Non-perturbative Dynamical Decoupling Control and Threshold

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Protection of quantum states in a noisy environment is essential for reliable quantum information processing. In this Letter, through a sequence of unoptimized periodical rectangular pulses we show that beyond perturbation, dynamical decoupling can be a powerful tool for the control of fidelity and entanglement of the open system in a non-Markovian bath. With this non-perturbative approach, we have found, for the first time, the threshold and pulse parameter regions in which the non-idealized pulses are effective in controlling quantum coherence.

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Introduction.— The ultimate information processor in modern quantum technologies is a quantum computer [1]. While a full-fledged quantum computer is still impractical at the present moment, research on quantum devices such as multiple-qubit simulators [2], the quantum cryptography device [3], quantum imaging [4], etc, has achieved momentous progress in recent decades. A simple, but vital quantum device called quantum memory is a quantum system that can be used to store quantum states in a noisy environment for a certain period of time. Therefore, critical to quantum information processing is the ability to efficiently protect quantum memory from the corruptive influence of the environmental noises either classical [5] or quantum [6].

A variety of theoretical strategies for combating the deleterious environmental noise have been proposed [7–15]. Among them, Bang-Bang control or dynamical decoupling control of system-environment interactions by external fields is widely discussed due to its simplicity and wide applicability [16–19]. Although the theoretical formalism for the dynamical decoupling is widely employed in combating decoherence and protecting information leakage, most analyses and numerical simulations are based on idealized pulses and the Trotter product formula [20], where it is assumed that these idealized pulses are effectively turned off when the pulses are applied to the system. More specifically, if the control Hamiltonian is described by \( H_c = J\sigma_z \), the total evolution operator in a short time \( \delta \) is \( U(\delta) = \exp(-iH\delta - iJ\delta\sigma_z) \approx -i\sigma_z \) when \( J\delta = \pi/2 \) and \( \delta \) goes to zero (The strength \( J \) goes to infinity). This amounts to taking the zero-order perturbation for an effective Hamiltonian \( H_{\text{eff}} = \pi/2\sigma_z + \delta H \) (setting \( h = 1 \)) with the small perturbation parameter \( \delta \). Without comparison with exact (numerical or analytical) solutions, it is not clear at all whether the zeroth-order perturbation or even perturbation theory is valid or not for any of models with realistic system-bath interactions.

For instance, if a phase transition takes place when \( \delta \) is zero, then the omission of the system-bath interaction may become problematic because of the singularity at the critical point, no matter how small \( \delta \) is, exemplified by the well-known failure of perturbation theory in treating the BCS pairing model. Specifically, if we set one of modes in the BCS model as the system of interest and the others as bath modes [21], the pairing correlation between this mode and others corresponds to the system-bath coupling and we discover a second-order quantum phase transition at \( \delta = 0 \). Therefore, it is highly desirable to have a non-perturbative theory to explore the non-perturbative regimes and provide the effective parameter regions of the control pulse.

In this Letter, we present a novel, non-perturbative dynamical control theory, based on the exact subsystem stochastic master equations obtained from combining the Feshbach projection technique [22–24] and the quantum state diffusion equation (QSD) [25–27]. Our approach employs the non-idealized pulses with finite widths \( \Delta \), periods \( \tau \) and finite pulse strengths \( \Psi/\Delta \) as characteristic parameters. Surprisingly, we have found the threshold and efficient parameter regions where the fidelity and entanglement dynamics can be controlled. We show that the characteristic quantity for quantum control is the ratio between the pulse period \( \tau \) and the width \( \Delta \). The other parameters, such as pulse strength, do not play an essential role in this scheme [23]. Our results show that the parameters for idealized fast-strong pulses control (\( \Delta \) and \( \tau \) are effectively taken as zeros) occupy a very small portion of the parameter region in which efficient quantum control can be realized. In addition, we will show that the non-Markovian environment is crucial for the dynamical control of quantum coherence.

Our non-perturbative approach can accommodate generic system-bath couplings, structured baths and varied numbers of environmental modes ranging from one mode, two modes to infinite number of modes. Our re-
results provide higher flexibility in the experimental implementation of the dynamical decoupling scheme since our scheme allows much larger parameter region.

