What can be learned from the Belle spectrum for the decay $\tau^- \rightarrow \nu_\tau K_S \pi^-$

Matthias Jamin$^1$, Antonio Pich$^2$, and Jorge Portolés$^2$

$^1$ Institució Catalana de Recerca i Estudis Avançats (ICREA) and Departament de Física Teòrica, IFAE, UAB, E-08193 Bellaterra, Barcelona, Spain

$^2$ Departament de Física Teòrica, IFIC, Universitat de València – CSIC, Apartat de Correus 22085, E-46071 València, Spain.

Abstract: A theoretical description of the differential decay spectrum for the decay $\tau^- \rightarrow \nu_\tau K_S \pi^-$, which is based on the contributing $K\pi$ vector and scalar form factors $F^K_{\pm}(s)$ and $F^K_0(s)$ being calculated in the framework of resonance chiral theory ($R\chi T$), additionally imposing constraints from dispersion relations as well as short distance QCD, provides a good representation of a recent measurement of the spectrum by the Belle collaboration. Our fit allows to deduce the total branching fraction $B[\tau^- \rightarrow \nu_\tau K_S \pi^-] = 0.427 \pm 0.024\%$ by integrating the spectrum, as well as the $K^*$ resonance parameters $M_{K^*} = 895.3 \pm 0.2$ MeV and $\Gamma_{K^*} = 47.5 \pm 0.4$ MeV, where the last two errors are statistical only. From our fits, we confirm that the scalar form factor $F^K_0(s)$ is required to provide a good description, but we were unable to further constrain this contribution. Finally, from our results for the vector form factor $F^K_\pm(s)$, we update the corresponding slope and curvature parameters $\lambda'_+ = (25.2 \pm 0.3) \cdot 10^{-3}$ and $\lambda''_+ = (12.9 \pm 0.3) \cdot 10^{-4}$, respectively.

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1 Introduction

An ideal system to study low-energy QCD under rather clean conditions is provided by hadronic decays of the $\tau$ lepton [1–5]. Detailed investigations of the $\tau$ hadronic width as well as invariant mass distributions allow to determine a plethora of QCD parameters, a most prominent example being the QCD coupling $\alpha_s$. Furthermore, the experimental separation of the Cabibbo-allowed decays and Cabibbo-suppressed modes into strange particles [6–8] opened a means to also determine the quark-mixing matrix element $|V_{us}|$ [9–11] as well as the mass of the strange quark [12–19], additional fundamental parameters within the Standard Model, from the $\tau$ strange spectral function.

The dominant contribution to the Cabibbo-suppressed $\tau$ decay rate arises from the decay $\tau^\pm \to \nu_\tau K^\mp \pi^\mp$. The corresponding distribution function has been measured experimentally in the past by ALEPH [8] and OPAL [7]. More recently, high-statistics data for the $\tau^\pm \to \nu_\tau K^\mp \pi^\mp$ spectrum became available from the Belle experiment [20], and results for the total branching fraction are also available from BaBar [21,22], with good prospects for results on the spectrum from BaBar and BESIII in the near future.

These new results call for a refined theoretical understanding of the $\tau^\pm \to \nu_\tau K^\mp \pi^\mp$ decay spectrum, and in ref. [23] we have provided a description based on the chiral theory with resonances (R$\chi$T) [24, 25], under the additional inclusion of constraints from dispersion relations. To start with, the general expression for the differential decay distribution takes the form [26]

$$
\frac{d\Gamma_{K\pi}}{d\sqrt{s}} = \frac{G^2_F|V_{us}|^2 M_\tau^3}{32\pi^3 s} S_{EW} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right)q_{K\pi}^3 |F_{K\pi}^{K\pi}(s)|^2 + \frac{3\Delta_{K\pi}^2}{4s} q_{K\pi} F_0^{K\pi}(s)^2\right],
$$

(1)

where we have assumed isospin invariance and have summed over the two possible decays $\tau^- \to \nu_\tau K^0 \pi^-$ and $\tau^- \to \nu_\tau K^- \pi^0$, with the individual decay channels contributing in the ratio 2 : 1 respectively. In this expression, $S_{EW}$ is an electro-weak correction factor, $F_{K\pi}^{K\pi}(s)$ and $F_0^{K\pi}(s)$ are the vector and scalar $K\pi$ form factors respectively which will be explicated in more detail in section 2. Furthermore, $\Delta_{K\pi} \equiv M_K^2 - M_{\pi}^2$, and $q_{K\pi}$ is the kaon momentum in the rest frame of the hadronic system,

$$
q_{K\pi}(s) = \frac{1}{2\sqrt{s}} \sqrt{s - (M_K + M_\pi)^2}\left(s - (M_K - M_\pi)^2\right) \cdot \theta\left(s - (M_K + M_\pi)^2\right). \tag{2}
$$

By far the dominant contribution to the decay distribution originates from the $K^*(892)$ meson. In the next section, we shall recall the effective description of this contribution to the vector form factor $F_{K\pi}^{K\pi}(s)$ in the framework of R$\chi$T that was presented in ref. [23], quite analogous to a similar description of the pion form factor given in refs. [27–29]. A second vector resonance, namely the $K^*(1410)$ meson, can straightforwardly be included in the effective chiral description. Finally, the scalar $K\pi$ form factor $F_0^{K\pi}(s)$ was calculated in the same R$\chi$T plus dispersive constraint framework in a series of articles [30–32], and the recent update of $F_0^{K\pi}(s)$ [33] will be incorporated in our work as well.

