SUBREGULAR $J$-RINGS OF COXETER SYSTEMS VIA QUIVER PATH ALGEBRAS

IVAN DIMITROV, CHARLES PAQUETTE, DAVID WEHLAU, AND TIANYUAN XU

Abstract. We study the subregular $J$-ring $J_C$ of a Coxeter system $(W, S)$, a subring of Lusztig’s $J$-ring. We prove that $J_C$ is isomorphic to a quotient of the path algebra of the double quiver of $(W, S)$ by a suitable ideal that we associate to a family of Chebyshev polynomials. As applications, we use quiver representations to study the category $\text{mod-}A_K$ of finite dimensional right modules of the algebra $A_K = K \otimes J_C$ over an algebraically closed field $K$ of characteristic zero. Our results include classifications of Coxeter systems for which $\text{mod-}A_K$ is semisimple, has finitely many simple modules up to isomorphism, or has a bound on the dimensions of simple modules. Incidentally, we show that every group algebra of a free product of finite cyclic groups is Morita equivalent to the algebra $A_K$ for a suitable Coxeter system; this allows us to specialize the classifications to the module categories of such group algebras.

1. Introduction

We study a subring of the $J$-ring of an arbitrary Coxeter system $(W, S)$. The $J$-ring was first introduced by Lusztig in [Lus87] in the case where $W$ is a Weyl or affine Weyl group to help study the Kazhdan–Lusztig cells in $W$. Later, Lusztig showed in [Lus14b] that the same construction of the $J$-ring is valid for arbitrary Coxeter systems, at least in the so-called “equal-parameter” case. In the “unequal-parameter” case, the validity of the construction relies on what has come to be known as Lusztig’s conjectures P1-P15; see [Bon17, Section 14.2]. We only deal with the equal-parameter case in this paper.

By definition, the $J$-ring equals the free abelian group $J = \oplus_{w \in W} \mathbb{Z} t_w$ as a group, and products in $J$ are given by the formula

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z}^{-1} t_z$$

where each coefficient $\gamma_{x,y,z}^{-1}$ is a certain nonnegative integer obtained via the Kazhdan–Lusztig basis of the Hecke algebra of $(W, S)$. The formula endows $J$ with the structure of an associative (but not necessarily unital)
Moreover, for each \( \text{two-sided Kazhdan–Lusztig cell} \) \( E \) of \( W \), the subgroup \( J_E := \bigoplus_{w \in E} \mathbb{Z} t_w \) of \( J \) is a subring of \( J \). In this paper, we focus on the ring \( J_C \) where \( C \) is a particular two-sided cell of \( W \) called the \textit{subregular cell}. This cell consists of all the non-identity elements in \( W \) that are \textit{rigid} in the sense that they each have a unique reduced expression; see [Lus83] and [Xu19]. We call \( J_C \) the \textit{subregular} \( J \)-ring and study the structure and representations of \( J_C \).

Also called the \textit{asymptotic Hecke algebra}, the \( J \)-ring may be viewed as a limit of the Hecke algebra of \( W \) in the sense of [Lus95]. As such, the \( J \)-ring has been an important tool for studying Hecke algebras and reductive groups; see, for example, [Lus89], [Gec98], [Gec07] and [Lus18]. Besides its applications, the structure of the \( J \)-ring itself has also been studied extensively. Notable results include the following: Bezrukavnikov, Finkelberg, and Ostrik studied a categorical version of the \( J \)-ring in [BFO09] and used it to compute explicitly the structure of the ring \( J_E \) for each two-sided cell \( E \) in \( W \); Braverman and Kazhdan showed in [BK18] that \( J \) is isomorphic to a certain subalgebra of the Harish-Chandra Schwartz algebra of a reductive group; by using a generalization of the Robinson–Schensted algorithm called the \textit{affine matrix-ball construction}, Kim and Pylyavskyy gave a canonical presentation for the \( J \)-ring in the special case where \( W \) is an (extended) affine symmetric group in [KP19], extending the work of Xi in [Xi02] for the same case.

It is worth noting that the results on the structure of the \( J \)-ring mentioned above are all restricted to Weyl or (extended) affine Weyl groups. On the other hand, the \( J \)-ring makes sense for an arbitrary Coxeter system, so it is natural to wonder what the structure of the \( J \)-ring is for more general Coxeter systems. Indeed, in Kazhdan–Lusztig theory it can often be interesting to study Coxeter systems in the full generality. One such indication comes from the proof of the famous “positivity conjecture” of Kazhdan and Lusztig, which states that all coefficients of so-called \textit{Kazhdan–Lusztig polynomials} are nonnegative integers. After its first appearance in [KL79] in 1979, the conjecture was proved along with other related deep results for Weyl and affine Weyl groups in the next two years by Kazhdan–Lusztig [KL80], Beilinson–Bernstein [BB81] and Brylinski–Kashiwara [BK81], via geometric methods involving local intersection cohomology of Schubert varieties, \( D \)-modules and perverse sheaves. The proof for the general case came much later: building upon the work of Soergel in [Soe90] [Soe92] [Soe07], Elias and Williamson proved the positivity conjecture for arbitrary Coxeter systems in [EW14] in 2014. In their work, they introduced a graphical calculus and a type of Hodge theory for the Soergel category, each of which is interesting in its own right; see [EW16] and [Wil18].

As was the case for the positivity conjecture, a disparity exists between what is known about the \( J \)-rings of Weyl or affine Weyl groups and the \( J \)-rings of other Coxeter systems. With the exception of Alvis’ work on the Coxeter group of type \( H_4 \) in [Alv08], the structures of the \( J \)-rings of
non-Weyl Coxeter groups remain largely unexplored. One obstacle to understanding $J$-rings in general is the difficulty in computing the structure constants of Hecke algebras with respect to their Kazhdan–Lusztig bases, which are necessary for obtaining the coefficients $\gamma_{x,y,z}^{-1}$ in Equation (1). As we will show, however, it is possible to circumvent this obstacle if we restrict from the $J$-ring to the subregular $J$-ring. In [Xu19], the last named author gave a description of products of the form $t_xt_y$ in $J_C$ that does not involve Kazhdan–Lusztig theory. In the present paper, we use this description to show that $J_C$ is isomorphic to certain quotients of the path algebra of a quiver, then use quiver representations to study representations of $J_C$. Roughly speaking, the reason why we can understand $J_C$ in full generality, in contrast to the entire $J$-ring, is that the rigidity of the elements of $C$ makes $C$ and $J_C$ more amenable to combinatorial analysis. It seems interesting that a similar contrast is also visible in the book [Bon17] by Bonnafé, where he singles out the subregular cell in Chapters 12 and 13 (he calls the cell the submaximal cell) and exploits its rich combinatorics in his investigation of various Kazhdan–Lusztig objects attached to the cell, including the so-called cell module of $C$ and its connection to the reflection representation of the Hecke algebra of $(W, S)$.

Let us elaborate on how $J_C$ relates to quivers. Recall that every Coxeter system $(W, S)$ corresponds to a unique Coxeter diagram $G$ with vertex set $S$ and edge set $\{a-b : a, b \in S, m(a, b) \geq 3\}$. We define the double quiver of $(W, S)$ to be the directed graph $Q = (Q_0, Q_1)$ with vertex set $Q_0 = S$ and edge set $Q_1 = \{a \rightarrow b : a, b \in S, m(a, b) \geq 3\}$, where we have a pair of arrows $a \rightarrow b$ and $b \rightarrow a$ arising from each edge $a \rightarrow b$ in $G$. Next, we consider the path algebra $\mathbb{Z}Q$ of $Q$ over $\mathbb{Z}$ and associate to each suitable family $\{f_n : n \in \mathbb{Z}_{\geq 2}\}$ of polynomials an ideal $I^Z_f$ in $\mathbb{Z}Q$ called an evaluation ideal of $\{f_n : n \in \mathbb{Z}_{\geq 2}\}$ (Definition 3.4). Our first main result, Theorem 3.6, establishes an algebra isomorphism between $J_C$ and the quotient $\mathbb{Z}Q/I^Z_u$ where $I^Z_u$ is the evaluation ideal of a family $\{u_n \in \mathbb{Z}[x] : n \geq 2\}$ of “Chebyshev polynomials”.

Fixing an algebraically closed field $K$ of characteristic zero, we extend the result that $J_C \cong \mathbb{Z}Q/I^Z_u$ in two ways (see Remark 4.17 for a discussion about assumptions on the field $K$). First, in Theorem 4.7, we show that upon an extension of scalars we may alter the family $\{u_n\}$ without changing the isomorphism type of the quotient of the path algebra by the evaluation ideal. More precisely, we show that for any two uniform families of polynomials $\{f_n\}, \{g_n\}$ over $K$ (Definition 3.3), we have $KQ/I_f \cong KQ/I_g$ where $KQ$ is the path algebra of $Q$ and $I_f, I_g$ are evaluation ideals of $KQ$ constructed from $\{f_n\}, \{g_n\}$. The Chebyshev polynomials $\{u_n\}$ form a uniform family, and the result enables us to realize the algebra $A_K := K \otimes_\mathbb{Z} J_C$ as a quotient $KQ/I_f$ where the ideal $I_f$ is generated by elements which can take very simple forms; see § 4.2. Together, Theorems 3.6 and 4.7 generalize Example 6.10 of Diaz-Lopez’s paper [DL15]. That paper cites the example as its main motivation and remarks that the example suggests a stronger connection
between path algebras and asymptotic Hecke algebras. We hope our result can be viewed as further support for such a connection.

In our second extension of Theorem 3.6, we develop a procedure to modify a quiver $Q$ to a new quiver $\bar{Q}$ such that the algebras $KQ/\mathcal{I}_f$ and $K\bar{Q}/\bar{\mathcal{I}}_f$ are Morita equivalent for any uniform family of polynomials $\{f_n\}$, where $\mathcal{I}_f$ is the evaluation ideal of $KQ$ associated to $\{f_n\}$; see Theorem 4.5. We call the procedure a quiver contraction and will often apply it iteratively, starting from the double quiver of a Coxeter diagram. Quiver contractions reveal certain interesting algebras that are Morita equivalent to algebras $A_K$ associated to Coxeter systems, such as the Laurent polynomial ring $K[t, t^{-1}]$ or group algebras of free products of finite cyclic groups; see Examples 4.10 and 4.12. In addition, we use quiver contractions to justify certain assumptions on Coxeter systems in the study of representations of $A_K$. For example, for any Coxeter system whose Coxeter diagram $G$ is a tree, quiver contractions allow us to assume that $G$ contains no simple edges when studying representations of $A_K$; see Example 4.8.

Theorem 3.6 and its extensions allow us to study representations of the subregular $J$-ring via quivers. More precisely, we use the double quiver $Q$ to study the category $\text{mod-}A_K$ of finite dimensional right modules of the algebra $A_K$. Representations of the $J$-ring and of rings of the form $J_E$ (where $E$ is a two-sided cell) are not only interesting on their own but also intimately related to representations of $W$ and its Hecke algebra; see [Lus14b], [Lus14a], [Lus18], [Gec07] and [Pie10]. On the other hand, quivers arise naturally in many areas of mathematics and have close connections to the representation theory of finite dimensional algebras, Kac–Moody algebras, quantum groups, and so on; see [Sav05] and [Sch14].

To study $\text{mod-}A_K$ via $Q$, we use the well-known fact that for each ideal $\mathcal{I}$ in $KQ$, the category of modules of the quotient $KQ/\mathcal{I}$ is equivalent to the category of representations of $Q$ that satisfy the relations in $\mathcal{I}$ (see § 2.4). Our main results are Theorems 5.1 and 5.2 which characterize in terms of the Coxeter diagram $G$ when the category $\text{mod-}A_K$ is semisimple, contains finite many simple modules, or has a bound on the dimensions of simple modules. In a sense, the characterizations are similar to those of the representation types of quivers given by the celebrated Gabriel’s Theorem (see [DDPW08]). Since we can use quiver contractions to show that every group algebra of a free product of finite cyclic groups is Morita equivalent to the algebra $A_K$ for a suitable Coxeter system (Example 4.12), Theorems 5.1 and 5.2 lead to similar characterizations for the module categories of such group algebras, which may be of independent interest as they are stated without mention of Coxeter systems or Kazhdan–Lusztig theory; see Remark 5.3 and Proposition 5.4.

The rest of the paper is organized as follows. In Section 2 we recall the relevant background on Coxeter systems, subregular $J$-rings, path algebras, and quiver representations. In Section 3 we define uniform families of polynomials $\{f_n\}$ and their associated evaluation ideals, then we realize $J_C$ and
the algebra $A_K$ as quotients of path algebras by suitable evaluation ideals via Theorems 3.6 and 3.7. Section 4 deals with quiver contractions and its main result is Theorem 4.5, which asserts that $KQ/I_f$ is Morita equivalent to $K\bar{Q}/\bar{I}_f$ if the quiver $\bar{Q}$ is obtained from $Q$ via a sequence of contractions. We define contractions in §4.1, give detailed examples of contractions in §4.2 and prove Theorem 4.5 in §4.3, then we analyze and give examples of representations of contracted quivers in §4.4. Finally, we state and prove the results on mod-$A_K$ in Section 5. Most of the examples from §4.2 and §4.4 will be used in the proofs.

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2. Background

2.1. Coxeter Systems. A Coxeter system is a pair $(W,S)$ where $S$ is a finite set and $W$ is the group given by the presentation

$$W = \langle S \mid (ab)^{m(a,b)} = 1 \text{ for all } a, b \in S \text{ with } m(a,b) < \infty \rangle,$$

where $m$ denotes a map $m : S \times S \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$ such that for all $a, b \in S$, we have $m(a,b) = m(b,a)$, and $m(a,b) = 1$ if and only if $a = b$. These conditions imply that $a^2 = 1$ for all $a \in S$ and that

$$(2) \quad aba \cdots = bab \cdots,$$

where both sides contain $m(a,b)$ factors, for every two distinct generators $a, b \in S$ with $m(a,b) < \infty$. We call each side of Equation (2) an $\{a,b\}$-braid and call the equation a braid relation.

Each Coxeter system $(W,S)$ can be encoded via its Coxeter diagram, the weighted, undirected graph $G$ whose vertex set is $S$, whose edge set is $\{\{a,b\} : m(a,b) \geq 3\}$, and where the weight of an edge $\{a,b\}$ is $m(a,b)$. An edge with weight $m$ in $G$ is simple if $m = 3$ and is heavy otherwise. When drawing $G$, we label each edge with its weight except for simple edges. A Coxeter system $(W,S)$ is said to be irreducible if its Coxeter diagram $G$ is connected and reducible otherwise.

For the rest of the paper, we let $(W,S)$ be an irreducible Coxeter system and let $G$ be its Coxeter diagram. The irreducibility assumption is made to simplify our statements, as the reducible case can be easily derived from the irreducible case for all the relevant results; see Remark 3.8.

2.2. The Subregular J-ring. Let $S^*$ be the free monoid generated by $S$. For each element $w \in W$, the words in $S^*$ that express $w$ and have minimal length are called the reduced words of $w$. The common length of these words, denoted $l(w)$, is called the length of $w$. By the well-known Matsumoto–Tits
theorem, every two reduced words of $w$ can be obtained from each other via a finite sequence of braid relations.

An element in $W$ is called rigid if it has a unique reduced word. In this paper we are particularly interested in the set

$$C = \{ w \in W : w \neq 1, w \text{ is rigid} \}.$$ 

The set $C$ is known to be a two-sided Kazhdan–Lusztig cell of $W$, and is called the subregular cell or submaximal cell of $W$ (see [Xu19] and [Bon17, Chapter 12]).

**Remark 2.1.** (a) By the Matsumoto–Tits theorem, a word $w \in S^*$ expresses an element in $C$ if and only if $w$ is nonempty and does not contain as a contiguous subword a word of the form $aa$ for any $a \in S$ or an $\{a,b\}$-braid for any distinct elements $a,b \in S$.

(b) Henceforth we will identify each element $w \in C$ with its unique reduced word. In particular, we will also use $w$ to denote the reduced word of the element (as in Propositions 2.2 and 3.9, for example).

To define the subregular $J$-ring, we first recall the construction of the $J$-ring, or the asymptotic Hecke algebra, of $(W,S)$. The construction is due to Lusztig, who defined the $J$-ring as the free abelian group $J := \oplus_{w \in W} \mathbb{Z}t_w$ and defined multiplication in $J$ by the formula

$$txty = \sum_{z \in W} \gamma_{x,y,z-1}tz,$$

where each coefficient $\gamma_{x,y,z-1}$ is a certain nonnegative integer extracted from the structure constants for the Kazhdan–Lusztig basis of the Iwahori–Hecke algebra of $(W,S)$; see [Lus87] and [Lus14b, Section 18.3]. Lusztig showed that for each two-sided cell $E$ of $W$, the subgroup $J_E := \oplus_{w \in E} \mathbb{Z}t_w$ is in fact a subring of $J$. We define the subregular $J$-ring to be the subring $J_C$ of $J$ arising from the subregular cell $C$ of $W$.

While the definition of $J$ relies heavily on Kazhdan–Lusztig theory, it is shown in [Xu19] that we can describe products in the subregular $J$-ring via simple manipulations of reduced words. To do so, for each pair of distinct generators $a,b \in S$, let us call an element $w \in C$ an $\{a,b\}$-element if $w$ lies in the subgroup of $W$ generated by $a$ and $b$. For two words $x = \ldots a_2a_1, y = b_1b_2 \ldots \in S^*$ with $a_1 = b_1$, let $x * y$ be the word $\ldots a_2b_1b_2 \ldots$, the result of concatenating $x$ and $y$ and deleting one duplicate copy of the letter $a_1 = b_1$. Then products in $J_C$ behave as follows:

**Proposition 2.2 ([Xu19] Corollary 4.2, Propositions 4.4 & 4.5).** Let $x,y$ be elements of $C$ with reduced words $x = \ldots a_2a_1$ and $y = b_1b_2 \ldots$, where we take $a_2$ and $b_2$ to be nonexistent when $l(x) = 1$ and $l(y) = 1$, respectively. Then the following holds.

(a) If $a_1 \neq b_1$, then $txty = 0$.

(b) If $a_1 = b_1$ and $a_2 \neq b_2$ (including the vacuous cases where $a_2$ or $b_2$ do not exist), then $txty = tx*ty$. 
(c) If $a_1 = b_1$ and $x, y$ are both $\{a, b\}$-elements for some $a, b \in S$, then $t_xt_y$ is a linear combination of the form $\sum_{z \in Z} t_z$ where $Z$ is a certain set of $\{a, b\}$-elements.

Note that the first two parts of the proposition imply that $J_C$ has a unit, namely, the element $\sum_{a \in S} t_a$. In the last part, the set $Z$ can be obtained via a truncated Clebsch–Gordan rule, but the exact description of $Z$ is not essential to this paper, so we omit it. Instead, we describe below the product $t_xt_y$ from Proposition 2.2(c) in the special case where $l(x) = 2$. The special case is in fact equivalent to the general case because one can deduce the latter from the former by induction.

**Proposition 2.3 ([Xu19 Corollary 4.2]).** Let $a, b \in S$ and let $m = m(a, b)$. Suppose that $m \geq 3$. For all $1 < i < m$, let $w_{a,i}$ be the $\{a, b\}$ element $aba\ldots$ of length $i$ and let $t_{a,i} = t_{w_{a,i}}$, then define $w_{b,i} = bab\ldots$ and $t_{b,i}$ similarly. Then for all $1 < i < m$, we have

$$t_{ab} t_{b,i} = \begin{cases} t_{a,i-1} + t_{a,i+1} & \text{if } i < m - 1; \\
 t_{a,i-1} & \text{if } i = m - 1. \end{cases}$$

The following example illustrates how Proposition 2.2 can be used to compute the product $t_xt_y$ for all $x, y \in C$: Suppose $(W, S)$ is a Coxeter system where $S = \{a, b, c\}$ and $m(a, b) = 3, m(a, c) = 4, m(b, c) = 5$. Let $x = abcb, y = bcac$. Then $x, y \in C$ by Remark 2.1(a). The first two parts of Proposition 2.2 imply that $t_y t_x = 0$ and

$$t_xt_y = (t_{ab} t_{bcb})(t_{bcb} t_{cac}) = t_{ab} (t_{bcb} t_{bcb}) t_{cac}.$$  

The product $t_{bcb} t_{bcb}$ can be computed using Part (c) and turns out to equal $t_{bc}$. Applying Part (b) again completes the computation:

$$t_xt_y = t_{ab} t_{bcb} t_{cac} = t_{abcac}.$$  

Intuitively, as the example shows, the reductions allowed by the first two parts of Proposition 2.2 mean that the most interesting multiplication in $J_C$ happen “locally”, for elements within subgroups of $W$ generated by two elements. This fact is a key reason why Theorem 3.6 holds.

### 2.3 Path Algebras.

In this and the next subsection, we recall the background on quivers, path algebras and quiver representations that is relevant to the paper. Our main reference is Sch14.

A **quiver** is a directed graph $Q = (Q_0, Q_1)$ where $Q_0$ is the set of vertices and $Q_1$ is the set of directed edges, or **arrows**. The sets $Q_0$ and $Q_1$ will be finite for all quivers in this paper. For each arrow $\alpha : a \to b$, we call $a$ and $b$ the **source** and the **target** of $\alpha$ and denote them by source($\alpha$) and target($\alpha$), respectively. An arrow $\alpha$ is called a **loop** at $a$ if source($\alpha$) = target($\alpha$) = $a$.

A **path** on $Q$ is an element of the form $p = \alpha_1 \alpha_2 \ldots \alpha_n$ where target($\alpha_i$) = source($\alpha_{i+1}$) for all $1 \leq i \leq n - 1$; we define the **source** of $p$ to be source($p$) := source($\alpha_1$) and the **target** of $p$ to be target($p$) := target($\alpha_n$). To each vertex
a \in Q_0$, we associate a special path $e_a$ called the stationary path at $a$; we consider it as a path that “stays at $a$”, so in particular we have source($e_a$) = target($e_a$) = $a$. The length of the path $p$, denoted by length($p$), is defined to be the number of arrows it traverses. In other words, each arrow has length 1, each stationary path has length 0, and we have length($p$) = $\sum_{i=1}^n$ length($\alpha_i$) for each path $p = \alpha_1 \ldots \alpha_n$.

