Subelliptic estimates for the $\bar{\partial}$-problem on complex algebraic surfaces with isolated singularities

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Abstract
We obtain subelliptic estimates for the $\bar{\partial}$-problem on complex algebraic surfaces embedded in $\mathbb{C}^n$ with isolated singularities. $W^\epsilon$ Sobolev norms of a form, $f$, for $0 < \epsilon < 1$ are estimated in terms of weighted $L^2$ norms of $\bar{\partial}f$ and $\bar{\partial}^*f$, with weights which vanish at the singularities, as well as weighted $L^2$ norms of $f$, with weights which blow up at the singularities.

1 Introduction
We let $X$ be an algebraic surface over $\mathbb{C}$ embedded in $\mathbb{C}^n$ with isolated singularities. The main goal of this article is to obtain subelliptic estimates for the $\bar{\partial}$-problem on $X$. Subelliptic estimates are an important topic in the theory of the $\bar{\partial}$-Neumann problem, and in particular provide regularity of the solution to the $\bar{\partial}$-problem. Whereas on smooth domains where such estimates are related to the geometry of the boundary, on complex spaces with isolated singularities little is known about the regularity of the $\bar{\partial}$-problem. There has however been significant progress made in the study of the $L^2$-cohomology for the $\bar{\partial}$-operator, \[9\] (see also \[11\]).

A study of subelliptic estimates on complex spaces with isolated singularities was initiated in \[8\] with the example $z^2 = xy$ in $\mathbb{C}^3$. The current article builds off the idea of the work on that example but achieves considerably more generality. Namely, we obtain results which apply to all complex algebraic surfaces with isolated singularities.

We work with some of the following simplifications, without any loss of generality. As the theory of the $\bar{\partial}$-Neumann problem is well established on
smooth manifolds, we will work in a neighborhood of an isolated singularity. We assume the singularity lies at the origin and $D^n(1) \cap (X - 0)$, where $D^n(1)$ is the ball of radius 1, contains no other singularities.

For our main result, let $f$ be a $(p, q)$-form ($0 \leq p, q \leq 2$) with support in $D^n(1) \cap X$ and with $f \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*)$, that is $\overline{\partial} f \in L^2_{(p, q+1)}(X)$ and $\overline{\partial}^* f \in L^2_{(p, q-1)}(X)$. For some $0 < \epsilon \leq 1$, we also suppose $(r^\epsilon \log(r))^{-1} f \in L^2_{(p, q)}(X)$, where $r = |z|$. We will often drop the designation of the form type in the notation of the Sobolev spaces; thus, $L^2(X)$ will also stand, for instance, for $L^2((0, 1))$ where appropriate. We establish the

**Main Theorem.** For $f$ as above we have $f \in W^\epsilon(X)$ with the estimates

$$\|f\|_{W^\epsilon(X)} \lesssim \|r^{1-\epsilon} \overline{\partial} f\|_{L^2(X)} + \|r^{1-\epsilon} \overline{\partial}^* f\|_{L^2(X)} + \frac{1}{r^\epsilon \log(r)} \|f\|_{L^2(X)}.$$  

The intermediate Sobolev norms are defined by interpolation (see [1]), and the first step in the proof is to establish $W^1(X)$ estimates (so that interpolation can follow). The works of [4] and the related [8] are essential in our use of coordinates which are particularly helpful in establishing estimates for the intermediate norms.

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### 2 Sobolev 1 estimate

If we first assume that a smooth form, $f$, is supported in a neighborhood of the origin, but away from the singularity at 0, we can follow the establishment of the Morrey-Kohn-Hömander identity (see for instance [2] Proposition 4.3.1 and Proposition 5.1.1), using integration by parts to show

$$\|f\|_{W^1(X)} \leq C_1 \left(\|\overline{\partial} f\|_{L^2(X)}^2 + \|\overline{\partial}^* f\|_{L^2(X)}^2\right) + C_2 \|f\|_{L^2(X)}.$$  

(2.1)

The constants, $C_1$ and $C_2$, in the above inequality, however, may depend on $f$; for instance, it is not known beforehand that a finite number of charts suffices to cover a neighborhood of the singularity, the constants of the above inequality depending on derivatives of local cutoffs subordinate to the charts.

Our strategy is to cover $D^n(1) \cap (X - 0)$, with neighborhoods such that each neighborhood is contained in a chart (to which we will refer here as a resolution chart) obtained by a resolution of the singularity at the origin. A resolution leads to a finite number of charts which cover $U = D^n(1) \cap (X - 0)$. The charts are of the form

$$z_1 = u^n v^m,$$

$$z_2 = f_2(z_1) + u^{n_2} v^{m_2},$$

$$z_i = f_i(z_1) + u^{n_i} v^{m_i},$$  

(2.2)

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where \( n_i \geq n_j, m_i \geq m_j \) for \( i > j \), with \( n_1 m_i - n_i m_1 \neq 0 \) for \( i \neq 1 \), \( f_i \) and \( f'_i \) are holomorphic functions of \( u \) and \( v \), and \( f_i = o(z_1) \), and the \( g_i \) are local units (are holomorphic with non-zero constant terms in their series expansions), see [4]. We refer the reader to [5] for background information on the resolution of singularities via quadratic transformations.

