Research Article

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Generalized derivatives and optimization problems for \( n \)-dimensional fuzzy-number-valued functions

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Abstract: In this paper, we present the concepts of generalized derivative, directional generalized derivative, subdifferential and conjugate for \( n \)-dimensional fuzzy-number-valued functions and discuss the characterizations of generalized derivative and directional generalized derivative by, respectively, using the derivative and directional derivative of crisp functions that are determined by the fuzzy mapping. Furthermore, the relations among generalized derivative, directional generalized derivative, subdifferential and convexity for \( n \)-dimensional fuzzy-number-valued functions are investigated. Finally, under two kinds of partial orderings defined on the set of all \( n \)-dimensional fuzzy numbers, the duality theorems and saddle point optimality criteria in fuzzy optimization problems with constraints are discussed.

Keywords: fuzzy \( n \)-cell number, \( n \)-dimensional fuzzy-number-valued function, directional generalized derivative, subdifferential, fuzzy optimization

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1 Introduction

In 1972, Chang and Zadeh [1] introduced the concept of fuzzy numbers with the consideration of the properties of probability functions. Since then the fuzzy numbers have been extensively studied by many authors. Fuzzy numbers are a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models. As part of the development of theories about fuzzy numbers and its applications, researchers began to study the differentiability and integrability of fuzzy mappings. Initially, Puri and Ralescu [2] defined the \( g \)-derivative of fuzzy mappings from an open subset of a normed space into the \( n \)-dimension fuzzy number space \( \mathbb{E}^n \) using the Hukuhara difference. In 1987, Kaleve [3] investigated fuzzy differential equations based on the \( g \)-derivative. In 2010, Farajzadeh et al. [4] proposed a numerical method for the solution of the fuzzy heat equation. Furthermore, Wang and Wu [5] defined the directional derivative of fuzzy mappings from \( \mathbb{R}^n \) into \( \mathbb{E}^1 \). However, the Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions and the \( H \)-difference of two fuzzy numbers does not always exist. The \( g \)-difference proposed in [6,7] overcomes these shortcomings of the above discussed concepts, and the \( g \)-difference of two fuzzy numbers always exists. Based on the generalizations of the Hukuhara difference for fuzzy sets, Bede and Stefanini [8] introduced and studied a new...
generalized differentiability concept for fuzzy-valued functions from $R$ into $E^1$ in 2013. Using the fuzzy $g$-difference introduced in Stefanini [7], in 2016, Hai et al. [9] defined and studied generalized differentiability for $n$-dimensional fuzzy-number-valued functions on $[a, b]$. In this paper, we generalize the concepts of generalized derivative and support-function-wise derivative for $n$-dimensional fuzzy-number-valued functions from $[a, b] \subseteq R$ to $M \subseteq R^n$. Furthermore, the directional generalized derivative, subdifferential and conjugate for fuzzy-number-valued functions are investigated, and we give characteristic theorems for the generalized derivative and directional generalized derivative for fuzzy-number-valued functions.

Recently, convexity has been increasingly important in the study of extremum problems in many areas of applied mathematics. In fact, convex analysis [10] is an important branch of mathematics, and it also has wide application in convex optimization. If the values of the objective function that is sought optimum solution are crisp real numbers, the optimization is a general crisp optimization [11]. But in reality, sometimes, the values of the objective function only are estimated values, so it is more suitable that the values are expressed with fuzzy numbers, and the optimization is a fuzzy optimization. In 1992, Nanda and Kar [12] introduced and discussed the concepts of convex fuzzy mappings from a vector space over the field $R$ into $E^3$, established criteria for convex fuzzy mappings. In 2005, Zhang et al. [13] discussed the convex fuzzy mappings and discussed the duality theory in fuzzy mathematical programming problems with fuzzy coefficients based on the ordering of two fuzzy numbers proposed in [12]. Under a general setting of partial ordering defined on the set of all fuzzy numbers, Wu [14] investigated the duality theorems and saddle point optimality conditions in fuzzy nonlinear programming problems based on two solution concepts for primal problem and three solution concepts for dual problem in 2007. A well-known fact in mathematical programming is that variational inequality problems have a close relation with the optimization problems. Similarly, the fuzzy variational inequality (inclusions) problems also have a close relation with fuzzy optimization problems. In 2009, Ahmad and Farajzadeh [15] investigated random variational inclusions with random fuzzy mappings and defined an iterative algorithm to compute the approximate solutions of random variational inclusion problem. However, very few studies have investigated the convexity and duality in fuzzy optimization of $n$-dimensional fuzzy-number-valued functions. The main reason is that there is almost no related research about the ordering and the difference of $n$-dimensional fuzzy numbers. In 2016, Gong and Hai introduced the concept of a convex fuzzy-number-valued function based on a new ordering $\leq_c$ of $n$-dimensional fuzzy numbers [16] and investigated differentiability for $n$-dimensional fuzzy-number-valued functions on $[a, b]$ and Karush-Kuhn-Tucker (KKT) conditions in fuzzy optimization problems based on the ordering $\leq_s$ in [9]. In 2019, Xie and Gong [17] investigated variational-like inequalities for $n$-dimensional fuzzy-vector-valued functions and obtained optimality conditions for fuzzy multiobjective optimization problems. In this paper, under the two kinds of partial orderings defined on the set of all $n$-dimensional fuzzy numbers, the duality theorems and saddle point optimality criteria in fuzzy optimization problems are discussed.

To make our analysis possible, we present the preliminary terminology used throughout this paper in Section 2. In Section 3, we present the concept of generalized derivative, directional generalized derivative, subdifferential and conjugate for fuzzy-number-valued functions and obtain characteristic theorems of the generalized derivative and directional generalized derivative for fuzzy-number-valued functions. Furthermore, the relations among generalized derivative, directional generalized derivative, subdifferential and convexity for $n$-dimensional fuzzy-number-valued functions are discussed. In Sections 4 and 5, the Lagrange duality theorem and the optimality conditions, including the KKT conditions and the saddle point optimality criteria, in fuzzy optimization problems with constraints for $n$-dimensional fuzzy-number-valued functions are derived, respectively, under the partial orderings $\leq_c$ and $\leq_s$ defined on the set of all $n$-dimensional fuzzy numbers. Section 6 concludes this paper.

2 Preliminaries

Throughout this paper, $R^n$ denotes the $n$-dimensional Euclidean space, $K^n$ and $K^n_\mathcal{F}$ denote the spaces of nonempty compact and compact convex sets of $R^n$, respectively. Let $\mathcal{F}(R^n)$ be the set of all fuzzy subsets on
A fuzzy set $u$ on $R^n$ is a mapping $u : R^n \to [0, 1]$, and $u(x)$ is the degree of membership of the element $x$ in the fuzzy set $u$. For each fuzzy set $u$, we denote its $r$-level set as $[u]^r = \{x \in R^n : u(x) \geq r\}$ for any $r \in (0, 1]$, and in some references also denoted by $u_r$, for short. The support of $u$ we denote by $\text{supp } u$ where $\text{supp } u = \{x \in R^n : u(x) > 0\}$. The closure of $\text{supp } u$ defines the 0-level of $u$, i.e., $[u]^0 = \text{cl}(\text{supp } u)$. Here $\text{cl}(M)$ denotes the closure of set $M$. Fuzzy set $u \in \mathcal{F}(R^n)$ is called a fuzzy number if

(i) $u$ is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(ii) $u$ is a convex fuzzy set, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0, 1]$,
(iii) $u$ is upper semicontinuous and
(iv) $[u]^0 = \text{cl}(\text{supp } u) = \text{cl}(\bigcup_{r \in [0,1]} [u]^r)$ is compact.

We use $E^n$ to denote the fuzzy number space $[18,19]$. When $n = 1$, $u$ is called a one-dimensional fuzzy number, and the fuzzy number space denoted by $E^1$ or $E$.

It is clear that each $u \in R^n$ can be considered as a fuzzy number $u$ defined by

$$u(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise}. \end{cases}$$

In particular, the fuzzy number $\bar{0}$ is defined as $\bar{0}(x) = 1$ if $x = 0$, and $\bar{0}(x) = 0$ otherwise.

**Example 2.1.** [17] Let $u \in E^2$ be defined by

$$u(x, y) = \begin{cases} \sqrt{1 - x^2 - y^2}, & x^2 + y^2 \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

then $[u]^r = \{(x, y) : x^2 + y^2 \leq 1 - r^2\}, r \in [0, 1]$.

A special kind of $n$-dimension fuzzy number is the fuzzy $n$-cell number proposed in [20]. If $u \in E^n$, and $[u]^r$ is a cell, i.e., $[u]^r$ can be represented by $[u]^r = \prod_{i=1}^{n} [u_i^r(r), u_i^r(r)] = [u_1^r(r), u_1^r(r)] \times [u_2^r(r), u_2^r(r)] \times \cdots \times [u_n^r(r), u_n^r(r)]$ for any $r \in [0, 1]$, where $u_i^r(r), u_i^r(r) \in R$ with $u_i^r(r) \leq u_i^r(r)$, $i = 1, 2, \ldots, n$, then we call $u$ a fuzzy $n$-cell number. And we denote the collection of all fuzzy $n$-cell numbers by $L(E^n)$.

**Example 2.2.** $F : [1, \infty) \to L(E^2)$ is a fuzzy 2-cell number function, which is defined by

$$F(s)(x_1, x_2) = \begin{cases} \sqrt{e^{2s} - x_1^2}, & 0 \leq x_1 \leq e^s, \ 0 \leq x_2 \leq 3, \\ 0, & \text{otherwise}, \end{cases}$$

where the parameter $s \in R$. Then for all $r \in [0, 1],$

$$F_r(s) = \{(x_1, x_2) : 0 \leq x_1 \leq \sqrt{1 - r^2} \cdot e^s, \ 0 \leq x_2 \leq 3\}.$$ 

**Theorem 2.3.** [3] If $u \in E^n$, then

(i) $[u]^r$ is a nonempty compact convex subset of $R^n$ for any $r \in (0, 1]$,
(ii) $[u]^0 \subseteq [u]^r$, whenever $0 \leq r_2 \leq r_1 \leq 1$,
(iii) if $r_n > 0$ and $r_n$ converging to $r \in [0, 1]$ is nondecreasing, then $\bigcap_{n=1}^{\infty} [u]^r_n = [u]^r$.

Conversely, suppose for any $r \in [0, 1]$, there exists an $A' \subseteq R^n$ which satisfies (i)-(iii), then there exists a unique $u \in E^n$ such that $[u]^r = A'$, $r \in (0, 1]$, $[u]^0 = \bigcup_{r \in [0,1]} [u]^r \subseteq A^0$.

Note that when $u \in E^1$, $[u]^r$ is a nonempty closed interval on $[0, 1]$ for any $r \in [0, 1]$; when $u \in L(E^n)$, $[u]^r$ is a nonempty $n$-dimensional closed polyhedron for any $r \in [0, 1]$. 

Theorem 2.4. [20] If \( u \in L(E^n) \), then for \( i = 1, 2, \ldots, n \), \( u_i^r(r) \) are real-valued functions on \([0, 1]\) and satisfy

(i) \( u_i^r(r) \) are non-decreasing and left continuous,
(ii) \( u_i^r(r) \) are non-increasing and left continuous,
(iii) \( u_i^r(r) \leq u_i^r(1) \) (it is equivalent to \( u_i^r(1) \leq u_i^r(1) \))
(iv) \( u_i^r(r), u_i^r(0) \) are right continuous at \( r = 0 \).

Conversely, if \( a(r), b(r), i = 1, 2, \ldots, n \), are real-valued functions on \([0, 1]\) which satisfy conditions (i)–(iv), then there exists a unique \( u \in L(E^n) \) such that \( \{u^r\} = \prod_{i=1}^n [a_i(r), b_i(r)] \) for any \( r \in [0, 1] \).

Let \( u, v \in E^n, k \in R \). For any \( x \in R^n \), the addition and scalar multiplication can be defined, respectively, as

\[
(u + v)(x) = \sup_{s+t=x} \min\{u(s), v(t)\},
\]

\[
(ku)(x) = u\left(\frac{x}{k}\right), \quad k \neq 0,
\]

\[
(0u)(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}
\]

It is well known that for any \( u, v \in E^n \) and \( k \in R \),

\[
[u + v]^r = [u]^r + [v]^r = \{x + y : x \in [u]^r, y \in [v]^r\},
\]

\[
[ku]^r = k[u]^r = \{kx : x \in [u]^r\}.
\]

If \( u, v \in L(E^n) \) and \( k \in R \), then for all \( r \in [0, 1], [20] [u + v]^r = [u]^r + [v]^r = \prod_{i=1}^n [u_i^r(r) + v_i^r(r)],
\]

\[
[ku]^r = k[u]^r = \prod_{i=1}^n [ku_i^r(r), ku_i^r(r)], \quad k \geq 0,
\]

\[
[ku]^r = k[u]^r = \prod_{i=1}^n [ku_i^r(r), ku_i^r(r)], \quad k < 0.
\]

Proposition 2.5. [19] If \( u, v \in E^n, k, k_1, k_2 \in R \), then

(i) \( k(u + v) = ku + kv \),
(ii) \( k_1(k_2u) = (k_1k_2)u \),
(iii) \( (k_1 + k_2)u = k_1u + k_2u \) when \( k_1 \geq 0 \) and \( k_2 \geq 0 \).

