ON THE EXISTENCE OF ABSOLUTELY SIMPLE
ABELIAN VARIETIES OF A GIVEN DIMENSION
OVER AN ARBITRARY FIELD

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Abstract. We prove that for every field $k$ and every positive integer $n$, there exists an absolutely simple $n$-dimensional abelian variety over $k$. We also prove an asymptotic result for finite fields: For every finite field $k$ and positive integer $n$, we let $S(k, n)$ denote the fraction of the isogeny classes of $n$-dimensional abelian varieties over $k$ that consist of absolutely simple ordinary abelian varieties. Then for every $n$ we have $S(F_q, n) \to 1$ as $q \to \infty$ over the prime powers.

1. Introduction

An abelian variety over a field $k$ is called simple if it has no proper nonzero sub-abelian varieties over $k$; it is called absolutely simple (or geometrically simple) if it is simple over the algebraic closure of $k$. In this paper we will prove the following theorem:

Theorem 1. Let $k$ be a field and let $n$ be a positive integer. Then there exists an absolutely simple $n$-dimensional abelian variety over $k$.

An easy reduction argument, similar in spirit to the one in Section 4 of [1], shows that an absolutely simple abelian variety over a field $k$ remains simple over every extension field of $k$, even the non-algebraic ones; thus, it suffices to prove Theorem 1 in the special case where $k$ is a prime field. Mori [3] (see also Zarhin [14]) provides examples of absolutely simple abelian varieties of arbitrary dimension over $\mathbb{Q}$, so we need only prove Theorem 1 for finite prime fields $k$. We will in fact prove that over such fields there exist absolutely simple ordinary abelian varieties of every dimension, and in addition we will prove an asymptotic result concerning arbitrary finite fields:

Theorem 2. For every integer $n \geq 0$ and finite field $k$ let $S(k, n)$ denote the fraction of the isogeny classes of $n$-dimensional abelian varieties over $k$ that consist of absolutely simple ordinary abelian varieties. Then for every $n$ we have $S(F_q, n) \to 1$ as $q \to \infty$ over the prime powers.

In fact, for every $n$ and $\epsilon$ we provide an explicit value of $M$ such that if $q > M$ then $0 < 1 - S(F_q, n) < \epsilon$; see Theorems 3 and 14 in Sections 6 and 7.

Suppose $A$ is an abelian variety over a finite field $k$. One can ask whether there exists an absolutely simple abelian variety over $k$ with the same formal isogeny type.
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(see [7]) as $A$. Theorem 1 shows that the answer to this question is yes when $A$ is ordinary; we have not considered the question for other formal isogeny types. Lenstra and Oort [7] considered the analogous question when $k$ is the algebraic closure of a finite field, and showed that the answer is yes when $A$ is not supersingular.

Theorem 1 leads to the question of whether there exist absolutely simple Jacobians of every dimension over a given field $k$. Chai and Oort [1] show that the answer is yes if $k$ is the algebraic closure of a finite field, and Mori [8] and Zarhin [14] show that the answer is also yes if $k$ has characteristic zero. If $k$ has positive characteristic $p$ but is not algebraic over $\mathbf{F}_p$, then results of Katz and Sarnak (see Sections 10.1 and 10.2 of [6]) can be used to show that once again the answer is yes — see also Mori [8] for some partial results for such fields. In fact, the examples provided by Katz and Sarnak, Mori, and Zarhin are Jacobians of explicitly-given hyperelliptic curves. However, the question seems to be open when $k$ is a finite field. The techniques we use in this paper do not help settle the general question for finite fields, but our results do at least show that over every finite field $k$ there are curves of genus 2 and 3 with absolutely simple Jacobians, as the following argument shows:

As we mentioned above, we show that for every $n$ and for every finite field $k$ there is an absolutely simple $n$-dimensional ordinary abelian variety over $k$, and in particular this is true for $n = 2$ and $n = 3$. But every absolutely simple ordinary abelian variety of dimension 2 or 3 over a finite field is isogenous to a principally polarized variety (see Corollary 12.6 and Theorem 1.2 of [3]). The main result of [2] shows that each such principally polarized variety is isomorphic (over the algebraic closure of $k$) to a Jacobian of a possibly reducible curve $C$, but since the Jacobian of $C$ is absolutely simple $C$ must be geometrically irreducible. Finally, a simple descent argument shows that $C$ has a model defined over $k$. Thus, for every finite field $k$ there are curves of genus 2 and 3 over $k$ with absolutely simple Jacobians.

Our paper is organized as follows. In Section 2, we briefly review the properties of Weil numbers and Weil polynomials that we will use in the proofs of Theorems 1 and 2. In Section 3 we give an easy-to-verify sufficient condition for an abelian variety over a finite field to be absolutely simple. We use this condition in Section 4 to prove Theorem 4, which shows how the characteristic polynomial of Frobenius of a simple ordinary abelian surface over a finite field can be used to quickly determine the splitting behavior of the surface over the algebraic closure. Theorem 4 allows us to give a very short proof of Theorem 1 in the case $n = 2$; we provide a proof for the case $n > 2$ in Section 5. In Sections 6 and 7 we prove Theorems 3 and 4, which are effective versions of Theorem 2 in the cases $n = 2$ and $n > 2$, respectively. Finally, in Section 8 we prove a lemma about polynomials with prescribed reduction modulo certain primes that is essential for our proof of Theorem 4.

**Conventions.** Suppose $A$ is an abelian variety over a field $k$ and suppose $\ell$ is an extension field of $k$. We will denote by $A_\ell$ the $\ell$-scheme $A \times_{\text{Spec} \, k} \text{Spec} \, \ell$. If $B$ is another abelian variety over $k$, then when we speak of a morphism from $A$ to $B$ we always mean a $k$-morphism; thus, we write $\text{End}_k A$ for what some authors would call $\text{End}_k A$.

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packages PARI/GP and MAGMA for some of the computations they performed in the course of writing this paper.

2. Weil numbers and Weil polynomials

Suppose $q$ is a power of a prime number $p$. A Weil $q$-number, or simply a Weil number if $q$ is clear from context, is an algebraic integer $\pi$ such that $|\varphi(\pi)| = q^{1/2}$ for every embedding $\varphi$ of $\mathbb{Q}(\pi)$ into the complex numbers. Suppose $k$ is a field with $q$ elements. To every abelian variety $A$ over $k$ we associate the characteristic polynomial $f_A \in \mathbb{Z}[x]$ of its Frobenius endomorphism (acting on the $\ell$-adic Tate modules of $A$); the polynomial $f_A$ is monic of degree twice the dimension of $A$. We call a polynomial $f$ a Weil $q$-polynomial, or simply a Weil polynomial, if there is an abelian variety $A$ over $k$ with $f = f_A$. Weil proved that all of the roots of a Weil polynomial are Weil numbers, and Honda showed that every Weil number is a root of some Weil polynomial. Furthermore, Tate showed that two abelian varieties over $k$ are isogenous if and only if their associated Weil polynomials are equal. If $A$ is a simple abelian variety over $k$ then $f_A$ is a power of an irreducible polynomial, and in fact the Honda-Tate theorem (see [13, Théorème 1]) says the map that sends $A$ to the set of roots (in $\overline{\mathbb{Q}}$) of $f_A$ induces a bijection between the set of isogeny classes of simple abelian varieties over $k$ and the set of Galois conjugacy classes of Weil numbers in $\mathbb{Q}$. The Honda-Tate theorem also provides a simple number-theoretic criterion for determining whether a polynomial, all of whose roots are Weil numbers, is a Weil polynomial. In addition, the theorem shows how the Weil polynomial of an abelian variety $A$ over $k$ determines the algebra $(\text{End} A) \otimes \mathbb{Q}$.

