CONSTANTS OF FORMAL DERIVATIVES OF NON-ASSOCIATIVE ALGEBRAS, TAYLOR EXPANSIONS AND APPLICATIONS

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Abstract. We study unitary multigraded non-associative algebras generated by an ordered set $X$ over a field $K$ of characteristic 0 such that the mappings $\partial_k : x_l \to \delta_{kl}$, $x_k, x_l \in X$, can be extended to derivations of $R$. The class of these algebras is quite large and includes free associative and Jordan algebras, absolutely free (non-associative) algebras, relatively free algebras in varieties of algebras, universal enveloping algebras of multigraded Lie algebras, etc.

There are Taylor-like formulas for $R$: Each element of $R$ can be uniquely presented as a sum of elements of the form $(\cdots (r_0 x_{j_1}) \cdots x_{j_{n-1}}) x_{j_n}$, where $r_0$ is a constant (i.e., $\partial_k(r_0) = 0$ for all $x_k \in X$) and $j_1 \leq \cdots \leq j_{n-1} \leq j_n$. We present methods for the description of the algebra of constants, including an approach via representation theory of the general linear or symmetric groups. As an application of the Taylor expansion for non-associative algebras, we consider the solutions of ordinary linear differential equations with constant coefficients from the base field.

Introduction

Let $K\{X\}$ be the (absolutely) free unitary non-associative algebra freely generated by an ordered set $X$ over a field $K$ of characteristic 0 ($K\{X\}$ is also called the free magma algebra over $X$). We consider factor algebras $R = K\{X\}/I$, where the ideal $I$ is multigraded and invariant under the formal derivatives $\partial_k = \frac{\partial}{\partial x_k}$, $x_k \in X$.

We shall denote the generators $x_j + I$ of $R$, $x_j \in X$, with the same symbols $x_j$. Since $\partial_k(I) \subseteq I$, the action of $\partial_k$ induces a derivation of $R$. Let $R_0$ be the algebra of constants of $R$, i.e., the set of all $r_0 \in R$ such that $\partial_k(r_0) = 0$ for all $x_k \in X$.

The description of $R_0$ is known in many cases. For example, an old result of Falk [F] describes the case when $R = K\{X\}$ is the free associative algebra. Then $R_0$ is generated by all Lie commutators $[\ldots [x_{j_1}, x_{j_2}], \ldots], x_{j_n}, n \geq 2$. Specht [Sp] applied products of such commutators in the study of algebras with polynomial identities, see the book of one of the authors [D3] for further application to the theory of PI-algebras. It is known, [G3], that in this case the algebra $R_0$ is free, see also [DK] for an explicit basis of $R_0$.

Any element $r \in R$ can be expressed in unique way in the form

$$r = \sum (\cdots (r_j x_{j_1}) \cdots x_{j_{n-1}}) x_{j_n}, \ r_j \in R_0, \ j_1 \leq \cdots \leq j_{n-1} \leq j_n.$$  

Such Taylor-like formulas were used for relatively free associative and Jordan algebras, see [D1][D2], and on skew polynomial constructions for the free associative

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algebra, see [GH], and on free magma algebras $K\{X\}$, see [GH]. We obtain more general results about factor algebras $R = K\{X\}/I$ in Section 2.

In Section 3, we consider the algebra of constants in the case of $K\{X\}$ and the case of free commutative non-associative algebras. We give different proofs for results of [GH], and we sketch methods to obtain a concrete description, including a method using representation theory of the general linear (or symmetric) groups, which follows ideas of Regev [R] and one of the authors [D1].

In the last section we consider ordinary linear differential equations of the form

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = f(x),$$

where the coefficients $a_1, \ldots, a_{n-1}, a_n$ are constants from the field $K$ and $f(x)$ is a formal power series from the completion $\hat{R}$ of the algebra $R = K\{x\}/I$ where $I$ is a homogeneous ideal invariant under the formal derivative $d/dx$. In this setup, we establish an analogue of the well known result about the general form of the solutions of such equations in terms of real and complex functions. A special case of our considerations is the non-associative exponential function defined in [DG].

1. Conventions

We fix a field $K$ of characteristic 0 and an ordered set $X = \{x_j \mid j \in J\}$.

Let $K\{X\}$ be the absolutely free non-associative and noncommutative unitary $K$-algebra freely generated by the set $X$. In the following, by an algebra we just mean a $K$-vector space $V$ together with a (not necessarily associative) binary operation $\cdot : V \times V \to V$. Every algebra is a factor algebra $R = K\{X\}/I$, where $I$ is an ideal of $K\{X\}$.

As a vector space $K\{X\}$ has a basis consisting of all non-associative words in the alphabet $X$. For example, we make a difference between $(x_1x_2)x_3$ and $x_1(x_2x_3)$ and even between $(xx)x$ and $x(xx)$. Words of length $n$ correspond to planar binary trees with $n$ leaves, see e. g. [H], [GH].

We call the elements of $K\{X\}$ polynomials in non-associative and noncommutative variables. We omit the parentheses when the products are left normed. For example, $uvw = (uv)w$ and $x^n = (x^{n-1})x$. The algebra $K\{X\}$ has a natural multigrading, counting the degree $\deg_j(u)$ of any non-associative word $u$ with respect to each free generator $x_j \in X$. Every mapping $\delta : X \to K\{X\}$ can be extended to a derivation of $K\{X\}$: If $u, v$ are monomials in $K\{X\}$, then we define $\delta(uv)$ inductively by $\delta(uv) = \delta(u)v + u\delta(v)$.

In this paper we are interested in the formal partial derivatives $\partial_k$ defined by

$$\partial_k(x_l) := \frac{\partial x_l}{\partial x_k} := \delta_{kl} = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

If $I$ is any multihomogeneous ideal of $K\{X\}$ such that $\partial_k(I) \subseteq I$ for all $x_k \in X$, then the factor algebra $R = K\{X\}/I$ admits the formal partial derivatives. In the sequel we shall consider such ideals $I$ and such algebras $R$ only. We shall use the same symbols $x_j, \partial_k$, etc. for the images in $R$ of the corresponding objects in $K\{X\}$. We denote by $R_0$ the subalgebra of $R$ consisting of all constants, i.e. $R_0$ is the intersection of the kernels $\ker \partial_k, x_k \in X$.