**Stochastic master equation for subsystem**—We start with the exact QSD equation describing a quantum trajectory of an open quantum system \( \psi_t \):

\[
\partial_t \psi_t = [-iH_{\text{sys}} + Lz^* - L\dot{O}(t, z^*)] \psi_t = H_{\text{eff}}(t) \psi_t, \tag{1}
\]

where \( H_{\text{sys}} \) and \( L \) are the system Hamiltonian and Lindblad operator, respectively. \( z^* \) is the Gaussian process describing the environmental influence, satisfying \( M[z^*_t] = M[z^*_t z^*_s] = 0 \) and \( M[z_t z^*_s] = \alpha(t, s) \), where \( M \) denotes the statistical mean over the noise \( z^*_t \) and \( \alpha(t, s) \) is an arbitrary environmental correlation function.

Using PQ-partitioning technique, we can divide the \( n \)-dimensional wave-function \( \psi_t \) into two parts: an interested \( m \)-dimensional vector \( P(t) \) and the rest part of the vector, \( Q(t) \), which is an \( (n-m) \)-dimensional vector. Accordingly, the stochastic state vector \( \psi_t \) and the stochastic effective Hamiltonian may be rewritten into a block form given by

\[
\psi_t = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}, \quad H_{\text{eff}} = \begin{pmatrix} h(t) & R(t) \\ W(t) & D(t) \end{pmatrix} \tag{2}
\]

Here the \( m \times m \) matrix \( h \) and the \( (n-m) \times (n-m) \) matrix \( D \) are the self-Hamiltonians living in the subspaces of \( P \) and \( Q \), respectively. \( W \) and \( R \) are the coupling terms. For the selected \( m \)-dimensional subsystem- \( P \), we can establish the following stochastic master equation:

\[
i\partial_t P(t) = h(t) P(t) + R(t) G(t, 0) Q(0) - i \int_0^t ds g(t, s) P(s), \tag{3}
\]

where \( G(t, s) = \mathcal{T}_- \left\{ \exp \left[ -i \int_s^t ds' D(s') \right] \right\} \) and \( g(t, s) = R(t) G(t, s) W(s) \). When \( m = 1 \), \( P \) is the amplitude of one basis vector, and Eq. (3) becomes a one-dimensional stochastic master equation [24]. As shown below, by choosing a proper basis and the order of the basis vectors, one can significantly simplify the numerical simulation and analytical calculations.

**Models and Analysis**—Here we consider two models: (i) a single-qubit system in dissipative bosonic environment with \( H_{\text{sys}} = E \sigma_z \) and \( L = \sigma_- \); (ii) a two-qubit system in a collective dissipative bosonic environment with \( H_{\text{sys}} = E (\sigma_z^1 + \sigma_z^2) \) and \( L = \sigma_-^1 + \sigma_-^2 \), where \( E \) may be a time-dependent function. The former corresponds to the rather general individual error considered in quantum information, and the latter to the collective error.

For simplicity, the non-Markovian environmental noise throughout the paper is described by the Ornstein-Uhlenbeck type noise with the correlation function \( \alpha(t, s) = \frac{2}{\gamma} e^{-\gamma |t-s|} \), where \( 1/\gamma \) characterizes the memory time of the non-Markovian environment. The correlation function has a well-defined Markov limit when \( \gamma \to \infty \).

For the single-qubit model, in the basis \( \{ |0\rangle, |1\rangle \} \), the effective Hamiltonian becomes

\[
H_{\text{eff}} = \begin{pmatrix} -E/2 & i z^*_t \\ 0 & E/2 - i F \end{pmatrix}, \tag{4}
\]

where \( F(t) \) is the coefficient function in the exact O-operator \( \dot{O}(t, z^*) = F(t) L \) [20] and satisfies

\[
\partial_t F(t) = \frac{\gamma}{2} + (-\gamma + iE) F + F^2, \tag{5}
\]

and \( F(0) = 0 \). When the initial state is chosen as \( |\psi_0\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle) \), by Eq. (3) and in the rotating frame of \( H_{\text{sys}} \), the fidelity measuring the survive probability of the initial state under the dissipative dynamics is easily obtained as

\[
F(t) = \frac{1 + \mathcal{R}[e^{-\int_0^t ds F(s)}]}{2}, \tag{6}
\]

where \( \mathcal{R}[\cdot] \) stands for the real part.

