A comparative analysis of multiple fractional solutions of generalized Couette flow of couple stress fluid in a channel

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Abstract

This study provides the exact solutions of couple stress fluid (CSF) in a channel. The CSF is bounded by two plates in which the lower plate is moving with constant velocity $U_0$ and the upper plate is fixed. The influence of the external pressure gradient is considered on the CSF fluid. To transform the classical CSF model, we have introduced three approaches to fractional derivatives, (a) Caputo (b) Caputo–Fabrizio (CF), and (c) Atangana–Baleanu (AB) fractional definitions. Exact solutions have been obtained using the Laplace and finite Fourier sine transforms jointly. Furthermore, the effect of different fractional derivatives is compared, and the results are shown in graphs. Moreover, parameters that affect the CSF motion are discussed in the graphs using the computational software MATHCAD. The most important outcome of the given study is the comparison of Caputo, CF, and AB fractional models with the classical CSF model. From the comparison, it can be noticed that AB fractional model discusses the dynamics of the CSF with a good memory effect as compared to Caputo and CF fractional operators. The CSF model can be reduced to Newtonian fluid as a limiting case and also investigated solutions in the absence of external pressure. Finally, Skin friction is evaluated for lower
Introduction

Different from a Newtonian fluid, the couple stress fluid (CSF) includes a new material constant, which is responsible for couple stress and the lubricant viscosity. In the momentum equation fourth order derivative term have a constant known as material constant which make them couple stress fluid. Couple-stress fluids (CSF) contain a new material constant, which makes them different from usual viscous fluids. The rheological properties of these fluids (CSFs) have many applications such as, in the extraction process of crude oils from petroleum products, electrostatics precipitations, aerodynamics heating phenomenon, solidification process of liquid crystals, cooling of the metallic plate, and so forth. CSF is a noted fluid in non-Newtonian fluids having enormous applications in fluid dynamics. The concept of CSF theory was first given by Stokes, and considers couple stresses in addition to the classical Cauchy stress. CSF has many real-world applications in engineering and biological sciences.

CSF models are very famous among non-Newtonian fluids. Due to the popularity of CSF, many researchers have investigated different situations. The CSF model has been considered by Akgül and Siddique in this study they discussed the novel applications of CSF flows between two parallel plates. The applications of MHD CSF flow between two parallel plates were investigated by Siddique et al. Kumar et al. also discussed unsteady couple stress nanofluid flow with Ohmic dissipation. Khan et al. analyzed the swirling of a CSF using a rotating disk. Hayat et al. studied CSF flow with Cattaneo-Christov and double diffusion. Ahmad et al. studied the CSF flow in an inclined rotating channel. Bég et al. calculated a mathematical model of CSF biofluid in an inclined plane. Yadav et al. discussed the analysis of internal heat generation in CSF in a saturated porous media. Furthermore, the analytical solutions of CSF flow in concentric cylinders were investigated by Devakar et al. Naeem studied some classes of CSF flow. Mekheimer analyzed the CSF flow in an annulus for endoscopic applications.

The idea of fractional calculus is getting more attention in the last three decades. This concept originated in 1695. After that many real-world problems and physical phenomena were explained by researchers using some suitable fractional order derivatives definitions. The use of fractional calculus is not only limited to engineering, but it has many applications in biological sciences and some other fields of sciences. Initially, researchers attempt many calculations and find some unique applications of fractional derivatives which are explained by Oldham and Spanier. The applications of fractional derivatives are not only limited to engineering problems, but it has many industrial applications which are discussed in Siddique and colleagues. Due to the uses and novel applications of fractional derivatives many definitions proposed related to fractional derivatives. Of these definitions, the most used is the Reimann-Liouville definition. There are two shortcomings in the Reimann-Liouville definition, firstly the derivative of some constant is not equal to zero, and secondly, the Laplace transform...
has some terms which have no physical meaning. After Reimann-Liouville Caputo develop a new fractional order derivative that overcomes the shortcomings of Reimann-Liouville’s definition, but still there is a problem of singularity in its kernel.

There are many fractional order derivatives that have been used for different real-world problems and scientific reasons. The present article focused to highlight the most recent definitions of fractional derivatives. After Reimann-Liouville definition Caputo\(^1\) developed a fractional derivative which is already used in many problems. After Caputo in 2015 Caputo and Fabrizio developed a fractional definition namely, Caputo and Fabrizio\(^2\) fractional order derivatives. Recently, Atangana and Baleanu\(^3\) introduced a modified definition that has many uses in science and engineering. The comparison of time fractional derivative of AB and CF developed by Arif et al.,\(^4\) to investigate the CSF model for generalized Couette flow. Akhtar\(^5\) studied a CSF using different fractional approaches. The main reason behind considering the fractional derivatives instead of classical one is that fractional derivatives have many applications in real-world problems. The big advantage of fractional derivatives is that we can formulate models describing the systems with memory effects. Memory in a system means how much information the system carries. This memory in different physical phenomena can be investigated by considering fractional derivatives.\(^6\)–\(^8\) Some other applications of fractional calculus is found in Imran et al.\(^9,10\) Fractional calculus has been used in engineering, physics, chemistry, and biological problem, and some of their applications can be found in Shah and colleagues.\(^11–13\)