Based on the theoretical expression (1) for the spectrum and the form factors discussed in section 2, in section 3, we shall perform fits of our description to the Belle data [20] for...
the decay $\tau^+ \to \nu_\tau K_S \pi^-$. From these fits it follows that both the scalar contribution and the second vector resonance are required in order to obtain a good description of the experimental spectrum. In addition, the fits allow to determine the resonance parameters of the charged $K^*(892)$ and $K^*(1410)$ mesons. Finally, integrating the distribution function $d\Gamma_{K^\pi}/d\sqrt{s}$, we are also in a position to present results for the total $B[\tau^- \to \nu_\tau K_S \pi^-]$ branching fraction.

2 The form factors

A theoretical representation of the vector form factor $F^{K\pi}_+(s)$, which is based on fundamental principles, has been developed in ref. [23], in complete analogy to the description of the pion form factor presented in refs. [27–29]. This approach employed our present knowledge on effective hadronic theories, short-distance QCD, the large-$N_C$ expansion as well as analyticity and unitarity. For the pion form factor the resulting expressions provide a very good description of the experimental data [27–29]. Precisely following the approach of ref. [27], in [23] we found the following representation of the form factor $F^{K\pi}_+(s)$:

$$F^{K\pi}_+(s) = \frac{M^2_{K^\ast} \cdot \text{Re}[\tilde{H}_{K^\pi}(s) + \tilde{H}_{K\eta}(s)]}{M^2_{K^\ast} - s - iM_{K^\ast}\Gamma_{K^\ast}(s)}.$$  \hspace{1cm} (3)

The one-loop function $\tilde{H}(s)$ is related to the corresponding function $H(s)$ of [34] by $\tilde{H}(s) \equiv H(s) - 2L_0^e s/(3F_0^2) \approx [sM^2(s) - L(s)]/(F_K F_\pi)$. Explicit expressions for $M^2(s)$ and $L(s)$ can be found in ref. [35]. The one-loop function $\tilde{H}(s)$ depends on the chiral scale $\mu$, and in eq. (3), this scale should be taken as $\mu = M_{K^\ast}$. In ref. [36], the off-shell width of a vector resonance was defined through the two-point vector current correlator, performing a Dyson-Schwinger resummation within $\chi$PT [24,25]. Following this scheme the energy-dependent width $\Gamma_{K^\ast}(s)$ is found to be

$$\Gamma_{K^\ast}(s) = \frac{G^2_V M_{K^\ast} s}{64\pi F_K^2 F_\pi^2} \left[ \frac{\sigma^3_{K^\pi}(s) + \sigma^3_{K\eta}(s)}{\sigma^2_{K^\pi}(2M_{K^\ast}^2) + \sigma^2_{K\eta}(2M_{K^\ast}^2)} \right],$$  \hspace{1cm} (4)

where $\Gamma_{K^\ast} \equiv \Gamma_{K^\ast}(M_{K^\ast}^2)$, and $G_V$ is the chiral vector coupling which appears in the framework of the $\chi$PT [24]. The phase space function $\sigma_{K^\pi}(s)$ is given by $\sigma_{K^\pi}(s) = 2q_{K^\pi}(s)/\sqrt{s}$, and $\sigma_{K\eta}(s)$ follows analogously with the replacement $M_\pi \to M_\eta$. Re-expanding eq. (3) in $s$ and comparing to the corresponding $\chi$PT expression [34], in the SU(3) symmetry limit one reproduces the short-distance constraint for the vector coupling $G_V = F_0/\sqrt{2}$ [25] which guarantees a vanishing form factor at $s$ to infinity, as well as the lowest-resonance estimate.

Since the $\tau$ lepton can also decay hadronically into the second vector resonance $K^* \equiv K^*(1410)$, this particle has been included in our parametrisation of the vector form factor.

