OPERATOR PRODUCT ON LOCALLY SYMMETRIC SPACES OF RANK ONE AND THE MULTIPLICATIVE ANOMALY

A. A. BYTSENKO\(^1\), E. ELIZALDE\(^2\), M. E. X. GUIMARÃES\(^3\)

1. Departamento de Física, Universidade Estadual de Londrina
   Caixa Postal 6001, Londrina–Paraná, Brazil
2. Institut d’Estudis Espacials de Catalunya,
   Consejo Superior de Investigaciones Científicas (IEEC/CSIC)
   Edifici Nexus, Gran Capità 2–4, 08034 Barcelona, Spain;
   Departament d’Estructura i Constituents de la Matèria, Facultat de Física,
   Universitat de Barcelona, Av. Diagonal 647, 08028 Barcelona, Spain
3. Universidade de Brasília, Departamento de Matemática
   CEP: 70910–900, Brasília–DF, Brazil

October 30, 2018

Abstract

The global multiplicative properties of Laplace type operators acting on irreducible rank one symmetric spaces are considered. The explicit form of the multiplicative anomaly is derived and its corresponding value is calculated exactly, for important classes of locally symmetric spaces and different dimensions.

1 Introduction

In theories of quantum fields (for example, in higher-derivative quantum gravity) one has to deal with the product of two (or more) elliptic differential operators. It is natural, therefore, to investigate multiplicative properties of the determinants of differential operators, in particular the so–called multiplicative anomaly \(^1\), \(^2\) (for the definition of this anomaly see Sect. 3 below). The multiplicative anomaly can be expressed by means of the non–commutative residue associated with a classical pseudo–differential operator, the Wodzicki residue \(^3\).

Recently, the important role of this residue has been recognized in physics. The Wodzicki residue, which is the unique extension of the Dixmier trace to the wider class of pseudo-differential operators \(^4\), \(^5\), has been considered within the non–commutative geometrical approach to the standard model of the electroweak interactions \(^6\), \(^7\), \(^8\), \(^9\), \(^10\). This residue is also used to write down
the Yang-Mills action functional. The residue formulas have also been employed for dealing with the structure of spectral functions related to operators acting in locally symmetric spaces \[12, 13\], singularity of the zeta functions \[14\], and the commutator anomalies of current algebras \[15\]. Other recent papers along these lines can be found in Refs. 15. The purpose of the present paper is to investigate the global multiplicative properties of invertible elliptic operators of Laplace type acting on a non-compact symmetric space, and related zeta functions.

2 The Spectral Functions

We shall be working with irreducible rank one symmetric space \(X = G/K\) of non-compact type. Thus \(G\) will be connected non-compact simple split rank one Lie group with finite center and \(K \subset G\) will be maximal compact subgroup. Let \(\Gamma \subset G\) be discrete, co-compact, torsion free subgroup. Let \(L : C^\infty (V(X)) \to C^\infty (V(X))\) be partial differential operators acting on smooth sections of vector bundles \(V(X)\). Let \(\chi\) be a finite-dimensional unitary representation of \(\Gamma\), let \(\{\lambda_\ell\}_{\ell = 0}^\infty\) be the set of eigenvalues of the second-order operator of Laplace type \(L = -\Delta_\Gamma\) acting on smooth sections of the vector bundle over \(\Gamma \backslash X\) induced by \(\chi\), and let \(n_\ell(\chi)\) denote the multiplicity of \(\lambda_\ell\).

We need further a suitable regularization of the determinant of a differential operator, since the naive definition of the product of eigenvalues gives rise to a badly divergent quantity. We make the choice of zeta-function regularization. The zeta function associated with the operators \(L \equiv L + b\) has the form

\[
\zeta(s|L) = \sum_\ell n_\ell(\chi)\{\lambda_\ell + b\}^{-s},
\]

here \(b\) is arbitrary constant (endomorphism of the vector bundle \(V(X)\)), called in the physical literature the potential term. \(\zeta(s|L)\) is a well-defined analytic function for \(\Re s > \dim(X)/2\), and can be analytically continued to a meromorphic function on the complex plane \(\mathbb{C}\), regular at \(s = 0\).