Set \( N = \partial U = \partial D^n(1) \cap (X - 0) \). From [4], there is a piecewise smooth diffeomorphism,

\[
h : N \times (0, 1] \rightarrow D^n(1) \cap (X - 0). \tag{2.3}
\]

The diffeomorphism is obtained by covering \( \partial U \) with neighborhoods, and in each neighborhood following flow lines from points on \( \partial U \) to the origin (the flow lines are piecewise smooth). The covering of \( \partial U \) is chosen so that each neighborhood and its trace along flow lines to the origin is contained in a coordinate chart obtained by resolving the singularity at \( 0 \) with repeated quadratic transformations, and so the traces along flow lines of the covering of \( \partial U \) constitute a covering of \( U \), written as \( U = \cup U_i \). From above, \( U_i \) is covered by a resolution chart, and we can write \( U_i = \cup_{j=1}^{l_i} U_{ij} \), where the \( U_{ij} \) are the regions on which \( h^{-1} \) is smooth, with the property that \( U_{ij} \cap U_{ij'} = \partial U_{ij} \cap \partial U_{ij'} \), for \( k \neq l \). Since the \( U_i \) constitute a finite covering, we have

\[
\|f\|_{W^1(U)} \simeq \sum_i \|f\|_{W^1(U_i)},
\]

and if we write \( L^i_1 \) and \( L^i_2 \) for the holomorphic vector fields on \( U_i \), we can then write for a function, \( f \),

\[
\|f\|_{W^1(U_i)} \simeq \sum_{j=1}^{j_i} \sum_{k=1}^{2} \|L^i_k f\|_{L^2(U_{ij})} + \|\mathcal{L}_k f\|_{L^2(U_{ij})} + \|f\|_{L^2(U_i)}.
\]

We thus have

\[
\|f\|_{W^1(U)} \simeq \sum_i \sum_{j=1}^{j_i} \sum_{k=1}^{2} \|L^i_k f\|_{L^2(U_{ij})} + \|\mathcal{L}_k f\|_{L^2(U_{ij})} + \|f\|_{L^2(U_i)}. \tag{2.4}
\]

Using a family of smooth cutoff functions, \( \{\varphi_i\} \), subordinate to the above covering of \( \partial U \), we then extend the cutoff functions along the flow lines in each \( U_i \). With this construction, each \( \varphi_i \) will depend only on variables \( \theta^i, x^i, y^i \) on \( N = \partial U \) (see also the next section), and derivatives of \( \varphi_i \) are bounded. We can thus write

\[
\|f\|_{W^2(U)} \simeq \sum_{i,j} \|\varphi_i f\|_{W^2(U_{ij})}.
\]

In what follows, we shall write \( f^i \) for \( \varphi_i f \). The \( \varphi_i \) functions allow us to assume without loss of generality that \( \text{supp}(f^i) \cap \partial U_i \subset \{0\} \), which we shall do in the next paragraph.

We now return to (2.1), which was obtained with the assumption of support away from the singularity. In order to obtain estimates for all \( f \), without restricting the support away from 0, we use cutoffs \( \mu_k \) as in [10]. \( \mu_k = \mu_k(|z|) \) is
a smooth function with the property $\mu_k = 0$ for $|z| < e^{-e^k + 1}$ and $\mu_k = 1$ for $|z| > e^{-e^k}$. Furthermore,

$$|d\mu_k(z)| \lesssim \frac{\chi_k(|z|^2)}{|z| \log |z|},$$

where $\chi_k$ is the characteristic function of $[e^{-e^k + 1}, e^{-e^k}]$. From (2.4), we have

$$\|\mu f\|_{W^1(U)}^2 \simeq \sum_{i,j} \sum_{l,k} \|L_{k} \mu_l f\|_{L^2(U_{ij})}^2 + \|\bar{L}_k \mu_l f\|_{L^2(U_{ij})}^2 + \|\mu f\|_{L^2(U)}.$$

The above inequalities for Sobolev norms on functions can be applied to the case of forms component-wise. Then following the proof of the Morrey-Kohn-H"ormander identity (see [2] Proposition 4.3.1 and Proposition 5.1.1), we can integrate by parts in an expression for

$$\|\bar{\partial} (\mu_k f)\|_{L^2(X)}^2 + \|\bar{\partial}^* (\mu_k f)\|_{L^2(X)}^2 + \|\mu_k \bar{\partial} f\|_{L^2(X)}^2 \lesssim \frac{\chi_k(r^2)}{r \log(r)} f \left\| \frac{1}{r \log(r)} f \right\|_{L^2(X)},$$

which also takes into account estimates away from the singularity (obtained by classical estimates on smooth manifolds).

