The maximally entangled set of multipartite quantum states

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Entanglement is a resource in quantum information theory when state manipulation is restricted to Local Operations assisted by Classical Communication (LOCC). It is, therefore, of paramount importance to decide which LOCC transformations are possible and, particularly, which states are maximally useful under this restriction. While the bipartite maximally entangled state is well known (it is the only state that cannot be obtained from any other and, at the same time, it can be transformed to any other by LOCC), no such state exists in the multipartite case. In order to scope with this fact, we introduce here the notion of the Maximally Entangled Set (MES) of $n$-partite states. This is the set of states which are maximally useful under LOCC manipulation, i.e., any state outside of this set can be obtained via LOCC from one of the states within the set and no state in the set can be obtained from any other state via LOCC. We determine the MES for three and four qubit states and provide a simple characterization for them. In both cases, infinitely many states are required. However, while the MES is of zero-measure for 3-qubit states, almost all 4-qubit states are in the MES. This is because, in contrast to the 3-qubit case, deterministic LOCC transformations are almost never possible among fully entangled four-partite states. We determine the zero-measure subset of the MES of LOCC convertible states. This is the only relevant class of states for entanglement manipulation.

Multiparticle entangled states constitute the essential ingredient for many fascinating applications within quantum computation and quantum communication 1,2. The theory of many-body states plays also an important role in other fields of physics 3. As the existence of those practical and abstract applications rests upon the subtle properties of multipartite entangled states, one of the main goals in quantum information theory is to gain a better understanding of the non-local properties of quantum states. Whereas the bipartite case is well understood, the theory of multipartite entanglement is still in its infancy 4.

In the context of quantum information theory, entanglement is a resource that allows to achieve certain information processing tasks. This has led to the development of entanglement theory, which deals with the qualification and quantification of entanglement and with the manipulation of this resource in general 5. Therein, the notion of Local Operations assisted by Classical Communication (LOCC) plays a central role as this is the most general form of manipulating a multipartite state by spatially separated parties. Thus, LOCC convertibility induces the natural ordering in the set of entangled states and the very fundamental condition for a function to be an entanglement measure is that it does not increase under LOCC. Hence, entanglement theory is a resource theory in which entanglement is a resource for manipulations restricted to LOCC. It is then a fundamental question to ask which states are maximally entangled, i.e., which states are maximally useful under the LOCC restriction. Nielsen 5 has characterized all LOCC transformations which are possible among arbitrary pure states in the bipartite case. From there, it follows that the Bell state (and its generalization to higher dimensions) is the maximally entangled state as this is the only state that can be transformed to any other by LOCC operations. Thus, it is not surprising that the Bell state plays a prominent role in most known bipartite quantum information protocols such as teleportation 6 or cryptography 7 and also in quantifying entanglement 8. Unfortunately, there is no straightforward extension of Nielsen’s theorem to the multipartite case and the investigation of LOCC transformations is very difficult due to their complicated mathematical structure 9. Indeed, LOCC convertibility is in general unknown in the multipartite case save for a few classes of states 10. For this among other reasons, multipartite entanglement has been classified according to other physically and/or mathematically motivated operations. Local unitary operations (LUs) are the only invertible LOCC transformations and, hence, they interconvert states with the same entanglement. Recently, necessary and sufficient conditions for the LU-equivalence of pure $n$-qubit state have been derived 11. Stochastic LOCC (SLOCC) operations identify states which can be interconverted by LOCC non-deterministically but with a non-zero probability of success. There exist two different SLOCC-classes for 3-qubit states 12 but there are infinitely many for more parties 13. Both LU and SLOCC identify fundamentally different forms of entanglement but they just define equivalence classes and cannot be utilized to identify which states are more useful than others. Thus, another approach has been investigating separable transformations (SEP) (see 14 and references therein), which although without a clear operational meaning, have a much simpler mathematical structure and they include LOCC. However, the inclusion is strict 15 and there exist SEP transformations that cannot be implemented by LOCC 16. Other authors have tried to identify multipartite maximally entangled states by extrapolating different particular properties of
the Bell state to the multipartite case [17]. However, let us stress here that, as argued above, LOCC transformations induce the only operationally meaningful ordering in the set of entangled states. Consequently, multipartite maximal entanglement can only be rigorously established on the grounds of maximal usefulness under LOCC. This is precisely the main aim of this Letter.

Despite all the difficulties one faces when investigating LOCC transformations, we introduce here the notion of the Maximally Entangled Set (MES) of \( n \)-partite states. A MES, \( S_n \), is the minimal set of \( n \)-partite states, such that any other truly \( n \)-partite entangled state can be obtained deterministically from a state in \( S_n \) via LOCC. In other words, a MES, \( S_n \), is a set of states with the following properties: (i) No state in \( S_n \) can be obtained from any other state via LOCC (excluding LU) and (ii) for any \( n \)-partite state, \( \Phi \notin S_n \), there exists a state in \( S_n \) such that \( \Phi \) can be obtained from it via LOCC. Thus, it is the set of states, not as in the bipartite case an individual state, which is maximally entangled. We consider the simplest nontrivial cases of few-qubit systems and we determine \( S_3 \) and \( S_4 \) [22]. Contrary to the bipartite case, the sets do not contain a single state but infinitely many. Nevertheless, \( S_3 \) is of measure zero in the full set of 3-qubit states and, hence, very few states are maximally useful under LOCC. Moreover, LOCC conversions are always possible. The situation changes drastically already in the four-partite case as we show that \( S_4 \) is of full-measure. Thus, perhaps surprisingly, almost all 4-qubit states are in the MES. The reason for this is that almost all states are isolated, i.e., they cannot be obtained from nor transformed to any other fully entangled state by deterministic LOCC (excluding LU). Hence, LOCC induces a trivial ordering in the set of entangled states, the possibility of LOCC conversions is very rare in the multipartite case and most states are useless from this perspective. However, we also identify a zero-measure subset of states in the MES which are LOCC convertible. Those are the most useful states regarding entanglement manipulation. Hence, the investigation of this class of states could, as was the case for the Bell state, lead to new multipartite applications of quantum information.

Throughout the paper we denote by \( 1, X, Y, Z \) as well as by \( \sigma_i \), where \( i = 0, 1, 2, 3 \) the identity operator and the Pauli operators. Moreover, \( W(\omega) = \exp(i\omega W) \) for \( W = X, Y, Z \). Whenever it does not lead to any confusion we ignore normalization. \( G \) denotes the set of local invertible (not necessarily determinant 1) operators and \( g, h \) denote elements of \( G \), e.g., \( g = g_1 \otimes \ldots \otimes g_n \), with \( g_i \in GL(2) \). Two states are said to be in the same SLOCC–class (LU–class) if there exists a \( g \in G \) (local unitary) which maps one state to the other.

Notice that when studying LOCC convertibility, we ignore pure LU transformations which can always be performed. Hence, in rigor, we characterize LOCC convertibility among the LU–equivalence classes and we only consider one representative for each class. Notice as well that we consider LOCC transformations among fully-entangled states which are, hence, only possible among states in the same SLOCC–class.

