The complex Hamiltonian systems and quasi-periodic solutions in the derivative nonlinear Schrödinger equations

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Abstract

The complex Hamiltonian systems with real-valued Hamiltonians are generalized to deduce quasi-periodic solutions for a hierarchy of derivative nonlinear Schrödinger (DNLS) equations. The DNLS hierarchy is decomposed into a family of complex finite-dimensional Hamiltonian systems by separating the temporal and spatial variables, and the complex Hamiltonian systems are then proved to be integrable in the Liouville sense. Due to the commutability of complex Hamiltonian flows, the relationship between the DNLS equations and the complex Hamiltonian systems is specified via the Bargmann map. The Abel-Jacobi variable is elaborated to straighten out the DNLS flows as linear superpositions on the Jacobi variety of an invariant Riemann surface. Finally, by using the technique of Riemann-Jacobi inversion, some quasi-periodic solutions are obtained for the DNLS equations in view of the Riemann theorem and the trace formulas.

KEYWORDS
complex Hamiltonian systems, derivative nonlinear Schrödinger equations, quasi-periodic solutions
1 | INTRODUCTION

The derivative nonlinear Schrödinger (DNLS) equation takes the form

\[ iu_t + \frac{1}{2}(u_x + iu|u|^2)_x = 0, \quad i^2 = -1, \]

which models the propagation of circularly polarized nonlinear Alfvén waves in plasmas,\(^1\) and the transmission of subpicosecond pulses in single mode optical fibers.\(^2\) The DNLS equation is rich of many explicit solutions, as it appears to be the compatibility condition of Lax pair\(^3\)

\[ \phi_x = U \phi, \quad U = -i \lambda^2 \sigma_1 + \lambda u \sigma_2 - \lambda \bar{u} \sigma_3, \quad \phi = (\phi_1, \phi_2)^T, \]

and

\[ \phi_t = V^{(2)} \phi, \]

\[ V^{(2)} = \left( -i \lambda^4 + \frac{1}{2} \lambda^2 |u|^2 \right) \sigma_1 + \left( \lambda^3 u + \frac{1}{2} \lambda (iu_x - u|u|^2) \right) \sigma_2 + \left( -\lambda^3 \bar{u} + \frac{1}{2} \lambda (i\bar{u}_x + \bar{u}|u|^2) \right) \sigma_3, \]

where \( \lambda \) is the complex spectral parameter independent of variables \( x \) and \( t \), \( \bar{u} \) is the complex conjugate of \( u \), and

\[ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

Followed by the isospectral nature, soliton equations always occur with a hierarchy of high-order candidates. In this paper, we dedicate to deduce quasi-periodic solutions simultaneously of the DNLS hierarchy stemmed from the Kaup-Newell (KN) spectral problem (under the reduction \( \nu = -\bar{u} \)) characterized by the DNLS equation (1).\(^3\)

The quasi-periodic (finite-gap, or \( N \)-phase) solutions are the natural analogues of multi-solitons in the sense that they can be constructed by the theory of finite-gap integration.\(^5\)\(^-\)\(^7\) The significance of quasi-periodic solutions resides in the fact that they are related to the elliptic function solutions and multi-solitons;\(^5\)\(^-\)\(^7\) also, the finite-gap method has been already used to study the rogue waves on the multi-phase solutions together with their magnification factors.\(^8\)\(^,\)\(^9\) More recently, as was shown in the 1-genus case,\(^10\)\(^-\)\(^12\) the implicit constants of integration of high-order stationary soliton (Novikov) equations\(^13\) are established, so that parameters of periodic waves are related to eigenvalues in the Lax spectrum. The isolated rogue waves on the periodic background are then obtained at the branch point of instable spectral-bands.

After the advent of Its-Matveev formula,\(^14\) Its, Kotlyarov, and Matveev first attained the finite-gap solutions for the NLS equation and its modified version, that is, the DNLS equation.\(^7\)\(^,\)\(^15\)\(^,\)\(^16\) Note that the DNLS equation (1) corresponds to a nonself-adjoint operator. Kamchatnov gave the one-phase periodic solution of the DNLS equation in improving the effectiveness for applications.\(^17\) Starting with a distinct version of the KN spectral problem, Geng et al arrived at all the quasi-periodic solutions to the KN hierarchy. Theoretically, the coupled DNLS equations can be reduced to the DNLS equation under the transformation \( \nu = i \bar{u} \) and the scaling \( t \rightarrow -2it \) (see eqs. 1.1 and 1.2 in Ref. 18), however, see their solutions (5.34) replacing \( t \) with \( -2it \), it cannot be confirmed that the second expression is the conjugate of the first formula multiplied by \( i \). This means that solutions of the DNLS hierarchy could not simply reduced from those solutions of the real KN hierarchy. Only recently, Wright...
deduced the squared amplitude of hyperelliptic solutions together with its upper bound for the standard DNLS equation (1). Zhao and Fan retrieved finite-gap solutions for the Gerdjikov-Ivanov equation of DNLS type (see eq. 8 in Ref. 20) in view of the algebro-geometric and the Riemann–Hilbert method.

For some other relevant works, the inverse scattering transformations with zero and nonzero boundary conditions were studied for the DNLS equation in Refs. 3 and 21. The multi-solitons and rogue wave solutions were presented by using the Bäcklund and Darboux transformation. In the context of KN spectral problem, Cao proposed a (2+1)-dimensional derivative Toda equation and gave its finite genus solution. Pelinovsky confirmed the existence of global solutions to the DNLS equation without the small-norm assumption in the framework of direct and inverse scattering transformations. Different from the above-mentioned treatments, we would like to set up the link between the complex finite-dimensional Hamiltonian systems (FDHSs) and the DNLS hierarchy for getting their quasi-periodic solutions in terms of Riemann theta functions.

The separation of variables is one of the most universal methods in solving soliton equations. As soliton equations can be represented as the compatibility condition of two linear spectral problems (Lax pair), the nonlinearization of Lax pair makes it possible to decompose soliton equations into FDHSs of solvable ordinary differential equations. The real FDHSs have been extensively exploited to obtain solitons, quasi-periodic solutions, as well as the rogue periodic waves for a number of completely integrable models. However, the complex FDHSs are not well studied, and are in the phase of collecting and classifying examples due to the complexity involving independent conjugates of eigenfunctions. Motivated by the above analysis, we develop a lucid algorithm not only to obtain a new formula of hyperelliptic solution of the DNLS equation (1) (compared with eqs. 98 and A13 in Ref. 19), but also to deliver all the quasi-periodic solutions for the DNLS hierarchy.

The main purpose of this work is to apply complex FDHSs for a simultaneous construction of quasi-periodic solutions of the DNLS hierarchy. Using the nonlinearization of Lax pair, the DNLS hierarchy is reduced to a family of complex FDHSs, so that simplifies the procedure for getting its explicit solutions. It follows from the Lax representations of DNLS hierarchy that a Lax matrix satisfied by the Lax equation is figured out, whose determinant gives rise to integrals of motion and a hyperelliptic curve for the complex FDHSs. With a set of quasi-Abel-Jacobi variables, the Liouville integrability of complex FDHSs is completed, which in a sense enriches the content of finite-dimensional integrable systems. Moreover, it turns out that involutive solutions of the complex FDHSs exactly yield finite parametric solutions of the DNLS hierarchy, and the used Bargmann map specifies a finite-dimensional invariant subspace to the DNLS flows. The Abel-Jacobi (or angle) variables are suitably elaborated to linearize the DNLS flows, which display the evolution behavior of associated flows on the Jacobi variety of an invariant Riemann surface. Resorting to the Riemann theorem, we apply the Riemann-Jacobi inversion to the Abel-Jacobi solutions of DNLS flows, and eventually arrive at quasi-periodic solutions of the DNLS hierarchy without any assumption on the periodic condition of potential.

The outline of this paper is as follows. In Section 2, the DNLS hierarchy is formulated into the zero-curvature pattern and further decomposed into a family of complex FDHSs. Section 3 focuses on the Liouville integrability of complex FDHSs, and the relationship between the DNLS hierarchy and the complex FDHSs is established in Section 4. Section 5 is devoted to the straightening out of the DNLS flows. In the last section, we briefly present the algebro-geometric construction of exact solutions for the DNLS hierarchy.
2 | THE DNLS HIERARCHY AND THE COMPLEX FDHSS

The Lax representations of soliton equations contain most information of exact solutions, and in particular the machinery of finite-gap integration may be used as the Lax representations are available. Let us first specify the Lax representation for each equation in the DNLS hierarchy.

2.1 | The Lax representations

Consider the stationary zero-curvature equation of the KN spectral problem with \( v = -\bar{u} \) in Ref. 3

\[
V_x = [U, V], \quad V = a_j \sigma_1 + b_j \sigma_2 + c_j \sigma_3 = \sum_{j=0}^{\infty} (a_{2j} \sigma_1 + b_{2j+1} \lambda^{-1} \sigma_2 + c_{2j+1} \lambda^{-1} \sigma_3) \lambda^{-2j},
\]

which results in

\[
\begin{align*}
a_{2j} &= \bar{u} a_{2j+1} + u c_{2j+1}, \\
b_{2j+1} &= -2i b_{2j+3} - 2u a_{2j+2}, \\
c_{2j+1} &= 2i c_{2j+3} - 2\bar{u} a_{2j+2},
\end{align*}
\]

Let \( a_0 = -i \), \( b_1 = u \), and \( c_1 = -\bar{u} \) be the initial datum. Then, up to constants of integration, all the \( a_{2j} \), \( b_{2j+1} \), and \( c_{2j+1} \) can be uniquely determined in virtue of the recursive formula (5). For example, we have

\[
\begin{align*}
a_2 &= \frac{i}{2} |u|^2, \\
b_3 &= \frac{1}{2} (iu_x - u |u|^2), \\
c_3 &= \frac{1}{2} (i\bar{u}_x + \bar{u} |u|^2), \\
a_4 &= \frac{1}{4} (u \bar{u}_x - u_x \bar{u}) - \frac{3}{8} l |u|^4, \\
b_5 &= -\frac{1}{4} u_{xx} - \frac{3}{4} i |u|^2 u_x + \frac{3}{8} u |u|^4, \\
c_5 &= \frac{1}{4} \bar{u}_{xx} - \frac{3}{4} i |u|^2 \bar{u}_x - \frac{3}{8} \bar{u} |u|^4.
\end{align*}
\]

