Rational approximation of P-wave kinematics – Part 2: orthorhombic media

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ABSTRACT

Orthorhombic anisotropy is a modern standard for 3D seismic studies in complex geologic settings. Several seismic data processing methods and wave propagation modeling algorithms in orthorhombic media rely on phase-velocity, group-velocity, and traveltime approximations. The algebraic simplicity of an approximate equation is an important factor in these media because the governing equations are more complicated than transversely isotropic media. To approximate the P-wave kinematics in acoustic orthorhombic media, we have developed a new 3D general functional equation that has a simple rational form. Using the general form, we adopt two versions of rational approximations for the phase velocity, group velocity, and traveltime. The first version uses a simpler functional form and parameter definition within the orthorhombic symmetry planes. The second version is more accurate, using one parameter that is defined out of the symmetry planes. For the phase velocity, we obtain another approximation that is no longer rational but is still algebraically simple, exact for 3D transversely isotropic media, and it is exact within the symmetry planes of orthorhombic media. We find superior accuracy in our approximations compared with previous ones, using numerical studies on multiple moderately anisotropic orthorhombic models. We investigate the effect of the negative anellipticity parameters on the accuracy and find that, in models in which the error of the existing most accurate approximations exceeds 2%, the error of the new approximations remains below 0.2%. The adopted approximations are algebraically simpler and stably more accurate than existing approximations; therefore, they may be considered as attractive alternatives for the existing approximations in many practical applications. We extend the applicability of our approximations by using them to obtain the equations of group direction as a function of phase direction and vice versa, which are useful in wave propagation modeling methods.

Keywords: Anisotropy, Acoustic properties, Traveltime, Phase, 3D

1 This paper is published by GEOPHYSICS, DOI: 10.1190/geo2020-0006.1
INTRODUCTION

Orthorhombic (ORT) anisotropy describes variations of wave propagation properties with both azimuthal and polar angles that are often seen in 3D seismic studies in anisotropic media. An ORT anisotropy may result from common geological settings, such as a set of vertical fractures penetrating a transversely isotropic (TI) layer (Schoenberg and Helbig, 1997), or multiple sets of orthogonal fractures in a layer (Bakulin et al., 2000). P-wave propagation in ORT media is fully described by six independent parameters, under the acoustic assumption (Alkhalifah, 2003).

In ORT media, the governing equations are algebraically more complicated than TI media because of the lower symmetry, therefore, the P-wave kinematics equations, including phase velocity, group velocity, and reflection traveltime (moveout) are approximated for practical purposes. These equations find application in most stages of seismic data processing (e.g., Tian and Zhang, 2019), and wave propagation modeling (e.g., Song and Alkhalifah, 2013). More examples of these applications are mentioned in Part I. For phase velocity, the exact equation is on hand (e.g. given in Appendix A in Abedi et al, 2019); however, it is algebraically complicated for some practical purposes. Therefore, the objective of a phase velocity approximation is to reduce the algebraic complexity. For group velocity and traveltime, the application of approximate equations is inevitable because there is neither an exact and explicit equation for group velocity as a function of group angle, nor traveltime as a function of offset.

The development of more accurate approximations for phase velocity, group velocity, and reflection traveltime in ORT media has been the subject of many research studies. Recent advances include the methods of Sripanich and Fomel (2015), Hao and Stovas (2016), Xu et al. (2017), Stovas and Fomel (2019), and Abedi et al. (2019). Besides the accuracy, the algebraic simplicity of an approximation is an influencing factor for practical purposes. The algebraic simplicity of the equation affects the computational cost of many modeling algorithms.

In this study, we propose new general functional equations for the approximation of P-wave kinematics in 3D acoustic ORT media that are highly accurate and algebraically simple. Moreover, within the proposed general framework, we obtain different approximations for each kinematics, which lets the user decide between more simplicity and higher accuracy.

THEORY

General form of approximations

Acoustic orthorhombic (ORT) media are characterized by three symmetry planes and are uniquely defined by six parameters. These parameters are three axial velocities \( v_i \), and three anellipticity
parameters \( (\eta_j) \) that are defined in each symmetry plane (each plane is identified by its normal axis as shown in Figure 1).

For the approximation of equations that define P-wave kinematics in acoustic ORT media, we propose a new general form. This form is rational and is built as a 3D extension of the general form for transversely isotropic (TI) media, which is proposed in Part I. In the phase domain, the 3D general functional form for kinematics approximations reads,

\[
v^2(n) = e + \frac{A_{ik}b_{ik} + A_{jk}b_{jk} + A_{ik}b_{jk} + Mc}{e + B_{ik}b_{ik} + B_{jk}b_{jk} + B_{ik}b_{jk} + Lc},
\]

where \( e(n) = n_1^2v_1^2 + n_2^2v_2^2 + n_3^2v_3^2 \) is the elliptical part, \( b_{ik}(n) = n_i^2n_k^2v_i^2v_k^2 \) are 2D cross-terms in each symmetry plane, \( c(n) = (n_i^2n_k^2v_i^2v_k^2)/e \) is a 3D cross-term that exists out of the symmetry planes. The phase direction vector is \( n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \), where \( \theta \) is the polar phase angle from \( x_3 \), and \( \phi \) is the azimuthal phase angle from the \( x_1 \) axis (Figure 1). The equation parameters, \( A_j, B_j, M, \) and \( L \) are called the anelliptic parameters, which are defined in the following section. \( A_j \) and \( B_j \) depend on the anellipticity parameter in each plane \( (\eta_j) \), and \( M \) and \( L \) depend on all the three anellipticity parameters. In this and all the following equations, the indices \( (i, j, k) \) have three possible combinations: \( (3,1,2), (1,2,3), (2,3,1) \).