\footnote{In our expressions, we have decided to replace all factors of $1/F_0^2$ by $1/(F_K F_\pi)$ since for the $K^\pi$ system it is to be expected that higher-order chiral corrections lead to the corresponding renormalisation of the meson decay constant.}
\( F_{+K\pi}(s) \). A parametrisation which is motivated by the R\( \chi \)T framework [24,25] can be written as follows:

\[
F_{+K\pi}(s) = \left[ \frac{M_{K^*}^2 + \gamma s}{M_{K^*}^2 - s - iM_{K^*}\Gamma_{K^*}(s)} - \frac{\gamma s}{M_{K^{*'}}^2 - s - iM_{K^{*'}}\Gamma_{K^{*'}}(s)} \right] e^{\frac{3}{2}Re[\bar{h}_{K\pi}(s) + \bar{h}_{K\pi}(s)]}.
\]

This parametrisation incorporates all known constraints from \( \chi \)PT and R\( \chi \)T. At low energies, it reproduces eq. (3) up to corrections proportional to \( \gamma s (M_{K^*} - M_{K^{*'}}) \). The relation of the parameter \( \gamma \) to the R\( \chi \)T couplings takes the form \( \gamma = F_V G_V / (F_K F_{\pi^0}) - 1 \), when one assumes a vanishing form factor at large \( s \) in the \( N_C \) to infinity limit. It is difficult, to a priori assess a precise value for \( \gamma \), but below we shall be able to fit it from the comparison of our description with the Belle spectrum. The width of the second resonance cannot be set unambiguously. Therefore, we have decided to endow the \( K^*(1410) \) contribution with a generic width as expected for a vector resonance. Hence, \( \Gamma_{K^{*'}}(s) \) will be taken to have the form

\[
\Gamma_{K^{*'}}(s) = \Gamma_{K^{*'}} \frac{s}{M_{K^{*'}}^2 - \frac{\sigma_{K\pi}^2(s)}{\sigma_{K\pi}^2(M_{K^{*'}}^2)}}.
\]

As a final ingredient for a prediction of the differential decay distribution of the decay \( \tau \to \nu_\tau K\pi \) according to eq. (1), we require the scalar form factor \( F_{0K\pi}(s) \). This form factor was calculated in a series of articles [30–32] in the framework of R\( \chi \)T, again also employing constraints from dispersion theory as well as the short-distance behaviour.\(^2\) Quite recently, the determination of \( F_{0K\pi}(s) \) was updated in [33] by employing novel experimental constraints on the form factor at the Callan-Treiman point \( \Delta_{K\pi} \), and in our fits below, we shall also make use of this update.

A remaining question is which value to use for the form factors \( F_{+K\pi}(s) \) and \( F_{+K\pi}(s) \) at the origin. However, inspecting eq. (1), one realises that what is needed is not \( F_{+K\pi}(0) = F_{+K\pi}(0) \) itself, but only the product \( |V_{us}| F_{+K\pi}(0) \). Once this normalisation is fixed, in the fits we only need to determine the shape of reduced form factors \( \bar{F}_{+K\pi}(s) \) and \( \bar{F}_{+K\pi}(s) \) which are normalised to one at the origin:

\[
\bar{F}_{+K\pi}(s) = \frac{F_{+K\pi}(s)}{F_{+K\pi}(0)}, \quad \bar{F}_{+K\pi}(s) = \frac{F_{+K\pi}(s)}{F_{+K\pi}(0)}.
\]

This also entails, that after fixing the normalisation of the decay spectrum by giving a value to \( |V_{us}| F_{+K\pi}(0) \), we are in a position to predict the total branching fraction \( B[\tau^- \to \nu_\tau K_S \pi^-] \) just from a fit of the shape of the form factors, independent of normalisation issues.

The product \( |V_{us}| F_{+K\pi}(0) \) is determined most precisely from the analysis of semi-leptonic kaon decays. The most recent average was presented by the FLAVIA.net kaon working group, and reads [37]

\[
|V_{us}| F_{+K_{0\pi^-}}(0) = 0.21664 \pm 0.00048.
\]

In what follows, we have renormalised our description for the form factors to one and have assumed the result (8) for the global normalisation. Incidentally, the value in (8) already

\(^2\)The original motivation for a precise description of \( F_{0K\pi}(s) \) was the determination of the strange quark mass \( m_s \) from scalar sum rules, also performed in [32].
corresponds to the $K^0\pi^−$ channel which was analysed by the Belle collaboration [20]. Therefore, possible isospin-breaking corrections to the normalisation are already properly taken into account.

