THE TOPOLOGY OF SPACES OF PROBABILITY MEASURES, I: FUNCTIONORS $P_\tau$ AND $\hat{P}$.

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For a Tychonoff space $X$, the constructions $\hat{P}(X)$ and $P_\tau(X)$ of the spaces of probability Radon measures and probability $\tau$-smooth measures on $X$ are considered. It is proved that the constructions $\hat{P}$ and $P_\tau$ determine functors in the category of Tychonoff spaces, which extend the functor $P$ of probability measures in the category of compacta. In this part we investigate general topological properties of the spaces $\hat{P}(X)$ and $P_\tau(X)$, as well as categorical properties of the functors $\hat{P}$ and $P_\tau$.

INTRODUCTION

The space of probability measures is a classical object which is studied from different points of view in Measure Theory, Functional Analysis, Probability Theory, Topology and Category Theory. This paper is the first part of a larger project (the results of which were announced in [1]) devoted to the study of spaces of probability measures on topological spaces, in particular, spaces of probability $\tau$-smooth measures and probability Radon measures. Our interests primarily touch on topological and categorical aspects of Measure Theory and are very much in line with the survey [2], where a functor $P : \text{Comp} \to \text{Comp}$ of the space of probability measures in the category of compacta is studied (we are going to use contemporary terminology, understanding a compact Hausdorff space under the term "compact").

The study of spaces of probability measures leads to the problem of extension of the functor $P$ from the category of compacta to wider categories, in particular, the category $\text{Tych}$ of Tychonoff spaces and their continuous maps. One of such extensions $P_\beta$ was suggested by By A.Ch. Chigogidze [3]: For a Tychonoff space $X$ let us consider the space $P_\beta(X) = \{ \mu \in P(\beta X) \mid \text{supp}(\mu) \subset X \subset \beta X \}$, where $\beta X$ is the Stone-Čech compactification of $X$, and $\text{supp}(\mu)$ is the support of the measure $\mu$. The structure $P_\beta(X)$ induces a functor $P_\beta : \text{Tych} \to \text{Tych}$, which extends the functor $P : \text{Comp} \to \text{Comp}$. Another construction was considered in [2] by V.V. Fedorchuk, who noted that the functor $P \circ \beta : \text{Tych} \to \text{Comp}$ assigning to each Tychonoff space $X$ the space $P(\beta X)$, also extends the functor $P : \text{Comp} \to \text{Comp}$.

However, the functors $P_\beta$ and $P \circ \beta$ have a number of drawbacks. In particular, the space $P_\beta(X)$ is very narrow and does not contain many natural countably additive measures on $X$ (i.e. measures non-compact supports), and, on the other hand, the space $P(\beta X)$ is very broad, and contains all finitely additive measures on $X$, and, as a result, the functor $P \circ \beta$ does not preserve many specific properties of the space $X$, in particular, it significantly raises the weight (although it does not raise the density).

Thus, it is natural to consider spaces of measures, which lie between the spaces $P_\beta(X)$ and $P(\beta X)$.

For that purpose, let us consider for a Tychonoff space $X$ the following two spaces of probability measures:

$$\hat{P}(X) = \{ \mu \in P(\beta X) \mid \mu_+(X) = 1 \} \quad \text{and} \quad P_\tau(X) = \{ \mu \in P(\beta X) \mid \mu^+(X) = 1 \},$$

where $\mu_+(X) = \sup\{ \mu(B) \mid X \supset B \text{ is a Borel subset of } \beta X \}$ and $\mu^+(X) = \inf\{ \mu(B) \mid X \supset B \text{ is a Borel subset of } \beta X \}$, which are, correspondingly, the upper and the lower $\mu$-measures of the set $X$ in $\beta X$ (as a tribute to historical tradition, we use the notation $\hat{P}(X)$ and $P_\tau(X)$, not $P_\beta(X)$ and $P^+(X)$, which seem to be more natural). Evidently,

$$P_\beta(X) \subset \hat{P}(X) \subset P_\tau(X) \subset P(\beta X)$$

The study of these two functors and their properties is the subject of this paper.

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for any Tychonoff space \( X \), and \( P_\beta(X) = \hat{P}(X) = P_\tau(X) = P(\beta X) \), if the space \( X \) is compact.

The measures belonging to the spaces \( \hat{P}(X) \) and \( P_\tau(X) \) can be equivalently described both in terms of countably additive measures on the space \( X \) and in terms of linear functionals on the Banach space \( C_0(X) \) of bounded continuous real-valued functions on \( X \). Before providing exact formulations, let us recall some definitions.

A countably additive finite measure \( \mu \), defined on the \( \sigma \)-algebra \( \mathcal{B}(X) \) of Borel subsets of a topological space \( X \), is called

(i) a probability measure, if \( \mu(X) = 1 \);

(ii) a regular measure, if \( \mu(A) = \sup\{\mu(Z) \mid A \supset Z \text{ is a closed subset of } X\} \) for every Borel subset \( A \subset X \);

(iii) \( \tau \)-smooth, if \( \mu(A) = \sup\{\mu(K) \mid A \supset K \text{ is a compact subset of } X\} \) for every Borel subset \( A \subset X \) (in [4] Radon measures are called dense measures);

(iv) \( \tau \)-smooth, if for any monotonically decreasing net \( \{Z_\alpha\} \) of closed subsets of \( X \) with empty intersection \( \bigcap_\alpha Z_\alpha \), the net \( \{\mu(Z_\alpha)\} \) of real numbers converges to zero (see [4]).

Further in this text, by measure on a topological space we shall understand a finitely additive Borel measure. One easily notices that every Radon measure on a Hausdorff space is regular and \( \tau \)-smooth. Moreover, a regular measure \( \mu \) on a Hausdorff space \( X \) is a Radon measure if and only if

\[
\mu(X) = \sup\{\mu(K) \mid K \text{ is a compact subset of } X\}.
\]

Let \( X \) be a Tychonoff space. For every measure \( \mu \in P_\tau(X) \) we will define a measure \( \tilde{\mu} \) on \( X \) by the formula

\[
\tilde{\mu}(A) = \mu^*(A) = \inf\{\mu(B) \mid A \subset B \text{ is a Borel subset of } \beta X\},
\]

where \( A \) is a Borel subset of \( X \). It is known [5], or [2, 1.11] (see also Remark 1.2) that the measure \( \tilde{\mu} \), defined in this way, is \( \tau \)-smooth on \( X \). Conversely, every probability \( \tau \)-smooth measure \( \tilde{\mu} \) on \( X \) determines a measure \( \mu \in P_\tau(X) \) by the formula \( \mu(A) = \tilde{\mu}(A \cap X) \), where \( A \in B(\beta X) \). Under this condition Radon measures (and only them) become measures on \( \beta X \), which belong to the set \( \hat{P}(X) \). Therefore, we will call measures from \( \hat{P}(X) \) Radon measures, and the measures from \( P_\tau(X) \) \( \tau \)-smooth measures.

By \( C_0(X) \) we denote the Banach space of all bounded continuous real-valued functions on \( X \), endowed with the norm \( \|f\| = \sup\{|f(x)| : x \in X\} \). Not getting into the definition of the integral, let us note that every regular probability measure \( \mu \) on \( X \) uniquely determines the integral \( \int_\mu f \), a non-negative linear functional on \( C_0(X) \) of norm 1 (the value of integral \( \int_\mu f \) on the function \( f \) will be denoted by \( \int_\mu f \), or simply \( \mu(f) \)). Under such identification we get the following equivalences (see [4]),

(i) the measure \( \mu \) is \( \tau \)-dense if and only if \( \mu(f_\alpha) \to 0 \) for any monotonically decreasing net \( \{f_\alpha\} \subset C_0(X) \) that pointwise converges to zero;

(ii) the measure \( \mu \) is Radon if and only if \( \mu(f_\alpha) \to 0 \) for any net of uniformly bounded sequence of functions \( \{f_\alpha\} \subset C_0(X) \) that converges to zero uniformly on compacta.

Similarly, a measure \( \mu \in P(\beta X) \) belongs to the set \( P_\tau(Y) \) if and only if \( \mu(f_\alpha) \to 0 \) for every monotonically decreasing net \( \{f_\alpha\} \subset C(\beta X) \) converging to zero pointwise on the set \( X \subset \beta X \).

Let us show that the constructions of the spaces \( P_\tau(X) \) and \( \hat{P}(X) \) are functorial in the category \( T_\text{ych} \).

Since \( \hat{P}(X) \subset P_\tau(X) \subset P(\beta X) \) for any Tychonoff space \( X \) and \( P \circ \beta : T_\text{ych} \to \text{Comp} \) is a functor on the category \( T_\text{ych} \) of Tychonoff spaces [2], in order to see that the constructions \( \hat{P} \) and \( P_\tau \) are functorial, it is sufficient to show that for any continuous map \( f : X \to Y \) of Tychonoff spaces \( P(\beta f)(P_\tau(X)) \subset P_\tau(Y) \) and \( P(\beta f)(\hat{P}(X)) \subset \hat{P}(Y) \), where \( \beta f : \beta X \to \beta Y \) is the Stone-Čech compactification of the map \( f \) (see [6, 3.60]). If \( \mu \in P_\tau(X) \), then \( \mu^*(X) = 1 \), and, consequently, \( \mu(B) = 1 \) for any Borel set \( B \subset \beta X \). Then for any Borel set \( B' \subset \beta Y \), \( P(\beta f)(B') = \mu((\beta f)^{-1}(B')) = 1 \), since \( (\beta f)^{-1}(B') \) is a Borel subset of \( \beta X \), which contains \( X \). This implies that \( P(\beta f)(\mu) \in P_\tau(Y) \), i.e. \( P(\beta f)(P_\tau(X)) \subset P_\tau(Y) \).

If \( \mu \in \hat{P}(X) \), then for any \( \varepsilon > 0 \) there exists a compact \( K \subset X \), such that \( \mu(K) > 1 - \varepsilon \). Then \( f(K) \subset Y \) is a compact in \( Y \) such that \( P(\beta f)(\mu (f(K))) = \mu((\beta f)^{-1}(f(K))) \geq \mu(K) > 1 - \varepsilon \). Therefore, the measure \( P(\beta f)(\mu) \) belongs to the set \( \hat{P}(Y) \), i.e. \( P(\beta f)(\hat{P}(X)) \subset \hat{P}(Y) \). Let us define \( P_\tau(f) = P(\beta f)|P_\tau(X) : P_\tau(X) \to P_\tau(Y) \) and \( \hat{P}(f) = P(\beta f)|\hat{P}(X) : \hat{P}(X) \to \hat{P}(Y) \). Thus, we have proved the following

**Theorem 0.1.** The constructions \( P_\tau \) u \( \hat{P} \) are covariant functors in the category \( T_\text{ych} \) of Tychonoff spaces and their continuous maps, which extend the functor \( P : \text{Comp} \to \text{Comp} \).
Let us note that we might as well have defined the functors $P_\tau : Tych \to Tych$ and $\hat{P} : Tych \to Tych$ from inside, without using Stone-Čech compactifications. In particular, for the Tychonoff space $X$ the space $P_\tau(X)$ consists of regular probability measures on $X$, and the topology on $P_\tau(X)$ is induced by a subbase consisting of sets of the form $\{\mu \in P_\tau(X) : |\mu(\varphi) - \mu_0(\varphi)| < 1\}$, where $\mu_0 \in P_\tau(X)$ and $\varphi \in C_b(X)$.

