ON THE LIFTING PROBLEM IN $\mathbb{P}^4$ IN CHARACTERISTIC $p$

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ABSTRACT. Given $\mathbb{P}^4_k$, with $k$ algebraically closed field of characteristic $p > 0$, and $X \subset \mathbb{P}^4_k$ an integral surface of degree $d$, let $Y = X \cap H$ be the general hyperplane section of $X$. We suppose that $h^0 \mathcal{F}_Y(s) \neq 0$ and $h^0 \mathcal{F}_X(s) = 0$ for some $s > 0$. This determines a nonzero element $\alpha \in H^1 \mathcal{F}_X(s)$ such that $\alpha \cdot H = 0$ in $H^1 \mathcal{F}_X(s)$. We find different upper bounds of $d$ in terms of $s$, $p$ and the order of $\alpha$ and we show that these bounds are sharp. In particular, we see that $d \leq s^2$ for $p < s$ and $d \leq s^2 - s + 2$ for $p \geq s$.

1. Introduction

Let $X \subset \mathbb{P}^4_k$, with $k$ algebraically closed field of positive characteristic $p$, be an integral surface of degree $d$. Let $Y = X \cap H$ be the general hyperplane section of $X$ and consider a surface of degree $s$ containing $Y$. In this paper we study the problem of lifting a surface of $H$ of degree $s$ containing $Y$ to a hypersurface in $\mathbb{P}^4_k$ of degree $s$ containing $X$. In particular, we suppose that $h^0 \mathcal{F}_Y(s) \neq 0$ and $h^0 \mathcal{F}_X(s) = 0$ for some $s > 0$.

In the case that $\text{char } k = 0$ the problem has been studied and solved by Mezzetti and Raspani in [14] and in [16], showing that $d \leq s^2 - s + 2$ and that this bound is sharp, and in [15] Mezzetti classifies the border case $d = s^2 - s + 2$. Other results concerning the lifting problem have been obtained in characteristic 0 for curves in $\mathbb{P}^3$ (see [11], Corollary p. 147, [5] and [22] Corollario 2) and for integral varieties of codimension 2 in $\mathbb{P}^n$ (see, for example, [16] for $n = 5$, [20] for $n = 6$ and [19] for the general case). In the case that $\text{char } p > 0$ the lifting problem has been studied for curves in $\mathbb{P}^3$ in [1].

In this paper, the starting point is that the non lifting section of $H^0 \mathcal{F}_Y(s)$ determines a nonzero element $\alpha \in H^1 \mathcal{F}_X(s)$ such that $\alpha \cdot H = 0$ in $H^1 \mathcal{F}_X(s)$. The order of $\alpha$ is the maximum integer $m \in \mathbb{N}$ such that $\alpha = \beta \cdot H^m$ for some $\beta \in H^1 \mathcal{F}_X(s - m - 1)$. For $p < s$ we need to relate $s$ and $m$. In particular, in Theorem 3 we suppose that $p < s$ and we show that:

(1) $d \leq s^2 - s + 2 + p^n$, if $s \geq 2m + 3$, with $p^n \leq m + 1$ and $p^{n+1} > m + 1$;
(2) $d \leq s^2$ if $s \leq 2m + 2$.

As a consequence, we see that for $p < s$ it must be $d \leq s^2$. For $p \geq s$ in Theorem 4 we show that $d \leq s^2 - s + 2$, which is the same bound as in the characteristic 0 case. In Example 5 we see that the bounds given in Theorem 3 and in Theorem 4 are sharp.

2. Hilbert function of points in $\mathbb{P}^2$

Let us denote by $X$ a zero-dimensional scheme in $\mathbb{P}^2_k$, where $k$ is an algebraically closed field of any characteristic. Let $H_X : \mathbb{N} \to \mathbb{N}$ be the Hilbert function of $X$ and
let us consider the first difference of $H_X$:

$$\Delta H(X, i) = H(X, i) - H(X, i - 1).$$

It is known [4] that there exist $a_1 \leq a_2 \leq t$ such that:

$$\Delta H(X, i) = \begin{cases} 
  i + 1 & \text{for } i = 0, \ldots, a_1 - 1 \\
  a_1 & \text{for } i = a_1, \ldots, a_2 - 1 \\
  < a_1 & \text{for } i = a_2 \\
  \text{non increasing} & \text{for } i = a_2 + 1, \ldots, t \\
  0 & \text{for } i > t.
\end{cases}$$

**Definition 1.** We say that $X$ has the Hilbert function of decreasing type if for $a_2 \leq i < j < t$ we have $\Delta H(X, i) > \Delta H(X, j)$.

The following is a result well known in characteristic 0 (see [6] and [10, Corollary 4.3]) and proved in any characteristic in [2, Corollary 4.3].

**Theorem 1.** Let $C \subset \mathbb{P}^3$ be an integral curve and let $X$ be its general plane section. Then $H_X$ is of decreasing type.

