ABSTRACT

We study the nonlinear growth of kinetic gyroresonance instability of cosmic rays (CRs) induced by large-scale compressible turbulence. This feedback of CRs on turbulence was shown to induce an important scattering mechanism in addition to direct interaction with the compressible turbulence. The linear growth is bound to saturate due to the wave–particle interactions. By balancing the increase of CR anisotropy via the large-scale compression and its decrease via the wave–particle scattering, we find the steady-state solutions. The nonlinear suppression due to the wave–particle scattering limits the energy range of CRs that can excite the instabilities and be scattered by the induced slab waves. The direct interaction with large-scale compressible modes still appears to be the dominant mechanism for isotropization of high-energy CRs (> 100 GeV).

Key words: cosmic rays – instabilities – magneto-hydrodynamics (MHD) – scattering – turbulence

Online-only material: color figures

1. INTRODUCTION

Cosmic rays (CRs) and turbulence are essential components in astrophysical systems. Their interactions are crucial for high-energy phenomena and dynamics of various systems. The resonant and non-resonant (e.g., transient time damping) interactions of CRs with MHD turbulence are the accepted principal mechanism to scatter and isotropize CRs (see Schlickeiser 2002). In addition, efficient scattering is essential for the acceleration of CRs. For instance, scattering of CRs back into the shock is a vital component of first-order Fermi acceleration (see Longair 1997). At the same time, stochastic acceleration by turbulence is entirely based on scattering.

It is generally accepted that the energy of turbulence is most probably due to supernovae explosions and cascaded down to small scales, where resonance with CRs of moderate energies happens. For years magnetic turbulence has been treated in an ad hoc manner, with variations that included plane Alfvén waves moving parallel to the magnetic field, a combination of these waves and so-called two-dimensional perturbations, isotropic magnetic turbulence with Kolmogorov spectrum, and anisotropic turbulence with constant degree of anisotropy. However, numerical simulation contradicted all the above models.

Unlike hydrodynamic turbulence, Alfvénic turbulence develops a scale-dependent anisotropy, with eddies elongated along the magnetic field (Goldreich & Sridhar 1995, henceforth GS95). The GS95 model describes incompressible Alfvénic turbulence, which formally means that plasma $\beta \equiv P_{\text{gas}} / P_{\text{mag}}$ (the ratio of gas pressure to magnetic pressure) is infinity. But it may be conjectured that GS95 scaling should be approximately true for moderately compressible plasma (Lithwick & Goldreich 2001). Studies in Cho & Lazarian (2002, henceforth CL02) showed that the coupling of Alfvénic and compressible modes is weak and that the Alfvénic modes follow the GS95 spectrum in a compressible medium. This is consistent with the analysis of observational data (Lazarian & Pogosyan 2000; Stanimirović & Lazarian 2001).

The turbulence injected on large scales may correspond to the GS95 model and its extensions to a compressible medium is less efficient in the scattering of CRs compared with the estimates made assuming that magnetic turbulence consists of plane waves moving parallel to the magnetic field. A cascade of Alfvénic perturbations initiated at large injection scales is shown to be inefficient for the CR scattering (Chandran 2000; Yan & Lazarian 2002).

At the same time, one should not disregard the possibilities of the generation of additional perturbations by CRs themselves. For instance, the slab Alfvénic perturbation can be created, e.g., via streaming instability (see Wentzel 1974; Cesarsky 1980). These perturbations are present for a range of CR energies (e.g., $\lesssim 100$ GeV in the interstellar medium for the streaming instability) owing to nonlinear damping arising from ambient turbulence (Yan & Lazarian 2002, 2004, henceforth YL02, YL04; Farmer & Goldreich 2004). Instabilities induced by anisotropic distribution of CRs were also suggested as a possible mechanism to scatter CRs (Lerche 1967; Melrose 1974).

Recent work by Lazarian & Beresnyak (2006, henceforth LB06) proposed that compressible turbulence can induce gyroresonance instability, which is an important feedback processes that can create slab modes to efficiently scatter CRs. This process is claimed to be more efficient in scattering CRs compared with the direct action of turbulent fluctuations.

This gyrokinetic instability is induced by the anisotropic distribution of CRs, which is caused by the compression arising from large-scale turbulence. The degree of anisotropy is determined by the compression on the scale of the CR mean free path in LB06. In their treatment, the growth of CRs needs to be balanced with steepening in order to prevent the diffusion from approaching the Bohm limit, where the mean free path of particles becomes comparable to the Larmor radius. In this paper, we shall investigate the nonlinear suppression of the
instability by considering the self-adjustment of anisotropy due to scattering with the magnetic perturbations. The instability reaches a stabilized growth rate due to the feedback of the increased perturbations on the anisotropy of CRs.

In what follows, in Section 2 we give a brief background on turbulence and CR scattering, in Section 3 we introduce the gyroresonance instability of CRs, and in Section 4 we formulate our approach to the problem. We outline our findings and the implications of our results in Section 5. In Section 6 we discuss our results. The summary is provided in Section 7.

2. MHD TURBULENCE AND COSMIC RAY SCATTERING

The propagation of CRs is mainly affected by their interaction with the magnetic field. The properties of turbulence are vital for the correct description of CR propagation. For instance, the scattering frequency is significantly dependent on the anisotropy of turbulence (Lerche & Schlickeiser 2001; Yan & Lazarian 2002).

Numerical simulations (see Cho & Vishniac 2000; Maron & Goldreich 2001; Müller & Biskamp 2000; Cho, Lazarian, & Vishniac 2002; Cho & Lazarian 2002, 2003; see also the book of Biskamp 2003, as well as Cho et al. 2003 and Elmegreen & Scalo 2004, for reviews), show that Alfvénic modes exhibit scale-dependent anisotropy. This anisotropy was first discussed in Goldreich & Sridhar (1995, henceforth GS95). However, the understanding came later that this anisotropy is observable only in the system of reference related to the local direction of the magnetic field on the scale of the eddy under consideration (Lazarian & Vishniac 1999; Cho & Vishniac 2000; Maron & Goldreich 2001). On the intuitive level, it can be explained as the result of the following fact: it is easier to mix the magnetic field lines perpendicular to the direction of the magnetic field rather than to bend them. However, one cannot do mixing in the perpendicular direction to very small scales without affecting the parallel scales. This is probably the major difference between the adopted model of Alfvénic perturbations and the reduced MHD (see Bieber et al. 1994). The corresponding scaling can be easily obtained from the critical balance condition, i.e., $k_∥\delta v_∥ \sim k⊥v_A$, where $k_∥$ and $k⊥$ are the parallel and perpendicular components of the wave vector $k$, $v_A$ is the Alfvén speed, and $\delta v_∥ \propto k^{-v}$ is the turbulence velocity. Throughout the paper, $∥$ and $⊥$ are defined in reference to the local magnetic field. As proven later by Cho et al. (2002), the mixing motion perpendicular to the magnetic field is essentially hydrodynamic in the turbulence cascade is conserved locally in phase space. From these arguments, the scale-dependent anisotropy $k_∥ \propto k^{-2/3}_⊥$ and $\delta v_∥ \propto k^{-1/3}_⊥ \propto k^{-1/2}_∥$ can be derived.

