Homotopy of unitaries in simple $C^*$-algebras with tracial rank one

Huaxin Lin

Abstract

Let $\epsilon > 0$ be a positive number. Is there a number $\delta > 0$ satisfying the following? Given any pair of unitaries $u$ and $v$ in a unital simple $C^*$-algebra $A$ with $[v] = 0$ in $K_1(A)$ for which

$$\|uv - vu\| < \delta,$$

there is a continuous path of unitaries $\{v(t) : t \in [0, 1]\} \subset A$ such that

$$v(0) = v, \ v(1) = 1 \text{ and } \|uv(t) - v(t)u\| < \epsilon \text{ for all } t \in [0, 1].$$

An answer is given to this question when $A$ is assumed to be a unital simple $C^*$-algebra with tracial rank no more than one. Let $C$ be a unital separable amenable simple $C^*$-algebra with tracial rank no more than one which also satisfies the UCT. Suppose that $\phi : C \to A$ is a unital monomorphism and suppose that $v \in A$ is a unitary with $[v] = 0$ in $K_1(A)$ such that $v$ almost commutes with $\phi$. It is shown that there is a continuous path of unitaries $\{v(t) : t \in [0, 1]\}$ in $A$ with $v(0) = v$ and $v(1) = 1$ such that the entire path $v(t)$ almost commutes with $\phi$, provided that an induced Bott map vanishes. Other versions of the so-called Basic Homotopy Lemma are also presented.

1 Introduction

Fix a positive number $\epsilon > 0$. Can one find a positive number $\delta$ such that, for any pair of unitary matrices $u$ and $v$ with $\|uv - vu\| < \delta$, there exists a continuous path of unitary matrices $\{v(t) : t \in [0, 1]\}$ for which $v(0) = v, \ v(1) = 1$ and $\|uv(t) - v(t)u\| < \epsilon$ for all $t \in [0, 1]$? The answer is negative in general. A Bott element associated with the pair of unitary matrices may appear. The hidden topological obstruction can be detected in a limit process. This was first found by Dan Voiculescu (143). On the other hand, it has been proved that there is such a path of unitary matrices if an additional condition, $\text{bott}_1(u, v) = 0$, is provided (see, for example, [2] and also 3.12 of [29]).

It was recognized by Bratteli, Elliott, Evans and A. Kishimoto (2) that the presence of such continuous path of unitaries in general simple $C^*$-algebras played an important role in the study of classification of simple $C^*$-algebras and perhaps plays important roles in some other areas. They proved what they called the Basic Homotopy Lemma: For any $\epsilon > 0$, there exists $\delta > 0$ satisfying the following: For any pair of unitaries $u$ and $v$ in $A$ with $\text{sp}(u) \delta$-dense in $\mathbb{T}$ and $[v] = 0$ in $K_1(A)$ for which

$$\|uv - vu\| < \delta \text{ and } \text{bott}_1(u, v) = 0,$$

there exists a continuous path of unitaries $\{v(t) : t \in [0, 1]\} \subset A$ such that

$$v(0) = v, \ v(1) = 1_A \text{ and } \|v(t)u - uv(t)\| < \epsilon$$

for all $t \in [0, 1]$, where $A$ is a unital purely infinite simple $C^*$-algebra or a unital simple $C^*$-algebra with real rank zero and stable rank one. Define $\phi : C(\mathbb{T}) \to A$ by $\phi(f) = f(u)$ for all
\( f \in C(\mathbb{T}) \). Instead of considering a pair of unitaries, one may consider a unital homomorphism from \( C(\mathbb{T}) \) into \( A \) and a unitary \( v \in A \) for which \( v \) almost commutes with \( \phi \).

In the study of asymptotic unitary equivalence of homomorphisms from an AH-algebra to a unital simple \( C^* \)-algebra, as well as the study of homotopy theory in simple \( C^* \)-algebras, one considers the following problem: Suppose that \( X \) is a compact metric space and \( \phi \) is a unital homomorphism from \( C(X) \) into a unital simple \( C^* \)-algebra \( A \). Suppose that there is a unitary \( u \in A \) with \([u] = 0\) in \( K_1(A) \) and \( u \) almost commutes with \( \phi \). When can one find a continuous path of unitaries \( \{u(t) : t \in [0,1]\} \subset A \) with \( u(0) = u \) and \( u(1) = 1 \) such that \( u(t) \) almost commutes with \( \phi \) for all \( t \in [0,1] \)?

Let \( C \) be a unital AH-algebra and let \( A \) be a unital simple \( C^* \)-algebra. Suppose that \( \phi, \psi : C \to A \) are two unital monomorphisms. Let us consider the question when \( \phi \) and \( \psi \) are asymptotically unitarily equivalent, i.e., when there is a continuous path of unitaries \( \{w(t) : t \in [0, \infty)\} \subset A \) such that

\[
\lim_{t \to \infty} w(t)^* \phi(c) w(t) = \psi(c) \text{ for all } c \in C.
\]

When \( A \) is assumed to have tracial rank zero, it was proved in [31] that they are asymptotically unitarily equivalent if and only if \([\phi] = [\psi]\) in \( KK(C, A) \), \( \tau \circ \phi = \tau \circ \psi \) for all tracial states \( \tau \) of \( A \) and a rotation map associated with \( \phi \) and \( \psi \) is zero. Apart from some direct applications, this result plays crucial roles in the study of the problem to embed crossed products into unital simple AF-algebras and in the classification of simple amenable \( C^* \)-algebras which do not have the finite tracial rank (see [44], [32] and [33]). One of the key machinery in the study of the above mentioned asymptotic unitary equivalence is the so-called Basic Homotopy Lemma concerning a unital monomorphism \( \phi \) and a unitary \( u \) which almost commutes with \( \phi \).

In this paper, we study the case that \( A \) is no longer assumed to have real rank zero, or tracial rank zero. The result of W. Winter in [44] provides the possible classification of simple finite \( C^* \)-algebras far beyond the cases of finite tracial rank. However, it requires to understand much more about asymptotic unitary equivalence in those unital separable simple \( C^* \)-algebras which have been classified. An immediate problem is to give a classification of monomorphisms (up to asymptotic unitary equivalence) from a unital separable simple AH-algebra into a unital separable simple \( C^* \)-algebra with tracial rank one. For that goal, it is paramount to study the Basic Homotopy Lemmas in a simple separable \( C^* \)-algebras with tracial rank one. This is the main purpose of this paper.

A number of problems occur when one replaces \( C^* \)-algebras of tracial rank zero by those of tracial rank one. First, one has to deal with contractive completely positive linear maps from \( C(X) \) into a unital \( C^* \)-algebra \( C \) with the form \( C([0,1],M_n) \) which are not homomorphisms but almost multiplicative. Such problem is already difficult when \( C = M_n \) but it has been proved that these above mentioned maps are close to homomorphisms if the associated \( K \)-theoretical data of these maps are consistent with those of homomorphisms. It is problematic when one tries to replace \( M_n \) by \( C([0,1],M_n) \). In addition to the usual \( K \)-theory and trace information, one also has to handle the maps from \( U(C)/CU(C) \) to \( U(A)/CU(A) \), where \( CU(C) \) and \( CU(A) \) are the closure of the subgroups of \( U(C) \) and \( U(A) \) generated by commutators, respectively. Other problems occur because of lack of projections in \( C^* \)-algebras which are not of real rank zero.

The main theorem is stated as follows: Let \( C \) be a unital separable simple amenable \( C^* \)-algebra with tracial rank one which satisfies the Universal Coefficient Theorem. For any \( \epsilon > 0 \) and any finite subset \( \mathcal{F} \subset C \), there exists \( \delta > 0 \), a finite subset \( \mathcal{G} \subset C \) and a finite subset \( \mathcal{P} \subset K(C) \) satisfying the following:

Suppose that \( A \) is a unital simple \( C^* \)-algebra with tracial rank no more than one, suppose
that $\phi : C \to A$ is a unital homomorphism and $u \in U(A)$ such that
\[ \|\phi(c), u\| < \delta \text{ for all } c \in G \text{ and } \text{Bott}(\phi, u)|_{\mathcal{P}} = 0. \quad \text{(e1.1)} \]

Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ such that
\[ u(0) = u, \ u(1) = 1 \text{ and } \|\phi(c), u(t)\| < \epsilon \text{ for all } c \in \mathcal{F} \quad \text{(e1.2)} \]

and for all $t \in [0,1]$.

We also give the following Basic Homotopy Lemma in simple $C^*$-algebra with tracial rank one (see [3.3] below):

Let $\epsilon > 0$ and let $\Delta : (0,1) \to (0,1)$ be a nondecreasing map. We show that there exists $\delta > 0$ and $\eta > 0$ (which does not depend on $\Delta$) satisfying the following:

Given any pair of unitaries $u$ and $v$ in a unital simple $C^*$-algebra $A$ with tracial rank no more than one such that $[v] = 0$ in $K_1(A),
\|u, v\| < \delta, \ \text{bott}_1(u, v) = 0 \text{ and } \mu_{\tau_0}(I_a) \geq \Delta(a)$

for all open arcs $I_a$ with length $a \geq \eta$, there exists a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset A$ such that
\[ v(0) = v, \ v(1) = 1 \text{ and } \|u, v(t)\| < \epsilon \text{ for all } t \in [0,1], \]

where $\iota : C(\mathbb{T}) \to A$ is the homomorphism defined by $\iota(f) = f(u)$ for all $f \in C(\mathbb{T})$ and $\mu_{\tau_0}$ is the Borel probability measure induced by the state $\tau \circ \iota$. It should be noted that, unlike the case that $A$ has real rank zero, the length of $\{v(t)\}$ cannot be controlled. In fact, it could be as long as one wishes.

In a subsequent paper, we use the main homotopy result (Theorem 8.4) of this paper and the results in [34] to establish a $K$-theoretical necessary and sufficient condition for homomorphisms from unital simple AH-algebras into a unital separable simple $C^*$-algebra with tracial rank no more than one to be asymptotically unitarily equivalent which, in turn, combining with a result of W. Winter, provides a classification theorem for a class of unital separable simple amenable $C^*$-algebras which properly contains all unital separable simple amenable $C^*$-algebras with tracial rank no more than one which satisfy the UCT as well as some projectionless $C^*$-algebras such as the Jiang-Su algebra.

2 Preliminaries and notation

2.1. Let $A$ be a unital $C^*$-algebra. Denote by $T(A)$ the tracial state space of $A$ and denote by $\text{Aff}(T(A))$ the set of affine continuous functions on $T(A)$.

Let $C = C(X)$ for some compact metric space $X$ and let $L : C \to A$ be a unital positive linear map. Denote by $\mu_{\tau_0 L}$ the Borel probability measure induced by the state $\tau \circ L$, where $\tau \in T(A)$.

2.2. Let $a$ and $b$ be two elements in a $C^*$-algebra $A$ and let $\epsilon > 0$ be a positive number. We write $a \approx_{\epsilon} b$ if $\|a - b\| < \epsilon$. Let $L_1, L_2 : A \to C$ be two maps from $A$ to another $C^*$-algebra $C$ and let $\mathcal{F} \subset A$ be a subset. We write
\[ L_1 \approx_{\epsilon} L_2 \text{ on } \mathcal{F}, \]

if $L_1(a) \approx_{\epsilon} L_2(a)$ for all $a \in \mathcal{F}$.

Suppose that $B \subset A$. We write $a \in_{\epsilon} B$ if there is an element $b \in B$ such that $\|a - b\| < \epsilon$. 
Let $G \subset A$ be a subset. We say $L$ is $\epsilon$-$G$-multiplicative if, for any $a, b \in G$,

$$L(ab) \approx \epsilon L(a)L(b)$$

for all $a, b \in G$.

2.3. Let $A$ be a unital $C^*$-algebra. Denote by $U(A)$ the unitary group of $A$. Denote by $U_0(A)$ the normal subgroup of $U(A)$ consisting of those unitaries in the path connected component of $U(A)$ containing the identity. Let $u \in U_0(A)$. Define

$$\text{cel}_A(u) = \inf \{\text{length}\{u(t)\} : u(t) \in C([0,1], U_0(A)), u(0) = u \text{ and } u(1) = 1_A\}.$$

We use $\text{cel}(u)$ if the $C^*$-algebra $A$ is not in question.

2.4. Denote by $CU(A)$ the closure of the subgroup generated by the commutators of $U(A)$. For $u \in U(A)$, we will use $\bar{u}$ for the image of $u$ in $U(A)/CU(A)$. If $\bar{u}, \bar{v} \in U(A)/CU(A)$, define

$$\text{dist}(\bar{u}, \bar{v}) = \inf\{|x - y| : x, y \in U(A) \text{ such that } \bar{x} = \bar{u}, \bar{y} = \bar{v}\}.$$

If $u, v \in U(A)$, then

$$\text{dist}(\bar{u}, \bar{v}) = \inf\{|uv^* - x| : x \in CU(A)\}.$$

2.5. Let $A$ and $B$ be two unital $C^*$-algebras and let $\phi : A \rightarrow B$ be a unital homomorphism. It is easy to check that $\phi$ maps $CU(A)$ to $CU(B)$. Denote by $\phi^\dagger$ the homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$ induced by $\phi$. We also use $\phi^\dagger$ for the homomorphism from $U(M_k(A))/CU(M_k(A))$ into $U(M_k(B))/CU(M_k(B))$ ($k = 1, 2, \ldots$).

2.6. Let $A$ and $C$ be two unital $C^*$-algebras and let $F \subset U(C)$ be a subgroup of $U(C)$. Suppose that $L : F \rightarrow U(A)$ is a homomorphism for which $L(F \cap CU(C)) \subset CU(A)$. We will use $L^\dagger : F/CU(C) \rightarrow U(A)/CU(A)$ for the induced map.

2.7. Let $A$ and $B$ be in 2.6 let $1 > \epsilon > 0$ and let $G \subset A$ be a subset. Suppose that $L$ is $\epsilon$-$G$-multiplicative unital completely positive linear map. Suppose that $u, u^* \in G$. Define $\langle L \rangle(u) = L(u)L(u^*u)^{-1/2}$.

**Definition 2.8.** Let $A$ and $B$ be two unital $C^*$-algebras. Let $h : A \rightarrow B$ be a homomorphism and let $v \in U(B)$ such that

$$h(g)v = vh(g) \text{ for all } g \in A.$$

Thus we obtain a homomorphism $\tilde{h} : A \otimes C(S^1) \rightarrow B$ by $\tilde{h}(f \otimes g) = h(f)g(v)$ for $f \in A$ and $g \in C(S^1)$. From the following splitting exact sequence:

$$0 \rightarrow SA \rightarrow A \otimes C(S^1) \xrightarrow{\alpha} A \rightarrow 0 \tag{e 2.3}$$

and the isomorphisms $K_i(A) \rightarrow K_{1-i}(SA)$ ($i = 0, 1$) given by the Bott periodicity, one obtains two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \rightarrow K_1(A \otimes C(S^1)) \tag{e 2.4}$$

$$\beta^{(1)} : K_1(A) \rightarrow K_0(A \otimes C(S^1)). \tag{e 2.5}$$

Note, in this way, one can write $K_1(A \otimes C(S^1)) = K_1(A) \bigoplus \beta^{(1-i)}(K_1-i(A))$. We use $\widehat{\beta^{(i)}} : K_i(A \otimes C(S^1)) \rightarrow \beta^{(1-i)}(K_{1-i}(A))$ for the the projection to the summand $\beta^{(1-i)}(K_{1-i}(A))$.

For each integer $k \geq 2$, one also obtains the following injective homomorphisms:

$$\beta^{(i)}_k : K_i(A, \mathbb{Z}/k\mathbb{Z}) \rightarrow K_{1-i}(A \otimes C(S^1), \mathbb{Z}/k\mathbb{Z}), i = 0, 1. \tag{e 2.6}$$
Thus we write
\[K_{1-i}(A \otimes C(S^1), \mathbb{Z}/k\mathbb{Z}) = K_{1-i}(A, \mathbb{Z}/k\mathbb{Z}) \bigoplus \beta_k^{(i)}(K_i(A, \mathbb{Z}/k\mathbb{Z})), \quad i = 0, 1.\] (e 2.7)

Denote by \(\beta_k^{(i)} : K_i(A \otimes C(S^1), \mathbb{Z}/k\mathbb{Z}) \to \beta_k^{(1-i)}(K_{1-i}(A, \mathbb{Z}/k\mathbb{Z}))\) similarly to that of \(\beta^{(i)}, i = 1, 2\).

If \(x \in K_i(A)\), we use \(\beta(x)\) for \(\beta^{(i)}(x)\) if \(x \in K_1(A)\) and for \(\beta_k^{(i)}(x)\) if \(x \in K_i(A, \mathbb{Z}/k\mathbb{Z})\). Thus we have a map \(\beta : K(A) \to K(A \otimes C(S^1))\) as well as \(\tilde{\beta} : K(A \otimes C(S^1)) \to \beta(K(A))\). Thus one may write \(K(A \otimes C(S^1)) = K(A) \bigoplus \beta(K(A))\).

On the other hand \(\bar{h}\) induces homomorphisms \(\bar{h}_{i,k} : K_i(A \otimes C(S^1)), \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}),\)

\(k = 0, 2, \ldots, \) and \(i = 0, 1\). We use \(\text{Bott}(h, v)\) for all homomorphisms \(\bar{h}_{i,k} \circ \beta_k^{(i)}\). We write

\[\text{Bott}(h, v) = 0,\]

if \(\bar{h}_{i,k} \circ \beta_k^{(i)} = 0\) for all \(k \geq 1\) and \(i = 0, 1\).

We will use \(\text{bott}_1(h, v)\) for the homomorphism \(\bar{h}_{1,0} \circ \beta^{(1)} : K_1(A) \to K_0(B),\) and \(\text{bott}_0(h, u)\) for the homomorphism \(\bar{h}_{0,0} \circ \beta^{(0)} : K_0(A) \to K_1(B)\).

Since \(A\) is unital, if \(\text{bott}_0(h, v) = 0\), then \(|v| = 0\) in \(K_1(B)\).

For a fixed finite subset \(\mathcal{P} \subset K(A)\), there exists \(\delta > 0\) and a finite subset \(\mathcal{G} \subset A\) such that, if \(v \in B\) is a unitary for which

\[\|h(a)v - vh(a)\| < \delta \quad \text{for all} \quad a \in \mathcal{G},\]

then \(\text{Bott}(h, v)\mid_{\mathcal{P}}\) is well defined. In what follows, whenever we write \(\text{Bott}(h, v)\mid_{\mathcal{P}}\), we mean that \(\delta\) is sufficiently small and \(\mathcal{G}\) is sufficiently large so it is well defined.

Now suppose that \(A\) is also amenable and \(K_i(A)\) is finitely generated \((i = 0, 1)\). For example, \(A = C(X)\), where \(X\) is a finite CW complex. When \(K_i(A)\) is finitely generated, \(\text{Bott}(h, v)\mid_{\mathcal{P}_0}\) defines \(\text{Bott}(h, v)\) for some sufficiently large finite subset \(\mathcal{P}_0\). In what follows such \(\mathcal{P}_0\) may be denoted by \(\mathcal{P}_A\). Suppose that \(\mathcal{P} \subset K(A)\) is a larger finite subset, and \(\mathcal{G} \supset \mathcal{G}_0\) and \(0 < \delta < \delta_0\).

A fact that we be used in this paper is that, \(\text{Bott}(h, v)\mid_{\mathcal{P}}\) defines the same map \(\text{Bott}(h, v)\) as \(\text{Bott}(h, v)\mid_{\mathcal{P}_0}\) defines, if

\[\|h(a)v - vh(a)\| \leq \delta \quad \text{for all} \quad a \in \mathcal{G},\]

when \(K_i(A)\) is finitely generated. In what follows, in the case that \(K_i(A)\) is finitely generated, whenever we write \(\text{Bott}(h, v)\), we always assume that \(\delta\) is smaller than \(\delta_0\) and \(\mathcal{G}\) is larger than \(\mathcal{G}_0\) so that \(\text{Bott}(h, v)\) is well-defined (see 2.10 of [29] for more details).

2.9. In the case that \(A = C(S^1)\), there is a concrete way to visualize \(\text{bott}_1(h, v)\). It is perhaps helpful to describe it here. The map \(\text{bott}_1(h, v)\) is determined by \(\text{bott}_1(h, v)([z]),\) where \(z\) is the identity map on the unit circle.

Denote \(u = h(z)\) and define

\[f(e^{2\pi it}) = \begin{cases} 1 - 2t, & \text{if } 0 \leq t \leq 1/2, \\ -1 + 2t, & \text{if } 1/2 < t \leq 1, \end{cases}\]

\[g(e^{2\pi it}) = \begin{cases} (f(e^{2\pi it}) - f(e^{2\pi it})^2)^{1/2} & \text{if } 0 \leq t \leq 1/2, \\ 0, & \text{if } 1/2 < t \leq 1 \end{cases}\]

and

\[h(e^{2\pi it}) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1/2, \\ (f(e^{2\pi it}) - f(e^{2\pi it})^2)^{1/2}, & \text{if } 1/2 < t \leq 1, \end{cases}\]
These are non-negative continuous functions defined on the unit circle. Suppose that \( uv = vu \).

Define
\[
\mathbf{b}(u, v) = \begin{pmatrix} f(v) & g(v) + h(v)u^* \\ g(v) + uh(v) & 1 - f(v) \end{pmatrix}
\] (e 2.8)

Then \( \mathbf{b}(u, v) \) is a projection. There is \( \delta_0 > 0 \) (independent of unitaries \( u, v \) and \( A \)) such that if \( \| [u, v] \| < \delta_0 \), the spectrum of the positive element \( \mathbf{p}(u, v) \) has a gap at 1/2. The bott element of \( u \) and \( v \) is an element in \( K_0(A) \) as defined in [11] and [12] which may be represented by
\[
bott_1(u, v) = [\chi_{[1/2, \infty)}(\mathbf{b}(u, v))] - \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right].
\] (e 2.9)

Note that \( \chi_{[1/2, \infty)} \) is a continuous function on \( \text{sp}(\mathbf{b}(u, v)) \). Suppose that \( \text{sp}(\mathbf{b}(u, v)) \subset (-\infty, a] \cup [1 - a, \infty) \) for some \( 0 < a < 1/2 \). Then \( \chi_{[1/2, \infty)} \) can be replaced by any other positive continuous function \( F \) for which \( F(t) = 0 \) if \( t \leq a \) and \( F(t) = 1 \) if \( t \geq 1/2 \).

**Definition 2.10.** Let \( A \) and \( C \) be two unital \( C^* \)-algebras. Let \( N : C_+ \setminus \{0\} \to \mathbb{N} \) and \( K : C_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\} \) be two maps. Define \( T = N \times K : C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\} \) by \( T(c) = (N(c), K(c)) \) for \( c \in C_+ \setminus \{0\} \). Let \( L : C \to A \) be a unital positive linear map. We say \( L \) is \( T \)-full if for any \( c \in C_+ \setminus \{0\} \), there are \( x_1, x_2, \ldots, x_{N(c)} \in A \) with \( \|x_i\| \leq K(c) \) such that
\[
\sum_{i=1}^{N(c)} x_i^* L(c) x_i = 1_A.
\]

Let \( \mathcal{H} \subset C_+ \setminus \{0\} \). We say that \( L \) is \( T \)-\( \mathcal{H} \)-full if
\[
\sum_{i=1}^{N(c)} x_i^* L(c) x_i = 1_A
\]
for all \( c \in \mathcal{H} \).

**Definition 2.11.** Denote by \( \mathcal{I} \) the class of unital \( C^* \)-algebras with the form \( \bigoplus_{i=1}^m C(X_i, M_{n(i)}) \), where \( X_i = [0, 1] \) or \( X_i \) is one point.

**Definition 2.12.** Let \( k \geq 0 \) be an integer. Denote by \( \mathcal{I}_k \) the class of all \( C^* \)-algebras \( B \) with the form \( B = PM_m(C(X))P \), where \( X \) is a finite CW complex with dimension no more than \( k \), \( P \) is a projection in \( M_m(C(X)) \).

Recall that a unital simple \( C^* \)-algebra \( A \) is said to have tracial rank no more than \( k \) (write \( TR(A) \leq k \)) if the following holds: For any \( \epsilon > 0 \), any positive element \( a \in A_+ \setminus \{0\} \) and any finite subset \( F \subset A \), there exists a non-zero projection \( p \in A \) and a \( C^* \)-subalgebra \( B \in \mathcal{I}_k \) with \( 1_B = p \) such that
\begin{enumerate}
  \item \( \|x_p - px\| < \epsilon \) for all \( x \in F \);
  \item \( pxp \in B \) for all \( x \in F \) and
  \item \( 1 - p \) is von Neumann equivalent to a projection in \( aAa \).
\end{enumerate}

If \( TR(A) \leq k \) and \( TR(A) \neq k - 1 \), we say \( A \) has tracial rank \( k \) and write \( TR(A) \leq k \).

It has been shown that if \( TR(A) = 1 \), then, in the above definition, one can replace \( B \) by a \( C^* \)-algebra in \( \mathcal{I} \) (see [19]). All unital simple AH-algebra with slow dimension growth and real rank zero have tracial rank zero (see [8] and also [22]) and all unital simple AH-algebras with no dimension growth have tracial rank no more than one (see [13], or, Theorem 2.5 of [28]). Note that all AH-algebras satisfy the Universal Coefficient Theorem. There are unital separable simple \( C^* \)-algebra \( A \) with \( TR(A) = 0 \) (and \( TR(A) = 1 \)) which are not amenable.
3 Unitary groups

**Lemma 3.1.** Let $H > 0$ be a positive number and let $N \geq 2$ be an integer. Then, for any unital $C^*$-algebra $A$, any projection $e \in A$ and any $u \in U_0(eAe)$ with $\text{cel}_{eAe}(u) < H$, 

$$\text{dist}(u + (1 - e), 1) < H/N,$$  

(e3.10)

if there are mutually orthogonal and mutually equivalent projections $e_1, e_2, ..., e_{2N} \in (1 - e)A(1 - e)$ such that $e_1$ is also equivalent to $e$.

