Risk of Cascading Failures in Platoon of Autonomous Vehicles with Delayed Communication

Guangyi Liu, Christoforos Somarakis, and Nader Motee

Abstract—We develop a framework to assess the risk of cascading failures (as collisions) in a platoon of vehicles in the presence of exogenous noise and communication time-delay. The notion of value-at-risk (VaR) is adopted to quantify the risk of collision between vehicles conditioned to the knowledge of multiple previously occurred failures in the platoon. We show that the risk of cascading failures depends on the Laplacian spectrum of the underlying communication graph, time-delay, and noise statistics. Furthermore, we exploit the structure of several standard graphs to show how the risk profile depends on the spatial location of the systemic events. Our theoretical findings are significant as they can be applied to design safe platoons. Finally, our results are validated with extensive simulations.

I. INTRODUCTION

Uncertainties, originating from the fundamental law of physics in nature, prevail in every application in the robotics field. Even when most robotic systems are designed to maximize controllability, there is always a chance that the system will be driven into an undesired and even dangerous state. The inherent stochasticity prevents one from predicting the exact future state of a system. However, one can still assess the chance or the estimated cost of the system falling into failure with the intuitive concept of “risk”. Quantifying the risk of failure plays an influential role in most robotic applications [13], since the evaluation of how “risky” a vehicle system is will help both researchers and users to gain adequate knowledge of the safety and reliability of the system of vehicles.

The risk analysis in robotic applications can be traced from when [11] built the algorithmic risk measure for surgical robots to when [24] showed how the involvement of risk-mitigation in robot operations could increase safety. A more systemic and analytical approach that adopts the widely used financial tool value-at-risk and the conditional-value-at-risk measure [16] has been proposed by both [13] and [22]. Our previous works [22, 18, 21, 19, 20] have shown that the value-at-risk measure is an effective tool to evaluate and analyze the safety in networked control systems, e.g., power grids and rendezvous problems for a team of vehicles.

In this work, we use the platoon problem of mobile vehicles as an example, which is omnipresent in both robotic research [8] and real-world applications such as the car platoon in a highway [23] or the platoon of cargo robots in the factory [9]. In either case, an effective communication network will be essential. Moreover, for any real-world application, there will always exist a time delay or noise for both communication and sensing. This phenomenon is also observed and considered in vehicle platoon problems [12, 23]. Hence, we will consider the existence of both time delay and noise in the dynamics of the mobile vehicle platoons.

The fundamental methodology of designing a multi-vehicle system always lies around the system’s reliability and safety. Using the primary value-at-risk measure can help avoid failures. However, systemic failure such as the malfunctioning of vehicles, modeled by a random variable, will eventually occur in reality. Therefore, it is meaningful to investigate how the system will react when inevitable failures occur, defined as the cascading or domino effect of the systemic fault [12]. Furthermore, the scale of existing failures is not restricted to one event, and many of them may co-occur. From the design perspective, one wants to see a resilient system that can endure failures and still perform normally in other locations in the system. Our work can quantify the risk of this situation and help the system’s design.

Our Contributions: This work is a generalization of the existing risk analysis of cascading failures in [12] from the occasion of one existing failure to an arbitrary amount of failures. It also stands upon our previous keystone works on the first- and second-order linear consensus networks [22, 18, 21, 19, 20]. The value-at-risk of cascading failure is examined using the steady-state distribution of a platoon of vehicles suffering from time-delayed and noisy communication. Using the statistics from the system output, we obtain the explicit formulas for the risk of systemic failure of a vehicle, given other malfunctioning vehicles (failures). From the communication graph perspective, we show the contribution and interaction from a pair of vehicles to another from the marginal variance and correlation, respectively, on some particular graph. Finally, we explore how the communication graph topology, the scale, and the location distribution of the malfunctioning vehicles will affect the risk among the entire platoon.

The rest of the paper is organized as follows. In §IV we present a few key results on the steady-state behavior and statistics of the closed-loop mobile vehicle platooning system. Then, we shape the risk formula of group cascading failures in §V which constitutes the major contribution of this work. Special graph topologies are examined in §VI where we investigate theoretically and by simulation the complexity of group cascading failure phenomena. All proofs...
of our theoretical results are presented in the Appendix B of this paper.

II. PRELIMINARIES

The \( n \)-dimensional Euclidean space with elements \( z = [z_1, \ldots, z_n]^T \) is denoted by \( \mathbb{R}^n \), where \( \mathbb{R}_+ \) will denote the positive orthant of \( \mathbb{R}^n \). We denote the vector of all ones by \( 1_n = [1, \ldots, 1]^T \). The set of standard Euclidean basis for \( \mathbb{R}^n \) is represented by \( \{e_1, \ldots, e_n\} \) and \( \bar{e}_i := e_{i+1} - e_i \) for all \( i = 1, \ldots, n-1 \).

Algebraic Graph Theory: A weighted graph is defined by \( G = (\mathcal{V}, \mathcal{E}, \omega) \), where \( \mathcal{V} \) is the set of nodes, \( \mathcal{E} \) is the set of edges (feedback links), and \( \omega : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+ \) is the weight function that assigns a non-negative number (feedback gain) to every link. Two nodes are directly connected if and only if \( (i,j) \in \mathcal{E} \).

**Assumption 1.** Every graph in this paper is connected. In addition, for every \( i,j \in \mathcal{V} \), the following properties hold:

- \( \omega(i,j) > 0 \) if and only if \( (i,j) \in \mathcal{E} \).
- \( \omega(i,j) = \omega(j,i) \), i.e., links are undirected.
- \( \omega(i,i) = 0 \), i.e., links are simple.

