Visualization of Complex Laplace’s Equations on a Hollow Rectangle

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ABSTRACT

In the investigation, the complex geometry domain is a hollow rectangle. The governing equations are expressed with complex Laplace’s equations. And the analysis is solved by point-matching method. Besides, the results of numerical calculation are obtained by using Visual C++. In the present paper, visualization and image processing obtained from mathematical formulation of the complex Laplace’s equations on a hollow rectangle. Then, the local values and the mean values of the function are also discussed in the article. We hope the results can further apply in the problem of fluid flow and heat conduction.

1. Introduction

The partial differential equations show an important role in the mathematical researches and analysis. Some significant efforts, thus, have been directed towards researches into related fields. For example, about the Laplace equation, Bungartz et al. [1] presented a simple model problem, the Laplace equation on the unit square with a Dirichlet boundary function. They present a proof of convergence for the so-called combination technique, a modern, efficient, and easily parallelizable sparse grid solver for elliptic partial differential equations. And Abouchabaka, et al. [2] presented the numerical approach of the free boundary by using shape optimization method. The numerical simulation is possible with the Laplace–Poisson model, which introduces two regions.

Besides, John G. Fikioris [3] used the method of Watson’s transformation to apply to a two-dimensional, orthogonal eigenfunction series of rectangular harmonic functions, and he provided the solution to a typical boundary value problem of Laplace’s equation. Furthermore, Wang [4] solved the diffusion across a corrugated saw-tooth plate with the Laplace equation. The transport properties and the theoretical increase in total flux due to corrugations were discussed. Recently, Antoine et al. [5] described a Monte Carlo method for the numerical computation of the principal eigenvalue of the Laplace operator in a bounded domain with Dirichlet conditions. Various tools of statistical estimation and different simulation schemes are developed to optimize the method. In the other related research, Reutskiy [6] presented the meshless radial basis function method for 2D steady-state heat conduction problems in anisotropic and inhomogeneous media. Next, Gal and Gal [7] solved the heat and Laplace-type equations with complex spatial variables in weighted bergman spaces and studied the classical heat and Laplace equations with real-time variable and complex spatial...
variable by the semigroup theory methods. Chang et al. [8] present a review of the current computational methods and applications of inverse heat conduction problems in different fields. In the present paper, there are two major solving categories of the issue: mesh methods and meshless algorithms of their strengths and weaknesses are also discussed in the study.

Although many researches about Laplace equation under different conditions had been discussed, the Laplace equation with complex domains is also worth discussing. The present paper, thus, will analyze a symmetric domain with complex Laplace equations under two kinds of boundary conditions in order to find local values and the mean values of the function. The analysis of two kinds of boundary conditions, case 1 and case 2, will be specified in the following mathematical formulation.

2. Mathematical formulation

The geometry domain in Figure 1a is a rectangle. The outer dimension of the rectangle is $2w \times 2h$, and there is a hollow part in the center, whose dimension is $2(w - d) \times 2b$.

2.1. The analysis of boundary conditions on Case 1

The boundary condition of the outer part is 0, and the condition of the hollow part is 1. The local function distribution in the outer solid part of the rectangle is $f(x, y)$.

Decompose the rectangle into four parts. Due to the symmetric character, only the left-bottom quarter of the geometry, i.e. the L-shaped region in Figure 1b, needs calculating.

The governing equation for the region is expressed with Laplace equation:

$$\nabla^2 f(x, y) = 0$$

Figures 1a, 1b. The figure of the main geometry domain. The L-shape region is composed of two rectangles; the governing equation for the left is one:

$$\nabla^2 f_1(x, y) = 0$$

The boundary conditions for Equation (2) are:

$$f_1(x, 0) = 0, \quad f_1(0, y) = 0, \quad \frac{\partial f_1(x, h)}{\partial y} = 0$$