The following representations of \(X\) up to local isomorphism can be chosen

\[
X = \begin{bmatrix}
SO_1(n, 1)/SO(n) \\
SU(n, 1)/U(n) \\
SP(n, 1)/(SP(n) \otimes SP(1)) \\
F_4(-20)/Spin(9)
\end{bmatrix},
\]

(2)

where \(\dim X = n, 2n, 4n, 16\), respectively. Then (see for detail \[17\])

\[
SO(p, q) \overset{def}{=} \left\{ g \in GL(p + q, \mathbb{R} \mid \begin{array}{c}
g^t I_{pq} g = I_{pq} \\
det g = 1\end{array} \right\},
\]

(3)

\[
SU(p, q) \overset{def}{=} \left\{ g \in GL(p + q, \mathbb{C} \mid \begin{array}{c}
g^t I_{pq} g = I_{pq} \\
det g = 1\end{array} \right\},
\]

(4)

\[
SP(p, q) \overset{def}{=} \left\{ g \in GL(2(p + q), \mathbb{C} \mid \begin{array}{c}
g^t J_{p+q} g = J_{p+q} \\
g^t K_{p+q} g = K_{p+q}\end{array} \right\},
\]

(5)
where $I_n$ is the identity matrix of order $n$ and

$$I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad K_{pq} = \begin{pmatrix} I_{pq} & 0 \\ 0 & I_{pq} \end{pmatrix}.$$

(6)

The groups $SU(p, q), SP(p, q)$ are connected; the group $SO_1(p, q)$ is defined as the connected component of the identity in $SO(p, q)$ while $F_4(-20)$ is the unique real form of $F_4$ with Dynkin diagram

$$\circ - \circ = \circ - \circ$$

(7)

for which the character $(\dim X - \dim K)$ assumes the value $(-20)$ (see Ref. 17). We assume that if $G = SO(m, 1)$ or $SU(q, 1)$ then $m$ is even and $q$ is odd.

The suitable Harish–Chandra–Plancherel measure is given as follows:

$$\lvert C(r) \rvert^{-2} = C_G \pi \tau P(r) \tanh (a(G)r),$$

(8)

$$a(G) = \begin{cases} \frac{\pi}{2} & \text{for } G = SO_1(2n, 1) \\ \frac{\pi}{2} & \text{for } G = SU(q, 1), \quad q \text{ odd} \\ \frac{\pi}{2} & \text{for } G = SP(m, 1), \quad F_4(-20) \end{cases},$$

(9)

while the constant $C_G$ and the polynomials $P(r)$ (are even polynomials with Miatello coefficients $a_{2\ell}$ [19, 20, 21, 22, 17, 13]) are given in the Table 1.

**Table 1. Structure of the Harish–Chandra–Plancherel measure**

| $G$            | $C_G$                                      | $P(r)$                                         |
|---------------|--------------------------------------------|-----------------------------------------------|
| $SO_1(n, 1)$, $n \geq 2$ | $\left[2^{2n-4} \frac{\Gamma (\frac{n}{2})}{2} \right]^{-1}$ | $\prod_{k=0}^{n-2} \left[ \frac{r^2}{4} + \frac{(2k+1)^2}{4} \right]^2$, $n = 2m$ |
|               |                                            | $\prod_{k=0}^{n} \left[ \frac{r^2}{4} + k^2 \right]$, $n = 2m + 1$ |
| $SU(n, 1)$, $n \geq 2$ | $\left[2^{2n-1} \frac{\Gamma (n)}{2} \right]^{-1}$ | $\prod_{k=1}^{n-1} \left[ \frac{r^2}{4} + \frac{(n-2k)^2}{4} \right]$ |
| $SP(n, 1)$, $n \geq 2$ | $\left[2^{4n+1} \frac{\Gamma (2n)}{2} \right]^{-1}$ | $\left[ \frac{r^2}{4} + \frac{1}{4} \right] \prod_{k=3}^{n+1} \left[ \frac{r^2}{4} + (n - k + \frac{3}{2})^2 \right]$ |
|               |                                            | $\times \left[ \frac{r^2}{4} + (n - k + \frac{5}{2})^2 \right]$ |
| $F_4(-20)$    | $\left[2^{21} \frac{\Gamma (8)}{2} \right]^{-1}$ | $\left[ \frac{r^2}{4} + \frac{1}{4} \right] \left[ \frac{r^2}{4} + \frac{9}{4} \right]$ |
|               |                                            | $\times \prod_{k=0}^{4} \left[ \frac{r^2}{4} + \left( \frac{2k+1}{4} \right)^2 \right]$ |

