On the Feynman Rules of Massive Gauge Theory in Physical Gauges

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For a massive gauge theory with Higgs mechanism in a physical gauge, the longitudinal polarization of gauge bosons can be naturally identified as mixture of the goldstone component and a remnant gauge component that vanishes at the limit of zero mass, making the goldstone equivalence manifest. In light of this observation, we re-examine the Feynman rules of massive gauge theory by treating gauge fields and their corresponding goldstone fields as single objects, writing them uniformly as 5-component “vector” fields. The gauge group is taken to be $SU(2)_L$ to preserve custodial symmetry. We find the derivation of gauge-goldstone propagators becomes rather trivial by noticing there is a remarkable parallel between massless gauge theory and massive gauge theory in this notation. We also derive the Feynman rules of all vertices, finding the vertex for self-interactions of vector (gauge-goldstone) bosons are especially simplified. We then demonstrate that the new form of the longitudinal polarization vector and the standard form give the same results for all the 3-point on-shell amplitudes. This on-shell matching confirms similar results obtained with on-shell approach for massive scattering amplitudes by Arkani-Hamed et al. in ref. [23]. Finally we calculate some $1 \rightarrow 2$ collinear splitting amplitudes by making use of the new Feynman rules and the on-shell match condition.
I. INTRODUCTION

In a massive gauge theory with Higgs mechanism, scattering amplitudes involving longitudinal polarizations have the famous problem of power counting\cite{1–3}: while single Feynman diagrams increase with energy, the S-matrix is well-behaved when taking into account the contribution of the Higgs boson. This failure of power counting causes many complications and confusion both practically and conceptually. The origin of the problem is the longitudinal polarization vector behave as $\epsilon^\mu_L \sim \frac{k^\mu}{m_W} + O(\frac{m_W}{E})$ in high energy limit. Another way to phrase it is that the longitudinal polarization vector and Feynman diagrams don’t have a smooth limit as $m_W \to 0$, thus it’s not clear how the theory approaches massless limit continuously. In future high energy colliders\cite{4, 5}, we will approach energy scales in which the EW symmetry will be effectively restored. This problem becomes even more severe.

Practically this problem is often solved by replacing the longitudinal vector bosons with the corresponding goldstone bosons, according to the so-called goldstone equivalence theorem (GET)\cite{11–15}, which states that scattering amplitudes involving longitudinal vector bosons can be approximated by the corresponding goldstone modes in high energy limit:

$$\mathcal{M}(W_L, W_L, W_L, ..., W_L) = (-i)^n \mathcal{M}(\phi, \phi, \phi, ..., \phi) + O\left(\frac{m_W}{\sqrt{s}}\right),$$

with $\sqrt{s}$ being the hard scale of the process. However, this solution is still not completely satisfactory, as GET is only an approximation with other terms suppressed in high energy limit. Although the approximation seems to work for naive power counting, it’s not utterly clear if the contributions of the terms neglected by GET are real subdominant. In fact, it was discovered in\cite{6} that there is a new class of splitting functions contributing to DGLAP evolution of EW PDFs and substructure of EW jets. Those new splitting functions originate precisely from the terms that are neglected by GET\cite{11–15}. It then becomes mandatory to account for all the terms that from the longitudinal polarization vector in calculation. Obviously we need a better solution for the power counting problem.

A physical gauge in a massive gauge theory can be defined by the gauge-fixing condition $n \cdot W = \frac{1}{2}$.

\footnote{The mistake of the naive power counting is that it neglects that a physical process is intrinsically multi-scaled. The terms neglected by GET has soft singularities (with infrared cut-offs provided by the masses) that give rise to contributions up to single logarithms, when the collinear scale $\lambda$ lies in $m_W \ll \lambda \ll \sqrt{s}$.}
0, with \( \mathbf{n} \) being any direction other than \( \mathbf{\hat{k}} \), is able to serve this purpose \[6–9\]. Heuristically we can argue this way: GET is the consequence of gauge symmetry. It can be derived from Ward identities of the theory. Nevertheless, there is also an alternative and a more direct way to prove GET, i.e. we can simply choose another gauge. Since in a physical gauge we only impose gauge-fixing on gauge fields, the gauge-goldstone mixing term in the Lagrangian remains, thus we are forced to identify gauge fields \( W^\mu \) and goldstone fields \( \phi \) as single fields, which we can denote as \( W^M = (W^\mu, \phi) \).

In the resulting gauge-goldstone propagator, goldstone modes and gauge modes obtain the same pole masses, the longitudinal polarization vectors are naturally identified as mixture of gauge components and goldstone components. We can write the longitudinal polarization vector as \( \epsilon^M_L = (\epsilon^\mu_n, -i) \), with \( \epsilon^\mu_n \sim -\frac{m_W}{E_n} n^\mu \) in high energy limit. Its specific form depends on the gauge direction \( n \). In this way, we obtain a precise formula of scattering amplitudes involving longitudinal vector bosons, which is a generalization of GET in Eq.(1),

\[
\mathcal{M}(W_L, W_L, W_L, \ldots, W_L) = (-i)^n \mathcal{M}(\phi, \phi, \ldots, \phi) + (-i)^{n-1} \mathcal{M}(W_n, \phi, \phi, \ldots, \phi) + \ldots + \mathcal{M}(W_n, W_n, W_n, \ldots, W_n)
\]

(2)

The polarization vectors of \( W_n \) are given by \( \epsilon_n \), which is usually neglected by GET. Thus physical gauges are vastly different from \( R_\xi \) gauge, in which the masses of the goldstone bosons are gauge-dependent. Of course, physical results cannot be gauge-dependent. The author in \[19\] obtained similar results as Eq.(2) based on Feynman gauge by making use of BRST symmetry to redefine the physical state. An earlier attempt along this line can be found in \[18\]. The longitudinal polarization and related scattering amplitudes in physical gauges agree precisely with those in \( R_\xi \) gauge in \[19\][18] if the gauge direction is chosen as \( n^\mu = (1, -\hat{k}), \) with \( \hat{k} = \frac{\vec{k}}{k} \). Thus the two approaches are equivalent with each other. Nevertheless, comparing to \( R_\xi \) gauge, physical gauges provide a much more clear physical picture as there is no gauge-dependent goldstone mass, no ambiguity in identifying physical states through LSZ reduction formula.

Although the power counting problem is overcome in a physical gauge, there is also a drawback: the Feynman rules become complicated, as we need to sum over all the terms from both gauge components and goldstone components. The problem becomes especially severe if the number of longitudinal states are multiplied. Besides, the derivation of the gauge-goldstone propagators

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2 Traditionally a physical gauge is defined for \( n^\mu \) along a fixed direction, e.g. \( n^\mu = (1, 0, 0, 1)^T \). Here we adopt a more general definition, which includes \( n \) being momentum dependent, e.g. Coulomb gauge. Other examples of momentum dependent “physical” gauge can be found in \[6\] and \[10\].
seems also to be complicated due to the gauge-goldstone mixing terms in the Lagrangian. The goal of this paper is to investigate and reorganize the Feynman rules of massive gauges in physical gauges by combining gauge fields and goldstone fields together. The strategy is, as mentioned above, to treat gauge fields and goldstone fields uniformly as 5-component fields $W^M = (W^\mu, \phi)$, and exploit possible underlying structures to simplify. The model we choose is the $\theta_W \to 0$ limit of the Standard Model of EW interactions, so the gauge group is $SU(2)_L$ instead of $SU(2)_L \times U(1)$. The motivation is that if custodial symmetry is preserved, both the gauge fields and goldstone fields transform as a triplet under $SU(2)$ global symmetry. It then becomes straightforwardly to combine gauge components and goldstone components. Additionally, it’s noteworthy that the new Feynman rules can also apply to Feynman gauge, in which goldstone bosons obtain the same masses as their corresponding gauge bosons. So we can make use of the Feynman rules describe here, if the longitudinal polarization vector is taken to be $\epsilon^M_L = (\epsilon_\mu, -i)$.

Apart from deriving and documenting the Feynman rules, we also investigate all the 3-point on-shell amplitudes. In recent years, the on-shell approach of scattering amplitudes using spinor-helicity \cite{21,22} has made remarkable progress. However, the success is still largely confined in massless particles. There have been many papers \cite{23,24,25,26} devoted to the massive case, but the topic still remains largely unexplored. 3-point on-shell amplitudes are the building blocks of on-shell approach to scattering amplitudes, thus one might hope that clearer understanding of them can shed some light in the direction. Our basic point is, now that we have two forms of longitudinal polarization vectors – one from gauge fields only, another from mixture of gauge fields and goldstone fields – the two forms should give the same amplitudes due to gauge invariance. This match between two ways of evaluating amplitudes should also be reflected on the 3-point on-shell amplitudes, which can be appropriately called “on-shell match”. This on-shell match gives a way to explain how the information of goldstone bosons are “encoded” in gauge fields for the case of 3-point amplitudes.

Another motivation for 3-point on-shell amplitudes is the calculation of collinear splitting functions, which can be reduced to the calculation of $1 \to 2$ collinear splitting amplitudes. Since collinear singularity emerges as the internal lines of the Feynman diagrams approach the mass poles, collinear splitting amplitudes are simply on-shell amplitudes. In light of this observation, we can simplify the calculation of splitting amplitudes – especially for massive particles – by making use of the new Feynman rules and the on-shell match condition.

The remaining of this paper is organised as following:

In Section \ref{sec:2} we write down the Lagrangian of the model, derive all the Feynman rules in physical gauges. We first derive the propagators and polarizations, we then derive all the vertices
systematically. All the Feynman rules at tree level are listed in the appendix A.

