NOTES ON GLOBAL STRESS AND HYPER-STRESS THEORIES

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Abstract. The fundamental ideas and tools of the global geometric formulation of stress and hyper-stress theory of continuum mechanics are introduced. The proposed framework is the infinite dimensional counterpart of statics of systems having finite number of degrees of freedom, as viewed in the geometric approach to analytical mechanics. For continuum mechanics, the configuration space is the manifold of embeddings of a body manifold into the space manifold. Generalized velocity fields are viewed as elements of the tangent bundle of the configuration space and forces are continuous linear functionals defined on tangent vectors, elements of the cotangent bundle. It is shown, in particular, that a natural choice of topology on the configuration space, implies that force functionals may be represented by objects that generalize the stresses of traditional continuum mechanics.

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1. Introduction

These notes provide an introduction to the fundamentals of global analytic continuum mechanics as developed in [ES80, Seg81, MH94, Seg86b, Seg86a, Seg16]. The terminology "global analytic" is used to imply that the formulation is based on the notion of a configuration space of the mechanical system as in analytic classical mechanics. As such, this review is complementary to that of [Seg13], which describes continuum mechanics on differentiable manifolds using a generalization of the Cauchy approach to flux and stress theory.

The setting for the basics of kinematics and statics is quite simple and provides an elegant geometric picture of mechanics. Consider the configuration space $\mathcal{Q}$ containing all admissible configuration of the system. Then, construct a differentiable manifold structure on the configuration space, define generalized (or virtual) velocities as tangent vectors, elements of $T\mathcal{Q}$, and define generalized forces as linear functions defined on the space of generalized velocities, elements of $T^*\mathcal{Q}$. The result of the action of a generalized force $F$ on a generalized velocity $w$ is interpreted as mechanical power. Thus, such a structure may be used to encompass both classical mechanics of mass particles and rigid bodies as well as continuum mechanics. The difference is that the configuration space for continuum mechanics and other field theories is infinite dimensional.

It is well known that the transition from the mechanics of mass particles and rigid bodies to continuum mechanics is not straightforward and requires the introduction of new notions and assumptions. The global analytic formulation explains this observation as follows. Linear functions, and forces in particular, are identically continuous when defined on a finite dimensional space. However, in the infinite dimensional situation, one has to specify exactly the topology on the infinite dimensional space of generalized velocities with respect to which forces should be continuous. Then, the properties of force functionals are deduced from the continuity requirement through a representation theorem. In other words, the properties of forces follow directly from the kinematics of the theory.

For continuum mechanics of a body $\mathcal{X}$ in space $\delta$, the basic kinematic assumption is traditionally referred to as the axiom of material impenetrability. A configuration of the body in space is specified by a mapping $\kappa : \mathcal{X} \rightarrow \delta$ which is assumed to be injective and of full rank at each point—an embedding. Hence, the configuration space for continuum mechanics should be the collection of embeddings of the body manifold into the space manifold. It turns out that the $C^1$-topology is the natural one to use in order to endow the collection of embeddings with a differentiable structure of a Banach manifold. The $C^r$-topologies for $r > 1$ are admissible also.

It follows that forces are continuous linear functionals on the space of vector fields over the body of class $C^r$, $r \geq 1$, equipped with the $C^r$-topology with a special role for the case $r = 1$. A standard procedure based on the Hahn–Banach
theorem leads to a representation theorem for a force functional in terms of
vector valued measures.

The measures representing a force generalize the stress and hyper-stress ob-
jects of continuum mechanics. On the one hand, as expected, a stress measure
is not determined uniquely by a force. This is in accordance with the inherent
static indeterminacy of continuum mechanics and it follows directly from the
representation procedure. While the case \( r = 1 \) leads to continuum mechanics
of order one, the cases \( r > 1 \) are extensions of higher order continuum me-
chanics. Thus, an existence theorem for hyper-stresses follows naturally. The
relation between a force and a representing stress object is a generalization of
the principle of virtual work in continuum mechanics and so it is analogous to
the equilibrium equations.

The representation of forces by stress measures is significant for two reasons.
First, the existence of the stress object as well as the corresponding equilibrium
condition are obtained for stress distributions that may be as singular as Radon
measures. In addition, while force functionals cannot be restricted to subsets of a
body, measures may be restricted to subsets. This reflects a fundamental feature
of stress distributions—they induce force systems on bodies. It is emphasized
that in no further assumptions of mathematical or physical nature are made.

The framework described above applies to continuum mechanics on general
differentiable manifolds without any additional structure such as a Riemannian
metric or a connection. The body manifold is assumed here to be a compact
manifold with corners. However, as described in [Mic20], it is now possible
to extend the applicability of this framework to a wider class of geometric ob-
ject—Whitney manifold germs.

Starting with the introduction of notation used in the manuscript in Section
2, we continue with the construction of the manifold structure on the space
of embeddings. Thus, Section 3 describes the Banachable vector spaces used
to construct the infinite dimensional manifold structure on the configuration
space and Section 4 is concerned with the Banach manifold structure on the set
of \( C^r \)-sections of a fiber bundle \( \xi : \mathcal{Y} \to \mathcal{X} \). This includes, as a special case,
the space \( C^r(\mathcal{X}, \mathcal{S}) \) of \( C^r \)-mappings of the body into space and also provides
a natural extension to continuum mechanics of generalized media. After describ-
ing the topology in \( C^r(\mathcal{X}, \mathcal{S}) \) in Section 5, we show in Section 6 that the set of
embeddings is open in \( C^r(\mathcal{X}, \mathcal{S}) \), \( r \geq 1 \). As such, it is a Banach manifold also
and the tangent bundle is inherited from that of \( C^r(\mathcal{X}, \mathcal{S}) \). In Section 7 we outline
the framework for the suggested force and stress theory as described roughly
above. Sections 8, 9 and 10 introduce relevant spaces of linear functionals on
manifolds, and present some of their properties. These include some standard
classes of functionals such as de Rham currents and Schwartz distributions on
manifolds. The representation theorem of forces by stress measures in consid-
ered in Section 11. Section 12 discusses the natural situation of simple forces
and stress, that is, the case \( r = 1 \).
2. Notation and Preliminaries

2.1. General notation. A collection of indices \( i_1 \cdots i_k \), \( i_r = 1, \ldots, n \) will be represented as a multi-index \( I \) and we will write \( |I| = k \), the length of the multi-index. In general, multi-indices will be denoted by upper-case roman letters and the associated indices will be denoted by the corresponding lower case letters. Thus, a generic element in a \( k \)-multilinear mapping \( A \in \mathbb{R}^k \) is given in terms of the array \( (A_I) \), \( |I| = k \). In what follows, we will use the summation convention for repeated indices as well as repeated multi-indices. Whenever the syntax is violated, e.g., when a multi-index appears more than twice in a term, it is understood that summation does not apply.

A multi-index \( I \) induces a sequence \( (I_1, \ldots, I_n) \) in which \( I_r \) is the number of times the index \( r \) appears in the sequence \( i_1 \cdots i_k \). Thus, \( |I| = \sum I_r \). Multi-indices may be concatenated naturally such that \( |I| = |I| + |J| \).

In case an array \( A \), \( A \in \mathbb{R}^k \), is symmetric, the independent components of the array may be listed as \( A_{i_1 \cdots i_r} \) with \( i_r \leq i_{r+1} \). A non-decreasing multi-index, that is, a multi-index that satisfies the condition \( i_l \leq i_{l+1} \), will be denoted by boldface, upper-case roman characters so that a symmetric tensor \( A \) is represented by the components \( (A_I) \), \( |I| = k \). In particular, for a function \( u : \mathbb{R}^n \to \mathbb{R} \), a particular partial derivative of order \( k \) is written in the form

\[
\partial_I u := \frac{\partial^{|I|} u}{(\partial X^1)^{I_1} \cdots (\partial X^n)^{I_n}},
\]

where \( I \) is a non-decreasing multi-index with \( |I| = k \).

The notation \( \partial I = \partial / \partial X^I \) will be used for both the partial derivatives in \( \mathbb{R}^n \) and for the elements of the basis of the tangent space \( T_X \mathcal{X} \) of a manifold \( \mathcal{X} \) at a point \( X \). The corresponding dual basis for \( T^*_X \mathcal{X} \) will be denoted by \( \{dX^I\} \).

Greek letters, \( \lambda, \mu, \nu \), will be used for strictly increasing multi-indices used in the representation of alternating tensors and forms, e.g.,

\[
\omega = \omega_\lambda dX^\lambda := \omega_{\lambda_1, \ldots, \lambda_p} dX^{\lambda_1} \wedge \cdots \wedge dX^{\lambda_p}.
\]

Given a strictly increasing multi-index \( \lambda \) with \( |\lambda| = p \), we will denote the strictly increasing \( (n-p) \)-multi-index that complements \( \lambda \) to \( 1, \ldots, n \) by \( \hat{\lambda} \). In this context, \( \hat{\mu}, \hat{\nu}, \) etc. will indicate generic increasing \( (n-p) \)-multi-indices. The Levi-Civita symbol will be denoted as \( \epsilon_I \) or \( \epsilon^I \), \( |I| = n \) so that for example \( dX^\lambda \wedge dX^{\hat{\lambda}} = \epsilon^{\lambda \hat{\lambda}} dX \), where we also set

\[
\partial_X := \partial_1 \wedge \cdots \wedge \partial_n, \quad dX := dX^1 \wedge \cdots \wedge dX^n.
\]

(Note that we view \( \lambda \) and \( \hat{\lambda} \) as two distinct indices so summation is not implied in a term such as \( dX^\lambda \wedge dX^{\hat{\lambda}} \).)
The following identifications will be implied for tensor products of vector spaces and vector bundles

\[ V^* \otimes U \cong L(V, U), \quad (V \otimes U)^* \cong V^* \otimes U^*. \] (2.4)

For vector bundles \( V \) and \( U \) over a manifold \( X \), let \( S \) be a section of \( V^* \otimes U \) and \( \chi \) a section of \( V \). The notation \( S \cdot \chi \) is used for the section of \( U \) given by

\[ (S \cdot \chi)(X) = S(X)(\chi(X)). \] (2.5)

2.2. Manifolds with corners. Our basic object will be a fiber bundle \( \xi : Y \to X \) where \( X \) is assumed to be an oriented manifold with corners. We recall (e.g., [DH73, Mic80, Mel96, Lee02, MRD08]) that an \( n \)-dimensional manifold with corners is a manifold whose charts assume values in the \( n \)-quadrant of \( \mathbb{R}^n \), that is, in

\[ \mathbb{R}^n_+ := \{X \in \mathbb{R}^n \mid X^i \geq 0, \ i = 1, \ldots, n\}. \] (2.6)

In the construction of the manifold structure, it is understood that a function defined on a quadrant is said to be differentiable if it is the restriction to the quadrant of a differentiable function defined on \( \mathbb{R}^n \). If \( \mathcal{X} \) is an \( n \)-dimensional manifold with corners, a subset \( \mathcal{Z} \subset \mathcal{X} \) is defined to be a \( k \)-dimensional, \( k \leq n \), submanifold with corners of \( \mathcal{X} \) if for any \( Z \in \mathcal{Z} \) there is a chart \((U, \varphi), Z \in U\), such that \( \varphi(\mathcal{Z} \cap U) \subset \{X \in \mathbb{R}^n_+ \mid X^i = 0, \ k < l \leq n\} \).

With an appropriate natural definition of the integral of an \((n-1)\)-form over the boundary of a manifold with corners, Stokes’s theorem holds for manifolds with corners (see [Lee02 pp. 363–370]).

Relevant to the subject at hand is the following result (see [DH73, Mic80, Mel96, Mic20]). Every \( n \)-dimensional manifold with corners \( \mathcal{X} \) is a submanifold with corners of a manifold \( \tilde{\mathcal{X}} \) without boundary of the same dimension. In addition, if \( \mathcal{X} \) is compact, it can be embedded as a submanifold with corners in a compact manifold without boundary \( \tilde{\mathcal{Y}} \) of the same dimension [Mel96 pp. 124–26]. Furthermore, \( C^k \)-forms defined on \( \mathcal{X} \), may be extended continuously and linearly to forms defined on \( \tilde{\mathcal{X}} \). Such a manifold \( \tilde{\mathcal{X}} \) is referred to as an extension of \( \mathcal{X} \). Each smooth vector bundle over \( \mathcal{X} \) extends to a smooth vector bundle over \( \tilde{\mathcal{X}} \). Each immersion (embedding) of \( \mathcal{X} \) into a smooth manifold \( \mathcal{Y} \) without boundary is the restriction of an immersion (embedding) of \( \tilde{\mathcal{X}} \) into \( \mathcal{Y} \).

It is emphasized that manifold with corners do not model some basic geometric shapes such as a pyramid with a rectangular base or a cone. However, much of material presented in this review is valid for a class of much more general objects, Whitney manifold germs as presented in [Mic20].
2.3. **Bundles, jets and iterated jets.** We will consider a fiber bundle \( \xi : Y \to X \), where \( X \) is \( n \)-dimensional and the typical fiber is \( m \)-dimensional. The projection \( \xi \) is represented locally by \((X^i, y^\alpha) \mapsto (X^i)\), \( i = 1, \ldots, n, \alpha = 1, \ldots, m \). Let
\[
T_\xi : T_Y \to T_X
\]
be the tangent mapping represented locally by
\[
(X^i, y^\alpha, \dot{X}^i, \dot{y}^\beta) \mapsto (X^i, \dot{X}^i). \tag{2.7}
\]
The *vertical sub-bundle* \( VY \) of \( TY \) is the kernel of \( T\xi \). An element \( v \in VY \) is represented locally as \((X^i, y^\alpha, 0, \dot{y}^\beta)\). With some abuse of notation, we will write both \( \tau : T_Y \to Y \) and \( \tau : VY \to Y \). For \( v \in VY \) with \( \tau(v) = y \) and \( \xi(y) = X \), we may view \( v \) as an element of \( T_y(Y_X) = T_y(\xi^{-1}(X)) \). In other words, elements of the vertical sub-bundle are tangent vectors to \( Y \) that are tangent to the fibers.

Let \( \kappa : X \to Y \) be a section and let
\[
\kappa^* \tau : \kappa^* VY \to X \tag{2.9}
\]
be the pullback of the vertical sub-bundle. Then, we may identify \( \kappa^* VY \) with the restriction of the vertical bundle to Image \( \kappa \).

2.3.1. **Jets.** We will denote by \( \xi^r : J^r(X, Y) \to X \) the corresponding \( r \)-jet bundle of \( \xi \). When no ambiguity may occur, we will often use the simpler notation \( \xi^r : J^r Y \to X \) and refer to a section of \( \xi^r \) as a section of \( J^r Y \). One has the additional natural projections \( \xi^r_i : J^r(X, Y) \to J^i(X, Y) \), \( i < r \), and in particular \( \xi^r_0 : J^r(X, Y) \to Y = J^0 Y \) [Sau89]. The jet extension mapping associates with a \( C^r \)-section, \( \kappa \), of \( \xi \), a continuous section \( J^r \kappa \) of the jet bundle \( \xi^r \).

Let \( \kappa : X \to Y \) be a section of \( \xi \) which is represented locally by
\[
X \mapsto (X, \kappa(X)), \quad \text{or,} \quad X^i \mapsto (X^i, \kappa^i(X^i)), \quad i = 1, \ldots, n, \alpha = 1, \ldots, m. \tag{2.10}
\]
Then, denoting the \( k \)-th derivative of \( \kappa \) by \( D^k \), a local representative of \( J^r \kappa \) is of the form
\[
X \mapsto (X, \kappa(X), \ldots, D^k \kappa(X)), \quad \text{or,} \quad X^i \mapsto (X^i, \kappa^i(X^i)), \quad 0 \leq |I| \leq r. \tag{2.11}
\]
Accordingly, an element \( A \in J^r(X, Y) \) is represented locally by the coordinates
\[
(X^i, A^i_I), \quad 0 \leq |I| \leq r. \tag{2.12}
\]

2.3.2. **Iterated (non-holonomic) jets.** Completely non-holonomic jets for the fiber bundle \( \xi : Y \to X \) are defined inductively as follows. Firstly, one defines the fiber bundles
\[
\hat{J}^0(X, Y) = Y, \quad \hat{J}^1(X, Y) := J^1(X, Y), \tag{2.13}
\]
and projections
\[
\hat{\xi}^1 = \xi^1 : \hat{J}(X, Y) \to X, \quad \hat{\xi}^1_{\xi^0} = \xi^1 : \hat{J}^1(X, Y) \to Y. \tag{2.14}
\]
Then, we define the **iterated r-jet bundle** as
\[
\hat{j}^r(\mathcal{X}, \mathcal{Y}) := j^1(\mathcal{X}, \hat{j}^{r-1}(\mathcal{X}, \mathcal{Y})),
\]
with projection
\[
\hat{\xi}^r = \hat{\xi}^{r-1} \circ \xi^{1,r}_{r-1} : \hat{j}^r(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathcal{X},
\]
where,
\[
\xi^{1,r}_{r-1} : j^r(\mathcal{X}, \mathcal{Y}) = j^1(\mathcal{X}, j^{r-1}(\mathcal{X}, \mathcal{Y})) \longrightarrow \hat{j}^{r-1}(\mathcal{X}, \mathcal{Y}).
\]
By induction, \( \hat{\xi}^r : \hat{j}^r(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{X} \) is a well defined fiber bundle..

When the projections \( \xi^{1,r}_{r-1} \) are used inductively \( l \)-times, we obtain a projection
\[
\hat{\xi}^r_{r-1} : j^r(\mathcal{X}, \mathcal{Y}) \longrightarrow \hat{j}^{r-l}(\mathcal{X}, \mathcal{Y}).
\]
Let \( \kappa : \mathcal{X} \rightarrow \mathcal{Y} \) be a \( C^l \)-section of \( \xi \). The iterated jet extension mapping
\[
j^r : C^r(\hat{\xi}) \longrightarrow C^0(\hat{\xi}^r)
\]
is naturally defined by
\[
j^1 = j^1 : C^1(\hat{\xi}) \longrightarrow C^{l-1}(\hat{\xi}^1), \quad \text{and} \quad j^r = j^1 \circ j^{r-1}.
\]
Note that we use \( j^1 \) here as a generic jet extension mapping, omitting the indication of the domain.

There is a natural inclusion
\[
l' : j^l(\mathcal{X}, \mathcal{Y}) \longrightarrow \hat{j}^l(\mathcal{X}, \mathcal{Y}), \quad \text{given by} \quad j^l \kappa(X) \longmapsto j^l \kappa(X).
\]
Let \( \pi : W \rightarrow \mathcal{X} \) be a vector bundle, then \( \hat{\pi}^1 = \pi^1 : \hat{j}^1 W = j^1 W \rightarrow \mathcal{X} \) is a vector bundle. Continuing inductively,
\[
\hat{\pi}^r : \hat{j}^r W \longrightarrow \mathcal{X}
\]
is a vector bundle. In this case, the inclusion \( l' : j^l \pi \rightarrow \hat{j}^l \pi \) is linear. Naturally, elements in the image of \( l' \) are referred to as **holonomic**.

