ON THE DISTRIBUTION OF FRACTIONS WITH
POWER DENOMINATOR

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Abstract. In this paper we obtain a sharp upper bound for the
number of solutions to a certain diophantine inequality involving
fractions with power denominator. This problem is motivated by a
conjecture of Zhao concerning the spacing of such fractions in short
intervals and the large sieve for power modulus. As applications
of our estimate we show Zhao’s conjecture is true except for a set
of small measure and give a new $\ell_1 \to \ell_2$ large sieve inequality
for power modulus. Our techniques are based on a variant of the
van der Corput method with origins in the work of Bombieri and
Iwaniec.

1. Introduction

Given integers $k$ and $N$ consider the set of fractions of the form

$$S_k(N) = \left\{ \frac{u}{n^k}, \ 1 \leq u \leq n^k, \ 1 \leq n \leq N, \ (n, q) = 1 \right\}. \tag{1.1}$$

In this paper we consider the distribution of $S_k(N) \subset [0, 1]$. This
problem first appears to be considered by Zhao [28] and is motivated
by large sieve type inequalities for characters to power moduli. For such
inequalities, one seeks to determine the smallest $\Delta_k(N, M)$ depending
on $k$, $N$ and $M$ such that for any sequence of complex numbers $\alpha_m$ we have

$$\sum_{1 \leq n \leq N} \sum_{a=1}^{n^k} \left| \sum_{K < m \leq K + M} \alpha_m e\left(\frac{am}{n^k}\right)\right|^2 \leq \Delta_k(N, M) \| \alpha \|_2^2. \tag{1.2}$$

We note the classical large sieve inequality states that

$$\Delta_1(N, M) \ll M + N^2,$$

and this implies the estimates

$$\Delta_k(N, M) \ll M + N^{2k} \quad \text{and} \quad \Delta_k(N, M) \ll NM + N^{k+1}. \tag{1.2}$$
In [28], Zhao provided some estimates for $\Delta_k(N, M)$ which go beyond (1.2) and conjectured that

\begin{equation}
\Delta_k(N, M) \ll (N^{k+1} + M)N^{o(1)},
\end{equation}

which is based on heuristics for the density of points of the form (1.1). One may establish quantitative relationships between $\Delta_k(N, M)$ and estimates for the number of points of $S_k(N)$ in short intervals. For real numbers $x$ and $Y$ we define

\begin{equation}
I_{k,N}(x, Y) = \left| \left\{ z \in S_k(N) : \| z - x \| \leq \frac{1}{Y} \right\} \right|.
\end{equation}

In [28], Zhao conjectured that for any $x$ we have

\begin{equation}
I_{k,N}(x, N^{k+1}) \ll N^{o(1)},
\end{equation}

and showed that (1.5) implies (1.3). There have been a number of improvements and extensions of the results of [28] to which we refer the reader to [6, 9, 18] for power moduli, to [1, 19–21] for more general sparse sets of moduli and to [2, 3, 5] for extensions to function fields and Gaussian integers. Such large sieve inequalities have had applications to a variety of different areas of number theory to which we refer the reader to [7, 10, 11, 22] for applications to the distribution of primes, to [12, 26] for elliptic curves and to [4, 15, 25] for arithmetic problems in finite fields and function fields.

In this paper we consider the distribution of points in $S_k(N)$. For integers $k, N, Y$ we let $I_k(N, Y)$ count the number of solutions to the inequality

\[ \left| \frac{u_1}{n_1^k} - \frac{u_2}{n_2^k} \right| \leq \frac{1}{Y}, \]

with variables satisfying

\[ 1 \leq n_1, n_2 \leq N, \quad 1 \leq u_1 \leq n_1^k, \quad 1 \leq u_2 \leq n_2^k, \]

and note that $I_k(N, Y)$ may be considered as an $\ell_2$-norm estimate of $I_{k,N}(x, Y)$ defined in (1.4). We give a sharp estimate for $I_k(N, Y)$ and in particular show that

\[ I_k(N, Y) \leq \left( \frac{N^{2k+2}}{Y} + N^{k+1} \right) N^{o(1)}. \]

As applications of our estimate we show Zhao’s conjecture (1.5) is true except for a set of small measure and provide a new large sieve type inequality for fractions with power denominator which may be thought of as an $\ell_1 \to \ell_2$ variant of conjecture (1.3). Our techniques are based
on the van der Corput method of exponential sums. For a typical application of the van der Corput method one iterates Poisson summation with a secondary process which reduces the amplitude of the exponential sums under consideration. One combines these two processes with the goal of reducing the length of summation until the resulting sums may be estimated trivially or by some special method. In their work on the Riemann zeta function \([13,14],\) Bombieri and Iwaniec showed how one may adapt the van der Corput method to obtain sharp estimates for certain diophantine inequalities. The basic idea is that when interpreting an inequality of the form

\[ |\phi(x) - \phi(y)| \leq \frac{1}{Y}, \quad x, y \in X, \]

in terms of moments of integrals we have

\[
\frac{1}{Y} \int_{-Y}^{Y} \left| \sum_{x \in X} e(y\phi(x)) \right|^2 dy \ll \frac{1}{X} \int_{-X}^{X} \left| \sum_{x \in X} e(y\phi(x)) \right|^2 dy,
\]

for any \(X \leq Y\) and this serves as a suitable process to iterate with Poisson summation in order to reduce the length of summation. We also refer the reader to [24] for a variation of this technique.

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2. Main results

**Theorem 2.1.** For integers \(k\) and \(N\) and a positive real number \(Y\) we let \(I_k(N, Y)\) count the number of solutions to the inequality

\[ \left| \frac{u_1}{n_1^k} - \frac{u_2}{n_2^k} \right| \leq \frac{1}{Y}, \]

with variables satisfying

\[ 1 \leq n_1, n_2 \leq N, \quad 1 \leq u_1 \leq n_1^k, \quad 1 \leq u_2 \leq n_2^k. \]

Then we have

\[ I_k(N, Y) \ll \left( \frac{N^{2k+2}}{Y} + N^{k+1} \right) N^{o(1)}. \]

As an immediate Corollary to Theorem 2.1, we have.

**Corollary 2.2.** Let \(k\) and \(N\) be integers, \(Y\) and \(x\) real numbers and define \(I_{k,N}(x, Y)\) as in (1.4). For any \(\varepsilon > 0\) have

\[ \mu \left( x \in [0, 1] : I_{k,N}(x, N^{k+1}) \geq N^{\varepsilon} \right) \leq N^{-\varepsilon + o(1)}, \]

where \(\mu\) denotes the Lebesgue measure.
Another consequence of Theorem 2.1 is the following \( \ell_1 \rightarrow \ell_2 \) large sieve inequality.

