Comparison of Two Stationary Spherical
Accretion Models

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Abstract

The general relativistic accretion onto a black hole is investigated in which the motion is steady and spherically symmetric, the gas being at rest at infinity. Two models with different equations of state are compared. Numerical calculations show that the predictions of the models are similar in most aspects. In the ultrarelativistic regime the allowed band of the asymptotic speed of sound and the mass accretion rate can be markedly different.

1 Introduction

The accretion of gas on compact objects (white dwarfs, neutron stars, black holes) has not been entirely investigated by now, even in the simplest case of spherically symmetric system. The study of accretion has its beginnings in the paper presented by Bondi (1952) [1]. He considered spherically symmetric accretion on the basis of Newtonian gravity. Further progress has been made by Michel (1972) [2] and Shapiro and Teukolsky (1983) [3] who gave a general relativistic version of the Bondi model (the \( p^m \) model). Another relativistic generalization was given by Malec (1999) [4] (the \( p^n \) model).

It is not clear which equation of state is appropriate in the description of relativistic collapsing gas. There are two commonly used polytropic equations of state: \( p = K \rho^m \) and \( p = C_n \rho^n \). Here \( p \) is the pressure, \( \rho \) is the density and \( n \) is the baryonic mass density. The intention of this paper is to compare predictions of both models concerning the sound velocity, wind velocity, density and mass accretion rate. Here the \( p^m \) model and the \( p^n \) model denotes the model with the equation of state given by \( p = K \rho^m \) and \( p = C_n \rho^n \), respectively.

We also show that the \( p^n \) model gives an upper bound for the asymptotic speed of sound [5].

The order of this work is as follows. In Sections 2 and 3 we briefly present
(p, n) and (p, m) models. Section 4 is dedicated to the derivation of the after-mentioned limits on $a_s^2$ and $a^2$ for the (p, n) model. In Section 5 we compare predictions of both models using the results of numerical calculations. In the course of the paper we set $G = c = 1$ everywhere.

2 The (p, n) Model of Stationary Accretion

Here we shall give a short briefing of this model following Shapiro and Teukolsky [3]. A polytropic equation of state is assumed

$$p = K n ;$$

where $K$ and $n$ are constant. The velocity of sound is given by

$$a^2 = \frac{\partial p}{\partial n};$$

The boundary conditions are as follows: the gas at rest is described by the baryonic mass density $n_1$ and total energy-mass density $\frac{1}{2}$. We omit details (can be found in [3]) and write down the final equations. One can find that at the sonic point $R = R_s$ the speed of sound $a$ and the infall velocity $u$ satisfy relation

$$u_s^2 = \frac{a_s^2}{1 + 3a_s^2} = \frac{M}{2R_s};$$

From the relativistic Euler equations in Schwarzschild coordinates we get

$$1 - \frac{2M}{R} + \frac{u^2}{1 + \frac{a^2}{a_s^2}} = 1 + \frac{a_{1s}^2}{a_i^2};$$

Making use of conservation of the baryonic mass: $4 \pi u^2 = \text{const}$, and rearranging Eq. (4) one finds that

$$u = \frac{1}{4R^2} \frac{a^2_i}{a^2} \frac{M}{R} + \frac{u^2}{1 + \frac{a^2}{a_s^2}} \frac{1 + 3a_s^2}{a_s^2};$$

Inverting both sides of Eq. (4) and evaluating at the sonic point with the aid of Eq. (3) one gets the key equation for our considerations:

$$1 + 3a^2_s \frac{a^2_i}{a^2_i} \frac{1}{1 + 3a^2_s} = \frac{a^2_i}{1};$$

Employing Eq. (6) one can describe mass accretion rate by

$$M_* = M^2 n_1 \frac{a^2}{a^2_i} \frac{P}{1 + 3a^2_s} \frac{1}{1 + 3a^2_s} \frac{3a^2_s}{a^2_s};$$
3 The (pM) Model of Stationary Accretion