If $f : X \to Y$ is a continuous map of Tychonoff spaces, then the map $P_\tau(f) : P_\tau(X) \to P_\tau(Y)$ is defined by $P_\tau(f)(\mu)(A) = \mu(f^{-1}(A))$, where $\mu \in P_\tau(X)$ and $A$ is a Borel subset of $Y$. Then $\hat{P}(X)$ is a subspace of $P_\tau(X)$, consisting of Radon probability measures, and $\hat{P}(f)$ is the restriction of the map $P_\tau(f)$ to the set $\hat{P}(X)$. Using the roundabout way (using Stone-Čech compactifications) we got rid of the necessity of checking that the constructions $P_\tau$ and $\hat{P}$, defined is such way, are functors in the category $Tych$ indeed.

Before moving on to the presentation of concrete results, let us note that in a number of spaces, for example, spaces $X$ which are Borel sets in their Stone-Čech compactification, every $\tau$-smooth measure is Radon. In this case the spaces $P_\tau(X)$ and $\hat{P}(X)$ coincide. More generally, this holds for the so called universally measurable spaces, that is, spaces $X$ which are measurable in some compactification $\gamma X$ with respect to any measure $\mu \in P(\gamma X)$. Besides absolute Borel spaces, analytic and coanalytic spaces are also absolutely measurable [7, 2.2.12].

1 Categorial properties of the functor $P_\tau$

In this section we shall investigate categorial properties of the functor $P_\tau$, and also some general topological properties of the spaces $P_\tau(X)$.

Let us start with the following simple remark.

**Lemma 1.1.** Let $X$ be a Tychonoff space. If $\mu \in P_\tau(X)$, then $\mu(A) = \mu(B)$ for any two Borel subsets $A, B \subset \beta X$ such that $A \cap X = B \cap X$.

**Proof.** Let $A, B \subset \beta X$ be Borel sets with $A \cap X = B \cap X$. Then

$$|\mu(A) - \mu(B)| = |\mu(A \cap B) + \mu(A \setminus B) - \mu(A \cap B) - \mu(B \setminus A)| =$$

$$= |\mu(A \setminus B) - \mu(B \setminus A)| \leq \mu(A \setminus B) + \mu(B \setminus A) = \mu((A \setminus B) \cup (B \setminus A)).$$

Since $A \cap X = B \cap X$, we have that $A \Delta B = (A \setminus B) \cup (B \setminus A) \subset \beta X \setminus X$. If $\mu \in P_\tau(X)$, then $\mu((\beta X \setminus X) = 0$, which implies that $\mu(A \Delta B) \leq \mu((\beta X \setminus X) = 0$, which means that $|\mu(A) - \mu(B)| \leq \mu(A \Delta B) = 0$, i.e. $\mu(A) = \mu(B)$. Thus, the lemma is proved.

**Remark 1.2.** Lemma 1.1 implies the following fact, which has already been mentioned in the introduction: every measure $\mu \in P_\tau(X)$ induces a probability measure $\hat{\mu}$ on $X$, according to the formula $\hat{\mu}(A) = \mu(B)$, where $B$ is any Borel subset $\beta X$ such that $B \cap X = A$, and $A$ is a Borel subset of $X$. Under these conditions, the measure $\hat{\mu}$ is $\tau$-smooth. Indeed, for every monotone decreasing net $\{Z_\alpha\}$ of non-empty closed subsets of $X$ with empty intersection, the net $\{\hat{\mu}(Z_\alpha)\}$ consisting of their closures in $\beta X$ also monotonically decreases.

Since $\{Z_\alpha\}$ is a centered family of closed subsets of the compact $\beta X$, it has a non-empty intersection $Z = \bigcap_\alpha Z_\alpha$ [6, 3.1.1]. Since $Z \cap X = (\bigcap_\alpha Z_\alpha) \cap X = \bigcap_\alpha (Z_\alpha \cap X) = \bigcap_\alpha Z_\alpha = \emptyset$, we have that $Z \subset \beta X \setminus X$. Keeping in mind that $\mu \in P_\tau(X)$, we get that $\hat{\mu}(Z) = 0$. The regularity of the measure $\mu$ implies that for any $\varepsilon > 0$ there exists an open set $U$, $Z \subset U \subset \beta X$ such that $\mu(U) < \varepsilon$. [6, 3.1.5] implies that since $\bigcap_\alpha Z_\alpha = Z \subset U$, $Z_\alpha \subset U$ for some $\alpha_0$. Consequently, $\hat{\mu}(Z_{\alpha_0}) = \mu(Z_{\alpha_0}) < \varepsilon$. Since the net $\{Z_\alpha\}$ monotonically decreases, $\hat{\mu}(Z_{\beta}) \leq \hat{\mu}(Z_{\alpha_0}) < \varepsilon$ for all $\beta \geq \alpha_0$. But this means that the real-valued net $\{\hat{\mu}(Z_{\alpha})\}$ converges to zero, thus, the measure $\hat{\mu}$ on $X$ is $\tau$-smooth.

Let us recall that a map $f : X \to Y$ between topological spaces is called perfect, if it is closed and the preimage $f^{-1}(y)$ of every point $y \in Y$ is compact.

**Theorem 1.3.** The functor $P_\tau : Tych \to Tych$ preserves the class of perfect maps.

**Proof.** Let $f : X \to Y$ be a perfect map of Tychonoff spaces. Then the extension $\beta f : \beta X \to \beta Y$ of the map $f$ (called the Stone-Čech compactification of the map $f$) has the following property: $\beta f(\beta X \setminus X) \subset \beta Y \setminus Y$ [6, 3.7.15]. Let us consider the map $P(\beta f) : P(\beta X) \to P(\beta Y)$. We are going to prove that $P(\beta f)(P(\beta X) \setminus P_\tau(X)) \subset P(\beta Y) \setminus P_\tau(Y)$. Indeed, let $\mu \in P(\beta X) \setminus P_\tau(X)$, i.e. $\mu^*(X) < 1$. This implies that there exists a compact $K \subset \beta X \setminus X$ such that $\mu(K) > 0$. Then $\beta f(K) \subset \beta Y \setminus Y$ is a compact subset with $P(\beta f)(\mu)(\beta f(K)) = \mu(\beta f^{-1}(\beta f(K))) \geq \mu(K) > 0$. Consequently, $P(\beta f)(\mu)^*(Y) < 1$, i.e. $P(\beta f)(\mu) \notin P_\tau(Y)$. Thus, $P(\beta f)(P(\beta X) \setminus P_\tau(X)) \subset P(\beta Y) \setminus P_\tau(Y)$. Since $P(\beta f) : P(\beta X) \to P(\beta Y)$ is a map between compacta, the last inclusion implies that the map $P_\tau(f) = P(\beta f)|P_\tau(X) : P_\tau(X) \to P_\tau(Y)$ is perfect. The theorem is proved.
Theorem 1.4. The functor $P_\tau : \mathcal{Tych} \to \mathcal{Tych}$ preserves the class of embeddings.

Proof. Let $f : X \to Y$ be a topological embedding of Tychonoff spaces and $\beta f : \beta X \to \beta Y$ be its Stone-Čech compactification. One can easily see that $\beta f(\beta X \setminus X) \subset \beta Y \setminus f(X)$. Let $A = \{\mu \in P(\beta Y) : |\mu^*(f(X))| = 1\}$. Similarly to the proof of Theorem 1.3, it can be shown that $P(\beta f)(P(\beta X) \setminus P_\tau(X)) \subset P(\beta Y) \setminus A$. Obviously, $P(\beta f)(P_\tau(X)) \subset A$. Thus, the map $P_\tau(f) = P(\beta f)|_{P_\tau(X)} : P_\tau(X) \to A$ is proper. Let us show that it is also injective, which will imply that $P_\tau(f) : P_\tau(X) \to P_\tau(Y)$ is an embedding.

Let $\mu, \eta \in P_\tau(X)$ be two distinct measures. The there exists a closed set $Z \subset \beta X$ such that $\mu(Z) \neq \eta(Z)$.

We state that $P_\tau(f)(\mu)(\beta f(Z)) \neq P_\tau(f)(\eta)(\beta f(Z))$, which will imply that the measures $P_\tau(f)(\mu), P_\tau(f)(\eta) \in P_\tau(Y)$ are distinct. Indeed, letting $Z' = (\beta f)^{-1}(\beta f(Z))$, let us observe that, by the definition, $P_\tau(f)(\mu)(\beta f(Z)) = \mu(Z')$ and $P_\tau(f)(\eta)(\beta f(Z)) = \eta(Z')$. Since $f$ is an embedding, $Z' \cap X = Z \cap X$. Then by Lemma 1.1

$$P_\tau(f)(\mu)(\beta f(Z)) = \mu(Z') = \mu(Z) \neq \eta(Z) = \eta(Z') = P_\tau(f)(\eta)(\beta f(Z)),$$

i.e. the measures $P_\tau(f)(\mu), P_\tau(f)(\eta) \in P_\tau(Y)$ are distinct. The theorem is proved. \qed

Theorems 1.3, 1.4 immediately imply

Corollary 1.5. The functor $P_\tau : \mathcal{Tych} \to \mathcal{Tych}$ preserves the class of closed embeddings.

Since the functor $P_\tau$ preserves embeddings, for a pair $X \subset Y$ of Tychonoff spaces we will treat the space $P_\tau(X)$ as a subset $\{\mu \in P_\tau(Y) : |\mu^*(X)| = 1\}$ of $P_\tau(Y)$. Let us note that when we do this, the set $\tilde{P}(X) \subset P_\tau(X)$ consisting of Radon probability measures on $X$ becomes the subset $\{\mu \in P_\tau(Y) : |\mu^*(X)| = 1\} \subset P_\tau(Y)$. It is also worthy of note that Theorem 1.4 implies that the construction of the space $P_\tau(X)$ in fact does not depend on the compactification of $X$, i.e. for any compactification $\gamma X$ of $X$ the space $\{\mu \in P(\gamma X) : |\mu^*(X)| = 1\}$ is naturally homeomorphic to $P_\tau(X)$. As we will see in §2, the image of the set under this homeomorphism $\{\mu \in P(\gamma X) : |\mu^*(X)| = 1\}$ is the space $\tilde{P}(X)$ of Radon probability measures on $X$.

Let us recall that the support of a measure $\mu \in P(X)$ on a compact space $X$ is the set $\text{supp}(\mu) = \cap\{F : F \subset X\}$. Under this condition $\mu(\text{supp}(\mu)) = 1$. i.e. the support of the measure $\mu$ is the smallest closed subset of $\mu$-measure one. If $X$ is a Tychonoff space, then by the support of a $\tau$-smooth probability measure $\mu \in P_\tau(X)$ on $X$ we will sometimes understand the set $\text{supp}(\mu) \cap X$.

The functor $P_\tau$ preserves neither injective nor surjective maps. To see that $P_\tau$ does not preserve injective maps, choose any non-measurable subset $Z \subset \beta X \cap \gamma Y$ of the closed interval $Y = [0,1]$ with lower and upper Lebesgue measures $\lambda_*(Z) = 0$ and $\lambda^*(Z) = 1$. Next, consider the subspace $X = Z \times \{0\} \cup ([0,1] \setminus Z) \times \{1\}$ of the plane $\mathbb{R}^2$, and let $f : X \to Y, f : (z,t) \mapsto z$, be the projection onto the first coordinate. It is clear that the map $f$ is bijective and continuous. It can be shown (see [20, Example 3]) that the Lebesgue measure $\lambda \in P(Y) = P_\tau(Y)$ has two preimages under the map $P_\tau(f) : P_\tau(X) \to P_\tau(Y)$, which means that the map $P_\tau(f)$ is not injective and hence $P_\tau$ does not preserve injective maps.

