Semidefinite Programming in Timetabling and Mutual-Exclusion Scheduling

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Abstract

In scheduling and timetabling applications, the mutual-exclusion constraint stipulates that certain pairs of tasks that cannot be executed at the same time. This corresponds to the vertex colouring problem in graph theory, for which there are well-known semidefinite programming (SDP) relaxations. In practice, however, the mutual-exclusion constraint is typically combined with many other constraints, whose SDP representability has not been studied.

We present SDP relaxations for a variety of mutual-exclusion scheduling and timetabling problems, starting from a bound on the number of tasks executed within each period, which corresponds to graph colouring bounded in the number of uses of each colour. In theory, this provides the strongest known bounds for these problems that are computable to any precision in time polynomial in the dimensions. In practice, we report encouraging computational results on random graphs, Kneser graphs, “forbidden intersection” graphs, the Toronto benchmark, and the International Timetabling Competition.

1 Introduction

Across many areas of combinatorial optimisation, semidefinite programming (SDP) (Wolkowicz, Saigal, & Vandenberghe\textsuperscript{[2000]}, Anjos & Lasserre\textsuperscript{[2011]}) has made it possible to derive strong lower bounds (Goemans & Rendl\textsuperscript{[2000]}), as well as to obtain very good solutions using randomised rounding (Lau, Ravi, & Singh\textsuperscript{[2011]}). Nevertheless, there seem to be only few applications to practical scheduling, timetabling, or rostering problems.

In scheduling and timetabling problems, one encounters extensions of the mutual-exclusion constraint, which stipulates that certain pairs of tasks or events cannot be executed at the same time. This corresponds to the graph colouring problem in graph theory, for which there are well-known semidefinite programming (SDP) relaxations.
The SDP representability of combinations of the mutual-exclusion constraint with other constraints has been an open problem.

Clearly, the work on graph colouring provides a test of infeasibility for timetabling and scheduling problems incorporating the mutual exclusion problem. Such an infeasibility test compares a lower bound on the optimum of bounded vertex colouring of the conflict graph against the number of periods available. Lower bounds obtained in ignorance of the extensions, especially the bound on the number of uses of each colour, are still perfectly valid, but generally weak.

In this paper, we set out to explore applicability of semidefinite programming in scheduling and timetabling problems, which extend graph colouring. In educational timetabling, this corresponds to considering room sizes, room features, room stability, and pre-allocated assignments. In transportation timetabling, these correspond to considering vehicle capacity, line-vehicle compatibility, etc. We show that semidefinite programming relaxations of a variety of such problems can be formulated, starting with bounded vertex colouring of the conflict graph.

Our paper is organised as follows: Following some preliminaries, we present our relaxations in Section 3. In Section 4 we specialise a well-known first-order method to solving the relaxations and showcase two algorithms for rounding in the relaxations. In Section 5 we analyse some properties of the relaxations and the performance of algorithms applied to them. In Section 6 we present results of extensive computational tests. On conflict graphs from the International Timetabling Competition 2007, the Toronto benchmark, as well as on random graphs, we show the relaxations often provide the best possible lower bound and make it possible to obtain very good solutions by randomised rounding. On “forbidden intersections” graphs, we show the strength and weakness of the bound. Further avenues for research are suggested in Section 7.

2 Notation and Related Work

2.1 Semidefinite Programming

Firstly, let us revisit the definition of semidefinite programming, which is a popular extension of linear programming. In linear programming (LP), the task is to optimise a linear combination \( c^T x \) subject to \( m \) linear constraints \( Ax = b \) subject to the element-wise restriction of variable \( x \) to non-negative real numbers. Notice \( c \) is an \( n \)-vector, \( x \) is a compatible vector variable, \( b \) is an \( m \)-vector, and \( A \) is an \( m \times n \) matrix. One can state the problem also as:

\[
\begin{align*}
    z &= \min_x c^T x \quad \text{s.t.} \quad \mathcal{A}_A(x) = b \quad \text{and} \quad x \geq 0 \\
    &= \max_y b^T y \quad \text{s.t.} \quad \mathcal{A}^*_A(y) \leq c
\end{align*}
\]

where linear operator \( \mathcal{A}_A(x) \) (parametrised by matrix \( A \)) maps vector \( x \) to a vector \( Ax \), and \( x \geq 0 \) denotes the element-wise non-negativity of \( x \), \( x \in (\mathbb{R}^+)^n \). The element-wise non-negativity of \( x \) should be seen as a restriction of vector \( x \) to the positive orthant, which is a convex cone, as all linear combinations with non-negative coefficients of element-wise non-negative vectors are element-wise non-negative vectors. Using a
variety of methods, linear programming can be solved to any fixed precision in polynomial time. These methods work for other symmetric convex cones as well.

Semidefinite programming (SDP, Bellman & Fan 1963; Alizadeh 1995; Wolkowicz et al. 2000, 2000; Anjos & Lasserre 2011) is a convex optimisation problem, which generalises linear programming. It replaces the vector variable with a square symmetric matrix variable and the polyhedral symmetric convex cone of the positive orthant with the non-polyhedral symmetric convex cone of positive semidefinite matrices. The primal-dual pair in the standard form is:

\[ z_p = \min_{X \in \mathbb{S}^n} \langle C, X \rangle \text{ s.t. } \mathcal{A}_A(X) = b \text{ and } X \succeq 0 \]  

(P STD)

\[ z_d = \max_{y \in \mathbb{R}^m, S \in \mathbb{S}^n} b^T y \text{ s.t. } \mathcal{A}_A^*(y) + S = C \text{ and } S \succeq 0 \]  

(D STD)

where \( X \) is a primal variable in the set of \( n \times n \) symmetric matrices \( \mathbb{S}^n \), \( y \) and \( S \) are the corresponding dual variables, \( b \) is an \( m \)-vector, \( C, A_i \) are compatible matrices, and linear operator \( \mathcal{A}_A(X) \) (parametrised by a symmetric matrix \( A \)) maps a symmetric matrix \( X \) to vectors in \( \mathbb{R}^n \), wherein the \( i \)th element \( \mathcal{A}_A(X)_i = \langle A_i, X \rangle \). \( \mathcal{A}_A^*(y) \) is again the adjoint operator. \( M \succeq N \) or \( M - N \succeq 0 \) denotes \( M - N \) is positive semidefinite. Note that an \( n \times n \) matrix, \( M \), is positive semidefinite if and only if \( y^T My \geq 0 \) for all \( y \in \mathbb{R}^n \). One can also treat inequalities explicitly in the primal-dual pair:

\[ z_p = \min_{X \in \mathbb{S}^n} \langle C, X \rangle \text{ s.t. } \mathcal{A}_A(X) = b \text{ and } \mathcal{A}_B(X) \geq d \text{ and } X \succeq 0 \]  

(P SDP)

\[ z_d = \max_{y \in \mathbb{R}^m, v \in \mathbb{R}^q, S \in \mathbb{S}^n} b^T y + d^T v \text{ s.t. } \mathcal{A}_A^*(y) + \mathcal{A}_B^*(v) + S = C \text{ and } S \succeq 0 \text{ and } v \geq 0 \]  

(D SDP)

where \( d \) is a \( q \)-vector and linear operator \( \mathcal{A}_B(X) \) maps \( n \times n \) matrices to \( q \)-vectors similarly to \( \mathcal{A}_A \) above. As all linear combinations with non-negative coefficients of positive semidefinite matrices are positive semidefinite, \( X \succeq 0 \) should again be seen as a restriction to a convex cone.

### 2.2 Semidefinite Programming Relaxations of Graph Colouring

Graph colouring, also known as vertex colouring, or partition into independent sets, is the problem of:

**GRAPH COLOURING**

Given an undirected graph \( G = (V, E) \) with vertices \( V = 1, 2, \ldots, n \) and edges \( E \subseteq \{(u, v) \text{ s.t. } 1 \leq u < v \leq n\} \), return a partition \( P = (P_i) \) of \( V \) of the smallest possible cardinality \( |P| \) so that for each partition \( P_i \), for no edge \((u, v) \in E\), there are both \( u \) and \( v \) in \( P_i \). As in any partition, \( \bigcup_i P_i = V \) and for all \( 1 \leq i < j \leq |P| \), we have \( P_i \cap P_j = \emptyset \).
The partitions are known as “colour classes” or “independent sets” in graph colour-
ing, or (assignment to) “time periods” in timetabling and scheduling. The optimum,
i.e., the smallest possible number \( |P| \) of colour classes, is denoted the “chromatic num-
ber” in graph colouring or minimum number of required time periods in timetabling,
or “makespan” in connection with certain mutual-exclusion problems (cf. Section 3.1)
in the scheduling literature.

