Constructive Modalities with Provability
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To the memory of Leo Esakia and Dito Pataraia

I overview the work of the Tbilisi school on intuitionistic modal logics of well-founded/scattered structures and its connections with contemporary theoretical computer science. Fixed-point theorems and their consequences are of particular interest.

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Disclaimer The paper is a modified version of [Lit14], which has been commissioned for the highly recommended volume [Bez14]. Please make clear which version you are quoting, as even the numeration of all environments is very different than in the version printed in the volume (this is by no means the only difference).
1 Introduction

The intended audience of [Bez14] is probably aware that much of Leo Esakia’s research concentrated on semantics for the intuitionistic logic $\text{IPC}$, the modal logic $\text{GL}^{cl}$ of Löb, its weakening $\text{wGrz}^{cl}$ and intuitionistic-modal systems like the logic $\text{KM}$ or its weakening $\text{mHC}$; see Table 2 for all definitions. $\text{GL}^{cl}$ is also known as the Gödel-Löb logic, but this name may suggest more personal involvement with the system than Gödel ever had; $\text{KM}$ or $\text{mHC}$ will be discussed in Section 4. A central feature of semantics for such systems is well-foundedness or scatteredness. While in the case of $\text{IPC}$ well-foundedness is a sufficient, but not necessary condition—intuitionistic logic is complete wrt well-founded or even finite partial orders, but sound wrt much bigger class of structures—$\text{GL}^{cl}$ and $\text{KM}$ require it even for soundness. This is due to the fact that the latter two systems include a form of an explicit induction axiom: in the case of $\text{GL}^{cl}$ the well-known Löb axiom (which here will be called the weak Löb axiom) and in the case of $\text{KM}$ the strong Löb axiom—less well-known to modal logicians, but as we are going to see, better known to type theorists. Scatteredness is the topological generalization of well-foundedness; Simmons [Sim82] provided tools necessary to define its point-free counterpart and the Tbilisi school noted that this notion also makes sense in the topos setting. In fact, the most generic way of defining scatteredness is via modal syntax: as validity of the Löb principle for a suitable “later” modality.

The interplay of relational, topological, point-free and algebraic aspects in the above paragraph should not feel unnatural to anybody familiar with Leo’s attitude to research. Let us look at an important example how results can travel from one setting to another. In the mid-1970’s, it was established that Löb-like logics enjoy the so-called Fixpoint Theorem. At first, the intention was to grasp the algebraic content of Gödel’s Diagonalization Lemma. Yet in its own right it turned out to be one of the most fascinating results ever proved about such systems. Section 3 gives an overview of some of its applications and consequences. For now, let us just mention that Leo Esakia used it, e.g., to characterize algebras for $\text{KM}$, see Theorem 14 and Corollary 15 here. Furthermore, it seems to have inspired the work on scattered toposes: [EJP00] Section 3 claims to present its topos-theoretic counterpart. However, as the result central for the topos version (Theorem 40 here) does not even include modalities in its formulation, the word counterpart has to be understood rather loosely.

As we will see, in hindsight [EJP00] turns out to be closely connected to very recent developments in Theoretical Computer Science, in particular the work of Birkedal et al. on the topos of trees [BMSS12], itself an example of a scattered topos. Thus, it seems particularly regrettable that the spadework of the Tbilisi school has not been carried further and is not more widely known.

The paper is structured as follows. Section 2 recalls syntactic and semantic basics of intuitionistic normal modal logics. Section 3 focuses on fixpoint results for Löb-like systems. Section 4 introduces the work of the Georgian school on extensions of $\text{mHC}$. Finally, Section 5 discusses scattered toposes, beginning with an overview of the topos logic.

While the work is intended as an overview and claims to novelty are minimal, they are perhaps not entirely non-existent. Theorem 14 is the most general form of [Esa06] Proposition 3; I could think of and Section 4.3 reproves results on extensions of $\text{mHC}$ using the framework of [WZ97, WZ98]; in fact, it seems that Corollary 31 is the first published proof of the corresponding extension of the Blok-Esakia Theorem announced in [Esa06].
Table 1: Axioms for $K_i$

| Axioms of the intuitionistic propositional calculus, see, e.g., CZ97 Sec. 1.3,(A1)-(A9) |
|-----------------------------------------------|
| (nrm) $(\square (A \rightarrow B)) \rightarrow (\square A \rightarrow \square B)$ |

Inference rule for $L_{int}$-fragment

| Inference rule for modality |
|----------------------------|
| MP $A \rightarrow B, A$ |
| NEC $A$ $\square A$ |

Remark 1. As a part of a larger project, I formalized most of syntactic derivations in the paper—in particular those relevant for Section 5.2—in the Coq proof assistant. Readers interested in this ongoing project are welcome to contact me. The formalization covers, in particular, most of material in [Esa98], which prepared the proof-theoretical background for [EJP00] and inspired the title of this paper.

2 A Primer on Intuitionistic Modalities

Modal formulas over a supply of propositional variables $\Sigma$ are defined by

$$A, B ::= \bot \mid p \mid A \rightarrow B \mid A \land B \mid A \lor B \mid \square A$$

where $p \in \Sigma$. The set is denoted by $L_{int} \Sigma$, but unless explicitly stated otherwise, I will keep $\Sigma$ fixed throughout and drop it from the notation. The purely intuitionistic language (i.e., without $\square$) will be denoted by $L_{int}$. Note that the syntax extended with a $\diamond$ operator, intuitionistically not definable from $\square$, is of no interest for us here.

$\Gamma \subseteq L_{int}$ is a normal $L_{int}$-logic or an intuitionistic normal modal logic if it is closed under rules and axioms from Table 1 plus the rule of substitution. For any $\Gamma, \Delta \subseteq L_{int}$, $\Gamma \oplus \Delta$ will denote the closure of $\Gamma \cup \Delta$ under substitution and the rules MP and NEC. In the case of $\Delta = \{\alpha\}$, I will also write $\Gamma \oplus \alpha$. Occasionally, I will write $\Gamma + \Delta$ for the closure under substitution and MP, but without NEC. This notation is analogous to the one used in CZ97.

$K_i$ is the smallest intuitionistic normal modal logic, i.e., $IPC \oplus (nrm)$. $IPC$—the intuitionistic propositional calculus—can be thus defined as the intersection of $K_i$ and $L_{int}$. Table 2 provides a list of additional axioms and logics which will be of interest to us. $GL_i$, $SL_i$, mHC and KM are of particular importance. As we see in Table 2 and Figure 1, there are several ways in which these and related systems can be axiomatized. In particular, we have

Theorem 2 (Ursini [Urs79], following Smorynski for the classical case). The following formalisms have the same set of theorems:

1. $GL_i$ as defined in Table 2
2. $K4_i \oplus \square A \rightarrow A$ $A$
3. $K4_i \oplus \square A \rightarrow A$ $\square A$
Table 2: $L_{\text{cl int}}$ axioms and logics. See, e.g., Sot84, WZ97, WZ98 for more (also in the syntax extended with a $\Diamond$ operator). $\Box A$ below abbreviates $A \land \Box A$

| (cl)   | $((B \to A) \to B) \to B$ | (em)   | $A \lor \neg A$ |
|--------|-----------------------------|--------|-----------------|
| $\text{Cl} = \text{IPC} + (\text{cl}) = \text{IPC} + (\text{em})$ |                |        |                 |
| (nm)   | $\Box (A \to B) \to (\Box A \to \Box B)$ | (opr)  | $\Box (A \land B) \leftrightarrow (\Box A \land \Box B)$ |
| $K' = \text{IPC} \oplus (\text{nm}) = \text{IPC} \oplus (\text{opr})$ | $K'^d = K' \oplus \text{Cl}$ |        |                 |
| (trns) | $\Box A \to (\Box A \land \Box A)$ | (bind) | $\Box \Box A \to \Box A$ |
| $K4' = K' \oplus (\text{trns})$ | $K4'^d = K4' \oplus \text{Cl}$ |        |                 |
| (r)    | $A \to (\Box A \lor \Box A)$ | (fmap) | $(A \to B) \to (\Box A \to \Box B)$ |
| $R' = K' \lor (r) = K' \land (\text{fmap})$ | Note that above $R'$, $\lor$ is the same as + |
| In using the symbol $R$, I follow FM97 |        |        |                 |
| (refl) | $\Box A \to A$ | $S4' = T' \lor K4'$ |
| $T' = K' \lor (\text{refl})$ | $\text{Triv}' = T' \lor R'$ |        |                 |
| (pll)  | $(A \lor \Box \Box A) \to \Box A$ | (henk) | $\Box (A \leftrightarrow \Box A) \to A$ |
| $\text{PL}L' = K' \lor (\text{pll}) = C4' \lor R'$ |        |        |                 |
| (wlöb) | $\Box \Box A \to A$ | (glb)  | $(\Box A \to A) \to \Box A$ |
| $\text{PLL}L' = K' \lor (\text{wlöb}) = K4' \lor (\text{glb}) = GL' \lor R'$ | The form (glb) comes from Goldblatt Gol81 |        |                 |
| (ufp)  | $\Box (A \leftrightarrow A) \to (\Box A \leftrightarrow A)$ | (gllb) | $\Box \Box A \to A$ |
| $\text{GL}L' = K' \lor (\text{ufp}) = K4' \lor (\text{gllb}) = GL' \lor R'$ | (see Theorem 2 below) | $\text{GL}L'^d = \text{GL}L' \oplus \text{Cl}$ |                 |
| (slöb) | $(\Box A \to A) \to A$ | (gb)   | $(\Box A \to A) \to \Box A$ |
| $\text{SL}L' = K' \lor (\text{slöb}) = K4' \lor (\text{gb}) = GL' \lor R'$ | The form (gb) comes from Goldblatt Gol81 |        |                 |
| (grz)  | $\Box \Box (A \to A) \to A$ | (sgrz) | $\Box (A \to \Box A) \to A$ |
| $\text{wGrz}^d = K4'^d \lor (\text{grz})$ | $\Box (A \to \Box A) \to A$ | $\text{wGrz}^d = K4'^d \lor (\text{sgrz})$ |                 |
| Note we only consider here classical variants of (grz) | (sgrz) |                |                 |
| (next) | $(\Box A \to ((B \to A) \to B) \to B)$ | (derv) | $\Box A \to ((B \to A) \lor B)$ |
| $\text{CB}L' = K' \lor (\text{next}) = K' \lor (\text{derv})$ | $\text{CB}L' \lor \text{GL}'$ | $\text{KM} = \text{CB}L' \lor \text{SL}'$ |                 |
| $\text{CB}L' \lor \text{GL}'$ | $\text{KM} = \text{CB}L' \lor \text{SL}'$ | $\text{mHC} = R' \lor \text{CB}L'$ |                 |
| $\text{KM} = \text{SL}' \lor \text{CB}L' = \text{SL}' \lor \text{mHC}$ | (gd) | $(A \to B) \lor (B \to A)$ |                 |
| $\Box (A \to B) \lor (\Box B \to A)$ | (gd) | $\text{LC} = \text{IPC} + (\text{gd})$ |                 |
| (ver)  | $\Box A$ | (boxbot) | $\Box \bot$ |
| $\Box A \lor \Box A \lor (\Box A \to \Box A)$ | (boxbot) | $\Box \bot$ |                 |
| (nv)   | $\neg \Box \bot$ | (nv)   | $\neg \Box \bot$ |
| $\text{NNV}'^d = K' \lor (\text{nv})$ | $\text{NV}'^d = K' \lor (\text{nv})$ | $\text{NNV}^d = K' \lor (\text{nv})$ | $\text{NV}^d = K' \lor (\text{nv})$ |
There are no normal logics in this area. $\mathbb{K}_i$ and $\mathbb{Triv}^d$ are the only maximal consistent ones above $\mathbb{K}_i$.

An isomorphic copy of $\langle IPC, Cl \rangle$, see [KM86], is also shown.

