Resonant Bose Condensate: Analog of Resonant Atom

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Abstract

The resonant formation of nonlinear coherent modes in trapped Bose-Einstein condensates is studied. These modes represent nonground-state Bose condensates. The methods of describing the spectrum of the nonlinear modes are discussed. The latter can be created by modulating the trapping potential with a frequency tuned close to the transition frequency between the two chosen modes. The requirement that the transition amplitudes be smaller than the transition frequency implies a constraint on the number of particles that can be transferred to an excited mode. The resonant Bose condensate serves as a collective analog of a single resonant atom. Such a condensate, displaying the coherent resonance, possesses several interesting features, among which are: mode locking, critical dynamics, interference patterns, interference current, atomic squeezing, and multiparticle entanglement.
1 Introduction

Dilute atomic gases, trapped and cooled down to temperatures when almost all atoms are in the Bose-condensed state, are described by the Gross-Pitaevskii equation (see reviews [1–3]). The mathematical structure of the latter is that of the nonlinear Schrödinger equation which, due to the presence of the trapping potential, possesses a discrete spectrum. The equilibrium Bose-Einstein condensate corresponds to the ground state associated with the lowest energy level of the spectrum. If not the ground but an excited state would be macroscopically populated, this would correspond to a nonground-state Bose condensate. The possibility of creating such a nonequilibrium condensate was advanced in Ref. [4]. This can be done by applying an alternating field with a frequency tuned to the transition frequency between the ground state and a chosen excited state. The latter states are called the nonlinear coherent modes and they are described by the stationary solutions to the Gross-Pitaevskii equation. The properties of these modes have been considered theoretically in several publications [4–13] and a nonlinear dipole mode was observed experimentally [14]. A known example of such a mode is a vortex that can be formed by means of a rotating laser spoon [15,16].

The aim of the present publication is threefold. First, we give a survey of the problems in characterizing the nonlinear coherent modes, especially in the accurate calculations of their spectrum. Second, we stress the analogy of the resonant Bose condensate with a resonant atom. And, third, we show that, because of its coherent collective nature, the resonant Bose condensate possesses a number of properties distinguishing it from a single resonant atom. We describe several interesting novel effects that can be observed in a nonequilibrium Bose condensate.

2 Nonlinear Coherent Modes

First of all, it is necessary to show how one could get an accurate description of nonlinear coherent modes. We assume that the system of $N$ Bose atoms is confined in a trap, with a trapping potential $U(r)$ tending to infinity at large $r \equiv |\mathbf{r}|$,

$$U(r) \to \infty \quad (r \to \infty).$$

Due to the presence of this confining potential, the spectrum of the stationary Gross-Pitaevskii equation is discrete, being defined by the eigenvalue problem

$$\hat{H}[\varphi_n](\mathbf{r}) = E_n \varphi_n(\mathbf{r}),$$

with the nonlinear Hamiltonian

$$\hat{H}[\varphi] \equiv -\frac{\hbar^2 \nabla^2}{2m_0} + U(\mathbf{r}) + N \int \Phi(\mathbf{r} - \mathbf{r}') |\varphi(\mathbf{r}')|^2 d\mathbf{r'},$$

where $m_0$ is atomic mass and $\Phi(\mathbf{r})$ is a potential of interatomic interactions. The eigenfunction $\varphi_n(\mathbf{r})$, labelled by a multi-index $n$, is a nonlinear coherent mode. It is assumed to be
normalized to unity, $||\varphi_n|| = 1$. Atomic interactions for dilute trapped gases are described by the Fermi contact potential

$$\Phi(r) = A_s \delta(r) , \quad A_s \equiv 4\pi \hbar^2 \frac{a_s}{m_0} ,$$

with $a_s$ being the $s$-wave scattering length. The trapping potential is commonly presented by the harmonic oscillator form

$$U(r) = \frac{m_0}{2} \left( \omega_x r_x^2 + \omega_y r_y^2 + \omega_z r_z^2 \right) .$$

Here, we shall consider the single-well potential (4), though the double-well and more complicated potentials [6,11,13] are possible, including periodic potentials corresponding to optical lattices [17–31].

In the case of cylindric symmetry, when $\omega_x = \omega_y \equiv \omega_r$, it is convenient to introduce the notation

$$\nu \equiv \frac{\omega_z}{\omega_r} , \quad l_r \equiv \sqrt{\frac{\hbar}{m_0\omega_r}} ,$$

for the frequency ratio $\nu$ and oscillator length $l_r$, which allows the usage of the dimensionless spatial variables

$$r_\perp \equiv \frac{\sqrt{r_x^2 + r_y^2}}{l_r} , \quad z \equiv \frac{r_z}{l_r} .$$

And the quantity

$$g \equiv 4\pi \frac{a_s}{l_r} N$$

is a dimensionless coupling parameter. Also, let us define the dimensionless Hamiltonian and the wave function, respectively,

$$\hat{H} \equiv \frac{\hat{H}[\varphi]}{\hbar\omega_r}, \quad \psi(r_\perp, \varphi, z) \equiv l_r^{3/2} \varphi(r) .$$

Then Eq. (2) reduces to

$$\hat{H} = -\frac{1}{2} \nabla^2 + \frac{1}{2} \left( r_\perp^2 + \nu^2 z^2 \right) + g|\psi|^2 ,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2_\perp} + \frac{1}{r_\perp} \frac{\partial}{\partial r_\perp} + \frac{1}{r^2_\perp} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} .$$

The eigenproblem (1) takes the form $\hat{H}\psi_{nmj} = E_{nmj}\psi_{nmj}$, in which $n = 0, 1, 2, \ldots$ is the radial quantum number, $m = 0, \pm 1, \pm 2, \ldots$ is the azimuthal quantum number, and $j = 0, 1, 2, \ldots$ is the axial quantum number.