An abelian variety $A$ over $k$ is ordinary if the rank of its group of $p$-torsion points over the algebraic closure of $k$ is equal to the dimension of $A$; a Weil polynomial is ordinary if it is the characteristic polynomial of Frobenius of an ordinary abelian variety; and a Weil number is ordinary if its minimal polynomial is an ordinary Weil polynomial. The Honda-Tate theorem simplifies considerably if one considers only ordinary varieties and ordinary Weil polynomials — see Section 3 of [3]. For example, a monic polynomial in $\mathbb{Z}[x]$ is an ordinary Weil $q$-polynomial if and only if it is of even degree $2n$, all of its roots are Weil numbers, and its middle coefficient (that is, the coefficient of $x^n$) is coprime to $q$. Furthermore, an ordinary abelian variety $A$ over $k$ is simple if and only if its Weil polynomial $f$ is irreducible. If $A$ is simple and ordinary then the algebra $(\text{End} A) \otimes \mathbb{Q}$ is generated by the Frobenius endomorphism of $A$, and so is isomorphic to the number field defined by $f$. Since the characteristic polynomial of Frobenius of $A$ has degree equal to twice the dimension of $A$, we see that the degree of the number field $K = (\text{End} A) \otimes \mathbb{Q}$ over $\mathbb{Q}$ is twice the dimension of $A$. In fact, the number field $K$ is a CM-field, which means that $K$ is a totally imaginary quadratic extension of a totally real field $K^+$. (A number field $L$ is totally imaginary if it cannot be embedded into $\mathbb{R}$, and it is totally real if every embedding of $L$ into $\mathbb{C}$ comes from an embedding of $L$ into $\mathbb{R}$.)

3. An easy test for absolute simplicity

In this section we will present an easy-to-verify sufficient condition for a simple abelian variety over a finite field to be absolutely simple. For ordinary varieties, the sufficient condition is also necessary. Throughout this section, $k$ will be a finite field, $\overline{F}$ its algebraic closure, $A$ a simple abelian variety over $k$, and $\pi$ its Frobenius
endomorphism. We let $\text{End}^0 A$ denote the algebra $(\text{End} A) \otimes \mathbb{Q}$. Note that the simplicity of $A$ implies that the subalgebra $\mathbb{Q}(\pi)$ of $\text{End}^0 A$ is a field.

**Proposition 3.** Let $D$ be the set of integers $d > 1$ such that either

(a) the minimal polynomial of $\pi$ lies in $\mathbb{Z}[x^d]$ or
(b) the field $\mathbb{Q}(\pi^d)$ is a proper subfield of $\mathbb{Q}(\pi)$ and there is a primitive $d$th root of unity $\zeta$ in $\mathbb{Q}(\pi)$ such that $\mathbb{Q}(\pi) = \mathbb{Q}(\pi^d, \zeta)$.

Then:

1. The set $D$ is empty if and only if $\mathbb{Q}(\pi^d) = \mathbb{Q}(\pi)$ for all $d > 0$.
2. If $\mathbb{Q}(\pi^d) = \mathbb{Q}(\pi)$ for all $d > 0$ then $A$ is absolutely simple. If $A$ is ordinary, then the converse is also true.

To prove this proposition we will need two elementary lemmas.

**Lemma 4.** Let $\ell$ be a finite extension of $k$. If $\mathbb{Q}(\pi^{[\ell:k]}) = \mathbb{Q}(\pi)$ then $A_\ell$ is simple. If $A$ is ordinary, then the converse is also true.

**Proof.** An abelian variety is simple if and only if its endomorphism ring contains no zero-divisors. Thus, if $A$ is simple and $A_\ell$ is not, there must exist an element of $\text{End}^0 A_\ell$ that does not come from $\text{End}^0 A$. But it follows from the Honda-Tate theorem [13] that $\text{End}^0 A_\ell = \text{End}^0 A$ if $\mathbb{Q}(\pi^{[\ell:k]}) = \mathbb{Q}(\pi)$. This proves the first statement of the lemma.

If $A$ is ordinary and $\mathbb{Q}(\pi^{[\ell:k]})$ is a proper subfield of $\mathbb{Q}(\pi)$, then it follows from the Honda-Tate theorem that $\text{End}^0 A_\ell$ is a matrix algebra over $\mathbb{Q}(\pi^{[\ell:k]})$. In particular, $\text{End}^0 A_\ell$ contains a zero-divisor, so that $A_\ell$ is not simple. $\square$

**Lemma 5.** Let $\alpha$ be an algebraic number with minimal polynomial $g \in \mathbb{Q}[x]$, and suppose that $d$ is a positive integer such that the field $L = \mathbb{Q}(\alpha^d)$ is a proper subfield of $K = \mathbb{Q}(\alpha)$ and such that $\mathbb{Q}(\alpha^r) = K$ for all positive $r < d$. Then either $g \in \mathbb{Q}[x^d]$ or there is a primitive $d$th root of unity $\zeta$ in $K$ such that $K = L(\zeta)$.

**Proof.** Let $\zeta$ be a primitive $d$th root of unity in an algebraic closure of $K$ and let $M = L(\zeta) \cap K$. Note that $M$ contains $L$. Since $L(\zeta)$ is a Galois extension of $L$ it is also a Galois extension of $M$, and it follows that $L(\zeta)$ and $K$ are linearly disjoint over $M$, so that $[K(\zeta) : L(\zeta)] = [K : M]$. Let $m = [K(\zeta) : L(\zeta)] = [K : M]$. Since $K(\zeta) = \mathbb{Q}(\alpha, \zeta)$ is a Kummer extension of $L(\zeta) = \mathbb{Q}(\alpha^d, \zeta)$, we see that $\alpha^m$ lies in $L(\zeta)$, and hence also in $M$.

Suppose $m > 1$. Then since $\mathbb{Q}(\alpha^m)$ is a subfield of the proper subfield $M$ of $K$, the lemma’s hypothesis shows we must have $m = d$. If we let $h$ be the minimal polynomial of $\alpha^d$ over $\mathbb{Q}$, then $g(x) = h(x^d)$.

Suppose $m = 1$. Then $K(\zeta) = L(\zeta)$, so $K/L$ is a subextension of the abelian extension $K(\zeta)/L$, and is therefore Galois. Let $G$ be its Galois group, and suppose $\sigma$ is a non-identity element of $G$. Let $\xi = \sigma(\alpha)/\alpha$, so that $\xi$ lies in the multiplicative group generated by $\zeta$. Suppose $r$ is a positive integer less than $d$. Then the hypothesis of the lemma shows that $K = \mathbb{Q}(\alpha^r)$, so we must have $\alpha^r \neq \sigma(\alpha^r) = \xi^r \alpha^r$. Thus $\xi$ must in fact be a primitive $d$th root of unity, which shows that $\zeta \in K$. It follows that $K = K(\zeta)$, and this last field is $L(\zeta)$ because $m = 1$. $\square$

**Proof of Proposition 3.** If $d$ is an integer in $D$ then clearly $\mathbb{Q}(\pi^d)$ is a proper subfield of $\mathbb{Q}(\pi)$. On the other hand, if there exists some $d > 0$ such that $\mathbb{Q}(\pi^d) \neq \mathbb{Q}(\pi)$ then there exists a smallest such $d$, and by Lemma 5 this $d$ lies in $D$. This proves the first statement of the proposition.
It is clear that $A$ is absolutely simple if and only if $A_\ell$ is simple for every finite extension $\ell$ of $k$. The second statement of the proposition follows from this fact and from Lemma 4.

