REFINEMENT OF INEQUALITIES AMONG MEANS

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Abstract. In this paper we shall consider some famous means such as arithmetic, harmonic, geometric, root-square means, etc. Some new means recently studied are also presented. Different kinds of refinement of inequalities among these means are given.

1. Mean of Order $t$

Let us consider the following well known mean of order $t$:

$$B_t(a, b) = \begin{cases} \left(\frac{a^t + b^t}{2}\right)^{1/t}, & t \neq 0 \\ \sqrt{ab}, & t = 0 \\ \max\{a, b\}, & t = \infty \\ \min\{a, b\}, & t = -\infty \end{cases}$$

for all $a, b, t \in \mathbb{R}$, $a, b > 0$.

In particular, we have

$$B_{-1}(a, b) = H(a, b) = \frac{2ab}{a + b},$$
$$B_0(a, b) = G(a, b) = \sqrt{ab},$$
$$B_{1/2}(a, b) = N_1(a, b) = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2,$$
$$B_1(a, b) = A(a, b) = \frac{a + b}{2},$$

and

$$B_2(a, b) = S(a, b) = \sqrt{\frac{a^2 + b^2}{2}}.$$

The means, $H(a, b)$, $G(a, b)$, $A(a, b)$ and $S(a, b)$ are known in the literature as harmonic, geometric, arithmetic and root-square means respectively. For simplicity we can call the measure, $N_1(a, b)$ as square-root mean. It is well know that the mean of order $s$ given in (1.1) is monotonically increasing in $s$, then we can write

$$H(a, b) \leq G(a, b) \leq N_1(a, b) \leq A(a, b) \leq S(a, b).$$

Dragomir and Pearce [3] (page 242) proved the following inequality:

$$\frac{a^r + b^r}{2} \leq \frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \leq \left(\frac{a + b}{2}\right)^r,$$

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for all $a, b > 0$, $a \neq b$, $r \in (0, 1)$. In particular take $r = \frac{1}{2}$ in (1.3), we get
\begin{equation}
\frac{\sqrt{a} + \sqrt{b}}{2} \leq \frac{2\left(b^{3/2} - a^{3/2}\right)}{3(b - a)} \leq \sqrt[3]{\frac{a + b}{2}}, \quad a \neq b.
\end{equation}

After necessary calculations in (1.5), we get
\begin{equation}
\left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^{2} \leq \frac{a + \sqrt{ab} + b}{3} \leq \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)\left(\sqrt[3]{\frac{a + b}{2}}\right).
\end{equation}

On the other side we can easily check that
\begin{equation}
\left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)\left(\sqrt[3]{\frac{a + b}{2}}\right) \leq \frac{a + b}{2}.
\end{equation}

Finally, the expressions (1.2), (1.5) and (1.6) lead us to the following inequality:
\begin{equation}
H(a, b) \leq G(a, b) \leq N_{1}(a, b) \leq N_{3}(a, b) \leq N_{2}(a, b) \leq A(a, b) \leq S(a, b),
\end{equation}
where
\[N_{2}(a, b) = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)\left(\sqrt[3]{\frac{a + b}{2}}\right),\]
and
\[N_{3}(a, b) = \frac{a + \sqrt{ab} + b}{3}.
\]

Moreover, we can write
\[N_{1}(a, b) = \frac{A(a, b) + G(a, b)}{2},\]
\[N_{2}(a, b) = \sqrt{N_{1}(a, b)A(a, b)},\]
and
\[N_{3}(a, b) = \frac{2A(a, b) + G(a, b)}{3}.
\]

Thus we have three new means, where $N_{1}(a, b)$ appears as a natural way. The $N_{2}(a, b)$
can be seen in Taneja [4, 5] and the mean $N_{3}(a, b)$ is known as Heron’s mean [2]. Some
studies on it can be seen in Zhang and Wu [6].

2. Difference of Means and Their Convexity

Let us consider the following difference of means:
\begin{equation}
M_{SA}(a, b) = S(a, b) - A(a, b),
\end{equation}
\begin{equation}
M_{SN_{2}}(a, b) = S(a, b) - N_{2}(a, b),
\end{equation}
\begin{equation}
M_{SN_{3}}(a, b) = S(a, b) - N_{3}(a, b),
\end{equation}
\begin{equation}
M_{SN_{1}}(a, b) = S(a, b) - N_{1}(a, b),
\end{equation}
\begin{equation}
M_{SG}(a, b) = S(a, b) - G(a, b),
\end{equation}
(2.6) \[ M_{SH}(a, b) = S(a, b) - H(a, b), \]

(2.7) \[ M_{AN_2}(a, b) = A(a, b) - N_2(a, b), \]

(2.8) \[ M_{AG}(a, b) = A(a, b) - G(a, b), \]

(2.9) \[ M_{AH}(a, b) = A(a, b) - H(a, b), \]

(2.10) \[ M_{N_2N_1}(a, b) = N_2(a, b) - N_1(a, b), \]

and

(2.11) \[ M_{N_2G}(a, b) = N_2(a, b) - G(a, b). \]

We easily check that

(2.12) \[ M_{AG}(a, b) = 2 \left[ N_1(a, b) - G(a, b) \right] := 2M_{N_1G}(a, b) \]

(2.13) \[ = 2 \left[ A(a, b) - N_1(a, b) \right] := 2M_{AN_1}(a, b) \]

(2.14) \[ = 3 \left[ A(a, b) - N_3(a, b) \right] := 3M_{AN_3}(a, b) \]

(2.15) \[ = \frac{3}{2} \left[ N_3(a, b) - G(a, b) \right] := \frac{3}{2}M_{N_3G}(a, b) \]

(2.16) \[ = 6 \left[ N_3(a, b) - N_1(a, b) \right] := 6M_{N_3N_1}(a, b). \]

Now, we shall prove the convexity of the means (2.1)-(2.11). It is based on the following lemma.

**Lemma 2.1.** Let \( f : I \subset \mathbb{R}_+ \to \mathbb{R} \) be a convex and differentiable function satisfying

\[ f(1) = f'(1) = 0. \]

Consider a function

(2.13) \[ \phi_f(a, b) = af \left( \frac{b}{a} \right), \quad a, b > 0, \]

then the function \( \phi_f(a, b) \) is convex in \( \mathbb{R}_+^2 \), and satisfies the following inequality:

(2.14) \[ 0 \leq \phi_f(a, b) \leq \left( \frac{b - a}{a} \right) \phi'_f(a, b). \]

**Proof.** It is well known that for the convex and differentiable function \( f \), we have the inequality

(2.15) \[ f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x), \]

for all \( x, y \in \mathbb{R}_+ \).