Owing to the existence of the parallel cascade, Alfvénic turbulence can cascade to the resonant scale of CRs, which is characterized by the parallel scale $k_∥,res \sim \Omega/v$, where $\Omega$ is the Larmor frequency and $v$ is the particle speed. The scattering efficiency is, however, substantially reduced because of the anisotropy of the Alfvénic turbulence so that we can completely neglect it (Yan & Lazarian 2002, 2004) except for ultra-high-energy CRs.

The distribution of energy between compressible and incompressible modes depends, in general, on the way turbulence is driven. Naturally a more systematic study of different types of driving is required to determine the energy partition. In the absence of this, in what follows we assume that equal amounts of energy are transferred into fast and Alfvén modes when driving is at large scales.

While particular aspects of the GS95 model, e.g., the particular value of the spectral index, are the subject of controversies (see Müller & Biskamp 2000; Boldyrev 2006; Gogoberidze 2007; Beresnyak & Lazarian 2009), we think that, at present, the GS95 model provides a good starting point for developing models of CR scattering. MHD turbulence can be decomposed into Alfvén, slow, and fast modes (Cho & Lazarian 2002, 2003). Slow modes are passive and follow the same GS95 scaling as do the Alfvén modes (Lithwick & Goldreich 2001; Cho & Lazarian 2002). Fast modes, instead, are marginally affected by Alfvén modes and follow an acoustic type cascade (Cho & Lazarian 2003). They have been identified by YL02, YL04, and Yan & Lazarian (2008, henceforth YL08) as the major source of CR scattering in interstellar medium and intracluster medium (ICM).

Until recently, test particle approximation was assumed in most of earlier studies and no feedback of CRs is included apart from the streaming instability. The turbulence cascade is established from large scales and no feedback of CRs is included. This may not reflect reality, as we know the energy of CRs is comparable to that in turbulence and the magnetic field (see Kulsrud 2005). It was suggested by LB06 that the gyroresonance instability of CRs can drain energy from the large-scale turbulence and cause instability on small scales by the turbulence compression induced anisotropy on CRs. And the wave generated on these scales, in turn, provides additional scattering to CRs. In this paper, we would like to provide quantitative studies based on the nonlinear theory of the growth of the instability to take into account more accurately the feedback of the growing waves on the distributions of CRs.

3. GYRORESONANCE INSTABILITY OF COSMIC RAYS

Plasma is known to be subject to numerous instabilities, one of them being gyroresonance instability (Gary et al. 1997). LB06 suggested that an analogous instability operates in CRs and is important for CRs subject to compressions arising from MHD turbulence.

3.1. Origin of Growth

In a nonuniform plasma, where magnetic field strength varies, instability can develop due to anisotropy of particles arising from the adiabatic invariant $\mu$, which is preserved when the variations happen on a length scale much larger than the particles’ Larmor radii and a timescale much longer than the Larmor period.

To understand the essence of the instability it is important to recall that the definition for the magnetic moment of a charged particle in the magnetic field is

$$\mu = \frac{qv_⊥r_L}{2c} = \frac{p_⊥v_⊥}{2B},$$

where $q$, $p$, $r_L$ are the charge, momentum, and Larmor radius of the particle, $c$ is the light speed, and $B$ is the strength of the magnetic field. For non-relativistic particles, $\mu = E_{k,⊥}/B$ is proportional to the kinetic energy of particles $E_{k,⊥}$; for relativistic particles, however, $p_⊥v_⊥$ deviates from the real kinetic energy. $\mu$ is an adiabatic invariant, indicating that $p_⊥v_⊥$ changes with the strength of the magnetic field. In the regions where the magnetic field increases, the perpendicular pressure of the particles is higher than the parallel one, inducing mirror and gyroresonance instability; in the places where the magnetic field decreases, the opposite is true, resulting in firehose instability. Waves are generated through the instabilities, enhancing the
scattering rates of the particles; their distribution will be relaxed to the marginal state of instability even in the collisionless environment. We define the degree of anisotropy of particle distribution as

$$A \equiv \frac{W_\perp}{W_\parallel} - 1,$$

where

$$W_\perp = \int d^3 p f(p) v_\perp p_\perp,$$

$$W_\parallel = \int d^3 p f(p) v_\parallel p_\parallel$$

is equivalent to the perpendicular and parallel components of the energy density of CRs.

In the presence of background compressible turbulence, the CR distribution is bound to be anisotropic because of the conservation of the adiabatic invariant $\mu$. Such anisotropic distribution is subjected to various instabilities. While the hydrodynamic instability requires certain threshold, the kinetic instability can grow very fast with small deviations from isotropy. In this paper, we shall focus on the gyroresonance instability. The rate at which the anisotropy $A$ varies is determined by turbulence compressional rate,

$$\left( \frac{\partial A}{\partial t} \right)_{\text{gr}} = \frac{1}{B} \frac{dB}{dt} = \omega \left( \frac{\delta v_\perp}{v_A} \right),$$

where $\omega \equiv v_{ph}/l_c$ is the frequency of the compressional modes at the compressional scale $l_c$, $\delta v_\perp \propto l_c^{-1}$ is the perpendicular component of the turbulence velocity. Since beyond mean free path, the anisotropy is effectively reduced because of scattering, only compression $\lesssim \lambda$, the mean free path of CRs, can alter the degree of anisotropy. As $\omega \delta v_\perp/v_A \sim l_c^{-1}$ and the power-law index of turbulence $v$ is less than 1, the compression rate in fact increases with the decrease of the scale. So different from LB06, we adopt the turbulence damping scale $l_c$ as the compression scale provided that $l_c \gg r_L$. For Kolmogorov scaling, $v = 1/3$. In the case of $\lambda < l_c$, as in some collisionless medium, the compression is still supplied from $l_c$, but with a reduced rate $\omega \delta v_\perp/v_A \propto l_c^{-1} \lambda l_c$. Using the compressional time $\omega^{-1}$ as the timescale for growth of the anisotropy, LB06 estimated the anisotropy $A \sim \delta v/v_A$. We note, however, that the growth time for the anisotropy should be constrained by the scattering time of CRs $\tau_{\text{scatt}}$. If $\tau_{\text{scatt}} \simeq \lambda/c$ were adopted, the anisotropy of CRs $A \sim \tau_{\text{scatt}} A/\partial t$ would not reach the threshold $v_A/v$ (see Kulsrud 2005) unless the mean free path of CRs is much larger than the turbulence compressional scale, which is limited by the damping.\(^6\) In fact, this constraint does not apply because we are dealing with an evolving system rather than an equilibrium one. The wave amplitude $\epsilon$ is growing at a rate $\Gamma_{gr}$ and so the mean free path is decreasing. Therefore, the scattering rate is decreased compared with the equilibrium value $\Omega/\epsilon \equiv \Omega B^2/8\pi v_A$. The reduced rate may be estimated as below. The variation of anisotropy $A$ can be expressed as

$$\frac{dA}{dt} \simeq v_A = \frac{1}{W_\parallel} \left( \frac{dW_\perp}{dt} - \frac{W_\perp}{W_\parallel} \frac{dW_\parallel}{dt} \right) \sim \Gamma_{gr} \epsilon / \beta_{CR}.$$

