AN ANALOG OF THE VARIATIONAL
DERIVATIVE AND CONSTRUCTIVE NECESSARY
INTEGRABILITY CONDITIONS FOR
HYPERBOLIC EQUATIONS

S. Ya. Startsev
Mathematical Institute
Ufa center of the Russian Academy of Sciences
Chernyshevsky street 112, 450000 Ufa, Russia
e-mail: starts@imat.rb.ru

Abstract

An algorithm is constructed which allows to express conserved
flows of hyperbolic equations in terms of corresponding conserved den-
sities and to eliminate these flows from conservation laws of hyperbolic
equations. The application of this algorithm to canonical conservation
laws gives constructive necessary integrability conditions of hyperbolic
equations in terms of the generalized Laplace invariants of these equa-
tions.

Let us consider the hyperbolic equation

\[ u_{xy} = F(x, y, u, u_x, u_y). \]  

(1)

The partial derivatives \( \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \) with \( i \cdot j \neq 0 \) can be eliminated by virtue of the equation (1) and its differential consequences. Therefore, we assume that all functions depend on the variables \( x, y, u, u_i = u_{x}^{i}, v_i = u_{y}^{i} \). We denote the total derivatives with respect to \( x \) and \( y \) by virtue of the equation
as $D_x$ and $D_y$, respectively. These total derivatives are expressed in the variables $x, y, u, u_i, v_i$ as

$$D_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \sum_{i=1}^{\infty} \left( u_{i+1} \frac{\partial}{\partial u_i} + D_y^{i-1}(F) \frac{\partial}{\partial v_i} \right),$$

$$D_y = \frac{\partial}{\partial y} + v_1 \frac{\partial}{\partial u} + \sum_{i=1}^{\infty} \left( v_{i+1} \frac{\partial}{\partial v_i} + D_x^{i-1}(F) \frac{\partial}{\partial u_i} \right).$$

Let us consider the differential operator

$$M = D_x D_y + aD_x + bD_y + c,$$  

(2)

where $a, b$ and $c$ are functions.

**Definition 1.** The functions

$$K_0 = D_y(b) + ab - c, \quad H_0 = D_x(a) + ab - c$$

are called the main Laplace $x$-invariant and $y$-invariant of the operator (2), respectively.

The subsequent Laplace $y$-invariants $H_i$ are defined by the formula

$$H_{i+1} = 2H_i - D_x(S_i) - H_{i-1},$$

where $H_{-1} = K_0$ and $S_i$ is a solution of the equation $H_i S_i = D_y(H_i)$.

The subsequent Laplace $x$-invariants $K_i$ are defined by the formula

$$K_{i+1} = 2K_i - D_y(R_i) - K_{i-1},$$

where $K_{-1} = H_0$ and $R_i$ is a solution of the equation $K_i R_i = D_x(K_i)$.

The Laplace invariants of the linearization operator

$$L = D_x D_y + F_{ux} D_x + F_{uy} D_y - F_u$$

(3)

of the equation (1) are called the Laplace invariants of this equation.

Let us denote

$$a_0 = a, \quad a_{i+1} = a_i - S_i; \quad b_0 = b, \quad b_{i+1} = b_i - R_i;$$

$$M_i = (D_y + a)(D_x + b_i) - K_i; \quad M_{-i} = (D_x + b)(D_y + a_i) - H_i;$$

$$\nabla_i = D_x + b_i; \quad \Delta_i = D_y + a_i;$$

$$A_{-1} = 1, \quad A_i = \Delta_i \circ A_{i-1};$$

$$B_{-1} = 1, \quad B_i = \nabla_i \circ B_{i-1};$$

$$C_0 = 1, \quad C_{i+1} = \Delta_{i+1} \circ C_i,$$
where $i \geq 0$.

It is easy to prove the relationship

$$C_i \circ M = \nabla_0 \circ A_i - iA_{i-1}, \quad i \geq 0$$  \hspace{1cm} (4)

by induction on $i$.

**Definition 2.** A function $f$ is called a symmetry of the equation (1) if $L(f) = 0$, where $L$ is defined by the formula (3).