### 2.3.3. Local representation of iterated jets

The local representatives of iterated jets are also constructed inductively. Hence, at each step, \( G \), to which we refer as **generation**, the number of arrays is multiplied. Hence powers of two are naturally used below. Thus, it is proposed to use multi-indices of the form \( I_p \), where \( p \), \( q \), etc. are binary numbers that enumerate the various arrays included in the representation. For example, a typical element of \( \hat{j}^3(\mathcal{X}, \mathcal{Y}) \), in the form
\[
(X^I; y_0^\alpha; y_1^\beta_1; y_2^\beta_2; y_3^\beta_3; y_4^\beta_4; y_5^\beta_5; y_6^\beta_6; y_7^\beta_7)
\]
is written as
\[
(X^I; y_0^\beta_0; y_1^\beta_1; y_2^\beta_2; y_3^\beta_3; y_4^\beta_4; y_5^\beta_5; y_6^\beta_6; y_7^\beta_7; y_{10}^\beta_{10}; y_{11}^\beta_{11});
\]
and for short
\[
(X^I; y_{10}^\beta_{10}), \quad \text{for all} \quad p \quad \text{with} \quad 0 \leq G_p \leq 3.
\]
Here, \( G_p \) is the generation where the \( p \)-th array appears and it is given by
\[
G_p = \lfloor \log_2 p \rfloor + 1,
\]
where \( \lfloor \log_2 p \rfloor \) denotes the integer part of \( \log_2 p \). In (2.23, 2.24) the generations are separated by semicolons. As indicated in the example above, with each \( p \) we associate a multi-index \( I_p = i_1 \ldots i_p \) as follows. For each binary digit 1 in \( p \) there is an index \( i_l \), \( l = 1, \ldots, p \). Thus, the total number of digits \( l \) in \( p \), which is denoted by \( |p| \), is the total number of indices, \( p \), in \( I_p \). In other words, the length, \( |I_p| \), of the induced multi-index \( I_p \) satisfies

\[
|I_p| = p = |p|.
\]

Note also that the expression \( \beta_p \), is not a multi-index since we use upper-case letters to denote multi-indices. Here, the subscript \( p \) serves for the enumeration of the \( \beta \) indices. If no ambiguity may arise, we will often make the notation somewhat shorter and write \( y_{I_p}^{\beta_p} \) for \( y_{I_p}^{p\beta_p} \). Continuing by induction, let a section \( A \) of \( \hat{j}^{r-1}(\mathcal{X}, \mathcal{Y}) \) be represented locally by \( (X^j; y_{I_p}^{p\alpha p}(X^j)) \), \( G_p \leq r - 1 \). Then, its 1-jet extension, a section of \( \hat{j}^r(\mathcal{X}, \mathcal{Y}) \), is of the form

\[
(X^j; y_{I_p}^{p\alpha p}(X^j); y_{I_p}^{p\beta_p}(X^j)), \quad G_p \leq r - 1,
\]

or equivalently,

\[
(X^j; y_{I_p}^{p\alpha p}(X^j); y_{I_p}^{1\alpha p}(X^j)), \quad G_p \leq r - 1,
\]

where \( 1p \) is the binary representation of \( 2^r + p \). It is noted that the array \( y_{I_p}^{1p} \) contains the derivatives of the array \( y_{I_p}^{p} \), and that \( G_{1p} = r \). Thus indeed, the number of digits 1 that appear in \( q \), i.e., \( |q| \), determine the length of the index \( I_q \).

It follows that an element of \( \hat{j}^r(\mathcal{X}, \mathcal{Y}) \) may be represented in the form

\[
(X^j; y_{I_q}^{q\alpha q}; y_{I_q}^{p\alpha p}), \quad G_q \leq r - 1
\]

or

\[
(X^j; y_{I_q}^{q\alpha q}), \quad \text{for all } q \text{ with } G_q \leq r.
\]

That is, for each \( p \) with \( G_p \leq r - 1 \), we have an index \( q = q(p) \) such that \( q = q(p) = 1p \) if \( G_q = r \) and \( q = q(p) = p \) if \( G_q < r \).

A similar line of reasoning leads to the expression for the local representatives of the iterated jet extension mapping. For a section \( \kappa : \mathcal{X} \rightarrow \mathcal{Y} \), the iterated jet extension \( B = j^r\kappa \), a section of \( \hat{j}^r \), the local representation \( (X^j; B_{I_q}^{q\alpha q}) \), \( G_q \leq r \), \( |I_q| = |q| \), satisfies

\[
B_{I_q}^{q\alpha q} = \kappa_{I_q}^{q\alpha q}, \quad |I_q| = |q|, \quad \text{independently of the particular value of } q.
\]

Indeed, if \( (X^j, B_{I_q}^{q\alpha q}) \), \( G_q \leq r - 1 \), with \( B_{I_q}^{q\alpha q} = \kappa_{I_q}^{q\alpha q} \) represent \( j^{r-1}\kappa \), then \( j^r\kappa \) is represented locally by

\[
(X^j, B_{I_q}^{q\alpha q}; B_{I_q}^{q\alpha q}), \quad G_q \leq r - 1.
\]

Thus, by induction, any \( p = p(q) \) with \( G_p = r \) and \( G_q < r \), may be written as \( p = 2^{r-1} + q \), \( I_p = I_q I_q \), so that \( B_{I_p}^{p\alpha p} = B_{I_q}^{q\alpha q} = \kappa_{I_p}^{p\alpha p} \).
Let an element $A \in \hat{J}(X, Y)$ be represented by $(X^i; y^p_{\mu})$, $G_p \leq r$, then $\hat{\xi}^r(A)$ is represented by $(X^i; y^p_{\mu})$, $G_p \leq l$.

2.4. Contraction. The right and left contractions of a $(p + r)$-form and a $p$-vector are given respectively by

\[(\theta \lrcorner \eta)(\eta') = \theta(\eta' \wedge \eta), \quad (\eta \lrcorner \theta)(\eta') = \theta(\eta \wedge \eta'),\]

(2.34)

for every $r$-vector $\eta'$. We will use the notation

\[C_\lrcorner: \mathcal{O}_p T^* X \otimes \mathcal{O}_{p + r} T^* X \longrightarrow \mathcal{O}_r T^* X,\]

and

\[C_\lrcorner: \mathcal{O}_p T^* X \otimes \mathcal{O}_{p + r} T^* X \longrightarrow \mathcal{O}_r T^* X,\]

(2.35)

(2.36)

for the mappings satisfying

\[C_\lrcorner(\xi \otimes \theta) = \theta \lrcorner \xi, \quad \text{and} \quad C_\lrcorner(\xi \otimes \theta) = \xi \lrcorner \theta,\]

(2.37)

respectively. The left and right contraction differ by a factor of $(-1)^{rp}$.

For the case $r + p = n$, $\dim(\mathcal{O}_p T^* X \otimes \mathcal{O}_{p + r} T^* X) = \dim \mathcal{O}_r T^* X$; as the mappings $C_\lrcorner$ and $C_\lrcorner$ are injective, they are invertible. Specifically, consider the mappings

\[e_\lrcorner: \mathcal{O}_n - p T^* X \longrightarrow \mathcal{O}_p T^* X \otimes \mathcal{O}_n T^* X \equiv L(\mathcal{O}_p T^* X, \mathcal{O}_n T^* X),\]

(2.38)

and

\[e_\lrcorner: \mathcal{O}_n - p T^* X \longrightarrow \mathcal{O}_p T^* X \otimes \mathcal{O}_n T^* X \equiv L(\mathcal{O}_p T^* X, \mathcal{O}_n T^* X),\]

(2.39)

given by

\[e_\lrcorner(\omega)(\psi) = \psi \wedge \omega, \quad \text{and} \quad e_\lrcorner(\omega)(\psi) = \omega \wedge \psi,\]

(2.40)

respectively. One can easily verify that these mappings are isomorphisms, and in fact, they are the inverses of the the contraction mappings defined above.

For example,

\[e_\lrcorner(C_\lrcorner(\xi \otimes \theta))(\psi) = C_\lrcorner(\xi \otimes \theta) \wedge \psi,\]

(2.41)

where we view $\xi$ s an element of the double dual. Thus,

\[e_\lrcorner = C_\lrcorner^{-1}, \quad \text{and} \quad e_\lrcorner = C_\lrcorner^{-1}.\]

(2.42)
3. Banachable Spaces of Sections of Vector Bundles over Compact Manifolds

For a compact manifold $X$, the infinite dimensional Banach manifold of mappings to a manifold $S$ and the manifold of sections of the fiber bundle $\xi : Y \to X$, are modeled by Banachable spaces of sections of vector bundles over $X$, as will be described in the next section. In this section we describe the Banachable structure of such a space of differentiable vector bundle sections and make some related observations. Thus, we consider in this section a vector bundle $\pi : W \to X$, where $X$ is a smooth compact $n$-dimensional manifold with corners and the typical fiber of $W$ is an $m$-dimensional vector space. The space of $C^r$-sections $w : X \to W$, $r \geq 0$, will be denoted by $C^r(\pi)$ or by $C^r(W)$ if no confusion may arise. A natural real vector space structure is induced on $C^r(\pi)$ by setting $(w_1 + w_2)(X) = w_1(X) + w_2(X)$ and $(cw)(X) = cw(X)$, $c \in \mathbb{R}$.

3.1. Precompact atlases. Let $K_a$, $a = 1, \ldots, A$, be a finite collection of compact subsets whose interiors cover $X$ such that for each $a$, $K_a$ is a subset of a domain of a chart $\varphi_a : U_a \to \mathbb{R}^n$ on $X$ and

$$ (\varphi_a, \Phi_a) : \pi^{-1}(U_a) \to \mathbb{R}^n \times \mathbb{R}^m, \quad v \mapsto (X^i, v^\alpha) $$ (3.1)

is some given vector bundle chart on $W$. Such a covering may always be found by the compactness of $X$ (using coordinate balls as, for example, in [Lee02, p. 16] or [Pal68, p. 10]). We will refer to such a structure as a precompact atlas. The same terminology will apply for the case of a fiber bundle.

3.2. The $C^r$-topology on $C^r(\pi)$. For a section $w$ of $\pi$ and each $a = 1, \ldots, A$, let

$$ w_a : \varphi_a(K_a) \to \mathbb{R}^m, $$ (3.2)

satisfying

$$ w_a(\varphi_a(X)) = \Phi_a(w(X)), \quad \text{for all} \quad X \in K_a, $$ (3.3)

be a local representative of $w$.

Such a choice of a vector bundle atlas and subsets $K_a$ makes it possible to define, for a section $w$,

$$ \|w\|^r = \sup_{a, \alpha, |I| \leq r} \left\{ \sup_{X \in \varphi_a(K_a)} \{ |(w_a^\alpha)_I(X)| \} \right\}. $$ (3.4)

Palais [Pal68] in particular, Chapter 4] shows that $\| \cdot \|^r$ is indeed a norm endowing $C^r(\pi)$ with a Banach space structure. The dependence of this norm on the particular choice of atlas and sets $K_a$, makes the resulting space Banachable, rather than a Banach space. Other choices will correspond to different norms. However, norms induced by different choices will induce equivalent topological vector space structures on $C^r(\pi)$. 
3.3. **The jet extension mapping.** Next, one observes that the foregoing may be applied, in particular, to the vector space \( C^0(\pi_r) = C^0(J^r W) \) of continuous sections of the \( r \)-jet bundle \( \pi_r : J^r W \to X \) of \( \pi \). As a continuous section \( B \) of \( \pi_r \) is locally of the form
\[
(X^i) \mapsto (X^i, B_{\alpha}^a(X^i)), \quad |I| \leq r,
\]
the analogous expression for the norm induced by a choice of a precompact vector bundle atlas is
\[
\|B\|^0 = \sup_{a, a, I \leq r} \left\{ \sup_{X \in \phi_a(K_a)} \{ |B_{\alpha}^a(X)| \} \right\}.
\]
Once, the topologies of \( C^r(\pi) \) and \( C^0(\pi_r) \) have been defined, one may consider the jet extension mapping
\[
j^r : C^r(\pi) \to C^0(\pi_r).
\]
For a section \( w \in C^r(\pi) \), with local representatives \( w_{\alpha}^a \), \( j^r w \) is represented by a section \( B \in C^0(\pi_r) \), the local representatives of which satisfy,
\[
B_{\alpha}^a = w_{\alpha}^a.
\]
Clearly, the mapping \( j^r \) is injective and linear. Furthermore, it follows that
\[
\|j^r w\|^0 = \sup_{a, a, I \leq r} \left\{ \sup_{X \in \phi_a(K_a)} \{ |w_{\alpha}^a(X)| \} \right\}.
\]
(Note that since we take the supremum of all partial derivatives, we could replace the non-decreasing multi-index \( I \) by a regular multi-index \( I \).) Thus, in view of Equation (3.4),
\[
\|j^r w\|^0 = \|w\|^r
\]
and we conclude that \( j^r \) is a linear embedding of \( C^r(\pi) \) into \( C^0(\pi_r) \). Evidently, \( j^r \) is not surjective as a section \( A \) in \( \pi_r \) need not be compatible, i.e., it need not satisfy (3.8), for some section \( w \) of \( \pi \). As a result of the above observations, \( j^r \) has a continuous right inverse
\[
(j^r)^{-1} : \text{Image}\ j^r \subset C^0(\pi_r) \to C^r(\pi).
\]

3.4. **The iterated jet extension mapping.** In analogy, we now consider the iterated jet extension mapping
\[
j^{r*} : C^r(\pi) \to C^0(\check{\pi^r}).
\]
Specializing Equation (2.31) for the case of the non-holonomic \( r \)-jet bundle
\[
\check{\pi^r} : J^r W \to X,
\]
a section \( B \) of \( \check{\pi^r} \) is represented locally in the form
\[
X^i \mapsto (X^i, B_{a \alpha q}(X^i)), \quad \text{for all } q \text{ with } G_q \leq r.
\]
Thus, the induced norm on $C^0(\tilde{\pi}')$ is given by
\[
\|B\|_0^0 = \sup \left\{ \|B_\alpha^\alpha q_q(X^i)\| \right\},
\]
where the supremum is taken over all $X \in \varphi_a(K_a)$, $a = 1, \ldots, A$, $\alpha_q = 1, \ldots, m$, $I_q$ with $|I| = |q|$, and $q$ with $G_q \leq r$.

Specializing (2.32) for the case of a vector bundle, it follows that if the section $B$ of $\tilde{\pi}'$, satisfies $B = j^r w$, its local representatives satisfy
\[
B_\alpha^\alpha q_q = w_\alpha^\alpha q_q |I_q| = |q|, \text{ independently of the particular value of } q. \tag{3.16}
\]
It follows that in
\[
\|j^r w\|_0^0 = \sup \left\{ \|w_\alpha^\alpha q_q(X^i)\| \right\}
\]
(where the supremum is taken over all $X^i \in \varphi_a(K_a)$, $a = 1, \ldots, A$, $\alpha = 1, \ldots, m$, $I_q$ with $|I| = |q|$, and $q$ with $G_q \leq r$), it is sufficient to take simply all derivatives $w_\alpha^\alpha q_q(X^i)$, for $|I| \leq r$. Hence,
\[
\|j^r w\|_0^0 = \sup \left\{ \|w_\alpha^\alpha_q(X^i)\| \right\}
\]
where the supremum is taken over all $X^i \in \varphi_a(K_a)$, $a = 1, \ldots, A$, $\alpha = 1, \ldots, m$, and $I$ with $|I| \leq r$. It is therefore concluded that
\[
\|j^r w\|_0^0 = \|j^r w\|_0^0 = \|w\|_r. \tag{3.19}
\]
In other words, one has a sequence of linear embeddings
\[
\begin{align*}
C^r(\pi) \xrightarrow{j^r} C^0(\pi') & \xrightarrow{C^0(\pi') \Rightarrow C^0(\tilde{\pi}')}
\end{align*}
\]
where, $j^r : j^r W \to \tilde{j}^r W$ is the natural inclusion (2.21) and $C^0(\pi')$ defined as $C^0(\pi')(A) := j^r \circ A$, for every continuous section $A$ of $j^r W$, is the inclusion of sections. These embeddings are not surjective. In particular, sections of $\hat{j}^r W$ need not have the symmetry properties that hold for sections of $j^r W$.

4. The Construction of Charts for the Manifold of Sections

In this section, we outline the construction of charts for the Banach manifold structure on the collection of sections $C^r(\xi)$ as in [Pal68]. (See a detailed presentation of the subject in this volume [Mic20].)

Let $\kappa$ be a $C^r$-section of $\xi$. Similarly to the construction of tubular neighborhoods, the basic idea is to identify points in a neighborhood of Image $\kappa$ with vectors at Image $\kappa$ which are tangent to the fibers. This is achieved by defining a second order differential equation, so that a neighboring point $y$ in the same fiber as $\kappa(X)$ is represented through the solution $c(t)$ of the differential equations with the initial condition $\varphi \in T_\kappa(X)(Y_\kappa)$ by $y = c(t)$. In other words, $y$ is the image of $\varphi$ under the exponential mapping.
To ensure that the image of the exponential mapping is located on the same fiber, \( Y_X \), the spray inducing the second order differential equation is a vector field

\[
\omega : V^2 Y \rightarrow T(V^2 Y)
\]

(4.1)

which is again tangent to the fiber in the sense that for

\[
T\tau_y : T(V^2 Y) \rightarrow T^2 Y, \quad \text{one has,} \quad T\tau_y \circ \omega \in V^2 Y.
\]

(4.2)

This condition, together with the analog of the standard condition for a second order differential equation, namely,

\[
T\tau_y \circ \omega(v) = v, \quad \text{for all} \quad v \in V^2 Y,
\]

(4.3)

imply that \( \omega \) is represented locally in the form

\[
(X^i, y^a, 0, \dot{y}^\beta) \mapsto (X^i, y^a, 0, \dot{y}^\beta, 0, 0, \tilde{\omega}^a(X^i, y^a, \dot{y}^\beta)).
\]

(4.4)

Finally, \( \omega \) is a bundle spray so that

\[
\tilde{\omega}^a(X^i, y^a, a_0\dot{y}^\beta) = a_0^2 \tilde{\omega}^a(X^i, y^a, \dot{y}^\beta).
\]

(4.5)

Bundle sprays can always be defined on compact manifolds using partitions of unity and the induced exponential mappings have the required properties.

The resulting structure makes it possible to identify an open neighborhood \( U \)—a vector bundle neighborhood—of Image \( \kappa \) in \( Y \) with

\[
V^2 Y|_{\text{Image} \ \kappa} \simeq \kappa^* V^2 Y.
\]

(4.6)

(We note that a rescaling is needed if \( U \) is to be identified with the whole of \( \kappa^* V^2 Y \). Otherwise, only an open neighborhood of the zero section of \( \kappa^* V^2 Y \) will be used to parametrize \( U \).)

Once the identification of \( U \) with \( \kappa^* V^2 Y \) is available, the collection of sections \( C^r(\xi, U) \) may be identified with \( C^r(\kappa^* V^2 Y) \), \( \kappa \in C^r(\xi) \). Thus, a chart into a Banachable space is constructed, where \( \kappa \) is identified with the zero section.

The construction of charts on the manifold of sections, implies that curves in \( C^r(\xi) \) in a neighborhood of \( \kappa \) are represented locally by curves in the Banachable space \( C^r(\kappa^* V^2 Y) \). Thus, tangent vectors \( w \in T_\kappa C^r(\xi) \) may be identified with elements of \( C^r(\kappa^* V^2 Y) \). We therefore make the identification

\[
T_\kappa C^r(\xi) = C^r(\kappa^* V^2 Y).
\]

(4.7)

5. The \( C^r \)-Topology on the Space of Sections of a Fiber Bundle

The topology on the space of sections of fiber bundles is conveniently described in terms of filters of neighborhoods (e.g., [Tre67]).
5.1. **Local representatives of sections.** We consider a fiber bundle $\xi : \mathcal{Y} \rightarrow \mathcal{X}$, where $\mathcal{X}$ is assumed to be a compact manifold with corners and the typical fiber is a manifold $\mathcal{S}$ without a boundary. Let $\{(U_a, \varphi_a, \Phi_a)\}, a = 1, \ldots, A$, and $K_a \subset U_a$, be a precompact (as in Section 3.1) fiber bundle trivialization on $\mathcal{Y}$. That is, the interiors of $K_a$ cover $\mathcal{X}$, and $(\varphi_a, \Phi_a) : \xi^{-1}(U_a) \rightarrow \mathbb{R}^n \times \mathcal{S}$. Let $\{(V_b, \psi_b)\}, b = 1, \ldots, B$, be an atlas on $\mathcal{S}$ so that $\{V_b\}$ cover $\mathcal{S}$.

