Exact relativistic models of perfect fluid disks in a magnetic field

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Using the well-known “displace, cut and reflect” method we construct thin disks made of a perfect fluid in presence of a magnetic field. The models are based in a magnetic Reissner-Nordstrom metric of Einstein-Maxwell equations for a conformastatic spacetime. The influence of the magnetic field on the matter properties of the disk are analyzed. We also study the motion of charged test particles around the disks. We construct models of perfect fluid disks satisfying all the energy conditions.

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I. INTRODUCTION

Axially symmetric exact solutions of Einstein equations describing the field of a thin disk are important in two context, in astrophysics as models of certain stars, galaxies, accretion disks, and the superposition of a black hole and a galaxy or an accretion disk as in the case of quasars, and in general relativity as sources of exact solutions of Einstein field’s equations. Thin disks in presence of magnetic field are also of astrophysical importance in the study of active galactic nuclei, neutron stars, white dwarfs and galaxy formation, or as sources of Einstein-Maxwell equations. For example, in the case of highly magnetized neutron stars, or magnetars, these fields are characterized by a strength of the order of $10^{14}$ G \[1–3\]. To describe the gravitational and electromagnetic fields of such configurations, is necessary to study the coupled Einstein-Maxwell equations.

Exact solutions of the Einstein equations representing the field of a static thin disks without radial pressure were first studied by Bonnor and Sackfield \[4\], and Morgan and Morgan \[5\], and with radial pressure also by Morgan and Morgan \[6\]. Several classes of exact solutions of the Einstein field equations corresponding to static thin disks with or without radial pressure have been obtained by different authors \[7–13\]. Rotating thin disks that can be considered as a source of a Kerr metric were presented by Bicák and Ledvinka \[14\], while rotating disks with heat flow were studied by González and Letelier \[15\]. The exact superposition of a disk and a static black hole was first considered by Lemos and Letelier in Refs. \[16–18\]. On the other hand, thin disks in presence of electromagnetic field have been discussed as sources for Kerr-Newman fields \[19\], conformastationary metrics \[20\], and magnetostatic axisymmetric fields \[21–25\]. Perfect fluid disks as sources of Taub-NUT-type spacetimes were presented in Ref. \[26\], and static charged perfect fluid disks were study in \[27\]. Also, conformastatic disk-haloes in Einstein-Maxwell gravity were considered in \[28\], variational thermodynamics of relativistic thin disks in \[29\], and the exact superposition of a disk and a static black hole in a magnetic field in \[30\].

In this work we consider exact relativistic models of thin disks made of perfect fluid in presence of a magnetic field. The models are constructed using the well-known “displace, cut and reflect” method. These disks are essentially of infinite extension. However, since the surface energy density of the disks decreases rapidly one can define a cut off radius, of the order of the galactic disk radius, and, in principle, to consider these disks as finite. Moreover, even though realistic disklike sources have thickness, in first approximation these astrophysical objects can be considered to be very thin, e.g., in our Galaxy the radius of the disk is 10 kpc and its thickness is 1 kpc. The disks also satisfy all the energy condition, which means that these structures are made of usual matter and no tachyonic or otherwise exotic matter.

The paper is organized as follows. In Sec. II we present the formalism to construct models of thin disks made of a perfect fluid and electric current for a conformastatic spacetime. We also analysis the motion of charged test particles around the disks. In Sec. IV a simple family of perfect fluid disks in presence of a magnetic field is considered based in a magnetic Reissner-Nordstrom metric of Einstein-Maxwell equations for a conformastatic spacetime. Finally, in Sec. V we summarize and discuss the results obtained.