In the group domain, the general form of our approximations reads,

\[
V^{-2}(N) = \hat{e} + \frac{\hat{A}_{ik}b_{ik} + \hat{A}_{jk}b_{jk} + \hat{A}_{ik}b_{jk} + \hat{M}c}{\hat{e} + \hat{B}_{ik}b_{ik} + \hat{B}_{jk}b_{jk} + \hat{B}_{ik}b_{jk} + \hat{L}c},
\]

where \( \hat{e}(N) = N_1^2/v_1^2 + N_2^2/v_2^2 + N_3^2/v_3^2 \) is the elliptical group velocity, \( \hat{b}_{ik}(N) = N_i^2N_k^2/v_i^2v_k^2 \) are 2D cross-terms in each symmetry plane, \( \hat{c}(N) = (N_i^2N_k^2N_j^2)/v_i^2v_j^2v_k^2 \) is a 3D cross-term. The group direction vector \( N = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta) \), where \( \Theta \) is the polar group angle from \( x_3 \), and \( \Phi \) is the azimuthal group angle from the \( x_1 \) axis. Parameters \( \hat{A}_j, \hat{B}_j, \hat{M}, \) and \( \hat{L} \) are the anelliptic parameters that are defined in the following sections (the sign \( \hat{\ } \) is used to distinguish the parameters in the group domain).

In each symmetry plane of ORT media, the exact acoustic properties are equivalent to those of TI media, and the proposed 3D general form in equations 1 and 2 will reduce to the proposed 2D general forms for TI media (in Part I). Therefore, the properties of the proposed approximation in Part I, such as the symmetrical dual fits through each in-plane parameter and the presence of the
Phase velocity

First, we use a simple version of the proposed general form without the 3D cross-terms and obtain a very simple phase velocity approximation. Our first rational phase velocity approximation for ORT media reads,

\[ v^2(n) = e + e \left( \frac{A_3 n_1^2 n_3^2 \nu_1^2 \nu_2^2 + A_2 n_1^2 n_3^2 \nu_1^2 \nu_3^2 + A_1 n_1^2 n_3^2 \nu_1^2 \nu_2^2}{e^2 + B_3 n_1^2 n_3^2 \nu_1^2 \nu_2^2 + B_2 n_1^2 n_3^2 \nu_1^2 \nu_3^2 + B_1 n_1^2 n_3^2 \nu_2^2 \nu_3^2} \right), \]  

(3)

where the equation parameters are defined by fitting in the symmetry planes as,

\[ A_j = -2\eta_j / (1 + 2\eta_j), \]
\[ B_j = -2 + 2\sqrt{1 + 2\eta_j}. \]  

(4)

The \( A_j \) parameters are obtained by means of fitting the second derivatives of equation 3 to their exact values along both axes in each symmetry plane (dual fits). We obtain \( B_j \), using the optimal rays in each symmetry plane to fit equation 3 and its first-order derivatives to their exact values. Setting \( n_2 = 0 \), equation 3 is reduced to the second-order phase-velocity approximation given in Part I, for TI media.

Figure 2 shows the absolute value of relative errors of equation 3 and the approximation of Stovas and Fomel (2019), for an ORT model defined in Table 1 as Model A. Similar to Stovas and Fomel (2019), all parameters of our first rational approximation are defined within the symmetry planes. Unlike Stovas and Fomel (2019), our first rational approximation is not exact in all directions within the symmetry planes; the functional form of Stovas and Fomel (2019) has three square roots, while our first rational approximation has one simple fraction.

As Abedi et al. (2019) note, the functional form should include 3D cross-terms to increase the accuracy of a kinematics approximation out of the symmetry planes of ORT media. Including one 3D cross-term of the general form (equation 1), we obtain our second rational phase velocity approximation,

\[ v^2(n) = e + \frac{e \left( A \cdot n_1^2 n_3^2 \nu_1^2 \nu_2^2 + A \cdot n_1^2 n_3^2 \nu_1^2 \nu_3^2 + A \cdot n_1^2 n_3^2 \nu_1^2 \nu_2^2 \right) + Mn_1^2 n_3^2 \nu_1^2 \nu_2^2 \nu_3^2}{e^2 + B n_1^2 n_3^2 \nu_1^2 \nu_2^2 + B n_1^2 n_3^2 \nu_1^2 \nu_3^2 + B n_1^2 n_3^2 \nu_2^2 \nu_3^2}, \]  

(5)

where \( e \) is the ellipsoidal part, \( A_j \), and \( B_j \) are in-plane parameters that are defined the same as in equation 4; \( M \) is the out of planes parameter defined as,
\[ M = 9(1 - u - w) + 3u^2, \] (6)

where,
\[ u = 1 + 2\sqrt{w} \cos \left[ \frac{1}{3} \arccos \left( w \left( (1 + A_1)(1 + A_2)(1 + A_3)w \right)^{-1/2} \right) \right], \]
\[ w = \frac{1}{3} (3 + A_1 + A_2 + A_3). \] (7)

The fitting assumption that results in equation 6 is explained in the following comment.