The following result will be useful in the proof of the main results of the paper.

**Proposition 1.** Let $X \subset \mathbb{P}^2$ be a 0-dimensional scheme whose Hilbert function is of decreasing type. Let us suppose that $h^0 \mathcal{I}_X(s - 1) = 0$ for some $s > 0$ and that one of the following conditions holds:

1. $h^0 \mathcal{I}_X(s) > 3$;
2. $h^0 \mathcal{I}_X(s) = 2$ and there exists $i \in \mathbb{N}$ such that $\Delta H_X(s + i) \leq s - i - 2$.

Then $\deg X \leq s^2 - s + i + 1$.

**Proof.** The proof is a straightforward computation and follows by the fact that the Hilbert function of $X$ is of decreasing type.

If $h^0 \mathcal{I}_X(s) > 3$, then we see that:

$$\deg X \leq \frac{s(s + 1)}{2} + \frac{(s - 2)(s - 1)}{2} = s^2 - s + 1 < s^2 - s + i + 1.$$ 

Let $h^0 \mathcal{I}_X(s) = 2$ and let us suppose that $i = \min\{k \in \mathbb{N} \mid \Delta H_X(s + k) \leq s - k - 2\}$. Since $H_X$ is of decreasing type, $\Delta H_X(s + k) = s - k - 1$ for $k \leq i - 1$ and $\Delta H_X(s + k) \leq s - k - 2$ for $k \geq i$. Then:

$$\deg X \leq \frac{s(s + 1)}{2} + \sum_{k=0}^{i-1} (s - k - 1) + \sum_{k=i}^{s-3} (s - k - 2) = s^2 - s + i + 1.$$ 

\]

### 3. Frobenius Morphism and Incidence Varieties

In this section we show some results about incidence varieties and Frobenius morphism. First let us recall the definition of absolute and relative Frobenius morphism:

**Definition 2.** The absolute Frobenius morphism of a scheme $X$ of characteristic $p > 0$ is $F_X : X \to X$, where $F_X$ is the identity as a map of topological spaces and on each $U$ open set $F_X^U : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is given by $f \mapsto f^p$ for each $f \in \mathcal{O}_X(U)$. Given $X \to S$ for some scheme $S$ and $X^{p/S} = X \times_S F_S S$, the absolute Frobenius
morphisms on $X$ and $S$ induce a morphism $F_{X/S}: X \rightarrow X^{p/S}$, called the Frobenius morphism of $X$ relative to $S$.

Given $\mathbb{P}^n$ for some $n \in \mathbb{N}$, let us consider the bi-projective space $\mathbb{P}^n \times \mathbb{P}^n$ and let $r \in \mathbb{N}$ be a non negative integer. Let $k[\mathbb{L}]$ and $k[\mathbb{Z}]$ be the coordinate rings of $\mathbb{P}^n$ and $\mathbb{P}^n$, respectively. Let $M_r \subset \mathbb{P}^n \times \mathbb{P}^n$ be the hypersurface of equation:

$$h_r := \sum_{i=0}^n t_i x_i^p = 0.$$

Note that in the case $r = 0$ $M_r$ is the usual incidence variety $M$ of equation $\sum_i t_i x_i = 0$. If $r \geq 1$, $M_r$ is determined by the following fibred product:

$$
\begin{array}{ccc}
M & \xrightarrow{(F_M)^*} & (P_{M_r})^* \\
\downarrow & & \downarrow \\
F_{M_r} & \xrightarrow{\pi} & M_r \\
\downarrow & & \downarrow \\
\mathbb{P}^n & \xrightarrow{p} & \mathbb{P}^n
\end{array}
$$

where $F: \mathbb{P}^n \rightarrow \mathbb{P}^n$ is the absolute Frobenius.

**Remark 1.** $M = \mathbb{P}(\mathcal{F}_p(-1))$, so that by [3] Lemma 1.5 $M_r = \mathbb{P}(F_{*-}(\mathcal{F}_p(-1)))$ and by [7] Ch.II, ex. 7.9 we see that $\text{Pic}(M_r) = \mathbb{Z} \times \mathbb{Z}$ for any $r \geq 0$. Moreover, since we have:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(-1, -p^r) \rightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \rightarrow \mathcal{O}_{M_r} \rightarrow 0$$

and $H^1\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(m, n) = 0$ for any $m, n \in \mathbb{Z}$ by the Künneth formula [17] Ch.VI, Corollary 8.13, then any hypersurface $V \subset M_r$ is the complete intersection given by $g = h_r = 0$ for some bi-homogeneous $g \in k[\mathbb{L}, \mathbb{Z}]$.