Figure 1: Relationships between systems introduced in Table 2. Ordinary arrows correspond to inclusion between systems. Lines joined by $\oplus$ denote joins: a join of two logics is of course the smallest normal modal logic containing both. Remember however that not all joins are shown, especially in case of joins with weak systems like $\mathbb{K}_n^i$ or $\mathbb{NNV}_i$. For more on $wGrz^d$, see [Esa06] and also [Lit07] and further references therein.

The observation that $\mathbb{Triv}^d$ and $\mathbb{Ver}^d$ are the only maximal consistent logics in $\mathbb{K}_i$ seems to have been made first by Vakarelov in [Vak81]. Wolter [Wol97] is one of most advanced studies of the lattice of extensions of $\mathbb{K}_i$; in particular, it solves a number of problem posed in [Vak81] and investigates the counterparts of $\Gamma^d$ above $\mathbb{K}_i$ for any given $\Gamma$. Finally, Wolter and Zakharyaschev [WZ97; WZ98] show how to reinterpret this lattice as the lattice of extensions of a certain bimodal classical logic. This is in many ways the most general of Blok-Esakia type results as will be discussed later in Section 4.3.
4. $K4^i \oplus (ufp) = \Box (B \leftrightarrow A[B/p]) \rightarrow (\Box (C \leftrightarrow A[C/p]) \rightarrow (B \leftrightarrow C))$

5. $K4^i \oplus (henk) = \Box (A \leftrightarrow \Box A) \rightarrow A$

A variable $p \in \Sigma$ is $\Box$-guarded in $A \in \mathcal{L}_{\text{int}}$ if all its occurrences are within the scope of $\Box$. This notion will be used repeatedly in connection with GL$^i$ and its extensions.

**Remark 3.** Equalities in Table 2 and joins in Figure 1 should be in fact treated as a large lemma on interderivability for a number of intuitionistic normal modal axioms. Of particular interest for this chapter are: the derivability of $(r)$ from $(sl\ddot{o}b)$, which mirrors derivability of $(trns)$ from $(wl\ddot{o}b)$; interderivability between $(glb)$ and $(sl\ddot{o}b)$; equivalence between either of these and the conjunction of $(wl\ddot{o}b)$ and $(r)$; two different ways of axiomatizing CB$^i$ by using $(\text{next})$ and $(\text{derv})$. All these statements are made assuming $K^i$.

All normal $\Box$-int-logics as defined above—more precisely, their associated global consequence relations—are strongly finitely algebraizable; see standard references like [BP89; Fon06; FJP03; FJP09; Ras74] for basic notions of algebraic logic and a more detailed discussion. Strong finite algebraizability also applies to normal logics in any fragment of $\Box$-int containing $\rightarrow$. This is due to the fact that all these systems are implicative logics in the sense of Rasiowa [Ras74]. Given a normal logic $\Gamma$, I will call the corresponding class of algebras obtained via the algebraization process $\Gamma$-algebras, e.g., $K^i$-algebras, GL$^i$-algebras, wGrz$^{cl}$-algebras etc. $K^i$-algebras are obviously special cases of HAOs—Heyting algebras with operators—namely HAOs with a single unary operator. Recall that an operator on a Heyting algebra is an operation preserving $\top$ and finite meets. An operator on a Heyting algebra which turns it into a $\Gamma$-algebra will be called a $\Gamma$-operator.

Finally, recall that for any algebra $\mathfrak{A}$, a $\mathfrak{A}$-polynomial is a term in the similarity type of $\mathfrak{A}$ enriched with a separate constant for each element of $\mathfrak{A}$ [BS81, Definition 13.3]. In my notation for polynomials, I will not distinguish between an element of $\mathfrak{A}$ and its corresponding constant. Moreover, I will use elements of $\Sigma$ (i.e., propositional variables) as variables of polynomials, consistent with the general policy of blurring the distinction between logical formulas and algebraic terms. The notion of $\Box$-guardedness for polynomials will be used in the same way as for formulas.

### 2.1 Relational Semantics

It is pretty obvious what should be the ingredients of a Kripke frame (a relational structure) for intuitionistic normal modal logic: it should be of the form $(W, \leq, \prec)$, where $\leq$ is a poset order used to interpret intuitionistic connectives, whereas $\prec$ is the modal accessibility relation used to interpret $\Box$. Valuations of propositional variables should be defined as as $V : \Sigma \rightarrow \up$ and the inductive extension to intuitionistic connectives using $\leq$ is standard. However, once we set to provide a precise interpretation of modalities, there are choices to be made. Here is what [Sim94, Section 3.3]—one of most comprehensive overviews—says on the subject:

One could take also the usual satisfaction clauses for the modalities in modal models . . . However, an essential feature of intuitionistic models is the monotonicity lemma . . . If the standard satisfaction clauses for the modalities are used then the monotonicity lemma does not hold. There are two possible remedies. (a) One is to modify the satisfaction clauses. This might be a reasonable thing to do, for
one might wish to use the partial order to give a more intuitionistic reading of the modalities. (b) The other remedy is to impose conditions on models that ensure that the monotonicity lemma does hold.

The monotonicity lemma referred to in the above is of course the fact that the denotation of any intuitionistic formula in any Kripke model is upward closed. The two solutions lead to two alternative satisfaction definitions:

- $\square_{(a)} A := \{ w \in W | \forall v, x. (w \leq v \text{ and } v \prec x \text{ implies } x \in A) \}$
- $\square_{(b)} A := \{ w \in W | \forall x. (w \prec x \text{ implies } x \in A) \}$

By definition, $\square_{(a)} A \in \mathcal{U}_p \subseteq (W)$ for any $A \subseteq W$. This, however, is easily seen to fail in general for (b), even for some $A \in \mathcal{U}_p \subseteq (W)$. This is precisely while we need to impose additional conditions on the interaction of $\leq$ and $\prec$:

**Lemma 4** ([BD84]). For any $(W, \preceq, \prec)$, the condition

for any $A \in \mathcal{U}_p \subseteq (W)$, $\square_{(b)} A \in \mathcal{U}_p \subseteq (W)$

(i.e., $\mathcal{U}_p \subseteq (W)$ is a subalgebra of $\mathcal{P}o(W)$ also with respect to $\square_{(b)}$) is satisfied iff the following condition holds:

\[ \leq ; \prec \subseteq \prec ; \leq \quad (1) \]

Curiously, condition (1), even though isolated in [BD84] and discussed in references such as [Sim94], does not seem to appear too often in intuitionistic modal literature. Most references, even those which adopt interpretation (b), assume conditions stronger than (1). Goldblatt [Gol81] requires

\[ \leq ; \prec \subseteq \prec \quad (2) \]

but in most of intuitionistic modal logic literature (see, e.g., [Sot84; WZ97; WZ98]) we have still stronger

\[ \leq ; \prec ; \leq = \prec \quad (3) \]

Obviously, (3) implies (2) and (2) implies (1), as $\leq$ is a partial order. But as noted in [BD84; Gol81]

\[ \prec ; \leq = \prec \quad (4) \]

(i.e., for any $x$, $\prec^{-1}[x]$ is upward closed) cannot be forced by any axiom, as any frame $(W, \preceq, \prec)$ can be shown to be semantically equivalent to $(W, \preceq, \prec')$ satisfying (4) by putting $\prec' = \prec ; \leq$.

In presence of (4), (1) implies (3) and hence all these conditions become equivalent. Thus, we can follow the others and assume (3), but from a logical point view, only (1) would be necessary.

**Remark 5.** A justification for adopting the strongest condition (3) is provided by completeness/canonicity result, for which the earliest references seem to be [BD84; Sot84]: when we define $\prec$ between prime filters of a Heyting algebra with a normal $\square$ in a standard way, the resulting Kripke frame will satisfy (3). This provides a justification of (3) and a bridge between algebraic and Kripke semantics, i.e., a completeness result for $K'$ wrt Kripke frames satisfying (3) with $\square$ read as dictated by (b).
Remark 6. What is the connection with (a)-semantics? It is easy to see that (a)-reading of \( \Box \) wrt \( \prec \) yields the same operator on \( \mathbb{U}_{\leq} (W) \) as (b)-reading wrt \( \prec \) modified to satisfy (2); that is, the closure of \( \prec \) under pre-composition with \( \preceq \). Note here that in order to obtain a relation satisfying (3), one needs to close \( \prec \) under both pre- and post-composition with \( \preceq \).

Thus, as stated in [Sim94], it turns out that both semantics induce the same intuitionistic modal logic. . . . Only when the \( \Diamond \) connective is added do the differences in the semantics become apparent.

The reason why (a) is often seen in the literature (see [Sim94] and references therein and also, e.g., [AMPR01; PS86]) is that it becomes pretty natural when intuitionistic modal logic is considered as a fragment of intuitionistic predicate logic—and, in close connection, when one wants to interpret \( \Diamond \) using the same relation as for \( \Box \). However, this is not what we are interested in here. Thus, in what follows, I stick to (b) reading of modality (henceforth (b) is dropped as a subscript) and impose condition (3) on all frames. This is justified not only by Remark 5 above, but also by the fact that intuitionistic modal semantics in practically all references relevant for further discussion satisfied (3) anyway. Let us summarize the discussion above with the following definition:

A Kripke frame or a relational structure is of the form \((W, \preceq, \prec)\), where

- \( \preceq \) is a poset order used to interpret intuitionistic connectives
- \( \prec \) is the modal accessibility relation used to interpret \( \Box \) and
- \( \preceq ; \prec \subseteq \prec \), where \( ; \) is relation composition

A valuation is a mapping \( V : \Sigma \to \mathbb{U}_{\leq} (W) \), where \( \mathbb{U}_{\leq} (W) \) is the Heyting algebra of upward closed sets of \( W \) and the inductive extension to \( \mathcal{L}_{\Box}^{\text{int}} \Sigma \) is standard.

Remark 7. When at least one of lattice connectives is removed, the situation at first sight appears more complicated. While the papers proving the separation property of IPC [Hor62; Mil10; Pra65] showed that its reducts remain complete wrt relational semantics, a Stone-type representation theorem for arbitrary algebras would seem more problematic; the one for Heyting algebras relies crucially on the fact that they have distributive lattice reducts. However, a series of papers beginning with [Köh81] and finishing with [BJ13] established that Brouwerian semilattices enjoy in fact Stone-, Priestley- and Esakia-type dualities.
Remark 8. It is worth discussing whether we can reconstruct this relational semantics from coalgebraic point of view, just like in the classical case. The relationship between the category of Heyting algebras (\(\mathbf{HA}\)) and the category of posets with bounded morphisms (say, \(\mathbf{PABM}\)) is the same as in the case of boolean algebras and sets (dual adjunction, restricting to equivalence in the finite case). Kripke frames for classical modal logic can be thought as coalgebras for covariant powerset functor. The counterpart of that functor in \(\mathbf{PABM}\) is the covariant upset functor \(\mathbf{Up}(-)\), which sends \((W, \leq)\) to \((\mathbf{Up}_\leq(W), \supseteq)\) and \(\mathbf{Up}_\leq(f)(A) = f[A]\). Then if we attempt to define relational structures as \(\mathbf{PABM}\)-morphisms, i.e., bounded morphisms from \((W, \leq)\) to \((\mathbf{Up}_\leq(W), \supseteq)\), the very choice of codomain of this morphism forces (4) and the forth condition of bounded morphism becomes precisely (2), so altogether we get just (3). However, the back condition seems to impose a rather special condition on \(\prec\): for any \(x \in W\) and any upward closed \(A \subseteq \prec^{-1}[x]\), there is \(y\) s.t. \(x \leq y\) and \(\prec^{-1}[y] = A\). There are many natural examples of structures satisfying (3) where this condition would fail and in general I do not know how many logics would be possibly complete with respect to frames of this kind. It is not necessarily harmless for some of the axioms considered here. However, when we can forget about the back condition for morphisms (i.e., in case of modalities over distributive lattices rather than over Heyting algebras), (3) emerges as the natural basic semantic condition also from the coalgebraic point of view.

