THE TWISTOR SPACE OF A QUATERNIONIC CONTACT MANIFOLD

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Abstract. We show that the CR structure on the twistor space of a quaternionic contact structure described by O. Biquard is normal if and only if the Ricci curvature of the Biquard connection commutes with the endomorphisms in the quaternionic structure of the contact distribution.

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1. Introduction

The notion of a quaternionic contact (QC) structure is introduced in [3] and it describes a type of geometrical structure that appears naturally as the conformal boundary at infinity of the quaternionic hyperbolic space. In general, a QC structure on a real $(4n+3)$-dimensional manifold $M$ is a codimension three distribution $H$, the contact distribution, locally given as the kernel of a $1$-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ such that the three $2$-forms $d\eta_i|_H$ are the fundamental $2$-forms of a quaternionic structure on $H$ (for more details see the next section).

It is a fundamental theorem of Biquard [3] that a QC-structure on a real analytic manifold $M^{4n+3}$ is always the conformal infinity of a quaternionic Kähler metric defined in the neighborhood of $M^{4n+3}$. This theorem generalizes an earlier result of LeBrun [17] that states that a real analytic conformal $3$-manifold is always the conformal infinity of a self-dual Einstein metric. From this point of view, one may regard the QC-geometry as a natural generalization to dimensions $4n + 3$ of the $3$-dimensional conformal Riemannian geometry. Moreover, the QC-geometry gives a natural setting for certain Yamabe-type problem [19, 11, 12, 14]. A particular case of this problem amounts to finding the extremals and the best constant in the $L^2$ Folland-Stein Sobolev-type embedding, [8] and [9], on the quaternionic Heisenberg group, see [10] and [12, 14].

The $1$-form $\eta$ that defines the QC-structure is determined up to a conformal factor and the action of $SO(3)$ on $\mathbb{R}^3$. Therefore $H$ is equipped with a conformal class $[g]$ of metrics and a $3$-dimensional quaternionic bundle $Q$. The associated $2$-sphere bundle $S^2(Q) \to M$ is called the twistor space of the QC-structure. The transformations preserving given QC structure $\eta$, i.e. the transformations of the type $\tilde{\eta} = \mu \Psi \cdot \eta$ for a positive smooth function $\mu$ and an $SO(3)$ matrix $\Psi$ with smooth functions as entries, are called quaternionic contact conformal (QC conformal) transformations. If the function
$\mu$ is constant we have quaternionic contact homothetic (QC homothetic) transformations. To every metric in the fixed conformal class $[g]$ on $H$ one can associate a linear connection preserving the QC structure, [3], which we shall call the Biquard connection. This connection is invariant under QC homothetic transformations but changes in a non-trivial way under QC conformal transformations.

Examples of QC manifolds can be found in [3, 4, 11, 7]. It is known that on the sphere $S^{4n+3}$, $n > 1$ there exist infinitely many different global QC-structures. Indeed, in [18] Lebrun has constructed an infinite-dimensional space of deformations of the standard hyperbolic quaternionic-Kähler metric on the open Ball $B^{4n+4}$ through complete quaternionic-Kähler metrics. After this, Biquard [3] has shown that each of the constructed metrics actually has as a conformal infinity a certain unique QC-structure on the boundary $S^{4n+3}$ of the ball. However, his construction does not give the QC-structures explicitly. The amount of known explicit examples remains very restricted.

As we have mentioned above and shall explain in the next paragraph, each QC-structures with a fixed metric in the conformal class $[g]$ determines a unique connection $\nabla$, the Biquard connection. This connection plays a role in the QC-geometry similar to the one played by the Levi-Civita connection in the 3-dimensional conformal geometry. The restriction to $H$ of the Ricci tensor of $(g, \nabla)$ gives rise to three quantities, namely the QC-scalar curvature $\text{Scal}$ and two symmetric trace-free $(0,2)$ tensor fields $T^0$ and $U$ defined on the contact distribution $H$. The tensors $T^0$ and $U$ determine the trace-free part of the Ricci tensor restricted to $H$ and can also be expressed in terms of the torsion endomorphisms of the Biquard connection [11] (see Section 2 for the details). According to [11], the vanishing of the torsion endomorphisms of the Biquard connection is equivalent to $T^0 = U = 0$ and if the dimension is at least eleven, then the function $\text{Scal}$ has to be constant. If in addition this constant is different from zero, then the QC-structure is locally QC-homothetic to a (positive or negative) 3-Sasakian structure (see also [16, 13]).

Explicit examples of QC manifolds with zero or non-zero torsion endomorphisms have been recently given in [1, 2]. The quaternionic Heisenberg group, the quaternionic sphere of dimension $4n + 3$ with its standard 3-Sasakian structure and the QC structures locally QC conformal to them are characterized in [15] by the vanishing of a tensor invariant under conformal transformations, the QC-conformal curvature defined in terms of the curvature and the torsion of the Biquard connection. Explicit examples of non QC conformally flat QC manifolds are constructed in [1, 2].

The twistor space $Z = S^2(Q)$ of a QC-structure is naturally equipped with a CR-structure [3, 6], which is invariant under the QC-conformal transformations. To each metric $g \in [g]$ one can naturally define a contact form $\eta^Z$ on the twistor space $Z$, compatible with the CR-structure there. The contact form depends on the choice of $g \in [g]$ and thus the whole construction is not QC-conformal invariant anymore but it remains however QC-homothetic invariant.

The purpose of the present notes is to show (Theorem 4.4) that the contact form $\eta^Z$ is normal if and only if the tensor $T^0$ vanishes. The latter condition is equivalent to the requirement that the Ricci tensor of the Biquard connection commutes with the endomorphisms in the quaternionic structure of $H$. Note that the normality of the contact manifold $(Z, \eta^Z)$ is equivalent to the condition that the product manifold $Z \times \mathbb{R}$ is a complex manifold with a certain naturally defined complex structure (cf. e.g. [5]).

Note also that every (negative or positive) 3-Sasakian manifold has constant QC-scalar curvature and satisfies the condition $T^0 = U = 0$ [11]. According to Theorem 4.4 the CR-structure on its twistor space is normal. It is shown in [13] that, in the case of zero torsion endomorphisms of the Biquard connection, the vector bundle $Q \to M$ admits a flat connection which implies that the corresponding bundle $\tilde{Q} \to \tilde{M}$ of the universal cover $\tilde{M}$ of $M$ is trivial. Thus in the case $T^0 = U = 0$ (plus the condition $\text{Scal}=\text{const}$ in dimension seven) we see that the twistor space $\tilde{Z}$ of the universal cover of $M$ is just the product $\tilde{Z} = \tilde{M} \times S^2$. 
To the best of our knowledge no explicit examples of QC structures with $T^0 = 0$ and non-constant QC scalar curvature are known.

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2. Quaternionic contact manifolds and the Biquard connection

In this section we will briefly review the basic notions of quaternionic contact geometry and recall certain results of [3] and [11].

A quaternionic contact structure (shortly, QC-structure) on a $(4n+3)$-dimensional smooth manifold $M$ consists of a rank $4n$ subbundle $H$ of $TM$, a positive definite metric $g$ on $H$ and a rank $3$ subbundle $Q$ of $End(H)$ such that, in a neighbourhood $U$ of each point of $M$, there are 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and a triple $\vartheta = (I_1, I_2, I_3)$ of sections of $Q$ with the following properties:

1. $H|U$ is the kernel of $\eta$;
2. The bundle $Q$ is locally generated by three almost complex structures $I_1, I_2, I_3$ on $H$ satisfying the identities of the imaginary unit quaternions, $I_1^2 = I_2^2 = I_3^2 = -Id_H$, $I_1I_2 = -I_2I_1 = I_3$.
3. $d\eta_s(X, Y) = 2g(I_sX, Y)$ for $X, Y \in H|U$.

Convention.

a) Throughout this paper, we shall use $X, Y, Z, U$ to denote vectors or sections of $H$;

b) $\{e_1, \ldots, e_{4n}\}$ denotes a local orthonormal basis of $H$;

c) The triple $(i, j, k)$ denotes any cyclic permutation of $(1, 2, 3)$.

d) $s, t$ will be any number from the set $\{1, 2, 3\}, \ s, t \in \{1, 2, 3\}$.

If $\eta = (\eta_1, \eta_2, \eta_3)$ and $\vartheta = (I_1, I_2, I_3)$ satisfy conditions (1), (2), (3), we shall say that $(\eta, \vartheta)$ is an admissible set for the QC-structure.