The Laplacian matrix of \( G \) is a \( n \times n \) matrix \( L = [l_{ij}] \) with elements

\[
l_{ij} := \begin{cases} -k_{i,j} & \text{if } i \neq j \\ k_{i,1} + \ldots + k_{i,n} & \text{if } i = j \end{cases}
\]

where \( k_{i,j} := \omega(i,j) \). Laplacian matrix of a graph is symmetric and positive semi-definite [27]. Assumption [1] implies the smallest Laplacian eigenvalue is zero with algebraic multiplicity one. The spectrum of \( L \) can be ordered as

\[ 0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n. \]

The eigenvector of \( L \) corresponding to \( \lambda_k \) is denoted by \( q_k \).

By letting \( Q = [q_1, \ldots, q_n] \), it follows that \( L = Q\Lambda Q^T \) with \( \Lambda = \text{diag}(0, \lambda_2, \ldots, \lambda_n) \). We normalize the Laplacian eigenvectors such that \( Q \) becomes an orthogonal matrix, i.e., \( Q^T Q = QQ^T = I_n \), with \( q_1 = \frac{1}{\sqrt{n}} 1_n \).

Probability Theory: Let \( \mathcal{L}(\mathbb{R}^q) \) be the set of all \( \mathbb{R}^q \)-valued random vectors \( z = [z^{(1)}, \ldots, z^{(q)}]^T \) of a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with finite second moments. A normal random variable \( y \in \mathbb{R}^q \) with mean \( \mu \in \mathbb{R}^q \) and \( q \times q \) covariance matrix \( \Sigma \) is represented by \( y \sim \mathcal{N}(\mu, \Sigma) \). The error function \( \text{erf} : \mathbb{R} \rightarrow (-1, 1) \) is \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \), which is invertible on its range as \( \text{erf}^{-1}(x) \). We employ standard notation \( d\xi_i \) for the formulation of stochastic differential equations.

III. PROBLEM STATEMENT

Suppose that a finite number of vehicles \( \mathcal{V} = 1, \ldots, n \) form a platoon along the horizontal axis. Vehicles are labeled in descending order, where the \( n \)'th vehicle is assumed to be the leading vehicle in the platoon. The \( i \)'th vehicle’s state is determined by \( [x_i^{(i)}, v_i^{(i)}]^T \), where \( x_i^{(i)} \) is the position and \( v_i^{(i)} \) is the velocity of vehicle \( i \in \mathcal{V} \). The \( i \)'th vehicle’s state evolves in time according to the following stochastic differential equation

\[
\begin{align*}
\frac{dx_i^{(i)}}{dt} &= v_i^{(i)} dt \\
\frac{dv_i^{(i)}}{dt} &= u_i^{(i)} dt + g d\xi_i^{(i)}
\end{align*}
\]

where \( u_i^{(i)} \in \mathbb{R} \) is the control input at time \( t \). The terms \( g d\xi_i^{(i)} \) for \( i = 1, \ldots, n \) represent white noise generators affecting dynamics of the vehicle and models the uncertainty diffused in the system. It is assumed that noise acts on every vehicle additively and independently from the other vehicles’ noises. The noise magnitude is represented through diffusion \( g \neq 0 \), assumed identical for all \( i \in \mathcal{V} \). The control objectives for the platoon are to guarantee the following two global behaviors: (i) pair-wise difference in position variables of every two consecutive vehicles converges to zero; and (ii) the platoon of vehicles attain the same constant velocity in steady state. To incorporate deficiencies of communication network, we assume all vehicles experience an identical and constant time-delay, \( \tau \in \mathbb{R}_+ \). It is known from [29] that the following feedback control law can achieve the platooning objectives

\[
\begin{align*}
u_i^{(i)} &= \sum_{j=1}^n k_{i,j} (v_j^{(j)} - v_i^{(i)}) \\
&\quad + \beta \sum_{j=1}^n k_{i,j} \left( x_j^{(j)} - x_i^{(i)} - (j-i)\tau \right) \cdot (j-i) d\tau.
\end{align*}
\]

The parameter \( \beta \in \mathbb{R}_+ \) balances the effect of the relative positions and the velocities. By defining the vector of positions, velocities, and noise inputs as \( x_t = [x_1^{(1)}, \ldots, x_n^{(n)}]^T \), \( v_t = [v_1^{(1)}, \ldots, v_n^{(n)}]^T \) and \( \xi_t = [\xi_1^{(1)}, \ldots, \xi_n^{(n)}]^T \), we denote the target distance vector as \( y = [d, 2d, \ldots, nd]^T \). Using the above control input, we represent the closed-loop dynamics as an initial value problem

\[
\begin{align*}
\frac{dx_t}{dt} &= v_t dt, \\
\frac{dv_t}{dt} &= -Lv_t dt - \beta L(x_t - y) dt + g d\xi_t,
\end{align*}
\]

for all \( t \geq 0 \) and given deterministic initial function of \( \phi_t^x \) and \( \phi_t^y \in \mathbb{R}^n \) for \( t \in [-\tau, 0] \). The existing classical results [14][6] assert that [3] generates a well-posed stochastic process \( \{(x_t, v_t)\}_{t \geq -\tau} \).