Table 3 lists semantic conditions corresponding to modal axioms. For \(\mathbf{GL}^i\) in particular, we have:

**Theorem 9** ([Urs79]). A structure \((W, \leq, \prec)\) validates \(\mathbf{GL}^i\) iff

- \(\prec\) is transitive, i.e., \(\prec; \prec \subseteq \prec\) and
- \(\prec\) is \(\mathbf{Up}_\leq(W)\)-Noetherian: for any \(A \in \mathbf{Up}_\leq(W)\), if \(A \neq W\), then there is \(w \in \Box A - A\), i.e., a \(\Box\)-maximal non-\(A\) point

**Proof (sketch).** We show only the ”if” direction. Assuming (wlöb) fails under a valuation \(V\), take \(B\) to be the extension of (wlöb) under \(V\) and show that \(\Box B \subseteq B\). This means that \(W - B\) witnesses the failure of Noetherianity. \(\Box\)

Note that Theorem 9 together with the \((r)\) row of Table 3 provides also the condition for \(\mathbf{SL}^i\): as already stated in Table 2 and Figure 1 \(\mathbf{SL}^i\) is obtained as \(\mathbf{GL}^i \oplus (r)\), so one simply takes the conjunction of the two conditions.

[Urs79] provides an interesting motivation for this semantics of \(\mathbf{GL}^i\) in terms of projects and streamlines in a research. It also provides other important results, such as the finite model property and decidability.

### 3 The Fixpoint Theorem

Theorem 2 above gave us several equivalent axiomatizations of \(\mathbf{GL}^i\). In particular, \((\text{ufp})\) forces uniqueness of fixed points.

**Definition 10.** Let \(B\) be a formula of \(L_{\text{Cint}}\) (its denotation in a given algebra and under a given valuation). \(B\) is a fixed point of (the term function associated with) \(A \in L_{\text{Cint}}\) relative
We only give a sketch of how to prove Theorem 11, sketch. For any $p$, there exists a constructively obtained formula $\text{diag}_p A$ s.t.

1. $\text{diag}_p A \leftrightarrow B \in \text{GL}^1$, where $B$ is obtained from $A$ by replacing all occurrences of $p$ under $\Box$ by $\text{diag}_p A$.

2. $A$ and $\text{diag}_p A$ have provably the same fixed points with respect to $p$, that is, for any $C$ not containing $p$ we have

$$\Box(C \leftrightarrow A(C/p)) \leftrightarrow \Box(C \leftrightarrow \text{diag}_p A(C/p)) \in \text{GL}^1$$

Clearly, if $p$ is $\Box$-guarded in $A$, then $B$ in the first clause is precisely $A[\text{diag}_p A/p]$ and $\text{diag}_p A$ does not contain $p$, hence being trivially its own fixed point wrt $p$. Thus, in such a situation $\text{diag}_p A$ itself is also the unique fixed-point of $A$ with respect to $p$!

**Proof of Theorem 11.** sketch. We only give a sketch of how $\text{diag}_p A$ is constructed. Any formula $A(p, \bar{q}) \in \mathcal{L}_{\int}$ in variables $p, \bar{q} \in \Sigma$ can be written as $B(\bigcirc C_1(p, \bar{q}), \ldots, \bigcirc C_k(p, \bar{q}), p, \bar{q})$, where $B \in \mathcal{L}_{\int}$ (i.e., is a formula without $\Box$) and $\bar{C} \in \mathcal{L}_{\int}$. Clearly, if $k = 0$, then $A$ itself belongs to $\mathcal{L}_{\int}$ and, in particular, there are no occurrences of $p$ under $\bigcirc$. Hence we can take $\text{diag}_p A$ to be $A$ itself. Otherwise, the proof can be conducted by induction on $k$, as we already have the base step. For any $i \leq k$, set

$$A_i := B(\bigcirc C_1(p, \bar{q}), \ldots, \bigcirc C_{i-1}(p, \bar{q}), \top, \bigcirc C_{i+1}(p, \bar{q}), \ldots, \bigcirc C_k(p, \bar{q}), p, \bar{q}).$$

### Table 3: Semantic counterparts of intuitionistic modal axioms

| Axiom | Semantic condition | Axiom | Semantic condition | Axiom | Semantic condition |
|-------|--------------------|-------|--------------------|-------|--------------------|
| (cl)  | $\leq = \Delta$   | (r)   | $\leq \subseteq \leq$ | (next) | $\leq \subseteq \leq$ |
|       | (trns) $\leq ; \leq \subseteq ;$ |       | (bind) $\leq \subseteq ;$ |
|       | (wob) see Theorem 9 |       | (nv) $W \subseteq W$ |

| A | Semantic condition | A | Semantic condition |
|---|--------------------|---|--------------------|
| $\lim$ | $\subseteq \leq W$ | $\top$ | $\leq W$ |
By definition, the inductive hypothesis applies to $A_i$. Now we set
\[
\text{diag}_p A := B(\Box C_1(\text{diag}_p A_1/p, \bar{q}), \ldots, \Box C_k(\text{diag}_p A_k/p, \bar{q}), p, \bar{q}).
\]

Remark 12. In fact, extensions of $\mathbf{SL}^1_i$—in particular $\mathbf{KM}$—allow a much simpler proof of Theorem 11 and a much simpler algorithm for computing these fixpoints: it is enough to substitute $\top$ for $p$. This follows already from observations made by Smoryński in [Smo79, Lemma 2.3] and has been discussed explicitly in [JV95, Propositions 4.2–4.6]. De Jongh and Visser describe $\mathbf{SL}^1_i$ as a kind of Kindergarten Theory in which all the well-known syntactical results of Provability Logic have extremely simple versions.

Remark 13. It is known that at least in the case of $\mathbf{GL}^1$ a non-constructive and non-explicit form of Theorem 11 can be obtained already from uniqueness of fixed-points combined with the Beth definability theorem for $\mathbf{GL}^1$, see, e.g., [Boo93, Smo79] for more information. However, as should be clear from the discussion below, the very fact that fixed points are obtained explicitly and constructively seems an advantage not to be given up lightly.

Theorem 11 has a nice algebraic corollary. I present it here as a more general version of [Esa06, Proposition 3].

Theorem 14. A $\mathbf{K4}^1_i$-algebra $\mathfrak{F}$ is a $\mathbf{GL}^1_i$-algebra iff every $\mathfrak{F}$-polynomial $t(p)$ in one $\Box$-guarded variable $p \in \Sigma$ has a fixed point.

Proof. The “only if” direction. This is a direct corollary of Theorem 11.

The “if” direction. Given any $h \in \mathfrak{F}$, consider the polynomial $t(p) = \Box p \rightarrow h$. As $p$ is $\Box$-guarded in it, it has a fixed point $i_h \in \mathfrak{F}$; that is, $i_h \equiv \Box i_h \rightarrow h$. By the fact that $\rightarrow$ is conjugate (or residual) to $\land$, one half of this equality is equivalent to $i_h \land \Box i_h \leq h$. On the other hand, $h \leq \Box i_h \rightarrow h = i_h$ by general implication laws. Taken together, these two inequalities imply $i_h \land \Box i_h = h \land \Box h$: the $\leq$ direction from the first inequality, normality and (trans), the $\geq$ direction from the second inequality and monotonicity of $\Box$. Using normality again, we get $\Box i_h \land \Box \Box i_h = \Box h \land \Box \Box h$ and using (trans) again, we arrive at $\Box i_h = \Box h$. Then we get $\Box(\Box h \rightarrow h) = \Box(\Box i_h \rightarrow h) = \Box i_h = \Box h$. As $h \in \mathfrak{F}$ was chosen arbitrarily, we have that $\mathfrak{F}$ is a $\mathbf{GL}^1_i$-algebra.

There is an analogy between the above result and alternative axiomatizations for $\mathbf{GL}^1$ presented in [Urs79].

Corollary 15.

• A $\mathbf{R}^1_i$-algebra $\mathfrak{F}$ is a $\mathbf{SL}^1_i$-algebra iff every $\mathfrak{F}$-polynomial $t(p)$ in one $\Box$-guarded variable $p \in \Sigma$ has a fixed point.

• [Esa06, Proposition 3] A $\mathbf{mHC}$-algebra $\mathfrak{F}$ is a $\mathbf{KM}$-algebra iff every $\mathfrak{F}$-polynomial $t(p)$ in one $\Box$-guarded variable $p \in \Sigma$ has a fixed point.
As stated above, Theorem 11 occurred first as an algebraization of Gödel’s Diagonalization Lemma. While the connection between $GL^i$ and Heyting Arithmetic $HA$ is not as tight as the one between $GL^cl$ and Peano Arithmetic $PA$ established by the completeness result of Solovay [Sol76] (see also [Boo93, Chapter 9]), [Sam76] notes that Theorem 11 yields a counterpart of the Diagonalization Lemma for any intuitionistic first-order theory with a canonical derivability predicate, including obviously $HA$. At any rate, the relevance of fixpoint results for Löb-like logics is not limited to arithmetic.

**Remark 16.** It is worth mentioning here that—unlike the case of $PA$—the search for a complete axiomatization of the provability logic of $HA$ is not over yet; [Iem01] gives a fascinating account. Regarding the arithmetical interpretation of $SL^i$, see the discussion of $HA^*$ in [JV95, Sec. 4–5].

To begin with, in the classical case one can use Theorem 11 to show that explicit smallest or greatest fixed-point operators are eliminable over $GL^cl$. In other words, adding $\mu$ or $\nu$ does not increase the expressive power of the classical modal logic of transitive and conversely well-founded structures; see [AF09; Ben06; Vis05]. Note that this includes all correctly formed expressions with $\mu$, without assuming that all occurrences of $p$ are $\square$-guarded: as usual, they only have to be positive. [CFL10, Section 3] discusses an application in the context of expressivity of navigational fragments of XML query languages.

While I am not aware of an exact analogue of the results in [AF09; Ben06; Vis05] in the intuitionistic context, Löb-like modalities—more specifically, variants of systems $GL^i$ and $SL^i$—have recently become rather popular in type theory. Examples include:

- **modality for recursion** [Nak00; Nak01]
- **approximation modality** [AMRV07]
- **guardedness type constructor** [AM13]
- **next-step modality/next clock tick** [KBH11a; KBH11b]
- **later operator** [BT09; BM13; BMSS12; JTS12]

One of reasons is precisely that such modalities guarantee existence and uniqueness of fixed-points of suitably guarded type expressions. However, the modal spadework of 1970’s seems rarely acknowledged. In [Nak00], which may be credited with introducing intuitionistic Löb-like modalities to the attention of this community, we find the following statement:

> Similar results concerning the existence of fixed points of proper type expressions could historically go back to the fixed point theorem of the logic of provability. . . . The difference is that our logic is intuitionistic, and fixed points are treated as sets of realizers [the emphasis is mine–T.L.].

This formulation suggests that Nakano was not aware that the intuitionistic fixed-point theorem had been already proved in [Sam76], not to mention improvements possible above $SL^i$ (cf. Remark 12). The only related references quoted in [Nak00] focus on classical $GL^cl$—e.g, [Boo93]—and in later papers even these are omitted. A valuable part of the logical tradition seems lost this way. Let us see what insights can be found in the work of the Tbilisi school.

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1It is a theme I am presently collaborating on with Albert Visser.
4 Leo Esakia and Extensions of mHC

4.1 mHC and Topological Derivative

Leo Esakia and collaborators devoted special attention to the system mHC and its extensions. [Esa06] is an excellent overview. The abbreviation mHC stood for modalized Heyting calculus. The reader may find the name surprising; after all, many natural intuitionistic modal systems are not subsystems of mHC. Esakia [Esa06] was perfectly aware of that:

The postulate (r) is not typical, while the postulate (deriv) stresses even more “nonstandardness” of the chosen basic system mHC and of its extension KM, which enables one to draw a conventional “demarcation line” between mHC and the standard intuitionistic modal logics.