For the two-qubit model, the exact O-operator contained in the stochastic Hamiltonian [11] can be explicitly derived [23] as \( \dot{O}(t, z^*) = F_1(t) O_1 + F_2(t) O_2 + i U(t, z^*) O_3 \), where \( O_1 = L \), \( O_2 = \sigma_3^1 \sigma^B_2 + \sigma^A_3 \sigma^B_2 \), \( O_3 = \sigma^A_1 \sigma^B_2 \), and \( U(t, z^*) = \int_0^t ds U(t, s) z^*_s \). These coefficient functions satisfy

\[
\begin{align*}
\partial_t F_1(t) &= \frac{\gamma}{2} + (-\gamma + iE) F_1 + F^2_1 + 3 F^2_2 - \frac{i}{2} \dot{U} \\
\partial_t F_2(t) &= (-\gamma + iE) F_2 - F^2_2 + 4 F_1 F_2 + F^2 - \frac{i}{2} \dot{U} \\
\partial_t \dot{U}(t) &= -2i \gamma F_2 - (2\gamma + 2iE) \dot{U} + 4F_1 \dot{U}, \tag{7}
\end{align*}
\]

with the notation \( \dot{U}(t) = \int_0^t ds \sigma_{0}(t, s) U(t, s) \). The boundary conditions are given by \( F_1(0) = F_2(0) = \dot{U}(0) = 0 \) and \( U(t, t) = -4i F_2(t) \).

It is advantageous to choose \( \{ |s\rangle, |a\rangle, |11\rangle, |00\rangle \} \) as basis, where \( |s\rangle = (1/\sqrt{2})(|10\rangle + |01\rangle) \) and \( |a\rangle = (1/\sqrt{2})(|10\rangle - |01\rangle) \). If \( P \) is chosen to be spanned by the first two basis vectors \( |s\rangle \) and \( |a\rangle \), the stochastic Hamiltonian can be written as

\[
H_{\text{eff}} = \begin{pmatrix} -2if & \sqrt{2}(z^*_t + U_z) \\ 0 & 0 \\ 0 & E - 2i(F_1 + F_2) \\ \sqrt{2}z^*_t & 0 \end{pmatrix}, \tag{8}
\]

where \( f(t) \equiv F_1(t) - F_2(t) \). From Eq. (7), we can check that

\[
\partial_t f(t) = \frac{\gamma}{2} + (-\gamma + iE) f + 2f^2, \tag{9}
\]

and \( f(0) = 0 \). Suppose that the initial state is the pure state \( \langle \psi_0 | = P_s(0)|0\rangle + P_a(0)|a\rangle + q_{11}(0)|11\rangle + q_{00}(0)|00\rangle \) with \( |P_s(0)|^2 + |P_a(0)|^2 = 1 \) and \( q_{11}(0) = q_{00}(0) = 0 \). From Eq. (8), we have \( P_s(t) = e^{-2f} \int_0^t ds f(s) P_s(0), P_a(t) = \)
\[ P_a(0) \) and \( |\psi_k\rangle = P_s(t)|s\rangle + P_a(t)|a\rangle + q_{00}(t)|00\rangle \). The concurrence \[ 34 \] for each trajectory is given by

\[
C(\psi_t) = |e^{-4\int_0^t ds f(s)} P_s^2(0) - P_a^2(0)|. \tag{10}
\]

In fact, Eqs. \[ 9 \] and \[ 10 \] do not contain stochastic noise explicitly. By Novikov theorem, ensemble average \( M[x(f)] = x(f) \). Thus \( M[C(\psi_t)] = C(\psi_t) \). The concurrence \( C(\rho_s) = |\rho_{ss} - \rho_{ss} - \rho_{aa} + \rho_{ss}| \), where \( \rho_{ss} = M[|P_s(t)|^2], \rho_{aa} = M[|P_a(t)|^2], \) and \( \rho_{ss} = \rho_{aa} = M[P_s(t)P_a(0)] \). Straightforward calculation shows that \( C(\rho_s) = M[C(\psi_t)] \). Moreover, \( C(\rho_s) \) is independent on \( \rho_{00} \) and any coherence terms involving \( |00\rangle \). Thus for the initial states with \( q_{11}(0) = 0 \), Eq. \[ 10 \] is valid and is always equal to \( C(\rho_s) \).

