On the evolution equation with a dynamic Hardy-type potential

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Abstract. Motivated by the celebrated paper of Baras and Goldstein (Trans Am Math Soc 284:121–139, 1984), we study the heat equation with a dynamic Hardy-type singular potential. In particular, we are interested in the case where the singular point moves in time. Under appropriate conditions on the potential and initial value, we show the existence, nonexistence and uniqueness of solutions and obtain a sharp lower and upper bound near the singular point. Proofs are given by using solutions of the radial heat equation, some precise estimates for an equivalent integral equation and the comparison principle.

1. Introduction and main results

We consider positive solutions of an evolution equation of the form

$$\begin{cases}
    u_t(x, t) = \Delta u(x, t) + V(x, t)u(x, t), & x \in \mathbb{R}^N \setminus \{\xi(t)\}, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \mathbb{R}^N \setminus \{\xi(0)\},
\end{cases} \tag{1.1}$$

where $N \geq 3$ and the singular point $\xi : [0, \infty) \to \mathbb{R}^N$ is a given Hölder-continuous function. We assume that the potential $V(x, t)$ is continuous in $x \in \mathbb{R}^N \setminus \{\xi(t)\}$ and $t \in [0, \infty)$ and is unbounded as $x \to \xi(t)$. The initial value $u_0(x)$ is continuous, nonnegative and nontrivial for $x \neq \xi(0)$ and is bounded for $|x - \xi(0)| > 1$. By a solution of (1.1), we mean a function $u = u(x, t)$ that satisfies (1.1) in the classical sense and is bounded for $|x - \xi(t)| > 1$. The aim of this paper is to study the existence, nonexistence and uniqueness of solutions, and a lower and upper bound near the singular point.

If $u_0 \in L^1_{loc}(\mathbb{R}^N)$, there exists a unique bounded solution $u = u_n$ of

$$\begin{cases}
    u_t(x, t) = \Delta u(x, t) + V_n(x, t)u(x, t), & x \in \mathbb{R}^N, \ t > 0, \\
    u(x, 0) = u_0(x), & x \in \mathbb{R}^N,
\end{cases} \tag{1.2}$$

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where \( V_n(x, t) := \min \{ V(x, t), n \} \). Then by the comparison principle, \( \{ u_n(x, t) \} \) is a monotone increasing sequence for every \((x, t)\). Hence, if the sequence is bounded, then

\[
u(x, t) := \lim_{n \to \infty} u_n(x, t)
\]

exists and is bounded for \( x \in \mathbb{R}^N \setminus \{ \xi(t) \} \). In this case, the standard parabolic regularity implies that the limiting function \( u(x, t) \) is a solution of (1.1). A solution defined in this way is called a minimal solution of (1.1). We note that this does not exclude the possibility that nonminimal solutions exist.

For the existence of a minimal solution of (1.1), Baras–Goldstein [2] (see also Cabré–Martel [4]) studied the case where the potential function is given by

\[
V(x, t) = \frac{\lambda}{|x|^2}.
\]

It was shown in [2] that the Hardy constant

\[
\lambda_c := \left( \frac{N - 2}{2} \right)^2 > 0.
\]

is critical in the following sense:

(i) If \( 0 < \lambda \leq \lambda_c \) and \(|x|^{-\alpha_1}u_0 \in L^1_{loc}(\mathbb{R}^N)\), then (1.1) has a time-global minimal solution \( u \). Moreover, for every \( \varepsilon > 0 \) and \( T > 0 \), there exists a constant \( c > 0 \) such that

\[
u(x, t) \geq c|x|^{-\alpha_1}, \quad 0 < |x| < 1, \quad \varepsilon \leq t \leq T.
\]

(ii) If \( \lambda > \lambda_c \), then (1.1) is not well-posed, that is, there exist no positive solutions.

We note that if \( \lambda < \lambda_c \), the quadratic equation

\[
\alpha^2 - (N - 2)\alpha + \lambda = 0
\]

has two positive roots given by

\[
\alpha_1 = \alpha_1(\lambda) := \sqrt{\lambda_c - \lambda_c - \lambda}, \quad \alpha_2 = \alpha_2(\lambda) := \sqrt{\lambda_c + \lambda_c - \lambda},
\]

which satisfy

\[
0 < \alpha_1(\lambda) < \frac{N - 2}{2} < \alpha_2(\lambda) < N - 2.
\]

If \( \lambda = \lambda_c \), then the equation has a double root \( \alpha_1(\lambda_c) = \alpha_2(\lambda_c) = \sqrt{\lambda_c} \).

The paper [2] has attracted much attention, and many results have been obtained later for equations with singular potentials [4, 9, 10, 12, 15, 16, 23, 24]. However, no results have been obtained when the position of the singularity depends on time. In
this paper, we consider the case where both the position and strength of the singularity
depend on time. See [7,14,20–22] for related works on such dynamic singularities.
For $r, R \in \mathbb{R}$ with $0 \leq r < R \leq \infty$, an interval $I \subset \mathbb{R}$ and $x_0 \in \mathbb{R}^N$, we introduce
the following notation:

\[
Q_{(r,R),I} := \left\{(x, t) \in \mathbb{R}^{N+1}; r < |x - \xi(t)| < R, \ t \in I \right\}, \\
A_R(x_0) := \{x \in \mathbb{R}^N; 0 < |x - x_0| < R\}, \\
B_R(x_0) := \{x \in \mathbb{R}^N; |x - x_0| < R\}.
\]

Throughout this paper, we assume that for $T > 0$ fixed, the following conditions hold
for $\xi, V$ and $u_0$:

(A1) There exist constants $\gamma \in (1/2, 1)$ and $C_\xi > 0$ such that
$|\xi(t) - \xi(s)| \leq C_\xi |t - s|^{\gamma}$ for all $t, s \in [0, T]$.

(A2) $V$ is nonnegative and continuous on $Q_{(0, \infty),[0,T]}$ and is bounded on $Q_{(1, \infty),[0,T]}$.

(A3) $u_0$ is nonnegative, nontrivial and continuous on $\mathbb{R}^N \setminus \{\xi(0)\}$ and is bounded on $\mathbb{R}^N \setminus B_1(\xi(0))$.

Note that $V(x, t)$ may not be differentiable. In fact, we shall consider potentials
such as $V(x, t) = \lambda |x - \xi(t)|^{-2}$, where $\xi(t)$ is not smooth in $t$.

First, we give a sufficient condition for the existence of a solution and its upper bound.

**Theorem 1.1.** Assume that $V$ satisfies

\[0 \leq V(x, t) \leq \lambda |x - \xi(t)|^{-2}, \quad (x, t) \in Q_{(0,R),[0,T]}\]

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. If

\[0 \leq u_0(x) \leq C_1 |x - \xi(0)|^{-k}, \quad x \in A_R(\xi(0))\]

with some $k \in (0, \alpha_2(\lambda) + 2)$ and $C_1 > 0$, then (1.1) has a solution satisfying

\[u(x, t) \leq C_2 |x - \xi(t)|^{-\alpha_1(\lambda) - \varepsilon}, \quad (x, t) \in Q_{(0,R),[\tau,T]},\]

where $\varepsilon > 0$ and $\tau \in (0, T)$ are arbitrary and $C_2 = C_2(\varepsilon, \tau) > 0$ is a constant.

This result with $\varepsilon = 0$ was proved by Baras–Goldstein [2, Theorem 2.2 (i)] when the
potential $V$ does not depend on $t$.

Next, we give a lower bound of solutions.

**Theorem 1.2.** Assume that $V$ satisfies

\[V(x, t) \geq \lambda |x - \xi(t)|^{-2}, \quad (x, t) \in Q_{(0,R),[0,T]}\]

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. Then any solution of (1.1) satisfies

\[u(x, t) \geq C |x - \xi(t)|^{-\alpha_1(\lambda) + \varepsilon}, \quad (x, t) \in Q_{(0,R),[\tau,T]},\]

where $\varepsilon > 0$ and $\tau \in (0, T)$ are arbitrary and $C = C(\varepsilon, \tau) > 0$ is a constant.
We remark that when $\xi(t)$ does not move, the lower bound was obtained in [2, Theorem 2.1 (i)] by using the method of [17, 18] (see also [5] for a related result). In this paper, we shall give a much simpler proof.

Next, we show that any two solutions of (1.1) coincide with each other if their difference is small in some sense.

**Theorem 1.3.** Assume that $V$ satisfies

$$0 \leq V(x, t) \leq \lambda |x - \xi(t)|^{-2}, \quad (x, t) \in Q(0, R), [0, T]$$

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. If $u_1$ and $u_2$ are solutions of (1.1) with the same initial value such that

$$|u_1(x, t) - u_2(x, t)| \leq C|x - \xi(t)|^{-\alpha_2(\lambda) + \epsilon}, \quad (x, t) \in Q(0, R), [0, T]$$

with some $C > 0$ and $\epsilon \in (0, \alpha_2(\lambda))$, then $u_1 \equiv u_2$ in $Q(0, \infty), [0, T]$.

In particular, this theorem implies that the solution given in Theorem 1.1 is unique, and hence, it must be a minimal solution. We remark that we consider solutions which are bounded outside a neighborhood of the singular point. In the context of uniqueness, we can relax this assumption for an exponential growth (see Theorem 5.1).

The following two theorems show that the conditions in Theorem 1.1 are sharp for the existence of a solution.

**Theorem 1.4.** Assume that $V$ satisfies

$$V(x, t) \geq \lambda |x - \xi(t)|^{-2}, \quad (x, t) \in Q(0, R), [0, T]$$

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. If

$$u_0(x) \geq C|x - \xi(0)|^{-k}, \quad x \in A_R(\xi(0))$$

with some $k > \alpha_2(\lambda) + 2$ and $C > 0$, then (1.1) has no solution.

**Theorem 1.5.** Assume that $V$ satisfies

$$V(x, t) \geq \lambda |x - \xi(t)|^{-2}, \quad (x, t) \in Q(0, R), [0, \tau]$$

with some $\lambda > \lambda_c$, $R > 0$ and $\tau \in (0, T)$. Then (1.1) has no solution for any $u_0$.

We remark that Theorems 1.4 and 1.5 are proved in [2] when $\xi(t)$ does not move. In this paper, we prove these results in a totally different manner.

Proofs of the above theorems are almost self-contained and are based on new ideas using some particular solutions of the radial heat equation, precise estimate of the integral representation formula of solutions and comparison principle. Finally we remark that the method used in this paper is applicable only when $\gamma > 1/2$ in (A1). In fact, the above theorems no longer hold if $\gamma < 1/2$ (see [19]).
This paper is organized as follows: In Sect. 2, we give some preliminary lemmas concerning the radial heat equation. In Sect. 3, we study the existence of a minimal solution. In Sect. 4, we give a lower bound of solutions. In Sect. 5, we prove the uniqueness of a solution, namely if the difference of two solutions are small, then they coincide with each other. In Sect. 6, we show the nonexistence for large initial value. In Sect. 7, we discuss the supercritical case.