Let $u \in R$. By $\lambda_u$ and $\rho_u$, respectively, we denote the operators of left and right multiplication defined by

$$\lambda_u : v \mapsto uv, \quad \rho_u : v \mapsto vu, \quad v \in R.$$
The left and right multiplications \( \lambda_u \) and \( \rho_v \) (all \( u, v \)) generate a subalgebra \( M(R) \) of the (associative) algebra of linear operators on the vector space \( R \). The associative algebra \( M(R) \) is called the algebra of multiplications of \( R \). The algebra \( M(R) \) inherits the multigrading of \( R \) and for any non-associative word \( u \in R \) we have \( \deg_j(\lambda_u) = \deg_j(\rho_u) = \deg_j(u), \ x_j \in X \).

2. Taylor Formulas and Constants of Derivatives

The following statement should be considered as a non-associative and noncommutative analogue of the Taylor formula.

**Proposition 2.1.** Let \( I \) be an ideal of \( K\{X\} \) which is invariant under the formal partial derivatives. Let \( R_0 \) be the subalgebra of all constants in the factor algebra \( R = K\{X\}/I \), i.e. \( r_0 \in R_0 \) if and only if \( \partial r_0/\partial x_k = 0 \) for all \( x_k \in X \). Then every element \( r \in R \) can be expressed in a unique way in the form

\[
r = \sum (\cdots (r_j x_{j1}) \cdots x_{jn-1}) x_{jn},
\]

where \( r_j \in R_0, j = (j_1, \ldots, j_n), \) are constants and \( x_{j1} < \cdots < x_{jn-1} < x_{jn} \).

If \( r \) depends on the variables \( x_1 < \cdots < x_m \) only, then

\[
r = \sum r_a \rho_1^{a1} \cdots \rho_m^{am}, \ r_a \in R_0.
\]

**Proof.** For \( R = K\{X\} \), this is Proposition 3.1 of [GH]. The proof for the case \( R = K\{X\}/I \) is similar. For the sake of completeness, we sketch it, following the main steps of the proof of [D2] Proposition 1.5.

Let \( r \in R \) depend on the variables \( x_1 < \cdots < x_m \) only. Consider the element

\[
r_0 = r - \frac{\partial r}{\partial x_m} \frac{\rho_m}{1!} + \frac{\partial^2 r}{\partial x_m^2} \frac{\rho_m^2}{2!} - \frac{\partial^3 r}{\partial x_m^3} \frac{\rho_m^3}{3!} + \cdots
\]

It is easy to see that \( \partial r_0/\partial x_m = 0 \). Since the total degree of \( \partial^p r/\partial x_m^p \) is lower than the degree of \( r \), by induction we obtain that the derivatives \( \partial^p r/\partial x_m^p \) already have the desired expression

\[
\frac{\partial^p r}{\partial x_m^p} = \sum r_{pb} \rho_m^b,
\]

with \( \partial r_{pb}/\partial x_m = 0 \). Hence \( r \) has the form

\[
r = \sum_{a \geq 0} r_a \rho_m^a, \ \frac{\partial r_a}{\partial x_m} = 0.
\]

Continuing with the next variables \( x_{m-1}, x_{m-2}, \ldots, x_1 \), we obtain the presentation.

In order to see that the presentation is unique, let

\[
r = \sum r_a \rho_1^{a1} \cdots \rho_m^{am}, \ r_a \in R_0,
\]

with some \( r_a \neq 0 \). We choose the maximal \( m \)-tuple \((k_1, \ldots, k_m)\) among all indices \((a_1, \ldots, a_m)\) with \( r_a \neq 0 \), with respect to the lexicographical order. Then

\[
0 = \frac{\partial^k_1 + \cdots + k_m}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}} = r_k \neq 0,
\]

which is a contradiction.

\[\square\]

**Corollary 2.2.** A multihomogeneous ideal \( I \) of \( K\{X\} \) is invariant under all formal partial derivatives if and only if \( I \) can be generated by constants.
Proof. If \( I \) is generated by constants \( r_j, j \in J \), then any element of \( I \) has the form
\[
r = \sum \alpha_{ju} r_j \mu_{u_1} \cdots \mu_{u_n},
\]
where \( \alpha_{ju} \in K \) and \( \mu_{u_p} \) is the operator of left or right multiplication by the monomial \( u_p \). Since
\[
\frac{\partial r}{\partial x_k} = \sum \alpha_{ju} \left( \frac{\partial r_j}{\partial x_k} \mu_{u_1} \cdots \mu_{u_n} + \sum_{p=1}^n r_j \mu_{u_1} \cdots \mu_{u_p} \frac{\partial \mu_{u_p}}{\partial x_k} \cdots \mu_{u_n} \right)
\]
and \( \frac{\partial r_j}{\partial x_k} = 0 \), we obtain that \( \frac{\partial r}{\partial x_k} \) also belongs to \( I \).

In the opposite direction, let, in the notation of Proposition 2.1,
\[
r = \sum r_a \rho_{a_1} \cdots \rho_{a_m} = 0, \ r_a \in R_0,
\]
be a generator of the ideal \( I \) invariant under all partial derivatives. We choose the maximal \( m \)-tuple \( (k_1, \ldots, k_m) \) among all indices \( (a_1, \ldots, a_m) \) with \( r_a \neq 0 \), with respect to the lexicographical order, and obtain
\[
\frac{\partial^{k_1 + \cdots + k_m} r}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}} = r_k \in I.
\]
Hence we can replace the generator \( r \) of \( I \) with the constants \( r_a \) which completes the proof. \( \square \)

Remark 2.3. The same arguments as above give that Corollary 2.2 is true if we replace \( K\{X\} \) with a factor algebra \( R = K\{X\}/I \) of a multihomogeneous ideal \( I \) of \( K\{X\} \) which is invariant under the formal derivatives.