The decision version of graph colouring appears on Karp’s original list (Karp, 1972) of NP-Complete problems. In polynomial time, one can obtain lower bounds
on the chromatic number, for instance using linear or semidefinite programming. Just
as there are a number of ways of formulating a lower bound on the chromatic number
in linear programming, there are a number of ways of formulating a lower bound on
the chromatic number using SDP. All are, in some sense, related to the inequality of
Wilf (Wilf, 1967), wherein the largest eigenvalue of an adjacency matrix of a graph
incremented by one bounds the chromatic number of a graph from the above. By con-
sidering the semi-definite programming lower bound on the largest-eigenvalue upper-
bound, one obtains a parameter of a graph, sometimes known as ‘theta’, \( \theta \) (Lovász
1979), which is at least as large as the clique number and no more than the chro-
matic number, yet is computable in polynomial time using SDP. The known bounds for
colouring form a hierarchy (Szegedy, 1994):

\[
\alpha(G) \leq \chi'(G) \leq \chi(G) \leq \chi^+(G) \leq \chi^+(\Delta)(G) \leq \chi(\overline{G}), \quad \text{or}
\]

\[
\omega(G) \leq \chi'(\overline{G}) \leq \chi(\overline{G}) \leq \chi^+(\overline{G}) \leq \chi^+(\Delta)(\overline{G}) \leq \chi(G),
\]

where \( \alpha \) is the size of the largest independent set, \( \omega \) is the size of the largest clique,
\( \chi \) is the chromatic number, \( \chi'(G) = \theta(\overline{G}) \) is the vector chromatic number (Lovász
1979; Karger, Motwani, & Sudan, 1998), \( \chi'(\overline{G}) = \theta_{1/2}(\overline{G}) \) is the strict vector chro-
matic number (Kleinberg & Goemans, 1998), \( \chi^+(G) = \theta_1(\overline{G}) \) is the strong vector
chromatic number (Szegedy, 1994), and bar indicates complementation of a graph. In
Figure 1 we summarise all three formulations for all three lower bounds in the vec-
tor programming notation. None of the formulations has, however, been extended to
bounded colouring, up to the best of our knowledge.
Figure 1: An overview of known vector programming (VP) and semidefinite programming (SDP) relaxations of vertex colouring, $\alpha(G) \leq \mathcal{X}'(G) \leq \mathcal{X}(G) \leq \mathcal{X}'(\overline{G})$ or $\omega(G) \leq \mathcal{X}'(G) \leq \mathcal{X}'(\overline{G}) \leq \mathcal{X}^+(G) \leq \chi(G)$.

**Lovász’s Bound as VP**

$\mathcal{X}(G) = \min t = \theta(\bar{G})$ \hspace{1cm} (2)

s.t. $\|v_i\|_2 = 1 \hspace{1cm} \forall i \in V$

$\langle v_i, v_j \rangle = -\frac{1}{t-1} \hspace{1cm} \forall \{i, j\} \in E$

Primal SDP for Lovász’s Bound

$\mathcal{X}(G) = \max \langle J, X \rangle = \theta(\bar{G})$ \hspace{1cm} (3)

s.t. $\text{trace}(X) = 1$

$X_{uv} = 0 \hspace{1cm} \forall \{u, v\} \in E$

$X \succeq 0$

Dual SDP of Lovász’s Bound

$\mathcal{X}(G) = \min t = \theta(\bar{G})$ \hspace{1cm} (4)

s.t. $U_{uv} = \frac{1}{t-1} \hspace{1cm} \forall \{u, v\} \in E$

$U \succeq 0, t \geq 2$

**Kleinberg’s Bound as VP**

$\mathcal{X}'(G) = \min t = \theta_{1,2}(\bar{G})$ \hspace{1cm} (5)

s.t. $\|v_i\|_2 = 1 \hspace{1cm} \forall i \in V$

$\langle v_i, v_j \rangle \leq -\frac{1}{t-1} \hspace{1cm} \forall \{i, j\} \in E$

Primal SDP for Kleinberg’s Bound

$\mathcal{X}'(G) = \max \langle J, X \rangle = \theta_{1,2}(\bar{G})$ \hspace{1cm} (6)

s.t. $\text{trace}(X) = 1$

$X_{uv} = 0 \hspace{1cm} \forall \{u, v\} \in E$

$X_{uv} \geq 0 \hspace{1cm} \forall \{u, v\} \in \overline{E}$

$X \succeq 0$

Dual SDP of Kleinberg’s Bound

$\mathcal{X}'(G) = \min t = \theta_{1,2}(\bar{G})$ \hspace{1cm} (7)

s.t. $U_{uv} = t \hspace{1cm} \forall u \in V$

$U_{uv} \leq \frac{1}{t-1} \hspace{1cm} \forall \{u, v\} \in \overline{E}$

$U \succeq 0, t \geq 2$

**Szegedy’s Bound as VP**

$\mathcal{X}^+(G) = \min t = \theta_{2}(\bar{G})$ \hspace{1cm} (8)

s.t. $\|v_i\|_2 = 1 \hspace{1cm} \forall i \in V$

$\langle v_i, v_j \rangle = -\frac{1}{t-1} \hspace{1cm} \forall \{i, j\} \in E$

$\langle v_i, v_j \rangle \geq -\frac{1}{t-1} \hspace{1cm} \forall \{i, j\} \in \overline{E}$

Primal SDP of Szegedy’s Bound

$\mathcal{X}^+(G) = \max \langle J, X \rangle = \theta_{2}(\bar{G})$ \hspace{1cm} (9)

s.t. $\text{trace}(X) = 1$

$X_{uv} \leq 0 \hspace{1cm} \forall \{u, v\} \in E$

$X \succeq 0$

Dual SDP of Szegedy’s Bound

$\mathcal{X}^+(G) = \min t = \theta_{2}(\bar{G})$ \hspace{1cm} (10)

s.t. $U_{uv} = t \hspace{1cm} \forall u \in V$

$U_{uv} \leq -\frac{1}{t-1} \hspace{1cm} \forall \{u, v\} \in \overline{E}$

$U \succeq 0, t \geq 2$
There are a number of ways of deriving and thinking about the SDP relaxations. In any case, the primal \( n \times n \) matrix variable can be seen as

\[
M_{u,v} = \begin{cases} 
1 & \text{if vertex } u \text{ is in the same colour class as } v \\
0 & \text{otherwise.} 
\end{cases}
\] (11)

Notice matrix \( M \) has the “hidden block diagonal” property:

**Proposition 1** For any value of \( M \), there exists a permutation matrix \( P \), such that

\[
M^{bd} = P^T MP = \begin{bmatrix}
J_{c_1} & 0 & 0 \\
0 & J_{c_2} & 0 \\
& & \ddots \\
0 & 0 & J_{c_s}
\end{bmatrix}
\] (12)

where \( J_c \) is the \( c \times c \) matrix of all ones and \( \sum_{i=1}^{s} c_i = n \). Such \( P^T MP \) is denoted a direct sum of \( J_c \).

One can hence derive the semidefinite programming relaxation from:

**Proposition 2** (Dukanovic and Rendl (2004)) For any symmetric 0-1 matrix \( M \) there exists a permutation matrix \( P \) such that \( P^T MP \) is the direct sum of \( s \) all-ones matrices if and only if there is a vector of all-ones on the diagonal and the rank of \( M \) is \( s \) and \( M \) is positive semidefinite.

by relaxing the rank constraint, as usual (Fazel, Hindi, & Boyd, 2004). Alternatively, one can see theta as an eigenvalue bound, where the largest eigenvalue \( \lambda_{\text{max}}(A) = \min \{ t \text{ s.t. } tI - A \succeq 0 \} \) for an identity matrix \( I \). Perhaps most “fundamentally”, one could see theta as a relaxation of the co-positive programming formulation of graph colouring, recently proposed by Bomze, Frommlet, and Locatelli (2010). A number of other derivations have been surveyed by Knuth (1994).

### 2.3 Related Applications

Within the job-shop scheduling, SDP relaxations of the maximum cut problem (MAX-CUT, Goemans & Williamson (1995) have been adapted to scheduling workload on two machines (Skutella, 2001; H. Yang, Ye, & Zhang, 2003) and home-away patterns in sports scheduling (Suzuka, Miyashiro, Yoshise, & Matsui, 2007). Bansal, Srinivasan, & Svensson (2016) extended this to scheduling with weighted completion time objectives on any number of unrelated machines, using a clever rounding technique in a lift-and-project relaxations. They have also shown that the relaxation of (Skutella, 2001) is in some sense weak (has 3/2 integrality gap). We are not aware of any applications of semidefinite programming to mutual-exclusion scheduling or timetabling, excepting two abstracts of the present authors (Burke, Marecek, & Parkes, 2010; Marecek & Parkes, 2012), which we build upon in this paper.
3 Old Problems and Novel Relaxations

3.1 Mutual-Exclusion Scheduling

In one of the prototypical problems in timetabling, scheduling, and staff rostering (Welsh & Powell, 1967; Burke, de Werra, & Kingston, 2004; Baker & Coffman, 1996), one needs to assign $n$ unit-time events, classes, tasks, or jobs (“vertices”), some of which must not be run, executed, or taught at the same time (“the mutual-exclusion constraint”), perhaps due to the use to some shared, renewable resource, to $m$ rooms, processors, machines, or employees (“uses of a colour”), so that the number of units of time required (“makespan”, “number of colours”) is as small as possible. It is natural to represent the elements being assigned by the elements of set $V = 1, 2, \ldots, n$, and to represent the mutual-exclusion constraint by a set of pairs of elements of $V$, which are not to be assigned to disjoint time-units (“conflict graph”).

Let us now consider:

$m$-Bounded Colouring

Given an undirected graph $G = (V,E)$ with vertices $V = 1, 2, \ldots, n$ and edges $E \subseteq \{(u,v) \text{ s.t. } 1 \leq u < v \leq n\}$, and an integer $m \leq |V|$, return a partition $P = (P_i)$ of $V$ of the smallest possible cardinality $|P|$ so that for each partition $P_i$, $|P_i| \leq m$ and for no edge $(u,v) \in E$ there are both $u$ and $v$ in $P_i$.

