Tilted Dirac cone effects in superconducting phase transitions in planar four-fermion models

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The chiral and superconducting gaps are studied in the context of a planar fermion model with four-fermion interactions. The effect of the tilt of the Dirac cone on both gaps is shown and discussed. The changes caused by the presence of a non-null chemical potential are also analyzed. Our results point to two different behaviors exhibited by planar fermionic systems. For values of the effective tilt parameter \( |\tilde{t}| < \tilde{t}^* \), where \( \tilde{t}^* \approx 0.55 \), the superconducting phase persists for negative values of the superconducting coupling constant. For positive values of the superconducting coupling constant, the induction of a superconducting gap by a chemical potential exists and which is similar as seen in graphene-like systems. For \( |\tilde{t}| > \tilde{t}^* \) and for a positive superconducting coupling constant, the superconducting phase can be present, but it is restricted to a smaller area in the phase portrait. Finally, our analysis shows that, in the case of positive values for the superconducting coupling constant, the induction of a superconducting gap in the presence of a chemical potential is ruled out and the increase of the chemical potential works in favor of the manifestation of a metallic phase.

I. INTRODUCTION

Since the seminal work of Gross and Neveu [1], where the authors use quantum field theory (QFT) tools to describe two-dimensional massless fermions with quartic interactions, much attention was expended to apply QFT techniques in low dimensional systems and with special attention to condensed matter problems. One of the most interesting examples of applications of QFT in condensed matter is to the study of graphene [2]. In this almost planar system, the electrons obey linearly dispersing relations and the fermionic excitations are well described by a relativistic Dirac equation in (2+1)-dimensions. The Lorentz symmetry is respected by the electrons in graphene due to its relativistic characteristics, but this feature is an exception compared to the majority of materials in condensed matter. Some of the condensed matter systems, where the dispersion in the proximity of band touching points can be generically linear and resemble the Weyl equation, do not respect Lorentz symmetry. Even though there are quasiparticles in the aforementioned systems that behave like Weyl fermions [3], these systems are described by Weyl-like Hamiltonians and, thus, these quasiparticles are by construction massless and more stable against gap formation in comparison to Dirac ones [4].

Our proposal in this paper is to study how the properties associated with Weyl fermions influence the formation of chiral and superconducting gaps. We will extend the usual Weyl Hamiltonian used in the description of Weyl semimetals [5, 6] by introducing two forms of four-fermion interactions that will allow for a chiral and a superconducting phases. We also analyze the properties of this system under the effects of a finite chemical potential, which in practice models the doping process. This will allow us to study the phase transitions in a (2+1)-dimensional GN-type model which describes the competition between the chiral symmetry breaking and superconductivity. These two phenomena will dispute the true ground state of the system through the intensity of the coupling constants and as a function of the chemical potential. Let us recall also that chiral symmetry and its breaking can be seen as a way of describing the metal-insulator phase transition in these planar systems. Thus, the study of chiral symmetry breaking in planar systems by GN-like four-fermion interactions has become a useful tool for qualitative analysis of the two-dimensional system and which have been used successfully already in many different contexts [7–25].

In this paper we also want the address the question of the production of a superconducting phase in the model and how the tilting of the Dirac cone affects it. The phenomenon of electron pairing in the vast majority of superconductors follows the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity. The BCS theory describes the condensation of electrons into pairs with antiparallel spins in a singlet state with a \( s \)-wave symmetry. The \( s \)-wave channel will be the superconducting channel that will be addressed in this paper. Several works have already indicated that superconductivity appears in planar systems, such as twisted bilayer graphene [26], normal trilayer graphene [27] and twisted trilayer graphene as well [28, 29]. However, to our knowledge, there has been so far no previous study of the effects caused by the tilt of the Dirac cone on the combined chiral and superconducting phases and how it might influence in particular the superconducting gap. This is the topic that we will be covering in this paper.

The remainder of this paper is organized as follows. In Sec. 11 we show the main properties of two-dimensional Dirac and Weyl semimetal systems. In Sec. 111 we present the extension of the model which describes the four-fermion interactions for the excitonic and superconducting channels and find the effective thermody-
The tilt vector we recover the Hamiltonian of the isotropic graphene. Points, denoted by of the non-null tilt term in Eq. (2.1) is that the Dirac cones in the Weyl semi-metal. A consequence of the Hamiltonian given by Eq. (2.1), one finds that the spectrum is described through the Hamiltonian

$$H_i(p) = v_F \left[ (\mathbf{t} \cdot \mathbf{p}) \tau^0 + (\xi_x p_x) \tau^x + (\xi_y p_y) \tau^y \right], \quad (2.1)$$

where $v_F$ is the Fermi velocity, $t$ is called the tilt vector and which describes the Dirac cone tilt, $\xi = (\xi_x, \xi_y)$ is the vector which describes the anisotropy of the material, $\tau^0 = \mathbb{1}$ is the $2 \times 2$ identity matrix and $\tau^x, \tau^y$ are the Pauli matrices. In the limit $t \to 0$ and $\xi_x = \xi_y = 1$, we recover the Hamiltonian of the isotropic graphene. The tilt vector $t$ is related to the separation between the Dirac cones in the Weyl semi-metal. A consequence of the non-null tilt term in Eq. (2.1) is that the Dirac points, denoted by $D$ and $D'$, no longer coincide with the Brillouin corners $K$ and $K'$ (see, e.g., Ref. [2]). In particular, type-I Weyl semi-metals are characterized by $|t| < 1$, while type-II ones by $|t| > 1$. From the Hamiltonian given by Eq. (2.1), one finds that the spectrum is given by

$$E_{\lambda}(p) = v_F \left[ (\mathbf{t} \cdot \mathbf{p}) + \lambda \sqrt{(\xi_x p_x)^2 + (\xi_y p_y)^2} \right], \quad (2.2)$$

where $\lambda = \pm 1$ represents the conduction and valence bands, respectively. Note that to be able to associate $\lambda = +1$ to a positive and $\lambda = -1$ to a negative energy state, it is required that $|t| < 1$, (2.3)

where $|t|$ is called the effective tilt parameter. In particular, when the condition Eq. (2.3) is not respected, the isoenergetic lines are no longer ellipses, but instead, they are hyperboles.