Let $\eta \in \mathbb{P}^n$ be the generic point and and consider $g_{M_r}: M_r \rightarrow \mathbb{P}^n$. Then $g_{M_r}^{-1}(\eta)$ is isomorphic to the hypersurface $H_r$ of $\mathbb{P}^n$ of degree $p^r$ such that, over the algebraic closure $\overline{k(\eta)}$ of $k(\eta)$, $(H_r)_{\text{red}}$ is the generic hyperplane $H$ of $\mathbb{P}^n$.

**Proposition 2.** $\Omega_{M_r/\mathbb{P}^n}|_H \cong F^{*}\mathcal{F}_H(-p^r)$.

**Proof.** The sheaf $\mathcal{E} = F_{*-}(\mathcal{F}_p(-1)) = F^{*}\mathcal{F}_p(-p^r)$ is determined by the exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-p^r) \rightarrow \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{E} \rightarrow 0$ and, since $M_r = \mathbb{P}(F^{*}(\mathcal{F}_p(-1)))$, by [7] Ch.III, Ex. 8.4(b) we have also $0 \rightarrow \Omega_{M_r/\mathbb{P}^n} \rightarrow p_{M_{r}}^{*}(F^{*}\mathcal{F}_p(-p^r)) \otimes \mathcal{O}_{M_r} \rightarrow \mathcal{O}_{M_r}(-1, 0) \rightarrow \mathcal{O}_{M_r} \rightarrow 0$. When we restrict to $H$, by the fact that the sequence locally splits it follows that the following sequence is exact:

$$0 \rightarrow \Omega_{M_r/\mathbb{P}^n}|_H \rightarrow p_{M_{r}}^{*}(F^{*}\mathcal{F}_p(-p^r)) \otimes \mathcal{O}_{M_r}(-1, 0)|_H \rightarrow \mathcal{O}_H \rightarrow 0$$

(2) $\Rightarrow 0 \rightarrow \Omega_{M_r/\mathbb{P}^n}|_H \rightarrow F^{*}\mathcal{F}_p(-p^r)|_H \rightarrow \mathcal{O}_H \rightarrow 0.$

Since $\mathcal{F}_p(-1)|_H \cong \mathcal{F}_H(-1) \otimes \mathcal{O}_H$, then $F^{*}\mathcal{F}_p(-p^r)|_H \cong F^{*}\mathcal{F}_H(-p^r) \otimes \mathcal{O}_H$. By the fact that $F^{*}\mathcal{F}_H$ is stable (see [18] Ch.II, Theorem 1.3.2 and [13] Theorem 2.1) and that $\mu(F^{*}\mathcal{F}_H(-p^r)) > 0$, it follows that $\text{Hom}(F^{*}\mathcal{F}_H(-p^r), \mathcal{O}_H) = 0$ and so by (2) also that $\Omega_{M_r/\mathbb{P}^n}|_H \cong F^{*}\mathcal{F}_H(-p^r)$. \qed
Now we prove some results about the projection from a hypersurface in \( M \) to \( \mathbb{P}^n \).

**Theorem 2.** Let \( V \subset \mathbb{P}^n \times \mathbb{P}^n \) be an integral hypersurface in \( M \) such that the projection \( \pi: V \to \mathbb{P}^n \) is dominant and not generically smooth. Then there exist \( r \geq 1 \) and \( V_r \subset M_r \) integral hypersurface such that \( \pi \) can be factored in the following way:

\[
\begin{array}{ccc}
V & \xrightarrow{\pi} & \mathbb{P}^n \\
\downarrow F_r & & \downarrow \pi_r \\
V_r & & 
\end{array}
\]

where the projection \( \pi_r \) is dominant and generically smooth and \( F_r \) is induced by the commutative diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{F_r} & V_r \\
\downarrow j & & \downarrow i \\
M & & M_r 
\end{array}
\]

**Proof.** The proof works as in [1, Theorem 3.3]. Indeed, the proofs ofLemma 3.1 and Proposition 3.2 in [1] works also for \( \mathbb{P}^n \times \mathbb{P}^n \). □

**Proposition 3.** Let \( V_r \subset M_r \) be an integral hypersurface given by:

\[
\begin{align*}
q(t, x) &= 0 \\
\sum_{i=0}^{n} t_i x_i^{p_i} &= 0
\end{align*}
\]

such that the projection \( \pi_r: V_r \to \mathbb{P}^n \) is generically smooth. Then \( \pi_r \) is not smooth exactly on the following closed subset of \( V_r \):

\[
V_r \cap V \left( x_i^{p_i} \frac{\partial q}{\partial t_i} - x_j^{p_j} \frac{\partial q}{\partial t_i}, i, j = 0, \ldots, n \right).
\]

**Proof.** Let \( P_0 = (a, b) \in V_r \) be such that \( V_r \) is not smooth in \( P_0 \). Then there exists \( \lambda \in k \) such that:

\[
\frac{\partial q}{\partial t_i}(P_0) = \lambda b_i^{p_i}
\]

for any \( i = 0, \ldots, n \).

