Einstein gravity in the Palatini first order formalism is shown to possess a vector supersymmetry of the type encountered in the topological gauge theories. A peculiar feature of the gravitational theory is the link of this vector supersymmetry with the field equation of motion of the Faddeev-Popov ghost associated to diffeomorphism invariance.

1 Introduction and Conclusions

I think it is relevant, in a conference organized to the memory of Efim Fradkin, who devoted an important part of his work to the understanding of fundamental symmetries, to talk about some not broadly known symmetry features of theories such as topological and gravitational theories. These theories are well known to be characterized by their invariance under the diffeomorphims of the space-time manifold (“general coordinate invariance”). Also, both types of theories are gauge theories, the gauge group of gravity being the Lorentz group. On the other hand, an interesting feature of topological theories such as Chern-Simons or $BF$ theory, is the presence of a “vector supersymmetry” – a supersymmetry whose generator is a vector valued operator [1]. In case the manifold admits isometries generated by Killing vectors – e.g. the space-time translations if the background metric is flat – the vector supersymmetry is a symmetry of the gauge-fixed action. It happens that its generator together with the BRST symmetry generator form an algebra which closes on the generators of the isometries of the translation type [2]. Vector supersymmetry has been shown to play a key role in the ultraviolet finiteness of the topological theories [3, 4].

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Talk given at the International Conference “Quantization, Gauge Theory, and Strings” dedicated to the memory of Professor Efim Fradkin, Moscow, 2000.
The question addressed in the present talk is about the possibility of such a vector supersymmetry in gravitation theory.

In the Palatini or first order formalism of gravitation theory, the independent dynamical variables are the vierbein 1-form $E$ (giving the metric $G$) and the connection 1-form $A$. The vierbein dependence of the connection is given by its field equation, whereas Einstein equation results from the field equation with respect to $E$. The action can be written, following \cite{6, 7}, in a “topological” form, i.e. in such a way that it can be interpreted as an action of the 1-form fields $E$ and $A$ on a differentiable manifold $M$, without reference to any a-priori background metric. The latter point is known \cite{5} to be an essential characteristic of topological theories, and trying to exploit this feature belongs to the spirit of the modern attempts towards a construction of quantum gravity (see \cite{8, 9} for reviews and further references).

Both local symmetries – diffeomorphism and local Lorentz invariances – have to be gauge fixed. We shall choose a gauge fixing of the Landau type, within the BRST formalism \cite{4}. Much in the same way as in topological theories, this requires the introduction of a nondynamical, background metric $g$. It should be clear that the background metric, being introduced only in the gauge fixing part of the theory, should not affect in any way the physical outcome, as it has been explicitly shown in \cite{2} for the – perturbative – quantum version of the Chern-Simons theory.

An interesting feature of gauge theories in a Landau type gauge is the so-called ghost equation \cite{10, 4}, which restricts the coupling of the ghosts and implies the nonrenormalization of their field amplitude\cite{4}. We shall see that the vector supersymmetry in the gravitational case is a direct consequence of the field equation of the Faddeev-Popov ghost associated to diffeomorphism invariance.

Vector supersymmetry will turn out to be maximal if the background metric is flat \cite{2}. Whereas, in this case, the supersymmetry generators of the topological theories are the components of a vector, the superalgebra closing on the translations, the supersymmetry generators of Einstein gravity in the Palatini formalism will be seen to be the components of one vector and one antisymmetric tensor: the full algebra here contains the ten Poincaré generators.

The present work is concerned only with the classical aspects of the theory. However, the results are of interest since they reveal the link between the construction of the observables via the $\delta$ operator of Sorella \cite{11} associated to vector symmetry, on one hand, and diffeomorphism invariance and the corresponding ghost equation, on the other hand.

A more detailed account of the results presented here may be found in \cite{12}.

