STABILITY ESTIMATES FOR THE X-RAY TRANSFORM ON SIMPLE
ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

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Abstract. We study the normal operator to the geodesic X-ray transform on functions
in the setting of simple asymptotically hyperbolic manifolds. We construct a parametrix
for the normal operator in the 0-pseudodifferential calculus and use it to show a stability
estimate.

1. Introduction

In this paper we consider the stability of geodesic X-ray transform
(1.1) \[ If(\gamma) = \int_\gamma f \, ds, \]
where \( \gamma \) is a unit speed geodesic, in the setting of asymptotically hyperbolic manifolds.
Our goal is to establish that small perturbations of the X-ray transform cannot originate
from large perturbations of the unknown function \( f \), and find appropriate spaces to measure
such perturbations. Sharp stability estimates are known for the X-ray transform in \( \mathbb{R}^n \) (see
[Nat86]) and stability has also been studied extensively on compact manifolds with boundary
under a variety of assumptions (see e.g. [Muk77], [MR78], [Sha94], [SU04], [FSU08], [UV16],
[HU18], [AS20] and [IM] for a survey).

Introducing our geometric setting, an \( n + 1 \)-dimensional Riemannian manifold \((M, g)\) will
be called asymptotically hyperbolic (AH) if \( \bar{M} \) is the interior of a smooth compact manifold
with boundary \( M \) such that for some (and thus any) boundary defining function \( x \in C^\infty(M) \)
it is the case that \( \bar{g} := x^2 g \) extends to a \( C^\infty \) Riemannian metric on \( M \) with \( ||dx||_{\partial M}^2 \equiv 1 \).
Recall that \( x \) is a boundary defining function if \( x|_{\partial M} \equiv 0 \), \( dx|_{\partial M} \not\equiv 0 \) and \( x > 0 \) on \( M \). AH
manifolds generalize the Poincaré model of hyperbolic space. An AH metric \( g \) determines
a conformal class of metrics on \( \partial M \), called the conformal infinity and given by \( [x^2 g]_{T\partial M} \).
As shown in [Maz86], AH manifolds are geodesically complete and their sectional curvatures
approach \( -\frac{1}{x^2} \) as \( x \to 0 \). Moreover, any unit speed geodesic \( \gamma(t) \) in an AH
manifold which eventually exits every compact set approaches a boundary point as \( t \to \infty \),
orthogonally to the boundary, and \( x \circ \gamma(t) = O(e^{-t}) \). One of the issues when studying the
X-ray transform on AH manifolds is that due to completeness the integral in (1.1) might
not converge unless some conditions are imposed on \( \gamma \) and \( f \); for instance, provided \( \gamma \)
does not spend infinite time in any compact set \( K \subset \bar{M} \) (i.e. it is not trapped), it suffices to
assume that \( f \in |\log x|^{\alpha} C^0(M), \alpha < -1 \). Parameterizing the space of geodesics in an AH
manifold is also a non-trivial task; this point will be discussed in more detail in Section 2.
In this paper we are concerned with simple AH manifolds, defined in [GGS+19]; those are by
definition non-trapping (i.e. they contain no trapped geodesics) and they have no boundary
conjugate points, that is, there exists no non-trivial Jacobi field \( Y \) along any unit-speed
geodesic such that \( \lim_{t \to \pm \infty} |Y(t)|_g \to 0 \). Using work of Knieper in [Kni18], the authors in
[GGS+19] showed that on a simple AH manifold there exist no interior conjugate points in
the usual sense, whereas in [EG20] it was shown that non-trapping AH manifolds with no conjugate points in the usual sense may have boundary conjugate points, i.e. they are not simple in general. A simple AH manifold is necessarily simply connected and diffeomorphic to a ball, via the exponential map at any point, though it might exhibit positive curvature. Simple AH manifolds are a natural geometric setting for the study of inverse problems such as tensor tomography and boundary rigidity. The study of those problems in the setting of AH manifolds was initiated in [GGS+19], and the present work is meant as a step in this direction.

Our approach to stability is mainly inspired by the works of Stefanov-Uhlmann [SU04] and of Berenstein-Casadio Tarabusi ([BC91]), both of which analyze the normal operator to the X-ray transform in different settings. On a simple compact manifold with boundary \((X, \tilde{g})\) (i.e. non-trapping, with no conjugate points and with strictly convex boundary), which is the setting of [SU04], the normal operator is given by

\[ N_{\tilde{g}} = I^* I \]

where \(I^* I \) is formal, however \(I^* \) can be interpreted as a formal adjoint for \(I\) using suitable inner products and function spaces (this is discussed in Section 2 for the AH case). In [SU04] it was shown that \(N_{\tilde{g}}\) extends to an elliptic pseudodifferential operator of order \(-1\) on \(\tilde{X}\), where \(\tilde{X}\) is an open domain slightly larger than \(X\) and of the same dimension, such that its closure is still simple. The ellipticity of \(N_{\tilde{g}}\) then implied the existence of a left parametrix (inverse up to compact error), and this yielded a stability estimate of the form

\[ \|u\|_{L^2(X)} \leq C\|N_{\tilde{g}} u\|_{H^1(\tilde{X})}, \quad u \in L^2(\tilde{X}), \supp u \subset X, \]

using injectivity of \(I\) on simple manifolds, which had already been established in the ’70s.

The construction of the normal operator carries over in the same way on hyperbolic space and it is well defined on \(C^\infty\) functions of suitable decay at infinity; the authors of [BC91] derived explicit inversion formulas for it using the spherical Fourier transform for radial distributions on hyperbolic space (see [Hel99]). Although they did not explicitly state a stability estimate, the estimate of Theorem 1 below for the special case of hyperbolic space follows immediately from their work using the machinery of the 0-calculus, which we will discuss shortly.

In [GGS+19] it was shown that on a simple AH manifold \((\tilde{M}, g), I\) is injective on \(x C^\infty(M)\), where \(x \in C^\infty(M)\) is a boundary defining function (in fact it is shown there that one can allow for trapped geodesics as well, provided that the trapped set is hyperbolic for the geodesic flow). The method of proof relied on showing that functions a priori in \(x C^\infty(M)\) which lie in the nullspace of \(I\) actually vanish to infinite order at \(\partial \tilde{M}\), and it does not yield stability. The main result of the present work is a stability estimate analogous to (1.2) on simple AH manifolds and a strengthened injectivity result. The normal operator on a simple AH manifold \((\tilde{M}, g)\) is defined similarly to the case of simple compact manifolds with boundary: we let

\[ N_{g} f = I^* I f(z) = \int_{S^* \tilde{M}} I f(\xi) d\mu_g(\xi), \quad f \in \dot{C}^\infty(M), \quad z \in \tilde{M}, \]
where $\hat{C}^\infty(M)$ denotes smooth functions vanishing to infinite order at the boundary and $d\mu_g$ is the measure induced on the fibers of $S^*\hat{M}$ by $g$, as before. In our setting, $\mathcal{N}_g$ turns out to be a well-behaved object within the framework of the $0$-calculus of pseudodifferential operators of Mazzeo and Melrose, which was introduced in [MM87] to analyze a modified resolvent of the Laplacian on AH spaces. $0$-pseudodifferential operators naturally act: we let $dV_g$ restrict to coordinates on $\partial M$, and for $h \in H^\delta(M, dV_g)$ then $H^\delta(M, dV_g)$ is defined by interpolation and for $s < 0$ by duality. Fixing vector fields $V_j \in \mathcal{V}_0$ in coordinate patches we can make sense of the norms $\| \cdot \|_{x^s H^\delta_0(M, dV_g)}$. As we will show in Section 3, it turns out that $I$ and $\mathcal{N}_g$ can be extended to operators on $x^\delta L^2(M, dV_g)$ for $\delta > -n/2$, bounded into appropriate weighted Sobolev spaces; specifically for $\mathcal{N}_g$ we have that it is bounded $x^\delta L^2(M, dV_g) \rightarrow x^{\delta'} H^1_0(M, dV_g)$ provided $\delta' \leq \delta$, $\delta > -n/2$ and $\delta' < n/2$. The main result of the paper is as follows:

Theorem 1. Let $(\hat{M}^{n+1}, g)$ be a simple AH manifold, $n \geq 1$. Then $I$ and $\mathcal{N}_g = I^{\star} I$ are injective on $x^\delta L^2(M, dV_g)$, $\delta > -n/2$. Moreover, one has the stability estimate:

$$\|u\|_{x^\delta H^\delta_0(M, dV_g)} \leq C\|\mathcal{N}_g u\|_{x^\delta H^{\delta+1}_0(M, dV_g)}, \quad \delta \in (-n/2, n/2), \quad s \geq 0.$$ 

Note that $x^C C^\infty(M) \subset x^\delta L^2(M, dV_g)$ provided $\delta < 1 - n/2$, so Theorem 1 includes the injectivity result of GGS+19 on simple AH manifolds as a special case. However, that result is used in an essential way in the proof, similarly to the way the injectivity of $I$ on simple compact manifolds with boundary was used to derive (1.2) in SU04.

As already mentioned, in the case of hyperbolic space a stability estimate as in Theorem 1 follows immediately from the work of BC91; moreover, the inversion of the hyperbolic Radon transform on the two dimensional hyperbolic space has been numerically implemented in a stable manner ([LP97], [FKL+00]). In the AH setting (under some assumptions) stability of the local X-ray transform follows from work in a forthcoming paper by C. Robin Graham and the author (see [Ept20, Chapter 1]).

We briefly outline the idea of the proof of Theorem 1. As we show in Section 4, $\mathcal{N}_g$ is an elliptic pseudodifferential operator in $\Psi^{-1, n, n}_0(M)$ in the large $0$-calculus (that is, it is a pseudodifferential operator of order $-1$ whose Schwartz kernel vanishes to order $n$ at the side faces of the $0$-stretched product, see Section 3). Its model operator can be identified with $\mathcal{N}_h$, where $h$ is the hyperbolic metric on the Poincaré ball; using the explicit inversion formulas for $\mathcal{N}_h$ derived in BC91 and methods developed in MM87 and Maz91a we construct a left parametrix for $\mathcal{N}_g$. In MM87 and Maz91a parametrices were constructed for elliptic 0 and edge differential operators, whereas here we apply those techniques to construct a parametrix for a pseudodifferential operator. We mention that Fredholm properties of certain classes of $0$-pseudodifferential operators were also studied in Lau03. The parametrix is used in two
ways: firstly, one obtains an estimate
\begin{equation}
\|u\|_{x^\delta H_0^s(M,dV_g)} \leq C \left( \|\mathcal{N}_g u\|_{x^\delta H_0^{s+1}(M,dV_g)} + \|K u\|_{x^\delta H_0^s(M,dV_g)} \right)
\end{equation}
for $\delta \in (-n/2, n/2)$ and $s \geq 0$, where $K : x^\delta H_0^s(M,dV_g) \to x^\delta H_0^s(M,dV_g)$ is a compact operator. Next, using the Mellin transform and the parametrix it can be shown that any function $u \in x^\delta L^2(M,dV_g)$ in the nullspace of $\mathcal{N}_g$, where $\delta > -n/2$, is smooth in $M$ and has a polyhomogeneous expansion at $\partial M$, vanishing there to order at least $n$. The author is indebted to Rafe Mazzeo for showing him this argument, which is similar in spirit to the constructions of polyhomogeneous expansions for elements in the nullspace of elliptic edge differential operators in \cite{Maz91b} §7, also see \cite{Maz91a}. In \cite{GGS19}, Proposition 3.15 it is shown that if $u \in xC^\infty(M)$ lies in the nullspace of $I$ then $u$ vanishes to infinite order at $\partial M$, and one checks that the proof also works for $u$ a priori assumed polyhomogeneous and vanishing to order at least 1 at $\partial M$. Since the nullspace of $\mathcal{N}_g$ agrees with that of $I$, it follows that $u$ is in the nullspace of the latter and polyhomogeneous, hence it vanishes to infinite order at $\partial M$. Once this has been established, the injectivity argument in \cite{GGS19} using Pestov identities applies to conclude that $u \equiv 0$. Finally the injectivity of $\mathcal{N}_g$ together with (1.3) yields Theorem 1 using a standard functional analysis result.

It would be interesting to explore whether stability still holds in the AH setting when one relaxes the simplicity assumption. In the compact manifold with boundary setting, presence of conjugate points in the interior of a compact Riemannian surface causes stability to fail in dimension 2 \cite{SU12, MS15}, and it is natural to expect an analogous behavior in the AH setting. However, in dimension 3 and higher, additional geometric assumptions can allow for stability even if there are conjugate points \cite{UV16, HU18} so it is likely that analogous results hold on AH manifolds. It would be especially interesting to investigate whether stability or instability holds in the presence of boundary conjugate points. It would also be interesting to study stability in the presence of trapped geodesics; as already mentioned, in the case when the trapped set is hyperbolic for the geodesic flow, injectivity of $I$ on $xC^\infty(M)$ is known by \cite{GGS19} and stability can be shown on compact manifolds with strictly convex boundary, no conjugate points and hyperbolic trapped set (see \cite{Gui17}).

The paper is organized as follows: in Section 2 we provide some background on the geodesic flow and the X-ray transform on AH manifolds following \cite{GGS19}. Section 3 contains background material on the 0-geometry and 0-calculus that will be needed later. In Section 4 we use the form of the distance function on a simple AH manifold (Proposition 4.2) to show that the normal operator $\mathcal{N}_g$ is an elliptic pseudodifferential operator in the 0-calculus. In Section 5 we identify the model operator of $\mathcal{N}_g$, which is invertible, as shown in \cite{BC91}. Finally, in Section 6 we construct a parametrix for $\mathcal{N}_g$, use it to show boundary regularity for elements in its nullspace, and prove Theorem 1. Throughout the paper we use Einstein notation, with Latin indices running from 0 to $n$ and Greek indices from 1 to $n$.

2. GEODESIC FLOW OF AH MANIFOLDS AND THE X-RAY TRANSFORM

In this section we recall facts related to the geodesic flow and the X-ray transform on AH manifolds, which were analyzed in \cite{GGS19}, and show a lemma which will be used in Section 4 to prove a mapping property for the X-ray transform. In this section $(M^{n+1}, g)$ is a non-trapping AH manifold, with $M$ the interior of a compact manifold with boundary $\partial M$.

Each representative $h$ in the conformal infinity of $(\hat{M}, g)$ determines a boundary defining function $x$ for $\partial M$, called geodesic boundary defining function associated to $h$, such that $x^2 g \big|_{\partial M} = h$ and $\|dx\|_{x^2 g} = 1$ near $\partial M$. Then via the flow of its gradient, $x$ induces a
product decomposition of a collar neighborhood of \( \partial M \) as \([0, \varepsilon)_x \times \partial M\), in terms of which the metric is written near \( \partial M \) in normal form

\[
g = \frac{dx^2 + h_x}{x^2},
\]

where \( h_x \) is a smooth 1-parameter family of metrics on \( \partial M \) satisfying \( h_0 = h \). Choosing coordinates \( g^\alpha \) for \( \partial M \) near a boundary point we can write \( g = \frac{dx^2 + (h_x)_{\alpha \beta} dy^\alpha dy^\beta}{x^2} \).

Parametrizing the space of geodesics on \( M \) is more involved than, e.g. on a compact manifold with boundary, due to their behavior near \( \partial M \) and the fact that they have infinite length. As shown in \([GGS^{+}19]\), it can be done by introducing an appropriate extension of the unit cosphere bundle \( S^*M = \{(z, \xi) \in T^*M : |\xi|_g = 1\} \) down to \( \partial M \). Recall that Melrose’s b-cotangent bundle \( bT^*M \) (see \([Mel93]\)) is a smooth bundle over \( M \) with natural projection \( \pi \), canonically isomorphic with \( T^*M \) over \( M \) and trivialized locally near the boundary by \((dx/x, dy^1, \ldots, dy^n)\). Viewed as a subset of \( (bT^*M)^0 \), \( S^*M \) can be written near \( \partial M \) as \( \{(z, \xi = \overline{\zeta} dx/x + \eta_\alpha dy^\alpha) \in (bT^*M)^0 : \overline{\zeta} + x^2|\eta|_{h_x}^2 = 1\} \). Thus the closure of \( S^*M \) in \( bT^*M \) is a smooth embedded non-compact submanifold of \( bT^*M \) with disconnected boundary; we denote it by \( S^*M \).

The Hamiltonian vector field \( X \) on \( S^*M \) associated with the metric Lagrangian \( L_g = |\xi|^2_g/2 \) can be written as \( X = x\overline{X} \), where \( \overline{X} \) extends to be smooth on \( S^*M \) and transversal to its boundary: in coordinates it takes the form

\[
\overline{X} = \overline{\zeta} \partial_x + x h^\alpha \partial_\alpha - (x|\eta|_{h_x}^2 + \frac{1}{2}x^2|\xi|_{h_x}^2) \partial_\xi - \frac{1}{2}x \partial_y^\alpha |\eta|_{h_x}^2 \partial_{h_\alpha}.
\]

The flow of \( \overline{X} \) is incomplete and, since \( X \) and \( \overline{X} \) are related by multiplication by a scalar function, their integrals curves in \( S^*M \) agree up to reparametrization. Orbits of the flow of \( \overline{X} \) can be parametrized by their “incoming” covector, that is, each orbit can be identified with its intersection with the connected component of \( \partial S^*M \) on which \( \overline{X} \) is inward pointing. This component, \( \partial_- S^*M := \{(z, \xi = \overline{\zeta} dx/x + \eta_\alpha dy^\alpha) \in bT^*M : \overline{\zeta} = 1\} \) is often referred to as the incoming boundary. The definition of the outgoing boundary \( \partial_+ S^*M \) is analogous, except \( \overline{\zeta} = -1 \) there. Both of those sets are invariant subsets of \( bT^*M \) independent of the choice of coordinates and of \( g \). Given a choice of conformal representative (which induces a geodesic boundary defining function), \( \partial_- S^*M \) can be identified with \( T^*\partial M \) via \( x^{-1}dx + \eta_\alpha dy^\alpha \leftrightarrow \eta_\alpha dy^\alpha \).