If \( P_0 \) is a regular point, then the projective tangent space \( T_{V_r, P_0} \) at \( P_0 \in V_r \) is given by the equations:

\[
\sum_{i=0}^{n} \frac{\partial q}{\partial x_i}(P_0)x_i + \sum_{i=0}^{3} \frac{\partial q}{\partial t_i}(P_0)t_i = \sum_{i=0}^{n} (a_i x_i + b_i t_i) = 0
\]

if \( r = 0 \) and by the equations:

\[
\sum_{i=0}^{n} \frac{\partial q}{\partial x_i}(P_0)x_i + \sum_{i=0}^{n} \frac{\partial q}{\partial t_i}(P_0)t_i = \sum_{i=0}^{n} b_i^{p_i} t_i = 0
\]
if $r \geq 1$. In both cases the projection on $\mathbb{T}_{\mathbb{P}^3}$ is not surjective if and only if there exists $\lambda \in k$ such that:

$$\frac{\partial q}{\partial t_i}(P_0) = \lambda b_i q^r \quad \forall i = 0, \ldots, n.$$  

This together with (3) proves the statement. □

4. Lifting problem

Let $X \subset \mathbb{P}^4$ be a scheme and, following the previous notation, consider the projections $p_{M_r}: M_r \rightarrow \mathbb{P}^4$ and $g_{M_r}: M_r \rightarrow \mathbb{P}^4$. Let $T_r = p_{M_r}^{-1}(X)$ and:

$$\mathcal{I}_r(m, n) = g_{M_r}^* (\mathfrak{O}_{\mathbb{P}^4}(m)) \otimes g_{M_r}^* (\mathfrak{I}_X(n))$$

for every $m, n \in \mathbb{Z}$.

Proposition 4. If $\mathcal{I}_r = \mathcal{I}_r(0, 0)$ and $\mathcal{I}_{T_r}$ is the ideal sheaf of $T_r$ in $M_r$, then $\mathcal{I}_r = \mathcal{I}_{T_r}$.

Proof. The proof works as in [1, Proposition 4.1]. □

Let $X \subset \mathbb{P}^4$ be an integral surface of degree $d$. Let $Y = X \cap H$ be the generic hyperplane section of $X$ and let $Z = Y \cap K$ be the generic plane section of $Y$. Let $\mathcal{I}_X$ be the ideal sheaf of $X$ in $\mathbb{P}^4$, $\mathcal{I}_Y$ the ideal sheaf of $Y$ in $H \cong \mathbb{P}^3$ and $\mathcal{I}_Z$ the ideal sheaf of $Z$ in $K \cong \mathbb{P}^2$. Let us consider for any $s \in \mathbb{N}$ the following maps:

$$\pi_s: H^0 \mathcal{I}_X(s) \rightarrow H^0 \mathcal{I}_Y(s) \quad \text{and} \quad \phi_s: H^1 \mathcal{I}_X(s-1) \rightarrow H^1 \mathcal{I}_X(s)$$

A sporadic zero of degree $s$ is an element $\alpha \in \ker(\pi_s) = \coker(\phi_s)$.

Definition 3. The order of a sporadic zero $\alpha$ is the maximum integer $m$ such that $\alpha = \beta \cdot H^m$, for some $\beta \in H^1 \mathcal{I}_X(s - m - 1)$.

Proposition 5. Let $\alpha$ be a sporadic zero of degree $s$ and let $h^0 \mathcal{I}_X(s) = 0$. Then one of the following conditions holds:

1. $\deg X \leq s^2 - s + 1$;
2. $h^0 \mathcal{I}_Y(s) = 1$ and $h^0 \mathcal{I}_Z(s) = 2$.

Proof. Let $q = \min \{ i \mid h^0 \mathcal{I}_Y(i) \neq 0 \}$. So $q \leq s$ and by hypothesis there is an integral surface of degree $q$ containing $Y$ that does not lift to an integral surface of degree $q$ containing $X$. In particular we have a sporadic zero of degree $q$ for $X$ and by [21, Theorem 2.1] we get a sporadic zero for $Y$ of degree $s' \leq q$. By [2, Theorem 4.1] this means that there is an integral curve of degree $s'$ in $K$ containing $Z$ that does not lift to a surface in $H$ of degree $s$ containing $Y$. However, by restricting the integral surface of degree $q$ containing $Y$ to $K$ we see that $d \leq qs'$.

If $s' < s$, then we see that $\deg X = \deg Z \leq s^2 - s$.

So we can suppose that $q = s' = s$, which implies that $h^0 \mathcal{I}_Z(s) \geq 1 + h^0 \mathcal{I}_Y(s) \geq 2$. If $h^0 \mathcal{I}_Z(s) \geq 3$, then by Theorem 1 and by Proposition 1 we get $\deg X = \deg Z \leq s^2 - s + 1$. So we can suppose that $h^0 \mathcal{I}_Z(s) = 2$, which implies also that $h^0 \mathcal{I}_Y(s) = 1$.