The equilibrium Bose-Einstein condensate corresponds to the ground-state solution to the stationary Gross-Pitaevskii equation, when $n = m = j = 0$. This equation is nonlinear and does not allow for an exact solution. If the coupling parameter (6) were small, $g \to 0,$
perturbation theory would be admissible. And in the opposite limit of asymptotically strong coupling, \( g \to \infty \), the Thomas-Fermi approximation could be used (see [1–3]). Here we demonstrate how one can construct an accurate solution for the ground state, which is valid for arbitrary coupling parameters \( g \in [0, \infty) \). For simplicity, we consider the spherical trap, when \( \nu = 1 \). Then the ground state \( \psi_0(r) \equiv \psi_{000}(r) \) depends on the spherical variable \( r \equiv \sqrt{r^2 + z^2} \).

To find an accurate approximate solution of a nonlinear differential equation, we employ the self-similar approximation theory [32–36], in the variant [36–38] designed for constructing self-similar crossover approximants satisfying the corresponding asymptotic conditions. To this end, we notice that at short radius \( r \to 0 \), when the nonlinear term in the Hamiltonian dominates, one has

\[
\psi_0(r) \simeq c_0 + c_2 r^2 + c_4 r^4 \quad (r \to 0).
\]

And at large \( r \to \infty \), where the harmonic term in Eq. (7) becomes dominant, the wave function tends to the Gaussian form

\[
\psi_0(r) \simeq C \exp \left( -\frac{1}{2} r^2 \right) \quad (r \to \infty).
\]

The self-similar crossover approximant, interpolating between the asymptotic limits (8) and (9) is

\[
\psi^*_0(r) = C \exp \left( -\frac{1}{2} r^2 \right) \exp \left\{ a r^2 \exp(-b r^2) \right\}.
\]

The coefficients \( a \) and \( b \) are obtained by expanding function (10) in powers of \( r \) and substituting this expansion in the stationary Gross-Pitaevskii equation, which yields

\[
a = \frac{1}{2} + \frac{1}{3} \left( gC^2 - E_0 \right), \quad b = \frac{1}{10a} \left( 1 - 2a \right) \left( E_0 - 1 + 2a \right) - \frac{1}{20a}.
\]

The ground-state energy \( E_0 \) and the normalization constant \( C \) are defined by the consistency conditions

\[
4\pi \int_0^\infty \psi_0(r) \hat{H} \psi_0(r) \, r^2 \, dr = E_0, \quad 4\pi \int_0^\infty \psi_0^2(r) r^2 \, dr = 1.
\]

The accuracy of an approximate solution can be characterized by calculating the local residual

\[
R(r) \equiv \sqrt{4\pi} \, r \left( \hat{H} - E_0 \right) \psi_0(r),
\]

showing the deviation of the radial wave function from the exact solution at each point \( r \). The behaviour of this residual for the self-similar approximant (10) is presented in Fig. 1 for different values of the coupling parameter (6), where it is compared with the residuals for the optimized Gaussian approximation and Thomas-Fermi approximation [3]. As is seen, the accuracy of the self-similar approximant (10) is essentially better. Figure 2 shows the atomic density

\[
n(r) \equiv 4\pi \psi^2_0(r)
\]

for the self-similar approximant (10), as compared with that for the optimized Gaussian and Thomas-Fermi approximations.
The way of constructing an accurate approximation for the wave function is suitable for describing the ground state in a spherical trap, but it becomes rather cumbersome for nonspherical traps and for nonground states. Moreover, the exact shape of wave functions is of less importance than the related energy values. Hence it is necessary to concentrate attention on the calculation of the spectrum of higher nonlinear modes. For this purpose, the most suitable is the optimized perturbation theory [39–41]. This approach has been successfully employed for a variety of physical problems. A brief survey of the method and many citations can be found in Refs. [35,36]. Recently, this approach was used for calculating the critical temperature of Bose-Einstein condensation in a dilute Bose gas [42–47]. The main idea of the optimized perturbation theory is the introduction of control functions that optimize the convergence of a calculational procedure [39–41]. Control functions can be introduced in different ways, and different calculational algorithms may be employed. Any kind of perturbation theory or iterative procedure can be used. The Rayleigh-Schrödinger and Dalgarno-Lewis perturbation theories can be invoked for the Schrödinger equation as well as for Green functions [48]. Semiclassical expansions are possible [49]. One may also prefer to work not with the Schrödinger equation itself but with an inverse Schrödinger equation [50].