II. EINSTEIN-MAXWELL EQUATIONS AND PERFECT FLUID DISKS

The metric for a conformastatic spacetime can be written as \[28\]

$$ds^2 = -e^{2\psi}dt^2 + e^{2\lambda}(R^2d\varphi^2 + dR^2 + dz^2),$$  \[1\]

where $(\varphi, R, z)$ are the usual cylindrical coordinates. For axially symmetric fields $\psi$ and $\lambda$ are functions of the coordinates $R$ and $z$ only. The vacuum Einstein-Maxwell equations, in geometrized units such that $G = c = 1$, are given by

$$R_{ab} = 8\pi T_{ab},$$ \[2a\]

$$\nabla_b F^{ab} = 0,$$ \[2b\]

where

$$T_{ab} = \frac{1}{4\pi} \left[ F_{ac}F^c_b - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right]$$ \[3\]

is the electromagnetic energy-momentum tensor,

$$F_{ab} = A_{b,a} - A_{a,b}$$ \[4\]

the electromagnetic field tensor, and $A_a$ the electromagnetic four potential. The other symbols have the usual meaning, i.e., $(\ )_a = \partial / \partial x^a$, $\nabla_b$ covariant derivate, etc.
Solutions of the Einstein-Maxwell equations (2a) - (2b) representing the field of a thin disk at \( z = 0 \) with electric current can be constructed assuming the components of the metric tensor and the electromagnetic potential continuous across the disk, and its first derivatives discontinuous in the direction normal to the disk. This can be written as

\[
\begin{align*}
\alpha_b &= [\alpha_{b,z}] = \alpha_{b,z}|_{z=0^+} - \alpha_{b,z}|_{z=0^-}, \\
\alpha_{ab} &= [\alpha_{ab,z}] = \alpha_{ab,z}|_{z=0^+} - \alpha_{ab,z}|_{z=0^-}.
\end{align*}
\]

The application of the formalism of distributions in curved spacetimes to the Einstein-Maxwell equations [31–35] give us

\[
\begin{align*}
R_{ab} &= 8\pi T_{ab}, \\
T_{ab} &= T_{ab}^{\text{el}} + T_{ab}^{\text{mat}} = T_{ab}^{\text{el}} + Q_{ab} \delta(z), \\
\nabla_b F_{ab} &= 4\pi J^a, \\
J^a &= j^a \delta(z),
\end{align*}
\]

where \( \delta(z) \) is the usual Dirac function with support on the disk, \( T_{ab}^{\text{el}} \) is the electromagnetic tensor (3),

\[
Q_{b} = \frac{1}{16\pi} \left\{ a^{zx} \delta{z}_b - b^{zx} \delta{b}_a + g^{az} b^z_b - g^{zz} b^a_b + b_c^z (g^{zz} \delta{a}_b - g^{az} \delta{z}_b) \right\}
\]

is the energy-momentum tensor on plane \( z = 0 \), and

\[
j^a = \frac{1}{4\pi} [F^{ab}] \delta{b}_a, \quad (8)
\]

is the electric current density on the disk. \([F^{ab}]\) means the jump of Maxwell tensor across the disk. The “true” surface energy-momentum tensor (SEMT) of the disk \( S_{ab} \) and the “true” surface current density \( j_a \) are given by

\[
\begin{align*}
S_{ab} &= \int T_{ab}^{\text{mat}} \, ds_n = \sqrt{g_{zz}} Q_{ab}, \\
j_a &= \int J_a \, ds_n = \sqrt{g_{zz}} j_a,
\end{align*}
\]

where \( ds_n = \sqrt{g_{zz}} \, dz \) is the “physical measure” of length in the direction normal to the disk. For the metric (1), the nonzero components of \( S_{ab} \) are

\[
\begin{align*}
S_{tt} &= \frac{1}{2\pi} e^{-\psi} \lambda, \\
S_{\varphi\varphi} &= S_{RR} = \frac{1}{4\pi} e^{-\psi + \lambda},
\end{align*}
\]

For axially symmetric magnetostatic fields the magnetic potential is given by \( A_a = \delta_a^\varphi A \) where \( A \) is functions of \( R \) and \( z \) only. The only nonzero component of the current density is

\[
J_\varphi = -\frac{1}{2\pi} e^{-\psi} A_{,z}.
\]

