Onset of $\eta$ nuclear binding

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Abstract. Recent studies of $\eta$ nuclear quasibound states by the Jerusalem-Prague Collaboration are reviewed, focusing on stochastic variational method self consistent calculations of $\eta$ few-nucleon systems. These calculations suggest that a minimum value $\text{Re} \ a_{\eta N} \approx 1 \text{ fm} \ (0.7 \text{ fm})$ is needed to bind $\eta^3\text{He}$ ($\eta^4\text{He}$).

1 Introduction

The $\eta N$ near-threshold interaction is attractive, owing to the $N^*(1535)$ resonance to which the $s$-wave $\eta N$ system is coupled strongly [1]. This has been confirmed in chiral meson-baryon coupled channel models that generate the $N^*(1535)$ dynamically, e.g. [2]. Hence $\eta$ nuclear quasibound states may exist [3] as also suggested experimentally by the near-threshold strong energy dependence of the $\eta^3\text{He}$ production cross sections shown in Fig. 1. However, the $\eta^3\text{He}$ scattering length deduced in Ref. [4], $a_{\eta^3\text{He}} = [-(2.23 \pm 1.29) + i(4.89 \pm 0.57)] \text{ fm}$, although of the right sign of its real part, does not satisfy the other necessary condition for a quasibound state pole: $-\text{Re} \ a > \text{Im} \ a$.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Near-threshold $\eta^3\text{He}$ production cross sections. Left: $d p \rightarrow \eta^3\text{He}$ [4]. Right: $\gamma^3\text{He} \rightarrow \eta^3\text{He}$ [5].

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Quite generally, experimental searches for \( \eta \) nuclear quasibound states in proton, pion or photon induced \( \eta \) production reactions are inconclusive. Regarding the onset of \( \eta \) nuclear binding, Krusche and Wilkin \cite{6} state: “The most straightforward (but not unique) interpretation of the data is that \( \eta d \) is unbound, \( \eta ^3 \text{He} \) is bound, but that the \( \eta ^3 \text{He} \) case is ambiguous.” Indeed, with \( \eta ^3 \text{He} \) almost bound, one might expect that the denser \( ^4 \text{He} \) nucleus should help forming a bound \( \eta ^4 \text{He} \). Nevertheless, a recent Faddeev-Yakubovsky evaluation \cite{7} of the scattering lengths \( a_{\eta ^A \text{He}} \) for both He isotopes, \( A = 3, 4 \), finds this not to be the case, with the denser \( ^4 \text{He} \) apparently leading to a stronger reduction of the subthreshold \( \eta N \) scattering amplitude than in \( ^3 \text{He} \).

The present overview reports and discusses recent few-body stochastic variational method (SVM) calculations of \( \eta NNN \) and \( \eta NNNN \) using several semi-realistic \( N \bar{N} \) interaction models together with two \( \eta N \) interaction models that, perhaps, provide sufficient attraction to bind \( \eta \) in the \( ^3 \text{He} \) and \( ^4 \text{He} \) isotopes \cite{8-10}.

\section{\( \eta N \) and \( N \bar{N} \) interaction model input}

![Graph showing real and imaginary parts of the \( \eta N \) cm scattering amplitude near threshold in two meson-baryon coupled channel models: GW \cite{11} and CS \cite{12}.](image)

Figure 2. Real and imaginary parts of the \( \eta N \) cm scattering amplitude near threshold in two meson-baryon coupled channel models: GW \cite{11} and CS \cite{12}.

