Approximating \((p, 2)\) flexible graph connectivity via the primal-dual method

Ishan Bansal∗ Joseph Cheriyan† Logan Grout‡ Sharat Ibrahimpur§

Abstract

We consider the Flexible Graph Connectivity model (denoted FGC) introduced by Adjiashvili, Hommelsheim and Mühenthaler [2, 1], and its generalization, \((p, q)\)-FGC, where \(p \geq 1\) and \(q \geq 0\) are integers, introduced by Boyd et al. [3, 4]. In the \((p, q)\)-FGC model, we have an undirected connected graph \(G = (V, E)\), non-negative costs \(c\) on the edges, and a partition \((S, U)\) of \(E\) into a set of safe edges \(S\) and a set of unsafe edges \(U\). A subset \(F \subseteq E\) of edges is called feasible if for any set \(F' \subseteq U\) with \(|F'| \leq q\), the subgraph \((V, F \setminus F')\) is \(p\)-edge connected. The goal is to find a feasible edge-set of minimum cost.

For the special case of \((p, q)\)-FGC when \(q = 2\), we give an \(O(1)\) approximation algorithm, thus improving on the logarithmic approximation ratio of [3, 4]. Our algorithm is based on the primal-dual method for covering an uncrossable family, due to Williamson et al. [7]. We conclude by studying weakly uncrossable families, which are a generalization of the well-known notion of an uncrossable family.
1 Introduction

Adjashvili, Hommelsheim and Mühlenenthaler [2, 1] introduced the model of Flexible Graph Connectivity that we denote by FGC. Recently, Boyd et al. [3, 4] introduced a generalization of FGC. Let \( p \geq 1 \) and \( q \geq 0 \) be integers. In an instance of the \((p,q)\)-Flexible Graph Connectivity problem, denoted \((p,q)\)-FGC, we have an undirected connected graph \( G = (V,E) \), a partition of \( E \) into a set of safe edges \( S \) and a set of unsafe edges \( U \), and nonnegative costs \( c \in \mathbb{R}^E_{\geq 0} \) on the edges. A subset \( F \subseteq E \) of edges is feasible for the \((p,q)\)-FGC problem if for any set \( F' \subseteq U \) with \( |F'| \leq q \), the subgraph \((V, F \setminus F')\) is \( p \)-edge connected. The algorithmic goal is to find a feasible solution \( F \) that minimizes \( c(F) = \sum_{e \in F} c_e \). Let \( n \) denote the number of nodes of the input graph.

We state two results of Boyd et al. [3, 4].

**Theorem 1** ([4], Theorem 1.4). There is a 4-approximation algorithm for \((p,1)\)-FGC.

**Theorem 2** ([4], Theorem 1.5). There is an \( O(q \log n) \)-approximation algorithm for \((p,q)\)-FGC.

In this paper, we focus on the special case of \((p,q)\)-FGC when \( q = 2 \), and we present constant-factor approximation algorithms. Thus, the algorithmic goal is to find a min-cost subset of the edges \( F \) such that for any pair of unsafe edges \( e, f \in F \cap U \), the subgraph \((V, F \setminus \{e, f\})\) is \( p \)-edge connected. For the case of odd \( p \), we prove an approximation ratio of 20, and for the case of even \( p \), we prove an approximation ratio of 6. Our algorithms are based on the primal-dual method for covering an uncrossable family, due to Williamson et al. [7].

Informally speaking, a family of subsets, \( \mathcal{F} \), of a ground-set \( V \) is called uncrossable if \( A, B \in \mathcal{F} \) implies that either \( A \cap B, A \cup B \in \mathcal{F} \) or else \( A \setminus B, B \setminus A \in \mathcal{F} \).

Our algorithms work in two stages. First, we compute a feasible solution \( F_1 \) to the \((p,1)\)-FGC problem on the same input graph, by applying the 4-approximation algorithm of \([3, 4]\), see Theorem 1. We then augment the subgraph \((V, F_1)\) using additional edges. Any cut in the subgraph \((V, F_1)\) either has (i) at least \( p \) safe edges or (ii) at least \( p + 1 \) edges. Thus the cuts that need to be augmented have exactly \( p + 1 \) edges with at least 2 unsafe edges. We call such cuts deficient. Augmenting all deficient cuts by at least one edge of any type will ensure that we have a feasible solution to \((p,2)\)-FGC. For the case of even \( p \), the (shores of the) deficient cuts form an uncrossable family (i.e., \( \mathcal{F} = \{ A \subseteq V : \delta(A) \text{ is a deficient cut} \} \) is an uncrossable family). We apply the primal-dual method of [7] to cover all the deficient cuts, and this incurs a cost of \( \leq 2 \cdot \text{OPT} \), where \( \text{OPT} \) denotes the cost of an optimal solution. See Section 3 for details.

Now, focus on the case of odd \( p \). When we try to apply the above method, we face a difficulty. The following example shows that when \( p \) is odd, then the family of deficient cuts need not form an uncrossable family.

