HORN PROBLEM FOR QUASI-HERMITIAN LIE GROUPS

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Abstract In this paper, we prove some convexity results associated to orbit projection of noncompact real reductive Lie groups.

Contents

1 Introduction 2806

2 The cone $\Delta_{\text{hol}}(\tilde{G},G)$: first properties 2810
   2.1. The holomorphic chamber 2810
   2.2. The cone $\Delta_{\text{hol}}(\tilde{G},G)$ is closed 2812
   2.3. Rational and weakly regular points 2813
   2.4. Weinstein’s theorem 2814

3 Holomorphic discrete series 2815
   3.1. Definition 2815
   3.2. Restriction 2816
   3.3. Discrete analogues of $\Delta_{\text{hol}}(\tilde{G},G)$ 2816
   3.4. Riemann–Roch numbers 2817
   3.5. Quantization commutes with reduction 2818

4 Proofs of the main results 2819
   4.1. Proof of Theorem A 2819
   4.2. The affine variety $\tilde{K}_C \times q$ 2821
   4.3. Proof of Theorem B 2822
   4.4. Proof of Theorem C 2822

5 Inequalities characterizing the cones $\Delta_{\text{hol}}(\tilde{G},G)$ 2823
   5.1. Admissible elements 2823
   5.2. Ressayre’s data 2824
This paper is concerned with convexity properties associated to orbit projection. Let us consider two Lie groups $G \subset \tilde{G}$ with Lie algebras $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ and corresponding projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}: \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$. A longstanding problem has been to understand how a coadjoint orbit $\mathcal{O} \subset \tilde{\mathfrak{g}}^*$ decomposes under the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$. For this purpose, we may define

$$\Delta_G(\tilde{\mathcal{O}}) = \{ O \in \mathfrak{g}^*/G ; \mathcal{O} \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}}) \}.$$ 

When the Lie group $G$ is compact and connected, the set $\mathfrak{g}^*/G$ admits a natural identification with a Weyl chamber $t^*_\geq 0$. In this context, we have the well-known convexity theorem [12, 1, 10, 16, 13, 35, 22].

**Theorem 1.1.** Suppose that $G$ is compact connected and that the projection $\pi_{\mathfrak{g},\tilde{\mathfrak{g}}}$ is proper when restricted to $\tilde{\mathcal{O}}$. Then $\Delta_G(\tilde{\mathcal{O}}) = \{ \xi \in t^*_\geq 0 ; G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{\mathcal{O}}) \}$ is a closed convex locally polyhedral subset of $t^*$.

When the Lie group $\tilde{G}$ is also compact and connected, we may consider

$$\Delta(\tilde{G},G) := \left\{ (\tilde{\xi},\xi) \in \tilde{t}^*_\geq 0 \times t^*_\geq 0 ; G\xi \subset \pi_{\mathfrak{g},\tilde{\mathfrak{g}}}(\tilde{G}\tilde{\xi}) \right\}. \quad (1)$$

Here is another convexity theorem [14, 17, 4, 2, 3, 25, 19, 20, 36].

**Theorem 1.2.** Suppose that $G \subset \tilde{G}$ are compact connected Lie groups. Then $\Delta(\tilde{G},G)$ is a closed convex polyhedral cone and we can parametrize its facets by cohomological means (i.e., Schubert calculus).

In this article, we obtain an extension of Theorems 1.1 and 1.2 in a case where $G$ and $\tilde{G}$ are both noncompact real reductive Lie groups.

Let us explain what framework we are considering. Let $\tilde{K}$ be a maximal compact subgroup of $\tilde{G}$. We suppose that $\tilde{G}/\tilde{K}$ is a Hermitian symmetric space of a noncompact type. Among the elliptic coadjoint orbits of $\tilde{G}$, some of them are naturally Kähler $\tilde{K}$-manifolds. These orbits are called the holomorphic coadjoint orbits of $\tilde{G}$. They are the strongly elliptic coadjoint orbits closely related to the holomorphic discrete series of Harish–Chandra. These orbits intersect the Weyl chamber $t^*_\geq 0$ of $\tilde{K}$ into a subchamber $\tilde{C}^\text{hol}$ called the holomorphic chamber. The basic fact here is that the union

$$\mathcal{C}^\text{0}_{\tilde{G}/\tilde{K}} := \bigcup_{\tilde{a} \in \tilde{C}^\text{hol}} \tilde{G}\tilde{a}$$

is an open invariant convex cone of $\tilde{\mathfrak{g}}^*$. See §2.1 for more details.
In this article, we work in the context where $\tilde{G}/\tilde{K}$ admits a sub-Hermitian symmetric space of a noncompact type $G/K$. For the convenience of the reader, we list below some examples of the pairs $(\tilde{G}, G)$:

| $\tilde{G}$ | $G$ |
|-------------|-----|
| $U(p,q)^s, s \geq 2$ | $U(p,q)$ |
| $Sp(n,\mathbb{R})$ | $Sp(p,\mathbb{R}) \times Sp(n-p,\mathbb{R})$ |
| $Sp(n,\mathbb{R})$ | $U(p,n-p)$ |
| $SO(2,2n)$ | $U(1,n)$ |
| $SO(2,n)$ | $SO(2,p) \times SO(n-p)$ |
| $SO^*(2n)$ | $U(p,n-p)$ |
| $SO^*(2n)$ | $SO^*(2p) \times SO^*(2n-2p)$ |
| $U(n,n)$ | $Sp(n,\mathbb{R})$ |
| $SO^*(2n)$ | $SO^*(2n)$ |
| $U(p,q)$ | $U(i,j) \times U(p-i,q-j)$ |

As the projection $\pi_{\tilde{g},\tilde{g}}: \tilde{g}^* \to g^*$ sends the convex cone $C^0_{\tilde{G}/\tilde{K}}$ inside the convex cone $C^0_{G/K}$, it is natural to study the following object reminiscent of equation (1):

$$\Delta_{\text{hol}}(\tilde{G}, G) := \left\{ (\tilde{\xi}, \xi) \in \tilde{C}_{\text{hol}} \times C_{\text{hol}}; \; G\xi \subset \pi_{\tilde{g},g}(\tilde{G}\tilde{\xi}) \right\}. \quad (2)$$

Let $\tilde{\mu} \in \tilde{C}_{\text{hol}}$. We will also give a particular attention to the intersection of $\Delta_{\text{hol}}(\tilde{G}, G)$ with the linear subspace $\tilde{\xi} = \tilde{\mu}$, that is to say

$$\Delta_{G}(\tilde{G}\tilde{\mu}) := \left\{ \xi \in C_{\text{hol}}; \; G\xi \subset \pi_{\tilde{g},g}(\tilde{G}\tilde{\mu}) \right\}. \quad (3)$$

Consider the case where $G$ is embedded diagonally in $\tilde{G} := G^s$ for $s \geq 2$. The corresponding set $\Delta_{\text{hol}}(G^s,G)$ is called the holomorphic Horn cone, and it is defined as follows:

$$\text{Horn}^s_{\text{hol}}(G) := \left\{ (\xi_1, \cdots, \xi_{s+1}) \in C^1_{\text{hol}}; \; G\xi_{s+1} \subset \sum_{j=1}^{s} G\xi_j \right\}. \quad (4)$$

The first result of this article is the following theorem.

**Theorem A.**

- $\Delta_{\text{hol}}(\tilde{G}, G)$ is a closed convex cone of $\tilde{C}_{\text{hol}} \times C_{\text{hol}}$.
- $\text{Horn}^s_{\text{hol}}(G)$ is a closed convex cone of $C^1_{\text{hol}}$ for any $s \geq 2$.

We obtain the following corollary which corresponds to a result of A. Weinstein [38].

**Corollary.** For any $\tilde{\mu} \in \tilde{C}_{\text{hol}}$, $\Delta_{G}(\tilde{G}\tilde{\mu})$ is a closed and convex subset of $C_{\text{hol}}$.

A first description of the closed convex cone $\Delta_{\text{hol}}(\tilde{G}, G)$ goes as follows. The quotient $q$ of the tangent spaces $T_eG/K$ and $T_e\tilde{G}/\tilde{K}$ has a natural structure of a Hermitian
The irreducible representations of $K$ (resp. $\tilde{K}$) are parametrized by a semi-group $\Lambda^*_+$ (resp. $\tilde{\Lambda}^*_+$). For any $\lambda \in \Lambda^*_+$ (resp. $\tilde{\lambda} \in \tilde{\Lambda}^*_+$), we denote by $V^K_{\lambda}$ (resp. $V^{\tilde{K}}_{\tilde{\lambda}}$) the irreducible representation of $K$ (resp. $\tilde{K}$) with highest weight $\lambda$ (resp. $\tilde{\lambda}$). If $E$ is a representation of $K$, we denote by $[V^K_{\lambda} : E]$ the multiplicity of $V^K_{\lambda}$ in $E$.

\textbf{Definition 1.3.}

1. $\Pi^G_q(\tilde{K}, K)$ is the semigroup of $\tilde{\Lambda}^*_+ \times \Lambda^*_+$ defined by the conditions:
   $$(\tilde{\lambda}, \lambda) \in \Pi^G_q(\tilde{K}, K) \iff [V^K_{\lambda} : V^{\tilde{K}}_{\tilde{\lambda}} \otimes \Sym(q)] \neq 0.$$

2. $\Pi_q(\tilde{K}, K)$ is the convex cone defined as the closure of $\mathbb{Q}^+ \cdot \Pi^G_q(\tilde{K}, K)$.

The second result of this article is the following theorem.

\textbf{Theorem B.} We have the equality

$$\Delta_{\text{hol}}(\hat{G}, G) = \Pi_q(\tilde{K}, K) \cap \tilde{C}_{\text{hol}} \times C_{\text{hol}}. \quad (4)$$

A natural question is the description of the facets of the convex cone $\Delta_{\text{hol}}(\hat{G}, G)$. In order to do that, we consider the group $\tilde{K}$ endowed with the following $\tilde{K} \times K$-action: $(\tilde{k}, k) \cdot \tilde{a} = \tilde{k}a\tilde{k}^{-1}$. The cotangent space $T^*\tilde{K}$ is then a symplectic manifold equipped with a Hamiltonian action of $\tilde{K} \times K$. We consider now the Hamiltonian $\tilde{K} \times K$-manifold $T^*\tilde{K} \times \mathfrak{q}$, and we denote by $\Delta(T^*\tilde{K} \times \mathfrak{q})$ the corresponding Kirwan polyhedron.

Let $W = N(T)/T$ be the Weyl group of $(K, T)$, and let $w_0$ be the longest Weyl group element. Define an involution $*: t^* \mapsto t^*$ by $\xi^* = -w_0\xi$. A standard result permits to affirm that $(\xi, \xi) \in \Pi_q(\tilde{K}, K)$ if and only if $(\xi, \xi^*) \in \Delta(T^*\tilde{K} \times \mathfrak{q})$ (see §4.2).

We obtain then another version of Theorem B.

\textbf{Theorem B, second version.} An element $(\tilde{\xi}, \xi)$ belongs to $\Delta_{\text{hol}}(\hat{G}, G)$ if and only if

$$(\tilde{\xi}, \xi) \in \tilde{C}_{\text{hol}} \times C_{\text{hol}} \quad \text{and} \quad (\tilde{\xi}, \xi^*) \in \Delta(T^*\tilde{K} \times \mathfrak{q}).$$

Thanks to the second version of Theorem B, a natural way to describe the facets of the cone $\Delta_{\text{hol}}(\hat{G}, G)$ is to exhibit those of the Kirwan polyhedron $\Delta(T^*\tilde{K} \times \mathfrak{q})$. In this later case, it can be done using Ressayre’s data (see §5).

The second version of Theorem B permits also the following description of the convex subsets $\Delta_G(\hat{G})$, $\tilde{K} \times \tilde{\mathfrak{q}}$ be the Kirwan polyhedron associated to the Hamiltonian action of $\tilde{K}$ on $\tilde{K} \hat{\mu} \times \tilde{\mathfrak{q}}$, where $\tilde{\mathfrak{q}}$ denotes the $\tilde{K}$-module $\mathfrak{q}$ with opposite complex structure.

\textbf{Theorem C.} For any $\tilde{\mu} \in \tilde{C}_{\text{hol}}$, we have $\Delta_G(\hat{G}) = \Delta_K(\tilde{K} \hat{\mu} \times \tilde{\mathfrak{q}})$.

Let us detail Theorem C in the case where $G$ is embedded in $\hat{G} = G \times G$ diagonally. We denote by $\mathfrak{p}$ the $K$-Hermitian space $T_eG/K$. Let $\kappa$ be the Killing form of the Lie algebra $\mathfrak{g}$. The vector space $\bar{\mathfrak{p}}$ is equipped with the symplectic 2-form $\Omega_{\bar{\mathfrak{p}}}(X, Y) = -\kappa(z, [X, Y])$ and the compatible complex structure $-\text{ad}(z)$. 
Let us denote by \( \Delta_K(K\mu_1 \times K\mu_2 \times \mathfrak{p}) \) and by \( \Delta_K(\mathfrak{p}) \) the Kirwan polyhedrons relative to the Hamiltonian actions of \( K \) on \( K\mu_1 \times K\mu_2 \times \mathfrak{p} \) and on \( \mathfrak{p} \). Theorem C says that, for any \( \mu_1, \mu_2 \in C_{\text{hol}} \), the convex set \( \Delta_G(G\mu_1 \times G\mu_2) \) is equal to the Kirwan polyhedron \( \Delta_K(K\mu_1 \times K\mu_2 \times \mathfrak{p}) \).

To any nonempty subset \( \mathcal{C} \) of a real vector space \( E \), we may associate its asymptotic cone \( \text{As}(\mathcal{C}) \subset E \) which is the set formed by the limits \( y = \lim_{k \to \infty} t_k y_k \), where \( (t_k) \) is a sequence of nonnegative reals converging to 0 and \( y_k \in \mathcal{C} \).