Terminology varies. In the scheduling community, the problem is known as scheduling of unit-length tasks on $m$ parallel machines with renewable resources (Blazewicz, Ecker, Pesch, Schmidt, & Weglarz, 2007, Chapter 12), Mutual-Exclusion Scheduling (Baker & Coffman, 1996), scheduling with incompatible objects (Leighton, 1979), or $Pm|\text{res},p_j=1|C_{\text{max}}$ in the notation of (Graham, Lawler, Lenstra, & Kan, 1979). In discrete mathematics (Kraráp & de Werra, 1982), some authors (Bodlaender & Jansen, 1993) refer to the problem as Partition into Bounded Independent Sets, while others use $m$-Bounded Colouring (Hansen, Hertz, & Kuplinsky, 1993) or just Bounded Graph Colouring. There is a simple transformation to Equitable Colouring (Bodlaender & Fomin, 2005), where the cardinality of colour classes can differ by at most one. In terms of complexity theory, $m$-Bounded Colouring is in P on trees (Chen & Lih, 1994), but NP-Hard on co-graphs, interval graphs, and bipartite graphs (Bodlaender & Jansen, 1993).

From the block-diagonal property of the matrix variable $\mathbf{12}$, it seems clear that we expect row-sums and column-sums in the binary-valued variable to be bounded from above by $m$: 
\begin{align*}
\min \ t \\
\text{s.t.:} \\
X_{vv} &= 1 \quad \forall v \in V \quad \text{(E1)} \\
X_{uv} &= 0 \quad \forall \{u, v\} \in E \quad \text{(E2)} \\
\text{rank}(X) &= t \\
\sum_{v \in V} X_{vv} &\leq m \quad \forall v \in V \quad \text{(IN)} \\
X &\succeq 0 \quad \text{(PSD)} \\
X_{uv} &\in \{0, 1\} \quad \text{(Binary)}
\end{align*}

where $X$ is an $n \times n$ matrix variable and $t$ is a scalar variable.

By dropping the element-wise integrality, relaxing a non-convex bound on $\text{rank}(X)$ to a convex bound on the $\text{trace}(X)$, and algebraic manipulations, we obtain the relaxation:

\begin{align*}
\min \ t \\
\text{s.t.:} \\
Y_{vv} &= t \quad \forall v \in V \quad \text{(E1)} \\
Y_{uv} &= 0 \quad \forall \{u, v\} \in E \quad \text{(E2)} \\
\sum_{u \in V} Y_{uv} &\leq tm \quad \forall v \in V \quad \text{(IN)} \\
Y - J &\succeq 0 \quad \text{(PSD)} \\
Y_{uv} &\geq 0 \quad \forall \{u, v\} \quad \text{(N)}
\end{align*}

where $Y$ is an $n \times n$ matrix variable and $t$ is a scalar variable. This can be seen as a spectahedron (E1 [PSD]) being intersected by a polyhedron given by the linear inequalities (IN).

Notice that additional inequalities $\sum_{v \in V} Y_{uv} \leq tm$ are not required due to symmetry. Notice also that the usual theta-like relaxations ($X(G)$, $X'(G)$, $X^+(G)$) cannot be used easily, as one cannot easily work with a graph’s complement, both due to its density and due to the constraints on the representability of our extensions.

The complication is the relaxation above is not a semidefinite program in the standard form (PSTD, DSTD). Notice scalar variable $t$ has been introduced only for clarity. As long as the entries on the diagonal of the matrix variable are constrained to be equal, any one of them can be used instead. One can either introduce new scalar slack variables and convert inequalities to equalities, or one can design solvers treating inequalities explicitly.

There remains the constraint $Y - J \succeq 0$ to deal with, as the standard form only allows to require a matrix, rather than expression, to be psd. The mechanistic approach, employed by automated model transformation tools (Lofberg [2004]), for instance, is to double the dimension by introducing new variable $X$, set $Y - J = Z, Z \succeq 0$. For an
arbitrary vertex \( w \in V \), one obtains:

\[
\begin{align*}
\text{max} & \quad Y_{w,w} \\
\text{s.t.} & \quad Z_{u,v} = -1 \quad \forall \{u,v\} \in E \quad (A1) \\
& \quad Z_{v,v} = Z_{w,w} \quad \forall v \in V \setminus \{w\} \quad (A2) \\
& \quad Y_{u,v} - Z_{u,v} = 1 \quad \forall u, v \in V \quad (W) \\
& \quad \sum_{w \in V} Y_{w,v} \leq tm \quad \forall v \in V \quad (B) \\
& \quad Z \succeq 0
\end{align*}
\]

An alternative approach is to optimise:

\[
\begin{align*}
\text{min} & \quad X_{w,w} + 1 \\
\text{s.t.} & \quad X_{u,v} = -1 \quad \forall \{u,v\} \in E \quad (A1) \\
& \quad X_{v,v} = X_{w,w} \quad \forall v \in V \setminus \{w\} \quad (A2) \\
& \quad |V| - mX_{w,w} - m - 1 + \sum_{v \in V} X_{v,v} \leq 0 \quad \forall v \in V \quad (B) \\
& \quad X \succeq 0
\end{align*}
\]

Any formulation involving \( Y - J \succeq \) in this paper can be easily transformed in this fashion.

### 3.2 Initial Assignment

In many applications, one has to deal with complicating constraints. In graph-theoretic terms, the most common complicating constraints are pre-existing assignments. Pre-existing assignments can be represented as subsets of \( V \), which need to be assigned to the same unit of time. This corresponds to:

\[
\text{m-Bounded Colouring with Pre-Colouring}
\]

Given an undirected graph \( G = (V,E) \) with vertices \( V = 1, 2, \ldots, n \) and edges \( E \subset \{(u,v)\ \text{s.t.}\ 1 \leq u < v \leq n\} \), an integer \( m \leq |V| \), and a family \( C = (C_i) \) of disjoint subsets of \( V \) with \( |C_i| \leq m \), return a partition \( P = (P_i) \) of \( V \) of the smallest possible cardinality \(|P|\) so that for each partition \( P_i \), \(|P_i| \leq m \) and for no edge \((u,v) \in E\) there are both \( u \) and \( v \) in \( P_i \) and for each set \( C_i \in C \), there is a partition in \( C_i \subseteq P_j \in P \). The graph \( G \) is called “conflict graph” and family \( C \) is called “pre-colouring”. The partition \( P \) corresponds to the “same-colour equivalence” and \( P_i \) are called “colour classes” or “independent sets”.

In terms of complexity theory, m-Bounded Colouring with Pre-Colouring is NP-Hard even on trees (Bodlaender & Fomin 2005). Even in the case of trees, however, there are fixed parameter tractable algorithms (Bodlaender & Jansen 1993) Bodlaender & Fomin 2005).
In terms of SDP representability, given a pre-colouring \( C = (C_i), C_i \subseteq V \), it suffices to set the corresponding elements of matrix variable \( Y \) to \( t \):

\[
\mathcal{Y}(G, m) = \max t \quad \text{s.t.} \quad Y_{vv} = t \quad \forall v \in V \\
Y_{uv} = 0 \quad \forall \{u, v\} \in E \\
\sum_u c_u Y_{uv} \leq tm \quad \forall v \in V \\
\sum_v c_v Y_{uv} \leq tm \quad \forall u \in V \\
Y - J \succeq 0
\]  

3.3 A Reformulation

As an aside, notice that \( m \)-BOUNDED COLOURING WITH PRE-COLOURING can be easily transformed into a problem without pre-colouring, but with certain weights on vertices:

\textit{c-WEIGHTED m-BOUNDED COLOURING}

Given an undirected graph \( G = (V, E) \) with vertices \( V = 1, 2, \ldots, n \) and edges \( E \subseteq \{(u, v) \mid 1 \leq u < v \leq n\} \), a vector of positive integers \( c \) of dimension \( |V| \), and an integer \( m \leq |V| \), return a partition \( P = (P_i) \) of \( V \) of the smallest possible cardinality \( |P| \) so that for each partition \( P_i \), \( cp_i \leq m \), where \( p_i \) is the \( 0 - 1 \) index vector corresponding to \( P_i \), and for no edge \( (u, v) \in E \) there are both \( u \) and \( v \) in \( P_i \).

For any non-empty \( C \), this leads to a reduction in the dimension of the matrix variable, compared to the simple relaxation (17):

\[
\mathcal{Y}(G, m) = \max t \quad \text{s.t.} \quad Y_{vv} = t \quad \forall v \in V \\
Y_{uv} = 0 \quad \forall \{u, v\} \in E \\
\sum_u c_u Y_{uv} \leq tm \quad \forall v \in V \\
\sum_v c_v Y_{uv} \leq tm \quad \forall u \in V \\
Y - J \geq 0
\]
While the reduction improves computational performance, when used with off-the-shelf solvers, it may render both the design of custom solvers and their analysis (cf. Section 5) more challenging.

3.4 Simple Laminar Timetabling

In timetabling applications, it is often necessary to consider assignment of events to both periods and rooms. In the relaxations above, vertices within a single colour class correspond to events taking place at the same time, but rooms are represented only by the $m$-bound, which corresponds to the number of room available. This may be insufficient: Consider, for example, a situation with two large lecture rooms, twenty periods per week, and forty large lectures. One could formulate this problem as a binary linear program with a variable with three indices (events, rooms, and periods) and apply the operators of [Lovász and Schrijver (1991)] or [Lasserre (2001)] to obtain semidefinite programming relaxations. We present a number of alternatives, where the matrix variable has a considerably lower dimension.