**Example 1.** We construct the graph \( G \) by starting with a 4-cycle \( v_1, v_2, v_3, v_4, v_1 \) and then replacing each edge of the 4-cycle by a pair of parallel edges; thus, we have a 4-regular graph with 8 edges; we designate the following four edges as unsafe (and the other four edges are safe): both copies of edge \( \{v_1, v_4\} \), one copy of edge \( \{v_1, v_2\} \), and one copy of edge \( \{v_3, v_4\} \). Clearly, \( G \) is a feasible instance of \((3,1)\)-FGC. On the other hand, \( G \) is infeasible for \((3,2)\)-FGC, and the cuts \( \delta(\{v_1, v_2\}) \) and \( \delta(\{v_2, v_3\}) \) are deficient. The family of deficient cuts does not form an uncrossable family (observe that the cuts \( \delta(v_2) \) and \( \delta(v_3) \) are not deficient).

A key idea is to focus on a generalization of the notion of an uncrossable family that we call a weakly uncrossable family, see Definition 1 for details. It turns out that a straight-forward application of the primal-dual method for covering a weakly uncrossable family fails to achieve an approximation ratio of \( O(1) \). See Section 5 for details. Fortunately, for \((p,2)\)-FGC, there is a way around this difficulty. By imposing additional conditions on a weakly uncrossable family, see Property (P1) in Theorem 7, we ensure sufficient “structure”
such that the primal-dual method of [7] achieves an approximation ratio of $O(1)$ for covering such a family. Thus, we have an 20-approximation algorithm for $(p,2)$-FGC for odd $p$, see Section 4 for details.

There are well-known polynomial-time algorithms for implementing all of the basic computations in this paper, see [6]. We state this explicitly in all relevant results, but we do not elaborate on this elsewhere.

2 Preliminaries

This section has definitions and preliminary results. Our notation and terms are consistent with [5, 6], and readers are referred to those texts for further information.

For a positive integer $k$, we use $[k]$ to denote the set $\{1, \ldots, k\}$. For a ground-set $V$ and a subset $S$ of $V$, the complement of $S$ (w.r.t. $V$) is denoted $V \setminus S$ or $\overline{S}$. Sets $A, B \subseteq V$ are said to cross denoted $A \bowtie B$ if each of the four sets $A \cap B$, $A \cup B$, $A \setminus B$, $B \setminus A$ is non-empty; on the other hand, if $A, B$ do not cross, then either $A \cup B = V$, or $A, B$ are disjoint, or one of $A, B$ is a subset of the other one. Clearly, if $A, B \subseteq V$ cross, then $\overline{A}, B$ cross, $A, \overline{B}$ cross, and $\overline{A}, \overline{B}$ cross.

2.1 Graphs, subgraphs, and related notions

Let $G = (V, E)$ be a (loop-free) multi-graph with non-negative costs $c \in \mathbb{R}_{\ge 0}^E$ on the edges. We take $G$ to be the input graph, and we use $n$ to denote $|V(G)|$. For a set of edges $F \subseteq E(G)$, $c(F) := \sum_{e \in F} c(e)$, and for a subgraph $G'$ of $G$, $c(G') := \sum_{e \in E(G')} c(e)$.

For any instance $H$, we use $\text{opt}(H)$ to denote the minimum cost of a feasible subgraph (i.e., a subgraph that satisfies the requirements of the problem). When there is no danger of ambiguity, we use $\text{opt}$ rather than $\text{opt}(H)$.

For a graph $H = (V, E)$ and any two disjoint node-sets $X, Y$ we use the notation $E(X, Y) := \{e = uv : u \in X, v \in Y\}$ to denote the set of edges with exactly one endpoint in each of $X$ and $Y$. For a graph $H$ and a set of nodes $S \subseteq V(H)$, $\delta_H(S)$ denotes the set of edges that have one end node in $S$ and one end node in $V(H) \setminus S$; moreover, $H[S]$ denotes the subgraph of $H$ induced by $S$, and $H - S$ denotes the subgraph of $H$ induced by $V(H) \setminus S$. For a graph $H$ and a set of edges $F \subseteq E(H)$, $H - F$ denotes the graph $(V(H), E(H) \setminus F)$. We may use relaxed notation for singleton sets, e.g., we may use $\delta_H(v)$ instead of $\delta_H(\{v\})$, and we may use $H - v$ instead of $H - \{v\}$, etc.

We may not distinguish between a subgraph and its node set; for example, given a graph $H$ and a set $S$ of its nodes, we use $E(S)$ to denote the edge set of the subgraph of $H$ induced by $S$.

A multi-graph $H$ is called $k$-edge connected if $|V(H)| \ge 2$ and for every $F \subseteq E(H)$ of size $< k$, $H - F$ is connected.

2.2 The Primal-Dual Algorithm for Uncrossable Functions

Let $G = (V, E)$, $c \in \mathbb{R}_{\ge 0}^E$ be an undirected graph with non-negative costs on the edges. A function $f : 2^V \to \{0, 1\}$ is called uncrossable if the following hold, see [7]: (i) $f(V) = 0$, (ii) for $A, B \subseteq V$, if $f(A) = f(B) = 1$ then either $f(A - B) = f(B - A) = 1$ or $f(A \cap B) = f(A \cup B) = 1$, (iii) $f(A) = f(V \setminus A)$ for $A \subseteq V$. Let $f : 2^V \to \{0, 1\}$ be an uncrossable function. In the $f$-augmentation problem, the goal is to find a minimum-cost edge-set $F \subseteq E$ such that $|\delta(S) \cap F| \ge f(S)$, for all non-empty, proper node-sets $S$ of $V$. We say that a set $S \subseteq V$ is violated with respect to an edge-set $F$ if $|\delta(S) \cap F| < f(S)$. Williamson et al. [7] present a 2-approximation algorithm for the $f$-augmentation problem for any uncrossable function $f$. 
Theorem 3 ([7], Lemma 2.1). Let \( f : 2^V \to \{0, 1\} \) be an uncrossable function that is given via a value oracle. Suppose that for any \( F \subseteq E \) we can compute all minimal violated sets (w.r.t. \( f \) and \( F \)) in polynomial time. Then, in polynomial time, we can compute a 2-approximate solution to the \( f \)-augmentation problem.

