INEQUALITIES OF THE EDMUNDSON–LAH–RIBARIČ TYPE FOR n-CONVEX FUNCTIONS WITH APPLICATIONS

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We deduce some Edmundson–Lah–Ribarič-type inequalities for positive linear functionals and n-convex functions. Our main results are applied to the generalized f-divergence functional. Examples with Zipf–Mandelbrot law are presented to illustrate the results.

1. Introduction

Let $E$ be a nonempty set and let $L$ be the vector space of real-valued functions $f: E \to \mathbb{R}$ with the following properties:

(L₁) $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in \mathbb{R}$;

(L₂) $1 \in L$, i.e., if $f(t) = 1$ for every $t \in E$, then $f \in L$.

We also consider positive linear functionals $A: L \to \mathbb{R}$. This means that we assume that:

(A₁) $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in \mathbb{R}$;

(A₂) $f \in L$, $f(t) \geq 0$ for every $t \in E$ $\Rightarrow A(f) \geq 0$ (A is positive).

Since it was proved, the famous Jensen inequality and its converses have been extensively studied by numerous authors and generalized in various directions. Jessen [17] proposed the following generalization of Jensen’s inequality for convex functions (see also [30, p. 47]):

**Theorem 1.1** [17]. Let $L$ satisfy the properties (L₁) and (L₂) on a nonempty set $E$. Assume that $f$ is a continuous convex function on an interval $I \subset \mathbb{R}$. If $A$ is a positive linear functional with $A(1) = 1$, then, for all $g \in L$ such that $f(g) \in L$, the following assertions are true: $A(g) \in I$ and

$$f(A(g)) \leq A(f(g)).$$

The following result is one of the most famous converses of the Jensen inequality known as the Edmundson–Lah–Ribarič inequality. It was proved by Beesack and Pečarić in [3] (see also [30, p. 98]).

**Theorem 1.2** [3]. Let $f$ be convex on the interval $I = [a, b]$ such that $-\infty < a < b < \infty$. Let $L$ satisfy conditions (L₁) and (L₂) on $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Then, for every
\[ g \in L \text{ such that } f(g) \in L \text{ (so that } a \leq g(t) \leq b \text{ for all } t \in E), \text{ the following inequality is true:} \]
\[
A(f(g)) \leq \frac{b - A(g)}{b - a} f(a) + \frac{A(g) - a}{b - a} f(b). \tag{1.2}
\]

For some recent results on the converses of the Jensen inequality, the reader is referred to \([7, 19, 20, 27, 29, 31]\).

Unlike the results presented in the above-mentioned papers, which require convexity of the involved functions, the main aim of the present paper is to obtain inequalities of the Edmundson–Lah–Ribarič type valid for \(n\)-convex functions, which also generalize the results from \([24, 25]\).

The definition of \(n\)-convex functions is characterized by the presence of \(n\)th order divided differences.

The \(n\)th order divided difference of a function \(f : [a, b] \to \mathbb{R}\) at mutually distinct points \(t_0, t_1, \ldots, t_n \in [a, b]\) is recursively defined by the formulas
\[
[t_i]f = f(t_i), \quad i = 0, \ldots, n, \\
[t_0, \ldots, t_n]f = \frac{[t_1, \ldots, t_n]f - [t_0, \ldots, t_{n-1}]f}{t_n - t_0}.
\]

The value \([t_0, \ldots, t_n]f\) is independent of the order of the points \(t_0, \ldots, t_n\).

The definition of divided differences can be extended to include the cases in which some or all points coincide (see, e.g., \([2, 30]\)):
\[
f[a, \ldots, a]_{\overbrace{\vdots}^{n \text{ times}}} = \frac{1}{(n - 1)!} f^{(n-1)}(a), \quad n \in \mathbb{N}.
\]

A function \(f : [a, b] \to \mathbb{R}\) is called \(n\)-convex \((n \geq 0)\) if and only if, for all choices of \((n + 1)\) distinct points \(t_0, t_1, \ldots, t_n \in [a, b]\), we have \([t_0, \ldots, t_n]f \geq 0\).

In the present paper, the results are obtained by using Hermite’s interpolating polynomial. Thus, it is necessary first to give its definition and present some of its properties (see \([2]\)).

Let \(-\infty < a < b < \infty\) and let \(a \leq a_1 < a_2 < \ldots < a_r \leq b\), where \(r \geq 2\), be given points. For \(f \in C^n([a, b])\), there exists a unique polynomial \(P_H(t)\) called Hermite’s interpolating polynomial, of degree \((n - 1)\) satisfying the following Hermite’s conditions:
\[
P_H^{(i)}(a_j) = f^{(i)}(a_j): \quad 0 \leq i \leq k_j, \quad 1 \leq j \leq r, \quad \sum_{j=1}^{r} k_j + r = n.
\]

Among other special cases, these conditions include type \((m, n-m)\) conditions, which are of especial interest for us:
\[
(r = 2, 1 \leq m \leq n - 1, k_1 = m - 1, k_2 = n - m - 1)
\]
\[
P_{mn}^{(i)}(a) = f^{(i)}(a), \quad 0 \leq i \leq m - 1,
\]
\[
P_{mn}^{(i)}(b) = f^{(i)}(b), \quad 0 \leq i \leq n - m - 1.
\]
To give a development of the interpolating polynomial in terms of divided differences, we first assume that the function \( f \) is also defined at a point \( t \neq a_j, \ 1 \leq j \leq r \). In [2], it was shown that

\[
f(t) = P(t) + R(t),
\]

where

\[
P(t) = f(a_1) + (t - a_1)f[a_1, a_2]
+ (t - a_1)(t - a_2)f[a_1, a_2, a_3]
+ \ldots + (t - a_1)\ldots(t - a_{r-1})f[a_1, \ldots, a_r]
\]

and

\[
R(t) = (t - a_1)\ldots(t - a_r)f[t, a_1, \ldots, a_r].
\]

In the case of \((m, n - m)\) conditions, relations (1.4) and (1.5) become

\[
P_{mn}(t) = f(a) + (t - a)f[a, a] + \ldots + (t - a)^{m-1}f[a, \ldots, a]_{m \text{ times}}
+ (t - a)^m f[a, \ldots, a; b]_{m \text{ times}} + (t - a)^m(t - b)f[a, \ldots, a; b, b]_{m \text{ times}}
+ \ldots + (t - a)^m(t - b)^{n-m-1}f[a, \ldots, a; b, b, \ldots, b]_{m \text{ times} \ (n-m) \text{ times}}
\]

and

\[
R_m(t) = (t - a)^m(t - b)^{n-m}f[t; a, \ldots, a; b, b, \ldots, b]_{m \text{ times} \ (n-m) \text{ times}}.
\]

The present paper is organized as follows. Our main results, i.e., inequalities of the Edmundson–Lah–Ribarič type for \( n \)-convex functions are given in Section 2. The application of the main results to the generalized \( f \)-divergence functional is given in Section 3. Finally, in Section 4, the results for the generalized \( f \)-divergence are applied to the Zipf–Mandelbrot law.