We apply the optimized perturbation theory, with the Rayleigh-Schrödinger algorithm, to the Gross-Pitaevskii equation (1). For the spectrum, in the first approximation, we find

\[ e(g) \equiv E(g, u(g), v(g)), \]

with the expression

\[ E(g, u, v) = \frac{p}{2} \left( u + \frac{1}{u} \right) + \frac{q}{4} \left( v + \frac{v^2}{u} \right) + u \sqrt{v} g I_{nmj}, \]

in which

\[ p \equiv 2n + |m| + 1, \quad q \equiv 2j + 1 \]

are the combinations of quantum numbers, and the integral

\[ I_{nmj} \equiv \frac{1}{u \sqrt{v}} \int_0^\infty r_\perp dr_\perp \int_0^{2\pi} d\varphi \int_{-\infty}^{+\infty} dz |\psi_{nmj}(r_\perp, \varphi, z)|^4 \]

contains the wave functions

\[ \psi_{nmj}(r_\perp, \varphi, z) = \left[ \frac{2n!}{(n + |m|)!} \right]^{1/2} r_\perp^{|m|} \exp \left( -\frac{u}{2} r_\perp^2 \right) \times \]

\[ \times L_n^{|m|} \left( u r_\perp^2 \right) \frac{e^{im\varphi}}{\sqrt{2\pi}} \left( \frac{v}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^j j!}} \exp \left( -\frac{v}{2} z^2 \right) H_j \left( \sqrt{v} z \right), \]

where \( L_n^{|m|}(\cdot) \) is a Laguerre polynomial and \( H_j(\cdot) \) is a Hermite polynomial. The control functions \( u = u(g) \) and \( v = v(g) \) are obtained from the optimization condition

\[ \left( \delta u \frac{\partial}{\partial u} + \delta v \frac{\partial}{\partial v} \right) E(g, u, v) = 0, \]
which results in the equations

\[ p \left(1 - \frac{1}{u^2}\right) + \frac{G}{p\nu} \sqrt{\frac{v}{q}} = 0, \quad q \left(1 - \frac{\nu^2}{v^2}\right) + \frac{uG}{p\nu} \sqrt{vq} = 0, \]  

(16)
in which

\[ G \equiv 2p \sqrt{q} I_{nmj} g\nu. \]  

(17)

Equations (15) to (17) define the spectrum (14) for all quantum numbers \( n, m, \) and \( j \) and for arbitrary values of the coupling parameter \( g \).

Instead of solving numerically Eqs. (14) to (17), we may derive explicit analytical expressions for the spectrum of nonlinear modes by resorting to the technique of self-similar root approximants \([36–38]\). To this end, we find the weak-coupling expansion of spectrum (14) in the form

\[ e(g) \simeq \sum_{n=0}^{k} a_n G^n \quad (G \to 0), \]  

(18)

with the coefficients

\[ a_0 = p + \frac{Q}{2}, \quad a_1 = \frac{1}{2pQ^{1/2}}, \quad a_2 = -\frac{p + 2Q}{16p^3Q^2}, \quad a_3 = \frac{(p + 2Q)^2}{64p^5Q^{7/2}} \quad (Q \equiv q\nu). \]

Also, we obtain the strong-coupling expansion

\[ e(g) \simeq \sum_{n=0}^{k} b_n G^{\beta_n} \quad (G \to \infty), \]  

(19)

where the coefficients are given by the equalities

\[ 4b_0 = 5, \quad 4b_1 = 2p^2 + Q^2, \]
\[ 20b_2 = -3p^4 + 2p^2Q^2 - 2Q^4, \quad 29b_3 = 2p^6 - p^4Q^2 - 2p^2Q^4 + 2Q^6, \]
\[ 500b_4 = -44p^8 + 22p^6Q^2 + 2p^4Q^4 + 78p^2Q^6 - 69Q^8, \]
\[ 12500b_5 = 1122p^{10} - 595p^8Q^2 - 70p^6Q^4 + 440p^4Q^6 - 3640p^2Q^8 + 2821Q^{10}, \]

and the powers are

\[ \beta_0 = \frac{2}{5}, \quad \beta_1 = -\frac{2}{5}, \quad \beta_2 = -\frac{6}{5}, \]
\[ \beta_3 = -2, \quad \beta_4 = -\frac{14}{5}, \quad \beta_5 = -\frac{18}{5}. \]

Interpolating the asymptotic limits (18) and (19) by means of the self-similar root approximants \([36–38]\), we obtain

\[ e_k(g) = a_0 \left(\ldots (1 + A_{k1}G)^{n_{k1}} + A_{k2}G^{n_{k2}} + \ldots A_{kk}G^k\right)^{n_{kk}}. \]  

(20)

