 1.2.** A function \( f \in \mathcal{A} \) is said to be in the class \( C_q(\alpha) \) if it satisfies

\[
\Re \left( \frac{D_q(zD_q f(z))}{D_q f(z)} \right) > \alpha, \quad (z \in \mathbb{U}).
\]

It is clear that \( \lim_{q \to 1-} S_q^L(\alpha) = S^L(\alpha) \) and \( \lim_{q \to 1-} C_q(\alpha) = C(\alpha) \), where \( S^L(\alpha) \) and \( C(\alpha) \) are, respectively, well-known starlike and convex functions of order \( \alpha \) in \( \mathbb{U} \).

We now introduce a new subclass of analytic functions defined by \( q \)-derivative operator \( D_q \).

**Definition 1.3.** A function \( f \in \mathcal{A} \) is said to be in the class \( C_q(\lambda, \alpha) \) if it satisfies

\[
\Re \left( \frac{\lambda z^3(zD_q f(z))'' + (2\lambda + 1)z^2(zD_q f(z))'' + z(zD_q f(z))'}{\lambda z^2(zD_q f(z))'' + z(zD_q f(z))'} \right) > \alpha, \quad (z \in \mathbb{U})
\]  

\[ (1.6) \]

where \( 0 \leq \alpha < 1, \ 0 \leq \lambda \leq 1 \).

We write

\[ \mathcal{T}C_q(\lambda, \alpha) = C_q(\lambda, \alpha) \cap \mathcal{T}. \]

A variable \( X \) is said to be Pascal distribution if it takes the values \( 0, 1, 2, 3, \ldots \) with probabilities

\[
(1 - s)^m, \quad \frac{s m (1 - s)^m}{1!}, \quad \frac{s^2 m (m + 1)(1 - s)^m}{2!}, \quad \frac{s^3 m (m + 1)(m + 2)(1 - s)^m}{3!}, \ldots
\]

respectively, where \( s \) and \( m \) are called the parameters, and thus

\[ P(X = k) = \binom{k + m - 1}{m - 1} s^k (1 - s)^m, \quad k = 0, 1, 2, 3, \ldots. \]

Very recently, El-Deeb et al. \cite{15} (see also, \cite{10, 34}) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

\[ \Psi^m_s(z) := z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1} (1 - s)^m z^n, \quad z \in \mathbb{U}, \]

where \( m \geq 1, 0 \leq s \leq 1 \), and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

\[ \Phi^m_s(z) := 2z - \Psi^m_s(z) = z - \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1} (1 - s)^m z^n, \quad z \in \mathbb{U}. \]  

\[ (1.7) \]

Let consider the linear operator \( \mathcal{I}^m_s : \mathcal{A} \to \mathcal{A} \) defined by the convolution or Hadamard product

\[ \mathcal{I}^m_s f(z) := \Psi^m_s(z) \ast f(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1} (1 - s)^m a_n z^n, \quad z \in \mathbb{U}, \]

where \( m \geq 1 \) and \( 0 \leq s \leq 1 \).

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions (see for example, \cite{8, 11, 21, 29, 42, 47}),
Lemma 1.4. A function \( f \) of the form (1.2) is in \( \mathcal{T}C_q(\lambda, \alpha) \) if and only if it satisfies

\[
\sum_{n=2}^{\infty} |n| p(n-\alpha)(\lambda n - \lambda + 1)|a_n| \leq 1 - \alpha,
\]

where \( 0 \leq \alpha < 1 \), \( 0 \leq \lambda \leq 1 \) and \( z \in \mathbb{U} \).

Lemma 1.4 can be proved using the same technique as in [27].

Lemma 1.5. [13] If \( f \in \mathcal{R}(A, B) \) is of the form (1.1), then

\[
|a_n| \leq (A - B) \frac{|\tau|}{n}, \quad n \in \mathbb{N}\setminus\{1\}.
\]

The result is sharp.