2. Results

Throughout the paper, whenever mentioning the interval \([a, b]\), we assume that \(-\infty < a < b < \infty\) holds.

Let \( L \) satisfy conditions \((L_1)\) and \((L_2)\) on a nonempty set \( E \), let \( A \) be any positive linear functional on \( L \) with \( A(1) = 1 \), and let \( g \in L \) be any function such that \( g(E) \subseteq [a, b] \). For a given function \( f : [a, b] \to \mathbb{R} \), we denote

\[
LR(f, g, a, b, A) = A(f(g)) - \frac{b - A(g)}{b - a} f(a) - \frac{A(g) - a}{b - a} f(b).
\]
The following representations of the left-hand side of the Edmundson–Lah–Ribarič inequality are obtained by using Hermite’s interpolating polynomials in terms of divided differences (1.6).

**Lemma 2.1.** Let $L$ satisfy conditions $(L_1)$ and $(L_2)$ on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Also let $f \in C^n([a, b])$ and let $g \in L$ be any function such that $f \circ g \in L$. Then the following identities hold:

\[
LR(f, g, a, b, A) = \sum_{k=2}^{n-1} f[a, b, \ldots, b] A[(g - a1)(g - b1)^{k-1}] + A(R_1(g)),
\]

\[
LR(f, g, a, b, A) = f[a, a; b] A[(g - a1)(g - b1)]
\]

\[
+ \sum_{k=2}^{n-2} f[a, a; b, \ldots, b] A[(g - a1)^2(g - b1)^{k-1}] + A(R_2(g)),
\]

\[
LR(f, g, a, b, A) = (A(g) - a) (f[a, a] - f[a, b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} A[(g - a1)^k]
\]

\[
+ \sum_{k=1}^{n-m} f[a, \ldots, a; b, b, \ldots, b] A[(g - a1)^{m}(g - b1)^{k-1}] + A(R_m(g)),
\]

where $m \geq 3$ and $R_m(\cdot)$ is defined in (1.7).

**Proof.** From representation (1.3), for every function $f \in C^n([a, b])$ and its Hermite interpolating polynomial of type $(m, n - m)$ conditions in terms of divided differences (1.6), we find

\[
f(t) = f(a) + (t - a)f[a, a] + \ldots + (t - a)^{m-1}f[a, \ldots, a]
\]

\[
+ (t - a)^m f[a, \ldots, a; b] + (t - a)^m(t - b)f[a, \ldots, a; b, b]
\]

\[
+ \ldots + (t - a)^m(t - b)^{n-m-1}f[a, \ldots, a; b, b, \ldots, b] + R_m(t),
\]

where $R_m(\cdot)$ is defined in (1.7). After necessary straightforward calculations, for different choices of $1 \leq m \leq n - 1$, from (2.5), we get the following relations:

for $m = 1$,

\[
LR(f, 1, a, b, \text{id}) = (t - a)(t - b)f[a; b, b] + (t - a)(t - b)^2f[a; b, b, b]
\]

\[
+ \ldots + (t - a)(t - b)^{n-2}f[a; b, b, \ldots, b] + R_1(t);
\]
for $m = 2$, 

$$LR(f, 1, a, b, \text{id}) = (t-a)(t-b)f[a, a; b] + (t-a)^2(t-b)f[a, a; b, b]$$

$$+ \ldots + (t-a)^2(t-b)^{n-3}f[a, a; b, b, \ldots, b] + R_2(t);$$ (2.7)

for $3 \leq m \leq n-1$,

$$LR(f, 1, a, b, \text{id}) = (t-a)(f[a, a] - f[a, b]) + \ldots +(t-a)^{m-1} f[a, \ldots, a]$$ \quad \text{$m$ times}

$$+ (t-a)^m f[a, \ldots, a; b] + (t-a)^m(t-a)^{m-1} f[a, \ldots, a; b, b]$$ \quad \text{$m$ times}

$$+ \ldots + (t-a)^m(t-b)^{n-m-1} f[a, \ldots, a; b, b, \ldots, b] + R_m(t).$$ (2.8)

Since $f \circ g \in L$, we conclude that $g(E) \subseteq [a, b]$ and, hence, we can replace $t$ with $g(t)$ in (2.6), (2.7), and (2.8). As a result, we obtain

$$LR(f, g, a, b, \text{id}) = \sum_{k=2}^{n-1} (g(t) - a) (g(t) - b)^{k-1} f[a; b, \ldots, b] + R_1(g(t)),$$

$$LR(f, g, a, b, \text{id}) = (g(t) - a)(g(t) - b)f[a, a; b]$$

$$+ \sum_{k=2}^{n-2} (g(t) - a)^2 (g(t) - b)^{k-1} f[a, a; b, \ldots, b] + R_2(g(t))$$

and

$$LR(f, g, a, b, \text{id}) = (g(t) - a)(f[a, a] - f[a, b]) + \sum_{k=3}^{m} (g(t) - a)^{k-1} f[a, \ldots, a]$$ \quad \text{$k$ times}

$$+ \sum_{k=1}^{n-m} (g(t) - a)^m (g(t) - b)^{k-1} f[a, \ldots, a; b, \ldots, b] + R_m(g(t)).$$

Identities (2.2), (2.3), and (2.4) are proved by applying the positive normalized linear functional $A$ to the indicated equalities, respectively.

Lemma 2.1 is proved.

**Lemma 2.2.** Let $L$ satisfy conditions $(L_1)$ and $(L_2)$ on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Also let $f \in C^n([a, b])$ and let $g \in L$ be any function such that $f \circ g \in L$. 
Then the following identities hold:

\[
LR(f, g, a, b, A) = \sum_{k=2}^{n-1} f[b; a_1, \ldots, a] \left( (g - b1)(g - a1)^{k-1} \right] + A(R^*_k(g)), \quad (2.9)
\]

\[
LR(f, g, a, b, A) = f[b, b; a]A[(g - b1)(g - a1)]
\]

\[
+ \sum_{k=2}^{n-2} f[b; a_1, \ldots, a] \left( (g - b1)^2(g - a1)^{k-1} \right] + A(R^*_2(g)), \quad (2.10)
\]

\[
LR(f, g, a, b, A) = (b - A(g))(f[a, b] - f[b, b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} A[(g - b1)^k]
\]

\[
+ \sum_{k=1}^{m-n} f[b_1, \ldots, b; a_1, \ldots, a] \left( (g - b1)^m(g - a1)^{k-1} \right] + A(R^*_m(g)), \quad (2.11)
\]

where \( m \geq 3 \) and

\[
A(R^*_m(g)) = A \left[ f[g; b1, \ldots, b1; a1, \ldots, a1] \left( (g - b1)^m(g - a1)^{n-m} \right) \right]. \quad (2.12)
\]