Here, depending on the approximation order \( k = 1, 2, \ldots \), we have in the first order

\[ A_{11} = \frac{1.746928}{a_0^{5/2}}, \quad n_{11} = \frac{2}{5}, \]
in the second order

\[ A_{21} = 2.533913 \frac{(2p^2 + Q^2)^{5/6}}{a_0^{25/6}}, \quad A_{22} = 3.051758 \frac{a^5}{a_0^5}, \quad n_{21} = \frac{6}{5}, \quad n_{22} = \frac{1}{5}, \]

and in the third order

\[ A_{31} = 1.405455 \frac{(8p^4 + 12p^2Q^2 + Q^4)^{5/6}}{a_0^{125/22}(2p^2 + Q^2)^{5/6}}, \quad A_{32} = 6.619620 \frac{(2p^2 + Q^2)^{10/11}}{a_0^{75/11}}, \]

\[ A_{33} = 5.331202 \frac{a_0^{15/2}}{a_0^{15/2}}, \quad n_{31} = \frac{6}{5}, \quad n_{32} = \frac{11}{10}, \quad n_{33} = \frac{2}{15}. \]

The accuracy of approximants (20), for vortex nonlinear modes with \( n = j = 0 \) and different \( m \), as compared to the numerical values of Eq. (14), are illustrated in Fig. 3. As we see, the approximants \( e^*_2(g) \) and \( e^*_3(g) \) are quite accurate.

The analytical expression (20) makes it possible a direct analysis of the spectrum with respect to varying quantum numbers and the coupling parameter. It also allows us to get a fast estimate of the critical parameter \( g_c \), and, hence, of the critical number of atoms above which the system of atoms with attractive interactions becomes unstable. For this purpose, we need to find the negative value of \( G_c \) at which the spectrum (20) becomes complex. To illustrate the idea, we limit ourselves by the first approximation \( e^*_1(g) \), which is complex for 

\[ G < G_c = -0.572433 a_0^{5/2}. \]

This, in view of notation (17), gives

\[ g_c \approx -0.05 \frac{(2p + q\nu)^{5/2}}{p\sqrt{q} I_{nmj\nu}}, \]

which, because of Eq. (6), yields the estimate for the critical number of atoms

\[ N_c \approx \frac{(2p + q\nu)^{5/2}}{300p\sqrt{q} I_{nmj\nu}} \frac{|l_r|}{a_s}. \]

This estimate is of order of that obtained by means of numerical solution [3,4]. A more accurate value of \( N_c \) can be derived from the consideration of the higher-order approximants (20).

### 3 Resonant Bose Condensate

Suppose that a trapped atomic gas is cooled down to temperatures when almost all atoms are in the Bose-Einstein condensate corresponding to the ground state energy \( E_0 \). As has been shown [4], applying an alternating external field, with the frequency close to the transition frequency between the ground state and an excited mode, it is possible to transfer atoms from the ground to this excited mode. Thus, a nonground state condensate can be created.
The latter, by applying one more resonant field can be transferred to another higher mode. In this way, we may consider resonant transitions between two energy levels, say $E_1$ and $E_2$, such that $E_1 < E_2$, with the transition frequency

$$\omega_{21} \equiv \frac{1}{\hbar} (E_2 - E_1) . \quad (21)$$

The transition is induced by an alternating field

$$\hat{V} = V_1(r) \cos \omega t + V_2(r) \sin \omega t , \quad (22)$$

whose frequency is close to frequency (21), so that the resonance condition

$$\left| \frac{\Delta \omega}{\omega} \right| \ll 1 , \quad \Delta \omega \equiv \omega - \omega_{21} \quad (23)$$

holds. This situation is similar to the resonant interaction of electromagnetic field with a single atom, though a Bose condensate is a many-particle system and the transition is between collective levels.

In the presence of the alternating field (22), we have to deal with the time-dependent Gross-Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t} \varphi(r,t) = \left( \hat{H}[\varphi] + \hat{V} \right) \varphi(r,t) , \quad (24)$$

with the nonlinear Hamiltonian (2). One may look for the solution to this equation in the form of the expansion

$$\varphi(r,t) = \sum_n c_n(t) \varphi_n(r) \exp \left( - \frac{i}{\hbar} E_n t \right) \quad (25)$$

over the nonlinear coherent modes $\varphi_n(r)$ defined by the eigenvalue problem (1). Note that, in general, one could look for a solution expanding the latter over an arbitrary basis [51]. However, the set \{\varphi_n(r)\} of the nonlinear coherent modes stands out as the sole natural basis, being physically distinguished from all other formally admissible expansion sets.

There are two types of transition amplitudes in the problem,

$$\alpha_{mn} \equiv A_s \frac{N}{\hbar} \int |\varphi_m(r)|^2 \left[ 2|\varphi_n(r)|^2 - |\varphi_m(r)|^2 \right] d\mathbf{r} ,$$

$$\beta_{mn} \equiv \frac{1}{\hbar} \int \varphi_m^*(r) [V_1(r) - iV_2(r)] \varphi_n(r) d\mathbf{r} , \quad (26)$$

the former being due to atomic interactions (3) and the latter, to the action of the modulating field (22). In analogy with the resonance in an atom, we need here that the transition amplitudes (26) be smaller than the transition frequency (21),

$$\left| \frac{\alpha_{12}}{\omega_{21}} \right| \ll 1 , \quad \left| \frac{\alpha_{21}}{\omega_{21}} \right| \ll 1 , \quad \left| \frac{\beta_{12}}{\omega_{21}} \right| \ll 1 , \quad (27)$$
in order that the transition frequency, and the resonance as such, be well defined. These conditions lead to the restriction

\[ |g\nu| < \frac{(2p^2 + Q^2)^{5/4}}{14p\sqrt{q} \, I_{nmj}} \]  

(28)
on the system parameters, which can be reformulated as the limitation

\[ N < \frac{(2p^2 + Q^2)^{5/4}}{56\pi\nu p\sqrt{q} \, I_{nmj}} \frac{|l_p|}{a_s} \]  

(29)on the number of particles that can be resonantly transferred to a chosen mode. Note that the limiting value in the right-hand side of Eq. (29) is close to \( N_c \).

