Generalized second law of thermodynamics in $f(T)$ gravity

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Abstract. We investigate the validity of the generalized second law (GSL) of gravitational thermodynamics in the framework of $f(T)$ modified teleparallel gravity. We consider a spatially flat FRW universe containing only the pressureless matter. The boundary of the universe is assumed to be enclosed by the Hubble horizon. For two viable $f(T)$ models containing $f(T) = T + \mu_1 (-T)^n$ and $f(T) = T - \mu_2 T (1 - e^{\beta T_0})$, we first calculate the effective equation of state and deceleration parameters. Then, we investigate the null and strong energy conditions and conclude that a sudden future singularity appears in both models. Furthermore, using a cosmographic analysis we check the viability of two models. Finally, we examine the validity of the GSL and find that for both models it is satisfied from the early times to the present epoch. But in the future, the GSL is violated for the special ranges of the torsion scalar $T$.

Keywords: modified gravity, cosmology of theories beyond the SM

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1K. Karami dedicated this paper to the 80 year Jubilee of Professor Yousef Sobouti.
1 Introduction

Observational data coming from the type Ia supernovae (SNeIa) surveys, large scale structure (LSS), and cosmic microwave background (CMB) anisotropy spectrum indicate that the expansion of our present universe is accelerating rather than slowing down [1]. This cosmic acceleration cannot be explained by the four known fundamental interactions in the standard models, which is the greatest challenge today in the modern physics. The proposals that have been put forward to explain this observed phenomenon can basically be classified into two categories. One is to assume that in the framework of Einstein’s general relativity (GR), an exotic component with negative pressure called dark energy (DE) is necessary to explain this observed phenomena. For a good review on the dynamics of different DE models see [2] and references therein. Another alternative to account for the current accelerating cosmic expansion is to modify GR theory. The well-known modified gravity theories are, for examples, $f(R)$ theory, scalar-tensor theory (including Brans-Dicke theory), braneworld scenarios (such as DGP, RSI and RSII), $f(G)$ theory ($G$ is the Gauss-Bonnet term), Horava-Lifshitz theory, MOdified Newtonian Dynamics (MOND), and so forth. For some relevant reviews see [3].

Recently, a new modified gravity theory namely the so-called $f(T)$ theory [4]–[13] was proposed to describe the present accelerating expansion of the universe without resorting to DE. It is a generalization of the teleparallel gravity (TG) [14] by replacing the so-called torsion scalar $T$ with $f(T)$. TG was originally developed by Einstein in an attempt of unifying gravity and electromagnetism. The basic variables in TG are tetrad fields $e_{i\mu}$, where the Weitzenbock connection rather than the Levi-Civita connection was used to define the covariant derivative. As a result, the spacetime has no curvature but contains torsion. A vector $V^\mu$ in TG is parallel transported along a curve if its projection $V_i = e_{i\mu}V^\mu$ remains constant, this is the so-called teleparallelism. The main advantage of $f(T)$ theory is that its field equations are the second order which are remarkably simpler than the fourth order equations of $f(R)$ theory [8].

The other interesting issue in modern cosmology is the thermodynamical description of the accelerating universe driven by DE or modified gravity. It was shown that by applying the first law of thermodynamics (Clausius relation) $-dE = T_AdS_A$ to the apparent horizon $\tilde{r}_A$, the Friedmann equation in the Einstein gravity can be derived if we take the Hawking
temperature $T_A = 1/(2\pi \tilde{r}_A)$ and the entropy $S_A = \frac{A}{4\ell_P}$ on the apparent horizon, where $A$ is the area of the horizon [15]. Here, $dE$ is the amount of energy flow through the fixed apparent horizon. The equivalence between the first law of thermodynamics and the Friedmann equation was also found for gravity with Gauss-Bonnet term, the Lovelock gravity theory and the braneworld scenarios [15–17].