Remark. A theorem of Silverberg [12] shows that if $A$ is an abelian variety over an arbitrary field $k$, then to check that $\text{End}^0 A = \text{End}^0 A_k$ it suffices to check that $\text{End}^0 A = \text{End}^0 A_\ell$ for a certain finite extension $\ell$ of $k$; in particular, if one chooses an integer $m > 2$ not divisible by the characteristic of $k$, Silverberg shows that one may take $\ell$ to be the smallest field over which every $m$-torsion point of $A$ is defined. The degrees of such $\ell$ over $k$ may be quite large, even when $k$ is a finite field. Lemmas 5 and the proof of Lemma 4 show that Silverberg’s general result can be improved in the special case where $k$ is finite.

4. Absolutely simple abelian surfaces

In this section we will prove a theorem that shows that, given the characteristic polynomial of Frobenius of a simple ordinary abelian surface over a finite field, it is quite easy to determine whether the surface is absolutely simple. At the end of the section we will use this theorem to prove the special case $n = 2$ of Theorem 1.

Suppose $k$ is a finite field with $q$ elements and $A$ is an abelian surface over $k$. Let $f$ be the characteristic polynomial of Frobenius for $A$. Then Weil’s “Riemann Hypothesis” shows that $f$ is of the form $x^4 + ax^3 + bx^2 + aqx + q^2$ for some integers $a$ and $b$. If neither $a$ nor $b$ is coprime to $q$ then one can use the Honda-Tate theorem to show that $A$ becomes isogenous to the square of a supersingular elliptic curve over a finite extension of $k$. If $a$ is coprime to $q$ but $b$ is not, then one can again use Honda-Tate to show that $A$ is absolutely simple if and only if it is simple, and that $A$ is simple if and only if $f$ is irreducible. The most interesting situation arises when $b$ is coprime to $q$, which is the case exactly when $A$ is an ordinary abelian variety. In this case, $A$ is simple if and only if $f$ is irreducible.

**Theorem 6.** Suppose $f = x^4 + ax^3 + bx^2 + aqx + q^2$ is the Weil polynomial of a simple ordinary abelian surface $A$ over $k$. Then exactly one of the following conditions holds:

(a) The variety $A$ is absolutely simple.
(b) We have $a = 0$.
(c) We have $a^2 = q + b$.
(d) We have $a^2 = 2b$.
(e) We have $a^2 = 3b - 3q$.

In cases (b), (c), (d), and (e), the smallest extension of $k$ over which $A$ splits is quadratic, cubic, quartic, and sextic, respectively.

**Proof.** Let $\pi$ be the Frobenius endomorphism of $A$ and let $K$ be the field $\mathbb{Q}(\pi)$. Because $A$ is ordinary, the field $K$ is a CM-field of degree 4 over $\mathbb{Q}$. The ordinarity of $A$ also implies that $\mathbb{Q}(\pi^d)$ is a CM-field for every positive integer $d$, and that $A$ splits over the degree-$d$ extension of $k$ if and only if $\mathbb{Q}(\pi^d)$ is a proper subfield of $K$.

Suppose $A$ is not absolutely simple. Then there is a positive integer $d$ such that $\mathbb{Q}(\pi^d)$ is a proper subfield of $K$; let us take $d$ to be the smallest such integer, and let $L$ be the imaginary quadratic field $\mathbb{Q}(\pi^d)$. By Lemma 3, either $d = 2$ and $f \in \mathbb{Z}[x^2]$, or $d = 4$ and $f \in \mathbb{Z}[x^4]$, or there is a primitive $d$th root of unity $\zeta$ in $K$.
such that $K = L(ζ)$. Let us first show that if the third possibility is the case and if $d > 4$ then $d$ must equal 6.

Suppose, to obtain a contradiction, that we are in the third case and that $d$ is greater than 4 but not equal to 6. Then the degree of $Q(ζ)$ over $Q$ is greater than 2, so $K$ must be $Q(ζ)$. Let $σ ∈ \text{Gal}(K/Q)$ be the nontrivial automorphism of $K$ that fixes $L$. The proof of Lemma 3 shows that we may choose our primitive root of unity $ζ$ so that $π^σ = ζπ$. Applying $σ$ to this equality, we find $π = ζ^σ π^σ = ζ^σ ζ π$, so that $ζ^σ ζ = 1$. The only element of the Galois group of the cyclotomic field with this property is complex conjugation. But then the fixed field $L$ of $σ$ must be totally real, and we have reached a contradiction.

So we must find, for $d = 2, 3, 4,$ and 6, the conditions on the coefficients $a$ and $b$ in the minimal polynomial of $π$ that are equivalent to $π^d$ lying in a quadratic subfield. Note that the characteristic polynomial of $π^d$ is of the form $x^4 + αx^3 + βx^2 + αq^d x + q^{2d}$, and that such a quartic polynomial is the square of a quadratic polynomial if and only if $α^2 - 4β + 8q^d = 0$. It is not difficult to explicitly calculate the characteristic polynomial of $π^d$ for each $d$ we are considering, and we find that

$$α^2 - 4β + 8q^d = \begin{cases} a^2(a^2 - 4b + 8q) & \text{if } d = 2; \\ (a^2 - b - q)(a^2 - 4b + 8q) & \text{if } d = 3; \\ a^2(a^2 - 2b)^2(a^2 - 4b + 8q) & \text{if } d = 4; \\ a^2(a^2 - b - q)^2(a^2 - 3b - 3q)(a^2 - 4b + 8q) & \text{if } d = 6. \end{cases}$$

We have assumed that $A$ is simple over $k$, so the characteristic polynomial for $π$ is irreducible; this means in particular that the quantity $a^2 - 4b + 8q$ is nonzero. Thus, if $A$ is not absolutely simple then one of the cases (b), (c), (d), or (e) must hold. Note that if two of these cases were to hold simultaneously, then $b$ would equal a multiple of $q$, contradicting our assumption that $b$ is coprime to $q$. Thus exactly one of the cases (a) through (e) must hold.

Finally, the formulas for $α^2 - 4β + 8q^d$ given above make it easy to verify the theorem’s statement about the degree of the minimal splitting field of $A$.

Using Theorem 3, it is easy to show that there exist absolutely simple ordinary abelian surfaces over every finite field. If $q$ is an arbitrary prime power, then Theorem 1.1 of [1] shows that the polynomial $x^4 + x^3 + x^2 + qx + q^2$ is an ordinary Weil polynomial. It is easy to check that this polynomial is irreducible, so it corresponds to an isogeny class of simple abelian varieties over the field $F_q$. Then Theorem 3 shows that the varieties in the isogeny class are absolutely simple.

5. THE EXISTENCE OF ABSOLUTELY SIMPLE ABELIAN VARIETIES OF HIGHER DIMENSION

In this section we will prove Theorem 7 in the case where $n > 2$. As we noted in the Introduction, it suffices to prove the theorem for finite prime fields $k$, but we will assume only that $k$ is finite. In fact, for such fields we will prove a result that is slightly stronger than Theorem 7.

**Theorem 7.** Let $k$ be a finite field and let $n > 2$ be an integer. Then there is an absolutely simple $n$-dimensional ordinary abelian variety over $k$.