Take \( y = \frac{b}{a} \) and \( x = 1 \) in (2.15) one gets

\[ f'(1) \left( \frac{b}{a} - 1 \right) \leq f \left( \frac{b}{a} \right) - f(1) \leq f'(1) \left( \frac{b}{a} \right) \left( \frac{b}{a} - 1 \right), \]

or equivalently,

(2.16) \[ f'(1) (b - a) \leq af \left( \frac{b}{a} \right) - af(1) \leq af' \left( \frac{b}{a} \right) \left( \frac{b - a}{a} \right). \]
Since $f(1) = f'(1) = 0$, then from (2.10) we get (2.14).

Now we shall show that the function $\phi_f(a, b)$ is jointly convex in $a$ and $b$. Since the function $f$ is convex, then for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_+^2$, $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = 1$ we can write

$$f\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 y_1 + \lambda_2 y_2}\right) = f\left(\frac{\lambda_1 y_1 x_1}{y_1 (\lambda_1 y_1 + \lambda_2 y_2)} + \frac{\lambda_2 y_2 x_2}{y_2 (\lambda_1 y_1 + \lambda_2 y_2)}\right).$$

(2.17)

\begin{align*}
\leq \frac{\lambda_1 y_1}{\lambda_1 y_1 + \lambda_2 y_2} f\left(\frac{x_1}{y_1}\right) + \frac{\lambda_2 y_2}{\lambda_1 y_1 + \lambda_2 y_2} f\left(\frac{x_2}{y_2}\right).
\end{align*}

Multiply (2.17) by $\lambda_1 y_1 + \lambda_2 y_2$ one gets

$$(\lambda_1 y_1 + \lambda_2 y_2) f\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 y_1 + \lambda_2 y_2}\right) \leq \lambda_1 y_1 f\left(\frac{x_1}{y_1}\right) + \lambda_2 y_2 f\left(\frac{x_2}{y_2}\right),$$

i.e.,

(2.18) \quad $\phi_f\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 y_1 + \lambda_2 y_2}\right) \leq \lambda_1 \phi_f\left(\frac{x_1}{y_1}\right) + \lambda_2 \phi_f\left(\frac{x_2}{y_2}\right),$

for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_+^2$. The expression (2.18) completes the required proof. \hfill \Box

Now we shall show that the difference of means given by (2.1)–(2.11) are convex in $\mathbb{R}_+^2$. Later in Section 3 we shall apply the convexity of these functions to establish improvement over the inequality (1.7).

**Theorem 2.1.** The difference of means given by (2.1)–(2.11) are nonnegative and convex in $\mathbb{R}_+^2$.

**Proof.** We shall write each measure in the form of generating function according to the measure (2.13), and then give their first and second order derivatives. It is understood that $x \in (0, \infty)$.

- **For $M_{SA}(a, b)$:**

\begin{align*}
\phi_{SA}(x) &= \sqrt{x^2 + 1} - \frac{x + 1}{2}, \\
\phi'_{SA}(x) &= \frac{x}{\sqrt{2x^2 + 1}} - \frac{1}{2}, \\
\phi''_{SA}(x) &= \frac{2}{(2x^2 + 2)^{3/2}} > 0.
\end{align*}

- **For $M_{SN_3}(a, b)$:**

\begin{align*}
\phi_{SN_3}(x) &= \sqrt{x^2 + 1} - \frac{x + \sqrt{x} + 1}{3}, \\
\phi'_{SN_3}(x) &= \frac{6x^{3/2} - (2\sqrt{x} + 1)\sqrt{2x^2 + 1}}{6\sqrt{2x(x^2 + 1)}},
\end{align*}
and
\[ f''_{SN_3}(x) = \frac{24x^{3/2} + (2x^2 + 2)^{3/2}}{12x^{3/2}(2x^2 + 2)^{3/2}} > 0. \]

- For \( M_{SN_2}(a, b) \):
  \[ f_{SN_2}(x) = \frac{2\sqrt{x^2 + 1} - (\sqrt{x} + 1) \sqrt{x + 1}}{2\sqrt{2}}, \]
  \[ f'_{SN_2}(x) = \frac{4x^{3/2} \sqrt{x + 1} - (2x + \sqrt{x} + 1) \sqrt{x^2 + 1}}{4\sqrt{2x(x + 1)(x^2 + 1)}}, \]
  and
  \[ f''_{SN_2}(x) = \frac{(\frac{3}{2} + 1)(x^2 + 1)^{3/2} + 8x^{3/2}(x + 1)^{3/2}}{8\sqrt{2}[x(x + 1)(x^2 + 1)]^{3/2}} > 0. \]

- For \( M_{SN_1}(a, b) \):
  \[ f_{SN_1}(x) = \frac{2\sqrt{2(x^2 + 1)} - (\sqrt{x} + 1)^2}{4}, \]
  \[ f'_{SN_1}(x) = \frac{4x^{3/2} - (\sqrt{x} + 1) \sqrt{2(x^2 + 1)}}{4\sqrt{2x(x^2 + 1)}}, \]
  and
  \[ f''_{SN_1}(x) = \frac{16x^{5/2} + x(2x^2 + 2)^{3/2}}{8x^{5/2}(2x^2 + 2)^{3/2}} > 0. \]

- For \( M_{SG}(a, b) \):
  \[ f_{SG}(x) = \sqrt{\frac{x^2 + 1}{2}} - \sqrt{x}, \]
  \[ f'_{SG}(x) = \frac{\sqrt{2x^{3/2} - \sqrt{x^2 + 1}}}{2\sqrt{x(x^2 + 1)}}, \]
  and
  \[ f''_{SG}(x) = \frac{1}{\sqrt{2(x^2 + 1)^{3/2}}} + \frac{1}{4x^{3/2}} > 0. \]