The problem is to quantify the risk of cascading collisions (failure) as a function of the communication graph topology, time-delay, and statistics of noise under some particular situations, where our goal is to calculate collision risk of a pair of vehicles subject to the knowledge that the other pairs in the platoon has already collided. To this end, we will develop a general framework to assess systemic risk.
and study the notion of cascading collisions (failures) using the steady-state statistics of the closed-loop system \( \mathbf{1} \) and \( \mathbf{3} \).

IV. PRELIMINARY RESULTS

To evaluate the risk of cascading collisions in a platoon, we briefly review some necessary concepts and results \( \mathbf{19} \).

A. Platooning State and Stability Conditions

Keeping a safe constant distance from each other while traversing with a constant velocity is commonly referred to as the target (or consensus) state in a platoon \( \mathbf{10}, \mathbf{5} \). We say that the group of vehicles with dynamics \( \mathbf{3} \) forms a platoon if

\[
\lim_{t \to \infty} |v_t^{(i)} - v_t^{(j)}| = 0 \text{ and } \lim_{t \to \infty} |x_t^{(i)} - x_t^{(j)} - (i-j)d| = 0
\]

for all \( i, j \in V \) and all initial functions. It is known \( \mathbf{29} \) that the deterministic time-delayed network of vehicles will converge and form a platoon if and only if \( (\lambda_i, \tau, \beta \tau) \in S \) for all \( i = 2, \ldots, n \) where

\[
S = \left\{ (s_1, s_2) \in \mathbb{R}^2 \mid s_1 \in (0, \frac{\pi}{2}), s_2 \in \left(0, \frac{a}{\tan(a)}\right) \right\}
\]

for \( a \in \left(0, \frac{\pi}{2}\right) \) the solution of \( a \sin(a) = s_1 \).

Throughout this paper, we assume that the underlying deterministic network of vehicles forms a platoon.

B. Steady-State Inter-vehicle Distance

In present of stochastic exogenous noise, inter-vehicle failures are mainly due to large deviations in inter-vehicle distances from a desired safe distance \( \mathbf{1} \). Let us denote the steady-state value of the relative distance between vehicles \( i \) and \( i+1 \) by \( d_i \) and \( d = [d_1, \ldots, d_{n-1}]^T \).

Lemma 1. \( \mathbf{12} \) Suppose that the network of vehicles forms a platoon. Then, random variable \( \tilde{d} \) has a multi-variate normal distribution in \( \mathbb{R}^{n-1} \) that is given by

\[
\tilde{d} \sim \mathcal{N}(d\mathbf{1}_n, \Sigma),
\]

where the elements of the covariance matrix \( \Sigma = [\sigma_{ij}] \) are

\[
\sigma_{ij} = g^2 \frac{3}{2\pi} \sum_{k=2}^{n} (\bar{e}_i^T \bar{e}_j) f(\lambda_k \tau, \beta \tau), \quad (4)
\]

for all \( i, j = 1, \ldots, n \) and

\[
f(s_1, s_2) = \int_{\mathbb{R}} \frac{d\tau}{(s_1 s_2 - r^2 \cos(\tau))^2 + r^2(s_1 - r \sin(\tau))^2}.
\]

For the simplicity of notations, we use \( \sigma_i^2 \) instead of \( \sigma_{ii} \).

C. Value-at-Risk of Collision

To quantify the uncertainty level encapsulated in the relative distances between the vehicles, we employ the notion of Value-at-Risk \( \mathbf{7}, \mathbf{6}, \mathbf{5}, \mathbf{19} \). The VaR indicates the chance of a random variable landing inside an undesirable set of values, i.e., a near-collision situation. The set of undesirable values is referred to a systemic set, which are denoted as \( C \subset \mathbb{R} \). In probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), the set of systemic events of random variable \( y : \Omega \to \mathbb{R} \) is define as

\[
\{\omega \in \Omega \mid (y(\omega) \in C) \}
\]

for any sequence \( \left\{ \delta_n \right\}_{n=1}^{\infty} \) that satisfy the following conditions

- \( C_{\delta_1} \subset C_{\delta_2} \) when \( \delta_1 > \delta_2 \).
- \( \lim_{n \to \infty} C_{\delta_n} = \bigcap_{n=1}^{\infty} C_{\delta_n} = C \).

In practice, one can tailor the super-sets to cover a suitable neighborhood of \( C \) to characterize alarm zones as a random variable approaches \( C \). For a given \( \delta > 0 \), the chance of \( \{y \in C_{\delta} \} \) indicates how close \( y \) can get to \( C \) in probability. For a given design parameter \( \varepsilon \in (0, 1) \), the VaR measure \( \mathcal{R}_\varepsilon : \mathcal{F} \to \mathbb{R}^+ \) is defined by

\[
\mathcal{R}_\varepsilon := \inf \left\{ \delta > 0 \mid \mathbb{P} \{y \in C_{\delta} \} < \varepsilon \right\},
\]

where a smaller \( \varepsilon \) indicates a higher level of confidence on random variable \( y \) to stay away from \( C_{\delta} \). Let us elaborate and interpret what typical values of \( \mathcal{R}_\varepsilon \) imply. The case \( \mathcal{R}_\varepsilon = 0 \) signifies that the probability of observing \( y \) dangerously close to \( C \) is less than \( \varepsilon \). We have \( \mathcal{R}_\varepsilon > 0 \) iff \( y \in C_{\delta} \) for some \( \delta > 0 \) (in fact, \( \delta > \mathcal{R}_\varepsilon \)) with probability greater than \( \varepsilon \). The extreme case \( \mathcal{R}_\varepsilon = \infty \) indicates that the event that \( y \) is to be found in \( C \) is assigned with a probability greater than \( \varepsilon \). In addition to several interesting properties (see for instance \( \mathbf{3}, \mathbf{4}, \mathbf{19} \)), the risk measure is non-increasing with \( \varepsilon \). We refer to Fig. 2 for an example.

