Spectral properties of the Ruelle operator on the Walters class over compact spaces

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Abstract
Recently the Ruelle–Perron–Frobenius theorem was proved for Hölder potentials defined on the symbolic space $\Omega = M^\Omega$, where (the alphabet) $M$ is any compact metric space. In this paper, we extend this theorem to the Walters space $W(\Omega)$, in similar general alphabets. We also describe in detail an abstract procedure to obtain the Fréchet analyticity of the Ruelle operator under quite general conditions and we apply this result to prove the analytic dependence of this operator on both Walters and Hölder spaces. The analyticity of the pressure functional on Hölder spaces is established. An exponential decay of the correlations is shown when the Ruelle operator has the spectral gap property.

A new (and natural) family of Walters potentials (on a finite alphabet derived from the Ising model) which do not have an exponential decay of the correlations is presented. Because of the lack of exponential decay, for such potentials there is an absence of the spectral gap for the Ruelle operator. The key idea in proving the lack of exponential decay of the correlations is the Griffiths–Kelly–Sherman inequalities.

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1. Introduction

The Ruelle–Perron–Frobenius theorem is one of the most important results in modern thermodynamic formalism. Nowadays the Ruelle operator has become a standard tool in many different areas in dynamical systems and other branches of mathematics and mathematical
physics. The literature about the Ruelle–Perron–Frobenius theorem is vast; the following is a partial list of books and papers on this subject [2, 6, 14, 15, 19, 30, 32, 38].

The classical thermodynamic formalism was originally developed in the Bernoulli space $M^N$, with $M$ being a finite alphabet (see [30]). This assumption with regard to $M$ allows one to conjugate, using Markov partitions, the shift maps in the Bernoulli space with uniform hyperbolic maps in differentiable manifolds.

The Ruelle operator formalism has also proved useful in a multifractal analysis context. Bowen, in the seminal work [7], established a relationship between the Hausdorff dimension of certain fractal sets and topological pressure in the context of conformal dynamics in one dimension. This technique is known as Bowen’s equation and in subsequent works it was extended by Manning and McCluskey for dynamical systems on surfaces, with the aim of computing the fractal dimension of horseshoes (for more details see [7, 28, 29]) and also the introductory texts [3, 31]).

The motivation for considering more general alphabets from a dynamical system point of view is given, for example, in [34, 36] where proposed models with an infinite alphabet $M = \mathbb{N}$ is used to describe non-uniformly hyperbolic maps, for example, the Manneville–Pomeau maps. In classical statistical mechanics uncountable alphabets show up, for example, in the so-called $O(n)$ models with $n \geq 2$. In these models the alphabet is the unit sphere $S^{n-1}$ in the Euclidean space $\mathbb{R}^n$ (for details see [17]). Unbounded alphabets as general standard Borel spaces, which include compact and non-compact, are considered in detail in [20]. We should mention that ergodic optimization problems are also being considered in infinite countable alphabets (see [4, 11, 18, 34]).

Walters’ work [40] marked the beginning of three decades of great activity in thermodynamic formalism where potentials less regular than Hölder potentials were considered. This class of potentials is called Walters class or alternatively Walters space. A rather complete theory in Walters space was developed for finite and countable alphabets (see [11]). In the Walters paper, the dynamical system is supposed to be defined on a class of compact sets, expansive and mixing, and the potentials can be any positive summable variation function.

The aim of this work is to extend some of the results obtained in [40] as the Ruelle–Perron–Frobenius theorem, the analytic dependence of the Ruelle operator with respect to the potential, spectral properties of this operator and consequences of either the presence or absence of the spectral gap, as in the pressure analyticity, for example. The main difficulty in carrying out the construction of the Ruelle operator for uncountable alphabets is overcome by the introduction of what we call an a priori probability measure on $M$, which is a common strategy in the theory of general DLR–Gibbs measures.

This paper is organized as follows. In section 2, we prove under quite general conditions the analyticity of the Ruelle operator and its dual with respect to the potential $f$. The result is more general; in fact we show that if the Ruelle operator $\mathcal{L} f$ leaves invariant Banach algebra $\mathcal{K} \subset C(\Omega)$ for all $f \in \mathcal{K}$, then the maps $\mathcal{K} \ni f \mapsto \mathcal{L} f$, $\mathcal{K}^* \ni f \mapsto \mathcal{L}^* f$ are analytic. We use this result in section 5 with $\mathcal{K} = C^n(\Omega)$ to obtain the analyticity of the topological pressure $P : C^n(\Omega) \to \mathbb{R}$, which extends the analogous results known in finite/discrete alphabets. We note, however, that even for a discrete case, despite this being a folkloric result, we were not able to find its rigorous proof in the literature.

Section 3 deals with $W(\Omega)$ the Walters space. In the discrete setting there are several known equivalent characterizations of the Walters condition; however, in a more general setting such as compact metric spaces, things are more subtle. We show that the natural generalization of the two most popular characterizations of the Walters class are not equivalent when the
alphabet is uncountable. We give an explicit example illustrating this fact. We introduce what we call weak and strong Walters conditions. Finally a generalization of [40] is proved for general compact alphabets. We note that this theorem is also a non-trivial generalization of one of the main theorems in [1, 26] where the Ruelle operator on an uncountable alphabet is taken into account.

In section 7 we introduce a new family of potentials for which the Ruelle operator is missing the spectral gap. This section is heavily based on the ideas borrowed from statistical mechanics and the Griffiths–Kelly–Sherman (GKS) inequalities ([22–25]). For the convenience of the reader, we precisely state all the theorems we need only in the needed generality but we also provide their classical references where general settings are presented. Some Ising model routine computations are also presented in detail in order to make our exposition self-contained for non-specialists in statistical mechanics. These potentials belong to a infinite-dimensional linear subspace of $C(\Omega, \mathbb{R})$ whose intersection with the Walters space is an infinite-dimensional linear subspace not contained in the Hölder space. On that space Dobrushin in [10], by using estimates of the mean value of exponential functionals of random processes, and later Cassandro and Olivieri [8], employing a renormalization group idea together with the cluster expansion, proved analyticity of the pressure. It worth mentioning that the subexponential decay obtained in section 5 cannot be recovered from the seminal Sarig work [35] about subexponential decay of correlations nor from the improvement provided by Gouëzel in [21]. The examples presented in section 5 can shed some light on potential applications of the GKS inequalities in order to study the absence of the spectral gap in other situations.

2. Basic definitions

In this section we set up the notation and present some preliminaries results. Let $M = (M, d)$ be a compact metric space, equipped with a Borel probability measure $\mu : \mathcal{B}(M) \to [0, 1]$ having the whole space $M$ as its support. We shall denote by $\Omega = M^\mathbb{N}$ the set of all sequences $x = (x_1, x_2, \ldots)$, where $x_i \in M$, for all $i \in \mathbb{N}$. We denote by $\sigma : \Omega \to \Omega$ the left-shift mapping which is given by $\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$. We consider the metric $d_\Omega$ on $\Omega$ given by

$$d_\Omega(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d(x_n, y_n)$$

The metric $d_\Omega$ induces the product topology and therefore follows from the Tychonoff theorem that $(\Omega, d_\Omega)$ is a compact metric space. The space of all the continuous real functions $C(\Omega, \mathbb{R})$ is denoted simply by $C(\Omega)$. For any fixed $0 \leq \gamma \leq 1$ we denote by $C^\gamma(\Omega)$ the space of all $\gamma$-Hölder continuous functions, i.e. the set of all functions $f : \Omega \to \mathbb{R}$ satisfying

$$\text{Hol}(f) = \sup_{x, y \in \Omega : x \neq y} \left| \frac{f(x) - f(y)}{d_\Omega(x, y)^\gamma} \right| < +\infty.$$ 

We equip $C^\gamma(\Omega)$, $0 \leq \gamma \leq 1$ with the norm $\| \cdot \|_\gamma$ which is defined for $\gamma = 0$ by $\| f \|_0 = \sup_{x \in \Omega} |f(x)|$ and for $0 < \gamma \leq 1$ by $\| f \|_\gamma = \| f \|_0 + \text{Hol}(f)$. We recall that $(C^\gamma(\Omega), \| \cdot \|_\gamma)$ is a Banach space for any $0 \leq \gamma \leq 1$.

Our potentials will be elements of $C(\Omega)$ and in order to have a well-defined Ruelle operator when $(M, d)$ is a general compact metric space we need to consider an a priori measure which is simply a Borel probability measure $\mu : \mathcal{B}(M) \to [0, 1]$, where $\mathcal{B}(M)$ denotes the Borel $\sigma$-algebra of $M$. For many of the most popular choices of an uncountable space $M$ there is a natural a priori measure $\mu$. Throughout this paper the a priori measure $\mu$ is supposed to have
the whole space $M$ as its support. The Ruelle operator $\mathcal{L}_f : C^\gamma(\Omega) \to C^\gamma(\Omega)$ is the mapping sending $\varphi \mapsto \mathcal{L}_f(\varphi)$ which is defined for any $x \in \Omega$ by the expression

$$\mathcal{L}_f(\varphi)(x) = \int_M e^{f(x)} \varphi(ax) d\mu(a),$$

where $ax$ denotes the sequence $ax = (a, x_1, x_2, \ldots) \in \Omega$.

This operator is a generalization of the classical Ruelle operator and has been appearing lately in the thermodynamic formalism literature (see for example [1, 26, 37]). The classical Ruelle operator can be recovered on this setting by considering $M = \{0, 1, \ldots, n\}$ and the a priori $\mu$ as the normalized counting measure. Our starting point is the following theorem.

**Theorem 2.1 (Ruelle–Perron–Frobenius).** Let $(M, d)$ be a compact metric space, $\mu$ a Borel probability measure of full support on $M$ and $f$ be a potential in $C^\gamma(\Omega)$, where $0 < \gamma < 1$. Then $\mathcal{L}_f : C^\gamma(\Omega) \to C^\gamma(\Omega)$ have a simple positive eigenvalue of maximal modulus $\lambda_f$, and there is a strictly positive function $h_f$ and a Borel probability measure $\nu$ on $\Omega$ such that

(i) the remainder of the spectrum of $\mathcal{L}_f : C^\gamma(\Omega) \to C^\gamma(\Omega)$ is contained in a disc with a radius strictly smaller then $\lambda_f$;

(ii) for all continuous functions $\varphi \in C(\Omega)$ we have

$$\lim_{n \to \infty} \frac{1}{n} \left\| \lambda_f^n \mathcal{L}_f^n \varphi - h_f \int_\Omega \varphi d\nu \right\|_0 = 0.$$

**Proof.** See [1] for the case $M = S^1$ and [26] for more general compact metric spaces. □

**Remark 2.2.** Strictly speaking, in [1] and [26] the item (ii) above was proved only for Hölder continuous potentials. However, this is enough since the space of the Hölder continuous potentials is dense in $(C(\Omega), \|\cdot\|_0)$. Therefore, a straightforward computation shows that the convergence on item (ii) holds for all $\varphi \in C(\Omega)$. The denseness of $C^\gamma(\Omega), 0 < \gamma < 1$ in $C(\Omega)$ is a consequence of the Stone–Weierstrass theorem. Indeed, $C^\gamma(\Omega)$ is an algebra of functions containing all the constant functions and if $x \neq y \in \Omega$, then the function $f$ given by $f(y) = d_\Omega(y, x)^\gamma$, separates $x$ and $y$ and $f \in C^\gamma(\Omega)$. Since $\Omega$ is compact, the result follows.

