Comparison of quantum field perturbation theory for the light front with the theory in lorentz coordinates

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The relationship between the perturbation theory in light-front coordinates and Lorentz-covariant perturbation theory is investigated. A method for finding the difference between separate terms of the corresponding series without their explicit evaluation is proposed. A procedure of constructing additional counter-terms to the canonical Hamiltonian that compensate this difference at any finite order is proposed. For the Yukawa model, the light-front Hamiltonian with all of these counter-terms is obtained in a closed form. Possible application of this approach to gauge theories is discussed.

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1. Introduction

The Hamiltonian approach to quantum field theory in light-front coordinates (LF), $x_\pm = \frac{1}{\sqrt{2}}(x_0 \pm x_3)$, $x^\perp = \{x^1, x^2\}$, is attractive as a possible method of solving strong interaction problems. In this approach, the formal triviality of the physical vacuum allows one to seek bound states without prior investigation of the complex vacuum structure. However, as is already known, canonical quantization in LF, i.e., on the $x^+ = \text{const}$ hypersurface, can result in a theory not quite equivalent to the Lorentz-invariant theory (i.e., to the standard Feynman formalism). This is due, first of all, to strong singularities at zero values of the "light-like" momentum variables $Q_- = \frac{1}{\sqrt{2}}(Q_0 - Q_3)$. To restore the equivalence with a Lorentz-covariant theory, one has to add unusual counter-terms to the formal canonical Hamiltonian for the LF, $H = P_+ = \frac{1}{\sqrt{2}}(P_0 + P_3)$ (the operator of a shift along the $x^+$-axis). These counter-terms can be found by comparing the perturbation theory based on the canonical LF formalism with Lorentz-covariant perturbation theory. This is done in the present paper. The light-front Hamiltonian thus obtained can then be used in nonperturbative calculations. It is possible, however, that perturbation theory does not provide all of the necessary additions to the canonical Hamiltonian, as some of these additions can be nonperturbative. In spite of this, it seems necessary to examine this problem within the framework of perturbation theory first.

For practical purposes a stationary noncovariant light-front perturbation theory, which is similar to the one applied in nonrelativistic quantum mechanics, is widely used. It was found \[1, 2, 3\] that the "light-front" Dyson formalism allows this theory to be transformed into an equivalent light-front Feynman theory (under an appropriate regularization). Then, by re-summing the integrands of the Feynman integrals, one can recast their form so that they become the same as in the Lorentz-covariant theory. (This is not the case for diagrams without external lines, which we do not consider here.) Then, the difference between the light-front and Lorentz-covariant approaches that persists is only due to the different regularizations and different methods of calculating the Feynman integrals (which is important because of the possible absence of their absolute convergence in pseudo-Euclidean space). In the present paper, we concentrate on the analysis of this difference.

A light-front theory needs not only the standard UV regularization, but also a special regularization of the singularities $Q_- = 0$. In our approach, this regularization eliminates the creation operators $a^+(Q)$ and annihilation operators $a(Q)$ with $|Q_-^i| < \varepsilon$ from the Fourier expansion of the field operators in the field representation. As a result, the integration w.r.t. the corresponding momentum $Q_-$ over the range $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$ is associated with each line before removing the $\delta$-functions. Different propagators are regularized independently, which allows the described re-arrangement of the perturbation theory series. On the other hand, this regularization is convenient for further nonperturbative numerical calculations with the light-front Hamiltonian, to which the necessary counter-terms are added (the "effective" Hamiltonian). We require that this Hamiltonian generate a theory equivalent to the Lorentz-covariant theory when the regularization is removed. Note that Lorentz-invariant methods of regularization (e.g., Pauli-Villars reg-
ularization) are far less convenient for numerical calculations and we shall only briefly mention them.

The specific properties of the light-front Feynman formalism manifest themselves only in the integration over the variables $Q_\pm = \frac{1}{\sqrt{2}}(Q_0 \pm Q_3)$, while integration over the transverse momenta $Q_\perp \equiv \{Q_1, Q_2\}$ is the same in the light-front and the Lorentz coordinates (though it might be nontrivial because it requires regularization and renormalization). Therefore, we concentrate on a comparison of diagrams for fixed transverse momenta (which is equivalent to a two-dimensional problem).

In the present paper, we propose a method that allows one to find the difference (in the limit $\varepsilon \to 0$) between any light-front Feynman integral and the corresponding Lorentz-covariant integral without having to calculate them completely. Based on this method, a procedure is elaborated for constructing an effective Hamiltonian in LF in any order of perturbation theory. The procedure can be applied to all nongauge field theories, as well as to Abelian and non-Abelian gauge theories in the gauge $A_- = 0$ with the vector meson propagator chosen according to the Mandelstam-Leibbrandt prescription [4, 5]. The question of whether the additional components of the Hamiltonian that arise can be combined into a finite number of counter-terms must be dealt with separately in each particular case.

Application of this formalism to the Yukawa model makes it possible to obtain the effective light-front Hamiltonian in a closed form. The result agrees with the conclusions of the work [1], where a comparison was made of the light-front and Lorentz-covariant methods via calculating self-energy diagrams in all orders of perturbation theory and other diagrams in lowest orders. Conversely, for gauge theories (both Abelian and non-Abelian), it was found that counter-terms of arbitrarily high order in field operators must be added to the effective Hamiltonian. This result may turn out to be wrong if the contributions to the counter-terms are mutually canceled. This calls for further investigation, but such possibility appears to be very unlikely.

What we have said above does not depreciate the light-front formalism as applied to gauge theories. This is because the only requirement concerning the light-front Hamiltonian is that it correctly reproduces all gauge-invariant quantities rather than the off-mass-shell Feynman integrals in a given gauge. However, renormalization of the light-front Hamiltonian turns out to be a difficult problem and it requires new approaches. We do not examine the possibilities of changing the light-front Hamiltonian by introducing new nonphysical fields by a method different from the Pauli-Villars regularization [6] or the possibilities of using gauges more general than $A_- = 0$ with the Mandelstam-Leibbrandt propagator. These points also need to be investigated further.
2. Reduction of light-front and Lorentz-covariant Feynman integrals to a form convenient for comparison

Let us examine an arbitrary IPI Feynman diagram. We fix all external momenta and all transverse momenta of integration, and integrate only over $Q_+$ and $Q_-:

$$F = \lim_{\varepsilon \to 0} \int \frac{\prod_i d^2Q_i^j f(Q_i^j, p_i^j)}{\prod_i (2Q_+^i Q_-^i - M_i^2 + i\varepsilon)}. \quad (1)$$

We assume that all vertices are polynomial and that the propagator has the form

$$\frac{z(Q)}{Q^2 - m^2 + i\varepsilon}, \quad \text{or} \quad \frac{z(Q) Q_+}{(Q^2 - m^2 + i\varepsilon)(2Q_+ Q_- + i\varepsilon)}, \quad (2)$$

where $z(Q)$ is a polynomial. A propagator of the second type in (2) arises in gauge theories in the gauge $A_- = 0$ if the Mandelstam-Leibbrandt formalism \cite{4, 5} with the vector boson propagator

$$\frac{1}{Q^2 + i\varepsilon} \left( g_{\mu\nu} - \frac{(\delta_\mu^+ Q_\nu + Q_\mu^+ \delta_\nu^+)}{2Q_+ Q_- + i\varepsilon} \right),$$

is used. In Eq. (3) either $M_i^2 = m_i^2 + Q_i^+ Q_i^- \neq 0$, where $m_i$ is the particle mass, or $M_i^2 = 0$.