V. RISK OF CASCADING FAILURES

In our recent work \( \mathbf{12} \), we evaluate the risk of a networked control system encountering only one failure event, e.g., inter-vehicle collision. However, in most multi-vehicle systems, the chance of encountering multiple systemic failures are not negligible, and it is desirable to design the
lemmas to assess the risk of cascading failures when an arbitrary number of failures have occurred in the system.

For the exposition of the next result, let us consider a generalized version of Lemma 1 in [12] and exploit the knowledge of an arbitrary number of existing systemic events to construct the risk measure of cascading failures. By considering that there are \( m \) failures in the system whose corresponding labels to their inter-vehicle distances are shown by \( \mathcal{I}_m = \{ i_1, \ldots, i_m \} \) for some \( m < n \), our objective is to evaluate the risk of collision between the \( j \)'th pair, where \( j \notin \mathcal{I}_m \). To evaluate the effect of the existing failures on the \( j \)'th pair of vehicles, we consider the block covariance matrix

\[
\hat{\Sigma} = \begin{pmatrix} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} \end{pmatrix},
\]

where \( \hat{\Sigma}_{11} = \sigma^2 \hat{\Sigma}_{12} = \hat{\Sigma}_{21} = [\sigma_{j_1, \ldots, j_m}] \), \( \hat{\Sigma}_{22} = [\sigma_{k_1, k_2}] \in \mathbb{R}^{m \times m} \) for all \( k_1, k_2 \in \mathcal{I}_m \). To calculate the effect of cascade, we consider the conditional probability distribution of \( \bar{d} \) given \( \bar{d}_m = \bar{d}_c \). in a multi-variate normal distribution, where \( \bar{d}_c = [\bar{d}_{c1}, \ldots, \bar{d}_{cm}]^T \) is the vector of failure states, e.g., the inter-vehicle collision between the \( i_1 \)'th pair can be characterized by \( \bar{d}_{c1} = 0 \).

**Lemma 2.** [25] Suppose that \( \bar{d} \sim \mathcal{N}(d1_n, \Sigma) \), the conditional distribution of \( \bar{d} \) given \( \bar{d}_m = \bar{d}_c \) is a normal distribution \( \mathcal{N}(\bar{\mu}, \bar{\sigma}) \) with

\[
\bar{\mu} = \bar{d} - \hat{\Sigma}_{12} \hat{\Sigma}^{-1}_{22} (\bar{d}_c - d1_m)
\]

and

\[
\bar{\sigma}^2 = \hat{\Sigma}_{11} - \hat{\Sigma}_{12} \hat{\Sigma}^{-1}_{22} \hat{\Sigma}_{21},
\]

where \( \hat{\Sigma} \) is defined by (5).

The above lemma provides the probability distribution of the \( j \)'th pair when the other \( \mathcal{I}_m \) pairs have already encountered systemic failures. The visualization of this idea is depicted in Fig. [2] where we calculate the risk of cascading collision for the \( j \)'th pair with respect to the following family of events

\[
\left\{ \bar{d}_j \in C \mid \bar{d}_m = \bar{d}_c \right\} \text{ with } C := \left( -\infty, \frac{d}{\delta + \varepsilon} \right)
\]

for \( \delta \in [0, \infty] \) and design parameter \( c \geq 1 \). The systemic set is \( C = (-\infty, 0) \). The value-at-risk measure for cascading collision is defined as

\[
\mathcal{R}^{\mathcal{I}_m, j}_{\varepsilon, C} := \inf \left\{ \delta > 0 \left| \mathbb{P} \left( \bar{d}_j \in C \mid \bar{d}_m = \bar{d}_c \right) < \varepsilon \right. \right\}
\]

with the confidence level \( \varepsilon \in (0, 1) \). In the following theorem, we present the closed-form representation for the generalized version of the cascading collision risk.

**Theorem 1.** Suppose that the vehicles form a platoon and it is given that pairs with labels \( i_1, \ldots, i_m \) have already encountered systemic failures. The risk of cascading collision of the \( j \)'th pair is

\[
\mathcal{R}^{\mathcal{I}_m, j}_{\varepsilon, C} = \begin{cases} 0, & \text{if } \frac{d-c\bar{\mu}}{\sqrt{2\bar{\sigma}c}} \leq \varepsilon \\
\frac{d}{\sqrt{2\bar{\sigma}c} + \bar{\mu}}, & \text{if } \varepsilon \in \left( \frac{d-c\bar{\mu}}{\sqrt{2\bar{\sigma}c}}, \frac{d-c\bar{\mu}}{\sqrt{2\bar{\sigma}c}} \right) \\
\infty, & \text{if } \frac{d-c\bar{\mu}}{\sqrt{2\bar{\sigma}c}} \geq \varepsilon
\end{cases}
\]

where \( \bar{\mu} \) and \( \bar{\sigma} \) are computed as in Lemma 2 and \( \varepsilon = \text{erf}^{-1}(2\varepsilon - 1) \).