Figure 2 shows \( \eta N \) s-wave scattering amplitudes \( F_{\eta N}(E) \) calculated in two meson-baryon coupled-channel models across the \( \eta N \) threshold where Re \( F_{\eta N} \) has a cusp. These amplitudes exhibit a resonance about 50 MeV above threshold, the \( N^*(1535) \). The sign of Re \( F_{\eta N} \) below the resonance indicates attraction which is far too weak to bind the \( \eta N \) two-body system. The threshold values \( F_{\eta N}(E_{\text{th}}) \) are given by the scattering lengths

\[
a_{\eta N}^{\text{GW}} = (0.96 + i0.26) \text{ fm}, \quad a_{\eta N}^{\text{CS}} = (0.67 + i0.20) \text{ fm},
\]

with lower values below threshold \((E_{\text{th}} = 1487 \text{ MeV})\). These free-space energy dependent subthreshold amplitudes are transformed to in-medium density dependent amplitudes, in terms of which optical potentials \( V_{\eta N}^{\text{opt}}(\rho) \) are constructed and used to calculate self consistently \( \eta \) nuclear quasibound states. This procedure was applied in Refs. \cite{13, 14} to several \( \eta N \) amplitude models, with results for \( 1s_\eta \) quasibound states in models GW and CS shown in Fig. 3 from \( ^{12} \text{C} \) to \( ^{208} \text{Pb} \).

Figure 3 demonstrates that in both of these \( \eta N \) amplitude models the \( 1s_\eta \) binding energy increases with \( A \), saturating in heavy nuclei. Model GW, with larger \( \eta N \) real and imaginary subthreshold amplitudes than in model CS, gives correspondingly larger values of \( B_\eta \) and \( \Gamma_\eta \). While model GW binds
potentials old amplitudes are transformed to in-medium density dependent amplitudes, in terms of which optical
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Real and imaginary parts of the
Figure 2. $F$ Faddeev-Yakubovsky evaluation [7] of the scattering lengths $B$ \split
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Figure 3 demonstrates that in both of these
The
binding energy increases $\eta$ also in nuclei lighter than $^{12}$C (not shown in the figure) this needs to be confirmed in few-body calculations.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Binding energies $B$ (left) and widths $\Gamma$ (right) of $1s_\eta$ quasibound states across the periodic table calculated self consistently [13, 14] using the GW and CS $\eta N$ scattering amplitudes of Fig. 2.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Real and imaginary parts of the strength function $b_\Lambda(E)$ of the effective $\eta N$ potential $v^{GW}_{\eta N}(E)$, Eq. (2), obtained from the scattering amplitude $F^{GW}_{\eta N}(E)$ of Fig. 2 below threshold for four values of the scale $\Lambda$ [9].}
\end{figure}

Few-body calculations, in distinction from optical model calculations, require the use of effective $\eta N$ potentials $v_{\eta N}$ which reproduce the free-space $\eta N$ amplitudes below threshold. Fig. 4 shows subthreshold values of the energy dependent strength function $b_\Lambda(E)$ for $v_{\eta N}$ of the form

$$v_{\eta N}(E; r) = -\frac{4\pi}{2m_{\eta N}}b_\Lambda(E)\delta_\Lambda(r), \quad \delta_\Lambda(r) = \left(\frac{\Lambda}{2\sqrt{\pi}}\right)^3 \exp\left(-\frac{\Lambda^2 r^2}{4}\right),$$

(2)
derived from the scattering amplitude $F^{GW}_{\eta N}(E)$ of Fig. 2 for several choices of inverse range $\Lambda$. The normalized Gaussian function $\delta_\Lambda(r)$ is perceived in $\pi$EFT (pionless EFT) as a single $\eta N$ zero-range
Dirac $\delta^{(3)}(r)$ contact term (CT), regulated by using a momentum-space scale parameter $\Lambda$. Regarding the choice of $\Lambda$, substituting the underlying short range vector-meson exchange dynamics by a single regulated CT suggests that the scale $\Lambda$ is limited to values $\Lambda \lesssim m_\rho (~4$ fm$^{-1}$).