2.1 The Miatello coefficients and explicit values of anomalies

The coefficients of the polynomial $P(r)$, will be denoted by $a_{2\ell}$:

$$P(r) = \sum_{\ell=0}^{n/2-1} a_{2\ell} r^{2\ell} \quad \text{for } G \neq SO_1(2m + 1, 1),$$
\[ = \sum_{\ell=0}^{m} a_{2\ell} r^{2\ell} \quad \text{for } G = SO_1(2m + 1, 1). \quad (10) \]

For the various rank one simple groups $G$ the Miatello coefficients of the polynomial $P(r)$, $a_{2\ell} = (1/2!)[(d^{2\ell}/dr^{2\ell})P(r)]|_{r=0}$, are given in the Table 2.

**Table 2. First Miatello coefficients**

| $G$   | $n$ | $a_{2\ell}$ |
|-------|-----|-------------|
| $SO_1(n, 1)$ | 2   | $a_0 = \frac{1}{2}$, $a_2 = 1$ |
|       | 4   | $a_0 = \frac{9}{16}$, $a_2 = \frac{7}{2}$, $a_4 = 1$ |
|       | 6   | $a_0 = \frac{225}{64}$, $a_2 = \frac{259}{16}$, $a_4 = \frac{35}{4}$, $a_6 = 1$ |
|       | 8   | $a_0 = \frac{11025}{256}$, $a_2 = \frac{3229}{16}$, $a_4 = \frac{287}{8}$, $a_6 = 21$, $a_8 = 1$ |
|       | 10  | $a_0 = \frac{893025}{1024}$, $a_2 = \frac{1057221}{256}$, $a_4 = \frac{86405}{32}$, $a_6 = \frac{4389}{8}$, $a_8 = \frac{165}{4}$, $a_{10} = 1$ |
| $SU(n, 1)$ | 2   | $a_0 = 0$, $a_2 = \frac{1}{4}$ |
|       | 3   | $a_0 = \frac{1}{16}$, $a_2 = \frac{1}{8}$, $a_4 = \frac{1}{16}$ |
|       | 4   | $a_0 = 0$, $a_2 = \frac{1}{4}$, $a_4 = \frac{1}{8}$, $a_6 = \frac{1}{16}$ |
|       | 5   | $a_0 = \frac{81}{256}$, $a_2 = \frac{45}{64}$, $a_4 = \frac{59}{128}$, $a_6 = \frac{5}{64}$, $a_8 = \frac{1}{256}$ |
|       | 6   | $a_0 = 0$, $a_2 = 4$, $a_4 = \frac{1}{2}$, $a_6 = \frac{1}{4}$, $a_8 = \frac{5}{128}$, $a_{10} = \frac{1}{1024}$ |
| $SP(n, 1)$ | 2   | $a_0 = \frac{9}{64}$, $a_2 = \frac{19}{64}$, $a_4 = \frac{11}{64}$, $a_6 = \frac{1}{64}$ |
|       | 3   | $a_0 = \frac{2025}{1024}$, $a_2 = \frac{4581}{1024}$, $a_4 = \frac{1565}{1024}$, $a_6 = \frac{309}{1024}$, $a_8 = \frac{45}{1024}$, $a_{10} = \frac{1}{1024}$ |
| $F_4(-20)$ |     | $a_0 = \frac{8047225}{16384}$, $a_2 = \frac{18445239}{16384}$, $a_4 = \frac{13020525}{16384}$, $a_6 = \frac{2864323}{16384}$, $a_8 = \frac{262075}{16384}$, $a_{10} = \frac{10437}{16384}$, $a_{12} = \frac{175}{16384}$, $a_{14} = \frac{1}{16384}$ |
The Multiplicative Anomaly and Associated One–Loop Contributions

The spectral zeta function associated with the product $\otimes L^j$ has the form

$$\zeta(s | \otimes L^j) = \sum_{\ell \geq 0} n_\ell \prod_j (\lambda_\ell + b_j)^{-s}. \quad (11)$$

We shall always assume that $b_1 \neq b_2$, say $b_1 > b_2$. If $b_1 = b_2$ then $\zeta(s | \otimes L^j) = \zeta(2s | L)$ is a well–known function. For $b_1, b_2 \in \mathbb{R}$, set $B_j := b_j + 1/4$.