The function $f$ involves the numerators of all propagators and all vertices with the necessary $\delta$-functions, that include the external momenta $p_i$ (the same expression without the $\delta$-functions is a polynomial, which we denote by $\tilde{f}$). We assume for the diagram $F$ and for all of its subdiagrams that the conditions

$$\omega_\parallel < 0, \quad \omega_+ < 0, \quad (3)$$

hold, where $\omega_+$ is the index of divergence w.r.t. $Q_+$ at $Q_-^i \neq 0 \ \forall i$, and $\omega_\parallel$ is the index of divergence in $Q_+$ and $Q_-$ (simultaneously); $Q_\pm = \frac{1}{\sqrt{2}} (Q_0 \pm Q_3)$. The diagrams that do not meet these conditions should be examined separately for each particular theory (their number is usually finite). We seek the difference between the value of integral (1) obtained by the Lorentz-covariant calculation and its value calculated in light-front coordinates (light-front calculation).

In the light-front calculation, one introduces and then removes the light-front cutoff $|Q_-| \geq \varepsilon > 0$:

$$F_{lf} = \lim_{\varepsilon \to 0} \lim_{\varepsilon' \to 0} \int_{V_\varepsilon} \prod_i dQ_i^- \int \prod_i dQ_i^+ \frac{f(Q_i^j, p_i^j)}{\prod_i (2Q_-^i Q_+^i - M_i^2 + i\varepsilon)},$$

where $V_\varepsilon = \prod_i ( (-\infty, -\varepsilon) \cup (\varepsilon, \infty) )$. Here (and in the diagram configurations to be defined below) we take the limit w.r.t. $\varepsilon$, but, generally speaking, this limit may not exist. In this case, we assume that we do not take the limit, but take the sum of all nonpositive
power terms of the Laurent series in $\varepsilon$ at the zero point. If conditions (3) are satisfied, Statement 2 from Appendix I can be used. This results in the equality

$$F_{\text{if}} = \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \int \prod_k dq^k_+ \int_{\varepsilon B_+} \prod_k dq^k_- \frac{\tilde{f}(Q^i, p^s)}{\Pi_i(2Q^i_+ Q^i_- - M^2_i + i\varepsilon)}. \quad (4)$$

From here on, the momenta of the lines $Q^i$ are assumed to be expressed in terms of the loop momenta $q^k$, $B_+$ is a sphere of a radius $L$ in the $q^k$-space, and $L$ depends on the external momenta. Now, using Statement 2 from Appendix I, we obtain

$$F_{\text{if}} = \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \lim_{\beta \to 0} \gamma \to 0 \int \prod_k dq^k_+ \int \prod_k dq^k_- \frac{\tilde{f}(Q^i, p^s) e^{-\gamma \sum Q^i_+ - \beta \sum Q^i_-}}{\Pi_i(2Q^i_+ Q^i_- - M^2_i + i\varepsilon)}. \quad (5)$$

To reduce the covariant Feynman integral to a form similar to (4), we introduce a quantity $\hat{F}$:

$$\hat{F} = \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \lim_{\beta \to 0} \gamma \to 0 \int \prod_k q^2 \frac{\tilde{f}(Q^i, p^s) e^{-\gamma \sum Q^i_+ - \beta \sum Q^i_-}}{\Pi_i(2Q^i_+ Q^i_- - M^2_i + i\varepsilon)}. \quad (6)$$

Let us prove that this quantity coincides with the result of the Lorentz-covariant calculation $F_{\text{cov}}$. To this end, we introduce the $\alpha$-representation in the Minkowski space of the propagator

$$\frac{z(Q^i)}{2Q^i_+ Q^i_- - M^2_i + i\varepsilon} = -iz \left( -i \frac{\partial}{\partial y_i} \right) \int_0^\infty e^{i\alpha_i (2Q^i_+ Q^i_- - M^2_i + i\varepsilon) + i(Q^i_+ y^i_+ + Q^i_- y^i_-) d\alpha_i} \bigg|_{y_i=0}. \quad (7)$$

Then we substitute (4) into (3). Due to the exponentials that cut off $q^k_+$, $q^k_-$ and $\alpha^i$ the integral over these variables is absolutely convergent. Therefore, one can interchange the integrations over $q^k_+$, $q^k_-$ and $\alpha^i$. As a result, we obtain the equality

$$\hat{F} = \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \lim_{\beta \to 0} \gamma \to 0 \int_0^\infty \prod_n d\alpha_i \varphi(\alpha_i, p^s, \gamma, \beta) e^{-\alpha \sum \alpha_i}, \quad (8)$$

where

$$\varphi(\alpha_i, p^s, \gamma, \beta) = (-i)^n \tilde{f} \left( -i \frac{\partial}{\partial y_i} \right) \times \int \prod_k q^2 \frac{e^{\sum_i \left[ i\alpha_i (2Q^i_+ Q^i_- - M^2_i) + i(Q^i_+ y^i_+ + Q^i_- y^i_-) - \gamma Q^i_+ - \beta Q^i_- \right]}}{\Pi_i(2Q^i_+ Q^i_- - M^2_i + i\varepsilon)} \bigg|_{y_i=0}. \quad (9)$$

For the Lorentz-covariant calculation in the $\alpha$-representation satisfying conditions (3), there is a known expression [3]

$$F_{\text{cov}} = \lim_{\varepsilon \to 0} \int_0^\infty \prod_n d\alpha_i \varphi_{\text{cov}}(\alpha_i, p^s) e^{-\alpha \sum \alpha_i}, \quad (10)$$
where

\[ \varphi_{\text{cov}}(\alpha_i, p^s) = (-i)^{\alpha} \left( -i \frac{\partial}{\partial y_i} \right) \times \]

\[ \lim_{\gamma, \beta \to 0} \int \prod_k d^2 q^k e^{\sum_i \left[ i\alpha_i(2Q^i_- - M^2_i) + (Q^i_+ y_i^+ + Q^i_- y_i^-) - \gamma Q^i_+ - \beta Q^i_- \right]} \Bigr|_{y_i=0}. \]

(11)

In Appendix 2, it is shown that in (8) the limits in \( \gamma \) and \( \beta \) can be interchanged, in turn, with the integration over \( \{\alpha_i\} \), and then with \( \tilde{f} \left( -i \frac{\partial}{\partial y_i} \right) \). After that, a comparison of relations (8), (9) and (10), (11), clearly shows that \( \hat{F} = F_{\text{cov}} \). Considering (6) and using Statement 1 from Appendix 1, we obtain the equality

\[ F_{\text{cov}} = \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \int \prod_k dq^k_{+} \int_{B_L} \prod_k dq^k_{-} \frac{\tilde{f}(Q^i, p^s)}{\Pi_i(2Q^i_+ Q^i_- - M^2_i + i\alpha)}. \]

(12)

Expression (12) differs from (4) only by the range of the integration over \( q^k_- \).