Similarly, a $\pi$EFT energy independent $v_{NN}(r)$ is derived at leading order (LO) by fitting a single regulated CT $\sim \delta_A(r)$ in each spin-isospin s-wave channel to the respective $NN$ scattering length. A $pp$ Coulomb interaction is included. To avoid $NNN$ and $\eta NN$ Thomas collapse in the limit $\Lambda \to \infty$, one introduces a three-body regulated CT for each of these three-body systems [9]:

$$V_{NNN}(r_{ij}, r_{jk}) = d_{\Lambda}^{\eta_{NN}}\delta_A(r_{ij}, r_{jk}), \quad V_{\eta NN}(r_{ij}, r_{hj}) = d_{\Lambda}^{\eta_{NN}}\delta_A(r_{ij}, r_{hj}), \quad (3)$$

where $\delta_A(r_{ij}, r_{jk}) = \delta_A(r_{ij})\delta_A(r_{jk})$. The three-nucleon CT $d_{\Lambda}^{\eta_{NN}}$ is fitted to $B_{\exp}(^3\text{He})$. With no further contact terms, $B_{\text{calc}}(^4\text{He})$ is found in this $\pi$EFT version [15] to vary moderately with $\Lambda$ and to exhibit renormalization scale invariance by approaching a finite value $B_{\Lambda \to \infty}(^4\text{He}) = 27.8 \pm 0.2$ MeV that compares well with $B_{\exp}(^4\text{He}) = 28.3$ MeV. In contrast, no $\eta$-related experimental datum is available for the $\eta NN$ CT $d_{\eta_{NN}}^{\Lambda}$ to be fitted to. Two versions for choosing this CT were tested: (i) $d_{\eta_{NN}}^{\Lambda} = d_{\eta_{NN}}^{\Lambda}$, and (ii) setting $d_{\eta_{NN}}^{\Lambda}$ so that $\eta d$ is just bound, i.e. $B_{\eta}(\eta d) = 0$. Added to $v_{\eta_{NN}}^{GW}(E)$, one finds that each of these versions prevents a potential collapse of $\eta d$, with calculated values of $B(\eta^4\text{He})$ that for $\Lambda \geq 4$ fm$^{-1}$ are nearly independent of the adopted version, as shown in Fig. 7 below.

### 3 Energy independent $\pi$EFT $\eta$ nuclear few-body calculations

**Figure 5.** Separation energies $B_\eta$ obtained in SVM calculations of $\eta^3\text{He}$ (left) and $\eta^4\text{He}$ (right) using $\pi$EFT $NN$ and $\eta N$ real interactions (2) fitted to values of $a_{\eta N} < 1$ fm, plus a universal $NNN$ and $\eta NN$ three-body CT (3), $d_{\eta_{NN}}^{\Lambda} = d_{\eta_{NN}}^{\Lambda}$, as a function of $1/\Lambda$.

Fig. 5 shows $\eta$ separation energies $B_\eta$ from $\pi$EFT SVM calculations of $\eta^3\text{He}$ and $\eta^4\text{He}$ using energy independent $\eta N$ potentials $v_{\eta N}(E=E_{\text{th}}, r)$ fitted to a given real values of $a_{\eta N}$ for a few values of $\Lambda$. The figure suggests that binding $\eta^3\text{He}$ ($\eta^4\text{He}$) requires that $a_{\eta N} \geq 0.55$ fm (0.45 fm), compatible with an effective value $\text{Re} a_{\eta N}' = 0.48 \pm 0.05$ fm derived for a nearly bound $\eta^3\text{He}$ [4]. For input values of $a_{\eta N}$ higher than shown in the figure, beginning at $a_{\eta N} \approx 1.2$ fm, the calculated binding energies $B_\eta^{\Lambda=\eta^3\text{He}}(\Lambda > 4$ fm$^{-1}$) diverge, apparently since $\eta d$ becomes bound then at $\Lambda = 4$ fm$^{-1}$ [8]. Qualitative arguments in support of this $\eta d$ onset-of-binding value of $a_{\eta N}$ are given here in Appendix A.
4 Energy dependence in \( \eta \) nuclear few-body systems

Having derived energy dependent \( \eta N \) potentials \( v_{\eta N}(E; r) \), see Eq. (2) and Fig. 4, a two-body subthreshold input energy \( \delta \sqrt{s} \equiv E - E_{\text{th}} \) needs to be chosen. However, \( \delta \sqrt{s} \) is not conserved in the \( \eta \) nuclear few-body problem, so the best one can do is to require that this choice agrees with the expectation value \( \langle \delta \sqrt{s} \rangle \) generated in solving the few-body problem, as given by [10]

\[
\langle \delta \sqrt{s} \rangle = -\frac{B}{A} - \frac{1}{A} \langle T_A \rangle + \frac{A-1}{A} \mathcal{E}_\eta - \xi_\eta A \xi_\eta \left( \frac{A-1}{A} \right)^2 \langle T_\eta \rangle.
\]