Let $\mathcal{P}$ be the set of all paths on $Q$, and let $R$ be a commutative ring. The path algebra of $Q$ over $R$, denoted by $RQ$, is the $R$-algebra with $\mathcal{P}$ as an $R$-basis and with multiplication induced by path concatenation: for paths $p = \alpha_1 \ldots \alpha_m, q = \beta_1 \ldots \beta_n \in \mathcal{P}$, we define $pq$ to be the path $\alpha_1 \ldots \alpha_m \beta_1 \ldots \beta_n$ if target($p$) = source($q$) and to be 0 otherwise. In particular, for any path $p$ with source $a$ and target $b$, we have $e_a p = p = pe_b$ in $RQ$. Consequently, $RQ$ contains the unit 1 = $\sum_{a \in Q_0}$ $e_a$, and we can describe $RQ$ as the algebra generated by the arrows and stationary paths in $Q$ subject only to the relations $e_a e_b = \delta_{a,b} e_a$ for all $a, b \in Q_0$ and $e_a \alpha = \alpha = \alpha e_b$ for each arrow $\alpha : a \to b$ in $Q_1$.

Among the elements of $RQ$, we will be especially interested in elements of the form $r = \sum_p c_p p \in RQ$ where the sum is taken over a finite set of paths on $Q$ which share the same source and the same target. Following [Sch14, Definition 3.1], we call such an element $r$ a uniform relation or simply a relation on $Q$. We define the source and target of $r$ to be the common source and common target of the paths appearing in it, respectively. Our first main theorem, Theorem 3.6, asserts that $J_C \cong ZQ/I_f$ for a suitable quiver $Q$ and a suitable ideal $I_f$ generated by a set of relations of the form $\mathcal{R} = \{r_u(\alpha) : \alpha \in Q_1\}$, where each relation corresponds to an arrow in $Q$.

2.4. Quiver Representations. Let $Q$ be a quiver and let $K$ be an arbitrary field. We recall below some basic facts about the representation theory of the path algebra $KQ$ and its quotients. All representations and modules we mention in this paper will be finite dimensional.

Let mod-$KQ$ be the category of finite dimensional right $KQ$-modules. It is well-known that mod-$KQ$ is naturally equivalent to the category $\text{rep}_K Q$ of finite dimensional representations of $Q$ over $K$. Here, a representation of a quiver $Q$ over $K$ is an assignment

$$M = (M_a, M_\alpha)_{a \in Q_0, \alpha \in Q_1}$$

of a $K$-vector space $M_a$ to each vertex $a$ of $Q$ and a linear map $M_\alpha : M_a \to M_b$ for each arrow $\alpha : a \to b$ in $Q$; the dimension of $M$ is defined by $\dim(M) := \sum_{a \in Q_0} \dim(M_a)$. A morphism $\varphi : M \to N$ between two representations $M, N$ of $Q$ consists of the data $\varphi = (\varphi_a)_{a \in Q_0}$ of linear maps $\varphi_a : M_a \to N_a$ for $a \in Q_0$ such that $\varphi_b \circ M_\alpha = N_\alpha \circ \varphi_a$ for every arrow $\alpha : a \to b$ in $Q$. The equivalence between the two categories can be established by two naturally defined quasi-inverse functors $\mathcal{F} : \text{mod}-KQ \to \text{rep}_K Q$ and $\mathcal{G} : \text{rep}_K Q \to \text{mod}-KQ$; see [Sch14, Chapter 5].
We can modify the equivalence between mod-$KQ$ and rep$_KQ$ to account for relations on $Q$. To do so, for each representation $M$ of $Q$, we set $M_a = \text{id}_{M_a}$ for all $a \in Q_0$ and associate to each path $p = \alpha_1 \ldots \alpha_n$ on $Q$ the map

$$M_p := M_{\alpha_1} \circ \cdots \circ M_{\alpha_n},$$

and we say that $M$ satisfies a relation $r = \sum c_pp$ if $\sum c_p M_p = 0$. For an ideal $\mathcal{I}$ of $KQ$ generated by a set of relations $\mathcal{R}$, define a representation of $Q$ to be a representation of $(Q, \mathcal{I})$ if it satisfies all relations in $\mathcal{R}$. Finally, let rep$_K(Q, \mathcal{I})$ be the full subcategory of rep$_KQ$ whose objects are the representations of $(Q, \mathcal{I})$. Then it is well-known that rep$_K(Q, \mathcal{I})$ is equivalent to mod-$KQ/\mathcal{I}$, the category of finite dimensional right modules of the quotient $KQ/\mathcal{I}$.

**Remark 2.4.** We introduce two types of shorthand notation to be used for the rest of the paper. First, for a category $C$, we introduce two types of shorthand notation to be used for $KQ/\mathcal{I}$ for relations on $Q$. First, for a category $C$, we introduce two types of shorthand notation to be used for $KQ/\mathcal{I}$ for relations on $Q$.

Familiar notions from mod-$KQ$ have obvious counterparts in rep$_KQ$: The zero representation in rep$_KQ$ is the representation $M$ with $M_a = 0$ for all $a \in Q_0$. A subrepresentation of a representation $M$ is an assignment $N = (N_a, N_0)_{a \in Q_0, a \in Q_1}$ such that for every arrow $\alpha : a \to b$ in $Q$, we have $N_a \subseteq M_a$, $M_0(N_a) \subseteq N_b$, and $N_0$ equals the restriction of $M_0$ to $N_0$. A representation is simple if it does not contain any proper, nonzero subrepresentation. The direct sum of two representations $M, N$ is the representation $M \oplus N$ where $(M \oplus N)_a = M_a \oplus N_a$ and $(M \oplus N)_{\alpha}(m, n) = (M_{\alpha}(m), N_{\alpha}(n))$ for every arrow $\alpha : a \to b$ and every element $(m, n) \in M_a \oplus N_a$. Finally, a representation is semisimple if it is a direct sum of simple representations, and each of rep$_KQ$ and rep$_K(Q, \mathcal{I})$ is semisimple if all representations in it are semisimple. These notions agree with their counterparts in mod-$KQ$ under the equivalences $\mathcal{F}$ and $\mathcal{G}$. For example, a representation $M \in \text{rep}_KQ$ is simple if and only if the module $G(M) \in \text{mod}-KQ$ is simple, and rep$_K(Q, \mathcal{I})$ is semisimple if and only if mod-$KQ/\mathcal{I}$ is semisimple. Indeed, the agreement of the notions can be attributed to the facts that mod-$KQ$ and rep$_KQ$ are abelian categories, that the definitions in rep$_KQ$ and mod-$KQ$ are specializations of the corresponding categorical notions, and that $\mathcal{F}, \mathcal{G}$ are equivalences of abelian categories.

### 3. Quiver Realizations

Henceforth, let $K$ be an algebraically closed field of characteristic zero, let $(W, S)$ be a Coxeter system, and let $G, C$ and $J_C$ be the Coxeter diagram, subregular cell and subregular $J$-ring of $(W, S)$, respectively. Let $A = A_K := K \otimes \mathbb{Z} J_C$. In this section, we associate a quiver $Q$ to $(W, S)$ and then show that $J_C \cong \mathbb{Z} Q/\mathcal{I}_z$ and $A \cong KQ/\mathcal{I}_f$ for suitable ideals $\mathcal{I}_z \subseteq \mathbb{Z} Q$ and $\mathcal{I}_f \subseteq KQ$. By §2.4, the latter isomorphism will allow us to study the category rep$_K A$ via the equivalent category rep$_K(Q, \mathcal{I}_f)$. 

SUBREGULAR J-RINGS VIA QUIVERS
3.1. **Statement of Results.** Let \( Q = (Q_0, Q_1) \) be the quiver with \( Q_0 = S \) and \( Q_1 = \{(a, b) : a, b \in S, m(a, b) \geq 3\} \). Each edge \( a - b \) in the Coxeter diagram \( G \) gives rise to a pair of arrows \( a \rightarrow b \) and \( b \rightarrow a \) in \( Q \), and all arrows of \( Q \) arise this way. For an arrow \( \alpha : a \rightarrow b \) in \( Q \), we call the arrow \( b \rightarrow a \) arising from the same edge in \( G \) the dual arrow of \( \alpha \) and denote it by \( \bar{\alpha} \); we define the weight of \( \alpha \) to be \( m(a, b) \) and denote it by \( m_{\alpha} \). We call the quiver \( Q \) the double quiver of \((W, S)\) or the double quiver of \( G \).

The ideal \( I_n^Z \) for which \( J_{C_1} \cong \mathbb{Z}Q/I_{n}^{Z} \) is generated by a set of (uniform) relations obtained via arrow evaluations of polynomials from suitable polynomial families. We first define arrow evaluations:

**Definition 3.1.** For each arrow \( \alpha \) in \( Q \), let \( \text{Eval}_\alpha : K[x] \rightarrow KQ \) be the unique \( K \)-linear map such that \( \text{Eval}_\alpha(1) = e_a \) where \( a = \text{source}(\alpha) \) and

\[
\text{Eval}_\alpha(x^n) = \alpha\bar{\alpha}\ldots,
\]

the product with \( n \) factors that start with \( \alpha \) and alternate in \( \alpha \) and \( \bar{\alpha} \), for all \( n > 0 \). For each polynomial \( f \in K[x] \), we write

\[
f(\alpha, \bar{\alpha}) := \text{Eval}_\alpha(f)
\]

and call \( f(\alpha, \bar{\alpha}) \) the \( \alpha \)-evaluation of \( f \).

By a “polynomial family” we mean a countable collection \( \{f_n : n \in \mathbb{Z}_{\geq 2}\} \) of polynomials in \( K[x] \). Note that for \( f(\alpha, \bar{\alpha}) \) to yield a uniform relation on \( Q \), the polynomial \( f \) needs to be either even or odd, therefore we will consider only polynomial families \( \{f_n\} \) where each \( f_n \) is either an even or an odd polynomial. To describe further conditions we would like to impose on \( \{f_n\} \), we need more notation:

**Definition 3.2.** For each even polynomial \( f = \sum c_i x^{2i} \in K[x] \), let

\[
\tilde{f} = \sum c_i x^i;
\]

for each odd polynomial \( f = \sum c_i x^{2i+1} \in K[x] \), let

\[
\tilde{f} = \sum c_i x^i.
\]

Note that when \( f \) is an even or odd polynomial of degree \( n \), the polynomial \( \tilde{f} \) has degree \( \lfloor n/2 \rfloor \) where \( \lfloor - \rfloor \) denotes the floor function; moreover, we have

\[
f(\alpha, \bar{\alpha}) = \begin{cases} 
\tilde{f}(\alpha\bar{\alpha}) & \text{if } f \text{ is even;} \\
\tilde{f}(\alpha\bar{\alpha}) \cdot \alpha = \alpha \cdot \tilde{f}(\bar{\alpha}\alpha) & \text{if } f \text{ is odd,}
\end{cases}
\]

where we evaluate a constant term \( c \) in \( \tilde{f} \) to \( ce_a \) for \( a = \text{source}(\alpha) \). For example, if \( f = x^3 - 2x \) then \( \tilde{f} = x - 2 \) and \( f(\alpha, \bar{\alpha}) = \alpha\bar{\alpha} - 2\alpha \), and if \( f = x^4 - 1 \) then \( \tilde{f} = x^2 - 1 \) and \( f(\alpha, \bar{\alpha}) = \alpha\bar{\alpha} - e_a \) where \( a = \text{source}(\alpha) \).

We are ready to define the polynomial families we need.

**Definition 3.3.** A uniform family of polynomials (over \( K \)) is a set

\[
\{f_n \in K[x] : n \in \mathbb{Z}_{\geq 2}\}
\]

such that for all \( n \in \mathbb{Z}_{\geq 2} \), we have
(a) $f_n$ has degree $n$, is even when $n$ is even, and is odd when $n$ is odd.
(b) zero is not a root of $\tilde{f}_n$, and no root of $\tilde{f}_n$ is repeated.

Given a uniform polynomial family, we assign one relation to each arrow and define an ideal $I_f$ of $KQ$ as follows:

**Definition 3.4.** Let $\{f_n : n \geq 2\}$ be a uniform family of polynomials.
(a) For each arrow $\alpha$ in $Q$, we set $m = m_\alpha$ and define

$$r_f(\alpha) = \begin{cases} 0 & \text{if } m = \infty; \\ f_{m-1}(\alpha, \bar{\alpha}) & \text{if } m < \infty. \end{cases}$$

(b) We define the *evaluation ideal* of $\{f_n\}$ to be the two-sided ideal

$$I_f := \langle r_f(\alpha) : \alpha \in Q_1 \rangle$$

of $KQ$ generated by the relations of the form $r_f(\alpha)$. More generally, if $f_n \in R[x]$ for all $n \geq 2$ for some subring $R$ of $K$, we define $I^R_f$ to be the two-sided ideal of $RQ$ given by

$$I^R_f := \langle r_f(\alpha) : \alpha \in Q_1 \rangle \subseteq RQ.$$

**Example 3.5.** Suppose $K = \mathbb{C}$, and consider the polynomials $u_n$ for $n \geq 0$ where

$$u_0 = 1, \quad u_1 = x, \quad \text{and} \quad u_n = xu_{n-1} - u_{n-2} \text{ for all } n \geq 2.$$  

These polynomials are normalizations of the *Chebyshev polynomials of the second kind*. It is easy to see by induction that for each $n \geq 2$, the polynomial $u_n$ has degree $n$, is even when $n$ is even, and is odd when $n$ is odd. Moreover, it is known that $u_n$ has $n$ distinct nonzero real roots $z_1, \ldots, z_n$ where

$$z_i = 2 \cos \left( \frac{i\pi}{n+1} \right)$$

for each $i$. The definition of the polynomial $\tilde{u}_n$ implies that $\tilde{u}_n$ has $\lfloor \frac{n}{2} \rfloor$ distinct nonzero roots, namely, the numbers $z_i^2$ where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. It follows that $\{u_n : n \geq 2\}$ forms a uniform family of polynomials over $\mathbb{C}$. Note that $u_n \in \mathbb{Z}[x]$ for all $n \geq 2$, so $I^\mathbb{Z}_u$ makes sense as an ideal of $\mathbb{Z}Q$.

We state our first two results below.

**Theorem 3.6.** Let $\{u_n : n \in \mathbb{Z}_{\geq 2}\}$ be as in Example 3.5. Then $J_C \cong \mathbb{Z}Q/I^\mathbb{Z}_u$ as unital rings.

**Theorem 3.7.** Let $K$ be an algebraically closed field of characteristic zero and let $\{f_n\}, \{g_n\}$ be two uniform families of polynomials over $K$. Then $KQ/I_f \cong KQ/I_g$ as $K$-algebras.

**Remark 3.8.** We can now explain why it suffices to deal with only irreducible Coxeter systems in this paper. Recall that if $(W,S)$ is reducible, then the connected components of its Coxeter diagram are the diagrams of Coxeter systems $(W_i, S_i)$ for $1 \leq i \leq k$ for some $k \geq 2$, and we have
\( S = \sqcup_i S_i, W = \Pi_i W_i \), where the symbols \( \sqcup \) and \( \Pi \) denote disjoint union and direct product, respectively. Now let \( C(i), Q(i) \) be the subregular cell and the double quiver of \((W_i, S_i)\) for each \( i \). Then \( C = \sqcup_i C(i) \) by definition and \( J_C = \Pi_i J_{C(i)} \) by Part (a) of Proposition 2.2. On the other hand, for any uniform polynomial family \( \{ f_n \} \) over \( K \) where \( f_n \in \mathbb{Z}[x] \) for all \( n \) (such as \( \{ u_n \} \)) it is easy to see that \( \mathbb{Z}Q/\mathcal{I}_f = \Pi_i \mathbb{Z}Q(i)/\mathcal{I}_{f(i)} \) and \( \mathbb{K}Q/\mathcal{I}_f \cong \Pi_i \mathbb{K}Q(i)/\mathcal{I}_{f(i)}, \) where for each \( i \) the ideals \( \mathcal{I}_{f(i)} \) and \( \mathcal{I}_f(i) \) are the evaluation ideals of \( \{ f_n \} \) in \( \mathbb{Z}Q(i) \) and \( \mathbb{K}Q(i), \) respectively. It follows that we can deduce Theorem 3.6 and Theorem 3.7 for reducible Coxeter systems from the irreducible cases by taking suitable direct products.

3.2. Proof of Theorem 3.6. In this section we prove Theorem 3.6 by constructing an explicit isomorphism \( \bar{\varphi} : \mathbb{Z}Q/\mathcal{I}_u \to J_c. \) To connect the two sides of the isomorphism, first observe that given any element \( w = s_1 s_2 \ldots s_k \in C, \) we must have \( m(s_i, s_{i+1}) \geq 3 \) for all \( 1 \leq i \leq k-1: \) otherwise we can exchange \( s_i \) and \( s_{i+1} \) to obtain another reduced word of \( w, \) contradicting the fact that \( w \) is rigid. It follows that the quiver \( Q \) contains an arrow \( \alpha_i : s_i \to s_{i+1} \) for all \( 1 \leq i \leq k \) as well as the path 
\[
p_w := \alpha_1 \alpha_2 \cdots \alpha_{k-1}.
\]
Recall the notation \( \mathcal{P} \) for the set of all paths on \( Q, \) and consider the map \( \iota : C \to \mathcal{P} \) which sends \( w \) to \( p_w \) for all \( w \in C. \) For each arrow \( \alpha : a \to b \) in \( Q \) with \( m_\alpha < \infty, \) let \( p_\alpha := \alpha \bar{\alpha} \alpha \cdots \) be the path of length \( m_\alpha - 1 \) obtained by concatenating \( \alpha \) and \( \bar{\alpha} \) repeatedly. Define a path \( p \in \mathcal{P} \) to be unbraided if it does not contain \( p_\alpha \) as a subpath, i.e., if we cannot write \( p = p_1 p_\alpha p_2 \) for some paths \( p_1, p_2 \in \mathcal{P}, \) for all \( \alpha \in Q_i \) with \( m_\alpha < \infty. \) Let \( \text{Unbr}(Q) \) be the set of unbraided paths in \( \mathcal{P}. \) Then by Remark 2.1(a), the image of \( \iota \) is exactly \( \text{Unbr}(Q). \) Since \( \iota \) is clearly injective, it gives a bijection from \( C \) to \( \text{Unbr}(Q). \) We will henceforth use \( \iota \) exclusively to denote this bijection. The definitions and notation of this paragraph are inspired by those from [Bon17, Chapter 12], where the bijection \( \pi : \text{Unbr}(Q) \to C \) is essentially the inverse of \( \iota. \)

Having connected \( C \) to \( \mathbb{Z}Q, \) let us next consider the effect of quotienting \( \mathbb{Z}Q \) by the ideal \( \mathcal{I}_u. \) Let \( \alpha \in Q_1 \) and let \( m = m_\alpha. \) Since the polynomial \( u_{m-1} \) has degree \( (m - 1), \) the relation \( r_f(\alpha) = r_{m-1}(\alpha, \bar{\alpha}) \in \mathcal{I}_u \) must be a linear combination of the alternating path \( q := \alpha \bar{\alpha} \alpha \cdots \) of length \( m - 1 \) and strictly shorter, unbraided paths in \( \mathbb{Z}Q. \) Since \( \alpha \) is arbitrary, it follows that modulo \( \mathcal{I}_u \) we can rewrite every path as a linear combination of unbraided paths. In other words, every element in the quotient \( \mathbb{Z}Q/\mathcal{I}_u \) can be represented in the form \( \sum_{p \in \text{Unbr}(Q)} c_p p \) where the coefficients \( c_p \in \mathbb{Z} \) are zero for all but finitely many paths.

The final tool we need concerns a natural filtration of \( J_C. \) For each \( i \in \mathbb{Z}_{\geq 0}, \) let \( C^{(i)} = \{ w \in C : l(w) \leq i + 1 \} \) and let \( J_C^{(i)} = \bigoplus_{w \in C^{(i)}} \mathbb{Z}l_w. \) As the example at the end of 2.2 illustrates, Propositions 2.2 and 2.3 imply that given elements \( x, y \in C \) with length \( l(x) = p + 1 \) and \( l(y) = q + 1 \) for
some \( p, q \geq 0 \), the product \( t_x t_y \) is always a linear combination of terms of the form \( t_z \) where \( l(z) \leq p + q + 1 \). It follows that the filtration

\[
0 \subseteq J_C^{(0)} \subseteq J_C^{(1)} \subseteq \ldots
\]
equips \( J_C \) with the structure of a filtered algebra. The same propositions also imply the following result.

**Proposition 3.9.** Let \( w = s_1 s_2 \ldots s_k \in C \). Then in \( J_C \), we have

\[
t_{s_1 s_2} t_{s_2 s_3} \ldots t_{s_{k-1} s_k} \in t_w + J_C^{(k-2)}
\]
where \( t_w + J_C^{(k-2)} = \{ t_w + z : z \in J_C^{(k-2)} \} \). In other words, the product is the sum of \( t_w \) and a linear combination of terms \( t_y \) for which \( l(y) < k \).

This proposition will be useful for proving that the map \( \bar{\phi} : \mathbb{Z}Q/\mathcal{I}_u \rightarrow J_C \) is an isomorphism, because we will examine several outputs of the map \( \bar{\phi} \) which have the form \( t_{s_1 s_2} t_{s_2 s_3} \ldots t_{s_{k-1} s_k} \). Rather than giving a formal proof of it, however, let us only sketch the main ideas needed with an example. The proposition follows from repeated application of Proposition 2.3 in the special case that \( w \) is an \( \{a, b\} \)-element for some \( a, b \in S \). The general case then reduces to the special case in the way illustrated by the following example: suppose \( w = abacbc \in C \) for some Coxeter system and let \( T = t_{aba} t_{acac} t_{cb} \). Then \( k = 7 \), and by the special case we have

\[
T = (t_{aba} t_{acac} t_{cb}) \in \left( t_{aba} + J_C^{(1)} \right) \left( t_{acac} + J_C^{(2)} \right) \left( t_{cb} + J_C^{(0)} \right)
\]
where the factors in parentheses correspond to the longest “dihedral” sub-words \( aba, acac, cb \) of \( w \). The filtration (7) implies that all terms \( t_w \) with \( l(w) = k \) which appear in \( T \) must come from the product \( t_{aba} t_{acac} t_{cb} \), where each factor is the “highest degree part” in a pair of parentheses. This product is nothing but \( t_w \) by Proposition 2.2(b), therefore

\[
T \in t_{aba} t_{acac} t_{cb} + J_C^{(5)} = t_w + J_C^{(k-2)},
\]
as desired.