The Hamiltonian Eq. (2.1) commutes with the chirality operator defined as

$$\mathcal{C} = \frac{(\xi_x p_x) \tau^x + (\xi_y p_y) \tau^y}{\sqrt{(\xi_x p_x)^2 + (\xi_y p_y)^2}}, \quad (2.4)$$

with the eigenvalues given by $\alpha = \pm 1$. Taking into account the two-fold degeneracy of the Dirac cones $D$ and $D'$, the free Weyl fermion can be described with a four-component spinor and a Dirac-like Lagrangian density can be written as follows (see also, e.g., Ref. [30]),

$$\mathcal{L} = \sum_{k=1}^{N} i \bar{\psi}_k M^{\mu \nu} \gamma^\mu \partial_\nu \psi_k, \quad (2.5)$$

where $\psi$ is a four-component Dirac fermion,

$$\gamma^\mu = \tau^\mu \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.6)$$

with $\tau^\mu = (\tau_z, i \tau_x, i \tau_y)$, $\bar{\psi} = \psi^\dagger \gamma^0$ and $\tau_z$ is the third Pauli matrix. The $\gamma$-matrices respect the algebra $\{\gamma^\mu, \gamma^\nu\} = 2 \eta^{\mu \nu}$, with $\eta^{\mu \nu} = \text{diag}(+,-,-)$. We also have that the matrix $M$ in Eq. (2.5) is given by

$$M = \begin{pmatrix} 1 & -v_F \xi_x & -v_F \xi_y \\ -v_F \xi_x & 0 & 0 \\ -v_F \xi_y & 0 & 0 \end{pmatrix}. \quad (2.7)$$

One should note that $M^{\mu \nu}$ contains the parameters that explicitly break the Lorentz symmetry. It is easy to show that the Lagrangian density Eq. (2.5) has a discrete chiral symmetry given by $\psi \to i \gamma_5 \psi$ and $\bar{\psi} \to -\bar{\psi} i \gamma_5$, with

$$i \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \quad (2.8)$$

Throughout the following sections one follows the Ref. [30] and choose the mass term which breaks the chiral symmetry as $\bar{\psi} \psi$.

III. CHIRAL AND DIFFERMION INTERACTIONS

In order to write a Lagrangian density that describes the (2+1)-dimensional Weyl semi-metal with both chiral symmetry breaking (excitonic pairing) and superconductivity (Cooper pairing), two forms of four-fermion interactions are introduced [13]: The four fermion interaction for the scalar fermion-antifermion and for the scalar dfermion channel. The complete model can then be written as

$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k (i M^{\mu \nu} \gamma_\mu \partial_\nu + \gamma_0 \mu) \psi_k + \frac{G_1 v_F}{2N} \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2 + \frac{G_2 v_F}{2N} \sum_{k=1}^{N} (\bar{\psi}_k C \psi_k) \sum_{j=1}^{N} (\bar{\psi}_j C \psi_j^j), \quad (3.1)$$

where $C$ is a chiral matrix.
where $C = i\tau^2$ is the charge conjugation matrix and $G_1$ and $G_2$ are the coupling constants for the chiral and dieron channels. The effective action of the model can be expressed as

$$
\exp(iS_{eff}) = \int D\hat{\psi} D\psi D\Delta D\Delta^* D\sigma
\times \exp\left\{ \int d^3x \left[ -\frac{N}{2G_1 v_F} \sigma^2 - \frac{N}{2G_2 v_F} \Delta^* \Delta
+ \sum_{k=1}^N \tilde{\sigma}_k (iM^{\mu\nu} \gamma_\mu \partial_\nu + \gamma^0 \mu + \sigma) \psi_k
+ \frac{\Delta^*}{2} \xi_k \bar{C} \psi_k + \frac{\Delta}{2} \bar{\psi}_k \tilde{C} \xi_k^T \right] \right\},
$$

(3.2)

where $\sigma = \frac{G_{eff}}{N} \sum_{j=1}^N \tilde{\psi}_j \tilde{\psi}_j$, $\Delta = \frac{G_{eff}}{N} \sum_{j=1}^N \bar{\psi}_j \tilde{C} \psi_j$ and $\Delta^* = \frac{G_{eff}}{N} \sum_{j=1}^N \bar{\psi}_j \bar{C} \psi_j$. We can explicitly integrate over the fermion field (for the technical details, see App. A) and the effective action can be rewritten as

$$
S_{eff}(\sigma, \Delta, \Delta^*) = N \int d^3x \Omega(\sigma, \Delta, \Delta^*),
$$

(3.9)

where $\Omega$ is the effective thermodynamics potential

$$
\Omega(\sigma, \Delta, \Delta^*) = -\left( \frac{1}{2G_1 v_F} \sigma^2 + \frac{1}{2G_2 v_F} \Delta^* \Delta \right)
+ \sum_{i=1}^2 \int \frac{d^3p}{(2\pi)^3} \ln \lambda_i(p),
$$

