Electrostatics 2
(1) $\int E \cdot dL = 0$
$$E_{1t}L - E_{2t}L = 0$$
$$E_{2t} = E_{1t}$$
$$a_n \times (E_2 - E_1) = 0$$

(2) $\int\int\int D \cdot dS = Q$
$$D_{2n}S - D_{1n}S = Q_{\text{inside volume}}$$
For no charge, $Q_{\text{inside volume}} = 0$; or for volume charge, $Q_{\text{inside volume}} = 0$ since volume $\to 0$ as $h \to 0$. This gives $D_{2n} = D_{1n}$.

For surface charge, $Q_{\text{inside volume}} = \rho_s S$  \[D_{2n} - D_{1n} = \rho_s\] or $a_n \cdot (D_2 - D_1) = \rho_s$
(1) Potential should be continuous across the boundary

\[ V_1 = V_2 \]

(2) \( \mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_S \)

\[ \mathbf{D}_2 = \varepsilon_2 \mathbf{E}_2 = -\varepsilon_2 \nabla V_2 \]
\[ \mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = -\varepsilon_1 \nabla V_1 \]

\[ (-\varepsilon_2 \frac{\partial V_2}{\partial n}) - (-\varepsilon_1 \frac{\partial V_1}{\partial n}) = \rho_S \]

\[ \varepsilon_1 \frac{\partial V_1}{\partial n} - \varepsilon_2 \frac{\partial V_2}{\partial n} = \rho_S \]

If there is no surface charge, \( \varepsilon_1 \frac{\partial V_1}{\partial n} = \varepsilon_2 \frac{\partial V_2}{\partial n} \).
Perfect Electric Conductor (PEC)

(1) \[ \mathbf{a}_n \times \mathbf{E} = 0 \]

\[ E_t = 0 \]

On PEC body (including boundary),

\[ V_{AB} = V_A - V_B = \int_{A}^{B} \mathbf{E} \cdot d\mathbf{L} = 0 \]

PEC is equal potential.

(2) \[ \mathbf{a}_n \cdot \mathbf{D} = \rho_S \]

or \[ \rho_S = \mathbf{a}_n \cdot \mathbf{D} = -\varepsilon \frac{\partial V}{\partial n} \]
\[ C = \frac{Q}{V} \]
Energy density

\[ w_e = \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \]

Total energy

\[ W_e = \iiint_V w_e dV \]
Poisson’s and Laplace’s Equations

From electrostatic equations:
\[
\begin{cases}
\nabla \times \mathbf{E} = 0 \\
\nabla \cdot \mathbf{D} = \rho_v
\end{cases}
\Rightarrow \mathbf{E} = -\nabla V
\]

In simple media \( \mathbf{D} = \varepsilon \mathbf{E} = -\varepsilon \nabla V \)

Then \( \nabla \cdot \mathbf{D} = \nabla \cdot (-\varepsilon \nabla V) = \rho_v \)

Or \( \nabla \cdot (\varepsilon \nabla V) = -\rho_v \)

For source free region, we have \( \nabla \cdot (\varepsilon \nabla V) = 0 \)

If the dielectric distribution is uniform
\[
\nabla^2 V = -\frac{\rho_v}{\varepsilon} \quad \text{Poisson’s Equation}
\]
\[
\nabla^2 V = 0 \quad \text{Laplace’s Equation}
\]
Uniqueness Theorem

For

\[ \nabla^2 V = -\frac{\rho_V}{\varepsilon} \quad \text{Poisson’s Equation} \]

or \[ \nabla^2 V = 0 \quad \text{Laplace’s Equation} \]

If on the boundary, \( V \) or \( \frac{\partial V}{\partial n} \) is given, the solution is unique.
Proof of Uniqueness Theorem (1)

Proof:

Assuming two solutions $V_1$ and $V_2$ satisfy Poisson's or Laplace's Equations, we have

$$\nabla^2 (V_1 - V_2) = 0$$

On the boundary, since either $V$ or $\frac{\partial V}{\partial n}$ is given, we have

$$\left. (V_1 - V_2) \right|_{\text{on boundary}} = 0 \text{ or } \left( \frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} \right)_{\text{on boundary}} = 0$$