Under conditions (27), the Gross-Pitaevskii equation (24) reduces to the system of equations

\[ i \frac{dc_1}{dt} = \alpha_{12}|c_2|^2c_1 + \frac{1}{2} \beta_{12}c_2e^{i\Delta\omega t}, \quad i \frac{dc_2}{dt} = \alpha_{21}|c_1|^2c_2 + \frac{1}{2} \beta_{12}^*c_1e^{-i\Delta\omega t} \]  

(30)
for the fractional population amplitudes. This reduction to an effective two-mode system has become possible due to the resonance condition (23) and inequalities (27). Such a situation is analogous to a resonant atom whose description also reduces to a two-level system. One more requirement is that the energy levels would not be equidistant [4]. This is always valid for electronic levels in an atom. Fortunately, owing to the nonlinear term in the Gross-Pitaevskii equation, this is also true for the spectrum of the nonlinear modes. There is as well another possibility of shifting the required levels by a slight modification of the trapping potential [52].

Equations (30), constituting a system of four equations for real functions, allow a further simplification. For this purpose, we may write

\[ c_1 = \sqrt{\frac{1-s}{2}} \exp(i\pi_1 t), \quad c_2 = \sqrt{\frac{1+s}{2}} \exp(i\pi_2 t), \]  

(31)
where \( \pi_1 = \pi_1(t) \) and \( \pi_2 = \pi_2(t) \) are the phases and

\[ s \equiv |c_2|^2 - |c_1|^2 \]  

(32)
is the population difference. We also introduce the phase difference

\[ x \equiv \pi_2 - \pi_1 + \gamma + \Delta\omega t, \]  

(33)
employ the notation

\[ \alpha_0 \equiv \frac{1}{2} (\alpha_{12} + \alpha_{21}), \quad \beta_{12} \equiv \beta e^{i\gamma}, \quad \beta \equiv |\beta_{12}|, \quad \delta \equiv \Delta\omega + \frac{1}{2} (\alpha_{12} - \alpha_{21}), \]  

and define the dimensionless parameters

\[ b \equiv \frac{\beta}{\alpha_0}, \quad \varepsilon \equiv \frac{\delta}{\alpha_0}. \]  

(34)
Then Eqs. (30) can be reorganized to the system of two equations
\[
\frac{ds}{dt} = -\beta \sqrt{1 - s^2} \sin x, \quad \frac{dx}{dt} = \alpha_0 s + \frac{\beta s}{\sqrt{1 - s^2}} \cos x + \delta
\] (35)
for the real functions (32) and (33), with the initial conditions \( s_0 = s(0) \) and \( x_0 = x(0) \).
Here we have considered the resonant field of the form (22), with the amplitudes depending on the spatial variable \( r \) but not depending on time \( t \). In general, we could keep in mind that the amplitudes \( V_i(r, t) \) are slow functions of time, that is, varying much slower than the oscillation period \( 2\pi/\omega \). If so, the value \( \beta \) in Eqs. (35) would be a function of time, which would permit us to study the behaviour of \( s(t) \) and \( x(t) \) under the influence of resonant pulses. This would enrich the variety of admissible solutions yielding the appearance of different transient phenomena, analogously to the case of transient coherent phenomena in systems of resonant atoms [53].

4 Dynamic Resonant Effects

The resonant Bose condensate, though being analogous to a resonant atom, possesses, because of its collective nonlinear nature, several features that make it rather different from the latter. Some of the interesting effects exhibited by the resonant Bose condensate, are surveyed below.

4.1 Mode locking

We shall say that the modes are locked if the fractional populations \( |c_i(t)|^2 \) oscillate in time so that they do not cross the line \( |c_1|^2 = |c_2|^2 = 1/2 \). Depending on initial conditions, it may be that \( |c_1|^2 > 1/2 \) and \( |c_2|^2 < 1/2 \), or that \( |c_2|^2 > 1/2 \) while \( |c_1|^2 < 1/2 \). With regard to the population difference (32), the mode locking means that, depending on initial conditions, either
\[
-1 \leq s(t) \leq 0 \quad (36)
\]
or
\[
0 \leq s(t) \leq 1 \quad . \quad (37)
\]
Such a mode-locked regime exists for some region of the parameters (34), in particular, when both \( b \) and \( \varepsilon \) are small [4].