3 Fits to the Belle $\tau \rightarrow \nu_\tau K\pi$ spectrum

For our fits to the decay spectrum of the $\tau^− \rightarrow \nu_\tau K_S\pi^−$ transition as obtained by the Belle collaboration [20], we make the following Ansatz:

$$\frac{1}{2} \cdot \frac{2}{3} \cdot 0.0115 \text{[GeV/bin]} \cdot \frac{1}{N_T} \cdot \frac{1}{\Gamma_\tau B_{K\pi}} \cdot \frac{d\Gamma_{K\pi}}{d\sqrt{s}}.$$  \hspace{1cm} (9)

The factors $1/2$ and $2/3$ come from the fact that the $K_S\pi^−$ channel has been analysed. Then, $11.5\text{ MeV}$ was the bin-width chosen by the Belle collaboration, and $N_T = 53110$ the total number of observed signal events. Finally, $\Gamma_\tau$ is the total decay width of the $\tau$ lepton and $\bar{B}_{K\pi}$ a remaining normalisation factor that will be deduced from the fits. The normalisation of our Ansatz (9) is taken such that for a perfect agreement between data and fit function, $\bar{B}_{K\pi}$ would correspond to the total branching fraction $B_{K\pi} \equiv B[\tau^− \rightarrow \nu_\tau K_S\pi^−]$ which is obtained by integrating the decay spectrum. Differences between $\bar{B}_{K\pi}$ and $B_{K\pi}$ point to imperfections of the fit, and will constitute one source of systematic uncertainties. As we shall see further below, for better fits also the agreement between $\bar{B}_{K\pi}$ and $B_{K\pi}$ improves as expected.

Before entering the details of our fits, let us discuss the numerical values of all input parameters. For the meson masses, we employ the physical masses corresponding to the decay channel in question, namely $M_{K_S} = 497.65\text{ MeV}$, $M_{\pi^−} = 139.57\text{ MeV}$ and $M_\eta = 547.51\text{ MeV}$ [38]. For the meson decay constants, we use the findings of the recent review [39], in our normalisation that is $F_\pi = 92.3\text{ MeV}$ and $F_K/F_\pi = 1.196$. For the electro-weak correction factor, we have utilised the result for inclusive hadronic $\tau$ decays, $S_{\text{EW}} = 1.0201$ [10] (and references therein). Even though the electro-weak correction factor for the exclusive decay in question need not be the same as $S_{\text{EW}}$, to the precision we are working this choice is supposedly sufficient. Besides, we are not aware of a published result for the correct factor in the case of the exclusive decay studied here. All remaining input parameters which have not been mentioned explicitly, are taken according to their PDG values [38].

As an initial step, only the central $K^*$ resonance region is fitted, in order to get an idea about the $K^*$ resonance parameters. For this fit, two forms of the dominant vector form factor $F_{K^*}^+(s)$ are used. On the one hand, we employ our description (3) as discussed in the last section. On the other hand, we also investigate a pure Breit-Wigner resonance shape as was used in the experimental work of the Belle collaboration. This later allows a better comparison to the findings of ref. [20]. The Breit-Wigner resonance factor is defined by

$$BW_{K^*}(s) \equiv \frac{M_{K^*}^2}{M_{K^*}^2 - s - iM_{K^*} \Gamma_{K^*}(s)} \cdot \Gamma_{K^*}(s) \left(\frac{s}{M_{K^*}^2}\right)^\frac{3}{2} \left(\frac{\sigma_{K^*}(s)}{\sigma_{K^*}(M_{K^*})}\right). \hspace{1cm} (11)$$
Thus, the $K^*$ width of (11) coincides with eq. (4) if the $K\eta$ contribution is neglected. Although our equations (10) and (11) are written in a form different from the one employed in [20], the expressions are in agreement. The Breit-Wigner version of the $K\pi$ vector form factor $F^K\pi_+(s)$ then reads

$$F^K\pi_+(s) = F^K\pi_+(0) \cdot BW_{K^*}(s).$$

(12)

In practice, as discussed above, for our fits we only require the reduced form factor $\bar{F}^K\pi_+(s)$ which in this case is equal to the Breit-Wigner factor $BW_{K^*}(s)$.

For our first fit, we employ the Belle data [20] in the range $0.808 – 1.015$ GeV (data points 16 – 34), where the vector form factor dominates and should provide a good description. The resulting fit parameters are presented as the left-hand column in table 1 for the Breit-Wigner fit, and the right-hand column for the chiral fit. Graphically, the corresponding fits are shown as the dotted and short-dashed lines in figure 1 respectively, together with the experimental data points. The fitted $K^*$ mass $M_{K^*}$ for the Breit-Wigner fit is close to the result by the Belle collaboration [20], while the width $\Gamma_{K^*}$ is found to be somewhat larger. Besides the normalisation factor $\bar{B}_{K\pi}$, which would be obtained when integrating the spectrum. The $\chi^2$/n.d.f. for this fit is found to be of order 2. Nevertheless, later we shall see that our final fit including all contributions will have a $\chi^2$/n.d.f. of order 1. So this is nothing to worry about at this point. From figure 1, one observes that the fit provides a reasonable description of the data in the fit region, but both, much below and much above the resonance peak marked deviations are clearly visible, implying missing contributions that will be discussed below.