Note that in the thermodynamics of the apparent horizon in the Einstein gravity, the geometric entropy is assumed to be proportional to its horizon area, $S_A = \frac{A}{4\ell_P}$ [15]. However, this definition is changed for other modified gravity theories. For instance, the geometric entropy in $f(R)$ gravity is given by $S_A = \frac{A}{4\ell_P}$ [18], where the subscript $R$ denotes a derivative with respect to the curvature scalar $R$. In $f(T)$ gravity, it was shown that when $f_{TT}$ is small, the first law of black hole thermodynamics is satisfied approximatively and the entropy of horizon is $S_A = \frac{A}{4\ell_T}$ [19], where the subscript $T$ denotes a derivative with respect to the torsion scalar $T$.

Besides examining the validity of the thermodynamical interpretation of gravity by expressing the gravitational field equations into the first law of thermodynamics in different spacetimes, it is also of great interest to investigate the validity of the generalized second law (GSL) of thermodynamics in the accelerating universe [20–28]. The GSL of thermodynamics like the first law is an accepted principle in physics.

Here, our aim is to investigate the GSL of thermodynamics in the framework of $f(T)$ gravity for a spatially flat Friedmann-Robertson-Walker (FRW) universe filled with the pressureless matter. To do this, in section 2, we briefly review the $f(T)$ gravity. In section 3, we investigate the GSL of thermodynamics on the dynamical apparent horizon with the Hawking temperature. In section 4, for two viable $f(T)$ models we first calculate the effective equation of state and deceleration parameters. Then, the null and strong energy conditions are investigated. Also the viability of both models is checked by cosmography. Finally, the validity of the GSL is examined. Section 5 is devoted to conclusions.

2 Brief review of $f(T)$ gravity

In the framework of $f(T)$ theory, the action of modified TG is given by [7]

$$I = \frac{1}{16\pi G} \int d^4 x \, e \left[ f(T) + L_m \right], \quad (2.1)$$

where $e = \det(e^i_\mu) = \sqrt{-g}$ and $L_m$ is the Lagrangian density of the matter inside the universe. Also $e^i_\mu$ is the vierbein field which is used as a dynamical object in TG.

The modified Friedmann equations in the framework of $f(T)$ gravity in the spatially flat FRW universe are given by [8, 10]

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_T), \quad (2.2)$$

$$\dot{H} + \frac{3}{2}H^2 = -4\pi G(p_m + p_T), \quad (2.3)$$

where

$$\rho_T = \frac{1}{16\pi G}(2Tf_T - f - T), \quad (2.4)$$

$$p_T = -\frac{1}{16\pi G}[-8\dot{H}T f_{TT} + (2T - 4\dot{H})f_T - f + 4\dot{H} - T], \quad (2.5)$$

$$T = -6H^2, \quad (2.6)$$
and $H = \dot{a}/a$ is the Hubble parameter. Here $\rho_m$ and $p_m$ are the energy density and pressure of the matter, respectively. Also $\rho_T$ and $p_T$ are the torsion contributions to the energy density and pressure. The energy conservation laws are still given by

$$
\dot{\rho}_m + 3H(\rho_m + p_m) = 0, 
\dot{\rho}_T + 3H(\rho_T + p_T) = 0. 
$$

(2.7) (2.8)

Note that if $f(T) = T$, from eqs. (2.4) and (2.5) we have $\rho_T = 0$ and $p_T = 0$ then eqs. (2.2) and (2.3) transform to the usual Friedmann equations in GR.

The effective equation of state (EoS) parameter due to the torsion contribution is defined as [8, 10]

$$
\omega_T = \frac{p_T}{\rho_T} = -1 - \frac{T}{3H} \left( \frac{2T f_{TT} + f_T - 1}{2T f_T - f - T} \right).
$$

(2.9)

Note that for the de Sitter universe, i.e. $\dot{H} = 0 = \dot{T}$, we have $\omega_T = -1$ behaving like the cosmological constant.