We finally get the following description of the asymptotic cone of \( \Delta_G(G\mu_1 \times G\mu_2) \).

**Corollary D.** For any \( \mu_1, \mu_2 \in C_{\text{hol}} \), the asymptotic cone of \( \Delta_G(G\mu_1 \times G\mu_2) \) is equal to \( \Delta_K(p) \).

In [29] §5, we explained how to describe the cone \( \Delta_K(\mathfrak{p}) \) in terms of strongly orthogonal roots.

Let us finish this introduction with few remarks on related works:

- When \( G \) is compact, equal to the maximal compact subgroup \( \tilde{K} \) of \( \tilde{G} \), the results of Theorems B and C were already obtained by G. Deltour in his thesis [6, 7]. He proved the equality \( \Delta_{\tilde{K}}(\tilde{G}\tilde{\mu}) = \Delta_K(\tilde{K}\tilde{\mu} \times \mathfrak{p}) \) by showing that the coadjoint orbit \( \tilde{G}\tilde{\mu} \) admits a \( \tilde{K} \)-equivariant symplectomorphism with \( \tilde{K}\tilde{\mu} \times \mathfrak{p} \), thus generalizing an earlier result of D. McDuff [26]. We explain in §7 a conjectural symplectomorphism that would lead to the relation \( \Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(\tilde{K}\tilde{\mu} \times \mathfrak{p}) \).

- In [9], A. Eshmatov and P. Foth proposed a description of the set \( \Delta_G(G\mu_1 \times G\mu_2) \). **But their computations do not give the same result as ours.** From their main result (Theorem 3.2), it follows that the asymptotic cone of \( \Delta_G(G\mu_1 \times G\mu_2) \) is equal to the intersection of the Kirwan polyhedron \( \Delta_T(\mathfrak{p}) \) with the Weyl chamber \( t^* \geq 0 \). But since \( \Delta_K(\mathfrak{p}) \neq \Delta_T(\mathfrak{p}) \cap t^* \geq 0 \) in general, it is in contradiction with Corollary D.

**Notations**

In this paper, we take the convention of A. Knapp [18]: A connected real reductive Lie group \( G \) means that we have a Cartan involution \( \Theta \) on \( G \) such that the fixed point set \( K := G^\Theta \) is a connected maximal compact subgroup. We have Cartan decompositions at the level of Lie algebras \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) and at the level of the group \( G \simeq K \times \exp(\mathfrak{p}) \). We denote by \( \mathfrak{b} \) a \( G \)-invariant nondegenerate bilinear form on \( \mathfrak{g} \) that is equal to the Killing form on \( [\mathfrak{g}, \mathfrak{g}] \), and that defines a \( K \)-invariant scalar product \( (X, Y) := -b(X, \Theta(Y)) \). We will use the \( K \)-equivariant identification \( \xi \mapsto \xi, \mathfrak{g}^* \simeq \mathfrak{g} \) defined by \( (\xi, X) := \langle \xi, X \rangle \) for \( \xi \in \mathfrak{g}^* \) and \( X \in \mathfrak{g} \).

When a Lie group \( H \) acts on a manifold \( N \), the stabilizer subgroup of \( n \in N \) is denoted by \( H_n = \{ g \in G : gn = n \} \) and its Lie algebra by \( \mathfrak{h}_n \). Let us define

\[
\dim_H(\mathcal{X}) = \min_{n \in \mathcal{X}} \dim(\mathfrak{h}_n)
\]

for any subset \( \mathcal{X} \subset N \).
2. The cone $\Delta_{\text{hol}}(\tilde{G}, G)$: first properties

We assume here that $G/K$ is a Hermitian symmetric space of a noncompact type, that is to say, there exists a $G$-invariant complex structure on the manifold $G/K$ or, equivalently, there exists a $K$-invariant element $z \in \mathfrak{k}$ such that $\text{ad}(z)|_\mathfrak{p}$ defines a complex structure on $\mathfrak{p}$: $(\text{ad}(z)|_\mathfrak{p})^2 = -\text{Id}_\mathfrak{p}$. This condition imposes that the ranks of $G$ and $K$ are equal.

We are interested in the following closed invariant convex cone of $\mathfrak{g}^*$:

$$C_{G/K} = \{ \xi \in \mathfrak{g}^*, \langle \xi, gz \rangle \geq 0, \forall g \in G \}.$$

2.1. The holomorphic chamber

Let $T$ be a maximal torus of $K$, with Lie algebra $\mathfrak{t}$. Its dual $\mathfrak{t}^*$ can be seen as the subspace of $\mathfrak{g}^*$ fixed by $T$. Let us denote by $\mathfrak{g}^*_e$ the set formed by the elliptic elements: In other words, $\mathfrak{g}^*_e := \text{Ad}^*(G) \cdot \mathfrak{t}^*$.

Following [38], we consider the invariant open subset $\mathfrak{g}^*_e = \{ \xi \in \mathfrak{g}^* | G_\xi \text{ is compact} \}$ of strongly elliptic elements. It is nonempty since the groups $G$ and $K$ have the same rank.

We start with the following basic facts.

Lemma 2.1.

- $\mathfrak{g}^*_e$ is contained in $\mathfrak{g}^*_e$.
- The interior $C^0_{G/K}$ of the cone $C_{G/K}$ is contained in $\mathfrak{g}^*_e$.

Proof. The first point is due to the fact that every compact subgroup of $G$ is conjugate to a subgroup of $K$.

Let $\xi \in C^0_{G/K}$. There exists $\epsilon > 0$ so that

$$\langle \xi + \eta, gz \rangle \geq 0, \quad \forall g \in G, \quad \forall \|\eta\| \leq \epsilon.$$

It implies that $|\langle \eta, gz \rangle| \leq \langle \xi, z \rangle, \forall g \in G_\xi$ and $\forall \|\eta\| \leq \epsilon$. In other words, the adjoint orbit $G_\xi \cdot z \subset \mathfrak{g}$ is bounded. For any $g = e^Xk$, with $(X,k) \in \mathfrak{p} \times K$, a direct computation shows that $\|gz\| = \|e^Xz\| \geq ||[z,X]|| = \|X\|$. Then, there exists $\rho > 0$ such that $\|X\| \leq \rho$ if $e^Xk \in G_\xi$ for some $k \in K$. It follows that the stabilizer subgroup $G_\xi$ is compact.

Let $\wedge^* \subset \mathfrak{t}^*$ be the weight lattice: By definition, $\alpha \in \wedge^*$ if and only if $i\alpha$ is the differential of a character of $T$. Let $\mathcal{R} \subset \wedge^*$ be the set of roots for the action of $T$ on $\mathfrak{g} \otimes \mathbb{C}$. We have $\mathcal{R} = \mathcal{R}_c \cup \mathcal{R}_n$, where $\mathcal{R}_c$ and $\mathcal{R}_n$ are, respectively, the set of roots for the action of $T$ on $\mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{p} \otimes \mathbb{C}$. We fix a system of positive roots $\mathcal{R}_c^+$ in $\mathcal{R}_c$, and we denote by $\mathfrak{t}_c^+$ the corresponding Weyl chamber.

We have $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where the $K$-module $\mathfrak{p}^\pm$ is equal to $\ker(\text{ad}(z) \mp i)$. Let $\mathcal{R}_n^{\pm,z}$ be the set of roots for the action of $T$ on $\mathfrak{p}^\pm$. The union

$$\mathcal{R}^+ = \mathcal{R}_c^+ \cup \mathcal{R}_n^{+,z}$$

defines then a system of positive roots in $\mathcal{R}$. We notice that $\mathcal{R}_n^{+,z}$ is the set of roots $\beta \in \mathcal{R}$ satisfying $\langle \beta, z \rangle = 1$. Hence, $\mathcal{R}_n^{+,z}$ is invariant relatively to the action of the Weyl group $W = N(T)/T$. 
Let us recall the following classical fact concerning the parametrization of the $G$-orbits in $C^0_{G/K}$ via the holomorphic chamber

$$C_{\text{hol}} := \{ \xi \in t^*_\geq 0, \langle \xi, \beta \rangle > 0, \forall \beta \in \mathfrak{h}_n^+ \}.$$ 

The elliptic coadjoint orbits of $G$, i.e., those contained in $\mathfrak{g}^*_e$, are parameterized by the Weyl chamber $t^*_\geq 0$. Thus, we have a projection $p: \mathfrak{g}^*_e \to t^*_\geq 0$, defined by the relations $G\xi \cap t^*_\geq 0 = \{ p(\xi) \}$, and that induces a bijection $\mathfrak{g}^*_e/G \simeq t^*_\geq 0$.

**Proposition 2.2.** The set $p(C^0_{G/K})$ is equal to $C_{\text{hol}}$. In other words, the map $p$ induces a bijective map between the set of $G$-orbits in $C^0_{G/K}$ and the holomorphic chamber $C_{\text{hol}}$.

**Proof.** Let us first prove that $p(C^0_{G/K}) = t^*_\geq 0 \cap C^0_{G/K}$ is contained in $C_{\text{hol}}$. Let $\xi \in t^*_\geq 0 \cap C^0_{G/K}$: We have to check that $\langle \xi, \beta \rangle > 0$ for any $\beta \in \mathfrak{h}_n^+$. Let $X, Y \in \mathfrak{g}$ such that $X + iY \in (p \otimes \mathbb{C})\beta$. We choose the following normalization: The vector $h_\beta := [X, Y]$ satisfies $\langle \beta, h_\beta \rangle = 1$. We see then that $\langle \xi, \beta \rangle = \frac{1}{\|h_\beta\|} \langle \xi, h_\beta \rangle$ for any $\xi \in \mathfrak{g}^*_e$. Standard computation [28] gives: $e^{t\text{ad}(X)}z = z + (\cosh(t) - 1)h_\beta + \sinh(t)Y$, $\forall t \in \mathbb{R}$. By definition, we must have $\langle \xi + \eta, e^{t\text{ad}(X)}z \rangle \geq 0$, $\forall t \in \mathbb{R}$, for any $\eta \in t^*$ small enough. It imposes that $\langle \xi, h_\beta \rangle > 0$. The first point is settled.

The other inclusion $C_{\text{hol}} \subset t^*_\geq 0 \cap C^0_{G/K}$ is a consequence of the next lemma. \hfill \square

**Lemma 2.3.** For any compact subset $\mathcal{K}$ of $C_{\text{hol}}$, there exists $c_\mathcal{K} > 0$ such that $\langle \xi, gz \rangle \geq c_\mathcal{K}\|gz\|$, $\forall g \in G$, $\forall \xi \in \mathcal{K}$.

**Proof.** Let us choose some maximal strongly orthogonal system $\Sigma \subset \mathfrak{h}_n^+$. The real span $\mathfrak{a}$ of the $X_\beta, \beta \in \Sigma$ is a maximal abelian subspace of $\mathfrak{g}$. Hence, any element $g \in G$ can be written $g = ke^{X(t)}k'$ with $X(t) = \sum_{\beta \in \Sigma} t_\beta X_\beta$ and $k, k' \in K$. We get

$$gz = k \left( z + \sum_{\beta \in \Sigma} (\cosh(t_\beta) - 1)h_\beta + \sum_{\beta \in \Sigma} \sinh(t_\beta)Y_\beta \right)$$

and

$$\langle \xi, gz \rangle = \langle k^{-1}\xi, z \rangle + \sum_{\beta \in \Sigma} (\cosh(t_\beta) - 1)\langle k^{-1}\xi, h_\beta \rangle.$$ 

For any $\xi \in C_{\text{hol}}$, we define $c_\xi := \min_{\beta \in \mathfrak{h}_n^+} \langle \xi, h_\beta \rangle > 0$. Let $\pi: t^* \to t^*$ be the projection. We have $\langle k^{-1}\xi, z \rangle = \langle \pi(k^{-1}\xi), z \rangle$ and $\langle k^{-1}\xi, h_\beta \rangle = \langle \pi(k^{-1}\xi), h_\beta \rangle$. The convexity theorem of Kostant tell us that $\pi(k^{-1}\xi)$ belongs to the convex hull of $\{ w_\xi, w \in W \}$. It follows that $\langle k^{-1}\xi, z \rangle \geq \langle \xi, z \rangle > 0$ and $\langle k^{-1}\xi, h_\beta \rangle \geq c_\xi > 0$ for any $k \in K$. We obtain then that $\langle \xi, gz \rangle \geq \frac{1}{2} \min(\langle \xi, z \rangle, c_\xi)\|X(t)\|$ for any $\xi \in C_{\text{hol}}$, where $\|X(t)\| = \sup_t |t_\beta|$. From equation (7), it is not difficult to see that there exists $C > 0$ such that $\|gz\| \leq C\|X(t)\|$ for any $g = ke^{X(t)}k' \in G$.

Let $\mathcal{K}$ be a compact subset of $C_{\text{hol}}$. Take $c_\mathcal{K} = \frac{1}{2c} \min(\min_{\xi \in \mathcal{K}} \langle \xi, z \rangle, \min_{\xi \in \mathcal{K}} c_\xi) > 0$. The previous computations show that $\langle \xi, gz \rangle \geq c_\mathcal{K}\|gz\|$, $\forall g \in G$, $\forall \xi \in \mathcal{K}$. \hfill \square
The following result is needed in §4.1.

Lemma 2.4. The map $p : C^0_{G/K} \to C_{\text{hol}}$ is continuous.

Proof. Let $(\xi_n)$ be a sequence of $C^0_{G/K}$ converging to $\xi_\infty \in C^0_{G/K}$. Let $\xi'_n = p(\xi_n)$ and $\xi'_\infty = p(\xi_\infty)$: We have to prove that the sequence $(\xi'_n)$ converges to $\xi'_\infty$. We choose elements $g_n, g_\infty \in G$ such that $\xi_n = g_n \xi'_n, \forall n$ and $\xi_\infty = g_\infty \xi'_\infty$.