Any two triples of sections of $Q$ satisfying condition (2) constitute frames of $Q$ which induce the same orientation, thus the bundle $Q$ has a canonical orientation.

Condition (3) implies that $g(IX, Y) = -g(X, Y)$ for any section $I$ of $Q$ and $X, Y \in H$ and, thus, $H$ is equipped with an $Sp(n)Sp(1)$-structure.

The metric $g$ induces a metric on the bundle $End(H)$ defined by $< A, B > = \frac{1}{4n} Trace A^tB$, where $A$ and $B$ are endomorphisms of a fibre of $H$ and $A^t$ is the adjoint of $A$ with respect to $g$. Any sections $I_1, I_2, I_3$ of $End(H)$ satisfying the imaginary quaternion relations form an orthonormal set with respect to the induced metric. Moreover, if $I$ and $J$ are sections of $Q$, then $< I, J > = 0$ if and only if $IJ = -JI$ and $< I, I > = 1$ exactly when $I^2 = -Id_H$. Note also that any oriented orthonormal frame of $Q$ consists of endomorphisms of $H$ satisfying the imaginary quaternion relations.

Given a distribution $H$ on a smooth manifold $M$ and a vector bundle $E$ over $M$, a partial connection on $E$ along $H$ is, by definition, a bilinear map $\nabla X \sigma$ defined for vector fields $X$ with values in $H$ and sections $\sigma$ of $E$ such that $\nabla f X \sigma = f \nabla X \sigma$ and $\nabla X (f \sigma) = X(f) \sigma + f \nabla X \sigma$ for every smooth function $f$ on $M$.

Let $H$ be a distribution of a manifold $M$ and $g$ be a metric on $H$. Biquard [3, Lemma II.1.1] has observed that, for any supplementary distribution $V$ of $H$ in $TM$, there is a unique partial connection $\nabla$ on $H$ along $H$ such that

(i) $\nabla g = 0$;

(ii) for any two sections $X, Y$ of $H$, the torsion $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ satisfies the identity $T(X, Y) = -[X, Y]_V$, where the subscript $V$ means "the component in $V";
Now let \((M, H, g, Q)\) be a quaternionic contact manifold. Fix a supplementary distribution \(V\) of \(H\) in \(TM\) and let \(\nabla\) be the associated connection on \(H\) along \(H\). The partial connection on \(\text{End}(H)\) along \(H\) induced by \(\nabla\) will be denoted also by \(\nabla\). Biquard [3, Lemma II.1.6, Proposition II.1.7] has shown that \(\nabla\) preserves the bundle \(Q\) if and only if, around any point of \(M\), there is an admissible set \((\eta, \vartheta)\) such that the frame \((\xi_1, \xi_2, \xi_3)\) of \(V\) dual to the frame \((\eta_1|V, \eta_2|V, \eta_3|V)\) satisfies the condition

\[
(\iota_{\xi_s}d\eta_t)H = -(\iota_{\xi_s}d\eta_t)H, \quad s, t = 1, 2, 3.
\]

where \(\iota\) denotes the interior multiplication. Note that, if condition (2.1) is satisfied for an admissible set \((\eta, \vartheta)\), then it holds for any other admissible set \((\eta', \vartheta')\). Indeed, we have \(\eta'_t = \sum_{s=1}^3 a_{ts}\eta_s\), where \([a_{ts}]\) is a non-singular \(3 \times 3\) matrix of smooth functions. In view of (1), \(d\eta'_s(X, Y) = \sum_{s=1}^3 a_{ts}(X, Y)\) for \(X, Y \in H\), hence \(I' = \sum_{s=1}^3 a_{ts}I_s\) by (3). The latter identity and (2) imply that \([a_{ts}] \in SO(3)\). Then \(\xi'_t = \sum_{s=1}^3 a_{ts}\xi_s\), where \((\xi'_1, \xi'_2, \xi'_3)\) is the dual frame of \((\eta'_1|V, \eta'_2|V, \eta'_3|V)\). This observation implies our claim.

Biquard [3, Théorème II.1.3] has proved that if \(\dim M > 7\), then there is a unique supplementary distribution \(V\) of \(H\) in \(TM\) for which the associated connection \(\nabla\) preserves the bundle \(Q\):

\[
(iii) \quad \nabla_X Q \subset Q \text{ for } X \in H.
\]

For any admissible set \((\eta, \vartheta)\), the frame \((\xi_1, \xi_2, \xi_3)\) of the bundle \(V\) dual to the frame \((\eta_1|V, \eta_2|V, \eta_3|V)\) will be called associated to \((\eta, \vartheta)\).

Given a section \(\xi\) of \(V\) and a section \(X\) of \(H\), set

\[
(iv) \quad \nabla_X \xi = [X, \xi]|_V.
\]

By [3, Proposition II.1.9], the latter formula defines a partial connection on \(V\) along \(H\) such that

\[
(v') \quad \nabla < \ldots > = 0.
\]

Let \((\eta, \vartheta)\) be an admissible set for the given quaternionic contact structure on \(M\) and let \((\xi_1, \xi_2, \xi_3)\) be the frame of \(V\) associated to \((\eta, \vartheta)\). Then the assignment

\[
(2.2) \quad \xi_s \to I_s, \quad s = 1, 2, 3,
\]

determines an bundle isomorphism \(\varphi : V \to Q\) that does not depend on the particular choice of the admissible set. The isomorphism \(\varphi\) has the property that \(\nabla_X \varphi = 0\) for \(X \in H\). Indeed, by (3) and (2.1), we have

\[
\nabla_X \varphi(\xi_t) = \nabla_X I_t = -\sum_{s=1}^3 d\eta_s(\xi_s, X) I_s = \sum_{s=1}^3 d\eta_s(\xi_t, X) I_s = \sum_{s=1}^3 \eta_s([X, \xi_t]|_V) I_s = \sum_{s=1}^3 \eta_s(\nabla_X \xi_t) \varphi(\xi_s) = \varphi(\nabla_X \xi_t)
\]

Set

\[
P = \{A \in \text{End}(H) \mid A \text{ is skew-symmetric and } AI = IA \text{ for every } I \in Q\}.
\]

This is a subbundle of \(\text{End}(H)\) of rank \(2n^2 + n\), orthogonal to \(Q\) and such that the commutator \([A_1, A_2]\) of two endomorphisms \(A_1, A_2 \in P\) is also in \(P\). Clearly, every fibre of \(P\) (resp. \(Q\)) is isomorphic to the Lie algebra \(sp(n)\) (resp. \(sp(1)\)).

It is shown in [3, Lemme II.2.1] that there is a unique partial connection \(\nabla\) on \(H\) along \(V\) such that

\[
(v) \quad \nabla g = 0;
\]

\[
(vi) \quad \text{The induced connection on } \text{End}(H) \text{ preserves the bundle } Q;
\]

\[
(vii) \quad \text{Setting } T(\xi, X) = \nabla_{\xi}X - \nabla_X \xi - [\xi, X] \text{ for } \xi \in V \text{ and } X \in H\text{, every endomorphism}
\]

\[
T_\xi : H \in X \to T(\xi, X) = \nabla_\xi X - [\xi, X]|_H \in H
\]

is an element of \((P \oplus Q)^\perp \subset \text{End}(H)\).
Note, that we have a bundle isomorphism $\{(P \oplus Q)^\perp \subset \text{End}(H)\} \cong \{(sp(n) \oplus sp(1))^\perp \subset g(4n)\}$.

Since $\nabla_\xi Q \subset Q$ for every $\xi \in V$, we can transfer $\nabla_\xi$ from $Q$ to $V$ via the isomorphism $\varphi : V \to Q$.

In this way get a partial connection on $V$ along $V$.

Combining the partial connections we have defined, we obtain a connection $\nabla$ on $TM$ having the properties $(i)$-$(vii)$ and the property $(viii)$ $\nabla \varphi = 0$.

We shall call $\nabla$ the Biquard connection of the QC-structure $(H, g, Q)$ on $M$.

In the case when the dimension of $M$ is seven, it is not always possible to find a supplement $V$ to $H$ for which condition (2.1) hold. Duchemin [6] has shown that if we assume that, around any point of $M$, there exists an admissible set for which we can find vector fields $\xi_1, \xi_2, \xi_3$ satisfying (2.1), then one can define a connection with the properties $(i)$-$(viii)$.