Following [1, 26] we define the entropy of a shift-invariant measure $\nu \in \mathcal{M}_\sigma$ and the pressure of the potential $f$, respectively, as follows:

$$h(\nu) = \inf_{f \in C^\gamma(\Omega)} \left\{ -\int f d\nu + \log \lambda_f \right\} \quad \text{and} \quad P(f) = \sup_{\nu \in \mathcal{M}_\sigma} \left\{ h(\nu) + \int_\Omega f d\nu \right\}.$$

**Proposition 2.3.** For each $f \in C^\gamma(\Omega)$ we have it for all $x \in \Omega$ that

$$P(f) = \lim_{n \to \infty} \frac{1}{n} \log \left[ \mathcal{L}_f^n (1)(x) \right] = \log \lambda_f.$$

**Proof.** See [26] corollary 1. □

**Definition 2.4.** Let $f \in C(\Omega)$ and consider $\mathcal{L}_f$ the associated Ruelle operator. We can say that the Ruelle theorem holds for $\mathcal{L}_f$ when $\mathcal{L}_f$ has the properties given by items (i) and (ii) of theorem 2.1.

If the Ruelle theorem holds for $\mathcal{L}_f$ then we can write the pressure as $P(f) = \log \lambda_f$. Another important property of the Ruelle operator is its analytic dependence (in the Fréchet sense) with respect to the potential. The lemma below state it precisely.
Lemma 2.5. The map $\Theta : C^r(\Omega) \to L(C^r(\Omega), C^r(\Omega))$ sending $f \in C^r(\Omega)$ to the Ruelle operator $\mathcal{L} f$ associated to the potential $f$, is an analytic map.

Proof. See [37] theorem 3.5.

One of the aims of this work is to extend the previously mentioned results to potentials on the Walters space $W(\Omega)$ (to be defined in the next section), where $\Omega$ is the infinite Cartesian product of a general compact metric space $(M, d)$.

We shall prove that the Ruelle operator and its dual depends analytically on the potential in $C^r(\Omega)$ and then derive the analyticity of the pressure. In order to formulate these results using an unified setting, we need to introduce some additional notation. Let $K \subset C(\Omega)$ be an arbitrary linear subspace of $C(\Omega)$, endowed with a norm $\| \cdot \|$. We use the notation $K^*$ to denote the topological dual of $(K, \| \cdot \|)$. As usual, we define the norm of an element $\phi \in K^*$ by $\| \phi \| = \sup \{ |\phi(f)| : f \in K \text{ and } \|f\| = 1 \}$. To lighten the notation, the space $L(K, K)$ of all the continuous (strong topology) linear operators acting on $K$ will be denoted by $V \equiv L(K, K)$.

Definition 2.6. Let $K \subset C(\Omega)$ be a linear subspace. We can say that $K$ is invariant for the Ruelle operator, if for all $f \in K$ we have $\mathcal{L} f \in K$.

The central examples of invariant subspaces for the Ruelle operator appearing here are the spaces $C^r(\Omega, \gamma)$, $0 < \gamma \leq 1$, and the Walters Space $W(\Omega)$.

The next proposition plays a key role in the study of the analyticity of the pressure. Its first statement is a simple generalization of theorem 3.5 in [37], which is presented here for the reader’s convenience. The second one follows from the first after some further work.

Proposition 2.7. Suppose that $K \subset C(\Omega)$ is equipped with a norm $\| \cdot \|$ so that $(K, \| \cdot \|)$ is a Banach algebra, $K$ is invariant for the Ruelle operator and for any $f \in K$ assume that $\mathcal{L} f \in V$. Then, both mappings $\Theta$ and $\Theta^*$ given by

$$K \ni f \mapsto \mathcal{L} f \in L(K, K) \quad \text{and} \quad K \ni f \mapsto \mathcal{L}^* f \in L(K^*, K^*)$$

define analytic functions.

Before proving the above proposition, we state an immediate corollary of it which is an important tool to obtain the analyticity of the pressure functional.

Corollary 2.8. For each $0 < \gamma \leq 1$, both mappings

$$C^r(\Omega) \ni f \mapsto \mathcal{L} f \in L(C^r(\Omega), C^r(\Omega)) \quad \text{and} \quad C^r(\Omega) \ni f \mapsto \mathcal{L}^* f \in L(C^r(\Omega)^*, C^r(\Omega)^*)$$

define analytic maps.

Proof. Notice that the subspace $C^r(\Omega)$ is invariant for the Ruelle operator and $(C^r(\Omega), \| \cdot \|)$ is a Banach algebra (see [37]), so we are finished.

Proof of the proposition 2.7. We first prove the analyticity of $\Theta$. Given $f, h \in K$ and $\varphi \in C(\Omega)$ for any $x \in \Omega$ we have the following equality:

$$\Theta(f + h)(\varphi)(x) - \Theta(f)(\varphi)(x) = \mathcal{L}_{f + h}(\varphi)(x) - \mathcal{L}_f(\varphi)(x) = \int_M e^{f(ax) + h(ax)} \varphi(ax) d\mu(a) - \int_M e^{f(ax)} \varphi(ax) d\mu(a)$$
$$= \int_M e^{f(ax)} \varphi(ax) (e^{h(ax)} - 1) d\mu(a)$$
$$= \int_M \left( e^{f(ax)} \varphi(ax) \sum_{n=1}^{\infty} \frac{(h(ax))^n}{n!} \right) d\mu(a).$$
As long as the Fubini theorem applies we get
\[ \Theta(f + h)(\varphi(x)) - \Theta(f)(h)(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathcal{M}} e^{f(ax)\varphi(ax)} [h(ax)]^n d\mu(a), \]
which is equivalent to \( \Theta(f + h)(\varphi)(x) - \Theta(f)(\varphi)(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \Theta(f)(\varphi \cdot h^n)(x). \) This equality can be rewritten, by omitting the dependence on \( x \) and \( \varphi \), simply as follows:
\[ \Theta(f + h) - \Theta(f) = \sum_{n=1}^{\infty} \frac{1}{n!} \Theta(f)(\cdot)[h^n]. \] (1)

To justify the applicability of the Fubini theorem in this case it is sufficient to prove that the above sum converges in \((\mathcal{K}, \|\cdot\|)\). We first observe that for any \( h_1, \ldots, h_k \) in \( \mathcal{K} \), the mapping \( \varphi \mapsto \Theta(f)(\varphi h_1 \cdots h_k) \) from \( \mathcal{K} \) to itself defines a continuous linear operator, i.e. \( \Theta(f)(\cdot)[h_1 \cdots h_k] \in V \) whose norm is bounded by
\[ \| \Theta(f)(\cdot)[h_1 \cdots h_k] \|_V \leq \| L_f^i \|_V \| h_1 \| \cdots \| h_k \|. \] (2)
which is proved by a routine computation given that \( L_f^i \in V \) and \((\mathcal{K}, \|\cdot\|)\) is a Banach algebra. As a consequence of this inequality we get
\[ \sum_{n=1}^{\infty} \frac{1}{n!} \| \Theta(f)(\cdot)[h^n] \|_V \leq \sum_{n=1}^{\infty} \frac{1}{n!} \| L_f^i \|_V (\| h \|)^n = \| L_f^i \|_V \{ e^{\| h \|} - 1 \} \]
which immediately implies that the series \( \sum_{n=1}^{\infty} (1/n!) \Theta(f)(\cdot)[h^n] \) converges in \( V \).

**Claim 1.** For any \( k \in \mathbb{N} \) and \( h_1, \ldots, h_k \in \mathcal{K} \), we have it that
\[ D^k \Theta(f)(h_1, \ldots, h_k) = \Theta(f)(\cdot)[h_1, \ldots, h_k]. \]

The verification will be carried out by induction on \( k \). In what follows \( L^k = L^k(\mathcal{K}, V) \) denotes the set of all continuous \( k \)-linear functions \( l : \mathcal{K} \times \cdots \times \mathcal{K} \to V \), from \( \mathcal{K} \times \cdots \times \mathcal{K} \) \((k-\text{copies of } \mathcal{K})\) into \( V \). The norm \( \|\cdot\|_{L^k} \) of \( L^k \) is given by
\[ \|l\|_{L^k} = \sup \|l(h_1, \ldots, h_k)\|_V, \quad l \in L^k. \]

Let us prove that the statement is true for \( k = 1 \): in fact; by using (1) we have,
\[ \Theta(f + h_1) - \Theta(f) = \Theta(f)(\cdot) h_1 + \mathcal{O}(h_1), \]
where, \( \mathcal{O}(h_1) = \sum_{n=2}^{\infty} (1/n!) \Theta(f)(\cdot)[h_1]^n \). The inequality (2) implies that \( \| \Theta(f)(\cdot) h_1 \|_V \leq \| L_f^i \|_V \| h_1 \| \) and thus the mapping \( h_1 \mapsto \Theta(f)(\cdot) h_1 \) is in \( L^1 \). Again, in view of the inequality (2) we have
\[ \| \mathcal{O}_1(h_1) \|_V = \left\| \sum_{n=2}^{\infty} \frac{1}{n!} \Theta(f)(\cdot)[h_1]^n \right\|_V \leq \sum_{n=2}^{\infty} \frac{1}{n!} \| L_f^i \|_V (\| h \|)^n \]
showing that \( (1/\| h_1 \|) \mathcal{O}_1(h_1) \xrightarrow{V} 0 \), when \( \| h_1 \| \to 0 \). Therefore, \( D^1 \Theta(f)(h_1) = \Theta(f)(\cdot) h_1 \) and the statement is true for \( k = 1 \). Now, let us suppose the statement is true for \( k = 1, \ k \geq 2 \), i.e.
We shall verify that the statement is true for $k$, i.e.
\[
D^k \Theta(f)(h_1, ..., h_{k-1}) = \Theta(f)((\cdot)h_1 \ldots h_{k-1}), \quad h_1, \ldots, h_{k-1} \in K.
\] (3)

By the induction hypothesis (3), given $h_1, \ldots, h_{k-1}, h_k$ and $h$ in $K$, we have
\[
D^{k-1} \Theta(f + h_k)(h_1, ..., h_{k-1})(h) - D^{k-1} \Theta(f)(h_1, ..., h_{k-1})(h) = \Theta(f + h_k)(hh_1 \ldots h_{k-1}) - \Theta(f)(hh_1 \ldots h_{k-1}).
\]

From (1) it follows that
\[
D^{k-1} \Theta(f + h_k)(h_1, ..., h_{k-1})(h) = \sum_{n=1}^{\infty} \frac{1}{n!} \Theta(f)(hh_1 \ldots h_{n-1}[h_k]^n).
\]

Clearly, the above equation shows that
\[
D^{k-1} \Theta(f + h_k)(h_1, ..., h_{k-1})(h) - D^{k-1} \Theta(f)(h_1, ..., h_{k-1})(h) = \Theta(f)((\cdot)h_1 \ldots h_{k-1}) + O_k(h_k)(h_1 \ldots h_{k-1}),
\]
where $O_k(h_k)$ is the element of $L^{k-1}$ given by
\[
O_k(h_k)(h_1 \ldots h_{k-1}) = \sum_{n=2}^{\infty} \frac{1}{n!} \Theta(f)((\cdot)h_1 \ldots h_{n-1}[h_k]^n).
\]

These upper bounds, together with the inequality (2), enable us to conclude that the map
\[
(h_1 \ldots h_k) \mapsto \Theta(f)((\cdot)h_1 \ldots h_k)
\]
is an element of $L^k$. The inequality (2) and the definition of $O_k(h_k)$ give us the upper bound
\[
\|O_k(h_k)(h_1 \ldots h_{k-1})\|_V \leq \sum_{n=2}^{\infty} \frac{1}{n!} \|L_f\| \|h_1\| \ldots \|h_{k-1}\| (\|h_k\|)^n
\]
and consequently $(1/\|h_k\|)\|O_k(h_k)\|_V \to 0$, when $\|h_k\| \to 0$. Therefore, $D^k \Theta(f)(h_1, \ldots, h_k) = \Theta(f)((\cdot)h_1 \ldots h_k)$ and the claim is proved.

By using claim 1 and the above estimates for the remainder, the analyticity of the mapping $K \ni f \mapsto \mathcal{L}_f \in V$ follows.