Let us now begin by reviewing the, for the present work relevant, results on the mathematically more tractable SEP presented in [14]. We denote by \( S(\Psi) = \{ S \in G : S|\Psi\rangle = |\Psi\rangle \} \) the set of stabilizers of \( |\Psi\rangle \). In [14] it has been shown that a state \( |\Psi_1\rangle = g |\Psi\rangle \) can be transformed via SEP to \( |\Psi_2\rangle = h |\Psi\rangle \) iff there exists a \( m \in N \) and a set of probabilities, \( \{ p_k \}_1^n \) \( (p_k \geq 0, \sum_{k=1}^m p_k = 1) \) and \( S_k \in S(\Psi) \) such that

\[
\sum_k p_k S_k^\dagger H S_k = rG. \tag{1}
\]

Here, \( H = h^\dagger h \equiv \bigotimes_i H_i \), and \( G = g^\dagger g \equiv \bigotimes_i G_i \) are local operators and \( r = \frac{n_k}{n_{k+1}} \) with \( n_{k+1} = |||\Psi_1|||^2 \). The local POVM elements accomplishing the task to transform \( |\Psi_1\rangle \) into \( |\Psi_2\rangle \) are given by \( M_k = \frac{hS_k g^{-1}}{\sqrt{r}} \). Note that if all \( S_k \) are unitary then Eq. \( 1 \) implies that \( rG \preceq H \). That is, the eigenvalues of the positive operator \( rG \) are majorized by the eigenvalues of \( H \). [13][14].

We first use this result to set necessary conditions for LOCC convertibility. The basic idea here is to show that any symmetry, which does not belong to the unitary group can be used to define a standard form up to LUs for the different SLOCC–classes. Once, this standard form is fixed, the only operations, which make a transformation possible are the unitary stabilizers [24]. Later, we show that these transformations are already so constrained that, whenever possible, one can also find a corresponding LOCC protocol. Moreover, as we will see, the LOCC protocols are always quite simple. In fact, each party needs to measure at most once.

Before we investigate the MES for three and four qubits, it will be useful to keep in mind that any state, which has the property that all single qubit reduced states are completely mixed is in the MES [18]. Thus, any connected graph state as well as any error correcting code is in the MES.

Let us now determine the MES, \( S_3 \), for three qubits. As there exist two inequivalent tripartite entangled SLOCC–classes, the GHZ–class and the W–class [12], \( S_3 \) must include at least two states. Notice that these are the only classes of multipartite states for which LOCC transformations have been characterized [10]. Since this exhausts all the possible classes (exclusively) for 3–qubit states, one could also determine \( S_3 \) from the results therein. However, in order to demonstrate our techniques we derive \( S_3 \) now independently from those results.

Let us begin with the GHZ–class. Since the single qubit reduced states of the GHZ state, \( |GHZ\rangle = |000\rangle + |111\rangle \) are completely mixed, we know that it is in \( S_3 \). It is well known that any element of the symmetry group of the GHZ–state can be written as \( |\Psi_i\rangle = X^k P_{\gamma_i} \), for \( k \in \mathbb{Z} \).
\( 0, 1 \). Here, \( P_\gamma = P_{\gamma_1} \otimes P_{\gamma_2} \otimes P_{\gamma_3} \) with \( P(\gamma) = diag(\gamma, \gamma^{-1}) \) and \( \gamma \in \mathcal{G} \) and \( \mathcal{X} \equiv X^{\otimes 3} \). We use this symmetry to determine a standard form of the states in the GHZ class. Since for any positive definite \( 2 \times 2 \) matrix \( g^1 g \) there exists a \( \gamma \) such that \( g^1 g = P_{\gamma} g_{\gamma} P_{\gamma} \)

where \( g_{\gamma} g_{\gamma} \in \text{span}(1, X) \) and \( \text{tr}(g^1 g_{\gamma} X) \geq 0 \), the states can be written (up to LUs [24]) as

\[
g^1 g^2 \otimes (g^3_3 P_z) \ket{\text{GHZ}},
\]

with \( z \in \mathcal{G} \). Here, and in the following, \( g^i \in \text{span}(1, W) \), for \( W \in \{X, Y, Z\} \) such that \( G^i = (g^i_0)_x = 1/2 (1 + g^i_1 W) \), where \( g^i_1 \in [0, 1/2] \) to ensure that the operators are invertible (otherwise entanglement is destroyed).

In order to simplify the notation we will in the following allow for negative values of \( g^i_1 \), even though the corresponding state would be of the form as in Eq. (2) for a properly chosen value of \( z \).

Using this standard form we show now that all states in the MES (apart from \( \ket{\text{GHZ}} \)) are of the form \( g^1 g^2 g^3 \ket{\text{GHZ}} \) (i.e. \( z = \pm 1 \) in Eq. (2)) with no trivial \( g^i_1 \) (for details see Appendix A). Let \( \ket{\Psi} = g \ket{\text{GHZ}} \) be an arbitrary initial state and \( \ket{\Psi_2} = h \ket{GHZ} \) an arbitrary final state, where \( g = g^1 \otimes g^2 \otimes g^3 \ket{\text{P}_3} \), with \( z_g = |z_g|e^{i\pi a} \), and similarly for \( h \). Due to Eq. (1) we have that \( \ket{\Psi_1} \) can be transformed to \( \ket{\Psi_2} \) via SEP if there exist probabilities, \( p_{k,\gamma} \) such that

\[
\sum_{k,\gamma} p_{k,\gamma} X^k P_{\gamma}^b H P_{\gamma}^c = r G.
\]

Multiplying both sides of this equation with \( (|l\rangle \langle m|)^{\otimes 3} \), for \( l, m = 0, 1 \), and taking the trace, it follows that if none of the \( h^i_1 \) is trivial, then a state with \( z_\gamma = \pm 1 \) can only be obtained from a state with \( z_\gamma = \pm 1 \).

On the other hand, any state with \( z_h \neq \pm 1 \) or at least one of the \( h^i_1 \) trivial can be obtained from a state with \( z_\gamma = \pm 1 \). In fact, in those cases, one can not only derive a SEP, but a LOCC protocol, which accomplishes this task. For instance, the final state, \( \ket{\Psi_2} = h^1 \otimes h^2 \otimes h^3 \ket{\text{P}_3} \ket{\Psi} \) (with \( z_h \neq \pm 1 \)) can be reached from the state \( \ket{\Psi_1} = h^1 \otimes h^2 \otimes g^3_3 \ket{\Psi} \), for a properly chosen operator \( g^3_3 \) via the following LOCC protocol: Party 3 applies the POVM \( \{ M_1 = h^3_3 P_{\text{z}_3} (g^3_3)^{-1}, M_2 = h^3_3 P_{\text{z}_3} X(g^3_3)^{-1} \} \) and in case outcome 1 (2) is obtained all other parties do nothing (apply a \( X \) operation). Since \( X \) commutes with \( h^i_1 \) it can be easily seen that the desired state is obtained for both outcomes. Thus, the only states in the MES in this class are of the form \( g^1 \otimes g^2 \otimes g^3_3 \ket{\Psi} \).

Let us now treat the \( W \)-class. Using the symmetry of the \( W \)-state, \( \ket{W} = (001) + (010) + (100) \) [19, Appendix A] it is easy to see that any state in the \( W \)-class can be written as \( g_1 \otimes g_2 \otimes 1 \ket{W} \), where

\[
\begin{align*}
g_1 &= \begin{pmatrix}
1 & 0 \\
0 & x_1(g)/x_3(g)
\end{pmatrix}, \\
g_2 &= \begin{pmatrix}
x_3(g) & x_0(g) \\
x_2(g) & 0
\end{pmatrix}.
\end{align*}
\]

We are going to show now that the states in this class are in the MES iff \( x_0(g) = 0 \) (for details see Appendix A).