We are interested in multiplicative properties of determinants, the multiplicative anomaly [3, 1, 2], associated with one–loop approximation in quantum theory. The partition function $\log Z \propto -\log \det (\otimes L^j)$ of the product of two elliptic differential operators for the simplest $O(2)$ invariant model of self–interacting charged fields [23] has been analyzed in Ref. 23. The loop approximation can be given in terms of the multiplicative anomaly $F(L_1, L_2)$, which has the form

$$F(L_1, L_2) = \det_\zeta[\otimes L_j][\det_\zeta(L_1)\det_\zeta(L_2)]^{-1}, \quad (12)$$

where we assume a zeta–regularization of determinants, i.e.

$$\det_\zeta(L_j) := \exp \left(-\frac{\partial}{\partial s} \zeta(s = 0 | L_j)\right). \quad (13)$$

Generally speaking, if the multiplicative anomaly related to elliptic operators is nonvanishing then the relation $\log \det(\otimes L_j) = \text{Tr} \log(\otimes L_j)$ does not hold.

3.1 The residue formula and the multiplicative anomaly

The value of $F(L_1, L_2)$ can be expressed by means of the non–commutative Wodzicki residue [2]. Let $O_j$ be invertible elliptic pseudo–differential operators of real non–zero orders $\alpha$ and $\beta$ such that $\alpha + \beta \neq 0$. Even if the zeta functions for operators $O_1, O_2$ and $O_1 \otimes O_2$ are well defined and if their principal symbols obey the Agmon–Nirenberg condition (with appropriate spectra cuts) one has in general that

$$F(O_1, O_2) \neq 1.$$  

For such invertible elliptic operators the formula for the anomaly of commuting operators holds [23]:

$$A(O_1, O_2) = A(O_2, O_1) = \log(F(O_1, O_2)) = \frac{\text{res} \left[\log(O_1^\beta \otimes O_2^{-\alpha})\right]^2}{2\alpha\beta(\alpha + \beta)}. \quad (14)$$

More general formulæ have been derived in Refs. 1, 2. In the case of the product of two operators the following result holds (see Ref. 12):

Theorem:
The explicit form of the multiplicative anomaly in the case $O_j \equiv L_j$ is

\[ A(L_1, L_2) = \frac{A}{2} \sum_{\ell=0}^{n/2-1} \frac{(-1)\alpha_{2\ell}}{(\ell+1)!} B_{2\ell+1} \sum_{k=1}^{\infty} \frac{\sigma_k(\ell + k + 1)!}{(k+1)!} \left( \frac{B_1 - B_2}{B_1} \right)^{k+1} \]

\[ + \frac{A}{2} \log \left( \frac{B_1}{B_2} \right) \sum_{\ell=0}^{n/2-1} \frac{(-1)^{\ell+1} \alpha_{2\ell}}{(\ell+1)!} \left( B_{1\ell+1} - B_{2\ell+1} \right), \] \hspace{1cm} (15)

where $A = (1/4) \chi(1) \text{Vol}(\Gamma \setminus G) \cdot C_G$ and $\sigma_{\ell} \overset{d\text{ef}}{=} \sum_{k=1}^{\ell+1} k^{-1}$.

Finally the numerical values of $A \equiv A(L_1, L_2) / (\chi(1) \text{Vol}(\Gamma \setminus G))$ related to the multiplicative anomaly $A(L_1, L_2)$ for various rank one groups $G$ and constants $[B_1; B_2]$ are given in Table 3.

