Mean of continuous variables observable \textit{via} measurement on single qubit

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It is shown that mean value of any observable with bounded spectrum can be uniquely determined from binary statistics of the measurement performed on single qubit ancilla coupled to a given system. The observable structure is fully encoded in the corresponding POVM. The method is generalised to the case of distant labs paradigm and discussed in the context of entanglement detection with few local measurements. The results are also discussed in the context of quantum measurement theory.

Pacs Numbers: 03.65.-w

One of the serious problems of quantum information theory is fragility of quantum entanglement and other fully quantum resources (like quantum gates) that are basic ingredient form most of quantum information phenomena.

However before using entanglement or other resource, one has to be sure that it really is present in the system (see [6] for some paradoxes). This domain involves seriously quantum measurement theory.

In particular it is important to be able to detect entanglement in distant labs scenario, where two observers are far apart and have restricted access to the composed system they share (see for instance [6]). There are many methods checking whether there is entanglement in the system (see [6, 8]). However they require prior state reconstruction i.e. full knowledge about density matrix of the system. Recently new paradigm was introduced [6] in which one asks about entanglement of unknown state produced by stationary source. The answers has been given in series of papers [6, 8, 11]. In particular for two qubits entanglement can be detected both qualitatively [6] and quantitatively [11] without state reconstruction. If partial information about the state is provided, then entanglement can be detected [11] in distant labs paradigm with minimal number of product observables locally estimated.

The problem is that, according to quantum measurement theory, even determining of mean value of single (sic) observable $A$ requires typically estimation of many parameters namely the probabilities of outcomes of von Neumann measurement (see Problem below). The problem is especially striking in continuous variables case [12] where any von Neumann measurement can be only approximate due to finite number of outcomes of any real experiment. In this context we address quite general:

Question. - \textit{Is there any way to determine mean value of given quantum observable $A$ from experimental estimation of single parameter?}

Here by estimation of single parameter we mean estimation of probability of some single outcome i.e. the number of clicks of single detector divided by the number of all runs of experiment. The simplest example is spin polarisation measurement along given axis: to get probability of being “up” we count “up” events and divide them by the number of all (“up” and “down”) results.

Surprisingly, the answer for the Question above is positive for any bounded observable, no matter whether it involves continuous variables or not. The nature of the associated effect seems to be quite fundamental, and it not been known in quantum measurement theory so far.

It can be explained as follows: if apart form our system we have single qubit and can control the system-qubit interaction then there exists general quantum measurement (POVM) with two outcomes such that mean value of $A$ can be immediately reconstructed from the POVM statistics. Because binary POVM corresponds to estimation of single parameter (see previous discussion) it happens that in the presented scheme estimation of mean value of single observable does correspond to single parameter. The mechanism of this effect can be roughly summarised with the statement that the observable has been encoded into the interaction (represented by POVM) between the system and the qubit ancilla.

In context of results of Ref. [11] we also pose similar issue in case of product observables measured by distant observers. It happens again that two binary POVM-s are sufficient but with data analysis refined to get apart form marginal statistics also one correlation probability (like in Bell-type experiments).

Let us note that as a byproduct of other investigations we have provided partial positive answer to the above Question [11]. The idea was to encode any spin-like observable $A$ into some state $\rho_A = \alpha I + \beta A$ (that can be viewed as a kind of program) of auxiliary system. Then the mean value of $A$ in given $\varrho$ was determined from the value $Tr(\varrho_A A)$ that was found to be easily measurable as a single parameter in some interferometric scheme [11]. However that scheme required complicated resources: any d-level system needs $(d+1)$ level ancilla. Moreover, as discussed subsequently, it can not be applied for continuous variables case [12].

Here we provide novel unified approach that has quite surprising advantages: (i) requires minimal ancilla - just single quantum bit (ii) is applicable for continuous variables case under the only assumption of boundness of the observable.

Furthermore the approach allows for very easy exten-
sion to LOCC schemes. The present analysis has also new general motivation: need of many parameters estimation in standard von Neumann measurement. As we already mentioned the LOCC scheme we provide here is especially important for local detection of unknown or partially unknown entanglement. In particular it provides additional justification to approaches from Ref. [1].

It is worth to mention that recently one developed ideas of quantum computing with quantum data structure [10,11], quantum programmable interferometric networks [12] or programmable quantum gates [13]. In this context we address natural (open) question: what observable can be implemented as a kind of quantum program, and, if so, how to do it in optimal way and how to quantify this process.

