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Contents

1 Introduction ................................. 7

2 Countably many disjoint sets ................................. 9

3 Independence as ternary relation ................................. 11
   3.1 Introduction .................................................. 11
     3.1.1 Independence .................................................. 11
     3.1.2 Overview .................................................... 12
     3.1.3 Discussion of some simple examples ......................... 12
       3.1.3.1 $X \times Z$ .................................................. 12
       3.1.3.2 $X \times Z \times W$ ........................................ 14
       3.1.3.3 $X \times Y \times Z$ ........................................ 15
       3.1.3.4 $X \times Y \times Z \times W$ ................................ 16
       3.1.3.5 A remark on generalization ................................. 17
       3.1.3.6 A remark on intuition .................................... 17
     3.1.4 Basic definitions .......................................... 18
   3.2 Probabilistic and set independence ......................... 18
     3.2.1 Probabilistic independence ................................. 18
       3.2.1.1 A side remark on preferential structures ................ 19
     3.2.2 Set independence ........................................... 21
   3.3 Basic results for set independence ......................... 23
     3.3.1 Example of a rule derived from the basic rules .............. 27
   3.4 Examples of new rules ........................................ 30
     3.4.1 New rules ................................................ 30
   3.5 There is no finite characterization ......................... 33
     3.5.1 Discussion .............................................. 33
| Section | Title | Page |
|---------|-------|------|
| 6.4     | The implicit extension of conjectures | 69   |
| 6.4.1   | Inference greed | 69   |
| 6.4.1.1 | A justification | 70   |
| 6.4.2   | Remarks on specificity | 70   |
| 6.4.3   | Induction | 71   |
| 6.5     | Modularity | 71   |
| 6.6     | Subset systems beyond principal filters | 71   |
| 6.7     | From propositional to first order logic | 72   |
| 6.8     | Validity of defaults and the best models | 74   |
| 6.8.1   | Asymmetric OR | 76   |
| 6.9     | Inheritance | 76   |
| 6.9.1   | Direct scepticism vs. intersection of extensions | 76   |
| 6.9.2   | The language of inheritance | 76   |
| References | | 79   |
Chapter 1

Introduction

We present here various results, which may one day be published in a bigger paper, and which we wish to make already available to the community.

We investigate several technical and conceptual questions.

Our main subject is the investigation of independence as a ternary relation in the context of non-monotonic logic. In the context of probability, this investigation was started by W. Spohn et al., and then followed by J. Pearl. We look at products of function sets, and thus continue our own investigation of independence in non-monotonic logic. We show that a finite characterization of this relation in our context is impossible, and indicate how to construct all valid rules.
Chapter 2

Countably many disjoint sets

We show here that - independent of the cardinality of the language - one can define only countably many inconsistent formulas.

The question is due to D. Makinson (personal communication).

We show here that, independent of the cardinality of the language, one can define only countably many inconsistent formulas.

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Example 2.0.1
There is a countably infinite set of formulas s.t. the defined model sets are pairwise disjoint.

Let $p_i : i \in \omega$ be propositional variables.

Consider $\phi_i := \bigwedge \{ \neg p_j : j < i \} \land p_i$ for $i \in \omega$.

Obviously, $M(\phi_i) \neq \emptyset$ for all $i$.

Let $i < i'$; we show $M(\phi_i) \cap M(\phi_{i'}) = \emptyset$. $M(\phi_{i'}) \models \neg p_i$, $M(\phi_i) \models p_i$.

\[\blacksquare\]

Fact 2.0.1
Any set $X$ of consistent formulas with pairwise disjoint model sets is at most countable

Proof
Let such $X$ be given.

(1) We may assume that $X$ consists of conjunctions of propositional variables or their negations.

Proof: Rewrite all $\phi \in X$ as disjunctions of conjunctions $\phi_j$. At least one of the conjunctions $\phi_j$ is consistent. Replace $\phi$ by one such $\phi_j$. Consistency is preserved, as is pairwise disjointness.
(2) Let $X$ be such a set of formulas. Let $X_i \subseteq X$ be the set of formulas in $X$ with length $i$, i.e., a consistent conjunction of $i$ many propositional variables or their negations, $i > 0$.

As the model sets for $X$ are pairwise disjoint, the model sets for all $\phi \in X_i$ have to be disjoint.

(3) It suffices now to show that each $X_i$ is at most countable; we even show that each $X_i$ is finite.

Proof by induction:

Consider $i = 1$. Let $\phi, \phi' \in X_1$. Let $\phi$ be $p$ or $\neg p$. If $\phi'$ is not $\neg \phi$, then $\phi$ and $\phi'$ have a common model. So one must be $p$, the other $\neg p$. But these are all possibilities, so $\text{card}(X_1)$ is finite.

Let the result be shown for $k < i$.

Consider now $X_i$. Take arbitrary $\phi \in X_i$. Without loss of generality, let $\phi = p_1 \land \ldots \land p_i$. Take arbitrary $\phi' \neq \phi$. As $M(\phi) \cap M(\phi') = \emptyset$, $\phi'$ must be a conjunction containing one of $\neg p_k$, $1 \leq k \leq i$.

Consider now $X_{i,k} := \{ \phi' \in X_i : \phi' \text{ contains } \neg p_k \}$. Thus $X_i = \{ \phi \} \cup \bigcup \{ X_{i,k} : 1 \leq k \leq i \}$. Note that all $\psi, \psi' \in X_{i,k}$ agree on $\neg p_k$, so the situation in $X_{i,k}$ is isomorphic to $X_{i-1}$. So, by induction hypothesis, $\text{card}(X_{i,k})$ is finite, as all $\phi' \in X_{i,k}$ have to be mutually inconsistent. Thus, $\text{card}(X_i)$ is finite. (Note that we did not use the fact that elements from different $X_{i,k}, X_{i,k'}$ also have to be mutually inconsistent; our rough proof suffices.)

\[ \Box \]

Note that the proof depends very little on logic. We needed normal forms, and used two truth values. Obviously, we can easily generalize to finitely many truth values.
Chapter 3

Independence as ternary relation

3.1 Introduction

3.1.1 Independence

Independence is a central concept of reasoning.
In the context of non-monotonic logic and related areas like theory revision, it was perhaps first investigated formally by R. Parikh and co-authors, see e.g. [Par96], to obtain “local” conflict solution.
The present authors investigated its role for interpolation in preferential logics in [GST0], and showed connections to abstract multiplication of size.
Independence plays also a central role for a FOL treatment of preferential logics, where problems like the “dark haired Swedes” have to be treated. This is still subject of ongoing research.
J. Pearl investigated independence in graphs and probabilistic reasoning, e.g. in [Pea88], also as a ternary relation, \langle X \mid Y \mid Z \rangle.
The aim of the present paper is to extend this abstract approach to the preferential situation. We should emphasize that this is only an abstract description of the independence relation, and thus not the same as independence for non-monotonic interpolation as examined in [GST0], where we used independence, essentially in the form of the multiplicative law \mu(X \times Y) = \mu(X) \times \mu(Y), which says that the \mu—function preserves independence.
We have not investigated if an interesting form of interpolation results from some application of \mu to situations described by \langle X \mid Y \mid Z \rangle, analogously to above application of \mu to situations described by \langle X \mid| Y \rangle.
3.1.2 Overview

We will first discuss simple examples, to introduce the main ideas.
We then present the basic definitions formally, for probabilistic and set independence.
We then show basic results for set independence as a ternary relation, and turn to our main results, absence of finite characterization, and construction of new rules for this ternary relation.

3.1.3 Discussion of some simple examples

We consider here \( X = Y = Z = W = \{0, 1\} \) and their products. We will later generalize, but the main ideas stay the same. First, we look at \( X \times Z \) (the Cartesian product of \( X \) with \( Z \)), then at \( X \times Z \times W \), at \( X \times Y \times Z \), finally at \( X \times Y \times Z \times W \). Elements of these products, i.e., sequences, will be written for simplicity 00, 01, 10, etc., context will disambiguate. General sequences will often be written \( \sigma, \tau, \) etc. We will also look at subsets of these products, like \( \{00, 11\} \subseteq X \times Z \), and various probability measures on these products.

As a matter of fact, the main part of this article concerns subsets \( A \) of products \( X_1 \times \ldots \times X_n \) and a suitable notion of independence for \( A \), roughly, if we can write \( A \) as \( A = A_1 \times \ldots \times A_m \). This will be made more precise and discussed in progressively more complicated cases in this section.

In the context of preferential structures, \( A \) is intended to be \( \mu(X_1 \times \ldots \times X_n) \), the set of minimal models of \( X_1 \times \ldots \times X_n \).

3.1.3.1 \( X \times Z \)

Let \( P : X \times Z \to [0, 1] \) be a (fixed) probability measure.
If \( A \subseteq X \times Z \), we will set \( P(A) := \Sigma \{ P(\sigma) : \sigma \in A \} \).
If \( A_x := \{ \sigma \in X \times Z : \sigma(X) = x \} \), we will write \( P(x) \) for \( P(A_x) \), likewise \( P(z) \) for \( P(A_z) \), if \( A_z := \{ \sigma \in X \times Z : \sigma(Z) = z \} \). When these are ambiguous, we will e.g. write \( A_{X=0} \) for \( \{ \sigma \in X \times Z : \sigma(X) = 0 \} \), and \( P(X = 0) \) for \( P(A_{X=0}) \), etc.
We say that \( X \) and \( Z \) are independent for this \( P \) iff for all \( xz \in X \times Z \) \( P(xz) = P(x) \cdot P(z) \).
We write then \( \langle X \mid\mid Z \rangle_P \), and call this and its variants probabilistic independence.

**Example 3.1.1**

\[ P(00) = P(01) = 1/6, \; P(10) = P(11) = 1/3. \]
Then \( P(X = 0) = 1/6 + 1/6 = 1/3 \), and \( P(X = 1) = 2/3, \; P(Z = 0) = 1/6 + 1/3 = 1/2 \), and \( P(Z = 1) = 1/2 \), so \( \langle X \mid Z \rangle_P \).

\[ P(00) = P(11) = 1/3, \; P(01) = P(10) = 1/6. \]
Then \( P(X = 0) = P(X = 1) = P(Z = 0) = P(Z = 1) = 1/2 \), but \( P(00) = 1/3 \neq 1/2 \cdot 1/2 = 1/4 \), so \( \neg \langle X \mid Z \rangle_P \).

**Definition 3.1.1**
Consider now $\emptyset \neq A \subseteq X \times Z$ for general $X, Z$.

Define the following probability measure on $X \times Z$:

$$P_A(\sigma) := \begin{cases} \frac{1}{\text{card}(A)} & \text{iff } \sigma \in A \\ 0 & \text{iff } \sigma \notin A \end{cases}$$

**Example 3.1.2**

(1)

$A := \{00, 01\},$

then $P_A(00) = P_A(01) = 1/2, P_A(10) = P_A(11) = 0, P_A(X = 0) = 1, P_A(X = 1) = 0, P_A(Z = 0) = P_A(Z = 1) = 1/2,$ and we have $(X \parallel Z)_P A.$

(2)

$A := \{00, 11\},$

then $P_A(00) = P_A(11) = 1/2, P_A(01) = P_A(10) = 0, P_A(X = 0) = P_A(X = 1) = 1/2, P_A(Z = 0) = P_A(Z = 1) = 1/2,$ but $P_A(00) = 1/2 \neq P_A(X = 0) * P_A(Z = 0) = 1/4,$ and we have $\neg(X \parallel Z)_P A.$

(3)

$A := \{00, 01, 11\},$

then $P_A(00) = P_A(01) = P_A(11) = 1/3, P_A(10) = 0, P_A(X = 0) = 2/3, P_A(X = 1) = 1/3, P_A(Z = 0) = 1/3, P_A(Z = 1) = 2/3,$ but $P_A(00) = 1/3 \neq P_A(X = 0) * P_A(Z = 0) = 2/3 * 1/3 = 2/9,$ and we have $\neg(X \parallel Z)_P A.$

Note that in (1) above, $A = \{0\} \times \{0, 1\},$ but neither in (2), nor in (3), $A$ can be written as such a product. This is no coincidence, as we will see now.

More formally, we write $(X \parallel Z)_A$ iff for all $\sigma \tau \in A$ there is $\rho \in A$ such that $\rho(X) = \sigma(X)$ and $\rho(Z) = \tau(Z),$ or, equivalently, that $A = \{\sigma(X) : \sigma \in A\} \times \{\sigma(Z) : \sigma \in A\},$ meaning that we can combine fragments of functions in $A$ arbitrarily.

We call this and its variants set independence.

**Fact 3.1.1**

Consider above situation $X \times Z.$ Then $(X \parallel Z)_P A$ iff $(X \parallel Z)_A.$

**Proof**

"$\Rightarrow$":

$A \subseteq \{\sigma(X) : \sigma \in A\} \times \{\sigma(Z) : \sigma \in A\}$ is trivial. Suppose $P_A(x, z) = P_A(x) * P_A(z),$ but there are $\sigma, \tau \in A,$ $\sigma(X)\tau(Z) \notin A.$ Then $P_A(x), P_A(z) > 0,$ but $P_A(x, z) = 0,$ a contradiction.

"$\Leftarrow$":

Case 1: $P_A(x) = 0,$ then $P_A(x, z) = 0,$ and we are done. Likewise for $P_A(Z) = 0.$
Case 2: $P_A(x), P_A(z) > 0$.

By definition and prerequisite,
\[
P_A(x) = \frac{\text{card}\{\sigma \in A : \sigma(X) = x\}}{\text{card}(A)},
\]
\[
P_A(z) = \frac{\text{card}\{\sigma \in A : \sigma(Z) = z\}}{\text{card}(A)},
\]
\[
P_A(x, z) = \frac{\text{card}\{\sigma \in A : \sigma(X) = x, \sigma(Z) = z\}}{\text{card}(A)}.
\]

By prerequisite again, $\text{card}(A) = \text{card}\{\sigma(X) : \sigma \in A\} = \text{card}\{\sigma(Z) : \sigma \in A\}$, so $\frac{\text{card}\{\sigma(Z) : \sigma \in A\}}{\text{card}(A)} = 1$.

\[\square\]

### 3.1.3.2 $X \times Z \times W$

Here, $W$ will not be mentioned directly.

Let $P : X \times Z \times W \rightarrow [0, 1]$ be a probability measure.

Again, we say that $X$ and $Z$ are independent for $P$, $(X \mid Z)_P$, iff for all $x \in X, z \in Z$ $P(x, z) = P(x) \cdot P(z)$.

**Example 3.1.3**

(1)

Let $P(000) = P(001) = P(010) = P(011) = 1/12, P(100) = P(101) = P(110) = P(111) = 1/6$, then $X$ and $Z$ are independent.

(2)

Let $P(100) = P(101) = P(010) = P(011) = 1/12, P(000) = P(001) = P(110) = P(111) = 1/6$, then

$P(X = 0) = P(X = 1) = P(Z = 0) = P(Z = 1) = 1/2$, but $P(X = 0, Z = 0) = 1/3 \neq 1/2 \cdot 1/2 = 1/4$, so $\neg(\gamma(X \mid Z)_P$.

As above, we define $P_A$ for $\emptyset \neq A \subseteq X \times Z \times W$.

**Example 3.1.4**

(1)

$A := \{000, 001, 010, 011\}$. Then $P_A(X = 0, Z = 0) = P_A(X = 0, Z = 1) = 1/2, P_A(X = 1, Z = 0) = P_A(X = 1, Z = 1) = 0, P_A(X = 0) = 1, P_A(X = 1) = 0, P_A(Z = 0) = P_A(Z = 1) = 1/2$, so $X$ and $Z$ are independent.

(2)

For $A := \{000, 001, 110, 111\}$, we see that $X$ and $Z$ are not independent for $P_A$.

Considering possible decompositions of $A$ into set products, we are not so much interested how many continuations into $W$ we have, but if there are any or none. This is often the case in logic, we are not interested how many models there are, but if there is a model at all.
Thus we define independence for $A$ again by:

$$\langle X \mid Z \rangle_A \iff \text{for all } \sigma \tau \in A \text{ there is } \rho \in A \text{ such that } \rho(X) = \sigma(X) \text{ and } \rho(Z) = \tau(Z).$$

The equivalence between probabilistic independence, $\langle X \mid Z \rangle_{P_A}$ and set independence, $\langle X \mid Z \rangle_A$ is lost now, as the second part of the following example shows:

**Example 3.1.5**

(1)

$A := \{000, 010, 100, 110\}$ satisfies both forms of independence, $\langle X \mid Z \rangle_{P_A}$ and set independence, $\langle X \mid Z \rangle_A$.

(2)

$A := \{000, 001, 010, 100, 110\}$. Here, we have $P_A(X = 0) = 3/5$, $P_A(X = 1) = 2/5$, $P_A(Z = 0) = 3/5$, $P_A(Z = 1) = 2/5$, but $P_A(X = 0, Z = 0) = 2/5 \neq 3/5 \times 3/5$.