**Proof.** Since $\text{cel}_{eAe}(u) < H$, there are unitaries $u_0, u_1, ..., u_N \in eAe$ such that

$$u_0 = u, \quad u_N = 1 \quad \text{and} \quad \|u_i - u_{i-1}\| < H/N, \quad i = 1, 2, ..., N. \quad (e3.11)$$

We will use the fact that 

$$(v \ 0) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v^* & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} v & 0 & 0 \\ 0 & v^* & 0 \end{pmatrix}.$$ 

In particular, 

$$(v \ 0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} (v^* \ 0 \ 0) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

is a commutator. Note that 

$$\|u \oplus u_1^* \oplus u_1 \oplus u_2^* \oplus \cdots \oplus u_N^* \oplus u_N - (u \oplus u_1^* \oplus u_1 \oplus u_2^* \oplus \cdots \oplus u_{N-1}^* \oplus u_N)\| < H/N. \quad (e3.12)$$

Since $u_N = 1$, $u \oplus u^* \oplus u_1 \oplus u_1^* \cdots \oplus u_{N-1}^* \oplus u_N$ is a commutator.

Now we write 

$$u \oplus e_1 \cdots \oplus e_{2N} = (u \oplus u_1^* \oplus u_1 \oplus \cdots \oplus u_N^* \oplus u_N)(e \oplus u_1 \oplus u_1^* \cdots \oplus u_N \oplus u_N^*).$$

We obtain $z \in CU((e + \sum_{i=1}^{2N} e_i)A(e + \sum_{i=1}^{2N} e_i))$ such that 

$$\|u \oplus e_1 \cdots \oplus e_{2N} - z\| < H/N.$$ 

It follows that 

$$\text{dist}(u + (1 - e), 1) < H/N.$$ 

□

**Definition 3.2.** Let $C = PM_k(C(X))P$, where $X$ is a compact metric space and $P \in M_k(C(X))$ is a projection. Let $u \in U(C)$. Recall (see [40]) that 

$$D_c(u) = \inf \{\|a\| : a \in C_{s.a.} \text{ such that } \det(\exp(ia) \cdot u)(x) = 1 \text{ for all } x \in X\}.$$ 

If no self-adjoint element $a \in A_{s.a.}$ exists for which $\det(\exp(ia) \cdot u)(x) = 1$ for all $x \in X$, define $D_c(u) = \infty$.

**Lemma 3.3.** Let $K \geq 1$ be an integer. Let $A$ be a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$, let $e \in A$ be a projection and let $u \in U_0(eAe)$. Suppose that $w = u + (1 - e)$ and suppose $\eta > 0$. Suppose also that 

$$[1 - e] \leq K[e] \text{ in } K_0(A) \text{ and } \text{dist}(\bar{w}, 1) < \eta. \quad (e3.13)$$

Then, if $\eta < 2$,

$$\text{cel}_{eAe}(u) < \frac{K \pi}{2} + 1/16 + 8\pi \text{ and } \text{dist}(\bar{u}, \bar{e}) < (K + 1/8)\eta,$$

and if $\eta = 2$,

$$\text{cel}_{eAe}(u) < \frac{K \pi}{2} \text{cel}(w) + 1/16 + 8\pi.$$
Proof. We assume that $(e3.13)$ holds. Note that $\eta \leq 2$. Put $L = cel(w)$.

We first consider the case that $\eta < 2$. There is a projection $e' \in M_2(A)$ such that
\[ [(1 - e) + e'] = K[e]. \]

To simplify notation, by replacing $A$ by $(1_A + e')M_2(A)(1_A + e')$ and $w$ by $w + e'$, without loss of generality, we may now assume that
\[ [1 - e] = K[e] \text{ and dist}(\bar{w}, 1) < \eta. \] (e3.14)

There is $R_1 > 1$ such that $\max\{L/R_1, 2/R_1, \eta \pi / R_1\} < \min\{\eta/64, 1/16\pi\}$.

For any $\frac{n}{32K(K+1)\pi} > \epsilon > 0$ with $\epsilon + \eta < 2$, since $TR(A) \leq 1$, there exists a projection $p \in A$ and a $C^*$-subalgebra $D \in \mathcal{I}$ with $1_D = p$ such that

1. $\|[p, x]\| < \epsilon$ for $x \in \{u, w, e, (1 - e)\}$,
2. $pwp, pup, pep, p(1 - e)p \in _\epsilon D$,
3. there is a projection $q \in D$ and a unitary $z_1 \in qDq$ such that $\|q - pep\| < \epsilon$, $\|z_1 - quq\| < \epsilon$, $\|z_1 \oplus (p - q) - pwp\| < \epsilon$ and $\|z_1 \oplus (p - q) - c_1\| < \epsilon + \eta$;
4. there is a projection $q_0 \in (1 - p)A(1 - p)$ and a unitary $z_0 \in q_0Aq_0$ such that $\|q_0 - (1 - p)e(1 - p)\| < \epsilon$, $\|z_0 - (1 - p)u(1 - p)\| < \epsilon$, $\|z_0 \oplus (1 - p - q_0) - (1 - p)w(1 - p)\| < \epsilon$, $\|z_0 \oplus (1 - p - q_0) - c_0\| < \epsilon + \eta$,
5. $[p - q] = K[q]$ in $K_0(D)$, $[(1 - p) - q_0] = K[q_0]$ in $K_0(A)$;
6. $2(K + 1)R_1[1 - p] < [p]$ in $K_0(A)$;
7. $cel((1 - p)A(1 - p))(z_0 \oplus (1 - p - q_0)) \leq L + \epsilon$,

where $c_1 \in CU(D)$ and $c_0 \in CU((1 - p)A(1 - p))$.

Note that $D_D(c_1) = 0$ (see (3.2)). Since $\epsilon + \eta < 2$, there is $h \in D_{s,0}$ with $\|h\| \leq 2 \arcsin(\frac{\epsilon + \eta}{2})$ such that (by (3) above)
\[ (z_1 \oplus (p - q)) \exp(\imath h) = c_1. \] (e3.15)

It follows that
\[ D_D((z_1 \oplus (p - q)) \exp(\imath h) = 0. \] (e3.16)

By (5) above and applying 3.3 of [40], one obtains that
\[ |D_qD_q(z_1)| \leq K^2 \arcsin(\frac{\epsilon + \eta}{2}). \] (e3.17)

If $2K \arcsin(\frac{\epsilon + \eta}{2}) \geq \pi$, then
\[ 2K(\frac{\epsilon + \eta}{2}) \frac{\pi}{2} \geq \pi. \]

It follows that
\[ K(\epsilon + \eta) \geq 2 \geq \text{dist}(\bar{z}_1, \bar{q}). \] (e3.18)

Since those unitaries in $D$ with $\det(u) = 1$ (for all points) are in $CU(D)$ (see, for example, 3.5 of [9]), from (e3.17), one computes that, when $2K \arcsin(\frac{\epsilon + \eta}{2}) < \pi$,
\[ \text{dist}(\bar{z}_1, \bar{q}) < 2 \sin(K \arcsin(\frac{\epsilon + \eta}{2})) \leq K(\epsilon + \eta). \] (e3.19)
By combining both (3.18) and (3.19), one obtains that
\[ \text{dist}(z_1, q) \leq K(\epsilon + \eta) \leq K\eta + \frac{\eta}{32(K+1)} \pi. \] (e 3.20)

By (3.17), it follows from 3.4 of [40] that
\[ \text{cel}_{qDq}(z_1) \leq 2K \arcsin \frac{\epsilon + \eta}{2} + 6\pi \leq K(\epsilon + \eta) \frac{\pi}{2} + 6\pi \leq \left( K\frac{\pi}{2} + \frac{1}{64(K+1)} \right) \eta + 6\pi. \] (e 3.21)

By (5) and (6) above,
\[ (K+1)[q] = [p - q] + [q] = [p] > 2(K+1)R_1[1-p]. \]

Since \( K_0(A) \) is weakly unperforated, one has
\[ 2R_1[1-p] < [q]. \] (e 3.22)

There is a unitary \( v \in A \) such that
\[ v^*(1 - p - q_0)v \leq q. \] (e 3.23)

Put \( v_1 = q_0 \oplus (1 - p - q_0)v \). Then
\[ v_1^*(z_0 \oplus (1 - p - q_0))v_1 = z_0 \oplus v^*(1 - p - q_0)v. \] (e 3.24)

Note that
\[ \|(z_0 \oplus v^*(1 - p - q_0)v)v_1^*v_0 - q_0 \oplus v^*(1 - p - q_0)v\| < \epsilon + \eta. \] (e 3.25)

Moreover, by (7) above,
\[ \text{cel}(z_0 \oplus v^*(1 - p - q_0)v) \leq L + \epsilon, \] (e 3.26)

It follows from (e 3.22) and Lemma 6.4 of [28] that
\[ \text{cel}_{(q_0 + q)A(q_0 + q)}(z_0 \oplus q) \leq 2\pi + (L + \epsilon)/R_1. \] (e 3.27)

Therefore, combining (e 3.21),
\[ \text{cel}_{(q_0 + q)A(q_0 + q)}(z_0 + z_1) \leq 2\pi + (L + \epsilon)/R_1 + \left( K\frac{\pi}{2} + \frac{1}{64(K+1)} \right) \eta + 6\pi. \] (e 3.28)

By (e 3.26), (e 3.22) and (3.1) in \( U_0((q_0 + q)A(q_0 + q))/CU((q_0 + q)A(q_0 + q)) \),
\[ \text{dist}(z_0 + q, q_0 + q) < \frac{(L + \epsilon)}{R_1}. \] (e 3.29)

Therefore, by (3.19) and (3.29),
\[ \text{dist}(z_0 \oplus z_1, q_0 + q) < \frac{(L + \epsilon)}{R_1} + K\eta + \frac{\eta}{32(K+1)} \pi < (K + 1/16)\eta. \] (e 3.30)

We note that
\[ \|e - (q_0 + q)\| < 2\epsilon \text{ and } \|u - (z_0 + z_1)\| < 2\epsilon. \] (e 3.31)
It follows that
\[ \text{dist}(\bar{u}, \bar{e}) < 4\epsilon + (K + 1/16)\eta < (K + 1/8)\eta. \] (e3.32)

Similarly, by (e3.28),
\[ \text{cel}_{eAe}(u) \leq 4\epsilon\pi + 2\pi + (L + \epsilon)/R_1 + (K\pi/2 + 1/64(K + 1))\eta + 6\pi \] (e3.33)
\[ < (K\pi/2 + 1/16)\eta + 8\pi. \] (e3.34)

This proves the case that \( \eta < 2 \).

Now suppose that \( \eta = 2 \). Define \( R = \text{cel}(\bar{w} + 1) \). Note that \( \frac{\text{cel}(\bar{w})}{R} < 1 \). There is a projection \( e' \in M_{R+1}(A) \) such that
\[ [(1 - e) + e'] = (K + RK)[e]. \]

It follows from Lemma 3.3 that
\[ \text{dist}(w \oplus e', 1A + e') < \frac{\text{cel}(w)}{R + 1}. \] (e3.35)

Put \( K_1 = K(R + 1) \). To simplify notation, without loss of generality, we may now assume that
\[ [1 - e] = K_1[e] \text{ and } \text{dist}(\bar{w}, \bar{1}) < \frac{\text{cel}(w)}{R + 1}. \] (e3.36)

It follows from the first part of the lemma that
\[ \text{cel}_{eAe}(u) < \frac{(K_1\pi/2 + 1/16)\text{cel}(w)}{R + 1} + 8\pi \] (e3.37)
\[ \leq \frac{K\pi\text{cel}(w)}{2} + 1/16 + 8\pi. \] (e3.38)

\[ \square \]

**Theorem 3.4.** Let \( A \) be a unital simple \( \text{C}^* \)-algebra with \( TR(A) \leq 1 \) and let \( e \in A \) be a non-zero projection. Then the map \( u \mapsto u + (1 - e) \) induces an isomorphism \( j \) from \( U(eAe)/CU(eAe) \) onto \( U(A)/CU(A) \).

**Proof.** It was shown in Theorem 6.7 of [28] that \( j \) is a surjective homomorphism. So it remains to show that it is also injective. To do this, fix a unitary \( u \in eAe \) so that \( \bar{u} \in \ker j \). We will show that \( u \in CU(eAe) \).

There is an integer \( K \geq 1 \) such that
\[ K[e] \geq [1 - e] \text{ in } K_0(A). \]

Let \( 1 > \epsilon > 0 \). Put \( v = u + (1 - e) \). Since \( \bar{u} \in \ker j, v \in CU(A) \). In particular,
\[ \text{dist}(\bar{v}, \bar{1}) < \epsilon/(K\pi/2 + 1). \]

It follows from Lemma 3.3 that
\[ \text{dist}(\bar{u}, \bar{e}) < (K\pi/2 + 1/16)(\epsilon/(K\pi/2 + 1)) < \epsilon. \]

It then follows that
\[ u \in CU(eAe). \]

\[ \square \]

**Corollary 3.5.** Let \( A \) be a unital simple \( \text{C}^* \)-algebra with \( TR(A) \leq 1 \). Then the map \( j : a \mapsto \text{diag} \langle a, 1, 1, \ldots, 1 \rangle \) from \( A \) to \( M_n(A) \) induces an isomorphism from \( U(A)/CU(A) \) onto \( U(M_n(A))/CU(M_n(A)) \) for any integer \( n \geq 1 \).
4 Full spectrum

One should compare the following with Theorem 3.1 of [42].

**Lemma 4.1.** Let $X$ be a path connected finite CW complex, let $C = C(X)$ and let $A = C([0, 1], M_n)$ for some integer $n \geq 1$. For any unital homomorphism $\phi : C \to A$, any finite subset $\mathcal{F} \subset C$ and any $\epsilon > 0$, there exists a unital homomorphism $\psi : C \to B$ such that

$$\|\phi(c) - \psi(c)\| < \epsilon \text{ for all } c \in \mathcal{F} \text{ and}$$

$$\psi(f)(t) = W(t)^* \begin{pmatrix} f(s_1(t)) \\ \vdots \\ f(s_n(t)) \end{pmatrix} W(t),$$

where $W \in U(A)$, $s_j \in C([0, 1], X)$, $j = 1, 2, ..., n$, and $t \in [0, 1]$.

**Proof.** To simplify the notation, without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of $C$. Since $X$ is also locally path connected, choose $\delta_1 > 0$ such that, for any point $x \in X$, $B(x, \delta_1)$ is path connected. Put $d = 2\pi/n$. Let $\delta_2 > 0$ (in place of $\delta$) be as required by Lemma 2.6.11 of [21] for $\epsilon/2$.

We will also apply Corollary 2.3 of [42]. By Corollary 2.3 of [42], there exists a finite subset $\mathcal{H}$ of positive functions in $C(X)$ and $\delta_3 > 0$ satisfying the following: For any pair of points $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$, if $\{h(x_i)\}_{i=1}^n$ and $\{h(y_i)\}_{i=1}^n$ can be paired to within $\delta_3$ one by one, in increasing order, counting multiplicity, for all $h \in \mathcal{H}$, then $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ can be paired to within $\delta_3/2$, one by one.

Put $\epsilon_1 = \min\{\epsilon/16, \delta_1/16, \delta_2/4, \delta_3/4\}$. There exists $\eta > 0$ such that

$$|f(t) - f(t')| < \epsilon_1/2 \text{ for all } f \in \phi(\mathcal{F} \cup \mathcal{H})$$

provided that $|t - t'| < \eta$. Choose a partition of the interval:

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

such that $|t_i - t_{i-1}| < \eta$, $i = 1, 2, ..., N$. Then

$$\|\phi(f)(t_i) - \phi(f)(t_{i-1})\| < \epsilon_1 \text{ for all } f \in \mathcal{F} \cup \mathcal{H},$$

and (e 4.41)

$i = 1, 2, ..., N$. There are unitaries $U_i \in M_n$ and $\{x_{i,j}\}_{j=1}^n$, $i = 0, 1, 2, ..., N$, such that

$$\phi(f)(t_i) = U_i^* \begin{pmatrix} f(x_{i,1}) \\ \vdots \\ f(x_{i,n}) \end{pmatrix} U_i.$$  (e 4.43)

By the Weyl spectral variation inequality (see [1]), the eigenvalues of $\{h(x_{i,j})\}_{j=1}^n$ and $\{h(x_{i-1,j})\}_{j=1}^n$ can be paired to within $\delta_3$, one by one, counting multiplicity, in decreasing order. It follows from Corollary 2.3 of [42] that $\{x_{i,j}\}_{j=1}^n$ and $\{x_{i-1,j}\}_{j=1}^n$ can be paired within $\delta_3/2$. We may assume that,

$$\text{dist}(x_{i,\sigma(j)}, x_{i-1,j}) < \delta_3/2,$$  (e 4.44)

where $\sigma_j : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ is a permutation. By the choice of $\delta_3$, there is continuous path $\{x_{i-1,j}(t) : t \in [t_{i-1}, (t_i + t_{i-1})/2]\} \subset B(x_{i-1}, \delta_3/2)$ such that

$$x_{i-1,j}(t_{i-1}) = x_{i-1,j} \text{ and } x_{i-1,j}((t_{i-1} + t_i)/2) = x_{i,\sigma(j)},$$  (e 4.45)
\[ j = 1, 2, \ldots, n. \] Put
\[ \psi(f)(t) = U_{i-1}^* \begin{pmatrix} f(x_{i,1}(t)) \\
\vdots \\
f(x_{i,n}(t)) \end{pmatrix} U_{i-1} \] for \( t \in [t_{i-1}, (t_{i-1} + t_i)/2] \) and for \( f \in C(X) \). In particular,
\[ \psi(f)\left(\frac{t_{i-1} + t_i}{2}\right) = U_{i-1}^* \begin{pmatrix} f(x_{i,\sigma(i)}(1)) \\
\vdots \\
f(x_{i,\sigma(n)}(1)) \end{pmatrix} U_{i-1} \] for \( f \in C(X) \). Note that
\[ \|\phi(f)(t_{i-1}) - \psi(f)(t)\| < \delta_2/4 \quad \text{and} \quad \|\psi(f)(t) - \phi(f)(t_i)\| < \delta_2/4 + \epsilon_1/2 < \delta_2/2 \] for all \( f \in \mathcal{F} \) and \( t \in [t_{i-1}, (t_{i-1} + t_i)/2] \). There exists a unitary \( W_i \in M_n \) such that
\[ W_i^* \psi(f)\left(\frac{t_{i-1} + t_i}{2}\right) W_i = \phi(f)(t_i) \] for all \( f \in C(X) \). It follows from (4.48) and (4.49) that
\[ \|W_i \psi(f)(\frac{t_{i-1} + t_i}{2}) - \psi(f)(\frac{t_{i-1} + t_i}{2}) W_i\| < \delta_2 \] for all \( f \in \mathcal{F} \). By the choice of \( \delta_2 \) and by applying Lemma 2.6.11 of [21], we obtain \( h_i \in M_n \) such that \( W_i = \exp(\sqrt{-1} h_i) \) and
\[ \|h_i \psi(f)(\frac{t_{i-1} + t_i}{2}) - \psi(f)(\frac{t_{i-1} + t_i}{2}) h_i\| < \epsilon/4 \quad \text{and} \quad (4.51)\]
\[ \|\exp(\sqrt{-1} th_i) \psi(f)(\frac{t_{i-1} + t_i}{2}) - \psi(f)(\frac{t_{i-1} + t_i}{2}) \exp(\sqrt{-1} th_i)\| < \epsilon/4 \] for all \( f \in \mathcal{F} \) and \( t \in [0, 1] \). From this we obtain a continuous path of unitaries \( \{W_i(t) : t \in [\frac{t_{i-1} + t_i}{2}, t_i]\} \subset M_n \) such that
\[ W_i\left(\frac{t_{i-1} + t_i}{2}\right) = 1, \quad W_i(t_i) = W_i \quad \text{(4.53)} \]
\[ \|W_i(t) \psi(f)(\frac{t_{i-1} + t_i}{2}) - \psi(f)(\frac{t_{i-1} + t_i}{2}) W_i(t)\| < \epsilon/4 \] for all \( f \in \mathcal{F} \) and \( t \in [\frac{t_{i-1} + t_i}{2}, t_i] \). Define \( \psi(f)(t) = W_i^*(t) \psi\left(\frac{t_{i-1} + t_i}{2}\right) W_i(t) \) for \( t \in [\frac{t_{i-1} + t_i}{2}, t_i] \), \( i = 1, 2, \ldots, N \). Note that \( \psi : C(X) \rightarrow A \). We conclude that
\[ \|\phi(f) - \psi(f)\| < \epsilon \quad \text{for all} \quad \mathcal{F}. \] Define
\[ U(t) = U_0 \quad \text{for} \quad t \in [0, \frac{t_1}{2}), \quad U(t) = U_0 W_1(t) \quad \text{for} \quad t \in [\frac{t_1}{2}, t_2), \quad (4.56) \]
\[ U(t) = U(t_i) \quad \text{for} \quad t \in [t_i, \frac{t_i + t_{i+1}}{2}), \quad U(t) = U(t_i) W_{i+1}(t) \quad \text{for} \quad t \in [\frac{t_i + t_{i+1}}{2}, t_{i+1}], \quad (4.57) \]
\( i = 1, 2, \ldots, N - 1 \) and define
\[ s_j(t) = x_{0,j}(t) \quad \text{for} \quad t \in [0, \frac{t_1}{2}), \quad s_j(t) = s_j\left(\frac{t_1}{2}\right) \quad \text{for} \quad t \in [\frac{t_1}{2}, t_2), \quad (4.58) \]
\[ s_j(t) = x_{i,\sigma(j)}(t) \quad \text{for} \quad t \in [t_i, \frac{t_i + t_{i+1}}{2}), \quad s_j(t) = s_j\left(\frac{t_i + t_{i+1}}{2}\right) \quad \text{for} \quad t \in [\frac{t_i + t_{i+1}}{2}, t_{i+1}], \quad (4.59) \]
for $f \in C(X)$. In fact, for $t \in [0, t_1]$, it is clear that (e 4.60) holds. Suppose that (e 4.60) holds for $t \in [0, t_i]$. Then, by (e 4.49), for $f \in C(X)$,

$$\psi(f)(t_i) = U(t_i)^* \begin{pmatrix} f(x_{i,1}) \\ \vdots \\ f(x_{i,n}) \end{pmatrix} U(t_i) = U_i^* \begin{pmatrix} f(x_{i,1}) \\ \vdots \\ f(x_{i,n}) \end{pmatrix} U_i. \quad (e 4.61)$$

Therefore, for $t \in [t_i, \frac{t_i + t_{i+1}}{2}]$,

$$\psi(f)(t) = U(t_i)^* \begin{pmatrix} f(x_{i,1}(t)) \\ \vdots \\ f(x_{i,n}(t)) \end{pmatrix} U(t_i)$$

$$= U(t_i)^* \begin{pmatrix} f(x_{i,1}(t)) \\ \vdots \\ f(x_{i,n}(t)) \end{pmatrix} U(t_i) \quad (e 4.63)$$

$$= U(t)^* \begin{pmatrix} f(s_1(t)) \\ \vdots \\ f(s_n(t)) \end{pmatrix} U(t). \quad (e 4.64)$$

For $t \in [\frac{t_i + t_{i+1}}{2}, t_{i+1}]$,

$$\psi(f)(t) = W_{i+1}(t)^* \psi(\frac{t_i + t_{i+1}}{2}) W_{i+1}(t) \quad (e 4.65)$$

$$= W_{i+1}(t)^* U(t_i)^* \begin{pmatrix} f(s_1(\frac{t_i + t_{i+1}}{2})) \\ \vdots \\ f(s_n(\frac{t_i + t_{i+1}}{2})) \end{pmatrix} U(t_i) W_{i+1}(t) \quad (e 4.66)$$

$$= U(t)^* \begin{pmatrix} f(s_1(t)) \\ \vdots \\ f(s_n(t)) \end{pmatrix} U(t). \quad (e 4.67)$$

This verifies (e 4.60).

\[ \square \]

Lemma 4.2. Let $X$ be a finite CW complex and let $A \in I$. Suppose that $\phi : C(X) \otimes C(\mathbb{T}) \to A$ is a unital homomorphism. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\}$ in $A$ such that

$$u(0) = \phi(1 \otimes 1), \quad u(1) = 1 \quad \text{and} \quad ||\phi(f \otimes 1), u(t)|| \leq \epsilon \quad (e 4.68)$$

for $f \in \mathcal{F}$ and $t \in [0, 1]$. 

13
Proof. It is clear that the general case can be reduced to the case that \( A = C([0, 1], M_n) \). Let \( q_1, q_2, ..., q_m \) be projections of \( C(X) \) corresponding to each path connected component of \( X \). Since \( \phi(q_i) A \phi(q_i) \cong C([0, 1], M_{n_i}) \) for some \( 1 \leq n_i \leq n, i = 1, 2, ..., m \), we may reduce the general case to the case that \( X \) is path connected and \( A = C([0, 1], M_n) \).

Note that we use \( z \) for the identity function on the unit circle.