where $\beta_{CR} \equiv 8\pi W_\parallel / B^2$ is the ratio of CR pressure to magnetic pressure. In the above equation, we utilized the fact that $A \ll 1$ and so $W_\perp / W_\parallel \simeq 1$ and that during the isotropization induced by the growing waves, the loss in $W_\parallel$ is greater than the gain in $W_\parallel$ so that wave can grow at the expense of the CRs’ energy. From the above equation, the isotropization rate can be estimated as

$$\tau_{\text{scatt}}^{-1} \sim \Gamma_{gr} \epsilon / \beta_{CR} A.$$  

By balancing the rate of decrease in anisotropy due to scattering and the growth due to compression, we get

$$\epsilon \sim \frac{\beta_{CR} \omega \delta v}{\Gamma_{gr} v_A},$$

$$\lambda = r_p / \epsilon.$$  

The magnetic field strength is altered by compressible modes. In low $\beta$ medium, the compression is mainly from fast modes. For fast modes, their cascade is slower than their wave relaxation. This means that they have periodical oscillations unlike slow modes turbulence. A compression/expansion is followed by another, therefore a growth rate must be faster than the wave frequency in order for the instability to grow (Yan 2009). In high $\beta$ medium, the situation is different and the contribution from slow modes is dominant.

In the above equations, we did not account for collisions. This is physical because the scale we consider is less than the mean free path. In LB06, the only feedback was passively incorporated in $\lambda$, which decreases as the wave grows. This process, however, will not stop until $\lambda$ reaches Larmor radius, namely, the Bohm diffusion regime. In this paper, we shall provide a quantitative treatment of the nonlinear suppression.

### 3.2. Growth Rate

The distribution function of CRs $f(x, p, t)$ is governed by the Vlasov equation (see, e.g., Kulsrud 2005):

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + q \left( E(x, t) + \frac{v \times B}{c} \right) \cdot \frac{\partial f}{\partial p} = 0,$$

where $v, p$ are velocity and the momentum of the particles, respectively. $E$ is the electric field. We can take a perturbation expansion for the distribution function

$$f = f_0 + f_1 + f_2 + \cdots, \quad f_{i+1} \ll f_i$$

if the field perturbation is small $\delta B \ll B_0$, where $B_0$ is taken in the $z$-direction.\(^7\)

Assuming a monochromatic Alfvén wave propagating along the $z$-direction, the linear perturbation $f_1$ can be obtained using the quasi-linear theory (see, e.g., Kulsrud 2005). The linearization of the Vlasov equation is

$$-i \omega f_1 + ikv_\perp f_1 + \frac{q}{c} (v \times B_0) \cdot \nabla_p f_1 = -q \left[ E + \frac{v \times (k \times E)}{\omega} \right] \cdot \nabla_p f_0.$$  

\(^6\) In collisionless environments, for instance, the dissipation scale of compressible modes can be very large (Yan & Lazarian 2004; Brunetti & Lazarian 2007).

\(^7\) The gyroradii are much smaller than the turbulence injection scale unless one is considering ultra-high-energy CRs. So the condition $\delta B \ll B_0$ is well justified for moderate energy CRs.
By integrating it, one can get

\[
f_{1}^{\pm} = \frac{i q v_{\perp}}{\omega} E_{\perp} \left( k \frac{\partial f_{0}}{\partial p_{\parallel}} + \omega - k_{j} v_{\perp} \frac{\partial f_{0}}{\partial p_{\perp}} \right) (k_{j} v_{\parallel} - \omega \mp \Omega)^{-1},
\]

where \( f_{1}^{+} \) corresponds to the right-moving CRs and \( f_{1}^{-} \) is for left-moving CRs. \( \omega, k \) are the wave frequency and the wave number. \( \Omega = \Omega_{0}/\gamma_{L} \) is the relativistic Larmor frequency and \( \Omega_{0} = q B/m c \), \( m \) is the proton mass. From the Maxwell equations, one can obtain

\[
E_{1} \left( 1 - \frac{k^{2} c^{2}}{\omega^{2}} \right) + \frac{4 \pi i J_{1}}{\omega} = 0,
\]

where

\[
J_{1} = \sum q \int d^{3} v v f_{1}
\]

is the current density arising from the perturbed distribution \( f_{1} \). Then, the dispersion relation can be obtained through Equations (11)–(13)

\[
1 - \frac{k^{2} c^{2}}{\omega^{2}} + \sum J_{j}^{\pm}(k, \omega) = 0
\]

and the perturbed part of dielectric tensor

\[
K_{j}^{\pm} = \frac{4 \pi i J_{j}}{\omega E_{1}} = \frac{2 \pi q^{2}}{\omega^{2}} \int d^{3} p \frac{v_{\perp}}{\omega - k_{j} v_{\parallel} \mp \Omega}
\]

\[
\times \left[ k v_{\perp} \frac{\partial f_{0}}{\partial p_{\parallel}} + (\omega - k_{j} v_{\parallel}) \frac{\partial f_{0}}{\partial p_{\perp}} \right],
\]

where the subscript “\( j \)” refers to different species. For \( \omega \ll \Omega \), \( K^{0} = c^{2}/v_{A}^{2} \), representing MHD waves parallel to the magnetic field with \( \omega_{0} = k v_{A} \). Insert \( \omega = \omega_{0} + \omega_{1} \) into the above equation, one can find the growth rate corresponding to the imaginary part of the perturbed \( K_{j} \),

\[
\Gamma^{\pm} = \Im(\omega_{1}) = -\frac{\omega_{0} v_{A}^{2}}{2 c^{2}} \Im(K_{j}^{\pm}). \tag{16}
\]

The growth rate of the gyroresonance instability can be easily derived from this equation:

\[
\Gamma^{\pm}_{gr} = \pi^{2} e^{2} v_{A}^{4} \int \frac{v^{2}}{c^{2}} \left( \frac{\partial f}{\partial p_{\parallel}} - \frac{v_{\parallel}}{v_{\perp}} \frac{\partial f}{\partial p_{\perp}} \right) \delta(k_{j} v_{\parallel} \mp \Omega) d^{3} p
\]

\[
= \frac{\omega_{p}^{2}}{n} \left( \frac{m \Omega_{0}/k_{j}^{2}}{\alpha} \right) (A - A_{cr}) Q,
\]

\[
= \frac{v_{A}}{L_{i}} (A - A_{cr}) \left( \frac{r_{p}}{r_{0}} \right)^{1-\alpha}, \tag{17}
\]

where \( n \) is the gas number density, \( \omega_{p} = \sqrt{4 \pi n e^{2}/m} \) is the proton plasma frequency, \( n_{cr} \propto p^{1-\alpha} \) is the number density of CRs, and (see e.g., LB06)

\[
A_{cr} = v_{A} / v, \quad Q = \frac{\pi}{16} (\alpha + 2)(\alpha - 1) Bt(3/2, \alpha/2),
\]

\[
L_{i} = 6.4 \times 10^{-7} \left( \frac{B}{5 \mu G} \right) \left[ \frac{4 \times 10^{-10} \text{ cm}^{-3}}{n_{cr}(r_{p} > r_{0})} \right] \text{ pc}, \tag{18}
\]

in which \( r_{0} \) is the gyroradius of 1 GeV particles, \( Bt \) represents a beta function (see Abramovitz & Stegun 1965).