(3.3)

with $\lambda_i$ denoting the eigenvalues of $B = CDC^{-1}D^T - |\Delta|^2$, with $D = M^{\mu\nu} \gamma_\mu \partial_\nu + \gamma^0 \mu - \sigma$, which are given by

$$
\lambda_{1,2} = \sigma^2 + (p_0 - v_F (t \cdot \mathbf{p}))^2 - v_F^2 |\mathbf{p}|^2 - \mu^2 - |\Delta|^2
+ 2\sqrt{\sigma^2((p_0 - v_F (t \cdot \mathbf{p}))^2 - v_F^2 |\mathbf{p}|^2) + v_F^2 \mu^2 |\mathbf{p}|^2}.
$$

(3.4)

Using the identity

$$
\int_{-\infty}^{\infty} dp_0 \ln(p_0 - A) = i\pi |A|,
$$

(3.5)

we find that

$$
\sum_{i=1}^2 \int \frac{d^3p}{(2\pi)^3} \ln \lambda_i(p) = -\int \frac{d^3p}{(2\pi)^3} (|\Sigma^+| + |\Sigma^-|),
$$

(3.6)

where

$$
\Sigma^\pm = v_F (t \cdot \mathbf{p}) + \sqrt{\mathbf{E}^2 + \mu^2 + |\Delta|^2 \pm 2\sqrt{\sigma^2 |\Delta|^2 + \mu^2 \mathbf{E}^2}},
$$

(3.7)

and $E^2 = v_F^2 |\mathbf{p}|^2 + \sigma^2$. Finally, for constant configurations $\sigma_0 = \langle \sigma \rangle$ and $\Delta_0 = \langle \Delta \rangle$, we find

$$
\Omega(\sigma_0, \Delta_0, \mu) = -\left( \frac{1}{2G_1 v_F} \sigma_0^2 + \frac{1}{2G_2 v_F} \Delta_0^2 \right)
- \int \frac{d^3p}{(2\pi)^3} (|\Sigma^+_0| + |\Sigma^-_0|).\tag{3.8}
$$

where $\Sigma^\pm_0 = \Sigma^\pm(\sigma = \sigma_0, \Delta = \Delta_0)$. Note that the momentum integral in Eq. (3.8) is divergent in the ultraviolet and, thus, the effective potential Eq. (3.8) requires to be renormalized. The renormalization of Eq. (3.8) is described below.

### A. Renormalization

Taking $\mu = 0$ in Eq. (3.8), we will have that $\Sigma^\pm_0 = t \cdot \mathbf{p} + \sqrt{\mathbf{p}^2 + (\sigma_0 + \Delta_0)^2}$ and, therefore,

$$
\Omega(\sigma_0, \Delta_0) = -\left( \frac{1}{2G_1 v_F} \sigma_0^2 + \frac{1}{2G_2 v_F} \Delta_0^2 \right)
- \int \frac{d^2p}{(2\pi)^2} \varepsilon_F (t \cdot \mathbf{p}) + \sqrt{v_F^2 |\mathbf{p}|^2 + (\sigma_0 + \Delta_0)^2},
$$

(3.9)

The linear term $t \cdot \mathbf{p}$ in Eq. (3.9) vanishes in the integration over the angular variable $\xi_{x,y}$ but the integral in Eq. (3.9) is still divergent. Thus, applying the re-scaling $\xi_{x,y} \rightarrow \kappa_{x,y}$ and integrating with the introduction of a momentum cutoff $\Lambda$, one defines the renormalization conditions,

$$
\frac{1}{g_1(m)} = \frac{d^2 \Omega(\sigma_0, \Delta_0)}{d\sigma_0^2} \bigg|_{\sigma_0=m, \Delta_0=0} = \frac{1}{G_1} + \frac{2m}{\varepsilon_F \xi_{x,y}} - \frac{\Lambda}{\varepsilon_F \xi_{x,y}},
$$

(3.10)

and

$$
\frac{1}{g_2(m')} = \frac{d^2 \Omega(\sigma_0, \Delta_0)}{d\Delta_0^2} \bigg|_{\sigma_0=0, \Delta_0=m'} = \frac{1}{G_2} + \frac{2m'}{\varepsilon_F \xi_{x,y}} - \frac{\Lambda}{\varepsilon_F \xi_{x,y}},
$$

(3.11)

where $m$ and $m'$ are regularization scales. Going further, defining the renormalized couplings $g_1$ and $g_2$ as

$$
\frac{1}{g_1} = \frac{1}{g_1(m)} - \frac{2m}{\varepsilon_F \xi_{x,y}},
$$

(3.12)

and

$$
\frac{1}{g_2} = \frac{1}{g_2(m')} - \frac{2m'}{\varepsilon_F \xi_{x,y}},
$$

(3.13)

the renormalized effective thermodynamic potential becomes

$$
\Omega^{\text{ren}}(\sigma_0, \Delta_0) = -\frac{1}{2g_1 v_F} \sigma_0^2 - \frac{1}{2g_2 v_F} \Delta_0^2
+ \frac{(\sigma_0 + \Delta_0)^3}{6\pi v_F^2 \xi_{x,y}} + \frac{|\sigma_0 - \Delta_0|^3}{6\pi v_F^2 \xi_{x,y}}.
$$

(3.14)

1 We use the identity $\int_{0}^{2\pi} d\theta |a \cos \theta + b| = 2\pi b \Theta(b-a)$, where $\Theta(x)$ is the Heaviside function, for $a > 0$ and $b > 0$. 
A. The pure chiral phase \((\Delta_0 = 0)\)