Give two subsets \( A, B \subseteq R^n \) and \( k \in R \), the Minkowski difference is given by \( A - B = A + (-1)B = \{a - b : a \in A, b \in B\} \). However, in general, \( A + (-A) \neq 0 \), i.e., the opposite of \( A \) is not the inverse of \( A \) in Minkowski addition (unless \( A = \{a\} \) is a singleton). The spaces \( K^n \) and \( K^n_0 \) are not linear spaces since they do not contain inverse elements and therefore subtraction is not defined. To partially overcome this situation, Hukuhara [21] introduced the following \( H \)-difference \( A \oplus B = C \Leftrightarrow A = B + C \) and an important property of \( \oplus \) is that \( A \oplus A = \{0\}, \forall A \in R^n \) and \( (A + B) \oplus B = A, \forall A, B \in R^n \). The \( H \)-difference is unique, but a necessary condition for \( A \oplus_B B \) to exist is that \( A \) contains a translation \( |C| + B \) of \( B \). In order to overcome this situation, Stefanini [22] defined the generalized Hukuhara difference of two sets \( A, B \in K^n \) as follows:

\[
A \oplus_{gh} B = C \Leftrightarrow \begin{cases} (1) & A = B + C, \\ (2) & B = A + (-1)C. \end{cases}
\]
For any \( A \subseteq \mathbb{R}^n \), the support function associated with \( A \) is \( s_A : \mathbb{R}^n \to \mathbb{R} \) defined by \( s_A(x) = \sup_{a \in A} \langle a, x \rangle \), \( x \in \mathbb{R}^n \). Let \( S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \) be the unit sphere of \( \mathbb{R}^n \) and \( \langle \cdot, \cdot \rangle \) be the inner product in \( \mathbb{R}^n \), i.e., 
\[
\langle x, y \rangle = \sum_{i=1}^n x_i y_i, \text{ where } x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n, y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n.
\]
If \( A \subseteq \mathbb{C}^n \), then \( s_A(x) = \sup_{a \in A} \langle a, x \rangle \) for any \( x \in S^{n-1} \). Let \( s_A(x), s_n(x) \), \( s_n(x) \), \( s_{-1:A}(x) \) and \( s_{-1:B}(x) \) be the support functions of \( A, B, C, (\sim)A \) and \( (\sim)B \), respectively. Then, for any \( x \in S^{n-1} \), we have [7]
\[
s_C(x) = \begin{cases} 
(1) & s_C(x) = s_n(x), \\
(2) & s_{-1:B}(x) = s_{-1:A}(x).
\end{cases}
\]

The generalized Hukuhara difference has been extended to the fuzzy case in [7]. For any \( u, v \in E^n \), the generalized Hukuhara difference (\( gH \)-difference for short) is the fuzzy number \( w \), if it exists, such that
\[
u \circ_{\text{gH}} v = w \Rightarrow \begin{cases} 
(1) & u = v + w, \\
(2) & v = u + (-1)w.
\end{cases}
\]

It is possible that the \( gH \)-difference of two fuzzy numbers does not exist. To solve this shortcoming, in [8] a new difference between fuzzy numbers was proposed. Using the convex hull (\( \text{conv} \)) the new difference was defined as follows.

**Definition 2.6.** [6,8] The generalized difference (\( g \)-difference for short) of two fuzzy numbers \( u, v \in E^n \) is given by its level set as
\[
[u \circ_g v]^r = \left( \text{conv} \bigcup_{\beta \in [r]} (|u|^\beta \circ_{\text{gH}} |v|^\beta) \right), \quad \forall r \in [0, 1],
\]
where the \( gH \)-difference \( \circ_{\text{gH}} \) is with interval operands \( |u|^\beta \) and \( |v|^\beta \).

**Definition 2.7.** [18] For \( u \in E^n, r \in [0, 1] \) and \( x \in S^{n-1} \), the support function of \( u \) is defined by
\[
u^r(x, r) = \sup_{a \in [u]^r} \langle a, x \rangle.
\]

Obviously, if \( u, v \in L(E^n) \), for any \( x \in S^{n-1} \) and \( r \in [0, 1] \), then
\[
u^r(x, r) = \sup_{a \in [u]^r} \langle a, x \rangle = \sum_{x_i \geq 0} x_i u_i^r(r) + \sum_{x_i < 0} x_i u_i^r(r).
\]

**Theorem 2.8.** [23] Suppose \( u \in E^n, r \in [0, 1] \), then
\[
[u]^r = \{ y \in \mathbb{R}^n : \langle y, x \rangle \leq u^r(x, r), x \in S^{n-1} \}.
\]

**Theorem 2.9.** [9] Let \( u, v \in E^n \). If the \( g \)-difference \( u \circ_g v \) of \( u \) and \( v \) exists, then for any \( r \in [0, 1] \) and \( x \in S^{n-1} \), we have
\[
(u \circ_g v)^r(x, r) = \begin{cases} 
(1) & \sup_{\beta \in [r]} (u^r(\beta, x) - v^r(\beta, x)), \\
(2) & \sup_{\beta \in [r]} (-v^r(\beta, x) - (-u)^r(\beta, x)).
\end{cases}
\]

**Remark 2.10.** Let \( u, v \in L(E^n) \), we have
\[
[u \circ_g v]^r = \prod_{i=1}^n \left[ \inf_{\beta \in [r]} (u_i^r(\beta) - v_i^r(\beta), u_i^r(\beta) - v_i^r(\beta), \sup_{\beta \in [r]} (u_i^r(\beta) - v_i^r(\beta), u_i^r(\beta) - v_i^r(\beta)) \right]
\]
\[
= \prod_{i=1}^n [u_i \circ_g v_i]^r.
\]
Proposition 2.11. [9] Let \( u, v \in E^n \). Then
(i) if the g-difference exists, it is unique,
(ii) \( u \oplus_g u = 0 \),
(iii) \( (u + v) \ominus_g v = u, (u + v) \ominus_g u = v \) and
(iv) \( u \ominus_g v = -(v \ominus_g u) \).

Given \( u, v \in E^n \), the distance \( D : E^n \times E^n \to [0, +\infty) \) between \( u \) and \( v \) is defined by the equation:

\[
D(u, v) = \sup_{r \in [0,1]} d([u]^r, [v]^r),
\]

where \( d \) is the Hausdorff metric given by

\[
d([u]^r, [v]^r) = \inf \{ \varepsilon : [u]^r \subset N([v]^r, \varepsilon), [v]^r \subset N([u]^r, \varepsilon) \} = \max \left\{ \sup_{a \in [u]^r} \inf_{b \in [v]^r} \|a - b\|, \sup_{b \in [v]^r} \inf_{a \in [u]^r} \|a - b\| \right\}.
\]

\( N([u]^r, \varepsilon) = \{ x \in R^n : d(x, [u]^r) = \inf_{x,y \in [u]^r} d(x, y) \leq \varepsilon \} \) is the \( \varepsilon \)-neighborhood of \( [u]^r \). Then, \((E^n, D)\) is a complete metric space and satisfies \( D(u + w, v + w) = D(u, v), D(\kappa u, \kappa v) = |\kappa|D(u, v) \) for any \( u, v, w \in E^n \) and \( k \in R \).

Remark 2.12. If \( u, v \in L(E^n) \), i.e., \( [u]^r = \bigcap_{i=1}^n [u_i(r), u_i^+(r)], [v]^r = \bigcap_{i=1}^n [v_i(r), v_i^+(r)] \), \( r \in [0,1] \),

\[
D(u, v) = \sup_{r \in [0,1]} d([u]^r, [v]^r) = \sup_{r \in [0,1]} \max \{ \|u_i^-(r) - v_i^+(r)\|, \|u_i^+(r) - v_i^-(r)\| \}.
\]

(2.16)

For \( u \in E^n \), we denote the centroid of \( [u]^r, r \in [0,1] \) as

\[
\tau(u) = \left( \frac{1}{r} \int_{\{r\}} \left[ \int_{\{r\}} x_1 dx_1 dx_2 \cdots dx_n \right. \right. \cdots \left. \left. \int_{\{r\}} x_n dx_1 dx_2 \cdots dx_n \right] \right)^T,
\]

where \( \int_{\{r\}} \cdots \int_{\{r\}} 1 dx_1 dx_2 \cdots dx_n \) is the solidity of \( [u]^r \), \( r \in [0,1] \) and \( \int_{\{r\}} \cdots \int_{\{r\}} x_i dx_1 dx_2 \cdots dx_n (i = 1, 2, \ldots, n) \) is the multiple integral of \( x_i \) on measurable sets \( [u]^r, r \in [0,1] \). Let \( \tau : E^n \to R^n \) be a real vector-valued function defined by

\[
\tau(u) = \left( \frac{1}{r} \int_{\{r\}} \left[ \int_{\{r\}} x_1 dx_1 dx_2 \cdots dx_n \right. \right. \cdots \left. \left. \int_{\{r\}} x_n dx_1 dx_2 \cdots dx_n \right] dr, \frac{1}{r} \int_{\{r\}} \left[ \int_{\{r\}} 1 dx_1 dx_2 \cdots dx_n \right. \right. \cdots \left. \left. \int_{\{r\}} 1 dx_1 dx_2 \cdots dx_n \right] dr, \ldots, \right)^T,
\]

(2.17)

where \( \int_{\{r\}} \cdots \int_{\{r\}} x_i dx_1 dx_2 \cdots dx_n dr (i = 1, 2, \ldots, n) \) is the Lebesgue integral of \( r \int_{\{r\}} \cdots \int_{\{r\}} x_i dx_1 dx_2 \cdots dx_n dr (i = 1, 2, \ldots, n) \) on \([0,1]\). The vector-valued function \( \tau \) is called a ranking value function defined on \( E^n \).

Definition 2.13. [16] Let \( u, v \in E^n \), \( C \subseteq R^n \) be a closed convex cone with \( 0 \in C \) and \( C \neq R^n \). We say that \( u \leq_C v \) (\( u \) precedes \( v \)) if

\[
\tau(v) \in \tau(u) + C.
\]

We say that \( u <_C v \) if \( u \leq_C v \) and \( \tau(u) \neq \tau(v) \). Sometimes we may write \( v \succ_C u \) (resp. \( v \succ_C u \)) instead of \( u \leq_C v \) (resp. \( u <_C v \)). If either \( u \leq_C v \) or \( v \leq_C u \), we say that \( u \) and \( v \) are comparable; otherwise, they are non-comparable. In addition, \( \tilde{e} \in E^n \) is said to be an arbitrary positive fuzzy-number if \( \tilde{e} \succeq 0 \) and \( D(\tilde{e}, \tilde{0}) < \varepsilon \), where \( \varepsilon \) is an arbitrary positive real number.
If \( u, v \in E^1 \), then \( \tau(u) = \int_0^1 r(u_1 + u_2) \, dr \), \( \tau(v) = \int_0^1 r(v_1 + v_2) \, dr \). Suppose \( C = R^+ = [0, +\infty) \subseteq R \), \( u \preceq v \) if and only if \( \tau(u) \leq \tau(v) \), i.e., \( \tau(v) \in \tau(u) + [0, +\infty) \), which coincides with the definition of ordering of \( u, v \) proposed by Goetschel and Voxman [10].

**Remark 2.14.**

1. Let \( u, v \in L(E^n) \), then \( \tau(u) = \left[ \int_0^1 r(u_1 + u_2) \, dr, \int_0^1 r(u_2 + u_3) \, dr, \ldots, \int_0^1 r(u_{n-1} + u_n) \, dr \right]^T \).

2. Let \( u, v \in L(E^n) \), for \( k_1, k_2 \in R \),

\[
\tau(k_1 u + k_2 v) = k_1 \tau(u) + k_2 \tau(v).
\]

3. Let \( u_1, u_2, v_1, v_2 \in L(E^n) \). If \( u_1 \preceq v_1 \) and \( u_2 \preceq v_2 \), then for \( k_1, k_2 \in [0, \infty) \),

\[
k_1 u_1 + k_2 u_2 \preceq k_1 v_1 + k_2 v_2.
\]

**Definition 2.15.** A subset \( S \) of \( E^n \) is said to be bounded above if there exists a fuzzy number \( u \in E^n \), called an upper bound of \( S \), such that \( v \preceq u \) for all \( v \in S \). Furthermore, a fuzzy number \( u_0 \in E^n \) is called the least upper bound for \( S \) if

1. \( u_0 \) is an upper bound of \( S \),
2. \( u_0 \preceq u \) for every upper bound \( u \) of \( S \).

Similarly, we can define the lower bound and the greatest lower bound of a subset of \( E^1 \).

We call \( F : M \rightarrow E^n \) a \( n \)-dimensional fuzzy-number-valued function. We denote its lower \( u \)-level set as \( L_u = \{ z \in M : F(z) \preceq u \} \), and the strict lower \( u \)-level set as \( \tilde{L}_u = \{ z \in M : F(z) < u \} \), and the epigraph as \( \text{epi}(F) = \{(x, u) : x \in M, u \in E^n, F(x) \preceq u \} \). For \( F : M \rightarrow L(E^n) \), we denote \( [F(x)]' = \prod_{i=1}^n \left[ F_i(x)(r), F_i(x)(r) \right] \) as \( [F(x)]' = \prod_{i=1}^n \left[ F_i(x)(r), F_i(x)(r) \right] \), \( r \in [0, 1] \).

**Definition 2.16.** [16] Let \( F : M \rightarrow E^n \). \( F \) is said to be lower semicontinuous at \( x_0 \) if for any \( \bar{\epsilon} \), a neighborhood \( U \) of \( x_0 \) exists when \( x_0 \in U \), and we have \( F(x_0) \preceq F(x) + \bar{\epsilon} \); \( F \) is said to be upper semicontinuous at \( x_0 \) if for any \( \bar{\epsilon} \), a neighborhood \( U \) of \( x_0 \) exists when \( x_0 \in U \), and we have \( F(x) \preceq F(x_0) + \bar{\epsilon} \). \( F \) is continuous at \( x_0 \in M \) if it is both lower semicontinuous and upper semicontinuous at \( x_0 \), and that it is continuous if and only if it is continuous at every point of \( M \).

Let \( \Phi : R^m \rightarrow R^n \) be a set-valued function. The graph of \( \Phi \) is the set \( \text{gr} \Phi = \{(x, y) \in R^m \times R^n : y \in \Phi(x)\} \). \( \Phi \) is said to be closed at \( x \in R^m \) if \( y \in \Phi(x) \) whenever there exists a sequence \( (x_k, y_k) \) contained in \( \text{gr} \Phi \) and converging to \( (x, y) \).