Define $X_k := X \cap \text{supp}(\mu_k)$. Note that

$$\|f\|_{W^1(X_k)}^2 \lesssim \|\mu_{k+1} f\|_{W^1(X)}^2$$

and for $f \in W^1(X)$,

$$\|f\|_{W^1(X_k)}^2 = \lim_{k \to \infty} \|f\|_{W^1(X_k)}^2$$

so that letting $k \to \infty$ in (2.5) leads to

$$\|f\|_{W^1(X)}^2 \lesssim \left( \|\bar{\partial} f\|_{L^2(X)}^2 + \|\bar{\partial}^* f\|_{L^2(X)}^2 \right) + \left\| \frac{1}{r \log(r)} f \right\|_{L^2(X)}^2.$$  \hspace{1cm} (2.6)

### 3 Useful coordinates

In (2.4), we estimate the $W^1(X)$ norm by finding holomorphic vector fields and estimating $L^2$ norms of the vector fields (and their conjugates) acting on a function (or form). In this section we use quasi-isometries proved in [4] and updated in [7] and [8] to define an equivalent norm to that in (2.4) which will aid in our comparison of intermediate (between 0 and 1) Sobolev norms with that of $W^1(X)$.  


With
\[ r = \left( \sum_{i=1}^{n} |z_i|^2 \right)^{1/2}, \]
a coordinate system over the region \( U_i \) may be chosen in which \( r \) is one co-ordinate and local coordinates \( \theta^i, x^i, y^i \) on \( N = \partial U \) form the other. This is the coordinate system used to describe the diffeomorphism, \( h \), in (2.3). The pullback of
\[ dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + \cdots + dz_n d\bar{z}_n, \tag{3.1} \]
under \( h \) is quasi-isometric to either a metric of Cheeger type [4],
\[ dr^2 + r^2 (d\theta^i)^2 + r^{2\alpha_i} ((dx^i)^2 + (dy^i)^2), \tag{3.2} \]
in a region \( U^i \), where \( \alpha_i \geq 1 \), or a metric of the form
\[ dr^2 + r^2 (d\theta^i)^2 + r^{2\alpha_i} (ds^i)^2 + (r^b_i + s^i)^2 (d\phi^i)^2) \tag{3.3} \]
[8] (with \( x^i = s^i \cos(\phi^i), y^i = s^i \sin(\phi^i) \)) for \( 0 < b^i < 1 \).

The respective volume forms over a region \( U_i \) have the property
\[ dV^i \simeq r^{2\alpha_i + 1} dr d\theta^i dx^i dy^i, \tag{3.4} \]
using (3.2), or
\[ dV^i \simeq r^{2\alpha_i + 1} (r^b_i + s^i) dr d\theta^i ds^i d\phi^i, \tag{3.5} \]
using (3.3).

We can thus estimate \( \|f\|_{W^1(X)} \) as defined by (2.4) with the use of vector fields
\[ \partial_r, \frac{1}{r} \partial_{\theta^i}, \frac{1}{r^{\alpha_i}} \partial_{x^i}, \frac{1}{r^{\alpha_i}} \partial_{y^i} \tag{3.6} \]
and the volume form in (3.4), respectively
\[ \partial_r, \frac{1}{r} \partial_{\theta^i}, \frac{1}{r^{\alpha_i}} \partial_{x^i}, \frac{1}{r^{\alpha_i}} \frac{1}{r^b_i + s^i} \partial_{\phi^i} \tag{3.7} \]
and the volume form in (3.5), over a region, \( U_i \).

4 Intermediate Sobolev spaces

In this section we look at intermediate Sobolev spaces, between \( L^2(X) \) and \( W^1(X) \), where \( W^1(X) \) is defined in (2.4). We obtain estimates for the intermediate spaces in terms of \( L^2(X) \) and \( W^1(X) \) norms.

Recall that \( U = D^n(1) \cap (X - 0) \), which is diffeomorphic, via \( h \) in (2.3), to \( \partial U \times (0, 1) \). We note that

**Lemma 4.1.** \( W^1(U) \) is dense in \( L^2(U) \).
Proof. For any \( \varepsilon > 0 \), denote \( U_\varepsilon = h(N \times (\varepsilon, 1)) \). The lemma follows from the density of \( C^\infty_0(U_\varepsilon) \) in \( L^2(U_\varepsilon) \) (\( U_\varepsilon \) is a complex manifold, without singularities) and the density of \( L^2(U_\varepsilon) \) in \( L^2(U) \), using extensions by zero to relate a function in \( L^2(U_\varepsilon) \) to one in \( L^2(U) \).