First of all, note that we only need to consider unitary stabilizers since any POVM element transforming the state, \( \ket{\Psi_1} = g_1 \otimes g_2 \otimes 1 \ket{W} \) into \( \ket{\Psi_2} = h_1 \otimes h_2 \otimes 1 \ket{W} \), would be of the form \( h_3 S_3^{-1} = h_3 S_3^{-1} \otimes h_2 S_3^{-1} \otimes S_3^{-1} \), which can only be implemented via LOCC if party 3 just applies a unitary, i.e. if \( S_3^{-1} \) is unitary. It can then be shown that in this case SEP is only possible if the whole stabilizer is unitary.

The solely unitaries leaving the \( W \)-state (up to a global phase) invariant, are of the form \( Z^{(\alpha)^{\otimes 3}} \) (\( \alpha \in \mathbb{R} \)). Suppose now that \( x_0(h) = 0 \). In this case, \( H_3 \) as well as \( H_1 \) commutes with any symmetry operator. Hence, \( \ket{\Psi_1} \) can be transformed into \( \ket{\Psi_2} \) via SEP (see Eq. (1)) iff

\[
H_1 H_2 \otimes 1 = r G_1 \otimes G_2 \otimes 1.
\]

Clearly this implies that \( x_0(g) = 0 \). Thus, states with \( x_0(h) = 0 \) can only be obtained from states with \( x_0(g) = 0 \). Moreover, in this case the states can only be transformed into each other if they are LU-equivalent since there is only one POVM element. This shows that states with \( x_0(g) = 0 \) are in the MES. Similarly to the GHZ case, one can construct a LOCC protocol which reaches any state with \( x_0(h) \neq 0 \) from one with \( x_0(g) = 0 \). This shows the following

**Theorem 1.** The MES of three qubits, \( S_3 \), is given by

\[
S_3 = \{ g^1 \otimes g^2 \otimes g^3_3 \ket{\text{GHZ}}, g_1 \otimes g_2 \otimes 1 \ket{W} \},
\]

where no \( g^i_3 \otimes 1 \) (except for the GHZ state) and \( g_1 \) and \( g_2 \) are diagonal.

Interestingly, \( S_3 \) has a very simple parametrization in terms of the decomposition of 3-qubit states some of us have introduced in [21]. While 3-qubit LU-classes are parameterized by 5 real parameters it can be shown that any state in \( S_3 \) belongs (up to LUs) to the three-parameter set (see Appendix A)

\[
\{ \ket{\Psi(a, \beta, \beta')} \equiv |0\rangle \ket{\Psi_s} + |1\rangle Y(\beta') \otimes Y(\beta) \ket{\Psi_s} \},
\]

where \( |\Psi_s\rangle = a |000\rangle + \sqrt{1 - a^2} |11\rangle \) has Schmidt decomposition and \( a, \beta, \beta' \in \mathbb{R} \). Using this form we show now that any state in \( S_3 \) can be mapped into some other state (outside of \( S_3 \)) via a non-trivial LOCC protocol, which also implies that no 3-qubit state is isolated. Note that \( X \otimes Z Y(-\beta') \otimes Z Y(-\beta) \) leaves any state \( \ket{\Psi(a, \beta, \beta')} \) invariant. Using this symmetry, it is easy to see that any \( \ket{\Psi(a, \beta, \beta')} \) can be transformed into any state \( A \otimes 1 \otimes 1 \ket{\Psi(a, \beta, \beta')} \), where \( A \) is such that \( \text{tr}(A^\dagger A X) = 0 \) and \( \text{tr}(A^\dagger A) = 1 \). The corresponding POVM is locally realizable and it is given by POVM elements \( M_1 = A \otimes 1 \otimes 1, M_2 = AX \otimes Z Y(-\beta') \otimes Z Y(-\beta) \).

Thus, in summary, for three qubits, although there are infinitely many states in the MES, it is of zero-measure and, furthermore, no state is isolated, i.e. LOCC entanglement manipulation from every state in the MES is always possible.

We move now to the 4-qubit case. Since there are infinitely many SLOCC-classes, \( S_4 \) has to contain infinitely many states. Generic states belong to the different SLOCC-classes known as \( G_{abcd} \) with representatives
where $a, b, c, d \in \mathcal{F}$ with $a \neq \pm b$ etc. We will call these states in the following seed states. Due to the normalization and the irrelevant global phase, the seed states are parameterized by 6 parameters. It can be easily seen that the stabilizer in this case is given by $\{\sigma_z^i\}_{i=0}^4$. For any state, $|\Phi\rangle$, in any of these SLOCC-classes there exists a local invertible matrix $g \in G$ and a seed state, $|\Psi\rangle$, such that $|\Phi\rangle = g |\Psi\rangle$. Without loss of generality we normalize the positive operators $G_i = g_i^4 g_i$ such that $\text{tr}(G_i) = 1$ and use the notation $G_i = 1/2^4 \sum_k g_k^{(i)} \sigma_k$, with $g_k^{(i)} \in \mathbb{R}$. Note that $0 \leq |g^{(i)}| < 1/2$, where $g^{(i)} = (g_1^{(i)}, g_2^{(i)}, g_3^{(i)})$ as we exclude here projectors in order to consider fully entangled 4-qubit states. Considering the trace of Eq. (11) we obtain $r = 1$. Note that for each $i$, the vector containing the eigenvalues of $G_i$, which are $1/2 \pm |g^{(i)}|$, must be majorized by the corresponding vector for $H_i$. Thus, $|g^{(0)}|$ cannot decrease under LOCC and therefore, these parameters have a monotonic behavior under LOCC. Since the symmetry of the seed states only allows to change simultaneously the sign of the same two parameters for all operators $G_i$, the matrices $G_i$ can be made unique and by sorting the coefficients in Eq. (6) this leads to a unique standard form. Let us note here, that this implies that two generic states are LU-equivalent iff their standard forms coincide. Since four-qubit LU equivalence classes are parameterized by 18 parameters [21], and the set of states considered here is parameterized by the 6 independent seed parameters and the 12 independent SLOCC parameters, $g_i^{(i)}$, for $i \in \{1, 2, 3, 4\}, j \in \{1, 2, 3\}$, the set is, as expected, of full-measure in the set of states.

Let us now study the MES, $S_4$. Notice that all single qubit reduced states of a seed state are completely mixed, which implies that all seed states are in $S_4$. We proceed now as in the three-partite case: first we derive necessary conditions for a state to be reachable via SEP, and then we derive the corresponding LOCC protocol (if it exists). Contrary to before we will see that almost no state can be reached via LOCC.