The proof of Theorem 7 depends on three lemmas, whose proofs we will postpone until after the proof of the theorem. The first lemma gives sufficient conditions for
an ordinary Weil number to correspond to an isogeny class of absolutely simple varieties.

**Lemma 8.** Let \( q \) be a prime power and let \( n > 2 \) be an integer. Suppose \( \pi \) is an ordinary Weil \( q \)-number, let \( K = \mathbb{Q}(\pi) \), let \( K^+ \) be the maximal real subfield of \( K \), and let \( n = [K^+:\mathbb{Q}] \). Suppose that

1. the minimal polynomial of \( \pi \) is not of the form \( x^{2n} + ax^n + q^n \),
2. the field \( K^+ \) has no proper subfields other than \( \mathbb{Q} \), and
3. the field \( K^+ \) is not the maximal real subfield of a cyclotomic field.

Then the isogeny class corresponding to \( \pi \) consists of absolutely simple varieties.

The second lemma shows that any polynomial satisfying a certain set of local conditions also satisfies the hypotheses of Lemma 8. We will use this lemma again in Section 7.

**Lemma 9.** Let \( q \) be a prime power and let \( n > 2 \) be an integer. Let \( g \in \mathbb{Z}[x] \) be a monic polynomial of degree \( n \), and let \( f \) be the polynomial given by \( f(x) = x^n g(x + q/x) \). Suppose that the following five conditions hold:

1. the polynomial \( f \) is not of the form \( x^{2n} + ax^n + q^n \),
2. all of the complex roots of \( g \) are real numbers of absolute value less than \( 2\sqrt{q} \),
3. the constant term of \( g \) is coprime to \( q \),
4. there exists a prime \( p_1 \) such that the reduction of \( g \) modulo \( p_1 \) is irreducible, and
5. there exists a prime \( p_2 \) such that the reduction of \( g \) modulo \( p_2 \) is a linear times an irreducible.

Then \( f \) is an irreducible ordinary Weil polynomial of degree \( 2n \), and its roots \( \pi \) satisfy the hypotheses of Lemma 8.

The third lemma gives us a way of producing polynomials that meet the hypotheses of Lemma 8.

**Lemma 10.** Let \( q \) be a prime power and let \( n > 2 \) be an integer. Then there is a monic polynomial \( g \) in \( \mathbb{Z}[x] \) that satisfies the following five conditions:

1. the polynomial \( g \) can be written
   \[ g = x^n + cx^{n-2} + \text{lower-order terms}, \]
   where either \( c \) is equal to \(-2n\) or \( c \) is not divisible by \( n \),
2. all of the complex roots of \( g \) are real numbers of absolute value less than \( 2\sqrt{2} \),
3. the constant term of \( g \) is coprime to \( q \),
4. the reduction of \( g \) modulo \( 2 \) is irreducible, and
5. the reduction of \( g \) modulo \( 3 \) is a linear times an irreducible.

**Proof of Theorem 7.** Let \( g \) be the polynomial whose existence is guaranteed by Lemma 10. Then \( g \) satisfies the last four of the five hypotheses of Lemma 8; we will show that it satisfies the first hypothesis as well.

First we will consider the case in which \( q > 2 \). Since

\[ g = x^n + cx^{n-2} + \text{lower-order terms}, \]
we find that the polynomial \( f \) defined in Lemma 8 may be written in the form

\[ f = x^{2n} + (qn + c)x^{2n-2} + \text{lower-order terms}. \]
Now, \( c \) is either \(-2n\) or is not a multiple of \( n \), so the coefficient of \( x^{2n-2} \) in \( f \) is nonzero. In particular, \( f \) is not of the form \( x^{2n} + ax^n + q^n \).

For the case in which \( q = 2 \) we use the easily-proven fact that the reduction of \( f \) modulo 2 is equal to \( x^n \) times the reduction of \( g \) modulo 2. Since \( g \) modulo 2 is irreducible, and since \( x^n + 1 \) is not irreducible over \( \mathbb{F}_2 \), the polynomial \( f \) must have an odd coefficient somewhere between \( x^{2n} \) and \( x^n \). Again we see that \( f \) is not of the form \( x^{2n} + ax^n + q^n \).

Thus \( g \) satisfies all the hypotheses of Lemma 3, so by Lemma 3 the roots of \( f \) are Weil numbers that correspond to an isogeny class of absolutely simple ordinary varieties over \( \mathbb{F}_q \).

Proof of Lemma 3. Suppose, to obtain a contradiction, that \( \pi \) corresponds to an isogeny class that is not absolutely simple. Then by Proposition 1 there is a positive integer \( d \) such that \( \mathbb{Q}(\pi^d) \) is a proper subfield of \( K \). Let \( d \) be the smallest positive integer with this property. Since \( \pi \) is ordinary, the field \( L = \mathbb{Q}(\pi^d) \) is a CM-field, and its maximal real subfield \( L^+ \) is a proper subfield of \( K^+ \). Hypothesis (2) shows that \( L^+ \) must be \( \mathbb{Q} \), so \( L \) is an imaginary quadratic field.

Lemma 3 shows that either the minimal polynomial \( f \) of \( \pi \) lies in \( \mathbb{Z}[x^d] \) or \( K = L(\zeta) \) for some primitive \( d \)th root of unity. The first possibility cannot occur, because it would imply that \( d = n \), contradicting hypothesis (1). Therefore the second possibility must be the case. We find that the maximal real subfield of \( \mathbb{Q}(\zeta) \) is a subfield of \( K^+ \), and since \( K^+ \) is not itself the maximal real subfield of a cyclotomic field (by assumption), we find that the maximal real subfield of \( \mathbb{Q}(\zeta) \) must be \( \mathbb{Q} \), so that \( \mathbb{Q}(\zeta) \) is either a quadratic field or \( \mathbb{Q} \) itself. But \( K \) is the compositum of \( L \) and \( \mathbb{Q}(\zeta) \), so the degree of \( K \) over \( \mathbb{Q} \) is at most 4. This contradicts our assumption that the degree of \( K \) over \( \mathbb{Q} \) is \( 2n \), where \( n > 2 \).

Proof of Lemma 4. Since \( g \) modulo \( p_1 \) is irreducible, \( g \) itself is irreducible in \( \mathbb{Z}[x] \), and since all of its complex roots are real, \( g \) defines a totally real number field \( K^+ \). Let \( \alpha \) be a root of \( g \) in \( K^+ \). The discriminant of the polynomial \( h = x^2 - \alpha x + q \) is totally negative because the roots of \( g \) all have magnitude less than \( 2\sqrt{q} \), so \( h \) defines a totally imaginary quadratic extension \( K \) of \( K^+ \). If \( \pi \) is a root of \( h \) in \( K \), then \( K = \mathbb{Q}(\pi) \) contains \( K^+ \) because \( \alpha = \pi + q/\pi \). Thus \( \pi \) is an algebraic number of degree \( 2n \). Furthermore, if \( \varphi \) is an embedding of \( K \) into \( \mathbb{C} \), then \( \varphi(\pi) \) is a root of \( x^2 - \varphi(\alpha)x + q \), and the quadratic formula shows that \( |\varphi(\pi)| = \sqrt{q} \). Thus \( \pi \) is in fact a Weil number of degree \( 2n \). Since \( \pi \) is a root of \( f \), the polynomial \( f \) must be the minimal polynomial of \( \pi \). This shows that \( f \) is an irreducible polynomial whose roots are Weil numbers, and to show that \( f \) is an ordinary Weil polynomial we need merely check that its middle coefficient is coprime to \( q \). But this follows from hypothesis (3), because the middle coefficient of \( f \) differs from the constant term of \( g \) by a multiple of \( q \).

