Phragmén–Lindelöf principles for generalized analytic functions on unbounded domains

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Abstract

We prove versions of the Phragmén–Lindelöf strong maximum principle for generalized analytic functions defined on unbounded domains. A version of Hadamard’s three-lines theorem is also derived.

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1 Introduction

Versions of the maximum principle for complex-valued functions defined on a domain in $\mathbb{C}$ have been of interest since the development of the classical maximum modulus theorem and Phragmén–Lindelöf principle for holomorphic functions (see, e.g. [10] Chap. V). It is important to distinguish between two types of result here. First, there is the weak maximum principle
asserting that under certain circumstances a nonconstant function \( f : \Omega \to \mathbb{C} \) cannot attain a local maximum in its domain \( \Omega \): thus if \( \Omega \) is bounded and \( f \) is continuous on \( \overline{\Omega} \) we have
\[
\sup_{z \in \Omega} |f(z)| = \sup_{z \in \partial \Omega} |f(z)|.
\]

Second – and this will be our main concern in this paper – there is the strong maximum principle or Phragmén–Lindelöf principle. This generally applies to unbounded domains, and generally a supplementary hypothesis on \( f \) is required for the conclusion (1) to hold. For example, if \( f : \Omega \to \mathbb{C} \) is analytic, where \( \Omega = \mathbb{C}_+ \), the right-hand half-plane \( \{ z \in \mathbb{C} : \text{Re} z > 0 \} \), then if \( f \) is known to be bounded we may conclude that (1) holds, whereas the example \( f(z) = \exp(z) \) shows that it does not hold in general.

We shall use the following standard notation:
\[
\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}(f_x - i f_y) \quad \text{and} \quad \overline{\partial f} = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f_x + i f_y).
\]

For quasi-conformal mappings \( f \), that is, those satisfying the Beltrami equation \( \overline{\partial f} = \nu \partial f \) with \( |\nu| \leq \kappa < 1 \), the weak maximum principle holds (see, for example [4]). This fact was used in [1, Prop. 4.3.1] to deduce a weak maximum principle for functions solving the conjugate Beltrami equation
\[
\overline{\partial f} = \nu \overline{\partial f}.
\]
(2)

Their argument is based on the fact that if \( f \) is a solution to (2), then it also satisfies a classical Beltrami equation \( \overline{\partial f} = \nu f \partial f \), where \( \nu f(z) = \nu(z) \overline{\partial f(z)/\partial f(z)} \), and hence \( f = G \circ h \) where \( G \) is holomorphic and \( h \) is a quasi-conformal mapping (cf. [12 Thm. 11.1.2]).

Carl [3] considered functions \( w \) satisfying equations of the form
\[
\overline{\partial w}(z) + A(z)w(z) + B(z)\overline{w(z)} = 0
\]
(3)
and deduced a weak maximum principle for such functions, analogous to (1), under certain hypotheses on the functions \( A \) and \( B \). We shall take this as our starting point.

For general background on generalized analytic functions (pseudo-analytic functions) we refer to the books [2, 9, 11]. The following definitions are taken from the recent paper [11].
Definition 1.1. Let $1 \leq p < \infty$. For $\nu \in W^{1,\infty}(\mathbb{D})$ (i.e., a Lipschitz function with bounded partial derivatives), the class $H^p_{\nu}$ consists of all measurable functions $f : \mathbb{D} \to \mathbb{C}$ satisfying the conjugate Beltrami equation (2) in a distributional sense, such that the norm

$$
\|f\|_{H^p_{\nu}} = \left( \text{ess sup}_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}
$$

is finite. Clearly for $\nu = 0$ we obtain the classical Hardy space $H^p(\mathbb{D})$. If instead $\nu$ is defined on an arbitrary subdomain $\Omega \subset \mathbb{C}$, we may define the class $H^\infty_{\nu}(\Omega)$ as the space of all bounded measurable functions satisfying (2), equipped with the supremum norm.

We may analogously define spaces $G^p_\alpha(\mathbb{D})$, where $\alpha \in L^\infty(\mathbb{D})$, and in general $G^\infty_\alpha(\Omega)$, where now, for a function $w$ we replace (2) by

$$
\overline{\partial}w = \alpha w. 
$$

(4)

Once again, the case $\alpha = 0$ is classical.