Using the density of \( W^1(U) \) in \( L^2(U) \), there exists a positive, self-adjoint operator, \( \Lambda \), such that

\[
(u, v)_{W^1(U)} = (\Lambda u, \Lambda v)_{L^2(U)} \quad \forall u, v \in W^1(U).
\]

Intermediate Sobolev norms can be defined in terms of powers of the operator \( \Lambda \) as in \[6\] (Section 2.1). We denote these spaces by \( W^\varepsilon(U) \):

\[
W^\varepsilon(U) = \text{dom}(\Lambda^\varepsilon)
\]

for \( 0 \leq \varepsilon \leq 1 \), with norm

\[
\|f\|_{W^\varepsilon(U)} = \|f\|_{L^2(U)} + \|\Lambda^\varepsilon f\|_{L^2(U)}.
\]

There are some techniques which allow one to interpolate between the spaces. Two methods, the "K-method" and the "J-method," can be used to interpolate between \( L^2(U) \) and \( W^1(U) \), each with equivalent norms. We refer the reader to \[1\] for a description of these methods. From Theorem 15.1 in \[6\] the spaces \( W^\varepsilon(U) \) as defined above are equivalent to those produced by using interpolation via the K-method, and give equivalent norms. Furthermore, we can use a norm described as in \[1\] to calculate the K-method norms. To calculate the \( W^\varepsilon \)-norm \((0 < \varepsilon < 1)\) of a function, \( f \), known to be in \( W^1(U) \), we will use that

\[
I(u) = \left( \int_0^\infty \|t^{1-\varepsilon}u(t)\|^2_{W^1(X)} t^{-1} dt \right)^{1/2} + \left( \int_0^\infty \|t^{1-\varepsilon}u'(t)\|^2_{L^2(X)} t^{-1} dt \right)^{1/2} \quad (4.1)
\]

taken over all

\[u : [0, \infty) \to L^2(X) + W^1(X)\]

with \( u(0) = f \) such that \( u(t) \) is locally integrable in \( W^1(X) \), and \( u'(t) \), defined in the distributional sense, is locally integrable in \( L^2(X) \), and such that the two terms on the right-hand side of (4.1) are finite (see Section 3.12, in particular Theorem 3.12.2, in \[1\]).

We use this to show the

**Theorem 4.2.** Let \( X \) be a complex surface with singularities. Let \( W^1(X) \) be defined by \[2.4\], and the intermediate Sobolev spaces \( W^\varepsilon(X) \) as above. Then for \( f \) a smooth function in \( U = D^n(1) \cap (X - 0) \), we have the estimates

\[
\|f\|_{W^\varepsilon(X)} \lesssim \|r^{1-\varepsilon}f\|_{W^1(X)} + \|r^{-\varepsilon}f\|_{L^2(X)},
\]

for \( 0 < \varepsilon < 1 \).

The estimate is obviously true for functions supported away from a singularity and so we prove Theorem 4.2 for functions supported in a neighborhood of the singularity.
Proof. As in Section 2, we assume \( x = 0 \in X \) is a singular point, and there are no other singularities in \( U \). We write \( U = \bigcup_i U_i \), where \( U_i \) is covered by a resolution chart. Further recall that we write \( U_i = \bigcup_{j=1}^{r_i} U_{ij} \), where the \( U_{ij} \) are the regions on which \( h^{-1} \) is smooth, where \( h \) is the diffeomorphism from Section 2.

We assume \( f \) is supported in \( D^n(s) \cap X \) for some \( s < 1 \). Over \( U_i \) we write \( f^i := f \big|_{U_i} \) in terms of coordinates \( r, \theta^i, x^i, y^i \): \( f^i = f_i^i(r, \theta^i, x^i, y^i) \). Define

\[
  u^i(t) = f^i(r + t, \theta^i, x^i, y^i) \eta(t),
\]

where \( \eta(t) \in C^\infty_0(\mathbb{R}^+) \) with the properties \( \eta(t) \equiv 1 \) near \( t = 0 \) and \( \eta(t) \equiv 0 \) for \( t > \delta \), where \( \delta \) is chosen small enough so that the support of \( f \) is contained in \( h(N \times (0, 1 - \delta)) \). Note that \( u^i(0) = f^i \) holds. We also use the properties of the volume form \( dV^i \) given in (3.4) and (3.5).

For the first term on the right-hand side of (4.1), we calculate

\[
  \int_0^\infty \|t^{1-\epsilon} u^i(t)\|^2_{W^1(X)} t^{-1} dt \lesssim \int_0^\delta \int_0^1 \|\partial_r u^i\|^2_{L^2(U^i_{ij})} r^{2\alpha_i+1} dr + \int_0^\delta \int_0^1 \|u^i\|^2_{W^1(U^i_{ij})} r^{2\alpha_i+1} dr dt.
\]

where \( L^1_i \) and \( L^2_i \) are as in Section 2.

Let \( U_i^\sigma \) denote the slice \( U_i \cap h(N \times \{\sigma\}) \) and \( U_{ij}^\sigma = U_{ij} \cap U_i^\sigma \). Thus, for instance, \( U^1_i \) is \( U_i \cap \partial U \). We first handle the case of (3.4). We separate the integrals over the tangential components using the properties of (3.4) and (3.5) for the volume element:

\[
  \int_{U_{ij}} \left( |L^1_k u^i|^2 + |L^2_k u^i|^2 \right) dV^j \simeq \int_0^1 \|\partial_r u^i\|^2_{L^2(U^i_{ij})} r^{2\alpha_i+1} dr + \int_0^1 \|u^i\|^2_{W^1(U^i_{ij})} r^{2\alpha_i+1} dr.
\]