The unit cosphere bundle \( S^*M \) has a natural measure \( d\lambda \) called the Liouville measure, induced by the restriction to \( S^*\tilde{M} \) of the \( 2n+1 \) form \( \lambda = \alpha \wedge (da)^n \), with \( \alpha \) the tautological 1-form on \( T^*\tilde{M} \). This measure decomposes as \( d\lambda = dV_g d\mu_g \), where \( d\mu_g \) is the measure induced by \( g \) on each fiber of \( S^*\tilde{M} \) and \( dV_g \) is the Riemannian volume density on \( \tilde{M} \). As shown in \([GGS^{+}19] \) Lemma 2.2], \( x d\lambda \) extends from \( S^*\tilde{M} \) to a smooth measure on \( S^*M \). Moreover, \( \iota_X \lambda \) extends to a smooth \( 2n \)-form on \( S^*M \), which restricts to a volume form on \( \partial_- S^*M \); the latter agrees with the canonical volume form on \( T^*\partial M \) (induced by the symplectic form there) under the identification described above. We will denote the corresponding measure on \( \partial_- S^*M \) by \( d\lambda_\partial \).

Now let \( f \in C_c^\infty(S^*\tilde{M}) \) and \( \varphi_t \) be the flow of the Hamiltonian vector field \( X \) on \( S^*\tilde{M} \), which is complete. We define the X-ray transform

\[
If(z, \xi) = \int_{-\infty}^{\infty} f(\varphi_t(z, \xi)) dt \in C^\infty_X(S^*\tilde{M}),
\]
where the space $C^\infty_X(S^\ast \hat{M})$ consists of smooth functions on $S^\ast \hat{M}$ constant along the orbits of $X$. Since $C^\infty_c(\hat{M})$ can be naturally viewed as a subset of $C^\infty_c(S^\ast \hat{M})$ via pullback, (2.1) reduces to the usual X-ray transform on $C^\infty_c(M)$, viewed as an element of $C^\infty_X(S^\ast \hat{M})$. Now as we mentioned before the vector field $\vec{X} = x^{-1}X$ extends to be smooth on $S^\ast M$ and transverse to $\partial S^\ast M$. This implies that any $u \in C^\infty_X(S^\ast \hat{M})$ extends smoothly to $S^\ast M$ down to $\partial_\pm S^\ast M$: by transversality, the flow of $\vec{X}$ running forward and backward can be used to identify a neighborhood of any point in $\partial_\pm S^\ast M$ respectively with a subset of $[0, \varepsilon) \times \partial_\pm S^\ast M$; then in terms of this decomposition $u$ is independent of $t$ and thus extends smoothly down to $t = 0$. Therefore the restriction $u|_{\partial_\pm S^\ast M} \in C^\infty(\partial_\pm S^\ast M)$ is well defined; conversely, any function in $C^\infty(\partial_\pm S^\ast M)$ can be extended off of $\partial_\pm S^\ast M$ to be constant along the orbits of $\vec{X}$ in $S^\ast M$, and hence also those of $X$ in $S^\ast \hat{M}$, thus yielding an element of $C^\infty_X(S^\ast \hat{M})$. This discussion implies that we have an isomorphism

$$\tag{2.2} C^\infty_X(S^\ast \hat{M}) \to C^\infty(\partial_\pm S^\ast M)$$

and both spaces are also isomorphic to $C^\infty_X(S^\ast M) = C^\infty(S^\ast M) \cap \ker \vec{X}$. Due to these facts, (2.1) can also be regarded as an element of $C^\infty_X(S^\ast M)$, and of $C^\infty(\partial_\pm S^\ast M)$ upon restricting. The range of $I$ is actually smaller than $C^\infty_X(S^\ast M)$ whenever acting on $C^\infty_c(S^\ast \hat{M})$ (or $C^\infty_c(M)$): by the discussion on short geodesics in [GGS+19, Section 2.2], given any compact set $K \subset S\hat{M}$ there exists a compact set $K' \subset \partial_\pm S^\ast M$ such that any integral curve of $\vec{X}$ starting at $(z, \xi) \not\in K'$ does not intersect $K$. Moreover, given a compact $K' \subset \partial_\pm S^\ast M$, the union of all integral curves of $\vec{X}$ starting at $K'$ forms a compact subset of $S^\ast M$. Thus for $f \in C^\infty_c(S^\ast \hat{M})$ (or $f \in C^\infty_c(M)$) (2.1) can be rewritten as

$$If(z, \xi) = \int_0^{\tau_+(z, \xi)} f(\varphi_\tau(z, \xi)) \frac{d\tau}{\nu_\tau(z, \xi)} \in C^\infty_c(S^\ast M).$$

One can identify a formal adjoint $I^*$ of $I$ on appropriate function spaces using suitably chosen inner products. By [GGS+19, Lemma 3.6], there is an analog of Santaló’s formula:

$$\tag{2.3} \int_{S^\ast \hat{M}} f \, d\lambda = \int_{\partial_\pm S^\ast M} If \, d\lambda_\partial, \quad f \in C^\infty_c(S^\ast \hat{M}).$$

Note that this implies that $I$ also extends continuously as an operator $I : L^1(S^\ast \hat{M}; d\lambda) \to L^1(\partial_\pm S^\ast M; d\lambda_\partial)$ (where the isomorphism (2.2) is used implicitly). We define an inner product on $C^\infty_c(S^\ast M)$: for $u_1, u_2 \in C^\infty_c(S^\ast M)$ let

$$\langle u_1, u_2 \rangle_\partial := \int_{\partial_\pm S^\ast M} u_1 \overline{u_2} \, d\lambda_\partial,$$

where on the right hand side $u_1, u_2$ are restricted to $\partial_\pm S^\ast M$; we will generally not write this restriction explicitly. Now consider the X-ray transform viewed as an operator $I : C^\infty_c(M) \to$
$C^\infty_{c,X}(S^*M)$ and define the operator $I^* : C^\infty_{c,X}(S^*M) \to C^\infty(M)$ by
\[
I^*u(z) = \int_{S^*_z M} u \, d\mu_g, \quad u \in C^\infty_{c,X}(S^*M).
\]

Considering real valued functions $f \in C^\infty_c(M)$ and $u \in C^\infty_{c,X}(S^*M)$, we use (2.3) to compute
\[
\langle u, If \rangle_\theta = \int_{\partial_- S^*M} u If \, d\lambda_\theta = \int_{\partial_- S^*M} I(uf) \, d\lambda_\theta
\]
\[
= \int_{S^*_\lambda M} uf \, d\lambda = \int_M \left( \int_{S^*_z M} uf \, d\mu_g \right) f(z) \, dV_g(z) = \langle I^*u, f \rangle_{L^2(M,dV_g)}.
\]

This computation implies that with the stated inner products and function spaces $I^*$ is a formal adjoint for $I$.

We will later need to consider the X-ray transform and the normal operator $N_g = I^*I$ acting on functions that live in weighted $L^2$ spaces. The target space of $I$ will also have to be an appropriately weighted $L^2$ space and as will become apparent soon it is more natural for this discussion to view $If$ as a function on $\partial_- S^*M$. Restriction to $\partial_- S^*M$ induces an isometry between $C^\infty_{c,X}(S^*M)$ and $C^\infty_c(\partial_- S^*M)$ with respect to the inner product $\langle \cdot, \cdot \rangle_\theta$ and the $L^2(\partial_- S^*M; d\lambda_\theta)$ inner product respectively, so (2.4) can also be rewritten as
\[
\langle u, If \rangle_{L^2(\partial_- S^*M,d\lambda_\theta)} = \langle I^*u, f \rangle_{L^2(M,dV_g)}, \quad f \in C^\infty_c(M), \quad u \in C^\infty_{c,X}(S^*M).
\]

By [GGS+19] Lemma 3.8 $I$ extends to a bounded operator $I : |\log x|^{-\beta}L^2(S^*\tilde{M}; d\lambda) \to L^2(\partial_- S^*M; d\lambda_\theta)$ provided $\beta > 1/2$. This also implies that $I : |\log x|^{-\beta}L^2(M,dV_g) \to L^2(\partial_- S^*M; d\lambda_\theta)$ is bounded. Hence (2.5) implies that $I^*$ extends to a bounded operator $I^* : L^2(\partial_- S^*M; d\lambda_\theta) \to |\log x|^\beta L^2(M,dV_g)$ for $\beta > 1/2$ (where the action of $I^*$ on a function $u \in L^2(\partial_- S^*M; d\lambda_\theta)$ is understood as an action on the extension of $u$ to $S^*M$ so that it is constant along the orbits of $X$). Thus (2.5) is valid for $u \in L^2(\partial_- S^*M; d\lambda_\theta)$ and $f \in |\log x|^{-\beta}L^2(M,dV_g)$.

Moreover, the normal operator is bounded
\[
N_g = I^*I : |\log x|^{-\beta}L^2(M,dV_g) \to |\log x|^\beta L^2(M,dV_g), \quad \beta > 1/2.
\]

Using the microlocal properties of $N_g$ that we prove in Section 4 and Lemma 2.1 below, we will obtain extensions of $I$ and $I^*I$ to larger spaces of functions, in Corollary 4.5.

The following lemma relates a weighted $L^2$ norm of functions in $C^\infty_{c,X}(S^*M)$ with a weighted $L^2$ norm of their restriction to $\partial_- S^*M$. We set $\langle \eta \rangle_\theta^\beta := \sqrt{1 + |\eta|^2_\theta}$.

**Lemma 2.1.** Let $\delta < 0$. Then there exists a $C = C_\delta > 0$ such that if $u \in C^\infty_{c,X}(S^*M) \cap x^\delta L^2(S^*\tilde{M}; d\lambda)$ one has, using the isomorphism (2.2),
\[
\frac{1}{C} \|u\|_{\langle \eta \rangle_\theta^{-\delta}L^2(\partial_- S^*M;d\lambda_\theta)} \leq \|u\|_{x^\delta L^2(S^*\tilde{M}; d\lambda)} \leq C \|u\|_{\langle \eta \rangle_\theta^{-\delta}L^2(\partial_- S^*M;d\lambda_\theta)} < \infty.
\]

**Proof.** First note that $C^\infty_{c,X}(S^*M) \cap x^\delta L^2(S^*\tilde{M}; d\lambda) \neq \emptyset$ for $\delta < 0$; indeed, let $f \in C^\infty_c(M)$, implying that $u = If \in C^\infty_{c,X}(S^*M) \subset C^\infty_{c,X}(S^*M)$. Since $x d\lambda$ is a smooth measure on $S^*M$, one sees that $x^{-\delta}C^\infty_{c,X}(S^*M) \subset L^2_{loc}(S^*M; d\lambda)$, implying the claim. Now by (2.3), since
implies that

\[ I(x^{-2\delta})(z, \xi) = \int_0^{\tau_+(z, \xi)} x^{-1-2\delta} \partial_\tau(z, \xi) d\tau = \int_0^{\tau_+(z, \xi)} (|\eta|^{-1}_h \sin(\alpha(z, \xi)(\tau)) + O(|\eta|^{-2}_h))^{-1-2\delta} d\tau \]

\[ = \int_0^\pi (|\eta|^{-1}_h \sin(s) + O(|\eta|^{-2}_h))^{-1-2\delta} \frac{ds}{|\eta|_h + O(1)} \]

\[ = |\eta|^{-2\delta}_h \int_0^\pi (\sin(s) + O(|\eta|^{-1}_h))^{-1-2\delta} \frac{ds}{1 + O(|\eta|^{-1}_h)}. \]

Since \[ \int_0^\pi (\sin(s) + O(|\eta|^{-1}_h))^{-1-2\delta} \frac{ds}{1 + O(|\eta|^{-1}_h)} = a_\delta + O(|\eta|^{-1}_h) \] with \( a_\delta > 0 \) for \( \delta < 0 \), we find that \( I(x^{-2\delta})(z, \xi) = a_\delta|\eta|^{2\delta}_h + O(|\eta|^{-1-2\delta}_h) \) as \( |\eta|_h \to \infty \). On the other hand, if \( |\eta|_h \leq C_0 \), \( I(x^{-2\delta}) \) is uniformly bounded above and below by positive constants depending on \( \delta \) and \( C_0 \). Thus (2.6) is comparable to \[ ||(\eta)^{\delta}_h u||_{L^2(\partial S^* M; d\lambda_0)} = ||u||_{H^\delta L^2(\partial S^* M; d\lambda_0)}. \]

3. The 0-Geometry and 0-Pseudodifferential Calculus

In this section we recall some background on the 0-geometry and the 0-calculus, which will be used throughout the paper. The main sources are [Maz86, MM87] and [Maz91a], also see [EMM91]. Throughout the section, \( M^{n+1} \) will be a compact manifold with boundary and \( (x, y^1, \ldots, y^n) \) are coordinates near a boundary point with \( x \) a boundary defining function.

3.1. The b- and 0-Tangent and Cotangent Bundles, Half Densities. We already introduced the b-cotangent bundle \( bT^* M \) of \( M \) in Section 2. It is the dual bundle of \( bTM \), which is the bundle over \( M \) whose local sections are smooth vector fields tangent to \( \partial M \) (denoted by \( \mathcal{V}_b \)) and which is trivialized by \( x \partial_x, \partial_{y^1}, \ldots, \partial_{y^n} \) locally near \( \partial M \). We denote by \( \Omega_b^{1/2}(M) \) the bundle over \( M \) with sections of the form \( x^{-1/2} \nu, \nu \in C^\infty(M; \Omega^{1/2}) \) (here \( \Omega^{1/2} \) is the smooth half density bundle). The 0-tangent bundle \( 0TM \) is the bundle over \( M \) whose local sections are smooth vector fields on \( M \) vanishing at \( \partial M \) (denoted by \( \mathcal{V}_0 \) as mentioned in the Introduction); it is trivialized locally near \( \partial M \) by \( x \partial_x, x \partial_{y^1}, \ldots, x \partial_{y^n} \). Its dual bundle, \( 0T^* M \), is trivialized locally near \( \partial M \) by \( dx/x, dy^1/x, \ldots, dy^n/x \). We let \( \Omega_0^{1/2}(M) \) be the smooth complex line bundle over \( M \) whose smooth local sections are of the form \( x^{-(n+1)/2} \nu, \nu \in C^\infty(M; \Omega^{1/2}) \). It is trivialized near \( \partial M \) by \( x^{-(n+1)/2} dx dy^1 \ldots dy^n/x \) and in case \( M \) is the compactification of an AH manifold \( (M^{n+1}, g) \) then \( \Omega_0^{1/2}(M) \) is the geometric half density bundle, globally trivialized by \( dV_g^{1/2} \). We will also occasionally use the notation \( \Omega_0^{1/2}(X) \) for \( X \) a manifold with corners related to \( M \) (such as \( X = M^2 \) or \( X = M_0^2 \), the
0-stretched product introduced later); in this case $\Omega^{1/2}_0(X)$ is trivialized by $\prod_j x_j^{-(n+1)/2}\nu$, $\nu \in C^\infty(X;\Omega^{1/2})$, with $x_j$ defining functions for the boundary faces of $X$. For a manifold with corners $X$ and for $\star \in \{0, b, 0\}$ we will write $C^\infty(X;\Omega^{1/2}_\star)$ for smooth sections of $\Omega^{1/2}_\star(X)$ whose derivatives of all orders vanish at $\partial X$ and $C^{-\infty}(X;\Omega^{1/2}_\star) := (C^\infty(X;\Omega^{1/2}_\star))^\prime$.

### 3.2. Polyhomogeneous Conormal Distributions.
We will make use of spaces of functions admitting asymptotic expansions at the boundary. Let $E \subset \mathbb{C} \times \mathbb{N}_0$ be an index set, that is, a discrete set with the properties
\[
|s_j, p_j| \to \infty \Rightarrow \text{Re}(s_j) \to \infty \quad \text{and}
\]
\[
(s_j, p_j) \in E \Rightarrow (s_j + m, p_j - \ell) \in E, \quad m \in \mathbb{N}_0 = \{0, 1, \ldots\}, \quad \ell = 0, 1, \ldots, p_j.
\]
If a discrete $E \subset \mathbb{C} \times \mathbb{N}_0$ satisfies \[3.1\] we write $\bar{E}$ to denote the smallest index set containing $E$. Now let $u \in C^{-\infty}(M)$; $u$ is polyhomogeneous conormal with index set $E$ if it admits an asymptotic expansion in a collar neighborhood $[0, \varepsilon)_x \times \partial M$ of the boundary of the form
\[
u \sim \sum_{(s_j, p_j) \in E} \sum_{k=0}^{p_j} x^{s_j} |\log x|^k a_{j,k}(y), \quad a_{j,k} \in C^\infty(\partial M).
\]
If $u$ satisfies \[3.3\] we write $u \in \mathcal{A}^E_{phg}$. By \[3.2\], the property $u \in \mathcal{A}^E_{phg}$ does not depend on the product decomposition chosen near $\partial M$. Note that the space $\mathcal{A}^E_{phg}$ is invariant under differentiation by vector fields in $\mathcal{V}_b(M)$ and that if $E_1 \subset E_2$ then $\mathcal{A}^{E_1}_{phg} \subset \mathcal{A}^{E_2}_{phg}$.

