INVARINTS OF PURE 2-DIMENSIONAL SHEAVES INSIDE THREEFOLDS AND MODULAR FORMS

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WITH AN APPENDIX WRITTEN JOINTLY WITH RICHARD THOMAS

ABSTRACT. Motivated by S-duality modularity conjectures in string theory, we study the Donaldson-Thomas type invariants of pure 2-dimensional sheaves inside a nonsingular threefold $X$ in three different situations:

1. $X$ is a K3 fibration over a curve. We study the Donaldson-Thomas invariants defined in [Tho00] of the 2-dimensional Gieseker stable sheaves in $X$ supported on the fibers. Analogous to the Gromov-Witten theory formula established in [MP07], we express these invariants in terms of the Euler characteristic of the Hilbert scheme of points on the K3 surface and the Noether-Lefschetz numbers of the fibration, and prove that the invariants have modular properties.

2. $X$ is the total space of the canonical bundle of $\mathbb{P}^2$. We study the generalized Donaldson-Thomas invariants defined in [ST11] of the moduli spaces of the 2-dimensional Gieseker semistable sheaves on $X$ with first Chern class equal to $k$ times the class of the zero section of $X$. When $k = 1, 2, 3$, and semistability implies stability, we express the invariants in terms of known modular forms. We prove a combinatorial formula for the invariants when $k = 2$ in the presence of the strictly semistable sheaves, and verify the BPS integrality conjecture of [ST11] in some cases.

3. (Joint with Richard Thomas) $X$ is a Calabi-Yau threefold and $L$ is a sufficiently positive line bundle. We define new invariants counting a restricted class of 2-dimensional torsion sheaves, enumerating pairs $Z \subset H$ in $X$. Here $H$ is a member of the linear system $|L|$ and $Z$ is a 1-dimensional subscheme of it. The associated sheaf is the ideal sheaf of $Z \subset H$, pushed forward to $X$ and considered as a certain Joyce-Song pair in the derived category of $X$. We express these invariants in terms of the MNOP invariants of $X$ [MNOP06].

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Date: May 9, 2013.
0. Introduction

0.1. Overview. We study the invariants virtually counting the configurations of a number of points and a vector bundle supported on the members of a system of divisors inside a nonsingular threefold. One of our motivations is that these invariants have been studied by the physicists [GSY07, GY07, OSV01] as a set of supersymmetric BPS invariants associated to D4-D2-D0 systems. By string theoretic considerations the generating series of these invariants are expected to be modular.

In this paper we interpret these invariants in terms of the moduli spaces of pure coherent sheaves with 2-dimensional support inside a smooth threefold $X$. Another motivation for considering pure 2-dimensional sheaves is to find a sheaf-theoretic analogue of the formulas proven in [MP07] that relate the Gromov-Witten invariants of a threefold to the Gromov-Witten invariants of a system of its divisors.

Purity of a 2-dimensional coherent sheaf $\mathcal{F}$ means that all the nonzero subsheaves of $\mathcal{F}$ have 2-dimensional supports. To construct a moduli space for the pure sheaves (possibly with additional structures) one needs a notion of stability. For the objects in this article we consider two types of stability: Gieseker stability and the stability of pairs [Pot93, JS11].

Definition 0.1. Let $(X, \mathcal{O}(1))$ be a smooth threefold with a polarization. For a pure sheaf $\mathcal{F}$ the Hilbert polynomial is defined to be $P_{\mathcal{F}}(m) = \chi(X, \mathcal{F}(m))$, and the reduced Hilbert polynomial of $\mathcal{F}$ is

$$p_{\mathcal{F}} = \frac{P_{\mathcal{F}}}{\text{leading coefficient of } P_{\mathcal{F}}}.$$
• \( \mathcal{F} \) is called Gieseker semistable if for any subsheaf \( \mathcal{G} \subset \mathcal{F} \) we have 
  \[ p_G(m) \leq p_F(m) \quad \text{for} \quad m \gg 0. \]
• \( \mathcal{F} \) is called Gieseker stable if the equality never holds for any proper subsheaf \( \mathcal{G} \).

For a fixed \( n \gg 0 \) a pair \((\mathcal{F}, s)\), where \( s \) is a nonzero section of \( \mathcal{F}(n) \), is called stable if
\begin{enumerate}
  \item \( \mathcal{F} \) is Gieseker semistable,
  \item if \( s \) factors through a proper subsheaf \( \mathcal{G}(n) \) then 
  \[ p_G(m) < p_F(m) \quad \text{for} \quad m \gg 0. \]
\end{enumerate}

Suppose now that \( \mathcal{M} \) is a proper moduli space of Gieseker stable sheaves \( \mathcal{F} \) or the moduli space of stable pairs \((\mathcal{F}, s)\) as above with fixed Hilbert polynomial \( P_F \). \( \mathcal{M} \) is usually singular and may have several components with different dimensions. To define (deformation invariant) invariants as integration over \( \mathcal{M} \) we need to have a virtual fundamental class of the moduli space constructed by means of a perfect obstruction theory on \( \mathcal{M} \). This can be obtained by studying the deformations and obstructions of the stable sheaves or the stable pairs [BF97, LT98, Tho00, HT10].

One of the richest examples of a case that the moduli space of sheaves over the threefold \( X \) admits a perfect obstruction theory is the moduli space of ideal sheaves of 1-dimensional subschemes in \( X \) which can be identified with a component of the Hilbert scheme of curves in \( X \) ([MNOP06, Tho00]). When \( X \) is a Calabi-Yau threefold, then the obstruction theory on the Hilbert scheme is symmetric and the corresponding invariants are expressible as a weighted Euler characteristic of the moduli space [Beh09]. The invariants are denoted by \( I_{\beta,n} \) depending on a curve class \( \beta \) in \( X \) and an integer \( n \) keeping track of the holomorphic Euler characteristic of the ideal sheaves. These invariants conjecturally determine all the Gromov-Witten invariants of \( X \) (see [MNOP06]). Neither set of invariants are expected to have modular properties for general \( X \).

Given this overview, we study the invariants arising from the pure 2-dimensional sheaves in three different situations depending on the geometry of \( X \) and what stability condition we impose. We are able to prove the modularity predicted by physicists in some cases. In the rest of this section we explain each situation separately and state our main results.

### 0.2. DT invariants of 2-dimensional sheaves

Let \( X \) be a smooth projective threefold over \( \mathbb{C} \) with a fixed polarization \( L \). Let \( P(m) \) be a degree 2 polynomial. We consider the moduli space \( \mathcal{M} = \mathcal{M}(X; P) \) of Gieseker semistable sheaves with Hilbert polynomial \( P \). By the assumption on the degree of \( P \), the support of any \( \mathbb{C} \)-point of \( \mathcal{M} \) is 2-dimensional. It is proven in [Tho00] that \( \mathcal{M} \) admits a perfect obstruction theory if the following conditions hold true:
\begin{itemize}
  \item There are no strictly semistable sheaves with Hilbert polynomial \( P \).
  \item \( \text{Ext}^3(\mathcal{F}, \mathcal{F})_0 = 0 \) for any stable sheaves with Hilbert polynomial \( P \)
    (the index 0 indicates the trace free part of \( \text{Ext}^3(\mathcal{F}, \mathcal{F}) \)). For stable
sheaves on any threefold with effective anti-canonical bundle this condition is satisfied by Serre duality.

**Definition 0.2.** [Tho00] Suppose that the conditions above hold true for $\mathcal{M}$. Then [Tho00] constructs a natural perfect obstruction theory $E^\bullet \to L^\bullet_{\mathcal{M}}$ such that

$$h^i(E^\bullet \vee) \cong \mathcal{E}xt^{-i+1}_{\pi_{\mathcal{M}}}(F, F)$$

for $i = 0, 1$. The corresponding virtual cycle $[\mathcal{M}]^{vir} = [\mathcal{M}, E^\bullet]^{vir}$ has dimension $d = \text{ext}^1(F, F) - \text{ext}^2(F, F)$. $d$ is called the virtual dimension of $\mathcal{M}$. The Donaldson-Thomas invariants are then defined by integrating a class $\alpha \in H^2(\mathcal{M}, \mathbb{Q})$ against $[\mathcal{M}]^{vir}$:

$$DT(X; P; \alpha) = \int_{[\mathcal{M}]^{vir}} \alpha.$$  

When the virtual dimension is zero, we define $DT(X; P) = DT(X; P; 1)$ to be the degree of $[\mathcal{M}]^{vir}$. The invariant $DT(X; P)$ is always an integer.

**Example 0.3.** (2-dimensional sheaves on $\mathbb{P}^3$) Let $X = \mathbb{P}^3$ with polarization $L = \mathcal{O}(1)$, and

$$P(m) = m^2/2 + (s + 3/2)m + d$$

for some $s, d \in \mathbb{Z}$. Then $\mathcal{M} = \mathcal{M}(\mathbb{P}^3; P)$ is the moduli space of semistable sheaves supported on the hyperplanes in $\mathbb{P}^3$. Denote the hyperplane class by $H$. Note that since $c_1(\mathcal{F}) = H$ is an irreducible class, then there are no strictly semistable sheaves with Hilbert polynomial $P$. Moreover, since $K_X = -4H$, Serre duality and the stability of $\mathcal{F}$ imply that $\text{Ext}^3(\mathcal{F}, \mathcal{F}) = 0$, and hence $\mathcal{M}$ is equipped with a perfect obstruction theory. In Section 1 we show that the virtual dimension of $\mathcal{M}$ is always 3 independent of $s$ and $d$. Furthermore, if $\alpha$ is the pull back of the class of a point from $(\mathbb{P}^3)^\vee$ (under the natural morphism $\mathcal{M} \to (\mathbb{P}^3)^\vee$ taking a sheaf to its support) then

$$DT(X; P; \alpha) = \int_{\text{Hilb}^n(\mathbb{P}^2)} c_{top}(\mathcal{E}),$$

where $\text{Hilb}^n(\mathbb{P}^2)$ is the Hilbert scheme of $n = s(s + 3)/2 - d + 1$ points on $\mathbb{P}^2$ and $\mathcal{E}$ is a locally free sheaf of rank $2n$ on $\text{Hilb}^n(\mathbb{P}^2)$. We will give a combinatorial formula for this integral in Section 1.

In case $X$ is a $K3$ fibration over a smooth curve, we study the moduli space $\mathcal{M}(X; P)$ of sheaves supported on the $K3$ fibers. Motivated by the ideas of [MP07] in Gromov-Witten theory, we express $DT(X; P)$ in terms of the Euler characteristic of the Hilbert scheme of points on the $K3$ surface and the Noether-Lefschetz numbers of the fibration [MP07, Bor99, Bor98, KM90]. Using the degeneration techniques of Wu and Li [LW11] and deformation invariance of the DT invariants, we extend this formula to the case where finitely many of the $K3$ fibers have rational double point singularities. In Section 2 we put the invariants $DT(X; P)$ into a generating function $Z(X, q)$ by letting the formal variable $q$ keep track of the constant
term of $P$. We completely determine the generating function $Z(X, q)$ and write it in terms of some known modular forms:

**Theorem 1.** Let $X$ be a K3 fibration over a smooth curve with at most finitely many of the fibers having nodal singularities. Let $\mathcal{M}(X; P)$ be a proper moduli space of Gieseker stable sheaves supported on the fibers. Assuming that the linear term of $P$ is nonzero then

$$Z(X; q) = \frac{\Phi(q)}{2\eta(q)^2},$$

where $\Phi(q)$ is a vector valued modular form keeping track of the Noether-Lefschetz numbers of a nonsingular model of $X$, and $\eta(q)$ is the Dedekind Eta function.

In the future we plan to use similar degeneration techniques to find the DT invariants of quintic Calabi-Yau threefolds (and other Calabi-Yau complete intersections) corresponding to 2-dimensional sheaves supported on hyperplane sections following ideas similar to those in [MP06a, PP12]. We hope to find a more general explanation of the modularity of the DT invariants of the quintic predicted by physicists [GSY07, GY07, OSV01].

0.3. **Generalized DT invariants of 2-dimensional sheaves.** Let $X$ be a Calabi-Yau threefold and let $P(m)$ be a degree 2 polynomial. If the moduli space of Gieseker semistable sheaves $\mathcal{M}(X; P)$ (as defined in Section 0.2) contains strictly semistable sheaves then one cannot define the invariants of $\mathcal{M}(X; P)$ by the methods of Section 0.2. Joyce and Song [JS11] instead define $\mathbb{Q}$-valued invariants for $\mathcal{M}(X; P)$ called the *generalized DT invariants* $\overline{DT}(X; P)$ by considering the contributions of strictly semistable sheaves using sophisticated *stack functions*. The generalized DT invariant is specialized to $\overline{DT}(X; P)$ of Section 0.2 if there are no strictly semistable sheaves and moreover, $\overline{DT}(X; P)$ is also deformation invariant.

We study in detail the case where $X$ is the total space of the canonical bundle of $\mathbb{P}^2$ and $\mathcal{M}(X; P)$ is the moduli space of semistable sheaves with Hilbert polynomial $P(m) = rm^2/2 + \ldots$. Any semistable sheaf $\mathcal{F}$ with Hilbert polynomial $P$ is (at least set theoretically) supported on the zero section of $X$, and $c_1(\mathcal{F})$ is equal to $r$ times the class of the zero section. We relate $\overline{DT}(X; P)$ to the topological invariants of the moduli space of torsion-free semistable sheaves on $\mathbb{P}^2$. Using the wall-crossing formula of Joyce-Song [JS11] and the toric methods of [Per04, Koo08] we find a formula for $\overline{DT}(X; P)$ when $r = 2$ in the presence of strictly semistable sheaves. To express the result, let $\mathcal{M}(\mathbb{P}^2; P)$ be the moduli spaces of Gieseker semistable rank 2 sheaves on $\mathbb{P}^2$ with Hilbert polynomial $P$ and let $\mathcal{M}^s(\mathbb{P}^2; P)$ be the open subset of stable sheaves. Denote by $\text{Hilb}^n(\mathbb{P}^2)$ the Hilbert scheme of $n$ points on $\mathbb{P}^2$. Then we prove

**Theorem 2.** Let $P(m) = m^2 + 3m + b$ where $b \in 2\mathbb{Z}$ then

$$\overline{DT}(X; P) = \chi(\text{Hilb}^{-b/2}(\mathbb{P}^2))/4 - \chi(\mathcal{M}^s(\mathbb{P}^2; P)) - C(b)$$
where \( C(b) \) is a combinatorial expression depending only on \( b \) and is defined in Section 3 (see Definition 3.7).

0.4. **Ideal sheaves on members of a linear system.**

Let \( X \) be a projective Calabi-Yau threefold such that \( H^1(X, \mathcal{O}_X) = 0 \), and fix a complete linear system \(|H|\) on \( X \) which is sufficiently positive in the sense of Definition B.2. Let \( Z \subset H \) be a 1-dimensional subscheme of a member of this linear system. The ideal sheaf of \( Z \) is a torsion-free rank 1 sheaf \( I_Z \) on \( H \). Letting \( i : H \hookrightarrow X \) be the natural inclusion, \( i_*I_Z \) is a pure 2-dimensional sheaf in \( X \). Suppose that \( \text{ch}_2(i_*\mathcal{O}_Z) = \beta \) and \( \chi(\mathcal{O}_Z) = n \), and let \( \mathcal{H} \to |H| \) be the universal hyperplane. Then there exists a relative Hilbert scheme

\[
\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)
\]

parameterizing these pairs \( Z \subset H \) in \( X \). We would like to consider it as parameterizing the coherent sheaves \( i_*I_Z \), but they may not be Gieseker stable (if \( H \) is reducible or not reduced) and there may be deformations of \( i_*I_Z \) which are not themselves pushforwards of ideal sheaves.

Notice that \( i_*I_Z \) is the cokernel of the map

\[
\mathcal{O}(-H) \xrightarrow{s_H} \mathcal{I}_Z,
\]

where \( s_H \) is the section of \( \mathcal{O}(H) \) vanishing on \( H \), and \( \mathcal{I}_Z \) is the ideal sheaf of \( Z \) when considered as a subscheme of \( X \) (rather than \( H \)). Therefore \( i_*I_Z \) is quasi-isomorphic to the complex

\[
I^\bullet := \mathcal{O}(-H) \xrightarrow{s_H} \mathcal{I}_Z,
\]

where we put \( \mathcal{I}_Z \) in degree 0. If we consider \( s_H \) to be a section of \( \mathcal{I}_Z(H) \) then \((\mathcal{I}_Z, s_H)\) is an example of a Joyce-Song pair [JS], and it is automatically stable as a pair in the sense of Definition 0.1. So pair stability defines a different notion of stability on the sheaves \( i_*I_Z \) that the complexes \( I^\bullet \) are quasi-isomorphic to, with respect to which all of these torsion sheaves are stable.

For \( H \) sufficiently ample, Joyce and Song show by deformation theory that the moduli space of their stable pairs is a locally complete moduli space of objects \( I^\bullet \) of \( D(X) \). In our situation this means that \( \text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \) is a locally complete moduli space of sheaves \( i_*I_Z \). In other words, the sheaves \( i_*I_Z \) do not deform to sheaves of a different form, for \( H \) sufficiently ample.

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1. This section is joint work with Richard Thomas.
2. In the sense of [HT10, Section 4.1].
**Theorem 3.** Let $X$ be a nonsingular projective Calabi-Yau threefold such that $H^1(X, O_X) = 0$. Given a curve class $\beta \in H_2(X; \mathbb{Z})$ and $n \in \mathbb{Z}$, let $H$ be sufficiently positive with respect to $\beta, n$ in the sense of Definition B.2. Then

\[ \text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \]

is a locally complete moduli space of torsion sheaves on $X$. It admits a symmetric perfect obstruction theory and so a virtual cycle of virtual dimension 0.

We can therefore define an invariant

\[ N_{\beta,n}^H := \int_{\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)^{vir}} 1. \]

Since the obstruction theory is symmetric, by [Beh09] this can also be written as a weighted Euler characteristic of $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$. The invariants $N_{\beta,n}^H$ are closely related to the MNOP invariants $I_{\beta,n}$ counting ideal sheaves $\mathcal{I}_Z$ of $X$.

**Theorem 4.** Under the conditions of Theorem 3 we have

\[ N_{\beta,n}^H = (-1)^{c-1} c \cdot I_{\beta,n}, \]

where $c = c(\beta, n, H, X)$ is the topological number

\[ \int_X \left( \frac{1}{6} H^3 + H^2 \text{td}_2(X) - H \cdot \beta \right) - n. \]

**Example 0.4.** Most of the examples worked out explicitly in [GY07, Sections 2.1 to 2.6] fit into the setting of this section and hence, the corresponding invariants are captured by the invariants $N_{\beta,n}^H$. As an illustration, we work out the following two simple cases for linear hyperplane sections $H$ of the quintic Calabi-Yau threefold $X$.