Recall, that if \( V = \bigoplus V^{(n_1, \ldots, n_m)} \) is a multigraded vector space where \( V^{(n_1, \ldots, n_m)} \) is the multihomogeneous component of degree \( (n_1, \ldots, n_m) \), then the Hilbert (or Poincaré) series of \( V \) is defined by
\[
\text{Hilb}(V, t_1, \ldots, t_m) = \sum \dim V^{(n_1, \ldots, n_m)} t_1^{n_1} \cdots t_m^{n_m}.
\]

Tracing the proof of Proposition 2.1 it is easy to see that if the multihomogeneous ideal \( I \) of \( K\{X\} \) is invariant under the formal partial derivatives, then the algebra of constants \( R_0 \) is also multigraded. The following statement is an analogue of the main theorem of \([D1]\), see also \([D3]\), Theorem 4.3.12. Our proof follows the idea of the proof given in \([D3]\).

Corollary 2.4. Let \( X = \{x_1, \ldots, x_m\} \) and let the ideal \( I \) of \( K\{X\} \) be invariant under the formal derivatives. Then the Hilbert series of the algebra \( R = K\{X\}/I \) and its subalgebra of constants \( R_0 \) are related by
\[
\text{Hilb}(R, t_1, \ldots, t_m) = \prod_{j=1}^m \frac{1}{1 - t_j} \text{Hilb}(R_0, t_1, \ldots, t_m).
\]

Proof. We fix a multihomogeneous basis \( \{r_n \mid n = 1, 2, \ldots \} \) of \( R_0 \). Let \( \rho_j \) be the right multiplication by \( x_j \), \( j = 1, \ldots, m \). Then the Taylor expansion of Proposition 2.1 shows that \( R \) has a basis
\[
r_n \rho_{a_1} \cdots \rho_{a_m}, \ n = 1, 2, \ldots, a_j \geq 0, \ j = 1, \ldots, m.
\]
Hence, as a multigraded vector space \( R \) is isomorphic to the tensor product \( R_0 \otimes_K K[x_1, \ldots, x_m] \) with isomorphism defined by
\[
r_n \rho_{a_1} \cdots \rho_{a_m} \rightarrow r_n \otimes x_1^{a_1} \cdots x_m^{a_m}.
\]
Since the Hilbert series of the tensor product is equal to the product of the Hilbert series of the factors, and the Hilbert series of the polynomial algebra $K[x_1, \ldots, x_m]$ is

$$\text{Hilb}(K[x_1, \ldots, x_m], t_1, \ldots, t_m) = \prod_{j=1}^{m} \frac{1}{1 - t_j},$$

we obtain immediately

$$\text{Hilb}(R, t_1, \ldots, t_m) = \text{Hilb}(R_0, t_1, \ldots, t_m) \text{Hilb}(K[x_1, \ldots, x_m], t_1, \ldots, t_m) = \prod_{j=1}^{m} \frac{1}{1 - t_j} \text{Hilb}(R_0, t_1, \ldots, t_m).$$

\[\square\]

**Remark 2.5.** For $X = \{x_1, \ldots, x_m\}$, there is a natural action of the general linear group $GL_m(K)$ on $K \{X\}$. More exactly, we can consider the vector space $K \{X\}(k)$ of the elements of total degree $k$ as a finite dimensional representation of $GL_m(K)$. Furthermore, for $n \leq m$, the symmetric group $S_n$ is embedded into $GL_m(K)$ (permuting the first $n$ variables and fixing the other variables). For the representations of symmetric and general linear groups, see [JK], [W].

The irreducible representations of $S_n$ and $GL_m(K)$ are symbolized by Young diagrams.

Under the conditions of Corollary 2.4 let us assume that the ideal $I$ of $K \{X\}$ is invariant with respect to the action of $GL_m(K)$. Thus $R^{(k)}$ is a representation of $GL_m(K)$ for each $k = 0, 1, 2, \ldots$.

The proof of Corollary 2.4 shows then that

$$R^{(k)} = \bigoplus_{j=0}^{k} R_0^{(j)} \otimes \begin{array}{c}
\cdots \\
\cdots \\
\cdots
\end{array}$$

as modules of the general linear group, where for simplicity of notation we have used the Young diagram instead of the corresponding $GL_m(K)$-module. An analogous formula (but using induced modules in the place of tensor products) holds if we look only at the multilinear parts $(n = m)$ for representations of the symmetric groups $S_n$.

**Examples 2.6.**

(i) Let $R = K \langle X \rangle$ be the free associative algebra. It is a homomorphic image of $K \{X\}$ modulo the ideal generated by the associators $(u, v, w) = (uv)w - u(vw)$, where $u, v, w$ run on the set of all non-associative words in the alphabet $X$. Since

$$\frac{\partial(u, v, w)}{\partial x_k} = \left( \frac{\partial u}{\partial x_k}, v, w \right) + \left( u, \frac{\partial v}{\partial x_k}, w \right) + \left( u, v, \frac{\partial w}{\partial x_k} \right),$$

this ideal is invariant under all partial derivatives. The algebra of constants of $K \langle X \rangle$ is spanned by all products of commutators $[x_{j_1}, \ldots, x_{j_p}] \cdots [x_{k_1}, \ldots, x_{k_q}]$, where $p, \ldots, q \geq 2$. See the comments in the introduction of the present paper as well as [D3], Section 4.3.

(ii) Let $\mathfrak{M}$ be a variety of unitary (associative or non-associative) algebras. The T-ideal $T(\mathfrak{M}) \subset K\{x_1, x_2, \ldots\}$ of all polynomial identities of $\mathfrak{M}$ is generated by constants. For associative algebras this is well known, see e. g. [D3], Section 4.3 for the proof. For non-associative algebras see [D2], Corollary 1.6.
(iii) Let \( L = L(X) \) be the free Lie algebra freely generated by the set \( X \). By the Witt theorem we may assume that \( L(X) \) is the Lie subalgebra of \( K(X) \) generated by \( X \) with respect to the bracket multiplication \([u,v] = uv - vu\). Let \( I_L \) be a multigraded ideal of \( L(X) \) and let \( G = L(X)/I_L \) be the corresponding multigraded Lie algebra. We assume that \( I_L \) does not contain elements of first degree, i.e., \( I_L \) is contained in the commutator ideal \( L'(X) \) of \( L(X) \). Clearly, the higher commutators 

\[ u = [x_{k_1}, \ldots, x_{k_n}] \]

which span \( L'(X) \) vanish under formal partial derivatives. By the Poincaré-Birkhoff-Witt theorem, the universal enveloping algebra of \( G \) is the associative algebra \( U(G) = K(X)/I \), where \( I \) is the ideal of \( K(X) \) generated by \( I_L \).