Williamson et al. [7] design and analyse a primal-dual algorithm based on an integer program (\( IP \)); their analysis is based on the dual \((D)\) of the LP relaxation of \((IP)\).

3 \((p, 2)\)-FGC: the case of even \( p \)

In this section, we obtain a 6-approximation algorithm for \((p, 2)\)-FGC when \( p \) is even, based on the primal-dual algorithm of [7].

Theorem 4. There is a 6-approximation algorithm for \((p, 2)\)-FGC when \( p \) is even.

Proof. Our algorithm works in two stages. First, we compute a feasible solution \( F_1 \) to the \((p, 1)\)-FGC problem on the same input graph, by applying the 4-approximation algorithm of [3, 4], see Theorem 1. We then augment the subgraph \((V, F_1)\) using additional edges. Any cut in the subgraph \((V, F_1)\) either has (i) at least \( p \) safe edges or (ii) at least \( p + 1 \) edges. Thus the cuts that need to be augmented have exactly \( p + 1 \) edges with at least 2 unsafe edges. We call such cuts deficient. Augmenting all deficient cuts by at least one edge of any type will ensure that we have a feasible solution to \((p, 2)\)-FGC. In the rest of the proof, we argue that the family of deficient cuts forms an uncrossable family. Hence, by applying the primal-dual method of [7] to cover the deficient cuts, see Theorem 3, the augmentation step costs \( \leq 2 \cdot \text{oPT} \). Overall, this leads to the claimed 6-approximation algorithm.

Consider two deficient cuts \( A, B \subseteq V \) in \((V, F_1)\). Without loss of generality, we may assume that both \( A \) and \( B \) contain a fixed node \( r \in V \). The following equations hold:

\[
\begin{align*}
|\delta(A)| &= |\delta(B)| = p + 1 \\
|\delta(A \cup B)| + |\delta(A \cap B)| + 2|E(A \setminus B, B \setminus A)| &= |\delta(A)| + |\delta(B)| \\
|\delta(A \setminus B)| + |\delta(B \setminus A)| + 2|E(A \cap B, V \setminus (A \cup B))| &= |\delta(A)| + |\delta(B)| \\
|\delta(A \setminus B)| + |\delta(A \cap B)| &= |\delta(A)| + 2|E(A \setminus B, A \cap B)|
\end{align*}
\]

Since all cuts in \((V, F_1)\) contain at least \( p \) edges, equations (1), (2) and (3) imply that \( |\delta(A \cup B)|, |\delta(A \cap B)|, |\delta(A \setminus B)|, |\delta(B \setminus A)| \in \{p, p + 1, p + 2\} \) and \( |E(A \setminus B, B \setminus A)|, |E(A \cap B, (A \cup B))| \leq 1 \). Furthermore, we also have the following parity-equations; observe that the last of these equations, (7), uses the assumption that \( p \) is even.

\[
\begin{align*}
|\delta(A \cup B)| &\equiv |\delta(A \cap B)| \mod 2 \\
|\delta(A \setminus B)| &\equiv |\delta(B \setminus A)| \mod 2 \\
|\delta(A \cap B)| &\equiv |\delta(A \setminus B)| + 1 \mod 2
\end{align*}
\]

Due to the above parity-equations, among the two pairs of cuts \( \{\delta(A \cup B), \delta(A \cap B)\} \) and \( \{\delta(A \setminus B), \delta(B \setminus A)\} \), one of the pairs consists of \((p + 1)\)-cuts (i.e., both cuts of the pair have size \( p + 1 \)), and the other pair has at least one cut of size \( p \). W.l.o.g. assume that \( \delta(A \cup B) \) and \( \delta(A \cap B) \) are \((p + 1)\)-cuts and that \( \delta(A \setminus B) \) is a \( p \)-cut.
The other cases are handled similarly. Clearly, \( \delta(A \setminus B) \) has no unsafe edges since \( F_1 \) is feasible for \((p, 1)\)-FGC. For \( A \) to be a deficient cut, either (i) \( E(A \cap B, B \setminus A) \) has two unsafe edges or (ii) \( E(A \cap B, B \setminus A) \) has one unsafe edge and \( E(A \cap B, (A \cup B)) \) has one unsafe edge (recall that the size of the latter edge-set is \( \leq 1 \)). For \( B \) to be a deficient cut, either (iii) \( E(B \setminus A, (A \cup B)) \) has two unsafe edges or (iv) \( E(B \setminus A, (A \cup B)) \) has one unsafe edge and \( E(A \cap B, (A \cup B)) \) has one unsafe edge. Now, observe that both the \((p+1)\)-cuts \( \delta(A \cap B) \) and \( \delta(A \cup B) \) are deficient, because, by (i) or (ii), \( \delta(A \cap B) \) has at least two unsafe edges, and, by (iii) or (iv), \( \delta(A \cup B) \) has at least two unsafe edges. This completes the proof. \( \square \)

4. \((p, 2)\)-FGC: the case of odd \( p \)

In this section, we obtain a 20-approximation algorithm for \((p, 2)\)-FGC when \( p \) is odd.