- For \( M_{SH}(a, b) \):
  \[ f_{SH}(x) = \sqrt{\frac{x^2 + 1}{2}} - \frac{2x}{x + 1}, \]
  \[ f'_{SH}(x) = \frac{x(x + 1)^2 - 2\sqrt{2(x^2 + 1)}}{(x + 1)^2 \sqrt{2(x^2 + 1)}}, \]
  and
  \[ f''_{SH}(x) = \frac{2 \left[(x + 1)^3 + 2(2x^2 + 2)^{3/2}\right]}{(x + 1)^3(2x^2 + 2)^{3/2}} > 0. \]
For $M_{AN_2}(a, b)$:

\[
\begin{align*}
    f_{AN_2}(x) &= \frac{2(x + 1) - (\sqrt{x} + 1) \sqrt{2(x + 1)}}{4}, \\
    f'_{AN_2}(x) &= \frac{2 \sqrt{2x(x + 1)} - (2x + \sqrt{x} + 1)}{4 \sqrt{2(x + 1)}}, \\
    f''_{AN_2}(x) &= \frac{x^{3/2} + 1}{4x^{3/2}(2x + 2)^{3/2}} > 0.
\end{align*}
\]

and

\[
\begin{align*}
    f''_{AN_2}(x) &= \frac{x^{3/2} + 1}{4x^{3/2}(2x + 2)^{3/2}} > 0.
\end{align*}
\]

For $M_{AG}(a, b)$:

\[
\begin{align*}
    f_{AG}(x) &= \frac{1}{2} (\sqrt{x} - 1)^2, \\
    f'_{AG}(x) &= \frac{\sqrt{x} - 1}{2 \sqrt{x}},
\end{align*}
\]

and

\[
\begin{align*}
    f''_{AG}(x) &= \frac{1}{4x^{3/2}} > 0.
\end{align*}
\]

For $M_{AH}(a, b)$:

\[
\begin{align*}
    f_{AH}(x) &= \frac{(x - 1)^2}{2(x + 1)}, \\
    f'_{AH}(x) &= \frac{(x - 1)(x + 3)}{2(x + 1)^2},
\end{align*}
\]

and

\[
\begin{align*}
    f''_{AH}(x) &= \frac{4}{(x + 1)^3} > 0.
\end{align*}
\]

For $M_{N_2N_1}(a, b)$:

\[
\begin{align*}
    f_{N_2N_1}(x) &= \frac{(\sqrt{x} + 1) \sqrt{2(x + 1)} - (\sqrt{x} + 1)^2}{4}, \\
    f'_{N_2N_1}(x) &= \frac{2x + \sqrt{x} + 1 - (\sqrt{x} + 1) \sqrt{2(x + 1)}}{4 \sqrt{2x(x + 1)}}, \\
    f''_{N_2N_1}(x) &= \frac{(2x + 2)^{3/2} - 2(x^{3/2} + 1)}{8x^{3/2}(2x + 2)^{3/2}}.
\end{align*}
\]

Since $(x + 1)^{3/2} \geq x^{3/2} + 1, \forall x \in (0, \infty)$ and $2^{3/2} \geq 2$, then obviously, $f''_{N_2N_1}(x) \geq 0, \forall x \in (0, \infty)$. 

Since $(x + 1)^{3/2} \geq x^{3/2} + 1, \forall x \in (0, \infty)$ and $2^{3/2} \geq 2$, then obviously, $f''_{N_2N_1}(x) \geq 0, \forall x \in (0, \infty)$. 

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Since $(x + 1)^{3/2} \geq x^{3/2} + 1, \forall x \in (0, \infty)$ and $2^{3/2} \geq 2$, then obviously, $f''_{N_2N_1}(x) \geq 0, \forall x \in (0, \infty)$. 

Since $(x + 1)^{3/2} \geq x^{3/2} + 1, \forall x \in (0, \infty)$ and $2^{3/2} \geq 2$, then obviously, $f''_{N_2N_1}(x) \geq 0, \forall x \in (0, \infty)$.
Remark 2.1. The inequality (1.7) also presents more nonnegative differences but we have considered only the convex ones. 

\[ f_{N_{2}G}(x) = \frac{(\sqrt{x} + 1)^{2(x+1)} - 4x}{4}, \]

\[ f'_{N_{2}G}(x) = \frac{2x + 1 + \sqrt{x} - 2\sqrt{2(x+1)}}{4\sqrt{2x(x+1)}}, \]

and

\[ f''_{N_{2}G}(x) = \frac{(2x + 2)^{3/2} - (x^{3/2} + 1)}{4x^{3/2}(2x + 2)^{3/2}}. \]

Since \((x + 1)^{3/2} \geq x^{3/2} + 1, \forall x \in (0, \infty)\) and \(2^{3/2} \geq 1, \) then obviously, \(f''_{N_{2}G}(x) \geq 0, \forall x \in (0, \infty).\)

We see that in all the cases the generating function \(f_{(\cdot)}(1) = f'_{(\cdot)}(1) = 0\) and the second derivative is positive for all \(x \in (0, \infty).\) This proves the nonnegativity and convexity of the means (2.1)-(2.16) in \(\mathbb{R}^+_1.\) This completes the proof of the theorem. □

**Remark 2.1.** The inequality (1.7) also presents more nonnegative differences but we have considered only the convex ones.

### 3. Inequality Among Difference of Means

In view of (1.7), the following inequalities are obviously true:

(3.1) \(M_{SA}(a, b) \leq M_{SN_2}(a, b) \leq M_{SN_3}(a, b) \leq M_{SG}(a, b) \leq M_{SH}(a, b),\)

(3.2) \(M_{AN_2}(a, b) \leq M_{AN_3}(a, b) \leq M_{AN_4}(a, b) \leq M_{AG}(a, b) \leq M_{AH}(a, b),\)

(3.3) \(M_{N_{2}N_3}(a, b) \leq M_{N_{2}N_4}(a, b) \leq M_{N_{2}G}(a, b) \leq M_{N_{2}H}(a, b),\)

(3.4) \(M_{N_{1}N_1}(a, b) \leq M_{N_{1}G}(a, b) \leq M_{N_{1}H}(a, b),\)

and

(3.5) \(M_{N_{1}G}(a, b) \leq M_{N_{1}H}(a, b),\)

In view of (1.7), (2.12) and (3.5), we can easily check that

(3.6) \(A(a, b) + H(a, b) \leq N_1(a, b) + N_3(a, b) \leq N_1(a, b) + N_2(a, b).\)

In this section we shall improve the inequalities (1.7) and then compare with the inequalities (3.1)-(3.5). This refinement is based on the following lemma.

