Asymptotic properties of the derivative of self-intersection local time of fractional Brownian motion

ARTURO JARAMILLO AND DAVID NUALART∗
Department of Mathematics
The University of Kansas
Lawrence, Kansas, 66045

Abstract
Let \( \{ B_t \}_{t \geq 0} \) be a fractional Brownian motion with Hurst parameter \( \frac{2}{3} < H < 1 \). We prove that the approximation of the derivative of self-intersection local time, defined as

\[
\alpha_\varepsilon = \int_0^T \int_0^t p_\varepsilon'(B_t - B_s)dsdt,
\]

where \( p_\varepsilon(x) \) is the heat kernel, satisfies a central limit theorem when renormalized by \( \varepsilon^{\frac{3}{2} - \frac{1}{2H}} \). We prove as well that for \( q \geq 2 \), the \( q \)-th chaotic component of \( \alpha_\varepsilon \) converges in \( L^2 \) when \( \frac{2}{3} < H < \frac{3}{4} \), and satisfies a central limit theorem when renormalized by a multiplicative factor \( \varepsilon^{1 - \frac{3}{4q}} \) in the case \( \frac{3}{4} < H < \frac{4}{3q-2} \).

Keywords: Fractional Brownian motion, self-intersection local time, Wiener chaos expansion, central limit theorem.
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1 Introduction
Let \( B = \{ B_t \}_{t \geq 0} \) be a one-dimensional fractional Brownian motion of Hurst parameter \( H \in (0, 1) \). Fix \( T > 0 \). The self-intersection local time of \( B \), formally defined by

\[
I(y) := \int_0^T \int_0^t \delta(B_t - B_s - y)dsdt,
\]

was first studied by Rosen in [11] in the planar case and it was further investigated using techniques from Malliavin calculus by Hu and Nualart in [5]. In particular, in [5] it is proved that for a \( d \)-dimensional fractional Brownian motion, \( I(0) \) exists in \( L^2 \) whenever the Hurst parameter \( H \) satisfies \( H < \frac{1}{d} \).

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Motivated by spatial integrals with respect to local time, developed by Rogers and Walsh in [10], Rosen introduced in [12] a formal derivative of $I(y)$, in the one-dimensional Brownian case, denoted by

$$\alpha(y) := d\frac{I}{dy}(y) = - \int_0^T \int_0^t \delta'(B_t - B_s - y)dsdt.$$ 

The random variable $\alpha := \alpha(0)$ is called the derivative of the self-intersection local time at zero, and is equal to the limit in $L^2$ of

$$\alpha_\varepsilon := \int_0^T \int_0^t \rho_\varepsilon'(B_t - B_s)dsdt, \quad (1.1)$$

where $\rho_\varepsilon(x) := (2\pi\varepsilon)^{-\frac{1}{2}} e^{-x^2/2\varepsilon}$. This random variable was subsequently used by Hu and Nualart [9], to study the asymptotic properties of the third spatial moment of the Brownian local time. In [8], Markowsky gave an alternative proof the existence of such limit by using Wiener chaos expansion.

Jung and Markowsky extended this result in [7] to the case $0 < H < \frac{2}{3}$ and conjectured that for the case $H > \frac{2}{3}$, $\varepsilon^{-\gamma(H)}\alpha_\varepsilon$ converges in law to a Gaussian distribution for some suitable constant $\gamma(H) > 0$, and at the critical point $H = \frac{2}{3}$, the variable $\lim_{\varepsilon \to 0} (1/\varepsilon)^{-\gamma} \alpha_\varepsilon$ converges in law to a Gaussian distribution for some $\gamma > 0$.

Let $\mathcal{N}(0, \gamma)$ denote a centered Gaussian random variable with variance $\gamma$. The primary goal of this paper is to analyze the behavior of the law of $\alpha_\varepsilon$ as $\varepsilon \to 0$, when $\frac{2}{3} < H < 1$. We will prove that when $\frac{2}{3} < H < 1$,

$$\varepsilon^{\frac{3}{2} - \frac{1}{3}H} \alpha_\varepsilon \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma^2), \quad \text{when} \quad \varepsilon \to 0,$$

for some constant $\sigma^2$ that will be specified later (see Theorem 4.1). Moreover, we will prove that for every $q \geq 2$ and $\frac{2}{3} < H < \frac{2}{3} + \frac{q-3}{q-2}$, $\lim_{\varepsilon \to 0} J_q[\alpha_\varepsilon]$ exists in $L^2$, where $J_q$ denotes the projection on the $q$-th Wiener chaos (see Theorem 4.2), while in the case $\frac{2}{3} < H < \frac{q-3}{q-2}$, the chaotic components $J_q[\alpha_\varepsilon]$ of $\alpha_\varepsilon$ satisfy

$$\varepsilon^{1 - \frac{1}{3q}} J_q[\alpha_\varepsilon] \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma^2_q), \quad \text{when} \quad \varepsilon \to 0,$$

for some constant $\sigma^2_q$ that will be specified latter (see Theorem 4.3). The proof of the central limit theorem for $\varepsilon^{\frac{3}{2} - \frac{1}{3}H} \alpha_\varepsilon$ follows easily from estimations of the $L^2$-norm of the chaotic components of $\alpha_\varepsilon$, while the proof of the central limit theorem for $\varepsilon^{1 - \frac{1}{3q}} J_q[\alpha_\varepsilon]$ relies on the multivariate version of the fourth moment theorem (see [3, 9]), as well as the a continuos version of the Breuer-Major theorem (11) proved in [2]. The behavior of $\alpha_\varepsilon$ in the critical case $H = \frac{2}{3}$, and the behavior of $J_q[\alpha_\varepsilon]$ in the critical cases $H = \frac{2}{3}$, $H = \frac{3}{4}$ and $H = \frac{4q-3}{4q-2}$ seems more involved and will not be discussed in this paper.

It is surprising to remark that the limit behavior of the chaotic components of $\alpha_\varepsilon$ is different from that of the whole sequence. This phenomenon was observed, for instance, in the central limit theorem for the second spatial moment of Brownian local
time increments (see [4]). However, in this case the limit of the whole sequence is a mixture of Gaussian distributions, whereas in the present paper the normalization of \( \alpha_\varepsilon \) converges to a Gaussian law. In our case, the projection on the first chaos of \( \alpha_\varepsilon \) is the leading term and is responsible for the Gaussian limit of the whole sequence.

The paper is organized as follows. In Section 2 we present some preliminary results on the fractional Brownian motion and the chaotic decomposition of \( \alpha_\varepsilon \). In Section 3 we compute the asymptotic behavior of the variances of the normalizations of the chaotic components of \( \alpha_\varepsilon \) as \( \varepsilon \to 0 \). The asymptotic behavior of the law of \( \alpha_\varepsilon \) and its chaotic components is presented in section 4. Finally, some technical lemmas are proved in Section 5.

2 Preliminaries

2.1 Fractional Brownian motion

Throughout the paper, \( B = \{B_t\}_{t \geq 0} \) will denote a fractional Brownian motion with Hurst parameter \( H \in (0,1) \), defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). That is, \( B \) is a centered Gaussian process with covariance function

\[
\mathbb{E}[B_t B_s] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).
\]

We will denote by \( \mathcal{H} \) the Hilbert space obtained by taking the completion of the space of step functions endowed with the inner product

\[
\langle 1_{[a,b]}, 1_{[c,d]} \rangle \mathcal{H} := \mathbb{E}[(B_b - B_a)(B_d - B_c)].
\]

The mapping \( 1_{[0,t]} \mapsto B_t \) can be extended to a linear isometry between \( \mathcal{H} \) and a Gaussian subspace of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \). We will denote by \( B(h) \) the image of \( h \in \mathcal{H} \) by this isometry. For any integer \( q \in \mathbb{N} \), we denote by \( \mathcal{H}^\otimes q \) and \( \mathcal{H}^\circ q \) the \( q \)th tensor product of \( \mathcal{H} \), and the \( q \)th symmetric tensor product of \( \mathcal{H} \), respectively. The \( q \)th Wiener chaos of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \), denoted by \( \mathcal{H}_q \), is the closed subspace of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) generated by the variables \( \{H_q(B(h)), h \in \mathcal{H}, \|h\|_\mathcal{H} = 1\} \), where \( H_q \) is the \( q \)th Hermite polynomial, defined by

\[
H_q(x) := (-1)^q e^{x^2} \frac{d^q}{dx^q} e^{-x^2}.
\]

The mapping \( I_q(h^\otimes q) = H_q(B(h)) \) provides a linear isometry between \( \mathcal{H}^\otimes q \) (equipped with the norm \( \sqrt{q} \|\cdot\|_\mathcal{H}^\otimes q \)) and \( \mathcal{H}_q \) (equipped with the \( L^2 \)-norm).