Hence we may apply Proposition 2.1. A concrete basis of the constants in \( U(G) \) can be obtained in the following way. Let us fix a basis of \( X \) and some higher commutators 

\[ u_j = [x_{k_{j1}}, \ldots, x_{k_{jp_j}}], \quad p_j \geq 2, \quad j = 1, 2, \ldots \]  

Then \( U(G) \) has a basis 

\[ \{u_1^{b_1} \cdots u_n^{b_n}, x_1^{a_1} \cdots x_m^{a_m} \mid a_j, b_k \geq 0 \} \]

and the algebra \( U_0 = (U(G))_0 \) of the constants of \( U(G) \) is spanned by the basis elements with \( a_1 = \cdots = a_m = 0 \).

(iv) A special case of (iii) is the free metabelian Lie algebra 

\[ M(X) = L(X)/L''(X). \]

It has a basis 

\[ X \cup \{[x_{k_1}, x_{k_2}, \ldots, x_{k_n}] \mid k_1 > k_2 \leq \cdots \leq k_n \} \]

and the higher commutators commute.

Hence the algebra of constants of \( U(M(X)) \) is isomorphic to the polynomial algebra generated by the “commutator variables” \([x_{k_1}, x_{k_2}, \ldots, x_{k_n}]\), where \( k_1 > k_2 \leq \cdots \leq k_n \). In [GR], Gerritzen used the completion of the algebra \( U(M(x,y)) \) in order to find a simple expression for the evaluation, modulo the metabelian identity \([[x_1, x_2], [x_3, x_4]] = 0\), of the Hausdorff series \( z = x + y + [x, y]/2 + \cdots \), where \( \exp(z) = \exp(x)\exp(y) \).

**Remark 2.7.** (cf. [GH], Proposition 2.3.) Let \( X = \{x_1, \ldots, x_m\} \) be any finite set and let the multihomogeneous ideal \( I \) of \( K\{X\} \) be invariant under the formal partial derivatives. Let \( R = K\{X\}/I \) be the factor algebra. The augmentation ideal \( \omega(R) \) of \( R \) is generated by the set \( X \) and consists of all elements of \( R \) without constant terms. The powers of \( \omega(R) \) induce a topology on \( R \) called the formal power series topology. It is defined by the property that the sets \( r + \omega^n(R), \quad r \in R, \quad n \geq 1, \) form a basis for the open sets. Clearly, the completion \( \hat{R} \) of \( R \) with respect to this topology consists of all formal power series in \( X \), i.e., formal infinite sums of the form

\[ r = \sum_{n \geq 0} r_n, \]

where \( r_n \in R \) are homogeneous elements of degree \( n \). Similarly, the completion \( \hat{R}_0 \) of the subalgebra \( R_0 \) of the constants in \( R \) consists of the formal power series 

\[ c = \sum_{k \geq 0} c_k, \quad c_k \text{ homogeneous constants of degree } k \]

The formal partial derivatives can be extended in a unique continuous way to derivatives of \( \hat{R} \) and it is clear that the algebra of constants of \( \hat{R} \) coincides with \( \hat{R}_0 \). Using the Taylor formula, every homogeneous \( r_n \in R \) can be expressed in the form

\[ r_n = \sum_{k \geq 0} c_{nk} \rho_1^{p_1} \cdots \rho_m^{p_m}, \]
where \( c_{nk} \in R_0 \) are homogeneous constants of degree \( k \), \( \rho_j \) is the operator of the right multiplication by \( x_j \) and \( p_j \geq 0, j = 1, \ldots, m \), \( p_1 + \cdots + p_m = n - k \). Hence we get a Taylor expansion

\[
    r = \sum_{p_j \geq 0} \left( \sum_{k \geq 0} c_{nk} \right) \rho_1^{p_1} \cdots \rho_m^{p_m} = \sum_{p_j \geq 0} c_p \rho_1^{p_1} \cdots \rho_m^{p_m}, \quad c_p \in \hat{R}_0,
\]

for every \( r \in \hat{R} \).

**Theorem 2.8.** Let \( R = K\{X\}/I \), where the multihomogeneous ideal \( I \) is invariant under the formal derivatives. For each \( x_j \in X \) and each \( k \geq 1 \) we fix an operator \( \mu_{jk} = \mu_{jk}(x_j) \) in the algebra of multiplications such that \( \mu_{jk} \) is homogeneous of degree \( k \), depends on \( x_j \) only and satisfies the property \( \mu_{jk}(1) \neq 0 \).

Then every element \( r \) of \( R \) has a presentation of the form

\[
    r = \sum r_a \mu_{1a_1} \cdots \mu_{ma_m}, \quad r_a \in R_0, \quad a_1, \ldots, a_m \geq 0,
\]

and this presentation is unique.

**Proof.** Since the algebra \( R \) is multihomogeneous, it is sufficient to consider the subalgebra generated by \( x_1, \ldots, x_m \), i.e., we may assume that \( X = \{x_1, \ldots, x_m\} \).

We have that \( \mu_{jk}(x_j) = \sum \alpha_{\nu, u} \nu_1, u_1 \cdots \nu_{n, u_n} \), where \( \alpha_{\nu, u} \in K \) and \( \nu_{n, u_n} = \lambda_{u_n} \) or \( \nu_{s,u} = \rho_u \), for some monomial \( u_s = u_s(x_j) \). The condition \( \mu_{jk}(1) \neq 0 \) just means \( \sum \alpha_{\nu, u} \neq 0 \).

It is easy to see that for \( r_0 \in R_0 \)

\[
    \frac{\partial^k r_0 \mu_{1k}(x_1)}{\partial x_1^k} = k! r_0 \mu_{1k}(1).
\]

We fix a multihomogeneous basis \( \{r_n | n = 1, 2, \ldots\} \) of \( R_0 \).

First, we want to show that the elements \( r_n \mu_{1a_1} \cdots \mu_{ma_m} \), \( n \geq 1 \), \( a_j \geq 0 \), are linearly independent. Let

\[
    r = \sum \beta_{n,a} r_n \mu_{1a_1} \cdots \mu_{ma_m} = 0,
\]

where some of the \( \beta_{n,a} \in K \) are nonzero.