Consider now $(X \mid Z)_A$: Take $\sigma, \tau \in A$, then for all possible values $\sigma(X)$, $\tau(Z)$, there is $\rho$ such that $\rho(X) = \sigma(X)$, $\rho(Z) = \tau(Z)$ - the value $\rho(W)$ is without importance.

We have, however:

**Fact 3.1.2**

$$\langle X \mid Z \rangle_{P_A} \Rightarrow \langle X \mid Z \rangle_A.$$

**Proof**

Let $\sigma, \tau \in A$, but suppose there is no $\rho \in A$ such that $\rho(X) = \sigma(X)$ and $\rho(Z) = \tau(Z)$. Then $P_A(\sigma(X)), P_A(\tau(Z)) > 0$, but $P_A(\sigma(X), \tau(Z)) = 0$. □

### 3.1.3.3 $X \times Y \times Z$

We consider now independence of $X$ and $Z$, given $Y$.

The probabilistic definition is:

$$\langle X \mid Y \mid Z \rangle_P \iff \text{for all } x \in X, y \in Y, z \in Z \text{ } P(x, y, z) * P(y) = P(x, y) * P(y, z).$$

As we are interested mainly in subsets $A \subseteq X \times Y \times Z$ and the resulting $P_A$, and combination of function fragments, we work immediately with these.

We have to define $\langle X \mid Y \mid Z \rangle_A$.

$$\langle X \mid Y \mid Z \rangle_A \iff \text{for all } \sigma, \tau \in A \text{ such that } \sigma(Y) = \tau(Y) \text{ there is } \rho \in A \text{ such that } \rho(X) = \sigma(X),$$

$$\rho(Y) = \sigma(Y) = \tau(Y), \quad \rho(Z) = \tau(Z).$$

When we set for $y \in Y A_y := \{\sigma \in A : \sigma(Y) = y\}$, we then have:

$$A_y = \{\sigma(X) : \sigma \in A_y\} \times \{y\} \times \{\sigma(Z) : \sigma \in A_y\}.$$

The following example shows that $\langle X \mid Y \mid Z \rangle_A$ and $\langle X \mid Z \rangle_A$ are independent from each other:
Example 3.1.6

(1)
\[ \langle X \mid Y \mid Z \rangle_A \text{ may hold, but not } \langle X \parallel Z \rangle_A : \]
Consider \( A := \{000, 111\} \). \( (X \mid Y \mid Z)_A \) is obvious, as only \( \sigma \) goes through each element in the middle. But there is no 0x1, so \( \langle X \parallel Z \rangle_A \) fails.

(2)
\[ \langle X \parallel Z \rangle_A \text{ may hold, but not } \langle X \mid Y \mid Z \rangle_A : \]
Consider \( A := \{000, 101, 110, 011\} \). Fixing, e.g., 0 in the middle shows that \( \langle X \mid Y \mid Z \rangle_A \) fails, but neglecting the middle, we can combine arbitrarily, so \( \langle X \parallel Z \rangle_A \) holds.

Example 3.1.7

This example show that \( \langle X \mid Y \mid Z \rangle_A \) does not mean that \( A \) is some product \( A_X \times A_Y \times A_Z : \)
Let \( A := \{000, 111\} \), then clearly \( \langle X \mid Y \mid Z \rangle_A \), but \( A \) is no such product.

We have again:

Fact 3.1.3

Let \( \emptyset \neq A \subseteq X \times Y \times Z \), then \( \langle X \mid Y \mid Z \rangle_A \) and \( \langle X \mid Y \mid Z \rangle_{P_A} \) are equivalent.

Proof

\begin{align*}
\text{“}\leq\text{“}:

\text{Suppose there are } \sigma, \tau \in A \text{ such that } \sigma(Y) = \tau(Y), \text{ but there is no } \rho \in A \text{ such that } \rho(X) = \sigma(X), \\
\rho(Y) = \sigma(Y) = \tau(Y), \rho(Z) = \tau(Z). \text{ Then } P_A(\sigma(X), \sigma(Y)), P_A(\tau(Y), \tau(Z)), P_A(\sigma(Y)) > 0, \text{ but } \\
P_A(\sigma(X), \sigma(Y) = \tau(Y), \tau(Z)) = 0.
\end{align*}

\text{“}\Rightarrow\text{“}:

Case 1: \( P_A(x, y) \text{ or } P_A(y, z) = 0 \), then \( P_A(x, y, z) = 0 \), and we are done.

Case 2: \( P_A(x, y), P_A(y, z) > 0 \). By definition and prerequisite, \\
\begin{align*}
P_A(x, y) &= \frac{\text{card}(\sigma \in A : \sigma(X) = x, \sigma(Y) = y)}{\text{card}(A)} \\
P_A(y, z) &= \frac{\text{card}(\sigma \in A : \sigma(Y) = y, \sigma(Z) = z)}{\text{card}(A)} \\
\end{align*}

so \\
\begin{align*}
P_A(x, y) * P_A(y, z) &= \frac{\text{card}(\sigma \in A : \sigma(Y) = y)}{\text{card}(A)^2} * \frac{\text{card}(\sigma \in A : \sigma(Z) = z)}{\text{card}(A)} \\
&= \frac{\text{card}(\sigma \in A : \sigma(Y) = y)}{\text{card}(A) * \text{card}(A)} \\
&= P_A(x, y) * P_A(y, z)
\end{align*}

\square

3.1.3.4 \( \times Y \times Z \times W \)

The definitions stay the same as for \( X \times Y \times Z \).

The equivalence between probabilistic independence, \( \langle X \mid Y \mid Z \rangle_{P_A} \) and set independence, \( \langle X \mid Y \mid Z \rangle_A \) is lost again, as the following example shows:
Example 3.1.8
A := \{0000, 0001, 0010, 1000, 1010\}.

Here, we have \( P_A(X = 0, Y = 0) = 3/5 \), \( P_A(X = 1, Y = 0) = 2/5 \), \( P_A(Y = 0, Z = 0) = 3/5 \), \( P_A(Y = 0, Z = 1) = 2/5 \), \( P_A(Y = 0) = 1 \), but \( P_A(X = 0, Y = 0, Z = 0) = 2/5 \neq 3/5 * 3/5 \).

Consider now \( (X \mid Y \mid Z)_A \) : Take \( \sigma, \tau \in A \), such that \( \sigma(Y) = \tau(Y) \), then for all possible values \( \sigma(X), \tau(Z) \), there is \( \rho \) such that \( \rho(X) = \sigma(X), \rho(Y) = \sigma(Y) = \tau(Y), \rho(Z) = \tau(Z) \) - the value \( \rho(W) \) is without importance.

We have, however:

Fact 3.1.4
\( (X \mid Y \mid Z)_{P_A} \Rightarrow (X \mid Y \mid Z)_A. \)

Proof
Let \( \sigma, \tau \in A \) such that \( \sigma(Y) = \tau(Y) \), but suppose there is no \( \rho \in A \) such that \( \rho(X) = \sigma(X), \rho(Y) = \sigma(Y) = \tau(Y), \rho(Z) = \tau(Z) \). Then \( P_A(\sigma(X), \sigma(Y)), P_A(\sigma(Y), \tau(Z)) > 0 \), but \( P_A(\sigma(X), \sigma(Y), \tau(Z)) = 0 \). \( \square \)

3.1.3.5 A remark on generalization

The \( X, Y, Z, W \) may also be more complicated sets, themselves products, but this will not change definitions and results beyond notation.

In the more complicated cases, we will often denote subsets by more complicated letters than A, e.g., by \( \Sigma \).

3.1.3.6 A remark on intuition

Consider set independence, where \( A := \mu(U), U = U_1 \times \ldots \times U_n \). Set \( \langle \ldots \rangle := \langle \ldots \rangle_{\mu(U)} \).

1. \( \langle X \mid \mid Z \rangle \) means then:
   1.1 all we know is that we are in a normal situation,
   1.2 if we know in addition something definite about \( Z \) (1 model!) we do not know anything more about \( X \), and vice versa.

2. \( \langle X \mid Y \mid Z \rangle \) means then:
   1.1 all we know is that we are in a normal situation,
   1.2 if we have definite information about \( Y \), we may know more about \( X \). But knowing something in addition about \( Z \) will not give us not more information about \( X \), and conversely.
(2) The restriction to $\mu(U)$ codes our background knowledge.

(3) Note that $X \cup Y \cup Z$ need not be $I$, e.g., $W$ might be missing. We did not count the continuations into $W$, but considered only existence of a continuation (if this does not exist, then there just is no such sequence).

This corresponds to multiplication with 1, the unit ALL on $W$, or, more generally, in the rest of the paper, with $1_{I-(X\cup Y\cup Z)}$. We may choose however we want, it has to be somewhere, in ALL.

3.1.4 Basic definitions

Definition 3.1.2
If $f$ is a function, $Y$ a subset of its domain, we write $f \upharpoonright Y$ for the restriction of $f$ to elements of $Y$.

If $F$ is a set of functions over $Y$, then $F \upharpoonright Y := \{ f \upharpoonright Y : f \in F \}$.

3.2 Probabilistic and set independence

3.2.1 Probabilistic independence

Independence as an abstract ternary relation for probability and other situations has been examined by W. Spohn, see [Spo80], A. P. Dawid, see [Daw79], J. Pearl, see, e.g., [Pea88], etc.

Definition 3.2.1
(1) Let $I \neq \emptyset$ be an arbitrary (index) set, for $i \in I$ arbitrary sets. Let $U := \Pi\{U_i : i \in I\}$, and for $X \subseteq I$ $U_X := \Pi\{U_i : i \in X\}$.

(2) Let $P : \mathcal{P}(U) \to [0, 1]$ be a probability measure. (We may assume that $P$ is defined by its value on singletons.)

By abuse of language, for $X \subseteq I$, $x \in U_X$, let $P(x) := P(\{u \in U : \forall i \in X u(i) = x(i)\})$, so $P(x) = P(\{u \in U : u \upharpoonright X = x\})$.

Analogously, for $X, Y \subseteq I$, $X \cap Y = \emptyset$, $x \in U_X$, $y \in U_Y$, let $P(x, y) := P(\{u \in U : u \upharpoonright X = x \text{ and } u \upharpoonright Y = y\})$.

Finally, for $X, Y, Z \subseteq I$ pairwise disjoint, $x \in U_X$, $y \in U_Y$, $z \in U_Z$, let $P(x \mid y) := \frac{P(x, y)}{P(y)}$, $P(x \mid y, z) := \frac{P(x, y, z)}{P(y, z)}$, etc.

(We have, of course, to pay attention that we do not divide by 0.)

Definition 3.2.2
\( P \) as above defines a 3-place relation of independence on pairwise disjoint \( X, Y, Z \subseteq I \langle X \mid Y \mid Z \rangle \_P \) by

\[
\langle X \mid Y \mid Z \rangle \_P \leftrightarrow \begin{cases} 
\forall x \in U_X, \forall y \in U_Y, \forall z \in U_Z(P(y, z) > 0 \rightarrow P(x \mid y) = P(x \mid y, z)), & \text{if } Y \neq \emptyset \\
i.e., P(x, y)/P(y) = P(x, y, z)/P(y, z), & \text{or} \\
P(x, y, z) \ast P(y) = P(x, y) \ast P(y, z) 
\end{cases}
\]

\[
\forall x \in U_X, \forall z \in U_Z(P(z) > 0 \rightarrow P(x) = P(x \mid z)), & \text{if } Y = \emptyset \\
i.e., P(x) = P(x, z)/P(z), & \text{or} \\
P(x, z) = P(x) \ast P(z) 
\]

If \( Y = \emptyset \), we shall also write \( (X \parallel Z)_P \) for \( (X \mid Y \mid Z)_P \).

Recall from Section 3.2.2 (page 18) that we call this notion probabilistic independence.

E.g., Pearl discusses the rules \((a) - (e)\) of Definition 3.2.3 (page 19) for the relation defined in Definition 3.2.2 (page 18).

**Definition 3.2.3**

(a) Symmetry: \( \langle X \mid Y \mid Z \rangle \leftrightarrow \langle Z \mid Y \mid X \rangle \)

(b) Decomposition: \( \langle X \mid Y \mid Z \cup W \rangle \rightarrow \langle X \mid Y \mid Z \rangle \)

(c) Weak Union: \( \langle X \mid Y \mid Z \cup W \rangle \rightarrow \langle X \mid Y \cup W \mid Z \rangle \)

(d) Contraction: \( \langle X \mid Y \mid Z \rangle \) and \( \langle X \mid Y \cup Z \mid W \rangle \rightarrow \langle X \mid Y \mid Z \cup W \rangle \)

(e) Intersection: \( \langle X \mid Y \cup W \mid Z \rangle \) and \( \langle X \mid Y \cup Z \mid W \rangle \rightarrow \langle X \mid Y \mid Z \cup W \rangle \)

(\emptyset) Empty outside: \( \langle X \mid Y \mid Z \rangle \) if \( X = \emptyset \) or \( Z = \emptyset \).

**Proposition 3.2.1**

If \( P \) is a probability measure, and \( \langle X \mid Y \mid Z \rangle \_P \) defined as above, then \((a) - (d)\) of Definition 3.2.3 (page 19) hold for \( (\ldots) = (\ldots)_P \), and if \( P \) is strictly positive, \((e)\) will also hold.

The proof is elementary, well known, and will not be repeated here.

Doch ein Beispiel geben?

### 3.2.1.1 A side remark on preferential structures

Being a minimal element is not upward absolute in general preferential structures, but in raked structures, provided the smaller set contains some element minimal in the bigger set.

**Fact 3.2.2**

In the probabilistic interpretation, the following holds:

Let \( U \) be a finite set, \( f : U \rightarrow \mathbb{R} \) such that \( \forall u \in U. f(u) \geq 0 \).
For all \( A \subseteq U \), such that \( \exists a' \in A. f(a') > 0 \) and all \( a \in A \)

\[ f_A(a) := \frac{f(a)}{\sum_{\{f(a') : a' \in A\}} f(a')} \]

defines a probability measure on \( A \).

For \( B \subseteq A \), define \( f_A(B) := \Sigma \{ f_A(b) : b \in B \} \). Then the following property holds:

(BASIC) For all \( D \subseteq B \subseteq A \subseteq U \) such that \( \exists b \in B. f(b) > 0 \)

\[ f_A(D) = f_A(B) * f_B(D). \]

**Proof**

For \( X \subseteq Y \subseteq U \) such that \( \exists y \in Y. f(y) > 0 \) we have

\[ f_Y(X) := \Sigma \{ f_Y(x) : x \in X \} = \frac{\Sigma \{ f(x) : x \in X \}}{\Sigma \{ f(y) : y \in Y \}}. \]

Thus, \( f_A(D) := \frac{\Sigma \{ f(d) : d \in D \}}{\Sigma \{ f(a) : a \in A \}} = \frac{\Sigma \{ f(b) : b \in B \}}{\Sigma \{ f(a) : a \in A \}} \cdot \frac{\Sigma \{ f(d) : d \in D \}}{\Sigma \{ f(b) : b \in B \}} = f_A(B) * f_B(D) \).

\[ \square \]

We have the following fact for \( \mu \) generated by a relation:

**Fact 3.2.3**

Let \( U \) be a finite preferential structure such that for \( A \subseteq U. \mu(A) = \emptyset \Rightarrow A = \emptyset \).

Then \( U \) is ranked iff (BASIC) as defined in Fact 3.2.2 (page 19) holds for \( f_A \).

**Proof**

\( \Rightarrow \):

Let \( D \subseteq B \subseteq A \subseteq U \), \( B \neq \emptyset \).

Case 1: \( D \cap \mu(A) = \emptyset \). Then \( f_A(D) = 0 \).

Case 1.1: If \( B \cap \mu(A) = \emptyset \), then \( f_A(B) = 0 \), and we are done.

Case 1.2: Let \( B \cap \mu(A) \neq \emptyset \). If \( D \cap \mu(B) = \emptyset \), then \( f_B(D) = 0 \), and we are done. Suppose \( D \cap \mu(B) \neq \emptyset \), so there is \( d \in D \cap \mu(B) \), so \( d \in D \cap \mu(A) \) by \( B \cap \mu(A) \neq \emptyset \) and rankedness, so \( f_A(D) \neq 0 \), contradiction.

Case 2: \( D \cap \mu(A) \neq \emptyset \).

Thus, by \( D \subseteq B \), \( B \cap \mu(A) \neq \emptyset \), and by rankedness \( \mu(B) = B \cap \mu(A) \). So by \( D \subseteq B \) again, \( D \cap \mu(A) = D \cap (B \cap \mu(A)) = D \cap \mu(B) \).

By definition, \( f_A(B) := \frac{\text{card}(\mu(A) \cap B)}{\text{card}(a(A))} \), \( f_A(D) := \frac{\text{card}(\mu(A) \cap D)}{\text{card}(a(A))} \), \( f_B(D) := \frac{\text{card}(\mu(B) \cap D)}{\text{card}(\mu(B))} \).

\( \Leftarrow \):

Then there are \( a, b, c \in U \), where \( a \) is incomparable to \( b \), and \( b \prec c \) but \( a \neq c \), or \( c \prec b \), but \( c \neq a \).