3. Reduction of the difference between the light-front and Lorentz-covariant Feynman integrals to a sum of configurations

Let us introduce a partition for each line,

\[ \left( \int_{-\infty}^{-\varepsilon} dQ_- + \int_{\varepsilon}^{\infty} dQ_- \right) = \left[ \int dQ_- + (-1) \int_{-\varepsilon}^{\varepsilon} dQ_- \right]. \]

(13)

We call a line with integration w.r.t. the momentum \( Q^-_i \) in the range \((-\varepsilon, \varepsilon)\) (before removing \( \delta \)-functions) a type-1 line, a line with integration in the range \((-\infty, -\varepsilon) \cup (\varepsilon, \infty)\) a type-2 line, and a line with integration over the whole range \((-\infty, \infty)\) a full line. In the diagrams, they are denoted as shown in Figs. 1a, b, and c, respectively.

Fig. 1: Notation for different types of lines in the diagrams: "a" is a type-1 line, "b" is a type-2 line, "c" is a full line, "d" is an \( \varepsilon \)-line, and "e" is a \( \Pi \)-line.

Let us substitute partition (13) into expression (4) for \( F_{lf} \) and open the brackets. Among the resulting terms, there is \( F_{\text{cov}} \) (expression (12)). We call the remaining terms "diagram configurations" and denote them by \( F_j \). Then we arrive at the relation \( F_{lf} - F_{\text{cov}} = \sum_j F_j \), where

\[ F_j = \lim_{\varepsilon \to 0} \lim_{\alpha \to 0} \int \prod_k dq^k_{+} \int_{\varepsilon \cap B_L} \prod_k dq^k_{-} \frac{\tilde{f}(Q^i, p^s)}{\Pi_i(2Q^i_+ Q^i_- - M^2_i + i\alpha)}, \]

(14)
and \( V^j \) is the region corresponding to the arrangement of full lines and type-1 lines in the given configuration.

Note that before taking the limit in \( \varepsilon \), Eqs. (12) and (14) can be used successfully: first, they are applied to a subdiagram and, then, are substituted into the formula for the entire diagram. This is admissible because, after the deformation of the contours described in the proof of Statement 1 from Appendix 1, the integral over the loop momenta \( \{ q^k \} \) of the subdiagram converges (after integration over the variables \( \{ q^k \} \) of this subdiagram) absolutely and uniformly with respect to the remaining loop momenta \( \{ q^{k'} \} \). Therefore, one can interchange the integrals over \( \{ q^k \} \) and \( \{ q^{k'} \} \).

Thus, the difference between the light-front and Lorentz-covariant calculations of the diagram is given by the sum of all of its configurations. A configuration of a diagram is the same diagram, but where each line is labeled as a full or type-1 line, provided that at least one type-1 line exists.

4. Behavior of the configuration as \( \varepsilon \to 0 \)

We assume that all external momenta \( p^s \) are fixed for the diagram in question and

\[ p^s \neq 0, \quad \sum_{s'} p^s' \neq 0, \quad (15) \]

where the summation is taken over any subset of external momenta; all of these momenta are assumed to be directed inward.

Let us consider an arbitrary configuration. We apply the term "\( \varepsilon \)-line" to all type-1 lines and those full lines for which integration over \( Q_- \) actually does not expand outside the domain \((-r\varepsilon, r\varepsilon)\), where \( r \) is a finite number (below, we explain when these lines appear). The remaining full lines are called \( \Pi \)-lines. In the diagrams, the \( \varepsilon \)-lines and \( \Pi \)-lines are denoted as shown in Figs. 1d and e, respectively. Note that the diagram can be drawn with lines "a" and "c" from Fig. 1 (this defines the configuration unambiguously), or with lines "d" and "e" (then the configuration is not uniquely defined).

If among the lines arriving at the vertex only one is full and the others are type-1 lines, this full line is an \( \varepsilon \)-line by virtue of the momentum conservation at the vertex. The remaining full lines form a subdiagram (probably unconnected). By virtue of conditions (13), there is a connected part to which all of the external lines are attached. All of the external lines of the remaining connected parts are \( \varepsilon \)-lines. Consequently, using Statement 1 from Appendix 1, we can see that integration over the internal momenta of these connected parts can be carried out in a domain of order \( \varepsilon \) in size, i.e., all of their internal lines are \( \varepsilon \)-lines. Thus, an arbitrary configuration can be drawn as in Fig. 2 and integral (14), with the corresponding integration domain, is associated with it.

Let us investigate the behavior of the configuration as \( \varepsilon \to 0 \). From here on, it is convenient to represent the propagator as

\[ \tilde{\bar{z}}(Q) = \frac{\zeta(Q)}{Q^2 - m^2 + i\varepsilon}, \quad \text{where} \quad \tilde{\bar{z}}(Q) = \frac{\zeta(Q)}{2Q_- + i\varepsilon/Q_+}. \quad (16) \]
rather than as \( (2) \). Then, in \( (1) \), \( M_2^i = m_i^2 + Q_i^2 \neq 0 \) and the function \( \tilde{f} \) is no longer a polynomial. If the numerator of the integrand consists of several terms, we consider each term separately (except when the terms arise from expressing the propagator momentum \( Q_i^- \) in terms of loop and external momenta).

We denote the loop momenta of subdiagram \( \Pi \) in Fig. 2 by \( q \) and the others by \( k \). We make following change of integration variables in \( (14) \):

\[
k^-_m \rightarrow \varepsilon k^m.
\]  

Then, the integration over \( k^m_+ \) goes within finite limits independent of \( \varepsilon \). We denote the power of \( \varepsilon \) in the common factor by \( \tau \) (it stems from the volume elements and the numerators when the transformation \( (17) \) is made). The contribution to \( \tau \) from the expression \( 1/(2Q^- + i\varepsilon/Q^+ \) (Eq. \( (16) \)), which is related to the \( \varepsilon \)-line, is equal to \(-1\). We divide the domain of integration over \( k^m_+ \) and \( q^l_+ \) into sectors such that the momenta of all full lines \( Q^i_+ \) have the same sign within one sector.

