The Equivalence Principle and Anomalous Magnetic Moment Experiments

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Abstract

We investigate the possibility of testing of the Einstein Equivalence Principle (EEP) using measurements of anomalous magnetic moments of elementary particles. We compute the one loop correction for the $g - 2$ anomaly within the class of non metric theories of gravity described by the $TH\epsilon\mu$ formalism. We find several novel mechanisms for breaking the EEP whose origin is due purely to radiative corrections. We discuss the possibilities of setting new empirical constraints on these effects.

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I Introduction and Summary

Metric theories of gravity offer the singular beauty of endowing spacetime with a symmetric, second-rank tensor field $g_{\mu\nu}$ that couples universally to all non-gravitational fields. This unique operational geometry is dependent upon the validity of the Einstein Equivalence Principle (EEP), which states that the outcomes of nongravitational test experiments performed within a local, freely falling frame are independent of the frame’s location (local position invariance, LPI) and velocity (local Lorentz invariance, LLI) in a background gravitational field. Non metric theories break this universality by coupling additional gravitational fields to matter, and so violate either LPI, LLI or both. Limits on LPI or LLI are imposed by gravitational redshift and atomic physics experiments respectively, by comparing atomic energy transitions that are sensitive to these symmetries. Laser experiments have set stringent limits on violations of LLI (to a precision of $\sim 10^{-22}$) \cite{1}, while the next generation of gravitational redshift experiments could reach a precision up to $10^{-9}$ \cite{2}.

These experiments probe transitions that are predominantly sensitive to nuclear electrostatic energy, although violations of WEP/EEP due to other forms of energy (virtually all of which are associated with baryonic matter) have also been estimated \cite{3}. However there exist many other physical systems, dominated by primarily non-baryonic energies, for which the validity of the EEP is comparatively less well understood \cite{4}. Included in the list of such systems are photons of differing polarization \cite{5}, antimatter systems \cite{6}, neutrinos \cite{7}, mesons \cite{8}, massive leptons, hypothesized dark matter, second and third generation matter, and quantum vacuum energies.

In order to establish the universal behavior of gravity, it is important to empirically confront the EEP with as diverse a range of non-gravitational interactions as is possible. Amongst the most intriguing of the list of physical systems noted above are those for which potential violations of the EEP can arise in quantities dependent upon vacuum energy shifts, which are peculiarly quantum-mechanical in origin (i.e. do not have a classical or semi-classical description). The empirical validity of the EEP in physical systems where radiative corrections are non-negligible remains an open question. Such systems provide an interesting empirical regime for a confrontation between gravitation and quantum mechanics.

Perhaps the most useful framework for further such testing of the EEP is provided by quantum electrodynamics (QED). We considered this approach in a previous paper \cite{9} by analyzing the behavior of Lamb shift transition energies within the context of a wide class of nonmetric theories of gravity. The Lamb shift, along with the anomalous
magnetic moments \((g-2)\) factors of elementary particles, presents the most compelling evidence in support of QED. It is therefore natural to extend the nonmetric analysis to the \(g-2\) anomaly.

We consider in this paper the possibility of using measurements of anomalous magnetic moments of elementary particles as a possible test of the EEP. The high precision attained in \(g-2\) experiments motivated the early work of Newman et al. \[10\] to use such experiments to set new bounds on the validity of special relativity. Similarly, we expect (and shall subsequently demonstrate) that such experiments could impose stringent and qualitatively new limits on the parameter space of nonmetric theories. Our interpretation of those experiments is substantially different from theirs as we explicitly include violations of the EEP in the computation of \(g-2\). These effects were not considered in ref. \[10\], which assumed that violations of special relativity arose only from the dynamical equation governing the motion of the fermion.

We consider the class of non-metric theories described by the \(TH\epsilon\mu\) formalism \[11\], following the approach given in ref. \[9\] in developing gravitationally modified (GM) QED. This formalism encompasses a wide class of nonmetric theories of gravity, and deals with the dynamics of charged particles and electromagnetic fields in a static, spherically symmetric gravitational field. It assumes that the (classical) non-gravitational laws of physics can be derived from an action:

\[
S_{NG} = -\sum_a m_a \int dt (T - H v_a^2)^{1/2} + \sum e_a \int dt v^\mu_a A_\mu(x^\nu_a) + \frac{1}{2} \int d^4x (\epsilon E^2 - B^2/\mu),
\]

where \(m_a, e_a,\) and \(x^\mu_a(t)\) are the rest mass, charge, and world line of particle \(a, x^0 \equiv t, v^\mu_a \equiv dx^\mu_a/dt, E \equiv -\nabla A_0 - \partial A/\partial t, B \equiv \nabla \times A\). The metric is assumed to be

\[
ds^2 = T(r)dt^2 - H(r)(dr^2 + r^2d\Omega^2)
\]

where \(T, H, \epsilon,\) and \(\mu\) are arbitrary functions of the (background) Newtonian gravitational potential \(U = GM/r\), which approaches unity as \(U \to 0\). For an arbitrary non-metric theory, these functions will depend upon the type of matter, \(i.e.\) the species of particle or field coupling to gravity. The functions \(\epsilon\) and \(\mu\) parameterize the ‘photon metric’, whereas \(T\) and \(H\) parameterize the ‘particle metric’ in the static, spherically symmetric case. Although we shall generically employ the notation \(T\) and \(H\) throughout this paper, it should be kept in mind that these functions shall in general have one set of values for electrons, another set for muons, another for protons, \(etc.\). Universality of gravitational coupling in the particle sector implies that the \(T\)
and $H$ functions are species independent. It is an empirical question as to whether or not such universality holds for all particle species. The stringent limits on universality violation set by previous experiments [1] have only been with regards to the relative gravitational couplings in the baryon/photon sector of the standard model. For the leptonic sector relevant to our considerations, relatively little is known [4].

In order to study atomic systems, this classical action can be generalized for quantum mechanical systems [12],

$$
S = \int d^4x \overline{\psi}(i \not \partial + e \not A - m)\psi + \frac{1}{2} \int d^4x (E^2 - c^2B^2), \tag{3}
$$

where local natural units are used, $A = \gamma_\mu A^\mu$, and $c^2 = H_0/T_0\epsilon_0\mu_0$ with the subscript “0” denoting the functions evaluated at $\vec{X} = 0$, the origin of the local frame of reference. This action emerges upon replacing the point-particle part of the action in (1) with the Dirac Lagrangian, expanding the $TH\epsilon\mu$ parameters about the origin, neglecting their spatial variation over atomic distance scales, and rescaling coordinates and fields. These operations have the effect of treating spatial and temporal derivatives and gauge couplings in the fermionic sector on the same footing, and so we need not add additional terms of the form $\overline{\psi}(i\gamma^0(\partial_0 + eA_0))\psi$ times an arbitrary constant, as such terms can be re-absorbed into a definition of the parameters which appear in (3) above, as noted below.