In particular, one can extend $m$-BOUNDED COLOURING WITH PRE-COLOURING to consider not only the number $m$ of available rooms (processors, machines, employees, or similar), but also their capacities and features. This corresponds to:

**SIMPLE TIMETABLING**

Given an undirected conflict graph $G = (V, E)$ where vertices $V$ are also denoted events, a vector of capacity-requirements $p \in \mathbb{R}^{|V|}$, an integer $m \leq |V|$, a vector of capacities $r \in \mathbb{R}^m$, a number of features $f \in \mathbb{N}$, set $F \subseteq V \times F$ detailing feature-requirements of events, set $G \subseteq \{1, 2, \ldots, m\} \times F$ detailing feature availability, and a family $C = (C_i)$ of disjoint subsets of $V$ with $|C_i| \leq m$, return a partition $P = (P_i)$ of $V$ of the smallest possible cardinality $|P|$ so that

- for each partition $P_i$, $|P_i| \leq m$
- for each partition $P_i$ and for no edge $(u, v) \in E$ there are both $u$ and $v$ in $P_i$
- for each partition $P_i$ and for each distinct capacity $c$ in $r$, the subset of $P_i$ with capacity greater or equal than $c$, according to $p$, is less than the number of elements in $r$ greater or equal to $c$
- for each partition $P_i$ and for each feature $1, 2, \ldots, f$, the subset of $P_i$ requiring the feature, according to $F$, is less than the number of rooms where it is available, according to $G$
- for each set $C_i \in C$, there is a partition in $C_i \subseteq P_j \in P$.

In an important special case, which we denote SIMPLE LAMINAR TIMETABLING, sets of events and rooms with certain feature-requirements and feature-availability.
are “laminar”. A collection of sets \( F \) is called “laminar” if \( A, B \in F \) implies that \( A \subseteq B, B \subseteq A \) or \( A \cap B = \emptyset \). The subsets of rooms requiring capacities larger than a certain value are naturally laminar, but naturally occurring features need not be. As \( m \)-Bounded Colouring with Pre-Colouring is a special case of Simple Laminar Timetabling, hardness results cited above apply also to Simple (Laminar) Timetabling. Such laminar timetabling and associated bounds can be of interest, for example, in planning the capacities of rooms cf. 

Initially, we restrict ourselves to the Simple Laminar Timetabling and extend the \( c \)-weighted relaxation above. Let us suppose \( n \) vertices \( V \) of the conflict graph correspond to \( n \) events attended by \( p_1, p_2, \ldots, p_n \) persons each, whereas the \( m \)-bound corresponds to rooms of capacities \( r_1, r_2, \ldots, r_m \). Let us denote distinct distinct numbers of persons attending an event \( P, L(p) \) the events of size \( s > p \) and \( R(p) \) the rooms of capacity \( r > p \). Clearly, one can add two constraints (PR1, PR2) for each element of \( P \):

\[
\tilde{R}(G, p, r) = \max t
\]

subject to

\[
Y_{vv} = t \quad \forall v \in V
\]

\[
Y_{uv} = 0 \quad \forall \{u, v\} \in E
\]

\[
\sum_{u \in L(p)} Y_{uv} \leq t |R(p)| \quad \forall p \in P \quad \forall v \in L(p)
\]

(PR1)

\[
\sum_{v \in L(p)} Y_{uv} \leq t |R(p)| \quad \forall p \in P \quad \forall u \in L(p)
\]

(PR2)

\[
Y - J \succeq 0
\]

Notice the newly added constraints (PR1, PR2) subsume the linear inequalities (L1, L2) of the relaxations above. There is always a feasible solution, and as will be shown in Section 5.3, one can efficiently produce feasible timetables satisfying those constraints from the value of the matrix variable.

Considering there is always a permutation matrix such that the matrix variable is block-diagonal (12), one can also add bounds obtained by counting arguments. The simple counting bound is \( \sum_{u,v \in V} Y_{uv} \leq m|V| \). Indeed, there are at most \( |V|/m \) blocks of
$m^2$ non-zeros each. One can generalise the bound to subsets $P$ of events:

$$\mathcal{R}(G, p, r) = \max t$$

s.t. $Y_{vv} = t \quad \forall v \in V$

$$Y_{uv} = 0 \quad \forall \{u, v\} \in E$$

$$\sum_{u \in L(p)} Y_{uv} \leq t|R(p)| \quad \forall p \in P \quad \forall v \in L(p) \quad \text{(PR1)}$$

$$\sum_{v \in L(p)} Y_{uv} \leq t|R(p)| \quad \forall p \in P \quad \forall u \in L(p) \quad \text{(PR2)}$$

$$\sum_{u \in L(p)} \sum_{v \in V} Y_{uv} \leq mt|R(p)| \quad \forall p \in P \quad \text{(CB1)}$$

$$\sum_{v \in L(p)} \sum_{u \in V} Y_{uv} \leq mt|R(p)| \quad \forall p \in P \quad \text{(CB2)}$$

$$Y - J \succeq 0$$

In SIMPLE LAMINAR TIMETABLING, one can include feature considerations similarly to capacity considerations. In a slight abuse of notation, we use $F(f)$ to denote the set of events requiring feature $f$ and $G(f)$ the set of rooms with feature $f$.

$$\mathcal{R}(G, p, r, f_{\text{max}}, F, G) = \max t$$

s.t. $Y_{vv} = t \quad \forall v \in V$

$$Y_{uv} = 0 \quad \forall \{u, v\} \in E$$

$$\sum_{u \in L(p)} Y_{uv} \leq t|R(p)| \quad \forall p \in P \quad \forall v \in L(p) \quad \text{(PR1)}$$

$$\sum_{v \in L(p)} Y_{uv} \leq t|R(p)| \quad \forall p \in P \quad \forall u \in L(p) \quad \text{(PR2)}$$

$$\sum_{u \in L(p)} \sum_{v \in V} Y_{uv} \leq mt|R(p)| \quad \forall p \in P \quad \text{(CB1)}$$

$$\sum_{v \in L(p)} \sum_{u \in V} Y_{uv} \leq mt|R(p)| \quad \forall p \in P \quad \text{(CB2)}$$

$$\sum_{u \in F(f)} Y_{uv} \leq t|G(f)| \quad \forall 1 \leq f \leq f_{\text{max}} \quad \forall v \in F(f) \quad \text{(FR1)}$$

$$\sum_{v \in F(f)} Y_{uv} \leq t|G(f)| \quad \forall 1 \leq f \leq f_{\text{max}} \quad \forall u \in F(f) \quad \text{(FR2)}$$

$$\sum_{u \in F(f)} \sum_{v \in V} Y_{uv} \leq mt|G(f)| \quad \forall 1 \leq f \leq f_{\text{max}} \quad \text{(FC1)}$$

$$\sum_{v \in F(f)} \sum_{u \in V} Y_{uv} \leq mt|G(f)| \quad \forall 1 \leq f \leq f_{\text{max}} \quad \text{(FC2)}$$

$$Y - J \succeq 0$$

Notice, however, the limitation to the laminar special case.
3.5 Simple Timetabling

A natural approach to formulating the Simple Timetabling Problem without laminarity requirements uses the additional variables:

\[
Z_{u,v,r} = \begin{cases} 
1 & \text{if vertex } u \text{ is in the same colour class as } v \text{ and } v \text{ is assigned (room) } r \\
0 & \text{otherwise.}
\end{cases}
\]  

(22)

The additional constraints follow. In timetabling language: an event \( v \) is assigned a room in \( Z_{u,v,r} \) for some \( r \) if and only if it is assigned time in \( Y_{u,v} \) (23), no event is assigned to two rooms (24), no two events share a room (25) and \( Z_{u,v,r} = 0 \) if the event-room combination does not match the event’s room-feature or capacity requirements (26, 27).

\[
\sum_{1 \leq r \leq m} Z_{u,v,r} = Y_{u,v} \quad \forall u, v \in V
\]  

(23)

\[
Z_{u,v,r} + Z_{u,v,r'} \leq t \quad \forall u, v \in V \forall 1 \leq r \leq m \forall r' \neq r
\]  

(24)

\[
Z_{u,v,r} + Z_{u,w,r} \leq t \quad \forall u, v, w \in V \setminus \{v\}
\]  

(25)

\[
Z_{u,v,r} = 0 \quad \forall v \in V \forall 1 \leq r \leq m, p_v \geq r_r
\]  

(26)

\[
Z_{u,v,r} = 0 \quad \forall v \in V \forall 1 \leq f \leq f_{\text{max}}, (v, f) \in F \forall 1 \leq r \leq r_r (r, f) \notin G
\]  

(27)

\[
Z_{u,v,r} \geq 0 \quad \forall v \in V \forall 1 \leq f \leq f_{\text{max}}
\]  

(28)

This results, however, in relaxations too large to be handled by solvers currently available. An alternative approach uses fewer additional variables:

\[
R_{v,r} = \begin{cases} 
1 & \text{if vertex } v \text{ is assigned (room) } r \\
0 & \text{otherwise.}
\end{cases}
\]  

(29)

The additional constraints follow. In timetabling language: each event is in exactly one room (30, 31), events in the same timeslot do not share rooms (32), and \( R_{v,r} = 0 \) if the event-room combination does not match the event’s room-feature or capacity requirements (33, 34).

\[
\sum_{1 \leq r \leq m} R_{v,r} = t \quad \forall v \in V
\]  

(30)

\[
R_{v,r} + R_{v,r'} \leq t \quad \forall v \in V \forall 1 \leq r, r' \leq m, r \neq r'
\]  

(31)

\[
R_{u,v} + R_{u,v'} + Y_{u,v} \leq 2t \quad \forall u, v \in V, u \neq v \forall 1 \leq r \leq m
\]  

(32)

\[
R_{v,r} = 0 \quad \forall v \in V \forall 1 \leq r \leq m, p_v \geq r_r
\]  

(33)

\[
R_{v,r} = 0 \quad \forall v \in V \forall 1 \leq f \leq f_{\text{max}}, (v, f) \in F \forall 1 \leq r \leq r_r (r, f) \notin G
\]  

(34)

\[
R_{v,r} \geq 0 \quad \forall v \in V \forall 1 \leq r \leq m
\]  

(35)

Notice that the encoding makes it possible to formulate room stability constraints and penalties, as it is invariant to “timeslot permutations”. For example, the hard constraint reads \( R_{v,r} + R_{v,r'} \leq t \) for all suitable \( v \neq v' \) and all \( r \neq r' \).
4 Algorithms

While our key contribution are the actual relaxations, we showcase how these can be used in state-of-the-art algorithms. Such algorithmic applications of the relaxations underlie not only our computational results, but also our analytical results in Section 5.