Now we must check that a root \( \pi \) of \( f \) satisfies the hypotheses of Lemma 3. The first of these hypotheses is identical to the first hypothesis of the lemma we are proving, and is therefore satisfied.

We will show that \( K^+ \) is not the maximal real subfield of a cyclotomic field. It will suffice to show that \( K^+ \) is not Galois over \( \mathbb{Q} \). The defining polynomial \( g \) of \( K^+ \) reduces modulo \( p_2 \) as a linear times an irreducible, so the prime \( p_2 \) splits in \( K^+ \) into two primes with different residue class degrees, so \( K^+ / \mathbb{Q} \) cannot be Galois.

Finally, we prove that \( K^+ \) has no proper subfields other than \( \mathbb{Q} \). For suppose \( K^+ \) had a proper subfield \( L \) other than \( \mathbb{Q} \). Let \( \mathfrak{p} \) be the prime of \( K^+ \) over \( p_2 \) whose
Table 1. Polynomials satisfying the conditions of Lemma 10 for small values of $n$.

| $n$ | $g$ |
|-----|-----|
| 3   | $x^3 - 5x + 1$ |
| 4   | $x^4 - 6x^2 - x + 1$ |
| 5   | $x^5 - 10x^3 + x^2 + 20x + 1$ |
| 6   | $x^6 - 12x^4 + 34x^2 + x - 1$ |
| 7   | $x^7 - 14x^5 + 56x^3 - 2x^2 - 57x + 1$ |
| 8   | $x^8 - 16x^6 + 81x^4 + x^3 - 129x^2 + 1$ |
| 9   | $x^9 - 18x^7 + 108x^5 + x^4 - 240x^3 - 9x^2 + 147x + 1$ |

Our proof of Lemma 10 depends on a result of Robinson concerning certain modified Chebyshev polynomials. Before starting on the proof of the lemma we will define these polynomials and present Robinson’s result.

For every positive integer $i$ let $t_i$ be the $i$th Chebyshev polynomial, so that $t_i(x) = \cos(i \cdot \arccos(x))$. For every positive integer $i$ let $T_i$ be the polynomial given by $T_i(x) = 2 \cdot 2^{i/2} \cdot t_i(x/2^{3/2})$. It is not hard to show that $T_i$ is a monic polynomial in $\mathbb{Z}[x]$ and that $T_i \equiv x^i \pmod{2}$. Let $T_0 = 1$.

**Lemma 11.** Suppose $a_1, \ldots, a_n$ are real numbers such that

$$
\left( \sum_{i=1}^{n-1} \frac{|a_i|}{2^{n/2}} \right) + \frac{1}{2} \frac{|a_n|}{2^{n/2}} < 1.
$$

Then every complex root of the polynomial

$T_n + a_1T_{n-1} + \cdots + a_{n-1}T_1 + a_nT_0$

is real and lies in the open interval $(-2\sqrt{2}, 2\sqrt{2})$.

**Proof.** This follows from the techniques of Robinson [10].

**Proof of Lemma 10.** If $n \leq 9$ we can simply choose the appropriate value of $g$ from Table 1, so let us assume that $n > 9$.

Lemma 12 (below) shows that there exist monic degree-$n$ polynomials $g_2$ in $\mathbb{F}_2[x]$ and $g_3$ in $\mathbb{F}_3[x]$ such that $g_2$ is irreducible, such that $g_3$ is a linear times an irreducible and has nonzero constant term, and such that the coefficients of
Lemma 12. Suppose $n \geq 10$. Then there exist monic degree-$n$ polynomials $g_2$ in $\mathbb{F}_2[x]$ and $g_3$ in $\mathbb{F}_3[x]$ such that $g_2$ is irreducible, such that $g_3$ is a linear times an irreducible and has nonzero constant term, and such that the coefficients of $x^{n-1}$ through $x^{n-6}$ of $g_2$ and $g_3$ are equal to the reductions modulo 2 and 3 of the corresponding coefficients of the modified Chebyshev polynomial $T_n$ defined above.

Proof. For $n \leq 18$ we choose $g_2$ and $g_3$ from Table 2. For $n > 18$ we argue as follows:

Corollary 3.2 (p. 94) of [3] shows that there exists a monic irreducible polynomial in $\mathbb{F}_2[x]$ of degree $n$ with zeroes for the first six coefficients after the leading $x^n$. We take this polynomial for our $g_2$. The same corollary shows that there is a monic irreducible polynomial $h$ in $\mathbb{F}_3[x]$ such that the first six coefficients of $(x - 1)h$ are equal to those of the reduction of $T_n$ modulo 3; we take $g_3$ to be $(x - 1)h$.

6. Asymptotic results for abelian surfaces

In this section we will prove Theorem 2 in the case $n = 2$. In fact, we will prove a more precise statement.

Theorem 13. Let $\epsilon$ be a positive real number. If $q$ is a prime power with $q > (659/\epsilon)^2$ then $S(\mathbb{F}_q, 2) > 1 - \epsilon$. 

of ordinary abelian surfaces that satisfy case (b) of Theorem 6 is at most 4. Theorem 1.2 of [2] shows that
not absolutely simple. For this we use Theorem 6. First note that if $x^4 + ax^3 + bx^2 + aqx + q^2$ is the Weil polynomial for an ordinary abelian surface over $F_q$, then $|a| < 4q^{1/2}$, and if $a = 0$ then $0 < |b| < 2q$. Thus, the number of Weil polynomials of ordinary abelian surfaces that satisfy case (b) of Theorem 6 is at most $4q$. Also, for every nonzero integer $d$ in the interval $(-4q^{1/2}, 4q^{1/2})$ there is at most one Weil polynomial with $a = d$ that satisfies case (c) of the theorem; for every nonzero integer $d$ in the interval $(-2q^{1/2}, 2q^{1/2})$ there is at most one Weil polynomial with $a = 2d$ that satisfies case (d) of the theorem; and for every nonzero integer $d$ in the

| $n$ | $g_2$ | $g_3$ |
|-----|-------|-------|
| 10  | $x^{10} + x^3 + 1$ | $x^{10} + x^8 - x^6 - x^4 + x^2 + x + 1$ |
| 11  | $x^{11} + x^2 + 1$ | $x^{11} - x^9 - x^7 - x^5 + 1$ |
| 12  | $x^{12} + x^3 + 1$ | $x^{12} + x^6 - x^2 + x + 1$ |
| 13  | $x^{13} + x^5 + x^2 + x + 1$ | $x^{13} + x^{11} - x^9 + x^2 + 1$ |
| 14  | $x^{14} + x^5 + 1$ | $x^{14} - x^{12} - x^{10} + x^2 + x + 1$ |
| 15  | $x^{15} + x + 1$ | $x^{15} - x^9 + x + 1$ |
| 16  | $x^{16} + x^6 + x^2 + x + 1$ | $x^{16} + x^{14} - x^{12} + x^{10} + x^2 + x - 1$ |
| 17  | $x^{17} + x^3 + 1$ | $x^{17} - x^{15} - x^{13} + x^{11} + x^3 + x^2 + 1$ |
| 18  | $x^{18} + x^3 + 1$ | $x^{18} + x^2 + x - 1$ |

Table 2. Polynomials satisfying the conditions of Lemma 12 for small values of $n$.