To see that the functor $P_\tau$ does not preserve surjective maps, consider the bijective map $f : D \to [0,1]$ of a discrete space $D$ onto the interval $[0,1]$. Then the standard Lebesgue measure on $[0,1]$ does not have a preimage under the map $P_\tau(f) : P_\tau(D) \to \tilde{P}(0,1]$. This is due to the fact that the set $D$, being open in its Stone-Čech compactification, is measurable with respect to any measure $\mu \in P(\beta D)$. Consequently, any $\tau$-smooth measure on $D$ is Radon and, since the space $D$ is discrete, it is also atomic (on atomic measures see [8, §2]). But the image of an atomic measure under the map $P_\tau(f)$ is also an atomic measure, and, consequently, it is not equal to the Lebesgue measure on $[0,1]$.

Nonetheless, the functor $P_\tau$ preserves one of the properties of maps, which in the case of compactness implies its surjectivity.

Proposition 1.6. Let $f : X \to Y$ be a map with dense image $f(X)$ in $Y$. Then the image $P_\tau(f)(P_\tau(X))$ is dense in $P_\tau(Y)$.

Proof. One can easily see that $P_\tau(f)(P_\tau(X))$ contains the set

$$P_\tau(f)(X) = \{\mu \in Y : |\text{supp}(\mu)| < \infty \text{ and } \text{supp}(\mu) \subset f(X)\},$$

which is dense in $P(\beta Y) \supset P_\tau(Y)$. \qed

Theorem 1.7. The functor $P_\tau$ preserves preimages, i.e. for any map $f : X \to Y$ between Tychonoff spaces and any subset $A \subset Y$ we have that $P_\tau(f)^{-1}(P_\tau(A)) = P_\tau(f^{-1}(A))$.

This example was added at the translation.
Proof. The inclusion \( P_\tau(f^{-1}(A)) \subset P_\tau(f^{-1}(P_\tau(A))) \) is trivial. Let us show that \( P_\tau(f^{-1}(P_\tau(A))) \subset P_\tau(f^{-1}(A)) \). This can be derived from the inclusion

\[
P_\tau(f)(P_\tau(X) \setminus P_\tau(f^{-1}(A))) \subset P_\tau(Y) \setminus P_\tau(A).
\]

Let \( \mu \in P_\tau(X) \setminus P_\tau(f^{-1}(A)) \), i.e. \( \mu^*(f^{-1}(A)) < 1 \). This means that there exists a compact \( K \subset X \setminus f^{-1}(A) \), such that \( \mu(K) > 0 \). Then \( f(K) \) is a compact subset of \( Y \setminus A \) such that \( P_\tau(f)(\mu(K)) = \mu(f^{-1}(f(K))) \geq \mu(K) > 0 \), i.e. \( P_\tau(f)(\mu)(A) < 1 \), and, as a consequence, \( P_\tau(f)(\mu) \notin P_\tau(A) \). The theorem is proved.

An embedding-preserving functor \( F : \mathcal{T}ych \to \mathcal{T}ych \) is said to preserve (closed) intersections, if for any Tychonoff space \( X \) and a family \( \{X_\alpha\}_{\alpha \in A} \) of its (closed) subsets we get \( F(\bigcap_{\alpha \in A} X_\alpha) = \bigcap_{\alpha \in A} F X_\alpha \).

**Remark 1.8.** Unlike the functor \( \tilde{P} \), which preserves countable intersections (see Theorem 2.15), the functor \( P_\tau \) does not preserve even finite intersections. This can be seen from the following example: let \( X \subset [0,1] \) be a subset of the interval such that \( \lambda^*(X) = 1 \) and \( \lambda_\alpha(X) = 0 \), where \( \lambda \) is the standard Lebesque measure on \([0,1] \). Then \( \lambda^*([0,1] \setminus X) = 1 \). Consequently, \( \lambda \in P_\tau(X) \cap P_\tau([0,1] \setminus X) \). However, \( P_\tau(X \cup ([0,1] \setminus X)) = P_\tau(\emptyset) = \emptyset \).

Yet, we have the following

**Proposition 1.9.** Let \( X \) be a Tychonoff space and \( A, B \subset X \) its two subsets, one of which is Borel. Then \( P_\tau(A \cap B) = P_\tau(A) \cap P_\tau(B) \).

**Theorem 1.10.** The functor \( P_\tau : \mathcal{T}ych \to \mathcal{T}ych \) preserves intersections of closed subsets, i.e. for any Tychonoff space \( X \) and closed subsets \( X_\alpha \subset X \), \( \alpha \in A \), we get

\[
P_\tau(\bigcap_{\alpha \in A} X_\alpha) = \bigcap_{\alpha \in A} P_\tau(X_\alpha).
\]

**Theorem 1.11.** The map \( R : P_\tau(\operatorname{lim} X_\alpha) \to \lim P_\tau(X_\alpha) \) is an embedding. If the bonding mappings (limit projections) \( p_\alpha : \operatorname{lim} X_\alpha \to X_\alpha, \alpha \in A \), are dense (i.e. \( \operatorname{lim} p_\alpha(X_\alpha) \) is dense in \( X_\alpha \)), then the image \( R(P_\tau(\operatorname{lim} X_\alpha)) \) is dense in \( \lim P_\tau(X_\alpha) \).
Proof. Let us consider the Stone-Čech compactification \(\{\beta X_\alpha, \beta(p_\alpha)\}\) of the system \(\{X_\alpha, p_\alpha\}\) and note that \(\lim X_\alpha\) can be embedded in \(\lim \beta X_\alpha\). Moreover, if the bonding mappings \(p_\alpha : \lim X_\alpha \to X_\alpha\) are dense, then the image of the space \(\lim X_\alpha\) is dense in \(\lim \beta X_\alpha\). By the continuity of the functor \(P : \text{Comp} \to \text{Comp}\), the corresponding mapping \(\tilde{R} : P(\lim \beta X_\alpha) \to P(\beta X_\alpha)\) is a homeomorphism. Applying the fact that the functor \(P\) preserves embeddings, we get that the map \(\tau : P(\lim X_\alpha) \to P(\beta X_\alpha)\) can be embedded in the homeomorphism \(\tilde{R}\) and, therefore, an embedding. Furthermore, since the functor \(P\) preserves maps with dense images, given that the bonding mappings \(p_\alpha\) are dense, the image of the space \(P(\lim X_\alpha)\) under the embedding \(R\) is dense in \(\lim P\tau(X_\alpha)\). The theorem is proved. \(\square\)

Now we are going to consider the property of preserving homotopies, which, in the compact case, is closely linked to the continuity of functors [10].

For Tychonoff spaces \(X\) and \(Y\) let us define a map \(j_{XY} : P\tau(X) \times Y \to P\tau(X \times Y)\) determined by the formula \(j_{XY}(\mu, y) = P\tau(i_y)(\mu)\), \(\mu \in P\tau(X), y \in Y\), where \(i_y : X \to X \times Y\) is an embedding of \(X\) in \(X \times Y\) as a fiber: \(i_y(x) = (x, y), x \in X\).

**Proposition 1.12.** The map \(j_{XY} : P\tau(X) \times Y \to P\tau(X \times Y)\) is a closed embedding.

**Proof.** Let \(X, Y\) be Tychonoff spaces and \(\beta X \cup \beta Y\) be their Stone-Čech compactifications. According to [9, VII.5.11 and VII.5.18], the map \(j_{\beta X, \beta Y} : P(\beta X) \times \beta Y \to P(\beta \times \beta Y)\) is an embedding of compacta. Now the claim follows from the obvious equality

\[
j_{\beta X, \beta Y}(P\tau(X) \times Y) = j_{\beta X, \beta Y}(P(\beta X) \times \beta Y) \cap P\tau(X \times Y).
\]

**Corollary 1.13.** The functor \(P\tau\) preserves homotopies, i.e., for any homotopy \(H_\tau : X \to Y\) the homotopy \(P\tau(H_\tau) : P\tau(X) \to P\tau(Y)\) is continuous as a map \(P\tau(H_\tau) : P\tau(X) \times [0, 1] \to P\tau(Y)\).

**Proof.** Let \(H : X \times [0, 1] \to Y\) be a homotopy. Then the map \(P\tau(H_\tau) : P\tau(X) \times [0, 1] \to P\tau(Y)\) is continuous as a composition \(P\tau(H_\tau) = P\tau(H) \circ j_{X \times [0, 1]}\) of continuous maps \(j_{X \times [0, 1]} : P\tau(X) \times [0, 1] \to P\tau(X \times [0, 1])\) and \(P\tau(H) : P\tau(X \times [0, 1]) \to P\tau(Y)\). \(\square\)

Let us recall the definition of a natural transformation of functors. Let \(F_i : C \to C', i = 1, 2\) be two covariant functors from a category \(C = (O, M)\) to a category \(C' = (O', M')\). A family \(\Phi = \{\varphi_X : F_1(X) \to F_2(X), X \in O\} \subset M'\) of morphisms is called a natural transformation of the functor \(F_1\) to the functor \(F_2\), if for any morphism \(f : X \to Y\) in the category \(C\) the following diagram is commutative:

\[
\begin{array}{ccc}
F_1(X) & \xrightarrow{\varphi_X} & F_2(X) \\
F_1(f) \downarrow & & \downarrow F_2(f) \\
F_1(Y) & \xrightarrow{\varphi_Y} & F_2(Y).
\end{array}
\]

For every Tychonoff space \(X\) let us define a map \(\delta_X : X \to P\tau(X)\) which assigns to each point \(x \in X\) the Dirac measure \(\delta_X(x)\) concentrated at the point \(x\).

**Theorem 1.14.** The family \(\delta = \{\delta_X\}\) defines a natural transformation of the identity functor \(\text{Id} : \mathcal{T}_{\text{ych}} \to \mathcal{T}_{\text{ych}}\) to the functor \(P\tau : \mathcal{T}_{\text{ych}} \to \mathcal{T}_{\text{ych}}\), and, moreover, every component \(\delta_X : X \to P\tau(X)\) is a closed embedding.

**Proof.** One can easily check that \(\delta = \{\delta_X\}\) is a natural transformation of the functor \(\text{Id}\) to the functor \(P\tau\). The fact that every map \(\delta_X : X \to P\tau(X)\) is a closed embedding implies from [4, II, §3]. \(\square\)

**Theorem 1.15.** The functor \(P\tau\) preserves the density of Tychonoff spaces, i.e., \(d(P\tau(X)) = d(X)\) for any infinite Tychonoff space \(X\).