We have four possible cases.

Let, in all cases, \( A := \{a, b, c\} \). We construct a contradiction to (BASIC).

Case 1, \( b \prec c \):

Case 1.1, \( a \) is incomparable to \( c \): Consider \( B := \{a, c\} \), \( D := \{a\} \). Then \( f_A(D) = \frac{1}{2} \), \( f_A(B) = \frac{1}{2} \), \( f_B(D) = \frac{1}{2} \).

Case 1.2, \( c \prec a \) (so \( \prec \) is not transitive): Consider \( B := \{a, b\} \), \( D := \{a\} \). Then \( f_A(D) = 0 \),
\[ f_A(B) = 1, \quad f_B(D) = \frac{1}{2}. \]

Case 2, \( c \preceq b \):

Consider \( B := \{a, b\}, \quad D := \{a\} \). Then \( f_A(D) = \frac{1}{2}, \quad f_A(B) = \frac{1}{2}, \quad f_B(D) = \frac{1}{2}. \)

Case 2.2, \( a \preceq c \) - similar to Case 1.2.

\[ \Box \]

**Remark 3.2.4**

Note that sets \( A \subseteq B \), where \( \mu(B) \cap A = \emptyset \), and sets where \( P(A) = 0 \) have a similar, exceptional role. This might still be important.

### 3.2.2 Set independence

We interpret independence here differently, but in a related way, as prepared in Section 3.1.3 (page 12).

**Definition 3.2.4**

We consider function sets \( \Sigma \) etc. over a fixed, arbitrary domain \( I \neq \emptyset \), into some fixed codomain \( K \).

(1)

For pairwise disjoint subsets \( X, Y, Z \) of \( I \), we define

\[ \langle X \mid Y \mid Z \rangle_\Sigma \text{ iff for all } f, g \in \Sigma \text{ such that } f \upharpoonright Y = g \upharpoonright Y, \text{ there is } h \in \Sigma \text{ such that } h \upharpoonright X = f \upharpoonright X, \ h \upharpoonright Y = f \upharpoonright Y = g \upharpoonright Y, \ h \upharpoonright Z = g \upharpoonright Z. \]

Recall from Section 3.1.3 (page 12) that we call this notion set independence.

\( Y \) may be empty, then the condition \( f \upharpoonright Y = g \upharpoonright Y \) is void.

Note that nothing is said about \( I - (X \cup Y \cup Z) \), so we look at the projection of \( U \) to \( X \cup Y \cup Z \).

When \( Y = \emptyset \), we will also write \( \langle X \mid Z \rangle_\Sigma \).

\( \langle X \mid Y \mid Z \rangle_\Sigma \) means thus, that we can piece functions together, or that we have a sort of decomposition of \( \Sigma \) into a product. This is an independence property, we can put parts together independently.

(2)

In the sequel, we will just write \( \langle \ldots \rangle \) for \( \langle \ldots \rangle_\Sigma \) when the meaning is clear from the context.

Recall that Example 3.1.3 (page 15) compares different forms of independence, the probabilistic and the set variant.

Obviously, we can generalize the equivalence results for probabilistic and set independence for \( X \times Z \) and \( X \times Y \times Z \) to the general situation with \( W \) in Section 3.1.3 (page 12), as long as we do not consider the full functions \( \sigma \), but only their restrictions to \( X, Y, Z, \sigma \upharpoonright (X \cup Y \cup Z) \). As we will
stop the discussion of probabilistic independence here, and restrict ourselves to set independence, this is left as an easy exercise to the reader.
3.3 Basic results for set independence

Notation 3.3.1
In more complicated cases, we will often write $ABC$ for $\langle A \mid B \mid C \rangle$, and $\neg ABC$ or $-ABC$ if $\langle A \mid B \mid C \rangle$ does not hold. Moreover, we will often just write $f(A)$ for $f \upharpoonright A$, etc.

For $\langle A \cup A' \mid B \mid C \rangle$, we will then write $(AA')BC$, etc.

If only singletons are involved, we will sometimes write $abc$ instead of $ABC$, etc.

When we speak about fragments of functions, we will often write just $A: \sigma$ for $\sigma \upharpoonright A$, $B: \sigma = \tau$ for $\sigma \upharpoonright B = \tau \upharpoonright B$, etc.

We use the following notations for functions:

Definition 3.3.1
The constant functions $0_c$ and $1_c$:

$0_c(i) = 0$ for all $i \in I$

$1_c(i) = 1$ for all $i \in I$

Moreover, when we define a function $\sigma: I \to \{0, 1\}$ argument by argument, we abbreviate $\sigma(a) = 0$ by $a = 0$, etc.

Sometimes, we also give (a fragment of) a function just by the sequence of the values, so instead of writing $a = 0$, $b = 1$, $c = 1$, we just write 011 - context will disambiguate.

Remark 3.3.1
This remark gives an intuitive justification of (some of) above rules in our context.

Rule (a) is trivial.

It is easiest to set $Y := \emptyset$ to see the intuitive meaning.

Rule (b) is a trivial consequence. If we can combine longer sequences, then we can combine shorter, too.

Rule (c) is again a trivial consequence. If we can combine arbitrary sequences, then we can also combine those which agree already on some part.

Rule (d) is the most interesting one, it says when we may combine longer sequences. Having just $\langle X \mid Z \rangle$ and $\langle X \mid W \rangle$ as prerequisite does not suffice, as we might lose when applying $\langle X \mid W \rangle$ what we had already by $\langle X \mid Z \rangle$. The condition $\langle X \mid Z \mid W \rangle$ guarantees that we do not lose this.

In our context, it means the following:

We want to combine $\sigma \upharpoonright X$ with $\tau \upharpoonright Z \cup W$. By $\langle X \mid Z \rangle$, we can combine $\sigma \upharpoonright X$ with $\tau \upharpoonright Z$. Fix $\rho$ such that $\rho \upharpoonright X = \sigma \upharpoonright X$, $\rho \upharpoonright Z = \tau \upharpoonright Z$. As $\rho \upharpoonright Z = \tau \upharpoonright Z$, by $\langle X \mid Z \mid W \rangle$, we can combine $\rho \upharpoonright X \cup Z$ with $\tau \upharpoonright W$, and have the result.

Note that we change the functions here, too: we start with $\sigma, \tau$, then continue with $\rho, \tau$.

We can use what we constructed already as a sort of scaffolding for constructing the rest.

Fact 3.3.2
Zusammenhang \( \langle X \mid Y \mid Z \rangle \) mit Produkten.

**Proof**

\[ \square \]

We show now that above Rules \((a) - (d)\) hold in our context, but \((e)\) does not hold.

**Fact 3.3.3**

In our interpretation,

1. rule \((e)\) does not hold,
2. all \( \langle X \mid Y \mid \emptyset \rangle \) (and thus also all \( \langle \emptyset \mid Y \mid Z \rangle \)) hold.
3. rules \((a) - (d)\) hold, even when one or both of the outside elements of the triplets is the empty set.

**Proof**

1. \((e)\) does not hold:

Consider \( I := \{ x, y, z, w \} \) and \( U := \{ 1111, 0100 \} \). Then \( x(yw)z \) and \( x(yz)w \) as for all \( \sigma \upharpoonright yw \) there is just one \( \tau \) this \( \sigma \) can be. The same holds for \( x(yz)w \). But for \( y = 1 \), there are two different paths through \( y = 1 \), which cannot be combined.

2. This is a trivial consequence of the fact that \( \{ f : f : \emptyset \rightarrow U \} = \{ \emptyset \} \).

3. Rules \((a), (b), (c)\) are trivial, by definition, also for \( X, Z = \emptyset \). In \((e)\), if \( W = \emptyset \), there is nothing to show.

Rule \((d)\): The cases for \( X, W, Z = \emptyset \) are trivial. Assume \( \sigma, \tau \) such that \( \sigma \upharpoonright Y = \tau \upharpoonright Y \), we want to combine \( \sigma \upharpoonright X \) with \( \tau \upharpoonright Z \cup W \). By \( \langle X \mid Y \mid Z \rangle \), there is \( \rho \) such that \( \rho \upharpoonright X = \sigma \upharpoonright X \), \( \rho \upharpoonright Y = \sigma \upharpoonright Y = \tau \upharpoonright Y \), \( \alpha \upharpoonright X = \rho \upharpoonright Z = \tau \upharpoonright Z \). Thus \( \rho \) and \( \tau \) satisfy the prerequisite of \( \langle X \mid Y \cup Z \mid W \rangle \), and there is \( \alpha \) such that \( \alpha \upharpoonright X = \rho \upharpoonright X = \sigma \upharpoonright X \), \( \alpha \upharpoonright X = \rho \upharpoonright Y = \sigma \upharpoonright Y = \tau \upharpoonright Y \), \( \alpha \upharpoonright W = \tau \upharpoonright W \).

\[ \square \]

Next, we give examples which shows that increasing the center set can change validity of the triplet in any way.

**Example 3.3.1**

1. This example shows that neither \( \langle X \mid Y \mid Z \rangle \) implies \( \langle X \parallel Z \rangle \), nor, conversely, \( \langle X \parallel Z \rangle \) implies \( \langle X \mid Y \mid Z \rangle \).

Consider \( I := \{ x, y \} \).
(1.1) Let $U := \{\langle 0, 0, 0 \rangle, \langle 1, 1, 1 \rangle, \langle 0, 1, 0 \rangle, \langle 1, 0, 1 \rangle, \langle 1, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$. Then $\langle x || z \rangle$, as all combinations for $x$ and $y$ exist, i.e. paths with the projections $\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 1, 1 \rangle$. Fix, e.g., $y = 1$. Then the paths through $y = 1$ are $\langle 1, 1, 1 \rangle, \langle 0, 1, 0 \rangle, \langle 1, 1, 0 \rangle$, but $\langle 0, 1, 1 \rangle$ is missing. So $\langle x \mid y \mid z \rangle$ does not hold.

(1.2) Let $U := \{\langle 0, 0, 0 \rangle, \langle 1, 1, 1 \rangle\}$. Then $\langle x || z \rangle$ trivially fails, but $\langle x \mid y \mid z \rangle$ holds.

(2) Consider $I := \{x, a, b, c, d, z\}$. Let $\Sigma := \{111111, 011110, 011101, 111100, 110111, 010000\}$. Then $\neg x(abc)z, x(abc)z, \neg x(ab)z$. For $\neg x(abcd)z$, fix $abcd = 1111$, then $111111, 011110 \in \Sigma$, but, e.g., $011111 \not\in \Sigma$.

For $x(abc)z$, the following combinations of $abc$ exist: $111, 101, 100$. The result is trivial for $101$ and $100$. For $111$, all combinations for $x$ and $z$ with 0 and 1 exist.

For $\neg x(ab)z$, fix $ab = 10$, then $110111, 010000 \in \Sigma$, but there is, e.g., no $110xy0 \not\in \Sigma$.

See Diagram 3.3.1 (page 25).
Heiteres

\[ \neg \langle x \mid abcd \mid z \rangle \] (1)

\[ \langle x \mid abc \mid z \rangle \] (2)

Add paths equal on abc, different on d, to compensate lacking paths in (1)

\[ \neg \langle x \mid ab \mid z \rangle \] (3)

Add paths different on ab, singletons on c, so they don’t disturb on abc: seen on abc, the added paths are singletons, so they respect automatically

\[ \langle x \mid abc \mid z \rangle \]
3.3.1 Example of a rule derived from the basic rules

We will use the following definition.

Definition 3.3.2
Given $\Sigma$ as above, set

$$\Sigma^\mu := \{ \langle X, Y, Z \rangle : X, Y, Z \text{ are pairwise disjoint subsets of } I, \langle X' \mid Y \mid Z \rangle \notin \Sigma, \text{ but for all } X' \subset X \text{ and all } Z' \subset Z \langle X' \mid Y \mid Z \rangle \in \Sigma \}.$$  

We will sometimes write $\langle X, X' \mid Y \mid Z \rangle$ etc. for $\langle X \cup X' \mid Y \mid Z \rangle$.

When we write $\langle X, X' \mid Y \mid Z \rangle$ etc., we will tacitly assume that all sets $X, X', Y, Z$ are pairwise disjoint.

Remark 3.3.4
(1) $\Sigma^\mu$ contain thus the minimal $X$ and $Z$ for fixed $Y$, such that $\langle X \mid Y \mid Z \rangle \notin \Sigma$.

(2) By rule (b), for all $\langle X \mid Y \mid Z \rangle \in \Sigma$, there is $\langle X', Y, Z' \rangle \in \Sigma^\mu$, $X \subseteq X'$, $Z \subseteq Z'$, unless all $\sigma, \tau$ such that $\sigma \upharpoonright Y = \tau \upharpoonright Y$ can be combined.

As the cases can become a bit complicated, it is important to develop a good intuition and representation of the problem. We do this now in the proof of the following fact, where we use the result we want to prove to guide our intuition.

Fact 3.3.5
Let $\Sigma$ be closed under rules $(a) - (d)$. Then, if $\langle X, X', X'' \mid Y \mid Z, Z' \rangle \in \Sigma^\mu$, then $\langle X, Z' \mid X', Y, Z'' \mid Z'', Z \rangle \notin \Sigma$.

Proof
The upper line is the final aim. Line (1) expresses that we can combine all parts except \( s_X \), by
\[
\langle X', X'' \mid Y \mid Z, Z', Z'' \rangle,
\]
which holds by \( \langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle \in \Sigma_\mu \), by similar arguments, we can combine as indicated in lines (2) – (6). We now assume \( \langle X, Z' \mid X', Y, Z'' \mid X'' \mid Z \rangle \in \Sigma \). So we have to look at fragments, which agree on \( X', Y, Z'' \). This is, for instance, true for (1) and (3).

We turn this argument now into a formal proof:

Assume

(A) \( \langle X, Z' \mid X', Y, Z'' \mid X'' \mid Z \rangle \in \Sigma \), and
(B) \( \langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle \in \Sigma_\mu \).
(C) \( \langle X, X' \mid Y \mid Z, Z', Z'' \rangle \) by (B), see line (3)
(D) \( \langle X \mid X', Y, Z', Z'' \mid X'' \mid Z \rangle \) by (A) and rule (c)
(E) \( \langle X \mid X', Y \mid Z, Z', Z'' \rangle \) by (C) and rule (c)
(F) \( \langle X \mid X', Y \mid Z', Z'' \rangle \) by (E) and (b)
(G) \( \langle X \mid X', Y \mid X'', Z, Z', Z'' \rangle \) by (D) and (F) and (d)
(K) \( \langle X \mid X', X'', Y \mid Z, Z', Z'' \rangle \) by (G) and (c)
(L) \( \langle X', X'' \mid Y \mid Z, Z', Z'' \rangle \) by (B), see line (1)
(M) \(\langle Z, Z', Z'' \mid X', X'', Y \mid X \rangle\) by (K) and (a)
(N) \(\langle Z, Z', Z'' \mid Y \mid X', X'' \rangle\) by (L) and (a)
(O) \(\langle Z, Z', Z'' \mid X, X', X'' \rangle\) by (M) and (N) and (d)
(P) \(\langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle\) by (O) and (a).

So we conclude \(\langle X, X', X'' \mid Y \mid Z, Z', Z'' \rangle \in \Sigma\), a contradiction.

Comment:
We first move \(Z', Z''\) to the right, and then \(X', X''\) to the left.

Moving \(Z', Z''\):
We use \(X''\) (or \(Z\)) on the right, which not be changed, therefore we can use line (3), resulting in
(C) \(\langle X, X' \mid Y \mid Z', Z'' \rangle\), or, directly
\((C') \langle X, X' \mid Y \mid Z', Z'' \rangle\), again by \(\Sigma\),
which is modified to
(F) \(\langle X \mid X', Y \mid Z', Z'' \rangle\), so we have on the right \(Z', Z''\) which we want to move.

We put \(Z'\) in the middle (\(Z''\) is there already) of (A), resulting in
(D) \(\langle X \mid X', Y, Z', Z'' \mid X'', Z \rangle\).

Now we can apply (d) to (D) and (F), and have moved \(Z', Z''\) to the right:
(G) \(\langle X \mid X', Y \mid X'', Z, Z', Z'' \rangle\).

We still have to move \(X'\) and \(X''\) to the left of (G), and do this in an analogous way.

\(\square\)

Note that our results stays valid, if some of the \(X', X'', Z', Z''\) are empty.

Aber resultat darf nicht links oder rechts \(\emptyset\) sein.

**Corollary 3.3.6**
Let \(\Sigma\) be closed under rules (a) – (d). Then, if \(\langle X, X', X'' \mid Y, Y', Y'' \mid Z, Z', Z'' \rangle \in \Sigma\), then
\(\langle X, Y', Z' \mid X', Y, Y'' \mid X'', Y'' \mid Z, Z', Z'' \rangle \notin \Sigma\).