- $\mathcal{I}_p$ is the ideal sheaf of a point $p \in X$, then $\beta = 0$, and $n = 1$, and hence
  \[ c = \text{td}_2(X) \cdot H + H^3/6 - 1 = 25/6 + 5/6 - 1 = 4. \]
  Also $I_{0,1} = -\chi(X) = 200$, and hence by $N_{0,1}^H = (-1)^3 \times 4 \times 200 = -800$.
- $\mathcal{I}_C$ is the ideal sheaf of a line $C \subset X$, so $\beta \cdot H = 1$, $n = 1$, and hence
  \[ c = 25/6 + 5/6 - 1 = 3. \]
  Also $I_{1,1} = 2875$, the number of rational degree 1 curves in a generic quintic, and hence $N_{1,1}^H = (-1)^2 \times 3 \times 2875 = 8625$.

**ACKNOWLEDGMENT**

We would like to thank Kai Behrend, Vincent Bouchard, Patrick Brosnan, Jim Bryan, Tudor Dimofte, Daniel Huybrechts, Dagan Karp, Sheldon Katz, Martijn Kool, Jun Li, Davesh Maulik, Yukinobu Toda and particularly Richard Thomas for many helpful discussions. The second author would
like to thank the University of British Columbia as well as Max Planck Institute für Mathematik for their hospitality during the time that this article was being completed. Special thanks go to Isaac Newton Institute for Mathematical Sciences in University of Cambridge for their hospitality during the time that the second author spent in 2010-2011 and was introduced to this project.

1. PROOF OF FORMULA \([1]\)

In this section we continue with the set up and notation of Example 0.3. We first prove (1), and then use torus localization to evaluate the integral appearing in (1). For any \(C\)-point \(F\) of \(M = \mathcal{M}(\mathbb{P}^3, P)\) we know that \(c_1(F) = H\). Therefore, by Hirzebruch-Riemann-Roch formula

\[
\chi(F, F) = \int_{\mathbb{P}^3} \text{ch}^\vee(F) \cdot \text{ch}(F) \cdot \text{td}(\mathbb{P}^3) = \\
\int_{\mathbb{P}^3} (H - \text{ch}_2(F) + \text{ch}_3(F)) \cdot (H + \text{ch}_2(F) + \text{ch}_3(F)) \cdot (1 + 2H + \ldots) = \\
\int_{\mathbb{P}^3} -2H^3 = -2.
\]

and hence the virtual dimension of \(M\) is 3, independent of \(s\) and \(d\). We denote the tangent and the obstruction sheaves of \(M\) by \(T\) and \(\text{Ob}\) respectively.

We have a natural morphism \(\pi: M \to (\mathbb{P}^3)^\vee \cong \mathbb{P}^3\) which, at the level of \(C\)-points, sends a sheaf \(F\) to its support. For any \(x \in \mathbb{P}^3\) the fiber of \(\pi\) over \(x\) is isomorphic to the Hilbert scheme of \(n\) points on \(\mathbb{P}^2\), denoted by \(\text{Hilb}^n(\mathbb{P}^2)\), where

\[
n = s(s + 3)/2 - d + 1.
\]

A \(C\)-point \(\mathcal{F}\) in this fiber is then the push forward of \(I(s) := I \otimes \mathcal{O}_{\mathbb{P}^2}(s)\) via the inclusion of the hyperplane \(\mathbb{P}^2 \hookrightarrow \mathbb{P}^3\), where \(I\) is the ideal sheaf of some length \(n\) subscheme of \(\mathbb{P}^2\). The fibers of \(T\) and \(\text{Ob}\) over the \(C\)-point \(\mathcal{F}\) fit into the exact sequence (see [Tho00, Lemma 3.42])

\[
0 \to T_{\mathcal{F}} \text{Hilb}^n(\mathbb{P}^2) \to T_{\mathcal{F}} \to \text{Hom}(I, I(1)) \to 0 \to \text{Ob}_{\mathcal{F}} \to \mathcal{E}_{\mathcal{F}} \to 0,
\]

where \(\mathcal{E}\) is a locally free sheaf of rank \(2n\) over \(\text{Hilb}^n(\mathbb{P}^2)\) whose fiber over \(I\) is identified with \(\text{Ext}^1_{\mathbb{P}^2}(I, I(1))\). From this description one can see that \(M\) is smooth of dimension \(2n + 3\) and moreover, \(\text{Ob}\) is locally free of rank \(2n\). So by [BF97, Proposition 5.6] we have

\[
[M]^{vir} = c_2n(\text{Ob}) \cap [M].
\]

Now if \([pt]\) is the class of a point on \(\mathbb{P}^3\) then

\[
DT(\mathbb{P}^3; P; \pi^*[pt]) = \int_{[M]^{vir}} \pi^*[pt] = \\
= \int_{[M]} c_2n(\text{Ob}) \cup \pi^*[pt] = \int_{[\text{Hilb}^n(\mathbb{P}^2)]} c_2n(\mathcal{E}).
\]
The last integral can be evaluated by localization method. \( C^{*2} \) acts on \( \mathbb{P}^2 \) via
\[
(t_1, t_2) \cdot (x_1 : x_2 : x_3) = (x_1 : t_1 x_2 : t_2 x_3).
\]
It is well-known that (see [ES87, Nak99]) under the induced action on \( \text{Hilb}^n(\mathbb{P}^2) \) the fixed point set is in bijection with the set of triples of 2d partitions as follows:
\[
\text{Fix}_n = \{ \Pi = (\pi_1, \pi_2, \pi_3) | \sum \# \pi_i = n \}.
\]
Now given a 2d partition \( \pi \), suppose \( b \in \pi \) is a box in the Young diagram associated to \( \pi \). Let \( a(b), l(b) \) be respectively the number of boxes above \( b \) and on the right side of \( b \) in the Young diagram of \( \pi \). Now given a fixed point \( Q \) corresponding to \( (\pi_1, \pi_2, \pi_3) \), the \( C^{*2} \)-character of the tangent space \( T_Q \text{Hilb}^n(\mathbb{P}^2) \) is given by
\[
\sum_{b \in \pi_1} t_1^{l(b)+1} t_2^{a(b)} + t_1^{-l(b)} t_2^{a(b)+1} +
\sum_{b \in \pi_2} t_1^{a(b)-l(b)-1} t_2^{a(b)} + t_1^{l(b)-a(b)-1} t_2^{a(b)+1} +
\sum_{b \in \pi_3} t_2^{a(b)-l(b)-1} t_1^{a(b)} + t_2^{l(b)-a(b)-1} t_1^{a(b)+1}.
\]

By Atiyah-Bott localization formula, the contribution of the fixed point \( Q \) corresponding to \( \Pi \in \text{Fix}_n \) to the integral \( \int_{\text{Hilb}^n(\mathbb{P}^2)} c_{2n}(\mathcal{E}) \) is
\[
C(\Pi) = \prod_{b \in \pi_2} \frac{((a - l - 2)s_1 - as_2)((l - a - 2)s_1 + (a + 1)s_2)}{((a - l - 1)s_1 - as_2)((l - a - 1)s_1 + (a + 1)s_2)} \times
\prod_{b \in \pi_3} \frac{((a - l - 2)s_2 - as_1)((l - a - 2)s_2 + (a + 1)s_1)}{((a - l - 1)s_2 - as_1)((l - a - 1)s_2 + (a + 1)s_1)},
\]
where \( s_i = c_1(t_i) \), and \( a = a(b), l = l(b) \).
So we have shown
\[
DT(\mathbb{P}^3; P; \pi^*[pt]) = \int_{\text{Hilb}^n(\mathbb{P}^2)} c_{2n}(\mathcal{E}) = \sum_{\Pi \in \text{Fix}_n} C(\Pi).
\]
Using this formula one can easily evaluate
\[
\int_{\text{Hilb}^1(\mathbb{P}^2)} c_2(\mathcal{E}) = 7,
\]
\[
\int_{\text{Hilb}^2(\mathbb{P}^2)} c_4(\mathcal{E}) = 35.
\]

2. Threefolds fibered over curves and proof of Theorem

Let \( C \) be a nonsingular projective curve and \( X \) be a nonsingular threefold that admits a projective surjective morphism \( \pi : X \to C \) with irreducible fibers. By Bertini’s theorem general fibers of \( \pi \) are nonsingular. In other words, we allow finitely many fibers to be possibly singular or nonreduced.

We denote by \( F \) the class of the general fibers of \( \pi \), and we fix a polarization \( L \) on \( X \) such that \( c_1(L) \) is a primitive class, and we let \( \ell = FL^2 \).

Suppose \( \mathcal{F} \) is a coherent sheaf on \( X \) with \( \text{ch}(\mathcal{F}) = (0, r, \gamma, \omega) \), then the Hilbert polynomial of \( \mathcal{F} \) is given by
\[
(3) \quad P(m) = (r\ell/2)m^2 + L(\gamma + rFc_1/2)m + \omega + c_1\gamma/2 + rF(c_1^2 + c_2)/12,
\]
where \( c_1 \) and \( c_2 \) are the first and second Chern classes of \( X \). We consider \( \mathcal{M} = \mathcal{M}(X; P) \), the moduli space of semistable 2-dimensional coherent sheaves with Hilbert polynomial \( P \). Any \( C \)-point of \( \mathcal{M} \) corresponds to an \( S \)-equivalence class of a coherent sheaf \( \mathcal{F} \) which is (at least set theoretically) supported on the fiber(s) of \( \pi \). We can be more specific with the scheme theoretic description of the support if \( \mathcal{F} \) turns out to be stable:

**Lemma 2.1.** Suppose that \( \mathcal{F} \) is a stable sheaf on \( X \) with \( c_1(\mathcal{F}) = rF \), then the support of \( \mathcal{F} \) is reduced and connected.

**Proof.** The support of \( \mathcal{F} \) is connected since \( \mathcal{F} \) is stable. To see the support of \( \mathcal{F} \) is reduced, denote by \( S \) the support of \( \mathcal{F} \) with the reduced induced structure, and let \( I_S \) be the ideal sheaf of \( S \) in \( X \). We have \( F = k[S] \) for some positive integer \( k \) because the fibers of \( X \) are irreducible. We also have that \( I_S|_S \) is a torsion invertible sheaf of order at most \( k \). Therefore, since \( c_1(\mathcal{F}) = rF \), we see that \( \mathcal{F} \) and \( \mathcal{F} \otimes I_S \) have the same Hilbert polynomial. Tensoring the short exact sequence \( 0 \to I_S \to \mathcal{O}_X \to \mathcal{O}_S \to 0 \) by \( \mathcal{F} \) we get the exact sequence
\[
\mathcal{F} \otimes I_S \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_S \to 0.
\]
By the stability of \( \mathcal{F} \) and \( \mathcal{F} \otimes I_S \), the first map in the sequence above is either an isomorphism or the zero map (see [HL97] Proposition 1.2.7). But the former is not possible, so we conclude that the second map in the sequence above is an isomorphism, and this finishes the proof. \( \Box \)

**Assumption 2.2.** (Condition on the Hilbert polynomial) Let \( P \) be the Hilbert polynomial of a sheaf \( \mathcal{F} \) with \( c_1(\mathcal{F}) = rF \). We assume throughout this section
that there are no strictly semistable sheaves with Hilbert polynomial $P$. \[ M = \mathcal{M}(X; P) \] is then the moduli space of stable sheaves supported on the fibers of $\pi$.

**Assumption 2.3.** (Condition on the canonical bundle $K_X$) We assume throughout this section that the restriction of $K_X$, the canonical bundle of $X$, to the fibers of $\pi$ is trivial. This implies that for any sheaf $\mathcal{F}$ with $c_1(\mathcal{F}) = rF$ we have $\mathcal{F} \otimes K_X \cong \mathcal{F}$ as $\mathcal{F}$ is supported on a fiber of $\pi$. Serre duality then implies that

\[ \text{Ext}^3(\mathcal{F}, \mathcal{F}) \cong \text{Hom}(\mathcal{F}, \mathcal{F})^\vee \quad \text{and} \quad \text{Ext}^2(\mathcal{F}, \mathcal{F}) \cong \text{Ext}^1(\mathcal{F}, \mathcal{F})^\vee. \]

By Assumptions 2.2 and 2.3 $\text{Ext}^3(\mathcal{F}, \mathcal{F})_0 = 0$ over $M(X; P)$. This ensures that $M(X; P)$ admits a perfect obstruction theory and hence a 0-dimensional virtual cycle $[M]^{\text{vir}}$ (see [Tho00]). The corresponding DT invariant is then defined to be the degree of this cycle:

\[ \text{DT}(X; P) = \int_{[M]^{\text{vir}}} 1. \]

**Notation.** We denote by $\pi_M : X \times M \to M$ and $\pi_X : X \times M \to X$ the natural projections. Let $\rho : M \to \mathbb{C}$ be the natural morphism which, at the level of $\mathbb{C}$-points, sends the sheaf $\mathcal{F}$ with support $S$ to $\pi(S) \in \mathbb{C}$. The morphism $\rho$ is well-defined by Assumption 2.2 and Lemma 2.1. Define

\[ S = X_{\pi \times \rho} M \overset{i_S}{\hookrightarrow} X \times M. \]

Then the universal sheaf $\mathcal{F}^\bullet$ is the push forward of a rank $r$ torsion-free sheaf $G$ supported on $S$ i.e., $\mathcal{F} = i_S \ast G$. We denote by $\pi_M'$ and $\pi_X'$ the compositions of $i_S$ with $\pi_M$ and $\pi_X$ respectively.

Using this notation we have

**Theorem 2.4.** ([Tho00]) There exists a perfect deformation-obstruction theory over $\mathcal{M}$ given by the following morphism in the derived category:

\[ E^\bullet := \left( \tau^{[1,2]} R\pi_{M!*} (R\mathcal{H}om_{X \times M}(\mathcal{F}, \mathcal{F}))^\vee \right)^{[-1]} \to \mathbb{I}_{\mathcal{M}}^\bullet. \]

**Example 2.5.** (The Enriques Calabi-Yau threefold [MP06b]) Let $Q = (S \times E)/\mathbb{Z}_2$ be the Enriques Calabi-Yau threefold obtained from $\mathbb{Z}_2$ acting by the Enriques involution on the K3 surface $S$ and by $-1$ on the elliptic curve $E$. The projection to the second factor then gives $\pi : Q \to \mathbb{P}^1$ which is a K3 fibration with four double Enriques fibers $T$ over $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$. $Q$ fits into the setting of this section. Since $Q$ is a Calabi-Yau threefold the obstruction theory above is symmetric.

---

3This is the case for example when $F$ is an irreducible class and $r = 1$.
4$\mathcal{F}$ may exist only as a twisted sheaf (see [Tho00] Section 3) for a relevant discussion).
If $r = 1/2$ then $\mathcal{M}(Q; P)$ is isomorphic to a union of the copies of $\text{Hilb}^n(T)$ for some $n$, and hence
\[
DT(Q; P) = 8 \sum_{h \in \mathbb{Z}} \sum_{\gamma \in H_2(X, \mathbb{Z}) \mid L \cdot \gamma = P(0)} \chi(\text{Hilb}^{h-P(0)}(T)).
\]

If $r = 1$ (recall that $P$ satisfies Assumption 2.2) then $\mathcal{M}(Q; P)$ is nonsingular at any point of $\rho^{-1}(p)$ where $p \neq p_i$. In fact, the general fibers of $\rho : \mathcal{M}(Q; P) \to \mathbb{P}^1$ are isomorphic to (a union of) $\text{Hilb}^n(S)$ for some $n$, and the fibers over $p_1, \ldots, p_4$ are isomorphic to the moduli space of rank 2 stable sheaves on $T$ with Hilbert polynomial $P$. Since $Q$ is a Calabi-Yau threefold we can write
\[
DT(Q; P) = -2 \sum_{h \in \mathbb{Z}} \sum_{\gamma \in H_2(X, \mathbb{Z}) \mid L \cdot \gamma = P(0)} \chi(\text{Hilb}^{h+1-P(0)}(S)) + 4\chi(\mathcal{M}(T; P), \nu_{\mathcal{M}(Q; P)}).
\]

Alternatively, one may use the degeneration and localization techniques (as in [MP06b]) to evaluate DT invariants of $Q$.

2.1. Smooth fibrations. We suppose that Assumptions 2.2 and 2.3 are both satisfied, and furthermore, we assume that the morphism
\[
\pi : X \to C
\]
is smooth. Then the fibers of $\pi$ are nonsingular projective surfaces with trivial canonical bundles (by Assumption 2.2). Suppose that the Hilbert polynomial $P$ satisfies Assumption 2.2. The moduli space $\mathcal{M}(X; P)$ can have several connected components:

Definition 2.6. We call a connected component $\mathcal{M}_c$ of $\mathcal{M} = \mathcal{M}(X; P)$ a type I component if $\rho(\mathcal{M}_c) = C$, otherwise we call $\mathcal{M}_c$ a type II component. If $\mathcal{M}_c$ is a type II component, then there exists a point $p \in C$ such that $\rho(\mathcal{M}_c) = p$; if the tangent sheaf of $\mathcal{M}_c$ is isomorphic to the tangent sheaf of $\mathcal{M}(\pi^{-1}(p); P)$ we call $\mathcal{M}_c$ an isolated type 2 component. In other words, an isolated type II component is isomorphic to the moduli space of rank $r$ stable sheaves with Hilbert polynomial $P$ on one of the fibers of $\pi$. We usually denote a type I component by $\mathcal{M}_0$ and an isolated type II component by $\mathcal{M}_{iso}$.

Mukai in [Muk84] proves that the moduli space of stable sheave on K3 or Abelian surfaces is smooth and irreducible. This immediately implies that

Lemma 2.7. If $\mathcal{M}_c$ is a type I or an isolated type II component of $\mathcal{M}(X; P)$ then $\mathcal{M}_c$ is smooth and irreducible.

In what follows we study the restriction of the virtual cycle of $\mathcal{M}$ (given by Theorem 2.4) to the type I and the isolated type II components. We know that for any fiber $S$ of $\pi$ and any sheaf $\mathcal{G}$ supported on $S$, $\text{Ext}_S^3(\mathcal{G}, \mathcal{G}) = 0$. 
This implies that a type I component $M_0$ of $\mathcal{M}$ admits a perfect $\rho$-relative obstruction theory. The following theorem proven in [Tho00, HT10] gives the natural $\rho$-relative perfect obstruction theory over a type I component $M_0$ (in the sense of [BF97]):

**Theorem 2.8.** There is a $\rho$-relative obstruction theory over a type I component $M_0$ given by the following morphism in the derived category:

$$\left(\tau^{\geq 1} R\pi_{M_0*}(R\mathcal{H}om_S(G,G))\right)^{\vee}[-1] \to \mathbb{L}_{M_0/C}^\bullet.$$ 

□

**Proposition 2.9.** The $\rho$-relative obstruction theory over a type I component $M_0$ given by Theorem 2.8 induces a perfect obstruction theory $F^\bullet_0 \to \mathbb{L}^\bullet_{M_0}$ over $M_0$.