\footnote{A review of the properties of topological theories mentioned above may be found in Chapters 6 and 7 of ref. \cite{4}.}
2 Action for Gravity in the Palatini Formalism

The Einstein gravity Lagrangian in the first order formalism of Palatini \cite{5} may be written as \cite{6}:

\[ S_{\text{inv}} = \frac{1}{4} \int_M \varepsilon_{IJKL} E^I \wedge E^J \wedge F^{KL}(A) + S_{\text{matter}}(E, A, \Phi) . \]  

(1)

The integral is taken over some differentiable manifold \( M \), \( E^I \) is a vierbein 1-form, with \( I = 0, \cdots, 3 \) a tangent plane Lorentz index. \( F^{KL} \) is the curvature 2-form

\[ F^{IJ}(A) = dA^{IJ} + A^{IK} \wedge A^{KJ} \]  

(2)

of a connection \( A^{IJ} \), the latter being taken as an independent variable \( 3 \). \( \varepsilon_{IJKL} \) is the rank four antisymmetric tensor, normalized by \( \varepsilon_{0123} = 1 \). In the following, the exterior multiplication symbol \( \wedge \) will be omitted. \( S_{\text{matter}} \) is some action for minimally coupled matter fields \( \Phi \), which we don’t need to specify. We shall in fact omit this part in the following, for the sake of simplicity and without loss of generality.

The field equations given by the variations of this action read

\[ \frac{\delta S_{\text{inv}}}{\delta E^I} = \frac{1}{2} \varepsilon_{IJKL} E^J F^{KL}, \]

\[ \frac{\delta S_{\text{inv}}}{\delta A^{IJ}} = \varepsilon_{IJKL} E^K D E^L, \]  

(3)

where \( D \) is the covariant exterior derivative: \( D E^I = dE^I - A^I J E^J \). It is known \cite{5, 6} that they lead to the usual specification of the connection as a function of the vierbein and to the Einstein equation, in a Riemann space-time with metric

\[ G_{\mu\nu} = E_\mu^I E_\nu^J \eta_{IJ}, \]  

(4)

where \( \eta_{IJ} \) is the flat metric used to lower and rise the tangent space indices \( I, J, \cdots \).

The action \( \square \) is invariant under the diffeomorphisms. In infinitesimal form, they read

\[ \delta_{(\xi)} \varphi = \mathcal{L}_\xi \varphi, \quad \varphi = E^I, A^{IJ}, \]  

(5)

where the infinitesimal parameter is a vector field \( \xi \) and \( \mathcal{L}_\xi \) is the Lie derivative along \( \xi \). The action is also invariant under the local Lorentz transformations which, in infinitesimal form, read

\[ \delta_{(\omega)} E^I = \omega^{IJ} E^J, \]

\[ \delta_{(\omega)} A^{IJ} = d\omega^{IJ} + \omega^K_J A^{KJ} + \omega^K_I A^{IK}, \]  

(6)

with local parameters \( \omega^{IJ} = -\omega^{JI} \).

\footnote{If the connection is self-dual, \( \square \) is the Ashtekar action \cite{13, 3}.}

\footnote{In a particular coordinate frame with \( x = (x^\mu, \mu = 0, \cdots, 3) \), \( E^I = E_\mu^I dx^\mu \), \( F^{IJ} = \frac{1}{2} F_{\mu\nu}^{IJ} dx^\mu \wedge dx^\nu \), etc.}
3 Three Dimensional BF Topological Theory