By considering the pure chiral phase, i.e., by considering \(\Delta_0 = 0\), one notices that

\[
\langle \Sigma_0 \rangle_{\Delta_0=0} = E_{\sigma} = v_F (\mathbf{t} \cdot \mathbf{p}) + |\tilde{E}_0| \pm |\mu|,
\]

where \(\tilde{E}_0 = \sqrt{v_F^2 |\mathbf{p}|^2 + \sigma_0^2}\). Assuming \(\mu > 0\), one finds in this case that the effective thermodynamic potential \((3.14)\) becomes

\[
\Omega^{ren}(\sigma_0, 0, \mu) = -\frac{\sigma_0^2}{2 g_1 v_F} + \frac{\sigma_0^3}{3 \pi v_F^2 \xi_x \xi_y} + \int \frac{d^2 p}{(2\pi)^2} \left(|E^{+}_\sigma| + |E^{-}_\sigma| - 2E_\sigma\right)
\]

\[
= -\frac{\sigma_0^2}{2 g_1 v_F} + \frac{\sigma_0^3}{3 \pi v_F^2 \xi_x \xi_y} - \int \frac{d^2 p}{(2\pi)^2} \left(\mu - E_\sigma + |\mu - \tilde{E}_\sigma|\right)
\]

Figure 1. Phase diagram of the model at \(\mu = 0\).

IV. EFFECTS DUE TO THE CHEMICAL POTENTIAL

Let us now turn on the chemical potential and analyze its effects on the different phases represented in Fig. 1.

B. Phase diagram of the system at \(\mu = 0\)

Let us first specialize the analysis of the effective thermodynamic potential and their properties for the case of a null chemical potential. In this perspective, we analyze the two sectors, the chiral and the superconductor ones, individually to extract the main characteristics of the model. After this analysis, we can compare the results and show where each phase will be mandatory in the system.

The minima of Eq. \((3.14)\) are given in terms of the gaps \(\sigma_c\) and \(\Delta_c\), which are defined as \(\sigma_c = \pi v_F \xi_x \xi_y / |g_1|\) and \(\Delta_c = \pi v_F \xi_x \xi_y / |g_2|\). The symmetric phase, where \(\sigma_0 = \Delta_0 = 0\), takes place for \(g_1 > 0\) and \(g_2 > 0\) and it is called phase I. The non-trivial minimum of the effective thermodynamic potential is given by \(\{\sigma_0 = \sigma_c, \Delta_0 = 0\}\) or \(\{\sigma_0 = 0, \Delta_0 = \Delta_c\}\). The chiral and superconducting phases have a non-trivial global minimum for \(g_1 < 0\) and \(g_2 < 0\), denoted as phases II and III, respectively. The different possible phases for the model, in the plane \((g_1, g_2)\), are sketched in Fig. 1 (see also Ref. [13] for a similar description for the non-tilted and isotropic system).
when \( g_2 > 0 \) and

\[
n(g_1 < 0) = -\frac{N(\mu^2 - \mu_0^2)}{2\pi v_F^2 E_x E_y (1 - |\mathbf{t}|^2)^{3/2}} \sum (\mu^2 - \mu_0^2),
\]

when \( g_1 < 0 \).

In the next subsection one turns to the analysis of the superconducting phase.

**B. The pure superconducting phase \((\sigma_0 = 0)\)**

In the case of a pure superconducting phase, i.e., considering now \( \sigma_0 = 0 \), and using the identity

\[
(S_0^\dagger)|_{\sigma_0=0} = \xi^2 \Delta = v_F(t \cdot p) + \sqrt{(v_F|\mathbf{p}| + \mu)^2 + \Delta_0^2},
\]

the effective thermodynamic potential \((4.14)\) can be written as (for details, see App. [2])

\[
\Omega^{ren}(0, \Delta_0, \mu) = -\frac{\Delta_0^2}{2g_2v_F} + \frac{\Delta_0^3}{3v_F^2 E_x E_y} + \sum_{\eta=\pm 1} \frac{1}{2\pi E_x E_y} I(\Delta_0, \mu),
\]

where \( I(\Delta_0, \mu) \) is derived explicitly in the App. [B] and given by Eq. (5.8). Performing the momentum integrals in Eq. (4.9), one finds

\[
\Omega^{ren}(0, \Delta_0, \mu) = -\frac{\Delta_0^2}{2g_2v_F} + \frac{(\mu^2 + \Delta_0^2)^{3/2}}{3v_F^2 E_x E_y} - \frac{\mu^2 \Delta_0^2}{2v_F^2 E_x E_y} \ln \left[ \frac{\mu + \sqrt{\mu^2 + \Delta_0^2}}{\Delta_0} \right] + \frac{1}{2\pi E_x E_y} I(\Delta_0, \mu).
\]

The effective thermodynamic potential Eq. (4.10) is plotted in Fig. 2 for both the cases of \( g_2 > 0 \) (panel a) and for \( g_2 < 0 \) (panel b). It is noted the emergence of a superconducting gap \( \Delta \neq 0 \) due to the combined effect of the tilt and chemical potential. Let us analyze in more details the contribution of the tilt parameter for the superconducting gap. New features generated by the tilt of the Dirac cone will influence the superconducting gap for both the \( g_2 > 0 \) and \( g_2 < 0 \) scenarios and are explained below.