The sign of \( \Gamma_{gr} \) determines whether the wave grows (“+”) or damps (“−”). It depends on the bracketed expression in Equation (15), which is a derivative of \( f \) along a circle centered at \( p_{z} = m v_{A} \), \( p_{\perp} = 0 \) (see Kulsrud 2005),

\[
\frac{d p_{\perp}}{d p_{\parallel}} = \frac{\omega - k v_{z}}{k v_{\perp}}.
\]
The wave grows if there are more and more particles along the way it propagates. Unlike with shifted distribution (streaming instability), only circularly polarized waves can be excited by the gyroresonance instability. First, we decide the directions of wave vector resonant with right and left moving CRs, respectively. Then we determine the polarization of the waves bearing in mind that CR protons always rotate clockwise with respect to the magnetic field. For an oblate distribution of particles with \( p_\perp > p_\parallel \), the left moving, right circularly polarized wave can resonate with the right moving CRs (with superscript “+”) and gain energy from them as it encounters more particles in the wave propagation direction (see Figure 1, right). So do the right moving, left circularly polarized waves with the left moving CRs (with superscript “−”). For a prolate distribution, the opposite is true. They are illustrated by Figure 1.

4. FEEDBACK PROCESSES AND STEADY-STATE GROWTH SOLUTIONS

4.1. Nonlinear Suppression

In this section, we shall provide a rigorous study of the suppression of the growth of the instability. The wave–particle interaction was treated using the second-order theory (SOT) in Gary & Tokar (1985) for bi-Maxwellian distribution of thermal particles. These particles have Maxwellian distribution characterized by \( T_1, T_\perp \), two different temperatures parallel and perpendicular to the local magnetic field. We shall modify the SOT for the CRs’ power-law distribution, i.e., \( f(p) \propto p^{-\alpha} \).

Assuming small-amplitude fluctuation, we can easily extend the Vlasov equation to the second order. The slowly varying second-order distribution of particles due to the stochastic interactions with the electro-magnetic perturbations can be described by the ensemble average of the second-order extension of the Vlasov equation,

\[
\frac{\partial (f_2)}{\partial t} + \frac{q (v \times B)}{c} \cdot \frac{\partial f_2}{\partial p} + q E_2 \cdot \frac{\partial f_0}{\partial p} = -e \left( \frac{1}{E_1 + B_1} \right) \cdot \frac{\partial f_1}{\partial p},
\]

(20)

The evolution of the anisotropy of the CRs \( A \) can be derived from this equation. The electric field \( E_1 \) is perpendicular to the magnetic field and therefore increases the perpendicular energy of CRs. The magnetic perturbation \( B_1 \) scatter particles and thus only causes exchange of particle energies in the parallel and perpendicular directions. The second moment of Equation (20) gives

\[
\frac{\partial W_\perp}{\partial t} = 2q \left( E_1 \cdot \int \nu_{\parallel} f_1 \, d^3p \right) + \frac{2q}{c} \left( B_1 \cdot \int d^3p \nu_{\perp} \times \nu_{\perp} f_1 \right),
\]

\[
\frac{\partial W_\parallel}{\partial t} = -\frac{2q}{c} \left( B_1 \cdot \int d^3p \nu_{\parallel} \times \nu_{\parallel} f_1 \right).
\]

(21)

The two terms on the right-hand side of the first equation correspond to the acceleration and scattering. They are given by (see the Appendix for the detailed derivation),

\[
q \left( E_1 \cdot \int \nu \, d^3p \right) = \int d\omega \frac{2}{|\omega|^2} [\omega^2 \omega^* (K^+ \varepsilon_+ + K^- \varepsilon_-)],
\]

\[
\frac{q}{c} \left( B_1 \cdot \int d^3p \nu_{\perp} \times \nu_{\perp} f_1 \right) = -\int d\omega \frac{2}{|\omega|^2} \left[ \omega_{\perp}^2 \omega^* (\Gamma^+ \varepsilon_+ + \Gamma^- \varepsilon_-) + 3[(\omega_\perp + \Omega) \omega_\perp^2 K^+ \varepsilon_+ + (\omega_\perp - \Omega) \omega_\perp^2 K^- \varepsilon_-] \right],
\]

(22)

where \( \omega_{\perp} \equiv \sqrt{4\pi n_{\perp} p > p_{\text{res}}} q^2/m \) is equivalent to the plasma frequency, but for CRs, \( \omega_r \) is the real part of \( \omega \),

\[
\varepsilon_\pm \equiv \frac{E_\pm^2 E_{\pm}}{8\pi}.
\]

(23)

For a symmetric distribution, we expect that \( \varepsilon_+ = \varepsilon_- \) and \( \Gamma^+ = \Gamma^- \). Then from Equations (21) and (22), we can get

\[
\frac{\partial W_\perp}{\partial t} = \int d\omega S_\perp(\omega) \frac{\partial \varepsilon_+(\omega)}{\partial t},
\]

\[
\frac{\partial W_\parallel}{\partial t} = \int d\omega S_\parallel(\omega) \frac{\partial \varepsilon_\pm(\omega)}{\partial t},
\]

(24)

where

\[
\varepsilon(\omega) \equiv \left| B_1 \right|^2 + \left| E_1 \right|^2 = \left( 1 + \left| \frac{k^2 c^2}{|\omega|^2} \right| (\varepsilon_+ + \varepsilon_-) \right)
\]

(25)

is the total energy density of the wave and \( S_\perp, S_\parallel \) are measures of the scattering efficiency, determining how fast is the response of the distribution of CRs to the wave perturbations. They increase with the decrease of wave frequency (see the Appendix for the detailed derivation). In general, their function is to reduce the anisotropy.

For gyroresonance instability, the growth of the wave happens at the gyroresonance frequency of the CRs, \( \Gamma_{\text{res}} \). Therefore in the first-order approximation the above evolution equation can be differentiated, and accordingly we obtain (see the Appendix for detailed derivation)

\[
\frac{\partial W_\perp(\omega)}{\partial t} = S_\perp(\omega) \frac{\partial \varepsilon_+(\omega)}{\partial t},
\]

\[
\frac{\partial W_\parallel(\omega)}{\partial t} = S_\parallel(\omega) \frac{\partial \varepsilon_\pm(\omega)}{\partial t},
\]

(26)

and

\[
S_\perp(\omega) \equiv -2 \left[ \frac{2(\alpha + 2)}{\omega_\perp^2 (\alpha + 1)(\alpha + 3)} \omega_\perp^2 + 2k^2 c^2 \right] / (|\omega|^2 + c^2 k^2),
\]

\[
\simeq -2 \left[ \frac{2(\alpha + 2)}{\omega_\perp^2 (\alpha + 1)(\alpha + 3)} \beta_{\text{CR}}(p > p_{\text{res}} + 2) \right].
\]

\[
S_\parallel(\omega) \equiv 2 \left[ \frac{\alpha(\alpha - 1)}{\omega_\perp^2 (\alpha + 2)} \omega_\perp^2 + k^2 c^2 \right] / (|\omega|^2 + c^2 k^2),
\]

\[
\simeq 2 \left[ \frac{\alpha(\alpha + 2)}{\omega_\perp^2 (\alpha + 1)(\alpha + 3)} \beta_{\text{CR}}(p > p_{\text{res}} + 1) \right].
\]

