Butterfly velocity in quadratic gravity

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Abstract

We present a systematic procedure of finding the shock wave equation in anisotropic spacetime of quadratic gravity with Lagrangian

\[ L = R + \Lambda + \alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2 + L_{\text{matter}}. \]

The general formula of the butterfly velocity is derived. We show that the shock wave equation in the planar, spherical or hyperbolic black hole spacetime of Einstein–Gauss–Bonnet gravity is the same as that in Einstein gravity if the space is isotropic. We consider the modified AdS spacetime deformed by the leading correction of the quadratic curvatures and find that the fourth order derivative shock wave equation leads to two butterfly velocities if \( 4\alpha + \beta < 0 \). We also show that the butterfly velocity in the \( D = 4 \) planar black hole is not corrected by the quadratic gravity if \( 4\alpha + \beta = 0 \), which includes the \( R^2 \) gravity. In general, the correction of butterfly velocity by the quadratic gravity may be positive or negative, depending on the values of \( \alpha, \beta, \gamma \) and temperature. We also investigate the butterfly velocity in the Gauss–Bonnet massive gravity.

Keywords: gravity/gauge duality, quantum chaos, holographics

1. Introduction

The notion of the out-of-time-order (OTO) correlator was introduced in the context of semiclassical methods in superconductivity many years ago [1]. Recently, it has received renewed attention in the context of the AdS/CFT correspondence, where the butterfly effects (quantum chaos) can be studied by looking at black holes. Many novel aspects and interesting properties have been found [2–10]. The OTO four-point function between a pair of local operators \( W(t, x), V(t) \) is defined by

\[ \langle [W(t, x), V(0)]^2 \rangle_T \sim e^{\lambda(t-t_*) - \frac{|x|}{v_B}}. \]

(1.1)

For the case of \( W(t) = x(t) \) and \( V(0) = p_x \), in the semiclassical limit where \( [x(x), p_x] = i\hbar \{x(t), p_t\} = i\hbar \frac{\partial x}{\partial t(0)} \) which describes how the final position depends on the small changes of the initial position, this correlator can be used to diagnose the quantum chaos. The
constant $t_*$ in (1.1) is the scrambling time at which the commutator grows to be $O(1)$. The butterfly velocity $v_B$ characterizes speed at which the perturbation grows; the Lyapunov exponent $\lambda$ measures the rate of growth of chaos.

It was found that the Lyapunov exponent is bounded by temperature $T$, $\lambda \leq 2\pi \beta$ where $\beta = \frac{1}{T}$ [8]. The inequality saturates for a thermal system that has a dual black hole described by the Einstein gravity. The quantum correction to the chaos, which modifies the Lyapunov was studied in [9]. Butterfly velocity has also an interesting property that it is related to the diffusion constant [11–14].

In the context of holography, the shock wave near the black hole horizon provides a description of the butterfly effect in the dual field theory and butterfly velocity is identified by the velocity of shock wave, which describes how the perturbation spreads in space [2, 3]. The method of finding the shock wave velocity for the general spacetime with matters was described in many years ago [15, 16]. Such a method has been used to obtain the butterfly velocity in the dual field theory by studying shock waves of the various black holes [17–35]. In [2], it was found that the speed of propagation is

$$v_B = \sqrt{\frac{D - 1}{2(D - 2)}},$$

where $D$ is the spacetime dimension of the bulk theory. Later investigations have extended this result to anisotropic black holes/branes [22, 26, 27, 29, 31, 33].

The butterfly velocities described by the higher derivative gravities, including 3D massive gravity and Gauss–Bonnet gravity, were discussed in [5, 23, 24, 26, 27, 33]. It was found that the Gauss–Bonnet term (in the absence of matters) modifies the screaming time but does not change the shock wave equation [5]. The butterfly velocity in four-derivative gravity for isotropic spacetimes was partially studied in [23].

Higher-derivative gravities find many applications in the study of holography. For instance, the authors of [38–41] considered a Lagrangian

$$\mathcal{L} = R + \Lambda + \alpha R_{abcd}R^{abcd} + \beta R_{ab}R^{ab} + \gamma R^2 + \mathcal{L}_{\text{matter}}$$

and found that for a small value of quadratic curvature the ratio of the shear viscosity $\eta$ to the entropy density $s$ is modified by $\eta/s = \frac{1}{4\pi} (1 - 4\alpha)$. The viscosity bound established in [42] is therefore violated in higher derivative gravities for $\alpha > 0$. Holographic superconductors with higher curvature corrections had been extensively studied [43–45]. Holographic shear sum rule in Einstein gravity corrected by squared curvature was studied in [46]. The higher derivative terms are shown to have a strong impact on the bound on charge diffusion [47, 48].

In the holography the addition of higher derivative corrections to gravity theories could be used to probe the dual physics moving away from infinite $N$. For example, in the study of viscosity bound violation Kats and Petrov [38] showed that $\alpha = \frac{1}{2N}$ for $N \gg 1$.

Note that at leading order of $\alpha$, $\beta$, $\gamma$ only $\alpha$ is unambiguous while $\beta$ and $\gamma$ can be arbitrarily altered by a field redefinition [38]. However, the coefficients $\beta$ and $\gamma$ in (1.3) are physical once the matter fields are turned on. For example, in an Einstein–Maxwell theory, shifting away the coefficients $\beta$ and $\gamma$ will generate new mixed terms of the form $RF^2$ and $R_{\mu\nu}F^{\mu\lambda}F_{\nu}^{\lambda}$, which is relevant in the studies of $R$-charged backgrounds [49].

1 The Gauss–Bonnet gravity is the simplest quadratic correction of the Einstein theory without introducing derivatives higher than second in the field equation. Higher derivative corrections arise due to stringy corrections of the classical action. In terms of AdS/CFT correspondence this corresponds to next-to-leading order corrections in the $1/N$ expansion of the dual CFT [36, 37].
Note also that in the context of string theory and holography the higher-order gravity is viewed as part of infinite series of corrections to the leading order string effective action. The action of higher-order gravity is not quantized, which makes the issue of ghostlike modes moot. Therefore, in despite of the existence of ghosts in the higher-order gravity except some special cases, such as Gauss–Bonnet gravity, we will study in this paper the general case with three coefficients of $\alpha$, $\beta$, $\gamma$. Since that the $D+1$ dimensional bulk theory will dual to $D$-dimensional boundary field theory the investigation in this paper can then be applied to any $D$-dimensional field theory.

In the previous paper [33], we had considered the Gauss–Bonnet gravity with arbitrary matter fields and derived a general formula of butterfly velocity. We calculated the butterfly velocity in planar, spherical, and hyperbolic black holes for the Gauss–Bonnet gravity, with Maxwell and scalar fields. The goal of this work is to consider butterfly velocity in anisotropic space of more general quadratic gravity (1.3) with arbitrary matters.

The paper is organized as follows.

In section 2, we briefly discuss some basic properties of quadratic gravity, including the $R^2$ gravity, Gauss–Bonnet gravity, conformal gravity and Gauss–Bonnet massive gravity.

In section 3, after describing general anisotropic space, we briefly discuss the Kruskal coordinates and derivation of the shock wave equation. The shock wave equation of Einstein gravity is presented in (3.19). The derivations were detailed in our previous paper.

In section 4, we first collect five relations and then use them to simplify the shock wave equation in the quadratic gravity. The equation is shown in (4.19) which has only six terms. Then, after calculations the exact formulas of the six term are presented. Using the formulas, as a simple example we obtain the formula of butterfly velocity in the anisotropic space of Gauss–Bonnet gravity theory. The formula is shown in (4.26) which, in the case of isotropic space, reproduces the equation (4.18) in our previous paper [33]. The double summations in the formula (4.26) means that the shock wave equation in the anisotropic space is a non-trivial extension of that in the isotropic space. Using the formula we prove that: in the $D$-dimensional planar, spherical or hyperbolic black hole spacetime the Einstein–Gauss–Bonnet gravity has the same shock wave equation as that in Einstein gravity if and only if the space is isotropic.

In section 5 we simplify the formula to case of isotropic spacetime in which the black hole is planar, spherical or hyperbolic. Then we apply the result to the metric in [38], which is an analytic solution in leading order of quadratic gravity, to calculate the butterfly velocity. The final formulas are shown in equations (5.26) and (5.27). In higher derivative gravity the shock wave equation is fourth order and the metric provides two sources for two operators in the context of AdS/CFT correspondence. Since that each operators results to a butterfly velocity and in general we have two butterfly velocities [23, 24]. However, we show that: Only if $4\alpha + \beta < 0$ could the second velocity appear. We also prove that: The butterfly velocity in $D = 4$ planar black hole does not be corrected by the quadratic gravity if $\beta + 4\alpha = 0$, which includes the $R^2$ gravity. We present various explicit butterfly velocity in the Gauss–Bonnet gravity, conformal gravity, and $R^2$ gravity respectively. We see that, depending on the values of $\alpha$, $\beta$ and $\gamma$ the velocity correction from the quadratic gravity may be from positive to negative or from negative to positive while increasing the temperature. We use our formula to check the two butterfly velocities in a special quadratic gravity which was first derived in Alishahiha’s paper [24]. The butterfly velocity in the Gauss–Bonnet massive gravity is also studied.