When \( p \) is odd, instead of equation \((7)\), we have \( |\delta(A \cap B)| \equiv |\delta(A \setminus B)| \pmod{2} \). Thus, it is possible that all the cuts involved are actually \((p+1)\)-cuts and the unsafe edges are positioned such that the cuts \( \delta(A \cup B) \) and \( \delta(A \setminus B) \) are deficient but the cuts \( \delta(A \cap B) \) and \( \delta(B \setminus A) \) are not deficient. See Example 1 for such a case.

As in the case of even \( p \), first we compute a feasible solution \( F_1 \) to the \((p, 1)\)-FGC problem, and then we augment the cuts of \((V, F_1)\) that violate the requirements of the \((p, 2)\)-FGC problem. Similarly to Section 3, it can be seen that the violated cuts of \((V, F_1)\) are cuts of size \( p+1 \) that have at least two unsafe edges. Let \( h^* : 2^V \rightarrow \{0, 1\} \) be the indicator function for these violated cuts. We prove some useful properties of the function \( h \) starting with some definitions.

**Definition 1.** Let \( h : 2^V \rightarrow \{0, 1\} \) be a binary function on the set of node-sets. We say that \( h \) is a weakly uncrossable function if (i) \( h(A) = h(B) = 1 \implies h(A \cup B) + h(A \cap B) + h(A \setminus B) + h(B \setminus A) \geq 2 \), (ii) \( h(A) = 1 \implies h(\overline{A}) = 1 \), and (iii) \( h(V) = 0 \). We refer to the support of \( h \) as a weakly uncrossable family of cuts.

**Definition 2.** Let \( h : 2^V \rightarrow \{0, 1\} \) be a weakly uncrossable function and \( F \subseteq E \). We say that a cut \( S \) is violated with respect to \( F \) if \( h(S) = 1 \) and \( \delta_F(S) := \delta(S) \cap F = \emptyset \). We will refer to the collection of violated cuts with respect to \( F \) as \( \mathcal{D}_F \) and to the inclusion-wise minimal elements of \( \mathcal{D}_F \) as \( \mathcal{C}_F \).

**Lemma 5.** The function \( h^* \) is a weakly uncrossable function.

**Proof.** Let \( A \) and \( B \) be two violated cuts. Notice that equations \((1)\) through \((6)\) are still true and \((7)\) becomes

\[
|\delta_F_1(A \cap B)| \equiv |\delta_F_1(A \setminus B)| \pmod{2} \tag{7'}
\]

Since all cuts contain at least \( p \) edges, equations \((1), (2)\) and \((3)\) imply that \( |\delta_F_1(A \cup B)|, |\delta_F_1(A \cap B)|, |\delta_F_1(A \setminus B)|, |\delta_F_1(B \setminus A)| \in \{p, p + 1, p + 2\} \). The parity conditions along with \((2)\) and \((3)\) imply that either all these four cuts are \( p+1 \)-cuts or at least one cut from each pair \((A \cup B, A \cap B)\) and \((A \setminus B, B \setminus A)\) is a \( p \)-cut. It is simple to check that in the latter case, it is not possible for \( A \) and \( B \) to both be violated since \( p \)-cuts do not have any unsafe edges. Hence all four cuts \((A \cup B, A \cap B, A \setminus B, B \setminus A)\) are \( p+1 \)-cuts. Now using the fact that \( \delta_F_1(A) \) and \( \delta_F_1(B) \) both contain at least two unsafe edges each, it is easy to verify that at least two of the four cuts \((A \cup B, A \cap B, A \setminus B, B \setminus A)\) contain at least two unsafe edges each. This completes the proof. \( \square \)

**Lemma 6.** Consider the function \( h^* \) and a subset of the edges \( F \subseteq E \). Suppose \( S_1, S_2 \in \mathcal{D}_F, C \in \mathcal{C}_F \) such that \( C \bowtie S_1, C \bowtie S_2 \) and \( S_1 \subsetneq S_2 \), then the set \( S_2 \setminus (S_1 \cup C) \) is either empty or lies in \( \mathcal{D}_F \).
Proof. Assume that $S_2 \setminus (S_1 \cup C)$ is non-empty. We will show that it is violated. Let $F_2 = F \cup F_1$ First let us analyze the structure of the crossing $C$ and $S_1$. In the proof of lemma 5 we have shown that all six cuts $C, S_1, C \cup S_1, C \cap S_1, C \setminus S_1, S_1 \setminus C$ are $p + 1$-cuts. This immediately implies that $|F_2(C \setminus S_1, C \cap S_1)| = |F_2(C \cap S_1, S_1 \setminus C)| = |F_2(S_1 \setminus C, C \cup S_1)| = |F_2(C \cup S_1, C \setminus S_1)| = (p + 1)/2$ and there are no other edges across any pair of these cuts. Additionally since no subset of $C$ is violated, it must be the case that $\delta F_2(C \setminus S_1, C \cap S_1)$ contains no unsafe edges, $\delta F_2(C \cap S_1, S_1 \setminus C)$ contains exactly one unsafe edge, $\delta F_2(S_1 \setminus C, C \cup S_1)$ contains at least two unsafe edges and $\delta F_2(C \cup S_1, C \setminus S_1)$ contains exactly one unsafe edge. Thus $C \cap S_1$ and $S_1 \setminus C$ are also violated. Note that $C$ and $S_2$ also cross similarly.