The following result, together with Proposition 4, provides us with the tools for the proof of the main results of this paper.
Lemma 1. Let $\alpha$ be a sporadic zero of degree $s$ and order $m$. Suppose that $\alpha$ determines a non-liftable integral surface $R$ in $H$ of degree $s$ containing $Y$ and that $I_R = \langle f \rangle$ for some $f \in H^0\mathcal{O}_H(s)$. Then for some $r \in \mathbb{N}$ such that $p^r \leq m + 1$ there exist:

1. $f_i \in H^0\mathcal{O}_H(s)$ for $i = 0, \ldots, 4$ such that the subscheme of $H$ associate to the ideal $(f, x_i p^r f_j - x_j p^r f_i|_H, i, j = 0, \ldots, 4)$ is a 1-dimensional scheme $E$ (which can have isolated or embedded 0-dimensional subschemes) such that $Y \subset E \subset R$;
2. a reflexive sheaf $\mathcal{N}$ of rank 3 such that we have the exact sequence:

\[
0 \to \mathcal{N} \to F^r \Omega_H(p^r) \to \mathcal{E}|_R(s) \to 0,
\]

being $\mathcal{E}|_R \subset \mathcal{O}_R$ the ideal sheaf of $E$.

Proof. Let $r \geq 0$ and let $\mathcal{E}_r = p_{M_r}^* \mathcal{F}_X$. Given the generic point $\eta \in \mathbb{P}^4$ and $g_{M_r} : M_r \to \mathbb{P}^4$, we have seen that $g_{M_r}^{-1}(\eta)$ is isomorphic to the hypersurface $H_r$ of $\mathbb{P}^4$ of degree $p^r$ such that, over $k(\eta)$, $(H_r)_{\text{red}} = H$.

By proceeding as in [1] Theorem 1.2, Step 1 and Step 2 and by Theorem 2 we see that there exist $r \geq 0$ and $V_r \subset M_r$ hypersurface given by:

\[
\begin{aligned}
q(\eta, \xi) &= 0 \\
\sum_{i=0}^4 t_i x_i p^r &= 0
\end{aligned}
\]

such that the projection $p_{V_r} : V_r \to \mathbb{P}^4$ is generically smooth and, given $g_{V_r} : V_r \to \mathbb{P}^4$, $g_{V_r}^{-1}(\eta)$ is the complete intersection of $H_r$ with a hypersurface of $\mathbb{P}^4$ of degree $s$ and it is such that $g_{V_r}^{-1}(\eta)_{\text{red}} \cong R$ over $k(\eta)$. This means that $m \geq p^r - 1$.

Let $U \subset V_r$ be the subscheme where $p_{V_r}$ is not smooth. Then by Proposition 3 we see that:

\[
U = V_r \cap V \begin{pmatrix} x_i p^r \frac{\partial q}{\partial t_i} - x_j p^r \frac{\partial q}{\partial t_j} \end{pmatrix} | i, j = 0, \ldots, 4.
\]

By proceeding as in [1] Theorem 1.2, Step 3 we see that $U \supseteq T_r$, $\dim U = 5$ and we have for some $b > 0$:

\[
0 \to \Omega_{V_r/\mathbb{P}^5} \to \Omega_{M_r/\mathbb{P}^4} \otimes \mathcal{O}_{M_r} \to \mathcal{E}|_{U^(b, s)} \to 0,
\]

with $\mathcal{E}|_{U^{(b, s)}} \subset \mathcal{O}_{V_r}$ ideal sheaf of $U$.

Restricting (4) to $H$, by Proposition 2 we get a surjective map $F^r \Omega_H(p^r) \otimes \mathcal{O}_H \to \mathcal{E}|_R(s)$, with $\mathcal{E}|_R \subset \mathcal{O}_R$ ideal sheaf of the 1-dimensional scheme $E = U \cap g_{M_r}^{-1}(\eta)_{\text{red}}$. Note that $E \supseteq T_r \cap g_{M_r}^{-1}(\eta)_{\text{red}} \cong Y$. The kernel of the map $F^r \Omega_H(p^r) \to \mathcal{E}|_R(s)$ is the sheaf $\mathcal{N}$ that determines the exact sequence (3) and it is torsion free and normal and so it is reflexive. Moreover, by (5) we get

\[
E = V \begin{pmatrix} q|_H, x_i p^r \frac{\partial q}{\partial t_i} - x_j p^r \frac{\partial q}{\partial t_j} |_H \end{pmatrix} | i, j = 0, \ldots, 4,
\]

where $q|_H = f$, and so the statement is proved by taking $f_i = \frac{\partial q}{\partial t_i}|_H$ for any $i = 0, \ldots, 4$. □

Now we can prove the first main result of the paper.
Theorem 3. Let $\alpha$ be a sporadic zero of degree $s$ and order $m$ and let $p < s$. Suppose that $h^0 \mathcal{F}_X(s) = 0$. Then:

1. if $s \geq 2m+3$, we have $d \leq s^2 - s + p^r + 1$, with $p^r \leq m + 1$ and $p^{r+1} > m + 1$;
2. if $s \leq 2m + 2$, we have $d \leq s^2$.