In a similar way, we get an explicit expression for the fidelity: \[ F \equiv \langle \psi_0|\rho_t|\psi_0\rangle \equiv \langle \psi_0|M[|\psi_t||\psi_t|]\psi_0\rangle = M[\langle \psi_0|\psi_t\rangle|\psi_t|\psi_0\rangle]. \]

Thus

\[
F(t) = |e^{-4\int_0^t ds f(s)}| P_s^2(0)| + |P_a^2(0)|^2. \tag{11}
\]

It should be noted that if \( |P_s(0)| = 1 \) and \( P_a(0) = 0 \), i.e. \( |\psi_0\rangle = |s\rangle \), \( C(\psi_t) = F(t) = e^{-4\int_0^t ds f(s)} \). In this situation, the fidelity can be used as a reliable measure for controlling entanglement.

Non-perturbative control of fidelity and entanglement. — The exact subsystem stochastic master equation \[ 3 \] is the starting point of our non-perturbative dynamical decoupling theory. The exact expressions in Eqs. \[ 6 \] and \[ 10 \] are determined by the differential equations given by Eq. \[ 3 \] and Eq. \[ 9 \] respectively, and the solutions to those equations are dependent on two physical quantities \( \gamma \) and \( E \). The parameter \( 1/\gamma \) is the correlation time of the environment and is non-tunable for a given environment. Our control is introduced through the system energy shift \( E(t) = \omega + c(t) \), where \( \omega \) is the bare frequency and \( c(t) \) is a time-dependent control. This is similar to coherent control theory, where phases of external field are the control parameters. In principle, the control \( c(t) \) can be an arbitrary function of time that can be realized by an external field applied to the system, including rectangular pulse sequences with arbitrary strength and frequency. The zeroth-order perturbation or idealized pulse approximation shows that if the frequency of the pulse is sufficiently high, the information leakage rate of subspace to the environment or the rest part of the system could be efficiently suppressed. In this Letter, we will go beyond perturbation and the idealized pulses to explore the non-perturbative regimes and control thresholds.

The control parameters consist of the period of the rectangular pulse \( \tau \), the duration \( \Delta \) and strength \( \Psi/\Delta \), respectively, i.e. \( c(t) = \Psi/\Delta \) for regions \( n\tau - \Delta < t \leq n\tau, \) \( n \geq 1 \) integral, otherwise \( c(t) = 0 \). The pulse frequency is determined by the dimensionless proportion parameter \( \tau/\Delta \). The smaller \( \tau/\Delta \), the more frequent pulses are applied.

**FIG. 1.** The fidelity at the moment \( \omega t = 10 \) of the single-qubit system under control. It is initially prepared in \( |\psi_0\rangle = (1/\sqrt{2})(|1\rangle + |0\rangle) \). The bottom triangular area (dark blue) has no physical meaning since \( \tau \) is always larger than \( \Delta \) by definition. We choose \( \Psi = \omega \) and \( \gamma = 0.5 \).

**FIG. 2.** The concurrence at the moment \( \omega t = 10 \) of a two-qubit system under control. It is initially prepared in \( |\psi_0\rangle = |s\rangle \). Again, the bottom triangular area (dark blue) has no physical meaning since \( \tau \) is always larger than \( \Delta \) by definition. We choose \( \Psi = \omega \) and \( \gamma = 0.5 \).

**FIG. 3.** The concurrence dynamics of a two-qubit system prepared in \( |\psi_0\rangle = |s\rangle \) with different \( \Delta \)'s and \( \tau \)'s. (a) \( \omega \Delta = 10^{-2} \); (b) \( \omega \Delta = 2 \times 10^{-2} \); (c) \( \omega \tau = 5 \times 10^{-2} \); (d) \( \omega \tau = 6 \times 10^{-2} \). We choose \( \Psi = \omega \) and \( \gamma = 0.5 \).
With different parameters, it allows efficient control of quantum coherence with initial subspace or state \(|\psi_0\rangle = |s\rangle\). We choose \(\Delta = 10^{-2}/\omega\) and \(\gamma = 0.5\).