Henceforth, by a quaternionic contact structure in dimension 7 we shall mean a QC-structure satisfying (2.1).

Let $\varphi : V \to Q$ be the isomorphism defined by (2.2). Using this isomorphism we transfer to $V$ the metric and the orientation of $Q$. Then any frame $\xi_1, \xi_2, \xi_3$ associated to an admissible set of the QC-structure is orthonormal and positively oriented. Putting together the metric of $V$ and the metric $g$ of $H$ we obtain a metric on $TM = H \oplus V$ for which $H$ and $V$ are orthogonal. This metric will be also denoted by $g$. It follows from properties $(v')$, $(v)$ and $(vi)$ that the connection $\nabla$ on $Q$ is compatible with the metric $<\cdot, \cdot>$. Therefore the metric $g$ on $TM$ is parallel with respect to the Biquard connection, $\nabla g = 0$.

The properties of the Biquard connection are encoded in the properties of the torsion endomorphisms $T_\xi = T(\xi, \cdot) : H \to H, \ \xi \in V$.

Any endomorphism $\Psi$ of $H$ can uniquely be decomposed with respect to the quaternionic structure $(Q, g)$ into four $Sp(n)$-invariant parts $\Psi = \Psi^{+++} + \Psi^{++} + \Psi^{++} + \Psi^{+-}$, where $\Psi^{+++}$ commutes with all three $I_1$, $\Psi^{++}$ commutes with $I_1$ and anti-commutes with the others two and so on. Explicitly,

$$4\Psi^{+++} = -I_1 \Psi I_1 - I_2 \Psi I_2 - I_3 \Psi I_3,$$

$$4\Psi^{++} = \Psi - I_1 \Psi I_1 + I_2 \Psi I_2 + I_3 \Psi I_3,$$

$$4\Psi^{++} = \Psi + I_1 \Psi I_1 - I_2 \Psi I_2 + I_3 \Psi I_3,$$

$$4\Psi^{--} = \Psi - I_1 \Psi I_1 + I_2 \Psi I_2 - I_3 \Psi I_3.$$

The two $Sp(n)Sp(1)$-invariant components are $\Psi^{+++}$ and $\Psi^{++} + \Psi^{+-}$. If $n = 1$, then the space of symmetric endomorphisms commuting with all $I_1$ is 1-dimensional, i.e. $\Psi^{+++}$ is proportional to the identity, $\Psi^{+++} = \frac{1}{4} I_1 d I_1 H$.

Decomposing the endomorphism $T_\xi \in (sp(n) + sp(1))^\perp$ into its symmetric part $T^{0}_\xi$ and skew-symmetric part $b_\xi, T_\xi = T^{0}_\xi + b_\xi$, Biquard has shown in [3] that the torsion $T_\xi$ is completely trace-free, $tr T_\xi = tr T^{0}_\xi = 0$, its symmetric part has the properties $T^{0}_\xi I_1 = -I_1 T^{0}_\xi, \ \ T^{0}_\xi I_2 = T^{0}_\xi I_3 = 0$. The skew-symmetric part can be represented as $b_\xi = u I_1$, where $u$ is a traceless symmetric $(1,1)$-tensor on $H$ which commutes with $I_1, I_2, I_3$. If $n = 1$ then the tensor $u$ vanishes identically, $u = 0$ and the torsion is a symmetric tensor, $T_\xi = T^{0}_\xi$.

As in [11], we define two symmetric 2-tensors $T^0$ and $U$ on $H$ setting

$$T^0(X, Y) = g((T^{0}_\xi I_1 + T^{0}_\xi I_2 + T^{0}_\xi I_3)X, Y), \ \ U(X, Y) = g(uX, Y).$$

It is easy to see that $T^0$ and $U$ are independent of the choice of the admissible set $(\eta, \vartheta)$, and that they have the following properties:

$$T^0(X, Y) + T^0(I_1 X, I_1 Y) + T^0(I_2 X, I_2 Y) + T^0(I_3 X, I_3 Y) = 0, \ \ tr g(T^0) = tr g(T^0 I_1) = 0$$

$$U(X, Y) = U(I_1 X, I_1 Y) = U(I_2 X, I_2 Y) = U(I_3 X, I_3 Y), \ \ tr g(U) = tr g(U I_1) = 0.$$

In dimension seven $(n = 1)$, the tensor $U$ vanishes identically, $U = 0$. 

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The identity $4q(T^0(\xi_s, X, Y) = -T^0(I_sX, Y) - T^0(X, I_sY)$, proved in [15, Proposition 2.3], together with the first equality in (2.3) implies the equivalence [11]
\[ T^0 = 0 \iff \{ T^0_{\xi_s} = 0, s = 1, 2, 3 \}. \]

The torsion of Biquard connection is given in terms of the tensors $T^0$ and $U$ by the formula
\[ g(T(\xi_s, X, Y) = g(T^0(\xi_s, X, Y) + U(I_sX, Y) = -\frac{T^0(I_sX, Y) + T^0(X, I_sY)}{4} + U(I_sX, Y). \]

Let $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ be the curvature tensor of the Biquard connection. The $QC$-Ricci curvature $\text{Ric}$, the $QC$-Ricci forms $\rho_s$ and the $QC$-scalar curvature $\text{Scal}$ are defined respectively by
\[ \text{Ric}(A, B) = \sum_{a, b=1}^{4n} g(R(e_b, A) B, e_b), \ A, B \in TM, \]
\[ \rho_s(A, B) = \frac{1}{4n} \sum_{a, b=1}^{4n} g(R(A, B) e_a, I_s e_a), \ \text{Scal} = \sum_{a, b=1}^{4n} g(R(e_b, e_a) e_a, e_b), \]
where $e_1, ..., e_{4n}$ is an orthonormal basis of $H$. The restriction of the Ricci curvature $\text{Ric}$ to $H$ is a symmetric 2-tensor ([3]) that could be $Sp(n)Sp(1)$-invariantly decomposed in exactly three components. It is shown in ([11]) that this three components are given by the 2-tensors $T^0$, $U$ and $\text{Scal} \cdot g$.

We have (see Theorem 3.12, [11]) :
\[ \text{Ric}(X, Y) = (2n + 2)T^0(X, Y) + (4n + 10)U(X, Y) + \frac{\text{Scal}}{4n} g(X, Y), \ X, Y \in H. \]

Since $V$ is preserved by $\nabla$ and $\nabla g = 0$, there exist local 1-forms $\alpha_1$, $\alpha_2$ and $\alpha_3$ such that
\[ \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j. \]

Set $\tau = \frac{\text{Scal}}{16n(n + 2)}$. Then, according to [11, Proposition 3.5 and Theorem 3.12], we have
\[ \alpha_1(\xi_s) = d\eta_s(\xi_j, \xi_k) - \delta_is(\tau + \frac{1}{2}d\eta_1(\xi_2, \xi_3) + \frac{1}{2}d\eta_2(\xi_3, \xi_1) + \frac{1}{2}d\eta_3(\xi_1, \xi_2)). \]

3. THE TWISTOR SPACE OF A QUATERNIONIC CONTACT MANIFOLD

Let $(M, H, g, Q)$ be a quaternionic contact manifold. Let $\pi : Q \to M$ be the projection onto $M$ of the bundle $Q$. Set
\[ \mathcal{Z} = \{ I \in Q \mid I^2 = -\text{Id}_H \}. \]

Then $\pi_{\mathcal{Z}} = \pi|\mathcal{Z} : \mathcal{Z} \to M$ is a subbundle of the vector bundle $Q$ called the twistor space of the given $QC$-manifold. As we have mentioned, the condition $I^2 = -\text{Id}_H$ for $I \in Q$ is equivalent to $< I, I > = 1$, thus $\mathcal{Z} = \{ I \in Q \mid < I, I > = 1 \}$.

Let $\nabla$ be the Biquard connection on $M$ and denote by $\mathcal{H}$ the horizontal subbundle of $TQ$ with respect to $\nabla$. For $I \in \mathcal{Z}$, the space $\mathcal{H}_I$ is tangent to the submanifold $\mathcal{Z}$ of $Q$ since $\mathcal{Z}$ is the unit-sphere bundle of the vector bundle $Q$ endowed with the metric $< ., . >$, parallel with respect to the connection $\nabla$ on $Q$. Further on, the restriction to $\mathcal{Z}$ of the horizontal bundle will be also denoted by $\mathcal{H}$. Let $\mathcal{V}$ be the vertical subbundle of $T\mathcal{Z}$. Then $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$.