**Proof.** Composing the morphism in the derived category in Theorem 2.8 with the second morphism in the exact triangle

$$\mathbb{L}^\bullet_{M_0} \to \mathbb{L}^\bullet_{M_0/C} \to \rho^*\omega_C[1],$$

we obtain a morphism in the derived category

$$\lambda : \left(\tau^{\geq 1} R\pi_{M_0*}(R\mathcal{H}om_S(G,G))\right)^{\vee}[-1] \to \rho^*\omega_C[1].$$

Now the two exact triangles below induce a morphism $g$ (the left vertical map) in the derived category.

\[
\begin{array}{ccc}
\text{Cone}(\lambda)[-1] & \longrightarrow & \left(\tau^{\geq 1} R\pi_{M_0*}(R\mathcal{H}om_S(G,G))\right)^{\vee}[-1] \longrightarrow \rho^*\omega_C[1] \\
\downarrow g & & \downarrow \text{id} \\
\mathbb{L}^\bullet_{M_0} & \longrightarrow & \mathbb{L}^\bullet_{M_0/C} \longrightarrow \rho^*\omega_C[1]
\end{array}
\]

Define $F^\bullet_0 := \text{Cone}(\lambda)[-1]$, then comparing the induced long exact sequences of cohomologies of the exact triangles in the first and second rows of diagram (2.1) proves that $g : F^\bullet_0 \to \mathbb{L}^\bullet_{M_0}$ defines a perfect deformation-obstruction theory for $M_0$. □

We use the following proposition to compare the obstruction theories given by Theorems 2.4 and 2.8 on a type I component $M_0$:

**Proposition 2.10.** There exists the following exact triangle in the derived category of $M_0$:

$$\tau^{\geq 1} R\pi_{M_0*} (i_S R\mathcal{H}om_S(G,G)) \to \tau^{[1,2]} R\pi_{M_0*} (R\mathcal{H}om_{X \times M_0}(F,F))$$

$$\to \tau^{-2} R\pi_{M_0*} (i_S R\mathcal{H}om_S(G,G \otimes \mathcal{O}_S(S))) [-1]$$

**Proof.** Applying the functor $R\mathcal{H}om(-,G)$ to the exact triangle

$$G \otimes \mathcal{O}_S(-S)[1] \to L\mathcal{L} \otimes i_S, G \to G,$$
we obtain
\[ R\mathcal{H}om_S(G, G) \rightarrow R\mathcal{H}om_S(Li^*_G i^*_S G, G) \rightarrow R\mathcal{H}om_S(G \otimes O_S(-S), G)[-1], \]
which by the adjoint property \( Li^*_G i^*_S \) gives
\[ i^*_S R\mathcal{H}om_S(G, G) \rightarrow R\mathcal{H}om_S(G \otimes O_S(S), G)[-1]. \]

The result follows by applying the functor \( R\pi_{M_0}^* \) and then truncating the last sequence above. Note that after applying \( R\pi_{M_0}^* \) to (5) the terms of the resulting exact triangle from left to right are respectively concentrated in degrees \([0, 2], [0, 3], \) and \([1, 3] \). Moreover, the 0th cohomologies of the first two terms, and the 3rd cohomologies of the last two terms are isomorphic. As a result, after truncation we still get an exact triangle. □

Theorem 2.4 and Proposition 2.9 produce the virtual cycles \([M_0, E^\bullet]^{vir}\) and \([M_0, F_0^\bullet]^{vir}\) (see [BF97]). The relation between these two cycles is given by

**Proposition 2.11.** Let \( M_0 \) be a type I component of \( \mathcal{M}(X; P) \) and let \( \text{Ob}_0 \) be the locally free sheaf on \( M_0 \) obtained by restricting \( \mathcal{E}xt^1_{\pi_M}(G, G(S)) \) to \( M_0 \) then
\[ [M_0, E^\bullet]^{vir} = c_{top}(\text{Ob}_0) \cap [M_0, F_0^\bullet]^{vir}. \]

**Proof.** By Proposition 2.10, \( \text{Ob}_0 \) fits into the short exact sequence of the vector bundle stacks over \( M_0 \) (see [BF97]):
\[ h^1 / h^0(F_0^\bullet) \xrightarrow{i} h^1 / h^0(E^\bullet) \xrightarrow{j} \text{Ob}_0. \]
Denote by \( 0_{\text{Ob}_0} \) and \( 0_{E^\bullet} \) the zero section embeddings of \( M_0 \). Then this gives the Cartesian diagram
\[
\begin{array}{ccc}
M_0 & \xrightarrow{0_{\text{Ob}_0}} & \text{Ob}_0 \\
\downarrow & & \downarrow j \\
h^1 / h^0(F_0^\bullet) & \xrightarrow{i} & h^1 / h^0(E^\bullet)
\end{array}
\]

Now if \( \mathfrak{c}_{M_0} \) is the intrinsic normal cone of \( M_0 \), from the diagram above, we get (see [Kre99], [BF97])
\[ [M_0, E^\bullet]^{vir} = 0_{E^\bullet}^* \mathfrak{c}_{M_0} = (i \circ 0_{F_0^\bullet})^* \mathfrak{c}_{M_0} = 0_{F_0^\bullet}^* \circ i^* \mathfrak{c}_{M_0} = c_{top}(\text{Ob}_0) \cap 0_{E^\bullet}^* \mathfrak{c}_{M_0} = c_{top}(\text{Ob}_0) \cap [M_0, F_0^\bullet]^{vir}. \]

For our later application we need to express \( c_{top}(\text{Ob}_0) \) in terms of the top Chern class of the tangent sheaf of \( M_0 \), where \( \text{Ob}_0 \) and \( M_0 \) are as in Proposition 2.10. The component \( M_0 \) is irreducible and smooth (see Lemma 2.7).
Lemma 2.12. Suppose that the dimension of the type I component \( \mathcal{M}_0 \) is \( n \) and \( \mathcal{T}_{\mathcal{M}_0/C} \), \( \mathcal{L}_0 \) and \( \mathcal{O}_b \) are given as above. Then we have the following relation in \( H^*(\mathcal{M}_0, \mathbb{Q}) \):

\[
c_{\text{top}}(\mathcal{O}_b) = c_n(\mathcal{T}_{\mathcal{M}_0/C}) + \sum_{j=1}^n A_j \cup (\rho^* \mathcal{L}_0)^j.
\]

for some \( A_j \in H^{n-j}(\mathcal{M}_0, \mathbb{Q}) \).

Proof. Grothendieck-Riemann-Roch formula gives:

\[
\begin{align*}
\chi & \left( \sum_{i=0}^2 (-1)^i \mathcal{E}xt^i_{\pi_{\mathcal{M}_0}}(G, G(S)) \right) \\
& = \pi_{\mathcal{M}_0*} \left( \chi(G)^{\vee} \cdot \chi(G) \cdot \chi(\mathcal{O}(S)) \cdot \pi_X \cdot \text{td}(X) \right) \\
& = \chi(\mathcal{E}xt_{\pi_{\mathcal{M}_0}}(G, G)) + \sum_{j=1}^n \frac{c_1(\rho^* \mathcal{L}_0)^j}{j!} \pi_{\mathcal{M}_0*} \left( \chi(G)^{\vee} \cdot \chi(G) \cdot \pi_X \cdot \text{td}(X) \right).
\end{align*}
\]

Note also that \( \chi(\mathcal{E}xt_{\pi_{\mathcal{M}_0}}(G, G(S))) \) and \( \chi(\mathcal{E}xt_{\pi_{\mathcal{M}_0}}(G, G)) \) are clearly polynomials in \( c_1(\rho^* \mathcal{L}_0) \) for \( i = 0, 2 \). The lemma is proven by an inductive argument on \( n \).

The type I component \( \mathcal{M}_0 \) is smooth and irreducible (see Lemma 2.7), so in particular it admits a fundamental class \([\mathcal{M}_0]\). To find the relation between the virtual cycle \([\mathcal{M}_0, F^*_{0}]^{\text{vir}}\) and \([\mathcal{M}_0]\), we note that the obstruction sheaf

\[
h^1(F_{0}^{\vee}) \cong R^2\pi_{\mathcal{M}_0*}R\mathcal{H}\text{oms}(G, G)
\]

is an invertible sheaf on \( \mathcal{M}_0 \). In fact the trace map defines an isomorphism

\[
R^2\pi_{\mathcal{M}_0*}R\mathcal{H}\text{oms}(G, G) \xrightarrow{\text{tr}} R^2\pi_{\mathcal{M}_0*}\mathcal{O}_S,
\]

since by Nakayama lemma it is enough to show that \( \text{tr} \) gives isomorphism on the level of fibers over the closed points of \( \mathcal{M}_0 \), and (fiberwise) the trace map is the Serre duality isomorphism. By Grothendieck-Verdier duality,

\[
R^2\pi_{\mathcal{M}_0*}\mathcal{O}_S \cong (R^0\pi_{\mathcal{M}_0*}\omega_{\pi_{\mathcal{M}_0}})^{\vee}.
\]

By the adjunction formula \( \omega_{\pi_{\mathcal{M}_0}} \cong \pi_X^* K_X \otimes \mathcal{O}_S(S) \) and hence, by Assumption 2.3, we have

\[
R^2\pi_{\mathcal{M}_0*}\mathcal{O}_S \cong \rho^* K^{\vee}
\]

for some invertible sheaf \( K \) on \( C \). The fiber of \( K \) over a closed point \( c \in C \) can be identified with \( H^0(S, K_X|_S) \) where \( S = \pi^{-1}(c) \).

Now by \([BF97\text{ Proposition 5.6}]\) we have
Proposition 2.13. Suppose that $M_0$ is a type I component of $\mathcal{M}(X; P)$, and $K$ is the invertible sheaf on $C$ defined as above. Then we have

$$[M_0, F_0]^{vir} = c_1(\rho^*K^\vee) \cap [M_0].$$

□

Remark 2.14. In more technical terms $[M_0] = [M_0, F_0]^{vir}_{red}$, where the latter is the reduced virtual class obtained by reducing the obstruction theory $F_0$ (see [MPT10, KL10, MP07]).

We finish this section by studying what happens when we restrict the virtual cycle $[M, E^{\bullet}]^{vir}$ to an isolated type II component $M_{iso}$. By Lemma 2.7 $M_{iso}$ is smooth and irreducible, and is isomorphic to a moduli space of stable rank $r$ torsion-free sheaves on a fiber $\pi^{-1}(p)$ for some $p \in C$. Let $[M_{iso}]$ be the fundamental class of $M_{iso}$.

Proposition 2.15. Suppose that $M_{iso}$ is an isolated type II component of $\mathcal{M}(X; P)$, and let $T_{iso}$ denote the tangent sheaf of $M_{iso}$. Then

$$[M, E^{\bullet}]^{vir}|_{M_{iso}} = c_{top}(T_{iso}) \cap [M_{iso}].$$

Proof. Let $p = \rho(M_{iso}) \in C$. Applying $R\pi_{M_{iso}}^*$ to the exact triangle (5) in the proof of Proposition 2.10 and taking cohomology, we get the following exact sequence on $M_{iso}$:

$$\mathcal{H}om_{\pi_{M_{iso}}^*}(G, G) \to \mathcal{E}xt^2_{\pi_{M_{iso}}}(G, G) \to \mathcal{E}xt^2_{\pi_{M_{iso}}}(F, F) \to \mathcal{E}xt^1_{\pi_{M_{iso}}}(G, G) \to 0.$$

Since $M_{iso}$ is a type II isolated component, by definition, we obtain the isomorphisms of the tangent sheaves

$$\mathcal{E}xt^1_{\pi_{M_{iso}}}(G, G) \to \mathcal{E}xt^1_{\pi_{M_{iso}}}(F, F),$$

from which we conclude that the first map in the sequence above is injective and hence an isomorphism. Hence, the exact sequence above implies that

$$h^1(E^{\bullet \vee})|_{M_{iso}} \cong \mathcal{E}xt^2_{\pi_{M_{iso}}}(F, F) \cong \mathcal{E}xt^1_{\pi_{M_{iso}}}(G, G).$$

But the right hand side is isomorphic to the tangent sheaf $T_{iso}$ and hence a locally free sheaf. Now the proposition follows from [BF97] Proposition 5.6. □

2.2. Smooth K3 fibrations. In this section we assume that the fibers of the smooth fibration (4) are K3 surfaces. Let $i : S \to X$ be the inclusion of the fiber of $\pi$ over a closed point $p \in C$, and suppose that a $C$-point $F$ of $\mathcal{M}(X; P)$ is supported on $S$. Then $F$ is the push forward of a stable rank $r$ torsion-free sheaf $G$ on $S$ with $c_1(G) = \beta$ and $c_2(G) = \tau$. Since $c_1(S) = 0$ and $c_2(S) = 24$, formula (5) is simplified to

$$P(m) = (r\ell/2)m^2 + dm + \beta^2/2 - \tau + 2r.$$
We have $P'(0) = d = i^*L \cdot \beta = L \cdot \gamma$ where $\gamma = \text{ch}_2(F)$, and $P(0) = \chi(F) = \beta^2/2 - \tau + 2r$ with $\beta^2/2 - \tau = \text{ch}_3(F)$. Let

\begin{equation}
\mathbf{v} = (r, \beta, \beta^2/2 - \tau + r)
\end{equation}

be the corresponding Mukai vector and $\mathcal{M}(S; \mathbf{v})$ be the moduli space of semistable sheaves on $S$ with Mukai vector $\mathbf{v}$. Then $\mathcal{M}(S; P)$, the moduli space of stable coherent sheaves on $S$ with Hilbert polynomial $P$ (with respect to $i^*L$), is a finite disjoint union of the moduli spaces $\mathcal{M}(S; \mathbf{v})$ over all possible Mukai vectors giving rise to $P$. By our assumption on the Hilbert polynomial we conclude that each Mukai vector $\mathbf{v}$ is primitive and hence, $\mathcal{M}(S; \mathbf{v})$ is smooth of dimension

$$2 - \int_S \mathbf{v} \cdot \mathbf{v}^\vee = 2r\tau - (r - 1)\beta^2 - 2(r^2 - 1) = 2 + 2r^2 + \beta^2 - 2rP(0).$$

The geometry of the moduli space $\mathcal{M}(S; \mathbf{v})$ has been thoroughly studied [KY00, HL95, Muk84, KY11]. The following result has been proven in [Muk84] and [HL97, Section 6]:

**Proposition 2.16.** Let $S$ be a K3 surface and $\mathbf{v}$ a primitive Mukai vector as in (7). Then $\mathcal{M}(S; \mathbf{v})$ is deformation invariant to $\text{Hilb}^n(S)$, the Hilbert scheme of $n$ points in $S$, where

$$n = r\tau - (r - 1)\beta^2/2 - r^2 + 1.$$ 

In particular,

$$\chi(\mathcal{M}(S; \mathbf{v})) = \chi(\text{Hilb}^n(S; \mathbf{v})).$$

We denote by $\mathcal{M}(X; \text{ch}_v)$ the component of $\mathcal{M}(X; P)$ corresponding to the Chern character vector

\begin{equation}
\text{ch}_v = (0, rF, \gamma = i_\ast \beta, \omega := \beta^2/2 - \tau) \in \bigoplus_{i=0}^3 H^{2i}(X, \mathbb{Q})
\end{equation}

assigned to the primitive Mukai vector (7).

**Remark 2.17.** For our later use, we need to extend the construction of $\mathcal{M}(X; \text{ch}_v)$ to the case where the K3-fibration $\pi : X \to \mathbb{C}$ is possibly not projective. We consider the case where there are finitely many K3 fibers $S$ of $\pi$ for which $i^*L$ is only a quasi-polarization (see [MP07]). Furthermore, we assume that there is a finite open affine cover $U_j$ of $\mathbb{C}$ such that $\pi|_{U_j}$ is projective. In this case for the compactly supported Chern character vector $\text{ch}_v^{cs}$ assigned to the primitive Mukai vector (7) the moduli spaces $\mathcal{M}(U_j; \text{ch}_v^{cs})$ can be constructed using the fiberwise polarizations. $\mathcal{M}(U_j; \text{ch}_v^{cs})$ and $\mathcal{M}(U_{ij}; \text{ch}_v^{cs})$ are canonically isomorphic over the overlaps $U_i \cap U_j$, and so they can be patched together to give a proper scheme $\mathcal{M}(X; \text{ch}_v)$ over $\mathbb{C}$. Two constructions above obviously give the same result when $X$ is projective over $\mathbb{C}$. The perfect obstruction theories and the virtual
cycles over $\mathcal{M}(U;\mathrm{ch}_v)$ constructed in the last section are also glued together to give the corresponding virtual cycles over $\mathcal{M}(X;\mathrm{ch}_v)$; we define $DT(X;\mathrm{ch}_v)$ as before, using the properness of $\mathcal{M}(X;\mathrm{ch}_v)$.