4.1 Solving the Relaxations

First, let us consider a first-order method based on the alternating-direction method of multipliers (ADMM) on an augmented Lagrangian, following the extensive literature (Malick, Povh, Rendl, & Wiegele, 2009; Burer & Vandenbussche, 2006; Zhao, Sun, & Toh, 2010; Povh, Rendl, & Wiegele, 2006; Wen, Goldfarb, & Yin, 2010; Goldfarb & Ma, 2010; L. Yang, Sun, & Toh, 2015). In order to distinguish between equality constraints reflecting the structure of the conflict graph ($A_1$) and the remainder of the equality constraints ($A_2$), let us consider the primal-dual pair:

$$z_p = \min_{X \in S^n} \langle C, X \rangle \text{ s.t. } A_1(X) = b_1 \text{ and } A_2(X) = b_2 \text{ and } A_3(X) \geq d \text{ and } X \succeq 0$$

$$z_d = \max_{y_1 \in R^m, y_2 \in R^p, v \in R^q, S \in S^n} b_1^T y_1 + b_2^T y_2 + d^T v \text{ s.t. } A_1^*(y_1) + A_2^*(y_2) + A_3^*(v) + S = C \text{ and } S \succeq 0 \text{ and } v \geq 0.$$ 

(36)

with the linear operator $A_3^*(X)$ mapping matrix $X$ and matrix $A$ to vector as in the definition of SDPs in Section 2.1. The augmented Lagrangian of the dual (36) is then:

$$L_\mu(X, y_1, y_2, v, S) = -b_1^T y_1 - b_2^T y_2 - d^T v + \langle X, A_1^*(y_1) + A_2^*(y_2) + A_3^*(v) + S - C \rangle + \frac{1}{2\mu} \|A_1^*(y_1) + A_2^*(y_2) + A_3^*(v) + S - C\|^2_F.$$ 

(37)

In an alternating direction method of multipliers, one minimises the augmented Lagrangian in $v, S,$ and $(y_1, y_2)$, in turns, as suggested in Algorithm Schema 1.

In general, Algorithm Schema 1 reduces the minimisation of one moderately complicated convex optimisation problem to solving three simpler convex optimisation subproblems. In Line 8, one can solve the linear system given by first-order Karush–Kuhn–Tucker optimality conditions of

$$\arg\min_{v \in R^q, v \geq 0} \left( B \left( X^k + \frac{1}{\mu} \left( A_1^T (y_1^{k+1}) + A_2^T (y_2^{k+1}) + S^k - C \right) \right) - d \right)^T v + \frac{1}{2\mu} v^T (BB^T)v.$$

(38)

In Line 9 it is important to realise that

$$\arg\min_{S \in S^n, S \geq 0} \left\| S - \left( C - A_1^T (y_1^{k+1}) - A_2^T (y_2^{k+1}) - B^T (v^{k+1}) - \mu X^k \right) \right\|_F^2$$

(39)
Finally, in Line 10, one can initialise the computation with:

\[
\begin{align*}
\gamma_1^{k+1} &= -(A_1 A_1^T)^{-1}(\mu(A_1(X^k) - b_1) + A_1(A_2^T y_2^k + B^T (v^k) + S^k - C)) \quad (40) \\
\gamma_2^{k+1} &= -(A_2 A_2^T)^{-1}(\mu(A_2(X^k) - b_2) + A_2(A_1^T y_1^{k+1} + B^T (v^k) + S^k - C)). \quad (41)
\end{align*}
\]

We refer to [L. Yang et al. 2015] for some excellent suggestions as to the implementation of the linear solver and spectral decomposition, as well as convergence properties of such as method.

### 4.2 Recovering an Assignment

Let us comment on the recovery of an upper bound from the lower bound provided by SDP. Since the seminal paper of Karger, Motwani, and Sudan [Karger et al. 1998], there has been a continuing interest in algorithms recovering a colouring from semidefinite relaxations. Typically, such algorithms are based on simple randomised iterative rounding of the semidefinite programming relaxation. One such algorithm, specialised to simple timetabling is displayed in Algorithm Schema 2.

Alternatively, one can consider methods solving a sequence of smaller semidefinite programming relaxations, inspired by the so-called iterated rounding in linear programming [Lau et al. 2011]. When applied to linear programming, the method fixes variables, whose values in the relaxation are close to 0 or 1, to 0 or 1, respectively, and resolves the smaller residual linear program. When applied to semidefinite programming, the method fixes eigenvectors whose corresponding eigenvalues are close to zero or one. Let us consider the example of [Morgenstern, Samadi, Singh, Tantipongpipat].
Algorithm Schema 2 Rounding($X$) based on Karger, Motwani, and Sudan

1: **Input:** Matrix variable $X$ of the solution to the SDP (17) of dimensions $n \times n$, bound $m$, number $a_{\max}$ of randomisations to test, plus the input to Simple Timetabling, if required

2: **Output:** Partition $P$ of the set $V = 1, 2, \ldots, n$

3: Compute vector $v = v^T v$ using Cholesky decomposition

4: **for** Each attempted randomisation $a = 1, \ldots, a_{\max}$ do

5: Initialise $P_a = \emptyset$, $i = 1$, $X = V$

6: **while** There are uncoloured vertices in $X$ do

7: Pick a suitable $c = \sqrt{\frac{2(k-2)}{k \log A}}$ for $\Delta$ being the maximum degree of the vertices in $X$

8: Generate a random vector $r$ of dimension $|X|$

9: Pick $R_i \subseteq X$ of at most $m$ elements in the descending order of $v_i r_i$, where (1) positive and (2) independent of previously chosen and, in Simple Timetabling, (3) the respective events fit within the rooms and (4) require only features available

10: Update $P_a = P_a \cup \{\{R_i\}\}$, $X = X \setminus R_i$, $i = i + 1$

11: **end while**

12: **end for**

13: Return $P_a$ of minimum cardinality

\& Vempala, 2019) starting from:

\[
\min \langle C, X \rangle \\
\text{s.t.} \quad \langle A_i, X \rangle \geq b_i \quad \forall 1 \leq i \leq m \\
\text{trace}(X) \leq d \\
0 \leq X \leq I_n,
\]

which can accommodate many of the SDP relaxations we have seen so far. There, (Morgenstern et al., 2019) initialise $F_0 = F_1 = \emptyset$ and $F = I_r$. In each iteration, subspaces spanned by eigenvectors corresponding to eigenvalues 0 or 1 are fixed and the corresponding standard basis vectors are moved from $F$ to $F_0$ and $F_1$, respectively. Thus, one increases the subspaces spanned by columns of $F_0$ and $F_1$, while maintaining pairwise orthogonality. To obtain new $F$, one solves a smaller semidefinite program in $r \times r$ symmetric matrix $X(r)$:

\[
\max \langle F^T C F, X(r) \rangle \\
\langle F^T A_i F, X(r) \rangle \geq b_i - F_i^T A_i F_i \quad i \in S \\
\text{trace}(X(r)) \leq d - \text{rank}(F_1) \\
0 \leq X(r) \leq I_r,
\]

which assures that, eventually, we can recover $X$ that is orthogonal to all vectors in subspace spanned by vectors in $F_0$, and whose eigenvectors corresponding to eigenvalue 1
Algorithm Schema 3 IterativeRounding($X$) based on Morgenstern et al.

1: **Input:** An $n \times n$ matrix $X$ of the solution to the SDP $\text{(42)}$, which has $m$ inequalities, alongside with the corresponding matrices $A_i$ for $i = 1, \ldots, m$
2: **Output:** Partition $P$ of the set $V = 1, 2, \ldots, n$
3: Initialize $F_0, F_1$ to be empty matrices and $F = I_n, S \leftarrow \{1, \ldots, m\}$.
4: Initialise $\delta > 0$ to be a threshold for rounding
5: while $F$ is non-empty do
6: Solve $\text{(43)}$ to obtain extreme point $X^* = \sum_{r=1}^{t} \lambda_j v_j v_j^T$ where $\lambda_j$ are the eigenvalues and $v_j \in \mathbb{R}^r$ are the corresponding eigenvectors.
7: For any eigenvector $v$ of $X^* = \sum_{r=1}^{t} \lambda_j v_j v_j^T$ with eigenvalue less than $\delta$, let $F_0 \leftarrow F_0 \cup \{Fv\}$.
8: For any eigenvector $v$ of $X^* = \sum_{r=1}^{t} \lambda_j v_j v_j^T$ with eigenvalue of more than $1 - \delta$, let $F_1 \leftarrow F_1 \cup \{Fv\}$.
9: Let $X_f = \sum_{j:0<\lambda_j<1} \lambda_j v_j v_j^T$. If there exists a constraint $i \in S$ such that $\langle F^T A_i F, X_f \rangle < \delta$, then $S \leftarrow S \setminus \{i\}$.
10: Update $F$ by taking every eigenvector $v$ of $X^* = \sum_{r=1}^{t} \lambda_j v_j v_j^T$ with eigenvalue within $[\delta, 1 - \delta]$, and taking $Fv$ to be the columns of $F$.
11: end while
12: From rank-$t$ matrix $F_1 F_1^T$ reconstruct partition $P$ by Cholesky decomposition

will be the columns of $F_1$. This is summarised in Algorithm Schema 3. As we will see in Section 5.3 this allows for non-trivial performance guarantees.