It follows from the recursive formulas (5) that the Lenard gradients \( \{g_j\} \) and the Lenard operator pair \( K \) and \( J \) are defined by

\[
K g_j = J g_{j+1}, \quad g_j = (g^1_j, g^2_j)^T = (c_{2j+1}, b_{2j+1})^T, \quad j \geq -1,
\]

with a supplementary definition \( g_{-1} = (0, 0)^T \), and

\[
K = \frac{1}{2} \begin{pmatrix}
\partial_x u \partial_x^{-1} u \partial_x & i \partial_x^2 - \partial_x \partial_x^{-1} \bar{u} \partial_x \\
-i \partial_x^2 - \partial_x \bar{u} \partial_x^{-1} u \partial_x & \partial_x \bar{u} \partial_x^{-1} \bar{u} \partial_x
\end{pmatrix}, \quad J = \begin{pmatrix}
0 & \partial_x \\
\partial_x & 0
\end{pmatrix},
\]

are two skew-symmetric operators, and \( \partial_x^{-1} \) is to denote the inverse operator of \( \partial_x = \partial / \partial x \) with the condition \( \partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x = 1 \), and \( \partial_x^2 = \partial^2 / \partial x^2 \), \( \partial_x^2 = \partial^2 / \partial x \partial t \), etc. Clearly, it is seen from (6) and (7) that

\[
g_0 = \begin{pmatrix}
-\bar{u} \\
u
\end{pmatrix}, \quad g_1 = \frac{1}{2} \begin{pmatrix}
i \bar{u}_x + \bar{u} |u|^2 \\
i u_x - u |u|^2
\end{pmatrix}, \quad g_2 = \frac{1}{8} \begin{pmatrix}
2 \bar{u}_{xx} - 6i |u|^2 \bar{u}_x - 3\bar{u} |u|^4 \\
-2u_{xx} - 6i |u|^2 u_x + 3u |u|^4
\end{pmatrix}.
\]
Let us introduce an auxiliary spectral problem in terms of the Lenard gradients \{g_j\}

\[
\varphi_{t_n} = V^{(n)} \varphi, \quad V^{(n)} = \lambda \partial_x^{-1}(ug^{(1)} + \bar{u}g^{(2)})\sigma_1 + g^{(2)}\sigma_2 + g^{(1)}\sigma_3, \quad n \geq 1,
\]

where

\[
g = (g^{(1)}, g^{(2)})^T = \sum_{j=0}^{n-1} g_j \lambda^{2n-2j-1}.
\]

The zero-curvature equation \(U_n - V_x^{(n)} + [U, V^{(n)}] = 0\) of spectral problems (2) and (10) gives rise to the DNLS hierarchy

\[
(u_n, -\bar{u}_n)^T = J g_{n-1} =: X_{n-1}, \quad n \geq 1
\]

together with a fundamental identity

\[
V_x^{(n)} - [U, V^{(n)}] = U_n[(K - \lambda^2 J)g],
\]

where

\[
U_n[\xi] = \frac{d}{d\xi} \bigg|_{\xi=0} U(u + \xi \xi_1, \bar{u} + \xi \xi_2), \quad \xi = (\xi_1, \xi_2)^T.
\]

It is easy to check that the first nontrivial member in (11) is the DNLS equation (1) with replacing the notation \(t_2 = t\). For the concreteness, the second equation of (11) reads

\[
u_t = -\frac{9}{8}|u|^4u_x - \frac{3}{4}u^2|u|^2 \bar{u}_x + \frac{3}{4}i(\bar{u}u_x^2 + |u|^2u_{xx} + u|u_x|^2) + \frac{1}{4}u_{xxx} = 0,
\]

which is the compatibility condition of spectral problems (2) and

\[
\varphi_{t_2} = V^{(3)} \varphi, \quad V^{(3)} = V^{(1)}_{11} \sigma_1 + V^{(1)}_{12} \sigma_2 + V^{(1)}_{21} \sigma_3,
\]

where

\[
V^{(1)}_{11} = -i\lambda^6 + \frac{i}{2}\lambda^4|u|^2 + \lambda^2\left[\frac{1}{4}(u\bar{u}_x - u_x\bar{u}) - \frac{3}{8}i|u|^4\right],
\]

\[
V^{(1)}_{12} = \lambda^5u + \frac{i}{2}\lambda^3(iu_x - u|u|^2) - \lambda\left[\frac{1}{4}u_{xx} + \frac{3}{4}i|u|^2u_x - \frac{3}{8}u|u|^4\right],
\]

\[
V^{(1)}_{21} = -\lambda^5\bar{u} + \frac{i}{2}\lambda^3(i\bar{u}_x + \bar{u}|u|^2) + \lambda\left[\frac{1}{4}\bar{u}_{xx} - \frac{3}{4}i|u|^2\bar{u}_x - \frac{3}{8}\bar{u}|u|^4\right].
\]

In general, the \(n\)th DNLS equation (11) allows the Lax representations (2) and \(\varphi_{t_n} = V^{(n)} \varphi\).

**Remark:** To fix \(n = 1\) in (10), we know from the spectral matrix \(V^{(1)} = U\) that the variable \(t_1\) is in fact the spacial variable \(x\).

### 2.2 The finite-dimensional integrable reduction

Let us now give some symbols and conventions to make our presentations self-contained. For the sake of convenience, we introduce the symbols \(p = (p_1, p_2, \ldots, p_N)^T, q = (q_1, q_2, \ldots, q_N)^T,\) and \(\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)\). The diamond bracket \(\langle \cdot, \cdot \rangle\) stands for the inner product in \(\mathbb{R}^N\). It is supposed
that \( \lambda_1, \lambda_2, \ldots, \lambda_N \) (\( \lambda_i \neq \lambda_j, \ i \neq j \)) are \( N \) nonzero complex eigenvalues to the spectral problem (2), and \((p_j, q_j)^T\) is the vector eigenfunction pertinent to \( \lambda_j, \ (1 \leq j \leq N) \).

Due to the symmetry of (2), \((\bar{q}_j, -\bar{p}_j)^T\) corresponds to the eigenvalue \( \bar{\lambda}_j \). Recalling the nonlinearization of Lax pair,\(^{26}\) we take \( N \) copies of spectral problem (2) together with their complex conjugates written as

\[
\begin{cases}
p_{j,x} = -i\lambda_j^2 p_j + \lambda_j u q_j, & q_{j,x} = -i\bar{\lambda}_j^2 \bar{q}_j + \bar{\lambda}_j u(-\bar{p}_j), \\
\tilde{q}_{j,x} = -\lambda_j \tilde{u} p_j + i\lambda_j^2 q_j, & \tilde{p}_{j,x} = -\bar{\lambda}_j \tilde{u} \tilde{q}_j + i\bar{\lambda}_j^2 (-\tilde{p}_j).
\end{cases}
\]  

\( \text{(15)} \)

From Refs. 26 and 34, a direct computation yields the functional gradients of \( \lambda_j \) and \( \bar{\lambda}_j \) on \( u \) and \( \tilde{u} \),

\[
\nabla \lambda_j = \begin{pmatrix} \frac{\delta \lambda_j}{\partial u} \\ \frac{\delta \lambda_j}{\partial \tilde{u}} \end{pmatrix} = \begin{pmatrix} -\lambda_j^2 q_j \\ \lambda_j p_j \end{pmatrix}, \quad \nabla \bar{\lambda}_j = \begin{pmatrix} \frac{\delta \bar{\lambda}_j}{\partial u} \\ \frac{\delta \bar{\lambda}_j}{\partial \tilde{u}} \end{pmatrix} = \begin{pmatrix} -\bar{\lambda}_j^2 \bar{q}_j \\ \bar{\lambda}_j \bar{q}_j \end{pmatrix}.
\]  

\( \text{(16)} \)

which is a special solution to the Lenard eigenvalue equations

\[
(K - \lambda_j^2 J)\nabla \lambda_j = 0, \quad (K - \bar{\lambda}_j^2 J)\nabla \bar{\lambda}_j = 0.
\]  

\( \text{(17)} \)

Consider the Bargmann (symmetric) constraint in the process of nonlinearization of Lax pair\(^{26}\)

\[
g_0 = \sum_{j=1}^N (\nabla \lambda_j + \nabla \bar{\lambda}_j),
\]  

\( \text{(18)} \)

which leads to a Bargmann map between the potential \( u \) and the eigenfunctions \((p, q)\)

\[
u = \langle \Lambda p, p \rangle + \langle \bar{\Lambda} \bar{q}, \bar{q} \rangle.
\]  

\( \text{(19)} \)

To progress further, on \( \mathbb{C}^{2N} \) we bring in the symplectic structure \( \omega^2 = dp \wedge dq + d\bar{p} \wedge d\bar{q} \) and define the Poisson bracket\(^{35}\)

\[
\{ f, g \} = \left( \frac{\partial f}{\partial q_j}, \frac{\partial g}{\partial p_j} \right) - \left( \frac{\partial f}{\partial p_j}, \frac{\partial g}{\partial q_j} \right) + \left( \frac{\partial f}{\partial \bar{q}_j}, \frac{\partial g}{\partial \bar{p}_j} \right) - \left( \frac{\partial f}{\partial \bar{p}_j}, \frac{\partial g}{\partial \bar{q}_j} \right).
\]

Simply substituting (19) back into the Lax representations (2), (3), (14), and using (15) leads to three complex FDHSs with real-valued Hamiltonians

\[
p_x = \{ p, H_0 \}, \quad q_x = \{ q, H_0 \}, \quad \bar{p}_x = \{ \bar{p}, H_0 \}, \quad \bar{q}_x = \{ \bar{q}, H_0 \},
\]

\( \text{(20)} \)

where

\[
H_0 = i \left( \langle \Lambda^2 p, q \rangle - \langle \bar{\Lambda}^2 \bar{p}, \bar{q} \rangle \right) - \frac{1}{2} \left( \langle \Lambda p, p \rangle + \langle \bar{\Lambda} \bar{q}, \bar{q} \rangle \right)^2,
\]

\( \text{(21)} \)

\[
p_t = \{ p, H_1 \}, \quad q_t = \{ q, H_1 \}, \quad \bar{p}_t = \{ \bar{p}, H_1 \}, \quad \bar{q}_t = \{ \bar{q}, H_1 \},
\]

\( \text{(22)} \)

where

\[
H_1 = i \left( \langle \Lambda^4 p, q \rangle - \langle \bar{\Lambda}^4 \bar{p}, \bar{q} \rangle \right) - \frac{i}{2} \left( \langle \Lambda^2 p, q \rangle - \langle \bar{\Lambda}^2 \bar{p}, \bar{q} \rangle \right) \left( \langle \Lambda p, p \rangle + \langle \bar{\Lambda} \bar{q}, \bar{q} \rangle \right)^2
\]

\[
- \frac{1}{2} \left( \langle \Lambda p, p \rangle + \langle \bar{\Lambda} \bar{q}, \bar{q} \rangle \right) \left( \langle \Lambda^3 q, q \rangle + \langle \bar{\Lambda}^3 \bar{p}, \bar{p} \rangle \right) - \frac{1}{2} \left( \langle \Lambda^3 p, p \rangle + \langle \bar{\Lambda}^3 \bar{q}, \bar{q} \rangle \right)
\]
\[
\times(\langle \Lambda q, q \rangle + \langle \Lambda \tilde{p}, \tilde{p} \rangle) + \frac{1}{8} |(\Lambda p, p) + \langle \Lambda \tilde{q}, \tilde{q} \rangle|^4, \tag{23}
\]
and
\[
p_{t_3} = \{ p, H_2 \}, \quad q_{t_3} = \{ q, H_2 \}, \quad \tilde{p}_{t_3} = \{ \tilde{p}, H_2 \}, \quad \tilde{q}_{t_3} = \{ \tilde{q}, H_2 \}, \tag{24}
\]
where
\[
H_2 = i(\langle \Lambda^6 p, q \rangle - \langle \Lambda^6 \tilde{p}, \tilde{q} \rangle) - \frac{1}{2} \left[ |\langle \Lambda^3 p, p \rangle + \langle \Lambda^3 \tilde{q}, \tilde{q} \rangle|^2 + (\langle \Lambda p, p \rangle + \langle \Lambda \tilde{q}, \tilde{q} \rangle) \times(\langle \Lambda^5 q, q \rangle + \langle \Lambda^5 \tilde{p}, \tilde{p} \rangle) + (\langle \Lambda q, q \rangle + \langle \Lambda \tilde{p}, \tilde{p} \rangle)(\langle \Lambda^5 p, p \rangle + \langle \Lambda^5 \tilde{q}, \tilde{q} \rangle) \right] + \frac{1}{2} |\langle \Lambda p, p \rangle + \langle \Lambda \tilde{q}, \tilde{q} \rangle|^2 \left[ (\langle \Lambda^2 p, q \rangle - \langle \Lambda^2 \tilde{p}, \tilde{q} \rangle)^2 - i(\langle \Lambda^4 p, q \rangle - \langle \Lambda^4 \tilde{p}, \tilde{q} \rangle) \right] + \frac{1}{4} \left[ |\langle \Lambda p, p \rangle + \langle \Lambda \tilde{q}, \tilde{q} \rangle|^2 - 2i(\langle \Lambda^2 p, q \rangle - \langle \Lambda^2 \tilde{p}, \tilde{q} \rangle) \right] \left[ |\langle \Lambda p, p \rangle + \langle \Lambda \tilde{q}, \tilde{q} \rangle|^2 - \frac{1}{16} |\langle \Lambda p, p \rangle + \langle \Lambda \tilde{q}, \tilde{q} \rangle|^6. \tag{25}
\]
As the DNLS equations (1) and (13) appear to be the compatibility condition of spectral problems (2), (3), and (14), on \((C^{2N}, \omega^2)\) they are indeed reduced to three complex FDHSs (20), (22), and (24) via separating the temporal and spatial variables.