2. Necessary and sufficient condition for \( \Phi_f^m \in \mathcal{T}C_q(\lambda, \alpha) \)

For convenience throughout in the sequel, we use the following identities that hold for \( m \geq 1 \) and \( 0 \leq s < 1 \):

\[
\sum_{n=0}^{\infty} \binom{n + m - 1}{m - 1} s^n = \frac{1}{(1-s)^m}, \quad \sum_{n=0}^{\infty} \binom{n + m - 2}{m - 2} s^n = \frac{1}{(1-s)^{m-1}},
\]

\[
\sum_{n=0}^{\infty} \binom{n + m}{m} s^n = \frac{1}{(1-s)^{m+1}}, \quad \sum_{n=0}^{\infty} \binom{n + m + 1}{m + 1} s^n = \frac{1}{(1-s)^{m+2}}.
\]

By simple calculations we derive the following relations:

\[
\sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1} = \sum_{n=0}^{\infty} \binom{n + m - 1}{m - 1} s^n - 1 = \frac{1}{(1-s)^m} - 1,
\]

\[
\sum_{n=2}^{\infty} (n-1) \binom{n + m - 2}{m - 1} s^{n-1} = s m \sum_{n=0}^{\infty} \binom{n + m}{m} s^n = s \frac{m}{(1-s)^{m+1}},
\]

\[
\sum_{n=3}^{\infty} (n-1)(n-2) \binom{n + m - 2}{m - 1} s^{n-1} = 2 s^2 \frac{m+1}{(1-s)^{m+2}},
\]

\[
\sum_{n=4}^{\infty} (n-1)(n-2)(n-3) \binom{n + m - 2}{m - 1} s^{n-1} = 6 s^3 \frac{m+2}{(1-s)^{m+3}}.
\]
and
\[
\sum_{n=5}^{\infty} (n-1)(n-2)(n-3)(n-4) \binom{n+m-2}{m-1} s^{n-1} = 24s^4 \frac{(m+3)}{(m-1)} \frac{\binom{m+1}{m-1}}{(1-s)^{m+4}}. \tag{2.5}
\]

Unless otherwise mentioned, we shall assume in this paper that \(0 \leq \alpha < 1\) and \(0 \leq \lambda \leq 1\), \(0 < q < 1\) and \(0 \leq s < 1\).

Firstly, we obtain the necessary and sufficient conditions for \(\Phi^m_s\) to be in the class \(\mathcal{T}C_q(\lambda, \alpha)\).

**Theorem 2.1.** Let \(m \geq 1\) and \(q \rightarrow 1-\). Then \(\Phi^m_s \in \mathcal{T}C_q(\lambda, \alpha)\) if and only if

\[
24\lambda \frac{\binom{m+3}{m-1}s^4}{(1-s)^{m+4}} + 6(\lambda(9-\alpha) + 1) \frac{\binom{m+2}{m-1}s^3}{(1-s)^{m+3}} + 2(4\lambda(2-\alpha) + 7 - 3\alpha) \frac{\binom{m+1}{m-1}s^2}{(1-s)^{m+2}}
\]
\[
(4\lambda(2-\alpha) + 7 - 3\alpha) \frac{s^2}{(1-s)^{m+1}} \leq 1 - \alpha.
\tag{2.6}
\]

**Proof.** Since \(\Phi^m_s\) is defined by (1.7), in view of Lemma 1.4 it is sufficient to show that

\[
P_q = \sum_{n=2}^{\infty} [n]_q n(n-\alpha)(\lambda n - \lambda + 1) \binom{n+m-2}{m-1} s^{n-1} (1-s)^n \leq 1 - \alpha.
\]