Choosing the maximal \( m \)-tuple \( (k_1, \ldots, k_m) \) among all indices \( (a_1, \ldots, a_m) \) with \( \beta_{n,a} \neq 0 \), with respect to the lexicographical order, we obtain for \( c = k_1 + \cdots + k_m \)

\[
    \frac{\partial^c r}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}} = k_1! \cdots k_m! \mu_{1k_1}(1) \cdots \mu_{mk_m}(1) \sum \beta_{n,k} r_n = 0.
\]

Hence \( \beta_{n,k} = 0 \) for all \( n \) which is a contradiction.

Now, let \( S \) be the vector subspace of \( R \) spanned by all \( r_n \mu_{1a_1} \cdots \mu_{ma_m} \), \( n \geq 1 \), \( a_j \geq 0 \). Since these elements are linearly independent, they form a multigraded basis of \( S \). Bearing in mind that \( \deg \mu_{jk} = k \), we obtain for the Hilbert series of \( S \) and \( R \)

\[
    \text{Hilb}(S, t_1, \ldots, t_m) = \text{Hilb}(R_0, t_1, \ldots, t_m) \sum t_1^{a_1} \cdots t_m^{a_m}
\]

\[
    = \text{Hilb}(R_0, t_1, \ldots, t_m) \text{Hilb}(K[x_1, \ldots, x_m], t_1, \ldots, t_m) = \text{Hilb}(R, t_1, \ldots, t_m).
\]

The coincidence of the Hilbert series of \( S \) and \( R \) gives that \( S = R \). \( \Box \)
Example 2.9. Let $SJ(X)$ be the free special Jordan algebra generated by the set $X$. This is the Jordan subalgebra of $K(X)$ generated by $X$ with respect to the multiplication $u \circ v = uv + vu$. In $K(X)$ we define the operators $\mu_{jk} = (\lambda_{x_j} + \rho_{x_j})_k$. Clearly $u \mu_{jk} = (\cdots (u \circ x_j) \circ \cdots) \circ x_j$ ($k$-times Jordan multiplication by $x_j$). If $SJ_0(X)$ is the subalgebra of constants, then Theorem 2.8 gives that the elements of $SJ(X)$ have the form $\sum (\cdots (r_j \circ x_{j_1}) \circ \cdots) \circ x_{j_n}$, where $x_{j_1} \leq \cdots \leq x_{j_n}$ and this presentation is unique.

3. Constants in free non-associative algebras

A variety $\mathfrak{M}$ of algebras satisfies the Nielsen-Schreier property if any subalgebra of the relatively free algebra $F(\mathfrak{M}) = K\{X\}/T(\mathfrak{M})$ is again relatively free in $\mathfrak{M}$. Examples of such varieties are the class of all algebras $K[\cdot]$, all commutative (non-associative) algebras and all anticommutative algebras, Shirshov [Sh], all Lie algebras, etc. See the paper by Umirbaev [U] for the description of the varieties with Nielsen-Schreier property in terms of necessary and sufficient conditions. (Of course, the free anticommutative algebra and the free Lie algebra do not satisfy our condition for invariantness under the formal partial derivatives because they are not unitary.) If the condition for invariantness under the formal partial derivatives is satisfied, we may consider the problem for the description of the algebra of constants and its free generating set.

Let $K\{x\}$ be the free non-associative algebra generated by a single element $x$. Recall that the number of all non-associative and noncommutative words in $x$ of length $n$ is equal to the $n$-th Catalan number $c_n = \frac{1}{n} \binom{2n-2}{n-1}$. The sequence $c_n$, $n = 1, 2, \ldots$, satisfies the relation $c_n = \sum_{p=1}^{n-1} c_p c_{n-p}$, $n = 2, 3, \ldots$ and the first few Catalan numbers are $c_1 = c_2 = 1, c_3 = 2, c_4 = 5, \ldots$.

If we define $c_0 = 1$, then the generating function of the sequence is $c(t) = \sum_{n=0}^{\infty} c_n t^n = 1 + \frac{1 - \sqrt{1 - 4t}}{2}$.

(It is also convenient to define $c_{-1} = 0$.)

Lemma 3.1. (See e.g. [G2], Proposition 1.1.) Let $X$ be a (multi)graded set with generating function $g(T) \in \mathbb{Z}[[T]]$, where $T = \{t_1, t_2, \ldots\}$ is some set of commuting variables. Then the Hilbert series of the absolutely free algebra $K\{X\}$ is

$$\text{Hilb}(K\{X\}, T) = c(g(T)) = 1 + \frac{1 - \sqrt{1 - 4g(T)}}{2}.$$
Proof: The considerations are standard and similar to those in \cite{G2}. We include them for completeness of the exposition only. Let \( \{ x \} \) be the set of all non-associative noncommutative words in one variable. If we replace any word \( n \) of length \( n \) with \( (x_{j_1} \cdots x_{j_p})(x_{j_{p+1}} \cdots x_{j_n}) \), where the \( x_j \)'s run on \( X \), we obtain a basis of \( K \{ X \} \). Hence the generating function of the words of length \( n \) in \( X \) is equal to \( c_n g^n(T) \) and the Hilbert series of \( K \{ X \} \) is

\[
\text{Hilb}(K \{ X \}, T) = \sum_{n \geq 0} c_n g^n(T) = c(g(T)) = 1 + \frac{1 - \sqrt{1 - 4g(T)}}{2}.
\]

\( \square \)

Remark 3.2. Let \( C = (K \{ x \})_0 \) be the algebra of constants of \( K \{ x \} \).

To determine \( \gamma(t) = \text{Hilb}(C, t) = \sum_{n \geq 0} \gamma_n t^n \), we can use Corollary 2.4 to get that

\[
\text{Hilb}(K \{ X \}, t) = \frac{1}{1-t} \text{Hilb}(C, t) = \left( 1 + \frac{1 - \sqrt{1 - 4t}}{2} \right).
\]

A theorem of Kurosh \cite{K} states that for any generating set \( X \) any subalgebra \( A \) of the algebra \( K \{ X \} \) is free. Of course, if \( A \) is graded with respect to any grading of \( K \{ X \} \), then the set of free generators of \( A \) is also graded. Let \( Y \) be a homogeneous system of free generators of \( C = K \{ Y \} \).