When $\nu$ is real (the most commonly-encountered situation), there is a link between the two notions: suppose that $\|\nu\|_{L^\infty(\Omega)}$ with $\|\nu\|_{\infty} \leq \kappa < 1$, and set $\sigma = \frac{1 - \nu}{1 + \nu}$ and $\alpha = \frac{\sigma}{2\sigma}$, so that $\sigma \in W^{1,\infty}_R(\Omega)$. Then $f \in L^p(\mathbb{D})$ satisfies (2) if and only if $w := \frac{f - \nu \overline{f}}{\sqrt{1 - \nu^2}}$ satisfies (4).

We shall mainly be considering the class $G^\infty_\alpha$, for which it is possible to prove a strong maximum principle and a generalization of the Hadamard three-lines theorem under mild hypotheses on $\alpha$, which are satisfied in standard examples. The referee has suggested that there may be a link between these assumptions and the strict ellipticity of $\sigma$, although we have not been able to show this.

2 Functions defined on unbounded domains

The following result is an immediate consequence of [3, Thm. 1], taking $A = 0$ and $B(z) = -\alpha(z)$ in (3) in order to obtain (4).
Proposition 2.1. Suppose that $\Omega$ is a bounded domain in $\mathbb{C}$ and that $w$ is a continuous function on $\overline{\Omega}$ such that (4) holds in $\Omega$, where $\alpha$ satisfies $2|\alpha|^2 \geq |\partial \alpha|$. Then $|w(z)| \leq \sup_{\zeta \in \partial \Omega} |w(\zeta)|$ for all $z \in \Omega$.

Proof. Taking $k = 2$ in [3, Thm. 1], we require that the matrix $M = (m_{ij})_{i,j=1}^2$ be negative semi-definite, where, with $a = -2|\alpha|^2$ and $b = -\partial \alpha$, we have

$$M = \begin{pmatrix} a + \text{Re} b & \text{Im} b \\ \text{Im} b & -b - \text{Re} a \end{pmatrix}.$$ 

On calculating $m_{11}$, $m_{22}$ (which must be non-positive) and $\text{det} M$ (which must be non-negative) we obtain the sufficient conditions $-2|\alpha|^2 \pm \text{Re} \partial \alpha \leq 0$ and $2|\alpha|^2 \geq |\partial \alpha|$: clearly the second condition implies the first.

Example 2.1. In the example $\sigma = 1/x$, occurring in the study of the tokamak reactor [5, 6], we have $\alpha(x) = -\frac{1}{4x}$ and $\partial \alpha = \frac{1}{8x^2}$; thus the inequality $2|\alpha|^2 \geq |\partial \alpha|$ is always an equality.

Note that by rescaling $z$ we may transform the equation (4) to one with $\alpha = -\frac{1}{\lambda} x$ for any $\lambda > 0$ (with the domain also changing); then the inequality requires that $2/\lambda^2 \geq 1/2\lambda$, so that if we take $0 < \lambda < 4$ the inequality is strict.

Now for $\varepsilon > 0$ we write $h_\varepsilon(z) = 1/(1 + \varepsilon z)$, and note that whenever $\Omega \subset \mathbb{C}_+$ is a domain, we have that the functions $h_\varepsilon$ satisfy

(i) For all $\varepsilon > 0$, $h_\varepsilon \in \text{Hol}(\Omega) \cap C(\overline{\Omega})$.

(ii) For all $\varepsilon > 0$, $\lim_{|z| \to \infty, z \in \Omega} h_\varepsilon(z) = 0$.

(iii) For all $z \in \Omega$, $\lim_{\varepsilon \to 0} |h_\varepsilon(z)| = 1$.

(iv) For all $\varepsilon > 0$, for all $z \in \partial \Omega$, $|h_\varepsilon(z)| \leq 1$.

Suppose that $\overline{\partial} w = \alpha \overline{w}$ and that $h$ is holomorphic; then $\overline{\partial}(hw) = \beta \overline{hw}$, where $\beta = \alpha h/\overline{h}$. Moreover,

$$\partial \beta = \partial(\alpha h)/\overline{h} = (\partial \alpha)(h/\overline{h}) + \alpha(\partial h)/\overline{h}.$$ 

That is, with $h = h_\varepsilon$, we have $|\beta| = |\alpha|$ and $|\partial \beta| \leq |\partial \alpha| + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$.