And the governing equation for the right rectangle is:

$$\nabla^2 f_2(x, y) = 0$$

The boundary conditions for Equation (4) are:

$$f_2(x, 0) = 0, \quad \frac{\partial f_2(w, y)}{\partial x} = 0, \quad f_2(x, h - b) = 1$$

With the boundary conditions Equations (3) and (5), the analytical solution to Equation (2) is $f_1(x, y)$, and to Equation (6) is $f_2(x, y)$. They are as follows:

$$f_1(x, y) = \sum_n A_n \sin(\alpha_n y) \left( e^{-\alpha_n (x+d)} - e^{-\alpha_n (x-d)} \right)$$

$$f_2(x, y) = \frac{y}{h - b} + \sum_m B_m \sin(\beta_m y) \left( e^{\beta_m (x-2w+d)} + e^{-\beta_m (x-d)} \right)$$

where the eigenvalues are:

$$\alpha = \frac{(2n-1)\pi}{2h}, \quad \beta = \frac{m\pi}{h - b}$$
Other boundary conditions for governing equations are:

\[ f_1(d, y) = 1; \quad h - b \leq y \leq h \]  

(9)

Substitute the boundary condition into Equation (6), and the following equation can be obtained:

\[ \sum_n A_n \sin(\alpha_n y_i)(e^{-2\alpha_n y_i} - 1) = 1 \]  

(10)

\( i = M + 1 \) to \( N \)

Next, the solutions to the two regions of the L-shape domain can be matched along the common boundary conditions [9]. The conditions can be expressed as:

\[ f_1(d, y) = f_2(d, y), \quad 0 \leq y \leq h - b \]  

(11)

\[ \frac{\partial f_1(d, y)}{\partial x} = \frac{\partial f_2(d, y)}{\partial x}, \quad 0 \leq y < h - b \]  

(12)

Substituting the boundary condition into Equations (6) and (7) can obtain the following equations:

\[ \sum_n A_n \sin(\alpha_n y_i)(e^{-2\alpha_n y_i} - 1) - \sum_m B_m \sin(\beta_m y_i)(1 + e^{-2\beta_m y_i}) = \frac{y_i}{h - b} \]  

(13)

\[ i = 1 \) to \( M \)

\[ \sum_n A_n \sin(\alpha_n y_i)(1 - e^{-2\alpha_n y_i}) - \sum_m B_m \sin(\beta_m y_i)(e^{-2\beta_m y_i} - 1) = 0 \]  

(14)

\( i = 1 \) to \( M \)

The mean value for \( f(x, y) \) is expressed as:

\[ f_{\text{mean}} = \frac{1}{hw - b(w - d)} \left[ \int_0^d \int_0^h f_1 dy dx + \int_0^d \int_0^h f_2 dy dx \right] \]  

(15)

Integrating Equation (15) can obtain the following equation:

\[ f_{\text{mean}} = \frac{1}{hw - b(w - d)} \left[ \sum_n \frac{2}{N} (2e^{-2\alpha_n d} - e^{-2\alpha_n d}) + \sum_m \frac{2}{B} [\cos(h - b)\beta - 1]e^{-2\beta_m (w - d)} - 1 \right] \]  

(16)

\[ \sum_n C_n \sin(\alpha_n y_i)(1 - e^{-2\alpha_n y_i}) = 1 \]  

(25)

\[ i = M + 1 \) to \( N \)

Next, the solutions to the two regions of the L-shape domain can be matched along the common boundary conditions. The conditions can be expressed as:

\[ f_1(d, y) = f_2(d, y), \quad 0 \leq y \leq h - b \]  

(26)

\[ \frac{\partial f_1(d, y)}{\partial x} = \frac{\partial f_2(d, y)}{\partial x}, \quad 0 \leq y \leq h - b \]  

(27)

2.2. The analysis of boundary conditions on Case 2

The geometry domain is also a rectangle. The outer dimension of the rectangle is \( 2w \times 2h \), and there is a hole part in the center, whose dimension is \( 2(w - d) \times 2b \). The boundary condition of the outer part is 1, and the condition of the hollow part is 0. The L-shape region is composed of two rectangles; the governing equation for the left rectangle is:

\[ \nabla^2 f_1(x, y) = 0 \]  