Introduce a bilinear generating function
\[
G_{\lambda} = \lambda g_{-1} + \sum_{j=1}^{N} \left( \frac{\lambda \Lambda \lambda_j}{\lambda^2 - \lambda_j^2} + \frac{\lambda \Lambda \lambda_j}{\lambda^2 - \lambda_j^2} \right) = \left( -\lambda Q_{\lambda}(\Lambda q, q) - \lambda Q_{\lambda}(\Lambda \tilde{p}, \tilde{p}) \right) + \lambda Q_{\lambda}(\Lambda p, p) + \lambda Q_{\lambda}(\Lambda \tilde{q}, \tilde{q}), \tag{26}
\]
which satisfies the Lenard eigenvalue equation corresponding to the spectral parameter \(\lambda\)
\[
(K - \lambda^2 J)G_{\lambda} = 0, \tag{27}
\]
where
\[
Q_{\lambda}(\xi, \eta) = \sum_{j=1}^{N} \frac{\xi_j \eta_j}{\lambda^2 - \lambda_j^2} = \sum_{k=0}^{\infty} \langle \Lambda^{2k} \xi, \eta \rangle \lambda^{-2k-2},
\]
\[
Q_{\lambda}(\xi, \eta) = \sum_{j=1}^{N} \frac{\xi_j \eta_j}{\lambda^2 - \lambda_j^2} = \sum_{k=0}^{\infty} \langle \Lambda^{2k} \xi, \eta \rangle \lambda^{-2k-2}
\]
under the condition \(|\lambda| > \max \{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|\}\). Substituting \(G_{\lambda}\) into the spectral matrix \(V^{(n)}\) delivers the desired Lax matrix
\[
V_{\lambda} = \begin{pmatrix}
-\lambda Q_{\lambda}(\Lambda^2 p, q) + Q_{\lambda}(\Lambda^2 \tilde{p}, \tilde{q}) & \lambda(Q_{\lambda}(\Lambda p, p) + Q_{\lambda}(\Lambda \tilde{q}, \tilde{q})) \\
-\lambda(Q_{\lambda}(\Lambda q, q) + Q_{\lambda}(\Lambda \tilde{p}, \tilde{p})) & i + Q_{\lambda}(\Lambda^2 p, q) - Q_{\lambda}(\Lambda^2 \tilde{p}, \tilde{q})
\end{pmatrix} = V_{\lambda}^{11} \sigma_1 + V_{\lambda}^{12} \sigma_2 + V_{\lambda}^{21} \sigma_3, \tag{28}
\]
which satisfies the Lax equation

$$(V_\lambda)_\lambda - [U, V_\lambda] = 0$$

(29)

in view of (12) and (27). From the Lax equation (29), with $|\lambda| > \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|\}$ we attain a generating function $\det V_\lambda$ of integrals of motion for the complex FDHSs (20)$^{36}$

$$\det V_\lambda = F_\lambda = 1 - 2iQ_\lambda(\Lambda^2 p, q) + i^2Q_\lambda(\Lambda p, p)Q_\lambda(\Lambda q, q) - Q_\lambda^2(\Lambda^2 p, q)$$

$$+2iQ_\lambda(\Lambda^2 \bar{p}, \bar{q}) + i^2Q_\lambda(\Lambda \bar{p}, \bar{p})Q_\lambda(\Lambda \bar{q}, \bar{q}) - Q_\lambda^2(\Lambda^2 \bar{p}, \bar{q})$$

$$+2Q_\lambda(\Lambda^2 p, q)Q_\lambda(\Lambda^2 \bar{p}, \bar{q}) + \lambda^2[Q_\lambda(\Lambda p, p)Q_\lambda(\Lambda \bar{p}, \bar{p}) + Q_\lambda(\Lambda q, q)Q_\lambda(\Lambda \bar{q}, \bar{q})]$$

$$= 1 + \sum_{k=0}^\infty F_k \lambda^{-2k-2},$$

(30)

where

$$F_0 = -2i(\langle \Lambda^2 p, q \rangle - \langle \Lambda^2 \bar{p}, \bar{q} \rangle) + \left| \langle \Lambda p, p \rangle + \langle \Lambda \bar{q}, \bar{q} \rangle \right|^2,$$

(31)

$$F_1 = -2i(\langle \Lambda^4 p, q \rangle - \langle \Lambda^4 \bar{p}, \bar{q} \rangle) - (\langle \Lambda^2 p, q \rangle - \langle \Lambda^2 \bar{p}, \bar{q} \rangle)^2 + (\langle \Lambda p, p \rangle + \langle \Lambda \bar{q}, \bar{q} \rangle)$$

$$\times(\langle \Lambda^3 p, q \rangle + \langle \Lambda^3 \bar{p}, \bar{p} \rangle) + (\langle \Lambda q, q \rangle + \langle \Lambda \bar{p}, \bar{p} \rangle)(\langle \Lambda^3 p, p \rangle + \langle \Lambda^3 \bar{q}, \bar{q} \rangle),$$

(32)

$$F_2 = -2i(\langle \Lambda^6 p, q \rangle - \langle \Lambda^6 \bar{p}, \bar{q} \rangle) + \left| \langle \Lambda^2 p, p \rangle + \langle \Lambda \bar{q}, \bar{q} \rangle \right|^2 + (\langle \Lambda p, p \rangle + \langle \Lambda \bar{q}, \bar{q} \rangle)$$

$$\times(\langle \Lambda^5 p, q \rangle + \langle \Lambda^5 \bar{p}, \bar{p} \rangle) + (\langle \Lambda q, q \rangle + \langle \Lambda \bar{p}, \bar{p} \rangle)(\langle \Lambda^5 p, p \rangle + \langle \Lambda^5 \bar{q}, \bar{q} \rangle)$$

$$-2(\langle \Lambda^2 p, q \rangle - \langle \Lambda^2 \bar{p}, \bar{q} \rangle)(\langle \Lambda^4 p, q \rangle - \langle \Lambda^4 \bar{p}, \bar{q} \rangle),$$

(33)

$$F_k = -2i(\langle \Lambda^{2k+2} p, q \rangle - \langle \Lambda^{2k+2} \bar{p}, \bar{q} \rangle)$$

$$+ \sum_{i=0}^k \left( \langle \Lambda^{2i+1} p, p \rangle + \langle \Lambda^{2i+1} \bar{q}, \bar{q} \rangle \right)(\langle \Lambda^{2i} \bar{q}, \bar{q} \rangle + \langle \Lambda^{2i+1} \bar{p}, \bar{p} \rangle)$$

$$- \sum_{i=0}^{k-1} (\langle \Lambda^{2i+2} p, q \rangle - \langle \Lambda^{2i+2} \bar{p}, \bar{q} \rangle)(\langle \Lambda^{2i} p, p \rangle - \langle \Lambda^{2i} \bar{q}, \bar{q} \rangle).$$

(34)

It is observed that

$$F_0 = -2H_0, \quad F_1 = -2H_1 + H_0^2, \quad F_2 = -2H_2 + 2H_0H_1.$$  

(35)

We introduce an infinite sequence of complex FDHSs described as

$$p_{k+1} = \{p, H_k\}, \quad q_{k+1} = \{q, H_k\}, \quad \bar{p}_{k+1} = \{\bar{p}, H_k\}, \quad \bar{q}_{k+1} = \{\bar{q}, H_k\}, \quad k \geq 3,$$

(36)
where the real-valued Hamiltonian $H_k$ is given by the recursive formula

$$H_k = -\frac{1}{2} F_k + \frac{1}{2} \sum_{k=0}^{k-1} H_i H_{k-1-i}, \quad k \geq 3,$$

(37)

which together with (35) can be put into the unified form

$$F_\lambda = (1 - H_\lambda)^2, \quad H_\lambda = \sum_{k=0}^{\infty} H_k \lambda^{-2k-2}.$$  

(38)

In what follows, we exhibit that the complex FDHSs (20), (22), (24), and (36) constitute the decompositions of the DNLS hierarchy (11), whose involutive solutions exactly generates finite parametric solutions to the DNLS hierarchy.

3 THE LIOUVILLE INTEGRABILITY

The integrability of a given complex FDHS is the existence of a set of $2N$ smooth functions, which is termed by a Liouville set if they are involutive in pairs and functionally independent on $(\mathbb{C}^{2N}, \omega^2)$. Recalling (31)-(34), we have a sequence of integrals of motion to the complex FDHSs $(H_0, \omega^2, \mathbb{C}^{2N})$.

Actually, they are integrals of motion for the complex FDHSs (22), (24), and (36) as well (see equation 45). The subsequent issue focuses on the involutivity and the functional independence of $\{F_k\}$ and $\{H_k\}$ ($k \geq 0$).

Regard $F_\lambda$ as a Hamiltonian on the symplectic space $(\mathbb{C}^{2N}, \omega^2)$, and denote the flow variables of $F_\lambda$, $H_\lambda$, $F_k$, and $H_k$ by $\tau_\lambda$, $t_\lambda$, $\tau_k$, and $t_{k+1}$ ($k \geq 0$), respectively. Complying with the definition of Poisson bracket, after a direct calculation we obtain two canonical Hamiltonian equations

$$\frac{d}{d\tau_\lambda} \left( p_j \begin{pmatrix} p_j \\ q_j \end{pmatrix} \right) = W(\lambda, \lambda) \left( p_j \begin{pmatrix} q_j \\ p_j \end{pmatrix} \right),$$

(39)

and

$$\frac{d}{d\tau_\lambda} \left( \bar{q}_j \begin{pmatrix} \bar{q}_j \\ -\bar{p}_j \end{pmatrix} \right) = W(\lambda, \bar{\lambda}) \left( \bar{q}_j \begin{pmatrix} -\bar{p}_j \\ \bar{q}_j \end{pmatrix} \right),$$

(40)

where

$$W(\lambda, \mu) = -\frac{2\mu}{\lambda^2 - \mu^2} \left( \mu V^{11}_\lambda \sigma_1 + \lambda V^{12}_\lambda \sigma_2 + \lambda V^{21}_\lambda \sigma_3 \right) \quad =: W^{11}_\lambda \sigma_1 + W^{12}_\lambda \sigma_2 + W^{21}_\lambda \sigma_3.$$  

(41)

Similar to the verifications implemented as the Proposition 3.1 in Ref. 28 or the Proposition 4.1 in Ref. 31, we obtain a key Lax equation and the involutivity of integrals of motion $\{F_k, k \geq 0\} \cup \{H_k, k \geq 0\}$.