We make a change of variables \( r' = r + t \), and with this change we estimate

\[
  \int_0^\delta \int_0^1 \|\partial_r u^i\|^2_{L^2(U^i_{ij})} r^{2\alpha_i+1} t^{1-2\epsilon} dt dr \
  \lesssim \int_0^\delta r^2(t) \int_0^1 \|\partial_{r'} f^i\|^2_{L^2(U^i_{ij})} (r' - t)^{2\alpha_i+1} t^{1-2\epsilon} dr' dt.
\]
We now change the order of the $r'$ and $t$ integrations, and estimate

$$
\int_0^\delta \int_0^{r'} \left\| \partial_{r'} f^i \right\|_{L^2(U_{r'})}^2 (r' - t)^{2\alpha_i + 1} r t^{1 - 2\varepsilon} dt dr' + \int_\delta^1 \int_0^{r'} \left\| \partial_{r'} f^i \right\|_{L^2(U_{r'})}^2 (r' - t)^{2\alpha_i + 1} r t^{1 - 2\varepsilon} dt dr'. \quad (4.4)
$$

We use

$$
\int_0^{r'} (r' - t)^{2\alpha_i + 1} r t^{1 - 2\varepsilon} dt \lesssim (r')^{2\alpha_i + 3 - 2\varepsilon}
$$

and

$$
\int_0^\delta (r' - t)^{2\alpha_i + 1} r t^{1 - 2\varepsilon} dt \lesssim (r')^{2\alpha_i + 3 - 2\varepsilon}
$$

since $\delta < r'$ in the second integral in (4.4).

The integrals in (4.3) can now be bounded by

$$
\int_0^1 \left\| \partial_{r'} f^i \right\|_{L^2(U_{r'})}^2 (r')^{2\alpha_i + 3 - 2\varepsilon} dr' = \int_0^1 \left\| \partial_{r} f^i \right\|_{L^2(U_{r})}^2 r^{2\alpha_i + 3 - 2\varepsilon} dr,
$$

and we have

$$
\int_0^\delta \int_0^1 \left\| \partial_{r'} u^i \right\|_{L^2(U_{r'})}^2 r^{2\alpha_i + 1} t^{1 - 2\varepsilon} dt dr \lesssim \int_0^1 \left\| \partial_{r} f^i \right\|_{L^2(U_{r})}^2 r^{2\alpha_i + 3 - 2\varepsilon} dr.
$$

We can similarly estimate

$$
\int_0^\delta \int_0^1 \left\| D^i_x u^i \right\|_{L^2(U_{r'})}^2 r^{2\alpha_i + 1} t^{1 - 2\varepsilon} dt dr,
$$

where $D^i_x$ is one of the last three vector fields in (3.6). For example, with $D^i_x = r^{-\alpha_i} \partial_{x^i}$, we have

$$
\int_0^\delta \int_0^1 \left\| D^i_x u^i \right\|_{L^2(U_{r'})}^2 r^{2\alpha_i + 1} t^{1 - 2\varepsilon} dt dr \lesssim \int_0^1 \left\| \partial_{x^i} f^i \right\|_{L^2(U_{r})}^2 r^{2 - 2\varepsilon} dr
$$

and proceeding as above leads to

$$
\int_0^\delta \int_0^1 \left\| \partial_{x^i} u^i \right\|_{L^2(U_{r'})}^2 t^{1 - 2\varepsilon} dr dt \lesssim \int_0^1 \left\| \partial_{x^i} f^i \right\|_{L^2(U_{r})}^2 r^{2 - 2\varepsilon} dr
$$

$$
\lesssim \int_0^1 \left\| r^{-\alpha_i} \partial_{x^i} f^i \right\|_{L^2(U_{r'})}^2 r^{2\alpha_i + 1} r^{2 - 2\varepsilon} dr.
$$

Thus the second integral on the right-hand side of (4.3) inserted into (4.2) is bounded by