For any \( \epsilon > 0 \) and any finite subset \( \mathcal{F} \subset C(X) \), by applying (4.1) one obtains a unital homomorphism \( \psi : C(X) \otimes C(\mathbb{T}) \to A \) such that

\[
\| \phi(g) - \psi(g) \| < \epsilon \text{ for all } g \in \{ f \otimes 1 : f \in \mathcal{F} \} \cup \{ 1 \otimes z \} \quad (e \, 4.69)
\]

\[
\psi(f)(t) = U(t)^* \begin{pmatrix}
  f(s_1(t)) \\
  \vdots \\
  f(s_n(t))
\end{pmatrix} U(t), \quad (e \, 4.70)
\]

for all \( f \in C(X \times \mathbb{T}) \), where \( U(t) \in U(C([0, 1], M_n)) \), \( s_j : [0, 1] \to X \times \mathbb{T} \) is a continuous map, \( j = 1, 2, ..., n \), and for all \( t \in [0, 1] \). There are continuous paths of unitaries \( \{ u_j(r) : r \in [0, 1] \} \subset C([0, 1]) \) such that

\[
u_j(0)(t) = (1 \otimes z)(s_j(t)), \quad u_j(1) = 1, \quad j = 1, 2, ..., n. \quad (e \, 4.71)
\]

Define

\[
u(r)(t) = U(t)^* \begin{pmatrix}
u_j(r)(t) \\
  \vdots \\
  u_n(r)(t)
\end{pmatrix} U(t). \quad (e \, 4.72)
\]

Then

\[
u(r) \psi(f \otimes 1) = \psi(f \otimes 1) u(r) \text{ for all } r \in [0, 1].
\]

It follows that

\[
\| [\phi(f \otimes 1), u(r)] \| < \epsilon \text{ for all } r \in [0, 1] \text{ and for all } f \in \mathcal{F}.
\]

\[\square\]

**Definition 4.3.** Let \( X \) be a compact metric space. We say that \( X \) satisfies property (H) if the following holds:

For any \( \epsilon > 0 \), any finite subsets \( \mathcal{F} \subset C(X) \) and any non-decreasing map \( \Delta : (0, 1) \to (0, 1) \), there exists \( \eta > 0 \) (which depends on \( \epsilon \) and \( \mathcal{F} \) but not \( \Delta \)), \( \delta > 0 \), a finite subset \( \mathcal{G} \subset C(X) \) and a finite subset \( \mathcal{P} \subset K(C(X)) \) satisfying the following:

Suppose that \( \phi : C(X) \to C([0, 1], M_n) \) is a unital \( \mathcal{G} \)-multiplicative contractive completely positive linear map for which

\[
\mu_{\tau \phi}(O_a) \geq \Delta(a) \quad (e \, 4.73)
\]

for any open ball \( O_a \) with radius \( a \geq \eta \) and for all tracial states \( \tau \) of \( C([0, 1], M_n) \), and

\[
[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \quad (e \, 4.74)
\]

where \( \Phi \) is a point-evaluation.

Then there exists a unital homomorphism \( h : C(X) \to C([0, 1], M_n) \) such that

\[
\| \phi(f) - h(f) \| < \epsilon \quad (e \, 4.75)
\]
for all $f \in \mathcal{F}$.

It is a restricted version of some relatively weakly semi-projectivity property. It has been shown in [34] that any $k$-dimensional torus has the property (H). So do those finite CW complexes $X$ with torsion free $K_0(C(X))$ and torsion $K_1(C(X))$, any finite CW complexes with form $Y \times T$ where $Y$ is contractive and all one-dimensional finite CW complexes.

**Theorem 4.4.** Let $X$ be a finite CW complex for which $X \times T$ has the property (H). Let $C = C(X)$ and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, $\eta > 0$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:

Suppose that $A$ is a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$, $\phi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c), u]\| < \delta \text{ for all } c \in \mathcal{G} \text{ and } \text{Bott}(\phi, u) = \{0\}. \quad (4.76)$$

Suppose also that there exists a unital contractive completely positive linear map $L : C \otimes C(T) \to A$ such that (with $z$ the identity function on the unit circle)

$$\|L(c \otimes 1) - \phi(c)\| < \delta, \|L(c \otimes z) - \phi(c)u\| < \delta \text{ for all } c \in \mathcal{G} \quad (4.77)$$

and $\mu_{\tau L}(O_a) \geq \Delta(a)$ for all $\tau \in T(A)$ and

$$\mu_{\tau L}(O_a) \geq \Delta(a) \quad (4.78)$$

for all open balls $O_a$ of $X \times T$ with radius $1 > a \geq \eta$, where $\mu_{\tau L}$ is the Borel probability measure defined by $L$. Then there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\}$ in $A$ such that

$$u(0) = u, \ u(1) = 1 \text{ and } ||[\phi(c), u(t)]|| < \epsilon \quad (4.79)$$

for all $c \in \mathcal{F}$ and for all $t \in [0, 1]$.

**Proof.** Let $\Delta_1(a) = \Delta(a)/2$. Denote by $z \in C(T)$ the identity map on the unit circle. Let $B = C \otimes C(T) = C(X \times T)$. Put $Y = X \times T$. Without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of $C$. Let $\mathcal{F}_1 = \{c \otimes 1 : c \in \mathcal{F}\} \cup \{1 \otimes z\}$.

Let $\eta_1 > 0$ (in place of $\eta$), $\delta_1 > 0$ (in place of $\delta$), $\mathcal{G}_1 \subset C(Y)$ (in place of $\mathcal{G}$) be a finite subset, $P_1 \subset K(C(Y))$ (in place of $\mathcal{P}$) and $U_1 \subset U(C(Y))$ be as required by Theorem 10.8 of [34] corresponding to $\epsilon/16$ (in place of $\epsilon$), $\mathcal{F}_1$ and $\Delta_1$ (in place of $\Delta$) above.

Without loss of generality, we may assume that

$$U_1 = \{\zeta_1 \otimes 1, ..., \zeta_{K_1} \otimes 1, \ p_1 \otimes z \oplus (1 - p_1 \otimes 1), ..., p_{K_2} \otimes z \oplus (1 - p_{K_2} \otimes 1)\}, \quad (4.80)$$

where $\zeta_k \in U(C)$, $k = 1, 2, ..., K_1$ and $p_j \in C$ is a projection, $j = 1, 2, ..., K_2$. Denote $z_i = p_i \otimes z \oplus (1 - p_i \otimes 1)$, $i = 1, 2, ..., K_2$. We may also assume that $U_1 \subset U(M_k(C(Y)))$.

For any contractively completely positive linear map $L'$ from $C(Y)$, we will also use $L'$ for $L' \otimes 1_{M_k}$.

Fix a finite subset $\mathcal{G}_2 \subset C(Y)$ which contains $\mathcal{G}_1$. Choose a small $\delta_1' > 0$. We choose $\mathcal{G}_2$ so large and $\delta_1'$ so small that, for any $\delta_1'-\mathcal{G}_2$-multiplicative map $L'$ form $C(Y)$ to a unital $C^*$-algebra $B'$, there are unitaries $w_i', w_j', ..., w_i'_{K_1}$ and $u_i', u_j', ..., u_i'_{K_2}$ in $M_k(B')$ such that

$$||L'w_i' - u_i'|| < \delta_1'/16 \text{ and } ||L'w_j' - u_j'|| < \delta_1'/16, \quad (4.81)$$

$i = 1, 2, ..., K_1$ and $j = 1, 2, ..., K_2$.

Let $\eta_2 > 0$ (in place of $\eta$), $\delta_2 > 0$ (in place of $\delta$), $\mathcal{G}_3 \subset C(Y)$ (in place of $\mathcal{G}$) be required by 10.7 of [34] for $\min\{\delta_1'/16, \delta_1'/16, \Delta_1(\eta_1)/16, \epsilon/16\}$ (in place of $\epsilon$), $\mathcal{G}_1 \cup \mathcal{F}_1$ and $\Delta_1$ (in place of $\Delta$) above. We may assume that $\mathcal{G}_3 \supset \mathcal{G}_2 \cup \mathcal{G}_1 \cup \mathcal{F}_1$ and $\mathcal{G}_3$ is in the unit ball of $C(Y)$. Moreover, we may further assume that $\mathcal{G}_3 = \{c \otimes 1 : c \in \mathcal{F}_2\} \cup \{z \otimes 1, 1 \otimes z, 1 \otimes 1\}$ for some finite subset $\mathcal{F}_2$. \[15\]
Suppose that $G \subset A$ is a finite subset which contains at least $F_2$. We may assume that $\delta_2 < \delta_1$. Let $\delta = \min\{\frac{\delta_1}{16}, \frac{\delta_1'}{16}, \frac{\Delta_1(\eta)}{4}\}$.

Let $A$ be a unital simple $C^*$-algebra with $TR(A) \leq 1$, let $\phi : C(X) \to A$ and $u \in U_0(A)$ be such that

$$\|\phi(c), u\| < \delta \text{ for all } c \in G \text{ and } \text{Bott}(\phi, u) = \{0\}.$$  \hspace{1cm} (e.4.82)

We may assume that (e.4.77) holds for $\eta = \eta_1/2$. We also assume that there is a $\delta$-$G_3$-multiplicative contractive completely positive linear map $L : C(Y) \to A$ such that

$$\|L(f \otimes 1) - \phi(f)\| < \delta, \|L(1 \otimes z) - u\| < \delta \text{ and }$$

$$\mu_{\tau \circ L}(O_a) \geq 2\Delta_1(a) \text{ for all } a \geq \eta,$$  \hspace{1cm} (e.4.83)

for all $\tau \in T(A)$ and for all $f \in F_2$. We will continue to use $L$ for $L \otimes \text{id}_{M_k}$.

By (e.4.81), one may also assume that there are unitaries $w_1, w_2, ..., w_{K_1}$ and unitaries $u_1, u_2, ..., u_{K_2}$ such that

$$\|L(\zeta_1 \otimes 1) - w_i\| < \delta_1/16 \text{ and } \|L(\zeta_j) - u_j\| < \delta_1/16,$$  \hspace{1cm} (e.4.85)

$i = 1, 2, ..., K_1$ and $j = 1, 2, ..., K_2$.

We note that, by (e.4.76), $[u_i] = 0$ in $K_1(A)$. Put

$$H = \max\{\text{cel}(u_i) : 1 \leq i \leq K_2\}.$$  \hspace{1cm} (e.4.86)

Let $N \geq 1$ be an integer such that

$$\frac{\max\{1, \pi, H + \delta_1 + \delta\}}{N} < \delta/4 \text{ and } \frac{1}{N} < \Delta_1(\eta)/4.$$  \hspace{1cm} (e.4.87)

For each $i$, there are self-adjoint elements $a_{i,1}, a_{i,2}, ..., a_{i,L(i)} \in A$ such that

$$u_i = \prod_{j=1}^{L(i)} \exp(\sqrt{-1}a_{i,j}) \text{ and } \sum_{i=1}^{L(i)} \|a_{i,j}\| \leq H + \delta/64,$$  \hspace{1cm} (e.4.88)

$i = 1, 2, ..., K_2$.

Put $\Lambda = \max\{L(i) : 1 \leq i \leq K_2\}$. Let $\epsilon_0 > 0$ such that if $\|p'q - pq'\| < \epsilon_0$ for any self-adjoint element $a$ and projection $p'$, $\|p' \exp(i\alpha) - \exp(i\alpha)p'\| < \delta_1/16\Lambda$.

By applying Corollary 10.7 of $\text{[33]}$, we obtain mutually orthogonal projections $P_0, P_1, P_2 \in A$ with $P_0 + P_1 + P_2 = 1$ and a $C^*$-subalgebra $D = \bigoplus_{j=1}^{s_2} C(X_j, M_{r(j)})$, where $X_j = [0, 1]$ or $X_j$ is a point, with $1_D = P_1$, a finite dimensional $C^*$-subalgebra $D_0 \subset A$ with $1_{D_0} = P_2$, a unital contractive completely positive linear map $L_0 : C(X) \to D_0$ and there exists a unital homomorphism $\Phi : C(Y) \to D$ such that

$$\|L(g) - (P_0L(g)P_0 + L_0(g) + \Phi(g))\| < \min\{\frac{\delta_1}{16}, \frac{\delta_1'}{16}, \frac{\Delta_1(\eta_1)}{4}, \frac{\epsilon}{16}\} \text{ for all } g \in G_2$$  \hspace{1cm} (e.4.89)

and $(2N + 1)\tau(P_0 + P_2) < \tau(P_1)$ for all $\tau \in T(A)$.

Moreover,

$$\|[x, P_0]\| < \min\{\epsilon_0, \delta_1/16, \delta_1'/16, \Delta_1(\eta_1)/4, \epsilon/16\}$$  \hspace{1cm} (e.4.90)

for all $x \in \{a_{i,j} : 1 \leq j \leq L(i), 1 \leq i \leq K_2\} \cup \{L(G_2)\}$.  \hspace{1cm} (16)
There exists a unitary \( u'_j \in M_k(P_0AP_0) \) and \( u''_j \in M_k(D_0) \) such that
\[
\| u'_j - P_0L(z_j)P_0 \| < \delta_1/16 \quad \text{and} \quad \| u''_j - L_0(z_j) \| < \delta_1/16 \quad \text{(e 4.91)}
\]
where \( P_0 = \text{diag}(P_0, P_0, \ldots, P_0) \). It follows from (e 4.90) and (e 4.87) that \([u'_j] = 0 \) in \( K_1(A) \) and
\[
\text{cel}(u'_j) \leq H + \delta_1/16 + \delta/64, \ j = 1, 2, \ldots, K_2 \quad \text{(e 4.92)}
\]
(in \( M_k(P_0AP_0) \)). Note also, since \( u''_j \in M_k(D_0) \),
\[
\text{cel}(u_j) \leq \pi, \ j = 1, 2, \ldots, K_2.
\]

By applying \([1.2]\) there exists a continuous path of unitaries \( \{v(t) : t \in [0, 1]\} \subset D \) such that
\[
v(0) = \Phi(1 \otimes z), \ v(1) = P_1 \quad \text{and} \quad \|\Phi(f \otimes 1), v(t)\| < \epsilon/4 \quad \text{(e 4.93)}
\]
for all \( t \in [0, 1] \). Define a contractive completely positive linear map \( L_1 : C(Y) = C(X) \otimes C(\mathbb{T}) \to A \) by
\[
L_1(f \otimes 1) = P_0L(f \otimes 1)P_0 + L_0(f \otimes 1) + \Phi(f \otimes 1) \quad \text{and} \quad L_1(1 \otimes g) = g(1) + P_0 + g(1) \cdot P_2 + \Phi(1 \otimes g(z)) \quad \text{(e 4.94, 4.95)}
\]
for all \( f \in C(X) \) and \( g \in C(\mathbb{T}) \). We compute (by choosing large \( G_2 \)) that
\[
\mu_{\tau_{o}L_1}(O_a) \geq \Delta_1(a) \quad \text{for all} \quad a \geq \eta \quad \text{(e 4.96)}
\]
and (by the fact that \( \text{Bott}(\phi, u) = \{0\} \))
\[
[L]|_{P_1} = [L_1]|_{P_1} \quad \text{(e 4.98)}
\]
We also have (by \([4.83]\) and \([4.88]\))
\[
\text{dist}(L_1^i(\zeta_i \otimes 1), L_1^i(\zeta_i \otimes 1)) < \delta_1/16, \ i = 1, 2, \ldots, K_1 \quad \text{(e 4.99)}
\]
Moreover, for \( j = 1, 2, \ldots, K_2 \),
\[
\text{dist}(L_1^j(z_j), L_1^j(z_j)) < \delta_1/16 + \text{dist}((u'_j + u''_j + \Phi(z_j))C_0 + C_2 + \Phi(z_j)), 1) \quad \text{(e 4.100)}
\]
\[
= \delta_1/16 + \text{dist}(u'_j + u''_j + \Phi(z_j))C_0 + C_2 + \Phi(z_j), 1) \quad \text{(e 4.101)}
\]
\[
< \delta_1/16 + \frac{\max \{\pi, H + \delta_1/16 + \delta_1/64\}}{N} \quad \text{(e 4.102)}
\]
\[
< \delta_1/16 + \delta/2 < \delta_1, \quad \text{(e 4.103)}
\]
where the third inequality follows from \([4.92]\) and \([3.1]\).

From \([4.96]\), \([4.97]\), \([4.98]\), and \([4.102]\), by applying Theorem 10.8 of \([34]\), one obtains a unitary \( W \in A \) such that
\[
\|\text{ad} W \circ L_1(g) - L(g)\| < \epsilon/16 \quad \text{for all} \ g \in \{c \otimes 1 : c \in \mathcal{F} \otimes 1\} \cup \{1 \otimes z\} \quad \text{(e 4.104)}
\]
Define
\[
u(t) = W^*(P_0 \otimes v(t))W \quad t \in [0, 1]. \quad \text{(e 4.105)}
\]

Then \( u'(0) = W^*(P_0 \oplus \Phi(1 \otimes z))W \) and \( u'(1) = 1 \). It follows from (e 4.93) and (e 4.104) that

\[
\|[(\phi(c), u'(t))] \| < \epsilon/2 \text{ for all } c \in \mathcal{F}
\] (e 4.106)

and for \( t \in [0, 1] \). Note that

\[
\|u'(0) - u\| < \epsilon/8.
\] (e 4.107)

One then obtains a continuous path \( \{u(t) : t \in [0, 1]\} \subset A \) by connecting \( u'(0) \) with \( u \) by a path with length no more than \( \epsilon/2 \). The theorem follows.

\[ \square \]

**Corollary 4.5.** Let \( C = C(X, M_n) \) where \( X = [0, 1] \) or \( X = \mathbb{T} \) and \( \Delta : (0, 1) \to (0, 1) \) be a non-decreasing map. For any \( \epsilon > 0 \) and any finite subset \( \mathcal{F} \subset C \), there exist \( \delta > 0, \eta > 0 \) and there exists a finite subset \( \mathcal{G} \subset C \) satisfying the following:

Suppose that \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), \( \phi : C \to A \) is a unital monomorphism and \( u \in A \) is a unitary and suppose that

\[
\|[(\phi(c), u)]\| < \delta \text{ for all } c \in \mathcal{G},
\] (e 4.108)

\[
bott_0(\phi, u) = \{0\} \text{ and } bott_1(\phi, u) = \{0\}.
\] (e 4.109)

Suppose also that there exists a unital contractive completely positive linear map \( L : C \otimes C(\mathbb{T}) \to A \) such that (with \( z \) the identity function on the unit circle)

\[
\|L(c \otimes 1) - \phi(c)\| < \delta, \quad \|L(c \otimes z) - \phi(c)u\| < \delta \text{ for all } c \in \mathcal{G} \text{ and } \mu_{\tau \circ L}(O_a) \geq \Delta(a)
\] (e 4.110)

for all open balls \( O_a \) of \([0, 1] \times \mathbb{T} \) with radius \( 1 > a \geq \eta \), where \( \mu_{\tau \circ L} \) is the Borel probability measure defined by restricting \( L \) on the center of \( C \otimes C(\mathbb{T}) \). Then there exists a continuous path of unitaries \( \{u(t) : t \in [0, 1]\} \) such that

\[
u(0) = u, \quad u(1) = 1 \quad \text{and} \quad \|[(\phi(c), u(t))]\| < \epsilon
\] (e 4.111)

for all \( c \in \mathcal{F} \) and for all \( t \in [0, 1] \).

**Corollary 4.6.** Let \( C = C([0, 1], M_n) \) and let \( T = N \times K : (C \otimes C(\mathbb{T}))_+ \setminus \{0\} \to N \times \mathbb{R}_+ \setminus \{0\} \) be a map. For any \( \epsilon > 0 \) and any finite subset \( \mathcal{F} \subset C \), there exists \( \delta > 0 \), a finite subset \( \mathcal{H} \subset (C \otimes C(\mathbb{T}))_+ \setminus \{0\} \) and there exists a finite subset \( \mathcal{G} \subset C \) satisfying the following:

Suppose that \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), \( \phi : C \to A \) is a unital monomorphism and \( u \in A \) is a unitary and suppose that

\[
\|[(\phi(c), u)]\| < \delta \text{ for all } c \in \mathcal{G} \text{ and } bott_0(\phi, u) = \{0\}.
\] (e 4.112)

Suppose also that there exists a unital contractive completely positive linear map \( L : C \otimes C(\mathbb{T}) \to A \) which is \( T \)-\( \mathcal{H} \)-full such that (with \( z \) the identity function on the unit circle)

\[
\|L(c \otimes 1) - \phi(c)\| < \delta \text{ and } \|L(c \otimes z) - \phi(c)u\| < \delta \text{ for all } c \in \mathcal{G}.
\] (e 4.114)

Then there exists a continuous path of unitaries \( \{u(t) : t \in [0, 1]\} \) in \( A \) such that

\[
u(0) = u, \quad u(1) = 1 \quad \text{and} \quad \|[(\phi(c), u(t))]\| < \epsilon
\] (e 4.115)

for all \( c \in \mathcal{F} \) and for all \( t \in [0, 1] \).
Lemma 5.1. Let $T = N \times K : \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$. Let $\Delta : (0,1) \to (0,1)$ be the non-decreasing map associated with $T$ as in Proposition 11.2 of [34]. Let $\mathcal{G} \subset C$, $\delta > 0$ and $\eta > 0$ be as required by [4.5] for $\epsilon$ and $\mathcal{F}$ given and the above $\Delta$.

It follows from 11.2 of [34] that there exists a finite subset $\mathcal{H} \subset (C \otimes C(\mathbb{T}))_+ \setminus \{0\}$ such that for any unital contractive completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$ which is $T$-$\mathcal{H}$-full, one has that

$$\mu_{\tau \circ L}(O_a) \geq \Delta(a)$$

(e 4.116)

for all open balls $O_a$ of $X \times \mathbb{T}$ with radius $a \geq \eta$.

The corollary then follows immediately from 4.5.

The following is an easy but known fact.

**Lemma 4.7.** Let $C = M_n$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F}$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C$ satisfying the following: For any unital $C^*$-algebra $A$ with $K_1(A) = U(A)/U_0(A)$ and any unital homomorphism $\phi : C \to A$ and any unitary $u \in A$ if

$$\|[\phi(c), u]\| < \delta \quad \text{and} \quad \text{bott}_0(\phi, u) = \{0\},$$

then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ such that

$$u(0) = u, \quad u(1) = 1 \quad \text{and} \quad \|[\phi(c), u(t)]\| < \epsilon$$

(e 4.118)

for all $c \in \mathcal{F}$ and $t \in [0,1]$.

**Proof.** First consider the case that $\phi(c)$ commutes with $u$ for all $c \in M_n$. Then one has a unital homomorphism $\Phi : M_n \otimes C(\mathbb{T}) \to A$ defined by $\Phi(c \otimes g) = \phi(c)g(u)$ for all $c \in C$ and $g \in C(\mathbb{T})$. Let $\{e_{i,j}\}$ be a matrix unit for $M_n$. Let $u_j = e_{j,j} \otimes z$, $j = 1,2,...,n$. The assumption $\text{bott}_0(\phi, u) = \{0\}$ implies that $\Phi_{*1} = \{0\}$. It follows that $u_j \in U_0(A)$, $j = 1,2,...,n$. One then obtains a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ such that

$$u(0) = u, \quad u(1) = 1 \quad \text{and} \quad \|[\phi(c), u(t)]\| = 0$$

for all $c \in C(\mathbb{T})$ and $t \in [0,1]$.

The general case follows from the fact that $C \otimes C(\mathbb{T})$ is weakly semi-projective.

**Remark 4.8.** Let $X$ be a compact metric space and let $A$ be a unital simple $C^*$-algebra. Suppose that $\phi : C(X) \to A$ is a unital injective completely positive linear map. Then it is easy to check (see 7.2 of [29], for example) that there exists a non-decreasing map $\Delta : (0,1) \to (0,1)$ such that

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a)$$

for all $a \in (0,1)$ and for all $\tau \in T(A)$.

5 Changing spectrum

**Lemma 5.1.** Let $n \geq 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset D \cong M_n$ satisfying the following:
Suppose that \( A \) is a unital \( C^* \)-algebra with \( T(A) \neq \emptyset \), \( D \subset A \) is a \( C^* \)-subalgebra with \( 1_D = 1_A \), suppose that \( \mathcal{F} \subset A \) is a finite subset and suppose that \( u \in U(A) \) such that

\[
\| [f, x] \| < \delta \text{ for all } f \in \mathcal{F} \text{ and } x \in \mathcal{G}, \quad \text{and} \quad (5.119)
\]

\[
\| [u, x] \| < \delta \text{ for all } x \in \mathcal{G}. \quad (5.120)
\]

Then, there exists a unitary \( v \in D \) and a continuous path of unitaries \( \{w(t) : t \in [0, 1]\} \subset D \) such that

\[
\| [u, w(t)] \| < n \delta < \epsilon, \quad \| [f, w(t)] \| < n \delta < \epsilon/2
\]

for all \( f \in \mathcal{F} \) and for all \( t \in [0, 1] \), (5.121)

\[
w(0) = 1, \ w(1) = v \text{ and } \mu_{\tau_0}(I_a) \geq \frac{2}{3n^2} \quad (5.122)
\]

for all open arcs \( I_a \) of \( \mathbb{T} \) with length \( a \geq 4\pi/n \) and for all \( \tau \in T(A) \), where \( \iota : C(\mathbb{T}) \to A \) is defined by \( \iota(f) = f(vu) \) for all \( f \in C(\mathbb{T}) \).

Moreover,

\[
\text{length}(\{w(t)\}) \leq \pi. \quad (5.124)
\]

If, in addition, \( \pi > b_1 > b_2 > \cdots > b_m > 0 \) and \( 1 = d_0 > d_1 > d_2 > \cdots > d_m > 0 \) are given so that

\[
\mu_{\tau_0}(I_{b_i}) \geq d_i \text{ for all } \tau \in T(A), \quad i = 1, 2, \ldots, m, \quad (5.125)
\]

where \( \iota_0 : C(\mathbb{T}) \to A \) is defined by \( \iota_0(f) = f(u) \) for all \( f \in C(\mathbb{T}) \), then one also has that

\[
\mu_{\tau_0}(I_{c_i}) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in T(A), \quad (5.126)
\]

where \( I_{b_i} \) and \( I_{c_i} \) are any open arcs with length \( b_i \) and \( c_i \), respectively, and where \( c_i = b_i + \epsilon_1 \), \( i = 1, 2, \ldots, m \).