First, we notice that $-b(\xi_n, \xi_n) = \|\xi'_n\|^2$; hence, the sequence $(\xi'_n)$ is bounded. We will now prove that the sequence $(g_n)$ is bounded. Let $\epsilon > 0$ such that $b(\xi_\infty + \eta, g z) \geq 0, \forall g \in G, \forall \|\eta\| \leq \epsilon$. If $\|\xi_0 - \xi_\infty\| \leq \epsilon/2$, we write $\xi = \frac{1}{2}(\xi_0 + \xi_\infty) + \frac{1}{2}\xi_\infty$, and then

$$\langle \xi, gz \rangle = \frac{1}{2}\langle \xi_0 + 2(\xi - \xi_\infty), gz \rangle + \frac{1}{2}\langle \xi_\infty, gz \rangle \geq \frac{1}{2}\langle \xi_\infty, gz \rangle, \quad \forall g \in G.$$ 

Now we have $\langle \xi'_n, z \rangle = \langle \xi_n, g_n z \rangle \geq \frac{1}{2}\langle \xi_\infty, g_n z \rangle$ if $n$ is large enough. This shows that the sequence $(\xi'_\infty, g_n z)$ is bounded. If we use Lemma 2.3, we can conclude that the sequence $(g_n)$ is bounded.

Let $(\xi'_{\phi(n)})$ be a subsequence converging to $\ell \in t^*_{>0}$. Since $(g_{\phi(n)})$ is bounded, there exists a subsequence $(g_{\phi \circ \psi(n)})$ converging to $h \in G$. From the relations $\xi_{\phi \circ \psi(n)} = g_{\phi \circ \psi(n)} \xi'_{\phi \circ \psi(n)}, \forall n \in N$, we obtain $\xi_\infty = h \ell$. Then $\ell = p(\xi_\infty) = \xi'_\infty$. Since every subsequence of $(\xi'_n)$ has a subsequential limit to $\xi'_\infty$, then the sequence $(\xi'_n)$ converges to $\xi'_\infty$. \hfill \Box

2.2. The cone $\Delta_{\text{hol}}(\tilde{G}, G)$ is closed.

We suppose that $G/K$ is a complex submanifold of a Hermitian symmetric space $\tilde{G}/\tilde{K}$. In other words, $\tilde{G}$ is a reductive real Lie group such that $G \subset \tilde{G}$ is a closed connected subgroup preserved by the Cartan involution, and $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$ containing $K$. We denote by $\check{g}$ and $\check{\mathfrak{k}}$ the Lie algebras of $\tilde{G}$ and $\tilde{K}$, respectively. We suppose that there exists a $\tilde{K}$-invariant element $z \in \mathfrak{t}$ such that $\text{ad}(z)|_{\check{\mathfrak{g}}}^2 = -I|_{\check{\mathfrak{g}}}$.

Let $C_{G/K} \subset \check{g}^*$ be the closed invariant cone associated to the Hermitian symmetric space $\tilde{G}/\tilde{K}$. We start with the following key fact.

Lemma 2.5. The projection $\pi_{\check{g}, \check{\mathfrak{g}}}: \check{g}^* \to \mathfrak{g}^*$ sends $C^0_{\tilde{G}/\tilde{K}}$ into $C^0_{G/K}$.

Proof. Let $\check{\xi} \in C^0_{\tilde{G}/\tilde{K}}$ and $\xi = \pi_{\check{g}, \check{\mathfrak{g}}} (\check{\xi})$. Then $\langle \check{\xi} + \check{\eta}, \check{g} z \rangle \geq 0, \forall \check{g} \in \check{G}$ if $\check{\eta} \in \check{g}^*$ is small enough. It follows that $\langle \xi + \pi_{\check{g}, \check{\mathfrak{g}}} (\check{\eta}), g z \rangle = \langle \check{\xi} + \check{\eta}, g z \rangle \geq 0, \forall g \in G$ if $\check{\eta}$ is small enough. Since $\pi_{\check{g}, \check{\mathfrak{g}}}$ is an open map, we can conclude that $\xi \in C^0_{G/K}$.

Let $\tilde{T}$ be a maximal torus of $\tilde{K}$, with Lie algebra $\check{\mathfrak{t}}$. The $\tilde{G}$-orbits in the interior of $C_{\tilde{G}/\tilde{K}}$ are parametrized by the holomorphic chamber $\check{C}_{\text{hol}} \subset \check{\mathfrak{t}}^*$. The previous lemma says that the projection $\pi_{\check{g}, \check{\mathfrak{g}}} (\check{O})$ of any $\tilde{G}$-orbit $\check{O} \subset C^0_{\tilde{G}/\tilde{K}}$ is the union of $G$-orbits $O \subset C^0_{G/K}$. So it is natural to study the following object:

$$\Delta_{\text{hol}}(\tilde{G}, G) := \left\{ (\check{\xi}, \xi) \in \check{C}_{\text{hol}} \times C_{\text{hol}}; \quad G \xi \subset \pi_{\check{g}, \check{\mathfrak{g}}} (\check{G} \check{\xi}) \right\}. \quad (8)$$

Here is a first result.
Proposition 2.6. \( \Delta_{\text{hol}}(\hat{G}, G) \) is a closed cone of \( \tilde{C}_{\text{hol}} \times C_{\text{hol}} \).

**Proof.** Suppose that a sequence \((\hat{\xi}_n, \xi_n) \in \Delta_{\text{hol}}(\hat{G}, G)\) converges to \((\hat{\xi}_\infty, \xi_\infty) \in \tilde{C}_{\text{hol}} \times C_{\text{hol}}\). By definition, there exists a sequence \((\hat{g}_n, g_n) \in \hat{G} \times G\) such that \(g_n \xi_n = \pi_{\hat{G}, G}(\hat{g}_n, \xi_n)\). Let \(\hat{h}_n := \hat{g}_n^{-1} \hat{g}_n\) so that \(\xi_n = \pi_{\hat{G}, 0}(\hat{h}_n, \xi_n)\) and \(\langle \hat{h}_n, \xi_n, z \rangle = \langle \xi_n, z \rangle\). We use now that the sequence \(\langle \xi_n, z \rangle\) is bounded and that the sequence \(\hat{\xi}_n\) belongs to a compact subset of \(\tilde{C}_{\text{hol}}\). Thanks to Lemma 2.3, these facts imply that \(\|\hat{h}_n^{-1} z\|\) is a bounded sequence. Hence, \(\hat{h}_n\) admits a subsequence converging to \(\hat{h}_\infty\). So we get \(\xi_\infty = \pi_{\hat{G}, 0}(\hat{h}_\infty, \xi_\infty)\), and that proves that \((\xi_\infty, \xi_\infty) \in \Delta_{\text{hol}}(\hat{G}, G)\). \(\square\)

### 2.3. Rational and weakly regular points

Let \((M, \Omega)\) be a symplectic manifold. We suppose that there exists a line bundle \(L\) with connection \(\nabla\) that prequantizes the 2-form \(\Omega\): In other words, \(\nabla^2 = -i \Omega\). Let \(K\) be a compact connected Lie group acting on \(L \to M\), and leaving the connection invariant. Let \(\Phi_K : M \to \mathfrak{k}^*\) be the moment map defined by Kostant’s relations

\[ L_X - \nabla_X = i \langle \Phi_K, X \rangle, \quad \forall X \in \mathfrak{k}. \]  

(9)

Here \(L_X\) is the Lie derivative acting on the sections of \(L \to M\).

Remark that relations (9) imply, via the equivariant Bianchi formula, the relations

\[ \iota(X_M)\Omega = -d\langle \Phi_K, X \rangle, \quad \forall X \in \mathfrak{k}, \]  

(10)

where \(X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} m\) is the vector field on \(M\) generated by \(X \in \mathfrak{k}\).

**Definition 2.7.** Let \(\dim_K(M) := \min_{m \in M} \dim \mathfrak{k}_m\). An element \(\xi \in \mathfrak{k}^*\) is a weakly regular value of \(\Phi_K\) if for all \(m \in \Phi_K^{-1}(\xi)\) we have \(\dim \mathfrak{k}_m = \dim_K(M)\).

When \(\xi \in \mathfrak{k}^*\) is a weakly regular value of \(\Phi_K\), the constant rank theorem tells us that \(\Phi_K^{-1}(\xi)\) is a submanifold of \(M\) stable under the action of the stabilizer subgroup \(K_\xi\). We see then that the reduced space

\[ M_\xi := \Phi_K^{-1}(\xi)/K_\xi \]  

(11)

is a symplectic orbifold.

Let \(T \subset K\) be a maximal torus with Lie algebra \(\mathfrak{t}\). We consider the lattice \(\wedge := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \to T)\) and the dual lattice \(\wedge^* \subset \mathfrak{t}^*\) defined by \(\wedge^* = \text{hom}(\wedge, \mathbb{Z})\). We remark that \(i\eta\) is a differential of a character of \(T\) if and only if \(\eta \in \wedge^*\). The \(\mathbb{Q}\)-vector space generated by the lattice \(\wedge^*\) is denoted by \(\mathfrak{t}_Q^*\). The vectors belonging to \(\mathfrak{t}_Q^*\) are designed as rational. An affine subspace \(V \subset \mathfrak{t}^*\) is called rational if it is affinely generated by its rational points.

We also fix a closed positive Weyl chamber \(\mathfrak{t}_{\geq 0}\). For each relatively open face \(\sigma \subset \mathfrak{t}_{\geq 0}\), the stabilizer \(K_\sigma\) of points \(\xi \in \sigma\) under the coadjoint action does not depend on \(\xi\) and will be denoted by \(K_\sigma\). The Lie algebra \(\mathfrak{k}_\sigma\) decomposes into its semisimple and central parts \(\mathfrak{k}_\sigma = [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma] \oplus \mathfrak{z}_\sigma\). The subspace \(\mathfrak{z}_\sigma^*\) is defined to be the annihilator of \([\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]\) or, equivalently, the fixed point set of the coadjoint \(K_\sigma\) action. Notice that \(\mathfrak{z}_\sigma^*\) is a rational subspace of \(\mathfrak{k}^*\) and that the face \(\sigma\) is an open cone of \(\mathfrak{z}_\sigma^*\).
We suppose that the moment map $\Phi_K$ is proper. The convexity theorem [1, 10, 16, 35, 22] tells us that $\Delta_K(M) := \text{Image}(\Phi_K) \cap t^*_0$ is a closed, convex, locally polyhedral set.

**Definition 2.8.** We denote by $\Delta_K(M)^0$ the subset of $\Delta_K(M)$ formed by the weakly regular values of the moment map $\Phi_K$ contained in $\Delta_K(M)$.

We will use the following remark in the next sections.

**Lemma 2.9.** The subset $\Delta_K(M)^0 \cap t^*_0$ is dense in $\Delta_K(M)$.

**Proof.** Let us first explain why $\Delta_K(M)^0$ is a dense open subset of $\Delta_K(M)$. There exists a unique open face $\tau$ of the Weyl chamber $t^*_{\geq 0}$ such as $\Delta_K(M) \cap \tau$ is dense in $\Delta_K(M)$: $\tau$ is called the principal face in [22]. The principal-cross-section theorem [22] tells us that $Y_\tau := \Phi^{-1}(\tau)$ is a symplectic $K_\tau$-manifold, with a trivial action of $[K_\tau, K_\tau]$. The line bundle $L_\tau := L|_{Y_\tau}$ prequantizes the symplectic structure on $Y_\tau$, and relations (10) show that $[K_\tau, K_\tau]$ acts trivially on $L_\tau$. Moreover, the restriction of $\Phi_K$ on $Y_\tau$ is the moment map $\Phi_\tau : Y_\tau \to z^*_\tau$ associated to the action of the torus $Z_\tau = \exp(z_\tau)$ on $L_\tau$.

Let $I \subset z^*_\tau$ be the smallest affine subspace containing $\Delta_K(M)$. Let $\mathfrak{z}_I \subset z_\tau$ be the annihilator of the subspace parallel to $I$: Relations (10) show that $\mathfrak{z}_I$ is the generic infinitesimal stabilizer of the $z_\tau$-action on $Y_\tau$. Hence, $\mathfrak{z}_I$ is the Lie algebra of the torus $Z_I := \exp(\mathfrak{z}_I)$.

We see then that any regular value of $\Phi_\tau : Y_\tau \to I$, viewed as a map with codomain $I$, is a weakly regular value of the moment map $\Phi_K$. It explains why $\Delta_K(M)^0$ is a dense open subset of $\Delta_K(M)$.

As the convex set $\Delta_K(M) \cap \tau$ is equal to $\Delta_{Z_\tau}(Y_\tau) := \text{Image}(\Phi_\tau)$, it is sufficient to check that $\Delta_{Z_\tau}(Y_\tau)^0 \cap t^*_Q$ is dense in $\Delta_{Z_\tau}(Y_\tau)$. The subtorus $Z_I \subset Z_\tau$ acts trivially on $Y_\tau$, and it acts on the line bundle $L_\tau$ through a character $\chi$. Let $\eta \in \wedge^* \cap t^*_\tau$ such that $d\chi = i\eta|_{\mathfrak{z}_I}$. The affine subspace $I$ which is equal to $\eta + (\mathfrak{z}_I)^\perp$ is rational. Since the open subset $\Delta_{Z_\tau}(Y_\tau)^0$ generates the rational affine subspace $I$, we can conclude that $\Delta_{Z_\tau}(Y_\tau)^0 \cap t^*_Q$ is dense in $\Delta_{Z_\tau}(Y_\tau)$.

**2.4. Weinstein’s theorem**

Let $\tilde{a} \in \tilde{C}_{\text{hol}}$. Consider the Hamiltonian action of the group $G$ on the coadjoint orbit $\tilde{G}\tilde{a}$. The moment map $\Phi_{\tilde{G}}^\tilde{a} : \tilde{G}\tilde{a} \to \mathfrak{g}^*$ corresponds to the restriction of the projection $\pi_{\mathfrak{g}, \tilde{a}}$ to $\tilde{G}\tilde{a}$. In this setting, the following conditions holds:

1. The action of $G$ on $\tilde{G}\tilde{a}$ is proper.
2. The moment map $\Phi_{\tilde{G}}^\tilde{a}$ is a proper map since the map $\langle \Phi_{\tilde{G}}^\tilde{a}, z \rangle$ is proper (see Lemma 2.3).