Thus, if, for given \(Y \cup Y' \cup Y''\), \(\langle X, X', X'' \mid Y, Y', Y'' \mid Z, Z', Z'' \rangle \in \Sigma\), then for no distribution of \(X \cup X' \cup X'' \cup Y \cup Y' \cup Y'' \cup Z \cup Z' \cup Z''\) such that the outward elements are non-empty,
\(\langle X, Y', Z' \mid X', Y, Y'' \mid X'', Y'' \mid Z, Z', Z'' \rangle \in \Sigma\).

**Proof**
Suppose \(\langle X, Y', Z' \mid X', Y, Z'' \mid X'', Y'' \mid Z \rangle \in \Sigma\). Then by rule (c) \(\langle X, Y', Y'', Z'' \mid X'', Z \rangle \in \Sigma\). Set \(Y_1 := Y \cup Y' \cup Y''\). Then \(\langle X, Y', Y_1, Z'' \mid X'', Z \rangle \in \Sigma\), and \(\langle X, X', X'' \mid Y_1 \mid Z, Z', Z'' \rangle \in \Sigma\), contradicting Fact 3.3.5 (page 27). \(\square\)
3.4 Examples of new rules

3.4.1 New rules

Above rules (a) – (d) are not the only ones to hold, and we introduce now more complicated ones, and show that they hold in our situation. Of the possibly infinitary rules, only (Loop1) is given in full generality, (Loop2) is only given to illustrate that even the infinitary rule (Loop1) is not all there is.

For warming up, we consider the following short version of (Loop1):

Example 3.4.1

\[ ABC, ACD, ADE, AEB \Rightarrow ABE. \]

We show that this rule holds in all \( \Sigma \).

Suppose \( A : \sigma, B : \sigma = \tau, C : \tau \), so by \( ABC \), there is \( \rho_1 \) such that

\[
A : \rho_1 = \sigma, B : \rho_1 = \sigma = \tau, C : \rho_1 = \tau.
\]

So by \( ACD \), there is \( \rho_2 \) such that

\[
A : \rho_2 = \sigma, C : \rho_2 = \rho_1 = \tau, D : \rho_2 = \tau.
\]

So by \( ADE \), there is \( \rho_3 \) such that

\[
A : \rho_3 = \sigma, D : \rho_3 = \rho_2 = \tau, E : \rho_3 = \tau.
\]

So by \( AEB \), there is \( \rho_4 \) such that

\[
A : \rho_4 = \sigma, E : \rho_4 = \rho_3 = \tau, B : \rho_4 = \tau = \sigma.
\]

So \( ABE \).

We abbreviate this reasoning by:

(1) \( ABC : A : \sigma, B : \sigma = \tau, C : \tau \)

(2) \( ACD : (1) + \tau \)

(3) \( ADE : (2) + \tau \)

(4) \( AEB : (3) + \tau \)

So \( ABE \).

It is helpful to draw a little diagram as in the following Table 3.4.1 (page 30).

We introduce now some new rules.

Definition 3.4.1
• (Bin1)
  \[ XYZ, XZ, Y(Z)Y' \Rightarrow X(Y'Y)'Z \]

• (Bin2)
  \[ XYZ, XZ, Y(Z)Y' \Rightarrow X(Y'Y)'Z \]

• (Loop1)
  \[ AB_1B_2, \ldots, AB_{i-1}B_i, AB_iB_{i+1}, AB_{i+1}B_{i+2}, \ldots, AB_{n-1}B_n, AB_nB_1 \Rightarrow AB_1B_n \]
  so we turn \( AB_nB_1 \) around to \( AB_1B_n \).
  When we have to be more precise, we will denote this condition \( (Loop1) \) to fix the length.

• (Loop2)
  \[ ABC, ACD, DAE, DEF, FDG, FGH, HFB \Rightarrow HBF \]

The complicated structure of these rules suggests already that the ternary relations are not the right level of abstraction to speak about construction of functions from fragments. This is made formal by our main result below, which shows that there is no finite characterization by such relations. In other words, the main things happen behind the screen.

**Fact 3.4.1**

The new rules are valid in our situation.

**Proof**

• (Bin1)
  \[ (1) \ XYZ : X : \sigma, Y : \sigma = \tau, Z : \tau \]
  \[ (2) \ XY'Z : X : \sigma, Y' : \sigma = \tau, Z : \tau \]
  \[ (3) \ Y(XZ)Y' : (1) + (2) \]
  So \( X(Y'Y)'Z \).

• (Bin2)
  Let \( X : \sigma, Y : \sigma = \tau, Y' : \sigma = \tau, Z : \tau \)
  \[ (1) \ XYZ : X : \sigma, Y : \sigma = \tau, Z : \tau \]
  \[ (2) \ XYZ' : (1) + \tau \]
  \[ (3) \ Y(XZ)Y' : (1) + (2) \]
  So \( X(Y'Y)'Z \).

• (Loop1)
  \[ (1) \ AB_1B_2 : A : \sigma, B_1 : \sigma = \tau, B_2 : \tau \]
  \[ (2) \ AB_2B_3 : (1) + \tau \]
  .....

\[ \text{Bin1} \]
\[ \text{Bin2} \]
\[ \text{Loop1} \]
\[ \text{Loop2} \]
(i-1) $AB_{i-1}B_i : (i - 2) + \tau$
(i) $AB_iB_{i+1} : (i - 1) + \tau$
(i + 1) $AB_{i+1}B_{i+2} : (i) + \tau$

....

(n - 1) $AB_{n-1}B_n : (n - 2) + \tau$
(n) $AB_nB_1 : (n - 1) + \tau$
So $AB_1B_n$.

• (Loop2)

Let

(1) $ABC : A : \sigma, B : \sigma = \tau, C : \tau$
(2) $ACD : 1 + \tau$
(3) $DAE : 2 + \sigma$
(4) $DEF : 3 + \sigma$
(5) $FDG : 4 + \tau$
(6) $FGH : 5 + \tau$
(7) $HFB : 6 + \sigma$
So $HFB$ by $B : \sigma = \tau$.

Note that we use here $B : \sigma = \tau, E : \sigma = \tau, H : \sigma = \tau$, whereas the other triplets are used for other functions.

\[ \square \]

Next we show that the full (Loop1) cannot be derived from the basic rules (a) – (d) and (Bin1), and shorter versions of (Loop1). (This is also a consequence of the sequel, but we want to point it out right away.)

**Fact 3.4.2**

Let $n \geq 1$, then (Loop$1_n$) does not follow from the rules (a) – (d), (\emptyset), (Bin1), and the shorter versions of (Loop1).

**Proof**

Consider the following set of triplets $L \cup L'$ over $I := \{a, b_1, \ldots, b_n\}$:

$L := \{ab_1b_2, \ldots, ab_ib_{i+1}, \ldots, ab_{n-1}b_n, ab_nb_1\}$,

$L' := \{\emptyset AB : A \cap B = \emptyset, A \cup B \subseteq I\}$,

and close this set under symmetry (rule (a)). Call the resulting set $A$.

Note that, on the outside, we have $\emptyset$ or singletons, inside singletons or $\emptyset$. If the inside is $\emptyset$, one of the outside sets must also be $\emptyset$. 
When we look at $L$, and define a relation $<$ by $x < y$ iff $axy \in L$, we see that the only $<$-loop is $b_1 < b_2 < \ldots < b_n < b_1$.

We show first that $A$ is closed under rules (a) – (d) (see Definition 3.2.3 (page 19)).

(a) is trivial.

(b) If $W = \emptyset$ or $Z = \emptyset$, this is trivial, if $W = Z$, this is trivial, too.

(c) If $Z \cup W = \emptyset$, this is trivial, if $Z \cup W$ is a singleton, so $Z = \emptyset$ or $W = \emptyset$ or $Z = W$. $Z = \emptyset$ or $W = \emptyset$ are trivial, otherwise $Z = W$ contradicts disjointness.

(d) $Z = \emptyset$ is trivial, so is $W = \emptyset$, otherwise $Z = W$ contradicts disjointness.

(Bin1) $X = \emptyset$ or $Z = \emptyset$ are trivial, otherwise $X = Z$ is excluded by disjointness. So we are in $L'$ for $Y(XZ)Y'$. So $Y = \emptyset$ or $Y' = \emptyset$ and it is trivial.

Obviously, $(Loop1_n)$ does not hold.

We show now that all $(Loop1_k)$, $0 \leq k < n$ hold.

The cases $n = 1$, $n = 2$ are trivial.

Consider the case $2 < k < n$.

This has the form $AB_1B_2, AB_2B_3, \ldots, AB_{k-1}B_k, AB_kB_1 \Rightarrow AB_1B_k$.

If $A = \emptyset$ or $B_k = \emptyset$, the condition holds.

So assume $A, B_k \neq \emptyset$. Thus, by above remark, descending to $B_{k-1}$ etc., we see that all $B_i \neq \emptyset$, $1 \leq i \leq k$. Thus, all prerequisites are in $L$. Moreover, $A$ has to be $a$, which is the only element occurring repeatedly on the outside. Consider now the relation $<'$ defined by $U <' V$ iff $AUV$ is among the prerequisites. We then have $B_1 <' B_2 <' \ldots <' B_k <' B_1$, where all $B_i$ are some $b_j$, we see that the resulting $<'$-loop is too short, so the prerequisites cannot hold, and we have a contradiction.

\[\square\]

### 3.5 There is no finite characterization

We turn to our main result.

#### 3.5.1 Discussion

Consider the following simple, short, loop for illustration:

$ABC, ACD, ADE, AEF, AFG, AGB \Rightarrow AGB$ - so we can turn $AGB$ around to $ABG$.

Of course, this construction may be arbitrarily long.

The idea is now to make $AGB$ false, and, to make it coherent, to make one of the interior conditions false, too, say $ADE$. We describe this situation fully, i.e. enumerate all conditions which hold in such a situation. If we make now $ADE$ true again, we know this is not valid, so any (finite) characterization must say “NO” to this. But as it is finite, it cannot describe all the interior tripels
of the type \( ADE \) in a sufficiently long loop, so we just change one of them which it does not “see” to FALSE, and it must give the same answer NO, so this fails.

Basically, we cannot describe parts of the loop, as the \(<|\,|\, >\)-language is not rich enough to express it, we see only the final outcome.

The problem is to fully describe the situation.

### 3.5.2 Composition of layers

A very helpful fact is the following:

**Definition 3.5.1**

Let \( \Sigma_j \) be function sets over \( I \) into some set \( K, j \in J \).

Let \( \Sigma := \{ f : I \to K^J : f(i) = \{ (f_j(i), j) : j \in J, f_j \in \Sigma_j \} \} \).

So any \( f \in \Sigma \) has the form \( f(i) = (f_1(i), f_2(i), \ldots, f_n(i)) \), \( f_m \in \Sigma_m \) (we may assume \( J \) to be finite).

Thus, given \( f \in \Sigma, f_m \in \Sigma_m \) is defined.

**Fact 3.5.1**

For the above \( \Sigma \langle A \mid B \mid C \rangle \) holds iff it holds for all \( \Sigma_j \).

Thus, we can destroy the \( \langle A \mid B \mid C \rangle \) independently, and collect the results.

**Proof**

The proof is trivial, and a direct consequence of the fact that \( f = f' \) iff for all components \( f_j = f'_j \).

Suppose for some \( \Sigma_k, k \in J, \neg \langle A \mid B \mid C \rangle \).

So for this \( k \) there are \( f_k, f'_k \in \Sigma_k \) such that \( f_k(B) = f'_k(B) \), but there is no \( f''_k \in \Sigma_k \) such that \( f''_k(A) = f_k(A), f''_k(B) = f'_k(B), f''_k(C) = f'_k(C) \) (or conversely). Consider now some \( h \in \Sigma \) such that \( h_k = f_k \), and \( h' \) is like \( h \), but \( h'_k = f'_k \), so also \( h' \in \Sigma \). Then \( h(B) = h'(B) \), but there is no \( h'' \in \Sigma \) such that \( h''(A) = h(A), h''(B) = h'(B), h''(C) = h'(C) \).

Conversely, suppose \( \langle A \mid B \mid C \rangle \) for all \( \Sigma_j \). Let \( h, h' \in \Sigma \) such that \( h(B) = h'(B) \), so for all \( j \in J h_j(B) = h'_j(B) \), where \( h_j \in \Sigma_j, h'_j \in \Sigma_j \), so there are \( h'' \in \Sigma_j \) with \( h''(A) = h_j(A), h''(B) = h'_j(B), h''(C) = h'_j(C) \) for all \( j \in J \). Thus, \( h'' \) composed of the \( h''_j \) is in \( \Sigma \), and \( h''(A) = h(A), h''(B) = h'(B), h''(C) = h'(C) \).

\( \square \)

### 3.5.3 Systematic construction

Recall the general form of (Loop1) for singletons:

\[
ab_1 b_2, \ldots, ab_{i-1} b_i, ab_i b_{i+1}, ab_{i+1} b_{i+2}, \ldots, ab_{n-1} b_n, ab_n b_1 \Rightarrow ab_1 b_n
\]
We will fully describe a model of above tripels, with the exception of \(ab_1b_n\) and \(ab_ib_{i+1}\) which will be made to fail, and all other \(\langle X \mid Y \mid Z \rangle\) which are not in above list of tripels to preserve, will fail, too (except for \(X = \emptyset\) or \(Z = \emptyset\)).

Thus, the tripels to preserve are:

\[ P := \{ab_1b_2, \ldots, ab_{i-1}b_i, \text{ (BUT NOT } ab_ib_{i+1}) \}, ab_{i+1}b_{i+2}, \ldots, ab_{n-1}b_n, ab_nb_1 \]

We use the following fact:

**Fact 3.5.2**

Let \(X \subseteq I\), \(\text{card}(X) > 1\), \(\Sigma_X := \{ \sigma : I \rightarrow \{0,1\} : \text{card}\{x \in X : \sigma(x) = 0\} \text{ is even } \}\)

Then \(\neg ABC\) iff \(A \cap X \neq \emptyset\), \(C \cap X \neq \emptyset\), \(X \subseteq A \cup B \cup C\).

**Proof**

\(\Leftarrow\):

Suppose \(A \cap X \neq \emptyset\), \(C \cap X \neq \emptyset\), \(X \subseteq A \cup B \cup C\).

Take \(\sigma\) such that \(\text{card}\{x \in X : \sigma(x) = 0\}\) is odd, then \(\sigma \notin \Sigma_X\). As \(X \not\subseteq A \cup B\), there is \(\tau \in \Sigma_X\) such that \(\sigma \upharpoonright A \cup B = \tau \upharpoonright A \cup B\). As \(X \not\subseteq B \cup C\), there is \(\rho \in \Sigma_X\) such that \(\rho \upharpoonright B \cup C = \sigma \upharpoonright B \cup C\). Thus, \(\tau \upharpoonright B = \rho \upharpoonright B\). If there were \(\alpha \in \Sigma_X\) such that \(\alpha \upharpoonright A \cup B = \tau \upharpoonright A \cup B\) and \(\alpha \upharpoonright B \cup C = \rho \upharpoonright B \cup C\), then \(\alpha \upharpoonright A \cup B \cup C = \sigma \upharpoonright A \cup B \cup C\), contradiction.

\(\Rightarrow\):

Suppose \(A \cap X = \emptyset\) or \(C \cap X = \emptyset\), or \(X \not\subseteq A \cup B \cup C\). We show \(ABC\).

Case 1: \(C \cap X = \emptyset\). Let \(\sigma, \tau \in \Sigma_X\) such that \(\sigma \upharpoonright B = \tau \upharpoonright B\). As \(C \cap X = \emptyset\), we can continue \(\sigma \upharpoonright A \cup B\) as we like.

Case 2, \(A \cap X = \emptyset\), analogous.

Case 3: \(X \not\subseteq A \cup B \cup C\). But then there is no restriction in \(A \cup B \cup C\).

\(\Box\)

We will have to make \(ab_1b_n\) false, but \(ab_nb_1\) true. On the other hand, we will make \(ab_1b_3\) false, but \(ab_3b_1\) need not be preserved.

This leads to the following definition, which helps to put order into the cases.

**Definition 3.5.2**

Suppose we have to destroy \(axy\). Then

\[dmin(axy) := \min\{d(\{a, x, y\}, \{a, u, v\}) : auv \text{ has to be preserved }\} - d \text{ the counting Hamming distance.}\]

Thus, \(dmin(ab_1b_n) = 0\) (as \(ab_nb_1\) has to be preserved), \(dmin(ab_1b_3) = 1\) (because \(ab_1b_2\) has to be preserved, but not \(ab_3b_1\)).

We introduce the following order defined from the loop prerequisites to be preserved.
Definition 3.5.3
Order the elements by following the string of sequences to be preserved as follows:
\[ b_{i+1} \prec b_{i+2} \prec \ldots \prec b_{n-1} \prec b_n \prec b_1 \prec b_2 \prec \ldots \prec b_{i-1} \prec b_i \]
Note that the interruption at \( ab_i b_{i+1} \) is crucial here - otherwise, there would be a cycle.
As usual, \( \preceq \) will stand for \( \prec \) or \( = \).