Here we describe the relativistic model shown in [4]. A suitable choice of an integral gauge condition leads to the comoving coordinates formulation that is particularly suitable for the description of self-gravitating uid. Spherically symmetric line element is given by

$$ds^2 = N^2dt^2 + adt^2 + R^2d\theta^2 + R^2\sin^2\theta d\phi^2;$$  \hspace{1cm} (8)

where $N$, $a$ and $R$ depend on $t$ (asymptotic time variable) and the radius $r$. The energy-momentum tensor of self-gravitating uid in comoving coordinates is given by

$$T \equiv (p + \rho)u \otimes u + pg;$$  \hspace{1cm} (9)

where $u \otimes u = 1$. Notice that $p = T_r^r = T_{\theta\theta}$. The rate of mass accretion $M_\bullet$ along orbits of a constant areal radius $R$ is equal to [3]

$$M_\bullet (R) = 4\pi R^2 u (p + \rho);$$  \hspace{1cm} (10)

where

$$u \equiv R = N;$$  \hspace{1cm} (11)

The effect of backreaction is neglected, that is the change of geometry caused by infalling gas is regarded to be negligible. Then one nds[4] that

$$N \equiv \frac{pR}{2} = 1 + \frac{2M}{R} + u^2;$$  \hspace{1cm} (12)

where $p$ is the mean curvature, and

$$a_s^2 = \frac{1}{2} \frac{3M}{2R_s} = u_s^2 = \frac{M}{2R_s};$$  \hspace{1cm} (13)

must hold in the sonic point. Assuming that the equation of state is given by

$$p = K;$$  \hspace{1cm} (14)

where the constant $K$ belongs to the interval $\frac{5}{3}$, and defining (as usual) the velocity of sound as $a^2 = \frac{\rho}{\rho}$ one arrives at [4]

$$a^2 = \frac{\rho}{g} + \frac{\rho a^2}{N};$$  \hspace{1cm} (15)

where $= (\frac{1}{3})$ and the integration constant $h$ is equal to the asymptotic velocity of sound at infinity.

One can nd that

$$u^2 = \frac{R^2 M}{2R_s^3} \frac{\rho^2}{2} \frac{1}{1 + R_s^2} = \frac{1}{1 + a^2} = \frac{2}{a^2};$$  \hspace{1cm} (16)
It should be emphasized that Eqs. 13, 15 and 16 form a purely algebraic system of equations describing the fluid accretion in a fixed space-time (Schwarzschild) geometry.

From the relation between pressure and energy density one obtains that

\[ \psi = \frac{1}{\psi} \left( a = a_1 \right)^2 = \frac{1}{\psi} \left( a = a_1 \right)^2 + \frac{a_1^2 + 1}{N} \]

where the constant \( \psi \) is the asymptotic mass density of the collapsing fluid. Substituting Eqs. 13 and 15 into rearranged 10 we find that the mass accretion rate can be described by means of the formula [2]:

\[ M = M_0 \frac{a_1^2}{a_1^2} \frac{a_2^2}{a_1^2} \left( \frac{a_1^2}{a_1^2} + \frac{a_1^2 + 1}{N} \right) \]

where \( a_1 \) is the asymptotic velocity of the collapsing fluid.

4 Limit on \( a_1^2 \) in the \((\psi, n)\) Model

It will be useful to transform Eq. 16 into

\[ 1 + 3a_2^2 = \frac{1}{\psi} \frac{a_2^2}{a_1^2} \left( \frac{a_1^2}{a_1^2} + \frac{a_1^2 + 1}{N} \right) \]

Let us assume that \( 0 < a_2^2 < a_{ax}^2 \) and \( X = (1 + 3a_{ax}^2) \). Thus values of the left-hand side of Eq. 16 belong to the range \([1, X]\). Hence

\[ 1 < \frac{a_2^2}{a_1^2} \left( \frac{a_1^2}{a_1^2} + \frac{a_1^2 + 1}{N} \right) \]

Solving 20 we get

\[ a_1^2 \quad a_2^2 \quad \left( \frac{a_1^2}{a_1^2} + \frac{a_1^2 + 1}{N} \right) \frac{1}{X} + \frac{a_1^2}{X} \]

and

\[ 2 \left( \frac{a_1^2}{a_1^2} + \frac{a_1^2 + 1}{N} \right) \frac{1}{X} + \frac{a_1^2}{X} = \]

These conditions can give us a restriction on the asymptotic velocity \( a_1^2 \). Simple calculations lead to the inequality [3]

\[ a_1^2 < (1) \]

5 Numerical Calculations

In this section we compare both models numerically referring to certain parameters important for the description of the process of accretion.
5.1 Evaluation of Parameter $a^2_s$