Following [MP07], let

$$\mathcal{V} : R^2\pi_*(\mathbb{Z}) \to \mathbb{C}$$

be the rank 22 local system determined by the K3-fibration $\pi$. Let $\mathcal{H}^\mathcal{V}$ denote the $\pi$-relative moduli space of Hodge structures as in [MP07] Section 1.4]. There exists a section map $\sigma : C \to \mathcal{H}^\mathcal{V}$ which is determined by the Hodge structures of the fibers of $\pi$:

$$\sigma(p) = H^0(X, K_S) \in \mathcal{H}^\mathcal{V}_p.$$  

For any $\gamma \in H_2(X, \mathbb{Z})$ and $h \in \mathbb{Z}$ let

$$\mathcal{V}_p(h, \gamma) = \{0 \neq \beta \in \mathcal{V}_p|\beta^2 = 2h - 2, i_*(\beta) = \gamma\},$$

and $B_p(h, \gamma) \subseteq \mathcal{V}_p(h, \gamma)$ be the subset containing $\beta \in \mathcal{V}_p(h, \gamma)$ where $\beta$ is a $(1, 1)$ class on $S$ (the fiber over $p$). Then $B_p(h, \gamma)$ is finite by Proposition 1 in [MP07]. $B(h, \gamma) = \bigcup_p B(h, \gamma) \subset \mathcal{V}$ can be decomposed into

$$B_1(h, \gamma) \bigsqcup B_{II}(h, \gamma)$$

where the first component defines a finite local system $\epsilon : B_1(h, \gamma) \to \mathbb{C}$, and the second component is an isolated set. Let $M_\epsilon$ be the connected component of $\mathcal{M}(X;\mathrm{ch}_v)$ corresponding to the local system $\epsilon$, and let

$$DT(X;\epsilon) = \int_{[M_\epsilon, E^\bullet]^{vir}} 1$$

be the contribution of this component to $DT(X;\mathrm{ch}_v)$. Note that $M_\epsilon$ is a type I component of $\mathcal{M}(X;\mathrm{ch}_v)$ in the sense of Definition 2.6 and $[M_\epsilon, E^\bullet]^{vir}$ is the restriction of the virtual cycle given by Theorem 2.4 to this component. Denote by $[M_\epsilon, E^\bullet_\epsilon]^{vir}$ the virtual cycle over this component given by Proposition 2.9. By Propositions 2.11 and 2.13 we have

$$[M_\epsilon, E^\bullet_\epsilon]^{vir} = c_{top}(\text{Ob}_\epsilon) \cap [M_\epsilon, E^\bullet_\epsilon]^{vir},$$

$$[M_\epsilon, F^\bullet_\epsilon]^{vir} = c_1(\rho^*K^\mathcal{V}) \cap [M_\epsilon],$$

where $[M_\epsilon]$ is the fundamental class of $M_\epsilon$, and $\text{Ob}_\epsilon$ is the locally free sheaf on $M_\epsilon$ defined as in Proposition 2.11.

On the other hand, if $\alpha \in B_{II}(h, \gamma)$ supported on the fiber $S$ is a result of a transversal intersection of $\sigma(C)$ with a Noether-Lefschetz divisor, then the corresponding connected component $M_\alpha$ of $\mathcal{M}(X;\mathrm{ch}_v)$ is an isolated type II component in the sense of Definition 2.6 and is isomorphic to $\mathcal{M}(S;\mathcal{v})$.

5 In [MP07] Section 1.4] this is denoted by $M^\mathcal{V}$.

6 i.e. with the intersection multiplicity 1
Denoting by \( T_{M_\alpha} \) and \([M_\alpha]\) the tangent sheaf and the fundamental class of \( M_\alpha \), by Proposition 2.15 we have

\[
[M_\alpha, E^\bullet]^{\text{vir}} = c_{\text{top}}(T_{M_\alpha}) \cap [M_\alpha],
\]

where \([M_\alpha, E^\bullet]^{\text{vir}}\) is the restriction of the virtual cycle given by Theorem 2.4 to \( M_\alpha \). We define

\[
D_T(X; \alpha) = \int_{[M_\alpha, E^\bullet]^{\text{vir}}} 1
\]

the contribution of \( D_T(X; \alpha) \) to this component.

For any integer \( h \in \mathbb{Z} \) the Noether-Lefschetz number \( NL_{h, \gamma}^{\pi} \) was defined in [MP07] by intersecting \( \sigma(C) \) with the \( \pi \)-relative Noether-Lefschetz divisor in \( H^\nu \) associated to \( h \) and \( \gamma \). Informally, \( NL_{h, \gamma}^{\pi} \) is the number of the fibers \( S \) of \( \pi \) for which there exists a \((1, 1)\) class \( \beta \in H^2(S, \mathbb{Z}) \) such that

\[
\beta^2 = 2h - 2 \quad \text{and} \quad i_\ast \beta = \gamma.
\]

It is proven in [MP07] that \( NL_{h, \gamma}^{\pi} \) vanishes if \( h > 1 + d^2/2\ell \), and

\[
NL_{h, \gamma}^{\pi} = NL_{h+d+\ell/2, \gamma}^{\pi+\ell \cdot L}.
\]

The following theorem which is the main result of this section is a sheaf theoretic version of [MP07, Theorem 1]:

**Theorem 2.18.** For a smooth K3 fibration \( \pi : X \to C \) and the Chern Character vector \( \text{ch}_v \) as in (8) we have

\[
D_T(X; \text{ch}_v) = \sum_{h \in \mathbb{Z}} \chi(Hilb^{h-r_0-r_2}(S)) \cdot NL_{h, \gamma}^{\pi} - k\delta_0, \gamma \cdot \chi(Hilb^{1-r_0-r_2}(S)),
\]

where \( \delta_0, \gamma \) is the Kronecker delta function, and \( k \) is the degree of the invertible sheaf \( K \) on \( C \) given in (6).

**Proof.** The proof follows the same ideas as the proof of Theorem 1 in [MP07]. We compare the contributions of \( B_I \) and \( B_{II} \) to \( D_T(X; \text{ch}_v) \) and the Noether-Lefschetz numbers. Suppose that \( \epsilon \) is a local system giving rise to \( B_I(h, \gamma) \) and the Noether-Lefschetz numbers. We can then write

\[
D_T(X; \epsilon) = \int_{[M_\epsilon]} c_{\text{top}}(\text{Ob}_\epsilon) \cdot c_1(\rho^*K^\vee) = \int_{[M(S,v)]} c_{\text{top}}(T_{M(S,v)}) \cdot \int_{B_I(h, \gamma)} c_1(\rho^*K^\vee) = \chi(Hilb^{h-r_0-r_2}(S)) \cdot \int_{B_I(h, \gamma)} c_1(\rho^*K^\vee),
\]

where the first equality is because of (10), the second equality holds by Lemma 2.12 and the last equality is due to Proposition 2.16.
By virtue of (9), it is shown in the proof of Theorem 1 in [MP07] (pp 22-23) that

\[
\int_{B_1(h, \gamma)} c_1(K^\vee)
\]
gives the contribution of \(B_1(h, \gamma)\) to \(NL_{h, \gamma}^{\pi}\).

Next, suppose that \(\alpha \in B_{II}(h, \gamma)\). Using the deformation invariance of DT invariants and the intersection numbers, we may assume that the corresponding component \(M_\alpha\) is an isolated type II component after possibly a small analytic perturbation of the section \(\sigma\) (which locally turns a multiplicity \(n\) intersection into \(n\) transversal intersections). Once this is done, the contribution of \(\alpha\) to \(NL_{h, \gamma}^{\pi}\) is exactly 1, and moreover we can use (11) and Proposition 2.16 to deduce

\[
DT(X; \alpha) = \int_{M_\alpha} c_{\text{top}}(T_{M_\alpha}) = \chi(\text{Hilb}^{h-r\omega-r^2}(S)).
\]

The proof of theorem is completed by adding the correction term involving the Kronecker delta function to take into account the contributions of \(\beta = 0\) to \(DT(X; \text{ch}_P)\).

\[\square\]

**Corollary 2.19.** For a smooth projective K3 fibration \(\pi : X \to C\) and the Hilbert polynomial \(P\) satisfying Assumption 2.2 we have

\[
DT(X; P) = \sum_{h \in \mathbb{Z}} \sum_{\gamma \in H_2(X) \text{ s.t. } L \cdot \gamma = P(0)} \chi(\text{Hilb}^{2+h-rP(0)}(S)) \cdot NL_{h, \gamma}^{\pi} - \delta_{(h, \gamma)} \cdot k \cdot \chi(\text{Hilb}^{r+1-rP(0)}(S)).
\]

\[\square\]

Now define

\[
NL_{h,d}^{\pi} = \sum_{\gamma \in H_2(X) \text{ s.t. } L \cdot \gamma = d} NL_{h, \gamma}^{\pi}.
\]

Then the formula in Corollary 2.19 for \(d \neq 0\) can be rewritten as

\[
DT(X; (r\ell/2)m^2 + dm + c) = \sum_{h \in \mathbb{Z}} \chi(\text{Hilb}^{2+h-rc}(S)) \cdot NL_{h,d}^{\pi}.
\]

By (12) we get the following symmetry among the DT invariants of \(X\):

\[
DT(X; (r\ell/2)m^2 + dm + c) = DT(X; (r\ell/2)m^2 + (d + \ell)m + c'),
\]

where \(c' = c - (2d + \ell)/2r\), provided that \(c'\) is an integer, and both Hilbert polynomials satisfy Assumption 2.2. This is true for example in the case \(r = 1\).
2.3. Nodal K3 fibrations. In this section we aim to extend Theorem 2.18 to the case where the K3 fibration has finitely many ordinary double point singularities (ODP). Using the deformation invariance, finding the DT invariants of $X$, when the singularities of the fibers are of more general type of rational double points (RDP), may be reduced to this case. We assume throughout this section that $r = 1$, so all the pure sheaves are automatically Gieseker stable.

Let $s_1, \ldots, s_k \in X$ be the singular points of the fibers of $X$, and assume that there are $k'$ singular fibers in $X$ over $c_1, \ldots, c_{k'} \in C$ where $k'$ is a non-negative integer. Moreover, we assume that $s_i$ is an ordinary double point (nodal) singularity for any $i$, and the other fibers of $X$ are smooth $K3$ surfaces. If $k'$ is even, define $c_0 = c_1$ and if $k'$ is odd, define $c_0$ to be an arbitrary point of $C$ distinct from $c_1, \ldots, c_{k'}$. Define $e : \tilde{C} \to C$ to be the double cover of $C$ branched over the points $c_0, \ldots, c_{k'}$. It can be seen that $e^∗(X)$ is a threefold with the conifold singularities. Denote by $\tilde{X}$ its small resolution with the exceptional nonsingular rational curves $e_1, \ldots, e_k$, and let $\tilde{\pi} : \tilde{X} \to \tilde{C}$ be the induced morphism. The normal bundle of $e_i$ in $\tilde{X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ [Ati58]. Moreover, let $e_i : C_i \to C$ be a double cover of $C$ branched at $k + 2\{k/2\}$ generic points of $C$ when $t \neq 0$ and $C_0 = \tilde{C}$. Define $X_t = e'_t(X)$. Our plan is to relate the DT invariants of $\tilde{X}$ and $X_t$ which differ by the conifold transitions. As in GW theory [LY04], [LR01] this can be done using the degeneration techniques. It is possible that $\tilde{X}$ is no longer projective in which case we use the modifications of Remark 2.17 to define the DT invariants. See Appendix A for a review of the degeneration techniques in DT theory.

We use the degenerations of the threefolds $\tilde{X}$ and $X_t$ to respectively

$$X' := Y \bigcup_{D_1, \ldots, D_k} \prod_{i=1}^k \mathbb{P}^1$$

and

$$X'' := Y \bigcup_{D_1, \ldots, D_k} \prod_{i=1}^k \mathbb{P}^2$$

where $Y$ is the threefold obtained by blowing up $e^∗(X)$ at

$$e^{-1}(s_1), \ldots, e^{-1}(s_k)$$

with the exceptional divisors $D_1 \prod \cdots \prod D_k$ (each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$),

$$\mathbb{P}_1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^2),$$

and $\mathbb{P}_2$ is a quadric in $\mathbb{P}^4$. The first degeneration is the degeneration to the normal cone $[Fu98]$ in which $D_i \subset Y$ is attached to the divisor at infinity $H_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^2)$ in the $i$-th copy of $\mathbb{P}_1$. The second degeneration is called the semistable reduction of a conifold degeneration [LY04] in which $D_i \subset X'$ is attached to a smooth hyperplane section $H_2$ in the $i$-th copy of $\mathbb{P}_2$. We denote by $X_1 \to \mathbb{A}^1$ and $X_2 \to \mathbb{A}^1$ the total spaces of the first and second degenerations above. In other words, $X_1$ (respectively, $X_2$) is a nonsingular 4-fold with the central fiber isomorphic to $X'$ (respectively $X''$). Let $X_i \to \mathcal{C}$ be the corresponding stack of expanded degenerations (see Appendix A).
Let \( \{V_i\}_{i=1}^{k'} \) be an open affine covering of \( \tilde{C} \), where
\[
V_i = \tilde{C} - \{c_1, \ldots, c_i, \ldots, c_{k'}\}.
\]
Define the invertible sheaves \( L_t \) and \( \bar{L} \) on \( X_t \) and \( \bar{X} \) as follows:
\[
L_t = \bar{e}_t^\ast(L), \quad \text{and} \quad \bar{L} = \bar{e}^\ast(L) - \sum_{i=1}^{k} \bar{D}_i
\]
where \( \bar{e} : \bar{X} \to X \) is the natural morphism, and \( \bar{D}_i \) is a divisor on \( \bar{X} \) with
\[
e_i \cdot \bar{D}_i = -2 \quad \text{and} \quad e \cdot \bar{D}_i = 0
\]
for any other curve \( e \) on \( U_i = \bar{\pi}^{-1}(V_i) \). \( L_t \) defines a polarization on \( X_t \) and \( \bar{L} \) defines a polarization on \( U_i \) for each \( i = 1, \ldots, k' \).

We also denote by \( F \) the class of fibers in the fibrations \( X_t \to C_t, \bar{X} \to \bar{C} \), and \( Y \to \bar{C} \). We denote by \( \mathcal{M}(\bar{X}; \bar{P}) \) the moduli space of 2-dimensional sheaves \( \mathcal{F} \) supported on the fibers of \( \bar{X} \to \bar{C} \) such that \( c_1(\mathcal{F}) = F \), with Hilbert polynomial \( \bar{P} \) with respect to \( \bar{L} \). Note that \( \bar{L} \) is a polarization on \( U_i \)'s and \( \mathcal{M}(\bar{X}; \bar{P}) \) is constructed by gluing the moduli spaces \( \mathcal{M}(U_i; \bar{P}) \) as in Remark 2.17.

Fix an invertible sheaf \( L_1 \) on \( X_1 \) whose restriction to a general fiber is \( \bar{L} \) and to \( Y \) is
\[
L' := b^\ast L - \sum_{i=1}^{k} D_i,
\]
where \( b : Y \to X \) is the natural map. The invertible sheaf \( L' \) defines a polarization on \( Y \) and we let \( \mathcal{M}(Y/D_1, \ldots, D_{k'}; P') \) be the relative moduli space of stable sheaves \( \mathcal{F} \) with Hilbert polynomial \( P' \) (with respect to \( L' \)) as in Definition A.2.

Similarly, fix a relatively ample invertible sheaf \( L_2 \) on \( X_2 \) whose restriction to a general fiber is \( L_t \) and to \( Y \) is \( L' \). Let \( \mathcal{M}(X_i/C_i; P_i) \) be the relative moduli space of stable 2-dimensional coherent sheaves with Hilbert polynomial \( P_i \) with respect to \( L_i \) (see Definition A.2).

Note that the general fibers of \( X_1 \) are not necessarily projective and hence, the moduli space is constructed by gluing the relative moduli spaces over the open cover \( \{U_i\} \) as in Remark 2.17.

**Lemma 2.20.** i) Suppose \( P_i \) is such that for any \( C \)-point \( \mathcal{F} \) of \( \mathcal{M}(X_i/C_i; P_i) \)
we have \( c_1(\mathcal{F}) = F \). Then \( \mathcal{M}(X_i/C_i; P_i) \) admits a perfect obstruction theory relative to \( C \) and a virtual cycle of dimension 1.

ii) Suppose \( P' \) is such that for any \( C \)-point \( \mathcal{F} \) of \( \mathcal{M}(Y/D_1, \ldots, D_{k'}; P') \) we have \( c_1(\mathcal{F}) = F \). Then \( \mathcal{M}(Y/D_1, \ldots, D_{k'}; P') \) admits a perfect deformation-obstruction theory and a virtual cycle of dimension 0.

iii) Suppose \( P' \) is such that for any \( C \)-point \( \mathcal{F} \) of \( \mathcal{M}(Y/D_1, \ldots, D_{k'}; P') \) we have \( c_1(\mathcal{F}) = F - D_i - \cdots - D_{i'} \). Then \( \mathcal{M}(Y/D_1, \ldots, D_{k'}; P') \) admits a perfect deformation-obstruction theory with virtual cycle equal to 0.

**Proof.** i) For any \( \mathcal{F} \) as in the proposition we have \( \text{Ext}^2(\mathcal{F}, \mathcal{F})_0 = 0 \) by Serre duality and the stability of \( \mathcal{F} \) and noting that \( \mathcal{F} \cong \mathcal{F} \otimes \omega_{X_i/A_i} \) where
\[ \omega_{X/\mathbb{A}^1} \] is the relative dualizing sheaf. Now the claim follows from the explanation after Definition A.2.

ii) As in part i) it suffices to show that \( \text{Ext}^3(F, F)_0 = 0 \) for the \( \mathbb{C} \)-points \( F \) in \( \mathcal{M}(Y/D_1, \ldots, D_k; P') \). We know that \( K_Y \cong D_1 + \cdots + D_k \) and \( F \cdot D_i = 0 \). Since \( c_1(F) = F \) by assumption, Serre duality and the stability of \( F \) imply that
\[
\text{Ext}^3(F, F) \cong \text{Hom}(F, F) = C,
\]
and hence \( \text{Ext}^3(F, F)_0 = 0 \).

iii) For simplicity we assume that \( c_1(F) = F - D_1 \). As in previous parts we prove \( \text{Ext}^3(F, F)_0 = 0 \). By Serre duality
\[
\text{Ext}^3(F, F) = \text{Hom}(F, F(D_1)).
\]
But \( F \) is rank 1 supported on a smooth K3 surface by our assumption, and hence \( F \) is isomorphic to an ideal sheaf of points \( I \) twisted by a invertible sheaf \( L \). Therefore,
\[
\text{Hom}(F, F(D_1)) \cong \text{Hom}(I, I(\hat{e}_1)),
\]
where \( \hat{e}_1 \cong \mathbb{P}^1 \) is given as the intersection of \( D_1 \) with the proper transform of the fiber containing the curve \( e_1 \). Since \( \hat{e}_1 \) is a \(-2\)-curve on a K3 surface, we know that \( H^0(O(\hat{e}_1)) = C \) which implies that \( \text{Hom}(I, I(\hat{e}_1)) \cong C \) as required.

To prove the vanishing of the virtual cycle, we show that the virtual dimension is negative. By the Hirzebruch-Riemann-Roch calculation:
\[
\text{ext}^0(F, F) - \text{ext}^1(F, F) + \text{ext}^2(F, F) - \text{ext}^3(F, F)
= \int_Y ((F - D_1) + \beta + n)(-(F - D_1) + \beta - n)(1 - \frac{1}{2} \sum_i D_i + \ldots)
= (F - D_1)^2 \cdot D_1/2 = 1,
\]
from which we get
\[
\text{ext}^1(F, F) - \text{ext}^2(F, F) = -1.
\]

\[ \square \]

**Lemma 2.21.** For \( i = 1, 2 \) let \( F_i \) be a pure 2-dimensional sheaf on \( P_i \) with \( c_1(F_i) = H_i \), and let \( Q_i \) be the Hilbert polynomial of \( F_i \) with respect to any polarization. The relative moduli space of stable sheaves \( \mathcal{M}(P_i/H_i; Q_i) \) admits a perfect deformation-obstruction theory.

