Solving the additive eigenvalue problem associated to a
dynamics of a 2D-traffic system

Nadir Farhi*

INRIA - Paris - Rocquencourt
Domaine de Voluceau, 78153, Le Chesnay, Cedex France.

Abstract

This is a technical note where we solve the additive eigenvalue problem associated to a dynamics of a 2D-traffic system. The traffic modeling is not explained here. It is available in [2]. It consists of a microscopic road traffic model of two circular roads crossing on one junction managed with the priority-to-the-right rule. It is based on Petri nets and minplus algebra. One of our objectives in [2] was to derive the fundamental diagram of 2D-traffic, which is the relation between the density and the flow of vehicles. The dynamics of this system, derived from a Petri net design, is non monotone and additively homogeneous of degree 1. In this note, we solve the additive eigenvalue problem associated to this dynamics.

1 Introduction

In this note we solve the additive eigenvalue problem (or the time-independent system) associated to the dynamics of a basic 2D-traffic model considered in [2]. It is a system of two circular roads crossing on one junction managed with the priority-to-the-right rule. The model is based on Petri nets and on minplus algebra [1]. It is an extension to an existing 1D-traffic model [3, 4], which gives the average speed of vehicles on one circular road as an eigenvalue of a minplus matrix, and thus allows the derivation of the fundamental diagram of 1D-traffic (the relation between the density and the flow of vehicles on the road).

We give a solution of the eigenvalue problem. We show that the eigenvalue is not necessarily unique, but is given in terms of two main quantities which are interpreted in terms of traffic as the density $d$ of vehicles in the system, and a parameter $r$ giving the ratio of the non priority road size with respect to the size of the whole system. We give a condition on the parameter $r$ such that the eigenvalue is unique and positive for non-high densities. In this case the eigenvalue problem can be written as a dynamic programming equation of a stochastic optimal control problem.

We use the minplus algebra notations, mainly for reason of compactness but also to use some classical results of this algebra [1]. In addition, the following notations are also used: $a/b$ denotes $a - b$, $\sqrt{a}$ denotes $a/2$, and $b^a$ denotes $ab$.

*Current address: University of Texas at Dallas, 800 West Campbell Road, Richardson, TX 75080, USA. nadir.farhi@utdallas.edu
The traffic dynamics is the following (see [2]):

\[ x_q^{k+1} = a_{q-1}x_q^k \oplus \bar{a}_q x_{q+1}^k, \quad q \in \{2, \ldots, n-1, n+2, \ldots, n+m-1\}, \quad (DS) \]

\[ x_n^{k+1} = \bar{a}_n^1 x_k^k + x_{n+1}^k \oplus a_{n-1} x_{n-1}^k, \quad \]

\[ x_{n+m}^{k+1} = \bar{a}_n x_{n+m}^k \oplus a_{n+m-1} x_{n+m-1}^k, \]

\[ x_1^{k+1} = a_n + \sqrt{x_n^k x_{n+m}^k + \bar{a}_2 x_2^k}, \]

\[ x_{n+1}^{k+1} = a_n \sqrt{x_n^k x_{n+m}^k + \bar{a}_{n+1} x_{n+2}^k}, \]

with the (traffic) constraints:

\[
\begin{align*}
0 &\leq a_i \leq 1, & i &= 1, 2, \ldots, n+m, \\
\bar{a}_i &= 1/a_i, & i &\neq n, n+m, \\
\bar{a}_n &= \bar{a}_{n+m} = 1/(a_n a_{n+m}), \\
a_n a_{n+m} &\leq 1.
\end{align*}
\]

For example, in the usual algebra, the equation (2) is written:

\[ x_n^{k+1} = \min \left\{ \bar{a}_n + x_1^k + x_{n+1}^k - x_{n+m}^k, a_{n-1} + x_{n-1}^k \right\}, \]

when the equation (4) is written:

\[ x_1^{k+1} = \min \left\{ a_n + \frac{x_n^k + x_{n+m}^k}{2}, \bar{a}_1 + x_2^k \right\}. \]

This system of equations is implicit but it is triangular, so its trajectory is unique.