One of the main differences between the four-partite and the three-partite case are that there are only finitely many symmetries and that all symmetries are unitary. This fact can be used to derive very simple necessary conditions for the existence of the LOCC protocol. The main idea here is to observe that for $|\Psi_1\rangle = g |\Psi\rangle$ and $|\Psi_2\rangle = h |\Psi\rangle$, Eq. (11) implies that

$$\mathcal{E}_4(H) = \mathcal{E}(H_1) \otimes \mathcal{E}(H_2) \otimes \mathcal{E}(H_3) \otimes \mathcal{E}(H_4),$$

where $\mathcal{E}_4$ is the completely positive map given in Eq. (11) and $\mathcal{E}(\rho) = \sum_k \rho_k \sigma_k \rho_k$. Note that Eq. (17) only depends on the state $|\Psi_2\rangle$ and is independent of $|\Psi_1\rangle$. Considering Eq. (17) for two systems, i.e. tracing over the other two, allows already to derive very strong necessary conditions on $|\Psi_2\rangle$ to be reachable via SEP. In Appendix B we show which of those states can indeed be reached via LOCC by constructing, similarly to the three-partite case, the corresponding LOCC protocol. With all that we obtain the following theorem (for details of the proof see Appendix B).

**Theorem 2.** A generic state, $h |\Psi\rangle$, is reachable via LOCC from some other state iff (up to permutations) either $h = h^1 \otimes h^2 \otimes h^3 \otimes h^4$, for $w \in \{x, y, z\}$ where $h^1 \neq h^2$, or $h = h^1 \otimes 1^4 \otimes 1$ with $h^1 \otimes 1$ arbitrary.

Since this is only a 12 parameter–family, only a zero-measure set of states can be reached via LOCC. Therefore, all the remaining generic states are necessarily in $S_4$, which is then of full-measure and contains almost all states.

Let us now characterize which states can indeed be used for entanglement manipulation, i.e. which of them can be transformed by LOCC into another state. Since a state $|\Psi_1\rangle = g |\Psi\rangle$ having this property can only be transformed into some state $|\Psi_2\rangle = h |\Psi\rangle$ given in Theorem 2 the conditions $\mathcal{E}(H_i) = G_i$ (see Eq. (11)) imply that $G_i = (g_i^w) g_i^w$ for $i = 2, 3, 4$. Indeed, one can easily show that any state obeying the conditions above allows for non-trivial LOCC transformations (see Appendix B). Thus, we have the following theorem, which shows that deterministic LOCC manipulations among fully entangled 4-qubit states are almost never possible.

**Theorem 3.** A generic state $g |\Psi\rangle$ is convertible via LOCC to some other state iff (up to permutations) $g = g^1 \otimes g^2_w \otimes g^3_w \otimes g^4_w$, with $w \in \{x, y, z\}$ and $g^1$ arbitrary.

Combining now Theorem 2 and Theorem 3 we see that every state that can be reached via LOCC can at the same time be transformed into another state and that all states that are not of the form given in Theorem 3 (which are almost all) are isolated. Moreover, the non-isolated generic states which are in the MES constitute a 10-parameter family of the form $G_w |\Psi\rangle$, with $G_w = g_w^1 \otimes g_w^2 \otimes g_w^3 \otimes g_w^4$, with $w \in \{x, y, z\}$ (excluding the case where $g_w^{1} \notin 1$ for exactly one $i$). Thus, the set of generic 4-qubit entangled states is divided into two subsets with very different physical properties: A zero-measure set of LOCC convertible states and a full-measure set of isolated states. Clearly, the first one appears to be the physically more relevant one. In particular, its intersection with the MES, i.e. the set $\{g_w^1 \otimes g_w^2 \otimes g_w^3 \otimes g_w^4 |\Psi\rangle\}$ with $w \in \{x, y, z\}$, gives rise to the most useful states under LOCC manipulation. Note that all these states can again be written in a very simple
form as $|0\rangle |\Psi_0\rangle + |1\rangle X^{\otimes 3} |\Psi_0\rangle$, with $|\Psi_0\rangle$ depending on the SLOCC parameters $\{g^{(i)}_j\}$ and the seed parameters.

In summary, we have introduced here the concept of the MES of $n$-qubit states, $S_n$, which is the analogue of the maximally entangled state in the bipartite case. We derived $S_3$ and have shown that almost all states are in $S_3$ by showing that all SEP transformations can be implemented by LOCC in this case. For more than two parties the MES contains infinitely many states; however, while $S_3$ is of measure zero, $S_2$ is of full-measure because almost all states are isolated. Those results implicate that almost all entangled 4–qubit states are incomparable according to the LOCC paradigm, which induces a rather trivial ordering in the set of entangled states and imply that almost all states are useless for entanglement manipulation. However, we determined the zero-measure subset of generic non–isolated states in the MES, which allows for a very simple decomposition. Understanding in more depth the physical and mathematical properties of this class of states might lead to new insights in multipartite entanglement and its applications. It will also be interesting to study the MES in the case of LOCC transformations among multiple copies of states. In a forthcoming article [22] we investigate all possible state transformations within the four qubit case. We anticipate that even in the non–generic case very few states in the MES allow for LOCC conversions.

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**APPENDIX A: 3–QUBIT MES**

We provide here the details of the proof of Theorem 1, which we restate here,

**Theorem 1.** The MES of three qubits, $S_3$, is given by

$$S_3 = \{g_1^x \otimes g_2^x \otimes g_3^x |GHZ\}, g_1 \otimes g_2 \otimes 1|W\}, \tag{8}$$

where, no $g_x^z \propto 1$ (except for the GHZ state) and $g_1$ and $g_2$ are diagonal.

The outline of the proof is as follows. Within each SLOCC–class we show first that none of the states in $S_3$ can be reached via separable maps (SEP). Since SEP includes LOCC this implies that these states cannot be obtained by any LOCC protocol either. Then we prove that all other states can be obtained via LOCC from states that are in the MES. In particular, we present the corresponding LOCC protocols.
Here, as in the main text, we use the notation \( H = h^j h^k \otimes H_i \), and \( G = g^j g^k \equiv G_i \) correspond to local operators and \( r = n_{q_2} / n_{q_1} \) with \( n_{q_1} = || |\Psi_1||^2 \).

In order to prove Theorem 1 we consider the two–partite SLOCC–classes, the GHZ–class and the W–class seperately, in Lemma 1 and 2 respectively. Let us start with the GHZ–class. We will use in the following the standard form for states in the GHZ–class that was introduced in the main text, i.e.

\[
g_1^1 \otimes g_2^2 \otimes (g_3^1 P_2)(GHZ),
\]

with \( P_2 = diag(z, z^{-1}) \) and \( z \in \mathbb{G} \). As mentioned before, every element of the symmetry group can be written as \( X^k P_2 \), with \( k \in \{0,1\} \). This follows from the fact that the symmetry group of the GHZ state is generated by \( P_1 \oplus P_2 \oplus g_{(\gamma_2)(\gamma_3)}^{-1} \equiv P_2 \), with \( \gamma_i \in \mathbb{G} \) and \( X \equiv X^{\otimes 3} \) and the fact that \( P_2 X = XP_2^{-1} \).

**Lemma 1.** The subset of states in \( S_3 \) which are in the GHZ–class is given by

\[
\{g_1^1 \otimes g_2^2 \otimes g_3^1(GHZ), |GHZ)\},
\]

where no \( g_x^i \propto 1 \).