Remark 17. Both axioms seem “nonstandard” mostly if one focuses on these intuitionistic modal logics which are obtained from popular classical systems. It is enough to look at Table 3 to realize why it must be so: (deriv) is trivially derivable in K^d (its consequent being a classical tautology), while the combination of (cl) and (r) yields that ≺ ⊆ Δ. That is, the only classical frames for R^d are disjoint unions of reflexive and irreflexive points. However, (r) is nowhere as pathological in a properly intuitionistic setting. There are many references on systems different from mHC and KM where nevertheless (r) is still derivable or even explicitly included as an axiom. A short and inexhaustive list includes [Aba07; Acz01; BBP98; BMSS12; Cur52; FM97; GA08; GP06; Gol81; Gol10; KB11b; MP08; Nak00; Nak01]. They can be split roughly into two main groups. The first one—e.g., [BMSS12; KB11b; Nak00; Nak01]—concerns SL^i and has already been mentioned in Section 3. The second one—e.g., [Acz01; BBP98; Cur52; FM97; GA08; GP06; Gol81; Gol10]—concerns the system which is denoted here as PLL^i (a.k.a. CL, see [BBP98]). [Gol03, Section 7.6] and [Gol10] are good if incomplete overviews of most relevant papers on this system—the most important omissions being perhaps [PD01, Section 7] and [Aba07; GP06]. See also [MP08] for a discussion of type systems with R^i modalities from programmer’s point of view.

[Esa06] gives the following reasons for the importance of mHC:

• Its connection to KM: mHC is KM minus the Löb axiom (slöb). Note that (wlös) and (slöb) are equivalent in the presence of (r)

• The connection with intuitionistic temporal logic “Always & Before” possessing rich expressive possibilities

• The fact that mHC can be obtained as a fragment of QINT (or, in Esakia’s notation, QHC)—quantified intuitionistic propositional calculus. This is similar to the encoding of mHC in the internal language of a topos, see the last point

• The topological connection with Cantor’s scattered spaces, notions of the limit and isolated point. This will be discussed at length in this section

• Finally, as mentioned above, mHC turns out to be a natural fragment of the topos logic. This last point builds on all the preceding ones and will be discussed in Sec. 5
that for any $w$. For any complete Heyting algebra, its point-free coderivative is an
\[ \text{Proposition 19.} \]
\[ \{ w \in W \mid \{ w \} \subseteq \{ w \} \subseteq A \}; \text{again, see Table 3 for notation. It is straightforward to note that for any } w \in W, w \in \# A \text{ iff for any } B \in \{ B \to A \} \cup B. \text{ This observation actually explains the shape of axiom (derv). Hence, given any complete Heyting algebra } \mathcal{H}, \text{ define its point-free coderivative } \mathcal{H}_{\text{deriv}} \text{ as } h := \bigwedge_{i \in \mathcal{H}} \{ (i \to h) \}. \]

\[ \text{Proposition 19. For any complete Heyting algebra, its point-free coderivative is an mHC-operator.} \]

\[ \text{Proof. A rather easy exercise for the reader; can be also extracted from the proof of [Esa06, Proposition 5].} \]

There is a slightly different description of $\#$. Take a Heyting algebra $\mathcal{H}$ and $h, i \in \mathcal{H}$ s.t. $i \leq h$. $h$ is $i$-dense or dense in $[i, \top]$ if for any $j \in \mathcal{H}$, we get that $h \land j = i$ implies $j = i$. Note that the standard topological notion of density can be considered a special case: an open set is topologically dense iff it is $\bot$-dense in the Heyting algebra of open sets of the space. The following was observed, e.g., in [EJP00].

\[ \text{Fact 20. For any Heyting algebra } \mathcal{H} \text{ and any } h \geq i \in \mathcal{H}, \text{ } h \text{ is } i \text{-dense iff there exists } j \in \mathcal{H} \text{ s.t. } h = j \lor (j \to i). \]

\[ \text{Corollary 21. For any complete Heyting algebra } \mathcal{H} \text{ and any } i \in \mathcal{H}, \]

\[ \# i = \bigwedge_{h \in \mathcal{H}} \{ h \in \mathcal{H} \mid h \geq i \text{ and } h \text{ is } i \text{-dense}. \]

Why coderivative? The reader is referred to a detailed account by Simmons [Sim14]. Briefly, recall that in topology the Cantor-Bendixson derivative of a set $A$ is the set of those $x$ whose every neighbourhood contains a point of $A$ other than $x$; the dual operator (hence co-derivative) consists of those $x$ which have an open neighbourhood entirely contained in $A \cup \{ x \}$ [Esa06]. As it turns out, this indeed coincides with $\#$ for practically all sensible topological spaces:
Theorem 22 (Simmons). For any $T_0$-space, its co-derivative operator coincides with the point-free coderivative $\#$ on the Heyting algebra of open sets.

Proof. For $\#$ defined as in Corollary 21, this was proved in [Sim82].

Obviously, as any intuitionistic Kripke frame with the Alexandroff topology given by $\text{Up}_{\leq} (W)$ is $T_0$, we get that $\#$ coincides with the dual of topological derivative of this topology. It is, in fact, easier to prove directly than by Simmons’ result.

Remark 23. One observation from [Sim82] is worth quoting here:

for non-$T_0$-spaces the usual definition of isolated point does not quite capture the intended notion

and hence for arbitrary spaces, the point-free definition of derivative given by $\#$ seems in fact more adequate than the standard one. The reader can verify this by extending the definition of intuitionistic Kripke frames to qosets rather than just posets and checking how both notions would fare in such a setting.

4.2 KM and Scatteredness

A complete Heyting algebra will be called scattered if its coderivative $\#$ is not only an mHC-operator, but a KM-operator. Recall Corollary 15 as an algebraic characterization of such a situation.

Proposition 24.

- For any topological space, its point-free coderivative is a KM-operator if the space is scattered in the usual sense: that is, if each non-empty subset has an isolated point
- For any $T_0$ topological space, its point-free coderivative is a KM-operator only if the space is scattered

Proof. A non-$T_0$-space can never be scattered, and for $T_0$-spaces, the point-free coderivative coincides with ordinary one as stated in Theorem 22. The remaining calculations are an exercise in point-set topology; in fact, the point-set part of this result has been shown first by Kuznetsov [Kuz79; Mur14]. One can use an alternative characterization of scatteredness here: for any open set $A$ distinct from the whole space, $\# A - A$ is non-empty.

Let us summarize some of the results above:

Corollary 25. A topological space $\mathfrak{T}$ is scattered iff in the complete Heyting algebra of open sets of $\mathfrak{T}$, every polynomial in one $\#$-guarded variable has a fixed point.

Corollary 26. The following are equivalent for any $(W, \leq, \prec)$:

- $\prec = \triangleleft$ and the Alexandroff topology $(W, \text{Up}_{\leq} (W))$ is scattered
- $(W, \leq, \prec)$ is a frame for KM
- $\prec = \triangleleft$ and it contains no infinite ascending chains
Proof. We only need to prove the equivalence of the last two conditions. \(\mathbf{\triangleleft} \subseteq \triangleleft\) is, as observed, enforced by \(\mathbf{mHC}\). Theorem 9 gives the corresponding semantic condition for (w6b). Quite obviously, \(\bigcup_{\mathbf{\leq}} (W)\)-Noetherianity forces irreflexivity of \(\triangleleft\). Thus, whenever \((W, \leq, \triangleleft)\) is a frame for \(\mathbf{KM}\) (that is, the join of \(\mathbf{mHC}\) and \(\mathbf{GL}\)), we have \(\triangleleft = \triangleleft\). Moreover, rewriting the condition of \(\bigcup_{\mathbf{\leq}} (W)\)-Noetherianity for \(\mathbf{mHC}\)-frames, we obtain that for any \(A \in \bigcup_{\mathbf{\leq}} (W) - \{W\}\), there is \(w \in A^\downarrow \uparrow - A\). Rewriting further, we obtain that any \(B \neq \emptyset\) s.t. \(B = B \uparrow \downarrow\) has a maximal element wrt \(\leq\). But this means that any nonempty subset of \(W\) has a maximal \(\leq\)-element. \(\square\)

4.3 Completeness, Lattice Isomorphisms and Bimodal Translations

Two important kinds of results have been missing from this overview so far. First, while I discussed Kripke correspondence for modal logics (Table 3, Theorem 9 and Corollary 26), I have not discussed completeness. Second, I have not said much about lattices of extensions of Löb-like logics and their relatives—in particular, about generalizations of the Blok-Esakia Theorem.

This section fixes both oversights. Rather than using original proofs of Kuznetsov, Muravitsky (for \(\mathbf{KM}\)) and Esakia (for \(\mathbf{mHC}\), we are going to use corollaries of Wolter and Zakharyaschev’s results on bimodal translations \(\mathbf{[WZ97; WZ98]}\), briefly discussed also in \(\mathbf{[WZ14]}\). Section 4). Their techniques allow to interpret intuitionistic modal logics as fragments of classical polymodal ones (cf. the discussion of \(\mathbf{implict vs. explicit epistemics}\) in \(\mathbf{[Ben91]}\)). In the case of the Blok-Esakia theorem for \(\mathbf{mHC}\), we will be able to see why axioms of \(\mathbf{mHC}\) and \(\mathbf{wGrz}\) have to look the way they look in order to allow the classical counterpart to be unimodal rather than polymodal, as it happens in the more general framework of \(\mathbf{[WZ97; WZ98]}\).

Take the bimodal language \(L_{i,m}\) with operators \([i]\) and \([m]\). For any formula \(A \in L_{\mathbf{clnt}}\), I will write \(A\) (respectively \(A\)) for \(A\) with all occurrences of \(\square\) (respectively \([m]\)). \([m]\) is the default counterpart of the original modality \(\square\) and \([i]\) encodes the intuitionistic poset order, hence the notation \(\mathbf{2}\). The logic \(S_{4i,m}\) is the normal logic axiomatized by the following axioms: (cl), (nrm\([i]\)), (refl\([i]\)), (trns\([i]\)) and (nrm\([m]\)); in other words, it is what modal logicians would describe as the fusion of \(S_{4i}\) and \(\mathbf{Km}\). The logic \(S_{4M,i,m}\) is \(S_{4i,m}\) extended with

\[
(mix) : [m]A \leftrightarrow [i][m][i]A.
\]

The logic \(sGrz_{4M,i,m}\) is \(S_{4i,m}\) extended with (mix) and \((sgrz\_[i])\). The translation \(b : L_{\mathbf{clnt}} \rightarrow L_{i,m}\) prefixes every subformula in \((\cdot)[m]\) with \([i]\). Of course, many occurrences of \([i]\) in the translation \(b\) can be removed relative to logics defined above:

**Fact 27.** The following equivalences belong to \(S_{4i,m}\): \(bA \leftrightarrow [i]bA\), \(b(A \land B) \leftrightarrow (bA \land bB)\), \(b(A \lor B) \leftrightarrow (bA \lor bB)\); in \(S_{4M,i,m}\), we moreover have \(b(\square A) \leftrightarrow [m]bA\).