|                  | BW form for $F^K\pi_+(s)$       | Chiral form for $F^K\pi_+(s)$     |
|------------------|----------------------------------|----------------------------------|
| $\bar{B}_{K\pi}$| $B_{K\pi}$ 0.3435 ± 0.0042 %    | $0.4658 ± 0.0057 % (0.4541 %)$   |
|                  | $M_{K^*}$ 895.59 ± 0.18 MeV      | 894.93 ± 0.18 MeV                |
|                  | $\Gamma_{K^*}$ 48.06 ± 0.45 MeV | 47.47 ± 0.44 MeV                 |
| $\chi^2$/n.d.f. | 30.3/16                         | 30.8/16                         |

Table 1: Fit to the Belle $\tau \to \nu_\tau K\pi$ spectrum in the $K^*$ resonance region with a pure vector resonance shape.

Performing in an analogous fashion the fit to the Belle data with the R$\chi$T form of $F^K\pi_+(s)$, the obtained fit parameters are listed in the right-hand column in table 1, and the fit curve is displayed as the short-dashed line in figure 1. The parameters obtained from both fits differ to some extent, especially the normalisation $\bar{B}_{K\pi}$, due to the different functional forms of the vector form factor. Still, we will postpone a detailed discussion of our numerical results until presenting the complete fit including all contributions below.\footnote{As the fit is practically insensitive to the parameter $r$ in the Blatt-Weisskopf barrier factor appearing in our previous parametrisation of the $K^*$ width [23], we have decided to set $r$ to zero, so that our fits are more directly comparable to the fits performed by the Belle collaboration [20], who have not applied such a factor. Employing the central result of our previous fit $r = 3.5$ GeV$^{-1}$ [23] would give practically the same $\chi^2$, but would result in a $K^*$ mass that is about 1.4 MeV lower and a $K^*$ width about 0.8 MeV lower. These values are somewhat lower than the Belle result but are still in agreement within the errors.} From figure 1, we see
that while both, the chiral and the Breit-Wigner fits give a similar spectrum below the $K^*$ resonance peak, above the peak there are substantial differences. This will play an important role below, when we shall aim at improving the fit by adding a second vector resonance $K^{*'}$ because it will certainly influence its fit parameters.

|                          | BW form for $F_{K\pi}^+(s)$ | Chiral form for $F_{K\pi}^+(s)$ |
|--------------------------|-----------------------------|---------------------------------|
| $B_{K\pi} (B_{K\pi})$   | $0.3575 \pm 0.0041\% (0.3518\%)$ | $0.4767 \pm 0.0056\% (0.4726\%)$ |
| $M_{K^*}$                | $895.56 \pm 0.18\text{ MeV}$ | $894.92 \pm 0.18\text{ MeV}$ |
| $\Gamma_{K^*}$          | $47.05 \pm 0.42\text{ MeV}$  | $46.94 \pm 0.42\text{ MeV}$  |
| $\chi^2$/n.d.f.         | $43.5/28$                    | $46.2/28$                      |

Table 2: Fit to the Belle $\tau \rightarrow \nu_\tau K\pi$ spectrum in the $K^*$ resonance region with a vector resonance shape for $F_{K\pi}^+(s)$ and the central scalar form factor $F_{0\pi}^+(s)$.

Thus far, we have completely omitted the contribution of the scalar $K\pi$ form factor conclusions are the same for both the Breit-Wigner or chiral form of the vector form factor $F_{K\pi}^+(s)$. This observation again indicates the fact that the precise functional form of the vector form factor matters for the resulting values of $K^*$ mass $M_{K^*}$ and width $\Gamma_{K^*}$.
$F_0^{K\pi}(s)$ to the differential $\tau \to \nu_\tau K\pi$ decay spectrum. When adding the corresponding contribution with the central parameters as presented in [33], it is found that the combined theoretical spectrum gives a good description also in the region below the $K^*$ resonance, with the exception of three data points in the range $0.682 - 0.705$ GeV (points 5, 6, 7). Therefore, as our next fit, we fit the entire low-energy region $0.636 - 1.015$ GeV while keeping the scalar form factor $F_0^{K\pi}(s)$ fixed but leaving out in the fit the problematic data points 5, 6 and 7. The resulting fit parameters in the case of the Breit-Wigner and chirally inspired vector form factor $F_+^{K\pi}(s)$ are tabulated in table 2, and the corresponding fit curves are plotted as the long-dashed and solid lines in figure 1 respectively. From table 2 one observes that $M_{K^*}$ is almost unchanged, the width $\Gamma_{K^*}$ is slightly decreased, and also the $\chi^2$/n.d.f. is somewhat reduced, although it is still larger than roughly 1.5. Nevertheless, it is clear that the scalar contribution is required in order to give a more satisfactory description of the region below the $K^*$ resonance.