All the quantities are evaluated at \( z = 0^+ \). In terms of the orthonormal tetrad \( e_{(a)}^b = \{V^b, W^b, X^b, Y^b\} \), where

\[
\begin{align*}
V^a &= e^{-\psi} \delta^a_t, \\
W^a &= e^{-\lambda} \delta^a_z / R, \\
X^a &= e^{-\lambda} \delta^a_R, \\
Y^a &= e^{-\lambda} \delta^a_z,
\end{align*}
\]

the SEMT can be written as

\[
S_{ab} = pg^{ab} + (p + \epsilon) V^a V^b, \quad (13)
\]

where

\[
\epsilon = -S_{tt}^t, \quad p = p_\varphi = p_R = S_{\varphi\varphi}.
\]
are, respectively, the surface energy density $\epsilon$ and the azimuthal and radial pressure $p$ on the disk. The azimuthal current density $j$ is given by

$$ j = W^\varphi j_\varphi. \tag{15} $$

Thus, these disks can be interpreted as a matter distribution made of a perfect fluid with electric current.

We will now analyze the motion of charged test particles around the disks. For equatorial circular orbits the 4-velocity $u^a$ of the particles with respect to the coordinates frame has components $u^a = u^0(1, \omega, 0, 0)$ where $\omega = u^1/u^0 = \frac{d\varphi}{dt}$ is the angular speed of the test particles. For the spacetime (1), the equation for the electrogeodesic motion of the particle is given by

$$ \frac{1}{2} g_{ab,R} u^a u^b = -\hat{\epsilon} F_{Ra} u^a, \tag{16} $$

where $\hat{\epsilon}$ is the specific electric charge of the particles. In the case magnetostatic the motion equation reads

$$ \frac{1}{2} u^0 (g_{\varphi\varphi,R} \omega^2 + g_{tt,R}) = -\hat{\epsilon} A_{R\omega}, \tag{17} $$

where $u^0$ obtains normalizing $u^a$, that is requiring $g_{ab}u^a u^b = -1$, so that

$$ (u^0)^2 = -\frac{1}{g_{\varphi\varphi} \omega^2 + g_{tt}}. \tag{18} $$

Thus, the angular speed $\omega$ is given by

$$ \omega^2 = \frac{-T_1 \pm \sqrt{T_1^2 - T_2 T_3}}{2 T_2}, \tag{19} $$

where

$$ T_1 = 2 g_{tt,R} g_{\varphi\varphi,R} + 4 \hat{\epsilon}^2 g_{tt} A_{R}^2, \tag{20a} $$
$$ T_2 = g_{\varphi\varphi,R}^2 + 4 \hat{\epsilon}^2 g_{\varphi\varphi} A_{R}^2, \tag{20b} $$
$$ T_3 = 4 g_{tt,R}^2. \tag{20c} $$

The positive sign corresponds to the direct orbits or co-rotating and the negative sign to the retrograde orbits or counter-rotating.

With respect to the orthonormal tetrad (12a)-(12b) the 3-velocity has components

$$ v^{(i)} = \frac{e^{(i)}_a u^a}{e^{(0)}_b u^b}. \tag{21} $$

For equatorial circular orbits the only nonvanishing velocity components is given by

$$ (v^{(\varphi)})^2 = v_\varphi^2 = -\frac{g_{\varphi\varphi}}{g_{tt}} \omega^2, \tag{22} $$

which represents the circular speed of the particle as seen by an observer at infinity.