Consider a $C^k$-section $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$. For any $a = 1, \ldots, A$, we can set $\tilde{\kappa}_a : U_a \rightarrow \mathcal{S}$, by $\tilde{\kappa}_a := \Phi_a \circ \kappa|_{U_a}$. (5.1)

Let

$$U_{ab} := U_a \cap \tilde{\kappa}_a^{-1}(V_b),$$

so that $\kappa(U_{ab}) \subset V_b$, and so, the local representatives of $\kappa$ are

$$\kappa_{ab} := \psi_b \circ \tilde{\kappa}_a \circ \varphi_a^{-1}|_{\varphi_a(U_{ab})} : \varphi_a(U_{ab}) \rightarrow \psi_b(V_b) \subset \mathbb{R}^m.$$ (5.3)

Thus, re-enumerating the subsets $\{U_{ab}\}$ and $\{V_b\}$ we may assume that we have a precompact trivialization $\{(U_a, \varphi_a, \Phi_a)\}, a = 1, \ldots, A, K_a \subset U_a$, on $\mathcal{Y}$, and an atlas $\{(V_a, \psi_a)\}$ on $\mathcal{S}$ such that $\tilde{\kappa}_a(U_a) \subset V_a$. The local representatives of $\kappa$ relative to these atlases are

$$\kappa_a := \psi_a \circ \tilde{\kappa}_a \circ \varphi_a^{-1}|_{\varphi_a(U_a)} : \varphi_a(U_a) \rightarrow \psi_a(V_a) \subset \mathbb{R}^m.$$ (5.4)

5.2. **Neighborhoods for $C^r(\xi)$ and the $C^r$-topology.** Let $\kappa \in C^r(\xi)$ be given. Consider sets of sections of the form $U_{\kappa, \varepsilon}$ induced by the collection of local representatives as above and numbers $\varepsilon > 0$ in the form

$$U_{\kappa, \varepsilon} = \{\kappa' \in C^r(\xi) \mid \sup \left\{ \left\| (\kappa_a'^{\alpha} - \kappa_a^\alpha) \right\|_{L^1(\mathcal{X})} \right\} < \varepsilon \},$$ (5.5)
where the supremum is taken over all

\[ X \in \varphi_a(K_a), \, \alpha = 1, \ldots, m, \, |l| \leq r, \, a = 1, \ldots, A. \]

The \( C^r \)-topology on \( C^r(\xi) \) uses all such sets as a basis of neighborhoods. Using the transformation rules for the various variables, it may be shown that other choices of precompact trivialization and atlas will lead to equivalent topologies. It is noted that we use here the compactness of \( \xi \) which implies that the weak and strong \( C^r \)-topologies (see [Hir76, p. 35]) become identical.

Remark 5.1. The collection of neighborhoods \( \{U_{\kappa, \epsilon}\} \) for the various values of \( \epsilon \) generate a basis of neighborhoods for the topology of \( C^r(\xi) \). If one keeps the value of \( a = 1, \ldots, A \), fixed, then the collection of sections

\[ U_{\kappa, \epsilon, a} = \{ \kappa' \in C^r(\xi) \mid \sup \{|(\kappa'_a - \kappa_a^\alpha).\tau(X)| \epsilon_{\alpha}\} \}, \]

where the supremum is taken over all

\[ X \in \varphi_a(K_a), \, \alpha = 1, \ldots, m, \, |l| \leq r, \]

is a neighborhood as it contains the open neighborhood \( U_{\kappa, \epsilon, a} \). In fact, since \( \xi \) is assumed to be compact, the collection of sets of the form \( \{U_{\kappa, \epsilon, a}\} \) is a sub-basis of neighborhoods of \( \kappa \) for the topology on \( C^r(\xi) \).

5.3. Open neighborhoods for \( C^r(\xi) \) using vector bundle neighborhoods.
In order to specialize the preceding constructions for the case where a vector bundle neighborhood is used, we first consider local representations of sections.

Let \( \kappa \in C^r(\xi) \) be a section and let

\[ \kappa^*\tau_y : \kappa^*V^\gamma \longrightarrow \mathcal{X} \]

be the vector bundle identified with an open subbundle \( U \) of \( \gamma \). (We will use the two aspects of the vector bundle neighborhood, interchangeably.) Since the typical fiber of \( \kappa^*V^\gamma \) is \( \mathbb{R}^m \), one may choose a precompact vector bundle atlas \( \{(U_a, \varphi_a, \Phi_a)\}, K_a \subset U_a, \, a = 1, \ldots, A \), on \( \kappa^*V^\gamma \), such that

\[ (\varphi_a, \Phi_a) : (\kappa^*\tau_y)^{-1}(U_a) \longrightarrow \mathbb{R}^n \times \mathbb{R}^m. \]

Thus, if we identify all open subsets \( V_a \) in Section 5.1 above with the typical fiber \( \mathbb{R}^m \), the representatives of a section are of the form

\[ \kappa_a := \Phi_a \circ \kappa|U_a \circ \varphi_a^{-1}|_{\varphi_a(U_a)} : \varphi_a(U_a) \longrightarrow \mathbb{R}^m. \]

A basic neighborhood of \( \kappa \) is given by Equation (5.5). However, from the point of view of a vector bundle neighborhood, \( \kappa \) is represented by the zero section and each \( \kappa' \) is viewed as a section of the vector bundle \( \kappa^*\tau_y \), which we may denote by \( w' \). Thus

\[ U_{\kappa, \epsilon} \simeq \{ w' \in C^r(\kappa^*\tau_y) \mid \sup \{|(w'_a)^\epsilon_\tau(X)| \epsilon_{\alpha}\} \}. \]

In other words, using the structure of a vector bundle neighborhood we have

\[ U_{\kappa, \epsilon} \simeq \{ w' \in C^r(\kappa^*\tau_y) \mid \|w'\|_r < \epsilon \}, \]

so that \( U_{\kappa, \epsilon} \) is identified with a ball of radius \( \epsilon \) in \( C^r(\kappa^*\tau_y) \) at the zero section.
It is concluded that the charts on $C^r(\xi)$ induced by the vector bundle neighborhood are compatible with the $C^r$-topology on $C^r(\xi)$.

6. The Space of Embeddings

The kinematic aspect of the Lagrangian formulation of continuum mechanics is founded on the notion of a configuration, an embeddings of a body manifold $\mathcal{X}$ into the space manifold $\mathcal{S}$. The restriction of configurations to be embeddings, rather than generic $C^r$-mappings of the body into space follows from the traditional principle of material impenetrability which requires that configurations be injective and that infinitesimal volume elements are not mapped into elements of zero volume.

It is noted that any configuration $\kappa : \mathcal{X} \to \mathcal{S}$ may be viewed as a section of the trivial fiber bundle $\xi : \mathcal{X} \to \mathcal{Y} = \mathcal{X} \times \mathcal{S}$. Thus, the constructions described above apply immediately to configurations in continuum mechanics. In this particular case, we will write $C^r(\mathcal{X}, \mathcal{S})$ for the collection of all $C^r$-mappings. Our objective in this section is to describe how the set of embeddings $\text{Emb}^r(\mathcal{X}, \mathcal{S})$ constitutes an open subset of $C^r(\mathcal{X}, \mathcal{S})$ for $r \geq 1$. In particular, it will follow that that at each configuration $\kappa$, $T_\kappa \text{Emb}^r(\mathcal{X}, \mathcal{S}) = T_\kappa C^r(\mathcal{X}, \mathcal{S})$. Since the $C^r$-topologies, for $r > 1$, are finer than the $C^1$-topology, it is sufficient to prove that $\text{Emb}^1(\mathcal{X}, \mathcal{S})$ is open in $C^1(\mathcal{X}, \mathcal{S})$. This brings to light the special role that the case $r = 1$ plays in continuum mechanics.

6.1. The case of a trivial fiber bundle—manifolds of mappings. It is observed that the definitions of Sections 5.1 and 5.2 hold with natural simplifications for the case of the trivial bundle. Thus, we use a precompact atlas $\{(U_a, \varphi_a)\}$, $a = 1, \ldots, A$, and $K_a \subset U_a$, in $\mathcal{X}$ (the interiors, $K_a^\circ$, cover $\mathcal{X}$). Given $\kappa \in C^1(\mathcal{X}, \mathcal{S})$, we can find an atlas $\{(V_a, \psi_a)\}$ on $\mathcal{S}$ such that $\kappa(U_a) \subset V_a$. The local representatives of $\kappa$ are of the form

$$\kappa_a = \psi_a \circ \kappa |_{U_a} \circ \varphi_a^{-1} : \varphi_a(U_a) \to \psi_a(V_a) \subset \mathbb{R}^m. \quad (6.1)$$

For the case $r = 1$, Equation (5.5) reduces to

$$U_{\kappa, \varepsilon} = \left\{ \kappa' \in C^1(\mathcal{X}, \mathcal{S}) \big| \sup \left\{ |(\kappa'_a)^\alpha(X) - \kappa_a^\alpha(X)|, |(\kappa'_a)'_j(X) - \kappa_a'^\alpha_j(X)| \right\} < \varepsilon \right\}, \quad (6.2)$$

where the supremum is taken over all

$$X \in \varphi_a(K_a), \quad \alpha = 1, \ldots, m, \quad a = 1, \ldots, A.$$

Remark 6.1. It is noted that in analogy with Remark 5.1, for a fixed $a = 1, \ldots, A$, a subset of mappings of the form

$$U_{\kappa, \varepsilon, a} = \left\{ \kappa' \in C^1(\mathcal{X}, \mathcal{S}) \big| \sup \left\{ |(\kappa'_a)^\alpha(X) - \kappa_a^\alpha(X)|, |(\kappa'_a)'_j(X) - \kappa_a'^\alpha_j(X)| \right\} < \varepsilon \right\}, \quad (6.3)$$

where the supremum is taken over all

$$X \in \varphi_a(K_a), \quad j = 1, \ldots, n, \quad \alpha = 1, \ldots, m,$$
is a neighborhood of $\kappa$ as it contains a neighborhood as defined above. The collection of such sets for various values of $a$ and $\epsilon$ form a subbasis of neighborhoods for the topology on $C^1(\mathcal{X}, \mathcal{S})$.

6.2. The space of immersions. Let $\kappa \in C^1(\mathcal{X}, \mathcal{S})$ be an immersion, so that $T_X \kappa : T_X \mathcal{X} \to T_{\kappa(X)} \mathcal{S}$ is injective for every $X \in \mathcal{X}$. We show that there is a neighborhood $U_\kappa \subset C^1(\mathcal{X}, \mathcal{S})$ of $\kappa$ such that all $\kappa' \in U_\kappa$ are immersions.

Note first, that since the evaluation of determinants of $n \times n$ matrices is a continuous mapping, the collection of $m \times n$ matrices for which all $n \times n$ minors vanish is a closed set. Hence, the collection $L_{in}(\mathbb{R}^n, \mathbb{R}^m)$ of all injective $m \times n$ matrices is open in $L(\mathbb{R}^n, \mathbb{R}^m)$. Let $\kappa$ be an immersion with representatives $\kappa_a$ as above. For each $a$, the derivative mapping

$$D\kappa_a : \varphi_a(U_a) \to L(\mathbb{R}^n, \mathbb{R}^m), \quad X \mapsto D\kappa_a(X),$$

(6.4)

is continuous, hence, $D\kappa_a(K_a)$ is a compact set of injective linear mappings. Choosing any norm in $L(\mathbb{R}^n, \mathbb{R}^m)$, one can cover $D\kappa_a(K_a)$ by a finite number of open balls all containing only injective mappings. In particular, setting

$$\|T\| = \max_{i,a} \{|T^i|\}, \quad T \in L(\mathbb{R}^n, \mathbb{R}^m),$$

(6.5)

let $\epsilon_a$ be the least radius of balls in this covering. Thus, we are guaranteed that any linear mapping $T$, such that $\|T - D\kappa_a(X)\| < \epsilon_a$ for some $X \in \varphi_a(K_a)$, is injective. Specifically, for any $\kappa' \in C^1(\mathcal{X}, \mathcal{S})$, if

$$\sup_{X \in \varphi_a(K_a)} |(\kappa'_a)^j_f(X) - \kappa^a_{a,j}(X)| \leq \epsilon_a,$$

(6.6)

$D\kappa'_a$ is injective everywhere in $\varphi_a(K_a)$. Letting $\epsilon = \min_a \epsilon_a$, any configuration in $U_{\kappa,\epsilon}$ as in [6.2] is an immersion.

6.3. Open neighborhoods of local embeddings. Let $\kappa \in C^1(\mathcal{X}, \mathcal{S})$ and $X \in \mathcal{X}$. It is shown below that if $T_X \kappa$ is injective, then there is a neighborhood of mappings $U_{\kappa,X}$ of $\kappa$ such that every $\kappa' \in U_{\kappa,X}$ is injective in some fixed neighborhood of $X$. Specifically, there is a neighborhood $W_X$ of $X$, and a neighborhood $U_{\kappa,X}$ of $\kappa$ such that for each $\kappa' \in U_{\kappa,X}$, $\kappa'|_{W_X}$ is injective.

Let $(U, \varphi)$ and $(V, \psi)$ be coordinate neighborhoods of $X$ and $\kappa(X)$, respectively, such that $\kappa(U) \subset V$. Let $X$ and $\kappa$ be the local representative of $X$ and $\kappa$ relative to these charts. Thus, we are guaranteed that

$$M := \inf_{|\nu|=1} |D\kappa(X)(\nu)| > 0.$$  

(6.7)

By a standard corollary of the inverse function theorem, due to the injectivity of $T_X \kappa$, we can choose $U$ to be small enough so that the restrictions of $\kappa$ and $\kappa$ to $U$ and its image under $\varphi$, respectively, are injective. Next, let $W_X$ be a neighborhood of $X$ such that $\varphi(W_X)$ is convex and its closure, $W_X$, is a compact subset of $U$. Thus, define the neighborhood $U_{\kappa,X} \subset C^1(\mathcal{X}, \mathcal{S})$ whose elements,
κ’, satisfy the conditions
\[ \kappa'(\overline{W}_X) \subset V, \quad \text{and} \quad |D\kappa'(X’) - D\kappa(X)| < \frac{M}{2}, \quad \text{for all} \quad X' \in \varphi(\overline{W}_X). \] (6.8)

By the definition of neighborhoods in \( C^1(\mathcal{X}, \mathcal{S}) \) in [6.2], \( U_{\kappa, X} \) contains a neighborhood of \( \kappa \), hence, it is also a neighborhood.

Next, it is shown that the fact that the values of the derivatives of elements of \( U_{\kappa, X} \) are close to the injective \( D\kappa(X) \) everywhere in \( \overline{W}_X \), implies that these mappings are close to the linear approximation using \( D\kappa(X) \), which in turn, implies injectivity in \( \overline{W}_X \) of these elements. Specifically, for and \( X_1, X_2 \in \overline{W}_X \), since
\[ \kappa'(X_2) - \kappa'(X_1) = \kappa'(X_2) - \kappa'(X_1) - D\kappa(X)(X_2 - X_1) + D\kappa(X)(X_2 - X_1), \] (6.9)
the triangle inequality implies that
\[ |\kappa'(X_2) - \kappa'(X_1)| \geq |D\kappa(X)(X_2 - X_1)| - |\kappa'(X_2) - \kappa'(X_1) - D\kappa(X)(X_2 - X_1)|, \]
\[ \geq M |X_2 - X_1| - |\kappa'(X_2) - \kappa'(X_1) - D\kappa(X)(X_2 - X_1)|. \] (6.10)

Using the mean value theorem, there is a point \( X_0 \in \varphi(\overline{W}_X) \) such that
\[ \kappa'(X_2) - \kappa'(X_1) = D\kappa'(X_0)(X_2 - X_1). \] (6.11)

Hence,
\[ |\kappa'(X_2) - \kappa'(X_1) - D\kappa(X)(X_2 - X_1)| = |(D\kappa'(X_0) - D\kappa(X))(X_2 - X_1)|, \]
\[ \leq |D\kappa'(X_0) - D\kappa(X)| |X_2 - X_1|, \]
\[ < \frac{M}{2} |X_2 - X_1|. \] (6.12)

It follows that
\[ |\kappa'(X_2) - \kappa'(X_1)| > \frac{M}{2} |X_2 - X_1|, \] (6.13)
which proves the injectivity.

6.4. Open neighborhoods of embeddings. Finally, it is shown how every \( \kappa \in \text{Emb}^1(\mathcal{X}, \mathcal{S}) \subset C^1(\mathcal{X}, \mathcal{S}) \) has a neighborhood consisting of embeddings only. It will follow that \( \text{Emb}^1(\mathcal{X}, \mathcal{S}) \) is an open subset of \( C^1(\mathcal{X}, \mathcal{S}) \). This has far-reaching consequences in continuum mechanics and it explains the special role played by the \( C^1 \)-topology in continuum mechanics.

Let \( \kappa \) be a given embedding. Using the foregoing result, for each \( X \in \mathcal{X} \) there is an open neighborhood \( W_X \) of \( X \) and a neighborhood \( U_{\kappa, X} \) of \( \kappa \), such that for each \( \kappa' \in U_{\kappa, X} \), \( \kappa'|_{W_X} \) is injective. The collection of neighborhoods \( \{W_X\}, X \in \mathcal{X} \), is an open cover of \( \mathcal{X} \) and by compactness, it has a finite subcover. Denote the finite number of open sets of the form \( W_X \) as above by \( W_a, \)
\[ a = 1, \ldots, A, \] so that \( K_a := \overline{W_a} \) is a compact subset of \( U_a, \kappa(U_a) \subset V_a \). For each \( a \), we have a neighborhood \( U_{\kappa, a} \) of \( \kappa \) such that each \( \kappa' \in U_{\kappa, a} \) satisfies the condition that \( \kappa'|_{K_a} \) is injective. Let \( N_1 = \bigcap_{a=1}^A U_{\kappa, a} \) so that for each \( \kappa' \in N_1 \),
κ’|_\mathcal{N}_a is injective for all a. Let \mathcal{N}_2 be a neighborhood of κ which contains only immersions as in Section 6.2. Thus, \mathcal{N}_0 = \mathcal{N}_1 \cap \mathcal{N}_2 contains immersions which are locally injective.

Let κ be an embedding and \mathcal{N}_0 as above. If there is no neighborhood of κ that contains only injective mappings, then, for each ν = 1, 2, ..., there is a κ_ν ∈ U_κ,ε_ν, ε_ν = 1/ν, and points X_ν, X'_ν ∈ \mathcal{X}, X_ν ≠ X'_ν, such that κ_ν(X_ν) = κ_ν(X'_ν). As \mathcal{N}_0 is a neighborhood of κ, we may assume that κ_ν ∈ \mathcal{N}_0 for all ν. By the compactness of \mathcal{X} and \mathcal{X} × \mathcal{X}, we can extract a converging subsequence from the sequence ((X_ν, X'_ν)) in \mathcal{X} × \mathcal{X}. We keep the same notation for the converging subsequences and let

\[(X_ν, X'_ν) \longrightarrow (X, X'), \quad \text{as } ν \longrightarrow \infty.\]  

(6.14)

We first exclude the possibility that X = X'. Assume X = X' ∈ K_\mathcal{A}_0, for some a_0 = 1, ..., A. Then, for any neighborhood U_κ,ε_ν of κ and any neighborhood of X = X', there is a configuration κ_ν such that κ_ν is not injective. This contradicts the construction of local injectivity above.