Before continuing with gravity, let us recall some facts about topological theories, specializing on the 3-dimensional BF model. The fields are a 1-form gauge potential $A$ and a 1-form field $B$ (in matrix notation):
\[
A = A_\mu(x) dx^\mu = A_\mu^a(x) T_a dx^\mu,  \\
B = B_\mu(x) dx^\mu = B_\mu^a(x) T_a dx^\mu,
\]
where the gauge group generators $T_a$ obey the algebra and the trace convention
\[
[T_a, T_b] = i f_{abc} T_c, \quad Tr(T_aT_b) = \delta_{ab}.
\]
Introducing the ghost fields $c$, $b$, as well as the antighost $\bar{c}$, $\bar{b}$ and the Lagrange multipliers fields $\pi$, $\lambda$, we may write the BRST transformations of the theory as
\[
sA_\mu = D_\mu c + [c, A_\mu], \quad sc = \frac{1}{2} \{c, c\} = c^2,  \\
sB_\mu = [c, B_\mu] + D_\mu b, \quad sb = \{c, b\},  \\
s\bar{c} = \pi, \quad s\pi = 0,  \\
s\bar{b} = \lambda, \quad s\lambda = 0.
\]
The ghost and antighost fields are anticommuting scalar fields, whereas the Lagrange multipliers are commuting. All fields belong to the adjoint representation of the gauge group:
\[
\varphi(x) \equiv \varphi^a(x) T_a
\]
We have directly written the gauge transformations of the theory in the form of BRST transformations. The usual gauge transformations of $A$ and $B$ are the transformations (9) with the ghost fields $b$ and $c$ replaced by ordinary local infinitesimal c-number parameters. The transformation corresponding to the ghost $b$ is specific of the BF models. The BRST transformations (9) are nilpotent:
\[
s^2 = 0.
\]
The BRST-invariant action reads
\[
S = S_{\text{inv}} + S_{gf},
\]
where the first term is the gauge invariant action of the BF theory [14]:
\[
S_{\text{inv}} = Tr \int_{R^3} BF = \frac{1}{2} Tr \int d^3x \varepsilon^{\mu\nu\rho} B_\mu F_{\nu\rho},
\]
with
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu],
\]
and the second term of (11) is the gauge fixing action

\[ S_{gf} = -s \text{Tr} \int d^3x \sqrt{g} g^{\mu\nu} \left( \partial_\mu \bar{c} A_\nu + \partial_\mu \bar{b} B_\nu \right) \]

\[ = \text{Tr} \int d^3x \sqrt{g} g^{\mu\nu} \left( -\partial_\mu \pi A_\nu - \partial_\mu \lambda B_\nu + \partial_\mu \bar{c} s A_\nu + \partial_\mu \bar{b} s B_\nu \right) , \]  

implementing the Landau gauge conditions \( \nabla_\mu A_\mu = 0 \) and \( \nabla_\mu B_\mu = 0 \), with \( \nabla \) the covariant derivative with respect to a background metric \( g_{\mu\nu} \).

A characteristic feature of topological theories such as the present model, in a Landau type gauge, is the invariance of the action under vector supersymmetry, a supersymmetry whose generators are components of a vector. The infinitesimal vector supersymmetry transformations read – we set here the background metric to be flat:

\[ \delta_{\text{vector susy}} \varphi = \varepsilon^\mu \delta_\mu \varphi , \]

where \( \varepsilon^\mu \) are three infinitesimal Grassmann parameters, and:

\[ \delta_\mu c = A_\mu , \quad \delta_\mu b = B_\mu , \]

\[ \delta_\mu A_\nu = \varepsilon_{\mu\rho\sigma} \partial_\sigma \bar{b} , \quad \delta_\mu B_\nu = \varepsilon_{\mu\rho\sigma} \partial_\sigma \bar{c} , \]

\[ \delta_\mu \pi = \partial_\mu \bar{c} , \quad \delta_\mu \lambda = \partial_\mu \bar{b} , \]

\[ \delta_\mu \bar{c} = 0 , \quad \delta_\mu \bar{b} = 0 . \]

One notes that these transformations lower the ghost number by one unit. Together with the BRST operator, they obey the Wess-Zumino-like superalgebra [1]

\[ \{ s, s \} = 0 , \quad \{ \delta_\mu , \delta_\nu \} = 0 , \quad \{ s , \delta_\mu \} = \partial_\mu . \]

Another typical feature – in fact shared by all gauge theories in Landau-type gauges – are the ghost equations [10]

\[ \int d^3x \left( \frac{\delta S}{\delta c} - \left[ \bar{c}, \frac{\delta S}{\delta \pi} \right] - \left[ \bar{b}, \frac{\delta S}{\delta \lambda} \right] \right) = 0 , \quad \int d^3x \left( \frac{\delta S}{\delta \bar{c}} - \left[ \bar{c}, \frac{\delta S}{\delta \pi} \right] - \left[ \bar{b}, \frac{\delta S}{\delta \lambda} \right] \right) = 0 , \]  

which characterize the coupling of the ghost fields. Together with vector supersymmetry, these equations imply the ultraviolet finiteness – or scale invariance – of the topological models where they hold (see [4] and references therein).