For $I \in \mathcal{Z}$, set $\xi_I = \varphi^{-1}(I)$ and denote by $\chi_I$ the horizontal lift of $\xi_I$ at the point $I$. Fix a $I_0 \in \mathcal{Z}$ and let $(\eta, \theta)$ be an admissible set of the $QC$-structure defined on a neighbourhood $U$ of the point $p = \pi(I_0)$. Denote by $(\xi_1, \xi_2, \xi_3)$ the frame of vector fields on $U$ associated to $(\eta, \theta)$. Then every $I \in \pi_{\mathcal{Z}}^{-1}(U)$ has a unique representation $I = x_1(I) I_1 + x_2(I) I_2 + x_3(I) I_3$ where $x_1, x_2, x_3$ are smooth functions such that $x_1^2 + x_2^2 + x_3^2 = 1$. We have $\chi = x_1 I_1^h + x_2 I_2^h + x_3 I_3^h$ on $\pi_{\mathcal{Z}}^{-1}(U)$ where the upper script $h$ means "the horizontal lift". This shows that $\chi$ is a smooth vector field on $\mathcal{Z}$.
Let $g^h$ be the lift of the metric $g$ on $TM$ to the horizontal bundle $\mathcal{H}$ of $\mathcal{Z}$. Denote by $W$ the orthogonal complement in $\mathcal{H}$ of the horizontal vector field $\chi$. Then $D = W \oplus V$ is a codimension 1 subbundle of $T\mathcal{Z}$ and, following [3], we shall define an almost complex structure $J$ on it as follows.

Any vertical space $V_I$, $I \in \mathcal{Z}$, is the tangent space at $I$ of the fibre $\mathcal{Z}_I$. The latter is the unit sphere in the 3-dimensional vector space $Q_I$, so $V_I = T\mathcal{Z}_I = \{ S \in Q \mid S, I >= 0 \} = \{ S \in Q_I \mid SI + IS = 0 \}$. We define $J|V_I$ to be the standard complex structure of the 2-sphere $\mathcal{Z}_I$. In other words, we set $\beta S = I \circ S$ for $S \in V_I$.

For $I \in \mathcal{Z}$, denote by $W_I$ the orthogonal complement in $V_I$ of $\xi_I$, $V_I$ being the fibre of $V$ at $p = \pi_2(I)$. We consider $W_I$ with the metric and the orientation induced by those of $V_I$. Since the dimension of the space $W_I$ is 2, there is a unique complex structure $J$ on it compatible with the metric and the orientation. If we denote by $\times$ the vector-cross product of the oriented Euclidean 3-dimensional vector space $V_I$, then $\hat{I}\zeta = \xi_I \times \zeta$ for $\zeta \in W_I$. Note that the isomorphism $\varphi : V \rightarrow Q$ sends $W_I$ onto $V_I \subset Q$ and

$$\varphi(\hat{I}\zeta) = \beta\varphi(\zeta), \quad \zeta \in W_I.$$ 

Now we define a complex structure $J_I$ on the space $H_p \oplus W_I$, $p = \pi_2(I)$, setting $J_I|H_p = I$ and $J_I|W_I = \hat{I}$. Then we define $\beta|W_I$ as the horizontal lift of $J_I$, i.e. $\beta|W_I$ is the pull-back of $J_I$ under the isomorphism $\pi_2* : W_I \rightarrow H_p \oplus W_I$.

In this way we obtain a CR manifold $(\mathcal{Z}, D, \beta)$. Recall that a Cauchy-Riemann (CR) structure (in wide sense) on a manifold $N$ is a pair $(D, \beta)$ of a subbundle $D$ of the tangent bundle $TN$ and an almost complex structure $\beta$ of the bundle $D$. For any two sections $X, Y$ of $D$, the value of $[X, Y] mod D$ at a point $p \in N$ depends only on the values of $X$ and $Y$ at $p$, and so we have a skew-symmetric bilinear form $\omega : D \times D \rightarrow TN/D$ defined by $\omega(X, Y) = [X, Y] mod D$; this form is called the Levi form of the CR-structure $(D, \beta)$. If the Levi form is $\beta$-invariant, we can define the Nijenhuis tensor of the CR-structure $(D, \beta)$ by

$$N^CR(X, Y) = -[X, Y]+[\beta X, \beta Y]-\beta([\beta X, Y]+[X, \beta Y]);$$

its value at a point $p \in N$ lies in $D$ and depends only on the values of the sections $X, Y$ at $p$. A CR-structure is said to be integrable if its Levi form is $\beta$-invariant and the Nijenhuis tensor vanishes. Let $D^C = D^{1,0} \oplus D^{0,1}$ be the decomposition of the complexification of $D$ into $(1, 0)$ and $(0, 1)$ parts with respect to $\beta$. If the CR-structure $(D, \beta)$ is integrable, then the bundle $D^1,0$ satisfies the following two conditions:

$$D^{1,0} \cap \overline{D^{1,0}} = 0, \quad [\Gamma(D^{1,0}), \Gamma(D^{1,0})] \subset \Gamma(D^{1,0})$$

where $\Gamma(D^{1,0})$ stands for the space of smooth sections of $D^{1,0}$. Conversely, suppose we are given a complex subbundle $E$ of the complexified tangent bundle $T^CN$ such that $\mathcal{E} \cap \overline{\mathcal{E}} = 0$ and $[\Gamma(E), \Gamma(E)] \subset \Gamma(E)$ (many authors called a bundle with these properties a "CR-structure"). Set $D = \{ X \in TN : X = Z + \bar{Z} \text{ for some (unique) } Z \in E \}$ and put $\beta X = -Im Z$ for $X \in D$. Then $(D, \beta)$ is an integrable CR-structure such that $D^{1,0} = \mathcal{E}$.

It is a result of Biquard [3, Theorem II.5.1] that the CR-structure $(D, \beta)$ on the twistor space $\mathcal{Z}$ is integrable and (up to isomorphism) is invariant under conformal changes of the metric $g$.

Let $(\eta, \vartheta)$ be an admissible set of the given QC-structure defined on an open subset $U$ of $M$. For $I = x_1I_1 + x_2I_2 + x_3I_3 \in \mathcal{Z}$, we set

$$\eta^\mathcal{Z}_I = x_1\pi^*_Z \eta_1 + x_2\pi^*_Z \eta_2 + x_3\pi^*_Z \eta_3.$$ 

The right-hand side of the latter formula does not depend on the choice of the admissible set $(\eta, \vartheta)$, thus we have a well-defined 1-form $\eta^\mathcal{Z}$ on $\mathcal{Z}$. It is clear that $\eta^\mathcal{Z}$ vanishes on $V$ and on the horizontal lift $H^h$ of the space $H$. Let $\xi_1, \xi_2, \xi_3$ be the frame of $V$ associated to $(\eta, \vartheta)$. Then $\xi_I = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$, hence

$$\eta^\mathcal{Z}(\chi_I) = x_1^2 + x_2^2 + x_3^2 = 1.$$
Moreover, if \( \zeta = \sum_{s=1}^{3} z_s \xi_s \in W_1 \), we have \( \eta^\zeta(\zeta^h_s) = \sum_{s=1}^{3} z_s z_s = 0 \). Therefore \( \text{Ker } \eta^\zeta = \mathcal{D} \). Define an endomorphism of the tangent bundle \( T\mathcal{Z} \) setting \( \Phi|\mathcal{D} = \mathcal{J} \) and \( \Phi(\chi) = 0 \). Then
\[
\Phi^2(A) = -A + \eta^\zeta(A)\chi, \quad A \in T\mathcal{Z}.
\]
Thus \((\Phi, \chi, \eta^\zeta)\) is an almost contact structure on the twistor space \( \mathcal{Z} \) with contact distribution \( \mathcal{D} \) and Reeb vector field \( \chi \).