The above theorem states that based on the selection of the confidence level 1 – \( \varepsilon \), the risk of cascading collision falls into three categories. In the first scenario, if \( \varepsilon \) is too large, i.e., the confidence level is too low, we have \( \mathbb{P} \{ \bar{d}_j \in C \mid \bar{d}_m = \bar{d}_c \} < \varepsilon \) for every \( \delta > 0 \). Hence, there is no risk of having further collision with the confidence level 1 – \( \varepsilon \); see the blue area in Fig. [3]. The third case indicates that if the \( \varepsilon \) is too small, i.e., the confidence level is too high, the risk of having the cascading collision will be infinitely large and no \( \delta \) can bound the value of \( \mathbb{P} \{ \bar{d}_j \in C \mid \bar{d}_m = \bar{d}_c \} \leq \varepsilon \); see the yellow area in Fig. [3].

To measure the cascading risk among the entire vehicle platoon, one may introduce the risk profile as follows

\[
\mathcal{R}^{\mathcal{I}_m, 0}_{\varepsilon, C} = \left[ \mathcal{R}^{\mathcal{I}_m, 1}_{\varepsilon, C}, \ldots, \mathcal{R}^{\mathcal{I}_m, n-1}_{\varepsilon, C} \right]^T
\]

in which \( \mathcal{R}^{\mathcal{I}_m, j}_{\varepsilon, C} = 0 \) if \( j \notin \mathcal{I}_m \).

**VI. CASCADING RISK IN SPECIAL GRAPH TOPOLOGIES**

The topology of the underlying communication graph plays an essential role in the quality of uncertainties propagation in networked systems [17]. This section investigates several graph topologies with certain symmetries to understand the spatial behavior of cascading failures in the vehicle platooning.

In many robotic applications, robots (vehicles) obtain an all-to-all communication as it is readily available and that can help achieve better performance; for example, when a MoCap system is used to localize vehicles in lab experiments as states information can be broadcast to all vehicles within the lab environment. Let us consider an unweighted complete communication graph, i.e., \( k_{i,j} = 1 \) for all \( i, j \in V \). The eigenvalues of the corresponding Laplacian matrix are: \( \lambda_1 = 0 \) and \( \lambda_2 \ldots = \lambda_n = n \). The closed-form representation of the steady-state variance of \( d \) is shown as follows.

**Lemma 3.** For a platoon with a complete communication graph, the steady-states distance is \( \bar{d} \sim \mathcal{N}(d1_n, \Sigma) \), where
the covariance matrix is defined element-wise by

\[ \sigma_{ij} := \begin{cases} 
\sigma_c, & \text{if } i = j \\
-\frac{\sigma_c}{\sqrt{2}c}, & \text{if } |i - j| = 1 \\
0, & \text{if } |i - j| > 1
\end{cases} \]

where \( \sigma_c = \frac{\sigma_c^2 + \sigma_c^2}{\pi} \) for all \( i, j = 1, \ldots, n - 1 \).

The above lemma states that the covariance matrix obtains a unique structure as a symmetric tridiagonal matrix, opening an opportunity to explore its risk profile in a vehicle platoon. For the exposition of our following result, we consider three cases of relative location for the \( j \)th pair w.r.t the group failures \( I_m \): (i) The \( j \)th pair is not adjacent to location of any of the failures, (ii) The \( j \)th pair is only adjacent to \( m' \) consecutive failures on one side, and (iii) The \( j \)th pair is surrounded by \( m_1 \) and \( m_2 \) consecutive failures. These three scenarios are illustrated in Fig. 4.

With a little abuse of notation, let us consider a new covariance matrix as in (5), but with only the adjacent failures of size \( \hat{m} \), which is denoted by a \((\hat{m} + 1) \times (\hat{m} + 1)\) matrix \( \hat{\Sigma} \). We also show the inverse of \( \hat{\Sigma}_{122} \) as \( \hat{\Sigma}_{122} = [\alpha_{ij}] \). In the second case, \( \hat{m} = m' \) and in the third case \( \hat{m} = m_1 + m_2 \).

**Theorem 2.** Suppose that the network of vehicles forms a platoon. If pairs with labels \( i_1, \ldots, i_m \) have experienced failures, then the risk of cascading collision between the vehicles in the \( j \)th pair is

\[ R_{\varepsilon,C} = \begin{cases} 
0, & \text{if } \frac{d - \hat{\mu}}{\sqrt{2}\hat{\sigma}} \leq \varepsilon \\
\frac{d}{\sqrt{2}\hat{\sigma}} - c, & \text{if } \varepsilon \in \left( \frac{-\hat{\mu}}{\sqrt{2}\hat{\sigma}}, \frac{d - \hat{\mu}}{\sqrt{2}\hat{\sigma}} \right) \\
\infty, & \text{if } \frac{-\hat{\mu}}{\sqrt{2}\hat{\sigma}} \geq \varepsilon
\end{cases} \]

where \( \hat{\mu} \) and \( \hat{\sigma} \) can be computed for each case as follows.

**Case (i):** When \( |k - j| > 1 \) for all \( k \in I_m \), \( \hat{\mu} = d \) and \( \hat{\sigma} = \sqrt{\hat{\sigma}c} \). The cascading risk falls into the simple collision risk, i.e., \( R_{\varepsilon,C} = R_{\varepsilon,C}^{i,j} \) as defined in [19].

