An example of noncommutative deformations

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Abstract
We compute the noncommutative deformations of a family of modules over the first Weyl algebra. This example shows some important properties of noncommutative deformation theory that separates it from commutative deformation theory.

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1 Introduction
Let $k$ be an algebraically closed field and let $A$ be an associative $k$-algebra. For any left $A$-module $M$, there is a flat commutative deformation functor

$$\text{Def}_M : l \to \text{Sets}$$

defined on the category $l$ of local Artinian commutative $k$-algebras with residue field $k$. We recall that for an object $R \in l$, a flat deformation of $M$ over $R$ is a pair $(M_R, \tau)$, where $M_R$ is an $A$-$R$ bimodule (on which $k$ acts centrally) that is $R$-flat, and $\tau : M_R \otimes_R k \to M$ is an isomorphism of left $A$-modules. Moreover, $(M_R, \tau) \sim (M'_R, \tau')$ as deformations in $\text{Def}_M(R)$ if there is an isomorphism $\eta : M_R \to M'_R$ of $A$-$R$ bimodules such that $\tau = \tau' \circ (\eta \otimes 1)$.

Laudal introduced noncommutative deformations of modules in Laudal [2]. For any finite family $M = \{M_1, \ldots, M_p\}$ of left $A$-modules, there is a noncommutative deformation functor

$$\text{Def}_M : a_p \to \text{Sets}$$

defined on the category $a_p$ of $p$-pointed Artinian $k$-algebras. We recall that an object $R$ of $a_p$ is an Artinian ring $R$, together with a pair of structural ring homomorphisms $f : k^p \to R$ and $g : R \to k^p$, such that $g \circ f = \text{id}$ and the radical $I(R) = \ker(g)$ is nilpotent. The morphisms of $a_p$ are ring homomorphisms that commute with the structural morphisms.

A deformation of the family $M$ over $R$ is a $(p + 1)$-tuple $(M_R, \tau_1, \ldots, \tau_p)$, where $M_R$ is an $A$-$R$ bimodule (on which $k$ acts centrally) such that $M_R \cong (M_i \otimes_k R_{ij})$ as right $R$-modules, and $\tau_i : M_R \otimes_R k_i \to M_i$ is an isomorphism of left $A$-modules for $1 \leq i \leq p$. By definition,

$$(M_i \otimes_k R_{ij}) = \bigoplus_{1 \leq i, j \leq p} M_i \otimes_k R_{ij}$$

with the natural right $R$-module structure, and $k_1, \ldots, k_p$ are the simple left $R$-modules of dimension one over $k$. Moreover, $(M_R, \tau_1, \ldots, \tau_p) \sim (M'_R, \tau'_1, \ldots, \tau'_p)$ as deformations in $\text{Def}_M(R)$ if there is an isomorphism $\eta : M_R \to M'_R$ of $A$-$R$ bimodules such that $\tau_i = \tau'_i \circ (\eta \otimes 1)$ for $1 \leq i \leq p$.\/
There is a cohomology theory and an obstruction calculus for $\text{Def}_M$, see Laudal [2] and Eriksen [1]. We compute the noncommutative deformations of a family $M = \{M_1, M_2\}$ of modules over the first Weyl algebra using the constructive methods described in Eriksen [1].

2 An example of noncommutative deformations of a family

Let $k$ be an algebraically closed field of characteristic 0, let $A = k[t]$, and let $D = \text{Diff}(A)$ be the first Weyl algebra over $k$. We recall that $D = k(t, \partial)/(\partial t - t \cdot \partial - 1)$. Let us consider the family $M = \{M_1, M_2\}$ of left $D$-modules, where $M_1 = D/D \cdot \partial \cong A$ and $M_2 = D/D \cdot t \cong k[\partial]$.

We shall compute the noncommutative deformations of the family $M$.