Let $f$ be a symmetry of the equation (1) and

$$p = D_x(\gamma), \quad p_{ui} = 0, \quad i > n.$$  \hspace{1cm} (5)

It is easy to verify that the vector field

$$\partial f = f \frac{\partial \psi}{\partial u} + \sum_{i=1}^{\infty} \left( D_i^y(f) \frac{\partial \psi}{\partial v_i} + \frac{\partial Q}{\partial u_i} D_i^x(f) \frac{\partial \psi}{\partial u_i} \right)$$

commutes with the total derivatives $D_x$ and $D_y$. Therefore, applying $\partial f$ to (5) we obtain that the operator $p_\ast - D_x \circ \gamma_\ast$, where $q_\ast$ denotes the linearization operator of the function $q$,

$$q_\ast = \frac{\partial q}{\partial u} + \sum_{i=1}^{\infty} \left( \frac{\partial q}{\partial v_i} D_i^y + \frac{\partial Q}{\partial u_i} D_i^x \right)$$,

maps any symmetry into zero. Hence,

$$p_\ast - D_x \circ \gamma_\ast = 0 \mod \{ L, D_x \circ L, D_y \circ L, \ldots \}.$$  \hspace{1cm} (6)

We rewrite

$$p_\ast = \xi_0 + \sum_{i=1}^{\infty} (\xi_i - iA_{i-1} + \xi_i B_{i-1}), \quad \gamma_\ast = \eta_0 + \sum_{i=1}^{\infty} (\eta_i - iA_{i-1} + \eta_i B_{i-1}).$$  \hspace{1cm} (7)

Using the formula (4) we rewrite (6) as

$$\xi_n = \eta_{n-1}, \quad \xi_i = (D_x - b_i)(\eta_i + \eta_{i-1}), \quad i = 1, n - 1, \quad \xi_{-i} = (D_x + F_{uy})(\eta_{-i} + \eta_{-(i+1)}H_i), \quad i \geq 0,$$  \hspace{1cm} (8)

where $H_i$ are the Laplace $y$-invariants of (1), $b_i = -F_{uy} - \sum_{j=0}^{i-1} R_j$.  

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Let us denote
\[
\begin{align*}
\delta_{n-1}(p) &= \xi_n, \\
\delta_i(p) &= \xi_{i+1} - (D_x - b_{i+1})(\delta_{i+1}(p)), \ i = -1, n - 2, \\
\delta_{-i}(p) &= H_{i-2} \ldots H_0 \xi_{1-i} - (D_x - \hat{b}_{i-1})(\delta_{1-i}(p)), \ i \geq 2,
\end{align*}
\] (9)
where \( \hat{b}_i = -F_{u_y} + \sum_{j=0}^{i-1} Q_j, \ D_x(H_j) = Q_j H_j \). The equations (8) imply in this notation that
\[
\begin{align*}
\eta_i &= \delta_i(p), \ i = 0, n - 1, \\
\eta_{-i} H_{i-1} \ldots H_0 &= \delta_{-i}(p), \ i > 0.
\end{align*}
\]
Since there exist \( k \) such that \( \gamma_{v_i} = 0 \) for all \( i > k \) and, therefore, \( \eta_{-i} = 0, \ i > k \), we prove the following

**Proposition 1.** If \( p \in \text{Im} \ D_x \), then there exist \( k \) such that \( \delta_{-i}(p) = 0 \) for all \( i > k \).

Let us consider a conservation law
\[ D_y(p) = D_x(\gamma), \ p_{u_i} = 0, \ i > n. \]
Repeating the slightly modified above calculation we obtain
\[
\begin{align*}
(D_y + F_{u_y})(\xi) &= \eta_{n-1}, \\
(D_y + F_{u_x})(\xi_i) + \xi_{i+1} K_i &= (D_x - b_i)(\eta_i) + \eta_{i-1}, \ i = 1, n - 1, \\
(D_y + F_{u_x})(\xi_0) + \xi_1 K_0 &= (D_x + F_{u_y})(\eta_0) + \eta_{-1} H_0, \\
(D_y - a_i)(\xi_0) + \xi_{1-i} &= (D_x + F_{u_y})(\eta_{-i}) + \eta_{-(i+1)} H_i, \ i \geq 1,
\end{align*}
\] (10)
where \( \xi_i \) and \( \eta_i \) are defined by (7), \( K_i \) are the Laplace invariants of the equation (1), \( a_i = -F_{ux} - \sum_{j=0}^{i-1} S_j \). Taking into account
\[
\begin{align*}
(D_x - b_{i-1}) \circ (L_i)^\top &= (L_{i-1})^\top \circ (D_x - b_i), \\
H_{i-1} \ldots H_0 (D_y - a_i) &= (D_y + F_{u_x}) \circ H_{i-1} \ldots H_0, \\
H_{i-1} \ldots H_0 (D_x + F_{u_y}) &= (D_x - \hat{b}_i) \circ H_{i-1} \ldots H_0, \\
(D_x - \hat{b}_i) \circ (L^\top)_{i-1} &= (L^\top)_{i} \circ (D_x - \hat{b}_{i-1}),
\end{align*}
\]
where \( i \geq 1, M^\top \) denotes the formally adjoint to \( M \) operator
\[
M^\top = D_x D_y - a D_x - b D_x + c - D_x(a) - D_y(b),
\]
we deduce from (10)
\[ \eta_{n-1} = (D_y + F_{ux})(\delta_{n-1}(p)) , \]
\[ \eta_i = (D_y + F_{ux})(\xi_i) - (L_i+1)\top(\delta_{i+1}(p)) , \quad i = 0, n-2 , \]
\[ \eta_{i-1}H_0 = (D_y + F_{ux})(\xi_0) - L_i\top(\delta_0(p)) , \]
\[ \eta_{i-1}H_{i-1} \cdots H_0 = (D_y + F_{ux})(H_{i-2} \cdots H_0 \xi_{i-1}) - (L_i)_{i-1}(\delta_{i-1}(p)) , \quad i \geq 2 , \]
where \( \delta_i(p) \) are defined by (9).