Since \([n]_q \rightarrow n\), when \(q \rightarrow 1-\), we get

\[
P_1 = \sum_{n=2}^{\infty} n^2(n-\alpha)(\lambda n - \lambda + 1) \binom{n+m-2}{m-1} s^{n-1} (1-s)^n
\]
\[
= \sum_{n=2}^{\infty} \left[\lambda n^4 + (1-\lambda - \alpha\lambda) n^3 + \alpha(\lambda - 1)n^2\right] \binom{n+m-2}{m-1} s^{n-1} (1-s)^n.
\]

Writing

\[
n^2 = (n-1)(n-2) + 3(n-1) + 1, \tag{2.7}
\]
\[
n^3 = (n-1)(n-2)(n-3) + 6(n-1)(n-2) + 7(n-1) + 1, \tag{2.8}
\]
\[
n^4 = (n-1)(n-2)(n-3)(n-4) + 10(n-1)(n-2)(n-3)
\]
\[
+ 25(n-1)(n-2) + 15(n-1) + 1, \tag{2.9}
\]

and using (2.2)–(2.5), we have
Let $m ≥ 1$ and $q → 1 − $. If $f ∈ R(A, B)$ and the inequality

$$(A - B)|τ| \left[ 6λs^3 \frac{(m+2)}{(m-1)} s^3 + 2λ(5 - α) + 1 \right] s^2 \frac{(m+1)}{(m-1)} s^2 \frac{(1-s)^2}{(1-s)^2} + (2λ(2 - α) + 3 - α) \frac{(m)}{1-s} + (1-α)(1-(1-s)^m) \right] \leq 1 - α.$$  

is satisfied then $I_s^m f ∈ TC_q(λ, α)$.  

Proof. According to Lemma 1.4 it is sufficient to show that

$$Q_q := \sum_{n=2}^{∞} [n]_q n(n-α)(λn - λ + 1) \left( \frac{n+m-2}{m-1} \right) s^{n-1}(1-s)^m |a_n| \leq 1 - α.$$  

3. Sufficient condition for $I_s^m (R(A, B)) ⊂ TC_q(λ, α)$

Making use of Lemma 1.5, we will study the action of the Pascal distribution series on the class $TC_q(λ, α)$.  

Theorem 3.1. Let $m ≥ 1$ and $q → 1 − $. If $f ∈ R(A, B)$ and the inequality

$$P_1 = \lambda \sum_{n=5}^{∞} (n-1)(n-2)(n-3)(n-4) \left( \frac{n+m-2}{m-1} \right) s^{n-1}(1-s)^m$$

$$= \frac{24λ}{(1-s)^4} + 6λ(9 - α) + 1 \frac{(m+2)}{(m-1)} s^3 + 2(4λ(2 - α) + 7 - 3α) \frac{(m+1)}{(1-s)^2}$$

$$+ (4λ(2 - α) + 7 - 3α) \frac{(m)}{1-s} + (1-α)(1-(1-s)^m).$$

but this last expression is upper bounded by $1 - α$ if and only if (2.6) holds.

Proof. According to Lemma 1.4 it is sufficient to show that

$$Q_q := \sum_{n=2}^{∞} [n]_q n(n-α)(λn - λ + 1) \left( \frac{n+m-2}{m-1} \right) s^{n-1}(1-s)^m |a_n| \leq 1 - α.$$  