**Corollary 2.3.** For any integers \( k, M, N \) and any sequence of complex numbers \( \alpha_m \) we have

\[
\sum_{1 \leq n \leq N} \sum_{a=1 \atop (a,n)=1}^{n^k} \frac{\alpha_m e\left(\frac{am}{n^k}\right)}{K < m \leq K + M} \ll \left(N^{k+1} + M^{1/2}N^{(k+1)/2}\right)N^{o(1)}\|\alpha\|_2.
\]

We note that the conjecture (1.3) implies Corollary 2.3 by the Cauchy-Schwarz inequality and that Corollary 2.3 is sharper than what one would obtain using results of [6,9,18].

3. Preliminary results

The following forms the basis of the van der Corput method of exponential sums, for a proof see [17, Theorem 8.16].

**Lemma 3.1.** For any real valued function \( f \) defined on an interval \([a, b]\) with derivatives satisfying

\[
\frac{T}{M^2} \ll f''(z) \ll \frac{T}{M^2}, \quad |f^{(3)}(z)| \ll \frac{T}{M^3}, \quad |f^{(4)}(z)| \ll \frac{T}{M^4}, \quad z \in [a, b],
\]

we have

\[
\sum_{a < n < b} e(f(n)) = \sum_{\alpha < m < \beta} f''(x_m)^{-1/2}e\left(f(x_m) - mx_m + \frac{1}{8}\right) + E,
\]

where \( \alpha = f'(a), \beta = f'(b), x_m \) is the unique solution to \( f'(x) = m \) for \( x \in [a, b] \) and

\[
E \ll \frac{M}{T^{1/2}} + \log(|f'(b) - f'(a)| + 2).
\]

Our main application of Lemma 3.1 will be when \( f \) is a monomial and in this case we have the following.

**Lemma 3.2.** Let \( N, \eta, \alpha, \) and \( y \) be real numbers satisfying

\[
\alpha \notin \mathbb{N}, \quad 1 \leq \eta \ll 1, \quad \rho > 0, \quad y > 0.
\]

Then we have

\[
\sum_{N < n < \eta N} e\left(\frac{y}{\alpha} \left(\frac{n}{N}\right)^\alpha\right) = \left(\beta - 1\right)y^{1/2} \sum_{c_1M < m < c_2M} \frac{1}{m} \left(\frac{m}{M}\right)^{\beta/2} e\left(\frac{1}{8} - \frac{y}{\beta} \left(\frac{m}{M}\right)^\beta\right) + O\left(\frac{N}{y^{1/2} + \log y}\right),
\]

\[
\sum_{a=1 \atop (a,n)=1}^{n^k} \frac{\alpha_m e\left(\frac{am}{n^k}\right)}{K < m \leq K + M} \ll \left(N^{k+1} + M^{1/2}N^{(k+1)/2}\right)N^{o(1)}\|\alpha\|_2.
\]
where $M$ and $\beta$ are defined by
\[
M = \frac{y}{N}, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1,
\]
and the constants $c_1$ and $c_2$ depend only on $\alpha$ and $\eta$.

We note that the constants $c_1$ and $c_2$ occurring in Lemma 3.2 can be given explicitly in terms of $\alpha$ although to state this result requires separating cases $\alpha > 0$ and $\alpha < 0$ and we have found the above form more convenient.

The following is a special case of [27, Lemma 2.1].

**Lemma 3.3.** For two intervals $I_1, I_2$, a real valued function
\[
\phi : I_1 \times I_2 \to \mathbb{R},
\]
and a positive real number $Y$, we let $J(I_1, I_2, \phi, Y)$ count the number of solutions to the inequality
\[
|\phi(u_1, n_1) - \phi(u_2, n_2)| \leq \frac{1}{Y},
\]
with integer variables
\[
n_1, n_2 \in I_1, \quad u_1, u_2 \in I_2.
\]
We have
\[
J(I_1, I_2, \phi, Y) \ll \frac{1}{Y} \int_{-Y}^{Y} \left| \sum_{n \in I_1} \sum_{u \in I_2} e(y\phi(n, u)) \right|^2 dy,
\]
and for any sequence of complex numbers $\theta(n, u)$ satisfying
(3.1)
\[
|\theta(n, u)| \leq 1,
\]
we have
\[
\frac{1}{Y} \int_{-Y}^{Y} \left| \sum_{n \in I_1} \sum_{u \in I_2} \theta(n, u)e(y\phi(n, u)) \right|^2 dy \ll J(I_1, I_2, \phi, Y).
\]

The next two results are consequences of Lemma 3.3.

**Lemma 3.4.** For intervals $I_1$ and $I_2$, a real valued function
\[
\phi : I_1 \times I_2 \to \mathbb{R},
\]
and positive real numbers $Y$ and $Z$ with $Z \leq Y$ we have
\[
\frac{1}{Y} \int_{-Y}^{Y} \left| \sum_{u \in I_1} \sum_{n \in I_2} e(y\phi(n, u)) \right|^2 dy \ll \frac{1}{Z} \int_{-Z}^{Z} \left| \sum_{n \in I_1} \sum_{u \in I_2} e(y\phi(n, u)) \right|^2 dy.
\]
**Lemma 3.5.** For intervals $I_1$ and $I_2$, a real valued function
\[ \phi : I_1 \times I_2 \to \mathbb{R}, \]
a positive real number $Y$ and a sequence of complex numbers $\theta(n, u)$ satisfying
\[ |\theta(n, u)| \leq 1, \]
we have
\[ \int_{-Y}^{Y} \left| \sum_{u \in I_1} \sum_{n \in I_2} \theta(n, u) e(y\phi(n, u)) \right|^2 dy \ll \int_{-Y}^{Y} \left| \sum_{u \in I_1} \sum_{n \in I_2} e(y\phi(n, u)) \right|^2 dy. \]

The following is [16, Lemma 6].