As a remark, we note that there are many other alternative rounding approaches within the Theoretical Computer Science literature. We refer to (Barak, Raghavendra, & Steurer, 2011; Raghavendra & Tan, 2012; Bansal et al., 2016; Abbasi-Zadeh et al., 2018) for notable examples. While they may not be directly applicable, they are based on important insights that would be applicable.

## 5 An Analysis

Next, let us analyse the strength of the bound and the complexity of computing it, both of which affect its practicality.

### 5.1 The Strength of the Bound

In terms of strength of the bound, one can extend a number of properties of relaxations of graph colouring to bounded colouring. For the sake of completeness, we reiterate some of them. For instance, one can show the sandwich-like:

**Proposition 3** For every graph $G$, there is an $m \geq 0$, such that

$$
\omega(G) \leq \chi_\text{SDP}(G) \leq \chi(G) \leq \chi(G, m) \leq Y(G, m) \leq Y_\text{Alt}(G, m) \leq \chi(G, m),
$$

(44)

$$
Y(G, m) \leq Y^+(G, m) \leq Y^+_{\Delta}(G, m) \leq \chi(G, m),
$$

(45)
where \( \omega \) is the size of the largest clique, \( \chi \) is the chromatic number, \( \kappa \) is a bound obtained by counting, \( \chi^m \) is the \( m \)-bounded chromatic number, the values of SDP relaxations follow the notation of Figure 1 and \( \mathcal{Y}^{+\Delta} \) is the strengthening of \( \mathcal{Y}^+(G,m) \) with triangle inequalities.

**Proof (sketch)** The relationship between the values of the successive relaxations of bounded colouring is clear. To show there is \( m \), such that \( \chi(G) \leq \kappa(G,m) \), let us study two cases: If there is \( r \geq 1 \) such that \( r \)-bounded graph colouring of \( G \) requires a strictly larger number of colour classes than the chromatic number, take \( m = r \). Otherwise, the graph cannot have independent sets larger than one, hence is a clique, and \( \omega(G) = \chi(G) = \chi(G,m) \) for any \( m \).

To see that (non-bounded) graph colouring relaxations \( (\mathcal{X}(G), \mathcal{X}^+(G), \mathcal{X}^+(G)) \) provide only very weak bounded graph colouring relaxations, consider empty graphs on \( n \) vertices and the constant function \( f(n) = 1 \):

**Proposition 4** There is an infinite family of graphs and \( f(n) \), where the chromatic number is \( O(1) \), the \( f(n) \)-bounded chromatic number is \( O(n) \).

In contrast, the value of the semidefinite programming relaxation of bounded colouring may match the bounded chromatic number on such graphs.

On random graphs where an edge between each pair of distinct vertices appears with probability \( p \), independent of any other edge, which are known as Erdős-Rényi \( G(n,p) \):

**Proposition 5** With probability \( 1 - o(1) \), graph \( G \) drawn randomly from \( G(n,p) \) has

\[
\mathcal{Y}(G) \geq \frac{\sqrt{n}}{2} \sqrt{\frac{1 - p}{p} + O(n^{\frac{2}{3}} \log n)},
\]

(46)

where the big-O notation hides lower-order terms.

**Proof (sketch)** The proof combines the sandwich-like Property 3 and the impressive result of Juhász (1982).

Computationally, this bound seems to be rather tight, as we show in Section 6.

### 5.2 The Structure of the Relaxations

For example, let us consider the formulation of bounded graph colouring (15) for a graph on \( n \) vertices and \( m \) edges. There equality constraints reflecting the structure of the conflict graph \( (A_1) \) have the cardinality of their support (number of non-zero elements) equal to the number of edges in the conflict graph and the remainder of the equality constraints \( (A_2) \) also have a very simple structure:

**Proposition 6** First \( m \) equalities \( (A_1) \) correspond to \( m \times n^2 \) matrix \( A_1 \). \( A_1A_1^T = I_m \), where \( I_m \) is the \( m \times m \) identity matrix.
Proposition 7 Further \( n - 1 \) equalities correspond to \( n - 1 \times n^2 \) matrix \( A_2 \). \( A_2A_2^T = J_{n-1} + I_{n-1} \), where \( I_{n-1} \) and \( J_{n-1} \) are \((n - 1) \times (n - 1)\) identity and all-ones matrices, respectively. \( (A_2A_2^T)^{-1} = -\frac{1}{n}J_{n-1} + I_{n-1} \). For \((n - 1)\)-vector \( y \), \( A_2^Ty \) is an \( n \times n \) matrix, with \( [(\sum y_i)(-y_1)(-y_2)\cdots(-y_{n-1})] \) on the diagonal and zeros elsewhere. For positive \( X \), \( A(A^T) = [2, 4, \cdots, 2(n-1)] \) of dimension \((n-1)\).

Proposition 8 Inequalities correspond to \( n \times n^2 \) matrix \( B = I \otimes j \), where \( j \) is the row-vector of \( n \) ones. Hence, \( BB^T = nI_n \), where \( I_n \) is the \( n \times n \) identity matrix. For an \( n \)-element column-vector \( v \), \( B^Tv = (v \otimes j)^T = [v_1j v_2j \cdots v_nj]^T \), where \( j \) is the row-vector of \( n \) ones.

Proposition 9 The elements of the objective matrix \( C \) are zeros except for \( C_{1,1} = 1 \). Hence \( \mathcal{A}_A(C) = 0 \), where 0 is the \( m \)-vector of zeros. \( \mathcal{A}_B(C) = j \), where \( j \) is the \((n-1)\)-vector of ones.

Across both:

- custom solvers, such as Algorithm Schema 1 and the three sub-problems in Lines 8–10 in particular, and
- general-purpose solvers allowing for the input of block-structured matrices with sparse and identity blocks, such as (Fujisawa, Fukuda, Kojima, & Nakata, 2000; Gondzio & Grothey, 2009).

It is possible to exploit Properties 6–8 so as to:

- not compute \( (A_1A_1^T)^{-1} \)
- compute \( A_1^Ty_1 \) in time \( O(m) \)
- compute \( (A_2A_2^T)^{-1} \) in time \( O(n^2) \)
- compute \( A_2^Ty_2 \) in time \( O(n) \)
- compute \( (AB^T)^{-1} \) in time \( O(n) \)
- compute \( B^Tv \) in time \( O(n) \)
- evaluate the augmented Lagrangian and its gradient at a given \( v \) in time \( n^2 \)

in relaxations of bounded graph colouring of a graph on \( n \) vertices, compared to \( O(n^6) \) run-time of methods not exploiting the structure.

5.3 The Recovery

Due to the hardness of approximation of colouring in a graph with large enough a chromatic number within the factor of \( n^\varepsilon \) for some fixed \( \varepsilon \) (Zuckerman, 2007), one cannot hope to guarantee reconstruction of a solution close to optimality in the worst case. Having said that, as we will illustrate in the next section, however, Algorithm Schema 2 performs rather well in practice.
One can also provide weaker guarantees. In particular, one could consider the so-called \(\varepsilon\)-solution, which satisfies linear constraints within an additive error of \(\varepsilon\), while being at most \(\varepsilon\) from the optimal objective. Notice that the fact that an \(\varepsilon\)-solution is obtainable in time polynomial in \(n\) and \(\log \frac{1}{\varepsilon}\) does not contradict the hardness of approximation results \cite{Zuckerman2007}, which consider the objective of solutions satisfying the constraints exactly.

**Proposition 10** There exists an \(\varepsilon > 0\) and an algorithm implementing Algorithm Schema 3 that, given any feasible solution to the SDP relaxation (21) of SIMPLE LAMINAR TIMETABLEING, runs in time polynomial in \(n\) and \(\log \frac{1}{\varepsilon}\) and returns an \(\varepsilon\)-feasible and \(\varepsilon\)-optimal solution to the SDP relaxation (21) of SIMPLE LAMINAR TIMETABLEING.

**Proof (sketch)** The proof extends the work of \cite{Morgenstern2019} on the number of fractional eigenvalues in any extreme point \(X\) of a suitable form of a semidefinite program with \(m\) linear inequalities and trace bounded by \(t\), which is

\[
t + \left[ \sqrt{2m + \frac{9}{4}} - \frac{3}{2} \right].
\]

Based on this bound, one can formulate a generic result on iterative rounding of SDPs, which we present in Proposition 11 below. The result applies to the SDP relaxation (21) of SIMPLE LAMINAR TIMETABLEING, because it can be cast into the suitable form (42). The bound on the run-time follows from the fact we solve at most \(n\) semidefinite programs in matrices at most \(n \times n\) and standard results on interior-point methods \cite{Alizadeh1995}.