From the effective thermodynamic potential one derives the gap equation,

\[
\text{sign}(g_2) + \frac{x^2}{\sqrt{x^2 + y^2}} - y \ln \left[ \frac{y + \sqrt{x^2 + y^2}}{x} \right] + \frac{1}{2x} \frac{\partial I(x, y)}{\partial x} = 0,
\]

with \( x = \Delta_0/\Delta_c \), \( y = \mu/\Delta_c \) and \( I(x, y) \) is given by

\[
I(x, y) = (2x^2 - y^2 + 2z_+^2 - yz_+) \sqrt{x^2 + (z_+ - y)^2} + 3x^2 y \tanh^{-1} \left( \frac{z_+ - y}{\sqrt{x^2 + (z_+ - y)^2}} \right) - (2x^2 - y^2 + 2z_-^2 - yz_-) \sqrt{x^2 + (z_- - y)^2} - 3x^2 y \tanh^{-1} \left( \frac{z_- - y}{\sqrt{x^2 + (z_- - y)^2}} \right),
\]

where \( x = \Delta_0/\Delta_c \), \( y = \mu/\Delta_c \) and \( I(x, y) \) is given by

\[
I(x, y) = (2x^2 - y^2 + 2z_+^2 - yz_+) \sqrt{x^2 + (z_+ - y)^2} + 3x^2 y \tanh^{-1} \left( \frac{z_+ - y}{\sqrt{x^2 + (z_+ - y)^2}} \right) - (2x^2 - y^2 + 2z_-^2 - yz_-) \sqrt{x^2 + (z_- - y)^2} - 3x^2 y \tanh^{-1} \left( \frac{z_- - y}{\sqrt{x^2 + (z_- - y)^2}} \right),
\]

with \( z_{\pm} \) defined as

\[
z_{\pm} = \frac{1}{1 - |\mathbf{t}|^2} \left[ y \pm Re \sqrt{|\mathbf{t}|^2 y^2 - (1 - |\mathbf{t}|^2) x^2} \right].
\]

In Fig. 3 the numerical results for the superconducting gap are shown as a function of the chemical potential and some representative values for the effective tilt parameter \( |\mathbf{t}| \) when \( g_2 < 0 \). In the case \( g_2 < 0 \), one can see in Fig. 3 that the tilt increases \( \Delta_0 \) for a given \( \mu > \mu_c^* \). In this special value \( \mu_c^* = \mu_c^*(|\mathbf{t}|) \) is where the tilt parameter starts to contribute to the superconducting gap and it is plotted in Fig. 4. For \( \mu > \mu_c^* \), the superconducting gap
takes the exact form $\Delta_c(\mu > \mu_c^*) = \Delta_t$, where

$$\Delta_t = \frac{|\tilde{t}| \mu}{\sqrt{1 - |\tilde{t}|^2}}. \quad (4.14)$$

For any value of the tilt parameter $|\tilde{t}| < \tilde{t}^*$, with $\tilde{t}^* \approx 0.55$, the effect of the tilt parameter in the superconducting gap vanishes for any $\mu$, and the superconducting gap of system obeys the solid gray curve shown in Fig. 3.

![Fig. 3. Superconducting gap for $g_2 < 0$ induced by the chemical potential and the tilt parameter in units of $\Delta_c$. For $|\tilde{t}| = 0$ (black curve), $|\tilde{t}| = 0.7$ (blue curve), $|\tilde{t}| = 0.8$ (green curve) and $|\tilde{t}| = 0.9$ (red curve).](image)

When $|\tilde{t}| > \tilde{t}^*$, the superconducting gap will be exactly $\Delta_c(|\tilde{t}| > \tilde{t}^*) = \Delta_t$.

![Fig. 4. Normalized critical chemical potential ($\mu_c^*/\Delta_c$) in which the tilt parameter drives the superconducting gap in the $g_2 < 0$ regime. For $|\tilde{t}| < 0.55$ the tilt contribution to the superconducting gap vanishes for any $\mu$. For $\mu > \mu_c^*$ the superconducting gap is given by $\Delta_c = \Delta_t$. The thin vertical dashed line on the left of the plot represents the critical value $|\tilde{t}| = 0.55$.](image)

Analyzing now the case for $g_2 > 0$, we are able to uncover another structure for the superconducting gap. As can be seen in Fig. 5, in this case we have two different situations: When $|\tilde{t}| < \tilde{t}^*$, the tilt parameter only contributes for $\mu < \mu_c^*$ (e.g., see blue curve in Fig. 5). This special value $\mu_c^* = \mu_c^*(|\tilde{t}|)$ sets a lower limit where the tilt parameter stops contributing to the superconducting gap. The behavior of $\mu_c^*$ is shown in Fig. 6.

![Fig. 5. Superconducting gap for $g_2 > 0$ induced by the tilt parameter in units of $\Delta_c$. For $|\tilde{t}| = 0$ (solid gray curve), $|\tilde{t}| = 0.4$ (blue curve), $|\tilde{t}| = 0.6$ (green curve) and $|\tilde{t}| = 0.8$ (red curve).](image)

![Fig. 6. Normalized chemical potential $\mu_c^*$ in which the tilt parameter stops to drive the superconducting gap in the $g_2 > 0$ regime. For $|\tilde{t}| < 0.55$ the tilt contribution to the superconducting gap vanishes for any $\mu$. For $\mu < \mu_c^*$ the superconducting gap is given by $\Delta_0 = \Delta_t$. The thin vertical dashed line on the right of the plot represents the critical value $|\tilde{t}| = 0.55$.](image)

Finally, the charge density, defined as

$$n = -N \frac{\partial \Omega^{ren}(0, \Delta_0, \mu)}{\partial \mu} \Bigg|_{\Delta_0 = (\Delta_0)} , \quad (4.15)$$

can be expressed through an exact expression and given by

$$n = \frac{N}{4\pi v_F^2 \xi_x \xi_y} \left( \mu \sqrt{\mu^2 + \Delta_0^2} + \Delta_0^2 \ln \left[ \frac{\mu + \sqrt{\mu^2 + \Delta_0^2}}{\Delta_0} \right] \right) \Bigg|_{\Delta_0 = (\Delta_0)} + N \frac{\partial I(\Delta_0, \mu)}{\partial \mu} \Bigg|_{\Delta_0 = (\Delta_0)} , \quad (4.16)$$
where \( \langle \Delta_0 \rangle \) is the solution of Eq. (4.11), which can be found numerically for both \( g_2 > 0 \) and \( g_2 < 0 \) cases. From the inequality \( \text{Re} \sqrt{\mu^2 - (1 - |\tilde{t}|^2)\Delta_0^2} \neq 0 \), one finds that the contribution for the charge density from \( I \) is non-null only for \( \Delta_0 < \Delta_t \). Thus, based on Fig. 3, this contribution is non-null only for \( \mu < \mu_+^* \). In the case \( g_2 > 0 \), on the other hand, from Fig. 4 the density will receive extra contributions only for \( \mu > \mu_+^* \).