**Proof.** We now define an auxiliary function \( F : [a, b] \to \mathbb{R} \) with

\[
F(t) = f(a + b - t).
\]

Since \( f \in C^n([a, b]) \), we immediately get \( F \in C^n([a, b]) \). Hence, we can apply (2.6), (2.7), and (2.8) to \( F \) and obtain, respectively,

\[
LR(F, 1, a, b, \text{id}) = \sum_{k=2}^{n-1} F[a; b, \ldots, b] \left( (t - a)(t - b)^{k-1} \right] + R_1(t), \quad (2.13)
\]

\[
LR(F, 1, a, b, \text{id}) = F[a, a; b](t - a)(t - b)
\]

\[
+ \sum_{k=2}^{n-2} F[a, a; b, \ldots, b] \left( (t - a)^2(t - b)^{k-1} \right] + R_2(t), \quad (2.14)
\]

\[
LR(F, 1, a, b, \text{id}) = (t - a)(F[a, a] - F[a, b]) + \sum_{k=2}^{m-1} \frac{F^{(k)}(a)}{k!} (t - a)^k
\]

\[
+ \sum_{k=1}^{m-n} F[a, \ldots, a; b, \ldots, b] \left( (t - a)^m(t - b)^{k-1} \right] + R_m(t). \quad (2.15)
\]
We can calculate divided differences of the function $F$ in terms of divided differences of the function $f$:

$$F\left[ a, \ldots, a; b, \ldots, b \right] = (-1)^{k+i-1} f\left[ b, \ldots, b; a, \ldots, a \right].$$

As a result, (2.13), (2.14), and (2.15) become

$$LR(F, 1, a, b, \text{id}) = \sum_{k=2}^{n-1} (-1)^k f[b; a, \ldots, a] (t-a)(t-b)^{k-1} + \tilde{R}_1(t),$$

$$LR(F, 1, a, b, \text{id}) = (-1)^2 f[b, b; a](t-a)(t-b)$$

$$+ \sum_{k=2}^{n-2} (-1)^{k+1} f[b, b; a, \ldots, a] (t-a)^2(t-b)^{k-1} + \tilde{R}_2(t),$$

$$LR(F, 1, a, b, \text{id}) = (t-a)(-f[b, b] + f[a, b]) + \sum_{k=2}^{m-1} \frac{(-1)^k f(k)(b)}{k!} (t-a)^k$$

$$+ \sum_{k=1}^{n-m} (-1)^{m+k-1} f[b, \ldots, b; a, \ldots, a] (t-a)^m(t-b)^{k-1} + \tilde{R}_m(t),$$

where

$$\tilde{R}_m(t) = (t-a)^m(t-b)^{n-m}(-1)^n f[a + b - t; b, \ldots, b; a, a, \ldots, a].$$

Let $g \in L$ be any function such that $f \circ g \in L$, that is, $a \leq g(t) \leq b$ for every $t \in E$. We define a function

$$\bar{g}(t) = a + b - g(t).$$

In a trivial way, we get

$$a \leq \bar{g}(t) \leq b \quad \text{and} \quad \bar{g} \in L.$$
we replace $t$ in relations (2.16), (2.17), and (2.18) with $g(t)$ and obtain

$$LR(f, g, a, b, \text{id}) = \sum_{k=2}^{n-1} (-1)^k f[b; a, \ldots, a] \frac{(b - g(t))(a - g(t))^{k-1} + \bar{R}_1(a + b - g(t))}{k \text{ times}},$$

$$LR(f, g, a, b, \text{id}) = (-1)^2 f[b; b; a](b - g(t))(a - g(t))$$

$$+ \sum_{k=2}^{n-2} (-1)^{k+1} f[b; b; a, \ldots, a] \frac{(b - g(t))^2(a - g(t))^{k-1} + \bar{R}_2(a + b - g(t))}{k \text{ times}},$$

$$LR(f, g, a, b, \text{id}) = (b - g(t))(-f[b, b] + f[a, b]) + \sum_{k=2}^{m-1} f[k(b)] \frac{(b - g(t))^k}{k!}$$

$$+ \sum_{k=1}^{n-m} (-1)^{m+k-1} f[b, \ldots, b; a, \ldots, a] \frac{(b - g(t))^m(a - g(t))^{k-1}}{m \text{ times} \quad k \text{ times}}$$

$$+ \bar{R}_m(a + b - g(t)).$$

Identities (2.9), (2.10), and (2.11) are deduced by applying a normalized positive linear functional $A$ to the previous equalities, respectively.

Lemma 2.2 is proved.

Our first result is an upper bound for the difference in the Edmundson–Lah–Ribarič inequality expressed by Hermite’s interpolating polynomials in terms of divided differences.

**Theorem 2.1.** Let $L$ satisfy conditions $(L_1)$ and $(L_2)$ on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Also let $f \in C^n([a, b])$ and let $g \in L$ be any function such that $f \circ g \in L$. If the function $f$ is $n$-convex and $n$ and $m \geq 3$ are of different parities, then

$$LR(f, g, a, b, A) \leq (A(g) - a)(f[a, a] - f[a, b]) + \sum_{k=2}^{m-1} f[k(a)] \frac{(g - a1)^k}{k!}$$

$$+ \sum_{k=1}^{n-m} f[a, \ldots, a; b, \ldots, b] \frac{(g - a1)^m(g - b1)^{k-1}}{m \text{ times} \quad k \text{ times}}. \tag{2.19}$$

Inequality (2.19) also holds when the function $f$ is $n$-concave and $n$ and $m$ are of the same parity. In the cases where the function $f$ is $n$-convex and $n$ and $m$ have the same parity or when the function $f$ is $n$-concave and $n$ and $m$ have different parities, the inequality sign in (2.19) is reversed.

**Proof.** We start with the representation of the left-hand side of the Edmundson–Lah–Ribarič inequality (2.4) with a special focus on the last term:

$$A(R(g)) = A \left( (g - a1)^m (g - b1)^{n-m} f[g; a1, \ldots, a1; b1, \ldots, b1] \right).$$
Since $A$ is positive, it preserves the sign. Hence, it is necessary to study the sign of the expression

$$(g(t) - a)^m (g(t) - b)^{n-m} f[g(t); a, \ldots, a; b, b, \ldots, b].$$

Since $a \leq g(t) \leq b$ for any $t \in E$, we have $(g(t) - a)^m \geq 0$ for every $t \in E$ and any choice of $m$. For the same reason, we get $(g(t) - b) \leq 0$. It is clear that $(g(t) - b)^{n-m} \leq 0$ if $n$ and $m$ are of different parities and $(g(t) - b)^{n-m} \geq 0$ if $n$ and $m$ are of the same parity.