The action (3) refers to the preferred frame, as defined by the rest frame of the external gravitational field $U$. In order to analyze effects in systems moving with velocity $\vec{u}$ with respect to that frame we need to transform the fields and coordinates in (3) via the corresponding Lorentz transformations. This gives

$$
S = \int d^4x \overline{\psi}(i \not \partial + e \not A - m)\psi + \int d^4x J_\mu A^\mu \\
+ \frac{1}{2} \int d^4x [(E^2 - B^2) \\
+ \xi \gamma^2 \left( \vec{u}^2 E^2 - (\vec{u} \cdot \vec{E})^2 + B^2 - (\vec{u} \cdot \vec{B})^2 + 2\vec{u} \cdot (\vec{E} \times \vec{B}) \right)], \tag{4}
$$

where $J^\mu$ is the electromagnetic 4-current associated with some external source, $\gamma^2 = (1 - \vec{u}^2)^{-1}$, and $\xi = 1 - c^2$ is a dimensionless (and species-dependent) parameter that measures the degree to which LPI/LLI is broken. This parameter scales with the magnitude of the dimensionless Newtonian potential, which is expected to be much smaller than unity for actual experiments. We are therefore able to compute effects of the terms in eq. (4) that break local Lorentz invariance via a perturbative analysis about the familiar and well-behaved $c \to 1$ or $\xi \to 0$ limit.
We therefore consider gravitationally modified Quantum Electrodynamics (GMQED) based on the action (4). The fermion sector and the interaction term do not change with respect to the metric case, and so neither do the fermion propagator and the vertex rule. All the non-metric effects are accounted for in the pure electromagnetic sector of the action, which in turns modifies the photon propagator. After a proper choice of the gauge fixing term involved in the quantization of the photon field, it can be shown [9] that the photon propagator takes the form up to first order in $\xi$ (in momentum space):

$$G_{\mu\nu} = -(1 + \xi)\frac{\eta_{\mu\nu}}{k^2} + \xi \frac{\gamma^2}{k^2} \left[ \eta_{\mu\nu} \frac{(\beta \cdot k)^2}{k^2} + \beta_{\mu} \beta_{\nu} \right], \quad (5)$$

where $\eta_{\mu\nu}$ is the Minkowski tensor with a signature (+ - - -); $\gamma^2 \equiv 1/(1 - \vec{u}^2)$ and $\beta^2 \equiv 1 - \vec{u}^2$.

Therefore eq. (5), along with the unmodified fermion propagator $S_F(p)$ and vertex rule, may be used as the basis of the Feynman rules of GMQED. The radiative corrections affecting these quantities are defined in terms of the photon self energy $\Pi^{\mu\nu}(k)$, fermion self energy $\Sigma(p)$, and vertex function $\Gamma^{\mu}$ respectively. These insertions involve the calculation of loop integrals as given by the Feynman rules up to a given order.

The addition of more parameters to the theory also entails new renormalizations beyond those of the wavefunctions, charge and mass of the fermion. The $TH\epsilon_{\mu}$ parameters appear as functions of $c_0^2 \equiv T_0/H_0$ and $c_\ast^2 \equiv 1/\mu_0 c_0$, and must then be correspondly redefined. In units where $c_0 \equiv 1 (c_\ast = c)$, EEP-violating corrections only appear in the electromagnetic sector of the action (as terms proportional to $\xi$). However a more general choice is $c_0 \neq 1$, for which the particle sector of the Lagrangian density is of the form

$$\mathcal{L}_D = \overline{\psi}(\not p - \not V - m)\psi + \xi_0 \overline{\psi}(p_0 - V_0)\gamma^0 \psi \quad (6)$$

with $\xi_0 \equiv 1 - c_0^{-1}$. In the moving frame this is (up to a constant)

$$\mathcal{L}'_D = \overline{\psi}(\not p - \not V - m)\psi + \xi_0 \gamma^2 \overline{\psi}(\beta \cdot p - \beta \cdot V) \beta \psi \quad (7)$$

From (7) we see that quantum corrections of the form

$$\delta \mathcal{L}_D = \overline{\psi}(\delta \xi_0^{(1)} \beta \cdot p - \delta \xi_0^{(2)} \beta \cdot V) \beta \psi \quad (8)$$
can still be expected. It is straightforward to show that gauge invariance will guarantee $\delta \xi^{(1)}_0 = \delta \xi^{(2)}_0 = \delta \xi_0$.

Hence, in order to renormalize the mass and the $T\!H\!\epsilon\!\mu$ parameters, we have to include counterterms of the form

$$\delta m + \delta \xi_0 \beta (\beta \cdot p - \beta \cdot V),$$

which consequently participate in the redefinition of the fermion self energy and vertex function. Additionally, the electromagnetic sector will induce quantum fluctuations of the form:

$$\delta L_{EM} = \delta \xi A \mu \{(k^2 - \gamma^2 (\beta \cdot k)^2 ) \eta_{\mu \nu} - \gamma^2 \beta_\mu \beta_\nu k^2 \} A^\nu$$

corresponding to the renormalization of the $T\!H\!\epsilon\!\mu$ parameters, or equivalently $\xi \equiv 1 - H_0/T_0\mu_0\epsilon_0$, entailing a renormalization of the photon self energy.

This summarizes the procedure for performing perturbative calculations in GMQED as employed previously in ref. [9] in computing the Lamb shift. This calculation was complicated by the boundedness of the electron to the hydrogenic atom under consideration. Here, in the $g-2$ case, we have to deal only with free leptonic states. In Sec. II we evaluate the (one loop) radiative corrections to the elastic scattering of a free electron by an external electromagnetic field. Sec. III relates the scattering amplitude with experimental observables describable in terms of the g-2 anomaly. This requires a derivation of the relativistic equation of motion for the electron spin in the presence of a magnetic field. This follows from a classical treatment for the electron and magnetic field. Quantum effects are introduced in the modified Hamiltonian only (as are non metric effects). The connection with possible LLI/LPI violating experiments is presented at the end of this section. Discussion of the results and general comments are given in Sec. IV. More details about the loop calculation are shown in the Appendix.

II (GM) Free Scattering

We shall consider the lowest order radiative correction to the elastic scattering of electrons by a static external field $A^\mu$. These one loop contributions can be summarized in terms of the Feynman diagrams illustrated in Fig. 1.

The Feynman amplitudes for the diagrams follow from the Feynman rules giving the result [13]:

$$\Lambda^\mu (p', p) = \pi(p') \left\{ \Gamma^\mu + P^\mu + L^\mu \right\} u(p)$$

(11)
Figure 1: One loop corrections to the elastic scattering of an electron by an external electromagnetic source

where

\[ \Gamma^\mu(p', p) = \frac{(ie)^2}{(2\pi)^4} \int d^4k \gamma^\alpha iS_F(p' - k) \gamma^\mu iS_F(p - k) \gamma^\beta iG_{\alpha\beta}(k) \] (12)

\[ P^\mu(p', p) = \gamma^\alpha iG_{\alpha\beta}(q)i\Pi^\beta\mu(q) \] (13)

\[ L^\mu(p', p) = i\Sigma(p')iS_F(p')\gamma^\mu + \gamma^\mu iS_F(p)i\Sigma(p) \] (14)

with

\[ i\Sigma(p) = \frac{(ie)^2}{(2\pi)^4} \int d^4k iG_{\alpha\beta}(k)\gamma^\alpha iS_F(p - k)\gamma^\beta \] (15)

\[ i\Pi^\beta\mu(q) = \frac{(ie)^2}{(2\pi)^4}(-)\text{Tr} \int d^4k \gamma^\beta iS_F(k + q)\gamma^\mu iS_F(k) \] (16)

and \( q \equiv p' - p \).

We refer to eqs. (12), (13), and (14) as the Vertex, Polarization, and Leg contributions, which respectively correspond to diagrams (a), (b) and (c) plus (d). We also note that expressions (12), (15), (16) represent the one loop corrections to the vertex, fermion and photon self energy parts respectively.
Given the form of the photon propagator it is convenient to introduce:

$$\Lambda^\mu = (1 + \xi)\Lambda_0^\mu + \gamma^2 \xi\Lambda_\xi^\mu$$  \hspace{1cm} (17)

where the subscript “0” denotes the (known) result coming from the standard part of the photon propagator, and “ξ” for the part proportional to $\gamma^2$ in (5)

$$G^\xi_{\mu\nu} = \frac{\beta^\mu \beta^\nu}{k^2} + \eta_{\mu\nu} \frac{(\beta \cdot k)^2}{k^4}$$  \hspace{1cm} (18)

In the remainder of this section we consider this part of the propagator only, omitting the “ξ” label in the corresponding expressions.