Thus, one should consider the situation for which X ≠ X'. Assume X ∈ K_\mathcal{A}_0 and X' ∈ K_\mathcal{A}_1 for a_0, a_1 = 1, ..., A. By the definition of U_κ,ε_ν, the local representatives of κ_ν|_K_\mathcal{A}_0 and κ_ν|_K_\mathcal{A}_1 converge uniformly to the local representatives of κ|_K_\mathcal{A}_0 and κ|_K_\mathcal{A}_1, respectively. This implies that

\[κ_ν(X_ν) \longrightarrow κ(X), \quad κ_ν(X'_ν) \longrightarrow κ(X'), \quad \text{as } ν \longrightarrow \infty.\]  

(6.15)

However, since for each ν, κ_ν(X_ν) = κ_ν(X'_ν), it follows that κ(X) = κ(X'), which contradicts the assumption that κ is an embedding.

It is finally noted that the set of Lipschitz embeddings equipped with the Lipschitz topology may be shown to be open in the manifold of all Lipschitz mappings \mathcal{X} → \mathcal{S}. See [FN05] and an application in continuum mechanics in [FS15].

7. The General Framework for Global Analytic Stress Theory

The preceding section implied that for the case where the kinematics of a material body \mathcal{X} is described by its embeddings in a physical space \mathcal{S}, the collection of configurations—the configuration space

\[\mathcal{Q} := \text{Emb}'(\mathcal{X}, \mathcal{S})\]  

(7.1)
—is an open subset of the manifold of mappings C'(\mathcal{X}, \mathcal{S}), for r ≥ 1. As a result, the configuration space is a Banach manifold in its own right and

\[T_κ\mathcal{Q} = T_κC'(\mathcal{X}, \mathcal{S}) = T_κC'(ξ)\]  

(7.2)

where ξ : \mathcal{X} × \mathcal{S} → \mathcal{X} is the natural projection of the trivial fiber bundle.

In view [4.7], \[T_κ\mathcal{Q} = C'(κ^*V\mathcal{Y}),\]  

where now

\[V\mathcal{Y} = \{v ∈ T\mathcal{Y} = T\mathcal{X} × T\mathcal{S} \ | \ Tξ(v) = 0 ∈ T\mathcal{X}\}.\]  

(7.3)

Hence, one may make the identifications

\[V\mathcal{Y} = \mathcal{X} × T\mathcal{S}\]  

(7.4)
and
\[(\kappa^*V\gamma)_X = (V\gamma)_{\kappa(X)} = T_{\kappa(X)}S.\] (7.5)
A section \(w\) of \(\kappa^*\tau: \kappa^*V\gamma \rightarrow \mathfrak{X}\) is of the form
\[X \mapsto w(X) \in T_{\kappa(X)}S\] (7.6)
and may be viewed as a vector field along \(\kappa\), i.e., a mapping
\[w : \mathfrak{X} \rightarrow TS, \quad \text{such that,} \quad \tau \circ w = \kappa.\] (7.7)
Thus, a tangent vector to the configuration space at the configuration \(\kappa\) may be viewed as a \(C^r\)-vector field along \(\kappa\). This is a straightforward generalization of the standard notion of a virtual velocity field and we summarize these observations by
\[T_\kappa\mathcal{Q} = C^r(\kappa^*V\gamma) = \{w \in C^r(\mathfrak{X}, TS) \mid \tau \circ w = \kappa\}.\] (7.8)
In the case of generalized continua, where \(\xi : \mathfrak{Y} \rightarrow \mathfrak{X}\) need not be a trivial vector bundle, this simplification does not apply of course. However, the foregoing discussion motivates the definition of the configuration space for a general continuum mechanical system specified by the fiber bundle \(\xi : \mathfrak{Y} \rightarrow \mathfrak{X}\) as
\[\mathcal{Q} = C^r(\xi), \quad r \geq 1.\] (7.9)
We note that the condition that configurations are embeddings is meaningless in the case of generalized continua.

The general framework for global analytic stress theory adopts the geometric structure for the statics of systems having a finite number of degrees of freedom. Once a configuration manifold \(\mathcal{Q}\) is specified, generalized or virtual velocities are defined to be elements of the tangent bundle, \(T\mathcal{Q}\), and generalized forces are defined to be elements of the cotangent bundle \(T^*\mathcal{Q}\). The action of a force \(F \in T^*\mathcal{Q}\) on a virtual velocity \(w \in T\mathcal{Q}\) is interpreted as virtual power and as such, the notion of power has a fundamental role in this formulation.

The foregoing discussion implies that a force at a configuration \(\kappa \in C^r(\xi)\) is an element of \(C^r(\kappa^*V\gamma)^*\)—a continuous and linear functional on the Banachable space of \(C^r\)-section of the vector bundle. Thus, in the following sections we consider the properties of linear functionals on the space of \(C^r\)-sections of a vector bundle \(W\). Of particular interest is the fact that our base manifold, or body manifold, is a manifold with corners rather than a manifold without boundary. The relation between such functionals, on the one hand, and Schwartz distribution and de Rham currents, on the other hand, is described. In Section 11 [11] we show that the notions of stresses and hyperstresses emerge from a representation theorem for such functionals and in Section 12 [12] we study further the properties of stresses.

8. Duals to Spaces of Differentiable Sections of a Vector Bundle: Localization of Sections and Functionals

As follows from the foregoing discussion, generalized forces are modeled mathematically as elements of the dual space \(C^r(\pi)^* = C^r(W)^*\) of the space of
8.1. Spaces of differentiable sections over a manifold without boundary and linear functionals. A comprehensive introduction to the subject considered here is available in the Ph.D thesis [Ste00] and the corresponding [GKOS01 Chapter 3]. See also [GS77, Chapter VI] and [Kor08].

Consider the space of $C^r$-sections of a vector bundle $\pi : W \to \mathcal{X}$, for $0 \leq r < \infty$. For manifolds without boundary that are not necessarily compact, the setting of Section 3.2 will not give a norm on the space of sections. Thus, one extends the settings used for Schwartz distributions and de Rham currents to sections on manifolds. (See, in particular, Section 8.5.) As an additional motivation for considering sections over manifolds without boundaries, it is observed that in both the Eulerian formulation of continuum mechanics and in classical field theories, the base manifold, either space or space-time, is usually taken as compact without boundary. We start with the case where $\mathcal{X}$ is a manifold without boundary and continue with the case where bodies are modeled by compact manifolds with corners.

Let $(U_a, \varphi_a, \Phi_a)_{a \in A}$ be a vector bundle atlas so that

$$(\varphi_a, \Phi_a) : \pi^{-1}(U_a) \longrightarrow \mathbb{R}^n \times \mathbb{R}^m, \quad v \longmapsto (X^i, v^\alpha).$$

and let $K$ be a compact subset of $\mathcal{X}$. Consider the vector subspace $C^r_{c,K}(\pi) \subset C^r(\pi)$ of sections, the supports of which are contained in $K$. Let $a_l \in A$, indicate a finite collection of charts such that $\{U_{a_l}\}$ cover $K$, and for each $U_{a_l}$ let $K_{a_l,\mu}$, $\mu = 1, 2, \ldots$, be a fundamental sequence of compact sets, i.e., $K_{a_l,\mu} \subset K_{a_l,\mu+1}$, covering $\varphi_{a_l}(U_{a_l}) \subset \mathbb{R}^n$. Then, for a section $w \in C^r_{c,K}(\pi)$, the collection of semi-norms

$$\|w\|_{K,\mu} = \sup_{a_l,\alpha,|I| \leq r} \left\{ \sup_{X \in K_{a_l,\mu}} \left\{ \left| (w^\alpha_{a_l})_I (X) \right| \right\} \right\},$$

induce a Fréchet space structure for $C^\infty_{c,K}(\pi)$. Since for each compact subset $K$, one has the inclusion mapping $i_K : C^r_{c,K}(\pi) \to C^r(\pi)$, one may define the topology on $C^r(\pi)$ as the inductive limit topology generated by these inclusions, i.e., the strongest topology on $C^r(\pi)$ for which all the inclusions are continuous. A sequence of sections in $C^r(\pi)$ converges to zero, if there is a compact subset $K \subset \mathcal{X}$ such that the supports of all sections in the sequence are contained in $K$ and the $r$-jets of the sections converge uniformly to zero in $K$.
A linear functional $T \in C^r_c(\pi)^*$ is continuous when it satisfies the following condition. Let $(\chi_j)$ be a sequence of sections of $\pi$ all of which are supported in a compact subset $K \subset U_a$ for some $a \in A$. In addition, assume that the local representatives of $\chi_j$ and their derivatives of all orders $k \leq r$ converge uniformly to zero in $K$. Then,

$$
\lim_{j \to \infty} T(\chi_j) = 0.
$$

(8.3)

Functionals in $C^r_c(\pi)^*$ for a finite value of $r$ are referred to as functionals of order $r$.

For a linear functional $T$, the support, $\text{supp } T$ is defined as follows. An open set $U \subset X$ is termed a null set of $T$ if $T(\chi) = 0$ for any section of $\pi$ with $\text{supp } \chi \subset U$. The union of all null sets, $U_0$ is an open set which is a null set also. Thus, one defines

$$
\text{supp } T := X \setminus U_0.
$$

(8.4)

8.2. Localization of sections and linear functionals for manifolds without boundaries. Let $\{(U_a, \varphi_a, \Phi_a)\}_{a \in A}$ be a locally finite vector bundle atlas on $W$ and consider

$$
E_{U_a} : C^r_c(\pi|_{U_a}) \longrightarrow C^r_c(\pi),
$$

(8.5)

the natural zero extension of sections supported in a compact subsets of $U_a$ to the space of sections that are compactly supported in $X$. This is evidently a linear and continuous injection of the subspace. On its image, the subspace of sections $\chi$ with $\text{supp } \chi \subset U_a$ we have a left inverse, the natural restriction

$$
\rho_{U_a} : \text{Image } E_{U_a} \longrightarrow C^r_c(\pi|_{U_a}),
$$

(8.6)

a surjective mapping. However, it is well known (e.g., [Sch63, Trè67, pp. 245–246]) that the inverse $\rho_{U_a}$ is not continuous.

The dual,

$$
E^*_{U_a} : C^r_c(\pi)^* \longrightarrow C^r_c(\pi|_{U_a})^*,
$$

(8.7)

is the restriction of functionals on $X$ to sections supported on $U_a$, and as $\rho_{U_a}$ is not continuous, $E^*_{U_a}$ is not surjective (loc. cit.). We will write

$$
T|_{U_a} := \tilde{T}_a := E^*_{U_a} T.
$$

(8.8)

We also note that the restrictions $\{\tilde{T}_a\}$ satisfy the condition

$$
\tilde{T}_a(\chi|_{U_a}) = \tilde{T}_b(\chi|_{U_b}) = T(\chi)
$$

(8.9)

for any section $\chi$ supported in $U_a \cap U_b$.

Consider the mapping

$$
s : \bigoplus_{a \in A} C^r_c(\pi|_{U_a}) \longrightarrow C^r_c(\pi)
$$

(8.10)

given by

$$
s(\chi_1, \ldots, \chi_a, \ldots) := \sum_{a \in A} E_{U_a}(\chi_a).
$$

(8.11)
Due to the overlapping between domains of definition, the mapping $s$ is not injective. However, $s$ is surjective because using a partition of unity, $\{u_a\}$, which subordinate to this atlas, for each section, $\chi$, $u_a\chi$ is a compactly supported in $U_a$ and $\chi = \sum_a u_a\chi$. Hence, the dual mapping,

$$s^* : C^*_c(\pi) \to \bigoplus_{a \in A} C^*_c(\pi|\{}_{U_a},$$

(8.12)
given by,

$$(s^*T)_a := E_{U_a}^* T = T|_{U_a}, \quad s^*T(\chi_1, \ldots) = T\left(\sum_{a \in A} E_{U_a}(\chi_a)\right),$$

(8.13)
is injective. In other words, a functional is determined uniquely by the collection of its restrictions. Note that no compatibility condition is imposed above on the local sections $\{\chi_a\}$.

Since $\{\tilde{T}_a\} \in \text{Image } s^*$ satisfy the compatibility condition (8.9), $s^*$ is not surjective. However, it is easy to see that Image $s^*$ is exactly the subspace of $\bigoplus_{a \in A} C^*_c(\pi|\{}_{U_a})$ containing the compatible collections of local functionals. For let $\{\tilde{T}_a\}$ be local functionals that satisfy (8.9) and $\{u_a\}$ a partition of unity. Consider the functional $T \in C^*_c(\pi)$ given by

$$T(\chi) = \sum_{a \in A} \tilde{T}_a(u_a\chi).$$

(8.14)

If $\chi$ is supported in $U_b$ for $b \in A$, then

$$T(\chi) = \sum_{a \in A} \tilde{T}_a(u_a\chi), \quad U_a \cap U_b \neq \emptyset,$n

$$= \sum_{a \in A} \tilde{T}_b(u_a\chi), \quad \text{by (8.9)},$$

$$= \tilde{T}_b\left(\sum_{a \in A} u_a\chi\right),$$

$$= \tilde{T}_b(\chi).$$

(8.15)

Thus, $T$ is a well defined functional on $\pi$ and it is uniquely determined by the collection $\{\tilde{T}_a\}$—its restrictions, independently of the partition of unity chosen.

As mentioned above, a partition of unity induces an injective right inverse to $s$ in the form

$$p : C^*_c(\pi) \to \bigoplus_{a \in A} C^*_c(\pi|\{}_{U_a}, \quad p(\chi) = \{(u_a\chi)|_{U_a}\},$$

(8.16)

that evidently satisfies $s \circ p = 1$. It is noted that $p$ is not a left inverse. In particular, for a section $\chi_a$ supported in $U_a$, with $\chi_b = 0$ for all $b \neq a$,

$$p \circ s(\chi_1, \ldots) = u_a\chi_a,$$

(8.17)

which need not be equal to $\chi_a$. Thus, $p$ depends on the partition of unity.
For the surjective dual mapping

\[ p^* : \bigoplus_{a \in A} C_c^r(\pi|_{U_a})^* \to C_c^r(\pi)^*, \quad (8.18) \]

we note that \( p^* \circ s^* = \text{Id} \) while \( s^* \circ p^* \neq \text{Id} \), in general. The surjectivity of \( p^* \) implies that every functional \( T \) may be represented by a non-unique collection \( \{T_a\} \) in the form

\[ T(\chi) = \sum_{a \in A} T_a((u_a\chi)|_{U_a}), \quad T = \sum_{a \in A} u_a T_a. \quad (8.20) \]

which depends on the partition of unity. Here, \( u_a T \) denotes the functional defined by \( u_a T(\chi) = T(u_a\chi) \).

Nevertheless, we may restrict \( p^* \) to the subspace of compatible local functionals, \( \text{Image } s^* \), \( i.e. \), those satisfying (8.9). Thus, the restriction

\[ p^*|_{\text{Image } s^*} : \text{Image } s^* \to C_c^r(\pi)^*, \quad (8.21) \]

is an isomorphism (which depends on the partition of unity). It follows that

\[ s^* \circ p^* = \text{Id} : \text{Image } s^* \to \text{Image } s^*. \quad (8.22) \]

(For additional details, see [Die72, p. 244–245], which is restricted to the case of de Rham currents, and [GKOS01, pp. 234–235].)

8.3. Localization of sections and linear functionals for manifolds with corners. In analogy with Section 8.2, we consider the various aspects of localization relevant to the case of compact manifolds with corners. Thus, the base manifold for the vector bundle \( \pi : W \to X \) is assumed to be a manifold with corners and we are concerned with elements of \( C^r(\pi)^* \) acting on sections that need not necessarily vanish together with their first \( r \) jets on the boundary of \( X \).

In [Pal68 pp 10–11], Palais proves what he refers to as the “Mayer-Vietoris Theorem”. Adapting the notation and specializing the theorem to the \( C^r \)-topology, the theorem may be stated as follows.

**Theorem 8.1.** Let \( X \) be a compact smooth manifold and let \( K_1, \ldots, K_A \) be compact \( n \)-dimensional submanifolds of \( X \) whose interiors cover \( X \) (such as in a precompact atlas). Given the vector bundle \( \pi \), set

\[ \tilde{C}^r(\pi) := \left\{ (\chi_1, \ldots, \chi_A) \in \bigoplus_{a=1}^A C^r(\pi|_{K_a}) \mid \chi_a|_{K_b} = \chi_b|_{K_a} \right\}, \quad (8.23) \]

and define

\[ \iota : C^r(\pi) \to \tilde{C}^r(\pi), \quad \text{by} \quad \iota(\chi) = (\chi|_{K_1}, \ldots, \chi|_{K_A}). \quad (8.24) \]

Then, \( \iota \) is an isomorphism of Banach spaces.
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We will refer to the condition in (8.23) as the compatibility condition for local representatives of sections. The most significant part of the proof is the construction of $i^{-1}$. Thus, one has to construct a field $w$ when a collection $(w_1, \ldots, w_A)$, satisfying the compatibility condition, is given. This is done using a partition of unity which is subordinate to the interiors of $K_1, \ldots, K_A$.

It is noted that the situation may be viewed as “dual” to that described in Section 8.2. For functionals on spaces of sections with compact supports defined on a manifold without boundary, there is a natural restriction of functionals, $E_{U_a}^*$, and the images $\{\tilde{T}_a\}$ of a functional $T$ under the restrictions satisfy the compatibility condition (8.9). The collection of restrictions determine $T$ uniquely.

Here, it follows from Theorem 8.1 that we have a natural restriction of sections, and the restricted sections satisfy the compatibility condition (8.23). The restrictions $\{\chi|K_a\}$ also determine the global section $\chi$, uniquely.

In Section 8.2, we observed that sections with compact supports on $X$ cannot be “restricted” naturally to sections with compact supports on the various $U_a$. Such restrictions depend on the chosen partition of unity. The analogous situation for functionals on manifolds with corners is described below.

Corollary 8.2. Let $T \in C^r(\pi)^*$, then $T$ may be represented (non-uniquely) by $(T_1, \ldots, T_A), T_a \in C^r(\pi|K_a)^*$, in the form

$$T(w) = \sum_{a=1}^A T_a(w|K_a).$$  (8.25)

Indeed, as $i$ in Theorem 8.1 is an embedding of $C^r(\pi)$ into a subspace of $\bigoplus_{a=1}^A C^r(\pi|K_a)^*$, one has a surjective

$$i^* : \bigoplus_{a=1}^A C^r(\pi|K_a)^* \rightarrow C^r(\pi)^*,$$  (8.26)
given by

$$i^*(T_1, \ldots, T_A)(w) = \sum_{a=1}^A T_a(w|K_a).$$  (8.27)

8.4. Supported sections, static indeterminacy and body forces. The foregoing observations are indicative of the fundamental problem of continuum mechanics—that of static indeterminacy. Given a force $F$ on a body as an element of $C^r(\pi)^*$ for some vector bundle $\pi : W \rightarrow X$, and a sub-body $R \subset X$, there is no unique restriction of $F$ to a force on $R$ in $C^r(\pi|_R)^*$. This problem is evident for standard continuum mechanics in Euclidean spaces and continues all the way to continuum mechanics of higher order on differentiable manifolds.