If the function $f$ is n-convex, then

$$f[g(t); a, \ldots, a; b, b, \ldots, b] \geq 0.$$  \hspace{1cm} (m \text{ times})

However, if the function $f$ is n-concave, then

$$f[g(t); a, \ldots, a; b, b, \ldots, b] \leq 0.$$  \hspace{1cm} (m \text{ times})

Thus, (2.19) easily follows from (2.1).

Theorem 2.1 is proved.

The following result provides us with a similar upper bound for the difference in the Edmundson–Lah–Ribarič inequality. It is obtained from Lemma 2.2.

**Theorem 2.2.** Let $L$ satisfy conditions (L1) and (L2) on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Also let $f \in C^n([a, b])$ and let $g \in L$ be any function such that $f \circ g \in L$. If the function $f$ is n-convex and $m \geq 3$ is odd, then

$$LR(f, g, a, b, A) \leq (b - A(g))(f[a, b] - f[b, b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} A[(g - b1)^k]$$

$$+ \sum_{k=1}^{n-m} f[b, \ldots, b; a, \ldots, a] A[(g - b1)^m(g - a1)^{k-1}].$$  \hspace{1cm} (2.20)

Inequality (2.20) also holds when the function $f$ is n-concave and $m$ is even. In the cases where the function $f$ is n-convex and $m$ is even or when the function $f$ is n-concave and $m$ is odd, the inequality sign in (2.20) is reversed.

**Proof.** As in the proof of the previous theorem, we start with the representation of the left-hand side of the Edmundson–Lah–Ribarič inequality (2.11) with a special focus on the last term:

$$A(R^*_m(g)) = A\left( f[g; b1, \ldots, b1; a1, \ldots, a1] (g - b1)^m(g - a1)^{n-m} \right).$$
As earlier, in view of the positivity of the linear functional $A$, we only need to study the sign of the expression:

$$(g(t) - b)^m(g(t) - a)^{n-m}f\left[g(t); b, \ldots, b; a, a, \ldots, a\right].$$

Since $a \leq g(t) \leq b$ for every $t \in E$, we have $(g(t) - a)^{n-m} \geq 0$ for every $t \in E$ and any choice of $m$. For the same reason, we get $(g(t) - b)^m \leq 0$. It is clear that $(g(t) - b)^m \leq 0$ if $m$ is odd and $(g(t) - b)^m \geq 0$ if $m$ is even.

If the function $f$ is $n$-convex, then its $n$th order divided differences are greater or equal to zero. At the same time, if the function $f$ is $n$-concave, then its $n$th order divided differences are less or equal to zero.

Thus, (2.20) easily follows from Lemma 2.2.

Theorem 2.2 is proved.

**Corollary 2.1.** Let $L$ satisfy conditions $(L_1)$ and $(L_2)$ on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Also let $n$ be an odd number, let $f \in C^n([a, b])$, and let $g \in L$ be any function such that $f \circ g \in L$. If the function $f$ is $n$-convex and $m \geq 3$ is odd, then

$$(A(g) - a)\left[[f[a, a] - f[a, b]] + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!}A[(g - a1)^k]\right]$$

$$+ \sum_{k=1}^{n-m} f\left[a, \ldots, a; b, \ldots, b\right]A[(g - a1)^m(g - b1)^{k-1}]$$

$$\leq LR(f, g, a, b, A)$$

$$\leq (b - A(g))\left[[f[a, a] - f[b, b]] + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!}A[(g - b1)^k]\right]$$

$$+ \sum_{k=1}^{n-m} f\left[b, \ldots, b; a, \ldots, a\right]A[(g - b1)^m(g - a1)^{k-1}].$$

Inequality (2.21) also holds when the function $f$ is $n$-concave and $m$ is even. In the cases where the function $f$ is $n$-convex and $m$ is even or the function $f$ is $n$-concave and $m$ is odd, the inequality signs in (2.21) should be reversed.

**Remark 2.1.** In [25] (Theorem 2.3), it is proved that, for 3-convex functions, we have

$$(A(g) - a)\left[f'(a) - \frac{f(b) - f(a)}{b - a}\right] + \frac{f''(a)}{2}A[(g - a1)^2]$$

$$\leq LR(f, g, a, b, A)$$

$$\leq (b - A(g))\left[f'(a) - \frac{f(b) - f(a)}{b - a} - f'(b)\right] + \frac{f''(b)}{2}A[(b1 - g)^2].$$
and, moreover, if the function $f$ is 3-concave, then the inequality signs should be reversed. It is obvious that inequalities (2.21) from Corollary 2.1 generalize the result stated above.

The next result gives upper and lower bounds for the difference in the Edmundson–Lah–Ribarič inequality expressed by Hermite’s interpolating polynomials in terms of divided differences. It is obtained from Lemma 2.1.

**Theorem 2.3.** Let $L$ satisfy conditions $(L_1)$ and $(L_2)$ on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Also let $f \in C^n([a, b])$ and let $g \in L$ be any function such that $f \circ g \in L$. If the function $f$ is $n$-convex and $n$ is odd, then

$$
\sum_{k=2}^{n-1} f[a; b, \ldots, b] A[(g - a 1)(g - b 1)^{k-1}] \leq LR(f, g, a, b, A)
$$

\[
\leq f[a, a; b] A[(g - a 1)(g - b 1)] + \sum_{k=2}^{n-2} f[a, a; b, \ldots, b] A[(g - a 1)^2(g - b 1)^{k-1}].
\tag{2.22}
\]

Inequalities (2.22) also hold if the function $f$ is $n$-concave and $n$ is even. In the cases where the function $f$ is $n$-convex and $n$ is even or the function $f$ is $n$-concave and $n$ is odd, the inequality signs in (2.22) should be reversed.

**Proof.** It follows from the discussion about positivity or negativity of the term $A(R_n(g))$ in the proof of Theorem 2.1 for $m = 1$ that

$$A(R_1(g)) \geq 0$$

if the function $f$ is $n$-convex and $n$ is odd or if $f$ is $n$-concave and $n$ even;

$$A(R_1(g)) \leq 0$$

if the function $f$ is $n$-concave and $n$ is odd or if $f$ is $n$-convex and $n$ even.

Thus, by using identity (2.2), we obtain

$$LR(f, g, a, b, A) \geq f[a; b, b] A[(g - a 1)(g - b 1)] + f[a; b, b, b]A[(g - a 1)(g - b 1)^2] + \ldots + f[a; b, b, \ldots, b] A[(g - a 1)(g - b 1)^{n-2}]$$

for $A(R_1(g)) \geq 0$. In the case where $A(R_1(g)) \leq 0$, the inequality sign is reversed.