**Proof.** We denote by \( |\Psi_1\rangle = |G|GHZ\) an arbitrary initial state and by \( |\Psi_2\rangle = |h|GHZ\) an arbitrary final state. Here, \( g = g_1^1 \otimes g_2^2 \otimes g_3^1 P_{x_2} \), with \( z_2 = |z_2|e^{i\alpha_2} \), and similarly for \( h \). Note that \( z_2 = \pm 1 \) implies that \( |\Psi_1\rangle \) is an element of the set given in Eq. (11). Using Eq. (10) we have that a state \( |\Psi_1\rangle \) can be transformed into \( |\Psi_2\rangle \) via SEP iff there exist finitely many probabilities, \( p_{k,\gamma} \) such that

\[
\sum_{k,\gamma} p_{k,\gamma} X^k P_{\gamma} H P_{\gamma} X^{-k} = rG.
\]

Considering now the necessary conditions \( tr(\sum_{k,\gamma} p_{k,\gamma} X^k P_{\gamma} H P_{\gamma} X^{-k}) = rtr(G(|III||II)) \) for \( r = 1 \), we have

\[
\sum_{k,\gamma} p_{k,\gamma} |z_2|^{2(-1)^k} = r|z_2|^2 \text{ and } \sum_{k,\gamma} p_{k,\gamma} |z_3|^{-2(-1)^k} = r|z_3|^{-2}.
\]

Let us now show that if \( |\Psi_2\rangle \) is as in Eq. (11) it cannot be obtained from any other state via LOCC. If \( |z_3\rangle = 1 \) the conditions above imply that \( |z_2\rangle = 1 \) and that \( r = 1 \), i.e. \( n_{q_2} = n_{q_1} \). The last condition is equivalent to \( h_1^{(1)} h_1^{(2)} h_3^{(1)} \cos(2\alpha_h) = g_1^{(1)} g_1^{(2)} g_1^{(3)} \cos(2\alpha_g) \).

Considering now \( tr(G|000\rangle \langle 111|) \) we obtain

\[
\sum_{k,\gamma} p_{k,\gamma} h_1^{(1)} h_1^{(2)} h_3^{(1)} e^{(-1)^k 2i\alpha_h} = g_1^{(1)} g_1^{(2)} g_1^{(3)} e^{2i\alpha_g}.
\]

Hence, if \( h_1^{(1)} h_1^{(2)} h_3^{(1)} \neq 0 \), the condition \( \alpha_h \in (0,\pi) \) implies that the same must be true for \( \alpha_g \). We used here that a state with \( \alpha_g = \pm \pi/2 \) is LU–equivalent to a state with \( \alpha_g \in (0,\pi) \). This shows that if \( h_1^{(1)} h_1^{(2)} h_3^{(1)} \neq 0 \) a state with \( z_2 = \pm 1 \) can only be obtained from a state with \( z_2 = \pm 1 \), which are precisely the states given in Eq. (11).

We will prove now that a state \( |\Psi_2\rangle = h_1^1 \otimes h_2^2 \otimes h_3^3(GHZ) \), with \( h_2^{(i)} \neq 1 \) for \( i \in \{1,2,3\} \) and \( z_k = \pm 1 \), can neither be obtained from any other state of this form nor from the GHZ state. Note that \( X \) commutes with \( H \). Thus, we only have to consider the symmetries of \( P_2 \). Due to the discussion above we have that \( r = 1 \) (see Eq. (3)) in this case. Taking now the trace of Eq. (9) leads to

\[
\sum_{k,\gamma} p_{k,\gamma} (|\gamma_1|^2 + |\gamma_1|^{-2}) (|\gamma_2|^2 + |\gamma_2|^{-2}) \times (|\gamma_1\gamma_2|^2 + |\gamma_1\gamma_2|^{-2}) = 8,
\]

which is satisfied iff \( |\gamma_i| = 1 \) for all \( \gamma_i \) occurring in the sum. Thus, we only have to consider unitary symmetries in Eq. (9). This implies that the vector containing the eigenvalues of \( G \) must be majorized by the corresponding vector of \( H \), which implies that \( |g_i^{(1)}| \) cannot decrease under LOCC. Using in addition that the norm of the two states coincides, i.e. \( |g_1^{(1)} g_1^{(2)} g_1^{(3)}| = h_1^{(1)} h_1^{(2)} h_3^{(3)} \) \( r = 1 \), we have that \( h_1^{(i)} = g_1^{(i)} \), \( i \in \{1,2,3\} \). Thus, \( |\Psi_1\rangle \) can only be transformed into \( |\Psi_2\rangle \) if the states are LU–equivalent. As already mentioned in the main text, the GHZ state is in \( S_3 \), since all the single qubit reduced states of the GHZ state are completely mixed.

We will prove now that a state is in the MES only if it is of the form given in Eq. (11), where no \( g_x^i \propto 1 \). In order to show that, we present the explicit LOCC protocols that allow to reach all the other states in the GHZ–class from a state in the MES. Let us start by showing that states of the form \( |\Psi_2\rangle = 1 \otimes h_2^2 \otimes h_3^3(GHZ) \), i.e. \( h_1^{(1)} h_1^{(2)} h_3^{(3)} = 0 \), can be obtained from the GHZ state via the following LOCC protocol. Party 2 applies the POVM \( \{h_x^2, h_x^2 Z\} \) and party 3 applies the POVM \( \{h_x^3, h_x^3 Z\} \). They communicate their outcomes, which we denote by \( i_2, i_3 \in \{0,1\} \) to party 1. Then, party 1 applies \( Z^{i_2 + i_3} \). Due to the symmetry of the GHZ state it can be easily seen that this LOCC protocol accomplishes the task.

The remaining class of states, namely those, where \( z_h \neq \pm 1 \), can be obtained from a state with \( z_2 = \pm 1 \), as can be seen as follows. We show that for an arbitrary final state, \( |\Psi_2\rangle = h_1^1 \otimes h_2^2 \otimes h_3^3 P_{x_h}(GHZ) \) (with \( z_h \neq \pm 1 \) there exists an operator \( g_{x_h}^3 \) such that the state \( |\Psi_1\rangle = h_1^1 \otimes h_2^2 \otimes g_{x_3}^3(GHZ) \) can be transformed into \( |\Psi_2\rangle \) via LOCC. Consider the POVM \( \{M_1, M_2\} \) with

\[
M_1 = \sqrt{p} \otimes h_2^2 P_{x_h}(g_{x_3}^3)^{-1}
\]
and
\[ M_2 = \sqrt{p} X \otimes 2 \otimes h_3^3 P_{x_1} X (g_3^3)^{-1}, \tag{13} \]
with \( p = 1/(|z_1|^2 + 1/|z_2|^2) \). It is easy to see that \( g_3^3 \) can be chosen so that \( G_2^3 \) constitutes a POVM. In particular, \( M_1^1 M_1 + M_2^1 M_2 = 1 \) iff
\[ P_{x_1}^1 H \otimes X P_{x_1}^3 P_{x_2} X = G_3^x. \tag{14} \]
The left hand side of this equation is equal to \( 1/2 \mathbb{I} + \tilde{b} X \), with \( \tilde{b} = 2p \cos(2\alpha_k) \). Thus, for any \( z, b \) one can choose \( G_2^3 = 1/2 \mathbb{I} + \tilde{b} X \) to satisfy the above condition. The LOCC protocol to transform \( |\Psi_2\rangle \) into \( |\Psi_1\rangle \) is then given by: Party 3 applies the POVM
\[ \{ h_3^3 P_{x_1} (g_3^3)^{-1}, h_3^3 P_{x_2} X (g_3^3)^{-1} \} \tag{15} \]
and communicates the outcome to the other parties. In case of outcome 1 all other parties do nothing, whereas in case of outcome 2 they apply a \( \mathcal{O} \) operation. Due to the fact that \( \mathcal{O} \) commutes with \( h_3^x \) and \( X \otimes 3 \) is an element of the symmetry group of the GHZ state the desired state is obtained for both outcomes.