**Table 3. Explicit numerical values of $A$**

| G       | $n$ | $A$ for some pairs of $[B_1; B_2]$ |
|---------|-----|----------------------------------|
| SO$_1(1, 1)$ | 4   | $[1; \frac{1}{4}] : 0.141936,$ |
|         |     | $[10; \frac{1}{4}] : 11.0951,$ |
|         |     | $[200; 100 + \frac{1}{4}] : 4.57424$ |
|         | 6   | $[1; \frac{1}{4}] : -0.0636199,$ |
|         |     | $[10; \frac{1}{4}] : -20.1404,$ |
|         |     | $[200; 100 + \frac{1}{4}] : -736.671$ |
|         | 8   | $[1; \frac{1}{4}] : 0.00716077,$ |
|         |     | $[10; \frac{1}{4}] : 14.3917,$ |
|         |     | $[200; 100 + \frac{1}{4}] : 25403.1$ |
|         | 10  | $[1; \frac{1}{4}] : -0.000551611,$ |
|         |     | $[10; \frac{1}{4}] : -6.66525,$ |
|         |     | $[200; 100 + \frac{1}{4}] : -599988.$ |
| G      | n     | A for some pairs of [B_1; B_2]                  |
|--------|-------|-----------------------------------------------|
| SU(n,1)| 4     | [1; \frac{1}{4}] : 0.00110888,                |
|        |       | [10; \frac{1}{4}] : 0.0866804,                |
|        |       | [200; 100 + \frac{1}{4}] : 0.0357362          |
|        | 6     | [1; \frac{1}{4}] : -0.000150991,              |
|        |       | [10; \frac{1}{4}] : -0.0488322,               |
|        |       | [200; 100 + \frac{1}{4}] : -1.79837           |
|        | 8     | [1; \frac{1}{4}] : 5.82224 \times 10^{-6},   |
|        |       | [10; \frac{1}{4}] : 0.0103493,                |
|        |       | [200; 100 + \frac{1}{4}] : 17.8461            |
|        | 10    | [1; \frac{1}{4}] : -1.13623 \times 10^{-7},   |
|        |       | [10; \frac{1}{4}] : -0.00102054,              |
|        |       | [200; 100 + \frac{1}{4}] : -85.8268           |
| SP(n,1)| 4     | [1; \frac{1}{4}] : 0.0000152281,              |
|        |       | [10; \frac{1}{4}] : 0.00119037,               |
|        |       | [200; 100 + \frac{1}{4}] : 0.000490762        |
|        | 6     | [1; \frac{1}{4}] : -2.63879 \times 10^{-8},   |
|        |       | [10; \frac{1}{4}] : -8.51064 \times 10^{-6},  |
|        |       | [200; 100 + \frac{1}{4}] : -0.000313153        |
### 4 Conclusions

In this paper, the multiplicative properties of operators of Laplace type and their related zeta functions have been studied. Explicit formulas for the multiplicative anomaly have been investigated. In a special case, namely for \( n = 2 \), Theorem 1 gives \( A(L_1, L_2) = 0 \). For any odd \( n \), the multiplicative anomaly is zero. This statement follows from the general theory of Laplace–type operators (see, for example, Ref. 23). Note that for the four-dimensional space with \( G = SO(4, 1) \), one derives from Theorem 1 the result \( A(L_1, L_2) = -A_G(b_1 - b_2)^2, n = 4 \), which also follows from Wodzicki’s formula (13), where \( A_G = Aa_2/4 \).

We have preferred to limit ourselves here to discuss in detail various particular cases and emphasize the general picture. It seems to us that the explicit results for the anomaly are not only interesting as mathematical results but are also of physical interest, in view of their applications to concrete problems in field theory and gravity, both at the classical and quantum levels. As we see from our numerical results, some nice patterns show up. It is interesting to notice that for all groups except \( F_{4(-20)}/Spin(9) \), there is a dependence of the sign of the anomaly with the dimension \( n = 2m \): for even \( m \) the anomaly is positive while for odd \( m \) it is negative. Also, generically, the absolute value of the anomaly diminishes with \( n \) for small values of \( B_1, B_2 \), but it can also get quite large, and increase, for bigger values of \( B_1 \) and \( B_2 \). Note also, that the spectral properties of products of differential operators related to higher-spin fields may differ, in principle, from the properties considered in this paper, what deserves