**Proof.** Let \(A \subset X\) be a dense subset of cardinality \(d(X)\) of an infinite Tychonoff space \(X\). Then the set \(B = \{\sum_{i=1}^n r_i \delta(x_i) \mid n \in \mathbb{N} \text{ and for every } 1 \leq i \leq n \text{ } r_i \text{ is rational and } x_i \in A\}\) is dense in \(P\tau(X)\). Moreover, it is clear that the cardinality of the set \(B\) is equal to \(d(X)\). \(\square\)

**Theorem 1.16.** The functor \(P\tau\) preserves the weight of Tychonoff spaces, i.e., \(w(P\tau(X)) = w(X)\) for any infinite Tychonoff space \(X\).
Lemma 1.19. Let $X$ be an infinite Tychonoff space. According to [6, 3.5.2], there exists a compactification $cX$ of the space $X$ such that $w(cX) = w(X)$. By Theorem 1.4, the space $P_\tau(X)$ can be embedded in the compact space $P(cX) = P(cX)$. Since the functor $P$ preserves weight [9, VII.3.9], $w(P(cX)) = w(cX) = w(X)$. Consequently, $w(P_\tau(X)) \leq w(X)$, and, since $X$ can be embedded in $P_\tau(X)$, $w(X) \leq w(P_\tau(X))$. That is, $w(P_\tau(X)) = w(X)$. The theorem is proved.

The following theorem follows from [4, II, §4] implies

Theorem 1.17. The functor $P_\tau$ preserve the class of metrizable spaces.

Let us recall that a $p$-paracompact space is a preimage of a metrizable space under a perfect map [11]. Theorems 1.3 and 1.7 imply

Theorem 1.18. The functor $P_\tau$ preserves the class of $p$-paracompact space.

Let $X$ be a topological space. For every countable ordinal number $\alpha$ we will define families $F_\alpha(X)$ and $G_\alpha(X)$ of Borel sets in the following way: the family $F_0(X)$ (family $G_0(X)$) consists of all closed (open) subsets of the space $X$, the family $F_\alpha(X)$ (the family $G_\alpha(X)$) consists of all countable unions (countable intersections) of sets from $\bigcup_{\xi<\alpha} F_\xi(X)$ (from $\bigcup_{\xi<\alpha} G_\xi(X)$) for odd ordinals $\alpha$ and all countable intersections (countable unions) of the sets from $\bigcup_{\xi<\alpha} F_\xi(X)$ (from $\bigcup_{\xi<\alpha} G_\xi(X)$) for even ordinals. It is obvious that for any $A \in F_\xi(X)$ ( $A \in G_\xi(X)$) we have that $X \setminus A \in G_\xi(X)$ ($X \setminus A \in F_\xi(X)$).

For a topological space $X$ by $B_0(X)$ we denote the $\sigma$-algebra of all Baire subsets of $X$, i.e. the smallest $\sigma$-algebra containing all functionally closed subsets of $X$. Baire subsets can be classified in the following way: $M_0(X)$ ($A_0(X)$) is the class of all functionally closed (functionally open) subsets of the space $X$. For every countable ordinal $\alpha$, $M_\alpha(X)$ ($A_\alpha(X)$) is the family of subsets of $X$, which can be presented as countable intersections (unions) of sets from $\bigcup_{\xi<\alpha} A_\xi(X)$ (from $\bigcup_{\xi<\alpha} M_\xi(X)$).

For a Borel subset $A$ of a Tychonoff space $X$ let us define a function $\chi_A : P_\tau(X) \to [0, 1]$ with the help of the formula $\chi_A(\mu) = \mu^*(A)$.

Lemma 1.19. Let $X$ be a Tychonoff space, $A$ be a subset of $X$, $\alpha$ be an even ordinal, $\xi$ be any ordinal, and $a \in \mathbb{R}$. Then

1. If $A \in M_\xi(X)$, then $\chi^{-1}_A((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(A) \geq a \} \in M_\xi(P_\tau(X)).$

2. If $A \in F_\alpha(X)$, then $\chi^{-1}_A((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(A) \geq a \} \in F_\alpha(P_\tau(X)).$

3. If $A \in A_\xi(X)$, then $\chi^{-1}_A((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(A) > a \} \in A_\xi(P_\tau(X)).$

4. If $A \in G_\alpha(X)$, then $\chi^{-1}_A((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(A) > a \} \in G_\alpha(P_\tau(X)).$

Proof. First, let us prove Statement (3) of this lemma. Let $U$ be a functionally open subset of $X$ and $\bar{U}$ be a functionally open subset of $\beta X$ such that $\bar{U} \cap X = U$. It is easy to construct a sequence $\{ f_n : \beta X \to [0, 1] \}_{n=1}^\infty$ of continuous functions converging to the characteristic function $\chi_\bar{U} : \beta X \to [0, 1]$ pointwise, such that $f_n|\beta X \setminus \bar{U} \equiv 0$, $f_n^{-1}\{1\} \subset f_{n+1}^{-1}\{1\}$, $n \in \mathbb{N}$, and $\bigcup_{n=1}^\infty f_n^{-1}\{1\} = \bar{U}$. By 1.1, $\mu^*(U) = \mu(\bar{U})$ for any $\mu \in P_\tau(X)$. Consequently,

$$\chi_\bar{U}^{-1}((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(U) > a \} = \{ \mu \in P_\tau(X) \mid \mu(\bar{U}) > a \} = \bigcup_{n=1}^\infty \{ \mu \in P_\tau(X) \mid \mu(f_n) > a \}$$

is a functionally open set in $P_\tau(X)$.

If $Z$ is a functionally closed subset of $X$, then the set $X \setminus Z$ is functionally open. Choose a functionally open subset $\bar{U}$ of $\beta X$ such that $\bar{U} \cap X = X \setminus Z$, and observe that the set $\{ \mu \in P_\tau(X) \mid \mu(\bar{U}) > 1 - a \}$ is functionally open in $P_\tau(X)$. Then

$$\chi_Z((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(Z) \geq a \} = \{ \mu \in P_\tau(X) \mid \mu(\beta X \setminus \bar{U}) \geq a \} = \{ \mu \in P_\tau(X) \mid \mu(\bar{U}) \leq 1 - a \}$$

being the complement to a functionally open set $\{ \mu \in P_\tau(X) \mid \mu(\bar{U}) > 1 - a \}$, is functionally closed in $P_\tau(X)$. 

Let us now show that for any closed set $F \subset X$, the set $\chi_F^{-1}([a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(F) \geq a \}$ is closed in $P_\tau(X)$. Let $F$ be the closure of the set $F$ in $\beta X$. Then (see [4]) the set $\{ \mu \in P_\tau(\beta X) \mid \mu(F) \geq a \}$ is closed in $P(\beta X)$. By Lemma 1.1, $\mu^*(F) = \mu(\bar{F})$ for each measure $\mu \in P_\tau(X)$. Therefore, the set

$$P_\tau(X) \cap \{ \mu \in P(\beta X) \mid \mu(\bar{F}) \geq a \} = \{ \mu \in P_\tau(X) \mid \mu^*(F) \geq a \} = \chi_F^{-1}([a, \infty))$$

is closed in $P_\tau(X)$. Switching to complements, we prove that for any open set $G$ in $X$ the set $\hat{\chi}_G^{-1}((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(G) > a \}$ is open in $P_\tau(X)$.

Thus, for $\xi = \alpha = 0$ the lemma is proved.

Now let $\xi$ be an ordinal. Prove for us any $a \in \mathbb{R}$ and $A \in \mathcal{M}_\xi(X)$, where $\xi' < \xi$, it has been proved that $\chi_A^{-1}([a, \infty)) \in \mathcal{M}_\xi(P_\tau(X))$. Let $A \in \mathcal{M}_\xi(X)$. Then $A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m}^\alpha$, where $A_{n,m}^\alpha \in \bigcup_{\xi' < \xi} \mathcal{M}_\xi(C)$. Without loss of generality, suppose that for every $n \in \mathbb{N}$, $A_1^\alpha \subset A_2^\alpha \subset \ldots$ and $\bigcup_{n=1}^{\infty} A_1^\alpha \supset \bigcup_{n=1}^{\infty} A_2^\alpha \supset \ldots$. One can easily observe that $\hat{\chi}_A^{-1}((a, \infty)) = \{ \mu \in P_\tau(X) \mid \mu^*(A) \geq a \} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{ \mu \in P_\tau(X) \mid \mu^*(A_{n,m}^\alpha) \geq a - \frac{1}{n} \}$. By the induction hypothesis for every $n, m \in \mathbb{N}$ $\{ \mu \in P_\tau(X) \mid \mu^*(A_{n,m}^\alpha) \geq a - \frac{1}{n} \} \in \mathcal{M}_\xi(P_\tau(X))$, where $\xi' < \xi$. Therefore, $\hat{\chi}_A^{-1}((a, \infty)) \in \mathcal{M}_\xi(P_\tau(X))$. In a similar fashion for any ordinal $\alpha$ we can prove that $\mu \in \mathcal{F}_\alpha(P_\tau(X))$ implies $\hat{\chi}_A^{-1}((a, \infty)) \in \mathcal{F}_\alpha(P_\tau(X))$.

The lemma is proved.

**Corollary 1.20.** The functor $P_\tau$ preserves Čech-complete spaces.

**Proof.** Let $X$ be a Čech-complete Tychonoff space. Then $X = \bigcap_{n=1}^{\infty} U_n$ is a $G_\delta$-set in $\beta X$ (here $U_n \subset \beta X$, $n \in \mathbb{N}$, are open sets in $\beta X$). Then

$$P_\tau(X) = \{ \mu \in P(\beta X) \mid \mu(U_n) = 1 \} = \bigcup_{n=1}^{\infty} \{ \mu \in P(\beta X) \mid \mu(U_n) > 1 - \frac{1}{n} \}.$$ 

By Lemma 1.18 the sets $\{ \mu \in P(\beta X) \mid \mu(U_n) > 1 - \frac{1}{n} \}$ are open in $P(\beta X)$, which means that $P_\tau(X)$ is a $G_\delta$-set in $P(\beta X)$. By [6, 3.9.1], the space $P_\tau(X)$ is Čech-complete.

**Corollary 1.21.** If $A$ is a Baire subset of a Tychonoff space $X$, then the function $\hat{\chi}_A : P_\tau(X) \to [0, 1]$ is measurable with respect to the $\sigma$-algebra of Baire subsets of $P_\tau(X)$.

**Theorem 1.22.** The functor $P_\tau$ preserves Baire subsets. Moreover, for any ordinal number $\xi$, if $A \in \mathcal{M}_\xi(X)$, then $P_\tau(A) \in \mathcal{M}_\xi(P_\tau(X))$; for any even ordinal $\alpha$, if $A \in \mathcal{F}_\alpha(X)$, then $P_\tau(A) \in \mathcal{F}_\alpha(P_\tau(X))$.

Let $X$ be a metrizable compact space. By $P(X)$ we denote the family of projective subsets of $X$, i.e. the smallest family containing the class $\mathcal{B}(X)$ of all Borel subsets of $X$ and satisfying the following conditions:

1. For any continuous map $f : A \to X$ of the set $A \in P(X)$ the image $f(A)$ belongs to the class $\mathcal{P}(X)$;
2. For any set $A \in P(X)$ its complement $X \setminus A$ belongs to the set $\mathcal{P}(X)$.

The family $P(X)$ can be presented as $P(X) = \bigcup_{n=0}^{\infty} P_n(X)$, where $P_0(X) = \mathcal{B}(X)$ is the family of Borel sets of $X$; projective sets of the class $P_{2n+1}(X)$ are continuous images of sets from the class $P_{2n}(X)$; projective sets of the class $P_{2n}(X)$ are complements to projective sets of the class $P_{2n+1}(X)$. Given this, for any $n \geq 0$ $P_{2n+1}(X) \subseteq P_{2n+3}(X) \cap P_{2n+4}(X)$ [12, §38]. Projective sets from classes $P_1(X)$ and $P_2(X)$ have specific names: they are called, respectively, analytic and coanalytic.