We denote by \( d \) the following quantity (which is interpreted in terms of traffic as the density of vehicles in the system):

\[ d = \frac{1}{n + m - 1} \sum_{i=1}^{n+m} a_i. \]

\[ (1) \]

\[ (2) \]

\[ (3) \]

\[ (4) \]

\[ (5) \]

\[ (6) \]

\[ (7) \]

2 Solving the additive eigenvalue problem

The additive eigenvalue problem corresponding to the dynamics (DS) is:

\[ \lambda x_i = a_{i-1} x_{i-1} + \bar{a}_i x_{i+1}, \quad i \in \{2, \ldots, n-1, n+2, \ldots, n+m-1\}, \quad (EV) \]

\[ \lambda x_n = \bar{a}_n x_{n+1}/(\lambda x_{n+m}) + a_{n-1} x_{n-1}, \]

\[ \lambda x_{n+m} = \bar{a}_n x_{n+1}/x_n + a_{n+m-1} x_{n+m-1}, \]

\[ \lambda x_1 = a_n \sqrt{x_n^k x_{n+m}^k + \bar{a}_2 x_2^k}, \]

\[ \lambda x_{n+1} = a_n \sqrt{x_n^k x_{n+m}^k + \bar{a}_{n+1} x_{n+2}^k}, \]

with the constraints (6) and the notation (7). Our aim in this note is to solve the system (EV).
Theorem 1. Solving the system \((EV)\) is equivalent to solving the following simplified system \((SS)\):

\[
x_i = (a_{i-1}/\lambda)x_{i-1} \oplus (\bar{a}_i/\lambda)x_{i+1},
\]

\[
i \in \{2, \ldots, n - 1, n + 2, \ldots, n + m - 1\},
\]

\[
x_n = (\bar{a}_n/\lambda^2)x_1x_{n+1}/x_{n+m} + (b_n/\lambda^{n-1})x_1,
\]

\[(SS): \quad x_{n+m} = (\bar{a}_{n+m}/\lambda)x_1x_{n+1}/x_n + (b_{m}/\lambda^{m-1})x_{n+1},
\]

\[
x_1 = (a_n/\lambda)\sqrt{x_nx_{n+m}} \oplus (\bar{b}_n/\lambda^{n-1})x_n,
\]

\[
x_{n+1} = (a_{n+m}/\lambda)\sqrt{x_nx_{n+m}} \oplus (\bar{b}_m/\lambda^{m-1})x_{n+m},
\]

where \(b_n = \bigotimes_{i=1}^{n-1} a_i\), \(\bar{b}_n = \bigotimes_{i=1}^{n-1} \bar{a}_i\), \(b_m = \bigotimes_{i=n+1}^{n+m-1} a_i\) and \(\bar{b}_m = \bigotimes_{i=n+1}^{n+m-1} \bar{a}_i\).

Proof. We proceed in two steps:

- First we show that if \((\lambda, x)\) is a solution of the system \((EV)\) then \(\lambda \leq 1/4\). Indeed, from the equations \((9), (11)\) and \((12)\), we have:

\[
\lambda x_n = \bar{a}_nx_1x_{n+1}/(\lambda x_{n+m}) \oplus a_{n-1}x_{n-1},
\]

\[
\lambda x_1 \leq a_n\sqrt{x_nx_{n+m}},
\]

\[
\lambda x_{n+1} \leq a_{n+m}\sqrt{x_nx_{n+m}}.
\]

Then by multiplying (standard adding) the terms of these inequalities, we obtain \(\lambda^4 \leq 1\), since \(a_{n+m}a_na_{n+m} = 1\).

- We see that if \(n = m = 2\) the systems \((EV)\) and \((SS)\) coincide. For \(n\) and \(m\) fixed, we denote by \(EV(n, m)\) and \(SS(n, m)\) the corresponding systems. By induction on \(n\) and \(m\), we suppose that \(EV(n, m) \Leftrightarrow SS(n, m)\), and we show that \(EV(n + 1, m) \Leftrightarrow SS(n + 1, m)\) and \(EV(n, m + 1) \Leftrightarrow SS(n, m + 1)\).

To show that \(EV(n + 1, m) \Leftrightarrow SS(n + 1, m)\), we eliminate the variable \(x_n\) in \(EV(n + 1, m)\) which gives a system \(EV(n, m)\), then we use the induction assumption.

Indeed the problem \(EV(n + 1, m)\) is written as follows:

\[
\lambda x_i = a_{i-1}x_{i-1} \oplus \bar{a}_ix_{i+1},
\]

\[
i \in \{2, \ldots, n, n + 3, \ldots, n + m\},
\]

\[
\lambda x_{n+1} = \bar{a}_{n+1}x_1x_{n+2}/(\lambda x_{n+1+m}) \oplus a_nx_n,
\]

\[
\lambda x_{n+1+m} = \bar{a}_{n+1+m}x_1x_{n+2}/x_{n+1} \oplus a_{n+m}x_{n+m},
\]

\[
\lambda x_1 = a_{n+1}\sqrt{x_{n+1}x_{n+1+m}} \oplus \bar{a}_1x_2,
\]

\[
\lambda x_{n+2} = a_{n+1+m}\sqrt{x_{n+1}x_{n+1+m}} \oplus \bar{a}_{n+2}x_{n+3},
\]

Using the expression of \(x_n\) in \((18)\), we replace it in the expression of \(x_{n+1}\) in \((19)\). We obtain:

\[
\lambda x_{n+1} = \bar{a}_{n+1}x_1x_{n+2}/(\lambda x_{n+1+m}) \oplus a_n[(a_{n-1}/\lambda)x_{n-1} \oplus (\bar{a}_n/\lambda)x_{n+1}],
\]
which gives:

\[ \lambda x_{n+1} = \bar{a}_{n+1}x_1x_{n+2}/(\lambda x_{n+1}+1) \oplus (a_n a_{n-1}/\lambda)x_{n-1}, \quad (23) \]

because \( \lambda x_{n+1} < (a_n a_n/\lambda)x_{n+1} \) since \( \lambda \leq 1/4 < 1/2 \) and \( a_n a_n = 1 \).