Theorem 2.1. Suppose that $\Omega \subset \mathbb{C}_+$ (not necessarily bounded) and that $w$ is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in $\Omega$ where $\alpha$ is a $C^1$ function satisfying $2|\alpha|^2 \geq |\partial \alpha| + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$ for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta \in \partial \Omega} |w(\zeta)|$ for all $z \in \Omega$.
Proof. Fix $\varepsilon > 0$ and $M = \sup_{z \in \partial \Omega} |w(z)|$. Suppose that $M > 0$. Then by property (ii) there is an $\eta > 0$ such that for all $z \in \Omega$ with $|z| \geq \eta$ we have $|w(z)h_\varepsilon(z)| \leq M$.

Now, by property (i) and Proposition 2.1 we have

$$\sup_{z \in \Omega} |w(z)| = \sup_{z \in \partial \Omega} |w(z)h_\varepsilon(z)|,$$

at least if $2|\alpha|^2 \geq |\partial \alpha| + |\alpha|/|\partial h_\varepsilon|/|h_\varepsilon|$.

Now $\partial (\Omega \cap D(0, \eta)) \subset (\partial \Omega \cap \overline{D(0, \eta)}) \cup (\partial D(0, \eta) \cap \overline{\Omega})$.

By hypothesis, $|w(z)| \leq M$ if $z \in \partial \Omega$, and by property (i), $|h_\varepsilon(z)| \leq 1$ for $z \in \partial \Omega$. So $\sup_{z \in \partial (\Omega \cap D(0, \eta))} |w(z)h_\varepsilon(z)| \leq M$.

By the definition of $\eta$ we also have $|w(z)h_\varepsilon(z)| \leq M$ if $|z| \geq \eta$ with $z \in \overline{\Omega}$, and in particular for $z \in \overline{\Omega} \cap \partial D(0, \eta)$.

We conclude that $\sup_{z \in \Omega \cap D(0, \eta)} |w(z)h_\varepsilon(z)| \leq M$. However, $|w(z)h_\varepsilon(z)| \leq M$ whenever $z \in \overline{\Omega}$ with $|z| \geq \eta$, and hence $\sup_{z \in \Omega} |w(z)h_\varepsilon(z)| \leq M$. Now, letting $\varepsilon$ tend to 0, and using property (iii), we have the result in the case $M > 0$.

If $M = 0$, then by the above we have that $\sup_{z \in \partial \Omega} |w(z)| \leq \gamma$ for all $\gamma > 0$, and the same holds for $z \in \Omega$ by the above. Letting $\gamma \to 0$ we conclude that $w$ is identically 0 on $\Omega$.

Example 2.2. Consider the case $\alpha = -\frac{1}{\lambda x}$ and $\partial \alpha = \frac{1}{2\lambda x^2}$. For the hypotheses of the theorem to be valid we require

$$\frac{2}{\lambda x^2} \geq \frac{1}{2\lambda x^2} + \frac{1}{\lambda x} \frac{x}{|1 + \varepsilon z|}.$$

If $\lambda = 1$ (and by rescaling the domain we can assume this) then this always holds, since $|1 + \lambda z| \geq \lambda x$.

In the following theorem, it will be helpful to note that we shall be considering composite mappings as follows:

$$\Lambda \stackrel{h}{\to} \Omega \stackrel{w}{\to} C \quad \text{and} \quad \Lambda \stackrel{h}{\to} \Omega \stackrel{\alpha}{\to} C.$$
Theorem 2.2. Suppose that $\Omega \subset \mathbb{C}$ is simply-connected and that the disc $D(a,r)$ is contained in $\mathbb{C} \setminus \overline{\Omega}$. Let $h : \mathbb{C} \to \mathbb{C}$ be defined by $h(z) = rz + a$, and let $\Lambda$ be a component of $h^{-1}(\Omega)$. Set $g_\varepsilon(z) = 1/(1 + \varepsilon g(z))$, where $g(z) = \log \left( \frac{z - a}{r} \right)$ is a single-valued inverse to $h$ defined on $\Omega$. Suppose that $w$ is a continuous bounded function on $\Omega$ such that (4) holds in $\Omega$ with $\alpha$ a $C^1$ function satisfying
\[
2|\alpha|^2 \geq |\partial \alpha| + |\alpha||\partial g_\varepsilon|/|g_\varepsilon| \tag{5}
\]
for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta \in \partial \Omega} |w(\zeta)|$ for all $z \in \Omega$.