The procedure for evaluating the loop integrals is equivalent to that of standard (or metric) QED. We need to regularize them first and then renormalize the parameters, which include the $TH\epsilon_\mu$ parameter along with the fermion charge and mass. The regularization of the photon propagator is carried out using

$$\frac{1}{k^2} \rightarrow -\int_{\mu^2}^{\Lambda^2} \frac{dL}{(k^2 - L)^2}, \quad \frac{1}{k^4} \rightarrow -2\int_{\mu^2}^{\Lambda^2} \frac{dL}{(k^2 - L)^3},$$  \hspace{1cm} (19)

with the assumed limits $\mu \rightarrow 0$ and $\Lambda \rightarrow \infty$, and the parameter renormalization by the inclusion of the corresponding counterterms to each loop integral. Details about this procedure and the corresponding calculations are given in the appendix. We quote the final result for the loop integrals:

$$\Sigma(p) = \frac{\alpha}{\pi} (p - m) \left\{ \frac{2}{3} \left( \frac{\beta \cdot p}{m^2} \right)^2 - \beta^2 \left( \frac{5}{24} \ln \left( \frac{\Lambda}{m} \right)^2 + \frac{133}{144} - \frac{1}{4} \ln \left( \frac{m}{\mu} \right)^2 \right) \right\}$$  \hspace{1cm} (20)

$$\Gamma^\mu(p', p) = \frac{\alpha}{\pi} \left\{ \gamma^\mu \left[ \frac{2}{3} \left( \frac{\beta \cdot p}{m^2} \right)^2 - \beta^2 \left( \frac{5}{24} \ln \left( \frac{\Lambda}{m} \right)^2 + \frac{133}{144} - \frac{1}{4} \ln \left( \frac{m}{\mu} \right)^2 \right) \right] + \frac{q^2}{m^2} \left( \frac{17}{144} \beta^2 + \frac{1}{12} + \ln \left( \frac{m}{\mu} \right)^2 \right) \right\}$$

$$- \frac{3}{2} \frac{\beta \cdot p}{m} q \cdot m \left( \frac{19}{36} \beta^2 - \frac{1}{6} \ln \left( \frac{m}{\mu} \right)^2 \right)$$

$$- \frac{q}{m} \gamma^\mu \beta \left( \frac{1}{6} \cdot \beta \cdot q + \frac{1}{12} \cdot m \right) + \left( \frac{1}{9} \cdot \ln \left( \frac{m}{\mu} \right) \right) \beta \beta^\mu$$  \hspace{1cm} (21)

$$\Pi^{\alpha\beta}(q) = -\frac{\alpha}{\pi} \left( \eta^{\alpha\beta} - q^{\alpha} q^{\beta} \right) \frac{q^2}{15 m^2} + O(q^6)$$  \hspace{1cm} (22)
where we have implicitly assumed that (21) is acting on free spinors.

The Ward identity
\[ \frac{\partial \Sigma(p)}{\partial p_\mu} = \Gamma^\mu(p, p). \] (23)

is a consequence of gauge invariance, and therefore it holds even in the absence of Lorentz invariance. It is straightforward to check that (20) and (21) satisfy (23).

The evaluation of (11) is also straightforward once the loop integrals have been calculated. We just comment on the computation of the Leg correction, which is ambiguous since it contains terms like “0/0”, which are indeterminate. To obtain an unambiguous result, we must explicitly introduce a damping factor, which is necessary for the correct definition of the initial and final “bare” states. Details of this adiabatic approach are presented in appendix B. The final result for the Leg correction is
\[
L^\mu = \alpha \frac{\pi}{\sqrt{2}} \left\{ \gamma^\mu \left[ \frac{2}{3} \left( \frac{(\beta \cdot p)^2}{m^2} + \frac{\beta \cdot p \beta \cdot q}{m} + \frac{1}{2} \frac{(\beta \cdot q)^2}{m^2} \right) \right] + \beta^2 \left( \frac{5}{24} \ln \left( \frac{\Lambda}{m} \right)^2 + \frac{133}{144} - \frac{1}{4} \ln \left( \frac{m}{\mu} \right)^2 \right) \right. \\
\left. + \frac{1}{3} \beta \beta^\mu - \frac{1}{6} \beta \beta^\mu + \frac{1}{6} \beta \beta^\mu \right\} 
\] (24)

Note that this part gives a contribution to the total amplitude that cannot be removed after renormalization. Furthermore, the gauge invariance of the Feynman amplitude which is manifest as
\[ q \cdot \Lambda = 0 \] (25)

requires the presence of such terms, a condition that is not satisfied by the vertex contribution only.

The final result for the scattering amplitude is
\[ \Lambda^\mu = F^\mu + G^\mu + I^\mu \] (26)

with
\[
F^\mu = \alpha \frac{\pi}{\sqrt{2}} \left\{ \gamma^\mu \left[ \frac{q^2}{m^2} \left( \frac{17}{144} \beta^2 + \frac{1}{12} + \ln \left( \frac{m}{\mu} \right)^2 \left( \frac{1}{8} - \frac{\beta^2}{24} \right) \right) + \frac{5 \beta \cdot p \beta \cdot q}{6 \ m \ m} \right] + \frac{(\beta \cdot q)^2}{m^2} \left( \frac{47}{180} - \frac{1}{6} \ln \left( \frac{m}{\mu} \right)^2 \right) \right. \\
\left. - \frac{4}{45} \beta \beta^\mu + \frac{1}{6} \beta \beta^\mu \right\} \\
G^\mu = \alpha \frac{\pi}{\sqrt{2}} \left\{ - \frac{q^2}{m^2} \beta \beta^\mu - \frac{\beta \cdot p}{6 \ m} - \frac{1}{3} \beta \beta^\mu - \frac{1}{3} \beta \beta^\mu + \frac{1}{6} \beta \beta^\mu + \frac{\beta^2}{24} + \frac{1}{6} \eta^\mu \right\} 
\] (27)
\[ I^\mu = \frac{\alpha}{\pi} \left\{ \frac{2}{3} \beta \cdot q \frac{\mu}{m} \beta + \left( \frac{1}{6} \beta \cdot p - \frac{37}{180} \beta \cdot q \right) \frac{q^\mu}{m} \beta - \left( \frac{\beta^2}{24} + \frac{1}{6} \right) \frac{q^\mu}{m} \right\} \]  

(29)

The various terms in (26) distinguish the different contributions to the scattering amplitude. In (27) we group terms of order \( q^2 \) or higher. \( G^\mu \) accounts for terms of order \( q \) at least, and \( I^\mu \) for the gauge terms or those who give no contribution to the amplitude. Note that the remaining infrared divergence in \( F^\mu \) can be understood in terms of soft photon radiation, analogous to the metric case.

In the next section we will use the above results to compute the \( g - 2 \) anomaly.

III  (GM)  \( g-2 \)

To lowest order the Feynman amplitude associated with the elastic scattering of an electron by a static external field is

\[ ie \pi(p') A(q) u(p) . \]  

(30)

The radiative correction of order \( \alpha \) to this process is given by

\[ ie \pi(p') \{(1 + \xi) \Lambda_0 \cdot A + \gamma^2 \xi \Lambda_\xi \cdot A\} u(p) \]  

(31)

where \( \Lambda_0 \) represents the (known) metric result and \( \Lambda_\xi \) represents the contribution from (26).

In the nonrelativistic limit of slowly moving particles (\( |q| \to 0 \)) and a static magnetic field, it is straightforward to show that

\[ e A(q) \to -\frac{e}{2m} \vec{B} \cdot \vec{\sigma} \]  

(32)

\[ e \Lambda_0 \cdot A \to -\frac{e}{2m} (\frac{\alpha}{2\pi}) \vec{B} \cdot \vec{\sigma} \]  

(33)

\[ e \Lambda_\xi \cdot A \to e G \cdot A \]  

(34)

with \( G^\mu \) given by (28), which is the dominant term as \( q \to 0 \).