2.2 Chaos decomposition for \( \alpha_\varepsilon \)

Proceeding as in [7] (also see [5]), we can determine the chaos decomposition of the random variable \( \alpha_\varepsilon \) defined in (1.1) as follows. First we write

\[
\alpha_\varepsilon = \int_0^T \int_0^t \alpha_{\varepsilon,s,t} ds dt,
\]

(2.1)
where $\alpha_{\varepsilon,s,t} := p'(B_t - B_s)$. We know that
\[
\alpha_{\varepsilon,s,t} = \sum_{q=1}^{\infty} I_{2q-1} (f_{2q-1,\varepsilon,s,t}),
\]
where
\[
f_{2q-1,\varepsilon,s,t}(x_1, \ldots, x_{2q-1}) := (-1)^q \beta_q (\varepsilon + (t - s)^2 H)^{-\frac{1}{2}} \prod_{j=1}^{2q-1} \mathbb{1}_{[a,t]}(x_j),
\]
and
\[
\beta_q := \frac{1}{2^{q-\frac{1}{2}} (q-1)! \sqrt{\pi}}
\]
As a consequence, the random variable $\alpha_{\varepsilon}$ has the chaos decomposition
\[
\alpha_{\varepsilon} = \sum_{q=1}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon}),
\]
where
\[
f_{2q-1,\varepsilon}(x_1, \ldots, x_{2q-1}) := \int_{\mathcal{R}} f_{2q-1,\varepsilon,s,t}(x_1, \ldots, x_{2q-1}) ds dt,
\]
and
\[
\mathcal{R} := \{(s, t) \in \mathbb{R}_+^2 \mid s \leq t \leq T\}.
\]
Let $T, \varepsilon > 0$, $\frac{2}{3} < H < 1$, and $q \in \mathbb{N}$ be fixed. Our first goal is to find the behavior as $\varepsilon \to 0$ of the variances of $\alpha_{\varepsilon}$ and $I_{2q-1}(f_{2q-1,\varepsilon})$. Before addressing this problem, we will introduce some notation. First notice that
\[
\mathbb{E} \left[ I_{2q-1}(f_{2q-1,\varepsilon})^2 \right] = (2q - 1)! \| f_{2q-1,\varepsilon} \|^2_{\mathcal{S}^{(2q-1)}}
\]
\[
= (2q - 1)! \left\langle \int_{\mathcal{R}} f_{2q-1,\varepsilon,s_1,t_1} ds_1 dt_1, \int_{\mathcal{R}} f_{2q-1,\varepsilon,s_2,t_2} ds_2 dt_2 \right\rangle_{\mathcal{S}^{(2q-1)}}
\]
\[
= 2(2q - 1)! \int_{\mathcal{S}} \left( f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2} \right)_{\mathcal{S}^{(2q-1)}} ds_1 ds_2 dt_1 dt_2,
\]
where the set $\mathcal{S}$ is defined by
\[
\mathcal{S} := \{(s_1, s_2, t_1, t_2) \in [0, T]^4 \mid s_1 \leq t_1, \ s_2 \leq t_2, \ \text{and} \ s_1 \leq s_2\}.
\]
We can write the set $\mathcal{S}$ as the union of the sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ defined by
\[
\mathcal{S}_1 := \{(s_1, s_2, t_1, t_2) \in [0, T]^4 \mid s_1 \leq s_2 \leq t_1 \leq t_2\},
\]
\[
\mathcal{S}_2 := \{(s_1, s_2, t_1, t_2) \in [0, T]^4 \mid s_1 \leq s_2 \leq t_2 \leq t_1\},
\]
\[
\mathcal{S}_3 := \{(s_1, s_2, t_1, t_2) \in [0, T]^4 \mid s_1 \leq t_1 \leq s_2 \leq t_2\}.
\]
Then, by (2.1),

\[ \mathbb{E}[\alpha_{\varepsilon}]^2 = \mathbb{E} \left[ \left( \int_{\mathbb{R}} \alpha_{\varepsilon,s,t} dsdt \right)^2 \right] = 2 \int_{\mathcal{S}} \mathbb{E} \left[ \alpha_{\varepsilon,s_1,t_1} \alpha_{\varepsilon,s_2,t_2} \right] ds_1 ds_2 dt_1 dt_2 = V_i(\varepsilon) + V_2(\varepsilon) + V_3(\varepsilon), \]

where

\[ V_i(\varepsilon) := 2 \int_{\mathcal{S}} \mathbb{E} \left[ \alpha_{\varepsilon,s_1,t_1} \alpha_{\varepsilon,s_2,t_2} \right] ds_1 ds_2 dt_1 dt_2, \quad i = 1, 2, 3. \]  (2.13)

Similarly, from (2.6) and (2.8), taking \( \varepsilon = 1 \), we get

\[ \mathbb{E} \left[ I_1 (f_{1,\varepsilon})^2 \right] = V_i^{(1)}(\varepsilon) + V_2^{(1)}(\varepsilon) + V_3^{(1)}(\varepsilon), \]

where

\[ V_i^{(1)}(\varepsilon) := 2 \int_{\mathcal{S}} \langle f_{1,\varepsilon,s_1,t_1}, f_{1,\varepsilon,s_2,t_2} \rangle_{\mathcal{H}} ds_1 ds_2 dt_1 dt_2, \quad i = 1, 2, 3. \]  (2.14)

As a consequence of (2.13) and (2.15), to determine the behavior of the variances of \( \alpha_{\varepsilon} \) and \( I_1 (f_{1,\varepsilon}) \) as \( \varepsilon \to 0 \), it suffices to determine the behavior of \( V_i(\varepsilon) \) and \( V_i^{(1)}(\varepsilon) \) respectively, for \( i = 1, 2, 3 \).

In order to describe the terms \( \langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2} \rangle_{\mathcal{H}^{(2q-1)}} \), we will introduce the following notation. For every \( x, u_1, u_2 > 0 \) define

\[ \mu(x, u_1, u_2) := \mathbb{E} \left[ B_{u_1}(B_{x+u_2} - B_x) \right]. \]  (2.17)

We can easily prove that for every \( s_1, s_2, t_1, t_2 \geq 0 \), such that \( s_1 \leq t_1, s_2 \leq t_2 \) and \( s_1 \leq s_2 \),

\[ \mathbb{E} \left[ (B_{t_1} - B_{s_1})(B_{t_2} - B_{s_2}) \right] = \mu(s_2 - s_1, t_1 - s_1, t_2 - s_2). \]  (2.18)

Using (2.13) and (2.18), for every \( 0 \leq s_1 \leq t_1, 0 \leq s_2 \leq t_2 \) such that \( s_1 \leq s_2 \), we can write

\[
\langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2} \rangle_{\mathcal{H}^{(2q-1)}} = \beta^2_q(\varepsilon + (t_1 - s_1)^{2H})^{-\frac{1}{2} - q}(\varepsilon + (t_2 - s_2)^{2H})^{-\frac{1}{2} - q} \\
\times \mu(s_2 - s_1, t_1 - s_1, t_2 - s_2) \\
= \beta^2_q(\varepsilon + (t_1 - s_1)^{2H})^{-\frac{1}{2} - q}(\varepsilon + (t_2 - s_2)^{2H})^{-\frac{1}{2} - q} \\
\times \mu(s_2 - s_1, t_1 - s_1, t_2 - s_2)^{2q-1}.
\]