In Section III we investigate all 3-point on-shell amplitudes. We first prove all those 3-point amplitudes satisfy on-shell gauge symmetry, then calculate collinear splitting amplitudes involving longitudinal vector bosons by making use of the Feynman rules obtained in this paper and on-shell match condition from on-shell gauge symmetry. Finally we have conclusions.

II. THE MODEL AND FEYNMAN RULES

A. Lagrangian

Our goal is to derive the Feynman rules of the Standard Model of Electroweak interactions by taking the $\theta_W \to 0$ limit. The gauge group is thus $SU(2)_L$ only. With the custodial symmetry, the Higgs potential has symmetry $SU(2)_L \times SU(2)_R$. We can parametrize the Higgs field by writing it as

$$\mathcal{H} = \frac{1}{\sqrt{2}}(i\sigma_2 \Phi^*, \Phi)$$

with $\Phi$ being

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}} - i(\phi_1 - i\phi_2) \\ h + i\phi_0 \end{pmatrix}$$

A more illuminating way to write Higgs doublet $\mathcal{H}$ is

$$\mathcal{H} = \frac{1}{\sqrt{2}}(i\sigma_2 \Phi^*, \Phi) = \frac{1}{2}(h - i\sigma^a \phi_a)$$

The would-be goldstone fields are isolated from the would-be Higgs field in this parametrization, which will be more convenient to treat the would-be goldstone bosons as the 5th component of the vector fields/states. The full Lagrangians are written as
\[ \mathcal{L}_{\text{Gauge}} = -\frac{1}{4}(W^a_{\mu\nu}W^a_{\mu\nu}) + \frac{1}{2\xi}[(n \cdot \partial n \cdot W^a)(n \cdot \partial n \cdot W^a)]^* \]

\[ \mathcal{L}_{\text{Higgs}} = \text{tr}[(D_\mu H)\dagger D^\mu H] - \frac{\lambda h}{4}(\text{Tr}[H\dagger H] - \frac{v^2}{2})^2 \]

\[ \mathcal{L}_{\text{Fermion}} = i \sum_{i=1,2,3} \bar{Q}^L_{iL} \gamma^\mu Q^L_{iL} + i \sum_{i=1,2,3} \bar{L}^R_{iL} \gamma^\mu L^R_{iL} \]

\[ \mathcal{L}_{\text{Yukawa}} = - \sum_{i,j=1,2,3} \sqrt{2} \bar{Q}^L_{iL} H Y^{ij}_{Q} Q^R_{ij} - \sum_{i,j=1,2,3} \sqrt{2} \bar{L}^R_{iL} H Y^{ij}_{L} L^R_{ij} + \text{h.c.} \]

Here \( W^a_\mu = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\epsilon^{abc}W^b_\mu W^c_\nu, \ D_\mu = \partial_\mu + ig\frac{2}{\xi}W^a_\mu, \ n^\mu \) can be either a fixed direction [7][8] (e.g. \( n^\mu = (1,0,0,1) \)), or an operator depending on the coordinates [6][9].

To make sure gauge-fixing parameter \( \xi \) is dimensionless, \( n^\mu \) is rescaled by \( n \cdot \partial \). Ghosts in physical gauges generally decouple from the theory, but don’t decouple if \( n \) is momentum-dependent [10]. Nevertheless, we restrict our focus on tree level in this paper.

For the Fermion sector and the Yukawa sector, \( Q^R_{iL/R} = (u^\prime_{iL/R}, d^\prime_{iL/R}) \) and \( L^R_{iL/R} = (\nu^\prime_{iL/R}, e^\prime_{iL/R}) \) denote quarks and leptons in flavor basis respectively. Indices \( i,j = 1,2,3 \) denotes generations. \( Y^{ij}_{Q} = \text{diag}(y_{u}^{ij}, y_{d}^{ij}) \) is Yukawa matrix for the quark sector in isospin space, \( Y^{ij}_{L} = \text{diag}(y_{\nu}^{ij}, y_{l}^{ij}) \) is Yukawa matrix for the lepton sector in isospin space.

The Lagrangian terms in Eq.(4) are invariant under

\[ \mathcal{H} \rightarrow e^{i\omega_{a\alpha}^a} \mathcal{H} \]
\[ Q^L_{iL} \rightarrow e^{i\omega_{a\alpha}^a} Q^L_{iL} \]
\[ L^L_{iL} \rightarrow e^{i\omega_{a\alpha}^a} L^L_{iL} \]  

(5)

Next we expand the Lagrangian terms in terms of \( W_\mu, h, \phi \). After symmetry breaking, the Higgs field has shift: \( h \rightarrow h + v \). Particles obtain masses, the relations between masses and \( v \) are

\[ m_W = \frac{gv}{2} \quad m_f = \frac{y_f v}{\sqrt{2}} \quad m_h^2 = \frac{\lambda h v^2}{2} \]  

(6)

We start with gauge sector and Higgs sector. For the gauge sector, the Lagrangian terms can be written as

\[ \mathcal{L}_{W_\mu} = -\frac{1}{2}\partial_\mu W^a_\nu \partial_\nu W_\mu^a + \frac{1}{2\xi}(n \cdot \partial n \cdot W^a)(n \cdot \partial n \cdot W^a)^* \]

\[ \mathcal{L}_{W_\mu} + \mathcal{L}_{W_\mu}^* = ge^{abc}\partial_\mu W_\nu^{ab} W_\mu^{bc} - \frac{g^2}{4}\epsilon^{abc}e^{afg}W^{bf}_\mu W^{cg}_\nu W^{df}_\mu W^{fg}_\nu. \]

(7)
For the Higgs sector, the covariant derivative on the Higgs doublet $H$ is written as

$$D_\mu H = (\partial_\mu + igW_{a\mu} \frac{\sigma^a}{2})((h + v)\frac{1}{2} - i\frac{\sigma^b}{2}\phi_b)$$

Making use of $\sigma^a\sigma^b = \delta^{ab}1 + i\epsilon^{abc}\sigma^c$ and separating $h$ and $\phi$, $D_\mu H$ can be written further as

$$D_\mu H = (\partial_\mu + igW_{a\mu} \frac{\sigma^a}{2}) \cdot ((h + v)\frac{1}{2} - i\frac{\sigma^b}{2}\phi_b) + \frac{1}{4}gW_{a\mu}\phi_a1.$$  \tag{8}

Then we plug in $D_\mu H$ into $\text{tr}(D_\mu H^\dagger D_\mu H)$. Combined with the Higgs potential $V(\text{Tr}(H^\dagger H))$, the Lagrangian terms for Higgs sector become

$$L_{h^2} = \frac{1}{2}\partial^\mu h \partial_\mu h - \frac{1}{2}m_h^2 h^2$$

$$L_{\phi^2 + W^2_{\mu} + \phi^2 W^2_{\mu}} = \frac{1}{2}\partial^\mu h \partial_\mu h - \frac{1}{2}m_h^2 h^2 + \frac{g^2}{2}h^2$$

$$L_{w^2 + \phi^2 W^2_{\mu} + h(\phi W^2_{\mu} + h)} = \frac{1}{2}\partial^\mu h \partial_\mu h - \frac{1}{2}m_h^2 h^2 + \frac{g^2}{2}h^2$$

$$L_{\phi^4} = \frac{1}{16}\lambda_h h^4 - \frac{1}{4}\lambda_h vh^2$$

$$L_{h^3 + h^4} = \frac{1}{8}\lambda_h h^2 \phi^a \phi_b$$

$$L_{\phi^2} = \frac{1}{16}\lambda_h h^2 \phi^a \phi_b$$

$$L_{\phi^4} = \frac{1}{16}\lambda_h h^2 \phi^a \phi_b$$

For the fermion sector, after symmetry breaking, $Q^i$ and $L^i$ are related to the mass basis by

$$Q^i = U_Q^{ij} Q^j$$

$$L^i = U_L^{ij} L^j$$

$$Y_{Q/L}^{ij} = U_Q^{ij} Y_{Q/L}^{jkl} U_L^{kl}$$

The Yukawa matrices and mass matrices are diagonalized by the mixing matrices $U_{Q/L}$.
\[ m^i_{Q/L} = U^{i \bar{j}}_{Q/L} Y^{\bar{j}k}_{Q/L} Y^{k \bar{l}}_{Q/L} \frac{\nu}{\sqrt{2}} = m^i_{Q/L} \delta_{il} \]  

(13)

with \( Y^i_Q = \text{diag}(y_{ui}, y_{di}) \), \( Y^i_L = \text{diag}(y_{vi}, y_{li}) \), \( m^i_Q = \text{diag}(m_{ui}, m_{di}) \), \( m^i_L = \text{diag}(m_{vi}, m_{li}) \). The Lagrangian terms for the fermion sector then become

\[
L_{f2} = i\overline{Q}_L \gamma^\mu (U^\dagger_Q \sigma_a U_Q) Q_L W_{\mu a} + \left( \frac{i}{\sqrt{2}} \overline{Q}_L (U^\dagger_Q \sigma_a Y_Q U_Q) W_{\mu a} + h.c. \right)
\]

\[
L_{fW} + L_{f\phi} = -\frac{g}{2} \overline{Q}_L \gamma^\mu (U^\dagger_Q \sigma_a U_Q) Q_L W_{\mu a} + \left( \frac{i}{\sqrt{2}} \overline{Q}_L (U^\dagger_Q \sigma_a Y_Q U_Q) W_{\mu a} + h.c. \right) + \frac{1}{\sqrt{2}} \overline{L}_L (U^\dagger_L \sigma_a Y_L U_L) L_R \phi^a + h.c.
\]

(14)

The generation indices have been suppressed.