Adopting the notation of [Mel96], denote by $\hat{C}^r(\pi)$ the space of sections of $\pi$, the $r$-jet extensions of which vanish on all the components of the boundary $\partial X$. Let $\hat{X}$ be a manifold without boundary extending $X$ and let

$$\hat{\pi} : \hat{W} \rightarrow \hat{X}$$  (8.28)
be an extension of $\pi$. Then, we may use zero extension to obtain an isomorphism

$$\hat{C}'(\pi) \equiv \{ \chi \in C'_c(\tilde{\pi}) \mid \text{supp } \chi \subset \tilde{\mathcal{X}} \}. \quad (8.29)$$

If $\mathcal{R}$ is a sub-body of $\mathcal{X}$, then, one has the inclusion

$$\hat{C}'(\pi|_{\mathcal{R}}) \hookrightarrow \hat{C}'(\pi). \quad (8.30)$$

The dual $\hat{C}'(\pi)^*$ to the space of sections supported in $\mathcal{X}$ is the space of extendable functionals. From [Mel96, Proposition 3.3.1] it follows that the restriction

$$\rho : C'(\pi)^* \longrightarrow \hat{C}'(\pi)^*. \quad (8.31)$$

is surjective and its kernel is the space of functionals on $\tilde{\mathcal{X}}$ supported in $\partial \mathcal{X}$.

Thus, if we interpret $T \in C'(\pi)^*$ as a force, $\rho(T) \in \hat{C}'(\pi)^*$ is interpreted as the corresponding body force. For a sub-body $\mathcal{R}$, using the dual of (8.30), one has

$$\rho|_{\mathcal{R}} : \hat{C}'(\pi)^* \longrightarrow \hat{C}'(\pi|_{\mathcal{R}})^*. \quad (8.32)$$

We conclude that even in this very general settings, body forces of any order may be restricted naturally to sub-bodies.

### 8.5. Supported functionals

Distributions on closed subsets of $\mathbb{R}^n$ have been considered by Glaeser [Gla58], Malgrange [Mal66, Chapter 7] and Oksak [Oks76]. The basic tool in the analysis of distributions on closed sets is Whitney’s extension theorem [Whi34] (see also [See64, Hör90]) which guarantees that a differentiable function on a closed subset of $\mathbb{R}^n$ may be extended to a compactly supported smooth function on $\mathbb{R}^n$. The extension mapping between the corresponding function spaces is continuous. The extension theorem implies that restriction of functions is surjective and so, the dual of the restriction mapping associates a unique distribution in an open subset of $\mathbb{R}^n$ with a linear functional defined on the given closed set. Distributions and functionals on manifold with corners have been considered by Melrose [Mel96, Chapter 3], whom we follow below.

Thus, let $T \in C'(\pi)^*$ and let $\tilde{\pi} : \tilde{W} \rightarrow \tilde{\mathcal{X}}$ be an extension of the vector bundle $\pi : W \rightarrow \mathcal{X}$, where $\tilde{\mathcal{X}}$ is a manifold without a boundary. The Whitney–Seeley extension

$$E : C'(\pi) \longrightarrow C'_c(\tilde{\pi}) \quad (8.33)$$

is a continuous injection. It follows that the natural restriction

$$\rho_\pi : C'_c(\tilde{\pi}) \longrightarrow C'(\pi), \quad (8.34)$$

its left inverse satisfying $\rho_\pi \circ E = \text{Id}$, is surjective and the inclusion

$$\rho_\pi^* : C'(\pi)^* \longrightarrow C'_c(\tilde{\pi})^* \quad (8.35)$$

is injective. In other words, each functional $T \in C'(\pi)^*$, determines uniquely a functional $\tilde{T} = \rho_\pi^* T$ satisfying

$$\tilde{T}(\tilde{\chi}) = T(\chi|_{\mathcal{X}}). \quad (8.36)$$
The last equation implies also that $\tilde{T}(\tilde{\chi}) = 0$ for any section $\tilde{\chi}$ supported in $\tilde{\mathcal{X}} \setminus \mathcal{X}$. Hence, $\tilde{T}$ is supported in $\mathcal{X}$.

Conversely, every $\tilde{T} \in C^*_c(\tilde{\mathcal{X}})$, with $\text{supp} \tilde{T} \subset \mathcal{X}$ represents a functional $T \in C^*(\pi')$, i.e., $\tilde{T} = \rho^*_\pi T$. This may be deduced as follows. For any such $\tilde{T}$, consider $T = E^*\tilde{T}$. One needs to show that $\tilde{T} = \rho^*_\pi \circ E^*(\tilde{T})$. Let $\tilde{\chi} \in C^*_c(\tilde{\mathcal{X}})$, then,

$$\begin{align*}
(\tilde{T} - \rho^*_\pi \circ E^*(\tilde{T}))(\tilde{\chi}) &= \tilde{T}(\tilde{\chi}) - \tilde{T}(E \circ \rho_\pi(\tilde{\chi})), \\
&= \tilde{T}(\tilde{\chi}) - E \circ \rho_\pi(\tilde{\chi}).
\end{align*}
$$

(8.37)

It is observed that $\tilde{\chi} - E \circ \rho_\pi(\tilde{\chi})$ vanishes on $\mathcal{X}$ so that $\text{supp}(\tilde{\chi} - E \circ \rho_\pi(\tilde{\chi})) \subset \tilde{\mathcal{X}} \setminus \mathcal{X}$. Since $\text{supp} \tilde{T} \subset \mathcal{X}$, $\tilde{T}(\tilde{\chi}') = 0$ for any section $\chi'$ supported in $\tilde{\mathcal{X}} \setminus \mathcal{X}$. However, approximating the section $\tilde{\chi} - E \circ \rho_\pi(\tilde{\chi})$, supported in the closure, $\mathcal{X} \setminus \mathcal{X}$, by sections supported in $\tilde{\mathcal{X}} \setminus \mathcal{X}$, one concludes that $\tilde{T}(\tilde{\chi} - E \circ \rho_\pi(\tilde{\chi})) = 0$ also.

Due to this construction, Melrose [Mel96, Chapter 3] refers to such functionals (distributions) as supported. It is noted that such functionals of compact support are of a finite order $r$.

8.6. Density dual and smooth functionals. A simple example for functionals on spaces of sections of a vector bundle $\pi : W \to \mathcal{X}$ is provided by smooth functionals. Consider, in analogy with the dual of a vector bundle, the vector bundle of linear mappings into another one-dimensional vector bundle, that of $n$-alternating tensors. Thus, for a given vector bundle, $W$, we use the notation (see Atiyah and Bott [AB67])

$$W' = L(W, \wedge^n T^* \mathcal{X}) \cong W^* \otimes \wedge^n T^* \mathcal{X}. \quad (8.38)$$

Let $A : W_1 \to W_2$ be a vector bundle morphism over $\mathcal{X}$. Then, in analogy with the dual mapping, one may consider

$$A' : W'_2 \to W'_1, \quad \text{given by} \quad f \mapsto f \circ A. \quad (8.39)$$

It is also noted that we have

$$(W')' = (W^* \otimes \wedge^n T^* \mathcal{X})'$$

$$= (W^* \otimes \wedge^n T^* \mathcal{X})^* \otimes \wedge^n T^* \mathcal{X}, \quad (8.40)$$

and as $\wedge^n T^* \mathcal{X} \otimes \wedge^n T^* \mathcal{X}$ is isomorphic with $\mathbb{R}$, one has a natural isomorphism

$$(W')' \cong W. \quad (8.41)$$

For the vector bundles $W, U$,

$$(W \otimes U)' \equiv W^* \otimes U^* \otimes \wedge^n T^* \mathcal{X} \equiv W^* \otimes U'. \quad (8.42)$$

We will refer to $W'$ as the density-dual bundle and to $A'$ as the density-dual mapping.

As an example, for the case $W = \wedge^p T^* \mathcal{X}$, we have an isomorphism (see Section 2.4),

$$e_\perp : \wedge^{n-p} T^* \mathcal{X} \to (\wedge^p T^* \mathcal{X})', \quad (8.43)$$
given by

$$e_\omega (\psi) = \omega \wedge \psi.$$  \hfill (8.44)

Smooth functionals may be induced by smooth sections of $W'$. For a section $S$ of $W'$, and a section $\chi$ of $W$, let $S \cdot \chi$ be the $n$-form

$$S \cdot \chi(X) = S(X)(\chi(X)).$$  \hfill (8.45)

The smooth functional $T_S$ induced by $S$ is defined by

$$T_S(\chi) := \int_S S \cdot \chi.$$  \hfill (8.46)

8.7. Generalized sections and distributions. Let $\pi_0 : W_0 \to \mathcal{X}$ be a vector bundle and consider the case where the vector bundle $\pi$ above is set to be $\pi : W'_0 := L(W_0, \wedge^n T^* \mathcal{X}) \cong W_0^* \otimes \wedge^n T^* \mathcal{X} \to \mathcal{X}$.  \hfill (8.47)

Thus, the corresponding functionals on sections of $\pi$ are elements of

$$C^r(\pi)^* = C^r(L(W_0, \wedge^n T^* \mathcal{X}))^* \cong C^r(W_0^* \otimes \wedge^n T^* \mathcal{X})^*.$$  \hfill (8.48)

In this case, smooth functionals are represented by smooth sections of

$$L(W, \wedge^n T^* \mathcal{X}) = L(W_0^* \otimes \wedge^n T^* \mathcal{X}, \wedge^n T^* \mathcal{X}),$$

$$\cong (W_0^* \otimes \wedge^n T^* \mathcal{X})^* \otimes \wedge^n T^* \mathcal{X},$$

$$\cong W_0 \otimes \wedge^n T \mathcal{X} \otimes \wedge^n T^* \mathcal{X},$$

$$\cong W_0.$$  \hfill (8.49)

One concludes that smooth functionals in $C^r(W_0^* \otimes \wedge^n T^* \mathcal{X})^*$ are represented by sections of $W_0$. It is natural therefore to refer to elements of

$$C^{-r}(W_0) := C^r(W_0^*)$$  \hfill (8.50)

as generalized sections of $W_0$ (see [AB67, GS77, GKOS01], [Kor08, p. 676]).

In the particular case where $W_0 = \mathcal{X} \times \mathbb{R}$ is the natural line bundle, smooth functionals are represented by real valued functions on $\mathcal{X}$. Consequently, elements of

$$C^r(\mathbb{R} \otimes \wedge^n T^* \mathcal{X})^* = C^r(\wedge^n T^* \mathcal{X})^*$$  \hfill (8.51)

are referred to as generalized functions.

The apparent complication in the definition of generalized sections using the density dual is justified in the sense that each element in $C^{-k}(W_0)$ may be approximated by a sequence of smooth functionals induced by sections of $W'_0$. (See [GKOS01] p. 241.)

In the literature, the term section distributions is used in different ways in this context. For example, in [GKOS01] and [Kor08] p. 676], $W_0$-valued distributions are defined as elements of $C^r(W_0^*)$, i.e., what are referred to here as generalized sections of $W_0$. (In [AB67, AS68] they are referred to as distributional sections.) On the other hand, in [GS77], distributions are defined as generalized sections of $\wedge^n T^* \mathcal{X}$—elements of $C^r(\mathcal{X})^*$. See further comments on this
9. de Rham currents

For a manifold without boundary $\mathcal{X}$, de Rham currents (see \cite{Rha84, Sch73, Fed69}) are functionals corresponding to the case of the vector bundle

$$\pi : \Lambda^p T^* \mathcal{X} \longrightarrow \mathcal{X}$$

so that test sections are smooth $p$-forms having compact supports. Thus, a $p$-current of order $r$ on $\mathcal{X}$ is a continuous linear functional on $C^r_c(\Lambda^p T^* \mathcal{X})$. A particular type of $p$-currents, smooth currents, are induced by differential $(n-p)$-forms. Such an $(n-p)$ form, $\omega$, induces the currents $\omega T$ and $T \omega = (-1)^p \omega T$ by

$$\omega T(\psi) = \int_{\mathcal{X}} \omega \wedge \psi, \quad T \omega(\psi) = \int_{\mathcal{X}} \psi \wedge \omega.$$  \hspace{1cm} (9.2)

Another simple $p$-current, $T_{\mathcal{Z}}$ is induced by an oriented $p$-dimensional submanifold $\mathcal{Z} \subset \mathcal{X}$. It is naturally defined by

$$T_{\mathcal{Z}}(\psi) = \int_{\mathcal{Z}} \psi.$$ \hspace{1cm} (9.3)

These two examples illustrate the two points of views on currents. On the one hand, the example of the current $\omega T$ suggests that a current in $C^r_c(\Lambda^p T^* \mathcal{X})^*$ is viewed as a generalized $(n-p)$-form. With this point of view in mind, elements of $C^r_c(\Lambda^p T^* \mathcal{X})$ are referred to as currents of degree $n-p$. Consequently, the space of $p$-currents on $\mathcal{X}$ is occasionally denoted by

$$C^{-r}(\Lambda^{n-p} T^* \mathcal{X}) = C^r_c(\Lambda^p T^* \mathcal{X})^*.$$ \hspace{1cm} (9.4)

On the other hand, the example of the current $T_{\mathcal{Z}}$ induced by a $p$-dimensional manifolds $\mathcal{Z}$, suggests that currents be viewed as a geometric object of dimension $p$. Thus, an element of $C^r_c(\Lambda^p T^* \mathcal{X})^*$ is referred to as a $p$-dimensional current.

9.1. Basic operations with currents. The contraction operations of a $(p+q)$-current $T$ and a $q$-form $\omega$, yields the $p$-currents defined by

$$(T \wedge \omega)(\psi) = T(\psi \wedge \omega) \quad \text{and} \quad (\omega \bullet T)(\psi) = T(\omega \wedge \psi),$$ \hspace{1cm} (9.5)

so that

$$T \wedge \omega = (-1)^p q \omega \bullet T.$$ \hspace{1cm} (9.6)

Note that our notation is different from that of \cite{Rha84} and different in sign form that of \cite{Fed69}. In particular, given a $p$-current $T$, any $p$-form $\psi$ induces naturally a zero-current

$$T \cdot \psi = T \wedge \psi,$$ \hspace{1cm} (9.7)

so that

$$T(\cdot \psi)(u) = T(u \psi).$$

The $p$-current $\omega T$ defined above can be expressed using contraction in the form

$$\omega T = \omega \bullet T_{\mathcal{X}}.$$ \hspace{1cm} (9.8)
For a $p$-current $T$ and a $q$-multi-vector field $\xi$, the $(p+q)$-currents $\xi \wedge T$ and $T \wedge \xi$ are defined by
\[
(\xi \wedge T)(\psi) := T(\xi \wedge \psi), \quad (T \wedge \xi)(\psi) := T(\psi \wedge \xi),
\]
for a $(p+q)$-form $\psi$. Using, $\xi \wedge \psi = (-1)^p \psi \wedge \xi$, one has
\[
\xi \wedge T = (-1)^p q T \wedge \xi,
\]
in analogy with the corresponding expression for multi-vectors. Thus, the wedge product of a $p$-current and an $r$-multi-vector is an $(r+p)$-current. Note that a real valued function $u$ defined on $X$ may be viewed both as a zero-form and as a zero-multi-vector. Hence, we may write $uT$ for any of the four operations defined above so that $(uT)(\psi) = T(u\psi)$.

The boundary operator
defined by
\[
\partial : C^{-r}(\wedge^{n-p} T^* X) \longrightarrow C^{-(r+1)}(\wedge^{n-(p-1)} T^* X),
\]
is a linear and continuous operator. In other words, the boundary of a $p$-current is a $(p-1)$-current. In particular, for a smooth current, $\omega T$ represented by the $(n-p)$-form $\omega$, one has
\[
\partial \omega T(\psi) = \int_X \omega \wedge d\psi,
\]
\[
= (-1)^{n-p} \int_X d(\omega \wedge \psi) - (-1)^{n-p} \int_X d\omega \wedge \psi,
\]
\[
= (-1)^{n-p+1} T_{d\omega}(\psi).
\]
Hence,
\[
\partial \omega T = (-1)^{n-p+1} T_{d\omega}.
\]
Similarly,
\[
\partial T_\omega = (-1)^{p+1} T_{d\omega}.
\]
In order to strengthen further the point of view that a $p$-current is a generalized $(n-p)$-form, the exterior derivative of a $p$-current $dT$ is defined by
\[
dT = (-1)^{n-p+1} \partial T.
\]
Thus, in the smooth case,
\[
d\omega T = T_{d\omega}.
\]
In addition, Stokes's theorem implies that for the $p$-current $T_\Sigma$ induced by the $p$-dimensional submanifold with boundary $\Sigma$, the boundary, a $(p-1)$-current, is given by
\[
\partial T_\Sigma = T_{d\Sigma}.
\]
It is quite evident, therefore, that the notion of a boundary generalizes and unites both the exterior derivative of forms and the boundaries of manifolds.

9.2. Local representation of currents. We consider next the local representation of de Rham currents in coordinate neighborhoods.
9.2.1. **Representation by 0-currents.** Let \( R = E^*_T \) be the restriction of a \( p \)-current \( T \) to forms supported in a particular coordinate neighborhood—a local representative of \( T \). Writing

\[
R(\psi) = R(\psi_\lambda dX^\lambda), \quad |\lambda| = p,
\]

(9.19)

we could have used \( R \cdot dX^\lambda \) just the same as the \( \psi_\lambda \) are real valued functions), one notes that locally

\[
R(\psi) = R(\psi_\lambda), \quad \text{where,} \quad R^\lambda := dX^\lambda \cdot R.
\]

(9.20)

Using the exterior product of a multi-vector field \( \xi \) and a current in (9.9), we may write

\[
R(\psi) = R^\lambda (\partial_\lambda \psi) = \partial_\lambda \wedge R^\lambda(\psi),
\]

(9.21)

and so a current may be represented locally in the form

\[
R = \partial_\lambda \wedge R^\lambda.
\]

(9.22)

This representation suggests that \( T \) be interpreted as a generalized multi-vector field (cf. [Whi57]).

In the sequel, when we refer to local representative of a current \( T \), we will often keep the same notation, \( T \), and it will be implied that we consider the restriction of \( T \) to forms (or sections, in general) supported in a generic coordinate neighborhood.