Therefore,

\[ \langle f_{2q-1,\varepsilon,s_1,t_1}, f_{2q-1,\varepsilon,s_2,t_2} \rangle_{\mathcal{H}^{(2q-1)}} = \beta^2_q C^{(q)}_{\varepsilon,s_2-s_1}(t_1 - s_1, t_2 - s_2), \]  (2.19)
where \(G_{\varepsilon,x}^{(q)}(u_1, u_2)\) is defined by
\[
G_{\varepsilon,x}^{(q)}(u_1, u_2) := (\varepsilon + u_1^{2H})^{-\frac{1}{2}-q} (\varepsilon + u_2^{2H})^{-\frac{1}{2}-q} \mu(x, u_1, u_2)^{2q-1}. \tag{2.20}
\]

Next we present some useful properties of the functions \(\mu(x, u_1, u_2)\) and \(G_{\varepsilon,x}^{(q)}(u_1, u_2)\). Taking into account that \(H > \frac{2}{3}\), we can write the covariance of \(B\) as
\[
\mathbb{E}[B_t B_s] = H(2H - 1) \int_0^t \int_0^s |v_1 - v_2|^{2H-2} dv_1 dv_2. \tag{2.21}
\]

In particular, this leads to
\[
\mu(x, u_1, u_2) = H(2H - 1) \int_0^{u_1} \int_x^{x+u_2} |v_2 - v_1|^{2H-2} dv_1 dv_2, \tag{2.22}
\]
which implies
\[
G_{\varepsilon,x}^{(q)}(u_1, u_2) \geq 0 \quad \text{for every } \varepsilon \geq 0. \tag{2.23}
\]

Using the chaos decomposition (2.22), as well as (2.19) and (2.23), we can check that for \(i = 1, 2, 3\), the terms \(V_i(\varepsilon), V_i^{(1)}(\varepsilon)\), defined by (2.14), (2.16), satisfy
\[
0 \leq V_i^{(1)}(\varepsilon) \leq V_i(\varepsilon). \tag{2.24}
\]

Further properties for the function \(G_{\varepsilon,x}^{(q)}(u_1, u_2)\) are described in the following lemma.

**Lemma 2.1.** Let \(G_{\varepsilon,x}^{(q)}(u_1, u_2)\) be defined by (2.20). There exists a constant \(K > 0\), depending on \(H\) and \(q\), such that for all \(x > 0\), and \(0 < v_1 \leq w_1\), \(0 < v_2 \leq w_2\) satisfying \(|v_i - w_i| \leq 1\),
\[
G_{\varepsilon,x}^{(q)}(v_1, v_2) \leq KG_{\varepsilon,x}^{(q)}(w_1, w_2).
\]

**Proof.** From (2.22) it follows that
\[
\mu(x, v_1, v_2) \leq \mu(x, w_1, w_2).
\]

As a consequence,
\[
G_{1,x}^{(q)}(v_1, v_2) = (1 + v_1^{2H})^{-\frac{1}{2}-q} (1 + v_2^{2H})^{-\frac{1}{2}-q} \mu(x, v_1, v_2)^{2q-1} \\
\leq (1 + v_1^{2H})^{-\frac{1}{2}-q} (1 + v_2^{2H})^{-\frac{1}{2}-q} \mu(x, w_1, w_2)^{2q-1} \\
= G_{1,x}^{(q)}(w_1, w_2) \left( \frac{(1 + v_1^{2H})(1 + v_2^{2H})}{(1 + v_1^{2H})(1 + v_2^{2H})} \right)^{q+\frac{1}{2}}.
\]

Using condition \(|v_i - w_i| \leq 1\), \(i = 1, 2\), we get
\[
G_{1,x}^{(q)}(v_1, v_2) \leq G_{1,x}^{(q)}(w_1, w_2) \left( \frac{(1 + (v_1 + 1)2H)(1 + (v_2 + 1)2H)}{(1 + v_1^{2H})(1 + v_2^{2H})} \right)^{q+\frac{1}{2}}. \tag{2.25}
\]

The second factor in the right-hand side of (2.25) is uniformly bounded for \(v_1, v_2 \geq 0\), which implies the desired result. \(\square\)
3 Behavior of the variances of $\alpha_\varepsilon$ and its chaotic components

The behavior of the variance of $\alpha_\varepsilon$ is described in the following lemma.

**Lemma 3.1.** Let $T > 0$ and $\frac{2}{3} < H < 1$ be fixed. Then,
\[
\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} \mathbb{E} \left[ \alpha_\varepsilon^2 \right] = \sigma^2,
\]
where $\sigma^2$ is defined by
\[
\sigma^2 := \frac{T^{2H}(2H - 1)}{4H \pi} B \left( \frac{1}{H}, \frac{3H - 2}{2H} \right)^2 B(2, 2H - 1),
\]
and $B(\cdot, \cdot)$ denotes the Beta function.

**Proof.** From (2.13) we have
\[
\varepsilon^{3 - \frac{2}{H}} \mathbb{E} \left[ \alpha_\varepsilon^2 \right] = \varepsilon^{3 - \frac{2}{H}} V_1(\varepsilon) + \varepsilon^{3 - \frac{2}{H}} V_2(\varepsilon) + \varepsilon^{3 - \frac{2}{H}} V_3(\varepsilon),
\]
where $V_1(\varepsilon)$, $V_2(\varepsilon)$ and $V_3(\varepsilon)$ are defined by (2.14). By Lemmas 5.3 and 5.4 we have $\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} V_1(\varepsilon) = 0$ and $\varepsilon^{3 - \frac{2}{H}} V_2(\varepsilon) = 0$, respectively. In addition, from Lemma 5.6 we have $\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} V_3(\varepsilon) = \sigma^2$, where $\sigma^2$ is defined by (3.2). This completes the proof of equation (3.1).

The behavior of the variance of the first chaotic component of $\alpha_\varepsilon$ is described by the following lemma.

**Lemma 3.2.** Let $T > 0$ be fixed. Define $f_{1, \varepsilon}$ as in equation (2.6). Then, for every $\frac{2}{3} < H < 1$, we have
\[
\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} \mathbb{E} \left[ I_1(f_{1, \varepsilon})^2 \right] = \sigma^2,
\]
where $\sigma^2$ is given by (3.2).

**Proof.** From (2.15) we have
\[
\varepsilon^{3 - \frac{2}{H}} \mathbb{E} \left[ I_1(f_{1, \varepsilon})^2 \right] = \varepsilon^{3 - \frac{2}{H}} V_1^{(1)}(\varepsilon) + \varepsilon^{3 - \frac{2}{H}} V_2^{(1)}(\varepsilon) + \varepsilon^{3 - \frac{2}{H}} V_3^{(1)}(\varepsilon),
\]
where $V_1^{(1)}(\varepsilon)$, $V_2^{(1)}(\varepsilon)$ and $V_3^{(1)}(\varepsilon)$ are defined by (2.16). By Lemmas 5.3 and 5.4 we have $\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} V_1^{(1)}(\varepsilon) = 0$ and $\varepsilon^{3 - \frac{2}{H}} V_2^{(1)}(\varepsilon) = 0$, respectively. Consequently, by (2.24) we get $\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} V_1^{(1)}(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} V_2^{(1)}(\varepsilon) = 0$. In addition, from Lemma 5.7 the term $V_3^{(1)}(\varepsilon)$ satisfies $\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} V_3^{(1)}(\varepsilon) = \sigma^2$, where $\sigma^2$ is given by (3.2). This completes the proof of equation (3.3).