Using the expression of \( x_n \) in (18), we replace it in the expression of \( x_{n-1} \) in (18) also. We obtain:

\[ \lambda x_{n-1} = a_{n-2}x_{n-2} \oplus \bar{a}_{n-1}[(a_{n-1}/\lambda)x_{n-1} \oplus (\bar{a}_n/\lambda)x_{n+1}], \]

which gives:

\[ \lambda x_{n-1} = a_{n-2}x_{n-2} \oplus (\bar{a}_{n-1} a_n/\lambda)x_{n+1}, \quad (24) \]

because \( \lambda x_{n-1} < (\bar{a}_{n-1} a_{n-1}/\lambda)x_{n-1} \), since \( \lambda \leq 1/4 < 1/2 \) and \( \bar{a}_{n-1} a_{n-1} = 1 \).

Let us denote by \( I_1 \) and \( I_2 \) the following sets of indexes:

\[ I_1 = \{1 \leq i \leq n + 1 + m, \quad i \neq 1, n + 1, n + 2, n + 1 + m\} , \]

\[ I_2 = \{1 \leq i \leq n + 1 + m, \quad i \neq 1, n - 1, n + 1, n + 2, n + 1 + m\} . \]

Thus we can conclude the following equivalence:

\[ \{18\}_{i \in I_1} \Leftrightarrow \{18\}_{i \in I_2}, \quad \{18\}_{i = n} \]. \quad (25) \]

The equations \( \{18\}_{i \in I_2} \) combined with the equations \( \{21\}, \{23\}, \{20\}, \{21\} \) and \( \{22\} \) form the following \( EV(n, m) \) system of variables \( x_i \), \( 1 \leq i \leq n + 1 + m \) and \( i \neq n \):

\[ \lambda x_i = a_{i-1}x_{i-1} \oplus \bar{a}_i x_{i+1}, \quad (26) \]

\[ i \in \{2, \ldots , n - 2, n + 3, \ldots , n + m\} , \]

\[ \lambda x_{n-1} = a_{n-2}x_{n-2} \oplus (\bar{a}_{n-1} a_n/\lambda)x_{n+1}, \quad (27) \]

\[ \lambda x_{n+1} = a_{n+1}x_1x_{n+2}/(\lambda x_{n+1}+1) \oplus (a_n a_{n-1}/\lambda)x_{n-1}, \quad (28) \]

\[ \lambda x_{n+1+m} = \bar{a}_{n+1+m}x_1x_{n+2}/x_{n+1} \oplus a_{n+m}x_{n+m}, \quad (29) \]

\[ \lambda x_1 = a_{n+1}x_{n+1}x_{n+1+m} \oplus \bar{a}_1 x_2, \quad (30) \]

\[ \lambda x_{n+2} = a_{n+1+m}x_{n+1}x_{n+1+m} \oplus \bar{a}_{n+2} x_{n+3} . \quad (31) \]

By using the induction assumption, this later system is equivalent to the following system:

\[ x_i = (a_{i-1}/\lambda)x_{i-1} \oplus (\bar{a}_i/\lambda)x_{i+1}, \quad (32) \]

\[ i \in \{2, \ldots , n - 1, n + 3, \ldots , n + m\} , \]

\[ x_{n+1} = (\bar{a}_{n+1}/\lambda^2)x_1x_{n+2}/x_{n+1} \oplus (b_{n+1}/\lambda^n)x_1, \quad (33) \]

\[ x_{n+1+m} = (\bar{a}_{n+1+m}/\lambda)x_1x_{n+2}/x_{n+1} \oplus (b_m/\lambda^{m-1})x_{n+2} , \quad (34) \]

\[ x_1 = (a_{n+1}/\lambda)x_{n+1}x_{n+1+m} \oplus (\bar{b}_{n+1}/\lambda^n)x_{n+1} , \quad (35) \]

\[ x_{n+2} = (a_{n+1+m}/\lambda)x_{n+1}x_{n+1+m} \oplus (\bar{b}_m/\lambda^{m-1})x_{n+1+m} . \quad (36) \]

By adding the equation \( \{18\}_{i = n} \) to the later system, we obtain \( SS(n + 1, m) \).