Proof. By Proposition 5 we can suppose that $h^0 \mathcal{F}_Y(s) = 1$ and $h^0 \mathcal{F}_Z(s) = 2$. In particular, if $s \leq 2m + 2$, we get the conclusion. So we suppose that $s \geq 2m + 3$ and we also see that the surface $R$ of degree $s$ containing $Y$ that can not be lifted to a hypersurface of degree $s$ containing $X$ is integral. Let $I_R = (f)$ in $H$ be the ideal of $R$.

By Lemma 1 we see that there exist $r \in \mathbb{N}$ with $p^r \leq m + 1$ and $f_i \in H^0 \mathcal{O}_H(s)$ for $i = 0, \ldots, 4$ such that the subscheme of $H$ associate to the ideal $(f, x_i p^r f_j - x_j p^r f_i |_H, i, j = 0, \ldots, 4)$ is a 1-dimensional scheme $E$, which can have isolated or embedded 0-dimensional schemes, such that $Y \subset E \subset R$. Moreover, there exists a reflexive sheaf $\mathcal{N}$ of rank 3 such that we have the exact sequence:

$$0 \to \mathcal{N} \to F^{r*} \Omega_H(p^r) \to \mathcal{F}_{E|R}(s) \to 0,$$

being $\mathcal{F}_{E|R} \subset \mathcal{O}_R$ the ideal sheaf of $E$. We want to prove that $d \leq s^2 - s + 1 + p^r$.

Note that $c_1(\mathcal{N}) = -p^r - s$ and

$$c_2(\mathcal{N}) = s^2 + p^r s + p^{2r} - \deg E.$$

So we see that if $\mathcal{N}$ is semistable, by the Bogomolov inequality for semistable reflexive sheaves and by the fact that $\deg E \geq \deg Y = \deg X$ we get the statement. So we can suppose that $\mathcal{N}$ is unstable. Moreover by Theorem 1 and by Proposition 1 we can suppose that $\Delta H_2(s+i) = s - i - 1$ for any $i \leq p^r$. Given $g \in H^0 \mathcal{O}_K(s)$ such that $f|_K$ and $g$ are generators of $I_Z$ in degree $s$, by [11 Proposition 1.4] we see that $f|_K$ and $g$ are the only generators of $I_Z$ in degree $\leq s + p^r$. By these assumptions we will get a contradiction.

Restricting (7) to $K$ we get:

$$0 \to \mathcal{N}|_K \to F^{r*} \Omega_K(p^r) \oplus \mathcal{O}_K \to \mathcal{F}_{E\cap K}|_{E\cap K}(s) \to 0.$$

Since $N$ is unstable of rank 3, $F^{r*} \Omega_H(p^r)$ is stable and $c_1(F^{r*} \Omega_H(p^r)) = -p^r < 0$, the maximal destabilizing subsheaf $\mathcal{F}$ of $\mathcal{N}$ has rank at most 2 and $c_1(\mathcal{F}) < 0$. By [12 Theorem 3.1] we see that $F|_K$ is still semistable and so it must be $h^0 \mathcal{N}|_K = 0$. By (9) we see that $h^0 \mathcal{F}_{E\cap K}|_{E\cap K}(s) \geq 1$, which implies that $h^0 \mathcal{F}_{E\cap K}(s) \geq 2$ and, since $E \cap K \supseteq Z$ and $h^0 \mathcal{F}_Z(s) = 2$, we get that $h^0 \mathcal{F}_{E\cap K}(s) = 2$. Since $E \cap K$ is integral of degree $s$ and $R \cap K \supseteq E \cap K$, we see that $\deg (E \cap K) \leq s^2$.