The behavior of the variance of the chaotic components of $\alpha_\varepsilon$ of order greater or equal to two and is described by the following lemma.
Lemma 3.3. Let $T, \varepsilon > 0$, $\frac{2}{3} < H < 1$ and $q \in \mathbb{N}$, $q \geq 2$ be fixed. Define $\beta_q, f_{2q-1, \varepsilon}$, and $G_{1,x}^{(q)}(u_1, u_2)$ by (2.24), (2.6) and (2.20) respectively. Then,

1. If $\frac{2}{3} < H < \frac{4q-3}{4q-2}$, then

$$
\lim_{\varepsilon \to 0} \varepsilon^2 \frac{2}{4q-3} \mathbb{E} \left[ I_{2q-1}(f_{2q-1, \varepsilon})^2 \right] = \sigma_q^2,
$$

where $\sigma_q^2$ is a finite constant given by

$$
\sigma_q^2 := 2(2q - 1)! \beta_q^2 T \int_{\mathbb{R}_+^3} G_{1,x}^{(q)}(u_1, u_2) \, dx \, du_1 \, du_2.
$$

2. In the case $\frac{2}{3} < H < \frac{3}{4}$, then

$$
\lim_{\varepsilon \to 0} \mathbb{E} \left[ I_{2q-1}(f_{2q-1, \varepsilon})^2 \right] = \overline{\sigma}_q^2,
$$

where $\overline{\sigma}_q^2$ is a finite constant given by

$$
\overline{\sigma}_q^2 := 2(2q - 1)! \beta_q^2 \int_{S} G_{0,s_2-s_1}^{(q)}(t_1 - s_1, t_2 - s_2) \, ds_1 \, ds_2 \, dt_1 \, dt_2,
$$

and $S$ is defined by (2.9).

Proof. First we prove (3.4) in the case $\frac{3}{4} < H < \frac{4q-3}{4q-2}$. By (2.8) and (2.19),

$$
\varepsilon^2 \frac{2}{4q-3} \mathbb{E} \left[ I_{2q-1}(f_{2q-1, \varepsilon})^2 \right] = 2(2q - 1)! \beta_q^2 \varepsilon^2 \frac{2}{4q-3} \int_{S} G_{\varepsilon,s_2-s_1}^{(q)}(t_1 - s_1, t_2 - s_2) \, ds_1 \, ds_2 \, dt_1 \, dt_2,
$$

where $S$ is defined by (2.9). Therefore, changing the coordinates $(s_1, s_2, t_1, t_2)$ by $(\varepsilon^{-\frac{1}{4q-3}} s_1, x := \varepsilon^{-\frac{1}{4q-3}} (s_2 - s_1), u_1 := \varepsilon^{-\frac{1}{4q-3}} (t_1 - s_1), u_2 := \varepsilon^{-\frac{1}{4q-3}} (t_2 - s_2))$, we get

$$
\varepsilon^2 \frac{2}{4q-3} \mathbb{E} \left[ I_{2q-1}(f_{2q-1, \varepsilon})^2 \right] = 2(2q - 1)! \beta_q^2 \varepsilon \frac{2}{4q-3} \int_{\mathbb{R}_+^4} \mathbb{1}_{(0, \varepsilon^{-\frac{1}{4q-3}} T)}(s_1 + u_1)
$$

$$
\times \mathbb{1}_{(0, \varepsilon^{-\frac{1}{4q-3}} T)}(s_1 + x + u_2) G_{1,x}^{(q)}(u_1, u_2) \, ds_1 \, dx \, du_1 \, du_2.
$$

Integrating with respect to the variable $s_1$ we get

$$
\varepsilon^2 \frac{2}{4q-3} \mathbb{E} \left[ I_{2q-1}(f_{2q-1, \varepsilon})^2 \right] = 2(2q - 1)! \beta_q^2 \varepsilon \frac{2}{4q-3} \int_{\mathbb{R}_+^4} (T - \varepsilon \frac{1}{4q-3} (u_1 \vee (x + u_2))) \mathbb{1}_{(0, \varepsilon^{-\frac{1}{4q-3}} T)}(u_1)
$$

$$
\times \mathbb{1}_{(0, \varepsilon^{-\frac{1}{4q-3}} T)}(s_1 + x + u_2) G_{1,x}^{(q)}(u_1, u_2) \, ds_1 \, dx \, du_1 \, du_2.
$$

From (2.23) we deduce that the integrand in the right-hand side of (3.8) is positive and increasing as $\varepsilon$ decreases to zero. Therefore, applying the monotone convergence theorem in relation (3.8) we obtain (3.4). The constant $\sigma_q^2$ is finite by Lemma 5.9.
To prove relation (3.6), notice that equations (2.8) and (2.19) imply that
\[
\mathbb{E} \left[ I_{2q-1}(f_{2q-1, \varepsilon})^2 \right] = 2(2q - 1)! \beta_q^2 \int S G_{t_1, \varepsilon}^q(t_1 - s_1, t_2 - s_2) ds_1 ds_2 dt_1 dt_2. \tag{3.9}
\]
Relation (3.6) follows by applying the monotone convergence theorem to (3.9). To prove that \( \sigma^2 \) is finite we change the coordinates \((s_1, s_2, t_1, t_2)\) by \((s_1, x := s_2 - s_1, u_1 := t_1 - s_1, u_2 := t_2 - s_2)\) in the integral of the right-hand side of (3.7), to get
\[
\int S G^q_{t_1, \varepsilon}(s_1, s_2, t_1, t_2) ds_1 ds_2 dt_1 dt_2 \leq \int_{[0,T]^4} G^q_{0, x}(u_1, u_2) ds_1 dx du_1 du_2
\]
\[
= T \int_{[0,T]^3} G^q_{0, x}(u_1, u_2) dx du_1 du_2.
\]
The latter integral is finite by Lemma 5.9. Therefore, the constant \( \sigma^2 \) is finite.

\section{Limit behavior of \( \alpha_\varepsilon \) and \( I_{2q-1}(f_{2q-1, \varepsilon}) \)}

The next result is a central limit theorem for \( \alpha_\varepsilon \) in case \( \frac{2}{3} < H < 1 \).

\textbf{Theorem 4.1.} Let \( T, \varepsilon > 0 \) and \( \frac{2}{3} < H < 1 \) be fixed. Then
\[
\varepsilon^{\frac{3}{2} - \frac{1}{H}} \alpha_\varepsilon \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma^2), \quad \text{when } \varepsilon \to 0, \tag{4.1}
\]
where \( \sigma^2 \) is defined by (3.2).

\textbf{Proof.} Let \( f_{2q-1, \varepsilon} \) be defined by (2.6). By equation (2.5),
\[
\varepsilon^{\frac{3}{2} - \frac{1}{H}} \alpha_\varepsilon = \varepsilon^{\frac{3}{2} - \frac{1}{H}} I_1(f_{1, \varepsilon}) + \varepsilon^{\frac{3}{2} - \frac{1}{H}} \sum_{q=2}^{\infty} I_{2q-1}(f_{2q-1, \varepsilon}).
\]
By Lemma 3.2, the variance of \( \varepsilon^{\frac{3}{2} - \frac{1}{H}} I_1(f_{1, \varepsilon}) \) converges to \( \sigma^2 \), where \( \sigma^2 \) is defined by (3.2). In addition, combining Lemmas 3.1 and 3.2, it follows that the term
\[
\varepsilon^{\frac{3}{2} - \frac{1}{H}} \sum_{q=2}^{\infty} I_{2q-1}(f_{2q-1, \varepsilon})
\]
converges to zero in \( L^2 \). Then (4.1) follows from the fact that \( \varepsilon^{\frac{3}{2} - \frac{1}{H}} I_1(f_{1, \varepsilon}) \) is Gaussian and its variance converges to \( \sigma^2 \). \qed

In the next result we describe the asymptotic behavior of the chaotic components of \( \alpha_\varepsilon \) in the case \( \frac{2}{3} < H < 1 \).