Proof. Let

\[
0 < \delta_0 < \min \{ \frac{\epsilon_1 d_i}{16n^2} : 1 \leq i \leq m \}.
\]

Let \( \{e_{i,j}\} \) be a matrix unit for \( D \) and let \( \mathcal{G} = \{e_{i,j}\} \). Define

\[
v = \sum_{j=1}^{n} e^{2\sqrt{-1}j\pi/n} e_{j,j}. \quad (5.127)
\]

Let \( f_1 \in C(\mathbb{T}) \) with \( f_1(t) = 1 \) for \( |t - e^{2\sqrt{-1}\pi/n}| < \pi/n \) and \( f_1(t) = 0 \) if \( |t - e^{2\sqrt{-1}\pi/n}| \geq 2\pi/n \) and \( 1 \geq f_1(t) \geq 0 \). Define \( f_{j+1}(t) = f_1(e^{2\sqrt{-1}j\pi/n} t), \ j = 1, 2, \ldots, n - 1 \). Note that

\[
f_i(e^{2\sqrt{-1}j\pi/n} t) = f_{i+j}(t) \text{ for all } t \in \mathbb{T} \quad (5.128)
\]

where \( i, j \in \mathbb{Z}/n\mathbb{Z} \).

Fix a finite subset \( \mathcal{F}_0 \subset C(\mathbb{T})_+ \) which contains \( f_i, \ i = 1, 2, \ldots, n \).

Choose \( \delta \) so small that the following hold:

1. there exists a unitary \( u_i \in e_{i,i}Ae_{i,i} \) such that \( \|e^{2\sqrt{-1}i\pi/n} e_{i,i}u e_{i,i} - u_i\| < \delta_0^2/16n^2, \ i = 1, 2, \ldots, n \);

2. \( \|e_{i,j}f(u) - f(u)e_{i,j}\| < \delta_0^2/16n^2 \) for all \( f \in \mathcal{F}_0 \).
(3) \( \|e_{i,j}f(vu) - e_{i,j}f(e^{2\sqrt{-1}\pi/n}u)\| < \delta_0^2/16n^2 \) for all \( f \in \mathcal{F}_0 \) and

(4) \( \|e_{i,j}^*f(u)e_{i,j} - e_{j,j}f(u)e_{j,j}\| < \delta_0^2/16n^2 \) for all \( f \in \mathcal{F}_0 \).

Fix \( k \). For each \( \tau \in T(A) \), by (2), (3) and (4) above, there is at least one \( i \) such that

\[
\tau(e_{j,j}f_i(u)) \geq 1/n^2 - \delta_0^2/16n^2. \tag{5.129}
\]

Choose \( j \) so that \( k + j = i \mod (n) \). Then,

\[
\tau(f_k(vu)) \geq \tau(e_{j,j}f_k(vu)) \geq \tau(e_{j,j}f_k(e^{2\sqrt{-1}\pi/n}u)) - \frac{\delta_0^2}{16n^2} \geq \tau(e_{j,j}f_k(u)) - \frac{\delta_0^2}{16n^2} \geq 1/n^2 - \frac{2\delta_0^2}{16n^2}. \tag{5.130}
\]

It follows that

\[
\mu_{\tau_{0\circ}}(B(e^{2\sqrt{-1}k\pi/n}, \pi/n)) \geq \frac{1}{n} - \frac{2\delta_0^2}{16n^2} \quad \text{for all} \quad \tau \in T(A) \tag{5.131}
\]

and for \( k = 1, 2, \ldots, n \).

It is then easy to compute that

\[
\mu_{\tau_{0\circ}}(I_a) \geq 2/3n^2 \quad \text{for all} \quad \tau \in T(A) \tag{5.132}
\]

and for any open arc with length \( a \geq 2(2\pi/n) = 4\pi/n \).

Note that if \( \|x, e_{i,j}\| < \delta \), then

\[
\|x, \sum_{i=1}^n \lambda_i e_{i,i}\| < n\delta < \epsilon/2 \quad \text{and} \quad \|u, \sum_{i=1}^n \lambda_i e_{i,i}\| < n\delta < \epsilon/2
\]

for any \( \lambda_i \in \mathbb{T} \). Thus, one obtains a continuous path \( \{w(t) : t \in [0, 1]\} \subset D \) with length(\{w(t)\}) \leq \pi \) and with \( w(0) = 1 \) and \( w(1) = v \) so that \( (5.121) \) holds.

Let \( \{x_1, x_2, \ldots, x_K\} \) be an \( \epsilon_1/64 \)-dense set of \( \mathbb{T} \). Let \( I_{i,j} \) be an open arc with center \( x_j \) and length \( b_j \), \( j = 1, 2, \ldots, K \) and \( i = 1, 2, \ldots, m \). For each \( j \) and \( i \), there is a positive function \( g_{j,i} \in C(\mathbb{T})_+ \) with \( 0 \leq g_{j,i} \leq 1 \) and \( g_{j,i}(t) = 1 \) if \( |t - x_j| < d_i \) and \( g_{j,i}(t) = 0 \) if \( |t - x_j| \geq d_i + \epsilon_1/64 \), \( j = 1, 2, \ldots, K, i = 1, 2, \ldots, m \). Put \( g_{i,j,k}(t) = g_{j,i}(e^{2\sqrt{-1}k\pi/n}t) \) for all \( t \in T, k = 1, 2, \ldots, n \). Suppose that \( \mathcal{F}_0 \) contains all \( g_{j,i} \) and \( g_{j,i,k} \). We have, by (2), (3) and (4) above,

\[
\tau(g_{j,i}(u)e_{l,l}), \tau(g_{j,i,k}(u)e_{l,l}) \geq \frac{d_i}{n} - \frac{\delta_0^2}{16n^2} \quad \text{for all} \quad \tau \in T(A), \tag{5.133}
\]

\( l = 1, 2, \ldots, n, j = 1, 2, \ldots, K \) and \( i = 1, 2, \ldots, m \). Thus

\[
\tau(e_{k,k}g_{j,i}(vu)) \geq \tau(e_{k,k}g_{j,i}(e^{2\sqrt{-1}k\pi/n}u)) - n\frac{\delta_0^2}{16n^2} \geq \frac{d_k}{n} - \frac{\delta_0^2}{8n} \quad \text{for all} \quad \tau \in T(A), \tag{5.134}
\]

\( k = 1, 2, \ldots, n, j = 1, 2, \ldots, K \) and \( i = 1, 2, \ldots, m \). Therefore

\[
\tau(g_{j,i}(vu)) \geq d_i - \frac{\delta_0^2}{8n} \geq (1 - \epsilon_1)d_i \quad \text{for all} \quad \tau \in T(A), \tag{5.135}
\]
\[ j = 1, 2, \ldots, K \text{ and } i = 1, 2, \ldots, m. \]

It follows that

\[ \mu_{\tau_0}(I_{ij}) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in T(A), \quad (e 5.139) \]

for all \( j = 1, 2, \ldots, K \) and \( i = 1, 2, \ldots, m \). Since \( \{x_1, x_2, \ldots, x_K\} \) is \( \epsilon_1/64 \)-dense in \( \mathbb{T} \), it follows that

\[ \mu_{\tau_0}(I_{ij}) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in T(A), \quad i = 1, 2, \ldots, m. \quad (e 5.140) \]

\[ \square \]

**Remark 5.2.** If the assumption that \( \|[f, x]\| < \delta \) for all \( f \in \mathcal{F} \) and for all \( x \in \mathcal{G} \) is replaced by for all \( x \in D \) with \( \|x\| \leq 1 \), then the conclusion can also be strengthened to \( \|[f, w(t)]\| < \delta \) for all \( f \in \mathcal{F} \) and \( t \in [0, 1] \).

The proof of the following is similar to that of 5.1.

**Lemma 5.3.** Let \( n \geq 64 \) be an integer. Let \( \epsilon > 0 \) and \( 1/2 > \epsilon_1 > 0 \). There exists \( \frac{1}{2n} > \delta > 0 \) and a finite subset \( \mathcal{G} \subset D \cong M_n \) satisfying the following:

Suppose that \( X \) is a compact metric space, \( \mathcal{F} \subset C(X) \) is a finite subset and \( 1 > b > 0 \). Then there exists a finite subset \( \mathcal{F}_1 \subset C(X) \) satisfying the following:

Suppose that \( A \) is a unital \( C^* \)-algebra with \( T(A) \neq \emptyset \), \( D \subset A \) is a \( C^* \)-subalgebra with \( 1_D = 1_A \), \( \phi : C(X) \rightarrow A \) is a unital homomorphism and suppose that \( u \in U(A) \) such that

\[ \|[x, u]\| < \delta \text{ and } \|[x, \phi(f)]\| < \delta \text{ for all } x \in \mathcal{G} \text{ and } f \in \mathcal{F}_1. \quad (e 5.141) \]

Suppose also that, for some \( \sigma > 0 \),

\[ \tau(\phi(f)) \geq \sigma \text{ for all } \tau \in T(A) \text{ and } (e 5.142) \]

for all \( f \in C(X) \) with \( 0 \leq f \leq 1 \) whose support contains an open ball of \( X \) with radius \( b \). Then, there exists a unitary \( v \in D \) and a continuous path of unitaries \( \{\psi(t) : t \in [0, 1]\} \subset D \) such that

\[ \|[u, \psi(t)]\| < n\delta < \epsilon \text{, } \|[\phi(f), \psi(t)]\| < n\delta < \epsilon \quad (e 5.143) \]

for all \( f \in \mathcal{F} \) and \( t \in [0, 1] \),

\[ v(0) = 1, \quad v(1) = v \text{ and } (e 5.144) \]

\[ \tau(\phi(f)g(vu)) \geq \frac{2\sigma}{3n^2} \text{ for all } \tau \in T(A) \quad (e 5.145) \]

for any pair of \( f \in C(X) \) with \( 0 \leq f \leq 1 \) whose support contains an open ball with radius \( 2b \) and \( g \in C(\mathbb{T}) \) with \( 0 \leq g \leq 1 \) whose support contains an open arc of \( \mathbb{T} \) with length at least \( 8\pi/n \). Moreover,

\[ \text{length} \{\{v(t)\}\} \leq \pi. \quad (e 5.146) \]

If, in addition, \( 1 > b_1 > b_2 > \cdots > b_k > 0 \), \( 1 > d_1 \geq d_2 \geq \cdots \geq d_k > 0 \) are given and

\[ \tau(\phi(f')g'(u)) \geq d_i \text{ for all } \tau \in T(A) \quad (e 5.147) \]

for any functions \( f' \in C(X) \) with \( 0 \leq f' \leq 1 \) whose support contains an open ball of \( X \) with radius \( b_i/2 \) and \( g' \in C(\mathbb{T}) \) with \( 0 \leq g' \leq 1 \) whose support contains an arc with length \( b_i \), then one also has that

\[ \tau(\phi(f'')g''(vu)) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in T(A), \quad (e 5.148) \]

where \( f'' \in C(X) \) with \( 0 \leq f'' \leq 1 \) whose support contains an open ball of radius \( c_i \) and \( g'' \in C(\mathbb{T}) \) with \( 0 \leq g'' \leq 1 \) whose support contains an arc with length \( 2c_i \) with \( c_i = b_i + \epsilon_1 \), \( i = 1, 2, \ldots, k \).
Proof. Let 0 < \delta_0 = \min\{\frac{i\delta}{16}n : i = 1, 2, \ldots, k\}.

Let \{e_{i,j}\} be a matrix unit for D and let \mathcal{G} = \{e_{i,j}\}. Define

\[ v = \sum_{j=1}^{n} e^{2\sqrt{-1}\pi/n}e_{i,j}. \]  

(e 5.150)

Let \(g_j \in C(\mathbb{T})\) with \(g_j(t) = 1\) for \(|t - e^{2\sqrt{-1}\pi/n}| < \pi/n\) and \(g_j(t) = 0\) if \(|t - e^{2\sqrt{-1}\pi/n}| \geq 2\pi/n\) and \(1 \geq g_j(t) \geq 0, j = 1, 2, \ldots, n\). As in the proof \(5.1\) we may also assume that

\[ g_i(e^{2\sqrt{-1}\pi/n}t) = g_{i+j}(t) \quad \text{for all } t \in \mathbb{T} \]  

(e 5.151)

where \(i, j \in \mathbb{Z}/n\mathbb{Z}\).

Let \(\{x_1, x_2, \ldots, x_m\}\) be a \(2/2\)-dense subset of \(X\). Define \(f_i \in C(X)\) with \(f_i(x) = 1\) for \(x \in B(x_i, b)\) and \(f_i(x) = 0\) if \(x \not\in B(x_i, 2b)\) and \(0 \leq f_i \leq 1, i = 1, 2, \ldots, m\).

Note that

\[ \tau(\phi(f_i)) \geq \sigma \quad \text{for all } \tau \in \mathcal{T}(A), \quad i = 1, 2, \ldots, m. \]  

(e 5.152)

Fix a finite subset \(\mathcal{F}_0 \subset C(\mathbb{T})\) which at least contains \(\{g_1, g_2, \ldots, g_n\}\) and \(\mathcal{F}_1 \subset C(X)\) which at least contains \(\mathcal{F}\) and \(\{f_1, f_2, \ldots, f_m\}\).

Choose \(\delta\) so small that the following hold:

1. there exists a unitary \(u_i \in e_i,iA\) such that \(\|e_i,i\phi(x)e_i,i - u_i\| < \delta_0^2/16n^4, i = 1, 2, \ldots, n;\)
2. \(\|e_{i,j}g(u) - g(u)e_{i,j}\| < \delta_0^2/16n^4, \|e_{i,j}\phi(f) - \phi(f)e_{i,j}\| < \delta_0^2/16n^4, \) for \(f \in \mathcal{F}_1\) and \(g \in \mathcal{F}_0, j, k = 1, 2, \ldots, n\) and \(s = 1, 2, \ldots, m;\)
3. \(\|e_{i,j}g(uv) - e_{i,j}g(e^{2\sqrt{-1}\pi/n}u)\| < \delta_0^2/16n^4\) for all \(g \in \mathcal{F}_0\) and
4. \(\|e_{i,j}^*g(u)e_{i,j} - e_{i,j}g(u)e_{i,j}\| < \delta_0^2/16n^4, \|e_{i,j}^*\phi(f)e_{i,j} - e_{i,j}\phi(f)e_{i,j}\| < \delta_0^2/16n^4\) for all \(f \in \mathcal{F}_1\) and \(g \in \mathcal{F}_0, j, k = 1, 2, \ldots, n\) and \(s = 1, 2, \ldots, m.\)

It follows from (4) that, for any \(k_0 \in \{1, 2, \ldots, m\},\)

\[ \tau(\phi(f_{k_0})e_{j,j}) \geq \sigma/n - n\delta_0^2/16n^4. \]  

(e 5.153)

Fix \(k_0\) and \(k\). For each \(\tau \in \mathcal{T}(A),\) there is at least one \(i\) such that

\[ \tau(\phi(f_{k_0})e_{j,j}g(u)) \geq \sigma/n^2 - \delta_0^2/16n^4. \]  

(e 5.154)

Choose \(j\) so that \(k + j = i \mod (n).\) Then,

\[ \tau(\phi(f_{k_0})g_k(uv)) \geq \tau(\phi(f_{k_0})e_{j,j}g_k(e^{2\sqrt{-1}\pi/n}u)) - \delta_0^2/16n^4 \]  

(e 5.155)

\[ = \tau(\phi(f_{k_0})e_{j,j}g_i(u)) - \delta_0^2/16n^4 \]  

(e 5.156)

\[ \geq \frac{\sigma}{n^2} - \frac{2\delta_0^2}{16n^4} \text{ for all } \tau \in \mathcal{T}(A). \]  

(e 5.157)

It is then easy to compute that

\[ \tau(\phi(f)g(uv)) \geq \frac{2\sigma}{3n^2} \text{ for all } \tau \in \mathcal{T}(A). \]  

(e 5.158)
and for any pair of $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball with radius $2b$ and $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ whose support contains an open arc of length at least $8\pi/n$.

Note that if $\|\phi(f), e_{i,i}\| < \delta$, then

$$\|\phi(f), \sum_{i=1}^{n} \lambda_i e_{i,i}\| < n\delta < \epsilon$$

for any $\lambda_i \in \mathbb{T}$ and $f \in \mathcal{F}_i$. We then also require that $\delta < \epsilon/2n$. Thus, one obtains a continuous path $\{v(t) : t \in [0,1]\} \subset D$ with length($\{v(t)\}$) $\leq \pi$ and with $v(0) = 1$ and $v(1) = v$ so that the second part of (e.5.143) holds.

Now we consider the last part of the lemma. Note also that, if $\mathcal{F}_0$ and $\mathcal{F}_1$ are sufficiently large, by the assumptions of the last part of the lemma, as in the proof of (e.5.1), we have

$$\tau(\phi(f)g(vu)) \geq \sum_{j=1}^{n} \tau(\phi(f)e_{i,j}g(vu)) - \frac{\delta^2}{16n^2} \text{ for all } \tau \in T(A),$$

where $g^{(j)}(t) = g(e^{2\sqrt{-1}j\pi/n} \cdot t)$ for $t \in \mathbb{T}$. If the support of $f$ contains an open ball with radius $b_i/2$ and that of $g$ contains open arcs with length at least $b_i$, so does that of $g^{(j)}$. So, if $\mathcal{F}_0$ and $\mathcal{F}_1$ are sufficiently large, by the assumptions of the last part of the lemma, as in the proof of (e.5.1) we have

$$\tau(\phi(f)g(vu)) \geq d_i - \frac{\delta^2}{16n^2} \text{ for all } \tau \in T(A)$$

for all $\tau \in T(A)$. As in the proof of (e.5.1) this lemma follows when we choose $\mathcal{F}_0$ and $\mathcal{F}_1$ large enough to begin with.

$$\square$$

**Lemma 5.4.** Let $C$ be a unital separable simple $C^*$-algebra with $TR(C) \leq 1$ and let $n \geq 1$ be an integer. For any $\epsilon > 0$, $\eta > 0$, any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, a projection $p \in A$ and a $C^*$-subalgebra $D \cong M_n$ with $1_D = p$ such that

$$\|xp\| < \epsilon \text{ for all } x \in \mathcal{F};$$

$$\|xp, y\| < \epsilon \text{ for all } x \in \mathcal{F} \text{ and } y \in D \text{ with } \|y\| \leq 1 \quad \text{(e.5.162)}$$

$$\tau(1-p) < \eta \text{ for all } \tau \in T(C).$$

**Proof.** Choose an integer $N \geq 1$ such that $1/N < \eta/2n$ and $N \geq 2n$. It follows from (the proof of ) Theorem 5.4 of [28] that there is a projection $q \in C$ and there exists a $C^*$-subalgebra $B$ of $C$ with $1_B = q$ and $B \cong \bigoplus_{i=1}^{L} M_{K_i}$ with $K_i \geq N$ such that

$$\|x, q\| < \eta/4 \text{ for all } x \in \mathcal{F};$$

$$\|x, q, y\| < \epsilon/4 \text{ for all } x \in \mathcal{F} \text{ and } y \in B \text{ with } \|y\| \leq 1 \quad \text{(e.5.165)}$$

$$\tau(1-q) < \eta/2n \text{ for all } \tau \in T(C).$$

Write $K_i = k_i n + r_i$ with $k_i \geq 1$ and $0 \leq r_i < n$ for some integers $k_i$ and $r_i$, $i = 1, 2, \ldots, L$. Let $p \in B$ be a projection such that the rank of $p$ is $k_i$ in each summand $M_{K_i}$ of $B$. Take $D_1 = pBp$. We have

$$\|xp\| < \epsilon/2 \text{ for all } x \in \mathcal{F};$$

$$\|xp, y\| < \epsilon \text{ for all } x \in \mathcal{F}, y \in D_1 \text{ with } \|y\| \leq 1 \quad \text{(e.5.166)}$$

$$\tau(1-p) < \eta/2n + n/N < \eta/2n + \eta/2 < \eta \text{ for all } \tau \in T(C).$$

$$\quad \text{(e.5.167)}$$

24
Note that there is a unital $C^*$-subalgebra $D \subset D_1$ such that $D \cong M_n$. 

**Lemma 5.5.** Let $n \geq 1$ be an integer with $n \geq 64$. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. Suppose that $A$ is a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$, suppose that $\mathcal{F} \subset A$ is a finite subset and suppose that $u \in U(A)$. Then, for any $\epsilon > 0$, there exists a unitary $v \in A$ and a continuous path of unitaries $\{w(t) : t \in [0, 1]\} \subset A$ such that

$$
\|[x, w(t)]\| < \epsilon \quad \text{for all } f \in \mathcal{F} \text{ and for all } t \in [0, 1],
$$

(e.5.171)

$$
w(0) = 1, \ w(1) = v \text{ and } \mu_{\tau_{w_0}}(I_0) \geq \frac{15}{24n^2}
$$

(e.5.172)

$$
\mu_{\tau_{v_0}}(I_{b_i}) \geq d_i \text{ for all } \tau \in T(A), \ i = 1, 2, \ldots, m,
$$

(e.5.175)

for all open arcs $I_{b_i}$ of $\mathbb{T}$ with length $a \geq 4\pi/n$ and for all $\tau \in T(A)$, where $\iota : C(\mathbb{T}) \to A$ is defined by $\iota(f) = f(vu)$. Moreover,

$$
\text{length}([w(t)]) \leq \pi.
$$

(e.5.174)

If, in addition, $\pi > b_1 > b_2 > \cdots > b_m > 0$ and $1 = d_0 > d_1 > d_2 > \cdots > d_m > 0$ are given so that

$$
\mu_{\tau_{w_0}}(I_{c_i}) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in T(A),
$$

(e.5.176)

where $I_{b_i}$ and $I_{c_i}$ are any open arcs with length $b_i$ and $c_i$, respectively, and where $c_i = b_i + \epsilon_1$, $i = 1, 2, \ldots, m$.

**Proof.** Let $\epsilon > 0$, and let $n \geq 64$ be an integer. Put $\epsilon_2 = \min\{\epsilon_1/16, 1/64n^2\}$. Let $\mathcal{F} \subset A$ be a finite subset and let $u \in U(A)$. Let $\delta_1 > 0$ (in place of $\delta$) be as in [5.4] for $\epsilon$, $\epsilon_2$ (in place of $\epsilon_1$) and let $\mathcal{G} = \{\epsilon_{i,j}\} \subset D \cong M_n$ be as required by [5.3].

Put $\delta = \delta_1/16$. By applying [5.4] there is a projection $p \in A$ and a $C^*$-subalgebra $D \cong M_n$ with $1_D = p$ such that

$$
\|[x, p]\| < \delta \text{ for all } x \in \mathcal{F};
$$

(e.5.177)

$$
\|[pxp, y]\| < \delta \text{ for all } x \in \mathcal{F} \text{ and } y \in D \text{ with } \|y\| \leq 1 \text{ and }
$$

(e.5.178)

$$
\tau(1 - p) < \epsilon_2 \text{ for all } \tau \in T(C).
$$

(e.5.179)

There is a unitary $u_0 \in (1 - p)A(1 - p)$ and a unitary $u_1 \in pAp$. Put $A_1 = pAp$ and $\mathcal{F}_1 = \{pxp : x \in \mathcal{F}\}$. We apply [5.1] to $A_1$, $\mathcal{F}_1$ and $u_1$. We check that lemma follows.

\[\square\]

The proof of the following lemma follows the same argument using [5.4] as in that of [5.3] but one applies [5.3] instead of [5.1].

**Lemma 5.6.** Let $n \geq 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. Suppose that $A$ is a unital simple $C^*$-algebra with $\text{TR}(A) \leq 1$, $X$ is a compact metric space, $\phi : C(X) \to A$ is a unital homomorphism, $\mathcal{F} \subset C(X)$ is a finite subset and suppose that $u \in U(A)$. Suppose also that, for some $\sigma > 0$ and $1 > b > 0$,

$$
\tau(\phi(f)) \geq \sigma \text{ for all } \tau \in T(A) \text{ and }
$$

(e.5.180)
for all \( f \in C(\mathbb{T}) \) with \( 0 \leq f \leq 1 \) whose supports contain an open ball with radius at least \( b \). Then, there exists a unitary \( v \in A \) and a continuous path of unitaries \( \{v(t) : t \in [0,1]\} \subset A \) such that \( v(0) = 1, v(1) = v \),
\[
\|\phi(f), v(t)\| < \epsilon \quad \text{and} \quad \|u, v(t)\| < \epsilon \quad \text{for all} \quad f \in \mathcal{F} \quad \text{and} \quad t \in [0,1],
\]
\[
\tau(\phi(f)g(vu)) \geq \frac{15\sigma}{24n^2} \quad \text{for all} \quad \tau \in T(A) \tag{e 5.182}
\]
for any \( f \in C(\mathbb{X}) \) with \( 0 \leq f \leq 1 \) whose support contains an open ball of radius at least \( 2b \) and any \( g \in C(\mathbb{T}) \) with \( 0 \leq g \leq 1 \) whose support contains an open arc of \( \mathbb{T} \) with length \( a \geq 8\pi/n \).