**Analyticity of $\Theta^*$.** Let $f$, $g$ and $h$ be potentials in $K$ and $\phi^* \in K^*$. From the expansion (1) for $\Theta(f + h)$ we get
\[
\Theta^*(f + h)(\phi^*)g = \phi^*(\Theta(f + h)(g)) = \phi^* \left( \sum_{n=0}^{\infty} \frac{1}{n!} \Theta(f)(g[h]^n)(\cdot) \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \phi^*(\Theta(f)(g[h]^n)(\cdot))
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} (\Theta^*(f)(\phi^*)(g[h]^n)(\cdot)).
\] (5)
Claim 2. Consider the derivative map \( D\Theta^*: \mathcal{K} \to L(\mathcal{K}, L(K^*, K^*)) \). Then, for any \( f \in \mathcal{K} \) and \( h \in \mathcal{K} \) we have it that \( D\Theta^*(f)(h)^* \) is given by \( (\Theta^*(f)(\phi^*))h = (\Theta^*(f)(\phi^*))(gh) \). Indeed, consider \( \mathcal{O}: \mathcal{K} \to L(\mathcal{K}^*, K^*) \) defined by \( \mathcal{O}(h)(\phi^*^*) = \sum_{n=2}^{\infty} (1/n!) \Theta^*(\phi^*)(\phi^*^* h)^n \). Then, we have

\[
\|\mathcal{O}(h)\|_{L(K^*, K^*)} = \sup_{\|\phi^*\|_1} \|\mathcal{O}(h)(\phi^*)\|_1 = \sup_{\|\phi^*\|_1} \left\| \sum_{n=2}^{\infty} \frac{1}{n!} \Theta^*(f)(\phi^*)(\phi^*^* h)^n \right\|
\leq \sup_{\|\phi^*\|_1} \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i=1}^{\infty} \frac{1}{n!} \Theta^*(f)(\phi^*)(\phi^*^* h)^n . \tag{6}
\]

The next step is to upper bound the quantity \( I \):

\[
I = \|\Theta^*(f)(\phi^*)(\phi^*^* h)^n\|_1 = \sup_{\|\xi\|_1} \|\Theta^*(f)(\phi^*^* g)^n\| = \sup_{\|\xi\|_1} \|\phi^*^* g^n\|_1 \leq \|\phi^*\|_1 \sup_{\|\xi\|_1} \|\Theta^*(f)(g h)^n\| 
\leq \|\phi^*\|_1 \sup_{\|\xi\|_1} \|\Theta^*(f)(g h)^n\| \leq \text{const.} \|h\|^n .
\]

By replacing (7) in (6) we get

\[
\|\mathcal{O}(h)\|_{L(K^*, K^*)} \leq \sum_{n=2}^{\infty} \frac{1}{n!} \text{Const.} \|h\|
\]

Therefore, \( \|\mathcal{O}(h)\|_{L(K^*, K^*)}/\|h\| \to 0 \) when \( \|h\| \to 0 \).

It is possible to show that the higher order derivatives \( D^k\Theta^*: \mathcal{K} \to L(\mathcal{K}^k, L(K^*, K^*)) \) for \( k \geq 2 \) are given by the following expression:

\[
D^k\Theta^*(f)(h_1, ..., h_k)^* = \Theta^*(f)^* \phi^*(h_1 \cdots h_k) = \phi^*(\mathcal{L}^k(h_1 \cdots h_k)).
\]

The proof is similar to the previous one and so it will be omitted.

3. The Ruelle operator on the Walters space

To simplify the notation for any \( f \in C(\Omega) \) and \( x, y \in \Omega \), we write \( S_n f(x) \equiv f(x) + f(\sigma(x)) + \cdots + f(\sigma^{n-1}(x)) \) and \( d_n(x, y) \equiv \max_{0 \leq k < n} d_0(\sigma^k x, \sigma^k y) \).

Definition 3.1. We can say that a continuous function \( f: \Omega \to \mathbb{R} \) is in the Walters class if given \( \epsilon > 0 \) there exists \( \eta > 0 \) such that

\[
\forall n \geq 1, \forall x, y \in \Omega, d_n(x, y) \leq \eta \implies |S_n f(x) - S_n f(y)| \leq \epsilon \tag{7}
\]

The space of all continuous functions satisfying the above condition is denoted by \( W(\Omega) \).

If a continuous function \( f: \Omega \to \mathbb{R} \) satisfies the condition (7), we can say that \( f \) satisfies the Walters condition.

Definition 3.2. Consider a continuous function \( f: \Omega \to \mathbb{R} \) and define \( C_f(x, y) \) by

\[
C_f(x, y) = \sup_{n \geq 1} \sup_{a \in M^n} S_n f(a x) - S_n f(a y) . \tag{8}
\]

2260
We can say that \( f \) satisfies the weak Walters condition if \( C_f(x, y) \to 0 \) when \( d_f(x, y) \to 0 \).

**Example.** Consider the metric space \((M, d)\) where \( M = [0, 1] \) and \( d = |\cdot| \). Now let \( f \) be the potential defined on \( \Omega = M^3 \) by \( f(x) = x_1 \), i.e. \( f \) depends only on the first coordinate. We claim that \( f \) satisfies the weak Walters condition, but not the (strong) Walters condition. In fact, we have it that \( S_nf(a) = S_nf(a) = \sum_{i=1}^{n}a_i \) for any \( a = (a_1, \ldots, a_n) \in M^n \). Therefore, \( f \) clearly satisfies the weak Walters condition. Now we show that \( f \) does not satisfy that Walters condition. Indeed, consider \( x = (0, 0 \ldots) \) the null vector and \( y = (\eta, \eta, \ldots) \) for a small \( \eta \). Notice that \( d_f(x, y) = d_n(x, y) = \eta \). On the other hand, where \( S_nf(x) = 0 \) and \( S_nf(y) = n \cdot \eta \), then

\[
|S_nf(x) - S_nf(y)| = n \cdot \eta.
\]

From this it is clear that \( f \) does not satisfy the Walters condition.

On the other hand, the opposite implication is always the truth:

**Proposition 3.3.** Let \( f \in C(\Omega) \), satisfying the Walters condition. Then, \( f \) satisfies the weak Walters condition.

**Proof.** Let \( f \in W(\Omega) \), then by definition, given \( \epsilon > 0 \), arbitrarily there exists \( \eta > 0 \) such that \( \forall n \geq 1, \forall z, w \in \Omega \), and with \( d_n(z, w) \leq \eta \) we have \( |S_nf(z) - S_nf(w)| \leq \epsilon \). Note that \( d_n(ax, ay) \leq d_n(x, y) \) for any \( a \in M^n \). Therefore, \( d(x, y) \leq \eta \Rightarrow d_n(ax, ay) \leq \eta \Rightarrow |S_nax - S_nf(ay)| \leq \epsilon, \forall a \in M^n, \forall n \geq 1 \). By taking the supremum over all \( a \in M^n \) and \( n \geq 1 \) the result follows.

The space \( W(\Omega) \) is clearly a linear space. Let \( S \) denote the expansivity constant of the mapping \( \sigma \). In [5] it was shown that for \( s \in (0, S) \) the following expression

\[
\|f\|_W = 2 \|f\|_0 + \sup_{n \geq 1} \max_{d(a, x) \in \mathbb{R}} |S_nf(x) - S_nf(y)|
\]

defines a family of equivalent norms and \((W(\Omega), \|\cdot\|_W)\) is a Banach space. Since the family of norms \((\|\cdot\|_W)_{0 \leq s \leq S} \) provides the same topology, there is no loss of generality in taking a particular value \( s \in (0, S) \) and developing the theory with the norm \( \|\cdot\|_W \equiv \|\cdot\|_W \).

En route to the proof of this work’s main theorem, we need an extra structure for this space which is the structure of the Banach algebra. This is the content of the next lemma.

**Lemma 3.4.** The space \( W(\Omega) \) with the norm \( \|\cdot\|_W \) is a Banach algebra over \( \mathbb{R} \), i.e. \( W(\Omega) \) is a real Banach space and for all \( f, g \in W(\Omega) \) we have \( \|fg\|_W \leq \|f\|_W \|g\|_W \).

**Proof.** Let \( f, g \in W(\Omega) \) and define \( I \equiv I(f, g) \) by

\[
I \equiv \sup_{n \geq 1} \max_{d(a, x) \leq \mathbb{R}} |S_n(fg)(x) - S_n(fg)(y)|
\]

\[
= \sup_{n \geq 1} \max_{d(a, x) \leq \mathbb{R}} \left| \sum_{j=1}^{n-1} (fg) \circ \sigma^j(x) - \sum_{j=1}^{n-1} (fg) \circ \sigma^j(y) \right|
\]

\[
= \sup_{n \geq 1} \max_{d(a, x) \leq \mathbb{R}} \left| \sum_{j=1}^{n-1} (fg) \circ \sigma^j(x) - \sum_{j=1}^{n-1} f \circ \sigma^j(x)g \circ \sigma^j(y) + \sum_{j=1}^{n-1} f \circ \sigma^j(x)g \circ \sigma^j(y) - \sum_{j=1}^{n-1} (fg) \circ \sigma^j(y) \right|
\].

2261
By applying the triangle inequality we get

\[
I \leq \sup_{n \geq 1} \max_{d(x,y) \leq t} \left| \sum_{j=1}^{n-1} f \circ \sigma^j(x) |g \circ \sigma^j(x) - g \circ \sigma^j(y)| \right|
+ \sup_{n \geq 1} \max_{d(x,y) \leq t} \left| \sum_{j=1}^{n-1} g \circ \sigma^j(y) [f \circ \sigma^j(x) - f \circ \sigma^j(y)] \right|
\leq \|f\|_0 \cdot \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_\sigma g(x) - S_\sigma g(y)| + \|g\|_0 \cdot \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_n f(x) - S_n f(y)|.
\]

The last upper bound readily implies that

\[
\|fg\|_w \leq 2\|f\|_0 + \|f\|_0 \cdot \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_\sigma g(x) - S_\sigma g(y)| + \|g\|_0 \cdot \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_n f(x) - S_n f(y)|.
\]

On the other hand, it follows from the definition of \(\|\cdot\|_w\) that

\[
\|fg\|_w \cdot \|g\|_w = 4\|f\|_0 \cdot \|g\|_0 + 2\|f\|_0 \cdot \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_\sigma g(x) - S_\sigma g(y)|
+ 2\|g\|_0 \cdot \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_n f(x) - S_n f(y)|
+ \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_\sigma g(x) - S_\sigma g(y)| \cdot \sup_{n \geq 1} \max_{d(x,y) \leq t} |S_n f(x) - S_n f(y)|.
\]

This identity and the previous estimates readily implies that \(\|fg\|_w \leq \|f\|_w \cdot \|g\|_w\).

**Proposition 3.5.** If \(f \in W(\Omega)\), then \(\mathcal{L}(W(\Omega)) \subseteq W(\Omega)\).

**Proof.** We can claim that for any fixed \(a \in \mathcal{M}\), if \(f \in W(\Omega)\) then the function \(x \mapsto f(ax)\) also belongs to \(W(\Omega)\). In fact, given \(\epsilon > 0\) we choose \(\eta > 0\) such that the Walters condition \((\ast)\) is satisfied for \(f\). Note that \(d_n(x,y) \leq \delta \Rightarrow d_n(ax,ay) \leq d_n(x,y) \leq \eta\). From the definition we have it that \(d_n(x,y) \leq \eta\) implies \(|S_n f(ax) - S_n f(ay)| \leq \epsilon\) for all \(n > 0\) and, therefore, the claim is proved.