If $X$ is a manifold with corners with boundary hypersurfaces $X_j$, $j = 1, \ldots, J$, denote by $\mathcal{E} = (E_1, \ldots, E_J)$ a $J$-tuple of of index sets. The space of polyhomogeneous distributions $\mathcal{A}^{E}_{phg}(X)$ is defined to be those which have the form \[3.3\] with $E$ replaced by $E_j$ near the interior of the boundary hypersurface $X_j$ for $j = 1, \ldots, J$ and which have product type expansions at the intersections of boundary hypersurfaces (for a more rigorous definition see [Maz91a]). We now list a few shorthand notations. If $E$ is an index set we will write $E + \ell = \{(s + \ell, p) : (s, p) \in E\}$. The notation $\text{Re}(E) > C$ will mean $\text{Re}(s) > C$ for all $(s, p) \in E$ and $\text{Re}(E) \geq C$ will mean that either $\text{Re}(E) > C$, or $\text{Re}(s) \geq C$ for all $(s, p) \in E$ and $E \cap \{(\text{Re } z = C) \times \{1, 2, \ldots\}\} = \emptyset$. Thus $\text{Re}(E) \geq 0$ implies that $u \in \mathcal{A}^E_{phg}$ is bounded.

If it is known that $E \subset \mathbb{R} \times \mathbb{N}_0$ we will often write $E \geq C$ or $E > C$. Whenever $u \in \mathcal{A}^{E}_{phg}(X)$ is smooth down to a boundary hypersurface $X_j$ and vanishing to order $k$ there we will be replacing $E_j$ in $\mathcal{E}$ by $k$: in this case $E_j \subset \mathbb{N}_0 \times \{0\}$.

If $E$ is a vector bundle over $X$ the discussion above can be used to define polyhomogeneous conormal sections, written as $\mathcal{A}^{E}_{phg}(X; E)$.

### 3.3. The Stretched Product.
Here we outline the construction of the 0-stretched product, which is a special case of a blow-up. For a detailed exposition see [Maz86], [MMS7]; more generally for the blow-up construction see [Mel].

If $M^{n+1}$ is a compact manifold with boundary, then the 0-stretched product $M^2_0 := [M^2; \partial \Delta t]$ is by definition the space obtained by blowing up the boundary of the diagonal $\Delta t = \{(z, z) : z \in M\}$ (see Figure 1). As a set, $M^2_0 = (M^2 \setminus \partial \Delta t) \bigsqcup SN^{++}(\partial \Delta t)$, where $SN^{++}(\partial \Delta t)$ is the inward pointing spherical normal bundle of $\partial \Delta t$ with fiber $SN^{++}_{(p, p)}(\partial \Delta t) = ((T^+_p M)^2 / T^+_p(p, p) \partial \Delta t) \setminus \{0\}/\mathbb{R}^+$ at $(p, p) \in \partial \Delta t$, where $T^+_p M$ is the inward pointing tangent space. $M^2_0$ is endowed with a natural smooth structure making it into a manifold with corners of codimension up to 3 such that the blow down map $\beta_0 : M^2_0 \to M^2$, $\beta_0 |_{(M^2, \partial \Delta t)} = id$, ...
\[ \beta_0|_{SN^{++}(\partial \Delta \ell)} = (p,p) \] is smooth. Under \( \beta_0 \), smooth vector fields on \( M^2 \) tangent to \( \partial \Delta \ell \) lift to be smooth and tangent to the boundary faces of \( M_0^2 \). The boundary face \( SN^{++}(\partial \Delta \ell) \subset M_0^2 \) is called the \emph{front face} and denoted by \( \text{ff} \); we also let \( \Delta \ell_0 = \beta_0^{-1}(\Delta \ell \setminus \partial \Delta \ell) \). The side faces are \( \ell f := \beta_0^{-1}(\partial M \times M) \) and \( rf := \beta_0^{-1}(M \times \partial M) \). We will use the notation \( x, x_r, x_f \) to refer to a defining function for \( \ell f \), \( rf \), \( \text{ff} \) respectively. Moreover, \( \mathcal{A}_{E \phi \mathbb{g}}(M_0^2) \) with \( E = (E_\ell, E_r, E_f) \) will denote polyhomogeneous distributions on \( M_0^2 \) with \( E_\ell, E_r, E_f \) index sets corresponding to \( \ell f \), \( rf \), \( \text{ff} \) respectively. Throughout the paper we will make use of the following projective coordinate systems on \( M_0^2 \): if \( (x, y) \) is a coordinate system near \( p \in \partial M \) with \( x \) a boundary defining function and \( (\tilde{x}, \tilde{y}) \) a copy of it on the right factor of \( M^2 \), the coordinate system

\[ (3.4) \quad (\tilde{x}, \tilde{y}, s = x/\tilde{x}, W = (y - \tilde{y})/\tilde{x}) \text{ is valid near } \ell f \cap \text{ff} \text{ and away from rf, whereas} \]

\[ (3.5) \quad (x, y, t = \tilde{x}/x, Y = (y - \tilde{y})/x) \text{ is valid near } rf \cap \text{ff} \text{ and away from } \ell f \].

In terms of (3.4) (resp. (3.5)) \( s \) (resp. \( t \)) is a defining function for \( \ell f \) (resp. \( rf \)) and \( \tilde{x} \) (resp. \( x \)) is a defining function for \( \text{ff} \); moreover, \( (s, W) \) (resp. \( (t, Y) \)) restrict to coordinates in the interior of the front face, smooth down to \( (\text{ff} \cap \ell f)^0 \) (resp. \( (\text{ff} \cap rf)^0 \)).

![Figure 1. The 0-stretched product.](image)

The fibers of the front face carry additional structure: fix \( p \in \partial M \) and let \( T^+_p M = \{v \in T_p M : dx(v) > 0\} \). The subgroup \( G_p \) of \( GL(T_p M) \) that preserves \( T^+_p M \) and fixes \( \partial(T^+_p M) \) pointwise induces an invariantly defined free and transitive action on \( T^+_p M \). Using linear coordinates \( (u, w) = (u, w^1, \ldots, w^n) \) on \( T_p M \) induced by coordinates \( (x, y) \) near \( p \), the action of \( (a, b) \in G_p \cong \mathbb{R}^+ \times \mathbb{R}^n \) is given by \( (a, b) \cdot (u, w) = (au, w + ub) \) and the group multiplication in \( G_p \) is given by \( (a, b) \cdot (a', b') = (aa', b' + a'b) \). The actions of \( G^\ell_p := G_p \times Id \) and \( G^r_p := Id \times G_p \) on \( (T^+_p M)^2 \) descend to the interior of the fiber \( SN^{++}(p,p) \partial \Delta \ell = \text{ff}_p \), defining transitive and free actions there. Moreover, each fiber of the front face has a canonically defined singled out point \( e_p \), given by \( \partial \Delta \ell_0|_{(p,p)} \). Thus one obtains diffeomorphic

identifications of \( \text{ff}_p^\circ \) with \( G^\ell_p \cong G^r_p \cong G_p \) and \( \text{ff}_p^\circ \) has two group structures, both canonically isomorphic to \( G_p \). The diffeomorphisms \( f^\ell_p, f^r_p : \text{ff}_p^\circ \rightarrow G_p \) obtained this way are given (using linear coordinates \( (u, w), (\tilde{u}, \tilde{w}) \) on \( (T^+_p M)^2 \)) as \( f^\ell_p([(u, w), (\tilde{u}, \tilde{w})]) = (u/\tilde{u}, (w - \tilde{w})/\tilde{u}) \), \( f^r_p([(u, w), (\tilde{u}, \tilde{w})]) = (\tilde{u}/u, (\tilde{w} - w)/u) \). Those diffeomorphisms have equivariance properties:
for $q \in \mathbb{G}_p$ write $q_l = (q, id) \in \mathbb{G}_p^l$ and $q_r = (id, q) \in \mathbb{G}_p^r$ to obtain as in [MM87] §3] that for $\omega \in \mathbb{F}_p^o$ one has
\begin{align}
&f_p^l(q_l \cdot \omega) = q : f_p^l(\omega), \quad f_p^r(q_r \cdot \omega) = q : f_p^r(\omega) \\
&f_p^l(q_r \cdot \omega) = f_p^r(\omega) \cdot q^{-1}, \quad f_p^r(q_l \cdot \omega) = f_p^r(\omega) \cdot q^{-1}.
\end{align}
(3.6)

If $g$ is an AH metric on $M$ and $p \in \partial M$ one obtains a canonical hyperbolic metric $h_p$ of curvature $-1$ on $T_p^+M$. If $x$ is a boundary defining function for $\partial M$ it is given by
\begin{align}
h_p|_v := (dx(v))^{-2}g|_p, \quad v \in T_p^+M,
\end{align}
(3.7)
where $\overline{g} = x^2g$ and the inner product $\overline{g}|_p$ on $T_v(T_pM)$ is naturally identified with an inner product on $T_v(T_pM)$ for any $v \in T_p^+M$. One checks that (3.7) does not depend on the choice of the boundary defining function $x$, and with an appropriate choice of coordinates $(x, y)$ near $p$ one can always arrange that $h_p = u^{-2}(du^2 + |dv|^2)$ in terms of induced linear coordinates $(u, w)$ on $T_p^+M$. The metric $h_p$ can be appropriately pulled back to $\mathbb{F}_p^o$ in two ways: since the action of $G_p$ on $T_p^+M$ is free and transitive, given $v \in T_p^+M$ one can define a diffeomorphism
\begin{align}
f_p^v : G_p \rightarrow T_p^+M, \quad G_p \ni q \mapsto q \cdot v \in T_p^+M.
\end{align}

Thus for each $v$ one obtains a hyperbolic metric $(f_p^v)^*h_p$ on $G_p$ which is right invariant with respect to the group structure of $G_p$, as can be checked in coordinates. Using this right invariance and the fact that for any two $v, v' \in T_p^+M$ there exists $\tilde{q} \in G_p$ such that $v' = \tilde{q} \cdot v$, one checks that the metric $(f_p^v)^*h_p =: h_{G_p}$ is in fact independent of $v$. Hence $h_p^v := (f_p^v)^*h_{G_p}$ and $h_p^r := (f_p^r)^*h_{G_p}$ are hyperbolic metrics on $\mathbb{F}_p^o$, which are invariant with respect to the left action of $G_p^r$, $G_p^l$ respectively, by (3.6). Again with an appropriate choice of coordinates we can arrange that $h_p^v = s^{-2}(ds^2 + |dW|^2)$ in terms of coordinates $(s, W)$ on $\mathbb{F}_p$ as in (3.4), and similarly that in terms of $(t, Y)$ in (3.5), $h_p^r = t^{-2}(dt + |dY|^2)$.

3.4. The 0-Calculus. We recall the definition and properties of the 0-calculus of pseudo-differential operators that we need later. As already mentioned, 0-differential operators of order $m \in \mathbb{N}$, denoted by $\text{Diff}_0^m(M)$, are those differential operators which can be written as finite sums of at most $m$-fold products of vector fields in $\mathcal{V}_0$: in coordinates $(x, y)$ near $\partial M, P \in \text{Diff}_0^m(M)$ can be written as
\begin{align}
P = \sum_{j+|\alpha| \leq m} a_{j, \alpha} (x, y)(x \partial_x)^j (x \partial_y)^\alpha, \quad a_{j, \alpha} \in C^\infty,
\end{align}
using multi-index notation.

Pseudodifferential operators in the 0-calculus are defined via the lifts of their Schwartz kernels to $M_0^2$. As shown in [MM87], smooth sections of $\Omega_0^{1/2}(M^2) \cong \pi_0^* \Omega_0^{1/2}(M) \otimes \pi_0^* \Omega_0^{1/2}(M)$ lift via $\beta_0$ to smooth sections of $\Omega_0^{1/2}(M_0^2)$. Thus for the Schwartz kernel $\kappa_P \in C^{-\infty}(M^2; \Omega_0^{1/2})$ of an operator $P : \hat{C}^\infty(M; \Omega_0^{1/2}) \rightarrow C^{-\infty}(M; \Omega_0^{1/2})$ we have $\beta_0^* \kappa_P \in C^{-\infty}(M_0^2; \Omega_0^{1/2})$. The small 0-calculus of order $m$, denoted by $\Psi_0^m(M)$, consists of operators whose Schwartz kernel $\kappa_P$ satisfies $\beta_0^* \kappa_P \in A_p^{\mathcal{E}}(M_0^2; \Delta_0^m; \Omega_0^{1/2})$ with $\mathcal{E} = (\emptyset, \emptyset, \emptyset)$. This means by definition that $\beta_0^* \kappa_P$ is a section of $\Omega_0^{1/2}(M_0^2)$ conormal of order $m$ to $\Delta_0$, smooth down to the front face away from $\Delta_0$, and vanishing to infinite order at the side faces. Recall that a distribution $u$ on a manifold with corners $X^d$ is said to be conormal of order $m \in \mathbb{R}$ with

\[\text{For any product space, } \pi_0, \pi_1, \text{ will generically denote projection onto the left and right factor respectively.}\]
respect to an interior $p$-submanifold $Y^{d-s}$ (see Maz91a) if whenever $Y$ is given locally as the zero set $\{y' = 0\}$ in terms of coordinates $(x, y', y'') \in [0, \infty)^k \times \mathbb{R}^s \times \mathbb{R}^{d-k-s}$ for $X^d$, we have $u(x, y) = \int_{S^2} e^{iy''\xi'} a(x, y'', \xi') d\xi'$. Here $a \in S^m \{([0, \infty)^k \times \mathbb{R}^{d-s-k}) \times \mathbb{R}^s\}$ is a symbol of order $m' = m + d/4 - s/2$ (following Hörmander's convention, see [Hör07 §18.2]); by definition the symbol $a$ satisfies symbol estimates

$$|\partial_{\xi'}^\alpha \partial_y^\beta \partial_x^\gamma a(x, y'', \xi')| \leq C_{\alpha, \beta, \gamma} (\xi')^{m'-|\gamma|},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, and $\alpha, \beta, \gamma$ are multi-indices. We write $u \in I^m(X, Y)$ for such conormal distributions; conormal sections of a bundle are defined similarly. To any operator $P \in \Psi^m_0(M)$ corresponds a principal symbol encoding the leading conormal singularity at $\Delta t_0$. This symbol is a symbolic section of a bundle over $N^*\Delta t_0$, but as it turns out, using the canonical identification of $N^*\Delta t_0 \leftrightarrow 0T^*M$, it can be canonically identified with a symbol $\sigma^m_0(P) \in S^m(0T^*M) := S^m(0T^*M)/S^{m-1}$ (see MM87, Lau03 for details), which is called the principal symbol of $P$. Provided there exists a symbol $a \in S^{l-m}(0T^*M)$ such that $\sigma^m_0(P) \cdot a \equiv 1$, $P \in \Psi^m_0(M)$ will be called elliptic. Below we set $\Psi^{-\infty}_0(M) := \bigcap_{m \in \mathbb{R}} \Psi^m_0(M)$.

The operators in the large 0-calculus have kernels with $\beta^*_a \kappa_P \in \mathcal{A}^E_{ph\beta} I^m(\mathbb{M}_0^2, \Delta t_0; \Omega^{1/2}_0)$, for $E = (E_t, E_r, E_f)$ and $m \in \mathbb{R}$ and are denoted by $\Psi^m_{0,E}(M)$. The definition means that $\beta^*_a \kappa_P (M^0) \in I^m((\mathbb{M}_0^0, \Delta \gamma_0; \Omega^{1/2}_0)$ and it has an asymptotic expansion of the form (3.3) at the boundary faces with index sets determined by $E$, with the coefficients $a_{jk}$ corresponding to the side faces being smooth in their interior and those corresponding to $\mathbb{E}$ being conormal of order $m + 1/4$ with respect to $\mathbb{E} \cap \partial \Delta t_0$ (the change in the order of conormality is due to Hörmander’s convention) [3]. The subspace $\Psi^{-\infty,E}_0(M)$ consists of operators whose kernels are smooth in $(\mathbb{M}_0^2)^0$ with polyhomogeneous expansions at the boundary faces. We will often write $\Psi^m_{0,E_t,E_r,E_f}(M)$ to imply that $E_f = \{(0, 0)\}$. In this case, $\Psi^m_{0,E_t,E_r,E_f}(M) = \Psi^m_0(M) + \Psi^{-\infty,E_t,E_r,E_f}(M)$. The rest of the shorthand notations for index sets outlined earlier will apply for $\Psi^m_{0,E};$ for instance $P \in \Psi^m_{0,a,E_t,E_f}(M)$, $a \in \mathbb{N}_0$, indicates that $\beta^*_a \kappa_P$ is smooth near the interior of $\mathbb{E}$ and vanishes at $\mathbb{E}$ to order $a$.