As the last step, now we also improve upon the description of the region above the $K^*$ resonance by including as a second vector resonance the $K^{*'}$. In the case of the Breit-Wigner form factor, the inclusion of the $K^{*'}$ resonance can be achieved by writing

$$F_+^{K\pi}(s) = \frac{F_+^{K\pi}(0)}{1 + \beta} \left[ \text{BW}_{K^*}(s) + \beta \text{BW}_{K^{*'}}(s) \right],$$

(13)

whereas in the case of the chiral resonance description, the corresponding expression for $F_+^{K\pi}(s)$ including the $K^{*'}$ is given above in eq. (5) and depends on the mixing parameter $\gamma$. Again, our fits are displayed in a graphical form in figure 2, where the solid line corresponds to the $R\chi T$ description, and the dashed line to the fit with a vector form factor according to eq. (13). For $R\chi T$, in addition we have separately displayed the contributions of the scalar form factor (dotted line) and of the $K^{*'}$ resonance (dashed-dotted line). The resulting fit

Table 3: Full fit to the Belle $\tau \to \nu_\tau K\pi$ spectrum with the two $K^*$ and $K^{*'}$ vector resonances in $F_+^{K\pi}(s)$ and the central scalar form factor $F_0^{K\pi}(s)$.

|               | BW form for $F_+^{K\pi}(s)$ | Chiral form for $F_+^{K\pi}(s)$ |
|---------------|-----------------------------|---------------------------------|
| $B_{K\pi} (B_{K\pi})$ | $0.423 \pm 0.012\% (0.421\%)$ | $0.430 \pm 0.011\% (0.427\%)$ |
| $M_{K^*}$     | $895.12 \pm 0.19$ MeV       | $895.28 \pm 0.20$ MeV           |
| $\Gamma_{K^*}$ | $46.79 \pm 0.41$ MeV        | $47.50 \pm 0.41$ MeV            |
| $M_{K^{*'}}$  | $1598 \pm 25$ MeV           | $1307 \pm 17$ MeV               |
| $\Gamma_{K^{*'}}$ | $224 \pm 47$ MeV           | $206 \pm 49$ MeV                |
| $\beta, \gamma$ | $-0.079 \pm 0.010$          | $-0.043 \pm 0.010$              |
| $\chi^2$/n.d.f. | 88.7/81                     | 79.5/81                         |

These three data points look as if there might be a bumpy structure, perhaps related to the $K_0^*(800)$. However, as has been discussed in section 7 of [30], the $K_0^*(800)$ is in fact present in our chiral description of the scalar form factor. As it is rather broad, we see no way how one could accommodate such a bump in the low-energy region below the $K^*$ resonance.
Figure 2: Main fit result to the Belle data [20] for the differential decay distribution of the decay $\tau^-\to \nu_{\tau}K_S\pi^-$. Our theoretical description includes the Breit-Wigner (dashed line) or $R\chi T$ (solid line) vector form factors with two resonances, as well as the scalar form factor according to ref. [33]. For $R\chi T$ also the scalar (dotted line) and $K^*$ (dashed-dotted) contributions are displayed.

parameters have been collected in table 3. We observe that the chirally inspired description of ref. [23] provides the better fit, and that as expected the $K^*$ mass $M_{K^*}$ turns out to be very different, though the $\Gamma_{K^*}$ widths (probably by chance) agree rather well. The mixing parameters $\beta$ and $\gamma$ also differ, but due to the different functional forms of our two descriptions for the vector form factor $F_{K\pi}^+(s)$, anyway they cannot be compared.

Up to now, in our fits we have only employed the central prediction for the scalar form factor $F_0^{K\pi}(s)$. Thus the question arises what happens if we modify $F_0^{K\pi}(s)$. As the normalisation of the form factors can be fixed by experiment, we only require the shape of $F_0^{K\pi}(s)$ and for this, in our dispersive approach [31–33], the dominant input parameter is the value of the ratio $F_0^{K\pi}(\Delta_{K\pi})/F_0^{K\pi}(0)$ at the Callan-Treiman point $\Delta_{K\pi} \equiv M_K^2 - M_\pi^2$, which has been discussed in detail in [33]. We can then introduce a fit parameter $\alpha$ which describes the change of shape of $F_0^{K\pi}(s)$ when $F_0^{K\pi}(\Delta_{K\pi})/F_0^{K\pi}(0)$ is modified. Let $\alpha = 0$ correspond to our central result of [33], $\alpha = 1$ to the scalar form factor which arises when $F_0^{K\pi}(\Delta_{K\pi})/F_0^{K\pi}(0)$ is larger by 1$\sigma$, and $\alpha = -1$ when $F_0^{K\pi}(\Delta_{K\pi})/F_0^{K\pi}(0)$ is smaller by 1$\sigma$. Adding $\alpha$ to our fit parameters, for the chirally inspired $F_+^{K\pi}(s)$ we obtain $\alpha = 4.4 \pm 1.9$, and for the pure Breit-Wigner form $\alpha = 6.3 \pm 2.7$, with only a slight change of the other parameters and a small improvement in the $\chi^2$/n.d.f. From this we conclude that the fit prefers a slightly larger
$F_0^{K\pi}(s)$, but the sensitivity to $\alpha$ is not very strong. Furthermore, the largest changes when leaving $\alpha$ free are in the parameters of the $K^*$, which entails that the found values for $\alpha$ are driven by the energy region above the $K^*$ resonance, where the theoretical description is less well founded. If the same exercise is repeated with the fits which only include the low-energy and $K^*$ resonance region (fits of table 2), then we obtain $\alpha = 4.7 \pm 7.9$ in the case of the RχT description. Hence, with the present precision of the data and in particular the open question about the three data points in the low-energy region, we are not able to further constrain the contribution of the scalar form factor $F_0^{K\pi}(s)$.

