Submanifolds of Sasakian Manifolds with Concurrent Vector Field

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Abstract. The submanifolds of Sasakian manifolds with a concurrent vector field have been studied. Applications of such submanifolds to Ricci solitons and Yamabe solitons has also been showed.

1 Introduction

Sasakian manifold $\bar{M}$ is a $(2n + 1)$-dimensional almost contact metric manifold such that [1]

\begin{align}
(\nabla_X \phi)Y &= g(X, Y)\xi - \eta(Y)X, \\
\nabla_X \xi &= -\phi X,
\end{align}

where $(\phi, \xi, \eta, g)$ is the almost contact metric structure and $\nabla$ is the Riemannian connection on $\bar{M}$. A vector field $X$ on $\bar{M}$ is said to be conformal if

$$L_X g = 2\alpha g,$$

where $\alpha \in C^\infty(\bar{M})$ and $L_X$ denotes the Lie derivative along $X$. In particular, if $\alpha = 0$ then $X$ is Killing. And $X$ is said to be concurrent if

$$\nabla_Z X = Z$$

for any $Z \in \chi(\bar{M})$.

Let $M$ be an $m$-dimensional submanifold of $\bar{M}$. A Ricci soliton on $M$ is a triplet $(g, W, \sigma)$ such that [12]

$$L_W g + 2S + 2\sigma g = 0,$$
where $S$ is the Ricci tensor on $M$, $W$ is the potential vector field and $\sigma \in \mathbb{R}$. An Yamabe soliton on $M$ is a triplet $(g, W, \lambda)$ such that

$$\frac{1}{2}L_W g = (r - \lambda)g,$$

(1.6)

where $r$ is the scalar curvature on $M$ and $\lambda \in \mathbb{R}$. If the dimension of $M$ is 2 then the notions of Ricci soliton and Yamabe soliton are equivalent. However, when the dimension of $M$ is greater than 2, they are different.

Chen and his co-author studied Euclidean submanifold whose canonical vector field are concurrent [4], concircular [11], conformal [10], torse-forming [9] and also in ([3], [5], [6]). Ricci soliton and Yamabe soliton whose canonical vector field are concurrent and conformal studied in ([2], [7], [8]).

The object of the present paper is to study of submanifolds of Sasakian manifolds with concurrent vector field. We also apply such submanifolds to Ricci solitons and Yamabe solitons.

2 Preliminaries

An odd dimensional smooth manifold $\tilde{M}^{2n+1}$ is said to be an almost contact metric manifold if the following relations hold: [1]

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0,$$

(2.1)

$$g(X, \xi) = \eta(X), \quad \phi \circ \eta = 0,$$

(2.2)

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

(2.3)

for all $X, Y \in \chi(M)$, where $\phi$ is a tensor of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is an 1-form and $g$ is a Riemannian metric on $\tilde{M}$.

Let $\nabla$ and $\nabla^\perp$ be the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

(2.4)

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V,$$

(2.5)

where $h$ and $A_V$ are second fundamental form and shape operator respectively for the immersion of $M$ into $\tilde{M}$ and they are related by the following equation, see [13]

$$g(h(X, Y), V) = g(A_V X, Y)$$

(2.6)
for any \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \). If \( h = 0 \), then \( M \) is said to be totally geodesic.

Let \( \{ e_i : 1 \leq i \leq m \} \) be an orthonormal basis to the tangent space at any point of \( M \). Then the mean curvature of \( M \) is

\[
H = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i). \tag{2.7}
\]

And \( M \) is said to be totally umbilical if

\[
h(X, Y) = g(X, Y)H. \tag{2.8}
\]

Again \( M \) is said to be umbilical with respect to \( V \in T^\perp M \) if

\[
g(h(X, Y), V) = \mu g(X, Y) \tag{2.9}
\]

for some function \( \mu \). In particular if \( g(h(X, Y), H) = \mu g(X, Y) \) holds then \( M \) is said to be pseudo-umbilical. Consider

\[
\phi X = PX + FX, \tag{2.10}
\]

where \( PX \) and \( FX \) are the tangential and normal components of \( \phi X \). And \( M \) is called generalized self-similar submanifold of \( \bar{M} \) if

\[
FX = fH, \tag{2.11}
\]

where \( f \in C^\infty(M) \).

3 Results

We now prove the followings:

**Theorem 3.1.** Let \( M \) be a submanifold of \( \bar{M} \) with a concurrent vector field \( X \) such that \( \xi \) is normal to \( M \). Then \( PX \) is conformal if and only if \( M \) is umbilical with respect to \( FX \).

**Proof.** Since \( X \) is concurrent vector field of \( \bar{M} \), we have from (1.4) that

\[
\phi Z = \phi \nabla_Z X
= \nabla_Z \phi X - (\nabla_Z \phi) X. \tag{3.1}
\]

Using (1.1), (2.4), (2.5) and (2.10) in (3.1) we have

\[
PZ + FZ = \nabla_Z (PX + FX) - g(X, Z)\xi
= \nabla_Z PX + h(Z, PX) + \nabla_Z^\perp FX - \nabla_Z^\perp FX - A_{FX}Z - g(X, Z)\xi
\]
Comparing the tangential component of (3.2) we have
\[ \nabla_Z PX = PZ + A_{FX} Z. \] (3.3)

Now we have
\[
(L_{PX} g)(Y, Z)
= g(\nabla_Y PX, Z) + g(Y, \nabla_Z PX)
= g(PY + A_{FX} Y, Z) + g(Y, PZ + A_{FX} Z)
= g(A_{FX} Y, Z) + g(A_{FX} Z, Y).
\] (3.4)

Using (2.6) in (3.4) we have
\[
(L_{PX} g)(Y, Z) = 2g(h(Y, Z), FX).
\] (3.5)

Suppose $PX$ is conformal. Then from (1.3) and (3.5) we have
\[ g(h(Y, Z), FX) = \alpha g(Y, Z), \] (3.6)
which implies that $M$ is umbilical with respect to $FX$.