In order to simplify this contribution, we consider a constant magnetic field \( \vec{B} \), that is \( \vec{A} = \frac{1}{2} \vec{r} \times \vec{B} \), in which case

\[ \beta \beta \cdot A \to -\frac{1}{2} (\vec{B} \cdot \vec{u} \vec{\sigma} \cdot \vec{u} - \vec{B} \cdot \vec{\sigma} \vec{u}^2) \]  

(35)

where we have neglected the terms that mix the large and small spinor components. Similarly, we can show

\[ \beta \cdot q \beta \ A \to -\frac{1}{2} (\vec{B} \cdot \vec{u} \vec{\sigma} \cdot \vec{u} - \vec{B} \cdot \vec{\sigma} \vec{u}^2) \]  

(36)
and in the non relativistic limit
\[
\frac{\mathbf{q} \cdot \mathbf{p}}{m} \approx \mathbf{q} \cdot \mathbf{p} \rightarrow -\mathbf{B} \cdot \mathbf{\sigma}
\] (37)
and
\[
\mathbf{q} \cdot \mathbf{A} \rightarrow -\mathbf{B} \cdot \mathbf{\sigma} .
\] (38)

If we put everything together in (28):
\[
e G \cdot \mathbf{A} \rightarrow -\frac{e}{2m} \alpha \{\mathbf{\bar{B}} \cdot \mathbf{\bar{\sigma}}(\frac{1}{12} + \frac{7}{12} \mathbf{\bar{u}}^2) - \frac{2}{3} \mathbf{\bar{B}} \cdot \mathbf{\bar{u}} \mathbf{\bar{\sigma}} \cdot \mathbf{\bar{u}}\}
\] (39)
As a cross-check on the above result, we take the limit \(u_i u_j \rightarrow -\delta_{ij}\) obtaining \(G \cdot \mathbf{A} \rightarrow -2\Lambda_0 \cdot \mathbf{A}\), which is the required limit consistent with the structure of eq. (18) in that case. The previous result is the contribution of (34) to (31), which added to (30), give us the relevant part of the Hamiltonian as
\[
H_{\sigma} = -\{\Gamma \mathbf{\bar{S}} \cdot \mathbf{\bar{B}} + \Gamma_\ast \mathbf{\bar{S}} \cdot \mathbf{\bar{u}} \mathbf{\bar{B}} \cdot \mathbf{\bar{u}}\} + O(\xi^2)O(\alpha^2) \equiv -\Gamma^{ij} S_i B_j
\] (40)
with
\[
\Gamma \equiv \frac{e}{2m} g \equiv \frac{e}{2m} \{2 + \frac{\alpha}{\pi}[1 + \xi(1 + \gamma^2/6(1 + 7\mathbf{\bar{u}}^2))]\}
\] (41)
\[
\Gamma_\ast \equiv \frac{e}{2m} g_\ast \equiv -\frac{e}{2m} \alpha \frac{4}{3} \gamma^2
\] (42)
where we have identified \(\mathbf{\bar{S}} \equiv \frac{\mathbf{\bar{\sigma}}}{2}\), and \(\mathbf{\bar{u}} = \mathbf{\bar{u}}/|\mathbf{\bar{u}}|\). The \(\Gamma\) parameters account for the coupling strength between the magnetic field and spin. We see that \(\Gamma_{ij}\) generalizes the gyromagnetic ratio of a fermion analogous to the manner in which the anomalous mass tensor generalizes the mass of a particle [14]. We therefore identify the parameters \(\Gamma_{ij} \equiv \Gamma_0 \delta_{ij} + \Gamma_\ast u^i u^j\) with the components of the anomalous gyromagnetic ratio tensor of the fermion in the class of \(TH\epsilon\mu\) theories.

Note that the presence of preferred frame effects induces a qualitatively new form of interaction between the spin and magnetic field which is quantified by \(\Gamma_\ast\). Here, instead of coupling with each other, they both couple independently to the fermion velocity relative to the preferred frame. This interaction stems purely from radiative corrections, and would be absent in any tree-level analysis of GMQED.

Hence, eq. (40) describes the interaction (as seen from the particle rest frame) between the particle spin and an external homogeneous magnetic field. From this we can extract the energy difference between electrons with opposite spin projection in the direction of the magnetic field as:
\[
\Delta E_\sigma = -\frac{eB}{2m} \left[ g + g_\ast u^2 \cos^2 \Theta \right]
\] (43)
where $\Theta$ is the angle between the magnetic field and the preferred frame velocity. The influence of the radiative corrections (coming from $g - 2$ and $g_*$) in (43) is negligible in comparison to the dominant factor of 2 in $g$. Since we want to single out the effects of the non-metric corrections, it is more interesting to study the precession of the spin or, more specifically, the oscillation of the longitudinal spin polarization. In the metric case, this frequency is proportional to the factor $g - 2$, and so it is a distinctive signature of radiative corrections.

The observable quantity in the $g - 2$ experiments is actually the electron polarization, which is proportional to the quantum mechanical expectation value of $\vec{S}$, that is, $\langle \vec{S} \rangle$. Using Ehrenfest’s theorem, a quantum mechanical solution for the motion of $\langle \vec{S} \rangle$ is obtained from the equation

$$
\frac{d\vec{S}}{dt}|_{\text{R.F.}} = -i[\vec{S}, H_o] = \vec{S} \times \left[ \Gamma \vec{B}' + \Gamma_\ast(\vec{B}' \cdot \vec{u})\vec{u} \right]
$$

(44)

where the primed variables are referred explicitly to the particle rest frame (R.F.). Note that the preferred frame effect will show distinctly as a temporal variation of the spin component parallel to the magnetic field.

In general we want to know the spin precession relative to some specific laboratory system, with respect to which the particle is moving with some velocity $\vec{\beta}$. This frame need not a-priori be the previously defined preferred frame, and so $\vec{\beta} \neq \vec{u}$.

Since the $TH\epsilon\mu$ formalism does not change (locally) the fermion electromagnetic field interaction, we expect that a charged particle in the presence of an homogeneous magnetic field will satisfy the equation

$$
\frac{d\vec{\beta}}{dt} = \vec{\beta} \times \vec{\Omega}_c
$$

(45)

with the cyclotron frequency $\vec{\Omega}_c = \frac{e}{mc} \vec{B}$ and $\gamma = (1 - \vec{\beta}^2)^{-1/2}$. Relating (44) to the laboratory system yields

$$
\frac{d\vec{S}}{dt}|_{\text{Lab}} = \frac{d\vec{S}}{dt}|_{\text{R.F.}} + \vec{\Omega}_T \times \vec{S}
$$

(46)

due to Thomas precession, with $\vec{\Omega}_T = \frac{\gamma^2}{\gamma + 1} \left( \frac{d\vec{\beta}}{dt} \times \vec{\beta} \right)$. This frequency is kinematic in origin and it is a consequence of the non-commutativity of the Lorentz transformations.

Relating the primed variables in (44) to the laboratory ones by a Lorentz transformation gives

$$
\frac{d\vec{S}}{dt}|_{\text{Lab}} = \vec{S} \times \vec{\Omega}_a
$$

(47)
with
\[ \bar{\Omega}_s = \Gamma \vec{B} + (1 - \gamma) \vec{\Omega}_c + \Gamma_s (\vec{B} \cdot \vec{u}) \vec{u} \]  
(48)
where we have set \( \vec{E} = 0 \) and considered (for simplicity) the case of orbital motion perpendicular to the magnetic field \((\vec{\beta} \cdot \vec{B} = 0)\) in the above. Note that the spin precession about \( \bar{\Omega}_s \) is no longer parallel to the magnetic field (axial direction), but has a component parallel to \( \vec{u} \) that comes from radiative and nonmetric effects.