Let us now come to a detailed discussion of our central fit results of table 3. The $\chi^2$/n.d.f. of both the chiral and the Breit-Wigner fits is of the order of one, but nevertheless the chiral fit provides the better description of the experimental data. For the complete fit including two vector resonances and the scalar contribution, within the fit uncertainties the normalisation $\hat{B}_{K\pi}$ and the branching fraction $B_{K\pi}$ are in very good agreement. In addition, as can be observed from table 3, also the branching fractions extracted from the two versions of parametrising $F_{\pm}^{K\pi}(s)$ display perfect consistency, once all contributions have been included in the fit. The remaining small difference can be traced back to the exponential factor in the numerator of the RχT expression (3). Since our chiral model for $F_{\pm}^{K\pi}(s)$ is theoretically better motivated and furthermore provides the better fit quality, as our central result for the branching fraction, we quote:

$$B[\tau^- \to \nu_\tau K^0 \pi^-] = 0.427 \pm 0.011 \pm 0.021\% = 0.427 \pm 0.024\%.$$  

(14)

The first quoted error corresponds to the statistical fit uncertainty. To be conservative, in the second error we made an attempt to estimate systematic uncertainties. To this end, we have performed analogous fits, where the chiral factors $1/F_0^2$ are taken to be $1/F_0^2$, which should give an idea of the importance of higher-order chiral corrections. (See footnote 1.) Then the branching fraction for the full RχT fit turns out to be $\hat{B}_{K\pi} = 0.448\%$, and we take the difference of this result to our main value as an additional systematic uncertainty. When comparing to previous determinations, within the uncertainties our result (14) is in agreement with the findings of the Belle collaboration $B[\tau^- \to \nu_\tau K^0 \pi^-] = 0.404 \pm 0.013\%$ [20], which are just based on a pure counting of events, as well as the Particle Data Group average for the related branching fraction $B[\tau^- \to \nu_\tau K^0 \pi^-] = 0.90 \pm 0.04\%$ [38]. When assuming isospin invariance, the above results can also be compared with the BaBar measurement $B[\tau^- \to \nu_\tau K^- \pi^0] = 0.416 \pm 0.018\%$ [21], showing very good overall consistency.

As far as the parameters of the charged $K^*$ resonance are concerned, within the uncertainties our value $M_{K^*} = 895.3 \pm 0.2$ is in very good agreement with the Belle result [20]. However, it is about $3.5$ MeV larger than the current PDG average [38].

\footnote{Funnily enough, it is in much better agreement with the PDG average for the neutral $K^*$ mass. For more details, the reader is referred to the related discussion of the $K^*$ mass in ref. [20].}

On the other hand, our finding for the width $\Gamma_{K^*} = 47.5 \pm 0.4$ MeV is significantly lower than the PDG average, but still roughly 1 MeV larger than the Belle result. The corresponding value of the chiral coupling $G_V$ which appears in eq. (4) is found to be $G_V = 72.0 \pm 0.6$ MeV. For the second vector resonance, the $K^*(1410)$, the obtained mass from our central chiral fit is about 100 MeV.
lower than the PDG average [38], while for the width, we find reasonable agreement to the PDG value. However, for the Breit-Wigner fit $M_{K^{*'}}$ was found to turn out much larger, which implies that the mass of the $K^*(1410)$ strongly depends on our parametrisation of the form factors and its determination is therefore not very reliable. As a general remark, we like to emphasise that one should not compare or average determinations done with different functional parametrisations.