**Proof.** The canonical bundle of \( P_i \) is \( K_{P_i} \cong -3H_i \). So for any \( \mathbb{C} \)-point of \( \mathcal{M} \) given by a pure 2 dimensional sheaf \( F \) on \( P_i \), we have \( \text{Ext}^3(F, F) \cong \text{Hom}(F, F, \otimes K_{P_i}) = 0 \) by the stability of \( F \) and Serre duality as before. \[ \square \]

Now we are ready to state the conifold transition formula for the DT invariants:
Proposition 2.22. (Conifold Transition Formula) Suppose that \( P \in \mathbb{Q}[m] \) is a degree 2 polynomial with the leading coefficient equal to \( FL^2 / 2 \). The DT invariants of \( X_t \) and \( \bar{X} \) are related by the following formula

\[
DT(X_t; P_t = P) = DT(\bar{X}, \bar{P} = P)
\]

where \( P_t \) is the Hilbert polynomial with respect to \( L_t \), and \( \bar{P} \) is the Hilbert polynomial with respect to \( \bar{L} \).

Proof. By Lemma 2.20, \( \mathcal{M}(X_1 / \mathcal{E}, P_1) \) admits a perfect obstruction theory. Applying the degeneration formula (17) which is followed from the naturality of the virtual cycle \([\mathcal{M}(X_1 / \mathcal{E}; P_1)]^{\text{vir}}\) (see Appendix A), part ii) of Lemma 2.20 and Lemma 2.21, we can express \( DT(\bar{X}, \bar{P} = P) \) in terms of the relative DT invariants of \( X \) and \( P_1 \). There are two possibilities for a \( C \)-point \( \mathcal{F} \) on the central fiber of \( \mathcal{M}(X_1 / \mathcal{E}, P_1) \). Either \( \mathcal{F} \) is completely supported on \( Y \) (and possibly its degenerations) or there are some \( i_1, \ldots, i_g \) such that \( c_1(\mathcal{F}|_Y) = F - D_{i_1} - \cdots - D_{i_g} \), and \( c_1(\mathcal{F}|_{P_1}) = H_{j_i} \) where \( P_1^j \) is the \( i \)-th copy of \( P_1 \). Only the former case contributes because of the vanishing of the virtual cycle proven in part iii) of Lemma 2.20. Therefore,

\[
DT(\bar{X}; \bar{P} = P) = DT(Y / D_{1}, \ldots, D_{k}; P^j = P)
\]

Similarly, using the degeneration formula (17) followed from the naturality of the virtual cycle \([\mathcal{M}(X_2 / \mathcal{E}; P_2)]^{\text{vir}}\), we can express \( DT(X_t; P_t = P) \) in terms of the relative DT invariants of \( Y \) and \( P_2 \). Again by the same argument as in the last paragraph by distinguishing two similar cases and using Lemmas 2.20 and 2.21 we get

\[
DT(X_t; P_t = P) = DT(Y / D_1, \ldots, D_k; P^j = P)
\]

Now the lemma follows from the last two identities.

Now we choose \( k' \) generic fibers \( S_1, \ldots, S_{k'} \) of \( X \to \mathbb{C} \). By our assumption \( S_t \) is a K3 surface. Let \( \mathcal{M}(X / S_1, \ldots, S_{k'}; P) \) be the relative moduli space of stable sheaves \( \mathcal{F} \) on \( X \) with \( c_1(\mathcal{F}) = F \) and Hilbert polynomial \( P \) (with respect to \( L \)) as in Definition A.2. By our conditions, one can see similar to the proof of Lemma 2.20 that \( \mathcal{M}(X / S_1, \ldots, S_{k'}; P) \) admits a perfect deformation-obstruction theory and virtual class of dimension zero.

Let \( X_t = S_t \times \mathbb{P}^1 \). Then \( X_t \) is a smooth K3-fibration over \( \mathbb{P}^1 \). Once more, we denote the class of the fibers by \( F \). Let \( \mathcal{M}(X_t / S_t; Q) \) be the relative moduli space of stable sheaves \( \mathcal{F} \) on \( X \) with \( c_1(\mathcal{F}) = F \) and Hilbert polynomial \( Q \) with respect to any polarization.

Lemma 2.23. \( DT(X_t / S_t; Q) = 0 \).

Proof. The invertible sheaf \( K \) associated to the fibration \( X_t \) as in (6) is easily seen to be trivial in this case, and hence \( DT(X_t, Q) = 0 \). Next, by the
2.4. Modularity of the DT invariants. In this section we assume the set up and notation of Section 2.3. Theorems 2.18 together with modularity property of the generating series of the invariants \( \chi(\text{Hilb}^n(S)) \) [Got90] and \( N_{h,J}^{\pi} \) [MP07, Bor98, Bor99, KM90] suggest the modularity of the generating series of \( DT(X; P) \). To explore this we first put the DT invariants into a generating function. We assume that \( r = 1 \) throughout this section. Let

\[
P(m) = (\ell/2)m^2 + dm + c
\]

with \( c,d \in \mathbb{Z} \). Since \( r = 1 \) then \( P(m) \) satisfies Assumption 2.2 and then for any \( \mathbb{C} \)-point \( \mathcal{F} \) of \( \mathcal{M}(X; P) \) we have \( \text{ch}_1(\mathcal{F}) = F, L \cdot \text{ch}_2(\mathcal{F}) = d \), and \( \text{ch}_3(\mathcal{F}) = c - 2 \). Our convention is that when \( X \) is a smooth K3 fibration (i.e. when \( k = 0 \)) then \( \tilde{X} \) is taken to be the disjoint union of two copies of \( X \). Define the generating series

\[
Z_d(X, q) = q^{1+d^2/2\ell} \sum_{c \in \mathbb{Z}} DT(X; (\ell/2)m^2 + dm + c)q^{-c},
\]

and let

\[
Z(X, q) = \sum_{d=0}^{\ell-1} Z_d(X; q)v_d \in \mathbb{C}[[q^{1/2\ell}]] \otimes \mathbb{C}[\mathbb{Z}/\ell\mathbb{Z}].
\]
Following [MP07], for $NL_{h,d} = \sum_{\gamma \in H_1(X)} \sum_{L \cdot \gamma = d} NL_{h,d}$, define

$$\Phi_d(q) = q^{1+d^2/2\ell} \sum_{h \in \mathbb{Z}} NL_{h,d} q^{-h},$$

and set

$$\Phi(q) = \sum_{d=0}^{\ell-1} \Phi_d(q) v_d \in \mathbb{C}[\mathbb{C}[q^{1/2\ell}]] \otimes \mathbb{C}[\mathbb{Z}/4\ell\mathbb{Z}]$$

noting that $NL_{h,d} = 0$ if $h > 1 + d^2/2\ell$. Then $\Phi(q)$ is a vector valued modular form of weight $21/2$ [MP07, Bor98, Bor99, KM90]. By [Got90] we know

$$\sum_{n \geq 0} \chi(Hilb^n(S)) \cdot q^n = \prod_{n \geq 1} \frac{1}{(1-q^n)^{24}},$$

so

$$\sum_{n \geq 0} \chi(Hilb^n(S)) \cdot q^{n-1} = \eta(q)^{-24},$$

where $\eta(q)$ is the Dedekind Eta function which is a modular form of weight $1/2$.

Combining these formulas with the results of the Sections 2.2 and 2.3 we can express the generating function of our DT invariants in terms of the product of two series with modular properties:

$$Z(X,q) = \frac{\Phi(q) - k v_0}{2\eta(q)^{24}}.$$  \hfill (13)

The formula in Theorem 1 is a special case of (13).

**Example 2.26.** (Lefschetz pencil of quartics [MP07])

Let $\pi : X_{4,2} \subset \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^1$ be a general hypersurface of type $(4, 2)$. $X_{4,2}$ is then a smooth Calabi-Yau 3-fold, and $\pi$ is a family of K3 surfaces with 216 nodal fibers over $\mathbb{P}^1$. The generating series $Z(X_{4,2}, q)$ is then given by (13) for which $\Phi$ is explicitly evaluated in [MP07] in terms of the modular form $\Theta$:

$$\Theta = -1 + 108q + 320q^9 + 5016q^{32} + \cdots$$

appeared for the first time in [KKRS04, MP07].

On the other hand, since $X_{4,2}$ is a Calabi-Yau 3-fold, one can also evaluate the DT invariants by means of the weighted Euler characteristic:

$$DT(X; P) = \chi(M) = \chi(M, v_M)\chi(\mathcal{M}, v_M)$$

Let $c_1, \ldots, c_{216} \in \mathbb{C}$ be such that $\pi^{-1}(c_i)$’s are the singular fibers of $\pi$. We denote by $S$ the general smooth fiber and $S_0$ the nodal fiber. We have $M_0 :=$...
the moduli spaces of rank 1 torsion-free sheaves on the nodal K3 surface \( S_0 \) with Hilbert polynomial \( P \).

\[
\rho^{-1}(c_i) \cong \mathcal{M}(S_0; P),
\]

the moduli spaces of rank 1 torsion-free sheaves on the nodal K3 surface \( S_0 \) with Hilbert polynomial \( P \).

\[
DT(X; P) = \overline{\chi}(\mathcal{M} - \{\rho^{-1}(c_1), \ldots, \rho^{-1}(c_{216})\}, \mathcal{M}) + \sum_{i=1}^{216} \overline{\chi}(\rho^{-1}(c_i), \mathcal{M})
= \chi(\mathcal{M} - \{\rho^{-1}(c_1), \ldots, \rho^{-1}(c_{216})\}) + \sum_{i=1}^{216} \overline{\chi}(\rho^{-1}(c_i), \mathcal{M})
= -214 \sum_{h \in \mathbb{Z}} \sum_{\gamma \in \text{Hilb}_2(\mathbb{C})} \chi(\text{Hilb}_2^{h+2-P(0)}(S)) + 216\overline{\chi}(\mathcal{M}_0, \mathcal{M}).
\]

From this one can express the generating function of the weighted Euler characteristic of \( \mathcal{M}(S_0, P) \) in terms of the modular forms of (13).

3. PROOF OF THEOREM

In this section we study the generalized DT invariants of the Gieseker semistable 2-dimensional sheaves of local \( \mathbb{P}^2 \). When we mention (semi)-stability of sheaves, unless otherwise is specified, we always mean Gieseker (semi)stability.

Let \( X \) be the total space of \( \mathcal{O}(-3) \) over \( \mathbb{P}^2 \). Then \( X \) is a quasiprojective Calabi-Yau threefold, called local \( \mathbb{P}^2 \). Let \( L \) be the pullback of \( \mathcal{O}(1) \) from \( \mathbb{P}^2 \), and let \( S \cong \mathbb{P}^2 \subset X \) denote the zero section. We identify the compactly supported cohomology groups of \( X \) with the cohomology groups of \( \mathbb{P}^2 \):

\[
H^*_c(X, \mathbb{Q}) \cong H^{*+2}(\mathbb{P}^2, \mathbb{Q}).
\]

Using this identification, let \( h \in H^2_{\text{G}}(X, \mathbb{Q}), h \in H^2_{\text{G}}(X, \mathbb{Q}), pt \in H^0_{\text{G}}(X, \mathbb{Q}) \) be respectively the classes of \( S \), a line and a point on \( S \). The Hilbert polynomial (with respect to \( L \)) of a 2-dimensional compactly supported coherent sheaf \( \mathcal{F} \) on \( X \) with the compactly supported Chern character

\[
\text{ch}_{\text{G}}(\mathcal{F}) = (0, kH, (3k/2 + a)h, (3k/2 + 3a/2 + b)pt) \in \bigoplus_{i=0}^3 H^i_{\text{G}}(X, \mathbb{Q})
\]

is given by

\[
P(m) = \frac{k}{2} m^2 + \frac{3k + 2a}{2} m + \frac{3k + 2a + 2b}{2}.
\]

Any such \( \mathcal{F} \) is set theoretically supported on \( S \). Moreover, we have

**Lemma 3.1.** If \( \mathcal{F} \) as above is semistable then \( \mathcal{F} \) is scheme theoretically supported on \( S \) and hence

\[
\mathcal{M}(X; P) \cong \mathcal{M}(\mathbb{P}^2; P),
\]

the moduli space of rank \( k \) semistable sheaves on \( \mathbb{P}^2 \) with Hilbert polynomial \( P \).

**Proof.** The ideal sheaf of \( S \) in \( X \) is isomorphic to \( L^3 \), hence we get the exact sequence

\[
\mathcal{F} \otimes L^3 \to \mathcal{F} \to \mathcal{F}|_S \to 0.
\]

Since \( \mathcal{F} \) is semistable, the first morphism in the sequence above is necessarily zero and hence \( \mathcal{F} \cong \mathcal{F}|_S \).

\( \square \)
Note that for any stable torsion-free sheaf $\mathcal{F}$ on $\mathbb{P}^2$ we have $\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ by Serre duality and the negativity of $K_{\mathbb{P}^2} \cong \mathcal{O}(-3)$. Therefore, if $P(m)$ is such that there are no strictly semistable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $P(m)$, then the moduli space $\mathcal{M} = \mathcal{M}(X; P) \cong \mathcal{M}(\mathbb{P}^2; P)$ is unobstructed and smooth of dimension

$$\dim \mathcal{M} = 1 - \chi(\mathcal{F}, \mathcal{F}) = -2kb + a^2 - k^2 + 1,$$

where as a sheaf on $\mathbb{P}^2$

$$\text{ch}_0(\mathcal{F}) = k, \quad \text{ch}_1(\mathcal{F}) = a \cdot h, \quad \text{ch}_2(\mathcal{F}) = b \cdot pt.$$

In this case the Behrend’s function is determined by $v_\mathcal{M} = (-1)^{\dim \mathcal{M}} \text{Beh}[\mathcal{M}]$, and hence

$$DT(X; P) = (-1)^{\dim \mathcal{M}} \chi(\mathcal{M}).$$

The generating function for the Euler characteristic of the moduli space of stable torsion-free sheaves on $\mathbb{P}^2$ is known for $k = 1, 2, 3$, by the results of [Koo09, Kly91, Got90, Man10, VW94] and they all have modular properties. Here is the summary of these results:

1. $k = 1$. By tensoring with $\mathcal{O}(-a)$ we can assume that $a = 0$. So then $\mathcal{M}(X; P) \cong \text{Hilb}^{-b}(\mathbb{P}^2)$, the Hilbert scheme of $-b$ points on $\mathbb{P}^2$, which is smooth of dimension $-2b$. Note that in this case there are no strictly semistable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $m^2 + 3m/2 + b + 1$, so by [Got90]

$$\sum_b DT(X; m^2/2 + 3m/2 + b + 1)q^{-b} = \prod_{n>0} \frac{1}{(1 - q^n)^3}.$$  

2. $k = 2$. By tensoring with $\mathcal{O}([-a/2])$ we can assume that either $a = 0$ or $a = 1$. If $a = 1$ then there are no strictly semistable sheaves with the corresponding Hilbert polynomial $m^2 + 4m + 7/2 + b$ and hence $\mathcal{M}(X; m^2 + 4m + 7/2 + b)$ is smooth of dimension $-4b - 2$ so by [Koo09, Corollary 4.2]

$$\sum_{\substack{b \in (1/2)\mathbb{Z} \\ b \neq \lfloor b \rfloor}} DT(X; m^2 + 4m + 7/2 + b)q^{1/2 - b} = \chi(\mathcal{M}(X; m^2 + 4m + 7/2 + b))$$

$$= \frac{1}{\prod_{n \geq 0} (1 - q^n)^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q^{mn}.$$  

When $a = 0$, there are strictly semistable sheaves with Hilbert polynomial $m^2 + 3m + 2 + b$ only when $b \in 2\mathbb{Z}$. If $b \notin 2\mathbb{Z}$ then $\mathcal{M}$ is smooth of dimension $-4b - 3$. So for $b = 2b' + 1$

$$DT(X; m^2 + 3m + 3 + 2b') = -\chi(\mathcal{M}(\mathbb{P}^2; m^2 + 3m + 3 + 2b')).$$

We will study the case $b \in 2\mathbb{Z}$ in more detail in what follows in this section.
(3) $k = 3$. We can assume again that $a = 0, 1$ or 2. In the latter two cases $\mathcal{M}$ has no strictly semistable sheaves and there is a modular formula for the generating function $DT(X; P)$ in terms of the generating function of the Euler characteristics of $\mathcal{M}(\mathbb{P}^2; P)$ computed in [Koo09, Section 4.3].

In the following we compute $DT(X; P)$ in the presence of semistable sheaves when $k = 2$. By the discussion above, we only need to consider $a = 0$ and $b \in 2\mathbb{Z}$. Let $P(m) = m^2 + 3m + b$ be the corresponding Hilbert polynomial. We use the moduli space of stable pairs in the sense of [JS11] (see Definition 0.1).

For $n \gg 0$ let $\mathcal{P}_n = \mathcal{P}_n(X; P)$ be the moduli space of stable pairs $(\mathcal{F}, s)$ where $\mathcal{F}$ is a semistable sheaf of rank 2 with Hilbert polynomial $m^2 + 3m + b$, and $s$ is a nonzero section of $\mathcal{F}(n)$. The stability of pairs (see Definition 0.1) further requires that if $G \neq 0$ is a proper subsheaf of $\mathcal{F}$, such that $s$ factors through $G(n)$, then the Hilbert polynomial of $G$ is strictly less than the Hilbert polynomial of $\mathcal{F}$. By [JS11] $\mathcal{P}_n$ admits a symmetric perfect obstruction theory. Let $PI_n = PI_n(X; \beta)$ be the corresponding pair invariants.

Note that, even though $X$ is not proper, $\mathcal{P}_n$ is proper (as all the semistable sheaves are supported on $\mathbb{P}^2 \cong S \subset X$) so $PI_n$ is well defined. Alternatively, $PI_n = \chi(\mathcal{P}_n, \nu_{\mathcal{P}_n})$.