Figure 2 compares the accuracy of equation 5 with equation 3 and Abedi et al. (2019). The approximation of Abedi et al. (2019) is an enhanced version of Stovas and Fomel (2019) by including a 3D cross-term. Comparing Figure 2d with Figure 2c, one can be seen that how well the out-of-planes accumulated error in equation 2 (the peak shape) is approximated and removed through the proposed 3D cross-term in the general form. Comparing our second rational phase velocity approximation with Abedi et al. (2019), our approximation in equation 5 is simpler but is not all exact within the symmetry planes.

**Comment**

Expanding the general functional form (equation 1) it in a generalized continued fraction up to infinite order, and defining all the in-planes parameters from different orders of derivatives along the axes,

\[ v^2(n) = e + \frac{A_1 b_{23} + A_2 b_{31} + A_3 b_{12} + Mc}{e + \frac{A_1 b_{23} + A_2 b_{31} + A_3 b_{12} + Mc}{e + \ddots}}, \] (8)

the expansion has a simple closed-form, which is exact in all directions along the symmetry planes.

The closed-form of the expansion in equation 8 is our third phase velocity approximation, which reads,

\[ v^2(n) = e + \sqrt{\left( \frac{e}{2} \right)^2 + A_1 b_{23} + A_2 b_{31} + A_3 b_{12} + Mc}, \] (9)

where, \( A_j \) are defined in equation 4, and \( M \) is defined in equation 6.

Figure 2 compares the accuracy of equation 9 with the aforementioned approximations. It is exact within the symmetry planes, and the most accurate approximation compared to the others. It is no longer rational but has one square root; therefore, it is still algebraically simpler than Stovas and Fomel (2019) and Abedi et al. (2019), which have three square roots.
In 3D models, a TI medium with vertical symmetry (VTI) or with horizontal symmetry (HTI) is a special case of ORT anisotropy. An acoustic ORT medium is reduced to VTI when $\eta_1 = \eta_2$, $\eta_3 = 0$, $v_1 = v_2$, and to an HTI medium that its symmetry axis is along the $x_1$ coordinate axis when $\eta_1 = 0$, $\eta_2 = \eta_3$, and $v_2 = v_3$. In 3D TI media, equation 9 gives the exact phase-velocity relation (note that the term $Mc$ vanishes).

The parameter $M$ in equation 6 is found by equating the closed-form approximation (equation 9) to the exact phase velocity at one out-of-planes direction $(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ defined as,

$$
\vec{n}_j = \left( v_j, \sqrt{\frac{1}{v_1^2} + \frac{1}{v_2^2} + \frac{1}{v_3^2}} \right)^{-1}.
$$

The $(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ approximates the direction of maximum error in equation 9 when $M = 0$. In the symmetry plane identified by index $j$, the direction $(\vec{n}_j, \vec{n}_k)$ points to the direction of the optimal ray in that plane, after normalization.

**Group velocity**

Our first group velocity approximation is based on the general form in equation 2, but for simplicity only uses the 2D cross terms. The first rational group velocity approximation is obtained as,

$$
\hat{V}^{-2}(\hat{N}) = \hat{e} + \frac{\left( \hat{A}_j N_j^2 N_j^2 v_i^2 + \hat{A}_j N_i^2 N_j^2 v_j^2 + \hat{A}_j N_2^2 N_j^2 v_3^2 \right) \hat{e}}{\hat{e}^2 v_1^2 v_2^2 v_3^2 + \hat{B}_i N_i^2 N_j^2 v_i^2 + \hat{B}_2 N_i^2 N_j^2 v_2^2 + \hat{B}_j N_i^2 N_j^2 v_3^2},
$$

where $\hat{e}(\hat{N})$ is the elliptical group velocity. The in-plane anelliptic parameters are defined in a similar way to the VTI media as,

$$
\hat{A}_j = 2\eta_j,
\hat{B}_j = 2(\eta_j - 1 + \sqrt{1 + 2\eta_j}).
$$

Parameters $\hat{A}_j$ are obtained by means of fitting the second derivatives of equation 11 to their exact values along both axes associated with each symmetry plane, and $\hat{B}_j$ are obtained by fitting equation 11 and its first-order derivatives to the exact values along the direction of the optimal rays in each symmetry plane (the optimal rays are defined in Part I). Figure 3 shows the absolute value of relative errors of equation 11 besides the approximation of Stovas and Fomel (2019) for the same ORT model as in Figure 2. Our first rational group velocity approximation is less accurate...
within the symmetry planes, but its functional form and parameters definitions are simpler than those of Stovas and Fomel (2019).