Let \((\eta, \vartheta)\) be an admissible set on an open subset \( U \) of \( M \) which is the domain of local coordinates \( u_1, \ldots, u_m \) of \( M, m = 4n+3 \). Set
\[
\bar{u}_r = u_r \circ \pi(I), \quad r = 1, \ldots, m, \quad x_s(I) = \langle I, I_s \rangle, \quad s = 1, 2, 3,
\]
for \( I \in \pi^{-1}(U) \subset Q \). Then \((\bar{u}_1, \ldots, \bar{u}_m, x_1, x_2, x_3)\) is a local coordinate system of the manifold \( Q \). For each vector field \( X = \sum_{r=1}^{m} X_r \frac{\partial}{\partial u_r} \) on \( U \), the horizontal lift \( X^h \) on \( \pi^{-1}(U) \) is given by
\[
X^h = \sum_{r=1}^{m} (X^r \circ \pi) \frac{\partial}{\partial \bar{u}_r} - \sum_{s,t=1}^{3} x_s(< \nabla_X I_s, I_t > \circ \pi) \frac{\partial}{\partial x_t}.
\]
(3.1)

It follows from (3.1) that
\[
[X^h, Y^h] = [X, Y]^h - \sum_{s,t=1}^{3} x_s (R(X, Y)I_s, I_t) \frac{\partial}{\partial x_t},
\]
where \( R(X, Y) \) is the curvature tensor of connection on \( \text{End}(H) \) induced by the Biquard connection.

Given a section \( a \) of \( Q \), we denote by \( \bar{a} \) the (local) vertical vector field on \( Q \) defined by \( \bar{a}_J = a_{\pi(J)} - \langle a_{\pi(J)}, J \rangle J > J \). This vector field is tangent to \( \mathcal{Z} \), thus its restriction to \( \mathcal{Z} \), denoted again by \( \bar{a} \), is a vertical vector field on the twistor space.

We shall use the above notation throughout this section and the following ones without further referring to it.

**Lemma 3.1.** For any \( I \in \mathcal{Z} \), a vector field \( X \) on \( M \) and a section \( a \) of \( Q \) near the point \( p = \pi_{\mathcal{Z}}(I) \), we have:
\[
[X^h, \bar{a}]_I = (\nabla_X a)_I, \quad [\chi, \bar{a}]_I = - (\varphi^{-1}(a))_I + (\nabla_{\xi_I} a)_I
\]
\[
[X^h, \chi]_I = (\nabla_{\xi_I} X + T_p(X, \xi_I))_I + R_p(X, \xi_I)I
\]
Proof. Let \( a = \sum_{s=1}^{3} a_s I_s \). Then \[
abla \tilde{a} = \sum_{s=1}^{3} \tilde{a}_s \frac{\partial}{\partial x_s}, \quad \text{where} \quad \tilde{a}_s = a_s - (\sum_{t=1}^{3} a_t x_t) x_s.
\]
We have \[
X^h = \sum_{r=1}^{m} X^r (p) \frac{\partial}{\partial a_r}, \quad \left[ X^h, \frac{\partial}{\partial a_r} \right] = 0, \quad \nabla_{X^r} a = \sum_{s=1}^{3} X_p (a_s) I_s (p)
\]
since \( \nabla I_s | p = 0, \) \( s = 1, 2, 3. \) Now the identity \( \left[ X^h, \tilde{a} \right] = (\nabla_X \tilde{a}) I \) follows easily from (3.1). The second formula stated in the lemma follows from the first one taking into account that (locally) \( \chi = \sum_{s=1}^{3} x_s \xi^h_s. \) Moreover, in view of (3.2), we have \[
[\chi, X^h] = \sum_{s=1}^{3} (x_s (I_s | \xi_s, X^h | R_p (\xi_s, X) I) = (\nabla_{\xi_s} X + T (X_p, \xi_s))^h + R_p (X, \xi_s) I).
\]
\( \square \)

**Lemma 3.2.** Let \( I \in \mathcal{Z}, \) \( X, Y \in T_p M \) and \( a, b \in \mathcal{V}. \) Let \( X_H, Y_H \) and \( X_V, Y_V \) be, respectively, the \( H \) - and the \( V \) - components of \( X, Y. \) Then
\[
d \eta^V (X^h_1, Y^h_1) = 2 g (I X_H, Y_{H}) - 2 r g (\xi_1 \times X, Y_{V}), \quad d \eta^V (X^h_1, a) = - g (X, \varphi^{-1} (a)), \quad d \eta^V (a, b) = 0.
\]

**Proof.** First, suppose that \( X, Y \in H_p, \) \( p = \pi (I). \) Since \( \nabla \) preserves \( H, \) we can extend the vectors \( X, Y \) to sections \( X, Y \) of \( \tau \) defined in a neighbourhood of \( p \) such that \( \nabla X | p = \nabla Y | p = 0. \)

We have \[
X^h (\eta^V (Y^h)) = X^h (g^h (Y^h, \chi)) = \sum_{s=1}^{3} X^h (x_s g (Y, \xi_s) \circ \pi) = \sum_{s=1}^{3} x_s X_p (g (Y, \xi_s)) = 0
\]
since \( X^h (x_s) = 0, \) \( \nabla Y | p = 0 \) and \( \nabla I_s | p = 0, \) \( s = 1, 2, 3. \) Similarly, \( Y^h (\eta^V (X^h)) = 0. \) Recall that the form \( \eta^V \) vanishes on the vertical vectors. Then, in view of (3.2), \( d \eta^V (X^h_1, Y^h_1) = \eta^V ([X, Y]^h_1) = - \eta^V ([X, Y]_h) = d \eta^V (X^h_1, Y^h_1) = 2 g_p (I_1 X, Y) = 2 g_p (I X, Y), \)

For any index \( s = 1, 2, 3, \) we have \( (\xi_s)^h_1 (\eta^V (X^h)) = 0 \) as above and \( X^h (\eta^V (\chi_s^h)) = \sum_{s=1}^{3} X^h (x_s \delta_s) = 0. \) It follows that \( d \eta^V (X^h_1, (\xi_s)^h_1) = - \eta^V ([X, \xi_s]^h_1) = \eta^V (T_p (X, \xi_s) = 0, s = 1, 2, 3, \) since \( T_p (X, \xi_s) \in H \) (property (vi) of the Biquard connection). Therefore \( d \eta^V (X^h_1, (\xi_s)^h_1) = 0 \) for \( X \in H_p \) and \( \xi \in \mathcal{V}. \)

We have also that \( d \eta^V ((\xi_s)^h_1, (\xi_t)^h_1) = - \eta^V ([\xi_s, \xi_t]_p) = d \eta^V (\xi_s, \xi_t) = 0, t = 1, 2, 3. \) If \( \alpha_i \) are the 1-forms defined by (2.8), then \( \alpha_i (\xi_s) = g (\nabla \xi_s, \xi_k) \) where \( (i, j, k) \) is a cyclic permutation of \( (1, 2, 3). \)

Therefore \( \alpha_i (\xi_s) = 0 \) at the point \( p \) and identity (2.9) implies that \( d \eta^V (\xi_1, \xi_2) = d \eta^V (\xi_3, \xi_1) = 0 \) at \( p. \)

Identity (2.9) gives also that
\[
2 d \eta^V (\xi_1, \xi_2, \xi_3) = 2 \tau - d \eta^V (\xi_2, \xi_3) - d \eta^V (\xi_1, \xi_2) = 0 \text{ at the point } p.
\]

Adding the three identities corresponding to the cyclic permutations of \( (1, 2, 3), \) we get from (3.3) that, at \( p, \)
\[
d \eta^V (\xi_2, \xi_3) + d \eta^V (\xi_3, \xi_1) + d \eta^V (\xi_1, \xi_2) = - 6 \tau. \]

The latter identity and identity (3.3) with \( (i, j, k) = (1, 2, 3) \) give \( d \eta^V (\xi_2, \xi_3) = - 2 \tau \) at \( p. \) It follows that, for every \( \xi, \xi \in \mathcal{V}, \) we have \( d \eta^V (\xi_1, \xi_2) = - 2 \tau g (\xi_1 \times \xi_2). \)

Now let \( X, Y \in T_p M \) be arbitrary tangent vectors. Writing \( X = X_H + X_V, \) \( Y = Y_H + Y_V \) and applying the preceding considerations we get the first formula stated in the lemma.

Next, take two sections of \( Q \) with values \( a \) and, respectively, \( b \) at the point \( p, \) and zero covariant derivatives at \( p. \) Denote these sections again by \( a \) and \( b. \) Extend \( X \) to a vector field for which \( \nabla X | p = 0. \) Then Lemma 3.1 implies that \( d \eta^V (X^h_1, a) = d \eta^V (X^h_1, \tilde{a}_1) = 0. \) The last formula stated in the lemma follows from the fact that \( \eta^V | \mathcal{V} = 0 \) and the bundle \( \mathcal{V} \) is closed under the Lie bracket.