**Proposition 1.** On $(\mathbb{C}^{2N}, \omega^2)$, the Lax matrix $V_\mu$ satisfies a Lax equation

$$\frac{dV_\mu}{d\tau_\lambda} = \left[ W(\lambda, \mu), V_\mu \right], \quad \forall \lambda, \mu \in \mathbb{C}, \ \lambda \neq \mu.$$  

(42)
Corollary 1.

\[ \{ F_\mu, F_\lambda \} = 0, \quad \{ F_j, F_k \} = 0, \quad j, k = 0, 1, 2, \ldots \]  \hspace{1cm} (43)

and

\[ \{ H_\mu, H_\lambda \} = 0, \quad \{ H_j, H_k \} = 0, \quad j, k = 0, 1, 2, \ldots \]  \hspace{1cm} (44)

Moreover, from (30), (38), and (44), a simple computation results in

\[
\frac{dF_\mu}{dt} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{dF_i}{dt} (\lambda \mu)^{-2i-2} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \{F_i, H_j\} (\lambda \mu)^{-2i-2} = \{F_\mu, H_\lambda\} = 2(H_\mu - 1)\{H_\mu, H_\lambda\} = 0,
\]

which signifies that \( \{F_k\} (k \geq 0) \) are involutive integrals of motion for all the complex FDHSs (20), (22), (24), and (36).

Conforming to the Liouville theorem, the other essential element of integrability is the functional independence of conserved quantities. The rest of this section is thus instructed to the functional independence of \( \{F_k\} \) and \( \{H_k\} (k \geq 0) \), which ensures that the complex FDHSs could be integrated completely.

It is noted that

\[
F_\lambda = -V^{12}_\lambda V^{21}_\lambda - (V^{11}_\lambda)^2 = 1 + \sum_{j=1}^{N} E_j + \tilde{E}_j + \sum_{j=1}^{N} \tilde{E}_j + \tilde{E}_j,
\]

where

\[
E_j = -2i\lambda_j^2 p_j q_j + \langle \Lambda p, p \rangle \lambda_j q_j^2 + \sum_{k=1, k \neq j}^{N} \frac{\lambda_j \lambda_k}{\lambda_j^2 - \lambda_k^2} (\lambda_j p_j q_k - \lambda_k p_k q_j)^2,
\]

\[
\tilde{E}_j = \langle \Lambda \bar{q}, \bar{q} \rangle \lambda_j \bar{q}_j^2 + \sum_{k=1}^{N} \frac{\lambda_j \bar{\lambda}_k}{\lambda_j^2 - \bar{\lambda}_k^2} (\lambda_j p_j \bar{\lambda}_k + q_j \bar{\lambda}_k \bar{\lambda}_k)^2,
\]

\[
\tilde{E}_j = \langle \Lambda p, p \rangle \lambda_j \bar{p}_j^2 + \sum_{k=1}^{N} \frac{\lambda_k \bar{\lambda}_j}{\lambda_j^2 - \bar{\lambda}_k^2} (\lambda_k p_k \lambda_j + q_k \lambda_j \bar{\lambda}_j)^2.
\]

We conclude from (28) and (46) that \( V^{12}_\lambda \) and \( F_\lambda \) are rational polynomial functions of \( \lambda \) with simple poles at \( \{ \pm \lambda_j, \pm \bar{\lambda}_j \} \ (j = 1, 2, \ldots, N) \). Writing \( \zeta = \lambda^2 \) for short, they can be factorized as

\[
F_\lambda = \frac{b(\zeta)}{a(\zeta)} = \frac{a(\zeta) b(\zeta)}{a^2(\zeta)} = \frac{R(\zeta)}{a^2(\zeta)},
\]

\[
V^{12}_\lambda = \lambda (Q_\lambda (\Lambda p, p) + Q_\lambda (\Lambda \bar{q}, \bar{q})) = \lambda \left( \langle \Lambda p, p \rangle + \langle \Lambda \bar{q}, \bar{q} \rangle \right) \frac{m(\zeta)}{a(\zeta)},
\]

where

\[
a(\zeta) = \prod_{k=1}^{N} (\zeta - \lambda_k^2)(\zeta - \bar{\lambda}_k^2), \quad b(\zeta) = \prod_{k=1}^{2N} (\zeta - \lambda_{N+k}^2),
\]
\[ m(\zeta) = \prod_{k=1}^{2N-1} (\zeta - \mu_k^2), \quad R(\zeta) = \prod_{k=1}^{N} (\zeta - \lambda_k^2)(\zeta - \overline{\lambda}_k^2) \prod_{k=1}^{2N} (\zeta - \lambda_{N+k}^2) \]

and \( \{ \mu_k \} \) (\( 1 \leq k \leq 2N - 1 \)) are the so-called elliptic variables to the complex FDHSs (20), (22), (24), and (36).

To set \( \zeta = \mu_k^2 \) (\( 1 \leq k \leq 2N - 1 \)) in (46), we know from (48) that

\[ V_{\lambda}^{11} \bigg|_{\lambda = \mu_k} = \frac{\sqrt{-R(\mu_k^2)}}{a(\mu_k^2)}, \quad 1 \leq k \leq 2N - 1. \]  

(49)

And then, the combination of formulas (41), (1,2)-entry of the Lax equations (42), (48), and (49) gives rises to the Dubrovin-type equation

\[ \frac{1}{4\sqrt{-R(\mu_k^2)}} \frac{d\mu_k^2}{d\tau_\lambda} = -\frac{\zeta m(\zeta)}{a(\zeta)(\zeta - \mu_k^2)m'(\mu_k^2)}, \quad 1 \leq k \leq 2N - 1, \]  

(50)

which determines the dynamics of \( \mu_k^2 \) along with the \( \tau_\lambda \)-flow over \( \mathbb{C}^{2N}, \omega^2 \). Multiplied by \( (\mu_k^2)^{2N-1-j} \) on both sides of (50), the summation regarding \( k \) from 1 to \( 2N - 1 \) results in

\[ \sum_{k=1}^{2N-1} \frac{(\mu_k^2)^{2N-1-j} d\mu_k^2}{4\sqrt{-R(\mu_k^2)}} d\tau_\lambda = -\frac{\xi^{2N-j}}{a(\zeta)}, \quad 1 \leq j \leq 2N - 1, \]  

(51)

with the help of the Lagrange interpolation formula.

Let us now turn to the basic theory of algebraic geometry. Introduce a hyperelliptic curve of Riemann surface \( \Gamma \) defined by the affine equation \( \overline{\xi}^2 + R(\zeta) = 0 \). The usual basis of holomorphic differentials on \( \Gamma \) can be chosen as

\[ \tilde{\omega}_j = \xi^{2N-1-j} \frac{d\xi}{4\sqrt{-R(\zeta)}}, \quad 1 \leq j \leq 2N - 1. \]  

(52)

Due to \( \deg R(\zeta) = 4N \), \( \Gamma \) has genus \( 2N - 1 \), which is in agreement with the number of elliptic variables \( \{ \mu_k^2 \} \). For any \( \zeta \neq \lambda_j^2, \overline{\lambda}_j^2 \) (\( 1 \leq j \leq N \)); \( \lambda_{N+k}^2 \) (\( 1 \leq k \leq 2N \)) \( \in \mathbb{C} \), there are two points \( P_\pm(\zeta) = (\zeta, \pm \sqrt{-R(\zeta)}) \) on the upper and lower sheets of \( \Gamma \). Specially, as \( \lambda = \infty \), \( \Gamma \) have two infinite points \( \infty_1 \) and \( \infty_2 \) that are not the branch points and can be described by \( (0, \mp 1) \) in the local coordinate \( z = \zeta^{-1} \). Let \( P_0(\neq \infty_0 \ (i = 1, 2); \lambda_j^2, \overline{\lambda}_j^2 \ (1 \leq j \leq N); \lambda_{N+k}^2 \ (1 \leq k \leq 2N)) \) be a fixed point on \( \Gamma \). We introduce a set of quasi-Abel-Jacobi variables

\[ \tilde{\phi}_j = \sum_{k=1}^{2N-1} \int_{P_0}^{P(\mu_k^2)} \tilde{\omega}_j, \quad 1 \leq j \leq 2N - 1. \]  

(53)

After a direct calculation, it is checked from (30) and (38) that

\[ \{ (p, q) - (\tilde{p}, \tilde{q}), F_\lambda \} = 2(H_\lambda - 1)\{(p, q) - (\tilde{p}, \tilde{q}), H_\lambda \} = 0. \]
Let us introduce one more integral of motion to all the resulted complex FDHSs

\[ F_{-1} = (1 + i(\langle p, q \rangle - \langle \bar{p}, \bar{q} \rangle))^2 = -2H_{-1} \tag{54} \]

with the flow variables \( \tau_{-1} \) for \( F_{-1} \) and \( t_{-1} \) for \( H_{-1} \), and temporarily extend \( F_\lambda \) and \( H_\lambda \) to be \( \tilde{F}_\lambda \) and \( \tilde{H}_\lambda \), where

\[ \tilde{F}_\lambda = (1 + i(\langle p, q \rangle - \langle \bar{p}, \bar{q} \rangle))^2 + \sum_{k=0}^{\infty} F_k \lambda^{-2k-2} = \sum_{k=-1}^{\infty} F_k \lambda^{-2k-2}, \quad \tilde{H}_\lambda = \sum_{k=-1}^{\infty} H_k \lambda^{-2k-2}, \]

whose flow variables were hence replaced with \( \tilde{\tau}_\lambda \) and \( \tilde{t}_\lambda \), respectively.

Resorting to (51) and (53), the quasi-Abel-Jacobi variable \( \tilde{\phi}_j \) straightens out the \( \tilde{F}_\lambda \)-flow under the assumption \(|\lambda| > \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|\}\)

\[ \frac{d\tilde{\phi}_j}{d\tilde{\tau}_\lambda} = \{\tilde{\phi}_j, \tilde{F}_\lambda\} = -\frac{\xi^{2N-j}}{a(\zeta)} = -\sum_{k=-1}^{\infty} \Lambda_{k+1-j} \xi^{-k-1}, \quad 1 \leq j \leq 2N - 1, \tag{55} \]

where

\[ \Lambda_{-k} = 0 \ (k \geq 1), \quad \Lambda_0 = 1, \quad \Lambda_1 = s_1, \]
\[ \Lambda_2 = \frac{1}{2}(s_2 + s_1^2), \quad \Lambda_3 = \frac{1}{6}(2s_3 + 3s_2s_1 + s_1^3), \]
\[ \Lambda_k = \frac{1}{k} \left( s_k + \sum_{i+j=k; i,j \geq 1} s_i \Lambda_j \right), \quad k \geq 4, \]
\[ s_k = \lambda_1^{2k} + \cdots + \lambda_N^{2k} + \lambda_1^{2k} + \cdots + \lambda_N^{2k}. \]

On the other hand, one gets from (30) that

\[ \frac{d\tilde{\phi}_j}{d\tilde{\tau}_\lambda} = \{\tilde{\phi}_j, \tilde{F}_\lambda\} = \sum_{k=-1}^{\infty} \{\tilde{\phi}_j, F_k\} \lambda^{-2k-2} = \sum_{k=-1}^{\infty} \frac{d\tilde{\phi}_j}{d\tau_k} \zeta^{-k-1}. \tag{56} \]

The combination of (55) and (56) immediately gives

\[ \frac{d\tilde{\phi}_j}{d\tau_k} = -\Lambda_{k+1-j}, \quad k \geq -1, \quad 1 \leq j \leq 2N - 1. \tag{57} \]

Let \( \tilde{\phi} \) be the column vector of \( (\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_{2N-1})^T \). It is clear to see from (57) that

\[ \frac{\partial(\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_{2N-1})}{\partial(\tau_0, \tau_1, \ldots, \tau_{2N-2})} = -\begin{pmatrix} 1 & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{2N-2} \\ 0 & 1 & \Lambda_1 & \cdots & \Lambda_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = -1. \tag{58} \]
**Proposition 2.** The integrals of motion \( \{ F_{-1}, F_0, \ldots, F_{2N-2} \} \) determined by (31)-(34) and (54) are functionally independent on \((C^{2N}, \omega^2)\).