We are ready to prove Theorem 3.6. Roughly speaking, the isomorphism holds for two main reasons: first, as we mentioned in §2.2, all interesting multiplications in \( J_C \) happen “locally” along individual edges of the Coxeter diagram, just as the relations generating \( \mathcal{I}_u^\mathbb{Z} \) are defined in the same fashion; second, via arrow evaluations, the recursion from Equation (3) which controls the local multiplication in \( J_c \) “agrees with” the recursive definition of \( \{u_n\} \) which controls the generators of \( \mathcal{I}_u^\mathbb{Z} \). We make these remarks more precise in the following proposition, where Theorem 3.6 appears as its last assertion.

**Proposition 3.10.** Let \((W, S)\) be an irreducible Coxeter system and let \( Q \) be its double quiver.
(a) There exists a unique algebra homomorphism \( \varphi : \mathbb{Z}Q \to J_C \) such that for every pair of dual arrows \( \alpha : a \to b \) and \( \beta : b \to a \) in \( Q \), we have

\[
\varphi(e_a) = t_a, \quad \varphi(e_b) = t_b, \quad \varphi(\alpha) = t_{ab}, \quad \varphi(\beta) = t_{ba}.
\]

Moreover, for all \( 1 \leq i \leq m := m(a,b) \), we have

\[
\varphi(u_{i-1}(\alpha, \beta)) = \begin{cases} 
  t_{a,i} & \text{if } i < m; \\
  0 & \text{if } i = m < \infty 
\end{cases}
\]

and similarly

\[
\varphi(u_{i-1}(\beta, \alpha)) = \begin{cases} 
  t_{b,i} & \text{if } i < m; \\
  0 & \text{if } i = m < \infty, 
\end{cases}
\]

where \( w_{a,i}, w_{b,i}, t_{a,i}, t_{b,i} \) are as in Proposition \([2.3]\).

(b) The map \( \varphi \) factors through the ideal \( I_u^Z \) and induces a homomorphism \( \bar{\varphi} : \mathbb{Z}Q/I_u^Z \to J_C \) given by \( \bar{\varphi}(p) = \varphi(p) \) for all \( p \in \text{Unbr}(Q) \).

(c) The map \( \bar{\varphi} \) is a unital algebra isomorphism, therefore \( J_C \cong \mathbb{Z}Q/I_u^Z \).

Proof. (a) Recall from Section \([2.3]\) that \( \mathbb{Z}Q \) is generated by the arrows and stationary paths of \( Q \) subject only to the relations \( e_u e_v = \delta_{u,v} e_u \) for all \( u, v \in Q_0 \) and \( e_a \alpha = \alpha = e_b \) for every arrow \( \alpha : a \to b \) in \( Q_1 \). On the other hand, in \( J_c \) we have \( t_u t_v = \delta_{u,v} t_u \) for all \( u, v \in Q_0 \) and \( t_a t_{ab} = t_{ab} = t_{ab} t_b \) for every arrow \( \alpha : a \to b \) in \( Q_1 \) by Proposition \([2.2]\). Thus, the relations satisfied by the generators of \( \mathbb{Z}Q \) are respected in the assignment \( e_a \mapsto t_a, \alpha \mapsto t_{ab} \) for all arrows \( \alpha : a \to b \) in \( Q_1 \). It follows that this assignment extends to a unique algebra homomorphism \( \varphi : \mathbb{Z}Q \to J_C \) which satisfies Equation \([8]\).

The homomorphism is unital since

\[
\varphi(1) = \varphi \left( \sum_{a \in Q_0} e_a \right) = \sum_{a \in Q_0} \varphi(e_a) = \sum_{a \in Q_0} t_a = 1.
\]

Note that for each \( w = s_1 s_2 \ldots s_k \in C \), Equation \([8]\) and the fact that \( \varphi \) is a homomorphism imply that \( \varphi(p_w) \) is exactly the element \( t_{s_1 s_2 s_3 \ldots t_{s_{k-1}} s_k} \).

It follows from Proposition \([3.3]\) that

\[
\varphi(p_w) \in t_w + J_C^{(l(w) - 2)}.
\]

To prove Equations \([9]\) and \([10]\), we induct on \( i \). For \( i \leq 2 \), the equations follow from the definition of \( \varphi \) and the fact that \( u_0 = 1, u_1 = x \). For \( i > 2 \), the recursion \( u_i = xu_{i-1} - u_{i-2} \) implies that

\[
u_i(\alpha, \beta) = \alpha u_{i-1}(\beta, \alpha) - u_{i-2}(\alpha, \beta),
\]

therefore

\[
\varphi(u_i(\alpha, \beta)) = \varphi(\alpha) \varphi(u_{i-1}(\beta, \alpha)) - \varphi(u_{i-2}(\alpha, \beta))
\]

\[
= t_{ab} t_{b,i} - t_{a,i-1}
\]

\[
= \begin{cases} 
  t_{a,i+1} & \text{if } i < m; \\
  0 & \text{if } i = m < \infty,
\end{cases}
\]
Lemma 3.11. This proves Equation (9); the proof of Equation (10) is similar.

(b) By construction, the ideal \( \mathcal{I}_u \) is generated by elements of the form \( r_u(\alpha) = u_{m_a-1}(\alpha, \alpha) \) where \( \alpha \) is an arrow in \( Q \) with finite weight. Such elements vanish via \( \varphi \) by Equation (9), therefore \( \varphi \) factors through \( \mathcal{I}_u^\mathbb{Z} \) and descends to the map \( \bar{\varphi} \) as claimed.

(c) The map \( \bar{\varphi} \) is unital since \( \varphi \) is unital. To show that \( \bar{\varphi} \) is surjective, we prove that \( t_w \in \text{im} \bar{\varphi} \) for all \( w \in C \) by induction on \( l(w) \). In the base case where \( l(w) = 1 \), we must have \( w = a \) for some \( a \in S \) and hence \( t_w = \bar{\varphi}(e_a) \in \text{im} \bar{\varphi} \). When \( l(w) > 1 \), we have

\[
\bar{\varphi}(p_w) = \varphi(p_w) \in t_w + J_C^{l(w)-2}
\]

by (11) and \( J_C^{l(w)-2} \subseteq \text{im} \bar{\varphi} \) by induction, therefore \( t_w \in \text{im} \bar{\varphi} \).

It remains to prove that \( \bar{\varphi} \) is injective. Let \( x = \sum_{p \in \text{Unbr}(Q)} c_p p \) be a nonzero element in \( \mathbb{Z}Q/\mathcal{I}_u^\mathbb{Z} \). We need to show that \( \bar{\varphi}(x) \neq 0 \). To do so, let \( k \) be the maximal length such that \( c_p \neq 0 \) for some unbraided path \( p \) of length \( k \), and let \( \{p_1, \ldots, p_n\} \) be the set of paths of length \( k \) appearing with nonzero coefficients in \( x \). Let \( w_i = t^{-1}(p_i) \) and write \( c_i := c_{p_i} \) for all \( 1 \leq i \leq n \). Then \( l(w_i) = k + 1 \) for all \( i \) and we have

\[
\bar{\varphi}(x) = \sum_{i=1}^n c_i \bar{\varphi}(p_i) \in \sum_{i=1}^n c_i t_{w_i} + J_C^{(k-1)}
\]

by (11). It follows that \( \bar{\varphi}(x) \neq 0 \), and the proof is complete.

3.3. Proof of Theorem 3.7: Dihedral Case. Let \( \{f_n\} \) be a uniform family of polynomials over \( K \). As the generators of the ideal \( \mathcal{I}_f \) correspond to individual pairs of dual arrows in \( Q \), we first prove Theorem 3.7 in the dihedral case, the case where \( |S| = 2 \). Let \( S = \{a, b\} \), let \( m = m(a, b) \geq 3 \), and denote the arrows \( a \rightarrow b \) and \( b \rightarrow a \) in \( Q \) by \( \alpha \) and \( \beta \), respectively. If \( m = \infty \), then \( \mathcal{I}_f = 0 \) and the theorem clearly holds, so until Corollary 3.14 we assume that \( m \) is finite. Under this assumption, we show that \( KQ/\mathcal{I}_f \) is semisimple and find its Artin–Wedderburn decomposition. We start with the category \( \text{rep}_K(Q, \mathcal{I}_f) \) in light of the equivalence between \( \text{mod}-KQ/\mathcal{I}_f \) and \( \text{rep}_K(Q, \mathcal{I}_f) \) (see §2.4). The simple modules of \( \text{rep}_K(Q, \mathcal{I}_f) \) turn out to have the following forms:

Lemma 3.11. (a) For each root \( \lambda \) of the polynomial \( \tilde{f}_{m-1} \), the assignment

\[
M(\lambda) := (M_a, M_b, M_{\alpha}, M_{\beta}) = (K, K, \text{id}, \lambda \cdot \text{id})
\]

defines a simple representation in \( \text{rep}(Q, \mathcal{I}_f) \). Moreover, if \( \lambda \) and \( \lambda' \) are distinct roots of \( \tilde{f}_{m-1} \), then \( M(\lambda) \not\cong M(\lambda') \).

(b) If \( m \) is even, then the assignments

\[
S(a) := (K, 0, 0, 0), \quad S(b) := (0, K, 0, 0)
\]

define two non-isomorphic simple representations in \( \text{rep}_K(Q, \mathcal{I}_f) \).
Proof. (a) Recall that \( \mathcal{I}_f \) is generated by the relations

\[
\begin{align*}
rf(\alpha) &= f_{m-1}(\alpha, \beta) = \begin{cases} 
\tilde{f}_{m-1}(\alpha \beta) & \text{if } m \text{ is odd;} \\
\tilde{f}_{m-1}(\alpha \beta)\alpha & \text{if } m \text{ is even,}
\end{cases} \\
r_f(\beta) &= f_{m-1}(\beta, \alpha) = \begin{cases} 
\tilde{f}_{m-1}(\beta \alpha) & \text{if } m \text{ is odd;} \\
\tilde{f}_{m-1}(\beta \alpha)\beta & \text{if } m \text{ is even.}
\end{cases}
\end{align*}
\]

The maps \( M_\alpha M_\beta \) and \( M_\beta M_\alpha \) both equal \( \lambda \cdot \text{id} \) as maps from \( K \) to \( K \). Since \( \lambda \) is a root of \( \tilde{f}_{m-1} \), it follows that \( \tilde{f}_{m-1}(M_\alpha M_\beta) = \tilde{f}_{m-1}(M_\beta M_\alpha) = 0 \), hence \( M(\lambda) \) satisfies the relations \( rf(\alpha) \) and \( rf(\beta) \) and forms a representation in \( \text{rep}(\mathcal{Q}, \mathcal{I}_f) \). Note that \( M(\lambda) \) is simple by basic linear algebra.

To check that \( M(\lambda) \not\cong M(\lambda') \) for distinct roots \( \lambda, \lambda' \), let \( M(\lambda') = (M'_a, M'_b, M'_a, M'_b) \). Then an isomorphism \( \phi : M_\alpha \to M_\mu \) must consist of two linear isomorphisms \( \phi_a : M_a \to M_\alpha, \phi_b : M_b \to M_b \) such that

\[
\phi_b M_a = M'_a \phi_a, \quad \phi_a M_b = M'_b \phi_b.
\]

The isomorphisms \( \phi_a, \phi_b \) must be multiplication by nonzero scalars \( x, y \), respectively, whence the above equations become \( y = x \) and \( \lambda y = \lambda' x \). This cannot happen, therefore \( M(\lambda) \not\cong M(\lambda') \).

(b) When \( m \) is even the assignments \( M_\alpha = M_\beta = 0 \) clearly satisfy the relations \( rf(\alpha) \) and \( rf(\beta) \), so \( S(a) \) and \( S(b) \) define representations in \( \text{rep}_K(\mathcal{Q}, \mathcal{I}_f) \). Moreover, the representations are simple and non-isomorphic by dimension considerations. \( \square \)

To prove \( \text{rep}_K(\mathcal{Q}, \mathcal{I}_f) \) is semisimple, we will use the following linear algebra facts to decompose every representation in \( \text{rep}_K(\mathcal{Q}, \mathcal{I}_f) \) into a direct sum of simple modules.

\begin{lemma}
Let \( h \in K[X] \) be a polynomial with degree \( k \geq 1 \) and with \( k \) distinct nonzero roots \( z_1, z_2, \ldots, z_k \) in \( K \). Let \( U \) and \( V \) be finite dimensional vector spaces, and let \( A : U \to V \) and \( B : V \to U \) be linear maps such that

\[
\begin{align*}
(14) \quad & h(BA) = 0_U \quad \text{and} \quad h(AB) = 0_V \\
(15) \quad & h(AB)A = 0_U \quad \text{and} \quad h(BA)B = 0_V.
\end{align*}
\]

Then the following results hold.

(a) Both \( AB \) and \( BA \) are diagonalizable; their eigenvalues lie in the set \( \{z_1, z_2, \ldots, z_k\} \) if \( 14 \) holds and in the set \( \{0, z_1, z_2, \ldots, z_k\} \) if \( 15 \) holds. In particular, we have eigenspace decompositions

\[
U = U_{z_1} \oplus U_{z_2} \oplus \ldots \oplus U_{z_k}, \quad V = V_{z_1} \oplus V_{z_2} \oplus \ldots \oplus V_{z_k}
\]

if \( 14 \) holds and

\[
U = U_0 \oplus U_{z_1} \oplus U_{z_2} \oplus \ldots \oplus U_{z_k}, \quad V = V_0 \oplus V_{z_1} \oplus V_{z_2} \oplus \ldots \oplus V_{z_k}
\]

if \( 15 \) holds.
\end{lemma}
if (15) holds, where $U_\lambda$ and $V_\lambda$ denotes the $\lambda$-eigenspace of $BA$ and $AB$ for each scalar $\lambda$, respectively.

(b) For all $1 \leq i \leq k$, the restrictions of $A$ to $U_{z_i}$ and of $z_i^{-1} \cdot B$ to $V_{z_i}$ form mutually inverse isomorphisms. When (15) holds, the restrictions of $A$ to $U_0$ and of $B$ to $V_0$ are both zero maps.

Proof. (a) The equations in (14) and in (15) imply that the minimal polynomials of both $AB$ and $BA$ divide $h$ and the polynomial $g := x \cdot h \in K[x]$, respectively. The result follows since the polynomials $h$ and $g$ have distinct roots in the sets $\{z_1, \ldots, z_k\}$ and $\{0, z_1, \ldots, z_k\}$, respectively.

(b) Let $1 \leq i \leq k$. Set $U_i = U_{z_i}, V_i = V_{z_i}$ and $B' = z_i^{-1} \cdot B$. Use $\mid$ to denote restriction of maps so that, for example, $A\mid_{U_i}$ stands for the restriction of $A$ to $U_i$. By direct computation, we have $Au \in V_i$ for all $u \in U_i$, $B'v \in U_i$ for all $v \in V_i$, and $B'A\mid_{U_i} = id_{U_i}, AB'\mid_{V_i} = id_{V_i}$. This proves the first claim. To prove the second claim, assume the equations in (15) hold and write $h = x \cdot \tilde{h} + c$ where $c$ is the constant term of $h$. Then $c \neq 0$ since $0$ is not a root of $h$, and we have

$$h(AB)A = A\tilde{h}(BA) = A(\tilde{h}BA) \cdot BA + c) = A\tilde{h}(BA) \circ BA + cA.$$  

Let $u \in U_0$. Then $BA(u) = 0$, therefore

$$0 = [h(AB)A](u) = [A(\tilde{h}BA)]BA(u) + cA(u) = cA(u).$$

where the first equality holds since $h(AB)A = 0_U$. It follows that $A(u) = 0$, so $A\mid_{U_0}$ is the zero map. The proof that $B\mid_{V_0}$ is the zero map is similar. \qed

Theorem 3.13. Let $(W, S)$ be an irreducible Coxeter system where $S = \{a, b\}$ and $3 \leq m := m(a, b) < \infty$.

(a) Suppose $m$ is odd. Then the category $\text{rep}_K(Q, I_f)$ is semisimple and has exactly $(m - 1)/2$ non-isomorphic simple representations, all of dimension 2. The algebra $KQ/I_f$ is semisimple, and is isomorphic to the direct product of $(m - 1)/2$ copies of the matrix algebra $M_{2 \times 2}(K)$.

(b) Suppose $m$ is even. Then the category $\text{rep}(Q, I_f)$ is semisimple and has exactly $(m - 2)/2 + 2$ non-isomorphic simple representations; two of these representations have dimension 1 and the other representations have dimension 2. The algebra $KQ/I_f$ is semisimple, and is isomorphic to the direct product of two copies of $K$ and $(m - 2)/2$ copies of $M_{2 \times 2}(K)$.

Proof. Let $M = (M_a, M_b, M_\alpha, M_\beta)$ be a representation in $\text{rep}_K(Q, I_f)$ where $\alpha$ and $\beta$ are the arrows $a \to b$ and $b \to a$ in $Q$, respectively. Set $h = \tilde{f}_{m-1}, U = M_a, V = M_b, A = M_\alpha$ and $B = M_\beta$. If $m$ is odd, then the equations in (14) hold by Equations (12) and (13). Using Lemma 3.12 we may then decompose $M$ into a direct sum where each summand is of the form $N(\lambda) := (U_\lambda, V_\lambda, A\mid_{U_\lambda}, B\mid_{V_\lambda})$ where $\lambda$ is one of the $(m - 1)/2$ roots of $h = \tilde{f}_{m-1}$ and $B\mid_{V_\lambda}A\mid_{U_\lambda} = \lambda \cdot id_{U_\lambda}, A\mid_{U_\lambda}B\mid_{V_\lambda} = \lambda \cdot id_{V_\lambda}$. It is easy to verify that $N(\lambda)$ is isomorphic to the representation $M(\lambda)$ from Lemma 3.11. The claims in Part (a) now follow from the Artin–Wedderburn theorem and the equivalence between $\text{rep}_K(Q, I_f)$ and mod-$KQ/I_f$. Similarly, if $m$ is even,
then the equations in \([15]\) hold and we may use Lemma \(3.12\) to decompose \(M\) into a direct sum of the representations

\[
\ker \alpha := (U_0, 0, 0, 0), \quad \ker \beta := (0, V_0, 0, 0)
\]

and simple representations isomorphic to \(M(\lambda)\) where \(\lambda\) is one of the \((m - 2)/2\) roots of \(f_m\). The representations \(\ker \alpha\) and \(\ker \beta\) further decompose into \(\dim(U_0)\) and \(\dim(V_0)\) copies of the modules \(S(a)\) and \(S(b)\) from Lemma \(3.11\) respectively. Part (b) follows.

We are ready to prove that in the case \(|S| = 2\), the isomorphism type of the algebra \(KQ/\mathcal{I}_f\) does not depend on the choice of the uniform family of polynomials \(\{f_n\}\). Note that we no longer assume that \(m\) is finite in the result below.

**Corollary 3.14.** Let \((W, S)\) be an irreducible Coxeter system where \(S = \{a, b\}\) and \(3 \leq m := m(a, b) \leq \infty\). Let \(\{f_n\}, \{g_n\}\) be two uniform families of polynomials over \(K\). Then there is an algebra isomorphism \(\Phi : KQ/\mathcal{I}_f \to KQ/\mathcal{I}_g\) such that \(\Phi(e_i) = e_i\) for \(i \in \{a, b\}\).

**Proof.** When \(m = m(a, b) = \infty\), we have \(\mathcal{I}_f = \mathcal{I}_g = 0\), so we may take \(\phi\) to be the identity map. Now assume \(m\) is finite. We first treat the case where \(m\) is even. Consider the direct product \(B = B_1 \times B_2 \times B_3 \times \cdots \times B_r\) where \(r = (m - 2)/2 + 2\), \(B_1 = B_2 = K\), and \(B_i\) equals the matrix algebra \(M_{2 \times 2}(K)\) for all \(3 \leq i \leq r\). By Theorem \(3.13\) there exist algebra isomorphisms \(\phi : KQ/\mathcal{I}_f \to B\) and \(\psi : KQ/\mathcal{I}_g \to B\).

Let \(x = \phi(e_1), y = \phi(e_2)\) and write \(x = (x_1, \ldots, x_r), y = (y_1, \ldots, y_r)\). Then we have:

(a) Since \(e_1, e_2\) are idempotents, \(x, y\) must be idempotents, therefore \(x_i, y_i\) are idempotents in \(B_i\) for all \(1 \leq i \leq r\). This implies that \(x_1, x_2, y_1, y_2 \in \{0, 1\}\) and that for all \(3 \leq i \leq r\), \(x_i, y_i\) must be each conjugate to the zero matrix, the identity matrix, or the matrix \(E_{11} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\).

(b) Since \(1 = e_1 + e_2\) in \(KQ/\mathcal{I}_f\), we must have \(x + y = 1\) and hence \(x_i + y_i = 1\) for all \(1 \leq i \leq r\).

(c) Since \(\phi\) is an isomorphism, we have

\[
\dim(xBx) = \dim(\phi(e_1)B\phi(e_1)) = \dim(e_1(KQ/\mathcal{I}_f)e_1).
\]

Here, we have \(\dim(xBx) = \sum_{i=1}^{r} \dim(x_iB_ix_i)\) in the direct product \(B\). We also have \(\dim(e_1(KQ/\mathcal{I}_f)e_1) = r - 1\) because it is easy to see that the classes of the elements \(e_1, \alpha \beta, (\alpha \beta)^2, \ldots, (\alpha \beta)^{r-2}\) form a basis of \(e_1(KQ/\mathcal{I}_f)e_1\). It follows that \(\sum_{i=1}^{r} \dim(x_iB_ix_i) = r - 1\). Similarly, we must have \(\sum_{i=1}^{r} \dim(y_iB_iy_i) = r - 1\). Notice that for all \(1 \leq i \leq r\), the dimensions of \(x_iB_ix_i\) and \(y_iB_iy_i\) depend only on the conjugacy classes of the idempotents \(x_i, y_i\), respectively.

By straightforward dimension considerations, the above three facts force that \(x_i, y_i\) are conjugate to \(E_{11}\) for all \(3 \leq i \leq r\) and that we either have...
\[ x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 0 \] or have \( x_1 = 1, x_2 = 0, y_1 = 0, y_2 = 1 \). Similarly, the same conclusions apply to the coordinates \( x', y' \) of the elements \( x' = (x'_1, x'_2, x'_3, \ldots, x'_l) = \psi(e_1) \) and \( y' = (y'_1, y'_2, y'_3, \ldots, y'_l) = \psi(e_2) \). Thus, we either have \( x \sim x', y \sim y' \) or have \( x \sim y', y \sim x' \) where \( \sim \) means two elements are conjugate. In both cases, it is easy to find an automorphism \( \eta \) of \( B \) such that \( \text{the map } \Phi := \psi^{-1} \eta \phi : KQ/I_f \to KQ/I_g \) is an isomorphism sending \( e_i \) to \( e_i \) for \( i \in \{1, 2\} \). This proves the corollary in the case where \( m \) is even.