Now consider the structure of the three sets $C, S_1, S_2$. Since $S_2$ and $S_1 \cup C$ are both violated, we know that $S_2 \setminus (S_1 \cup C)$ is a $p + 1$-cut and $|F_2(S_2 \setminus (S_1 \cup C), C \cup S_2)| = (p + 1)/2$. Additionally, from the structure of $C$ and $S_2$, we know that $F_2(S_2 \setminus C, C \cup S_2)$ has exactly $(p + 1)/2$ edges at least two of which are unsafe. This implies that at least two of the edges in $\delta F_2(S_2 \setminus (S_1 \cup C), C \cup S_2)$ are unsafe. Hence, $S_2 \setminus (S_1 \cup C)$ is a $p + 1$-cut with at least two unsafe edges and hence is violated. This completes the proof.

Consider the problem of augmenting a weakly uncrossable function $h$, i.e., finding a subset of edges $F \subseteq E$ such that no cut $S \subseteq V$ is violated with respect to $F$. The IP and its corresponding dual can be written as

$$
\begin{align*}
\text{min} \sum_{e \in E} c_e x_e & \quad (IP_h) \text{s.t.} \sum_{e \in \delta(S)} x_e \geq h(S) \quad \forall \emptyset \neq S \subseteq V \\
x_e \in \{0, 1\} & \quad \forall e \in E
\end{align*}
$$

$$
\begin{align*}
\text{max} \sum_{\emptyset \neq S \subseteq V} h(S) y_S & \quad \text{s.t.} \sum_{S \subseteq V : e \in \delta(S)} y_S \leq c_e \quad \forall e \in E \\
y_S \geq 0 & \quad \forall \emptyset \neq S \subseteq V
\end{align*}
$$

**Theorem 7.** Suppose $h$ is a weakly uncrossable function with the following property

(P1) For any $F \subseteq E$, $S_1, S_2 \in \mathcal{D}_F$, $C \in \mathcal{C}_F$ such that $C \preceq S_1, C \preceq S_2$ and $S_1 \subseteq S_2$, implies the set $S_2 \setminus (S_1 \cup C)$ is either empty or lies in $\mathcal{D}_F$.

Then, the primal dual algorithm outputs a feasible augmentation of $h$ whose cost is at most 16 times the cost of an optimal solution.

**Proof.** Let $F'$ be the output of the algorithm. We know that for any edge $e \in F'$, $\sum_{S : e \in \delta(S)} y_S = c_e$. Thus the cost of the solution $F'$ can be written as

$$
c(F') = \sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S : e \in \delta(S)} y_S = \sum_{S \subseteq V} y_S \cdot |\delta F'(S)|
$$

We will show that

$$
\sum_{S \subseteq V} y_S \cdot |\delta F'(S)| \leq 16 \sum_{S \subseteq V} h(S) y_S
$$

Consider the collection $\mathcal{C}$ of active sets at the start of some iteration. The left hand side of the inequality above will increase by $\sum_{C \in \mathcal{C}} c \cdot |\delta F'(C)|$ in this iteration, while the right hand side will increase by $16c \cdot |\mathcal{C}|$. Hence, it is sufficient to show that in every iteration,

$$
\sum_{C \in \mathcal{C}} |\delta F'(C)| \leq 16|\mathcal{C}|
$$
For the remainder of this proof we will focus on the active sets $C$ during a particular iteration of the algorithm. Let $H = \cup_{C \in \mathcal{C}} \delta_{F^v}(C)$.

**Lemma 8.** For any edge $e \in H$, there exists a witness set $S_e \subset V$ such that

(i) $h(S_e) = 1$ and $S_e$ is violated in the current iteration, and

(ii) $\delta_{F^v}(S_e) = \{e\}$. 

**Proof.** This can be proved by the same arguments as in the proof of [7, Lemma 5.1]. ⊓⊔

**Lemma 9.** Suppose $S_1$ is a witness for edge $e_1$ and $S_2$ is a witness for edge $e_2$ such that $S_1 \not\succeq S_2$, then we can find $S_1'$ and $S_2'$ satisfying the following properties:

(i) $S_1'$ is a valid witness for edge $e_1$ and $S_2'$ is a valid witness for edge $e_2$ and $S_1'$ does not cross $S_2'$.

(ii) $S_1', S_2' \in \{S_1, S_2, S_1 \cup S_2, S_1 \cap S_2, S_1 \setminus S_2, S_2 \setminus S_1\}$.

(iii) either $S_1' = S_1$ or $S_2' = S_2$.

**Proof.** We perform an exhaustive case analysis to check that the lemma is true. Note that at least two of the four sets $S_1 \cup S_2, S_1 \cap S_2, S_1 \setminus S_2, S_2 \setminus S_1$ must be violated in the current iteration.

*Case 1.* Suppose $S_1 \cup S_2$ and $S_1 \cap S_2$ are violated or $S_1 \setminus S_2$ and $S_2 \setminus S_1$ are violated, then the proof of Lemma 5.2 in [7] can be applied.