**Proof.** It follows from Ref. 35 that we only need to verify the linear independence of differentials \(dF_{-1}, dF_0, \ldots, dF_{2N-2} \) at each cotangent space \( T^*_x C^{2N} \) \((\forall x \in C^{2N})\). Assume that there exist \(2N \) constants \( \gamma_{-1}, \gamma_0, \ldots, \gamma_{2N-2} \) satisfying the identity \( \sum_{k=-1}^{2N-2} \gamma_k dF_k = 0 \). Recalling the relation between the Poisson bracket and the symplectic structure \( \omega^2 \), one has

\[
0 = \sum_{k=-1}^{2N-2} \gamma_k dF_k = \sum_{k=-1}^{2N-2} \gamma_k \omega^2(I dF_k, I d\phi_j) = \sum_{k=-1}^{2N-2} \gamma_k \{ \phi_j, F_k \} = \sum_{k=0}^{2N-2} \gamma_k \frac{d\phi_j}{d\tau_k}
\]

in view of \( \frac{d\phi_j}{d\tau_{-1}} = \{ \phi_j, F_{-1} \} = -\Lambda_{-j} = 0 \). Note that the coefficient matrix in the rightmost-hand of (59) is nondegenerate. We conclude that \( \gamma_0 = \gamma_1 = \ldots = \gamma_{2N-2} = 0 \). Also, we arrive at \( \gamma_{-1} dF_{-1} = 0 \), and thus \( \gamma_{-1} = 0 \) due to \( dF_{-1} \neq 0 \), which completes the proof. \( \blacksquare \)

**Proposition 3.** The real-valued Hamiltonians \( \{ H_{-1}, H_0, \ldots, H_{2N-2} \} \) given by (21), (23), (25), (37), and (54) are functionally independent on \((C^{2N}, \omega^2)\).

**Proof.** From (37), we have

\[
\begin{pmatrix}
  dF_{-1} \\
  dF_0 \\
  dF_1 \\
  : \\
  dF_{2N-2}
\end{pmatrix} = \begin{pmatrix}
  -2 & 0 & 0 & \ldots & 0 \\
  0 & -2 & 0 & \ldots & 0 \\
  0 & 2H_0 & -2 & \ldots & 0 \\
  : & : & : & \ddots & : \\
  0 & H_{2N-3} & H_{2N-4} & \ldots & -2
\end{pmatrix} \begin{pmatrix}
  dH_{-1} \\
  dH_0 \\
  dH_1 \\
  \vdots \\
  dH_{2N-2}
\end{pmatrix},
\]

which leads to the functional independence of \( \{ H_{-1}, H_0, \ldots, H_{2N-2} \} \) by using Proposition 2. \( \blacksquare \)

Followed by Corollary 1 and Proposition 3, we claim the Liouville integrability for all the complex FDHSs (20), (22), (24), and (36).

**Theorem 1.** The complex FDHSs \( (H_k, \omega^2, C^{2N}) \), \( k = 0, 1, \ldots, \) are completely integrable in the Liouville sense.

Based on Theorem 1, we exhibit that arbitrary two complex FDHSs \( (H_i, \omega^2, C^{2N}) \), \( i \geq 0 \) and \( (H_j, \omega^2, C^{2N}) \), \( j \geq 0 \) are consistent with each other. This means that the existence of a smooth function in variables \( t_i \) and \( t_j \) generates an involutive solution for the associated complex FDHSs. In the next step, from the commutability of complex Hamiltonian flows, we turn to the relationship between the complex FDHSs and the DNLS hierarchy.

### 4 | THE FINITE PARAMETRIC SOLUTIONS

The real FDHSs have been adapted to deduce exact solutions in quite a few cases.\(^{10,27–31}\) The key point lies in the fact that the relationship between real FDHSs and soliton equations is established, in which involutive solutions of real FDHSs naturally produce finite parametric solutions of soliton equations and the used symmetric constraint specifies a finite-dimensional invariant subspace to soliton equations. Based on the above observation, in this section we show that the Bargmann map (19) gives not only finite parametric solutions of the DNLS hierarchy, but also the finite-gap potential of the complex Novikov equation.
To deal with the DNLS hierarchy simultaneously, the generating function method is adopted in a series of later arguments. Consider one more generating function of Lenard gradients \( \{ g_j \} \)

\[
g_\lambda = \left( g_1^\lambda, g_2^\lambda \right)^T = \lambda g_{-1} + \sum_{k=0}^{\infty} g_k \lambda^{-2k-1}, \tag{60}
\]

which satisfies the Lenard eigenvalue equation \((K - \lambda^2 J)g_\lambda = 0\) by virtue of the recurrence chain (7). Acting the operator \( J^{-1}K \) on the Bargmann constraint (18) \( k \) times, together with (17) we have

\[
\sum_{j=1}^{N} \left( \lambda_j^{2k} \nabla \lambda_j + \bar{\lambda}_j^{2k} \nabla \bar{\lambda}_j \right) = g_k + \hat{c}_2 g_{k-1} + \cdots + \hat{c}_{k+1} g_0, \quad k \geq 1, \tag{61}
\]

where \( \hat{c}_2, \hat{c}_3, \ldots, \hat{c}_{k+2} \) are some constants of integration. By means of (60) and (61), under \( |\lambda| > \max\{ |\lambda_1|, |\lambda_2|, \ldots, |\lambda_N| \} \) the bilinear generating function \( G_\lambda \) (26) can be rewritten as

\[
G_\lambda = (G_1^\lambda, G_2^\lambda)^T = \sum_{k=0}^{\infty} \sum_{j=1}^{N} \left( \lambda_j^{2k} \nabla \lambda_j + \bar{\lambda}_j^{2k} \nabla \bar{\lambda}_j \right) \lambda^{-2k-1}
\]

\[
= \sum_{k=0}^{\infty} (g_k + \hat{c}_2 g_{k-1} + \cdots + \hat{c}_{k+1} g_0) \lambda^{-2k-1} = \hat{c}_\lambda g_\lambda, \tag{62}
\]

where

\[
\hat{c}_\lambda = 1 + \sum_{k=0}^{\infty} \hat{c}_{k+2} \lambda^{-2k-2}. \tag{63}
\]

Due to the rapidly decaying condition of \( u \) as \( x \to \infty \), it follows from (10), (38), (62), and (63) that

\[
F_\lambda = \hat{c}_\lambda^2 = (1 - H_\lambda)^2, \quad \hat{c}_\lambda = 1 - H_\lambda, \quad \hat{c}_{k+2} = -H_k. \tag{64}
\]

Let \( f \) be an arbitrary test function. By using the Leibniz rule of Poisson bracket and the formulas (38) and (64), we have

\[
\frac{df}{d\tau_\lambda} = \{ f, F_\lambda \} = -2(1 - H_\lambda)\{ f, H_\lambda \} = -2\hat{c}_\lambda \frac{df}{dt_\lambda},
\]

which together with (30) and (38) results in

\[
\frac{d}{dt_\lambda} = \frac{1}{2\hat{c}_\lambda} \frac{d}{d\tau_\lambda}, \quad \frac{d}{dt_{k+1}} = -\frac{1}{2\hat{c}_\lambda} \frac{d}{d\tau_k}. \tag{65}
\]

**Proposition 4.** Let \( (p(x, t_n), q(x, t_n))^T, (n \geq 2) \), be an involutive solution of the complex integrable FDHSs \((H_0, \bar{\omega}^2, \mathbb{C}^{2N})\) and \((H_{n-1}, \bar{\omega}^2, \mathbb{C}^{2N})\). Then,

\[
u = \langle \Lambda p(x, t_n), p(x, t_n) \rangle + \langle \bar{\Lambda} q(x, t_n), \bar{q}(x, t_n) \rangle \tag{66}
\]

is a finite parametric solution of the \((n-1)th\) DNLS equation (11).
Proof. On one hand, by (39), (40), and (66), a straightforward computation gives
\[
\frac{du}{d\tau} = 4i\left(Q_\lambda(\Lambda^3 p, p) + Q_\lambda(\bar{\Lambda}^3 \bar{q}, \bar{q}) \right) + 4\left(Q_\lambda(\Lambda^2 p, q) - Q_\lambda(\bar{\Lambda}^2 \bar{p}, \bar{q}) \right)
\times \left( \left(Q_\lambda(\Lambda^3 p, p) + Q_\lambda(\bar{\Lambda}^3 \bar{q}, \bar{q}) \right) - \lambda^2 \left(Q_\lambda(\Lambda p, p) + Q_\lambda(\bar{\Lambda} q, \bar{q}) \right) \right).
\]
(67)

On the other hand, by (15), a direct calculation yields
\[
\partial_x \left[ Q_\lambda(\Lambda p, p) + Q_\lambda(\bar{\Lambda} \bar{q}, \bar{q}) \right] = 2 \left( \langle \Lambda p, p \rangle + \langle \bar{\Lambda} \bar{q}, \bar{q} \rangle \right) \left(Q_\lambda(\Lambda^2 p, q) - Q_\lambda(\bar{\Lambda}^2 \bar{p}, \bar{q}) \right)
- 2i \left(Q_\lambda(\Lambda^3 p, p) + Q_\lambda(\bar{\Lambda}^3 \bar{q}, \bar{q}) \right).
\]
(68)

Recalling the formulas (8), (11), (26), (60), (62), (65), (67), and (68), we arrive at
\[
\frac{d}{d\lambda} \left( \frac{u}{-\bar{u}} \right) = -\frac{1}{2\hat{c}_\lambda} \frac{d}{d\tau} \left( \frac{u}{-\bar{u}} \right) = \frac{JG_\lambda}{\hat{c}_\lambda} = \frac{Jg_\lambda}{\hat{c}_\lambda} = \sum_{k=0}^\infty Jg_{k\lambda}^{-2k-2} = \sum_{k=0}^\infty X_k \lambda^{-2k-2},
\]
in view of the identity \( Q_\lambda(\Lambda^3 \xi, \eta) = \lambda^2 Q_\lambda(\Lambda \xi, \eta) = \langle \Lambda \xi, \eta \rangle \), which means that the finite parametric solution (66) solves the DNLS hierarchy (11).

Corollary 2. Let \((p(x, t), q(x, t))^T\) be an involutive solution of the complex integrable FDHSs (20) and (22). Thus,
\[
u = \langle \Lambda p(x, t), p(x, t) \rangle + \langle \bar{\Lambda} \bar{q}(x, t), \bar{q}(x, t) \rangle
\]
(69)
solves the DNLS equation (1).