At this point it becomes necessary to define the preferred coordinate system. There are several candidates (such as the rest frame of the cosmic microwave background) for this frame \[2\]. To study this issue it is sufficient to assume that the laboratory system (Earth) moves with a non-relativistic velocity \((\vec{V})\) with respect to the preferred frame, and so we can identify
\[ \vec{u} = \vec{V} + \vec{\beta} \]

In order to single out the effects of radiative corrections, we study the spin precession relative to the rotational motion of the electron, that is:
\[ \frac{d\vec{S}}{dt}|_{\text{rot}} = \vec{S} \times \bar{\Omega}_D \]  
(49)
with \( \vec{\Omega}_D = \bar{\Omega}_s - \bar{\Omega}_c \) and \( \vec{S} = (S_{\perp}^\parallel, S_{\perp}^\bot, S_{\parallel}) \), where the first two components are perpendicular to \( \vec{B} \) (lower index) but parallel and perpendicular to \( \vec{\beta} \) (upper index), and the last one parallel to \( \vec{B} \). In the following we refer to the difference frequency \((\Omega_D)\) as the anomalous frequency (given its connection with the anomalous magnetic moment in the metric case). It is convenient to rewrite:
\[ \bar{\Omega}_D = \bar{\Omega}_a + \Omega_a^* \cos \Theta (\vec{V}_\perp + \vec{\beta}) \]  
(50)
with
\[ \bar{\Omega}_a = \frac{e}{2m} \left( g + g_s V^2 \cos^2 \Theta - 2 \right) \vec{B} \]  
(51)
and \( \Omega_a^* = \frac{e}{2m} g_s BV \); where \( \Theta \) represents the angle between \( V \) and the magnetic field, and \( V_\perp \) the component of the velocity perpendicular to \( B \). In \( \bar{\Omega}_a \) we group all the terms parallel to the magnetic field that contribute to the anomalous frequency (including nonmetric effects). The remaining terms perpendicular to \( B \) arise from nonmetric effects only, and produce a temporal variation of the spin component parallel to the magnetic field. This effect is absent in the metric case, and so represents a qualitatively new manifestation of possible EEP violation.
In general we are interested in solving (49) for the cases $\beta >> V$ or $\beta << V$ so that $\gamma(u) \simeq \gamma(\beta)$ or $\gamma(V)$, but is otherwise constant. Since $\Omega_a$ is proportional to $\xi$, we can perturbatively solve for each component in (49). Taking, for example, the initial condition $\vec{S}(0) = S\hat{\beta}$ we find

$$
\begin{align*}
S_\parallel &= S \cos \Omega_a t \quad S_\perp = S \sin \Omega_a t \\
S_\| &= S \frac{\Omega_a}{\Omega_a} \beta \cos \Theta (1 - \cos \Omega_a t) + S \frac{\Omega_a}{\Omega_a + \Omega_c} \frac{2V}{2} \left[ \cos(\Omega_a + \Omega_c) t - 1 \right]
\end{align*}
$$

where we have chosen a coordinate system where $\hat{E} = \hat{z}$ so that $\hat{V} = \hat{x} \sin \Theta + \hat{z} \cos \Theta$, $\hat{\beta} = \hat{y} \cos \Omega_c t - \hat{x} \sin \Omega_c t$ and assumed that any rotation related to $\Theta$ is negligible in comparison to other frequencies involved in the problem ($\Omega_a$ or $\Omega_c$).

The fact that $\Omega_a$ was (in the metric case) proportional to $g - 2$, motived the very precise $g - 2$ experiments which were designed to specifically measure that anomalous frequency. We see that this frequency is modified by from its metric value by the additional terms present in (51). If we assume that the EEP-violating contributions to $\Omega_a$ are bounded by the current level of precision for anomalous magnetic moments [15], then the discrepancy between the best empirical and theoretical values for the electron yields the bounds

$$
|\xi_e^-| < 3.5 \times 10^{-8} \quad \text{and} \quad |\xi_e^- - \xi_e^+| < 10^{-9}
$$

the latter following from a comparison of positron and electron magnetic moments. For muons, a similar analysis yields

$$
|\xi_{\mu^-}| < 10^{-8} \quad \text{and} \quad |\xi_{\mu^-} - \xi_{\mu^+}| < 10^{-8} .
$$

Even though the accuracy of the muon anomaly is lower than the electron one, the slightly stronger bound in (55) arises because the experiments are carried out for high-velocity muons [16]. To our knowledge these bounds on violation of gravitational universality are the most stringent yet noted for leptonic matter.

Newman et. al. analyzed the $g - 2$ experiments [10] in order to find new bounds for the validity of special relativity. They assumed that the parameter $\gamma$ involved in the electron motion had a different value ($\tilde{\gamma}$) from that which arises kinematically (in Thomas precession and Lorentz transformations). The equivalent equation for (51) is in that case

$$
\Omega_a^{NFRS} = \frac{eB}{m} \left( \frac{g}{2} - \gamma \right)
$$
and by comparing with two electron $g - 2$ experiments, one at electron relativistic energy ($\beta = 0.57$) and the other nearly at rest ($\beta = 5 \times 10^{-5}$), they obtained the constraint $\delta \gamma / \tilde{\gamma} < 5.3 \times 10^{-9}$. Our approach is qualitatively different from theirs, in that we assume $\gamma = \tilde{\gamma}$ but include preferred frame effects in the evaluation of the anomalous magnetic moment. A similar analysis in our case yields the weaker bounds of $|\xi_e| < 7 \times 10^{-6}$ for electrons, and $|\xi_\mu| < 2 \times 10^{-7}$ for muons. In the latter we used the $g - 2$ muon experiments carried at $\beta = 0.9994$ ($\gamma = 29$) [16], and $\beta = 0.92$ ($\gamma = 12$) [17].

Preferred effects not only modify the anomalous frequency according to (51), but also induce oscillations in the spin component parallel to $B$. As stated above, this is a qualitatively new consequence of EEP violations due solely to radiative corrections in GMQED. Searching for such oscillations therefore provides a new null test of the EEP. We can estimate the magnitude of such effects by taking the temporal average of $S_\parallel$ over the main oscillation given by $\Omega_a$, which gives

$$\delta = \frac{\langle S_\parallel \rangle}{S} \sim \xi V \beta \cos \Theta \gamma^2$$

(57)

This effect is enhanced in highly relativistic situations, and can be estimated by considering a typical experiment with $V \sim 10^{-3}$. For electrons $\beta \sim 0.5$, and so $\delta_e \sim 10^{-11}$; for muons $\beta = 0.9994$, yielding $\delta_\mu \sim 10^{-8}$. In both cases we used the corresponding present constraints for $\xi$ given above.

The novelty of the $S_\parallel$ oscillation suggests the possibility of putting tighter constraints on the non-metric parameter, once appropriate experiments are carried out. The same goes for the analysis of $\Omega_a$ at different values of $\Theta$ (the angle between the magnetic field and the velocity of the laboratory system with respect to the preferred frame). The rotation of the Earth will turn this orientation dependence into a time-dependence of the anomalous magnetic moment, with a period related to that of the sidereal day.