*Case 2.* Suppose $S_1 \cup S_2$ and $S_1 \setminus S_2$ are violated, then consider where the endpoints of the edges $e_1$ and $e_2$ lie. If $e_1 \in E(S_1 \setminus S_2, S_1 \cup S_2)$ and $e_2 \in E(S_1 \setminus S_2, S_1 \cap S_2)$, then we can set $S_1' = S_1 \cup S_2$ and $S_2' = S_2$. The other possibilities for $e_1$ and $e_2$ are handled similarly.

*Case 3.* Suppose $S_1 \cap S_2$ and $S_1 \setminus S_2$ are violated, again consider where the endpoints of the edges $e_1$ and $e_2$ lie. If $e_1 \in E(S_1 \setminus S_2, S_1 \cup S_2)$ and $e_2 \in E(S_1 \setminus S_2, S_1 \cap S_2)$, then we can set $S_1' = S_1 \cap S_2$ and $S_2' = S_2$. The other possibilities for $e_1$ and $e_2$ are handled similarly. ⊓⊔

**Lemma 10.** There exists a laminar witness family

**Proof.** We show that any witness family can be uncrossed. We prove this by induction on the size of the witness family $\ell$. For the base case, suppose $\ell = 2$, then one application of lemma 9 is sufficient. Assume that any witness family of size $\ell \leq k$ can be uncrossed and consider $\ell = k + 1$. Thus we have witness sets $S_1, \ldots, S_{k+1}$. By the inductive hypothesis, we can uncross all the witness sets $S_1, \ldots, S_k$ to obtain witness sets $S_1', \ldots, S_k'$ that form a laminar family. Now suppose $S_{k+1}$ does not cross some $S_i'$, say $S_1'$ then we can apply the inductive hypothesis to the $k$ sets $S_2', \ldots, S_k', S_{k+1}$ and we will obtain a laminar family of witness sets, none of which cross $S_1'$ either and so we are done. Thus assume $S_{k+1}$ crosses all the sets $S_1', \ldots, S_k'$. Now suppose for some $S_i'$ say $S_1'$, applying lemma 9 to the pair $S_1', S_{k+1}$ gives $S_1', S_{k+1}'$ (that is keeps $S_1'$ intact), then $S_1'$ does not cross any of the witness sets $S_2', \ldots, S_k'$ and hence applying the inductive hypothesis to these $k$ sets will give us a laminar witness family. Thus assume that for every $S_i'$, applying lemma 9 to the pair $S_i', S_{k+1}$ keeps $S_{k+1}$ intact and returns a pair $S_i'', S_{k+1}$, Then after doing this for every $S_i'$, we will end up with the witness family $S_1'', \ldots, S_k'', S_{k+1}$ with the property that $S_{k+1}$ does not cross any of the other sets. Applying the inductive hypothesis to the $k$ sets $S_1'', \ldots, S_k''$ completes the proof. ⊓⊔
Let $L$ be the laminar witness family and augment this family with the node-set $V$. The family can be viewed as defining a tree $T$ with a node $v_S$ for each $S \in L$ and an edge directed from $v_S$ to $v_T$ if $S$ is the smallest element of $L$ properly containing $T$. To each active set $C \in \mathcal{C}$, we correspond the smallest set $S \in L$ that contains it. We will call a node $v_S$ active if it is associated with some active set $C$. We will color such nodes red. Additionally, we will call a non-active node $v_S$ a high-degree node and color it green, if it has at least three neighbors. Similar to [7], it is easy to check that every leaf of $T$ is colored red.

Consider now the following charging scheme that works in two phases:

- In the first phase, for $C \in \mathcal{C}$ and $e \in \delta_F(C)$, assign a new charge to the node $v_{S_e}$ corresponding to the witness set $S_e$ for the edge $e$.
- In the second phase, if an uncolored node has any charges, pass it down along its child until the charges arrive at a colored node. Note that this operation is well-defined since uncolored nodes have degree 2.

**Lemma 11.** On any directed 4-path in $T$, there is at least one colored node. Hence, The maximum number of charges on any node in $T$ is 8.

**Proof.** For the sake of contradiction, assume that there exists an uncolored 4-path in $T$ say $v_{S_1} \rightarrow \cdots \rightarrow v_{S_{16}}$. Let $S_i$ be the witness for edge $e_i$. Consider now an active set $C$ such that $e_1 \in \delta_F(C)$. We first claim that $C$ is not a subset of $S_1$. Suppose that $C$ is a subset of $S_1$, then the smallest set $S_{C'} \in L$ containing $C$ must have $e_1 \in \delta_F(S_{C'})$. But then due to property (ii) of witness sets in Lemma 8, $S_{C'} = S_1$ and so $v_{S_1}$ is colored red in $T$. This contradiction arose due to the assumption that $C$ is a subset of $S_1$.

Next, we claim that all the sets $S_2, S_3$ and $S_4$ cross the active set $C$. Let nodes $a$ and $b$ be the end points of the edge $e_1$ with $a \in S_1$. Then, $a \in S_2$ and since $e_1 \notin \delta_F(S_2)$, the node $b$ also lies in $S_2$. Notice that $C$ contains at most one of a and b since $e_1 \in \delta_F(C)$, hence $C \cap S_2 \neq \emptyset$ and $S_2 \not\subseteq C$. Additionally, $C \not\subseteq S_2$ as otherwise using the fact that $C \not\subseteq S_1$, $v_{S_2}$ would be colored red in $T$. Hence $C$ crosses $S_2$ and similar arguments can be made about $S_3$ and $S_4$.