**Lemma 3.6.** Let $V, L, \nu$ and $\lambda$ be real numbers satisfying
\[ 0 < L \leq V \leq \nu V < \lambda L, \]
and let $a_v$ be a sequence of complex numbers satisfying $|a_v| \leq 1$. Then we have
\[ \sum_{V < v \leq \nu V} a_v = \frac{1}{2\pi} \int_{-L}^{L} \left( \sum_{L < \ell \leq \lambda L} a_{\ell} e^{-it\ell} \right) V^{it} (\nu^{it} - 1)t^{-1} dt + O(\log (2 + L)). \]

**Lemma 3.7.** Let $Y, N, U, \alpha, \rho, \gamma$ and $\eta$ be positive real numbers satisfying
\[ Y \geq 2, \quad \alpha \not\in \mathbb{N}, \quad \eta, \rho > 1. \]
There exists real numbers $Y_1, Y_2, c_1, c_2$ satisfying
\[ Y \ll Y_1 \ll Y_2 \ll Y, \quad 1 \ll c_3 \ll c_4 \ll 1, \]
and a sequence of complex numbers $\theta(n, u)$ satisfying
\[ |\theta(n, u)| \leq 1, \]
such that
\[ \int_{Y}^{2Y} \left| \sum_{N \leq n < \eta N} \sum_{U \leq u < \rho U} e \left( y \left( \frac{u}{U} \right)^{\alpha} \left( \frac{n}{N} \right)^{\gamma} \right) \right|^2 dy \ll \frac{U^2 (\log Y)^2}{Y} \int_{Y_1}^{Y_2} \left| \sum_{N \leq n < \eta N} \sum_{U \leq u \leq c_4 Y/U} \theta(n, u) e \left( y \left( \frac{n}{N} \right)^{\gamma(1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) \right|^2 dy \]
\[ + U^2 N^2 (\log Y)^2, \]
with implied constants depending only on $\alpha, \rho, \gamma$ and $\eta$ and $\beta$ is given by
\[ \frac{1}{\alpha} + \frac{1}{\beta} = 1. \]
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Proof. By Lemma 3.2, for fixed $Y \leq y \leq 2Y$ and $N \leq n < \eta N$, we have

$$
\sum_{U \leq u < \rho U} e \left( y \left( \frac{u}{U} \right)^{\alpha} \left( \frac{n}{N} \right)^{\gamma} \right) = \left( \frac{(\beta - 1)\alpha yn^{\gamma}}{N^{\gamma}} \right)^{1/2}
$$

$$
\times \sum_{c_1 yn^{\gamma} U^{-1} N^{-\gamma} \leq v < c_2 yn^{\gamma} U^{-1} N^{-\gamma}} \frac{1}{v} \left( \frac{v}{yn^{\gamma} U^{-1} N^{-\gamma}} \right)^{\beta/2} e \left( \frac{1}{8} - \frac{\alpha yn^{\gamma}}{N^{\gamma} \beta} \left( \frac{v}{yn^{\gamma} U^{-1} N^{-\gamma}} \right)^{\beta} \right)
$$

$$
+ O \left( \frac{U \log Y}{Y^{1/2}} \right),
$$

where $\beta$ is given by

$$
\frac{1}{\alpha} + \frac{1}{\beta} = 1,
$$

and the constants $c_1$ and $c_2$ depend only on $\alpha$. Summing the above over $N \leq n < \eta N$ and taking absolute values, we see that

$$
\left| \sum_{N \leq n < \eta N} \sum_{U \leq u < \rho U} e \left( y \left( \frac{u}{U} \right)^{\alpha} \left( \frac{n}{N} \right)^{\gamma} \right) \right| \ll \frac{U}{Y^{1/2}}
$$

$$
\left| \sum_{N \leq n < \eta N} \sum_{c_1 yn^{\gamma} V/Y^{\gamma} \leq v < c_2 yn^{\gamma} V/Y^{\gamma}} \theta(n, v) e \left( \frac{\alpha y}{\beta} \left( \frac{Y}{y} \right)^{\beta} \left( \frac{n}{N} \right)^{\gamma(1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) \right|
$$

$$
+ \frac{UN \log Y}{Y^{1/2}},
$$

where

$$
V = \frac{Y}{U},
$$

and

$$
\theta(n, v) = \left( \frac{n}{N} \right)^{\gamma/2} \frac{Y}{vU} \left( \frac{v}{yn^{\gamma} U^{-1} N^{-\gamma}} \right)^{\beta/2},
$$

so that if $V \ll v \ll V$ and $N \leq n < \eta N$ we have

$$
|\theta(n, v)| \ll 1.
$$

Defining

$$
W(y) = \left| \sum_{N \leq n < \eta N} \sum_{c_1 yn^{\gamma} V/Y^{\gamma} \leq v < c_2 yn^{\gamma} V/Y^{\gamma}} \theta(n, v) e \left( \frac{\alpha y}{\beta} \left( \frac{Y}{y} \right)^{\beta} \left( \frac{n}{N} \right)^{\gamma(1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) \right|,
$$
the above simplifies to
\[
(3.3) \quad \left| \sum_{N \leq n < \eta N} \sum_{U \leq u < \rho U} e \left( y \left( \frac{u}{U} \right)^{\alpha} \left( \frac{n}{N} \right)^{\gamma} \right) \right| \ll \frac{U}{Y^{1/2}} W(y) + O \left( \frac{UN(\log Y)}{Y^{1/2}} \right).
\]

We next remove the dependence on \( n \) and \( y \) in summation over \( v \) in \( W(y) \). There exists absolute constants \( c_3 \) and \( c_4 \) depending only on \( c_1, c_2 \) and \( \eta \) such that whenever \( Y \leq y \leq 2Y \) and \( N \leq n \leq \eta N \) we have
\[
\left[ \frac{c_1 yn^{\gamma} V}{Y^{N\gamma}}, \frac{c_2 yn^{\gamma} V}{Y^{N\gamma}} \right] \subseteq [c_3 V, c_4 V],
\]
and hence by Lemma 3.6, for some \( c \) depending only on \( c_1 \) and \( c_2 \)
\[
W(y) = \frac{1}{2\pi} \int_{-V}^{V} \left( \sum_{N \leq n < \eta N} \sum_{c_3 V \leq v \leq c_4 V} \theta(n, v)e \left( \frac{\alpha y}{\beta} \left( \frac{Y}{y} \right)^{\beta} \left( \frac{n}{N} \right)^{\gamma (1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) v^{-it} \right)
\times \left( \frac{yn^{\gamma} V}{Y^{N\gamma}} \right)^{it} (c^it - 1)t^{-1} dt + O \left( N(\log (V + 2)) \right).
\]