The previous analysis was concerned with effects related to spatial anisotropy. We turn now to considering possible violations of local position invariance. The position dependence in the former section was implicit in the redefinitions of charge, mass and fields. These quantities were rescaled in terms of the $TH\epsilon_\mu$ functions, which were considered constant throughout the computation. LPI violating experiments are of two types. One of these entails the measurement of a given frequency at two different points in a gravitational field (where differences in the gravitational potential could be significant) within the same reference system. The other type involves a comparison of frequencies arising from two different forms of energy (i.e. two different clocks)
at the same point in a gravitational potential. We parameterize the gravitational dependence on a given frequency as:

$$\Omega = \Omega^0 \left[ 1 - U + \Xi^{i j} U_{i j} \right] + \cdots$$

(58)

where \( U_{i j} \) represents the external gravitational tensor, satisfying \( U_{i i} = U \), and the ellipsis represents higher order terms (going as either \( U^2 \) or velocity times \( U \)) in the gravitational potential or terms independent of it.

The measured redshift parameter related to this frequency may be written as

$$Z = \Delta U \left( 1 - \Xi \right), \quad \Xi = \Xi^{i j} \frac{\Delta U_{i j}}{\Delta U}$$

(59)

where \( \Xi^{i j} \) will depend upon the specific frequency measured in the experiments. Note that this tensor is equivalent to the anomalous passive gravitational mass tensor introduced for the study of atomic transitions.

In \( g - 2 \) experiments the relevant frequency is \( \Omega_a \), which describes the precession of the longitudinal polarization in the presence of a constant magnetic field. Using the \( TH\epsilon\mu \) formalism (see eq. (51)) we obtain

$$\Omega_a = \frac{e B}{2m} \left[ g - 2 \right] + \cdots = \frac{e B \alpha}{2m \pi} \left[ 1 + \xi \frac{7}{6} \right] + \cdots$$

(60)

where we have omitted terms proportional to velocities, which eventually will contribute as \( O(v^2 U) \) terms at most.

In order to carry out the loop calculation, the \( TH\epsilon\mu \) dependence was absorbed into the definition of the parameters under the rescaling

$$\Omega \to \Omega/c_0 \quad m \to m \sqrt{T_0/c_0} \quad \alpha \to \alpha/\epsilon_0 c_0$$

(61)

with \( c_0 = (T_0/H_0)^{1/2} \) as the limiting speed of the massive particles, the subscript ‘0’ denoting the \( TH\epsilon\mu \) functions evaluated locally at \( \vec{X} = 0 \). Although the product \( eB \) remains invariant under this rescaling, the expression for the constant magnetic field still depends on the \( TH\epsilon\mu \) parameters once it is written solely in terms of atomic parameters. This can be seen clearly by considering the magnetic field produced by a long solenoid of length \( L \), with \( N \) turns and carrying a current \( I \). The gravitationally modified Maxwell equation to solve is:

$$\vec{\nabla} \times (\mu^{-1} \vec{B}) = 4\pi \vec{J}$$

(62)

and so we find the non-vanishing magnetic field inside the solenoid to be \( \vec{B} = 4\pi \mu_0 I N/L \). Again we assume that the \( TH\epsilon\mu \) functions are constant throughout
the size of the experimental device. In terms of fundamental atomic parameters, $L$ is proportional to an integer times the Bohr radius (the interatomic spacing), which is known to rescale as $a_0 \rightarrow a_0 c_0^2 \sqrt{T_0}$. If we now write $I = \int \vec{J} \cdot d\vec{S}$, where $J$ can be expressed in terms of a density charge $\rho$ in motion ($v$) through a volume $V$, and then relate the Bohr radius to each spatial dimension along with the limiting particle velocity $c_0$ to the velocity distribution $v$, we can show $I \rightarrow I \sqrt{T_0/\epsilon_0 c_0}$, and so $B \rightarrow B \mu_0 T_0/\epsilon_0 c_0^3$. Along with (61), this gives the position dependence of (60) to be

$$\Omega_a = \Omega_a^0 \sqrt{T_0/\epsilon_0 c_0} (1 + \frac{\xi}{6})$$

with $\Omega_a^0 = eB\alpha/2m\pi$ (recall $\xi = 1 - 1/\mu_0 \epsilon_0 c_0^2$).

Note that the $TH\epsilon\mu$ functions are evaluated at some representative point of the system, which we have chosen to be the origin $\vec{X} = 0$. In order to determine how $\Omega_a$ changes as the position of the system varies, we expand the $TH\epsilon\mu$ functions in (63)

$$T(U) = T_0 + T_0' \vec{g}_0 \cdot \vec{X} + O(\vec{g}_0 \cdot \vec{X})^2$$

where $\vec{g}_0 = \nabla U|_{\vec{X}=0}$, $T_0 = T|_{\vec{X}=0}$, and $T_0' = dT/dU|_{\vec{X}=0}$. It is useful to redefine the gravitational potential $U$ by

$$U \rightarrow -\frac{1}{2} \frac{T_0'}{H_0} \vec{g}_0 \cdot \vec{X}$$

whose gradient yields the test-body acceleration $\vec{g}$. This finally yields

$$\Omega_a = \Omega_a^0 \left[1 - U + \left(\frac{11}{6} \Gamma_0 - \frac{13}{6} \Lambda_0\right)U\right]$$

where we have rescaled again according to (61), and omitted terms proportional to $\xi$, since the main position dependence parameterization is given in terms of:

$$\Gamma_0 = \frac{2T_0}{T_0'} \left(\frac{\epsilon_0'}{\epsilon_0} + \frac{T_0'}{2T_0} - \frac{H_0'}{2H_0}\right), \quad \Lambda_0 = \frac{2T_0}{T_0'} \left(\frac{\mu_0'}{\mu_0} + \frac{T_0'}{2T_0} - \frac{H_0'}{2H_0}\right)$$

By comparing eq. (66) with (68), we can identify

$$\Xi^g = 11/6 \Gamma_0 - 13/6 \Lambda_0$$

as the LPI-violating parameter. Note that this depends on the anomalous frequency related to the longitudinal polarization of the beam. It is also species-dependent, with the value of $\Gamma_0$ and $\Lambda_0$ for the electron differing from that of the muon. A search for possible position dependence of anomalous spin precession frequencies provides another qualitatively new test of LPI sensitive to radiative corrections.
Actually the most precise $g - 2$ experiments for electron measure the ratio $a = \Omega_a / \Omega_c$ at non relativistic electron energies ($\beta \sim 10^{-5}$), and so $\Omega_c \simeq eB/m$. This is interesting because by following the former parameterization we can write:

$$\Omega_c = \Omega_c^0 \left[ 1 - U + (2\Gamma_0 - \Lambda_0)U \right]$$

(69)

or by taking the ratio of (66) to (69):

$$a = a^0(1 + U \Xi^a), \quad \Xi^a = \frac{1}{6} \Gamma_0 + \frac{7}{6} \Lambda_0$$

(70)

and then by identifying $a$ with the most precise experimental value [18] and $a^0$ with the theoretical one[15], we can constrain through the resulting theoretical/experimental errors $|U \Xi^a| < 3 \times 10^{-8}$. This result is sensitive to the absolute value of the total local gravitational potential [4], whose magnitude has recently been estimated to be as large as $3 \times 10^{-5}$ due to the local supercluster [8]. Hence measurements of this type can provide us with empirical information sensitive to radiative corrections that constrains the allowed regions of $(\Gamma_0, \Lambda_0)$ parameter space, giving in this case:

$$|\frac{1}{6} \Gamma_0 + \frac{7}{6} \Lambda_0| < 10^{-3}$$

(71)

For muons the analogous constraint is $|U \Xi^a_\mu| < 10^{-5}$, and so nothing conclusive is obtained.