The next step is to prove that the function \(r : \Omega \to \mathbb{R}\) given by \(r(x) = \int_M h(ax) d\mu(a)\) belongs to \(W(\Omega)\) whenever \(h \in W(\Omega)\). From the previous claim it follows that \(x \mapsto h(ax)\) is in \(W(\Omega)\) since \(h \in W(\Omega)\). Given \(\epsilon > 0\) we now choose \(\eta > 0\) such that the condition \((\ast)\) is satisfied for \(x \mapsto h(ax)\). Since \(d_n(x,y) \leq \eta \Rightarrow d_n(ax,ay) \leq \eta\), the Walters condition for \(r\) follows from the inequality

\[
|S_n r(x) - S_n r(y)| \leq \int_M |S_n h(ax) - S_n h(ay)| d\mu(a) \leq \epsilon \cdot \mu(M).
\]

By hypothesis, the potential \(f \in W(\Omega)\), since the Walters space is a Banach algebra we have it that \(\exp(f) \in W(\Omega)\). For the same reason, for any \(\varphi \in W(\Omega)\) we have \(\varphi \cdot \exp(f) \in W(\Omega)\). As argued above, for any \(a \in \mathcal{M}\), the mapping \(x \mapsto \varphi(ax) \cdot \exp(f(ax))\) belongs to \(W(\Omega)\). Using the result proved above for the function \(r\), with \(h(x) = \varphi(x) \cdot \exp(f(x))\), it follows that the mapping \(x \mapsto \int_M \varphi(ax) \exp(f(ax)) d\mu(a) = \mathcal{L}(\varphi)(x)\)

is in the Walters space for any \(\varphi \in W(\Omega)\), which finishes the proof. 

\[2262\]
4. The Ruelle theorem on Walters space

The proof of this version of the Ruelle theorem is inspired by the original proof presented in Walters work [39].

To prove the Ruelle theorem, we concentrate on a certain subclass of $C(\Omega)$ which is given by

$$G_0(\Omega) = \left\{ g \in C(\Omega) : g > 0 \text{ and } \int_M g(ax) d\mu(a) = 1 \quad \forall x \in \Omega \right\}.$$ 

If $f : \Omega \to \mathbb{R}$ is a potential given by $f = \log g$, where $g \in G_0(\Omega)$, then the weak Walters condition for $f$ can be rephrased in terms of $g$ by saying that

$$D_\epsilon^g(x, y) = \sup \sup_{n \geq 1} \left| \prod_{i=0}^{n-1} \frac{g(\sigma^i ax)}{g(\sigma^i ay)} - 1 \right| \leq D_\epsilon - 1$$

for all $x, y$ with $d_\Omega(x, y) < \epsilon_0$ and $D_\epsilon^g(x, y) \to 0$ when $d_\Omega(x, y) \to 0$.

**Theorem 4.1.** Let $g \in G_0(\Omega)$ be a function such that $\log g$ satisfies the weak Walters condition. Then there is a probability measure $\nu : \mathcal{B}(\Omega) \to [0, 1]$ such that

$$\mathcal{L}_{\log g} \nu = \nu$$

for all $\varphi \in C(\Omega)$. Moreover, $\nu$ is the unique probability measure satisfying $\mathcal{L}_{\log g} \nu = \nu$.

**Proof.** The proof is based on a simple modification of the arguments given in [40] and it is presented here for the reader’s convenience.

Let us introduce the temporary notation $\mathcal{L}$ for $\mathcal{L}_{\log g}$. We begin by proving that $\{ \mathcal{L}^n \varphi, n \geq 0 \}$ is an equicontinuous family for any fixed $\varphi$ satisfying the weak Walters condition. Indeed, from the definition of the Ruelle operator we have

$$|\mathcal{L}^n \varphi(x) - \mathcal{L}^n \varphi(y)| \leq \int_M \left| \exp(S_n \log g(ax)) \varphi(ax) - \exp(S_n \log g(ay)) \varphi(ay) \right| \prod_{i=0}^{n-1} d\mu(a_i)$$

$$\leq \int_M \prod_{i=0}^{n-1} g(\sigma^i(ax)) [\varphi(ax) - \varphi(ay)] \prod_{i=0}^{n-1} d\mu(a_i)$$

$$+ \int_M \varphi(ay) \left[ \prod_{i=0}^{n-1} g(\sigma^i(ax)) - \prod_{i=0}^{n-1} g(\sigma^i(ay)) \right] \prod_{i=0}^{n-1} d\mu(a_i)$$

The two terms on the rhs above can be bounded by
Since \( g \in G_0(\Omega) \), it follows from the Fubini theorem that the iterated integral on the first term above is equal to one. The second term can be bounded similarly by using the definition of \( D_x^N(x, y) \), which gives us the following inequality:

\[
|\mathcal{L}_x^N \varphi(x) - \mathcal{L}_y^N \varphi(y)| \leq \sup_{a \in M^*} |\varphi(ax) - \varphi(ay)| + \|\varphi\|_0 \cdot D_x^N(x, y).
\]

Since \( \varphi \) is a continuous function and \( \log g \) satisfies the weak Walters condition, the above inequality ensures that the family \( \{\mathcal{L}_x^N \varphi, \ n \geq 0\} \) is equicontinuous.

Recalling that \( g \in G_0(\Omega) \), we get from the definition of the Ruelle operator for all \( n \in \mathbb{N} \) that:

\[
\|\mathcal{L}_x^n \varphi\|_0 \leq \|\varphi\|_0
\]

for any \( \varphi \in C(\Omega) \). This inequality implies that the closure in the uniform topology of \( \{\mathcal{L}_x^n \varphi, \ n \geq 0\} \) is uniformly bounded in \( C(\Omega) \). Therefore, we can apply the Arzelà–Ascoli theorem for the family \( \{\mathcal{L}_x^n \varphi, \ n \geq 0\} \) to guarantee that there exists a subsequence \( (n_i) \subset \mathbb{N} \) and function \( \varphi^* \in C(\Omega) \) so that \( \mathcal{L}_x^{n_i} \varphi \rightarrow \varphi^* \) uniformly.

Let us proceed by showing that \( \varphi^* \) is a constant function. Notice that the identity \( \mathcal{L}(1) = 1 \) implies the following inequalities: \( \min(\varphi) \leq \min(\mathcal{L}(\varphi)) \leq \cdots \leq \min(\varphi^*) \).

**Claim 1.** For any \( k \in \mathbb{N} \) we have \( \min(\mathcal{L}_x^k \varphi^*) = \min(\varphi^*) \). Indeed, we have it that:

\[
\min(\mathcal{L}_x^k \varphi^*) = \min(\mathcal{L}_x^k(\lim_n \mathcal{L}_x^n \varphi^*)) = \lim_n \min(\mathcal{L}_x^{n+k} \varphi^*) = \lim_n \min(\mathcal{L}_x^{n+k} \varphi) = \min(\varphi^*),
\]

where the last equality follows from the monotonicity of the sequence \( \min \mathcal{L}_x^k \varphi \) and \( \min \mathcal{L}_x^k \varphi \rightarrow \min \varphi^* \). Given \( \epsilon > 0 \) we choose \( x \in \Omega \) and \( N \in \mathbb{N} \) such that \( \min(\mathcal{L}_x^N \varphi^*) = \mathcal{L}_x^N \varphi^*(x) \) and \( \{ax, \ a \in M^N\} \) is \( \epsilon \)-dense in \( \Omega \).

**Claim 2.** For all \( y \in \sigma^{-N}x \) we have \( \varphi^*(y) = \min(\varphi^*) \). From claim 1 and the choice of \( x \), we have \( \mathcal{L}_x^N \varphi^*(x) = \min(\varphi^*) \). Let \( z \in \Omega \) be such that \( \varphi^*(z) = \min(\varphi^*) \), then:

\[
\int_{M^N} g(ax) g(\sigma ax) \cdots g(\sigma^{-1}ax) \varphi^*(ax) \prod_{i=1}^N d\mu(a_i) = \mathcal{L}_x^N \varphi^*(x) = \min(\varphi^*) = \varphi^*(z).
\]

By using the identity \( 1 = \mathcal{L}_x^N 1(x) \), it follows from the above equation that:

\[
0 = \int_{M^N} g(ax) g(\sigma ax) \cdots g(\sigma^{-1}ax) [\varphi^*(ax) - \varphi^*(z)] \prod_{i=1}^N d\mu(a_i).
\]

By using the continuity of \( \varphi^* \) and the assumption \( \text{supp}(\mu) = M \) it is easy to see that \( \varphi^*(ax) = \varphi^*(z) \) for any \( a \in M^N \). Since \( \varphi^* \) is continuous and constant over \( \{ax, \ a \in M^N\} \), which is \( \epsilon \)-dense in \( \Omega \), it follows that \( \varphi^* \) is a constant function.

We now show the existence and uniqueness of the fixed point for \( \mathcal{L}^* = \mathcal{L}_{\log g}^* \). Define the linear functional \( F : C(\Omega) \rightarrow \mathbb{R} \) by \( F(\varphi) = \varphi^* \). The functional \( F \) sends the cone of positive continuous functions to itself and satisfies \( F(1) = 1 \). Then, it follows from the Riesz–Markov theorem that there exists a unique Borel probability measure \( \nu \in \mathcal{M}(\Omega) \) that represents \( F \). It is
a simple matter to show that $\mathcal{L}^n\nu = \nu$. For the uniqueness, suppose that there exists another probability measure $\gamma \in \mathcal{M}(\Omega)$ such that $\mathcal{L}^n\gamma = \gamma$. Of course, $(\mathcal{L}^n)^n\gamma = \gamma$ for every $n \in \mathbb{N}$ and

$$\int_{\Omega} \varphi \, d\gamma = \int_{\Omega} \varphi \, d[(\mathcal{L}^n)^n\gamma] = \int_{\Omega} \mathcal{L}^n\varphi \, d\gamma = \int_{\Omega} \lim_{n \to \infty} \mathcal{L}^n\varphi \, d\gamma = \int_{\Omega} \varphi^\ast \, d\gamma = \int_{\Omega} \varphi \, d\nu.$$

Since $\varphi \in C(\Omega)$ is arbitrary, it follows that $\gamma = \nu$.

**Lemma 4.2.** Let $f$ satisfy the weak Walters condition. Then, there exists $N > 0$ and $a \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ and $\varphi \in \sigma^{-n}x \cap B_d(x, \epsilon)$ with $S_n f(w) \geq a$.

**Proof.** See [40] page 126.

**Lemma 4.3.** Let $f \in C(\Omega)$ be a potential. Then, there exists a real number $\lambda > 0$ and a Borel probability measure $\nu \in \mathcal{M}(\Omega)$ such that $\mathcal{L}^\nu f = \lambda \nu$.

**Proof.** The mapping $\gamma \mapsto \mathcal{L}^\gamma f(\mathcal{L}^\gamma f)(1)$ define a continuous function from $\mathcal{M}(\Omega)$ to itself. The Schauder–Tychonoff fixed-point theorem ensures the existence of a fixed point $\nu$ for this mapping. By taking $\lambda = (\mathcal{L}^\nu f)(1)$, the lemma follows.

We are now able to prove the main theorem of this section, which is the Ruelle theorem for Walters potentials defined over an infinite Cartesian product of general metric compact spaces.

**Theorem 4.4.** Let $f$ be a potential satisfying the weak Walters condition and consider the Ruelle operator $\mathcal{L}_f : C(\Omega) \to C(\Omega)$ associated to $f$. Then, there are a real number $\lambda_f > 0$, a strictly positive continuous function $h_f$ and a unique Borel probability measure $\nu_f$ such that

(i) $\mathcal{L}_f h_f = \lambda_f h_f$, $\mathcal{L}_f \nu_f = \lambda_f \nu_f$.

(ii) For any $\varphi \in C(\Omega)$ we have

$$\left\| \left( \lambda_f^{1/n} \mathcal{L}_f^{n} \varphi - h_f \int_{\Omega} \varphi \, d\nu_f \right) \right\| \to 0,$$

when $n \to \infty$.

**Proof.** The proof will be divided into three steps:

**Claim 1.** Let $\nu$ and $\lambda$, as given by the lemma 4.3. Then, for any $f$ satisfying the weak Walters condition and $\epsilon_0 > 0$ the set

$\Lambda = \{ \varphi \in C(\Omega) : \varphi \geq 0, \nu(\varphi) = 1$ and $\varphi(x) \leq \exp(C_f(x, y))\varphi(y)$ if $d_\Omega(x, y) < \epsilon_0 \}$

is convex, closed, bounded and uniformly equicontinuous.