For any intuitionistic normal modal logic \(\Gamma \supseteq S_{4i,m}\) and any bimodal normal logic \(\Delta \subseteq L_{i,m}\), let

- \(\tau^{\mathbf{mix}}\Gamma := S_{4M,i,m} \oplus \{bA \mid A \in \Gamma\}\)
- \(\sigma^{\mathbf{mix}}\Gamma := \tau^{\mathbf{mix}}\Gamma \oplus (sgrz\_[i])\)
- \(\rho^{\mathbf{mix}}\Delta := \{A \in L_{\mathbf{clnt}} \mid bA \in \Delta\}\)

\(^2\)The reader has to be warned that the notation in this section differs somewhat from that in references like \(\mathbf{[WZ97; WZ98; WZ14]}\).
\[ \Delta \text{ is a } L_{i,m}\text{-companion of } \Gamma \text{ if for any } A \in L_{\text{int}}, A \in \Gamma \iff \flat A \in \Delta, \text{ i.e., iff } \rho_{\text{m}} \Delta = \Gamma. \]

**Theorem 28** ([WZ97; WZ98; WZ14]). Let \( \Delta \supseteq S4_{i,m}^{cl} \) be a normal bimodal logic and \( \Gamma \subseteq L_{\text{int}} \) be an intuitionistic normal logic. Then

(A) \( \rho_{\text{m}} \Delta \) is an intuitionistic normal modal logic

(B) \( \tau^\text{mix} \Gamma \) and \( \sigma^\text{mix} \Gamma \) are, respectively, the smallest and the greatest \( L_{i,m}\)-companions of \( \Gamma \) containing \( \text{(mix)} \)

(C) \( \rho_{\text{m}} \) preserves decidability, Kripke completeness and the finite model property. If \( \text{(mix)} \in \Delta, \rho_{\text{m}} \) also preserves canonicity

(D) \( \tau^\text{mix} \) preserves canonicity

(E) \( \sigma^\text{mix} \) preserves the finite model property

(F) \( \sigma^\text{mix} \) is an isomorphism from the lattice of normal extensions of \( K_i^{\text{cl}} \) onto the lattice of normal extensions of \( sGrzM_{i,m}^{cl} \)

(G) \( \Gamma \supseteq K4^i \) has the finite model property whenever its \( L_{i,m}\)-companions over \( S4M_{i,m}^{cl} \oplus (\text{trns})_{[m]} \) include a canonical subframe logic

Proof. (A) can be easily proved from Fact [27] note that we need the assumption we are above \( S4_{i,m}^{cl} \). (B) is a consequence of Theorem 27 in [WZ97]. (C), (D) and (E) are consequences of Proposition 29 and Theorem 30 in [WZ97] and Theorems 11 and 12 in [WZ98]. (F) is a consequence of Corollary 28 in [WZ97]. (G) is a consequence of Corollary 18 in [WZ98].

[WZ97; WZ98] illustrate on many examples how powerful these results are. As it turns out, they also have corollaries of immediate interest for us.

**Corollary 29.** \( \text{mHC} \) is canonical and has the finite model property.

Proof. First, note that

\[
S4M_{i,m}^{cl} \oplus b(r) = S4M_{i,m}^{cl} \oplus ([i]A \rightarrow [m]A) \supseteq S4M_{i,m}^{cl} \oplus (\text{trns})_{[m]}. 
\]

Clearly, \([i]A \rightarrow [m]A\) is a Sahlqvist formula with an universal FO counterpart. Furthermore, \( S4M_{i,m}^{cl} \oplus b(\text{deriv}) \) is the same logic as the extension of \( S4M_{i,m}^{cl} \) with

\[
[m]B \land (i)C \rightarrow [i]((i)C \lor [i]B). 
\]

The latter is a simple Sahlqvist implication (cf. e.g., [BdV01, Definition 3.47]). Applying the algorithm in the proof of Theorem 3.49 in [BdV01] and doing some FO-preprocessing, we get an universal formula

\[
\forall y,z,w.(x \trianglelefteq y \land x \trianglelefteq z \rightarrow (z \leq y \lor (z \leq w \rightarrow x < w))) 
\]

(6)

(where \( \trianglelefteq \) is the accessibility relation corresponding to \([i]\) and \( < \) is the accessibility relation corresponding to \([m]\)). Thus, \( \tau^{\text{mix}} \text{mHC} \) is a canonical subframe logic over \( S4M_{i,m}^{cl} \oplus (\text{trns})_{[m]} \).

Now, canonicity of \( \text{mHC} \) follows from (C) and the fmp from (G) of Theorem 28.
Remark 30. It is worth noting that the semantic counterpart of (next) from Table 3, i.e., \( \preceq \subseteq \prec \) is equivalent to (6) above. For one direction, substitute \( x = y \) in (6) and use poset properties. For the other direction, note that whenever \( x \preceq y \) and \( x \preceq z \) but \( \neg (z \preceq y) \), then \( x \prec z \), ergo \( x \prec z \). Now whenever \( z \preceq w \), we can use the interaction between \( \prec \) and \( \preceq \) as expressed by (mix) (in fact, even a weaker axiom would do).

Canonicality of \( \text{mHC} \) has been noted, e.g., in [Esa06; Gol81]. I was unable to locate references where the finite model property has been explicitly claimed. The following corollary can appear more surprising, as bimodal logics over \( L_i \), \( m \) do not even occur in its statement.

Corollary 31. The lattice of normal extensions of \( \text{mHC} \) is isomorphic to the lattice of normal extensions of \( \text{wGrz}^{cl} \). The sublattice of normal extensions of \( \text{KM} \) is isomorphic to the lattice of normal extensions of \( \text{GL}^{cl} \).

Proof. The heart of the proof is to notice that \([i]A \leftrightarrow [m]A \land A \land (\text{grz})_{[m]} \) are derivable in \( \sigma^\text{mix}\text{mHC} \); in fact, these two formulas axiomatize this logic over \( L_i \oplus (\text{trans})_{[m]} \). Let us derive the first of them. For convenience, we will do it in the algebraic setting:

\[
\begin{align*}
[m]A \land A \land (i)\neg A & \leq \text{ by (refl)}_{[i]} \\
[i]((i)(A \land (i)\neg A) \lor [i]A) & \leq \text{ by (5)} \\
[i]((i)(A \land (i)\neg A) \lor A) & = \\
[i](i)(A \rightarrow [i]A \rightarrow A) & \leq [i]A \text{ by (grz)}_{[i]}.
\end{align*}
\]

We get that \( \sigma^\text{mix}\text{mHC} \) is just a notational variant of \( \sigma^\text{mix}\text{wGrz}^{cl} \), with \([m]\) being \( \Box \) and \([i]\) being \( \lozenge \). This yields the first statement by clause \((F)\) of Theorem 28. For the second, it is enough to add the observation that over \( \sigma^\text{mix}\text{mHC} \), adding \( \blacklozenge(\text{slöb}) \) is equivalent to adding \( \text{(wlöb)}_{[m]} \).

The second statement of Corollary 31 above was first proved by Kuznetsov and Muravitsky in mid-1980's, see [KM86; Mur14]. The first statement was announced in [Esa06] as follows:

Finally let us note that ...the lattice \( \text{Lat(mHC)} \) of all extensions of \( \text{mHC} \) is isomorphic to the lattice \( \text{Lat(K4.Grz)} \) of all normal extensions of the modal system \( \text{K4.Grz} \). However, a proof of this result requires additional considerations as the above algebraic machinery does not suffice for it.

It seems that the proof has not been published so far.

Corollary 32 \([\text{Gol81; KM86; Mur81}]\). \( \text{KM} \) has the finite model property.

Proof. The proof of Corollary 31 has established that \( \sigma^\text{mix}\text{KM} \) is just a notational variant of \( \text{GL}^{cl} \), with \([m]\) being \( \Box \) and \([i]\) being \( \lozenge \). Now use clause \((C)\) of Theorem 28 and the finite model property for \( \text{GL}^{cl} \) (see, e.g., [BdV01; Boo93; CZ97; Fin85; Mos07] for references).
Remark 33. Note that we could also prove Corollary 29 in an analogous way to Corollary 32 using the fmp of \(wGrz^d\) established explicitly by Amerbauer [Ame96]. The latter is actually a direct consequence of \(wGrz^d\) being a transitive subframe logic [Fin85; Lit07]. However, I believe that the proof of Corollary 29 provided above has some additional value: we obtained a convenient form of \(\flat(\text{deriv})\) — which we actually used in the proof of Corollary 31 — together with its FO translation, which also provides some additional insight, as discussed in Remark 30.

Remark 34. It could be an interesting exercise — and very much in the spirit of the Tbilisi school — to show that the above-discussed results of [WZ97; WZ98; WZ14] survive when the base bimodal logic is weakened from \(S4i_m\) to the fusion of \(K^i \oplus (A \land [i]A \rightarrow [i][i]A)\) with \(K^m_m\) and the translation \(\flat\) is modified to \(\flat^*\) replacing every subformula \(A\) with \(\flat^*A \land [i]\flat^*A\). On the other hand, it is not obvious how much generality would be really gained in this way. Note that using Wolter and Zakharyaschev’s original results we were able to investigate lattices of logics which are not extensions of \(S4^d\), such as \(wGrz^d\) in Corollary 31 above.

5 Scattered Toposes

We are ready to discuss the topos of trees of [BMSS12], scattered toposes of [EJP00] and the relationship between fixpoint results in both papers.

5.1 Preliminaries on Topos Logic

Just like Section 4.3 assumed certain familiarity with technicalities of modal logic, this section in turn assumes some familiarity with basics of category theory — mostly with the notions of a ccc (cartesian closed category), a functor and a natural transformation. Those readers who know more than that, in particular understand well the internal logic of a topos, can probably skip this subsection. Due to obvious space constraints, the presentation has to be rather abstract and example-free; see [Gol06; Joh02; MM92] for more examples and motivation. Furthermore, like any presentation of topos theory by logicians and for logicians, it can be accused of neglecting spatial intuitions. See, e.g., [McL90] for a passionate polemic with the view that toposes were invented to generalize set-theoretical foundations of mathematics. Nevertheless, applications of toposes in fields like algebraic geometry or foundations of physics or their actual historical origins are not directly relevant here. My aim is a minimalist presentation focusing on the contrast between the logic of a topos and that of a ccc, but also making clear how the Beth-Kripke-Joyal semantics is related to more familiar ones for the intuitionistic predicate logic. Of all accounts in the literature, the one in [LS86] is probably closest to this goal.

Speaking of [McL90], footnote 4 provides an argument that the plural form intended by Grothendieck was toposes rather than topoi. I stick to the same convention, also because — as a quick Google search shows — the form toposes is used mostly by mathematicians, whereas topoi seems prevalent for unrelated notions in the humanities. Besides, this was the form used by Leo, Mamuka and Dito.
Let $\mathcal{E}$ be a ccc with the terminal object $1$ and for any $Y \in \mathcal{E}$, let $\text{fin}^0_Y$ be the unique element of $\mathcal{E}[Y, 1]$. I use the obvious notation for (finite) products, coproducts (whenever they exist, but in a topos they always do), their associated morphisms and I denote the ccc evaluation mapping $B^A \times A \to B$ as $\text{eval}_{A,B}$. Recall that $\mathcal{E}$ is an \textbf{elementary topos} if there exists an object $\Omega \in \mathcal{E}$ s.t. $(\Omega, 1 \to \Omega)$ is a \textbf{subobject classifier}, i.e., for any monic (left-cancellable morphism) $Y \xrightarrow{\chi} X$ there exists exactly one mapping $X \xrightarrow{\chi|} \Omega$ s.t. we have a pullback diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{\chi} & X \\
\downarrow & & \downarrow \\
\text{fin}^0_Y & \xrightarrow{\chi|} & \Omega \\
\text{true} & \xrightarrow{1} & \Omega
\end{array}
$$

As observed by C. Juul Mikkelsen, this definition already implies that $\mathcal{E}$ is \textbf{bicartesian closed}, where the latter notion is defined as in, e.g., [LS86]; see [Gol06 Section 4.3] for references.

Before we proceed with formal definitions, some general discussion can be helpful. In every category, topos or not, (equivalence classes of) monics into $X$ are abstract counterparts of subsets of $X$; in fact, they are called \textbf{subobjects of $X$}, just like morphisms $1 \to X$ are \textbf{elements of $X$}. Global elements (or equivalence classes thereof) can be considered as special cases of subobjects: think of the usual identification of an element $x \in X$ with the subset $\{x\}^\sharp$. We can go further and define a \textbf{generalized element} of $X$ as any morphism $A \to X$, which is then called $A$\textbf{-based} or \textbf{defined over $A$}. See [MM92 Section V] for a lucid and brief discussion of those notions.