4.2 Critical dynamics

When the amplitude of the resonant field \( b \) or detuning \( \varepsilon \) increase in the magnitude, they may reach the values at which the dynamics of the fractional populations experience dramatic changes [5,8,12]. This happens on the critical line of the parametric manifold. More precisely, for each given set of initial conditions, \( s_0, x_0 \), there is a separate critical line on the plane \( \{b, \varepsilon\} \), described by the equation
\[
\frac{s_0^2}{2} - b\sqrt{1 - s_0^2} \cos x_0 + \varepsilon s_0 = |b| \quad . \quad (38)
\]
The critical behaviour of fractional populations, when crossing the critical line (38), has been investigated numerically [5,8,12]. The dynamic critical phenomena appear when the motion changes from the mode-locked type to mode-unlocked type. This happens when a given starting point of a trajectory is crossed by a saddle separatrix on the plane of the variables $s(t), x(t)$. Thus, for the starting point $s_0 = -1, x_0 = 0$, and the detuning $\varepsilon = -0.1$, the critical pumping amplitude, defined by Eq. (38), is $b_c = 0.39821$. The motion, starting at $s_0 = -1, x_0 = 0$, is mode locked for $b < b_c$, and becomes mode unlocked for $b > b_c$. The phase portrait for $b = 0.51 > b_c$ is shown in Fig. 4, where it is seen that the motion beginning at $s_0 = -1, x_0 = 0$ is mode unlocked, i.e. the trajectory lies in the interval $-1 \leq s \leq 1$.

Another way of explaining the change from the mode-locked regime to the mode-unlocked one can be as follows. Let us introduce the function

$$h \equiv 2c_1^*c_2e^{i(\Delta \omega t + \gamma)} = \sqrt{1 - s^2} e^{ix}. \quad (39)$$

And let us measure time in units of $\alpha_0$. Then Eqs. (30) can be presented in the form

$$\frac{dh}{dt} = i(s + \varepsilon)h + ibs, \quad \frac{ds}{dt} = \frac{i}{2} b(h - h^*). \quad (40)$$

From the first of these equations, we have

$$h = \frac{1}{s + \varepsilon} \left( bs + i \frac{dh}{dt} \right).$$

Substituting this in the second of Eqs. (40) gives

$$2(s + \varepsilon) \frac{ds}{dt} = b \frac{d}{dt} (h + h^*).$$

The latter equation is easily integrated yielding

$$(s + \varepsilon)^2 - b(h + h^*) = k_0,$$

with the integration constant

$$k_0 \equiv (s_0 + \varepsilon)^2 - b(h_0 + h_0^*) = (s_0 + \varepsilon)^2 - 2b \sqrt{1 - s_0^2} \cos x_0.$$

Differentiating the second of Eqs. (40), we get

$$\frac{d^2s}{dt^2} = -\frac{b}{2} (s + \varepsilon) (h + h^*) - bs^2,$$

which can be transformed to

$$\frac{d^2s}{dt^2} = -\frac{1}{2} (s + \varepsilon)^3 + \frac{k_0}{2} (s + \varepsilon) - b^2s.$$

From here, for the shifted population difference

$$\overline{s} \equiv s + \varepsilon,$$
we find
\[
\left( \frac{d\bar{s}}{dt} \right)^2 = -\frac{1}{4} \bar{s}^4 + \frac{k_1}{2} \bar{s}^2 + 2b^2 \varepsilon \bar{s} + k_2 ,
\] (41)
with the integration constants
\[
k_1 \equiv k_0 - 2b^2 , \quad k_2 \equiv \dot{s}_0^2 + \frac{1}{4} s_0^4 - \frac{k_1}{2} s_0^2 - 2b^2 \varepsilon s_0 ,
\]
where
\[
\dot{s}_0 = \frac{i}{2} b (h_0 - h_0^* ) = -b \sqrt{1 - s_0^2} \sin x_0 .
\]
The exact solution to Eq. (41) can be expressed [54] as a ratio of the Jakobi elliptic functions [55].

To simplify the consideration, let us set the zero detuning \( \varepsilon = 0 \). Then \( \bar{s} = s \). For the integration constants, we get
\[
k_1 = s_0^2 - 2b \sqrt{1 - s_0^2} \cos x_0 - 2b^2 , \quad k_2 = \frac{1}{4} s_0^4 - \frac{k_1}{2} s_0^2 + b^2 \left( 1 - s_0^2 \right) \sin^2 x_0 .
\]
Equation (41) reduces to
\[
\left( \frac{ds}{dt} \right)^2 = -\frac{1}{4} s^4 + \frac{k_1}{2} s^2 + k_2 .
\] (42)
The polynomial in the right-hand side of Eq. (42) has the roots
\[
s^2 = k_1 \pm \sqrt{k_1^2 + 4k_2} ,
\] (43)
which prescribe the admissible region of varying \( s \). For instance, in the case of the initial conditions \( s_0 = \pm 1 \), we obtain the roots
\[
s_{1,2} = \pm 1 , \quad s_{3,4} = \pm \sqrt{1 - 4b^2} .
\] (44)
The left-hand side of Eq. (42) is always nonnegative, hence it should be that
\[
1 - 4b^2 \leq s^2 \leq 1 .
\] (45)
The analysis of the inequalities (45) tells us that for \( b^2 \leq 1/4 \) the mode-locking regime is realized, since the motion is locked in one of the regions
\[
\begin{align*}
-1 & \leq s \leq -(1 - 4b^2) \\
1 - 4b^2 & \leq s \leq 1
\end{align*}
\] (46)
depending on initial conditions \( s_0 = \pm 1 \). But for higher \( b^2 \), the motion becomes unlocked, with the trajectory wandering in the whole admissible region
\[
-1 \leq s \leq 1 \quad \left( b^2 > \frac{1}{4} \right) .
\] (47)
The change of the mode-locked regime to the mode-unlocked one happens at the critical value \( b_c^2 = 1/4 \).
4.3 Interference effects