V. PHASE STRUCTURE FOR \( \mu \neq 0 \)

Previous works \cite{13, 16, 17} have shown that it is sufficient to analyze the chiral-superconducting phase structure by comparing the vacuum properties in the \( \sigma_0 = 0 \) and \( \Delta_0 = 0 \) axes. Through this analysis of the local minimum in each axis, we can compare them and find the global minimum which defines the real phase of the system. For instance, as shown in the previous section, for fixed \( q_1 < 0 \), there is a critical chemical potential \( \mu_+^* (g_2) \). The value of \( \mu_+^* (g_2) \) defines the lower bound for the chemical potential such that for \( \mu > \mu_+^* (g_2) \) the system is in the superconducting phase (phase III) and for \( \mu < \mu_+^* (g_2) \), the system is in the chiral symmetry breaking phase (phase II). The critical point \( \mu = \mu_+^* (g_2) \) defines a first-order transition between the phases II and III. In the case of \( g_2 < 0 \), there is another critical value for the chemical potential, \( \mu_-^* \), such that for \( \mu > \mu_-^* \) the superconducting phase stops to drive the system in favor of the chiral phase. The opposite happens when \( g_2 > 0 \), in which case \( \mu_-^* \) becomes an upper bound and for \( \mu < \mu_-^* \) is when the superconducting phase stops to drive the system in favor of the chiral phase. Finally, the chiral symmetry will be restored for \( \mu > \sqrt{1 - |\tilde{t}|^2}\sigma_c \).

Let us now show the different phase portraits that will display the above structure relating the chemical potential with the superconducting phase portraits that will display the above structure relating the chemical potential with the superconducting phase stops to drive the system in favor of the chiral phase. The opposite happens when \( g_2 > 0 \), in which case \( \mu_-^* \) becomes an upper bound and for \( \mu < \mu_-^* \) is when the superconducting phase stops to drive the system in favor of the chiral phase. Finally, the chiral symmetry will be restored for \( \mu > \sqrt{1 - |\tilde{t}|^2}\sigma_c \).

\[ (\mu^*, g_2) |_{g_2 > 0, |\tilde{t}| < t^*} = \left( \frac{1 - |\tilde{t}|^2}{|g_1|}, \frac{1 - |\tilde{t}|^2}{\mu_-^*} \right). \quad (5.1) \]

The presence of the tricritical point as a consequence of the tilt of the Dirac cone is one of our main results, showing a quite different behavior when compared to the results in the non-tilted case \cite{13}.

\[ (\mu^*, g_2) |_{g_2 < 0, |\tilde{t}| > t^*} = \left( \frac{1 - |\tilde{t}|^2}{|g_1|}, \frac{1 - |\tilde{t}|^2}{\mu_-^*} \right). \quad (5.1) \]

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Going further, looking at the case for \( g_2 < 0 \) and \(|\tilde{t}| > \tilde{t}^*\), which is shown in Fig. 8, one notices a much stronger change in the phase portrait as compared to the region with \( g_2 < 0 \) shown in Fig. 4. The presence of the tilt effectively causes the superconducting gap to stop to drive the system for \( \mu > \mu_\zeta^* \) and phase I now takes place for \( \mu > \sqrt{1 - \frac{|\tilde{t}|^2}{|g_1|}} \). The phase portrait in this case displays a much restricted area for the superconducting phase. The superconducting phase occurs only for values of \(|g_2|/|g_1| \leq 1\). In this case, one tricritical point also appears and it is found to be given by

\[
(m^t, g_2^t)_{|g_2|<0,|\tilde{t}|>\tilde{t}^*} = \left( \frac{\sqrt{1 - |\tilde{t}|^2}}{|g_1|} - \frac{1}{\mu_\zeta^*} \right).
\]  

(5.2)

Finally, looking at the region where \( g_2 > 0 \) shown in Fig. 8 the induction of a superconducting phase due to chemical potential is ruled out for any value of the chemical potential and the phase transition occurs between the phases I and II. The increase of the chemical potential in the presence of a tilt satisfying \(|\tilde{t}| > \tilde{t}^*\), in this case both effects work in favor of the chiral symmetric phase. This can be seen by the enlarged region for phase I shown in Fig. 8 when compared to the non-tilted case. This is our other main result that is extracted from the phase portrait. It shows once more the effect of the tilt on hindering the formation of gaps in the system and, in this case, the formation of an induced gap due to the presence of the chemical potential.

VI. CONCLUDING REMARKS

In this paper, we have analyzed the chiral and superconducting gaps in a planar fermion model with four-fermion interactions. The interactions describe both the excitonic and the Cooper pairings in the presence of tilted Dirac cones. We have obtained two different behaviors depending on the value of the effective tilt parameter \(|\tilde{t}|\). One finds a critical value \( \tilde{t}^* \), given by \( \tilde{t}^* \approx 0.55 \), where the system behaves completely different under the formation of the chiral and superconducting gaps when compared to the non-tilted case. In the case where \( |\tilde{t}| < \tilde{t}^* \), the superconducting phase persists for a negative superconducting coupling constant, which is responsible for the attractive interaction in the Cooper channel. A first-order phase transition occurs for a critical chemical potential represented by the black curve shown in Fig. 7. This feature is similar to the results for graphene and other two-dimensional materials [13]. One also sees that for \( g_2 > 0 \) the induction of a superconducting gap due to the presence of a chemical potential exists, although this induction happens only for stronger values of the coupling constant \( g_2 \), since the metallic phase appears for small values of the superconducting coupling constant. Due to the presence of a metallic phase, we were able to find the expression for the tricritical point, given by Eq. (5.1), which is one of the important new results shown in this work.