First we analyse formula $a^2_s (a^2_1)$ in the (p n) model given by Eq. (6). The analytical bound on $a^2_s$ shown in previous section has been confirmed by numerical calculations. In Fig. 1 we show all solutions of Eq. (6) for $\lambda = 1.3$. As one can see there are four families of solutions corresponding to various combinations of the branches (1) - (4). We eliminate from our considerations the branches (3) and (4) which describe the case $a^2_s < a^2_1$. It is not clear that the branch (2) should be rejected. However it seems peculiar for us that the asymptotic speed of sound decreases while the speed of sound in the sonic point increases (e.g., for $a^1_1 = 0$; $a^2_s$ has a nonzero value). Notice that this branch is not represented at the Newtonian level since in the Bondi model $a^2_s (a^2_1 = 0) = 0$. In the (p n) model it is assumed that for large $R$ the flow satisfies condition $\hat{u} \leq 1$ and is subsonic with $\hat{u}^2 < a^2$ (e.g. as $R \to 1$, $\hat{u} \to 0$, $a \to a_1$). For the branch (2) numerical calculations of the behaviour of parameter $u$ as a function of $R$ yield relativistic values of the parameter $u$, even if $R$ is very big (Tab. 1). Therefore we exclude the branch (2) from what follows. Next we analyse the solutions of Eq. (6) for certain values of $\lambda$ and compare them to the solutions of Eq. (15) in the (p n) model. Our calculations are shown in Fig. 2. Numerical results confirm previous analytical estimations of the cut-off of the parameter $a^2_s$. As one can see in the (p n) model the greater $\lambda$ is the greater value of $\lambda$ is reached. In contrast no such limitation appears in the (p n) model.
Figure 2: Plot of $a^2_{\infty}$ in terms of $a^2_{1\infty}$ for three different values of the adiabatic index: $\Gamma = 1.1$, $\Gamma = 1.2$, and $\Gamma = 1.6$. Dotted and solid curves refer to the $(p\ n)$ model and the $(p\ )$ model, respectively.

| $u(R)$ (c) | R  | M  |
|------------|----|----|
| 0.981071   | 5  |    |
| 0.873213   | 10 |    |
| 0.776209   | 50 |    |
| 0.763217   | 100|    |
| 0.751332   | 100|    |
| 0.750134   | 100|    |
| 0.750014   | 100|    |
| 0.750002   | 100|    |

Table 1: The values of $u(R)$ for the supercritical branch (2).

5.2 Fluid Velocity

Fluid velocity as a function of a distance is described by Eqs. (5) and (16) for the $(p\ n)$ model and the $(p\ )$ model, respectively. It rises monotonically as the radius tends to the event horizon. Comparing both models we assume the same asymptotic sound velocity $a^2_{1\infty}$. As one can see (Figs. 3 and 4) both models predict similar values of $u$.

We noticed that the greater the slower fluid velocity is at the given distance $R$.

Next conclusion is the confirmation of the fact (previously stated in [4]) that the value of $u$ near the horizon strongly depends on the location of the sonic
The further the sonic point the larger the fluid velocity and the closer to the speed of light at $R = 2M$.

![Plot of fluid velocity $u$ as a function of a radius $R$ for $\alpha = 1.1$ and $a_i^2 = 0.99$. Dotted and solid curves refer to the ($p\ nu$) model and the ($p\ n$) model, respectively.](image)

**Figure 3:** Plot of fluid velocity $u$ as a function of a radius $R$ for $\alpha = 1.1$ and $a_i^2 = 0.99$. Dotted and solid curves refer to the ($p\ nu$) model and the ($p\ n$) model, respectively.

### 5.3 Density Profile

We recall here that the main difference between the ($p\ nu$) model and the ($p\ n$) model lies in the equations of state: $p = Cn$ and $p = K$, respectively. We can relate $n$ and $\nu$ by

$$n = \exp \left( \frac{1}{1 + K} \right)$$

that can be integrated with the result:

$$n = 1 + K \quad 1^{\nu = 1} = 1 + a_i^2 \quad 1^{\nu = 1}$$

Given the ($p\ nu$) polytropic model one can always find $n$. And conversely, one can find $n$, given the polytropic ($p\ n$) model. The preceding equation yields $n_1 = 1$ if $a_i^2 \ll 1$; the same is true in the alternative description ($p\ n$) under the condition $p = (1 + K)$. According to the numerical calculations when matter approaches the horizon its density increases. We also noticed that the location of the sonic point $R_s$ plays a very important role. If it is situated close to the horizon the density of matter there increases approximately to $10^{11}$. On the other hand if $R_s 

$2M$ the density of
Figure 4: Plot of fluid velocity $u$ as a function of a radius $R$ for $a_1^2 = 1.4$ and $a_1^2 = 0.099$. Dotted and solid curves refer to the ($\rho n$) model and the ($\rho$) model, respectively.