Recently, three fractional order derivatives have been developed which are applied to different real-world problems and find the accurate solutions to those problems. In this study, we collect some combined applications of these fractional derivatives. Motivated by the above-mentioned literature this paper presents the comparison among these fractional order derivatives on the CSF fluid model in a channel. The classical CSF model is transformed by applying first Caputo, Caputo-Fabrizio (CF), and then Atangana and Baleanu (AB) fractional approaches. The exact solutions have been recovered by applying the Fourier and Laplace transforms. The obtained solutions for Caputo, CF, and AB have been compared with the classical couple stress model. From the comparison, we found that AB fractional model discusses the dynamics of the CSF with a good memory effect as compared to Caputo and CF fractional operators.

2 | MATHEMATICAL MODELING AND SOLUTION OF THE PROBLEM

This section provides mathematical modeling and exact solutions for CSF in the channel. The fluid is taken in a channel bounded by two plates the distance between them is taken \(d\). The governing equations of the present study are given as follows\(^1\):

\[\nabla \cdot \vec{V} = 0,\]

\[
\rho \frac{\partial \vec{V}}{\partial t} = -\nabla p - \mu \nabla \times \nabla \times \vec{V} - \chi \nabla \times \nabla \times \nabla \times \nabla \times \vec{V} + \rho \vec{b}_1.
\]

Unidirectional CSF is taken in the present paper. Therefore, \(\vec{V}\) shows the velocity field in the x-direction. The components form of CSF can be written as.\(^14\)
\[ \rho \frac{\partial u(y, t)}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u(y, t)}{\partial y^2} - \chi \frac{\partial^4 u(y, t)}{\partial y^4}. \] (3)

2.1 The generalized Couette flow

In the present analysis, we have considered the CSF flow in a channel bounded by two parallel plates separated by a distance \( d \). It is worth noting that Couette flow is the type of channel flow and can be defined as “The flow between two parallel plates in which one plate is at rest and the other one is moving with constant velocity is called Couette flow.” Furthermore, when an external pressure gradient is applied to the Couette flow the resultant flow is called “generalized Couette flow.” In the present analysis, we have considered that one plate is fixed and the second one is moving with constant velocity \( U_0 \). Therefore, the flow is called Couette flow. Moreover, we have used the word generalized Couette flow because when an external pressure gradient is considered in the fluid direction then the Couette flow becomes generalized Couette flow. The present research investigated the CSF flow in the channel in the presence of external pressure. The pressure is applied along the direction of the fluid which accelerates the fluid motion. The physical geometry of the given model is shown in Figure 1.

The equations which govern the flow can be written as

\[ \rho u_t = G^x + \mu u_{yy} - \chi u_{yyyy}. \] (4)

The IC's and BC's are given as

\[
\begin{align*}
0 & \leq y \leq h \quad \text{and} \quad t = 0, \quad u(y, t) = 0, \\
y & = 0 \quad \text{and} \quad t > 0, \quad u(y, t) = H(t) U_0, \\
y & = h \quad \text{and} \quad t > 0, \quad u(y, t) = 0, \\
y & = 0 \quad \text{and} \quad y = h, \quad u_{yy}(y, t) = 0.
\end{align*}
\] (5)

**FIGURE 1** Physical sketch of the problem
To get the dimensionless governing equations using the following nondimensional quantities:

\[ \xi = \frac{y}{h}; \eta = \frac{u}{U_0}; \tau = \frac{U_0 t}{h}. \]

The dimensionless initial and boundary value problem is given below:

\[ \lambda \eta_{\xi\xi\xi\xi} - \eta_{\eta\eta} - G + Re \eta_\tau = 0. \] (6)

The following are the dimensionless IC's and BC's:

\[
\eta(\xi, \tau) = 0, \quad \text{for} \quad 0 \leq \xi \leq 1 \quad \text{and} \quad \tau = 0, \\
\eta(\xi, \tau) = 1, \quad \text{for} \quad \xi = 0 \quad \text{and} \quad \tau > 0, \\
\eta(\xi, \tau) = 0, \quad \text{for} \quad \xi = 1 \quad \text{and} \quad \tau > 0, \\
\frac{\partial^2 \eta(\xi, \tau)}{\partial \xi^2} = 0, \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = 1. \] (7)

\[ Re = \frac{U_0 d}{v}; \lambda = \frac{\chi}{\mu d^2} \quad \text{and} \quad G = \frac{d^2}{\mu U} G^*. \]

Here \( Re \) and \( G \) is the Reynolds number and pressure \( \lambda \) is particles size effect of CSF.

## 3 | EXACT SOLUTION

In the present article the three fractional operators like, Caputo, CF, and AB definitions are applied to a simple CSF model in order to compare the results of these fractional approaches. Solutions of CSF model with Caputo fractional approach are given in section 3.1, solutions of CSF model with CF is given in 3.2, and solutions of CSF model with AB fractional derivative are given in Section 3.3.