- We show with the same manipulations that \( EV(n, m + 1) \Leftrightarrow SS(n, m + 1) \).
Lemma 1. (Baccelli et al. [1]) Given $A$ a $(m \times m)$ minplus matrix, if the weights of all the circuits of the graph $G(A)$ associated to $A$ are positive, then the equation $x = A \otimes x \oplus b$ admits a unique solution $x = A^* \otimes b$, where

$$A^* = \bigoplus_{n=0}^{\infty} A^n = \bigoplus_{n=0}^{m-1} A^n.$$ 

Corollary 1. Solving the system $(EV)$ is equivalent to solve the following system of four variables:

$$x_n = (\bar{a}_n/\lambda^2)x_1x_{n+1}/x_{n+m} \oplus (b_n/\lambda^{n-1})x_1, \quad (37)$$

$$(S) : \quad x_{n+m} = (\bar{a}_{n+m}/\lambda)x_1x_{n+1}/x_n \oplus (b_m/\lambda^{n-1})x_{n+1}, \quad (38)$$

$$x_1 = (a_n/\lambda)\sqrt{x_nx_{n+m}} \oplus (\bar{b}_n/\lambda^{n-1})x_n, \quad (39)$$

$$x_{n+1} = (a_{n+m}/\lambda)\sqrt{x_nx_{n+m}} \oplus (\bar{b}_m/\lambda^{n-1})x_{n+m}, \quad (40)$$

Proof. Taking into account the equivalence $(EV) \Leftrightarrow (SS)$, suppose that $\lambda$, $x_1$, $x_n$, $x_{n+1}$, $x_{n+m}$ are known. To determine the other variables i.e. $x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_{n+m-1}$ we have to solve the system of equation $(S)$. This system is written:

$$x = A \otimes x \oplus b, \quad (41)$$

where

$$x = \langle x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_{n+m-1} \rangle,$$

$$A = (ef\lambda) \begin{bmatrix} A_1 & \varepsilon \\ \varepsilon & A_2 \end{bmatrix},$$

with

$$A_1 = \begin{bmatrix} \varepsilon & \bar{a}_2 & \varepsilon & \cdots & \varepsilon \\ a_2 & \varepsilon & \bar{a}_3 & \cdots & \varepsilon \\ \varepsilon & a_3 & \varepsilon & \cdots & \cdots \\ \vdots & \varepsilon & \cdots & \cdots & \bar{a}_{n-1} \\ \varepsilon & \varepsilon & \cdots & a_{n-1} & \varepsilon \end{bmatrix},$$

and

$$A_2 = \begin{bmatrix} \varepsilon & \bar{a}_{n+1} & \varepsilon & \cdots & \varepsilon \\ a_{n+1} & \varepsilon & \bar{a}_{n+2} & \cdots & \varepsilon \\ \varepsilon & a_{n+2} & \varepsilon & \cdots & \cdots \\ \vdots & \varepsilon & \cdots & \cdots & \bar{a}_{n+m-1} \\ \varepsilon & \varepsilon & \cdots & a_{n+m-1} & \varepsilon \end{bmatrix},$$

and

$$b = \langle a_1x_1, e, \ldots, e, a_{n-1}x_n, a_{n+1}x_{n+1}, e, \ldots, e, \bar{a}_{n+m-1}x_{n+m} \rangle.$$ 

Using Lemma 1 all the circuits of the graph associated to the matrix $A$ have the average weight $1/\lambda^2$ which is positive because $\lambda \leq 1/4 < 1/2$. Thus the matrix $A^*$ exists and the solution of the system $(S)$ is given by: $x = A^* \otimes b$.
3 Solving the system \((S)\):

Let us use the notations:

\[ r = \frac{n}{n + m - 1}, \]
\[ \rho = \frac{1}{n + m - 1} = r/n. \]

**Theorem 2.** There exists a solution \((\lambda, x)\) of \(S\) such that \(\lambda\) satisfies:

\[ 0 = \max \left\{ \min \left\{ \frac{d - (1 + \rho)\lambda}{4} - \lambda, r - d - (2r - 1 + \rho)\lambda \right\}, -\lambda \right\}. \]

**Remark 1.** Before we give the proof of Theorem 2 let us explain it. Using the notations:

\[ d_1 = \frac{(n + m)/[4(n + m - 1)]}{(1 + \rho)(1/4)}, \]
\[ d_2 = \frac{(3n + m - 2)/[4(n + m - 1)]}{(2r + 1 - \rho)/4}, \]

the result can be explained as follows (see Figure 1):

- If \(0 \leq d \leq d_1\) then \((S)\) admits a solution \((\lambda, x)\) such that: \(\lambda = \frac{d}{1 + \rho}\),
- If \(d_1 \leq d \leq d_2\) the \((S)\) admits a solution \((\lambda, x)\) such that: \(\lambda = 1/4\),
- If \(d_2 < d \leq r\) or \(r \leq d < d_2\) which cases correspond respectively to \(r > 1/2\) or \(r < 1/2\) then \((S)\) admits a solution \((\lambda, x)\) such that: \(\lambda = \frac{(r - d)/(2r - 1 + \rho)}{,}\)
- If \(r \leq d \leq 1\) then \((S)\) admits a solution \((\lambda, x)\) such that: \(\lambda = 0\).