(4)

Here \( \xi_{N(\eta)} = m_{N(\eta)}/(m_N + m_\eta) \), \( \xi_A = \Lambda m_N/(\Lambda m_N + m_\eta) \), \( T_A \) and \( T_\eta \) denote the nuclear and \( \eta \) kinetic energy operators in appropriate Jacobi coordinates, \( B \) is the total binding energy, and \( \mathcal{E}_\eta = \langle H - H_N \rangle \) with each Hamiltonian defined in its own cm frame. Self consistency (SC), \( \langle \delta \sqrt{s} \rangle = \delta \sqrt{s} \), is imposed in our calculations, as demonstrated graphically in Fig. 6 (left). Applications of SC to meson-nuclear systems are reviewed in Ref. [16]. For recent \( K^- \text{-atom} \) and nuclear applications see Refs. [17, 18]. More recently, Hoshino et al. [19] argued in a \( K^- \text{-atom} \) study that by applying this procedure one violates the requirement of total momentum conservation. In Appendix B here we show specifically for \( A = 2 \) that our choice of SC Eq. (4) is not in conflict with any conservation law.

Finally, we note that Eq. (4) in the limit \( A >> 1 \) coincides with the optical model downward energy shift (supplemented by a Coulomb term) used in recent \( K^- \text{-atom} \) and nuclear studies [17, 18]:

\[
\langle \delta \sqrt{s} \rangle = -B_N \frac{\rho}{\hat{\rho}} - \xi_N \mathcal{B}_N \frac{\rho}{\rho_0} - \xi_N T_N \left( \frac{\rho}{\hat{\rho}} \right)^{2/3} + \xi_\eta \text{Re} \mathcal{V}_\eta^{\text{opt}} (\delta \sqrt{s}),
\]

(5)

where \( T_N = \langle T_A \rangle/A = 23.0 \text{ MeV} \) at the average nuclear density \( \hat{\rho} \), \( B_N = B_{\text{sub}}/A \approx 8.5 \text{ MeV} \) is an average nucleon binding energy and \( B_\eta \) denotes the calculated \( \eta \) separation energy. All terms here are negative, thereby leading to a downward energy shift.

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Figure 6. Left: \( \eta^4 \text{He} \) bound state energy \( E \) (red, squares) and the expectation value \( \langle \delta \sqrt{s} \rangle \) (blue, circles), calculated using the AV4' \( NN \) potential (denoted here AV4p), as a function of the input energy argument \( \delta \sqrt{s} \) of the GW \( \eta N \) potential with \( \Lambda = 4 \text{ fm}^{-1} \). The dotted vertical line marks the self consistent output values of \( \langle \delta \sqrt{s} \rangle \) and \( E \). The horizontal dashed line denotes the calculated \( ^4 \text{He} \) g.s. energy, marking the threshold of \( \eta \) binding. The green curve shows the expectation value \( \langle H_N \rangle \) of the nuclear core energy. Right: subthreshold \( \eta N \) energies \( \delta \sqrt{s} = E - E_{\text{th}} \) probed by the \( \eta \) nuclear optical potential as a function of the relative nuclear RMF density in Ca. Each of the two curves was calculated self consistently for a particular \( \eta N \) subthreshold amplitude model.
The SC procedure is demonstrated in Fig. 6 (left) for $^4\text{He}$ binding energy calculated using the AV4' $NN$ potential and GW $\eta N$ potential with $\Lambda$=4 fm$^{-1}$. The $^4\text{He}$ bound state energy $E$ (excluding rest masses) and the output expectation value $\langle \delta \sqrt{s} \rangle$, where $\delta \sqrt{s}$ stands for the $\eta N$ cm energy with respect to its threshold value $E_{th}$, are plotted as a function of the subthreshold input energy argument $\delta \sqrt{s}$ of the potential $\epsilon_{\eta N}^{\text{GW}}$. The SC condition requires $\delta \sqrt{s} = \langle \delta \sqrt{s} \rangle$ which is satisfied at $-32.4$ MeV. The corresponding value of $E(\langle \delta \sqrt{s} \rangle)$ then represents the SC energy of $^4\text{He}$, with $B_{\eta}^{\text{SC}} = 3.5$ MeV, considerably less than the value $B_{\eta}^{\text{th}} = 13$ MeV obtained by disregarding the energy dependence of $\epsilon_{\eta N}^{\text{GW}}$ and using its threshold value corresponding to $\delta \sqrt{s} = 0$.