We pause to compare this result to an analogous result obtained for hyperfine transitions (maser clocks). In this case the baryonic and leptonic gravitational parameters appear simultaneously. The atomic hyperfine splitting comes from the interaction between the magnetic moments of the electron and proton (nucleus). The proton metric appears only in the latter, and so it does not affect the principal and fine structure atomic energy levels. Non-metric effects imply a shift in the hyperfine energy $E_{hf}$ which is [2]

$$\Delta E_{hf} = \mathcal{E}_{hf}(1 - U_B) + \mathcal{E}_{hf}U_B \Xi^{hf}$$

(72)

with

$$\Xi^{hf} = 3\Gamma_B - \Lambda_B + \Delta$$

(73)

where $U_B$, $\Gamma_B$ and $\Lambda_B$ are the baryonic analogues of the parameters appearing in (67). In (72) we rescaled the atomic parameters to absorb the $TH\epsilon\mu$ functions and chose units such that $c_B = 1$. The quantity $\Delta$ is given by

$$\Delta = 2 \frac{T_B}{T_B'} \left[ 2\left( \frac{H_B'}{H_B} - \frac{H_0'}{H_0} \right) - \frac{T_B'}{T_B} + \frac{T_0'}{T_0} \right]$$

(74)
and would vanish under the assumption that the leptonic and baryonic $T H \epsilon \mu$ parameters were the same. The gravity probe A experiment [19], employing hydrogen-maser clocks, was able to constrain the corresponding LPI violating parameter related to hyperfine transitions, obtaining

$$|\Xi_{Hf}| = |3\Gamma_B - \Lambda_B + \Delta| < 2 \times 10^{-4}$$

(75)

for the most stringent bound to date on $\Xi_{Hf}$.

We note that a similar analysis could be carried out for the energy shift defined in (43), which can be used as a frequency test to look for position or frame dependence. This can be done by following the same procedure as for atomic energy shifts, where the anomalous passive and inertial gravitational tensor are introduced in order to relate non-metric effects to redshift and time dilation parameters. Since radiative corrections are irrelevant in that energy shift, we omit that procedure here.

IV Concluding Remarks

Refined measurements of anomalous magnetic moments can provide an interesting new arena for investigating the validity of the EEP in physical systems where radiative corrections are important. We have considered this possibility explicitly for the class of non-metric theories described by the $T H \epsilon \mu$ formalism. The non-universal character of the gravitational couplings in such theories affects the one loop corrections to the scattering amplitude of a free fermion in an external electromagnetic field in a rather complicated way, giving rise to several novel effects.

An evaluation of the one-loop diagrams reveals that the leg corrections, which in the metric case give no contribution to the total amplitude after a proper renormalization of mass and spinor field, provide contributions which cannot be removed after renormalization. Moreover they are essential in ensuring the gauge invariance of the scattering amplitude, which is not fulfilled by the vertex correction alone. The consistency of the calculation is verified explicitly through the Ward identity, which furnishes a cross-check between the fermion self energy and the vertex correction. The non metric corrections to the scattering amplitude also have an infrared divergence, which could be understood in terms of inelastic soft photon radiation, as in the metric case. This does not affect the term associated with the anomalous magnetic moment.

The presence of preferred frame effects induces a new type of coupling between the magnetic field and the spin as described by (10). This interaction stems purely from radiative corrections, and generalizes the gyromagnetic ratio of a fermion to a
tensorial coupling described by $\Gamma_{ij}$. We emphasize that qualitatively new information on the validity of the EEP will be obtained by setting new empirical bounds on this coupling, as it is associated with purely leptonic matter. Comparatively little is known about such empirical limits on EEP-violation relative to the baryonic sector \cite{4}, for which previous experiments have set the limit $|\xi_B| \equiv |1 - c_B^2| < 6 \times 10^{-21}$ where $c_B$ is the ratio of the limiting speed of baryonic matter to the speed of light. We can therefore safely neglect any putative effects of $\xi_B$ in our analysis.

Consequently, discussion of a $g - 2$ contribution to the magnetic moment no longer makes sense, and we instead refer to the anomalous frequency as the main connection with experiment. Note that this frequency, defined as the relative electron spin precession with respect to its velocity, comes from radiative corrections and it becomes proportional to $g - 2$ in the metric case. This frequency shows an explicit dependence on both the preferred frame velocity and its relative direction with respect to the external magnetic field. There is also a dependence on the electron velocity, which makes the other contributions negligible at relativistic electron energies. Two $g - 2$ experiments on the electron (one at relativistic energies and the other almost at rest) may then be used to limit the preferred frame parameter to be no larger than $10^{-5}$, analogous to the work of Newman et al.. Constraining any possible EEP violation to be no larger than the present discrepancy between theory and experiment we found the most stringent bounds for $\xi$ yet obtained for leptonic matter, as given in (54) and (55).

We expect that new experiments which probe the anisotropic character (or angular dependence) of the frequency could be used to impose stronger limits in different physical regimes. For example, as the Earth rotates, the spatial orientation of the magnetic field changes – this should in turn diminish the experimental errors involved in the comparison between two energetically different $g - 2$ experiments.

The relativistic generalization of the spin polarization equation (47), followed the same procedure as for the metric case, where non-metric effects where included in the interaction only (eq. (40)). This yields an equation of motion for the spin (as seen from the rest frame) which is qualitatively different from that expected from its classical counterpart, where the angular momentum rate is related to the torque applied on the system. This approach for dealing with violations of Lorentz invariance is dynamical; from a kinematical viewpoint we assume that standard Lorentz transformations relate coordinates and fields from one system to another.

Perhaps the most remarkable feature of the non-metric effects is that of the oscillations of the component of spin polarization parallel to the magnetic field. Since
this component remains constant in the metric case, an experiment which searches for such oscillations is a new null test of the equivalence principle that is uniquely sensitive to radiative corrections in the leptonic sector. Hence an empirical investigation of its behavior will provide qualitatively new information about the validity of EEP, and could constrain even further the limits on the preferred frame parameters.

Finally, we analyzed the behavior of the anomalous frequency in the context of redshift experiments, which can put constraints on the LPI-violating parameters ($\Gamma_0$, $\Lambda_0$) once the corresponding experiments are carried out. This region of parameter space is qualitatively different from that probed by either Lamb-shift or hyperfine effects. In the electron sector a bound on the magnitude of $U^a$ can be obtained by demanding that it be no larger than the error bounds in the discrepancy between the experimental and theoretical values of the ratio $a = \Omega_a/\Omega_c$. Assuming the local potential to be as large as that estimated from the local supercluster, we obtain a bound on $|\Xi^a|$ that is comparable the limit on an analogous quantity in the baryonic sector obtained from redshift experiments [19]. However this latter experiment is proportional to changes in the local potential, which are $\sim 10^{-10}$. More direct limits on $|\Xi^a|$ must be set by performing a similar sort of redshift experiment on anomalous magnetic moments [20]. The logistics and higher precision demanded by such an experiment will be a major challenge to undertake.

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Appendices

A  Loop integrations

We show the main steps leading to Eqs. (21), (20), and (22). Details are given throughout the computation by considering only the first term of the photon propagator (18), that is

$$G_{\xi}^{\mu\nu} = \frac{\beta_\mu \beta_\nu}{k^2} + \cdots$$  \hspace{1cm} (A1)$$

with the remaining term in (18) contributing in a similar manner.
We solve for the fermion self energy by replacing (A1) in (15), and using (19) along with the Feynman parameters

\[
\frac{1}{a^2b} = 2 \int_0^1 \frac{dz}{|az+b(1-z)|^3}.
\]

After integrating we obtain

\[
\Sigma(p) = \frac{\alpha}{4 \pi} \left( \frac{5}{2} + \ln(\frac{A}{m})^2 \right) - (\not p - m) \frac{\beta^2}{2} \left( \ln(\frac{A}{m})^2 + \frac{1}{2} \right)
\]

with \( \Delta = p^2 - m^2 \). We consider \( \Delta/m^2 \ll 1 \), and expand the above to obtain after some manipulation

\[
\Sigma(p) = \frac{\alpha}{4 \pi} \left\{ \beta \cdot p \left( \frac{5}{2} + \ln(\frac{A}{m})^2 \right) - (\not p - m) \frac{\beta^2}{2} \left( \ln(\frac{A}{m})^2 + \frac{1}{2} \right) \right\} - m \frac{\beta^2}{2} \left( \frac{1}{2} - \ln(\frac{A}{m})^2 \right) + O\left( (\not p - m)^2 \right) + \cdots
\]

where we have kept the leading terms as \( \mu \to 0 \) and \( \Lambda \to \infty \), and \( O\left( (\not p - m)^2 \right) \) stands for the terms satisfying

\[
S_F(p) O\left( (\not p - m)^2 \right) u(\vec{p}) = 0.
\]

We renormalize \( \Sigma(p) \) by subtracting

\[
\delta \Sigma = \delta m + \delta \xi_0 \beta \beta \cdot p
\]

where the counterterms respectively account for mass and \( TH\epsilon_m \) -parameter renormalization.

