Langlands parameters for Heisenberg modules.

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Introduction

Below we define a “spectral decomposition” of the category of representations of the Heisenberg Lie algebra; the spectral parameters are moduli of de Rham local systems. This is a very particular case (the group $G$ is a torus) of the following general conjecture on Langlands parameters in the de Rham-Kac-Moody (split) setting. The key device here is chiral Hecke algebra (we assume its definition is known to the reader; otherwise he should either consider it as a black box or simply skip the introduction). When $G$ is a torus the chiral Hecke algebra is the same as a usual lattice Heisenberg vertex algebra.

General idea: Representations of a p-adic group have a canonical “spectral decomposition” over the spectrum of Bernstein’s center. According to the local Langlands conjecture these parameters can be identified roughly with the moduli of Galois representations with values in the Langlands dual group. A possible imitation of this picture in the de Rham setting: replace the p-adic group by a Kac-Moody Lie algebra and the Galois representation by a de Rham local system. The fun is that the usual Bernstein center is trivial here, which corresponds to the fact that the moduli of de Rham local systems have no non-constant global functions. In the de Rham setting however, contrary to the p-adic situation, there is a direct way to relate the Galois side of the picture with “automorphic” one provided by the Hecke chiral algebras.

One proceeds as follows. Below $F \simeq k((t))$ is a local $k$-field, $O \subset F, O \sim k[[t]]$ the local ring (here $k$ is our base field of char 0). Let $G$ be a reductive group over $k$, $c$ an integral level for $G$ which is less than critical. Let $\mathcal{H}_c$ be the chiral Hecke algebra of level $c$. The Langlands dual group $G^\vee$ acts on $\mathcal{H}_c$, so you can twist $\mathcal{H}_c$ by any $G^\vee$-local system $\phi$ (in the de Rham sense) on Spec$F$ getting a chiral algebra $\mathcal{H}_c^\phi$ on Spec$F$. When $\phi$ varies we get a family of chiral algebras $\mathcal{H}_{\mathcal{L}S}^\phi$ parametrized by the moduli stack $\mathcal{L}S = \mathcal{L}S_{G^\vee}$ of $G^\vee$-local systems on Spec$F$. Consider the category of $\mathcal{H}_{\mathcal{L}S}^\phi$-modules. Every such animal $V$ is an $O$-module on $\mathcal{L}S$ equipped with an extra structure. In particular, it carries an action of the “constant” (with respect to $\phi$) vertex algebra of $G^\vee$-invariants in $\mathcal{H}_c^\phi$, which is the enveloping algebra of the Kac-Moody Lie algebra $\mathfrak{g}(F)^c$ of level $c$. Thus $\Gamma(\mathcal{L}S, V)$ is a $\mathfrak{g}(F)^c$-module.

Conjecture. The functor $\Gamma(\mathcal{L}S, \cdot) : \{\mathcal{H}_{\mathcal{L}S}^\phi\text{-modules}\} \to \{\mathfrak{g}(F)^c\text{-modules}\}$ is an equivalence of categories.

Roughly speaking, this means that $\mathcal{H}_{\mathcal{L}S}^\phi$-modules equipped with an action of $\text{Aut}\phi$ are the same as certain special $\mathfrak{g}(F)^c$-modules; $\mathfrak{g}(F)^c$-modules coming from $\mathcal{H}_{\mathcal{L}S}^\phi$’s with different $\phi$’s are different; and every $\mathfrak{g}(F)^c$-module can be presented as an

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1In truth representations of the p-adic group correspond not to individual Kac-Moody modules but rather to categories of those invariant with respect to the $G(F)$-action, see Remark (i) below.

2supported at the puncture of Spec$F$.

3We consider $\mathfrak{g}(F)^c$ as a topological Lie algebra, so $\mathfrak{g}(F)^c$-modules are assumed to have the property that every vector is killed by $t^n \mathfrak{g}(O) \subset \mathfrak{g}(F)^c$ for sufficiently large $n$. The central element $1 \in k \subset \mathfrak{g}(F)^c$ acts as identity.
“integral” of modules coming from \( \mathcal{H}_G^c \)-modules. Of course, a precise formulation of the above conjecture requires an explanation of what are \( \mathcal{O} \)-modules on \( \mathcal{LS} \) (which is an ind-algebraic stack in a broad sense), etc.

**Remarks.** (i) The chiral algebra \( \mathcal{H}_c^c \) on \( \text{Spec } F \) carries a natural \( G(F) \)-action commuting with the \( G^\vee \)-action, so \( G(F) \) acts on \( \mathcal{H}_{c, \mathcal{LS}}^c \) as well. Thus \( G(F) \) acts on the category of \( \mathcal{H}_{c, \mathcal{LS}}^c \)-modules; it acts also on the category of \( \mathfrak{g}(F)^c \)-modules (via the adjoint action of \( G(F) \) on \( \mathfrak{g}(F)^c \)). The functor \( \Gamma(\mathcal{LS}, \cdot) \) commutes with the \( G(F) \)-action.

(ii) Assuming the above conjecture, one defines the (Galois) support of a \( \mathfrak{g}(F)^c \)-module as the support of the corresponding \( \mathcal{H}_{c, \mathcal{LS}}^c \)-module. It would be very interesting to describe explicitly the categories of \( \mathfrak{g}(F)^c \)-modules with support in a given subspace of \( \mathcal{LS} \). For example, one can hope that the category of \( \mathfrak{g}(F)^c \)-modules supported on the substack of regular singular connections with nilpotent residue is equivalent to the product of several copies of the category of \( \mathcal{D} \)-modules on the affine flags space \( \mathcal{F}_G \) of \( G(F) \).\(^4\) The copies are labeled by the orbits of the \( c \)-affine action of the affine Weyl group on lattice of weights. As in the finite-dimensional situation, the functor that corresponds to an orbit assigns to a \( \mathcal{D} \)-module \( M \) the vector space of global sections \( \Gamma_G(M) := \Gamma(\mathcal{F}_G, M \otimes \mathcal{L}) \) where \( \mathcal{L} \) is the (positive) line bundle on \( \mathcal{F}_G \) that corresponds to the (only) negative weight from the orbit; this is a \( \mathfrak{g}(F)^c \)-module in an obvious way. It is known that the functor \( \Gamma_G \) is exact and fully faithful; a construction due to Gaitsgory provides a canonical lifting of \( \Gamma_G \) to a functor with values in \( \mathcal{H}_{c, \mathcal{LS}}^c \)-modules. More generally, the category of \( \mathfrak{g}(F)^c \)-modules supported on the substack of connections with regular singularities with fixed “semi-simple part of the monodromy” should be equivalent to the product of several copies of the category of appropriately twisted \( \mathcal{D} \)-modules on \( \mathcal{F}_G \). It would be very interesting to guess what happens in the case of irregular singularities.

(iii) Denote by \( T \) the Cartan group of \( G \), so \( T^\vee \subset G^\vee \) is the Cartan torus of \( G^\vee \). Let \( V \) be a \( \mathfrak{g}(F)^c \)-module, \( \phi \) a \( G^\vee \)-local system which comes from a \( T^\vee \)-local system \( \phi_{T^\vee} \). One can hope that \( \phi \) is in the support of \( V \) if and only if \( \phi_{T^\vee} \) belongs to the support of one of the Heisenberg modules \( \mathcal{H}^\infty/2^+ \ast (\mathfrak{n}(F), V) \) where \( \mathfrak{n} \) is the Lie algebra of a maximal nilpotent subgroup \( N \subset G \).

In this note we check the toy case of the above conjecture when \( G = T \) is a torus, hence the Hecke chiral algebra is the lattice Heisenberg chiral algebra. The result is essentially straightforward (the key point is that \( \mathcal{LS}_{T^\vee} \) is covered by \( \text{Spec}(\text{Sym}(O)) = \text{Spec}(\mathfrak{g}(O)) \)). The pages below are mostly a review of the known definitions and constructions. §1 considers a class of Heisenberg groups we play with. In §2 we transplant the picture to a curve. §3 deals with lattice vertex algebras. In case of a non-degenerate level we establish a natural bijection between symmetric Heisenberg groups and symmetric lattice vertex algebras (the construction is essentially the same as in [K] 5.4, 5.5 though the presentation may look different) and identify the corresponding categories of representations (the latter subject is essentially contained in [D], the proof below is similar to Dong’s). We also show that the twist by a local system of a lattice chiral algebra does not affect (locally) the chiral

\(^4\) \( \mathcal{F}_G \) is a formally smooth ind-proper ind-scheme.

\(^5\) D. Gaitsgory found recently another, more natural, proof.
algebra structure but changes the Heisenberg Lie algebra embedding. In §4 the structure of the moduli space of local systems is spelled out and the promised fact on Langlands parameters is stated and proved.

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§1 Symmetric Heisenberg extensions

1.1 Suppose $A$, $C$ are abelian groups, $H$ a central $C$-extension of $A$. Denote by \{ , \} the corresponding commutator pairing $A \times A \to C$. For $a \in A$ we denote its preimage in $H$ by $H_a$; this is a $C$-torsor. We write elements of $H_a$ as $\tilde{a}$.

A symmetric structure on $H$ is an automorphism $\sigma$ of $H$ such that $\sigma^2 = \text{id}_H$, $\sigma|_C = \text{id}_C$, and $\sigma \text{mod} C$ is the involution $a \mapsto a^{-1}$ of $A$. We call $(H, \sigma)$ a symmetric (central) $C$-extension of $A$.

Example. Let $s : A \to H$ be a set-theoretic section such that the corresponding cocycle $a_1, a_2 \mapsto \gamma_s(a_1, a_2) := s(a_1)s(a_2)s(a_1a_2)^{-1}$ has property $\gamma_s(a_1^{-1}, a_2^{-1}) = \gamma_s(a_1, a_2)$. Then $s$ defines a symmetric structure such that $\sigma s(a) = s(a^{-1})$. In other words, the extension defined by a two cocycle with the above property is symmetric.

All central $C$-extensions of $A$ form a groupoid. The Baer product defines on it the structure of a Picard groupoid. Same is true for the groupoid of symmetric extensions. The automorphism group of a central extension equals $\text{Hom}(A, C)$, the one of a symmetric extension is $\text{Hom}(A, C^2) \subset \text{Hom}(A, C)$.

1.2 Denote by $H^{(2)}$ the Baer product of two copies of $H$. We have a canonical homomorphism $H \to H^{(2)}$, $h \mapsto h^{(2)}$, which lifts $\text{id}_A$ and such that $c^{(2)} = c^2$ for $c \in C$. A symmetric structure $\sigma$ on $H$ yields a set-theoretic section $s_\sigma : A \to H^{(2)}$. Namely, for $a \in A$ our $s_\sigma(a)$ is the trivialization of the $C$-torsor $H^{(2)}_a$ defined by the identification $H^{(2)}_a := H_a \cdot H_a \sim C$, $h \cdot h' \mapsto h\sigma(h') = \sigma(h)h' \in C \subset H$.

Lemma. The map $\sigma \mapsto s_\sigma$ is a bijection between the set of symmetric structures on $H$ and the set of set-theoretic sections $s : A \to H^{(2)}$ which satisfy an equation $s(a)s(b) = s(ab)\{a, b\}$.

Proof. One has $s_\sigma(a) = [\tilde{a}\sigma(\tilde{a})]^{-1}\tilde{a}^{(2)}$ where $\tilde{a} \in H$ is a lifting of $a$ and $[\tilde{a}\sigma(\tilde{a})] \in C$. Therefore $s_\sigma(a)s_\sigma(b) = [\tilde{a}\sigma(\tilde{a})]^{-1}\tilde{a}^{(2)}[\tilde{b}\sigma(\tilde{b})]^{-1}\tilde{b}^{(2)} = [\tilde{a}\sigma(\tilde{a})bh\sigma(\tilde{b})]^{-1}(\tilde{a}\tilde{b})^{(2)} = \{a, b\}[(\tilde{a}\tilde{b})\sigma(\tilde{a}\tilde{b})]^{-1}(\tilde{a}\tilde{b})^{(2)} = \{a, b\} s_\sigma(ab)$, so $s_\sigma$ satisfies our equation. Conversely, for a given $s$ we define $s_\sigma : H \to H$ as $s_\sigma(\tilde{a}) := [\tilde{a}^{(2)}s(a)^{-1}]\tilde{a}^{-1}$ where $[\tilde{a}^{(2)}s(a)^{-1}] \in C$. We leave it to the reader to check that $\sigma$ is a symmetric structure on $H$.

In other words, for a fixed skew-symmetric pairing $\{ , \} : C \times C \to A$ a symmetric $C$-extension with the commutant pairing equal to $\{ , \}$ is the same as a Baer square root of the $C$-extension of $A$ defined by the 2-cocycle $\{ , \}$.

1.3 We see that the Picard groupoid of commutative symmetric $C$-extensions of $A$ is canonically equivalent to the Picard groupoid $\mathcal{E}xt(A, C^2)$ of commutative

\footnote{We identify $f : A \to C$ with the automorphism $\tilde{a} \mapsto \tilde{a}f(a)$.}

\footnote{Here $C_2 \subset C$ is the subgroup of elements of order 2.}
$C_2$-extensions of $A$. If $A$ is a free commutative group then this is the category of $\text{Hom}(A,C_2)$-torsors.

For a fixed pairing $\{ , \} : A \times A \to C$ the groupoid of symmetric $C$-extensions of $A$ having $\{ , \}$ as the commutator pairing carries the Baer sum action of the Picard groupoid of commutative symmetric extensions, i.e., that of $\text{Ext}(A,C_2)$. If our groupoid is non-empty then it is an $\text{Ext}(A,C_2)$-torsor.