In Fig. 6 (right) we present the $\eta N$ downward energy shift $\delta \sqrt{s} = E - E_{th}$ as a function of the relative nuclear density $\rho/\rho_0$ in Ca, evaluated self consistently via Eq. (5) in the CS and GW models. The energy shift at $\rho_0$ is $-55 \pm 10$ MeV, about twice larger than the SC condition $\delta \sqrt{s} = -B_{\eta}$ applied in some other works, e.g. [20]. The GW shift exceeds the CS shift owing to the stronger GW amplitude of Fig. 2 and both were incorporated in the calculation of $1s_\eta$ quasibound nuclear states, Fig. 3.

5 Results of $\eta$ nuclear few-body calculations

Our fully self consistent $\eta NN$, $\eta NNN$ and $\eta NNNN$ bound-state calculations [8–10] use the following nuclear core models: (i) EFT including a three-body contact term [15], (ii) AV4p, a Gaussian basis adaptation of the Argonne AV4' $NN$ potential [21], and (iii) MNC, the Minnesota soft core $NN$ potential [22]. Models GW [11] and CS [12] were used to generate energy dependent $\eta N$ potentials which prove too weak to bind any $\eta NN$ system when using AV4p or MNC for the nuclear core model. Calculated $\eta$ separation energies $B_{\eta}$ are shown in Figs. 7 and 8.

![Figure 7](https://doi.org/10.1051/epjconf/201818101011)

**Figure 7.** $B_{\eta}(^3\text{He})$ (left) and $B_{\eta}(^4\text{He})$ (right) as a function of $1/\Lambda$ from EFT few-body calculations [9] using $\epsilon_{\eta N}^{\text{GW}}$, with (squares) & without (circles) imposing self consistency. Solid lines: $d_{\eta NN} = d_{\eta NNN}^\Lambda$, dashed lines: $d_{\eta NN}^\Lambda$ ensuring that $B_{\eta}(\eta d) = 0$.

Fig. 7 demonstrates in EFT the moderating effect that imposing SC (red, squares) by using $\epsilon_{\eta N}^{\text{GW}}(E_{th})$, rather than using threshold values $\epsilon_{\eta N}^{\text{GW}}(E_{th})$ (blue, circles), bears on the calculated $B_{\eta}$ values and their $\Lambda$ scale dependence [9]. Near $\Lambda$=4 fm$^{-1}$, imposing SC lowers $B_{\eta}(^3\text{He})$ by close to 5 MeV and $B_{\eta}(^4\text{He})$ by close to 10 MeV. The figure demonstrates that $B_{\eta}(^4\text{He})$ is always larger than $B_{\eta}(^3\text{He})$. Focusing on scale parameters near $\Lambda$=4 fm$^{-1}$ one observes that $^3\text{He}$ is hardly bound by a fraction of MeV, whereas $^4\text{He}$ is bound by a few MeV. The choice of three-body CT $d_{\eta NN}^\Lambda$ hardly matters for $\Lambda > 4$ fm$^{-1}$, becoming substantial at $\Lambda < 4$ fm$^{-1}$. 