To clarify the action of $P$ in terms of coordinates, locally near $\partial M$ write $\gamma_0 = (x^{-n-1}dy)_{1/2} \in C^\infty(M; \Omega^{1/2}_0)$, $\kappa_P = K_P \cdot \pi^*_r \gamma_0 \otimes \pi^*_s \gamma_0$ and use the notation $\beta^*_a K'_P$ for $\beta^*_0 K_P$ expressed in terms of coordinates $(x, y, s, W)$ on $M^0_2$. Then for $P \in \Psi^m_{0,E}(M)$ and $f \in \mathcal{C}^\infty(M)$ we have

$$P(f \cdot \gamma_0)(x, y) = \int \beta^*_a K'_P \left( \frac{x}{s}, y - \frac{W}{s}, x, s, W \right) f \left( \frac{x}{s}, y - \frac{W}{s} \right) \frac{|dsdW|}{s} \cdot \gamma_0. \tag{3.8}$$

Operators in the large 0-calculus can be composed under compatibility assumptions. The following proposition follows from the proof of Maz91a, Theorem 3.15] with a change in normalizations. As stated below it can also be found in [Alb].

**Proposition 3.1.** Let $P \in \Psi^m_{0,E}(M)$, $P' \in \Psi^{m',F}(M)$. If $\text{Re}(E_r + F_t) > n$ then the composition $P \circ P'$ is defined and $P \circ P' \in \Psi^{m+m',W}(M)$, where $W$ is given by

$$W_t = (F_t + E_f) \cup E_t, \quad W_r = (E_r + F_f) \cup F_r, \quad W_f = (E_f + F_r) \cup (E_f + F_f);$$

\footnote{Our definition of the large calculus is consistent with the one given in MM87 (except for the fact that we demand that the kernels be polyhomogeneous conormal to the boundary faces of the stretched product and not merely conormal, see [MM87] but it differs from the one in Maz91a, where the author defines the large edge calculus, of which the 0-calculus is a special case, as $\Psi^m_{0,E} = \Psi^m_0 + \Psi^{-\infty,E}$.}
here the sum and extended union respectively of the index sets $E$, $E'$ are given by

$$E + E' = \{(s, p) + (s', p') : (s, p) \in E, (s', p') \in E'\},$$

$$E \cup E' = E \cup E' \cup \{(s, p + p' + 1) : \text{there exist} (s, p) \in E, (s, p') \in E'\}.$$

We also state results regarding mapping properties on polyhomogeneous functions and on half densities in Sobolev spaces. For a proof of the following, see [Alb], [EMM91], [Maz91a]:

**Proposition 3.2.** Let $u \in \mathcal{A}^{F}_p(M; \Omega_0^{1/2})$ and $P \in \Psi_0^{m, \mathcal{E}}(M)$, $m \in \mathbb{R}$. If $\text{Re}(E_r + F) > n$ then $Pu \in \mathcal{A}^{F}_p(M; \Omega_0^{1/2})$, where $F' = E_r \overline{\cup}(E_f + F)$.

The mapping properties of the large 0-calculus in terms of weighted Sobolev half densities, denoted by $x^m H^s_0(M; \Omega_0^{1/2})$, will be important later. We remark that $C_c^\infty(M; \Omega_0^{1/2})$ (hence also $C^\infty(M; \Omega_0^{1/2})$) is dense in $x^s H^m_0(M; \Omega_0^{1/2})$ for $s \geq 0$ (see [Lee06] Lemma 3.9). Also, the inclusion $x^m H^s_0'(M; \Omega_0^{1/2}) \hookrightarrow x^s H^m_0(M; \Omega_0^{1/2})$ is compact provided $m' > m$ and $\delta' > \delta$.

**Proposition 3.3.** Let $P \in \Psi_0^{m, \mathcal{E}}(M)$, $m \in \mathbb{R}$. Provided $s \in \mathbb{R}$, $\text{Re}(E_r) > n/2 - \delta$, $\text{Re}(E_f) \geq \delta' - \delta$ and $\text{Re}(E_d) > \delta' + n/2$ one has that

$$P : x^s H^s_0(M; \Omega_0^{1/2}) \to x^{s'} H^{s'-m}_0(M; \Omega_0^{1/2})$$

is bounded. In particular, if $m < 0$, $\text{Re}(E_r) > n/2 - \delta$, $\text{Re}(E_f) > 0$ and $\text{Re}(E_d) > \delta + n/2$ then $P : x^s H^s_0(M; \Omega_0^{1/2}) \to x^s H^s_0(M; \Omega_0^{1/2})$ is compact.

### 3.5. The Model Operator.

Given $p \in \partial M$, an operator $P \in \Psi_0^{m, \mathcal{E}}(M)$ with $\text{Re}(E_f) \geq 0$ determines an invariantly defined operator $N_p(P)$ on $T^+_p M$, which we call the *model operator*. It is closely related to the group structures on $\mathbb{F}_p$ and captures the 0-th order behavior of the Schwartz kernel of $P$ at the front face. Consider neighborhoods $U' \subset T^+_p M$ and $U \subset M$ of 0 and $p$ respectively, and diffeomorphism $\varphi : U' \to U$ with $\varphi(0) = p$, $d\varphi|_0 = Id$, and $\varphi(T_p \partial M) \subset \partial M$. Also let $R_r : T^+_p M \to T^+_p M$, $r \in (0, \infty)$, be the canonical radial action. If $P \in \text{Diff}^m_\infty(M)$ and $f \in C_c^\infty(T^+_p M)$ the model operator is defined by

$$N_p(P)f = \lim_{r \to 0} R_{r^*}\varphi^*P(\varphi^{-1})^*R_{r^*}.\tag{3.9}$$

As shown in [MM87], $N_p(P)$ is independent of the choice of $\varphi$ and given by freezing the coefficients of $P$ at $p$. For $P \in \Psi_0^{m, \mathcal{E}}(M)$ we still define $N_p(P)$ by (3.9), but with $f \in C_c^\infty(T^+_p M; \Omega_0^{1/2})$ and the limit in the space of distributional half densities. Here $\Omega_0^{1/2}(T^+_p M)$ is the half density bundle trivialized by a global section of the form $u^{-n+1/2} |dudw|^{1/2}$ in linear coordinates as above; henceforth we denote such a section by $\gamma_p$. If an AH metric $g$ is chosen on $\hat{M}$, it also determines a hyperbolic metric on $T^+_p M$ and the various density bundles are naturally trivial, so (3.9) is also naturally defined for $P \in \text{Diff}^m_\infty(M)$ and $f$ a half density.

The model operator can be appropriately interpreted as a convolution operator. As already mentioned, the interior $\mathbb{F}^p$ of each fiber of the front face carries two group structures normalized operator. Despite not following the usual convention for its name, we maintain the traditional notation $N_p$.\footnote{As already mentioned, the more common name for the model operator is normal operator.}
isomorphic to the group $G_p \subset GL(T_p M)$ and hence $\mathbb{f}^0_p$ acts on $T_p^+ M$ from the left in two ways. For the rest of the section we use the map $f^\ell$ to identify $\mathbb{f}^0_p$ with $G_p$ without writing it explicitly. Given $v \in T_p^+ M$, a section $\gamma_p$ pull back via the map $f^\ell_p \ni q \mapsto f^\ell_p(q^{-1}) \in T_p^+ M$ to a left-invariant half density $\gamma_H$ on $\mathbb{f}^0_p$. In coordinates $(s, W)$ on $\mathbb{f}^0_p$ (see (3.4)) $\gamma_H$ is a constant multiple of $|s^{-1} ds dW|^{1/2}$. We denote by $(\mathbb{O}^{1/2}_H(\mathbb{f}^0_p))$ the bundle spanned by $\gamma_H$ over $C^\infty(\mathbb{f}^0_p)$. Given a distributional half density $\tilde{u} = (u \cdot \gamma_H) \otimes \gamma_p \in (C^\infty(\mathbb{f}^0_p; \mathbb{O}^{1/2}_H))^0 \otimes C^\infty(\overline{T_p^+ M; \mathbb{O}^{1/2}_0})$ one can define an operator on $T^+_p M$ by left convolution, i.e. by

$$
(3.10) \quad \tilde{u} \ast (f \cdot \gamma_p)(v) := \int u(q) f(q^{-1} \cdot v) \gamma_H^2(q) \cdot \gamma_p, \quad f \cdot \gamma_p \in C^\infty(T^+_p M; \mathbb{O}^{1/2}_0).
$$

Since the lifted kernel of an operator $P \in \Psi^m_{\infty, E}(M)$ with $\text{Re}(E_f) \geq 0$ is continuous down to the front face with values in distributional sections of $\mathbb{O}^{1/2}_0(M^2)$ conormal to $\Delta_0$, and $\Delta_0$ is transversal to $\mathbb{f}^0_p$, $P$ determines a distributional half density on $\mathbb{f}^0_p$ by restriction:

$$
(3.11) \quad \beta^*_{0, \kappa_p}|_{\mathbb{f}^0_p} \in A^E(\mathbb{f}^0_p) \cap \mathcal{M}_p^{+1/2}((\mathbb{f}^0_p, \{e_p\}'; (\pi^*_p \mathbb{O}^{1/2}_0(M) \otimes \pi^*_p \mathbb{O}^{1/2}_0(M))((p, p)));
$$

in (3.11) $E_\ell$ (resp. $E_r$) corresponds to the expansion at $\mathbb{f} \cap \mathbb{f}$ (resp. $\mathbb{r} \cap \mathbb{f}$). Note that fiber elements in $\gamma_0|_{\mathbb{f}^0_p} \in \mathbb{O}_0^{1/2}(M)|_{\mathbb{f}^0_p}$ can be identified with constant multiples of $\gamma_p$ after pulling back $\gamma_0$ by $\varphi \circ R_r$ and taking a limit as $r \to 0$. Moreover, the diffeomorphism $\mathbb{f}^0_p \times T^+_p M \ni (q, v) \mapsto (v, q^{-1} \cdot v) \in (T^+_p M)^2$ can be used to pull back $\pi^*_p \gamma_p \otimes \pi^*_r \gamma_p$ to a constant multiple of $\gamma_H \otimes \gamma_p$, so under those identifications we let

$$
(3.12) \quad F_p(P) := \beta^*_{0, \kappa_p}|_{\mathbb{f}^0_p} \in A^E(\mathbb{f}^0_p) \cap \mathcal{M}_p^{+1/2}(\mathbb{f}^0_p, \{e_p\}'; \mathbb{O}_H^{1/2}) \otimes \text{span}_C\{\gamma_p\}.
$$

By [MMS7], Proposition 5.19, for operators with smooth kernel down to the interior of the front face there exists a short exact sequence

$$
(3.13) \quad 0 \to \Psi^m_{\infty, E_\ell, E_r, 1}(M) \to \Psi^m_{\infty, E_\ell, E_r}(M) \to A^E(E_\ell, E_r)(\mathbb{f}^0_p, \ell \mathbb{O}_H^{1/2}) \otimes \text{span}_C\{\gamma_p\} \to 0.
$$

Unraveling the definitions above and using the coordinate expression (3.5) (assuming that the coordinates $(x, y)$ are centered at $p$) one checks that

$$
N_p(P)(f \cdot \gamma_p)(u, w) = \int \beta^*_{0, K^\ell_p}(0, 0, s, W)f(u - W/u, w - W/w) \frac{|ds dW|}{W} \cdot \gamma_p = (F_p(P) \ast (f \cdot \gamma_p))(u, w),
$$

which implies the following:

**Lemma 3.4.** Let $P \in \Psi^m_{\infty, E}(M)$ with $\text{Re}(E_f) \geq 0$. Then for each $p \in \partial M$ and $f \cdot \gamma_p \in C^\infty(T^+_p M; \mathbb{O}^{1/2}_0)$ one has $N_p(P)(f \cdot \gamma_p) = F_p(P) \ast (f \cdot \gamma_p)$.

Under suitable assumptions the model operator is a homomorphism under composition. The following proposition is stated and proved in [MMS7] in the case $P \in \text{Diff}^m_0(M)$. It is also stated in [Alb] and in [EMM91] that the homomorphism property holds, with no assumptions explicitly mentioned. A detailed proof can be found in [Ept20].

**Proposition 3.5.** Let $P$, $P'$ be as in Proposition 3.1, with the additional assumptions $\text{Re}(E_f) \geq 0$, $\text{Re}(F_f) \geq 0$ and $\text{Re}(E_\ell + F_r) > 0$. Then for each $p \in \partial M$ one has $N_p(P \circ P') = N_p(P) \circ N_p(P')$.

**Remark 3.6.** The assumptions $\text{Re}(E_f) \geq 0$, $\text{Re}(F_f) \geq 0$ and $\text{Re}(E_\ell + F_r) > 0$ guarantee that $P$, $P'$ and $P \circ P'$ have well defined model operators (see Proposition 3.1).
If \( \tilde{M} \) is endowed with an AH metric \( g \), by writing the hyperbolic metric \( h_p \) in (3.7) as \( h_p = u^{-2}(du^2 + |dw|^2) \), \( (T^+_p M, h_p) \) is identified with \( (\mathbb{H}^{n+1} = \{(u, w) \in \mathbb{R}^+ \times \mathbb{R}^n\}, h) \), the hyperbolic upper half space model. Via the map \((f_p^0)^{-1} : T^+_p M \rightarrow \mathbb{H}^n_{p} \) (recall that we use \( f_p^0 \) to identify \( f_p^0 \) and \( G_p \)) the integration in (3.10) can be pulled back to \( T^+_p M \), and \( N_p(P) \) can be regarded as an operator on \( \mathbb{H}^{n+1} \) written as

\[
N_p(P)(f \cdot \gamma_p)(u, w) = \int_{\mathbb{H}^{n+1}} \beta_0^P K_p^0 \left( 0, 0, \frac{u - w}{u} \right) f(u, w) \frac{|dudw|}{u^{n+1}} \cdot \gamma_p.
\]

In (3.14) and henceforth, whenever an AH metric \( g \) has been chosen on \( \tilde{M} \) it will be assumed that \( \gamma_0 = |\sqrt{\det g} dx dy|^{1/2} \) in coordinates and that \( \gamma_p = |\sqrt{\det h_p} du dw|^{1/2} \). Conjugating by the Cayley transform, \( N_p(P) \) can be interpreted as an operator on the Poincaré ball \( (\mathbb{B}^{n+1}, \frac{4|dz|^2}{(1-|z|^2)^2}) \) and one checks that if \( P \in \Psi^m_0(\mathbb{M}) \) with \( \text{Re}(E_f) \geq 0 \) then one has \( N_p(P) \in \Psi^m_0(\mathbb{B}^{n+1}) \) with \( E' = (E_f, E_r, \{(0,0)\}) \); thus the model operator also extends to appropriate weighted Sobolev spaces on \( \mathbb{B}^{n+1} \) according to Proposition 3.3.

4. The Pseudodifferential Property

For the rest of the paper we assume a simple AH manifold \( (\tilde{M}^{n+1}, g) \). In this section we show that the normal operator \( N_g \) is a 0-pseudodifferential operator, namely that \( N_g \in \Psi^{-1,0,0}_0(M) \). As an intermediate step we study the distance function induced by \( g \). Once the pseudodifferential property of \( N_g \) has been established, we use it to extend \( I \) to larger weighted \( L^2 \) spaces than those of Section 2.

By following the proof of [CH16, Proposition 19] one can show the following technical lemma (a detailed proof also appears in [Ept20]).

**Lemma 4.1.** Let \( (\tilde{M}, g) \) be a simple AH manifold. The map \( \Phi : T^*\tilde{M} \rightarrow M^3_0; (z, \xi) \mapsto (z, \exp_z(\xi^\#)) \) extends smoothly to a map \( \tilde{\Phi} : 0T^*\tilde{M} \rightarrow M^3_0 \), where we are using the canonical identification of \( 0T^*M \big|_{\tilde{M}} = (0T^*M)^\circ \) and \( T^*\tilde{M} \). Here \( \# \) raises an index with respect to the metric \( g \). Moreover, the differential of \( \tilde{\Phi} \) at \( (z, 0) \in 0T^*\tilde{M}\big|_{\partial M} \) has full rank.

The behavior of the distance function \( \rho \) on AH manifolds away from the diagonal has been studied by various authors, see for instance [SW16], [CH16] and [GGS+19], and also [MSVI14] for small perturbations of hyperbolic metric. As Proposition 4.2 below indicates, provided \( (\tilde{M}, g) \) is simple, the lift of the distance function to \( M^3_0 \) is smooth away from \( \Delta_0 \) and the side faces, however our analysis of \( N_g \) will also require smoothness of \( \beta_0^\rho \rho^2 \) in a neighborhood of \( \Delta_0 \), all the way to the front face. We are not aware of this fact explicitly stated in the literature, so we provide a proof.

**Proposition 4.2.** Let \( (\tilde{M}, g) \) be a simple AH manifold and let \( \rho : \tilde{M}^2 \rightarrow \mathbb{R} \) be the geodesic distance function. There exists \( \alpha \in C^\infty(M^3_0 \setminus \Delta_0) \) such that

\[
\beta_0^\rho \rho = \alpha - \log(x_\ell) - \log(x_r),
\]

where \( x_\ell \) and \( x_r \) are defining functions for the left and right face of \( M^3_0 \) respectively. Moreover, \( \beta_0^\rho \rho^2 \) extends to a smooth function on \( M^3_0 \setminus (lf \cup rf) \).