B. Propagator

In this section we are deriving the propagator of vector bosons, which has intrinsic mixing between gauge modes and goldstone modes. Combining the kinetic terms in Eq.(7), Eq.(9) and Eq.(10), the quadratic Lagrangian terms for gauge fields and goldstone fields are

\[
L_{W^2} = -\frac{1}{2} \partial^\mu W^\nu_a \partial_\mu W_{\nu a} + \frac{1}{2} \partial^\mu W_{\alpha a} \partial^\nu W_{\alpha a} + \frac{1}{2} m^2_{W} W_{\alpha a} W^{\alpha a} + \frac{1}{2} \xi (n \cdot \partial n \cdot W_a) (n \cdot \partial n \cdot W_a)^* 
\]

\[
L_{W^a} = -m_W W^{\alpha a} \partial_\mu \phi_a 
\]

\[
L_{\phi_2} = \frac{1}{2} (\partial^\mu \phi_a)^2 
\]

(15)

We note that Eq.(15) is not only true for the \( SU(2)_L \) theory, but applies to any model with Higgs mechanism. We write gauge-goldstone fields as 5-component vector fields \( W^a_M = (W^\mu_a, \phi_a) \), then the kinetic Lagrangian terms can be written as following up to terms with total derivative,

\[
L_{W^2} = -\frac{1}{2} \partial_M W^\alpha_N \partial^M W_{\alpha N} + \frac{1}{2} (\partial_M W^a_M)^2 + \frac{1}{2} \xi (n \cdot \partial n_M W^a_M) (n \cdot \partial n_M W^a_M)^* 
\]

(16)
with \( n^M = (n^\mu, 0) \), \( W^\alpha_M = (W^\alpha_\mu, \phi^\alpha) \), \( \partial^M = (\partial^\mu, -m_W) \), \( g_{MN} = g^{MN} = \text{diag}(1, -1, -1, -1) \).

It looks the same as the kinetic Lagrangian terms of massless gauge theory, except \( \mu \) becomes \( M \). This similarity is not just a nice way of writing all the Lagrangian terms. Indeed, noticing the Fourier transformation of \( \partial^M = (\partial^\mu, -m_W) \) gives \( k^M = (k^\mu, -im_W) \) for inwards momentum, and \( k^*M = (k^\mu, im_W) \) for outwards momentum, we could write the dot product of \( k^M \) as

\[
k \cdot k^* = g_{MN} k^M k^*_N = k^2 - m_W^2 \tag{17}
\]

This equals 0 when on-shell, just as \( k \cdot k = k^2 = 0 \) when on-shell for massless case. Thus all the algebra with the tensor \( g_{\mu\nu} \) and \( k^\mu \), could be applied straightforwardly to \( g_{MN} \) and \( k^M / k^*_N \), with \( n^M = (n^\mu, 0) \). For massless gauge fields, the kinetic Lagrangian after gauge fixing is

\[
\mathcal{L}_{\text{kinetic}} = -\frac{1}{2} \partial_\mu W^\alpha_\nu \partial^\mu W^\alpha_\nu + \frac{1}{2} (\partial \cdot W) (\partial \cdot n \cdot W) + \text{total derivative} \tag{18}
\]

the propagator of the gauge bosons can be easily evaluated to be

\[
<W^\mu_a W^\nu_b> = \frac{-i\delta_{ab}(g^{\mu\nu} - \frac{n^\alpha k^\nu + k^\mu n^\nu}{n \cdot k} + \frac{n^2 k^\mu k^\nu}{(n \cdot k)^2} + \frac{\xi k^\mu k^\nu}{k^2 + i\varepsilon})}{k^2 + i\varepsilon} \tag{19}
\]

Following the arguments above, a direct analogue to the massless propagator in Eq.\(^{19}\) gives us the gauge-goldstone propagator in massive gauge theory,

\[
<W^M_a W^N_b> = \frac{-i\delta_{ab}(g^{MN} - \frac{n^M k^*_N + k^M n^*_N}{n \cdot k} + \frac{n^2 k^M k^*_N}{(n \cdot k)^2} + \frac{\xi k \cdot k^*}{k^2 + i\varepsilon})}{k \cdot k^* + i\varepsilon} \tag{20}
\]

By writing gauge components \( M = \mu \) and \( M = 4 \) component separately, the propagator of vector boson becomes

\[
<(W^\mu_a, \phi_a), (W^\nu_b, \phi_b) > = \frac{i\delta_{ab}}{k^2 - m_W^2 + i\varepsilon} \begin{pmatrix}
-(g^{\mu\nu} - \frac{n^\alpha k^\nu + k^\mu n^\nu}{n \cdot k} + n^2 \frac{k^\mu k^\nu}{(n \cdot k)^2}) & \frac{i m_W}{n \cdot k} (n^\mu - n^2 \frac{k^\mu}{n \cdot k}) \\
-i m_W (n^\nu - n^2 \frac{k^\nu}{n \cdot k}) & 1 - \frac{n^2 m_W^2}{(n \cdot k)^2}
\end{pmatrix}
\]

When \( k^2 = m_W^2 \) or \( k \cdot k^* = 0 \), the numerator of the propagator can be written as sum of the polarizations,
\[ < W_a^M W_b^* N > = \frac{i \delta_{ab} \sum_{s=\pm, L} \epsilon_s^M \epsilon_s^{N*}}{k \cdot k^* + i \epsilon} \]

In the 5-component notation, the transverse and longitudinal polarizations are

\[ g^{44} = -1 : \quad \epsilon_{\pm L}^{M} = \begin{pmatrix} \epsilon_{\pm}^{\mu} \\ 0 \end{pmatrix} \quad \epsilon_{L}^{M} = \frac{1}{\sqrt{1 - \frac{n^2 m_{W}^2}{(n \cdot k)^2}}} \begin{pmatrix} -\frac{m_{W}}{n \cdot k} (n^{\mu} - \frac{n^2 k^{\mu}}{n \cdot k}) \\ i(1 - \frac{n^2 m_{W}^2}{(n \cdot k)^2}) \end{pmatrix} \] (21)

They satisfy the transverse condition and normalization condition

\[ \epsilon_s \cdot \epsilon_s^* = -1 \cdot \delta_{ss'} \]
\[ k^* \cdot \epsilon_s = k \cdot \epsilon_s^* = 0 \]
\[ n \cdot \epsilon_{s=\pm} = 0 \] (22)

The amplitudes involving longitudinal vector bosons are evaluated by summing over the contributions from both gauge components and goldstone components:

\[ i\mathcal{M}(L) = ig^{MN} \mathcal{M}_M \epsilon_N = i\mathcal{M}^{\mu} \epsilon_{n\mu} - i\mathcal{M}^4 \epsilon_4 \] (23)

Notice the “metric” \[ g^{MN} = \text{diag}(1, -1, -1, -1, -1) \] induces a minus sign between amplitudes involving gauge component and goldstone component. In practical calculations it’s more convenient to absorb this minus sign into the polarization vectors, which become

\[ g^{44} = 1 : \quad \epsilon_{L}^{M} = \frac{1}{\sqrt{1 - \frac{n^2 m_{W}^2}{(n \cdot k)^2}}} \begin{pmatrix} -\frac{m_{W}}{n \cdot k} (n^{\mu} - \frac{n^2 k^{\mu}}{n \cdot k}) \\ -i(1 - \frac{n^2 m_{W}^2}{(n \cdot k)^2}) \end{pmatrix} \] (24)

so that the amplitude is evaluated by simply summing over the diagrams involving gauge components and goldstone components, i.e.

\[ i\mathcal{M}(L) = i\mathcal{M}^{\mu} \epsilon_{n\mu} + i\mathcal{M}^4 \epsilon_4 \] (25)

Our results agree with Ref.[6] by choosing \[ n^{\mu} = (1, -\frac{k}{|k|}) \] up to a global phase which is unphysical.
C. Vertices

In this section we apply the 5-component treatment further to vertices. Our goal is to recombine the Lagrangian terms for interactions, so that the Feynman rules for vector bosons is given by Lagrangian of \( W^a_M = (W^a_\mu, \phi^a) \). We start from the gauge sector and Higgs sector, which give rise to vertices of W-W-W and W-W-W-W. The corresponding Lagrangian terms are written as

\[
\mathcal{L}_{W^3_M} = g\epsilon^{abc}\partial_\mu W^a_N W^b_\rho W^c_K g^{\mu\rho} g^{NK}
\]

\[
\mathcal{L}_{W^4_M} = -\frac{g^2}{4}\epsilon^{abc}\epsilon^{aef} W^b_\mu W^e_\nu W^f_\rho W^g_\sigma g^{\mu\nu} g^{\rho\sigma} + \frac{g^2}{8} W^a_\mu W^b_\nu \phi^a \phi^b g^{\mu\nu} - \frac{1}{16} \lambda h \phi^a \phi^a \phi^b \phi^b
\]

with \( g^{NK} = \text{diag}(1, -1, -1, -1, -1/2) \). The \( g^{NK} \) here is not to be confused with \( g^{NK} \) appearing say in Eq.(23): \( g^{NK} \) in Eq.(26) is for bookkeeping the relative coefficient between the different Lagrangian terms, whereas \( g^{NK} \) in Eq.(23) is to keep track of the relative phase between (sub)diagrams of gauge components and goldstone components, which can be absorbed into the definition of polarization vectors.