(27)

From this, we can get

\[
\frac{\partial A}{\partial t} = \partial \left( \frac{W_\perp}{W_\parallel} \right) / \partial t = \frac{1}{W_\parallel} \left[ S_\perp - (A + 1) S_\parallel \right] \varepsilon_\perp \Gamma_{\text{gr}}.
\]

(28)

By equating Equations (4) and (28), we get

\[
S_\parallel \Gamma_{\text{gr}} \varepsilon_N + \beta_{\text{CR}} \omega \left( \frac{\delta v_\perp}{v_A} \right) - (A + 1) S_\parallel \Gamma_{\text{gr}} \varepsilon_N = 0.
\]

(29)

Since \( S_\parallel, S_\perp \) are of order 1 (see Figure 2), we see that the results from the above equation is comparable to our earlier estimate in Equation (7). There are two unknowns \( A, \varepsilon_N \) in Equation (29).
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There has to be another relation to determine them. We assume here that the system relaxes to the marginal state of instability, and \( A = A_{\text{cr}} = v_A / \nu \). For the compression by fast modes, \( \omega = v_A / l_c, \delta V_\perp \approx V(l_c / L)^{\nu}, V \) is the injection velocity of turbulence at the injection scale \( L \). We get for the compression by fast modes

\[
\varepsilon_N = \beta_{\text{CR}} M_A \left( \frac{l_c}{L} \right)^{\nu} \frac{L_i}{l_c A} \left( \frac{r_p}{r_0} \right)^{a-1} \frac{1}{[S_\| (1 + A) - S_\perp]}, \text{ if } \lambda > l_c,
\]

(30)

\[
\varepsilon_N = \beta_{\text{CR}} M_A \frac{r_p}{l_c} \left( \frac{l_c}{L} \right)^{\nu} \frac{L_i}{l_c A} \left( \frac{r_p}{r_0} \right)^{a-1} \frac{1}{[S_\| (1 + A) - S_\perp]}, \text{ if } \lambda < l_c,
\]

(31)

where \( M_A \equiv V/v_A \) is the Alfvénic Mach number, \( \nu = 1/3 \) for the Kolmogorov turbulence and \( \nu = 1/4 \) for the Iroshnikov–Kraichnan turbulence. We adopt Kolmogorov scaling\(^8\) for the fast modes in this paper.

The parallel damping scale of slow modes in high \( \beta \) medium is the proton diffusion scale \( \beta^{1/2} l_{\text{infp}} \), where \( l_{\text{infp}} \) is the mean free path of thermal particles (Barnes 1966). For slow modes in high \( \beta \) medium, \( \omega = v_A / l_c, \delta V_\perp / v_A = \delta V_A = v_A L / (l_c / L)^{1/2} \), where \( l_A = L / M_A^2 \) in the case of \( M_A > 1 \). Inserting it into Equation (29), we get

\[
\varepsilon_N = \beta_{\text{CR}} \left( \frac{l_c}{l_A} \right)^{\nu} \frac{L_i}{l_A A} \left( \frac{r_p}{r_0} \right)^{a-1} \frac{1}{[S_\| (1 + A) - S_\perp]}, \text{ if } \lambda > l_c,
\]

(32)

\[
[\beta_{\text{CR}} \frac{r_p}{l_c} \left( \frac{l_c}{L} \right)^{\nu} \frac{L_i}{l_c A} \left( \frac{r_p}{r_0} \right)^{a-1} \frac{1}{[S_\| (1 + A) - S_\perp]}], \text{ if } \lambda < l_c.
\]

4.2. Bottleneck for the Growth Due to Energy Constraint

The creation of the slab waves through the CR resonant instability is another channel to drain the energy of large-scale turbulence. This process, on one hand, can damp the turbulence. On the other hand, it means that the growth rate is limited by the turbulence cascade. The energy growth rate \( \Gamma_{\text{gr}} \) cannot be larger than the turbulence energy cascading rate, which is \( 1/2 \rho v^2 / v_A / L \) for fast modes in low \( \beta \) medium and \( \rho v_A^3 / l_A \) for slow modes in high \( \beta \) medium. This places a constraint on the growth:

\[
\Gamma_{\text{gr}} \varepsilon_N \leq \begin{cases} M_A^2 v_A / L, & \beta < 1, \\ v_A / l_A, & \beta > 1. \end{cases}
\]

(34)

Thus, the upper limit of wave energy is given by

\[
\varepsilon_N^u = \begin{cases} M_A^2 L_i / (L A) (r_p / r_0)^{a-1}, & \beta < 1, \\ L_i / (l_A A) (r_p / r_0)^{a-1}, & \beta > 1. \end{cases}
\]

(35)

The growth is induced by the compression at scales \( \lesssim \lambda \). Therefore, in the case that \( \Gamma_{\text{gr}} \varepsilon_N \) reaches the energy cascading rate, fast modes are damped at the corresponding maximum turbulence pumping scale \( \lambda_{fb} = r_p / \varepsilon_N \). If \( \lambda_{fb} \) is larger than the original damping scale \( l_c \), then \( l_c \) should be replaced by \( \lambda_{fb} \), and one needs to iterate until the results are self-consistent.

4.3. Summary of Procedures

The gyroresonance instability can be driven by CRs with anisotropic distribution which is induced by compressible turbulence. Fast and slow modes dominate the compression in low and high \( \beta \) medium, respectively. The compressional rate is maximized at the smallest scale-damping scale of the turbulence. Since CRs do not “remember” the perturbation of the magnetic field and its anisotropy beyond its mean free path, compressions larger than the mean free path of CRs will not induce anisotropy. Therefore, we adopt \( \min(\lambda, l_c) \) as the compressional scale that drives the anisotropy of CRs. We first calculate the damping scale of the compressible modes. If the damping scale is larger than the original mean free path \( \lambda_{fb} \), one should use Equations (30) and (32) to obtain the wave amplitude \( \varepsilon_N \) and the new mean free path \( \lambda = r_p / \varepsilon_N \). If the damping scale is smaller than the mean free path \( \lambda_{fb} \), we use Equations (31) and (33) to get \( \varepsilon_N \) and \( \lambda = r_p / \varepsilon_N \).

The resulting energy growth rate \( \Gamma_{\text{gr}} \varepsilon_N \) should be compared with the large-scale turbulence energy cascading rate \( V^3 / L \). If it is larger than \( V^3 / L \), the growth is limited by the energy budget, rather than the nonlinear feedback. Then Equation (35) can be used to estimate the growth (see Figure 3).