**Lemma 3.1.** Let \(f_1, f_2 : I \subset \mathbb{R}^+_1 \rightarrow \mathbb{R}\) be two convex functions satisfying the assumptions:

(i) \(f_1(1) = f'_1(1) = 0, f_2(1) = f'_2(1) = 0;\)

(ii) \(f_1\) and \(f_2\) are twice differentiable in \(\mathbb{R}^+_1;\)

(iii) there exists the real constants \(\alpha, \beta\) such that \(0 \leq \alpha < \beta\) and

(3.7) \(\alpha \leq \frac{f''_1(x)}{f''_2(x)} \leq \beta, f''_2(x) > 0,\)

for all \(x > 0\) then we have the inequalities:

(3.8) \(\alpha \phi_{f_2}(a, b) \leq \phi_{f_1}(a, b) \leq \beta \phi_{f_2}(a, b),\)

for all \(a, b \in (0, \infty).\)
Proof. Let us consider the functions

\[ k(x) = f_1(x) - \alpha f_2(x) \]

and

\[ h(x) = \beta f_2(x) - f_1(x), \]

where \( \alpha \) and \( \beta \) are as given by (3.7).

In view of item (i), we have \( k(1) = h(1) = 0 \) and \( k'(1) = h'(1) = 0 \). Since the functions \( f_1(x) \) and \( f_2(x) \) are twice differentiable, then in view of (3.7), we have

\[
(3.9) \quad k''(x) = f_1''(x) - \alpha f_2''(x) = f_2''(x) \left( \frac{f_1''(x)}{f_2''(x)} - \alpha \right) \geq 0,
\]

and

\[
(3.10) \quad h''(x) = \beta f_2''(x) - f_1''(x) = f_2''(x) \left( M - \frac{f_1''(x)}{f_2''(x)} \right) \geq 0,
\]

for all \( x \in (0, \infty) \).

In view of (3.9) and (3.10), we can say that the functions \( k(\cdot) \) and \( h(\cdot) \), are convex on \( I \subset \mathbb{R}_+ \).

According to (2.14), we have

\[
(3.11) \quad a k \left( \frac{b}{a} \right) = a \left[ f_1 \left( \frac{b}{a} \right) - \alpha f_2 \left( \frac{b}{a} \right) \right] = a f_1 \left( \frac{b}{a} \right) - \alpha a f_2 \left( \frac{b}{a} \right) \geq 0,
\]

and

\[
(3.12) \quad a h \left( \frac{b}{a} \right) = a \left[ \beta f_2 \left( \frac{b}{a} \right) - f_1 \left( \frac{b}{a} \right) \right] = \beta a f_2 \left( \frac{b}{a} \right) - f_1 \left( \frac{b}{a} \right) \geq 0,
\]

Combining (3.11) and (3.12) we have the proof of (3.8). \( \square \)

Theorem 3.1. The following inequalities among the mean differences hold:

\[
(3.13) \quad M_{SA}(a, b) \leq \frac{1}{3} M_{SH}(a, b) \leq \frac{1}{2} M_{AH}(a, b) \leq \frac{1}{2} M_{SG}(a, b) \leq M_{AG}(a, b).
\]

Proof. In order to prove the above theorem, we shall prove each part separately.

Let us consider

\[
g_{SA,SH}(x) = \frac{f_{SA}'(x)}{f_{SH}'(x)} = \frac{(x + 1)^3}{(x + 1)^3 + 4\sqrt{2}(x^2 + 1)^{3/2}}, \quad x \in (0, \infty),
\]

This gives

\[
(3.14) \quad g'_{SA,SH}(x) = -\frac{24(x - 1)(x^2 + 1)(x + 1)^2}{\sqrt{2}(x^2 + 1) \left[ (x + 1)^3 + 4\sqrt{2}(x^2 + 1)^{3/2} \right]^2} \begin{cases} \geq 0, & x \leq 1 \\ \leq 0, & x \geq 1 \end{cases}
\]

In view of (3.14) we conclude that the function \( g_{SA,SH}(x) \) increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence

\[
(3.15) \quad \beta = \sup_{x \in (0, \infty)} g_{SA,SH}(x) = g_{SA,SH}(1) = \frac{1}{3}.
\]

Applying (3.8) for the difference of means \( M_{SA}(a, b) \) and \( M_{SH}(a, b) \), and using (3.15), we get

\[
(3.16) \quad M_{SA}(a, b) \leq \frac{1}{3} M_{SH}(a, b).
\]
Let us consider
\[ g_{SH\cdot AH}(x) = \frac{f''_{SH}(x)}{f''_{AH}(x)} = \frac{(x + 1)^3 + 4\sqrt{2}(x^2 + 1)^{3/2}}{4\sqrt{2}(x^2 + 1)^{3/2}}, \quad x \in (0, \infty), \]

This gives
\[ (3.17) \quad g'_{SH\cdot AH}(x) = \frac{-3(x - 1)(x + 1)^2}{4\sqrt{2}(x^2 + 1)^{5/2}} \begin{cases} \geq 0, & x \leq 1 \\ \leq 0, & x \geq 1 \end{cases}. \]

In view of (3.17), we conclude that the function \(g_{SH\cdot AH}(x)\) is increasing in \(x \in (0, 1)\) and decreasing in \(x \in (1, \infty)\), and hence
\[ (3.18) \quad \beta = \sup_{x \in (0, \infty)} g_{SH\cdot AH}(x) = g_{SG\cdot AH}(1) = \frac{3}{2}. \]

Applying (3.8) for the difference of means \(M_{SH}(a, b)\) and \(M_{AH}(a, b)\), and using (3.18), we get
\[ (3.19) \quad M_{SH}(a, b) \leq \frac{3}{2} M_{AH}(a, b). \]

Let us consider
\[ g_{SG\cdot AH}(x) = \frac{f''_{SG}(x)}{f''_{AH}(x)} = \frac{(x + 1)^3 \left[4x^{3/2} + \sqrt{2} (x^2 + 1)^{3/2}\right]}{16\sqrt{2} (x^2 + 1)^{5/2}}, \quad x \in (0, \infty), \]

This gives
\[ (3.20) \quad g'_{SG\cdot AH}(x) = \frac{3(x + 1)^4(x - 1) \left[\sqrt{2} (x^2 + 1)^{5/2} - 8x^{5/2}\right]}{32\sqrt{2} [x(x + 1)]^{5/2}} \begin{cases} \geq 0, & x \geq 1 \\ \leq 0, & x \leq 1 \end{cases}, \]

where we have used the fact that \(x^2 + 1 \geq 2x, \forall x \in (0, \infty)\).