### 4 4-Dimensional Gravity

Whereas gravity in 3-dimensional space-time is equivalent to a topological \( BF \) theory, the gauge invariance being local Lorentz invariance, this is no more the case in 4 dimensions and higher. However, as announced in the Introduction, some features
of topological theories are kept, such as the ghost equations and vector supersymmetry. Let us first write the BRST transformations leaving the Palatini action (1) invariant [15]:

\[ sE^I = L \xi E^I + \omega^I J E^J , \]
\[ sA^{IJ} = L \xi A^{IJ} + d \omega^{IJ} + \omega^I K A^{KJ} + \omega^J K A^{IK} , \]
\[ s\xi = \frac{1}{2} \{ \xi, \xi \} , \; \text{ (or: } s\xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu ) , \]
\[ s\omega^I J = L \omega^I J + \omega^I K \omega^K J , \]

(18)

with the nilpotency property \( s^2 = 0 \). They correspond to local Lorentz invariance and diffeomorphism invariance. The infinitesimal parameters \( \xi^\mu (x) \) - the components of the vector \( \xi \) – and \( \omega^I J (x) \) of the respective symmetries are now Grassmann (i.e. anticommuting) number fields – the Faddeev-Popov ghosts. The bracket \( \{ , \} \) is the Lie bracket \( \{ u, v \}^\mu = u^\lambda \partial_\lambda v^\mu \pm v^\lambda \partial_\lambda u^\mu \), with the sign + if both \( u \) and \( v \) are odd, and the sign − otherwise.

In order to gauge fix the theory with respect to its local symmetries – diffeomorphism and local Lorentz invariances – we introduce the respective antighost \( \bar{\xi}^I \), \( \bar{\omega}^I J \) and Lagrange multipliers \( \lambda_I, b_{IJ} \), with the following nilpotent BRST transformations:

\[ s\bar{C}^i = \Pi_i , \; s\Pi_i = 0 , \; i = 1, 2 , \]

(19)

where we are using the condensed notation

\[ \{ \bar{C}^i, i = 1, 2 \} = \{ \xi^I, \bar{\omega}^I J \} , \; \{ \Pi_i, i = 1, 2 \} = \{ \lambda_I, b_{IJ} \} , \]
\[ \{ A^I \mu, i = 1, 2 \} = \{ E^I \mu, A^I \mu \} , \]

(20)

The gauge fixing part of the action is then defined as:

\[ S_{gf} = -s \int_M d^4 x \sqrt{-g} \sum_{i=1}^2 \partial_\mu \bar{C}^i A^i, \]
\[ = \int_M d^4 x \sqrt{-g} \sum_{i=1}^2 \left( -\partial_\mu \Pi_i A^i + \partial_\mu \bar{C}^i sA^i \right) , \]

(21)

which is automatically BRST invariant. Note that in order to contract the world indices \( \mu, \nu \) we had to introduce a background metric \( g_{\mu\nu} \) – not to be confounded with the physical, dynamical metric \( G_{\mu\nu} \) defined in (4).

It is worthwhile to remark that the gauge fixing action is completely determined by its BRST invariance and the “gauge condition”

\[ \frac{\delta S}{\delta \Pi_i} = \partial_\mu \left( \sqrt{-g} g^{\mu\nu} A^i_\nu \right) , \]

(22)

where \( S \) is the total action

\[ S = S_{inv} + S_{gf} . \]