### 3.1 | Exact solutions using caputo definition

To developed the generalized CSF model by applying Caputo fractional derivative, we multiply \( \ell = \frac{U_0 t}{h} \) to Equation (4), and by replacing \( \ell \) by \( \ell^\gamma \) and partial derivatives on the left sides with respect to \( \tau \) by \( C D_\tau^\gamma \):

\[ \ell^\gamma C D_\tau^\gamma \rho u_\tau = \ell G^* + \ell \mu u_{yy} - \ell \chi u_{yyyy}. \] (8)

Using the following dimensionless variables

\[ \xi = \frac{y}{h}; \eta = \frac{u}{U_0}; \tau = \frac{U_0 t}{h}, \lambda = \frac{U_0 t}{h}. \]

After dimensionalities we get the following equation:

\[ \lambda \eta_{\xi\xi\xi\xi} - \eta_{\eta\eta} - G + Re \eta_\tau = 0. \]
where \( C^D_\tau e \) is the Caputo definition of order \( \gamma \) which is defined as\(^{33,34}\)

\[
C^D_\tau e(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^\tau (\tau - t)^{-\gamma} e'(t) \, dt, \quad 0 \leq \gamma < 1.
\]  

(10)

Applying first the Laplace and then Fourier sine transforms to Equation (9) using the conditions from Equation (7), our transformed solution is given as

\[
Re^p e \bar{\eta}_s(n, p) = \frac{G(1 - (-1)^n)}{p\sigma_n} + \frac{\sigma_n}{p} - \sigma_n^2 \bar{\eta}_s(n, p) + \lambda \frac{\sigma_n^4}{p} - \lambda \sigma_n^4 \bar{\eta}_s(n, p),
\]  

(11)

from above we can write as

\[
\bar{\eta}_s(n, p) = \left( \frac{G(1 - (-1)^n) + \sigma_n^2 + \lambda \sigma_n^4}{p\sigma_n(Re^p + \sigma_n^2 + \lambda \sigma_n^4)} \right).
\]  

(12)

After some mathematical calculation our solution is given as

\[
\bar{\eta}_s(n, p) = \left( \frac{G(1 - (-1)^n) + \sigma_n^2 + \lambda \sigma_n^4}{\sigma_n Re} \right) \times \left( \frac{1}{H_1 p} + \frac{1}{p^{1-\gamma}(p^\gamma + H_1)} \right).
\]  

(13)

Applying both the inverse Laplace and Fourier transforms our solution is given in the following form\(^{35,36}\)

\[
\eta(\xi, \tau) = 1 - \left( \frac{2 - Gd^2}{2d} \right) \xi - G - \frac{G \xi^2}{2} + G \left( \frac{\cosh \left( \frac{d - 2d\xi}{2} \right)}{\cosh \left( \frac{d}{2} \right)} \right)
\]

\[
+ \frac{2}{H_1} \sum_{n=1}^{\infty} \frac{d_n h(t) e^{-(H_1 + \tau)n}}{d^n} \sin(\sigma_n \xi),
\]

where \( H_1 = \frac{\sigma_n^2 + \lambda \sigma_n^4}{Re} \).

\[
L^{-1}\left( \frac{1}{s^{1-\gamma}} \right) = h(t) = \frac{1}{\Gamma(1-\gamma)},
\]  

(15)

\[
F_\gamma(-H_1, \tau) = L^{-1}\left( \frac{1}{p^\gamma + H_1} \right) \sum_{n=0}^{\infty} \frac{(-H_1)^n e^{-(n+1)\gamma-1}}{\Gamma((n+1)\gamma)}.
\]  

(16)

\( F_\gamma(.,.) \) is defined in \( \)Lorenzo and Hartley\(^{37} \) and called Robotnov and Hartleys' function.
\[ X_0 = \frac{G(1 - (-1)^n) + \sigma_n^2 + \lambda \sigma_n^4}{\sigma_n \text{Re}}. \]

The steady-state solution of the given model is given by

\[ \eta_p(\xi) = 1 - \left( \frac{2 - Gd^2}{2d} \right) \xi - G - \frac{G\xi^2}{2} + G \left\{ \frac{\cosh \left( \frac{d-2\xi}{2} \right)}{\cosh \left( \frac{d}{2} \right)} \right\}, \quad (17) \]

and the transient solution \( \eta_t(\xi, \tau) \) is given as

\[ \eta_t(\xi, \tau) = \frac{2}{h} \sum_{n=1}^{\infty} \chi_0 h(t)^n \exp(-H_1, \tau) \sin(\sigma_n \xi). \quad (18) \]

### 3.2 Exact solutions using CF definition

Classical CSF is converted into a fractional model by applying CF fractional approach we get the following fractional model of CSF:

\[ ^{CF}D^\alpha_t \text{Re} \eta(\xi, \tau) = G + \eta_{\xi\xi} - \lambda \eta_{\xi\xi\xi\xi}, \quad (19) \]

where \(^{CF}D^\alpha_t(\cdot)\) CF fractional derivatives which are defined as\(^{17}\)

\[ ^{CF}D^\alpha_t(\tau) = \frac{1}{1 - \alpha} \int_0^\tau \exp \left( \frac{-\alpha(\tau - t)}{1 - \alpha} \right) f'(t)dt. \quad (20) \]