In view of (3.20), we conclude that the function \(g_{SG\cdot AH}(x)\) is decreasing in \(x \in (0, 1)\) and increasing in \(x \in (1, \infty)\), and hence
\[ (3.21) \quad \alpha = \inf_{x \in (0, \infty)} g_{SG\cdot AH}(x) = g_{SG\cdot AH}(1) = 1. \]

Applying (3.8) for the difference of means \(M_{AH}(a, b)\) and \(M_{SG}(a, b)\), and using (3.21), we get
\[ (3.22) \quad M_{AH}(a, b) \leq M_{SG}(a, b). \]

Let us consider
\[ g_{SG\cdot AG}(x) = \frac{f''_{SG}(x)}{f''_{AG}(x)} = \frac{4x^{3/2} + \sqrt{2} (x^2 + 1)^{3/2}}{\sqrt{2} (x^2 + 1)^{3/2}}, \quad x \in (0, \infty), \]

This gives
\[ (3.23) \quad g'_{SG\cdot AG}(x) = \frac{-6(x - 1)(x + 1)\sqrt{x}}{\sqrt{2}(x^2 + 1)^{5/2}} \begin{cases} \geq 0, & x \leq 1 \\ \leq 0, & x \geq 1 \end{cases}. \]

In view of (3.23), we conclude that the function \(g_{SG\cdot AG}(x)\) is increasing in \(x \in (0, 1)\) and decreasing in \(x \in (1, \infty)\), and hence
\[ (3.24) \quad M = \sup_{x \in (0, \infty)} g_{SG\cdot AG}(x) = g_{SG\cdot AG}(1) = 2. \]
Applying (3.2) for the difference of means $M_{SG}(a, b)$ and $M_{AG}(a, b)$, and using (3.21), we get

$$\frac{1}{2} M_{SG}(a, b) \leq M_{AG}(a, b)$$

(3.25)

Combining the results (3.16), (3.19), (3.22) and (3.25) we get the proof of the inequality (3.13). □

Corollary 3.1. The following inequalities hold:

$$H(a, b) \leq G(a, b) \leq \frac{2H(a, b) + S(a, b)}{3} \leq \frac{A(a, b) + H(a, b)}{2}$$

$$\leq \frac{S(a, b) + G(a, b)}{2} \leq \frac{H(a, b) + 2S(a, b)}{3}$$

$$\leq A(a, b) \leq S(a, b) + H(a, b) - G(a, b)$$

$$\leq S(a, b) \leq 3[A(a, b) - G(a, b)] + H(a, b).$$

Proof. Simplifying the results given in (3.16), (3.19), (3.22) and (3.25) we get the required result. □

Remark 3.1. The inequalities (3.26) are the improvement over the following well known result

$$\min \{a, b\} \leq H(a, b) \leq G(a, b) \leq A(a, b) \leq S(a, b) \leq \max \{a, b\}.$$ (3.27)

In the following corollary, we shall give a further improvement over the inequalities (3.26).

Corollary 3.2. The following inequalities hold:

$$H(a, b) \leq \frac{2A(a, b)H(a, b)}{A(a, b) + H(a, b)} \leq G(a, b) \leq \frac{2H(a, b) + S(a, b)}{3}$$

$$\leq \frac{A(a, b) + H(a, b)}{2} \leq \frac{(A(a, b))^2 + (H(a, b))^2}{2} \leq \frac{S(a, b) + G(a, b)}{2}$$

$$\leq \frac{H(a, b) + 2S(a, b)}{3} \leq A(a, b) \leq S(a, b) + H(a, b) - G(a, b)$$

$$\leq S(a, b) \leq 3[A(a, b) - G(a, b)] + H(a, b).$$

Proof. Replace $a$ by $A(a, b)$ and $b$ by $H(a, b)$ in (3.27) we get

$$\min \{A(a, b), H(a, b)\} \leq H(A(a, b), H(a, b)) \leq G(A(a, b), H(a, b))$$

$$\leq A(A(a, b), H(a, b)) \leq S(A(a, b), H(a, b)) \leq \max \{A(a, b), H(a, b)\}.$$

This gives

$$H(a, b) \leq \frac{2A(a, b)H(a, b)}{A(a, b) + H(a, b)} \leq G(a, b) \leq \frac{A(a, b) + H(a, b)}{2}$$

$$\leq \frac{(A(a, b))^2 + (H(a, b))^2}{2} \leq A(a, b) \leq S(a, b)$$

(3.29)

The inequality (3.29) gives a different kind of improvement over the inequality (3.27).
Let us consider

\[ K(a, b) = \frac{S(a, b) + G(a, b)}{2} - \sqrt{\frac{A(a, b)^2 + H(a, b)^2}{2}} \]

Now we shall show that

\[ \left( \frac{S(a, b) + G(a, b)}{2} \right)^2 - \frac{A(a, b)^2 + H(a, b)^2}{2} \geq 0. \]

For it, let us consider

\[ k(x) = \left[ \frac{\sqrt{2(x^2 + 1)}}{4} + \frac{\sqrt{x}}{2} \right]^2 - \frac{1}{2} \left[ \left( \frac{x + 1}{2} \right)^2 + \left( \frac{2x}{x + 1} \right)^2 \right] \]

\[ = \frac{8x^2 + (x + 1)^2 \sqrt{2x(x^2 + 1)}}{4(x + 1)^2} > 0, \quad \forall x \in (0, \infty). \]

Now the expression (3.32) together with (2.13) give us (3.31), or equivalently, we can say that

\[ \frac{\sqrt{A(a, b)^2 + H(a, b)^2}}{2} \leq \frac{S(a, b) + G(a, b)}{2}. \]

Finally, the inequalities (3.26), (3.29) and (3.33) give us the proof of the inequalities (3.29). This completes the proof of the corollary.