Fig. 8 demonstrates in non-EFT calculations the dependence of $B_\eta$, calculated self consistently, on the choice of $NN$ and $\eta N$ interaction models. Using the more realistic AV4' $NN$ interaction results in less $\eta$ binding than using the soft-core MNC $NN$ interaction. For $v_\eta^\text{GW}$ near $\Lambda=4$ fm$^{-1}$ the difference amounts to about 0.3 MeV for $^3$He and about 1.5 MeV for $^4$He; $^3$He appears then barely bound whereas $^4$He is bound by a few MeV. The weaker $v_\eta^\text{CS}$ does not bind $^3$He and barely binds $^4$He using the MNC $NN$ interaction, implying that $^4$He is unlikely to bind for the more realistic AV4' $NN$ interaction. For smaller, but still physically acceptable values of $\Lambda$ down to $\Lambda = 2$ fm$^{-1}$, $^3$He becomes unbound and $^4$He is barely bound using the AV4' $NN$ and GW $\eta N$ interactions.

The $B_\eta$ values calculated in Refs. [8–10] were calculated assuming real Hamiltonians, justified by Im $v_\eta N \ll \text{Re} v_\eta N$ from Fig. 4. This approximation is estimated to add near threshold less than 0.3 MeV to $B_\eta$. Perturbatively-calculated widths $\Gamma_\eta$ of weakly bound states amount to only few MeV, outdating those reported in Ref. [8].

In future work it will be interesting to extend the present SVM few-body calculations to heavier nuclei, beginning with light $p$-shell nuclei. This represents highly non-trivial task. In Fig. 9 we
present preliminary results for $\eta^6$Li, using the central Minnesota $NN$ and GW $\eta N$ potentials. In this calculation the $^6$Li nuclear core consisted of a single $S = 1, T = 0$ spin-isospin configuration, yielding $B(^6\text{Li}) = 34.66$ MeV which is short by almost 2 MeV with respect to a calculation reported in Ref. [24] that used the same $NN$ interaction while including more spin-isospin configurations. The figure suggests that $\eta^6$Li is comfortably bound, even for as low value of scale parameter as $\Lambda = 2$ fm$^{-1}$.

6 Summary

Based mostly on the A V4′ results in Fig. 8, which are close to the $\pi$EFT results in Fig. 7, we conclude that $\eta^3$He becomes bound for $\text{Re } a_{\eta N} \sim 1$ fm, as in model GW, while $\eta^4$He binding requires a lower value of $\text{Re } a_{\eta N} \sim 0.7$ fm, almost reached in model CS. These $\text{Re } a_{\eta N}$ onset values, obtained by incorporating the requirements of $\eta N$ subthreshold kinematics, are obviously larger than those estimated in Sect. 3 upon calculating with $\eta N(E = E_{\text{th}}, r)$ threshold input. Finally, $\text{Re } a_{\eta N} < 0.7$ fm if $\eta^4$He is unbound, as might be deduced from the recent WASA-at-COSY search [23].

Appendix A: Onset of $\eta d$ binding

Here we apply the Brueckner formula [25], expressing the $\eta d$ scattering length in terms of the $\eta N$ scattering length, to discuss qualitatively the onset of $\eta d$ binding. This formula was originally proposed for a system of a light meson ($\pi$ meson) and two heavy static nucleons. More recently it was used to estimate the $K^-d$ scattering length (see derivation and discussion in Ref. [26]) where the meson-nucleon mass ratio is similar to that for $\eta N$. For $\eta d$ the Brueckner formula assumes the form

$$ a_{\eta d} = \int a_{\eta d}(r)|\psi_d(r)|^2 dr, \quad (6) $$

$$ a_{\eta d}(r) = \left(1 + \frac{m_\eta}{m_d}\right)^{-1}\tilde{a}_\eta + \tilde{a_n} + 2\tilde{a}_\eta\tilde{a}_n/r, \quad (7) $$

where $\tilde{a} = (1 + m_\eta/m_d)a$, with $a_\eta$ and $a_n$ standing for $a_{\eta p}$ and $a_{\eta n}$ respectively in the $\eta N$ cm system. The numerator in the Brueckner formula consists of single- and double-scattering terms, whereas the denominator provides for the renormalization of these terms by higher-order scattering terms. Since $a_\eta = a_n$ for the isoscalar $\eta$ meson, Eq. (7) reduces to a simpler form,