\textbf{Theorem 4.2.} Let \( T, \varepsilon > 0 \) and \( q \in \mathbb{N}, q \geq 2 \) be fixed. Define \( f_{2q-1, \varepsilon} \) by (2.6). If \( \frac{2}{3} < H < \frac{3}{4} \), then \( I_{2q-1}(f_{2q-1, \varepsilon}) \) converges in \( L^2 \) when \( \varepsilon \to 0 \).
Proof. Define $f_{2q-1,\varepsilon,s,t}$ by (2.3). For every $\varepsilon, \eta > 0$ we have
\[
\mathbb{E}\left[(I_{2q-1}(f_{2q-1,\varepsilon}) - I_{2q-1}(f_{2q-1,\eta}))^2\right] = \mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})^2\right] + \mathbb{E}\left[I_{2q-1}(f_{2q-1,\eta})^2\right] - 2\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})I_{2q-1}(f_{2q-1,\eta})\right].
\]
Define $R$ and $S$ by (2.7) and (2.9), respectively. Then we have
\[
\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})I_{2q-1}(f_{2q-1,\eta})\right] = (2q-1)\int f_{2q-1,\varepsilon,S,T}d\mathbb{R},
\]
where
\[
\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})I_{2q-1}(f_{2q-1,\eta})\right] = (2q-1)\int G_{\varepsilon,\eta}(t_1 - s_1, t_2 - s_2)d\mathbb{S}d\mathbb{T}dt.
\]
Substituting (2.19) into (4.2), yields
\[
\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})I_{2q-1}(f_{2q-1,\eta})\right] = 2(2q-1)!\beta_q^2 \int G_{\varepsilon,\eta}(t_1 - s_1, t_2 - s_2)d\mathbb{S}d\mathbb{T}dt,
\]
where $G_{\varepsilon,\eta}(u_1, u_2)$ is defined by (2.20). Since $G_{\varepsilon,\eta}(u_1, u_2)$ is nonnegative (see equation (2.23)), the integral in the right-hand side of the previous identity is positive and decreasing in the variables $\varepsilon, \eta$. Hence, by the monotone convergence theorem, as $\varepsilon, \eta \to 0$, the terms $\mathbb{E}\left[I_{2q-1}(f_{2q-1,\varepsilon})I_{2q-1}(f_{2q-1,\eta})\right]$, $\mathbb{E}\left[I_{2q-1}(f_{2q-1,\eta})^2\right]$ and $\mathbb{E}\left[I_{2q-1}(f_{2q-1,\eta})^2\right]$ converge to
\[
2(2q-1)!\beta_q^2 \int G_{\varepsilon,\eta}(t_1 - s_1, t_2 - s_2)d\mathbb{S}d\mathbb{T}dt.
\]
The previous quantity is finite thanks to Lemma 3.3. From the previous analysis we conclude that the sequence $\{I_{2q-1}(f_{2q-1,\varepsilon})\}_{\varepsilon \in \mathbb{N}}$ is Cauchy in $L^2$, for any sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0,1]$ such that $\varepsilon_n \to 0$ as $n \to \infty$, which implies the desired result.

The next result is a central limit theorem for $I_{2q-1}(f_{2q-1,\varepsilon})$ in the case $\frac{3}{4} < H < \frac{4q-3}{4q-2}$.

**Theorem 4.3.** Let $T, \varepsilon > 0$ and $q \in \mathbb{N}$, $q \geq 2$ be fixed. Define $f_{2q-1,\varepsilon}$ by (2.6). Then, for every $\frac{3}{4} < H < \frac{4q-3}{4q-2}$ we have
\[
\varepsilon^{1-rac{3}{4H}} I_{2q-1}(f_{2q-1,\varepsilon}) \xrightarrow{\text{Law}} N(0, \sigma_q^2), \quad \text{when } \varepsilon \to 0,
\]
where $\sigma_q^2$ is the finite constant defined by (3.5).

**Proof.** Define $f_{2q-1,\varepsilon,s,t}$, for $0 \leq s \leq t$, by (2.3) and $R$ by (2.7). By (2.6),
\[
\varepsilon^{1-rac{3}{4H}} I_{2q-1}(f_{2q-1,\varepsilon}) = (-1)^q\varepsilon^{1-rac{3}{4H}} \int R \beta_q(\varepsilon + (t-s)^{2H})^{\frac{1}{2}-q} I_{2q-1}\left(\frac{1}{4}\right) \text{d}sdt.
\]
Then, using the self-similarity of the fractional Brownian motion we get
\[ \varepsilon^{-\frac{3}{2H}} I_{2q-1}(f_{2q-1,\varepsilon}) \]
\[ \xrightarrow{\text{Law}} (-1)^q \varepsilon^{-\frac{3}{2H}} \int_{\mathbb{R}} \beta_q(\varepsilon + (t-s)^{2H})^{-\frac{1}{2}-q} I_{2q-1} \left( \left( \sqrt{\varepsilon} \mathbb{1}_{\varepsilon^{-\frac{1}{2H}}[s,t]} \right)^{\otimes(2q-1)} \right) \, ds \, dt. \]

Therefore, changing the coordinates \((s, t)\) by \((\varepsilon^{-\frac{1}{2H}} s, \varepsilon^{-\frac{1}{2H}} t)\) we get
\[ \varepsilon^{-\frac{3}{2H}} I_{2q-1}(f_{2q-1,\varepsilon}) \]
\[ \xrightarrow{\text{Law}} (-1)^q \varepsilon^{-\frac{3}{2H}} \int_{\varepsilon^{-\frac{1}{2H}} \mathbb{R}} \beta_q(1 + (t-s)^{2H})^{-\frac{1}{2}-q} I_{2q-1} \left( \mathbb{1}_{[s,t]}^{\otimes(2q-1)} \right) \, ds \, dt \]
\[ = \varepsilon^{-\frac{3}{2H}} \int_{\varepsilon^{-\frac{1}{2H}} \mathbb{R}} I_{2q-1}(f_{2q-1,1,s,t}) \, ds \, dt. \] (4.5)

Changing the coordinates \((s, t)\) by \((s, u := t - s)\) in (4.5), and defining \(N := \varepsilon^{-\frac{1}{2H}}\), we obtain
\[ \varepsilon^{-\frac{3}{2H}} I_{2q-1}(f_{2q-1,\varepsilon}) \xrightarrow{\text{Law}} \frac{1}{\sqrt{N}} \int_{0}^{NT} \int_{0}^{NT-s} I_{2q-1}(f_{2q-1,1,s,s+u}) \, du \, ds. \] (4.6)

From (4.6) it follows that the convergence (4.3) is equivalent to
\[ \frac{1}{\sqrt{N}} \int_{0}^{NT} \int_{0}^{NT-s} I_{2q-1}(f_{2q-1,1,s,s+u}) \, du \, ds \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_q^2), \quad \text{as} \quad N \to \infty. \] (4.7)

The proof of (4.7) will be done in several steps.