Moreover,
\[
\text{length}(\{v(t)\}) \leq \pi. \tag{e 5.183}
\]

If, in addition, \( 1 > b_1 > b_2 > \cdots > b_k > 0, 1 > d_1 \geq d_2 \geq \cdots \geq d_k > 0 \) are given and
\[
\tau(\phi(f')g'(u)) \geq d_i \quad \text{for all} \quad \tau \in T(A) \tag{e 5.184}
\]
for any functions \( f' \in C(\mathbb{X}) \) with \( 0 \leq f' \leq 1 \) whose support contains an open ball with radius \( b_i/2 \) and any function \( g' \in C(\mathbb{T}) \) with \( 0 \leq g' \leq 1 \) whose support contains an arc with length \( b_i \), then one also has that
\[
\tau(\phi(f'')g''(vu)) \geq (1 - \epsilon_1)d_i \quad \text{for all} \quad \tau \in T(A), \tag{e 5.185}
\]
where \( f'' \in C(\mathbb{X}) \) with \( 0 \leq f'' \leq 1 \) whose support contains an open ball with radius \( c_i \) and \( g'' \in C(\mathbb{T}) \) with \( 0 \leq g'' \) whose support contains an arc with length \( 2c_i \), where \( c_i = b_i + \epsilon_1, i = 1, 2, \ldots, k \).

5.7. Define
\[
\Delta_{00}(r) = \frac{1}{2(n+1)^2} \quad \text{if} \quad 0 < \frac{8\pi}{n+1} + \frac{4\pi}{2^{n+2}(n+1)} < r \leq \frac{8\pi}{n} + \frac{4\pi}{2^{n+1}n} \tag{e 5.186}
\]
for \( n \geq 64 \) and
\[
\Delta_{00}(r) = \frac{1}{2(65)^2} \quad \text{if} \quad r \geq 8\pi/64 + \frac{4\pi}{2^{65}(64)}. \tag{e 5.187}
\]

Let \( \Delta : (0, 1) \to (0, 1) \) be a non-decreasing map. Define
\[
D_0(\Delta)(r) = \Delta(\pi/n)\Delta_{00}(r) \quad \text{if} \quad 0 < \frac{8\pi}{n+1} + \frac{4\pi}{2^{n+2}(n+1)} < r \leq \frac{8\pi}{n} + \frac{4\pi}{2^{n+1}n} \tag{e 5.189}
\]
for \( n \geq 64 \) and
\[
D_0(\Delta)(r) = D_0(\Delta)(4\pi/64) \quad \text{if} \quad r \geq 8\pi/64 + \frac{4\pi}{2^{65}(64)}. \tag{e 5.191}
\]

**Lemma 5.8.** Suppose that \( A \) is a unital separable simple \( C^* \)-algebra with \( TR(A) \leq 1 \), suppose that \( \mathcal{F} \subset A \) is a finite subset and suppose that \( u \in U(A) \). For any \( \epsilon > 0 \) and any \( \eta > 0 \), there exists a unitary \( v \in U_0(A) \) and a continuous path of unitaries \( \{w(t) : t \in [0,1]\} \subset U_0(A) \) such that
\[
w(0) = 1, \quad w(1) = v, \quad \|[f, w(t)]\| < \epsilon \quad \text{for all} \quad f \in \mathcal{F} \quad \text{and} \quad t \in [0,1], \quad \text{and} \quad \epsilon (5.192)
\]
\[
\mu_{\tau o r}(I_a) \geq \Delta_{00}(a) \quad \text{for all} \quad \tau \in T(A) \tag{5.193}
\]
for any open arc \( I_a \) with length \( a \geq \eta \), where \( v : C(\mathbb{T}) \to A \) is defined by \( v(g) = g(vu) \) for all \( g \in C(\mathbb{T}) \) and \( \Delta_{00} \) is defined in 5.7.
Proof. Define
\[
\Delta_{00,n}(r) = \frac{7}{12(k+1)^2} - \sum_{m=1}^{n} \frac{2}{9 \cdot 2^{m+1}(m+1)^2} \quad (e\ 5.194)
\]
if \(0 < \frac{4\pi}{k+1} \sum_{m=1}^{n} \frac{4\pi}{2m+1}(m+1)^2 < r \leq \frac{4\pi}{k} + \sum_{m=1}^{n} \frac{4\pi}{2m+1}(m+1)^2 \quad (e\ 5.195)
\]
if \(n \geq k \geq 32\), and \(\Delta_{00,n}(r) = \Delta_{00,n}(4\pi/32 + \frac{4\pi}{2k+1})\) if \(r \geq \frac{4\pi/32}{2k+1}\).

Without loss of generality, we may assume that \(\eta = 4\pi/n\) for some \(n \geq 32\). We will use the induction to prove the statement which is exactly the same as that of Lemma 5.8 but replace \(\Delta_{00}\) by \(\Delta_{00,k}\) for \(k \geq 32\). It follows from \(5.5\) by choosing small \(\epsilon_1\), the statement holds for \(k = 32\).

Now suppose that the statement holds for all integers \(m\) with \(k \geq m \geq 32\). Thus we have a continuous path of unitaries \(\{w'(t) : t \in [0,1]\} \subset U_0(A)\) such that
\[
w'(0) = 1, \ w'(1) = v', \ ||f, w'(t)|| < \epsilon/2 \text{ for all } t \in [0,1] \quad (e\ 5.196)
\]
\[
\mu_{\tau \circ k}(I_a) \geq \Delta_{00,k} \text{ for all } \tau \in T(A), \quad (e\ 5.197)
\]
for all open arcs with length \(a \geq 4\pi/k\), where \(\tau_k : C(\mathbb{T}) \rightarrow A\) is defined by \(\tau_k(g) = g(v'u)\) for all \(g \in C(\mathbb{T})\).

Let
\[
b_j = \frac{4\pi}{j+1} + \sum_{m=j}^{k} \frac{4\pi}{2m+1}(m+1)^2 \text{ and } d_j = \frac{7}{12(j+1)^2} - \sum_{m=j}^{k+1} \frac{2}{9 \cdot 2^{m+1}(m+1)^2} \quad (e\ 5.198)
\]
\(j = 32, 33, \ldots, k\). Choose \(\epsilon_1 = \frac{2}{9 \cdot 2^{k+1}}\). By applying \(5.5\) we obtain a continuous path of unitaries \(\{w''(t) : t \in [0,1]\} \subset U_0(A)\) such that
\[
w''(0) = 1, \ w''(1) = v'', \ ||f, w''(t)|| < \epsilon/2 \text{ for all } t \in [0,1] \quad (e\ 5.199)
\]
\[
\mu_{\tau \circ (k+1)}(I_b) \geq \frac{15\pi}{4(k+1)^2} \text{ for all } \tau \in T(A) \quad (e\ 5.200)
\]
for all open arcs \(I_b\) with length \(b \geq \frac{4\pi}{k+1}\), where \(\tau_{k+1} : C(\mathbb{T}) \rightarrow A\) is defined by \(\tau_{k+1}(g) = g(v''(v'u))\) for all \(g \in C(\mathbb{T})\). Moreover, for any open arc \(I_{c_j}\) with length \(c_j\),
\[
\tau \circ \tau_{k+1}(I_{c_j}) \geq (1 - \epsilon_1)d_j \geq \frac{7}{12(j+1)^2} - \sum_{m=j}^{k+1} \frac{2}{9 \cdot 2^{m+1}(m+1)^2} \text{ for all } \tau \in T(A), \quad (e\ 5.201)
\]
\(j = 32, 33, \ldots, k\). Now define \(w(t) = w''(t)w'(t)\) for \(t \in [0,1]\). Then
\[
w(0) = 1, \ w(1) = v''v' \text{ and } ||f, w(t)|| < \epsilon \text{ for all } t \in [0,1]. \quad (e\ 5.202)
\]
This shows that the statement holds for \(k + 1\). By the induction, this proves the statement.

Note that \(\Delta_{00,n}(r) \geq \Delta_{00}(r)\) for all \(r \geq 4\pi/n = \eta\). The lemma follows immediately from the statement.

\[\square\]

**Corollary 5.9.** Let \(C\) be a unital separable simple amenable \(C^*\)-algebra with \(TR(C) \leq 1\) which satisfies the UCT. Let \(\epsilon > 0, F \subset C\) be a finite subset and let \(1 > \eta > 0\).

Suppose that \(A\) is a unital simple \(C^*\)-algebra with \(TR(A) \leq 1\), \(\phi : C \rightarrow A\) is a unital homomorphism and \(u \in U(A)\) is a unitary with
\[
||\phi(c), u|| < \epsilon \text{ for all } c \in F. \quad (e\ 5.203)
\]
Then there exist a continuous path of unitaries \( \{u(t) : t \in [0,1]\} \subset U(A) \) such that
\[
u(0) = u, \ u(1) = w \quad \text{and} \quad \|([\phi(f), u(t)]\| < 2\epsilon \quad \text{(e 5.204)}
\]
for all \( f \in \mathcal{F} \) and \( t \in [0,1] \). Moreover, for any open arc \( I_a \) with length \( a \),
\[
\mu_{\tau \circ \phi}(I_a) \geq \Delta_0(a) \quad \text{for all } a \geq \eta \quad \text{(e 5.205)}
\]
where \( \iota : C(\mathbb{T}) \to A \) is defined by \( \iota(f) = f(w) \) for all \( f \in C(\mathbb{T}) \).

**Proof.** Let \( \epsilon > 0 \) and \( \mathcal{F} \subset C \) be as described. Put \( \mathcal{F}_1 = \phi(\mathcal{F}) \). The corollary follows from 5.8 by taking \( u(t) = \tau(t)u \).

The proof of the following lemma follows from the same argument used in that of 5.8 by applying 5.10 instead.

**Lemma 5.10.** Let \( \Delta : (0,1) \to (0,1) \) be a non-decreasing map, let \( \eta > 0 \), let \( X \) be a compact metric space and let \( \mathcal{F} \subset C(X) \) be a finite subset. Suppose that \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), suppose that \( \phi : C(X) \to A \) is a unital homomorphism and suppose that \( u \in U(A) \) such that
\[
\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \quad \text{for all } \tau \in T(A) \quad \text{(e 5.206)}
\]
for any open ball with radius \( a \geq \eta \). For any \( \epsilon > 0 \), there exists a unitary \( v \in U_0(A) \) and a continuous path of unitaries \( \{v(t) : t \in [0,1]\} \subset U_0(A) \) such that
\[
v(0) = 1, \ v(1) = v \quad \text{(e 5.207)}
\]
\[
\|[\phi(f), v(t)]\| < \epsilon, \|[u, v(t)]\| < \epsilon \quad \text{for all } f \in \mathcal{F} \text{ and } t \in [0,1] \quad \text{(e 5.208)}
\]
\[
\tau(\phi(f)g(vu)) \geq D_0(\Delta)(a) \quad \text{for all } \tau \in T(A) \quad \text{(e 5.209)}
\]
for any \( f \in C(X) \) with \( 0 \leq f \leq 1 \) whose support contains an open ball with radius \( a \geq 4\eta \) and any \( g \in C(\mathbb{T}) \) with \( 0 \leq g \leq 1 \) whose support contains an open arc with length \( a \geq 4\eta \), where \( D_0(\Delta) \) is defined in 5.7.

## 6 The Basic Homotopy Lemma for \( C(X) \)

In this section we will prove Theorem 6.2 below. We will apply the results of the previous section to produce the map \( L \) which was required in Theorem 4.5 by using a continuous path of unitaries.

**Lemma 6.1.** Let \( X \) be a compact metric space, let \( \Delta : (0,1) \to (0,1) \) be a non-decreasing map, let \( \epsilon > 0 \), let \( \eta > 0 \) and let \( \mathcal{F} \subset C(X) \) be a finite subset. There exists \( \delta > 0 \) and a finite subset \( \mathcal{G} \subset C(X) \) satisfying the following:

Suppose that \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), suppose that \( \phi : C(X) \to A \) and suppose that \( u \in U(A) \) such that
\[
\|[\phi(f), u]\| < \delta \quad \text{for all } f \in \mathcal{G} \text{ and} \quad \text{(e 6.210)}
\]
\[
\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \quad \text{for all } \tau \in T(A) \quad \text{(e 6.211)}
\]
for any open balls $O_b$ with radius $b \geq \eta/2$. There exists a unitary $v \in U_0(A)$, a unital completely positive linear map $L : C(X \times T) \to A$ and a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset U_0(A)$ such that

\[
v(0) = u, \quad v(1) = v, \quad \|\phi(f), v(t)\| < \epsilon \quad \text{for all } f \in \mathcal{F} \text{ and } t \in [0,1],
\]

\[
\|L(f \otimes z) - \phi(f)v\| < \epsilon, \quad \|L(f \otimes 1) - \phi(f)\| < \epsilon \quad \text{for all } f \in \mathcal{F} \text{ and } t \in \mathcal{T}(A)
\]

\[
\mu_{\tau \circ L}(O_a) \geq (2/3)D_0(\Delta)(a/2) \quad \text{for all } \tau \in \mathcal{T}(A)
\]

for any open balls $O_a$ of $X \times T$ with radius $a \geq 5\eta$, where $D_0(\Delta)$ is defined in [5.7].

**Proof.** Fix $\epsilon > 0, \eta > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let $\mathcal{F}_1 \subset C(X)$ be a finite subset containing $\mathcal{F}$. Let $\epsilon_0 = \min\{\epsilon/2, D_0(\Delta)(\eta)/4\}$. Let $\mathcal{G} \subset C(X)$ be a finite subset containing $\mathcal{F}$, $1_{C(X)}$ and $z$. There is $\delta_0 > 0$ such that there is a unital completely positive linear map $L' : C(X \times T) \to B$ (for unital $C^*$-algebra $B$) satisfying the following:

\[
\|L'(f \otimes z) - \phi'(f)u\| < \epsilon_0 \quad \text{for all } f \in \mathcal{F}_1
\]

for any unital homomorphism $\phi' : C(X) \to B$ and any unitary $u' \in B$ whenever

\[
\|[\phi'(g), u']\| < \delta_0 \quad \text{for all } g \in \mathcal{G}.
\]

Let $0 < \delta < \min\{\delta_0/2, \epsilon/2, \epsilon_0/2\}$ and suppose that

\[
\|[\phi(g), u]\| < \delta \quad \text{for all } g \in \mathcal{G}.
\]

It follows from [5.10] that there is a continuous path of unitaries $\{z(t) : t \in [0,1]\} \subset U_0(A)$ such that

\[
z(0) = 1, \quad z(1) = v_1,
\]

\[
\|[\phi(f), z(t)]\| < \delta/2, \quad \|[u, z(t)]\| < \delta/2 \quad \text{for all } t \in [0,1] \quad \text{and}
\]

\[
\tau(\phi(f)g(v_1u)) \geq D_0(\Delta)(a)
\]

for any $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball with radius $4\eta$ and $g \in C(T)$ with $0 \leq g \leq 1$ whose support contains open arcs with length $a \geq 4\eta$.

Put $v = v_1u$. Then we obtain a unital completely positive linear map $L : C(X \times T) \to A$ such that

\[
\|L(f \otimes z) - \phi(f)v\| < \epsilon_0 \quad \text{and} \quad \|L(f \otimes 1) - \phi(f)\| < \epsilon_0 \quad \text{for all } f \in \mathcal{F}_1.
\]

If $\mathcal{F}_1$ is sufficiently large (depending on $\eta$ only), we may also assume that

\[
\mu_{\tau \circ L}(B_a \times J_a) \geq (2/3)D_0(\Delta)(a/2)
\]

for any open ball $B_a$ with radius $a$ and open arcs with length $a$, where $a \geq 5\eta$.

Theorem 6.2. Let $X$ be a finite CW complex so that $X \times T$ has the property (H). Let $C = PC(X, M_n)P$ for some projection $P \in C(X, M_n)$ and let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0, \eta > 0$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:

Suppose that $A$ is a unital simple $C^*$-algebra with $TR(A) \leq 1$, $\phi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary and suppose that

\[
\|[\phi(c), u]\| < \delta \quad \text{for all } c \in \mathcal{G} \quad \text{and} \quad \text{Bott}(\phi, u) = \{0\}.
\]
Suppose also that

\[ \mu_{\tau_0 \phi}(O_a) \geq \Delta(a) \]  

(\text{e 6.224})

for all open balls \( O_a \) of \( X \) with radius \( 1 > a \geq \eta \), where \( \mu_{\tau_0 \phi} \) is the Borel probability measure defined by restricting \( \phi \) on the center of \( C \). Then there exists a continuous path of unitaries \( \{ u(t) : t \in [0,1] \} \) in \( A \) such that

\[ u(0) = u, \; u(1) = 1 \; \text{and} \; \| [\phi(c), u(t)] \| < \epsilon \]  

(\text{e 6.225})

for all \( c \in \mathcal{F} \) and for all \( t \in [0,1] \).

\begin{proof}
First it is easy to see that the general case can be reduced to the case that \( C = C(X, M_n) \). It is then easy to see that this case can be further reduced to the case that \( C = C(X) \). Then the theorem follows from the combination of \([8, 3]\) and \([6, 1]\).
\end{proof}

**Corollary 6.3.** Let \( k \geq 1 \) be an integer, let \( \epsilon > 0 \) and let \( \Delta : (0,1) \rightarrow (0,1) \) be any non-decreasing map. There exist \( \delta > 0 \) and \( \eta > 0 \) (\( \eta \) does not depend on \( \Delta \)) satisfying the following:

For any \( k \) mutually commutative unitaries \( u_1, u_2, \ldots, u_k \) and a unitary \( v \in U(A) \) in a unital separable simple \( C^* \)-algebra \( A \) with tracial rank no more than one for which

\[ \|[u_i, v]\| < \delta, \; \text{bott}_j(u_i, v) = 0, \; j = 0, 1, i = 1, 2, \ldots, k, \; \text{and} \; \mu_{\tau_0 \phi}(O_a) \geq \Delta(a) \]  

for all \( \tau \in T(A) \),

for any open ball \( O_a \) with radius \( a \geq \eta \), where \( \phi : C(T^k) \rightarrow A \) is the homomorphism defined by

\[ \phi(f) = f(u_1, u_2, \ldots, u_k) \]  

for all \( f \in C(T^k) \), there exists a continuous path of unitaries \( \{ v(t) : t \in [0,1] \} \subset A \) such that \( v(0) = v, \; v(1) = 1 \) and

\[ \|[u_i, v(t)]\| < \epsilon \]  

for all \( t \in [0,1], \; i = 1, 2, \ldots, k. \)

\end{corollary}

**Remark 6.4.** In \([6, 3]\) if \( k = 1 \), the condition that \( \text{bott}_0(u_1, v) = 0 \) is the same as \( v \in U_0(A) \). Note that in Theorem \([6, 2]\) the constant \( \delta \) depends not only on \( \epsilon \) and the finite subset \( \mathcal{F} \) but also depends on the measure distribution \( \Delta \). As in section 9 of \([29]\), in general, \( \delta \) can not be chosen independent of \( \Delta \).

Unlike the Basic Homotopy Lemma in simple \( C^* \)-algebras of real rank zero, in Theorem \([6, 2]\) as well as in \([6, 3]\) the length of \( \{ u(t) \} \) (or \( \{ v_t \} \)) can not be possibly controlled. To see this, one notes that, it is known (see \([39]\)) that \( \text{cel}(A) = \infty \) for some simple AH-algebras with no dimension growth. It is proved (see \([13]\), or Theorem 2.5 of \([28]\) ) that all of these \( C^* \)-algebras \( A \) have tracial rank one. For those simple \( C^* \)-algebras, let \( k = 1 \). For any number \( L > \pi \), choose \( u = v \) and \( v \in U_0(A) \) with \( \text{cel}(v) > L \). This gives an example that the length of \( \{ v_t \} \) is longer than \( L \). This shows that, in general, the length of \( \{ v_t \} \) could be as long as one wishes.

However, we can always assume that the path \( \{ u(t) : t \in [0,1] \} \) is piece-wise smooth. For example, suppose that \( \{ u(t) : t \in [0,1] \} \) satisfies the conclusion of \([6, 2]\) for \( \epsilon/2 \). There are \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) such that

\[ \| u(t_i) - u(t_{i-1}) \| < \epsilon/32, \; i = 1, 2, \ldots, n. \]

There is a selfadjoint element \( h_i \in A \) with \( \| h_i \| \leq \epsilon/8 \) such that

\[ u(t_i) = u(t_{i-1}) \exp(\sqrt{-1}h_i), \; i = 1, 2, \ldots, n. \]

Define

\[ w(t) = u(t_{i-1}) \exp(\sqrt{-1}(t - t_{i-1})/t_i - t_{i-1})h_i) \]  

for all \( t \in [t_{i-1}, t_i), \; i = 1, 2, \ldots, n. \) Note that

\[ \|[\phi(c), w(t)]\| < \epsilon \]  

for all \( t \in [0,1] \).

On the other hand, it is easy to see that \( w(t) \) is continuous and piece-wise smooth.
7 An approximate unitary equivalence result

The following is a variation of some results in [15]. We refer to [15] for the terminologies used in the following statement.

Theorem 7.1. (cf. Theorem 1.1 of [15]) Let $C$ be a unital separable amenable $C^*$-algebra satisfying the UCT. Let $b \geq 1$, let $T : \mathbb{N}^2 \to \mathbb{N}$, $L : U(M_\infty(C)) \to \mathbb{R}_+$, $E : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ and $T_1 = N \times K : C_+ \setminus \{0\} \to N \times \mathbb{R}_+ \setminus \{0\}$ be four maps. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, a finite subset $\mathcal{G} \subset C$, a finite subset $\mathcal{H} \subset C_+ \setminus \{0\}$, a finite subset $\mathcal{P} \subset K(C)$, a finite subset $\mathcal{U} \subset U(M_\infty(C))$, an integer $l > 0$ and an integer $k > 0$ satisfying the following:

for any unital $C^*$-algebra $A$ with stable rank one, $K_0$-divisible rank $T$, exponential length divisible rank $E$ and $\text{cer}(M_m(A)) \leq b$ (for all $m$), if $\phi, \psi : C \to A$ are two unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear maps with

$$[\phi]|_\mathcal{F} = [\psi]|_\mathcal{F} \quad \text{and} \quad \text{cel}([\phi](u)|^*([\psi](u)) \leq L(u) \quad (e \ 7.226)$$

for all $u \in \mathcal{U}$, then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\theta : C \to M_l(A)$ which is also $T$-$\mathcal{H}$-full, there exists a unitary $u \in M_{lk+1}(A)$ such that

$$\|u^*\text{diag}(\phi(a), \theta(a), \theta(a), \ldots, \theta(a))u - \text{diag}(\psi(a), \theta(a), \theta(a), \ldots, \theta(a))\| < \epsilon \quad (e \ 7.227)$$

for all $a \in \mathcal{F}$.

Proof. Suppose that the theorem is false. Then there exists $\epsilon_0 > 0$ and a finite subset $\mathcal{F} \subset C$ such that there are a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \downarrow 0$, an increasing sequence of finite subsets $\{\mathcal{G}_n\}$ whose union is dense in $C$, an increasing sequence of finite subsets $\{\mathcal{H}_n\} \subset C_+ \setminus \{0\}$ whose union is dense in $C_+$, a sequence of finite subsets $\{\mathcal{P}_n\}$ of $K(C)$ with $\bigcup_{n=1}^\infty \mathcal{P}_n = K(C)$, a sequence of finite subsets $\{\mathcal{U}_n\} \subset U(M_\infty(C))$, two sequences of $\{l(n)\}$ and $\{k(n)\}$ of integers (with $\lim_{n \to \infty} l(n) = \infty$), a sequence of unital $C^*$-algebra $A_n$ with stable rank one, $K_0$-divisible rank $T$, exponential length divisible rank $E$ and $\text{cer}(M_m(A_n)) \leq b$ (for all $m$) and sequences $\{\phi_n\}$, $\{\psi_n\}$ of $\mathcal{G}_n$-$\mathcal{G}_n$-multiplicative contractive completely positive linear maps from $C$ into $A_n$ with

$$[\phi_n]|_\mathcal{F} = [\psi_n]|_\mathcal{F} \quad \text{and} \quad \text{cel}([\phi](u)|^*([\psi](u)) \leq L(u) \quad (e \ 7.228)$$

$u \in \mathcal{U}_n$ satisfying the following:

$$\inf\{\sup\{\|v^*\text{diag}(\phi_n(a), S_n(a))v - \text{diag}(\psi_n(a), S_n(a))\| : a \in \mathcal{F}\} \geq \epsilon_0 \quad (e \ 7.229)$$

where the infimum is taken among all unital $T_1$-$\mathcal{H}_n$-full and $\mathcal{G}_n$-$\mathcal{G}_n$-multiplicative contractive completely positive linear maps $\sigma_n : C \to M_{l(n)}(A_n)$ and where

$$S_n(a) = \text{diag}(\sigma_n(a), \sigma_n(a), \ldots, \sigma_n(a)),$$

and among all unitaries $v$ in $M_{l(n)k(n)+1}(A_n)$.

Let $B_0 = \bigoplus_{n=1}^\infty A_n$, $B = \prod_{n=1}^\infty B_n$, $Q(B) = B/B_0$ and $\pi : B \to Q(B)$ be the quotient map. Define $\Phi, \Psi : C \to B$ by $\Phi(a) = \{\phi_n(a)\}$ and $\Psi(a) = \{\psi_n(a)\}$ for $a \in C$. Note that $\pi \circ \Phi$ and $\pi \circ \Psi$ are homomorphisms.