$$ a_{\eta d}(r) = \frac{2}{1 + \frac{m_\eta}{m_d}} \frac{\tilde{a}_\eta N}{1 - \tilde{a}_\eta N/r}, \quad (8) $$

which leads to the following approximate expression:

$$ a_{\eta d} = \frac{2}{1 + \frac{m_\eta}{m_d}} \frac{\tilde{a}_\eta N}{1 - \tilde{a}_\eta N(1/r)_d}, \quad (9) $$

with expansion parameter $\tilde{a}(1/r)_d$, where $(1/r)_d \approx 0.45$ fm$^{-1}$ for a realistic deuteron wavefunction [27]. Hence, this multiple scattering series faces divergence for sufficiently large $\eta N$ scattering length, say $a > 1.4$ fm.

Several straightforward applications of Eq. (9) are as follows:
• For Re \(a_{\eta N}^{GW} = 0.96\) fm, suppressing Im \(a_{\eta N}^{GW}\), one gets \(a_{nd} = 7.46\) fm. Increasing this GW input value of \(a_{\eta N}\), a critical value \(a_{\eta N}^{cr} = 1.40\) fm is reached at which the denominator in Eq. (9) vanishes, signaling the appearance of a zero-energy \(\eta d\) bound state.

• The LO \(\pi\)EFT nuclear calculations [15] yield a more compact deuteron, \(r_{rms} = 1.55\) fm for \(A \to \infty\) compared to the ’experimental’ value \(r_{rms} = 1.97\) fm. Scaling the value \((1/r)_{d} = 0.45\) used in Eq. (9) by 1.97/1.55, one gets \(a_{nd} = 18.1\) fm and \(a_{\eta N}^{cr} = 1.10\) fm.

• For the fully complex scattering length \(a_{\eta N}^{GW} = 0.96 + i0.26\) fm, one gets \(a_{nd} = 4.66 + i4.76\) fm. Increasing Re \(a_{\eta N}\) at a frozen value of Im \(a_{\eta N}\), Re \(a_{nd}\) reverses its sign at Re \(a_{\eta N}^{cr} = 1.35\) fm while Im \(a_{nd}\) keeps positive all through.

• At Re \(a_{\eta N}^{cr} = 1.59\) fm, |Re \(a_{nd}\)| becomes larger than Im \(a_{nd}\), which signals a threshold \(\eta d\) bound state.

**Appendix B: \(\eta N\) subthreshold kinematics**

Here we outline the choice of the \(\eta N\) subthreshold energy shift \(\delta \sqrt{s} \equiv \sqrt{s_{\eta N}} - (m_{N} + m_{\eta})\) applied in our \(\eta\) nuclear few-body works [8–10], see Eq. (4), with emphasis on the three-body \(\eta d\) system. Since the \(\eta N\) effective potential \(v_{\eta N}\) discussed in Sect. 2 is energy dependent, one needs to determine as consistently as possible a fixed input value \(\delta \sqrt{s}\) at which \(v_{\eta N}\) should enter the \(\eta\) nuclear few-body calculation. The two-body Mandelstam variable \(\sqrt{s_{\eta N}} = \sqrt{(E_{\eta} + E_{N})^{2} - (\vec{p}_{\eta} + \vec{p}_{N})^{2}}\) which reduces to \((E_{\eta} + E_{N})\) in the \(\eta N\) two-body cm system is not a conserved quantity in the \(\eta\) nuclear few-body problem since spectator nucleons move the interacting \(\eta N\) two-body subsystem outside of its cm system. We proceed to evaluate the expectation value of output values of \(\delta \sqrt{s}\), replacing \(\sqrt{s_{\eta N}}\) by \((1/A)\sum_{i=1}^{A} \sqrt{(E_{\eta} + E_{i})^{2} - (\vec{p}_{\eta} + \vec{p}_{i})^{2}}\) due to the antisymmetry of the nuclear wavefunction. Expanding about the \(\eta N\) threshold, one gets in leading order of \(p^{2}\)