3.5.4 The cases to consider
The elements to consider are: \( a, b_1, \ldots, b_n \).
Recall that the triads to preserve are:
\[ P := \{ ab_1 b_2, \ldots, ab_{i-1} b_i, \text{ (BUT NOT } ab_i b_{i+1}, b_{i+1}, \ldots, ab_n b_n, ab_n b_1 \} \]
The \( \langle X \mid Y \mid Z \rangle \) to destroy are (except when \( X = \emptyset \) or \( Z = \emptyset \)):
(1) all \( \langle X \mid \mid Z \rangle \)
(2) all \( \langle X \mid Y \mid Z \rangle \) such that \( X \cup Y \cup Z \) has \( > 3 \) elements
(3) all triads which do not have \( a \) on the outside, e.g. \( bgc \)
(4) and the following triads:
   (the \( 0 \) will be explained below - for the moment, just ignore it)
   \[ ab_1 b_2, \ldots, ab_{i-1} b_i, ab_1 b_n \ (0) \]
   \[ ab_2 b_1 \ (0), ab_2 b_4, \ldots, ab_2 b_n \]
   \[ ab_3 b_1, ab_3 b_2 \ (0), ab_3 b_5, \ldots, ab_3 b_n \]
   . . .
   \[ ab_i b_1, ab_i b_2, \ldots, \text{ ALSO } ab_i b_{i+1}, \ldots, ab_i b_n \]
   . . .
   \[ ab_{n-2} b_1, \ldots, ab_{n-2} b_{n-3} \ (0), ab_{n-2} b_n \]
   \[ ab_{n-1} b_1, \ldots, ab_{n-1} b_{n-2} \ (0), \]
   \[ ab_n b_1, \ldots, ab_n b_{n-1} \ (0) \]

3.5.5 Solution of the cases
We show how to destroy all triads mentioned above, while preserving all triads in \( P \).

(1) all \( \langle X \mid Y \mid Z \rangle \) where \( X \cup Y \cup Z \) has \( > 3 \) elements:
   See Fact 3.5.2 (page 65) with the \( X \) there with \( 4 \) elements, for all such \( X, Y, Z \) separately, so all triads in \( P \) are preserved.
(2) all \( \langle X \mid Y \mid Z \rangle \) with 1 element: -
(3) all \(X \parallel Z\):
This can be done by considering \(\Sigma_j := \{0_c, 1_c\}\). Then, say for \(a, c\), we have to examine the fragments 00 and 11, but there is no 10 or 01. For \(\langle a \mid b \mid c \rangle\) this is no problem, as we have only the two 000, 111, which do not agree on \(b\).

(4) all \(\langle X \mid Y \mid Z \rangle\) with 2 elements: eliminated by \(\langle X \parallel Z \rangle\)

(5) all \(\langle X \mid Y \mid Z \rangle\) with 3 elements:

(5.1) \(a\) is not on the outside

(5.1.1) \(a\) is in the middle, we need \(\neg xay\):
Consider \(\Sigma\) with 2 functions, \(0_c\), and the second defined by \(a = 0\), and all \(u = 1\) for \(u \neq a\). Obviously, \(\neg xay\). Recall that all tripels to be preserved have \(a\) on the outside, and some other element \(x\) in the middle. Then the two functions are different on \(x\).

(5.1.2) \(a\) is not in \(xyz\), we need \(\neg xyz\):
Consider \(\Sigma\) with 2 functions, \(0_c\), and the second defined by \(a = y = 0\), all \(u = 1\) for \(u \neq a, u \neq y\). As \(a\) is neither \(x\) nor \(z\), \(\neg xyz\). If some \(uvw\) has \(a\) on the outside, say \(u = a\), then both functions are 000 or 0vw on this tripel, so \(uvw\) holds.

(5.2) \(a\) is on the outside, we destroy \(ayz\):

(5.2.1) Case \(d_{\min}(ayz) > 0\):
Take as \(\Sigma\) the set of all functions with values in \(\{0, 1\}\), but eliminate those with \(a = y = z = 0\). Then \(\neg ayz\) (we have 100, 001, 101, but not 000), but for all \(auv\) with \(d(\{a, y, z\}, \{a, u, v\}) > 0\) \(auv\) has all possible combinations, as all combinations for \(ay\) and \(az\) exist.

(5.2.2) Case \(d_{\min}(ayz) = 0\):
The elements with \(d_{\min} = 0\) are:
\(ab_1b_n, abyb_1, \ldots, ab_ibi-1, NOT ab_i+1b_i, ab_i+2b_{i+1}, \ldots, ab_{i-1}b_{n-2}, ab_nb_{n-1}\), they were marked with (0) above.
\(\Sigma\) will again have 2 functions, the first is always \(0_c\).
The second function: Always set \(a = 1\).
We see that the tripels with \(d_{\min} = 0\) to be destroyed have the form \(ayz\), where \(z\) is the immediate \(\prec\)-predecessor of \(y\) in above order - see Definition 3.5.3 (page 36).
Conversely, those to be preserved (in \(P\)) have the form \(azy\), where again \(z\) is the immediate \(\prec\)-predecessor of \(y\).
We set \(z' = 1\) for all \(z' \leq z\), and \(y' = 0\) for all \(y' \geq y\). Recall that \(z < y\), so we have the picture \(b_{i+1} = 1, \ldots, z = 1, y = 0, \ldots, b_1 = 0\).
Then \(\neg ayz\), as we have the fragments 000, 101. But \(azy\), as we have the fragments 000, 110. Moreover, considering the successors of the sequence, we give the values 11, or 10, or 00. This results in the function fragments for \(auv\) as 111, or 110, or 100. But the resulting fragment sets (together with \(0_c\)) are then: \(\{000, 111\}, \{000, 110\}, \{000, 100\}\). They all make \(auv\) true. Thus, all tripels in \(P\) are preserved.
3.6 Systematic construction of new rules

This section is an outline - not a formal proof - for constructing a complete rule set for our scenario. We give here a general way how to construct new rules of the type ABC, DEF, ... ⇒ XYZ which are valid in our situation.

3.6.1 Consequences of a single tripel

Let \((XX'X'')Y(ZZ'Z'')\) be a tripel, then all consequences of this single tripel have the form \(X(X'Y'Z')Z\) (up to symmetry).

Obviously, such \(X(X'Y'Z')Z\) are consequences, using rules (b) and (c).

We now give counterexamples to other forms, to show that they are not consequences in our setting. We always assume that the outside is not \(\emptyset\). We consider \(A = B = C = \{0, 1\}\), and subsets of \(A \times B \times C\).

1. \(Y\) decreases:
   Consider \(\{000, 111\}\), then ABC, but not \(A\emptyset C\).

2. \(Z\) increases:
   Consider \(\{000, 101\}\), then \(A\emptyset B\), but not \(A\emptyset (BC)\).

3. \(X\) goes from left to right:
   Consider \(\{000, 110\}\), then \((AB)C\), but not \(A(BC)\)

4. \(Y\) increases by some arbitrary \(W\):
   Consider \(\{000, 101, 110, 011\}\), then \(A\emptyset C\), but not ABC.

3.6.2 Construction of function trees

We can construct new functions from two old functions using tripels ABC, so, in a more general way, we have a binary function construction tree, where the old functions are the leaves, and the new function is the root. The form of such a tree is obvious, the tripels used are either directly given, or consequences of such tripels. In Example 3.6.3 (page 49), for instance, in the construction of \(\rho_2\), we used ACD, but we could also have used e.g. \(AC(DD')\), for some \(D'\).

3.6.3 Derivation trees

Not all such function construction trees are proof trees for a rule \(T_1, \ldots, T_n \Rightarrow T\), where the \(T_i\) and \(T\) are tripels.

We have to look at the logical structure of the tripels to see what we need. In order to show \(T = ABC\), we assume given two arbitrary functions \(\sigma\) and \(\tau\), which agree on \(B\), and construct \(\rho\) such that on \(A\) \(\rho = \sigma\), on \(B\) \(\rho = \sigma = \tau\) (the latter, \(\sigma = \tau\) by prerequisite), and on \(C\) \(\rho = \tau\). We will write this as \(A: \rho = \sigma, B: \rho = \sigma = \tau, C: \rho = \tau\).
Thus, we have no functions at the beginning, except $\sigma$ and $\tau$, so all leaves in a proof tree for $T_1, \ldots, T_n \Rightarrow T$ have to be $\sigma$ or $\tau$. Moreover, all we know about $\sigma$ and $\tau$ is that they agree on $B$. Thus, we can only use some $T' = A'B'C'$ on $\sigma$ and $\tau$ if $B' \subseteq B$. Likewise, in the interior of the tree, we can only use $\sigma | B = \tau | B$, and, of course, all equalities which hold be construction. E.g., in Example 3.6.3 (page 49), in the construction of $\rho_2$, by construction of $\rho_1$, $C : \rho_1 = \tau$, so we can use $ACD$ to construct $\rho_2$ from $\rho_1$ and $\tau$.

3.6.4 Universal trees

3.6.4.1 A proof for XYZ

The following is a universal proof for XYZ:

It is a binary tree, whose leaves are all $f$ or $g$.

It uses as prerequisite only $Y : f = g$ (and equalities constructed on the way).

It makes $f, g$, and all other functions as different as possible.

For instance, in Example 3.6.3 (page 49), where we show that $ABC, ACD, ADE, AEB \Rightarrow ABE$, let us assume all sets $A$, etc. are singletons, we then set: $\sigma = 00000$, $\tau = 10111$, $\rho_1 = 00122$, $\rho_2 = 03113$, $\rho_3 = 04411$, $\rho_4 = 00551$. So each new node has a new default value ($2, 3, 4, 5$ here).

Then we have no chance equalities, but only those we constructed. In particular, if we write the equalities with $\sigma, \tau$ for every $\rho_i$ thus constructed, we can read off the derived equalities. There are no others.

The root of the tree must be a function $h$, which agrees on $X, Y$ with $f$, and on $Y, Z$ with $g$.

This is a universal proof tree, as it works for any other pair $f, g$, and any other internally constructed $\rho_i$, too.

3.6.4.2 Requirements for a proof for XYZ

Suppose we have a proof for XYZ.

We cannot assume we have anything but $f, g$ to start with.

The proof must be a binary tree, as the proof will be constructive, and we have no other construction principles but the combination of 2 functions.

So it is a binary tree, with leaves $f, g$.

It must also work for $f, g$ maximally different, i.e. outside $Y$, they may be different. It must also work for the internal functions $\rho_i$ maximally different. So we can only assume that $f, g$ agree on $Y$, and all other equalities must be by construction. Thus, it must also work for the universal choice
as done above. Assume now we have constructed this way $h$ such that $h = f$ on $X, Y$, $h = g$ on $Y, Z$. This cannot be by coincidence, but it has to be a new function, constructed by the tree.

3.6.4.3 Summary: proofs for XYZ

To show XYZ, construct all universal trees for XYZ:

Begin with $f, g$ which agree at most on $Y$, make them different everywhere else.

Make all internal nodes different from each other by enumerating them, and giving their number as default values to all other arguments.

Check if the root can be seen as the construction of a $h$ s.t. $h = f$ on $XY$, $h = g$ on $YZ$.

If so, we have a proof of XYZ.

All proofs of XYZ have this form, as they must work for the universal tree.
3.6.5 Examples
Diagram 3.6.1

Example 3.6.1

Example 3.6.2 and 3.6.4
Example 3.6.3

\[ \begin{align*}
\rho_1 & \quad \tau \quad \rho_1, \text{ using } ABC - A: \sigma, B: \sigma = \tau, C: \tau \\
\rho_2 & \quad \tau \quad \rho_2, \text{ using } ACD - A: \rho_1 = \sigma, C: \rho_1 = \tau, D: \tau \\
\rho_3 & \quad \tau \quad \rho_3, \text{ using } ADE - A: \rho_2 = \sigma, D: \rho_2 = \tau, E: \tau \\
\rho_4 & \quad \rho_4, \text{ using } AEB - A: \rho_3 = \sigma, E: \rho_3 = \tau, B: \sigma = \tau \\
\end{align*} \]

Interpretation: ABE, common part B: \( \sigma = \tau \)
Explanation:
By “prerequisite” of $\rho_i$ we mean the set $X$ we used in the construction, where $X : \sigma = \tau$. For instance, in the construction of $\rho_2$ in Example 3.6.1 (page 48), we used only that $B \cup C : \rho_1 = \tau$ by the construction of $\rho_1$, no additional use of some $\sigma = \tau$ was made.

By “common part” of $\rho_i$ we mean the set $X$ such that $X : \rho_i = \sigma = \tau$.

Example 3.6.1
(Contraction), ABC, $A(BC)D \rightarrow AB(CD)$:
(See Diagram 3.6.1 (page 48) upper part.)

- $\rho_1 : A : \sigma, B : \sigma = \tau, C : \tau$
  generated by $ABC$ from $\sigma, \tau$
  prerequisite $B$,
  common part: $B$
  $\rho_1$ can be interpreted as the (trivial) derived tripel $ABC$
- $\rho_2 : A : \rho_1 = \sigma, B : \rho_1 = \sigma = \tau, C : \rho_1 = \tau, D : \tau$
  generated by $A(BC)D$ from $\rho_1, \tau$
  prerequisite -, 
  common part: $B$.
  $\rho_2$ can be interpreted as a derived tripel by $AB(CD)$.
  $\rho_2$ can also be interpreted as a derived tripel by $A(BC)D$ or $A(BD)C$. Note that these possibilities can be derived from $AB(CD)$ by rule (c), Weak Union.

Example 3.6.2
(Bin1), XYZ, $XY'Z, Y(XZ)Y' \Rightarrow X(YY')Z$:
(See Diagram 3.6.1 (page 48) lower part.)

- $\rho_1 : X : \sigma, Y : \sigma = \tau, Z : \tau$
  generated by $XYZ$ from $\sigma, \tau$
  prerequisite $Y$
  common part: $Y$
- $\rho_2 : X : \sigma, Y' : \sigma = \tau, Z : \tau$
  generated by $XY'Z$ from $\sigma, \tau$
  prerequisite $Y'$
  common part: $Y'$
- $\rho_3 : Y : \rho_1 = \sigma = \tau, X : \rho_1 = \rho_2 = \sigma, Z : \rho_1 = \rho_2 = \tau, Y' : \rho_2 = \sigma = \tau$
  generated by $Y(XZ)Y'$ from $\rho_1, \rho_2$
prerequisites -
common part: \( YY' \)
\( \rho_3 \) can be interpreted as a derived triple by \( X(YY')Z \).

**Example 3.6.3**

(Loop 1) ABC, ACD, ADE, AEB \( \Rightarrow \) ABE:

(See Diagram 3.6.2 (page 41.).)

- \( \rho_1 : A : \sigma, B : \sigma = \tau, C : \tau \)
generated by \( ABC \) from \( \sigma, \tau \)
prerequisite B
common part B

- \( \rho_2 : A : \rho_1 = \sigma, C : \rho_1 = \tau, D : \tau \)
generated by \( ACD \) from \( \rho_1, \tau \)
prerequisite -
common part -
\( \rho_2 \) cannot be interpreted as a derived triple, as there was a prerequisite used in its derivation (B), but the common part in \( \rho_2 \) is \( \emptyset \).

- \( \rho_3 \) similar to \( \rho_2 \):
\( \rho_3 : A : \rho_2 = \sigma, D : \rho_2 = \tau, E : \tau \)
generated by \( ADE \) from \( \rho_2, \tau \)
prerequisite -
common part -
\( \rho_3 \) cannot be interpreted as a derived triple, as there was a prerequisite used in its derivation (B), but the common part in \( \rho_3 \) is \( \emptyset \).

- \( \rho_4 : A : \rho_3 = \sigma, E : \rho_3 = \tau, B : \sigma = \tau \)
generated by \( AEB \) from \( \rho_3, \tau \)
prerequisites -
common part B
\( \rho_4 \) can be interpreted as the common part \( B \) contains all prerequisites used in its derivation. \( ABE \) is the only non-trivial derived triple.

Note that we could, e.g., also have replaced ACD by \( AC'(DC'') \), where \( C = C' \cup C'' \), using rule (c), Weak Union.

**Example 3.6.4**

\( BA(CD), DF(CE), (AB)(CD)(EF) \Rightarrow B(ADF)(CE) \):

(See Diagram 3.6.1 (page 41) lower part.)

This example shows that we may need an assumption in the interior of the tree (in the construction of \( \rho_3 \), we use \( D : \sigma = \tau \)).
\( \rho_1 : A : \sigma = \tau, B : \sigma, C : \tau, D : \tau \)
generated by \( BA(CD) \) from \( \sigma, \tau \)
prerequisites \( A \)
common part \( A \)

\( \rho_2 : C : \tau, D : \sigma, E : \tau, F : \sigma = \tau \)
generated by \( DF(CE) \) from \( \sigma, \tau \)
prerequisite \( F \)
common part \( F \)

\( \rho_3 : A : \rho_1 = \sigma = \tau, B : \rho_3 = \sigma, C : \rho_1 = \rho_2 = \sigma = \tau, D : \rho_1 = \rho_2 = \sigma = \tau, E : \rho_2 = \tau, F : \rho_2 = \sigma = \tau \)
generated by \( (AB)(CD)(EF) \) from \( \rho_1, \rho_2 \)
prerequisite \( D \)
common part \( ADF \)

So \( \rho_3 \) can be seen as the derived tripel \( B(ADF)(CE) \) (but NOT as \( (AB)(DF)(CE) \) etc., as \( DF \) does not contain \( ADF \).)