**Theorem 3.2.** The following inequalities hold:

\[ 1/8 M_{AH}(a, b) \leq M_{N_2N_1}(a, b) \leq 1/3 M_{N_2G}(a, b) \leq 1/4 M_{AG}(a, b) \leq M_{AN_2}(a, b). \]

**Proof.** In order to prove the above theorem, we shall prove each part separately.

Let us consider

\[ g_{AH,N_2N_1}(x) = \frac{f''_{AH}(x)}{f''_{N_2N_1}(x)} = \frac{32x^{5/2}(2x + 2)^{3/2}}{(x + 1)^3 [-2x - 2x^{5/2} + x(2x + 2)^{3/2}], \quad x \in (0, \infty).} \]

This gives

\[ g'_{AH,N_2N_1}(x) = -\frac{48 \sqrt{2x(x + 1)}}{(x + 1)^4 [-2x - 2x^{5/2} + x(2x + 2)^{3/2}]^2} \times \left[ 4x^2(1 - x^{5/2}) + x^2(x - 1)(2x + 2)^{5/2} \right] \]

\[ = \frac{48x^2(x + 1)(1 - \sqrt{x}) \sqrt{2x(x + 1)}}{(x + 1)^4 [-2x - 2x^{5/2} + x(2x + 2)^{3/2}]^2} \times \left[ \sqrt{2} (\sqrt{x} + 1)(x + 1)^{3/2} - (x^2 + x^{3/2} + x + \sqrt{x} + 1) \right]. \]

Since \( \sqrt{2(x + 1)} \geq \sqrt{x + 1}, \quad \forall x \in (0, \infty), \) then this implies that

\[ \sqrt{2}(x + 1)^{3/2} (\sqrt{x} + 1) \geq (\sqrt{x} + 1)^2 (x + 1) \]

\[ \geq x^2 + x^{3/2} + x + \sqrt{x} + 1 \]
Thus we conclude that
\begin{equation}
(3.35) 
\begin{cases}
g'_{AHN_2N_1}(x) < 0, & x > 1 \\
g'_{AHN_2N_1}(x) > 0, & x < 1.
\end{cases}
\end{equation}

In view of (3.35), we conclude that the function \( g_{AHN_2N_1}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\begin{equation}
(3.36) 
\beta = \sup_{x \in (0, \infty)} g_{AHN_2N_1}(x) = g_{AHN_2N_1}(1) = 8.
\end{equation}

Applying (3.8) for the difference of means \( M_{AH}(a, b) \) and \( M_{N_2N_1}(a, b) \) along with (3.36), we get
\begin{equation}
(3.37) 
\frac{1}{8} M_{AH}(a, b) \leq M_{N_2N_1}(a, b)
\end{equation}

Let us consider
\[ g_{N_2N_1N_2G} = \frac{f''_{N_2N_1}}{f''_{N_2G}}(x) = \frac{-2x - 2x^{5/2} + x(2x + 2)^{3/2}}{2x [1 + x^{3/2} - (2x + 2)^{3/2}]} \], \( x \in (0, \infty) \).

This gives
\begin{equation}
(3.38) 
g'_{N_2N_1N_2G_1} = \frac{3x^2 \sqrt{2x + 2} (1 - \sqrt{x})}{2x^2 [-1 - x^{3/2} + (2x + 2)^{3/2}]} \begin{cases}
< 0, & x > 1, \\
> 0, & x < 1.
\end{cases}
\end{equation}

In view of (3.38), we conclude that the function \( g_{N_2N_1N_2G}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\begin{equation}
(3.39) 
\beta = \sup_{x \in (0, \infty)} g_{N_2N_1N_2G}(x) = g_{N_2N_1N_2G}(1) = \frac{1}{3}.
\end{equation}

Applying (3.8) for the difference of means \( M_{N_2N_1}(a, b) \) and \( M_{N_2G}(a, b) \) along with (3.39), we get
\begin{equation}
(3.40) 
M_{N_2N_1}(a, b) \leq \frac{1}{3} M_{N_2G}(a, b).
\end{equation}

Let us consider
\[ g_{N_2GAG} = \frac{f''_{N_2G}}{f''_{AG}}(x) = \frac{-1 + x^{3/2} - (2x + 2)^{3/2}}{(2x + 2)^{3/2}} \], \( x \in (0, \infty) \).

This gives
\begin{equation}
(3.41) 
g'_{N_2GAG} = \frac{3(1 - \sqrt{x})}{(2x + 2)^{5/2}} \begin{cases}
\leq 0, & x \geq 1, \\
\geq 0, & x \leq 1.
\end{cases}
\end{equation}

In view of (3.41), we conclude that the function \( g_{AHN_2N_1}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\begin{equation}
(3.42) 
\beta = \sup_{x \in (0, \infty)} g_{N_2GAG}(x) = g_{N_2GAG}(1) = \frac{3}{4}.
\end{equation}

Applying (3.8) for the difference of means \( M_{N_2G}(a, b) \) and \( M_{AG}(a, b) \) along with (3.42), we get
\begin{equation}
(3.43) 
M_{N_2G}(a, b) \leq \frac{3}{4} M_{AG}(a, b).
\end{equation}
Let us consider 
\[ g_{AG\cdot AN_2}(x) = \frac{f''_{AG}(x)}{f''_{AN_2}(x)} = \frac{(2x + 2)^{3/2}}{(\sqrt{x} + 1)(x - \sqrt{x} + 1)}, \quad x \in (0, \infty). \]

This gives
\[ g'_{AG\cdot AN_2}(x) = 3 \frac{(1 - \sqrt{x}) \sqrt{2x + 2}}{(\sqrt{x} + 1)^2 (x - \sqrt{x} + 1)^2} \begin{cases} \leq 0, & x \geq 1 \\ \geq 0, & x \leq 1 \end{cases} \]

In view of (3.44), we conclude that the function \( g_{AG\cdot AN_2}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\[ \beta = \sup_{x \in (0, \infty)} g_{AG\cdot AN_2}(x) = g_{AG\cdot AN_2}(1) = 4. \]

Applying (3.8) for the difference of means \( M_{AG}(a, b) \) and \( M_{AN_2}(a, b) \) along with (3.45) we get the required result.
\[ \frac{1}{4} M_{AG}(a, b) \leq M_{AN_2}(a, b). \]

Combining the results (3.37), (3.40), (3.43) and (3.46) we get the proof of the inequalities (3.34).