\[
\langle \delta \sqrt{s} \rangle \approx \frac{1}{A} \left( \sum_{i=1}^{A} (E_{\eta} + E_{i}) - \sum_{i=1}^{A} \frac{(\vec{p}_{\eta} + \vec{p}_{i})^{2}}{2(m_{N} + m_{\eta})} \right),
\]

where \(E_{\eta} = E_{\eta} - m_{\eta}\) and \(E_{i} = E_{i} - m_{N}\). Since \(\sum_{i=1}^{A} E_{i}\) is naturally identified with the expectation value of the nuclear Hamiltonian \(H_{N}\), \(\sum_{i=1}^{A} E_{i} = \langle H_{N} \rangle = E_{nuc} = -B_{nuc}\), it is natural and also consistent to identify \(E_{\eta}\) with the expectation value of \((H - H_{N})\), \(E_{\eta} = \langle H - H_{N} \rangle\). Furthermore, recalling that \(E_{\eta} + \sum_{i=1}^{A} E_{i} = E = -B\), where \(E = \langle H \rangle\) is the total \(\eta\) nuclear energy and \(B\) is the total binding energy, the sum over the momentum independent part in Eq. (10) gives \([-B + (A - 1)E_{\eta}]/A\), thereby reproducing two of the four terms in Eq. (4). Note that \(E_{\eta}\) is negative and its magnitude exceeds the \(\eta\) separation energy \(B_{\eta}\). The sum over the momentum dependent part of Eq. (10) yields the other two terms of Eq. (4), which we demonstrate for \(\eta d\), \(A = 2\).

Since the \(\eta d\) calculation employs translationally invariant coordinate sets, the total momentum vanishes sharply: \((\vec{p}_{\eta} + \vec{p}_{1} + \vec{p}_{2}) = 0\). We then substitute \(\vec{p}_{1}^{2}\) for \((\vec{p}_{\eta} + \vec{p}_{2})^{2}\) and \(\vec{p}_{2}^{2}\) for \((\vec{p}_{\eta} + \vec{p}_{1})^{2}\) in the momentum dependent part in Eq. (10), resulting in momentum dependence proportional to \(\vec{p}_{1}^{2} + \vec{p}_{2}^{2}\). This is rewritten as

\[
\vec{p}_{1}^{2} + \vec{p}_{2}^{2} = \frac{1}{2} [(\vec{p}_{1} - \vec{p}_{2})^{2} + (\vec{p}_{1} + \vec{p}_{2})^{2}] = 2\vec{p}_{N:N}^{2} + \frac{1}{2} \vec{p}_{\eta}^{2},
\]

where \(\vec{p}_{N:N}\) is the nucleon-nucleon relative momentum operator. To obtain the \(\eta\) momentum operator \(\vec{p}_{\eta}\) on the r.h.s. we used again total momentum conservation. Finally, transforming \(\vec{p}_{N:N}^{2}\) and \(\vec{p}_{\eta}^{2}\) to
intrinsic kinetic energies, $T_{N:N}$ for the internal motion of the deuteron core and $T_\eta$ for that of the $\eta$ meson with respect to the NN cm, one gets for this $A = 2$ special case

$$\langle \delta \sqrt{s} \rangle_{nd} \approx - \frac{1}{2} \left( B - E_\eta + \xi_N \langle T_{N:N} \rangle + \xi_{A=2} \xi_\eta \frac{1}{2} \langle T_\eta \rangle \right),$$

(12)

which agrees with Eq. (4) for $A = 2$ upon realizing that $T_{N:N}$ here coincides with $T_{A=2}$ there. To get an idea of the relative importance of the various terms in this expression, we assume a near-threshold $\eta d$ bound state for which both $E_\eta$ and $\langle T_\eta \rangle$ are negligible (fraction of MeV each) and $B \rightarrow B_d \approx 2.2$ MeV. With $\langle T_{N:N} \rangle \rightarrow \langle T_d \rangle$, and with a deuteron kinetic energy $\langle T_d \rangle$ in the range of 10 to 20 MeV, this term provides the largest contribution to the downward energy shift which is then of order $-5$ MeV for the diffuse deuteron nuclear core.

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