Choosing the counterterms so that

\[
\overline{\pi}(\vec{p}) \Sigma(p) u(\vec{p}) = 0,
\]

and so

\[
\delta m = -\frac{\alpha}{4 \pi} \frac{\beta^2}{2} \left( \frac{1}{2} - \ln(\frac{A}{m})^2 \right) \quad \text{(A5)}
\]

\[
\delta \xi_0 = \frac{\alpha}{4 \pi} \left( \frac{5}{2} + \ln(\frac{A}{m})^2 \right) \quad \text{(A6)}
\]

the regularized result is then

\[
\Sigma(p) = \frac{\alpha}{\pi} (\not p - m) \left\{ \frac{(\beta \cdot p)^2}{m^2} \left[ \frac{1}{2} \ln(\frac{m}{\mu})^2 - \frac{3}{2} \right] - \beta^2 \left[ \frac{1}{8} \ln(\frac{A}{m})^2 + \frac{1}{16} \right] \right\} + O\left( (\not p - m)^2 \right) \quad \text{(A7)}
\]
Note that the remaining ultraviolet divergence related to this term could be removed after charge renormalization. We find it convenient to leave it in order to cross-check the calculation, since a similar term from the vertex part should cancel it, thereby removing the divergence from the resulting scattering amplitude.

The evaluation of the vertex function follows a similar procedure, giving the result

$$
\Gamma^\mu = \frac{\alpha}{\pi} \int \frac{dx}{p_x^2} \left\{ \gamma^\mu \left[ \frac{1}{2} \beta \cdot p \gamma^\alpha \beta + (1 - x) \frac{\beta}{2} \right] \right\}
$$

with

$$
p_x^2 = m^2 - x(1 - x)q^2,
$$

with $q = p' - p$, and so expand

$$
m^2 = 1 + x(1 - x)q^2/m^2 + O(q^4),
$$

which after some algebra reduces (A8) to

$$
\Gamma^\mu(p', p) = \frac{\alpha}{\pi} \left\{ \gamma^\mu \left[ -\beta^2 \left( \frac{1}{16} + \frac{1}{8} \ln(m^2/\Lambda^2) \right) + \frac{(\beta \cdot p)^2}{m^2} \left( \frac{1}{2} \ln(m^2/\mu^2) - 3 \right) \right]
$$

+ \frac{q^2}{m^2} \left( \frac{1}{12} \ln(m^2/\mu^2)^2 - \frac{1}{3} \right) + \frac{\beta \cdot p \beta \cdot q}{m} \left( \frac{1}{2} \ln(m^2/\mu^2) - \frac{5}{4} \right)
$$

+ \frac{q^2}{m^2} \left( \frac{1}{2} \ln(m^2/\mu^2) - \frac{5}{4} \right) + \frac{5}{24} \beta \cdot q \beta \cdot m
$$

\frac{\beta^2 m^2}{2} \left( \frac{1}{4} \beta \cdot p + \frac{1}{8} \beta \cdot q \right)
$$

- \frac{1}{8} \frac{q^2}{m^2} \beta \cdot q \beta \cdot m
$$

- \frac{\beta^2 q^2}{m^2} \beta \cdot q \beta \cdot m
$$

+ O(q^3)
$$

where the vertex function has been renormalized by subtracting a term like

$$
\delta \Gamma^\mu = \delta \xi_0 \beta \beta^\mu
$$

with $\delta \xi_0$ is given by (A10). We recall that gauge invariance forces this coefficient to be equal to the one participating in the renormalization of the fermion self energy.
B Adiabatic hypothesis

In order to describe how self energy effects convert the incident electron from a bare particle to a physical one, it is convenient to introduce a damping function, $g(t)$, which adiabatically switches off the coupling between fields, such that the interaction lagrangian is replaced by

$$\mathcal{L}_I = e g(t) \overline{\psi}(x) A(x) \psi(x) \quad (B1)$$

It is assumed that the time $T$ over which $g(t)$ varies is very long compared to the duration of the scattering process. In momentum space

$$g(t) = \int G(\Omega_0) \exp(i \Omega \cdot x) d\Omega_0 \quad (B2)$$

with $\Omega \equiv (\Omega_0, 0)$, and $g(0) = 1$. It is supposed that $G(\Omega_0)$ is almost a delta function, being large for $\Omega_0$ in a range of about $T^{-1}$.

In the presence of an external field $A_\mu$, eq. (14) will now read

$$L \cdot A \rightarrow \int G(\Omega_0) G(\Omega'_0) d\Omega_0 d\Omega'_0 A(p' - p - \Omega - \Omega') i S_F(p - \Omega - \Omega') i \Sigma(p - \Omega) + \cdots \quad (B3)$$

where $\cdots$ represents the equivalent second term from (14).

As $T \to \infty$, and $\Omega_0, \Omega'_0 \to 0$, the fermion propagator reduces to

$$S_F(p - \Omega - \Omega') \Sigma(p - \Omega) = \frac{1}{\not{p} - m} \frac{\not{\Omega} - \not{\Omega'} - m}{2p_0(\Omega_0 + \Omega'_0)} \quad (B4)$$

where we used $p^2 = m^2$. This implies that we can expand $\Sigma$ up to order $\Omega$ only, since higher terms vanish after taking the previous limit. Here we employ the relation

$$\frac{1}{A - B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \cdots \quad (B5)$$

to expand

$$\Sigma(p - \Omega) \simeq \Sigma(p) - \frac{\partial \Sigma(p)}{\partial p^\mu} \Omega^\mu \quad . \quad (B6)$$

After renormalization, $\Sigma$ takes the form

$$\Sigma(p) = (\not{p} - m)(A + B(\beta \cdot p)^2) + C(\beta \cdot p)\left\{ \beta(\not{p} - m) + (\not{p} - m) \beta \right\} + \cdots \quad (B7)$$

where the constants $A, B, \text{and } C$ can be obtained from eq. (20).

Let us introduce $\Omega \equiv \Omega + \Omega'$, and symmetrize $\Omega$ by $\frac{1}{2}(\Omega + \Omega')$ in (B3), to write

$$S_F(p - \Omega - \Omega') \Sigma(p - \Omega) = \frac{1}{\not{p} - m} \frac{\not{\Omega} - m}{2} \left( \Sigma(p) - \frac{1}{2} \frac{\partial \Sigma(p)}{\partial p^\mu} \Omega^\mu \right) \quad (B8)$$
which after using (B7) can be written as
\[
\frac{1}{2}(A + B(\beta \cdot p)^2) + C\beta \cdot p \beta \tag{B9}
\]
where we have used that \(\Sigma(p - \Omega)\) is acting on a free spinor, and therefore terms of the form \((p' - m)u(p)\) vanish. Now, the final evaluation of (14) follows directly from (B9).

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