Lemma 3.1 and direct calculations show that

\[
\gamma(t) = (1-t) \left( 1 + \frac{1 - \sqrt{1 - 4t}}{2} \right) = 1 + \frac{1 - \sqrt{1 - 4t}}{2},
\]

\[
g(t) = -3t^2 + t^3 + 3t(1-t) \frac{1 - \sqrt{1 - 4t}}{2} = t^3 + 3 \sum_{n \geq 4} (c_{n-1} - c_{n-2}) t^n.
\]

Thus we have obtained a different proof of \cite{GH} Proposition 2.9 and \cite{GH} Proposition 2.10(ii):

**Proposition 3.3.** (cf. \cite{GH}, Prop. 2.9. and 2.10.) Let \( C = (K \{ x \})_0 \) be the algebra of constants of \( K \{ x \} \) with respect to the formal derivative \( d/dx \).

(i) The Hilbert series of \( C \) is

\[
\text{Hilb}(C, t) = (1-t) \left( 1 + \frac{1 - \sqrt{1 - 4t}}{2} \right) = \sum_{n \geq 0} (c_n - c_{n-1}) t^n.
\]

(ii) Let \( Y \) be a homogeneous system of free generators of \( C = K \{ Y \} \). Then the generating function of \( Y \) is

\[
g(t) = -3t^2 + t^3 + 3t(1-t) \frac{1 - \sqrt{1 - 4t}}{2} = t^3 + 3 \sum_{n \geq 4} (c_{n-1} - c_{n-2}) t^n.
\]

\( \square \)

**Remark 3.4.** A linear ordering \( \prec \) on the set \( \{ x \} \) of words can be defined by:

(i) If \( u, v \in \{ x \} \) and \( \deg u < \deg v \), then \( u \prec v \).

(ii) If \( \deg u = \deg v > 1 \), where \( u = u_1 v_2 \), \( v = v_1 v_2 \) and \( u_2 \prec v_2 \), then \( u \prec v \).

(iii) If \( \deg u = \deg v > 1 \), where \( u = u_1 w \), \( v = v_1 w \) and \( u_1 \prec v_1 \), then \( u \prec v \).
For example,
\[ x < x^2 < (xx)x < x(xx) < (xx)(xx) < x((xx)x) < x(x(xx)) < \cdots. \]

For any nonzero element
\[ f = \sum_{k=1}^{p} \alpha_k u_k \in K\{x\}, \quad 0 \neq \alpha_k \in K, \ u_k \in \{x\}, \ u_1 < \cdots < u_p, \]
we define the leading term of \( f \) as \( \text{lt}(f) = \alpha_p u_p \). Clearly, if \( f \) and \( g \) are two nonzero polynomials in \( K\{x\} \), then \( \text{lt}(fg) = \text{lt}(f)\text{lt}(g) \).

Using this ordering one can show, see [GH] Proposition 2.9., that \( C \) has a vector space basis consisting of all polynomials
\[ u - \frac{u'}{1!} + \frac{u''}{2!} - \frac{u'''}{3!} + \cdots, \]
where \( u \) runs on the set of all words in \( \{x\} \) which are not in the form \( u = v \cdot x \), \( v \in \{x\} \).

One of the basic properties of the free algebras in any variety \( \mathfrak{M} \) satisfying the Nielsen-Schreier property is the following. If we define an ordering which agrees with the multiplication in \( F(\mathfrak{M}) \), and if some nonzero polynomials \( f_1, \ldots, f_m \in F(\mathfrak{M}) \) satisfy a nontrivial relation \( \omega(f_1, \ldots, f_m) = 0 \), then the leading term \( \text{lt}(f_i) \) of one of the polynomials belongs to the subalgebra generated by the other leading terms \( \text{lt}(f_1), \ldots, \text{lt}(f_{i-1}), \text{lt}(f_{i+1}), \ldots, \text{lt}(f_m) \).

Applying this argument in the case of \( K\{x\} \) one can deduce that a free algebra basis of \( C \) is given by the set \( Y \) consisting of those
\[ u - \frac{u'}{1!} + \frac{u''}{2!} - \frac{u'''}{3!} + \cdots, \]
which have the property that the word \( u \) has one of the forms
\[ u = v \cdot (x^2), \ u = x \cdot w, \ u = (x^2) \cdot w, \ u = (v_1 \cdot x)w, \ u = v(w_1 \cdot x), \]
where \( \deg v, \deg w, \deg v_1, \deg w_1 \geq 2 \); see [GH] Proposition 2.10.(i).

**Remark 3.5.** Let now \( X = \{x_1, \ldots, x_m\} \) and \( C = (K\{X\})_0 \) be the algebra of constants of \( K\{X\} \). We have already observed, see Remark 3.4, that
\[ K\{X\}^{(k)} = \bigoplus_{j=0}^{k} C^{(j)} \otimes \begin{array}{ccc} \cdots \end{array} \]
as modules of the general linear group. (Note that the conditions given in Remark 2.5 are trivially fulfilled.)

The representation given by the component \( K\{X\}^{(k)} \) of degree \( k \) of the free associative algebra is well-known, and \( K\{X\}^{(k)} \) consists just of \( c_k \) copies.

We can apply methods described by Regev (cf. [R]), and of [D1], [D3], to recursively determine \( C^{(k)} \) for \( k = 0, 1, 2, \ldots \), starting with \( C^{(0)} = K \), \( C^{(1)} = 0 \). The methods make use of the Young rule and Littlewood-Richardson rule.

**Example 3.6.** From \( K\{X\}^{(2)} = K\{X\}(2) = \begin{array}{ccc} \cdots \end{array} \oplus C^{(2)} \) we get that \( C^{(2)} \)
corresponds to the sign representation \( \begin{array}{ccc} \cdots \end{array} \), provided \( m \geq 2 \). (The generators are the commutators \([x_i, x_j]\).) In the following, let \( m \geq k \) always.
In degree 3 we have to compare $c_2 = 2$ copies of $K\langle X \rangle^{(3)}$ with

\[
\begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \left( \begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} \right) \oplus C^{(3)}.
\]

By Young’s rule,

\[
\begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} = \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]

Therefore

\[
C^{(3)} = \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 3 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]

It is easily seen that the multiplicity of the trivial representation in $C^{(k)}$ is always $c_k - c_{k-1}$ (we already know this from the case $X = \{x\}$).