In the same way, for $m = 2$ we obtain

$$A(R_2(g)) \leq 0$$

when the function $f$ is $n$-convex and $n$ is odd or when $f$ is $n$-concave and $n$ even;

$$A(R_2(g)) \geq 0$$

when the function $f$ is $n$-concave and $n$ is odd or when $f$ is $n$-convex and $n$ even.

In this case, identity (2.3) for $A(R_2(g)) \leq 0$ implies that

$$LR(f, g, a, b, A) \leq f[a, a; b] A[(g - a 1)(g - b 1)]$$
\[ f[a, a; b, b]A[(g - a1)^2(g - b1)] + \ldots + f[a, a;b,b,\ldots,b]A[(g - a1)^2(g - b1)^{n-3}] \]

At the same time, for \( A(R_2(g)) \geq 0 \), the inequality sign should be reversed.

Thus, combining the two results presented above, we get exactly (2.22).

Theorem 2.3 is proved.

By using Lemma 2.2, we can get similar bounds for the difference in the Edmundson–Lah–Ribarič inequality valid not only for the odd numbers but for all \( n \in \mathbb{N} \).

**Theorem 2.4.** Let \( L \) satisfy conditions \((L_1)\) and \((L_2)\) on a nonempty set \( E \) and let \( A \) be any positive linear functional on \( L \) with \( A(1) = 1 \). Also let \( f \in \mathcal{C}^n([a, b]) \) and let \( g \in L \) be any function such that \( f \circ g \in L \). If the function \( f \) is \( n \)-convex, then

\[
f[b, b; a]A[(g - b1)(g - a1)] + \sum_{k=2}^{n-2} f[b, b, a, \ldots, a]A[(g - b1)^2(g - a1)^{k-1}] \leq LR(f, g, a, b, A) \leq \sum_{k=1}^{n-1} f[b, a, \ldots, a]A[(g - b1)(g - a1)^{k-1}]. \tag{2.23}
\]

If the function \( f \) is \( n \)-concave, then the inequality signs in (2.23) are reversed.

**Proof.** We now return to the discussion about positivity or negativity of the term \( A(R_m^*(g)) \) in the proof of Theorem 2.2. Thus, for \( m = 1 \), we have

\[
(g(t) - b1)^1(g(t) - a1)^{n-1} \leq 0 \quad \text{for every} \quad t \in E.
\]

Hence, \( A(R_m^*(g)) \geq 0 \) if the function \( f \) is \( n \)-concave and \( A(R_m^+(g)) \leq 0 \) if the function \( f \) is \( n \)-convex. Thus, by using identity (2.9) for a \( n \)-convex function \( f \), we obtain

\[
LR(f, g, a, b, A) \geq f[b, b; a]A[(g - b1)(g - a1)] + f[b, b; a, a]A[(g - b1)^2(g - a1)] + \ldots + f[b, b, a, a, \ldots, a]A[(g - b1)^2(g - a1)^{n-3}] \]

At the same time, if the function \( f \) is \( n \)-concave, the inequality sign should be reversed.

Similarly, for \( m = 2 \), we have

\[
(g(t) - b1)^2(g(t) - a1)^{n-2} \geq 0 \quad \text{for every} \quad t \in E.
\]
Therefore, $A(R_n^+(g)) \geq 0$ if the function $f$ is $n$-convex and $A(R_n^-(g)) \leq 0$ if the function $f$ is $n$-concave. In this case, identity (2.10) for an $n$-convex function $f$ gives

$$LR(f, g, a, b, A) \leq f[b; a, a]A[(g - b1)(g - a1)]$$

$$+ f[b; a, a, a]A[(g - b1)(g - a1)^2]$$

$$+ \ldots + f[b; a, a, \ldots, a]A[(g - b1)(g - a1)^{n-2}].$$

If the function $f$ is $n$-concave, then the inequality sign should be reversed.

Combining the two results presented above, we get exactly inequality (2.23).

Theorem 2.4 is proved.

**Remark 2.2.** Note that

$$f[a; b, b] = \frac{1}{b - a} \left( f'(b) - \frac{f(b) - f(a)}{b - a} \right),$$

$$f[a, a; b] = \frac{1}{b - a} \left( f'(b) - \frac{f(b) - f(a)}{b - a} \right).$$

Thus, if we take $n = 3$ in (2.22) or (2.23), then we get

$$\frac{A[(g - a1)(g - b1)]}{b - a} \left( f'(b) - \frac{f(b) - f(a)}{b - a} \right)$$

$$\leq LR(f, g, a, b, A) \leq \frac{A[(g - a1)(g - b1)]}{b - a} \left( f'(b) - \frac{f(b) - f(a)}{b - a} \right)$$

(2.24)

for a 3-convex function. At the same time, for a 3-concave function the inequality signs should be reversed. Inequalities (2.24) were proved in [25] (Theorem 2.1). Hence, it is possible to conclude that Theorem 2.3 and Theorem 2.4 generalize the result from [25].

### 3. Applications to Csiszár Divergence

We denote the set of all finite discrete probability distributions by $\mathbb{P}$, i.e., we say that $p = (p_1, \ldots, p_r) \in \mathbb{P}$ if $p_i \in [0, 1]$ for $i = 1, \ldots, r$ and $\sum_{i=1}^{r} p_i = 1$.

Numerous theoretical divergence measures between two probability distributions have been introduced and comprehensively studied. Their applications can be found in the analysis of contingency tables [13], in the approximation of probability distributions [8, 22], in signal processing [18], and in pattern recognition [4, 6].

Csiszár [9–10] introduced the $f$-divergence functional as follows:

$$D_f(p, q) = \sum_{i=1}^{r} q_i f\left( \frac{p_i}{q_i} \right),$$

(3.1)

where $f : [0, +\infty)$ is a convex function. It represents a “distance function” on the set of probability distributions $\mathbb{P}$. 
Numerous theoretical divergences are special cases of the Csiszár $f$-divergence for different choices of the function $f$.

As in [10], we interpret undefined expressions as follows:

$$f(0) = \lim_{t \to 0^+} f(t), \quad 0 \cdot f \left( \frac{0}{0} \right) = 0,$$

$$0 \cdot f \left( \frac{a}{0} \right) = \lim_{\epsilon \to 0^+} \epsilon \cdot f \left( \frac{a}{\epsilon} \right) = a \cdot \lim_{t \to \infty} \frac{f(t)}{t}.$$

In this section, our aim is to deduce mutual bounds for the generalized $f$-divergence functional in the outlined setting. In this way, we obtain some new reverse relations for the generalized $f$-divergence functional corresponding to the class of $n$-convex functions. This is a generalization of the results obtained in [25]. Throughout this section, when speaking about the interval $[a, b]$, we always assume that $[a, b] \subseteq \mathbb{R}_+$. For an $n$-convex function $f : [a, b] \to \mathbb{R}$, we give the following definition of generalized $f$-divergence functional:

$$\tilde{D}_f(p, q) = \sum_{i=1}^{r} q_i f \left( \frac{p_i}{q_i} \right).$$

(3.2)

The first result in this section is valid by virtue of our Theorem 2.1.