Conditions 1 and 2 impose that the image of $\Phi_{\tilde{G}}^\tilde{a}$ is contained in the open subset $\mathfrak{g}_{se}$ of strongly elliptic elements [31]. Thus, the $G$-orbits contained in the image of $\Phi_{\tilde{G}}^\tilde{a}$ are parametrized by the following subset of the holomorphic chamber $C_{\text{hol}}$:

$$\Delta_G(\tilde{G}\tilde{a}) := \text{Image}(\Phi_{\tilde{G}}^\tilde{a}) \cap t^*_0.$$ 

We notice that $\Delta_{\text{hol}}(\tilde{G},G) = \bigcup_{\tilde{a} \in \tilde{C}_{\text{hol}}} \{\tilde{a}\} \times \Delta_G(\tilde{G}\tilde{a})$. 
Like in Definition 2.7, an element $\xi \in \mathfrak{g}^*$ is a weakly regular value of $\Phi^\mu_G$ if for all $m \in (\Phi^\mu_G)^{-1}(\xi)$ we have $\dim \mathfrak{g}_m = \min_{x \in \tilde{G}\hat{a}} \dim(\mathfrak{g}_x)$. We denote by $\Delta_G(\tilde{G}\hat{a})^0$ the set of elements $\xi \in \Delta_G(\tilde{G}\hat{a})$ that are weakly regular for $\Phi^\mu_G$.

**Theorem 2.10** (Weinstein). For any $\hat{a} \in \tilde{C}_{\text{hol}}$, $\Delta_G(\tilde{G}\hat{a})$ is a closed convex subset contained in $\tilde{C}_{\text{hol}}$.

**Proof.** We recall briefly the arguments of the proof (see [38] or [31][§2]). Under Conditions 1 and 2, one checks easily that $Y_{\hat{a}} := (\Phi^\mu_G)^{-1}(\mathfrak{t}^*)$ is a smooth $K$-invariant symplectic submanifold of $\tilde{G}\hat{a}$ such that

$$G\hat{a} \simeq G \times_K Y_{\hat{a}}. \tag{12}$$

The moment map $\Phi^\mu_K : Y_{\hat{a}} \to \mathfrak{t}^*$, which corresponds to the restriction of the map $\Phi^\mu_G$ to $Y_{\hat{a}}$, is a proper map. Hence, the convexity theorem tells us that $\Delta_K(Y_{\hat{a}}) := \text{Image}(\Phi^\mu_K) \cap \mathfrak{t}^*_{\geq 0}$ is a closed, convex, locally polyhedral set. Thanks to the isomorphism (12), we see that $\Delta_G(\tilde{G}\hat{a})$ coincides with the closed convex subset $\Delta_K(Y_{\hat{a}})$. The proof is completed. \hfill $\square$

The next lemma is used in §4.1.

**Lemma 2.11.** Let $\tilde{a} \in \tilde{C}_{\text{hol}}$ be a rational element. Then $\Delta_G(\tilde{G}\tilde{a})^0 \cap \mathfrak{t}_Q^*$ is dense in $\Delta_G(\tilde{G}\tilde{a})$.

**Proof.** Thanks to equation (12), we know that $\Delta_G(\tilde{G}\tilde{a}) = \Delta_K(Y_{\tilde{a}})$. Relation (12) shows also that $\Delta_G(\tilde{G}\tilde{a})^0 = \Delta_K(Y_{\tilde{a}})^0$. Let $N \geq 1$ such that $\tilde{\mu} = N \tilde{a} \in \wedge^* \cap \tilde{C}_{\text{hol}}$. The stabilizer subgroup $\tilde{G}_{\tilde{\mu}}$ is compact, equal to $\tilde{K}_{\tilde{\mu}}$. Let us denote by $\mathbb{C}_{\tilde{\mu}}$ the one-dimensional representation of $\tilde{K}_{\tilde{\mu}}$ associated to $\tilde{\mu}$. The convex set $\Delta_G(\tilde{G}\tilde{a})$ is equal to $\frac{1}{N}\Delta_G(\tilde{G}\tilde{\mu})$, so it is sufficient to check that $\Delta_G(\tilde{G}\tilde{\mu})^0 \cap \mathfrak{t}_Q^* = \Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}_Q^*$ is dense in $\Delta_G(\tilde{G}\tilde{\mu}) = \Delta_K(Y_{\tilde{\mu}})$. The coadjoint orbit $\tilde{G}\tilde{\mu}$ is prequantized by the line bundle $\tilde{G} \times_{\tilde{K}_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}$, and the symplectic slice $Y_{\tilde{\mu}}$ is prequantized by the line bundle $L_{\tilde{\mu}} := \tilde{G} \times_{\tilde{K}_{\tilde{\mu}}} \mathbb{C}_{\tilde{\mu}}|_{Y_{\tilde{\mu}}}$. Thanks to Lemma 2.9, we know that $\Delta_K(Y_{\tilde{\mu}})^0 \cap \mathfrak{t}_Q^*$ is dense in $\Delta_K(Y_{\tilde{\mu}})$: The proof is complete. \hfill $\square$

3. Holomorphic discrete series

3.1. Definition

We return to the framework of §2.1. We recall the notion of holomorphic discrete series representations associated to a Hermitian symmetric spaces $G/K$. Let us introduce

$$C^\rho_{\text{hol}} := \{ \xi \in \mathfrak{t}^*_{\geq 0} | (\xi, \beta) \geq (2\rho_n, \beta), \forall \beta \in \mathfrak{h}^+_{\mathbb{R}} \},$$

where $2\rho_n = \sum_{\beta \in \mathfrak{h}^+_{\mathbb{R}}} \beta$ is $W$-invariant.

**Lemma 3.1.**

1. We have $C^\rho_{\text{hol}} \subset \tilde{C}_{\text{hol}}$.

2. For any $\xi \in \tilde{C}_{\text{hol}}$, there exists $N \geq 1$ such that $N\xi \in C^\rho_{\text{hol}}$.

**Proof.** The first point is due to the fact that $(\beta_0, \beta_1) \geq 0$ for any $\beta_0, \beta_1 \in \mathfrak{h}^+_{\mathbb{R}}$. The second point is obvious. \hfill $\square$
We will be interested in the following subset of dominant weights:

$$\hat{G}_{\text{hol}} := \bigwedge^* \cap C_{\text{hol}}^\rho.$$ 

Let Sym($p$) be the symmetric algebra of the complex $K$-module ($p, \text{ad}(z)$).

**Theorem 3.2** (Harish–Chandra). For any $\lambda \in \hat{G}_{\text{hol}}$, there exists an irreducible unitary representation of $G$, denoted by $V^G_\lambda$, such that the vector space of $K$-finite vectors is $V^G_\lambda|_K := V^K_\lambda \otimes \text{Sym}(p)$. 

The set $V^G_\lambda, \lambda \in \hat{G}_{\text{hol}}$ corresponds to the holomorphic discrete series representations associated to the complex structure $\text{ad}(z)$.

### 3.2. Restriction

We come back to the framework of §2.2. We consider two compatible Hermitian symmetric spaces $G/K \subset \tilde{G}/\tilde{K}$, and we look after the restriction of holomorphic discrete series representations of $\tilde{G}$ to the subgroup $G$.

Let $\tilde{\lambda} \in \hat{G}_{\text{hol}}$. Since the representation $V^G_{\tilde{\lambda}}$ is discretely admissible relatively to the circle group $\exp(\mathbb{R}z)$, it is also discretely admissible relatively to $G$. We can be more precise [15, 24, 21]:

**Proposition 3.3.** We have an Hilbertian direct sum

$$V^G_{\tilde{\lambda}}|_G = \bigoplus_{\lambda \in \hat{G}_{\text{hol}}} m^\lambda_{\tilde{\lambda}} V^G_\lambda,$$

where the multiplicity $m^\lambda_{\tilde{\lambda}} := [V^G_\lambda : V^G_{\tilde{\lambda}}]$ is finite for any $\lambda$.

The Hermitian $\tilde{K}$-vector space $\tilde{p}$, when restricted to the $K$-action, admits an orthogonal decomposition $\tilde{p} = p \oplus q$. Notice that the symmetric algebra Sym($q$) is an admissible $K$-module.

In [15], H. P. Jakobsen and M. Vergne obtained the following nice characterization of the multiplicities $[V^G_\lambda : V^G_{\tilde{\lambda}}]$. Another proof is given in [31], §4.4.

**Theorem 3.4** (Jakobsen–Vergne). Let $(\tilde{\lambda}, \lambda) \in \hat{G}_{\text{hol}} \times \hat{G}_{\text{hol}}$. The multiplicity $[V^G_\lambda : V^G_{\tilde{\lambda}}]$ is equal to the multiplicity of the representation $V^K_\lambda$ in $\text{Sym}(q) \otimes V^K_{\tilde{\lambda}}|_K$.

### 3.3. Discrete analogues of $\Delta_{\text{hol}}(\tilde{G}, G)$

We define the following discrete analogues of the cone $\Delta_{\text{hol}}(\tilde{G}, G)$:

$$\Pi^G_{\text{hol}}(\tilde{G}, G) := \left\{ (\tilde{\lambda}, \lambda) \in \hat{G}_{\text{hol}} \times \hat{G}_{\text{hol}} \mid [V^G_\lambda : V^G_{\tilde{\lambda}}] \neq 0 \right\},$$

and

$$\Pi^Q_{\text{hol}}(\tilde{G}, G) := \left\{ (\tilde{\xi}, \xi) \in \hat{C}_{\text{hol}} \times \hat{C}_{\text{hol}} \mid \exists N \geq 1, (N\xi, N\tilde{\xi}) \in \Pi^G_{\text{hol}}(\tilde{G}, G) \right\}.$$ 

We have the following key fact.
Proposition 3.5.

- $\Pi_{\text{hol}}^Z(\tilde{G}, G)$ is a subset of $\lambda^* \times \wedge^*$ stable under the addition.
- $\Pi_{\text{hol}}^Q(\tilde{G}, G)$ is a $\mathbb{Q}$-convex cone of the $\mathbb{Q}$-vector space $t_{\mathbb{Q}}^* \times t_{\mathbb{Q}}^*$.

Proof. Suppose that $a_1 := (\lambda_1, \lambda_1)$ and $a_2 := (\lambda_2, \lambda_2)$ belongs to $\Pi_{\text{hol}}^Z(\tilde{G}, G)$. Thanks to Theorem 3.4, we know that the $K$-modules $\text{Sym}(q) \otimes (V^K_{\lambda_j})^* \otimes V^K_{\lambda_j} K$ possess a nonzero invariant vector $\phi_j$, for $j = 1, 2$.

Let $X := K/T \times \tilde{K}/T$ be the product of flag manifolds. The complex structure is normalized so that $T(\varphi_1, \varphi_2)$, where $\varphi_j = \sum_{\alpha < 0} (tC)_{\alpha}$ and $\alpha_{\mathbb{C}} = \sum_{\alpha > 0} (tC)_{\alpha}$. We associate to each data $a_j$, the holomorphic line bundle $L_j := K \times_T C_{-\lambda_j} \otimes K \times_T C_{-\lambda_j}$ on $X$. Let $H^0(X, L_j)$ be the space of holomorphic sections of the line bundle $L_j$. The Borel–Weil theorem tells us that $H^0(X, L_j) \simeq (V^K_{\lambda_j})^* \otimes V^K_{\lambda_j} K$, $\forall j \in \{1, 2\}$.

We have $\phi_j \in [\text{Sym}(q) \otimes H^0(X, L_j)]^K_j$, $\forall j$, and then $\phi_1 \phi_2 \in \text{Sym}(q) \otimes H^0(X, L_1 \otimes L_2)$ is a nonzero invariant vector. Hence, $[\text{Sym}(q) \otimes (V^K_{\lambda_1 + \lambda_2})^* \otimes V^K_{\lambda_1 + \lambda_2} K] \neq 0$. Thanks to Theorem 3.4, we can conclude that $a_1 + a_2 = (\lambda_1 + \lambda_2, \lambda_1 + \lambda_2)$ belongs to $\Pi_{\text{hol}}^Z(\tilde{G}, G)$. The first point is proved. From the first point, one checks easily that

- $\Pi_{\text{hol}}^Z(\tilde{G}, G)$ is stable under addition,
- $\Pi_{\text{hol}}^Z(\tilde{G}, G)$ is stable by expansion by a nonnegative rational number.

The second point is settled. \hfill \square

3.4. Riemann–Roch numbers

We come back to the framework of §2.3.

We associate to a dominant weight $\mu \in \wedge^*_+ \cong \lambda_0^*$ the (possibly singular) symplectic reduced space $M_\mu := \Phi^{-1}_K(\mu)/K_\mu$ and the (possibly singular) line bundle over $M_\mu$:

$$L_\mu := \left( L \mid_{\Phi^{-1}_K(\mu) \otimes \mathbb{C}_{-\mu}} \right)/K_\mu.$$

Suppose first that $\mu$ is a weakly regular value of $\Phi_K$. Then $M_\mu$ is an orbifold equipped with a symplectic structure $\Omega_\mu$, and $L_\mu$ is a line orbibundle over $M_\mu$ that prequantizes the symplectic structure. By choosing an almost complex structure on $M_\mu$ compatible with $\Omega_\mu$, we get a decomposition $\wedge \mathbf{T}^* M_\mu \otimes \mathbb{C} = \oplus_{i,j} \wedge^i \mathbf{T}^* M_\mu$ of the bundle of differential forms. Using Hermitian structures in the tangent bundle $\mathbf{T} M_\mu$ of $M_\mu$ and in the fibers of $L_\mu$, we define a Dolbeault–Dirac operator

$$D_\mu^+ : \mathcal{A}^{0,+}(M_\mu, L_\mu) \longrightarrow \mathcal{A}^{0,-}(M_\mu, L_\mu),$$

where $\mathcal{A}^{i,j}(M_\mu, L_\mu) = \Gamma(M_\mu, \wedge^{i,j} \mathbf{T}^* M_\mu \otimes L_\mu)$.