The above considerations make sense if $H$ is a central extension of an abelian group $A$ by any Picard category; in particular, we can play with superextensions of $A$.

1.4 All schemes below are over the base field $k$ of char 0. We will consider symmetric $\mathbb{G}_m$-extensions or superextensions of certain group ind-schemes.

**Remark.** Let $A$ be a group ind-scheme equipped with a parity homomorphism $A \to \mathbb{Z}/2$ and a (super) skew-symmetric pairing $\{ , \} : A \times A \to \mathbb{G}_m$. As follows from 1.2, symmetric superextensions of $A$ having $\{ , \}$ as the commutator pairing are in bijective correspondence with those for the reduced ind-scheme $A_{\text{red}}$.

Let $F \simeq k((t))$ be a local $k$-field, $O \simeq k[[t]]$ its local ring, $m_x \simeq tk[[t]]$ the maximal ideal, $x \in \text{Spec}(O)$ is the closed point.

Let $T = \Gamma \otimes \mathbb{G}_m$ be a torus, $t = \Gamma \otimes k$ its Lie algebra. We fix a level $c$ which is an integral symmetric bilinear form on $\Gamma$. We have the commutative group ind-scheme $T(F) = \Gamma \otimes F^\times$. For $\gamma \in \Gamma$, $f \in F^\times$ we set $f^\gamma := \gamma \otimes f \in T(F)$. The Lie algebra of $T(F)$ equals $t(F) = \Gamma \otimes F$; for $\gamma \in \Gamma$, $\varphi \in F$ we set $\varphi \cdot \gamma := \gamma \otimes \varphi \in t(F)$. The valuation $v : F^\times \to \mathbb{Z}$ yields a projection $T(F) \to \Gamma$; for $\gamma \in \Gamma$ the preimage of $\gamma$ is $T(F)^\gamma \subset T(F)$. Notice that $T(F)^0_{\text{red}}$ is the group scheme $T(O)$.

**Definition.** A Heisenberg extension of level $c$ is a central $\mathbb{G}_m$-superextension $T(F)^\circ$ of $T(K)$ equipped with a splitting $i : T(O) \to T(F)^\circ$ which satisfies the following two properties:

(i) The $T(F)^\circ$-parity of $a \in T(F)_\gamma$ equals $c(\gamma, \gamma) \mod 2$.

(ii) The $T(F)^\circ$-commutator pairing $T(F) \times T(F) \to \mathbb{G}_m$ for $T(F)^\circ$ is

$$f_1^{\gamma_1}, f_2^{\gamma_2} \mapsto \{f_1^{\gamma_1}, f_2^{\gamma_2}\}^c := \{f_1, f_2\}^{c(\gamma_1, \gamma_2)}$$

where $\{f_1, f_2\} \in \mathbb{G}_m$ is the Contou-Carrère symbol.

We say that $T(F)^\circ$ is symmetric it is equipped with a symmetric structure $\sigma$ which fixes the splitting $i$.

1.5 Remarks. (i) The Baer product of (symmetric) Heisenberg extensions of levels $c, c'$ is a (symmetric) Heisenberg extension of level $c + c'$.

(ii) The Lie algebra $t(F)^\circ$ of a Heisenberg extension of level $c$ is a central $k$-extension of the commutative Lie algebra $t(F)$. The corresponding commutator pairing $t(F) \times t(F) \to k$ is $\varphi_1 \cdot \gamma_1, \varphi_2 \cdot \gamma_2 \mapsto c(\gamma_1, \gamma_2) \text{Res}(\varphi_2 d \varphi_1)$. A symmetric structure defines a $k$-linear continuous splitting $i : t(F) \to t(F)^\circ$ (the only splitting

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8See [BBE] §2 for the terminology.
9See [CC] or [BBE] §3.
fixed by $\sigma$) which equals $i$ on $t(O)$. Thus $t(F)^\gamma$ is uniquely defined: as a vector space it equals $t(F) \oplus k$ with bracket defined by the above formula. We call $t(F)^\gamma$ the symmetric Heisenberg Lie algebra of level $c$. The adjoint action of the group $T(F)$ on $t(F)^\gamma$ is $\text{Ad} f \cdot (\varphi \cdot \gamma') = \varphi \cdot \gamma' + c(\gamma, \gamma')\text{Res}(\varphi \log f)$.

(iii) Let $T^\vee = \Gamma^\vee \otimes \mathbb{G}_m$ be the dual torus. The group ind-scheme $T^\vee(F)$ acts on $T(F)^\gamma$ according to formula $g^\gamma(f^\gamma) = \{f, g\}^{(\gamma, \gamma)}f^\gamma$ where $f, g \in F^\times$, $\gamma, \tilde{\gamma} \in \Gamma^\vee$, and $\tilde{f}^\gamma \in T(F)^\gamma$ is a lifting of $f^\gamma$. One can view $c$ as a homomorphism $T \to T^\vee$. Then for any $f^\gamma \in T(F)$ the adjoint action of $f^\gamma$ on $T(F)^\gamma$ coincides with the action of $c(f^\gamma) \in T^\vee(F)$.

(iv) As follows from 1.2 for a symmetric Heisenberg extension $T(F)^\gamma$ the action of the group ind-scheme $\text{Aut}(F)$ on $T(F)$ lifts uniquely to an action on $T(F)^\gamma$ of the two-sheeted covering $\text{Aut}^{1/2}(F)$ which preserves $\sigma$ and $i$. In particular, $T(F)^\gamma$ carries a canonical action of the Lie algebra $\Theta(F) := \text{Der}(F)$ of vector fields on $F$.

Here is an explicit formula:

1.6 Lemma. For $f \in F^\times$, $\gamma \in \Gamma$, $\theta \in \Theta(F)$, and a lifting $\tilde{f}^\gamma \in T(F)^\gamma$ one has

$$\theta(\tilde{f}^\gamma) = [i(\frac{\theta(f)}{f} \cdot \gamma) + \frac{c(\gamma, \gamma)}{2} \text{Res}(\theta(f) \frac{df}{f^2})] \tilde{f}^\gamma = \tilde{f}^\gamma [i(\frac{\theta(f)}{f} \cdot \gamma) - \frac{c(\gamma, \gamma)}{2} \text{Res}(\theta(f) \frac{df}{f^2})].$$

Proof. The action of $\text{Aut}^{1/2}(F)$ is compatible with the projection $T(F)^\gamma \to \mathbb{G}_m$, $\tilde{f}^\gamma \mapsto \tilde{f}^\gamma \sigma(\tilde{f}^\gamma)$, see 1.2. Therefore $(1 + c(\gamma)) \tilde{f}^\gamma = (1 + \epsilon i(\frac{\theta(f)}{f} \cdot \gamma)) \tilde{f}^\gamma (1 + \epsilon i(\frac{\theta(f)}{f} \cdot \gamma))$. Now use (1.4.1). \qed

Denote by $\mathcal{H} s^e$ the groupoid of symmetric Heisenberg extensions of level $c$. By 1.2 and 1.5(i) $\mathcal{H} s^0$ is a Picard groupoid canonically equivalent to that of $\text{Hom}(\Gamma, \mu_2) = \Gamma^\vee \otimes \mu_2$-torsors.\textsuperscript{10} Thus for any $c$ the groupoid $\mathcal{H} s^c$ is a $\Gamma^\vee \otimes \mu_2$-gerbe (the fact that $\mathcal{H} s^c$ is non-empty follows from, say, Remark in 1.7 below).

1.7 Here is another description of $\mathcal{H} s^c$. Let $T(F)^\gamma$ be a symmetric Heisenberg extension. For $\gamma \in \Gamma$ let $\lambda^\gamma$ be a superline defined as follows. Consider $T(F)^\gamma$ as a superline $\lambda$ over $T(F)$ equivariant with respect to the left and right $T(F)^\gamma$-translations. Restrict $\lambda$ to $T(F)^\gamma_{\text{red}}$ and consider the right translation action of $T(O) \xrightarrow{i} T(F)^\gamma$. Our $\lambda^\gamma$ is the superline of $T(O)$-equivariant sections.\textsuperscript{11}

Denote by $\omega_x$ the cotangent $k$-line $m_x/m_x^2$.

Lemma. (i) For every $\gamma_1, \gamma_2 \in \Gamma$ there is a canonical isomorphism $\mu = \mu_{\gamma_1\gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \xrightarrow{\gamma_1} \lambda^{\gamma_1+\gamma_2} \otimes \omega_x^{c(\gamma_1, \gamma_2)}$, $\ell_1 \otimes \ell_2 \mapsto \mu(\ell_1, \ell_2)$, and for every $\gamma \in \Gamma$ a one $\sigma = \sigma_\gamma : \lambda^\gamma \xrightarrow{\gamma} \lambda^{-\gamma}$ such that

(a) $\mu$ is associative: for every $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and $\ell_1 \in \lambda^{\gamma_1}$ one has $\mu(\mu(\ell_1, \ell_2), \ell_3) = \mu(\ell_1, \mu(\ell_2, \ell_3)) \in \lambda^{\gamma_1+\gamma_2+\gamma_3} \otimes \omega_x^{c(\gamma_1, \gamma_2)+c(\gamma_2, \gamma_3)+c(\gamma_1, \gamma_3)}$.

(b) One has $\mu_{\gamma_1\gamma_2} = (-1)^{c(\gamma_1, \gamma_2)}\mu_{\gamma_2\gamma_1}$ where $\delta : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \xrightarrow{\gamma} \lambda^{\gamma_2} \otimes \lambda^{\gamma_1}$ is the commutativity constraint.

(c) $\sigma^2 = \text{id}$, i.e., $\sigma_\gamma = \sigma^{-1}_{-\gamma}$, and $\mu$ commutes with $\sigma$.

\textsuperscript{10}Here $\Gamma^\vee$ is the dual lattice.

\textsuperscript{11}We use notation $\lambda^\gamma$ instead of $\lambda^{-\gamma}$ since $T(O)$ acts on it by the character $c(\gamma)$, see 3.2, 3.3.
(ii) The above construction identifies $\mathcal{H}s^c$ with the groupoid of triples $\{\lambda^\gamma, \mu, \sigma\}$ where $\lambda^\gamma, \gamma \in \Gamma$, is a collection of superlines such that each $\lambda^\gamma$ has parity $c(\gamma, \gamma)$ mod 2, and $\mu, \sigma$ are data of isomorphisms as in (i) above.

Remark (cf. [K] 5.5). Choose a square root $\lambda_x$ of $\omega_x$ and consider it as an odd superline. For a datum as in Lemma set $\lambda^\gamma := \lambda^\gamma \otimes \lambda_x^{c(\gamma, \gamma)}$. Then $\mu$ and $\sigma$ make $\{\lambda^\gamma\}$ a symmetric $G_m$-extension of $\Gamma$ with the commutator pairing $\gamma_1, \gamma_2 \mapsto (-1)^{c(\gamma_1, \gamma_2) + c(\gamma_1, \gamma_1)c(\gamma_2, \gamma_2)}$. Thus a choice of $\lambda_x$ identifies $\mathcal{H}s^c$ with the category of such symmetric $G_m$-extensions of $\Gamma$.

Proof of Lemma. (i) The isomorphisms $\sigma$ come from the symmetric structure on $T(F)^\gamma$. In order to define $\mu$ let us choose a parameter $t$ in $F$ (so $t \in F, v(t) = 1$). Then $\lambda^\gamma \sim \lambda_{t^{-\gamma}}$, $t^\gamma \mapsto t^\gamma_t := t^\gamma(t^{-\gamma})$, and we define $\mu$ by formula $(\mu(t^1, t^2)(dt_0)^{-c(\gamma_1, \gamma_2)})_t := t^{\gamma_1}_t \cdot t^{\gamma_2}_t$ where $\cdot$ is the product map in $T(F)^\gamma$. It remains to check that $\mu$ so defined does not depend on the choice of $t$. Take another parameter $at, a \in O^\times$. Then $t^\gamma_{at} = t^\gamma_{at} \cdot i(a^{-\gamma})$ hence $t^{\gamma_1}_{at} \cdot t^{\gamma_2}_{at} = (t^{\gamma_1}_{t} \cdot t^{\gamma_2}_{t}) \cdot i(a^{-\gamma_1} - \gamma_2).$ Since $d(at)_0 = a_0 dt_0$ this implies the independence.

Properties (a)–(c) are immediate.

(ii) Assume we have a datum $\{\lambda^\gamma, \mu, \sigma\}$. By Remark in 1.4 a symmetric Heisenberg extension of $T(F)$ amounts to a similar extension of $T(F)^{\gamma}_{red}$, so it suffices to define $T(F)^{\gamma}_{red}$. To do this we track back the construction of (i). As a mere $T(O)$-equivariant superline $T(F)^{\gamma}_{red}$ is constant over each $T(F)^{\gamma}_{red}$ with fiber $\lambda^\gamma$. We also have its trivialization $i$ over $T(O)$ and the symmetry $\sigma$. To recover the product let us fix a parameter $t$ in $F$. Then $\mu$ recovers the multiplication law over the subgroup $t^\gamma \subset T(F)^{\gamma}_{red}$ according to formula $t^{\gamma_1}_t \cdot t^{\gamma_2}_t := (\mu(t^1, t^2)(dt_0)^{-c(\gamma_1, \gamma_2)})_{t^{\gamma_1} + \gamma_2}.$ It extends in a unique way to the product over the whole $T(F)^{\gamma}_{red}$ if we demand that the $T(O)$-equivariant structure coincides with the right multiplication via $i$ and the commutator pairing equals (1.4.1). The independence of $t$ follows from a straightforward computation as in (i). 