From math formula

$$\nabla \cdot (\phi \nabla \phi) = \phi \nabla^2 \phi + (\nabla \phi)^2$$

if we let $\phi = V_1 - V_2$, we have

$$\nabla \cdot [(V_1 - V_2) \nabla (V_1 - V_2)] = (V_1 - V_2) \nabla^2 (V_1 - V_2) + [\nabla (V_1 - V_2)]^2$$

$$= [\nabla (V_1 - V_2)]^2$$
Proof of Uniqueness Theorem (2)

Integrating the above equation over volume, we have

\[ \iiint (\nabla (V_1 - V_2))^2 dV = \iiint \nabla \cdot [(V_1 - V_2) \nabla (V_1 - V_2)] dV \]

\[ = \iint_S (V_1 - V_2) \nabla (V_1 - V_2) \cdot dS \quad \text{(from Gauss's Theorem)} \]

\[ = \iint_S (V_1 - V_2) \left( \frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} \right) dS = 0 \quad \text{(from boundary condition)} \]

Then we have

\[ \nabla (V_1 - V_2) = 0 \]

or

\[ V_1 - V_2 = \text{constant} \]

The constant can be easily evaluated to be zero since on the boundary \( V_1 - V_2 = 0 \) when \( V \) is given. Therefore,

\[ V_1 = V_2 \] giving two identical solutions.
Laplace’s Equations

$$\nabla^2 V = 0$$

- Rectangular (Cartesian) Coordinate
  
  $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- Polar (Cylindrical) Coordinate
  
  $$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- Spherical Coordinate
  
  $$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$
Example 1 (1)

Find the potential distribution between two parallel plates. Assume the two plates are large enough so that

$$\frac{\partial V}{\partial x} = 0 \text{ and } \frac{\partial V}{\partial y} = 0$$
Example 1 (2)

Solution:

From Laplace’s equation \( \nabla^2 V = 0 \) we have
\[
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{in Cartesian coordinate.}
\]

Assuming the two plates are large enough so that
\[
\frac{\partial V}{\partial x} = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 0
\]
we have
\[
\frac{d^2 V}{dz^2} = 0 \quad \text{(1)}
\]

The boundary conditions are:
\[
V(z_1) = V_1 \quad \text{and} \quad V(z_2) = V_2 \quad \text{(2)}
\]
Example 1 (3)

The general solution for (1) is  

\[ V(z) = Az + B \tag{3} \]

Inserting (2) into (3) results in

\[
\begin{align*}
Az_1 + B &= V_1 \\
Az_2 + B &= V_2
\end{align*}
\]

from which, we have

\[
A = \frac{V_2 - V_1}{z_2 - z_1} \quad \text{and} \quad B = \frac{V_1 z_2 - V_2 z_1}{z_2 - z_1}
\]

Therefore

\[
V(z) = \frac{V_2 - V_1}{z_2 - z_1} z + \frac{V_1 z_2 - V_2 z_1}{z_2 - z_1}
\]

If \( z_1 = 0 \) and \( V_1 = 0 \), we have

\[
V(z) = \frac{V_2}{d} z \rightarrow \text{linear distribution}
\]
Find capacitance between two parallel plates.

\[ V(z) = \frac{V_2}{d} z \]

\[ E = -\nabla V = -\frac{V_2}{d} \mathbf{a}_z \]

\[ D = \varepsilon E = -\varepsilon \frac{V_2}{d} \mathbf{a}_z \]

\[ \rho_{S_{\text{top}}} = (-\mathbf{a}_z) \cdot (-\varepsilon \frac{V_2}{d} \mathbf{a}_z) = \varepsilon \frac{V_2}{d} \]

\[ Q = \rho_{S_{\text{top}}} S = \varepsilon \frac{V_2}{d} S \]

Then \[ C = \frac{Q}{V_2} = \varepsilon \frac{S}{d} \]
Example 2

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0
\]

\[\frac{\partial}{\partial r} = 0, \quad \frac{\partial}{\partial \phi} = 0\]

\[\frac{d}{d\theta} \left[ \sin \theta \frac{dV}{d\theta} \right] = 0\]

\[V(\theta_1) = 0\]

\[V(\theta_2) = V_0\]

To use `dsolve`, we need to change the equation to:

\[
\sin \theta \frac{d^2 V}{d\theta^2} + \cos \theta \frac{dV}{d\theta} = 0
\]