The Lagrangian terms for h-W-W and h-h-W-W are

\[
\mathcal{L}_{hW^2_M} = \frac{g}{2}(\partial_\mu h W^a_\nu \phi^a g^{\mu\nu} - \partial_\mu \phi^a W^a_\mu h g^{\mu\nu}) + \frac{g^2}{4} g_{MN} h W^a_\mu W^a_\nu g^{\mu\nu} - \frac{1}{4} \lambda h \phi^a \phi^a
\]

\[
\mathcal{L}_{h^2W^2_M} = \frac{g^2}{8} h^2 W^a_\mu W^a_\nu g^{\mu\nu} - \frac{1}{8} \lambda h^2 \phi^a \phi^a
\]

To obtain Feynman rules we need the last step of writing \( W^a \) in the basis \( (W^3_M, W^3_M) \), with

\[
W^\pm_M = \frac{1}{\sqrt{2}}(W^1_M \mp iW^2_M).
\]

This identity are useful

\[
\sigma^a W^a_M = W^3_M T_3 + \sqrt{2}(W^+_M T^+ + W^-_M T^-).
\]

\( T_3 \) and \( T^\pm \) are separately,

\[
T^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \hspace{1cm} T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \hspace{1cm} T^- = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
Also define $V_{\text{CKM}}$ and $V_{\text{PMNS}}$ as

$$V_{\text{CKM}} = U^\dagger_u U_d \quad V_{\text{PMNS}} = U^\dagger_\nu U_l$$

(30)

as well as

$$W_M^1 W_N^2 - W_M^2 W_N^1 = i(-W_M^+ W_N^- + W_M^- W_N^+)$$

$$W_M^1 W_N^1 + W_M^2 W_N^2 = W_M^+ W_N^- + W_M^- W_N^+$$

(31)

Writing all Lagrangian terms in the physical basis. The Lagrangian terms giving rise to vertices $W-W-W$ and $W-W-W-W$ are

$$L_{W_3^M} = -ig(\partial_\mu W_N^3 W_\mu^+ W_\nu^-)g^{\mu\rho}g^{NK} + \text{cyclic in (3, +, -)}$$

$$L_{W_4^M} = \frac{g^2}{2} \left[ g^{\mu\nu} g^{\sigma\tau} W_\mu^+ W_\nu^- W_\sigma^+ W_\tau^- - g^{\mu\nu} g^{P\Sigma} W_\mu^+ W_\nu^- W_P^+ W_{\Sigma^-} \right] - \frac{\lambda_h}{4}(\phi^+ \phi^-)^2$$

$$- \frac{g^2}{2} \left[ g^{\mu\nu} g^{P\Sigma} W_\mu^+ W_\nu^- W_\Sigma^+ W_3^3 - g^{\mu\nu} g^{P\Sigma} W_\mu^+ W_\nu^- W_\Sigma^+ W_3^- W_3^+ + (\mu \leftrightarrow P, \nu \leftrightarrow \Sigma) \right]$$

$$- \frac{\lambda_h}{4}(\phi^+ \phi^-)^2$$

$$+ \frac{g^2}{8} W_\mu^3 W_\nu^3 \phi^3 \phi^3 g^{\mu\nu} - \frac{\lambda_h}{16}(\phi^3)^4$$

(32)

The Lagrangian terms giving rise to vertices $h-W-W$ and $h-h-W-W$ are

$$L_{hW_3^M} = \frac{g}{2} (\partial_\mu h W_\nu^3 \phi^3 - \partial_\nu \phi^3 W_\mu h)g^{\mu\nu} + \frac{1}{2} g m_N h W_\mu^3 W_\mu^3 - \frac{\lambda_h}{4} v h (\phi^3)^2$$

$$\left[ \frac{g}{2} g^{\mu\nu} (\partial_\mu h W_\nu^+ \phi^- - \partial_\nu \phi^- W_\mu^+ h) + (+ \leftrightarrow -) \right] + g m_N h W_\mu^+ W_\mu^- - \frac{1}{2} \lambda_h v h (\phi^+ \phi^-)$$

(33)

$$L_{h^2 W_3^M} = \frac{g^2}{8} g^{\mu\nu} h^2 (2W_\mu^+ W_\nu^- + W_\mu^3 W_\nu^3) - \frac{\lambda_h}{8} h^2 (2\phi^+ \phi^- + (\phi^3)^2)$$

(34)

The Lagrangian terms for the quark sector become

$$L_{fW_3^M} = -\frac{g}{2} (\bar{u}_L W_3^3 u_L - \bar{d}_L W_3^3 d_L) + \left[ \frac{i}{\sqrt{2}} (y_u \bar{u}_L u_R \phi^3 - y_d \bar{d}_L d_R \phi^3) + \text{h.c.} \right]$$

$$-\frac{g}{\sqrt{2}} (\bar{u}_L W_3^+ V_{\text{CKM}} d_L + \text{h.c.}) + \left[ \frac{i}{\sqrt{2}} (y_u \bar{u}_L V_{\text{CKM}} d_R \phi^+ + y_d \bar{d}_L V_{\text{CKM}}^\dagger u_R \phi^-) + \text{h.c.} \right]$$

$$L_{ffh} = -\frac{1}{\sqrt{2}} (y_u \bar{u}_L u_R h + y_d \bar{d}_L d_R h) + \text{h.c.}$$
For the lepton sector we simply make the replacement \((u \rightarrow \nu, d \rightarrow l)\) and \(V_{\text{CKM}} \rightarrow V_{\text{PMNS}}\).

Finally we comment on a subtlety in the derivation of Feynman rules: since \(W\) and \(\phi\) are simply different components of the same physical fields, it’s necessary to sum over all possible contractions in deriving Feynman rules. This operation gives an additional overal factor to the Feynman rules. For example, the overal factor of vertex \(h h h h\) is \(4! = 24\). However, sometimes it requires writing different terms explicitly. This is especially true in the case of gauge-goldstone fields, in which the gauge components and goldstone components appear to be different fields. For example, in \(L^{h W^2}_M\) of Eq.(27), the term \(\frac{g}{2}(\partial_\mu h W^a_\nu \phi^a_g g^{\mu\nu} - \partial_\mu \phi^a W^a_\nu h g^{\mu\nu})\) gives rise to an overal factor 2, which is taken into account by adding a different term with \(\phi^a \leftrightarrow W^a_\mu\) in the Feynman rule. While for \(W^3_\pm M\) they have been written out explicitly, for \(W^3_3 M\) we still need to take into account the term with \(\phi^3 \leftrightarrow W^3_\mu\).

### III. ON-SHELL MATCH FOR 3-POINT AMPLITUDES

#### A. On-shell Gauge Symmetry

Having derived all the Feynman rules in a physical gauge. We proceed to analyse the on-shell gauge symmetry, especially how they are reflected in 3-point amplitudes. So far we have concluded that a longitudinal polarization in a physical gauge is composed of gauge components \(\epsilon^\mu_n\) and goldstone component \(i/ - i\), as in Eq.(21). We choose \(n^2 = 0\), i.e. light-cone gauge, the longitudinal polarization vector in 5-component becomes:

\[
\epsilon^L_M = \begin{pmatrix} \epsilon^\mu_{n_1} \\ i \end{pmatrix}
\]

(35)

for incoming particles, and \(\epsilon^L_M\) for outgoing particles. Here \(\epsilon^\mu_{n_1} = -\frac{m_W}{n_1 \cdot k} n_1^\mu\). On the other hand, we already have the standard form for the longitudinal polarization vector, which can be written in the 5-component format as

\[
\epsilon^L_M = \begin{pmatrix} \epsilon^\mu_{n_2} + \frac{k^\mu}{m_W} \\ 0 \end{pmatrix}
\]

(36)

with the 5th component to be 0 and \(n_2^\mu = (1, -\frac{k}{|k|})\). If we choose \(n_1\) in Eq. (35) as \(n_1 = n_2\), \(\epsilon^M_{1L}\) and \(\epsilon^M_{2L}\) are related with each other by
Meanwhile since S-matrix is gauge-invariant, the two forms of longitudinal polarizations have to give the same S-matrix, i.e.,

$$\varepsilon_{2L}^M = \varepsilon_{1L}^M - \frac{k^M}{m_W} \tag{37}$$

Plugging in Eq. (37), Eq. (38) is equivalent to

$$k_1^{M_1} ... k_i^{M_i} S_{M_1 ... M_i}(k_1 ... k_i ... ) = 0. \tag{39}$$

Interestingly, Eq. (39) share similar form as the on-shell gauge symmetry for massless gauge theory, with $M \rightarrow \mu$. Thus, we can appropriately call it “on-shell gauge symmetry” for massive gauge theory.

Ignoring the terms of $O\left(\frac{m_W}{n \cdot k}\right)$, Eq. (38) reduced to

$$\varepsilon_{2L}^M S_{M_1 ... M_i}(k_1 ... k_i ... ) = S(\phi_1 ... \phi_i ... ) + O\left(\frac{m_W}{n \cdot k}\right) \tag{40}$$

So on-shell gauge symmetry reduces to goldstone equivalence theorem as in Eq. (1).

In this paper we don’t intend to give a complete proof of on-shell gauge symmetry Rather, we only set to prove that Eq. (39) is satisfied for all on-shell 3-point amplitudes: $WWW$, $hWW$ and $ff'W$.