Definition 3.6. Let $\mu \in \wedge^*_+ \cong \lambda_0^*$ be a weakly regular value of the moment map $\Phi_K$. The Riemann–Roch number $RR(M_\mu, L_\mu) \in \mathbb{Z}$ is defined as the index of the elliptic operator $D_\mu^+ : RR(M_\mu, L_\mu) = \dim(\ker(D_\mu^+)) - \dim(\text{coker}(D_\mu^+))$.

Suppose that $\mu \notin \Delta_K(M)$. Then $M_\mu = \emptyset$, and we take $RR(M_\mu, L_\mu) = 0$. 

Suppose now that \( \mu \in \Delta_K(M) \) is not (necessarily) a weakly regular value of \( \Phi_K \). Take a small element \( \epsilon \in \mathfrak{t}^* \) such that \( \mu + \epsilon \) is a weakly regular value of \( \Phi_K \) belonging to \( \Delta_K(M) \). We consider the symplectic orbifold \( M_{\mu+\epsilon} \). If \( \epsilon \) is small enough,
\[
\mathcal{L}_{\mu,\epsilon} := \left( \mathcal{L}|_{\Phi_K^{-1}(\mu+\epsilon) \otimes \mathbb{C}_{-\mu}} \right)/K_{\mu+\epsilon}.
\]
is a line orbi-bundle over \( M_{\mu+\epsilon} \).

We have the following important result (see §3.4.3 in [34]).

**Proposition 3.7.** Let \( \mu \in \Delta_K(M) \cap \land^* \). The Riemann–Roch number \( RR(M_{\mu+\epsilon}, \mathcal{L}_{\mu,\epsilon}) \) do not depend on the choice of \( \epsilon \) small enough so that \( \mu + \epsilon \in \Delta_K(M) \) is a weakly regular value of \( \Phi_K \).

We can now introduce the following definition.

**Definition 3.8.** Let \( \mu \in \land^*_+ \). We define
\[
Q(M_\mu, \Omega_\mu) = \begin{cases} 
0 & \text{if } \mu \notin \Delta_K(M), \\
RR(M_{\mu+\epsilon}, \mathcal{L}_{\mu,\epsilon}) & \text{if } \mu \in \Delta_K(M) \text{ and } ||\epsilon|| << 1.
\end{cases}
\]
Above, \( \epsilon \) is chosen small enough so that \( \mu + \epsilon \in \Delta_K(M) \) is a weakly regular value of \( \Phi_K \).

Let \( n \geq 1 \). The manifold \( M \), equipped with the symplectic structure \( n\Omega \), is prequantized by the line bundle \( \mathcal{L}^\otimes n \): The corresponding moment map is \( n\Phi_K \). For any dominant weight \( \mu \in \land^*_+ \), the symplectic reduction of \( (M,n\Omega) \) relatively to the weight \( n\mu \) is \( (M_\mu,n\Omega_\mu) \). Like in Definition 3.8, we consider the following Riemann–Roch numbers
\[
Q(M_\mu, n\Omega_\mu) = \begin{cases} 
0 & \text{if } \mu \notin \Delta_K(M), \\
RR(M_{\mu+\epsilon}, (\mathcal{L}_{\mu,\epsilon})^\otimes n) & \text{if } \mu \in \Delta_K(M) \text{ and } ||\epsilon|| << 1.
\end{cases}
\]

The Kawasaki–Riemann–Roch formula shows that \( n \geq 1 \mapsto Q(M_\mu, n\Omega_\mu) \) is a quasi-polynomial map \([37, 23]\). When \( \mu \) is a weakly regular value of \( \Phi_K \), we denote by \( \text{vol}(M_\mu) := \frac{1}{d_\mu} \int_{M_\mu} \frac{\Omega_\mu}{2\pi} \frac{\dim M_\mu}{2} \) the symplectic volume of the symplectic orbifold \( (M_\mu, \Omega_\mu) \). Here, \( d_\mu \) is the generic value of the map \( m \in \Phi_K^{-1}(\mu) \mapsto \text{cardinal}(K_m/K_m^0) \).

The following proposition is a direct consequence of the Kawasaki–Riemann–Roch formula (see [23] or §1.3.4 in [30]).

**Proposition 3.9.** Let \( \mu \in \Delta_K(M) \cap \land^*_+ \) be a weakly regular value of \( \Phi_K \). Then we have \( Q(M_\mu, n\Omega_\mu) \sim \text{vol}(M_\mu) n^{\frac{\dim M_\mu}{2}} \) when \( n \to \infty \). In particular, the map \( n \geq 1 \mapsto Q(M_\mu, n\Omega_\mu) \) is nonzero.

### 3.5. Quantization commutes with reduction

Let us explain the “quantization commutes with reduction” theorem proved in [31].

We fix \( \lambda \in \widehat{G}_{\text{hol}} \). The coadjoint orbit \( \widehat{G}\lambda \) is prequantized by the line bundle \( \widehat{G} \times_{K_\lambda} \mathbb{C}_\lambda \), and the moment map \( \Phi^\lambda_G : \widehat{G}\lambda \to \mathfrak{g}^* \) corresponding to the \( G \)-action on \( \widehat{G} \times_{K_\lambda} \mathbb{C}_\lambda \) is equal to the restriction of the map \( \pi_{\mathfrak{g},\widehat{G}} \) to \( \widehat{G}\lambda \).
The symplectic slice $Y_{\lambda} = (\Phi_{\tilde{G}})^{-1}(t^*)$ is prequantized by the line bundle $L_{\lambda} := \tilde{G} \times_{K_{\lambda}} \mathbb{C}_{\lambda}|_{Y_{\lambda}}$. The moment map $\Phi_{\tilde{K}} : Y_{\lambda} \rightarrow t^*$ corresponding to the $K$-action is equal to the restriction of $\Phi_{\tilde{G}}$ to $Y_{\lambda}$.

For any $\lambda \in \hat{\tilde{G}}_{\text{hol}}$, we consider the (possibly singular) symplectic reduced space $X_{\tilde{\lambda}, \lambda} := (\Phi_{\tilde{G}})^{-1}(\lambda)/K_{\lambda}$ equipped with the reduced symplectic form $\Omega_{\tilde{\lambda}, \lambda}$, and the (possibly singular) line bundle $L_{\tilde{\lambda}, \lambda} := (L_{\tilde{G}}|_{{\Phi}_{\tilde{G}}^{-1}(\lambda)} \otimes \mathbb{C}_{-\lambda})/K_{\lambda}$.

Thanks to Definition 3.8, the geometric quantization $Q(X_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda}) \in \mathbb{Z}$ of those compact symplectic spaces $(X_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda})$ are well-defined even if they are singular. In particular, $Q(X_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda}) = 0$ when $X_{\tilde{\lambda}, \lambda} = \emptyset$.

The following theorem is proved in [31].

**Theorem 3.10.** Let $\tilde{\lambda} \in \hat{\tilde{G}}_{\text{hol}}$. We have an Hilbertian direct sum

$$V_{\tilde{\lambda}}^{\tilde{G}}|_{G} = \bigoplus_{\lambda \in \hat{G}_{\text{hol}}} Q(X_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda}) V_{\lambda}^{G}. $$

It means that, for any $\lambda \in \hat{\tilde{G}}_{\text{hol}}$, the multiplicity of the representation $V_{\lambda}^{G}$ in the restriction $V_{\tilde{\lambda}}^{\tilde{G}}|_{G}$ is equal to the geometric quantization $Q(X_{\tilde{\lambda}, \lambda}, \Omega_{\tilde{\lambda}, \lambda})$ of the (compact) reduced space $X_{\tilde{\lambda}, \lambda}$.

**Remark 3.11.** Let $(\tilde{\lambda}, \lambda) \in \hat{\tilde{G}}_{\text{hol}} \times \hat{\tilde{G}}_{\text{hol}}$. Theorem 3.10. shows that

$$[V_{n\lambda}^{G} : V_{n\tilde{\lambda}}^{\tilde{G}}] = Q(X_{\tilde{\lambda}, \lambda}, n\Omega_{\tilde{\lambda}, \lambda})$$

for any $n \geq 1$.

### 4. Proofs of the main results

We come back to the setting of §2.2: $G/K$ is a complex submanifold of a Hermitian symmetric space $\tilde{G}/\tilde{K}$. It means that there exits a $\tilde{K}$-invariant element $z \in t$ such that $\text{ad}(z)$ defines complex structures on $\tilde{p}$ and $p$. We consider the orthogonal decomposition $\tilde{p} = p \oplus q$, and we denote by $\text{Sym}(q)$ the symmetric algebra of the complex $K$-module $(q, \text{ad}(z))$.

#### 4.1. Proof of Theorem A

The set $\Delta_{\text{hol}}(\tilde{G}, G)$ is equal to $\bigcup_{\tilde{a} \in \tilde{c}_{\text{hol}}} \{\tilde{a}\} \times \Delta_{\text{G}}(\tilde{G}\tilde{a})$. We define

$$\Delta_{\text{hol}}(\tilde{G}, G)^0 := \bigcup_{\tilde{a} \in \tilde{c}_{\text{hol}}} \{\tilde{a}\} \times \Delta_{\text{G}}(\tilde{G}\tilde{a})^0.$$ We start with the following result.
Lemma 4.1. The set $\Delta_{\text{hol}}(\hat{G},G)^0 \cap \hat{t}_Q^* \times t_Q^*$ is dense in $\Delta_{\text{hol}}(\hat{G},G)$.

Proof. Let $(\tilde{\xi},\xi) \in \Delta_{\text{hol}}(\hat{G},G)$; take $\tilde{g} \in \hat{G}$ such that $\xi = \pi_{\hat{g}^{-1}}(\tilde{g}\tilde{\xi})$. We consider a sequence $\xi_n \in \hat{C}_{\text{hol}} \cap t_Q^*$ converging to $\xi$. Then $\xi_n := \pi_{\hat{g}_n^{-1}}(\tilde{g}\xi_n)$ is a sequence of $C_{G/K}^0$ converging to $\xi \in \hat{C}_{\text{hol}}$. Since the map $p : C_{G/K}^0 \to \hat{C}_{\text{hol}}$ is continuous (see Lemma 2.4), the sequence $\eta_n := p(\xi_n)$ converges to $p(\xi) = \xi$. By definition, we have $\eta_n \in \Delta_G(\hat{G}\xi_n)$ for any $n \in \mathbb{N}$. Since $\xi_n$ are rational, each subset $\Delta_G(\hat{G}\xi_n) \cap t_Q^*$ is dense in $\Delta_G(\hat{G}\xi_n)$ (see Lemma 2.11). Hence, for each $\eta_n \in \Delta_G(\hat{G}\xi_n)$, there exists $\zeta_n \in \Delta_G(\hat{G}\xi_n) \cap t_Q^*$ such that $\|\zeta_n - \eta_n\| \leq 2^{-n}$. Finally, we see that $(\xi_n, \zeta_n)$ is a sequence of rational elements of $\Delta_{\text{hol}}(\hat{G},G)^0$ converging to $(\xi, \xi)$.

The main purpose of this section is the proof of the following theorem.

Theorem 4.2. For any rational element $(\tilde{\mu}, \mu)$ of the holomorphic chamber $\hat{C}_{\text{hol}} \times C_{\text{hol}}$, the following statements hold:

- If $\mu \in \Delta_G(\hat{G}\tilde{\mu})^0$, then $(\tilde{\mu},\mu) \in \Pi_{\text{hol}}^Q(\hat{G},G)$.
- If $(\tilde{\mu},\mu) \in \Pi_{\text{hol}}^Q(\hat{G},G)$, then $\mu \in \Delta_G(\hat{G}\tilde{\mu})$.

In other words, we have the following inclusions:

$$\Delta_{\text{hol}}(\hat{G},G)^0 \cap \hat{t}_Q^* \times t_Q^* \subset \Pi_{\text{hol}}^Q(\hat{G},G) \subset \Delta_{\text{hol}}(\hat{G},G).$$

Lemma 4.1 and Theorem 4.2 gives the important corollary.

Corollary 4.3. $\Pi_{\text{hol}}^Q(\hat{G},G)$ is dense in $\Delta_{\text{hol}}(\hat{G},G)$.

Proof of Theorem 4.2. Let $(\tilde{\mu},\mu) \in \Pi_{\text{hol}}^Q(\hat{G},G)$: There exists $N \geq 1$ such that $(N\tilde{\mu}, N\mu) \in \Pi_{\text{hol}}^Z(\hat{G},G)$. The multiplicity $[V_{N\tilde{\mu}}^G : V_{N\mu}^\hat{G}]$ is nonzero, and thanks to Theorem 3.10, it implies that the reduced space $X_{N\tilde{\mu}, N\mu}$ is nonempty. In other words, $(N\tilde{\mu}, N\mu) \in \Delta_{\text{hol}}(\hat{G},G)$. The inclusion (2) is proven.