The paper is organised as follows. First we pose the problem with standard von Neumann measurement and we show how solve it encoding given observable into binary POVM. Then we provide similar result for product observable in LOCC paradigm. Finally we briefly discuss the result especially in context of recently considered computing with quantum data structure and related issues.

The Problem .- Consider arbitrary quantum observable $A = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$. If it has more than two different eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n, n > 2$ then usual procedure to get mean value of $A$ in given state $\varrho$

$$\langle A \rangle_\varrho = \text{Tr}(A \varrho) \quad (1)$$

requires von Neumann measurement with $n$ outcomes corresponding to $n$ eigenvectors of the state $\varrho$: $\psi_1, \psi_2, ..., \psi_n$. The measurement rely on estimation of $n-1$ parameters that are probabilities of outcomes $p_1 = \langle \psi_1 | \varrho | \psi_1 \rangle$, $p_2 = \langle \psi_2 | \varrho | \psi_2 \rangle$, ..., $p_{n-1} = \langle \psi_{n-1} | \varrho | \psi_{n-1} \rangle$ (the last parameter $p_n$ can be inferred from normalisation condition). Finally we multiply probabilities by eigenvalues and calculate the sum $\sum_{i=1}^n p_i \lambda_i$ which is equivalent just to the mean value $\langle A \rangle_\varrho$ we were looking for. Clearly if $n$ is greater than two we need estimation of more than one parameter in the sense that (apart from counting runs of our experiment) we have to count clicks corresponding to more than one outcomes. Moreover if the observable correspond to continuous variables case ($n = \infty$ above) than there is no way to measure it directly and any indirect measurement must be approximate.

We shall see however that one can overcome those disadvantages under two assumptions: (i) boundness of the observable spectrum (ii) additional resource: well controlled interaction with single quantum bit.

Solution for single observable .- Let us first assume now that the observable has the spectrum bounded and its lower and upper bound correspond to $a_{\text{min}}$ and $a_{\text{max}}$ respectively. Let us define the non negative number $a_- \equiv \text{min} [0, -a_{\text{min}}]$. Then the following operator $D = a_- I + A$ is positive ($D \geq 0$) ie. has no negative eigenvalue. Now we define the new operator hermitian operator $D' = D/(a_+) (a_+ = \text{max}[a_- + a_{\text{max}}])$ satisfies the property $0 \leq D' \leq 1$ which means that it all eigenvalues belong to the interval $[0, 1]$. Consider now the following operators

$$V_0 = \sqrt{D'} = \sqrt{(a_- I + A)/a_+},$$
$$V_1 = \sqrt{I - V_0^\dagger V_0}. \quad (2)$$

They satisfy the following condition $\sum_{i=0}^1 V_i^\dagger V_i = I$ so they represent so called generalised quantum measurement and can be easily implemented on our system. It has two outcomes $i = 0, 1$ with probabilities $p_0 = \text{Tr}(V_0^\dagger V_0 \varrho), p_1 = \text{Tr}(V_1^\dagger V_1 \varrho) = 1 - p_0$. Note that only single parameter $p_0$ describes this binary statistics. Now it is elementary to see that because of hermiticity of $V_0$ (which means that $V_0 = V_0^\dagger$) one has $p_0 = \text{Tr}((a_- I + A) \varrho)/a_+$ and finally, because of $\text{Tr}(A \varrho) = \langle A \rangle_\varrho$ this leads to the main conclusion

$$\langle A \rangle_\varrho = a_+ p_0 - a_- \quad (3)$$

Thus we have reproduced mean value of arbitrary observable $A$ with bounded spectrum with help of single parameter $p_0$ coming from binary generalised quantum measurement (POVM).

It is remarkable that the above POVM can be performed on the system if only we have one qubit ancilla (additional physical system) and can control interaction between our system and the ancilla. Indeed this is all what binary POVM requires (18, see Appendix of 13 for tutorial review). We prepare our ancilla qubit in the pure state $|0\rangle$. Then we perform unitary evolution $U$ of our global system: “system in state $\varrho$ + ancilla”. Finally we measure observable $\sigma_i$ on our ancilla. If we get result “up” (ancilla state unchanged ie. remains in initial $|0\rangle$) this corresponds to result $i = 0$, if we get result “down” (ancilla state changed into $|1\rangle$) this corresponds to the result $i = 1$ both occurring with probabilities $p_0, p_1$ defined above.