**Proof.** The first statement follows from work in [SW16], [CH16] and [GGS+19] (see [GGS+19, Remark 7]). We show the second statement. Assume without loss of generality that \( x_\ell, x_r \equiv 1 \) in a neighborhood of \( \Delta_0 \). Since \( \rho^2 \) is smooth near \( \Delta_\ell \cap \tilde{M}^2 \) and thus \( \beta_0^\rho \rho^2 \) extends to a
function in $C^\infty(M^2_0 \setminus (\partial \Delta_0 \cup \ell \cup \text{rfl}))$, it is enough to show that $\beta_0^* \rho^2$ extends to be smooth in a neighborhood of $\partial \Delta_0$. By the Inverse Function Theorem, Lemma 4.1 implies that $\tilde{\Phi}$ restricted to a neighborhood of a point $(p,0) \in 0^T M|_{\partial M}$ is invertible. The inverse, defined in a neighborhood $U \subset M^2_0$ of $\partial \Delta_0|_{(p,p)}$, is smooth down to the front face. In $U \cap (M^2_0)^\circ$

\begin{equation}
\beta_0^* \rho^2(z,\bar{z}) = |\exp_x^{-1}(z,\bar{z})|^2_g = \left|\Phi^{-1}(z,\bar{z})\right|_{g^{-1}}^2
\end{equation}

using the identification $(M^2_0)^\circ \leftrightarrow \tilde{M}^2$. Since $g$ induces a non-degenerate quadratic form on the fibers of $0^T M$, smooth all the way to the boundary, (4.1) extends smoothly to $\partial \Delta_0$. □

The proof of the following lemma, which uses the Gauss Lemma, is contained in [SU04].

**Lemma 4.3.** Let $(M,g)$ be a simple AH manifold and let $z = (z^0, \ldots, z^n), \bar{z} = (\bar{z}^0, \ldots, \bar{z}^n)$ two copies of the same (possibly global) coordinate system in each of the two factors of $\tilde{M}^2$. In the set where $(z,\bar{z})$ are valid coordinates for $\tilde{M}^2$, the kernel of $N_g$, viewed as a section of $\Omega^1_0(\tilde{M}^2)$, is given by $K_{N_g}(z,\bar{z}) \cdot \gamma_0(z) \otimes \gamma_0(\bar{z})$, where

\begin{equation}
K_{N_g}(z,\bar{z}) = \frac{2|\det(\partial_{\bar{z}} \rho^2/2)|}{\rho^n(z,\bar{z}) \sqrt{\det g(z) \det g(\bar{z})}}.
\end{equation}

Recall that on an AH manifold $\gamma_0(z) := dV_g^{1/2} = |\sqrt{\det g(z)}dz|^{1/2}$.

We now prove the following key proposition:

**Proposition 4.4.** Let $(\tilde{M}^{n+1},g)$ be a simple AH manifold. Then $\mathcal{N}_g \in \Psi^{-1,n,n}(M)$. Moreover, it is elliptic.

**Proof.** We examine the Schwartz kernel of $N_g$ on $\tilde{M}^2$ and on the stretched product $M^2_0$. As noted in [SU04], (4.2) implies that on $\tilde{M}^2$ the kernel of $N_g$ agrees with the kernel of a pseudodifferential operator of order $-1$ with principal symbol $C_n|\xi|^{-1}$. Since smooth sections of $\Omega^1_0(\tilde{M}^2)$ lift to smooth sections of $\Omega^1_0(M^2_0)$ it suffices to study the behavior of $K_{N_g}(z,\bar{z})$ in (4.2) and its pullback to $M^2_0$ as $z,\bar{z} \rightarrow \partial M$, both away from, and near the diagonal. Throughout the proof, $z = (x,y), \bar{z} = (\bar{x},\bar{y})$ are representations in terms of two copies of the same coordinate system in each factor of $M^2$ such that $x,\bar{x}$ are boundary defining functions.

First note that by the Gauss Lemma $|\det(\partial_{\bar{z}} \rho^2/2)| = |\det(d_z \exp_x^{-1}(z,\bar{z}))|$, so by simplicity $\det(\partial_{\bar{z}} \rho^2/2) \neq 0$ on $\tilde{M}^2$ and the absolute value can be ignored in the process of examining the smoothness properties of $K_{N_g}$ and $\beta_0^* K_{N_g}$. Moreover, we have a simplification of (4.2) away from the diagonal: note that $\partial^2_{\bar{z}} \rho^2/2 = \rho \partial^2_{\bar{z}\bar{z}} \rho + \partial_z \rho \otimes \partial_{\bar{z}} \rho$. Since for $H \in \mathbb{R}^{d \times d}$ and $u,v \in \mathbb{R}^d$ one has $\det(H + u \otimes v) = \det(H) + (\text{adj}(H)u) \cdot v$ by the matrix determinant lemma, where $\text{adj}(H)$ is the adjugate matrix of $H$ and $\cdot$ denotes the Euclidean dot product, we have

\begin{equation}
\det(\partial^2_{\bar{z}} \rho^2/2) = \rho^{n+1} \det(\partial^2_{\bar{z}\bar{z}} \rho) + \rho^n(\text{adj}(\partial^2_{\bar{z}\bar{z}} \rho) \partial_z \rho) \cdot \partial_{\bar{z}} \rho.
\end{equation}

Observe now that the first term vanishes away from the diagonal. Indeed, if $z \neq \bar{z}$ the Gauss Lemma yields $|d_z \rho(z,\bar{z})|_g = 1$, thus the rank of the map $d_z \rho(z,\cdot) : \tilde{M}\{z\} \rightarrow S^*_z \tilde{M}$ is at most $n$. Therefore, $\det(\partial^2_{\bar{z}\bar{z}} \rho) = 0$ and thus away from the diagonal we have

\begin{equation}
K_{N_g}(z,\bar{z}) = \frac{2|\text{adj}(\partial^2_{\bar{z}\bar{z}} \rho) \partial_z \rho| \cdot \partial_{\bar{z}} \rho}{\sqrt{\det g(z) \det g(\bar{z})}}.
\end{equation}
We first examine \( K_{\mathcal{N}_g}(z, \overline{z}) \) on \( M^2 \) away from the diagonal when \( z \to \partial M \) or \( \overline{z} \to \partial M \). By Proposition 4.2 for \( z, \overline{z} \) away from the diagonal we have

\[
\rho(z, \overline{z}) = \alpha(x, y, \overline{x}, \overline{y}) - \log(x) - \log(\overline{x}),
\]

where \( \alpha \in C^\infty(M^2 \setminus \Delta_t) \). Here without loss of generality we can take \( (x, y), (\overline{x}, \overline{y}) \) to be global coordinate systems on \( M \) since \((M, g)\) is simple. Since \( \partial^2_{zz} \rho = \partial^2_{\overline{z}\overline{z}} \alpha, \text{adj}(\partial^2_{zz} \rho) \in C^\infty(M^2 \setminus \Delta_t) \). Moreover, \( \sqrt{\det g(z)} = x^{-n-1} \sqrt{\det g(\overline{z})} \) and \( \sqrt{\det g(\overline{z})} = \overline{x}^{-n-1} \sqrt{\det g(z)} \) with \( \det g(z), \det g(\overline{z}) \in C^\infty(M) \) and non-vanishing. Finally, \( \partial_z \rho \in x^{-1}C^\infty(M^2 \setminus \Delta_t) \) and similarly for \( \partial_{\overline{z}} \rho \), thus

\[
K_{\mathcal{N}_g}(z, \overline{z}) \in x^n \overline{x}^n C^\infty(M^2 \setminus \Delta_t).
\]

Now we examine the pullback \( \beta^*_0 K_{\mathcal{N}_g} \) of \( K_{\mathcal{N}_g} \) to \( M^2 \). First let \( \mathcal{U} \) be a neighborhood of \( \text{ff} \setminus \partial \Delta_{t_0} \), disjoint from the diagonal and \( \text{rf} \). On \( \mathcal{U} \) we use the projective coordinates \( (3, 4) \). By Proposition 4.2 in \( \mathcal{U} \) we have \( \beta^*_0 \rho = \tilde{\alpha} - \log(s), \tilde{\alpha} \in C^\infty(\mathcal{U}) \). Thus the chain rule yields

\[
\begin{align*}
\beta^*_0 (\partial_x \rho, \partial_y \rho) &= (\overline{x}^{-1} (\partial_x \tilde{\alpha} - s^{-1}), \overline{x}^{-1} \partial_y \tilde{\alpha}) = s^{-1} \overline{x}^{-1} \phi, \\
\beta^*_0 (\partial_{\overline{z}} \rho, \partial_{\overline{y}} \rho) &= (\partial_{\overline{z}} \tilde{\alpha} - s \overline{x}^{-1} (\partial_y \tilde{\alpha} - s^{-1}) - \overline{x}^{-1} W^\sigma \partial_{W^\sigma} \tilde{\alpha}, \partial_{\overline{y}} \tilde{\alpha} - \overline{x}^{-1} \partial_y \tilde{\alpha}) = \overline{x}^{-1} \phi,
\end{align*}
\]

where \( \phi, \phi' \) have components in \( C^\infty(\mathcal{U}) \), and further

\[
\begin{align*}
\beta^*_0 \partial^2_{xx} \rho &= \overline{x}^{-2} \left( -\partial_x \tilde{\alpha} + \overline{x} \partial^2_{s\overline{s}} \tilde{\alpha} - so^2_{s\overline{s}} \tilde{\alpha} - W^\lambda \partial^2_{sW^\lambda} \tilde{\alpha} \right) = \overline{x}^{-2} \psi_{00}, \\
\beta^*_0 \partial^2_{x\overline{y}} \rho &= \overline{x}^{-2} \left( \overline{x} \partial^2_{s\overline{s}} \tilde{\alpha} - \partial^2_{sW^\lambda} \tilde{\alpha} \right) = \overline{x}^{-2} \psi_{0\overline{r}}, \\
\beta^*_0 \partial^2_{y\overline{y}} \rho &= \overline{x}^{-2} \left( \overline{x} \partial^2_{W^\sigma W^\sigma} \tilde{\alpha} - \partial^2_{W^\sigma W^\sigma} \tilde{\alpha} \right) = \overline{x}^{-2} \psi_{\overline{r}\overline{r}}, \\
\beta^*_0 \partial^2_{x\overline{z}} \rho &= \overline{x}^{-2} \left( -\partial_{W^\sigma} \tilde{\alpha} + \overline{x} \partial^2_{W^\sigma W^\lambda} \tilde{\alpha} - s \partial^2_{W^\sigma s\overline{s}} \tilde{\alpha} - W^\lambda \partial^2_{W^\sigma W^\lambda} \tilde{\alpha} \right) = \overline{x}^{-2} \psi_{0\overline{z}},
\end{align*}
\]

where \( \psi_{ij} \in C^\infty(\mathcal{U}) \). Note that \( \partial_x, \partial_y \) have different meanings in the left and right hand sides of the above equations. Since for \( H \in \mathbb{R}^{d \times d} \) and \( \lambda \in \mathbb{R} \) we have \( \text{adj}(\lambda H) = \lambda^{d-1} \text{adj}(H) \), \( \beta^*_0 \left( \text{adj}(\partial^2_{zz} \rho) \right) \in \overline{x}^{-2n}C^\infty(\mathcal{U}; \mathbb{R}^{(n+1) \times (n+1)}) \). On the other hand, \( \beta^*_0 \sqrt{\det g(z)} = \overline{x}^{-n-1} \tilde{g}_1 \) and \( \beta^*_0 \sqrt{\det g(\overline{z})} = s^{-n-1} \overline{x}^{-n-1} \tilde{g}_2 \) with \( \tilde{g}_j \in C^\infty(\mathcal{U}) \) and non-vanishing for \( j = 1, 2 \). By (4.4) we conclude that \( \beta^*_0 K_{\mathcal{N}_g} \) has the claimed behavior away from \( \text{rf} \) and \( \Delta_{t_0} \); moreover, the fact that (4.2) is symmetric implies that this is also true away from \( \ell f \) and \( \Delta_{t_0} \).

We now examine \( \beta^*_0 K_{\mathcal{N}_g} \) in a neighborhood \( \mathcal{W} \) of a point in \( \ell f \cap \mathcal{F} \cap \text{rf} \) away from \( \Delta_{t_0} \). Near such a point we have \( |y - \overline{y}| \neq 0 \), hence at least one of the functions \( y^2 - \overline{y}^2 \) does not vanish. We may assume without loss of generality that \( y^n - \overline{y}^n > 0 \) and use coordinates

\[
r = y^n - \overline{y}^n, \quad \theta = \frac{x}{r}, \quad \overline{\theta} = \frac{\overline{x}}{r}, \quad \hat{\gamma} = \frac{y^\lambda - \overline{y}^\lambda}{r}, \quad \lambda = 1, \ldots, n - 1,
\]

which are valid in \( \mathcal{W} \). In terms of these, \( r \) is a defining function for \( \text{ff} \) and \( \theta, \overline{\theta} \) are defining functions for \( \ell f, \text{rf} \) respectively. A computation using the chain rule yields

\[
\begin{align*}
\beta^*_0 \partial_x &= r^{-1} \partial_y, \quad \beta^*_0 \partial_{x\overline{r}} = r^{-1} \left( (r \partial_r - \tilde{\theta} \partial_{\overline{y}}) \partial^2_{x\overline{r}} + V_r \right), \quad V_r \in \mathcal{V}_0(M^2_0), \quad \tilde{\theta}(V_r) = dr(V_r) = 0 \\
\beta^*_0 \partial_{\overline{z}} &= r^{-1} \partial_{\overline{y}}, \quad \beta^*_0 \partial_{\overline{y}\overline{r}} = r^{-1} \left( - (r \partial_r - \tilde{\theta} \partial_{\overline{y}}) \partial^2_{\overline{y}\overline{r}} + \overline{V}_r \right), \quad \overline{V}_r \in \mathcal{V}_0(M^2_0), \quad \overline{\theta}(\overline{V}_r) = dr(\overline{V}_r) = 0.
\end{align*}
\]
We conclude that $\beta^0 \rho = \tilde{a} - \log(\theta) - \log(\tilde{\theta})$, $\tilde{a} \in C^\infty(W)$, so

$$\beta^0_0(\partial_z \rho, \partial_{\tilde{y}} \rho) = -r(\theta)^{-1}, 0 + r^{-1}\tilde{\phi}' \rho,$$

where $\tilde{\phi}, \tilde{\phi}' \in C^\infty(W; \mathbb{R}^{n+1})$. Further, $(r^2 - \theta \tilde{\theta})((r\theta)^{-1}) = 0$ and $(r \partial_r - \tilde{\theta} \partial_{\tilde{r}})((\tilde{r} \theta)^{-1}) = 0$ imply $\beta^0_0 \partial^2 \tilde{z} \rho \in r^{-2}C^\infty(W; \mathbb{R}^{n+1} \times (n+1))$, hence $\beta^0_0 \partial^2 \tilde{z} \rho \in r^{-2n}C^\infty(W; \mathbb{R}^{n+1} \times (n+1))$. Noting that $\beta^0 \sqrt{\det g(z)} = r^{-n-1} \theta - n-1 \tilde{g}_1$ and $\beta^0_0 \sqrt{\det g(\tilde{z})} = r^{-n-1} \tilde{\theta} - n-1 \tilde{g}_2$ with $\tilde{g}_j \in C^\infty(W)$ and non-vanishing, we use (4.4) again and (4.8) to find that $\beta^0_0 K_{\tilde{N}_p} \in \theta^n \tilde{\theta}^n C^\infty(W)$. We conclude that $\beta^0_0 K_{\tilde{N}_p} \subset C^\infty(M_0^a \setminus \Delta_{t0})$ and vanishes to order $n$ on $\ell f$ and $rf$.