Let $(\tilde{\mu},\mu) \in \Delta_{\text{hol}}(\hat{G},G)^0 \cap \hat{t}_Q^* \times t_Q^*$. There exists $N_0 \geq 1$ such that $\lambda := N_0\mu \in \hat{G}_{\text{hol}}$, $\lambda := N_0\tilde{\mu} \in \hat{G}_{\text{hol}}$ and $\lambda \in \Delta_G(\hat{G}\lambda)^0$: The element $\lambda$ is a weakly regular value of the moment map $\hat{G}\lambda \to g^*$. Theorem 3.10 tells us that, for any $n \geq 1$, the multiplicity $[V_{n\lambda}^G : V_{n\lambda}^\hat{G}]$ is equal to Riemann–Roch number $Q(X_{\lambda, \lambda}, n\Omega_{\lambda, \lambda})$. Since the map $n \mapsto Q(X_{\lambda, \lambda}, n\Omega_{\lambda, \lambda})$ is nonzero (see Proposition 3.9), we can conclude that there exists $n_0 \geq 1$ such that $[V_{n_0\lambda}^G : V_{n_0\lambda}^\hat{G}] \neq 0$. In other words, we obtain $n_0 N_0 (\tilde{\mu},\mu) \in \Pi_{\text{hol}}^Z(\hat{G},G)$ and so $(\tilde{\mu},\mu) \in \Pi_{\text{hol}}^Q(\hat{G},G)$. The inclusion (1) is settled.

Now we can terminate the proof of Theorem A.

Thanks to Proposition 3.5, we know that $\Pi_{\text{hol}}^Q(\hat{G},G)$ is a $Q$-convex cone. Since $\Delta_{\text{hol}}(\hat{G},G)$ is a closed subset of $C_{\text{hol}} \times C_{\text{hol}}$ (see Proposition 2.6), we can conclude, by a density argument, that $\Delta_{\text{hol}}(\hat{G},G)$ is a closed convex cone of $\hat{C}_{\text{hol}} \times C_{\text{hol}}$. 


4.2. The affine variety $\tilde{K}_C \times q$

Let $\tilde{\kappa}$ be the Killing form on the Lie algebra $\tilde{g}$. We consider the $\tilde{K}$-invariant symplectic structures $\Omega_\tilde{p}$ on $\tilde{p}$, defined by the relation

$$\Omega_\tilde{p}(\tilde{Y},\tilde{Y}') = \tilde{\kappa}(z,[\tilde{Y},\tilde{Y}']), \forall \tilde{Y},\tilde{Y}' \in \tilde{p}.$$ 

We notice that the complex structure $\text{ad}(z)$ is adapted to $\Omega_\tilde{p}$: $\Omega_\tilde{p}(\tilde{Y},\text{ad}(z)\tilde{Y}) > 0$ if $\tilde{Y} \neq 0$.

We denote by $\Omega_q$ the restriction of $\Omega_\tilde{p}$ on the symplectic subspace $q$. The moment map $\Phi_q$ associated to the $K$-action on $(q,\Omega_q)$ is defined by the relations $\langle \Phi_q(Y),X \rangle = \frac{-1}{2}\tilde{\kappa}([X,Y],[z,Y])$, $\forall (X,Y) \in p \times q$. In particular, $\langle \Phi_q(Y),z \rangle = \frac{-1}{2}\|Y\|^2$, $\forall Y \in q$, so the map $\langle \Phi_q,z \rangle$ is proper.

The complex reductive group $\tilde{K}_C$ is equipped with the following action of $\tilde{K} \times K$: $(\tilde{k},k) \cdot a = \tilde{k}ak^{-1}$. It has a canonical structure of a smooth affine variety. There is a diffeomorphism of the cotangent bundle $T^*\tilde{K}$ with $\tilde{K}_C$ defined as follows. We identify $T^*\tilde{K}$ with $\tilde{K} \times \mathfrak{t}^*$ by means of left-translation and then with $\tilde{K} \times \mathfrak{t}$ by means of an invariant inner product on $\mathfrak{t}$. The map $\varphi : K \times \mathfrak{t} \to \tilde{K}_C$ given by $\varphi(a,X) = ae^{iX}$ is a diffeomorphism. If we use $\varphi$ to transport the complex structure of $\tilde{K}_C$ to $T^*\tilde{K}$, then the resulting complex structure on $T^*\tilde{K}$ is compatible with the symplectic structure on $T^*\tilde{K}$ so that $T^*\tilde{K}$ becomes a Kähler Hamiltonian $\tilde{K} \times K$-manifold (see [11], §3). The moment map relative to the $\tilde{K} \times K$-action is the proper map $\Phi_{\tilde{K} \oplus K} : T^*\tilde{K} \to \mathfrak{t}^* \oplus \mathfrak{t}^*$ defined by $\Phi_{\tilde{K}}(\tilde{a},\tilde{\eta}) = -\tilde{a}\tilde{\eta}$ and $\Phi_K(\tilde{a},\tilde{\eta}) = \pi_{\mathfrak{t},\mathfrak{t}}(\tilde{\eta})$. Here $\pi_{\mathfrak{t},\mathfrak{t}} : \mathfrak{t}^* \to \mathfrak{t}^*$ is the projection dual to the inclusion $\mathfrak{t} \hookrightarrow \mathfrak{t}$ of Lie algebras.

Finally, we consider the Kähler Hamiltonian $\tilde{K} \times K$-manifold $T^*\tilde{K} \times q$, where $q$ is equipped with the symplectic structure $\Omega_q$. Let us denote by $\Phi : T^*\tilde{K} \times q \to \mathfrak{t}^* \oplus \mathfrak{t}^*$ the moment map relative to the $\tilde{K} \times K$-action:

$$\Phi(\tilde{a},\tilde{\eta},Y) = \left( -\tilde{a}\tilde{\eta}, \pi_{\mathfrak{t},\mathfrak{t}}(\tilde{\eta}) + \Phi_q(Y) \right).$$

(15)

Since $\Phi$ is proper map, the convexity theorem tells us that

$$\Delta(T^*\tilde{K} \times q) := \text{Image}(\Phi) \bigcap \mathfrak{t}_{\geq 0}^* \times \mathfrak{t}_{\geq 0}^*$$

is a closed convex locally polyhedral set.

We consider now the action of $\tilde{K} \times K$ on the affine variety $\tilde{K}_C \times q$. The set of highest weights of $\tilde{K}_C \times q$ is the semigroup $\Pi^\mathbb{Z}(\tilde{K}_C \times q) \subset \mathbb{A}^*_+ \times \mathbb{A}^*_+$ consisting of all dominant weights $(\lambda,\bar{\lambda})$ such that the irreducible $\tilde{K} \times K$-representation $V_{\bar{\lambda}}^{\tilde{K}} \otimes V_\lambda^K$ occurs in the coordinate ring $\mathbb{C}[\tilde{K}_C \times q]$. We denote by $\Pi^\mathbb{Q}(\tilde{K}_C \times q)$ the $\mathbb{Q}$-convex cone generated by the semigroup $\Pi^\mathbb{Z}(\tilde{K}_C \times q)$: $(\xi,\bar{\xi}) \in \Pi^\mathbb{Q}(\tilde{K}_C \times q)$ if and only if $\exists N \geq 1$, $N(\xi,\bar{\xi}) \in \Pi^\mathbb{Z}(\tilde{K}_C \times q)$.

The following important fact is classical (see Theorem 4.9 in [35]).

**Proposition 4.4.** The Kirwan polyhedron $\Delta(T^*\tilde{K} \times q)$ is equal to the closure of the $\mathbb{Q}$-convex cone $\Pi^\mathbb{Q}(\tilde{K}_C \times q)$. 

A direct application of the Peter–Weyl theorem gives the following characterization:

\[
(\tilde{\lambda}, \lambda) \in \Pi^Z(\bar{K}_C \times q) \iff \left[ V^K_\lambda : V^K_\lambda \otimes \text{Sym}(q) \right]^K \neq 0
\]

\[
\iff \left[ V^K_\lambda : V^K_\lambda \otimes \text{Sym}(q) \right] \neq 0
\]

\[
\iff (\tilde{\lambda}, \lambda^*) \in \Pi^Z_q(\bar{K}, K).
\]

4.3. **Proof of Theorem B**

Consider the semigroup $\Pi^Z_q(\bar{K}, K)$ of $\mathbb{Z}^* \times \mathbb{Q}^+$ (see Definition 1.3) and the $\mathbb{Q}$-convex cone $\Pi^Q_q(\bar{K}, K) := \{ (\tilde{\xi}, \xi) \in t^*_0 \times t^*_0 \mid \exists N \geq 1, N(\tilde{\xi}, \xi) \in \Pi^Z_q(\bar{K}, K) \}$.

The Jakobsen–Vergne theorem says that $\Pi^Z_q(\bar{K}, K)$ is diffeomorphic, as a $\bar{\xi}$-symplectic manifold, to the symplectic vector space $(\bar{\xi}, \xi, \xi^*)$. By definition $\tilde{\lambda} = \Phi(t, \tilde{\lambda}) \in \Pi^Z_q(\bar{K}, K)$ is contained in $\Pi^Z_q(\bar{K}, K)$.

4.4. **Proof of Theorem C**

We denote by $\bar{q}$ the $K$-vector space $q$ equipped with the opposite symplectic form $-\Omega_q$ and opposite complex structure $-\text{ad}(z)$. The moment map relative to the $K$-action on $\bar{q}$ is denoted by $\Phi_q = -\Phi_q$.

**Lemma 4.5.** Any element $(\tilde{\xi}, \xi) \in t^*_0 \times t^*_0$ satisfies the equivalence

\[
(\tilde{\xi}, \xi^*) \in \Delta(T^* \bar{K} \times q) \iff \xi \in \Delta_K(\bar{K} \tilde{\xi} \times \bar{q}).
\]

**Proof.** Thanks to equation (15), we see immediately that $\exists (\tilde{a}, \tilde{a}, Y) \in T^* \bar{K} \times q$ such that $(\tilde{\xi}, \xi^*) = \Phi(\tilde{a}, \tilde{a}, Y)$ if and only if $\exists (\tilde{b}, Z) \in \bar{K} \times q$ such that $\xi = \pi_{\tilde{t}, \tilde{t}}(\tilde{b}, \tilde{\xi}) + \Phi_q(Z)$.

At this stage, we know that $\Delta_K(\bar{K} \tilde{\mu} \times \bar{q}) = \Delta_{\bar{t}}(\bar{K} \tilde{\mu} \times \bar{q}) \cap C_{\text{hol}}$. Hence, Theorem C will follow from the next result.

**Proposition 4.6.** For any $\tilde{\mu} \in \bar{C}_{\text{hol}}$, the Kirwan polyhedron $\Delta_{\bar{t}}(\bar{K} \tilde{\mu} \times \bar{q})$ is contained in $C_{\text{hol}}$.

**Proof.** By definition $C_{\text{hol}} = C^0_G/K \cap t^*_{\geq 0}$, so we have to prove that $\pi_{\tilde{t}, \tilde{t}}(\tilde{K} \tilde{\mu}) + \text{Image}(\Phi_q)$ is contained in $C^0_{G/K}$. By definition $\tilde{K} \tilde{\mu} \subset C^0_{G/K}$, and then $\pi_{\tilde{t}, \tilde{t}}(\tilde{K} \tilde{\mu}) \subset C^0_{G/K}$. Since $C^0_{G/K} + C_{G/K} \subset C^0_{G/K}$, it is sufficient to check that $\text{Image}(\Phi_q) \subset C_{G/K}$. Let $\Phi_{\tilde{p}}$ be the moment map relative to the action of $\tilde{K}$ on $(\tilde{p}, \Omega_{\tilde{p}})$. As $\text{Image}(\Phi_q) \subset \pi_{\tilde{t}, \tilde{t}}(\text{Image}(-\Phi_{\tilde{p}}))$, the following lemma will terminate the proof of Proposition 4.6.

**Lemma 4.7.** The image of the moment map $-\Phi_{\tilde{p}}$ is contained in $\bar{C}_{\text{hol}}$.

**Proof.** Let $z^* \in \tilde{t}^*$ such that $\langle z^*, \tilde{X} \rangle = -\hat{\xi}(z, \tilde{X})$, $\forall \tilde{X} \in \tilde{g}$. Consider the coadjoint orbit $\tilde{O} = Gz^*$ equipped with its canonical symplectic structure $\Omega_{\tilde{O}}$. The symplectic vector space $T_{z^*} \tilde{O}$ is canonically isomorphic to $(\tilde{p}, -\Omega_{\tilde{p}})$. In [26], McDuff proved that $(\tilde{O}, \Omega_{\tilde{O}})$ is diffeomorphic, as a $\tilde{K}$-symplectic manifold, to the symplectic vector space $(\tilde{p}, -\Omega_{\tilde{p}})$.
(see [6, 8] for a generalization of this fact). McDuff’s results show in particular that Image(−Φ_p) = π_bar,t(Ŵ). Our proof is completed if we check that π_bar,t(Ŵ) ⊂ C_{G/˜K}. In other words, if ⟨π_bar,t(˜g_0 z^*), ˜g_1 z⟩ ≥ 0, ∀ ˜g_0, ˜g_1 ∈ ˜G. But
\[ 2⟨π_bar,t(˜g_0 z^*), ˜g_1 z⟩ = ⟨˜g_0 z^*, ˜g_1 z + Θ(˜g_1)z⟩ = −κ(z, ˜g_0^{-1} ˜g_1 z) − κ(z, ˜g_0^{-1} Θ(˜g_1)z). \]

With equation (7) in hand, it is not difficult to see that −κ(z, ˜g z) ≥ 0 for every ˜g ∈ ˜G. We thus verified that π_bar,t(Ŵ) ⊂ C_{G/˜K}. □

5. Inequalities characterizing the cones Δ_hol(˜G, G)

We come back to the framework of §4.2. We consider the Kähler Hamiltonian ˜K × K-manifold T^* ˜K × q. The moment map, Φ : T^* ˜K × q → ˜t^* ⊕ t^*, relative to the ˜K × K-action, is defined by equation (15).

In this section, we adapt to our case the result of §6 of [32] concerning the parametrization of the facets of Kirwan polyhedrons in terms of Ressayre’s data.

5.1. Admissible elements

We choose maximal tori ˜T ⊂ ˜K and T ⊂ K such that T ⊂ ˜T. Let R_o and R be, respectively, the set of roots for the action of T on (ĝ/g) ⊗ C and g ⊗ C. Let R̂ be the set of roots for the action of ˜T on ĝ ⊗ C. Let R^+ ⊂ R and ˜R^+ ⊂ ˜R be the systems of positive roots defined in equation (6). Let W, ˜W be the Weyl groups of (T, K) and (˜T, ˜K). Let w_o ∈ W be the longest element.