In particular, the global elements of $\Omega$ can be identified with logical constants, $X$-based generalized elements of $\Omega$ with predicates over $X$ (i.e., formulas with a single free variable from $X$) and $n$-ary propositional connectives with morphisms $\Omega^n \to \Omega$. Set $\text{false} := \chi_{\text{fin}^0_0}$, $\neg := \chi(\text{false})$, $\land := \chi(\text{true,true})$ and $\lor := \chi(\langle\text{true}_0, \text{id}_0\rangle, \langle\text{id}_0, \text{true}_0\rangle)$. Recall that for any $X \in \mathcal{E}$, $\text{true}_X$ stands for $\text{true} \circ \text{fin}^0_X$ and $\text{eq}_X$ stands for $\chi(\langle\text{id}_X, \text{id}_X\rangle)$. The latter allows to define \textbf{internal equality for generalized elements of type $X$} as $\sigma \approx \tau := \text{eq}_X \circ \langle\sigma, \tau\rangle$. That is, if $A \xrightarrow{\sigma} X$ and $B \xrightarrow{\tau} X$ are generalized elements of $X$, then $\sigma \approx \tau$ is a generalized element of $\Omega$ defined over $A \times B$. For $\Omega$, we can define not only $\text{eq}_\Omega$, but also $\text{leq}_\Omega$ as the equalizer of $\Omega \times \Omega \xrightarrow{\text{true} \land \Pi_1} \Omega$. Implication, the only remaining intuitionistic connective, can be now defined as $\to := \chi_{\text{leq}_\Omega}$.

Thus, in toposes one can reduce reasoning about the poset of subobjects of any given object $X \in \mathcal{E}$ (in fact, whenever $\mathcal{E}$ is a topos, this poset is always a lattice and even a Heyting algebra—see [MM92 Theorem IV.8.1]) to reasoning about $\mathcal{E}[X, \Omega]$ and further still to reasoning about an \textbf{internal Heyting algebra} provided by a suitable exponential object. What this means is: in any category, monics into $X$ have a natural preorder defined as $f \subseteq g$ if $f$ factors through $g$, i.e., there is a morphism $h$ s.t. $f = g \circ h$. Dividing by equivalence classes with respect to $\subseteq$, we get a category-theoretic generalization of the poset of subsets of $X$ ordered by inclusion. In general, without understanding the global structure of $\mathcal{E}$, we are not likely to learn much about these posets of subobjects. But in a topos, the poset of subobjects of $X$ is isomorphic to something more tangible: namely, to $\mathcal{E}[X, \Omega]$, i.e., $X$-based generalized elements of $\Omega$. Think of the usual identification of subsets of $X$ with elements of

\footnote{Note that toposes very rarely happen to mimic sets in having enough global elements to determine all subobjects; such special toposes are called \textit{well-pointed}.}
$2^X$. Here is also where first- and higher-order aspects of the internal logic come into play.

If $\mathcal{E}$ is a category with products, a power object of $X \in \mathcal{E}$ is a pair $(\mathcal{P}X, \exists X \xrightarrow{r} \mathcal{P}X \times X)$ s.t. for any $(Y, \mathcal{R} \xrightarrow{r} Y \times X)$ there exists exactly one $Y \xrightarrow{f_r} \mathcal{P}X$ for which there is a pullback

$$
\begin{array}{c}
R \xrightarrow{r} Y \times X \\
\downarrow \\
\exists X \xrightarrow{r} \mathcal{P}X \times X
\end{array}
\begin{array}{c}
f_r \circ \text{id}_X \\
\downarrow
\end{array}

\Rightarrow \quad X \xrightarrow{2^X} \mathcal{P}X \times X
$$

As shown, e.g., in [Gol06, Theorem 4.7.1], in any topos we can take $\mathcal{P}X$ to be $\Omega^X$ and the subobject $\exists X \xrightarrow{3^X} \Omega^X \times X$ can be obtained by pulling back $\text{true}$ along $\Omega^X \times X \xrightarrow{\text{eval}_{X,\Omega}} \Omega$. Thus, we see that in a topos, the notions of power object, subobject classifier and exponential object are indeed well-matched and we can define the membership predicate $\text{true}$ for a pair of generalized elements $(A \xrightarrow{\sigma} X, B \xrightarrow{\tau} \mathcal{P}X)$ as $\text{eval}_{X,\Omega} \circ (\sigma, \tau)$. We are now ready for a single definition formalizing the whole discussion above and more (see [MM92, Sec. VI.5-7] and also [Cro93, LS86]):

**Definition 35** (The Mitchell-Bénabou language). Consider a topos $\mathcal{E}$. The collection of ground types and the signature of the Mitchell-Bénabou language of $\mathcal{E}$ are defined, respectively, as

$$
\text{Ground}(\mathcal{E}) := \{E \ | E \in \mathcal{E}\}
$$

$$
\text{Sg}(\mathcal{E}) := \{f : F_1, \ldots, F_n \rightarrow E \ | f \in \mathcal{E}[F_1, \ldots, F_n, E]\}
$$

(Instead of $k : 1 \rightarrow E$ I will write $k : E$) and the full collection of types $\text{Types}(\mathcal{E})$ is $A, B := E \ | 1 \ | \Omega \ | A \times B \ | B^A \ | \text{eval}_{E,\Omega}$ where $E \in \text{Ground}(\mathcal{E})$. $\Omega^A$ can be also written as $\mathcal{P}A$. Fix, moreover, a supply of term variables $x, y, z \in \text{tVar}$. The collection of terms $\text{Terms}(\mathcal{E})$ over $\text{Sg}(\mathcal{E})$ is defined as

$$
M, N := x \ | \overline{fM} \ | \text{fin} \ | M \approx N \ | \{M, N\} \ | \Pi_1 M \ | \Pi_2 M \ | \lambda x : A.M \ | M\{N
$$

where $x \in \text{tVar}$ and $f \in \text{Sg}(\mathcal{E})$ is of suitable arity. The typing rules and some standard abbreviations (including all logical connectives) of the language are defined in Table 4. Interpretation of types, contexts and terms-in-context in $\mathcal{E}$ is given in Table 3.

**Definition 36** (Forcing for an elementary topos). Assume $\Gamma = x_1 : F_1, \ldots, x_n : F_n$ and $\Gamma \vdash \phi : \Omega$. By $\{x_1 \ldots x_n\} \in F_1 \times \ldots \times F_n \ | \phi$, I will denote the pullback of the following diagram:

$$
\begin{array}{c}
\Gamma \\
\downarrow [\Gamma \vdash \phi : \Omega] \\
1 \xrightarrow{\text{true}} \Omega
\end{array}
$$

Now for $F \xrightarrow{f_1} F_1, \ldots, F \xrightarrow{f_n} F_n$ write $F, f_1, \ldots, f_n \vdash \phi$ if $F \xrightarrow{f_1, \ldots, f_n} [\Gamma]$ factors through $\{x_1 \ldots x_n\} \in F_1 \times \ldots \times F_n \ | \phi \rightarrow [\Gamma]$. In what follows, $f_1 \circ g, \ldots, f_n \circ g$ will be denoted by $f \circ g$. Moreover, let $[\Gamma \vdash \phi : \Omega] = \text{true}[\Gamma]$ be written as $\Gamma \vdash \phi$.

**Fact 37.**
\[\Gamma \vdash \mathcal{M} : A \quad \Gamma \vdash N : A \quad \Gamma \vdash M : A \quad \Gamma \vdash N : B\]

| \[\Gamma, x : A \vdash x : A\] | \[\Gamma \vdash M \approx N : \Omega\] | \[\Gamma \vdash \text{fin} : 1\] | \[\Gamma \vdash \langle M, N \rangle : A \times B\] |
|---|---|---|---|
| \[\Gamma, x : A \vdash M : B\] | \[\Gamma \vdash M : B^A\] | \[\Gamma \vdash N : A\] | \[\Gamma \vdash M : A \times B\] |
| \[\Gamma \vdash \lambda x : A.M : B^A\] | \[\Gamma \vdash \pi_{1}M : A\] | \[\Gamma \vdash \pi_{2}M : B\] | \[\Gamma \vdash \pi_{1}N : B\] |
| \[f : F_{1}, \ldots, F_{n} \to E \in \text{Sg}(\mathcal{E})\] | \[\Gamma \vdash \lambda x : A.M : \Pi_{1}\] | \[\Gamma \vdash \lambda x : A.M : \Pi_{2}\] | \[\Gamma \vdash \lambda x : A.M : \Lambda\] |

\[\Gamma \vdash \lambda x : A.M : \Pi_{1}\] \[\Gamma \vdash \lambda x : A.M : \Pi_{2}\] \[\Gamma \vdash \lambda x : A.M : \Lambda\]

\[\text{true} := \text{fin} \approx \text{fin}\]

\[\forall x : A.\phi := \lambda x : A.\phi \approx \lambda x : A.\text{true}\]

\[\exists x : A.\phi := \forall t : \Omega.((\forall x : A.\phi \approx t) \approx t)\]

\[\phi \lor \psi := \forall t : \Omega.((\phi \approx t) \lor (\psi \approx t) \approx t)\]

\[\neg \phi := \phi \approx \text{false}\]

Table 4: Typing rules and defined abbreviations of the Mitchell-Bénabou language of \(\mathcal{E}\)

| \[\mathcal{E}\] := E \[1\] := 1 \[\Omega\] := \Omega \[A \times B\] := \[A\] \times \[B\] \[B^A\] := \[B]\[^A\] |
|---|---|---|---|
| \[x_1 : A_1, \ldots, x_n : A_n\] := \[A_1\] \times \cdots \times \[A_n\] |

\[\Gamma, x : A \vdash x : A\] \[\Gamma \vdash \mathcal{M} : A\] \[\Gamma \vdash N : A\] \[\Gamma \vdash M : A \times B\] \[\Gamma \vdash \text{fin} : 1\] \[\Gamma \vdash \langle M, N \rangle : A \times B\]

| \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] | \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] |

| \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] | \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] |

| \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] | \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] |

| \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] | \[\Gamma \vdash \lambda x : A.M : B^A\] := \text{curry}^{[\Pi_{1}[A][B]}(\sigma) : \[\Gamma\] \to \[B][^A\] |

\[\Gamma \vdash \lambda x : A.M : B^A\] \[\Gamma \vdash \lambda x : A.M : B^A\] \[\Gamma \vdash \lambda x : A.M : B^A\]

Table 5: Interpretation of types, contexts and terms-in-context

22
\begin{itemize}
\item \( F, \bar{f} \models \phi \iff [\Gamma \vdash \phi : \Omega] \circ \bar{f} = \text{true}_F \)
\item \( \Gamma \models \phi \iff \text{for any } F \xrightarrow{\bar{f}} [\Gamma], \text{ it holds that } F, \bar{f} \models \phi \)
\end{itemize}

The following result, which can be found as Theorem VI.6.1 in [MM92] or Theorem II.8.4 in [LS86], connects the definition of forcing given above with more standard intuitionistic semantics:

**Theorem 38 (Beth-Kripke-Joyal semantics in an elementary topos).** Assume \( F \xrightarrow{\bar{f}_1} E_1, \ldots, F \xrightarrow{\bar{f}_n} E_n \) and \( \Gamma = x_1 : E_1, \ldots, x_n : E_n \).

\begin{itemize}
\item \( F, \bar{f} \models \phi \land \psi \iff F, \bar{f}_1 \models \phi \) and \( F, \bar{f} \models \psi \)
\item \( F, \bar{f} \models \phi \lor \psi \) iff there are arrows \( G \xrightarrow{\bar{g}} F \) and \( H \xrightarrow{\bar{h}} F \) s.t. \( G + H \xrightarrow{[\bar{g}, \bar{h}]} F \) is epi, \( G, \bar{f} \circ g \models \phi \) and \( H, \bar{f} \circ h \models \psi \)
\item \( F, \bar{f} \models \phi \to \psi \) iff for any \( G \xrightarrow{\bar{g}} F \) it holds that \( G, \bar{f} \circ g \models \psi \) whenever \( G, \bar{f} \circ g \models \phi \)
\end{itemize}