At a first glance the present scheme has close similarity to that of universal quantum estimator detecting nonlinear state functions [16] as in the latter only single control qubit is finally measured. There is a significant difference however, that makes the present scheme successful for CV case. We shall discuss that issue in the conclusion part of the paper.

Note that the above scheme allows to detect “mean value” of nonhermitian operator $X$ defined by $\langle X \rangle_\varrho$ with help of decomposing $X$ into hermitian and antihemitian part ( cf. 14) and detecting the corresponding observables with help of two binary POVM-s.

\footnote{We say that $A \geq B$ if for all $\Psi$ one has $\langle \Psi | A - B | \Psi \rangle \geq 0$.}
Product observables and distant labs paradigm. Suppose now that Alice and Bob are in distant labs paradigm i.e. they are far apart ad they share some bipartite quantum state $\varrho_{AB}$. This is like in quantum teleportation process where they shared single state (here we allow $\varrho$ to be mixed). In such case Alice and Bob are allowed to perform local operations (LO) and communicate classically (CC). Suppose now that they want to detect mean value of some entanglement witness $W = \sum_{k=1}^{m} A_k \otimes B_k$ with its structure and number chosen properly (see [1]). Because of LOCC restrictions this can be achieved only by measurement of local measurements and exchange of information. Usually it is done like in standard Bell inequalities (for similarity of entanglement witnessed formalism to Bell inequalities theory see [3]). Namely this corresponds to local measurements of observables $A_k, B_k$ (for each fixed $k$) but with keeping the record of results and finally establishing the mean value form joint statistics. However there is more outcomes in general so again there is a question whether we can reduce the above scheme to binary experiments. The answer is “Yes”, though the solution is not so simple as it was before. Suppose that Alice and Bob want to measure mean value

$$\langle A \otimes B \rangle_{\varrho_{AB}} = Tr(A \otimes B \varrho_{AB})$$

(4)

of product observable $A \otimes B$ on shared state $\varrho_{AB}$. Then they should perform local POVM-s corresponding to local observables as defined in previous section but they should use the data in more sophisticated way. Let Alice POVM be $\{V_0, V_1\}$ (as before) while by Bob’s POVM we denote $\{W_0, W_1\}$. They have pairs of possible local outcomes $i_A, i_B = 0, 1$ where $i_A$ ($i_B$) corresponds to Alice (Bob) outcome respectively. Then, performing measurements on their ancillas they should not only estimate parameters $p_0 = Tr(W_0 \otimes W_0 \varrho_{AB}), q_0 = Tr(W_0^\dagger \otimes W_0^\dagger \varrho_{AB})$ which corresponds to normalised numbers of outcomes $i_A = 0, i_B = 0$ respectively. In addition they should also communicate classically and count all cases when they got the results $i_A = i_B = 0$ correlated i.e. coming from the copy of the state $\varrho_{AB}$. Normalising the resulting number of the cases i.e. dividing it by the number of all measurements they got joint correlation probability

$$p_{00} = Tr(V_0 \otimes W_0 \varrho_{AB})$$

(5)

of getting the same outcome $i_A = i_B = 0$ on both sides from the same copy of the state.

The above process is equivalent to estimation of mean values $\langle \sigma_z^{(A)} \rangle$, $\langle \sigma_z^{(B)} \rangle$, and $\langle \sigma_z^{(A)} \otimes \sigma_z^{(B)} \rangle$ on Alice and Bob local ancillas that were needed to implement the POVM.

Thus the process is virtually identical to what happens in usual Bell-type inequality on two spin-1/2 particle where also marginal and correlation probabilities are determined. Summarising Alice and Bob need to determine probabilities $p_0, q_0$ and $p_{00}$ of standard Pauli $\sigma_z$ measurements on their ancillas. It is easy to see that from the probabilities they easily get the needed mean value as follows

$$\langle A \otimes B \rangle_{\varrho_{AB}} = a_+ b_+ p_{00} + a_- b_- - [a_+ b_- p_0 + a_- b_+ q_0]$$

(6)

where $b_\pm$ are defined with respect to observable $B$ in full analogy to $a_\pm$. Thus again we have reduced LOCC measurement of $A \otimes B$ to two binary POVM-s with more careful data analysis leading not only to binary marginal distributions (determined by probabilities $p_0, q_0$) but also to correlation probability $p_{00}$. Finally let us note that the above reasoning can be generalised to multipartite LOCC scheme. Then only the proper hierarchy of correlation probabilities must be taken into account.