Taking absolute values and applying the triangle inequality gives
\[
|W(y)| \ll \int_{-V}^{V} \left| \sum_{N \leq n < \eta N} \sum_{c_3 V \leq v \leq c_4 V} \theta(n, v, t)e \left( \frac{\alpha y}{\beta} \left( \frac{Y}{y} \right)^{\beta} \left( \frac{n}{N} \right)^{\gamma (1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) \right| G(t) dt
+ O \left( N(\log Y) \right),
\]
where \( G(t) \) is the unique continuous function on \( \mathbb{R} \) defined for nonzero \( t \) by
\[
G(t) = \frac{|c^it - 1|}{|t|},
\]
so that
\[
\int_{-V}^{V} G(t) dt \ll \log (V + 2) \ll \log Y,
\]
and \( \theta(n, v, t) \) is given by
\[
\theta(n, v, t) = \theta(n, v)v^{-itn^{\gamma}}.
\]
This implies that for some \( |t_0| \leq V \) we have
\[
|W(y)| \ll \log Y \left| \sum_{N \leq n < \eta N} \sum_{c_3 V \leq v \leq c_4 V} \theta(n, v, t_0)e \left( \frac{\alpha y}{\beta} \left( \frac{Y}{y} \right)^{\beta} \left( \frac{n}{N} \right)^{\gamma (1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) \right|
+ N(\log Y),
\]
and hence by (3.3)

\[
\int_Y^{2Y} \left| \sum_{N \leq n < \eta N} \sum_{U \leq u < \rho U} e \left( y \left( \frac{u}{U} \right)^{\alpha} \left( \frac{n}{N} \right)^{\gamma} \right) \right|^2 dy \ll \frac{U^2 (\log Y)^2}{Y} T(Y) + N^2 U^2 (\log Y)^2,
\]

where

\[
T(Y) = \int_Y^{2Y} \left| \sum_{N \leq n < \eta N} \sum_{c_3 V \leq u \leq c_4 V} \theta(n, v, t_0) e \left( \frac{\alpha y}{\beta} \left( \frac{Y}{y} \right)^{\gamma} \left( \frac{n}{N} \right)^{(1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) \right|^2 dy.
\]

The change of variable

\[
z = \frac{\alpha Y^\beta y^{1-\beta}}{\beta},
\]

in the above integral implies that there exists \( Y_1 \) and \( Y_2 \) satisfying

\[Y \ll Y_1 \ll Y_2 \ll Y,\]

such that

\[T(Y) \ll \int_{Y_1}^{Y_2} \left| \sum_{N \leq n < \eta N} \sum_{c_3 V \leq u \leq c_4 V} \theta(n, v, t_0) e \left( z \left( \frac{n}{N} \right)^{(1-\beta)} \left( \frac{v}{V} \right)^{\beta} \right) \right|^2 dz,
\]

and the result follows from (3.4).

\[\square\]

**Lemma 3.8.** For positive real numbers \( W, M, V \) and integer \( k \) we define

\[T_k(W, N, U) = \int_W^{2W} \left| \sum_{N \leq n < 2N} \sum_{U \leq u < 2U} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy.
\]

There exists absolute constants \( c, c_2 \) and \( c_3 \) such that if \( W \gg 1 \) and

\[1 \leq \Delta \leq W,
\]

then we have

\[T_k(W, N, U) \ll \Delta W (\log W)^4 \max_{1 \leq X \leq W/\Delta} \frac{1}{X} \int_0^X \left| \sum_{N \leq n < 2N} \sum_{U \leq u < c_2 U / W} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy
\]

\[+ N^2 U^2 (\log W)^2 + \Delta N^2 W (\log W)^5.
\]
Proof. By Lemma 3.7

\[
\int_{W}^{2W} \left| \sum_{n \leq N < 2N} \sum_{U \leq u < 2U} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 \ dy \ll N^2 U^2 (\log W)^2 + \]

(3.5)

\[
\frac{U^2 (\log W)^2}{W} \int_{W_1}^{W_2} \left| \sum_{n \leq N < 2N} \sum_{V_2 \leq u < V_1} \theta(n, v) e \left( y \left( \frac{n}{N} \right)^{k/2} \left( \frac{v}{V} \right)^{1/2} \right) \right|^2 \ dy,
\]

where

(3.6) \( W \ll W_1 \ll W_2 \ll W, \quad V = \frac{W}{U}, \quad V \ll V_1 \ll V_2 \ll V, \)

and \( \theta(n, u) \) is a sequence of complex numbers satisfying \( |\theta(n, u)| \leq 1. \)

By Lemma 3.4 and Lemma 3.5

\[
\int_{W_1}^{W_2} \left| \sum_{n \leq N < 2N} \sum_{c_3 V \leq v \leq c_4 V} \theta(n, v) e \left( y \left( \frac{n}{N} \right)^{k/2} \left( \frac{v}{V} \right)^{1/2} \right) \right|^2 \ dy \ll \]

(3.7) \[ \Delta \int_{0}^{W_2/\Delta} \left| \sum_{n \leq N < 2N} \sum_{c_3 \leq v \leq c_4} e \left( y \left( \frac{n}{N} \right)^{k/2} \left( \frac{v}{V} \right)^{1/2} \right) \right|^2 \ dy,
\]

and after performing a dyadic subdivision of the last integral, we get

\[
\int_{W_1}^{W_2} \left| \sum_{n \leq N < 2N} \sum_{c_3 V \leq v \leq c_4 V} \theta(n, v) e \left( y \left( \frac{n}{N} \right)^{k/2} \left( \frac{v}{V} \right)^{1/2} \right) \right|^2 \ dy \ll \Delta N^2 V^2
\]

(3.8)

\[ + \Delta (\log W) \int_{X}^{2X} \left| \sum_{n \leq N < 2N} \sum_{c_3 V \leq v \leq c_4 V} e \left( y \left( \frac{n}{N} \right)^{k/2} \left( \frac{v}{V} \right)^{1/2} \right) \right|^2 \ dy,
\]

for some

(3.9) \( C \leq X \ll W_2/\Delta, \)

where \( C \) is a sufficiently large constant. By Lemma 3.7 we have

\[
\int_{X}^{2X} \left| \sum_{n \leq N < 2N} \sum_{c_3 V \leq v \leq c_4 V} e \left( y \left( \frac{n}{N} \right)^{k/2} \left( \frac{v}{V} \right)^{1/2} \right) \right|^2 \ dy \ll \]

\[
\frac{V^2 (\log W)^2}{X} \int_{X_1}^{X_2} \left| \sum_{n \leq N < 2N} \sum_{U_3 \leq u \leq U_4} \theta(n, u) e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 \ dy
\]

\[ + N^2 V^2 (\log W)^2, \]
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which combined with (3.8) gives

\[
\int_{W_1}^{W_2} \left| \sum_{N \leq n < 2N} \sum_{v \leq c_4 v} \theta(n, v) e \left( y \left( \frac{n}{N} \right)^{k/2} \left( \frac{v}{v} \right)^{1/2} \right) \right|^2 dy \ll
\]

\[
\frac{\Delta (\log W)^3 V^2}{X} \int_{X_1}^{X_2} \left| \sum_{N \leq n < 2N} \sum_{XU_3/W \leq u \leq XU_4/W} \theta(n, u) e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy
\]

\[
+ \Delta N^2 V^2 (\log W)^2,
\]

and hence by (3.5), (3.6) and (3.7)

\[
\int_{W}^{2W} \left| \sum_{N \leq n < 2N} \sum_{U \leq u < 2U} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy \ll
\]

\[
\frac{\Delta W (\log W)^5}{X} \int_{X_1}^{X_2} \left| \sum_{N \leq n < 2N} \sum_{XU_3/W \leq u \leq XU_4/W} \theta(n, u) e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy
\]

\[
+ N^2 U^2 (\log W)^2 + \Delta N^2 W (\log W)^5.
\]