4 Conclusions

Let us briefly summarise our findings before drawing further conclusions. From a description of the $K\pi$ vector and scalar form factors $F^{K\pi+}_+(s)$ and $F^{K\pi+}_0(s)$ in the framework $\chi T$, additionally imposing constraints from dispersion relations as well as short distance QCD, we were able to obtain a good fit to the recent Belle data [20] for the spectrum of the decay $\tau^- \rightarrow \nu_\tau K^0_S \pi^-$. From our fit we could extract the corresponding branching fraction and the parameters of the $K^*$ resonance

$$B[\tau^- \rightarrow \nu_\tau K^0_S \pi^-] = 0.427 \pm 0.024 \%,$$

$$M_{K^*} = 895.3 \pm 0.2 \text{ MeV}, \quad \Gamma_{K^*} = 47.5 \pm 0.4 \text{ MeV},$$

where the quoted errors for $M_{K^*}$ and $\Gamma_{K^*}$ only include the statistical fit uncertainties. Besides, we observe a substantial model dependence of these parameters. (See footnote 3.) This model dependence is even more pronounced for the second included resonance, the $K^*(1410)$, and therefore we are unable to make a reliable prediction for $M_{K^{*'}}$ and $\Gamma_{K^{*'}}$.

As far as the scalar form factor $F^{K\pi+}_0(s)$ is concerned, below the $K^*$ resonance it is obvious that this contribution is required in order to provide a satisfactory description of the data. (With the exception of three data points which appear to form a small bump.) Trying to also fit the scalar part, it is seen that the data prefer a slightly larger contribution, but on the basis of the present data this is statistically not significant. Above the $K^*$, we have the well established $K^*_0(1430)$ resonance, but here it interferes with higher vector resonances. Due to these correlations and the strong model dependence of the higher vector resonances, it will be difficult to disentangle scalar and vector contributions without a dedicated analysis of angular correlations [26, 40].

An independent investigation of the Belle $\tau^- \rightarrow \nu_\tau K^0_S \pi^-$ decay spectrum on the basis of Mushkelishvili-Omnés integral equations, also incorporating chiral constraints at low energies as well as QCD short-distance constraints at high energies was recently published in ref. [41]. A visual inspection of the corresponding fit results presented in figure 5 of [41] suggests that the quality of the fit is not as good as in our case, though no further details, e.g. a $\chi^2$, were provided in [41]. Still, it would be interesting to see, if somehow the approaches used in ref. [41] and in our work could be merged, to be able to impose as many theoretical constraints as possible on the employed form factors.

Already in ref. [23], from our description of the vector form factor $F^{K\pi+}_+(s)$, we deduced slope and curvature of the form factor close to $s = 0$, which are important parameters in the
determination of $|V_{us}|$ from $K_{l3}$ decays. Let us define a general expansion of the reduced form factor $\tilde{F}_+^{K \pi}(s)$ as:

$$\tilde{F}_+^{K \pi}(s) \equiv 1 + \lambda_+ s + \frac{1}{2} \lambda''_+ s^2 + \frac{1}{6} \lambda'''_+ s^3 + \ldots,$$

(17)

where $\lambda_+, \lambda''_+$ and $\lambda'''_+$ are the slope, curvature and cubic expansion parameter respectively. On the basis of our fit results of table 3, we are now in a position to update these quantities, also estimating the corresponding uncertainties, which yields:

$$\lambda_+ = (25.20 \pm 0.33) \cdot 10^{-3}, \quad \lambda''_+ = (12.85 \pm 0.31) \cdot 10^{-4}, \quad \lambda'''_+ = (9.56 \pm 0.28) \cdot 10^{-5}. \quad (18)$$

In an attempt to estimate systematic uncertainties from higher orders in the chiral expansion, like in the last section we have again also investigated the case $F_K = F_\pi$, which contributes the largest part of the error quoted in (18). The next important source of uncertainty stems from the mixing parameter of the $K^{*'}$ resonance $\gamma$, for which we have used the fit result of table 3. Besides, the vector masses $M_{K^{*}}$ and $M_{K^{*'}}$ have been varied by 1 MeV and 100 MeV respectively, but these modifications only have a small impact on the uncertainties for the expansion parameters of $\tilde{F}_+^{K \pi}(s)$. Comparing to the most recent determination of $\lambda_+$ and $\lambda''_+$ from an average of current experimental results for $K_{l3}$ decays [37] (where also detailed references to the individual experiments can be found), we observe that both determinations are in very good agreement, though for the time being our theoretical extraction is more precise.

To conclude, our R$\chi$T description of the $K\pi$ vector and scalar form factors provides a good representation of the experimental data of the Belle collaboration for the spectrum of the decay $\tau^- \to \nu_\tau K_S \pi^-$ [20], thereby allowing to deduce many parameters of this approach. The used method can also be applied to $\tau$ decay channels which involve three final state hadrons, and this has already been performed successfully for the decays $\tau \to \nu_\tau \pi\pi\pi$ [42] as well as $\tau \to \nu_\tau K K \pi$ [43]. In the near future, we plan to return to the still missing decay mode $\tau \to \nu_\tau K \pi\pi$, which is the most interesting one in view of getting a better handle on the hadronic $\tau$ decay rate into strange final states.

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