1.8 Example (will not be used below). Assume that $\Gamma = Z$, so $T(F) = F^\times$, and $c = 1$, i.e., $c(\gamma_1, \gamma_2) = \gamma_1 \gamma_2$. Choose a square root $L$ of the $F$-line $\omega(F)$. The group $F^\times$ acts on the Tate $k$-vector space $L$ by homotheties, so we have the Heisenberg group $F^\times_{(L)}$ defined as the pull-back of the Tate superextension of the group $GL(L)$ of all continuous $k$-automorphisms of $L$ (see e.g. [BBE] 3.7). Now $F^\times_{(L)}$ carries a canonical symmetric structure. Indeed, the pairing $L \times L \rightarrow k, \ell, \ell' \mapsto \text{Res}(\ell \ell')$, makes $L$ a self-dual Tate $k$-vector space, and the symmetric structure $\sigma$ comes from the canonical isomorphism $GL^\gamma(L) \overset{\sim}{\rightarrow} GL^\gamma(L)$ which lifts the automorphism $g \mapsto g^{-1}$ (see [BBE] (2.16.1)). Notice that the functor $L \mapsto F^\times_{(L)}$ is a morphism, hence an equivalence, of the $\mu_2$-gerbes.

1.9 We use notation of 1.5(iii). Set

(1.9.1) $Z^c := \text{Ker}(c : T \rightarrow T^\gamma) = \text{Hom}(\text{Coker}(c : \Gamma \rightarrow \Gamma^\gamma), G_m).$

We consider $Z^c$ as a subgroup of $T \subset T(O)$. The level $c$ is non-degenerate if and only if $Z^c$ is finite.

Denote by $T(F)$ the category of $T(F)^\gamma$-modules, i.e., $k$-vector (super)spaces
equipped with an action of $T(F)^\circ$ (the subgroup $\mathbb{G}_m \subset T(F)^\circ$ is assumed to act by standard homotheties) and by $Z$-module the category of $Z^c$-modules.

Assume that $c$ is non-degenerate.

**Lemma.** Every set-theoretic section $s : \Gamma^\circ/c(\Gamma) \to \Gamma^\circ$ defines an equivalence of categories $e_s : T(F)\mod \xrightarrow{\sim} Z$-mod. The equivalences $e_s$ for different $s$ are (non-canonically) isomorphic.

**Proof.** For any $T(F)$-module $M$ the action of $T \subset T(F)$ defines a $\Gamma^\circ$-grading $M = \oplus M^\gamma$ such that $T(F)\gamma$ acts by operators of degree $c(\gamma)$. The corresponding $\Gamma^\circ/c(\Gamma)$-grading is compatible with the $T(F)^\circ$-action; it is the decomposition of $M$ by $Z^c$-isotypical components (one has $\Gamma^\circ/c(\Gamma) = \text{Hom}(Z^c, \mathbb{G}_m)$).

Set $M^s := \bigoplus_{\chi \in \Gamma^\circ/c(\Gamma)} M^s(\chi)$ and $e_s(M) := (M^s)^T(m)$ where $T(m) := \text{Ker}(T(O) \to T)$ is the unipotent radical of $T(O)$. Then $T = T(O)/T(m)$ acts on a $\chi$-isotypical component of $e_s(M)$ by the characters $s(\chi)$, and $M$ is equal to $\text{Ind}_{T(O)}^{T(F)}e_s(M)$.\footnote{It suffices to show that $M^s = \text{Ind}_{T(O)}^{T(F)}e_s(M)$. One has $T(F)\circ = T \times (T(F)^\circ/T)$, and $T(F)^\circ/T$-modules are the same as the corresponding Lie algebra modules such that $\text{Lie}T(m)$ acts by locally nilpotent operators. So our assertion follows, say, from the usual Kashiwara’s lemma.}

This induction is the functor inverse to $\text{Ind}_{T(O)}^{T(F)}$.\footnote{It suffices to show that $M^s = \text{Ind}_{T(O)}^{T(F)}e_s(M)$. One has $T(F)\circ = T \times (T(F)^\circ/T)$, and $T(F)^\circ/T$-modules are the same as the corresponding Lie algebra modules such that $\text{Lie}T(m)$ acts by locally nilpotent operators. So our assertion follows, say, from the usual Kashiwara’s lemma.}

This induction is the functor inverse to $e_s$.

Different $s'$ differ by a map $\Gamma^\circ/c(\Gamma) \to \Gamma \xrightarrow{\gamma} \Gamma^\circ$. An isomorphism between the $e_s$’s is given by a lifting of this map to $T(F)^\circ$ (here $\Gamma = T(F)/T(F)^0$).

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## 2 Moving points

2.1 Let $X$ be a smooth curve. Then for every $x \in X$ we have a local field $F_x$ of Laurent series at $x$. The group ind-schemes $T(F_x)$ are fibers of a formally smooth commutative group ind-scheme $T(F_X)$ over $X$. For a $k$-algebra $R$ one has $T(F_X)(R)$ is the group of pairs $(x, f^\gamma)$ where $x$ is an $R$-point of $X$ and $f^\gamma$ is a $T$-point with values in the ring of functions on the punctured formal tubular neighborhood of the graph of $x$. We have a group subscheme $T(O_X) \subset T(F_X)$ affine over $X$ and the quotient ind-scheme $\text{Gr}_X := T(F_X)/T(O_X)$ which is ind-finite, $\text{Gr}_X = \sqcup \text{Gr}_X^\gamma$.

Infinitesimal automorphisms of $X$ act on $T(F_X)$ by transport of structure. Thus $T(F_X)$ carries a canonical action of the Lie algebroid of vector fields on $X$; for $\theta \in \Theta(X)$ its action is denoted by $\text{Lie}_\theta$. Our $T(F_X)$ also carries a canonical connection $\nabla$ along $X$. For $\theta \in \Theta(X)$ the vertical vector field $\text{Lie}_\theta - \nabla_\theta$ on $T(F_X)$ acts on every fiber $T(F_x)$ as the restriction of $\theta$ to $F_x$, see 1.5(iv). These three actions preserve $T(O_X)$, so they also act on $\text{Gr}_X$.

It is clear what a symmetric Heisenberg extension of level $c$ in the present setting is. Such an extension yields a superline $\lambda_{\text{Gr}_X} := T(F_X)^\circ/i(T(O_X))$ on $\text{Gr}_X$. Notice that the restriction of $\lambda_{\text{Gr}_X}$ to $\text{Gr}_X^{\text{red}} = X$ is the superline $\lambda_X$ from Lemma 1.7. The latter Lemma remains true obviously modified, so symmetric Heisenberg extensions form a $\Gamma^\circ \otimes \mathbb{H}_2$-gerbe on $X$ which is always trivial (but has no canonical trivialization). As in 1.5(iv) we see that the above three actions of $\Theta(X)$ on $T(F_X)$ lift canonically to any symmetric Heisenberg extension $T(F_X)^\circ$.

2.2 As every ind-schemes of jets and meromorphic jets, $T(O_X)$ and $T(F_X)$ have
canonical ind-factorization structures (see [BD] 3.4). So for every $n \geq 1$ there is a canonical ind-affine group ind-scheme $T(F_{X^n})$ on $X^n$ equipped with a flat connection and its affine group subscheme $T(O_{X^n})$ whose fibers $T(F_{x_1,..,x_n})$ over a $k$-point $(x_1,..,x_n) \in X^n$ are products of fibers $T(F_x)$, $T(O_x)$ for $x \in \{x_1,..,x_n\} \subset X$. In particular, on $X \times X$ we have $T(O_{X \times X}) \subset T(F_{X \times X})$ whose pull-back to the diagonal $\Delta : X \rightarrow X \times X equals T(O_X) \subset T(F_X)$ and to the complement of the diagonal $j : U \rightarrow X \times X equals T(O_X) \times T(O_X) \subset T(F_X) \times T(F_X)$.

Recall the construction. First one has canonical ind-factorization commutative ring scheme and ind-scheme $O_X \subset F_X$ equipped with connections. Their $R$-points can be described as follows. For $(x_1,..,x_n) \in X^n(R)$ denote by $O_{(x_1,..,x_n)}$ the ring of functions of the normal completion of $X \times \text{Spec}R$ at the union of graphs $Y_{x_i}$ of $x_i$'s, and by $F_{(x_1,..,x_n)}$ the localization of $O_{(x_1,..,x_n)}$ with respect to the equations of the divisors $Y_{x_i}$. These $O_{(x_1,..,x_n)} \subset F_{(x_1,..,x_n)}$ are fibers of $F_{X^n}(R) \subset F_{X^n}(R)$ over $(x_1,..,x_n)$. The connections and the factorization identifications are obvious.

Now $T(O_{X^n})(R) := T(O_{X^n}(R))$ and $T(F_{X^n})(R) := T(F_{X^n}(R))$.

The schemes $T(O_{X^n})$ are flat over $X^n$, and ind-schemes $T(F_{X^n})$ ind-flat.

2.3 The Contou-Carrère symbol satisfies the factorization property, i.e., for every $n \geq 1$ we have a pairing $\{ , \}_n : F_{X^n} \times F_{X^n} \rightarrow O_{X^n}$ that satisfy the usual compatibilities. Indeed, the definition of $\{ , \}$ from [BBE] §2 immediately extends to the above situation; the factorization compatibilities are clear.

A factorization structure on a symmetric Heisenberg extension $T(F)^\gamma_X$ is formed by symmetric superextensions $T(F)^{\gamma_X}$ of $T(F)^{\gamma_X}$ connected by the usual factorization identifications (so the restriction of $T(F)^{\gamma_X}$ to the fiber $\pi T(F_x)$ as above is the Baer product of the symmetric Heisenberg extensions $T(F)^{\gamma}$. The corresponding commutator pairing is $\{ f_1^{\gamma_1}, f_2^{\gamma_2} \}_n = \{ f_1, f_2 \}_n^{\gamma_1,\gamma_2}$. As follows from Remark in 1.4, every symmetric Heisenberg extension admits a unique factorization structure. The canonical flat connection and the actions of $\Theta(X)$ on $T(F_{X^n})$ lift canonically to $T(F_{X^n})$ in the way compatible with the factorization structure.

The Lie algebra of $T(F_{X^n})$ is the symmetric Heisenberg extension $t(F_{X^n})$ of $t(F_{X^n}) = \Gamma \otimes F_{X^n}$.

2.4 The above factorization structures yield a factorization structure on $Gr_X$ and the superline $\lambda$ on it. The action of the factorization group ind-scheme $T(F_{X^n})$ on $Gr_{X^n}$ lifts to a (left) action of $T(F_{X^n})$ on $\lambda_{Gr_{X^n}}$.

Every $Gr_{X^n}$ is inductive limit of subschemes finite and flat over $X^n$.

The factorization ind-scheme $Gr_X$ can be also understood as the affine Grassmannian for the group $T$. Namely, $R$-points of $Gr_{X^n}$ are the same as triples $(x_1,..,x_n, L, \gamma)$ where $(x_1,..,x_n) \in X^n(R)$, $L$ is a $T$-torsor over $X \times \text{Spec}R$, $\gamma$ is a trivialization of $L$ over the complement to the graphs of $x_i$.

Remarks. (i) The distinguished horizontal section 1 of $Gr_X$ is a unit for the factorization structure, i.e., for any (local) section $\phi$ of $Gr_X$ the section $\phi \times 1$ of $Gr_{X \times X}$ defined a priori on $U$ extends to a section over $X \times X$ whose restriction

\footnote{Since $T(F)^{\gamma_X}$ is formally smooth and connected it suffice to check this over the complement to the diagonal divisor; now use the factorization property.}
to the diagonal equals \( \phi \). The superline \( \lambda_{Gr,X} \) is canonically trivialized over 1, and this trivialization is compatible with the factorization structure on \( \lambda_{Gr,X} \).

(ii) The section 1 of \( T(O_X) \) is not a unit for the factorization structure on \( T(O_X) \).

**2.5** (the reader can skip it). The affine Grassmannian description makes it clear that for every \( n \geq 2 \) there is a natural morphism \( \pi^n : Gr^n_X \rightarrow Gr^n_X \) which equals the factorization isomorphism over the complement to the diagonal divisor in \( X^n \). Namely, \( \pi^n \) sends \( ((x_1, L_1, \gamma_1), \ldots, (x_n, L_n, \gamma_n)) \) to \( ((x_1, \ldots, x_n), L_1 \cdot \cdot L_n, \gamma_1 \cdot \gamma_n) \) where \( \cdot \) is the Baer product.

Lemma. The factorization isomorphism yields an isomorphism of superlines

\[ \mu_n : \lambda_{Gr,X}^n \otimes \ldots \otimes \lambda_{Gr,X}^n \rightarrow (\pi^n \ast \lambda_{Gr,X^n})(-\sum_{i < j} c(\gamma_i, \gamma_j) \Delta_{ij}) \]

on \( Gr^n_X \times \ldots \times Gr^n_X \) (here \( \Delta_{ij} \) are the diagonal divisors \( x_i = x_j \)). The restriction of \( \mu_2 \) to \( Gr^2_X \otimes X \) is the isomorphism \( \mu_{\gamma_1, \gamma_2} \) from Lemma 1.7.