Using eqs. (2.2), (2.4) and (2.6) we have

$$
\rho_m = \frac{1}{16\pi G} (f - 2T f_T).
$$

(2.10)

Here, we consider a spatially flat FRW universe filled with the pressureless matter, i.e. $p_m = 0$. We ignore the contribution of radiation which is compatible with the observations [29]. From eqs. (2.2) to (2.5) for $p_m = 0$, one can obtain

$$
\dot{H} = -\frac{4\pi G \rho_m}{f_T + 2T f_{TT}}.
$$

(2.11)

Substituting eq. (2.10) into (2.11) and using $\dot{T} = -12H\dot{H}$, one can get

$$
\dot{T} = 3H \left( \frac{f - 2T f_T}{f_T + 2T f_{TT}} \right).
$$

(2.12)

With the help of above relation, the effective EoS parameter (2.9) can be rewritten as

$$
\omega_T = -\frac{f/T - f_T + 2T f_{TT}}{(f_T + 2T f_{TT})(f/T - 2f_T + 1)}.
$$

(2.13)

For the deceleration parameter

$$
q = -1 - \frac{\dot{H}}{H^2},
$$

(2.14)

using eqs. (2.6) and (2.12) one can obtain

$$
q = 2 \left( \frac{f_T - T f_{TT} - \frac{3f}{4}}{f_T + 2T f_{TT}} \right).
$$

(2.15)

For the case $f(T) = T$, we get $q = 0.5$ corresponding to the matter dominated phase.
3 GSL in $f(T)$ gravity

Here, we investigate the validity of the GSL for a spatially flat FRW universe filled with the pressureless barionic matter (BM) and dark matter (DM). According to the GSL, the entropy of matter inside the horizon plus the entropy of the horizon must not decrease in time [15].

We assume that the boundary of the universe to be enclosed by the dynamical apparent horizon $\tilde{r}_A$. Also the Hawking temperature on the apparent horizon $\tilde{r}_A$ is given by [15]

$$T_A = \frac{1}{2\pi \tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right), \quad (3.1)$$

where $\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} < 1$ ensures that the temperature is positive.

The entropy of the universe including BM and DM inside the dynamical apparent horizon is given by Gibb’s equation [20]

$$T_A dS_m = dE_m + p_m dV, \quad (3.2)$$

where $p_m = 0$ and $V = \frac{4\pi \tilde{r}_A^3}{3}$ is the volume containing the pressureless matter with the radius of the dynamical apparent horizon $\tilde{r}_A$. Also

$$E_m = \frac{4\pi \tilde{r}_A^3}{3} \rho_m. \quad (3.3)$$

Taking time derivative of both sides of eq. (3.3) and using (2.7), Gibb’s equation (3.2) yields

$$T_A \dot{S}_m = 4\pi \tilde{r}_A^3 \rho_m (\dot{\tilde{r}}_A - H\tilde{r}_A), \quad (3.4)$$

where the dot denotes a time derivative. Substituting eq. (2.10) into (3.4) gives

$$T_A \dot{S}_m = \frac{\tilde{r}_A^3}{4G} (\dot{\tilde{r}}_A - H\tilde{r}_A) (f - 2Tf_T). \quad (3.5)$$

Now we need to calculate the contribution of the apparent horizon entropy. Following [19], when $f_{TT}$ is small, the horizon entropy in $f(T)$ gravity is given by

$$S_A = A f_T \frac{\tilde{r}_A^3}{4G}, \quad (3.6)$$

where $A = 4\pi \tilde{r}_A^3$. For the special case $f(T) = T$, eq. (3.6) recovers the Bekenstein-Hawking entropy $S_A = A f_T \frac{\tilde{r}_A^3}{4G}$ in the Einstein gravity.

Using eq. (3.1), the evolution of horizon entropy (3.6) is obtained as

$$T_A \dot{S}_A = \frac{1}{2G} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right) \left( 2\tilde{r}_A f_T + \dot{\tilde{r}}_A f_{TT} \right). \quad (3.7)$$

Inserting eq. (2.12) into (3.7) gives

$$T_A \dot{S}_A = \frac{1}{2G} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right) \left[ \frac{(2\tilde{r}_A - 3H\tilde{r}_A)2Tf_T f_{TT} + 3H\tilde{r}_A f(T) f_{TT} + 2\dot{\tilde{r}}_A f_T^2}{f_T + 2Tf_{TT}} \right]. \quad (3.8)$$