The growth rate should also overcome the damping by background turbulence (YL02; YL04; Farmer & Goldreich 2004; Beresnyak & Lazarian 2008). If the growth rate is lower than the damping rate

\[
\Gamma_d = V_{\text{LM}} / \sqrt{L_{\text{MR}} r_p},
\]

(36)

---

\(^8\) At a given scale \( l_c \), the compression and wave growth will be stronger if IK scaling is used.
the instability does not operate. $L_M, V_{LM}$ are the injection scale of MHD turbulence. $L_M = l_A, V_{LM} = v_A$ if $M_A > 1$, and $L_M = L M_A^2, V_{LM} = v_A M_A$ if $M_A < 1$.

For fast modes, one also needs to take into account the turbulence compression rate at the dissipation scale $\omega = v_A/l_c$. If the growth rate is smaller than the compression, the subsequent expansion will smear out the anisotropy, the trigger of the kinetic instability, and the growth will not happen.

We show results of our study in Figures 4–6. In the plots, we also demonstrate several relevant dimensionless parameters: degree of anisotropy $A$ and the ratio of CR pressure to magnetic pressure, $\beta_{CR}$, as well as the growth and $\omega$. The mean free path of CRs is plotted together with the earlier estimates from LB06. Compared to LB06, the degree of anisotropy $A$ is much smaller and set by the marginal state of instability. This results in a much lower growth rate, and therefore smaller range of energies of CRs that can induce the instability because of the constraint by damping by background turbulence. In addition, the nonlinear feedback due to the scattering by the growing waves is more efficient in saturating the growth at small amplitude ($\varepsilon_N < 1$) compared with the feedback considered in LB06. We do not need additional wave steepening to stop the wave from growing to a large amplitude (or the diffusion to reach the Bohm limit). Our treatment is self-consistent.

5. ASTROPHYSICAL IMPLICATIONS

5.1. Importance of the Instability

Our present work confirms the finding in LB06 that the gyroresonance instability plays an important role for the propagation of CRs in turbulent astrophysical fluids. We note that the coupling of the compressible and incompressible motions reported in Cho & Lazarian (2003) between compressible and Alfvénic modes is strong enough at the scale of driving for diverting about 20% of energy into compressible cascade, even if the driving is purely solenoidal. While we still do not know many details of the MHD cascade (see Beresnyak & Lazarian 2009), the fact that a noticeable portion of energy is going into large-scale compressions is generally accepted. It is those compressions that the gyroresonance instability requires to induce substantial changes of the dynamics of the astrophysical fluids.

In view of the above understandings, it seems incorrect that present-day numerical codes ignore the effects of CRs. We may expect an appreciable change of the compressible dynamics if CRs drain energy from the compressible motions. The consequences of the process for the ISM and molecular cloud formation will be discussed elsewhere.

At the same time, the plane–parallel Alfvén waves with $k_\parallel \gg k_\perp$ are expected to interact efficiently with CRs. Apart from scattering, this should induce CR acceleration via the second-order Fermi process. The consequence of this important process is still to be evaluated quantitatively for different astrophysical environments.

Our quantitative study shows that LB06 in their simplified treatment overestimated the growth rate and therefore the range of energies of CRs over which the instability is active. The reason is that the degree of anisotropy $A$ is overestimated in LB06. The nonlinear feedback that we study in this paper limits the growth through the scattering and therefore $A$ is maintained at the level of marginal instability. Due to the same scattering, our steady-state wave energy/mean free path is smaller/larger than that in LB06 (see the right panels of Figures 4–6). The quenching of growth they considered by only reducing the mean free path is not adequate, and in fact does not apply here as we consider the compression at the minimum scale of turbulence instead of the mean free path. The turbulence energy cascading rate is another constraint that sometimes is more stringent than the scattering.10

5.2. Particular Cases of Astrophysical Environments

5.2.1. Galaxy

In interstellar medium, compressible turbulence is subjected to various damping processes. See Table 1 for the particular cases and the corresponding parameters that we consider in this paper. In the case of the ionized phase, the damping of fast modes varies with the wave pitch angle $\theta$ (YL02; YL04; YL08). The turbulence truncation scale increases with the pitch angle. Approaching to $90^\circ$, the truncation scale is the largest partially because of the field line wandering (YL04; YL08; see Figure 4, left), where the mean free path of CRs $\lambda$ can be smaller than the truncation scale $l_c$. In the mean time, at smaller pitch angles, there is less damping and the mean free path $\lambda$ is larger than the truncation scale. In the low $\beta$ medium, the fast modes are the dominant source for the compression of magnetic fields and we focus on fast modes for the low $\beta$ Galactic halo. Using the compression by the fast modes at the mean free path, we obtain a fairly large growth of the waves $\varepsilon_0$ with amplitude larger than that allowed by the energy budget of the turbulence (see Figure 4(a)). Turbulence is thus damped at the maximum mean free path of CRs $\lambda_{\text{max}} = r_p/\varepsilon_N$. Compression below this scale is in fact reduced and Equation (30) should be adopted to recalculate the amplitude of the wave. We show the results in Figure 4.

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9 In the solar and magnetospheric community it is accepted to talk about the energetic particles rather than cosmic rays. This distinction is not important for the basic processes we discuss.

10 The effect on turbulence cascade was considered in LB06, but not the feedback on the wave growth.
Figure 4. (a) The amplitude of the growing wave using different approaches. \( \epsilon_{N,0} \) is obtained with pumping at the mean free path, which exceeds the maximum value \( \epsilon_r \) allowed by the energy of turbulence and is therefore unrealistic; \( \epsilon_r \) is calculated using the reduced compression from the damping scale set by the maximum mean free path obtained through \( \epsilon_{N,0} \); the real amplitude is the minimum of \( \epsilon_r = \min(\epsilon_{N,0}, \epsilon_r) \) (see the text for the details). (b) The ratio of cosmic ray to magnetic pressure \( \beta \), the degree of anisotropy \( A \) and the wave energy of the slab modes \( \epsilon_N \) at the steady state vs. kinetic energy of CRs in Galactic halo. (c) The steady-state wave growth rate, turbulence compressional rate, and damping rate. Growth only happens if the growth rate is larger than the damping rate. For the compression by fast modes, compressional rate is another threshold. (d) The mean free paths, obtained in this paper and in LB06. (A color version of this figure is available in the online journal.)

Figure 5. Same as Figures 4(b)–(d), but for HIM. (A color version of this figure is available in the online journal.)

Figure 6. Same as Figures 4(b)–(d), but for ICM. (A color version of this figure is available in the online journal.)

### Table 1

| ISM Phase | \( T \) (K) | \( n \) (cm\(^{-3}\)) | \( B \) (\( \mu \)G) | \( M_H \) | \( L \) (pc) | \( n_{cr} \) \((10^{-10} \text{ cm}^{-3})\) | \( l_c \) (cm) | \( \lambda_{fb} \) (cm) |
|-----------|-------------|-----------------|-----------------|-----|-----|----------------|-------|-------|
| Halo      | \( 2 \times 10^6 \) | 0.001           | 5               | 1   | 30  | 4              | \( 2 \times 10^{19} \) | \( 1.84 \times 10^{17} \) |
| HIM       | \( 10^6 \) | 0.005           | 2.5             | 1   | 30  | 4              | \( 4.9 \times 10^{16} \) | \( \cdots \) |
| ICM       | \( 10^6 \) | 0.001           | 1               | 10  | \( 3 \times 10^6 \) | 4              | \( 4.2 \times 10^{19} \) | \( \cdots \) |

Note. HIM: hot ionized medium and ICM: general intracluster medium.
Another case we consider is a hot ionized medium (HIM), which differs from the halo in the sense that $\beta > 1$. In the high $\beta$ medium, the dominant compression originates from slow modes. In the collisionless medium, the damping cut-off scale, i.e., the proton diffusion scale, is quite large. We use therefore Equation (33) to calculate the wave amplitude. We check the consistency by comparing the final mean free path with the damping scale. The results are shown in Figure 5.