**Proof.** It suffices to consider the case of \( n = 2 \) and restrict our picture to the reduced schemes. One has \( Gr^2_X \rightarrow X \), so we have superlines \( \lambda_X^2 := \lambda_{Gr,X}^2 \) on \( X \) and \( \lambda_{X \times X} \) on \( X \times X \) defined as the restriction of \( \pi^2 \ast \lambda_{Gr_X} \times \lambda_{Gr_X} \). The factorization yields an isomorphism \( \phi : \lambda_X^1 \otimes \lambda_X^2 \rightarrow \lambda_{X \times X}(\Delta) \); we want to show that \( ? = -c(\gamma_1, \gamma_2) \) and compute the restriction of \( \phi \) to the diagonal.

Let \( t \) be a local coordinate on \( X \), \( x \) the corresponding coordinate on another copy of \( X \), \( x_1, x_2 \) the local coordinates on \( X \times X \). Recall that \( \lambda_{Gr,X}^1 \) is formed by sections of \( T(F_X)^\ast \) over \( T(F_X)^\ast \)_red which are right invariant with respect to the \( i(T(O_X)) \)-action. Therefore \( \lambda_X^1 \) identifies with the pull-back of \( T(F_X)^\ast \) by the section \( x \mapsto (t - x)^\gamma_i \) of \( T(F_X)^\ast \). Similarly, the superlines \( \pi^i \lambda_X^i \) on \( X \times X \) identify with the pull-back of \( T(F_{X \times X}) \) by the sections \( (x_1, x_2) \mapsto (t - x_1)^\gamma_i \) of \( T(F_{X \times X}) \), and \( \lambda_{X \times X} \) is the pull-back by the product of these sections. These identifications together with the product in \( T(F_{X \times X})^\ast \) provide an isomorphism \( \psi = \psi_t : \lambda_X^1 \otimes \lambda_X^2 \rightarrow \lambda_{X \times X} \).

It remains to compute the function \( \phi/\psi \). Let \( \ell^i \) be invertible (local) sections of \( \lambda_X^i \) considered as sections of the pull-back of \( T(F_X)^\ast \) by \( x \mapsto (t - x)^\gamma_i \). Our \( \ell^i \) yields an invertible section \( \hat{\ell}^i \) of \( \pi^i \lambda_X^i \) identified with the pull-back of \( T(F_{X \times X})^\ast \) by \( x \mapsto (t - x)^\gamma_i \). The product \( \hat{\ell}^1 \cdot \hat{\ell}^2 \) is a section of the pull-back of \( T(F_{X \times X})^\ast \) by \( x \mapsto (t - x_1)^\gamma_i (t - x_2)^\gamma_i \). From the point of view of the factorization isomorphism outside of the diagonal our sections are \( \ell^1_{(x_1, x_2)} = \ell^1_{x_1} i(t - x_1)^\gamma_i, l^2_{(x_1, x_2)} = i(t - x_2)^\gamma_i \ell^2_{x_2}, \) and \( (\hat{\ell}^1 \cdot \hat{\ell}^2)(x_1, x_2) = (\ell^1 i(t - x_2)^\gamma_i l^2_{x_1}) \cdot i(t - x_1)^\gamma_i \ell^2_{x_2} \). To compute \( \phi/\psi \) we have to extend \( \ell^1 \otimes \ell^2 \) and \( \hat{\ell}^1 \cdot \hat{\ell}^2 \) to sections of \( T(F_{X \times X})^\ast \) invariant with respect to the right \( i(T(O_X \times X)) \)-translations and take the ratio. Thus \( \phi/\psi = \ell^2_{x_2}/Ad_i(t - x_1)^\gamma_i \ell^2_{x_2} = (x_2 - x_1)^\gamma_i \). This implies Lemma.

\[ \square \]

§3 The vertex counterpart

**3.1** We use the framework of [BD], so for us a vertex algebra \( A \) is the same as a universal chiral (super)algebra. We consider it as a chiral algebra \( A_\Omega \) over \( \text{Spec}(O) \) equipped with an action of \( \text{Aut}^{1/2}(O) \) (so \( A = A_\Omega/\mathfrak{m}_x A_\Omega \)). Sometimes
we prefer to consider the corresponding chiral algebra $A_X$ on a curve $X$. Similarly, “vertex Lie algebras” from [FBZ] or “conformal algebras” from [K] are the same as universal Lie* algebras. For such a fellow $L$ we have the corresponding Lie algebras $h_F(L) \supset h_O(L)$ of $\theta_i$-coivariants on $L_F \supset L_O$. We always have a distinguished central element $1_L \in L$; the enveloping chiral algebra is denoted by $U(L)$.\footnote{Our $U(L)$ is the plain chiral envelope modulo the relation $1_L = 1$.}

We denote by $t_\gamma$ the symmetric Heisenberg Lie* algebra of level $c$. Then $h_F(t_\gamma)$ is the symmetric Heisenberg Lie algebra $t(F)^{-}$ from 1.5(ii).

3.2 Definition. A symmetric lattice vertex algebra of level $c$ is a $\Gamma$-graded vertex algebra $A = \oplus A^\gamma$ equipped with a morphism of Lie* algebras $\alpha : t_\gamma \to A^0$ and an automorphism $\sigma$ such that:

- $\alpha$ is compatible with $\sigma$, sends $1 \in t_\gamma$ to $1 \in A$, and yields $U(t_\gamma) \sim A^0$.
- Each $A^\gamma$ has parity $c(\gamma, \gamma) \mod 2$. As a $t(F)^-\text{-module}$ $A^\gamma$ is isomorphic to the representation Ind$(\gamma)$ induced by the character $c(\gamma) \in \Gamma^\ast = \text{Hom}(T(O), G_m)$ of the Lie subalgebra $t(O) \sim t(F)^-$.
- The chiral product on every pair of components $A^{\gamma_1}, A^{\gamma_2}$ does not vanish.
- One has $\sigma^2 = \text{id}_A$ and $\sigma(A^\gamma) = A^{-\gamma}$.

Example. The algebra $A$ of functions on the scheme of $T^\vee$-valued jets is a symmetric Heisenberg vertex algebra of level 0. Here the $\Gamma$-grading comes from the translation action of $T^\vee$, $\sigma$ comes from the involution $\tau \mapsto \tau^{-1}$ of $T^\vee$, and $\alpha$ from the embedding $t \otimes \omega_\tau^{-1} = \Gamma \otimes \theta_\tau^{-1} \hookrightarrow A$, $\gamma \otimes \partial_x \mapsto e^{-\gamma} \partial_x e^{\gamma}$ where $e^{\gamma}$ denotes the character of $T^\vee$ corresponding to $\gamma \in \Gamma$.

Remark. If $c$ is non-degenerate then representations Ind$(\gamma)$ are irreducible and pairwise non-isomorphic, so the datum of $\Gamma$-grading is redundant.

3.3 Let $T(F)^-$ be a symmetric Heisenberg extension of level $c$. Denote by $A$ the representation of $T(F)^-$ induced from the trivial representation of $T(O) \sim t(F)^-$. We have $T(F)\sim T(F)^-$.

Proposition. $A$ is a symmetric lattice vertex algebra in a natural way.

Proof. A comment on the definition of $A$. As in 2.1 set $Gr := T(F)/T(O)$; this is an ind-finite ind-scheme whose connected components are labeled by $\gamma \in \Gamma$. Denote by $\lambda_{Gr}$ the superline $T(F)^\gamma/i(T(O))$ over $Gr$; it is a $T(F)^-$-equivariant superline in the obvious way. Let $\delta_{Gr}$ be the dualizing $\mathcal{O}$-module on $Gr$. Equivalently, $\delta_{Gr}$ is the cofree $(T(F)^-)\text{-equivariant} \mathcal{O}$-module on $Gr$ such that $i^!_1 \delta_{Gr} = k$. Set $A^\gamma := \Gamma(Gr^{-\gamma}, \lambda_{Gr} \otimes \delta_{Gr})$ and $A := \oplus A^\gamma$.

We see that $A$ is $\Gamma$-graded and is equipped with an action of $\sigma$ and $\text{Aut}^{1/2}(O)$ (they act on $T(F)^-$ preserving $T(O)$). In particular, the Lie algebra of vector fields $\Theta(O)$ acts on $A$. Each $A^\gamma$ contains the superline $\lambda^\gamma := \Gamma(Gr^{-\gamma}, \lambda_{Gr})$ which equals $\lambda^\gamma$ from 1.7.

For a curve $X$ and a symmetric Heisenberg extension $T(F_X)^-$ on $X$ we get a $\mathcal{D}_X$-module $A_X$. A vertex algebra structure on $A$ amounts to a natural chiral algebra structure on $A_X$. Now the factorization structure on $Gr$ and $\lambda_{Gr}$ as defined in 2.4
define a factorization structure on $A_X$ which amounts to a chiral algebra structure
(see [BD] 3.4). Thus $A$ is a vertex algebra with unit $1_A = \lambda^0$.

The group ind-schemes $T(F_{X^n})$ act on $O_{X^n}$-modules $A_{X^n}$ (see 2.4) in a way compatible with the $D$-module and factorization structures.

Notice that $A^0$ equals the chiral envelope of $t_{D}$.\(^{15}\) so 3.2(a) is valid. Properties 3.2(c),(d) are clear. To see 3.2(b) note that $\Delta^*$ acts on $A$ in two ways: first as the Lie algebra of $T(F)^{-}$ and second as $h_F(t_{D}) \subset h_F(A)$ by the adjoint chiral algebra action. The two actions coincide. For the first action it is clear that $t(0) \subset T(F)^{-}$ acts on the superline $\lambda^7 \subset A^7$ by the character $c(\gamma)$, and $A^7$ is the corresponding induced $t(F)^{-}$-module.

\[3.4\] Here is another description of the chiral product on $A_X$. The action of $T(F_X)^{-}$ on $A_X$ yields, via the factorization structure on $T(F_X)^{-}$, an action of $T(F_{X^2})^{-}$ on $j_2^*(A_X \boxtimes A_X)$ and $\Delta_* A_X$ compatible with the connections.\(^{16}\) Now the embedding $1_A : O_X \hookrightarrow A_X$ identifies $A_X$ with the $T(F_X)^{-}$-module induced from the trivial $T(O_X)$-module $O_X$. So $1^\circ_A : j_2j^*(A_X \boxtimes A_X) \hookrightarrow j_2^*(A_X \boxtimes A_X)$ identifies $j_2j^*(A_X \boxtimes A_X)$ with the $T(F_{X^2})^{-}$-module induced from the trivial $T(O_{X^2})$-module $j_2^*O_{X \times X}$. Therefore the canonical morphism $j_2j^*O_{X \times X} \to \Delta_* O_X$ extends canonically to a morphism of $T(F_{X^2})^{-}$-modules

\[\mu_A : j_2j^*(A_X \boxtimes A_X) \to \Delta_* A_X.\]

**Lemma.** The chiral product on $A_X$ equals $\mu_A$.

**Proof.** Recall (see 3.3) that $T(F_{X^2})^{-}$ acts naturally on $A_{X^2}$, hence on $j_2j^*A_{X^2}$ and $j_2j^*A_{X^2}/A_{X^2}$. The factorization isomorphisms $j_2j^*A_{X^2} \xrightarrow{\sim} j_2j^*(A_X \boxtimes A_X)$, $j_2j^*A_{X^2}/A_{X^2} \xrightarrow{\sim} \Delta_* A_X$ are compatible with the $T(F_{X^2})^{-}$-actions. Thus the chiral product on $A_X$ is compatible with the $T(F_{X^2})^{-}$-action. On $1_A$ it equals the canonical morphism $j_2j^*O_{X \times X} \to \Delta_* O_X$. We are done since these two properties characterize $\mu_A$. \(\square\)

\[3.5\] Let us deduce the usual explicit formula for vertex operators. For a point $x \in X$ consider the operator product $\circ : A_{O_x} \otimes A_x \to F_x \hat{\otimes} A_x$, $\phi \otimes a \mapsto \phi \circ a$, with fixed second variable equal to $x$. An invertible section $\ell \in \chi_{O_x}$ defines then the vertex operator $V_\ell^\gamma : A_x \to F_x \hat{\otimes} A_x$, $V_\ell^\gamma(a) := \ell \circ a$, which we want to compute.

Denote by $F_{O_x}$ the localization of $O_x \hat{\otimes} O_x$ with respect to an equation of the diagonal. One has obvious morphisms $F_{O_x} \to F_x \hat{\otimes} O_x := \varprojlim F_x \otimes (O_x/m_x^n)$ and $F_{O_x} \to O_x \hat{\otimes} F_x := \varprojlim (O_x/m_x^n) \otimes F_x$ which we denote by $f \mapsto f_+, f_-$. \(\varprojlim\)

Let $g \in T(F_{O_x})$ be any element such that its order of pole along the diagonal divisor equals $\gamma$; such $g$ is defined up to multiplication by an element of $T(O_x \hat{\otimes} O_x)$. So $g$ is a section of $T(F_X)^{-\gamma}$ over $\text{Spec } O_x$. Thus $\chi_{O_x}$ equals the pull-back of $T(F_X)^{-\gamma}$ by $g$, so our $\ell$ defines a lifting $\tilde{g} := \ell(g) \in T(F_X)^{-\gamma}$ of $g$.

Two points of view on $g$: \(^{15}\)See [BD] 3.7.5-3.7.11, 3.7.20. \(^{16}\)Notice that for a $D_X$-module $M$ a $T(F_X)^{-\gamma}$-action on $\Delta_* M$ compatible with the connections amounts to a $\Delta^* T(F_X)^{-\gamma}$-action on $M$ compatible with the connections.
(i) We have $g_+ \in T(F_x)(O_x) := \lim T(F_x)(O_x/m_x^\infty)$. In other words, $g_+$ is the restriction of $g$ to the formal neighbourhood of $x$ identified with an $O_x$-point of $T(F_x)$ by means of the connection on $T(F_X)$. Similarly, we have $\tilde{g}_+ \in T(F_x)(O_x)$.