The last section is a short summary. Many detailed tensor calculations are collected in the appendices.
2. Quadratic gravity and Gauss–Bonnet massive gravity

2.1. Quadratic gravity

We are interested in the general higher-derivative gravity described in (1.3). The generalized gravitational equation is

\[ G_{ab} = T_{ab}, \]  

where the generalized Einstein tensor is defined by

\[ G_{ab} = R_{ab} + K_{ab} + D_{ab} - \frac{1}{2} \delta_{ab} \left( \mathcal{L} - (\beta + 4\gamma) \Box R \right), \]  

with

\[ K_{ab} = 2\alpha R_{ade} R_b^{de} + 2(2\alpha + \beta) R_{acbd} R^{cd} - 4\alpha R_{ab} R_c^c + 2\gamma R_{ab} R, \]  

\[ D_{ab} = (4\alpha + \beta) \Box R_{ab} - (2\alpha + \beta + 2\gamma) \nabla_a \nabla_b R. \]

Note that \( D_{ab} = 0 \) when \( \beta = -4\alpha, \alpha = -\gamma \): this is the Gauss–Bonnet gravity where terms with higher derivatives (more than second-derivative) do not appear.

The Gauss–Bonnet gravity is non-renormalizable but ghost free. It can be shown that the inclusion of the GB term entails causality violation [50]. However, this does not totally rule out its uses in the holographic context. In the case \( \alpha = -\beta = -\gamma \), one has conformal/Weyl gravity which has ghosts, though it is renormalizable [51, 52]. The \( R^2 \) gravity is a simple modification to the Einstein gravity with various interesting applications. One can, for instance, include \( R^2 \) to study the late-time expansion of the cosmic acceleration [55]. We will consider butterfly velocities in all these cases. More discussions about the higher-derivative gravities can be found in [56].

2.2. Gauss–Bonnet massive gravity

In recent a new class of nontrivial massive black holes in AdS spacetime was studied in [57–59]. In the theory of the massive gravity, the optical conductivity shows an emergent scaling law which is consistent with that found earlier by Horowitz et al. [60] who introduced an explicit inhomogeneous lattice into the system.

Note that the mass terms of the gravitons will be plagued by various instabilities sometimes at the non-linear level. The authors of [61–63] constructed a theory where the Boulware–Deser ghost [64] was eliminated by introducing higher order interaction terms into the Lagrangian which is

\[ \mathcal{L} = R + \Lambda + m^2 \sum_{i=1}^{4} c_i \mathcal{U}_{i}(g, f) + \mathcal{L}_{\text{matters}} \]  

\[ \mathcal{U}_1 = [\mathcal{K}], \quad \mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \quad \mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3] \]  

\[ \mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 8[\mathcal{K}][\mathcal{K}^3] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4] \]

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2 We are grateful to Edelstein for pointing it out.

3 It was discussed in [53, 54] that Einstein gravity may emerge from conformal gravity upon imposing suitable boundary conditions eliminating ghosts.
where $c_i$ are constants, and $\mathcal{K}_\mu^\nu = \sqrt{g_{\mu\nu}}$ with $[\mathcal{K}] = \mathcal{K}_{\mu}^{\mu}$ for a reference metric $f_{\mu\nu}$. For a recent review of massive gravity in this context, see [65].

In the papers [66–70] the thermodynamics of black hole in Gauss–Bonnet massive gravity theory were studied and in this paper, for the sake of completeness, we will adopt the metric solutions therein to calculate the associated butterfly velocity.

### 3. Kruskal coordinate and shock wave equation

For self-consistent we briefly describe the Kruskal coordinate and sketch the shock wave equation.

#### 3.1. Kruskal coordinate

We will derive the formula of butterfly velocity in the following anisotropic background:

$$\text{d}s^2 = -a(r)f(r)\text{d}t^2 + \frac{\text{d}r^2}{b(r)f(r)} + \sum_{S=1}^{n} h^{(S)}(r) \bar{g}^{(S)}_{ij}(x) \text{d}x_i^{(S)}\text{d}x_j^{(S)}.$$ (3.1)

The horizon locates at $r = r_H$ then $f(r_H) = 0$ while $a(r_H) \neq 0$ and $b(r_H) \neq 0$. The associated temperature of black hole or black brane is

$$T = \frac{f'(r_H)\sqrt{a(r_H)b(r_H)}}{4\pi}. \quad (3.2)$$

Defining the tortoise coordinate $r_*$ the line element of time and radial parts can be expressed as

$$\text{d}s^2 = -a(r)f(r)\text{d}t^2 + \frac{\text{d}r^2}{f(r)\sqrt{a(r)b(r)}}. \quad (3.3)$$

$$\text{d}r_* = \frac{\text{d}r}{f(r)\sqrt{a(r)b(r)}}. \quad (3.4)$$

The metric can be written in Kruskal coordinate

$$\text{d}s^2 = 2A(UV)dUdV + \sum_{S} h^{(S)}(UV) \bar{g}^{(S)}_{ij}(x)\text{d}x_i^{(S)}\text{d}x_j^{(S)}.$$ (3.5)

where

$$A(UV) = \frac{2a(r)f(r)}{f'(r_H)^2a(r_H)b(r_H)e^{-\sqrt{a(r_H)b(r_H)}f'(r_H)r_*}}.$$ (3.6)

$$U = e^{\sqrt{a(r)b(r)}f'(r_H)\ln(UV)}(-t+r_*). \quad (3.7)$$

$$V = e^{\sqrt{a(r)b(r)}f'(r_H)\ln(UV)}(t+r_*). \quad (3.8)$$

$$r_* = \frac{1}{\sqrt{a(r_H)b(r_H)}f'(r_H)}\ln(UV). \quad (3.9)$$

In the tortoise coordinate $r_*(r_H) = -\infty$ and thus on the horizon $U_H = 0$. Above definitions imply following useful relations
A(U_H) = \frac{2r_H}{f'(r_H) b(r_H)} \tag{3.10}

h'(U_H) = \frac{dR(UV)}{dV} \frac{dR}{dr} \bigg|_{r=r_H} = r_H h'(r_H) \tag{3.11}

A'(U_H) = \left( \frac{dA(UV)}{dV} \frac{dR}{dr} \right)_{r=r_H} = \frac{r_H^2}{b(r_H) f'(r_H)} \left( \frac{3d'(r_H)}{a(r_H)} + \frac{b'(r_H)}{b(r_H)} + \frac{2f'(r_H)}{f(r_H)} \right). \tag{3.12}

Higher derivative terms \( h''(U_H) \) and \( A''(U_H) \) can be evaluated in the similar way.

3.2. Shock wave equation

In the Kruskal coordinate the generalized gravitational equation can be expressed as

\[ G = T_{\text{matter}} = 2T_{UU}(U, V, x) dU dV + T_{UV}(U, V, x) dU dV + T_{VV}(U, V, x) dV dV + \sum_{S} T^{(s)}_{ij}(U, V, x) dx^{i}_{(S)} dx^{j}_{(S)}. \tag{3.13} \]

Along the arguments of Dray and t’Hooft [15], after adding a small null perturbation of asymptotic energy \( E \)

\[ T_{(\text{shock})U} = \frac{E e^{2\pi i/3}}{a(\bar{x})} \delta(\bar{U}) \] \tag{3.14}

the spacetime is still described by (3.5) but \( V \) is shifted by

\[ V \rightarrow V + \alpha(x). \tag{3.15} \]

Through analysis we can find that in terms of the new coordinates [16]

\[ \bar{U} = U, \quad \bar{V} = V + \Theta(U) \alpha(x) \tag{3.16} \]

where \( \Theta = \Theta(U) \) is a step function, the metric can be expressed by

\[ ds^2 = 2\bar{A}(\bar{U}, \bar{V}) d\bar{U} d\bar{V} + \sum_{S} \bar{g}^{(S)}_{ij}(\bar{U}, \bar{V}, \bar{x}) dx^{i}_{(S)} dx^{j}_{(S)} - 2\bar{A}(\bar{x}) \delta(\bar{U}) d\bar{U}^2 \tag{3.17} \]

the generalized gravitational equation, after dropping hat notation, becomes

\[ G^{(1)}_{UU} + 2G^{(0)}_{UV} \alpha(x) \delta(U) = E e^{2\pi i/3} a(x) \delta(U). \tag{3.18} \]

Term \( G^{(1)}_{UU} \) and \( G^{(0)}_{UV} \) are the first-order correction and zero-order generalized Einstein tensor in the metric (3.17) respectively. This is the shock wave equation. Using above formulation we will present a systematic procedure to find the differential equation of \( \alpha(x) \) for the quadratic gravity in the anisotropic spacetime (3.17) and then obtain the associated formula of butterfly velocity.