We now examine the behavior of the pullback of (4.2) by $\beta_0$ near $\partial \Delta_{t0}$. First note that $\beta^0_0 \rho^2 \big|_{(M_0^a)^o}$ vanishes exactly on $\Delta_{t0} \cap (M_0^a)^o$. By Proposition 4.2, $\beta^0_0 \rho^2$ is smooth in a neighborhood of $\Delta_{t0}$, hence it also vanishes at $\partial \Delta_{t0}$; moreover, it vanishes nowhere else on $\mathbb{R}^n$: this follows from [CH16, Proposition 24], according to which for every $p \in \partial M$, $\beta_0 \rho_p \big|_{\mathbb{R}^n} = \rho_h^u \cdot e_p$, where $\rho_h^u$ is the hyperbolic distance induced by $h^u$ on $\mathbb{R}^n$. We now use a variant of the coordinates given by (3.4) near $\Delta_{t0}$ and away from $rf$; we use $(\tilde{z}, Z) = (z, z - \tilde{z})/\tilde{x}$, $(\tilde{x}, y, s - 1, W)$. In terms of those coordinates $\tilde{x}$ is a defining function for $ff$ and $\Delta_{t0}$ is expressed as $\{Z = 0\}$. Observe that on $(M_0^a)^o$ one has

$$\beta^0_0 \rho^2 \big|_{Z = 0} = 0, \quad \partial_{Z_{\tilde{z}}}((\beta^0_0 \rho^2) \big|_{Z = 0}) = 2\bar{x} \beta^0_0 \left(\partial_{Z_{\tilde{z}}}((\rho^2) \big|_{Z = 0})\right),$$

$$\partial_{Z_{\tilde{z}}}((\beta^0_0 \rho^2) \big|_{Z = 0}) = 2\bar{x}^2 \beta^0_0 \left(\partial_{Z_{\tilde{z}}}((\rho^2) \big|_{Z = 0})\right).$$

By the smoothness of $\beta^0_0 \rho^2$ near $\Delta_{t0}$, (4.9) holds all the way to the front face. Thus by Taylor’s Theorem, viewing $\tilde{z}$ as parameters, we write

$$\beta^0_0 \rho^2 \bigg|_{\Delta_{t0}} = 2|\partial(\beta^0_0 \rho^2)\big|_{Z = 0} \bar{x} + b_{klm}(\tilde{z}, Z) Z^k Z^l Z^m,$$

where $b_{klm}(\tilde{z}, Z)$ is smooth. We now show that the expression $\hat{\beta}((z, \tilde{z}) := -2|\partial(\beta^0_0 \rho^2)\big|_{Z = 0} \bar{x}$ pulls back to a non-vanishing smooth function in a neighborhood of $\Delta_{t0}$, all the way to the front face. Using computations as in (4.6) with $\rho$ replaced by $\rho^2$, one can conclude that $\bar{x}^2 \beta^0_0 \partial_{\tilde{z}}((\rho^2)/2)$ is a smooth matrix valued function in a neighborhood of $\partial \Delta_{t0}$ (note here that its behavior near $s = 0$ is irrelevant for this computation). Also, for $z, \tilde{z} \in \bar{M}$ one has $|\det(\partial_{\tilde{z}} \rho^2)\big|_{Z = 0} = |\det g(\tilde{z})|$. Therefore, since $\bar{x}^{2n+2} \beta^0_0 \left(\sqrt{\det g(z)} \sqrt{\det g(\tilde{z})}\right)$ is smooth and non-vanishing near $\Delta_{t0}$, $\beta^0_0 \hat{K}$ is smooth in a neighborhood of the lifted diagonal. Moreover, $\hat{K}|_{\Delta_{t0}} = 2$ implies that $\beta^0_0 \hat{K}|_{\Delta_{t0}} = 2$.

The facts in the preceding paragraph together with a standard argument involving the Fourier transform imply that $\beta^0_0 K_{\tilde{N}_p} \in L^{-1}(M_0^a, \Delta_{t0})$, with principal symbol $\sigma^0_0(\tilde{N}_p) = C_n|\tilde{\xi}|^{-1}$ for $\tilde{\xi} \neq 0$, where $\tilde{\xi}$ is the fiber variable for $N^* \Delta_{t0}$; near the zero section the principal symbol is smooth and modifying it in compact subsets of the fiber does not change the operator modulo $\Psi_0^{-\infty}(M)$. Using the identification of $N^* \Delta_{t0}$ with $0^* T^* M$ and the fact that the latter is trivialized by $\{dz^2/x^2\}$ near $\partial M$, we can write invariently $\sigma^0_0(\tilde{N}_p)(z, \tilde{\xi}) = C_n|\tilde{\xi}|^{-1}, (z, \tilde{\xi}) \in 0^* T^* M \setminus 0$; this agrees with the principal symbol computed in [SU04]. Since $g$ defines a non-degenerate quadratic form in the fibers of $0^* T^* M$, $\sigma^0_0(\tilde{N}_p)$ is invertible on $0^* T^* M$. We have thus shown that $\tilde{N}_p \in \Psi_0^{-1, n, n}(M)$ and is elliptic, completing the proof. □
By Propositions 4.4 and 3.3 it follows immediately that for \( s \geq 0 \)
\[
N_g : x^\delta H^1_0(M; \Omega_0^{1/2}) \to x^{\delta'} H^{s+1}_0(M; \Omega_0^{1/2})
\]
is bounded if \( \delta > -n/2, \delta' < n/2 \) and \( \delta' \leq \delta \). We can now prove a boundedness property for the X-ray transform showing that one can extend it to larger weighted \( L^2 \) spaces than the ones that appeared in Section 2. We use notations as in Lemma 2.1.

**Corollary 4.5.** Let \( (\mathbb{M}^{n+1}, g) \) be a simple AH manifold. If \( \delta' < \delta, \delta' < 0 \) and \( \delta > -n/2 \) the X-ray transform is bounded:
\[
I : x^\delta L^2(M, dV_g) \to \langle \eta \rangle_h^{-\delta'} L^2(\partial - S^* M, d\lambda_0).
\]

**Proof.** We will show that for \( \delta, \delta' \) as in the statement there exists a constant \( C \) such that for any \( f \in C^\infty_c(M) \) one has
\[
(4.10) \quad \|If\|_{x^{\delta'} L^2(SM; d\lambda)} \leq C\|f\|_{x^{\delta} L^2(M; dV_g)}.
\]

Since for \( f \in C^\infty_c(M) \) one has \( If \in C^\infty_c(S^* M) \cap x^{\delta'} L^2(S^* M; d\lambda) \) as explained in the proof of Lemma 2.1, the latter applies for \( If \) to show that if (4.10) is known then one has
\[
\|If\|_{\langle \eta \rangle_h^{-\delta'} L^2(\partial - S^* M; d\lambda_0)} \leq C\|f\|_{x^{\delta} L^2(M; dV_g)},
\]
yielding the result by density.

First let \( 0 < \varepsilon < \min\{\delta + n/2, \delta - \delta'\} \) and note that for each fixed \( \varepsilon \) the expression
\[
I(x^{2\varepsilon})(z, \xi) = \int_\mathbb{R} x^{-2\varepsilon} \circ \varphi_t(z, \xi) dt
\]
is uniformly bounded on \( S^* M \), by the proof of Lemma 2.1.

Now for \( f \in C^\infty_c(M) \) apply Cauchy-Schwarz to find
\[
\|If\|_{x^{\delta'} L^2(SM; d\lambda)}^2 = \int_{S^* M} x^{-2\delta'} |If(z, \xi)|^2 d\lambda = \int_{S^* M} x^{-2\delta'} \int_\mathbb{R} |f(\varphi_t(z, \xi))| dt d\lambda
\]
\[
\leq \int_{S^* M} x^{-2\delta'} \int_\mathbb{R} x^{2\varepsilon} \circ \varphi_t(z, \xi) dt \int_\mathbb{R} |(x^{-\varepsilon} f)(\varphi_t(z, \xi))|^2 dt d\lambda
\]
\[
\leq C \int_{S^* M} x^{-2\delta'} \int_\mathbb{R} |(x^{-\varepsilon} f)(\varphi_t(z, \xi))|^2 dt d\lambda = C \int_{M} x^{-2\delta'} \int_{S^* M} I(x^{-2\varepsilon} |f|^2) H_d d\lambda_0
\]
\[
= C\|N_g(x^{-2\varepsilon} |f|^2)\|_{x^{\delta'} L^1(M, d\lambda_0)}.
\]

Now if \( \delta'' := \delta - \varepsilon \) the choice of \( \varepsilon, \delta \) and \( \delta' \) imply that \( \delta'' < 0 \), \( \delta'' < -n \) and \( 2\delta'' \leq 2\delta'' \).

On the other hand, again an argument similar (but simpler) to the one of used in Maz'ya (1991) to show Proposition 3.3 (also see Heisler 1999 Appendix 1), shows that if \( P \in \Psi^{-1, n, n}_0(M) \) with \( \text{Re}(E_0) > n + \sigma', \text{Re}(E_0) > -\sigma \) and \( \sigma'' - \sigma \leq \text{Re}(E_0) \) then \( P : x^{\sigma} L^1(M, dV_g) \to x^{\sigma'} L^1(M, dV_g) \) is bounded. Hence the fact that \( N_g \in \Psi^{-1, n, n}_0(M) \) implies that
\[
\|N_g(x^{-2\varepsilon} |f|^2)\|_{x^{\delta'} L^1(M, d\lambda_0)} \leq C\|x^{-2\varepsilon} |f|^2\|_{x^{\delta''} L^1(M, d\lambda_0)} = C\|f\|_{x^{\delta''} L^2(M, d\lambda_0)}^2
\]
and this finishes the proof. \( \square \)

**Remark 4.6.** By Corollary 4.5 and (2.4) one also has that \( I^* : \langle \eta \rangle_h^\delta L^2(\partial - S^* M, d\lambda_0) \to x^{-\delta} L^2(M, dV_g) \) is bounded for \( \delta' < \delta, \delta' < 0 \) and \( \delta > -n/2 \).

5. The Model Operator

In this section we show that the model operator of \( N_g \) at a point \( p \in \partial M \) can be identified with the normal operator \( N_h \) on the Poincaré hyperbolic ball \( (\mathbb{R}^{n+1}, h) \). This operator was studied in (1991) and an explicit inversion formula was computed for it using the spherical Fourier transform; using this formula we will show that \( N_h^{-1} \in \Psi^{0, n+1, n+1}_0(\mathbb{R}^{n+1}) \).
In what follows we always assume that a choice of coordinates has been made with respect to a point of interest \( p \in \partial M \), such that the hyperbolic metric \( h_p \) in (3.7) takes the form \( h_p = u^{-2}(du^2 + |dw|^2) \) with respect to induced linear coordinates \((u, w)\) on \( T^*_p M \).

The following is an analog of Proposition 2.17 in [MM87], which shows that for each \( p \in \partial M \) the model operator of the Laplacian corresponding to an AH metric \( g \) on \( M \) is the hyperbolic Laplacian on \((T^*_p M, h_p)\):

**Proposition 5.1.** For any \( p \in \partial M \) the model operator \( N_p(\mathcal{N}_p) \) on \( T^*_p M \) is given by \( N_{h_p} \), the normal operator corresponding to the X-ray transform on \( (T^*_p M, h_p) \).

**Proof.** We will show that in coordinates \((\tilde{x}, \tilde{y}, s, W)\) (see (3.4)) we have

\[
\beta^*_p(K_N \langle z, \tilde{z} \rangle \gamma_0(z) \otimes \gamma_0(\tilde{z}))|_{\mathcal{T}_p} \quad (\text{for } u, \lambda = \langle (s, W), (1, 0) \rangle)
\]

with the interpretation of (3.12), where \( h^\ell_p \) denotes the \( h^\ell_p \)-distance function on \( \mathcal{I}^\circ_p \).

This will imply that in linear coordinates \( v = (u, w) \), \( \tilde{v} = (u, w) \) on \( T^*_p M \) we have as in (3.14)

\[
N_p(\mathcal{N}_p)(f \cdot \gamma_p)(v) = \int_{T^*_p M} \frac{2|\det(g^{p}_{\tilde{x}}p^2_{\tilde{y}})/2)| \rho^n_p \rho^s_p \pi^s_p \sqrt{\det h_p} \pi^s_p \sqrt{\det h_p}}{f(\tilde{v})dV_{\tilde{h}_p}(\tilde{v}) \cdot \gamma_p = N_{h_p}(f \cdot \gamma_p).}
\]

To see the second equality, it suffices to show that under the transformations \((u, w, \tilde{u}, \tilde{w}) \mapsto (u, w + a, \tilde{u}, \tilde{w} + a), a \in \mathbb{R}^n, (u, \tilde{u}, w, \tilde{w}) \mapsto (\lambda u, \lambda w, \lambda \tilde{u}, \lambda \tilde{w}), \lambda \in \mathbb{R} \).

\[F(v, \tilde{v}) \mapsto (F, \tilde{F}) \text{ on } (\mathbb{R}^n \times \mathbb{R}^n)^2 \text{ has this property exactly when it is annihilated by } \partial_{w^s} + \partial_{\tilde{w}^s} \text{ and } u \partial_u + w^s \partial_{w^s} + \tilde{u} \partial_{\tilde{u}} + \tilde{w} \partial_{\tilde{w}^s}; \text{ the fact that } [u \partial_u, u \partial_{w^s}] = u \partial_{w^s} - u \partial_{u^s} \text{ can be used to show that if } F \text{ has this property, the same is true of } u \partial_u F, \tilde{u} \partial_{\tilde{u}} F. \]

By the explicit formula \( \cosh \rho_{h_p}((u, w), (\tilde{u}, \tilde{w})) = 1 + \frac{|w - \tilde{w}|^2 + |u - \tilde{u}|^2}{2u} \) for the \( h_p \)-distance on \( T^*_p M \) when \( h_p = u^{-2}(du^2 + |dw|^2) \), \( \rho_{h_p} \) has the required invariance, thus \( u \partial_u^2 \tilde{w} \rho_{h_p}^2/2 \) also does, and the same is true of \( (w + 1)w^{n+1} \rho_{h_p}^n \rho^s_p \pi^s_p \sqrt{\det h_p} \).

We now show (5.1). By [CH16, Proposition 24], \( \beta^*_p \beta^*_p |_{\mathcal{T}_p}(g) = \rho_{h_p}(g, e_p) \).

As mentioned earlier, it can be arranged that in terms of coordinates \((s, W) \) on \( \mathcal{I}_p \) as in (3.4) we have \( h_p^\ell = s^{-2}(ds^2 + |dW|^2) \), hence again \( \rho_{h_p^\ell}((s, W), (\tilde{s}, \tilde{W})) = \rho_{h_p^\ell}((s, W), (\tilde{s}, \tilde{W})/s (W - \tilde{W}), (1, 0)) \).

Using this, one checks that at \( q, e_p = ((s, Y), (1, 0)) \) we have \( \partial_s \rho_{h_p^\ell} = -s \partial_s \rho_{h_p^\ell} - W \partial_W \rho_{h_p^\ell}, \partial_W \rho_{h_p^\ell} = -\partial_W \rho_{h_p^\ell} \).

Those facts together with (4.6) yield that at \( q = (s, W) \)

\[
\beta^*_p(\tilde{x}(\partial_\tilde{z} \rho, \partial_\tilde{y} \rho))|_{\mathcal{T}_p} = (\partial_s \rho_{h_p^\ell}, \partial_W \rho_{h_p^\ell})|_{\mathcal{T}_p}, \quad \beta^*_p(\tilde{x}(\partial_\tilde{z} \rho, \partial_y \rho))|_{\mathcal{T}_p} = (\partial_s \rho_{h_p^\ell}, \partial_W \rho_{h_p^\ell})|_{\mathcal{T}_p},
\]

\[
\beta^*_p(\tilde{x}(\partial_\tilde{y} \rho, \partial_\tilde{y} \rho))|_{\mathcal{T}_p} = (\partial_s \rho_{h_p^\ell}, \partial_W \rho_{h_p^\ell})|_{\mathcal{T}_p}, \quad \beta^*_p(\tilde{x}(\partial_\tilde{y} \rho, \partial_y \rho))|_{\mathcal{T}_p} = (\partial_s \rho_{h_p^\ell}, \partial_W \rho_{h_p^\ell})|_{\mathcal{T}_p}.
\]
constant, where $\Delta$ denotes the hyperbolic Laplacian with principal symbol $-|\xi|^2$. On the other hand, we have $\beta^*(\bar{x}^{2n+2}\sqrt{\det g(\bar{z})\det g(\bar{z})})|_{\bar{x}=0} = s^{-n-1} = \pi^*_h\sqrt{\det h^\ell_p} \pi^*_h\sqrt{\det h^\ell_p}|(s,W),(1,0))$. Since $\pi^*_h\gamma_0 \otimes \pi^*_h\gamma_0|(p,p)$ can be identified with $\frac{\det h^\ell_p}{\det h^\ell_p}|1/2 \frac{\det h^\ell_p}{\det h^\ell_p}|1/2$ as explained in Section 3.3, we obtain (5.1).

As mentioned in §3.5 for each $p \in \partial M$ the model operator $N_{h_p}$ can be equivalently realized as an operator on the Poincaré ball $(B^{n+1}, h = \frac{4|dz|^2}{(1-|z|^2)^2})$. The following proposition is essentially an immediate consequence of the results in [BC91]. Henceforth we write $B$ instead of $B^{n+1}$ (i.e. without a superscript for the dimension).

**Proposition 5.2.** For any $p \in \partial M$ the model operator $N_p(N_g)$ can be identified with the operator $N_h : C_c^\infty(B; \Omega^1_0) \to C_c^\infty(B; \Omega^1_0)$ on $(B, h)$, which for $\delta \in (-n/2, n/2)$ extends continuously to an operator $N_h : x^\delta L^2(B; \Omega^1_0) \to x^\delta H^1_0(B; \Omega^1_0)$. The operator $N_h$ has a two sided inverse $N_h^{-1} \in \Psi^{-n,1,1,n}_0(B)$ such that $N_h^{-1}N_h = N_hN_h^{-1} = Id$ on $x^\delta L^2(B; \Omega^1_0)$ for $\delta \in (-n/2, n/2)$.

**Proof.** For each $p \in \partial M$, $N_p(N_g) = N_{h_p}$ on $(T^+_p M, h_p)$ by Proposition 5.1 and $N_{h_p}$ can be identified with $\mathcal{N}_h$ on $(B, h)$ as explained above. Thus by Proposition 4.4, $N_h \in \Psi^{-1,n,n}_0(B)$ and the extension statement follows from Proposition 3.3. It was observed in [BC91] that $N_h$ can be expressed as

\[(5.2) \quad N_h f(z) = \int_B R(\rho_h(z, \bar{z})) f(\bar{z}) dV_h(\bar{z}), \quad f \in C_c^\infty(B) \]

where $\rho_h$ is the geodesic distance function with respect to the hyperbolic metric and $R(r) = 2 \sinh^{-n}(r)$. $\bar{z}$ can be interpreted as convolution by a locally integrable radial function on the homogeneous space $B \cong G/H_o$, where $G = O^+(1, n + 1)$ is the isometry group of $B$ and $H_o \cong O(n + 1)$ is the isotropy group of the origin $o \in B$. For $f \in C_c^\infty(B)$

\[N_h f(gH_o) = R * f(gH_o) = \int_B R(\rho_h^{-1}(gH_o)) f(gH_o) dV_h(gH_o), \quad z = gH_o, \quad \bar{z} = gH_o,\]

where above and in what follows, slightly abusing notation, we identify radial functions and distributions on $B$ with ones on $[0, \infty)$, writing for instance $R(z) = R(\rho_h(z, o))$ for $z \in B$.