2. Radial heat equation

We start from a simple equation

$$u_t = \Delta u + \frac{\lambda}{|x|^2} u, \quad x \in \mathbb{R}^N \setminus \{0\}.$$  

If the initial value is radially symmetric, then this equation is reduced to

$$v_t = v_{rr} + \frac{N-1}{r} v_r + \frac{\lambda}{r^2} v, \quad r > 0. \quad (2.1)$$  

When $0 < \lambda < \lambda_c$, we set $v(r, t) = r^{-\alpha_1(\lambda)} w(r, t)$ to obtain the radial heat equation

$$w_t = w_{rr} + d - 1 \frac{d-1}{r} w_r, \quad r > 0, \quad (2.2)$$  

where $d = N - 2\alpha_1(\lambda) > 2$ corresponds to the spatial dimension. See [1,3,11,13] for the analysis of the radial heat equation.

Lemma 2.1. If $d > 2$, then any nonnegative and nontrivial solution $w$ of (2.2) satisfies $w(0, t) > 0$ for $t > 0$.

Proof. Let $\tilde{w}_0 \in C^\infty_0((0, \infty))$ satisfy $0 \leq \tilde{w}_0 \leq w(\cdot, 0)$ and $\tilde{w}_0 \neq 0$. Define

$$\tilde{w}(r, t) = \int_0^\infty G(r, s, t) \tilde{w}_0(s) ds,$$

where $G$ is the fundamental solution of (2.2) with $w_r(+0, t) = 0$ (see [1,3,11]). Then $\tilde{w}(0, t) > 0$ for $t > 0$. For $\epsilon > 0$, we define a subsolution of (2.2) by

$$w^-_{\epsilon}(r, t) = \max\{\tilde{w}(r, t) - \epsilon r^{-d+2}, 0\}.$$  

Then by comparison, any nonnegative and nontrivial solution $w$ of (2.2) satisfies $w(r, t) \geq w^-_{\epsilon}(r, t)$ for $r > 0$ and $t > 0$. Taking the limit as $\epsilon \downarrow 0$, we obtain $w(r, t) \geq \tilde{w}(r, t)$. Since $\tilde{w}(0, t) > 0$, the proof is complete. \qed

Next, let us consider forward self-similar solutions of (2.2) of the form

$$w(r, t) = \frac{1}{\sigma(t) t^{1/2}} \varphi(\rho), \quad \rho = \frac{r}{t^{1/2}}, \quad (2.3)$$  

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\end{itemize}
where \( l > 0 \) is a constant and \( \sigma(l) \) is given in Lemma 2.2(iii). Substituting this in (2.2), we see that \( \varphi \) must satisfy
\[
\varphi_{\rho\rho} + \frac{d - 1}{\rho} \varphi_{\rho} + \frac{\rho}{2} \varphi_{\rho} + \frac{l}{2} \varphi = 0, \quad \rho > 0.
\]
(2.4)

**Lemma 2.2.** For \( d > 2 \) and \( l > 0 \), there exists a unique solution of (2.4) subject to the initial condition \( \varphi(0) = 1 \) and \( \varphi_{\rho}(0) = 0 \), and the solution has the following properties:

(i) If \( 0 < l < d \), then \( \varphi(\rho) > 0 \) and \( \varphi_{\rho}(\rho) < 0 \) for all \( \rho > 0 \).

(ii) If \( l = d \), then \( \varphi(\rho) = e^{-\rho^2/4} \).

(iii) If \( 0 < l < d \), there exists a constant \( \sigma(l) > 0 \) such that \( \varphi(\rho) = \sigma(l) \rho^{-l} + o(\rho^{-l}) \) as \( \rho \to \infty \).

(iv) \( \sigma(l) \downarrow 0 \) as \( l \uparrow d \).

**Proof.** The existence and uniqueness of a solution of (2.4) with \( \varphi(0) = 1 \) and \( \varphi_{\rho}(0) = 0 \) can be proved in the same manner as [6, Lemma 3.1].

(i) Suppose that
\[
z := \inf \{ \rho > 0; \varphi(\rho) \leq 0 \} < \infty.
\]
Then \( \varphi(z) = 0 \) and \( \varphi_{\rho}(z) \leq 0 \). We rewrite (2.4) as
\[
(\rho^{d-1} \varphi)_{\rho} + \frac{\rho^{d-l}}{2} (\rho^l \varphi)_{\rho} = 0,
\]
and integrate this over \([0, z]\) to obtain
\[
0 \geq z^{d-1} \varphi_{\rho}(z) = -\frac{1}{2} \int_{0}^{z} \rho^{d-l} (\rho^l \varphi)_{\rho} d\rho = \frac{d - l}{2} \int_{0}^{z} \rho^{d-1} \varphi d\rho > 0.
\]
This contradiction implies that \( \varphi(\rho) > 0 \) for all \( \rho > 0 \). Moreover, we see from (2.4) that \( \varphi \) does not attain a positive local minimum. Hence, \( \varphi_{\rho}(\rho) < 0 \) for all \( \rho > 0 \).

(ii) This part can be verified by direct computation.

(iii) We take \( \gamma \in (l, l + 2) \) and \( \theta > 1 \) such that
\[
\theta^{-2} |\gamma(\gamma + 2 - d)| < \frac{\gamma - l}{4}, \quad \theta^{\gamma - l} \frac{\gamma - l}{4} - |l(l + 2 - d)| > 0,
\]
and define a function
\[
g(\rho) := \rho^{-l} - \theta^{\gamma - l} \rho^{-\gamma}.
\]
Note that $g(\theta) = 0$ and $g(\rho) > 0$ for $\rho > \theta$. By direct substitution, we have

$$g_{\rho \rho} + \frac{d - 1}{\rho} g_{\rho} + \frac{\rho}{2} g_{\rho} + \frac{l}{2} g = l(l + 2 - d)\rho^{-l-2} - \theta^{\gamma-l}(\gamma + 2 - d)\rho^{-\gamma-2}$$

$$+ \theta^{\gamma-l} \frac{\gamma - l}{2} \rho^{-\gamma}$$

$$\geq -|l(l + 2 - d)|\rho^{-l-2} - \theta^{\gamma-l} |\gamma(\gamma + 2 - d)|\rho^{-\gamma}$$

$$+ \theta^{\gamma-l} \frac{\gamma - l}{2} \rho^{-\gamma}$$

$$\geq -|l(l + 2 - d)|\rho^{-l-2} + \theta^{\gamma-l} \frac{\gamma - l}{4} \rho^{-\gamma}$$

$$\geq \left\{ \theta^{\gamma-l} \frac{\gamma - l}{4} - |l(l + 2 - d)| \right\} \rho^{-l-2}.$$ 

Hence, we obtain

$$g_{\rho \rho} + \frac{d - 1}{\rho} g_{\rho} + \frac{\rho}{2} g_{\rho} + \frac{l}{2} g > 0, \quad \rho \geq \theta. \quad (2.7)$$

We rewrite (2.4) as

$$(\rho^{d-1} e^{\rho^2/4} \varphi_{\rho})_{\rho} + \frac{l}{2} \rho^{d-1} e^{\rho^2/4} \varphi = 0.$$ 

Integrating this on $[0, \rho]$, we easily see that $\varphi_{\rho}(\rho) = O(\rho)$ as $\rho \to 0$. Multiplying this by $g(\rho)$, integrating it on $(\theta, \rho)$ by parts and using (2.7), we obtain

$$\rho^{d-1} e^{\rho^2/4} (\varphi_{\rho}(\rho) g(\rho) - \varphi(\rho) g_{\rho}(\rho)) + \theta^{d-1} e^{\theta^2/4} \varphi(\theta) g_{\rho}(\theta) < 0.$$ 

Since $\varphi(\theta) > 0$ and $g_{\rho}(\theta) > 0$, we obtain $\varphi_{\rho} g - \varphi g_{\rho} < 0$ for $\rho > \theta$. This implies that $\varphi(\rho)/g(\rho)$ is decreasing in $\rho \in (\theta, \infty)$.

On the other hand, using $g(\rho) \leq \rho^{-l}$, multiplying (2.5) by $\rho^{l-d}$, integrating it on $(0, \tilde{\rho})$ for $\tilde{\rho} > \theta$ and applying (i), we obtain

$$\frac{\varphi(\tilde{\rho})}{2 g(\tilde{\rho})} \geq \frac{1}{2} \tilde{\rho} \varphi(\tilde{\rho}) = - \int_{0}^{\tilde{\rho}} \rho^{l-d} (\rho^{d-1} \varphi_{\rho})_{\rho} d\rho$$

$$= -(d - l) \int_{0}^{\tilde{\rho}} \rho^{l-2} \varphi_{\rho} d\rho - \left[ \rho^{l-1} \varphi_{\rho} \right]_{0}^{\tilde{\rho}}$$

$$\geq -(d - l) \int_{0}^{\theta} \rho^{l-2} \varphi_{\rho} d\rho > 0.$$ 

(Here $\rho^{l-1} \varphi_{\rho} \to 0$ as $\rho \to 0$ and $\rho^{l-2} \varphi_{\rho}$ is integrable on $[0, \tilde{\rho}]$, because $l > 0$ and $\varphi_{\rho}(\rho) = O(\rho)$ as $\rho \to 0$.) Hence, $\varphi(\rho)/g(\rho)$ is bounded below on $(\theta, \infty)$, so that $\varphi(\rho)/g(\rho)$ converges to a positive constant as $\rho \to \infty$.

(iv) Assume that $d - \delta < l < d$ for some small $\delta > 0$. Then we can take $\theta$ and $\gamma$ in (2.6) to be independent of $l$. For any $\varepsilon > 0$, we take $\rho_{0} > \theta$ such that

$$e^{-\rho_{0}^2/4} \leq \varepsilon g(\rho_{0}).$$
Then by (ii) and the continuity of \( \varphi \) with respect to \( l \), there exists \( \delta > 0 \) such that 
\[
\varphi(\rho_l) \leq 2g(\rho_0) \quad \text{for} \quad l \in (d - \delta, d).
\]
Since \( \varphi(\rho)/g(\rho) \) is decreasing, we obtain \( \sigma(l) \leq 2\varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, the proof is complete. \( \square \)

**Lemma 2.3.** If \( 0 < \lambda < \lambda_c \) and \( 0 \leq k < \alpha_2(\lambda) + 2 \), then 
\[
v(r, t) = v(r, t; k) = \frac{r^{-k}}{\sigma(l)} \left( \frac{r}{t^{1/2}} \right)^l \varphi \left( \frac{r}{t^{1/2}} \right),
\]
satisfies (2.1), where \( l = k - \alpha_1(\lambda) \) and \( \sigma \) and \( \varphi \) are given in Lemma 2.2 with \( d = N - 2\alpha_1(\lambda) \). Moreover, the function \( v \) has the following properties:

(i) \( v(r, t; k) \to r^{-k} \) as \( t \downarrow 0 \) uniformly on \( (\delta, \infty) \), where \( \delta > 0 \) is arbitrary.