**Step I**

Define the random variable
\[ Y_N := \frac{1}{\sqrt{N}} \int_{0}^{NT} \int_{NT-s}^{\infty} I_{2q-1}(f_{2q-1,1,s,s+u}) \, du \, ds. \]

First we show that \(Y_N\) converges to zero in \(L^2\) as \(N \to \infty\). Notice that
\[
\mathbb{E}\left[Y_N^2\right] = \frac{2}{N} \int_{0}^{NT} \int_{0}^{NT} \int_{NT-s}^{\infty} \int_{NT-s_1}^{\infty} \mathbb{1}_{\{s_1 \leq s_2\}} \times \mathbb{E}\left[I_{2q-1}\left(f_{2q-1,1,s_1,s_1+u_1}\right) I_{2q-1}\left(f_{2q-1,1,s_2,s_2+u_2}\right)\right] \, du_1 \, du_2 \, ds_1 \, ds_2
\]
\[ = \frac{2(2q-1)!}{N} \int_{0}^{NT} \int_{0}^{NT} \int_{NT-s_2}^{\infty} \int_{NT-s_1}^{\infty} \mathbb{1}_{\{s_1 \leq s_2\}} \times \left(f_{2q-1,1,s_1,s_1+u_1}, f_{2q-1,1,s_2,s_2+u_2}\right)^{\otimes(2q-1)} \, du_1 \, du_2 \, ds_1 \, ds_2. \] (4.8)
Define the function \( G^{(q)}_{1,x}(v_1, v_2) \), \( x, v_1, v_2 \geq 0 \), as in (2.20). Substituting equation (2.19) in (1.8), and changing the order of integration, we get

\[
\mathbb{E}[Y^2_N] = \frac{2 (2q - 1)! \beta^2}{N} \int_0^\infty \int_0^\infty \int_0^{NT} \int_0^{NT} G^{(q)}_{1,x}(u_1, u_2) ds_1 ds_2 du_1 du_2.
\]

Changing the coordinates \((s_1, s_2, u_1, u_2)\) by \((s_1, x \equiv s_2 - s_1, u_1, u_2)\) in the right hand side of (4.9), we get

\[
\mathbb{E}[Y^2_N] \leq 2 (2q - 1)! \beta^2 \int_\mathbb{R}_+ \int_0^{NT} G^{(q)}_{1,x}(u_1, u_2) ds_1 dx du_1 du_2.
\]

and then integrating the \(s_1\) variable,

\[
\mathbb{E}[Y^2_N] \leq 2 (2q - 1)! \beta^2 \int_\mathbb{R}_+ \left( T - \frac{0 \vee (NT - u_1)}{N} \right) G^{(q)}_{1,x}(u_1, u_2) dx du_1 du_2. \tag{4.10}
\]

The integrand in (4.10) converges to zero pointwise, and is dominated by the function

\[
2 (2q - 1)! \beta^2 T G^{(q)}_{1,x}(u_1, u_2).
\]

By condition \( H < \frac{4q^3 - 3}{4q^2 - 2} \) and Lemma 5.8, the function \( G^{(q)}_{1,x}(u_1, u_2) \) is integrable in \( \mathbb{R}_+^2 \). Hence, applying the dominated convergence theorem to (4.10), we obtain \( \mathbb{E}[Y^2_N] \to 0 \), as \( N \to \infty \) as required.

**Step II**

Since \( Y_N \to 0 \) in \( L^2 \) as \( N \to \infty \), to prove the convergence (4.7) it suffices to show that the random variable

\[
J_{2q-1,N} := \frac{1}{\sqrt{N}} \int_0^{NT} I_{2q-1} \left( f_{2q-1,1,s,s+u} \right) du ds,
\]

converges in law to a Gaussian distribution with variance \( \sigma_q^2 \) as \( N \to \infty \). For \( M \in \mathbb{N}, M \geq 1 \) fixed, consider the following Riemann sum approximation for \( J_{2q-1,N} \)

\[
\tilde{J}_{2q-1,M,N} := \frac{1}{2^M} \sum_{k=2}^{M^2} \frac{1}{\sqrt{N}} \int_0^{NT} I_{2q-1} \left( f_{2q-1,1,s,s+u(k)} \right) ds,
\]

where \( u(k) := \frac{k}{2^M}, \) for \( k = 2, \ldots, M^2 \). We will prove that \( \tilde{J}_{2q-1,M,N} \to J_{2q-1,N} \) in \( L^2 \) as \( M \to \infty \) uniformly in \( N > 1 \), and \( \tilde{J}_{2q-1,M,N} \to \mathcal{N}(0, \tilde{\sigma}_q^2 \alpha_{q,M}) \) as \( N \to \infty \) for some constant \( \tilde{\sigma}_q \alpha_{q,M} \) satisfying \( \tilde{\sigma}_q \alpha_{q,M} \to \sigma_q^2 \) as \( M \to \infty \). The result will then follow by a standard approximation argument. We will separate the argument in the following steps.
Step III

Next we prove that prove that \( \tilde{J}_{2q-1,M,N} \to J_{2q-1,N} \) in \( L^2 \) as \( M \to \infty \) uniformly in \( N > 1 \), namely,

\[
\lim_{M \to \infty} \sup_{N > 1} \left\| J_{2q-1,N} - \tilde{J}_{2q-1,M,N} \right\|_{L^2} = 0. \tag{4.11}
\]

For \( M \in \mathbb{N} \) fixed, we decompose the term \( J_{2q-1,N} \) as

\[ J_{2q-1,N} = J_{2q-1,N}^{(1)} + J_{2q-1,N}^{(2)}, \tag{4.12} \]

where

\[
J_{2q-1,M,N}^{(1)} := \frac{1}{\sqrt{N}} \int_0^{NT} \int_{2-M}^M (f_{2q-1,1,s,s+u}) \, du \tag{4.13}
\]

and

\[
J_{2q-1,M,N}^{(2)} := \frac{1}{\sqrt{N}} \int_0^{NT} \int_0^\infty 1_{(0,2^{-M}) \cup (M,\infty)}(u) f_{2q-1,1,s,s+u) \, du. \tag{4.14}
\]

From (4.12) we deduce that relation (4.11) is equivalent to

\[
\lim_{M \to \infty} \sup_{N > 1} \left\| J_{2q-1,N}^{(1)} - \tilde{J}_{2q-1,M,N} \right\|_{L^2} = 0,
\]

provided that

\[
\lim_{M \to \infty} \sup_{N > 1} \left\| J_{2q-1,N}^{(2)} \right\|_{L^2} = 0. \tag{4.15}
\]

To prove (4.14) we proceed as follows. First we write

\[
\left\| J_{2q-1,M,N}^{(2)} \right\|_{L^2}^2 = \frac{2(2q-1)!}{N} \int_{\mathbb{R}^2} \int_{[0,NT]^2} 1_{(0,2^{-M}) \cup (M,\infty)}(u_1) 1_{(0,2^{-M}) \cup (M,\infty)}(u_2) \times 1\{s_1 \leq s_2\} \langle f_{2q-1,1,s_1+u_1}, f_{2q-1,1,s_2,s_2+u_2} \rangle_{S^0(2q-1)} \, ds_1 ds_2 du_1 du_2. \tag{4.16}
\]

Let \( G_{1,v}(v_1, v_2), x, v_1, v_2 \in \mathbb{R}_+ \) be defined by (2.20). Applying identity (2.19) in (4.15), and then changing the coordinates \( (s_1, s_2, u_1, u_2) \) by \( (s_1, x := s_2 - s_1, u_1, u_2) \) in (4.15), we get

\[
\left\| J_{2q-1,M,N}^{(2)} \right\|_{L^2}^2 \leq \frac{2(2q-1)! \beta_2^2}{N} \int_{\mathbb{R}^4} \int_0^{NT} 1_{(0,2^{-M}) \cup (M,\infty)}(u_1) \times 1_{(0,2^{-M}) \cup (M,\infty)}(u_2) G_{1,x}(u_1, u_2) \, ds_1 dx du_1 du_2. \tag{4.17}
\]
Integrating the variable $s_1$ in (4.10) we obtain
\[
\left\| J_{2q-1,M,N}^{(2)} \right\|_{L^2}^2 \leq 2T(2q-1)! \beta_q^2 \int_{\mathbb{R}_+^3} \mathbb{I}_{(0,2^{-M})\cup(M,\infty)}(u_2) \times \mathbb{I}_{(0,2^{-M})\cup(M,\infty)}(u_2) G_{1,x}^{(q)}(u_1, u_2) dx du_1 du_2. 
\] (4.17)

The integrand is dominated by the function $2(2q-1)! \beta_q^2 T G_{1,x}^{(q)}(u_1, u_2)$, which is integrable by the condition $H < \frac{2q-3}{4q-2}$, and Lemma 5.8. Hence, applying the dominated convergence theorem to (4.17), we get (4.14).