Binary Bell-like inequalities. Note that any Bell inequality involving arbitrary bounded observables can be reduced in this way to the “binary Bell-like inequality” involving joint probabilities of binary events (like $p_0, q_0, p_{00}$ and their multipartite analogs). However validity of such inequality assumes validity of quantum mechanics (quantum interaction). Whether and when it is possible to remove the latter assumption in such modified ”binary Bell inequality” is an open question. In any case it seems that there is possibility of some universal translation of arbitrary Bell inequalities into the legitimate binary ones along the above lines. Discussion and conclusions. We have discussed the problem of whether measurement of single observable with many eigenvalues can be restricted to estimation of single parameter. We have shown that it is always possible if (i)the observable is bounded, i.e. has upper and lower bounds on its spectrum and (ii) one has well controlled interaction with single qubit ancilla.

We have constructed the corresponding POVM and pointed out that it can be achieved with only one additional quantum bit. Namely the estimated parameter corresponds to probability of getting one outcome of two possible in measurement on Pauli operator $\sigma_z$ on single qubit ancilla.

We have also considered the issue of detecting partially known entanglement with minimal number of estimated parameters in context of Ref. [1]. In this case it happens that number of local observables involved in the measurement is equal to the corresponding binary POVM-s that can supersede them. The result of POVM-s, however should be used in more detailed way to get not only marginal (single parameter) binary statistics $\{p_0, 1-p_0\}, \{(q_0, 1-q_0)\}$ on Alice (Bob) side but also join correlation probability $p_{00}$. It can be generalised to multipartite systems and leads to compression of usual Bell inequalities into “binary Bell-like inequalities” involving only joint probabilities of binary events.

Let us observe that in CV case the measurement of general observable is impossible - due to infinite num-
ber of outcomes one can only measure some approximated ("digitalised") observable instead of the original one. The present binary POVM method seem to be the only one that provides the mean value of the observable itself rather than its approximation. The present result has some similarity to recent interferometric method [10] where final estimation comes form measurement of Pauli matrix $\sigma_z$. However its fundamental difference can be seen easily when one realises that the interferometric approach by no means can work for infinite-dimensional scenario called continuous variables (CV) case. The observable is there encoded by affine transformation $A \rightarrow \varrho_A = \alpha I + \beta A$ where $I$ stands for identity operator. Hence for CV $\varrho_A$ is no longer a quantum state (as the interferometric method of [10] requires) because has no finite trace. There is no problems like that for the present method. Even the presence of square root in formulas defining $V_0, V_1$ does not mean that the discussed difference is equivalent to that between probabilities and amplitudes in quantum theory. There is a deeper reason: in the present method the observable $A$ is encoded directly in global dynamics (ancilla-system interaction Hamiltonian that can be inferred from the POVM) rather than "programmed" into the "static" ancilla as it was proposed in Ref. [10].

There is, however, some important point that links the present approach with that of Ref. [10]. Let us recall that some kind of "quantum programs" that implement some physical observable in physical system (ancilla) has been already presented in previous approach [10]. Moreover there is general idea of quantum programming with quantum data structure [10] and systematic way of quantum gates programming [17].

In the above context an intriguing question arises naturally: is it possible to implement given observable as a kind of "program", and if so, what is the most optimal way to do that? Form the present analysis we already know that some CV observables can not be realised as a sort of "program" (state of the ancilla). But one can imagine the scenario: where observable parameters are "split" into the programmable part and the one that is non programmable but can be encoded into dynamics. In this context one would need measures that would quantify both parts. It seems that to characterise the second part the entangling power concept can be important [18] as well as quantum gates programming [17] and gates cost [19]. Also, in case of continuous variables, the concept of both classical and quantum complexity of observable parameters will have to be taken into account.

Finally, it may be interesting to consider application of the present result in context of Bell inequalities tests for continuous variables systems [20].

The author thanks A. Ekert, D. Oi, C. M. Alves, Ch. Fuchs, M. Horodecki and R. Horodecki for interesting discussions. The work is partially supported by the project EQUIP, contract No. IST-1999-11053.

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