(ii) We have $g_- \in T(O_x)(F_x)$ hence $\tilde{g}_- := \tilde{t}(g_-) \in T(O_x)(F_x)$.

Consider $\tilde{g}$ as a section of the restriction of $T(F_X)$ to $\text{Spec}O_x \times \{x\} \subset X \times X$. Then $\tilde{g}$ acts on $A_{F_x} \otimes A_x$ (which is the restriction of $j_*j^*(A_X \otimes A_X)$) as $\tilde{g} \otimes \tilde{g}_-$, and on $(F_x/O_x) \otimes A_x$ (which is the restriction of $\Delta_x A_X$) as $\tilde{g}_+$. Since the chiral product $\mu_A$, so the op product, commutes with the action of $\tilde{g} \in T(F_X)^\gamma$ (see 3.4) we get, applying them to $1_A \otimes \tilde{g}_-^{-1} a \in F_x \cdot 1_A \otimes A_x \subset A_{F_x} \otimes A_x$, the formula

\[
V_\ell^\gamma = \tilde{g}_+ \tilde{g}_-^{-1}.
\]

This is, indeed, the usual formula. To see this choose a coordinate $t$ at $x$; let $z$ be the same coordinate on another copy of $X$. Pick any non-zero $\ell_x \in \lambda_x^\gamma$; let $\ell \in \lambda^\gamma_{O_x}$ be the corresponding translation invariant\(^{17}\) section. Take $g := (t - z)^{-\gamma} \in T(F_{O_x})$; as above, $\ell$ yields its translation invariant lifting $\tilde{g} := \tilde{t}(g)$ which at $z = 0$ takes value $\tilde{t}^{-\gamma} := \ell_x (t^{-\gamma}) \in T(F_x)^\gamma$. Set $h_\gamma^\gamma := i(t^\alpha \cdot \gamma) \in t(F_x)^\gamma$ (see 1.5, 1.6). Then

\[
\tilde{g}_+(z) = \exp\left(\sum_{n>0} h_\gamma^\gamma \frac{z^n}{n}\right) t^{-\gamma}, \quad \tilde{g}_-(z) = \exp\left(-\sum_{n>0} h_\gamma^\gamma \frac{z^n}{n}\right) \left(-z\right)^{-h_\gamma^\gamma}.
\]

\[
V_\ell^\gamma = \exp\left(\sum_{n>0} h_\gamma^\gamma \frac{z^n}{n}\right) t^{-\gamma} \exp\left(-\sum_{n>0} h_\gamma^\gamma \frac{z^n}{n}\right) \left(-z\right)^{h_\gamma^\gamma}.
\]

\[\text{(3.5.2)}\]

\[\text{3.6 Remarks. (a) For a coordinate $t$ as above set $L_\alpha := t^{n+1} \partial_t \in \Theta(F)$.} \]

Now, by 1.6, for $\ell \in \lambda_x^\gamma \subset A^\gamma$ one has

\[
(3.6.1) \quad L_{-1} \ell = -h_\gamma^{\gamma \ell}, \quad L_0 \ell = -\frac{1}{2} c(\gamma, \gamma) \ell, \quad L_{\geq 1} \ell = 0.
\]

These formulas, together with the compatibility of the $\Theta(F)$-action with the $t(F)^\gamma$-action recover the $\Theta(F)$-action on $A$ via the $t(F)^\gamma$-action.

The first of the above equations implies that the operator $V_\ell^\gamma$ from (3.5.2) satisfies the following differential equation:\(^{18}\)

\[
(3.6.2) \quad z \partial_z V_\ell^\gamma = \sum_{n>0} h_\gamma^\gamma n V_\ell^\gamma z^n + \sum_{n \geq 0} V_\ell^\gamma h_\gamma^\gamma n z^{-n}.
\]

(b) Let $x_1, x_2$ be the coordinates on $X \times X$ that correspond to $t$. As follows from (3.5.2), for every $\gamma_1, \gamma_2 \in \Gamma$ and $\ell_i \in \lambda_{\gamma_i}$ the operator product $\ell_1 \circ \ell_2$ belongs to $(x_1 - x_2)^{c(\gamma_1, \gamma_2)} A_{\gamma_1, \gamma_2}$. The top coefficient lies, in fact, in $(x_1 - x_2)^{c(\gamma_1, \gamma_2)} \lambda_{\gamma_1, \gamma_2}$ and it equals $\mu(\ell_1, \ell_2)$ where $\mu$ was defined in 1.7.\(^{19}\)

(c)\(^{20}\) The chiral algebra structure on $A$ can be described in terms of the $t(F)^\gamma$-action as follows. It suffices to recover $A_{\gamma_1, \gamma_2}^{\gamma} := A_{\gamma_1} \cap j_* j^* A_{\gamma_1}^\gamma \otimes A_{\gamma_2}^\gamma \subset j_* j^* A_{\gamma_1}^\gamma \otimes A_{\gamma_2}^\gamma$.

\[\text{\footnotesize 17}\text{ i.e., Lie}_{\gamma}\text{-invariant.}\]

\[\text{\footnotesize 18}Which is clear also from the formula (3.5.2) itself.\]

\[\text{\footnotesize 19}\text{Precisely, the top coefficient is } (x_1 - x_2)^{c(\gamma_1, \gamma_2)} \mu(\ell_1, \ell_2) dt^{-c(\gamma_1, \gamma_2)}.\]

\[\text{\footnotesize 20}\text{Need not for the rest of the text.}\]
together with identifications $\Delta^*A_{X^2}^{\gamma_2} \cong A_X^{\gamma_1+\gamma_2}$. Consider the sheaves of Lie $\mathcal{O}_{X^2}$-algebras $t(O_X) \subset t(F_X) \subset j_*j^*t(F_X) \times t(F_X)$ equipped with a connection $\nabla$ and the symmetric Heisenberg extension $t(F_X)^\gamma$ (see 2.2).

Lemma. $A_{X^2}^{\gamma_2}$ is the $(t(F_X)^\gamma)$-submodule of $j_*j^*A_X^{\gamma_1} \boxtimes A_X^{\gamma_2}$ generated by the superline $\lambda_{X^2}^{\gamma_2} := \lambda^{\gamma_1}_X \boxtimes \lambda^{\gamma_2}_X(c(\gamma_1,\gamma_2)\Delta)$. The isomorphism $\Delta^*A_{X^2}^{\gamma_2} \cong A_{X^2}^{\gamma_1+\gamma_2}$ is the morphism of $(t(F_X)^\gamma)$-modules induced by the morphism $\mu_{\gamma_1,\gamma_2} : \Delta^*\lambda_{X^2}^{\gamma_2} = \lambda^{\gamma_1}_X \boxtimes \lambda^{\gamma_2}_X \otimes \omega_X^{-c(\gamma_1,\gamma_2)} \cong \lambda^{\gamma_1+\gamma_2}_X$ between the generators (see 1.7).

Proof. By 2.5 $A_{X^2}^{\gamma_1,\gamma_2}$ contains the superline $\lambda_{X^2}^{\gamma_1,\gamma_2}$ hence the $(t(F_X)^\gamma)$-submodule $A_{X^2}^{\gamma_1,\gamma_2}$ generated by it. $A_{X^2}^{\gamma_1,\gamma_2}$ equals the $(t(F_X)^\gamma)$-module obtained from the $(t(O_X)^\gamma)$-module $\lambda_{X^2}^{\gamma_1,\gamma_2}$ on which $t(O_X)^\gamma(t_{x_1,x_2} \mapsto \gamma_1\phi(x_1)x_1,x_2 + \gamma_2\phi(x_2)x_1,x_2)$. So $\Delta^*A_{X^2}^{\gamma_1,\gamma_2}$ is the $\Delta^*(t(F_X)^\gamma)$-module induced from the $(t(O_X)^\gamma)$-module $\Delta^*\lambda_{X^2}^{\gamma_1,\gamma_2} = \lambda^{\gamma_1}_X \boxtimes \lambda^{\gamma_2}_X \otimes \omega_X^{-c(\gamma_1,\gamma_2)}$ on which $t(O_X)$ acts by $\gamma_1 + \gamma_2$.

To show that $A_{X^2}^{\gamma_1,\gamma_2} = A_{X^2}^{\gamma_2,\gamma_1}$ it suffices to check that $A_{X^2}^{\gamma_1,\gamma_2}$ is a $\mathcal{D}_{X^2}$-submodule (since both $A_{X^2}^{\gamma_1,\gamma_2}$, $A_{X^2}^{\gamma_2,\gamma_1}$ have the same pull-back to the diagonal) which in turn amounts to the fact that the action of vector fields sends $\lambda_{X^2}^{\gamma_2}$ to $A_{X^2}^{\gamma_1,\gamma_2}$. Let $t$ be a local coordinate on $X$, $x_1$, $x_2$ the corresponding coordinates on $X^2$. For translation invariant sections $\ell_i \in \lambda^{\gamma_i}_X$ one has $\partial_{x_1}((x_1 - x_2)^{-c(\gamma_1,\gamma_2)}\ell_1 \boxtimes \ell_2) = -c(\gamma_1,\gamma_2)(x_1 - x_2)^{-c(\gamma_1,\gamma_2)} - \ell_1 \boxtimes \ell_2 + (x_1 - x_2)^{-c(\gamma_1,\gamma_2)}(h^{\gamma_1}_X \ell_1) \boxtimes \ell_2$. Consider $\phi := (t(x_1)-1)^\gamma \in t(F_X)^\gamma$; one has $\phi (\ell_1 \boxtimes \ell_2) = (h^{\gamma_1}_X \ell_1) \boxtimes \ell_2 - c(\gamma_1,\gamma_2)(x_2 - x_1)^{-1} \ell_1 \boxtimes \ell_2$. Therefore $\partial_{x_1}((x_1 - x_2)^c(\gamma_1,\gamma_2)\ell_1 \boxtimes \ell_2) = \phi (x_1 - x_2)^c(\gamma_1,\gamma_2)\ell_1 \boxtimes \ell_2)$.

The last statement of Lemma follows from 2.5. \qed

3.7 Symmetric lattice vertex algebra of level $c$ form a groupoid $\mathcal{V}s^c$. In 3.3 we have defined a functor $\mathcal{H}s^c \rightarrow \mathcal{V}s^c$, $T(F)^\gamma \mapsto A = A(T(F)^\gamma)$.

Lemma. If $c$ is non-degenerate then our functor $\mathcal{H}s^c \rightarrow \mathcal{V}s^c$ is an equivalence.

Proof. Let $A$ be a symmetric lattice vertex algebra of level $c$. Set $\lambda^\gamma := (A^\gamma)^{T(m)}$; this is a superline on which $T(O)$ acts by the character $c(\gamma)$, and $A^\gamma$ is the corresponding induced $(t(F)^\gamma)$-module.

(i) Let us check that formulas (3.6.1) remain valid. Set $t_a := \tilde{t}(t^a) \subset t(F)^\gamma$; for $a \neq 0$ this is the $a$-eigenspace for the $L_0$-action on $t(F)^\gamma$. Take a non-zero $\ell \in \lambda^\gamma$. It is clear that $L_{\geq 1}\ell = 0$ and $L_0\ell = w\ell$ for some $w \in k$. Since $[L_{-1}, L_0] = L_{-1}$ we see that $L_{-1}\ell$ is an eigenvector of $L_0$ of eigenvalue $w - 1$, hence $L_{-1}\ell = \phi\ell$ for some $\phi \in t_{-1}$. For every $\psi \in t_{1}$ one has $\psi L_{-1}\ell = -(L_{-1}(\psi))\ell = [\psi, \phi]\ell$. Since $c$ is non-degenerate this implies that $L_{-1}\ell = -h^{\gamma}_0\ell$. Now, since $L_0 = \frac{1}{2}[L_{-1}, L_2]$ and $L_2(h^{\gamma}_0) = -h^{\gamma}_0$, one has $L_0\ell = \frac{1}{2}L_2h^{\gamma}_0\ell = -\frac{1}{2}h^{\gamma}_0\ell = -\frac{1}{2}c(\gamma, \gamma)\ell$, and we are done.

(ii) Let us check that 3.6(b) remains valid. Since $\circ$ is compatible with the Heisenberg action we see that that the top non-zero coefficient of $\ell_1 \circ \ell_2$ belongs to $\lambda^{\gamma_1+\gamma_2}$. By (i) we know how $L_0$ acts on $\lambda^\gamma$, therefore the compatibility of $\circ$ with the $L_0$-action implies that this coefficient has degree $-\frac{1}{2}c(\gamma_1, \gamma_1) - \frac{1}{2}c(\gamma_2, \gamma_2) + \frac{1}{2}c(\gamma_1 + \gamma_2, \gamma_1 + \gamma_2) = c(\gamma_1, \gamma_2)$.

\footnote{Here “translation” is the action of Lie$\partial_t$.}
(iii) The above top non-zero coefficients form a system of isomorphisms $\mu : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \xrightarrow{\sim} \lambda^{\gamma_1+\gamma_2} \otimes \omega_x^{c(\gamma_1,\gamma_2)}$. Together with identifications $\sigma : \lambda^{\gamma} \xrightarrow{\sim} \lambda^{-\gamma}$ they form the datum of 1.7, so we get the symmetric Heisenberg extension $T(F)^\gamma$.