In this example, we use the methods described in Eriksen [1] to compute noncommutative deformations. In particular, we use the cohomology $\text{YH}^n(M_j, M_i)$ of the Yoneda complex

$$\text{YC}^p(M_j, M_i) = \prod_{m \geq 0} \text{Hom}_D(L_{m,j}, L_{m-p,i})$$

for $1 \leq i, j \leq 2$, where $(L_{s,i}, d_{s,i})$ is a free resolution of $M_i$, and an obstruction calculus based on these free resolutions. We recall that $\text{YH}^n(M_j, M_i) \cong \text{Ext}_D^n(M_j, M_i)$.

Let us compute the cohomology $\text{YH}^n(M_j, M_i)$ for $n = 1, 2, 1 \leq i, j \leq 2$. We use the free resolutions of $M_1$ and $M_2$ as left $D$-modules given by

$$0 \leftarrow M_1 \leftarrow D \xleftarrow{\partial} D \leftarrow 0$$
$$0 \leftarrow M_2 \leftarrow D \xleftarrow{\partial} D \leftarrow 0$$

and the definition of the differentials $\text{YC}^0(M_j, M_i) \rightarrow \text{YC}^1(M_j, M_i) \rightarrow \text{YC}^2(M_j, M_i) = 0$ in the Yoneda complex, and obtain

$$\text{YH}^1(M_1, M_1) \cong \text{Ext}_D^1(M_1, M_1) = 0 \quad \text{YH}^1(M_1, M_2) \cong \text{Ext}_D^1(M_1, M_2) = k \cdot \xi_{21}$$
$$\text{YH}^1(M_2, M_1) \cong \text{Ext}_D^1(M_2, M_1) = k \cdot \xi_{12} \quad \text{YH}^1(M_2, M_2) \cong \text{Ext}_D^1(M_2, M_2) = 0$$

The base vector $\xi_{ij}$ is represented by the 1-cocycle given by $D \xrightarrow{1} D$ in $\text{YC}^1(M_j, M_i)$ when $i \neq j$. Since $\text{YC}^2(M_j, M_i) = 0$ for all $i, j$, it is clear that $\text{YH}^2(M_j, M_i) \cong \text{Ext}_D^2(M_j, M_i) = 0$ for $1 \leq i, j \leq 2$.

We conclude that $\text{Def}_M$ is unobstructed. Hence, in the notation of Eriksen [1], the pro-representing hull $H$ of $\text{Def}_M$ is given by

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \cong \begin{pmatrix} k[[s_{12}s_{21}]] & \langle s_{12} \rangle \\ \langle s_{21} \rangle & k[[s_{21}s_{12}]] \end{pmatrix}$$

where $\langle s_{12} \rangle = H_{11} \cdot s_{12} \cdot H_{22}$ and $\langle s_{21} \rangle = H_{22} \cdot s_{21} \cdot H_{11}$.

In order to describe the versal family $M_H$ of left $D$-modules defined over $H$, we use $M$-free resolutions in the notation of Eriksen [1]. In fact, the $D$-$H$ bimodule $M_H$ has an $M$-free resolution of the form

$$0 \leftarrow M_H \leftarrow \begin{pmatrix} D \otimes_k H_{11} & D \otimes_k H_{12} \\ D \otimes_k H_{21} & D \otimes_k H_{22} \end{pmatrix} \xrightarrow{d^H} \begin{pmatrix} D \otimes_k H_{11} & D \otimes_k H_{12} \\ D \otimes_k H_{21} & D \otimes_k H_{22} \end{pmatrix} \leftarrow 0$$

where $d^H = (\partial \otimes e_1 - (-1) \otimes s_{12} - (-1) \otimes s_{21} + (t) \otimes e_2$. This means that for any $P, Q \in D$, we have that $d^H(P \otimes e_1) = (P \cdot \partial) \otimes e_1 - (P \cdot 1) \otimes s_{21}$ and $d^H(Q \otimes e_2) = (Q \cdot t) \otimes e_2 - (Q \cdot 1) \otimes s_{12}$.