**Definition 2.17.** [16] Let \( M \subseteq R^m \) be a convex set and \( F : M \rightarrow E^n \).
1. \( F \) is said to be convex on \( M \) if for any \( x_1, x_2 \in M \) and \( \lambda \in [0, 1] \)

\[
F(\lambda x_1 + (1 - \lambda) x_2) \preceq \lambda F(x_1) + (1 - \lambda) F(x_2).
\]
2. \( F \) is said to be quasi-convex on \( M \) if for any \( x_1, x_2 \in M \) and \( \lambda \in [0, 1] \)

\[
F(\lambda x_1 + (1 - \lambda) x_2) \preceq \max\{F(x_1), F(x_2)\}.
\]

**Proposition 2.18.** [16] Let \( F_i : M \rightarrow E^n \) be \( i = 1, 2, \ldots, k \) be convex. If \( a_1, a_2, \ldots, a_k > 0 \), then \( F(x) = \sum_{i=1}^k a_i F_i(x) \) is a convex fuzzy-number-valued function.

**Proposition 2.19.** (Interpolation property) Let \( F : M \rightarrow L(E^n) \) be an \( n \)-cell fuzzy-number-valued function. \( F \) is convex if and only if \( \forall x_i, x_j \in M, \lambda \in [0, 1] \) and \( \forall u, v \in L(E^n) \) with \( F(x_i) \preceq u, F(x_j) \preceq v \),

\[
F(\lambda x_i + (1 - \lambda) x_j) \preceq \lambda u + (1 - \lambda) v.
\]

(2.21)
Theorem 2.23. Let $F : M \to E^n$ and $u \in E^n$. Then $F$ is quasi-convex on $M$ if and only if its lower $u$-level set $L_u$ (or the strict lower $u$-level set $\tilde{L}_u$) is a convex subset of $R^m$.

Definition 2.23. Let $F : M \to E^n$ be quasi-convex. The normal operator and strict normal operator of $F$ at $x$ are set-valued functions, which are defined as the normal cones to $L_{F(x)}$ and $\tilde{L}_{F(x)}$ at $x$, respectively, i.e.,

$$N_{F_{x_0}}(x) = \{x' \in R^m : (x')^T(z - x) \leq 0, \forall z \in L_{F(x)}\},$$

$$\tilde{N}_{F_{x_0}}(x) = \{x' \in R^m : (x')^T(z - x) \leq 0, \forall z \in \tilde{L}_{F(x)}\}. (2.23)$$

Obviously, $N_{F_{x_0}}(x)$ and $\tilde{N}_{F_{x_0}}(x)$ are convex sets.

Theorem 2.22. Let $F : M \to E^n$ be quasi-convex. If $F$ is lower semicontinuous at $x$, then $\tilde{N}_{F_{x_0}}(x)$ is a closed convex set.

Proof. Let $(x_k, x'_k)$ be a sequence contained in $gr\tilde{N}$ and converge to $(x, x')$.

Let $z \in \tilde{L}_{F(x)}$, i.e., $F(z) \prec F(x)$. Since $F$ is lower semicontinuous at $x$, then $F(z) \prec F(x_k)$ for $k$ large enough. Therefore, $x'_k \in \tilde{N}_{F_{x_0}}(x_k)$ implies that $(x'_k)^T(z - x_k) \leq 0$. Let $k \to \infty$, we have $(x')^T(z - x) \leq 0$, thus, $\tilde{N}_{F_{x_0}}(x)$ is closed at $x$. □

Definition 2.22. Let $F : R^m \to E^n$ and $S^* = \left\{ y \in R^{m \times n} : \sup_{x \in R^m} \{y^T x - \tau(F(x))\} < \infty \right\}$. The function $F^* : S^* \to E^n$, defined as

$$F^*(y) = \sup_{x \in R^m} \{y^T x - \tau(F(x)) = \tau(u)\}, (2.24)$$

is called the conjugate of the fuzzy-number-valued function $F$.

Obviously, $\tau(F(x)) + \tau(F^*(y)) = y^T x, \forall x \in R^m, y \in S^*$.

Theorem 2.24. Let $F : R^m \to L(E^n)$. Then $F^*$ is convex on $S^*$.

Proof. $S^* = \left\{ y \in R^{m \times n} : \sup_{x \in R^m} \{y^T x - \tau(F(x))\} < \infty \right\}$ is a convex set. In fact, $\forall a^*, b^* \in S^*$,

$$\sup_{x \in R^m} \{(a^*)^T x - \tau(F(x))\} < \infty, \sup_{x \in R^m} \{(b^*)^T x - \tau(F(x))\} < \infty,$$
thus,
\[
\sup_{x \in \mathbb{R}^n} ((\lambda a^* + (1 - \lambda) b^*)^T x - \tau(F(x))) = \sup_{x \in \mathbb{R}^n} ((\lambda a^* + (1 - \lambda) b^*)^T x - \lambda \tau(F(x)) - (1 - \lambda) \tau(F(x))) \\
= \sup_{x \in \mathbb{R}^n} (\lambda ((a^*)^T x - \tau(F(x))) + (1 - \lambda) ((b^*)^T x - \tau(F(x)))) \\
\leq \lambda \sup_{x \in \mathbb{R}^n} ((a^*)^T x - \tau(F(x))) + (1 - \lambda) \sup_{x \in \mathbb{R}^n} ((b^*)^T x - \tau(F(x))) < \infty,
\]

therefore, \((\lambda a^* + (1 - \lambda) b^*) \in S^*\), that is, \(S^*\) is a convex set.

For any \(y_1, y_2 \in \mathbb{R}^{m \times n}\),
\[
F^*(y_1) = \sup_{x \in \mathbb{R}^n} \{u_1 \in L(E^n) | y_1^T x - \tau(F(x)) = \tau(u_1)\},
\]
\[
F^*(y_2) = \sup_{x \in \mathbb{R}^n} \{u_2 \in L(E^n) | y_2^T x - \tau(F(x)) = \tau(u_2)\},
\]
then for any \(\lambda \in [0, 1]\), by (2.19), we have
\[
F^*(\lambda y_1 + (1 - \lambda)y_2) = \sup_{x \in \mathbb{R}^n} \{u \in L(E^n) | (\lambda y_1 + (1 - \lambda)y_2)^T x - \tau(F(x)) = \tau(u)\} \\
= \sup_{x \in \mathbb{R}^n} \{u \in L(E^n) | \lambda y_1^T x - \lambda \tau(F(x)) + (1 - \lambda)y_2^T x - (1 - \lambda) \tau(F(x)) = \tau(u)\} \\
= \sup_{x \in \mathbb{R}^n} \{u \in L(E^n) | \lambda \tau(u_1) + (1 - \lambda) \tau(u_2) = \tau(u)\} \\
= \sup_{x \in \mathbb{R}^n} \{u_1 + (1 - \lambda) u_2 \in L(E^n) | y_1^T x - \tau(F(x)) = \tau(u_1), y_2^T x - \tau(F(x)) = \tau(u_2), u_1, u_2 \in L(E^n)\} \\
\leq_c \sup_{x \in \mathbb{R}^n} \{u_1 \in L(E^n) | y_1^T x - \tau(F(x)) = \tau(u_1)\} + (1 - \lambda) \sup_{x \in \mathbb{R}^n} \{u_2 \in L(E^n) | y_2^T x - \tau(F(x)) = \tau(u_2)\} \\
= \lambda F^*(y_1) + (1 - \lambda) F^*(y_2),
\]

therefore, \(F^*\) is convex on \(S^*\).

Proposition 2.25. Let \(F, G : \mathbb{R}^n \rightarrow L(E^n)\). We denote \(S^*_f = \{y \in \mathbb{R}^{m \times n} : \sup_{x \in \mathbb{R}^n} \{y^T x - \tau(F(x))\} < \infty\}\), \(S^*_g = \{y \in \mathbb{R}^{m \times n} : \sup_{x \in \mathbb{R}^n} \{y^T x - \tau(F(x))\} < \infty, \sup_{x \in \mathbb{R}^n} \{y^T x - \tau(G(x))\} < \infty\}\).

(1) If \(F(x) \leq_c G(x), \forall x \in \mathbb{R}^n\), then \(F^*(y) \geq_c G^*(y)\), \(y \in S^*_f\).

(2) \((kF)^*(y) = k F^*(k^{-1}y)\), \(y \in S^*_f, k \in R\).

Proof.

(1) Since \(F(x) \leq_c G(x), \forall x \in \mathbb{R}^n\), then \(\tau(G(x)) - \tau(F(x)) \in C\), where \(C \subseteq \mathbb{R}^n\) is a closed convex cone with \(0 \in C\) and \(C \neq \mathbb{R}^n\). It follows that
\[
\tau(G(x)) - \tau(F(x)) = (y^T x - \tau(F(x))) - (y^T x - \tau(G(x))) \in C, \forall x \in \mathbb{R}^n, y \in S^*_f,
\]
thus, let \(u, v \in L(E^n)\), for \(\tau(u) = y^T x - \tau(F(x))\), \(\tau(v) = y^T x - \tau(G(x))\), we have \(\tau(u) \in \tau(v) + C\), that is, \(u \geq_c v\). Therefore,
\[
F^*(y) = \sup_{x \in \mathbb{R}^n} \{u \in L(E^n) | y^T x - \tau(F(x)) = \tau(u)\} \leq_c \sup_{x \in \mathbb{R}^n} \{v \in L(E^n) | y^T x - \tau(G(x)) = \tau(v)\} = G^*(y).
\]

(2) Since \(\sup_{x \in \mathbb{R}^n} \{y^T x - \tau(F(x))\} = k \sup_{x \in \mathbb{R}^n} \{k^{-1}y^T x - \tau(F(x))\}\), then \(y \in S^*_f \Rightarrow k^{-1}y \in S^*_f\).

Therefore, for \(k \in R\), we have
\[
(kF)^*(y) = \sup_{x \in \mathbb{R}^n} \{u \in L(E^n) | y^T x - \tau(F(x)) = \tau(u)\} = \sup_{x \in \mathbb{R}^n} \{k^{-1}u \in L(E^n) | k^{-1}y^T x - \tau(F(x)) = \tau(k^{-1}u)\} = k \sup_{x \in \mathbb{R}^n} \{k^{-1}u \in L(E^n) | k^{-1}y^T x - \tau(F(x)) = \tau(k^{-1}u)\} = k F^*(k^{-1}y).
\]

□
For any \( u_i \in E^n, i = 1, 2, \ldots, n \), we call the ordered \( n \)-dimensional fuzzy number class \( u_1, u_2, \ldots, u_n \) (i.e., the Cartesian product of \( n \)-dimensional fuzzy number \( u_1, u_2, \ldots, u_n \)) a \( n \)-dimensional fuzzy vector, denoted it as \( (u_1, u_2, \ldots, u_n) \), and call the collection of all \( n \)-dimensional fuzzy vectors (i.e., the Cartesian product \( E^n \times E^n \times \cdots \times E^n \)) \( n \)-dimensional fuzzy vector space, and denote it as \((E^n)^n\).

For any \( n \)-dimensional fuzzy vectors \( u = (u_1, u_2, \ldots, u_n)^T \in (E^n)^n \) and \( v = (v_1, v_2, \ldots, v_n)^T \in (E^n)^n \), let \( k = (k_1, k_2, \ldots, k_n)^T \in R^n \), we define \( u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \), \( k^T u = \sum_{i=1}^{n} k_i u_i \), where + is the fuzzy addition, \( \cdot \) is the fuzzy multiplication, and \( [u]^r = ([u_1]^r, [u_2]^r, \ldots, [u_n]^r) \), \( r \in [0, 1] \). We use the following convention for equalities and inequalities: (i) \( u \leq v \iff u_i \leq v_i \), \( i = 1, 2, \ldots, n \), (ii) \( u = v \iff u_i = v_i \), \( i = 1, 2, \ldots, n \) and (iii) \( u < v \iff u_i < v_i \), \( i = 1, 2, \ldots, n \).

3 Directional \( g \)-derivative and subdifferential for fuzzy-number-valued functions

In this section, we generalize the concepts of \( g \)-derivative proposed by Hai et al. [9] from \([a, b] \subset R \) to \( M \subset R^m \), and the directional \( g \)-derivative and subdifferential for fuzzy-number-valued functions are investigated. In addition, we give some kinds of definitions of convexity for fuzzy-number-valued functions, so that we can more conveniently discuss convex fuzzy mappings and convex fuzzy programming.

Let \( M \subset R^m \), \( F : M \rightarrow E^n \) be the \( n \)-dimensional fuzzy-number-valued function (fuzzy-number-valued functions for short). The fuzzy-number-valued functions in the following arguments are assumed to be comparable.

**Definition 3.1.** Let \( F : M \rightarrow E^n \) be a fuzzy-number-valued function, \( x_0 = (x_0^1, x_0^2, \ldots, x_0^m) \in M \) and \( h \in R \) with \( (x_0^1, x_0^2, \ldots, x_0^m) \in M \). If \( g \)-difference \( F(x_0^1, x_0^2, \ldots, x_0^m) \circ g F(x_0^1, x_0^2, \ldots, x_0^m) \) exists and there exists \( u_j \in E^n \) \( (j = 1, 2, \ldots, m) \) such that

\[
\lim_{h \rightarrow 0} \frac{F(x_0^1, x_0^2, \ldots, x_0^m) \circ g F(x_0^1, x_0^2, \ldots, x_0^m)}{h} = u_j,
\]

then we say that \( F \) has the \( j \)-th partial generalized derivative (\( g \)-derivative for short) at \( x_0 \), denoted by \( u_j = \frac{\partial F(x_0)}{\partial x_j} \). If all the partial \( g \)-derivatives at \( x_0 \) exist, then we say \( F \) is generalized differentiable (\( g \)-differentiable for short) at \( x_0 \), denoted by \( F'_g(x_0) \).

Here the limit is taken in the metric space \((E^n, D)\). If \( F \) is \( g \)-differentiable at any point of \( M \), then \( F \) is said to be \( g \)-differentiable on \( M \). The fuzzy vector \((u_1, u_2, \ldots, u_m) \in (E^n)^m \) is said to be the gradient of \( F \) at \( x_0 \), denoted by \( \nabla_g F(x_0) \), i.e., \( \nabla_g F(x_0) = (u_1, u_2, \ldots, u_m)^T = \left( \frac{\partial F(x_0)}{\partial x_1}, \frac{\partial F(x_0)}{\partial x_2}, \ldots, \frac{\partial F(x_0)}{\partial x_m} \right)^T \).

