A constant equation of state for quintessence?

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Abstract

Quintessence is often invoked to explain the universe’s acceleration suggested by type Ia supernovae observations. The aim of this letter is to study the validity of using a constant equation of state for quintessence models. We shall show that this hypothesis strongly constrains the form of the scalar potential.

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1 Introduction

The Standard Cosmological Model (SCM) can only describe decelerated universe models and thus cannot reproduce the observations coming from the recent type Ia supernovae, CMB anisotropies, ... in favour of a presently accelerated universe (see e.g. [1]). But, as the SCM can give a satisfactory explanation to other observational properties of the present Universe (e.g. primordial nucleosynthesis, extragalactic sources’redshifts, cosmic microwave radiation), the tendency is to consider the SCM as incomplete rather than incorrect.

The SCM can be transformed in an accelerated model by adding a new ingredient which behaves as a fluid with a negative pressure. The oldest and most studied candidate for this missing component is the cosmological constant \( \Lambda \) which is equivalent to a perfect fluid with constant density and pressure related by the equation of state \( p = -\rho \) \cite{2}. However, this does not constitute the only possibility: among all the other candidates, this missing energy can be associated with a dynamical time-dependent and spatially (in)homogeneous scalar field \( \phi \) evolving slowly down its potential \( V(\phi) \). The resulting cosmological models are known as quintessence models \cite{3}. In these models, the scalar field can be interpreted as a fluid with a negative pressure given by \( p = w \rho \) \((-1 \leq w < 0)\). Quintessence allows a wide range of models including a constant or a time-varying \( w \). Nevertheless some of these models admit an attractive property, called tracker solution \cite{4}, that permits to solve the cosmic coincidence problem\cite{4}. This is why solution admitting tracker behavior leads to the most favoured models. As they are characterized by an approximatively constant \( w \), in what follows, we shall focus on equations of state with a constant \( w \).

There are several reasons that lead us to favour a scalar field candidate. First of all, while the cosmological constant does not yet possess a completely satisfactory physical interpretation, the scalar field appears naturally in the field equations of a large number of alternative theories to general relativity. Moreover, in some of these alternative theories (e.g. superstring theory),

\footnote{“Cosmic coincidence problem” refers to an initial conditions’problem. Indeed, since quintessence and matter energy densities evolve at different rates with the universe’s expansion, conditions in the early universe must be set very carefully in order for them to be comparable to the ones existing today. In a tracker model, a very wide range of initial conditions yields to a common evolution for today and so the problem can be avoided.}
the scalar terms play an important physical role and consequently cannot be neglected.

In this letter, we shall consider the hypothesis of a constant equation of state for quintessence and show that, in this case, the set of field equations and conservation laws does not allow to use any potential form.

2 The constraint on the potential form

The field equations of a FLRW spacetime filled with ordinary matter non-coupled with a homogeneous scalar field are

\[
\frac{\ddot{R}}{R^2} + \frac{k}{R^2} = \frac{1}{3} \kappa^2 \rho + \frac{1}{3} \kappa^2 \rho_M
\]  
\[\text{(1)}\]

\[
2 \frac{\dddot{R}}{R} + \frac{\ddot{R}}{R} + \frac{k}{R^2} = -\kappa^2 p_\phi
\]  
\[\text{(2)}\]

\[
\dddot{\phi} + 3 \phi \frac{\dot{R}}{R} = -V'
\]  
\[\text{(3)}\]

where we have defined

\[
\kappa^2 \rho_\phi \equiv \frac{1}{2} \phi^2 + V(\phi)
\]  
\[\text{(4)}\]

\[
\kappa^2 p_\phi \equiv \frac{1}{2} \phi^2 - V(\phi)
\]  
\[\text{(5)}\]

and where we have taken as equation of state for the ordinary fluid: \(p_M = 0\). In all our equations, the dot denotes the derivative with respect to the time coordinate and the prime, the derivative with respect to the scalar field.

The fundamental assumption at the basis of the quintessential hypothesis is to consider that the scalar field behaves like a fluid with as equation of state

\[
p_\phi = w \rho_\phi
\]  
\[\text{(6)}\]

where \(w\) is lying between \(-1\) and 0, the limit \(w = -1\) corresponding to the cosmological constant.