Corollary 3. Let \((p(x, t_3), q(x, t_3))^T\) be an involutive solution of the complex integrable FDHSs (20) and (24). Thus,
\[
u = \langle \Lambda p(x, t_3), p(x, t_3) \rangle + \langle \bar{\Lambda} \bar{q}(x, t_3), \bar{q}(x, t_3) \rangle
\]
(70)
solves the second DNLS equation (13).

A solution is said to be a finite-gap potential if it satisfies a high-order stationary soliton (Novikov) equation. To solve the DNLS hierarchy, we propose the conception of complex Novikov equation, which specifies a finite-dimensional invariant subset for the DNLS flows.

Theorem 2. Let \((p(x), q(x))^T\) be a solution of the complex Hamiltonian system (20). Then
\[
u = \langle \Lambda p(x), p(x) \rangle + \langle \bar{\Lambda} \bar{q}(x), \bar{q}(x) \rangle
\]
(71)
is a finite-gap solution of the complex Novikov (high-order stationary DNLS) equation
\[
X_{2N} + \hat{c}_2 X_{2N-1} + \cdots + \hat{c}_{2N+1} X_0 = 0,
\]
(72)
where
\[
\hat{c}_j = \sum_{k=0}^{j-1} a_k \hat{c}_{j-k}, \quad j = 2, 3, \ldots, 2N + 1,
\]
\[ a_0 = 1, \quad a_1 = - \sum_{j=1}^{N} (\lambda_j^2 + \bar{\lambda}_j^2), \quad a_2 = \sum_{1 \leq i < j \leq N} (\lambda_i^2 \lambda_j^2 + \bar{\lambda}_i^2 \bar{\lambda}_j^2) + \sum_{i,j=1}^{N} \lambda_i^2 \bar{\lambda}_j^2, \]

\[ a_j = (-1)^j \sum_{1 \leq l_1 < \cdots < l_j \leq N} \left( \lambda_{l_1}^2 \cdots \lambda_{l_j}^2 + \bar{\lambda}_{l_1}^2 \cdots \bar{\lambda}_{l_j}^2 \right) + \sum_{1 \leq k_1 < \cdots < k_j \leq N} \lambda_{k_1}^2 \cdots \lambda_{k_j}^2 \lambda_{l_{j-1}}^2 \cdots \bar{\lambda}_{l_{j-1}}^2, \quad 3 \leq j \leq N, \]

\[ a_j = (-1)^j \sum_{j-N \leq s \leq j} \lambda_{k_1}^2 \cdots \lambda_{k_s}^2 \lambda_{l_1}^2 \cdots \lambda_{l_{j-s}}^2, \quad N+1 \leq j \leq 2N, \]

and \( \hat{c}_2, \hat{c}_3, \ldots, \hat{c}_{2N+1} \) are some constants of integration with a supplementary definition \( \hat{c}_1 = 1 \).

**Proof.** Expanding the polynomial \( a(\zeta) \) in powers of \( \zeta \) gives rise to

\[ a(\zeta) = \prod_{j=1}^{N} (\zeta - \lambda_j^2)(\zeta - \bar{\lambda}_j^2) = a_0 \zeta^{2N} + a_1 \zeta^{2N-1} + \cdots + a_{2N}. \]

Resorting to (18) and (61), we have

\[
0 = \sum_{j=1}^{N} \left[ a(\lambda_j^2) \nabla \lambda_j + a(\bar{\lambda}_j^2) \nabla \bar{\lambda}_j \right] \\
= \sum_{j=1}^{N} \left[ \left( \lambda_j^{4N} \nabla \lambda_j + \bar{\lambda}_j^{4N} \nabla \bar{\lambda}_j \right) + a_1 \left( \lambda_j^{4N-2} \nabla \lambda_j + \bar{\lambda}_j^{4N-2} \nabla \bar{\lambda}_j \right) + \cdots + a_{2N} \left( \nabla \lambda_j + \nabla \bar{\lambda}_j \right) \right] \\
= \left( g_{2N} + \hat{c}_2 g_{2N-1} + \cdots + \hat{c}_{2N+1} g_0 \right) + a_1 \left( g_{2N-1} + \hat{c}_2 g_{2N-2} + \cdots + \hat{c}_{2N} g_0 \right) + \cdots + a_{2N} g_0 \\
= g_{2N} + \hat{c}_2 g_{2N-1} + \cdots + \hat{c}_{2N+1} g_0. \quad (73)
\]

By applying the Lenard operator \( J \) on (73), we derive the complex Novikov equation (72). \( \blacksquare \)

### 5 | The Evolution of DNLS Flows

The most attractive point of the Liouville-Arnold theory is to find action-angle variables for linearizing various flows on a complex invariant torus.\(^{35}\) In fact, as for the classical integrable systems subjected to the inverse scattering transformation, the standard construction of action-angle variables using poles of the Baker-Akhiezer function is equivalent to the separation of variables.\(^{37}\) Note that the DNLS hierarchy has been reduced to a family of complex integrable FDHSs in Section 4. In the next step, we present a systematical way of constructing the Abel-Jacobi (or angle) variable to straighten out the DNLS flows on the Jacobi variety of an invariant Riemann surface.
Let \((\hat{a}_j, \hat{b}_j)_{j=1}^{2N-1}\) be a set of canonical basis of cycles on \(\Gamma\) with the intersection numbers
\[
\hat{a}_i \circ \hat{a}_j = \hat{b}_i \circ \hat{b}_j = 0, \quad \hat{a}_i \circ \hat{b}_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, 2N - 1.
\]
For later use, we denote
\[
\mathcal{A}_{ij} = \int_{\hat{a}_j} \omega_i, \quad \mathcal{A} = (\mathcal{A}_{ij})_{2N-1 \times 2N-1}, \quad 1 \leq i, j \leq 2N - 1.
\]
It follows from the Riemann bilinear relation\(^{32}\) that the matrix \(\mathcal{A}\) is nondegenerate. And then, its inverse exists \(\mathcal{A}^{-1} =: C = (C_1, C_2, \ldots, C_{2N-1})\), which normalize the basis of holomorphic differential \(\omega = (\omega_1, \omega_2, \ldots, \omega_{2N-1})^T\) as follows
\[
\omega = (\omega_1, \omega_2, \ldots, \omega_{2N-1})^T, \quad \omega_j = \sum_{l=1}^{2N-1} C_{jl} \omega_l, \quad 1 \leq j \leq 2N - 1,
\]
with the normalization
\[
\int_{\hat{a}_i} \omega_j = \sum_{l=1}^{2N-1} C_{jl} \int_{\hat{a}_i} \omega_l = \sum_{l=1}^{2N-1} C_{jl} \mathcal{A}_{li} = \delta_{ji} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}
\]
The integral of \(\omega\) along with the cycles \(\hat{a}_j\) and \(\hat{b}_j\) determines \(4N - 2\) periodic vectors
\[
\delta_j = \int_{\hat{a}_j} \omega, \quad B_j = \int_{\hat{b}_j} \omega, \quad 1 \leq j \leq 2N - 1,
\]
whose components are \(\delta_{ij} = \int_{\hat{a}_j} \omega_i\) and \(B_{ij} = \int_{\hat{b}_j} \omega_l\). It is known that \(\delta = (\delta_1, \delta_2, \ldots, \delta_{2N-1})\) is a unit matrix, and \(B = (B_1, B_2, \ldots, B_{2N-1})\) is a symmetric matrix with positive-definite imaginary part.\(^{32,33}\) Moreover, the matrix \(B\) is usually used to introduce the Riemann theta function\(^{32,33}\)
\[
\theta(\zeta; B) = \sum_{z \in \mathbb{Z}^{2N-1}} \exp \pi i (\langle B z, z \rangle + 2 \langle \zeta, z \rangle), \quad \zeta \in \mathbb{C}^{2N-1}.
\]
Let \(T\) be a lattice spanned by the \(4N - 2\) periodic vectors \(\{\delta_j, B_j\}\) in the complex space \(\mathbb{C}^{2N-1}\). The quotient space of \(\mathbb{C}^{2N-1}\) over \(T\) is called the Jacobi variety \(J(\Gamma) = \mathbb{C}^{2N-1}/T\) of \(\Gamma\). For a fixed point \(P_0 (\neq \infty)\) \((i = 1, 2)\); \(\lambda^2_j, \lambda^2_j (1 \leq j \leq N); \lambda^2_{N+k} (1 \leq k \leq 2N) \in \Gamma\), the Abel map \(A : \text{Div}(\Gamma) \to J(\Gamma)\) from the divisor group \(\text{Div}(\Gamma)\) of \(\Gamma\) to the Jacobi variety \(J(\Gamma)\) is defined by
\[
A(P) = \int_{P_0}^P \omega, \quad A \left( \sum_{k=1}^{2N-1} n_k P_k \right) = \sum_{k=1}^{2N-1} n_k A(P_k).
\]
By selecting the special divisor \(\sum_{k=1}^{2N-1} P(\mu^2_k)\), we suitably introduce the Abel-Jacobi variable
\[
\phi = A \left( \sum_{k=1}^{2N-1} P(\mu^2_k) \right) = \sum_{k=1}^{2N-1} \int_{P_0}^{P(\mu^2_k)} \omega = C \tilde{\phi}, \quad P(\mu^2_k) = (\mu^2_k, \xi(\mu^2_k)), \quad (74)
\]
which will be used to straighten out the complex Hamiltonian flows and the DNLS flows. To progress further, it is assumed that

$$S_k = \sum_{j=1}^{N} (\lambda_j^{2k} + \bar{\lambda}_j^{2k}) + \sum_{j=1}^{2N} \lambda_{j+N}^{2k}, \quad \tilde{R}(\zeta^{-1}) = \prod_{k=1}^{N} (1 - \lambda_k^2 \zeta^{-1})(1 - \overline{\lambda}_k^2 \zeta^{-1}) \prod_{k=1}^{2N} (1 - \lambda_{N+k}^2 \zeta^{-1}).$$

By the expansion of power series, we arrive at

$$\frac{1}{\sqrt{\tilde{R}(\zeta^{-1})}} = \sum_{k=0}^{\infty} A_k \zeta^{-k}, \quad |\lambda| > \max\{|\lambda_1|, \ldots, |\lambda_N|, |\lambda_{N+1}|, \ldots, |\lambda_{3N}|\}, \quad (75)$$

where

$$A_{-k} = 0 \ (k \geq 1), \quad A_0 = 1, \quad A_1 = \frac{1}{2} S_1, \quad A_2 = \frac{1}{8} (2S_2 + S_1^2),$$

$$A_k = \frac{1}{2k} \left(S_k + \sum_{i+j=k, i,j \geq 1} S_i A_j\right), \quad k \geq 3.$$

**Theorem 3.** Under the Abel-Jacobi variable $\phi$, the evolution behavior of complex Hamiltonian flows is described by

$$\frac{d\phi}{dt_\lambda} = \sum_{k=0}^{\infty} \Omega_k \lambda^{-2k-2}, \quad \frac{d\phi}{dt_{k+1}} = \Omega_k, \quad k \geq 0, \quad (76)$$

where

$$\Omega_0 = \frac{1}{2} C_1, \quad \Omega_1 = \frac{1}{2} (A_1 C_1 + C_2),$$

$$\Omega_k = \frac{1}{2} (A_k C_1 + \cdots + A_{k+1} C_{k+1}), \quad 2 \leq k \leq 2N - 2,$$

$$\Omega_k = \frac{1}{2} (A_k C_1 + \cdots + A_{k-2N+2} C_{2N-1}), \quad k \geq 2N - 1,$$

are the evolution velocities independent of flow variables.