For $k = 4$, $c_k$ is 5, and

\[
3 \bigoplus_{j=0} C^{(j)} \otimes \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}
\]

is computed using that

\[
\begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} = \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]

Then one gets that $C^{(4)}$ is given by

\[
3 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 10 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 7 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 10 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 4 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]

**Remark 3.7.** The free commutative non-associative algebra $R$ in $m$ variables has a nice basis and allows the formal derivatives $d/dx_1$. Its Hilbert series $\text{Hilb}(R, t)$ given by

\[
\text{Hilb}(R, t) = mt + \frac{1}{2}\text{Hilb}(R, t^2) + \frac{1}{2}\text{Hilb}(R, t^4)
\]

can be written in the form

\[
\text{Hilb}(R, t) = 1 - \sqrt{1 - \text{Hilb}(R, t^2) - 2mt}
\]

\[
= 1 - \sqrt{1 - \text{Hilb}(R, t^4) - 2mt^2 - 2mt} = \cdots
\]

(see [P]).

Using the techniques described above it is possible to determine the first components of $R_0$. 

- \[
\begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} \oplus C^{(3)}.
\]

- \[
\begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} = \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]

- \[
C^{(3)} = \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 3 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]

- \[
3 \bigoplus_{j=0} C^{(j)} \otimes \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}
\]

- \[
\begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} \otimes \text{\textbullet}
\end{bmatrix} = \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]

- \[
C^{4} \text{ is given by:}
\]

\[
3 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 10 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 7 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 10 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix} \oplus 4 \begin{bmatrix}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{bmatrix}.
\]
Example 3.8. We can recursively compute $R_0^{(k)}$ for the free commutative non-associative algebra $R$ generated by $m$ variables, assuming $m \geq k$ always. Clearly $R_0^{(0)} = K$ and $R_0^{(1)} = 0$. Since

$$R_0^{(0)} \otimes \square$$

is already $R^{(2)}$, we get that $R_0^{(2)} = 0$, i.e. there are no constants in degree 2.

In degree 3, we observe that $R_0^{(3)}$ is

$\square \oplus \square$

and the second summand must be equal to $R_0^{(3)}$.

Now

$$\bigoplus_{j=0}^{3} R_0^{(j)} \otimes \square$$

as

$$\square \oplus \square \oplus \square \oplus \square$$

by Young’s rule.

Since $R_0^{(4)}$ is given by

$$\bigoplus_{j=0}^{4} R_0^{(j)} \otimes \square$$

we get that $R_0^{(4)}$ must be equal to

$$\square \oplus \square \oplus \square \oplus \square$$

4. Linear Differential Equations

We consider ordinary linear differential equations of the form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x),$$

where the coefficients $a_1, \ldots, a_{n-1}, a_n$ are constants from the base field $K$ and $f(x)$ is a formal power series from the completion $\hat{R}$ of the algebra $R = K\{x\}/I$ where $I$ is a homogeneous ideal invariant under the formal derivative $d/dx$.

Proposition 4.1. Let the homogeneous ideal $I$ of $K\{x\}$ be invariant under the formal derivative $d/dx$ and $R = K\{x\}/I$. For any constants $c_0(x), c_1(x), \ldots, c_{n-1}(x)$ in the algebra $\hat{R}$ of formal power series the linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x),$$

...
f(x) ∈ \hat{R}, a_1, \ldots, a_{n-1}, a_n ∈ K, has a unique solution y(x) of the form
\[ y(x) = c_0 + c_1\frac{\rho}{1!} + c_2\frac{\rho^2}{2!} + \cdots + c_{n-1}\frac{\rho^{n-1}}{(n-1)!} + c_n\frac{\rho^n}{n!} + \cdots, \]
where \( \rho \) is the operator of right multiplication by \( x \) and \( c_n(x), c_{n+1}(x), \ldots \) belong to \( \hat{R}_0 \).

Proof. Let \( f(x) = f_0 + f_1\rho/1! + f_2\rho^2/2! + \cdots \), where \( f_k \in \hat{R}_0 \). We are looking for a solution of the form \( y(x) = c_0 + c_1\rho/1! + c_2\rho^2/2! + \cdots \), \( c_k \in \hat{R}_0 \), where the first \( n \) coefficients \( c_0, c_1, \ldots, c_{n-1} \) coincide with the prescribed ones. Since \( c'_k = 0 \), it is directly to see that
\[ y^{(j)}(x) = c_j + c_{j+1}\frac{\rho}{1!} + c_{j+2}\frac{\rho^2}{2!} + \cdots \]
and the differential equation has the form
\[ \sum_{j≥ 0}(c_{j+n} + a_1c_{j+n-1} + \cdots + a_{n-1}c_{j+1} + a_nc_j)\rho^j = \sum_{j≥ 0}f_j\rho^j. \]
Comparing the coefficients of the power series in \( \rho \), we obtain that
\[ c_{j+n} + a_1c_{j+n-1} + \cdots + a_{n-1}c_{j+1} + a_nc_j = f_j. \]
Since \( c_0, c_1, \ldots, c_{n-1} \) are already fixed, and hence known, this allows to define step-by-step and in a unique way the other coefficients \( c_n, c_{n+1}, \ldots \).

As in the case of functions in one real or complex variable, all solutions of any ordinary linear differential equation are obtained as sums of a given partial solution of the given equation and all the solutions of the corresponding homogeneous equation. In this setup, we establish an analogue of the well known result about the general form of the solutions of the homogeneous equation. For simplicity of the exposition, we assume that \( K = \mathbb{C} \).