The result follows since by Lemma 3.5

\[
\int_{X_1}^{X_2} \left| \sum_{N \leq n < 2N} \sum_{XU_3/W \leq u \leq XU_4/W} \theta(n, u) e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy \ll
\]

\[
\int_{0}^{X_2} \left| \sum_{N \leq n < 2N} \sum_{XU_3/W \leq u \leq XU_4/W} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy.
\]

\[\square\]

Lemma 3.9. For integers \(N\) and \(U\) and a real number \(Y > 0\) we let \(J_k (N, U, Y)\) count the number of solutions to the inequality

\[
(3.10) \quad \left| \frac{u_1}{n_1^k} - \frac{u_2}{n_2^k} \right| \leq \frac{1}{Y},
\]

with integer variables satisfying

\[
(3.11) \quad N \leq n_1, n_2 < 2N, \quad U \leq u_1, u_2 < 2U.
\]

If \(Y\) and \(U\) satisfy

\[
Y \geq N^k, \quad U \geq N,
\]
then for any $\delta > 0$ we have

\[
J_k(N,U,Y) \ll (\log N)^5 N^\delta \max_{N^k/Y \leq \rho \leq 1} \rho \times J_k \left( N, c\gamma U, \left( \frac{\rho}{\gamma} \right) \frac{Y}{c^2} \right)
+ \frac{UN^{k+2}(\log N)^3}{Y} + N^{2+\delta}(\log N)^6.
\]

Proof. If $n_1, n_2, u_1$ and $u_2$ satisfy (3.10) and (3.11) then

\[
\left| \frac{n_1}{u_1} - \frac{n_2}{u_2} \right| \leq \frac{4N^{2k}}{YU^2}.
\]

Define

(3.12) \quad Z = \frac{YU^2}{4N^{2k}},

and let $J_k^*(N,V,Z)$ count the number of solutions to the inequality

(3.13) \quad \left| \frac{n_1^k}{u_1} - \frac{n_2^k}{u_2} \right| \leq \frac{1}{Z},

with integer variables satisfying (3.11). The above implies that

(3.14) \quad J_k(N,U,Y) \leq J_k^*(N,U,Z).

Conversely, suppose $n_1, n_2, u_1, u_2$ satisfy (3.11) and (3.13), then

\[
\left| \frac{u_1}{n_1^k} - \frac{u_2}{n_2^k} \right| \leq \frac{4U^2}{ZN^{2k}},
\]

and hence for any $Z > 0$

(3.15) \quad J_k^*(N,V,Z) \leq J_k \left( N, V, \frac{ZN^{2k}}{4U^2} \right).

Considering $J_k^*(N,U,Z)$, by Lemma 3.3 we have

\[
J_k^*(N,U,Z) \ll \frac{1}{Z} \int_{-Z}^{Z} \left| \sum_{N \leq n \leq 2N} \sum_{U \leq u \leq 2U} \frac{e\left( \frac{yn^k}{u} \right)}{u^k} \right|^2 dy
\ll \frac{U}{N^{k+1}Z} \int_0^{Z^{N^k}} \left| \sum_{N \leq n \leq 2N} \sum_{U \leq u \leq 2U} e\left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy.
\]
Performing a dyadic partition of the above integral and isolating the contribution from $y = O(1)$, by (3.12) we have

$$J^*_k(N, U, Z) \ll \frac{U N^{k+2} \log N}{Y} + \frac{U \log N}{N^k Z} \max_{c \leq W \leq ZN^k/U} \int_W^{2W} \left| \sum_{N \leq n < 2N} \sum_{U \leq u < 2U} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy,$$

for some sufficiently large constant $C$. By Lemma 3.12

$$J^*_k(N, U, Z) \ll \frac{U N^{k+2} \log N}{Y} + \frac{U \log N}{N^k Z} \max_{u \leq W \leq ZN^k/U} \int_W^{2W} \left| \sum_{N \leq n < 2N} \sum_{U \leq u < 2U} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy,$$

and hence with notation as in Lemma 3.8, we see that

$$J^*_k(N, U, Z) \ll \frac{U N^{k+2} \log N}{Y} + \frac{U \log Z}{N^k Z} T_k(W, N, U),$$

for some

$$U \leq W \leq \frac{Z N^k}{U}.$$  

We apply Lemma 3.8 with

$$\Delta = N^6,$$

to get

$$J^*_k(N, U, Z) \ll \frac{U N^{k+2} \log N}{Y} + \Delta N^2 \log N^6$$

$$+ \frac{\Delta W U \log N}{N^k Z} \frac{1}{X} \int_0^X \left| \sum_{N \leq n < 2N} \sum_{c_2 X U/W \leq u \leq c_3 X U/W} e \left( y \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy.$$ 

for some

$$X \ll W/\Delta.$$ 

Partitioning summation over $u$ into dyadic intervals and using (3.14) and (3.15) we have

$$J_k(N, U, Y) \ll \Delta \log N^5 \left( \frac{U W}{N^k Z} \right) J_k \left( N, \frac{c_4 X U}{W}, \frac{N^k W^2}{c_2 UX} \right)$$

$$+ \frac{U N^{k+2} \log N^3}{Y} + \Delta N^2 \log N^6.$$
for some constant $c_2 \leq c \leq c_3$. By (3.12), (3.16) and (3.18) we may write

$$W = \eta U, \quad \frac{X}{W} = \gamma,$$

for some

$$1 \leq \eta \leq \frac{Y}{N^k} \quad \text{and} \quad \gamma \leq \Delta^{-1},$$

and hence

$$J_k(N, U, Y) \ll (\log N)^5 N^\delta \max_{N^k \leq \rho \leq 1} \rho \times J_k\left(N, c \gamma U, \left(\frac{\rho}{\gamma}\right) \left(\frac{Y}{c^2}\right)\right)$$

$$+ \frac{UN^{k+2}(\log N)^3}{Y} + N^{2+\delta}(\log N)^6,$$

on recalling (3.17). \qed

**Lemma 3.10.** For integers $U_1, N_1, U_2, N_2, Y$ and $k$ we let $J_k(U_1, N_1, U_2, N_2, Y)$ count the number of solutions to the inequality

$$\left| \frac{u_1}{n_1^k} - \frac{u_2}{n_2^k} \right| \leq \frac{1}{Y},$$

with integer variables satisfying

$U_1 \leq u_1 < 2U_1, \quad N_1 \leq n_1 < 2N_1, \quad U_2 \leq u_2 < 2U_2, \quad N_2 \leq n_2 < 2N_2,$

and when $U_1 = U_2$ and $N_1 = N_2$ we write

$$J_k(U_1, N_1, U_2, N_2, Y) = J_k(U_1, N_1, Y).$$

We have

$$J_k(U_1, N_1, U_2, N_2, Y) \ll J_k(U_1, N_1, Y)^{1/2} J_k(U_2, N_2, Y)^{1/2}.$$