\( \square \)

**Corollary 3.3.** \( d \eta^V (A, \chi) = 0 \) for every \( A \in T \mathcal{Z}. \)
Proof. If $A = \chi_h$ for a vector $X \in T_\infty M$, then, by Lemma 3.2, $d\eta^Z(A, \chi) = -2g(X_V \times \xi_t, \xi_t) = 0$. If $A$ is a vertical vector, then $d\eta^Z(A, \chi) = g(\xi_t, \varphi^{-1}(A)) = -\varphi(\xi_t), A = -\varphi, A = 0$. □

Set

\begin{equation}
G(A, B) = \frac{1}{2} d\eta^Z(A, \Phi B) + \eta^Z(A)\eta^Z(B), \quad A, B \in TZ.
\end{equation}

Obviously, $G(\chi, \chi) = 1$. By Corollary 3.3, $G(A, \chi) = G(\chi, A) = 0$ for every $A \in TZ$. Moreover, it is easy to check by means of Lemma 3.2 that $G$ is a symmetric non-degenerate tensor. It is an observation of Biquard [3, Theorem II.5.1] that the symmetric form $d\eta^Z(A, \Phi B)$ on the space $\ker \eta^Z$ is of signature $(2n + 2, 2)$. Therefore $G$ is of signature $(4n + 2, 2)$; see also Corollary 4.1 below. Thus, $G$ is a pseudo-Riemannian metric on $\mathcal{Z}$ such that $G(\Phi A, \Phi B) = G(A, B) - \eta^Z(A)\eta^Z(B)$ and $d\eta^Z(A, B) = 2G(\Phi A, B), A, B \in TZ$. Therefore $(\Phi, \chi, \eta^Z, G)$ is a contact metric structure on the twistor space $\mathcal{Z}$.

4. Normality of the contact structure on the twistor space of a QC-manifold

The condition for the contact structure $(\Phi, \chi, \eta^Z)$ to be normal is the vanishing of the following tensor (cf., for example, [5])

\begin{equation}
N^1(A, B) = \Phi^2[A, B] + [\Phi A, \Phi B] - \Phi([\Phi A, B)] + [A, \Phi B]) + 2d\eta^Z(A, B)\chi, \quad A, B \in TZ.
\end{equation}

Let $A, B$ be vector fields with values in $\ker \eta^Z = \mathcal{D}$. Then

\begin{equation}
N^1(A, B) = N^{CR}(A, B) + \eta^Z([A, B]\chi + d\eta^Z(A, B)\chi = N^{CR}(A, B).
\end{equation}

But, as we have mentioned, $N^{CR}(A, B) = 0$ for $A, B \in \ker \eta^Z$ and to prove that $N^1 = 0$ it remains to show that $N^1(A, \chi) = 0$ for $A \in \ker \eta^Z$. The latter identity is equivalent to

\begin{equation}
(\mathcal{L}_A G)(A, B) = 0 \quad A, B \in \ker \eta^Z = \mathcal{D},
\end{equation}

where $\mathcal{L}$ stands for the Lie derivative. Indeed, we have $\mathcal{L}_A d\eta = (d\mathcal{L}_A + \mathcal{L}_A d)\eta = d(\mathcal{L}_A d)\eta = 0$ since $\mathcal{L}_A d\eta = 0$. On the other hand, $(\mathcal{L}_A d\eta)(A, \Phi B) = -\chi G(A, B) + G([\chi, A], B) - G(A, \Phi [\chi, B])$ for $A, B \in \mathcal{D}$. Hence $\chi G(A, B) - G([\chi, A], B) = -G(A, \Phi [\chi, B])$. Then $(\mathcal{L}_A G)(A, B) = -G(A, \Phi [\chi, B]) - G([\chi, B]) = G(\mathcal{L}_A \Phi(B), A) = -G(N^1(B, \chi), A)$. This proves our claim since the form $G$ is non-degenerate.

Lemma 3.2 implies the following.

Corollary 4.1. Let $I \in \mathcal{Z}, X, Y \in T_{\pi(I)}M$ and $a, b \in \mathcal{V}_I$. Let $X_H, Y_H$ and $X_V, Y_V$ be, respectively, the $H$- and the $V$-components of $X, Y$. Then

\begin{equation}
G(X^h_I, Y^h_I) = g(X_H, Y_H) - \tau g(X_V, Y_V) + (\tau + 1)g(X, \xi_I)g(Y, \xi_I),
\end{equation}

\begin{equation}
G(X^h_I, a) = \frac{1}{2} g(\xi_I \times X_V, \varphi^{-1}(a)), \quad G(a, b) = 0.
\end{equation}

Proof. We have $Y^h_I = (Y_H)^h_I + (Y_V - g(Y, \xi_I)\xi_I)^h_I + g(Y, \xi_I)\eta$, hence $\Phi^1^h_I = (IY)^h_I + (\xi_I \times Y_V)^h_I$ and $\eta^Z(Y^h_I) = g(Y, \xi_I)$. Now the statement follows from (3.4) and Lemma 3.2. □

Corollary 4.2. Let $A, B \in T_{\pi(I)} \mathcal{Z}$ and let

\begin{equation}
A = X^h + \sum_{s=1}^{3} u_s(\xi_s)^h + x_2 I_2 + x_3 I_3, \quad B = Y^h + \sum_{s=1}^{3} v_s(\xi_s)^h + y_2 I_2 + y_3 I_3,
\end{equation}

where
where $X, Y \in H$. Then

$$G(A, B) = g(X, Y) + u_1v_1 - \tau(u_2v_2 + u_3v_3) - \frac{1}{2}(u_3y_2 + v_3x_2) + \frac{1}{2}(u_2y_3 + v_2x_3).$$

In particular, $G$ is of signature $(4n+2, 2)$.

We shall use Lemma 3.1 and Corollary 4.1 to prove the following.

**Lemma 4.3.** Let $I \in \mathcal{I}$, $X, Y \in H_p$, $p = \pi(I)$, and $a, b \in \mathcal{V}_I$. Then

$$(\mathcal{L}_G)(X^h, Y^h) = 2g_p(T^0_{\xi_1}(X), Y), \quad (\mathcal{L}_G)(X^h, (\xi_s)^h) = \rho_s(X, \xi_1)p + g_p([\xi_s, \xi_1], X), \quad s = 2, 3,$$

$$(\mathcal{L}_G)((\xi_s)^h, (\xi_t)^h) = -d\tau(\xi_1)p\delta_{st} + \rho_s(\xi_t, \xi_1)p + \rho_t(\xi_s, \xi_1)p, \quad s, t = 2, 3,$$

$$(\mathcal{L}_G)(X^h, a) = 0, \quad (\mathcal{L}_G)((\xi_s)^h, a), \quad s = 2, 3,$$

$$(\mathcal{L}_G)(a, b) = 0.$$

**Proof.** Extend the vectors $X, Y$ to sections $X, Y$ of $H$ for which $\nabla X|_p = \nabla Y|_p = 0$. By Lemma 3.1 and Corollary 4.1 we have

$$(\mathcal{L}_G)(X^h, Y^h) = \xi_1(g(X, Y)) - g(T(X, \xi_1), Y) - g(T(Y, \xi_1), X) = 2g(T^h_{\xi_1}(X), Y).$$

Lemma 3.1 implies also that

$$(\mathcal{L}_G)(X^h, (\xi_s)^h) = \xi_1^h(g(X^h, (\xi_s)^h)) - G([\xi, X^h]_I, (\xi_s)^h) - G(X^h, [\xi, (\xi_s)]_I)$$

$$= \xi_1^h(g(X^h, (\xi_s)^h)) - G((T(X, \xi_1))^h, (\xi_s)^h) - G(R(X, \xi_1)I, (\xi_s)^h)$$

$$- G((T(\xi_s, \xi_1))^h, X^h) - G(R(\xi_s, \xi_1)I, X^h).$$

The first, second and last terms in the latter identity vanish by Corollary 4.1 since $X \in H$ and $T(\xi_t, X) \in H$. Moreover, the fourth term is equal to $-g(T(\xi_s, \xi_1), X) = g([\xi_s, \xi_1], X)$ and for the third term we have, that when $s \neq 1$,

$$G(R(X, \xi_1)I, (\xi_s)^h) = \frac{1}{2}g_p(\xi_1 \times \xi_s, \varphi^{-1}(R(X, \xi_1)I)) = \frac{1}{2} < R(X, \xi_1)I, I_sI_s>_p =$$

$$= \frac{1}{2} \frac{1}{4n} \sum_{a=1}^{4n}(-g(R(X, \xi_1)I_1e_a, I_sI_s)e_a) = -\rho_s(X, \xi_1),$$

where $e_1, \ldots, e_{4n}$ is an orthonormal basis of $H_p$. This proves the second formula stated in the lemma.