The condition of “on-shell” needs some extra comments, as it’s not always kinematically possible to put all particles on-shell for 3-point amplitudes. This constraint is usually bypassed by analytical continuation of making momenta complex. Nevertheless, for $hWW$ and $ff'W$, we can also put amplitudes on-shell through the analytical continuation of parameters in the theory. For example, for a decay process $h \rightarrow W^+W^-$, the Higgs mass has to satisfy the on-shell condition $m_h \geq 2m_W$, with $m_h \sim \lambda_h v$, $m_W \sim g v$. We can think of the amplitude as the function of parameters of theory: $\mathcal{M} = \mathcal{M}(\lambda_h, v, g)$. We can first choose the parameters to satisfy the on-shell condition, but the resulting amplitude will have to be the same for all possible values of parameters within the perturbative limits. The same argument can be applied to $f \rightarrow f'W$, whose amplitude can be
seen as the function of \( y_f, y_{f'}, g, v \): \( \mathcal{M} = \mathcal{M}(\lambda_f, \lambda_{f'}, g, v) \). Notice this argument doesn’t apply to \( WWW \), since the amplitude is controlled by only one coupling \( g \). It’s not possible to adjust \( g \) to put the all the particles on-shell.

For the convention, we absorb the intrinsic minus sign between gauge components and goldstone components in amplitudes to the definition of polarization vectors, which are given by Eq.(24) with \( g^{44} = 1, n^2 = 0 \),

\[
g^{44} = 1 : \quad \epsilon^M_L = \begin{pmatrix} \epsilon^\mu_n \\ -i \end{pmatrix} \quad \epsilon^*_L = \begin{pmatrix} \epsilon^\mu_n \\ i \end{pmatrix}
\]

with \( \epsilon_n^\mu = -\frac{m_W}{n_k} n^\mu \) satisfying \( k \cdot \epsilon_n = -m_W \).

In proving the on-shell gauge symmetry for 3-point amplitudes, \( k^M/k^*_M \) also need to be redefined to

\[
k^M_L = \begin{pmatrix} k^\mu \\ im_W \end{pmatrix} \quad k^*_M_L = \begin{pmatrix} k^\mu \\ -im_W \end{pmatrix}
\]

However, in the transverse condition and on-shell condition for \( k^M \), the intrinsic minus sign between gauge components and goldstone component still need to be taken into account, i.e. we have

\[
\begin{align*}
k^M \epsilon^*_M &= k^*_M \epsilon^*_M = k \cdot \epsilon_n + m_W = 0 \\
k^M k^*_M &= k^*M k_M = k^2 - m_W^2 = 0
\end{align*}
\]

We also extract the factor “i” out for S-matrix by defining \( S = i\mathcal{M} \). Our convention is that all particles are incoming.

**h-W-W**

The on-shell gauge symmetry for 3-point amplitudes can be written as

\[
i\mathcal{M}(1^s; 2^s 3^s)|_{\epsilon^*_M \rightarrow \frac{k^*_M}{m_W}} = 0
\]
i denotes any particles being $W$ bosons. To prove it, we start with $\mathcal{M}(hWW)$, with only one $W$ replaced by $\frac{k^M}{m_W}$. With one particle being the Higgs and another particle being $\frac{k^M}{m_W}$, there are two cases for the polarizations of particle 2: a) transverse b) longitudinal. We start with case a) with $s_2 = \pm$. In this case $\epsilon^4_{s_2} = 0$, so there is no goldstone component contribution from particle 2, but $\frac{k^4}{m_W} = i$.

\[
i\mathcal{M}(1^{h2s_2=\pm}3^{s_3})|_{\epsilon^M_3 \rightarrow \frac{k^M}{m_W}} = ig m_W \epsilon^\pm_2 \cdot \frac{k^3}{m_W} + \frac{g}{2}((k^1 - k_3) \cdot \epsilon^\pm_2)(i)
\]
\[
= ig(k^3 \cdot \epsilon^\pm_2 + \frac{1}{2}(-k^2 - 2k^3) \cdot \epsilon^\pm_2)
\]
\[
= 0
\]

In the second step we used energy-momentum conservation $k^1 + k^2 + k^3 = 0$, in the third step we used the transverse condition for particle 2: $k^2 \cdot \epsilon^\pm_2 = 0$.

Then we turn to case b) $s_2 = L$, so the polarization vector has both gauge components $\epsilon^n_{2n}$ and goldstone component $\epsilon^4_2 = -i$, the amplitude is

\[
i\mathcal{M}(1^{h2s_2=L}3^{s_3})|_{\epsilon^M_3 \rightarrow \frac{k^M}{m_W}} = ig m_W \epsilon^n_2 \cdot \frac{k^3}{m_W} + \frac{g}{2}((k^1 - k_3) \cdot \epsilon^n_2)(i) + \frac{g}{2}(k^1 - k_2) \cdot \frac{k^3}{m_W}(-i) - ig \frac{m^2_h}{2m_W} \cdot i \cdot (-i)
\]
\[
= ig(\epsilon^n_2 \cdot k_3 - \epsilon^n_2 \cdot k_3 - \frac{1}{2}k^2 \cdot \epsilon^n_2 - \frac{(k^1 - k_2)(k^1 + k_2)}{2m_W} - \frac{m^2_h}{2m_W})
\]

To further simplify, we need first to make use of on-shell condition for $k^1$ and $k^2$:

\[
(k^1 - k_2)(k^1 + k_2) = k^2_1 - k^2_2 = m^2_h - m^2_W
\]
as well as the transverse condition for $\epsilon^M_{2L}$: $k^2 \cdot \epsilon^n_2 = -m_W$, which is another expression of $k^* \cdot \epsilon^M_{2L} = 0$. Plugging in, the amplitude becomes

\[
i\mathcal{M}(1^{h2s_2=L}3^{s_3})|_{\epsilon^M_3 \rightarrow \frac{k^M}{m_W}} = ig \left(\frac{m_W}{2} + \frac{m^2_h - m^2_W}{2m_W} - \frac{m^2_h}{2m_W}\right)
\]
\[
= 0,
\]

$f-f'-W$
We then proceed to prove the on-shell gauge symmetry Eq. (41) for the amplitude of $ffW$,

$$iM(1^{s_1}2^{s_2}3^{s_3})|_{\epsilon^{\mu}_3 \to k_3^\mu / m_W} = 0$$

with particle 1 and 2 being fermions, particle 3 being W boson. Since only particle 3 is W boson, the identity is relatively easy to prove. Writing the amplitude with gauge components and the amplitude with the goldstone component separately, Eq. (41) becomes,

$$iM(1^{s_1}2^{s_2}3^{s_3})|_{\epsilon^{\mu}_3 \to k_3^\mu / m_W} = M(1^{s_1}2^{s_2}3^{s_3})|_{\epsilon^{\mu}_3 \to 0, \epsilon^\gamma_3 \to -i} = 0 \quad (42)$$

The first term is the amplitude from the fermion-fermion-gauge vertex, the second term is from the fermion-fermion-goldstone vertex. Let’s check Eq. (42) explicitly. First look at the first term, to fix all the momenta to be incoming, we choose particle 1 to be a d-type anti-quark, particle 2 to be a d-type quark, then the W boson has to be $W^+$, we also set the CKM matrix to be $I$, the amplitude becomes

$$iM|_{\epsilon^{\mu}_3 \to k_3^\mu / m_W, \epsilon^\gamma_3 \to 0} = -ig \sqrt{2} \bar{\nu}_L \gamma^\mu u_L^s_2 k_3^\mu / m_W$$

$$= ig \sqrt{2} \bar{\nu}_L ((\hat{k}_1 + \hat{k}_2) u_L^s_2 / m_W$$

$$= ig \sqrt{2} (-m_d / m_W \bar{\nu}_R^s_2 u^s_2 + m_u / m_W \bar{\nu}_L^s_2 u_R^s_2) \quad (43)$$

In the second equality, we made use of the energy-momentum conservation $k_1 + k_2 + k_3 = 0$, in the third equality we made use of equation of motion for fermions: $\hat{k}_2 u_{L/R}(k_2) = m_u u_{L/R}(k_2)$ as well as $\bar{\nu}_{L/R}(k_1) \bar{k}_1 = -\bar{\nu}_{R/L}(k_1) m_d$. We then look at the second term in Eq. (42), with $\epsilon^\gamma_3 = -i$, we have

$$iM|_{\epsilon^{\mu}_3 \to 0, \epsilon^\gamma_3 \to -i} = -i \cdot (-i) g \sqrt{2} \left( \frac{m_d}{m_W} \bar{\nu}_R^s_1 u_L^s_3 - \frac{m_u}{m_W} \bar{\nu}_L^s_1 u_R^s_3 \right)$$

$$= -g \sqrt{2} (-m_d / m_W \bar{\nu}_R^s_1 u_L^s_3 + m_u / m_W \bar{\nu}_L^s_1 u_R^s_3) \quad (44)$$

Here we made use of $y_f = \frac{m_f}{\sqrt{2} m_W}$. Combining Eq. (43) and Eq. (44), we finish the proof of Eq. (42). Although we only went through the example of $\bar{d}_1 u_2 W^+_3$, it can be checked straightforwardly that Eq. (42) is satisfied for all other cases. Having ignoring the CKM matrix $\bar{u}_1 d_2 W^-_3$ is identical.
to $d_1 u_2 W^+_3$. For neutral current, i.e. $W$ being $W^3$, the proof is also identical except we need to replace $\gamma^\mu$ with $\frac{1}{\sqrt{2}} \gamma^\mu T_3$ for the fermion-fermion-gauge vertex, and replace $\gamma^5$ with $\frac{1}{\sqrt{2}} \gamma^5 T_3$ for the fermion-fermion-goldstone vertex. It’s also not hard to see that the conclusion also applies to the SM with the gauge group being $SU(2)_L \times U(1)_Y$, in which case the only difference is the neutral current case. For $ff\gamma$, $i\mathcal{M}|_{\gamma^\mu \rightarrow k^\mu} = 0$ according to ward identity. For $ffZ$, the fermion-fermion-goldstone vertex is identical to the $SU(2)$ case; for the fermion-fermion-gauge vertex, ignoring the overall factor difference, there is an additional term of vector current $Q_f \sin^2 \theta_W$ relative to the $SU(2)$ case. Nevertheless, this term gives 0 when $k^\mu$ dots into the S-matrix because the interaction is vector-like. So we conclude Eq. (42) is satisfied for the SM too. Indeed, since the argument is very general, we expect Eq. (42) applies to any massive gauge theory with Higgs mechanism.