Including one 3D cross-term from the proposed general form to account for the out-of-planes error without affecting the approximation within the symmetry planes, our second rational group velocity approximation is obtained as,

\[
V^{-2}(\mathbf{N}) = \hat{e} + \frac{\left(\hat{A}_1 N_2^2 N_3 \hat{v}_1^2 + \hat{A}_2 N_1^2 N_3 \hat{v}_2^2 + \hat{A}_3 N_1^2 N_3 \hat{v}_3^2 \right) \hat{e} + \hat{M} N_2^2 N_3 \hat{v}_2^2}{\hat{v}_1^2 \hat{v}_2^2 \hat{v}_3^2 + \hat{B}_1 N_2^2 N_3 \hat{v}_1^2 + \hat{B}_2 N_1^2 N_3 \hat{v}_2^2 + \hat{B}_3 N_1^2 N_3 \hat{v}_3^2}. \tag{13}
\]

To obtain the parameter \( \hat{M} \), we use the phase direction given in equation 10 and fit the approximation at the group direction that this phase vector is associated with,

\[
\hat{M} = (V^{-2} - \hat{e}) \frac{\hat{v}_1^2 \hat{v}_2^2 \hat{v}_3^2}{\hat{N}_1^2 \hat{N}_2^2 \hat{N}_3^2} - \sum_{j=1}^{3} \left( \left( \hat{A}_j + \hat{B}_j \right) \hat{e} - \hat{B}_j V^{-2} \right) \hat{N}_j^2, \tag{14}
\]

where \( V = V(\bar{n}) \), is given in equation A-1, \( \hat{N}_j = N_j(\bar{n}) \) is given in equation A-3, and \( \bar{n} = (\bar{n}_1, \bar{n}_2, \bar{n}_3) \) is given in equation 10 (Note that equation A-1 and A-3 are exact at \( \bar{n} \)).

Alternatively, one can use the group direction that is defined by the intersection of the optimal directions in each symmetry plane in the group domain,

\[
\hat{N}_j = \frac{\nu_j}{\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2}}, \tag{15}
\]

where \( \nu_j \) are the three axial velocities of ORT medium. Equation 15 approximates the location of maximum relative error in equation 11. Figure 3 compares the accuracy of equation 13 with equation 11 and the group velocity approximation in Abedi et al. (2019). The Abedi et al. (2019) approximation is an enhanced version of Stovas and Fomel (2019) by including a 3D cross-term, different from the proposed terms in here. Comparing Figure 3d with Figure 3c, it can be seen that the out-of-planes accumulated error in equation 11 (the peak shape) is well approximated and removed by including the 3D cross-term in equation 13. The remaining errors are extensions of the in-planes errors. Comparing our second rational group velocity approximation with Abedi et al. (2019), our approximation in equation 13 is still algebraically simpler.

**Traveltime**

A group velocity approximation can be converted to a reflection traveltime (moveout) approximation using a simple geometrical relation (e.g., equation 32 in Sripanich and Fomel, 2015). From equation 11, our first rational traveltime approximation is obtained as,
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\[ t^2(x_1, x_2) = H + \frac{(\hat{A}t_0^2x_1^2 + \hat{A}t_0^2x_2^2 + \hat{A}x_1^2x_2^2)H}{H^2v_1^2v_2^2 + \hat{B}t_0^2x_1^2v_2^2 + \hat{B}t_0^2x_2^2v_1^2 + \hat{B}x_1^2x_2^2}, \]

where \( H = t_0^2 + \frac{x_1^2}{v_1^2} + \frac{x_2^2}{v_2^2} \), is the hyperboloid part of the 3D moveout, and the parameters are defined in equation 12. In homogeneous ORT media, \( v_1 = v_{n2}\sqrt{1+2\eta_2} \) and \( v_2 = v_{n1}\sqrt{1+2\eta_1} \) (note that the normal moveout velocity \( v_{n} \) and anellipticity \( \eta \) are in-plane parameters, numbered by the plane’s normal axis, Figure 1). The second rational traveltime approximation is obtained as,

\[ t^2(x_1, x_2) = H + \frac{(\hat{A}t_0^2x_1^2 + \hat{A}t_0^2x_2^2 + \hat{A}x_1^2x_2^2)H + \hat{M}t_0^2x_1^2x_2^2}{H^2v_1^2v_2^2 + \hat{B}t_0^2x_1^2v_2^2 + \hat{B}t_0^2x_2^2v_1^2 + \hat{B}x_1^2x_2^2}, \]

where \( \hat{M} \) is defined by equation 14, or alternatively, by fitting equation 17 to the exact traveltime at a finite mid-offset along the direction \( p_2 = p_1 \frac{v_1}{v_2} \), which is expressed in terms of the components of horizontal slowness (\( p_1 \) and \( p_2 \)). Having \( p_1 \) and \( p_2 \), the exact reference traveltime \( \hat{r} \) and offset coordinates \((\vec{x}_1, \vec{x}_2)\) are calculated, using the parametric traveltime and offset equations (equation 12 of Stovas (2015)). Then, \( \hat{M} \) is found from equating the approximation and exact reference traveltime at reference offset coordinate \((t^2(\vec{x}_1, \vec{x}_2) = \hat{r}\)). Figure 4 compares the accuracy of the proposed traveltime approximations, identified as the first and second rational approximations, with four former methods. The error level of the first rational approximation is comparable with Sripanich and Fomel (2015) and Stovas and Fomel (2019). Note that the moveout approximation of Stovas and Fomel (2019) that we use is given in equation 12 of Abedi et al. (2019), which uses the optimal rays for parameter definition. The accuracy of the second rational approximation is comparable with Xu et al. (2017; Case C) and Abedi et al. (2019; given in equation 13). The second rational approximation and Abedi et al. (2019) are the methods that use 3D cross-terms; but the employed cross-term in the proposed second rational approximation better matches the out of planes error in the first rational, than the cross-term of Abedi et al. (2019) matches the error in Stovas and Fomel (2019). This is more prominent when model parameters include negative anellipticity parameters.