For any $u \in \mathcal{U}_n$, since $A_n$ has stable rank one, when $n \geq m$,

$$\langle \phi_n(u)\rangle\langle \psi_n(u)\rangle^* \in U_0(A_n) \quad \text{and} \quad \text{cel}([\phi_n(u)|^*([\psi_n(u)]) \leq L(u). \quad (e \ 7.230)$$
It follows that, for all \( n \geq m \), (by Lemma 1.1 of [15] for example), there is a continuous path \( \{U(t) \in \prod_{n=m}^{\infty} A_n : t \in [0,1] \} \) such that

\[
U(0) = \{ (\phi_n(u)) \}_{n \geq m} \quad \text{and} \quad U(1) = \{ (\psi_n(u)) \}_{n \geq m}.
\]

Since this holds for each \( m \), it follows that

\[
(\pi \circ \Phi)_{*1} = (\pi \circ \Psi)_{*1} \quad (e \, 7.231)
\]

It follows from (2) of Corollary 2.1 of [15] that

\[
K_0(B) = \prod B_0(B_n) \quad \text{and} \quad K_0(Q(B)) = \prod B_0(B_n)/B_0(B_n).
\]

Then, by \((e \, 7.228)\) and by using the fact that each \( B_n \) has stable rank one again, one concludes that

\[
(\pi \circ \Phi)_{*0} = (\pi \circ \Psi)_{*0} \quad (e \, 7.233)
\]

Moreover, with the same argument, by \((e \, 7.228)\) and by applying (2) of Corollary 2.1 of [15],

\[
[\pi \circ \Phi]|_{K_i(C,\mathbb{Z}/k\mathbb{Z})} = [\pi \circ \Psi]|_{K_i(C,\mathbb{Z}/k\mathbb{Z})}, \quad k = 2, 3, ..., \quad \text{and} \quad i = 0, 1.
\]

Since \( C \) satisfies the UCT, by \([7]\),

\[
[\pi \circ \Phi] = [\pi \circ \Psi] \quad \text{in} \quad KL(C, Q(B)).
\]

On the other hand, since each \( \sigma_n \) is \( \delta_n \)-\( \mathcal{G}_n \)-multiplicative and \( T_1 \)-\( \mathcal{H}_n \)-full, we conclude that \( \pi \circ \Sigma \) is a full homomorphism, where \( \Sigma : C \to B \) is defined by \( \Sigma(c) = \{ \sigma_n(c) \} \) for \( c \in C \).

It follows from Theorem 3.9 of [25] that there exists an integer \( N \) and a unitary \( \bar{W} \in Q(B) \) such that

\[
\| \bar{W}^* \text{diag}(\pi \circ \Phi(c), \pi \circ \Sigma(c), \cdots, \pi \circ \Sigma(c)) \bar{W}^\text{N} - \text{diag}(\pi \circ \Psi(c), \pi \circ \pi \circ \Sigma(c), \cdots, \pi \circ \Sigma(c)) \| < \epsilon_0/2 \quad (e \, 7.237)
\]

for all \( c \in \mathcal{F} \). There exists a unitary \( u_n \in A_n \) for each \( n \) such that \( \pi(\{ u_n \}) = \bar{W} \). Therefore, by \((e \, 7.238)\), for some large \( n_0 \geq 0 \),

\[
\| u_n^* \text{diag}(\phi_n(c), \sigma_n(c), \cdots, \sigma_n(c)) u_n^\text{N} - \text{diag}(\psi_n(c), \sigma_n(c), \cdots, \sigma_n(c)) \| < \epsilon_0 \quad (e \, 7.239)
\]

for all \( c \in \mathcal{F} \). This contradicts with \((e \, 7.229)\).

\[ \square \]

**Remark 7.2.** Suppose that \( U(C)/U_0(C) = K_1(C) \). Then, from the proof, one sees that we may only consider \( U \subset U(C) \).
Theorem 7.3. Let $C$ be a unital separable simple amenable $C^*$-algebra with $TR(C) \leq 1$ satisfying the UCT and let $D = C \otimes C(\mathbb{T})$. Let $T = N \times K : D_+ \setminus \{0\} \to \mathbb{N}_+ \times \mathbb{R}_+ \setminus \{0\}$.

Then, for any $\epsilon > 0$ and any finite subset $F \subset D$, there exists $\delta > 0$, a finite subset $G \subset D$, a finite subset $H \subset D_+ \setminus \{0\}$, a finite subset $P \subset K(C)$ and a finite subset $U \subset U(D)$ satisfying the following: Suppose that $A$ is a unital simple $C^*$-algebra with $TR(A) \leq 1$ and $\phi, \psi : D \to A$ are two unital $\delta$-$G$-multiplicative contractive completely positive linear maps such that $\phi, \psi$ are $T$-$H$-full,

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in G$$  \hspace{1cm} (e 7.240)

for all $\tau \in T(A)$,

$$[\phi]|_P = [\psi]|_P \text{ and }$$  \hspace{1cm} (e 7.241)

$$\text{dist}(\phi^l(\bar{w}), \psi^l(\bar{w})) < \delta$$  \hspace{1cm} (e 7.242)

for all $w \in U$. Then there exists a unitary $u \in U(A)$ such that

$$ad \, u \circ \psi \approx_\epsilon \phi \text{ on } F.$$  \hspace{1cm} (e 7.243)

Proof. Let $\epsilon > 0$ and a finite subset $F \subset D$. Fix $T = N \times K$ as given. Let $T_1 = N \times 2K$. Let $1 > \delta_1 > 0$, $G_1 \subset D$, $H_1 \subset D_+ \setminus \{0\}$, $P_1 \subset K(D)$, $U_1 \subset U(M_\infty(D))$, integer $l$ and $k$ as required by (7.244) for $\epsilon/8$, $F$ and $T$ as well as for $b = 2$, $T(u, m) = 1$, $L(u) = 2c(\epsilon) + 3\pi + 1$, $E(l, k) = 8\pi + l/k$. We may assume that $\delta_1 < \min\{\epsilon/8, 1/8\pi\}$ and $k \geq 2$. Without loss of generality, we may assume that $G_1 = \{g \otimes 1, g \in G_0\} \cup \{1 \otimes z\}$, where $G_0 \subset C$ and $z$ is the identity function on $\mathbb{T}$, the unit circle. Note that $K_1(D) = K_1(A) \bigoplus K_0(A)$. It is clear that $K_1(D)$ is generated by $u \otimes 1$ and $(p \otimes z) + (1 - p) \otimes 1$ for $u \in U(A)$ and projections $p \in A$. In particular, $K_1(D) = U(D)/U_0(D)$. Thus (see the remark 7.22), we may assume that $U_1 \subset U(A)$.

Since $TR(C) \leq 1$, for any $\delta_2 > 0$, there exists a projection $e \in C$ and a $C^*$-subalgebra $C_0 \subset \mathcal{I}$ with $1_{C_0} = e$ and a contractive completely positive linear map $j_1 : C \to C_0$ such that

(1) $\| [x, e]\| < \delta_2$ for $x \in G_0$;

(2) $\text{dist}(ex, j_1(x)) < \delta_2/4$ for $x \in G_0$ and

(3) $(2kl + 1)\tau(1 - e) < \tau(e)$ and $\tau(1 - e) < \delta_2/(2kl + 1)$ for all $\tau \in T(C)$.

Put $z_0 = (1 - e) \otimes z$, $z_1 = e \otimes z$ and $j_0(e) = (1 - e)c(1 - e)$ for $c \in C$. We may also assume that $\delta_2 < \delta_1/4$. Put $G_{00} = j_1(G_0)$. Thus

$$\text{dist}(ex, G_{00}) < \delta/4 \text{ for all } x \in G_0.$$  \hspace{1cm} (e 7.244)

Let $D_0 = C_0 \otimes C(\mathbb{T})$. Let $T' = T|_{(D_0)_+ \setminus \{0\}}$. Let $\delta_3 > 0$ (in place $\delta$), let $G_2$ (in place of $G$) be a finite subset of $D_0$, let $H_2 \subset (D_0)_+ \setminus \{0\}$, let $P_2$ (in place of $P$) be a finite subset of $K(D_0)$ and let $U_2$ (in place of $U$) be a finite subset of $U(M_\infty(D_0))$ required by Theorem 11.5 of [31] for $\delta_1/4$ (in place of $\epsilon$), $G_{00} \cup \{z_1\}$ (in place of $F$) and $(D_0$ in place of $C$). Here we identify $e$ with $1_{D_0}$. Let $J = j_1 \otimes \text{id}_{C(\mathbb{T})} : D_0 \to D$ be the obvious embedding and $J_0 = j_0 \otimes \text{id}_{C(\mathbb{T})}$. Let $P_2 \subset K(D)$ be the image of $P_2$ under $[J]$.

Now let $\delta = \min\{\delta_2/(8kl + 1), \delta_3/(8kl + 1)\}, G = G_1 \cup G_2 \cup \{e, (1 - e)\}$. Here we also view $G_2$ as a subset of $D$. Let $H = H_1 \cup H_2$, let $P = P_1 \cup P_2$ and $U = U_1 \cup \{(e + j_0)(u) : u \in U_1\} \cup \{(1 - e) + v : v \in U_2\}$.

Suppose that $\phi$ and $\psi$ satisfy the assumptions of the theorem for the above $G$, $H$, $P$ and $U$. Let $\phi' = \phi \circ J$, $\psi' = \psi \circ J$. There is a unitary $u_0 \in A$ such that

$$u_0 \psi'(e)u_0 = \phi'(e) = e_0 \in A.$$
Put $A_1 = e_0 A e_0$. We have $[\text{ad } u_0 \circ \psi']|_{P_2} = [\phi']|_{P_2}$ and, for $g \in G$,
\[ |t \circ \text{ad } u_0 \circ \psi'(g) - t \circ \phi'(g)| < \frac{\delta}{1 - \delta/(2klt + 1)} < \delta_3 \text{ for all } t \in T(e A e). \] (e 7.245)

Moreover, by the first part of 5.3,
\[ \text{dist}((\text{ad } u_0 \circ \psi')^i(\bar{w}), (\phi')^i(\bar{w})) < (2 + 1)\delta < \delta_3 \] (e 7.246)
for all $w \in U_2$.

By the choice of $G_2, H_2, U_2$ and $P_2$, and by applying 11.5 of [34], there is a unitary $u_1 \in A_1$ such that
\[ \text{ad } u_1 \circ \text{ad } u_0 \circ \psi' \approx_{\epsilon/2} \phi' \text{ on } G_0. \] (e 7.247)

Let $G_0' \subseteq G_0$ be a finite subset containing $G_0 \cup j(H_1)$ and $\delta_4 > 0$. Since $TR(A_1) \leq 1$, by Lemma 5.5 of [28], there are mutually orthogonal projections $q_0, q_1, q_2, \ldots, q_{8kl + 4}$ with $[q_0] \leq [q_1]$ and $[q_1] = [q_i], i = 1, 2, \ldots, 8kl + 4$, and there are unital $\delta_4, G_0'$ multiplicative contractive completely positive linear maps $L_0 : D_0 \rightarrow q_0 A q_0$ and $L_i : D_0 \rightarrow q_i A q_i$ ($i = 1, 2, \ldots, 8kl + 4$) such that
\[ \phi' \approx_{\delta_4} L_0 \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_{8kl+4} \text{ on } G_0', \] (e 7.248)
and there exists a unitary $W_i \in (q_1 + q_i) A(q_1 + q_i)$ such that
\[ \text{ad } W_i \circ L_i = L_1, \ i = 1, 2, \ldots, 8kl. \] (e 7.249)

Since $\phi$ is $T\mathcal{H}$-full, with sufficiently small $\delta_4$ and sufficiently large $G_0'$ we may also assume that each $L_i \circ j_1$ is also $T_1\mathcal{H}_1$-full and $\delta_4 < \delta/4$. Define $Q_i = \sum_{j=1}^{8i+4} q_i, Q_0 = \sum_{i=1}^{4} q_i$ and $\Phi_i = \sum_{j=1}^{8i+4} L_i, i = 1, 2, \ldots, kl$. Note by (e 7.249), $\Phi_i$ are unitarily equivalent to $\Phi_i$.

Since $K_0(A)$ is weakly unperforated (see Theorem 6.11 of [19]), we check that
\[ [p_0 + q_0 + Q_0] \leq [Q_i] \text{ and } 2[p_0 + q_0 + Q_0] \geq [Q_i], \ i = 1, 2, \ldots, kl. \] (e 7.250)

Put $\phi_0 = \phi \circ J_0 \oplus L_0 \circ J \oplus \sum_{i=1}^{4} L_i \circ J$ and $\psi_0 = \psi \circ J_0 \oplus L_0 \circ J \oplus \sum_{i=1}^{4} L_i \circ J$. By 3.3 we compute that
\[ \text{dist}(\phi_0^i(\bar{w}), \psi_0^i(\bar{w})) < \delta_1 \text{ for all } w \in U \] (e 7.251)
It follows Lemma 6.9 of [28] that
\[ \text{cel}(\langle \phi_0(u) \rangle \langle \psi_0(u) \rangle^*) < 8\pi + 1 \text{ for all } u \in U. \] (e 7.252)

We also have
\[ [\phi_0]|_{P_1} = [\psi_0]|_{P_1}. \] (e 7.253)

Since $\Phi_1 \circ j$ is $T_1\mathcal{H}_1$-full, by applying 7.1, we obtain a unitary $w \in U(A)$,
\[ \|w^* \text{diag}(\psi_0(c), \Phi_1 \circ J(c), \ldots, \Phi_1 \circ J(c)) w - \text{diag}(\phi_0(c), \Phi_1 \circ J(c), \ldots, \Phi_1 \circ J(c))\| < \epsilon/8 \] (e 7.254)
for all $c \in \mathcal{F}$. Since $\Phi_1 \circ j_1$ is unitarily equivalent to $\Phi_1 \circ j_1$, there is a unitary $w' \in U(A)$ such that
\[ \|(w')^* \text{diag}(\psi_0(c), \Phi_1 \circ J(c), \ldots, \Phi_{kl} \circ J(c)) w' - \text{diag}(\phi_0(c), \Phi_1 \circ J(c), \ldots, \Phi_{kl} \circ J(c))\| < \epsilon/8 \] (e 7.255)
for all \(c \in \mathcal{F}\). It follows that
\[
\| (w')^* \text{diag}(\psi \circ J_0(c), L_0 \circ J(c), \phi'(c))w' - \text{diag}(\phi \circ J_0(c), L_0 \circ J(c), \phi'(c)) \| < \epsilon/4 \quad (e\,7.257)
\]
for all \(c \in \mathcal{F}\).

Let \(u = ((1 - e_0) \oplus e_0 u_0 u_1)w'\). Then, by [e\,7.241], we have
\[
\| u^* \text{diag}(\psi \circ J_0(c), \psi'(c))u - \text{diag}(\phi \circ J_0(c), \phi'(c)) \| < \epsilon/2 \quad (e\,7.258)
\]
for all \(c \in \mathcal{F}\). It follows that
\[
\text{ad} \, u \circ \psi \approx_\epsilon \phi \text{ on } \mathcal{F}. \quad (e\,7.259)
\]

\[\square\]

**Corollary 7.4.** Let \(C\) be a unital separable amenable simple \(C^*\)-algebra with \(\text{TR}(C) \leq 1\) which satisfies the UCT, let \(D = C \otimes C(\mathbb{T})\) and let \(A\) be a unital simple \(C^*\)-algebra with \(\text{TR}(A) \leq 1\). Suppose that \(\phi, \psi : D \to A\) are two unital monomorphisms. Then \(\phi\) and \(\psi\) are approximately unitarily equivalent, i.e., there exists a sequence of unitaries \(\{u_n\} \subset A\) such that
\[
\lim_{n \to \infty} \text{ad} \, u_n \circ \psi(d) = \phi(d) \text{ for all } d \in D,
\]
if and only if
\[
[\phi] = [\psi] \text{ in } KL(D, A), \tau \circ \phi = \tau \circ \psi \text{ for all } \tau \in T(A) \text{ and } \psi^\dagger = \phi^\dagger.
\]

## 8 The Main Basic Homotopy Lemma

**Lemma 8.1.** Let \(C\) be a unital separable simple \(C^*\)-algebra with \(\text{TR}(C) \leq 1\) and let \(\Delta : (0,1) \to (0,1)\) be a non-decreasing map. There exists a map \(T = N \times K : D_+ \setminus \{0\} \to \mathbb{N}_+ \times \mathbb{R}_+ \setminus \{0\}\), where \(D = C \otimes C(\mathbb{T})\), satisfying the following:

For any \(\epsilon > 0\), any finite subset \(\mathcal{F} \subset C\) and any finite subset \(\mathcal{H} \subset D_+ \setminus \{0\}\), there exists \(\delta > 0\), \(\eta > 0\) and a finite subset \(\mathcal{G} \subset C\) satisfying the following: for any unital separable unital simple \(C^*\)-algebra \(A\), any unital homomorphism \(\phi : C \to A\) and any unitary \(u \in A\) such that
\[
\| [\phi(c), u] \| < \delta \text{ for all } c \in \mathcal{G} \text{ and } \quad (e\,8.260)
\]
\[
\mu_{\tau_0}(O_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \quad (e\,8.261)
\]
and for all open balls \(O_a\) with radius \(a \geq \eta\), where \(\iota : C(\mathbb{T}) \to A\) is defined by \(\iota(f) = f(u)\), there is a unital contractive completely positive linear map \(L : D \to A\) such that
\[
\| L(c \otimes 1) - \phi(c) \| < \epsilon \quad \| L(c \otimes z) - \phi(c)u \| < \epsilon \text{ for all } c \in \mathcal{F} \quad (e\,8.262)
\]
and \(L\) is \(T\)-\(\mathcal{H}\)-full.

**Proof.** We identify \(D\) with \(C(\mathbb{T}, C)\). Let \(f \in D_+ \setminus \{0\}\). There is positive number \(b \geq 1\), \(g \in D_+\) with \(0 \leq g \leq b \cdot 1\) and \(f_1 \in D_+ \setminus \{0\}\) with \(0 \leq f_1 \leq 1\) such that
\[
g f g f_1 = f_1. \quad (e\,8.263)
\]

There is a point \(t_0 \in \mathbb{T}\) such that \(f_1(t_0) \neq 0\). There is \(r > 0\) such that
\[
\tau(f_1(t)) \geq \tau(f_1(t_0))/2
\]
for all $\tau \in T(C)$ and for all $t$ with $\text{dist}(t, t_0) < r$.

Define $\Delta_0(f) = \inf\{\tau(f_1(t_0))/4 : \tau \in T(C)\} \cdot \Delta(r)$. There is an integer $n \geq 1$ such that

$$n \cdot \Delta_0(f) > 1.$$  \hfill (e 8.264)

Define $T(f) = (n, b)$. Put

$$\eta = \inf\{\Delta_0(f) : f \in \mathcal{H}\}/2$$

and $\epsilon_1 = \min\{\epsilon, \eta\}$.

We claim that there exists an $\epsilon_1$-$\mathcal{F} \cup \mathcal{H}$-multiplicative contractive completely positive linear map $L : D \to A$ such that

$$\|L(c \otimes 1) - \phi(c)\| < \epsilon$$

for all $c \in \mathcal{F}$, $\|L(1 \otimes z) - u\| < \epsilon$ and

$$|\tau \circ L(f_1) - \int_T \tau(\phi(f_1(s))) d\mu_{\tau \circ L}(s)| < \eta$$

for all $\tau \in T(A)$.

and for all $f \in \mathcal{H}$. Otherwise, there exists a sequence of unitaries $\{u_n\} \subset U(A)$ for which

$$\mu_{\tau \circ L}(O_a) \geq \Delta(a)$$

for all $\tau \in T(A)$ and for any open balls $O_a$ with radius $a \geq a_n$ with $a_n \to 0$.

and for which

$$\lim_{n \to \infty} \|[\phi(c), u_n]\| = 0$$

for all $c \in C$ and suppose for any sequence of contractive completely positive linear maps $L_n : D \to A$ with

$$\lim_{n \to \infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0$$

for all $a, b \in D$, $\lim_{n \to \infty} \|L_n(c \otimes f) - \phi(c)f(u_n)\| = 0$,

for all $c \in C, f \in C(\mathbb{T})$ and

$$\liminf_n \{\max\{|\tau \circ L_n(f_1) - \int_T \tau(\phi(f_1(s))) d\mu_{\tau \circ L_n}(s)| : f \in \mathcal{H}\}\} \geq \eta$$

for some $\tau \in T(A)$, where $\iota_n : C(\mathbb{T}) \to D$ is defined by $\iota_n(f) = f(u_n)$ for $f \in C(\mathbb{T})$ (or no contractive completely positive linear maps $L_n$ exists that $\iota_n(f)$ for $f \in C(\mathbb{T})$ (or no contractive completely positive linear maps $L_n$ exists that

Put $A_n = A, n = 1, 2, \ldots$, and $Q(A) = \bigoplus_n A_n/\bigoplus_n A_n$. Let $\pi : \bigoplus_n A_n \to Q(A)$ be the quotient map. Define a linear map $L' : D \to \bigoplus_n A_n$ by $L(c \otimes 1) = \{\phi(c)\}$ and $L'(1 \otimes z) = \{u_n\}$. Then $\pi \circ L' : D \to Q(A)$ is a unital homomorphism. It follows from a theorem of Effros and Choi (3) that there exists a contractive completely positive linear map $L : D \to \bigoplus_n A_n$ such that $\pi \circ L = \pi \circ L'$. Write $L = \{L_n\}$, where $L_n : D \to A_n$ is a contractive completely positive linear map. Note that

$$\lim_{n \to \infty} \|L_n(a)L_n(b) - L_n(ab)\| = 0$$

for all $a, b \in D$.

Fix $\tau \in T(A)$, define $\iota_n : \bigoplus_n A_n \to C$ by $\iota_n(\{d_n\}) = \tau(d_n)$. Let $t$ be a limit point of $\{t_n\}$. Then $t$ gives a state on $\bigoplus_n A_n$.

Note that if $\{d_n\} \in \bigoplus_n A_n$, then $t_n(\{d_n\}) \to 0$. It follows that $t$ gives a state $\tilde{t}$ on $Q(A)$.

Note that (by (e 8.267))

$$\tilde{t}(\pi \circ L(c \otimes 1)) = \tau(\phi(c))$$

for all $c \in C$. It follows that

$$\tilde{t}(\pi \circ L(f)) = \int_T \tilde{t}(\pi \circ L(f(s) \otimes 1)) d\mu_{\tilde{t} \circ L_{1\otimes C(\mathbb{T})}} = \int_T \tau(\phi(f(s))) d\mu_{\tilde{t} \circ L_{1\otimes C(\mathbb{T})}}$$

(e 8.271)
for all \( f \in C(T,C) \). Therefore, for a subsequence \( \{n(k)\} \),

\[
|\tau \circ L_{n(k)}(f_1) - \int_T \tau(\phi(f_1(s)))d\mu_{\tau \circ L_{n(k)}}(s)| < \eta/2 \tag{e 8.272}
\]

for all \( f \in \mathcal{H} \). This contradicts with (e 8.270). Moreover, from this, it is easy to compute that

\[
\mu_{\circ \sigma \circ L_{|T|}}(O_a) \geq \Delta(a)
\]

for all open balls \( O_a \) of \( t \) with radius \( 1 > a \). This proves the claim.

Note that

\[
\int_T \tau \circ \phi(f_1(s))d\mu_{\tau \circ \pi}(s) \geq (\tau(\phi(f_1(t_0))/2)) \cdot \Delta(r)
\]

for all \( \tau \in T(A) \). It follows that

\[
\tau(L(f_1)) \geq \inf\{t(f_1(t_0))/2 : t \in T(C)\} - \eta/2 \geq (4/3)\Delta_0(f) \tag{e 8.273}
\]

for all \( f \in \mathcal{H} \).

By Corollary 9.4 of [34], there exists a projection \( e \in L(f_1)AL(f_1) \) such that

\[
\tau(e) \geq \Delta_0(f) \text{ for all } \tau \in T(A). \tag{e 8.274}
\]

It follows from (e 8.274) that there exists a partial isometry \( w \in M_n(A) \) such that

\[
w^* \text{diag}(e,e,\ldots,e)w \geq 1_A.
\]

Thus there \( x_1, x_2, \ldots, x_n \in A \) with \( \|x_i\| \leq 1 \) such that

\[
\sum_{i=1}^n x_i^* e x_i \geq 1. \tag{e 8.275}
\]

Hence

\[
\sum_{i=1}^n x_i^* g f g x_i \geq 1. \tag{e 8.276}
\]

It then follows that there are \( y_1, y_2, \ldots, y_n \in A \) with \( \|y_i\| \leq b \) such that

\[
\sum_{i=1}^n y_i^* f y_i = 1. \tag{e 8.277}
\]

Therefore \( L \) is \( T-H \)-full.

**Lemma 8.2.** Let \( C \) be a unital separable amenable simple \( C^* \)-algebra with \( TR(C) \leq 1 \) satisfying the UCT. For \( 1/2 > \sigma > 0 \), any finite subset \( G_0 \) and any projections \( p_1, p_2, \ldots, p_m \in C \). There is \( \delta_0 > 0 \), a finite subset \( G \subseteq C \) and a finite subset of projections \( P_0 \subseteq C \) satisfying the following: Suppose that \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), \( \phi : C \rightarrow A \) is a unital homomorphism and \( u \in U_0(A) \) is a unitary such that

\[
\|\phi(c), u\| < \delta < \delta_0 \text{ for all } c \in G \cup G_0 \text{ and } \text{bott}_0(\phi, u)|_{P_0} = \{0\}, \tag{e 8.278}
\]

37
where \( P_0 \) is the image of \( P_0 \) in \( K_0(C) \). Then there exists a continuous path of unitaries \( \{u(t) : t \in [0,1] \} \) in \( A \) with \( u(0) = u \) and \( u(1) = w \) such that

\[
\|\phi(c), u(t)\| < 3\delta \text{ for all } c \in G \cup G_0 \text{ and }
\]

\[
w_j \oplus (1 - \phi(p_j)) \in CU(A),
\]

where \( w_j \in U_0(\phi(p_j)A\phi(p_j)) \) and

\[
\|w_j - \phi(p_j)w\phi(p_j)\| < \sigma,
\]

\( j = 1, 2, ..., m \).