Applying both the Laplace and Fourier transforms jointly to Equation (19) we get:

\[ \bar{\eta}_p(n, p) = \frac{G(1 - (-1)^n) + \sigma_n^2 + \lambda \sigma_n^4}{\sigma_n p \left( \text{Re} H_2 + \sigma_n^2 + \lambda \sigma_n^4 \right)(p + H_4)}. \quad (21) \]

After some simplification we get the given equation in more appropriate form as

\[ \bar{\eta}_p(n, p) = \frac{G(1 - (-1)^n) + \sigma_n^2 + \lambda \sigma_n^4}{\sigma_n \left( \sigma_n^2 + \lambda \sigma_n^4 \right)p} + \frac{G(1 - (-1)^n) + \sigma_n^2 + \lambda \sigma_n^4}{\sigma_n \left( \text{Re} H_2 + \sigma_n^2 + \lambda \sigma_n^4 \right)(p + H_4)} \]

\[ - \frac{G(1 - (-1)^n) + \sigma_n^2 + \lambda \sigma_n^4}{\sigma_n \left( \sigma_n^2 + \lambda \sigma_n^4 \right)(p + H_4)}, \quad (22) \]

Applying the Laplace and Fourier inverse transform we get the following solutions\(^{35,36}\):
\[
\eta(\xi, \tau) = 1 - \left(\frac{2 - Gd^2}{2d}\right)\xi - G\frac{G\xi^2}{2} + G\left\{\frac{\cosh\left(\frac{d - 2\xi}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right\}
\]
\[+ \frac{2}{d} \sum_{n=1}^{\infty} A_1 \sin(\delta_n \xi)\exp(-H_4 \tau) - \frac{2}{d} \sum_{n=1}^{\infty} A_2 \sin(\delta_n \xi)\exp(-H_4 \tau).\] (23)

The above solution can be written as
\[
\eta(\xi, \tau) = 1 - \left(\frac{2 - Gd^2}{2d}\right)\xi - G\frac{G\xi^2}{2} + G\left\{\frac{\cosh\left(\frac{d - 2\xi}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right\}
\]
\[+ \frac{2}{d} \sum_{n=1}^{\infty} (A_1 - A_2)\sin(\delta_n \xi)\exp(-H_4 \tau).\] (24)

From the above solution, the steady is given by
\[
\eta(\xi, \tau) = 1 - \left(\frac{2 - Gd^2}{2d}\right)\xi - G\frac{G\xi^2}{2} + G\left\{\frac{\cosh\left(\frac{d - 2\xi}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right\}.\] (25)

The transient solution is given as follows:
\[
\eta_c(\tau) = \frac{2}{d} \sum_{n=1}^{\infty} (A_1 - A_2)\sin(\delta_n \xi)\exp(-H_4 \tau),\] (26)

where
\[
H_2 = \frac{1}{1 - \alpha} \delta_n = \frac{n\pi}{d}, \quad H_3 = \frac{\alpha}{1 - \alpha} \quad \text{and} \quad H_4 = \frac{(\delta_n^2 + \lambda\delta_n^4)H_1}{ReH_2 + \delta_n^2 + \lambda\delta_n}, \quad A_1 = \frac{G(1 - (-1)^n) + \delta_n^2 + \lambda\delta_n^4}{\delta_n(\delta_n^2 + \lambda\delta_n^4)}.
\]

\[
A_2 = \left[\frac{G(1 - (-1)^n) + \delta_n^2 + \lambda\delta_n^4}{\delta_n(\delta_n^2 + \lambda\delta_n^4)}\right]^{-1}.
\]

### 3.3 Exact solutions using AB definition

The classical CSF model is transformed into a fractional model by applying AB fractional approach we get:

\[
{^A^B}D_\tau^\beta A\eta(\xi, \tau) = G + \eta_{\text{steady}} - \lambda\eta_{\text{steady}}.\] (27)

AB time fractional derivatives is defined as\(^3^8\)
\[ AB \mathbb{D}_t^\beta(\tau) = \frac{N(\beta)}{(1-\beta)} \int_0^\tau E_\beta^\beta \left( -\beta (\tau-t)^\beta \right) f'(\tau) d\tau, \]  

(28)

where \( E_\beta \) Mittag–Leffler function is used in AB fractional derivative.\(^{39}\)

\[ E_\beta(-t^\beta) = \sum_{k=0}^\infty \frac{(-t)^{\beta k}}{\Gamma(\beta k + 1)}. \]  

(29)

Applying the Laplace and Fourier transform we obtain:

\[ \tilde{\eta}_2(n, p) = \frac{[G(1-(-1)^n) + \sigma_n^2 + \lambda \sigma_n^4]}{\sigma_n(\sigma_n^2 + \lambda \sigma_n^4) p} + \frac{[G(1-(-1)^n) + \sigma_n^2 + \lambda \sigma_n^4](M_2 - M_3)}{\sigma_n(ReM_1 + \sigma_n^2 + \lambda \sigma_n^4)(-M_3)^{\beta}(p^\beta + M_3)}. \]  