We start by introducing the notion of admissible elements. The group hom(U(1), T) admits a natural identification with the lattice ∧ := \frac{1}{2π} ker(exp : t → T). A vector γ ∈ t is called rational if it belongs to the Q-vector space t_Q generated by ∧.

We consider the ˜K × K-action on N := T^* ˜K × q. We associate to any subset X ⊂ N, the integer dim_{K×K}(X) (see equation (5)).

**Definition 5.1.** A nonzero element (γ, γ) ∈ ˜t × t is called admissible if the elements ˜γ and γ are rational and if dim_{K×K}(N(γ)) − dim_{K×K}(N) ∈ {0, 1}.

If γ ∈ t, we denote by R_o ∩ γ⊥ the subsets of weight vanishing against γ. We start with the following lemma whose proof is left to the reader (see §6.1.1 of [32]).

**Lemma 5.2.**

1. N(γ) ≠ ∅ if and only if ˜γ ∈ ˜W γ.
2. dim_{K×K}(N(γ)) = dim_T(\tilde{g}/g) = dim(t) − dim(Vect(R_o)).
3. For any ˜w ∈ ˜W, dim_{K×K}(N(˜w, γ)) = dim_T(\tilde{g}^γ/\tilde{g}^\gamma) = dim(t) − dim(Vect(R_o ∩ γ⊥)).

The next result is a direct consequence of the previous lemma.
Lemma 5.3. The admissible elements relative to the $\hat{K} \times K$-action on $T^*\hat{K} \times q$ are of the form $(\hat{w} \gamma, \gamma)$, where $\hat{w} \in \hat{W}$ and $\gamma$ is a nonzero rational element satisfying $\text{Vect}(\mathfrak{R}_0 \cap \gamma^\perp) = \text{Vect}(\mathfrak{R}_0 \cap \gamma^\perp)$.

5.2. Ressayre’s data

Definition 5.4.

1. Consider the linear action $\rho : G \to \text{GL}_C(V)$ of a compact Lie group on a complex vector space $V$. For any $(\eta, a) \in \mathfrak{g} \times \mathbb{R}$, we define the vector subspace $V^{\eta=a} = \{v \in V, d\rho(\eta)v = iav\}$. Thus, for any $\eta \in \mathfrak{g}$, we have the decomposition $V = V^{\eta>0} \oplus V^{\eta=0} \oplus V^{\eta<0}$, where $V^{\eta>0} = \sum_{a>0} V^{\eta=a}$, and $V^{\eta<0} = \sum_{a<0} V^{\eta=a}$.

2. The real number $\text{Tr}_\eta(V^{\eta>0})$ is defined as the sum $\sum_{a>0} a \dim(V^{\eta=a})$.

We consider an admissible element $(\hat{w} \gamma, \gamma)$. The submanifold of $N \simeq \hat{K}_C \times q$ fixed by $(\hat{w} \gamma, \gamma)$ is $N(\hat{w} \gamma, \gamma) = \hat{w} \hat{K}_C \times q^\gamma$. There is a canonical isomorphism between the manifold $N(\hat{w} \gamma, \gamma)$ equipped with the action of $\hat{w} \hat{K}_C \times q^\gamma$ with the manifold $\hat{K}_C \times q^\gamma$ equipped with the action of $K^\gamma \times K^\gamma$. The tangent bundle $(T_N|_{N(\hat{w} \gamma, \gamma)})(\hat{w} \gamma, \gamma)>0$ is isomorphic to $N^{\nu_0} \times \tilde{\xi}_C^{\gamma>0} \times q^{\gamma>0}$.

The choice of positive roots $\mathfrak{R}^+$ (resp. $\tilde{\mathfrak{R}}^+$) induces a decomposition $\tilde{\xi}_C = n \oplus \mathfrak{t}_C \oplus \tilde{n}$ (resp. $\tilde{\xi}_C = \tilde{n} \oplus \tilde{\xi}_C \oplus \tilde{n}$), where $n = \sum_{\alpha \in \mathfrak{R}^+} (\mathfrak{t} \otimes \mathbb{C})_\alpha$ (resp. $\tilde{n} = \sum_{\tilde{\alpha} \in \tilde{\mathfrak{R}}^+} (\tilde{\xi} \otimes \mathbb{C})_{\tilde{\alpha}}$). We consider the map

$$\rho^{\hat{w} \gamma} : \hat{K}_C^\gamma \times q^\gamma \to \text{hom}\left(\hat{\mathfrak{n}}^{\hat{w} \gamma>0} \times n^{\gamma>0}, \hat{\xi}_C^{\gamma>0} \times q^{\gamma>0}\right)$$

defined by the relation

$$\rho^{\hat{w} \gamma}(\hat{x}, v) : (\hat{X}, X) \mapsto ((\hat{w} \hat{x})^{-1} \hat{X} - X ; X \cdot v)$$

for any $(\hat{x}, v) \in \hat{K}_C^\gamma \times q^\gamma$.

Definition 5.5. $(\gamma, \hat{w}) \in t \times \hat{W}$ is a Ressayre’s datum if

1. $(\hat{w} \gamma, \gamma)$ is admissible,
2. $\exists (\hat{x}, v)$ such that $\rho^{\hat{w} \gamma}(\hat{x}, v)$ is bijective.

Remark 5.6. In [32], the Ressayre’s data were called regular infinitesimal $B$-Ressayre’s pairs.

Since the linear map $\rho^{\hat{w} \gamma}(\hat{x}, v)$ commutes with the $\gamma$-actions, we obtain the following necessary conditions.

Lemma 5.7. If $(\gamma, \hat{w}) \in t \times \hat{W}$ is a Ressayre’s datum, then

- Relation (A): $\dim(\hat{\mathfrak{n}}^{\hat{w} \gamma>0}) + \dim(n^{\gamma>0}) = \dim(\hat{\xi}_C^{\gamma>0}) + \dim(q^{\gamma>0})$.
- Relation (B): $\text{Tr}_{\hat{w} \gamma}(\hat{\mathfrak{n}}^{\hat{w} \gamma>0}) + \text{Tr}_\gamma(n^{\gamma>0}) = \text{Tr}_\gamma(\hat{\xi}_C^{\gamma>0}) + \text{Tr}_\gamma(q^{\gamma>0})$. 
Lemma 5.8. Relation (B) is equivalent to
\[ \sum_{\substack{\alpha \in \mathfrak{m}^+ \\ \langle \alpha, \gamma \rangle > 0}} \langle \alpha, \gamma \rangle = \sum_{\alpha \in \mathfrak{h}^+} \sum_{\substack{\tilde{\alpha} \in \mathfrak{h}^+ \\ \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle. \] (17)

Proof. First, one sees that
\[ \text{Tr}_\gamma(q^{\gamma > 0}) = \text{Tr}_\gamma(\tilde{\mu}^{\gamma > 0}) - \text{Tr}_\gamma(p^{\gamma > 0}) = \sum_{\alpha \in \mathfrak{m}^+} \sum_{\substack{\tilde{\alpha} \in \mathfrak{h}^+ \\ \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle. \]

Relation (B) is equivalent to
\[ \text{Tr}_\gamma(n^{\gamma > 0}) + \sum_{\alpha \in \mathfrak{m}^+} \sum_{\substack{\tilde{\alpha} \in \mathfrak{h}^+ \\ \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle. \] (18)

Since \( \tilde{\mathfrak{h}}_n^+ \) is invariant under the action of the Weyl group \( \tilde{W} \), the right-hand side of equation (18) is equal to \( \sum_{\alpha \in \mathfrak{m}^+} \sum_{\substack{\tilde{\alpha} \in \mathfrak{h}^+ \\ \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle > 0}} \langle \tilde{\alpha}, \omega_0 \bar{w} \gamma \rangle. \) Since the left-hand side of equation (18) is equal to \( \sum_{\alpha \in \mathfrak{m}^+} \langle \alpha, \gamma \rangle \), the proof of the lemma is complete. \( \square \)

5.3. Cohomological characterization of Ressayre’s data

Let \( \gamma \in \mathfrak{t} \) be a nonzero rational element. We denote by \( B \subset K_\mathfrak{C} \) and by \( \tilde{B} \subset \tilde{K}_\mathfrak{C} \) the Borel subgroups with Lie algebra \( \mathfrak{b} = \mathfrak{t}_\mathfrak{C} \oplus \mathfrak{n} \) and \( \mathfrak{b} = \mathfrak{t}_\mathfrak{C} \oplus \tilde{\mathfrak{n}} \). Consider the parabolic subgroup \( P_\gamma \subset K_\mathfrak{C} \) defined by
\[ P_\gamma = \{ g \in K_\mathfrak{C}, \lim_{t \to \infty} \exp(-it\gamma)g\exp(it\gamma) \text{ exists} \}. \] (19)

Similarly, one defines a parabolic subgroup \( \tilde{P}_\gamma \subset \tilde{K}_\mathfrak{C} \).

We work with the projective varieties \( \mathcal{F}_\gamma := K_\mathfrak{C}/P_\gamma \), \( \tilde{\mathcal{F}}_\gamma := \tilde{K}_\mathfrak{C}/\tilde{P}_\gamma \) and the canonical embedding \( \iota : \mathcal{F}_\gamma \to \tilde{\mathcal{F}}_\gamma \). We associate to any \( \tilde{w} \in \tilde{W} \), the Schubert cell \( \tilde{X}_{\tilde{w},\gamma}^0 := \tilde{B}[\tilde{w}] \subset \tilde{\mathcal{F}}_\gamma \) and the Schubert variety \( \tilde{X}_{\tilde{w},\gamma} := \overline{\tilde{X}_{\tilde{w},\gamma}^0} \). If \( \tilde{W}_\gamma \) denotes the subgroup of \( \tilde{W} \) that fixes \( \gamma \), we see that the Schubert cell \( \tilde{X}_{\tilde{w},\gamma}^0 \) and the Schubert variety \( \tilde{X}_{\tilde{w},\gamma} \) depend only of the class of \( \tilde{w} \) in \( \tilde{W}/\tilde{W}_\gamma \).

On the variety \( \mathcal{F}_\gamma \), we consider the Schubert cell \( \mathcal{X}_\gamma^0 := B[e] \) and the Schubert variety \( \mathcal{X}_\gamma := \overline{\mathcal{X}_\gamma^0} \).

We consider the cohomology\(^1\) ring \( H^*(\tilde{\mathcal{F}}_\gamma, \mathbb{Z}) \) of \( \tilde{\mathcal{F}}_\gamma \). If \( Y \) is an irreducible closed subvariety of \( \tilde{\mathcal{F}}_\gamma \), we denote by \( [Y] \in H^{2n_Y}(\tilde{\mathcal{F}}_\gamma, \mathbb{Z}) \) its cycle class in cohomology: Here \( n_Y = \text{codim}_{\mathfrak{C}}(Y) \). Let \( \iota^* : H^*(\tilde{\mathcal{F}}_\gamma, \mathbb{Z}) \to H^*(\mathcal{F}_\gamma, \mathbb{Z}) \) be the pull-back map in cohomology. Recall that the cohomology class \( [pt] \) associated to a singleton \( Y = \{ pt \} \subset \mathcal{F}_\gamma \) is a basis of \( H_{\text{max}}^*(\mathcal{F}_\gamma, \mathbb{Z}) \).

\(^1\)Here, we use singular cohomology with integer coefficients.
To an oriented real vector bundle $E \to N$ of rank $r$, we can associate its Euler class $\text{Eul}(E) \in H^{2r}(N, \mathbb{Z})$. When $\mathcal{V} \to N$ is a complex vector bundle, then $\text{Eul}(\mathcal{V}_R)$ corresponds to the top Chern class $c_p(\mathcal{V})$, where $p$ is the complex rank of $\mathcal{V}$, and $\mathcal{V}_R$ means $\mathcal{V}$ viewed as a real vector bundle oriented by its complex structure (see [5], §21).

The isomorphism $q^{\gamma>0} \simeq q/q^{\leq 0}$ shows that $q^{\gamma>0}$ can be viewed as a $P_{\gamma}$-module. Let $[q^{\gamma>0}] = K_G \times P_{\gamma} q^{\gamma>0}$ be the corresponding complex vector bundle on $\mathcal{F}_\gamma$. We denote simply by $\text{Eul}(q^{\gamma>0})$ the Euler class $\text{Eul}([q^{\gamma>0}]_R) \in H^*(\mathcal{F}_\gamma, \mathbb{Z})$.

The following characterization of Ressayre’s data was obtained in [32], §6. Recall that $\mathfrak{R}_o$ denotes the set of weights relative to the $T$-action on $(\mathfrak{g}/\mathfrak{g}) \otimes \mathbb{C}$.

**Proposition 5.9.** An element $(\gamma, \tilde{w}) \in t \times \tilde{W}$ is a Ressayre’s datum if and only if the following conditions hold:

- $\gamma$ is nonzero and rational.
- $\text{Vect}(\mathfrak{R}_o \cap \gamma^+) = \text{Vect}(\mathfrak{R}_o) \cap \gamma^+$.
- $[X_\gamma] \cdot t^*([X_{\tilde{w}}, \gamma]) \cdot \text{Eul}(q^{\gamma>0}) = k[pt]$, $k \geq 1$ in $H^*(\mathcal{F}_\gamma, \mathbb{Z})$.
- $\sum_{\alpha \in \mathfrak{R}_o^+} \langle \alpha, \gamma \rangle = \sum_{\tilde{\alpha} \in \mathfrak{R}_o^+} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w}_0 \gamma \rangle$.

### 5.4. Parametrization of the facets

We can finally describe the Kirwan polyhedron $\Delta(T^* \tilde{K} \times q)$ (see [32], §6).