Proof. According to Lemma 1.4 it is sufficient to show that
Since $f \in \mathcal{R}^q(A, B)$, using Lemma 1.5 we have
\[ |a_n| \leq \frac{(A - B)|r|}{n}, \quad n \in \mathbb{N} \setminus \{1\}, \]
therefore
\[
Q_1 \leq (A - B)|r| \left[ \sum_{n=2}^{\infty} n(n-\alpha)(\lambda n - \lambda + 1) \left( \frac{n + m - 2}{m - 1} \right)^{s^{n-1}} (1 - s)^m \right]
\]
\[
= (A - B)|r| \left[ \sum_{n=2}^{\infty} \left[ \lambda n^3 + (1 - \lambda - \alpha \lambda) n^2 + \alpha(\lambda - 1)n \right] \left( \frac{n + m - 2}{m - 1} \right)^{s^{n-1}} (1 - s)^m \right].
\]
Writing $n^2, n^3$ as given in (2.7) and (2.8), $n = n - 1 + 1$, and making use of (2.2)–(2.5), we get
\[
Q_1 \leq (A - B)|r| \left[ A \sum_{n=4}^{\infty} (n-1)(n-2)(n-3) \left( \frac{n + m - 2}{m - 1} \right)^{s^{n-1}} (1 - s)^m 
\right.
\]
\[+ (\lambda(5 - \alpha) + 1) \sum_{n=3}^{\infty} (n-1)(n-2) \left( \frac{n + m - 2}{m - 1} \right)^{s^{n-1}} (1 - s)^m 
\]
\[+ (2 \lambda (2 - \alpha) + 3 - \alpha) \sum_{n=2}^{\infty} (n-1) \left( \frac{n + m - 2}{m - 1} \right)^{s^{n-1}} (1 - s)^m 
\]
\[+ (1 - \alpha) \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right)^{s^{n-1}} (1 - s)^m \right]
\[
= (A - B)|r| \left[ 6 A s^3 \frac{(m+2)}{(m-1)^3} + 2(\lambda(5 - \alpha) + 1)s^2 \frac{(m+1)}{(m-1)} \frac{1}{1-s} 
\right.
\]
\[+ (2 \lambda (2 - \alpha) + 3 - \alpha) \frac{(m+1)}{1-s} + (1 - \alpha)(1 - (1-s)^m) \right].
\]
but this last expression is upper bounded by $1 - \alpha$ if and only if (3.1) holds. \qed

4. Integral operator

Theorem 4.1. Let $m \geq 1$ and $q \to 1 -$. If the integral operator $G_s^m$ is given by
\[
G_s^m(z) := \int_0^z \frac{\Phi_s^m(t)}{t} dt, \quad z \in U,
\]
then $G_s^m \in TC_q(\lambda, \alpha)$ if and only
\[
6 A s^3 \frac{(m+2)}{(m-1)^3} + 2(\lambda(5 - \alpha) + 1)s^2 \frac{(m+1)}{(m-1)} \frac{1}{1-s} 
\]
\[+ (2 \lambda (2 - \alpha) + 3 - \alpha) \frac{(m+1)}{1-s} \]
\[
\leq 1 - \alpha.
\]
Proof. According to (1.7) it follows that
\[ G^m_q(z) = z - \sum_{n=2}^{\infty} \left( \frac{n+m-2}{m-1} \right) s^{n-1} (1-s)^{m-1}, \quad z \in \mathbb{U}. \]

Using Lemma 1.4, the function \( G^m_q(z) \) belongs to \( TC_q(\lambda, \alpha) \) if and only if
\[ R_q := \sum_{n=2}^{\infty} [n]_q n(n-\alpha)(\lambda n - \lambda + 1) \times \frac{1}{n} \left( \frac{n+m-2}{m-1} \right) s^{n-1} (1-s)^m \leq 1 - \alpha, \]

Now,
\[ R_1 = \sum_{n=2}^{\infty} [\lambda n^3 + (1-\lambda - \alpha \lambda) n^2 + \alpha(\lambda - 1) n] \left( \frac{n+m-2}{m-1} \right) s^{n-1} (1-s)^m \]

By a similar proof like those of Theorem 3.1 we get that \( G^m_q f \in TC_q(\lambda, \alpha) \) if and only if (4.2) holds. \( \square \)

5. Corollaries and consequences

**Corollary 5.1.** Let \( m \geq 1 \) and \( q \rightarrow 1 - \). Then \( \Phi^m_q \in TC_q(0, \alpha) \), if and only if
\[ 6 \left( \frac{m-1}{m-1} \right) s^3 \frac{1}{(1-s)^{m+3}} + 2(7-3\alpha) \left( \frac{m-1}{m-1} \right) s^2 \frac{1}{(1-s)^{m+2}} + (7-3\alpha) \left( \frac{m}{m-1} \right) s \frac{1}{(1-s)^{m+1}} \leq 1 - \alpha. \]