Thus, we prove

**Proposition 2.** If \( D_y(p) \in \text{Im} \ D_x \), then there exist \( k \) such that \( (L_i)_{i}(\delta_i(p)) = 0 \) for all \( i > k \).

**Proposition 3.** Let the equation (1) admits a symmetry \( f \), \( f_{v_k} \neq 0 \), \( f_{v_i} = 0 \) for all \( i > k \). Then
\[
D_x(f_{v_k}) = 0 \quad \text{if} \ k \geq 1 ;
\]
\[
D_y(F_{ux}) = D_x \left( \frac{f_{v_{k+1}}}{k_{v_k}} \right) \quad \text{if} \ k \geq 3 \quad (11) ;
\]
\[
D_y \left( \sqrt{f_{v_k}(F_u + F_{ux}F_{uy})} \right) \in \text{Im} \ D_x \quad \text{if} \ k \geq 5 .
\]

Differentiating the relationship \( L(f) = 0 \) with respect to \( v_{k+1}, v_k \) and \( v_{k-1} \) we easily prove this proposition.

It is easy to verify by induction on \( i \) that
\[
\delta_{-i}(F_u) = h_{i-1}H_{i-2} \cdots H_0 ,
\]
where \( h_{i-1} = (H_{i-1})_{v_i} \), \( i \geq 1 \).

The application of proposition 2 to the canonical conservation law (11) and the formula \( (L_i)_{i}\circ H_{i-1} \cdots H_0 = H_{i-1} \cdots H_0(L_{-i})\top \) give

**Theorem.** If the equation (1) has a symmetry \( f \), \( f_{v_k} \neq 0 \), \( f_{v_i} = 0 \) for all \( i > k, k \geq 3 \), then
\[
(L_{-k})\top \left( \frac{h_{k-1}}{H_{k-1}} \right) = 0
\]
or
\[
H_i = 0 \quad \text{for some} \ i, 0 \leq i \leq k - 1 .
\]

Moreover, we can express \( \gamma_{u_i} \) and \( \gamma_{v_i} \) in terms of \( \eta_i \) and obtain a first order partial differential system on \( \gamma \). The compatibility conditions of this system corresponding to a canonical conservation law give additional integrability conditions of the equation (1).
References

[1] I. M. Anderson and N. Kamran. The variational bicomplex for second order scalar partial differential equations in the plane. CRM technical report, September, 1994.

[2] A. V. Zhiber, V. V. Sokolov and S. Ya. Startsev. On the Darboux integrable nonlinear hyperbolic equations. Doklady RAN, 1995, Vol. 343, No. 6, p. 746-748 (in Russian).

[3] V. V. Sokolov and A. V. Zhiber. On the Darboux integrable hyperbolic equations. Phis. Lett. A., 1995, Vol. 208, p. 303-308.

[4] S. Ya. Startsev. Differential substitutions and symmetries of hyperbolic equations. In book ”Asymptotics and symmetries of nonlinear dynamical systems”, Mathematical Institute of Ufa center of the Russian Academy of Sciences, Ufa, 1995, p. 80-95 (in Russian).

[5] M. Juras and I. M. Anderson. Generalized Laplace invariant and the Method of Darboux. Preprint, University of Montreal, June, 1995.