$W-W-W$

Next we proceed to prove Eq. (42) for the amplitude of $WWW$. Stripping of the overall factor of $-ig$, the general amplitude of $WWW$ can be written as

$$i\mathcal{M}(1s_1^2s_2^2T_3^3) = (\epsilon_1 \cdot \epsilon_2 - \frac{1}{2} \epsilon_1^A \cdot \epsilon_2^A) [(p_1 - p_2) \cdot \epsilon_3] + \text{cyclic} \quad (45)$$

Replacing one of the polarizations are replaced by $\frac{k^M}{m_W}$, there are three different cases for the other two polarizations: a) two transverse; b) one transverse and one longitudinal; c) two longitudinal. We start from case a), since both particle 1 and particle 2 are transverse their goldstone components are 0: $\epsilon_1^4 = \epsilon_2^4 = 0$. Consequently, there is no goldstone contribution in this case, so we have

$$i\mathcal{M}(1s_1^T_2 s_2^T_2 T_3^3)\bigg|_{\epsilon_3^M \rightarrow \frac{k^M}{m_W}} = i\mathcal{M}(1s_1^T_2 s_2^T_2 T_3^3)\bigg|_{\epsilon_3^M \rightarrow \frac{k^M}{m_W}, \epsilon_3^4 \rightarrow 0} \quad (46)$$

This means the on-shell gauge symmetry for $1^T_2 2^T_2 T_3^3$ is directly analogue to the massless case, with only gauge vertex contributing.

$$i\mathcal{M}(1s_1^T_2 s_2^T_2 T_3^3)\bigg|_{\epsilon_3^M \rightarrow k^M, \epsilon_3^4 \rightarrow 0} = (\epsilon_1^T \cdot \epsilon_2^T) [(k_1 - k_2) \cdot (k_1 - k_2)] + \epsilon_2^T \cdot k_3 [(k_2 - k_3) \cdot \epsilon_1^T]$$
$$+ k_3 \cdot \epsilon_1^T [(k_3 - k_1) \epsilon_2^T]$$
$$= 0 - 2 \epsilon_2^T \cdot k_3 k_3 \cdot \epsilon_1^T + 2 k_3 \cdot \epsilon_1^T \cdot k_3 \cdot \epsilon_2^T$$
$$= 0 \quad (47)$$
In the second step we made use of energy-momentum conservation, on-shell conditions and transverse conditions for particle 1 and 2 respectively,

\[ k_1 + k_2 + k_3 = 0 \]
\[ k_1^2 - k_2^2 = m_W^2 - m_W^2 = 0 \]
\[ k_1 \cdot \epsilon_1^T = k_2 \cdot \epsilon_2^T = 0 \]  

(48)

Next we turn to case b) with particle 1 being transverse and particle 2 being longitudinal, we get

\[
i\mathcal{M}(1^{s_1=T^2}=2^{s_2=L^3}:s_3)\bigg|_{\epsilon_3^M \rightarrow -\frac{1}{2} \epsilon_3^M} = \epsilon_1^T \cdot \epsilon_2^n \frac{(k_1 - k_2) \cdot k_3^3}{m_W} + (\epsilon_2^n \cdot \frac{k_3}{m_W} - \frac{1}{2} (k_2 - k_3) \cdot \epsilon_1^T + \frac{k_3 \cdot \epsilon_1^T}{m_W} (k_3 - k_1) \cdot \epsilon_2^n \\
= 0 + \frac{\epsilon_2^n \cdot k_3}{m_W} (-k_1 - 2k_3) \cdot \epsilon_1^T - \frac{1}{2} (-k_1 - 2k_3) \cdot \epsilon_1^T + \frac{k_3 \cdot \epsilon_1^T}{m_W} (2k_3 + k_2) \cdot \epsilon_2^n \\
= \frac{\epsilon_2^n \cdot k_3}{m_W} 2k_3 \cdot \epsilon_1^T + \frac{\epsilon_2^n \cdot k_3}{m_W} - k_3 \cdot \epsilon_1^T + k_3 \cdot \epsilon_1^T \\
= 0
\]  

(49)

Again we have used the conditions of all the particles being on-shell. For the longitudinal state \( \epsilon_2^L \), the on-shell condition implies \( k_2 \epsilon_2^L = -m_W \).

Finally, we turn to case c) with both particle 1 and 2 being longitudinal polarizations, we have

\[
i\mathcal{M}(1^{s_1=L^2}=2^{s_2=L^3}:s_3)\bigg|_{\epsilon_3^M \rightarrow \frac{1}{2} \epsilon_3^M} = (\epsilon_1^n \cdot \epsilon_2^n + \frac{1}{2} (k_1 - k_2) \cdot k_3^3 m_W) + (\epsilon_2^n \cdot \frac{k_3}{m_W} - \frac{1}{2} (-i) \cdot \epsilon_1^T (k_2 - k_3) \cdot \epsilon_1^n \\
+ (\frac{k_3 \cdot \epsilon_1^n}{m_W} - \frac{1}{2} (-i) \cdot \epsilon_1^T (k_3 - k_1) \cdot \epsilon_2^n \\
= 0 + \frac{\epsilon_2^n \cdot k_3}{m_W} (-k_1 - 2k_3) \cdot \epsilon_1^n - \frac{1}{2} (-k_1 - 2k_3) \cdot \epsilon_1^n \\
+ \frac{k_3 \cdot \epsilon_1^n}{m_W} (2k_3 + k_2) \cdot \epsilon_2^n - \frac{1}{2} (2k_3 + k_2) \cdot \epsilon_2^n \\
= \frac{k_3 \cdot \epsilon_1^n}{m_W} (-2k_3) \cdot \epsilon_1^n + k_3 \cdot \epsilon_1^n + \frac{1}{2} m_W + k_3 \cdot \epsilon_1^n \\
+ \frac{k_3 \cdot \epsilon_1^n}{m_W} 2k_3 \cdot \epsilon_1^n - k_3 \cdot \epsilon_1^n + \frac{1}{2} m_W - k_3 \cdot \epsilon_2^n \\
= 0
\]  

(50)

Thus combining case a), case b) and case c), we have proved that on-shell gauge symmetry is satisfied for all possible 3-point amplitudes: \( h-W-W, f-f'-W, W-W-W \). However, our proof
still has two loop holes: the first one is only one W state is replaced by \( \frac{k^M}{m_W} \), the second one is particles are assumed to be incoming. Here we demonstrate neither of the two assumptions affect the conclusion. Starting with the first one, to prove the general case of arbitrary number of polarizations of W being replaced by \( \frac{k^M}{m_W} \), we need only notice that by replacing the gauge components \( \epsilon_n^\mu \) with \( k^\mu \), the transverse condition for the longitudinal polarization

\[
k^*M \epsilon_{LM} = 0
\]

turns to the on-shell condition for \( k^M \),

\[
k^*M k_M = 0.
\]

Since in the proof above, the only conditions we used are on-shell condition for \( k^M \) and transverse conditions for \( \epsilon_n^M \), the proof of Eq. (41) is exactly the same for multiple polarization vectors being replaced by \( \frac{k^M}{m_W} \).

The second loophole is automatically fixed if crossing symmetry is satisfied. Under \( k \rightarrow -k \), the incoming longitudinal state becomes \( \epsilon_L^M(-k) = -\epsilon_L^{*M}(k) \), as \( \epsilon_n^\mu(-k) = -\epsilon_n^\mu(k) \). So we get the longitudinal polarization vector for the outgoing state up to a minus sign. Energy-momentum conservation becomes

\[
\sum_i k_i + k = 0 \rightarrow \sum_i k_i + (-k) = 0
\]

Thus we obtain the amplitude for one particle being outgoing if its momentum is under \( k \rightarrow -k \). So crossing symmetry is indeed satisfied. Moreover, \( k^M/k^{*M} \) turns to \( -k^M/-k^M \) under \( k \rightarrow -k \). Therefore, we finished our proof of on-shell gauge symmetry for all 3-point amplitudes.

**B. 1 \rightarrow 2 Splitting Amplitudes**

In this section we are demonstrating how to use the new Feynman rules and the on-shell match condition from on-shell gauge symmetry to do calculations. Our examples are \( 1 \rightarrow 2 \) collinear
splitting amplitudes involving longitudinal vector bosons: \( W_L \rightarrow W_L W_L \), \( h \rightarrow W_L W_L \) and \( f \rightarrow f' W_L \). Those splitting amplitudes have been calculated in [6]. However, as we will see, with the new prescriptions the calculations become largely simplified.