**Theorem 3.1.** Let $[a, b] \subseteq \mathbb{R}$ be an interval such that $a \leq 1 \leq b$, let $f \in C^n([a, b])$, and let $p = (p_1, \ldots, p_r)$ and $q = (q_1, \ldots, q_r)$ be probability distributions such that $p_i/q_i \in [a, b]$ for every $i = 1, \ldots, r$. If the function $f$ is $n$-convex and $n$ and $3 \leq m \leq n - 1$ have different parities, then

$$\frac{b - 1}{b - a} f(a) + \frac{1 - a}{b - a} f(b) - \tilde{D}_f(p, q)$$

$$\leq (1 - a) \left( f[a, a] - f[a, b] \right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{r} \frac{(p_i - a q_i)^k}{q_i^{k-1}}$$

$$+ \sum_{k=1}^{n-m} f \left[ a, \ldots, a; b, \ldots, b \right] \sum_{i=1}^{r} \frac{(p_i - a q_i)^m (p_i - a q_i)^{k-1}}{q_i^{m+k-2}}. \quad (3.3)$$

Inequality (3.3) also holds when the function $f$ is $n$-concave and $n$ and $m$ are of the same parity. If the function $f$ is $n$-convex and $n$ and $m$ are of the same parity or if the function $f$ is $n$-concave and $n$ and $m$ are of different parities, then the inequality sign in (3.3) is reversed.

**Proof.** Let $x = (x_1, \ldots, x_r)$ be such that $x_i \in [a, b]$ for $i = 1, \ldots, r$. In relation (2.19), we can make the replacement

$$g \leftrightarrow x \quad \text{and} \quad A(x) = \sum_{i=1}^{r} p_i x_i.$$

In this way, we get

$$\frac{b - \bar{x}}{b - a} f(a) + \frac{\bar{x} - a}{b - a} f(b) - \sum_{i=1}^{r} p_i f(x_i)$$

$$\leq (\bar{x} - a) \left( f[a, a] - f[a, b] \right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{r} p_i (x_i - a)^k$$

$$+ \sum_{k=1}^{n-m} f \left[ a, \ldots, a; b, \ldots, b \right] \sum_{i=1}^{r} \frac{(p_i - a q_i)^m (p_i - a q_i)^{k-1}}{q_i^{m+k-2}}.$$
Thus, after calculating
\[ \bar{x} = \sum_{i=1}^{n} p_i x_i \]
we get (3.3).

Theorem 3.1 is proved.

By using Theorem 2.2 in the same way as above, we get an Edmundson–Lah–Ribarič-type inequality for
the generalized \( f\)-divergence functional (3.2) that does not depend on the parity of \( n \). This is done in the following theorem:

**Theorem 3.2.** Let \( [a, b] \subseteq \mathbb{R} \) be an interval such that \( a \leq 1 \leq b \), let \( f \in \mathcal{C}^{m}([a, b]) \), and let \( p = (p_1, \ldots, p_r) \) and \( q = (q_1, \ldots, q_r) \) be probability distributions such that \( p_i/q_i \in [a, b] \) for every \( i = 1, \ldots, r \). If the function \( f \) is \( n \)-convex and \( 3 \leq m \leq n - 1 \) is odd, then
\[
\frac{b - 1}{b - a} f(a) + \frac{1 - a}{b - a} f(b) - \hat{D}_f(p, q) \\
\leq (b - 1)(f[a, b] - f[b, b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b) - f^{(k)}(a)}{k!} \\
+ \frac{1}{m - 1} \frac{1}{k} \sum_{i=1}^{r} (p_i - bq_i)^m (p_i - aq_i)^{k-1} q_i^{m+k-2} \tag{3.4}
\]

Inequality (3.4) also holds when the function \( f \) is \( n \)-concave and \( m \) is even. In the case where the function \( f \) is \( n \)-convex and \( m \) is even or the function \( f \) is \( n \)-concave and \( m \) is odd, the inequality sign in (3.4) is reversed.

Another generalization of the Edmundson–Lah–Ribarič inequality, which gives lower and upper bounds for the generalized \( f\)-divergence functional, is given by the following theorem:

**Theorem 3.3.** Let \( [a, b] \subseteq \mathbb{R} \) be an interval such that \( a \leq 1 \leq b \), let \( f \in \mathcal{C}^{m}([a, b]) \), and let \( p = (p_1, \ldots, p_r) \) and \( q = (q_1, \ldots, q_r) \) be probability distributions such that \( p_i/q_i \in [a, b] \) for every \( i = 1, \ldots, r \). If the function \( f \) is \( n \)-convex and \( n \) is odd, then
\[
\sum_{k=2}^{n-1} \frac{f[a; b, b, \ldots, b]}{k} \sum_{i=1}^{r} (p_i - aq_i)(p_i - bq_i)^{k-1} q_i^{k-1} \\
\leq \frac{b - 1}{b - a} f(a) + \frac{1 - a}{b - a} f(b) - \hat{D}_f(p, q)
\]
\[
\leq f[a, a; b] \sum_{i=1}^{r} \frac{(p_i - aq_i)(p_i - bq_i)}{q_i} \\
+ \sum_{k=2}^{n-2} f[b, b; a, \ldots, a] \sum_{k \text{ times}} \frac{(p_i - aq_i)^2(p_i - bq_i)^{k-1}}{q_i^k}.
\] (3.5)

Inequalities (3.5) also hold when the function \( f \) is \( n \)-concave and \( n \) is even. In the cases where the function \( f \) is \( n \)-convex and \( n \) is even or the function \( f \) is \( n \)-concave and \( n \) is odd, the inequality signs in (3.5) are reversed.

**Proof.** We first consider inequalities (2.22) and follow the steps of the proof of Theorem 3.1.

By using Theorem 2.4 in an analogous way, we get similar bounds for the generalized \( f \)-divergence functional valid not only for the odd numbers but for all \( n \in \mathbb{N} \).