**Proof.** According to (47) and (64), a simple calculation yields

$$\sqrt{R(\zeta)} = \tilde{c} \lambda a(\zeta). \quad (78)$$

Additionally, by combining the formulas (65), (74), (51), (53), (78), and (75), we get

$$\frac{d\phi}{dt_\lambda} = -\frac{1}{2\tilde{c}_\lambda} \frac{d\phi}{d\tau_\lambda} = -\frac{1}{2\tilde{c}_\lambda} C \frac{d\phi}{d\tau_\lambda}$$

$$= -\frac{1}{2\tilde{c}_\lambda} \left(C_1, C_2, \ldots, C_{2N-1}\right) \left(\frac{d\phi_1}{d\tau_\lambda}, \frac{d\phi_2}{d\tau_\lambda}, \ldots, \frac{d\phi_{2N-1}}{d\tau_\lambda}\right)^T$$

$$= \frac{1}{2\tilde{c}_\lambda} \left(C_1, C_2, \ldots, C_{2N-1}\right) \left(\frac{\zeta^{2N-1}}{a(\zeta)}, \frac{\zeta^{2N-2}}{a(\zeta)}, \ldots, \frac{\zeta}{a(\zeta)}\right)^T$$

$$= \frac{\zeta^{2N}}{2\tilde{c}_\lambda a(\zeta)} \sum_{j=1}^{2N-1} C_j \zeta^{-j} = \frac{1}{2\sqrt{R(\zeta^{-1})}} \sum_{j=1}^{2N-1} C_j \zeta^{-j}$$
\[ \frac{1}{2} \sum_{k=0}^{\infty} A_k \zeta^{-k} \sum_{j=1}^{2N-1} C_j \zeta^{-j} = \sum_{k=0}^{\infty} \Omega_k \zeta^{-k-1} = \sum_{k=0}^{\infty} \Omega_k \lambda^{-2k-2}, \]  

(79)

which signifies that the first formula of (76) holds true. From the definition of Poisson bracket and (38), we derive

\[ \frac{d\phi}{dt} = \{\phi, H_\lambda\} = \sum_{k=0}^{\infty} \{\phi, H_k\} \lambda^{-2k-2} = \sum_{k=0}^{\infty} \frac{d\phi}{dt} \lambda^{-2k-2}. \]  

(80)

The comparison of (79) and (80) of the same powers in \( \lambda \) leads to the second formula of (76).

6 | THE RIEMANN-JACOBI INVERSION

Apart from solitons, quasi-periodic solutions are another class of interesting exact solutions, which enjoys a much richer structure due to its connection to algebraic geometry inherent in its construction. The Bargmann map (19) delivers both the finite parametric solutions of the DNLS hierarchy and the finite-gap potential of the high-order stationary DNLS (or complex Novikov) equation. The significance of finite-gap is to specify a finite-dimensional invariant subspace from the infinite-dimensional function space, similar to the sufficiently fast-decaying initial data in the inverse scattering transformation. Moreover, the Lax matrix (28) determines a hyperelliptic curve of Riemann surface, whose genus coincides with the number of elliptic variables. We are now in a position to complete the Riemann-Jacobi inversion \( \phi \iff \{\mu_2^1, \mu_2^2, \ldots, \mu_2^{2N-1}\} \), such that the potential \( u \) is represented by Riemann theta functions.

To do so, first we represent the amplitude of potential \( u \) as a symmetric function of elliptic variables \( \{\mu_1^2, \mu_2^2, \ldots, \mu_2^{2N-1}\} \).

**Lemma 1.** Let \( |\lambda| > \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|\} \). Then,

\[ \partial_x \ln |u| = i \sum_{j=1}^{2N-1} \left( \mu_j^2 - \bar{\mu}_j^2 \right). \]  

(84)
Proof. Multiplied by \( a(\zeta) \) on both sides of (48), the expansion in \( \zeta \) of the right-hand side (RHS) of (48) gives

\[
\text{RHS} = (\langle \Lambda p, p \rangle + \langle \tilde{\Lambda} \tilde{q}, \tilde{q} \rangle) \left( \varepsilon^{2N-1} - \sum_{j=1}^{2N-1} \mu_j^2 \varepsilon^{2N-2} + \sum_{i<j} \mu_i^2 \mu_j^2 \varepsilon^{2N-3} - \cdots - \prod_{j=1}^{2N-1} \mu_j^2 \right), \tag{85}
\]
while the left-hand side (LHS) of (48) reads

\[
\text{LHS} = \sum_{k=0}^{\infty} (\langle \Lambda^{2k+1} p, p \rangle + \langle \tilde{\Lambda}^{2k+1} \tilde{q}, \tilde{q} \rangle) \varepsilon^{-k-1} \left[ \varepsilon^{2N} - \sum_{j=1}^{N} (\lambda_j^2 + \tilde{\lambda}_j^2) \varepsilon^{2N-1} + \left( \sum_{i<j} (\lambda_i^2 \lambda_j^2 + \tilde{\lambda}_i^2 \tilde{\lambda}_j^2) + \sum_{i,j=1}^{N} \lambda_i^2 \tilde{\lambda}_j^2 \right) \varepsilon^{2N-2} - \cdots - \prod_{j=1}^{N} |\lambda_j|^2 \right]. \tag{86}
\]

By comparing the coefficient of power \( \varepsilon^{2N-2} \) in (85) and (86), we derive

\[
\sum_{j=1}^{2N-1} \mu_j^2 = \sum_{j=1}^{N} (\lambda_j^2 + \tilde{\lambda}_j^2) - \frac{\langle \Lambda^3 p, p \rangle + \langle \tilde{\Lambda}^3 \tilde{q}, \tilde{q} \rangle}{\langle \Lambda p, p \rangle + \langle \tilde{\Lambda} \tilde{q}, \tilde{q} \rangle}. \tag{87}
\]

On the other hand, from (15) and (19), we have

\[
\partial_x \ln u = -2i \frac{\langle \Lambda^3 p, p \rangle + \langle \tilde{\Lambda}^3 \tilde{q}, \tilde{q} \rangle}{\langle \Lambda p, p \rangle + \langle \tilde{\Lambda} \tilde{q}, \tilde{q} \rangle} + 2 \left( \langle \Lambda^2 p, q \rangle - \langle \tilde{\Lambda}^2 \tilde{p}, \tilde{q} \rangle \right). \tag{88}
\]

The combination of (87), (88), and their conjugates immediately gives rise to (84). \( \blacksquare \)

Followed by Lemma 1, we do not need to figure out each elliptic variable \( \mu_j^2 \) (\( 1 \leq j \leq 2N - 1 \)), but the symmetric function of \( \{ \mu_k^2 \}_{k=1}^{2N-1} \). As a matter of the above fact, we turn to the Riemann theorem,\(^3\) for the Abel-Jacobi variable \( \phi \) defined by (74), there exists a vector of Riemann constant \( M = (M_1, M_2, \ldots, M_{2N-1})^T \in \mathbb{C}^{2N-1} \) such that

- \( f(\zeta) = : \theta(\mathcal{A}(P(\zeta)) - \phi - M) \) has \( 2N - 1 \) simple zeros at \( \{ \mu_1^2, \mu_2^2, \ldots, \mu_{2N-1}^2 \} \).

To make the function \( f(\zeta) \) single value, the Riemann surface \( \Gamma \) is suitably cut along with the cycles \( \{ a_j, b_j \}_{j=1}^{2N-1} \) to form a simply connected region, whose boundary \( \gamma \) is composed of \( 8N - 4 \) edges in the order \( a_1^+ b_1^+ a_1^- b_1^- a_2^+ b_2^+ a_2^- b_2^- \cdots a_{2N-1}^+ b_{2N-1}^+ a_{2N-1}^- b_{2N-1}^- \), where the symbols + and − represent the orientation. According to the calculation of residues of \( f(\zeta) \) at \( \infty_{1,2} \), the symmetric function of elliptic variables \( \{ \mu_k^2 \}_{k=1}^{2N-1} \) can be worked out in virtue of the inversion formula

\[
\sum_{j=1}^{2N-1} \mu_j^{2k} = I_k(\Gamma) - \sum_{s=1}^{2} \text{Res} \zeta^k d \ln f(\zeta), \quad k \geq 1, \tag{89}
\]
where

\[
I_k(\Gamma) = \frac{1}{2\pi i} \oint_{\gamma} \zeta^k d \ln f(\zeta) = \sum_{j=1}^{2N-1} \int_{\partial_j} \zeta^k \omega_j
\]
is a constant independent of the Abel-Jacobi variable \( \phi.\(^6\)
Let $z = \zeta^{-1}$ be a local coordinate and $\tilde{\xi} = z^{4N} \xi$. The affine equation of $\Gamma$ is thus placed into the pattern $\tilde{\xi}^2 + \tilde{R}(z) = 0$, and $\infty_{1,2}$ on the upper ($s = 2$) and lower ($s = 1$) sheets of $\Gamma$ are transformed to be two zeros

$$\infty_s = \left. \left( z, (-1)^s \sqrt{-\tilde{R}(z)} \right) \right|_{z=0} = (0, (-1)^s i), \quad s = 1, 2.$$  

**Lemma 2.** Near $\infty_s$ ($s = 1, 2$), $\omega$ has the asymptotic expansion ($z = \zeta^{-1}$)

$$\omega = \frac{(-1)^{s-1}}{2i} \sum_{k=0}^{\infty} \Omega_k z^k dz,$$

and $\mathcal{A}(P(\zeta))$ has the asymptotic expansion

$$\mathcal{A}(P(\zeta)) = \mathcal{A}(P(z^{-1})) = -\chi_s + \frac{(-1)^{s-1}}{2i} \sum_{k=0}^{\infty} \Omega_k z^{k+1}, \quad \chi_s = \int_{\infty_s}^{P_0} \omega. \quad (91)$$

**Proof.** By (74), (52), (75), and (79), after a direct calculation we have

$$\omega = C\tilde{\omega} = (C_1, C_2, \ldots, C_{2N-1})(\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_{2N-1})^T$$

$$= \frac{\tilde{\xi} 2N \tilde{\xi}^j}{4 \sqrt{-\tilde{R}(\xi)}} \sum_{j=1}^{2N-1} C_j \tilde{\xi}^{-j} = \frac{d\xi}{4i\xi} \frac{1}{\sqrt{R(\xi^{-1})}} \sum_{j=1}^{2N-1} C_j \xi^{-j}$$

$$= \frac{d\xi}{4i\xi} \sum_{k=0}^{\infty} \sum_{j=1}^{2N-1} C_j \xi^{-j} = \frac{\Omega_k}{2i} \sum_{k=0}^{\infty} \Omega_k z^{k+1} \xi^{-k+1}.$$  

Substituting (92) back into the expression $\mathcal{A}(P(\xi))$ results in (91), which completes the proof. \qed