To prove (4.13) we proceed as follows. For $k = 2, \ldots, M2^M$ define the interval $I_k := (\frac{k-1}{2^M}, \frac{k}{2^M}]$. Notice that $J_{2q-1,M,N}^{(1)}$ and $\tilde{J}_{2q-1,M,N}$ can be written, respectively, as
\[
J_{2q-1,M,N}^{(1)} = \frac{1}{\sqrt{N}} \int_0^{NT} \int_{\mathbb{R}_+^M} I_{2q-1} (f_{2q-1,1,s,s+u}) \mathbb{I}_{I_k}(u) duds, 
\] (4.18)

and
\[
\tilde{J}_{2q-1,M,N} = \frac{1}{\sqrt{N}} \int_0^{NT} \int_{\mathbb{R}_+^M} I_{2q-1} (f_{2q-1,1,s,s+u(k)}) \mathbb{I}_{I_k}(u) duds. 
\] (4.19)

Applying (2.19), we can prove that
\[
\left\| J_{2q-1,M,N}^{(1)} - \tilde{J}_{2q-1,M,N} \right\|_{L^2}^2 = \frac{2(2q-1)! \beta_q^2}{N} \int_{\mathbb{R}_+^3} \int_{[0,NT]^2} \sum_{k_1,k_2=2}^{M2^M} \mathbb{I}_{I_{k_1}}(u_1) \mathbb{I}_{I_{k_2}}(u_2) \times \mathbb{I}_{(s_1 \leq s_2)} \Theta_{k_1,k_2}^{(q)} (s_2 - s_1, u_1, u_2) ds_1 ds_2 du_1 du_2, 
\] (4.20)

where the function $\Theta_{k_1,k_2}^{(q)}$ is defined by
\[
\Theta_{k_1,k_2}^{(q)}(x, u_1, u_2) := \left( G_{1,x}^{(q)}(u_1, u_2) - G_{1,x}^{(q)}(u(k_1), u_2) - G_{1,x}^{(q)}(u(1), u(k_2)) + G_{1,x}^{(q)}(u(k_1), u(k_2)) \right). 
\]

Changing the coordinates $(s_1, s_2, u_1, u_2)$ by $(s_1, x := s_2 - s_1, u_1, u_2)$, and then integrating the $s_1$ variable in (4.20), we obtain
\[
\left\| J_{2q-1,M,N}^{(1)} - \tilde{J}_{2q-1,M,N} \right\|_{L^2}^2 = \frac{2(2q-1)! \beta_q^2}{N} \int_{\mathbb{R}_+^3} \int_0^{NT} \sum_{k_1,k_2=2}^{M2^M} \mathbb{I}_{I_{k_1}}(u_1) \mathbb{I}_{I_{k_2}}(u_2) \times \left( T - \frac{x}{N} \right) \Theta_{k_1,k_2}^{(q)}(x, u_1, u_2) dx du_1 du_2. 
\]
As a consequence,
$$\left\| J_{2q-1,M,N}^{(1)} - \tilde{J}_{2q-1,M,N} \right\|_{L^2}^2 \leq 2(2q - 1)! \beta_2^2 T \int_{\mathbb{R}^3_+} \sum_{k_1,k_2=2}^{M^2} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \times \Theta_{k_1,k_2}^{(q)}(x,u_1,u_2) \, dx \, du_1 \, du_2.$$ 

By the continuity of $G_{1,x}(u_1,u_2)$, the term
$$\sum_{k_1,k_2=2}^{M^2} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \Theta_{k_1,k_2}^{(q)}(x,u_1,u_2)$$
converges to zero as $M \to \infty$. Next we prove that this term is dominated by an integrable function. Let $u_1 \in I_{k_1}, u_2 \in I_{k_2}$ be fixed. Notice that $u_i, u(k_i) \leq u_i + 2^{-M} \leq u_i + 1$ for $i = 1, 2$. Hence, applying Lemma 2.1, we deduce that the terms $G_{1,x}^{(q)}(u_1,u_2), G_{1,x}^{(q)}(u(k_1),u(k_2))$ and $G_{1,x}^{(q)}(u(k_1), u(k_2))$ are bounded by $K G_{1,x}^{(q)}(u_1+1,u_2+1)$, for some constant $K > 0$ only depending on $H$ and $q$. As a consequence,
$$\sum_{k_1,k_2=2}^{M^2} \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2) \Theta_{k_1,k_2}^{(q)}(x,u_1,u_2) \leq 4K G_{1,x}^{(q)}(u_1+1,u_2+1),$$
for some constant $K$ only depending on $H$ and $q$. Therefore, the right-hand side of the previous identity is integrable over $x, u_1, u_2 > 0$ due to Lemma 5.8 since
$$\int_{\mathbb{R}^3_+} G_{1,x}^{(q)}(u_1+1,u_2+1) \, dx \, du_1 \, du_2 = \int_{[1,\infty)^2} G_{1,x}^{(q)}(u_1,u_2) \, dx \, du_1 \, du_2$$
$$\leq \int_{\mathbb{R}^3_+} G_{1,x}^{(q)}(u_1,u_2) \, dx \, du_1 \, du_2 < \infty. \quad (4.21)$$

This finishes the proof of (4.13).

**Step IV**

Next we prove that
$$\lim_{N \to \infty} \mathbb{E} \left[ \tilde{J}_{2q-1,M,N}^2 \right] = \tilde{\sigma}_{q,M}^2, \quad (4.22)$$
where $\tilde{\sigma}_{q,M}^2$ is the finite constant defined by
$$\tilde{\sigma}_{q,M}^2 := (2q - 1)! \beta_2^2 2^{1-2M} T \sum_{k_1,k_2=2}^{M^2} \int_0^\infty G_{1,x}^{(q)}(u(k_1), u(k_2)) \, dx. \quad (4.23)$$

In addition, we will prove that $\tilde{\sigma}_{q,M}^2$ satisfies
$$\lim_{M \to \infty} \tilde{\sigma}_{q,M}^2 = \sigma_q^2, \quad (4.24)$$
where $\sigma^2_q$ is defined by (3.5). In order to prove (4.22) and (4.24) we proceed as follows. From (4.19), we can prove that

$$
\mathbb{E} \left[ J_{2q-1,M,N}^2 \right] = \int_{\mathbb{R}^3} Q_{M,N}(x, u_1, u_2) dx \, du_1 \, du_2,
$$

where

$$
Q_{M,N}(x, u_1, u_2) := 2(2q - 1)! \, \beta^2_q T \sum_{k_1, k_2 = 2}^{M2^M} \left( T - \frac{x}{N} \right) \, G^{(q)}_{1,x}(u(k_1), u(k_2)) \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2).
$$

Notice that $Q_{M,N}$ satisfies

$$
\lim_{N \to \infty} Q_{M,N}(x, u_1, u_2) = Q_M(x, u_1, u_2), \tag{4.25}
$$

where $Q_M$ is defined by

$$
Q_M(x, u_1, u_2) := 2(2q - 1)! \beta^2_q T \sum_{k_1, k_2 = 2}^{M2^M} \left( T - \frac{x}{N} \right) \, G^{(q)}_{1,x}(u(k_1), u(k_2)) \mathbb{1}_{I_{k_1}}(u_1) \mathbb{1}_{I_{k_2}}(u_2).
$$