9.2.2. **Representation by \( n \)-currents.** Alternatively (cf. [Rha84, p. 36]), for a \( p \)-current \( R \) defined in a coordinate neighborhood and \( \hat{\lambda} \) with \( |\hat{\lambda}| = n - p \), consider the \( n \)-currents

\[
R_{\hat{\lambda}} := \partial_{\hat{\lambda}} \wedge R, \quad \text{so that} \quad R_{\hat{\lambda}}(\theta) = R(\partial_{\hat{\lambda}} \Lambda \theta).
\]

(9.23)

Then, for every \( p \)-form \( \omega \),

\[
(dX^{\hat{\lambda}} \cdot R_{\hat{\lambda}})(\omega) = R_{\hat{\lambda}}(dX^{\hat{\lambda}} \wedge \omega),
\]

\[
= R(\partial_{\hat{\lambda}} \wedge (dX^{\hat{\lambda}} \wedge \omega)),
\]

\[
= R_{\hat{\lambda}}(\epsilon_{\hat{\lambda} \mu} \omega_\mu dX),
\]

(9.24)

where we used

\[
dX^{\hat{\lambda}} \wedge \omega = \epsilon_{\hat{\lambda} \mu} \omega_\mu dX.
\]

(9.25)

Also,

\[
(\partial_{\hat{\lambda}} dX)(\partial_\mu) = dX(\partial_{\hat{\lambda}} \wedge \partial_\mu),
\]

\[
= \epsilon_{\hat{\lambda} \mu},
\]

(9.26)

\[
= \epsilon_{\hat{\lambda} \nu} dX^\nu(\partial_\mu),
\]

implies

\[
\partial_{\hat{\lambda}} dX = \epsilon_{\hat{\lambda} \nu} dX^\nu,
\]

(9.27)
and so, 
\[ \partial_{\hat{\lambda}} \wedge (dX^{\hat{\lambda}} \wedge \omega) = \omega_{\hat{\lambda}} dX^{\hat{\lambda}} = \omega, \] (9.28)
as expected. Hence, 
\[ (dX^{\hat{\lambda}} \wedge R_{\hat{\lambda}})(\omega) = R(\omega), \] (9.29)
and we conclude that \( R \) may be represented by the \( n \)-currents 
\[ R_{\hat{\lambda}} := \partial_{\hat{\lambda}} \wedge R, \quad \text{in the form} \quad R = dX^{\hat{\lambda}} \wedge R_{\hat{\lambda}}, \] (9.30)
with 
\[ R(\omega) = R_{\hat{\lambda}}(dX^{\hat{\lambda}} \wedge \omega) = \epsilon^{\hat{\lambda}\mu} R_{\hat{\lambda}}(\omega_{\mu} dX) = \sum_{\hat{\lambda}} \epsilon^{\hat{\lambda}\lambda} R_{\hat{\lambda}}(\omega_{\lambda} dX). \] (9.31)
This representation suggests again that a \( p \)-current \( T \) be interpreted as a generalized \((n - p)\)-form. In particular, an \( n \)-current is a generalized function and is often referred to as a distribution on the manifold (e.g., \cite[Chapter 3]{Mel96}).

**Remark 9.1.** It is noted that one may set 
\[ R'_{\hat{\lambda}} := R \wedge \partial_{\hat{\lambda}}. \] (9.32)
Using (9.10) for the \((n - p)\)-multi-vector \( \partial_{\hat{\lambda}} \) 
\[ R'_{\hat{\lambda}} = (-1)^{p(n-p)} R_{\hat{\lambda}}. \] (9.33)
In addition, by (9.30) and (9.6), for the \( n \)-current \( R_{\hat{\lambda}} \) and the \((n - p)\)-form \( dX^{\hat{\lambda}} \), 
\[ R = R'_{\hat{\lambda}} \wedge dX^{\hat{\lambda}}, \] (9.34)
and 
\[ R(\omega) = R'_{\hat{\lambda}}(\omega \wedge dX^{\hat{\lambda}}) = \sum_{\hat{\lambda}} \epsilon^{\hat{\lambda}\lambda} R'_{\hat{\lambda}}(\omega_{\lambda} dX). \] (9.35)

### 10. Vector-valued currents

A natural extension of the notions of generalized sections and de Rham currents yields vector valued vector valued currents that will be used to model stresses. Vector valued currents and their local representations will be considered in this section.

**10.1. Vector valued forms.** Let \( \pi : W \to \mathcal{X} \) be a given vector bundle whose typical fiber is \( m \)-dimensional. We will refer to sections of 
\[ L(W, \wedge^p T^* \mathcal{X}) \cong W^* \otimes \wedge^p T^* \mathcal{X}. \] (10.1)
as vector valued \( p \)-forms, which is short for the more appropriate vector bundle valued \( p \)-form (cf. \cite[p. 340]{Sch73}). Thus in particular, sections of the density dual, \( W^* = W^* \otimes \wedge^n T^* \mathcal{X} \) are vector valued forms. In the mechanical context, we will also be concerned with co-vector valued forms, that is, sections of 
\[ L(W^*, \wedge^p T^* \mathcal{X}) \cong W \otimes \wedge^p T^* \mathcal{X}. \] (10.2)
The terminology follows from the observation that using the isomorphism induced by transposition, i.e., \( \wedge^p T^* X \otimes W \cong W \otimes \wedge^p T X \), a co-vector valued form may be viewed as a section of

\[
\wedge^p T^* X \otimes W \cong \wedge^p (T X, W) \cong U(\wedge^p T^* X, W).
\]  

(10.3)

Given a co-vector valued \( p \)-form, \( \chi \), and a vector valued \( (n-p) \)-form, \( f \), one can define the bilinear action \( f \wedge \chi \) by setting

\[
(g \otimes \omega) \wedge (w \otimes \psi) := g(\omega) \wedge \psi,
\]

for sections \( g, w, \omega, \psi \) of \( W^* \), \( W \), \( \wedge^{n-p} T^* X \), \( \wedge^p T X \), respectively. Thus, \( \wedge \) induces a bilinear mapping

\[
\hat{\chi} : (W^* \otimes \wedge^{n-p} T^* X) \times (W \otimes \wedge^p T X) \rightarrow \wedge^n T^* X,
\]

(10.5)
or a linear

\[
\hat{\chi} : W^* \otimes \wedge^{n-p} T^* X \otimes W \otimes \wedge^p T X \rightarrow \wedge^n T^* X.
\]

(10.6)
The mapping \( \hat{\chi} \) gives rise to an extension of the isomorphism \( e_{\wedge} \) considered above to an isomorphism (we keep the same notation)

\[
e_{\wedge} : W^* \otimes \wedge^{n-p} T^* X \rightarrow (W \otimes \wedge^p T X)' \quad \text{and} \quad e_{\wedge}(f)(\chi) = f \wedge \chi.
\]

(10.7)

Let \( \{ (U_a, \phi_a, \Phi_a) \}_{a \in A} \) be a vector bundle trivialization for the vector bundle \( \pi : W \rightarrow X \) so that

\[
\Phi_a : \pi^{-1}(U_a) \rightarrow U_a \times W,
\]

(10.8)
where \( W \) is the \( m \)-dimensional typical fiber. Given a basis in \( W \), let \( \{ e_a \}_{a=1}^m \) and \( \{ e^a \}_{a=1}^m \) be the local bases and dual bases induced by \( \Phi_a^{-1} \) on \( \pi^{-1}(U_a) \). Then, a co-vector valued form \( \chi \) and a vector valued \( p \)-form \( f \) are represented locally in the forms

\[
\chi^a e_a \otimes dX^\lambda, \quad \text{and} \quad f_{a\lambda} e^a \otimes dX^\lambda, \quad |\lambda| = p,
\]

(10.9)
respectively.

10.2. **Vector valued currents.** We now substitute the vector bundle \( W^* \otimes \wedge^p T^* X \) for the vector bundle \( W_0 \) in definition (8.50) of generalized sections. Thus,

\[
C^{-\tau}(W^* \otimes \wedge^p T^* X) = C'(W^* \otimes \wedge^p T^* X)^*.
\]

(10.10)
Using the isomorphism \( e_{\wedge} \) as defined above, it is concluded that we may make the identifications

\[
C^{-\tau}(W^* \otimes \wedge^p T^* X) = C'(W \otimes \wedge^{n-p} T^* X)^* \quad \text{(see [Sch73 p. 340]).}
\]

Comparing the last equation to (10.1) we may refer to elements of these spaces as *generalized vector valued \( p \)-forms* or as *vector valued \((n-p)\)-currents*.

A smooth vector valued \((n-p)\)-current may be represented by a \( W^* \otimes \wedge^{n-p} T X \) valued \( n \)-form—a smooth section \( S \) of \( W^* \otimes \wedge^{n-p} T X \otimes \wedge^n T^* X \) by

\[
\chi \mapsto \int_X S \cdot \chi,
\]

(10.12)
where it is noted that $S \cdot \chi$ is an $n$-form. Locally, for $|\mu| = n - p$,
\[ S = S_\alpha^\mu e^\alpha \otimes \partial_\mu \otimes dX, \quad S \cdot \chi = S_\alpha^\mu \chi^\alpha dX. \quad (10.13) \]

Alternatively, a smooth element of $C^{-r}(W^* \wedge \wedge^p T^*\mathcal{X})$ is induced by a section $\tilde{S}$ of $W \otimes \wedge^p T^*\mathcal{X}$ in the form
\[ \chi \mapsto \int_{\mathcal{X}} \tilde{S} \wedge \chi, \quad (10.14) \]
for every $C^r$-section $\chi$ of $W \otimes \wedge^{n-p} T^*\mathcal{X}$. Locally, for $|\tilde{\mu}| = p$, $|\lambda| = n - p$,
\[ \tilde{S} = \tilde{S}_\alpha^\mu e^\alpha \otimes dX^\mu, \quad \tilde{S} \wedge \chi = \tilde{S}_\alpha^\mu \chi^\alpha dX^\mu \wedge dX^\lambda = \sum_\lambda \varepsilon_{\lambda \lambda} \tilde{S}_\alpha^\mu \chi^\alpha dX^\mu. \quad (10.15) \]
Comparing the last two expressions for the resulting densities, one concludes that
\[ \tilde{S}_\alpha^\lambda = \varepsilon_{\lambda \lambda} S_\alpha^\lambda. \quad (10.16) \]
Globally, it follows that
\[ \tilde{S} = C_{\alpha}(S), \quad (10.17) \]
where
\[ C_{\alpha} : W^* \wedge \wedge^{n-p} T^*\mathcal{X} \otimes \wedge^{n-p} T^*\mathcal{X} \to W^* \wedge \wedge^p T^*\mathcal{X} \quad (10.18) \]
is induced by the right contraction $\theta_{\alpha} \eta_1(\eta_2) = \theta(\eta_2 \wedge \eta_1)$. What determined the direction of the contraction is the choice of action of $\tilde{S}$ in $\text{[10.7]}$ as in Remark 9.1.

10.3. **Local representation of vector valued currents.** We now consider the local representation of the restriction of a vector valued $p$-current to vector valued forms supported in some given vector bundle chart. We introduce first some basic operations.

10.3.1. **The inner product of a vector valued current and a vector field.** Given a vector valued current $T$ in $C'(W \otimes \wedge^p T^*\mathcal{X})^*$ and a $C^r$-section $w$ of $W$, we define the (scalar) $p$-current $T \cdot w$ by
\[ T \cdot w(\omega) = T(w \otimes \omega). \quad (10.19) \]
For local representation, one may consider the $p$-currents
\[ T_a := T \cdot e_a. \quad (10.20) \]
Thus, in analogy with $\text{[9.19, 9.20]}$ we have
\[ T(w \otimes \omega) = T(w^a e_a \otimes \omega), \]
\[ = (T \cdot e_a)(w^a \omega), \]
\[ = T_a(w^a \omega). \quad (10.21) \]
10.3.2. *The tensor product of a current and a co-vector field.* A (scalar) $p$-current, $T$, and a $C^r$-section of $W^*$, $g$, induce a vector valued current $g \otimes T \in C^r (W \otimes \wedge^p T^* \mathfrak{X})^*$ by setting

\[(g \otimes T)(w \otimes \omega) := T((g \cdot w) \omega).\] (10.22)

In particular, locally,

\[(e^a \otimes T)(w \otimes \omega) := T(w^a \omega).\] (10.23)

Utilizing this definition, one may write for local representatives

\[e^a \otimes T_a (w \otimes \omega) = T_a (w^a \omega),\] (10.24)

and so, complementing (10.21), one has

\[T = e^a \otimes T_a.\] (10.25)

10.3.3. *Representation by 0-currents.* Proceeding as in Section 9.2, the $p$-current $T$ may be represented by the 0-currents

\[T^\lambda := dX^\lambda \wedge T_a = dX^\lambda \wedge (T \cdot e_a),\] in the form \[T(\chi) = T^\lambda \chi^\lambda.\] (10.26)

Using (9.9) and (9.22), we finally have

\[T = e^a \otimes (\partial_\lambda \wedge T^\lambda_a).\] (10.27)

In the case where the 0-currents $T^\lambda_a$ are represented locally by smooth $n$-forms $S^\lambda_a dX$, one has

\[T(\chi) = \int_U S^\lambda_a \chi^\lambda dX\] (10.28)

in accordance with (10.13).

10.3.4. *The exterior product of a vector valued current and a multi-vector field.* Next, in analogy with Section 9.2, for a vector valued $p$-current $T$ and a $q$-multi-vector $\eta$, $q \leq n - p$, consider the vector valued $(p + q)$-current $\eta \wedge T$ defined by

\[(\eta \wedge T)(w \otimes \omega) := T(w \otimes (\eta \wedge \omega)).\] (10.29)

In particular, for multi-indices $\hat{\lambda}$, $|\hat{\lambda}| = n - p$, we define locally the vector valued $n$-currents

\[T_{\hat{\lambda}} := \partial_{\hat{\lambda}} \wedge T,\quad T_{\hat{\lambda}} (w \otimes \theta) = T(w \otimes (\partial_{\hat{\lambda}} \wedge \theta)),\] (10.30)

so that

\[T_{\hat{\lambda}} (w \otimes dX) = T(w \otimes (\partial_{\hat{\lambda}} \wedge dX)).\] (10.31)
10.3.5. The contraction of a vector valued current and a form. Also, for a vector valued \( p \)-current \( T \) and a \( q \)-form \( \psi \), \( q \leq p \), define the vector valued \((p - q)\)-currents \( \psi \downarrow T \) and \( T \downarrow \psi \) as
\[
(\psi \downarrow T)(w \otimes \omega) := T(w \otimes (\psi \wedge \omega)) \tag{10.32}
\]
and
\[
(T \downarrow \psi)(w \otimes \omega) := T(w \otimes (\omega \wedge \psi)), \tag{10.33}
\]
so that \( T \downarrow \psi = (-1)^{pq} \psi \downarrow T \). In the case where \( q = p \), we obtain an element \( \omega \downarrow T \in C'(W)^* \), a vector valued 0-current, satisfying
\[
(\omega \downarrow T)(w) = T(w \otimes \omega). \tag{10.34}
\]
Locally, one may consider the functionals—vector valued 0-currents,
\[
T^\lambda := dX^\lambda \downarrow T, \quad T^\lambda(w) = T(w \otimes dX^\lambda), \quad |\lambda| = p. \tag{10.35}
\]
Hence,
\[
T(w \otimes \omega) = T^\lambda(\omega_\lambda w). \tag{10.36}
\]
is a local representation of the action of \( T \) using vector valued 0-currents. It is implied by the identity \( T^\lambda(\omega_\lambda w) = \partial_\lambda \wedge T^\lambda(w \otimes \omega) \), that
\[
T = \partial_\lambda \wedge T^\lambda. \tag{10.37}
\]

10.3.6. Representation by \( n \)-currents. Next, for a local basis \( dX^\lambda \) of \( \wedge^{n-p} T^* X \), using \( (10.30) \) and \( (10.34) \),
\[
(dX^\lambda \downarrow T^\lambda)(w \otimes \omega) = T^\lambda(w \otimes (dX^\lambda \wedge \omega)), \tag{10.38}
\]
\[
= T(w \otimes (\partial_\lambda \downarrow (dX^\lambda \wedge \omega))).
\]
Following the same procedure as that leading to \( (9.30) \) and \( (9.31) \), one concludes that the vector valued \( p \)-current \( T \) may be represented locally by the vector valued \( n \)-currents \( T^\lambda \) in the form
\[
T = dX^\lambda \downarrow T^\lambda, \tag{10.39}
\]
and
\[
T(w \otimes \omega) = T^\lambda(w \otimes (dX^\lambda \wedge \omega)) = \sum_\lambda \epsilon^\lambda T^\lambda(\omega_\lambda w \otimes dX). \tag{10.40}
\]
Using \( (10.19) \) and \( (10.25) \), we may define the (scalar) \( n \)-currents
\[
T_{a\lambda} := T^\lambda \cdot e_a, \quad \text{so that} \quad T^\lambda = e^a \otimes T_{a\lambda}, \tag{10.41}
\]
and
\[
T_{a\lambda}(\theta) = T^\lambda(e_a \otimes \theta) = T(e_a \otimes (\partial_\lambda \downarrow \theta)). \tag{10.42}
\]
Considering the \( p \)-currents \( T_\alpha \) in (10.20), the local components \((T_\alpha)_{\hat{\lambda}}\) are defined by \((T_\alpha)_{\hat{\lambda}} = \partial_{\hat{\lambda}} \wedge (T \cdot e_\alpha)\), hence,

\[
(T_\alpha)_{\hat{\lambda}}(\theta) = \partial_{\hat{\lambda}} \wedge (T \cdot e_\alpha)(\theta),
= (T \cdot e_\alpha)(\partial_{\hat{\lambda}} \wedge \theta),
= T(e_\alpha \otimes (\partial_{\hat{\lambda}} \wedge \theta)),
\]

and we conclude that

\[
T_{a\hat{\lambda}} = (T_\alpha)_{\hat{\lambda}}.
\]

Thus, Equations (10.39) and (10.40) may be rewritten as

\[
T = dX^{\hat{\lambda}} \cdot (e^\alpha \otimes T_{a\hat{\lambda}}),
\]

and

\[
T(w \otimes \omega) = T_{a\hat{\lambda}}(w^\alpha (dX^{\hat{\lambda}} \wedge \omega)),
= \sum_{\lambda} \epsilon^{\hat{\lambda} \lambda} T_{a\hat{\lambda}}(\omega w^\alpha dX),
= \sum_{\hat{\lambda}} \epsilon^{\hat{\lambda} \lambda} dX^{\hat{\lambda}} T_{a\hat{\lambda}}(\omega w^\alpha),
\]

Comparing the last equation with (10.26) we arrive at

\[
dX^{\hat{\lambda}} T_{a\hat{\lambda}} = \epsilon^{\hat{\lambda} \lambda} T_{a\hat{\lambda}}^{\lambda},
T_{a\hat{\lambda}} = \epsilon_{\hat{\lambda}\lambda} \partial X \wedge T_{\alpha}^{\hat{\lambda}}.
\]

In the smooth case, the \( n \)-currents \( T_{a\hat{\lambda}} \) are represented by functions \( \widehat{S}_{a\hat{\lambda}} \) that make up the vector valued \((n-p)\)-form

\[
\widehat{S} = \widehat{S}_{a\hat{\lambda}} e^\alpha \otimes dX^\hat{\lambda}
\]
as in (10.14) (10.15).

**Remark 10.1.** In summary, the representation by zero currents (e.g., (10.28) corresponds to viewing the vector valued current as an element of \( C^r(W \otimes \wedge^p T^*X)^* = C^{r-1}(W^* \otimes \wedge^p T^*X \otimes \wedge^n T^*X) \). On the other hand, the representation as in (10.28) (10.48) corresponds to the point of view that by the isomorphism of \( \wedge^p T^*X \otimes \wedge^n T^*X \) with \( \wedge^n -^p T^*X \),

\[
C^r(W \otimes \wedge^p T^*X)^* \equiv C^{r-1}(W^* \otimes \wedge^n -^p T^*X)
\]

**Remark 10.2.** In the foregoing discussion we have made special choices and used, for example, the definitions, \( T_{\hat{\lambda}} := \partial_{\hat{\lambda}} \wedge T \) and \( T^{\hat{\lambda}} := dX^{\hat{\lambda}} \wedge T \) rather than \( T'_{\hat{\lambda}} := T \wedge \partial_{\hat{\lambda}} \) and \( T'^{\hat{\lambda}} := T \cdot dX^{\hat{\lambda}} \), respectively. The correspondence between the two schemes is a natural extension of Remark 9.1. In particular, \( \epsilon_{\hat{\lambda}\lambda} \) will be replaced by \( \epsilon_{\hat{\lambda}\lambda} \).
11. The Representation of Forces by Hyper-Stresses and Non-Holonomic Stresses

11.1. Stresses and non-holonomic stresses. We recall that the tangent space $T_κ C^r(ξ)$ to the Banach manifold of $C^r$-sections of the fiber bundle $ξ : Y → X$ at the section $κ : X → Y$ may be identified with the Banachable space $C^r(κ^∗V Y)$ of sections of the pullback vector bundle $κ^∗τ Y : κ^∗V Y → X$. Elements of the tangent space at $κ$ to the configuration manifold represent generalized velocities of the continuous mechanical system. Consequently, a generalized force is modeled mathematically by and element $F ∈ C^r(κ^∗V Y)^∗$. The central message of this section is that although such functionals cannot be restricted naturally to subbodies of $X$, as discussed in Section 8.3, forces may be represented, non-uniquely, by stress objects that enable restriction of forces to sub-bodies.