(ii) \( v(r, t; k) \to \infty \) as \( k \uparrow \alpha_2(\lambda) + 2 \) uniformly on any compact subset of \( (0, \infty) \times (0, \infty) \).

**Proof.** (i) If \( w(r, t) \) satisfies (2.2), then \( v(r, t) = r^{-\alpha_1(\lambda)}w(r, t) \) satisfies (2.1). By \( \lambda < \lambda_c \), we have \( d > 2 \) and \( l < d \). Hence, (2.2) has a positive solution given by (2.3). On the other hand, by Lemma 2.2 (iii), we have 
\[
v(r, t; k) = \frac{r^{-k}}{\sigma(l)} \left( \frac{r}{t^{1/2}} \right)^l \varphi \left( \frac{r}{t^{1/2}} \right) \to r^{-k} \quad \text{as} \quad t \downarrow 0.
\]

(ii) We remark that \( l \uparrow d \) is equivalent to \( k \uparrow N - \alpha_1(\lambda) = \alpha_2(\lambda) + 2 \). Then by Lemma 2.2(iv), we have \( \sigma(l) \to 0 \) and \( \varphi(\rho) \to e^{-\rho^2/4} \) as \( k \uparrow \alpha_2(\lambda) + 2 \). \( \square \)

3. Existence in the subcritical case

In this section, we prove Theorem 1.1 by using a uniform upper estimate of the solution \( u_n \) for the cut-off problem. We use solutions of a simple equation \( v_t = \Delta v + \lambda'|x|^{-2}v \quad (0 < \lambda' \leq \lambda_c) \).

**Proof of Theorem 1.1.** For \( n \geq 1 \), set \( V_n := \min\{V, n\} \). Let \( u_n \) be a unique solution of the approximate problem (1.2). Note that \( u_n \) satisfies 
\[
u_n(x, t) = \int_{\mathbb{R}^N} G(x - y, t)u_0(y)dy 
+ \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)V_n(y, s)u_n(y, s)dyds. \tag{3.1}
\]

For a while, we fix \( n \), and set \( \tilde{u}(x, t) := u_n(x + \xi(t), t) \) and \( \tilde{V}_n(x, t) := V_n(x + \xi(t), t) \). Then \( \tilde{u} \) satisfies \( \tilde{u}(x, t) = \Phi[\tilde{u}](x, t) \), where 
\[
\Phi[\tilde{u}](x, t) := \int_{\mathbb{R}^N} G(x - y + \xi(t) - \xi(0), t)u_0(y + \xi(0))dy 
+ \int_0^t \int_{\mathbb{R}^N} G(x - y + \xi(t) - \xi(s), t - s)\tilde{V}_n(y, s)\tilde{u}(y, s)dyds. \tag{3.2}
\]
Let $\varepsilon > 0$ be arbitrarily given. For $k < \alpha_2(\lambda) + 2$, we take constants $\delta = \delta(\varepsilon)$ and $0 < T_\delta \leq T$ independent of $n$ such that
\[
c_0 \lambda < \lambda_c, \quad k < \alpha_2(c_0 \lambda) + 2, \quad \alpha_1(c_0 \lambda) < \alpha_1(\lambda) + \varepsilon,
\]
\[
c_0 := (1 - \delta)^{-\frac{2}{2 \gamma}} \exp \left( \frac{(1 - \delta) C_\xi^2}{4 \delta T_\delta^{2 \gamma - 1}} \right) > 1,
\]
where $C_\xi$ is a constant given by (A1).

We claim that there exist $w_n$ and $w^+$ on $t \in [0, T_\delta]$ such that $w_n = \Phi[w_n]$ and $w_n \leq w^+$ for $t \in [0, T_\delta]$. Once we prove this claim, by the uniqueness of solutions of (1.2), we obtain
\[
\begin{aligned}
u_n(x, t) &= \tilde{u}(x - \xi(t), t) = w_n(x - \xi(t), t) \leq w^+(x - \xi(t), t), \\
(x, t) &\in Q_{(0, \infty), [0, T_\delta]},
\end{aligned}
\]
where $Q_{(0, \infty), [0, T_\delta]}$ is given in (1.3). We prove the claim by using solutions of a simple problem
\[
\begin{aligned}
u_t &= \Delta \nu + \frac{c_0 \lambda}{|x|^2} \nu, \quad (x, t) \in A_\infty(0) \times (0, \infty), \\
\nu(x, 0) &= |x|^{-k}, \quad x \in A_\infty(0),
\end{aligned}
\]
for which we denote a solution by $\nu = \nu(x, t; k)$. Note that the existence of a solution $\nu$ follows from $k < \alpha_2(c_0 \lambda) + 2$ and Lemma 2.3.

By (A2), (A3) and the assumption on $V$, there exists a constant $M > 0$ independent of $n$ such that
\[
\begin{aligned}
\tilde{V}_n(x, t) &\leq \lambda |x|^{-2} + M, \quad (x, t) \in A_\infty(0) \times [0, T], \\
u_0(x + \xi(0)) &\leq M(|x|^{-k} + 1), \quad x \in A_\infty(0).
\end{aligned}
\]

Define
\[
w^+(x, t) := c_0 M e^{c_0 M t} \{ v((1 - \delta)^{1/2} x, t; k) + v((1 - \delta)^{1/2} x, t; 0) \}.
\]

We will see that
\[
\Phi[w^+](x, t) \leq w^+(x, t), \quad (x, t) \in A_\infty(0) \times [0, T_\delta].
\]

Direct computations show that $w^+$ is a solution of the following problem:
\[
\begin{aligned}
w^+_t &= \frac{1}{1 - \delta} \Delta w^+ + \frac{c_0 \lambda}{(1 - \delta)|x|^2} w^+ + c_0 M w^+, \quad (x, t) \in A_\infty(0) \times (0, \infty), \\
w^+(x, 0) &= c_0 M ((1 - \delta)^{-\frac{k}{2}} |x|^{-k} + 1), \quad x \in A_\infty(0),
\end{aligned}
\]
which is equivalent to
\[
\begin{aligned}
w^+(x, t) &= \int_{\mathbb{R}^N} G\left(x - y, \frac{t}{1 - \delta} \right) c_0 M ((1 - \delta)^{-\frac{k}{2}} |y|^{-k} + 1) dy \\
&\quad + \int_t^0 \int_{\mathbb{R}^N} G\left(x - y, \frac{t - s}{1 - \delta} \right) \left\{ \frac{c_0 \lambda}{(1 - \delta)|y|^2} + c_0 M \right\} w^+(y, s) dy ds.
\end{aligned}
\]
By (A1), for $0 \leq t \leq T_\delta$, we have
\[
|x - y + \xi(t) - \xi(s)|^2 \geq (1 - \delta)|x - y|^2 - \frac{1 - \delta}{\delta} |\xi(t) - \xi(s)|^2
\]
\[
\geq (1 - \delta)|x - y|^2 - \frac{(1 - \delta)C^2}{\delta}(t - s)^2. \tag{3.6}
\]
\[
G(x - y + \xi(t) - \xi(s), t - s) \leq c_0 G \left( x - y, \frac{t - s}{1 - \delta} \right).
\]
These computations together with (3.2) and (3.4) yield
\[
\Phi[w^+](x, t) \leq \int_{\mathbb{R}^N} c_0 G \left( x - y, \frac{t}{1 - \delta} \right) u_0(y + \xi(0))dy
\]
\[
+ \int_0^t \int_{\mathbb{R}^N} c_0 G \left( x - y, \frac{t - s}{1 - \delta} \right) \tilde{V}(y, s)w^+(y, s)dyds
\]
\[
\leq \int_{\mathbb{R}^N} G \left( x - y, \frac{t}{1 - \delta} \right) c_0 M (|y|^{-k} + 1)dy
\]
\[
+ \int_0^t \int_{\mathbb{R}^N} G \left( x - y, \frac{t - s}{1 - \delta} \right) \left( \frac{c_0 \lambda}{|y|^2} + c_0 M \right) w^+(y, s)dyds
\]
\[
\leq w^+(x, t)
\]
for $t \in (0, T_\delta]$. This proves (3.5).

We inductively define a sequence \{\(w_i\)\} by
\[
w_0(x, t) := \int_{\mathbb{R}^N} G(x - y + \xi(t) - \xi(0), t)u_0(y + \xi(0))dy
\]
and \(w_i := \Phi[w_{i-1}] (i \geq 1)\). By (3.2), (3.4) and (3.6), we have $0 \leq w_0 = \Phi[0] \leq w^+$, and so $\Phi[0] \leq \Phi[w_0] \leq \Phi[w^+]$. Then $\Phi[w^+] \leq w^+$ implies $0 \leq w_0 \leq w_1 \leq w^+$. Similarly, we have
\[
0 \leq \Phi[0] \leq \Phi[w_0] = w_1 \leq \Phi[w_1] = w_2 \leq \Phi[w^+] \leq w^+.
\]
Repeating this procedure yields
\[
0 \leq w_1(x, t) \leq w_2(x, t) \leq \cdots \leq w_i(x, t) \leq \cdots \leq w^+(x, t) < \infty
\]
for \((x, t) \in Q(0, \infty), [0, T_\delta]\), and the limit $w_n(x, t) := \lim_{i \to \infty} w_i(x, t)$ exists for each \((x, t) \in Q(0, \infty), [0, T_\delta]\). Then $w_n$ is the desired function stated in the claim, and hence, (3.3) follows.

Since $w^+$ is independent of $n$, we see from (3.3) that the limit $u(x, t) := \lim_{n \to \infty} u_n(x, t)$ exists and satisfies $u(x, t) \leq w^+(x - \xi(t), t)$ for each \((x, t) \in Q(0, \infty), [0, T_\delta]\). In addition, the standard regularity theory for linear parabolic equations implies that the limiting function $u$ satisfies (1.1) on $[0, T_\delta]$. 