**Ray-traced parameterization**

In the previous parts, the parameters of the proposed equations are derived for acoustic ORT media. In this part, we present the general definition of parameters, using the properties of mid-angle (or finite-offset) rays. We use one mid-angle ray in each symmetry plane, and one out of the planes. In the symmetry planes, we use kinematic properties and their first-order derivatives to
define $A_j$, and $B_j$. Out of the planes, $M$ is found by fitting the approximation to each kinematic property. For the rational phase-velocity approximations in equations 3 and 5, we obtain,

$$
A_j = -\frac{\left(\tilde{n}_i^\lambda v_i^2 + \tilde{n}_k^\lambda v_k^2 - \tilde{v}_j^2\right) \left(\tilde{n}_i^\lambda v_i^4 - \tilde{n}_k^\lambda v_k^4\right)}{\tilde{n}_i^\lambda \tilde{n}_k^\lambda \tilde{v}_i \tilde{v}_k \tilde{\gamma}_j \left(\left(\tilde{n}_i^\lambda v_i^2 + \tilde{n}_k^\lambda v_k^2\right)\tilde{v}_j^2 - \tilde{n}_i^\lambda \left(\tilde{v}_k^2 - v_k^2\right)\tilde{v}_j^2\right)},
$$

$$
B_j = \left(\frac{\tilde{v}_j^2}{\tilde{v}_j^2 - \tilde{n}_i^\lambda v_i^2 - \tilde{n}_k^\lambda v_k^2} - 1\right) A_j - \frac{\left(\tilde{n}_i^\lambda v_i^4 + \tilde{n}_k^\lambda v_k^4\right)^2}{\tilde{n}_i^\lambda \tilde{n}_k^\lambda \tilde{v}_i \tilde{v}_k \tilde{\gamma}_j^2},
$$

(18)

where $\tilde{v}_j$ is the exact phase velocity, and $\tilde{v}_j^\prime$ is its first derivative with respect to angle at a reference mid-angle phase direction $\tilde{n}$ within the symmetry plane identified by index $j$. For equation 5, the out-of-planes parameter is obtained as,

$$
M = \left(9 + B_1 + B_2 + B_3\right) \left(\tilde{v}^{-2} \tilde{v}_j^2 + \tilde{v}_j^2 \tilde{v}_j^2 + \tilde{v}_j^2 \tilde{v}_j^2\right) - 3 \left(A_1 + A_2 + A_3\right),
$$

(19)

where $\tilde{v}$ is the exact phase velocity at the phase direction $\tilde{n}$ defined in equation 10.

For the rational group velocity approximations in equations 11 and 13, we obtain,

$$
\hat{A}_j = \frac{\left(\tilde{N}_i^\lambda v_i^2 + \tilde{N}_k^\lambda v_k^2\right) \left(\tilde{N}_j^\lambda \tilde{V}_j^2 v_i^2 + \left(\tilde{N}_j^\lambda \tilde{V}_j^2 - v_i^2\right) v_i^2\right) \left(\tilde{N}_i^\lambda v_i^4 - \tilde{N}_k^\lambda v_k^4\right)}{\tilde{N}_i^\lambda \tilde{N}_k^\lambda \tilde{V}_i \tilde{V}_k \tilde{\gamma}_j^2 \left(\tilde{V}_j^2 v_i^2 - v_i^2 + \tilde{V}_j^2 \tilde{V}_j^2\left(\tilde{V}_j^2 v_i^2 + \tilde{V}_j^2 v_k^2\right)\right)},
$$

$$
\hat{B}_j = \left(\frac{\tilde{V}_j^2 v_k^2}{\tilde{V}_j^2 v_k^2 - \tilde{V}_j^2} \left(\tilde{N}_k^\lambda v_i^2 + \tilde{N}_k^\lambda v_k^2\right) - 1\right) \hat{A}_j - \frac{\left(\tilde{N}_i^\lambda v_i^4 + \tilde{N}_k^\lambda v_k^4\right)^2}{\tilde{N}_i^\lambda \tilde{N}_k^\lambda \tilde{V}_i \tilde{V}_k \tilde{\gamma}_j^2},
$$

(20)

where $\tilde{V}_j$ is the exact group velocity, and $\tilde{V}_j^\prime$ is its first derivative with respect to group angle at a reference mid-angle group direction $\tilde{N}_j$ within the symmetry plane identified by index $j$. For equation 13, the out-of-planes parameter is obtained as defined in equation 14, but at an arbitrary mid-angle group direction $\left(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3\right)$ that is associated with the exact velocity $\tilde{V} \left(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3\right)$ out of the symmetry planes.