In order to simplify the notation, we will consider a general vector bundle $π : W → X$, as in Section 3, and the notation introduced there will be used throughout. The construction is analogous to representation theorem for distributions of finite order (e.g., [Sch73, p. 91] or [Trè67, p. 259]).

Consider the jet extension linear mapping

$$j^r : C^r(π) → C^0(π^r)$$

as in Section 3.3. As noted, $j^r$ is an embedding and under the norm induced by an atlas, it is even isometric. Evidently, due to the compatibility constraint, Image $j^r$ is a proper subset of $C^0(π^r)$ and its complement is open. Hence, the inverse

$$(j^r)^{-1} : \text{Image } j^r → C^r(π)$$

is a well defined linear homeomorphism. Given a force $F ∈ C^r(π)^∗$, the linear functional

$$F ∘ (j^r)^{-1} : \text{Image } j^r → \mathbb{R}$$

is a continuous and linear functional on Image $j^r$. Hence, by the Hahn–Banach theorem, it may be extended to a linear functional $ζ ∈ C^0(π^r)^∗$. In other words, the linear mapping

$$j^r∗ : C^0(π^r)^∗ → C^r(π)^∗$$

is surjective.

By the definition of the dual mapping, $ζ$ represents a force $F$, i.e.,

$$j^r∗ζ = F$$

if and only if,

$$F(ω) = ζ(j^rω)$$

for all $C^r$-virtual velocity fields $ω$. The object $ζ ∈ C^0(π^r)^∗$ is interpreted as a generalization of the notion of hyper-stress in higher-order continuum mechanics and will be so referred to. For $r = 1$, $ζ$ is a generalization of the standard stress tensor. The condition (11.6), resulting from the representation theorem, is a generalization of the principle of virtual work as it states that the power
expended by the force $F$ for a virtual velocity field $w$ is equal to the power expended by the hyper-stress for $j^r w$—containing the first $r$ derivatives of the velocity field. Accordingly, Equation (11.5) is a generalization of the equilibrium equation of continuum mechanics.

It is noted that $\varsigma$ is not unique. The non-uniqueness originates from the fact that the image of the jet extension mapping, containing the compatible jet fields, is not a dense subset of $C^0(\pi^r)$. Thus, the static indeterminacy of continuum mechanics follows naturally from the representation theorem.

In view of (3.19), the same procedure applies if we use the non-holonomic jet extension $\hat{j} : C^r(\pi) \to C^0(\hat{\pi}^r)$. A force $F$ may then be represented by a non-unique, non-holonomic stress $\hat{\varsigma} \in C^0(\hat{\pi}^r)^*$ in the form

$$F = j^r \hat{\varsigma}. \quad (11.7)$$

The mapping $C^0(i^r)$ of Section 3.4 is an embedding. Hence, a hyper-stress $\varsigma$ may be represented by some non-unique, non-holonomic stress $\hat{\varsigma}$ in the form

$$\varsigma = C^0(i^r)^*(\hat{\varsigma}), \quad (11.8)$$

and in the following commutative diagram all mappings are surjective.

$$\begin{array}{c}
C^r(\pi) \\
j^r \searrow \nearrow j^r^* \\
\overset{\hat{j}^r}{\searrow} C^0(\pi^r) \overset{\hat{C}^0(i^r)^*}{\searrow} C^0(\hat{\pi}^r) \overset{\hat{j}^r^*}{\nearrow}
\end{array} \quad (11.9)$$

11.2. **Smooth stresses.** In view of the discussion in Section 8.7, hyper-stresses are elements of

$$C^0(\pi^r)^* = C^0(\pi^r)^* = C^0(\pi^r)^* \otimes \wedge^n T^* X, \quad (11.10)$$

and so they may be approximated by smooth sections of $\pi^r \otimes \wedge^n T^* X$, i.e., $n$-forms valued in the dual of the $r$-jet bundle.

Similarly, non-holonomic stresses are elements of

$$C^0(\hat{\pi}^r)^* = C^0(\hat{\pi}^r)^* \otimes \wedge^n T^* \hat{X}, \quad (11.11)$$

and smooth non-holonomic stresses are $n$-forms valued in the dual of the $r$-iterated jet bundle.

11.3. **Stress measures.** Analytically, stresses are vector valued zero-currents that are representable by integration. (See [Fed69, Section 4.1], for the scalar case).

As noted in Section 8.2, given a vector bundle atlas $\{(U_a, \varphi_a, \Phi_a)\}_{a \in A}$, a linear functional is uniquely determined by its restrictions to sections supported in the various domains $\varphi_a(U_a)$—its local representatives. In particular, for the case of an $m$-dimensional vector bundle $\pi : W \to X$, and the space of functionals $C^0(\pi)^*$, a typical local representative is an element $T_a \in (C^0_c(\varphi_a(U_a))^*)^m$.

Thus, each component $(T_a)_{\alpha} \in C^0_c(\varphi_a(U_a))^*$ is a Radon measure or a distribution
representable by integration. We will use the same notation for the measure. Consequently, for a section \( w_a \) compactly supported in \( \varphi_a(U_a) \), we may write

\[
T_a(w_a) = \int_{\varphi_a(U_a)} w_a \cdot dT_a := \int_{\varphi_a(U_a)} w_a^\alpha dT_{aa}.
\]  

(11.12)

Given a partition of unity \( \{ u_a \} \) subordinate to the atlas, one has

\[
T(w) = \int_{\mathcal{X}} w \cdot dT := \sum_{a \in A} \int_{\varphi_a(U_a)} \Phi_a(u_a w) \cdot dT_a = \sum_{a \in A} \int_{\varphi_a(U_a)} u_a w_a^\alpha dT_{aa}.
\]  

(11.13)

For the case of stresses, one has to replace \( W \) by \( J^* W, T \) by \( \varsigma \), and \( T_{aa} \) by \( \varsigma_{aa}^l \), \( |I| \leq r \). In addition, as \( \mathcal{X} \) is a manifold with corners, representing measures may be viewed as measures on the extension \( \hat{\mathcal{X}} \) which are supported in \( \mathcal{X} \). Thus, for a section \( \chi \) of \( J^* W \), represented locally by \( \chi_{aa}^\alpha \),

\[
\varsigma(\chi) = \int_{\mathcal{X}} \chi \cdot d\varsigma := \sum_{a \in A} \int_{\varphi_a(U_a)} \Phi_a(u_a \chi) \cdot d\varsigma_a = \sum_{a \in A} \int_{\varphi_a(U_a)} u_a \chi_{aa}^\alpha d\varsigma_a^l.
\]  

(11.14)

We note that the components \( \varsigma_{aa}^l \) have the same symmetry under permutations of \( I \) as sections of the jet bundle. If \( w \) is a section of a vector bundle \( W_0 \), then,

\[
\varsigma(j^* w) = \int_{\mathcal{X}} j^* w \cdot d\varsigma = \sum_{a \in A} \int_{\varphi_a(U_a)} u_a w_a^\alpha j^* d\varsigma_a^l.
\]  

(11.15)

The same reasoning applies to the representation by non–holonomic stresses, only here we consider sections \( \hat{\chi} \) of the iterated jet bundle represented locally by \( \hat{\chi}_{aa}^{P\alpha} \), \( G_p \leq r \). The local non–holonomic stress measures have components \( \hat{\varsigma}_{aa}^{P\alpha} \) and

\[
\hat{\varsigma}(\hat{\chi}) = \int_{\mathcal{X}} \hat{\chi} \cdot d\hat{\varsigma} := \sum_{a \in A} \int_{\varphi_a(U_a)} \Phi_a(u_a \hat{\chi}) \cdot d\hat{\varsigma}_a = \sum_{a \in A} \int_{\varphi_a(U_a)} u_a \hat{\chi}_{aa}^{P\alpha} d\varsigma_a^{P\alpha},
\]  

(11.16)

\[
\hat{\varsigma}(j^* w) = \int_{\mathcal{X}} j^* w \cdot d\hat{\varsigma} = \sum_{a \in A} \int_{\varphi_a(U_a)} u_a w_a^{P\alpha} j^* d\varsigma_a^{P\alpha}.
\]  

(11.17)

where summation is implied on all values of \( \alpha_p, \, I_p \), for all values of \( p \) such that \( G_p \leq r \).

It is concluded that for a given force \( F \), there is some non–unique vector valued hyper–stress measure \( \varsigma \) and a non–holonomic stress measure \( \hat{\varsigma} \), such that

\[
F(w) = \int_{\mathcal{X}} j^* w \cdot d\varsigma = \int_{\mathcal{X}} j^* w \cdot d\hat{\varsigma}.
\]  

(11.18)

11.4. **Force system induced by stresses.** It was noted in Section 8.3 that given a force on a body \( \mathcal{X} \), a manifold with corners, there is no unique way to restrict it to an \( n \)–dimensional submanifold with corners, a sub-body \( \mathcal{R} \subset \mathcal{X} \). We view this as the fundamental problem of continuum mechanics—static indeterminacy.
Stress, though not determined uniquely by a force, provides means for inducing a force system, the assignment of a force $F_R$ to each sub-body $R$. Indeed, once a stress measure is given, be it a hyper-stress or a non-holonomic stress, integration theory makes it possible to consider the force system given by

$$ F_R(w) = \int_R J^r w \cdot d\zeta = \int_R J^r w \cdot d\hat{\zeta} $$

for any section $w$ of $\pi|_R$.

Further details on the relation between hyper-stresses and force systems are available in [SD91]. It is our opinion that the foregoing line of reasoning captures the essence of stress theory in continuum mechanics accurately and elegantly.

### 12. Simple Forces and Stresses

We restrict ourselves now to the most natural setting for continuum mechanics, the case $r = 1$—the first value for which the set of $C^r$-embeddings is open in the manifold of mappings. (See [FS15] for consideration of configurations modeled as Lipschitz mappings.) Evidently, hyper-stresses and non-holonomic stresses become identical now, and therefore, it is natural in this case to use the terminology simple forces and stresses.

#### 12.1. Simple stresses

A simple stress $\zeta$ on a body $\mathfrak{X}$ is an element of

$$ C^0(J^1W)^* =: C^0((J^1W)^* \otimes \wedge^n T^* \mathfrak{X}) $$

which implies that smooth stress distributions are sections of

$$(J^1W)^* \otimes \wedge^n T^* \mathfrak{X} = L(J^1W, \wedge^n T^* \mathfrak{X}).$$

Following the discussion in Section 8.5, $\zeta$ may be viewed as a generalized section of $(J^1\hat{W})^* \otimes \wedge^n T^* \hat{\mathfrak{X}}$, which is supported in $\mathfrak{X}$, where we use the extension of the vector bundle to a vector bundle $\hat{\pi} : \hat{W} \to \hat{\mathfrak{X}}$ over a compact manifold without boundary $\hat{\mathfrak{X}}$.

A typical local representative of a section of the jet bundle is of the form

$$ \chi = \chi^a e_\alpha + \chi^i dX^i \otimes e_\alpha $$

so that locally,

$$ \zeta(\chi) = \zeta(\chi^a e_\alpha + \chi^i dX^i \otimes e_\alpha), $$

$$ = \zeta_\alpha(\chi^a) + \zeta^i_a(\chi^a). $$

Here, $\zeta_\alpha$ and $\zeta^i_a$ are 0-currents defined by

$$ \zeta_\alpha(u) := (\zeta \cdot e_\alpha)(u) = \zeta(ue_\alpha), $$

$$ \zeta^i_a(u) := (\zeta \cdot (dX^i \otimes e_\alpha))(u) = \zeta(u dX^i \otimes e_\alpha). $$
In the smooth case, \( \varsigma \) is represented by a section \( S \) of \((J^1W)^* \otimes \wedge^nT^*\mathcal{X}\) in the form
\[
\varsigma(\chi) = \int_{\mathcal{X}} S \cdot \chi. \tag{12.7}
\]
Locally, such a vector valued form is represented as
\[
S = (S_a e^a + S^i_a \partial_i \otimes e^a) \otimes dX \tag{12.8}
\]
so that, for the domain of a chart, \( U \), and a section \( \chi \) with \( \text{supp} \chi \subset U \),
\[
\varsigma(\chi) = \int_U (S_a \chi^a + S^i_a \chi^a_i) dX. \tag{12.9}
\]

12.2. The vertical projection. The vertical sub-bundle
\[
V \pi^1 : V J^1W \to \mathcal{X} \tag{12.10}
\]
is kernel of the natural projection
\[
\pi_0^1 : J^1W \to W. \tag{12.11}
\]
In other words, elements of the vertical sub-bundle at a point \( X \in \mathcal{X} \) are jets of sections that vanish at \( X \). Thus, if a typical element of \( J^1W \) is represented locally in the form \((\chi^\alpha, \chi_i^\beta)\), an element of the vertical sub-bundle, \( V J^1W \subset J^1W \) has the form \((0, \chi_i^\beta)\) in any adapted coordinate system. The vertical sub-bundle may be identified with the vector bundle \( T^*\mathcal{X} \otimes W \). Denoting the natural inclusion by
\[
i_V : V J^1W \to J^1W, \tag{12.12}
\]
one has the induced inclusion
\[
C^0(i_V) : C^0(V \pi^1) \to C^0(\pi^1), \quad \chi \mapsto i_V \circ \chi. \tag{12.13}
\]
Clearly, \( C^0(i_V) \) is injective and a homeomorphism onto its image. Hence, its dual
\[
C^0(i_V)^* : C^0(\pi^1)^* \to C^0(V \pi^1)^* \cong C^0(\pi^*\mathcal{X} \otimes W)^* = C^{-0}(\pi^*\mathcal{X} \otimes W^* \otimes \wedge^nT^*\mathcal{X}), \tag{12.14}
\]
is a well defined surjection. Simply put, \( C^0(i_V)^* \) is the restriction of the stress \( \varsigma \) to sections of the vertical sub-bundle. Accordingly, we will refer to an element of \( C^0(\pi^*\mathcal{X} \otimes W)^* \) as a vertical stress and to \( C^0(i_V)^* \) as the vertical projection.

In case the stress \( \varsigma \) is represented locally by the 0-currents \((\varsigma^a, \varsigma_i^\beta)\) as in \(12.4\), then \( C^0(i_V)^* \) is represented by \((\varsigma_i^\beta)\).

Let \( \varsigma^+ \in C^0(\pi^*\mathcal{X} \otimes W)^* \) be a vertical stress and let \( w \in C^0(\pi) \). Then, \( \varsigma^+ \cdot w \) defined by
\[
(\varsigma^+ \cdot w)(\varphi) := \varsigma^+(\varphi \otimes w), \quad \varphi \in C^0(\pi^*\mathcal{X}) \tag{12.15}
\]
is a 1-current. This is an indication of the fact that \( \varsigma^+ \) may be viewed as a vector valued 1-current.

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We may use the local representation of currents as in Section 10.3 to represent \( \varsigma^+ \) by the scalar 1-currents \( \varsigma^+_\alpha = (\varsigma^+ \cdot e_\alpha) \) given as
\[
\varsigma^+_\alpha(\varphi) = (\varsigma^+ \cdot e_\alpha)(\varphi) = \varsigma^+(\varphi \otimes e_\alpha)
\]  
(12.16)
so that
\[
\varsigma^+(\varphi \otimes w) = \varsigma^+(\varphi^i w^i dX^i \otimes e_\alpha),
\]
\[
= \varsigma^+_\alpha(\varphi^i w^i dX^i),
\]
\[
= (w^\alpha \downarrow \varsigma^+_\alpha)(\varphi).
\]  
(12.17)
Similarly, the 1-current \( \varsigma^+ \cdot w \) may be represented locally as in Section 9.2 by 0-currents \( (\varsigma^+ \cdot w)^i \) given by
\[
(\varsigma^+ \cdot w)^i(u) = (dX^i \downarrow (\varsigma^+ \cdot w))(u) = (\varsigma^+ \cdot w)(udX^i)
\]  
(12.18)
in the form
\[
(\varsigma^+ \cdot w)(\varphi) = (\varsigma^+ \cdot w)^i(\varphi^i).
\]  
(12.19)
Evidently,
\[
(\varsigma^+ \cdot w)^i(u) = \varsigma^+(u w^\alpha dX^i \otimes e_\alpha),
\]
\[
= \varsigma^+_\alpha(u w^\alpha),
\]
\[
= (w^\alpha \downarrow \varsigma^+_\alpha)(u),
\]  
(12.20)
and so
\[
(\varsigma^+ \cdot w)^i = w^\alpha \downarrow \varsigma^+_\alpha.
\]  
(12.21)
It is concluded that
\[
\varsigma^+ = e^\alpha \otimes (\partial_i \wedge \varsigma^+_i), \quad \varsigma^+_i = dX^i \downarrow (\varsigma^+ \cdot e_\alpha),
\]  
(12.22)
\[
\varsigma^+(\varphi \otimes w) = \varsigma^+_\alpha(\varphi^i w^\alpha)
\]  
(12.23)
where it is recalled that in case \( \varsigma^+ = C^0(i_U)^*(\varsigma) \), then, \( \varsigma^+_\alpha = \varsigma^\alpha \).