We remark that there is a natural algebraization $S$ of the pro-representation hull $H$, given by

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \cong \begin{pmatrix} k[[s_{12}s_{21}]] & \langle s_{12} \rangle \\ \langle s_{21} \rangle & k[[s_{21}s_{12}]] \end{pmatrix}$$

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In other words, $S$ is an associative $k$-algebra of finite type such that the $J$-adic completion $\hat{S} \cong H$ for the ideal $J = (s_{12}, s_{21}) \subseteq S$. The corresponding algebraization $M_S$ of the versal family $M_H$ is given by the M-free resolution

$$0 \leftarrow M_S \leftarrow \left( \begin{array}{cc} D \otimes_k S_{11} & D \otimes_k S_{12} \\ D \otimes_k S_{21} & D \otimes_k S_{22} \end{array} \right) \xrightarrow{d^S} \left( \begin{array}{cc} D \otimes_k S_{11} & D \otimes_k S_{12} \\ D \otimes_k S_{21} & D \otimes_k S_{22} \end{array} \right) \leftarrow 0$$

with differential $d^S = (\partial) \otimes e_i - (i+1) \otimes s_{12} - (i) \otimes s_{21} + (i) \otimes e_2$.

We shall determine the $D$-modules parameterized by the family $M_S$ over the noncommutative algebra $S$ — this is much more complicated than in the commutative case. We consider the simple left $S$-modules as the points of the noncommutative algebra $S$, following Laudal [3], [4]. For any simple $S$-module $T$, we obtain a left $D$-module

$$M_T = M_S \otimes_S T$$

Therefore, we consider the problem of classifying simple $S$-modules of dimension $n \geq 1$.

Any $S$-module of dimension $n \geq 1$ is given by a ring homomorphism $\rho : S \rightarrow \text{End}_k(T)$, and we may identify $\text{End}_k(T) \cong M_n(k)$ by choosing a $k$-linear base $\{v_1, \ldots, v_n\}$ for $T$. We see that $S$ is generated by $e_1, s_{12}, s_{21}$ as a $k$-algebra (since $e_2 = 1 - e_1$), and there are relations

$$s_{12}^2 = s_{21}^2 = 0, \quad e_1^2 = e_1, \quad e_1 s_{12} = s_{12}, \quad s_{21} e_1 = s_{21}, \quad s_{12} e_1 = e_1 s_{21} = 0$$

Any $S$-module of dimension $n$ is therefore given by matrices $E_1, S_{12}, S_{21} \in M_n(k)$ satisfying the matrix equations

$$S_{12}^2 = S_{21}^2 = 0, \quad E_1^2 = E_1, \quad E_1 S_{12} = S_{12} E_1 = S_{21} E_1 = S_{21}, \quad S_{12} E_1 = E_1 S_{21} = 0$$

The $S$-modules represented by $(E_1, S_{12}, S_{21})$ and $(E_1', S_{12}', S_{21}')$ are isomorphic if and only if there is an invertible matrix $G \in M_n(k)$ such that $G E_1 G^{-1} = E_1', \quad G S_{12} G^{-1} = S_{12}', \quad G S_{21} G^{-1} = S_{21}'$.

Using this characterization, it is a straightforward but tedious task to classify all $S$-modules of dimension $n$ up to isomorphism for a given integer $n \geq 1$.

Let us first remark that for any $S$-module of dimension $n = 1$, $\rho$ factorizes through the commutatization $k^2$ of $S$. It follows that there are exactly two non-isomorphic simple $S$-modules of dimension one, $T_{1,1}$ and $T_{1,2}$, and the corresponding deformations of $M$ are

$$M_{1,i} = M_S \otimes_S T_{1,i} \cong M_i \quad \text{for } i = 1, 2$$

This reflects that $M_1$ and $M_2$ are rigid as left $D$-modules.