**Theorem 5.10.** An element $(\xi, \xi) \in t_{\geq 0}^* \times t_{\geq 0}^*$ belongs to $\Delta(T^* \tilde{K} \times q)$ if and only if

$$\langle \tilde{\xi}, \tilde{w} \rangle + \langle \xi, \gamma \rangle \geq 0$$

for any Ressayre’s datum $(\gamma, \tilde{w}) \in t \times \tilde{W}$.

Theorem 5.10 and Theorem B permit us to give the following description of the convex cone $\Delta_{\text{hol}}(G, G)$.

**Theorem 5.11.** An element $(\tilde{\xi}, \xi)$ belongs to $\Delta_{\text{hol}}(G, G)$ if and only if $(\tilde{\xi}, \xi) \in \tilde{C}_{\text{hol}} \times C_{\text{hol}}$ and

$$\langle \tilde{\xi}, \tilde{w} \gamma \rangle \geq \langle \xi, w_0 \gamma \rangle$$

for any $(\gamma, \tilde{w}) \in t \times \tilde{W}$ satisfying the following conditions:

- $\gamma$ is nonzero and rational.
- $\text{Vect}(\mathfrak{R}_o \cap \gamma^+) = \text{Vect}(\mathfrak{R}_o) \cap \gamma^+$.
- $[X_\gamma] \cdot t^*([X_{\tilde{w}}, \gamma]) \cdot \text{Eul}(q^{\gamma>0}) = k[pt]$, $k \geq 1$ in $H^*(\mathcal{F}_\gamma, \mathbb{Z})$.
- $\sum_{\alpha \in \mathfrak{R}_o^+} \langle \alpha, \gamma \rangle = \sum_{\tilde{\alpha} \in \mathfrak{R}_o^+} \langle \tilde{\alpha}, \tilde{w}_0 \tilde{w}_0 \gamma \rangle$.

### 6. Example: the holomorphic Horn cone $\text{Horn}_{\text{hol}}(p, q)$

Let $p \geq q \geq 1$. We consider the pseudo-unitary group $G = U(p, q) \subset GL_{p+q}(\mathbb{C})$ defined by the relation: $g \in U(p, q)$ if and only if $g \text{Id}_{p, q} \cdot q^* = \text{Id}_{p, q}$, where $\text{Id}_{p, q}$ is the diagonal matrix $\text{Diag}(\text{Id}_p, -\text{Id}_q)$. 


We work with the maximal compact subgroup \( K = U(p) \times U(q) \subset G \). We have the Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \), where \( \mathfrak{p} \) is identified with the vector space \( M_{p,q} \) of \( p \times q \) matrices through the map
\[
X \in M_{p,q} \mapsto \begin{pmatrix} 0 & X \end{pmatrix}. 
\]

We work with the element \( z_{p,q} = \frac{i}{2} \text{Id}_{p,q} \) which belongs to the center of \( \mathfrak{k} \). The adjoint action of \( z_{p,q} \) on \( \mathfrak{p} \) corresponds to the standard complex structure on \( M_{p,q} \).

The trace on \( \mathfrak{gl}_{p+q}(\mathbb{C}) \) defines an identification \( g \simeq g^* = \text{hom}(g, \mathbb{R}) \): To \( X \in g \) we associate \( \xi_X \in g^* \) defined by \( \langle \xi_X, Y \rangle = -\text{Tr}(XY) \). Thus, the \( G \)-invariant cone \( C_{G/K} \) defined by \( z_{p,q} \) can be viewed as the following cone of \( g \):
\[
C(p,q) = \left\{ X \in g, \ \text{Im} \left( \text{Tr}(gXg^{-1} \text{Id}_{p,q}) \right) \geq 0, \ \forall g \in U(p,q) \right\}.
\]

Let \( T \subset U(p) \times U(q) \) be the maximal torus formed by the diagonal matrices. The Lie algebra \( \mathfrak{t} \) is identified with \( \mathbb{R}^p \times \mathbb{R}^q \) through the map \( \mathfrak{d} : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathfrak{u}(p) \times \mathfrak{u}(q) \): \( \mathfrak{d}_x = \text{Diag}(ix_1, \cdots, ix_p, ix_{p+1}, \cdots, ix_{p+q}) \). The Weyl chamber is
\[
t_{\geq 0} = \{ x \in \mathbb{R}^p \times \mathbb{R}^q, x_1 \geq \cdots \geq x_p \text{ and } x_{p+1} \geq \cdots \geq x_{p+q} \}.
\]

Proposition 2.2 tells us that the \( U(p,q) \) adjoint orbits in the interior of \( C(p,q) \) are parametrized by the holomorphic chamber
\[
C_{p,q} = \{ x \in \mathbb{R}^p \times \mathbb{R}^q, x_1 \geq \cdots \geq x_p > x_{p+1} \geq \cdots \geq x_{p+q} \} \subset t_{\geq 0}.
\]

**Definition 6.1.** The holomorphic Horn cone \( \text{Horn}_{\text{hol}}(p,q) := \text{Horn}^2_{\text{hol}}(U(p,q)) \) is defined by the relations
\[
\text{Horn}_{\text{hol}}(p,q) = \{ (A,B,C) \in (C_{p,q})^3, U(p,q)\mathfrak{d}_C \subset U(p,q)\mathfrak{d}_A + U(p,q)\mathfrak{d}_B \}.
\]

Let us detail the description given of \( \text{Horn}_{\text{hol}}(p,q) \) by Theorem B. For any \( n \geq 1 \), we consider the semigroup \( \land_{+} = \{ (\lambda_1, \cdots, \lambda_n) \} \subset \mathbb{Z}^n \). If \( \lambda = (\lambda', \lambda'') \in \land_{+} \times \land_{+} \), then \( V_{\lambda} := V_{\lambda'}^{U(p)} \otimes V_{\lambda''}^{U(q)} \) denotes the irreducible representation of \( U(p) \times U(q) \) with highest weight \( \lambda \). We denote by \( \text{Sym}(M_{p,q}) \) the symmetric algebra of \( M_{p,q} \).

**Definition 6.2.**

1. \( \text{Horn}^{\mathbb{Z}}(p,q) \) is the semigroup of \( (\land_{+} \times \land_{+})^3 \) defined by the conditions:
\[
(\lambda, \mu, \nu) \in \text{Horn}^{\mathbb{Z}}(p,q) \iff [V_\nu : V_\lambda \otimes V_\mu \otimes \text{Sym}(M_{p,q}) \neq 0].
\]

2. \( \text{Horn}(p,q) \) is the convex cone of \( (t_{\geq 0})^3 \) defined as the closure of \( \mathbb{Q}^{>0} \cdot \text{Horn}^{\mathbb{Z}}(p,q) \).

Theorem B asserts that
\[
\text{Horn}_{\text{hol}}(p,q) = \text{Horn}(p,q) \cap (C_{p,q})^3. \tag{20}
\]

In another article [33], we obtained a recursive description of the cones \( \text{Horn}(p,q) \). This allows us to give the following description of the holomorphic Horn cone \( \text{Horn}_{\text{hol}}(2,2) \).
Example 6.3. An element \((A,B,C) \in (\mathbb{R}^4)^3\) belongs to \(\text{Hornhol}(2,2)\) if and only if the following conditions hold:

\[
\begin{align*}
    a_1 &\geq a_2 > a_3 \geq a_4 \\
    b_1 &\geq b_2 > b_3 \geq b_4 \\
    c_1 &\geq c_2 > c_3 \geq c_4 \\
\end{align*}
\]

\[
\begin{align*}
    a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 &= c_1 + c_2 + c_3 + c_4 \\
    a_1 + a_2 + b_1 + b_2 &\leq c_1 + c_2 \\
    \begin{align*}
    a_2 + b_2 &\leq c_2 \\
    a_2 + b_1 &\leq c_1 \\
    a_1 + b_2 &\leq c_1 \\
    a_3 + b_3 &\geq c_3 \\
    a_3 + b_4 &\geq c_4 \\
    a_4 + b_3 &\geq c_4 \\
    a_2 + a_4 + b_2 + b_4 &\leq c_1 + c_4 \\
    a_2 + a_4 + b_2 + b_4 &\leq c_2 + c_3 \\
    a_2 + a_4 + b_1 + b_4 &\leq c_1 + c_3 \\
    a_1 + a_4 + b_2 + b_4 &\leq c_1 + c_3 \\
    a_2 + a_4 + b_2 + b_3 &\leq c_1 + c_3 \\
    a_2 + a_3 + b_2 + b_4 &\leq c_1 + c_3 \\
\end{align*}
\]

7. A conjectural symplectomorphism

Let \(\tilde{\mu} \in \tilde{\mathcal{C}}_{\text{hol}}\). In this section, we are interested in the geometry of the coadjoint orbit \(\tilde{G}\tilde{\mu}\) viewed as a Hamiltonian \(G\)-manifold with proper moment map \(\Phi_{\tilde{G}}^{\tilde{\mu}} : \tilde{G}\tilde{\mu} \to \mathfrak{g}^*\).

We start with a decomposition that we have already used. The pullback \(Y_{\tilde{\mu}} = (\Phi_{\tilde{G}}^{\tilde{\mu}})^{-1}(\mathfrak{t}^*)\) is a symplectic submanifold of \(\tilde{G}\tilde{\mu}\) which is stable under the \(K\)-action: Let \(\Omega_{\tilde{\mu}}\) be the corresponding two form on \(Y_{\tilde{\mu}}\). The action of \(K\) on \((Y_{\tilde{\mu}}, \Omega_{\tilde{\mu}})\) is Hamiltonian, with a proper moment map \(\Phi_{K}^{\tilde{\mu}} : Y_{\tilde{\mu}} \to \mathfrak{t}^*\) equal to the restriction of \(\Phi_{\tilde{G}}^{\tilde{\mu}}\) to \(Y_{\tilde{\mu}}\).

The map \([g,x] \mapsto gx\) defines a symplectomorphism

\[
G \times_K Y_{\tilde{\mu}} \simeq \tilde{G}\tilde{\mu}
\]

so that \(\Phi_{G}^{\tilde{\mu}}([g,x]) = g \cdot \Phi_{K}^{\tilde{\mu}}(x)\) [31]. This allows us to see that the Kirwan polytope \(\Delta_G(\tilde{G}\tilde{\mu})\) relative to the \(G\)-action on \(\tilde{G}\tilde{\mu}\) is equal to the Kirwan polytope \(\Delta_K(Y_{\tilde{\mu}})\) relative to the \(K\)-action on \(Y_{\tilde{\mu}}\).
We consider the orthogonal decomposition $\tilde{p} = p \oplus q$. Mostow's decomposition theorem [27] says that the map $\psi : p \times q \times \tilde{K} \to \tilde{G}, (X,Y,\tilde{k}) \mapsto e^X e^Y \tilde{k}$ is a diffeomorphism. This leads to the following result.

**Lemma 7.1.** We have the following $G$-equivariant diffeomorphisms:

\[
\psi_o : G \times_K \left( q \times \tilde{K} \right) \to \tilde{G},
\begin{bmatrix} g; Y, \tilde{k} \end{bmatrix} \mapsto ge^Y \tilde{k},
\]

\[
\psi_\mu : G \times_K \left( q \times \tilde{K} \tilde{\mu} \right) \to \tilde{G}\tilde{\mu},
\begin{bmatrix} g; Y, \xi \end{bmatrix} \mapsto ge^Y \xi.
\]

We obtain the following geometric information on the $K$-manifold $Y_{\tilde{\mu}}$.

**Corollary 7.2.** There exists a $K$-equivariant diffeomorphism $q \times \tilde{K} \tilde{\mu} \simeq Y_{\tilde{\mu}}$.

**Proof.** Thanks to the diffeomorphisms (21) and $\psi_\mu$, we know that the manifolds $G \times_K Y_{\tilde{\mu}}$ and $G \times_K (q \times \tilde{K} \tilde{\mu})$ admit a $G$-equivariant diffeomorphism. Our result follows from this.

Let $\check{\kappa}$ be the Killing form on the Lie algebra $\tilde{g}$. We consider the $\tilde{K}$-invariant symplectic structures $\Omega_{\tilde{p}}$ on $\tilde{p}$, defined by the relation $\Omega_{\tilde{p}}(Y,Y') = \check{\kappa}(z,[Y,Y']), \forall Y,Y' \in \tilde{p}$. We denote by $\Omega_q$ the restriction of $\Omega_{\tilde{p}}$ on the symplectic subspace $q$.

We consider the following symplectic structure $-\Omega_q \times \Omega_{\tilde{K} \tilde{\mu}}$ on $q \times \tilde{K} \tilde{\mu}$. Knowing that $\Delta_G(\tilde{G} \tilde{\mu}) = \Delta_K(Y_{\tilde{\mu}})$, the following conjectural result would give another proof of Theorem C.

**Conjecture 7.3.** There exists a $K$-equivariant symplectomorphism between $(Y_{\tilde{\mu}}, \Omega_{\tilde{\mu}})$ and $(q \times \tilde{K} \tilde{\mu}, -\Omega_q \times \Omega_{\tilde{K} \tilde{\mu}})$.

This conjecture generalizes some results obtained when $G = \tilde{K}$:

1. In [26], McDuff showed that $\tilde{G} \tilde{\mu} \simeq \tilde{G} / \tilde{K}$ admit a $\tilde{K}$-equivariant symplectomorphism with $(\tilde{p}, -\Omega_{\tilde{p}})$ when $\tilde{\mu}$ is a central element of $\tilde{k}^*$.
2. In [8], Deltour extended the result of McDuff by showing that $\tilde{G} \tilde{\mu}$ admits a $\tilde{K}$-equivariant symplectomorphism with $(\tilde{p} \times \tilde{K} \tilde{\mu}, -\Omega_{\tilde{p}} \times \Omega_{\tilde{K} \tilde{\mu}})$ for any $\tilde{\mu} \in \tilde{C}_{\text{hol}}$.

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Horn Problem for Quasi-Hermitian Lie Groups

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