For the rational traveltime approximations in equations 16 and 17, the definitions of anelliptic parameters from the finite offset match are obtained as,
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\begin{equation}
\hat{A}_j = - \frac{\left( (t_0^2 - \tilde{t}_j^2)^2 + \tilde{x}_i^2 \right) \left( t_0^2 \tilde{v}_i^2 - \tilde{x}_i^2 \right)}{t_0^2 \tilde{v}_i^2 \tilde{x}_i^2 \tilde{t}_j \left( \tilde{x}_i \tilde{t}_j - \tilde{p}_j \left( t_0^2 \tilde{v}_i^2 + \tilde{x}_i^2 \right) \right)},
\end{equation}

\begin{equation}
\hat{B}_j = \left( \frac{\tilde{t}_j \tilde{v}_i^2}{(\tilde{t}_j^2 - t_0^2) \tilde{v}_i^2 - \tilde{x}_i^2} - 1 \right) \hat{A}_j - \frac{\left( t_0^2 \tilde{v}_i^2 + \tilde{x}_i^2 \right)^2}{t_0^2 \tilde{v}_i^2 \tilde{x}_i^2}.
\end{equation}

where \( \tilde{t}_j \) is exact traveltime, and \( \tilde{p}_j \) is its first derivative (ray-parameter) at a reference mid-offset \( \tilde{x}_i \) within the symmetry plane that is identified by index \( j \). For equation 17, the out-of-planes parameter is obtained at an arbitrary mid-offset location \( (\tilde{x}_1, \tilde{x}_2) \) out of the symmetry planes that have exact traveltime \( \tilde{t} \),

\begin{equation}
\hat{M} = (\tilde{t}^2 - H) \left( \frac{B_1 \tilde{v}_1^2}{\tilde{x}_1^2} + \frac{B_2 \tilde{v}_2^2}{\tilde{x}_2^2} + \frac{B_3}{t_0^2} + \frac{H \tilde{v}_i^2}{t_0^2 \tilde{x}_i^2 \tilde{v}_i^2} \right) - \frac{1}{H} \left( \frac{A_1 \tilde{v}_1^2}{\tilde{x}_1^2} + \frac{A_2 \tilde{v}_2^2}{\tilde{x}_2^2} + \frac{A_3}{t_0^2} \right).
\end{equation}

The four rays used in this part define the anelliptic parameters from mid-angle or finite-offset matches. Depending on a model, other rays may be needed to define the three orthogonal \( v_i \) velocity parameters.

**NUMERICAL ANALYSIS**

We have shown error surfaces of the proposed approximations and compared them with other methods in Figure 2-4. Here, we repeat this comparison on a second model that has one negative \( \eta \) parameter. This ORT model, which is based on parameter estimation in a physical sample, is explained in Mah and Schmitt (2003), and presented in Table 1 as Model B. Figure 5 replicates Figure 2 but for the second ORT model. The main point is the change of the out-of-plane errors in the presence of a negative \( \eta \) parameter. This change is prominent in the Stovas and Fomel (2019) approximation, as a result, the added cross-term by Abedi et al. (2019) could not properly compensate for it. On the other hand, the negative \( \eta \) parameter has no evident effect on the form of error in the first rational approximation. As a result, the second rational and the closed-form approximations have properly compensated the error out of the planes. Similar observations are made for group velocity, and traveltime approximations from comparing Figure 3 and 4, respectively. Figure 7 also shows an issue with another classical moveout approximation in the presence of negative \( \eta \) parameters; the Xu et al. (2017; Case C) approximation becomes numerically unstable at certain offset coordinates and produces large errors (the Xu et al. (2017) errors are clipped in Figure 7).

An application of the proposed approximations is to derive the corresponding group direction at a specified phase direction and vice versa. Equations of \( N(n) \) and \( n(N) \) are calculated based on the
proposed phase and group velocity approximations, and presented in Appendix A. Figure 8 shows the accuracy of phase-to-group and Figure 9 shows the accuracy of the group-to-phase conversion. These figures show the errors in the approximated $N(n)$ and $n(N)$ in terms of polar and azimuthal angles. The conversion from phase to group domain is more accurate.

To more comprehensively study the accuracy of the proposed approximations, we employ a variety of ORT models and calculate the maximum relative error of different approximations for these models. In Part I, we analytically show that the anellipticity parameter is the only influencing parameter on the maximum value of relative errors of the proposed kinematics approximations in TI media. It is still true about the proposed approximations for ORT media, but can only be investigated numerically. The properties of the multiple models that we use here are presented in Table 1. We specifically intend to study the effect of negative $\eta$, therefore, our models differ based on different combinations of $\eta_1, \eta_2, \eta_3 \in \{-0.25, -0.15, -0.05, 0.05, 0.15, 0.25\}$. Figure 10 shows the maximum errors of different approximations that are proposed in this study, alongside other well-known and recent approximations. In Figure 10, the model parameters are sorted in a way that the models that have three negative anellipticity parameters are on the left, the ones that have three positive anellipticity parameters are on the right, and the rest that has a combination of negative and positive anellipticity parameters are in the middle. Generally, the errors of all approximations increase when the models have one or more negative anellipticity parameters.