To prove the third formula, we note first that

$$(\mathcal{L}_G)((\xi_s)^h, (\xi_t)^h) = \xi_1(\tau\delta_{st}) + \tau g_p(T(\xi_s, \xi_1), \xi_t) - \frac{1}{2}g_p(\xi_1 \times \xi_t, \varphi^{-1}(R(\xi_s, \xi_1)I)) +$$

$$g_p(T(\xi_t, \xi_1), \xi_s) = \frac{1}{2}g_p(\xi_1 \times \xi_s, \varphi^{-1}(R(\xi_s, \xi_1)I)).$$

Next, $g_p(T(\xi_s, \xi_1), \xi_1) = -g_p([\xi_s, \xi_1], \xi_1) = d\eta_s(\xi_s, \xi_1)$ and, similarly, $g_p(T(\xi_t, \xi_1), \xi_s) = d\eta_s(\xi_t, \xi_1)$. By (2.8) and (2.9) we have

$$0 = g_p(\nabla \xi_2, \xi_2) = \alpha_3(\xi_2)_p = d\eta_2(\xi_1, \xi_2)_p, \quad 0 = g_p(\nabla \xi_3, \xi_1) = \alpha_2(\xi_1)_p = d\eta_3(\xi_3, \xi_1)_p$$

$$0 = g_p(\nabla \xi_3, \xi_1) + g_p(\nabla \xi_2, \xi_1) = \alpha_2(\xi_3)_p + \alpha_3(\xi_2)_p = d\eta_2(\xi_3, \xi_1)_p - d\eta_3(\xi_2, \xi_3)_p.$$

It follows that

$$g_p(T(\xi_s, \xi_1), \xi_t) + g_p(T(\xi_t, \xi_1), \xi_s) = d\eta_s(\xi_3, \xi_1)_p + d\eta_s(\xi_2, \xi_1)_p = 0, \quad s, t = 2, 3.$$
On the other hand
\[ \frac{1}{2} g_p(\xi_1 \times \xi_t, \varphi^{-1}(R(\xi_s, \xi_1)I_1)) = \frac{1}{2} < R(\xi_s, \xi_1)I_1, I_1I_1 > = -\rho_t(\xi_s, \xi_1)_p \]
\[ \frac{1}{2} g_p(\xi_1 \times \xi_s, \varphi^{-1}(R(\xi_t, \xi_1)I_1)) = \frac{1}{2} < R(\xi_t, \xi_1)I_1, I_1I_1 > = -\rho_j(\xi_t, \xi_1)_p. \]

Thus
\[ (\mathcal{L}_X G)((\xi_s)^h_t, (\xi_t)^h_s) = -d\tau(\xi_1)\delta_{st} + \rho_s(\xi_t, \xi_1) + \rho_t(\xi_s, \xi_1). \]

Now, extend the vertical vectors \( a \) and \( b \) to sections of the bundle \( Q \). Then, by Lemma 3.1 and Corollary 4.1, we have
\[ (\mathcal{L}_X G)(X^h_I, a) = (\mathcal{L}_X G)(X^h_I, \tilde{a}_I) = -G((T(X, \xi_1))^h_I, a) = 0 \]
since \( T(X, \xi_1) \in H_p. \)

Applying Lemma 3.1 and Corollary 4.1, we get easily that
\[ (\mathcal{L}_X G)((\xi_s)^h_I) = \xi_1(g(\xi_1 \times \xi_s, \varphi^{-1}(\tilde{a})) - \frac{1}{2} g_p(\xi_1 \times T(\xi_s, \xi_1)\nu, \varphi^{-1}(a)) - \tau(p)g_p(\xi_s, \varphi^{-1}(a)), \quad s = 2, 3. \]

In the case when \( a = I_1(p) \), \( t = 2, 3 \), the latter formula takes the form
\[ (\mathcal{L}_X G)((\xi_s)^h_I, I_1(p)) = \xi_1(g(\xi_1 \times \xi_s, \xi_t) + \frac{1}{2} g_p(\xi_1 \times \xi_t, T(\xi_s, \xi_1)) - \tau(p)g_p(\xi_s, \xi_t) = -\frac{1}{2} g_p(\xi_1 \times \xi_t, [\xi_s, \xi_t]) - \tau(p)\delta_{st}. \]

If \( s = t = 2 \) or \( s = t = 3 \), then \( g_p(\xi_1 \times \xi_t, [\xi_s, \xi_t]) \) is equal to \( d\eta_1(\xi_1, \xi_2)_p \) or to \( d\eta_2(\xi_3, \xi_1)_p \), respectively. We have seen in the proof of Lemma 3.2 that \( d\eta_1(\xi_2, \xi_3) = -2\tau \) at the point \( p \). Similar simple arguments show that \( d\eta_2(\xi_3, \xi_1) = -2\tau \) and \( d\eta_3(\xi_1, \xi_2) = -2\tau \) at \( p \). Therefore
\[ (\mathcal{L}_X G)((\xi_s)^h_I, I_1(p)) = 0 \text{ for } s = t = 2 \text{ or } s = t = 3. \]

This identity holds also for \( s = 2, t = 3 \) and \( s = 3, t = 2 \) since \( g_p(\xi_1 \times \xi_t, [\xi_s, \xi_1]) \) is equal to \( d\eta_2(\xi_2, \xi_1)_p = 0 \) by (4.1) in the first case and is equal to \( d\eta_3(\xi_1, \xi_3)_p = 0 \) in the second one. It follows that \( (\mathcal{L}_X G)((\xi_s)^h_I, a) = 0 \).

Finally,
\[ (\mathcal{L}_X G)(a, b) = \chi(G(\tilde{a}, \tilde{b})) - G([\chi, \tilde{b}], \tilde{a}) - G(a, [\chi, \tilde{b}]_I) = \]
\[ \frac{1}{2}[g_{\xi_t} \times \varphi^{-1}(a), \varphi^{-1}(b)] + g(\xi_t \times \varphi^{-1}(b), \varphi^{-1}(a)) = 0. \]

\[ \square \]

**Theorem 4.4.** The contact structure \( (\Phi, \chi, \eta^Z) \) on the twistor space \( Z \) is normal if and only if the tensor \( T^0 \) on \( M \) vanishes.

**Proof.** Recall that the normality condition for the structure \( (\Phi, \chi, \eta^Z) \) is equivalent to the condition
\[ (\mathcal{L}_X G)(A, B) = 0 \text{ for } A, B \in Ker \eta^Z = D. \]

Thus, if this structure is normal, then, by Lemma 4.3, \( T^0_{\xi_s} = 0, s = 1, 2, 3 \). Hence, \( T^0 = 0 \) because of (2.5).