3.3. Shock wave equation in Einstein gravity

For Einstein gravity theory the tensor calculation in our previous paper [33] gives

\[ G^{(1)}_{UU} + 2G^{(0)}_{UV} \alpha(x) \delta(U) = \frac{A}{2} \sum_{S} \left( \frac{2}{h^{(S)}} \Delta^{(S)} \alpha(x) - \frac{d^{(S)} h^{(S)}}{Ah^{(S)}} \alpha(x) \right) \delta(U). \tag{3.19} \]
and, in the case with local source \(a(x) = \delta(x_{i}^{(Q)})\), the shock wave equation becomes

\[
\[A^{(Q)} - h^{(Q)}(U_H) \sum_{s} \frac{d^{(s)} h^{(s)}(U_H)}{2A(U_H) h^{(s)}(U_H)} \alpha(t, x_{i}^{(Q)}) = E e^{2\pi i/\beta} \frac{h^{(Q)}(U_H)}{A(U_H)} \delta(x_{i}^{(Q)}). \tag{3.20}
\]

The butterfly velocity along the direct \(x_{i}^{(Q)}\) is

\[
v_{h}^{(Q)} = \frac{2\pi kT}{M_{(Q)}}, M_{(Q)}^{2} = h^{(Q)}(r_{H}) \sum_{s} d^{(s)} \frac{b(r_{H}) f'(r_{H}) h^{(s)}(r_{H})}{4h^{(s)}(r_{H})} \tag{3.21}
\]

where we have used the relations of (3.10) and (3.11). Formulas (3.19) and (3.20) were derived by us in equations (2.28) and (2.30) in [33], respectively.

### 4. Shock wave equation in quadratic gravity

To investigate the theory with quadratic gravity we first collect following primary relations which can be proved with the help of appendices A–C. Note that we denote coordinate index by \(a, b, c, d\) and these not \(U, V\) by \(i, j, k, m, n\).

#### 4.1. Five relations

**Relation 1:** On the horizon the non-zero values of \(R^{a}_{bcd}\) are

\[
R^{ij}_{UV} = -R^{ij}_{UV} = -\frac{A'(0)}{A(0)}, \quad R^{ij}_{VU} = -R^{ij}_{VU} = -\frac{\hat{g}^{(S)}(x) h^{(i)}(0)}{2A(0)} \tag{4.1}
\]

\[
R^{iV}_{VV} = -R^{iV}_{VV} = -\frac{A'(0)}{A(0)}, \quad R^{iV}_{VU} = -R^{iV}_{VU} = -\frac{\hat{g}^{(S)}(x) h^{(i)}(0)}{2A(0)} \tag{4.2}
\]

\[
R^{(S)}_{ij} = -R^{(S)}_{ij} = R^{(S)}_{ij} = -R^{(S)}_{ij} = \frac{h^{(i)}(0)}{2h^{(i)}(0)}, R^{(S)}_{jkm} = \tilde{R}^{(S)}_{jkm} \neq 0. \tag{4.3}
\]

**Relation 2:** On the horizon the non-zero values of \(R_{ab}\) and \(R\) are

\[
R_{UV} = R_{UV} = -\frac{A'(0)}{A(0)} - \sum_{s} d^{(s)} h^{(s)}(0) \tag{4.4}
\]

\[
R^{(S)}_{ij} = -\hat{R}^{(S)}_{ij} - \frac{\hat{g}^{(S)}(x) h^{(i)}(0)}{A(0)} \tag{4.5}
\]

\[
R = -\frac{2A'(0)}{A(0)^{2}} + \sum_{s} \frac{\tilde{R}^{(S)}(0)}{h^{(S)}(0)} - \frac{2d^{(S)} h^{(S)}(0)}{A(0) h^{(S)}(0)} \tag{4.6}
\]

where the superscript \((S)\) is used to specify the coordinate \(dx_{(S)}^{i}\) in metric (3.5). The notation \(\tilde{0}\) is used to emphasize that the value is calculated on the horizon. Notice that the index \((S)\) in \(\tilde{R}^{(S)}_{jkm}\) means that indices \(i, j, k, m\) have to be on the same \((S)\) otherwise the tensor is zero.
We use $\bar{R}^{(S)}_{\text{bulk}}, \bar{R}^{(S)}_\gamma$ and $\bar{R}^{(S)}$ to denote the curvature evaluated in metric $\text{d}s^2 = \bar{g}^{(S)}_\gamma(x)\text{d}x^i\text{d}x^j$. Note that $d^{(S)} = \bar{g}^{(S)}_\gamma$ is the dimension of space $\text{d}x^{(S)}(x)$ in (3.5). The relation between bulk dimension $D$ and dimension $d^{(S)}$ is

$$D = 2 + \sum S d^{(S)}.$$  

(4.7)

In isotropic space above relation reduces to $D = 2 + d$ while other literature denotes $D = 1 + d$. Our notation is convenient when space is anisotropic.

Relation 3: On the horizon the non-zero values of $\delta R^{a}_{b c d}$ are

$$\delta R^{V}_{U W} = -\delta R^{V}_{U V U} = -\frac{2\alpha(x)A'}{A} \delta(U)$$  

(4.8)

$$\delta R^{V}_{y U} = -\delta R^{y V}_{U} = \delta(U) \bar{\nabla}_i \bar{\nabla}_j \alpha(x) - \frac{1}{2\Lambda} \bar{g}_{y h} \alpha(x) \delta(U)$$  

(4.9)

$$\delta R^{U}_{V J} = -\delta R^{U J}_{V} = \frac{h'(x)}{2h} \delta(U) \delta^j + \frac{A\delta(U)}{h} \bar{\nabla}_i \bar{\nabla}_j \alpha(x)$$  

(4.10)

$$\delta R^{U}_{V J} = -\delta R^{U J}_{V} = \frac{h'(x)}{2h} \delta(U) \delta^j + \frac{A\delta(U)}{h} \bar{\nabla}_i \bar{\nabla}_j \alpha(x).$$  

(4.11)

Relation 4: On the horizon the non-zero values of $\delta R_{ab}$ are $\delta R_{UU}$ and $\delta R = 0$.

$$\delta R_{U V} = \left(\frac{2A'}{A} + \sum S d^{(S)} \frac{h^{(S)}}{2h^{(S)}}\alpha(x) \delta(U) + \delta(U) \sum S \frac{A}{h^{(S)}} \Delta^{(S)} \alpha(x)\right)$$  

(4.12)

$$\delta R = 0$$  

(4.13)

where the Laplacian is defined by

$$\Delta^{(S)} \alpha(x) = \frac{1}{\sqrt{\bar{g}^{(S)}}} \delta^{(S)} \left(\sqrt{\bar{g}^{(S)}} \bar{g}^{(S) y} \delta^{(S)} \alpha(x)\right)$$  

(4.14)

and $\bar{\nabla}_i$ is the covariant derivative in the space with metric $\bar{g}^{(S)}$. With the helps of relation 1 $\sim$ relation 4, we can prove following relation.

Relation 5: On the horizon

$$\delta(R_{abcd}R^{abcd}) = \delta(R_{ab}R^{ab}) = \delta(R^2) = \delta(\square R) = 0.$$  

(4.15)

Last relation plays a central role to obtain the simplified shock wave equation of quadratic gravity in below.

### 4.2. Simplified shock wave equation

Using these relations we begin to evaluate $G^{(1)}_{U V} + 2 G^{(0)}_{U V} \alpha(x) \delta(U)$ in quadratic gravity theory. After explicitly expansion we find that
After substituting the explicitly forms of \( R_{UV} \) and substituting them into equation (4.17) we can find following four formulas:

\[
\begin{align*}
G_U^{(1)} + 2G_U^{(0)} &= \alpha(x) \delta(U) \\
&= 2\alpha \delta(R_{Ude}^{R_{de}}) + 2(2\alpha + \beta)\delta(R_{Ude}^{R_{de}}) - 4\alpha \delta(R_{Ue}^{R_{e}}) \\
&+ 2\gamma \delta(R_{Ue}^{R_{e}}) + (4\alpha + \beta)\delta(\Box R_{UV}) - (2\alpha + \beta + 2\gamma)\delta(\nabla_U \nabla_V R)
\end{align*}
\]

To obtain the last relation we have used the properties of proposition 5 to conclude that the operator \( \delta \) in first bracket of (4.16) only produces \( \delta g_{UV} \). After substituting the explicitly forms of \( \delta g_{UU} = 2A(UV)\alpha(x)\delta(U) \) in first bracket and \( g_{UV} = A(UV) \) in second bracket we see that they are canceled to each other and we have the last relation (4.17).

Equation (4.17) has six zero-order terms and six first-order terms. It is interesting to see that we could furthermore simplify it to contain only six first-order terms in another forms.

Using the metric properties in (3.5) and (3.17) we can find the following simple relation, which can be applied to any tensor \( F_{UV} \):

\[
\delta F_{UU} = \delta(g_{UV}F_{U}^{a}) = (\delta g_{UV})F_{UU} + g_{UV}\delta(F_{U}^{a}) = (\delta g_{UV})F_{U}^{a} + g_{UV}\delta(F_{U}^{a}) = (\delta g_{UV})g^{UV}F_{UU} + g_{UV}\delta(F_{U}^{a}) = -2F_{UU}\alpha(x)\delta(U) + g_{UV}\delta(F_{U}^{a}).
\]

After identifying \( F_{UU} \) as \( R_{Ude}^{R_{de}} \). \( R_{Ude}^{R_{de}} \). \( R_{Ude}^{R_{de}} \). \( \cdots \) and substituting them into equation (4.17) we find that

\[
G_U^{(1)} + 2G_U^{(0)} = \alpha(x) \delta(U) = g_{UV} \left( 2\alpha \delta(R_{Ude}^{R_{de}}) + 2(2\alpha + \beta)\delta(R_{Ude}^{R_{de}}) - 4\alpha \delta(R_{Ue}^{R_{e}}) \right)
\]

\[
+ 2\gamma \delta(R_{Ue}^{R_{e}}) + (4\alpha + \beta)\delta(\Box R_{UV}) - (2\alpha + \beta + 2\gamma)\delta(\nabla_U \nabla_V R) \right).
\]

Now, it remains only six terms. Among them four terms are quadratic curvature and two terms are derivative of curvature. With the help of appendix and after calculations we collect the formulas of these six terms in below.