**Proposition 11** (Theorem 7 of \cite{Morgenstern2019}) Let \(C\) be a \(n \times n\) matrix and \(\{A_1, \ldots, A_m\}\) be a collection of \(n \times n\) real matrices, \(d \leq n\), and \(b_1, \ldots, b_m \in \mathbb{R}\). Suppose the semi-definite program (42) with a trace bounded by \(d\) and \(m\) other constraints has a nonempty feasible set and let \(X^*\) denote an optimal solution. There is an algorithm that given a matrix \(X_0\) that is a strictly feasible solution, returns a matrix \(\tilde{X}\) such that

1. rank of \(\tilde{X}\) is at most \(d\),
2. \(\langle C, \tilde{X} \rangle \leq \langle C, X^* \rangle\), and
3. for each index \(1 \leq i \leq m\) of a constraint we have

\[
\langle A_i, \tilde{X} \rangle \geq b_i - \max_{S \subseteq [m]} \left\lfloor \sqrt{2 |S|} + 1 \right\rfloor \sum_{i=1}^{\left\lfloor \sqrt{2 |S|} + 1 \right\rfloor} \sigma_i(S),
\]

where \(\sigma_i(S)\) is the \(i^{th}\) largest singular of the average of matrices \(\frac{1}{|S|} \sum_{i \in S} A_i\) for any subset of matrices defining the constraints, \(S \subseteq \{1, \ldots, m\}\).

In the violation bound (48), the quantity \(\sigma_i(S)\) is non-trivial to reason about, but it is clear that it is rather modest, because the singular values are at most 1 and the summation goes over at most \(\sqrt{2m} + 1\) values.
6 Computational Experience

To corroborate our analytical results in Section 5, we have conducted a variety of computational tests. Most of these have been driven by YALMIP (Löfberg, 2004) scripts running within MathWorks Matlab R2017b on a laptop with Intel Core Duo i5 at 2.7 GHz with 8 GB of RAM, which also had IBM ILOG CPLEX 12.8 and SeDuMi 1.3 (Sturm, 1999) installed. Let us refer to it as a laptop. When explicitly mentioned, we also present results obtained on a machine equipped with 80 cores of Intel Xeon E7-8850 at 2.00 GHz and 700 GB of RAM, which had MathWorks Matlab R2016b, IBM ILOG CPLEX 12.6.1, and SeDuMi 1.3 (Sturm, 1999) installed. Let us refer to it as a large-memory machine.

6.1 A Motivating Example

As a first concrete computational example, we consider a small conflict graph from a standard collection of benchmark problems in timetabling. Specifically, we take the instance sta-f-83 from the Toronto examination timetabling benchmarks. There are 139 events, but the conflict graph has three connected components of 30, 47 and 62 vertices. Here, we use the 47-vertex component. The results are given in Table 1, with bounded chromatic numbers obtained using the most straightforward integer linear programming formulation solved using the default settings of IBM ILOG CPLEX on a laptop.

Firstly, note that \( m = 1 \) gives precisely the number of nodes, as would be expected. Secondly, note that \( \Psi_m \) is generally much tighter than the lower bound \( |V|/m \) obtained by simple counting arguments. Accidentally, \( \Psi_m \) lower bounds actually happen to match the optima in this particular instance. For example, at \( m = 5 \), counting cannot rule out a 10-colouring, but the SDP bound shows that at least 14 colours are required. As far as we know, SDP relaxations are the only way to get such information in polynomial time, considering that the 14-colouring together with a certificate of its optimality can be obtained using CPLEX, but not in polynomial time.

6.2 Random Graphs

Next, we show that the same behaviour can be observed on a large sample of random graphs.

First, we demonstrate the improved strength of the lower bound obtained from semidefinite programming as the restriction on the number of uses of a colour is tightened (i.e., cardinality of a colour class is bounded from above by progressively smaller numbers). In general, we compute the best possible vertex colouring, without any bound on the number of uses of a colour, and take the size of the largest colour class to be \( C \). Subsequently, we obtain lower bounds, upper bounds, and optima for \((C-1)\)-bounded colouring, \((C-2)\)-bounded colouring, etc., of the same graph. In particular, we use random graphs with constant probability 0.5 of an edge appearing between a pair of distinct vertices and the varying numbers \( n \) of vertices, which are known as \( G(n, \frac{1}{2}) \).

\footnote{See ftp://ftp.mie.utoronto.ca/pub/carter/testprob/ and http://www.cs.nott.ac.uk/~rxq/data.htm}

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Table 1: An illustration of the effects of bounding the $m$-bounded chromatic number of the instance sta-f-83: Column $\chi^m$ lists the $m$-bounded chromatic number obtained using integer linear programming, within time listed under “$\chi^m$ Runtime” in seconds. Column $Y^m$ lists the bounds obtained using semidefinite programming and rounding up, within time listed under “$Y^m$ Runtime” in seconds. Column $|V|/m$ lists the lower bound on the colours obtained by simple counting arguments and rounding up. Dash denotes the omission of the $m$-bounding constraint, giving $X$ instead of $Y^m$.

| $m$ | $\chi^m$ | $\chi^m$ Runtime | $Y^m$ | $Y^m$ Runtime | $|V|/m$ |
|---|---|---|---|---|---|
| 1 | 47 | 0.09 | 47 | 3.46 | 47 |
| 2 | 26 | 2.88 | 26 | 2.92 | 24 |
| 3 | 20 | 2.67 | 20 | 3.34 | 16 |
| 4 | 16 | 7.22 | 16 | 3.70 | 12 |
| 5 | 14 | 11.10 | 14 | 3.24 | 10 |
| 6 | 13 | 2.67 | 13 | 3.12 | 8 |
| 7 | 12 | 8.77 | 12 | 3.26 | 7 |
| 8 | 11 | 2.89 | 11 | 3.40 | 6 |
| 9 | 11 | 3.39 | 11 | 3.14 | 6 |
| 47 | 11 | 0.35 | 11 | 3.92 | 1 |
| --- | 11 | 0.34 | 11 | 3.45 | --- |

For each number $n$ of vertices, we have generated 100 random graphs, computed the true chromatic numbers and the size of the largest colour class $C$ using CPLEX, lower bounds on the bounded colouring using SeDuMi, and upper bounds by rounding the semidefinite programming relaxation, all running on a laptop. In Figure 2, the true value is plotted in a solid line, while a semi-transparent region spans the lower and upper bounds. Notice that: In Figure 2, the true value is plotted in a solid line, while a semi-transparent region spans the lower and upper bounds. Notice that:

- the upper bounds obtained by rounding the semidefinite programming relaxation coincides with the true value obtained by ILOG
- for unbounded and $(C - 1)$-bounded colouring, there is a considerable gap between the SDP-based lower bound and the true value
- for $(C - 3)$-bounded colouring, the SDP-based lower bound and the true value coincide in the majority of cases. (That is: One can round the upper bound up, as it has to be integral. The average over 100 samples need not be integral, though.)
- for $(C - 3)$-bounded colouring, the SDP-based lower bound is essentially tight.

Second, we illustrate the practicality of the approach by illustrating the dependence of the run-time on dimensions of the graph. Figure 3 presents the results on instances, which fit within the memory of a laptop. It suggests that run-time of commonly used first-order methods for solving semidefinite-programming relaxations increases linearly with the number of vertices of the graph, while the run-time of commonly used second-order methods for solving semidefinite-programming relaxations
increases quadratically with the number of vertices of the graph. This is surprising. Consider the fact that the dimension of the matrix variable increases quadratically with the number of vertices and the number of elements in the Hessian matrix considered in second-order methods increases quadratically in the dimension. In both cases, the observed run-time is due to the ability of the respective methods to exploit the structure of Section 5.2.

Figure 4 presents the corresponding results on instances, which no longer fit within the memory of a laptop, as run on a large-memory machine. We note that already on a random graph on 200 vertices, \( G(200, 0.5) \), SeDuMi regularly consumes over 24 GB physical memory, with further 13 GB in swap, in solving the SDP relaxation of bounded graph colouring. Although the run-times are longer, considering the sheer amounts of data processed, the evolution of run-time as a function of the number of vertices seems similar to Figure 3.

Figure 5 illustrates that commonly used methods do not exhibit a major increase in run-time as the density of the graph increases, due to their ability to exploit the structure of Section 5.2. Again, this is surprising. Consider that the number of edges in the conflict graph asymptotically approaches the square of the number of vertices in a dense graph. If there were no structure, the cubic increase of run-time with each of the quadratic number of constraints may render the approach impractical. In particular, we use random graphs of varying densities, all on 100 vertices. The data are again available on-line. For each of the densities \( p = 0.1, 0.2, \ldots, 0.9\%), we have generated 100 graphs \( G(40, p) \). Subsequently, we have obtained SDP-based lower bounds using 4 different methods:

- **IPM** a standard implementation of a primal-dual interior point method (IPM) by (Sturm, 1999)
- **DimRed** a dimension-reduction procedure of (Löfberg, 2004) followed by the IPM of (Sturm, 1999)
- **SparseCoLo** a sparsity-exploiting procedure of (Kim, Kojima, Mevissen, & Yamashita, 2011) followed by the IPM of (Sturm, 1999)
- **AugLag** an implementation of a first-order method considering the augmented Lagrangian, based on (Zhao et al., 2010; L. Yang et al., 2015)

For each of the methods, and each of the densities \( p \), we report the average run-time over 100 graphs.