In Statement 1 of Appendix 1, it is shown that for each sector, the contours of integration over \( q^l_+ \) and \( k^m_+ \) can be bent in such a way that absolute convergence in \( q^l_+ \), \( k^m_+ \), \( q^l_- \) and \( k^m_- \) takes place. Since, in this case, the momenta \( Q^-_i \) of \( \Pi \)-lines are separated from zero by an \( \varepsilon \)-independent constant, the corresponding \( \Pi \)-line-related propagators and factors from the vertices can be expanded in a series in \( \varepsilon \). This expansion commutes with integration.

It is also clear that the denominators of the propagators allow the following estimates under an infinite increase in \( |Q^+| \):

\[
\left| \frac{1}{2Q^+_+Q^- - M^2 + i\varepsilon} \right| < \begin{cases} 
\frac{1}{c |Q^+|} & \text{for } \Pi \text{-lines}, \\
\frac{1}{\tilde{c} \varepsilon |Q^+|} & \text{for } \varepsilon \text{-lines},
\end{cases}
\]

(18)

(19)

Here \( c \) and \( \tilde{c} \) are \( \varepsilon \)-independent constants. Note that for fixed finite \( Q^+ \), the estimated expressions are bounded as \( \varepsilon \rightarrow 0 \). After transformation \( (17) \) and release of the factor \( \frac{1}{\varepsilon} \) (in
accordance with what was said about the contribution to \( \tau \), the \( \varepsilon \)-line-related expression from (16) becomes

\[
\frac{1}{2Q_- + i\varepsilon/\Lambda_+} \to \frac{1}{2Q_- + i\varepsilon/(Q_+\varepsilon)} \leq \frac{1}{2|Q_-|},
\]

where a \( Q_+ \)-independent quantity was used for the estimate (this quantity is meaningful and does not depend on \( \varepsilon \) because the value of \( Q_- \) is separated from zero by an \( \varepsilon \)-independent constant).

We integrate first over \( q_\perp^l, k_\perp^m \) within one sector and then over \( q_\perp^l, k_\perp^m \) (the latter integral converges uniformly in \( \varepsilon \)). Let us examine the convergence of the integral over \( q_\perp^l, k_\perp^m \) with canceled denominators of the \( \varepsilon \)-lines (which is equivalent to estimating expressions (19) by a constant). If it converges, then the initial integral is obviously independent of \( \varepsilon \) and the contribution from this sector to the configuration is proportional to \( \varepsilon \tau \).

Let us show that if it diverges with a degree of divergence \( \alpha \), the contribution to the initial integral is proportional to \( \varepsilon^{-\alpha} \) up to logarithmic corrections. To this end, we divide the domain of integration over \( q_\perp^l, k_\perp^m \) into two regions: \( U_1 \), which lies inside a sphere of radius \( \Lambda/\varepsilon \) (\( \Lambda \) is fixed), and \( U_2 \), which lies outside this sphere (recall that in our reasoning, we deal with each sector separately). Now we estimate (18) (like (19)) in terms of \( \frac{1}{\varepsilon|Q_+|} \) (which is admissible) and change the integration variables as follows:

\[
q_\perp^l \to \frac{1}{\varepsilon} q_\perp^l, \quad k_\perp^m \to \frac{1}{\varepsilon} k_\perp^m.
\]

After \( \varepsilon \) is factored out of the numerator and the volume element, the integrand becomes independent of \( \varepsilon \). Thus, the integral converges.

One can choose such \( \Lambda \) (independent of \( \varepsilon \)) that the contribution from the domain \( U_2 \) is smaller in absolute value than the contribution from the domain \( U_1 \). Consequently, the whole integral can be estimated via the integral over the finite domain \( U_1 \). Now we make an inverse replacement in (20) and estimate (19) by a constant (as above). Since the size of the integration domain is \( \Lambda/\varepsilon \) and the degree of divergence is \( \alpha \), the integral behaves as \( \varepsilon^{-\alpha} \) (up to logarithmic corrections), q.e.d. This reasoning is valid for each sector and, thus, for the configuration as a whole. Obviously,

\[
\alpha = \max_r \alpha_r,
\]

where \( \alpha_r \) is the subdiagram divergence index and the maximum is taken over all subdiagrams \( D_r \) (including unconnected subdiagrams for which \( \alpha_r \) is the sum of the divergence indices of their connected parts). In the case under consideration, \( \alpha_r = \omega_r^+ + \nu^r \), where \( \nu^r \) is the number of internal \( \varepsilon \)-lines in the subdiagram \( D_r \). The quantities \( \omega_r^+ \) are the UV divergence indices of the subdiagram \( D_r \) w.r.t. \( Q_+ \).

Above, we introduced a quantity \( \tau \), which is equal to the power of \( \varepsilon \) that stems from the numerators and volume elements of the entire configuration. We can write \( \tau = \omega^- - \mu^r + \nu^r + \eta^r \), where \( \mu^r \) is the index of the UV divergence in \( Q_- \) of a smaller subdiagram.
(probably, a tree subdiagram or a nonconnected one) consisting of $\Pi$-lines entering $D_r$. The term $\eta^r$ is the power of $\varepsilon$ in the common factor, which, during transformation (17), stems from the volume elements and numerators of the lines that did not enter $D_r$. (It is implied that the integration momenta are chosen in the same way as when calculating the divergence indices of $D_r$.) Then, up to logarithmic corrections, we have

$$F_j \sim \varepsilon^\sigma, \quad \sigma = \min_r (\tau, \omega_r^r - \omega_r^r - \mu^r + \eta^r).$$

(22)

Consequently, for $\varepsilon \to 0$, the configuration is equal to zero if $\sigma > 0$. Relation (22) allows all essential configurations to be distinguished.

5. Correction procedure and analysis of counter-terms

We want to build a corrected light-front Hamiltonian $H_{lf}^{\text{cor}}$ with the cutoff $|Q_i^\perp| > \varepsilon$, which would generate Green’s functions that coincide in the limit $\varepsilon \to 0$ with covariant Green’s functions within the perturbation theory. We begin with a usual canonical Hamiltonian in the light-front coordinates $H_{lf}$ with the cutoff $|Q_i^\perp| > \varepsilon$. We imply that the integrands of the Feynman diagrams derived from this light-front Hamiltonian coincide with the covariant integrands after some resummation [1, 2, 3]. However, a difference may arise due to the various methods of doing the integration, e.g., due to different auxiliary regularizations. As shown in Sec. 3, this difference (in the limit $\varepsilon \to 0$) is equal to the sum of all properly arranged configurations of the diagram. One should add such correcting counter-terms to $H_{lf}$, which generates additional "counter-term" diagrams, that reproduce nonzero (after taking limit w.r.t. $\varepsilon$) configurations of all of the diagrams. Were we able to do this, we would obtain the desired $H_{lf}^{\text{cor}}$. In fact, we can only show how to seek the $H_{lf}^{\text{cor}}$ that generates the Green’s functions coinciding with the covariant ones everywhere except the null set in the external momentum space (defined by condition (15)). However, this restriction is not essential because this possible difference does not affect the physical results.