Figure 10a shows the maximum errors of the proposed phase velocity approximations, alongside two approximations from Stovas and Fomel (2019), and Abedi et al. (2019). In this figure, the second rational and the closed-form approximations (equations 5 and 9) are the most accurate approximations in all models. The first rational approximation (equation 3) is more accurate than Stovas and Fomel (2019) when all anellipticity parameters have the same sign, but in other models, their relative higher accuracy is model dependent. While the approximation of Abedi et al. (2019) is highly accurate when anellipticity parameters are all positive, its accuracy considerably decreases in presence of negative $\eta_j$ in a model but remains more accurate than Stovas and Fomel (2019). To further investigate the robustness of the proposed approximations in the presence of negative anellipticity parameters, the direction of maximum errors are plotted in Figure 11. The out-of-plane direction that is used to define the parameter $M$ (equation 10) is closer to the maximum errors of equation 3 than to the maximum errors of Stovas and Fomel (2019).

Figure 10b shows the maximum error of the proposed group velocity approximations, alongside two other approximations from Stovas and Fomel (2019), and Abedi et al. (2019). Similar to the phase velocity, the accuracy of the second rational group velocity approximation (equation 13) is consistently high while other methods fluctuate between high and low errors in different models. The presented errors of the first rational group velocity approximation (equation 11) are comparable to Stovas and Fomel (2019), while there is no clear preference. In a few models with
negative $\eta_j$ the maximum error of Abedi et al. (2019) surpasses the Stovas and Fomel (2019), while it was intended as an enhancement.

Figure 10c compares the maximum errors of the proposed traveltime approximations, alongside approximations from Sripanich and Fomel (2015), Stovas and Fomel (2019), and Abedi et al. (2019). We omit Xu et al. (2017) because it is highly unstable in the presence of negative $\eta_j$ and produces high peak errors. Again, the accuracy of the second rational approximation (equation 17) is consistently high in all models. The errors of the first rational approximation (equation 16) and Sripanich and Fomel (2015) are almost identical in most of the models. The errors of Abedi et al. (2019) are highly model dependent when include negative $\eta_j$ in models.

Figure 10d and e show the maximum errors of calculated angles in phase-to-group and group-to-phase direction conversion, using the proposed approximations. In Figure 10, the maximum errors of the most accurate approximations of $N(n)$ and $n(N)$ are 0.23° and 1.74°, respectively; but, for $\eta_j > 0$, the maximum value of those errors become 0.03° and 0.13°, respectively.

**Conclusions**

We propose flexible functional forms in phase and group domains for the approximation of P-wave kinematics in acoustic orthorhombic media. Different versions of approximations for phase velocity, group velocity, and traveltime are obtained. The first rational approximation in each part is very simple with all parameters being defined within the symmetry planes, but show an accumulated error out of the symmetry planes. The second rational kinematics approximations include a 3D cross-term and an extra parameter to compensate for the errors out of the planes. Therefore, in an application, a user can decide between the desired accuracy and parameter definition simplicity. For phase velocity, we obtain the third approximation as the closed-form resulted from the expansion of our rational approximations in a generalized continued fraction up to infinite order. This closed-form approximation is algebraically simple, exact for 3D transversely isotropic media and within the symmetry planes of orthorhombic. Numerical studies on multiple orthorhombic models show that in models that have negative anellipticity parameters the errors of the previous approximations are amplified. This is while the second rational approximation in each part, alongside the closed-form approximation for phase velocity, remains consistently the most accurate approximations. Therefore, our approximations in here, are simple, stable, and highly accurate, because of using the suitable general functional form and the robust parameter definitions both within and out of the symmetry planes of orthorhombic media. We use the proposed approximations in phase-to-group and group-to-phase conversion, where the importance of accuracy and algebraic simplicity becomes more prominent because the derivatives of the proposed approximations are used. Potential applications include various forward modeling and ray tracing algorithms.
ACKNOWLEDGMENTS

I am grateful to the journal reviewers I. Ravve, V. Singh, and an anonymous reviewer and to the associate editor I. Jones and the editor in chief J. Shragge for their insightful comments and suggestions. Gratitude is extended to D. Pardo for helpful discussions. This work has received funding from the European Union’s Horizon 2020 research and innovation program under grant no. 777778 (MATHROCKS), the European POCTEFA 2014-2020 Project PIXIL (EFA362/19) by the European Regional Development Fund (ERDF) through the Interreg V-A Spain-France-Andorra program, the BCAM “Severo Ochoa” accreditation of excellence (SEV-2017-0718), the Basque government through the BERC 2018-2021 program and “Artificial Intelligence in BCAM no. EXP. 2019/00432,” the Project of the Spanish Ministry of Science and Innovation with reference PID2019-108111RB-I00 (FEDER/AEI).