(23)
As in the BF model of last section, the ghost $\xi$ and $\omega$ obey ghost equations. Let us concentrate on the equation for $\xi$:

$$
\delta S \frac{\delta}{\delta \xi^\mu} = \sum_{i=1,2} \left( -\sqrt{-g} g^\rho\lambda \partial_\chi \bar{C}_i \partial_\mu A_i^\rho + \partial_\nu \left( \sqrt{-g} g^{\nu\lambda} \partial_\lambda \bar{C}_i A_i^\mu \right) \right).
$$

(24)

What is interesting with this ghost equation, is that it may be interpreted as a Ward identity for a vector supersymmetry, at least for some class of background metrics. Let us first consider the simpler case of a flat background metric and write the space-time integral of the ghost equation, having used integrations by parts and the gauge condition (22):

$$
\delta \mu S \equiv \int_M d^4x \left( \frac{\delta}{\delta \xi^\mu} + \sum_{i=1,2} \partial_\mu \bar{C}_i \frac{\delta}{\delta \Pi_i} \right) S = 0.
$$

(25)

Indeed, the last equation expresses the invariance of the action under the infinitesimal transformations

$$
\begin{align*}
\delta_\mu \xi^\nu &= \delta_\nu^\mu, \\
\delta_\mu \Pi_i &= \partial_\mu \bar{C}_i, \\
\delta_\mu \varphi &= 0, \quad \varphi \neq \xi^\mu, \quad \Pi_i,
\end{align*}
$$

(26)

The supersymmetry operators $\delta_\mu$, together with the BRST operator $s$, obey the superalgebra

$$
\{s, s\} = 0, \quad \{\delta_\mu, \delta_\nu\} = 0, \quad \{s, \delta_\mu\} = \partial_\mu,
$$

(27)

where the partial derivative $\partial_\mu$ is the infinitesimal generator of the translations.

In the case of a general non-flat background metric, we define the vector supersymmetry transformations

$$
\delta_\varepsilon^S \equiv \varepsilon^\mu \delta_\mu,
$$

(28)

with $\delta_\mu$ given by (24), and where the infinitesimal parameter $\varepsilon^\mu$ is a vector field – taken as commuting, to the contrary of $\xi^\mu$. The supersymmetry operators $\delta_\varepsilon^S$, together with the BRST operator $s$, obey now the superalgebra

$$
\{s, s\} = 0, \quad \{\delta_\varepsilon^S, \delta_\varepsilon'^S\} = 0, \quad \{s, \delta_\varepsilon^S\} = \partial_\mu,
$$

(29)

where the Lie derivative $L_\varepsilon$ is the infinitesimal generator of the diffeomorphisms along the vector field $\varepsilon$.

It turns out [12] that the action (23) is invariant under the vector supersymmetry (28) provided the infinitesimal parameter $\varepsilon$ is a Killing vector of the background metric, i.e. satisfies to the condition

$$
L_\varepsilon g^{\mu\nu} = 0, \quad \text{or:} \quad \nabla_\mu^{(g)} \varepsilon_\nu + \nabla_\nu^{(g)} \varepsilon_\mu = 0.
$$

(30)
Thus, vector supersymmetry holds if the background metric admits Killing vectors. The number of independent Killing vectors is maximum for a flat background metric. In this case, the general solution of the condition (30) reads
\[ \varepsilon^\mu = a^\mu + b^{\mu\nu} x_\nu, \quad \text{with} \quad a^\mu, b^{\mu\nu} = -b^{\nu\mu} \text{ constant parameters}. \] (31)

The right hand side of the anticommutator in (29) is then an infinitesimal rigid Poincaré transformation of parameters \( a^\mu \) and \( b^{\mu\nu} \). This last result is in contrast to the one obtained in the case of the topological models (see e.g. [2] for the 3-dimensional Chern-Simons theory), where namely only the vector supersymmetry associated to translation invariance is obtained in the flat limit.

Let me finally mention that, as in the Yang-Mills case, the vector symmetry operator may be expressed in the form of the so-called operator \( \delta \) of Sorella [11] used to construct the invariants of the theory, and characterized by the algebraic relation
\[ [\delta, s] = d, \] (32)
where \( d \) is the exterior derivative. A construction of the operator \( \delta \) is given in [10]. More details may be found in [12].

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