Our correction procedure is similar to the renormalization procedure. We assume that the perturbation theory parameter is the number of loops. We carry out the correction by steps: first, we find the counterterms to the Hamiltonian that generate all nonzero configurations of the diagrams up to the given order and, then, pass to the next order. We take into account that this step involves the counter-term diagrams that arose from the counter-terms added to the Hamiltonian for lower orders. Thus, at each step, we introduce new correcting counter-terms that generate the difference remaining in this order. Let us show how to successfully look for the correcting counter-terms.

We call a configuration nonzero if it does not vanish as $\varepsilon \to 0$. We call a nonzero configuration "primary" if $\Pi$ is a tree subdiagram in it (see Fig. 2). Note that for this configuration, breaking any $\Pi$-line results in a violation of conditions (15); then, the resulting diagram is not a configuration. We say that the configuration is changed if all of the $\Pi$-lines in the related integral (14) are expanded in series in $\varepsilon$ (see the reasoning above Eq. (18) in Sec. 4) and only those terms that do not vanish in the limit $\varepsilon \to 0$ after
the integration are retained. As mentioned above, developing this series and integration are interchangeable operations. Thus, in the limit \( \varepsilon \to 0 \), the changed and unchanged configurations coincide. Therefore, we always require that the Hamiltonian counter-terms generate changed configurations, as this simplifies the form of the counter-terms. Using additional terms in the Hamiltonian, one can generate only counter-term diagrams, which are equal to zero for external momenta meeting the condition \(|p^e_s| < \varepsilon\), because with the cutoff used (see the Introduction), the external lines of the diagrams do not carry momenta with \(|p^e_s| < \varepsilon\). We bear this in mind in what follows.

We seek counter-terms by the induction method. It is clear that, in the first order in the number of loops, all nonzero configurations are primary. We add the counter-terms that generate them to the Hamiltonian. Now, we examine an arbitrary order of perturbation theory. We assume that in lower orders, all nonzero configurations that can be derived from the counter-terms, accounting for the above comment, have already been generated by the Hamiltonian.

Let us proceed to the order in question. First, we examine nonzero configurations with only one loop momentum \(k\) and a number of momenta \(q\) (see the notation above Eq. (17)). We break the configuration lines one by one without touching the other lines (so that the ends of the broken lines become external lines). The line break may result in a structure that is not a configuration (if conditions (15) are violated); a line break may also result in a zero configuration or in a nonzero configuration. If the first case is realized for each broken line, then the initial configuration is primary and it must be generated by the counter-terms of the Hamiltonian in the order under consideration. If breaking of each line results in either the first or the second case, we call the initial configuration real and it must be also generated in this order.

Assume that breaking a line results in the third case. This means that the resulting configuration stems from counter-terms in the lower orders. Then, after restoration of the broken line (i.e., after the appropriate integration), it turns out that the counter-terms of the lower orders have generated the initial configuration (we take into account the comment on successive application of Eq. (14); see the end of Sec. 3) with the following distinctions: (i) the broken line (and, probably, some others, if a nonsimply connected diagram arises after breaking the line) is not a \(\Pi\)-line but a type-2 line, due to the conditions \(|p^e_s| > \varepsilon\); (ii) if, after restoration of the broken line, the behavior at small \(\varepsilon\) becomes worse (i.e., \(\sigma\) decreased), then fewer terms than are necessary for the initial configuration were considered in the above-mentioned series in \(\varepsilon\). We expand these arising type-2 lines by formula (13) and obtain a term where all of these lines are replaced by \(\Pi\)-lines or other terms where some (or all) of these lines have become type-1 lines. In the latter case, one of the momenta \(q\) becomes the momentum \(k\). We call these terms "repeated parts of the configuration" and analyze them together with the configurations that have two momenta \(k\). In the former case, we obtain the initial configuration up to distinction (ii). We add a counter-term to the Hamiltonian that compensates this distinction (the counter-term diagrams generated by it are called the compensating diagrams).

If there is only one line for which the third case is realized, it turns out that, in the given order, it is not necessary to generate the initial configuration by the counter-terms,
except for the compensating addition and the repeated part that is considered at the next step. If there are several lines for which the third case is realized, the initial configuration is generated in lower orders more than once. For compensation, it should be generated (with the corresponding numerical coefficient and the opposite sign) by the Hamiltonian counter-terms in the given order. We call this configuration a secondary one. Next, we proceed to examine configurations with two momenta \( k \) and so on up to configurations with all momenta \( k \), which are primary configurations.

Thus, the configurations to be generated by the Hamiltonian counter-terms can be primary (not only the initial primary configurations but also the repeated parts analogous to them, called primary-like), real, compensating, and secondary. If the theory does not produce either the loop consisting only of lines with \( Q_+ \) in the numerator (accounting for contributions from the vertices) or a line with \( Q_+^n \) in the numerator for \( n > 1 \), then real configurations are absent because a line without \( Q_+ \) in the numerator can always be broken without increasing \( \sigma \) (see Eq. (22)). It is not difficult to demonstrate that if each appearing primary, real, and compensating configuration has only two external line, then there are no secondary configurations at all.

The dependence of the primary configuration on external momenta becomes trivial if its degree of divergence \( \alpha \) is positive, the maximum in formula (21) is reached on the diagram itself, and \( \sigma = 0 \). Then, only the first term is taken into account in the above-mentioned series. Thus, not all of the \( \Pi \)-line-related propagators and vertex factors depend on \( k^m \) and they can be pulled out of the sign of the integral w.r.t. \( \{k^m\} \) in (14). We then obtain

\[
F_j^{\text{prim}} = \lim_{\varepsilon \to 0} \lim_{\sigma \to 0} \int \prod_m dk^m_+ \frac{j'(k^m, p^s)}{\prod_i (2Q_i^+ Q_i^- - M_i^2 + i\varepsilon)} \times \prod_m dk^m_- \frac{j''(k^m)}{\prod_i (2Q_i^+ Q_i^- - M_i^2 + i\varepsilon)},
\]

where \( V_\varepsilon \) is a domain of order \( \varepsilon \) in size. Let us carry out transformations (17) and (20). For the denominator of the \( \Pi \)-line, we obtain

\[
\frac{1}{2(\frac{1}{\varepsilon} \sum k_+ + \sum p_+)(\sum p_-) - M^2 + i\varepsilon} \to \frac{\varepsilon}{2(\sum k_+)(\sum p_-)}.
\]

Here we neglect terms of order \( \varepsilon \) in the denominator because the singularity at \( k^m_+ = 0 \) is integrable under the given conditions for \( \alpha \) and everything can be calculated in zero order in \( \varepsilon \) at \( \sigma = 0 \). Thus, the dependence on external momenta can be completely collected into an easily obtained common factor.