**Proof.**

- If \(0 \leq d \leq d_1\), then a solution \((\lambda, x)\) is given by:

\[
\lambda = \frac{d}{1 + \rho} = \frac{n + m - 1}{n + m} d,
\]

\[
\begin{bmatrix}
  x_n \\
  x_{n+m} \\
  x_1 \\
  x_{n+1}
\end{bmatrix} = \begin{bmatrix}
  b_n/\lambda^{n-1} \\
  \lambda^{n+1}/a_n^2/b_n \\
  e/a_{n+m}/a_n
\end{bmatrix},
\]

which is a solution of:

\[
\begin{cases}
  x_n = (b_n/\lambda^{n-1})x_1, \\
  x_{n+m} = (b_m/\lambda^{m-1})x_{n+1}, \\
  x_1 = (a_n/\lambda)\sqrt{x_nx_{n+m}}, \\
  x_{n+1} = (a_{n+m}/\lambda)\sqrt{x_nx_{n+m}}.
\end{cases}
\]

Indeed:

\[
[(\bar{a}_n/\lambda^2)x_1x_{n+1}/x_{n+m}]x_n = 1/\lambda^4, \quad \text{because} \quad \bar{a}_n = 1/(a_n a_{n+m}),
\]

\[
\geq e, \quad \text{because} \quad \lambda = \frac{n + m - 1}{n + m} d \leq \frac{n + m - 1}{n + m} 1/4 \leq 1/4.
\]
\[( \bar{a}_{n+m}/\lambda )x_1x_{n+1}/x_{n+m} = 1/\lambda^3, \text{ because } \bar{a}_{n+m} = 1/(a_n a_{n+m}), \]
\[\geq e, \text{ because } \lambda \leq 1/4 < 1/3.\]

\[( \bar{b}_n/\lambda^{n-1})x_n/x_1 = (1/\lambda^2)^n-1, \text{ because } \bar{b}_n = 1^{n-1}, \]
\[\geq e, \text{ because } \lambda \leq 1/4 < 1/2.\]

\[( \bar{b}_m/\lambda^{m-1})x_{n+m}/x_{n+1} = 1^{m-1}/(b_m b_n a_n a_{n+m}) \lambda^{n-m+2}, \text{ because } \bar{b}_m = 1^{m-1}/b_m, \]
\[= (1/\lambda^2)^m-1, \text{ because } b_m b_n a_n a_{n+m} = d^{n+m-1} = \lambda^{n+m}, \]
\[\geq e, \text{ because } \lambda \leq 1/4 < 1/2.\]

- If \( d_1 \leq d \leq d_2 \) then a solution \((\lambda, x)\) is given by:

\[
\lambda = 1/4
\]

\[
\begin{bmatrix}
x_n \\
x_{n+m} \\
x_1 \\
x_{n+1}
\end{bmatrix}
= \begin{bmatrix}
\lambda^{m-3}\bar{a}_n/b_m \\
b_m a_{n+m}/a_n/\lambda^{m-1} \\
e \\
a_{n+m}/a_n
\end{bmatrix},
\]

which is a solution of:

\[
\begin{cases}
x_n = (\bar{a}_n/\lambda^2)x_1x_{n+1}/x_{n+m}, \\
x_{n+m} = (b_m/\lambda^{m-1})x_{n+1}, \\
x_1 = (a_n/\lambda)\sqrt{x_n x_{n+m}}, \\
x_{n+1} = (a_{n+m}/\lambda)\sqrt{x_n x_{n+m}}.
\end{cases}
\]

Indeed:

\[(b_n/\lambda^{n-1})x_n = (a_n a_{n+m} b_n b_m)/1/\lambda^{n+m-4}, \text{ because } \bar{a}_n = 1/(a_n a_{n+m}), \]
\[= d^{n+m-1}/1/(1/4)^{n+m-4}, \]
\[\text{because } a_n a_{n+m} b_n b_m = d^{n+m-1}, \text{ and } \lambda = 1/4, \]
\[\geq \frac{n+m}{4} - 1 - \frac{n+m-4}{4} = e, \text{ because } d \geq d_1.\]