For the flat FRW metric, the dynamical apparent horizon is same as the Hubble horizon given by [30]

$$\tilde{r}_A = \frac{1}{H}. \quad (3.9)$$
Taking time derivative of eq. (3.9), using (2.6), (2.10) and (2.11) one can get

\[
\dot{r}_A = -\frac{3}{2T} \left( \frac{f - 2T f_T}{f_T + 2T f_{TT}} \right).
\]  

(3.10)

Replacing eqs. (3.9)–(3.10) into (3.5) and (3.8) and using (2.6), one can obtain

\[
T_A \dot{S}_m = \frac{3}{4G T^2} \left( \frac{f - 2T f_T}{f_T + 2T f_{TT}} \right) \left( 4T^2 f_{TT} - 4T f_T + 3f \right),
\]  

(3.11)

\[
T_A \dot{S}_A = \frac{3}{4G T^2} \left( \frac{f - 2T f_T}{f_T + 2T f_{TT}} \right) \left[ \frac{4T^2 f^2_{TT} - T f_T + \frac{3}{2} f}{(f_T + 2T f_{TT})^2} \left( 2T^2 f^2_{TT} - f^2_T + T f_T f_{TT} \right) \right],
\]  

(3.12)

Adding eqs. (3.11) and (3.12) yields the GSL in \( f(T) \) gravity as

\[
T_A \dot{S}_\text{tot} = \frac{9}{8G} \left( \frac{f - 2T f_T}{f_T + 2T f_{TT}} \right) \left[ 4f_{TT} + \left( \frac{f - 2T f_T}{f_T + 2T f_{TT}} \right) \left( \frac{f_T + 5T f_{TT}}{T^2} \right) \right],
\]  

(3.13)

where \( S_\text{tot} = S_m + S_A \) is the total entropy due to different contributions of the matter and the horizon. Note that in the Einstein TG, i.e. \( f(T) = T \), the GSL (3.13) yields

\[
T_A \dot{S}_\text{tot} = \frac{9}{8G} > 0,
\]  

(3.14)

which always holds. In what follows, we investigate the validity of the GSL, i.e. \( T_A \dot{S}_\text{tot} \geq 0 \), for two viable \( f(T) \) models introduced in the literature.

4 Two viable \( f(T) \) models

Here, we consider two viable \( f(T) \) models proposed by [9, 11] to explain the present cosmic accelerating expansion of the universe. The first model is a power law

\[
f(T) = T + \mu_1 (-T)^n, \quad \text{Model 1},
\]  

(4.1)

where \( \mu_1 \) and \( n \) are constants [9, 11]. The second model has an exponential dependence on the torsion scalar as

\[
f(T) = T - \mu_2 T \left( 1 - e^{\frac{T_0}{T}} \right), \quad \text{Model 2},
\]  

(4.2)

where \( \mu_2 \) and \( \beta \) are two model parameters, and \( T_0 = -6H_0^2 \) [9].

The parameters \( \mu_1 \) and \( \mu_2 \) can be obtained by inserting eqs. (4.1) and (4.2) into the modified Friedmann eq. (2.2). Solving the resulting equations for the present time gives

\[
\mu_1 = \left( \frac{1 - \Omega_{m0}}{2n - 1} \right) (6H_0^2)^{1-n},
\]  

(4.3)

\[
\mu_2 = \frac{1 - \Omega_{m0}}{1 - (1 - 2\beta)e^\beta},
\]  

(4.4)

where \( \Omega_{m0} = \frac{8\pi G \rho_{m0}}{3H_0^2} \) is the dimensionless matter energy density and the index 0 denotes the value of a quantity at the present. Note that both models can unify a number of interesting extensions of gravity beyond the standard GR. For instance, model 1 for the cases \( n = 0 \)
and \( n = 1/2 \) reduces to the ΛCDM and DGP [31] models, respectively. Model 2 acts like the ΛCDM for \( \beta = 0 \). Also both models can satisfy the condition

\[
\lim_{T \to \infty} \frac{f}{T} = 1,
\]

at high redshift which is compatible with the primordial nucleosynthesis and CMB constraints [9, 11]. For model 1 to be a viable model compared to current data one needs \( n \ll 1 \) [9, 11]. The joint analysis of the astronomical data from SNeIa+BAO+CMB gives the best fit values \( (n = 0.04^{+0.22}_{-0.33}, \Omega_m = 0.272^{+0.036}_{-0.032}) \) for model 1 and \( (\beta = -0.02^{+0.31}_{-0.20}, \Omega_m = 0.272^{+0.036}_{-0.034}) \) for model 2 at the 95% confidence level (CL) [9].