**Case (ii):** When \( |k - j| = 1 \) for only one \( k \in I_m \) and \( |k' - j| > 1 \) for all \( k' \in I_m \cap k' \neq k \), one has

\[ \hat{\sigma} = \sqrt{\frac{\sigma_i^2 - \sigma_c m'}{2(m' + 1)}}, \quad \text{and} \quad \hat{\mu} = d + \frac{\sigma_c}{2} \sum_{k=1}^{m'} \alpha_{1,k}(d_{e_k} - d). \]

**Case (iii):** When \( k = j - 1 \) and \( k' = j + 1 \) for some \( k, k' \in I_m \), one has

\[ \hat{\sigma} = \sqrt{\frac{\sigma_i^2 - \sigma_c 4m_1m_2 + m_1 + m_2}{2(m_1 + m_2 + 1)}}, \]

\[ \hat{\mu} = d + \frac{\sigma_c}{2} \sum_{k=1}^{m_1} \alpha_{1,k}(d_{e_k} - d). \]

The above theorem asserts that when the locations of failures \( I_m \) are not adjacent to the \( j \)th pair, the level of \( R_{\varepsilon,C}^{i,j} \) will remain the same as the basic collision risk [19] as the cross-correlation vanishes when the pairs are not adjacent. One can also see this in Lemma 3 when \( |i - j| > 1 \) or, in Fig. 5, \( R_{\varepsilon,C}^{i,j} \) remains the same value with \( R_{\varepsilon,C}^{i,j} \) when it is not adjacent to any existing systemic failures.

In the view of the second case, when the \( j \)th pair is adjacent to only one “group” of failures with size \( m' \), the magnitude of \( R_{\varepsilon,C}^{i,j} \) will only depend on the dimension of the failure group, and the results are the same when it is located at the front or back of the failure group. This phenomenon is also depicted in Fig. 5(a), where the pairs adjacent to the failures assume different risk values. In the last case, the \( j \)th pair is surrounded by two “groups” of failures with the size of \( m_1 \) and \( m_2 \), which is depicted in Fig. 4. Due to the collective impact from both failure groups, the cascading risk depends on the value of \( m_1 \) and \( m_2 \); this is shown by red dots in Fig. 5 and it assumes a higher value than the risk in the second case.

**VII. CASE STUDIES**

In the case studies, we consider the failures as some vehicles could not maintain the target platoon distance \( d \) and to be found staying at \( 1.1d \), i.e., \( d_{e} = [1.1d, \ldots, 1.1d]^{T} \). We consider \( n = 50, c = 1, d = 2, \) and \( \varepsilon = 0.1 \) for all case studies.
A. Risk of Cascading Collision

We evaluate the risk profile of cascading collision $R_{\varepsilon,C}^{I_m}$ for all pairs of vehicles in the platoon with the closed-form representation derived in the previous sections (see Fig. 6).

Complete Graph: We assume the all to all communication is available, and set $g = 10$, $\tau = 0.02$, and $\beta = 1$. It is shown that the vehicle pair that is adjacent to the failure group obtains a higher value than the remaining pairs. The remaining pairs obtain the same value as $R_{\varepsilon,C}^{I_m}$, this phenomenon is also observed in Theorem 2.

Path Graph [27]: We assume vehicles can only communicate with their front and back neighbors in the platoon. This type of communication structure can be interpreted as a car platoon on the highway. We set $g = 0.4$, $\tau = 0.05$, and $\beta = 4$. The results indicate that the impact from the failures in a path graph will first exasperate and then dilute as the distance to the failure increases.

p-Cycle Graph [27]: We assume vehicles form the platoon as a loop, and they can communicate to their $p$ immediate neighbors. We set $g = 0.1$, $\tau = 0.01$, and $\beta = 2$. Unlike any of the previously observed risk profiles, $R_{\varepsilon,C}^{I_m,j}$ remains low for the 1-cycle graph when $j$ is close to $I_m$. As the pair of the interest going further from the failures group, $R_{\varepsilon,C}^{I_m,j}$ exaggerates. However, the pattern is not completely revealed in this setting, and one can observe the full pattern in the next subsection. Another intuitive finding is that the risk profile pattern gets more similar to a complete graph as $p$ increases.

B. Impact from Failures' Properties

The impact originating from malfunctioning vehicles gets complicated since the dimension of the failure has been lifted from one to an arbitrary $m < n$. Hence, one should expect the domino effect from systemic failures to be a function of the scale of malfunctioning robots in the system, the location of those robots, and the communication graph topology.

1) Scale of Failures: It is instinctive to notice that the scale of existing systemic failures affects $R_{\varepsilon,C}^{I_m,j}$ since the overall systemic uncertainty and the statistics of the conditional distribution of $d_j$ are highly dependent on the value of $m$. To reveal this relation, we assume there exists $1, \ldots, 20$ failures at the beginning of the platoon, and the risk profile is presented over different communication graph topologies, depicted in Fig. 7(a) - (d). There are a few key observations that are worth reporting. First, in a complete or a $p$-cycle graph, the overall risk increases as the scale of the failure grow. Second, in a complete graph, the pair that is adjacent to the failure group $I_m$ is more likely to fail, which matches the result in Theorem 2. Third, if adopting $p$-cycle graph topology, the pair in the middle of the functioning robots is the most dangerous. Finally, a counter-intuitive observation shows that the risk profile decreases as the scale of malfunctioning robots increases in a path graph due to its unique structure, indicating that the robot platoon with a path communication graph can endure larger scale failures while it is vulnerable to small-scale malfunctions.