Let us first prove that $\Lambda$ is not empty. Indeed, for any $x, y \in \Omega$ we have

$$\mathcal{L}_f^1(x) = \int_M e^{f(ax)} \, d\mu(a) = \int_M e^{f(ay)} e^{f(ax) - f(ay)} \, d\mu(a) \leq \exp(\sup_{a \in M} f(ax) - f(ay)) \int_M e^{f(ay)} \, d\mu(a) \leq \exp(C_f(x, y)) \mathcal{L}_f^1(x).$$

The set $\Lambda$ is clearly closed and convex. Now we shall prove that $\Lambda$ is a bounded set. Let $x, y \in \Omega$. By the lemma 4.2, given $\epsilon > 0$ and $a \in \mathbb{R}$, there is $N \in \mathbb{N}$ and $y_0 = a_0 x$, where $a_0 = a_1 \ldots a_N$, such that $d_\Omega(x, y) < \epsilon$ and $S_N f(y_0) \geq a$. Given $\epsilon > 0$, it follows from the continuity of $\varphi$ that
we can choose $\delta > 0$ such that for any point $a$ in the closed ball $B[a_0, \delta] \subseteq M^N$ we have $S_t f(ax) \geq a - \delta t$. In particular, we can choose $\delta$ such that $B[0, \delta] \subseteq B[y, \varepsilon]$. Therefore, it follows from the definition of the Ruelle operator and the choice of $\delta$ that

$$L^N \varphi(x) = \int_{M^N} e^{S_t f(ax)} \varphi(ax) \prod_{i=1}^N d\mu(a_i)$$

$$= \int_{M^N \cap \partial[a_0, \delta]} e^{S_t f(ax)} \varphi(ax) \prod_{i=1}^N d\mu(a_i) + \int_{B[a_0, \delta]} e^{S_t f(ax)} \varphi(ax) \prod_{i=1}^N d\mu(a_i)$$

$$\geq \int_{B[a_0, \delta]} e^{S_t f(ax)} \varphi(ax) \prod_{i=1}^N d\mu(a_i)$$

$$\geq \mu \times \cdots \times \mu(B[a_0, \delta]) e^{\alpha_t - \delta t} \varphi(w_0),$$

where $w_0$ minimizes the function $(a_1, \ldots, a_N) \mapsto \varphi(a_1 \cdots a_N)$ in $B[y, \varepsilon]$. Now, observe that $w_0 \in B[y, \varepsilon]$; by using the compactness of $\Omega$ and the definition of $\Lambda$ for all $x, y \in \Omega$ we get the following inequality: $\varphi(y) \leq \text{Const.} \cdot e^{\alpha_t - \delta t} L^N \varphi(x)$. Recalling that $L^f \nu = \lambda \nu$ and $\nu(\varphi) = 1$, by integrating both sides of the previous inequality we obtain:

$$\varphi(y) \leq \text{Const.} \cdot e^{\alpha_t - \delta t} L^N \varphi(x) \Rightarrow \varphi(y) \leq \text{Const.} \cdot e^{\alpha_t - \delta t} L^N \varphi(x) = \text{Const.} \cdot e^{\alpha_t - \delta t} \lambda^N.$$ 

Hence $\lambda$ is bounded. The uniform equicontinuity of $\Lambda$ is proved as in [40, p 129] mutatis mutandis.

**Claim 2.** The operator $\lambda^{-1} L_f$ maps $\Lambda$ into $\Lambda$.

Let $\varphi \in \Lambda$ and $x, y \in \Omega$ with $d_\Omega(x, y) < \varepsilon_0$. Then,

$$\frac{1}{\lambda} L_f \varphi(x) = \frac{1}{\lambda} \int_M e^{f(ax)} \varphi(ax) \, d\mu(a)$$

$$\leq \frac{1}{\lambda} \int_M e^{f(ax)} \varphi(ax) \left( e^{(f(ax) - f(\sigma ax))} C_f(ax, ay) \right) \, d\mu(a)$$

$$\leq \frac{1}{\lambda} \int_M e^{f(ax)} \varphi(ax) \left( e^{(f(ax) - f(\sigma ax))} C_f(x, y) \right) \, d\mu(a)$$

$$= \frac{1}{\lambda} L_f \varphi(y) e^{C_f(x, y)}$$

where the inequality $e^{f(ax) - f(\sigma ax)} C_f(ax, ay) \leq e^{C_f(x, y)}$ is justified by observing that $C_f(ax, ay)$ is equal to

$$\sup_{n \geq 1} \sup_a (f(ax) - f(\sigma ax) + f(\sigma ax) - f(\sigma^2 ax) + \cdots + f(\sigma^n ax) - f(\sigma^n ax)).$$

From this, it is clearly shown that $C_f(ax, ay) + (f(ax) - f(ay)) \leq C_f(x, y)$.

**Claim 3.** If $g = e^{h(h \circ T)}$, then $g \in G_0(\Omega)$ and $D_g^*(x, y) \to 0$ when $d_\Omega(x, y) \to 0$. The proof is similar to the one given by [40, p 130].

Claims 1 and 2 allow us to use the Schauder–Tychonoff fixed-point theorem to obtain a fixed point $h \in \Lambda$ for the operator $\lambda^{-1} L$. This fixed point $h$ satisfies $L^t h = \lambda^t h$, $\nu(h) = 1$ and $h(x) \leq e^{C_f(x, x)} \varphi(x)$ whenever $d_\Omega(x, y) < \varepsilon_0$. We shall now show that $h > 0$. Suppose that there exists some $x \in \Omega$ such that $h(x) = 0$. For all $n \in \mathbb{N}$ we have $L^n h(x) = \lambda^n h(x) = 0$, so $h$ must be 0 on the set $\{ \sigma^{-n} x, n \in \mathbb{N} \}$, which is dense, given that $\mu$ has full support we have $h \equiv 0$, contradicting the fact that $\nu(h) = 1$. 
By using the theorem 4.1 we have it that \( \mathcal{L}_{\log g}^{n} \phi \xrightarrow{\|h\|} \mu(\phi) \) for all \( \phi \in C^0(\Omega) \), where \( \mu \in \mathcal{M}(\Omega) \) is the fixed point of \( \mathcal{L}_{\log g}^{n} \) in \( \mathcal{M}(\Omega) \). On the other hand, if 
\[
\frac{1}{\lambda} |x|^n \mathcal{L}_{\log g}^{n} \phi(x) = h(x)(\mathcal{L}_{\log g}^{n} \phi)(h(x))
\]
then it follows that \( I/\lambda^n \mathcal{L}_{\log g}^{n} \phi \xrightarrow{\|h\|} h : \mu(\phi) \). We shall show that \( \mu(\phi) = \nu(\phi) \). Let \( m \in \mathcal{M}(\Omega) \) defined by \( m(\phi) = \nu(h(\phi)) \). Then,
\[
m(\mathcal{L}_{\log g}^{n} \phi) = \nu(h \cdot \mathcal{L}_{\log g}^{n} \phi) = \frac{1}{\lambda} = \nu(\mathcal{L}_{f}^{n} \phi \cdot h) = m(\phi).
\]

5. Spectral gap and analyticity of the pressure

By ‘presence of the spectral gap’ in the Ruelle operator, we mean the existence of a single isolated eigenvalue of maximum modulus. The presence of the spectral gap in the Ruelle operator is the key property to prove analyticity of the pressure and also implies the exponential decay of correlations with respect to the Gibbs measures. These are classical results and very well known for Hölder potentials when the state space \( M \) is finite. Here, we shall analyze the generalizations of these results in the sense of the state space \( M \) and the regularity of the potential.

The main difference between the Ruelle theorem operator for \( f \in W(\Omega) \) and for \( f \in C^\gamma(\Omega) \) is the fact that in the first case we do not have much information about the spectrum of \( \mathcal{L}_{f} \). In particular, it seems hard to decide whether we have the presence or absence of the spectral gap for the Ruelle operator \( \mathcal{L}_{f} \), which is a crucial property for gaining a deep understanding of the associated Gibbs measure.

**Theorem 5.1.** Let \( K \subset C(\Omega) \) be an invariant subspace of \( C(\Omega) \). A sufficient condition for the analyticity of the pressure is the analyticity of the map \( K \ni f \mapsto \mathcal{L}_{f} \in K \), and the Ruelle theorem holds for \( \mathcal{L}_{f}^{n} \) and \( f^{\gamma} \in K \) and the presence of the spectral gap in the spectrum of the Ruelle operator.

In order to prove the theorem 5.1 we shall use the following lemma, which seems to be a well-known fact in the scientific community. We decided to give a proof for this lemma in order to keep the text as self-contained as possible and because we were not able to find its reference.

Recall that in this paper we are using the expression simple eigenvalue to refer to an eigenvalue \( \lambda \) of an operator \( T : X \to X \) such that the image of the spectral projector \( \pi_\lambda = \int_{\lambda_0}^{\lambda}(I - L)^{-1}d\lambda \) is some one-dimensional subspace of \( X \).

**Lemma 5.2.** Let \( T : X \to X \) be a bounded linear operator possessing an isolated simple eigenvalue \( \lambda \in \mathbb{C} \). Let \( D \) be a closed disc centered at \( \lambda \) such that \( D \cap \text{spec}(T) = \{\lambda\} \). Then, there is a neighborhood \( U \) of \( T \) in \( L(X, X) \) so that the mapping \( U \ni L \mapsto \lambda(L) \in \mathbb{C} \), where \( \lambda(L) \) is the unique point in \( \text{spec}(L) \cap \text{int}(D) \), is well defined and, moreover, this mapping is analytic.

**Corollary 5.3.** For any fixed \( 0 < \gamma \leq 1 \) both mappings
\[
C^\gamma(\Omega) \ni f \mapsto h_f \in C^\gamma(\Omega) \quad \text{and} \quad C^\gamma(\Omega) \ni f \mapsto \nu_f \in (C^\gamma(\Omega))^\prime
\]
are analytic.
Proof. Choose \( f \) arbitrarily in \( C^1(\Omega) \). Let \( U \) and \( \tilde{U} \) be neighborhoods of \( L_f \) and \( L_{\tilde{f}} \), respectively such that for all \( \lambda \in U \) and \( \lambda \in \tilde{U} \) the images of the spectral projectors \( \pi_{\lambda} \) and \( \pi_{\lambda}^{\tilde{f}} \) are the one-dimensional spaces associated to the eigenvalue \( \lambda \). Consider a neighborhood \( W \subset C^1(\Omega) \) of \( f \) so that \( L_{\tilde{f}} \in U \) and \( L_{\tilde{g}} \in \tilde{U} \), respectively whenever \( g \in W \). Therefore, we have it that \( \pi_{\lambda}^{\tilde{f}} = c \cdot \nu_\lambda \). Then, \( c = c \cdot \int_{\Omega} \mathrm{d}g \equiv c \cdot \langle \nu_\lambda, 1 \rangle = \langle \pi_{\lambda}^{\tilde{f}} \nu_\lambda, 1 \rangle \) so \( \nu_\lambda = \left( \pi_{\lambda}^{\tilde{f}} \nu_\lambda, 1 \right)^{-1} \cdot \pi_{\lambda}^{\tilde{f}} \nu_\lambda \). Since the rhs of the last expression is a composition of analytic functions, it follows that \( g \mapsto \nu_\lambda \) is analytic in a neighborhood of \( f \). On the other hand, \( \pi_{\lambda}^{hf} = C \cdot h_\lambda \), so by a suitable choice of the eigenfunction it follows that \( C = C \cdot \langle \nu_\lambda, h_\lambda \rangle = \langle \nu_\lambda, \pi_{\lambda}^{hf} \rangle \). Therefore, \( h_\lambda = \left( \nu_\lambda, \pi_{\lambda}^{hf} \right)^{-1} \cdot \pi_{\lambda}^{hf} \) which is again a composition of analytic functions and so \( g \mapsto h_\lambda \) is analytic in a neighborhood of \( f \). \( \square \)

Now we are ready to state and prove the main result of this section.

Proof of the theorem 5.1.