An exact left inverse for $N_h$ is computed in [BC91] Theorems 4.2, 4.3, 4.4: one has

\[C_n p(\Delta) S_n N_h = Id \quad \text{on } C_c^\infty(B),\]

where $\Delta$ denotes the hyperbolic Laplacian with principal symbol $-|\xi|^2$, $C_n$ is an explicit constant, $p(t) = -(t + n - 1)$ and $S_n$ is given by convolution with the locally integrable radial kernel

\[S_n(r) = \begin{cases} \coth(r) - 1, & n = 1 \\ \sinh^{-n}(r) \cosh(r), & n \geq 2 \end{cases};\]

that is,

\[S_n f(z) = S_n * f(z) = \int_B S_n(\rho_h(z, \bar{z})) f(\bar{z}) dV_h(\bar{z}), \quad f \in C_c^\infty(B).\]

The fact that $C_n p(\Delta) S_n$ is also a right inverse for $N_h$ follows by tracing through the proofs of Theorems 4.2.4-4.4 in [BC91]. The authors use the spherical Fourier transform of a
radial distribution, given by \( \tilde{f}(\lambda) = \int_{\mathbb{B}} f(\bar{z}) \phi_{-\lambda}(\bar{z}) dV_h(\bar{z}) \) for \( \lambda \in \mathbb{R} \), where \( \phi_\lambda \) is the radial eigenfunction of \( \Delta \) with eigenvalue \(-n^2/4 - \lambda^2\) that satisfies \( \phi_\lambda(o) = 1 \). The spherical Fourier transform is well defined pointwise whenever \( f(\bar{z}) \phi_{-\lambda}(\bar{z}) \) is integrable; the reason why the formula corresponding to \( n = 1 \) in (5.3) differs from the one corresponding to \( n \geq 2 \) is exactly to ensure that \( \hat{S}_1 \) is well defined. Their strategy is to show that \( (C_n p(\Delta)S_n * \mathcal{R}) \) implies that \( \hat{\delta} \), where \( \delta \) is the delta distribution at the origin. Thus the claim reduces to showing that \( (\mathcal{R} * C_n p(\Delta)S_n) \) \( \hat{=} \delta \). This in turn follows from the fact that for radial distributions \( \mathcal{U}, \mathcal{V} \) one has \( \mathcal{U} \ast \mathcal{V}(\lambda) = \mathcal{U}(\lambda) \mathcal{V}(\lambda) \), and also \( p(\Delta)\mathcal{U}(\lambda) = -p(-n^2/4 - \lambda^2)\mathcal{U}(\lambda) \), provided the expressions make sense.

Now let \( \varphi(x) \in C_0^{\infty}([0, \infty)) \) be identically 1 on \([0, 1]\) and identically 0 on \([0, 2]\] and let

\[
S_{n,1} f(z) = \int_{\mathbb{B}} \varphi(\rho_h(z, \bar{z})) S_n(\rho_h(z, \bar{z})) f(\bar{z}) dV_h(\bar{z})
\]

and

\[
S_{n,2} f(z) = \int_{\mathbb{B}} (1 - \varphi(\rho_h(z, \bar{z}))) S_n(\rho_h(z, \bar{z})) f(\bar{z}) dV_h(\bar{z})
\]

for \( f \in C_0^{\infty}(\mathbb{B}^n) \), so that \( S_n = S_{n,1} + S_{n,2} \). Using Proposition 1.2 one sees that the Schwartz kernel of \( S_{n,1} \) vanishes identically near the left and right faces of the 0-stretched product \( (\mathbb{B}^2)^n \); thus the \( O(r^{-n}) \) leading order singularity of \( S_n(r) \) at \( r = 0 \) implies that \( S_{n,1} \in \Psi_{1}^{-1}(\mathbb{B}) \) and hence \( p(\Delta)S_{n,1} \in \Psi_{0}^{0}(\mathbb{B}) \) since \( p(\Delta) \) \( \in \) \( \Diff_{0}^{1}(\mathbb{B}) \).

The hyperbolic Laplacian acting on radial distributions is given in terms of geodesic polar coordinates by \( \Delta = \partial_t^2 + n \coth(t) \partial_t \) and one checks that for \( n \geq 1 \)

\[
p(\Delta)S_n(r) = - (\Delta + n - 1) \sinh^{-n}(r) \cosh(r) = -n \sinh^{-n-2}(r) \cosh(r).
\]

Since for \( f \in C_0^{\infty}(\mathbb{B}) \)

\[
p(\Delta) S_{n,2} f(z) = \int_{\mathbb{B}} p(\Delta)((1 - \varphi(r)) S_n(r)) \big|_{r = \rho_h(z, \bar{z})} f(\bar{z}) dV_h(\bar{z}),
\]

(5.4) and Proposition 4.2 imply that \( p(\Delta)S_{n,2} \in \Psi_{0}^{-n-1}(\mathbb{B}^{n+1}) \). We conclude that \( p(\Delta)S_n \in \Psi_{0}^{-n,n+1}(\mathbb{B}) \) for all \( n \geq 1 \) and the spaces on which the inversion is valid follow again from Proposition 3.3 by density.

6. Parametrix construction and Stability Estimates

We begin this section with the construction of a parametrix for \( \mathcal{N}'_g \):

**Proposition 6.1.** Let \( (\hat{M}^{n+1}, g) \) be a simple AH manifold. There exists an operator \( B \) such that for \( \delta \in (-n/2, n/2) \) and \( s \geq 0 \)

\[
B : x^\delta H_0^{s+1}(M; \Omega_0^{1/2}) \to x^\delta H_0^{s}(M; \Omega_0^{1/2})
\]

is bounded and on \( x^\delta H_0^{s}(M; \Omega_0^{1/2}) \) one has

\[
BN_g = Id - K, \quad K \in \Psi_{0}^{-\infty}(\mathcal{F}(M)), \quad F_{\ell} \geq 1, \quad F_{\ell}, F_r \geq n.
\]

In particular, \( K : x^\delta H_0^{s}(M; \Omega_0^{1/2}) \to x^\delta H_0^{s}(M; \Omega_0^{1/2}) \) is compact for such \( \delta \) and \( s \).

**Proof.** We write \( \mathcal{N}_g = A_1 + A_2 \), where \( A_1 \in \Psi_{1}^{-1}(M) \), \( A_2 \in \Psi_{0}^{-n,n}(M) \). By the ellipticity of \( A_1 \) (and hence of \( A_1 \)), Theorem 3.8] implies the existence of \( B_1 \in \Psi_{1}^{0}(M) \) such that

\[
B_1 A_1 = Id - K_1, \quad K_1 \in \Psi_{1}^{-\infty}(M).
\]
Note that $K_1$ is not compact on any weighted Sobolev space $\pi^s H^s_0(M; \Omega^{1/2}_0)$ since its kernel does not vanish at $\partial f$. Using Proposition 3.1, we reach

$$B_1 N_g = Id - K_2, \quad K_2 := K_1 - B_1 A_2 \in \Psi_0^{-\infty, n, n}(\mathbb{B}).$$

We now improve the error term to ensure that its kernel vanishes at the front face. For each $p \in \partial M$ we have $N_p (K_2) \in \Psi_0^{-\infty, n, n}(\mathbb{B})$, under the identification of $(T^*_p M, h_p)$ with $(\mathbb{B}, h)$, according to the remarks following (3.14); this identification depends smoothly on $p$. Propositions 5.2 and 3.1 imply that $N_p (K_2) N_p (N_g)^{-1} = N_p (N_g)^{-1} = N_p (B_1) \in \Psi_0^{1, n+1, n+1}(\mathbb{B})$.

Thus $N_p (K_2) N_p (N_g)^{-1} \in \Psi_0^{-\infty, \ell}(\mathbb{B}) \cap \Psi_0^{1, n+1, n+1}(\mathbb{B}) \subset \Psi_0^{-\infty, n+1, n+1}(\mathbb{B})$. Again using the identification $(T^*_p M, h_p) \leftrightarrow (\mathbb{B}, h)$, the convolution kernel of $N_p (K_2) N_p (N_g)^{-1}$ is a polyhomogeneous (in fact smooth) half density in $\mathcal{A}_p^{n+1, n+1}(\mathbb{B}) \cap \mathcal{S}^{n+1, n+1}(\mathbb{B})$. By (3.13), it can be extended off of the front smoothly to produce an operator $B_2 \in \Psi_0^{-\infty, n+1, n+1}(M)$ such that at each $p \in \partial M$, $F_p (B_2)$ agrees with the convolution kernel of $N_p (K_2) N_p (N_g)^{-1}$.

By Lemma 3.4 and Proposition 3.5 this implies that $F_p (B_2 N_g) = F_p (K_2)$. Setting $B = B_1 + B_2 \in \Psi_0^{1, n+1, n+1}(M)$ and using Proposition 3.1, we find

$$B N_g = Id - K, \quad K \in \Psi_0^{-\infty, \mathcal{F}}(M),$$

$$F_f = \{(1, 0)\} \cup \{(2n + 1, 1)\}, \quad F_\ell = F_r = \{(n, 0)\} \cup \{(n + 1, 1)\}.$$  

As already stated earlier, by Proposition 3.3 one has that for $s \geq 0$, $N_g : x^\delta H^s_0(M; \Omega^{1/2}_0) \to x^{\delta'} H^{s+1}_0(M; \Omega^{1/2}_0)$ is bounded provided $\delta > -n/2$, $\delta' < n/2$ and $\delta \leq \delta'$. Moreover, $B : x^\delta H^{s+1}_0(M; \Omega^{1/2}_0) \to x^{\delta''} H^{s}_0(M; \Omega^{1/2}_0)$ is bounded provided $\delta'' > -n/2 - 1$, $\delta'' < n/2 + 1$ and $\delta'' \leq \delta'$. Hence choosing $\delta = \delta' = \delta'' \in (-n/2, n/2)$ we obtain (6.1) and (6.2). Moreover, for such choice of $\delta$, choose $\delta \bar{s}$ such that $\delta < \delta < \min\{n/2, \delta + 1\}$, and $\bar{s} > s$ to guarantee that $K : x^\delta H^{s}_0(M; \Omega^{1/2}_0) \to x\bar{s} H^{\bar{s}}_0(M; \Omega^{1/2}_0)$ is bounded, implying that $K : x^\delta H^{s}_0(M; \Omega^{1/2}_0) \to x^{\delta} H^{s}_0(M; \Omega^{1/2}_0)$ is compact, as claimed. \hfill \square

Proposition 6.1 together with Proposition 3.3 imply that for $\delta \in (-n/2, n/2)$

(6.3) $x^\delta L^2(M; \Omega^{1/2}_0) \cap \ker N_g \subset \bigcap_{m \in \mathbb{R}} x^\delta H^m_0(M; \Omega^{1/2}_0) =: x^\delta H^\infty_0(M; \Omega^{1/2}_0) \subset C^\infty(M; \Omega^{1/2}_0)$.

We will now show that elements of $x^\delta L^2(M; \Omega^{1/2}_0) \cap \ker N_g$ also have polyhomogeneous expansions at the boundary. Henceforth we work with functions as opposed to half densities for convenience. We start by showing tangential regularity.

**Lemma 6.2.** Let $u \in x^\delta L^2(M, dv_g) \cap \ker N_g$, with $\delta \in (-n/2, n/2)$. Then

$$u \in x^\delta H^\infty_0(M, dv_g) := \{u \in x^\delta L^2(M, dv_g) : Pu \in x^\delta L^2(M, dv_g), P \in \text{Diff}^m_0(M), m \geq 0\};$$

Here $\text{Diff}^m_0(M)$ (by analogy with $\text{Diff}^m(M)$) stands for differential operators consisting of finite sums of at most $m$-fold products of vector fields in $V_0(M)$.

**Proof.** Smoothness of $u$ in $\tilde{M}$ was already remarked in (6.3). We will show that for any $m \geq 0$, if it is the case that $Pu \in x^\delta L^2(M, dv_g)$ for every $P \in \text{Diff}^m_0(M)$, then $P' u \in x^\delta L^2(M, dv_g)$ for every $P' \in \text{Diff}^{m+1}_0(M)$. Since $u \in x^\delta L^2(M, dv_g)$ by assumption, this suffices to prove
the lemma. We will first prove an auxiliary fact, namely that if \( P \in \text{Diff}^{m+1}_b(M) \), \( m \geq 0 \), then \( Pu \) can be written as a finite sum

\[
(6.4) \quad Pu = \sum_j Q_j^{(m)} P_j^{(m)} u, \quad Q_j^{(m)} \in \Psi_0^{-\infty,F_i,F_r,F_j}(M) \text{ and } P_j^{(m)} \in \text{Diff}^m_b(M),
\]

where \( F_i' = F_f - 1 \geq 0 \) and \( F_i, F_r, F_r \) are as in Proposition 6.1. Once it has been established, \( (6.4) \) will immediately imply the claim using Proposition 3.3.

By Proposition 3.30 in [Maz91a], \( [Q,V] \) and show that it has the required form. We find \( \psi \in \text{Diff}^m_b(M) \) and

\[
\text{VPu} = \sum_j VQ_j^{(m)} P_j^{(m)} u = \sum_j (Q_j^{(m)} VP_j^{(m)} u - [Q_j^{(m)}, V] P_j^{(m)} u).
\]

By Proposition 3.30 in [Maz91a], \( [Q,V] \in \Psi_0^{-\infty,F_i,F_r,F_j}(M) \) for \( Q \in \Psi_0^{-\infty,F_i,F_r}(M) \) and \( V \in \mathcal{V}_b(M) \). Thus \( [Q_j^{(m)}, V] \in \Psi_0^{-\infty,F_i,F_r,F_j}(M) \) for all \( j \). Since \( VP_j^{(m)}, P_j^{(m)} \in \text{Diff}^{m+1}_b(M) \) we obtain \( (6.4) \) for \( m + 1 \), finishing the proof.

In Proposition 6.6 below we will use the Mellin transform to show the existence of polyhomogeneous expansion at the boundary for elements in the nullspace of \( \mathcal{N}_g \). We briefly recall its definition and main properties. Below we write \( \mathbb{R}^+ = (0, \infty) \).

**Definition 6.3.** If \( f \in C_c^\infty(\mathbb{R}^+) \) and \( \zeta \in \mathbb{C} \) we define the Mellin Transform of \( f \) by

\[
f_M(\zeta) = \int_0^\infty x^\zeta f(x) \frac{dx}{x}.
\]

By the fact that \( f_M(\zeta) = \mathcal{F}(f(\exp(\cdot)))(i\zeta) \) for \( \zeta \) imaginary, we see that for \( f \in C_c^\infty(\mathbb{R}^+) \), \( f_M \) is rapidly decaying along each line \( \zeta = \alpha + i\eta \), as \( \mathbb{R} \ni \eta \to \pm \infty \) where \( \alpha \in \mathbb{R} \) is constant. Moreover, the Mellin transform induces an isomorphism

\[
\mathcal{M} : x^\delta L^2(\mathbb{R}^+, \frac{dx}{x}) \to L^2(\{\text{Re}(\zeta) = -\text{Re}(\delta)\}, |d\zeta|)
\]

with inverse given by

\[
u(x) = \frac{1}{2\pi} \int_{\text{Re}(\zeta) = -\text{Re}(\delta)} x^{-\zeta} u_M(\zeta) |d\zeta|.
\]

By the Paley-Wiener Theorem, if \( u \in x^\delta L^2(\mathbb{R}^+, \frac{dx}{x}) \) and \( \text{supp} u \subset [0,1) \) then \( u_M \) extends to a holomorphic function on the half plane \( \{\text{Re}(\zeta) > -\text{Re}(\delta)\} \), uniformly in \( L^2(\{\text{Re}(\zeta) = \alpha\}, |d\zeta|) \) for \( \alpha \geq -\text{Re}(\delta) \). By analogy with the Fourier transform, we also have \( (x\partial_x u)_M(\zeta) = -\zeta u_M(\zeta) \) on the half plane \( \{\text{Re}(\zeta) \geq -\text{Re}(\delta)\} \) provided \( \text{supp} u \subset [0,1) \) and \( u, x\partial_x u \in x^\delta L^2(\mathbb{R}^+, \frac{dx}{x}) \). Moreover, if \( \varphi \in C_c^\infty([0,\infty)) \) is identically 1 near 0 then \( (x^\delta |\log(x)|^k \varphi)_M(\zeta) \) is holomorphic on the half plane \( \{\text{Re}(\zeta) > -\text{Re}(\delta)\} \) for \( k \) non-negative integer, and using an integration by parts one sees that it extends meromorphically on \( \mathbb{C} \), with a pole of order \( k+1 \).
at \( \zeta = -\delta \). If \( M \) is a compact manifold with boundary one can use a product decomposition \([0, \varepsilon_0) \times \partial M \) of a collar neighborhood of \( \partial M \) and compute the Mellin transform in the \( x \) variable for polyhomogeneous conormal functions supported near \( \partial M \). If \( \varphi \in C^\infty(M) \) is supported near \( \partial M \) and \( u \in \mathcal{A}_{phg}^E(M) \), then \((u \varphi(x))_M \) is meromorphic on \( \mathbb{C} \) with poles of order \( p + 1 \) at \( \zeta = -s - \ell \) and values in \( C^\infty(\partial M) \) for each \((s, p) \in E \) and for \( \ell \in \mathbb{N}_0 = \{0, 1, \ldots\} \). The fact that the space \( \mathcal{A}_{phg}^E(M) \) is invariantly defined, as already remarked earlier, implies that the analyticity properties of \((\varphi u)_M \) are invariantly defined.