Moreover, when looking at the case where \(|\tilde{t}|\) exceeds the critical value \( \tilde{t}^* \), the superconducting phase now becomes restricted to a smaller area in the phase portrait. Indeed, in this case, the superconducting phase occurs only for regions with small and negative superconducting coupling constant. Through a first-order phase transition, the metallic phase takes place for a sufficient large superconducting coupling constant and chemical potential. One also finds the analytic expression for the tricritical point in this case, which is given by Eq. (5.2). Finally, one shows that our qualitative analysis point to the fact that for \( g_2 > 0 \) the superconducting gap induced by chemical potential is ruled out and a first-order phase transition occurs between phases I and II at the critical chemical potential represented by the black curve shown in Fig. 8.

We can try to explore the consequences of the results we have obtained for some known planar systems which have been currently studied in laboratory experiments. For example, using the experimental data obtained from the 2-D organic conductor \( \alpha -(\text{BEDT-TTF})_2I_3 \) [33, 34], the estimated effective tilt parameter is found to be \( \tilde{t} \approx 0.76 \) (see, e.g., Ref. [30]). This case falls in the situation where \( |\tilde{t}| > \tilde{t}^* \approx 0.55 \) and which we have discovered in this paper. From our results, this implies that the inducing of a superconducting gap should be absent in this material. It would be interesting to prove this prediction using this type of material in the laboratory. By also accounting the results obtained from the analysis of Ref. [30], we can also conclude that this same system should exhibit a metallic phase, which would become very strong under doping. On the other hand, we can also compare with the predictions that our results would imply for the case of quinoid-type graphene under uniaxial strain [5]. In this case, the estimated values for the effective tilt parameter are such that \( \tilde{t} \lesssim 0.06 \) for moderate deformations. From our results, we can conclude that for this material the properties of the superconducting gap should be similar to the graphene case, which includes the induction of a superconducting gap by a chemical potential.

The study of such two-dimensional fermionic systems can be exploited in several directions. For instance, the presence of an anomalous Hall effect [30, 32] in the 2D Weyl semimetal indicates the possibility that the tilt of the Dirac cones could modify the superconducting gap under the presence of an external magnetic field. Furthermore, since the tilt of the Dirac cone introduces a special direction in the system, the analysis of the \( p \)-wave superconducting gap properties in this context could bring new features. This can be another problem of interest that can be a target of further investigation. These problems are possible lines of study that our results motivate and we hope to address them in the future.
Appendix A: Performing the path integral over the fermion in Eq. (3.2)

Let us show here some of the details of the path integral over the fermions in Eq. (3.2) and which leads to the effective thermodynamic potential. Adopting the procedure described in Ref. [13], we assume two anti-commuting four component Dirac spinor fields $q(x)$ and $\bar{q}(x)$. Then, Eq. (3.2) can be rewritten as

$$I = \int DqD\bar{q} \, e^{i \int d^4x \left[ \bar{q} \mathcal{O} C q - \bar{q} \gamma^\tau C q - \frac{\Delta}{2} \bar{q} C q \right]},$$  \hspace{1cm} (A1)

where $\mathcal{O} = iM^{\mu \nu} \gamma_\mu \partial_\nu + \mu \gamma^0 - \sigma$ and $C = i \gamma^2$ is the charge conjugation matrix. Using the Gaussian path integral identities

$$\int \mathcal{P} \, e^{i \int d^3x \left[ -\frac{1}{2} p^\tau A p + \eta^\tau p \right]} = (\det A)^{1/2} e^{-\frac{i}{2} \int d^3x \langle A^{-1} \eta^\tau \rangle \langle \eta \rangle}, \hspace{1cm} (A2)$$

and

$$\int \mathcal{P} \, e^{i \int d^3x \left[ -\frac{1}{2} A p^\tau + \eta^\tau \right]} = (\det A)^{1/2} e^{-\frac{i}{2} \int d^3x \delta A^{-1} \eta^\tau}, \hspace{1cm} (A3)$$

and by also considering $A = \Delta C$, $\bar{q} \mathcal{O} \eta = \eta^\tau$, $\mathcal{O}^\tau \eta = \eta$, one finds, after integrating over $q$ and $\bar{q}$, the result

$$I = \int DqD\bar{q} \, e^{i \int d^3x \left\{ \bar{q} \mathcal{O} C[q \mathcal{O} C]^\dagger \right\}}$$

$$= (\det \Delta C)^{1/2} \int Dq \, e^{i \int d^3x \left\{ \bar{q} \mathcal{O} C[q \mathcal{O} C]^\dagger \right\}}$$

$$= (\det \Delta C)^{1/2} \left[ (\det (\Delta^* C + \mathcal{O}(\Delta C)^{-1} \mathcal{O}^T)) \right]^{1/2}$$

$$= [\det (\Delta^2 + \mathcal{O}^\dagger \mathcal{O})]^{1/2},$$  \hspace{1cm} (A4)

where we have assumed $\Delta = \Delta^*$ in the last step (we are not interested in the phase of the superconducting order parameter, but solely on its absolute (modulus) value). Using the relations $C^{-1} \gamma^\tau T C = -\gamma_\mu$ and $\partial^\tau = -\partial_\mu$ one finds that