Since the spatial distribution of two different nonlinear modes $\varphi_1(r)$ and $\varphi_2(r)$ is different, there appears the interference pattern described by

$$\rho_{\text{int}}(r, t) \equiv \rho(r, t) - \rho_1(r, t) - \rho_2(r, t), \quad (48)$$

where

$$\rho(r, t) = |c_1(t)\varphi_1(r)e^{-iE_1t/\hbar} + c_2(t)\varphi_2(r)e^{-iE_2t/\hbar}|^2,$$
$$\rho_j(r, t) = |c_j(t)\varphi_j(r)|^2 \quad (j = 1, 2).$$

The properties of the pattern (48) were studied in Ref. [12].

Different spatial shapes of the nonlinear modes and their different related energies result in the appearance of the interference current

$$\mathbf{j}_{\text{int}}(r, t) \equiv \mathbf{j}(r, t) - \mathbf{j}_1(r, t) - \mathbf{j}_2(r, t), \quad (49)$$

in which $\mathbf{j}(r, t)$ is the total current in the system, while $\mathbf{j}_i(r, t)$, with $i = 1, 2$ are the partial currents of the corresponding modes [12].

4.4 Atomic squeezing

The evolution equations (30) for the effective two-mode system may be rewritten in the spin representation as the equations for the collective spin operators

$$S_\alpha \equiv \sum_{i=1}^N S_\alpha^i, \quad S_\pm \equiv S_x \pm iS_y,$$

where $\alpha = x, y, z$. Thence, we may consider atomic squeezing connected with the spin operators, because of which it is called spin squeezing [56–58]. Thus, the squeezing of $S_z$ with respect to $S_\pm$ is described by the squeezing factor

$$Q_z \equiv \frac{2\Delta^2(S_z)}{|\langle S_\pm \rangle|}, \quad (50)$$

in which $\Delta^2(S_z) \equiv \langle S_z^2 \rangle - \langle S_z \rangle^2$. For the resonant Bose condensate, we find

$$Q_z = \sqrt{1 - s^2}, \quad (51)$$

with $s$ being the population difference (32). This shows that for all $s \neq 0$, the squeezing factor $Q_z < 1$, which implies that $S_z$ is squeezed with respect to $S_\pm$. The latter, in physical parlance, means the squeezing of atomic states.
4.5 Multiparticle entanglement

To study the possible entanglement for a multiparticle system, one has to consider the reduced density matrices. A $p$-particle reduced density matrix is given by

$$
\rho_p(r_1 \ldots r_p, r'_1 \ldots r'_p, t) \equiv \text{Tr} \psi(r_1) \ldots \psi(r_p) \hat{\rho}(t) \psi^\dagger(r'_p) \ldots \psi^\dagger(r'_1),
$$

(52)

where $r_i$ are Cartesian coordinates, $t$ is time, $\psi(r)$ and $\psi^\dagger(r)$ are field operators, $\hat{\rho}(t)$ is a statistical operator, and the trace is taken over the Fock space. As a basis for calculating the trace, we may accept coherent states [48] or Fock-Hartree states [59,60]. At low temperature, when practically all $N$ atoms are Bose-condensed, the trace (52) is concentrated on the coherent states normalized to the number of particles $N$. These coherent states have the form of the column

$$
|\eta_{n\alpha} > = \left[ e^{-N/2} \prod_{k=1}^k \eta_{n\alpha}(r_i) \right],
$$

(53)

in which

$$
\eta_{n\alpha}(r) = \sqrt{N} \varphi_n(r) e^{i\alpha} \quad (0 \leq \alpha \leq 2\pi),
$$

(54)

$\varphi_n(r)$ being the solution to the eigenproblem (1). The phase $\alpha$ here is random, because of which the vector (53) may be called the random-phase coherent state or mixed coherent state [48,61]. These states are asymptotically orthogonal,

$$
< \eta_{m\alpha} | \eta_{n\beta} > \simeq \delta_{mn} \delta_{\alpha\beta} \quad (N \gg 1).
$$

The closed linear envelope

$$
\mathcal{F}_0 \equiv \overline{\mathcal{L}\{|\eta_{n\alpha} >\}}
$$

(55)

forms a subspace of the Fock space. On this subspace, one has the asymptotic, as $N \to \infty$, resolution of unity

$$
\sum_n \int |\eta_{n\alpha}><\eta_{n\alpha}| \frac{d\alpha}{2\pi} \simeq \hat{1},
$$

(56)

understood in the weak sense.