Let $\tau \in (0, T_\delta)$ and let $\sigma(\cdot;k)$ and $\varphi(\cdot;k)$ be as in Lemma 2.2 with $l = k - \alpha_1(c_0\lambda)$ and $d = N - 2\alpha_1(c_0\lambda)$. Then by Lemma 2.3, there exists a constant $C = C(\delta, \tau) > 0$ such that

$$w^+(x, t) = c_0Me^{c_0Mt}\left[\frac{\{(1 - \delta)^{\frac{1}{2}}|x|\}^{\alpha_1(c_0\lambda)}\sigma(k - \alpha_1(c_0\lambda); k)}{\sigma(k - \alpha_1(c_0\lambda); k)}t^{k-\alpha_1(c_0\lambda)}\varphi(t^{\frac{1}{2}}(1 - \delta)^{\frac{1}{2}}|x|; k)\right.$$  

$$+ \frac{\{(1 - \delta)^{\frac{1}{2}}|x|\}^{\alpha_1(c_0\lambda)}\sigma(-\alpha_1(c_0\lambda); 0)}{\sigma(-\alpha_1(c_0\lambda); 0)}t^{\alpha_1(c_0\lambda)}\varphi(t^{\frac{1}{2}}(1 - \delta)^{\frac{1}{2}}|x|; 0)\right]$$  

$$\leq C|x|^{-\alpha_1(c_0\lambda)}$$

for $(x, t) \in A_R(0) \times [\tau, T_\delta]$. Recall that $\alpha_1(c_0\lambda) < \alpha_1(\lambda) + \varepsilon$ and that $\delta$ is determined by $\varepsilon$. Then there exists a constant $C = C(\varepsilon, \tau) > 0$ such that

$$u(x, t) \leq w^+(x - \xi(t), t) \leq C|x - \xi(t)|^{-\alpha_1(\lambda) - \varepsilon}$$

for $(x, t) \in Q(0, R), [\tau, T_\delta]$. Since the equation is linear and $T_\delta$ is determined by $\varepsilon$, we see that $u$ has a continuation as a solution of (1.1) for $t \in [0, T]$ satisfying

$$u(x, t) \leq C|x - \xi(t)|^{-\alpha_1(\lambda) - \varepsilon}, \quad (x, t) \in Q(0, R), [\tau, T]$$

where $\varepsilon > 0$ and $\tau \in (0, T)$ are arbitrary and $C = C(\varepsilon, \tau) > 0$ is a constant.

**Remark 3.1.** For Theorem 1.1, we can relax (A1) as follows. Let

$$h(t) := \limsup_{s \downarrow 0} \frac{\left|\xi(t + s) - \xi(t)\right|}{s^{1/2}}$$

which is well-defined if $\xi(t)$ is at least 1/2-Hölder continuous. If $h(t)$ is sufficiently small for $t > 0$, then the proof given above holds true. If $\xi(t)$ is $\gamma$-Hölder continuous in $t \in [0, \infty)$ with some $\gamma > 1/2$, then $h(t)$ is identically equal to 0.

**4. Lower bound of solutions**

In this section, assuming the existence of a minimal solution of (1.1), we shall derive a lower bound of solutions by using a solution of a simple equation $v_t = \Delta v + \lambda' |x|^{-2}v$ ($0 < \lambda' < \lambda_e$).

**Proof of Theorem 1.2.** Fix $\tau \in (0, T), t_1 \in [\tau, T]$ and $\varepsilon > 0$. We take constants $\delta = \delta(\varepsilon) > 0$ and $T_\delta \in (0, \tau)$ independent of $t_1$ such that

$$\alpha_1(c_1\lambda) > \alpha_1(\lambda) - \varepsilon, \quad c_1 := (1 + \delta)^{-\frac{N}{2}} \exp\left(-\frac{(1 + \delta)C_\xi^2}{4\delta T_\delta^{2\gamma - 1}}\right) < 1.$$

Let $u$ be a minimal solution of (1.1). Recall that $u$ is obtained by $u(x, t) = \lim_{n \to \infty} u_n(x, t)$, where $u_n$ is a unique solution of the integral equation (3.1). We
set \( \tilde{u}_n(x, t) := u_n(x + \xi(t), t) \). Then \( \tilde{u}_n \) satisfies \( \tilde{u}_n = \Phi[\tilde{u}_n] \), where \( \Phi \) is defined by (3.2). From the same computations as in (3.6), it follows that

\[
|x - y + \xi(t) - \xi(s)|^2 \leq (1 + \delta)|x - y|^2 + \frac{(1 + \delta)C^2 \delta}{\delta}(t - s)^{2\gamma},
\]

\[
G(x - y + \xi(t) - \xi(s), t - s) \geq c_1 G\left(x - y, \frac{t - s}{1 + \delta}\right)
\]

for \( t_0 := t_1 - T_\delta \leq s < t \leq t_1 \). Then for \( t_0 < t \leq t_1 \), by writing \( \tilde{V}_n(y, s) := V_n(y + \xi(s), s) \), we have

\[
\tilde{u}_n(x, t) \geq \tilde{v}_n(x, t) := \int_{\mathbb{R}^N} c_1 G\left(x - y, \frac{t}{1 + \delta}\right) u_0(y + \xi(0)) dy
\]

\[
+ \int_{t_0}^t \int_{\mathbb{R}^N} c_1 G\left(x - y, \frac{t - s}{1 + \delta}\right) \tilde{V}_n(y, s) \tilde{u}_n(y, s) dy ds.
\]

Direct computations show that

\[
\begin{cases}
\tilde{v}_n t = \frac{1}{1 + \delta} \Delta \tilde{u}_n + c_1 \tilde{V}_n \tilde{u}_n \geq \frac{1}{1 + \delta} \Delta \tilde{u}_n + c_1 \tilde{V}_n \tilde{v}_n, & (x, t) \in A_{\infty}(0) \times (t_0, t_1), \\
\tilde{v}_n(x, t_0) = \int_{\mathbb{R}^N} c_1 G\left(x - y, \frac{t_0}{1 + \delta}\right) u_0(y + \xi(0)) dy > 0, & x \in A_{\infty}(0).
\end{cases}
\]

Note that similar argument is used in [8]. By (A2) and the assumption on \( V \), there exists a constant \( M > 0 \) independent of \( n \) such that

\[
\tilde{V}(x, t) \geq \lambda|x|^{-2} - M, \quad (x, t) \in A_{\infty}(0) \times [0, T],
\]

(4.1)

Hence, we obtain

\[
c_1 \tilde{V}_n \tilde{v}_n \geq \min \left\{ \frac{c_1 \lambda}{(1 + \delta)|x|^2} - c_1 M, c_1 n \right\} \tilde{v}_n.
\]

Let \( \tilde{v}_n^- \) be a unique solution of

\[
\begin{cases}
\tilde{v}_n t = \frac{1}{1 + \delta} \Delta \tilde{v}_n^- + \min \left\{ \frac{c_1 \lambda}{(1 + \delta)|x|^2} - c_1 M, c_1 n \right\} \tilde{v}_n^- , & x \in A_{\infty}(0), \quad t > t_0, \\
\tilde{v}_n^- (x, t_0) = \tilde{v}_n^- (|x|), & x \in \mathbb{R}^N.
\end{cases}
\]

Here the initial value \( \tilde{v}_n^- = \tilde{v}_n^- (|x|) \) is a positive and radially symmetric function satisfying \( \tilde{v}_n^- (|x|) \leq \tilde{v}_n(x, t_0) \) and \( (\tilde{v}_n^-)_r (0) = 0 \). We see that \( \tilde{v}_n^- \) is a supersolution of the problem for \( \tilde{v}_n^- \), and hence, the comparison principle for bounded functions implies \( \tilde{v}_n \geq \tilde{v}_n^- \). This together with \( \tilde{u}_n \geq \tilde{v}_n \) yields \( \tilde{u}_n \geq \tilde{v}_n^- \). Letting \( n \rightarrow \infty \), we obtain

\[
\tilde{u} \geq \tilde{v}^-.
\]

(4.2)
Here $\tilde{u}(x, t) := u(x + \xi(t), t)$ and $\tilde{v}^-$ is a minimal solution of

$$
\begin{cases}
\tilde{v}^-_t = \frac{1}{1 + \delta} \Delta \tilde{v}^- + \left\{ \frac{c_1 \lambda}{(1 + \delta)|x|^2} - c_1 M \right\} \tilde{v}^-, & x \in A_\infty(0), \ t > t_0, \\
\tilde{v}^-(x, t_0) = \tilde{v}_0^-(|x|), & x \in \mathbb{R}^N.
\end{cases}
$$

Since the initial value $\tilde{v}_0^-$ is radially symmetric, the solution $\tilde{v}_n^-$ is also radially symmetric, and so the limit $\tilde{v}^-$ is radially symmetric. Thus we can rewrite $\tilde{v}^-(x, t) = \tilde{u}^-(1 + \delta)^{1/2} r, t - t_0)$ $(r = |x|)$, where $\tilde{u}^-$ is a solution of the corresponding problem

$$
\begin{cases}
\tilde{u}^-_t = \tilde{u}_{rr} + \frac{N - 1}{r} \tilde{u}_r + \left( \frac{c_1 \lambda}{r^2} - c_1 M \right) \tilde{u}^-, & r > 0, \ t \in (0, T_\delta], \\
\tilde{u}^-(r, 0) = \tilde{v}_0^-(1 + \delta)^{-\frac{1}{2}} r), & r \geq 0.
\end{cases}
$$

Set $w(r, t) := e^{c_1 M t r \alpha_1(c_1 \lambda)} \tilde{u}^-(r, t)$. From

$$(\alpha_1(c_1 \lambda))^2 - (N - 2)\alpha_1(c_1 \lambda) + c_1 \lambda = 0,$$

it follows that $w$ satisfies

$$w_t = w_{rr} + \frac{N - 2\alpha_1(c_1 \lambda) - 1}{r} w_r, \quad r > 0, \ t \in (0, T_\delta].$$

By Lemma 2.1, we have $w(0, t) > 0$ for $t \in (0, T_\delta]$. Then there exists a constant $C = C(\delta, T_\delta) > 0$ independent of $t_1$ such that $w(r, t) \geq C$ for $(r, t) \in [0, R] \times [T_\delta/2, T_\delta]$, and hence,

$$\tilde{u}^-(r, t) \geq C e^{c_1 M t r^{-\alpha_1(c_1 \lambda)}}, \quad (r, t) \in (0, R] \times [T_\delta/2, T_\delta].$$

This together with (4.2) and $\alpha_1(c_1 \lambda) > \alpha_1(\lambda) - \epsilon$ shows that

$$u(x, t) = \tilde{u}(x - \xi(t), t) \geq \tilde{v}^-((1 + \delta)^{1/2}|x - \xi(t)|, t - t_0)$$

$$\geq C \frac{e^{-c_1 M (t - t_0)}}{(1 + \delta)^{\alpha_1(c_1 \lambda)/2}} |x - \xi(t)|^{-\alpha_1(c_1 \lambda)}$$

$$\geq C e^{-c_1 M T_\delta} |x - \xi(t)|^{-\alpha_1(c_1 \lambda)} \geq C |x - \xi(t)|^{-\alpha_1(\lambda) + \epsilon}$$

for $(x, t) \in Q(0, R),[t_0 + T_\delta/2, t_0 + T_\delta]$, where the above constants $C$ may have different values within the same line. In particular, since $t_1 = t_0 + T_\delta$ and $T_\delta$ depends on $\tau$, there exists a constant $C = C(\delta, \tau) > 0$ such that $u(x, t_1) \geq C|x - \xi(t_1)|^{-\alpha_1(\lambda) + \epsilon}$ for $x \in A_R(\xi(t_1))$. Since $t_1 \in [\tau, T]$ is arbitrary, $C$ is independent of $t_1$ and $\delta$ is determined by $\epsilon$, we obtain

$$u(x, t) \geq C |x - \xi(t)|^{-\alpha_1(\lambda) + \epsilon}, \quad (x, t) \in Q(0, R),[\tau, T],$$

where $\epsilon > 0$ and $\tau \in (0, T)$ are arbitrary and $C = C(\epsilon, \tau) > 0$ is a constant. \qed
5. Uniqueness of solutions

We prove the uniqueness of solutions. The proof is based on nontrivial modifications of the method of Marchi [16, pp. 1075–1079]. In that paper, the position of the singularity does not move in time, while in our paper, the singularity moves in time and its motion may not be smooth.