**Proof.** For integer $j$ let $J_1(j)$ count the number of solutions to the inequality

$$\frac{j}{Y} \leq \frac{u}{n^k} < \frac{j + 1}{Y},$$

with variables satisfying

$U_1 \leq u_1 < 2U_1, \quad N_1 \leq n_1 < 2N_1,$

and let $J_2(j)$ count the number of solutions to the inequality (3.19) with variables satisfying

$U_2 \leq u_1 < 2U_2, \quad N_2 \leq n_1 < 2N_2.$
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We have

\[ J_k(U_1, N_1, U_2, N_2, Y) \leq \sum_{j_1, j_2 \mid |j_1 - j_2| \leq 1} J_1(j_1) J_2(j_2), \]

and hence by the Cauchy-Schwarz inequality

\[ J_k(U_1, N_1, U_2, N_2, Y)^2 \ll \left( \sum_j J_1(j)^2 \right) \left( \sum_j J_2(j)^2 \right) \ll J_k(U_1, N_1, Y) J_k(U_2, N_2, Y). \]

□

The following is known as the Kusmin-Landau inequality, for a proof see [17, Corollary 8.11].

Lemma 3.11. Let \( f \) be a continuously differentiable function on some interval \( I \) with derivative satisfying

\[ \|f'(x)\| \geq \lambda, \quad x \in I. \]

Then we have

\[ \sum_{n \in I} f(n) \ll \lambda^{-1}. \]

Lemma 3.12. For integers \( k, N, U \) and a real number \( W \), if

\( W \ll U, \)

(3.20)

for a sufficiently small constant then

\[ \int_{W}^{2W} \left| \sum_{N \leq n < 2N} \sum_{U \leq u < 2U} e \left( z \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy \ll \frac{U^2 N^2}{W}. \]

Proof. By Lemma 3.11 and (3.20), for each \( W \leq z \leq 2W \) and \( N \leq n \leq 2N \) we have

\[ \sum_{U \leq u < 2U} e \left( z \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \ll \frac{U}{W}, \]

and hence

\[ \int_{W}^{2W} \left| \sum_{N \leq n < 2N} \sum_{U \leq u < 2U} e \left( z \left( \frac{n}{N} \right)^k \left( \frac{U}{u} \right) \right) \right|^2 dy \ll \frac{U^2 N^2}{W}. \]

□
Lemma 3.13. For integers \( k, N, M, U \) and \( V \) satisfying
\[
M \ll N, \quad V \ll U, \quad U \ll N^k, \tag{3.21}
\]
and a real number \( Y \) satisfying
\[
Y \leq \frac{N^k}{2}, \tag{3.22}
\]
we have
\[
\frac{1}{Y} \int_{-Y}^{Y} \left| \sum_{N<n \leq M} \sum_{U<u \leq V} e \left( \frac{yu}{n^k} \right) \right|^2 dy \ll \frac{N^2 U^2}{Y} + \frac{N^{2k+2} (\log N)^2}{Y}.
\]

Proof. Bounding the contribution from \( y \in [-1, 1] \) in the above integral trivially gives
\[
\frac{1}{Y} \int_{-Y}^{Y} \left| \sum_{N<n \leq M} \sum_{U<u \leq V} e \left( \frac{yu}{n^k} \right) \right|^2 dy \leq \frac{N^2 U^2}{Y} + \frac{1}{Y} \int_{1}^{Y} \left| \sum_{N<n \leq M} \sum_{U<u \leq V} e \left( \frac{yu}{n^k} \right) \right|^2 dy.
\]
For any fixed \( 1 \leq y \leq Y \) we have
\[
\sum_{N<n \leq M} \sum_{U<u \leq V} e \left( \frac{yu}{n^k} \right) \ll \sum_{N<n \leq M} \min \left( U, \frac{1}{\|y/n^k\|} \right),
\]
and for any \( N < n \leq M \) and \( 1 \leq y \leq Y \), by (3.22) we have
\[
\frac{y}{n^k} \leq \frac{1}{2},
\]
and for any \( N < n_1 < n_2 \leq M \)
\[
\frac{y}{n_1} - \frac{y}{n_2} \gg \frac{y(n_2 - n_1)}{N^{k+1}},
\]
so that
\[
\sum_{N<n \leq M} \min \left( U, \frac{1}{\|y/n^k\|} \right) \ll \frac{N^{k+1}}{y} \sum_{1 \leq n \leq M} \frac{1}{n} \ll \frac{N^{k+1} (\log N)}{Y},
\]
and hence
\[
\frac{1}{Y} \int_{1}^{Y} \left| \sum_{N<n \leq M} \sum_{U<u \leq V} e \left( \frac{yu}{n^k} \right) \right|^2 dy \ll \frac{N^{2k+2} (\log N)^2}{Y},
\]
from which the desired result follows. \( \square \)

Lemma 3.14. Let \( N \) be an integer, \( Y \) and \( \varepsilon \) positive real numbers satisfying
\[
Y \leq N^{k+\varepsilon}. \tag{3.23}
\]
Then with $I_k(N, Y)$ as in Theorem 2.1, we have

$$I_k(N, Y) \ll \frac{N^{2k+2+\varepsilon} (\log N)^4}{Y}$$

**Proof.** With notation as in Lemma 3.10, we have

$$I_k(N, Y) \leq \sum_{i,j} \sum_{\ell,s} J_k(2^j, 2^i, 2^\ell, 2^s, Y)$$

$$\leq \left( \sum_{i,j} J_k(2^j, 2^i, Y)^{1/2} \right)^2,$$

and hence

$$I_k(N, Y) \ll (\log N)^2 \max_{M \leq 2N \atop V \leq 2M^k} J_k(M, V, Y). \quad (3.24)$$

Fix some $M \leq 2N$ and $V \leq 2N^k$ and consider $J_k(M, V, Y)$. By Lemma 3.3 we have

$$J_k(M, V, Y) \ll \frac{1}{Y} \int_{-Y}^{Y} \left| \sum_{M < n \leq 2M} \sum_{V < \nu \leq 2V} e \left( \frac{yu}{n^k} \right) \right|^2 dy.$$ 