**Lemma 3.2.** $\overline{DT}(X; P) = DT(X; P/2)^2 \cdot P(n)/8 - PI_n(X; P)/P(n)$.

**Proof.** This is a direct corollary of the wall-crossing formula [JS11, 5.17] by noting two facts. Firstly, the only decomposable semistable sheaves with Hilbert polynomial $P$ are of the form $\mathcal{I}_{Z_1} \oplus \mathcal{I}_{Z_2}$ where $\mathcal{I}_{Z_1}$ and $\mathcal{I}_{Z_2}$ are the push forwards to $X$ of ideal sheaves of the 0-dimensional subschemes $Z_1, Z_2 \subset \mathbb{P}^2$ of length $-b/2$. Secondly, the Euler form $\chi(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2}) = 0$. □

**Remark 3.3.** Note the polynomial in the right hand side of Lemma 3.2 is a rational number independent of $n \gg 0$.

There is a natural morphism $\mathcal{P}_n \to \mathcal{M}$ that sends a stable pair $(\mathcal{F}, s)$ to the $S$-equivalence class of $\mathcal{F}$. Note that $\mathcal{M}$ is singular at a point corresponding to a strictly semistable sheaf. However, we have

**Proposition 3.4.** $\mathcal{P}_n(X; P)$ is a smooth scheme of dimension $P(n) - 4b - 4$.

**Proof.** We denote by $I^\bullet$ the 2-term complex $O(-n) \xrightarrow{s(-n)} \mathcal{F}$ corresponding to a stable pair $(\mathcal{F}, s)$. By the stability of pairs $\mathcal{F}$ has to be a semistable sheaf and hence Lemma 3.1 implies that $\mathcal{P}_n(X; P) \cong \mathcal{P}_n(\mathbb{P}^2; P)$, the moduli space of the stable pairs on $\mathbb{P}^2$. The Zariski tangent space and the obstruction space at a $C$-point $(F, s) \in \mathcal{P}_n$ are then identified with
\( \text{Hom}_{\mathbb{P}^2}(I^*, F) \) and \( \text{Ext}^1_{\mathbb{P}^2}(I^*, F) \) respectively. Consider the following natural exact sequence:

\[
0 \to \text{Hom}_{\mathbb{P}^2}(F, F) \to \text{Hom}_{\mathbb{P}^2}(\mathcal{O}(\mathcal{O}(-n), F) \to \text{Hom}_{\mathbb{P}^2}(I^*, F) \to \text{Ext}^1_{\mathbb{P}^2}(F, F) \\
\to \text{Ext}^1_{\mathbb{P}^2}(\mathcal{O}(\mathcal{O}(-n), F) \to \text{Ext}^1_{\mathbb{P}^2}(I^*, F) \to \text{Ext}^2_{\mathbb{P}^2}(F, F)
\]

Since \( n \gg 0 \), we have \( \text{Ext}^2_{\mathbb{P}^2}(\mathcal{O}(\mathcal{O}(-n), F) \cong H^1(\mathbb{P}^2, F(n)) = 0 \) for \( i > 0 \).

We also know that \( \text{Ext}^2_{\mathbb{P}^2}(F, F) = 0 \) by Serre duality and the semistability of \( F \). So the exact sequence above firstly implies that \( \text{Ext}^1(I^*, F) = 0 \) which means that \( \mathcal{P}_n \) is unobstructed and hence smooth, and secondly

\[
\dim \text{Hom}(I^*, F) = \chi_{\mathbb{P}^2}(\mathcal{F}(n)) - \chi_{\mathbb{P}^2}(\mathcal{F}, F) = P(n) - 4b - 4.
\]

By Proposition 3.4 and noting that \( P(n) \in 2\mathbb{Z} \), we have

**Corollary 3.5.** \( P_n(X; P) = \chi(\mathcal{P}_n(X; P)) = \chi(\mathcal{P}_n(\mathbb{P}^2; P)) \).

In the following we will find \( \chi(\mathcal{P}_n(\mathbb{P}^2; P)) \) using toric techniques. According to [Koo09], a torsion-free \( T = \mathbb{C}^* \)-equivariant sheaf \( F \) on \( \mathbb{P}^2 \) corresponds to three compatible \( \sigma \)-families, \( F_1, F_2, F_3 \) one for each of the standard \( T \)-invariant open subsets \( U_1, U_2, U_3 \) of \( \mathbb{P}^2 \). For any element \( m \) of the character group of \( T \) identified with \( \mathbb{Z}^2 \), \( F_i(m) = \Gamma(U_i, F)_m \), the eigenspace corresponding to \( m \) in the space of sections of \( F \) on \( U_i \). In this way, a \( T \)-equivariant rank 1 torsion-free sheaf \( I \) on \( \mathbb{P}^2 \) is determined by three integers \( u, v, w \) and three 2d partitions \( \pi_1', \pi_2', \pi_3' \). See the diagrams below indicating the corresponding \( \sigma \)-families \( I_1, I_2, I_3 \) over \( U_1, U_2, U_3 \) respectively:

![Diagram](image)

For any \( j = 1, 2, 3 \) we have \( I_j(m) = 0 \) if \( m \) is below the horizontal axis, on the left of the vertical axis, or inside the partition \( \pi_j' \). Otherwise, \( I_j(m) = \mathbb{C} \). In terms of these data we have

\[
\text{ch}_1(I) = -u - v - w,
\]

\[
\text{ch}_2(I) = (u + v + w)^2 / 2 - \#\pi_1' - \#\pi_2' - \#\pi_3'.
\]

Similarly, an indecomposable \( T \)-equivariant rank 2 torsion-free sheaf \( F \) on \( \mathbb{P}^2 \) up to a T-isomorphism is determined by an integer \( A \), positive integers \( \Delta_1, \Delta_2, \Delta_3 \), three distinct 1-dimensional subspaces \( p, q, r \subset \mathbb{C}^2 \), and six 2d partitions \( \pi_1, \pi_2, \pi_3 \) for \( j = 1, 2, 3 \). See the diagrams below indicating the corresponding \( \sigma \)-families \( F_1, F_2, F_3 \) on respectively \( U_1, U_2, U_3 \) and:
The points indicated by • have the coordinates (0,0), (0,A), (A,0), respectively. The partitions \( \pi_j \) are respectively placed at the points (indicated by ◦) with the coordinates

\[ (0, \Delta_2), (\Delta_1, 0), (0, A + \Delta_3), (\Delta_2, A), (A, \Delta_1), (A + \Delta_3, 0). \]

We denote by \( S_j^1, S_j^2 \) the vertical and the horizontal strips made by two vertical and two horizontal lines in each diagram. We also denote by \( R_1, R_2, R_3 \) the areas located above the horizontal strip and to the right of the vertical strip. For any \( j = 1, 2, 3 \) we then have

(F1) \( F_j(m) = 0 \) if either \( m \in \pi_j^1 \cap \pi_j^2 \), \( m \in S_j^1 \cap S_j^2 \), \( m \in S_j^1 \cap \pi_j^1 \), \( m \) is on the left of the strip \( S_j^1 \), or \( m \) is below the strip \( S_j^2 \).

(F2) \( F_j(m) = \mathbb{C}^2 \) if \( m \) is in \( R_j - \pi_j^1 - \pi_j^2 \).

(F3) \( F_1(m) = p \) if \( m \in S_1^1 - \pi_1^1 \) or \( m \) belongs to a connected component of \( \pi_1^2 - \pi_1^1 \) adjacent to a member of \( S_1^1 - \pi_1^1 \); and \( F_1(m) = q \) if \( m \in S_1^2 - \pi_1^2 \), or \( m \) belongs to a connected component of \( \pi_1^2 - \pi_1^2 \) adjacent to a member of \( m \in S_1^2 - \pi_1^2 \).

(F4) \( F_2(m) = q \) if \( m \in S_2^1 - \pi_2^1 \) or \( m \) belongs to a connected component of \( \pi_2^2 - \pi_2^1 \) adjacent to a member of \( m \in S_2^2 - \pi_2^1 \); and \( F_2(m) = r \) if \( m \in S_2^2 - \pi_2^2 \) or \( m \) belongs to a connected component of \( \pi_2^2 - \pi_2^2 \) adjacent to a member of \( m \in S_2^2 - \pi_2^2 \).

(F5) \( F_3(m) = r \) if \( m \in S_3^1 - \pi_3^1 \) or \( m \) belongs to a connected component of \( \pi_3^2 - \pi_3^1 \) adjacent to a member of \( m \in S_3^2 - \pi_3^1 \); and \( F_3(m) = p \) if \( m \in S_3^2 - \pi_3^2 \) or \( m \) belongs to a connected component of \( \pi_3^2 - \pi_3^2 \) adjacent to a member of \( m \in S_3^2 - \pi_3^2 \).

(F6) \( F_j(m) = s \subset \mathbb{C}^2 \) where \( s \) is an arbitrary 1-dimensional subspace of \( \mathbb{C}^2 \) for all \( m \) in any connected component of \( \pi_j^1 \cap \pi_j^2 - \pi_j^1 \cap \pi_j^2 \) other than the ones mentioned in (3),(4),(5).
In terms of these data
\[
\begin{align*}
\text{ch}_1(\mathcal{F}) &= -2A - \Delta_1 - \Delta_2 - \Delta_3, \\
\text{ch}_2(\mathcal{F}) &= A^2/2 + (A + \sum_i \Delta_i)^2/2 - \sum_{i<j} \Delta_i \Delta_j - \sum_{i,j} \# \pi_{ij}.
\end{align*}
\]
Given \(\mathcal{F}\) as above, we define the rank 1 torsion-free \(T\)-equivariant sheaves \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\), to be the maximal subsheaves of \(\mathcal{F}\) respectively with \(u = 0, v = \Delta_2, w = A + \Delta_3\), generated by \(p\)
\[
\begin{align*}
u = \Delta_1, v = 0, w = A + \Delta_3, \quad \text{generated by} \quad q \\
u = \Delta_1, v = \Delta_2, w = A, \quad \text{generated by} \quad r.
\end{align*}
\]

We are only interested in the case where the Hilbert polynomial of \(\mathcal{F}\) is \(P(m) = m^2 + 3m + b\), so we must have
\[
\Delta_1 + \Delta_2 + \Delta_3 = -2A,
\]
and
\[
b = A^2 - \sum_{i<j} \Delta_i \Delta_j - \sum_{i,j} \# \pi_{ij}.
\]
Then one can see that \(\mathcal{F}\) is (semi)stable if the Hilbert polynomials of \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\) are less than (less than or equal to) \(P/2\) (see [Koo08, Proposition 3.19]). Using this set up we have

**Proposition 3.6.** Suppose \(\mathcal{F}\) is a rank 2 semistable \(T\)-equivariant sheaf on \(\mathbb{P}^2\) with Hilbert polynomial \(P(m) = m^2 + 3m + b\) given by the data above. Then the contribution of \(\mathcal{F}\) to the pair invariants \(PI_n(X; P)\), denoted by \(PI_n(\mathcal{F})\) is as follows:

1. If \(\mathcal{F} \cong \mathcal{I}_{Z_1} \oplus \mathcal{I}_{Z_2}\), where \(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2}\) are the ideal sheaves of the \(T\)-invariant 0-dimensional subschemes \(Z_1, Z_2 \subset \mathbb{P}^2\) such that the Hilbert polynomials of \(\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2}\) are equal to \(P/2\) then

\[
PI_n(\mathcal{F}) = \begin{cases} 
P(n)(P(n) - 2)/8 & \text{if } Z_1 = Z_2, \\
P(n)^2/4 & \text{if } Z_1 \neq Z_2.
\end{cases}
\]

2. If \(\mathcal{F}\) is indecomposable and strictly semistable then \(PI_n(\mathcal{F}) = P(n)/2\).

3. If \(\mathcal{F}\) is stable then \(PI_n(\mathcal{F}) = P(n)\).

**Proof.** Given \(\mathcal{F}\) as in the statement of Proposition 3.6 \(PI_n(\mathcal{F})\) is equal to the Euler characteristic of the space of \(T\)-equivariant sections of \(\mathcal{F}(n)\) satisfying the stability condition for the pairs. This number can be obtained by counting the number of the lattice points in the toric description of \(\mathcal{F}\) given as above corresponding to these sections. If \(m\) is such a lattice point and \(F_j(m)\) is 2-dimensional then \(m\) corresponds to a \(\mathbb{P}^1\) worth of \(T\)-equivariant sections and hence, \(m\) must be counted with multiplicity 2 to account for the Euler characteristic of \(\mathbb{P}^1\). If \(F_j(m)\) is 1-dimensional then \(m\) is counted with multiplicity 1. \(\square\)
Before stating the next theorem we give the following definition:

**Definition 3.7.** Let $D_1, D_2 \in \mathbb{Z}_+$ be positive integers and let $\pi_1, \pi_2, \ldots, \pi_3 \in \mathcal{P}$ be $2d$ partitions. We say that the 8-tuple 

$$(D_1, D_2, \pi_1, \pi_2, \ldots, \pi_3) \in \mathbb{Z}_+^2 \times \mathcal{P}^6$$

is $P$-admissible if

$$b = -D_1 D_2 - \#\pi_1 - \#\pi_2 - \cdots - \#\pi_3,$$

$$b/2 = -\#\pi_1 - \#\pi_2 - \#\pi_3 \cap \pi_3,$$

Define

$$C(b) = (3/2) \cdot \# \{ \Omega \in \mathbb{Z}_+^2 \times \mathcal{P}^6 | \Omega \text{ is } P\text{-admissible} \}.$$
$\mathcal{DT}(X; P)$ is in general a rational number in the presence of semistable sheaves. Joyce and Song in \cite[Section 6.2]{JS11} define the corresponding BPS invariants denoted by $\hat{\mathcal{DT}}(X; P)$ by the following formula:

$$
\hat{\mathcal{DT}}(X; P) = \sum_{d \geq 1, \, d | P(m)} \frac{1}{d^2} \mathcal{DT}(X; P/d).
$$

Joyce and Song conjecture that $\hat{\mathcal{DT}}(X; P)$ is an integer. In the case that there are no strictly semistable sheaves with Hilbert polynomial $P$ we have $\hat{\mathcal{DT}}(X; P) = \mathcal{DT}(X; P)$.

**Corollary 3.9.** Using the notation of Theorem 3.8, we assume that $b$ is an even number then

$$
\hat{\mathcal{DT}}(X; P) = -\chi(\mathcal{M}^b(\mathbb{P}^2; P))
$$

$$
- (3/2)\# \{ \Omega \in \mathbb{Z}_+^2 \times \mathcal{P}^6 \mid \Omega \text{ is } P\text{-admissible} \}.
$$

In particular to show $\hat{\mathcal{DT}}(X; P) \in \mathbb{Z}$ one needs to show that

$$
\# \{ \Omega \in \mathbb{Z}_+^2 \times \mathcal{P}^6 \mid \Omega \text{ is } P\text{-admissible} \} \in 2\mathbb{Z}.
$$

**Example 3.10.** In this example we work out the cases $b = 0$, $b = -2$, and $b = -4$. In the first two cases we provide more details to make the proof of Theorem 3.8 clearer.

- **$b = 0$.** In this case the only semistable sheaf with Hilbert polynomial $P(m) = m^2 + 3m$ is isomorphic $O \oplus O$. Therefore, by Proposition 3.6 part (1) we have $P\text{.}_n(X; P) = P(n)(P(n) - 2)/8$, and hence by Lemma 3.2 and noting that $\mathcal{DT}(X; P/2) = 1$ we get

$$
\hat{\mathcal{DT}}(X; P) = P(n)/8 - (P(n) - 2)/8 = 1/4
$$

in agreement with the result of Theorem 3.8. We can easily see that $\hat{\mathcal{DT}}(X; P) = 0$.

- **$b = -2$.** Suppose that $\mathcal{F}$ is a $\mathbb{T}$-equivariant sheaf with Hilbert polynomial $m^2 + 3m - 2$. Then either $\mathcal{F} \cong \mathcal{I}_{Z_1} \oplus \mathcal{I}_{Z_2}$ as in Proposition 3.6 part (1), or $\mathcal{F}$ is strictly semistable and indecomposable as in Proposition 3.6 part (2). So we have

$$
P\text{.}_n(X; P) = 3P(n)^2/4 + 3P(n)(P(n) - 2)/8 + 6P(n)/2.
$$

The first term is the sum of the contributions of $\mathcal{I}_{Z_1} \oplus \mathcal{I}_{Z_2}$ where $Z_1$, $Z_2$ are two distinct $\mathbb{T}$-fixed points of $\mathbb{P}^2$. The second term is the sum of the contributions $\mathcal{I}_Z \oplus \mathcal{I}_Z$ where $Z$ is a $\mathbb{T}$-fixed point of $\mathbb{P}^2$, and the last term is the sum of the contributions of the $\mathbb{T}$-equivariant strictly semistable indecomposable sheaves which are determined by the following data:

$$
\Delta_1 = 2, \Delta_2 = \Delta_3 = 1, \quad \#\pi^1_i = 1, \#\pi^2_j = 0 \quad (i, j) \neq (1, 1),
$$
\[\Delta_1 = 2, \Delta_2 = \Delta_3 = 1, \quad \#\pi^2_i = 1, \#\pi^2_j = 0 \quad (i, j) \neq (2, 1),\]
\[\Delta_2 = 2, \Delta_1 = \Delta_3 = 1, \quad \#\pi^2_i = 1, \#\pi^2_j = 0 \quad (i, j) \neq (1, 2),\]
\[\Delta_2 = 2, \Delta_1 = \Delta_3 = 1, \quad \#\pi^2_i = 1, \#\pi^2_j = 0 \quad (i, j) \neq (2, 2),\]
\[\Delta_3 = 2, \Delta_1 = \Delta_2 = 1, \quad \#\pi^2_i = 1, \#\pi^2_j = 0 \quad (i, j) \neq (1, 3),\]
\[\Delta_3 = 2, \Delta_1 = \Delta_2 = 1, \quad \#\pi^2_i = 1, \#\pi^2_j = 0 \quad (i, j) \neq (2, 3),\]

Now using the fact that \(DT(X; P/2) = 3\) from [14], by Lemma 3.2 we get
\[\mathcal{DT}(X; P) = 9P(n)/8 - 3P(n)/4 - 3(\frac{P(n) - 2}{8}) - 3 = -9/4\]

in agreement with the result of Theorem 3.8. We can easily see that
\[\Delta T(X; P) = -3.\]

\[\bullet \quad b = -4.\] In this case \(\chi(M^s(X; P)) = 36\) by Remark 3.11, \(\chi(Hilb^2(\mathbb{P}^2)) = 9\) by [14]. It can be seen that the elements of the set
\[\{ \Omega \in \mathbb{Z}_+^2 \times \mathcal{P}^6 \mid \Omega \text{ is } P\text{-admissible} \}\]
are as follows:
\[
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square],
(1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square], (1, 1, [\square, \square, \ldots, \square].