Proof. Let $r$ be the arithmetic function defined by $r(x) = \varphi(x)/x$, where $\varphi$ is Euler’s $\varphi$-function, let $I$ be the number of isogeny classes of abelian surfaces over $F_q$, let $O_{\text{simple}}$ be the number of isogeny classes of simple ordinary abelian surfaces, and let $O_{\text{abs.simple}}$ be the number of isogeny classes of absolutely simple ordinary abelian surfaces. Theorem 1.2 of [2] shows that

$$I < \frac{32}{3}r(q)q^{3/2} + 3473q + 8359q^{1/2};$$

this upper bound is obtained by combining the estimates that Theorem 1.2 gives for the number of ordinary and non-ordinary isogeny classes of abelian surfaces.

The same theorem shows that the number of isogeny classes of ordinary abelian surfaces over $F_q$ is at least

$$\frac{32}{3}r(q)q^{3/2} - 8359q^{1/2}.$$ 

The isogeny classes of ordinary elliptic curves over $F_q$ correspond to the integers $t$ such that $|t| < 2q^{1/2}$ and $(t, q) = 1$, so there are at most $4q^{1/2}$ such isogeny classes. A non-simple isogeny class of ordinary abelian surfaces is determined by its two factors, so there are at most $4q^{1/2}(4q^{1/2} + 1)/2 = 8q + 2q^{1/2}$ such reducible isogeny classes. Thus we have

$$O_{\text{simple}} > \frac{32}{3}r(q)q^{3/2} - 8q - 8361q^{1/2}.$$ 

Now we must estimate the number of simple ordinary isogeny classes that are not absolutely simple. For this we use Theorem 6. First note that if $x^4 + ax^3 + bx^2 + aqx + q^2$ is the Weil polynomial for an ordinary abelian surface over $F_q$ then $|a| < 4q^{1/2}$, and if $a = 0$ then $0 < |b| < 2q$. Thus, the number of Weil polynomials of ordinary abelian surfaces that satisfy case (b) of Theorem 6 is at most $4q$. Also, for every nonzero integer $d$ in the interval $(-4q^{1/2}, 4q^{1/2})$ there is at most one Weil polynomial with $a = d$ that satisfies case (c) of the theorem; for every nonzero integer $d$ in the interval $(-2q^{1/2}, 2q^{1/2})$ there is at most one Weil polynomial with $a = 2d$ that satisfies case (d) of the theorem; and for every nonzero integer $d$ in the
interval \((-4/3)q^{1/2}, (4/3)q^{1/2}\) there is at most one Weil polynomial with \(a = 3d\) that satisfies case (e) of the theorem. We find that there are at most \(15q^{1/2}\) simple Weil polynomials \(x^4 + ax^3 + bx^2 + aqx + q^2\) with \(a \neq 0\) that are not absolutely simple.

Combining these estimates with the lower bound for \(O_{\text{simple}}\) given above, we find that
\[
O_{\text{abs.simple}} > \frac{32}{3}r(q)q^{3/2} - 12q - 8376q^{1/2}.
\]

Now suppose \(\epsilon\) is given. If \(\epsilon \geq 1\) then the conclusion of the theorem is clearly true for all \(q\), so we may assume that \(\epsilon < 1\) and that \(q > 659^2\). With this lower bound for \(q\), our bounds for \(I\) and \(O_{\text{abs.simple}}\) show that
\[
I < \frac{32}{3}r(q)q^{3/2} + 3486q
\]
and
\[
O_{\text{abs.simple}} > \frac{32}{3}r(q)q^{3/2} - 25q.
\]
Thus
\[
\frac{O_{\text{abs.simple}}}{I} > \left(1 - \frac{75}{32r(q)q^{1/2}}\right) \left(1 - \frac{10458}{32r(q)q^{1/2}}\right).
\]
The denominator is less than 2, so we have
\[
\frac{O_{\text{abs.simple}}}{I} > \left(1 - \frac{75}{32r(q)q^{1/2}}\right) \left(1 - \frac{10458}{32r(q)q^{1/2}}\right)
> 1 - \frac{10533}{32r(q)q^{1/2}} > 1 - \frac{659}{q^{1/2}} > 1 - \epsilon,
\]
as was to be shown.

7. ASYMPTOTIC RESULTS FOR ABELIAN VARIETIES OF HIGHER DIMENSION

In this section we will prove Theorem 3 in the case \(n > 2\) by proving a more precise result, whose statement requires us to introduce some notation. First we define constants \(c_1, c_2,\) and \(c_3\) by setting
\[
c_1 = \sqrt{3}/6 \approx 0.288675,
\]
\[
c_2 = \exp(3/2) \cdot 2(1 + \sqrt{2})\sqrt{3(1 + \sqrt{3}/162)})^3/3 \approx 12.898608,
\]
and
\[
c_3 = c_2/(1 + \sqrt{2}) \approx 5.342778.
\]
Next, for every positive integer \(n\) we let
\[
v_n = \frac{2^n}{n!} \prod_{j=1}^{n} \left(\frac{2j}{2j-1}\right)^{n+1-j}
\]
and we let
\[
G_n = \frac{1}{v_n} 6^n c_1^n c_3^n \frac{n(n+1)}{(n-1)!}.
\]
Finally, if \( n > 1 \) is an integer and if \( \epsilon \) is a positive real, we let \( k_{n, \epsilon} \) denote the smallest positive integer \( k \) such that
\[
\left(1 - \frac{1}{2n}\right)^k \leq \frac{\epsilon}{8},
\]
we let \( m_{n, \epsilon} \) be the product of the first \( k_{n, \epsilon} \) prime numbers, and we let
\[
M_{n, \epsilon} = \left(\frac{8G_nm_{n, \epsilon}}{\epsilon}\right)^2.
\]

Recall that \( S(F_q, n) \) denotes the fraction of isogeny classes of \( n \)-dimensional abelian varieties over \( F_q \) that are ordinary and absolutely simple.

**Theorem 14.** Let \( n > 2 \) be an integer and let \( \epsilon \) be a positive real number. If \( q > M_{n, \epsilon} \) then \( S(F_q, n) > 1 - \epsilon \).

For every prime power \( q \) and non-negative integer \( n \) we let \( \mathcal{I}(q, n) \) denote the set of isogeny classes of \( n \)-dimensional abelian varieties over \( F_q \) and we let \( \mathcal{O}(q, n) \) and \( \mathcal{N}(q, n) \) denote the sets of ordinary and non-ordinary isogeny classes in \( \mathcal{I}(q, n) \), respectively. Also, we let \( \mathcal{O}_{\text{simple}}(q, n) \) and \( \mathcal{O}_{\text{abs.simple}}(q, n) \) denote the sets of simple and absolutely simple isogeny classes in \( \mathcal{O}(q, n) \), respectively. As in Section 3 we let \( r \) be the arithmetic function defined by \( r(x) = \varphi(x)/x \), where \( \varphi \) is Euler’s \( \varphi \)-function. Our proof of Theorem 14 breaks into two parts. First we will give an upper bound for \( \#\mathcal{I}(q, n) \).

**Proposition 15.** Let \( n > 2 \) be an integer and let \( \epsilon \) be a positive real number with \( \epsilon \leq 1 \). If \( q > M_{n, \epsilon} \) then \( \#\mathcal{I}(q, n) < (1 + \epsilon/8)v_n r(q)q^{n(n+1)/4} \).