Proof. The proof is based on the analyticity of both functions \( \mathcal{C} \ni f \mapsto L_f \) and \( U \ni T \mapsto \lambda(T) \in \mathbb{R} \). The analyticity of the first map is the content of the proposition (2.7). The analyticity of the second mapping follows from the existence of spectral gap hypothesis and the Lemma 5.2. To finish the proof it is enough to observe that, the Ruelle theorem holds for \( L_f \) by hypothesis, so the mapping \( \mathcal{C} \ni f \mapsto P(f) \in \mathbb{R} \) is given by \( P(f) = \log \lambda_f \), which is a composition of the following analytic mappings: \( \log, \mathcal{C} \ni f \mapsto L_f \) and \( U \ni T \mapsto \lambda(T) \in \mathbb{R} \). \( \square \)

An immediate consequence of the above theorem is the following corollary:

Corollary 5.4. The function defined by \( C^1(\Omega) \ni f \mapsto P(f) \in \mathbb{R} \) is a real analytic function.

Proof of the lemma 5.2. The argument is based on the following two claims:

Claim 1. There is a neighborhood \( U \) of \( T \) in \( L(X, X) \) such that \( \text{spec}(L) \cap \partial D = \emptyset \) for all \( L \in U \). This claim is proved by contradiction. Suppose that for each \( n \) there exists \( L_n \in B(T, 1/n) \subset L(X, X) \) so that \( \lambda_{L_n} \in \text{spec}(L_n) \cap \partial D \) and the operator \( \lambda L_n I - L_n \) is not invertible. Since \( \partial D \) is compact, we can find a convergent subsequence \( \{ \lambda_{L_n} \} \subset \partial D \) so that \( \lambda_{L_n} \to \lambda_L \in \partial D \). Since \( L_n \to T \), we have it that \( (\lambda_{L_n} I - L_n) \to (\lambda_L I - T) \) in the strong topology. Since \( \lambda_L I - T \) is invertible and the space of the invertible bounded linear operators is open, we thus have a contradiction.

Claim 2. We can shrink \( U \) so that for all \( L \in U \) we have it that \( \text{spec}(L) \cap \text{int}(D) \) is a simple eigenvalue of \( L \). In fact, for each \( L \in U \) let \( \pi_L \) be the spectral projector given by

\[
\pi_L = \int_{\partial D} (M - L)^{-1} \lambda \, d\lambda.
\]

Notice that the mapping \( U \ni L \mapsto \pi_L \in L(X, X) \) is continuous, so by the proposition B.5, if necessary one can shrink \( U \) so that for all \( L \in U \) the application \( \pi_L \) has the same rank as \( \pi_T \). This fact, together with the remark B.2, implies that the portion of the spectrum of \( L \in U \) which lies in \( \text{int}(D) \) is not empty; call it \( \Sigma(L) \). Define \( X_{\Sigma(L)} = \pi_L X \) and \( T_{\Sigma(L)} = T_{\Sigma(L)} \). It is well known [27, p 98] that \( \text{spec}(T_{\Sigma(L)}) = \Sigma(L) \). If \( \Sigma(L) \) is not a unitary set, then \( X_{\Sigma(L)} \) is not a uni-dimensional subspace and, therefore, \( 1 \neq \dim(X_{\Sigma(L)}) = \dim(\pi_T X) = 1 \). This contradiction shows that there is a unique simple eigenvalue \( \lambda(L) \) of \( L \) inside \( D \).
By using the two previous claims one has a well-defined mapping \( U \ni L \mapsto \lambda(L) \in \mathbb{C} \), where \( \lambda(L) \) is the unique simple eigenvalue of \( L \) inside \( \text{int}(D) \). Now we proceed to the proof that this mapping is analytic. Fix \( v \in X \) such that \( \pi_L v \) is a non-zero vector and choose \( w \in X^* \) such that \( w(\pi_L v) = w(\pi_L v) = 0 \) (Hahn Banach) for all \( L \) in a small enough neighborhood of \( T \). According to proposition B.3 the operator \( L \) commutes with \( \pi_L \) so then we get

\[
\lambda(L) = \frac{\langle w, \pi_L(Lv) \rangle}{\langle w, \pi_L(v) \rangle}
\]

From definition B.1 and the above equality we obtain the analyticity of the mapping \( U \ni L \mapsto \lambda(L) \in \mathbb{C} \).

\[\square\]

6. Spectral gap and exponential decay

In this section we will closely follow [2].

**Definition 6.1.** Consider the probability space \( (\Omega, \mathcal{F}, \nu) \). Let \( \sigma \) be the left shift on \( \Omega \). For each \( \varphi_1, \varphi_2 \in L^2(\Omega, \nu) \) we define the correlation function \( C_{\varphi_1,\varphi_2} : \mathbb{Z} \to \mathbb{R} \) by

\[
C_{\varphi_1,\varphi_2}(n) = \int_\Omega (\varphi_1 \circ \sigma^n) \varphi_2 \, d\nu - \int_\Omega \varphi_1 \, d\nu \int_\Omega \varphi_2 \, d\nu.
\]

**Theorem 6.2.** Suppose that \( f \in W(\Omega) \) is a potential for which the Ruelle operator \( \mathcal{L}_f \) has the spectral gap property. Consider the measure \( \mu = \mathcal{L}_f \mu \) where \( \mu \) is the eigenmeasure given by theorem 4.4. Then, the correlation function \( C_{\varphi_1,\varphi_2}(n) \) decays exponentially fast. More precisely, there are \( \tilde{\tau}, \tau > 0 \) such that for all \( \varphi_1, \varphi_2 \in W(\Omega) \) the correlation function satisfies

\[
|C_{\varphi_1,\varphi_2}(n)| = \left| \int_\Omega (\varphi_1 \circ \sigma^n) \varphi_2 \, d\mu - \int_\Omega \varphi_1 \, d\mu \int_\Omega \varphi_2 \, d\mu \right| \leq C \tilde{\tau}^n.
\]

Before proving the above theorem, we present two auxiliary lemmas:

**Lemma 6.3.** The spectral projection \( \pi_f \equiv \pi_{\mathcal{L}_f} \) is given by \( \pi_f(\varphi) = \left( \int_\Omega \varphi \, d\nu \right) \cdot h_f \).

**Proof.** We know that \( \pi_f \) and \( \mathcal{L}_f \) commutes. From the Ruelle theorem (theorem 4.4) we have it that \( \lim_{n \to \infty} (1/\lambda^n) \mathcal{L}_f^n \varphi = h_f \int_\Omega \varphi \, d\nu \) uniformly. Since \( \pi_f \) is bounded we get

\[
\left\| \pi_f(\lambda^{-n} \mathcal{L}_f^n \varphi - h_f \int_\Omega \varphi \, d\nu) \right\|_b \leq \| \pi_f \| \left\| \lambda^{-n} \mathcal{L}_f^n \varphi - h_f \int_\Omega \varphi \, d\nu \right\|_b \to 0,
\]

when \( n \to \infty \). Since \( \pi_f(\lambda^{-n} \mathcal{L}_f^n \varphi)) = \lambda^{-n} \mathcal{L}_f^n \pi_f(\varphi) = \lambda^n \pi_f(\varphi) = \pi_f(\varphi) \) we get

\[
\pi_f(\varphi) = \pi_f(h_f \int_\Omega \varphi \, d\nu) = \int_\Omega \varphi \, d\nu \cdot \pi_f(h_f) = \int_\Omega \varphi \, d\nu \cdot h_f.
\]

**Lemma 6.4.** Let be \( \varphi_1, \varphi_2 \in W(\Omega) \) then \( \mathcal{L}_f^n (\varphi_1 \circ \sigma^n \cdot \varphi_2 \cdot h_f) = \varphi_1 \mathcal{L}_f^n (\varphi_2 h_f) \).

\[\square\]
Proof. The proof is an easy calculation. Let \( x \in \Omega \) and \( \varphi \in W(\Omega) \). Then, we have that 
\[
\mathcal{L}_f^\sigma(\varphi) = \int_{\Omega^f} \varphi(\mathbf{a}) e^{\beta(\mathbf{a}x)} d\mu(\mathbf{a}).
\]
Since \( \varphi_1 \circ \sigma^n(\mathbf{a}) = \varphi(x) \), \( \forall \mathbf{a} \in M^n \) and \( \forall x \in \Omega \) we get 
\[
\mathcal{L}_f^\sigma(\varphi_1 \circ \sigma^n \varphi_2 h_f)(x) = \int_{\Omega^f} \varphi_1 \circ \sigma^n \varphi_2 h_f(\mathbf{a}) e^{\beta(\mathbf{a}x)} d\mu(\mathbf{a}) = \varphi(x) \int_{\Omega^f} (\varphi_2 h_f)(\mathbf{a}) e^{\beta(\mathbf{a}x)} d\mu(\mathbf{a}).
\]
It follows from the definition of the correlation function that 
\[
|C_{\varphi_1 \varphi_2 \varphi_3}(n)| = \left| \int_{\Omega} (\varphi_1 \circ \sigma^n) \varphi_2 h_f \; d\gamma - \int_{\Omega} \varphi_2 h_f \; d\gamma \int_{\Omega} \varphi_3 h_f \; d\gamma \right|.
\]
Notice that \( (\mathcal{L}_f^\sigma)\nu_f = \lambda_f^n \nu_f \) and therefore the rhs above is equal to 
\[
\left| \int_{\Omega} \lambda_f^n \mathcal{L}_f^\sigma((\varphi_1 \circ \sigma^n) \varphi_2 h_f) \; d\gamma - \int_{\Omega} \varphi_2 h_f \; d\gamma \int_{\Omega} \varphi_3 h_f \; d\gamma \right|
\]
By using the lemma 6.4 and performing simple algebraic computations we get 
\[
|C_{\varphi_1 \varphi_2 \varphi_3}(n)| \leq \left( \int_{\Omega} |\varphi_1| \; d\gamma \right) \left| \lambda_f^n \mathcal{L}_f^\sigma(\varphi_2 h_f - h_f \int_{\Omega} \varphi_2 h_f \; d\gamma) \right|_0. \tag{12}
\]
We are supposing that the spectrum of \( \mathcal{L}_f^\sigma : W(\Omega) \to W(\Omega) \) consists of a simple eigenvalue \( \lambda_f > 0 \) and a subset of a disc with a radius strictly smaller than \( \lambda_f \). Set \( \tau = \sup \{ |z| : |z| < 1 \} \) and \( \tilde{\tau} = \lambda_f - \tau \). The existence of the spectral gap guarantees that \( \tau < 1 \). Let \( \pi_f \) be the spectral projection associated to eigenvalue \( \lambda_f \); then by the proposition B.4, the spectral radius of the operator \( \mathcal{L}_f^\sigma(I - \pi_f) \) is exactly \( \tau \cdot \lambda_f \). Since the commutator \( [\mathcal{L}_f^\sigma, \pi_f] = 0 \), we get \( \forall n \in \mathbb{N} \) such that \( [\mathcal{L}_f^\sigma(I - \pi_f)]^n = \mathcal{L}_f^\sigma(I - \pi_f) \). From the spectral radius formula (B.1) it follows that for each choice of \( \tilde{\tau} > \tau \) there is \( n_0 = n_0(\tilde{\tau}) \in \mathbb{N} \) so that for all \( n \geq n_0 \) we have 
\[
\| \mathcal{L}_f^\sigma(\varphi - \pi_f \varphi) \| \leq \lambda_f^n \tilde{\tau} \| \varphi \|, \quad \forall \varphi \in W(\Omega).
\]
There is a constant \( C(\tilde{\tau}) > 0 \) such that for every \( n \geq 1 \)
\[
\| \mathcal{L}_f^\sigma(\varphi - \pi_f \varphi) \| \leq C(\tilde{\tau}) \lambda_f^n \tilde{\tau} \| \varphi \|, \quad \forall \varphi \in W(\Omega).
\]
By using the lemma 6.3 and the above upper bound in the inequality (12) we obtain 
\[
|C_{\varphi_1 \varphi_2 \varphi_3}(n)| \leq \left( \int_{\Omega} |\varphi_1| \; d\gamma \right) \left| \lambda_f^n \mathcal{L}_f^\sigma(\varphi_2 h_f - h_f \int_{\Omega} \varphi_2 h_f \; d\gamma) \right|_0. \tag{12}
\]

7. Absence of the spectral gap in the Walters space

In this section we present the so-called long-range Ising model of the lattice \( \mathbb{N} \) in the thermodynamic formalism setting. The goal is to explicitly exhibit a potential in the Walters space for which the associated Ruelle operator does not have the spectral gap.