(iv) It follows directly from the construction of 3.3 that the functor $A \mapsto T(F)^\gamma$ is left inverse to the functor from 3.3. Let us show that it is right inverse as well. Let $A$ be as above, $T(F)^\gamma$ the Heisenberg extension just defined, and let $A'$ be the lattice vertex algebra defined by $T(F)^\gamma$ as in 3.3. Then there is a canonical isomorphism $A \xrightarrow{\sim} A'$ of $t(F)^\gamma$-modules which is identity map on the generators $\lambda^\gamma$. We have to show that this isomorphism is compatible with the vertex algebra structures (the compatibility with the $\Gamma$-gradings and $\sigma$ is clear).

We already know that $A \xrightarrow{\sim} A'$ is compatible with the $\Theta(O)$-action. Denote by $\sigma'$ the ope product on $A$ that comes from the one of $A'$. To show that $\sigma = \sigma'$ it suffices to prove that $\ell_1 \circ \ell_2 = \ell_1 \circ' \ell_2 \in A((z))$ for $\ell_i \in \lambda^{\gamma_i}$. We know that these formal power series have the same order of pole, the same leading coefficient, and satisfy the same differential equation $z \partial_z p(z) = \sum_{n>0} h_{n1} p(z) z^n + c(\gamma_1, \gamma_2) p(z)$ which arises from the first equality in (3.6.1). Hence they are equal, q.e.d.

3.8 Representations. For $x \in X$ we have the category of $T(F_x)^\gamma$-modules $T(F_x)^\gamma$-mod (see 1.9) and the category $M_x(A)$ of $A_x$-modules supported at $x$.

Lemma. There is a canonical fully faithful embedding $T(F_x)^\gamma$-mod $\rightarrow M_x(A)$.

Proof. The construction, parallel to 3.4, goes as follows. Notation: $j_x : U_x := X \setminus \{x\} \rightarrow X$, $i_x : \{x\} \rightarrow X$.

Let $O_{X,x} \subset F'_{X,x} \subset F_{X,x}$ be the commutative ring (ind)-schemes over $X$ with fibers over $y \in X(R)$ equal, respectively, to $O_{(y,x)}$, $F'_{y,x}$: the localization of $O_{(y,x)}$ with respect to the equation of $x$, and $F_{(y,x)}$ (see 2.2). They carry an obvious canonical connection. There is an obvious projection $F_{X,x} \rightarrow (F_x)_x$ and the factorization identifications $i_x^* F_{X,x} = i_x^* F_{X,x} = F_x$, $j_x^* F_{X,x} = j_x^* O_X \times F_x$, $i_x^* F_{X,x} = i_x^* F_x \times F_x$.

We have the corresponding group ind-schemes $O_{X,x} \subset F'_{X,x} \subset T(F_{X,x})$. The latter one is the restriction of $T(F_{X,x})$ to $X \times \{x\} \subset X \times X$, so we have the corresponding superextension $T(F_{X,x})$. The restriction of $T(F_{X,x})$ to $x \in X$ equals $T(F_x)^\gamma$. The pull-back $T'(F_{X,x})$ of $T(F_x)^\gamma$ by the projection $T'(F_{X,x}) \rightarrow T(F_{x})$ coincides with the restriction of $T(F_{X,x})$ to $T(F_{X,x})$.

Let $M_x$ be a $T(F_x)^\gamma$-module. We will construct a canonical chiral action of $A$ on $i_x^* M_x$ which defines (3.6.1). $T(F_{X,x})$ acts on $j_x^* O_X \otimes M_x$ via $T(F_{X,x})$. The corresponding induced $T(F_{X,x})$-module equals $j_x^* A_X \otimes M$. Notice that $T(F_{X,x})$ acts on the $D_X$-module $i_x^* M_x$ in a way compatible with connections, so the obvious morphism $j_x^* O_X \otimes M_x \rightarrow i_x^* M_x$ extends to a morphism of $T(F_{X,x})$-modules $\mu_M : j_x^* j_x^* A_X \otimes M_x \rightarrow i_x^* M_x$. This is the promised chiral $A_X$-action on $i_x^* M_x$.

It is clear that the chiral action of the Lie Heisenberg $t_D$ on $i_x^* M_x$ coincides with the $t(F_x)^\gamma$-action. Repeating the argument of 3.5 we see that $\lambda X \subset A_X$ acts on $i_x^* T(F_{X,x})^\gamma$-action which we have since $i_x^* T(F_{X,x})^\gamma = T(F_x)^\gamma$.

\footnote{To see this use e.g. Remark in 1.4.}

\footnote{Such action amounts to an $i_x^* T(F_{X,x})$-action which we have since $i_x^* T(F_{X,x}) = T(F_x)$.}
on \(i_{x*}M_x\) according to formula (3.5.1). These operators determine completely the action of \(T(F_x)^-\) on \(M_x\), so our functor is fully faithful. □

3.9 Proposition. If \(c\) is non-degenerate then the functor of 3.8 is an equivalence.

Proof. Let \(M\) be a vector space. Suppose we have an \(A_X\)-module structure on \(i_{x*}M\). We want to show that it comes from a \(T(F_x)^-\)-module structure.

(a) The \(A_X^0\)-action is the same as a chiral \(t_2\)-action = a \(T(F_x)^-\)-module structure on \(M\). Let us show that our \((T(F_x)^-)\)-action is integrable, i.e., it comes from a \((T(F_x)^-)\)-action. Pick some \(\gamma \in \Gamma\); we want to check that all operators \(h_{\gamma n}, n > 0\), act on \(M\) in a locally nilpotent way, and \(h_0^\gamma\) is semi-simple with integral eigenvalues.

(b) We follow the notation from the end of 3.5. For \(\ell\) as in loc.cit. we have the vertex operator \(V^\gamma_\ell(z):= \ell \circ \cdot : M \to M((z))\) acting on \(M\). Then \([h_{\gamma n}^\gamma, V^\gamma_\ell(z)] = c(\gamma, \gamma')z^nV^\gamma_\ell(z)\) and, by the first equation from (3.6.1), our \(V^\gamma_\ell(z)\) satisfies differential equation (3.6.2).

Consider the modified operator \(\tilde{V}^\gamma_\ell(z) := \exp(\sum_{n > 0} h_{\gamma n}^\gamma \frac{z^n}{n})V^\gamma_\ell(z) : M \to M((z)).\)

One has:

- (i) \(\tilde{V}^\gamma_\ell\) commutes with every \(h_{\gamma'}^\gamma, \gamma' \in \Gamma, n > 0\).
- (ii) \([h_{\gamma n}^\gamma, \tilde{V}^\gamma_\ell(z)] = c(\gamma, \gamma')z^n\tilde{V}^\gamma_\ell(z)\) for \(n \leq 0\).
- (iii) \(z\partial_z \tilde{V}^\gamma_\ell(z) = \tilde{V}^\gamma_\ell(z)(\sum_{n \geq 0} h_{\gamma n}^\gamma z^{-n}) = (\sum_{n \geq 0} h_{\gamma n}^\gamma z^{-n} - c(\gamma, \gamma))\tilde{V}^\gamma_\ell(z)\).

(c) Local nilpotency of \(h_{\gamma 0}^\gamma\): Consider \(M\) as a \(k[h_1^\gamma, h_2^\gamma, ..]\)-module. We want to check that it is supported at \(0 \in \text{Spec}k[h_1^\gamma, h_2^\gamma, ..] := \cup \text{Spec}k[h_1^\gamma, h_2^\gamma, ..]\). Take any \(m \in M\). Set \(v(z) = \sum v_i z^i := \tilde{V}^\gamma_\ell(z)m\). Then the support of \(v(z)\) (as the union of supports of \(v_i\)’s) is equal to that of \(m\). Indeed, \(\text{Supp} v(z) \subset \text{Supp} m\) by (i) above.

To see the opposite inclusion notice that for appropriate \(\ell' \in \lambda^{-\gamma}\) one has \((z - z')^c(\gamma, \gamma')V^\gamma_\ell(z')V^\gamma_\ell(z)(m) = m + (z - z')\phi(z, z')\) where \(\phi \in M[[z', z]][z'^{-1}, z^{-1}]\). Since \(V^\gamma_\ell(z')V^\gamma_\ell(z)m = \exp(\sum_{n \geq 0} h_{\gamma n}^\gamma \frac{z^n}{n})V^\gamma_\ell\exp(\sum_{n \geq 0} h_{\gamma n}^\gamma \frac{z^n}{n})v(z) \in M((z))((z))\) we see that \(m = \sum \alpha_i v_i\) where \(\alpha_i\) are differential operators acting on \(M\) almost all equal to 0. Thus \(\text{Supp} m \subset \text{Supp} v(z)\).

Assume that \(m\) is killed by \(h_{\gamma N}^\gamma\); let \(\eta \in \text{Spec}k[h_1^\gamma, .., h_N^\gamma]\) be a generic point of the support of \(m\). We want to show that \(\eta \neq 0\). Changing \(m\) if necessary we can assume that the maximal ideal of \(\eta\) kills \(m\). Let us localize \(M\) (as a \(k[h_1^\gamma, .., h_N^\gamma]\)-module) at \(\eta\). By above, \(v_\eta(z)\) is a non-zero element of \(M_\eta((z))\) killed by the maximal ideal of the local ring. Differential equation (iii) shows that \(z\partial_z v_\eta(z) = \sum h_{\gamma n}^\gamma z^{-n} - c(\gamma, \gamma))v_\eta(z)\) where \(h_{\gamma n}^\eta \in k_\eta\) are images of \(h_{\gamma n}^\gamma\) in the residue field.

An immediate induction by \(n\) shows that \(h_{\gamma n}^\eta = .. = h_{\gamma 1}^\eta = 0\).

(d) \(h_0^\gamma\) is semi-simple with integral eigenvalues: By (c) the operator \(\bar{V}^\gamma_\ell(z) := \tilde{V}^\gamma_\ell(z)\exp(\sum_{n \geq 0} h_{\gamma n}^\gamma \frac{z^n}{n})\) is well-defined. Since \(z\partial_z \bar{V}^\gamma_\ell(z) = (h_0^\gamma - c(\gamma, \gamma))\bar{V}^\gamma_\ell(z)\) we see

\[\text{3.5 itself we consider the case } M = A_x.\]

\[\text{with respect to the } k[h_1^\gamma, h_2^\gamma, ..]\text{-module structure on } M.\]
that each component $\tilde{V}_t^\gamma$ sends $M$ to the $c(\gamma, \gamma) - n$-eigenspace of $h_0^\gamma$. It is easy to see\footnote{Here $\tilde{V}_t^\gamma(z) = \sum \bar{V}_n^\gamma z^{-n}$.} that the image of $\tilde{V}_t^\gamma(z)$ generates $M$, and we are done.

(e) Now we can define the action of $T(F)^*$ on $M$. We already know how the connected component $T(F)^0$ acts. We define the action of elements $t^{-\gamma}$ as $\tilde{V}_t^\gamma(-z)^{-h_0^\gamma}$ (cf. (3.5.2)); these operators do not depend on $z$. One check immediately that this provides a $T(F)^*$-action on $M$; the corresponding vertex module structure coincides with the initial one by (3.5.2).

\[ \square \]

### 3.10 Twists and rigidity.
Recall that the category of chiral algebras on $X$ is a tensor category. So we know what is a coalgebra in our tensor category, its (co)action on some chiral algebra $A_X$, and the chiral subalgebra of $A_X$ of invariants of this action. We need the case when our coalgebra is actually (the algebra of functions on) an affine group $D_X$-scheme $\mathcal{G}_X$. Here we say that $\mathcal{G}_X$ acts on $A_X$ and denote the invariants by $A_X^{\mathcal{G}_X}$. Suppose now that we have a $D_X$-scheme $\mathcal{G}_X$-torsor $\mathcal{F} = \text{Spec} R_X$ (i.e., a $\mathcal{G}_X$-torsor with connection). The $\mathcal{F}$-twist of $A_X$ is, by definition, $A_X^\mathcal{F} := (R_X \otimes A_X)^{\mathcal{G}_X}$.

Our situation: Let $A$ be a lattice Heisenberg vertex algebra of a non-degenerate level $c$, $A_X$ the corresponding chiral algebra on $X$.

(a) Consider the $\Gamma$-grading of $A$ as a $T^\vee$-action. It yields an action of the constant group $D_X$-scheme $T_X^\Gamma$ on $A_X$. So for every de Rham $T^\vee$-local system $(\mathfrak{F}, \nabla)$ (:= a $D_X$-scheme $T_X^\Gamma$-torsor) we have the twisted chiral algebra $A_X^{(\mathfrak{F}, \nabla)}$ on $X$. The $\Gamma$-grading on $A$ yields the one on $A_X^{(\mathfrak{F}, \nabla)}$.

(b) On the other hand, $T(O)$ acts on $A$ via $i : T(O) \hookrightarrow T(F)^*$. The subgroup $Z^c \subset T(O)$ (see 1.9) acts trivially, so, since $c : T(O)/Z \twoheadrightarrow T^\vee(O)$,\footnote{Look at the composition $V_t^\gamma V_t^{-\gamma}$ and note that $\tilde{V}_t^\gamma$ commutes with $h_0^\gamma$, cf. (c).} we have a $T^\vee(O)$-action on $A$. This provides an action of the group $D_X$-scheme $\mathcal{J}_X T^\vee$ of $T^\vee$-valued jets on the chiral algebra $A_X$. So every $D_X$-scheme $\mathcal{J}_X T^\vee$-torsor $\mathfrak{F}_J$ yields the twisted chiral algebra $A_X^{\mathfrak{F}_J}$ on $X$. The $T^\vee(O)$-action on $A$ preserves the $\Gamma$-grading, so $A_X^{\mathfrak{F}_J}$ carries a $\Gamma$-grading.