The proof for the case where \( m \) is odd is similar but simpler: let \( B = B_1 \times \cdots \times B_r \) where \( r = (m - 1)/2 \) and \( B_i = M_{2 \times 2}(K) \) for all \( 1 \leq i \leq r \), then consider the isomorphisms \( \phi, \psi \) guaranteed by Theorem 3.13 as before. This time, facts similar to (a), (b), (c) will force all coordinates of \( \phi(\varepsilon_1), \phi(\varepsilon_2), \psi(\varepsilon_1), \psi(\varepsilon_2) \) to be conjugate to \( E_{11} \), allowing us to form an isomorphism \( \Phi \) with the desired properties as before.

**3.4. Proof of Theorem 3.7.** General Case. We now prove Theorem 3.7 for a general Coxeter system \( (W, S) \). The rough idea is to notice that each edge in the Coxeter diagram corresponds to a dihedral system, so we can take the “local” isomorphisms provided by Corollary 3.14 and then assemble them to a “global” isomorphism between the quotients of \( KQ \).

**Proof of Theorem 3.7.** Let \( E \) be the set of edges of the Coxeter diagram of \( (W, S) \). For each \( e \in E \) of the form \( a \rightarrow b \), let \( \alpha : a \rightarrow b \) and \( \beta : b \rightarrow a \) be the dual arrows arising from \( e \) in \( Q \) and consider the subquiver \( Q_e = \{\{a, b\}, \{\alpha, \beta\}\} \) of \( Q \). Let \( I_f(e), I_g(e) \) be the evaluation ideals of \( \{f_n\} \) and \( \{g_n\} \) in \( Q_e \), respectively. Fix an isomorphism \( \Phi_e : KQ_e/I_f(e) \to KQ_e/I_g(e) \) such that \( \Phi_e(\varepsilon_a) = \varepsilon_a, \Phi_e(\varepsilon_b) = \varepsilon_b \) for \( e \); such an isomorphism exists by Corollary 3.14. Note that \( KQ_e/I_g(e) \) naturally embeds into \( KQ/I_g \), so we can naturally view an element of \( KQ_e/I_g(e) \) as an element in \( KQ/I_g \). We will do so without further comment.

Let \( Q^{\leq 1} = \{\varepsilon_a : a \in Q_0\} \cup Q_1 \) be the set of stationary paths and arrows of \( Q \). Consider the function \( \phi : Q^{\leq 1} \to KQ/I_g \) such that for every edge \( e = \{a, b\} \) in \( G \) and the arrows \( \alpha : a \rightarrow b, \beta : b \rightarrow a \) in \( Q \), we have

\[
\phi(\varepsilon_a) = \Phi_e(\varepsilon_a), \quad \phi(\varepsilon_b) = \Phi_e(\varepsilon_b), \quad \phi(\alpha) = \Phi_e(\alpha), \quad \phi(\beta) = \Phi_e(\beta).
\]

This function is well-defined because even if a vertex \( a \) in \( G \) is incident to two distinct edges \( e, e' \) in \( G \), the maps \( \Phi_e \) and \( \Phi_{e'} \) both send \( \varepsilon_a \) to \( \varepsilon_a \), causing no ambiguity for the value of \( \phi(\varepsilon_a) \). Next, recall again that the path algebra \( KQ \) is generated by \( Q^{\leq 1} \) subject only to the relations that \( e_u e_v = \delta_{u,v} e_u \) for \( u, v \in Q_0 \) and the relations \( e_a \alpha = \alpha e_b \) for each arrow \( \alpha : a \rightarrow b \) in \( Q_1 \), and note that the map \( \phi \) respects these relations: we have \( \phi(e_a) \phi(e_v) = e_u e_v = e_a e_v = \delta_{u,v} e_u = \delta_{u,v} \phi(e_u), \phi(e_u) \phi(\alpha) = \Phi_e(e_u) \Phi_e(\alpha) = \Phi_e(e_a \alpha) = \Phi(\alpha) \phi(\alpha), \phi(\alpha) \phi(\beta) = \phi(\alpha) \phi(\beta) \). It follows that \( \phi \) extends to a unique homomorphism \( \Phi : KQ \to KQ/I_g \) with \( \Phi(x) = \phi(x) \) for all \( x \in Q^{\leq 1} \). Finally, for each edge \( e : a \rightarrow b \) in \( Q \) and the corresponding arrows \( \alpha : a \rightarrow b, \beta : b \rightarrow a \), the restriction of \( \Phi \) to \( KQ_e \) agrees with \( \Phi_e \),
therefore Φ sends both \( r_f(α) \) and \( r_f(β) \) to zero because \( Φ_e \) does. It follows that Φ factors through \( ℐ_f \) to induce a homomorphism \( Φ : KQ/ℐ_f \to KQ/ℐ_g \). Starting from the collection \( \{Ψ_e : e ∈ E \} \) where \( Ψ_e = Φ_e^{-1} \) for all \( e ∈ E \), we may obtain in the same way a homomorphism \( Ψ : KQ/ℐ_g \to KQ/ℐ_f \), and it is clear that Ψ and Φ are mutual inverses, therefore \( KQ/ℐ_f \cong KQ/ℐ_g \). □

4. Quiver Contractions

Let \( J_C \) be the subregular \( J \)-ring of an arbitrary Coxeter system \((W,S)\), let \( K \) be an algebraically closed field of characteristic zero, and let \( A = A_K = K ⊗_Z J_C \). Let \( \text{mod}-A \) be the category of finite dimensional right \( A \)-modules. The rest of the paper is dedicated to the study of \( \text{mod}-A \).

In this section we introduce a procedure to modify a quiver \( Q \) to a new quiver \( \bar{Q} \) such that the algebra \( KQ/ℐ_{\bar{Q}} \) is Morita equivalent to \( KQ/ℐ_f \) for any uniform family of polynomials \( \{f_n\} \) over \( K \), where \( ℐ_f \) is the evaluation ideal of \( \{f_n\} \) in \( KQ \). We call the procedure a quiver contraction. In applications, we will often iterate contractions to obtain sequences of the form

\[
Q^{(0)} := Q \to Q^{(1)} \to \ldots \to Q^{(n)}
\]

where \( Q \) is the double quiver of \((W,S)\). Denote the evaluation ideal of \( \{f_n\} \) in \( KQ^{(i)} \) by \( ℐ_f^{(i)} \) for each \( i \). Then \( A \cong KQ/ℐ_f \) by Theorems 3.6 and 3.7, therefore \( \text{mod}-A \) is equivalent to \( \text{rep}_K(Q^{(i)}, ℐ_f^{(i)}) \) for all \( 0 \leq i \leq n \). For this reason, we shall develop tools for studying the last type of category in this section to prepare for the study of \( \text{mod}-A \) in Section § 5.

4.1. Definition of Quiver Contractions. Consider the following generalization of double quivers of Coxeter systems:

**Definition 4.1.** A generalized double quiver is a triple \((Q, d, m)\) consisting of a quiver \( Q = (Q_0, Q_1) \), a map \( d : Q_1 \to Q_1 \), and a map \( m : Q_1 \to \mathbb{Z}_{≥1} \cup \{∞\} \) such that

(a) \( d(Q_{ab}) = Q_{ba} \) for all \( a, b ∈ Q_0 \), where \( Q_{c,d} \) denotes the set of all arrows in \( Q \) from \( c \) to \( d \) for all \( c, d ∈ Q_0 \).

(b) \( d^2(α) = α \) for all \( α ∈ Q_1 \);

(c) \( m(α) = m(d(α)) \) for all \( α ∈ Q_1 \).

Given such a triple, we also call \( Q \) a generalized double quiver. We say that two arrows \( α, β ∈ Q_1 \) are dual to each other if \( β = d(α) \), and call \( m(α) \) the weight of \( α \) for all \( α ∈ Q_1 \).

Note that we may (and will) naturally view the double quiver \( Q \) of a Coxeter system as a generalized double quiver by setting \( m(α) = m_α \) and \( d(α) = ̄α \) for all \( α ∈ Q_1 \).

We now define quiver contractions as operations on generalized double quivers \((Q, d, m)\). Roughly speaking, we will define a contraction along a suitable pair of arrows \( α : a → b \) and \( β : b → a \) where \( a, b \) are distinct
vertices in $Q$. The contraction will identify $a,b$ by collapsing them into a
new vertex $v_{ab}$, replace $\alpha,\beta$ by a loop at $v_{ab}$, and reroute all other arrows
incident to $a$ or $b$ in $Q$ to new arrows incident to $v_{ab}$. The assignments of
duals and weights of arrows in the new quiver will be naturally inherited
from $d$ and $m$.

**Definition 4.2.** Let $(Q,d,m)$ be a generalized double quiver. A pair of
arrows $\{\alpha,\beta\}$ is called **contractible** if they are of the form $\alpha : a \to b, \beta : b \to a$
where $a,b$ are distinct vertices in $Q$, $\beta = d(\alpha)$, and $m(\alpha) = m(\beta)$ is an odd
integer that is at least 3.

For a contractible pair of arrows $\{\alpha : a \to b, \beta : b \to a\}$, the **contraction**
of $(Q,d,m)$ along $\{\alpha,\beta\}$ is the generalized double quiver $(\bar{Q},\bar{d},\bar{m})$ where
(a) The vertex set of the quiver $\bar{Q}$ is
$$Q_0 := Q_0 \setminus \{a,b\} \cup \{v_{ab}\},$$
where $v_{ab} \notin Q_0$ is a newly introduced vertex. The arrow set $\bar{Q}_1$ of $\bar{Q}$ is
defined as follows: write
$$c' := \begin{cases} v_{ab} & \text{if } c \in \{a,b\}; \\ c & \text{otherwise} \end{cases}$$
for all $c \in Q_0$ and define $\gamma'$ to be the arrow $u' \to v'$ for each arrow
$\gamma : u \to v$ in $Q_1$, then let
$$\bar{Q}_1 := \{\gamma' : \gamma \in Q_1 \setminus \{\alpha,\beta\}\} \cup \{\varepsilon_{ab}\},$$
where $\varepsilon_{ab} \notin Q_1$ is a newly introduced loop at $v_{ab}$.
(b) $\bar{d}$ is defined by $\bar{d}(\varepsilon_{ab}) = \varepsilon_{ab}$ and $\bar{d}(\gamma') = d(\gamma)'$ for all $\gamma \in Q_1 \setminus \{\alpha,\beta\}$.
(c) $\bar{m}$ is defined by $\bar{m}(\varepsilon_{ab}) = m(\alpha)$ and $\bar{m}(\gamma') = m(\gamma)$ for all $\gamma \in Q_1 \setminus \{\alpha,\beta\}$.

Note that quiver contractions introduce loops and may lead to multiple pairs
of arrows between two distinct vertices in the resulting quiver (see the quiver
$Q^{(2)}$ in Example 4.10), features that cannot be present in double quivers of
Coxeter diagrams. This is the reason why we do not forbid these features in
Definition 4.3. On the other hand, the generalization from double quivers to
generalized ones is mild enough that we can extend the definition of
evaluation ideals easily, at least for the cases we are interested in:

**Definition 4.3.** Let $\{f_n : n \geq 2\}$ be a uniform family of polynomials over
$K$, and let $(Q,d,m)$ be a generalized double quiver. We define the **evaluation ideal**
of $\{f_n\}$ in $KQ$ to be the two-sided ideal $I_f \subseteq KQ$ given by
$$I_f := \langle r_f(\alpha) : \alpha \in Q_1 \rangle$$
where
$$r_f(\alpha) = \begin{cases} 0 & \text{if } m = \infty; \\ f_{m-1}(\alpha,d(\alpha)) & \text{if } m < \infty \text{ and } d(\alpha) \neq \alpha; \\ \tilde{f}_{m-1}(\alpha) & \text{if } m < \infty \text{ and } d(\alpha) = \alpha, \end{cases}$$
where $m = m(\alpha)$. Here, as in Equation (1), the evaluation of $\alpha$ through a constant term $c$ in $\hat{f}_{m-1}(\varepsilon)$ returns $ce_a$ for $a = \text{source}(\alpha)$. For example, if $\alpha$ is a self-dual loop $\varepsilon : a \to a$ with $m = 5$, then $r_f(\alpha) = \varepsilon^2 - e_a$ if $f_4 = x^4 - 1$ and $r_f(\alpha) = 2\varepsilon^2 - \varepsilon - 3e_a$ if $f_4 = 2x^4 - x^2 - 3$.

**Remark 4.4.** In this paper we are only interested in generalized double quivers $Q$ obtained from the double quiver of a Coxeter diagram via iterated contractions. In this case, every self-dual arrow $\alpha$ in $Q$ must be either a loop of the form $\varepsilon = \varepsilon_{ab}$ at a vertex $v = v_{ab}$ introduced during a contraction of a quiver $\tilde{Q}$ along a dual pair of arrows $\gamma : a \to b$, $\delta : b \to a$ or a reroute of such a loop. In particular, $m$ must be a finite, odd integer, so the third case in the definition of $r_f(\alpha)$ applies and gives $r_f(\alpha) = \hat{f}_{m-1}(\varepsilon)$. The relation $r_f(\alpha) = \hat{f}_{m-1}(\varepsilon)$ in $K\tilde{Q}$ mirrors the relation $r_f(\varepsilon) = \hat{f}_{m-1}(\varepsilon)$ in the evaluation ideal $\mathcal{I}_f$ of $KQ$ via the replacements $\gamma \delta \mapsto \varepsilon$ and $a \mapsto v$. For example, if $m = 5$ and $f_4 = x^4 - 1$, then the relation $r_f(\varepsilon) = \hat{f}_4(\varepsilon) = \varepsilon^2 - e_v$ is mirrored by the relation $r_f(\varepsilon) = \hat{f}_4(\varepsilon) = \varepsilon^2 - e_v$.

Our main result on contractions is the following theorem.

**Theorem 4.5.** Let $(Q, d, m)$ be a generalized double quiver and let $(\tilde{Q}, \tilde{d}, \tilde{m})$ be a contraction of $(Q, d, m)$ along a contractible pair of arrows $\{\alpha, \beta\}$. Let $\{f_n : n \geq 2\}$ be a uniform family of polynomials over $K$, and let $\mathcal{I}_f$ and $\tilde{\mathcal{I}}_f$ be the evaluation ideal of $\{f_n\}$ in $KQ$ and $\tilde{KQ}$, respectively. Then the algebras $KQ/\mathcal{I}_f$ and $\tilde{KQ}/\tilde{\mathcal{I}}_f$ are Morita equivalent.

We postpone the proof of the theorem to §4.3. Before the proof, we discuss several detailed examples of quiver contractions and some consequences of the theorem in the next subsection.

**4.2. Examples of Quiver Contractions.** Throughout this subsection, $G$ denotes the Coxeter diagram of a Coxeter system $(W, S)$ and $Q$ stands for the double quiver of $G$. When drawing generalized double quivers, we label each pair of dual arrows $\{\alpha, d(\alpha)\}$ with their common weight $m(\alpha)$, except when $m(\alpha) = 3$, including for the case where $\alpha$ is a self-dual loop of the form $\varepsilon_{ab}$ introduced by a contraction. For convenience, we consider only the polynomials $\{f_n : n \geq 2\}$ where

\[
(17) \quad f_n = \begin{cases} x^n - 1 & \text{if } n \text{ is even;} \\ x^n - x & \text{if } n \text{ is odd.} \end{cases}
\]

for all $n \geq 2$. Note that $\{f_n\}$ is a uniform family over $K$ since $K$ is algebraically closed (see Remark 4.17).

**Example 4.6.** Suppose that $G$ and $Q$ are as shown at the top of Figure 1. The arrows $\alpha$ and $\beta$ are dual to each other in $Q$ and have weight 3, therefore they form a contractible pair. The contraction along $\{\alpha, \beta\}$ results in the quiver $\tilde{Q}$ shown in the bottom right corner of the figure, where $v = v_{ab}$ and $\varepsilon = \varepsilon_{ab}$.
Let us examine the effect of the contraction. The three pairs of arrows \{\gamma, \delta\}, \{\eta, \zeta\} and \{\kappa, \lambda\} in \(Q\) have weights 3, 5, 4 and give rise to the elements
\[
\begin{align*}
    r_1 &= \gamma \delta - e_b, \\
    r_2 &= \delta \gamma - e_c, \\
    r_3 &= (\zeta \eta)^2 - e_c, \\
    r_4 &= (\eta \zeta)^2 - e_d, \\
    r_5 &= \kappa \lambda \kappa - \kappa, \\
    r_6 &= \lambda \kappa \lambda - \lambda
\end{align*}
\] of the evaluation ideal \(I_f\). The contraction reroutes the arrows \(\gamma, \delta, \kappa, \lambda\) since they are incident to \(a\) or \(b\), but the rerouting preserves weights by definition, so the rerouted arrows give rise to “duplicates” of the relations \(r_1, r_2, r_5, r_6\) in the ideal \(\bar{I}_f\), namely, the relations
\[
\begin{align*}
    r'_1 &= \gamma' \delta' - e_v, \\
    r'_2 &= \delta' \gamma' - e_c, \\
    r'_5 &= \kappa' \lambda' \kappa' - \kappa', \\
    r'_6 &= \lambda' \kappa' \lambda' - \lambda'.
\end{align*}
\]
The arrows \(\zeta\) and \(\eta\) and their weights remain unchanged in the contraction since they are not incident to \(a\) or \(b\). Consequently, they contribute the same relations \(r_3\) and \(r_4\) to \(\bar{I}_f\) just as they do to \(I_f\). Finally, the arrows \(\alpha, \beta\) are replaced by a single loop at \(v\). Since \(m := m(\alpha) = m(\beta)\) is odd, \(\alpha\) and \(\beta\) contribute the relations
\[
\begin{align*}
    r_f(\alpha) &= (\alpha \beta)^k - e_a, \\
    r_f(\beta) &= (\beta \alpha)^k - e_b
\end{align*}
\] to \(\bar{I}_f\), and their replacement \(\varepsilon\) contributes a single relation
\[
    r_f(\varepsilon) = r_f(d(\varepsilon)) = \varepsilon^k - e_v
\] to \(\bar{I}_f\); here, we have \(m = 3\) and \(k = (m - 1)/2 = 1\).

By Theorem 4.5, the algebra \(A\) is Morita equivalent to the quotient \(K\bar{Q}/\bar{I}_f\) where \(\bar{I}_f = \langle r'_1, r'_2, r'_3 = r_3, r'_4 = r_4, r'_5, r'_6, r_f(\varepsilon) \rangle\). The relation
$r_f(\varepsilon) = \varepsilon - e_v$ implies that $\varepsilon = e_v$ in the quotient, therefore the quotient is isomorphic to the quotient $K\hat{Q}/\hat{I}_f$ where $\hat{Q}$ is obtained from $Q$ by removing the loop $\varepsilon$ and $\hat{I}_f = \langle r_i : 1 \leq i \leq 6 \rangle \subseteq K\hat{Q}$. More generally, we define a quiver contraction with respect to a pair of arrows $\{\alpha, \beta\}$ to be simple if $m(\alpha) = m(\beta) = 3$. By the above discussion, Theorem 4.5 remains true for a simple contraction if we omit the loop $\varepsilon$ in the construction of $\hat{Q}$. We shall do so from now on.

For a double quiver $Q$ of a Coxeter diagram $G$, a contraction of $Q$ along a pair of arrows $\{\alpha : a \to b, \beta : b \to a\}$ is simple if and only if the corresponding edge $a - b$ in $G$ is simple. When this is the case, we may define a simple contraction of $G$ by “contracting” the edge $a - b$ until $a, b$ are identified as a new vertex $v := v_{ab}$, thus effectively rerouting all edges incident to $a$ or $b$ to $v$. More precisely, we may define a weighted graph $\tilde{G}$ whose vertex set is $S \setminus \{a, b\} \cup \{v\}$ where $v \notin S$ and whose edge set is $\{e' : e$ is an edge in $G$ other than $a - b\}$, where $e' = e$ if $e$ is not incident to $a$ or $b$ and $e'$ is the edge $v - c$ whenever $e$ is of the form $a - c$ or $b - c$ for a vertex $c \in S \setminus \{a, b\}$; the weight $m(e')$ of $e'$ is defined to be the same as that of $e$. We call $\tilde{G}$ the simple contraction of $G$ along $a - b$. For the above example, the contraction of $G$ along $a - b$ results in the graph $\tilde{G}$ shown in the lower left corner of Figure 1. Note that if $a, b$ share no neighbor in $G$, which is the case for our example and must be the case if $G$ has no cycles, then the graph $\tilde{G}$ can again be viewed as the Coxeter diagram of a Coxeter system $(\bar{W}, \bar{S})$. In this case, the double quiver of $\tilde{G}$ makes sense, and it is clear that the double quiver of the simple contraction $\tilde{G}$ of $G$ coincides with the simple contraction $\tilde{Q}$ of the double quiver $Q$ of $G$ (once we ignore the loop $\varepsilon = e_{ab}$). This phenomenon is manifest in our example: the diagram in Figure 1 commutes once we ignore the loop $\varepsilon$.

**Remark 4.7.** Maintain the assumptions and notation of the previous paragraph. In particular, assume that $a - b$ is a simple edge in $G$ where $a, b$ have no common neighbor. Then Theorem 4.5 implies that the algebra $A = K \otimes_{\mathbb{Z}} J_C$ associated to the Coxeter system $(W, S)$ is Morita equivalent to the algebra $\hat{A} := K \otimes_{\mathbb{Z}} \hat{J}_C$ associated to the system $(\bar{W}, \bar{S})$. Note that we may state the equivalence purely in terms of contractions of Coxeter diagrams, with no reference to quivers. Also note that the equivalence implies that when we study the category mod-$A$, we may assume $G$ has no simple edges whenever $G$ is a tree. The reason is that, since no two vertices can share a neighbor in a tree, we may repeatedly remove all simple edges in $G$ by simple contractions.

The following example illustrates the reduction allowed by Remark 4.7. After the example, we record an application of the reduction for future use.

**Example 4.8.** Suppose that $G$ is the tree shown on the left in Figure 2. By Remark 4.7, the algebra $A$ associated to $(W, S)$ is Morita equivalent to the algebra $A' := K \otimes J_C'$ where $J_C'$ is the subregular $J$-ring of the Coxeter
system whose diagram is the graph $G'$ obtained by contracting all simple edges in $G$; the graph $G'$ is shown on the right on Figure 2.