So by Theorem 3.8 \(DT(X; P) = 9/4 - 36 - 3\cdot 2\cdot 30 = -315/4.\) Also
\[\Delta T(X; P) = -81\] by Corollary 3.9.

\[\circ\]

**Remark 3.11.** In [Koo09 Corollary 4.2] the Euler characteristic of
\[\mathcal{M}^{\mu_s}(X; P) \subseteq \mathcal{M}^s(X; P),\]
the moduli space of \(\mu\)-stable rank 2 sheaves, is computed. Let \(P(m) = m^2 + 3m + 2 + b\) be the Hilbert polynomial corresponding to \(a = 0\) then
\[
\sum_{c \in \mathbb{Z}} \chi(\mathcal{M}^{\mu_s}(\mathbb{P}^2; m^2 + 3m + 2 + b))q^b = \frac{1}{\prod_{n > 0}(1 - q^n)^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q^{mn + m + n} \sum_{1 - q^{m+n}}.
\]
In order to find $\chi(\mathcal{M}^s(\mathbb{P}^2; P))$ appearing in Theorem 3.8 we need to add to the formula above the contribution of the stable sheaves which are not $\mu$-stable. Here

$$\mathcal{M}^s(\mathbb{P}^2; m^2 + 3m + 2 + b) \subseteq \mathcal{M}(\mathbb{P}^2; m^2 + 3m + 2 + b)$$

is the open subset of the stable sheaves (see [Koo09]). Using the toric method above one can see that this contribution is given by

$$3\# \left\{ \Omega = (D_1, D_2, \pi_1^1, \ldots, \pi_3^3) \in \mathbb{Z}_+^2 \times \mathcal{P}^6 \mid \Omega \text{ satisfies } (15) \right\}.$$

(15)

$$b = -D_1 D_2 - \#\pi_1^1 - \#\pi_2^2 - \cdots - \#\pi_3^2,$$

$$b/2 > -\#\pi_1^1 - \#\pi_3^2 - \#\pi_2^1 \cap \pi_2^2.$$

Note that if $j$ is the number of the connected components satisfying (F6) in the toric description of a stable toric sheaf $\mathcal{F}$, then $\mathcal{F}$ belongs to a $T$-fixed component of the moduli space of stable sheaves isomorphic to $(\mathbb{P}^1)^j$. The combinatorial expression above, satisfying (15), accounts for such non-isolated components in a subtle way.

**Remark 3.12.** It would be very interesting to investigate the possible modular properties of the generating series of the invariants

$$\overline{DT}(X; m^2 + 3m + b) \text{ or } \hat{DT}(X; m^2 + 3m + b).$$
APPENDIX A. RELATIVE MODULI SPACE OF SHEAVES

In this appendix we review part of the construction of Li and Wu in [LW11] needed in Section 2.3. Let \( q : W \rightarrow \mathbb{A}^1 \) be a good degeneration of the projective threefolds, i.e.

i) \( W \) is smooth,

ii) all the fibers except \( \pi^{-1}(0) \) are smooth projective threefolds,

iii) \( \pi^{-1}(0) = W_1 \cup_D W_2 \) where \( W_i \) is a smooth threefold, \( D \subset W_i \) is smooth divisor, and \( \pi^{-1}(0) \) is a normal crossing divisor in \( W \).

Li and Wu in [LW11] construct the Artin stack of expanded degenerations

\[
\begin{array}{ccc}
\mathfrak{W} & \overset{p}{\longrightarrow} & W \\
\downarrow & & \downarrow q \\
\mathcal{C} & \overset{r}{\longrightarrow} & \mathbb{A}^1.
\end{array}
\]

Away from \( r^{-1}(0) \) the family \( \mathfrak{W} \) is isomorphic to the original family

\( W \setminus \pi^{-1}(0) \rightarrow \mathbb{A}^1 \setminus 0 \).

The central fiber \( \pi^{-1}(0) \) of the original family \( W \rightarrow \mathbb{A}^1 \) is replaced in \( \mathfrak{W} \) by the union over all \( k \) of the \( k \)-step degenerations

\[ W[k] = W_1 \cup_D \mathbb{P}(\mathcal{O} \oplus N_D^0) \cup_D \mathbb{P}(\mathcal{O} \oplus N_D^0) \cup_D \cdots \cup_D \mathbb{P}(\mathcal{O} \oplus N_D^0) \cup_D W_2 \]

together with the automorphisms \( C^k \) induced from the \( C^* \)-action along the fibers of the standard ruled variety \( \mathbb{P}(\mathcal{O} \oplus N_D^0) \).

Similarly, for a pair of a smooth projective threefold \( Y \) and a smooth divisor \( D \subset Y \), the Artin stack of expanded degenerations

\[
\begin{array}{ccc}
\mathfrak{Y} & \overset{p}{\longrightarrow} & Y \\
\downarrow & & \downarrow \\
\mathfrak{X} & \longrightarrow & \text{spec } \mathbb{C}
\end{array}
\]

is constructed in [LW11] using the \( k \)-step degeneration

\[ Y[k] = Y \cup_D \mathbb{P}(\mathcal{O} \oplus N_D^0) \cup_D \mathbb{P}(\mathcal{O} \oplus N_D^0) \cup_D \cdots \cup_D \mathbb{P}(\mathcal{O} \oplus N_D^0) \]

together with the automorphisms \( C^k \) as above.

**Definition A.1.** ([LW11] Definition 3.1, 3.9, 3.12) Let \( \mathcal{F} \) be a coherent sheaf on a \( \mathbb{C} \)-scheme \( T \) of finite type, and suppose that \( Z \subset T \) is a closed subscheme. \( \mathcal{F} \) is called **normal** to \( \mathcal{F} \) if \( \text{Tor}^1_T(\mathcal{F}, \mathcal{O}_Z) = 0 \). A coherent sheaf \( \mathcal{F} \) on \( W[k] \) is called **admissible** if \( \mathcal{F} \) is normal to all singular loci \( D_i \cong D \) of \( W[k] \). Similarly, a coherent sheaf \( \mathcal{F} \) on \( Y[k] \) is called **relative** if \( \mathcal{F} \) is normal to all the singular loci \( D_i \cong D \) as well as the divisor at infinity of the last copy of \( \mathbb{P}(\mathcal{O} \oplus N_D^0) \). \( \square \)
Suppose that $V$ is a locally free sheaf on $W$ (respectively on $Y$), and $\mathcal{O}(1)$ be an ample invertible sheaf on $W \to \mathbb{A}^1$ (respectively $Y$). Li and Wu construct the Quot schemes $\text{Quot}^{V,P}_{W/\mathcal{E}}$ (respectively $\text{Quot}^{V,P}_{Y}$) of quotients $\phi : p^*V \to \mathcal{F}$ on $W[k]$ (respectively on $Y[k]$) satisfying

i) $\mathcal{F}$ is admissible (respectively relative),

ii) $\phi$ has finitely many automorphisms covering the automorphisms $C_i^k$,

iii) The Hilbert polynomial of $\mathcal{F}$ with respect to $p^*\mathcal{O}(1)$ is $P$.

Moreover, they show that $\text{Quot}^{V,P}_{W/\mathcal{E}}$ (respectively $\text{Quot}^{V,P}_{Y/\mathcal{D}}$) is a separated, proper over $\mathbb{A}^1$ (respectively separated, proper), Deligne-Mumford stack of finite type ([LW11] Theorems 4.14 and 4.15).

**Definition A.2.** For any $C$-point $\phi : p^*V \to \mathcal{F}$ of $\text{Quot}^{V,P}_{W/\mathcal{E}}$ (respectively $\text{Quot}^{V,P}_{Y/\mathcal{D}}$), we say that $\mathcal{F}$ is Gieseker (semi)stable if

i) the restriction of $\mathcal{F}$ to $W_i$ (respectively $Y_i$) is Gieseker (semi)stable,

ii) the restriction of $\mathcal{F}$ to each of the components $\mathbb{P}(\mathcal{O} \oplus N^\vee_Y)$ is Gieseker (semi)stable,

iii) the restriction of $\mathcal{F}$ to each of the singular loci $D_i \cong D$ of $W[k]$ is Gieseker (semi)stable (see Definition 0.1).

Suppose now that the Hilbert polynomial $P$ is such that for any $\mathcal{F}$ as above the Gieseker semistability implies Gieseker stability. Let $R$ be the open substack of $\text{Quot}^{V,P}_{W/\mathcal{E}}$ (respectively $\text{Quot}^{V,P}_{Y/\mathcal{D}}$) consisting of the quotients as above such that $\mathcal{F}$ is Gieseker stable and the induced map $H^0(V(N)) \to H^0(\mathcal{F}(N))$ is an isomorphism. We define the moduli stack $\mathcal{M}(\mathfrak{M}/\mathscr{C}; P)$ (respectively $\mathcal{M}(Y/D; P)$) by taking the quotient stack of $R$ by the natural action of $GL(N)$, where

$$
\mathcal{V} = \bigoplus_{i=1}^{P(N)} \mathcal{O}(-N)
$$

for some fixed $N \gg 0$. By our assumption all $C$-points of $\mathcal{M}(\mathfrak{M}/\mathscr{C}; P)$ (respectively $\mathcal{M}(Y/D; P)$) have the stabilizer group $C_i$. We define the relative moduli space of stable sheaves

$$
\mathcal{M}(\mathfrak{M}/\mathscr{C}; P) \quad (\text{respectively } \mathcal{M}(Y/D; P))
$$

by rigidifying $\mathcal{M}(\mathfrak{M}/\mathscr{C}; P)$ (respectively $\mathcal{M}(Y/D; P)$) [AGV08].

$\mathcal{M}(\mathfrak{M}/\mathscr{C}; P)$ (respectively $\mathcal{M}(Y/D; P)$) is then a separated Deligne-Mumford stack of finite type. Moreover, assume that the Hilbert polynomial $P$ is such that for any $C$-point $\phi : p^*V \to \mathcal{F}$ of $\text{Quot}^{V,P}_{W/\mathcal{E}}$ (respectively $\text{Quot}^{V,P}_{Y/\mathcal{D}}$), the purity of $\mathcal{F}$ implies that $\mathcal{F}$ is Gieseker stable, then $\mathcal{M}(\mathfrak{M}/\mathscr{C}; P)$ (respectively $\mathcal{M}(Y/D; P)$) is proper over $\mathbb{A}^1$ (respectively proper).

From now on we assume that $\mathcal{M} = \mathcal{M}(\mathfrak{M}/\mathscr{C}; P)$ (respectively $\mathcal{M} = \mathcal{M}(Y/D; P)$) can be constructed as in Definition A.2. If for any $C$-point $\mathcal{F}$ of $\mathcal{M}$ we have $\text{Ext}^3(\mathcal{F}, \mathcal{F})_0 = 0$ then, by the result of [Tho00, HT10, MPT10], there is a perfect obstruction theory on $\mathcal{M}$ relative to $\mathcal{C}$ (respectively $\mathfrak{A}$) given by
(16) \[
\left( \tau^{[1,2]} R\pi_\ast (R\mathcal{H}om(\mathcal{F}, \mathcal{F})) \right)^\vee [-1],
\]
where \(\mathcal{F}\) is the universal sheaf.

This defines a virtual cycle

\[\mathcal{M}(\mathfrak{M}/\mathfrak{C}; P)^{\text{vir}} \in A_{n+1}(\mathcal{M}(\mathfrak{M}/\mathfrak{C}; P))\]
(respectively \(\mathcal{M}(Y/D; P)^{\text{vir}} \in A_n(\mathcal{M}(Y/D; P))\), where \(n\) is the rank of the relative obstruction theory above. Moreover, the restriction of the virtual cycle \(\mathcal{M}(\mathfrak{M}/\mathfrak{C}; P)^{\text{vir}}\) to a general fiber \(W_t\) of \(q : W \to \mathbb{A}^1\) is the virtual cycle

\[\mathcal{M}(W_t; P)^{\text{vir}} \in A_n(\mathcal{M}(W_t; P)).\]

The degeneration formula of Li-Wu follows from the naturality of the virtual cycle \(\mathcal{M}(\mathfrak{M}/\mathfrak{C}; P)^{\text{vir}}\). In case \(n = 0\) and assuming properness (see Definition A.2), this can be written as

\[(17) \deg [\mathcal{M}(W_t; P)]^{\text{vir}} = \deg [\mathcal{M}(\mathfrak{M}_0^t/\mathfrak{C}_0^t; P)]^{\text{vir}},\]

where

\[
\begin{array}{ccc}
\mathfrak{M}_0^t & \longrightarrow & \mathfrak{M} \\
\downarrow & & \downarrow \\
\mathfrak{C}_0^t & \longrightarrow & \mathfrak{C}
\end{array}
\]

is the stack of node marking objects in \(\mathfrak{M}_0\) as defined in [LW11, Section 2.4].

Suppose now that the moduli space of stable sheaves on the smooth surface \(D\), containing all the restrictions \(\mathcal{F}|_D\) (where \(\mathcal{F}\) is as above) with the induced Hilbert polynomial, is smooth. Then, the virtual cycle \(\mathcal{M}(\mathfrak{M}_0^t/\mathfrak{C}_0^t; P)^{\text{vir}}\) on the right hand side of (17) can be expressed as a finite weighted sum of the products

\[\mathcal{M}(W_1/D; P_1)^{\text{vir}} \times \mathcal{M}(W_2/D; P_2)^{\text{vir}}\]

over all possible decompositions of the Hilbert polynomial \(P\), provided that all such \(\mathcal{M}(W_i/D; P_i)^{\text{vir}}\) exist (see [LW11, MPT10]).

In case the dimension of the virtual cycle \(\mathcal{M}(Y/D; P)^{\text{vir}}\) is zero (i.e. when \(n = 0\) in above discussion), and \(\mathcal{M}(Y/D; P)\) is proper, we define the relative DT invariant

\[DT( Y/D; P) = \deg [\mathcal{M}(Y/D; P)]^{\text{vir}}.\]
Appendix B.

COUNTING CURVES IN SURFACES IN CALABI-YAU 3-FOLDS

by Amin Gholampour, Artan Sheshmani and Richard Thomas

Fix a Calabi-Yau 3-fold $X$ with $H^1(X, \mathcal{O}_X) = 0$. In this appendix we describe a way to count pairs of the form

(18) \[ Z \subset H \quad \text{in} \quad X, \]

where $H$ is a suitably ample hypersurface and $Z$ is a one-dimensional sub-scheme of $H$. While we demand that $H$ be pure of dimension 2 (i.e. a divisor in $X$), the subscheme $Z$ need not be pure, so is in general the union of a curve and a 0-dimensional subscheme.

B.1. Outline. Since $H^1(X, \mathcal{O}_X) = 0$, all deformations of $H$ are in the same linear system

\[ |H| := \mathbb{P}(H^0(\mathcal{O}_X(H))). \]

This carries a universal hyperplane

(19) \[
\begin{array}{ccc}
\mathcal{H} & \subset & X \times |H| \\
\downarrow & & \\
|H|. & & 
\end{array}
\]

The data (18) are naturally parameterized by the relative Hilbert scheme of 1-dimensional subschemes of the fibers of the family (19). That is, if we fix $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, then the moduli space of pairs (18) is the relative Hilbert scheme

(20) \[ \text{Hilb}_{\beta, n}(|H|) \]

of 1-dimensional subschemes $Z$ of the universal hyperplane whose push-forward to $X$ has fundamental class $\beta = [Z]$ and holomorphic Euler characteristic $n = \chi(\mathcal{O}_Z)$.

We will produce a symmetric perfect obstruction theory on (20) by considering it as a moduli space of sheaves of the form

(21) \[ i_*I_Z \in \text{Coh}(X). \]

Here $i: H \hookrightarrow X$ is the inclusion, and $I_Z$ denotes the ideal sheaf of $Z$ considered as a subscheme of $H$. So (21) is a torsion sheaf on $X$ with rank 1 on its 2-dimensional support $H$. When $Z \subset H$ is a Cartier divisor, it is the pushforward of a line bundle on $H$.

There are two obvious problems with forming a moduli space of sheaves such as (21):

- They need not be (Gieseker) semistable when $H$ is reducible or nonreduced.
- Deformations of (21) need not be sheaves of the same form. They could be arbitrary torsion sheaves of the same topological type, like...
the pushforward \( i_* L \) of a general line bundle on a hyperplane \( H \) rather than one which is a subsheaf of \( O_H \).

We circumvent these by using certain Joyce-Song pairs [JS11]. These come with a different notion of stability which gets round the first problem. And for \( H \gg 0 \), they allow us to see that deformations of (21) are also push forwards of ideal sheaves, so that \( \text{Hilb}_{\beta,n}(H/|H|) \) is indeed an open and closed subscheme of the stack of all coherent sheaves on \( X \).

Our Joyce-Song pairs are of the form

\[
\left( I_Z, s \in H^0(I_Z(H)) \right),
\]

where \( Z \) and \( H \) are as before, but \( I_Z \) denotes the ideal sheaf of \( Z \) when considered as a subscheme of \( X \) (rather than \( H \): we always use straight \( I \)s to denote ideal sheaves on \( H \) and curly \( I \)s for ideal sheaves on \( X \)). Since the torsion-free rank-1 sheaf \( I_Z \) is automatically Gieseker stable, the only further stability condition we need is that \( s \) should be nonzero (see Definition 0.1). Therefore we get an injection

\[
O(-H) \xrightarrow{s} I_Z
\]

whose cokernel \( i_* I_Z \) is the torsion sheaf (21). More globally we will show (Proposition B.1) that the observation that the Joyce-Song complexes (22) are quasi-isomorphic to the torsion sheaves (21) gives an isomorphism between

- an open and closed subscheme of the moduli space of Joyce-Song pairs (22), and
- the relative Hilbert scheme (20) parameterizing the sheaves (21).

Finally, for \( H \gg 0 \), Joyce and Song show that the deformation theory of the complex (22) in \( D(X) \) gives this space a symmetric perfect obstruction theory. Since (22) is quasi-isomorphic to \( i_* I_Z \in D(X) \), this shows that the deformation theory of the sheaf \( i_* I_Z \) indeed endows \( \text{Hilb}_{\beta,n}(H/|H|) \) with a symmetric perfect obstruction theory.

B.2. Details. Fix a Calabi-Yau 3-fold \( X \) with \( H^1(X, O_X) = 0 \) and an ample class \( H \). A Joyce-Song stable pair [JS11] is a pair

\[
\left( E, s \in H^0(E(H)) \right),
\]

such that

- \( E \) is coherent sheaf on \( X \), Gieseker semistable with respect to \( H \), and
- \( s \) is a section which does not factor through any destabilizing subsheaf of \( E \).

A family of such pairs over a base scheme \( B \) is a sheaf \( \mathcal{E} \) over \( X \times B \), flat over \( B \), and a section of \( \mathcal{E} \otimes \pi^*_X O(H) \) such that the restriction of \( (\mathcal{E}, s) \)
to any fiber $X \times \{b\}$ is a stable pair in the above sense. Fixing a Chern character $c \in H^{ev}(X, \mathbb{Q})$, there is a projective scheme $J_c(X)$ representing the moduli functor which assigns to a scheme $B$ the set of isomorphism classes of Joyce-Song stable pairs over $B$.