**Remark 3.2.** We give an equivalent form of Definition 3.1. Let \( F : M \rightarrow E^n \) be a fuzzy-number-valued function, \( x_0 \in M \) and \( h \in R \) with \( x_0 + h \in M \). If \( g \)-difference \( F(x_0 + h) \circ g F(x_0) \) and there exists a fuzzy vector \( u = (u_1, u_2, \ldots, u_m) \in (E^n)^m \) such that

\[
\lim_{x \rightarrow x_0} \frac{D(F(x) \circ g F(x_0), (x - x_0)^T u)}{d(x, x_0)} = 0,
\]

then we say \( F \) is \( g \)-differentiable at \( x_0 \), and such \( u \) is called the gradient of \( F \) at \( x_0 \), denoted by \( \nabla_g F(x_0) \).

**Definition 3.3.** Let \( F : M \rightarrow E^n \) be a fuzzy-number-valued function, \( x \in M \). The one-sided directional \( g \)-derivative of \( F \) at \( x \) with respect to a vector \( y \in R^m \) is defined to be the limit

\[
F'_g(x, y) = \lim_{\lambda \rightarrow 0^+} \frac{F(x + \lambda y) \circ g F(x)}{\lambda},
\]
if it exists. Note that
\[-F_g'(x, -y) = \lim_{h \to 0^-} \frac{F(x + Ay) \ominus_g F(x)}{A}.
\]

We say the one-sided directional g-derivative \(F_g'(x, y)\) is two-sided if and only if \(F_g'(x, -y)\) exists and
\[F_g'(x, y) = -F_g'(x, -y).
\]

We also say \(F\) is g-differentiable in the direction \(y\) at \(x\). Here the limit is taken in the metric space \((E^n, D)\).

**Theorem 3.4.** Let \(F : M \to E^n\) be a fuzzy-number-valued function. If \(F\) is g-differentiable at \(x\), then the directional g-derivatives \(F_g'(x, y)\) are two-sided, and
\[F_g'(x, y) = y^T \nabla_F F(x), \quad \forall y \in R^m.
\]

**Proof.** Since \(F\) is g-differentiable at \(x\), then for any \(y \in R^m\) and \(y \neq 0\),
\[
0 = \lim_{h \to 0} \frac{D(F(x + Ay) \ominus_g F(x), (Ay)^T \nabla_F F(x))}{|y|}
= \frac{1}{|y|} \lim_{h \to 0} \frac{D(F(x + Ay) \ominus_g F(x), (Ay)^T \nabla_F F(x))}{|y|}
= \frac{1}{|y|} D(F'(g, x), y^T \nabla_F F(x)).
\]

Thus, \(F_g'(x, y)\) exists and \(F_g'(x, y) = y^T \nabla_F F(x), \forall y \in R^m\). Similarly, we can prove \(F_g'(x, y)\) are two-sided.

\(\square\)

**Theorem 3.5.** Let \(F : M \to L(E^n)\), \(x_0 = (x_0^0, x_0^1, \ldots, x_0^m) \in M\). Then \(F\) is g-differentiable at \(x_0\) if and only if for any \(r \in [0, 1]\), the real-valued functions \(F_i(x, r)\) and \(F_{i'}(x, r)\), \(i = 1, 2, \ldots, n\), are differentiable at \(x_0\), and
\[
\inf_{\beta \leq r} \left\{ \frac{\partial F_i(x_0, \beta)}{\partial x_j^i}, \frac{\partial F_{i'}(x_0, \beta)}{\partial x_j^{i'}} \right\}
\text{and}
\sup_{\beta \leq r} \left\{ \frac{\partial F_i(x_0, \beta)}{\partial x_j^i}, \frac{\partial F_{i'}(x_0, \beta)}{\partial x_j^{i'}} \right\}
\]
satisfy conditions (1)–(4) of Theorem 2.4, where \(\frac{\partial F_i(x_0, r)}{\partial x_j^i}\) and \(\frac{\partial F_{i'}(x_0, r)}{\partial x_j^{i'}}\), \(j = 1, 2, \ldots, m\), are the partial derivatives of \(F_i(x, r)\) and \(F_{i'}(x, r)\) at \(x_0\) with respect to the \(j\)th component, and
\[
\left[ \frac{\partial F(x_0)}{\partial x_j^i} \right] = \prod_{i=1}^n \left[ \inf_{\beta \leq r} \left\{ \frac{\partial F_i(x_0, \beta)}{\partial x_j^i}, \frac{\partial F_{i'}(x_0, \beta)}{\partial x_j^{i'}} \right\}, \sup_{\beta \leq r} \left\{ \frac{\partial F_i(x_0, \beta)}{\partial x_j^i}, \frac{\partial F_{i'}(x_0, \beta)}{\partial x_j^{i'}} \right\} \right].
\]

**Proof.** \(F\) is g-differentiable at \(x_0\) if and only if there exists \(u_j \in L(E^n)\), \(j = 1, 2, \ldots, m\), such that
\[
\lim_{h \to 0} \frac{F(x_0^0, \ldots, x_0^0 + h, \ldots, x_0^0)}{h} \ominus_g F(x_0^0, \ldots, x_0^0) = u_j.
\]

If and only if
\[
\lim_{h \to 0} D \left( \frac{F(x_0^0, \ldots, x_0^0 + h, \ldots, x_0^0) \ominus_g F(x_0^0, \ldots, x_0^0)}{h}, u_j \right) = 0.
\]

If and only if by (2.16)
\[
\lim_{h \to 0} \sup_{r \in [0, 1]} \left\| \left[ \frac{F(x_0^0, \ldots, x_0^0 + h, \ldots, x_0^0) \ominus_g F(x_0^0, \ldots, x_0^0)}{h} \right]_j^r - u_j^r \right\|_r = 0.
\]
If and only if by (2.14)

\[
\lim \sup_{h \to 0 \atop r \in [0, 1]} \sup_{\beta \in \mathbb{R}^r} \inf_{i \leq m} \left\{ \frac{F_i(x_i^0, \ldots, x_j^0 + h, \ldots, x_m^0, \beta) - F_i(x_i^0, \ldots, x_j^0, \ldots, x_m^0, \beta)}{h} \right\} - u_i^\prime(r) \right\} = 0
\]

\[
\lim \inf_{h \to 0 \atop r \in [0, 1]} \inf_{\beta \in \mathbb{R}^r} \left\{ \frac{F_i(x_i^0, \ldots, x_j^0 + h, \ldots, x_m^0, \beta) - F_i(x_i^0, \ldots, x_j^0, \ldots, x_m^0, \beta)}{h} \right\} - u_i^\prime(r) \right\} = 0,
\]

\[
\lim \sup_{h \to 0 \atop r \in [0, 1]} \sup_{\beta \in \mathbb{R}^r} \left\{ \frac{F_i(x_i^0, \ldots, x_j^0 + h, \ldots, x_m^0, \beta) - F_i(x_i^0, \ldots, x_j^0, \ldots, x_m^0, \beta)}{h} \right\} - u_i^\prime(r) \right\} = 0,
\]

\[i = 1, 2, \ldots, n\]

\[\Rightarrow\]

\[
\inf_{\beta \in \mathbb{R}^r} \left\{ \lim_{h \to 0} \frac{F_i(x_i^0, \ldots, x_j^0 + h, \ldots, x_m^0, \beta) - F_i(x_i^0, \ldots, x_j^0, \ldots, x_m^0, \beta)}{h} \right\} - u_i^\prime(r) \right\} = 0,
\]

\[
\lim_{h \to 0} \frac{F_i(x_i^0, \ldots, x_j^0 + h, \ldots, x_m^0, \beta) - F_i(x_i^0, \ldots, x_j^0, \ldots, x_m^0, \beta)}{h} = u_i^\prime(r),
\]

\[
\sup_{\beta \in \mathbb{R}^r} \left\{ \lim_{h \to 0} \frac{F_i(x_i^0, \ldots, x_j^0 + h, \ldots, x_m^0, \beta) - F_i(x_i^0, \ldots, x_j^0, \ldots, x_m^0, \beta)}{h} \right\} = u_i^\prime(r),
\]

\[i = 1, 2, \ldots, n, r \in [0, 1]\]

\[\Rightarrow\]

For any \(r \in [0, 1]\), \(F_i'(x, r), F_i'(x, r), i = 1, 2, \ldots, n\), are differentiable at \(x_0\), \(\inf_{\beta \in \mathbb{R}^r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0}, \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0} \right\} = u_i^\prime(r)\) and \(\sup_{\beta \in \mathbb{R}^r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0}, \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0} \right\} = u_i^\prime(r), j = 1, 2, \ldots, m\), satisfy conditions (1)–(4) of Theorem 2.4, and

\[
\left[ \frac{\partial F_i'(x_0)}{\partial x_j^0} \right]^r = \prod_{i=1}^n \left\{ \inf_{\beta \in \mathbb{R}^r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0}, \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0} \right\}, \sup_{\beta \in \mathbb{R}^r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0}, \frac{\partial F_i'(x_0, \beta)}{\partial x_j^0} \right\} \right\}.
\]

\(\square\)
Remark 3.6. For \( r \in [0, 1] \), we denote
\[
\nabla_y F'(x_0, r) = \left( \prod_{i=1}^{n} \inf_{\beta \in r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_i^0}, \frac{\partial F_i'(x_0, \beta)}{\partial x_i^1} \right\} \right), \quad \prod_{i=1}^{n} \inf_{\beta \in r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_i^2}, \frac{\partial F_i'(x_0, \beta)}{\partial x_i^3} \right\} \right),
\]
\[
\nabla_y F'(x_0, r) = \left( \prod_{i=1}^{n} \sup_{\beta \in r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_i^0}, \frac{\partial F_i'(x_0, \beta)}{\partial x_i^1} \right\} \right), \quad \prod_{i=1}^{n} \sup_{\beta \in r} \left\{ \frac{\partial F_i'(x_0, \beta)}{\partial x_i^2}, \frac{\partial F_i'(x_0, \beta)}{\partial x_i^3} \right\} \right).
\]

Theorem 3.7. Let \( F : M \to L(E^n), x \in M, y \in R^m \). Then \( F \) is \( g \)-differentiable in the direction \( y \) at \( x \) if and only if for any \( r \in [0, 1] \), the real-valued functions from \( M \) into \( R \), \( F'_i(x, r) \) and \( F''_i(x, r), i = 1, 2, ..., n \), are differentiable in the direction \( y \) at \( x \), and \( \inf_{\beta \in r} F_i'(x, y, \beta), F_i''(x, y, \beta) \) and \( \sup_{\beta \in r} F_i'(x, y, \beta), F_i''(x, y, \beta) \) satisfy conditions (i)–(iv) of Theorem 2.4, and
\[
[F'_i(x, y)] = \left[ \sup_{\beta \in r} \left\{ \frac{F_i'(x, y, \beta)}{\partial x_i^0}, \frac{F_i''(x, y, \beta)}{\partial x_i^1} \right\} \right], \quad \inf_{\beta \in r} \left\{ \frac{F_i'(x, y, \beta)}{\partial x_i^2}, \frac{F_i''(x, y, \beta)}{\partial x_i^3} \right\} \right).
\]

Proof. For \( u \in L(E^n) \),
\[
\lim_{\lambda \to 0} D\left( F(x + \lambda y) \cdot F(x), \lambda, u \right) = 0
\]
\[
\Rightarrow \lim_{\lambda \to 0} \sup_{t \in \{0, 1\}} \left\{ \inf_{\beta \in r} \left\{ \frac{F_i'(x + \lambda y, \beta) - F_i'(x + \lambda y, \beta)}{\lambda} \right\} \right\} - u_i(r), \quad \sup_{\beta \in r} \left\{ \frac{F_i'(x + \lambda y, \beta) - F_i'(x + \lambda y, \beta)}{\lambda} \right\} - u_i^+(r) \right\} = 0
\]
\[
\Rightarrow \lim_{\lambda \to 0} \sup_{t \in \{0, 1\}} \left\{ \inf_{\beta \in r} \left\{ \frac{F_i'(x + \lambda y, \beta) - F_i'(x + \lambda y, \beta)}{\lambda} \right\} \right\} - u_i(r), \quad \sup_{\beta \in r} \left\{ \frac{F_i'(x + \lambda y, \beta) - F_i'(x + \lambda y, \beta)}{\lambda} \right\} - u_i^+(r) \right\} = 0
\]
\[
\Rightarrow \inf_{\beta \in r} \left\{ \lim_{\lambda \to 0} \frac{F_i'(x + \lambda y, \beta) - F_i'(x + \lambda y, \beta)}{\lambda}, \lim_{\lambda \to 0} \frac{F_i''(x + \lambda y, \beta) - F''_i(x + \lambda y, \beta)}{\lambda} \right\} = u_i(r), \quad \sup_{\beta \in r} \left\{ \lim_{\lambda \to 0} \frac{F_i'(x + \lambda y, \beta) - F_i'(x + \lambda y, \beta)}{\lambda}, \lim_{\lambda \to 0} \frac{F_i''(x + \lambda y, \beta) - F''_i(x + \lambda y, \beta)}{\lambda} \right\} = u_i^+(r),
\]
\[
i = 1, 2, ..., n, \quad r \in [0, 1]
\]
For any \( r \in [0, 1] \), \( F_{i}(x, r) \) and \( F_{i}(x, r) \), \( i = 1, 2, \ldots, n \), are differentiable in the direction \( y \) at \( x \), and
\[
\inf_{\beta \in \mathbb{R}} \min \{ F_{i}^{\beta}(x, y, \beta), F_{i}^{\beta}(x, y, \beta) \} = u_{i}^{\beta}(r) \quad \text{and} \quad \sup_{\beta \in \mathbb{R}} \max \{ F_{i}^{\beta}(x, y, \beta), F_{i}^{\beta}(x, y, \beta) \} = u_{i}^{\beta}(r)
\]
 satisfy conditions (i)–(iv) of Theorem 2.4, and
\[
[F_{i}'(x, y)]' = \prod_{i=1}^{n} \left[ \inf_{\beta \in \mathbb{R}} \min \{ F_{i}^{\beta}(x, y, \beta), F_{i}^{\beta}(x, y, \beta) \}, \sup_{\beta \in \mathbb{R}} \max \{ F_{i}^{\beta}(x, y, \beta), F_{i}^{\beta}(x, y, \beta) \} \right].
\]

**Definition 3.8.** Let \( M \subseteq \mathbb{R}^{m} \) be a convex set and \( F : M \rightarrow L(\mathbb{E}^{n}) \) be an \( n \)-cell fuzzy-number-valued function. If for any \( x_{1}, x_{2} \in M \) and \( \lambda \in [0, 1] \), we have
\[
F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r) \leq \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r), \quad i = 1, 2, \ldots, n,
\]
and
\[
F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r) \leq \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r), \quad i = 1, 2, \ldots, n,
\]
uniformly for \( r \in [0, 1] \), that is, for any fixed \( r \in [0, 1] \), \( F_{i}(x, r) \) and \( F_{i}(x, r) \) are all convex functions of \( x \), then \( F \) is said to be endpoint-wise convex (e-convex for short) on \( M \).