### 4.3. Six formulas

Using the propositions 1–4 we can find following four formulas:

**Formula 1:**

\[
\delta(R_{Ude}^{R_{de}}) = \sum_S \frac{d^{(S)}\alpha(x)(h^{(S)})^2}{A^2(h^{(S)})^2}\delta(U) - 2\delta(U) \frac{h^{(S)}}{A(h^{(S)})^2} \Delta^{(S)}\alpha(x).
\]
We now apply above formulas to the simplest case of Gauss–Bonnet gravity in which α = γ_{GB}, β = -4γ_{GB}, γ = γ_{GB}. The equation (4.19) now has a simple form.
is introduced to make the metric to be AdS space asymptotically. We consider $G^{(1)}_{UU} + 2 G^{(2)}_{UV} \alpha(x) \delta(U)$
\[
= 2g_{UV} \gamma_{GB} \left( \delta(R^{\alpha \beta}_{\alpha \beta} R^{\alpha \beta}_{\alpha \beta}) - 2 \delta(R^{\alpha \beta}_{\alpha \beta} R^{\alpha \beta}_{\alpha \beta}) - 2 \delta(R^{\alpha \beta}_{\alpha \beta} R^{\alpha \beta}_{\alpha \beta}) + \delta(R^{\alpha \beta}_{\alpha \beta} R^{\alpha \beta}_{\alpha \beta}) \right) \\
= 2 \delta(U) \gamma_{GB} \left[ -2 \left( \sum_{s} \frac{A}{h^{(s)}} \right) \delta^{(s)} \nabla^{(s)} \nabla^{(s)} \alpha(x) - \frac{R^{(s)} h^{(s)}}{2h^{(s)}} \alpha(x) \right] \\
+ \left( \sum_{s} \frac{A}{h^{(s)}} \nabla^{(s)} \alpha(x) - \frac{d(S) h^{(s)}}{2h^{(s)}} \alpha(x) \right) \left( \sum_{s} \delta^{(s)} \right). 
\]
(4.26)

Let us make three comments about the result:

1. In the case of isotropic space we can remove the summations over $S$ and $\tilde{S}$, then formula (4.26) reproduces the formula (4.18) in our previous paper [33].

2. Due to the appearance of double summations the formula (4.26) is not just that adding a simple summation over $S$ to the formula (4.18) in [33], in which only isotropic space was analyzed. Thus the shock wave equation in anisotropic space is a non-trivial extension of that in isotropic space.

3. Consider the the planar, spherical, or hyperbolic black hole metric in (5.2) and using the relations (5.3) and (5.4) we find that

\[
\frac{A}{(h^{(s)})^2} \left( \tilde{R}^{(s)} \nabla^{(s)} \nabla^{(s)} \right) \alpha(x) - \frac{R^{(s)} h^{(s)}}{2h^{(s)}} \alpha(x) = k^{(s)} (d^{(s)} - 1) \left( \frac{A}{h^{(s)}} \nabla^{(s)} \alpha(x) - \frac{d(S) h^{(s)}}{2h^{(s)}} \alpha(x) \right). 
\]
(4.27)

Substituting this relation into (4.26) we see that the shock wave equation of Einstein–Gauss–Bonnet gravity and that of Einstein gravity, i.e. (3.19), obey the same differential equation when the space is isotropic. The double summations in the formula (4.26) will ruin this property when the space is anisotropic.

Thus, we conclude that in the $D$-dimensional planar, spherical or hyperbolic black hole spacetime the Einstein–Gauss–Bonnet gravity has the same shock wave equation as that in Einstein gravity if and only if the space is isotropic.

5. Butterfly velocity in isotropic spaces of quadratic gravity

5.1. Formula

In this section we will find the simplified formula of shock wave equation for the quadratic gravity in the following spacetime\(^4\)

\[
d^2 = -N_0^2 f(r) dr^2 + \frac{dr^2}{f(r)} \left[ \sum_{i,j} h^{ij} \delta_{ij}(x) dx_i dx_j. 
\]
(5.1)

The constant $N_0^2$ is introduced to make the metric to be AdS space asymptotically. We consider $2 + d$ dimensional planar, spherical or hyperbolic black holes. The general metric is

\(^4\)Through coordinate transformation the space can become a general form: $d^2 = -a(\tilde{r}) \tilde{f}(\tilde{r}) d\tilde{r}^2 + \frac{d\tilde{r}^2}{a(\tilde{r}) \tilde{f}(\tilde{r})} + \tilde{h}(\tilde{r}) \tilde{g}_{ij}(x) dx_i dx_j$ and property found in this paper is very general.
\[\tilde{g}_{\alpha}(x)dx^\alpha dx^\beta = \begin{cases} \frac{df^2}{\alpha} + \frac{d\theta_1^2}{\alpha} + \cdots + \frac{d\theta_k^2}{\alpha}, & k = 0 \\ \frac{d\theta_1^2}{\alpha} + \sin^2 \theta_1 (d\theta_2^2) + \sin^2 \theta_2 (d\theta_3^2) + \cdots + \sin^2 \theta_{d-1} (d\theta_d^2), & k = 1 \\ \frac{d\theta_1^2}{\alpha} + \sin h^2 \theta_1 (d\theta_2^2) + \sin h^2 \theta_2 (d\theta_3^2) + \cdots + \sin h^2 \theta_{d-1} (d\theta_d^2), & k = -1 \end{cases}\]

which implies
\[\tilde{R}^{\alpha} \bar{\nabla}_\alpha \nabla_\alpha = k (d-1) \Delta \alpha(x)\]  
(5.2)

\[\tilde{R} = kd (d-1).\]  
(5.3)

Note that the shock wave equation has two parts, one is from Einstein gravity (EG) and another is from quadratic gravity (QG). The results are
\[G_{UU}^{(1)} + 2G_{U\bar{U}}^{(0)} \alpha(x) \delta(U) = [G_{UU}^{(1)} + 2G_{U\bar{U}}^{(0)} \alpha(x) \delta(U)]_{EG} + [G_{UU}^{(1)} + 2G_{U\bar{U}}^{(0)} \alpha(x) \delta(U)]_{QG}\]  
(5.5)

where
\[G_{UU}^{(1)} + 2G_{U\bar{U}}^{(0)} \alpha(x) \delta(U)]_{EG} = \frac{A}{r_H} \left( \Delta \alpha(x) - \frac{d}{2} r_H f'(r_H) \alpha(x) \right) \delta(U)\]  
(5.6)

\[G_{UU}^{(1)} + 2G_{U\bar{U}}^{(0)} \alpha(x) \delta(U)]_{QG} = A \left( 2\alpha \delta(R^V_{\nu\mu} R_{\nu\mu}^{,\nu}) + 2(2\alpha + \beta) \delta(R^V_{\nu\mu} R_{\nu\mu}^{,\nu}) - 4\alpha \delta(R^V_{\nu\mu} R_{\nu\mu}^{,\nu}) \right) + 2\gamma \delta(R^V_{\nu\mu}) + (4\alpha + \beta) \delta(\Box R^V_{\nu\mu}) - (2\alpha + \beta + 2\gamma) \delta(\bar{\nabla}^V \bar{\nabla}_U R)\]  
(5.7)

and
\[\delta(R^V_{bcd} R_{U}^{,bcd}) = -\frac{2f'(r_H)}{r_H^3} \left( \Delta \alpha(x) - \frac{d}{2} r_H f'(r_H) \alpha(x) \right)\]  
(5.8)

\[\delta(R^V_{\nu\mu} R_{\nu\mu}^{,\nu}) = \frac{2\alpha \delta(R^V_{\nu\mu} R_{\nu\mu}^{,\nu}) + 2(2\alpha + \beta) \delta(R^V_{\nu\mu} R_{\nu\mu}^{,\nu}) - 4\alpha \delta(R^V_{\nu\mu} R_{\nu\mu}^{,\nu}) \right) + 2\gamma \delta(R^V_{\nu\mu}) + (4\alpha + \beta) \delta(\Box R^V_{\nu\mu}) - (2\alpha + \beta + 2\gamma) \delta(\bar{\nabla}^V \bar{\nabla}_U R)\]  
(5.9)

which are calculated from (4.20)–(4.23) respectively. And
\[\delta(\Box R^V_{\nu\mu}) = \frac{\Delta \Delta \alpha(x)}{r_H^3} - \frac{f'(r_H)}{r_H^3} \left( d + r_H f'(r_H) \right) \Delta \alpha(x) + \frac{df'(r_H)}{4r_H^3} \left( 2r_H f''(r_H) + df'(r_H) \right) \alpha(x)\]  
(5.10)

\[\delta(\bar{\nabla}^V \bar{\nabla}_U R) = -\frac{2(\Delta \alpha(x))}{r_H^3} \left( 2(2\alpha + \beta) + 2(2\alpha + \beta + 2\gamma) \delta(\bar{\nabla}^V \bar{\nabla}_U R) \right)\]  
(5.11)

which are calculated from (4.24) and (4.25) respectively. Note that (4.20)–(4.23) have a common factor ‘\(\Delta \alpha(x) - \frac{d}{2} r_H f'(r_H) \alpha(x)\)’. This factor is just that appears in the Einstein gravity (3.19).
We now apply these formulas to the spacetime (5.1) with

\[ f(r) = r^2 \left( 1 - \left( \frac{r_0}{r} \right)^{d+1} + \delta + \eta \left( \frac{r_0}{r} \right)^{2(d+1)} \right), \quad b(r) = r^2 \]