6. Application to the Yukawa model

The Yukawa model involves diagrams that do not satisfy condition (3). These are displayed in Figs. 3a and b. We have \( \omega_1 = 0 \) for diagram "a" and \( \omega_+ = 0 \) for diagram "b".
Nevertheless, these diagrams can be easily included in the general scheme of reasoning. To this end, one should subtract the divergent part, independent of external momenta, in the integrand of the logarithmically divergent (in two-dimensional space, with fixed internal transverse momenta) diagram ”a”. We obtain an expression with $\omega_+ < 0$ (i.e., which converges in two-dimensional space) and $\omega_+ = 0$, as in diagram ”b”. This means that the integral over $q_+$ converges only in the sense of the principal value (and it is this value of the integral that should be taken in the light-front coordinates to ensure agreement with the stationary noncovariant perturbation theory). This value can be obtained by distinguishing the $q_+$-even part of the integrand.

Two approaches are possible. One is to introduce an appropriate regularization in transverse momenta and to imply integration over them; then, it is convenient to distinguish the part that is even in four-dimensional momenta $q$. The other is to keep all transverse momenta fixed; then, the part that is even in longitudinal momenta $q_\parallel$ can be released. For the Yukawa theory, we use the first approach. For the transverse regularization, we use a ”smearing” of vertices, which is equivalent to dividing each propagator by $1 + Q_\perp^2/\Lambda_\perp^2$. In four-dimensional space, diagram ”a” diverges quadratically. Under introduction and subsequent removal of the transverse regularization, the divergent part, which was previously subtracted from this diagram, acquires the form $C_1 + C_2 p_\perp^2$.

After separating the even part of the regularized expression, we fix all of the transverse momenta again. Then it turns out that diagrams ”a” and ”b” in Fig. 3 meet conditions (3) and one can show that after all of the operations mentioned, the exponent $\sigma$ (see (22)) does not decrease for any of their configurations. Hence, they can be included in the general scheme without any additional corrections.

Let us first analyze the primary configurations (see the definition in Sec. 5). In the numerators, $k_-$ appears only in the zero or one power and there are no loops where the numerators of all of the lines contain $k_-$. Consequently, one always has $\tau > 0$, $\mu^r \leq 0$, and $\eta^r \geq 0$ (see the definitions in Sec. 4). Analyzing the properties of the expression $\omega^- - \omega_+^r$ for the Yukawa model diagrams, we conclude from (22) that $\sigma \geq 0$ always holds. The general form of the nonzero primary configurations with $\sigma = 0$ is depicted in Fig. 4. Note that they are all configurations with two external lines.

Further, it is clear that there are no nonzero real configurations (see the comment at the end of Sec. 3), and it can be shown by induction that there are no nonzero compensating or secondary configurations either (the definitions are given in Sec. 4 also). Thus, only
primary or primary-like configurations can be nonzero and all of them have the form shown in Fig. 4. It can be shown that their degree of divergence \( \alpha \) is positive and the maximum in formula (21) is reached for the diagram itself. Thus, the reasoning above and below formula (23) applies to them. Then, denoting the configurations displayed in Figs. 4a-d by \( D_a - D_d \), we arrive at the equalities \( D_a = \frac{\gamma^+}{p_-} C_a \), \( D_b = \frac{\gamma^+}{p_-} C_b \), \( D_c = C_c \) and \( D_d = C_d \), where the expressions \( C_a - C_d \) depend only on the masses and transverse momenta, but not on the external longitudinal momenta, and have a finite limit as \( \varepsilon \to 0 \).

Now we assume that \( D_a - D_d \) are not single configurations but are the sums of all configurations of the same form and that integration over the internal transverse momenta has already been carried out, (with the above-described regularization). In four-dimensional space, the diagrams \( D_a \) and \( D_b \) diverge linearly while \( D_c \) and \( D_d \) diverge quadratically.

Therefore, because of the transverse regularization, the coefficients \( C_c \) and \( C_d \) in the limit for removing this regularization take the form \( C_1 + C_2 p_\perp^2 \), where \( C_1 \) and \( C_2 \) do not depend on the external momenta (neither do \( C_a \), \( C_b \)). Thus, to generate all nonzero configurations by the light-front Hamiltonian, only the expression

\[
H_c = \tilde{C}_1 \varphi^2 + \tilde{C}_2 p_\perp^2 \varphi^2 + \tilde{C}_3 \bar{\psi} \frac{\gamma^+}{p_-} \psi,
\]

should be added, where \( \varphi \) and \( \psi \) are the boson and fermion fields, respectively, and \( \tilde{C}_i \), are the constant coefficients.

Comparing (24) with the initial canonical light-front Hamiltonian, one can easily see that the found counter-terms are reduced to a renormalization of various terms of the Hamiltonian (in particular the boson mass squared and the fermion mass squared without changing the fermion mass itself). The explicit Lorentz invariance is absent, which compensates the violation of the Lorentz invariance inherent, in the light-front formalism.

Note that in the framework of the second approach, mentioned at the beginning of this section, one can obtain the same results. The only difference is that in two-dimensional space, the contributions from the configurations displayed in Fig. 3 would additionally depend on external transverse momenta. However, this dependence disappears after integration over internal transverse momenta with the introduction and subsequent removal
of an appropriate regularization.

In the Pauli Villars regularization, it is easy to verify that the expression $\omega_r^r - \omega_r^r - \mu^r + \eta^r$ from (22) increases. This is because the number of terms in the numerators of the propagator increases. Then, the contribution from the $\varepsilon$-lines does not change, while the $\Pi$-lines belonging to $D_r$ make zero contribution to $\omega_r^r - \omega_r^r$ and $\eta^r$, but $-1$ contribution to $\mu^r$. Since $\tau > 0$, this regularization makes it possible to meet the condition $\sigma > 0$ for the configurations that were nonzero (one additional boson field and one additional fermion field are enough). Then it turns out that the canonical light-front Hamiltonian cannot be corrected at all.

7. Application to gauge theories

Let us consider a gauge theory (e.g., QED or QCD) in the gauge $A_\perp = 0$. The boson propagator in the Mandelstam-Leibbrandt prescription has the form

$$\frac{1}{Q^2 + i\varepsilon} \left( g_{\mu\nu} - \frac{Q_\mu \delta_\nu^+ Q_+ + Q_\nu \delta_\mu^+ Q_+}{2Q_+ Q_- + i\varepsilon} \right).$$

All of the above reasoning was organized such that it could be applied to a theory like this (with fixed transverse momenta $Q_\perp \neq 0$). It turns out that there are nonzero configurations with arbitrarily large numbers of external lines. An example of such a configuration is given in Fig. 5.

Fig. 5: Nonzero configuration with an arbitrarily large number of external lines in a gauge theory. The symbols $\gamma^\perp$ on the lines and the symbols $+$ or $\perp$ by the vertices indicate that the corresponding terms $\gamma^+$ or $\gamma^\perp$ are taken in the numerators of propagators and in the vertex factors.