**Proposition 3.9.** Let \( F : M \rightarrow L(\mathbb{E}^{n}) \) be e-convex, then \( F \) is convex.

**Proof.** Since \( F \) is e-convex, then for any \( x_{1}, x_{2} \in M \), \( \lambda \in [0, 1] \) and for \( i = 1, 2, \ldots, n \), we have
\[
F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r) \leq \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r), \quad r \in [0, 1],
\]
and
\[
F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r) \leq \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r), \quad r \in [0, 1].
\]
It follows that for \( i = 1, 2, \ldots, n \),
\[
F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r) + F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r) \leq \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r) + \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r).
\]
Since \( r \in [0, 1] \), then for \( i = 1, 2, \ldots, n \),
\[
\int_{0}^{1} r(F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r) + F_{i}(\lambda x_{1} + (1 - \lambda) x_{2}, r)) \, dr \\
\leq \int_{0}^{1} \left( \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r) + \lambda F_{i}(x_{1}, r) + (1 - \lambda) F_{i}(x_{2}, r) \right) \, dr,
\]
thus,
\[
\tau(F(\lambda x_{1} + (1 - \lambda) x_{2})) \leq \tau(\lambda F(x_{1}) + (1 - \lambda) F(x_{2})).
\]
Let \( C = R^{m} = \{ (x_{1}, x_{2}, \ldots, x_{n})' \in R^{m} : x_{1} \geq 0, x_{2} \geq 0, \ldots, x_{n} \geq 0 \} \subseteq R^{m}, \) we obtain
\[
\tau(\lambda F(x_{1}) + (1 - \lambda) F(x_{2})) \in \tau(F(\lambda x_{1} + (1 - \lambda) x_{2})), \text{ that is, } F(\lambda x_{1} + (1 - \lambda) x_{2}) \leq_{e} \lambda F(x_{1}) + (1 - \lambda) F(x_{2}),
\]
therefore, \( F \) is convex.

**Definition 3.10.** Let \( F : M \rightarrow L(\mathbb{E}^{n}) \), then we say \( F \) is endpoint-wise differentiable (e-differentiable for short) at \( x_{0} \), that is, if there exists \( u_{ij}, u_{ij} \in R \), \( i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m \), such that
\[
\lim_{h \to 0} \frac{F_{i}(x_{0}, \ldots, x_{0} + h, \ldots, x_{m}^{0}, r) - F_{i}(x_{0}, \ldots, x_{0}, \ldots, x_{m}^{0}, r)}{h} = u_{ij}
\]
and
\[
\lim_{h \to 0} \frac{F_{i}(x_{0}, \ldots, x_{0} + h, \ldots, x_{m}^{0}, r) - F_{i}(x_{0}, \ldots, x_{0}, \ldots, x_{m}^{0}, r)}{h} = u_{ij}^{*},
\]
uniformly for \( r \in [0, 1] \), then we say \( F \) has \( j \)th partial e-differentiable at \( x_{0} \), denoted by \( \frac{\partial F_{i}(x_{0}, \ldots, x_{0}, r)}{\partial x_{0}^{j}} = u_{ij}, \)
\( \frac{\partial F_{i}(x_{0}, \ldots, x_{0})}{\partial x_{0}^{j}} = u_{ij}^{*} \). If all the partial e-derivatives at \( x_{0} \) exist, then we say \( F \) is e-differentiable at \( x_{0} \), denoted by
The endpoint-wise gradients of $F$ at $x_0$, denoted by $\nabla_e F_i(x_0)$, $i = 1, \ldots, n$, are
\[
\nabla_e F_i(x_0, r) = (u_{i1}, u_{i2}, \ldots, u_{im})^T = \left( \frac{\partial F_i(x_0, r)}{\partial x_1}, \frac{\partial F_i(x_0, r)}{\partial x_2}, \ldots, \frac{\partial F_i(x_0, r)}{\partial x_m} \right)^T, \quad i = 1, 2, \ldots, n.
\]
and
\[
\nabla_e F_i(x_0, r) = (u_{i1}^+, u_{i2}^+, \ldots, u_{im}^+) = \left( \frac{\partial F_i(x_0, r)}{\partial x_1^+}, \frac{\partial F_i(x_0, r)}{\partial x_2^+}, \ldots, \frac{\partial F_i(x_0, r)}{\partial x_m^+} \right)^T, \quad i = 1, 2, \ldots, n.
\]

**Theorem 3.11.** Let $M \subseteq \mathbb{R}^m$ be a convex set and $F : M \to E^n$ be an e-differentiable fuzzy-number-valued function on $M$. $F$ is e-convex on $M$ if and only if for any $x_0, x_1 \in M$,
\[
F_i(x_0, r) \geq F_i(x_1, r) + \nabla_e F_i(x_0)^T (x_1 - x_0)
\]
and
\[
F_i(x_0, r) \geq F_i(x_1, r) + \nabla_e F_i(x_1)^T (x_0 - x_1),
\]
uniformly for $r \in [0, 1], i = 1, 2, \ldots, n$.

The proof is similar to the proof of Theorem 6.1.2 in the study of Mangasarian [24].

**Definition 3.12.** Let $M \subseteq \mathbb{R}^m$ be a convex set and $F : M \to E^n$ be convex. Then a fuzzy vector $x \in (E^n)^m$ is said to be a subgradient of $F$ at a point $x \in M$ if
\[
F(x) \succ_c F(z) + \xi^T (z - x), \quad \forall z \in M.
\]
The set of all subgradients of $F$ at $x$ is called the subdifferential of $F$ at $x$ and is denoted by $\partial F(x)$. The multivalued mapping $\partial F : x \to \partial F(x)$ is called the subdifferential of $F$. If $\partial F(x)$ is not empty, $F$ is said to be subdifferentiable at $x$.

**Proposition 3.13.** Let $F : M \to L(E^n)$ be a convex fuzzy n-cell number, $x \in M$. If $\xi \in \partial F(x)$, then
\[
F'_e(x, y) \succ_c \xi^T y, \quad \forall y \in \mathbb{R}^m.
\]

**Proof.** Setting $z = x + \lambda y, \lambda > 0, \forall y \in \mathbb{R}^m$. Since $\xi \in \partial F(x)$, then we have by definition
\[
F(x + \lambda y) \succ_c F(x) + \lambda \xi^T y, \quad \forall y \in \mathbb{R}^m, \lambda > 0.
\]
By Proposition 2.11, we have $F(x) + F(x + \lambda y) / \lambda \succ_c F(x) + \lambda \xi^T y$. According to (2.20),
\[
\frac{F(x)}{\lambda} + \frac{F(x + \lambda y)}{\lambda} \succ_c \frac{F(x)}{\lambda} + \xi^T y,
\]
that is,
\[
\frac{F(x)}{\lambda} + \frac{F(x + \lambda y)}{\lambda} / \lambda \succ_c \frac{F(x)}{\lambda} + \xi^T y + \frac{F(x)}{\lambda} + \xi^T y
\]
and we obtain by definition $F'_e(x, y) \succ_c \xi^T y, \forall y \in \mathbb{R}^m$. □
Theorem 3.14. Let $F : M \to L(E^n)$ be convex and $x$ be a vector in $M$. Then $\partial F(x)$ is a convex set. Furthermore, if $F$ is lower semicontinuous at $x$, then $\partial F(x)$ is a closed set.

Proof. If $\partial F(x)$ is empty, the theorem is trivial. If $\partial F(x)$ is not empty, then suppose $\xi_i, \xi_j \in \partial F(x)$, for all $\lambda \in [0, 1]$, we have

$$AF(z) \succ_c AF(x) + \lambda (z - x)^T \xi_i, \forall z \in M,$$

$$(1 - \lambda) F(z) \succ_c (1 - \lambda) F(x) + (1 - \lambda) (z - x)^T \xi_j, \forall z \in M.$$ By (2.20), we have $F(z) \succ_c F(x) + (z - x)^T (\lambda \xi_i + (1 - \lambda) \xi_j), \forall z \in M$. Therefore, $\lambda \xi_i + (1 - \lambda) \xi_j \in \partial F(x)$, that is, $\partial F(x)$ is convex.

On the other hand, for $\xi_k = (\xi^1_k, \xi^2_k, \ldots, x^m_k) \in (E^n)^m, k \in N^*$, $\xi = (\xi^1, \xi^2, \ldots, \xi^m)$, suppose that $\xi_n \in \partial F(x)$, and $\xi_k \to \xi$ ($k \to \infty$), that is, $\xi^i_k \to \xi^i$ ($k \to \infty$), $i = 1, 2, \ldots, m$. Since $\xi^i_k \in E^n$ ($i = 1, 2, \ldots, m$) and $(E^n, D)$ is a complete metric space, then $\xi^i \in E^n, i = 1, 2, \ldots, m$, thus, $\xi = (\xi^1, \xi^2, \ldots, \xi^m) \in (E^n)^m$. Let $\{(x_k, \xi_k)\}$ be a sequence contained in $\text{gr} \partial F$ and converge to $(x, \xi)$. Since $\xi_k \in \partial F$, then $F(z) \succ_c F(x_k) + \xi^T_k (z - x_k), \forall z \in M$. Since $F$ is lower semicontinuous at $x$, then we obtain $F(z) \succ_c F(x) + \xi^T (z - x), \forall z \in M$. Let $k \to \infty$, then we have $F(z) \succ_c F(x) + \xi^T (z - x), \forall z \in M$, thus, $\xi \in \partial F(x)$. Therefore, $\partial F(x)$ is a closed convex set. \hfill \Box

Proposition 3.15. Let $F, G : M \to L(E^n)$ be convex and $x$ be a vector in $M$. Then for all $\lambda \geq 0$,

$$\partial (AF)(x) = \lambda \partial F(x), \quad \partial (F + G)(x) = \partial F(x) + \partial G(x).$$

(3.8)

It is not difficult to prove by (2.20) and Proposition 2.18.

Let $M \subseteq R^n$ be a convex set and $F : M \to E^n$. Consider the following fuzzy optimization problem with no constraints (FOP)

$$\begin{align*}
\text{minimize} & \quad F(x) \\
\text{subject to} & \quad x \in M.
\end{align*}$$

(3.9)

A point $x \in M$ is called a feasible solution to (FOP). Let $x^* \in M$, if $F(x^*) \preceq_c F(x)$ for any $x \in M$, then $x^*$ is said to be an optimal solution to the problem.

Theorem 3.16. Let $F : M \to E^n$ be convex. Then $x^*$ is an optimal solution to (FOP) if and only if $0 \in \partial F(x^*)$.

Proof. By Definition 3.12, $0 \in \partial F(x^*)$ if and only if

$$F(x) \succeq_c F(x^*) + (z - x)^T 0 = F(x^*), \forall x \in M.$$ \hfill \Box

4 Duality optimality conditions and saddle point optimality criteria in fuzzy optimization problems with constraints

Under the ordering $\prec_c$, for $n$-cell fuzzy-number-valued functions, the linear properties (2.19) and (2.20) hold. In this section, we investigate the Lagrange duality and the optimality conditions, including the KKT conditions and the saddle point optimality criteria, in fuzzy optimization problems with constraints for $n$-cell fuzzy-number-valued functions.

Let $M \subseteq R^n$, $F : M \to L(E^n)$ be the objective function, and $g_k(x), h_s(x) : M \to L(E^n), k = 1, 2, \ldots, l, s = 1, 2, \ldots, t$, be the constraint conditions. The fuzzy optimization problem with constraints (FOP1) is defined as

$$\begin{align*}
\text{minimize} & \quad F(x) \\
\text{subject to} & \quad g_k(x) \preceq_c 0, \quad k = 1, 2, \ldots, l, \\
& \quad h_s(x) = 0, \quad s = 1, 2, \ldots, t,
\end{align*}$$

(4.1)
with variable $x \in M$.

$D = \{ x \in M : g_k(x) \leq c_0, h_s(x) = 0, k = 1, 2, \ldots, l, s = 1, 2, \ldots, t \}$ is said to be the feasible set of (FOP1). Let $x^* \in D$, if $F(x^*) \leq c_0$ for each $x \in S$, then $x^*$ is said to be an optimal solution to (FOP1), and we denote the optimal value in (FOP1) by $p^*$, i.e., $p^* = F(x^*)$. We assume $D$ is nonempty. In the following, we denote $G(x) = (g_1(x), g_2(x), \ldots, g_s(x))^T$, $H(x) = (h_1(x), h_2(x), \ldots, h_l(x))^T$.

**Definition 4.1.** The Lagrangian $L : R^n \times R^l \times R^t \rightarrow L(E^n)$ associated with problem 4.1 is defined as

$$L(x, a, \beta) = F(x) + a^T G(x) + \beta^T H(x),$$

(4.2)

with $\text{dom} L = D \times R^l \times R^t$, where $a = (a_0, a_0, \ldots, a_0)^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_1)^T$. We refer to $a_k$ as the Lagrange multiplier associated with the $k$th inequality constraint $g_k(x) \leq c_0$; similarly, we refer to $\beta_k$ as the Lagrange multiplier associated with the $s$th equality constraint $h_s(x) = 0$. The vectors $a$ and $\beta$ are called the dual variables or Lagrange multiplier vectors associated with problem 4.1.