The other SLOCC-class for truly entangled three-party states is the \( W \)-class. The symmetry of the \( W \)-state \( |W\rangle = |001\rangle + |010\rangle + |100\rangle \), is given by
\[ S = S_1 \otimes S_2 \otimes S_3 \equiv \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \otimes \begin{pmatrix} z & y \\ 0 & 1/z \end{pmatrix} \otimes \begin{pmatrix} x & -y - z \\ 0 & 1/x \end{pmatrix}. \tag{16} \]
where \( x, y, z \in \mathbb{C} \) and any state in this SLOCC-class can be written as \( x_0|000\rangle + x_1|100\rangle + x_2|010\rangle + x_3|001\rangle \), with \( x_i \geq 0 \). Another way of presenting an arbitrary state in the \( W \)-class, which we will use in the following, is \( g_1 \otimes g_2 \otimes \mathbb{I} |W\rangle \), where
\[ g_1 = \begin{pmatrix} 1 & 0 \\ 0 & x_1 g_1(x)/x_3(g) \end{pmatrix}, \quad g_2 = \begin{pmatrix} x_3(g) & x_0(g) \\ 0 & x_2(g) \end{pmatrix}. \tag{17} \]

**Lemma 2.** The subset of states in \( S_3 \) which are in the \( W \)-class is given by
\[ \{ g_1 \otimes g_2 \otimes \mathbb{I} |W\rangle \}, \tag{18} \]
where \( g_1 \) and \( g_2 \) are diagonal.

**Proof.** In order to prove that the states in this class are in the MES if \( x_0(g) = 0 \), let us first show that we only need to consider unitary stabilizers. Recall from the main text that any element of a POVM that can be implemented must be such that the operation applied by party 3, \( S_3^3 \), is unitary. This implies that the corresponding symmetries obey \( z = -y \) and \( |x| = 1 \) (see Eq. \ref{eq:15}). Inserting these symmetries, as well as the expressions for \( G \) and \( H \), where \( x_0(h) = 0 \), in Eq. \ref{eq:14} and tracing over the third party results in
\[ \sum_{x,y} p_{x,y} x y \begin{pmatrix} 1 \\ y^* x \end{pmatrix} \frac{x^* y}{x_3^*(g)} + |y|^2 \otimes \begin{pmatrix} (x_3(h)^2)^{1/2} b^* \\ b \end{pmatrix} \]
\[ = r \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{x_3(g)^2}{|x_3(g)|^2} \end{pmatrix} \otimes \begin{pmatrix} x_0(g) x_3^*(g) \\ x_0^*(g) x_3(g) \end{pmatrix} |x_3(g)|^2 + |x_2(g)|^2, \]
where \( b = -x^* y |x_3(h)|^2 \) and \( c = |y|^2 |x_3(h)|^2 + |x_2(h)|^2 \). Applying \( |0 \rangle \langle 1| \otimes |0\rangle \langle 0| \) to both sides of the previous equation and taking the trace leads to \( \sum_{x,y} p_{x,y} |y|^2 = 0 \). Thus, \( y \) has to be zero and therefore the stabilizer is unitary (see Eq. \ref{eq:15}).

The unitaries that leave the \( W \) state (up to a global phase) invariant, are given by \( Z(\alpha)^{\otimes 3} \), with \( \alpha \in \mathbb{R} \). Thus, \( |\Psi_1\rangle \) can be transformed into \( |\Psi_2\rangle \) via SEP iff there exist finitely many probabilities \( p_\alpha \) such that
\[ \sum_\alpha p_\alpha H_1 \otimes Z(-\alpha) H_2 Z(\alpha) \otimes \mathbb{I} = r G_1 \otimes G_2 \otimes \mathbb{I}, \tag{19} \]
where we used that \( H_1 \) commutes with any \( Z(\alpha) \). Note that if \( x_0(h) = 0 \), then also \( H_2 \) commutes with \( Z(\alpha) \) and therefore in this case we have \( H_1 \otimes H_2 \otimes \mathbb{I} = r G_1 \otimes G_2 \otimes \mathbb{I} \), which implies the condition \( x_0(g) = 0 \). Hence, states with \( x_0(h) = 0 \) can only be obtained via LOCC from states with \( x_0(g) = 0 \). Moreover, since there is only one POVM element in this case the states can only be transformed into each other if they are LU–equivalent. From this it follows that states with \( x_0(g) = 0 \) are in MES.

Let us now show that these are the only states of the \( W \)-class in MES. In order to do so, we show that any state, \( |\Psi_2\rangle = h_1 \otimes h_2 \otimes \mathbb{I} |W\rangle \), with \( x_0(h) \neq 0 \) can be obtained from a state \( |\Psi_1\rangle = h_1 \otimes g_2 \otimes \mathbb{I} |W\rangle \) with \( x_0(g) = 0 \). One can choose \( g_2 \) such that the POVM
\[ \{ M_1 \equiv \mathbb{I} \otimes h_2 g_2^{-1} \otimes \mathbb{I}, M_2 \equiv Z \otimes h_2 Z g_2^{-1} \otimes Z \} \tag{20} \]
accomplishes this task. This can be seen as follows. The condition \( M_1^1 M_1 + M_2^1 M_2 = \mathbb{I} \) is equivalent to \( G_2 = H_2 \otimes Z H_2 Z \), where the right hand side of this equation is equal to
\[ 2 \begin{pmatrix} x_3(h)^2 & 0 \\ 0 & x_2(h)^2 + x_3(h)^2 \end{pmatrix}. \tag{21} \]
Thus, choosing \( G_2 \) as in the equation above (with \( x_0(g) = 0 \)) ensures that \( \{ M_1, M_2 \} \) is a POVM. Hence, the LOCC protocol transforming \( |\Psi_1\rangle \) into \( |\Psi_2\rangle \) is as follows: Party 2 applies the POVM \( \{ h_2 g_2^{-1}, h_2 Z g_2^{-1} \} \) and communicates the outcome of the measurement to the other parties. In case outcome 1 (2) is obtained, all remaining parties apply the \( Z \) (2) respectively to obtain the desired state. Thus, the states in MES in this class are of the form
\[ |\Psi_W\rangle = g_1 \otimes g_2 \otimes \mathbb{I} |W\rangle, \tag{22} \]
where \( g_1 \) and \( g_2 \) are diagonal, i.e. \( x_0(g) = 0 \). \( \square \)
In summary, we have characterized the MES in Lemma 1 and 2 for each of the three–partite entangled SLOCC–classes separately. Combining these results directly proves Theorem 1. Note that the non–unitary symmetry was only used to derive a standard form within the different SLOCC–classes. The transformations which are then possible are only possible due to the Pauli operators.

Writing the states that are in the MES in the computational basis it is easy to show that any state in $S_3$ is an element of the set

$$\tilde{S}_3 = \{ |\Psi(a, \beta, \beta') \rangle, a \geq 0, |Y(\beta') \rangle \otimes |Y(\beta) \rangle \},$$

where $|\Psi_s \rangle = a |00 \rangle + \sqrt{1 - a^2} |1 \rangle$ has Schmidt decomposition and the parameters $a$, $\beta$ and $\beta'$ are simple functions of the parameters $\{g_{i}^{(i)}\}_{i=1}^{3}$ for the GHZ–class and $\{x_i\}_{i=1}^{3}$ for the $W$–class.