**Corollary 5.2.** Let \( m \geq 1 \) and \( q \rightarrow 1 - \). If \( f \in R^\alpha(A, B) \) and the inequality
\[ (A - B)|\tau| 2 \left( \frac{m-1}{m-1} \right) s^2 \frac{1}{(1-s)^2} + (3-\alpha) \left( \frac{m-1}{m-1} \right) s \frac{1}{1-s} + (1-\alpha)(1-(1-s)^m) \leq 1 - \alpha. \]

is satisfied then \( I^m_q f \in TC_q(0, \alpha) \).

**Corollary 5.3.** Let \( m \geq 1 \) and \( q \rightarrow 1 - \). If the integral operator \( G^m_q \) is given by (4.1), then \( G^m_q \in TC_q(0, \alpha) \) if and only
\[ 2s^2 \left( \frac{m}{m-1} \right) + (3-\alpha) \left( \frac{m}{m-1} \right) s \frac{1}{1-s} \leq 1 - \alpha. \]

6. Conclusions

In this paper, we find the necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in a subclass of analytic functions defined by \( q \)-derivative operator. Basic (or \( q \)-) series and basic (or \( q \)-) polynomials, especially the basic (or \( q \)-) hypergeometric functions and basic (or \( q \)-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [45], pp.350–351) and [44], p.328). Moreover, in this recently-published survey-cum-expository review article by Srivastava [44], the so-called \((p,q)\)-calculus was exposed to be a rather trivial and inconsequential variation of the classical \( q \)-calculus, the additional parameter \( p \) being redundant (see, for details, [44], p.340]). This observation by Srivastava [44] will indeed apply also to any attempt to produce the rather straightforward \((p,q)\)-variations of the results which we have presented in this paper.

AIMS Mathematics Volume 6, Issue 5, 5008–5019.
Acknowledgements

This research was funded by Universiti Kebangsaan Malaysia, grant number GUP-2019-032. The authors would like to thank the referees for their helpful comments and suggestions.

Conflicts of interest

The authors declare no conflict of interest.