(5.14)

\[ \delta = \frac{(d-2)}{d} \left[ (d+1) \left( (d+2) \gamma + \beta \right) + 2 \alpha \right], \quad \eta = (d-1)(d-2) \alpha \]

(5.15)

\[ N_B^2 = 1 + \delta. \]

(5.16)

The black hole horizon and temperature are

\[ r_H = r_0 \left( 1 - \frac{\delta + \eta}{d+1} \right) \]

(5.17)

\[ T = \frac{(d+1)r_H}{4\pi} \left[ 1 - \gamma (d-2)(d-1) + \frac{(d-2) \left( (d+1)(\alpha(d+2) + \beta) + 2 \gamma \right)}{2d} \right]. \]

(5.18)

Above metric was first derived in [38]. It had been used by [38, 39] to show the viscosity bound violation and shear sum rule in higher derivative gravity theories [46]. We will use this metric to study the effect of quadratic gravity on the butterfly velocity.

After the calculation the shock wave equation (3.18) becomes

\[ C_2 \Delta \Delta \alpha(x) + C_1 \Delta \alpha(x) + C_0 \alpha(x) = E e^{2\pi \beta/\beta} a(x) \]

(5.19)

where

\[ C_2 = (4\alpha + \beta) \frac{1}{r_H^2} \]

(5.20)

\[ C_1 = -(1 + d)^2(4\alpha + \beta) + \frac{1}{r_H^2} \left( 1 - 2(-2 + d + 3d^2)\alpha - 4\gamma - 2d^2(\beta + \gamma) + 2d(\beta + 3\gamma) \right) \]

(5.21)

\[ C_0 = (1 + d) \left[ (-2d + 2(1 + d)(4 + d^2))\alpha + (1 + d)(4 + d(2 + d))\beta + 2(1 + d)(2 + d)^2 \gamma \right]. \]

(5.22)

The appearing of the term \( \Delta \Delta \alpha(x) \), which is fourth-order derivative of \( \alpha(x) \), is the general property after introducing the quadratic gravity.

To proceed we can follow the paper [24] to find the two butterfly velocities therein. In general the solution can be written as

\[ \alpha(x) \sim e^{\frac{2\pi}{T} (t - t_0 - \frac{\omega}{\omega_0})} - \frac{v_B^{(2)}}{v_B^{(1)}} e^{\frac{2\pi}{T} (t - t_0 - \frac{\omega}{\omega_0})} \]

(5.23)

where the butterfly velocity is defined by [2]

\[ v_B^{(i)} = N_T \frac{2\pi T}{M^{(i)}} \]

(5.24)

\[ 5 \text{ This is the planar black solution.} \]

\[ 6 \text{ The temperature form in (5.18) is shown in [38], which is that without factor } N_T. \]
$M^{(i)}$ are calculated from the following equation

$$C_2 \Delta \Delta \alpha(x) + C_1 \Delta \alpha(x) + C_0 \alpha(x) = C_2 \left( \Delta - (M^{(1)})^2 \right) \left( \Delta - (M^{(2)})^2 \right).$$  \tag{5.25}

The details are described by Alishahiha et al in [24].

After calculation the final formula of the holographic butterfly velocity propagating in the space (5.14) becomes

$$v_B^{(1)} = \sqrt{\frac{d+1}{2d}} \left[ 1 - 8 \pi^2 (\beta + 4 \alpha) T^2 - \frac{1}{2} (d-2) \left( (d-1) \alpha + (d+1)(\beta + 4 \alpha) + (3d+1)(\gamma - \alpha) \right) \right] \nonumber$$

$$+ O\left( (\alpha, \beta, \gamma)^3 \right)$$ \tag{5.26}

$$v_B^{(2)} = \frac{(d+1)}{2} \sqrt{\frac{(\beta + 4 \alpha)}{2}} + O\left( (\alpha, \beta, \gamma)^3/2 \right)$$ \tag{5.27}

in which $(\alpha, \beta, \gamma)^2$ represents any function of second order of variables $\alpha, \beta, \gamma$. We now use above relations to discuss the various properties of butterfly velocity in quadratic gravity.

5.2. Example: butterfly velocity in quadratic gravity

1. The second velocity becomes zero if $4 \alpha + \beta = 0$. This is the result that $C_2 = 0$ when $4 \alpha + \beta = 0$ and the shock wave equation becomes second-order derivative differential equation. It is interesting to see that second velocity can appear only if $4 \alpha + \beta < 0$.

2. In the case of $\beta + 4 \alpha = 0$, this including the $R + \gamma R^2$ gravity, the second velocity is zero and the first velocity becomes

$$v_B^{(1)} = \sqrt{\frac{d+1}{2d}} \left[ 1 - \frac{1}{2} (d-2) \left( (d-1) \alpha + (3d+1)(\gamma - \alpha) \right) \right].$$ \tag{5.28}

In this case the butterfly velocity in $D = 4$ black hole, i.e. $d = 2$, does not have correction by the quadratic curvatures.

3. The quantities $(\beta + 4 \alpha)$ and $(\gamma - \alpha)$ in equation (5.26) measure the deviations from the Gauss–Bonnet gravity. The case of both quantities being vanish corresponds to the Einstein–Gauss–Bonnet gravity and

$$v_B^{(GB)}(\alpha) = \left[ 1 - \frac{1}{2} \alpha (d-1)(d-2) \right] v_B^{(GB)}(0), \quad v_B^{(GB)}(0) = \sqrt{\frac{d+1}{2d}}$$ \tag{5.29}

which was first found in [5]. The factor $\left[ 1 - \frac{1}{2} \alpha (d-1)(d-2) \right]$ is from the constant value of $N_c$, which defined in (5.1), while the another factor $\sqrt{\frac{2d}{2d}}$ is from the shock wave equation in the Einstein gravity.

4. When $d = 2$, i.e. $D = 4$, the the butterfly velocities are functions of $(\beta + 4 \alpha)$ which is zero in Gauss–Bonnet gravity. This reveals that $D = 4$ Gauss–Bonnet term is topological.

5. In the case of $R + \gamma R^2$ gravity the second velocity is zero and the first velocity becomes

$$v_B^{(R^2)} = \sqrt{\frac{d+1}{2d}} \left[ 1 - \frac{(d-2)(3d+1)\gamma}{2} \right].$$ \tag{5.30}

This means that for the $D = 4$ planar black hole the $R^2$ gravity does not give any correction to the butterfly velocity. Otherwise the correction maybe positive or negative, depending on the values of $\gamma$. 

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6. In the Einstein-conformal gravity in which $\beta = -2\alpha$ and $\gamma = \frac{1}{2}\alpha$ the second butterfly velocity becomes

$$v_B^{(1)} = \sqrt{\frac{d+1}{2d}} \left[ 1 - \left( 16\pi^2 T^2 + \frac{(d-2)(3d+1)}{6} \alpha \right) \right]$$  \hspace{1cm} (5.31)

which shows a different behavior form that in $R^2$ gravity, since that for the $D = 4$ planar black hole the conformal gravity will correct the butterfly velocity.

7. At high temperature we have a simple relation

$$v_B^{(1)} \approx \sqrt{\frac{d+1}{2d}} \left[ 1 - \frac{1}{2}(d-2)\left( (d-1)\alpha + (d+1)(\beta + 4\alpha) + (3d+1)(\gamma - \alpha) \right) \right]$$  \hspace{1cm} (5.32)

which is independent of the values of $\gamma$. This means that, at high temperature the correction of butterfly velocity is independent of $R^3$ term. $R^3$ term can correct the butterfly velocity at order $O(T^0)$.

8. At low temperature we have a simple relation

$$v_B^{(1)} \approx \sqrt{\frac{d+1}{2d}} \left[ 1 - \frac{1}{2}(d-2)\left( (d-1)\alpha + (d+1)\beta + (3d+1)(\gamma - \alpha) \right) \right]$$  \hspace{1cm} (5.33)

Comparing above equation to the case of high temperature expansion we see that, depending on the values of $\alpha$, $\beta$, $\gamma$ and $d$, the velocity correction from the quadratic gravity may be from positive to negative or from negative to positive while increasing the temperature.