Throughout this section we assume the metric space \((M, d)\) is given by \((-1, 1), |.|\), where \(|.|\) is the modulus function and the a priori probability measure \( \nu = (1/2)[\delta_{-1} + \delta_{1}] \). Fix \( \alpha > 1 \) and consider the potential \( f : \Omega \to \mathbb{R} \) given by 
\[
f(x) = - \sum_{n \geq 2} \frac{\lambda_n x_n}{(n - 1)^\alpha}.
\]
This potential is not $\gamma$-Hölder-continuous for any $0 < \gamma \leq 1$ (see [9]). When $1 < \alpha < 2$, Dyson [12] proved that this model has spontaneous magnetization for sufficiently low temperatures. This fact for these models implies non-uniqueness of the DLR-Gibbs measures at such temperatures and also that the pressure cannot be Fréchet-differentiable on a suitable Banach space. For $\alpha = 2$, this phase transition result was proved by Fröhlich and Spencer [16]. On the other hand, when $\alpha > 2$ the potential $f$ is in the Walters class. Indeed, for any choice of $n, p \in \mathbb{N}$ we have $\var_{n+p}(f(x) + f(\sigma(x)) + \ldots + f(\sigma^{n-1}(x))) = (n + p)^{-\alpha + 1} + (n + p - 1)^{-\alpha + 1} + \ldots + p^{-\alpha + 1}$ which implies that

$$\sup_{n \in \mathbb{N}} [\var_{n+p}(f(x) + f(\sigma(x)) + \ldots + f(\sigma^{n-1}(x)))] \sim \sum_{j=p}^{\infty} j^{-\alpha + 1} \sim p^{-\alpha + 2},$$

and then the Walters condition. For this reason, in what follows we assume that $\alpha > 2$. In this case, as mentioned in the introduction, the potential $f$ belongs to an infinite-dimensional subspace of $C(\Omega)$ as defined in [8] where the pressure is Fréchet-analytic. Note that the previous computation implies that this space cannot be contained in the Hölder space.

In the statistical mechanics setting the potential $f$ is normally replaced/constructed by the absolutely uniformly summable interaction $\Phi = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} A_{n,m} \delta_{x^n \cap x^m \neq \emptyset}$, given by

$$A_{n,m} := |n - m|^\alpha \quad \text{if} \quad \{n, m\} \subset \mathbb{N} \text{ and } m \neq n;$$

$$0, \quad \text{otherwise.}$$

The relationship between the potential $f$ and the interaction $\Phi$ is described in [9] and expressed by the following equality:

$$H_\phi(x) = \sum_{A \in \mathcal{A}_n \cap \Lambda_\alpha \neq \emptyset} \Phi_A(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1} x). \quad (13)$$

Following [9, 20, 33] we construct a DLR-Gibbs measure as follows. Fix a potential $f$, and a boundary condition, which here, for convenience, will be chosen as $y = (1, 1, \ldots) \in \Omega$. Now we take any cluster point (with respect to the weak topology) of the sequence $(\nu_n^F)_{n \in \mathbb{N}}$ in $\mathcal{P}(\Omega, \mathcal{F})$, where $\nu_n^F : \mathcal{F} \to [0, 1]$ is the probability measure defined for each $F \in \mathcal{F}$ by the following expression:

$$\nu_n^F(F) = \frac{1}{Z_n} \sum_{x \in \Omega_1 \cap \sigma^y(x) = \sigma^y(y)} I_F(x) \exp(H_\phi(x)), \quad \text{where} \quad Z_n^F = \sum_{x \in \Omega_1 \cap \sigma^y(x) = \sigma^y(y)} \exp(H_\phi(x)).$$

Since $\alpha > 2$, it is well-known that the sequence $(\nu_n^F)_{n \in \mathbb{N}}$ has a unique cluster point which will be denoted by $\nu_\Omega$. A proof of this classical fact, with a dynamical system point of view, using a consequence of the Dobrushin uniqueness theorem and also the Ruelle operator formalism is presented in [9].

Our next step is to construct a probability measure $\nu_\Omega$ on the symbolic space $\hat{\Omega} \equiv \{-1, 1\}^\mathbb{Z} \equiv \{-1, 1\}^{\mathbb{Z} \cap (-\infty, 0)} \times \{-1, 1\}^\mathbb{N}$ such that

$$\nu_\Omega(F) = \nu_\Omega((-1, 1)^{\mathbb{Z} \cap (-\infty, 0)} \times F), \quad \forall F \in \mathcal{F}. \quad (14)$$

Let us denote $\text{Diag}(\mathbb{Z} \times \mathbb{Z}) \equiv \{(r, r) : r \in \mathbb{Z}\}$ and $\mathbb{M} \equiv \mathbb{Z} \times \mathbb{Z} \setminus \text{Diag}(\mathbb{Z} \times \mathbb{Z})$. We define a linear space $J \subset \mathbb{R}^\mathbb{M} \equiv \{J_j \in \mathbb{R} : (i, j) \in \mathbb{M}\}$ as being the set of points in $\mathbb{R}^\mathbb{M}$ satisfying
sup \{ \sum_j \sum_{j \neq j} |j| \} < \infty. \text{ Let } J_z \text{ and } J_{\tilde{z}} \text{ be two points in } \mathbb{J} \text{ defined by } (J_z)_{ij} = |i - j|^{-\alpha} \text{ for all } (i, j) \in \mathbb{M} \text{ and } (J_{\tilde{z}})_{ij} = (J_z)_{ij} \text{ if } i, j \in \mathbb{N} \text{ with } i \neq j \text{ and } (J_{\tilde{z}})_{ij} = 0 \text{ otherwise. For each } n \in \mathbb{N} \text{ and } J \in \mathbb{J}, \text{ we define the function } \mathcal{M}_n : \hat{\Omega} \times \mathbb{J} \rightarrow \mathbb{R} \text{ by }

\mathcal{M}_n(z, J) = \sum_{i = -n}^{n} \sum_{j \in \mathbb{Z} \setminus \{0\}} J_{ij} \cdot \sum_{j \neq j} z_J z_{\tilde{z}} j.

(15)

For any fixed } J \in \mathbb{J} \text{ and } \hat{y} = (\ldots, 1, 1, 1, \ldots) \in \hat{\Omega} \text{ we can define, similarly to above, a probability measure } \nu_{J} \text{ such that for each Borelian } F \text{ of } \hat{\Omega} \text{ we have }

\nu_{J}(F) = \frac{1}{Z_{n}^{J}} \sum_{z \in \hat{\Omega} : z = -n \cdots n} l_{f}(z) \exp(\mathcal{M}_n(z, J)), \text{ where } Z_{n}^{J} = \sum_{z \in \hat{\Omega} : z = -n \cdots n} \exp(\mathcal{M}_n(z, J)).

(16)

By straightforward computations we obtain, for each } n \in \mathbb{N} \text{ and } z \in \hat{\Omega} \text{ fixed, the following identities:}

1. \mathcal{M}_n(z, J_{0}) = H_{0}(z, z, \ldots);
2. \ Z_{n}^{J_{0}} = 2^{n+1} Z_{n}^{J};
3. l_{((-1)^{n}z \cdots z J)}(z) = l_{f}(z, z, \ldots).

Using the above three identities one can immediately see that

\nu_{J}(\{-1, 1\}^{\mathbb{Z} \setminus \{0\}} \times F) = \frac{1}{2^{n+1} Z_{n}^{J}} \sum_{z \in \hat{\Omega} : z = -n \cdots n} l_{f}(z, z, \ldots) \exp(-H_{0}(z, z, \ldots))

= \nu_{J}(F).

(17)

Recalling that } \alpha > 2, \text{ it follows from classical results about DLR-Gibbs measures that the sequence } (\nu_{J})_{\in \mathbb{N}} \text{ has a unique cluster point which we call } \nu_{\infty}. \text{ From the previous equality it is easy to conclude that } (14) \text{ is valid.

According to the definition of } \nu_{J}, \text{ for any fixed measurable set } F \text{ and } n \in \mathbb{N}, \text{ the function } \mathbb{J} \ni J \mapsto \nu_{J}(F) \text{ is Fréchet-analytic, since it is just a finite sum of analytic functions. A straightforward computation shows that for each fixed } (i, j) \in \mathbb{M} \text{ we have }

\frac{\partial}{\partial J_{ij}} Z_{n}^{J} = \sum_{z \in \hat{\Omega} : z = -n \cdots n} \exp(\mathcal{M}_n(z, J)) \frac{\partial}{\partial J_{ij}} (\mathcal{M}_n(z, J)) = \sum_{z \in \hat{\Omega} : z = -n \cdots n} \exp(\mathcal{M}_n(z, J)) \cdot z_{J} z_{\tilde{z}} j.

By multiplying and dividing the rhs above by } Z_{n}^{J} \text{ and using the definition of the Lebesgue integral we get }

\frac{\partial}{\partial J_{ij}} Z_{n}^{J} = Z_{n}^{J} \int_{\hat{\Omega}} z_{J} z_{\tilde{z}} j \psi_{n}(z).

Performing similar computations and using the quotient rule for any bounded measurable function, we have } \psi : \hat{\Omega} \rightarrow \mathbb{R}.
Before proceeding, we state the GKS-II inequality, but only in the generality required in this section. For more general cases, see [13, 22, 24, 25].

**Theorem 7.1 (GKS-II Inequality [24, 25]).** Fix $n \in \mathbb{N}$ and $\{n_1, n_2, \ldots, n_k\}$ as an arbitrary subset of $\{-n, \ldots, n\}$. If $J \in \mathbb{J}$ satisfies $J_{ij} \geq 0$ for all $(i, j) \in \mathbb{M}$, then

$$\int_{\Omega} z_{m_1} \cdots z_{m_k} \, d\nu^{n, J}_{n}(z) - \int_{\Omega} z_{m_1} \cdots z_{m_k} \, d\nu^{n, J}_{n}(z) \geq 0,$$

where $\nu^{n, J}_{n}$ denotes the probability measure defined in (16).

From now on we use $\varphi$ to denote the function $\varphi(z) = z_k$. Strictly speaking $\varphi$ is defined on $\hat{\Omega}$, but we will abuse notation and also use $\varphi(x)$ to denote the projection on the first coordinate of an element in $\Omega$. If $J \in \mathbb{J}$ is such that $J_{ij} \geq 0$ for all $(i, j) \in \mathbb{M}$ it follows from (18) and GKS-II inequality that

$$\frac{\partial}{\partial J_{ij}} \nu^{n, J}_{n}(\varphi) \bigg|_{J = J} \geq 0.$$  

This inequality implies that the mapping $\mathbb{J} \ni J \mapsto \nu^{n, J}_{n}(\varphi)$ is, coordinatewise, non-decreasing in $\mathbb{J} \cap [0, +\infty)^{\mathbb{M}}$. This monotonicity, together with the inequalities $(J_{n})_{ij} \leq (J_{m})_{ij}$ immediately implies that

$$\int_{\Omega} \varphi(z) \, d\nu^{n, J}_{n}(z) \leq \int_{\Omega} \varphi(z) \, d\nu^{n, J}_{n}(z).$$

Since $\varphi$ is a simple function taking only the values $-1$ and 1, the lhs above is (from the definition of the Lebesgue integral and the identity (17)) equal to

$$\int_{\Omega} \varphi(z) \, d\nu^{n, J}_{n}(z) = \nu^{n, J}_{n}(\{z \in \hat{\Omega} : z_1 = 1\}) - \nu^{n, J}_{n}(\{z \in \hat{\Omega} : z_1 = -1\})$$

$$= \nu^{n}_{n}(\{x \in \Omega : x_1 = 1\}) - \nu^{n}_{n}(\{x \in \Omega : x_1 = -1\})$$

$$= \int_{\Omega} \varphi(x) \, d\nu^{n}_{n}(x).$$

Replacing this last expression in the above inequality, we arrive at

$$\int_{\Omega} \varphi(x) \, d\nu^{n}_{n}(x) \leq \int_{\Omega} \varphi(z) \, d\nu^{n, J}_{n}(z).$$

One can prove that $\nu^{n, \Omega}_{n} \to \nu$ and $\nu^{n}_{n} \to \nu_{n}$ (see [13]). We note that $\nu$ is different from $\nu_{\Omega}$, defined above. By using the previous inequality, we get from the definition of the weak convergence

$$\int_{\Omega} \varphi(x) \, d\nu_{n}(x) \leq \int_{\Omega} \varphi(z) \, d\nu(z).$$
To prove that the lhs above is non-negative we will use the GKS-I inequality. Again, the statement is given only in the needed generality. Its proof as well as its more general version can be found in [13, 22, 23].