Note that $\tilde{S}_3$ also includes biseparable states, e.g. $a = 1, \beta = 0$ and $\beta'$ arbitrary. Thus, $\tilde{S}_3$ is strictly larger than $S_3$. However, $S_3$ can be obtained from $\tilde{S}_3$, by excluding state like those in the GHZ–class, with $g_{i}^{(i)} = 0$ for some $i$ (excluding the GHZ state), and states which belong to the same LU–equivalence class as one element within the set.

**APPENDIX B: GENERIC 4–QUBIT MES**

In this section we consider LOCC transformations among generic 4–qubit states and we prove Theorems 2 and 3 in the main text which we restate below for readability. As discussed in the main text, these theorems imply that almost all 4–qubit states are both, in $S_4$ and isolated. Moreover, they allow to characterize the subset of the MES of LOCC convertible states.

As we have seen before, for an LOCC transformation from a generic 4–qubit state $|\Psi_1 \rangle = g |\Psi \rangle$ to another $|\Psi_2 \rangle = h |\Psi \rangle$ (here $|\Psi \rangle$ is any of the seed states given in Eq. (6) in the main text) to be possible it must hold that

$$\sum_{k} p_k \sigma_k \otimes 4 H \sigma_k \otimes 4 = G,$$  \hspace{1cm} (23)

where, $H = h^1 \otimes 4 H_1$, and $G = g^1 \otimes 4 G_1$ for each party $i = 1, 2, 3, 4$. Each of these positive operators, e. g. $G_1 = g_1^1 g_1$, is chosen such that $tr(G_1) = 1$ and, therefore, $G_1 = \frac{1}{2^4 I} + \sum_{k} g_{k}^{(i)} \sigma_k$, with $g^{(i)}_{k} = (g_{1}^{(i)}, g_{2}^{(i)}, g_{3}^{(i)}, g_{4}^{(i)}) \in \mathbb{R}^4$ fulfilling $0 \leq |g_{k}^{(i)}| < 1/2$. As mentioned before, Eq. (23) implies that

$$\mathcal{E}_4(H) = \mathcal{E}(H_1) \otimes \mathcal{E}(H_2) \otimes \mathcal{E}(H_3) \otimes \mathcal{E}(H_4).$$  \hspace{1cm} (24)

Like in the main text the Equation 4 is the completely positive map given in Eq. (23) and $\mathcal{E}(\rho) = \sum_{k=0}^{3} p_k \sigma_k \rho \sigma_k$. Similarly, we will use the notation $\mathcal{E}_l(\rho) = \sum_{k=0}^{3} p_k \sigma_k^{(l)} \rho S_k$, where $S_k = \sigma_k^{(l)}$ is acting on $l$ systems and $\rho$ describes $l$ systems.

Note that Eq. (24) only depends on $|\Psi_2 \rangle = h |\Psi \rangle$ and is independent of $|\Psi_1 \rangle = g |\Psi \rangle$. Like in the main text we use the notation $h_{w}^{l} \in span \{ I, W \}$, where $W \in \{ X, Y, Z \}$ such that $H_1 = (h_{w}^{l})^{l} h_{w}^{l} = 1/2 I + h_{w}^{l} W$.

**Theorem 1.** A generic state $h |\Psi \rangle$ is reachable via LOCC from some other state iff either

1. $h = h^{1} \otimes 4 \beta$ with $h^{1} \neq 0$ arbitrary or
2. $h = h^{1} \otimes h^{2} \otimes h^{3} \otimes h^{4}$, for $w \in \{ x, y, z \}$ where $h^{1} \neq h^{w}$.

**Proof.** Throughout the proof we use lowercase indices $i \in \{ 1, 2, 3 \}$ for the coordinates of the SLOCC parameters and uppercase indices $I \in \{ 1, 2, 3, 4 \}$ for the parties, e. g. $h^{i}$ denotes the $i$th component of the operator $h$ acting on system $I$.

Only if: We first show that the conditions stated in the Theorem are necessary for a state $h |\Psi \rangle$ to be obtainable via LOCC from some other state $g |\Psi \rangle$. From Eq. (24) it follows that $\mathcal{E}_4(H_1 \otimes H_2) = \mathcal{E}(H_1) \otimes \mathcal{E}(H_2)$ must hold (and similarly for other pairs of parties). This is equivalent to

$$h^{(1)} (h^{(2)})^T \otimes (N_1 - N_2) = 0,$$  \hspace{1cm} (25)

where $\otimes$ is the Hadamard product (i. e. entry-wise matrix multiplication), $h^{(i)} = (h_{1}^{(i)}, h_{2}^{(i)}, h_{3}^{(i)}, h_{4}^{(i)})^T$, for any system $I$ and $N_1 = \eta_0 \eta^T$, with $\eta = (\eta_0, \eta_2, \eta_3, \eta_1)$ and

$$N_2 = \begin{pmatrix} \eta_0 & \eta_3 & \eta_2 \\ \eta_3 & \eta_0 & \eta_1 \\ \eta_2 & \eta_1 & \eta_0 \end{pmatrix}. \hspace{1cm} (26)$$

We used here the notation $\eta_0 = \sum_{k=0}^{3} p_k = 1$ and $\eta_i = p_i - (p_j + p_k)$ (where $i, j, k$ are assumed here to be all different).

Notice that if more than one $\eta_i^2$ equals 1, then only one $p_k$ differs from zero, which implies that the initial and the final states are LU–equivalent. Thus, we are looking for solutions of Eq. (24) where there exists at most one $i \in \{ 1, 2, 3 \}$ such that $\eta_i^2 = 1$.

Clearly, if $h^{(1)} (h^{(2)})^T = 0$, which is the case iff $h^{(1)} = 0$ and/or $h^{(2)} = 0$, Eq. (25) is satisfied. The same reasoning extends to other pairs of parties and this leads to the states of case (i) of the Theorem, where we need to take into account that for one party it must hold that $h^{(i)} \neq 0$ since we know that seed states are in the MES. In case $h^{(1)} (h^{(2)})^T \neq 0$ there can be at most one $i$ such that $h_i^{(1)} h_i^{(2)} \neq 0$, since otherwise more than one $\eta_i^2$ must be 1, which can be seen by looking at the diagonal entries of Eq. (24). Let us consider first the case in which no $\eta_i^2 = 1$. Then, $h^{(1)} = (0, h_2^{(1)}, h_3^{(1)})$ and $h^{(2)} = (h_2^{(2)}, 0, 0)$ up to permutations of the parties and/or entries of the vectors. Moreover, imposing Eq. (24) for parties (1,2) and 3 and parties (1,2) and 4 leads to $h^{(3)} = h^{(4)} = 0$ (otherwise $\eta_i^2 = 1$). Those states are included in case
(ii) of the Theorem. Finally, let us consider the case in which \( q_i^2 = 1 \) for exactly one \( i \). Say, without loss of generality, \( q_i^2 = 1 \). Then, by looking at Eq. \( 25 \) for other pairs of parties it must hold that \( h^{(1)} = \) arbitrary, \( h^{(2)} = (h^{(2)}_1, 0, 0), h^{(3)} = (h^{(3)}_1, 0, 0) \) and \( h^{(4)} = (h^{(4)}_1, 0, 0) \) up to permutations, which corresponds to the case of states of (ii) of the Theorem. However, it remains to show that \( h^{(1)} \) is not completely arbitrary as the case \( h^{(1)} = (h^{(1)}_1, 0, 0) \) must be excluded. To see this, notice that \( E(H_I) = G_I \) for all parties \( I \) and \( q_i^2 = 1 \) imposes that \( |h^{(1)}_i| = |g^{(1)}_i| \). This, together with the fact that \( |g^{(i)}| \) cannot decrease under deterministic LOCC transformations proven in the main text, imposes that \( h^{(1)}_w \otimes h^{(2)}_w \otimes h^{(3)}_w \otimes h^{(4)}_w |\Psi\rangle \) cannot be reached from any other LU inequivalent state.