**Corollary 3.3.** The inequalities hold:
\[ (3.47) \quad H(a, b) \leq G(a, b) \leq \frac{G(a, b) + H(a, b) + 3N_2(a, b)}{5} \]
\[ \leq \frac{G(a, b) + 2N_2(a, b)}{3} \leq \frac{N_1(a, b)}{2} \leq \frac{2A(a, b) + 7N_1(a, b)}{9} \leq N_2(a, b) \]
\[ \leq A(a, b) + N_1(a, b) \leq \frac{7A(a, b) + H(a, b)}{8} \leq A(a, b). \]

**Proof.** Follows in view of (3.32), (3.35), (3.38), (3.41) and (3.6).

**Remark 3.2.** The inequalities (3.47) can be considered as an improvement over the following inequalities:
\[ (3.48) \quad H(a, b) \leq G(a, b) \leq N_1(a, b) \leq N_2(a, b) \leq A(a, b). \]

**Theorem 3.3.** The following inequalities hold:
\[ (3.49) \quad M_{SA}(a, b) \leq \frac{4}{5} M_{SN_2}(a, b) \leq 4M_{AN_2}(a, b), \]
\[ (3.50) \quad M_{SH}(a, b) \leq 2M_{SN_1}(a, b) \leq \frac{3}{2} M_{SG}(a, b), \]
and
\[ (3.51) \quad M_{SA}(a, b) \leq \frac{3}{4} M_{SN_1}(a, b) \leq \frac{2}{3} M_{SN_1}(a, b). \]

**Proof.** In order to prove the above theorem, we shall prove each part separately.

Let us consider
\[ g_{SA\cdot SN_2}(x) = \frac{f''_{SA}(x)}{f''_{SN_1}(x)} = \frac{8x^{3/2}(2x + 2)^{3/2}}{8x^{3/2}(2x + 2)^{3/2} + (1 + x^{3/2})(2x^{3/2} + 2)^{3/2}}, \quad x \in (0, \infty). \]
This gives
\[ g'_{SA_{SN_2}}(x) = -\frac{96 \sqrt{x(x^2 + 1)(x + 1)}}{\left[8x^{3/2}(2x + 2)^{3/2} + (1 + x^{3/2})(2x^{3/2} + 2)^{3/2}\right]^2} \times \left[(x^2 - 1)(x^{5/2} + 1) + 2x(x^{5/2} - 1)\right] \]
\[ = -\frac{96 (\sqrt{x} - 1) \sqrt{x(x^2 + 1)(x + 1)}}{\left[8x^{3/2}(2x + 2)^{3/2} + (1 + x^{3/2})(2x^{3/2} + 2)^{3/2}\right]^2} \times \left[(\sqrt{x} + 1)(x + 1)(x^{5/2} + 1) + 2x (x^2 + x^{3/2} + x + \sqrt{x} + 1)\right]. \]

Thus, we have
\begin{align*}
(3.52) \quad g'_{SA_{SN_2}}(x) \begin{cases} 
\geq 0, & x \geq 1, \\
\leq 0, & x \leq 1.
\end{cases}
\end{align*}

In view of (3.52), we conclude that the function \( g_{SA_{SN_2}}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\begin{align*}
(3.53) \quad \beta = \sup_{x \in (0, \infty)} g_{SA_{SN_2}}(x) = g_{SA_{SN_2}}(1) = \frac{4}{5}.
\end{align*}

Applying (3.8) for the difference of means \( M_{SA}(a, b) \) and \( M_{SN_2}(a, b) \) along with (3.53), we get
\begin{align*}
(3.54) \quad M_{SA}(a, b) \leq \frac{4}{5} M_{SN_2}(a, b).
\end{align*}

Let us consider
\[ g_{SN_2_{AN_2}}(x) = \frac{f''_{SN_2}(x)}{f''_{AN_2}(x)} = \frac{8x^{3/2}(2x + 2)^{3/2} + (1 + x^{3/2})(2x^{3/2} + 2)^{3/2}}{(2x^2 + 2)(x^{3/2} + 1)}, \quad x \in (0, \infty), \]
This gives
\[ g'_{SN_2_{AN_2}}(x) = -\frac{12 \left[x(x + 1)^{9/2} \left[(x^2 - 1)(1 + x^{5/2}) + 2x(x^{5/2} - 1)\right]\right]}{(x^2 + 1)^{5/2}x^4(x + 1)^4(x^{3/2} + 1)^2} \]
\[ = -\frac{12 (x(x + 1)^{9/2} (\sqrt{x} - 1))}{(x^2 + 1)^{5/2}x^4(x + 1)^4(x^{3/2} + 1)^2} \times \left[(\sqrt{x} + 1)(x + 1)(x^{5/2} + 1) + 2x (x^2 + x^{3/2} + x + \sqrt{x} + 1)\right]. \]

Thus we have
\begin{align*}
(3.55) \quad g'_{SN_2_{AN_2}}(x) \begin{cases} 
\geq 0, & x \leq 1, \\
\leq 0, & x \geq 1.
\end{cases}
\end{align*}

In view of (3.55), we conclude that the function \( g_{SN_2_{AN_2}}(x) \) is increasing in \( x \in (0, 1) \) and decreasing in \( x \in (1, \infty) \), and hence
\begin{align*}
(3.56) \quad \beta = \sup_{x \in (0, \infty)} g_{SN_2_{AN_2}}(x) = g_{SN_2_{AN_2}}(1) = \frac{4}{5}.
\end{align*}

Applying (3.8) for the difference of means \( M_{SN_2}(a, b) \) and \( M_{AN_2}(a, b) \) along with (3.56), we get
\begin{align*}
(3.57) \quad \frac{1}{5} M_{SN_2}(a, b) \leq M_{AN_2}(a, b).
\end{align*}
Combining the results (3.54) and (3.57) we get the proof of the inequalities (3.49). Now we shall give the proof of (3.50).

Let us consider

\[ g_{SH,SN_1}(x) = \frac{f''_{SH}(x)}{f''_{SN_1}(x)} = \frac{16x^{3/2}[(x+1)^3+2(2x^2+2)^{3/2}]}{(x+1)^3[16x^{3/2}+(2x^2+2)^{3/2}]}, \quad x \in (0, \infty), \]

This gives

\[ g'_{SH,SN_1}(x) = -\frac{48\sqrt{2x^2+2}}{x^2(x+1)^4[16x^{3/2}+(2x^2+2)^{3/2}]^2} \times \]
\[ \times \left[64x^{9/2}(1-x) + 5x^4(x-1) + 4x^3(x^4-1) \right. \]
\[ + x^2(x^2-1) + x^2(x-1)(2x^2+2)^{5/2} \]
\[ = -\frac{1536x^2(x-1)\sqrt{2x^2+2}}{x^2(x+1)^4[16x^{3/2}+(2x^2+2)^{3/2}]^2} \times \]
\[ \times \left\{ \left[ \frac{(x+1)}{2} - \sqrt{x} \right]^5 + \left[ \frac{(x+1)}{2} \right]^5 - (\sqrt{x})^5 \right\}. \]