### III. DISKS FROM A MAGNETIC REISSNER-NORDSTROM SOLUTION

A simple exact solution of Maxwell-Equations for the spacetime (1) is given by the metric

$$ ds^2 = -\left(1 - \frac{\tilde{a}}{4\tau}\right)^2 \left(1 + \frac{\tilde{a} + \tilde{b}}{2\tau}\right)^2 dt^2 + \left(1 + \frac{\tilde{a} - \tilde{b}}{2\tau}\right)^2 \left(1 + \frac{\tilde{a} - \tilde{b}}{2\tau}\right)^2 (R^2 d\varphi^2 + dR^2 + dz^2), \tag{23a} $$
$$ A = \tilde{b} \left(\frac{z}{r} + 1\right), \tag{23b} $$
\[ r' = r \left( \frac{1 + \frac{\tilde{a} + \tilde{b}}{2r}}{1 + \frac{\tilde{a} - \tilde{b}}{2r}} \right), \]  

one find that this spacetime is the magnetic Reissner-Nordstrom metric

\[ ds^2 = \left( 1 - \frac{2\tilde{a}}{r} + \frac{\tilde{a}^2}{r^2} \right) dt^2 + \left( 1 - \frac{2\tilde{a}}{r} + \frac{\tilde{a}^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2, \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \). The solution is the magnetostatic counterpart of the Reissner-Nordstrom solution and can be obtained using the well-known “displace, cut and reflect” method that was first used by Kuzmin and Toomre to construct Newtonian models of disks, and later extended to general relativity. Given a solution of the Einstein-Maxwell equation, this procedure is mathematically equivalent to apply the transformation \( z \rightarrow |z| + z_0 \), with \( z_0 \) constant. The method is an adaptation of the method of images of electrostatic.

From (14)-(15), the main physical quantities associated with the system are

\[ \dot{\varepsilon} = \frac{4\alpha(2\alpha r + 1)}{\pi(4\alpha^2 + 2a\alpha + 1)^{1/2}}, \]  

\[ \tilde{p} = \frac{2\alpha(4\alpha^2 + 4a\alpha + 1)}{\pi(4\alpha^2 - 1)(4\alpha^2 + 2a\alpha + 1)^{1/2}}, \]  

\[ j = -\frac{8b\alpha^3}{\pi(4\alpha^2 + 2a\alpha + 1)^{3/2}}, \]

where \( \varepsilon = k\alpha, \tilde{p} = kp, \) and \( \tilde{r} = \sqrt{R^2 + a^2} \), with \( R = R/k \) and \( \alpha = z_0/k \). Since \( \varepsilon \geq 0 \) these disks satisfy the weak energy condition for all values of parameter and we have always pressure (stress positive) for \( \alpha > 1/2 \). It follows that the strong energy condition \( \varepsilon + 2p \geq 0 \) is also satisfied. The dominant energy condition requires that \( p \leq \varepsilon \) which for \( \alpha > 1/2 \) reads \( (2\varepsilon^2)/(8\alpha^2) + 4\varepsilon^2 + 3 \geq 8\alpha^2 \). It follows that these disks also satisfy this condition.

On the other hand, the lines of force of the magnetic field are given by the equation \( A(r, z) = C \), being \( C \) a constant. Calling \( C = C_A \), the lines satisfy the expression

\[ R = \frac{\sqrt{C_A(2kb - 1)}}{C_A - kb}(|z| + z_0). \]

Thus, to order to have well defined lines \( kb \geq 1/2 \) and \( C_A > kb \). The lines of force of the gravitational field can be represent by drawing the equipotential lines which are given by \( g_{00} = C \), being \( C \) also a constant. Gravitational field lines are perpendicular to equipotential lines. Making \( C = -C_G^2 \), the equipotential lines satisfy the relation

\[ R^2 + (|z| + z_0)^2 = \tilde{C}_G^2, \]

where

\[ \tilde{C}_G = -aC_G \pm \frac{a^2C_G^2 + 4k^2(1 - C_G^2)}{4(C_G - 1)}, \]

with \(-1 \leq C_G < 1\).