Finally, we can directly apply the general formula derived in previous subsection to a simple case in which the metric solution is the Schwarzschild–AdS black hole solution

$$ds^2 = -r^2 \left( 1 - \frac{r_H^2}{r^2} \right) dt^2 + \frac{dr^2}{r^2 \left( 1 - \frac{r_H^2}{r^2} \right)} + r^2(dx^2 + dy^2)$$  \hspace{1cm} (5.34)

since that any solution of the pure Einstein theory continues to be a solution of the theory with the quadratic modifications. This is the spacetime considered by Alishahiha et al in [24]. The fourth order differential equation of shock wave equation becomes

$$\frac{4\alpha + \beta}{r_H^4} \Delta \Delta \alpha(x) + \frac{1 - 3(8\alpha + 4\beta + 3\beta^2(4\alpha + \beta) + 8\gamma)}{r_H^4} \Delta \alpha(x) + (-3 + 36\alpha + 27\beta + 72\gamma) \alpha(x) = E e^{2\pi t/\beta} a(x).$$  \hspace{1cm} (5.35)

After the calculations the butterfly velocities becomes

$$v_B^{(1)} = \frac{\sqrt{3}}{2} \left( 1 - \frac{8\pi^2(4\alpha + \beta)}{1 - 6\beta - 24\gamma} \right) T^2 + O(T^4)$$  \hspace{1cm} (5.36)

$$v_B^{(2)} = \frac{3}{2} \sqrt{\frac{(4\alpha + \beta)}{12\alpha + 9\beta + 24\gamma}} \left( 1 + \frac{8\pi^2(4\alpha + \beta)}{1 - 6\beta - 24\gamma} \right) T^2 + O(T^4)$$  \hspace{1cm} (5.37)

which consists with equation (22) in [24] when $\alpha = T = 0$. The second velocity is real if $\frac{(4\alpha + \beta)}{12\alpha + 9\beta + 24\gamma} > 0$. The condition reduces to $4\alpha + \beta < 0$ for small values of $\alpha$, $\beta$, $\gamma$. 


5.3. Example: butterfly velocity in Gauss–Bonnet massive gravity

Since that $U_i$ in (2.5) are functions of metric $g_{\mu\nu}$ and reference metric $f_{\mu\nu}$ while do not depend on the Riemann curvature we can regard them as some kinds of the extra matter fields. Thus the formula derived in previous section, which analyze the variation with respect to the Riemann curvature, can be directly applied. Thus we conclude that:

*In the D-dimensional planar, spherical or hyperbolic black hole spacetime the Einstein–Gauss–Bonnet massive gravity has the same shock wave equation as that in Einstein gravity if and only if the space is isotropic.*

Therefore, in the case of isotropic space we can quickly calculate the butterfly velocity in the Gauss–Bonnet massive gravity theories, following the method described in our previous paper [33], i.e. (3.21). We consider the $(d+2)$ dimensional Maxwell–Gauss–Bonnet massive gravity. The Lagrangian is described by (2.5) where $L_{\text{matter}}$ is the Maxwell field and we add the Gauss–Bonnet curvature with coefficient $\alpha$. The charged black hole solution found in [66] is

$$d^2 s^2 = -N_{\sharp}^2 f(r) dr^2 + \frac{dr^2}{f(r)} + r^2 \delta_{ij} dx^i dx^j, \quad i,j = 1, 2, 3,...d$$  \hspace{1cm} (5.38)

$$F_a = \frac{Q}{r^d}$$  \hspace{1cm} (5.39)

$$f(r) = k + \frac{r^2}{2\alpha d_1 d_2} \left( 1 - \sqrt{1 + \frac{8\alpha d_1 d_2}{d_1 d_2} \left[ \Lambda + \frac{d_1 d_2 m_0}{2 r^d} - \frac{Q^2 d_1}{dr^{2d_2}} + \Upsilon \right]} \right)$$  \hspace{1cm} (5.40)

$$\Upsilon = -m^2 d_1 d_2 \left[ \frac{d_3 d_4 c_4 c_4}{2 r^4} + \frac{d_3 d_4 c_3 c_3}{2 r^4} + \frac{c_2^2 c_2}{2 r^2} + \frac{c c_1}{2 d_2 r} \right].$$  \hspace{1cm} (5.41)

The reference metric is chosen to be $f_{\mu\nu} = (0,0,c^2\delta_{ij})$. The notation $d_i = d + 2 - i$ is used. The constant $N_{\sharp}^2$ is

$$N_{\sharp}^2 = \frac{1}{2} \left( 1 + \sqrt{1 - 2\alpha(d-1)(d-2)} \right)$$  \hspace{1cm} (5.42)

after substituting the conventional value of $\Lambda = -\frac{(d+1)}{4}$. The horizon defined by $f(r_H) = 0$ leads to relation

$$1 + \frac{2k\alpha d_1 d_2}{r_H} = \sqrt{1 + \frac{8\alpha d_3 d_4}{d_1 d_2} \left[ \Lambda + \frac{d_1 d_2 m_0}{2 r^d_H} - \frac{Q^2 d_1}{dr^{2d_2}_H} + \Upsilon_H \right]}$$  \hspace{1cm} (5.43)

and black hole temperature is

$$4\pi T = - \frac{2k}{r_H} + \frac{d_1 m_0}{r_H^{d_1-1}} - \frac{4Q^2}{d_3 r_H^{d_3-1}} - m^2 \left[ \frac{4d_3 d_4 c_4 c_4}{r_H} + \frac{3d_3 d_4 c_3 c_3}{r_H} + \frac{2c_2^2 c_2}{r_H} + \frac{c c_1}{d_2} \right]$$  \hspace{1cm} (5.44)

Above relation can be used to express $r_H$ as a function of temperature while does not explicitly depend on $\alpha$. Therefore, after using the basic formula $v_B = N_{\sharp} \sqrt{\frac{4\pi T}{2\alpha r}}$ the butterfly velocity has an exact relation
\[ v_m^H(\alpha) = \left( \frac{1}{2} \left( 1 + \sqrt{1 - 2\alpha(d - 1)(d - 2)} \right) \right)^{1/2} v_m^H(0). \] \hspace{1cm} (5.45)

The ratio between \( v_m^H(\alpha) \) and \( v_m^H(0) \) had appeared in previous literature \([6, 33]\) and \((5.29)\)\(^7\). Since this paper is to see how the quadratic curvature affect the butterfly velocity the property of \( v_m^H(0) \) is left to reader to analyze. Note that above relation can also appear in Gauss–Bonnet massive gravity with Born–Infeld electrodynamics \([69]\) or in the presence of power-Maxwell field \([70]\), since that \( r_H \) in these cases still are a function of temperature while does not explicitly depend on \( \alpha \).

6. Conclusions

In this paper we continue previous work \([33]\) to study the butterfly velocity in general quadratic gravity with Lagrangian \( L = \alpha R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^2 + L_{\text{matter}} \). Contrast to the case of Gauss–Bonnet theory, in which \( \alpha = \gamma = -\frac{\beta}{4} \), the quadratic gravity can correct the shock wave equation. After the detailed tensor calculations the general formula of shock wave equation in the general anisotropic spacetime is derived. We use the formula to prove that in the D-dimensional planar, spherical or hyperbolic black hole spacetime the shock wave equation in the Einstein–Gauss–Bonnet gravity has the same form as that in Einstein gravity only if the space is isotropic.

We consider the example of a simple spacetime, which is the solution in leading order of \( \alpha, \beta \) and \( \gamma \). We obtain a simple formula of butterfly velocity in equations \((5.26)\) and \((5.27)\). Using the formula we find that the fourth-derivative shock wave equation therein could lead to two butterfly velocities if and only if \( 4\alpha + \beta < 0 \). We also see that the \( D = 4 \) planar black hole does not give correction to the butterfly velocity in the quadratic gravity with \( \beta + 4\alpha = 0 \), which includes the \( R^2 \) gravity. We also see that, depending on the values of \( \alpha, \beta, \gamma \), and the black-hole shape the velocity correction from the quadratic gravity may be from positive to negative or from negative to positive while increasing the temperature. The butterfly velocity in the theory of Gauss–Bonnet massive gravity is also studied.

Since our formula collected in section \(4\) is very general it can be applied to general anisotropic space with arbitrary matter fields. While the application in section \(5\) is in a simple isotropic space it is interesting to applied it to more complex space with matters and to see how the butterfly velocity will be in there. It hopes that our formula is helpful in studying the butterfly velocity in quadratic gravity.

We make four comments to conclude this paper.

1. As mentioned in section \(1\) that the value of \( \alpha \) is related to the \( \frac{1}{2N} \). Also the values of \( \beta \) and \( \gamma \) are related the \( R \) charges of the theory. Thus the calculations in section \(5\) tell us that correction of butterfly velocity, which describes how the perturbation spreads, may be positive or negative depending on the values of \( \frac{1}{2N} \), \( R \)-charges and temperature of the theory.

2. It is known that there is a simple relation between diffusion constant and butterfly velocity: \( D_c \sim v_m^H T \) \([11]\). Thus, using this relation we can read the property of how the diffusion constant will depends on the values of \( \frac{1}{2N} \), \( R \)-charges and temperature of the theory.