DATA AND MATERIAL AVAILABILITY

Data and codes associated with this research are available and can be accessed at: https://www.mathworks.com/matlabcentral/fileexchange/75545-rational-approximation-of-p-wave-kinematics-part-2-ort

APPENDIX A

CONVERSION BETWEEN THE PHASE AND GROUP DIRECTIONS

The exact relations for velocity conversion from phase-to-group and group-to-phase domain in ORT media are given in Stovas et al. (2018). Based on these relations, the proposed phase and group velocity approximations can be used to derive the equation of group direction as a function of phase directions, and vice versa. Initially, the equation of group velocity as a function of phase direction is calculated,

\[ V^-2(n) = \frac{4v^2}{d_1^2 + d_2^2 - (d_1n_1 + d_2n_2)^2 + 4v^4}, \]  

(A-1)

where \( v^2(n) \) is the phase velocity squared that is approximated in the phased velocity section, \( d_1 \) and \( d_2 \) are its derivatives with respect to \( n_1 \) and \( n_2 \), approximated as,

\[ d_i = n_i w_i + \frac{2A_i b_i n_i^2 + 2A_j b_j (n_i^2 - n_j^2) - n_j^2 (2A_j b_j + en_j^2 w_i)}{n_i n_j^2 (e - 2v^2)} + 2Mc \frac{e (n_i^2 - n_j^2) + n_j^2 n_i^2 w_i}{e^2 n_i n_j^2 (e - 2v^2)^2}, \]  

(A-2)

where \( i, j \in (1, 2) \), \( w_i = v_i^2 - v_j^2 \), and all other terms are defined the same as presented for the phase velocity. Equation A-2 is the derivative of the closed-form phase velocity (equation 9), which is our most accurate approximation. However, since the closed-form approximation has one square
root, in its derivative the square root term will be repeated. We have replaced that term with \( e^{-2\nu^2} \). Therefore, using the proposed rational approximations as \( \nu^2 \), equation A-2 results in a rationalized approximation that is more accurate than the derivative of the rational equations. Compared to the derivative of the exact phase velocity equation that is calculated from the Christoffel equation, equation A-2 is much simpler.

Then, using equation A-2, we calculate the equation of group direction (N) as a function of phase direction (n) as,

\[
N_i^2(n) = \frac{\left( d_i \left( n_i^2 - 1 \right) + n_i \left( d_j n_j - 2\nu^2 \right) \right)}{d_2^2 + d_1^2 - \left( d_1 n_1 + d_2 n_2 \right)^2 + 4\nu^4},
\]

where \( i, j \in 1, 2 \), and \( N_3^2 = 1 - N_1^2 - N_2^2 \).

Finally, we present the equation of phase direction (n) as a function of group direction (N) as,

\[
n_i^2(N) = \frac{\left( \hat{d}_i \left( N_i^2 - 1 \right) + N_i \left( \hat{d}_j N_j - 2\nu^2 \right) \right)}{\hat{d}_2^2 + \hat{d}_1^2 - \left( \hat{d}_1 N_1 + \hat{d}_2 N_2 \right)^2 + 4\nu^{-4}},
\]

where \( i, j \in 1, 2 \), and \( \nu^{-2}(N) \) is the inverse of group velocity squared that is approximated in the group velocity section, \( \hat{d}_i \) and \( \hat{d}_j \) are its derivatives with respect to \( N_1 \) and \( N_2 \), calculated as,

\[
\hat{d}_i(N) = 2N_i\hat{\nu}_i + \frac{2}{N_i N_3^2 (\hat{e}^2 + u_b)}[\left( \hat{A} \hat{\nu}_j + \hat{A} \hat{\nu}_3 \right) e N_3^2 - \left( \hat{A} \hat{\nu}_j + \hat{A} \hat{\nu}_3 \right) \hat{e} N_i^2 \left( \hat{e}^2 + u_b \right) + u_A \left( \hat{b}_j \hat{B}_j + \hat{b}_3 \hat{B}_3 \right) e N_i^2 - \left( \hat{b}_j \hat{B}_j + \hat{b}_3 \hat{B}_3 \right) \hat{e} N_i^2 N_3^2 \left( u_b - \hat{e}^2 \right) \hat{\nu}_i] (A-5)
\]

where \( u_A = \hat{A} \hat{\nu}_1 + \hat{A} \hat{\nu}_2 + \hat{A} \hat{\nu}_3, u_b = \hat{B}_1 \hat{B}_3 + \hat{B}_2 \hat{B}_3 + \hat{B}_3 \hat{B}_3, \hat{\nu}_i = \nu_i^{-2} - \nu_i^{-2} \), \( n_3^2 = 1 - n_1^2 - n_2^2 \) and all other terms are defined the same presented for the group velocity. Equation A-5 presents the derivatives of the second rational group velocity approximation. For the first rational parameter \( \hat{M} = 0 \).
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|          | $v_1$ | $v_2$ | $v_3$ | $\eta_1$ | $\eta_2$ | $\eta_3$ |
|----------|-------|-------|-------|----------|----------|----------|
| **Model A** | 3     | 2.5   | 3.5   | 0.3      | 0.35     | 0.15     |
| **Model B** | 3.99  | 3.94  | 3.33  | -0.044   | 0.035    | 0.077    |
| **Multiple Models** | 3     | 3.5   | 4     | -0.25, -0.15, -0.05, 0.05, 0.15, 0.25 | -0.25, -0.15, -0.05, 0.05, 0.15, 0.25 | -0.25, -0.15, -0.05, 0.05, 0.15, 0.25 |

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