**Theorem 3.4.** Let \([a, b] \subset \mathbb{R}\) be an interval such that \( a \leq 1 \leq b \), let \( f \in C^n([a, b]) \), and let \( p = (p_1, \ldots, p_r) \) and \( p = (q_1, \ldots, q_r) \) be probability distributions such that \( p_i/q_i \in [a, b] \) for every \( i = 1, \ldots, r \). If the function \( f \) is \( n \)-convex, then

\[
f[b, b; a] \sum_{i=1}^{r} \frac{(p_i - aq_i)(p_i - bq_i)}{q_i} \\
+ \sum_{k=2}^{n-2} f[b, b; a, a, \ldots, a] \sum_{k \text{ times}} \frac{(p_i - aq_i)^2(p_i - bq_i)^{k-1}}{q_i^k} \leq \frac{b - 1}{b - a} f(a) + \frac{1 - a}{b - a} f(b) - \tilde{D}_f(p, q) \leq n \sum_{k=2}^{n-1} f[b, a, \ldots, a] \sum_{k \text{ times}} \frac{(p_i - aq_i)^{k-1}(p_i - bq_i)}{q_i^{k-1}}.
\] (3.6)

If the function \( f \) is \( n \)-concave, then the inequality signs in (3.6) are reversed.

**Example 3.1.** Let \( p = (p_1, \ldots, p_r) \) and \( p = (q_1, \ldots, q_r) \) be probability distributions.

The Kullback–Leibler divergence of the probability distributions \( p \) and \( q \) is defined as

\[D_{KL}(p, q) = \sum_{i=1}^{r} q_i \log \frac{q_i}{p_i},\]

while the corresponding generating function is \( f(t) = t \log t, \ t > 0 \). We can compute

\[f^{(n)}(t) = (-1)^n(n - 2)!t^{-(n-1)}.\]

It is clear that this function is \((2n - 1)\)-concave and \((2n)\)-convex for any \( n \in \mathbb{N} \).

The Hellinger divergence of the probability distributions \( p \) and \( q \) is defined as

\[D_H(p, q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{q_i} - \sqrt{p_i})^2,\]
whereas the corresponding generating function is

\[ f(t) = \frac{1}{2}(1 - \sqrt{t})^2, \quad t > 0. \]

It is easy to see that

\[ f^{(n)}(t) = (-1)^n \frac{(2n - 3)!!}{2^n} t^{-2n-1}. \]

Hence, the function \( f \) is \((2n - 1)\)-concave and \((2n)\)-convex for any \( n \in \mathbb{N} \).

The harmonic divergence of the probability distributions \( p \) and \( q \) is defined as

\[ D_{H_0}(p, q) = \sum_{i=1}^{n} \frac{2p_i q_i}{p_i + q_i}, \]

and the corresponding generating function is \( f(t) = \frac{2t}{1 + t} \). We can compute

\[ f^{(n)}(t) = 2(-1)^{n+1} n!(1 + t)^{-(n+1)}. \]

It is necessary to consider the following two cases:

- if \( t < -1 \), then the function \( f \) is \( n \)-convex for any \( n \in \mathbb{N} \);
- if \( t > -1 \), then the function \( f \) is \((2n)\)-concave and \((2n - 1)\)-convex for any \( n \in \mathbb{N} \).

The Jeffreys divergence of the probability distributions \( p \) and \( q \) is defined as

\[ D_{J}(p, q) = \frac{1}{2} \sum_{i=1}^{n} (q_i - p_i) \log \frac{q_i}{p_i}, \]

and the corresponding generating function is

\[ f(t) = (1 - t) \log \frac{1}{t}, \quad t > 0. \]

After necessary calculations, we conclude that

\[ f^{(n)}(t) = (-1)^{n+1} t^{-n} (n - 1)! (1 + nt). \]

Obviously, this function is \((2n - 1)\)-convex and \((2n)\)-concave for any \( n \in \mathbb{N} \).

It is clear that all results obtained in the present section can be applied to the special types of divergences mentioned in this example.

4. Examples with Zipf and Zipf–Mandelbrot Law

The Zipf law [33, 34] has significant applications in a broad variety of scientific disciplines – from astronomy to demography, software structures, economics, zoology, and even warfare [12]. This is one of the basic laws in the information science and bibliometrics but it is also often used in linguistics. In typical cases, it is customary
to deal with integer-valued observables (numbers of objects, people, cities, words, animals, and corpses) and the frequency of their occurrence.

The probability mass function of the Zipf law with parameters $N \in \mathbb{N}$ and $s > 0$ is as follows:

$$f(k; N, s) = \frac{1}{k^s}, \quad \text{where} \quad H_{N,s} = \sum_{i=1}^{N} \frac{1}{i^s}.$$  

In 1966, B. Mandelbrot proposed an improvement of the Zipf law for the count of the low-rank words. Various scientific fields use this law for different purposes. Thus, the information sciences use it for indexing [11, 32]; the ecological field studies apply it for the predictability of ecosystems [26]; in music, it is used to detect aesthetically pleasant music [23].

The Zipf–Mandelbrot law is a discrete probability distribution with parameters $N \in \mathbb{N}$ and $q, s \in \mathbb{R}$ such that $q \geq 0$ and $s > 0$, possible values $\{1, 2, \ldots, N\}$, and a probability mass function

$$f(i; N, q, s) = \frac{1}{(i + q)^s}, \quad \text{where} \quad H_{N,q,s} = \sum_{i=1}^{N} \frac{1}{(i + q)^s}. \quad (4.1)$$

Let $p$ and $q$ be the Zipf–Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$ and $s_1, s_2 > 0$, respectively. We denote

$$H_{N,q_1,s_1} = H_1, \quad H_{N,q_2,s_2} = H_2,$$

$$a_{p,q} := \min \left\{ \frac{p_i}{q_i} \right\} = \frac{H_2}{H_1} \min \left\{ \frac{(i + q_2)^{s_2}}{(i + q_1)^{s_1}} \right\}, \quad (4.2)$$

$$b_{p,q} := \max \left\{ \frac{p_i}{q_i} \right\} = \frac{H_2}{H_1} \max \left\{ \frac{(i + q_2)^{s_2}}{(i + q_1)^{s_1}} \right\}.$$

In this section, we apply the results obtained for the Csiszár divergence in the previous section in order to get different inequalities for the Zipf–Mandelbrot law. The following results are special cases of Theorems 3.1, 3.2, 3.3, and 3.4, respectively. They give an Edmundson–Lah–Ribarič-type inequality for the generalized $f$-divergence of the Zipf–Mandelbrot law.