$$
\int_0^1 \left\| f^i \right\|_{W^2(U_{r})}^2 r^{2\alpha_i + 1} r^{2 - 2\varepsilon} dr.
$$
Similar estimates for the second integral on the right hand side of (4.2) yield
\[
\int_0^\delta t^{1-2\varepsilon} \|u^i\|^2_{L^2(U_{ij})} dt \lesssim \|r^{-1-\varepsilon} f^i\|_{L^2(U_{ij})}^2.
\]
Putting all this together in (4.2) yields
\[
\int_0^\infty \|t^{1-\varepsilon} u(t)\|^2_{W^1(U_{ij})} t^{-1} dt \lesssim \int_0^1 \|\partial_s f^i\|^2_{L^2(U_{ij})} r^{2\alpha + 1 + 2 - 2\varepsilon} dr + \int_0^1 \|f^i\|^2_{W^1(U_{ij})} r^{2\alpha + 1 + 2 - 2\varepsilon} dr \lesssim \|r^{-1-\varepsilon} f\|_{W^1(U_{ij})} + \|r^{-\varepsilon} f\|_{L^2(U_{ij})}.
\]
In the case the metric is isometric to (3.3), we proceed in a similar manner. We have
\[
\sum_i \int_{U_{ij}} \left( |L_i^k u^i|^2 + |\mathcal{T}_k u^i|^2 \right) dV^i \simeq \sum_{k=1}^4 \int_{U_{ij}} |D_k^i u^i|^2 dV^i,
\]
where $D_k^i$ are the vector fields in (3.7). Let us handle the cases $D_k^i = \frac{1}{r^\alpha} \partial_s^i$, and $D_k^i = \frac{1}{r^\alpha} \partial_{s_i}^i$. We first estimate
\[
\int_0^\delta \int_0^1 \int \left( \frac{1}{r^\alpha} \partial_s u^i \right)^2 (r^b + s^i) r^{2\alpha + 1 + 2 - 2\varepsilon} ds d\varphi d\theta dt dr d\theta.
\]
As above, we change the order of integration and estimate
\[
\int_0^\delta \int_0^1 \int \left( \partial_s u^i \right)^2 ((r' - t)^b + s^i)(r' - t)^{1-2\varepsilon} ds d\varphi d\theta dt dr d\theta.
\]
We use
\[
\int_0^\delta \int_0^1 \int \left( \partial_s u^i \right)^2 ((r' - t)^b + s^i)(r' - t)^{1-\varepsilon} dt dr d\theta.
\]
and
\[
\int_{\delta}^{1} \int_{0}^{1} |\partial_{x} u|^{2} ((r' - t)^{b_i} + s^{i}) (r' - t) t^{1 - \epsilon} dt d\delta' \lesssim \\
\int_{\delta}^{1} |\partial_{r} f|^{2} ((r' - t)^{b_i} + s^{i}) (r')^{3 - 2\epsilon} d\delta'.
\]

We have
\[
\int_{0}^{\delta} \int_{0}^{1} \left| \frac{1}{r^{\alpha_{i}}} \partial_{\varphi} u \right|^{2} (r^{b_{i}} + s^{i}) t^{1 - 2\epsilon} ds d\varphi^{i} d\theta^{i} dr \lesssim \\
\int_{0}^{\delta} \int_{0}^{1} \left| \partial_{r} f \right|^{2} (r^{b_{i}} + s^{i}) r^{3 - 2\epsilon} dr ds d\varphi^{i} d\theta^{i} \\
= \int_{0}^{\delta} \int_{0}^{1} \left| \frac{1}{r^{\alpha_{i}}} \partial_{\varphi} f \right|^{2} (r^{b_{i}} + s^{i}) t^{2 - \epsilon} ds d\varphi^{i} d\theta^{i} \\
\sim \int_{0}^{\delta} \left| \frac{1}{r^{\alpha_{i}}} \partial_{\varphi} f \right|^{2} r^{2 - \epsilon} dV^{i}.
\]

Similarly, in the case \( D_{k}^{i} = \frac{1}{r^{\alpha_{i}}}, \frac{1}{r^{\alpha_{i}}} \partial_{\varphi} \) we estimate
\[
\int_{0}^{\delta} \int_{0}^{1} \left| \partial_{\varphi} u \right|^{2} (r^{b_{i}} + s^{i})^{-1} rt^{1 - 2\epsilon} ds d\varphi^{i} d\theta^{i} dr dt \\
= \int_{0}^{\delta} \int_{0}^{1} \left| \partial_{r} u \right|^{2} \frac{(r' - t)}{(r' - t)^{b_{i}} + s^{i}} t^{1 - 2\epsilon} ds d\varphi^{i} d\theta^{i} dr dt.
\]

As above, we change the order of integration and estimate
\[
\int \int_{0}^{\delta} \int_{0}^{1} \left| \partial_{\varphi} u \right|^{2} \frac{(r' - t)}{(r' - t)^{b_{i}} + s^{i}} t^{1 - 2\epsilon} dr dt ds d\varphi^{i} d\theta^{i} \\
+ \int \int_{0}^{\delta} \int_{0}^{1} \left| \partial_{r} u \right|^{2} \frac{(r' - t)}{(r' - t)^{b_{i}} + s^{i}} t^{1 - 2\epsilon} dr dt ds d\varphi^{i} d\theta^{i}.
\]

We use
\[
\frac{(r' - t)}{(r' - t)^{b_{i}} + s^{i}} \lesssim \frac{r^{b_{i}}}{(r' - t)^{b_{i}} + s^{i}}
\]
(for \( r' > t \) and \( 0 < b_{i} < 1 \)) in both integrals to estimate
\[
\int_{0}^{\delta} \int_{0}^{1} \left| \partial_{\varphi} u \right|^{2} (r^{b_{i}} + s^{i})^{-1} rt^{1 - 2\epsilon} ds d\varphi^{i} d\theta^{i} dr dt \\
\lesssim \int_{0}^{1} \left| \partial_{r} f \right|^{2} \frac{r}{r^{b_{i}} + s^{i}} t^{2 - 2\epsilon} dr ds d\varphi^{i} d\theta^{i} \\
\lesssim \int \left| \frac{1}{r^{\alpha_{i}}} \frac{1}{(r^{b_{i}} + s^{i})} \partial_{\varphi} f \right|^{2} r^{2 - 2\epsilon} dV^{i}.
\]
The other vector fields in (3.7) are handled similarly and we obtain in the case the metric over \( U_i \) is quasi-isometric to (3.3)

\[
\int_0^\infty \| t^{1-\epsilon} u(t) \|_{W^1(U_{ij})}^2 t^{-1} dt \lesssim \| r^{1-\epsilon} f \|_{W^1(U_{ij})} + \| r^{-\epsilon} f \|_{L^2(U_{ij})}^2
\]
as above.