We define the Lagrange dual function $\varphi : R^l \times R^t \rightarrow E^n$ as the minimum value of the Lagrangian over $x$, i.e., for $a \in R^l$, $\beta \in R^t$,

$$\varphi(a, \beta) = \inf_{x \in D} L(x, a, \beta) = \inf_{x \in D} (F(x) + a^T G(x) + \beta^T H(x)).$$

(4.3)

When the Lagrangian is unbounded below in $x$, the dual function takes on the value $-\infty$.

**Proposition 4.2.** The dual function yields lower bounds on the optimal value $p^*$ of problem 4.1, i.e., for any $a \geq 0$ and any $\beta$, we have

$$\varphi(a, \beta) \leq c_0 p^*. \quad (4.4)$$

**Proof.** Suppose $\bar{x}$ is a feasible solution to problem 4.1, i.e., $g_k(\bar{x}) \leq c_0$ and $h_s(\bar{x}) = 0$, and $a \geq 0$, then

$$a^T G(\bar{x}) + \beta^T H(\bar{x}) \leq 0.$$

Since each term in the first sum is nonpositive, and each term in the second sum is zero, then by (2.20)

$$L(\bar{x}, a, \beta) = F(\bar{x}) + a^T G(\bar{x}) + \beta^T H(\bar{x}) \leq c_0 F(\bar{x}).$$

Hence, $\varphi(a, \beta) = \inf_{x \in D} L(x, a, \beta) \leq c_0 L(\bar{x}, a, \beta) \leq c_0 F(\bar{x})$. Since $\varphi(a, \beta) \leq c_0 F(\bar{x})$ holds for every feasible point $\bar{x}$, inequality 37 follows. This completes the proof.

For each pair $(a, \beta)$ with $a \geq 0$, the Lagrange dual function gives us a lower bound on the optimal value $p^*$ of the fuzzy optimization problem 4.1. Thus, we have a lower bound that depends on some parameters $a, \beta$. A natural question is what is the best lower bound that can be obtained from the Lagrange dual function. This leads to the optimization problem

$$\text{maximize} \quad \varphi(a, \beta)$$

$$\text{subject to} \quad a \geq 0.$$  

(4.5)

This problem is called the Lagrange dual problem associated (DFOP1) with the problem (FOP1). The original problem (FOP1) is also called the primal problem.

The set $D_2 = \{ (a, \beta) : a \geq 0, \varphi(a, \beta) > -\infty \}$ is said to be the dual feasible set of the primal problem (FOP1), that is, it is the feasible set of the dual problem (DFOP1). Let $(a^*, \beta^*) \in D_2$, if $\varphi(a^*, \beta^*)$ for each $(a, \beta) \in D_2$, then we refer to the pair $(a^*, \beta^*)$ as the dual optimal solution or optimal Lagrange multipliers, and the optimal value of the Lagrange dual problem denoted by $d^*$, i.e., $d^* = \varphi(a^*, \beta^*)$.

**Theorem 4.3.** (Weak Duality Theorem) If $x$ and $(a, \beta)$ are feasible solutions to the primal problem (FOP1) and the Lagrange dual problem (DFOP1), respectively, then weak duality holds:
\[ F(x) \geq c \cdot \varphi(a, \beta). \quad (4.6) \]

**Proof.** By definition, we have
\[ \varphi(a, \beta) = \inf_{x \in \mathcal{D}} L(x, a, \beta) \leq c F(x) + a^T G(x) + \beta^T H(x) \preceq c F(x). \]
\[ \square \]

**Corollary 4.4.** If \( x^* \) and \((a^*, \beta^*)\) are the optimal solutions to the primal problem (FOP1) and the Lagrange dual problem (DFOP1), respectively, then
\[ F(x^*) \preceq c \varphi(a^*, \beta^*) \quad (i.e., \ p^* \preceq c d^*). \quad (4.7) \]

We say that strong duality holds if
\[ p^* = d^*. \quad (4.8) \]

**Remark 4.5.** Let \( x^* \) be a primal optimal solution, \((a^*, \beta^*)\) a dual optimal solution with strong duality. Then the complementary slackness condition holds:
\[ a^*_k g_k(x^*) = 0, \quad (4.9) \]

namely, \( a^*_k > 0 \Rightarrow g_k(x^*) = 0 \) or \( g_k(x^*) \prec 0 \Rightarrow a^*_k = 0. \)

**Proof.** By definition, we have
\[ F(x^*) = \varphi(a^*, \beta^*) = \inf_{x \in \mathcal{D}} \left( F(x) + \sum_{k=1}^{l} a^*_k g_k(x) + \sum_{s=1}^{l} \beta^*_s h_s(x) \right) \]
\[ \preceq c F(x^*) + \sum_{k=1}^{l} a^*_k g_k(x^*) + \sum_{s=1}^{l} \beta^*_s h_s(x^*) \preceq c F(x^*). \]

It follows that
\[ \sum_{k=1}^{l} a^*_k g_k(x^*) + \sum_{s=1}^{l} \beta^*_s h_s(x^*) = 0. \]

According to \( h_s(x^*) = 0, \ s = 1, 2, \ldots, t, \) we have \( \sum_{k=1}^{l} a^*_k g_k(x^*) = 0, \) and since \( a^*_k \geq 0, \ g_k(x^*) \preceq c 0, \) \( k = 1, 2, \ldots, l, \)
we obtain \( a^*_k g_k(x^*) = 0. \)

Now we investigate optimality conditions, which are called the KKT conditions, for the solutions to be primal and dual optimal, when the primal problem is e-convex.

Setting \( C = R^{n^+} = \{ (x_1, x_2, \ldots, x_n)^T \in R^n : x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \} \subseteq R^n, \) the constraint conditions are equivalent to
\[ G(x) = 0 \Leftrightarrow r(g_k(x)) \leq 0 \ (0 \in R^m), \quad k = 1, 2, \ldots, l, \]
where \( r(g_k(x)) = \left( \int_0^1 r(g_{k1}(x)(r) + g_{k2}(x)(r)) dr, \ldots, \int_0^1 r(g_{k2}(x)(r) + g_{k21}(x)(r)) dr \right)^T, \)
\( k = 1, 2, \ldots, l, \ r \in [0, 1], \) thus, we obtain
\[ 0 \leq G(x) \Leftrightarrow \int_0^1 r(g_{k1}^r(x)(r) + g_{k2}^r(x)(r)) dr \leq 0 \ (0 \in R^m), \quad k = 1, 2, \ldots, l, \ j = 1, 2, \ldots, n. \]

Similarly, we have
\[ 0 \leq H(x) = \int_0^1 r(h_{ij}^r(x)(r) + h_{ij}^r(x)(r)) dr \leq 0 \ (0 \in R^m), \quad s = 1, 2, \ldots, t, \ j = 1, 2, \ldots, n. \]
We denote $G_k(x) = \int_0^1 r g_{k_l}(x)(r) + g_{k_r}(x)(r))dr$, $H_k(x) = \int_0^1 r h_{k_l}(x)(r) + h_{k_r}(x)(r))dr$, $k' = 1, 2, ..., l \times n$, $s' = 1, 2, ..., t \times n$, and denote $l \times n = p$, $t \times n = q$, then the fuzzy optimization problem (FOP1) can be transformed into the following fuzzy optimization problem (FOP1')

\[
\begin{align*}
\text{minimize} & \quad F(x) \\
\text{subject to} & \quad G_k(x) \leq 0, \\
& \quad H_k(x) = 0,
\end{align*}
\]

where $x \in M$, $F : M \rightarrow L(E^p)$, $G_k$, $H_k : M \rightarrow R$. Obviously, the feasible set of (FOP1') is equivalent to the feasible set of (FOP1).

Let

\[ q'(\alpha, \beta) = \inf_{x \in \partial} L(x, \alpha, \beta) = \inf_{x \in \partial} (F(x) + \alpha(r)T \mathbf{G}(x) + \beta(r)T \mathbf{H}(x)), \quad r \in [0, 1], \]

where $\mathbf{G}(x) = (G_1(x), G_2(x), ..., G_p(x))^T$, $\mathbf{H}(x) = (H_1(x), H_2(x), ..., H_q(x))^T$. The Lagrange dual problem (DFOP1') associated with the problem (FOP1') is

\[
\begin{align*}
\text{maximize} & \quad q'(\alpha, \beta) \\
\text{subject to} & \quad \alpha \geq 0,
\end{align*}
\]

We refer to $\alpha(r) = (\alpha_1(r), \alpha_2(r), ..., \alpha_p(r))^T \in R^{p+}$ and $\beta(r) = (\beta_1(r), \beta_2(r), ..., \beta_q(r))^T \in R^q$ as the Lagrange multiplier vectors containing parameter.

We now assume that the feasible set of (FOP1') $D' = \{ x \in M : G_k(x) \leq 0, H_k(x) = 0, k' = 1, 2, ..., p, s' = 1, 2, ..., q \} \subseteq K^2$, the real-valued functions $G_k(x)$ are convex and differentiable on $M$, and $H_k(x)$ are affine functions.

**Theorem 4.6. (KKT conditions)** Let $F$ be e-convex and e-differentiable on $M$. If $x^* = (x_1^*, x_2^*, ..., x_m^*)$ is an optimal solution to (FOP1') and $(\alpha^*(r), \beta^*(r))$ ($r \in [0, 1]$) is an optimal solution to (DFOP1') with strong duality, then $x^*$, $\alpha^*$, $\beta^*$ satisfy the following conditions:

\[
\begin{align*}
\frac{\partial F_i(x^*, r)}{\partial x_i} & + \frac{\partial F_i(x^*, r)}{\partial x_i^*} + \sum_{k' = 1}^p \alpha^*_{k'}(r) \frac{\partial G_k(x^*)}{\partial x_i} + \sum_{s' = 1}^q \beta^*_{s'}(r) \frac{\partial H_s(x^*)}{\partial x_i^*} = 0, \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., m, \\
\alpha^*_{k'}(r) G_k(x^*) & = 0, \quad k' = 1, 2, ..., p, \\
G_k(x^*) & \leq 0, \quad H_k(x^*) = 0, \quad k' = 1, 2, ..., p, \quad s' = 1, 2, ..., q, \\
\alpha^*_{k'}(r) & \geq 0, \quad k' = 1, 2, ..., p.
\end{align*}
\]

Conversely, if $x^*$, $\alpha^*$, $\beta^*$ are any points that satisfy the KKT conditions (4.12–4.15), then $x^*$ and $(\alpha^*, \beta^*)$ are primal solution and dual optimal solution, and strong duality holds.

**Proof.** \( \forall r \in [0, 1] \), we denote $\bar{F}_i(x, r) = F_i(x, r) + F_i^*(x, r)$, $i = 1, 2, ..., n$. Since $F$ is e-convex and e-differentiable on $M$, then the real-valued function $\bar{F}_i(x, r)$ and $F_i^*(x, r)$, $i = 1, 2, ..., n$, are convex and differentiable on $M$. Thus, $\forall r \in [0, 1]$, $\bar{F}_i(x, r)$ is convex on $M$ and differentiable at $x^*$, furthermore, we have

\[
\frac{\partial \bar{F}_i(x^*, r)}{\partial x_i^*} = \frac{\partial F_i(x^*, r)}{\partial x_i^*} + \frac{\partial F_i^*(x^*, r)}{\partial x_i^*}, \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., m.
\]

Therefore, $\forall r \in [0, 1]$, (4.12) of the KKT conditions is equivalent to

\[
\begin{align*}
\frac{\partial \bar{F}_i(x^*, r)}{\partial x_i^*} & + \sum_{k' = 1}^p \alpha^*_{k'}(r) \frac{\partial G_k(x^*)}{\partial x_i^*} + \sum_{s' = 1}^q \beta^*_{s'}(r) \frac{\partial H_s(x^*)}{\partial x_i^*} = 0, \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., m,
\end{align*}
\]

thus, (FOP1') is equivalent to the problem of which the objective function is the real-valued function containing parameter $\bar{F}_i(x, r)$ under the constraint conditions (4.13–4.16), and its Lagrangian is

\[
L(x, \alpha^*, \beta^*) = \inf_{x \in \partial} (\bar{F}_i(x, r) + \alpha(r)T \mathbf{G}(x) + \beta(r)T \mathbf{H}(x)), \quad r \in [0, 1].
\]
Now we prove the necessity. Since $x^\ast$ is an optimal solution to (FOP1), $x^\ast$ minimizes $L(x, \alpha^\ast, \beta^\ast)(r)$ over $x$, it follows that its gradient must vanish at $x^\ast$, we obtain (4.16) and equivalently have (4.12). Since strong duality holds, by (4.9), it is not difficult to obtain (4.13). Conditions (4.14) and (4.15) hold since they are constraint conditions of (FOP1) and (DFOP1), respectively.

Conversely, since $x^\ast, \alpha^\ast, \beta^\ast$ satisfy the KKT conditions (4.12-4.15), then $x^\ast, \alpha^\ast, \beta^\ast$ also satisfy the KKT conditions (4.13-4.16) with the objective function is the real-valued function $F_i(x, r)$, thus, $x^\ast$ is an optimal solution to the optimization problem and $(\alpha^\ast, \beta^\ast)$ is an optimal solution to its Lagrange dual problem, that is, $\forall x \in \text{int} M$ and $\forall (\alpha, \beta) \in (\mathbb{R}^n)^\ast$,

$$
F_i(x^\ast, r) \leq F_i(x, r), \quad i = 1, 2, \ldots, n, \quad (4.17)
$$
$$
\varphi'(\alpha^\ast, \beta^\ast) \geq \varphi'(\alpha, \beta). \quad (4.18)
$$

By reductio ad absurdum, suppose that $x^\ast$ is not an optimal solution of (FOP1'), then there exists $x' \in \text{int} M$ such that $F(x') < F(x^\ast)$. Let $C = \mathbb{R}^n \subseteq \mathbb{R}^n$, according to Definition 2.13, we have

$$
\int_0^1 rF_i(x', r) + F_i(x', r)dr < \int_0^1 rF_i(x^\ast, r) + F_i(x^\ast, r)dr, \quad i = 1, 2, \ldots, n,
$$

that is, $\int_0^1 rF_i(x', r)dr < \int_0^1 rF_i(x^\ast, r)dr$, $i = 1, 2, \ldots, n$, which is in contradiction to (4.17). Therefore, $x^\ast$ is an optimal solution to (FOP1). Equation (4.18) can be proved similarly. This completes the proof. □

**Theorem 4.7.** (Strong Duality Theorem) If Slater’s condition holds in e-convex problem (FOP1), i.e., there exists $\bar{x} \in \text{relint} D$ with $g_k(\bar{x}) < c$, then strong duality holds.