| \( G \) | \( n \) | \( A \) for some pairs of \([B_1; B_2]\) |
|---|---|---|
| 8 | \( [1; \frac{1}{4}] \) | \( 8.70798 \times 10^{-12} \), |
| | \( [10; \frac{1}{4}] \) | \( 1.5324 \times 10^{-8} \), |
| | \( [200; 100 + \frac{1}{4}] \) | \( 0.0000263649 \), |
| 10 | \( [1; \frac{1}{4}] \) | \( -1.30145 \times 10^{-15} \), |
| | \( [10; \frac{1}{4}] \) | \( -1.13401 \times 10^{-11} \), |
| | \( [200; 100 + \frac{1}{4}] \) | \( -9.46242 \times 10^{-7} \), |
| \( F_{4(-20)} \) | \( [1; \frac{1}{4}] \) | \( -2.09552 \times 10^{-11} \), |
| | \( [10; \frac{1}{4}] \) | \( -6.76666 \times 10^{-9} \), |
| | \( [200; 100 + \frac{1}{4}] \) | \( -2.49078 \times 10^{-7} \), |
further study. We hope to return soon to this problem elsewhere.

References

[1] M. Kontsevich and S. Vishik, "Determinants of Elliptic Pseudo-Differential Operators", Preprint MPI/94-30 (1994).

[2] M. Kontsevich and S. Vishik, Geometry of Determinants of Elliptic Operators, e–Print arXiv [hep-th/9406140].

[3] M. Wodzicki, in Lecture Notes in Mathematics, ed. Yu. I. Manin (Springer–Verlag, Berlin, 1987), Vol. 1289, p. 320.

[4] A. Connes, Commun. Math. Phys. 117, 673 (1988).

[5] D. Kastler, Commun. Math. Phys. 166, 633 (1995).

[6] A. Connes and J. Lott, Nucl. Phys. B 18, 29 (1990).

[7] A. Connes, Non–Commutative Geometry (Academic Press, New York, 1994).

[8] A. Connes, Commun. Math. Phys. 182, 155 (1996).

[9] A. H. Chamseddin and A. Connes, Phys. Rev. Lett. 77, 4868 (1996).

[10] A. H. Chamseddine and A. Connes, Commun. Math. Phys. 186, 731 (1997).

[11] C. P. Martin, J. M. Gracia–Bondia and J. C. Varilly, Phys. Reports 294, 363 (1998).

[12] A. A. Bytsenko and F. L. Williams, JMP 39, 1075 (1998).

[13] A. A. Bytsenko and F. L. Williams, J. Phys. A: Math. Gen. 32, 5773 (1999).

[14] E. Elizalde, J. Phys. A 30, 2735 (1997).

[15] J. Mickelsson, in Lecture Notes in Phys. (Springer, Berlin, 1994), Vol. 436, pp. 123–135.

[16] G. Cognola, E. Elizalde and S. Zerbini, Dirac Functional Determinant in Terms of the Eta Invariant and the Noncommutative Residue, Commun. Math. Phys., to appear;
E. Elizalde, JHEP 9907, 015 (1999);
E. Elizalde, G. Cognola and S. Zerbini, Nucl. Phys. B532, 407 (1998).

[17] F.L. Williams, J. Math. Phys. 38, 796 (1997).

[18] S. Helgason, Differential Geometry and Symmetric Spaces, Pure and Applied Math. Ser. 12, Academic Press (1962).
[19] R. Miatello, *The Minakshisundaram–Pleijel Coefficients for the Vector-Valued Heat Kernel on Compact Locally Symmetric Spaces of Negative Curvature*, PhD Thesis, Rutgers University (1976).

[20] R. Miatello, *Manuscripta Math.* **29**, 249 (1979).

[21] R. Miatello, *Trans. Am. Math. Soc.* **260**, 1 (1980).

[22] A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, *Phys. Reports* **266**, 1 (1996).

[23] K. Benson, J. Bernstein and S. Dodelson, *Phys. Rev. D* **44**, 2480 (1991).

[24] E. Elizalde, L. Vanzo and S. Zerbini, *Commun. Math. Phys.* **194**, 613 (1998).