As there is no interaction between the matter field and the scalar field, we have to impose the conservation law on these two fields separately:

\[
\kappa^2 \rho_M = \kappa^2 \rho_{M,0} \left( \frac{R_0}{R} \right)^3 \quad \kappa^2 \rho_\phi = \kappa^2 \rho_{\phi,0} \left( \frac{R_0}{R} \right)^{3(1+w)}
\]  
\[\text{(7)}\]
where the subscript "0" means "the current value". The supernovae observations being given in terms of the density parameters, it is convenient to introduce the following dimensionless quantities:

$$\Omega_k \equiv \frac{-k}{R_0^2 H_0^2}, \quad \Omega_\phi \equiv \frac{\kappa^2 \rho_{\phi,0}}{3 H_0^2}, \quad \Omega_M \equiv \frac{\kappa^2 \rho_{M,0}}{3 H_0^2} \tag{8}$$

constrained by $\Omega_0 \equiv 1 - \Omega_k = \Omega_M + \Omega_\phi$. Introducing the definitions (6)-(8) in the field equation (1), we obtain the following differential equation for the scale factor $R(t)$:

$$\dot{R} = R_0^2 H_0^2 \left[ \Omega_\phi \left( \frac{R_0}{R} \right)^{3w+1} + \Omega_M \frac{R_0}{R} + \Omega_k \right] \tag{9}$$

We shall now transform this relation in a differential equation for the scalar potential $V(\phi)$. Using combinations of eqs. (4)-(7), we find the relation between $R(t)$ and $V(\phi)$:

$$\frac{R}{R_0} = \left( \frac{V}{V_0} \right)^{-\frac{1}{3(1+w)\phi}} \tag{10}$$

where $V_0$, defined by $V_0 \equiv 3(1-w)H_0^2 \Omega_\phi / 2$, represents the current value for the scalar potential $V(\phi)$. The derivative of (10) with respect to $\phi$ leads to an expression of $\dot{R}(t)$ as a function of $V(\phi)$ and $V'(\phi)$:

$$\frac{\dot{R}}{R_0 H_0} = \left( \frac{V}{V_0} \right)^{-\frac{3w+5}{3(1+w)\phi}} \frac{V'}{V'} \tag{11}$$

where $V_0'$, defined by $V_0' \equiv \pm 3H_0 \sqrt{(1-w^2)V_0/2}$, is the current value of $V'(\phi)$. Using (10) and (11), we can write relation (1) in terms of $V(\phi)$ and $V'(\phi)$ and find the following constraint on the scalar potential:

$$\frac{V'}{V_0'} = \left[ \Omega_\phi \left( \frac{V}{V_0} \right)^2 + \Omega_M \left( \frac{V}{V_0} \right)^{\frac{w+2}{w}} + \Omega_k \left( \frac{V}{V_0} \right)^{\frac{3w+5}{3(1+w)\phi}} \right] \tag{12}$$

This relation has been found assuming that $w \neq -1$. For $w = -1$, we are in the case of the cosmological constant which implies that the scalar field and its potential are constant, so that eq. (12) loses its meaning. As we can see from (12), any form of potential $V(\phi)$ with any value of $w$ is not consistent with the field equations and the conservation laws.

We have been able to solve the constraint (12) only in some peculiar cases:
1. For \( k = 0 \) (\( \forall w \)) (flat FLRW model):

\[
V(\phi) = V_0 \left[ \sqrt{\frac{\Omega_M}{\Omega_\phi}} \sinh [\pm \beta_0 (\phi - \phi_0) + \alpha_0] \right]^{2(w+1)/w}
\]

(13)

with \( \alpha_0 \equiv \text{arcsinh} \left[ \sqrt{\Omega_\phi / \Omega_M} \right] \) and \( \beta_0 \equiv \frac{w}{2} \sqrt{3/(w + 1)} \). Quintessence model with this potential has already been considered by Chimento and Jakubi in [5] and more recently by Ureno-Lopez and Matos in [6].

2. For \( w = -2/3 \):

\[
V(\phi) = V_0 \left[ \sqrt{\frac{\Omega_M}{\Omega_\phi}} \sinh [\pm (\phi - \phi_0) + \delta_0] + \frac{1}{4} \frac{\Omega_k}{\Omega_\phi} \left( \frac{\Omega_k}{\Omega_M} e^{\pm (\phi - \phi_0) - \sigma_0 - 2} \right) \right]^{-1}
\]

(14)

with \( e^{\sigma_0} = \left( 2 \Omega_\phi + 2 \sqrt{\Omega_\phi + \Omega_k} \right) / \Omega_M \) and \( \delta_0 = \sigma_0 + \ln \sqrt{\Omega_\phi / (4 \Omega_\phi)} \).

3. For \( w = -1/3 \):

- For \( \Omega_\phi + \Omega_k = 0 \):

\[
V(\phi) = V_0 \left( \frac{\phi_0}{\phi} \right)^4
\]

(15)

- For \( \Omega_\phi + \Omega_k \neq 0 \), it is

\[
V(\phi) = V_0 \left[ \sqrt{\frac{\Omega_M}{\Omega_\phi + \Omega_k}} \sinh [\pm \epsilon_0 (\phi - \phi_0) + \nu_0] \right]^{-4}
\]

(16)

where \( \epsilon_0 \equiv \sqrt{(\Omega_\phi + \Omega_k) / (8 \Omega_M)} \) and \( \nu_0 \equiv \text{arcsinh} \left[ 2 \sqrt{2} \epsilon_0 \right] \).