For the second integral in (4.1) we use

\[
\partial_t u^i(t) = (\partial_t f^i) \eta(t) + f^i \eta'(t) = (\partial_r f^i) \eta(t) + f^i \eta'(t).
\]

Therefore,

\[
\int_0^\infty \| t^{1-\epsilon} \partial_t u^i(t) \|_{L^2(X)}^2 t^{-1} dt \lesssim \sum_{ij} \int_0^\delta \int_{U_{ij}} |\partial_r f^i|^2 t^{1-2\epsilon} dV^i dt
\]

\[
+ \sum_{ij} \int_0^\delta \int_{U_{ij}} |f^i|^2 t^{1-2\epsilon} dV^i dt. \quad (4.5)
\]

Using a change of coordinates \( r = r' + t \) as above in the first integral above yields

\[
\int_0^\delta \int_{U_{ij}} |\partial_r f^i|^2 t^{1-2\epsilon} dV^i dt \lesssim \int |\partial_r f^i|^2 t^{2-2\epsilon} dV^i
\]

\[
\lesssim \| r^{1-\epsilon} f \|_{W^1(U_{ij})} + \| r^{-\epsilon} f \|_{L^2(U_{ij})}.
\]

Summing over \( U_{ij} \) shows the first integral on the right-hand side of (4.5) is bounded by

\[
\| r^{1-\epsilon} f \|_{W^1(X)} + \| r^{-\epsilon} f \|_{L^2(X)}.
\]

Estimates for the second term on the right of (4.5) follow as those above, and in terms of integrals over \( X \), we have

\[
\int_0^\delta \| t^{1-\epsilon} \partial_t u^i(t) \|_{L^2(X)}^2 t^{-1} dt \lesssim \| r^{1-\epsilon} f \|_{W^1(X)} + \| r^{-\epsilon} f \|_{L^2(X)}.
\]  

\[\square\]

5 Approximation by smooth forms

Theorem 4.2 was proved under the condition of smoothness of the function to be estimated. We use an approximation argument in this section to broaden the class of functions (or forms) to which the theorem applies.

Let us define the weighted \( L^2 \) norms

\[
L^{2,-C}(X) := \{ f \in L^2(X) : r^{-C} f \in L^2(X) \}
\]
for $C > 1$. The motivation for the weights, $r^{-C}$, for $C > 1$ comes from the $L^2$ norms on the right-hand side of (2.6). Recall that weighted norms for forms are defined by the norm for functions applied component-wise. Thus, for instance, $L^2_{(p,q)}(X)$ is defined to consist of those $(p,q)$-forms such that each component is in $L^2_{2,-C}(X)$.

We use the cutoffs $\mu_k$ from Section 2. Recall $\mu_k = \mu_k(|z|)$ is a smooth function such that $\mu_k = 0$ for $|z| < e^{-Ck^+}$ and $\mu_k = 1$ for $|z| > e^{-C}$, and with the property

$$|d\mu_k(z)| \lesssim \frac{\chi_k(|z|^2)}{|z| \log |z|},$$

where $\chi_k$ is the characteristic function of $[e^{-Ck^+}, e^{-C}]$. With $f_k := \mu_k f$ we have

**Lemma 5.1.** Let $C > 1$ and $f$ with support near the origin be such that $f \in L^2_{(p,q)}(X)$, $\bar{\partial} f \in L^2_{(p,q+1)}(X)$, and $\bar{\partial}^* f \in L^2_{(p,q-1)}(X)$. Then

$$r^{-C} f_k \to r^{-C} f \quad \text{in} \quad L^2_{(p,q)}(X),$$

and

$$\bar{\partial} f_k \to \bar{\partial} f \quad \text{in} \quad L^2_{(p,q+1)}(X)$$

$$\bar{\partial}^* f_k \to \bar{\partial}^* f \quad \text{in} \quad L^2_{(p,q-1)}(X).$$

Furthermore,

$$f_k \overset{W^1_{2,1}}{\to} f.$$  

**Proof.** The limit (5.2) is clear. To show (5.3) we estimate

$$\|\bar{\partial} \mu_k f\|_{L^2(X)} \leq \|\mu_k \bar{\partial} f\|_{L^2(X)} + \|\bar{\partial} \mu_k \wedge f\|_{L^2(X)}$$

$$\lesssim \|\mu_k \bar{\partial} f\|_{L^2(X)} + \left\| \frac{\chi_k(r^2)}{r \log(r)} f_k \right\|_{L^2(X)},$$

$$\lesssim \|\mu_k \bar{\partial} f\|_{L^2(X)} + \left\| r^{-C} f_k \right\|_{L^2(X)}$$

using (5.1).