Then we will give a lower bound for \( \#\mathcal{O}_{\text{abs.simple}}(q, n) \).

**Proposition 16.** Let \( n > 2 \) be an integer and let \( \epsilon \) be a positive real number with \( \epsilon \leq 1 \). If \( q > M_{n, \epsilon} \) then \( \#\mathcal{O}_{\text{abs.simple}}(q, n) \geq (1 - 7\epsilon/8)v_n r(q)q^{n(n+1)/4} \).

Clearly these two propositions provide a proof of Theorem 14.

**Proof of Proposition 15.** Combining the estimates for \( \#\mathcal{O}(q, n) \) and \( \#\mathcal{N}(q, n) \) given in Theorem 1.2 of [2], we find that the quantity \( \#\mathcal{I}(q, n) - v_n r(q)q^{n(n+1)/4} \) is less than or equal to
\[
6^n c_1^2 c_2 \frac{n(n+1)}{(n-1)!} q^{n(n-1)/4} + \left(v_n + 6^n c_1^2 c_3 \frac{n(n+1)}{(n-1)!}\right) q^{(n+2)(n-1)/4},
\]
so
\[
\frac{\#\mathcal{I}(q, n)}{v_n r(q)q^{n(n+1)/4}} \leq 1 + \frac{c_2 G_n}{c_3 r(q)q^{n/2}} + \frac{1}{r(q)q^{1/2}} + \frac{G_n}{r(q)q^{1/2}}.
\]

An easy induction shows that \( G_n > 2 \), and certainly \( c_2/c_3 < 2.5 \), so we have
\[
\frac{\#\mathcal{I}(q, n)}{v_n r(q)q^{n(n+1)/4}} < 1 + \frac{G_n}{r(q)q^{1/2}} \left(\frac{c_2}{c_3} + 1/2 + 1\right)
\]
\[
< 1 + \frac{4G_n}{r(q)q^{1/2}}.
\]

Since \( q > M_{n, \epsilon} \) we have \( q^{1/2} > 8G_nm_{n, \epsilon}/\epsilon \), and combining this with the fact that \( r(q) \geq 1/2 \) we find that
\[
\frac{\#\mathcal{I}(q, n)}{v_n r(q)q^{n(n+1)/4}} \leq 1 + \frac{\epsilon}{m_{n, \epsilon}}.
\]
But $m_{n,e}$ is greater than 8 for $\epsilon \leq 1$, so the right-hand side is at most $1 + \epsilon/8$. This proves the inequality of the proposition. $
abla$

Our proof of Proposition 16 is based upon Lemmas 8 and 9. We will compute

- a lower bound on the number of degree-$n$ polynomials satisfying hypotheses (2), (4), and (5) of Lemma 9,
- an upper bound on the number of degree-$n$ polynomials satisfying hypothesis (2) but failing hypothesis (3) of Lemma 9, and
- an upper bound on the number of degree-$n$ polynomials satisfying hypothesis (2) but failing hypothesis (1) of Lemma 9.

Subtracting the sum of the latter two estimates from the first estimate will give us a lower bound on the number of degree-$n$ polynomials satisfying all the hypotheses of Lemma 9. By Lemma 8, this lower bound will also be a lower bound on $\#\mathcal{O}_{abs,simple}(q,n)$. The computation of the lower bound on the number of polynomials satisfying hypotheses (2), (4), and (5) of Lemma 8 will depend on the following lemma, whose proof we will postpone until the next section.

**Lemma 17.** Let $n > 2$ be a positive integer, let $\epsilon$ be a real number between 0 and 1, and let $m = m_{n,e}$ be as defined at the beginning of this section. Then there are at least $m^n(1 - \epsilon/4)$ monic degree-$n$ polynomials in $(\mathbb{Z}/m\mathbb{Z})[x]$ such that

1. there exists a prime divisor $p_1$ of $m$ such that the reduction of $g$ modulo $p_1$ is irreducible, and
2. there exists a prime divisor $p_2$ of $m$ such that the reduction of $g$ modulo $p_2$ is a linear times an irreducible.

Before we proceed to the proof of Proposition 16 we should mention a basic correspondence that we will use repeatedly in our argument. Fix our prime power $q$. Suppose $g$ is a monic polynomial of degree $n$ with integer coefficients, say

$$g = x^n + b_1x^{n-1} + \cdots + b_n,$$

and let $f$ be the polynomial defined by $f(x) = x^ng(x + q/x)$, so that

$$f = (x^{2n} + q^n) + a_1(x^{2n-1} + q^n - x) + \cdots + a_{n-1}(x^{n+1} + qnx^n) + a_nx^n$$

for some integers $a_i$. Then the linear map $\Omega$ from $\mathbb{Z}^n$ to $\mathbb{Z}^n$ that sends a vector $b = (b_1, \ldots, b_n)$ to the vector $a = (a_1, \ldots, a_n)$ is invertible — in fact, it is represented by a matrix with integer entries that is lower-triangular with 1’s on the diagonal. Thus, if we let $b$ range over a set of vectors that reduces modulo some integer $m$ to the entire set $(\mathbb{Z}/m\mathbb{Z})^n$, then $\Omega(b)$ will also range over such a set, and conversely, if $a$ ranges over such a set, then so will $\Omega^{-1}(a)$.

Note that if $g$ and $f$ are related as above, then $g$ satisfies hypothesis (2) of Lemma 9 if and only if the roots of $f$ in the complex numbers all have magnitude $q^{1/2}$ and the roots of $f$ in the real numbers all have even multiplicity. Also, the roots of $f$ meet this last condition if and only if the vector $(a_1q^{-1/2}, a_2q^{-1}, \ldots, a_nq^{-n/2})$ lies in the region $V_n$ of $\mathbb{R}^n$ defined in 5. Thus we will be interested in estimating the sizes of the intersections of certain lattices with $V_n$.

Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. Our arguments will involve two lattices in $\mathbb{R}^n$: The first lattice, denoted $\Lambda$, is generated by the vectors $q^{-i/2}e_i$, and the second, denoted $\Lambda'$, is generated by the same set of vectors, except with $q^{-n/2}e_n$ replaced with $pq^{-n/2}e_n$. Thus $\Lambda \supset \Lambda'$.
Proof of Proposition 3.3.1. Let $m = m_{n, \epsilon}$ and let $\Lambda''$ denote the lattice $m\Lambda$. If $\ell$ is a point in $\Lambda''$ let $B_{\ell}$ denote the “brick”

$$\ell + \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \forall i: 0 \leq x_i < mq^{-i/2}e_i\}.$$ 

Let $S$ denote the set of all $\ell \in \Lambda''$ such that $B_{\ell} \subseteq V_n$. The proof of Proposition 2.3.1 of [3] (see especially p. 435) shows that

$$\#S \geq \frac{\text{volume } V_n}{\text{covolume } \Lambda''} - 6^n c_1^2 c_3 \frac{n(n+1)}{(n-1)!} d \frac{1}{\text{covolume } \Lambda''},$$

where $d$ is the mesh of $\Lambda''$ (see p. 434 of [3]), which is $mq^{-1/2}$. Since the covolume of $\Lambda''$ is $m^n q^{-n(n+1)/4}$, we find that

$$\#S \geq m^{-n} v_n q^{n(n+1)/4} - 6^n c_1^2 c_3 \frac{n(n+1)}{(n-1)!} m^{-n+1} q^{n(n+2)}/4.$$ 