2) Location Distribution of Failures: Considering that the scale of failures is fixed in the system, the location distribution of those malfunctioning robots also profoundly impacts the overall risk. As expected, the risk increases when the malfunctioning robots are closer to the center of the platoon.
Fig. 7: The risk profile over different scale and location distribution of malfunctioning robots.
impacts the risk profile. We measure this effect by first assuming $m = 10$ failures occur at the beginning of the platoon, then separate them into two groups, each contains 5 failures, and gradually increase their distances, depicted in Fig. 7(e) - (h). One should focus more on the adjacent pair to $I_m$ in a complete graph and the pair in the middle of the functioning robots if using a $p$-cycle graph. When using a path graph communication topology, the overall risk profile decreases as the grouped failures get more distributed in the platoon. Hence, in applications like highway cars or robots platoon, one wants to avoid congregate failures and keep them distributed if failures are inevitable.

### B. Proofs

1) **Proof of Lemma 2** The result follows directly after Lemma [1] and the conditional distribution of a multi-variate normal random variable as in [25]. □

2) **Proof of Theorem 2** Following Lemma 2, one can rewrite (8) as

\[ \inf \left\{ \delta > 0 : \int_{-\infty}^{\infty} \exp \left( -t^2 \right) dt < \sqrt{\pi \varepsilon} \right\}. \]

There are three cases on the risk value, let us first consider $R_{x,c}^{2,m,j} = 0$, which is equivalent to

\[ \int_{-\infty}^{d-\mu(\delta+c)} \exp \left( -t^2 \right) dt < \sqrt{\pi \varepsilon} \Leftrightarrow \text{erf} \left( \frac{d-c\mu}{2\sigma C} \right) < 2\varepsilon - 1, \]

one can then consider $\text{erf}^{-1}(\cdot)$ to conclude. Similar argument works for the case $\delta \to \infty$ and using the term $\sigma C \sqrt{\varepsilon}$ instead.

For the intermediate branch, one can obtain the solution of $\delta > 0$ by solving

\[ \int_{-\infty}^{d-\mu(\delta+c)} e^{-t^2} dt = \sqrt{\pi \varepsilon}, \]

where the uniqueness of this solution is obtained with the the monotonicity of $\frac{d-\mu(\delta+c)}{\sqrt{2\varepsilon}}$. □

3) **Proof of Lemma 3** Following Lemma 1 given that the eigenvalues $\lambda_k$ are identical for all $k \in \{2, \ldots, n\}$,

\[ \sigma_{ij} = g^2 \frac{r^3}{2\pi} \int_{0}^{\infty} f(n\tau, \beta \tau) \sum_{k=2}^{n} \phi(i, j, k), \]

in which $q_{i,j}$ denotes the element of the matrix $Q = [q_{1} | \ldots | q_{n}]$, and

\[ \phi(i, j, k) = \left( q_{k+1, i}q_{k+1, j} - q_{k+1, i}q_{k, j} - q_{k, i}q_{k+1, j} + q_{k, i}q_{k, j} \right). \]

From the orthogonality property of the normalized eigenvectors, it is known that $\sum_{k} q_{i,k}q_{k,j} = \delta_{ij}$, in which $\delta_{ij}$ denotes the Kronecker delta. We have if $i = j$, then

\[ \sigma_{ij} = g^2 \frac{r^3}{2\pi} f(n\tau, \beta \tau) \left( 1 - 0 + 0 + 1 \right) = \frac{1}{\pi} f(n\tau, \beta \tau)g^2 r^3. \]

Using the same line of arguments, one has for $|i - j| = 1$, which is equivalent to $j = i + 1$ or $i = j + 1$,

\[ \sigma_{ij} = -\frac{1}{\pi} f(n\tau, \beta \tau) g^2 r^3. \]

For $|i - j| > 1$, it follows immediately that $\sigma_{ij} = 0$. □

4) **Proof of Theorem 2** Observing the structure of $\Sigma_{22}$ is a tridiagonal (Jacobi) matrix. Using the result from [26], considering $\theta_i = 2^{-i}\sigma_c(i + 1)$, the inversion $\Sigma_{22}^{-1} = [\alpha_{ij}]$ is given by:

\[ \alpha_{ij} = \begin{cases} \prod_{i} \left( \frac{\sigma_{i+1}}{\sigma_{i}} \right), & i < j \\ \prod_{i} \left( \frac{\sigma_{i}}{\sigma_{i-1}} \right), & i = j \\ \prod_{i} \left( \frac{\sigma_{i-1}}{\sigma_{i}} \right), & i > j \end{cases}. \]

Case 1: The result is immediate since $\Sigma_{12} = \Sigma_{21}^T = \left[ 0, \ldots, 0 \right]$, the conditional mean value and the variance remains unchanged.

Case 2: Since the existing failures have no impact for the non-adjacent robot pairs, we only consider the adjacent failure group with size $m' < m$. Given that $\Sigma_{12} = \Sigma_{21} = \left[ -\sigma_c/2, 0, \ldots, 0 \right] [0, \ldots, 0, -\sigma_c/2]$. Using the symmetry, the conditional variance can be written as

\[ \sigma^2 = \sigma^2 - \frac{\sigma_c^2}{4} \alpha_{11} = \sigma^2 - \frac{\sigma_c^2}{4} \alpha_{m'm} = \sigma^2 - \frac{\sigma_c m'}{2(m' + 1)}, \]

and the value of $\mu$ can be obtained with the same lines of argument.