**Theorem 1.23.** The functor $P_\tau$ preserves projective subsets of metrizable compacta. Moreover, for any metrizable compact space $X$ and any $n \geq 1$, if $A \in P_{2n-1}(X)$, then $P_\tau(A) \in P_{2n+2}(P(X))$. 

Proof. Let $X$ be a metrizable compact space. By $\exp(X)$ we denote the hyperspace of non-empty closed subsets of $X$ endowed with the Vietoris topology [9]. Let us note that for any $0 < a \leq 1$ the set $R(a) = \{(\mu, K) \in P(X) \times \exp(X) \mid \mu(K) \geq a\}$ is closed in $P(X) \times \exp(X)$. Indeed, let $(\mu, K) \in P(X) \times \exp(X)$ be a limit point of the set $R(a)$. By the definition, $\mu(K) = \inf\{\mu(f) \mid f \in C(X), \ f \geq 0, \ f|K \equiv 1\}$. Let $f : X \to [0,1]$ be any function with $f|K \equiv 1$. Then for any $\varepsilon > 0$ the set
\[
\langle f^{-1}(1 - \varepsilon/2, 1) \rangle = \{B \in \exp(X) \mid B \subset f^{-1}(1 - \varepsilon/2, 1)\}
\]
is open in $\exp(X)$. Since $(\mu, K)$ is a limit point of the set $R(a)$, there exists a pair $(\eta, C) \in R(a)$ such that $|\eta(f) - \mu(f)| < \varepsilon$ and $C \subset f^{-1}(1 - \varepsilon/2, 1)$. Then $\mu(f) > \eta(f) - \frac{\varepsilon}{2} = \int_X f \, d\eta - \frac{\varepsilon}{2} \geq \int_C f \, d\eta - \frac{\varepsilon}{2} > (1 - \frac{\varepsilon}{2})\eta(C) - \frac{\varepsilon}{2} \geq (1 - \frac{\varepsilon}{2})a - \frac{\varepsilon}{2} \geq a - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\mu(f) \geq a$, which implies that $\mu(K) \geq a$ and $(\mu, K) \in R(a)$.

Now let $n \geq 1$ and $A \in P_{2n-1}(X)$. Then
\[
P(X) \setminus P_n(A) = \{\mu \in P(X) \mid \mu(A) < 1\} = \{\mu \in P(X) \mid \mu(X) \setminus A > 0\} = \bigcup_{m=1}^{\infty} \{\mu \in P(X) \mid \mu(K) \geq \frac{1}{m}\} \text{ for some compact } K \subset X \setminus A = \bigcup_{m=1}^{\infty} \text{pr}_1(G_m),
\]
where
\[
G_m = \{(\mu, K) \in P(X) \times \exp(X) \mid K \subset X \setminus A, \ \mu(K) \geq \frac{1}{m}\} = R(\frac{1}{m}) \cap (P(X) \times \exp(X) \setminus A),
\]
and $\text{pr}_1 : P(X) \times \exp(X) \to P(X)$ is the projection onto the first factor. Since $A \in P_{2n-1}(X)$, then $X \setminus A \in P_{2n}(X)$, and, according to [13], $\exp(X \setminus A) \in P_{2n}(\exp(X))$. Consequently, $P(X) \setminus P_n(A) \in P_{2n+1}(P(X))$ and $P_n(A) \in P_{2n+2}(P(X))$. The theorem is proved.

Remark 1.24. Theorem 1.22 implies that $P_n(A) \in P_0(P(X))$ for any $A \in P_0(X)$. Moreover, Theorem 2.32 implies that for any coanalytic subset $A \subset X$, $P_n(A) \in P_0(P(X))$. This follows from the fact that coanalytic subsets of metric compacta are measurable with respect to any measure and, therefore, for a coanalytic set $A \subset X$ the equality $P_n(A) = \hat{P}(A)$ holds.

Let us now recall the notion of monad, introduced by S. Eilenberg and T. Moore [14].

Definition 1.25. A monad on a category $C$ is a triple $T = (T, \delta, \psi)$ consisting of a covariant functor $T : C \to C$ and natural transformations $\delta : \text{Id} \to T$ (identity) and $\psi : T^2 \to T$ (multiplication) which satisfy the following conditions:
\[
\psi \circ T\delta = \psi \circ \delta T = \text{id}_T \quad \text{and} \quad \psi \circ \psi T = \psi \circ T\psi.
\]
A functor $T$ that can be included in a triple $T$ is called monadic in the category $C$.

It is well-known [2] that the functor $P$ is monadic in the category $\text{Comp}$ of compacta. In fact, it can be included in the monad $\mathbb{P} = (P, \delta, \psi)$, where $\delta$ is the Dirac transformation, and the component $\psi_X : P^2(X) \to P(X)$ of multiplication $\psi$ is determined by the formula $\psi_X(M)(f) = M(F_f)$ for $f \in C(X)$, $M \in P^2(X)$, where $F_f : P(X) \to \mathbb{R}$ is a continuous function such that $F_f(\mu) = \mu(f)$, $\mu \in P(X)$.

Our aim is to show that the functor $P_n : \text{Ych} \to \text{Ych}$ can also be included in a monad. For this purpose, apparently, it is sufficient to show that for any Tychonoff space $X$ $\delta_{\beta X}(X) \subset P_n(X)$ and $\psi_{\beta X}(P^2_n(X)) \subset P_n(X)$, where $\delta_{\beta X}$ and $\psi_{\beta X}$ are components of natural transformations included in the triple $\mathbb{P} = (P, \delta, \psi)$ (here and further in the text the symbol $P^2_n$ denotes the composition of functors $P_n \circ P_n$). The first inclusion $\delta_{\beta X}(X) \subset P_n(X)$ follows from Theorem 1.14. In order to prove the second inclusion, let us assume that $M \in P^2_n(X) \subset P^2(\beta X)$, i.e. $M^*(P_n(X)) = 1$. Let $\{\varphi_\alpha \subset C(\beta X)\}$ be a monotonically decreasing net of continuous functions on $\beta X$ converging to zero on the set $X \subset \beta X$ pointwise. According to [4], $\psi_{\beta X}(M) \in P_n(X)$ provided we show that the real-valued net $\{\psi_{\beta X}(M)(\varphi_\alpha)\}$ converges to zero. Observe that $\{F_{\beta n} : P(\beta X) \to \mathbb{R}\}$ is a monotonically decreasing net of continuous functions on $P(\beta X)$, converging to zero on the set $P_n(X) \subset P(\beta X)$ pointwise. Since $M$ is a $\tau$-smooth measure on $P_n(X)$, by [4] we have that, $\{M(F_{\beta n})\} \to 0$. But $\psi_{\beta X}(M)(\varphi_\alpha) = M(F_{\beta n})$ for every $\alpha$. Therefore, the net $\{\psi_{\beta X}(M)(\varphi_\alpha)\}$ converges to zero, i.e. $\psi_{\beta X}(P^2_n(X)) \subset P_n(X)$. Let us choose $\delta_X = \delta_{\beta X}|X : X \to P_n(X)$ and $\psi_X = \psi_{\beta X}|P^2_n(X) : P^2_n(X) \to P_n(X)$. One can easily see that $\mathbb{P} = (P, \delta, \psi)$ is a monad on the category $\text{Ych}$. Thus, we have proved

Theorem 1.26. The functor $P_n : \text{Ych} \to \text{Ych}$ is a monad on the category $\text{Ych}$ extending the monad $P : \text{Comp} \to \text{Comp}$. 
Lemma 1.27. For any Tychonoff space X and its subset Y the equality \( \psi_X^{-1}(P_r(Y)) = P_r^2(Y) \) holds.

**Proof.** The inclusion \( \psi_X(P_r^2(Y)) \subset P_r(Y) \) follows from the fact that the functor \( P_r \) is a monad. Let us now prove the inverse inclusion. Let \( M \in P^2(\beta X) \) be a measure such that \( \psi_{\beta X}(M) \notin P_r(Y) \). Then there exists a compact \( K \subset \beta Y \setminus Y \) such that \( \psi_{\beta X}(M)(K) > 0 \). Consider a family of functions \( \Phi = \{ f \in C(\beta X) \mid 0 \leq f \leq 1, f|K \equiv 1 \} \), equipped with a natural partial order \( \preceq \). The family \( \Phi \) is downward-directed, i.e. for any functions \( f, g \in \Phi \) \( \min(f, g) \in \Phi \), and, if treated as a net, it converges to zero on the set \( Y \) pointwise. Then the net \( \{ F_f : P(\beta X) \to \mathbb{R} \}_{f \in \Phi} \) monotonically decreases and converges to zero on the set \( P_r(Y) \). If the measure \( M \) would belong to the set \( P_r^2(Y) \), the net \( \{ M(F_f) \}_{f \in \Phi} \) would converge to zero. But \( M(F_f) = \psi_{\beta X}(M)(f) \geq \psi_{\beta X}(M)(K) > 0 \). This contradiction shows that \( M \notin P_r^2(Y) \). Thus \( \psi_X^{-1}(P_r(Y)) = P_r^2(Y) \).

**Corollary 1.28.** For every Tychonoff space \( X \) the component \( \psi_X : P_r^2(X) \to P_r(X) \) of multiplication is an open and perfect map.

The proof follows from Lemma 1.27 and from the fact that \( \psi_{\beta X} : P^2(\beta X) \to P(\beta X) \) is an open mapping of compacta [2, 7.8], or [15].

Let us recall that a map \( p : X \to Y \) between topological spaces is called soft for the class of metric spaces if for any metric space \( A \), its closed subset \( B \subset A \) and maps \( g : A \to Y \) and \( f : B \to X \) such that \( p \circ g = g|B \) there exists a map \( F : A \to X \) such that \( F|B = f \) and \( p \circ F = g \).

**Theorem 1.29.** For every metric space \( X \) the map \( \psi_X : P_r^2(X) \to P_r(X) \) is soft for the class of metric spaces.

The proof follows from Corollary 1.28, metrizability of the space \( P_r^2(X) \) (see Theorem 1.17), Michael’s selection theorems [16, §1.4 and Ex. 1.4.2] and the fact that the preimage \( \psi_X^{-1}(\mu) \subset P_r^2(X) \) of any measure \( \mu \in P_r(X) \) is a convex compact in \( P_r^2(X) \).

We will say that the map \( p : E \to B \) is homeomorphic to a trivial \( Q \)-fibration if there exists a homeomorphism \( f : E \to B \times Q \) such that \( pr_B \circ f = p \), where \( pr_B : B \times Q \to B \) is a natural projection. Here \( Q = [-1, 1]^\omega \) stands for the Hilbert cube.

**Theorem 1.30.** For a metrizable separable space \( X \) that contains more than one point, the map \( \psi_X|P_r^2(X) \setminus \delta(X) : P_r^2(X) \setminus \delta(X) \to P_r(X) \setminus \delta(X) \) is homeomorphic to a trivial \( Q \)-fibration.

**Proof.** Let \( cX \) be a metric compactification of the space \( X \neq \{ * \} \). In [15] it is proved that the map \( \psi_X|P_r^2(cX) \setminus \delta(cX) : P_r^2(cX) \setminus \delta(cX) \to P(cX) \setminus \delta(cX) \) is a trivial \( Q \)-fibration. Now the theorem follows from Lemma 1.27.

## 2 CATEGORICAL PROPERTIES OF THE FUNCTOR \( \hat{P} \)

In this section we shall investigate categorical properties of the functor \( \hat{P} : \mathcal{T}_{ych} \to \mathcal{T}_{ych} \) of Radon probability measures.