Thus

$$m^n \#S \geq v_n q^{n(n+1)/4}(1 - mGnq^{-1/2}),$$

and using the fact that $q > M_{n, \epsilon}$ we find that

$$m^n \#S \geq v_n q^{n(n+1)/4}(1 - \epsilon/8).$$

Now suppose $\ell$ is a lattice point in $S$, and consider a typical element $x = (a_1 q^{-1/2}, a_2 q^{-1}, \ldots, a_n q^{-n/2})$ of $\Lambda \cap B_{\ell}$. As $x$ ranges over all of $\Lambda \cap B_{\ell}$, the vector $a = (a_1, \ldots, a_n)$ ranges over a set of $m^n$ elements of $\mathbb{Z}^n$ that reduces modulo $m$ to all of $(\mathbb{Z}/m\mathbb{Z})^n$. Lemma 3 above shows that of the $m^n$ polynomials $g$ we obtain from the vectors $\Omega(a)$ arising from elements of $\Lambda \cap B_{\ell}$, at least $m^n(1 - \epsilon/4)$ satisfy hypotheses (4) and (5) of Lemma 3. So for each element of $S$ we obtain at least $m^n(1 - \epsilon/4)$ polynomials satisfying hypotheses (2), (4), and (5) of Lemma 3. Thus the total number of such polynomials is at least $m^n \#S(1 - \epsilon/4)$, and by the results of the preceding paragraph this number is at least

$$v_n q^{n(n+1)/4}(1 - \epsilon/4)(1 - \epsilon/8),$$

which is greater than $v_n q^{n(n+1)/4}(1 - 3\epsilon/8)$.

Next we estimate the number of polynomials $g$ that satisfy hypothesis (2) of Lemma 3, but that fail to satisfy hypothesis (3). There is a bijection between the set of such polynomials and the set $\Lambda' \cap V_n$, and Proposition 2.3.1 of [2] gives upper and lower bounds for the size of the latter set; in particular, we find that the number of such polynomials differs from $(1/p)v_n q^{n(n+1)/4}$ by at most

$$\frac{q^{n(n+1)/4}}{pq^{1/2}} - 6^n c_1^2 c_3 \frac{n(n+1)}{(n-1)!},$$

which is $v_n q^{n(n+1)/4}G_n/pq^{1/2}$. Since $q$ is at least $M_{n, \epsilon}$, this last quantity is at most $v_n q^{n(n+1)/4}\epsilon/(4pm)$, which is less than $v_n q^{n(n+1)/4}\epsilon/32$, because $m > 4$ when $\epsilon \leq 1$. Thus the number of polynomials that satisfy hypothesis (2) but not hypothesis (3) is at most

$$v_n q^{n(n+1)/4}\left(\frac{1}{p} + \frac{\epsilon}{32}\right).$$

Finally we estimate the number of polynomials $g$ that satisfy hypothesis (2) of Lemma 3, but that fail to satisfy hypothesis (1). Now, a polynomial $x^{2n} + ax^n + q^n$ has all of its roots on the circle $|z| = q^{1/2}$ if and only if $|a| \leq 2q^{n/2}$, so there are at most $4q^{n/2} + 1$ polynomials meeting hypothesis (2) but failing hypothesis (1). It is very easy to show that $4q^{n/2} + 1 < v_n q^{n(n+1)/4}\epsilon/32$ when $q > M_{n, \epsilon}$.
Now, the number of polynomials meeting all five hypotheses of Lemma 9 is at least as large as the number that satisfy hypotheses (2), (4), and (5), less the number that satisfy hypothesis (2) but that fail either hypothesis (1) or hypothesis (3). We find that the number of polynomials meeting all five hypotheses is at least

\[
v_n q^{n(n+1)/4} \left( 1 - \frac{3\epsilon}{8} \right) - v_n q^{n(n+1)/4} \left( \frac{1}{p} + \frac{\epsilon}{32} \right) - v_n q^{n(n+1)/4} \frac{\epsilon}{32} = v_n q^{n(n+1)/4} \left( r(q) - \frac{7\epsilon}{16} \right) \geq v_n q^{n(n+1)/4} \left( 1 - \frac{7\epsilon}{8} \right)
\]

and this is the statement of Proposition 13.

8. Proof of Lemma 17

In this section we will prove Lemma 17. We continue to use the notation set at the beginning of Section 7.

For the moment, let us write \( A_{n,p} \) for the set of monic degree-\( n \) irreducible polynomials in \( \mathbb{F}_p[x] \) and \( B_{n,p} \) for the set of monic degree-\( n \) polynomials in \( \mathbb{F}_p[x] \) that factor as a linear polynomial times an irreducible.

**Lemma 18.** Let \( p \) be a prime. For all \( n > 0 \) we have \( \# A_{n,p} \geq p^n/(2n) \), and for all \( n > 1 \) we have \( \# B_{n,p} \geq p^n/(2n-2) \).

**Proof.** The lemma follows easily from the well-known exact formula

\[
\# A_{n,p} = \frac{1}{n} \sum_{d|n} p^d \mu \left( \frac{n}{d} \right),
\]

where \( \mu \) is the Möbius function. \( \square \)

Suppose \( n > 1 \). We see from Lemma 18 that if we choose a monic degree-\( n \) polynomial \( f \) at random from \( \mathbb{F}_p[x] \) (with the uniform distribution), then

\[
\text{Prob}(f \notin A_{n,p}) \leq 1 - \frac{1}{2n}
\]

and

\[
\text{Prob}(f \notin B_{n,p}) \leq 1 - \frac{1}{2n-2}.
\]

Suppose that \( \epsilon \) is given, with \( 0 < \epsilon < 1 \). Let \( k = k_{n,\epsilon} \) and \( m = m_{n,\epsilon} \) be as at the beginning of Section 7, so that \( m \) is the product of the first \( k \) prime numbers. Now suppose we choose a monic degree-\( n \) polynomial \( f \) at random from \( (\mathbb{Z}/m\mathbb{Z})[x] \). By the Chinese remainder theorem, making such a choice is equivalent to choosing a monic degree-\( n \) polynomial \( f \) at random from \( \mathbb{F}_p[x] \) for each of the first \( k \) primes \( p \). Thus we see that

\[
\text{Prob}(\forall p \mid m : (f \mod p) \notin A_{n,p}) \leq \left( 1 - \frac{1}{2n} \right)^k
\]

and

\[
\text{Prob}(\forall p \mid m : (f \mod p) \notin B_{n,p}) \leq \left( 1 - \frac{1}{2n-2} \right)^k
\]

This completes the proof.
and it follows that

\[ \text{Prob}(\exists p_1, p_2 \mid m : (f \mod p_1) \in A_{n,p_1} \text{ and } (f \mod p_2) \in B_{n,p_2}) > 1 - \left(1 - \frac{1}{2n}\right)^k - \left(1 - \frac{1}{2n-2}\right)^k > 1 - 2\left(1 - \frac{1}{2n}\right)^k. \]

But the definition of \(k_{n,\epsilon}\) shows that

\[ \left(1 - \frac{1}{2n}\right)^k \leq \epsilon/8, \]

so

\[ \text{Prob}(\exists p_1, p_2 \mid m : (f \mod p_1) \in A_{n,p_1} \text{ and } (f \mod p_2) \in B_{n,p_2}) > 1 - \frac{\epsilon}{4}. \]

Thus the number of monic degree-\(n\) polynomials in \(\mathbb{Z}/m\mathbb{Z}[x]\) that satisfy the two conditions of Lemma 17 is at least \(m^n(1 - \epsilon/4)\), as was to be shown.

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