Moreover,

\[
\text{cel}(w_j \oplus (1 - \phi(p_j))) \leq 8\pi + 1/4, \quad j = 1, 2, ..., m.
\]

**Proof.** It follows from the combination of Theorem 4.8 and Theorem 4.9 of [9] and theorem 10.10 of [23] that one may write \( C = \lim_{n \to \infty} (C_n, \psi_n) \), where each \( C_n = \bigoplus_{j=1}^{R(n)} P_{n,j}C(X_{n,j}, M_{r(n,j)})P_{n,j} \) and where \( P_{n,i} \in C(X_{n,i}, M_{r(n,i)}) \) is a projection and \( X_{n,i} \) is a connected finite CW complex of dimension no more than two with torsion free \( K_1(C(X_{n,i})) \) and \( K_0(C(X_{n,i})) = \mathbb{Z} \oplus \mathbb{Z}/s(j)\mathbb{Z} \) \((s(i) \geq 1)\) and with positive cone \( \{(0,0) \cup (m,x) : m \geq 1\} \) (when \( s(j) = 1 \), we mean \( K_0(C(X_{n,i})) = Z \)), or \( X_{n,i} \) is a connected finite CW complex of dimension three with \( K_0(C(X_{n,i})) = Z \) and torsion \( K_1(C(X_{n,i})) \). Let \( d(j) \) be the rank of \( P_{n,j} \). It is known that one may assume that

\[
d(j) \geq \prod_{i=1}^{R(n)} s(j) + 6, \quad j = 1, 2, ..., R(n).
\]

This can be seen, for example, from Lemma 2.2, 2.3 (and the proof of Theorem 2.1) of [10].

Without loss of generality, we may assume that \( G_0 \subset C_\infty(C_n) \) and that there are projections \( p_{i,0} \in C_n \) such that \( \psi_n(p_{i,0}) = p_i, \quad i = 1, 2, ..., m \). Choose, for each \( j \), mutually orthogonal rank one projections \( q_{j,0}^{(0)}, q_{j,1}^{(0)} \in P_{n,j}(C(X_{n,j}, M_{r(n,j)}))P_{n,j} \) such that

\[
[q_{j,0}^{(0)}] = (1, 0) \text{ and } [q_{j,1}^{(0)}] = (1, 1) \in \mathbb{Z} \bigoplus \mathbb{Z}/s(j)\mathbb{Z},
\]

or \( q_{j,1}^{(0)} = 0 \), if \( K_0(C(X_{n,j})) = Z \), \( j = 1, 2, ..., R(n) \). Put \( q_{j,i} = \psi_n, (q_{j,i}^{(0)}) \) and \( q_{j,i} = \phi(q_{j,i}^{(0)}) \), \( i = 0, 1 \) and \( j = 1, 2, ..., R(n) \). Clearly, in \( C \), \( p_k \) may be written as \( W_j^*Q_jW_j \), where \( Q_j \) is a finite orthogonal sum of \( q_{j,0} \) and \( q_{j,1} \), \( j = 1, 2, ..., R(n) \).

By choosing a sufficiently large \( G \) which contains \( G_0 \) (and which contains \( Q_j, q_{j,i} \) as well as \( W_j \), among other elements) and sufficiently small \( \delta_0 > 0 \), one sees that it suffices to show the case that \( \{p_1, p_2, ..., p_m\} \subset \{q_{j,0}, q_{j,1} : j = 1, 2, ..., R(n)\} \). Thus we obtain a finite subset \( G' \) and \( \delta_0' \) so that when \( G \supset G' \) and \( \delta_0 < \delta_0' \) one can make the assumption that \( \{p_1, p_2, ..., p_m\} \subset \{q_{j,i} : i = 0, 1, \quad j = 1, 2, ..., R(n)\} \). In particular, \( \{q_{j,i} : i = 0, 1, \quad j = 1, 2, ..., R(n)\} \subset G' \).

Let \( G_0' = G_0 \cup \{q_{j,0}^{(0)}, q_{j,1}^{(0)} : j = 1, 2, ..., R(n)\} \). Fix \( 0 < \eta < \min\{\sigma/4, \delta_0'/2, 1/16\} \). Note that \( P_{n,j} \) is locally trivial in \( C(X_{n,j}, M_{r(n,j)}) \). Since \( TR(C) \leq 1 \), it has (SP) (see [19]). It is then easy to find a projection \( e_j \in \psi_n, (P_{n,j})C_\infty, (P_{n,j})C_\infty, (P_{n,j}) \) and \( B_j \equiv M_{d(j)} \subset \psi_n, (P_{n,j})C_\infty, (P_{n,j}) \) with \( 1_{B_j} = e_j \) such that

\[
\|[x, e_j]\| < \eta \text{ for all } x \in G_0',
\]

\[
dist(e_jxe_j, B_j) < \eta \text{ for all } x \in G_0' \text{ and } e_jq_{j,0}'e_j, e_jq_{j,1}'e_j \neq 0,
\]

\( j = 1, 2, ..., R(n) \). Furthermore, one may require that there is a projection \( q_{j,0}'e_j \in B_j \) with rank one in \( B_j \) such that

\[
\|q_{j,0}'e_j - e_jq_{j,0}'e_j\| < 2\eta, \quad i = 0, 1, \quad j = 1, 2, ..., R(n).
\]
To simplify notation further, by replacing $q'_{j,i}$ by one of its nearby projections, we may assume that 
$q'_{j,i} = q'_{j,i} + (q'_{j,i} - q'_{j,i})$ and $q'_{j,i} \geq q'_{j,i}, i = 0, 1$ and $j = 1, 2, ..., R(n)$. Since $s(j)[q'_{j,1}] = s(j)[q'_{j,0}]$, there is a unitary $Y_j \in P_{n,j}C(X_{n,j}, M_{R(n,j)})P_{n,j}$ such that

$$Y_j^* \text{diag}(q'_{j,1}, q'_{j,1}, ..., q'_{j,0}, q'_{j,0}, q'_{j,0}) Y_j = \text{diag}(q'_{j,0}, q'_{j,0}, ..., q'_{j,0}).$$  \hspace{1cm} (e 8.286)

(Note that $d(j) \geq \prod_{j=1}^{R(n)} s(j) + 6$ and each $q'_{j,i}$ has rank one.)

Let $\{e_{i,k}\}$ be a matrix unit for $B_j, j = 1, 2, ..., R(n)$. We choose a finite subset $G$ which contains $G'_0$ as well as $\{e_{i,k}\}, q'_{j,0}, q'_{j,1}$ and $\{Y_j, Y_j^*\}, j = 1, 2, ..., R(n)$. Suppose that $v_{j,0} \in U_0(\phi(e_{1,1}) A \phi(e_{1,1}))$ and

$$v_j = \text{diag}(v_{j,0}, v_{j,0}, ..., v_{j,0}), j = 1, 2, ..., R(n).$$ \hspace{1cm} (e 8.287)

Then

$$\phi(x)v_j = v_j \phi(x) \text{ for all } x \in B_j, j = 1, 2, ..., R(n).$$ \hspace{1cm} (e 8.288)

Choose

$$P_0 = \{[q'_{j,0}], [q'_{j,1}], [e_{j,i}], [q'_{j,0}], [q'_{j,1}], j = 1, 2, ..., R(n)\}.$$  \hspace{1cm}

Put $\phi = \phi(q'_{j,0}), i = 0, 1$ and $j = 1, 2, ..., R(n)$. We choose $\delta_0 > 0$ such that $\text{bott}_0(\phi, u)|P_0$ is well-defined which is zero and there is a unitary $u'_{j,i} \in U_0(q_{j,i} A q_{j,i})$ such that

$$\|u'_{j,i} - q_{j,i} q_{j,i}\| < 2\delta_0, i = 0, 1, j = 1, 2, ..., R(n),$$  \hspace{1cm}

whenever, $\|\phi(c), u\| < \delta_0$ for all $c \in G$.

Let $\delta_0 = \{1/32, \delta_0/4, \delta_0'/4, \sigma/8\}$. Suppose that $\{e_{2.28}\}$ holds for the above $G, P_0$ and $0 < \delta < \delta_0$. One obtains a unitary $u''_{j,i} \in U_0(q_{j,i} A q_{j,i})$ and a unitary $u''_{j,i} \in U_0((q_{j,i} - q_{j,i}) A (q_{j,i} - q_{j,i}))$ such that

$$\|u''_{j,i} - q_{j,i} u''_{j,i}\| < 2\delta,$$ \hspace{1cm} (e 8.289)

where $u_{j,i} = u'_{j,i} + u''_{j,i}, i = 0, 1$ and $j = 1, 2, ..., R(n)$. It follows $3.4$ (see also Theorem 6.6 of [28]) that there is $v_{j,0} \in U_0(\phi(e_{1,1}) A \phi(e_{1,1}))$ such that

$$d(j)(v_{j,0} + (1 - \sum_{i=2}^{d(j)} \phi(e_{i,i})) = u_{j,0}, \text{ in } U_0(A)/CU(A), j = 1, 2, ..., R(n).$$ \hspace{1cm} (e 8.290)

Put $v_j$ as in $\{8.287\}$. It follows from $\{8.290\}$ that

$$\left(v_j \oplus (1 - \phi(e_{j,0})))w_{j,0} \oplus (1 - q_{j,0}) = 1, \right.$$ \hspace{1cm} (e 8.291)

$j = 1, 2, ..., R(n)$. Since $v_{j,0} \in U_0(\phi(e_{1,1}) A \phi(e_{1,1}))$, one has a continuous path of unitaries $\{v_{j,0}(t) : t \in [0, 1]\}$ such that $v_{j,0}(0) = \phi(e_{1,1})$ and $v_{j,0}(1) = v_{j,0}, j = 1, 2, ..., R(n)$. Put

$$v_j(t) = \text{diag}(v_{j,0}(t), v_{j,0}(t), ..., v_{j,0}(t)), j = 1, 2, ..., R(n).$$
It follows that
\[ \phi(x)v_j(t) = v_j(t)\phi(x) \text{ for all } x \in B_j \]
and \( t \in [0,1] \), \( j = 1,2,\ldots,R(n) \). Put
\[ u(t) = (\sum_{j=1}^{R(n)} v_j(t) + (1 - \sum_{j=1}^{R(n)} \phi(e_j))u \text{ for } t \in [0,1]. \]

Note that, \( u(0) = u \) and, if \( \eta \) is sufficiently small,
\[ \| \phi(c), u(t) \| < 2(\delta + \eta) < 3\delta \text{ for all } c \in \mathcal{G}. \]

Put
\[ w = u(1), \ w_{j,0} = (v_j \oplus (1 - \phi(e_j)))(u_{j,0} \oplus (1 - q_{j,0})) \text{ and } \]
\[ w_{j,1} = (v_j \oplus (1 - \phi(e_j)))(u_{j,1} \oplus (1 - q_{j,1}), w_{j,i} = v_ju_{j,i}, \]

\( i = 0,1 \) and \( j = 1,2,\ldots,R(n) \). Define
\[ \bar{w}_j = (v_j \oplus (1 - \phi(e_j))(u_{j,0} \oplus u_{j,1} \oplus (1 - q_{j,0} - q_{j,1})). \]

We have that
\[ \bar{w}_jq_{j,i} = w_{j,i}q_{j,i} = v_ju_{j,i} = w_{j,i}' = q_{j,i}w_{j,i}, \]

\( i = 0,1 \) and \( j = 1,2,\ldots,R(n) \). Note that, by \( (e \, 8.292), (e \, 8.283) \) and \( (e \, 8.285) \),
\[ \| w_{j,i}q_{j,i} - q_{j,i}w_{j,i} \| < \sigma \]

\( i = 0,1, j = 1,2,\ldots,R(n) \). By \( (e \, 8.291) \),
\[ w_{j,0} \in CU(A), \ j = 1,2,\ldots,R(n). \]

It follows from Lemma 6.9 of \([28]\) that
\[ cel(w_{j,0}) \leq 8\pi + 1/4. \]

Put
\[ E_j = 1 - \phi(Y_j^*)\text{diag}(q_{j,1}',q_{j,1}',\ldots,q_{j,1}',q_{j,0}',q_{j,0}',q_{j,0}')Y_j). \]

It follows from \( (e \, 8.286) \) that in \( U_0(A)/CU(A), \)
\[ \sum_{j=1}^{s(j)} w_{j,1}^{s(j)}w_{j,0}^3 = \text{diag}(\bar{w}_jq_{j,1},\bar{w}_jq_{j,1},\ldots,\bar{w}_jq_{j,1},w_{j,0},w_{j,0},w_{j,0}) \oplus E_j \]
\[ = \phi(Y_j^*)\text{diag}(\bar{w}_jq_{j,1},\bar{w}_jq_{j,1},\ldots,\bar{w}_jq_{j,1},w_{j,0},w_{j,0},w_{j,0}) \phi(Y_j) \oplus E_j \]
\[ = \text{diag}(w_{j,0},w_{j,0},\ldots,w_{j,0}) \oplus E_j = \mathbf{1}, \]

\( j = 1,2,\ldots,R(n) \). By \( (e \, 8.298) \), the above implies that
\[ w_{j,1}^{s(j)} = \mathbf{1}, \ j = 1,2,\ldots,R(n). \]

40
It follows from Theorem 6.11 that
\[ w_{j,1} \in CU(A), \ j = 1, 2, ..., R(n). \]  

(e 8.304)

It follows from Lemma 6.9 of \([28]\) that
\[ \text{cel}(w_{j,1}) \leq 8\pi + 1/4, \ j = 1, 2, ..., R(n). \]

Lemma 8.3. Let \(C\) be a unital separable simple amenable C*-algebra with \(\text{TR}(C) \leq 1\) satisfying the UCT. Let \(\Delta : (0,1) \rightarrow (0,1)\) be a non-decreasing map. Then, for any \(\epsilon > 0\) and any finite subset \(\mathcal{F} \subset C\), there exists \(\delta > 0, \eta > 0\), a finite subset \(\mathcal{G} \subset C\) and a finite subset \(\mathcal{P} \subset K(C)\) satisfying the following:

For any unital simple C*-algebra \(A\) with \(\text{TR}(A) \leq 1\), any unital homomorphism \(\phi : C \rightarrow A\) and any unitary \(u \in U(A)\) with
\[
\| [\phi(f), u] \| < \delta, \quad \text{Bott}(\phi, u)_p = \{0\} \text{ and } \mu_{\tau_0}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta, \tag{e 8.305}
\]

where \(\iota : C(\mathbb{T}) \rightarrow A\) is defined by \(\iota(f) = f(u)\) for all \(f \in C(\mathbb{T})\), there exists a continuous path of unitaries \(\{u(t) : t \in [0,1]\} \subset A\) such that
\[
u(0) = u, \ \nu(1) = 1 \text{ and } \| [\phi(f), \nu(t)] \| < \epsilon \tag{e 8.307}
\]
for all \(f \in \mathcal{F}\) and \(t \in [0,1]\).

Proof. Let \(\Delta_1 : (0,1) \rightarrow (0,1)\) be defined by \(\Delta_1(a) = \Delta(a)/2\) for all \(a \in (0,1)\). Put \(D = C \otimes C(\mathbb{T})\). Let \(T = N \times K : D_+ \setminus \{0\} \rightarrow N \times \mathbb{R}^+ \setminus \{0\}\) be associated with \(\Delta\) as in (8.1) and \(T' = N' \times K' : D_+ \setminus \{0\} \rightarrow N' \times \mathbb{R}^+ \setminus \{0\}\) be associated with \(\Delta_1\) as in (8.1). Let \(N_1 = \max\{N, N'\}\) and \(K_1 = \max\{K, K'\}\). Define \(T_0(h) = N_1(h) \times K_1(h)\) for \(h \in D_+ \setminus \{0\}\).

Let \(\epsilon > 0\) and \(\mathcal{F} \subset C\) be a finite subset. Let \(\mathcal{F}_1 = \{f \otimes g : f \in \mathcal{F} \cup \{1_C\}, \ g \in \{z, 1_{C(\mathbb{T})}\}\}\). Let \(\delta_1 > 0\) (in place of \(\delta\)), \(\mathcal{G}_1 \subset D\) (in place of \(\mathcal{G}\)), \(\mathcal{H}_0 \subset D_+ \setminus \{0\}\), \(\mathcal{P}_1 \subset K(D)\) (in place of \(\mathcal{P}\)) and \(\mathcal{U} \subset U(D)\) be as required by (7.3) for \(\epsilon/256\) (in place of \(\epsilon\)), \(\mathcal{F}_1\) and \(T_0\) (in place of \(T\)). We may assume that \(\delta_1 < \epsilon/256\).

To simplify notation, without loss of generality, we may assume that \(\mathcal{H}_0\) is in the unit ball of \(D\) and \(\mathcal{G}_1 = \{c \otimes g : c \in \mathcal{G}_1', \ g = 1_{C(\mathbb{T})}\}, \ g = z\), where \(1_{C(\mathbb{T})} \in \mathcal{G}_1'\) is a finite subset of \(C\). Without loss of generality, we may assume that \(\mathcal{U} = \mathcal{U}_1 \cup \{z_1, z_2, ..., z_n\}\), where \(\mathcal{U}_1 \subset \{w \otimes 1_{C(\mathbb{T})} : w \in U(C)\}\) is a finite subset and \(z_i = q_i \otimes z \oplus (1 - q_i) \otimes 1_{C(\mathbb{T})}, \ i = 1, 2, ..., n\) and \(\{q_1, q_2, ..., q_n\}\) is a set of projections. We write \(K(D) = K(C) \oplus \beta(K(C))\) (see (2.3)). Without loss of generality, we may also assume that \(\mathcal{P}_1 = \mathcal{P}_0 \cup \beta(\mathcal{P}_1)\), where \(\mathcal{P}_0, \mathcal{P}_2 \subset K(C)\) are finite subsets. Furthermore, we assume that \(q_j \in \mathcal{G}_1'\) and \([q_j] \in \mathcal{P}_2, \ j = 1, 2, ..., n\). Let \(\delta_0 > 0\) and let \(\mathcal{G}_0 \subset C\) be a finite subset such that there is a unital completely positive linear map \(L' : D \rightarrow A\) such that
\[
\|L'(c \otimes g) - \phi(c)g(u)\| < \delta_1/2 \text{ for all } c \in \mathcal{G}_1' \text{ and } g = 1 \text{ or } g = z, \tag{e 8.308}
\]
whenever there is a unitary \(u \in A\) such that \(\|\phi(c), u\| < \delta_0\) for all \(c \in \mathcal{G}_0\). By applying (8.1) we may assume that, \(L'\) is \(T'\)-\(\mathcal{H}_0\)-full if, in addition,
\[
\mu_{\tau_0}(O_a) \geq \Delta_1(a)
\]
for all open balls \(O_a\) of \(\mathbb{T}\) with radius \(a \geq \eta_0\) for some \(\eta_0 > 0\) and for all \(\tau \in T(A)\), where \(\iota : C(\mathbb{T}) \rightarrow A\) is defined by \(\iota(g) = g(u)\) for all \(g \in C(\mathbb{T})\).
We may assume that
\[ [L']|_{\mathcal{P}_0} = [L'']|_{\mathcal{P}_0} \]  
for any pair of unital completely positive linear maps \( L', L'': C \otimes C(\mathbb{T}) \to A \) for which \( \delta_1 \) holds for both \( L' \) and \( L'' \) and
\[ L' \cong_\delta_1 L'' \text{ on } \mathcal{G}_1'. \]  
(8.310)

Choose an integer \( K_0 \geq 1 \) such that \( \frac{K_0 - 1}{\delta_1} \geq 128/\delta_1 \). In particular, \( (8\pi + 1)/[K_0 - 1] < \delta_1 \).

Since \( TR(C) \leq 1 \), there is a projection \( p \in C \) and a \( C^* \)-subalgebra \( B = \bigoplus_{j=1}^k C(X_j, M_{r(j)}) \), where \( X_j = [0,1] \), or \( X_j \) is a point, with \( 1_B = p \) such that
\[ \|x, p\| < \min\{\epsilon/256, \delta_0/4, \delta_1/16\} \text{ for all } x \in \mathcal{G}_1' \cup \mathcal{G}_0 \]  
(8.311)
\[ \text{dist}(pxp, B) < \min\{\epsilon/256, \delta_0/4, \delta_1/16\} \text{ for all } x \in \mathcal{G}_1' \cup \mathcal{G}_0 \text{ and} \]  
(8.312)
\[ \tau(1-p) < \min\{\delta_1/K_0, \Delta(\delta_0)/4, \delta_0/4\} \text{ for all } \tau \in T(C). \]  
(8.313)

We may also assume that there are projections \( q_1, q_2, \ldots, q_n \in (1-p)C(1-p) \) such that
\[ \|q_i - (1-p)q_i(1-p)\| < \min\{\epsilon/16, \delta_0/4, \delta_1/16\}, \quad i = 1, 2, \ldots, n. \]  
(8.314)

To simplify notation, without loss of generality, we may assume that \( p \) commutes with \( \mathcal{G}'_1 \cup \mathcal{G}_0 \).

Moreover, we may assume that there is a unital completely positive linear map \( L_{00} : C \to pCp \to B \) (first sending \( c \) to \( pcp \) then to \( B \)) such that
\[ \|x - ((1-p)x(1-p) + L_{00}(x))\| < \min\{\epsilon/16, \delta_0/2, \delta_1/4\} \text{ for all } x \in \mathcal{G}_1. \]  
(8.315)

Put \( L'_0(c) = (1-p)c(1-p) \) and \( L_0(c) = L'_0(c) + L_{00}(pcp) \) for all \( c \in C \). We may further assume that \([L_{00}]|_{\mathcal{P}_2} \) and \([L'_0]|_{\mathcal{P}_2} \) are well-defined and
\[ [L_0]|_{\mathcal{P}_0 \cup \mathcal{P}_2} = [id_C]|_{\mathcal{P}_0 \cup \mathcal{P}_2}. \]  
(8.316)

Put \( \mathcal{P}_3 = [L_0]|_{\mathcal{P}_2} \cup \{[q_i] : 1 \leq i \leq n\} \cup \mathcal{P}_0 \) and \( \mathcal{P}_4 = [L_{00}]|_{\mathcal{P}_2} \). From the above, \( x = [L'_0]|_{\mathcal{P}_2} \cup \mathcal{P}_4 \).

We also assume that
\[ [L']|_{\mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4} = [L'']|_{\mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4} \]  
(8.317)
for any pair of unital completely positive linear maps from \( C \otimes C(\mathbb{T}) \to A \) such that
\[ L_1 \cong_\delta_2 L_2 \text{ on } \mathcal{G}_2'. \]  
(8.318)

and items in \( \langle 8.317 \rangle \) are well-defined for some \( \delta_2 > 0 \) and a finite subset \( \mathcal{G}_2' \).

Let \( \delta_2 > 0 \) (in place of \( \delta_0 \)), \( \mathcal{G}_2 \subset (1-p)C(1-p) \) and \( P_0 \subset (1-p)C(1-p) \) be as required by \( 8.2 \) for \( C = (1-p)C(1-p), \sigma = 1/16, \mathcal{G}_1' \cup \mathcal{G}_0 \) (in place of \( \mathcal{G}_0 \)) and \( q_1, q_2, \ldots, q_n \) (in place of \( p_1, p_2, \ldots, p_m \)). Note that we may assume that \( P_0 \subset \mathcal{G}_2 \).

Put \( \mathcal{P}_3' = [L_0]|_{\mathcal{P}_2} \cup \{[q] : q \in P_0\} \). Note again that elements in \( \mathcal{P}_3' \) are represented by elements in \( (1-p)C(1-p) \). We may assume that
\[ \text{Bott}(\phi, u)|_{\mathcal{P}_3'} = \text{Bott}(\phi, u')|_{\mathcal{P}_3'} \]  
(8.319)
for any pair of unitaries \( u \) and \( u' \) in \( A \) for which
\[ \|\phi(c), u\| < \min\{\delta_1, \delta_0\}, \quad \|\phi(c), u'\| < 2\min\{\delta_1, \delta_0\} \]  
(8.318)
and for which there exists a continuous path of unitaries \( \{W(t) : t \in [0, 1]\} \subset (1 - \phi(p))A(1 - \phi(p)) \) with

\[
\|W(0) - (1 - \phi(p))u(1 - \phi(p))\| < \min\{\delta_1, \delta_0\} \quad \text{and} \quad (e\, 8.320)
\]

\[
\|W(1) - (1 - \phi(p))u'(1 - \phi(p))\| < \min\{\delta_1, \delta_0\}, \quad (e\, 8.321)
\]

and

\[
\|\phi(c), W(t)\| < \min\{\delta_1, \delta_0\}
\]

for all \( c \in \mathcal{G}_2 \) and \( t \in [0, 1] \).

Write \( p_i = 1_{C(X_i, M_{r(i)})} \in B_i \), \( i = 1, 2, \ldots, k \). Let \( \mathcal{F}_{0,i} = \{p_i x p_i : x \in \mathcal{F}\}, i = 1, 2, \ldots, k \). We may assume that \( X_j = [0, 1], j = 1, 2, \ldots, k, \) and \( X_j \) is a point for \( i = k_0 + 1, k_0 + 2, \ldots, k \).

Put \( D_j = C(X_j, M_{r(j)}) \otimes C(\mathbb{T}) \). Define \( \mathcal{T}_i = N_{\{(D_j)_+ \setminus \{0\}} \times 2R \left|_{(D_j)_+ \setminus \{0\} \times \mathbb{T}} \right. \), \( j = 1, 2, \ldots, k_0 \). Let \( \delta_{0,i} > 0 \) (in place of \( \delta \)), \( \mathcal{H}_i \subset (D_i)_+ \setminus \{0\} \) and \( \mathcal{G}_{0,i} \subset C(X_i, M_{r(i)}) \) be required by \( 4.6 \) for \( \epsilon/256k \) and \( \mathcal{F}_{0,i} \) and \( \mathcal{T}_i \), \( i = 1, 2, \ldots, k_0 \). Let \( \delta_{0,i} > 0 \) (in place of \( \delta \)), \( \mathcal{G}_{0,i} \subset M_{r(i)} \) be required by \( 4.7 \) for \( \epsilon/256k \) and \( \mathcal{F}_{0,i}, i = k_0 + 1, k_0 + 2, \ldots, k \).