\(^7\)(5.29) is leading order of \( \alpha \).
3. It remains to find a simple explanation of why the shock wave equation of Einstein gravity does not be modified in the Gauss–Bonnet gravity with any matters for the planar, spherical or hyperbolic black hole spacetime in the case of isotropic space?

4. It is known that the bound (KSS) violation of viscosity in higher derivative gravity have been explained to relate to the Weyl anomaly and central charge [38, 39, 41]. So, does butterfly velocity corrected by higher derivative gravity has any simple explanation? The investigations in [24] and [27] had related it to the conformal dimension and central charge. The more general property is worthy to be studied in detain.

The answers to the problems could help us to understand the intrinsic properties of the quantum chaos.

Appendix A. Exact forms of the Christoffel symbol $\Gamma^{a}_{bc}$ and $\delta \Gamma^{a}_{bc}$

Tensor calculations of this paper begin with the exact forms of the Christoffel symbol $\Gamma^{a}_{bc}$ and $\delta \Gamma^{a}_{bc}$ in the metric (3.5), which we present in below.

**Properties of $\Gamma^{a}_{bc}$:** Non-trivial values of the Christoffel symbols $\Gamma^{a}_{bc}$ are

$$\Gamma^{U}_{UU} = \frac{VA'(UV)}{A(UV)} \hat{g}_{U} = 0, \quad \Gamma^{U}_{Uj} = -\frac{Uh'(UV)}{2A(UV)} \hat{g}_{j} = 0 \quad \text{(A.1)}$$

$$\Gamma^{V}_{VV} = \frac{UA'(UV)}{A(UV)} \hat{g}_{V} = 0, \quad \Gamma^{V}_{Vj} = -\frac{Vh'(UV)}{2A(UV)} \hat{g}_{j} = 0 \quad \text{(A.2)}$$

$$\Gamma^{i}_{Uj} = \Gamma^{j}_{Ui} = -\frac{Vh'(UV)}{2h(UV)} \hat{g}_{j} = 0 \quad \text{(A.3)}$$

$$\Gamma^{i}_{Vj} = \Gamma^{j}_{Vi} = -\frac{Uh'(UV)}{2h(UV)} \hat{g}_{j} = 0 \quad \text{(A.4)}$$

$$\Gamma^{i}_{jk} = \frac{1}{2} \hat{g}^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}) = \Gamma^{i}_{jk} \hat{g} = 0 \quad \text{(A.5)}$$

and all other components are exact zero. Note the notation $\hat{g} =$ used to represent the value that calculated on the horizon while the notation $\hat{g} =$ is used to emphasize that the relation has not yet been put on horizon. Values of not being on horizon are necessary in some calculations.

**Properties of $\delta \Gamma^{a}_{bc}$:** Non-trivial values of the Christoffel symbols $\delta \Gamma^{a}_{bc}$ are

$$\delta \Gamma^{U}_{UU} = \frac{UA'}{A} \alpha(x) \delta(U) = 0 \quad \text{(A.6)}$$

$$\delta \Gamma^{V}_{VV} = \frac{1}{U} \alpha(x) \delta(U) + \frac{VA'}{A} \alpha(x) \delta(U) \quad \text{(A.7)}$$

$$\delta \Gamma^{V}_{UV} = \delta \Gamma^{U}_{VU} = -\frac{UA'}{A} \alpha(x) \delta(U) = 0 \quad \text{(A.8)}$$

$$\delta \Gamma^{i}_{Uj} = \delta \Gamma^{j}_{Ui} = -\delta(U) \partial_{i} \alpha(x) \quad \text{(A.9)}$$
\[ \delta \Gamma^V_U = \frac{U h^I}{A} \alpha(x) \delta(U) g_{ij} = 0 \] (A.10)

\[ \delta \Gamma^V_{LU} = g^jA \delta(U) \partial_j \alpha(x). \] (A.11)

**Appendix B. Exact forms of \( \nabla_a \nabla_b \delta g_{cd} \)**

**Properties of \( \nabla_a \nabla_b \delta g_{cd} \):** On the horizon the non-zero values of \( \nabla_a \nabla_b \delta g_{cd} \) are

\[ \nabla_a \nabla_b \delta g_{UU} \equiv - 2A \alpha(x) \delta''(U) \] (B.1)

\[ \nabla_a \nabla_b \delta g_{UV} \equiv \nabla_a \nabla_b \delta g_{UV} \equiv - 2A \delta'(U) \partial_a \alpha(x) \] (B.2)

\[ \nabla_a \nabla_b \delta g_{UU} \equiv 4A' \alpha(x) \delta(U) \] (B.3)

\[ \nabla_a \nabla_b \delta g_{UV} \equiv - 2A \delta(U) \nabla_a \nabla_b \alpha(x) + g_{ab} \delta'(U) \delta(U) \] (B.4)

\[ \nabla_a \nabla_b \delta g_{UU} \equiv \delta'(U) \delta(U) g_{ij} \] (B.5)

where \( \nabla_i \) is the covariant derivative in the space with metric \( \tilde{g}^{(5)}_{ij} \). Using the above relations we can obtain the propositions 3 and 4.

**Appendix C. Calculation of \( \nabla^2 \delta g_{UU} \)**

We also need the relation

\[ \nabla^2 \delta g_{UU} \equiv \partial^a \partial_a \delta g_{UU} - 2 \Gamma^a_{ab} \partial^b \delta g_{UU} - \Gamma^a_{bc} \partial_a \delta g_{UU} - 2 \Gamma^a_{ab} \partial_a \delta g_{UU} - 2 \delta_{ab} \delta^{(5)} \delta g_{UU} \]

\[ \equiv 2g^{ab} \partial_a \partial_b \delta g_{UU} + g^{ab} \partial_a \partial_b \delta g_{UU} - g^{ab} \Gamma^b_\alpha \partial_a \delta g_{UU} - g^{ab} \Gamma^b_\alpha \partial_a \delta g_{UU} - 2g^{ab} \delta_{ab} \delta^{(5)} \delta g_{UU} \]

\[ \equiv \left( \frac{4A'}{A} + \sum S \delta^{(5)} \right) \alpha(x) \delta(U) - 2A(\sum S \Delta^{(5)} \alpha(x) \right) \] (C.1)

where the Laplacian is defined by

\[ \Delta^{(5)} \alpha(x) = \frac{1}{\sqrt{g^{(5)}}} \delta^{(5)} \left( \sqrt{g^{(5)}} \right) \frac{g^{(5)} \delta^{(5)} \alpha(x)}{\sqrt{g^{(5)}}} \] (C.2)

and we have used the following property:

\[ \partial_a \partial_b \left[ A \left( U \delta(U) \right) \right] = A' \delta(U) + UA' \delta(U) = \left( A' \delta(U) + UA' \frac{\delta(U)}{U} \right) = 0. \] (C.3)

It is interesting to note that despite \( \delta g_{UU} \) is not a scalar field we find a simple relation

\[ \nabla^2 \delta g_{UU} = \frac{1}{\sqrt{g}} \partial_a \left( \sqrt{g} g^{ab} \partial_b (\delta g_{UU}) \right). \] (C.4)

The properties of \( \delta g_{ab} = \delta g_{UU} \delta^U_a \delta^K_b \) and metric form in (3.5) are the necessary conditions to have the above relation.
Appendix D. Exact forms of $R_{ab}$, $R$, $\delta R_{ab}$ and $\delta R$

We need the following exact forms, which are off the horizon, to find formulas 5 and formula 6.

\[
R_{UU} = -V^2 \sum_s \frac{d^S h''(S)}{2h(S)} + \frac{V^2 A'}{A} \sum_s \frac{d^S h'(S)}{2h(S)} + V^2 \sum_s \frac{d^S (h'(S))^2}{4(h^S)^2}
\]  \hspace{1cm} (D.1)

\[
R_{VV} = -U^2 \sum_s \frac{d^S h''(S)}{2h(S)} + \frac{U^2 A'}{A} \sum_s \frac{d^S h'(S)}{2h(S)} + U^2 \sum_s \frac{d^S (h'(S))^2}{4(h^S)^2}
\]  \hspace{1cm} (D.2)

\[
R_{UV} = \frac{A'}{A} \left( \sum_s \frac{d^S h'(S)}{2h(S)} \right) + UV \left( \frac{(A')^2}{A^2} - \frac{A''}{A} - \sum_s \frac{d^S h''(S)}{2h(S)} + \sum_s \frac{d^S (h'(S))^2}{4(h^S)^2} \right)
\]  \hspace{1cm} (D.3)

\[
R_{U\bar{g}} = 0
\]  \hspace{1cm} (D.4)

\[
R_{\bar{g}g} = \bar{R}_{\bar{g}g} - \frac{\bar{H}'}{\bar{A}} \bar{g}_{\bar{g}} = UV \left( \frac{h''(S)}{A} + \frac{(d^S - 2)(h'(S))^2}{2Ah} \right) \frac{\bar{g}(S)}{\bar{g}_{\bar{g}}}
\]  \hspace{1cm} (D.5)

\[
R = -\frac{2A'}{A^2} + \sum_s \left( \frac{\bar{R}^S(S)}{\bar{h}(S)} - \frac{2d^S h'(S)}{Ah(S)} \right)
+ UV \left( \frac{2(A')^2}{A^3} - \frac{2A''}{A^2} - \sum_s \frac{2d^S h''(S)}{Ah(S)} - \frac{d^S (d^S - 3)(h'(S))^2}{2Ah(S)^2} \right)
\]  \hspace{1cm} (D.6)

in which we use $\bar{R}^{-S}(S)$ and $\bar{R}^S(S)$ to denote the curvature evaluated in the metric $d\bar{x}^2 = \bar{g}(S)(\bar{x})d\bar{x}\bar{x}'$. Note that $d^S = \bar{g}(S)\bar{g}_{\bar{g}}$ is the dimension of space $d\bar{x}^i(S)$ in (3.5). On the horizon above results reduce to proposition 2.