**Example 3.6.5**

\( (AA')BC, AD(CD'), (AB')C(C'D), (A'B'C)(C'D'), (AD)(B'C'C')(A'D'), BC(ADD') \Rightarrow A(BD)(CD') \):

(See Diagram 3.6.3 (page 45).)

This example shows that we may need an equality (here \( \alpha \) and \( \beta \) in the construction of \( \rho_5 \)) which is not related to \( \sigma \) and \( \tau \). Of course, we cannot use it as an assumption, but we know the equality by construction.

\( \alpha \) and \( \beta \) will not be known, they are fixed, unknown fragments.

\( \rho_1 : A : \sigma, A' : \sigma, B : \sigma = \tau, B' : \alpha, C : \tau \)
generated by \( (AA')BC \) from \( \sigma, \tau \)
prerequisites \( B \)
common part \( B \)

\( \rho_2 : A : \sigma, C : \tau, C' : \beta, D : \sigma = \tau, D' : \tau \)
generated by \( AD(CD') \) from \( \sigma \) and \( \tau \)
prerequisite \( D \)
common part \( D \)

\( \rho_3 : A : \sigma, B' : \alpha, C : \tau, C' : \beta, D : \sigma = \tau \)
generated by \( (AB')C(C'D) \) from \( \rho_1 \) and \( \rho_2 \)
prerequisite -
common part \( D \)
• $\rho_4 : A' : \sigma, B' : \alpha, C : \tau, C' : \beta, D' : \tau$
  Generated by $(A'B')C(C'D')$ from $\rho_1$ and $\rho_2$
  prerequisites -
  common part -

• $\rho_5 : A : \sigma, A' : \sigma, B' : \alpha, C : \tau, C' : \beta, D : \tau, D' : \tau$
  generated by $(AD)(B'C'C')(A'D')$ from $\rho_3$ and $\rho_4$
  prerequisites - (note that equality on $B'$ and $C'$ is by construction of $\rho_3$ and $\rho_4$, and not by a prerequisite on $\sigma$ and $\tau$)
  common part: $D$

• $\rho_6 : A : \sigma, B : \sigma = \tau, C : \tau, D : \sigma = \tau, D' : \tau$
  generated by $BC(ADD')$ from $\rho_1$ and $\rho_5$
  prerequisites -
  common part: $BD$
  Thus, $\rho_6$ may be seen as derived tripel $A(BD)(CD')$
Chapter 4

Subideal cases

4.1 The problem and the outline of a solution
One of the advantages of defeasible inheritance systems is the ability to treat subideal cases.

In the left hand diagram, (see Diagram 4.1.1 (page 53)), $C$ inherits from $B$ $A$ and $A'$. This is the ideal case. In the right hand diagram, $C$ does not have property $A$, the direct link $C \not\rightarrow A$ prevents this, but it still inherits $A'$ from $B$, this is the subideal case.

When we interpret $A$ by “blond”, $A'$ by “tall”, $B$ by “Swede”, $C$ by a subset of “Swedes”, which are not blond, we have the classical dark haired Swedes problem. Even dark haired Swedes should be tall. Preferential structures have a problem with this, as they do not say anything about subideal cases (where not all properties which hold in the minimal models, are valid).

Inheritance systems are modular in the following sense: the conditions which are inherited are clearly and separately spelled out, $A$ and $A'$ here. In preferential structures, we have - in principle - one tight knot of ideal cases, and no way to separate the different properties - without additional machinery. It is this machinery we want to examine here.
In inheritance systems, in principle, all combinations are possible: $A \land A' \land B \land C$, $A \land A' \land B \land \neg C$, 
$\ldots$, $\neg A \land \neg A' \land \neg B \land \neg C$. We might not mention all, but there is no contradiction to add nodes and arrows to make them visible. E.g., we can introduce $D$ to one of the diagrams, with the arrows $D \rightarrow A$, $D \not\rightarrow A'$, $D \rightarrow B$, $D \not\rightarrow C$, etc. So, the nodes code implicitly logically independent possibilities, and we use this idea for preferential structures.

Suppose we have a language $p, q, r$, and a preferential structure where $\text{True} \models p \land q \land r$. Intuitively, we want to “decompose” this into 3 rules: prefer $p$ over $\neg p$, $q$ over $\neg q$, $r$ over $\neg r$. Note that we can describe $\mu(\text{True})$ by $p \land q \land r$, but also by the conjunction of 7 rules, excluding all other models one by one: $\neg(pq\neg r)$, etc. But these rules are not independent: There are cases with $\neg(pq\neg r) \land \neg(p\neg qr)$, but there is no case with $(pq\neg r) \land (p\neg qr)$.

So, the solutions seems to be, roughly: Find the finest (this exists, see Fact 3.4 in [GS09b]) independent factorization $f_1, \ldots, f_n$ describing $\mu(X)$, and for $X' \subseteq X$ with $X' \cap \mu(X) = \emptyset$ (when $X' \cap \mu(X) \neq \emptyset$, preferential structures take care of this), apply as many of the $f_i$ to $X'$ as possible. The “as many” should probably be determined by the subset relation, and not by counting, as it is not sure that we are prepared to compensate the failure of one $f_i$ by the validity of another $f_{i'}$. So, we “know” how to inherit properties to subideal cases in preferential structures.

Another basic idea of inheritance systems is specificity: Conflicts are, if possible, solved by specificity. Tweety the penguin inherits egg-laying from birds, but not-flying from penguins, and not flying from birds, as penguins are more specific than birds. We have to carry this over to our approach to preferential structures. The general situation is as follows: We have a set $X$, and inherit from $Y_1, \ldots, Y_n$ factors $f_{i,1}, \ldots, f_{i,m_1}, \ldots, f_{n,1}, \ldots, f_{n,m_n}$, where the $f_{i,j}$ are the factors of $\mu(Y_i)$. The $Y_i$ are partially ordered, and it seems natural to do some “merger” of the $f_{i,j}$, respecting priority determined by specificity. It is probably adequate to take an “axiom based” approach, taking a suitable subset of the $f_{i,j}$, as formalisms coming up with some compromise (e.g. determined by some distance between models) are not only different from the inheritance formalism, but will probably give unexpected results.

The situation is more complicated than in inheritance, as the $f_{i,j}$ need not be independent when considering different $i$’s. Some approach like the following is probably reasonable:

1. Consider the strongest $Y_i$, ordered by specificity, and their $f_{i,j}$.
2. Consider all $f_{i,j}$ for those $Y_i$. Identify minimal inconsistent sets of those $f_{i,j}$, and erase all $f_{i,j}$ involved (this corresponds to direct scepticism in inheritance), until a consistent set of $f_{i,j}$ is obtained.
3. Consider the next strongest $Y_i$, and add similar to step (2) new $f_{i',j}$, while preserving the $f_{i,j}$ already chosen, and considering consistency together with the $f_{i,j}$ already chosen.
4. Etc., until all $Y_i$ are done with.

### 4.2 Comments

1. We have here essentially a multi-valued approach. Not only classical validity as maximally strong, and preferential structure as next strongest, but, partially ordered by specificity, arbitrarily many levels of strength.
2. Note that preferential structures take care automatically of specificity for the ideal case,
basically, as we can handle all sets independently. Here, we have to add a formalism to handle specificity.

(3) Higher preferential structures, see [GS08], can code our approach, but it is not sure that the coding would be natural.

(4) Independence as discussed above might need to be refined. For instance, we might consider independence inside $X$ when considering $\mu(X)$.

(5) We can also ask whether we should not perhaps consider independence of $X - \mu(X)$, instead of independence of $\mu(X)$. The following example gives an answer:

**Example 4.2.1**

Consider the language $\{p, q\}$.

(1) Let $\mu(pq, \neg pq, p\neg q, \neg p\neg q) = pq$. We have two rules, $p < \neg p$, $q < \neg q$, and apply both.

(2) $\mu(pq, \neg pq, p\neg q, \neg p\neg q) = (pq, \neg pq, p\neg q)$, we avoid $\neg p\neg q$. But we do not avoid $\neg p$ and $\neg q$, the rule is to avoid one of them. This does not seem to be such a good rule. In particular, factorization as above does not work, contrary to the symmetric case (1).

(6) Note that we can see the factorization of $\mu(X)$ as an approximation of the ideal case $\mu(X)$ by a set of rules.
Chapter 5

Coding graphs by multisets

5.1 Introduction

This is a short comment on [AGS09].

We examine here the coding of graphs by sets and multisets.

In the following, we abbreviate a set of labels or elements like \{a, b, c\} by abc, etc.

5.2 Even the case with simple (not multi) sets is quite complicated

We consider here graphs generated by a subset of some powerset (with the natural ordering by inclusion), and show that we need a certain number of atomic labels to represent them.

The examples show that it is probably quite difficult to come up with a minimal number of elements - let alone working with multisets. Example 5.2.1 (page 57) shows how complicated things can become. There is an interplay of chains up and down, and antichains involved.

Thus, I am quite sceptical about a good solution to the problem.

Moreover, Example 5.3.1 (page 58) shows that an inductive construction is impossible.

Example 5.2.1

Consider the graph generated by the subset \{abcde.f, abcd, abc, ab, a, bcf, ef, e, f\} of \mathcal{P}(abcdef).

(A label of the node corresponding to) abcd has to have at least 4 elements, by a \prec ab \prec abc \prec abcd, and e, f have to have at least 4 elements less than the top node abcd.f.

All maximal antichains have 3 elements, e.g., \{abcd, e, f\}. The longest chains have 5 elements.
But we cannot use only 5 elements for labels, as any antichain containing a node with 4 elements can have size at most 2.

5.3 There is no inductive algorithm by the natural ordering for the simple set case

Example 5.3.1

This example shows that, in general, an inductive procedure is impossible.

Recall that, for a given set of \( n \) elements, the number of subsets of size \( m < n \) is \( \frac{n!}{(n-m)!m!} \).

Take now a structure consisting of one antichain with 20 elements, and nothing else. This can be represented with 6 elements and subsets of size 3, as \( \frac{6!}{3!3!} = 20 \).

Take a structure with 4 antichains, each of size 20, and one above the other. Thus, the size of the representing sets will increase at least by 1 from the lowest antichain to the next, etc.

As we are not allowed to look ahead, we begin again with subsets of size 3 of a set of 6 atomic labels for the lowest antichain. So the next antichain must consist of sets of at least 4 elements, the next of 5, the final of 6. Thus, we need at least 8 elements for representation, as \( \frac{8!}{6!2!} = 28 \), 7 elements will not do.

If, however, we had begun with subsets of size 2 of a 7 element set, by \( \frac{7!}{5!2!} = 21 \), we could have in the top layer only 5 element subsets, and this is again possible, so 7 elements will do. But we have to look at the whole structure to see this.
5.4 The multiset case

We work with a set of atomic labels $L := \{a\} \cup B$, where $a$ may occur several times, this will be written $a^n$ for $n$ times $a$, etc.

We have the following trivial fact:

Fact 5.4.1

(1) Let $B' \subseteq B$, then $a^n B'$ and $a^m B'$ are comparabel, as $n \leq m$ or $m \leq n$.

(2) Let $B' \subset B'' \subseteq B$, then $a^n B'$ is not comparabel to $a^m B''$ iff $n > m$.

Corollary 5.4.2

(1) To code an antichain of size $2^n$, we need $B$ of size at least $n$.

(2) We can code an antichain of size $2^n$ with $B$ of size $n$.

Proof

(1) Suppose $B$ is smaller, then $card(\mathcal{P}(B)) < 2^n$, so two elements of the antichain are coded by the same $B' \subseteq B$, contradicting Fact 5.4.1 (page 59), (1).

(2) Let $(B')$ be the exponent of the (unique by Fact 5.4.1 (page 59), (1)) $a^{(B')} B'$. Code the elements of the antichain by $\{a^{(B')} B' : B' \subseteq B\}$, where $B' \subset B''$ implies $(B'') < (B')$. Then the codes are pairwise incomparable by Fact 5.4.1 (page 59), (2). Note that $(\emptyset)$ is the biggest exponent, and $(B)$ the smallest. (The idea is that, if the $B$-part of two codes is comparabel, we make the $a$-part comparabel in the other direction, so the whole codes are incomparabel.)

Example 5.4.1

Consider the structure $X \prec Y$ and an isolated $Z$.

Obviously, we need at least one $b$. We may code this by $a < a^2$, $b$, or, by $b < ab$, $a^2$, and we have two, non-isomorphic, codings.

In the following, we will code a bottom antichain of size $2^n$ by $\{a^{(B')} B' : B' \subseteq B\}$, where $B$ has $n$ elements.

5.5 There is no inductive algorithm by the natural ordering for the multiset case

We now show that an upward inductive algorithm, using the natural ordering, is impossible. For this, we discuss progressively more complicated examples. The last one, Example 5.5.3 (page 61), is perhaps the most interesting, as it shows that we have to consider an arbitrarily deep and wide substructure (with non-trivial interior nodes), to see that a decision taken lower down cannot be upheld.
Example 5.5.1

Consider an antichain of 4 elements at the bottom, say $A, B, C, D$.

We might code this with the labels $abc, a^2b, a^2c, a^3$.

In the next level, we have an antichain of 2 elements, say $X, Y$, and they have both the same predecessors, say $A, B$.

Suppose $A$ was coded by $a^3$, $B$ by $a^2b$. Then we can code $X$ by $a^3b$, $Y$ by $a^4bc$, and need no new label.

Suppose now that $A$ was coded by $abc$, $B$ by $a^2b$. Then $X$ and $Y$ must include $a^2bc$, we may for instance make $X$ $a^4bc$, but now we have to introduce a new variable, say $d$, and make $Y$ $a^3bcd$.

So, we have to look ahead. But it can be much more complicated. Take again above example.

Suppose we have now two antichains, $X, Y$, and $X', Y'$, one is above $A, B$, the other above $C, D$. Which one will have the $abc$? If $X, Y$ is higher than $X', Y'$, then we might have needed already $d$ elsewhere, so we can use it without additional cost. But $X$ might also be higher than $X', Y'$ higher than $Y$. What shall we do?

Example 5.5.2

This example shows that even the initial step of coding 4 elements with 3 labels, as done above, might not always work: again, we have to look ahead.

Consider again an antichain of 4 elements at the bottom, say $A, B, C, D$. Again, we might code this with the labels $abc, a^2b, a^2c, a^3$.

Suppose we have in the second layer one new point above each pair from $A, B, C, D$. One of the bottom nodes will be coded by $a^2bc$, another by $a^3b$, another by $a^3c$. Suppose $n' \geq n''$. Let $X$ be above the bottom elements coded by $a^2b$ and $a^2bc$. Then it will also be above the bottom element coded by $a^n c$. But this is not wanted.

Thus, in this situation, we need a new label, say $d$, to code the element coded by $a^2bc$.

(We could also put the second layer nodes $X$ on "stilts", so they will have arbitrary height, like $a^n b \prec a^{n+1} b \prec a^{n+2} b \prec \ldots \prec X$, etc., so we have to climb up arbitrarily high to see the problem.)

Fact 5.5.1

Consider a bottom antichain with elements $a(B')B'$, where $B' \subseteq B$.

Fix now $D \subseteq B$, let $D' := B - D$, and consider $X := \{a(D'E)D'E : E \subseteq D\}$. Then, of course, $(D') = (D'\emptyset) > (D'E)$ for all $E \neq \emptyset$.

Let $X = X' \cup X''$, where $X', X''$ are disjoint and have the same cardinality, and introduce two new nodes, $B'$ and $B''$, such that $B' \supset A'$ for all $A' \in X'$, $B'' \supset A''$ for all $A'' \in X''$ but for no $A'' \in X'' B' \supset A'$.

Suppose without loss of generality $a(D')D' \in X'$ Then there is $b \in D$ such that for no $a(D'E)D'E \in X', b \in E$. (Otherwise, by maximality of $(D')$, all $x \in X$ would be below $A'$.)

On the other hand, for cardinality reasons, there cannot be two such $b \in D$. □
Example 5.5.3

Using \( n+1 \) atomic labels, \( L := \{a, b_0, \ldots, b_{n-1}\} \), we can code a bottom antichain \( X_{0,0}, \ldots, X_{0,2^n-1} \) as follows: Work in the binary system. Set \( B := \{b_0, \ldots, b_{n-1}\} \), and code \( B' \subseteq B \) by \( c(B') := \sum \{2^i : b_i \in B'\} \). This gives a natural total order on \( \mathcal{P}(B) \), and we use the inverse of this order for the exponent of \( a \). Thus, as it should be, \( (\emptyset) = (0, \ldots, 0) \) is the biggest exponent, and \( (B) = (1, \ldots, 1) \) the smallest exponent.