Using the statefinder geometrical analysis and \( Om(z) \) diagnostic method, it was shown that both model 1 and model 2 evolve from the standard cold DM (SCDM) to a de Sitter phase [9]. Also the effective DE for model 2 with \( \beta \neq 0 \) is similar to the ΛCDM both in the high redshift regimes and in the future, while for model 1 with \( n \neq 0 \) this similarity occurs only in the future [9].

The evolution of the effective EoS parameter, eq. (2.13), for model 1 and model 2 is plotted in figures 1 and 2, respectively. Figures show that from the early times to the present epoch, both models behave like quintessence DE, i.e. \( \omega_T > -1 \), and their effective EoS parameters can not cross the phantom divide line [32]. This result has been already obtained by [9, 11]. At early times \( (T/T_0 \to +\infty) \), for model 1 and model 2 we have \( \omega_T \simeq -0.96 \) and \(-1 \), respectively. Model 2 behaves like the ΛCDM at high redshift. At the present time \( (T/T_0 = 1) \), for model 1 and model 2 we obtain \( \omega_{T_0} \simeq -0.989 \) and \(-0.992 \), respectively. An interesting result which is absent in [9, 11] is that in the future \((1 < T/T_0 < 0)\) for model 1 and model 2 at \( T/T_0 \simeq 0.718 \) and 0.719, respectively, we have a transition from the quintessence state, \( \omega_T > -1 \), to the phantom regime, \( \omega_T < -1 \).

The evolution of the deceleration parameter, eq. (2.15), for model 1 and model 2 is plotted in figures 3 and 4, respectively. Figures illustrate that for model 1 and model 2
in the past $T/T_0 \approx 2.194$ and 2.199, respectively, we have a cosmic deceleration $q > 0$ to acceleration $q < 0$ transition which is compatible with the observations [33]. At early times ($T/T_0 \to +\infty$), for both models we obtain $q \approx 0.5$ which indicates that the universe was experiencing a phase of deceleration at the early stage of its evolution due to the domination of the matter component. At the present time ($T/T_0 = 1$), for model 1 and model 2 we get $q_0 \approx -0.580$ and $-0.583$, respectively, which are in good agreement with the recent observational result $-1.4 \leq q_0 \leq -0.3$ [33].

The evolution of $\rho_T + p_T$ versus $T/T_0$ for model 1 and model 2 is plotted in figures 5 and 6, respectively. Figures show that the null energy condition (NEC), i.e. $\rho_T + p_T \geq 0$, is violated for model 1 when $T/T_0 \in (0.025, 0.718)$ and for model 2 when $T/T_0 \in (0, 0.017)$ and
Figure 4. Same as figure 3, for model 2. Auxiliary parameters as in figure 2.

Figure 5. The evolution of $\rho_T + p_T$ versus $T/T_0$ for model 1. Auxiliary parameters as in figure 1.

(0.138, 0.719). When the EoS parameter of DE is less than $-1$, the universe reaches a Big Rip singularity within a finite time. In this case, the NEC is violated. Barrow [35] showed that a different type of future singularity, the so-called sudden singularity, can appear at a finite time even when the strong energy condition (SEC), i.e. $\rho_T + 3p_T \geq 0$ and $\rho_T + p_T \geq 0$, is satisfied. This type of future singularity corresponds to the one in which the pressure density $p_T$ diverges at $T = T_s$, but the energy density $\rho_T$ is finite. The properties of different future singularities in the DE universe have been investigated in ample detail by [36]. Our numerical results show that the sudden future singularity appears in model 1 and model 2 when $(T_s/T_0 = 0.025, 16\pi G\rho_T = 20754)$ and $(T_s/T_0 = 0.138, 16\pi G\rho_T = 19822)$, respectively. Also the SEC for model 1 and model 2 is satisfied only when $T/T_0 \in (0, 0.025)$ and $T/T_0 \in (0.018, 0.138)$, respectively.
4.1 Cosmographic analysis