Next, observe that active sets in $\mathcal{C}$ are pair-wise disjoint. Suppose not and $C_1, C_2 \in \mathcal{C}$ and $C_1 \cap C_2 \neq \emptyset$, then due to the property (i) of weakly uncrossable functions, some subset of either $C_1$ or $C_2$ is also violated. This is a contradiction.

Now due to Property (P1) in Theorem 7, $S_3 \setminus (S_2 \cup C)$ is either empty or is violated. Suppose it is violated, then it must contain an active set disjoint from $S_2$. Hence the node $v_{S_3}$ in $T$ is either colored red or has a child distinct from $v_{S_2}$ implying that it is colored green. Thus, $S_3 \setminus (S_2 \cup C)$ is empty. Similarly, $S_4 \setminus (S_3 \cup C)$ is empty. Now consider the edge $e_3$. Due to property (ii) of witness sets in Lemma 8, one end-point of $e_3$ must lie in $S_3 \setminus S_2$ and the other end-point must lie in $S_1 \setminus S_3$. But since $S_3 \setminus (S_2 \cup C)$ and $S_4 \setminus (S_3 \cup C)$ are both empty, the end-points of the edge $e_3$ must both lie inside $C$. But now, the active set $C'$ with $e_3 \in \delta_F(C')$ must intersect $C$ and this is a contradiction.

We now show that the maximum number of charges on any node in $T$ is at most 8. First, observe that during the first phase of the charging scheme, any node of $T$ receives at most two charges. This is because an edge $e \in H$ can be incident to at most two active sets since active sets are pair-wise disjoint. During the second phase of the charging scheme, a node in $T$ could receive additional charges from its uncolored ancestors. But due to lemma 11, it can have at most three uncolored ancestors that push charges onto it. Thus, at the end of the second phase of charging, any node can have at most 8 charges in total.

Clearly the total charge on all the nodes is equal to $\sum_{C \in \mathcal{C}} |\delta_F(C)|$. Let $|R|$ be the number of red nodes and $|G|$ be the number of green nodes. Let $|L|$ be the number of leaves. Note that $|G| \leq |L|$ since the number of
leaves in a tree is at least the number of nodes of degree 3 or more. Additionally \(|L| \leq |R|\) since every leaf of \(T\) is colored red. We have then

\[
\sum_{C \in \mathcal{C}} |\delta_{\mathcal{F}'}(C)| \leq 8|R| + 8|G| \leq 16|R| \leq 16|\mathcal{C}|
\]

\[\square\]

5 Weakly Uncrossable families: an example

Example \[\square\] shows that when \(p\) is odd, the family of deficient cuts need not form an uncrossable family. However, one can observe that whenever the cuts with shores \(A, B\) are deficient, at least two of the four cuts with shores \(A \cup B, A \cap B, A \setminus B, B \setminus A\) will be deficient. A natural question arises: does the extension of the primal-dual algorithm of Williamson et al. for covering a weakly uncrossable family achieve an approximation ratio of \(O(1)\)? The next construction shows that the primal-dual algorithm of Williamson et al. cannot achieve an approximation ratio of \(O(1)\) on the problem of augmenting the cuts of a weakly uncrossable family of sets. We start with a basic lemma.

Lemma 12. Suppose \(\mathcal{F}'\) is a family of node-sets such that \(A, B \in \mathcal{F}'\) implies that at least two of the four sets \(A \cup B, A \cap B, A \setminus B, B \setminus A\) also belong to \(\mathcal{F}'\). Then the family \(\mathcal{F} = \mathcal{F}' \cup \{\overline{A} | A \in \mathcal{F}'\}\) is a weakly uncrossable family.

Now, we describe the construction: We have node-sets \(C_{ij}\) for \(i = 1, 2, \ldots, k\) and \(j = 0, 1, 2, \ldots, k\). All these \(C\)-sets are pair-wise disjoint. Additionally, for \(j' = 1, \ldots, k\), we have node-sets \(T_{1j'} \subseteq T_{2j'} \subseteq \cdots \subseteq T_{kj'}\) and each of these are disjoint from all the \(C\)-sets. Additionally, we have at least one node \(v_0\) lying outside the union of all these \(C\)-sets and \(T\)-sets. See Figure [\square]. We designate \(\mathcal{F}'\) to be the following family of node-sets that consists of two types of sets:

\[
\begin{align*}
(I) & \quad C_i := \bigcup_{j=0}^{k} C_{ij} \quad \text{for } i \in [k] \\
(II) & \quad T_{i'j'} \cup \bigcup_{(i,j) \in R'} C_{ij} \quad \text{for } i' \in [k], j' \in [k] \text{ and } R' \subseteq R(i',j') \text{ where } R(i',j') = \{(i,j) | 1 \leq i < i', 0 \leq j \leq k\}
\end{align*}
\]