\[(\bar{a}_{n+m}/\lambda)x_1 x_{n+1}/x_{n+m} = \lambda = 1/4 \geq e.\]
\[
\left[ \left( \frac{b_n}{\lambda^{n-1}} \right) x_n \right] x_1 = 1^n \left( a_n a_{n+m} b_n b_m \right) / \lambda^{m-n-2},
\]

because \( b_n = 1^{n-1}/b_n \), and \( \bar{a}_n = 1/\left(a_n a_{n+m}\right) \),

\[
= 1^n / d^{n+m-1}/\lambda^{n-m+2},
\]

because \( a_n a_{n+m} b_n b_m = d^{n+m-1} \),

\[
\geq n - \frac{3n+m-2}{4} - \frac{n-m+2}{4} = e, \quad \text{because} \quad d \leq d_2.
\]

\[
\left[ \left( \frac{b_m}{\lambda^{m-1}} \right) x_{n+m} \right] / x_{n+1} = (1/\lambda^2)^{m-1}, \quad \text{because} \quad b_m b_m = 1^{m-1},
\]

\[
= \frac{m-1}{2} > e, \quad \text{because} \quad \lambda = 1/4.
\]

- If \( d_2 < d \leq r \) or \( r \leq d < d_2 \), then

\[
\lambda = \frac{r - d}{2r - 1 + \rho} = \frac{n}{n - m + 2} - \frac{n + m - 1}{n - m + 2} d
\]

\[
\begin{bmatrix}
  x_n \\
  x_{n+m} \\
  x_1 \\
  x_{n+1}
\end{bmatrix} =
\begin{bmatrix}
  \lambda^{n-1} / \bar{b}_n \\
  b_m^2 a_{n+m}^2 / \bar{b}_n / \lambda^{m-n+1} \\
  e \\
  b_m a_{n+m}^2 / \bar{b}_n / \lambda^{m-n+2}
\end{bmatrix},
\]

which is a solution of:

\[
\begin{align*}
  x_n &= (\bar{a}_n / \lambda^2) x_1 x_{n+1} / x_{n+m}, \\
  x_{n+m} &= (b_m / \lambda^{m-1}) x_{n+1}, \\
  x_1 &= (\bar{b}_n / \lambda^{n-1}) x_n, \\
  x_{n+1} &= (a_{n+m} / \lambda) \sqrt{x_n x_{n+m}},
\end{align*}
\]

Indeed, \( \lambda \) given by (42) satisfies \( 0 \leq \lambda \leq 1/4 \) because:

1. If \( d_2 \leq d \leq r \), which corresponds to \( n - m + 2 > 0 \), then \( n + m - 1 \over n - m + 2 \) \( > 0 \), and we can check that: \( d_2 \leq d \leq r \Rightarrow 0 \leq \lambda \leq 1/4 \).

2. If \( r \leq d \leq d_2 \), which corresponds to \( n - m + 2 < 0 \), then \( n + m - 1 \over n - m + 2 \) \( < 0 \), and we can check that: \( r \leq d \leq d_2 \Rightarrow 0 \leq \lambda \leq 1/4 \).

Then:

\[
\left[ \left( \frac{b_n}{\lambda^{n-1}} \right) x_1 \right] x_n = \left( 1/\lambda^2 \right)^{n-1}, \quad \text{because} \quad b_n \bar{b}_n = 1^{n-1},
\]

\[
\geq e, \quad \text{because} \quad \lambda \leq 1/4 < 1/2.
\]
\[ [(\bar{a}_{n+m}/\lambda)x_1x_{n+1}/x_n]/x_{n+m} = 1^n \lambda^{m-n-1}/a_n/a_{n+1}/b_m/b_n, \]

because \( \bar{a}_{n+m} = 1/a_n/a_{n+m} \) and \( \bar{b}_n = 1^{n-1}/b_n, \)

\[ = (1^n \lambda^{n-m+2}/d^{n+m-1})\lambda, \]

because \( a_n a_{n+m} b_m b_n = d^{n+m-1}, \)

\[ = \lambda \geq e, \quad \text{because} \quad \lambda^{n-m+2} = 1^n/d^{n+m-1}. \]

\[ [(a_n/\lambda)\sqrt{x_n x_{n+m}}]/x_1 = a_n a_{n+m} b_m b_n/1^{n-1}\lambda^{n-m-2}, \quad \text{because} \quad \bar{b}_n = 1^{n-1}/b_n, \]

\[ = (d^{m-1}/1^n \lambda^{n-m-2}) (1/\lambda^4), \quad \text{because} \quad a_n a_{n+m} b_m b_n = d^{n+m-1}, \]

\[ = 1/\lambda^4 \quad \text{because} \quad \lambda^{n-m+2} = 1^n/d^{n+m-1}, \]

\[ \geq e, \quad \text{because} \quad \lambda \leq 1/4. \]

\[ [(\bar{b}_m/\lambda^{m-1})x_{n+m}]/x_{n+1} = 1^{m-1}/\lambda^{2m-2}, \quad \text{because} \quad b_m \bar{b}_m = 1^{m-1}, \]

\[ = (1/\lambda^2)^{m-1}, \]

\[ \geq e, \quad \text{because} \quad \lambda \leq 1/4 < 1/2. \]