5.2.2. Clusters of Galaxies

Observations of galaxy clusters show nonthermal component together with the thermal component in the ICM. Turbulence may be induced by a cluster of mergers and accretion of matter at the virial radius. Both observation and simulations suggest an appreciable amount of energy resides in turbulence. The typical velocity of turbulence at the injection scale is expected to be around 500–1000 km s$^{-1}$, which is highly super-Alfvénic because of high plasma $\beta$. In this case, the magnetic field correlation length $l_B$ could be smaller than the Coulomb mean free path. The effective mean free path should be the minimum of the two if not accounting for further scattering of thermal particles due to plasma instabilities.

Because of the similarity with the HIM, the results for ICM can be obtained with a similar procedure. They are shown in Figure 6. Note that because of the plasma instabilities, the real thermal proton mean free path is less than the one we use. This means that the mean free paths for CRs that we obtain are an upper limit and practically, they can be even smaller.

6. DISCUSSION

We demonstrate that the steady-state amplitude of the slab wave increases with the CR energy. This is because among all the parameters, the growth rate changes ($\propto \gamma_1^{-1-\alpha}$) most rapidly with the energy. And since the scattering rate linearly depends on the growth rate, the wave amplitude, inversely proportional to the energy. And since the scattering rate linearly depends on the growth rate, the wave amplitude, inversely proportional to the energy (see Equation (35))

We show that the gyroresonance instability operates only for CRs of a limited energy range. On the low energy end, the turbulence energy can limit the growth of the instability (as in the case of the halo). On the high-energy end, the growth is constrained by the feedback of the growing slab waves on the anisotropy of the CRs’ distribution. The mechanism we discussed earlier (YL02; YL04; YL08), namely, CR scattering by large-scale fast modes is, therefore, still dominant for higher energy CRs.

Indeed the damping by background turbulence is the major constraint for the growth of the instability. In the environments where Alfvénic turbulence is weak or damped, the instability can extend to higher energies. For instance, in the regions where the ionization degree is low, the slab wave arising from the instability will protrude to a scale of $l_A^{1/3}/L_1^{1/3}$ (LB06).

In the calculation on collisionless plasma, we simply used the Coulomb mean free path. The effective viscosity in collisionless plasma should be less because of additional scattering of thermal particles caused by plasma instabilities. The consideration of this complex processes is beyond the scope of this paper. However, one thing we can be certain is that with the reduced viscosity, higher compressional efficiency can be achieved and the mean free path of CRs is further reduced and CRs are better confined, which is crucial for their acceleration.

Our results show that turbulence compression provides an additional channel for CR scattering and the feedback of CRs is important for the cascade of turbulence, confirming LB06’s findings although our quantitative conclusions are very much different. In future MHD simulations, it is necessary to take into account the contribution from CRs.

7. SUMMARY

We have considered the gyroresonance instability of CRs in MHD turbulence. It arises from the compression from the compressible turbulence. The growth of the instability is self-adjusted due to the feedback of the slab wave on the anisotropy of CRs. We find that

1. Compression by large-scale turbulence results in anisotropic distribution of CRs, which induces gyroresonance instability.
2. The growth of the instability is balanced by the feedback of the slab waves on the anisotropy of CRs. The high-energy cutoff is determined by the balance between the growth rate and the damping of the wave by the large-scale turbulence.
3. The slab waves generated through gyroresonance instability limits the degree of the anisotropy of CRs and is an efficient isotropization mechanism for CRs $< 100$ GeV.
4. The feedback of the instability on the large-scale turbulence should be taken into account. In some cases, turbulence cascade is truncated because of the drain of energy to the small-scale waves.

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APPENDIX

DERIVATION OF SCATTERING EFFICIENCIES $S_\perp, S_\parallel$

We define

$$D = \frac{\omega - kv_x}{v_\perp} \frac{\partial f_0}{\partial p_\perp} + k \frac{\partial f_0}{\partial p_\parallel},$$

$$2e\mathbb{H} \langle \mathbf{E} \cdot \mathbf{\Gamma} \rangle = 2e\mathbb{H} \left\{ \mathbf{E} \cdot \int d^3p \mathbf{v} f_1 \right\}$$

$$= -\varepsilon_\perp \int d^3p \frac{8\pi e^2}{\omega} \frac{v_\perp^2}{kv_\parallel - \omega} D$$

$$= \frac{4}{|\omega|^2} \langle v^2 \hat{\omega} \cdot K \rangle \varepsilon_\perp,$$

where we applied Equations (11) and (15) and the fact $\mathbb{H} \langle A(x, t)B(x, t) \rangle = 1/2\mathbb{H} \langle A^*(\mathbf{k}, t)B(\mathbf{k}, t) \rangle$. We use $K$ to express the result as it is directly associated with the growth rate (Equation (16))
\[-\frac{2e}{c} \Re \left( \int d^3 p v_{\perp} \cdot [v \times B_{\perp}] f_1 \right) = -2e \Re \left( \int d^3 p v_{\perp} \cdot \left[ v \times i \nabla \times E \right] \frac{\omega}{\omega_e} f_1 \right) = -e \Re \left( \int d^3 p k v_{\perp} \frac{v_{\perp} \cdot E}{\omega_e} f_1 \right) = - \varepsilon_{\perp} k^2 \int d^3 \frac{8 \pi e^2}{|\omega|^2} \frac{D v_{\perp}^2}{|v_{\perp}|^2} \frac{k v_{\perp}}{|v_{\perp}|} - \omega + \Omega^2 = - \varepsilon_{\perp} k^2 \int d^3 \frac{8 \pi e^2}{|\omega|^2} \frac{D v_{\perp}^2}{|v_{\perp}|^2} \left( 1 + \frac{\omega}{k v_{\perp}} - \omega + \Omega^2 \right) = - \varepsilon_{\perp} k^2 \int d^3 \frac{8 \pi e^2}{|\omega|^2} \frac{v_{\perp} \partial f_0}{\partial p_{\perp}} \mathcal{L}(\omega) + \frac{4}{|\omega|^2} [\mathcal{L}(\omega^2)(\omega + \Omega)^2]^{\pm} \varepsilon_{\perp}. \quad (A3)\]