**Theorem 7.2 (GKS-I Inequality [23]).** Fix a natural number \( n \geq 1 \) and subset \( \{n_1, n_2, \ldots, n_k\} \subset \{-n, \ldots, n\} \) and \( J \in \mathbb{J} \) satisfying \( J_{ij} \geq 0 \) for all \((i, j) \in \mathcal{M}\). If \( \nu_n^{J} \) denotes the probability measure defined in (16), then

\[
\int_{\Omega} z_{m_1} \cdots z_{m_n} \cdot z_i \cdot z_j \; d\nu_n^{J}(z) \geq 0.
\]

By applying GKS-I inequality to the lhs of (19) we get

\[
0 \leq \int_{\Omega} \varphi(z) \; d\nu_n^{J}(z) \leq \int_{\Omega} \varphi(x) \; d\nu_n(x).
\]

By taking the weak limit, when \( n \to \infty \), in the second integral above and using (21), it follows that

\[
0 \leq \int_{\Omega} \varphi(x) \; d\nu_n(x) \leq \int_{\Omega} \varphi(z) \; d\nu(z).
\]

Since \( \alpha > 2 \), there is a theorem ensuring that \( \int_{\Omega} \varphi(z) \; d\nu(z) = 0 \) (see [13]). Therefore, we have proved that

\[
\int_{\Omega} \varphi(x) \; d\nu_n(x) = 0. \tag{22}
\]

To get the lower bound we are interested in, we consider for \( n \geq 1 \) the element \( J^{1,n+1} \in \mathbb{J} \), given by \( (J^{1,n+1})_{ij} = n^{-\alpha} \) if \((i, j) = (1, n + 1)\) and \((J^{1,n+1})_{ij} = 0\) otherwise. Similarly as above, by another application of the GKS-II inequality we obtain the coordinatewise monotonicity of the mapping \( \mathbb{J} \ni J \mapsto \nu_{m}^{J}(z_{n+1\Omega}) \), whenever \( m > n \); therefore, we can conclude that

\[
\int_{\Omega} z_{m+1\Omega} \; d\nu_{m}^{J^{1,n+1}}(z) \leq \int_{\Omega} z_{n+1\Omega} \; d\nu_{m}^{J}(z).
\]

Notice that the lhs above can be explicitly computed as follows (and its value is independent of \( m \)):

\[
\int_{\Omega} z_{m+1\Omega} \; d\nu_{m}^{J^{1,n+1}}(z) = \sum_{z_{n+1}=\pm 1} \sum_{z_{n}=\pm 1} z_{n+1\Omega} \exp\left( \frac{z_{n+1\Omega}}{n^\alpha} \right) \left( \sum_{z_{n+1}=\pm 1} \sum_{z_{n}=\pm 1} \exp\left( \frac{z_{n+1\Omega}}{n^\alpha} \right) \right)^{-1}
\]

\[
= \frac{2 \exp(n^{-\alpha}) - 2 \exp(n^{-\alpha})}{2 \exp(n^{-\alpha}) + 2 \exp(n^{-\alpha})}
\]

\[\tag{10}
= \tanh(n^{-\alpha}).
\]

On the other hand, by using the previous equality and (17) for any \( m > n \) we get

\[
\tanh(n^{-\alpha}) \leq \int_{\Omega} z_{m+1\Omega} \; d\nu_{m}^{J^{1,n+1}}(z) \leq \int_{\Omega} x_{m+1x} \; d\nu_{m}^{J}(x) = \int_{\Omega} (\varphi \circ \sigma^n) \varphi \; d\nu_m.
\]

By Taylor expanding the hyperbolic tangent and taking the weak limit when \( n \to \infty \), for some constant \( C > 0 \) we get the following inequality:

\[
\tanh(n^{-\alpha}) \leq \int_{\Omega} z_{m+1\Omega} \; d\nu_{m}^{J^{1,n+1}}(z) \leq \int_{\Omega} x_{m+1x} \; d\nu_{m}^{J}(x) = \int_{\Omega} (\varphi \circ \sigma^n) \varphi \; d\nu_m.
\]
\[
\int_{\Omega} (\varphi \circ \sigma^n) \varphi \, d\nu_N \geq \tanh(n^{-\alpha}) \geq \frac{C}{|n|^{\alpha}}.
\]

Piecing together the previous inequality and (22) we finally arrive at
\[
\frac{C}{|n|^{\alpha}} \leq \int_{\Omega} (\varphi \circ \sigma^n) \varphi \, d\nu_N = \int_{\Omega} \varphi \, d\nu_N \int_{\Omega} \varphi \, d\nu_N = C_{\varphi, \varphi, \nu}(n). \tag{23}
\]

It was shown in [9] that \(d\mu_f \equiv h_f \, d\nu_f\), where \(\nu_f\) and \(h_f\) is given by theorem 4.4, belongs to \(G^{DLR}(f)\). The authors have also shown that, for \(\alpha > 2\), the set \(G^{DLR}(f)\) is a singleton and, therefore, \(\mu_f = \nu_f\). This fact, together with the continuity of \(h_f\) and the previous inequality, shows that \(C_{\varphi, \varphi, \nu}(n)\) cannot decay exponentially quickly. Since \(\varphi(x) = x_1\) is in the Walters class, it follows from theorem 6.2 that \(\mathcal{L}_{\varphi}\) does not have the spectral gap property.

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Appendix A. Analyticity on Banach spaces

**Definition A.1.** Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be Banach spaces and \(U\) an open subset of \(X\). For each \(k \in \mathbb{N}\), a function \(F : U \rightarrow Y\) is said to be \(k\)-differentiable in \(x\) if for \(j \in \{1, \ldots, k\}\), there exists a \(j\)-linear bounded transformation \(D^jF(x) : X^j \rightarrow Y\) such that
\[
D^{j-1}F(x + v)(v_1, \ldots, v_{j-1}) - D^{j-1}F(x)(v_1, \ldots, v_{j-1}) = D^jF(x)(v_1, \ldots, v_j) + o_j(v)
\]
where \(o_j : X \rightarrow Y\) is such that \(\lim_{v \rightarrow 0} \|o_j(v)\|_Y/\|v\|_X = 0\).

We can say that \(F\) has derivatives of all orders in \(U\), if for any \(k \in \mathbb{N}\), and any \(x \in U\), \(F\) is \(k\)-differentiable in \(x\).

**Definition A.2.** Let \(X\) and \(Y\) be Banach spaces and \(U\) an open subset of \(X\). A function \(F : U \rightarrow Y\) is called analytic on \(U\), when \(F\) has derivatives of all orders in \(U\), and for each \(x \in U\), there exists an open neighborhood \(U_x\) of \(x\) in \(U\) such that for all \(v \in U_x\), we have it that
\[
F(x + v) - F(x) = \sum_{j=1}^{\infty} \frac{1}{n!} D^jF(x)v^j,
\]
where \(D^jF(x)v^j = D^jF(x)(v, \ldots, v)\) and \(D^jF(x)\) is the \(j\)-th derivative of \(F\) in \(x\).

If \(F : U \rightarrow Y\) is analytic on \(U\), then for each \(n \in \mathbb{N}\), the Taylor expansion of order \(n\) is
\[
F(x + v) = F(x) + D^1F(x)v + \frac{D^2F(x)v^2}{2} + \frac{D^3F(x)v^3}{6} + \ldots + \frac{D^nF(x)v^n}{n!} + o_{n+1}(v), \tag{A.1}
\]
where \(o_{n+1}(v) = \sum_{j=n+1}^{\infty} (1/j!)|D^jF(x)v^j|\) satisfies \(\lim_{v \rightarrow 0} \|o_{n+1}(v)\|_Y/\|v\|_X^n = 0\).
Appendix B. Some background on spectral theory

In this section we list some classical results of spectral theory. For more details and proofs, see [27]. Let $X$ be a Banach space and $T : X \to X$ a bounded operator. We define the spectrum of the operator $T$ by

$$\text{spec}(T) = \{ \lambda \in \mathbb{C}; (\lambda I - T)^{-1} \text{ do not exists} \}.$$  

The resolvent set $\rho(T)$ of $T$ is defined as the complement of $\text{spec}(T)$. The resolvent set of a bounded operator is an open set, while the spectrum is a compact set. The spectral radius of the operator $T$ is defined as

$$r = \sup_{\lambda \in \text{spec}(T)} |\lambda|.$$  

The spectral radius has the following characterization:

$$r = \lim_{n \to \infty} \|T^n\|^{1/n}. \quad (B.1)$$

It is also known that $\text{spec}(T) \subseteq B(0, r(T))$ and $\text{spec}(T) = \text{spec}(T^*)$, where $T^* : X^* \to X^*$ is the adjoint of $T$.

**Definition B.1.** Let $T : X \to X$ be a bounded linear operator and $\gamma$ a rectifiable Jordan curve that lies in $\rho(T)$. Then, we define the spectral projection $\pi_T : X \to X$ as follows:

$$\pi_T = \frac{1}{2\pi i} \int_\gamma (\lambda I - T)^{-1} d\lambda.$$  

**Remark B.2.** If the interior of $\gamma$ lies in the interior of $\rho(T)$, then $\pi_T = 0$. On the other hand, if $\text{spec}(T)$ lies entirely in the interior of $\gamma$, then $\pi_T = \text{Id}$.

**Proposition B.3.** If $T : X \to X$ is bounded, then $\pi_T$ is a projection, i.e. $\pi_T^2 = \pi_T$. Moreover, $\pi_T$ commutes with $T$.

A subset of $\text{spec}(T)$ which is both open and closed in $\text{spec}(T)$ is called a spectral set. Let $\Sigma(T) \subseteq \text{spec}(T)$ be a spectral set, and $\gamma$ a rectifiable Jordan curve which lies in $\rho(T)$ containing $\Sigma(T)$ in its interior. Denote by $\pi_{T,\Sigma(T)}$ the spectral projection associated with $T$ and $\gamma$, i.e.

$$\pi_{T,\Sigma(T)} = \frac{1}{2\pi i} \int_\gamma (\lambda I - T)^{-1} d\lambda,$$

where $\gamma$ is any rectifiable Jordan curve surrounding the spectral set $\Sigma(T)$ completely contained in $\rho(T)$ and such that any other point in the spectrum is outside $\gamma$.

We use the notation $X_{\Sigma(T)} = \pi_{T,\Sigma(T)} X$ and $T_{\Sigma(T)} = T|_{X_{\Sigma(T)}}$.

**Proposition B.4.** Let $\Sigma(T)$ be a spectral set of $\text{spec}(T)$; then $\text{spec}(T_{\Sigma(T)}) = \Sigma(T)$.

**Proposition B.5.** Let $\pi_1, \pi_2 : X \to X$ be linear projections. Then, there exists $\epsilon > 0$ such that if $\|\pi_1 - \pi_2\| < \epsilon$ then $\pi_1$ and $\pi_2$ have the same rank, i.e. $\dim \pi_1(X) = \dim \pi_2(X)$.

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