As it was mentioned in the introduction, there exist two equivalent approaches to defining the space \( \hat{P}(X) \), where \( X \) is a Tychonoff space. The first is via embedding in compact spaces: \( \hat{P}(X) = \{ \mu \in P(\beta X) \mid \mu_*(X) = 1 \} \subset P(\beta X) \). In the second approach, \( \hat{P}(X) \) is defined as the space of all Radon probability measures on \( X \). Further in the text, depending on the situation, without any specific caveats we will use either the first or the second approach to the description of the space \( \hat{P}(X) \).

**Theorem 2.1.** The functor \( \hat{P} \) preserves the class of injective maps.

**Proof.** Let \( f : X \to Y \) be an injective map and \( \mu_1, \mu_2 \in \hat{P}(X) \), \( \mu_1 \neq \mu_2 \). Then \( \mu_1(A) \neq \mu_2(A) \) for some Borel set \( A \subset X \). Since the measures \( \mu_1, \mu_2 \) are Radon, there exists a compact \( K \subset A \) such that \( \mu_1(K) \neq \mu_2(K) \). Then \( f(K) \) is a compact in \( Y \) and

\[ \hat{P}(f)(\mu_1)(f(K)) = \mu_1(f^{-1}(f(K))) = \mu_1(K) \neq \mu_2(K) = \hat{P}(f)(\mu_2)(f(K)), \]

i.e. \( \hat{P}(f)(\mu_1) \neq \hat{P}(f)(\mu_2) \). The theorem is proved.

**Theorem 2.2.** The functor \( \hat{P} \) preserves the class of perfect maps.
Proof. Let \( f : X \to Y \) be a perfect map of Tychonoff spaces. Then the extension \( \beta f : \beta X \to \beta Y \) of the map \( f \) has the following property: \( \beta f(\beta X \setminus Y) \subset \beta Y \setminus Y \) [6, 3.7.15]. We will show that \( P(\beta f)(P(\beta X) \setminus \hat{P}(X)) \subset P(\beta Y) \setminus \hat{P}(Y) \). Let \( \mu \in P(\beta X) \) and \( P(\beta f)(\mu) \in \hat{P}(Y) \). Then for any \( \varepsilon > 0 \) there exists a compact \( K \subset Y \) such that \( P(\beta f)(\mu)(K) > 1 - \varepsilon \). Since \( f \) is a proper map, \( (\beta f)^{-1}(K) = f^{-1}(K) \) is a compact in \( X \) (see [6, 3.7.2]). Then \( P(\beta f)(\mu)(K) = P(\beta f)(f^{-1}(K)) > 1 - \varepsilon \). Therefore, \( \mu \in \hat{P}(X) \) and \( P(\beta f)(P(\beta X) \setminus \hat{P}(X)) \subset P(\beta Y) \setminus \hat{P}(Y) \). Since \( P(\beta f) : P(\beta X) \to P(\beta Y) \) is a map between compacta, the last inclusion implies that the map \( \hat{P}(f) = P(\beta f)P(Y) : \hat{P}(X) \to \hat{P}(Y) \) is perfect. The theorem is proved. \( \square \)

Corollary 2.3. The functor \( \hat{P} \) preserves the class of closed embeddings.

Since \( \hat{P} \) is a subfunctor of the functor \( P_\tau \), Theorem 1.4 implies

Theorem 2.4. The functor \( \hat{P} \) preserves the class of topological embeddings.

Thus, the functor \( \hat{P} \) preserves the class of injective maps and the class of (closed) embeddings. But in the case of surjective maps it is not so straightforward.

We say that a map \( f : X \to Y \) has the property of Borel selection if there exists a map \( s : Y \to X \) (not necessarily continuous) such that \( f \circ s = \text{id}_Y \), and for any open set \( U \subset X \) \( s^{-1}(U) \) is a Borel subset of the space \( Y \).

The map \( f : X \to Y \) has local Borel selections, if for every open set \( U \subset X \) there exists a Borel selection \( s : Y \to X \) of the map \( f \) such that \( s(f(U)) \subset U \).

Example 2.5. Let \( p : \varepsilon \to [0,1] \) be a bijective continuous mapping of a discrete space \( \varepsilon \) onto an interval. Then the map \( \hat{P}(p) : \hat{P}(\varepsilon) \to \hat{P}([0,1]) \) is not surjective (the Lebesque measure on \([0,1] \) does not have a preimage).

At the same time, the following is also true:

Proposition 2.6. Let \( f : X \to Y \) be a map between separable metric spaces that has a Borel selection. Then the map \( \hat{P}(f) : \hat{P}(X) \to \hat{P}(Y) \) is surjective.

Proof. Let \( s : Y \to X \) be a Borel selection of the map \( f \). For every measure \( \mu \in \hat{P}(Y) \) let us choose \( \eta \), a countably additive probability measure on \( X \) defined by the condition \( \eta(A) = \mu(f(A \cap s(Y))) \) for every Borel set \( A \subset X \).

Let us show that \( \eta \) is a Radon measure. It is sufficient to show that for any \( \varepsilon > 0 \) there exists a compact \( K \subset X \) such that \( \eta(K) > 1 - \varepsilon \). Let us fix \( \varepsilon > 0 \). Since the map \( s : Y \to X \) is Borel measurable, the Lusin theorem [7, 2.3.5] implies that there exists a closed subset \( C \subset Y \) such that \( \mu(C) > 1 - \varepsilon/2 \) and the map \( s|C : C \to X \) is continuous. Since the measure \( \mu \) on \( Y \) is Radon, there exists such a compact \( K \subset C \) that \( \mu(C \setminus K) < \frac{\varepsilon}{2} \). Then \( s(K) \subset C \) is compact. Furthermore, \( \eta(s(K)) = \mu(f(s(K) \cap s(Y)) \mu(f(s(K))) = \mu(K) > 1 - \varepsilon \). Thus the measure \( \eta \) on \( X \) is Radon and \( \hat{P}(f)(\eta) = \mu \). The theorem is proved. \( \square \)

Corollary 2.7. Let \( f : X \to Y \) be a bijective continuous mapping of a separable Borel space \( X \) onto a metric space \( Y \). Then the map \( \hat{P}(f) : \hat{P}(X) \to \hat{P}(Y) \) is bijective.

Proof. The injectivity of the map \( \hat{P}(f) \) follows from Theorem 2.1. The surjectivity of \( \hat{P}(f) \) follows from Proposition 2.6, since the map \( f^{-1} : Y \to X \) is Borel measurable [12, §39, IV]. \( \square \)

The condition that a Borel selection should exist is crucial here. Indeed, let us consider

Example 2.8. Let \( Z \subset [0,1] \) be a subset of the interval with inner Lebesque measure \( \lambda_*(Z) \neq 0 \) and outer measure \( \lambda^*(Z) = 1 \). Let \( X = Z \times \{0\} \cup ([0,1] \setminus Z) \times \{1\} \) and \( f : X \to [0,1] \) be a projection onto the first factor. Then the Lebesque measure \( \lambda \) on \([0,1] \) does not have a preimage under the map \( \hat{P}(f) : \hat{P}(X) \to \hat{P}([0,1]) \).

At the same time, the functor \( \hat{P} \) preserves a feature of maps which implies surjectivity in the compact case.

Proposition 2.9. Let \( f : X \to Y \) be a map such that the image \( f(X) \) is dense in \( Y \). Then the image \( \hat{P}(f)(\hat{P}(X)) \) is dense in \( \hat{P}(Y) \).

The proof is similar to the proof of Proposition 1.6.

Proposition 2.10. Let \( f : X \to Y \) be an open map between separable metric spaces that has local Borel selections. Then the map \( \hat{P}(f) : \hat{P}(X) \to \hat{P}(Y) \) is surjective and open.
Proof. By Proposition 2.6, the map \( \hat{P}(f) \) is surjective. Let us now show that it is open. According to [4, II, §1], the system of sets \( \mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon) = \{ \{ \tilde{v} \in \tilde{X} : |\mu_1(\tilde{v}) - \mu_0(\tilde{v})| > \varepsilon \} \} \), \( \varepsilon > 0 \), \( \mu_0 \in \hat{P}(X) \) and \( U_1, \ldots, U_n \) are open sets in \( X \), forms a base for the topology on \( \hat{P}(X) \). Let us fix a base set \( \mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon) \) and show that its image \( \hat{P}(f)(\mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon)) \) is a neighborhood of the measure \( \eta_0 = \hat{P}(f)(\mu_0) \in \hat{P}(Y) \). To achieve this, we will first find a base neighborhood \( \mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon') \subset \mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon) \) such that \( V_i \), \( 1 \leq i \leq m \), are pairwise disjoint open subsets of \( X \).

By \( n \) we will denote the \( n \)-element set \( \{ 1, \ldots, n \} \). We will equip the set \( \exp(n) \) of all non-empty subsets of \( n \) with a linear order such that for any \( A, B \subset n \), if \( A \supset B \), then \( A \leq B \) (see [17, §2.4, Theorem 4]). Let us note that \( |\exp(n)| < 2^n \). Fix \( \varepsilon' = \varepsilon'_{/2^{m+1}} \). For every \( A \subset n \) we choose \( U_A = \bigcap_{i \in A} U_i \). By induction, for every \( A \subset n \) find an open set \( V_A \subset X \) such that \( \hat{V}_A \subset U_A \setminus \bigcup_{B \subset A} \hat{V}_B \) and \( \mu_0(V_A) > \mu_0(U_A) - \sum_{B \subset A} \mu_0(V_B) + 2^{n+1}\varepsilon' \). One can easily see that for any \( A \subset n \), \( \mu_0(V_A) > \mu_0(U_A) - \sum_{B \subset A} \mu_0(V_B) + 2^{n+1}\varepsilon' \). Also, it is obvious that \( \hat{V}_A \cap \hat{V}_B = \emptyset \) for any \( A \neq B \).

We claim that \( \mathcal{N}^*(\mu_0, \{ V_A : A \subset n \}, \varepsilon') \subset \mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon) \). Indeed, if \( \mu \in \mathcal{N}^*(\mu_0, \{ V_A : A \subset n \}, \varepsilon') \), then \( \mu(U_i) = \sum_{A \subset i} \mu(V_A) + \mu(U_i \setminus \bigcup_{A \subset i} \hat{V}_A) \geq \sum_{A \subset i} \mu(V_A) > \mu_0(V_A) - \varepsilon' > \sum_{A \subset i} \mu_0(V_A) - 2^{n+1}\varepsilon' > \mu_0(U_i) - 2^{n+1}\varepsilon' = \mu_0(U_i) - \varepsilon, \, 1 \leq i \leq n \). That is, \( \mu \in \mathcal{N}^*(\mu_0, U_1, \ldots, U_n, \varepsilon) \).