In Figs. 1(a)-1(c), we show the surface energy density \( \dot{\varepsilon} \), the pressure \( \tilde{p} \) and the azimuthal electric current density \( j \) for disks with cut parameter \( \alpha = 2 \) and values of the parameter of magnetic field \( b = 0, 0.5, 1, 1.5, \) and 2, as functions of \( R \). We can see that the energy is everywhere positive, its maximum occurs in the center of the disk, and it vanishes sufficiently fast as \( r \) increases. This permits to define a cut off radius and, in principle, to model these structures as a compact object. We observe that the pressure and the electric current have a behavior similar to the energy. We find that the presence of magnetic field increases the energy density and the electric current density while the pressure...
is lowered everywhere on the disks. We also computed these functions for other values of the parameters within the allowed range and in all cases we found a similar behavior.

In Figs. 2(a) and 2(b), we illustrate the behavior of the circular speed for direct orbit \( v_+^2 \) and retrograde orbits \( v_-^2 \) of charged test particles with \( \tilde{e} = 1, b = 0 \) (dashed curves), 0.5, 1, 1.5, and 2 (dash-dotted curves) and the same value of \( \alpha \). We observe that circular speed of particles is always a quantity less than the speed of light and that the magnetic field makes more relativistic these orbits. In Figs. 2(c) and 2(d), we also plot the circular speed for the same value of \( \alpha, b = 1 \), and different values of the specific electric charge \( \tilde{e} = 0 \) (dashed curves), 0.5, 1, 1.5, and 2 (dash-dotted curves). We find that for direct orbits the specific charge increases the speed of the particles whereas for retrograde orbit the contrary occurs. We also find that for neutral particles \( \tilde{e} = 0 \) the speed of particles is the same for both direct and indirect orbits, whereas for charged particles are different.

In Figs. 3(a) and 3(b), we have drawn the magnetic field lines and equipotential lines of the gravitational field for \( z_0 = 2, b = 1, C_A = 2, 3, 4, 5, 6 \) , and \( C_G = 0.9, 0.92, 0.93, 0.94, 0.95 \). Since gravitational field lines are perpendicular to equipotential lines, we conclude that they have a similar behavior to the magnetic ones.

**IV. DISCUSSION**

A simple family of perfect fluid disks in presence of a magnetic field was presented based in a magnetic Reissner-Nordstrom solution. The models were constructed using the well-known displace, cut and reflect method. The disks satisfy all the energy conditions.

We found that the the presence of magnetic field increases the energy density and the electric current density while the pressure is lowered everywhere on the disks. The same behavior was observed for all the values of parameter.

We also analyzed the electrogeodesic equatorial circular motion of charged test particles around of the disks. We found that the presence of magnetic field provides the possibility of to find relativist charged particles moving in both direct and retrograde direction. We also observer that the circular speed of particles is always a quantity less than the speed of light and that the magnetic field makes more relativistic these orbits.

Finally, other magnetostatic axially symmetric solutions that can be written in conformastatic form are being investigated.

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FIG. 1. (a) The energy density $\tilde{\epsilon}$ and (b) the pressure $\tilde{p}$ for perfect fluid disk in a magnetic field with $\alpha = 2$ and values of magnetic field parameter $b = 0$ (dashed curves), 0.5, 1, 1.5, and 2 (dash-dotted curves), as functions of $\tilde{R}$. (c). The azimuthal electric current density $j$ for $b = 0$ (axis $\tilde{R}$), 0.5, 1, 1.5, and 2 (top curve) and the same value of the parameter $\alpha$. 
FIG. 2. The circular speed (a) $v_+^2$ and (b) $v_-^2$ for for \( \alpha = 2, \tilde{c} = 1, b = 0 \) (dashed curves), 0.5, 1, 1.5, and 2 (dash-dotted curves). Again the circular speed (c) $v_+^2$ and (d) $v_-^2$ for the same value of \( \alpha, b = 1, \tilde{c} = 0 \) (dashed curves), 0.5, 1, 1.5, and 2 (dash-dotted curves).

FIG. 3. Magnetic field lines (a) and equipotential lines of the gravitational field (b) for \( z_0 = 2, b = 1, C_A = 2, 3, 4, 5, 6, C_G = 0.9, 0.92, 0.93, 0.94, 0.95 \). Note that the gravitational field lines have the same behavior that the magnetic one.