\bullet \quad \text{If} \quad r \leq d \leq 1, \quad \text{then} \quad \lambda = 0

\[
\begin{bmatrix}
  x_n \\
  x_{n+m} \\
  x_1 \\
  x_{n+1}
\end{bmatrix} =
\begin{bmatrix}
  e/\bar{b}_n \\
  1^{n+1}/a_n^2/b_n \\
  e \\
  1/a_{n+m}/a_n
\end{bmatrix},
\]

which is a solution of:

\[
\begin{align*}
  x_n &= (\bar{a}_n/\lambda^2)x_1x_{n+1}/x_{n+m}, \\
  x_{n+m} &= (\bar{a}_{n+m}/\lambda)x_1x_{n+1}/x_n, \\
  x_1 &= (\bar{b}_n/\lambda^{n-1})x_n, \\
  x_{n+1} &= (a_{n+m}/\lambda)\sqrt{x_n x_{n+m}},
\end{align*}
\]

Indeed:

\[ [(b_n/\lambda^{n-1})x_1]/x_n = 1^{n-1} \geq e, \quad \text{because} \quad b_n b_n = 1^{n-1}, \]

\[ [(b_m/\lambda^{m-1})x_{n+1}]/x_{n+m} = a_n a_{n+m} b_n b_m/1^n, \]

\[ = d^{m-1}/1^n, \quad \text{because} \quad a_n a_{n+m} b_n b_m = d^{n+m-1}, \]

\[ \geq e, \quad \text{because} \quad d \geq r. \]
Corollary 3. For large values of \( n \) and \( m \) such that \( n > m - 2 \) (which is the case \( r \geq 1/2 \)), a non negative eigenvalue \( \lambda \) of (S) is given by:

\[
\lambda = \max \left\{ \min \left\{ \frac{1}{1+\rho} \left( \frac{1}{2} \right), \frac{r-d}{2r-1} \right\}, 0 \right\}.
\]

Proof. follows directly from Theorem 2.

Corollary 2. In the case \( r \geq 1/2 \), a non negative eigenvalue \( \lambda \) of (S) is given by:

\[
\lambda = \max \left\{ \min \left\{ \frac{1}{1+\rho} d, \frac{r-d}{2r-1} \right\}, 0 \right\}.
\]

Proof. follows directly from Corollary 3.

Remark 2. We can check that as soon as we assume \( m > 1 \), we get \( d_1 < d_2 \), so we have: \( 0 < d_1 < d_2 < 1 \). The position of \( r \) with respect to \( d_1 \) and \( d_2 \) gives three cases and divides the interval \([0, 1]\), in each case in four regions, where the eigenvalues \( \lambda \) of (S) satisfy (see Figure 1):

A. \( d \in [0, \min(d_1, r)] \) \( \Rightarrow \lambda = \frac{1}{1+\rho} d, \)

B. \( d \in [\min(d_1, r), d_1[ \) \( \Rightarrow \lambda = \frac{1}{1+\rho} d, \) or \( \lambda = \frac{r-d}{2r-1+\rho}, \)

C. \( d \in [d_1, \min(d_2, r)] \) \( \Rightarrow \lambda = \frac{1}{1+\rho}, \)

D. \( d \in [\max(d_1, r), d_2[ \) \( \Rightarrow \lambda = \frac{r-d}{2r-1+\rho}, \) or \( \lambda = 0. \)

E. \( d \in [d_2, \max(d_2, r)] \) \( \Rightarrow \lambda = \frac{r-d}{2r-1+\rho}, \)

F. \( d \in [\max(d_2, r), 1] \) \( \Rightarrow \lambda = 0. \)

Lemma 2. If \( \lambda > 0 \) then the system (S) (solved on \( (\lambda, z) \)) is equivalent to the following system (solved on \( (\lambda, z) \)):

\[
(SZ) : \quad \begin{align*}
zs &= (\bar{a}_n/\lambda)r^{\lambda z}z_1 + (b_n/\lambda^{n+m-2})z_1, \\
zs+n+m &= b_m z_{n+1}, \\
zs_1 &= (a_n/\lambda)\sqrt{z_n z_{n+m}} + (\bar{b}_n/\lambda^{n-m})z_n, \\
zs_{n+1} &= (a_n+m/\lambda)\sqrt{z_n z_{n+m}} + (\bar{b}_m/\lambda^{2m-2})z_{n+m}.
\end{align*}
\]
Figure 1: The curve of $\lambda$ given in Theorem 2 depending on $d$.