Note that this potential that may mimic a negative spatial curvature of the Universe was introduced e.g. by Starobinsky in [7].

In these three cases, \( \phi_0 \) is the current value of the scalar field \( \phi(t) \) so that we can always write \( V(\phi_0) = V_0 \).
3 The potentials used in the literature

We shall now consider the three most used potential forms in the quintessential literature, namely the inverse power-law form $V(\phi) = V_0 \left( \frac{\phi}{\phi_0} \right)^{a_0}$ (with $a_0 > 1$) suggested by SUSY models [8], the exponential form $V(\phi) = V_0 e^{\phi - \phi_0}$ invoqued in the context of Kaluza-Klein or superstring theories [9] and the cosine form $V(\phi) = \frac{1}{2} V_0 \left[ \cos([\phi - \phi_0]/f) + 1 \right]$ motivated by the physics of pseudo-Nambu-Goldstone bosons [10].

The introduction of the inverse power-law potential in the constraint (12) gives

$$a_0^2 V_0^2 \frac{(\phi_0/\phi)}{\phi_0^2 V_0^2} = \frac{2(a_0 + 1)}{a_0} \left( \frac{\phi_0}{\phi} \right)^{2a_0} + \Omega_M \left( \frac{\phi_0}{\phi} \right)^{w+2/a_0} + \Omega_k \left( \frac{\phi_0}{\phi} \right)^{3w+5/(a_0(w+1))} a_0$$

The only way to satisfy this relation is to identify the terms that can have the same exponent of $(\phi_0/\phi)$. In this case, we have two possibilities for those identifications:

$$2(a_0 + 1) = a_0 (3w+5)/3(w+1)$$
$$a_0^2 V_0^2 / \phi_0^2 V_0^2 = \Omega_k$$
$$2a_0 = a_0 (w+2)/(w+1)$$
$$\Omega_\phi + \Omega_M = 0$$

or

$$2(a_0 + 1) = a_0 (w+2)/(w+1)$$
$$a_0^2 V_0^2 / \phi_0^2 V_0^2 = \Omega_M$$
$$2a_0 = a_0 (3w+5)/3(w+1)$$
$$\Omega_\phi + \Omega_M = 0$$

It is easy to see from (20) that no value of $w$ allows one to satisfy the first set of relations. The second set can be satisfied only for one value of $w$: $w = -1/3$. We also find the following constraints on the other constants: $a_0 = 4$, $\Omega_M = 1$ and $\Omega_\phi = -\Omega_k = \phi_0^2 / 8$. The negative value of $\Omega_k$ means that the corresponding universe model is closed ($k = 1$). Note also that the value $a_0 = 4$ has already often been considered in the context of quintessence...
However, as it is well known, the value $\Omega_M = 1$ found in this model is completely incompatible with those deduced from primordial nucleosynthesis and supernovae observations ($\Omega_M \approx 0.25$) which leads us to admit that an inverse power-law potential as we have used is not coherent with constraint (12).

With the exponential potential, constraint (12) becomes

$$\left( \frac{V_0}{V_0} \right)^2 \left( \frac{V}{V_0} \right)^2 = \Omega_\phi \left( \frac{V}{V_0} \right)^2 + \Omega_M \left( \frac{V}{V_0} \right)^{w+2} + \Omega_k \left( \frac{V}{V_0} \right)^{3w+5}$$

while with the cosine form, it can be written as

$$\left( \frac{V_0}{V_0} \right)^2 \left[ \frac{V}{V_0} - \left( \frac{V}{V_0} \right)^2 \right] = \Omega_\phi \left( \frac{V}{V_0} \right)^2 + \Omega_M \left( \frac{V}{V_0} \right)^{w+2} + \Omega_k \left( \frac{V}{V_0} \right)^{3w+5}$$

By trying to make similar identifications, one can easily see that there is no value of $w$ able to satisfy the constraint in those two cases. So we are led again to the same conclusion as in the first subcase of the inverse power-law potential: all these scalar potential forms are not consistent with the set of field and conservation equations issued from quintessential hypothesis.

### 4 Conclusion

In this letter, we have attempted to establish that the field equations and the conservation laws strongly constrain the scalar potential form invoked in the framework of the quintessence. As an application, we have studied the three most used quintessential potentials. In all cases, it appeared that they were incompatible with the constraint we found on the potential. This can cast some doubt on the way the quintessential hypothesis has been presently implemented in the theoretical framework. However, as it has already been mentioned in the introduction, our results have been obtained making the hypothesis of a constant scalar field equation of state which is a good approximation only for a tracker solution. So the case of a variable $w$ deserves further theoretical investigations [11]. As suggested by Huterer and Turner [12], it should be soon possible to discriminate between a constant and a time-varying $w$ using future SNeIa observations and also high precision measurements of the multipole power spectrum expected from the MAP and
Planck Surveyor satellites.

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