Indeed, using formula (22), we can see that for the configuration in Fig. 5, $\tau = 0$ and, thus, $\sigma \leq 0$, i.e., this is a nonzero configuration. It is also clear that introduction of the Pauli-Villars regularization does not improve the situation because it does not affect $\tau$.

Thus, within the framework of the above-described method for correcting the canonical light-front Hamiltonian of the gauge theory, an infinite number of counter-terms must be added to the Hamiltonian. Note, however, that the formulated conditions for the vanishing of the configuration are sufficient, but, generally speaking, not necessary. Because of this
and because of the possible cancellation of different configurations after integration w.r.t. transverse momenta, the number of necessary counter-terms may be smaller.

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Appendix 1

Statement 1. If conditions (3) are satisfied, then, for fixed external momenta \( p^s \) and \( p^s_\neq 0 \ \forall s \), the equality

\[
\lim_{\beta \to 0} \lim_{\gamma \to 0} \int \prod_k dq^k_+ \int \prod_k dq^k_- \frac{f(Q^i, p^s) e^{-\gamma \sum_i Q^i_+^2 - \beta \sum_i Q^i_-^2}}{\prod_i (2Q^i_+ Q^i_- - M_i^2 + i\alpha)} = \int \prod_k dq^k_+ \int \prod_k dq^k_- \frac{\tilde{f}(Q^i, p^s)}{\prod_i (2Q^i_+ Q^i_- - M_i^2 + i\alpha)}, \tag{A.1.1}
\]

holds while the expressions appearing in (A.1.1) exist and the integral over \( \{q^k_-\} \) on the right-hand side is absolutely convergent. It is assumed that the momenta of lines \( Q^i \) are expressed in terms of loop momenta \( q^k \), \( V_\varepsilon \) is the domain corresponding to the presence of full lines, type-1 lines, and type-2 lines (the definitions are given following formula (13)), \( B_L \) is the sphere of radius \( L \), where \( L \geq S \max_s |p^s_-| \), and \( S \) is a number depending on the diagram structure.

Let us prove the statement. For each type-1 line in (A.1.1), we perform the following partitioning:

\[
\int_{-\varepsilon}^{\varepsilon} dQ^i_- = \left( \int_{-\varepsilon}^{\varepsilon} dQ^i_- + (-1) \left( \int_{-\infty}^{-\varepsilon} dQ^i_- + \int_{\varepsilon}^{\infty} dQ^i_- \right) \right).
\]

Then both sides of Eq. (A.1.1) become the sum of expressions of the same form in which, however, the domain \( V_\varepsilon \) corresponds to the presence of only full and type-2 lines. It is clear that by proving the statement for this \( V_\varepsilon \): (which is done below), we prove the original statement as well.

Let \( \tilde{B} \) be a domain such that the surfaces on which \( Q^i_- = 0 \) are not tangent to the boundary \( \tilde{B} \). First, we prove that in the expression

\[
\int \prod_k dq^k_+ \int \prod_k dq^k_- \frac{\tilde{f}(Q^i, p^s) e^{-\beta \sum_i Q^i_-^2}}{\prod_i (2Q^i_+ Q^i_- - M_i^2 + i\alpha)} \tag{A.1.2}
\]

the integral over \( \{q^k_+\} \) is absolutely convergent (here the integral over \( \{q^k_-\} \) is finite because \( \alpha > 0, \beta > 0 \)). This becomes obvious (considering conditions (3) and the fact that, in
type-2 lines, the momentum $Q^i_-$ is separated from zero) if the contours of the integration over $\{q^k_+\}$ can be deformed in such a way that the momenta $Q^i_-$ of the full lines are separated from zero by a finite quantity (within the domain $V_\varepsilon \cap B$). In this case, we can repeat the well-known Weinberg reasoning \[8\]. What can prevent deformation is either a "clamping" of the contour or the point $Q^i_- = 0$ falling on the integration boundary.

Let us investigate the first alternative. We divide the domain of integration over $q^k_+$ into sectors such that the momenta of all full lines $Q^i_+$ have a constant sign within one sector. Let us examine one sector. We take a set of full lines whose $Q^i_-$ may simultaneously vanish. In the vicinity of the point where $Q^i_-$ from this set vanish simultaneously, we bend the contours of the integration over $\{q^k_+\}$ such that these contours pass through the points $Q^i_- = i B^i$ and the momenta $Q^i_-$ of the type-2 lines do not change. Let $B^i$ be such that $B^i Q^i_+ \geq 0$ for the lines from the set (for $Q^i_+$ from the sector under consideration). It is easy to check that this bending is possible. (Since the contours of integration over $q^k_-$ are bent and $Q^i_-$ are expressed in terms of $q^k_-$, one should only check that such $b^k$ exist, where the necessary $B^i$ are expressed in the same way, i.e., that $B^i$ obey the conservation laws and flow only along the full lines). With this bending, rather small in relation to the deviation and the size of the deviation region, the contours do not pass through the poles because, for the denominator of each line from the set in question, we have

$$\left(2Q^i_+Q^i_- + M^2_i + i\varepsilon q^i_+ - M^2_i + i\varepsilon q^i_- \right) \rightarrow \left(2Q^i_+ \left(Q^i_- + iB^i\right) - M^2_i + i\varepsilon \right), \quad Q^i_+ B^i \geq 0,$$

and for the other denominators, the bending takes place in a region separated from the point where the corresponding momenta $Q^i_-$ are equal to 0. Repeating the reasoning for all sets, we can see that there is no contour "clamping".

The other alternative is excluded by the above condition for $\tilde{B}$. To make this clear, one should introduce such coordinates $\xi^\alpha$ in the $q^k$-space that the boundary of the domain $\tilde{B}$ is determined by the equation $\xi^1 = \alpha = \text{const}$ and then argue as above for the coordinates $\xi^\alpha$ with $\alpha \geq 2$.

After bending the contours, integral (A.1.2) is absolutely convergent in $q^k_+$, $q^k_-$ if the integration in $q^k_+$ is carried out within the sector under consideration. On pointing out that the result, of internal integration in (A.1.2) does not depend on the bending, we add the integrals over all sectors and conclude that (A.1.2) converges in $\{q^k_+\}$ absolutely.

Now let us prove that if $\tilde{B}$ is a quite small, finite vicinity of the point $\{\dot{q}^k_+\}$ that lies outside the sphere $B_L$, then expression (A.1.2) is equal to zero. We consider the momentum $Q^i_-$ of one line. Flowing along the diagram, it can ramify or it can merge with other momenta. Clearly, two situations are possible: either it flows away completely through external lines, or, probably, after long wandering, part of it, $\dot{Q}_-$, makes a complete loop. The former situation is possible only if $|Q^i_-| \leq \sum |p^i_-|$, where all external momenta leaving the diagram (but not entering it) are summed. Obviously, $S$ can be chosen such that for $\{q^k_+\}$ from $\dot{B}$, a line exists whose momentum violates this condition.