Informally speaking, the sets \(C_1, C_2, \ldots, C_k\) can be viewed as (pairwise-disjoint) “cylinders”, see Figure [\square] the (first) index \(i\) is associated with one of these cylinders, and note that \(C_{i0} = C_i \setminus (\bigcup_{j=1}^{k} C_{ij})\); the (second) index \(j\) is associated with a “layer” (i.e., a horizontal plane), and the sub-family \(T_{1j} \subseteq T_{2j} \subseteq \cdots \subseteq T_{kj}\) forms a nested family on layer \(j'\), see Figure [\square] observe that a set of type (II) is the union of one set \(T_{i'j'}\) of the nested family of layer \(j'\) together with the sets of an arbitrary sub-family of each of the “cylinder families” \(\{C_{i0}, C_{i1}, C_{i2}, \ldots, C_{ik}\}\) with (first) index \(i \leq i'\). We claim that the family \(\mathcal{F}'\) satisfies the condition of Lemma [\square]. Indeed consider \(A, B \in \mathcal{F}'\). If \(A\) and \(B\) are of type (I), then they are disjoint. If \(A\) is of type (I) and \(B\) is of type (II) such that they intersect, then \(A \cup B\) and \(B \setminus A\) are sets of type (II) so both belong to \(\mathcal{F}'\). If \(A\) and \(B\) are of type (II) such that both have the same (second) index \(j'\), then both \(A \cup B\) and \(A \cap B\) are sets of type (II). On the other hand, if \(A\) and \(B\) are of type (II) such that the (second) index \(j'\) is different for \(A, B\), then \(A \setminus B\) and \(B \setminus A\) are sets of type (II). Thus, adding in all the complements of the sets of this family gives us a weakly uncrossable family \(\mathcal{F}\).
The edges of the graph are of two types:

(A) For \( i' = 1, \ldots, k \) and \( j' = 1, \ldots, k \), we have an edge from \( T_{i'j'} \setminus T_{(i'-1)j'} \) to \( C_{ij} \) where \( i = i' \) and \( j = j' \).

(B) For \( j = 1, \ldots, k \), we have an edge from \( v_0 \) to \( T_{1j} \). For \( i = 1, \ldots, k \), we have an edge from \( v_0 \) to \( C_{i0} \).

When the primal-dual algorithm is applied to this instance, then it picks all the edges of type (A); note that there are \( k^2 \) edges of type (A). Let us sketch the working of the primal-dual algorithm on this instance.

- Initially, the active sets are the type (I) sets \( C_1, \ldots, C_k \) and the smallest \( T \)-sets \( T_{11}, \ldots, T_{1k} \). The algorithm increases the dual variables of each of these sets by \( 1/2 \) and then picks all the type (A) edges between \( T_{1j} \) and \( C_{1j} \) for \( j \in [k] \).

- In the next iteration, the (new) active sets are the type (I) sets \( C_2, \ldots, C_k \) and the type (II) sets of the form \( T_{2j} \cup C_{1j} \) for \( j \in [k] \). The algorithm increases the dual variables of each of these sets by \( 1/4 \) and then picks all the type (A) edges between \( T_{2j} \) and \( C_{2j} \) for \( j \in [k] \).

- Similarly, in the \( i^{th} \) iteration, the active sets are the type (I) sets \( C_i, \ldots, C_k \) and the type (II) sets of the form \( T_{ij} \cup C_{1j} \cup \cdots \cup C_{(i-1)j} \) for \( j \in [k] \). The algorithm increases the dual variables of each of these sets by \( 1/2^i \) and then picks all the type (A) edges between \( T_{ij} \) and \( C_{ij} \) for \( j \in [k] \).

- Finally, the reverse-delete step does not delete any edges.

On the other hand, all the edges of type (B) form a feasible solution of this instance, and there are \( \leq 2k \) such edges. In more detail, each of the type (I) sets \( C_i \) is covered by the type (B) edge between \( v_0 \) and \( C_{i0} \), and each of the type (II) sets containing \( T_{i'j'} \) is covered by the type (B) edge between \( v_0 \) and \( T_{1j'} \). Thus, the optimal solution picks \( \leq 2k \) edges.

![Figure 1: A bad example for the primal-dual method for augmenting a weakly uncrossable family.](image-url)
Figure 2: Edges of type (A) and type (B) in a bad example for the primal-dual method for augmenting a weakly uncrossable family.

References

[1] David Adjiashvili, Felix Hommelsheim, and Moritz Mühlenhalter. Flexible Graph Connectivity. In Proceedings of the 21st Integer Programming and Combinatorial Optimization Conference, volume 12125 of Lecture Notes in Computer Science, pages 13–26, 2020.

[2] David Adjiashvili, Felix Hommelsheim, and Moritz Mühlenhalter. Flexible Graph Connectivity. Mathematical Programming, pages 1–33, 2021.

[3] Sylvia C. Boyd, Joseph Cheriyan, Arash Haddadan, and Sharat Ibrahimpur. Approximation algorithms for flexible graph connectivity. In Mikolaj Bojanczyk and Chandra Chekuri, editors, 41st IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2021, December 15-17, 2021, Virtual Conference, volume 213 of LIPIcs, pages 9:1–9:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[4] Sylvia C. Boyd, Joseph Cheriyan, Arash Haddadan, and Sharat Ibrahimpur. Approximation algorithms for flexible graph connectivity. CoRR, abs/2202.13298, 2022.

[5] R. Diestel. Graph Theory (4th ed.). Graduate Texts in Mathematics, Volume 173. Springer-Verlag, Heidelberg, 2010.

[6] Alexander Schrijver. Combinatorial Optimization: Polyhedra and Efficiency, volume 24 of Algorithms and Combinatorics. Springer-Verlag Berlin Heidelberg, 2003.

[7] David P. Williamson, Michel X. Goemans, Milena Mihail, and Vijay V. Vazirani. A Primal-Dual Approximation Algorithm for Generalized Steiner Network Problems. Combinatorica, 15(3):435–454, 1995.