**Proof.** Consider the real-valued convex optimization problem (OP1)

$$
\begin{align*}
\text{minimize} & \quad F_i(x, r) \\
\text{subject to} & \quad G_k(x) \leq 0, \\
& \quad H_i(x) = 0,
\end{align*}
\quad (4.19)
$$

where $x \in M$, $r \in [0, 1]$, $i = 1, 2, \ldots, n$. For $\bar{x} \in \text{relint} D$, since $g_k(\bar{x}) < c \Rightarrow G_k(\bar{x}) < 0$, the convex optimization problem (OP1) satisfies Slater’s condition, therefore, for (OP1) and its Lagrange dual problem, the strong duality holds [25]. Since the feasible set of (FOP1) 4.1 is equivalent to the feasible set of (OP1) 4.19, then for (FOP1) and its Lagrange dual problem, the strong duality holds. □

**Definition 4.8.** Let $f : \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^l \rightarrow \mathbb{R}^n, X \subseteq \mathbb{R}^m, W \subseteq \mathbb{R}^l, Z \subseteq \mathbb{R}^l$. A vector triplet $(\bar{x}, \bar{a}, \bar{b})$ with $\bar{x} \in X, \bar{w} \in W, \bar{z} \in Z$ is said to be a saddle point of $f$ if

$$
f(\bar{x}, \bar{w}, \bar{z}) \leq_c f(\bar{x}, \bar{w}, \bar{z}) \leq_c f(x, \bar{w}, \bar{z}) \quad (4.20)
$$

holds for all $x \in X, w \in W, z \in Z$.

In other words, $x$ minimizes $f(x, \bar{w}, \bar{z})$ over $x \in X$ and $(\bar{w}, \bar{z})$ maximizes $f(x, \bar{w}, \bar{z})$ over $(\bar{w}, \bar{z}) \in (W, Z)$, i.e.,

$$
f(\bar{x}, \bar{w}, \bar{z}) = \inf_{x \in X} f(x, \bar{w}, \bar{z}), \quad f(\bar{x}, \bar{w}, \bar{z}) = \sup_{w \in W} f(\bar{x}, w, \bar{z}).
$$

**Theorem 4.9.** (Saddle Point Theorem) If $x^\ast$ and $(\alpha^\ast, \beta^\ast)$ are primal and dual optimal solutions to (FOP1) in which strong duality obtains, respectively, then $(x^\ast, \alpha^\ast, \beta^\ast)$ forms a saddle point for the Lagrangian. Conversely, if $(\bar{x}, \bar{a}, \bar{b})$ is a saddle point of the Lagrangian, then $\bar{x}$ is a primal optimal solution, $(\bar{a}, \bar{b})$ is a dual optimal solution, and strong duality holds.
Proof. For any $x \in R^n$,

\[
\sup_{a \geq 0} L(x, \alpha, \beta) = \sup_{a \geq 0} \left\{ F(x) + \sum_{k=1}^l a_k g_k(x) + \sum_{s=1}^t \beta_s h_s(x) \right\} = \begin{cases} F(x), & x \in D, \\ \infty, & \text{otherwise}. \end{cases} \tag{4.21}
\]

Indeed, if $x$ is a feasible solution, i.e., $g_k(x) \leq 0$, $h_s(x) = 0$, $k = 1, 2, \ldots, l$, $s = 1, 2, \ldots, t$, then the optimal choice $\alpha$ is $\alpha = 0$ for any $\beta$, thus $\sup_{a \geq 0} L(x, \alpha, \beta) = F(x)$. If $x$ is not a feasible solution, then there exists $g_k(x) > 0$ for some $k$, let $\alpha_m = 0$, $m \neq k$ and $\alpha_k \to \infty$, we obtain $\sup_{a \geq 0} L(x, \alpha, \beta) = \infty$.

Similarly, for any $\alpha \in R^l, \beta \in R^t$,

\[
\inf_{x \in D} L(x, \alpha, \beta) = \inf_{x \in D} \left\{ F(x) + \sum_{k=1}^l a_k g_k(x) + \sum_{s=1}^t \beta_s h_s(x) \right\} = \begin{cases} \inf_{x \in D} \left( F(x) + \sum_{k=1}^l a_k g_k(x) + \sum_{s=1}^t \beta_s h_s(x) \right), & a_k \geq 0, k = 1, 2, \ldots, l, \\ -\infty, & \text{otherwise}. \end{cases} \tag{4.22}
\]

Therefore, we can express the optimal value of the primal problem as

\[
p^* = \inf_{x \in D} \sup_{a \geq 0} L(x, \alpha, \beta). \tag{4.23}
\]

If $x^*$ is a primal optimal solution, then we obtain $p^* = \sup_{a \geq 0} L(x^*, \alpha, \beta)$. If the pair $(\alpha^*, \beta^*)$ is a dual optimal solution, then we have $d^* = \varphi(\alpha^*, \beta^*) = \sup \varphi(\alpha, \beta) = \sup \inf_{x \in D} L(x, \alpha, \beta) = \inf L(x, \alpha^*, \beta^*)$.

Since strong duality holds, then $p^* = d^*$, thus, we have

\[
\inf_{x \in D} L(x, \alpha^*, \beta^*) = L(x^*, \alpha^*, \beta^*) = \sup_{a \geq 0} L(x^*, \alpha, \beta), \quad \forall x \in D, \quad \alpha \in R^l, \quad \beta \in R^t,
\]

that is, $L(x^*, \alpha, \beta) \leq c L(x^*, \alpha^*, \beta^*) \leq c L(x, \alpha^*, \beta^*)$. Therefore, $(x^*, \alpha^*, \beta^*)$ is a saddle point for the Lagrangian.

Conversely, since $(\tilde{x}, \tilde{\alpha}, \tilde{\beta})$ is a saddle point of the Lagrangian, then

\[
\inf_{x \in D} L(x, \tilde{\alpha}, \tilde{\beta}) = \sup_{a \geq 0} L(\tilde{x}, \alpha, \beta), \quad \forall x \in D, \quad \alpha \in R^l, \quad \beta \in R^t.
\]

It follows from (4.21) and (4.22) that $\tilde{x} \in S$, $\tilde{\alpha} \in R^l$, $\tilde{\beta} \in R^t$, $L(x, \tilde{\alpha}, \tilde{\beta}) = F(\tilde{x})$. For any $x \in D$, we have $\tilde{a}_k g_k(x) \leq 0$, $k = 1, 2, \ldots, l$, and consequently $L(x, \tilde{\alpha}, \tilde{\beta}) \leq c F(x)$, where $a \in R^l$. Therefore,

\[
\inf_{x \in D} L(x, \tilde{\alpha}, \tilde{\beta}) \leq c \inf_{x \in D} F(x) = p^* \leq c F(\tilde{x}).
\]

It follows that $\inf L(x, \tilde{\alpha}, \tilde{\beta}) = p^* = F(\tilde{x})$. Thus, $\tilde{x}$ is a primal optimal solution, $(\tilde{\alpha}, \tilde{\beta})$ is a dual optimal solution, and strong duality holds.

Example 4.10. A company operates a training program for all new employees and without loss of generality. During the training program, we have to characterize the working state of one person. It is well known that the working state of one person changes with time. It is not appropriate to characterize the working efficiency of one person based only on their production speed because we also need to consider the quality of their products. If we denote the production speed and the qualification rate by $x_1$ and $x_2$, respectively, then the working state of the person can be characterized by a two-dimensional quantity $(x_1, x_2)$. However, the quantity is only an estimated quantity, then using a two-dimensional fuzzy number-valued function $u(x_1, x_2)$ to express the quantity is more appropriate than using a crisp two-dimensional quantity. Suppose that one person’s production speed and qualification rate are about 100 and 0.95,
respectively, then the person’s working state can be expressed by the two-dimensional fuzzy number-valued function \( u(x, y) \), which is defined as follows:

\[
\begin{align*}
\text{if } & 20x - 18, & 0.9 \leq x \leq 0.95, & 200x - 90 \leq x \leq 290 - 200x, \\
& -0.1x + 11, & 100 \leq x \leq 110, & 1.45 - 0.005x \leq x \leq 0.45 + 0.005x, \\
& -20x + 20, & 0.95 \leq x \leq 1, & 290 - 200x \leq x \leq 200x - 90, \\
& 0.1x - 9, & 90 \leq x \leq 100, & 0.45 + 0.005x \leq x \leq 1.45 - 0.005x, \\
& 0, & \text{otherwise}. 
\end{align*}
\]

Let \( C = R^2 \subseteq R^2 \), then \( u_r = [90 + 10r, 100 - 10r] \times [0.9 + 0.05r, 1 - 0.05r] \), \( r \in [0, 1] \). By (2.18), we have \( r(u) = (95, 0.95)^T \). Suppose that \( F(t)(x, y) = f(t) \cdot u(x, y) \), where \( f(t) = e^t - t^2, t \in [1, \infty) \). Consider the following fuzzy optimization problem

\[
\begin{align*}
\text{minimize} & \quad F(t) \\
\text{subject to} & \quad G(t) = t^2 - 36 \leq 0, \\
& \quad G_2(t) = -t + 1 \leq 0. 
\end{align*}
\]

Obviously, \( G_1(t) \) and \( G_2(t) \) are convex, and differentiable and continuous at \( t = 1 \). There exist \( a_1^1 = 0 \) and \( a_2^2 = e - 2 \), satisfy the condition of Theorem 4.6, therefore, \( t = 1 \) and the above fuzzy optimization problem.

**Example 4.11.** Let \( u_1, u_2 : R^1 \rightarrow E^1 \) be triangular fuzzy numbers, and \( u_1 = (x_1 - 1, x_1, x_1 + 1) \), \( u_2 = (x_2 - 1, x_2, x_2 + 1) \) and \( \bar{a} = (3, 4, 5) \) is a triangular fuzzy number. Consider the following fuzzy optimization problem

\[
\begin{align*}
\text{minimize} & \quad F(x, y) = u_1 \cdot u_1 + u_2 \cdot u_2 \\
\text{subject to} & \quad G(x, y) = u_1 + u_2 + (-\bar{a}) \pm 0, 
\end{align*}
\]

where \( x_1, x_2 \geq 1 \).

Since for \( u_1, u_2 \in E^1 \), and \( r \in [0, 1] \), \( [u \cdot v]^r = [\min[u^{-1}(r) \cdot v^{-1}(r), u^{-1}(r) \cdot v^{-1}(r), u^{-1}(r) \cdot v^{-1}(r), u^{-1}(r) \cdot v^{-1}(r)]] \\
\max[u^{-1}(r) \cdot v^{-1}(r), u^{-1}(r) \cdot v^{-1}(r), u^{-1}(r) \cdot v^{-1}(r), u^{-1}(r) \cdot v^{-1}(r)]] \\
= [\min[x_1 - 1 - r, x_1 + (1 - r)], \\
[x_2 - 1 - r, x_2 + (1 - r)], \\
[\bar{a}]^r = [4 - (1 - r), 4 + (1 - r)], \\
[F(x, y)]^r = [(x_1 - (1 - r))^2 + (x_2 - (1 - r))^2, (x_1 + (1 - r))^2 + (x_2 + (1 - r))^2], \\
[G(x, y)]^r = [x_1 - 1 - r + x_2 - (1 - r) - 3 - r, x_1 + (1 - r) + x_2 + (1 - r) - 5 + r] \\
= [x_1 + x_2 + r - 5, x_1 + x_2 - r - 3],
\]

thus, we obtain

\[
\begin{align*}
[\nabla F(x, y)]^r = \left((2(x_1 - 1 + r), 2(x_1 + 1 - r)), (2(x_2 - 1 + r), 2(x_2 + 1 - r))\right)^T, \\
[\nabla G(x, y)]^r = \left([1, 1], [1, 1]\right)^T, 
\end{align*}
\]

therefore, according to Theorem 4.6, the KKT conditions are as follows:

\[
\begin{align*}
& \begin{cases}
2x_1 - \alpha = 0, \\
2x_2 - \alpha = 0, \\
\alpha(x_1 + x_2 - 4) = 0, \\
\alpha \geq 0.
\end{cases}
\end{align*}
\]

If \( \alpha = 0 \), then \( x_1 = 0, x_2 = 0 \); if \( \alpha = 4 \), then \( x_1 = 2, x_2 = 2 \). Since \( x_1, x_2 \geq 1 \), then \( x = (2, 2)^T \) is an optimal solution to (4.24).
5 Discussion

Based on another fuzzy ordering $\leq$ proposed in [9], since the linear properties hold for $n$-dimensional fuzzy-number-valued functions, we can similarly investigate the Lagrange duality and the optimality conditions.