When external states of a process become collinear with each other, one internal lines of one of the Feynman diagrams will approach its pole or mass singularity. The amplitude can then be factorised in the following way

\[
iM = \sum_s iM_{\text{split}} \cdot \frac{i}{k_3^2 - m_3^2} \cdot iM_0 + \text{power suppressed} \tag{51}
\]

Thus the splitting amplitude \( M_{\text{split}} \) should be evaluated on-shell. The collinear splitting amplitudes \( M_{\text{split}} \) are related to collinear splitting functions in the following way [6],

\[
\frac{dP}{dz dk_T^2} \propto |M_{\text{split}}|^2 \tag{52}
\]

So evaluating collinear splitting functions is reduced to evaluating splitting amplitudes.

\( h \rightarrow W_L^+ W_L^- \)

The splitting amplitude for \( h(k_3) \rightarrow W_L^+(k_1)W_L^-(k_2) \) can be more conveniently calculated using the polarization vector \( \epsilon^\mu_L = \frac{k^\mu}{m_W} - \frac{m_W n^\mu}{n \cdot k} \), and evaluating the splitting amplitude by treating all particles “on-shell”.

\[
iM(h \rightarrow W_L^+ W_L^-) = igm_W \left( \frac{k_2^\mu}{m_W} + \epsilon_2^\mu \right) \left( \frac{k_1^\mu}{m_W} + \epsilon_1^\mu \right) = igm_W \left( \frac{k_3^2 - k_2^2 - k_1^2}{2m_W^2} \right) + \frac{k_2 \cdot \epsilon_1 + k_1 \cdot \epsilon_2 + \epsilon_1 \cdot \epsilon_2}{m_W} \]

\[
\text{onshell} = igm_W \left( \frac{m_2^2 - 2m_W^2}{2m_W^2} \right) - \frac{k_2 \cdot n_1}{k_1 \cdot n_2} - \frac{k_1 \cdot n_2}{k_2 \cdot n_1} + m_W^2 \frac{n_2 \cdot n_1}{(n_2 \cdot k_2)(n_1 \cdot k_1)} \]

\[
\text{onshell} = igm_W \left( \frac{m_2^2 - 2m_W^2}{2m_W^2} \right) - \frac{z}{z} = \frac{n \cdot k_1}{n \cdot k_3} \tag{53}
\]

In the third step we made use of on-shell conditions \( k_3^2 = m_h^2 \), \( k_1^2 = m_W^2 \) and \( k_2^2 = m_W^2 \); in the final step we choose \( n_1 = n_2 = n_3 = n \), and define energy fraction of \( k_1/k_3 \) to \( k_2/k_3 \) as

\[
z = \frac{n \cdot k_1}{n \cdot k_3} \quad \bar{z} = \frac{n \cdot k_2}{n \cdot k_3} \tag{54}
\]
In the limit of particles 1, 2 and 3 are massless, as well as $k_1$, $k_2$, $k_3$ are collinear with each other, we have $\bar{z} = 1 - z$.

After reorganization, we have

$$iM(h \rightarrow W^+_L W^-_L) = igm_W \frac{1}{z\bar{z}} \left( \frac{m_h^2}{2m_W^2} z\bar{z} - (1 - z\bar{z}) \right)$$

(55)

$f \rightarrow f'W^+_L$

Similar to $h \rightarrow W^+_L W^-_L$, the splitting amplitude can be evaluated “on-shell” using the polarization vector $\epsilon^\mu_L = \frac{k_\mu}{m_W} - \frac{m_W^2}{n_\mu k} n^\mu$. For the interaction between fermion current and gauge boson is given by $L = \frac{g}{\sqrt{2}} \bar{\psi}_1 \gamma^\mu P_L \psi_2 W^\mu$, we have the splitting amplitude to be

$$iM(f^{s_1} \rightarrow f'^{s_2}W^+_L) = i \frac{g}{\sqrt{2}} \bar{u}^s_L (k_2) \gamma^\mu u^s_L (k_3) \cdot \left( \frac{k_{1\mu}}{m_W} + \epsilon_{n_\mu} \right)$$

$$= i \frac{g}{\sqrt{2}m_W} \bar{u}^s_L (k_2)(\gamma^\mu(k_3 - k_2)u^s_L (k_3)) - i \frac{g m_W}{\sqrt{2}n_1 \cdot k_1} \bar{u}_L (k_2)\gamma^\mu u_L (k_3)$$

$$\text{onshell} = i \frac{g}{\sqrt{2}m_W} (m_2 \bar{u}^s_R (k_2)u^s_L (k_3) - m_3 \bar{u}^s_L (k_2)u^s_R (k_3))$$

$$-i \frac{g m_W}{\sqrt{2}n_1 \cdot k_1} \bar{u}_L (k_2)\gamma^\mu u_L (k_3)$$

(56)

In the second line, we made use of equations of motion for the fermions. The first two terms give the contribution of the goldstone component, as can be seen by the factor $\frac{m_W}{\sqrt{2}m_W} = y_f$, with $i = 2, 3$. To continue the calculation, we need the explicit form of the fermion wave function:

$$u^{-\frac{1}{2}}_L (k) = u^{-\frac{1}{2}}_R (k) = \sqrt{n \cdot k} \xi$$

$$u^{\frac{1}{2}}_R (k) = u^{\frac{1}{2}}_L (k) = \sqrt{n \cdot k} \xi$$

$$\bar{u}^{-\frac{1}{2}}_L (k) = \bar{u}^{\frac{1}{2}}_R (k) = \sqrt{n \cdot k} \xi^\dagger$$

$$\bar{u}^{\frac{1}{2}}_L (k) = \bar{u}^{-\frac{1}{2}}_R (k) = -\sqrt{n \cdot k} \xi^\dagger$$

(57)

Here $n^\mu = (1, -\frac{i\xi}{k})$.

We take $s_1 = s_2 = -\frac{1}{2}$ as the example, at the collinear limit $k_1 \simeq k_2 \simeq k_3$, we have $n_1 \simeq n_2 \simeq n_3 = n = (1, 0, 0, -1)$ with $z$ direction along $k_3$, and $\xi_2 \xi_3 \simeq \xi^\dagger \xi = 1$

$$\bar{u}^{-\frac{1}{2}}_R (k_2)u^{-\frac{1}{2}}_L (k_3) = m_2 \sqrt{\frac{n \cdot k_3}{n \cdot k_2}}\xi_2 \xi_3 = m_2 \frac{1}{\sqrt{z}}$$

$$\bar{u}^{-\frac{1}{2}}_L (k_2)u^{-\frac{1}{2}}_R (k_3) = m_1 \sqrt{\frac{n \cdot k_2}{n \cdot k_3}}\xi_2 \xi_3 = m_3 \sqrt{z}$$

(58)
Here we have used the definition of $z/\tilde{z}$ as in Eq. (54). We also need,

\[
\bar{u}_L^{\frac{1}{2}}(k_2)f_{n_1}^{-\frac{1}{2}}u_L^{\frac{1}{2}}(k_3) = -\frac{m_1}{n \cdot k_1} \sqrt{(n \cdot k_3)(n \cdot k_2)} \xi^\dagger n \cdot \sigma \xi = -2m_1 \frac{\sqrt{z}}{z} \tag{59}
\]

Here we have used

\[
\xi^\dagger \frac{1}{2} n \cdot \sigma \xi = \xi^\dagger (1 - (-1)) \xi = 2
\]

Plug Eq. (58) and Eq. (59) into Eq. (56), we get,

\[
iM(f^{-\frac{1}{2}} \rightarrow f'^{-\frac{1}{2}} W_L^+) = i g \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\tilde{z}z}} \left( \frac{m_2^2}{m_W} z - \frac{m_2^2}{m_W} z \tilde{z} - 2m_W \tilde{z} \right)
\]

\[
= i(y_{f_2} m_2 \frac{1}{\sqrt{z}} - y_{f_1} m_1 \frac{1}{\sqrt{z}} - \frac{g}{\sqrt{2}} 2m_W \frac{\sqrt{z}}{z}) \tag{60}
\]

Similarly, for $s_2 = s_3 = \frac{1}{2}$, we have

\[
iM(f^\frac{1}{2} \rightarrow f'^{\frac{1}{2}} W_L^+) = i g \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\tilde{z}z}} \left( \frac{m_2^2}{m_W} z \tilde{z} - \frac{m_2^2}{m_W} z - 2m_W \tilde{z} \right)
\]

\[
= i(-y_{f_1} m_1 \frac{1}{\sqrt{z}} + y_{f_2} m_2 \frac{1}{\sqrt{z}} - \frac{g}{\sqrt{2}} 2m_W \frac{\sqrt{z}}{z}) \tag{61}
\]

Based on the results above, it's also straightforwardly to work out the splitting amplitudes if the
gauge boson couples to right-handed fermion current, i.e. $L = \frac{g}{\sqrt{2}} \bar{\psi}_1 \gamma^\mu P_R \psi_2 W_\mu$. For $s_2 = s_3 = -\frac{1}{2}$, and $s_2 = s_3 = \frac{1}{2}$ respectively, the splitting amplitudes are

\[
iM(f^{-\frac{1}{2}} \rightarrow f'^{-\frac{1}{2}} W_L^+) = i g \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\tilde{z}z}} \left( \frac{m_3^2}{m_W} z \tilde{z} - \frac{m_3^2}{m_W} z - 2m_W \tilde{z} \right)
\]

\[
iM(f^\frac{1}{2} \rightarrow f'^{\frac{1}{2}} W_L^+) = i g \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\tilde{z}z}} \left( \frac{m_3^2}{m_W} z - \frac{m_3^2}{m_W} z \tilde{z} - 2m_W \tilde{z} \right) \tag{62}
\]

With splitting amplitudes for gauge boson coupling to left-handed current and right-handed current, we are able to calculate the splitting amplitudes given by the Lagrangian $L = \frac{g}{\sqrt{2}} \bar{\psi}_1 \gamma^\mu (Q_L P_L + Q_R P_R) \psi_2 W_\mu$, with arbitrary $Q_L$ and $Q_R$. 