Case 3: We consider the $j$’th pair is between two groups with the sizes as $m_1$ and $m_2$. Let us form a new $(m_1 + m_2) \times (m_1 + m_2)$ covariance matrix $\tilde{\Sigma}$, and the rest of the result follows similarly as the previous case. □

### REFERENCES

[1] A. Ali, G. Garcia, and P. Martinet. “Enhanced flatbed tow truck model for stable and safe platooning in the presences of lags, communication and sensing delays”. In: 2015 IEEE International Conference on Robotics and Automation (ICRA). IEEE, 2015.

[2] G. Antonelli and S. Chiaverini. “Fault tolerant kinematic control of platoons of autonomous vehicles”. In: IEEE International Conference on Robotics and Automation, 2004. Proceedings. IEEE. 2004.

[3] P. Artzner. “Thinking coherently”. In: Risk (1997), pp. 68–71.

[4] P. Artzner et al. “Coherent measures of risk”. In: Mathematical Finance 9.3 (1999), pp. 203–228.

[5] B. Banjerdpongdej et al. “Coherence in large-scale networks: Dimension-dependent limitations of local feedback”. In: IEEE Transactions on Automatic Control 57.9 (2012), pp. 2235–2249.

[6] L. Evans, An Introduction to Stochastic Differential Equations. American Mathematical Society, Dec. 2013.

[7] H. Füllmer and A. Schied. Stochastic Finance. De Gruyter, July 2016.

[8] M. G. C. Grunewald, R. Gasper, and D. Abel. “Energy-based control for platoons of nonholonomic vehicles”. In: IFAC Proceedings Volumes 43.14 (2010), pp. 635–640.

[9] L. Hu et al. “iRobot-Factory: An intelligent robot factory based on cognitive manufacturing and edge computing”. In: Future Generation Computer Systems 90 (2019).

[10] G. Klančar, D. Matko, and S. Blažič. “A control strategy for platoons of differential drive wheeled mobile robot”. In: Robotics and Autonomous Systems 59.2 (2011), pp. 57–64.

[11] W Korb et al. “Risk analysis and safety assessment in surgical robotics: A case study on a biopsy robot”. In: Minimally Invasive Therapy & Allied Technologies 14.1 (2005), pp. 23–31.
[12] G. Liu, C. Somarakis, and N. Motee. “Risk of Cascading Failures in Time-Delayed Vehicle Platooning”. In: *IEEE Conference on Decision and Control*. 2021.
[13] A. Majumdar and M. Pavone. “How should a robot assess risk? towards an axiomatic theory of risk in robotics”. In: *Robotics Research*. Springer, 2020, pp. 75–84.
[14] S. E. A. Mohammed and A. M. Salah-El Din. *Stochastic functional differential equations*. Vol. 99. Pitman Advanced Publishing Program, 1984.
[15] R. T. Rockafellar and S. Uryasev. “Optimization of Conditional Value-at-Risk”. In: *Portfolio The Magazine Of The Fine Arts* 2 (1999), pp. 1–26.
[16] R. Tyrrell Rockafellar and Stanislav Uryasev. “Conditional value-at-risk for general loss distributions”. In: *Journal of Banking and Finance* 26.7 (2002), pp. 1443–1471.
[17] Milad Siami and Nader Motee. “Fundamental limits and tradeoffs on disturbance propagation in linear dynamical networks”. In: *IEEE Transactions on Automatic Control* 61.12 (2016), pp. 4055–4062.
[18] C. Somarakis, Y. Ghaedsharaf, and N. Motee. “Aggregate fluctuations in time-delay linear consensus networks: A systemic risk perspective”. In: *Proceedings of the American Control Conference*. 2017.
[19] C. Somarakis, Y. Ghaedsharaf, and N. Motee. “Risk of Collision and Detachment in Vehicle Platooning: Time-Delay-Induced Limitations and Tradeoffs”. In: *IEEE Transactions on Automatic Control* 65.8 (2020).
[20] C. Somarakis, Y. Ghaedsharaf, and N. Motee. “Risk of collision in a vehicle platoon in presence of communication time delay and exogenous stochastic disturbance”. In: *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 4487–4492.
[21] C. Somarakis, Y. Ghaedsharaf, and N. Motee. “Time-delay origins of fundamental tradeoffs between risk of large fluctuations and network connectivity”. In: *IEEE Transactions on Automatic Control* 64.9 (2019).
[22] C. Somarakis, M. Siami, and N. Motee. “Interplays Between Systemic Risk and Network Topology in Consensus Networks”. In: *IFAC-PapersOnLine*. Vol. 49. 22. 2016.
[23] H. Tan, R. Rajamani, and W. Zhang. “Demonstration of an automated highway platoon system”. In: *Proceedings of the 1998 American control conference*. Vol. 3. IEEE. 1998.
[24] A. Terra et al. “Safety vs. efficiency: Ai-based risk mitigation in collaborative robotics”. In: *International Conference on Control, Automation and Robotics (ICCAR)*. IEEE. 2020.
[25] Y. L. Tong. *The multivariate normal distribution*. Springer Science & Business Media, 2012.
[26] R. A. Usmani. “Inversion of a tridiagonal Jacobi matrix”. In: *Linear Algebra and its Applications* (1994).
[27] P. Van Mieghem. *Graph spectra for complex networks*. Cambridge University Press, 2010.
[28] C. K. Verginis et al. “Decentralized 2-D control of vehicular platoons under limited visual feedback”. In: *2015 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE. 2015.
[29] W. Yu, G. Chen, and M. Cao. “Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems”. In: *Automatica* 46.6 (2010), pp. 1089–1095.