References

1. O. P. Ahuja, A. Çetinkaya, Y. Polatoglu, Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of q-convex and q-close-to-convex functions, *J. Comput. Anal. Appl.*, **26** (2019), 639–649.
2. H. Aldweby, M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, *Abstr. Appl. Anal.*, **2014** (2014), 1–6.
3. H. Aldweby, M. Darus, A new subclass of harmonic meromorphic functions involving quantum calculus, *J. Classical Anal.*, **6** (2015), 153–162.
4. H. Aldweby, M. Darus, Coefficient estimates of classes of q-starlike and q-convex functions, *Adv. Stud. Contemp. Math.*, **26** (2016), 21–26.
5. H. Aldweby, M. Darus, On Fekete-Szego problems for certain subclasses defined by q-derivative, *J. Funct. Spaces*, **2017** (2017), 1–5.
6. H. Aldweby, M. Darus, A note on q-integral operators, *Electron. Notes Discrete Math.*, **67** (2018), 25–30.
7. M. K. Aouf, A. O. Mostafa, F. Y. Al-Quhali, Properties for class of B-uniformly univalent functions defined by Salagean type q-difference operator, *Int. J. Open Probl. Complex Anal.*, **11** (2019), 1–16.
8. M. K. Aouf, A. O. Mostafa, H. M. Zayed, Some constraints of hypergeometric functions to belong to certain subclasses of analytic functions, *J. Egypt. Math. Soc.*, **24** (2016), 361–366.
9. S. Araci, U. Duran, M. Acikgoz, H. M. Srivastava, A certain (p, q)-derivative operator and associated divided differences, *J. Inequal. Appl.*, **2016** (2016), 301.
10. S. Çakmak, S. Yalçın, Ş. Altminkaya, Some connections between various classes of analytic functions associated with the power series distribution, *Sakarya Univ. J. Sci.*, **23** (2019), 982–985.
11. N. E. Cho, S. Y. Woo, S. Owa, Uniform convexity properties for hypergeometric functions, *Fract. Calc. Appl. Anal.*, **5** (2002), 303–313.
12. H. E. Darwish, A. Y. Lashin, E. M. Madar, On subclasses of uniformly starlike and convex functions defined by Struve functions, *Int. J. Open Probl. Complex Anal.*, **1** (2016), 34–43.
13. K. K. Dixit, S. K. Pal, On a class of univalent functions related to complex order, *Indian J. Pure Appl. Math.*, **26** (1995), 889–896.
14. R. M. El-Ashwah, W. Y. Kota, Some condition on a Poisson distribution series to be in subclasses of univalent functions, *Acta Univ. Apulensis Math. Inform.*, **51** (2017), 89–103.
15. S. M. El-Deeb, T. Bulboacă, J. Dziok, Pascal distribution series connected with certain subclasses of univalent functions, *Kyungpook Math. J.*, 59 (2019), 301–314.
16. B. A. Frasin, On certain subclasses of analytic functions associated with Poisson distribution series, *Acta Univ. Sapientiae, Math.*, 11 (2019) 78–86.
17. B. A. Frasin, Subclasses of analytic functions associated with Pascal distribution series, *Adv. Theory Nonlinear Anal. Appl.*, 4 (2020), 92–99.
18. B. A. Frasin, I. Aldawish, On subclasses of uniformly spirallike functions associated with generalized Bessel functions, *J. Funct. Space.*, 2019 (2019), 1–6.
19. B. A. Frasin, M. M. Gharaibeh, Subclass of analytic functions associated with Poisson distribution series, *Afr. Mat.*, 31 (2020), 1167–1173.
20. B. A. Frasin, G. Murugusundaramoorthy, A subordination results for a class of analytic functions defined by $q$-differential operator, *Ann. Univ. Paedagog. Cracov. Stud. Math.*, 19 (2020), 53–64.
21. B. A. Frasin, T. Al-Hawary, F. Yousef, Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions, *Afr. Mat.*, 30 (2019), 223–230.
22. B. A. Frasin, F. Yousef, T. Al-Hawary, I. Aldawish, Application of generalized Bessel functions to classes of analytic functions, *Afr. Mat.*, (2020), 1–9.
23. F. H. Jackson, On $q$-functions and a certain difference operator, *Earth Env. Sci. T. R. So.*, 46 (1909), 253–281.
24. T. Janani, G. Murugusundaramoorthy, Inclusion results on subclasses of starlike and convex functions associated with Struve functions, *Ital. J. Pure Appl. Math.*, 32 (2014), 467–476.
25. T. Al-Hawary, F. Yousef, B. A. Frasin, *Subclasses of analytic functions of complex order involving Jackson's $(p, q)$-derivative*, Proc. Int. Conf. Fract. Differ. Appl. (ICFDA), 2018.
26. M. E. H. Ismail, E. Merkes, D. Steyr, A generalization of starlike functions, *Complex Var. Theory*, 14 (1990), 77–84.
27. M. Kamali, S. Akbulut, On a subclass of certain convex functions with negative coefficients, *J. Appl. Math. Comput.*, 145 (2002), 341–350.
28. S. Kanas, D. Răducanu, Some class of analytic functions related to conic domains, *Math. Slovaca*, 64 (2014), 1183–1196.
29. E. Merkes, B. T. Scott, Starlike hypergeometric functions, *Proc. Am. Math. Soc.*, 12 (1961), 885–888.
30. S. Mahmood, J. Sokol, New subclass of analytic functions in conical domain associated with Ruscheweyh $q$-differential operator, *Res. Math.*, 71 (2017), 1345–1357.
31. A. Mohammed, M. Darus, A generalized operator involving the $q$-hypergeometric function, *Mat. Vesn.*, 65 (2013), 454–465.
32. G. Murugusundaramoorthy, Subclasses of starlike and convex functions involving Poisson distribution series, *Afr. Mat.*, 28 (2017), 1357–1366.
33. G. Murugusundaramoorthy, T. Janani, An application of generalized Bessel functions on certain subclasses of analytic functions, *Turkish J. Anal. Number Theory*, 3 (2015), 1–6.
34. G. Murugusundaramoorthy, B. A. Frasin, T. Al-Hawary, Uniformly convex spiral functions and uniformly spirallike function associated with Pascal distribution series, *Mat. Bohem.*, In press.