(17)

The boundary conditions for Equation (17) are:

\[ f_1(x, 0) = 1 \]

\[ f_1(0, y) = 1 \]

\[ \frac{\partial f_1(x, h)}{\partial y} = 0 \]  

(18)

And the governing equation for the right rectangle is:

\[ \nabla^2 f_2(x, y) = 0 \]  

(19)

The boundary conditions for Equation (19) are:

\[ f_2(x, 0) = 1 \]

\[ f_2(x, h - b) = 0 \]

\[ \frac{\partial f_2(x, w)}{\partial y} = 0 \]  

(20)

With the boundary conditions Equations (18) and (20), the analytical solution to Equation (17) is \( f_1(x, y) \), and to Equation (19) is \( f_2(x, y) \). They are as follows:

\[ f_1(x, y) = 1 + \sum_n C_n \sin(\alpha_n y_i)(e^{-\alpha_n (x+d)} - e^{-\alpha_n (x-d)}) \]  

(21)

\[ f_2(x, y) = 1 - \frac{y}{h - b} + \sum_m D_m \sin(\beta_m y_i)(e^{\beta_m (x-2w+d)} + e^{-\beta_m (x-d)}) \]  

(22)

where the eigenvalues are:

\[ \alpha = \frac{(2n - 1)\pi}{2h}, \beta = \frac{m\pi}{h - b} \]  

(23)

Other boundary conditions for governing equations are:

\[ f_1(d, y) = 0; \quad h - b \leq y \leq h \]  

(24)

Substitute the boundary condition into Equation (21), and the following equation can be obtained:

\[ \sum_n C_n \sin(\alpha_n y_i)(1 - e^{-2\alpha_n y_i}) = 1 \]  

(25)
\[
\sum_{n} C_n \sin(\alpha_n y_i) (e^{-2\beta_n x_i} - 1) - \sum_{m} D_m \sin(\beta_m x_i) (1 + e^{-2\beta_n y_i}) \frac{y_i}{b-h} = 0
\]  
(28)

\[
\sum_{n} C_n \alpha_n \sin(\alpha_n y_i) (-1 - e^{-2\alpha_n x_i}) - \sum_{m} D_m \beta_m \sin(\beta_m x_i) (e^{-2\beta_m y_i} - 1) = 0
\]  
(29)

\[
f_{\text{mean}} = \frac{1}{hw - b(w - d)} \left[ \int_{0}^{d} \int_{0}^{h} f_1 \, dy \, dx + \int_{d}^{w} \int_{0}^{h} f_2 \, dy \, dx \right]  
\]  
(30)

Integrating Equation (15) can obtain the following equation:

\[
f_{\text{mean}} = \frac{1}{hw - b(w - d)} \left[ dh + \sum_{n} \frac{C_n}{\alpha_n} (2e^{-\alpha_n x} - e^{-2\alpha_n x} - 1) + \sum_{m} \frac{D_m}{\beta_m} (\cos(h - b)\beta - 1)[e^{-2\beta_m y} - 1] \right]
\]  
(31)

3. Numerical methods

The following steps of numerical methods are estimated by using Visual C++:

1. Give the constants \( h, b, w \) and \( d \).
2. Set \( N, M = \text{int}[1/(1-b/h)] \) and \( y_i = ih/N, 1 \leq i \leq N \).
3. Equations (10), (13) and (14) are expressed as the linear system of \((N + M)\) equations to solve coefficients \( A_n \) and \( B_m \).
4. Substitute coefficients \( A_n \) and \( B_m \) into equations \( f_j \) Equation (6) and \( f_j \) Equation (7). This process is repeated at all nodes within the range, i.e. \( 0 \leq y \leq h, 0 \leq x \leq w \)
5. Map the \( f(x, y) \) on the entire domain.
6. The average values of function can be calculated from Equation (16).
7. Repeat the previous methods can estimate the results of Case 2.