Denote by \( \{e_{2,i}^{(j)}\} \) a matrix unit for \( M_{r(i)} \), \( i = 1, 2, \ldots, k \). Put

\[
\bar{R} = \max\{N(h)R(h) : h \in \mathcal{H}_i, \ i = 1, 2, \ldots, k_0\}.
\]

Let \( \delta_3 = \min\{\epsilon/512, \delta_2/2, \delta_1/16, \delta_{0,1}/2, \ldots, \delta_{0,k}/2\} \). Let \( \mathcal{G}_3 = \mathcal{G}_3' \cup \mathcal{G}_3'' \cup \mathcal{G}_2 \cup \{1 - p, p\} \cup \bigcup_{i=1}^{k_0} \mathcal{G}_{0,i} \). Let \( \mathcal{H} = \mathcal{H}_0 \cup \{p : h \in \mathcal{H}_0 \} \cup \bigcup_{i=1}^{k_0} \mathcal{H}_i \) and let \( \mathcal{P} = \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \{[1 - p], [p], [e_{2,i}^{(j)}], [p_i], i = 1, 2, \ldots, k\} \).

It follows from \( 8.1 \) that there exists \( \delta_4 > 0, \eta > 0 \) and a finite subset \( \mathcal{G}' \subset C \) satisfying the following: there exists a contractive completely positive linear map \( L : D \to A \) which is \( T \)-\( \mathcal{H} \)-full such that

\[
\|L(c \otimes 1) - \phi(c)\| < \delta_3/16k\bar{R} \quad \text{and} \quad \|L(c \otimes z) - \phi(c)w\| < \delta_3/16k\bar{R} \quad \text{for all} \ c \in \mathcal{G}_3 \quad (e\, 8.322)
\]

and \( [L]_{\mathcal{P}_1 \cup \mathcal{G}(\mathcal{P}_2')} \) is well-defined, provided that \( w \in A \) is a unitary with

\[
\|[\phi(b), w]\| < 3\delta_4 \quad \text{for all} \ b \in \mathcal{G}' \quad \text{and} \quad (e\, 8.323)
\]

\[
\mu_{\tau\mathcal{O}}(O_a) \geq \Delta(a) \quad \text{(e\, 8.324)}
\]

for all open balls \( O_a \) of \( \mathbb{T} \) with radius \( a \geq \eta \) for all \( \tau \in T(A) \), where \( \iota : C(\mathbb{T}) \to A \) is defined by \( \iota(f) = f(w) \) for \( f \in C(\mathbb{T}) \). We may assume that \( \eta < \eta_0 \) and \( \delta_4 < \epsilon/256 \).

Note that, for \( h \in \mathcal{H}_i \),

\[
L(h) \leq L(\|h\|p_i) \leq \|h\|L(p_i), \ i = 1, 2, \ldots, k. \quad (e\, 8.325)
\]

Therefore, we may assume that (with a smaller \( \delta_4 \)),

\[
\|L(h) - \phi(p_i)L(\phi(p_i))\| < \delta_3/2k\bar{R} \quad (e\, 8.326)
\]

for any \( h \in \mathcal{H}_i, i = 1, 2, \ldots, k_0 \). We may also assume that

\[
\|\phi(p_i)L(c \otimes z)\phi(p_i) - \phi(c)w_i'\| < \delta_3/16k\bar{R} \quad \text{for all} \ c \in p_i\mathcal{G}_3p_i, \quad (e\, 8.327)
\]

provided that \( w_i' \in U(\phi(p_i)A\phi(p_i)) \) such that

\[
\|w_i' - \phi(p_i)u\phi(p_i)\| < 3\delta_4, \ i = 1, 2, \ldots, k. \quad (e\, 8.328)
\]
For any function \( g \in C(\mathbb{T}) \) with \( 0 \leq g \leq 1 \) and for any unitary \( u \in U(A) \),

\[
\tau(g(u)) = \tau(\phi(p)g(u)\phi(p)) + \tau((1 - \phi(p))g(u)(1 - \phi(p))) \quad \text{and} \quad (e \ 8.329)
\]

\[
\tau(\phi(p)g(u)\phi(p)) \geq \tau(g(u)) - \tau(1 - \phi(p)) \quad \text{for all} \quad \tau \in T(A). \quad (e \ 8.330)
\]

Thus, we may assume (by choosing smaller \( \delta_4 \)) that

\[
\mu_{\tau_{O_a}}(O_a) \geq \Delta(a)/2 \quad (e \ 8.331)
\]

for all \( a \geq \eta \) and \( \tau \in T(A) \), where \( \iota' : C(\mathbb{T}) \to A \) is defined by \( \iota'(f) = f(w) \) (for \( f \in C(\mathbb{T}) \)) for any \( w \in U(A) \) for which \( w = w_0 \oplus w_1 \), where \( w_0 \in U((1 - \phi(p))A(1 - \phi(p))) \) and \( w_1 \in U(\phi(p)A\phi(p)) \), such that

\[
\|w_1 - \phi(p)u\phi(p)\| < 2\delta_4, \quad (e \ 8.332)
\]

where \( u \) and \( \phi \) satisfy (e 8.322) and (e 8.323). Put \( \delta = \min\{\delta_4/12, \delta_3/12\} \) and \( G_4 = G' \cup G_3 \). Put \( G = G_4 \cup \{(1 - p)g(1 - p) : g \in G_3\} \cup \{e_{i,0}, [g_{i,i}]\} \).

Now suppose that \( \phi \) and \( u \in A \) satisfy the assumptions of the lemma for the above \( \delta, \eta, G \) and \( P \). In particular, \( u \in U_0(A) \). To simplify notation, without loss of generality, we may assume that all elements in \( G \) and in \( H \) have norm no more than 1.

By applying (8.2) one obtains a continuous path of unitaries \( \{w_0(t) : t \in [0, 1]\} \subset (1 - \phi(p))A(1 - \phi(p)) \) and unitaries \( w'_j \in U_0(\phi(q'_j)A\phi(q'_j)) \) such that

\[
\|[(\phi(c), w_0(t))]\| < 3\delta \quad \text{for all} \quad c \in pGp \quad \text{and} \quad (e \ 8.333)
\]

for all \( t \in [0, 1] \),

\[
\|w_0(0) - (1 - \phi(p))u(1 - \phi(p))\| < \delta_1/16, \quad (e \ 8.334)
\]

\[
\|w'_j - \phi(q'_j)w_0(1)\phi(q'_j)\| < \delta_1/16 \quad \text{and} \quad (e \ 8.335)
\]

\[
w'_j \oplus (1 - \phi(p) - \phi(q'_j)) \in CU((1 - \phi(p))A(1 - \phi(p))), \quad (e \ 8.336)
\]

\( j = 1, 2, \ldots, n \). Define \( w = w_0(1) \oplus w_1 \) for some unitary \( w_1 \) for which (e 8.332) holds.

We compute (by (e 8.305), (e 8.332) and (e 8.319)) that

\[
\text{Bott}(\phi, w)|_P = \{0\}. \quad (e \ 8.337)
\]

By (e 8.332), one also has that

\[
\mu_{\tau_{O_a}}(O_a) \geq \Delta_1(a) \quad \text{for all} \quad \tau \in T(A) \quad (e \ 8.338)
\]

and for any open balls \( O_a \) of \( \mathbb{T} \) with radius \( a \geq \eta \), where \( \iota' : C(\mathbb{T}) \to A \) is defined by \( \iota'(g) = g(w) \) for all \( g \in C(\mathbb{T}) \).

Let \( L : D \to A \) be a unital contractive completely positive linear map which satisfies (e 8.322). We may also assume that \( [L]|_P \) is well-defined

\[
[L]|_{\mathcal{P}_0} = [\phi]|_{\mathcal{P}_0} \quad \text{and} \quad [L]|_{\mathcal{G}(\mathcal{P})} = \{0\} \quad \text{by} \quad (e \ 8.339).
\]

There is a unital completely positive linear map \( \Phi : (1 - p)\mathcal{C}(1 - p) \otimes C(\mathbb{T}) \to (1 - \phi(p))A(1 - \phi(p)) \) such that

\[
\|\Phi(c \otimes g(z)) - \phi(c)g(w_0(1))\| < \delta_1/2 \quad (e \ 8.340)
\]

for all \( c \in G'_1 \cup G_0 \) and \( g = 1_{C(\mathbb{T})} \) and \( g = z \).
Define $L_1, L_2 : C \otimes C(T) \to A$ as follows:

\[
L_1(c \otimes g(z)) = \Phi((1 - p)c(1 - p) \otimes g) \oplus \phi(p)L(c \otimes g)\phi(p) \quad \text{and} \quad (e \text{8.341})
\]
\[
L_2(c \otimes g) = \phi((1 - p)c(1 - p))g(1) \oplus \phi(p)L(c \otimes g)\phi(p) \quad \text{(e \text{8.342})}
\]

for all $c \in C$ and $g \in C(T)$. By (e 8.349), we compute that,

\[
[L]|_{\mathcal{P}_0} = [L_1]|_{\mathcal{P}_0} = [L_2]|_{\mathcal{P}_0}.
\]

(e 8.343)

It is easy to see that

\[
[\phi(1 - p)L_2\phi(1 - p)]|_{\mathcal{P}_2} = \{0\}.
\]

(e 8.344)

One also has, by (e 8.310),

\[
[L_2]|_{\mathcal{P}_2} = [L_1]|_{\mathcal{P}_2} = \text{Bott}(\phi, u)|_{\mathcal{P}_2} = \{0\}.
\]

(e 8.345)

Combining (e 8.344) and (e 8.346), one obtains that

\[
[L_2]|_{\mathcal{P}_2} = \{0\}.
\]

(e 8.347)

From (e 8.319), one computes that

\[
[L_1]|_{\mathcal{P}_2} = [L]|_{\mathcal{P}_2} = \text{Bott}(\phi, u)|_{\mathcal{P}_2} = \{0\}.
\]

(e 8.348)

It follows that

\[
[L_1]|_{\mathcal{P}_1} = [L]|_{\mathcal{P}_1}.
\]

(e 8.349)

It is routine to check that

\[
|\tau \circ L(g) - \tau \circ L_1(g)| < \delta_1 \quad \text{for all} \quad g \in G_1 \quad \text{for all} \quad \tau \in T(A).
\]

(e 8.350)

If $v \in U_1$, since $\|\phi(v) - L_1(v \otimes 1)\| < \delta_1/2$ and $\|\phi(v) - L_2(v \otimes 1)\| < \delta_1/2$, it follows that

\[
\text{dist}(L_1^\dagger(\bar{v}), L_2^\dagger(\bar{v})) < \delta_1.
\]

(e 8.351)

If $\zeta_j = q_j \otimes z, \ j = 1, 2, ..., n$, by (e 8.334), (e 8.335) and (e 8.336), by the choice of $K_0$ and by applying 3.1 one has that

\[
\text{dist}((L_1^\dagger(\bar{\zeta}_j)), L_2^\dagger(\bar{\zeta}_j)) < \delta_1.
\]

(e 8.352)

Note also that, by (e 8.338) and by 8.1 both $L_1$ and $L_2$ are $T_0 \cdot H_0$-full. It then follows from (e 8.346), (e 8.347), (e 8.351), (e 8.352) and 7.3 that there exists a unitary $W \in U(A)$ such that

\[
\text{ad} W \circ L_2 \approx_{\epsilon/256} L_1 \quad \text{on} \quad \mathcal{F}_1.
\]

(e 8.353)

We may assume that

\[
\|u_i - \phi(p_i)u\phi(p_i)\| < 2\delta \quad \text{and} \quad w_1 = \sum_{i=1}^{k} u_i
\]

(e 8.354)

for some $u_i \in U(\phi(p_i)A\phi(p_i)), \ i = 1, 2, ..., R(n)$ and

\[
u_i \in U_0(\phi(p_i)A\phi(p_i)), \ i = 1, 2, ..., k
\]

(e 8.355)
(since $\text{Bott}(\phi, u)|_p = \{0\}$). There is a positive element $a_i \in \phi(p_i)A\phi(p_i)$ such that

$$a_iL(p_i)a_i = \phi(p_i) \quad \text{and} \quad \|a_i - \phi(p_i)\| < \delta_3/8k \bar{R}, \quad i = 1, 2, \ldots, k. \quad (e.8.356)$$

Let $\Psi_i : D_i \to \phi(p_i)A\phi(p_i)$ be defined by $\Psi_i(a) = a_i\phi(p_i)L(a)\phi(p_i)a_i$ for all $a \in D_i$, $i = 1, 2, \ldots, k$. Thus

$$\|\Psi_i(h) - \phi(p_i)L(h)\phi(p_i)\| < \delta_3/4k \bar{R} \quad (e.8.357)$$

for all $h \in H_i$, $i = 1, 2, \ldots, k$. Note also that (by $e.8.327$)

$$\|\Psi_i(c \otimes 1) - \phi(c)\| < \delta + \delta_3/4k \bar{R} \quad \text{and} \quad \|L_i(c \otimes z) - \phi(c)u_i\| < \delta_3/4k \bar{R} \quad (e.8.358)$$

for all $c \in G_{0,i}$, $i = 1, 2, \ldots, k$. Note also that

$$\text{bott}_0(\phi|_{C(X_i, M_{\phi(i)})}) \colon u_i = \{0\}, \quad i = 1, 2, \ldots, k. \quad (e.8.359)$$

Furthermore, for each $h \in H_i$, there exist $x_1(h), x_2(h), \ldots, x_N(h)(h)$ with and $\|x_j\| \leq R(h)$, $j = 1, 2, \ldots, N(h)$ such that

$$\sum_{j=1}^{N(h)} x_j(h)^*L(h)x_j(h) = 1_A. \quad (e.8.360)$$

It follows from $(e.8.357)$ that

$$\|\sum_{j=1}^{N(h)} x_j(h)^*\Psi_i(h)x_j(h) - 1_A\| < N(h)R(h)(\frac{\delta_3}{4k \bar{R}}) < \delta_3/4k \quad (e.8.361)$$

Therefore that there exists $y(h) \in A_+$ with $\|y(h)\| \leq \sqrt{2}$ such that

$$\sum_{j=1}^{N(h)} y(h)(x_j(h))^*\Phi_i(h)(x_j(h))y(h) = \phi(p_i). \quad (e.8.362)$$

It follows that $\Phi_i$ is $T_r$-$H_i$-full, $i = 1, 2, \ldots, k$.

It follows from $[4.6]$ and $[4.7]$ that there is a continuous path of unitaries $\{u_i(t) : t \in [0, 1]\} \subset \phi(p_i)A\phi(p_i)$ such that

$$u_i(0) = u_i, \quad u_i(1) = p_i \quad \text{and} \quad ||[\Psi_i(c), u_i(t)]|| < \epsilon/k256 \quad \text{for all} \quad c \in F_{0,i} \quad (e.8.363)$$

and for all $t \in [0, 1], \quad i = 1, 2, \ldots, k$.

Define a continuous path of unitaries $\{z(t) : t \in [0, 1]\} \subset A$ by

$$z(t) = (1 - \phi(p)) \oplus \sum_{i=1}^{k} u_i(t) \quad \text{for all} \quad t \in [0, 1].$$

Then $z(0) = (1 - \phi(p)) + \sum_{i=1}^{k} u_i$ and $z(1) = 1_A$. By $(e.8.364)$, $(e.8.357)$ and $(e.8.322)$,

$$||[\phi(c), z(t)]|| < \epsilon/128 \quad \text{for all} \quad c \in F. \quad (e.8.365)$$
The lemma follows when one connects than (\(\epsilon/W\)). Moreover, by (e 8.353),
a and for all open balls of radius \(G \subset C\) satisfies the UCT. For any
Fix \(P \subset C\) such that
\(u\) unital homomorphism and \(\phi\) is a unital simple
\(A\to A\) is a unital homomorphism and \(u \in U(A)\) such that
\([\phi(c), u(t)]\| < \epsilon/2\) for all \(c \in F\) and \(t \in [0, 1]\).
Theorem 8.4. Let \(C\) be a unital separable amenable simple \(C^*\)-algebra with \(\text{TR}(C) \leq 1\) which satisfies the UCT. For any \(\epsilon > 0\) and any finite subset \(F \subset C\), there exists \(\delta > 0\), a finite subset
\(G \subset C\) and a finite subset \(P \subset K(C)\) satisfying the following:
Suppose that \(A\) is a unital simple \(C^*\)-algebra with \(\text{TR}(A) \leq 1\), suppose that \(\phi : C \to A\) is a unital homomorphism and \(u \in U(A)\) such that
\([\phi(c), u]\| < \delta\) for all \(c \in G\) and \(\text{Bott}(\phi, u)_{\mathcal{P}} = 0\).
Then there exists a continuous and piece-wise smooth path of unitaries \(\{u(t) : t \in [0, 1]\}\) such that
\(u(0) = u,\ u(1) = 1\) and \([\phi(c), u(t)]\| < \epsilon\) for all \(c \in F\)
and for all \(t \in [0, 1]\).
Proof. Fix \(\epsilon > 0\) and a finite subset \(F \subset C\). Let \(\delta_1 > 0\) (in place of \(\delta\)), \(\eta > 0\), \(G_1 \subset C\) (in place of \(G\)) be a finite subset and \(P \subset K(C)\) be finite subset as required by (8.3) for \(\epsilon\), \(F\) and \(\Delta = \Delta_{\phi}\). We may assume that \(\delta_1 < \epsilon\).
Let \(\delta = \delta_1/2\). Suppose that \(\phi\) and \(u\) satisfy the conditions in the theorem for the above \(\delta, G\) and \(P\). It follows from (5.9) that there is a continuous path of unitaries \(\{v(t) : t \in [0, 1]\} \subset U(A)\) such that
\(v(0) = u,\ v(1) = u_1\) and \([\phi(c), v(t)]\| < \delta_1\)
for all \(c \in G_1\) and for all \(t \in [0, 1]\), and
\(\mu_{\tau_{\phi}}(O_{\alpha}) \geq \Delta(a)\) for all \(\tau \in T(A)\)
and for all open balls of radius \(a \geq \eta\).
By applying (5.3) there is a continuous path of unitaries \(\{w(t) : t \in [0, 1]\} \subset A\) such that
\(w(0) = u_1,\ w(1) = 1\) and \([\phi(c), w(t)]\| < \epsilon\)
for all \(c \in F\) and \(t \in [0, 1]\). Put
\(u(t) = v(2t)\) for all \(t \in [0, 1/2]\) and \(u(t) = w(2t - 1/2)\) for all \(t \in [1/2, 1]\).
Remark 8.3 shows that we can actually require, in addition, the path is piece-wise smooth.
\(\square\)
References

[1] R. Bhatia, Perturbation Bounds for matrix Eigenvalues, Pitman Research Notes in Mathematics, 162, Longman, London, 1987.

[2] O. Bratteli, G. A. Elliott, D. E. Evans and A. Kishimoto, Homotopy of a pair of approximately commuting unitaries in a simple C*-algebra, J. Funct. Anal. 160 (1998), 466–523.

[3] M-D. Choi and E. Effros, The completely positive lifting problem for C*-algebras, Ann. of Math. 104 (1976), 585–609.

[4] J. Cuntz, K-theory for certain C*-algebras, Ann. of Math. 113 (1981), 181–197.

[5] M. Dădărlat, Approximately unitarily equivalent morphisms and inductive limit C*-algebras, K-Theory 9 (1995), 117–137.

[6] M. Dădărlat and T. A. Loring, The K-theory of abelian subalgebras of AF algebras, J. Reine Angew. Math. 432 (1992), 39–55.

[7] M. Dădărlat and T. Loring, A universal multicoefficient theorem for the Kasparov groups, Duke Math. J. 84 (1996), 355–377.

[8] G. A. Elliott and G. Gong, On the classification of C*-algebras of real rank zero. II, Ann. of Math. 144 (1996), 497–610.

[9] G. A. Elliott, George, G. Gong and L. Li, On the classification of simple inductive limit C*-algebras. II. The isomorphism theorem, Invent. Math. 168 (2007), 249–320.

[10] G. A. Elliott, George, G. Gong and L. Li, Approximate divisibility of simple inductive limit C*-algebras, Operator algebras and operator theory (Shanghai, 1997), 87–97, Contemp. Math., 228, Amer. Math. Soc., Providence, RI, 1998.

[11] R. Exel and T. Loring, Invariants of almost commuting unitaries, J. Funct. Anal. 95 (1991), 364–376.

[12] R. Exel and T. Loring, Almost commuting unitary matrices, Proc. Amer. Math. Soc. 106 (1989), 913–915.

[13] G. Gong, On the classification of simple inductive limit C*-algebras. I. The reduction theorem, Doc. Math. 7 (2002), 255–461 (electronic).

[14] G. Gong and H. Lin, Almost multiplicative morphisms and almost commuting matrices, J. Operator Theory 40 (1998), 217–275.

[15] G. Gong and H. Lin, Almost multiplicative morphisms and K-theory, Internat. J. Math. 11 (2000), 983–1000.

[16] S. Hu, H. Lin, Y. Xue, Limits of homomorphisms with finite-dimensional range, Internat. J. Math. 16 (2005), 807–821.

[17] D. Husemoller, Fibre Bundles, McGraw-Hill, New York, 1966, reprinted in Springer-Verlag Graduate Texts in Mathematics.

[18] H. Lin Embedding an AH-algebra into a simple C*-algebra with prescribed KK-data, K-Theory 24 (2001), 135–156.
[19] H. Lin, *Tracial topological ranks of $C^*$-algebras*, Proc. London Math. Soc., **83** (2001), 199-234.

[20] H. Lin, *Classification of simple $C^*$-algebras and higher dimensional noncommutative tori*, Ann. of Math. (2) 157 (2003), no. 2, 521–544.

[21] H. Lin, *An Introduction to the Classification of Amenable $C^*$-Algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xii+320 pp. ISBN: 981-02-4680-3.

[22] H. Lin, *Simple AH-algebras of real rank zero*, Proc. Amer. Math. Soc. **131** (2003), 3813–3819

[23] H. Lin, *Classification of simple $C^*$-algebras with tracial topological rank zero*, Duke Math. J., **125** (2004), 91-119.

[24] H. Lin, *Traces and simple $C^*$-algebras with tracial topological rank zero*, J. Reine Angew. Math. 568 (2004), 99–137.

[25] H. Lin, *An approximate universal coefficient theorem*, Trans. Amer. Math. Soc. **357** (2005), 3375–3405.

[26] H. Lin, *A separable Brown-Douglas-Fillmore theorem and weak stability*, Trans. Amer. Math. Soc. **356** (2004), 2889–2925.

[27] H. Lin, *Classification of homomorphisms and dynamical systems*, Trans. Amer. Math. Soc. **359** (2007), 859-895.

[28] H. Lin, *Simple nuclear $C^*$-algebras of tracial topological rank one*, J. Funct. Anal. **251** (2007), 601–679.

[29] H. Lin, *Approximate homotopy of homomorphisms from $C(X)$ into a simple $C^*$-algebra*, preprint, arxiv.org/ OA/0612125.

[30] H. Lin, *AF-embeddings of crossed products of AH-algebras by $\mathbb{Z}$ and asymptotic AF-embeddings*, Indiana Univ. Math. J. **57** (2008), 891–944.

[31] H. Lin, *Asymptotically unitarily equivalence and asymptotically inner automorphisms*, preprint, arXiv: math.OA/0703610v2, 2007.

[32] H. Lin *AF-embedding of the crossed products of AH-algebras by finitely generated abelian groups*, Inter Math. Res. Papers (2008) Vol. 2008 : article ID rpn007, 67 pages, doi:10.1093/imrp/rpn007.

[33] H. Lin, *Localizing the Elliott Conjecture at Strongly Self-absorbing $C^*$-algebras*, – an appendix, preprint, arXiv:math.OA/0709.1654.

[34] H. Lin *Approximate Unitary Equivalence in Simple $C^*$-algebras of Tracial Rank One*, preprint, arXiv:0801.2929.

[35] H. Lin, *Asymptotically unitary equivalence and classification of simple amenable $C^*$-algebras*, preprint, arXiv:0806.0636.

[36] H. Lin and N. C. Phillips, *Almost multiplicative morphisms and the Cuntz algebra $O_2$*, Internat. J. Math. **6** (1995), 625–643.
[37] T. Loring, *K*-theory and asymptotically commuting matrices, Canad. J. Math. 40 (1988), 197-216.

[38] H. Matui, AF embeddability of crossed products of AT algebras by the integers and its application, J. Funct. Anal. 192 (2002), 562–580.

[39] N. C. Phillips, A survey of exponential rank, C*-algebras: 1943–1993 (San Antonio, TX, 1993), 352–399, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994.

[40] N. C. Phillips, Reduction of exponential rank in direct limits of C*-algebras, Canad. J. Math. 46 (1994), 818–853.

[41] M. Rieffel, The homotopy groups of the unitary groups of noncommutative tori, J. Operator Theory 17 (1987), 237–254.

[42] H. Su, On the classification of C*-algebras of real rank zero: inductive limits of matrix algebras over non-Hausdorff graphs, Mem. Amer. Math. Soc. 114 (1995), no. 547.

[43] D. Voiculescu, Asymptotically commuting finite rank unitary operators without commuting approximants. Acta Sci. Math. (Szeged) 45 (1983), 429–431.

[44] W. Winter, Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, preprint. [arXiv:0708.0283](https://arxiv.org/abs/0708.0283)