**Corollary 4.1.** Let $p$ and $q$ be the Zipf–Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$, and $s_1, s_2 > 0$, respectively, and let $H_1$, $H_2$, $a_{p,q}$, and $a_{p,q}$ be defined in (4.2). Also let $f \in C^n([a_{p,q}, b_{p,q}])$ be an $n$-convex function. If $n$ and $3 \leq m \leq n - 1$ are of different parities, then

$$\frac{b_{p,q} - 1}{b_{p,q} - a_{p,q}} f(a_{p,q}) + \frac{1 - a_{p,q}}{b_{p,q} - a_{p,q}} f(b_{p,q}) - \hat{D}_f(p, q)$$

$$\leq (1 - a_{p,q}) \left( f'(a_{p,q}) - f[a_{p,q}, b_{p,q}] \right)$$

$$+ \sum_{k=2}^{m-1} \frac{f^{(k)}(a_{p,q})}{H_2 k!} \sum_{k=1}^{r} \frac{H_2(i + q_2)^{s_2}}{H_1(i + q_1)^{s_1}} - a_{p,q}^k$$

$$\leq \left( \frac{H_2(i + q_2)^{s_2}}{H_1(i + q_1)^{s_1}} - a_{p,q}^k \right)$$
\[ + \sum_{k=1}^{n-m} f[a_{p,q}, \ldots, a_{p,q}; b_{p,q}, \ldots, b_{p,q}] \]

\[ \times \sum_{i=1}^{r} \frac{\left( H_2(i + q_2)^{s_2} - a_{p,q} \right)^m \left( H_2(i + q_2)^{s_2} - b_{p,q} \right)^{k-1}}{H_2(i + q_2)^{s_2}}. \]

This inequality also holds if the function \( f \) is \( n \)-concave and \( n \) and \( m \) are of the same parity. In the cases where the function \( f \) is \( n \)-convex and \( n \) and \( m \) are of the same parity, or the function \( f \) is \( n \)-concave and \( n \) and \( m \) are of different parities, the inequality sign is reversed.

**Corollary 4.2.** Let \( p \) and \( q \) be the Zipf–Mandelbrot laws with parameters \( N \in \mathbb{N}, q_1, q_2 \geq 0, \) and \( s_1, s_2 > 0, \) respectively, and let \( H_1, H_2, a_{p,q} \) and \( a_{p,q} \) be defined in (4.2). Also let \( f \in C^n([a_{p,q}, b_{p,q}]) \) be an \( n \)-convex function and let \( 3 \leq m \leq n-1 \) be of different parities. Then

\[ \frac{b_{p,q} - 1}{b_{p,q} - a_{p,q}} f(a_{p,q}) + \frac{1 - a_{p,q}}{b_{p,q} - a_{p,q}} f(b_{p,q}) - \tilde{D}_f(p, q) \]

\[ \leq (b_{p,q} - 1) \left( f[a_{p,q}, b_{p,q}] - f'(b_{p,q}) \right) \]

\[ + \sum_{k=2}^{m-1} \frac{f(k)(b_{p,q})}{H_2 k!} \sum_{i=1}^{r} \frac{\left( H_2(i + q_2)^{s_2} - a_{p,q} \right)^k}{H_2(i + q_2)^{s_2}} \]

\[ + \sum_{k=1}^{n-m} f[a_{p,q}, \ldots, b_{p,q}; b_{p,q}, \ldots, a_{p,q}] \]

\[ \times \sum_{i=1}^{r} \frac{\left( H_2(i + q_2)^{s_2} - b_{p,q} \right)^m \left( H_2(i + q_2)^{s_2} - a_{p,q} \right)^{k-1}}{H_2(i + q_2)^{s_2}}. \]

This inequality also holds if the function \( f \) is \( n \)-concave and \( m \) is even. In the cases where the function \( f \) is \( n \)-convex and \( m \) is even or the function \( f \) is \( n \)-concave and \( m \) is odd, the inequality sign is reversed.

**Corollary 4.3.** Let \( p \) and \( q \) be the Zipf–Mandelbrot laws with parameters \( N \in \mathbb{N}, q_1, q_2 \geq 0, \) and \( s_1, s_2 > 0, \) respectively, and let \( H_1, H_2, a_{p,q} \) and \( a_{p,q} \) be defined in (4.2). Also let \( f \in C^n([a_{p,q}, b_{p,q}]) \) be an \( n \)-convex function. If \( n \) is odd, then

\[ \sum_{k=2}^{n-1} f[a_{p,q}; b_{p,q}, \ldots, b_{p,q}] \]

\[ \sum_{i=1}^{r} \frac{\left( H_2(i + q_2)^{s_2} - a_{p,q} \right)^m \left( H_2(i + q_2)^{s_2} - b_{p,q} \right)^{k-1}}{H_2(i + q_2)^{s_2}} \]

\[ \leq \frac{b_{p,q} - 1}{b_{p,q} - a_{p,q}} f(a_{p,q}) + \frac{1 - a_{p,q}}{b_{p,q} - a_{p,q}} f(b_{p,q}) - \tilde{D}_f(p, q) \]
If the function $f$ is $n$-concave and $n$ is even. In the case where the function $f$ is $n$-concave and $n$ is odd, the inequality signs are reversed.

**Corollary 4.4.** Let $p$ and $q$ be the Zipf–Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1, q_2 \geq 0$, and $s_1, s_2 > 0$, respectively, and let $H_1, H_2, a_{p,q}$ and $a_{p,q}$ be defined in (4.2). Also let $f \in C^n([a_{p,q}, b_{p,q}])$ be a $n$-convex function. Then

\[
\begin{align*}
\leq f[a_{p,q}, a_{p,q}; b_{p,q}] & \sum_{i=1}^{r} \frac{(H_2(i + q_2)^{s_2} - a_{p,q})}{H_1(i + q_1)^{s_1}} \frac{(H_2(i + q_2)^{s_2} - b_{p,q})}{H_2(i + q_2)^{s_2}} \\
+ \sum_{k=2}^{n-2} f \left[ a_{p,q}, a_{p,q}; \underbrace{b_{p,q}, \ldots, b_{p,q}}_{k \text{ times}} \right] & \sum_{i=1}^{r} \frac{(H_2(i + q_2)^{s_2} - a_{p,q})}{H_1(i + q_1)^{s_1}} \frac{(H_2(i + q_2)^{s_2} - b_{p,q})}{H_2(i + q_2)^{s_2}}^{k-1} \\
& \leq \frac{b - 1}{b - a} f(a) + \frac{1 - a}{b - a} f(b) - D_f(p, q) \\
& \leq \sum_{k=2}^{n-1} f \left[ a_{p,q}, a_{p,q}; \ldots, a_{p,q} \right] \sum_{i=1}^{r} \frac{(H_2(i + q_2)^{s_2} - a_{p,q})}{H_1(i + q_1)^{s_1}} \frac{(H_2(i + q_2)^{s_2} - b_{p,q})}{H_2(i + q_2)^{s_2}}^{k-1}.
\end{align*}
\]

If the function $f$ is $n$-concave, then the inequality signs are reversed.

**Remark 4.1.** By analyzing Example 3.1, we conclude that the general results from this section can be easily applied to any of the following divergences: the Kullback–Leibler divergence, the Hellinger divergence, the harmonic divergence, or the Jeffreys divergence.

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