For the case of quantified formulas, note that \( \Gamma \vdash \forall x_{n+1} : E_{n+1} \phi \iff \Gamma, x_{n+1} : E_{n+1} \vdash \phi \). Same holds for \( \Gamma \vdash \exists x_{n+1} : E_{n+1} \phi \). Then we have:

\begin{itemize}
\item \( F, \bar{f} \models \forall x_{n+1} : E_{n+1} \cdot \phi \) iff for every \( G \xrightarrow{\bar{g}} F \) and every \( G \xrightarrow{\bar{g}'} E_{n+1} \) it holds that \( G, f_1 \circ g, \ldots, f_n \circ g, g' \models \phi \)
\item \( F, \bar{f} \models \exists x_{n+1} : E_{n+1} \cdot \phi \) iff there exist \( G \xrightarrow{\bar{g}} E_{n+1} \) and an epi \( G \xrightarrow{\bar{g}'} F \) s.t. \( G, f_1 \circ g, \ldots, f_n \circ g, g' \models \phi \)
\item \( F, \bar{f} \models \sigma \approx \tau \iff [\Gamma \vdash \sigma : E] \circ \bar{f} = [\Gamma \vdash \tau : E] \circ \bar{f} \)
\end{itemize}

The clauses for \( \exists \) and \( \lor \) above resemble those of intuitionistic Beth semantics. This is why “Beth-Kripke-Joyal” seems a more appropriate name in the general case of an arbitrary elementary topos; see, e.g., [Gol06, Section 14.6]. However, when the topos happens to be the topos of presheaves, i.e., covariant functors into \( \textbf{Set} \), on a given small category \( \mathcal{R} \)—in particular, a poset taken as a category—the definition of forcing can be significantly simplified.\footnote{Reader should be warned that in most of categorical literature, presheaves are assumed to be contravariant, but see, e.g., [Ghi89] for an example of the covariant convention.}

Perhaps the most straightforward account of this simplification can be found in [LS86]. First, the clause for disjunction can be “kripkified” for indecomposable objects and the clause for existential quantifiers—for projective ones [LS86, Proposition 8.7]. Second, the second clause of Fact [37] suggests that to check validity of a given judgement-in-context \( \Gamma \vdash \phi : \Omega \) in a topos, it is enough to restrict attention to those \( F \) which belong to a generating set for a given topos. Third, by the Yoneda Lemma, in a topos of presheaves \( \textbf{Set}^{\mathcal{R}} \) for an arbitrary small category \( \mathcal{R} \), objects of the form \( \text{hom}^C_{\mathcal{R}} := \mathcal{R}[C, -] \) for any given \( C \in \mathcal{R} \) satisfy all these conditions: they are projective, indecomposable and do form a generating set. Moreover, also by the Yoneda Lemma, elements of \( \text{Set}^{\mathcal{R}}[\text{hom}^C_{\mathcal{R}}, F] \) are in 1–1 correspondence with elements of \( F(C) \):

\[
\text{Set}^{\mathcal{R}}[\text{hom}^C_{\mathcal{R}}, F] \ni f \rightarrow \bar{f} := f_C(\text{id}_C) \hfill \\
F(C) \times \text{hom}^C_{\mathcal{R}} \ni (c, h) \rightarrow \bar{c}(h) := F(h(c))
\]
Note also that in clauses like the one for \( \leadsto \), we can restrict attention to those \( G \xrightarrow{\beta} F \) whose source \( G \) lies in the generating set. In the case of \( \text{Set}^g \), this means replacing \( G \xrightarrow{\beta} \text{hom}_F^C \) with elements of \( \text{Set}^g[\text{hom}_F^C, \text{hom}_F^C] \). But, by the Yoneda Lemma again, these can be replaced with arrows in \( \mathcal{R}[C, D] \) (note the change of direction!).

Taking all this into account, we can obtain the following modified version of the semantics—this time properly "Kripkean" (see [LS86, Proposition 9.3]).

**Corollary 39** (Kripke-Joyal semantics in a topos of presheaves). Let \( \mathcal{R} \) be a small category, \( F_1, \ldots, F_n \in \text{Set}^g \), \( C \in \mathcal{R} \), \( c_1 \in F_1(C) \), \( \ldots \), \( c_n \in F_n(C) \) , \( \Gamma = c_1 : E_1, \ldots, c_n : E_n \) and \( \Gamma \vdash \phi : \Omega \). Write \( C, \tau \vDash \phi \) for \( \text{hom}_F^C, c_1, \ldots, c_n \vdash \phi \). Given any \( f \in \mathcal{R}[C, D] \), write \( f(\tau) \) for \( \hat{c}_1(f), \ldots, \hat{c}_n(f) \)—that is, \( F f(c_1), \ldots, F f(c_n) \). Then we have:

- \( C, \tau \vdash \phi \land \psi \iff C, \tau \vdash \phi \) and \( C, \tau \vdash \psi \)
- \( C, \tau \vdash \phi \lor \psi \iff \text{if } C, \tau \vdash \phi \text{ or } C, \tau \vdash \psi \)
- \( C, \tau \vdash \phi \rightarrow \psi \iff \text{for any } f \in \mathcal{R}[C, D] \), \( D, f(\tau) \vdash \psi \) whenever \( D, f(\tau) \vdash \phi \)
- \( C, \tau \vdash \forall x. \phi \iff \text{for any } f \in \mathcal{R}[C, D] \), it does not hold that \( D, f(\tau) \vdash \phi \)
- \( C, \tau \vdash \exists x. \phi \iff \text{there exist } d \in F_{n+1}(C) \text{ s.t. } C, \tau, d \vDash \phi \)

[Ghi89] uses toposes of presheaves as a generalization of Kripke semantics for the intuitionistic first-order logic to prove incompleteness results. Of numerous follow-ups of that work, let me just mention [Ni97, Sk12]. Let us also note that the derivation of Corollary 39 from Theorem 38 takes a somewhat more roundabout route in [MM92]: toposes of presheaves are handled there as a subclass of toposes of sheaves on a site.

### 5.2 Non-expansive Morphisms, Fixpoints and Scattered Toposes

Let \( \mathcal{E} \) be an elementary topos. Call an endomorphism \( f \in \mathcal{E}[X, X] \) **unchanging** or **non-expansive** if

\[
\vdash_{\mathcal{E}} \forall x, y : X. (f x \approx f y \leadsto x \approx y) \leadsto x \approx y.
\]

As noted in [EJP00], in a boolean setting **non-expansive** means just **constant**: negate the sentence and play with boolean laws. Obviously then a classical proof that a non-expansive endomorphism on a non-empty set has a unique fixed point does not carry much computational content. In a constructive setting, however, the situation is different.

Assume \( \vdash \phi, \psi : \Omega^X \) and \( f \in \mathcal{E}[X, X] \) and define:

\[
\begin{align*}
\text{SubTe}(\phi) & := \forall x, y : X. (\phi x \land \phi y \leadsto x \approx y) \\
\phi \subseteq \psi & := \forall x : \text{SubTe}(\phi) \land \forall \alpha : \Omega^X. (\text{SubTe}(\alpha) \land \phi \subseteq \alpha \leadsto \alpha \subseteq \phi) \\
\text{MaxST}(\phi) & := \text{SubTe}(\phi) \land \forall \alpha : \Omega^X. (\text{SubTe}(\alpha) \land \phi \subseteq \alpha \leadsto \alpha \subseteq \phi) \\
\text{NonExp}(f) & := \forall x, y : X. (f x \approx f y \leadsto x \approx y) \leadsto x \approx y \\
\text{fix}_{\text{So}} & := \lambda x : X. (x \approx f x)
\end{align*}
\]

With this apparatus, we can state the main Theorem of Section 3 of [EJP00]:

24
Theorem 40. Assume $\mathcal{E}$ is an elementary topos and $f \in \mathcal{E}[X, X]$. Then

$$\models_\mathcal{E} \text{Non}_\text{exp}(f) \leadsto \text{MaxST}(\text{fix}_\so f).$$

Remark 41. A proof formalized in the Coq proof assistant is available from the author, see Remark 2. Those who would like to try a manual yet rigorous proof in the Mitchell-Bénabou language should do first Exercise 5 in [LS86, p. 139] and then formalize the proof in [EJP00, p. 105] using all the abbreviations given above.

Remark 42. Some of derivations in [Esa98; EJP00], especially when fully formalized in a proof assistant, make explicit an interesting fact: certain axioms governing quantification which are not generally intuitionistically valid (e.g., the Kuroda axiom and its generalization due to Casari), become valid in presence of a KM modality, even though these axioms do not involve any modalities at all in their statement. Some related observations are made by [Bir12] (a reference I became aware of having written the published version of this overview).

In words, this result says: the fixpoints of a non-expansive endomorphism form a maximal subterminal subobject $^6$. The syntactic shape of $\text{SubTe}_f$ easily suggests that subterminality is the internal counterpart of “being of cardinality at most one”, i.e., uniqueness of fixpoints. However, the situation with existence is more complicated. First of all, toposes of presheaves can differ significantly from the topos of sets in having non-trivial objects with no global elements whatsoever. More importantly, even being inhabited is not enough to ensure maximal subterminal objects are global elements.

Example 43 ([EJP00]). Consider the topos of presheaves on $(\omega + 1, \succeq)$, where $\succeq$ is the converse of the standard ordinal order. Presheaf $X$ defined as $X(n) = n + 1$ and $X(\omega) = \omega$ with $X(\beta \succeq \alpha)(n) = \min(n, \alpha)$ is clearly inhabited. Furthermore, $f : X \to X$ defined as $f_n(i) = \min(i + 1, n)$ and $f_\omega(i) = i + 1$ is a non-expansive endomorphism. Yet it fails to have a fixpoint—i.e., a global element $1 \xrightarrow{\text{fin}} X$ s.t. $f \circ c = c$.

Of course, we can do better in special cases.

Corollary 44. Whenever $X \in \mathcal{E}$ is s.t. any maximal subterminal subobject of $X$ is a global element (for example, $X$ is an injective object), there exists $\text{fin}! \xrightarrow{\text{fin}} X$ s.t. $f \circ c = c$ for any non-expansive $f \in \mathcal{E}[X, X]$.

We could try to express unique existence in the internal logic using the standard abbreviation $\exists !$ for “exists exactly one”. However, as kindly pointed out by Thomas Streicher, this abbreviation works as intended in toposes of presheaves, but not necessarily in arbitrary ones.

But where is the place for a modality in all this? Say that $\oplus : \Omega \to \Omega$ is a strong Löb operator if $\models_\mathcal{E} \forall p : \Omega. (\oplus p \leadsto p) \leadsto p$. Also, call a morphism $f \in \mathcal{E}[X, Z]$ $\oplus$-contractive if $\models_\mathcal{E} \forall x, y : X. (x \oplus y) \leadsto (f x \approx f y)$:

$^6$The corresponding theorem in [EJP00] contained also an additional statement about density of the support of the fixed-point subobject, but this does not seem essential for us here.
Corollary 45. Let $\odot : \Omega \to \Omega$ be a strong L"ob operator, $f \in \mathcal{E}[X, X]$, and assume that $f$ is $\odot$-contractive. Then $f$ is non-expansive and hence its subobject of fixed points is a maximal subterminal one.

Proof. We have that $\models_\mathcal{E} \forall p, q : \Omega (\odot p \rightsquigarrow (p \lor q)) \rightsquigarrow ((q \lor p) \rightsquigarrow p)$. In fact, this is an equivalent form of (slöb)—cf. the proof of Theorem 2(iv) in [EJP00]. Now substitute $x \approx y$ for $p$ and $fx \approx fy$ for $q$ to get the result. \hfill \Box

\textbf{EJP00} states the result only for a special case of contractiveness and a special subclass of toposes (introduced below) but this generalization is straightforward. As before, we can derive the conclusion about the existence of unique fixed points whenever every maximal subterminal object of $X$ happens to be a global element—e.g., whenever $X$ is injective.

Define $\odot \phi := \forall t : \Omega (t \lor (t \rightsquigarrow \phi))$, i.e., an internalized coderivative. We have the following counterpart of Proposition 19:

Proposition 46. In any elementary topos $\mathcal{E}$, we have $\models_\mathcal{E} \forall p : \Omega p \rightsquigarrow \odot p$ and $\models_\mathcal{E} \forall p, q : \Omega \odot p \rightsquigarrow (q \lor (q \rightsquigarrow p))$.