![Figure 2](image)

**Proposition 4.9.** Let $(W,S)$ be a Coxeter system whose Coxeter graph $G$ is a tree, has no edge with infinite weight, and has at most one heavy edge. Then the category $\text{mod-}A$ associated to $(W,S)$ is semisimple.

**Proof.** Let $e$ be an edge of maximal weight in $G$. In other words, let $e$ be the unique heavy edge if $G$ has one, and let $e$ be any simple edge in $G$ otherwise. By contracting all edges different from $e$ in $G$ if necessary, we may assume that $e$ is the only edge in $G$ and hence $|S| = 2$. Theorem 3.13 then implies that $\text{mod-}A$ is semisimple. □

The next two examples involve iterated quiver contractions.

**Example 4.10.** Suppose that $G = C_n(m)$ is a cycle with at most one heavy edge, where $n$ is the number of vertices in $G$ and $m$ is the maximal edge weight. In other words, suppose that $G$ has $n$ vertices $v_1, v_2, \ldots, v_n$ for some $n \geq 3$ and has exactly $n$ edges $v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n - v_1$, then assume, without loss of generality, that $G$ has edge weights $m(v_2, v_3) = m(v_3, v_4) = \cdots = m(v_{n-1}, v_n) = m(v_n, v_1) = 3$ and $m(v_1, v_2) = m \geq 3$. Note that when $m = 3$, the Coxeter group arising from the Coxeter diagram $C_n(m) = C_n(3)$ is exactly the affine Weyl group of type $\tilde{A}_{n-1}$ for all $n \geq 3$.

By repeated simple contractions of Coxeter diagrams, we may reduce $G$ to a triangle whose double quiver is the quiver $Q^{(1)}$ shown on the left of Figure 3. Contracting $Q^{(1)}$ along the arrows $\alpha_3, \beta_3$ results in the quiver $Q^{(2)}$ in the same figure, where $a = v_{v_1,v_3}$ and the loop $\varepsilon_v$ is omitted as usual. Furthermore, since $m(\alpha_2') = m(\alpha_2) = 3$, we may further contract $Q^{(2)}$ along the arrows $\alpha_2', \beta_2'$ to obtain the quiver $Q^{(3)}$, where $b = v_{v_2v_3}$, the loop $\varepsilon_b$ is omitted, and the arrows $\alpha_1'', \beta_1''$ are dual to each other and have weight $m$. Note that as the last contraction demonstrates, given a contractible pair of arrows $\alpha : a \rightarrow b, \beta : b \rightarrow a$ in a generalized double quiver $Q$, every pair of dual arrows of the form $\gamma : a \rightarrow b, \delta : b \rightarrow a$ where $\gamma \neq \alpha$ is rerouted to a pair of distinct loops $\gamma', \delta'$ at $v_{ab}$ which are dual to each other and have the same weight as $\gamma$ and $\delta$.

By Theorem 4.5, the algebra $A$ associated to the Coxeter system whose Coxeter diagram is the cycle $G$ is Morita equivalent to the algebra $\tilde{A} :=$
\[ KQ^{(3)}/I_f^{(3)} \text{ where } I_f^{(3)} \text{ is the evaluation ideal of } \{f_n\} \text{ in } KQ^{(3)}. \]  

The path algebra \( KQ^{(3)} \) is isomorphic to the free unital associative algebra \( K\langle x, y \rangle \) on two variables via the identification \( e_b \mapsto 1, \alpha''_1 \mapsto x, \beta''_1 \mapsto y, \) and \( I_f^{(3)} \) is given by

\[ I_f^{(3)} := \langle (\alpha''_1 \beta''_1)^k - e_b, (\beta''_1 \alpha''_1)^k - e_b \rangle \]

where \( k = (m - 1)/2, \) so \( \tilde{A} \) is isomorphic to the algebra

\[ T_k := K\langle x, y \rangle/((xy)^k = (yx)^k = 1). \]

In particular, if \( m = 3, \) then \( k = 1 \) and \( \tilde{A} \) is isomorphic to the Laurent polynomial algebra \( K[t, t^{-1}] \) via the identification \( e_b \mapsto 1, \alpha''_1 \mapsto t, \beta''_1 \mapsto t^{-1}. \)

To summarize, we have just proved the following result.

**Proposition 4.11.** Let \( n \geq 3. \) Let \( m \geq 3 \) be an odd integer, let \( k = (m - 1)/2, \) and let \((W, S)\) be the Coxeter system with Coxeter diagram \( G = C_n(m). \) Then the algebra \( A \) associated to \((W, S)\) is Morita equivalent to the algebra \( T_k \) defined by Equation (21). In particular, the Morita equivalence class of \( A \) does not depend on the value of \( n, \) and if \( m = 3, \) i.e., if \((W, S)\) is of type \( A_{n-1}, \) then \( A \) is Morita equivalent to \( K[t, t^{-1}]. \)

**Example 4.12.** Apart from the algebras of the form \( T_k \) from Equation (21), group algebras of free products of finite cyclic groups can also be realized as the algebras of the form \( A \) associated to Coxeter systems up to Morita equivalence. To see this, let \( A_k \) be the \( K\)-algebra given by the presentation

\[ A_k := \langle x : x^k = 1 \rangle \]

for each integer \( k > 1, \) and let

\[ A_k = \langle x_j : 1 \leq j \leq n, x_j^k = 1 \rangle \]

for each tuple \( k = (k_1, \ldots, k_n) \) where \( n \geq 1 \) and \( k_i \in \mathbb{Z}_{\geq 1} \) for all \( 1 \leq i \leq n. \) Then \( A_k \) is isomorphic to the group algebra of the cyclic group \( C_k \) of order \( k, \) and \( A_k \) is isomorphic to the group algebra of the free product \( C_k := C_{k_1} * \cdots * C_{k_n} \) of \( C_{k_1}, \ldots, C_{k_n}. \) For each tuple \( k = (k_1, \ldots, k_n), \) let \((W, S)\) be the Coxeter system where \( S = \{0, 1, 2, \ldots, n\}, \) \( m_j := m(0, j) = 2k_j + 1 \) for each \( 1 \leq j \leq n, \) and \( m(i, j) = 2 \) for all \( 1 \leq i < j \leq n. \) Let \( Q \) be the double quiver of \((W, S), \) and for each \( 1 \leq j \leq n \) denote the arrows \( 0 \rightarrow j \) and \( j \rightarrow 0 \)
in $Q$ by $\alpha_j$ and $\beta_j$, respectively. It is easy to see that by starting from the quiver $Q$ and performing successive contractions, first along $\{\alpha_1, \beta_1\}$, then along (the reroutes of) $\{\alpha_2, \beta_2\}$, then along (the reroutes of) $\{\alpha_3, \beta_3\}$, and so on, we can transform $Q$ to a generalized double quiver $\bar{Q}$ with a single vertex $v$ and $n$ self-dual loops $\varepsilon_1, \ldots, \varepsilon_n$ at $v$ of weight $m_1, m_2, \ldots, m_n$, respectively.

Figure 4 demonstrates the construction of $\bar{Q}$ from the Coxeter diagram $G$ of $(W, S)$ when $n = 4$.

By Theorem 4.5, the algebra $A$ is Morita equivalent to the quotient $K\bar{Q}/\bar{I}_f$ where $\bar{I}_f$ is the evaluation ideal of $\{f_n\}$ in $K\bar{Q}$. The following proposition is now immediate.

**Proposition 4.13.** Let $(W, S)$ be as described in the above paragraph. Then the algebra $A$ associated to $(W, S)$ is Morita equivalent to the algebra $A_k$.

**Proof.** Note that $K\bar{Q}$ is the free unital associative algebra generated by the loops $\varepsilon_j$ where $1 \leq j \leq n$ and that $\bar{I}_f = \langle \varepsilon_j^k - e_v : 1 \leq j \leq n \rangle$. It follows that we may induce an algebra isomorphism $\varphi: A_k \rightarrow K\bar{Q}/\bar{I}_f$ from the assignment $\varphi(x_j) = \varepsilon_j$ for all $1 \leq j \leq n$. $\square$

**Remark 4.14.** A pleasant feature of iterated quiver contractions is that for every sequence of the form (16), since we reduce the number of vertices in the quiver with every contraction, representations in the category $\text{rep}_K(Q^{(n)}, I_f^{(n)})$ are often relatively easy to describe. For instance, in Example 4.12 a representation in $\text{rep}_K(Q^{(n)}, I_f^{(n)})$ is simply the data of a vector space $M_v$ and endormorphisms $\phi_1, \ldots, \phi_n$ of $M_v$ where $\phi_j^{k_j} = \text{id}$ for all $1 \leq j \leq n$. In Example 4.10 to define a representation in $\text{rep}_K(Q^{(3)}, I_f^{(3)})$ it suffices to specify a space $M_v$ and two endomorphisms $M_{\alpha''_1}, M_{\beta''_1}$ of $M_v$ satisfying the relations $f_{m-1}(\alpha''_1, \beta''_1)$ and $f_{m-1}(\beta''_1, \alpha''_1)$. Finally, for the Coxeter system $(W, S)$ from Example 4.8 by repeated contractions along arrows corresponding to the edges of weight 5 and 7 in $G'$, we may transform the double quiver of $G'$ to a quiver $Q$ of the form shown in Figure 5 where the loop $\varepsilon_1$ is self-dual and has weight 7, the loop $\varepsilon_2$ is self-dual and has weight 5, and $\alpha, \beta$ are dual to each other and have weight 4. It follows that $\text{mod}-A$
is equivalent to the category $\text{rep}_K(\bar{Q}, \bar{I}_f)$ where

$$\bar{I}_f = \langle \varepsilon_1^2 - e_v, \varepsilon_2^3 - e_v, \alpha \beta \alpha - \alpha, \beta \alpha \beta - \beta \rangle.$$ 

A representation in the latter category is then simply the data of two vector spaces $M_v, M_z$, two operators $M_{\varepsilon_1}, M_{\varepsilon_2}$ on $M_v$ and two maps $M_{\alpha} : M_v \to M_z, M_{\beta} : M_z \to M_v$ which satisfy the four relations in the above equation. In all these examples, the representations are much easier to describe than those in the category $\text{rep}_K(Q, I_f)$ attached to the original double quiver $Q$. Endomorphism like $\phi_j, M_{\alpha}M_{\beta}, M_{\beta}M_{\alpha}$ and $M_{\varepsilon_1}, M_{\varepsilon_2}$ will be key tools in our study of $\text{mod-}A$; we will elaborate on their use in §4.4.

![Diagram](image_url)  

**Figure 5.**

4.3. **Proof of Theorem 4.5.** Let $(Q, d, m)$ be a generalized double quiver and let $(\bar{Q}, \bar{d}, \bar{m})$ be its contraction along a contractible pair of arrows $\{\alpha : a \to b, \beta : b \to a\}$. We prove Theorem 4.5 in this subsection. To begin, we introduce an intermediate generalized double quiver $(Q', d', m')$ where

(a) $Q_0' = Q_0$, and $Q_1'$ is defined as follows: write

$$c' = \begin{cases} a & \text{if } c = b; \\ c & \text{otherwise} \end{cases}$$

for all $c \in Q_0$, let

$$\gamma' = \begin{cases} \gamma & \text{if } \gamma \in \{\alpha, \beta\}; \\ (c' \to d') & \text{otherwise, if } \gamma \text{ is of the form } c \to d \end{cases}$$

for each arrow $\gamma \in Q_1$, then set

$$Q_1' = \{\gamma' : \gamma \in Q_1\}.$$ 

(b) $d'$ is defined by $d'(\gamma') = [d(\gamma')]$ for all $\gamma \in Q_1'$;

(c) $m'$ is defined by $m'(\gamma') = m(\gamma)$ for all $\gamma \in Q_1$.

Intuitively, we consider $Q'$ an intermediate rerouted version of $Q$ similar to $\bar{Q}$: in $Q$, we identify $a, b$ with $v = v_{ab}$ and “transfer” all data relevant to $a$ or $b$ in $Q$ to $v$ by rerouting all arrows in $Q$ incident to $a$ or $b$ to $v$; in $Q'$, however, we transfer almost all data relevant to $b$ to $a$ except for the arrows $\alpha$ and $\beta$. We may thus think of $a \in Q_0'$ as a partial copy of $a, b \in Q_0$ and $v \in Q_0$ as a complete copy of $a, b$. To obtain $\bar{Q}$ from $Q'$, it remains to rename $a$ as $v_{ab}$, rename each arrow $\gamma \in Q_1' \setminus \{\alpha, \beta\}$ incident to $a$ as $\gamma'$, replace $\alpha', \beta'$ with $e_{ab}$, and remove $b$. For the quiver $Q$ from Example 4.6.
this procedure, along with the construction of $Q'$ from $Q$, is illustrated in Figure 6, where $v = v_{ab}$ and $\varepsilon = \varepsilon_{ab}$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (1,0) {$b$};
  \node (d) at (0,-1) {$d$};
  \node (c) at (1,-1) {$c$};
  \draw[->] (a) -- (b);
  \draw[->] (a) -- (d);
  \draw[->] (a) -- (c);
  \draw[->] (b) -- (c);
  \node (4) at (0,1) {$Q : 4$};

  \node (a') at (2,0) {$a'$};
  \node (b') at (3,0) {$b'$};
  \node (d') at (2,-1) {$d'$};
  \node (c') at (3,-1) {$c'$};
  \draw[->] (a') -- (b');
  \draw[->] (a') -- (d');
  \draw[->] (a') -- (c');
  \draw[->] (b') -- (c');
  \node (4') at (2,1) {$Q' : 4'$};

  \node (a'') at (4,0) {$a''$};
  \node (b'') at (5,0) {$b''$};
  \node (d'') at (4,-1) {$d''$};
  \node (c'') at (5,-1) {$c''$};
  \draw[->] (a'') -- (b'');
  \draw[->] (a'') -- (d'');
  \draw[->] (a'') -- (c'');
  \draw[->] (b'') -- (c'');
  \node (4'') at (4,1) {$\tilde{Q} : 4''$};

\end{tikzpicture}
\caption{Figure 6.}
\end{figure}

Maintain the notation of Theorem 4.5 and let $I'_f$ be its evaluation ideal of $KQ'$ associated to the polynomials $\{f_n\}$. We show below that the algebra $KQ'/I'_f$ is both isomorphic to $KQ/I_f$ and Morita equivalent to $KQ/\tilde{I}_f$; Theorem 4.5 immediately follows. Note that by the last paragraph, we “fa-

\begin{proposition}
\textbf{Proposition 4.15.} Maintain the setting of Theorem 4.5. Then the algebra $KQ'/I'_f$ is isomorphic to $KQ/I_f$.
\end{proposition}

\textbf{Proof.} We will construct mutually inverse homomorphisms $\Phi : KQ/I_f \to KQ'/I'_f$ and $\Psi : KQ'/I'_f \to KQ/I_f$ to prove the proposition. To do so, we use presentations of the algebras as usual: since $KQ$ is the algebra generated by the set $Q^\leq 1 = \{e_u : u \in Q_0\} \cup Q_1$ subject only to the relations that $e_u e_v = \delta_{u,v} e_u$ for all $u,v \in Q_0$ and the relations $e_u \gamma = \gamma = \gamma e_v$ for each arrow $\gamma : u \to v$ in $Q_1$, the algebra $KQ/I_f$ is generated by the same set $Q^\leq 1$ subject to the above relations and the relations $r_f(\alpha)$ for all $\alpha \in Q_1$. We may therefore construct $\Phi$ by inducing it from a function $\varphi : Q^\leq 1 \to KQ'/I'_f$ which respects all the necessary relations. Similarly, we may construct the homomorphism $\Psi$ from a function $\psi : Q^\leq 1 = \{e_s : s \in Q'_0\} \cup Q'_1 \to KQ/I_f$ which respects the necessary conditions.

Let $m = m(a,b)$. Then $m \geq 3$ and $m$ is odd by Definition 4.2. By scaling if necessary, we may assume the polynomial $f_{m-1}$ has constant term $-1$, in which case the relations $r_f(\alpha), r_f(\beta) \in I_f$ must be of the form

\begin{align*}
f_{m-1}(\alpha, \beta) &= g(\alpha \beta) \alpha \beta - e_a, \\
f_{m-1}(\beta, \alpha) &= g(\beta \alpha) \beta \alpha - e_b = \beta g(\alpha \beta) \alpha - e_b
\end{align*}
for some polynomial $g \in K[x]$, respectively. Let
\[\sigma_1 = g(\alpha\beta), \quad \sigma_2 = \beta.\]
Then $\sigma_1, \sigma_2$ make sense in both $KQ$ and $KQ'$, and we have $r_f(\alpha) = \sigma_1\sigma_2 - e_a, r_f(\beta) = \sigma_2\sigma_1 - e_b$, so that
\[
(22) \quad \sigma_1\sigma_2 = e_a, \quad \sigma_2\sigma_1 = e_b
\]
in both $KQ/I_f$ and $KQ'/I_f$. To define the functions $\varphi : Q^{\leq 1} \to KQ/I_f$ and $\psi : Q^{\leq 1} \to KQ/I_f$, first let
\[
X_+ = \{\gamma \in Q_1 : \text{source}(\gamma) = b\} \setminus \{\beta\},
\]
and
\[
X_- = \{\gamma \in Q_1 : \text{target}(\gamma) = b\} \setminus \{\alpha\}.
\]
Note that the set $X_+ \cap X_-$ consists of all loops at $b$ in $Q_1$, and each loop in it is rerouted to a loop at $b$ in $Q'$. Next, let $\varphi(e_a) = e_a$ for all $u \in Q_0 = Q'_0$. Finally, recall that $Q'_1 = \{\gamma' : \gamma \in Q_1\}$ and define $\varphi$ and $\psi$ on $Q_1$ and $Q'_1$ by letting

\[
\varphi(\gamma) = \begin{cases}
\sigma_2\gamma'/\gamma' & \text{if } \gamma \in X_+ \cap X_-; \\
\sigma_2\gamma' & \text{if } \gamma \in X_+ \setminus X_-; \\
\gamma'\gamma & \text{if } \gamma \in X_- \setminus X_+; \\
\gamma & \text{otherwise,}
\end{cases}
\]
and
\[
\psi(\gamma') = \begin{cases}
\sigma_1\gamma & \text{if } \gamma \in X_+ \cap X_-; \\
\sigma_1\gamma & \text{if } \gamma \in X_+ \setminus X_-; \\
\gamma & \text{otherwise,}
\end{cases}
\]
for each $\gamma \in Q_1$. Using the relations in (22), it is straightforward to verify that $\varphi$ and $\psi$ respect all necessary relations mentioned in the previous paragraph and induce mutually inverse algebra homomorphisms $\Phi : KQ/I_f \to KQ'/I_f$ and $\Psi : KQ'/I_f \to KQ/I_f$, as desired.

**Proposition 4.16.** Maintain the setting of Theorem 4.5. Then the algebra $KQ/I(R)$ is Morita equivalent to $KQ'/I(R')$. 

**Proof.** Let $\Lambda = KQ'/I_f$ and let $\sigma_1, \sigma_2$ be as in the proof of Proposition 4.15. Since $\sigma_1\sigma_2 = e_a$ and $\sigma_2\sigma_1 = e_b$ in $\Lambda$, the maps $\phi_1 : e_a\Lambda \to e_b\Lambda, x \mapsto \sigma_2x$ and $\phi_2 : e_b\Lambda \to e_a\Lambda, y \mapsto \sigma_1y$ give mutually inverse isomorphisms between the projective modules $\Lambda$-modules $e_a\Lambda$ and $e_b\Lambda$. Set $V_b = Q_0 \setminus \{b\}$ and let
\[
e = 1 - e_b = \sum_{u \in V_b} e_u \in \Lambda.
\]
Since $e_a\Lambda \cong e_b\Lambda$, the submodule $\Lambda' := e\Lambda = \oplus_{u \in V_b} (e_u \Lambda)$ of the regular module $\Lambda$ is a progenator in the category of $\Lambda$-modules, therefore $\Lambda$ is Morita equivalent to the endomorphism algebra $\text{End}_\Lambda(\Lambda')$. We have $\text{End}_\Lambda(\Lambda') \cong e\Lambda e$ since $e$ is an idempotent, so to prove the proposition it suffices to show that $KQ/I(R)$ is isomorphic to $e\Lambda e$. We will do so by inducing a homomorphism $\Phi : KQ/I(R) \to e\Lambda e$ from a function $\phi : Q^{\leq 1} := \{e_u : u \in Q_0\} \cup Q_1 \to e\Lambda e$ and then showing that $\Phi$ is bijective.

Let $v = v_{ab}$ and $e = e_{ab}$. To define the function $\phi : \{e_u : u \in Q_0\} \cup Q_1 \to e\Lambda e$, let $\varphi(v) = e_a$ and let $\phi(e_u) = e_a$ for all $u \in Q_0 \setminus \{v\}$, then let $\phi(e) = \alpha\beta$. 

and let \( \phi(\gamma) = \gamma \) for all \( \gamma \in Q_1 \setminus \{\varepsilon\} \). Viewing \( K\bar{Q}/\bar{I}_f \) as the algebra generated by \( Q^{\leq 1} \) subject to the suitable relations as usual, we can again check that \( \phi \) respects all these relations: indeed, by the definitions of \( Q \) and \( Q' \), it suffices to check only the relations involving the loop \( \varepsilon \in Q_1 \), i.e., the relation \( e_v \varepsilon = \varepsilon = \varepsilon e_v \) and the relation \( r_f(\varepsilon) = \tilde{f}_{m(\alpha)-1}(\varepsilon) \). These relations are respected by \( \phi \) since \( \phi(e_v) = e_v, \phi(\varepsilon) = \alpha \beta, e_{\alpha \beta} = \alpha \beta = \alpha \beta e_a \), and \( \tilde{f}_{m(\alpha)-1}(\alpha \beta) = r_f(\alpha) \in I_f \) (see Remark 4.4). It follows that \( \Phi \) induces a unique algebra homomorphism \( \Phi : K\bar{Q}/\bar{I}_f \to e\Lambda e \).