In more detail, code \( X_{0,i} \) by \( a^{(i)}i \), where \( i \) is written in binary, \( i \) coding as above a subset of \( B \). Thus, \( X_{0,0} \) is coded by \( a^{(0)}0 \), \( X_{0,1} \) by \( a^{(0\ldots1)}0 \ldots 1 = a^{(b_0)}b_0 \), \( X_{0,2} \) by \( a^{(0\ldots10)}0 \ldots 10 = a^{(b_1)}b_1 \), etc., up to \( X_{0,2^n-1} = a^{(b_{n-1}\ldots b_0)}b_{n-1}\ldots b_0 \).

Then create new nodes above the bottom level, etc., always grouping successive lower nodes together, as follows:

\[
X_{1,0}, \ldots, X_{1,2^{n-1}-1} \\
X_{1,i} \succ X_{0,i+2}, X_{0,i+2+1} \\
X_{k,0}, \ldots, X_{k,2^{n-k}-1} \\
X_{k,i} \succ X_{k-1,i+2}, X_{k-1,i+2+1}
\]

up to \( k = n - 1 \) (included).

The labelling of the new nodes is made by taking the union of lower labels. Our ordering of the exponents shows that this is possible, with exactly the relations as defined. See Diagram 5.5.1 (page 61) for an example with \( n = 3 \).
For instance, \((d)\) is the highest exponent in the right half, but all exponents on the left half are bigger than \((d)\). Thus, all nodes on the right half are below \(a^{(d)bd}c\), and none on the left is below \(a^{(d)bcd}\). For \(a^{(0)bc}\), all nodes on the right half contain \(d\), so they are not below \(a^{(0)bc}\), etc. We add now two additional nodes, \(a^{(0)bd}\), and \(a^{(c)bcd}\). The latter will have more nodes below it than intended - see the broken line in the diagram. Consider first the node labelled \(a^{(0)bd}\). The nodes below \(a^{(cd)bcd}\) contain \(c\), so they are not concerned, the same holds for those below \(a^{(c)bc}\). But it is impossible to add the node \(a^{(c)bcd}\): By \((c) > (d) > (bd)\), we see that \(a^{(d)d} < a^{(c)bcd}\) and \(a^{(bd)bd} < a^{(c)bcd}\), a contradiction.

This is no accident, it does not depend on the specific choice and distribution of the base labels, as we show now. "(labelled \ldots)" refers to the example for \(n = 3\), described in Diagram 5.5.1 (page 61).

Consider, for an arbitrary labelling, \(X_{n-1,0}\) (labelled by \(a^{(0)bc}\) and \(X_{n-1,1}\) (labelled by \(a^{(d)bcd}\)) (these are all which are on level \(n\)). One of them has to be above \(a^{(0)}\), without loss of generality, let this be \(X_{n-1,0}\). Note that \((\emptyset)\) has to be the strictly biggest exponent, otherwise we have no antichain. One of the atomic labels, say \(b_j\) (\(d\) in the diagram) does not occur in the labelling of \(X_{n-1,0}\), otherwise, all bottom nodes would be below \(X_{n-1,0}\). For cardinality reasons, all others have to occur in the labelling of \(X_{n-1,0}\), see Fact 5.5.1 (page 60). Moreover, \(b_j\) occurs in all labels of the bottom nodes below \(X_{n-1,1}\), and all combinations of the other \(b_k\) occur below \(X_{n-1,1}\). In particular, we have \(a^{(b_j)}b_j\) and \(a^{(b_{n-1}...b_0)}b_{n-1}...b_0\) below \(X_{n-1,1}\), and, by the same reasoning, \((b_j)\) is the strictly biggest exponent below \(X_{n-1,1}\).

We split now \(X_{n-1,0}\) into \(X_{n-2,0}\) (labelled \(a^{(0)b}\)) and \(X_{n-2,1}\) (labelled \(a^{(c)bc}\)) and repeat the argument, using again Fact 5.5.1 (page 60).

Suppose, without loss of generality, \(a^{(0)}\emptyset\) is below \(X_{n-2,0}\), so there must be some \(a^{(b_j)}b_j\) (labelled \(a^{(c)c}\)) below \(X_{n-2,1}\). As \(a^{(b_j)}b_j\) is not below \(X_{n-1,1}\), \((b_j) > (b_j)\). Split now \(X_{n-1,1}\) into \(X_{n-2,2}\) (labelled \(a^{(d)bd}\)) and \(X_{n-2,3}\) (labelled \(a^{(d)bcd}\)), and suppose without loss of generality \(a^{(b_j)}b_j\) is below \(X_{n-2,2}\). Create a new node \(X\) (labelled \(a^{(c)bcd}\)) above \(X_{n-2,1}\) and \(X_{n-2,3}\). Then it is bigger than \(a^{(b_j)}b_j\), so its label has the exponent \((b_j)\), but it is also above \(a^{(b_{n-1}...b_0)}b_{n-1}...b_0\) (labelled \(a^{(bcd)bcd}\)), so it is also above \(a^{(b_j)}b_j\), a contradiction by \((b_j) > (b_j)\). But we detect this only at level \(n - 2\), and we have to look at arbitrarily big subsets of the construction (in width and depth!) to find a contradiction. Thus, in a strong sense, a recursion is impossible.

Note that we may modify above example, e.g., introduce a smallest node with label \(\emptyset\), and then lift the whole construction by adding everywhere a new set of labels, so we can embed it into an arbitrary diagram. Thus, the problem is not only with the base level.
5.6 Generalization

We identify the different situations or objects (cameras, etc.) with propositional models, and the properties with propositional variables. The models may be defined only partially.

To distinguish different models, we name them. Thus, we might have different models with the same properties, but with different names. We assume that all values can only be 0/1 (the bull example needs more values).

I think there are different ways to treat the situation:

1. We have only a local ranking, which is based on the values of the propositional variables. Based on this ranking, we try to complete the partially defined models. Gaps are permitted (undefined values), if there is a gap, we just forget this value for the ranking. If \( m \prec m' \prec m'' \), and \( m'(p) \) is undefined, then we try to complete it, so that \( m(p) \leq m'(p) \leq m''(p) \).

2. We have, in addition, a global ranking, where model \( m \) may be considered better than model \( m' \), for some external reason.

   In this case, we try to complete the undefined values according to local and global ranking.

3. We have, in addition, a ranking of the propositional variables, where \( p \) might be stronger than \( p' \), etc. In this case, we can work within one model, e.g., as follows: If \( m(p) \) is “positive”, and \( m(p') \) unknown, then we assume that \( m(p') \) is positive, too.

We then see the following:

1. We have a structure on the language, as 1 is better than 0. In the third case above, we have an order on the variables, too, so even more structure. See p. 10 of our new book.

2. We may have a “soft” ranking, where some properties might be unknown, then the known properties determine the ranking.

   In this case, we fill in the unknown properties to coincide with the soft ranking.

3. I do not see why it is necessary to have only one (?) in the matrix.

   In particular, we may sometimes split 1 big matrix with two (?) into 2 small matrices with 1 (?) each.

4. This way of ordering reminds me of the ordering in deontic logic, where situations may be better in several aspects.

5. It might be possible to generalize from elementary properties (propos. variables) to formulas.

6. The locality of reasoning makes it likely that we have interpolation - if we find a nice way to express it.

7. What are the laws of this reasoning? If we modify the matrices, what stays constant, what changes, and how?

8. If we admit, say, 2 holes, we can examine Cumulativity: Is the result the same, when we fill both at the same time, or, first 1, then with the new matrix, 2?
(9) I think we can see this as a special case of preferential structures: Replace the (?) with branching into 2 models, then prefer the one which fits in better.

(10) Vielleicht kann ich auch pref. Modelle wie oben als Matrix sehen, und dann geometrisch arbeiten?

(11) Mit Implikationen machen?

(12) aus Bahnfahrt:
- wieso nicht learning/detecting regularity?
- hat an force bei $a$ gedacht, nicht an min. labels, drum die vielen Fehler
- Ist das nicht detecting causality?
- detect order, tendency
- Ist Ansatz 0/1 einzusetzen, um zu sehen, was besser passt, gerechtfertigt? Koennte das nicht eine Tendenz verschleiern?

Dov, I have a few questions and remarks, which we might discuss on the phone:

(1) The problem differs from an interpolation problem, as, in the latter, the order is give, here it has to be found. Correct?

(2) Is finding regularities in one dimension (product, or model) really the same as finding them in the other dimension (properties)?

(3) I am not sure that the coding of “force” by $a^n$ is really what you want, and if the multiset approach is the right one. Do you have more on this?

(4) Detecting regularities is traditionally a learning problem, I think. Is there a reason why this is not mentioned? Perhaps, we should work with someone from the learning community?

(5) You examine which of the possibilities give a better fit, 0 or 1 in the place of?. Does this always correspond to finding regularities? This sounds like a stupid question, but I am not sure your answer is always true. If so, it might need a proof.

Karl
Chapter 6

Re-considering some principles of non-monotonic logics

6.1 Introduction

We try to take a fresh look at some fundamental ideas of non-monotonic logics. In particular, we

1. examine the step from “normally . . . ” to “normal”
2. differentiate the consistency criterion of Reiter defaults
3. look at the “inference greed” of Reiter defaults, and other formalisms like inheritance, and give it an intuitive semantics through tentative theory formation, and connect it to inductive reasoning
4. describe that specificity is not always a good criterion
5. suggest a more modular approach a la inheritance
6. examine subset systems more general than principal filters used in preferential structures
7. describe how to generalize from propositional to first order defaults
8. introduce a notion of validity of a default in a classical model, and describe how to use it to solve conflicts and determine “good” models
9. finally, take a closer look at inheritance and motivate the use of direct scepticism or of the intersection of extensions, and also re-consider the translation of inheritance to other systems by examining their language.
We stress those aspects which seem elementary, “first principles” to us, and try to translate pro-
cedural aspects into a more declarative content. The text is more questions and problems than
answers.

6.2 General remarks

6.2.1 Not all defaults are about normality

Medical students are told: “if you hear hoofbeat, think horses, not zebras”. The meaning is,
of course, first think of normal, usual situations, and not exotic illnesses. When we walk in
the country, and hear the hissing of a snake, the advice might be: “think rattle snakes, not
garter snakes”, though the latter might be more common. The reason is, to treat first potentially
dangerous situations.

Both describe default reasoning, but for different purposes (they can, however, both be summarized
as “useful” reasoning, the first to treat common situations, the second to avoid dangers). For the
moment, we treat both as advice for acting (reasoning), or rules, and will write (hoofbeat:horse)
and (hissing: rattler). They are justified by different reasons, we have, so far, no formal justification
or semantics, and no way to treat a system of such rules. But we are aware that the rules are
“rough”, it might be a zebra, it might be a garter snake, after all.

Note that the default rule we chose to apply may depend on the context. When we walk in
the countryside, we use the cautious snake rule, when we observe from a safe position, we may
use the rule that garter snakes are more common after all, so we conjecture it is a garter snake,
(hissing: garter-snake).

6.2.2 Systems of rules, subideal cases

We have many rules for birds, (birds: feathers), (birds: fly), (birds: lay-eggs), etc. When we write
down all rules about birds, it might be that no single bird satisfies all, the total set of rules for birds
behaves like the lottery paradox. We may also have a mixture of rules with different motivations.
In medical diagnosis, one rule might be to check for a common and not so serious illness, another
rule to exclude a rare, but dangerous and rapidly developing one. We will probably decide about
the latter first, then turn to the common illness, and if both are wrong, investigate further. Note
that we do not have here just “normal” and “abnormal” cases, but three classes - just as we
sometimes have three cases to consider for a mathematical proof.

6.3 Clarification of notions: Normality and consistency

6.3.1 Normality

6.3.1.1 From “normally” to “normal”

There is an important - but often overlooked, see the author’s own work - change from “normally,
birds fly” to “normal birds fly”. The latter presupposes that normal birds, the ideal bird case,
exist, the former does not, it considers also partially normal birds. The ideal case need not exist, as the lottery paradox shows. The intersection of the bird sets with “normal” properties might be empty - or meaninglessly small.

6.3.1.2 The behaviour of “normal” vs. finding normal elements

 Preferential structures and their abstract treatment are about the normal case. They investigate the properties of normality, of the ideal case. They do not investigate the subideal case, where only some properties of the ideal case are satisfied. This is done, implicitly, by Reiter defaults, defeasible inheritance, etc., where we preserve as many normal properties as possible. Preferential structures also do not investigate which elements (in the first order case) are normal, or as normal as possible. This is done by first order Reiter defaults, where as many elements as possible are made as normal as possible.

6.3.2 The consistency criterion for Reiter defaults (and other formalisms)

 A Reiter default is allowed to fire unless the consistency criterion is violated. But the inconsistency might be against a classical background theory, or against another default, or a combination of other defaults, etc. In particular, criteria like specificity might be important. Thus, a whole theory of elimination of inconsistencies may be necessary to solve conflicts - as it is brought to light in defeasible inheritance. In the first order case, which elements are normal, and to which degree, is also solved by an, implicitly, complicated theory.

 Note that preferential structures have total control of minimal elements, so there is no room for downward inheriting properties - unless we want to work with special structures - and potential conflicts are obvious.

6.4 The implicit extension of conjectures

6.4.1 Inference greed

 Reiter defaults (and, e.g., inheritance networks) are “inference greedy” in the following triple sense:

 (1) The default ( φ ) will “fire”, even if we know already ψ , ( φ ) is implicitly broken down to subsets - contrary to preferential structures, where we do not have this homogeneity.

 (2) In the default set { ( φ ), ( φ′ )}, if ( φ ) cannot fire (as ¬φ holds), ( φ′ ) may still be able to fire - in preferential structures, we know nothing beyond classical logic about not totally normal, ideal, elements, whereas defaults can also treat the subideal case.

 (3) Open defaults ( φ(x) ) make as many elements as possible normal, i.e. satisfy φ(x).
6.4.1.1 A justification

It seems difficult to find a semantics in the usual sense for this behaviour. Why should the world “feel” a pressure for normality? Why should there be a direction towards maximal possible normality in the world?

The only idea the present author had was to give an (informal) semantics of both the world and our theory building about the world. My, certainly naive, idea is in the platonic tradition. We make a theory about the world, knowing that it is only an approximation, but try to extend it as far as possible (until contradictions - to be elaborated, see above, Section 6.3.2 (page 69)). The basic assumption is that the world is regular, and we can, in principle, describe it in simple terms, but our description will not be perfect. It is an assumption about homogeneity of the world, and independence of properties, unless proven otherwise. (It is also an exploratory approach: we explore the world, and try to be conservative, in the sense of simplicity. As such, it has much in common with inductive reasoning.)

Thus, we have a pragmatic view, make as many defaults hold as possible, also for subsets, and for as many elements as possible in the first order case. We do not seek “best” knowledge, about absolutely normal cases, but, more modestly, distinguish between levels of knowledge, probabilities, like truth values in inheritance networks. This can then be formalized by a simple relation of “better” between models and elements, forgetting the human element of extending knowledge.

6.4.2 Remarks on specificity

The specificity criterion for deciding conflicts is one of the basic tenets of non-monotonic reasoning. If Tweety is a penguin, we conclude that the more specific information, that penguins don’t fly, will win over the more general information that birds fly. If there is no conflict, we assume that subsets behave like supersets - see above.

The specificity criterion is fine for classification, as we assume that many properties will be inherited from super- to subclass, but not all. Subclasses may have a somewhat modified “building plan”. But specificity is irrelevant for other properties - for example for “destructive” properties. We will not try to find out if dead penguins can still walk, once we understood that dead animals cannot walk. Something in the “construction” of the animal has gone wrong, and we do not assume normal life functioning any more. Thus, we have to distinguish properties which “feel” specificity, and those which do not. (Likewise, we will not investigate how the dead specimens of a newly discovered bird behave - we know it already, it is a “transverse” property, and no inductive reasoning is necessary.)

This distinction goes beyond classical logic, as we distinguish different types of predivates (or propositional variables, in the propositional case).

“Penguin” is not a capacity like flying, but a complex of properties. Similarly, we do diagnosis, e.g., for an illness, with distinctive properties, which serve as indicators.

Note that specificity can be seen as an approximation: a more specific set $B$ is a better approximation to $A$ than less specific set $C : A \subseteq B \subseteq C$. But we do not really work with specificity as a set-wise relation: Tweety, a kolibri, a blackbird, is a small set, but it seems useless. We need “well defined” small sets, like penguins, we need property-wise or class-wise (like penguin) approximation.
6.4.3 Induction

The justification for the inference greedy behaviour of defaults makes a connection to inductive logic plausible. Inductive reasoning is also inference greedy, we try to push our knowledge as far as possible. Of course, the reasoning goes upward, towards the more general case, and not downward to subsets. Still, one should explore further if there are common points. In particular:

(1) Is induction only inverse to the downward extension of knowledge of defaults, or are there deeper differences?

(2) Can we transfer results and rules from one domain to the other?

(3) Can we define inductive reasoning by the generalization which is best extended downward in default reasoning (or vice versa)? So one will be a reflection of the other?