Let us also define a Hilbert space $\mathcal{H}$ as a closed linear envelope $\mathcal{H} \equiv \overline{\mathcal{L}\{\varphi_n(r)\}}$ supplemented with a scalar product. And let us introduce a $p$-fold tensor product $\mathcal{H}^p \equiv \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$. Then the functions

$$
\varphi_n^p(r_1, \ldots, r_p) \equiv \prod_{i=1}^p \varphi_n(r_i)
$$

are approximately orthogonal on $\mathcal{H}^p$ in the following sense. The scalar product

$$
(\varphi_m^p, \varphi_n^p) = \prod_{i=1}^p (\varphi_m, \varphi_n)
$$

has the property

$$
0 \leq |(\varphi_m^p, \varphi_n^p)| \leq 1,
$$

14
being equal to unity if and only if $m = n$. But for $m \neq n$, one has
\[
\lim_{p \to \infty} |(\varphi_m^p, \varphi_n^p)| = 0 \quad (m \neq n).
\]

Assuming that the trace in Eq. (52) is concentrated on the condensate subspace (55), we find
\[
\rho_p(r_1 \ldots r_p, r'_1 \ldots r'_p, t) \simeq \sum_n D_n(t) \prod_{i=1}^p \varphi_n(r_i) \varphi^*_n(r'_i) ,
\] (57)
where the coefficients are
\[
D_n(t) \equiv <\eta_n | \hat{\rho}(t) | \eta_n > .
\] (58)
The latter can be defined from the normalization conditions
\[
\int \rho_p(r_1 \ldots r_p, r'_1 \ldots r'_p, t) \, dr_1 \ldots dr_p = \frac{N!}{(N-p)!} , \quad \sum_n |c_n(t)|^2 = 1 ,
\]
which suggests
\[
D_n(t) = \frac{N!}{(N-p)!} |c_n(t)|^2 .
\] (59)

For the resonant Bose condensate, with $N \gg 1$, one gets
\[
D_1(t) \simeq \frac{1-s}{2} \, N^p , \quad D_2(t) \simeq \frac{1+s}{2} \, N^p .
\] (60)

The density matrix (57) cannot be presented as a product of single-particle density matrices, which implies that the state of the system is entangled [62,63]. This happens for all $|s| \neq 1$. Maximal entanglement occurs at $s = 0$, when $D_1(t) = D_2(t) \simeq N^p/2$.

5 Conclusion

The method of resonant formation of nonlinear coherent modes makes it possible to create a novel type of systems, the resonant Bose condensate. The latter shares many analogies with a resonant atom. But, being a collective system, the resonant Bose condensate also displays the features essentially distinguishing it from a single resonant atom. First of all, the spectrum of the nonlinear coherent modes is described by the stationary Gross-Pitaevskii equation, which is a nonlinear Schrödinger equation. These modes, though formally looking as single-particle functions, correspond to collective states of a coherent multiatomic system. It would be admissible to say that such coherent modes are single-quasiparticle wave functions, but not single-particle ones. Here, one should imply a quasiparticle in the sense of Landay, that is, an effective dressed single object inside a multiparticle ensemble.

The evolution equations for the resonant Bose condensate can be reduced to the system of two equations for the complex amplitudes of two fractional populations. This reduction to an effective two-mode problem is analogous to that happening for a resonant atom. As in the case of any two-mode, two-level, or two-component system, the problem allows for the usage of quasispin representation. Employing this representation, one could talk about quasispin
dynamics, qiasispin waves [64], having some formal properties similar to the dynamics of real spins [65], and so on. To study phase portraits, it is also convenient to work in the representation employing two real quantities: population difference and phase difference.

The collective nonlinear nature of the resonant Bose condensate yields the existence of several novel effects making this system rather different from a single resonant atom. The most interesting of these effects, we have investigated, are: mode locking, critical dynamics, interference effects, atomic squeezing, and multiparticle entanglement. These effects can find a broad range of various applications.

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Figure captions

**Fig. 1.** Local residual (12) as a function of the dimensionless radial variable $r$ for the self-similar approximant (solid line), optimized Gaussian approximation (dotted line), and Thomas-Fermi approximation (dashed line) for different values of the coupling parameter: (a) $g=25$; (b) $g=250$. The residual for the Thomas-Fermi approximation diverges at the edge of the atomic cloud.

**Fig. 2.** Atomic density (13) as a function of $r$ for the self-similar approximant (solid line), optimized Gaussian approximation (dotted line), and Thomas-Fermi approximation (dashed line), at the coupling parameter $g = 2500$.

**Fig. 3.** Percentage errors of the self-similar root approximants $e^*_1(g)$ (solid line), $e^*_2(g)$ (dashed line), and $e^*_3(g)$ (dotted line), as functions of the coupling parameter $g$, for vortex nonlinear modes with $n = j = 0$ and different winding numbers $m$ and for different trap aspect ratio: (a) $\nu = 1, m = 1$; (b) $\nu = 1, m = 10$; (c) $\nu = 10, m = 0$; (d) $\nu = 10, m = 1$; (e) $\nu = 10, m = 10$; (f) $\nu = 100, m = 1$; (g) $\nu = 100, m = 10$.

**Fig. 4.** Phase portrait on the plane of the variables $s(t)$ and $x(t)$ for the detuning $\varepsilon = -0.1$ and the amplitude $b = 0.51$ that is larger than the critical value $b_c = 0.39821$ associated with the initial conditions $s_0 = -1, x_0 = 0$. 