Here, following Capozziello et al. [37] we use a cosmographic analysis to check the viability of model 1 and model 2 without the need of explicitly solving the field equations and fitting the data. In this approach, the parameters of a given \( f(T) \) model must be chosen in such a way that the model-independent constraints on the cosmographic parameters \( (h, q_0, j_0, s_0, l_0) \) obtained by fitting to SNeIa Hubble diagram and BAO data are satisfied. Here, \( h \) is the Hubble constant and \( q_0, j_0, s_0, l_0 \) are the deceleration, jerk, snap, and lerk parameters, respectively.

Let us first start with model 2. Using eqs. (4.21) and (4.22) in [37] one can find

\[
\mu_2 = \left( \frac{1}{\beta} - 1 \right) (1 - \Omega_{m0}). \tag{4.6}
\]

Now the \( f_i = f^{(i)}(T_0)/(6H_0^2)^{-(i-1)} \) value for \( i = (2, 3, 4, 5) \) where \( f^{(i)}(T) = d^i f/dT^i \) can be expressed as function of \( \beta \) only when we fix \( \Omega_{m0} = 0.1329/h^2 \) from the WMAP7 data. Following [37] for each \( f_2 \) value of the sample obtained above from the cosmographic parameters analysis, we solve \( \hat{f}_2(\beta) = f_2 \). This yields \( \beta = -0.2 \) which takes place in the 95% CL from the model-dependent constraints [9]. Then, we estimate the theoretically expected values for the other derivatives \( (f_3, f_4, f_5) \). The median and 68% and 95% confidence ranges are obtained as

\[
\begin{align*}
    f_3 &= 0.371^{+0.117}_{-0.351} + 0.148 \\
    f_4 &= 1.385^{+0.380}_{-1.302} + 0.473 \\
    f_5 &= 6.447^{+1.498}_{-0.035} + 1.832
\end{align*}
\tag{4.7}
\]

Now we compare the above results with the model-independent constraints on the \( f_i \) values given in table II in [37]. Following [37] we use only the 68% CL which we compare the above constraints to. Our comparison shows that the values of \( (f_3, f_4, f_5) \) take place in the 68% CL in table II in [37]. Therefore, we conclude that model 2 is favored by the observational data.

For model 1, its cosmographic analysis has been already done by Capozziello et al. [37]. They showed that for the best fit value of the cosmographic parameters, solving \( \hat{f}_2(n) = f_2 \)
yields $n = -0.011$. Note that the value of $n$ takes place in the 95% CL from the model-dependent constraints [9]. Capozziello et al. [37] found quite small values for $(f_3, f_4, f_5)$ as expected for the ΛCDM model. Here, the disagreement with the constraints in table II in [37] may be due to this fact that depending on the value of $n$, assuming $f^{(i)}(T_0) = 0$ for $i \geq 6$ in Taylor expanding $f(T)$ can fail for model 1 so that the constraints on $f_i$ should not be considered reliable [37].

4.2 Examining the GSL for model 1 and model 2

Here, we examine the validity of the GSL, eq. (3.13), for both models. First, we need to check the validity of the horizon entropy relation (3.6). Because as we already mentioned, eq. (3.6) is valid only when $f_{TT}$ is small [19]. To check this, we plot $f_{TT}$ versus $T/T_0$ for model 1 and model 2 in figures 7 and 8, respectively. Figures show that the $f_{TT}$ is very small throughout history of the universe. This confirms the validity of eq. (3.6) for both models.