To prove that \( \Phi \) is bijective, we keep the notation from Definition 4.2 and from the definition of \( Q' \). Let \( X = Q_1 \setminus \{\alpha, \beta\} \). Then

\[
\bar{Q}_1 = \{\gamma' : \gamma \in X\} \cup \{\varepsilon\}, \quad Q'_1 = \{\gamma' : \gamma \in X\} \cup \{\alpha, \beta\},
\]

and \( \Phi(\gamma') = \gamma' \) for all \( \gamma \in X \) (where the \( \gamma' \)'s stand for their respective images in \( K\bar{Q}/\bar{I}_f \) and \( e\Lambda e \)). Let \( P_b \) be the set of all paths on \( Q' \) which both start and end at a vertex in \( V_b \). Then \( e\Lambda e \) is spanned by the classes of paths in \( P_b \). Now, since \( \alpha, \beta \) are the only arrows in \( Q' \) with \( b \) as its target and source, respectively, if a path \( p \in P_b \) passes \( b \) at any point then it must have traveled to \( b \) from \( a \) via \( \alpha \) and then immediately traveled back to \( a \) via \( \beta \). Consequently, \( p \) is a product of \( \alpha \beta = \Phi(\varepsilon) \) and arrows from the set \( X = \Phi(X) \subseteq \text{im} \Phi \). It follows that \( p \in \text{im} \Phi \), so \( \Phi \) is surjective.

It remains to prove that \( \Phi : K\bar{Q}/\bar{I}_f \to e\Lambda e \) is injective. Since \( \Lambda = KQ'/I_f \), it suffices to show that \( e\bar{I}_f e \subseteq \Phi(\bar{I}_f) \). Since \( Q'_1 = \{\alpha, \beta\} \cup \{\gamma' : \gamma \in X\} \), the set \( e\bar{I}_f e \) is spanned by nonzero elements of the form

\[
y_1 = p_1[r_f(\alpha)]q_1 = p_1[\Phi(f(\varepsilon, \varepsilon))]q_1, \tag{23}
\]

\[
y_2 = p_2[r_f(\beta)]q_2, \tag{24}
\]

and

\[
y_3 = p_3[r_f(\gamma')]q_3 = p_3[\Phi(r_f(\gamma))]q_3 \tag{25}
\]

where \( \gamma \in X \) and \( p_i, q_i \) are paths in \( KQ' \) for all \( i \in \{1, 2, 3\} \). We need to show that \( y_1, y_2, y_3 \in \Phi(\bar{I}_f) \). Note that the following holds for all \( i \in \{1, 2, 3\} \):

(a) Since \( e = \sum_{u \in V_b} e_u \) and \( y_i \neq 0 \), we have \( \text{source}(p_i), \text{target}(q_i) \in V_b \).

(b) Let \( r_i \) be the bracketed relation in \( y_i \) in Equations (23)-(25). Then \( \text{source}(p_i) = \text{source}(r_i), \text{target}(r_i) = \text{source}(q_i) \) since \( y_i \neq 0 \). In particular, we have \( \text{source}(p_i) = \text{source}(q_1) = a \in V_b \) and \( \text{target}(p_3) \), \( \text{source}(q_3) \in V_b \) since the rerouted arrow \( \gamma' \) cannot be incident to \( b \).

(c) By (a) and (b), \( p_3, q_3 \in P_b \subseteq \text{im} \Phi \) where the last containment holds by the last paragraph. Furthermore, we have \( r_1 = r_f(\alpha) = \Phi(r_f(\varepsilon)) \) and \( r_3 = r_f(\gamma') = \Phi(r_f(\gamma)) \) by the definition of \( \Phi \). It follows that \( y_1, y_3 \in \Phi(\bar{I}_f) \), as desired.

(d) By (b) we have \( \text{target}(p_2) = b = \text{source}(q_2) \), but since \( \alpha, \beta \) are the only arrows in \( Q' \) with \( b \) as its target and source, respectively, we must have \( p_2 = p_2' \alpha \) and \( q_2 = \beta q_2'' \) for some paths \( p_2', q_2'' \in P_b \). It follows that

\[
y_2 = p_2'[r_f(\beta)]\beta q_2'' = p_2'[r_f(\alpha)]\alpha \beta q_2'' = p_2'[\Phi(r_f(\varepsilon, \varepsilon))]q_2'' \tag{26}
\]
where \( q_2' = \alpha \beta q_2'' \). Note that \( p_2', q_2' \in \mathcal{P}_b \subseteq \text{im } \Phi \), therefore \( y_2 \in \Phi(\bar{\mathcal{I}}_f) \).

The proof is now complete. \( \square \)

**Remark 4.17.** We have assumed that the field \( K \) is algebraically closed in Theorem 4.7, Theorem 4.13, Theorem 4.5, Proposition 4.15, and Proposition 4.16. However, it is worth noting these results also hold, by the exact same proofs, if we assume instead that \( K \) is an arbitrary field of characteristic zero and that \( \{f_n\} \) is a family of polynomials which all split over \( K \) and satisfy Conditions (a) and (b) of Definition 3.3. The purpose of the assumption that \( K \) is algebraically closed is to guarantee that the polynomials \( \{f_n\} \) defined by Equation 17 split and hence form a uniform family over \( K \); the simple forms of these polynomials will greatly simplify the study of representations in categories of the form \( \text{rep}_K(\bar{Q}, \bar{\mathcal{I}}_f) \) appearing in § 4.2 and the remaining parts of the paper.

### 4.4. Representations of Contracted Quivers

Let \((\bar{Q}, \bar{d}, \bar{m})\) be a generalized double quiver obtained from the double quiver \( Q \) of a Coxeter system \((W, S)\) via a sequence of contractions. Let \( \bar{\mathcal{I}}_f \) be the evaluation ideal of a uniform family \( \{f_n\} \) of polynomials over \( K \). Then the category \( \text{mod-}A \) is equivalent to the category \( \text{rep}_K(\bar{Q}, \bar{\mathcal{I}}_f) \). We develop tools for constructing and analyzing representations in \( \text{rep}_K(\bar{Q}, \bar{\mathcal{I}}_f) \) in this subsection.

Let \( M = (M_a, M_\alpha)_{a \in \bar{Q}, \alpha \in Q_1} \) be a representation in \( \text{rep}_K \bar{Q} \). The definition of \( \bar{\mathcal{I}}_f \) implies that \( M \) is a representation in \( \text{rep}_K(\bar{Q}, \bar{\mathcal{I}}_f) \) if and only if for every arrow of the form \( \alpha : a \to b \) in \( Q \), the set of assignments

\[
M_{\{\alpha, \beta\}} := \{M_a, M_b, M_\alpha, M_\beta\}
\]

where \( \beta = \bar{d}(\alpha) \) satisfies the relations \( r_f(\alpha) \) and \( r_f(\beta) \) in the sense that the maps

\[
f_{m-1}(M_\alpha, M_\beta) := \begin{cases} f_{m-1}(M_\alpha M_\beta) & \text{if } m \text{ is odd;} \\ f_{m-1}(M_\alpha M_\beta)M_\alpha & \text{if } m \text{ is even} \end{cases}
\]

and

\[
f_{m-1}(M_\beta, M_\alpha) := \begin{cases} f_{m-1}(M_\beta M_\alpha) & \text{if } m \text{ is odd;} \\ f_{m-1}(M_\alpha M_\beta)M_\alpha & \text{if } m \text{ is even} \end{cases}
\]

where \( m = m(\alpha) \) both equal 0. Call a set of the form \( M_{\{\alpha, \beta\}} \) a local representation for \( \{\alpha, \beta\} \) if it satisfies the equations (27) and (28). Then to construct a representation \( M \in \text{rep}_K(\bar{Q}, \bar{\mathcal{I}}_f) \) it suffices to assemble a collection of local representations

\[
\mathcal{M} := \{M_{\{\alpha, \beta\}} \mid \alpha \in Q_1, \beta = \bar{\alpha}\}
\]

that is consistent in the sense that for every vertex \( a \in \bar{Q}_0 \), there is a common vector space \( V_a \) such that \( M_a = V_a \) for every local representation \( M_{\alpha, \beta} \in \mathcal{M} \) where \( \alpha \) is incident to \( a \). Here, we assemble a consistent collection \( \mathcal{M} \) into a representation \( M \in \text{rep}_K(\bar{Q}, \bar{\mathcal{I}}_f) \) as follows: first, for each vertex \( a \in \bar{Q}_0 \), pick any arrow \( \alpha \in \bar{Q}_1 \) incident to \( a \), denote \( M_{\{\alpha, \bar{\alpha}\}} \) by \( M' \), then let
Let \( M \in \text{rep}_K \bar{Q} \). Let \( \alpha : a \to b \) be an arrow in \( \bar{Q}_1 \), let \( \beta = \bar{d}(\alpha) \), and let \( m = \bar{m}(\alpha) \). Let \( M_{\{\alpha, \beta\}} = \{M_a, M_b, M_\alpha, M_\beta\} \). Then the following results hold.

(a) If \( m = \infty \), then \( M_{\{\alpha, \beta\}} \) is automatically a local representation for \( \{\alpha, \beta\} \).

(b) If \( M_{\{\alpha, \beta\}} \) is a local representation whenever \( M_\alpha M_\beta \) and \( M_\beta M_\alpha \) are diagonalizable maps whose eigenvalues are roots of \( f_{m-1} \).

Proof. Part (a) follows from Equations (27) and (28). Part (b) follows from the same equations and Lemma 3.12.(a). \( \square \)

The following specializations of Proposition 4.18(a) will be very useful:

Corollary 4.19. Let \( M, \alpha, \beta, a, b, m \) and \( M_{\{\alpha, \beta\}} \) be as in Proposition 4.18. Let \( \{f_n\} \) be the polynomial family defined by (17). Then the following holds for any positive integer \( n \).

(a) If \( M_a = M_b = K^n \) and \( M_\alpha = M_\beta = \text{id} \), then \( M_{\{\alpha, \beta\}} \) defines a local representation for \( \{\alpha, \beta\} \).

(b) Suppose that \( m \geq 5 \) and \( \beta \neq \alpha \). If \( M_a = M_b = K^n \) and \( (M_\alpha M_\beta)^2 = (M_\beta M_\alpha)^2 = \text{id} \), then \( M_{\{\alpha, \beta\}} \) defines a local representation for \( \{\alpha, \beta\} \).

Proof. The results follow from Proposition 4.18(a), the fact that 1 is a root for \( f_{m-1} \), and the fact that \( x^2 - 1 \) divides \( f_{m-1} \) whenever \( m \geq 5 \). \( \square \)

To prove results on mod-\( A \) in Section \S \ 5 we will often need to not only construct a suitable representation \( M \) in \( \text{rep}_K (\bar{Q}, \bar{R}) \) but also prove that \( M \) is simple or not semisimple. The proofs typically proceed in the following way. Consider a specific vertex \( a \in Q_0 \) and a number of paths \( p_1, p_2, \ldots, p_k \) on \( Q \) that both start and end at \( a \). Recall from \S \ 2.4 that these paths give rise to endomorphisms \( \phi_1 := M_{p_1}, \phi_2 := M_{p_2}, \ldots \) and \( \phi_k := M_{p_k} \) of \( M_a \), and that a subrepresentation \( N \) of \( M \) must assign to \( a \) a vector space \( N_a \subseteq M_a \) that satisfies the invariance condition \( \phi_i(N_a) \subseteq N_a \) for all \( 1 \leq i \leq k \). Together, these invariance conditions force \( N_a \) to take certain forms, which in turn
force $M$ to satisfy certain properties such as being simple. We refer to the analysis of what form $N_a$ can take as *subspace analysis at* $a$.

We now explain how we will construct representations to facilitate successful subspace analysis via examples. All the examples will be used in the proofs of Section 3 and $\{f_n\}$ stands for the uniform family of polynomials defined by Equation (17) throughout the examples. Our first method starts with an irreducible representation $\rho : G \to \text{GL}(V)$ of a group $G$:

**Example 4.20.** We construct a simple representation $M$ in $\text{rep}_K(Q, \mathcal{I}_f)$ for the generalized double quiver $\bar{Q}$ in Figure 5 from Remark 4.14. To start, consider the symmetric group $G = S_q$ where $q \geq 8$ and any irreducible representation $\rho : G \to \text{GL}(V)$ of $G$. By [Mil01], the group $S_q$ can be generated by two elements $\sigma, \tau$ of orders 2 and 3, respectively. It follows that $\rho(\sigma)^2 = \rho(\sigma^2) = \rho(e) = \text{id}_V$ and similarly $\rho(\tau)^3 = \text{id}_V$.

To define $M$, first let $M_\sigma = V, M_\tau = \rho(\sigma)$ and $M_{e_2} = \rho(\tau)$. This defines local representations for the sets $\{\varepsilon_1\}$ and $\{\varepsilon_2\}$ because $M_\varepsilon_1, M_\varepsilon_2$ satisfy the relations $r_f(\varepsilon_1) = \varepsilon_1^2 - v, r_f(\varepsilon_2) = \varepsilon_2^3 - v$, respectively. To finish the definition of $M$, it remains to assign a local representation for the dual arrows $\{\alpha, \beta\}$ that is consistent with these two local representations. By Corollary 4.19, it suffices to define $M_\alpha = V$ and $M_\beta = \text{id}_V$.

Let $N$ be a subrepresentation of $M$. Since the representation $V$ is irreducible and $\sigma, \tau$ generate $G$, the only subspaces of $M_\sigma$ that are invariant under both $M_\sigma = \rho(\sigma)$ and $M_\tau = \rho(\tau)$ are 0 and $M_\sigma$ itself, therefore we have $N_\sigma = 0$ or $N_\sigma = M_\sigma$. Since $M_\alpha, M_\beta$ are isomorphisms, in these two cases we must have $N_\sigma = 0$ or $N = M$, respectively, therefore $M$ is simple.

The endomorphisms $\phi_1, \ldots, \phi_k$ of $M_\sigma$ mentioned above are often all diagonalizable in our examples. This makes the following well-known fact from linear algebra very useful for subspace analysis: let $V$ be a vector space and let $\phi$ be a diagonalizable endomorphism of $V$. Suppose $\phi$ has $d$ distinct eigenvalues and $V = \oplus_{i=1}^d E_i$ is the corresponding eigenspace decomposition. Then a subspace $W$ of $V$ is invariant under $\phi$ if and only if $W$ is *compatible with the eigenspace decomposition* in the sense that $W = \oplus_{i=1}^d (W \cap E_i)$. We use this characterization in the following two examples.

**Example 4.21.** Let $(W, S)$ be the Coxeter system whose Coxeter diagram $G$ is shown on the left of Figure 7, where $m_1, m_2 \in \mathbb{Z}_{\geq 4} \cup \{\infty\}$. The double quiver $Q$ of $G$ is shown on the right of the same figure. We construct a representation $M$ in $\text{rep}_K(Q, \mathcal{I}_f)$ and apply subspace analysis at $b$ to show that $\text{rep}_K(Q, \mathcal{R})$, hence mod-$A$, is not semisimple.

$$G : \begin{array}{ccc} a & \overset{m_1}{\longrightarrow} & b & \overset{m_2}{\longrightarrow} & c \\ \end{array} \quad \rightarrow \quad Q : \begin{array}{ccc} a & \overset{\alpha}{\longleftarrow} & b & \overset{\gamma}{\longleftarrow} & c \\ \end{array}$$

**Figure 7.**

We begin by constructing the local representations $M_{\{\alpha, \beta\}}$ and $M_{\{\gamma, \delta\}}$ based on the values of $m_1$ and $m_2$, respectively. For $\{\alpha, \beta\}$, first let $M_\alpha = \text{id}_V =$...
$M_b = K^2$. Let $\lambda_2 = 1$. Take $\lambda_1 = -1$ if $m_1 = \infty$, $\lambda = 0$ if $m_1 = 4$, and $\lambda = z$, a root of $\hat{f}_{m-1}$ different from $\lambda_2$, if $4 < m_1 < \infty$. Next, set $M_\beta$ to be the map given by the matrix

$$
\begin{bmatrix}
0 & 0 \\
\lambda_1 & 1 \\
0 & \lambda_2
\end{bmatrix}
$$

if $m_1 = 4$ and to be the identity map otherwise, then set $M_\alpha = \lambda_1 1 0 
\begin{bmatrix}
0 & 0 \\
\lambda_1 & 1 \\
0 & \lambda_2
\end{bmatrix}$. It is straightforward to check that $M_{\{\alpha, \beta\}} := \{M_\alpha, M_b, M_\alpha, M_\beta\}$ forms a local representation: if $m_1 = \infty$, then the relations $r_f(\alpha), r_f(\beta)$ are zero and there is nothing to check; if $4 < m_1 < \infty$, then the relations $r_f(\alpha), r_f(\beta)$ are satisfied by Proposition 4.18 because $M_\alpha M_\beta = M_\beta M_\alpha = M_\alpha$, a diagonalizable map whose eigenvalues are roots of $\hat{f}_{m-1}$; finally, if $m_1 = 4$, then the relations $r_f(\alpha) = \alpha \beta \alpha - \alpha, r_f(\beta) = \beta \alpha \beta - \beta$ are satisfied because $M_\alpha M_\beta M_\alpha = M_\alpha, M_\beta M_\alpha M_\beta = M_\beta$ by direct computation. We may similarly define a local representation for $\{\gamma, \delta\}$ by letting $M_b = M_\gamma = K^2$, defining numbers $\mu_2, \mu_1$ and the map $M_\beta$ based on $m_2$ in the same way we defined $\lambda_2, \lambda_1$ and $M_\beta$ based on $m_1$, and defining the map $M_\delta$ to be given by the matrix $\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$. The two local representations are consistent because they assign the same vector space $K^2$ to the vertex $b$, therefore they can be assembled to a representation $M \in \text{rep}_K(Q, \mathcal{I}_f)$.

Consider the operators $\phi_1 := M_\alpha \circ M_\beta$ and $\phi_2 := M_\delta \circ M_\gamma$ on $M_b$. Then

$$
\phi_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},
$$

so the eigenspace decompositions of $M_b$ with respect to $\phi_1, \phi_2$ are given by

$$
M_b = \langle e_1 \rangle \oplus \langle e_1 + (\lambda_2 - \lambda_1)e_2 \rangle = \langle e_1 \rangle \oplus \langle e_2 \rangle
$$

where $\langle v \rangle$ stands for the span of $v$ for each vector $v$ and $e_1, e_2$ denote the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of $K^2$, respectively. Let $N$ be a nonzero subrepresentation of $M$. Then the vector space $N_b$ must be invariant under both $\phi_1$ and $\phi_2$. Consequently, $N_b$ must be compatible with the decompositions in Equation (29) in the sense that

$$
N_b = (N_b \cap \langle e_1 \rangle) \oplus (N_b \cap \langle e_2 \rangle) = (N_b \cap \langle e_1 \rangle) \oplus (N_b \cap \langle e_1 + (\lambda_2 - \lambda_1)e_2 \rangle).
$$

But each intersection in the above equation is either trivial or of dimension 1, so $N_b$ must be $\langle e_1 \rangle$. This implies that $N$ cannot have a complement in $M$, so $\text{rep}_K(Q, \mathcal{I}_f)$ is not semisimple.

**Example 4.22.** Let $Q$ and $M$ be as in the previous example. We modify $M$ to produce an infinite family of simple representations in $\text{rep}_K(Q, \mathcal{I}_f)$ to be used in Section 5. To begin, define a representation $M^x$ for each scalar $x \in K \setminus \{\lambda_1, \lambda_2\}$ as follows: let $M^x_\alpha = M^x_b = M^x_\gamma = M^x = K^2, M^{x}_\gamma = M_\gamma$ and $M^{x}_\delta = M_\delta$, then let $M^x_\beta = \text{id}$ and let $M^x_\alpha$ be the map given by the matrix

$$
B_x = \begin{bmatrix} x & x(\lambda_1 + \lambda_2 - x) - \lambda_1 \lambda_2 \\ 1 & \lambda_1 + \lambda_2 - x \end{bmatrix}.
$$
As before, this defines local representations $M_{\{\gamma, \delta\}} := \{M_{h}^{x}, M_{e}^{x}, M_{i}^{x}, M_{d}^{x}\}$ and $M_{\{\alpha, \beta\}} := \{M_{a}^{x}, M_{b}^{x}, M_{c}^{x}, M_{d}^{x}\}$ for $\{\gamma, \delta\}$ and $\{\alpha, \beta\}$, respectively. Moreover, the two local representations are consistent and assemble to a representation $M^{x}$. Let $\phi_{1}^{x} := M_{a}^{x} \circ M_{b}^{x}$ and $\phi_{2}^{x} := M_{d}^{x} \circ M_{d}^{x}$. The matrix $B_{x}$ guarantees that the map $\phi_{1}^{x}$ has an eigenvector $v_{1} := (x - \lambda_{1})e_{1} + e_{2}$ with eigenvalue $\lambda_{1}$ as well as an eigenvector $v_{2} := (x - \lambda_{2})e_{1} + e_{2}$ with eigenvalue $\lambda_{2}$, so the eigenspace decompositions of $M_{b}^{x}$ with respect to $\phi_{1}^{x}$ and $\phi_{2}^{x}$ are given by

$$M_{b}^{x} = \langle v_{1} \rangle \oplus \langle v_{2} \rangle = \langle e_{1} \rangle \oplus \langle e_{2} \rangle.$$  

It follows that in any subrepresentation $N$ of $M^{x}$ we must have

$$N_{b} = N_{b} \cap \langle v_{1} \rangle + N_{b} \cap \langle v_{2} \rangle = N_{b} \cap \langle e_{1} \rangle + N_{b} \cap \langle e_{2} \rangle.$$  

The vectors $v_{1}, v_{2}, e_{1}, e_{2}$ are pairwise distinct since $\lambda_{1} \neq \lambda_{2}, \mu_{1} \neq \mu_{2}$ and $x \notin \{\lambda_{1}, \lambda_{2}\}$, therefore the above equation holds only if $N_{b} = 0$ or $N_{b} = K^{2}$. If $m_{2} \neq 4$, then $\mu_{1} \neq 0$ and $M_{b}^{x}$ is an isomorphism, therefore we must have $N = 0$ or $N = M^{x}$, which in turn implies that $M^{x}$ is simple. If $m_{2} = 4$, it is easy to check that $M^{x}$ contains a simple module $N^{x}$ such that $N_{a}^{x} = N_{b}^{x} = K^{2}$ and $N_{c}^{x} = \langle e_{2} \rangle$. Set $N^{x} = M^{x}$ when $m_{2} \neq 4$. Then $N^{x}$ is a simple representation in $\text{rep}_{K}(Q, I_{f})$ for all $x \in K \setminus \{\lambda_{1}, \lambda_{2}\}$ regardless of the value of $m_{2}$.

Observe that if $x \neq y$ then $N^{x} \not\cong N^{y}$, for there cannot exist linear isomorphisms $\phi_{a} : N_{a}^{x} \rightarrow N_{a}^{y}, \phi_{b} : N_{b}^{x} \rightarrow N_{b}^{y}$ such that $\phi_{b}N_{a}^{x} = N_{b}^{y}\phi_{a}$ and $\phi_{a}N_{\beta}^{x} = N_{\beta}^{y}\phi_{b}$ simultaneously: the latter equation holds only if $\phi_{a} = \phi_{b}$ as maps from $K^{2}$ to $K^{2}$ because $N_{a}^{x} = N_{b}^{y} = \text{id}$, but then the first equation implies that the matrices $B_{x}, B_{y}$ are equal, which cannot happen if $x \neq y$.