(c) The canonical morphism $\mathcal{J}_X T^\vee \rightarrow T_X^\Gamma$ of group $X$-schemes identifies the sheaf of horizontal sections of $\mathcal{J}_X T^\vee$ with $T^\vee(O_X)$. So, by (b), $T^\vee(O_X)$ acts on $A_X$ as a plain sheaf of groups. Thus any $T^\vee$-torsor $\mathfrak{F}$ on $X$ yields a $\Gamma$-graded chiral algebra $A_X^\mathfrak{F} :=$ the twist of $A_X$ by the $T^\vee(O_X)$-torsor of sections of $\mathfrak{F}$.

**Remarks.** (i) Assume we are in situation (a). Since $T^\vee$ acts on $A^0$ trivially we have a canonical identification of chiral algebras

\[ A_X^0 \simeq A_X^{(\mathfrak{F}, \nabla)^0}. \]

(ii) The constructions of (b) and (c) are essentially equivalent. Indeed, there is a canonical equivalence of groupoids

\[ \{ T^\vee\text{-torsors on } X \} \rightarrow \{ D_X \text{-scheme } \mathcal{J}_X T^\vee\text{-torsors on } X \} \]

\[ (3.10.1) \]

\[ (3.10.2) \]
which assigns to a $T^\nu$-torsor $\mathcal{F}$ on $X$ the $\mathcal{D}_X$-scheme $\mathcal{J}_X T^\nu$-torsor of jets $\mathcal{J}_\mathcal{F}$; the inverse functor is the push-out for the canonical homomorphism of the group $X$-schemes $\mathcal{J}_X T^\nu \to T^\nu_X$. It identifies the $T^\nu(\mathcal{O}_X)$-torsor of sections of $\mathcal{F}$ with that of horizontal sections of $\mathcal{J}_\mathcal{F}$, so one has a canonical identification

$$A^\mathcal{F}_X \xrightarrow{\sim} A^{\mathcal{J}_\mathcal{F}}_X.$$  

**3.11 Lemma.** (i) For every de Rham $T^\nu$-local system $(\mathcal{F}, \nabla)$ there is a canonical isomorphism of chiral algebras

$$\phi : A^\mathcal{F}_X \xrightarrow{\sim} A^\mathcal{F}_X.$$  

(ii) A section $s \in \mathcal{F}$ yields $\nu := \nabla \log s \in t^\nu \otimes \omega(X)$ and an identification of chiral algebras $\beta_s : A_X \xrightarrow{\sim} A^\mathcal{F}_X$. Now (3.10.1) equals the composition of $\phi^{-1}\beta_s$ with the automorphism of $A^0_X = U(t_\mathcal{F})$ coming from an automorphism $\tilde{a} \mapsto \tilde{a} + (\nu, a)$ of $t_\mathcal{F}$.

*Proof.* Key remark: the action of $T^\nu$ from 3.10 (a) coincides, via $T^\nu \subset T^\nu(\mathcal{O})$, with the action from 3.10 (b). Therefore $A^\mathcal{F}_X$ coincides with the twist of $A_X$ by the $\mathcal{D}_X$-scheme $\mathcal{J}_X T^\nu$-torsor induced from $(\mathcal{F}, \nabla)$ by the embedding of group $\mathcal{D}_X$-schemes $T^\nu_X \hookrightarrow \mathcal{J}_X T^\nu$. By (3.10.2) the latter torsor equals $\mathcal{J}_\mathcal{F}$, so (i) comes from (3.10.3).

In more details, consider the canonical morphism of group $X$-schemes $\mathcal{J}_X T^\nu \to T^\nu_X$ and the one of torsors $\mathcal{J}_\mathcal{F} \to \mathcal{F}$. The “constant” connection on $T^\nu_X$ yields a section $\epsilon^\nabla : \mathcal{F} \hookrightarrow \mathcal{J}_\mathcal{F}$ which is an embedding of group $\mathcal{D}_X$-schemes, and $\nabla$ yields a section $\epsilon^\nabla : \mathcal{F} \hookrightarrow \mathcal{J}_\mathcal{F}$ which is an embedding of group $\mathcal{D}_X$-schemes compatible with the actions of $T^\nu_X \hookrightarrow \mathcal{J}_X T^\nu$. This $\epsilon^\nabla$ identifies $\mathcal{J}_\mathcal{F}$ with the $\mathcal{D}_X$-scheme $\mathcal{J}_X T^\nu$-torsor induced from $(\mathcal{F}, \nabla)$ via the embedding of the group $\mathcal{D}_X$-schemes $T^\nu_X \hookrightarrow \mathcal{J}_X T^\nu$. Our $\phi$ is the composition $A^\mathcal{F}_X \xrightarrow{\epsilon^\nabla} A^{\mathcal{J}_\mathcal{F}}_X$ (3.10.3) $\iff$ $A^\mathcal{F}_X$.

Let us write down $\phi$ in terms of $s$ as in (ii). Let $\mathcal{J} s$ be the horizontal section of $\mathcal{J}_\mathcal{F}$ defined by $s$. The projection $\mathcal{J}_\mathcal{F} \twoheadrightarrow \mathcal{F}$ maps $\mathcal{J} s$ to $s$, hence

$$e^\nabla(s) = \kappa_\nu \mathcal{J} s$$

where $\kappa_\nu$ is a section of $\mathcal{J}_X T^\nu$ killed by $\mathcal{J}_X T^\nu \to T^\nu_X$ uniquely determined by the condition $d \log \kappa_\nu = \nu \in t^\nu \otimes \omega_X \subset \mathcal{J}_X t^\nu \otimes \omega_X$.

As a mere $\mathcal{O}_X$-module $A^\mathcal{F}_X$ is the $\mathcal{F}$-twist of $A_X$ with respect to the action of $T^\nu$ given by the $G$-grading (see 3.10 (a)), so $s$ yields an identification $\alpha_s : A_X \xrightarrow{\sim} A^\mathcal{F}_X$ of $\mathcal{O}_X$-modules. Similarly, $\mathcal{J} s$ yields an identification of chiral algebras $\alpha_{\mathcal{J} s} : A_X \xrightarrow{\sim} A^{\mathcal{J}_\mathcal{F}}_X$ which equals the composition of $\beta_s$ (see 3.11(ii)) and $A^0_X$. By (3.11.2) one has

$$\phi \alpha_s = \beta_s \kappa_\nu : A_X \xrightarrow{\sim} A^\mathcal{F}_X.$$  

On $A^0_X = U(t_\mathcal{F})$ our $\alpha_s$ equals (3.10.1) and $\kappa_\nu$ acts according to the automorphism $\tilde{a} \mapsto \tilde{a} + (\nu, a)$ of $t_\mathcal{F}$ (see 1.5(ii)). This implies (ii).
§4 Moving $T'$-local system

4.1 Let $\mathcal{LS} = \mathcal{LS}_{T'}$ be the moduli stack of de Rham $T'$-local systems on $\text{Spec} F$. By definition, an $R$-point of $\mathcal{LS}$ is a pair $(\mathfrak{F}_R, \nabla)$ where $\mathfrak{F}_R$ is a $T'$-torsor on $\text{Spec} F_R$, $F_R := F \otimes R = \lim F/m^n \otimes R \simeq R[[t]]$, and $\nabla$ is an $R$-relative continuous connection on $\mathfrak{F}_R$. We always assume that $\mathfrak{F}_R$ comes from a $T'$-torsor on $\text{Spec} F_R$, $O_R := O \otimes R \simeq R[[t]]$, étale locally on $\text{Spec} R$.

According to Drinfeld, the latter unpleasant assumption becomes redundant if we consider instead of étale topology a finer one (a version of cdh topology), but this seems to be irrelevant for what follows.

4.2 Here are some convenient descriptions of $\mathcal{LS}$:

Set $\omega(F) := \{1\text{-forms on } \text{Spec} F\}$; this is an ind-scheme. The ind-scheme $\mathcal{C}$ of connections on the trivialized $T'$-bundle identifies canonically with $t' \otimes \omega(F),\nu \mapsto \nabla_\nu := \partial_t + \nu$. Notice that

\[(4.2.1) \quad \mathcal{C} = \text{Spec}(\text{Sym} t(F)) := \lim \text{Spec}(\text{Sym} t(F)/t(m^n))\]

where $a \in t(F)$ is identified with a linear function $\phi(a) : \nu \mapsto \text{Res}(a, \nu)$ on $\mathcal{C}$.

The group ind-scheme $T'(F)$ of automorphisms of a $T'$-bundle acts on $\mathcal{C}$; the corresponding gauge action on $t' \otimes \omega(F)$ is $g(\nu) = \nu + d\log(g)$. Thus

\[(4.2.2) \quad \mathcal{LS} = \mathcal{C}/T'(F) = t' \otimes \omega(F)/T'(F).\]

Set $\Phi := T'(F)/T'(m)$. Thus $T' = T'(O)/T'(m)$ is a subgroup of $\Phi$ and $\Phi/T' = Q \times \Gamma'$ where $Q$ is the formal group whose Lie algebra equals $t' \otimes (F/O)$. Set $\omega(F)^- := \omega(F)/\omega(O)$, $\bar{\mathcal{C}} := t' \otimes \omega(F)^-$; one has $\bar{\mathcal{C}} = \text{Spec}(\text{Sym}(t(O))) := \lim \text{Spec}(\text{Sym}(t(O)/m^n))$ (see (4.2.1)). Since $d\log$ yields an isomorphism $T'(m) \rightarrow t' \otimes \omega(O)$ one has $\bar{\mathcal{C}} = \mathcal{C}/T'(m)$ and

\[(4.2.3) \quad \mathcal{LS} = \bar{\mathcal{C}}/\Phi.\]

There is a canonical decomposition $\omega(F)^- = \omega(F)^{\text{irr}} \times k$ where the projection $\omega(F)^- \rightarrow k$ is the residue map, and $k \hookrightarrow \omega(F)^-$ is the subspace of forms with pole of order one. Hence $\bar{\mathcal{C}} = \bar{\mathcal{C}}^{\text{irr}} \times t'$. The morphism $d\log : \Phi \rightarrow t' \otimes \omega(F)^-$ kills $T'$, so it yields a morphism $Q \times \Gamma \rightarrow (t' \otimes \omega(F)^{\text{irr}}) \times t'$. This is an embedding compatible with the product decomposition; it identifies $Q$ with the formal completion of $t' \otimes \omega(F)^{\text{irr}}$, and $\Gamma' \hookrightarrow t'$ is the usual embedding. Therefore, by (4.2.3), we have a canonical projection

\[(4.2.4) \quad \mathcal{LS} \rightarrow (\bar{\mathcal{C}}^{\text{irr}}/Q) \times (t'/\Gamma')\]

which makes $\mathcal{LS}$ a $T'$-gerbe over $(\bar{\mathcal{C}}^{\text{irr}}/Q) \times (t'/\Gamma')$. Any splitting of the extension $0 \rightarrow T' \rightarrow \Phi \rightarrow Q \times \Gamma \rightarrow 0$ yields a trivialization of this gerbe, i.e., an identification

\[(4.2.5) \quad \mathcal{LS} \cong (\bar{\mathcal{C}}^{\text{irr}}/Q) \times (t'/\Gamma') \times BT'.\]
4.3 The definition of 3.10(a) works for families of de Rham $T^\vee$-local systems, so we have an $\mathcal{LS}$-family $A_{F}^{\mathcal{LS}}$ of twisted chiral algebras over $\text{Spec}F$ equipped with a $\Gamma$-grading.

Namely, recall that a Spec$R$-family of chiral algebras on Spec$F$ is a flat $F_R$-module $A_{F_R}$ equipped with an $R$-relative connection and a chiral product operation which amounts to an $R$-bilinear operation $A_{F_R} \otimes A_{F_R} \rightarrow A_{F_R,1}((t_1 - t_2))$ which satisfies the usual properties.

For $(\mathcal{F}_R, \nabla) \in \mathcal{LS}(R)$ one defines the $(\mathcal{F}_R, \nabla)$-twisted chiral lattice algebra $A_{F_R}^{(\mathcal{F}_R, \nabla)}$ as in 3.10(a). This construction is compatible with the $R$-base change. The datum of $A_{F_R}^{(\mathcal{F}_R, \nabla)}$ together with the base change isomorphisms forms $A_{F}^{\mathcal{LS}}$.

The pull-back of $A_{F}^{\mathcal{LS}}$ to $\mathcal{C} = t^\vee \otimes \omega(F)$ is a $\mathcal{C}$-family of chiral algebras $A_{C}^{\mathcal{LS}}$ on Spec$F$ equivariant with respect to the action of the group ind-scheme $T^\vee(F)$.

4.4 Denote by $\mathcal{M}(A_{F}^{\mathcal{LS}})$ the abelian $k$-category of $A_{F}^{\mathcal{LS}}$-modules, i.e., of $\mathcal{O}^!$-modules on $\mathcal{LS}$ equipped with an action of $A_{F}^{\mathcal{LS}}$. Explicitly, an $A_{F}^{\mathcal{LS}}$-module $M$ amounts to a discrete module $M_C$ over the topological algebra $\mathcal{O}(C) = \mathcal{O}(t^\vee \otimes \omega(F)) = \lim Sym_k(t \otimes F/m^n)$ equipped with compatible actions of the group ind-scheme $T^\vee(F)$ and the chiral algebra $A_{C}^{\mathcal{LS}}$.