For a power-law distribution \( f_0 \propto p^{-\alpha-2}, v_{\perp} \partial f_0 / \partial p_{\perp} = -(\alpha + 2) c^2 p_{\perp}^{\alpha-5} \). Insert it into the second to the last term into Equation (A3), we get

\[- \varepsilon_{\perp} \int d^3 \frac{8 \pi e^2}{|\omega|^2} \frac{v_{\perp} \partial f_0}{\partial p_{\perp}} \mathcal{L}(\omega) \approx \left[ \frac{16 \pi e^2}{(\alpha + 1)(\alpha + 3)|\omega|^2} \right] \int d \left( \frac{\rho}{\rho_{\text{res}}} \right) \frac{4 \omega_{\text{pr}}}{\gamma_{\perp} m} \mathcal{L}(\omega) = \frac{2(\alpha + 2)}{(\alpha + 1)(\alpha + 3)} \int d \left( \frac{\rho}{\rho_{\text{res}}} \right) \frac{4 \omega_{\text{pr}}^2}{|\omega|^2} \mathcal{L}(\omega) \varepsilon_{\perp}. \quad (A4)\]

Therefore,

\[- \frac{2q}{c} \Re \left\{ \int d^3 p v_{\perp} \cdot \nabla f_1 \right\} \approx \frac{2(\alpha + 2)}{(\alpha + 1)(\alpha + 3)} \int d \left( \frac{\rho}{\rho_{\text{res}}} \right) \frac{\omega^2 pr}{|\omega|^2} \mathcal{L}(\omega) \varepsilon_{\perp} + \frac{4e \varepsilon_{\perp}}{|\omega|^2} \mathcal{L}(\omega) \varepsilon_{\perp}. \quad (A5)\]

From Equations (21), (A2), and (A6), we get

\[\frac{\partial W_{\perp}}{\partial t} = -4 \sum \frac{\omega^2 pr}{|\omega|^2} \left( \frac{\omega}{|\omega|^2} (2i \Omega \pm \Omega^2) \varepsilon_{\pm} \mathcal{L}(\omega) \varepsilon_{\perp} \right), \quad \frac{\partial W_{\parallel}}{\partial t} = 4 \sum \frac{\omega^2 pr}{|\omega|^2} \left( \frac{\omega}{|\omega|^2} (2i \Omega \pm \Omega^2) \varepsilon_{\pm} \mathcal{L}(\omega) \varepsilon_{\perp} \right), \quad (A6)\]

where \( \sum \) is performed over the right moving CRs ("=") and left moving CRs ("-"). In the above equations,

\[\omega^2 = (\omega_r + i \gamma)^2 = (\omega_r^2 + \gamma^2) + 2i \omega_r \Gamma, \quad K^\pm = c^2/\varepsilon_{\pm} \left( 1 - 2i \Gamma/\omega_0 \right). \quad (A7)\]

Taking into account that \( \gamma \ll \omega_r \) and \( \omega_r \approx \omega_0 \), we get

\[S_{\perp}(k) = \frac{W_{\perp}(k)}{\varepsilon_{\perp}} = -2 \left[ \frac{2(\alpha + 2)}{(\alpha + 1)(\alpha + 3)} \mathcal{L}(\omega) \varepsilon_{\perp} \right], \quad S_{\parallel}(k) = \frac{W_{\parallel}(k)}{\varepsilon_{\perp}} = 2 \left[ \frac{2(\alpha + 2)}{(\alpha + 1)(\alpha + 3)} \mathcal{L}(\omega) \varepsilon_{\perp} \right]. \quad (A8)\]

where we used the fact that \( \delta \varepsilon / \delta t = 2 \Gamma_e (1 + k^2 c^2 / |\omega|^2) (\varepsilon_+ + \varepsilon_-) \).

REFERENCES
Abrahamowitz, M., & Stegun, I. A. (ed.) 1965, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (New York: Dover)
Barnes, A. 1966, Phys. Fluids, 9, 1483
Beresnyak, A., & Lazarian, A. 2008, ApJ, 682, 1070
Beresnyak, A., & Lazarian, A. 2009, ApJ, 702, 1190
Bieber, J. W., Matthaeus, W. H., Smith, C. W., Wanner, W., Kallenrode, M., & Wibberenz, G. 1994, ApJ, 420, 294
Biskamp, D. (ed.) 2003, Magnetohydrodynamic Turbulence (Cambridge: Cambridge Univ. Press), (310)
Budyrev, S. 2006, Phys. Rev. Lett., 96, 115002
Brunetti, G., & Lazarian, A. 2007, MNRAS, 378, 245
Cesarsky, C. J. 1980, ARA&A, 18, 289
Chandran, B. G. D. 2000, Phys. Rev. Lett., 85, 4656
Cho, J., & Lazarian, A. 2002, Phys. Rev. Lett., 88, 245001
Cho, J., & Lazarian, A. 2003, MNRAS, 345, 352
Cho, J., Lazarian, A., & Vishniac, E. T. 2002, ApJ, 564, 291
Cho, J., Lazarian, A., & Vishniac, E. T. 2003, in Turbulence and Magnetic Fields in Astrophysics, ed. E. Falgarone & T. Passot (Lecture Notes in Physics, Vol. 614; Berlin: Springer), 56
Cho, J., & Vishniac, E. T. 2000, ApJ, 539, 273
Elmegreen, B. G., & Scalo, J. 2004, ARA&A, 42, 211
Farmer, A. J., & Goldreich, P. 2004, ApJ, 604, 671
Gary, S. P., & Tokar, R. L. 1985, J. Geophys. Res., 90, 65
Gary, S. P., Wang, J., Winske, D., & Fossey, S. A. 1997, J. Geophys. Res., 102, 27159
Gogoberidze, G. 2007, Phys. Plasmas, 14, 022304
Goldreich, P., & Sridhar, S. 1995, ApJ, 438, 763
Kulsrud, R. M. (ed.) 2005, Plasma Physics for Astrophysics (Princeton, NJ: Princeton Univ. Press)
Lazarian, A., & Beresnyak, A. 2006, MNRAS, 373, 1195
Lazarian, A., & Pogosyan, D. 2000, ApJ, 537, 720
Lazarian, A., & Vishniac, E. T. 1999, ApJ, 517, 700
Lerche, I. 1967, ApJ, 147, 689
Lerche, I., & Schlickeiser, R. 2001, A&A, 366, 1008
Lighthick, Y., & Goldreich, P. 2001, ApJ, 562, 279
Longair, M. S. 1997, High Energy Astrophysics, Vol. 2: Stars, the Galaxy and the Interstellar Medium (Cambridge: Cambridge Univ. Press)
Maron, J., & Goldreich, P. 2001, ApJ, 554, 1175
Melrose, D. B. 1974, Sol. Phys., 37, 353
Müller, W., & Biskamp, D. 2000, Phys. Rev. Lett., 84, 475
Schlickeiser, R. (ed.) 2002, Cosmic Ray Astrophysics (Astrophysics Library; Berlin: Springer)
Stanimirović, S., & Lazarian, A. 2001, ApJ, 551, L53
Wentzel, D. G. 1974, ARA&A, 12, 71
Yan, H. 2009, MNRAS, 397, 1093
Yan, H., & Lazarian, A. 2002, Phys. Rev. Lett., 89, B1102
Yan, H., & Lazarian, A. 2004, ApJ, 614, 757
Yan, H., & Lazarian, A. 2008, ApJ, 673, 942