(30)

Now, applying the inverse Laplace and sine-Fourier transform we obtain the following form.\(^{35,36}\)

\[ \eta(\xi, \tau) = 1 - \left( \frac{2 - Gd^2}{2d} \right) \xi - G - \frac{G\xi^2}{2} + G \left\{ \frac{\cosh \left( \frac{d-2\xi}{2} \right)}{\cosh \left( \frac{d}{2} \right)} \right\} + \frac{2}{d} \sum_{n=0}^\infty A_3 \sin(\delta_n \xi). \]  

(31)

Steady solution is given as follows:

\[ \eta(\xi, \tau) = 1 - \left( \frac{2 - Gd^2}{2d} \right) \xi - G - \frac{G\xi^2}{2} + G \left\{ \frac{\cosh \left( \frac{d-2\xi}{2} \right)}{\cosh \left( \frac{d}{2} \right)} \right\}, \]  

(32)

unsteady solutions is given by

\[ \eta_\epsilon(\tau) = \frac{2}{d} \sum_{n=0}^\infty A_3 \sin(\delta_n \xi), \]  

(33)

where

\[ M_1 = \frac{1}{1-\beta}, \delta_n = \frac{n\pi}{d}, M_2 = \frac{\beta}{1-\beta} \quad \text{and} \quad M_3 = \frac{(\delta_n^2 + \lambda \delta_n^4)M_2}{ReM_1 + \delta_n^2 + \lambda \delta_n^4}. \]

\[ L^{-1} \left( \frac{1}{s^\beta + z} \right) = F_\beta(-z, t) = \sum_{n=0}^\infty \frac{(-z)^n t^{(n+1)\beta - 1}}{\Gamma((n+1)\beta)}. \]

\[ A_3 = \frac{[G(1-(-1)^n) + \delta_n^2 + \lambda \delta_n^4](M_2 - M_3)}{\delta_n(ReM_1 + \delta_n^2 + \lambda \delta_n^4)(-M_3)^{\beta}} \times \sum_{k=0}^\infty \frac{(-M_3)^k t^{(k+1)\beta - 1}}{\Gamma((k+1)\beta)}. \]
4 | LIMITING CASES

4.1 | Newtonian viscous fluid

By putting the CSF parameter \( \lambda = 0 \) the following Newtonian solution is obtained:

\[
\eta(\xi, \tau) = 1 - G - \left( 1 - \frac{Gd}{2} \right) \frac{G \xi^2}{2} + G \left\{ \cosh \left( \frac{d}{2} - \xi \right) \right\} \left( \frac{d}{2} \right) + G \left( -\frac{G \xi^2}{2} + \frac{G \xi^2}{2} + \frac{G \xi^2}{2} \right)
\]

\[
+ \frac{2}{d} \sum_{n=0}^{\infty} \left[ \frac{G(1 - (-1)^n) + \delta_n^2}{\delta_n(ReN_1 + \delta_n^2)}(N_2 - N_3) \right] \times \sum_{k=0}^{\infty} \left( -\frac{N_3}{\delta_n(ReN_1 + \delta_n^2)} \right)^{k+1} \sin(\delta_n \xi),
\]

where

\[
N_1 = \frac{1}{1 - \beta}, \quad \delta_n = \frac{n \pi}{d}, \quad N_2 = \frac{\beta}{1 - \beta} \quad \text{and} \quad N_3 = \frac{(\delta_n^2) N_2}{ReN_1 + \delta_n^2}.
\]

5 | CLASSICAL CSF FLUID

By putting \( \beta \to 1 \) our fractional CSF model reduces to the solutions of Akhtar and Shah.\(^{32}\)

By using the property we have:

\[
\lim_{\beta \to 1} AB D_\beta^\xi u(\xi, \tau) = \lim_{\beta \to 1} L \left\{ AB D_\beta^\xi u(\xi, \tau) \right\} = L \left\{ \lim_{\beta \to 1} \left\{ q^\beta u(\xi, q) - u(\xi, 0) \right\} \right\} = L \left\{ q^\beta u(\xi, q) - u(\xi, 0) \right\} = u'(\xi, \tau).
\]

\[
\eta'(\xi, \tau) = 1 - G - \left( 1 - \frac{Gd}{2} \right) \frac{G \xi^2}{2} + G \left\{ \cosh \left( \frac{d}{2} - \xi \right) \right\} \left( \frac{d}{2} \right) + G \left( -\frac{G \xi^2}{2} + \frac{G \xi^2}{2} + \frac{G \xi^2}{2} \right)
\]

\[
- \frac{2}{d} \sum_{n=1}^{\infty} \left[ \frac{G(1 - (-1)^n)}{\delta_n^3(1 + \delta_n^2)} + \frac{1}{\delta_n} \right] \sin(\delta_n \xi) e^{\left( \frac{\delta_n^2 + \delta_n^4}{Re} \right) \tau}.
\]

The above solution is reduced to the published results which verify our solutions and used as a limiting case.
6 | SPECIAL CASE