Proof. First we identify the equation satisfied by $v = w \circ h$, where $h$ is holomorphic. Namely,
\[
\bar{\partial} v = \bar{\partial}(w \circ h) = \bar{\partial}(\overline{w \circ h}) = (\overline{\partial w \circ h})(\overline{\partial h}) = (\overline{\partial w \circ h})(\overline{\partial h}) = (\alpha \circ h)(\overline{\partial h}) = (\alpha \circ h)(\overline{\partial h}) = \beta \pi,
\]
where $\beta = (\alpha \circ h)(\overline{\partial h})$. Note that $\partial \beta = (\partial \alpha \circ h)|\partial h|^2$, since $\partial(\overline{\partial h}) = 0$.

The condition
\[
2|\beta|^2 \geq |\partial \beta| + |\beta||\partial h_\varepsilon|/|h_\varepsilon| \tag{6}
\]
at a point of $\Lambda$ can be rewritten

\[
2|\alpha \circ h|^2|\partial h|^2 \geq |\partial \alpha \circ h||\partial h|^2 + |\alpha \circ h||\partial h||\partial h_\varepsilon|/|h_\varepsilon|.
\]

Now $g_\varepsilon = h_\varepsilon \circ g$; thus $\partial h_\varepsilon = (\partial g_\varepsilon \circ h)(\partial h)$.

That is, (6) is equivalent to

\[
2|\alpha \circ h|^2|\partial h|^2 \geq |\partial \alpha \circ h||\partial h|^2 + |\alpha \circ h||\partial h|^2|\partial g_\varepsilon \circ h|/|g_\varepsilon \circ h|,
\]
or
\[
2|\alpha \circ h|^2 \geq |\partial \alpha \circ h| + |\alpha \circ h||\partial g_\varepsilon \circ h|/|g_\varepsilon \circ h|.
\]

The set $\Lambda$ is open, and thus $\partial \Lambda \cap \Lambda = \emptyset$ and also $h(\partial \Lambda) \cap \Omega = \emptyset$. Moreover, since $h(\partial \Lambda) \subset h(\overline{\Lambda}) \subset \overline{h(\Lambda)}$, we get $h(\partial \Lambda) \subset \overline{\Omega} \setminus \Omega = \partial \Omega$.

Since $w$ is bounded on $\Omega$, the function $v = w \circ h$ is bounded on $\Lambda$, and using the calculations above and Theorem 2.1 with condition (6), we see that
\[
\sup_{z \in \Lambda} |v(z)| = \sup_{z \in \partial \Lambda} |v(z)|.
\]
Since \( h(\Lambda) = \Omega \), \( \sup_{z \in \Lambda} |v(z)| = \sup_{z \in \Omega} |w(z)| \). Moreover, since \( h(\partial \Lambda) \subset \partial \Omega \), we have also
\[
\sup_{z \in \partial \Lambda} |v(z)| \leq \sup_{z \in \partial \Omega} |w(z)|.
\]
It follows that \( \sup_{z \in \Omega} |w(z)| \leq \sup_{z \in \partial \Omega} |w(z)| \) and we obtain equality.

We now provide a generalization of the three-lines theorem of Hadamard (see, for example [8, Thm. 9.4.8] for the classical formulation with \( \alpha = 0 \)).

**Theorem 2.3.** Suppose that \( a \) and \( b \) are real numbers with \( 0 < a < b \), and let \( \Omega = \{ z \in \mathbb{C} : a < \Re z < b \} \). Suppose that \( w \) is a continuous bounded function on \( \Omega \) such that (4) holds in \( \Omega \) where \( \alpha \) is a \( C^1 \) function satisfying
\[
2|\alpha|^2 \geq |\partial \alpha| + |\alpha||\log(M(a)/M(b))|/b - a + |\alpha||\partial \alpha|/h_{\varepsilon}\]
for each \( \varepsilon > 0 \). Then the function \( M \) defined on \([a, b]\) by
\[
M(x) = \sup_{y \in \mathbb{R}} |w(x + iy)|
\]
satisfies, for all \( x \in (a, b) \),
\[
M(x)^{b-a} \leq M(a)^{b-a} M(b)^{b-a}.
\]
That is, \( \log M \) is convex on \((a, b)\).