\[
\delta R_{UU} = \left[ \frac{2A'}{A} + \sum_s \frac{d^S h'(S)}{2h} \right] + UV \left( \frac{2A''}{A} - \frac{2(A')^2}{A^2} + \sum_s \frac{d^S A^\prime h'(S)}{Ah(S)} \right) \alpha(x) \delta(U)
\]  \hspace{1cm} (D.7)

\[
\delta R_{UV} = U^2 \left[ \frac{(A')^2}{A^3} - \frac{A''}{A} - \sum_s \frac{d^S A^\prime h'(S)}{2Ah(S)} \right] \alpha(x) \delta(U)
\]  \hspace{1cm} (D.8)

\[
\delta R_{\bar{g}g} = -U^2 \left[ \frac{h''(S)}{A} + \frac{(d^S - 2)(h'(S))^2}{2Ah(S)} \right] \frac{\bar{g}(S)}{\bar{g}_{\bar{g}}} \alpha(x) \delta(U)
\]  \hspace{1cm} (D.9)

\[
\delta R = \delta(g^{ab}R_{ab}) = \delta(g^{UU}R_{UU} + 2g^{UV}R_{UV} + g^{\bar{g}\bar{g}}\delta(R_{\bar{g}g}))
\]  \hspace{1cm} (D.10)

\[
= U^2 \left[ \frac{2(A')^2}{A^3} - \frac{2A''}{A^2} - \sum_s \frac{2d^S h'(S)}{2Ah(S)} + d^S (d^S - 3)(h'(S))^2 \right] \alpha(x) \delta(U).
\]

On the horizon above results reduce to proposition 4.
Appendix E. Calculations of $\delta(\Box R_{UV})$ and $\delta(\nabla^V \nabla_u R)$

Use the above results we can find the two curvature derivative terms:

E.1. Calculation of $\delta(\Box R_{UV})$

We use the following expansion

$$\delta(\Box R_{UV}) = \delta(\Box (g^{VU} R_{UV})) = (\delta g^{VU}) (\Box R_{UV}) + g^{VU} \delta(\Box R_{UV})$$  \hspace{1cm} (E.1)

where

$$\Box R_{UV} = \partial^a \partial_a R_{UV} - R_{UV} \partial^a \Gamma_a^{bV} - R_{UV} \partial^b \Gamma_a^{bV}$$

$$= 2g^{UV} \partial_a R_{bV} - R_{UV} g^{UW} (\partial_a \Gamma_{UV} + \partial_a \Gamma_{UV})$$

$$= 6(A')^2 \frac{4A''}{A^2} + \sum_s d(s) (\frac{A'h(s)}{A^2 h(s)} - \frac{2h''(s)}{Ah(s)} + \frac{3(h''(s))^2}{2A(h(s))^2}).$$  \hspace{1cm} (E.2)

Next, we expand

$$\delta(\Box R_{UV}) = (\delta \Box) R_{UV} + \Box (\delta R_{UV})$$  \hspace{1cm} (E.3)

in which $(\delta \Box)$ is the Laplacian operator while the Christoffel symbol shall be replaced by $\Gamma'^a_{bc} \rightarrow \Gamma'^a_{bc} + \delta \Gamma'^a_{bc}$ and keeps first order in $\alpha(x)$. The first term becomes

$$(\delta \Box) R_{UV} = 6 \delta g^{ab} \partial_a \partial_b R_{UV} - 2 \delta \Gamma'^{ab}_{uc} \partial_a R_{bU} - 2 \delta \Gamma'^{ba}_{uc} \partial_b R_{aU} - 2 \delta \Gamma'^{ba}_{Uc} \partial_R R_{bU} - 2 \delta \Gamma'^{ab}_{Uc} \partial_R R_{aU}$$

$$+ 2 \delta \Gamma'^{ab}_{uc} \Gamma'^{ac}_{Ud} \partial_R R_{bd} + 2 \delta \Gamma'^{ac}_{Ud} \Gamma'^{bc}_{Ue} \partial_R R_{ed} + 2 \delta \Gamma'^{ab}_{Uc} \Gamma'^{ac}_{Ud} \partial_R R_{bd}$$

$$= \left( \frac{8A''}{A^2} - \frac{8(A')^2}{A^3} + \sum_s \frac{2d(s)h''(s)}{Ah(s)} - \sum_s \frac{2d(s)h''(s)}{Ah(s)^2} \right) \alpha(x) \delta(U)$$

$$+ \left( \sum_s \frac{2\Delta(s)\alpha(x)}{h(s)} \left( \frac{A'}{A} + \sum_s \frac{d(s)h''(s)}{2h(s)} \right) \delta(U) \right]$$

$$- \left( \sum_s \frac{2\Delta(s)\alpha(x)}{h(s)} \left( \frac{A'}{A} + \sum_s \frac{d(s)h''(s)}{2h(s)} \right) \delta(U) \right].$$  \hspace{1cm} (E.4)

The second term becomes

$$\Box (\delta R_{UV}) = \partial^a \partial_a \delta R_{UV} - 2 \delta \Gamma'^{ab}_{uc} \partial_a R_{bU} - \delta \Gamma'^{ab}_{uc} \partial_a \delta R_{UV} - 2 \delta \Gamma'^{ab}_{uc} \partial_a \delta R_{UV} - 2 \delta \Gamma'^{ab}_{uV} \partial_a R_{bU}$$

$$+ 2 \delta \Gamma'^{ab}_{uc} \Gamma'^{ac}_{Ud} \partial_R R_{bd} + 2 \delta \Gamma'^{ac}_{Ud} \Gamma'^{bc}_{Ue} \partial_R R_{ed} + 2 \delta \Gamma'^{ab}_{uV} \Gamma'^{ac}_{Ud} \partial_R R_{bd}$$

$$= \delta(U) \sum_{s,S} \frac{A}{h(s)h(S)} \Delta(s) \Delta(S) \alpha(x) - \frac{4}{A} \left( \frac{A'}{A} + \sum_s \frac{d(s)h''(s)}{4h(s)^2} \right) \alpha(x) \delta(U).$$  \hspace{1cm} (E.5)

Collecting all we then find the formula 5.
\[ E.2. \text{Calculation of } \delta(\nabla^V \nabla_U R) \]

We make following expansion

\[ \delta(\nabla^V \nabla_U R) = \delta(g^{VU} \nabla_V \nabla_U R) = (\delta g^{VV}) \nabla_V \nabla_U R + g^{VV} \delta \nabla_V \nabla_U R \]

\[ = (\delta g^{VV}) \nabla_V \nabla_U R - g^{VU} (\delta \Gamma^a_U \nabla_a \nabla R) + 8 g^{VU} \nabla_V \nabla U (\delta R). \quad (E.6) \]

Using the values of \( R \) and \( \delta R \) in (D.6) and (D.10) above three terms become

- \[ \nabla_V \nabla_U R = \delta R \partial_V R - \Gamma^a_{VU} \partial_a R \]

\[ \approx \frac{6(A')^2}{A^3} - \frac{4A''}{A^2} - \sum_s \left( \frac{R^{(s)} h^{(s)}}{(h^{(s)})^2} + \frac{4d^{(s)} h^{(s)}}{Ah^{(s)}} - \frac{2d^{(s)} A' h^{(s)}}{A^2 h^{(s)}} + \frac{d^{(s)} (d^{(s)} - 7)(h^{(s)})^2}{2A(h^{(s)})^2} \right) \quad (E.7) \]

- \[ (\delta \Gamma^a_{VU}) \nabla_V R \approx (\delta \Gamma^a_{VU}) \nabla_V R + (\delta \Gamma^a_{VU}) \nabla_U R \]

\[ \approx \left[ \frac{6(A')^2}{A^3} - \frac{4A''}{A^2} - \sum_s \left( \frac{R^{(s)} h^{(s)}}{(h^{(s)})^2} + \frac{4d^{(s)} h^{(s)}}{Ah^{(s)}} - \frac{2d^{(s)} A' h^{(s)}}{A^2 h^{(s)}} + \frac{d^{(s)} (d^{(s)} - 7)(h^{(s)})^2}{2A(h^{(s)})^2} \right) \right] \delta_a R \quad (E.8) \]

\[ \nabla_U \nabla_V (\delta R) = \partial_V \partial_U (\delta R) - \Gamma^a_{VU} \partial_a (\delta R) \approx \partial_V \partial_U (\delta R) = 0 \quad (E.9) \]

where \( \partial \tilde{\Omega} = g^{ij} \partial_j \). Collecting all we then find the formula 6.

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