Figure 5: Plot of nondimensional density profile $\rho/\rho_{\infty}$ as a function of the distance $R$ for the ($\rho$) model. The asymptotic velocity $a_1^2 = 0.099$ for both $a_1^2 = 1.4$ (solid curve) and $a_1^2 = 1.4$ (dotted curve).

Matter approaching the horizon becomes few orders of magnitude greater than the asymptotic density. The predictions of the two models agree in the full
Figure 6: Plot of nondimensional baryon density number profile $n=n_0$ as a function of the distance $R$ for the $(p\ n)$ model. The asymptotic velocity $a_2^2 = 0.99$ for both $= 1/d$ (solid curve) and $n = 1$ (dotted curve).

5.4 Mass Accretion Rate

In this subsection we compare the most important parameter to the description of accretion: mass accretion rate $M_{ac}$. For simplicity we introduce the parameter which is defined as the ratio of mass accretion rate in relativistic model and the mass accretion rate predicted by the Bondi model: $M_{ac} = M_{ac}/M_{ac}$. Hence, it can be interpreted as relativistic correction factor.

In the $(p\ n)$ model this parameter, with help of $\xi$, is expressed by

$$a^n_2 = a^3_2 \left( \frac{a^2_2}{a^2_1} \right)^{\frac{5}{2}} \left( 1 + 3a^2_2 \right) \left( \frac{3}{2} \right) \left( \frac{3}{2} \right) ; \quad (25)$$

while in the $(p\ )$ model using $\xi$, we get

$$a^n_2 = \left( \frac{5}{2a^2_1} \right)^{\frac{3}{2}} \left( 1 + 3a^2_2 \right) \left( 1 + \frac{a^2_2}{a^2_1} \right) ; \quad (26)$$

The comparison of the parameter for the two models (Figs. 6 and 8) leads to the conclusion that they slightly differ in a full range of allowed $\xi$, but it should be emphasized that the accretion in the $(p\ n)$ model is more efficient.

Next we compare the relativistic correction factors as functions of the adiabatic index. We consider here an ultrarelativistic regime, i.e., we assume them axi...
possible value of $a_2^2$.

In [4] it was shown that for the $(p)$ model the relativistic correction factor satisfies

$$4 \left( 1 + \frac{1}{1 - \sqrt{5}} \right)$$

$$1 \leq 1 + \frac{1}{\sqrt{5}}.$$ (27)

We confirm here that the values of a parameter belong to the range defined by (27). However, we revealed earlier [5], that the factor is not a monotonic function of and for $1 \leq 46$ it has a minimum of a value $4.77$. This is again confirmed by the present calculations. For the $(p,n)$ model, rises monotonically as increases (Fig. 7).

It should be mentioned that in non-relativistic case $a_2^2 = 1$ the relativistic correction factor is close to 1 (Fig. 8), in full agreement with theoretical expectations [4].

6 Conclusions

We examined two models of stationary and spherically symmetrical accretion of gas onto a black hole. We show that both models essentially agree as concerns quantities such as fluid velocity $u$, density profile and the mass accretion rate $M_\ast$.

What drastically differs the models is the bound on the sound velocity $c_\ast$, which has been found in the $(p,n)$ model. No such restriction appears in the $(p)$ model.
Figure 8: Plot of relativistic correction factor as a function of asymptotic velocity of sound for fixed value of $\omega = 1\omega$. Dotted and solid curves refer to the $(p\ n)$ model and the $(p\ )$ model, respectively.

Figure 9: Plot of the relativistic correction factor as a function of asymptotic velocity of sound for fixed value of $\omega = 1\omega$. Dotted and solid curves refer to the $(p\ n)$ model and the $(p\ )$ model, respectively.

... model. It makes this model more advantageous especially for the values of adiabatic index close to 1 where the $(p\ n)$ model provides the solutions...
only in a very narrow range. Another interesting difference can be observed in the ultrarelativistic regime and for close to $5=3$; the relativistic correction is significantly larger in the case of the $(\rho \cdot n)$ model than in the $(\rho)$ model.

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