FIG. 4. \(\tau/\Delta - \Psi\) phase diagram of the concurrence at the moment \(\omega t = 10\) of a two-qubit system under control. It is initially prepared in \(|\psi_0\rangle = |s\rangle\). We choose \(\Delta = 10^{-2}/\omega\) and \(\gamma = 0.5\).

FIG. 5. The concurrence damping time \(T(C = 0.95)/\omega t\) of entanglement of a two-qubit system prepared in \(|\psi_0\rangle = |s\rangle\) vs. \(\gamma\) with different \(\tau/\Delta\)'s. We choose \(\Delta = 0.02\omega t\) and \(\Psi = \omega\).

In what follows, we will explore the parameter regions that allow efficient control of quantum coherence with more practical pulses. For the single qubit model, we plot the contour of the fidelity at \(\omega t = 10\) as the function of the pulse period \(\tau\) and the pulse duration \(\Delta\), which may be called a \(\tau - \Delta\) phase diagram. The result of ideal pulse sequence \((\Delta \to 0)\) is at the left-bottom corner in Fig. 1. Surprisingly, our calculations show that there is a large parameter region for the non-idealized pulses, where the initial subspace or state \(|\psi_0\rangle = (1/\sqrt{2})(|1\rangle + |0\rangle)\) can be protected with the same high fidelity as the idealized pulses, as shown in the red zone of Fig. 1 and characterized by \(1 < \tau/\Delta \lesssim 3\). More transparently, we see that the fidelity remains above 0.95 at time \(\omega t = 10\) for all the parameters satisfying \(1 < \tau/\Delta \lesssim 3\). However, in the parameter region characterized by \(\tau \gg 3\Delta\), the effective control becomes impossible. It turns out that what matters for dynamical control is the ratio \(\tau/\Delta\) rather than \(\Delta\) or \(\tau\) alone.

For the two-qubit model, our initial state is \(|\psi_0\rangle = |s\rangle\).

Without control, the state will lose its entanglement in time. Fig. 4 plots \(\tau - \Delta\) phase diagram of concurrence at the time point \(\omega t = 10\). The results clearly show there exists a threshold that dictates where the efficient entanglement control is possible \((C \geq 0.95)\). More specifically, the entanglement can be protected in the region of \(1 < \tau/\Delta \lesssim 8/3\), we see again that the idealized-pulse approximation is only a small portion of the large permissible parameter region.

Fig. 5 shows the control condition of dynamics. In Fig. (a) and (b), we fix \(\Delta\) and use different \(\tau\) values to show the effect of \(\tau\); while in Fig. (c) and (d), we show the effect of \(\Delta\) by fixing \(\tau\). For a long time scale \(\omega t \leq 30\), each value of \(\Delta\) constrains the value of \(\tau\) to yielding a good entanglement control \([\text{that is}\ C(t) \text{is close to } C(0)]\) and vice versa. We therefore conclude that as long as \(\tau/\Delta\) is not too large, a quantum state or its entanglement can be well protected. This is the only condition for dynamical control to work. In addition, we have checked the pulse strength \([\text{See Fig. } (1)]\). Interestingly, we see that the pulse strength does not play an essential role in quantum coherence control in contrast to the idealized pulse case where the pulse strength is assumed to be effectively infinite.

Note that Figs. (1)–(4) are plotted with \(\gamma = 0.5\), which corresponds to a strongly non-Markovian regime. Fig. 5 is used to show how the environmental memory times affect our quantum control scheme. We use \(T\) to denote the time after which the concurrence decays below 0.95. Fig. 5 is plotted in a logarithmic scale to give rise to a better view of the entanglement change against \(\gamma\). We emphasize that the non-Markovian bath is essential for the effective quantum control as shown in Fig. 6.

Conclusion—We have established a non-perturbative approach to dynamical control theory based on the Feshbach PQ-partitioning technique and the non-Markovian quantum trajectory. Surprisingly, we have found the threshold and parameter regions of non-ideal pulse sequence where dynamical decoupling can be used to protect the entanglement and fidelity of the subsystem. We show that the idealized pulse approximation merely occupies a tiny part of the large permissible parameter region that is able to suppress the noise. This suggests that the zeroth-order approximation may not be necessary and thus prompts a more flexible experimental implementation of the dynamical decoupling scheme. Additionally, we also testify to the environmental memory effect on the decoupling process.

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