Before we show the existence of an asymptotic expansion for elements in the nullspace of \( N_g \) we show a lemma about index sets.

**Lemma 6.4.** Let \( E_1, E_2, F \subset \mathbb{C} \times \mathbb{N}_0 \) be index sets. Then \((E_1 \cap E_2) + F \subset (E_1 + F) \cap (E_2 + F)\).

**Proof.** First note that \((E_1 \cap E_2) + F = (E_1 + F) \cap (E_2 + F) \subset (E_1 + F) \cap (E_2 + F)\). Now suppose that \((s, p_1 + p_2 + 1) \in E_1 \cap E_2 \), where \((s, p_1) \in E_1 \) and \((s, p_2) \in E_2 \) and let \((\tilde{s}, \tilde{p}) \in F\). Then \((s + \tilde{s}, (p_1 + \tilde{p}) + (p_2 + \tilde{p}) + 1) \in (E_1 + F) \cap (E_2 + F)\), so it is also the case that \((s, p_1 + p_2 + 1) + (\tilde{s}, \tilde{p}) = (s + \tilde{s}, p_1 + p_2 + \tilde{p} + 1) \in (E_1 + F) \cap (E_2 + F)\) by (3.2) and we have shown the claim.

**Remark 6.5.** In general one does not have \((E_1 \cap E_2) + F = (E_1 + F) \cap (E_2 + F)\). For instance consider the index sets \( E_1 = \{(1, 10)\}, E_2 = \{(1/2, 0)\} \) and \( F = \{(1/2, 5), (0, 0)\}\). Then \((1, 16) \in (E_1 + F) \cap (E_2 + F)\).

**Proposition 6.6.** Let \( u \in x^\delta L^2(M, dv_g) \cap \ker N_g \), with \( \delta \in (-n/2, n/2) \). Then \( u \in \mathcal{A}_{phg}^E(M) \) with \( E = \bigcup_{j \geq 0} (F_t + jF_j) \), where \( F_t, F_j \) are the index sets in \( \Omega_6 \) and \( jF_j = \sum_{i=1}^{j} F_j \). Note that \( F_t + jF_j \geq n + j \) and hence \( E \) is an index set.

**Proof.** By (6.3), any \( u \) as in the statement is smooth in \( M \), hence it suffices to show the existence of an asymptotic expansion at the boundary for \( u \). We first show that if \( u \in x^\delta L^2(M, dv_g) \cap \ker N_g \) for some \( \delta \in (-n/2, n/2) \) then \( u \in x^{\delta'} H_\infty(M, dv_g) \) for all \( \delta' < n/2 \). Since \( u = Ku \), the mapping properties of \( K \) (by \( \Omega_6 \) and Proposition 3.3) imply that \( u \in x^{\delta} H_\infty(M, dv_g) \) provided \( \delta_1 < n/2, \delta_1 \leq \delta + 1 \), that is, the existence of a parametrix allows us to obtain an improvement in the decay of \( u \). Using \( j \) times the improved decay and \( u = Ku \), we inductively find that \( u \in x^{\delta} H_\infty(M, dv_g) \), provided \( \delta_j < n/2 \) and \( \delta_j \leq \delta + j \), so taking \( j \) sufficiently large we conclude that \( u \in x^{\delta} H_\infty(M, dv_g) \) for \( \delta' < n/2 \). By Lemma 6.2, \( u \in x^{\delta} H_\infty(M, dv_g) \) for \( \delta' < n/2 \), hence \( u \in x^{\delta'} H_\infty(M, dm_b) \) for \( \tau < n \); the latter is the same space as in the statement of Lemma 6.2 with \( dv_g \) replaced by \( dm_b \), a measure on \( M \) induced by a section of \( \Omega_b(M) \), of the form \( x^{-1}|dxdy| \) locally near \( \partial M \). (In this proof we used this measure due to its more natural behavior with respect to the Mellin transform.)

Functions in \( x^{\tau} H_\infty(M, dm_b) \) supported near \( \partial M \) can be identified with functions which lie in \( x^{\tau} H_\infty(dx/x; H^\ell(\partial M)) \) for \( k, \ell \in \mathbb{N}_0 \); the latter is the space of \( \nu : \mathbb{R}^+ \to H^\ell(\partial M) \) which are \( k \) times Fréchet differentiable almost everywhere on \( \mathbb{R}^+ \) (and supported near \( 0 \)), and \( \|x^{-\tau}(x \partial_x)^j \nu\|_{H^j(\partial M)} \in L^2(dx/x) \) for \( j = 0, \ldots, k \). Therefore, if \( \varphi \in C^\infty(M) \) is supported in a small neighborhood of \( \partial M \) and identically 1 near \( \partial M \), by taking the Mellin transform in \( x \) we find that \((\varphi u)_M(\zeta) \) is holomorphic in the half plane \( \{\text{Re}(\zeta) > -\tau\} \), with values in functions smooth in \( y \) and with the \( L^2(\{\text{Re}(\zeta) = \alpha\}, |d\zeta|) \) norm of \( \|((\varphi u)_M)|_{H^j(\partial M)} \) being uniformly bounded for \( \alpha > -\tau \) for each \( \ell \).

We now recover the leading order term in the expansion of \( u \) at \( \partial M \). Observe that localizing \( K \) near the boundary from the left does not alter its index sets: that is, if \( \varphi \) is smooth and supported near \( \partial M \) with \( \varphi \equiv 1 \) near \( \partial M \), then \( \varphi K \in \Psi_0^{-\infty, F}(M), \) with \( F = \)
that $F_\ell = \bigcup_{j \geq 0} \{(n + j, p_j)\}$ with $p_0 = 0$; denote $F_\ell^k = \bigcup_{j \geq k} \{(n + j, p_j)\}$ for $k \in \mathbb{N}_0$. Now let $P_0 = (x\partial_x - n) \in \text{Diff}^1_0(M)$; $P_0$ lifts to $M_2^0$ to a $C^\infty$ vector field that takes the form $(s\partial_s - n)$ near $\ell f$ in terms of coordinates $(3.4)$. Then $K_0 := P_0(\varphi K) \in \Psi_{-\infty}^{-\infty,F_\ell,F,F_\ell}(M)$, that is, the term of order $n$ in the expansion of $\varphi K$ at the left face of $M_2^0$ is removed. We can now see that $K_0u \in x^\tau H^{\infty}_b(M; d\mu_b)$ for $\tau < n + 1$: indeed, since $u \in x^\tau H^{\infty}_b(M; d\mu_b)$ for $\tau' < n$, $P_0u \in x^\tau L^2(M; d\mu_b)$ for $P \in \text{Diff}^1_0(M)$, $m \geq 0$. By an inductive argument using commutators as in Lemma 6.2, one sees that if $P' \in \text{Diff}^m_0(M)$, $m \geq 0$, then $P'K_0u = \sum_{j=1}^J Q_j P_j u$, where $P_j \in \text{Diff}^m_0(M)$ and $Q_j \in \Psi_{-\infty}^{-\infty,F_\ell,F,F_\ell}(M)$. Thus Proposition 3.3, $F_\ell^1 \geq n + 1 - \varepsilon$ for all $\varepsilon > 0$, and $F_\ell \geq 1$ imply that if $\tau < n + 1$ then for $P' \in \text{Diff}^{m+1}_0(M)$ we have $P'K_0u \in x^\tau L^2(M; d\mu_b)$ for all $m \geq 0$.

In other words, $K_0u \in x^\tau H^{\infty}_b(M; d\mu_b)$ for such $\tau$, as claimed. This implies that $(K_0u)_M(\zeta)$ is holomorphic on the half plane $\{\text{Re}(\zeta) > -n - 1\}$ with values in functions smooth in $y$. Since $(P_0(\varphi u))_M = (-\zeta - n)(\varphi u)_M$, $(\varphi u)_M = (-\zeta - n)^{-1}(K_0u)_M$ and we conclude that $(\varphi u)_M$ extends meromorphically on the half plane $\{\text{Re}(\zeta) > -n - 1\}$, with a pole of order $1$ at $\zeta = -n$ and values in smooth functions on $\partial M$. Computing the inverse Mellin transform on the line $\{\text{Re}(\zeta) = -n + 1 + \varepsilon\}$, where $\varepsilon > 0$ is small (note that on such a line $(-\zeta - n)^{-1}(K_0u)_M$ depends smoothly on $y$ and is in $L^2(\{\text{Re}(\zeta) = -n + 1 + \varepsilon\}; |d\zeta|)$ for each $y$), we recover the leading term of the expansion of $u$: near $\partial M$

$$u(x, y) = a_{00}(y)x^n + v, \quad a_{00} \in C^\infty(\partial M), v \in x^{\tau}H^{\infty}_b(M; d\mu_b), \tau < n + 1.$$

Now suppose that for $m \geq 1$ we have recovered the asymptotic expansion of $u$ up to the $m$-th exponent appearing in the index set $E = \bigcup_{j \geq 0}(F_\ell + jF_\ell)$ and corresponding to powers of $x$. It will be convenient to write $E = \bigcup_{j \geq 0} \{(n + j, r_j)\}$, where $r_j$ is the highest power of a logarithmic factor multiplying $x^{n+j}$ in the expansion induced by $E$. Similarly, write $F_\ell + mF_\ell = \bigcup_{j \geq m} \{(n + j, r_{j;m})\}$, so that $r_j + 1 = \sum_{m=0}^{j} r_{j;m} + 1$. Note that $r_{j;m} = p_j$ and also $r_0 = r_{0,0} = p_0 = 0$. So suppose that we have

$$u = u_m + v \quad \text{where} \quad v \in x^{\tau}H^{\infty}_b(M; d\mu_b), \quad \tau < n + m,$$

and near $\partial M$

$$u_m(x, y) = \sum_{j=0}^{m-1} \sum_{k=0}^{r_j} a_{jk}(y)x^{n+j} |\log x|^k, \quad a_{jk} \in C^\infty(\partial M).$$

We will show that (6.5) holds for $m + 1$; then by induction we will done. By (6.5), $(\varphi u)_M$ is meromorphic on the half plane $\{\text{Re}(\zeta) > -n - m\}$ with poles of order $r_j + 1$ at $\zeta = -n - j$ for $0 \leq j \leq m - 1$. Set $P_j = (x\partial_x - n - j) \in \text{Diff}^1_0(M)$ and write $K_m := \prod_{j=0}^{m} P_j^{p_j+1}(\varphi K)$, then $K_m \in \Psi_{-\infty}^{-\infty,F_\ell+1,F,F_\ell}(M)$ as before. Now by (6.5)

$$\prod_{j=0}^{m} P_j^{p_j+1}(\varphi u) = K_m u_m + K_m v.$$

Since $K_m \in \Psi_{-\infty}^{-\infty,F_\ell+1,F,F_\ell}(M)$ with $F_\ell \geq 1$ and $F_\ell^{p_j+1} \geq n + m + 1 - \varepsilon$ for all $\varepsilon > 0$, the fact that $v \in x^{\tau}H^{\infty}_b(M; d\mu_b)$, for $\tau < n + m$ implies that $K_m v \in x^{\tau}H^{\infty}_b(M; d\mu_b)$ for $\tau < n + m + 1$ using the same commutator argument as before and Proposition 3.3. Moreover, it follows by Proposition 3.2 that $K_m u_m \in A_{p_\text{hys}}^G(M)$, where

$$G = F_\ell^{p_j+1} \bigcup \left( \sum_{j=0}^{m-1} (F_\ell + jF_\ell) + F_\ell \right) \subset F_\ell^{p_j+1} \bigcup \left( \sum_{k=1}^{m} (F_\ell + kF_\ell) \right) =: G'.$$
where the inclusion follows from Lemma 6.4. Thus \( K_m u_m \in \mathcal{A}_{phg}^E(M) \). Upon taking the Mellin transform in (6.6),

\[
\prod_{j=0}^{m} (-\zeta - n - j)^{p_j + 1} (\varphi u)_{M}(\zeta) = (K_m u_m)_{M}(\zeta) + (K_m v)_{M}(\zeta),
\]

(6.8)

where \((K_m v)_{M}(\zeta)\) is holomorphic in \( \{ \text{Re}(\zeta) > -n - m - 1 \} \) (with values in \( C^\infty(\partial M) \)). On the other hand, for \( 1 \leq j \leq m \), \((K_m u_m)_{M}(\zeta)\) has a pole of order \( \sum_{k=1}^{j} (r_{j,k} + 1) \) at \( \zeta = -n - j \). Note that the index set \( F^{m+1}_\ell \) in (6.7) does not contribute any poles in the open half plane \( \{ \text{Re}(\zeta) > -n - m - 1 \} \). Thus upon dividing we find that \((\varphi u)_{M}(\zeta)\) is meromorphic on the half plane \( \{ \text{Re}(\zeta) > -n - m - 1 \} \) with values in \( C^\infty(\partial M) \) and poles of order \( p_j + 1 + \sum_{k=1}^{j} (r_{j,k} + 1) = (r_{j,0} + 1) + \sum_{k=1}^{j} (r_{j,k} + 1) = r_j + 1 \) at \( \zeta = -n - j, 0 \leq j \leq m \).

Taking the inverse Mellin transform of (6.8) on a vertical line \( \{ \text{Re}(\zeta) = -n - m - 1 + \varepsilon \} \) for small \( \varepsilon > 0 \) similarly to the first inductive step we obtain (6.5) for \( m + 1 \).

\[\square\]

**Remark 6.7.** It follows from (6.7) that the index set \( E \) in the statement of Proposition (6.6) includes higher powers of logarithmic factors than it needs to, but its form suffices for our needs.

We will need the following standard result from functional analysis (see [SU04] for a proof):

**Lemma 6.8.** Let \( X, Y, Z \) be Banach spaces, and let \( A : X \rightarrow Y \) be bounded and injective. If there exists a compact operator \( K : X \rightarrow Z \) such that

\[\|u\|_X \leq C (\|Au\|_Y + \|Ku\|_Z), \quad u \in X \]

for some constant \( C \), then there exists a constant \( C' \) such that

\[\|u\|_X \leq C' \|Au\|_Y, \quad u \in X.\]

We now prove the main theorem:

**Proof of Theorem 7.** Let \( u \in x^\delta L^2(M, dV_g) \cap \ker(N_g), \delta \in (-n/2, n/2). \) We claim that \( u = 0 \). Note that by Corollary 4.5, the X-ray transform is well defined on such a \( u \) in the sense that \( Iu \in (\eta h)^{-\delta} L^2(\partial_\delta S^* M; d\lambda_0), \delta' < \min\{\delta, 0\}. \) By Proposition 6.6 \( u \in \mathcal{A}_{phg}^E(M), E \geq n. \) In particular, \( u \in x^\delta L^2(M, dV_g) \) for \( \delta < n/2. \) Now by (2.5) and the discussion immediately after it we find

\[0 = (N_g u, u)_{L^2(M, dV_g)} = (I^*Iu, u)_{L^2(M, dV_g)} = \|Iu\|_{L^2(\partial_\delta S^* M; d\lambda_0)}^2.\]

This implies that \( Iu = 0. \) Then one checks that the proof of Theorem 1 in [GGS+19], which shows injectivity of \( I \) on \( xC^\infty(M) \), also applies for polyhomogeneous functions in \( \mathcal{A}_{phg}^E(M), E \geq 1. \) More specifically, by the proof of [GGS+19] Proposition 3.15 it follows that for \( u \in \mathcal{A}_{phg}^E(M) \cap \ker I \) one has the stronger result \( u \in \dot{C}^\infty(M) \) (i.e. \( u \) vanishes to infinite order at the boundary). Then the injectivity argument using Pestov identities in the proof of Theorem 1 in the same paper yields \( u \equiv 0. \) We have shown that \( N_g \) is injective on \( x^\delta L^2(M, dV_g), \delta > -n/2. \) Now by Proposition 6.1 we have

\[\|u\|_{x^\delta H^s_0(M, dV_g)} \leq C \left( \|N_g u\|_{x^\delta H^{s+1}_0(M, dV_g)} + \|Ku\|_{x^\delta H^s_0(M, dV_g)} \right), \quad \delta \in (-n/2, n/2), s \geq 0,\]

where \( K : x^\delta H^s_0(M, dV_g) \rightarrow x^\delta H^s_0(M, dV_g) \) is compact. Thus Lemma 6.8 implies

\[\|u\|_{x^\delta H^s_0(M, dV_g)} \leq C' \|N_g u\|_{x^\delta H^{s+1}_0(M, dV_g)}, \quad \delta \in (-n/2, n/2), s \geq 0,\]

which is the claimed estimate. \[\square\]
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