Recall that for any $i, j = 0, \ldots, 4$:

$$x_i p^r f_j - x_j p^r f_i |_H \in H^0 \mathcal{F}_E(s + p^r) \Rightarrow x_i p^r f_j - x_j p^r f_i |_K \in H^0 \mathcal{F}_Z(s + p^r)$$

where $p^r \leq m + 1$. By the assumption that $f|_K$ and $g$ generate $I_Z$ in degree $\leq s + p^r$ we can say that:

$$x_i p^r f_j - x_j p^r f_i |_K = h_{ij} f|_K + l_{ij} g,$$

for some $h_{ij}, l_{ij} \in H^0 \mathcal{O}_K(p^r)$. So:

$$E \cap K = V (f|_K, l_{ij} g | i, j = 0, \ldots, 4).$$

So $E \cap K$ contains the complete intersection of 2 curves of degree $s \cdot \deg(f|_K, g)$, but we have seen that $\deg(E \cap K) \leq s^2$. This implies that $E \cap K$ is the complete
intersection $V(f|_K, g)$ and so $\mathcal{I}_{E \cap K | R \cap K} \cong \mathcal{O}_{R \cap K}(-s)$. So by (9) we have:

$$0 \to \mathcal{N}|_K \to F^{r*} \Omega_K(p^r) \oplus \mathcal{O}_K \to \mathcal{O}_{R \cap K} \to 0. \tag{11}$$

By the fact that $h^0 \mathcal{N}|_K = 0$, that $R \cap K$ is integral and by the commutative diagram:

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
\mathcal{N}|_K \\
\downarrow \\
F^{r*} \Omega_K(p^r) \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
\mathcal{O}_K \\
\downarrow \\
\mathcal{O}_K \\
\downarrow \\
\mathcal{O}_K \\
\downarrow \\
\mathcal{O}_K \\
\downarrow \\
\mathcal{O}_K
\end{array}
\quad
\begin{array}{c}
\mathcal{N}|_K \\
\downarrow \\
F^{r*} \Omega_K(p^r) \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}
\]

we get the exact sequence:

$$0 \to \mathcal{O}_K(-s) \to \mathcal{N}|_K \to F^{r*} \Omega_K(p^r) \to 0. \tag{12}$$

By the exact sequence:

$$0 \to F^{r*} \Omega_K(p^r) \to \mathcal{O}_K^\oplus 3 \to \mathcal{O}_K(p^r) \to 0$$

and by the fact that $p^r \leq m + 1 < \frac{s}{2}$ we see that $\text{Ext}^1(F^{r*} \Omega_K(p^r), \mathcal{O}_K(-s)) = 0$ and so $\mathcal{N}|_K \cong F^{r*} \Omega_K(p^r) \oplus \mathcal{O}_K(-s)$. Since $F^{r*} \Omega_K(p^r)$ is stable and:

$$\mu(F^{r*} \Omega_K(p^r)) = \frac{-p^r}{2} > \mu(\mathcal{O}_K(-s)) = -s,$$

we see that the maximal destabilizing subsheaf of $\mathcal{N}|_K$ is $F^{r*} \Omega_K(p^r)$. So, since $\mathcal{N}$ is unstable of rank 3, by [12, Theorem 3.1] the maximal destabilizing subsheaf of $\mathcal{N}$ must be a reflexive sheaf $\mathcal{F}$ of rank 2 such that:

$$\mathcal{F}|_K \cong F^{r*} \Omega_K(p^r). \tag{13}$$

So, being $\mathcal{F}$ the maximal destabilizing sheaf of $\mathcal{N}$, we have the following commutative diagram:
where \( \mathcal{I}_T \) is the ideal sheaf in \( H \) of a zero-dimensional scheme \( T \) and \( \mathcal{Q} \) is a rank 1 sheaf such that \( c_1(\mathcal{Q}) = 0 \). Since \( \mathcal{Q}|_K \cong \mathcal{O}_K \), \( \mathcal{Q} \) must be torsion free and so \( \mathcal{Q} = \mathcal{I}_W \) for some zero-dimensional scheme \( W \). So we get:

\[
0 \to \mathcal{I}_T(-s) \to \mathcal{I}_W \to \mathcal{I}_{E|R}(s) \to 0,
\]

by which we get that \( W \neq \emptyset \), because \( h^0\mathcal{I}_Y(s) = 1 \). Moreover:

\[
h^1\mathcal{E}(n) = h^1\mathcal{I}_{E|R}(n) = \deg W - \deg T
\]

for any \( n < s \) and:

\[
h^1\mathcal{E}(s) = h^1\mathcal{I}_{E|R}(s) = \deg W - \deg T - 1,
\]

because \( h^0\mathcal{I}_{E|R}(s) = 0 \).

Let \( F \subset E \) be the equidimensional component of dimension 1. Then there exists a sheaf \( \mathcal{K} \) of finite length determining the following exact sequence:

\[
0 \to \mathcal{I}_E \to \mathcal{I}_F \to \mathcal{K} \to 0.
\]

Then we see that \( h^1\mathcal{E}(n) = h^0\mathcal{K} \) for \( n \ll 0 \), so that by (14) we see that \( h^0\mathcal{K} = \deg W - \deg T \). Moreover:

\[
h^0\mathcal{E}(s) - h^0\mathcal{F}(s) + h^0\mathcal{K} - h^1\mathcal{E}(s) + h^1\mathcal{F}(s) = 0
\]

and so, since \( Y \subset F \subset E \), \( h^0\mathcal{E}(s) = h^0\mathcal{F}(s) = 1 \) and by (15) we get:

\[
h^1\mathcal{F}(s) = h^1\mathcal{E}(s) - h^0\mathcal{K} = -1.
\]

This is impossible and so we get a contradiction. \( \square \)

**Corollary 1.** Let \( h^0\mathcal{I}_Y(s) \neq 0 \) and let \( p < s \). If \( \deg X > s^2 \), then \( h^0\mathcal{I}_X(s) \neq 0 \).