Forgetting the section $s$ and passing to the $S$-equivalence class of the sheaf $E$ gives a morphism

$$J_c(X) \to M_c(X)$$

to the moduli space of Gieseker semistable sheaves $E$ of Chern character $c$.

We take $c = (1, 0, -\beta, -n)$.

Then the Hilbert scheme $I_n(X, \beta)$ of ideal sheaves $I_Z$ defines an open and closed subscheme of $M_c(X)$ (see for instance the proof of Theorem 2.7 in [PT09] for a careful proof of this fact). Thus its inverse image

$$J_n(X, \beta) := J_c(X) \times_{M_c(X)} I_n(X, \beta)$$

is both open and closed in $J_c(X)$. It therefore has the same deformation theory as $J_c(X)$, so we can use the results of Joyce and Song.

**Proposition B.1.** $J_n(X, \beta) \cong \text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$.

**Proof.** $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ carries a pair of universal subschemes

$$Z \subset \mathcal{H} \subset X \times \text{Hilb}_{\beta,n}(\mathcal{H}/|H|).$$

Letting $i$ denote the second inclusion, we get the right hand column and central row of the following commutative diagram of short exact sequences.

$$
\begin{array}{cccccc}
0 & 0 \\
0 & \mathcal{O}_{X \times \text{Hilb}}(-\mathcal{H}) & \mathcal{O}_Z & \mathcal{O}_Z \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_{X \times \text{Hilb}}(-\mathcal{H}) & \mathcal{O}_{X \times \text{Hilb}} & \mathcal{O}_{Z} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_Z & \mathcal{O}_Z & 0 \\
& 0 & 0 & 0
\end{array}
$$

---

8 Isomorphism is meant in the strict sense: two families of stable pairs $(E_i, s_i)$ are isomorphic if there is an isomorphism $E_1 \to E_2$ taking $s_1$ to $s_2$. Stable pairs have no automorphisms so there is no need to tensor by line bundles pulled back from the base (as one does to define isomorphism for families of stable sheaves, for example: stable sheaves have $\text{Aut} = \mathbb{C}^\times$).

9 In fact it is all of $M_c(X)$ when $H^2(X, \mathbb{Z})$ is torsion-free.
Filling in the diagram gives the top row, producing the flat family of stable pairs

\[ \mathcal{O} \xrightarrow{s} \mathcal{I}_Z(\mathcal{H} - \pi_X^* \mathcal{H}) \otimes \pi_X^* \mathcal{O}(\mathcal{H}) \]

over the base \( \text{Hilb}_{\beta,n}(\mathcal{H}/|\mathcal{H}|) \). This is classified by a map from the base to the moduli space of stable pairs:

\[ \text{Hilb}_{\beta,n}(\mathcal{H}/|\mathcal{H}|) \rightarrow J_n(X, \beta). \]

Similarly, \( X \times J_n(X, \beta) \) carries a universal stable pair

\[ \mathcal{O} \rightarrow \mathcal{E} \otimes \pi_X^* \mathcal{O}(\mathcal{H}), \]

flat over \( J_n(X, \beta) \). Since the restriction of \( \mathcal{E} \) to each \( X \)-fiber is torsion-free of rank 1, its double dual \( \mathcal{E}^{\vee \vee} \) is locally free by [Kol90, Lemma 6.13]. Therefore it defines a map from \( J_n(X, \beta) \) to \( \text{Pic}(X) \) which takes closed points to the trivial line bundle \( \mathcal{O}_X \). But \( H^1(X, \mathcal{O}_X) = 0 \) so \( \text{Pic}(X) \) is a union of discrete reduced points and the map is constant. Pulling back a Poincaré line bundle shows that \( \mathcal{E}^{\vee \vee} \) is the pullback \( \pi_J^* \mathcal{L} \) of some line bundle \( \mathcal{L} \) on \( J_n(X, \beta) \). Therefore \( \mathcal{E} \subset \mathcal{E}^{\vee \vee} \) must take the form

\[ \mathcal{E} \cong \pi_J^* \mathcal{L} \otimes \mathcal{I}_Z \]

for some subscheme \( Z \subset X \times J_n(X, \beta) \). Since \( \mathcal{E} \) is flat over \( J_n(X, \beta) \), so \( Z \) must be too.

Composing the section \( (27) \) with \( \mathcal{E} \rightarrow \mathcal{E}^{\vee \vee} \) gives a section

\[ \mathcal{O}_{X \times J_n(X, \beta)} \rightarrow \mathcal{O}(\mathcal{H}) \boxtimes \mathcal{L}, \]

nonzero on each \( X \)-fiber by stability. Its zero locus is a divisor

\[ \mathcal{H} \in |\mathcal{O}(\mathcal{H}) \boxtimes \mathcal{L}|, \]

giving a classifying map \( J_n(X, \beta) \rightarrow |\mathcal{H}| \) such that the pullback of \( \mathcal{O}_{X \times |\mathcal{H}|}(\mathcal{H}) \) is \( \mathcal{O}(\mathcal{H}) \boxtimes \mathcal{L} \). Since the section \( (29) \) factors through \( (28) \) it follows that

\[ Z \subset \mathcal{H}, \]

giving us a classifying map from our base \( J_n(X, \beta) \) to \( \text{Hilb}_{\beta,n}(\mathcal{H}/|\mathcal{H}|) \). By inspection it is the inverse of the map \( (26) \).

**Definition B.2.** Assume that \( \beta \in H_2(X; \mathbb{Z}) \) and \( n \in \mathbb{Z} \) are given. We say \( H \) is sufficiently positive with respect to \( \beta, n \) if

\[ H^i(X, \mathcal{I}_Z(\mathcal{H})) = 0 \]

for \( i > 0 \) for any ideal sheaf \( \mathcal{I}_Z \in I_n(X, \beta) \). For fixed \( (\beta, n) \) the ideal sheaves \( \mathcal{I}_Z \in I_n(X, \beta) \) form a bounded family, so \( (30) \) is satisfied for all \( H \) sufficiently positive.
Corollary B.3. Suppose that $H^1(X, \mathcal{O}_X) = 0$ and $\beta, n, H$ satisfy (30). Then, using the same notation as in (23), the deformation-obstruction theory of the sheaves $i_{\ast}I_Z$ (21) given by [HT10] Section 4.4,

$$\tau^{[1,2]} R\pi_{H_{\ast}} R\mathcal{H}om(i_{\ast}I_Z, i_{\ast}I_Z)[2] \to \mathcal{L}_{\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)},$$

defines a symmetric perfect obstruction theory on $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$.

In (31), $\pi_H$ denotes the projection from $X \times \text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ to its second factor. We recall the definition of the arrow [HT10] Section 4. Project the Atiyah class

$$\text{At}(i_{\ast}I_Z) \in \text{Ext}^1(i_{\ast}I_Z, i_{\ast}I_Z \otimes \mathcal{L}_{X \times \text{Hilb}})$$

and consider the resulting element as a map

$$\pi_{H_{\ast}}^! \mathcal{L}_{\text{Hilb}}^{\vee} \to R\mathcal{H}om(i_{\ast}I_Z, i_{\ast}I_Z)[1].$$

By adjunction this induces

$$\mathcal{L}_{\text{Hilb}}^{\vee} \to R\pi_{H_{\ast}} R\mathcal{H}om(i_{\ast}I_Z, i_{\ast}I_Z)[1].$$

Next we project to the $\tau^{\geq 0}$ truncation of the right hand side. In turn this receives a map from the truncation $\tau^{[0,1]}$, and in [HT10] Section 4.4 it is shown that the map from $\mathcal{L}_{\text{Hilb}}^{\vee}$ lifts uniquely to it. Dualizing and using Serre duality down the fibers of $\pi_H$ gives (31).

Corollary B.3 completes the proof of Theorem B.3.

Proof of Corollary B.3. For $H \gg 0$, Joyce and Song [STI] Theorem 12.20 give a symmetric perfect obstruction theory on $I_{c}(X)$:

$$\tau^{[1,2]} R\pi_{I_{c}} R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)[2] \to \mathcal{L}_{I_{c}(X)}.$$  

Here $\pi_{I_{c}}$ is the projection $X \times I_{c}(X) \to I_{c}(X)$ and $\mathbb{I}^\bullet \in D(X \times I_{c}(X))$ is the 2-term complex

$$\mathcal{O}_{X \times I_{c}(X)}(-\pi_{X_{\ast}}H) \to \mathcal{E}$$

defined by the universal stable pair $\mathcal{O} \to \mathcal{E} \otimes \pi_{X_{\ast}}\mathcal{O}(H)$. (From now on we suppress the $\pi_{X_{\ast}}$.) The arrow in (34) is the composition of Serre duality for $R\pi_{I_{c}} R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)$ and the Atiyah class of $\mathbb{I}^\bullet$, just as in (33).

We restrict their result to $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ using Proposition B.1. Their precise $H \gg 0$ condition is that the sheaves in their stable pairs are $H$-regular; this becomes the cohomology vanishing condition (30). The first row of the diagram (24) gives the quasi-isomorphism

$$\mathbb{I}^\bullet \otimes \mathcal{O}(H - \mathcal{H}) = \{ \mathcal{O}_{X \times \text{Hilb}_{\beta,n}(\mathcal{H}/|H|)}(-\mathcal{H}) \to \mathcal{I}_Z \} \simeq i_{\ast}I_Z.$$

Using the line bundle $\mathcal{O}(1)$ pulled back from the projective space $|H|$, we have the isomorphism

$$\mathcal{O}(\mathcal{H}) \cong \mathcal{O}(H) \boxtimes \mathcal{O}(1).$$
In particular, \( \mathcal{I}^* \simeq i_*\mathcal{I}_Z(\mathcal{H} - H) \approx i_*\mathcal{I}_Z(1) \), so (34) can be written
\[
\tau^{[1,2]} R\pi_{H*} R\text{Hom}(i_*\mathcal{I}_Z(1), i_*\mathcal{I}_Z(1))[2] \rightarrow \mathcal{I}_{\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)}.
\]
Again the arrow is given by the composition of Serre duality and the Atiyah class of \( i_*\mathcal{I}_Z(1) \).

This is almost identical to (31, 33) except for the twist by \( \mathcal{O}(1) \). Now
\[
\text{At}(i_*\mathcal{I}_Z(1)) = \text{At}(i_*\mathcal{I}_Z) + \text{id}_{i_*\mathcal{I}_Z} \otimes \text{At}(\mathcal{O}(1))
\]
in \( \text{Ext}^1(i_*\mathcal{I}_Z(1), i_*\mathcal{I}_Z(1) \otimes \pi^*_H\mathcal{I}_{\text{Hilb}}) \cong \text{Ext}^1(i_*\mathcal{I}_Z, i_*\mathcal{I}_Z \otimes \pi^*_H\mathcal{I}_{\text{Hilb}}) \). That is, tensoring with \( \mathcal{O}(1) \) changes the map (32) by addition of the following composition
\[
\pi^*_H \mathcal{I}_{\text{Hilb}} \xrightarrow{\text{At}(\mathcal{O}(1))} \mathcal{O}_{X \times \text{Hilb}}[1] \xrightarrow{\text{id}_{i_*\mathcal{I}_Z}} R\text{Hom}(i_*\mathcal{I}_Z, i_*\mathcal{I}_Z)[1].
\]
Therefore (33) changes by addition of the composition
\[
\mathcal{I}_{\text{Hilb}} \xrightarrow{\mathcal{O}(\mathcal{O}(1))} R\pi_{H*} \mathcal{O}_{X \times \text{Hilb}}[1] \xrightarrow{\text{id}_{i_*\mathcal{I}_Z}} R\pi_{H*} R\text{Hom}(i_*\mathcal{I}_Z, i_*\mathcal{I}_Z)[1].
\]
The truncation procedure gives unique lifts to the \( \tau^{[0,1]} \) truncation of the central and right hand terms. But our \( H^1(X, \mathcal{O}_X) = 0 \) condition means that \( \tau^{[0,1]} \) applied to the central term is zero. \( \square \)

Thus we can define invariants counting the pairs (18). Then we will relate them to the MNOP invariants \( I_{\beta,n} \) counting subschemes \( Z \subset X \).

**Definition B.4.** Let \( X \) be a Calabi-Yau 3-fold with \( H^1(X, \mathcal{O}_X) = 0 \) and suppose that \( H \) is sufficiently positive with respect to \( \beta, n \) in the sense of Definition B.2.

The perfect obstruction theories of Corollary B.3 and [Tho00] respectively endow \( \text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \) and \( I_n(X, \beta) \) with virtual cycles of dimension zero. We define
\[
N^H_{\beta,n} := \int_{[\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)]_{\text{vir}}} 1, \quad I_{\beta,n} := \int_{[I_n(X, \beta)]_{\text{vir}}} 1.
\]
Since these obstruction theories are symmetric, by [Beh09] the invariants can also be written as weighted Euler characteristics
\[
N^H_{\beta,n} = \chi(\text{Hilb}_{\beta,n}(\mathcal{H}/|H|), v_{\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)}), \quad I_{\beta,n} = \chi(I_n(X, \beta), v_{I_n(X, \beta)}),
\]
where \( v_M \) denotes the Behrend function [Beh09] of a scheme \( M \).

By the usual arguments [HT10, JS11, Tho00] these invariants are unchanged under smooth deformation of \( X \).
B.3. Relation to MNOP invariants. There is an obvious forgetful map
\[(36) \text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \rightarrow I_n(X,\beta), \quad (Z \subset H) \mapsto Z.\]
The fiber over $Z \subset X$ is $\mathbb{P}(H^0(I_Z(H)))$. More globally we have the following result. We use the notation
\[Z \subset X \times I_n(X,\beta) \xrightarrow{\pi_I} I_n(X,\beta)\]
for the universal subscheme and the projection to the second factor.

**Lemma B.5.** Supposing again that $\beta, n, H$ satisfy \[(30),\]
then the map \[(36)\] is the projective bundle
\[\text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \cong \mathbb{P}(\pi_I^*I_Z(H)).\]

**Proof.** The cohomology vanishing condition \[(30)\] ensures that $\pi_I^*I_Z(H)$ is a vector bundle. Its pullback to $\mathbb{P}(\pi_I^*I_Z(H))$ carries a tautological subbundle
\[O(-1) \hookrightarrow \pi_I^*I_Z(H),\]
where again we have suppressed the pullback map. By adjunction we get
\[\pi_I^*O(-1) \hookrightarrow I_Z(H),\]
and so a nowhere vanishing section of $I_Z(H) \boxtimes O(1) = I_Z(H)$. This is a family of stable pairs over the base $\mathbb{P}(\pi_I^*I_Z(H))$, classified by a map
\[(37) \quad \mathbb{P}(\pi_I^*I_Z(H)) \rightarrow I_n(X,\beta) \cong \text{Hilb}_{\beta,n}(\mathcal{H}/|H|).\]

Conversely, $X \times \text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ carries the universal family of stable pairs \[(25).\] Twisting by $O(-1)$ (pulled back from $|H|$) gives
\[O(-1) \xrightarrow{s} I_Z(H).\]
Pushing down to $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ gives
\[(38) \quad O(-1) \hookrightarrow \pi_H^*I_Z(H).\]
Now $\pi_H^*I_Z(H)$ on $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|)$ is the pullback of the bundle $\pi_I^*I_Z(H)$ on $I_n(X,\beta)$ via the map \[(36).\] Therefore the line subbundle \[(38)\] is classified by a map $\text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \rightarrow \mathbb{P}(\pi_I^*I_Z(H))$, the inverse of \[(37).\]

**Corollary B.6.** For $H^1(X,O_X) = 0$ and $\beta, n, H$ satisfying \[(30),\]
the invariants of Definition B.4 satisfy
\[N^H_{\beta,n} = (-1)^{c-1} c \cdot I_{\beta,n},\]
where $c = c(\beta,n,H,X)$ is the topological number
\[c = \chi(I_Z(H)) = \int_X \left(\frac{1}{6}H^3 + H^2 \text{td}_2(X) - H \cdot \beta\right) - n.\]
Proof. By Lemma B.5, the map (36)

\[ \text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \rightarrow I_n(X,\beta) \]

is smooth of relative dimension \( \chi(I_Z(H)) - 1 \). Therefore by [Beh09, Proposition 1.5(i)], the Behrend function of \( \text{Hilb}_{\beta,n}(\mathcal{H}/|H|) \) is the pullback of that of \( I_n(X,\beta) \), multiplied by \((-1)^{\chi(I_Z(H)) - 1}\). Since the fibers of the map (36) have Euler characteristic \( \chi(I_Z(H)) \), the formulas (35) give the result. \( \square \)

Corollary B.6 finishes the proof of Theorem 4.

B.4. S-duality and modularity. Our motivation for defining the invariants \( N_{\beta,n}^H \) was to try to understand how to define the “supersymmetric BPS invariants associated to D4-D2-D0 systems” studied by string theorists [GSY07, GY07, OSV01]. Their S-duality conjecture is that the generating series of these putative BPS invariants should be modular.

Most of the examples of D4-D2-D0 systems studied in [GY07 Sections 2.1–2.6] are of the form (21) above, which is what led to our definition. However it seems that in general one should count all (semi)stable torsion sheaves of the right topological type supported on hyperplanes, not just those of the form (21). (Thanks to Tudor Dimofte and Davesh Maulik for discussions on this point.)

Sometimes all of the sheaves (21) are stable, for instance when any member of the linear system \(|H|\) is reduced and irreducible. (The hyperplane sections of the quintic threefold have this property, by the Lefschetz hyperplane theorem.) In that case one can think of \( N_{\beta,n}^H \) as the contribution of the component \( \text{Hilb}_{\beta,n}(X) \) of the moduli space of torsion sheaves to the physicists’ numbers.

More generally one would expect to be able to relate our invariants \( N_{\beta,n}^H \) to invariants counting more general torsion sheaves via a sequence of wall crossings. \(^{10}\) Ideally these would be in the space of Bridgeland stability conditions on \( D(X) \), starting from a stability condition that approximates Joyce-Song stability for the complexes (22), and ending with one approximating Gieseker stability for the quasi-isomorphic sheaves (21).

In combination with Corollary B.6, this would express the MNOP invariants \( I_{\beta,n} \) in terms of modular forms. We plan to return to this in future work.

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\(^{10}\) In [Tod11] Toda studies some different but remarkable wall crossings. These relate the counting of torsion sheaves to the counting of both ideal sheaves [MNOP06] and stable pairs [PT09]. There may be some connection: a hint is provided by the fact that Fujita’s conjecture (which would determine which \( H \) are sufficiently positive for given \( \beta, n \)) enters into his analysis.
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