Now letting $k \to \infty$ we have $\|\mu_k \bar{\partial} f\|_{L^2} \to \|\bar{\partial} f\|_{L^2}$ from the dominated convergence theorem. Thus $\|\bar{\partial} f_k\|_{L^2} \to \|\bar{\partial} f\|_{L^2}$ and a similar argument shows $\bar{\partial}^* f_k \overset{L^2}{\to} \bar{\partial}^* f$. When combined with (2.6), (5.2) and (5.3) yield (5.4).
the use of mollifiers so that

\[ r^{-C} f_k^\delta \to r^{-C} f_k \]
\[ \bar{\partial} f_k^\delta \to \bar{\partial} f_k \]
\[ \bar{\partial}^* f_k^\delta \to \bar{\partial}^* f_k \]

(as \( \delta \to 0 \)).

Letting \( \varphi = f_k^\delta \) in the above estimates and letting \( \delta \to 0 \), gives

\[ \| f_k \|_{W^1_p(X)} \lesssim \| \bar{\partial} f_k \|_{L^2_p(X)} + \| \bar{\partial}^* f_k \|_{L^2_p(X)} + \left\| \frac{1}{r \log(r)} f_k \right\|_{L^2_p(X)}. \]

We can now apply Lemma 5.1 and let \( k \to \infty \). We obtain for \( f \in L^{2-C}_{(p,q)}(X) \) \((0 \leq p, q \leq 2)\), with \( \bar{\partial} f \in L^2_{(p,q+1)}(X) \), and \( \bar{\partial}^* f \in L^2_{(p,q-1)}(X) \), that \( f \in W^1_{(p,q)}(X) \) with estimates

\[ \| f \|_{W^1_p(X)} \lesssim \| \bar{\partial} f \|_{L^2_p(X)} + \| \bar{\partial}^* f \|_{L^2_p(X)} + \left\| \frac{1}{r \log(r)} f \right\|_{L^2_p(X)}. \]  (5.5)

Combining (5.5) with Theorem 4.2, we have

**Theorem 5.2.** Let \( C > 1 \). For \( f \) supported near the origin such that \( f \in L^{2-C}_{(p,q)}(X) \) and \( f \in L^2_{(p,q)}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*) \) we have

\[ \| f \|_{W^\epsilon_p(X)} \lesssim \| r^{1-\epsilon} \bar{\partial} f \|_{L^2_p(X)} + \| r^{1-\epsilon} \bar{\partial}^* f \|_{L^2_p(X)} + \left\| \frac{1}{r \log(r)} f \right\|_{L^2_p(X)} \]

for \( 0 < \epsilon \leq 1 \).

If we fix \( \epsilon \), we can replace the hypothesis that \( f \in L^{2-C}_{(p,q)}(X) \) with \( \frac{1}{r \log(r)} f \in L^2_{(p,q)}(X) \) as in the Main Theorem. The theorem can easily be extended to the case of forms whose support contains multiple singularities of \( X \).

As a final corollary we relate \( \| \frac{1}{r \log(r)} f \|_{L^2_p(X)} \) estimates to \( L^p \)-estimates.

With

\[ \frac{1}{p} + \frac{1}{p'} = 1 \]

we have for a function, \( f \in L^{2p}(X) \) with support near 0,

\[ \left\| \frac{1}{r \log(r)} f \right\|_{L^2_p(X)}^2 \lesssim \| r^{-\delta} f \|_{L^2_p(X)}^2 \]

\[ \lesssim \sum_i \left( \int_{U_i} |r^{-2p' \delta}| dV^i \right)^{1/p'} \left( \int_{U_i} |f|^2 p dV^i \right)^{1/p} \]

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for $\delta > \epsilon$. Using (3.4) for each $U_i$, the integration of $r^{-2p/\delta}$ converges for $p' < 2/\delta$, which corresponds to $p > 2/(2 - \delta)$.

We apply the above with a $\delta > 1$. We conclude that with $f \in L^{2p}(X)$ for $p > 2$, we have that $f$ is also in $L^{2, -C}(X)$ for some $C > 1$. If in addition, $\bar{\partial}f \in L^2(X)$, and $\bar{\partial}^* f \in L^2(X)$, then Theorem 5.2 applies, and, since $\|r^{-\epsilon} f\| \lesssim \|r^{-\delta} f\|$, we have the following

**Corollary 5.3.** For a form, $f$, supported near the origin such that $f \in L^{p}(X) \cap \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$ with $p > 4$,

$$\|f\|_{W^{1, \epsilon}(X)} \lesssim \left\| r^{1-\epsilon} \bar{\partial} f \right\|_{L^2(X)} + \left\| r^{1-\epsilon} \bar{\partial}^* f \right\|_{L^2(X)} + \|f\|_{L^p(X)}$$

for $0 < \epsilon \leq 1$.

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