6.1 | CSF fluid without pressure

By putting \((G = 0)\) we get the following solution:

\[
\eta(\xi, \tau) = 1 - \left(\frac{1}{d}\right)^{\xi} + \frac{2}{d} \sum_{n=0}^{\infty} \left( \frac{[\delta_n^2 + \lambda \delta_n^4](M_2 - M_3)}{\delta_n(ReM_1 + \delta_n^2 + \lambda \delta_n^4)(-M_3)^{\frac{1}{\beta}}} \times \sum_{k=0}^{\infty} \frac{(-M_3)^k \tau^{(k+1)\beta-1}}{\Gamma((k + 1)\beta)} \right) \sin(\delta_n^{\xi}).
\]

7 | SKIN FRICITION

The skin friction is given by

\[
C_f(\xi, \tau) = \left( \frac{\partial \eta}{\partial \xi} - \lambda \frac{\partial^3 \eta}{\partial \xi^3} \right).
\]

The skin friction for lower plate:

\[
C_{f_{lp}}(0, \tau) = \left( \frac{\partial \eta}{\partial \xi} - \lambda \frac{\partial^3 \eta}{\partial \xi^3} \right).
\]

The skin friction for upper plate:

\[
C_{f_{up}}(1, \tau) = \left( \frac{\partial \eta}{\partial \xi} - \lambda \frac{\partial^3 \eta}{\partial \xi^3} \right).
\]

8 | RESULTS AND DISCUSSION

The aim of this section is to analyze the results through graphs. In the given study we have considered CSF in an open channel bounded by two parallel plates. Figure 1 shows the physical sketch of the problem. The influence of various physical parameters of interest is shown in Figures 2–10 for a clear understanding of the behavior of these physical parameters.

We have considered three different time fractional derivatives Caputo, CF and AB derivative. The comparison of these fractional operators is highlighted in Figure 2. From the graph, it can be seen that CSF velocity shows a good memory effect for AB fractional derivatives as compared to Caputo and CF fractional derivatives and classical CSF.

It is necessary to discuss here that in the results and discussion section in all the graphs we have compared the classical CSF with time-fractional Caputo, CF, and AB fractional derivatives. Furthermore, in this study, we have considered \(\gamma, \alpha\) and \(\beta\) as fractional operators for Caputo, CF, and AB derivatives respectively.
The different values of Caputo fractional operator $\gamma$ on CSF velocity are highlighted in Figure 3. From the graph, the variation in CSF velocity can be noticed. From the figure, it is depicted that CSF velocity of fractional order is between zero and one, while for classical CSF the value is ($\gamma = 1$) are plotted to see the differences clearly. From the graph, it is clear that for different values of fractional parameter $\gamma$ variation in CSF velocity can be observed. The fractional solution of CSF is the more general solution for the classical solutions when $\gamma = 1$ our obtained result is quite identical to solutions which verifies our solutions.

**FIGURE 2** Comparison of Caputo, CF, AB and classical CSF velocity when $\eta = 1.2$, $G = 8$, $Re = 0.6$, $t = 0.8$, $\alpha = \beta = \gamma = 0.5$. AB, Atangana–Baleanu; CF, Caputo–Fabrizio; CSF, couple stress fluid.

**FIGURE 3** The impact of various values of Caputo fractional parameter $\gamma$ on CSF velocity when $\lambda = 1.2$, $G = 10$, $Re = 0.6$, $t = 0.8$. CSF, couple stress fluid.
The increasing values of CF fractional operator $\alpha$ on CSF velocity are highlighted in Figure 4. From the graph, the variation in CSF velocity can be noticed. From the figure, it is depicted that CSF velocity of fractional order is between zero and one, while for classical CSF the value is ($\alpha = 1$) are plotted to see the differences clearly. From the graph, it is clear that for different values of fractional parameter $\alpha$ variation in CSF velocity can be observed. The fractional solution of CSF is the more general solution for the classical solutions when $\alpha = 1$ our obtained result is quite identical to solutions$^{32}$ that verify our solutions.

The change in CSF velocity for different values of AB fractional operator $\beta$ is highlighted in Figure 5. From the figure, it can be observed that different values of AB fractional operator $\beta$
result in a variation in CSF velocity. From the figure, it is depicted that CSFs of fractional order and classical CSF are compared to observe the differences clearly. It can be noted that for all the three classical CSF cases when \( \alpha = \beta = \gamma = 1 \) our obtained result in all cases reduced to solutions obtained in Akhtar and Shah\(^{32} \) which verifies our solutions.

Figure 6 highlights the comparison of a CSF velocity profile for C, CF, AB, and classical velocity with the variation of external pressure gradient \( G \) when \( Re = 0.4, \tau = 0.7, \alpha = \beta = \gamma = 0.5, \) and \( \lambda = 0.8. \) From this figure it is found that increasing the values of \( G \) from \( G = 8 \) to \( G = 10 \) CSF velocity increases for C, CF, AB, and for the case of classical CSF. This behavior of \( G \) on CSF velocity is obvious because external pressure accelerates fluid motion.