4. Results and discussion

Case 1:

Figures 2a and 2b show the contour plot for the domain. Figure 2a is a rectangle \((w < h)\), and Figure 2b is also a rectangle \((w > h)\). In the figures, \( w, b, h \) and \( d \) values are different. With the boundary condition Equations (3) and (5) and governing equation \( f_j (x, y) \) and \( f_j (x, y) \), the function values distributing from the inner region of the domain, \( f = 1.0 \) to the edges, \( f = 0 \) gradually decrease.

Figures 3a and 3b show the local function values for the entire domain \((x = 0 \text{ to } x = 2w, y = 0 \text{ to } y = 2h) \) as the figures show, the inner region has the maximum function values, i.e. \( f = 1.0 \), and then function values gradually decrease from inner region to the edges of the outer rectangle, \( f = 0 \). Furthermore, because of the different values of \( w, b, h \) and \( d \), the width and the height of the rectangle, \( d \) and \( (h-b) \), will influence the gradients of functions. The smaller the width and the height, the larger the gradients are.

Figure 4 presents the results of Equation (16), and shows the influence of the values \( w \) on the mean values of \( f(x, y) \) under five different \( d \) values. Observation of the figure shows that for a fixed value of \( d \), the mean values increase as the values of \( w \) increase. As a result, the increase in \( d \) leads to a decrease in the mean values.

Figure 5 shows the influence of the values \( d \) on the mean values of \( f(x, y) \) under six different \( w \) values. Observation of the figure shows that the mean values increase as the values of \( d \) increase. And the increase in \( w \) leads to a decrease in the average values.

Figure 6 shows the influence of \( h \) on the mean values of \( f(x, y) \) under three different \( b \) values. As the figure shows, from \( h = 0.5 \) to \( h = 2.0 \), the mean values of function \( f(x, y) \) increase from \( h = 0.5 \), and then the mean values reach a maximum value. Beyond the maximum value, the mean values gradually decrease to \( h = 2.0 \). Thus, under a particular range of the height and the width, the present study can get the maximum mean values of the function.
Figure 8 shows the local function values for the entire domain \((x = 0 \text{ to } x = 2w, y = 0 \text{ to } y = 2h)\). As the figure shows, the inner region has the minimum function values, i.e., \(f = 0\), and then function values gradually increase from inner region to the edges of the outer rectangle, \(f = 1.0\).

**Case 2:**

Figure 7 shows the contour plot for the domain. Figure 7 is a square \((w = h)\). The function values distributing from the inner region of the domain, \(f = 0\) to the outer boundary, \(f = 1.0\) gradually increase.

Figure 2b. Contour plot for \(h = 0.5, b = 0.25, d = 0.4, w = 1.2\).

Figure 3a. The distribution of local function values for \(h = 1.2, b = 0.8, d = 0.3, w = 0.8\).

Figure 3b. The distribution of local function values for \(h = 0.5, b = 0.25, d = 0.4, w = 1.2\).
Figure 9 shows the influence of $h$ on the mean values of $f(x,y)$ under three different $b$ values. As the figure shows, from $h = 0.5$ to $h = 2.0$, the mean values of function $f(x,y)$ decrease from $h = 0.5$, and then the mean values reach a minimum value. Beyond the minimum value, the mean values gradually increase to $h = 2.0$.

5. Conclusions

The following conclusions may be drawn from the results of this study:
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Figure 9. Mean function values for $w=1.0$, $d=0.5$ – effect on values.

(a) The present paper can find the influence of the height and the width of the geometry domain on the function mean values. Furthermore, under a particular range of the height and the width, the present study can get the maximum and minimum mean values of the function.

(b) The present paper used the analytical solution of point match methods and numerical methods can easily compute coefficient $A_n$, $B_m$, $C_n$, $D_m$ and function $f(x,y)$.