We obtain the following list of $S$-modules of dimension $n = 2$, up to isomorphism. We have used that, without loss of generality, we may assume that $E_1$ has Jordan form:

1. $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
2. $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
3. $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
4. $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
5. $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
6. $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S_{21} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ for $a \in k^*$
We shall write $T_{2,1} - T_{2,5}$ and $T_{2,6,a}$ for the corresponding $S$-modules of dimension two. Notice that $T_{2,6,a}$ is simple for all $a \in k^*$, while $T_{2,1} - T_{2,5}$ are extensions of simple $S$-modules of dimension one. The corresponding deformations of $\mathcal{M}$ are given by

$$M_{2,6,a} = M_S \otimes_S T_{2,6,a} \quad \text{for } a \in k^*$$

In fact, one may show that $M_{2,6,a} \cong D/D \cdot (t\partial - a)$ for any $a \in k^*$. In particular, $M_{2,6,a}$ is a simple $D$-module if $a \notin \mathbb{Z}$, and in this case $M_{2,6,a} \cong M_{2,6,b}$ if and only if $a - b \in \mathbb{Z}$. Furthermore, $M_{2,6,-1} \cong D/D \cdot \partial t$, $M_{2,6,n} \cong M_1$ for $n = 1, 2, \ldots$, and $M_{2,6,-n} \cong M_2$ for $n = 2, 3, \ldots$.

We obtain the following list of $S$-modules of dimension $n = 3$, up to isomorphism. We have used that, without loss of generality, we may assume that $E_1$ has Jordan form:

|   | $E_1$ | $S_{12}$ | $S_{21}$ |
|---|-------|---------|---------|
| 1 | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 2 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 3 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 4 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 5 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 6 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 7 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$ for $b \in k^*$ |
| 8 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 9 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 10 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 11 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ |
| 12 | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$ for $c \in k^*$ |
We shall write $T_{3,1} - T_{3,6}$, $T_{3,7,2}$, $T_{3,8} - T_{3,11}$, and $T_{3,12,c}$ for the corresponding $S$-modules of dimension three. Notice that all $S$-modules of dimension three are extensions of simple $S$-modules of dimension one and two, so there are no simple $S$-modules of dimension $n = 3$.

We remark that if $T$ is a simple $S$-module, then $\rho : S \to \text{End}_k(T)$ is a surjective ring homomorphism. Hence it seems unlikely that there are any simple $S$-modules of dimension $n > 3$.

Finally, we remark that the commutative deformation functor $\text{Def}_M : 1 \to \text{Sets}$ of the direct sum $M = M_1 \oplus M_2$ has pro-representing hull $(H = k[[s_{12}, s_{21}]], M_H)$, and an algebraization $(S = k[s_{12}, s_{21}], M_S)$. It is not difficult to find the family $M_S$ in this case. In fact, for any point $(\alpha, \beta) \in \text{Spec} S = \mathbb{A}_k^2$, the left $D$-module $M_{\alpha, \beta} = M_S \otimes_S S/(s_{12} - \alpha, s_{21} - \beta)$ is given by

$$M_{0,0} \cong M_1 \oplus M_2$$
$$M_{\alpha,0} \cong D/D \cdot (\partial t) \quad \text{for } \alpha \neq 0$$
$$M_{\alpha,\beta} \cong D/D \cdot (t \partial - \alpha \beta) \quad \text{for } \beta \neq 0$$

We see that we obtain exactly the same isomorphism classes of left $D$-modules as commutative deformations of $M = M_1 \oplus M_2$ as we obtained as noncommutative deformations of the family $\mathcal{M} = \{M_1, M_2\}$. However, the points of the pro-representing hull

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \cong \left( k[[s_{12}, s_{21}]] \langle s_{12} \rangle \langle s_{21} \rangle k[[s_{12}, s_{21}]] \right)$$

of the noncommutative deformation functor $\text{Def}_\mathcal{M}$ give a much better geometric picture of the local structure of the moduli space of left $D$-modules.

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