The vector space of sections $\Gamma(M) := \Gamma(\mathcal{LS},M)$ is defined as a BRST-type combination of invariants and coinvariants:

\[(4.4.1) \quad \Gamma(M) := (M_C^{T^\vee(O_F)})_{T^\vee(F)/T^\vee(O_F)}.\]

Since $A_{F}^{\mathcal{LS}0} \subset A_{F}^{\mathcal{LS}}$ is a $C$-constant family of chiral algebras equal $U(t_D)_C$ (see (3.10.1)) the Heisenberg Lie algebra $t(F)^\sim$ acts on $M_C$ and $\Gamma(M)$.

4.5 Theorem. The functor $\Gamma: \mathcal{M}(A_{F}^{\mathcal{LS}}) \rightarrow t(F)^\sim$-mod is an equivalence of categories.

Proof. (a) Recall that $T^\vee(F)$ acts on $T(F)^\sim$ (see 1.5(iii)); denote by $(T^\vee \times T)(F)^\sim$ the corresponding semi-direct product. This is a symmetric Heisenberg extension for the torus $T^\vee \times T$ whose level is of determinant 1. According to 1.9 the functor $P \mapsto P^{T^\vee(O) \times T(O)}$ from the category of $(T^\vee \times T)(F)^\sim$-modules to (super) vector spaces is an equivalence of categories.\(^{29}\)

(b) The group $(T^\vee \times T)(F)^\sim$ acts on $\mathcal{C}$ via the projection to $T^\vee(F)$, so we have the category $\mathcal{P}$ of $(T^\vee \times T)(F)^\sim$-equivariant discrete $\mathcal{O}(C)$-modules.\(^{30}\)

Lemma. (i) For every $M \in \mathcal{M}(A_{F}^{\mathcal{LS}})$ there is a natural $T(F)^\sim$-action on $M_C$ which, together with the structure $\mathcal{O}(C)$- and $T^\vee(F)$-actions, makes $M_C$ an object of $\mathcal{P}$. The functor $\mathcal{M}(A_{F}^{\mathcal{LS}}) \rightarrow \mathcal{P}$ is an equivalence of categories.

(ii) The $t(F)^\sim$-action on $M_C$ from 4.4 is $\tilde{a}m = a(\tilde{a}) \star m$ where $\star$ is the $t(F)^\sim$-action coming from the $T(F)^\sim$-module structure from (i) and $a$ is a $C$-family of automorphisms of $(F)^\sim$; $a(\tilde{a}) := \tilde{a} + \phi(a)$ (see (4.2.1)).

\(^{29}\)The inverse functor is induction from $T^\vee(O) \times T(O)$ to $(T^\vee(F) \times T(F))^\sim$.

\(^{30}\)As always, the center $G_m$ of $(T^\vee \times T)(F)^\sim$ acts on our modules by standard homotheties.
Proof of Lemma. Notice that Lemma 3.11, as well as its proof, remains valid for families of de Rham $T^\vee$-local systems; one can also replace the curve $X$ by Spec$F$. By 3.11(i) our $C$-family of chiral algebras $A^C_F$ on Spec$F$ can be canonically identified with the constant $C$-family $A_{FC}$. This isomorphism is compatible with the $T^\vee(F)$-actions (here the $T^\vee(F)$-action on $A_{FC}$ is the product of the action on $A_F$ from 3.10(c) and the gauge action on $C$).

Therefore an $A^C_F$-module $M$ amounts to a $T^\vee(F)$-equivariant discrete $O(C)$-module $M_C$ equipped with an $A_F$-action which commutes with the $O(C)$-action and is compatible with the $T^\vee(F)$ one. According to 3.8, 3.9 an $A_F$-action is the same as a $T(F)$-action. As follows from the construction of 3.8, the compatibility with the $T^\vee(F)$-action just means that the $T^\vee(F)$- and $T(F)$-actions form an action of $(T^\vee \times T)(F)^\vee$. This proves (i), and (ii) follows from 3.11(ii).

(c) Set $H := T(F)^0/T$, $H^\vee := T^\vee(F)^0/T^\vee$. Since $T$ is a central subgroup of $T(F)^0$ and $T^\vee \subset T^\vee(F)^0$ acts trivially on $T(F)^0$ we have a central $\mathbb{G}_m$-extension $H^0 := T(F)^0/T$ of $H$ and the corresponding semi-direct product $(H^\vee \times H)^\vee$ of $H^\vee$ and $H$. Since $T^\vee(F)^0$ acts on $C$ through its quotient $H^\vee$ we have the category $P^0$ of $(H^\vee \times H)^\vee$-equivariant discrete $O(C)$-modules. As follows from (a) the obvious functor $P \to P^0$, $M_C \to M^0_C := (M_C)T^\vee \times T$, is an equivalence of categories.

(d) As we have seen in 4.2, $C$ is a $T^\vee(m)$-torsor over $\tilde{C}$ (see 4.2). The $T^\vee(m)$-action on $H^\vee$ defines the corresponding twisted group ind-scheme on $\tilde{C}$ which we denote by $H^\vee_\tilde{C}$; this is a central $\mathbb{G}_m$-extension of the constant group ind-scheme $H_\tilde{C}$. The action of $H^\vee$ on $C$ and $H$ defines an action of the formal group ind-scheme $Q := T^\vee(F)^0/T^\vee(O) = H^\vee/T^\vee(m)$ on $\tilde{C}$ and $H^\vee_\tilde{C}$.

Let $\tilde{P}$ be the category of discrete $O(\tilde{C})$-modules $M_\tilde{C}$ equipped with an $H^\vee_\tilde{C}$-action and equivariant with respect to the $Q$-action. There is an obvious descent equivalence $P^0 \simeq \tilde{P}$, $N_C \mapsto N_\tilde{C} := N_C T^\vee(m)$.

(e) The $C$-family of automorphisms of $t(F)^\vee$ from (b)(ii) has property $g(\alpha(\tilde{a}),\nu) = \alpha(\tilde{a}g\nu)$ for every $g \in T^\vee(F)$, $\nu \in C$, $\tilde{a} \in t(F)^\vee$. Therefore it yields a canonical morphism of Lie algebras $\tilde{\alpha} : t(F)^\vee_\tilde{C} \to h^\vee_\tilde{C}$ on $\tilde{C}$ equivariant with respect to the $Q$-action. Here $t(F)^\vee_\tilde{C}$ is the constant $\tilde{C}$-family and $h^\vee_\tilde{C}$ is the Lie algebra of $H^\vee_\tilde{C}$. Thus every $N_\tilde{C} \in \tilde{P}$ is a $t(F)^\vee$-module in a natural way. The $t(F)^\vee$-action commutes with the $Q$-action, so the space of $Q$-coinvariants $(N_\tilde{C})_Q$ is also a $t(F)^\vee$-module.

We have an equivalence $M(A^C_F) \simeq \tilde{P}$, $M \mapsto M^0_C$, defined as the composition of the equivalences from (b)(i), (c), and (d). By (4.4.1), (b)(ii) there is a canonical identification of $t(F)^\vee$-modules $\Gamma(M) = (M^0_C)_Q$. Therefore our theorem amounts to the fact that the functor $\tilde{P} \to t(F)^\vee$-mod, $N_\tilde{C} \mapsto (N_\tilde{C})_Q$, is an equivalence of categories.

(f) The morphism $\tilde{\alpha} : t(F)^\vee_\tilde{C} \to h^\vee_\tilde{C}$ is surjective and $H^\vee_\tilde{C}$ is connected, so for $N_\tilde{C} \in \tilde{P}$ the $H^\vee_\tilde{C}$-action on it is uniquely determined by the $t(F)^\vee$-action. Therefore $\tilde{P}$ is a full subcategory of the category of $Q$-equivariant discrete $O(\tilde{C})$-modules equipped with a $t(F)^\vee$-action.

Notice that $H$ is an extension of a formal group by a unipotent one $T(m)$. Thus $H$-modules are the same as $h$-modules such that the Lie algebra of $T(m) \subset H$ acts
in a locally nilpotent way. Same is true for $H^*_C$-modules.

Let $\beta: t(O)\bar{C} \to t(F)\bar{C}$ be the morphism $\beta(a) := a - \phi(a)$. Then $\text{Ker } \bar{\alpha} = \beta(t_c)$ and $\bar{\alpha}$ identifies $\beta(t(\mathfrak{m})\bar{C})$ with the Lie algebra of $T(\mathfrak{m}) \subset H^*_C$.

We see that objects of $\bar{\mathcal{P}}$ are exactly those $Q$-equivariant discrete $\mathcal{O}(\bar{C})$-modules $N^\wedge_C$ equipped with a $(t(F))^{-}\text{action}$ which satisfy the following conditions:

(i) $\beta(t)$ kills $N^\wedge_C$,

(ii) $\beta(t(\mathfrak{m})\bar{C})$ acts on $N^\wedge_C$ in a locally nilpotent way.

(g) It is convenient to view $(t(F))^{-}\text{modules}$ in the following “geometric” way.

According to 4.2 the embedding $\phi: t(O) \hookrightarrow \mathcal{O}(\bar{C})$ identifies $\bar{C}$ with $\text{Spec}(\text{Sym}(O))$. Let $\pi: C \to t^\vee$ be the projection that corresponds to the embedding $t \hookrightarrow t(O)$. The Lie algebra $t(F)$ acts along the fibers of $\pi$: namely, for $a \in t(F)$, $b \in t(O) \subset \mathcal{O}(\bar{C})$ one has $a(b) := \text{Res}_{C}(b, da)$. Let $\Theta^\wedge_\pi$ be a Lie algebroid on $\bar{C}$ equipped with a Lie algebra homomorphism $\rho: t(F) \to \Theta^\wedge_\pi$ such that:

(i) $\Theta^\wedge_\pi$ is an extension of the relative tangent algebroid $\Theta_\pi = \Theta_{C/\text{Spec}(C)}$ by $\mathcal{O}_C$,

(ii) $\rho$ lifts the above action of $t(F)$ on $\bar{C}$, and $\rho|_{t(O)}$ equals $\phi: t(O) \to \mathcal{O}(\bar{C}) \subset \Theta^\wedge_\pi$.

Such $\Theta^\wedge_\pi$ exists and is unique.

By definition, a $\Theta^\wedge_\pi$-module any discrete $\mathcal{O}(\bar{C})$-module $L^\wedge_C$ equipped with a (right)$^{33}$ action of $\Theta^\wedge_\pi$ such that $1 \in \mathcal{O}_\bar{C} \subset \Theta^\wedge_\pi$ acts as identity. Then $L^\wedge_C$ is a $(t(F))^{-}\text{module}$ via $\rho$. The functor $\Theta^\wedge_\pi\text{-mod} \to (t(F))^{-}\text{mod}$ is an equivalence of categories.

So $(t(F))^{-}\text{modules}$ are the same as (twisted) $\mathcal{D}$-modules along the fibers of $\pi$.

(h) Now we are ready to prove the theorem. Returning to (f) notice that $Q$ is a formal group which acts formally simply transitively along the fibers of $\pi$. Therefore a $Q$-equivariant structure on an $\mathcal{O}(\bar{C})$-module is the same as a (right) $\Theta^\wedge_\pi$-action. Such object equipped with an $(t(F))^{-}\text{action}$ amounts, by (g), to a $\Theta^\wedge_\pi \times \Theta^\wedge_\pi$-module, i.e., a discrete $\mathcal{O}(\bar{C} \times \bar{C})$-module $N^\wedge_{\bar{C} \times \bar{C}}$ equipped with a (right) action of $\Theta^\wedge_\pi \times \Theta^\wedge_\pi$ such that $1 \in \mathcal{O}_{\bar{C} \times \bar{C}} \subset \Theta^\wedge_{\bar{C} \times \bar{C}}$ acts as identity. Condition (f)(i) means that $N^\wedge_{\bar{C} \times \bar{C}}$ is supported (scheme-theoretically) on $\bar{C} \times \bar{C} \subset \bar{C} \times \bar{C}$. Condition (f)(ii) means that $N^\wedge_{\bar{C} \times \bar{C}}$ is supported set-theoretically on the diagonal $\bar{C} \subset \bar{C} \times \bar{C}$.

According to the Kashiwara lemma such animals are the same as $\Theta^\wedge_\pi$-modules on the diagonal $\bar{C} \subset \bar{C} \times \bar{C}$. The identification can be given by the integration functor along the first copy of $\bar{C}$ which is the same as taking $Q$-coinvariants. Returning from $\Theta^\wedge_\pi$-modules to $(t(F))^{-}\text{modules}$ we get the theorem. $\square$

References.

[BBE] A. Beilinson, S. Bloch, H. Esnault. $\mathcal{E}$-factors for Gauß-Manin determinants. alg-geom 2001.

[BD] A. Beilinson, V. Drinfeld. Chiral Algebras.

$^{31}$Here $\phi(a) \in \mathcal{O}(\bar{C})$ is the linear function on $\bar{C}$ that corresponds to $a$, see 4.2.

$^{32}$ $\pi$ can be also described as the residue map $\bar{C} = t^\vee \otimes (\omega(F)/\omega(O)) \to t^\vee$, see 4.2.

$^{33}$Since $\bar{C}$ is an ind-scheme only right actions make sense.
[CC] C. E. Contou-Carrère. Jacobienne locale, groupe de bivecteurs de Witt universel et symbole local modéré. C. R. Acad. Sci. Paris, t. 318, Série I (1994), 743-746.

[D] C. Dong. Vertex algebras associated with even lattices. J. Algebra 161 (1993) 245–265.

[FBZ] E. Frenkel, D. Ben-Zvi. Vertex algebras and algebraic curves. AMS 2001.

[K] V. Kac. Vertex algebras for beginners. University Lecture Series vol. 10, AMS, 1998.