Let us present the set \( \mathcal{N}^*(\mu_0, \{ V_A : A \subset n \}, \varepsilon') \) as \( \mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon') \), where \( m = |\exp(n)| \). By \( m \) we will denote the \( m \)-element set \( \{ 1, \ldots, m \} \). Let us equip the set \( \exp(m) \) with a linear order such that for any \( A, B \subset m \), if \( A \supset B \), then \( A \leq B \). For every \( A \subset m \) fix \( W_A = \bigcap_{\varepsilon < A} f(V_\varepsilon) \). Since \( f \) is an open map, the sets \( W_A \subset Y \) are open. Fix \( \delta = \delta_{/2^{m+1}} \). By induction, for every \( A \subset m \) find an open set \( W_A \subset Y \) such that \( W_A \subset W_A \setminus \bigcup_{\varepsilon < A} \hat{W}_{\varepsilon} \) and \( \eta_0(W_A) > \eta_0(W_A \setminus \bigcup_{\varepsilon < A} \hat{W}_{\varepsilon}) - \delta \). One can easily observe that for any distinct \( A, B \subset m \) the sets \( \hat{W}_A, \hat{W}_B \) are disjoint. Furthermore, for any \( A \subset m \), \( \eta_0(W_A \setminus \bigcup_{\varepsilon < A} \hat{W}_{\varepsilon}) - \delta < \eta_0(W_A) \). We claim that \( \mathcal{N}^*(\eta_0, \{ W_A : A \subset m \}, \delta) \subset \hat{P}(f)(\mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon')) \).

Indeed, let \( \eta \in \mathcal{N}^*(\eta_0, \{ W_A : A \subset m \}, \delta) \). For every \( A \subset m \) and every \( i \in A \), fix a Borel selection \( s_{A,i} : Y \to X \) of the map \( f \) such that \( s_{A,i}(W_A) \subset V_i \). Let \( \alpha^A_i, \, i \in A \) be non-negative numbers such that for any \( C \subset m \) we get \( \sum_{i \in A} \alpha^A_i = 1 \) and \( \alpha^A_i \eta_0(W_{\varepsilon}) \geq \mu_0(f^{-1}(W_{\varepsilon} \cap V_i)) \). Fix an arbitrary Borel selection \( s_0 : Y \to X \) of the map \( f \). Let \( \mu \) be a measure on \( X \) such that for any Borel set \( C \subset X \)

\[
\mu(C) = \eta(f(s_0(Y \setminus \bigcup_{A \subset m} W_A)) \cap C) + \sum_{A \subset m} \sum_{i \in A} \alpha^A_i \eta(f(s_{A,i}(W_A)) \cap C).
\]

Similarly to the proof of Proposition 2.6, it can be shown that \( \mu \) is a Radon probability on \( X \), i.e. \( \mu \in \hat{P}(X) \), and \( \hat{P}(f)(\mu) = \eta \). We will show that \( \mu \in \mathcal{N}^*(\mu_0, V_1, \ldots, V_m, \varepsilon') \). Indeed, \( \mu(V_i) \geq \sum_{A \subset i} \alpha^A_i \eta_0(W_A) - 2^{n+1}\delta = \sum_{A \subset i} \alpha^A_i \eta_0(W_A) - 2^{n+1}\delta \). That remains to be done is to assess the value \( \eta_0(W_A) \). By the definition of sets \( W_A \subset \bigcup_{\varepsilon < A} W_A = W^*_i \). Then \( \eta_0(W_A) \geq \mu_0(W_A) - \delta \). Finally, we get that \( \mu(V_i) > \mu_0(V_i) - 2^{n+1}\delta = \mu_0(V_i) - \delta' \).

Proposition 2.11. Let \( f : X \to Y \) be a map between Tychonoff spaces. If \( \hat{P}(f) : \hat{P}(X) \to \hat{P}(Y) \) is an open map, then the map \( f \) is also open.

The proof literally repeats the proof of Proposition 4.1 [18].

Remark 2.12. Under the assumption of the continuum hypothesis (\( \aleph_1 = \mathfrak{c} \)), the condition of separability in Propositions 2.6, 2.10 can be omitted. This follows from [12, §31, X.8] and the fact that the support of any Radon measure is separable.

Question 2.13. Let \( f : X \to Y \) be an open surjective map between separable Borel spaces. Will the map \( \hat{P}(f) \) be open?

In [8, §3] this question will be answered in the affirmative in the case when \( X \) is metrizable by a complete metric.

Let \( A \) be a subset of a Tychonoff space \( X \). Since the functor \( \hat{P} \) preserves embeddings, we will threat the space \( \hat{P}(A) \) as a subset of the space \( \hat{P}(X) \).

Theorem 2.14. The functor \( \hat{P} \) preserves preimages, i.e. for any map \( f : X \to Y \) between Tychonoff spaces and any subset \( A \subset Y \) we get \( \hat{P}(f^{-1}(\hat{P}(A))) = \hat{P}(f^{-1}(A)) \).

This question was answered affirmatively in [19].
Proof. The inclusion $\hat{P}(f^{-1}(A)) \subset \hat{P}(f^{-1}(\hat{P}(A)))$ is simple. We will show that $\hat{P}(f^{-1}(\hat{P}(A))) \subset \hat{P}(f^{-1}(A))$. Let $\mu \in \hat{P}(X)$ be a measure satisfying the condition $\hat{P}(f)(\mu) \in \hat{P}(A)$. Let us fix $\varepsilon > 0$. Since $\hat{P}(f)(\mu) \in \hat{P}(A)$, there exists a compact $K \subset A$ such that $\hat{P}(f)(\mu)(K) > 1 - \frac{\varepsilon}{2}$. The set $f^{-1}(K) \subset X$ is closed, with $\mu(f^{-1}(K)) = \hat{P}(f)(\mu)(K) > 1 - \frac{\varepsilon}{2}$. Since the measure $\mu$ is Radon, there exists a compact $C \subset f^{-1}(K)$ such that $\mu(f^{-1}(K) \setminus C) < \frac{\varepsilon}{2}$. Therefore, $C \subset f^{-1}(A)$ and $\mu(C) > 1 - \varepsilon$, i.e. $\mu \in \hat{P}(f^{-1}(A))$. The theorem is proved.

**Theorem 2.15.** The functor $\hat{P}$ preserves countable intersections, i.e. for any Tychonoff space $X$ and its subsets $X_n \subset X$, $n \in \mathbb{N}$, $\hat{P}(\bigcap_{n \in \mathbb{N}} X_n) = \bigcap_{n \in \mathbb{N}} \hat{P}(X_n)$.

Proof. The inclusion $\hat{P}(\bigcap_{n \in \mathbb{N}} X_n) \subset \bigcap_{n \in \mathbb{N}} \hat{P}(X_n)$ is obvious. Now let $\mu \in \bigcap_{n \in \mathbb{N}} \hat{P}(X_n)$. We will show that $\mu \in \hat{P}(\bigcap_{n \in \mathbb{N}} X_n)$. Fix $\varepsilon > 0$. Since $\mu \in \hat{P}(X_n)$, $n \in \mathbb{N}$, for every $n \in \mathbb{N}$ there exists a compact $K_n \subset X_n$ such that $\mu(K_n) > 1 - \varepsilon/2^n$. Let $K = \bigcap_{n \in \mathbb{N}} K_n$. One can easily check that $K \subset \bigcap_{n \in \mathbb{N}} X_n \mu(K) > 1 - \varepsilon$, i.e. $\mu \in \hat{P}(\bigcap_{n \in \mathbb{N}} X_n)$. The theorem is proved.

**Remark 2.16.** Theorem 2.15 does not hold for an arbitrary number of indices. Indeed, let $X = [0, 1)$ and $X_\alpha = [0, 1] \setminus \{\alpha\}$, where $\alpha \in [0, 1]$. Then for every $\alpha \in [0, 1]$ the Lebesque measure $\lambda$ belongs to the set $\hat{P}(X_\alpha)$. But $\bigcap_{\alpha \in [0, 1]} X_\alpha = \varnothing$. That is $\hat{P}(\bigcap_{\alpha \in [0, 1]} X_\alpha) \neq \bigcap_{\alpha \in [0, 1]} \hat{P}(X_\alpha)$.

**Lemma 2.17.** Let $X$ be a Tychonoff space and $B \subset X$ be a Borel subset of $X$. Then $\hat{P}(B) = P(\hat{B}) \cap \hat{P}(X) \subset P(\hat{X} \setminus B)$.

Proof. The inclusion $\hat{P}(B) \subset P(\hat{B}) \cap \hat{P}(X)$ is obvious. Let $\mu \in P(\hat{B}) \cap \hat{P}(X)$. Then $\mu^*(B) = 1$ and $\mu_*(X) = 1$. Choose a Borel subset $B' \subset \beta X$ such that $B \cap X = B'$. Then $\mu(B') \geq \mu^*(B) = 1$. Since the measure $\mu$ is regular, for any $\varepsilon > 0$ there exists a compact $K_1 \subset B$ such that $\mu(B \setminus K_1) < \varepsilon/2$. By definition, $\mu_*(X) = 1$ implies that there exists a compact subset $K_2 \subset X \subset \beta X$ such that $\mu(K_2) > 1 - \varepsilon/2$. Then $K = K_1 \cap K_2 \subset B \cap X = B$ is a compact subset of $B$ satisfying $\mu(\beta X \setminus K) \leq \mu(\beta X \setminus K_1) + \mu(\beta X \setminus K_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, which, by the arbitrariness of $\varepsilon > 0$ implies that $\mu_*(B) = 1$ and $\mu \in \hat{P}(B)$. The lemma is proved.

**Theorem 2.18.** The functor $\hat{P}$ preserves intersections of closed subsets, i.e. for any Tychonoff space $X$ and its closed subsets $X_\alpha$, $\alpha \in A$, $\hat{P}(\bigcap_{\alpha \in A} X_\alpha) = \bigcap_{\alpha \in A} \hat{P}(X_\alpha)$.

Proof. Theorem 1.10 implies that $P(\bigcap_{\alpha \in A} X_\alpha) = \bigcap_{\alpha \in A} P(\hat{X}_\alpha)$. Then, by Lemma 2.17, $\hat{P}(\bigcap_{\alpha \in A} X_\alpha) = \hat{P}(\bigcap_{\alpha \in A} X_\alpha) \cap \hat{P}(X) = \bigcap_{\alpha \in A} \hat{P}(X_\alpha) \cap \hat{P}(X) = \bigcap_{\alpha \in A} \hat{P}(X_\alpha)$. The theorem is proved.

Now let us consider the question of continuity of the functor $\hat{P}$. Let $A$ be a directed partially ordered set (which means that for any $\alpha, \beta \in A$ there exists a $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$).

Let $\{X_\alpha, p^\beta_\alpha\}$ be an inverse system indexed by the set $A$ and consisting of Tychonoff spaces. By $\lim X_\alpha$ we denote the limit of that system, and by $p_\alpha : \lim X_\alpha \to X_\alpha$, $\alpha \in A$, the bonding maps.

The inverse system $\{X_\alpha, p^\beta_\alpha\}$ induces the inverse system $\{\hat{P}(X_\alpha), \hat{P}(p^\beta_\alpha)\}$, whose limit is denoted by $\lim \hat{P}(X_\alpha)$, and the limit projections by $p_\alpha : \lim \hat{P}(X_\alpha) \to \hat{P}(X_\alpha)$. The maps $\hat{P}(p_\alpha) : \hat{P}(\lim X_\alpha) \to \hat{P}(X_\alpha)$ induce a map $R : \hat{P}(\lim X_\alpha) \to \hat{P}(X_\alpha)$.

It is well-known that if all $X_\alpha$ are compact, then the map $R$ is a homeomorphism. This follows from the continuity of the functor $\hat{P}$ in the category of compacta [9, VII.3.11].