A scattered topos is defined analogously to scattered locales or Heyting algebras in Section 4.2 by the validity of the only remaining KM law, i.e., the axiom $\forall p : \Omega (\odot p \rightsquigarrow p) \rightsquigarrow p$. Thus, scattered toposes are those where $\odot$ is a strong L"ob operator. This notion turns out to have several equivalent characterizations, see [EJP00]. Let us discuss in detail here another one for the special case of Set$^\mathcal{E}$:

Theorem 47. Let $\mathcal{R} = (W, \preceq)$ be a poset. Then Set$^\mathcal{E}$ is scattered iff $(W, \preceq, \triangleleft)$ satisfies any of the equivalent conditions in Corollary 29.

Proof. For a direct proof, it is useful to compute the semantic meaning of $\odot$. Define $\odot : \Omega \to \Omega$ as $[p : \Omega \vdash \odot p : \Omega]$. Recall also that in a topos of the form Set$^\mathcal{E}$ for $\mathcal{R} = (W, \preceq)$, $\Omega(w)$ is equal to \{ $A \cap \{w\} \preceq \uparrow | A \in \forall_{\forall \subseteq} (W)$\}. A morphism $f : \Omega \to \Omega$ is a natural transformation: a family of mappings $\{ f_w : \Omega(w) \to \Omega(w) | w \in W \}$ satisfying

$f_z (A \cap \{z\} \preceq \uparrow) = f_w (A) \cap \{z\} \preceq \uparrow$ for any $A \in \Omega(w), z \preceq w$.

Now let us note the following

Fact 48. For any topos of the form Set$^\mathcal{E}$ where $\mathcal{R} = (W, \preceq)$, for any $w \in W$ and for any $A \in \Omega(w)$ (i.e., $A$ an upward closed subset of $\{w\} \preceq \uparrow$),

$\odot w(A) = \{ z \preceq w | \{z\} \preceq \uparrow \subseteq A \}$.

The reader may want to consult Section 4 and Table 3 for the notation used above; in particular, recall that $\{z\} \triangleleft \uparrow \ast \{z\} \preceq \uparrow - \{z\}$. Note also that we can add an atomic clause to Corollary 39 in the preceding section:

Fact 49. $w, A \models t$ (where $\Gamma = t : \Omega$ and $A \in \Omega(w)$) iff $A = \{w\} \preceq \uparrow$.

This fact, while rather basic, is worth an explicit proof, as it helps to put together several definitions and propositions above:

Proof of Fact 49. $w, A \models t$ is an abbreviation for $\hom_{\forall \subseteq}^w A \models t$, while this in turn can be reformulated as $\hat{A} = \text{true} \circ \text{fin}_{\hom_{\forall \subseteq}^w}$ (Fact 37). In particular, $\hat{A}(w \preceq w) = \{w\} \preceq \uparrow$. But $\hat{A}(w \preceq w) = A$. \hfill \Box
Fact 49 can be generalized with variable \( t \) on the right hand side of the turnstile replaced with arbitrary \( \phi(t) \). Somewhat informally speaking, \( w, A \models \phi(t) \) iff the value of \( \phi(A) \) contains \( \{ w \} ^{\leq} \) (think of \( \phi \) here as a polynomial on the Heyting algebra of upward closed subsets of \( \{ w \} ^{\leq} \)).

Putting all this together, we get that \( w, A \models (\circ p \Rightarrow p) \Rightarrow p \) iff for any \( z \geq w \), it holds that \( z, A \cap \{ z \} ^{\leq} \models (\circ p \Rightarrow p) \Rightarrow p \). That is, \( \{ z' \geq z \mid \{ z' \} ^{\leq} \subseteq A \} \subseteq A \) only if \( \{ z \} ^{\leq} \subseteq A \) and in order for \( \text{Set}^{(W, \leq)} \) to be scattered this has to hold for any \( w \in W \), any \( z \geq w \) and any \( A \in \Omega(w) \). But then the reasoning can be completed just like in the case of Corollary 26. This finishes the proof of Theorem 47.

5.3 Topos of Trees and Its Generalizations

\[ \text{BMSS12} \] introduced the topos of trees (or forests) \( S \), i.e., the topos of presheaves on \((\omega, \supseteq)\), where \( \supseteq \) is the converse of the usual order on \( \omega \). Let us begin with the observation that an axiomatization of the mHC-logic of the underlying frame of \( S \) can be obtained by means of techniques introduced in Section 4.3.

**Theorem 50.** \( \text{KM} \oplus \text{LC} \) is the propositional logic of \((\omega, \supseteq, \supset)\).

**Proof (sketch).** Just like in the first sentence of the proof of Corollary 32, we note that \( \sigma^\text{mix}(\text{KM} \oplus \text{LC}) \) is a just a notational variant of \( \text{GL}^{\text{cl}, 3} := \text{GL}^{\text{cl}} \oplus 3 \). This logic in turn can be shown (using standard modal techniques) to be the logic of natural numbers with reverse strict order taken as a classical Kripke frame. The result now follows from properties of \( \sigma^\text{mix} \), which are not quoted explicitly in Theorem 28, but can be extracted immediately from the original references, for example \[ \text{WZ97, Prop. 21} \].

Reasoning analogous to those in proofs of Corollary 39 and Theorem 47 yields then that \( \text{KM} \oplus \text{LC} \) coincides with the set of \( S \)-validities in the Mitchell-Bénabou language restricted to (topos-theoretical counterparts of) connectives in \( \mathcal{L}_{\text{int}} \) together with \( \odot \), the latter interpreted in \( \text{Set}^{(\omega, \leq)} \) by \( \odot \) as specified by Fact 48 above. Similar observations underly recent proof-theoretical investigations of this logic in \[ \text{CG14} \].

Theorem 47 shows that the topos of trees is scattered. \( \Omega \)-endomorphism “\( \supseteq \)” (this notation here would risk clashing with the one for a strict partial order and its converse) defined in \[ \text{BMSS12} \] is easily seen to coincide with \( \odot \). The Internal Banach Fixpoint Theorem 2.9 of \[ \text{BMSS12} \] shows that \( \odot \)-contractive mappings on arbitrary inhabited objects in \( S \) do have (unique) fixpoints.

Now, Example 43 above shows that such a strong statement is not valid in arbitrary scattered toposes of presheaves, even quite similar to \( S \). The crucial Lemma 2.10 in \[ \text{BMSS12} \] is not amenable to far-reaching generalizations.

However, \[ \text{BMSS12, Section 8} \] discusses a whole class of toposes together with a notion of a contractiveness guaranteeing fixpoint’s existence. The class in question are sheaves on complete Heyting algebras with a well-founded basis \[ \text{DM04} \] rather than just presheaves on Noetherian partial orders—crucially, \( S \) can be also seen as such a sheaf topos—and the required notion of contractiveness is stronger than the one expressible in the internal logic.

Let us elaborate on the last point. As we saw, toposes allow an internal interpretation of modalities as morphisms \( \Omega \to \Omega \). Actually, from the “propositions as predicates” perspective, any operation on subobjects of a given object is a “local” candidate for a modality. However,
constructive or categorical logic is mostly about “propositions as types”; see, e.g., [AMPR01; BPR01; PR11; BP00; PD01; PR11] for modal aspects. This perspective works even with mild assumptions about the underlying category. In particular, algebraic type theories require only finite products, whereas ccc’s correspond to functional type theories [Cro93]: those whose type system is in fact that of Brouwerian semilattices of Remark 7 above. To see the details of this correspondence, just remove the rules for Ω ≈ and all abbreviations using these from Tables 4 and 5 then interpret conjunctions as products and implication as exponentiation.

From this perspective, modalities correspond to endofunctors. In particular, ND systems for PLL and S4 are interpreted by, respectively, monads and comonads—see, e.g., references in Remark 17—and SL yields a special subclass of pointed or applicative functors. [MP08]

More precisely, one obtains a variant of [BMSS12, Definition 6.1]. Possible differences are: the modal assumption of normality forces only being monoidal wrt cartesian structure (cf. [BPR01; PR11]) rather than preservation of all finite limits as in the second clause of that definition; furthermore, the assumption of uniqueness in the first clause would rely on exact reduction and conversion rules of the proof system. A systematic study of such endofunctors in a cartesian setting—i.e., assuming only the presence conjunction among propositional connectives—has been undertaken by [ML13] under the name of guarded fixpoint categories. Other possible names for such endofunctors include contraction, delay, (strong) Löb, SL and MGRT, the last being an abbreviation of the original name in [BMSS12].

One can relate these two views on modalities. Whenever \( F : \mathcal{E} \to \mathcal{E} \) is monic-preserving and \( \mathcal{E} \) has pullbacks, associate with a \( F \)-coalgebra \( C \) a modality \([F]\) on subobjects \( M \xrightarrow{m} C \):

\[
\begin{array}{ccc}
[F.L]M & \xrightarrow{(F.L)_M} & C \\
\downarrow & & \downarrow \gamma \\
FM & \xrightarrow{Fm} & FC
\end{array}
\]

(see [AMMS12] for the history of this diagram in papers on well-founded coalgebras). Furthermore, whenever \( F \) is pointed (applicative), i.e., a \( \Omega \)-endofunctor with \((F)_0 = 1\) being the point or unit of \( F \), \((F)_CM\) is a subcoalgebra of \((F)_C\) for any \( M \xrightarrow{m} C \) and hence \( m \leq [F.(F)_C]m \), i.e., the “local” translation of \((r)\) is universally valid.

**Remark 51.** [BMSS12, Theorem 6.8] allows to isolate sufficient conditions ensuring that the operator \([F.(F)_C]\) induced by a SL\(^i\)-endofunctor \( F : \mathcal{E} \to \mathcal{E} \) is a SL\(^i\)-modality: pullback-preservation of \( F \) and \( \mathcal{E} \) being a topos. While it is not mentioned in [BMSS12], one can find natural counterexamples when such conditions are dropped!

The operation on subobjects of \( C \in \mathcal{S} \), induced by \( \odot : \Omega \to \Omega \) from Fact 48 is defined in an alternative way in [BMSS12]: as \([\odot.(\odot)_C]\) for a delay endofunctor \( \odot : \mathcal{S} \to \mathcal{S} \), whose action on objects is \((\odot C)(0) := 1\) and \((\odot C)(n + 1) := (\odot C)(n)\). In a sense, \( \odot \) can be called the Cantor-Bendixson endofunctor. Factoring through it is the desired “external” notion of contractivity ensuring fixpoint’s existence. Both notions nicely complement each other:

... the external notion provides for a simple algebraic theory of fixed points for not only morphisms but also functors (see Section 2.6), whereas the internal notion is
useful when working in the internal logic. [BMSS12]

Could [EJP00] have had more impact if the authors had a) employed the external perspective on modalities in addition to the internal one and b) had the hindsight of [DM04]? This is rather too counterfactual a question to consider. Note also that what matters from the point of view of [BMSS12]—and Theoretical CS in general—is the use made of these external and internal L"ob modalities. [BMSS12, Section 3] constructs a model of a programming language with higher-order store and recursive types entirely inside the internal logic of $\mathcal{S}$. [BMSS12, Section 4] provides semantic foundation for dependent type theories extended with a $\mathcal{SL}$ modality and guarded recursive types; this can be regarded as an extension of fixpoint results along the lines of Section 3 above to predicate and higher-order constructive logics. [BMSS12, Section 5] shows that a class of (ultra-)metric spaces commonly used in modelling corecursion on streams is equivalent to a subcategory of $\mathcal{S}$.

Clearly, this is a large area rather overlooked by researchers on (intuitionistic) modal logic side. There is no space here to discuss my own work in progress, e.g., on the Curry-Howard interpretation of $\text{mHC}$, but let us conclude with a question from participants of ToLo III: is there a natural subclass of internal modalities in toposes (endomorphisms $\Omega \rightarrow \Omega$) inducing external modalities (endofunctors) in a generic way?

**Remark 52.** Earlier incarnations of this paper included also a question by Lars Birkedal: what are additional logical principles which would allow a scattered topos to model not only guarded (co-)recursion, but also, e.g., countable nondeterminism? However, a recent work [BBML14] addressed the issue: expressing the additional property needed for stating the adequacy of the logical relation requires more than logical connectives introduced so far. It can be only stated in presence of an additional modality right adjoint to double negation.

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