35. G. Murugusundaramoorthy, K. Vijaya, S. Porwal, Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series, *Hacet. J. Math. Stat.*, **45** (2016), 1101–1107.

36. S. R. Mondal, A. Swaminathan, Geometric properties of generalized Bessel functions, *Bull. Malays. Math. Sci. Soc.*, **35** (2012), 179–194.

37. S. Porwal, An application of a Poisson distribution series on certain analytic functions, *J. Complex Anal.*, **2014** (2014), 1–3.

38. C. Ramachandran, T. Soupramanien, B. A. Frasin, New subclasses of analytic function associated with q-difference operator, *Eur. J. Pure Appl.*, **10** (2017), 348–362.

39. M. S. U. Rehman, Q. Z. Ahmad, H. M. Srivastava, N. Khan, M. Darus, B. Khan, Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions, *AIMS Math.*, **6** (2021), 1110–1125.

40. T. M. Seoudy, M. K. Aouf, Convolution properties for certain classes of analytic functions defined by q-derivative operator, *Abstr. Appl. Anal.*, **2014** (2014), 1–7.

41. T. M. Seoudy, M. K. Aouf, Coefficient estimates of new classes of q-starlike and q-convex functions of complex order, *J. Math. Inequal.*, **10** (2016), 135–145.

42. H. Silverman, Starlike and convexity properties for hypergeometric functions, *J. Math. Anal. Appl.*, **172** (1993), 574–581.

43. H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inf. Sci.*, **5** (2011), 390–444.

44. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. A: Sci.*, **44** (2020), 327–344.

45. H. M. Srivastava, *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*, In: *Univalent Functions, Fractional Calculus, and Their Applications*, (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester); John Wiley and Sons: New York, Chichester, Brisbane and Toronto, (1989), 329–354.

46. H. M. Srivastava, D. Bansal, Close-to-convexity of a certain family of q-Mittag-Leffler functions, *J. Nonlinear Var. Anal.*, **1** (2017), 61–69.

47. H. M. Srivastava, G. Murugusundaramoorthy, S. Sivasubramanian, Hypergeometric functions in the parabolic starlike and uniformly convex domains, *Integr. Transf. Spec. Funct.*, **18** (2007), 511–520.

48. H. M. Srivastava, Q. Zahoor, N. Khan, N. Khan, B. Khan, Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, *Mathematics*, **7** (2019), 181.

49. H. M. Srivastava, N. Raza, E. S. A. Abu Jarad, G. Srivastava, M. H. AbuJarad, Fekete-Szegő inequality for classes of (p; q)-starlike and (p; q)-convex functions, *RACSAM*, **113** (2019), 3563–3584.
50. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general classes of $q$-starlike functions associated with the Janowski functions, *Symmetry*, 11 (2019), 1–14.

51. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general families of $q$-starlike functions associated with the Janowski functions, *Filomat*, 33 (2019), 2613–2626.

52. R. Srivastava, H. M. Zayed, Subclasses of analytic functions of complex order defined by $q$-derivative operator, *Stud. Univ. Babes-Bolyai Math.*, 64 (2019), 69–78.

© 2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)