Note that, in the definition of solutions of (1.1), we assume the boundedness of solutions for \(|x - \xi(t)| \geq 1\). However, for uniqueness, this can be relaxed as follows, and so we prove the following result which is stronger than Theorem 1.3.

**Theorem 5.1.** Assume that \(V\) satisfies

\[
0 \leq V(x, t) \leq \lambda|x - \xi(t)|^{-2}, \quad (x, t) \in Q_{(0,R),[0,T]}
\]

with some \(\lambda \in (0, \lambda_c)\) and \(R > 0\). If \(u_1\) and \(u_2\) are solutions of (1.1) with the same initial data such that

\[
|u_1(x, t) - u_2(x, t)| \leq C|x - \xi(t)|^{-\alpha_2(\lambda) + \epsilon} e^{C|x - \xi(t)|^2}, \quad (x, t) \in Q_{(0,\infty),[0,T]}
\]

with some \(C > 0\) and \(\epsilon \in (0, \alpha_2(\lambda))\), then \(u_1 \equiv u_2\) on \(Q_{(0,\infty),[0,T]}\).

**Proof.** Let \(\epsilon \in (0, \alpha_2(\lambda))\). We prepare constants \(\delta = \delta(\epsilon)\) and \(0 < T_\delta \leq T\) similarly to the proof of Theorem 1.1,

\[
c_0 \lambda < \lambda_c, \quad \alpha_1(c_0 \lambda) < \alpha_1(\lambda) + \frac{\epsilon}{2}, \quad k := \alpha_1(\lambda) + \frac{\epsilon}{2} < \alpha_2(c_0 \lambda) + 2,
\]

\[
c_0 := (1 - \delta)^{-\frac{N}{2}} \exp \left( \frac{(1 - \delta)C_{\delta}^2}{4\delta T_\delta^{-2\gamma - 1}} \right) > 1.
\]

In addition, we take \(T_\delta\) so small that

\[
T_\delta^{-\frac{N}{2}} \leq e^{\frac{1}{16T_\delta}}. \tag{5.1}
\]

Fix \(t_0 \in (0, T_\delta]\). Since \(u_1\) and \(u_2\) have the same initial data, we have

\[
\int_0^{t_0} \int_{\mathbb{R}^N} (u_1 - u_2)(\phi_t + \Delta \phi + V \phi)dxdt = \int_{\mathbb{R}^N} (u_1(x, t_0) - u_2(x, t_0)) \phi(x, t_0)dx \tag{5.2}
\]

for any \(\phi \in C^2_0(\mathbb{R}^N \times [0, t_0])\). We will construct a suitable test function \(\phi\).

Let \(d > 0\) and let \(\theta \in C^\infty_0(B_d(\xi(t_0)))\) be nonnegative. We write \(d_0 := \sup_{t \in [0, T]} |\xi(t)|\) and fix \(R_0 := d_0 + d + 1\). For \(R > 2R_0\), we consider the problem

\[
\begin{cases}
w_{n,t} + \Delta w_n + V_n(x, t) w_n = 0, \quad (x, t) \in B_R(0) \times (0, t_0), \\
w_n(x, t) = 0, \quad (x, t) \in \partial B_R(0) \times (0, t_0), \\
w_n(x, t_0) = \theta(x), \quad x \in B_R.
\end{cases}
\]
where $V_n := \min \{ V, n \}$. By taking $\tilde{w}_n(x, t) = w_n(x, t) \ (t = t_0 - t)$, we see that this problem is equivalent to

$$
\begin{align*}
\tilde{w}_{n, \tau} &= \Delta \tilde{w}_n + V_n(x, t_0 - \tau) \tilde{w}_n, \quad (x, \tau) \in B_R(0) \times (0, t_0), \\
\tilde{w}_n(x, \tau) &= 0, \quad (x, \tau) \in \partial B_R(0) \times (0, t_0), \\
\tilde{w}_n(x, 0) &= \theta(x), \quad x \in B_R.
\end{align*}
$$

(5.3)

First, we estimate $w_n$ as follows. In order to construct a supersolution of (5.3), we consider a unique solution $u_n$ of

$$
\begin{align*}
u_{n, \tau} &= \Delta u_n + V_n(x, t_0 - \tau)u_n, \quad x \in A_\infty(\xi(t_0 - \tau)), \ t \in (0, t_0], \\
u_n(x, 0) &= \theta(x), \quad x \in \mathbb{R}^N.
\end{align*}
$$

Setting $\tilde{u}_n(x, \tau) := u_n(x + \xi(t_0 - \tau), \tau)$ and $\tilde{V}_n(x, \tau) := V_n(x + \xi(t_0 - \tau), t_0 - \tau)$, we have

$$
\begin{align*}
\tilde{u}_n(x, \tau) &= \int_{\mathbb{R}^N} G(x - y + \xi(t_0 - \tau) - \xi(t_0), \tau) \theta(y + \xi(t_0)) dy \\
&\quad + \int_0^\tau \int_{\mathbb{R}^N} G(x - y + \xi(t_0 - \tau) - \xi(t_0 - s), \tau - s) \tilde{V}_n(y, s) \tilde{u}_n(y, s) dy ds.
\end{align*}
$$

Then by the same argument as in the proof of Theorem 1.1, we can prove that

$$
u_n(x, \tau) = \tilde{u}_n(x - \xi(t_0 - \tau), \tau)
\leq c_0 M e^{c_0 \alpha_1 \lambda} \frac{(1 - \delta) \frac{1}{2} \vert x - \xi(t_0 - \tau) \vert^{\alpha_1(\xi(t_0 - \tau))}}{\sigma(k - \alpha_1(\xi(t_0 - \tau)))} \tau^{\frac{k - \alpha_1(\xi(t_0 - \tau))}{2}} \varphi
\left( \tau^{\frac{1}{2} (1 - \delta) \frac{1}{2} \vert x - \xi(t_0 - \tau) \vert} \right)
$$

for $x \in A_\infty(\xi(t_0 - \tau))$ and $\tau \in (0, t_0)$, where $\sigma$ and $\varphi$ are given in Lemma 2.2 with $l = k - \alpha_1(\xi(t_0 - \tau))$ and $d = N - 2\alpha_1(\xi(t_0 - \tau))$, and $M$ is a constant independent of $n$ such that

$$
\begin{align*}
\tilde{V}_n(x, \tau) &\leq \lambda \vert x \vert^{-2} + M, \quad (x, \tau) \in A_\infty(0) \times [0, T], \\
\theta(x + \xi(t_0)) &\leq M \vert x \vert^{-k}, \quad x \in A_\infty(0).
\end{align*}
$$

By $l < d$ and Lemma 2.2(iii), there exists a constant $C = C(\delta, T_\delta) > 0$ such that

$$
u_n(x, \tau) \leq C \vert x - \xi(t_0 - \tau) \vert^{l - \alpha_1(\xi(t_0 - \tau))}, \quad (x, \tau) \in A_\infty(\xi(t_0 - \tau)), \ \tau \in [0, t_0].
$$

Since $l = \alpha_1(\lambda) + (\varepsilon/2) - \alpha_1(\xi(t_0 - \tau))$ and $u_n$ is a supersolution of (5.3), we obtain

$$
\begin{align*}
w_n(x, t) &= \tilde{w}_n(x, t_0 - t) \leq u_n(x, t_0 - t) \leq C \vert x - \xi(t) \vert^{-\alpha_1(\lambda) - \frac{\varepsilon}{2}}, \\
(x, t) &\in B_R(0) \times [0, t_0].
\end{align*}
$$

(5.4)

Let us next show that there exists a constant $C > 0$ which depends on $R_0$ but is independent of $n, \delta$ and $R$ such that

$$
\vert \nabla w_n \vert \leq C e^{-\frac{(R - 1)^2}{16T_\delta}} \quad \text{on } \partial B_R(0) \times (0, t_0).
$$

(5.5)
Since $\theta = 0$ in $\mathbb{R}^N \setminus B_{R_0}(0)$, we see that $\tilde{w}_n$ is a solution of

$$
\begin{cases}
  g_\tau = \Delta g + V_n(x, t_0 - \tau)g, & (x, \tau) \in (B_R(0) \setminus \overline{B_{R_0}(0)}) \times (0, t_0), \\
  g(x, \tau) = 0, & (x, \tau) \in \partial B_R(0) \times (0, t_0), \\
  g(x, \tau) = \tilde{w}_n(x, \tau), & (x, \tau) \in \partial B_{R_0}(0) \times (0, t_0), \\
  g(x, 0) = 0, & x \in B_R(0) \setminus \overline{B_{R_0}(0)}.
\end{cases}
$$

(5.6)

By (A2) and (5.4), we can choose positive constants $C_1, C_2$ and $C_3$ independent of $n$, $\delta$ and $R$ such that

$$
sup_{(\mathbb{R}^N \setminus \overline{B_{R_0}(0)}) \times [0, T_\delta)} V \leq C_1, \quad sup_{\partial B_{R_0} \times (0, t_0)} \tilde{w}_n \leq C_2, \quad C_3 e^{C_1 T_3} (8\pi T_\delta)^{-\frac{N}{2}} e^{-\frac{k_0^2}{4T_\delta}} \geq C_2.
$$

We define

$$
W_1^+(x, \tau) := C_3 e^{C_1 (T_3 + \tau)} G(x, T_\delta + \tau).
$$

Then we can check that $W_1^+$ is a supersolution of (5.6), and hence, $\tilde{w}_n \leq W_1^+$ in $(B_R(0) \setminus \overline{B_{R_0}(0)}) \times (0, t_0)$. In particular,

$$
\tilde{w}_n \leq W_1^+ \quad \text{on} \ \partial B_{R-1}(0) \times (0, t_0).
$$