The latter situation results in the existence of a loop, where the inequality $Q^i_- > \dot{Q}_-$ holds for all momenta of its lines and the positive direction of the momenta is along the loop. Then the integral over $q^k_+$ of the loop in question can be interchanged with the
integrals over \( \{ q_i^k \} \) (because it is absolutely and uniformly convergent for all \( q_i^k \)) and the residue formula can be used to perform this integration. Since, for the loop in question, the momenta \( Q^i_\neq \) of the lines of this loop are separated from zero and are of the same sign, the result is zero. This has a simple physical meaning. If we pass to stationary noncovariant perturbation theory, we find that only quanta with positive \( Q^i_\neq \) can exist. In this case, external particles with positive \( p^\neq \) are incoming and those with negative \( p^\neq \) are outgoing. Then, the momentum conservation law favors the occurrence of the first situation.

The entire outside space for \( \tilde{B} \) can be composed of the above domains \( B_L \) (everything converges well at infinity due to the factor \( \exp(-\beta \sum_i Q^i_\neq^2) \)). Thus, on the left-hand side of (A.1.1), one can substitute the integration domain \( V_\varepsilon \cap B_L \) for \( V_\varepsilon \), set the limit in \( \gamma \) because of its absolute convergence, and also set the limit in \( \beta \) under the integration sign because the domain of the integration over \( \{ q_i^k \} \) is bounded. Thus, we obtain the right-hand side. The statement is proved.

Statement 2. If \( V_\varepsilon \) corresponds to the presence of type-2 lines alone, then, under the same conditions as in Statement 1, the equality

\[
\int_{V_\varepsilon} \prod_k dq^+_k \int_{\{ q_i^k \}} dq^+_k \frac{\tilde{f}(Q^i, p^i)}{\prod_i(2Q^i_+Q^i_- - M^2_i + i\varepsilon)} = \int_{V_\varepsilon \cap B_L} \prod_k dq^+_k \frac{\tilde{f}(Q^i, p^i)}{\prod_i(2Q^i_+Q^i_- - M^2_i + i\varepsilon)}.
\]

is valid.

The proof of this statement is analogous to the second part of the proof of Statement 1.

Appendix 2

Statement. If conditions (3) are satisfied, the limits in \( \gamma \) and \( \beta \) in (3) can be interchanged (in turn) with the sign of the integral over \( \{ \alpha_i \} \) and then with \( \int (\frac{\partial}{\partial y^i}) \).

To prove this, we define the vectors \( \{ q^+_{1, q^1_-, q^1_+, q^1_-} \} \equiv S, \{ Q^1_+, Q^1_-, \ldots, Q^n_+, Q^n_- \} \equiv \mu S + P, \) and \( \{ y^+_1, y^-_1, \ldots, y^+_n, y^-_n \} \equiv Y, \) where the vector \( P \) is built only from external momenta and \( \mu \) is an \( l \times n \) matrix of rank \( l, \mu^2_{2k-1} = \mu^2_{2k-1} = 0, \mu^2_{2k} = \mu^2_{2k-1} \). Next, we introduce the following notation:

\[
\tilde{\Lambda}_i = \begin{pmatrix} \gamma & -i\alpha_i \\
-\alpha_i & \beta \end{pmatrix}, \quad \Lambda = \text{diag}\{ \tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_n \}, \quad A = \mu^t \Lambda \mu,
\]

\[
B = \mu^t \Lambda P - \frac{1}{2} i \mu^t Y, \quad C = -P^t \Lambda P + i Y^t P - i \sum_i \alpha_i M^2_i.
\]

Then it follows from (3) that

\[
\hat{\phi}(\alpha_1, p^8, \gamma, \beta) = (-i)^n \tilde{f} \left( -i \frac{\partial}{\partial y_i} \right) \int d^{2l} S e^{-S^t A S - 2B^t S + C} \bigg|_{y_i=0} =
\]
\[ (-i)^n \tilde{f} \left( -i \frac{\partial}{\partial y_i} \right) e^{B^*A^{-1}B+C} \frac{\pi^l}{\sqrt{\det A}} \bigg|_{y_i=0}. \]  

(A.2.1)

The function \( \tilde{f} \) is a polynomial and we consider each of its terms separately. Up to a factor, each term has the form \( \frac{\partial}{\partial y_{i_1}} \ldots \frac{\partial}{\partial y_{i_r}} \). These derivatives act on \( C \) and \( B \). The action on \( C \) results in the constant factor \( iN^tP \), the action on \( B \) results in the factor \(-\frac{1}{2}i N^t \mu A^{-1}B \) or \(-\frac{1}{4}N_1^t \mu A^{-1} \mu N_2 \) (the latter is the result of the action of two derivatives; \( N, N_1, \) and \( N_2 \) are constant vectors).

It is necessary to prove the correctness of the following three procedures: (i) setting the limit in \( \gamma \) under the integral sign for fixed \( \beta > 0 \); (ii) setting the limit in \( \beta \) for \( \gamma = 0 \); (iii) setting the limits in \( \gamma \) and \( \beta \) under the signs of differentiation with respect to \( Y \). In cases (i) and (ii), one must obtain the bounds

\[ |\hat{\varphi}(\alpha_i, p^s, \gamma, \beta)| \leq \varphi'(\alpha_i, p^s, \beta), \]  

(A.2.2)

\[ |\hat{\varphi}(\alpha_i, p^s, 0, \beta)| \leq \varphi''(\alpha_i, p^s), \]  

(A.2.3)

where \( \varphi' \) and \( \varphi'' \) are functions integrable (for \( \varphi' \) if \( \beta > 0 \)) in any finite domain over \( \alpha_i \), with \( \alpha_i \geq 0 \). Then, for case (i), we have

\[ |\hat{\varphi}(\alpha_i, p^s, \gamma, \beta) e^{-\varphi' \sum \alpha_i}| \leq \varphi'(\alpha_i, p^s, \beta) e^{-\varphi' \sum \alpha_i}, \]

i.e., a limit on the integrated function arises, and, thus, the limit in \( \gamma \) can be put under the integral sign. The situation is similar for case (ii). It is evident from (A.2.1) that the function \( \hat{\varphi} \) can be singular only if the eigenvalues of matrix \( A \) become zero. On finding the lower bound of these eigenvalues, one can prove through rather long reasoning that bounds (A.2.2), (A.2.3) exist if condition (3) is satisfied.

After the limits in \( \gamma \) and \( \beta \) are put under the integral sign, it is not difficult to interchange them with the differentiation with respect to \( Y \). One need do it only for \( \alpha_i > 0 \) (for each \( i \)) and, in this case, one can show that the eigenvalues of the matrix \( A \) are nonzero and \( \hat{\varphi} \) is not singular.

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