Conversely, suppose that \( T^0 = 0 \). In view of Lemma 4.3, to show that the structure \( (\Phi, \chi, \eta^Z) \) is normal, we should prove that the following identities hold on \( M \).
\[ T^0_{\xi_1} = 0; \]
\[ \rho_s(X, \xi_1)_p + g_p([\xi_s, \xi_1], X) = 0, \quad s = 2, 3, \quad X \in H; \]
\[ 2\rho_s(\xi_s, \xi_1)_p = d\tau(\xi_1)_p, \quad s = 2, 3; \]
\[ \rho_2(\xi_3, \xi_1)_p + \rho_3(\xi_2, \xi_1)_p = 0. \]

We shall prove that this system of equations follows from the single equation \( T^0 = 0 \). The equation (4.2) follows from (2.5).
To prove (4.3), note first that, according to [11, formula (4.3)], we have
\[ \rho_i(X, \xi_j) = \frac{1}{2} d_{nj}([\xi_j, \xi_k]_H, X) = g(I_j[\xi_j, \xi_k]_H, X), \]
\[ \rho_i(X, \xi_k) = \frac{1}{2} d_{nk}([\xi_j, \xi_k]_H, X) = g(I_k[\xi_j, \xi_k]_H, X), \]
where the subscript \( H \) means "the component in \( H \)". It follows that condition (4.3) is equivalent to the identities
\[ I_1[\xi_1, \xi_1]_H = [\xi_1, \xi_2]_H, \quad I_1[\xi_1, \xi_2]_H = [\xi_1, \xi_3]_H \]
It has been shown in [15, formula (3.7)] that
\[ 3(2n + 1)\rho_i(I_3X, \xi_j) = -\frac{(2n + 1)(2n - 1)}{16(n + 2)} X(Scal) \]
\[ + \frac{1}{4} \sum_{a=1}^{4n} (\nabla_{e_a} T^0)(4n + 1)(e_a, X) + 3(I_1e_a, I_1X)] + 2(n + 1) \sum_{a=1}^{4n} (\nabla_{e_a} U)(X, e_a). \]
Thus we get
\[ 3(2n + 1)g(I_i[\xi_j, \xi_k], X) = -\frac{(2n + 1)(2n - 1)}{16(n + 2)} X(Scal) + 2(n + 1) \sum_{a=1}^{4n} (\nabla_{e_a} U)(X, e_a). \]
The right hand-side of this identity does not depend on the indices \( i, j, k \), therefore \( I_1[\xi_2, \xi_3]_H = I_2[\xi_3, \xi_1]_H = I_3[\xi_1, \xi_2]_H \) for every \( (\xi_1, \xi_2, \xi_3) \). Then \( I_1[\xi_3, \xi_1]_H = -I_3[\xi_2, \xi_1]_H = -I_2[\xi_3, \xi_2]_H = [\xi_1, \xi_2]_H \).
Moreover, writing the identity \( I_1[\xi_2, \xi_3]_H = I_3[\xi_1, \xi_2]_H \) for \( (\xi_3, \xi_1, \xi_2) \), we have \( I_3[\xi_1, \xi_2]_H = I_2[\xi_3, \xi_1]_H \), hence \( I_1[\xi_1, \xi_2]_H = I_2[\xi_3, \xi_1]_H = I_3[\xi_1, \xi_2]_H = [\xi_1, \xi_2]_H \). This proves (4.6).
Recall that, by definition, \( \rho_a(\xi_i, \xi_j) = \frac{1}{4n} \sum_{a=1}^{4n} g(R(\xi_i, \xi_j)e_a, I_ae_a) \). According to [15, formula (3.6)], we have
\[ g(R(\xi_i, \xi_j)e_a, I_ae_a) = (\nabla_{\xi_i} U)(I_je_a, I_ae_a) - (\nabla_{\xi_j} U)(I_ie_a, I_ae_a) \]
\[ -\frac{1}{4}(\nabla_{e_a} T^0)(I_je_a, I_ae_a) - (\nabla_{\xi_i} T^0)(e_a, I_je_a) + \frac{1}{4}(\nabla_{\xi_j} T^0)(I_je_a, I_ae_a) \]
\[ + \frac{1}{4}(\nabla_{e_a} T^0)(e_a, I_je_a) - (\nabla_{e_a} \rho_k)(I_je_a, I_ae_a) - \frac{Scal}{8n(n + 2)} g(T(\xi_k, e_a), I_ae_a) \]
\[ - \sum_{b=1}^{4n} g(T(\xi_j, e_a), e_b) g(T(\xi_i, e_b), I_ae_a) + \sum_{b=1}^{4n} g(T(\xi_j, e_b), I_ae_a) g(T(\xi_i, e_b), e_b). \]
In view of (2.6) and \( T^0 = 0 \), we obtain from (4.8) that
\[ g(R(\xi_i, \xi_j)e_a, I_ae_a) = (\nabla_{\xi_i} U)(I_je_a, I_ae_a) - (\nabla_{\xi_j} U)(I_ie_a, I_ae_a) \]
\[ -\frac{1}{4}(\nabla_{e_a} T^0)(I_je_a, I_ae_a) - \frac{Scal}{8n(n + 2)} U(I_je_a, I_ae_a) \]
\[ - \sum_{b=1}^{4n} U(I_je_a, e_b) U(I_ie_a, I_ae_a) + \sum_{b=1}^{4n} U(I_je_a, I_ae_a) U(I_je_a, I_ae) \]
Now, identities (4.9), (4.10) and the fact that \( U \) is completely trace-free imply that

\[
4n\rho_i(\xi_i, \xi_j) = \sum_{a=1}^{4n} g(R(\xi_i, \xi_j)e_a, I_i e_a) = \sum_{a=1}^{4n} (\nabla_{e_a} \rho_k)(e_a, \xi_i)
\]

(4.11)

\[
4n\rho_j(\xi_i, \xi_j) = \sum_{a=1}^{4n} g(R(\xi_i, \xi_j)e_a, I_j e_a) = -\sum_{a=1}^{4n} (\nabla_{e_a} \rho_k)(I_k e_a, \xi_i)
\]

(4.12)

\[
4n\rho_k(\xi_i, \xi_j) = \sum_{a=1}^{4n} g(R(\xi_i, \xi_j)e_a, I_k e_a) = \sum_{a=1}^{4n} (\nabla_{e_a} \rho_k)(I_j e_a, \xi_i) - 2||U||^2.
\]

(4.13)

Next, according to (4.7), we have

\[
3(2n + 1)\rho_k(X, \xi_i) = (2n + 1)(2n - 1)d\tau(I_j X) + 2(n + 1) \sum_{a=1}^{4n} (\nabla_{e_a} U)(I_j X, e_a), \quad X \in H.
\]

(4.14)

Let \( X \in H_p \). Extend \( X \) to a local vector field such that \( \nabla X|_p = 0 \). Then we get from (4.14) that

\[
3(2n + 1)\nabla X \rho_k)(X, \xi_i) = (2n + 1)(2n - 1)X(d\tau(I_j X)) + 2(n + 1) \sum_{a=1}^{4n} X((\nabla_{e_a} U)(I_j X, e_a)).
\]

This identity and identities (4.11), (4.12), (4.13) imply that at the point \( p \) we have:

\[
12(2n + 1)\rho_i(\xi_i, \xi_j) = \sum_{a=1}^{4n} [(2n + 1)(2n - 1)e_a(d\tau(I_j e_a)) + 2(n + 1) \sum_{a,b=1}^{4n} e_a((\nabla_{e_a} U)(I_j e_a, e_b))],
\]

(4.15)

\[
12(2n + 1)\rho_j(\xi_i, \xi_j) = -\sum_{a=1}^{4n} [(2n + 1)(2n - 1)e_a(d\tau(I_i e_a)) + 2(n + 1) \sum_{a,b=1}^{4n} e_a((\nabla_{e_a} U)(I_i e_a, e_b))]
\]

(4.16)

\[
3(2n + 1)4n\rho_k(\xi_i, \xi_j) = -\sum_{a=1}^{4n} [(2n + 1)(2n - 1)e_a(d\tau(e_a)) + 2(n + 1) \sum_{a,b=1}^{4n} e_a((\nabla_{e_a} U)(e_a, e_b))] - 6(2n + 1)||U||^2.
\]

(4.17)

The right hand-side of the last identity does not depend on the indices \( i, j, k \), therefore \( \rho_3(\xi_1, \xi_2)_p = \rho_2(\xi_3, \xi_1)_p \), which proves (4.5).

Identities (4.15) and (4.16) imply that \( \rho_3(\xi_3, \xi_1)_p = -\rho_2(\xi_1, \xi_2)_p \), and so \( \rho_3(\xi_3, \xi_1)_p = \rho_2(\xi_2, \xi_1)_p \). On the other hand, we have \( \rho_3(\xi_3, \xi_1) + \rho_2(\xi_2, \xi_1) = \xi_1(\tau) \) by [11, formula (4.6)]. Therefore \( 2\rho_2(\xi_2, \xi_1)_p = \rho_3(\xi_3, \xi_1)_p = d\tau(\xi_1)_p \), i.e. condition (4.4) also holds.

\[\square\]

Theorem 4.4. (2.7) and (2.4) imply the following

**Theorem 4.5.** The contact structure \((\Phi, \chi, \eta^Z)\) on the twistor space \( \mathcal{Z} \) is normal if and only if the QC Ricci tensor commutes with the quaternionic structure on the contact distribution,

\[\text{Ric}(IX, IY) = \text{Ric}(X, Y), \quad X, Y \in H, I \in Q.\]
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