(5.7)

On the other hand, $\tilde{w}_n$ is also a solution of

$$
\begin{cases}
  h_\tau = \Delta h + V_n(x, t_0 - \tau)h, & (x, \tau) \in (B_R(0) \setminus \overline{B_{R-1}(0)}) \times (0, t_0), \\
  h(x, \tau) = 0, & (x, \tau) \in \partial B_R(0) \times (0, t_0), \\
  h(x, \tau) = \tilde{w}_n(x, \tau), & (x, \tau) \in \partial B_{R-1}(0) \times (0, t_0), \\
  h(x, 0) = 0, & x \in B_R(0) \setminus \overline{B_{R-1}(0)}.
\end{cases}
$$

(5.8)

We define

$$
W_2^+(x, \tau) := e^{C_1 (T_3 + \tau)} \left(C_4 |x|^{2-N} - C_5\right),
$$

where $C_4 > 0$ and $C_5 > 0$ are positive constants such that

$$
R^{2-N} C_4 - C_5 = 0, \quad (R - 1)^{2-N} C_4 - C_5 = \sup_{\partial B_{R-1}(0) \times (0, t_0)} W_1^+.
$$

Note that

$$
C_4 = \frac{1}{(R - 1)^{2-N} - R^{2-N}} \sup_{\partial B_{R-1}(0) \times (0, t_0)} W_1^+.
$$

By (5.7), we see that $W_2^+$ is a supersolution of (5.8) so that $\tilde{w}_n \leq W_2^+$. This together with $\tilde{w}_n = W_2^+ = 0$ on $\partial B_R(0) \times (0, t_0)$ shows that $|\nabla \tilde{w}_n| \leq |\nabla W_2^+|$ on $\partial B_R(0) \times (0, t_0)$. By the definition of $W_2^+$ and $W_1^+$ and using (5.1), we have
\[ |\nabla W_2^+| = \frac{(N - 2)R^{-1}(R - 1)^{N-2}}{R^{N-2} - (R - 1)^{N-2}} C_3 e^{2C_1(T_\delta + \tau)} \{4\pi (T_\delta + \tau)\}^{-\frac{N}{2}} e^{-\frac{(R-1)^2}{4(R-1)T_\delta}} \]
\[ \leq (N - 2)C_3 e^{4C_1T_\delta} (4\pi T_\delta)^{-\frac{N}{2}} e^{-\frac{(R-1)^2}{8T_\delta}} \]
\[ \leq (N - 2)C_3 e^{4C_1T} (4\pi)^{-\frac{N}{2}} e^{-\frac{(R-1)^2}{16T_\delta}} \]
for \((x, \tau) \in \partial B_R(0) \times (0, t_0)\). Recall \(\bar{w}_n(x, \tau) = w_n(x, t)\) \((\tau = t_0 - t)\). Then there exists a constant \(C > 0\) independent of \(n, \delta\) and \(R\) such that (5.5) holds.

Next, for \(r > 0\) small, we take \(\psi_r \in C_0^\infty(B_R(0))\) such that
\[
\left\{ \begin{array}{ll}
\psi_r = 1 & \text{for } |x| \leq R - 2r, \\
\psi_r = 0 & \text{for } |x| \geq R - r, \\
0 \leq \psi_r \leq 1, & |\nabla \psi_r| \leq Cr^{-1}, \quad |\Delta \psi_r| \leq Cr^{-2},
\end{array} \right.
\]
where \(C > 0\) is a constant independent of \(R\) and \(r\). Then \(\phi_{r,n} := w_n \psi_r\) satisfies
\[
(\phi_{r,n})_t + \Delta \phi_{r,n} + V\phi_{r,n} = w_n \Delta \psi_r + 2\nabla w_n \cdot \nabla \psi_r + w_n \psi_r (V - V_n),
\]
so that (5.2) with \(\phi = \phi_{r,n}\) implies
\[
\int_{\mathbb{R}^N} (u_1(x, t_0) - u_2(x, t_0)) \theta(x) dx = I_{r,n} + J_{r,n}
\]
\[
= \int_0^{t_0} \int_{R-2r \leq |x| \leq R-r} (u_1 - u_2) (w_n \Delta \psi_r + 2\nabla w_n \cdot \nabla \psi_r) \, dx \, dt
\]
\[
+ \int_0^{t_0} \int_{\mathbb{R}^N} (u_1 - u_2) w_n \psi_r (V - V_n) \, dx \, dt.
\]
We first estimate \(I_{r,n}\). Note that
\[
\sup_{t \in (0,t_0)} w_n \leq Cr \quad \sup_{t \in (0,t_0)} |\nabla w_n(x,t)|
\]
with some \(C > 0\) independent of \(r, n\) and \(R\). Since \(|u_1 - u_2| \leq Ce^{2C_0^2} e^{2C|x|^2}\) for \(x \in \mathbb{R}^N \setminus B_1(\xi(t))\) and \(t \in [0, T]\), we have
\[
\lim_{r \to 0} \sup_{t \in (0,t_0)} |I_{r,n}| \leq Ce^{2C_0^2} \lim_{r \to 0} \int_0^{t_0} \int_{R-2r \leq |x| \leq R-r} \left( r^{-2} w_n + r^{-1} |\nabla w_n| \right) \, dx \, dt
\]
\[
\leq Ce^{2C_0^2} \lim_{r \to 0} \left( r^{-1} \sup_{t \in (0,t_0)} \int_0^{t_0} \int_{R-2r \leq |x| \leq R-r} |\nabla w_n(x,t)| \, dx \, dt \right).
\]
From (5.5), it follows that
\[
\lim_{r \to 0} \sup_{t \in (0,t_0)} |I_{r,n}| \leq Ce^{2C_0^2} \sup_{|x| = R, t \in (0,t_0)} |\nabla w_n(x,t)| \lim_{r \to 0} (r^{-1} Cr R^{-1}) \leq CR^{-1} e^{-\frac{(R-1)^2}{16T_\delta}} + 2CR^2 \leq \bar{C} R^{-1} e^{-\frac{(R-1)^2}{64T_\delta}} + \bar{C} R^2
\]
with some constants $C, \tilde{C} > 0$ independent of $n$ and $R$. Thus, by replacing $T_\delta > 0$ with a small constant depending on $\tilde{C}$, we have

$$\limsup_{r \to 0} |I_{r,n}| \leq CR^{N-1}e^{-\frac{R^2}{128T_\delta}}$$

with some constant $C > 0$ depending on $T_\delta$ but not on $n$ and $R$, and hence,

$$\lim_{R \to 0} \left( \limsup_{n \to \infty} \left( \limsup_{r \to 0} I_{r,n} \right) \right) = 0.$$

Finally, we estimate $J_{r,n}$. From (5.4) and the assumption on $|u_1 - u_2|$, it follows that

$$|J_{r,n}| \leq C \int_0^{t_0} \int_{B_R(0)} |x - \xi(t)|^{-\alpha_2(\lambda)+\varepsilon}|x - \xi(t)|^{-\alpha_1(\lambda)-\frac{\varepsilon}{2}} |V - V_n| dx dt,$$

where $C > 0$ is a constant independent of $r$ and $n$. Thus

$$\limsup_{r \to 0} |J_{r,n}| \leq C \int_0^{t_0} \int_{\mathbb{R}^N} |x - \xi(t)|^{-\alpha_2(\lambda)+\frac{\varepsilon}{2}} |V - V_n| \chi_{B_R(0)}(x) dx dt.$$

Here $\chi_{B_R(0)}$ is the characteristic function on $B_R(0)$. By $\alpha_1(\lambda) + \alpha_2(\lambda) = N - 2$, $0 \leq V_n \leq V$, (A2) and (A3), we have

$$|x - \xi(t)|^{-\alpha_1(\lambda)-\alpha_2(\lambda)+\frac{\varepsilon}{2}} |V - V_n| \chi_{B_R(0)}(x) \leq C |x - \xi(t)|^{-N+\frac{\varepsilon}{2}} \chi_{B_R(0)}(x)$$

for a constant $C > 0$ independent of $r$ and $n$. The right-hand side of this inequality is independent of $n$ and belongs to $L^1(\mathbb{R}^N \times (0, t_0))$. Hence by $V_n(x, t) \to V(x, t)$ as $n \to \infty$ and the Lebesgue-dominated convergence theorem, we obtain

$$\lim_{n \to \infty} \left( \limsup_{r \to 0} |J_{r,n}| \right) = 0.$$

The left-hand side of (5.9) is independent of $r, n$ and $R$, and so by taking $\limsup_{r \to 0}$, $\limsup_{n \to \infty}$ and $\lim_{R \to 0}$ in (5.9), we obtain

$$\int_{\mathbb{R}^N} (u_1(x, t_0) - u_2(x, t_0)) \theta(x) dx = 0.$$

Since $t_0 \in (0, T_\delta]$ and $\theta$ are arbitrarily chosen, $u_1 \equiv u_2$ on $Q(0, \infty), [0, T_\delta]$. We can repeat the same argument as above to obtain $u_1 \equiv u_2$ on $Q(0, \infty), [0, T]$. □

6. Nonexistence for large initial data

In this section, we show the nonexistence of a solution when the initial value is large. Our idea is to compare the solution of (1.1) with a self-similar solution of the radial heat equation (2.1). In the proof, Lemma 2.2(iv) plays an essential role.
Proof of Theorem 1.4.} We take constants $\delta$ and $0 < T_\delta \leq T$ such that
\[ \alpha_2(\lambda) + 2 < \alpha_2(c_1 \lambda) + 2 < k, \]
\[ c_1 := (1 + \delta)^{-\frac{N}{2}} \exp \left( -\frac{(1 + \delta) C^2 \xi^2}{4\delta T^2 y_{\delta}^{-1}} \right) < 1. \]

Without loss of generality, it suffices to show the nonexistence of solutions of (1.1) with $u_0$ satisfying that $u_0(x) \geq C|x - \xi(0)|^{-\alpha_2(c_1 \lambda) - 2}$ for $x \in A_\infty(\xi(0))$.