(4) Can we learn from “real” science, how physicists, or researchers in life sciences, determine if a theory is thought to be sufficiently corroborated? What does “practical philosophy of science” say? How do they exclude “disturbing influences”? What does this mean for default reasoning? Can we reflect this to default reasoning?

(5) Can the degree of inconsistency of Section 6.8 (page 74) be generalized to induction?

6.5 Modularity

An attractive feature of inheritance systems is their modularity. Modularity corresponds also to the description of information as approximation. We have several “aims”, building blocks of a description, and put them together as well as possible, in a principled way, based on a basically modular world itself.

If we take this idea seriously, we have not one big language and theory, but small fragments of non-monotonic theories, and - non-monotonic - operators on those fragments, which combine them, similar to a revision of non-monotonic logics by non-monotonic logics. (Combining different languages is, e.g., a multiplication of models, etc.)

6.6 Subset systems beyond principal filters

Preferential structures (in the minimal version) generate principal filters on sets, \( \mathcal{F}(X) := \{ A : \mu(X) \subseteq A \subseteq X \} \), together with coherence properties between filters over different sets, \( X, X' \), etc. They have an intuitive interpretation by the notion of size. The minimal elements are the ideal cases, and everything non-minimal is negligible, or small.

The lottery paradox and the limit version of preferential structures motivate to consider more general filters, or even weak filters.

Default systems also generate subset systems. E.g., \{(: \phi),(: \psi)\} generate the “good” subsets \{m : m \models \phi\}, \{m : m \models \psi\}, \{m : m \models \phi \land \psi\}, and perhaps \{m : m \models \phi \lor \psi\}. Considering the default system \{(: \phi \lor \psi),(: \psi)\} shows that \{\{m : m \models \phi \lor \psi\}, \{m : m \models \psi\}\} and \{\{m : m \models \psi\}\} should not be considered equivalent. In the latter, only \{m : m \models \psi\} is “good”, in the former,
also \( \{ m : m \models \phi \lor \psi \} \) will be considered good, though not as good as \( \{ m : m \models \psi \} \). This is intuitive, as the default \((\psi)\) might not be able to fire, but the default \((\phi \lor \psi)\) may - the system \(\{(\phi \lor \psi), (\psi)\}\) is not equivalent to the system \(\{(\psi)\}\). If we interpret \(\psi\) as the ideal case, then both describe the same ideal case, or limit, but not the same subideal cases. (This is like contrary-to-duty conditionals.)

Let \(\mathcal{N}(X)\) denote such abstract systems.

The following questions arise about \(\mathcal{N}(X)\):

1. What are reasonable closure properties of \(\mathcal{N}(X)\)?
   - A first idea is to proceed as for deontic logic: Take all model sets derived from single defaults, and close under union and intersection.
   - If \(\bigcap \mathcal{N}(X) = \emptyset\), we should probably consider only non-empty intersections.
   - Should \(X \in \mathcal{N}(X)\)? Probably not.
   - Is a system like \(\{A, X - A\}\) reasonable? Are systems with \(\bigcup \mathcal{N}(X) = X\) reasonable?
   - Can different closure properties code different intuitions?
   - In which cases does \(\mathcal{N}(X)\) describe an approximation of ideal cases?

2. Can we compare two different \(\mathcal{N}(X)\), \(\mathcal{N}'(X)\), e.g., if \(\forall A \in \mathcal{N}(X) \exists A' \in \mathcal{N}'(X). A' \subseteq A\), then \(\mathcal{N}'(X)\) is at least as sharp as \(\mathcal{N}(X)\) is?

3. What are reasonable coherence conditions between \(\mathcal{N}(X)\) and \(\mathcal{N}(X')\)?

4. Can we find an intuitive interpretation of such systems, as we can interpret \(\mu(X)\) by size?

5. Can we generate such systems locally by a relation, as we did for \(\mu(X)\)? By higher order, reactive, relations?

6. If \(\mathcal{N}(X)\) is generated by a probability (as for the lottery paradox), are there special laws, resulting from substitution and sums?
   E.g.: If \(\{x, y\} \not\in \mathcal{N}(X), \{x, y'\}, \{x', y\} \in \mathcal{N}(X),\) then \(\{x', y'\} \in \mathcal{N}(X)\)?

7. Are there intuitive ways to combine \(A \in \mathcal{N}(X)\) with \(A' \in \mathcal{N}(X')\) to \(A \times A' \in \mathcal{N}(X \times X')\), etc.?

Given \(\mathcal{N}(X)\), we can compare \(x, x' \in X\):

**Definition 6.6.1**

Define \(U(x) := \{ A \in \mathcal{N}(X) : x \in A \}\), and \(S(x) := \bigcap U(x)\).

Let \(x \prec x'\) iff \(S(x) \subseteq S(x')\) (alternatively: \(\text{card}(S(x)) < \text{card}(S(x'))\))

This generalizes the comparison in preferential relations, minimal elements are not comparable among each other.

**Remark 6.6.1**

1. This is a special case of a preferential relation, as minimal elements stay minimal, it is about subideal elements.
(2) It is robust under weakenings like in \( \{ N(\phi), N(\phi \lor \psi) \} \).

(3) What are the properties of the resulting relation, coherence conditions?

(4) Can we find a complete set of such properties (representation)?

(5) Transitivity of defaults is treated correctly: for \((\phi : \psi), (\psi : \rho)\), the best \(\phi\)-models satisfy \(\psi\), and the best \(\psi\)-models satisfy \(\rho\), so the overall best \(\phi\)-models satisfy \(\rho\). This is not surprising, as we pushed defaults into the order, where we work with the best possible elements, as in preferential structures.

**Remark 6.6.2**

A remark on reasoning dynamics:

The full system \(P\) has no dynamics, because of Cumulativity. In the lottery paradox, once we concluded that \(n\) will now win, \(n'\) has become more likely to win. But, it could also be otherwise. If we conclude that a bird will probably fly, the flying birds might even be more likely to have feathers than the not flying ones. Thus, drawing conclusions might also make further conclusions more secure. In inheritance, upward chaining adds new conclusions, but they become less certain, as longer paths of reasoning offer more possibilities of attack.

There does not seem to exist a fully general theory of the dynamics of reasoning - but this might also be too general a problem.

### 6.7 From propositional to first order logic

In propositional logic, every (complete) possibility exists exactly once. In 1st order logic, a predicate \(p(.)\) may have 0, 1, many elements, likewise combinations of predicates, like \(p(.) \land \neg q(.)\). The combinations of properties correspond to propositional models. Here, we treat these combinations, as if they were classical models. Then, we put as many elements into the “good” combinations, and compare all models as in the propositional case. Thus, we try to put as many penguins as possible into the non-flying set, and the others into the flying set. So, given a fixed universe \(U\), we prefer those structures where more elements are “good”.

(1) More precisely, as in the propositional case, all cases are possible, like \(b(x) \land f(x), b(x) \land \neg f(x)\), etc., but they need not have the same cardinality. E.g., \(b(x) \land \neg f(x)\) might have 3 elements, \(b(x) \land f(x)\) 1 element, or, vice versa. We prefer the latter, as the “better” case \(b(x) \land f(x)\) has more elements than the “less good” case \(b(x) \land \neg f(x)\).

Again, this is still up to interpretation for the right preference relation. This preference relation should certainly satisfy: If, in structure \(S\), every \(x\) in the universe satisfies a default set \(X_x\) which is at least as good as the default set \(X_x'\) satisfied in structure \(S'\), then \(S\) should be preferred to \(S'\). More complicated relations may be considered, e.g., taking into account cardinalities, like: More \(x\) in \(S\) satisfy “good” default sets than in \(S'\), etc., see Section 6.8 (page 74).

(2) Suppose we have birds, penguins, sparrows. Penguins cannot be flying birds, but sparrows should be. Sparrows are not penguins, so, flying sparrows are better than not-flying sparrows. Flying sparrows satisfy both defaults (not being penguins, see Section 6.8 (page 74)), but not-flying sparrows violate the “fly” default, and satisfy the \((\text{penguin} : \neg \text{fly})\) default, so they are worse. We choose sparrows so that they fall into the normal birds set, or, more precisely,
among the most normal birds. Names should be treated as unary predicates, interpreted by as normal as possible elements.

6.8 Validity of defaults and the best models

Consider the propositional case, and a non-nested default \((\phi : \psi)\), i.e., \(\phi\) and \(\psi\) are classical formulas. We treat the default similarly to the classical implication \(\phi \to \psi\), and define for a classical model \(m\):

\[ m \models (\phi : \psi) \text{ iff } m \models \neg \phi \text{ or } m \models \phi \land \psi. \]

We refine this. In classical logic, validity is absolute, 0 or 1. We differentiate the strength of validity for defaults:

- \(m \models (\phi : \psi)\) holds with strength 1 (the strength of \(m \models \neg \phi\)) if \(m \models \neg \phi\).
- \(m \models (\phi : \psi)\) holds with strength \(M(\phi)\) if \(m \models \phi \land \psi\).
- \(m \models (\phi : \psi)\) fails with strength \(M(\phi)\) if \(m \models \phi \land \neg \psi\).

The strength \(M(\phi)\) takes care of specificity - the smaller \(M(\phi)\), the bigger the strength, this gives a partial order on strength.

For a full picture, we have to extend this definition to nested defaults.

Example 6.8.1

Consider birds, penguin, ravens. Birds (including ravens) fly, penguins don’t, penguins are birds, etc. A penguin Tweety which does not fly, fails \((\text{bird} : \neg \text{fly})\) with strength “bird”, and satisfies \((\text{penguin} : \neg \text{fly})\) with strength “penguin”. A penguin Tweety’ which flies, satisfies \((\text{bird} : \text{fly})\) with strength “bird”, and fails \((\text{penguin} : \neg \text{fly})\) with strength “penguin”. Tweety is a better model of the whole theory than Tweety’ is, as Tweety fails for less strong defaults than Tweety’ does. Blacky, the flying raven, satisfies \((\text{birds} : \text{fly})\) with strength “bird”, and \((\text{penguin} : \neg \text{fly})\) with strength 1, as it is no penguin. Thus, Blacky is the best model of the theory (among Tweety, Tweety’, Blacky).

We turn to the treatment of contradictions, this can be done in several ways, defining a partial relation between models. We outline requirements and possibilities, considering a theory \(T\) with classical information \(\phi, \ldots\) and default information \((\phi : \psi), \ldots\)

1. Models which contradict classical information \(\phi\) are the worst.

2. Models which contradict neither classical nor default information are the best.

3. Fix a classical model \(m\). Let \(S(m)\) be the (multi-) set of strengths of defaults which \(m\) fails. E.g., if \(m \models \phi \land \phi’\), \(m \models \neg \psi \land \neg \rho \land \neg \psi’\), and \(T\) consists of the defaults \((\phi : \psi), (\phi : \rho), (\phi’ : \psi’),\) then \(S(m) = \{M(\phi), M(\phi), M(\phi’), M(\phi’), \}.\) (We suppose that \(\not\models \psi \leftrightarrow \rho\) - this has to be refined to account for \(\psi, \rho\) which are not independent.)

4. A comparison of \(m\) and \(m’\) will be via a comparison of \(S(m)\) with \(S(m’).\)

There are many possibilities:
(4.1) We can treat $S(m)$ as a set, and forget multiple occurrences of the same strength. This is probably unsatisfactory, as we will treat a model which fails one default the same way as a model which fails many defaults - as long as they have the same strength. It results in usual preferential structures, which are unable to treat subideal cases. $m \models \phi \land \psi \land \neg \rho$ will be then considered equivalent to $m' \models \phi \land \neg \psi \land \neg \rho$ (when we consider just the defaults $(\phi : \psi)$, $(\phi : \rho)$).

(4.2) We can consider $\bigcap S(m)$, and if $\bigcap S(m) \subset \bigcap S(m')$, conclude that $m$ fails in a worse way than $m'$ does.

(4.3) We can combine (4.2) with a multiset approach, and "count" only if (4.2) will not decide between $m$ and $m'$.

(4.4) We can use any other reasonable way to order a set of partially ordered multisets.

The following questions arise:

(4.1) These are special preferential relations, do additional properties hold?

(4.2) Is there an abstract description, characterization, of such relations?

(5) The first order case:

We use above partial order between propositional models to treat (open) FOL defaults. A propositional model corresponds to a subset of the universe, like $X := \{x : p(x) \land \neg q(x)\}$. If two such subsets $X, X'$ of the universe are comparable by above order, we prefer the model which has more elements in the preferred $X$ - all other things being equal.

This is then a straightforward extension of the propositional case, and handled in the same spirit.

Example 6.8.2

(1) Consider the default set $(p : q), (p : \neg q)$.

Any $\neg p$-model satisfies both defaults, any $p$-model one, but not the other. So the globally best models are the $\neg p$-models, the best models of $T = \{p, (p : q), (p : \neg q)\}$ are all $p$-models.

(2) This also gives an answer to the inconsistent default $(p : \neg p)$: The globally best models are the $\neg p$-models, the best models for $p$ are all $p$-models, being all equally bad.

(3) Consider the default set $(p : r), (: q), (: r)$, and the background theory $(p \land q \land r) \lor (r \land \neg p \land \neg q)$. Then the model $p \land q \land \neg r$ satisfies 2 defaults, the model $r \land \neg p \land \neg q$ only one. We decide by cardinality, so the former model is better.

(We need here that the defaults are “decomposed”, e.g., not $(p \land q)$ instead of $(p), (q)$. A finer treatment might be needed to cover cases like $(p \land q)$.)

(4) We treat the Nixon diamond similarly. $(p : r), (q : \neg r)$. Consider $T := \{p \land q, (p : r), (q : \neg r)\}$, the two models $p \land q \land r$ and $p \land q \land \neg r$ are equally good (or bad), so none is preferred - we are directly sceptical, we have no result about $r$.

(5) We use specificity. For $(b : f), (p : \neg f), p \rightarrow b$, we have the globally best models: $\neg b$-models and $b \land \neg p$ $f$-models (they satisfy both defaults), among the $p$-models, (by $p \rightarrow b$, $b$ holds) all fail one default, and the $\neg f$-models are better by specificity.

(6) Consider $(\phi), (: \phi \lor \psi)$. The best models are those which satisfy $\phi$, the second best satisfy $\phi \lor \psi$ but not $\phi$, the worst satisfy neither.
6.8.1 Asymmetric OR

We have treated \( (\phi : \psi) \) above similarly to the classical implication \( \phi \rightarrow \psi \). It is natural to try and extend this, by translating “somehow” \( (\phi : \psi) \) to \( \phi \rightarrow (\neg \text{normal}(\phi) \lor \psi) \). But this is then an asymmetric “OR”, as in most cases, normality and \( \psi \) will hold. In particular, we will prefer to try and make normality hold, e.g., in a mechanical proof system.

We may extend this idea to asymmetric theory revision, where \( K * (\phi \lor \psi) \) is preferably achieved by making \( \phi \) true.

6.9 Inheritance

Remark 6.9.1

Inheritance diagrams allow to treat subideal cases, but only with information of differing strength; penguins still inherit “feathers” from birds, although they cannot fly. Preclusion might override weaker information. But we have an unrestricted AND for information of same maximal strength (the “Garbage In” rule). So we cannot treat the Lottery Paradox.

6.9.1 Direct scepticism vs. intersection of extensions

We have to distinguish whether inheritance systems are to speak about the state of the world, or about our knowledge of the world. We may not know whether Nixon was a pacifist or not (direct scepticism), but he was one of the two, so one of the extensions represents reality (leading to the intersection of extensions approach). Thus, the distinction between state of the world and knowledge, and between elements and sets, provides an answer to the direct scepticism vs. intersection of extensions question.

In addition, if “Nixon” were a set, and not one element, there is even a third possibility: (almost) all Nixons are pacifists, (almost) all Nixons are not pacifists, and there is no majority for either. (In knowledge terms, we may know the latter holds, the latter or the first holds, etc.)

Remark 6.9.2

The existence of copies in classical preferential structures may code our ignorance - we do not know which \( x \) is smaller than \( x' \), we only know that it is one of the \( x \in X \). We have all possibilities in one structure, this expresses scepticism. Alternatively, we may work with many structures in parallel, see \cite{Sain2000}, this corresponds to an extensions approach.

6.9.2 The language of inheritance

(1) The (implicit) language of inheritance is not sets and arrows, but the atoms are arrows, and the results are valid paths. Only in a latter step, valid paths are transformed into (soft) arrows. When \( p \rightarrow q \) is an arrow in the diagram, neither \( q \rightarrow p \) nor \( q \not\rightarrow p \) need be in the language. Thus, a comparison (soundness and completeness) with the reasoning with small sets etc. must only be about the information which can be expressed in the language of the diagram. Here, \( q \not\rightarrow p \) may well be a result of reasoning with corresponding small sets, but we cannot compare it, as it is not in the language.
(2) We see this (the language) also by the fact that we may have several paths resulting in the same conclusion, but one might be destroyed by further reasoning, and the other not.

(3) We can define the language using admissible paths (concatenations of arrows pointing in the same directions, with at most one negative arrow, at the end), and/or their conclusions.
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