**Theorem 4.2.** Let \( a_1, \ldots, a_{n-1}, a_n \in \mathbb{C} \) and let \( \lambda_1, \ldots, \lambda_p \in \mathbb{C} \) be all pairwise different solutions of the algebraic equation
\[ \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0 \]
with multiplicity \( k_1, \ldots, k_p \), respectively. Let the homogeneous ideal \( I \) of \( \mathbb{C}\{x\} \) be invariant under the formal derivative \( d/dx \) and let \( R = \mathbb{C}\{x\}/I \). Then all solutions of the homogeneous linear differential equation
\[ y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0 \]
in the algebra \( \hat{R} \) of formal power series are given by the formula
\[ y(x) = (c_{10} + c_{11}\rho + \cdots + c_{1,k_1-1}\rho^{k_1-1})\exp(\lambda_1\rho) + \cdots + (c_{p0} + c_{p1}\rho + \cdots + c_{p,k_p-1}\rho^{k_p-1})\exp(\lambda_p\rho), \]
where \( c_{ij} \) are arbitrary constants in \( \hat{R}_0 \), \( \rho \) is the operator of right multiplication by \( x \) and \( \exp(\lambda\rho), \lambda \in \mathbb{C} \), is the element of the completion \( \mathcal{M}(\hat{R}) \) of the algebra of multiplications \( \mathcal{M}(R) \) defined by
\[ \exp(\lambda\rho) = 1 + \lambda\frac{\rho}{1!} + \lambda^2\frac{\rho^2}{2!} + \lambda^3\frac{\rho^3}{3!} + \cdots \]
Proof. If \( c(x) \in \hat{R}_0 \) is any constant and \( s(\rho) = \sum_{j \geq 0} \alpha_j \rho^j, \alpha_j \in \mathbb{C} \), is a formal power series in the operator of multiplication \( \rho \), then, since \( c'(x) = 0 \),
\[
\frac{dc(x)s(\rho)}{dx} = \sum_{j \geq 0} \alpha_j \frac{dc(x)\rho^j}{dx} = c(x) \sum_{j \geq 0} \alpha_j j \rho^j = c(x)s'(\rho).
\]

Hence \( y(x) = c(x)s(\rho) \) is a solution of the equation \( y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \) if and only if \( s(t) \) is a solution of the equation \( s^{(n)} + a_1 s^{(n-1)} + \cdots + a_{n-1} s' + a_n s = 0 \). The standard theory of homogeneous linear differential equations with constant coefficients gives that the \( n \) formal power series
\[
s_{ij}(t) = t^i \exp(\lambda_i t), \ j = 0, 1, \ldots, k_i - 1, \ i = 1, \ldots, p,
\]
are solutions of the equation \( s^{(n)}(t) + a_1 s^{(n-1)}(t) + \cdots + a_{n-1} s'(t) + a_n s(t) = 0 \) in \( \mathbb{C}[[t]] \). Hence
\[
y_{ij} = y_{ij}(c_{ij}, x) = c_{ij} t^i \exp(\lambda_i t), \ j = 0, 1, \ldots, k_i - 1, \ i = 1, \ldots, p,
\]
are solutions of the equation \( y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \).

The functions \( s_{ij}(t), j = 0, 1, \ldots, k_i - 1, i = 1, \ldots, p, \) form a fundamental system of solutions of the equation \( s^{(n)}(t) + a_1 s^{(n-1)}(t) + \cdots + a_{n-1} s'(t) + a_n s(t) = 0 \). If we fix \( s_0, s_1, \ldots, s_{n-1} \), there exists a unique solution \( s(t) = s_0 + s_1 t/1! + \cdots + s_{n-1} t^{n-1}/(n-1)! + t^n u(t) \) for some \( u(t) \in \mathbb{C}[[t]] \) (because \( s_j = s^{(j)}(0) \)) and a unique system of constants \( \gamma_{ij} \in \mathbb{C} \) such that \( c(t) = \sum \gamma_{ij} c_{ij}(t) \). Hence the partial sum of the first \( n \) summands of the series
\[
(\gamma_{10} + \gamma_{11} t + \cdots + \gamma_{1,k_i-1} t^{k_i-1}) \exp(\lambda_1 t) + \cdots + (\gamma_{p0} + \gamma_{p1} t + \cdots + \gamma_{p,k_p-1} t^{k_p-1}) \exp(\lambda_p t)
\]
coincides with \( s_0 + s_1 t/1! + \cdots + s_{n-1} t^{n-1}/(n-1)! \). If we compare the coefficients of \( t^j, j = 0, 1, \ldots, n - 1 \), we obtain a system of \( n \) linear equations with unknowns \( \gamma_{ij} \), and the determinant of the system is different from 0.

By Proposition 1.1. the only solution \( y(x) = \sum_{j \geq 0} c_{ij}(x) \rho^j \) with \( c_0(x) = c_1(x) = \cdots = c_{n-1}(x) = 0 \) is the zero formal power series. Hence, in order to see that an arbitrary solution is a linear combination of solutions of the form \( y_{ij}(c_{ij}, x) \), it is sufficient to show that for any \( n \) constants \( c_0(x), c_1(x), \ldots, c_{n-1}(x) \in \hat{R}_0 \) there exist constants \( c_{ij}(x) \) such that the partial sum of the first \( n \) summands of the series
\[
(c_{10} + c_{11} \rho + \cdots + c_{1,k_i-1} \rho^{k_i-1}) \exp(\lambda_1 \rho) + \cdots + (c_{p0} + c_{p1} \rho + \cdots + c_{p,k_p-1} \rho^{k_p-1}) \exp(\lambda_p \rho)
\]
coincide with \( c_0 + c_1 \rho + \cdots + c_{n-1} \rho^{n-1} \). Again, we consider the coefficients from \( \hat{R}_0 \) of \( \rho^j, j = 0, 1, \ldots, n - 1 \), form a system of \( n \) linear equations with unknowns \( c_{ij} \), and the determinant of this system is the same as the determinant of the corresponding system for \( \mathbb{C}[[t]] \). Since this determinant is nonzero, the system has a unique solution and we can find the desired constants \( c_{ij}(x) \in \hat{R}_0 \).

A special case of our considerations is the non-associative exponential function introduced in [DG]. We define the exponent \( E(x) \) as the formal power series in \( K(\{x\}) = \hat{K}(\{x\}) \) satisfying the conditions \( E'(x) = E(x) \), \( E(0) = 1 \) and \( E(x) E(x) = E(2x) \). By Theorem 1.2 all solutions of the equation \( E'(x) = E(x) \) are \( c(x) \exp(\rho) \), where \( c(x) \in (\hat{K}(\{x\}))_0 \). There are many solutions satisfying the condition \( E(0) = 1 \).
(which simply means that $c(0) = 1$), and only the second condition $E(x)E(x) = E(2x)$ determines $E(x)$ in a unique way.

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