Contrary to the conclusion, suppose that there exists a solution of (1.1) for $t \in [0, T_\delta]$ with an initial data satisfying $u_0(x) \geq C|x - \xi(0)|^{-\alpha_2(c_1 \lambda) - 2}$ for $x \in A_\infty(\xi(0))$ with some constant $C > 0$. Then there exists $C_0 > 0$ such that, for each $0 < \tilde{k} < \alpha_2(c_1 \lambda) + 2$, (1.1) also has a solution $u_{\tilde{k}}$ if
\[ u_0(x) \geq C_0|x - \xi(0)|^{-\tilde{k}}, \quad x \in A_\infty(\xi(0)). \] (6.1)

Let $u_n$ be a unique solution of the integral equation (3.1) with $u_0$ satisfying (6.1). We set $\tilde{u}_n(x, t) := u_n(x + \xi(t), t)$. Then $\tilde{u}_n$ satisfies $\tilde{u}_n = \Phi[\tilde{u}_n]$, where $\Phi$ is defined by (3.2). By the same argument as in Sect. 4, we have
\[ \tilde{u}_n(x, t) \geq \tilde{v}_n(x, t) := \int_{\mathbb{R}^N} c_1 G \left( x - y, \frac{t}{1 + \delta} \right) u_0(y + \xi(0)) dy + \int_0^t \int_{\mathbb{R}^N} c_1 G \left( x - y, \frac{t - s}{1 + \delta} \right) \tilde{V}_n(y, s) \tilde{u}_n(y, s) dy ds \]
for $t \in (0, T_\delta)$. Then $\tilde{v}_n$ satisfies
\[
\begin{cases}
\tilde{v}_{n,t} \geq \frac{1}{1 + \delta} \Delta \tilde{v}_n + \min \left\{ \frac{c_1 \lambda}{(1 + \delta)|x|^2} - c_1 M, c_1 n \right\} \tilde{v}_n, & (x, t) \in A_\infty(0) \times (0, T_\delta), \\
\tilde{v}_n(x, 0) \geq c_1 C_0|x|^{-\tilde{k}}, & x \in A_\infty(0),
\end{cases}
\]
where $M > 0$ is a constant given by (4.1). Let $\tilde{v}_n^-$ be a unique solution of
\[
\begin{cases}
\tilde{v}_{n,t}^- = \frac{1}{1 + \delta} \Delta \tilde{v}_n^- + \min \left\{ \frac{c_1 \lambda}{(1 + \delta)|x|^2} - c_1 M, c_1 n \right\} \tilde{v}_n^-, & (x, t) \in A_\infty(0) \times (0, T_\delta), \\
\tilde{v}_n^-(x, 0) = c_1 C_0|x|^{-\tilde{k}}, & x \in A_\infty(0).
\end{cases}
\]

By the comparison principle for bounded functions, we have $\tilde{v}_n^- \leq \tilde{v}_n$. This together with $\tilde{v}_n \leq \tilde{u}_n$ gives $\tilde{v}_n^- \leq \tilde{u}_n$. Letting $n \to \infty$, we obtain $\tilde{v}^- \leq \tilde{u}$, where $\tilde{u}(x, t) := u_{\tilde{k}}(x + \xi(t), t)$ and $\tilde{v}^-$ is a minimal solution of
\[
\begin{cases}
\tilde{v}^- = \frac{1}{1 + \delta} \Delta \tilde{v}^- + \left\{ \frac{c_1 \lambda}{(1 + \delta)|x|^2} - c_1 M \right\} \tilde{v}^-, & (x, t) \in A_\infty(0) \times (0, T_\delta), \\
\tilde{v}^-(x, 0) = c_1 C_0|x|^{-\tilde{k}}, & x \in A_\infty(0).
\end{cases}
\]

Since $\tilde{v}_n^- (\cdot, 0)$ is radially symmetric, $\tilde{v}^-$ is also radially symmetric. Then we rewrite
\[ \tilde{v}^-(x, t) = c_1 C_0 e^{-c_1 M t} v \left( \frac{t}{1 + \delta} \right), \quad r = |x|, \]
where \( v \) is a minimal solution of
\[
\begin{align*}
v_t &= v_{rr} + \frac{N-1}{r} v_r + \frac{c_1 \lambda}{r^2} v, \quad r > 0, \ t > 0, \\
v(r, 0) &= r^{-\tilde{k}}, \quad r > 0.
\end{align*}
\]

By \( c_1 \lambda < \lambda_c, \tilde{k} < \alpha_2(c_1 \lambda) + 2 \), Lemma 2.3 and Theorem 1.3, the minimal solution \( v = v(r, t; \tilde{k}) \) can be written as
\[
v(r, t; \tilde{k}) = \frac{r^{-\alpha_1(c_1 \lambda)}}{\sigma(\tilde{k} - \alpha_1(c_1 \lambda))} t^{-\frac{\tilde{k}-\alpha_1(c_1 \lambda)}{2}} \varphi \left( \frac{r}{t^{1/2}} \right),
\]
where \( \sigma \) and \( \varphi \) are given in Lemma 2.2 with \( l = \tilde{k} - \alpha_1(c_1 \lambda) \) and \( d = N - 2\alpha_1(c_1 \lambda) \).

By Lemma 2.2(iv), \( v(r, t; \tilde{k}) \to \infty \) as \( \tilde{k} \uparrow \alpha_2(c_1 \lambda) + 2 \) uniformly on any compact subset of \( (0, \infty) \times (0, \infty) \). Hence
\[
u(x, t) = \tilde{u}(x - \xi(t), t) \geq v^{-}(x - \xi(t), t) = c_1 C_0 e^{-c_1 M^t} v \left( x - \xi(t), \frac{t}{1 + \delta}; \tilde{k} \right) \to \infty
\]
as \( \tilde{k} \uparrow \alpha_2(c_1 \lambda) + 2 \) uniformly on any compact subset of \( Q_{(0,\infty),(0,(1+\delta)^{-1}T_3)} \). This is a contradiction. \( \square \)

7. Nonexistence in the supercritical case

In this section, we consider the supercritical case and show the nonexistence of a solution. In order to prove Theorem 1.5, it suffices to study the case where there exist constants \( \lambda_0 \in (\lambda_c, \lambda_c + 1), r_0 \in (0, 1) \) and \( \bar{t} \in (0, 1) \) such that
\[
V(x, t) \geq \frac{\lambda_0}{|x - \xi(t)|^2}
\]
for \( x \in A_{r_0}(\xi(t)) \) and \( t \in [0, \bar{t}] \). Our idea for the proof is to use the lower estimate by Theorem 1.2 and derive a contradiction from the equivalent integral equation by assuming the existence of a solution of (1.1).

We prepare a lemma and then prove Theorem 1.5.

**Lemma 7.1.** Let \( \underline{t} < \bar{t} \). Then there exist positive constants \( C_1 = C_1(N, C_\xi, \gamma) \) and \( C_2 = C_2(N) \) independent of \( \underline{t} \) and \( \bar{t} \) such that
\[
\int_{\underline{t}}^{t} G(z - \xi(s), t - s) ds \geq \frac{\Gamma(N-2)}{4\pi N/2} |z - \xi(t)|^{2-N}
\]
\[-C_1 |z - \xi(t)|^{1+2\gamma-N} - C_2 (t - \underline{t})^{-N-2} \]
for \( z \in A_1(\xi(t)) \) and \( t \in (\underline{t}, \bar{t}) \).
Proof. We consider the following $I_1, I_2$ and $I_3$:

$$
\int_{L}^{t} G(z - \xi(s), t - s)ds = \int_{-\infty}^{t} G(z - \xi(t), t - s)ds - \int_{-\infty}^{t} G(z - \xi(t), t - s)ds
$$

$$
+ \int_{L}^{t} (G(z - \xi(s), t - s) - G(z - \xi(t), t - s))ds
$$

$$
=: I_1 - I_2 + I_3.
$$

One can easily show that

$$
I_1 = \frac{\Gamma(N - 2)}{4\pi N/2} |z - \xi(t)|^{2 - N}, \quad |I_2| \leq C(t - t)^{\frac{N - 2}{2}}.
$$

Let us estimate $I_3$. We have

$$
|I_3| \leq C \int_{L}^{t} \frac{|z - \xi(t)|^2 - |z - \xi(s)|^2}{(t - s)^{N/2 + 1}} e^{-\frac{|z - \xi(t)|^2}{4(t - s)}} \int_{0}^{1} e^{\frac{\theta}{4(t - s)}} |z - \xi(t)|^2 - |z - \xi(s)|^2 |d\theta| ds.
$$

From (A1), it follows that

$$
|z - \xi(t)|^2 - |z - \xi(s)|^2 \leq 2|z - \xi(t)||\xi(t) - \xi(s)| + |\xi(t) - \xi(s)|^2
$$

$$
\leq \left\{ \begin{array}{l}
\frac{1}{2}|z - \xi(t)|^2 + 3C^{2}(t - s)^{2\gamma}, \\
2C^{2}|z - \xi(t)|(t - s)^{\gamma} + C^{2}(t - s)^{2\gamma}.
\end{array} \right.
$$

Then by $t - s \leq \bar{t} < 1$ and $1/2 < \gamma \leq 1$, we have

$$
|I_3| \leq C \int_{L}^{t} \frac{|z - \xi(t)|(t - s)^{\gamma} + (t - s)^{2\gamma} e^{-\frac{|z - \xi(t)|^2}{8(t - s)}}}{(t - s)^{N/2 + 1}} ds
$$

$$
\leq C \int_{L}^{t} \left\{ |z - \xi(t)|(t - s)^{-\frac{N}{2} + \gamma} + (t - s)^{-\frac{N}{2} + 2\gamma} \right\} e^{-\frac{|z - \xi(t)|^2}{8(t - s)}} ds
$$

$$
\leq C \int_{-\infty}^{t} \left\{ |z - \xi(t)|(t - s)^{-\frac{N}{2} + \gamma} + (t - s)^{-\frac{N}{2} + 2\gamma} \right\} e^{-\frac{|z - \xi(t)|^2}{8(t - s)}} ds
$$

$$
\leq C(|z - \xi(t)|^{1 + 2\gamma - N} + |z - \xi(t)|^{2\gamma - N}) \leq C|z - \xi(t)|^{1 + 2\gamma - N}
$$

for $z \in A_{1}(\xi(t))$. Thus, the lemma follows. \hfill \Box

Proof of Theorem 1.5. Contrary to the conclusion, suppose that (1.1) has a nontrivial solution $u$ for $t \in (0, \bar{t})$. Fix $\alpha$ and $\varepsilon$ such that

$$
N - 2 - \frac{2}{N - 2} \lambda_0 < \alpha < \frac{N - 2}{2}, \quad 0 < 4\varepsilon < \frac{2\lambda_0}{(N - 2 - \alpha)(N - 2) - 1}.
$$

(7.1)

Let $t_1 \in (0, \bar{t})$. By Theorem 1.2 and the comparison principle, there exist constants $r_1 \in (0, r_0)$ and $c > 0$ such that

$$
u(x, t) \geq c|x - \xi(t)|^{-\alpha}, \quad x \in A_{r_1}(\xi(t)), \quad t \in [t_1, \bar{t}].$$
Then there exists a constant \( r_2 \in (0, r_1/2) \) such that

\[
\frac{4\pi^{N/2}(\lambda_c + 1)C_1(2r_2)^{2\gamma-1}}{\lambda_c \Gamma\left(\frac{N-2}{2}\right)} < \varepsilon, \quad 2N\omega_N(\lambda_c + 1)C_2 \frac{r_2^{N-3}}{N-2} < \varepsilon
\]  