### 6.3 Conflict Graphs from Timetabling Benchmarks

As a further illustration of the strength of SDP lower bounds, we present lower bounds for conflict graphs from two timetabling benchmarks. From instances used in Track 3 of International Timetabling Competition 2007, we have extracted course-based conflict graphs, where there is edge between two vertices, if there is a curriculum prescribing the enrollment in both corresponding courses, or if a single teacher teaches both courses. For details, please see (Bonatti, De Cesco, Di Gaspero, & Schaerf, 2012) or (Burke, Mareček, Parkes, & Rudová, 2012). From Toronto Examination Timetabling
Figure 2: The effects of tightening the bound on the number of uses of a colour on the strength of the lower bound: For a random graph $G(n, 0.5)$, where the size of the largest colour class in an optimal colouring is $C$, the mean lower bounds, upper bounds, and optima for unbounded colouring, $(C - 1)$-bounded colouring, $(C - 2)$-bounded colouring, etc., are computed from a sample of $N = 100$ for each number of vertices $n$ and restriction on the size of the colour class.

Benchmark, we have extracted exam-based conflict graphs, where there is edge between two vertices, if there is a student who should sit both corresponding exams. For details, please see (Qu, Burke, McCollum, Merlot, & Lee, 2009). For each graph, we have computed the best possible vertex colouring, without any bound on the number of uses of a colour, and took the size of the largest colour class to be $C$. Subsequently, we have obtained lower bounds, upper bounds, and optima for $(C - 1)$-bounded colouring, $(C - 2)$-bounded colouring, etc.
Figure 3: The run-time of the presented methods on a laptop as a function of the number of vertices: Sample mean run-times of an interior point method (IPM) and an augmented Lagrangian (AugLag) method on relaxations for $G(n, 0.5)$ on a laptop for $N = 100$ samples per each number $n$ of vertices.

6.4 Two Examples of Theoretical Interest

To illustrate the weakness of the bound on certain graphs, we present the results for Knesser graphs of Lovász (Lovász, 1978) and the “forbidden intersections” graphs of Frankl and Rödl (Frankl & Rödl, 1987). Knesser graph $K(n, k)$, $n > k > 1$, has $\binom{n}{k}$ vertices, corresponding to subsets of $\{1, 2, \ldots, n\}$ of cardinality $k$. Two vertices are adjacent if the corresponding subsets are disjoint. Lovász has shown (Lovász, 1978) the chromatic number of $K(n, k)$ is exactly $n - 2k + 2$, despite the fact $K(n, k)$ has no triangle for $n > 3k$. Similarly, forbidden intersections graph $F(m, \gamma)$, $m \geq 1, 0 < \gamma < 1$, such that $(1 - \gamma)m$ is an even integer, has $2^m$ vertices, corresponding to sequences of $m$ bits (zeros and ones). Two vertices are adjacent, if the corresponding sequences differ in precisely $(1 - \gamma)m$ bits. It is known the theta bound of Lovász and related semidefinite programming relaxations of graph colouring perform poorly on both “forbidden intersections” (Charikar, 2002) and Knesser graphs (Karger et al., 1998): the lower bound
Figure 4: The run-time of the augmented Lagrangian (AugLag) method on a large-memory machine, as a function of the number of vertices $n$ in $G(n, 0.5)$. We restrict ourselves to $N = 1$ sample per each number $n$ of vertices, due to the run-time of YALMIP constructing the SDP instances.

is $O(1)$ as $n$ grows, whereas the actual chromatic number grow $O(n)$ with $n$. Table 3 shows the lower bound gets tighter as the bound on the number of uses of a colour gets tighter.

It should be noted that there is a large difference between clique and chromatic numbers in both Knesser and forbidden intersection graphs, which makes them quite unlike conflict graphs encountered in timetabling applications. Although semidefinite programming lower bounds for graph colouring are weak on these graphs, they do tighten, as the bound on the number of uses of colours tightens. Nevertheless, the proposed lower bound is far from tight, in the worst case.
Figure 5: The run-time of the presented methods on a laptop, as a function of graph’s density: For random graphs $G(25,p)$, sample mean run-times of an interior point (IPM), possibly with a with dimension reduction (DimRed) and exploitation of sparsity (SparseCoLo), compared against the run-times of an augmented Lagrangian (AugLag) method, for $N = 100$ samples per each density $p = 0.1, 0.2, \ldots, 0.9$.

7 Conclusions

This paper has explored the limits of representability of extensions of graph colouring in semidefinite programming (SDP). SDP clearly provides some of the strongest known relaxations in timetabling. In particular, relaxations of simple timetabling problems related to Lovász theta provide useful lower bounds on the number of periods required in the timetable, considering the conflict graph, the number of rooms, the capacities of rooms and special equipment available therein, and a pre-assignment of certain events to certain periods.

In such low-dimensional SDP relaxations, the colour assignment is not represented directly, but only in terms of the classes of equivalence of nodes assigned the same
Table 2: Results for instances from Track 3 (comp) of the International Timetabling Competition 2007.

| Graph     | \(X^0\) | Runtime | \(X^m\) | Runtime | Rounded |
|-----------|---------|---------|---------|---------|---------|
| comp01.course | 4.00   | 4 s     | 5.00   | 0 s     | 7       |
| comp02.course | 5.98   | 2 s     | 6.00   | 7 s     | 12      |
| comp03.course | 6.94   | 1 s     | 7.00   | 7 s     | 14      |
| comp04.course | 4.98   | 1 s     | 5.00   | 3 s     | 12      |
| comp05.course | 8.00   | 1 s     | 7.99   | 3 s     | 14      |
| comp06.course | 5.99   | 3 s     | 6.00   | 8 s     | 14      |
| comp07.course | 6.00   | 5 s     | 6.55   | 21 s    | 17      |
| comp08.course | 6.99   | 2 s     | 6.98   | 7 s     | 11      |
| comp09.course | 4.99   | 1 s     | 5.00   | 5 s     | 10      |
| comp10.course | 6.00   | 3 s     | 6.39   | 11 s    | 16      |
| comp11.course | 5.00   | 1 s     | 6.00   | 0 s     | 8       |
| comp12.course | 9.91   | 4 s     | 9.96   | 15 s    | 18      |
| comp13.course | 5.98   | 1 s     | 6.00   | 7 s     | 8       |
| comp14.course | 6.00   | 2 s     | 6.00   | 10 s    | 14      |
| comp15.course | 6.94   | 1 s     | 7.00   | 7 s     | 15      |
| comp16.course | 6.00   | 4 s     | 5.99   | 9 s     | 15      |
| comp17.course | 5.98   | 3 s     | 6.00   | 14 s    | 12      |
| comp18.course | 4.99   | 1 s     | 5.22   | 1 s     | 8       |
| comp19.course | 6.00   | 1 s     | 6.00   | 5 s     | 11      |
| comp20.course | 6.00   | 4 s     | 6.37   | 14 s    | 13      |
| comp21.course | 8.00   | 2 s     | 8.00   | 14 s    | 13      |

Table 3: For Kneser graphs \(K(n, 2)\) and forbidden intersection graphs \(F(n, \gamma)\), where the size of the largest colour class in an optimal colouring is \(C\), lower bounds \(Y^m\) and optima \(\chi^m\) for \((C - m)\)-bounded colouring are shown. For \(m = 0\), no bounds were applied.

| Graph     | \(Y^0\) | \(\chi^0\) | \(Y^{-1}\) | \(\chi^{-1}\) | \(Y^{-2}\) | \(\chi^{-2}\) | \(Y^{-3}\) | \(\chi^{-3}\) |
|-----------|---------|------------|------------|------------|------------|------------|------------|------------|
| \(K(5, 2)\) | 2.50   | 3         | 3.33       | 4         | 5.00       | 5          | 10.00      | 1          |
| \(K(6, 2)\) | 3.00   | 4         | 3.75       | 4         | 5.00       | 5          | 7.50       | 8          |
| \(K(7, 2)\) | 3.50   | 5         | 4.20       | 5         | 5.25       | 6          | 7.00       | 7          |
| \(K(8, 2)\) | 4.67   | 6         | 5.60       | 6         | 7.00       | 7          | 9.33       | 10         |
| \(FI(6, 0.50)\) | 2.00   | 2         | 2.03       | 3         | 2.06       | 3          | 2.13       | 3          |
| \(FI(6, 0.67)\) | 6.40   | 7         | 7.11       | 8         | 8.00       | 8          | 9.14       | 10         |
| \(FI(6, 0.83)\) | 2.00   | 2         | 2.03       | 3         | 2.06       | 3          | 2.13       | 3          |
| \(FI(6, 1.00)\) | 2.00   | 2         | 2.00       | 2         | 2.06       | 3          | 2.13       | 3          |
under “colour permutations” and so the “same colour” representation is no longer sufficient. The matrix variable will need to capture the assignment of events to rooms as well as periods, and hence be constrained so that there is only a single event in each room-period pair. This gives a constraint on the rank of the matrix variable, which can be relaxed in a SDP. Despite the higher dimension of such relaxations, relaxations of rank-minimisation have proven very successful in many other fields (Fazel et al., 2004), and may turn out to be applicable also in timetabling. Modelling further and progressively more complex problems in semidefinite programming, with particular focus on relaxations one can solve fast, offers ample space for future work.

In theory, one may wonder whether the relaxations as the best one can obtain in polynomial time assuming the unique games conjecture (Khot, 2005). One could also seek approximation results for the problems we describe, either for the relaxations and rounding procedures of this paper, or for novel ones. For example, one could obtain so-called lifted relaxations, e.g., using the method of moments of (Lasserre, 2015) applied to the copositive formulation, and to analyse the rounding therein, as (Bansal et al., 2016) have done for job-shop scheduling. These would be an important advances in our understanding of scheduling and timetabling.

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