Figure 7 shows the variation of Reynolds number \( Re \) on velocity profile when \( G = 8, \alpha = \beta = \gamma = 0.5, \tau = 0.7, \) and \( \lambda = 0.8. \) From this figure, one can notice that for higher Reynolds number \( Re \) from \( Re = 0.4 \) to \( Re = 0.6 \) CSF velocity decreases. This shows that CSF velocity is controlled by \( Re \) in an open channel.

Figure 8 depicts the comparison of CSF velocity profile for C, CF, AB, and classical velocity with the variation of time \( \tau \) when \( G = 8, \alpha = \beta = \gamma = 0.5, \) \( Re = 0.4, \) and \( \lambda = 0.8. \) From the graph, we see that for the higher values of time from \( \tau = 0.7 \) to \( \tau = 0.9 \) the fractional and classical CSF velocities increase. As we have considered the unsteady CSF in a channel that is time-dependent. Increasing time CSF velocity of course increases.

Figure 9 displays the variation of CSF parameter \( \lambda \) when \( G = 8, \alpha = \beta = \gamma = 0.5, \tau = 0.7, \) and \( Re= 0.8. \) From the figure, CSF velocity is lowered for higher values of \( \lambda. \) The couple stress parameter increases from \( \lambda = 0.8 \) to \( \lambda = 1.2 \) as a result fluid velocity decreases. From this graph, it is clear that CSFs are more viscous as compared to Newtonian viscous fluids.

The CSF flow with and without \( G \) is highlighted in Figure 10. From the graphical analysis, it is quite clear that CSF velocity with \( G \) is higher than CSF velocity without \( G. \) This figure clearly shows the effect of \( G \) on CSF velocity that \( G \) is responsible to accelerate the CSF motion in a channel.

The Newtonian viscous fluid is compared with CSF fluid which is highlighted in Figure 11. From the graphical results, we have CSF velocity with couple tress parameter \( \lambda \) this lowered as

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**FIGURE 6**  The impact of external pressure gradient \( G \) on CSF velocity for C, CF, AB, and classical velocity when \( Re = 0.4, \tau = 0.7, \alpha = \beta = \gamma = 0.5, \) and \( \lambda = 0.8. \) AB, Atangana–Baleanu; CF, Caputo–Fabrizio; CSF, couple stress fluid.
compared to Newtonian viscous fluid velocity with $\lambda = 0$. In other words, the velocity of simple Newtonian fluid is higher than CSF velocity.

Table 1 shows the skin friction variation for the lower plate. In this table, the numerical values of skin fraction of C, CF, AB, and classical results are evaluated and presented in Table 1. The bold values given in Table 1 shows the impact of that specific parameter on the skin friction by keeping all other parameters fixed. Similarly, in Table 2 the numerical values of skin friction variation at the upper plate are listed. In this table also the numerical values of skin fraction of C, CF, AB, and classical results are evaluated and presented in Table 2. The bold values in this table also provide the effect of that specific parameter on the skin friction keeping all other values constant. In both of the tables, the bold numerical values show the skin friction variation for $G$, $Re$, $\tau$, $\lambda$, $\alpha$, $\beta$, and $\gamma$ in that specific row.

**FIGURE 7** The impact of Reynolds number $Re$ on CSF velocity for C, CF, AB, and classical velocity when $G = 8$, $\alpha = \beta = \gamma = 0.5$, $\tau = 0.7$, and $\lambda = 0.8$. AB, Atangana–Baleanu; CF, Caputo–Fabrizio; CSF, couple stress fluid.

**FIGURE 8** The impact of time $\tau$ on velocity profile for C, CF, AB and classical CSF when $G = 8$, $\alpha = \beta = \gamma = 0.5$, $Re = 0.4$, and $\lambda = 0.8$. AB, Atangana–Baleanu; CF, Caputo–Fabrizio; CSF, couple stress fluid.
The purpose of the present work is to evaluate and obtained the closed form solutions for the CSF in an open channel using the integral transforms (Laplace and Fourier). The results obtained for Caputo, CF, AB, and classical CSF are compared through graphs. Furthermore, the effect of some physical parameters is depicted which effect the CSF motion. Finally, some special cases were reproduced to published results which verify our obtained solutions. Additionally, from the comparative analysis of C, CF, AB, and classical CSF through graphs we can observe that CSF velocity for AB fractional derivative shows a good memory effect of fluid dynamics as compared to C and CF and the classical CSF velocity.

Some important points are noted during this study which are given below.
The influence of C, CF, AB fractional derivatives, and classical CSF is displayed in the velocity profile. From the figure, it can be seen that CSF velocity for AB fractional derivative is maximum as compared to all other velocities. It is also noted that classical CSF velocity has minimum magnitude.

- Increasing $G$ CSF velocity increases.
- Increasing $Re$ the magnitude of CSF velocity decreases in cases of C, CF, AB, and classical CSF.
- Increasing $\tau$ velocity of the CSF increase.
- AB fractional derivatives have a good memory effect as compared to C, CF, and Classical CSF.