**Proof.** From the equations (37) and (38) we obtain:

$$
\begin{cases}
  x_n x_{n+m} \leq (\bar{a}_n / \lambda^2) x_1 x_{n+1}, \\
  x_n x_{n+m} \leq (\bar{a}_{n+m} / \lambda) x_1 x_{n+1}.
\end{cases}
$$

Since $\lambda > 0$ we have: $x_n x_{n+m} < (\bar{a}_{n+m} / \lambda) x_1 x_{n+1}$. Thus:

$$
x_{n+m} = (b_m / \lambda^{m-1}) x_{n+1}.
$$

(47)

By replacing $x_{n+m}$ in (37), we obtain:

$$
x_n = [(\bar{a}_n / b_m \lambda^{m-3} \oplus b_n / \lambda^{n-1})] x_1.
$$

(48)

The system $(S)$ is then equivalent to the system $\{ (48), (47), (39), (40) \}$. On this later system we use the following changing variable:

\[
z_n = x_n, \\
z_{n+m} = x_{n+m} \lambda^{2m-2}, \\
z_1 = x_1 \lambda^{m-1}, \\
z_{n+1} = x_{n+1} \lambda^{m-1},
\]

and we obtain the system $(SZ)$

**Theorem 3.** If $r > 1/2$ (that is $n \geq m$), and for the densities $d$ satisfying $0 < d < r$, the system $(S)$, and thus the eigenvalue problem $(EV)$, admit a unique positive eigenvalue $\lambda$ given by:

$$
\lambda = \min \left\{ \frac{1}{1+\rho} d , \frac{1}{4} , \frac{r-d}{2r-1+\rho} \right\} > 0.
$$

(49)

This situation corresponds to the phases A, C and E of the case 3 on Figure 4 of Remark 3.

**Proof.**

- Let $\lambda$ be positive. Lemma 2 gives the equivalence of the systems $(S)$ and $(SZ)$. The later system
is the eigenvalue problem associated to the following dynamical system:

\[
\begin{align*}
    z_n^k &= (a_n f_m) z_n^{k-2} \oplus b_m z_n^{k-(n+m-2)}, \\
    z_{n+m}^k &= b_m z_{n+1}^k, \\
    z_1^k &= a_n \sqrt{z_n^{k-1} z_{n+1}^{k-1} \oplus b_m z_n^{k-(n-m)}}, \\
    z_{n+1}^{k+1} &= a_{n+m} \sqrt{z_n^{k-1} z_{n+1}^{k-1} \oplus b_m z_n^{k-(2m-2)}}.
\end{align*}
\]

If \( r > 1/2 \), that is \( n \geq m \), then this dynamical system is implicit but triangular. Indeed, an iteration of the dynamics is to compute \( z_n^k \) and \( z_{n+1}^k \) in parallel, then compute \( z_{n+m}^k \), and finally compute \( z_1^k \). So the system \((SZ)\) can be interpreted as a dynamic programming equation of a stochastic optimal control problem, where \( \lambda \) is the average optimal cost by unit of time. Since \( \lambda \) is supposed to be positive, and from Corollary 2, we obtain (49).

- Let \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) be two positive eigenvalues of \((S)\). Lemma 2 tells us that both of \( \lambda_1 \) and \( \lambda_2 \) are eigenvalues of the system \((SZ)\). Since in the case when \( r > 2 \), the system \((SZ)\) is a dynamic programming equation of a stochastic optimal control problem, which thus admits a unique eigenvalue, we conclude that \( \lambda_1 = \lambda_2 \). ■

4 Conclusion

The results of this note give a solution to the additive eigenvalue problem associated to a dynamics of an elementary 2D-traffic system (two circular roads crossing at one junction, managed by the priority-to-the-right-rule). The eigenvalue \( \lambda \), which is not necessarily unique, is given as a function of two main quantities which are interpreted in terms of traffic as the density \( d \) of vehicles in the system, and the ratio \( r \) between the non priority road size and the size of the whole system. Moreover, we showed that when \( r \) satisfies \( r > 1/2 \), that is when the size of the non priority road is bigger than the size of the priority road, the uniqueness of \( \lambda \), which is positive in this case, is proved for densities satisfying \( d < r \).

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat : Synchronization and Linearity, Wiley, 1992.

[2] N. Farhi, Modélisation Minplus et Commande du Trafic de Villes Régulières, PhD Thesis Paris 1 University, 2008.

[3] N. Farhi, M. Goursat, J.-P. Quadrat : Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and Petri net modeling, in Proceedings of the 44th IEEE - CDC, Sevilla, 2005.

[4] P. Lotito, E. Mancinelli and J.P. Quadrat A Minplus Derivation of the Fundamental Car-Traffic Law, IEEE Transactions on Automatic Control V.50, N.5, p.699-705 May 2005.