$W_L^+ \rightarrow W_L^+ W_L^0$
Following the Feynman rules in appendix A and the momenta for $W_L^+ \rightarrow W_L^+ W_L^0$ are $k_3 \rightarrow k_1 k_2$.

The splitting amplitude for is given by the cubic vertex for vector bosons,

$$i\mathcal{M}(W_L^+ \rightarrow W_L^0 W_L^+) = -ig + \frac{i^2}{2}(-k_1 + k_2) \cdot \epsilon_{n_3}(k_3)$$

$$+ [\epsilon_{n_1}(k_2) \cdot \epsilon_{n_3}(k_3) - \frac{i(-i)}{2}(-k_2 - k_3) \cdot \epsilon_{n_1}(k_1)]$$

$$+ [\epsilon_{n_3}(k_3) \cdot \epsilon_{n_1}(k_1) - \frac{i+i}{2}(k_3 + k_1) \cdot \epsilon_{n_2}(k_2)] \epsilon_{n_1} \cdot \epsilon_{n_j}$$

is suppressed by both the factor of $\frac{m_W}{2E_k}$ and $\theta$, so they are negligible. Indeed, the simplest way is to choose $n_1 = n_2 = n_3 = n$, which corresponds to the conventional light-cone gauge. This choice leads to $\epsilon_{n_1} \cdot \epsilon_{n_j} = 0$.

The splitting amplitude then becomes

$$i\mathcal{M}(W_L^+ \rightarrow W_L^0 W_L^+) = \frac{ig}{2} m_W \left[-\frac{(k_1 - k_2) \cdot n}{n \cdot k_3} + \frac{(k_2 + k_3) \cdot n}{n \cdot k_1} + \frac{-(k_3 + k_1) \cdot n}{n \cdot k_2} \right] \quad (63)$$

We also write $m_W = \frac{2\sqrt{2} \alpha}{\sqrt{2}}$, plug all in. After organization, and making use of the definition of energy fraction $z/\bar{z}$ in Eq. (54), the amplitude finally becomes

$$i\mathcal{M}_{W_L^+ \rightarrow W_L^+ W_L^0} = \frac{ig^2 v}{2} \frac{z - \bar{z}}{z \bar{z}} (1 + \frac{z \bar{z}}{2}) \quad (64)$$

IV. CONCLUSIONS

In this paper we derived the Feynman rules of massive gauge theory in physical gauges. The model is $\theta_W \rightarrow 0$ limit of the standard model with gauge group $SU(2)_L$. The main novelty is that we treat gauge fields and goldstone fields uniformly as 5-component vector fields: $W^M = (W^\mu, \phi)$.

Making use of the new notation, we derived the propagator for vector bosons. We noticed there is a remarkable similarity between massless gauge theory and massive gauge theory in the algebra level, making the derivation almost trivial. We also derived the Feynman rules for vertices. Especially, we found that gauge-gauge-gauge vertex and goldstone-goldstone-gauge vertex can be combined into single $W$-$W$-$W$ vertex with a common factor $\epsilon^{abc}$, which is obviously due to the remaining custodial symmetry in the scalar potential.
We also investigated the structure of 3-point on-shell amplitudes. We demonstrated that all 3-point on-shell amplitudes – $W-W-W$, $h-W-W$, $f-f'-W$ – satisfy on-shell gauge symmetry, which is a reflection of on-shell gauge symmetry for general S-matrix. This on-shell gauge symmetry ensures that amplitudes calculated with the new Feynman rules and with the usual Feynman rules are equivalent. We call this equivalence on-shell match condition. Finally, making use of the new Feynman rules and on-shell match condition for 3-point amplitudes, we calculated some collinear splitting amplitudes in massive gauge theory.

**Acknowledgements** The author thanks the discussions with Tao Han, Brock Tweedie and Kaoro Hagiwara.

**Appendix A: Feynman Rules**

Convention:

\[
g'_{MN} = g'_{MN} = \text{diag}(g_{\mu\nu}, -1/2) \quad g^M_N = g_{MN} = \text{diag}(g_{\mu\nu}, -1)
\]

\[
k^M = \begin{pmatrix} k^\mu \\ -im_W \end{pmatrix} \quad k^*M = \begin{pmatrix} k^\mu \\ im_W \end{pmatrix} \quad n^M = \begin{pmatrix} n^\mu \\ 0 \end{pmatrix}
\]

\[
n^2 = 0 \quad k \cdot k^* = g_{MN}k^M k^{*M}
\]
Propagators

\[ W^\pm = \frac{-i}{k \cdot k^* + i\epsilon} (g^{MN} - \frac{n^M k^*N + k^M n^*N}{n \cdot k - i\epsilon} + \xi \frac{k \cdot k^*}{(n \cdot k)^4} k^M k^*N) \]

\[ W^0 = \frac{-i}{k \cdot k^* + i\epsilon} (g^{MN} - \frac{n^M k^*N + k^M n^*N}{n \cdot k - i\epsilon} + \xi \frac{k \cdot k^*}{(n \cdot k)^4} k^M k^*N) \]

\[ h = \frac{i}{k^2 - m_f^2 + i\epsilon} \]

\[ h = \frac{i}{k^2 - m_h^2 + i\epsilon} \]

(A1)
Gauge-goldstone Sector

\[ = -ig \left( g^{MN}(k_1 - k_2)^\rho + g^{NK}(k_2 - k_3)^\mu + g^{KM}(k_3 - k_1)^\nu \right) \]

\[ = ig^2 \left( 2g^{\mu\rho}g^{\nu\sigma} - g^{MN}g^{\rho\Sigma} - g^{MS}g^{NP} \right) - i\left( \lambda_h - \frac{g^2}{2} \right)g^Mg^Ng^Pg^Sg^A \]

\[ = -ig^2 \left( 2g^{MN}g^{\rho\Sigma} - g^{MP}g^{\rho\Sigma} - g^{MS}g^{PN} \right) - i\lambda_h \frac{g^Mg^Ng^Pg^Sg^A}{2} \]

\[ \frac{-ig^2}{2} \left( g^{\mu\nu}g^{P4}g^A + g^{M4}g^{N4}g^{\rho\sigma} + g^{\mu\rho}g^{N4}g^A + g^{M4}g^{P4}g^{\nu\rho} \right) + g^{M4}g^{P4}g^{\nu\rho} + g^{N4}g^{P4}g^{\mu\sigma} + g^{M4}g^{S4}g^{\nu\rho} - i\frac{3\lambda_h}{2}g^{M4}g^{N4}g^{P4}g^{S4} \]
Higgs Sector and “VEV” Sector

\[
\begin{align*}
    h & \rightarrow q W^- M^- \\
    k_1 & \rightarrow k_2 W_+^N
\end{align*}
\]

\[
= -\frac{g}{2} \left( g^{N_4} (k_1^\mu - q^\mu) + g^{M_4} (k_2^\nu - q^\nu) \right) + i g_{W} g^{\mu\nu} - i \frac{\lambda_{h^0}}{2} g^{M_4} g^{N_4}
\]

\[
\begin{align*}
    h & \rightarrow q W_0^M \\
    k_1 & \rightarrow k_2 W_0^N
\end{align*}
\]

\[
= -\frac{g}{2} \left( (k_1 - q)^\mu g^{N_4} + g^{M_4} (k_2 - q)^\nu \right) + i g_{W} g^{\mu\nu} - i \frac{\lambda_{h^0}}{2} g^{M_4} g^{N_4}
\]

\[
\begin{align*}
    h & \rightarrow W_+^N \\
    h & \rightarrow W_{-+}^\Sigma
\end{align*}
\]

\[
= -i \frac{g^2}{2} g^{\nu\sigma} - i \frac{\lambda_{h^4}}{2} g^{N_4} g^{\Sigma_4}
\]

\[
\begin{align*}
    h & \rightarrow W_0^N \\
    h & \rightarrow W_{0-}^\Sigma
\end{align*}
\]

\[
= -i \frac{g^2}{2} g^{\nu\sigma} - i \frac{\lambda_{h^4}}{2} g^{N_4} g^{\Sigma_4}
\]

\[
\begin{align*}
    h & \rightarrow W_{-}^\Sigma \\
    h & \rightarrow W_{0}^\Sigma
\end{align*}
\]

\[
= -i \frac{3 \lambda_{h^0}}{2}
\]

\[
\begin{align*}
    h & \rightarrow h h \\
    h & \rightarrow h h
\end{align*}
\]

\[
= -i \frac{3 \lambda_{h}}{2}
\]
Fermion Sector

\begin{align*}
W^M_{+} & = (-i \frac{g}{\sqrt{2}} \gamma^\mu P_L - (y_d P_R - y_u P_L) g^M_4) V_{ij} \\
W^M_{-} & = (-i \frac{g}{\sqrt{2}} \gamma^\mu P_L - (y_u P_R - y_d P_L) g^M_4) V^*_{ij} \\
W^N_{0} & = -ig \gamma^\mu (T^3_3 P_L) - y_f \gamma_5 T_3 g^M_4
\end{align*}

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