### TABLE 1  The skin friction at lower plate for Caputo, CF, AB and classical CSF

| $G$ | $Re$ | $\tau$ | $\lambda$ | $\gamma$ | $\alpha$ | $\beta$ | $Cf_\gamma$ | $Cf_\alpha$ | $Cf_\beta$ | $Cf_{\text{classic}}$ |
|-----|------|--------|------------|-----------|----------|---------|-------------|-------------|------------|-----------------|
| 8   | 0.6  | 0.8    | 1.2        | 0.5       | 0.5      | 0.5     | 0.356       | 1.372       | 2.79       | 0.41            |
| 9   | 0.6  | 0.8    | 1.2        | 0.5       | 0.5      | 0.5     | 0.805       | 2.586       | 5.103      | 0.259           |
| 8   | **0.8** | 0.8    | 1.2        | 0.5       | 0.5      | 0.5     | 0.144       | 0.261       | 0.887      | 0.603           |
| 8   | 0.6  | **0.9** | 1.2        | 0.5       | 0.5      | 0.5     | 0.996       | 2.751       | 5.413      | 0.148           |
| 8   | 0.6  | 0.8    | **1.4**    | 0.5       | 0.5      | 0.5     | 0.33        | 1.39        | 2.902      | 0.442           |
| 8   | 0.6  | 0.8    | 1.2        | **0.7**   | 0.5      | 0.5     | 0.09        | 1.372       | 2.79       | 0.41            |
| 8   | 0.6  | 0.8    | 1.2        | 0.5       | **0.7**  | 0.5     | 0.356       | 0.269       | 2.79       | 0.41            |
| 8   | 0.6  | 0.8    | 1.2        | 0.5       | 0.5      | **0.7**  | 0.356       | 1.372       | 0.767      | 0.41            |

Abbreviations: AB, Atangana–Baleanu; CF, Caputo–Fabrizio; CSF, couple stress fluid.
The bold values show the impact of that specific parameter on the skin friction by keeping all other parameters fixed.
| $G$ | $Re$ | $\tau$ | $\lambda$ | $\gamma$ | $\alpha$ | $\beta$ | $Cf_\gamma$ | $Cf_\alpha$ | $Cf_\beta$ | $Cf_{classic}$ |
|-----|------|-------|-----------|---------|--------|--------|-----------|----------|----------|-------------|
| 8   | 0.6  | 0.8   | 1.2       | 0.5     | 0.5    | 0.5    | 1.611     | 2.1      | 2.778    | 1.244       |
| 9   | 0.6  | 0.8   | 1.2       | 0.5     | 0.5    | 0.5    | 1.821     | 2.675    | 3.879    | 1.312       |
| 8   | 0.7  | 0.8   | 1.2       | 0.5     | 0.5    | 0.5    | 1.471     | 1.767    | 2.193    | 1.192       |
| 8   | 0.6  | 0.9   | 1.2       | 0.5     | 0.5    | 0.5    | 1.912     | 2.754    | 4.027    | 1.364       |
| 8   | 0.6  | 0.8   | 1.3       | 0.5     | 0.5    | 0.5    | 1.61      | 2.112    | 2.817    | 1.245       |
| 8   | 0.6  | 0.8   | 0.7       | 0.5     | 0.5    | 0.5    | 1.396     | 2.1      | 2.778    | 1.244       |
| 8   | 0.6  | 0.8   | 1.2       | 0.5     | 0.7    | 0.5    | 1.611     | 1.571    | 2.778    | 1.244       |
| 8   | 0.6  | 0.8   | 1.2       | 0.5     | 0.7    | 0.5    | 1.611     | 2.1      | 1.81     | 1.244       |

Abbreviations: AB, Atangana–Baleanu; CF, Caputo–Fabrizio; CSF, couple stress fluid.
The bold values show the impact of that specific parameter on the skin friction by keeping all other parameters fixed.

NOMENCLATURE

CSF  couple stress fluid
AB  Atangana–Baleanu
CF  Caputo–Fabrizio
$\rho$  density
$u$  velocity
$\vec{V}$  velocity in vector form
$\mu$  dynamic viscosity
$p$  pressure
$b_1$  body forces
$\lambda$  couple stress parameter
$d$  distance between the plates
$U_0$  constant velocity
$H(\tau)$  heaviside step function
$Re$  Reynolds number
$G$  constant pressure gradient
$\tau$  time
$AB^{\beta}(\cdot)$  Atangana–Baleanu fractional derivative
$CF^{\gamma}(\cdot)$  Caputo–Fabrizio fractional derivative
$C^{\alpha}(\cdot)$  Caputo fractional derivative
$N(\beta)$  normalization function
$\beta$  AB fractional operator of
$\alpha$  CF fractional operator
$\gamma$  Caputo fractional operator
$E_\beta$  Mittag–Leffler function
$F_\alpha(\cdot)$  Robotnov and Hartleys’ function
$\eta_p(\xi)$  steady-state velocity
$\eta_r(\xi, \tau)$  unsteady velocity
$\alpha$  fractional operator of Caputo–Fabrizio
$Cf(\xi, \tau)$  skin friction
$C_\tau(0, \tau)$ skin friction for the lower plate

$C_\tau(1, \tau)$ skin friction for the upper plate

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CONFLICT OF INTEREST
The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT
All data used in this manuscript have been presented within the manuscript. No data is hidden or restricted.

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