Proposition 5.1. [9,18,19,23] Suppose $u \in E^n$, then
(1) $u'(r, x + y) \leq u'(r, x) + u'(r, y)$,
(2) $u'(r, x) \leq \sup_{||y||} ||u'|| ||y||$, i.e., $u'(r, x)$ is bounded on $S^{n-1}$ for each fixed $r \in [0,1]$,
(3) $u'(r, x)$ is nonincreasing and left continuous in $r \in [0,1]$, right continuous at $r = 0$, for each fixed $x \in S^{n-1}$,
(4) $u'(r, x)$ is Lipschitz continuous in $x$, i.e., $|u'(r, x) - u'(r, y)| \leq (\sup_{\|y\|} \|u'|| ||y||) \|x - y\|,$
(5) if $u, v \in E^n, r \in [0,1]$, then $d([u]'', [v]'') = \sup_{x \in S^n-} |u'(r, x) - u'(r, x)|$,
(6) $(u + v)'(r, x) = u'(r, x) + v'(r, x)$,
(7) $(k u)'(r, x) = k u'(r, x)$, for any $k \geq 0$,
(8) $-u'(r, -x) \leq u'(r, x),$
(9) $(-u)'(r, x) = u'(r, -x)$.

Definition 5.2. Let $F : M \to E^n$ be a fuzzy-number-valued function, $x_0 = (x_0^0, x_0^1, \ldots, x_0^m) \in M$ and $h \in R$ with $(x_0^0, \ldots, x_0^j, h, \ldots, x_0^m) \in M$. If there exists $u_j \in E^n (j = 1, 2, \ldots, m)$ such that
$$\lim_{h \to 0} \frac{F(x_0^0, \ldots, x_0^j + h, \ldots, x_0^m)'(r, x) - F(x_0^0, \ldots, x_0^j, \ldots, x_0^m)'(r, x)}{h} = u_j'(r, x)$$
uniformly for $r \in [0,1]$ and $x \in S^{n-1}$, then we say that $F$ has the $j$th partial support-function-wise derivative (s-derivative for short) at $x_0$, denoted by $\frac{\partial F(x_0)}{\partial x^j} = u_j$. If all the partial s-derivatives at $x_0$ exist, then we say $F$ is support-function-wise differentiable (s-differentiable for short) at $x_0$, denoted by $F'(x_0)$.

If $F$ is s-differentiable at any point of $M$, then $F$ is said to be s-differentiable on $M$. The fuzzy vector $(u_1, u_2, \ldots, u_m) \in (E^n)^m$ is said to be the support-function-wise gradient of $F$ at $x_0$, denoted by $\nabla_x F(x_0)$,
$$\nabla_x F(x_0) = (u_1, u_2, \ldots, u_m)^T = \left( \frac{\partial F(x_0)}{\partial x^0}, \frac{\partial F(x_0)}{\partial x^1}, \ldots, \frac{\partial F(x_0)}{\partial x^m} \right)^T.$$

Theorem 5.3. Let $F : M \to E^n$ be s-differentiable at $x_0 \in M$. If there exists $\delta > 0$ such that $x_0 + h \in M$ and $g$-difference $F(x_0 + h) \circ g F(x_0)$ exists for any $h < \delta$, then $F$ is g-differentiable at $x_0$ and we have
$$F'(x_0)'(r, x) = \begin{cases} (1) \sup_{\beta \leq r} F'(x_0)'(\beta, x), \\ (2) \sup_{\beta \leq r} (-F'(x_0)'(\beta, x)). \end{cases} \quad (5.1)$$

The proof is similar to the proof of Theorem 4.2 in the study of Hai et al. [9].

Definition 5.4. Let $M \subseteq R^m$ be a convex set and $F : M \to E^n$ be a fuzzy-number-valued function. If for any $x_1, x_2 \in M$ and $\lambda \in [0,1]$,
$$F(\lambda x_1 + (1 - \lambda) x_2)'(r, x) \leq \lambda F(x_1)'(r, x) + (1 - \lambda) F(x_2)'(r, x)$$
uniformly for $r \in [0,1]$ and $x \in S^{n-1}$, then $F$ is said to be support-function-wise convex (s-convex) on $M$.

Theorem 5.5. Let $M \subseteq R^m$ be a convex set and $F : M \to E^n$ be an s-differentiable fuzzy-number-valued function on $M$. $F$ is s-convex on $M$ if and only if for any $x_1, x_2 \in M$,
$$F(x_1)'(r, x) - F(x_2)'(r, x) \geq \nabla_x F(x_0)'(r, x)^T(x_1 - x_2),$$
uniformly for $r \in [0,1]$ and $x \in S^{n-1}$. 
Note that for \( u = (u_1, u_2, \ldots, u_n)^T \in (E^n)^n, u^*(r, x)^T = (u_1^*(r, x), u_2^*(r, x), \ldots, u_n^*(r, x))^T \). The proof is similar to the proof of Theorem 6.1.2 in the study of Mangasarian [24].

**Definition 5.6.** [9] For \( u, v \in E^n \), we say that \( u \leq v \) if for any \( r \in [0, 1] \) and \( x \in S^{n-1} \), \( u^*(r, x) \leq v^*(r, x) \). We say that \( u \prec v \) if \( u \leq v \) and there exists \( r_0 \in [0, 1] \) with \( u^*(r_0, x) < v^*(r_0, x) \), or there exists \( x_0 \in S^{n-1} \) with \( u^*(r, x_0) < v^*(r, x_0) \).

Let \( M \subseteq R^m \) be a convex set, \( F(x), g_k(x), h_p(x), k = 1, 2, \ldots, l, p = 1, 2, \ldots, q \), be \( s \)-convex and \( s \)-differentiable fuzzy-number-valued functions on \( M \). We consider the following optimization problem (FOP2):

\[
\begin{align*}
\text{minimize} & \quad F(x) \\
\text{subject to} & \quad g_k(x) \leq 0, \quad k = 1, 2, \ldots, l, \\
& \quad h_p(x) = 0, \quad p = 1, 2, \ldots, q.
\end{align*}
\]

\( S = \{x \in M : g_k(x) \leq 0, h_p(x) = 0, \quad k = 1, 2, \ldots, l, \quad p = 1, 2, \ldots, q\} \) is said to be the feasible set of (FOP2). Let \( x^* \in S \), if \( F(x^*) \leq F(x) \) for each \( x \in S \), then \( x^* \) is said to be an optimal solution to (FOP2).

We define

\[
\varphi(\alpha, \beta) = \inf_{x \in S} L(x, \alpha, \beta) = \inf_{x \in S} (F(x) + \alpha^T G(x) + \beta^T H(x)),
\]

where \( \alpha = (a_1, a_2, \ldots, a_l)^T \in R^l, \beta = (\beta_1, \beta_2, \ldots, \beta_q)^T \in R^q, G(x) = (g_1(x), g_2(x), \ldots, g_l(x))^T, H(x) = (h_1(x), h_2(x), \ldots, h_q(x))^T \).

The Lagrange dual problem (DFOP2) associated with (FOP2) is

\[
\begin{align*}
\text{maximize} & \quad \varphi(\alpha, \beta) \\
\text{subject to} & \quad \alpha \geq 0.
\end{align*}
\]

**Theorem 5.7.** (Weak Duality Theorem) If \( x \) and \( (\alpha, \beta) \) are feasible solutions to the primal problem (FOP2) and the Lagrange dual problem (DFOP2), respectively, then weak duality holds: \( F(x) \geq \varphi(\alpha, \beta) \).

**Theorem 5.8.** (KKT Conditions) Let \( F \) be \( s \)-convex and \( s \)-differentiable on \( M \). If \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \) is an optimal solution to (FOP2) and \( (\tilde{\alpha}, \tilde{\beta}) \) is an optimal solution to (DFOP2) with strong duality, then \( \tilde{x}, \tilde{\alpha}, \tilde{\beta} \) satisfy the following conditions:

\[
\begin{align*}
\frac{\partial F(\tilde{x})}{\partial \tilde{x}_j} + \sum_{k=1}^l \tilde{\alpha}_k \frac{\partial (g_k)(\tilde{x})}{\partial \tilde{x}_j} + \sum_{p=1}^q \tilde{\beta}_p \frac{\partial (h_p)(\tilde{x})}{\partial \tilde{x}_j} &= 0, \quad j = 1, 2, \ldots, m, \\
\tilde{\alpha}_k g_k(\tilde{x}) &= 0, \quad k = 1, 2, \ldots, l, \\
g_k(\tilde{x}) &\leq 0, \quad h_p(\tilde{x}) = 0, \quad k = 1, 2, \ldots, l, \quad p = 1, 2, \ldots, q, \\
\tilde{\alpha}_k &\geq 0, \quad k = 1, 2, \ldots, l.
\end{align*}
\]

Conversely, if \( \tilde{x}, \tilde{\alpha}, \tilde{\beta} \) are any points that satisfy the KKT conditions (5.4)–(5.7), then \( \tilde{x} \) and \( (\tilde{\alpha}, \tilde{\beta}) \) are primal solution and dual optimal solution, and strong duality holds.

**Proof.** If \( \tilde{x} \) is an optimal solution to (FOP2) and \( (\tilde{\alpha}, \tilde{\beta}) \) is an optimal solution to (DFOP2) with strong duality, since under the ordering \( \leq_s \), the support function of a fuzzy number is a real-valued function, then analogous to Theorem 4.6, it is not difficult to prove that \( \tilde{x}, \tilde{\alpha}, \tilde{\beta} \) satisfy conditions (5.4)–(5.7).

Conversely, for any \( x \in S \), we have

\[
g_k(x) \leq 0, \quad h_p(x) = 0, \quad k = 1, 2, \ldots, l, \quad p = 1, 2, \ldots, q.
\]
If \( \bar{x}, \bar{a}, \bar{\beta} \) are any points that satisfy the KKT conditions (5.4)–(5.7), then by Theorem 5.5, (5.4)–(5.7) and (5.8),

\[
F(x)^*(r, t) - F(\bar{x})^*(r, t) \\
\geq \left( \frac{\partial F_1(\bar{x})}{\partial \bar{x}_1}, \frac{\partial F_1(\bar{x})}{\partial \bar{x}_2}, \ldots, \frac{\partial F_1(\bar{x})}{\partial \bar{x}_m} \right)^T (x - \bar{x}) \\
= - \left( \sum_{k=1}^q \bar{a}_k \frac{\partial (h_k)(\bar{x})}{\partial \bar{x}_1} \right)^* (r, t) + \sum_{p=1}^q \bar{\beta}_p \frac{\partial (h_p)(\bar{x})}{\partial \bar{x}_1} (r, t),
\]

is an optimal solution to the primal problem. If Slater’s condition holds, i.e., there exists \( \bar{x} \in \text{relint } D \) with \( g_k(\bar{x}) < 0 \), then strong duality holds.

If \( \bar{x}, \bar{a}, \bar{\beta} \) are any points that satisfy the KKT conditions (5.4)–(5.7), then by Theorem 5.5, (5.4)–(5.7) and (5.8),

\[
F(x)^*(r, t) - F(\bar{x})^*(r, t) \\
\geq \left( \frac{\partial F_1(\bar{x})}{\partial \bar{x}_1}, \frac{\partial F_1(\bar{x})}{\partial \bar{x}_2}, \ldots, \frac{\partial F_1(\bar{x})}{\partial \bar{x}_m} \right)^T (x - \bar{x}) \\
= - \left( \sum_{k=1}^q \bar{a}_k \frac{\partial (h_k)(\bar{x})}{\partial \bar{x}_1} \right)^* (r, t) + \sum_{p=1}^q \bar{\beta}_p \frac{\partial (h_p)(\bar{x})}{\partial \bar{x}_1} (r, t),
\]

are any points that satisfy the KKT conditions (5.4)–(5.7), then strong duality holds.

Theorem 5.9. (Strong Duality Theorem) For s-convex problem (FOP2), if Slater’s condition holds, i.e., there exists \( \bar{x} \in \text{relint } D \) with \( g_k(\bar{x}) < 0 \), then strong duality holds.

Theorem 5.9. (Strong Duality Theorem) For s-convex problem (FOP2), if Slater’s condition holds, i.e., there exists \( \bar{x} \in \text{relint } D \) with \( g_k(\bar{x}) < 0 \), then strong duality holds.

Proof. For any \( r \in [0, 1] \), \( t \in S^{n-1} \), consider the real-valued convex optimization problem (OP2)

\[
\begin{align*}
\text{minimize} & \quad F(x)^*(r, t) \\
\text{subject to} & \quad g_k(x)^*(r, t) \leq 0, \\
& \quad h_p(x)^*(r, t) = 0,
\end{align*}
\]

where \( x \in M \), the real-valued functions \( F(x)^*(r, t) \), \( g_k(x)^*(r, t) \) are convex and differentiable on \( M \), and \( H_p(x)^*(r, t) \) are affine functions. For \( \bar{x} \in \text{relint } D \), since \( g_k(\bar{x}) < 0 \), \( g_k(x)^*(r, t)(\bar{x}) < 0 \), the convex optimization problem (OP2) satisfies Slater’s condition, therefore, for (OP2) and its Lagrange dual problem, the
strong duality holds (see [25]). Since the feasible set of (FOP2) (5.2) is equivalent to the feasible set of (OP2) (5.9), then for (FOP2) and its Lagrange dual (DOP2) (5.3), the strong duality holds.

**Theorem 5.10.** (Saddle Point Theorem) If \( x^* \) and \((\alpha^*, \beta^*)\) are primal and dual optimal solutions to (FOP2) in which strong duality obtains, respectively, then \((x^*, \alpha^*, \beta^*)\) forms a saddle point for the Lagrangian. Conversely, if \((\tilde{x}, \tilde{\alpha}, \tilde{\beta})\) is a saddle point of the Lagrangian, then \( \tilde{x} \) is a primal optimal solution, \((\tilde{\alpha}, \tilde{\beta})\) is a dual optimal solution, and strong duality holds.

6 Conclusion

We present the concepts of generalized derivative, directional generalized derivative, subdifferential and conjugate for \( n \)-dimensional fuzzy-number-valued functions from \( R^m \) to \( E^n \) and give the characteristic theorems of generalized derivative and directional generalized. We examine the relations among generalized derivative, directional generalized derivative, subdifferential and convexity for \( n \)-dimensional fuzzy-number-valued functions. Additionally, under two kinds of partial orderings defined on the set of all \( n \)-dimensional fuzzy numbers, we discuss the duality theorems and saddle point optimality criteria in fuzzy optimization problems with constraints. The next step for the continuation of the research direction proposed here is to investigate the fuzzy optimization problems under non-differentiable case.

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