If: Let us now show that the states given in the Theorem can indeed be reached via LOCC. Let us first treat the states belonging to case (ii) where, without loss of generality, we choose \( w = x \). Consider \( |\Psi_1\rangle = g^1_1 \otimes g^2_w \otimes g^3_x \otimes g^4_w |\Psi\rangle \) for some \( g^1_1 \), which is specified below, and the two-outcome POVM \( \{M_1, M_2\} \) with

\[
M_1 = \frac{1}{\sqrt{2}} h^{(1)} g^{(1)} - 1 \otimes I^\otimes 3, \\
M_2 = \frac{1}{\sqrt{2}} h^{(1)} g^{(1)} - 1 \otimes X^\otimes 3.
\]

Since \( 1/2(\sqrt{2}X H_1 X + H_1) = 1/2 I + h^{(1)}_X |\Psi\rangle \), \( \{M_1, M_2\} \) is a valid POVM whenever \( h^{(1)}_w = g^{(1)}_w \). Moreover, the POVM can be implemented by LOCC: party 1 implements the POVM \( \{h^{(1)}(g^1) - 1/\sqrt{2}, h^{(1)}(g^1) - 1/\sqrt{2}\} \), and in case of the second outcome the other parties implement the LU \( X \). Since \( h^{(1)} X = 0 \) and \( X^\otimes 3 |\Psi\rangle = |\Psi\rangle \) for any seed state \( |\Psi\rangle \), both branches of the protocol lead to the same outcome. Thus, any state \( h^{(1)} \otimes h^{(2)} \otimes h^{(3)} \otimes h^{(4)}_w |\Psi\rangle \) with arbitrary \( h^{(1)} \neq h^{(2)} \) can be obtained by LOCC from \( |\Psi_1\rangle \) if \( h^{(1)} = g^{(1)}_w \) holds.

Let us now consider the case of the states of case (i), where \( h^{(1)} \) is arbitrary, including in particular the case \( h^{(1)} = h^{(2)} \), and all the other \( h^{(i)} \) are proportional to the identity. This state can be obtained form the seed state \( |\Psi\rangle \) via the POVM \( \{M_i\}_{i=0}^3 \), with

\[
M_i = \frac{1}{\sqrt{2}} h^{(i)} \sigma_i \otimes \sigma_i^\otimes 3.
\]

Arguing as above, it is clearly seen that this POVM can also be implemented by LOCC. \( \square \)

**Theorem 2.** A generic state \( g |\Psi\rangle \) is is convertible via LOCC to some other state if (up to permutations) \( g = g^1_1 \otimes g^2_w \otimes g^3_w \otimes g^4_w \) with \( w \in \{x, y, z\} \) and \( g^{(i)}_w \) arbitrary.

*Proof. If:* Let us choose without loss of generality \( w = x \), since the same argument applies to any other choice. Following the same methods as in the proof of Theorem \( \ref{thm:main} \), it can be readily seen that \( |\Psi_1\rangle = g |\Psi\rangle \) with \( g = g^1_1 \otimes g^2_w \otimes g^3_w \otimes g^4_w \) can be transformed into some (LU–inequivalent) state \( |\Psi_2\rangle = h |\Psi\rangle \) with \( h = h^1 \otimes h^2_w \otimes h^3_w \otimes h^4_w \). For that, choose a similar POVM \( \{M_1, M_2\} \) to that of Eq. \( 27 \) where now

\[
M_1 = \sqrt{p} h^{(1)}(g^1_1)^{-1} \otimes I^\otimes 3, \\
M_2 = \sqrt{1-p} h^{2} X(g^1_1)^{-1} \otimes X^\otimes 3
\]

with \( 1 > p > 0 \) arbitrary. This is indeed a POVM if \( (1-p)X H_1 X + pH_1 = G_1 \), which amounts to \( h^{(1)} = g^{(1)}_1 \) and \( (2p-1)h^{(1)}_{2,3} = g^{(1)}_{2,3} \). Since \( 0 \leq |g^{(i)}|, |h^{(i)}| < 1/2 \) is the only condition for the states, given any state \( |\Psi_1\rangle \) of the above form, there is always some value of \( p \) so that it can be transformed by LOCC to a non-LU equivalent state \( |\Psi_2\rangle \) of the above form by increasing (by the same proportion) the parameters \( g^{(1)}_{2,3} \) and \( g^{(1)}_w \).

Similarly, one can show that if \( |\Psi_1\rangle = g |\Psi\rangle \) with \( g = g^1_1 \otimes I^\otimes 3 \), then the state can be transformed by LOCC into another state. This is any state of the form \( |\Psi_2\rangle = h^1 \otimes I^\otimes 3 |\Psi\rangle \) with \( h^1 \) such that there exists probabilities \( p_k \) and therefore values of \( q_k \) such that \( G_1 = E(H_1) = 1/2 I + \sum_k h_k^{(1)} q_k \sigma_i \). The corresponding POVM is \( \{M_i\}_{i=1}^4 \) with

\[
M_i = \sqrt{p} h^1 \sigma_i (g^1_1)^{-1} \otimes (\sigma_i)^\otimes 3,
\]

which completes the proof of the if-part.

**Only if:** Due to Theorem \( \ref{thm:main} \) we know that the only states which can be reached via LOCC are of the form \( |\Psi_2\rangle = h |\Psi\rangle \) with either \( h = h^1 \otimes I^\otimes 3 \) with \( h^1 \neq 1 \) arbitrary (case (i)) or \( h = h^1 \otimes h^2_w \otimes h^3_w \otimes h^4_w \), for \( w \in \{x, y, z\} \) where \( h^1 \neq h^2_w \) (case (ii)). Thus, any LOCC convertible state, \( |\Psi_1\rangle = g |\Psi\rangle \), can only be transformed into one of these states. It is easy to see now that \( |\Psi_1\rangle \) must be of the form given in Theorem \( \ref{thm:main} \) since Eq. \( 23 \) implies that \( E(H_i) = 1/2 I + \sum_k q_k h_k^{(i)} \sigma_i = G_i \). Thus, a component of \( G_i \) can only be non-vanishing if the corresponding component of \( H_i \) is non–vanishing, which proves the statement. \( \square \)

[1] G. Gour and N.R. Wallach, New J. Phys. 13, 073013 (2011).
[2] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A 65, 032308 (2002).
[3] Note that we can choose here \( w = z \neq 0 \), since we are considering fully entangled states.