Conversely, assume that $M$ is umbilical with respect to $FX$. Then from (2.9) and (3.5) we have
\[
(L_{PX} g)(Y, Z) = 2\mu g(Y, Z),
\] (3.7)
which means that $PX$ is conformal.

**Theorem 3.2.** Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$. Then $X$ is a homothetic vector field.

**Proof.** Since $X$ is a concurrent vector field, so we have from (1.4) and (2.4) that
\[ \nabla_Z X + h(X, Z) = Z. \] (3.8)

Equating tangential and normal components of (3.8) we get
\[ \nabla_Z X = Z, \ h(X, Z) = 0. \] (3.9)

Now we have
\[
(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).
\] (3.10)

Using (3.9) in (3.10) we have
\[
(L_X g)(Y, Z) = 2g(Y, Z),
\] (3.11)
which implies that $X$ is conformal vector field of $M$ with constant function $\alpha = 1$, i.e. $X$ is homothetic.
**Theorem 3.3.** Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$. If $(g, X, \sigma)$ is a Ricci soliton on $M$ then $M$ is Einstein and such a soliton is shrinking.

*Proof.* Since $(g, X, \sigma)$ is a Ricci soliton on $M$, we have the equation (1.5). Using (3.11) in (1.5) we get $S(Y, Z) = -(\sigma + 1)g(Y, Z)$, which implies that $M$ is Einstein.

By virtue of (3.9) we get

$$R(Y, Z)X = \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y, Z]} X = 0,$$

and hence $S(Y, X) = 0$. So, $\sigma + 1 = 0$, i.e., $\sigma = -1$. Hence the given Ricci soliton is shrinking. \hfill \Box

**Theorem 3.4.** Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$. If $(g, X, \lambda)$ is a Yamabe soliton on $M$ then such soliton is shrinking, steady and expanding according as $r < 1$, $r = 1$ and $r > 1$ respectively.

*Proof.* Since $(g, X, \lambda)$ is an Yamabe soliton on $M$, we have the equation (1.6). Using (3.11) in (1.6) we get $\lambda = r - 1$. Hence the result. \hfill \Box

**Theorem 3.5.** Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$ such that $\xi$ is normal to $M$. If $(g, PX, \lambda)$ is an Yamabe soliton on $M$, then $PX$ is conformal.

*Proof.* Let $(g, PX, \lambda)$ be an Yamabe soliton on $M$. Then from the equation (1.6), we get

$$\frac{1}{2}(\mathcal{L}_{PX} g)(Y, Z) = (r - \lambda)g(Y, Z).$$

(3.12)

From (3.5) and (3.12) we have

$$g(h(Y, Z), FX) = (r - \lambda)g(Y, Z)$$

(3.13)

for all $Y, Z \in \Gamma(TM)$, which implies that $M$ is umbilical with respect to $FX$. Then by virtue of Theorem 3.1, it follows that $PX$ is conformal. \hfill \Box

**Theorem 3.6.** Let $M$ be a generalized self-similar submanifold of $\bar{M}$ with a concurrent vector field $X$ such that $\xi$ is normal to $M$. Then $PX$ is conformal vector field if and only if $M$ is pseudo-umbilical.

*Proof.* Let $M$ be a generalized self-similar submanifold of $\bar{M}$, then we have the equation (2.11). If $PX$ is conformal vector field, then we have the equation (3.6). From (2.11) and (3.6) we can say that $M$ is pseudo-umbilical.

Conversely, if $M$ is pseudo umbilical submanifold then from equation (2.11) we say that $M$ is umbilical with respect to $FX$. So, by virtue of Theorem 3.1 it follows that $PX$ is conformal vector field. \hfill \Box
Theorem 3.7. Let \( M \) be a submanifold of \( \bar{M} \) with a concurrent vector field \( X \) such that \( \xi \) is normal to \( M \). Then \((g, PX, \sigma)\) is a Ricci soliton on \( M \) if and only if the following condition holds:

\[
S(Y, Z) = -\sigma g(Y, Z) - g(h(Y, Z), FX)
\]

(3.14)

for any \( Y, Z \) tangent to \( M \).

Proof. Using (3.5) in (1.5), we get the equation (3.14). \( \square \)

Theorem 3.8. Let \( M \) be a submanifold of \( \bar{M} \) with a concurrent vector field \( X \) such that \( \xi \) is normal to \( M \) and \((g, PX, \sigma)\) is a Ricci soliton on \( M \). Then \( PX \) is conformal if and only if \( M \) is umbilical.

Proof. Since \((g, PX, \sigma)\) is a Ricci soliton on \( M \), then we have (3.14). Also since \( PX \) is conformal, using (3.7) in (1.5) we have

\[
S(Y, Z) = -\sigma g(Y, Z) - \mu g(Y, Z).
\]

(3.15)

From (3.14) and (3.15) we can say that \( M \) is umbilical.

Conversely, suppose \( M \) is umbilical. Then we have the equation (2.9). Using (2.9) in (3.14) we get

\[
S(Y, Z) = -\sigma g(Y, Z) - \mu g(Y, Z).
\]

(3.16)

Using (3.16) in (1.5), we obtain

\[
(\mathcal{L}_{PX} g)(Y, Z) = 2\mu g(Y, Z),
\]

(3.17)

which means that \( PX \) is conformal. \( \square \)

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