(7.2)

and that

\[
c_0' < (1 + \varepsilon) c_0,
\]

(7.3)

where positive constants \( c_0 \) and \( c_0' \) are given by

\[
c_0 := \inf\{|x - \xi(t)|^\alpha u(x, t); \ |x - \xi(t)| < r_2, \ t < \bar{t}\},
\]

\[
c_0' := \inf\{|x - \xi(t)|^\alpha u(x, t); \ |x - \xi(t)| < r_2, \ t + r_2 < t < \bar{t}\}.
\]

Let \((x, t)\) satisfy \(|x - \xi(t)| < r_2\) and \(t + r_2 < t < \bar{t}\). Then,

\[
u(x, t) = \int_{\mathbb{R}^N} G(x - y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s)V(y, s)u(y, s)dyds
\]

\[
\geq \lambda_0 c_0 \int_{\mathbb{R}^N} \int_{|y - \xi(s)| < r_2} G(x - y, t - s)|y - \xi(s)|^{-\alpha - 2}dyds
\]

\[
= \lambda_0 c_0 \int_{|y| < r_2} |y|^{-\alpha - 2} \left( \int_{t_0}^t G(x - y - \xi(s), t - s)ds \right) dy.
\]

This together with Lemma 7.1 and \( t > t + r_2 \) shows that

\[
u(x, t) \geq \lambda_0 c_0 \int_{|y| < r_2} |y|^{-\alpha - 2}|x - y - \xi(t)|^{2-N}dy
\]

\[- \lambda_0 c_0 C_1 \int_{|y| < r_2} |y|^{-\alpha - 2}|x - y - \xi(t)|^{1+2\gamma-N}dy
\]

\[- \lambda_0 c_0 C_2 (t - t_0)^{-\frac{N-2}{2}} \int_{|y| < r_2} |y|^{-\alpha - 2}dy
\]

\[
\geq c_0 \left\{ \lambda_0 \Gamma\left(\frac{N-2}{2}\right) - \lambda_0 C_1(2r_2)^{2\gamma-1} \right\} \int_{|y - \xi(t)| < |x - \xi(t)|} |x - y|^{2-N}dy
\]

\[- \frac{N\omega_N \lambda_0 C_2}{N - 2 - \alpha} c_0 r_2^{-\alpha}.
\]

Set \( r = |x - \xi(t)| \). We claim that

\[
\int_{|y - \xi(t)| < r} \frac{|x - y|^{2-N}}{|y - \xi(t)|^{2+\alpha}}dy = \frac{8\pi^{N/2}}{(N - 2 - \alpha)(N - 2)\Gamma\left(\frac{N-2}{2}\right)} r^{-\alpha}.
\]

(7.5)

For \( y \in B_r(\xi(t)) \), we take \( \rho > 0 \) and \( \theta \in (-\pi/2, \pi/2) \) as

\[
\rho = |x - y|, \quad |y - \xi(t)|^2 = \rho^2 + r^2 - 2\rho r \cos \theta.
\]
and then
\[
\int_{B_r(\xi(t))} |x - y|^{2-N} |y - \xi(t)|^{2+\alpha} \, dy = 2 \int_0^{\pi/2} \int_0^{2r \cos \theta} \rho^{2-N} \left( \frac{\rho^2 + r^2 - 2 \rho r \cos \theta}{\rho^2 + r^2 - 2 \rho r \cos \theta} \right)^{\frac{N-1}{2}} \, d\rho d\theta \\
\cdot 2 \frac{\pi^{N-1}}{\Gamma \left( \frac{N-1}{2} \right)} \cdot \rho (\rho \sin \theta)^{N-2} \, d\rho d\theta
\]
\[
= \frac{4\pi^{N-1}}{\Gamma \left( \frac{N-1}{2} \right)} \int_0^{\pi/2} \sin^{N-2} \theta \int_0^{2r \cos \theta} \rho \left\{ (\rho - r \cos \theta)^2 + r^2 \sin^2 \theta \right\}^{\frac{N-1}{2}} \, d\rho d\theta.
\]

By the change of variables \( \rho = r \cos \theta + r \sin \theta \tan \phi \), the last integral is written as
\[
\int_0^{2r \cos \theta} \frac{\rho}{(\rho - r \cos \theta)^2 + r^2 \sin^2 \theta}^{\frac{N-1}{2}} \, d\rho = r^{-\alpha} \cos \theta \sin^{1+\alpha} \theta \int_{(\frac{\pi}{2}-\theta)}^{\frac{\pi}{2}} \cos^\alpha \phi d\phi.
\]
This together with Fubini’s theorem shows
\[
\int_{B_r(\xi(t))} |x - y|^{2-N} |y - \xi(t)|^{2+\alpha} \, dy = \frac{8\pi^{N-1}}{\Gamma \left( \frac{N-1}{2} \right)} r^{-\alpha} \int_0^{\pi/2} \cos \phi \int_0^{\frac{\pi}{2}-\phi} \sin^{N-3-\alpha} \theta \cos \theta d\theta d\phi
\]
\[
= \frac{8\pi^{N-1}}{\Gamma \left( \frac{N-1}{2} \right)} r^{-\alpha} \times \frac{\pi^{\frac{1}{2} \Gamma \left( \frac{N-1}{2} \right)} 2(N - 2 - \alpha) \Gamma \left( \frac{N}{2} \right)}{2(N - 2 - \alpha) \Gamma \left( \frac{N}{2} \right)},
\]
so that (7.5) follows.

By the estimate (7.4) together with (7.5), (7.1), \( \lambda_0 < \lambda_c + 1 \) and (7.2), we have
\[
|x - \xi(t)|^\alpha u(x, t) \geq c_0 \left\{ \frac{2\lambda_0}{(N - 2 - \alpha)(N - 2)} - \frac{8\pi^{N/2} \lambda_0 C_1 (2r_2)^{2\gamma - 1}}{(N - 2 - \alpha)(N - 2) \Gamma \left( \frac{N-1}{2} \right)} \right\}
\]
\[
- \frac{N \omega_N \lambda_0 C_2}{N - 2 - \alpha} \frac{r_2^{N-2}}{2} c_0
\]
\[
\geq c_0 \left\{ 1 + 4 \epsilon - \frac{4\pi^{N/2} (\lambda_c + 1) C_1 (2r_2)^{2\gamma - 1}}{\lambda_c \Gamma \left( \frac{N-1}{2} \right)} \right\} - \frac{2N \omega_N (\lambda_c + 1) C_2}{N - 2} \frac{r_2^{N-2}}{2} c_0
\]
\[
\geq c_0 (1 + 2 \epsilon).
\]
Then, taking the infimum over \( \{(x, t); |x - \xi(t)| < r_2, \ t + r_2 < t < \bar{t}\} \), we obtain
\[
c'_0 \geq (1 + 2 \epsilon) c_0.
\]

However, this contradicts (7.3).

\[\Box\]

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REFERENCES
[1] O. Arena, On a singular parabolic equation related to axially symmetric heat potentials. Ann. Mat. Pura Appl. (4) 105 (1975), 347–393.
[2] P. Baras and J. Goldstein, The heat equation with a singular potential. Trans. Amer. Math. Soc. 284 (1984), 121–139.

[3] L. R. Bragg, The radial heat equation with pole type data. Bull. Amer. Math. Soc. 73 (1967), 133–135.

[4] X. Cabré and Y. Martel, Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier. C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), 973–978.

[5] F. Chiarenza and R. Serapioni, A remark on a Harnack inequality for degenerate parabolic equations. Rend. Sem. Mat. Univ. Padova 73 (1985), 179–190.

[6] M. Fila, J. R. King, M. Winkler and E. Yanagida, Optimal lower bound of the grow-up rate for a supercritical parabolic equation. J. Differential Equations 228 (2006), 339–356.

[7] M. Fila, J. Takahashi and E. Yanagida, Solutions with moving singularities for equations of porous medium type. Nonlinear Anal. 179 (2019), 237–253.

[8] Y. Fujishima and K. Ishige, Blowing up solutions for nonlinear parabolic systems with unequal elliptic operators. J. Dynam. Differential Equations 32 (2020), 1219–1231.

[9] V. A. Galaktionov and I. V. Kamotski, On nonexistence of Baras-Goldstein type for higher-order parabolic equations with singular potentials. Trans. Amer. Math. Soc. 362 (2010), 4117–4136.

[10] J. P. García Azorero and I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems. J. Differential Equations 144 (1998), 441–476.

[11] D. T. Haimo, Functions with the Huygens property. Bull. Amer. Math. Soc. 71 (1965), 528–532.

[12] D. Hirata and M. Tsutsumi, On the well-posedness of a linear heat equation with a critical singular potential. Differential Integral Equations 14 (2001), 1–18.

[13] K. Ishige and A. Mukai, Large time behavior of solutions of the heat equation with inverse square potential. Discrete Contin. Dyn. Syst. 38 (2018), 4041–4069.

[14] T. Kan and J. Takahashi, On the profile of solutions with time-dependent singularities for the heat equation. Kodai Math. J. 37 (2014), 568–585.

[15] V. Liskevich, A. Shishkov and Z. Sobol, Singular solutions to the heat equations with nonlinear absorption and Hardy potentials. Commun. Contemp. Math. 14 (2012), 1250013, 28 pp.

[16] C. Marchi, The Cauchy problem for the heat equation with a singular potential. Differential Integral Equations 16 (2003), 1065–1081.

[17] J. Moser, A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations. Comm. Pure Appl. Math. 13 (1960), 457–468.

[18] J. Moser, On Harnack’s theorem for elliptic differential equations. Comm. Pure Appl. Math. 14 (1961), 577–591.

[19] I. Okada and E. Yanagida On the heat equation with a dynamic Hardy-type potential: probabilistic approach, preprint.

[20] S. Sato and E. Yanagida, Solutions with moving singularities for a semilinear parabolic equation. J. Differential Equations 246 (2009), 724–748.

[21] J. Takahashi and E. Yanagida, Time-dependent singularities in the heat equation, Commun. Pure Appl. Anal. 14 (2015), 969–979.

[22] J. Takahashi and E. Yanagida, Time-dependent singularities in a semilinear parabolic equation with absorption. Commun. Contemp. Math. 18 (2016), 1550077, 27 pp.

[23] J. Vancostenoble and E. Zuazua, Null controllability for the heat equation with singular inverse-square potentials. J. Funct. Anal. 254 (2008), 1864–1902.

[24] J. L. Vázquez and E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. J. Funct. Anal. 173 (2000), 103–153.
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