**Theorem 2.19.** The map $R : \hat{P}(\lim X_\alpha) \to \hat{P}(X_\alpha)$ is an embedding. If the limit projections $p_\alpha : \lim X_\alpha \to X_\alpha$ are dense, then the image $R(\hat{P}(\lim X_\alpha))$ is dense $\hat{P}(X_\alpha)$. If the index set $A$ is countable, then $R$ is a homeomorphism.

Proof. The first two statements are proved in a similar fashion to corresponding statements of Theorem 1.11. Let us assume that the set $A$ is countable and show that the map $R : \hat{P}(\lim X_\alpha) \to \hat{P}(X_\alpha)$ is a homeomorphism. For this purpose it is sufficient to prove the surjectivity of the map $R$. Like in the proof of 1.11, let us embed the map $R$ in the homeomorphism $\hat{R} : \hat{P}(\lim \beta X_\alpha) \to \lim \hat{P}(\beta X_\alpha)$.

Fix a thread $p_{\alpha \in A} \in \lim \hat{P}(\beta X_\alpha)$. Let us show that $\mu = \hat{R}^{-1}(\{\mu_\alpha\}_{\alpha \in A}) \in \hat{P}(\lim X_\alpha) \subset \hat{P}(\lim \beta X_\alpha)$. Choose an $\varepsilon > 0$. Fix a bijection $\xi : A \to \mathbb{N}$. For every $\alpha \in A$ choose a compact $K_\alpha \subset X_\alpha$ satisfying $\mu_\alpha(K_\alpha) > 1 - \varepsilon/2^\xi(\alpha)$. One can easily observe that the set $K = \{x_\alpha(\alpha \in A) \lim X_\alpha | p_{\alpha}(x_\alpha) \in K_\alpha, \alpha \in A\}$ is compact. Furthermore, $\mu(\lim X_\alpha \setminus K) \leq \bigcup_{\alpha \in A} \mu(p^\alpha_{-1}(X_\alpha \setminus K_\alpha)) = \bigcup_{\alpha \in A} \mu_\alpha(X_\alpha \setminus K_\alpha) \leq \sum_{\alpha \in A} \varepsilon/2^\xi(\alpha) = \varepsilon$. Therefore, the map $R$ is surjective and the theorem is proved.
Corollary 1.13 implies

**Proposition 2.20.** The functor \( \hat{P} \) preserves homotopies, i.e. for any homotopy \( H_t : X \to Y \) the homotopy \( \hat{P}(H_t) : \hat{P}(X) \to \hat{P}(Y) \) is continuous as a map \( \hat{P}(H(\cdot)) : \hat{P}(X) \times [0,1] \to \hat{P}(Y) \).

Now we will consider the operation of tensor product of Radon probability measures. It is well-known (see [9, VIII, §1]) that given a family \( \{X_\alpha\}_{\alpha \in A} \) of compacts, for any probability measures \( \mu_\alpha \in \hat{P}(X_\alpha) \), \( \alpha \in A \), there exists a unique measure \( \otimes_{\alpha \in A} \mu_\alpha \in \hat{P}(\prod_{\alpha \in A} X_\alpha) \) (which is called the tensor product of measures \( \mu_\alpha \)) on the product \( \prod_{\alpha \in A} X_\alpha \) satisfying the following condition: for any finite \( B \subset A \) and any Borel sets \( Y_\alpha \subset X_\alpha, \alpha \in A \), where \( Y_\alpha = X_\alpha \), if \( \alpha \notin B \), the following holds: \( \otimes_{\alpha \in A} \mu_\alpha(Y_\alpha) = \prod_{\alpha \in A} \mu_\alpha(Y_\alpha) \).

**Proposition 2.21.** Let \( \{X_\alpha\}_{\alpha \in A} \) be a family of Tychonoff spaces, \( \{cX_\alpha\}_{\alpha \in A} \) be a family of their compactifications and \( \mu_\alpha \in \hat{P}(cX_\alpha) \subset \hat{P}(cX_\alpha) \), \( \alpha \in A \), be a family of Radon probability measures. If the index set \( A \) is at most countable, then \( \otimes_{\alpha \in A} \mu_\alpha \in \hat{P}(\prod_{\alpha \in A} X_\alpha) \subset \hat{P}(\prod_{\alpha \in A} cX_\alpha) \).

**Proof.** Let us show that the measure \( \otimes_{\alpha \in A} \mu_\alpha \) belongs to the set \( \hat{P}(\prod_{\alpha \in A} X_\alpha) \subset \hat{P}(\prod_{\alpha \in A} cX_\alpha) \). For this purpose fix an arbitrary \( \varepsilon > 0 \). As the set \( A \) is at most countable, there exists an injection \( \xi : A \to \mathbb{N} \). Since every measure \( \mu_\alpha \in \hat{P}(cX_\alpha) \) is Radon, for every \( \alpha \in A \) there exists a compact \( K_\alpha \subset X_\alpha \subset cX_\alpha \) such that \( \mu_\alpha(cX_\alpha \setminus K_\alpha) < \varepsilon/2^{-\xi(\alpha)} \). Then for the compact space \( \prod_{\alpha \in A} K_\alpha \subset \prod_{\alpha \in A} X_\alpha \) we get:

\[
(\otimes_{\alpha \in A} \mu_\alpha)(\prod_{\alpha \in A} cX_\alpha \setminus \prod_{\alpha \in A} K_\alpha) \leq \sum_{\alpha \in A} \mu_\alpha(cX_\alpha \setminus K_\alpha) < \sum_{\alpha \in A} \varepsilon/2^{-\xi(\alpha)} \leq \varepsilon.
\]

Consequently, \( \otimes_{\alpha \in A} \mu_\alpha \in \hat{P}(\prod_{\alpha \in A} X_\alpha) \).

**Remark 2.22.** Proposition 2.21 and well-known facts about the tensor product of probability measures on compact spaces implies that for any at most countable set of Radon probability measures \( \mu_\alpha \in \hat{P}(X_\alpha), \alpha \in A \), on Tychonoff spaces \( X_\alpha \), there exists a unique Radon probability measure \( \otimes_{\alpha \in A} \mu_\alpha \in \hat{P}(\prod_{\alpha \in A} X_\alpha) \) (which is called the tensor product of measures \( \mu_\alpha \)) such that for any Borel sets \( Y_\alpha \subset X_\alpha, \alpha \in A \), we have the following equality: \( (\otimes_{\alpha \in A} \mu_\alpha)(\prod_{\alpha \in A} Y_\alpha) = \prod_{\alpha \in A} \mu_\alpha(Y_\alpha) \).

**Remark 2.23.** Proposition 2.21 does not hold if the index set \( A \) is uncountable. Indeed, if every measure \( \mu_\alpha \in X_\alpha, \alpha \in A \), has a non-compact support \( \supp_{cX_\alpha}(\mu_\alpha) \cap X_\alpha \), then it can be shown that the measure \( (\otimes_{\alpha \in A} \mu_\alpha)(K) \) of any compact set \( K \subset \prod_{\alpha \in A} X_\alpha \subset \prod_{\alpha \in A} cX_\alpha \) is zero.

For every Tychonoff space \( X \) let us define a map \( \delta_X : X \to \hat{P}(X) \) assigning to each point \( x \in X \) the Dirac measure \( \delta_X(x) \), concentrated at the point \( x \).

**Theorem 1.14** implies

**Theorem 2.24.** The family \( \delta = \{\delta_X\} \) defines a unique natural transformation of the identity functor \( \text{Id} : \mathcal{T}\text{ych} \to \mathcal{T}\text{ych} \) to the functor \( \hat{P} : \mathcal{T}\text{ych} \to \mathcal{T}\text{ych} \), whose components \( \delta_X : X \to \hat{P}(X) \) are a closed embeddings.

In a similar fashion to Theorem 1.15 we can prove

**Theorem 2.25.** The functor \( \hat{P} \) preserves the density of Tychonoff spaces, i.e. \( d(\hat{P}(X)) = d(X) \) for any infinite Tychonoff space \( X \).

Theorem 1.16—1.22, and also Lemma 2.17 imply

**Theorem 2.26.** The functor \( \hat{P} \) preserves the weight of Tychonoff spaces, i.e. \( w(\hat{P}(X)) = w(X) \) for any infinite Tychonoff space \( X \).

**Theorem 2.27.** The functor \( \hat{P} \) preserves the class of metrizable spaces.

**Proposition 2.28.** The functor \( \hat{P} \) preserves \( \check{C}ech \)-complete spaces.

**Proposition 2.29.** If \( A \) is a Baire subset of a Tychonoff space \( X \), then the function \( \check{\chi}_A : \hat{P}(X) \to [0,1], \) where \( \check{\chi}_A(\mu) = \mu(A), \mu \in \hat{P}(X) \), is measurable with respect to the \( \sigma \)-algebra of Baire subsets of \( \hat{P}(X) \).
Theorem 2.30. The functor $\hat{P}$ preserves Baire subsets. Moreover, for any ordinal number $\xi$, if $A \in M_\xi(P(X))$, then $P(A) \in M_\xi(\hat{P}(X))$; for every even ordinal number $\alpha$, if $A \in F_\alpha(P(X))$, then $\hat{P}(A) \in F_\alpha(\hat{P}(X))$.

Theorems 2.2 and 2.27 imply

Theorem 2.31. The functor $\hat{P}$ preserves the class of p-paracompact spaces.

Theorem 2.32. The functor $\hat{P}$ preserves projective subsets of metrizable compacta. Furthermore, for every $n \geq 0$, if $A \in P_{2n}(X)$, then $\hat{P}(A) \in P_{2n+1}(\hat{P}(X))$.

Proof. Let $X$ be a metric compact. For $n = 0$ the statement of the theorem follows from Theorem 2.30. By $\exp(X)$ we denote the hyperspace of non-empty closed subsets of $X$, equipped with the Vietoris topology.

Now let $n \geq 1$ and $A \in P_{2n}(X)$. In this case $\hat{P}(A) = \{\mu \in P(X) \mid \mu_*(A) = 1\} = \{\mu \in P(X) \mid \text{for any } m \geq 1 \text{ there exists a compact } K \subset A \text{ such that } \mu(K) \geq 1 - \frac{1}{m}\} = \bigcap_{m=1}^{\infty} \text{pr}_1(E_m)$, where $E_m = \{\mu, K \in P(X) \times \exp(X) \mid K \subset A \text{ and } \mu(K) \geq 1 - \frac{1}{m}\}$ and $\text{pr}_1 : P(X) \times \exp(X) \to P(X)$, the projection onto the first factor. Let $\exp(A) = \{C \in \exp(X) \mid C \subset A\}$. In [13] it was proved that $\exp(A) \in P_{2n}(\exp(X))$. Then for any $m \in \mathbb{N}$, $E_m = R(1 - \frac{1}{m}) \cap (P(X) \times \exp(X))$. Since the set $R(1 - \frac{1}{m}) \subset P(X) \times \exp(X)$ is closed (see the proof of Theorem 1.22), $E_m \in P_{2n}(P(X) \times \exp(X))$. Consequently, $\text{pr}_1(E_m) \in P_{2n+1}(P(X))$ and $\hat{P}(A) = \bigcap_{m=1}^{\infty} \text{pr}_1(E_m) \in P_{2n+1}(\hat{P}(X))$. [12, §38, III, Theorem 3].

Remark 2.33. If $A$ is an analytic subset of the a compact space $X$, then $\hat{P}(A) \in P_4(P(X))$. This follows from Theorem 1.23 and the equality $\hat{P}(A) = P_2(A)$, which is the result of the fact that analytic subsets of metric compacta are measurable with respect to any measure.

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