In the next example, we combine the ideas of the last three examples to construct simple representations in a more flexible way.

**Example 4.23.** Let $(W, S), G$, and $Q$ be as in Example 4.21. We construct a representation $M \in \text{rep}_{K}(Q, I)$ in the case that $m_{1} \geq 4, m_{2} \geq 6$ and prove that $M$ contains a simple subrepresentation $N$.

Consider the irreducible representation $\rho : S_{q} \rightarrow \text{GL}(V)$ and the elements $\sigma, \tau \in S_{q}$ from Example 4.20. Using minimal polynomials as we did in the proof of Lemma 3.12 we may deduce from the fact $\rho(\sigma)^{2} = \text{id}_{V}$ that $\rho(\sigma)$ is a diagonalizable map whose eigenvalues are from the set $\{-1, 1\}$, so with respect to $\rho(\sigma)$ we have an eigenspace decomposition $V = E_{1} \oplus E_{2}$ where $E_{1}, E_{2}$ are the eigenspaces for the eigenvalues $1$ and $-1$, respectively. Similarly, since $\rho(\tau)^{3} = \text{id}_{V}$, the map $\rho(\tau)$ is diagonalizable and its eigenvalues lie in the set $\{\omega_{1}, \omega_{2}, \omega_{3}\}$ containing the three third roots of unity, therefore we have an eigenspace decomposition $V = F_{1} \oplus F_{2} \oplus F_{3}$ of $V$ where each $F_{i}$ is the eigenspace for the eigenvalue $\omega_{i}$. Note that $0$ and $V$ are the only subspaces of $V$ compatible with both these decompositions, because they are the only subspaces invariant under both $\rho(\sigma)$ and $\rho(\tau)$ by Example 4.20.

To define $M$, first set $M_{a} = M_{b} = M_{c} = V$. Next, assign the linear maps $M_{a}, M_{\beta}$ based on the value of $m_{1}$: if $m_{1} > 4$, then $\tilde{f}_{m_{1}-1}$ contains at least
two nonzero roots $\lambda_1, \lambda_2$, and we set

$$M_\alpha = \text{id}_{E_1} \oplus \text{id}_{E_2}, \quad M_\beta = (\lambda_1 \cdot \text{id}_{E_2}) \oplus (\lambda_2 \cdot \text{id}_{E_2})$$

where the notation means, for example, that $M_\beta$ restricts to $\lambda_1 \cdot \text{id}$ on $E_1$ and to $\lambda_2 \cdot \text{id}$ on $E_2$; if $m_1 = 4$, we set

$$M_\alpha = \text{id}_{E_1} \oplus 0_{E_2}, \quad M_\beta = (\lambda_1 \cdot \text{id}_{E_2}) \oplus 0_{E_2}$$

where $\lambda_1$ is the unique nonzero root of $\tilde{f}_3$. Similarly, if $m_2 > 6$ then $\tilde{f}_{m_2 - 1}$ has at least three nonzero roots $\mu_1, \mu_2, \mu_3$ and we set

$$M_\gamma = \text{id}_{F_1} \oplus \text{id}_{F_2} \oplus \text{id}_{F_3}, \quad M_\delta = (\mu_1 \cdot \text{id}_{F_1}) \oplus (\mu_2 \cdot \text{id}_{F_2}) \oplus (\mu_3 \cdot \text{id}_{F_3}),$$

while if $m_2 = 6$ then we set

$$M_\gamma = \text{id}_{F_1} \oplus \text{id}_{F_2} \oplus 0_{F_3}, \quad M_\delta = (\mu_1 \cdot \text{id}_{F_1}) \oplus (\mu_2 \cdot \text{id}_{F_2}) \oplus 0_{F_3}.$$

where $\mu_1, \mu_2$ are the two nonzero roots of $\tilde{f}_{m_2 - 1}$. By Parts (a) and (b) of Proposition 4.13 the assignments define a representation in $\text{rep}_K(Q, \mathcal{I}_f)$. Moreover, the eigenspace decompositions of $M_b$ with respect to the maps $\phi_1 = M_b \phi_3$ and $\phi_2 = M_b \phi_3$ coincide with the eigenspace decompositions of $V$ with respect to the maps $\rho(\sigma)$ and $\rho(\tau)$, respectively, therefore a subrepresentation of $M$ must assign the space 0 or $V$ to the vertex $b$. It follows that $M$ contains a simple representation $N$ with $N_b = V$ and

$$N_a = \begin{cases} 
M_a = E_1 \oplus E_2 & \text{if } m_1 > 4; \\
E_1 & \text{if } m_1 = 4, \\
E_1 \oplus F_1 & \text{if } m_1 = 6.
\end{cases}$$

$$N_c = \begin{cases} 
M_c = F_1 \oplus F_2 \oplus F_3 & \text{if } m_1 > 6; \\
F_1 \oplus F_2 & \text{if } m_1 = 6.
\end{cases}$$

**Remark 4.24.** Given a vector space $V$ and operators $\phi_1, \ldots, \phi_k$ on $V$, the study of subspaces of $V$ that are simultaneously compatible with the eigenspace decompositions of all the operators is closely related to enumerative geometry and Schubert calculus. For example, if $K = \mathbb{C}, V = K^4, n = 2$ and the operators $\phi_1, \phi_2$ yield eigenspace decompositions $M_b = E_1 \oplus E_2$ and $M_b = F_1 \oplus F_2$ where $E_1, E_2, F_1, F_2$ all have dimension 2, then “generically” there exists a subspace $W \subseteq V$ with dimension 2 that is simultaneously compatible with both $\phi_1$ and $\phi_2$, because a classical result in Schubert calculus asserts that generically, given four lines in the projective 3-space $\mathbb{P}^3$, there are two lines that intersect all these four lines. On the other hand, when the dimensions of the eigenspaces of $V$ with respect to $\phi_1, \ldots, \phi_k$ are known, we can often show that no proper, nontrivial subspace of $V$ can be simultaneously compatible with the corresponding eigenspace decompositions by certain codimension computations involving products of Schubert classes. For instance, to construct a simple representation in $\text{rep}_K(Q, \mathcal{I}_f)$ in Example 4.20, it is possible to specify for every positive integer $n$ a representation $M \in \text{rep}_K(Q, \mathcal{I}_f)$ in such a way that $\dim(M_b) = 6n$, the maps $M_\alpha$ and $M_\beta$ are isomorphisms, $\phi_1 := M_{e_1}$ is diagonalizable with two eigenspaces $E_1, E_2$ of dimension $3n$, and $\phi_2 := M_{e_2}$ is diagonalizable with three eigenspaces $F_1, F_2, F_3$ of dimension $2n$. A codimension computation using Schubert calculus guarantees that generically $M_b$ has no subspace compatible with both
$M_α$ and $M_β$ other than 0 and $M_β$ itself, so $M$ is simple (generically). This yields an alternative proof of the existence of certain simple representations in $\text{rep}_K(\bar{Q}, \bar{R})$. In the interest of space, however, we omit details of the codimension computation and the necessary background on Schubert calculus. In particular, we will not make precise what the word “generically” means in this paragraph.

We end this subsection by proving a proposition to be used in §5.2 under the following setting: Let $Q$ be the double quiver of a Coxeter diagram $G$ and let $\{f_\alpha\}$ be as defined in Equation 17. Suppose that $G$ contains a vertex $v$ which is adjacent to a unique vertex in $G$. Let $u$ be that unique vertex, let $m = m(u,v)$, and let $\hat{G}$ be the graph obtained from $G$ by removing the vertex $v$ and the edge $v - u$. Let $\hat{Q}$ be the double quiver of $\hat{G}$, then let $\hat{I}_f$ be the evaluation ideal of $\{f_\alpha\}$ in $K\hat{Q}$. The proposition allows us to “enlarge” certain representations in $\text{rep}_K(\hat{Q}, \hat{I}_f)$ to a simple representation in $\text{rep}_K(\bar{Q}, \bar{I}_f)$:

**Proposition 4.25.** Let $Q, R, \bar{Q}, \bar{R}$ and $u, v$ be as described above. Suppose that $\{S(1), S(2), \ldots, S(k)\}$ is a nonempty set of pairwise non-isomorphic simple representations in $\text{rep}_K(\bar{Q}, \bar{R})$ and let $S = \bigoplus_{i=1}^k S(i)$. If $m > 3$, then there is a simple representation $M$ in $\text{rep}_K(Q, I_f)$ such that $M_a = S_a$ for all $a \in \hat{Q}_0$.

We prove the proposition via subspace analysis at the vertex $u$, by using the following lemma:

**Lemma 4.26.** Let $Λ$ be an arbitrary ring. Let $\{S(1), \ldots, S(k)\}$ be a set of pairwise non-isomorphic simple right $Λ$-modules, let $S = \bigoplus_{i=1}^k S(i)$, and let $x = \sum_{1 \leq i \leq k} x_i \in S$ where $x_i \in S(i)$ for each $i$. If $x_i \neq 0$ for all $1 \leq i \leq k$, then the submodule generated by $x$ equals $S$.

**Proof.** We use induction on $k$. The claim clearly holds when $k = 1$. For $k > 1$, let $I_i = \text{Ann}(x_i) := \{r \in Λ : x_ir = 0\}$ for each $i$. Then $I_1, \ldots, I_k$ are distinct maximal right ideals of $Λ$ since $S(1), \ldots, S(k)$ are pairwise non-isomorphic simple right $Λ$-modules. Let $r \in I_1 \setminus I_2$ and let $y_i = x_ir$ for all $i$. Then $y_1 = 0$ and $y_2 \neq 0$. Let $J = \{1 \leq j \leq k : y_j \neq 0\}$ and let $y = \sum_{j \in J} y_j$. Applying the inductive hypothesis on the module $S' := \bigoplus_{j \in J} S(j)$, we conclude that $S' \subseteq yΛ$. Furthermore, since $y = \sum_{j \in J} y_j = \sum_{1 \leq j \leq k} y_j = \sum_{1 \leq j \leq k} x_j r = x r$, we have $yΛ \subseteq xΛ$. It follows that $S'' \subseteq xΛ$. In particular, we have $\sum_{j \in J} x_j \in xΛ$ and hence $\sum_{j \notin J} x_j \in xΛ$. By the inductive hypothesis, the element $\sum_{j \notin J} x_j$ generates the module $S'' := \bigoplus_{j \notin J} S(j)$, so $S'' \subseteq xΛ$. It follows that $S = S' \oplus S'' \subseteq xΛ$. □

**Proof of Proposition 4.25.** Denote the arrows $u \rightarrow v$ and $v \rightarrow u$ in $Q$ by $α$ and $β$, respectively. By §4.4 to construct a representation $M \in \text{rep}_K(Q, I_f)$ it suffices to extend $S$ by a local representation $M_{\{α, β\}} = \{M_u, M_v, M_α, M_β\}$
for the set \( \{ \alpha, \beta \} \) such that \( M_u = S_u \). We do so by setting \( M_u = S_u, M_v = K \) and setting \( M_\alpha, M_\beta \) as follows:

(a) If \( m = 4 \), let \( d = \text{dim}(S_u) \) and let \( B_S = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k \) be a basis of \( S_u \) where \( B_i \) is a basis of \( S(i)_u \) for all \( 1 \leq i \leq k \). Consider the \( 1 \times d \) and \( d \times 1 \) matrices

\[
X = [1 \ 1 \ldots \ 1], \quad Y = [d \ -1 \ldots \ -1]^T.
\]

Define \( M_\alpha : S_u \to K \) and \( M_\beta : K \to S_u \) to be the maps whose matrices with respect to \( B_S \) and \{1\} (considered a basis of \( K \)) are given by \( X \) and \( Y \), respectively. Since \( M_\alpha M_\beta = \text{id} \), the assignments for \( M_\alpha \) and \( M_\beta \) satisfy the relations \( r_f(\alpha) = \alpha \beta \alpha - \alpha \) and \( r_f(\beta) = \beta \alpha \beta - \beta \), so \( M \) is indeed a representation in \( \text{rep}_K(Q, I_f) \).

(b) If \( m \geq 5 \), then let \( d \) and \( X, Y \) be as before, set \( M_\alpha = \text{id} \), and define \( M_\beta \) to be the whose matrix with respect to \( B_S \) is \( I_d - 2YX \). These assignments ensure that \( (M_\alpha M_\beta)^2 = (M_\beta M_\alpha)^2 = M_\beta^2 = \text{id} \), so they define a representation in \( \text{rep}_K(Q, I_f) \) by Lemma 4.19(b).

The representation \( M \) satisfies the condition that \( M_a = S_a \) for all \( a \in \hat{Q}_0 \) by definition, so it remains to show that \( M \) is simple. To this end, let \( N \) be a subrepresentation of \( M \) in \( \text{rep}_C(Q, I) \), and let \( x = (x_i)_{1 \leq i \leq k} \) be any nonzero vector in \( S_u = \bigoplus_{i=1}^k S(i)_u \). Then the set \( J := \{ 1 \leq j \leq n : x_j \neq 0 \} \) is nonempty. Invoke the equivalence between the categories \( \text{rep}_K(Q, I_f) \) and \( \text{mod-}K\hat{Q}/\hat{I}_f \) to identify \( S(1), \ldots, S(k) \) and \( S \) as modules of the algebra \( K\hat{Q}/\hat{I}_f \). Then Lemma 4.26 implies that the submodule of generated by \( x \) in \( S \) must contain the direct sum \( \bigoplus_{j \in J} S(j) \). Invoking the same equivalence again, we conclude that \( N_u \) contains a basis vector \( e \) from the basis \( B_j \) of \( S(j) \) for some \( j \in J \). By direct computation, the element \( x' = M_\beta M_\alpha(e) \in M_u \) must have nonzero entries at all coordinates, therefore \( x' \) generates all of \( M_u \) in \( N \), i.e., we have \( N_u = M_u \), by Lemma 4.26. Since \( S(1), \ldots, S(k) \) are simple and \( M_\alpha \) is surjective, it follows that \( M \) is simple.

\[ \square \]

5. Results on \text{mod-}A

We maintain the setting of Section 4 and study the category \( \text{mod-}A \) in this section. Recall that \( A = A_K = K \otimes_\mathbb{Z} J_C \) where \( K \) is an algebraically closed field with characteristic zero and \( J_C \) is an irreducible Coxeter system \((W,S)\) with Coxeter diagram \( G \) and subregular cell \( C \).

5.1. Results. Our first main result characterizes in terms of the Coxeter diagram \( G \) when \( \text{mod-}A \) is semisimple, as well as when \( \text{mod-}A \) has finitely many simples, i.e., when it contains finitely many simple modules up to isomorphism.

**Theorem 5.1.** The following conditions are equivalent:

(a) The graph \( G \) is a tree, has no edge with infinite weight, and has at most one heavy edge.

(b) The category \( \text{mod-}A \) is semisimple.
(c) The category $\text{mod-}A$ has finitely many simples.

Our second main result gives a similar characterization of when $\text{mod-}A$ has bounded simples in the sense that there exists an upper bound on the dimensions of the simple modules of $\text{mod-}A$. Since the simple modules of $\text{mod-}A$ are certainly bounded if there are only finitely many of them, we start with the assumption that the conditions of Theorem 5.1 do not hold:

**Theorem 5.2.** Suppose that $G$ contains a cycle or has an edge with infinite weight or has at least two heavy edges. Then the dimensions of the simple modules of $\text{mod-}A$ are bounded above if and only if one of the following mutually exclusive conditions holds:

(a) $G$ contains a unique cycle, and all edges in $G$ are simple.

(b) $G$ is a tree and contains exactly two heavy edges; moreover, each of those two edges has weight 4 or weight 5.

Here and henceforth, a cycle in a graph means a tuple $C = (v_1, v_2, \ldots, v_n)$ of $n \geq 3$ vertices in the graph such that $v_1 - v_2, v_2 - v_3, \ldots, v_n - v_1$ are all edges in $G$.

Note that Theorems 5.1 and 5.2 has the following consequence:

**Remark 5.3.** Recall from Example 4.12 that for every tuple $k = (k_1, \ldots, k_n)$ of positive integers, the algebra $A_k = \langle x_j : 1 \leq j \leq n, x_j^{k_j} = 1 \rangle$ is isomorphic to the group algebra of the free product $C_k = C_{k_1} \ast \cdots \ast C_{k_n}$ where each $C_{k_i}$ is the cyclic group of order $k_i$. Note that if $k_j = 1$ for some $j$ then $C_{k_j}$ is the trivial group and makes trivial contribution to the free product in the sense that $C_k \cong C_{k_1} \ast \cdots \ast C_{k_{j-1}} \ast C_{k_{j+1}} \ast \cdots \ast C_{k_n}$, so we assume from now on that $k_j > 1$ for all $1 \leq j \leq n$. In the example, we showed that $A_k$ is Morita equivalent to the algebra $A$ associated with a Coxeter system whose Coxeter diagram is a tree and has a heavy edge of weight $m_j := 2k_j + 1$ for each $1 \leq j \leq n$; in particular, under the assumption that $k_j > 1$ for all $j$, the weight $m_j$ is an odd number greater than 3, and we have $m_j = 5$ if and only if $k_j = 2$. Theorems 5.1 and 5.2 now imply the following result:

**Proposition 5.4.** Suppose $k = (k_1, \ldots, k_n)$ where $k_i \in \mathbb{Z}_{>1}$ for all $1 \leq i \leq n$, and let $\text{mod-}A_k$ be the category of finite dimensional right modules of $A_k$.

(a) The category $\text{mod-}A_k$ is semisimple if and only it contains finitely many isomorphism classes of simple modules. Moreover, these two conditions are satisfied if and only if $n = 1$, i.e., if and only if $C_k$ has a single factor and is a finite cyclic group.

(b) Suppose the category $\text{mod-}A$ has infinitely many pairwise non-isomorphic simple modules. Then the simple modules of $\text{mod-}A_k$ have bounded dimensions if and only if $k = (2, 2)$, i.e., if and only if $C_k$ is isomorphic to the free product $C_2 \ast C_2$.

Let us explain our strategy for proving Theorems 5.1 and 5.2. For Theorem 5.1 first recall that (a) implies (b) by Proposition 4.9, thanks to simple graph contractions. It is well-known that Condition (a) is equivalent to the
condition that the cell $C$ is finite (see [Lus83]), therefore (a) also implies (c), since $\dim(A) = |C|$ and a finite dimensional semisimple algebra has finitely many simple modules. To prove the theorem, it remains to prove that (b) implies (a) and that (c) implies (a). We will prove the contrapositives of these two implications:

**Proposition 5.5.** If $G$ contains a cycle or has an edge with infinite weight or has at least two heavy edges, then $\text{mod-}A$ contains a module which is not semisimple.

**Proposition 5.6.** If $G$ contains a cycle or has an edge with infinite weight or has at least two heavy edges, then $\text{mod-}A$ contains an infinite set of pairwise non-isomorphic simple modules.

To prove Theorem 5.2 we will first prove the “if” implication:

**Proposition 5.7.** If $G$ satisfies either Condition (a) or Condition (b) in Theorem 5.2, then $\text{mod-}A$ has bounded simples.

To prove the “only if” implication of Theorem 5.2, we again prove its contrapositive. Doing so requires describing the situations where Conditions (a) and (b) in the theorem fail under the assumption that $G$ is not a tree, has an edge with infinite weight, or has at least two heavy edges. A moment’s thought reveals that we may formulate the contrapositive as follows:

**Proposition 5.8.** The dimensions of the simple modules in $\text{mod-}A$ have no upper bound if $G$ satisfies one of the following conditions:

(a) $G$ contains a unique cycle as well as a heavy edge;
(b) $G$ contains at least two cycles;
(c) $G$ is a tree and has exactly two heavy edges; moreover, one of these heavy edges has weight at least 6;
(d) $G$ is a tree and has at least three heavy edges.

We have reduced the proofs of Theorems 5.1 and 5.2 to the proofs of Propositions 5.5, 5.6, 5.7, and 5.8, which we will give in § 5.2. Note that all these theorems and propositions are stated without any reference to any quivers. On the other hand, the proofs in § 5.2 will all use quiver representations and rely heavily on the techniques and examples of Section 4. It is also worth noting that part of the proof of Proposition 5.8 will use Proposition 5.6 to obtain desired simple representations for the former proposition, we will sometimes form direct sums of simple representations promised by latter proposition and then “enlarge” the direct sums using Proposition 4.25.

### 5.2. Proofs

Let $Q$ be the double quiver of $G$, let $\{f_n\}$ be the uniform family of polynomials defined by Equation (17), and let $I_f$ be the evaluation ideal of $\{f_n\}$ in $KQ$. We prove Propositions 5.5, 5.6, 5.7, and 5.8 by proving the same conclusions for the equivalent category $\text{rep}_K(Q, I_f)$ in this subsection. Of the four propositions, we first prove Proposition 5.7. The other three propositions all state
that mod-$A$ contains modules with certain properties, so we will prove them by explicit construction of suitable representations in $\text{rep}_K(Q, I_f)$. Since the properties of mod-$A$ that we are interested in, namely, being semisimple, having finitely many simples, and having bounded simples, are all preserved under Morita equivalences, when dealing with $\text{rep}_K(Q, I_f)$ we may assume that certain contractions have been performed on $Q$ and thus effectively deal with a category of the form $\text{rep}_K(\bar{Q}, I_f)$ from § 4.4. For instance, by Example 4.8 and Remark 4.14, if the Coxeter diagram $G$ is the tree from Figure 2 then we may study mod-$A$ via the category $\text{rep}_K(\bar{Q}, I_f)$ for the generalized double quiver $\bar{Q}$ from Figure 5.