**Proof.** Consider the function \( g \) defined on \( \Omega \) by
\[
h(z) = M(a)^{(z-b)/(b-a)} M(b)^{(a-z)/(b-a)},
\]
where quantities of the form \( M^\omega \) are defined for \( M > 0 \) and \( \omega \in \mathbb{C} \) as \( \exp(\omega \log M) \), taking the principal value of the logarithm.

Now \( v := hw \) satisfies \( |v(z)| \leq 1 \) for \( z \in \partial \Omega \), since \( |h(a + iy)| = 1/M(a) \) and \( |h(b + iy)| = 1/M(b) \).

Given that \( \overline{\partial w} = \alpha \overline{w} \) and that \( h \) is holomorphic, then, as we have seen,
\[
\overline{\alpha} \overline{w} = \beta \overline{w}, \quad \text{where} \quad \beta = \alpha \overline{h}/h.
\]
Moreover, \( \partial \beta = \partial(\alpha h)/h = (\partial \alpha)(h/\overline{h}) + \alpha(\partial h)/h \).

Now \( \log h = \frac{z-b}{b-a} \log M(a) + \frac{a-z}{b-a} \log M(b) \), and so
\[
\left| \frac{\partial h}{h} \right| = \left| \frac{\log M(a)/M(b)}{b-a} \right|.
\]
Thus the condition (7) on $\alpha$ implies that $\beta$ satisfies $2|\beta|^2 \geq |\partial \beta| + |\beta||\partial h_\varepsilon|/|h_\varepsilon|$. Hence we can apply Theorem 2.1 to $v$, and the result follow.

\[ \square \]

**Remark 2.1.** As in Example 2.2, rescaling $z$ is helpful here, since if $z$ is reparametrized as $\lambda z$, then $\partial \alpha$ is divided by $\lambda$ and $b-a$ is also divided by $\lambda$: thus the inequality (7) becomes easier to satisfy.

## 3 Weights depending on one variable

We look at two cases here, for functions defined on a subdomain of $\mathbb{C}_+$, namely weights $\alpha = \alpha(x)$ and radial weights $\alpha = \alpha(r)$. We revisit Theorem 2.1.

Since we now have $\partial \alpha = \alpha'/2$, we obtain the following corollary.

**Corollary 3.1.** Suppose that $\Omega \subset \mathbb{C}_+$ (not necessarily bounded) and that $w$ is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in $\Omega$ where $\alpha = \alpha(x)$ is a $C^1$ function satisfying $2|\alpha|^2 \geq |\alpha'|/2 + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$ for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta \in \partial \Omega} |w(\zeta)|$ for all $z \in \Omega$.

Likewise, in polar coordinates $(r, \theta)$ we have

$$ \partial = \frac{1}{2} \left( e^{-i\theta} \partial_r - \frac{i e^{-i\theta}}{r} \partial_\theta \right), $$

giving the following result.

**Corollary 3.2.** Suppose that $\Omega \subset \mathbb{C}_+$ (not necessarily bounded) and that $w$ is a continuous bounded function on $\overline{\Omega}$ such that (4) holds in $\Omega$ where $\alpha = \alpha(r)$ is a $C^1$ function satisfying $2|\alpha|^2 \geq |\alpha'|/2 + |\alpha||\partial h_\varepsilon|/|h_\varepsilon|$ for all $\varepsilon > 0$. Then $|w(z)| \leq \sup_{\zeta \in \partial \Omega} |w(\zeta)|$ for all $z \in \Omega$.

Suppose now that $\alpha(x) = ax^\mu$. The condition we require is then

$$ 2|a|^2 x^{2\mu} \geq |a\mu|x^{\mu-1}/2 + |a|x^\mu \frac{\varepsilon}{|1 + \varepsilon z|}, $$

which is only possible for $\mu = -1$. However, it is easy to write down polynomials in $x$ that do not vanish at 0 but which satisfy the conditions of Corollary 3.2.
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