Now we can calculate the GSL, eq. (3.13), for both models. For model 1, the resulting GSL is

$$T_A \dot{S}_{\text{tot}} = \frac{9}{8} \frac{1 + \mu_1 (1 - 2n) (-T)^{n-1}}{(1 + \mu_1 n (1 - 2n) (-T)^{n-1})^2} \left[1 + (1 - n (2 + n)) \mu_1 (-T)^{n-1} + n (1 - 2n) \left(4 - n (3 - 4n) \right) \mu_1^2 (-T)^{2n-2} \right], \quad (4.8)$$

and for model 2 we obtain

$$T_A \dot{S}_{\text{tot}} = \frac{I}{8T \left[ (1 - \mu_2) T^2 + \mu_2 (T^2 - \beta T_0 T + 2 \beta^2 T_0^2) e^{\beta T_0} \right]^2}, \quad (4.9)$$

where

$$I = 9 (1 - \mu_2)^3 T^5 + 9 \mu_2 \left[ (3 - 2 \mu_2) T^5 - 5 \beta T_0 (1 + \mu_2 - \mu_2^2) T^4 \right. \left. + \beta^2 T_0^2 \left( 1 + 17 \mu_2 - 9 \mu_2^2 \right) T^3 - 4 \mu_2 \beta^3 T_0^3 \left( (5 - 2 \mu_2) T + \beta T_0 (1 - 2 \mu_2) \right) T \right] + 16 \mu_2^2 T_0^5 \beta^5 e^{\beta T_0}. \quad (4.10)$$
Figure 8. Same as figure 7, for model 2. Auxiliary parameters as in figure 2.

Figure 9. The evolution of the GSL, eq. (4.8), versus $\frac{T}{T_0}$ for model 1. Auxiliary parameters as in figure 1.

The evolutions of the GSL (4.8) and (4.9) are plotted in figures 9 and 10, respectively. Figures clear that for both model 1 and model 2, the GSL is satisfied from the early times to the present epoch. At early times ($T/T_0 \to +\infty$), for both models we have $G T_A \dot{S}_{\text{tot}} \simeq 1.125$. At the present time ($T/T_0 = 1$), for model 1 and model 2 we get $G T_A \dot{S}_{\text{tot}} \simeq 0.116$ and 0.010, respectively. Note that in the future ($1 < T/T_0 < 0$), figure 9 clears that the GSL for model 1 is violated, i.e. $G T_A \dot{S}_{\text{tot}} < 0$, for $T/T_0 \in (0, 0.133)$ and $(0.005, 0.718)$. Also in the future, according to figure 10 the GSL for model 2, like model 1, is violated for $T/T_0 \in (0.005, 0.017)$, $(0.039, 0.321)$ and $(0.603, 0.719)$. Figures 9 and 10 don’t show the negative values of the GSL, because the axis of $G T_A \dot{S}_{\text{tot}}$ is based on a logarithmic scale.
5 Conclusions

Here, we studied the GSL in the framework of $f(T)$ gravity. Among other approaches related with a variety of DE models, a very promising approach to DE is related with the modified TG known as $f(T)$ gravity, in which DE emerges from the modification of torsion. The class of $f(T)$ gravity theories is an intriguing generalization of Einstein’s new GR, taking a curvature-free approach and using a connection with torsion. It is analogous to the $f(R)$ extension of the Einstein-Hilbert action of standard GR, but has the advantage of the second order field equations \[11\]. We investigated the GSL on the Hubble horizon with the Hawking temperature for a spatially flat FRW universe filled with the pressureless matter. For two viable $f(T)$ models containing $f(T) = T + \mu_1( - T)^n$ and $f(T) = T - \mu_2T(1 - e^{\beta T})$, we first calculated the effective EoS and deceleration parameters. Interestingly enough, we found that for both models there is a transition from the quintessence state, $\omega_T > -1$, to the phantom regime, $\omega_T < -1$, in the future. Also both models showed a cosmic deceleration $q > 0$ to acceleration $q < 0$ transition in the near past which is compatible with the observations \[33\]. Furthermore, we investigated the NEC and SEC and concluded that both models show a sudden future singularity. Also our cosmographic analysis cleared that model 1 is unable to predict the observationally motivated $(f_2, f_3, f_4, f_5)$ values but model 2 is favored by the observational data. Finally, we examined the validity of the GSL for the selected $f(T)$ models. We concluded that the GSL is satisfied for both models from the early times to the present epoch. But in the future, the GSL is violated for model 1 when $T/T_0 \in (0, 0.133)$ and $(0.569, 0.718)$ and for model 2 when $T/T_0 \in (0.005, 0.017)$, $(0.039, 0.321)$ and $(0.603, 0.719)$.

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