Since \( S(a,b) \geq A(a,b) \geq G(a,b) \), one gets

(3.58)

\[ g'_{SH,SN_1}(x) \begin{cases} 
\geq 0, & x \leq 1, \\
\leq 0, & x \geq 1.
\end{cases} \]

In view of (3.58) we conclude that the function \( g_{SH,SN_1}(x) \) increasing in \( x \in (0,1) \) and decreasing in \( x \in (1,\infty) \), and hence

(3.59)

\[ \beta = \sup_{x \in (0,\infty)} g_{SH,SN_1}(x) = g_{SH,SN_1}(1) = 2. \]

Applying (3.8) for the difference of means \( M_{SH}(a,b) \) and \( M_{SN_1}(a,b) \) along with (3.59) we get

(3.60)

\[ M_{SH}(a,b) \leq 2M_{SN_1}(a,b). \]

Let us consider

\[ g_{SN_1,SG}(x) = \frac{f''_{SN_1}(x)}{f''_{SG}(x)} = \frac{8x^{3/2}+(x^2+1)\sqrt{2x^2+2}}{2[4x^{3/2}+(x^2+1)\sqrt{2x^2+2}]}, \quad x \in (0, \infty), \]

This gives

(3.61)

\[ g'_{SN_1,SG}(x) = -\frac{3(x^2-1)\sqrt{2x^3+2x}}{[4x^{3/2}+(x^2+1)\sqrt{2x^2+2}]^2} \begin{cases} 
\geq 0, & x \leq 1, \\
\leq 0, & x \geq 1.
\end{cases} \]

In view of (3.61), we conclude that the function \( g_{SN_1,SG}(x) \) is increasing in \( x \in (0,1) \) and decreasing in \( x \in (1,\infty) \), and hence

(3.62)

\[ \beta = \sup_{x \in (0,\infty)} g_{SN_1,SG}(x) = g_{SN_1,SG}(1) = \frac{3}{4}. \]

Applying (3.8) for the difference of means \( M_{SN_1}(a,b) \) and \( M_{SG}(a,b) \) along with (3.62) we get

(3.63)

\[ M_{SN_1}(a,b) \leq \frac{3}{4} M_{SG}(a,b). \]
Combining the results given in (3.60) and (3.63) we get the proof of the inequalities (3.50). Let us prove now the inequalities (3.51).

Let us consider

$$g_{SA, SN_3}(x) = \frac{f''_{SA}(x)}{f''_{SN_3}(x)} = \frac{24x^{3/2}}{24x^{3/2} + (2x^2 + 2)^{3/2}}, \ x \in (0, \infty),$$

This gives

(3.64) $$g'_{SA, SN_3}(x) = -\frac{72(x - 1)(x + 1)\sqrt{2x(x^2 + 1)}}{[24x^{3/2} + (2x^2 + 2)^{3/2}]^2} \begin{cases} \geq 0, & x \leq 1, \\ \leq 0, & x \geq 1. \end{cases}$$

In view of (3.64), we conclude that the function $g_{SA, SN_3}(x)$ is increasing in $x \in (0, 1)$ and decreasing in $x \in (1, \infty)$, and hence

(3.65) $$\beta = \sup_{x \in (0, \infty)} g_{SA, SN_3}(x) = g_{SA, SN_3}(1) = \frac{3}{4}.$$ 

Applying (3.8) for the difference of means $M_{SA}(a, b)$ and $M_{SN_3}(a, b)$ along with (3.66) we get

(3.66) $$M_{SA}(a, b) \leq \frac{3}{4}M_{SN_3}(a, b).$$

Let us consider

$$g_{SN_3, SN_1}(x) = \frac{f''_{SN_3}(x)}{f''_{SN_1}(x)} = \frac{2 [24x^{3/2} + (2x^2 + 2)^{3/2}]}{3 [16x^{3/2} + (2x^2 + 2)^{3/2}]], \ x \in (0, \infty),$$

This gives

(3.67) $$g'_{SN_3, SN_1}(x) = -\frac{(x - 1)(x + 1)\sqrt{2x(x^2 + 1)}}{[16x^{3/2} + (2x^2 + 2)^{3/2}]^2} \begin{cases} \geq 0, & x \leq 1, \\ \leq 0, & x \geq 1. \end{cases}$$

In view of (3.67), we conclude that the function $g_{SN_3, SN_1}(x)$ is increasing in $x \in (0, 1)$ and decreasing in $x \in (1, \infty)$, and hence

(3.68) $$\beta = \sup_{x \in (0, \infty)} g_{SN_3, SN_1}(x) = g_{SN_3, SN_1}(1) = \frac{3}{4}.$$ 

Applying (3.8) for the difference of means $M_{SN_3}(a, b)$ and $M_{SN_1}(a, b)$ along with (3.68) we get

(3.69) $$M_{SN_3}(a, b) \leq \frac{8}{9}M_{SN_1}(a, b).$$

Combining the results given in (3.66) and (3.69) we get the proof of the inequalities (3.54). This completes the proof of the theorem. □

Corollary 3.4. The following inequalities hold:

(3.70) $$G(a, b) \leq \frac{S(a, b) + 3G(a, b)}{4} \leq N_1(a, b) \leq \frac{S(a, b) + 8N_1(a, b)}{9} \leq N_3(a, b) \leq N_2(a, b) \leq \frac{A(a, b) + N_1(a, b)}{2} \leq \frac{S(a, b) + 2N_1(a, b)}{3} \leq \frac{S(a, b) + 4N_2(a, b)}{5} \text{ or } \frac{S(a, b) + 3N_3(a, b)}{4} \leq A(a, b)$$
and

\begin{equation}
G(a, b) \leq \frac{S(a, b) + 2H(a, b)}{2} \leq N_1(a, b) \leq \frac{S(a, b) + H(a, b)}{2} \leq N_2(a, b).
\end{equation}

Proof. The inequalities (3.49)-(3.51) lead us to (3.70) and (3.71).

\[ \blacksquare \]

Remark 3.3. The inequalities (3.70) can be considered as refinement over the inequality (1.7). Thus we have three different kind of refinements given by (3.28), (3.47) and (3.70) for the inequality (1.7). The inequalities (3.71) gives alternative improvement among the means \( G(a, b), N_1(a, b) \) and \( N_2(a, b) \).

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