In the smooth case, the vertical projection of the stress is represented by a section \( S^+ \) of \( TX \otimes W^* \otimes \Lambda^n T^* X \) so that
\[
C^0(i_U)^*(\varsigma)(\chi) = \int_X S^+ \cdot \chi,
\]  
(12.24)
where
\[
(S^+ \cdot \chi)(v_1, \ldots, v_n)(X) = S^+(X)(\chi(X) \otimes (v_1(X) \wedge \cdots \wedge v_n(X))).
\]  
(12.25)
If \( \varsigma \) is represented by a section \( S \) of \( (j^1 W)^* \otimes \Lambda^n T^* X \) then, \( C^0(i_U)^*(\varsigma) \) is represented locally by
\[
S^+ = S^i_\alpha \partial_i \otimes e^\alpha \otimes dX.
\]  
(12.26)
For a vertical stress \( \varsigma^+ \) which is represented by \( S^i_\alpha \) as above and a field \( w \), the 1-current \( \varsigma^+ \cdot w \) is given locally by
\[
\varphi \mapsto \int_U S^i_\alpha w^\alpha \varphi_i dX,
\]  
(12.27)
for a 1-form \( \varphi \) supported in \( U \). In other words, if the vertical stress \( \zeta^+ \) is represented by the section \( S^+ \mathcal{T} \mathcal{X} \otimes W \otimes \wedge^n T^* \mathcal{X} \), the 1-current \( \zeta^+ \cdot w \) is represented by the density \( S^+ \cdot w \), a section of \( T^* \mathcal{X} \otimes \wedge^n T^* \mathcal{X} \) given by

\[
(S^+ \cdot w)(\varphi) = S^+ (\varphi \otimes w). \tag{12.28}
\]

12.3. **Traction stresses.** Using the transposition \( \text{tr} : W \otimes T^* \mathcal{X} \rightarrow T^* \mathcal{X} \otimes W \), one has a mapping on the space of vertical stresses

\[
C^0(\text{tr})^* : C^0(T^* \mathcal{X} \otimes W)^* \rightarrow C^0(W \otimes T^* \mathcal{X})^*. \tag{12.29}
\]

We define traction stress distributions to be elements of

\[
C^0(W \otimes T^* \mathcal{X})^* = C^{-0}(W^* \otimes \wedge^{n-1} T^* \mathcal{X}). \tag{12.30}
\]

Thus, it is noted that a traction stress is not much different than a vertical stress distribution but transposition enables its representation as a vector valued current. Using local representation in accordance with Section 10.3.6, a traction stress \( \sigma \) is represented locally by \( n \)-currents \( \sigma_\alpha \hat{1} \), \( |\hat{1}| = n - 1 \), in the form

\[
\sigma = dX^i \wedge \sigma_i = e^\alpha \otimes \sigma_\alpha = dX^i \wedge (e^\alpha \otimes \sigma_\alpha), \tag{12.31}
\]

where,

\[
\sigma_i := \partial_i \wedge \sigma, \quad \sigma_\alpha := \sigma \cdot e_\alpha, \quad \sigma_\alpha := \sigma_i \cdot e_\alpha = (\sigma_\alpha)_i = (\partial_i \wedge \sigma) \cdot e_\alpha. \tag{12.32}
\]

Hence,

\[
\sigma(w \otimes \varphi) = \sum_i \epsilon^i \sigma_\alpha_i (\varphi_i w^\alpha dX),
= \sum_i (-1)^{n-1} \sigma_\alpha_i (\varphi_i w^\alpha dX),
= \sum_i \sigma_\alpha_i (w^\alpha \varphi \wedge dX^i),
= \sum_i \sigma_\alpha_i \wedge dX^i (w^\alpha \varphi). \tag{12.33}
\]

Introducing the notation

\[
p_\sigma := C^0(\text{tr})^* \circ C^0(\text{tr})^*: C^0(\pi^1)^* \rightarrow C^0(W \otimes T^* \mathcal{X})^*, \tag{12.34}
\]

a simple stress distribution \( \zeta \) induces a traction stress distribution \( \sigma \) by

\[
\sigma = p_\sigma (\zeta). \tag{12.35}
\]

Let \( \sigma = p_\sigma (\zeta) \), then, comparing the last equation with \( 12.23 \), it is concluded that, in accordance with \( 10.47 \), locally,

\[
(-1)^{n-1} \sigma_\alpha_i (u dX) = \zeta^j (u) \tag{12.36}
\]

for any function \( u \), and so

\[
dX^i \wedge \sigma_\alpha_i = (-1)^{n-1} \zeta^j, \quad \sigma_\alpha_i = (-1)^{n-1} dX \wedge \zeta^j. \tag{12.37}
\]
Remark 12.1. Continuing Remark 10.2 it is observed that one may consider
\[ \sigma' := \sigma \wedge \partial_i, \quad \sigma_{ai}' := \sigma_i' \cdot e_a = (\sigma_i')_a = (\sigma \wedge \partial_i) \cdot e_a, \] (12.38)
so that
\[ \sigma = \sigma' \wedge dX^i = (e^a \otimes \sigma_{ai}') \wedge dX^i. \] (12.39)
Thus,
\[ \sigma(w \otimes \varphi) = (e^a \otimes \sigma_{ai}') \wedge dX^i(w \otimes \varphi), \]
\[ = \sigma_{ai}'(w^a \varphi \wedge dX^i), \]
\[ = \sum_i (-1)^{i-1} \sigma_{ai}'(\varphi_i w^a dX), \] (12.40)
and comparison with (12.23) implies that
\[ \sigma_{ai}' \wedge dX = (-1)^{i-1} \sigma_i'^a, \quad \sigma_{ai}' = (-1)^{i-1} \sigma_i'^a \wedge \partial_X = (-1)^{n-1} \sigma_{ai}, \] (12.41)
where it is observed that as \( \varsigma_{ai} \) are zero currents, the order of the contraction and wedge product in the last equation is immaterial.

12.4. Smooth traction stresses. In the smooth case, we adapt (10.15) to the current context. The traction stress \( \sigma \) is represented by a section \( s \) of \( W^* \otimes \bigwedge^{n-1} T^*X \) so that
\[ \sigma(\chi) = \int_X s \wedge \chi. \] (12.42)
Locally,
\[ s = s_{ai} e^a \otimes dX^i, \quad s \wedge \chi = \sum_i \varepsilon_i s_{ai} \chi^a dX \] (12.43)
so that the local components of \( s \) are the \( n \)-currents—functions—that represent \( \sigma \) locally.

Let \( \varsigma \) be a smooth stress represented by the vector valued form \( S \), a section of \( (J^1W)^* \otimes \bigwedge^{n-1} T^*X \) as in (12.8, 12.9) and let \( S^+ \) be its vertical component as in (12.26). In view of (10.17, 10.16), \( p_\sigma(S) \) is represented by the section \( s \) of \( W^* \otimes \bigwedge^{n-1} T^*X \) with
\[ s = C_L (S^+ \circ \text{tr}), \quad s_{ai} = (-1)^{n-1} S'^i_{ai}, \] (12.44)
The terminology, traction stress, originates from the fact that a traction stress \( \sigma \) represented by a smooth section \( s \) of \( W^* \otimes \bigwedge^{n-1} T^*X \), induces the analog of a traction field on oriented hyper-surfaces in the body as follows. (See [Seg13] for further details.) We consider, for any given section \( s \) of \( W^* \otimes \bigwedge^{n-1} T^*X \) and a field \( w \), the \( (n-1) \)-form \( \sigma \cdot w \) given by
\[ (s \cdot w)(\eta) = s(w \otimes \eta) \] (12.45)
for sections \( \eta \) of \( \bigwedge^{n-1} T^*X \). Consider an \( (n-1) \)-dimensional oriented smooth submanifold \( \mathcal{E} \subset X \). Let
\[ \nu_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{X} \] (12.46)
be the natural inclusion and
\[ t_Z^*: C^\infty(\wedge^{n-1} T^* X) \rightarrow C^\infty(\wedge^{n-1} T^* Z), \]  
(12.47)
the corresponding restriction of \((n - 1)\)-forms. Combining the above, one may define a linear mapping
\[ \rho_Z : C^\infty(W^* \otimes \wedge^{n-1} T^* X) \rightarrow C^\infty(W^* \otimes \wedge^{n-1} T^* Z) \]  
(12.48)
whereby
\[ \rho_Z(s) \cdot w = t_Z^*(s \cdot w) \in \wedge^{n-1} T^* Z. \]  
(12.49)
A section \( t \) of \( W^* \otimes \wedge^{n-1} T^* Z \) is interpreted as a surface traction distribution on the hyper-surface \( Z \). Its action \( t \cdot w \) is interpreted as the power density of the corresponding surface force, and may be integrated over \( Z \). In particular, the condition
\[ t = (-1)^{n-1} \rho_Z(s) \]  
(12.50)
is a generalization of Cauchy’s formula for the relation between traction fields and stresses.

**Remark 12.2.** The factor \((-1)^{n-1}\) that appears in (12.50) above, and is absent in [Seg13], originates from our use of the choice to use exterior multiplication on the left as in Remarks 9.1 and 10.2. Evidently, if we represented \( \sigma \) by the smooth vector valued form \( s' \) such that
\[ \sigma(\chi) = \int_X \chi \wedge s' \]  
(12.51)
instead of (12.42), the factor \((-1)^{n-1}\) would not appear in the analogous computation and
\[ t = \rho_Z(s'). \]  
(12.52)
In addition, the second of Equations (12.44), may be rewritten as
\[ s'_{ai} = (-1)^{n-1} S_{ai}. \]  
(12.53)

**12.5. The generalized divergence of the stress.** Let \( \zeta \) be a stress distribution, \( \sigma = p_\sigma(\zeta) \), and \( w \in C^1(\pi) \). We compute using (12.33) a local expression for the boundary of the \( 1 \)-current \( \sigma \cdot w \) as
\[
\partial(\sigma \cdot w)(u) = (\sigma \cdot w)(du),
\]
\[= \sigma(w \otimes du),
\]= \( \zeta^I_{ai}(u, w^\alpha) \),
\[= \zeta^I_{ai}((uw^\alpha)_i) - \zeta^I_{ai}(uw^\alpha_i),
\]= \( \zeta^I_{ai}(\partial_i \cdot d(uw^\alpha)) - \zeta^I_{ai}(uw^\alpha_i) - \zeta_\alpha(uw^\alpha) + \zeta_\alpha(uw^\alpha),
\]= \( \partial_i \wedge \zeta_\alpha(d(uw^\alpha)) - (\zeta \cdot j^1 w)(u) + (w^\alpha \cdot \zeta_\alpha)(u),
\]= \( \partial(\partial_i \cdot \zeta_\alpha)(uw^\alpha) - (\zeta \cdot j^1 w)(u) + (w^\alpha \cdot \zeta_\alpha)(u),
\]= \( [\partial_i \zeta^I_{ai} \cdot w^\alpha] - (\zeta \cdot j^1 w) + (w^\alpha \cdot \zeta_\alpha)(u). \)  
(12.54)
Here, for a 0-current $T = \zeta^i$, $\partial_i T$ is the “partial boundary” operator or the dual to the partial derivative, a 0-current defined by
\[
\partial_i T(u) := \partial(\partial_i T)(u), \\
= (\partial_i T)(du), \\
= T(u_i).
\] (12.55)

Since $\sigma \cdot w$ and $\sigma_\alpha$ are 1-currents, definition (9.16) implies that the exterior derivatives satisfy
\[
d(\sigma \cdot w) = (-1)^n \partial(\sigma \cdot w), \\
d\sigma_\alpha = (-1)^n \partial\sigma_\alpha,
\] (12.56)
and
\[
d(\sigma \cdot w)(u) = (d\sigma_\alpha \cdot w^\alpha)(u) + (-1)^{n-1}[\zeta \cdot j^1 w - w^\alpha \cdot \zeta_\alpha](u).
\] (12.57)

The computations above imply that there are “dual” linear differential operators
\[
\tilde{\partial} : C^{-0}(W^* \otimes \wedge^{n-1} T^* \mathcal{X}) \longrightarrow C^{-1}(W^* \otimes \wedge^n T^* \mathcal{X}) \equiv C^1(W \otimes \mathbb{R})^*,
\] (12.58)
and
\[
\tilde{\partial} : C^{-0}(W^* \otimes \wedge^{n-1} T^* \mathcal{X}) \longrightarrow C^{-1}(W^* \otimes \wedge^n T^* \mathcal{X}) \equiv C^1(W \otimes \mathbb{R})^*,
\] (12.59)

such that
\[
\tilde{\partial}\sigma(w \otimes u) := \sigma(w \otimes du) = \partial(\sigma \cdot w)(u) = (\tilde{\partial}\sigma \cdot w)(u), \\
\tilde{\partial}\sigma := (-1)^n \tilde{\partial}\sigma.
\] (12.60)

Consequently, we define the generalized divergence, a differential operator
\[
\text{div} : C^{-0}(J^1 W^* \otimes \wedge^n T^* \mathcal{X}) \longrightarrow C^{-1}(W^* \otimes \wedge^n T^* \mathcal{X}),
\] (12.61)
by
\[
\text{div} \zeta = -\tilde{\partial}(p_\sigma \zeta) - j^1 \zeta.
\] (12.62)

Here, we view the various terms as vector valued 0-currents, elements of $C^{-1}(W^* \otimes \wedge^n T^* \mathcal{X}) = C^1(W \otimes \mathbb{R})^*$, so that each may be contracted with $w$ to give an element of $C^{-1}(\wedge^n T^* \mathcal{X})$. Thus,
\[
(div \zeta \cdot w)(u) = -(\tilde{\partial}(p_\sigma \zeta) \cdot w)(u) - (\zeta \cdot j^1 w)(u), \\
= -p_\sigma \zeta(w \otimes du) - (\zeta \cdot j^1 w)(u).
\] (12.63)

The local expression for the generalized divergence in a coordinate neighborhood $U$ is obtained using (12.54). For a smooth function $u$ having a compact support in $U$,
\[
(div \zeta \cdot w)(u) = [-\partial_i \zeta^i_\alpha \cdot w^\alpha - w^\alpha \cdot \zeta_\alpha](u),
\] (12.64)
so that
\[
\text{div} \zeta \cdot w = -\partial_i \zeta^i_\alpha \cdot w^\alpha - w^\alpha \cdot \zeta_\alpha.
\] (12.65)
In the smooth case
\[
\partial_i S^i_\alpha \cdot w^\alpha (u) = \partial_i S^i_\alpha (w^\alpha u),
\]
\[
= S^i_\alpha ((w^\alpha u), i),
\]
\[
= \int_U S^i_\alpha (w^\alpha u), i dX,
\]
\[
= \int_U (S^i_\alpha w^\alpha u), i dX - \int_U S^i_{\alpha, i} w^\alpha u dX,
\]
\[
= - \int_U S^i_{\alpha, i} w^\alpha u dX,
\]
where we have used
\[
\int_U (S^i_\alpha w^\alpha u), i dX = \int_U (S^i_\alpha w^\alpha u), i dX \wedge dX^i \eta_{i\bar{i}},
\]
\[
= \int_U d(S^i_\alpha w^\alpha u) \eta_{i\bar{i}},
\]
\[
= \int_{\partial U} S^i_\alpha w^\alpha u \eta_{i\bar{i}},
\]
\[
= 0,
\]
(12.66)
as \( u \) is compactly supported in \( U \). Thus, noting that in the smooth case \( \zeta_\alpha \) are represented by the \( n \)-forms \( S_\alpha dX \), we conclude that locally
\[
\text{div} \zeta \cdot w = \int_U (S^i_{\alpha, i} - S_\alpha) w^\alpha dX.
\]
(12.68)
In other words, locally, the vector valued 0-current is represented by the vector valued form
\[
\text{div} S := (S^i_{\alpha, i} - S_\alpha) e^\alpha \otimes dX.
\]
(12.69)

12.6. The balance equation. We define now the body force current \( b \) and the boundary force current \( t \) corresponding to the stress \( \zeta \), elements of \( C^{-1}(W^* \otimes \wedge^n T^*X) \), by
\[
b := - \text{div} \zeta, \quad t := - \tilde{\partial}(p_\sigma \zeta) = (-1)^{n-1} \tilde{\partial}(p_\sigma \zeta).
\]
(12.70)
From the definition of the divergence in (12.63) we deduce
\[
\zeta \cdot j^1 w = b \cdot w + t \cdot w,
\]
(12.71)
The last equation is yet another generalization of the principle of virtual work in continuum mechanics.
For the smooth case, $\sigma$ is represented by the smooth vector valued form $s$ as in (12.42) and we can compute, for any differentiable function $u$ defined on $\mathcal{X}$,
\[
\partial(\sigma \cdot w)(u) = \int_{\mathcal{X}} s \lambda (w \otimes du),
\]
\[
= \int_{\mathcal{X}} (s \cdot w) \wedge du,
\]
\[
= \left(-1\right)^{n-1} \int_{\mathcal{X}} d((s \cdot w) \wedge u) - \left(-1\right)^{n-1} \int_{\mathcal{X}} d(s \cdot w) \wedge u,
\]
\[
= \left(-1\right)^{n-1} \int_{\partial\mathcal{X}} (s \cdot w)u - \left(-1\right)^{n-1} \int_{\mathcal{X}} d(s \cdot w)u,
\]
\[
= \int_{\partial\mathcal{X}} (t \cdot w)u - \left(-1\right)^{n-1} \int_{\mathcal{X}} d(s \cdot w)u,
\]
where Stokes’s theorem was utilized in the fourth line and (12.50) was used in the fifth line. Thus, in the smooth case, $\partial(\sigma \cdot w)$ contains, upon appropriate choices of $u$, information regarding the action of the surface force.

12.7. Application to non-holonomic stresses. In spite of numerous attempts (see [Seg86a, Seg17, SS18b, SS18a]), for the general geometry of manifolds, we were not able to extend the foregoing analysis to hyper-stresses, even for the case of stresses represented by smooth densities. Yet, the introduction of non-holonomic stresses makes it possible to carry out one step of the reduction.

Let $W_0$ be a vector bundle over $\mathcal{X}$ and consider forces in $C^r(W_0)^*$. Using the representation by non-holonomic stresses as in (11.7) in Section 11.1, let
\[
W_{r-1} := j^{r-1}W_0.
\]
(12.73)

Then, a force $F \in C^r(W_0)^*$ is represented by an element
\[
\hat{\xi} \in C^0(j^rW_0)^* = C^0(j^1W_{r-1})^*
\]
(12.74)
in the form
\[
F(w) = j^r*(\hat{\xi})(w),
\]
\[
= \hat{\xi}(j^r w),
\]
\[
= \hat{\xi}(j^1(j^{r-1} w)),
\]
\[
= j^1\hat{\xi}(j^{r-1} w).
\]
(12.75)

Thus, one may apply the foregoing analysis of simple stresses to the study the action $\hat{\xi}(j^1 \chi) = j^1\hat{\xi}(\chi)$ for elements
\[
\chi \in C^1(j^{r-1})^* = C^1(W_{r-1})^*.
\]
(12.76)

In other words, the analysis of simple stresses is used where we substitute $W_{r-1}$ and $\chi$ for $W$ and $w$ above respectively. In particular, the balance equations for this reduction will yield
\[
F(w) = \hat{\xi}(j^r w) = b(j^{r-1}w) + t(j^{r-1}w),
\]
(12.77)
where

\[ t, b \in C^{-1}(W_{r-1}^* \otimes \wedge^n T^* X) \]  

are interpreted as hyper surface traction and hyper body force distributions, respectively.

13. Concluding Remarks

The foregoing text is meant to serve as an introduction to global geometric stress and hyper-stress theory. We used a simple geometric model of a mechanical system in which forces are modeled as elements of the cotangent bundle of the configuration space and outlined the necessary steps needed in order to use it in the infinite dimensional case of continuum mechanics. The traditional choice of configurations as embeddings of a body in space, led us to the natural \( C^1 \)-topology which determined the properties of forces as linear functionals. In particular, the stress object emerges from a representation theorem for force functionals.

The general stress object we obtain preserves the basic feature of the stress tensor—it induces a force system on the body and its sub-bodies as described in Section 11.4. Further details of the relation between hyper-stresses and force systems are presented in \[SD91\] for the general case where stresses are as irregular as measures.

Generalizing continuum mechanics to differentiable manifolds implies that derivatives can no longer be decomposed invariantly from the values of vector fields and jets, combining the values of the field and its derivatives, are used. As a result, simple stresses mix both components dual to the values of the velocity fields, \( \varsigma_\alpha \), and components dual to the derivatives, \( \varsigma^\alpha_\mu \). This distinction form the classical stress tensor may be treated if additional mathematical structure is introduced. It is noted that no conditions of equilibrium, which are equivalent to invariance of the virtual power under the action of the Euclidean group, were imposed. In the general case, one may assume the action of a Lie group on the space manifold and obtain corresponding balance laws (see \[Seg94\]).

Another subject that has been omitted here is that of constitutive relations. Constitutive relations, in particular the notion of locality have been considered from the global point of view in \[Seg88\]. Roughly speaking, it is shown in \[11.4\] that a local constitutive relation, viewed from the global point of view as a mapping that assigns a stress distribution to a configuration, which is continuous relative to the \( C^r \)-topology, is a constitutive relation for a material of grade \( r \). Thus, the notion of locality is tied in with that of continuity.

A framework for the dynamics of a continuous body, for the geometry of differentiable manifolds, was suggested in \[KOS17\]. The dynamics of the system is specified using a Riemannian metric on the infinite dimensional configuration space.
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