In the following theorem we see that for \( p \geq s \) the bound for \( d \) is independent on the order of the sporadic zero \( \alpha \) and coincides with the bound of the characteristic zero case (see [14] and [16]).

**Theorem 4.** Let \( h^0\mathcal{I}_Y(s) \neq 0 \), \( h^0\mathcal{I}_X(s) = 0 \) and let \( p \geq s \). Then \( \deg X \leq s^2 - s + 2 \).
This implies that $\mathcal{I}_{X \cap H_r}(s) \to 0$, where $\mathcal{I}_{X \cap H_r} \subset \mathcal{O}_{H_r}$ is the ideal sheaf of $X \cap H_r$. Since $h^0(\mathcal{I}_{X \cap H_r}(s)) \neq 0$ and $h^0(\mathcal{I}(s)) = 0$, it must be $h^1(\mathcal{I}(s - p^r) \neq 0$. By the fact that $X$ is integral we see that it must be $p^r < s$ and so $r = 0$ and $p^r = 1$.

Now we show that the bounds given in Theorem 3 and Theorem 4 are sharp.

**Example 1.** Let $r$, $p$, $s \in \mathbb{N}$ such that $s \geq 2p^r$. Let us consider $E = \mathcal{O}_{\mathbb{P}^4}(p^r - 2s) \oplus \mathcal{O}_{\mathbb{P}^4}(-p^r - s)$ and $\mathcal{F} = F^r \Omega_{\mathbb{P}^4}(p^r - 1)$. Then, since $E^r \otimes F$ is generated by global sections, by [5] the dependency locus of a general homomorphism $\varphi \in \text{Hom}(E, \mathcal{F})$ is a smooth surface $X \subset \mathbb{P}^4$, and it is determined by the sequence:

\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^4}(p^r - 2s) \oplus \mathcal{O}_{\mathbb{P}^4}(-p^r - s) \to F^r \Omega_{\mathbb{P}^4}(p^r - 1) \to \mathcal{I}_X \to 0.
\end{equation}

Together with:

\begin{equation}
0 \to F^r \Omega_{\mathbb{P}^4}(p^r) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(p^r) \to 0
\end{equation}

this implies that $h^1(\mathcal{I}_X) = 0$, so that $h^0(\mathcal{O}_X) = 1$ and $X$ is connected and, being smooth, $X$ is integral. Moreover, $h^3(\mathcal{I}_X(s)) = 0$ and by a computation with Chern classes we see that $\deg X = s^2 - p^r s + 2p^{2r}$.

Let $H \subset \mathbb{P}^4$ be a general hyperplane and let $H_r \subset \mathbb{P}^4$ be the nonreduced hypersurface of degree $p^r$ such that $H_{r,\text{red}} = H$. Then, $(F^r)^{-1}(H) = H_r$. This shows that we have a commutative diagram:

\[
\begin{array}{ccc}
H_r & \xrightarrow{i} & H \\
\downarrow{\pi} & & \downarrow{\jmath} \\
\mathbb{P}^4 & \to & \mathbb{P}^4
\end{array}
\]

So we have:

\[i^*(F^r \Omega_{\mathbb{P}^4}(p^r)) = i^*(F^r \Omega_{\mathbb{P}^4}(1)) = \pi^*(j^*(\Omega_{\mathbb{P}^4}(1))) \cong \pi^*(\Omega_H(1)) \otimes \mathcal{O}_{H_r}.
\]

This implies that $h^0(\mathcal{O}_{\mathbb{P}^4}(p^r)) \geq 1$. In particular, by [10] we see that $h^0(\mathcal{I}_{X \cap H_r}(s)) = 0$, so that $h^0(\mathcal{I}_X(s)) = 0$. Moreover, by [16] and by [17] we see that $h^1(\mathcal{I}_X(s - p^r - 1) = 0$. This shows that $X$ has a sporadic zero of degree $s$ and order $m = p^r - 1$. So:

1. if $r = 0$ and $s \geq 2$, then $p^r = 1$, $m = 0$ and $\deg X = s^2 - 2s + 2$;
2. if $s = 2p^r + 1$, then $s = 2m + 3$ and $\deg X = s^2 - \frac{2p^r}{2} = s^2 - s + p^r + 1$;
3. if $s = 2p^r$, then $s = 2m + 2$ and $\deg X = s^2$.

This shows that the bounds in Theorem 3 and Theorem 4 are sharp.

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