If $Y < M^k/2$ then by Lemma 3.13

$$J_k(M, V, Y) \ll \frac{M^2 V^2}{Y} + \frac{N^{2k+2} (\log N)^2}{Y} \ll \frac{N^{2k+2} (\log N)^2}{Y}, \quad (3.25)$$

and if $M^k/2 < Y$ then by Lemma 3.4 and Lemma 3.13

$$J_k(M, V, Y) \ll \frac{1}{M^k} \int_{-M^k/4}^{M^k/4} \left| \sum_{M < n \leq 2M} \sum_{V < \nu \leq 2V} e \left( \frac{yu}{n^k} \right) \right|^2 dy$$

$$\ll \frac{M^2 V^2}{M^k} + (\log M)^2 M^{k+2} \ll (\log M)^2 M^{k+2}.$$ 

By (3.23) we have

$$M^{k+2} \leq \frac{YM^{k+2}}{Y} \leq \frac{N^{2k+2+\varepsilon}}{Y},$$

which gives

$$J_k(M, V, Y) \ll \frac{N^{2k+2+\varepsilon} (\log N)^2}{Y}.$$
Combining the above with (3.24) and (3.25) we get

\[ I_k(N, Y) \ll \frac{N^{2k+2+\varepsilon} (\log N)^4}{Y}, \]

and completes the proof. \(\square\)

4. PROOF OF THEOREM 2.1

By Lemma 3.14 we may suppose

\( Y \geq N^{k+\varepsilon}, \)

for a sufficiently small \( \varepsilon \). Arguing as in the proof of Lemma 3.14, we have

\( I_k(N, Y) \ll (\log N)^2 J_k(M, U, Y), \)

for some

\[ M \leq 2N, \quad U \leq 2M^k. \]

Fixing some sufficiently small \( \delta \) and iterate Lemma 3.9 to obtain a sequence of pairs \((\rho_j, \gamma_j)\) satisfying

\[ \frac{N^k}{Y} \ll \rho_1 \ll 1, \quad \gamma_1 \ll N^{-\delta}, \]

and

\[ \gamma_j \ll N^{-\delta}, \quad \frac{\gamma_j^{-1}N^k}{\rho_j^{-1}Y} \ll \rho_j \ll 1, \]

with implied constants depending at most on \( j \) such that

\[ J_k(M, U, Y) \ll (\log N)^{5j} (\rho_1 \ldots \rho_j N^{j\gamma}) J_k \left( M, \gamma_1 \ldots \gamma_j U, \left( \frac{\rho_1 \ldots \rho_j}{\gamma_1 \ldots \gamma_j} \right)^{Y} \right) \]

\[ + \frac{UN^{k+2}(\log N)^{5j}}{Y} + \left( \sum_{i=0}^{j} \rho_1 \ldots \rho_i N^{i\delta} \right) N^{2+\delta} (\log N)^{6j}. \]

Let \( j \) be the smallest integer such that either

\( \gamma_1 \cdots \gamma_j \leq \frac{N^{2\delta}}{U}, \)

or

\( \left( \frac{\rho_1 \ldots \rho_j}{\gamma_1 \ldots \gamma_j} \right) \geq \frac{2N^{2k}}{Y}, \)

so that

\( j \ll 1, \)
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with implied constant depending only on \( k \) and \( \delta \). We consider the following two subcases

\[
\gamma_1 \ldots \gamma_j \geq \frac{N^{2\delta}}{U}, \quad \left( \frac{\rho_1 \ldots \rho_j}{\gamma_1 \ldots \gamma_j} \right) \geq \frac{2N^{2k}}{Y}, \quad \left( \frac{\rho_1 \ldots \rho_i}{\gamma_1 \ldots \gamma_i} \right) \leq \frac{2N^{2k}}{Y},
\]

for each \( 1 \leq i \leq j - 1 \) or

\[
\left( \frac{\rho_1 \ldots \rho_i}{\gamma_1 \ldots \gamma_i} \right) \leq \frac{2N^{2k}}{Y}, \quad \gamma_1 \ldots \gamma_j \leq \frac{N^{2\delta}}{U}, \quad \gamma_1 \ldots \gamma_{j-1} \geq \frac{N^{2\delta}}{U},
\]

for each \( 1 \leq i \leq j \). These exhaust all possibilities for either (4.4) or (4.5) to hold. If this were false then we would have both

\[
\left( \frac{\rho_1 \ldots \rho_i}{\gamma_1 \ldots \gamma_i} \right) \geq \frac{2N^{2k}}{Y} \text{ for some } 1 \leq i \leq j - 1,
\]

and

\[
\gamma_1 \ldots \gamma_{j-1} \leq \frac{N^{2\delta}}{U},
\]

contradicting the assumption that \( j \) is the smallest integer satisfying either (4.4) or (4.5). Suppose first that (4.6) holds. We see that \( J_k \left( M, \gamma_1 \ldots \gamma_j U, \left( \frac{\rho_1 \ldots \rho_j}{\gamma_1 \ldots \gamma_j} \right) Y \right) \) is bounded by the number of solutions to the inequality

\[
\left| u_1 n_2^k - u_2 n_1^k \right| \ll 1,
\]

with variables satisfying

\[
\gamma_1 \ldots \gamma_j U \ll u_1, u_2 \ll \gamma_1 \ldots \gamma_j U, \quad M \leq n_1, n_2 \leq 2M.
\]

Fixing \( u_1 \) and \( n_2 \) and using estimates for the divisor function, we see that

\[
J_k \left( M, \gamma_1 \ldots \gamma_j U, \left( \frac{\rho_1 \ldots \rho_j}{\gamma_1 \ldots \gamma_j} \right) Y \right) \ll \gamma_1 \ldots \gamma_j U M N^{o(1)},
\]

and hence

\[
J_k(M, U, Y) \ll (\log N)^5 \rho_1 \ldots \rho_j U M N^{o(1)}
\]

\[
+ \frac{U N^{k+2} (\log N)^{5j} \rho_1 \ldots \rho_j}{Y} + \left( \sum_{i=0}^{j} \frac{\rho_1 \ldots \rho_i}{\gamma_1 \ldots \gamma_i} \right) N^{2+\delta} (\log N)^{6j},
\]

which by (4.3) and (4.6) implies that

\[
J_k(M, U, Y) \ll (\log N)^{5j} U N^{1+o(1)} + \frac{U N^{k+2} (\log N)^{5j}}{Y} + \frac{N^{2k+2+3\delta}}{Y} (\log N)^{6j}.
\]