As final preparation for our proofs, we fix some notation and terminology for cycles in the Coxeter diagram $G$. Given a cycle $C = (v_1, v_2, \ldots, v_n)$ with $n \geq 3$ vertices in $G$, we say $C$ has length $n$, set $v_{n+1} := v_1$, define $V_C = \{v_1, \ldots, v_n\}$, and define

$$E_C = \{v_i - v_{i+1} : 1 \leq i \leq n\}.$$ 

We call the edges in $E_C$ the sides of $C$ and define a diagonal in $C$ to be a edge in $G$ that connects two vertices in $C$ but does not lie in $E_C$. We say $C$ is a minimal cycle in $G$ if $C$ has no diagonals (a diagonal in $C$ would break $C$ into two shorter cycles). For each $1 \leq i \leq n$, we let $m_{C,i} = m(v_i, v_{i+1})$ and denote the arrows $v_i \rightarrow v_{i+1}$ and $v_{i+1} \rightarrow v_i$ in the double $Q$ of $G$ by $\alpha(C, i)$ and $\beta(C, i)$, respectively. For a representation $M \in \text{rep}_K Q$, we let

$$M_C := M_{\alpha(C,n)} \circ \cdots \circ M_{\alpha(C,2)} \circ M_{\alpha(C,1)},$$

$$M_C := M_{\beta(C,1)} \circ M_{\beta(C,2)} \circ \cdots \circ M_{\beta(C,n)}.$$ 

If it is clear what $C$ is from context, then we omit $C$ and write $m_i, \alpha_i, \beta_i, M_i$ and $M_i$ for $m_{C,i}, \alpha(C, i), \beta(C, i), M_{\alpha(C, i)}$ and $M_{\beta(C, i)}$, respectively.

**Proof of Proposition 5.7.** It suffices to show that mod-$A$ or $\text{rep}_K(Q, I_f)$ has bounded simples if $G$ satisfies one of the conditions in Theorem 5.2. To do so, first assume that $G$ satisfies Condition (a), i.e., that $G$ contains a unique cycle and has only simple edges. Let $C$ be the unique cycle. Then $C$ is necessarily minimal. By applying simple graph contractions if necessary, we may assume that $G$ is exactly $C$ in the sense that $V_C$ contains all the vertices of $G$ and $E_C$ contains all the edge of $G$. But then the algebra $A$ is Morita equivalent to the Laurent polynomial ring $A = K[t, t^{-1}]$ by Example 4.10. As $A$ is commutative, every simple module of $A$ has dimension 1, so mod-$A$ has bounded simples.

Next, suppose that $G$ satisfies Condition (b), i.e., that $G$ is a tree with exactly two heavy edges and the edges have weights $m_1, m_2 \in \{4, 5\}$. Applying simple graph contractions on $G$ if necessary, we may assume that $G$ and $Q$ are as pictured in Figure 7. Let $M \in \text{rep}_K(Q, I_f)$. Let $\alpha, \beta, \gamma, \delta$ be as in Figure 7 and let $\phi_1 = M_\alpha M_\beta, \phi_2 = M_\delta M_\gamma$. Let $i \in \{1, 2\}$. Then $\phi_i$ are diagonalizable by Proposition 4.18(c). Since $f_3 = x^3 - x$ and $f_4 = x^4 - 1$, it follows that if $m_i = 4$, then $\phi_i^4 = \phi_i$ and the eigenvalues of $\phi_i$ lie in the
set \{0, 1\}; if \(m_i = 5\), then \(\phi_i^2 = \text{id}\) and the eigenvalues of \(\phi_i\) lie in the set \{-1, 1\}. Now let \(n = \dim(M_b)\) and suppose the eigenspace decomposition of \(M_b\) relative to \(\phi_1\) and \(\phi_2\) are \(M_b = E_1 \oplus E_2\) and \(M_b = F_1 \oplus F_2\), respectively, with \(E_1, F_1\) being the eigenspaces for the eigenvalue 1 and \(E_2, F_2\) being the eigenspaces for the eigenvalue 0 or -1. We claim that \(M\) contains a submodule of dimension at most 6. The claim would imply that \(\text{rep}_K(Q, I_f)\) has bounded simples, as desired.

To prove the claim, first note that if \(E_i \cap F_j \neq 0\) for some \(i, j \in \{1, 2\}\), then any nonzero vector \(v \in E_i \cap F_j \subseteq M_b\) must generate a submodule \(N\) of \(M\) where \(N_a, N_b, N_c\) are the spans of \(M_\beta(v), v, M_\gamma(v)\), respectively. The module \(N\) has dimension at most 3, proving the claim. Otherwise, we must have \(n = 2k\) for some positive integer \(k\) and \(\dim(E_1) = \dim(E_2) = \dim(F_1) = \dim(F_2) = k\). In this case, we may choose a suitable basis \(B\) for \(M_b\) so that the matrices of \(\phi_1\) relative to \(B\) is the block diagonal matrix

\[
[\phi_1]_B = \begin{bmatrix} I_k & 0 \\ 0 & D \end{bmatrix}
\]

where \(I_k\) is the \(k \times k\) identity matrix, \(D = 0\) if \(m_1 = 4\), and \(D = -I_k\) if \(m_1 = 5\). Further, by choosing a basis \(B'\) of \(M_b\) for which the change-of-basis matrix \(P\) from \(B'\) to \(B\) is of the block diagonal form

\[
P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}
\]

with suitable \(k \times k\) matrices \(P_1, P_2\), we can ensure that

\[
[\phi_1]_{B'} = P^{-1}[\phi_1]_B P = [\phi_1]_B, \quad [\phi_2]_{B'} = P^{-1}[\phi_2]_B P = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]

where each \(A_{ij}\) is \(k \times k\) and \(A_{11}, A_{14}\) are in Jordan canonical form. Suppose \(B' = \{v_1, \ldots, v_n\}\). Then \(\phi_1(v_1) = v_1\) and \(\phi_2(v_1) = \lambda v_1 + v\) where \(\lambda\) is the top left entry in \(A_{11}\) and \(v\) in the span \(\langle v_{k+1}, v_{k+2}, \ldots, v_n \rangle\). It follows that \(\langle v_1, \phi_2(v_1) \rangle = \langle v_1, v \rangle\). Moreover, since either \(\phi_2^2 = \phi_2\) or \(\phi_2^2 = 1\), we have

\[
\phi_2(v) = \phi_2(\phi_2(v_1) - \lambda v_1) = \phi_2^2(v_1) - \lambda \phi_2(v_1) \in \langle v_1, \phi_2(v_1) \rangle = \langle v_1, v \rangle.
\]

It follows that the space \(V := \langle v_1, v \rangle\) is invariant under both \(\phi_1\) and \(\phi_2\), so it generates a subrepresentation \(N\) of \(M\) such that \(N_b = V\) and \(\dim(N) \leq 6\). This completes the proof. \(\square\)

Proof of Proposition 5.3. We need to construct a non-semisimple representation \(M \in \text{rep}_K(Q, I_f)\) when \(G\) contains a cycle, an edge with infinite weight, or at least two heavy edges. We first deal with the case that \(G\) contains an edge with infinite weight or a cycle. Let \(\{a, b\}\) be an edge with infinite weight in \(G\) if such an edge exists; otherwise, let \(a, b\) be the vertices \(v_1, v_2\) from a cycle \(C = (v_1, v_2, \ldots, v_n)\) in \(G\), respectively. Denote the arrow \(a \rightarrow b\) by \(\alpha\) and the arrow \(b \rightarrow a\) by \(\beta\). To construct \(M\), first let \(M_s = K^2\) for all \(s \in Q_0\). Let \(m = m(a, b)\), let \(\lambda_m\) be a root of the polynomial \(\tilde{f}_{m-1}\) if
$m < \infty$, let $x$ be an arbitrary nonzero scalar in $K$, and let
\[ J_x = \begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}, \quad L = \begin{cases} I_2 & \text{if } m = \infty; \\ \lambda_m \cdot J^{-1} & \text{if } m < \infty. \end{cases} \]

Let $M_{\alpha}, M_{\beta}$ be the maps given by $J_x$ and $L$, respectively, then let $M_{\gamma} = \text{id}$ for all arrows $\gamma \in Q_1 \setminus \{\alpha, \beta\}$. The assignment $M := (M_s, M_{\gamma})_{s \in Q_0, \gamma \in Q_1}$ defines a representation in $\text{rep}_K(Q, I_f)$ by Proposition 4.18 and Corollary 4.19. It is clear that $M^x$ has a subrepresentation $N$ with $N_s = \langle e_1 \rangle$, the span of the first standard basis vector, for all $s \in Q_0$. On the other hand, if we set $\phi = M_{\beta}M_{\alpha}$ in the case $m = \infty$ and set $\phi = M_C$ otherwise, then $\phi$ is an endomorphism of $M_a$ given by the matrix $J_x$ which is in Jordan canonical form and has a single $2 \times 2$ Jordan block, therefore the subspace $N_a$ of $M_a$ cannot have a complement in $M_a$ that is invariant under $\phi$. It follows that $N$ has no complement in $M$ as a subrepresentation, so $M$ is not semisimple.

It remains to consider the case where $G$ has no cycles or edges of infinite weight but has at least two heavy edges. By applying graph contractions, we may ensure that $G$ contains a subgraph of the form shown in Figure 7 and $Q$ contains a subquiver of the form shown in the same Figure. We may define a representation $M \in \text{rep}_K(Q, I_f)$ by setting $M_{a}, M_{b}, M_{c}, M_{\alpha}, M_{\beta}, M_{\gamma}, M_{\delta}$ as in Example 4.21 and setting $M_s = K^2$ and $M_{\zeta} = \text{id}$ for all vertices $s \in Q_0 \setminus \{a, b, c\}$ and all arrows $\zeta \in Q_1 \setminus \{\alpha, \beta, \gamma, \delta\}$, because doing so amounts to assembling a consistent collection of local representations in the sense of 4.4. Moreover, it is clear that $M$ has a subrepresentation $L$ with $L_s = \langle e_1 \rangle$ for all $s \in Q_0$. By the subspace analysis in Example 4.21, any subrepresentation $N$ of $M$ must have $N_b = \langle e_1 \rangle = L_b$, therefore $N$ has no complement in $M$. It follows that $M$ is not semisimple, and we are done.

\textbf{Proof of Proposition 5.6.} We need to find infinitely many simple representations in $\text{rep}_K(Q, I_f)$ when $G$ contains a cycle, an edge with infinite weight, or at least two heavy edges. We keep the notation from the previous proof and start with the case that $G$ contains an edge with infinite weight or a cycle. In this case, let $M$ and $N$ be as in the previous proof. Denote $N$ by $N^x$ to reflect the fact that $N$ depends on the value of the scalar $x$ because the matrix $J_x$ does. Then $N^x$ is clearly simple, so to show that $\text{rep}_K(Q, I_f)$ has infinitely many simples it suffices to verify that $N^x \not\cong N^y$ whenever $x \neq y$. If $m = \infty$, then we can do so by using basic linear algebra to show that there do not exist linear isomorphisms such that $\phi_bN^x = N^y_{\phi_a}$ and $\phi_aN^x_{\phi} = N^y_{\phi} \phi_b$ simultaneously. If $m \neq \infty$, then by definition we have $a = v_1, b = v_2$ for vertices $v_1, v_2$ in a cycle $C = (v_1, v_2, \ldots, v_n)$, and we can show that $N^x \not\cong N^y$ if $x \neq y$ by showing that no linear map $\phi_1 : M_a \to M_a$ can satisfy $N^y_C\phi_1 = \phi_1N^y_C$ when $x \neq y$. We omit the details.

It remains to deal with the case where $G$ has no cycle or edges of infinite weight but has at least two heavy edges. As in the previous proof, we may assume that $G$ contains the Coxeter diagram in Figure 7 as a subgraph and
assume \( G \) as in the proof of Proposition 5.7, up to simple graph contractions we may \( \{a, b, c\} \) and all arrows \( \{\alpha, \beta, \gamma, \delta\} \). This specifies a representation in \( \text{rep}_K(Q, \mathcal{I}_f) \), and the extended representation \( N^x \) is still simple since all the maps \( N^x_\zeta \) are isomorphisms.

(b) If \( N^x_c = K \), then note that since \( G \) contains no cycle, removing the edge \( b - c \) from \( G \) must result in a graph with two connected components, one containing \( b \) and the other containing \( c \). Let \( V_b, V_c \) be the sets of vertices in the first and second component, respectively. Then we may extend \( N^x \) by setting \( N^x_s = K^2 \) for all \( s \in V_b \setminus \{a, b\} \), setting \( N^x_s = K \) for all \( s \in V_c \setminus \{c\} \), and setting \( N^x_\zeta = \text{id} \) for all \( \zeta \in Q_1 \setminus \{\alpha, \beta, \gamma, \delta\} \). It is easy to see that the extension gives a simple representation in \( \text{rep}_K(Q, \mathcal{I}_f) \) as in Case (a).

To finish the proof, it suffices to show that \( N^x \not\cong N^y \) whenever \( x \neq y \). This holds by the same argument used at the end of Example 4.22.

**Proof of Proposition 5.8.** Let \( n \) be an arbitrary positive integer larger than 7. We prove the proposition by constructing a simple representation \( M \in \text{rep}_K(Q, \mathcal{I}_f) \) with \( \dim(M) > n \) when any of the Conditions (a)-(d) holds:

(a) Suppose \( G \) contains a unique cycle \( C = (v_1, v_2, \ldots, v_k) \) and a heavy edge. We first consider the case where the set \( E_C \) contains a heavy edge. As in the proof of Proposition 5.7, up to simple graph contractions we may assume \( G \) is exactly \( C \). Without loss of generality, suppose that the edge \( \{v_1, v_2\} \) is heavy and let \( m = m(v_1, v_2) \). Depending on whether \( m > 4 \) or \( m = 4 \), we construct \( M \) in one of two ways:

(i) If \( m > 4 \), then let \( q = n + 1 \), consider the symmetric group \( G = S_q \), and consider the partition \((n, 1)\) of \( q \). By the theory of Specht modules, the partition gives rise to an irreducible representation \( \rho : G \to \text{GL}(V) \) of \( G \) over \( K \) where \( \dim(V) = n \). Recall from Example 4.20 that since \( n > 7 \) there exist elements \( \sigma, \tau \in G \) which generate \( G \) with orders 2 and 3, respectively, and that consequently we have \( \rho(\sigma)^2 = \rho(\tau)^3 = \text{id}_V \). To define \( M \), let \( M_v = V \) for all \( v \in Q_0 \), let

\[
M_{\alpha_1} = \rho(\sigma), \quad M_{\alpha_2} = \rho(\tau)\rho(\sigma)^{-1}, \quad M_{\beta_2} = M_{\alpha_2}^{-1},
\]

and let \( M_\beta = \text{id} \) for all \( \gamma \in Q_1 \setminus \{\alpha_1, \alpha_2, \beta_2\} \). This defines a representation by Lemma 4.19 and clearly we have \( \dim(M) > n \). To see that \( M \) is simple, note that the operator \( M_{\beta_1}M_{\alpha_1} : M_1 \to M_1 \) equals \( \rho(\sigma) \) and the operator \( M_C : M_1 \to M_1 \) equals \( \rho(\tau) \). By subspace analysis at \( v_1 \) similar to the analysis at \( y \) in Example 4.20 any subrepresentation \( N \) of \( M \) must have either \( N_{v_1} = 0 \) or \( N_{v_1} = M_{v_1} \). Since \( M_{v_1} \) is an isomorphism for all \( \gamma \in Q_1 \), it follows that \( M \) is simple.

(ii) Now suppose \( m = 4 \). By Example 4.10 up to contractions we may assume that \( Q \) is the generalized double quiver denoted \( Q^{(3)} \) in Figure 3.
In other words, after relabelling vertices and arrows we may assume that $Q$ is the quiver with a unique vertex $v$ along with two loops $\alpha, \beta : v \to v$ and that

$$\mathcal{I}_f = \{f_{m-1}(\alpha, \beta) = \alpha\beta\alpha - \alpha, f_{m-1}(\beta, \alpha) = \beta\alpha\beta - \beta\}.$$ 

To construct $M$, let $M_n = K^n$, let $B = \{e_1, e_2, \ldots, e_n\}$ be the standard basis of $K^n$, and let $M_\alpha, M_\beta$ be the unique linear maps defined by

$$M_\alpha(e_i) = \begin{cases} e_{i+1} & \text{if } 1 \leq i < n; \\ 0 & \text{if } i = n; \end{cases} \quad M_\beta(e_i) = \begin{cases} e_{i-1} & \text{if } 1 < i \leq n; \\ 0 & \text{if } i = 1. \end{cases}$$

Intuitively, we may think of $M_\alpha$ as a “raising” operator on $K^n$ and $M_\beta$ as a “lowering” operator in light of their effects on the standard basis elements. It is easy to check that the above assignments define a representation in $\text{rep}_K(Q, \mathcal{I}_f)$. Moreover, given any nonzero vector $v \in M_v$, we may use the maps $M_\alpha, M_\beta$ to obtain any basis vector in $B$ up to a scalar, therefore $M$ must be simple.

It remains to deal with the case that all edges in $C$ are simple but some edge in $G$ not in $V_C$ is heavy. Applying simple graph contractions if necessary, we may assume that some vertex $u \in V_C$ is incident to a heavy edge $\{u, v\}$ of weight $m = m(u, v) \geq 4$ for some $v \notin V_C$. Without loss of generality, we may also assume that $u = v_1$. View $C$ as a graph $G''$ with vertex set $V_C$ and edge set $E_C$, define $G'$ to be the subgraph of $G$ obtained by adding the vertex $v$ and the edge $\{u, v\}$ to $G''$, and let $Q''$ and $Q'$ be the double quivers of $G''$ and $G'$, respectively. Let $\mathcal{I}_f''$ and $\mathcal{I}_f'$ be the evaluation ideals of $\{f_n\}$ in $KQ''$ and $KQ'$. Then by the proof of Proposition 5.6, the category $\text{rep}_K(Q'', \mathcal{I}_f'')$ contains $(n + 1)$ pairwise non-isomorphic simple representations $S(1), S(2), \ldots, S(n), S(n + 1)$. Let $S = \oplus_{i=1}^{n+1} S(i)$. Then Proposition 4.25 implies that the category $\text{rep}_K(Q', \mathcal{I}_f')$ contains a simple representation $M$ with $\dim(M_\alpha) = \dim(S_\alpha)$. Finally, we may extend $M$ to a representation $M \in \text{rep}_K(Q, \mathcal{I})$, by using the same idea as in “Case (b)” in the proof of Proposition 5.6, since $C$ is the only cycle in $G$, removing the edges in $E_C$ from $G$ results in a graph with $k$ connected components with vertex sets $V_1, V_2, \ldots, V_k$ such that $v_i \in V_i$ for all $1 \leq i \leq k$; we can then extend $M$ by setting $M_a = M_v$ for all $a \in V_1 \setminus \{v_1, v\}$, setting $M_a = M_v$ for all $a \in V_i \setminus \{v_i\}$ for each $2 \leq i \leq k$, and setting $M_\gamma = \text{id}$ for all $\gamma \in Q_1 \setminus Q'_1$. It is clear that the extended representation is still simple and has dimension larger than $n$.

(b) Suppose $G$ contains two cycles. In light of Part (a), to construct the desired representation $M$ we may assume that all edges in $G$ are simple. By considering minimal cycles and applying simple graph contractions if necessary, we may assume that $G$ contains two minimal cycles which share a vertex, i.e., two minimal cycles of the form $C = (v_1, v_2, \ldots, v_k)$ and $C' = (w_1, w_2, \ldots, w_l)$ where $v_1 = w_1$. Furthermore, while $C$ and $C'$ may share an edge, we may write the tuples $C, C'$ in such a way that $v_2 \neq w_2$. Denote
the arrows \(v_1 \to v_2, v_2 \to v_1, w_1 \to w_2\) and \(w_2 \to w_1\) by \(\alpha_1, \beta_1, \alpha'_1\) and \(\beta'_1\), respectively. To construct the desired representation \(M \in \text{rep}_K(Q, I_f)\), consider the representation \(\rho : S_q \to \text{GL}(V)\) from Part (a).(i), let \(\sigma, \tau\) be the same elements in \(S_q\) as before, let \(M_s = V\) for all \(s \in Q_0\), let \(M_{\alpha_1} = \rho(\sigma), M_{\beta_1} = \rho(\sigma)^{-1}, M_{\alpha'_1} = \rho(\tau), M_{\beta'_1} = \rho(\tau)^{-1}\), and let \(M_\gamma = \text{id}\) for all arrows \(\gamma \in Q_1 \setminus \{\alpha_1, \beta_1, \alpha'_1, \beta'_1\}\). This defines a representation \(M \in \text{rep}_{K}(Q, I)\) by Lemma 4.19 and it is obvious that \(\dim(M) > n\). The endomorphisms \(M_{\beta_1}M_{\alpha_1}\) and \(M_{\beta'_1}M_{\alpha'_1}\) of \(M_v\) equal \(\rho(\sigma)\) and \(\rho(\tau)\), respectively, therefore \(M\) is simple by the same arguments as before.

(c) Suppose \(G\) is a tree and has exactly two heavy edges, one of which has weight at least 6. Using simple graph contractions if necessary, we may assume that \(G\) is of the form shown in Figure 7, with \(m_1 \geq 4\) and \(m_2 \geq 6\). Let \(q, G\) and \(V\) be as in Part (a).(i). Then by Example 4.23 there exists a simple representation \(M \in \text{rep}_K(Q, I_f)\) such that \(\dim(M) > \dim(V) = n\), as desired.

(d) Suppose that \(G\) is a tree and has at least three heavy edges. Using simple graph contractions if necessary, we may assume that \(G\) contains a subgraph of one of the forms shown in Figure 8. In both cases, let \(G''\) be the subgraph of \(G\) induced by the vertices \(x, y, u, v\), let \(G'\) be the subgraph of \(G\) obtained by adding the vertex \(v\) and the edge \(\{u, v\}\) to \(G''\), then define \(Q'', I_f', Q', R'\) as we did in Part (a). We may produce a representation \(M \in \text{rep}_K(Q, I_f)\) in the same fashion as in Part (a): first, use the proof of Proposition 5.6 to find \((n + 1)\) pairwise non-isomorphic simple representations in \(\text{rep}_K(Q'', I_f')\); second, use Proposition 4.25 to extend the direct sum of these \((n + 1)\) simple representations to a simple representation \(M \in \text{rep}_K(Q, I_f)\); finally, further extend \(M\) to a representation in \(\text{rep}_K(Q, I_f)\) where \(M_{\gamma} = \text{id}\) for all \(\gamma \in Q_1 \setminus Q'_1\). As before, the extended representation \(M \in \text{rep}_K(Q, I)\) must be simple and satisfy \(\dim(M) > n\). This completes the proof. □

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Department of Mathematics and Statistics, Queen’s University
Email address: dimitrov@queensu.ca

Department of Mathematics and Computer Science, Royal Military College of Canada
Email address: charles.paquette.math@gmail.com

Department of Mathematics and Computer Science, Royal Military College of Canada
Email address: wehlau@rmc.ca

Department of Mathematics, University of Colorado Boulder
Email address: tianyuan.xu@colorado.edu