For the brevity of huge formula, we adopt the Einstein summation convention, and denote the $j$th component of $f(\xi)$ by $f_j$, and $\partial_j = \partial / \partial f_j$, $\partial^2_j = \partial^2 / \partial f_j \partial f_k$, etc. In the neighborhood of $\lambda = \infty_s$ ($s = 1, 2$), by using Lemma 2 and (81), as $z = \zeta^{-1}$ the Riemann theta function $f(\xi)$ has the Maclaurin expansion

$$f(\xi) \big|_{\xi=\infty_s} = f(z^{-1}) \big|_{z=0_s} = \theta_s^{(\infty)}(\mathcal{A}(P(z^{-1})) - \phi - M)$$

$$= \theta_s^{(\infty)}(\phi + M + \chi_s) + \frac{(-1)^s}{2i} \Omega_0 \partial_j \theta_s^{(\infty)} z$$

$$- \left( \frac{1}{8} \Omega_0 \Omega_{0k} \partial^2_{jk} \theta_s^{(\infty)} - \frac{(-1)^s}{4i} \Omega_1 \partial_j \theta_s^{(\infty)} \right) z^2$$

$$+ \left( \frac{(-1)^{s+1}}{48i} \Omega_0 \Omega_{0k} \partial^3_{jkl} \theta_s^{(\infty)} - \frac{1}{8} \Omega_0 \Omega_{1k} \partial^2_{jk} \theta_s^{(\infty)} \right) z^3 + O(z^4)$$

$$= \theta_s^{(\infty)}(\phi + M + \chi_s) + \frac{(-1)^s}{2i} \partial_j \theta_s^{(\infty)} z - \frac{z^2}{8} \left( (-1)^s 2i \partial_j \theta_s^{(\infty)} + \partial^2_z \theta_s^{(\infty)} \right)$$
\[
\frac{d}{d\zeta} \ln f(\zeta) \bigg|_{\zeta=\infty} = \frac{-1}{2l} \partial_x \ln \theta^{(\infty)}_1 - \frac{z}{4} \left( -1 \right)^y 2i \partial_t \ln \theta^{(\infty)}_1 + \frac{1}{2} \partial_x \ln \theta^{(\infty)}_1 + O(z^2),
\]

which leads to

\[
\frac{d \ln f(\zeta)}{d\zeta} = \frac{-1}{2l} \partial_x \ln \theta^{(\infty)}_1 - \frac{z}{4} \left( -1 \right)^y 2i \partial_t \ln \theta^{(\infty)}_1 + \frac{1}{2} \partial_x \ln \theta^{(\infty)}_1 + O(z^2),
\]

in view of the property of even function in reference to the Riemann theta function. Combining (89) with (94), we attain the required trace formulas

\[
\sum_{j=1}^{2N-1} \mu_j^2 = I_1(\Gamma) + \frac{1}{2l} \partial_x \ln \frac{\theta^{(\infty)}_1}{\theta^{(\infty)}_0},
\]

\[
\sum_{j=1}^{2N-1} \mu_j^4 = I_2(\Gamma) + \frac{1}{4} \partial_x^2 \ln \theta^{(\infty)}_1 \theta^{(\infty)}_2 + \frac{i}{2} \partial_t \ln \frac{\theta^{(\infty)}_2}{\theta^{(\infty)}_1}.
\]

Based on Proposition 4, Theorems 2 and 3, substituting (95) into (84) generates the amplitude of quasi-periodic solutions to the DNLS hierarchy (11) in the form of Riemann theta functions

\[
|u(x, t_n)| = |u(0, t_n)| \left| \frac{\theta(\Omega_{0}x + \Omega_{n-1}t_n + \alpha_1) \theta(\Omega_{n-1}t_n + \alpha_2)}{\theta(\Omega_{0}x + \Omega_{n-1}t_n + \alpha_2) \theta(\Omega_{n-1}t_n + \alpha_1)} \right| e^{-2i \text{Im}(I_1(\Gamma)) x}, \quad n \geq 2,
\]

where \( \text{Im}(I_1(\Gamma)) \) stands for the imaginary part of \( I_1(\Gamma) \) and \( \alpha_i = \phi_0 + M + \chi_i \) \( i = 1, 2 \). Similar to the calculation of Theorem 11.1 in Ref. 28, with the help of equations (1), (13), and the complex Hamiltonian flow (24), particularly we achieve the Riemann theta function representation of the amplitude of quasi-periodic solution for the DNLS equation (1)

\[
|u(x, t)| = |u(0, 0)| \left| \frac{\theta(\Omega_{0}x + \Omega_{1}t + \alpha_1) \theta(\alpha_2)}{\theta(\Omega_{0}x + \Omega_{1}t + \alpha_2) \theta(\alpha_1)} \right| e^{-2i \text{Im}(I_1(\Gamma)) x + K t},
\]

where \( K = \frac{d}{dt} \left|_{t=0} \ln |u(0, t)| \right| \) is a constant independent of variables \( x \) and \( t \).

Compared with the derived \( N \)-phase solution of DNLS equation (see eqs. 98 and A13 in Ref. 19), the expression of (97) is simplified to be the amplitude of finite-gap potential with an explicit \( N \)-phase function, instead of the squared amplitude of solution to the DNLS equation. Moreover, followed by the commutability of complex Hamiltonian flows on \( (C^{2N}, \omega^2) \), we present a new formula for the squared amplitude of quasi-periodic solution of the DNLS equation (1).

It is checked from (4), (28), and (62) that

\[
a_{\lambda} \hat{c}_{\lambda} = -i - Q_{\lambda}(\Lambda^2 p, q) + Q_{\lambda}(\Lambda^2 \bar{p}, \bar{q}),
\]

which together with (19), (39), (40), and (65) result in

\[
\frac{d \ln u}{dt_{\lambda}} + 2a_{\lambda} = \frac{-2i \bar{m}(z)}{\sqrt{R(z)}}, \quad \bar{m}(z) = \prod_{k=1}^{2N-1} (1 - z \mu_k^2).
\]

It is noted in the neighborhood of \( z = 0 \) \( s = 1, 2 \) that

\[
\ln \frac{\bar{m}(z)}{\sqrt{R(z)}} = \ln \bar{m}(z) - \frac{1}{2} \ln R(z) = - \sum_{k=1}^{\infty} \frac{L_k}{k} z^k,
\]
where
\[ L_k = \frac{1}{2} \left[ \sum_{j=1}^{N} (\lambda_j^{2k} + \bar{\lambda}_j^{2k}) + \sum_{j=1}^{2N} \lambda_j^{2k} \right] - \sum_{j=1}^{2N-1} \mu_j^{2k}, \quad k \geq 1, \]

and in particular, recalling (95), the first two members \( L_1 \) and \( L_2 \) are of the forms
\[
L_1 = -\frac{1}{2i} \partial_x \ln \frac{\theta^{(\infty)}}{\theta_2^{(\infty)}} + D_1, \quad L_2 = -\frac{1}{4} \partial_x^2 \ln \theta_1^{(\infty)} \theta_2^{(\infty)} - \frac{i}{2} \partial_t \ln \frac{2}{\theta_1^{(\infty)}} + D_2. \quad (101)
\]

where
\[
D_k = \frac{1}{2} \left[ \sum_{j=1}^{N} (\lambda_j^{2k} + \bar{\lambda}_j^{2k}) + \sum_{j=1}^{2N} \lambda_j^{2k} \right] - I_k(\Gamma), \quad k = 1, 2, \ldots.
\]

Hence, by using (4), (38), (99), and (100), we arrive at
\[
-2i + \sum_{k=1}^{\infty} T_k z^k = -2i \left[ 1 + \sum_{k=1}^{\infty} \frac{L_k}{k} z^k + \frac{1}{2} \left( \sum_{k=1}^{\infty} \frac{L_k}{k} z^k \right)^2 + \cdots \right], \quad (102)
\]

where \( T_k = \frac{d \ln u}{dt_k} + 2a_{2k} \) is determined by the recursive formulas
\[
T_1 = -2i L_1, \quad T_2 = -i(L_2 + L_1^2), \quad kT_k = -2i L_k + \sum_{l+m=k \atop l, m \geq 1} T_l L_m, \quad k \geq 3, \quad (103)
\]

in terms of the expansion of power series at \( z = 0 \) (\( s = 1, 2 \)).

Resorting to the Lax equation (42), one gets
\[
\frac{dV_{\mu}^{11}}{d\tau_{\lambda}} = -2\lambda\mu \begin{vmatrix} V_{\lambda}^{12} & V_{\mu}^{12} \\ V_{\lambda}^{21} & V_{\mu}^{21} \end{vmatrix} = \frac{dV_{\lambda}^{11}}{d\tau_{\mu}},
\]

due to the symmetry of spectral parameters \( \lambda \) and \( \mu \). In addition, it is noted from \( V_{\lambda}^{12} = \hat{e}_\lambda g_{\lambda}^2, V_{\mu}^{21} = \hat{e}_\mu g_{\mu}^1 \) and (65) that
\[
\frac{d a_\mu}{dt_\lambda} = \frac{\lambda \mu}{\lambda^2 - \mu^2} \begin{vmatrix} g_{\lambda}^2 & g_{\mu}^2 \\ g_{\lambda}^1 & g_{\mu}^1 \end{vmatrix} = \frac{d a_\mu}{dt_\mu}, \quad (104)
\]

Simply substituting \( a_\lambda = \sum_{j=0}^{\infty} a_{2j} \lambda^{-2j} \) and \( \frac{d}{dt_\lambda} = \sum_{k=0}^{\infty} \frac{d}{dt_{k+1}} \lambda^{-2k-2} \) into (104) leads to
\[
\partial_{t_j} a_{2k} = \partial_{t_k} a_{2j}, \quad j, k \geq 1 \quad (105)
\]

because \( a_0 = -i \) is a complex constant. Specially, as \( j = 1 \) and \( k = 2 \), we obtain \( \partial_{t_1} a_4 = \partial_{t_2} a_2 \) because of the aforementioned notation \( t_1 = x \) and \( t_2 = t \).
Taking a partial derivative regarding $t$ on the first formula of (103), we have

$$\partial^2_{xt} \ln u + 2 \partial_x a_2 = \partial^2_{xt} \ln \frac{\theta^{(\infty)}_1}{\theta^{(\infty)}_2},$$

which yields

$$\partial_t \ln u + 2a_4 - \partial_t \ln \frac{\theta^{(\infty)}_1}{\theta^{(\infty)}_2} = \beta,$$  \hspace{1cm} (106)

where we have used the fact $\partial_x a_4 = \partial_x a_2$, and $\beta$ is independent of $x$. By using another partial derivative regarding $x$ on the first formula of (103) and (106), we have

$$(2i\partial_t - \partial^2_x) \ln \frac{\theta^{(\infty)}_1}{\theta^{(\infty)}_2} = 2i\partial_x \ln u + 4ia_4 - \partial^2_x \ln u - 2a_{2x} - 2i\beta.$$  \hspace{1cm} (107)

On the other hand, from the second formula of (103), a direct calculation gives rise to

$$(2i\partial_t - \partial^2_x) \ln \frac{\theta^{(\infty)}_1}{\theta^{(\infty)}_2} = 2\partial^2_x \ln \theta^{(\infty)}_2 + 4i\partial_t \ln u + 8ia_4 + T^2_1 - 4D_2.$$  \hspace{1cm} (108)

To progress further, from (1), (6), (103), (107), and (108), we arrive at

$$2\partial^2_x \ln \theta^{(\infty)}_2 + 2i\beta - 4D_2 = -2i\partial_t \ln u - 4ia_4 - T^2_1 - \partial^2_x \ln u - 2a_{2x}$$

$$= -iu\tilde{a}_x - \frac{1}{2}|u|^4.$$  \hspace{1cm} (109)

Again, the combination of (84), (95), (109), and its conjugate generates a new formula of the squared amplitude of quasi-periodic solution for the standard DNLS equation (1)

$$|u(x,t)|^2 = -4\text{Im}(\partial_x \ln \theta(\Omega_0x + \Omega_1 t + a_2)) + 4(2\text{Im}(D_2) - \text{Re}(\beta))x + \eta,$$  \hspace{1cm} (110)

where $\text{Re}(\beta)$ represents the real part of $\beta$, and $\eta$ is independent of $x$.

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