$$I = [\det(-\Delta^2 + \mathcal{O}_+ \mathcal{O}_-)]^{1/2} = (\det B)^{1/2},$$  \hspace{1cm} (A5)

with $\mathcal{O}_\pm = iM^{\mu \nu} \gamma_\mu \partial_\nu \pm \mu \gamma^0 - \sigma$. Finally, using the identity $\det B = \exp(Tr \ln B)$ one finds

$$\ln I = \frac{1}{2} \text{tr}(\ln B) = \int d^3x \sum_{i=1}^{2} \int \frac{d^3p}{(2\pi)^3} \ln \lambda_i(p),$$  \hspace{1cm} (A6)

where

$$\lambda_{1,2} = \sigma^2 + [p_0 - v_F(t \cdot \mathbf{p})]^2 - v_F^2 \mathbf{p}^2 - \mu^2 - |\Delta|^2$$

$$\pm 2 \sqrt{\sigma^2 [(p_0 - v_F(t \cdot \mathbf{p}))^2 - v_F^2 \mathbf{p}^2]} + v_F^2 \mu^2 \mathbf{p}^2,$$

are the eigenvalues of $B$.

Appendix B: The effective thermodynamic potential

In this section one shows some of the details for the derivation of the effective thermodynamic potential. From Eq. (4.9), we obtain

$$\Omega^{\text{ce}}(0, \Delta_0, \mu) = \frac{\Delta_0^2}{2g_2 v_F} + \frac{\Delta_0^3}{3\pi v_F^2 \xi_x \xi_y} - i_+ - i_-. \hspace{1cm} (B3)$$

It can now be shown that for $\mu > 0$, $\mathcal{E}_+ > 0$ for all $p > 0$. Hence, after some algebraic steps, one finds

$$i_+ = \frac{1}{2\pi v_F^2 \xi_x \xi_y} \int_0^\infty dp \left( \sqrt{(p + \mu)^2 + \Delta_0^2} - \sqrt{p^2 + \Delta_0^2} \right).$$  \hspace{1cm} (B4)

For $\mathcal{E}_-$, one has that $\mathcal{E}_- > 0$ only for $p < p_-$ and for $p > p_+$, where

$$p_{\pm} = \frac{1}{2} \left( 1 - |t|^2 \right) \left[ \mu \pm \text{Re} \sqrt{|t|^2 \mu^2 - (1 - |t|^2) \Delta_0^2} \right],$$  \hspace{1cm} (B5)

where $\text{Re}$ means the real part. From the above expressions, then, it follows that

$$\int \frac{d^2p}{(2\pi)^2} |\mathcal{E}_-| = \frac{1}{2\pi v_F^2 \xi_x \xi_y} \int_0^\infty dp \int_0^{2\pi} \left[ |t| p \cos \theta + \sqrt{(p - \mu)^2 + \Delta_0^2} \right]$$

$$= \frac{1}{2\pi v_F^2 \xi_x \xi_y} \int_0^\infty dp \int_0^{2\pi} \left[ |t| p \cos \theta + \sqrt{(p - \mu)^2 + \Delta_0^2} \right]$$

$$\times \Theta \left( \sqrt{(p - \mu)^2 + \Delta_0^2} - |t| p \right),$$  \hspace{1cm} (B6)

where we have used the identity $\int_0^{2\pi} d\theta \Theta(b - a) = \Theta(b(a - a))$, for $a > 0$ and $b > 0$ in the last step and $\Theta(x)$ is the Heaviside function. Finally, the inequality $\sqrt{(p - \mu)^2 + \Delta_0^2} - |t| p > 0$ is respected when $p < p_-$ and
\[ p > p_+ \text{. In particular, one notes that in the limit } |\vec{t}| \to 0, \text{ one finds } p_+ = p_- = \mu. \text{ Therefore, using the fact that } \int_{0}^{p_-} + \int_{0}^{\infty} = \int_{0}^{p_-} - \int_{p_-}^{\infty} \text{ it follows that} \]

\[
\begin{align*}
I(\Delta_0, \mu) &= \frac{1}{2\pi v_F^2 \xi_x \xi_y} \int_{p_-}^{p_+} dp p \sqrt{(p - \mu)^2 + \Delta_0^2} \\
&= (2\Delta_0^2 - \mu^2 + 2p_+^2 - \mu p_-) \sqrt{\Delta_0^2 + (p_+ - \mu)^2} \\
&+ 3\Delta_0^2 \mu \tanh^{-1}\left(\frac{p_+ - \mu}{\sqrt{\Delta_0^2 + (p_+ - \mu)^2}}\right) \\
&- (2\Delta_0^2 - \mu^2 + 2p_-^2 - \mu p_+) \sqrt{\Delta_0^2 + (p_- - \mu)^2} \\
&- 3\Delta_0^2 \mu \tanh^{-1}\left(\frac{p_- - \mu}{\sqrt{\Delta_0^2 + (p_- - \mu)^2}}\right). \\
\end{align*}
\]

Finally, after integration over the momentum \( p \), one can write the renormalized effective thermodynamic potential for the superconducting phase (when \( \sigma_0 = 0 \)) as

\[
\Omega^{ren}(0, \Delta_0, \mu) = \frac{\Delta_0^2}{2g_{2}v_F} + \frac{(\mu^2 + \Delta_0^2)^{3/2}}{3\pi v_F^2 \xi_x \xi_y} \\
- \frac{\mu^2 \sqrt{\mu^2 + \Delta_0^2}}{2\pi v_F^2} \ln\left(\frac{\mu + \sqrt{\mu^2 + \Delta_0^2}}{\Delta_0}\right) \\
+ \frac{1}{2\pi \xi_x \xi_y v_F^2} I(\Delta_0, \mu). \\
\]

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