(4.8)
Suppose next that (4.6) holds. Then \( J_k \left( M, \gamma_1 \ldots \gamma_j U, \left( \frac{\rho_1 \ldots \rho_j}{\gamma_1 \ldots \gamma_j} \right) Y \right) \) is bounded by the number of solutions to the inequality

\[
\left| \frac{u_1}{n_1^k} - \frac{u_2}{n_2^k} \right| \leq \frac{\gamma_1 \ldots \gamma_j}{\rho_1 \ldots \rho_j Y},
\]

with variables satisfying

\[
M \leq n_1, n_2 \leq 2M, \quad N^{2\delta} \ll u_1, u_2 \ll N^{2\delta}.
\]

Fixing variables \( n_2, u_1, u_2 \), we have

\[
\left| \frac{n_1^k - u_2 n_2^k}{u_1} \right| \ll \frac{\gamma_1 \ldots \gamma_j M^{2k}}{\rho_1 \ldots \rho_j Y},
\]

and hence

\[
J_k \left( M, \gamma_1 \ldots \gamma_j U, \left( \frac{\rho_1 \ldots \rho_j}{\gamma_1 \ldots \gamma_j} \right) Y \right) \ll N^{2\delta} M \left( 1 + \frac{\gamma_1 \ldots \gamma_j M^{k+1}}{\rho_1 \ldots \rho_j Y} \right).
\]

This gives

\[
J_k(M, U, Y) \ll (\log N)^{5j} \left( \rho_1 \ldots \rho_j N^{2\delta} \right) N^{2\delta} M + \frac{\gamma_1 \ldots \gamma_j N^{2\delta + 2\delta} M^{k+2}}{Y} + \frac{U N^{k+2}(\log N)^{5j}}{Y} + \left( \sum_{i=0}^{j} \rho_1 \ldots \rho_i N^{i\delta} \right) N^{2\delta} (\log N)^{6j},
\]

which combined with (4.3) and (4.7) implies

\[
J_k(M, U, Y) \ll (\log N)^{5j} U M N^{2\delta} + \frac{M^{k+2+2\delta}}{Y} + \frac{U N^{k+2}(\log N)^{5j}}{Y} + (\log N)^{6j} \frac{N^{2k+2+3\delta}}{Y}.
\]

Using the above, (4.2) and (4.8) we get

\[
I_k(N, Y) \ll \left( \frac{N^{2k+2}}{Y} + N^{k+1} \right) N^{o(1)},
\]

on taking \( \delta \) sufficiently small.

5. Proof of Corollary 2.3

We first recall a special case of the duality principle to which we refer the reader to [23, Lemma 2] for a proof.
Lemma 5.1. Let \( c_{n,m} \) be a sequence of complex numbers. Suppose that for every sequence of complex numbers \( \alpha_n \) we have
\[
\left( \sum_m \left| \sum_n \alpha_n c_{n,m} \right|^2 \right)^{1/2} \leq D \| \alpha \|_{\infty}.
\]

Then for any sequence of complex numbers \( \alpha_m \) we have
\[
\sum_n \left| \sum_m \alpha_m c_{n,m} \right| \leq D \| \alpha \|_2.
\]

By Lemma 5.1, in order to prove Corollary 2.3 it is sufficient to show that for any sequence of complex numbers \( \alpha_{a,n} \) satisfying
\[
|\alpha_{a,n}| \leq A,
\]
we have
\[
(5.1) \sum_{K < m \leq K + M} \left| \sum_{1 \leq n \leq N \atop 1 \leq a \leq n^k \atop (a,n) = 1} \alpha_{a,n} e\left(\frac{am}{n^k}\right) \right|^2 \ll (N^{2k+2} + M N^{k+1}) N^{o(1)} A^2.
\]

Let \( \phi \) be a positive valued Schwartz function satisfying
\[
\phi(x) \geq 1, \quad |x| \leq 1 \quad \text{and} \quad \text{supp}(\hat{\phi}) \subseteq [-2,2],
\]
where \( \hat{\phi} \) denotes the Fourier transform. We have
\[
\sum_{K < m \leq K + M} \left| \sum_{1 \leq n \leq N \atop 1 \leq a \leq n^k \atop (a,n) = 1} \alpha_{a,n} e\left(\frac{am}{n^k}\right) \right|^2 \leq \sum_{m \in \mathbb{Z}} \phi \left(\frac{m - K}{M}\right) \sum_{1 \leq n \leq N \atop 1 \leq a \leq n^k \atop (a,n) = 1} \alpha_{a,n} e\left(\frac{am}{n^k}\right)^2.
\]
which after expanding the square and interchanging summation gives

\[
\sum_{K < m \leq K + M} \left| \sum_{1 \leq n \leq N} \sum_{1 \leq a \leq n^k \atop (a, n) = 1} \alpha_{a, n} e \left( \frac{am}{n^k} \right) \right|^2 \leq A^2 \sum_{1 \leq n_1, n_2 \leq N} \sum_{1 \leq a_1 \leq n_1^k \atop 1 \leq a_2 \leq n_2^k} \sum_{m \in \mathbb{Z}} \phi \left( \frac{m - K}{M} \right) e \left( \left( \frac{a_1}{n_1^k} - \frac{a_2}{n_2^k} \right) m \right) e \left( \left( \frac{a_1}{n_1^k} - \frac{a_2}{n_2^k} \right) m \right) \left( \frac{M}{n_1^k} - \frac{M}{n_2^k} \right) \leq 4M,
\]

Applying Poisson summation and using the fact that \( \text{supp}(\hat{\phi}) \subseteq [-2, 2] \) we see that

\[
\sum_{m \in \mathbb{Z}} \phi \left( \frac{m - K}{M} \right) e \left( \left( \frac{a_1}{n_1^k} - \frac{a_2}{n_2^k} \right) m \right) \ll M \quad \text{if} \quad \left| \frac{a_1}{n_1^k} - \frac{a_2}{n_2^k} \right| \leq \frac{4}{M},
\]

and

\[
\sum_{m \in \mathbb{Z}} \phi \left( \frac{m - K}{M} \right) e \left( \left( \frac{a_1}{n_1^k} - \frac{a_2}{n_2^k} \right) m \right) = 0 \quad \text{otherwise},
\]

and hence with notation as in Theorem 2.1

\[
\sum_{K < m \leq K + M} \left| \sum_{1 \leq n \leq N} \sum_{1 \leq a \leq n^k \atop (a, n) = 1} \alpha_{a, n} e \left( \frac{am}{n^k} \right) \right|^2 \ll A^2 M I_k(N, 4M) \ll A^2 \left( N^{2k+2} + MN^{k+1} \right) N^{o(1)},
\]

which establishes (5.1) and completes the proof.

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