In turn, $Q_M$ satisfies

$$
\lim_{M \to \infty} Q_M(x, u_1, u_2) = Q(x, u_1, u_2), \tag{4.26}
$$

where $Q$ is defined by

$$
Q(x, u_1, u_2) := 2(2q - 1)! \beta^2_q T G^{(q)}_{1,x}(u_1, u_2).
$$

Let $x > 0$ and $2 \leq k_1, k_2 \leq M2^M$ be fixed, and take $u_i \in I_{k_i}$, $i = 1, 2$. Since $u(k_i) \leq u_i + 2^{-M} \leq u_i + 1$, by Lemma 2.1 there exists a constant $K > 0$, only depending on $q$ and $H$, such that

$$
G^{(q)}_{1,x}(u(k_1), u(k_2)) \leq KG^{(q)}_{1,x}(u_1 + 1, u_2 + 1),
$$

As a consequence, there exists a constant $K$ only depending on $q, H$ and $T$ such that

$$
Q_{M,N}(x, u_1, u_2) \leq KG^{(q)}_{1,x}(u_1 + 1, u_2 + 1), \tag{4.27}
$$

and, hence,

$$
Q_M(x, u_1, u_2) \leq KG^{(q)}_{1,x}(u_1 + 1, u_2 + 1). \tag{4.28}
$$
The function \( G_{1,x}^{(q)}(u_1 + 1, u_2 + 1) \) is integrable with respect to the variables \( x, u_1, u_2 > 0 \) thanks to (4.21). Hence, taking into account (4.25) and (4.26), as well as the estimates (4.27) and (4.28), we can apply the dominated convergence theorem twice, to obtain

\[
\lim_{M \to \infty} \lim_{N \to \infty} E\left[J_{2q-1,M,N}^2\right] = \lim_{M \to \infty} \int_{\mathbb{R}_+^3} Q_M(x, u_1, u_2) dx du_1 du_2 = \int_{\mathbb{R}_+^3} Q(x, u_1, u_2) dx du_1 du_2. \tag{4.29}
\]

Equations (4.22) and (4.24) then follow from (4.29).

**Step V**

Next we prove the convergence in law of \( J_{2q-1,N} \) to a Gaussian random variable with variance \( \sigma_q^2 \), which we will denote by \( \mathcal{N}(0, \sigma_q^2) \). Let \( y \in \mathbb{R} \) be fixed. Notice that

\[
|\mathbb{P}[J_{2q-1,N} \leq y] - \mathbb{P}[\mathcal{N}(0, \sigma_q^2) \leq y]| \leq \sup_{N>1} |\mathbb{P}[J_{2q-1,N} \leq y] - \mathbb{P}[\tilde{J}_{2q-1,M,N} \leq y]|
+ |\mathbb{P}[\tilde{J}_{2q-1,M,N} \leq y] - \mathbb{P}[\mathcal{N}(0, \sigma_{q,M}^2) \leq y]|
+ |\mathbb{P}[\mathcal{N}(0, \sigma_{q,M}^2) \leq y] - \mathbb{P}[\mathcal{N}(0, \sigma_q^2) \leq y]|. \tag{4.30}
\]

Therefore, if we prove that for \( M > 0 \) fixed

\[
\tilde{J}_{2q-1,M,N} \xrightarrow{Law} \mathcal{N}(0, \sigma_{q,M}^2) \quad \text{as} \quad N \to \infty, \tag{4.31}
\]

then from (4.30) we get

\[
\limsup_{N \to \infty} |\mathbb{P}[J_{2q-1,N} \leq y] - \mathbb{P}[\mathcal{N}(0, \sigma_q^2) \leq y]| \leq \sup_{N>1} |\mathbb{P}[J_{2q-1,N} \leq y] - \mathbb{P}[\tilde{J}_{2q-1,M,N} \leq y]|
+ |\mathbb{P}[\tilde{J}_{2q-1,M,N} \leq y] - \mathbb{P}[\mathcal{N}(0, \sigma_{q,M}^2) \leq y]|
+ |\mathbb{P}[\mathcal{N}(0, \sigma_{q,M}^2) \leq y] - \mathbb{P}[\mathcal{N}(0, \sigma_q^2) \leq y]|, \tag{4.32}
\]

and hence, from relations (4.11), (4.24) and (4.32), we conclude that

\[
\limsup_{N \to \infty} |\mathbb{P}[J_{2q-1,N}^2 \leq y] - \mathbb{P}[\mathcal{N}(0, \sigma_q^2) \leq y]| = 0, \tag{4.33}
\]

and the proof will then be complete. Therefore, it suffices to show (4.31) for \( M \) fixed. To prove this first we show that the random vector

\[
Z^{(N)} = \left( Z^{(N)}_{k,N} \right)_{k=2}^{M^2} = \left( \frac{1}{\sqrt{N}} \int_0^{NT} J_{2q-1} f_{2q-1,1,s,s+u(k)} ds \right)_{k=2}^{M^2}
\]

converges to a multivariate Gaussian distribution. By the Peccati-Tudor criterion (see [2]), it suffices to prove that the components of the vector \( Z^{(N)} \) converge to a Gaussian distribution, and the covariance matrix of \( Z^{(N)} \) is convergent.
In order to prove that the covariance matrix of $Z^{(N)}$ is convergent we proceed as follows. First, for $2 \leq j, k \leq M 2^M$, we write

$$
E \left[ Z_k^{(N)} Z_j^{(N)} \right] = \frac{1}{N} \int_{[0,NT]^2} E \left[ I_{2q-1} \left( f_{2q-1,1,s_1,s_1+u(k)} \right) I_{2q-1} \left( f_{2q-1,1,s_2,s_2+u(j)} \right) \right] ds_1 ds_2.
$$

Then, using (2.19) we get

$$
E \left[ Z_k^{(N)} Z_j^{(N)} \right] = \frac{(2q - 1)! \beta_q^2}{N} \int_{[0,NT]^2} G_{1,s_2-s_1}^{(q)}(u(k), u(j)) ds_1 ds_2,
$$

where in the last equality we used the notation $G_{1,y}(v_1, v_2) := G_{1,y}(v_2, v_1)$, for $y, v_1, v_2 > 0$. Changing the coordinates $(s_1, s_2)$ by $(s_1, x := s_2 - s_1)$ in relation (4.34) and integrating the $s_1$, yields

$$
E \left[ Z_k^{(N)} Z_j^{(N)} \right] = (2q - 1)! \beta_q^2 \int_{-NT}^{NT} \left( T - \frac{|x|}{N} \right) G_{1,x}^{(q)}(u(k), u(j)) dx.
$$

Finally, applying the monotone convergence theorem in (4.35), we get

$$
\lim_{N \to \infty} E \left[ Z_k^{(N)} Z_j^{(N)} \right] = (2q - 1)! \beta_q^2 T \int_{\mathbb{R}} G_{1,x}^{(q)}(u(k), u(j)) dx,
$$

which is clearly finite. Thus, we have proved that the covariance matrix of $Z^{(N)}$ converges to the matrix $\Sigma = (\Sigma_{k,j})_{2 \leq k, j \leq M 2^M}$, where

$$
\Sigma_{k,j} := T(2q - 1)! \beta_q^2 \int_{\mathbb{R}} G_{1,x}^{(q)}(u(k), u(j)) dx.
$$

Next, for $2 \leq k \leq M 2^M$ fixed, we prove the convergence of $Z_k^{(N)}$ to a Gaussian law. By (2.3),

$$
Z_k^{(N)} = \frac{C_{q,k}}{\sqrt{N}} \int_0^{NT} I_{2q-1} \left( \frac{1}{s_1 \wedge s_2 + u_k} \right) ds,
$$

where $C_{q,k} = (-1)^q \beta_q (1 + u_k^{2H})^{-\frac{1}{2}}$. Hence, by the self-similarity of the fractional Brownian motion we can write

$$
Z_k^{(N)} \overset{\text{Law}}{=} \frac{C_{q,k}}{\sqrt{N}} \int_0^{NT} I_{2q-1} \left( \left( u_k^H \mathbb{1}_{[\frac{r}{N}]} \right)^{\otimes (2q-1)} \right) ds.
$$

Making the change of variables $r := \frac{s_1 + s_2}{N u_k}$ in the right hand side of (4.36), we get

$$
Z_k^{(N)} \overset{\text{Law}}{=} \frac{C_{q,k}}{\sqrt{N}} \int_0^{\frac{2q}{N u_k}} I_{2q-1} \left( \left( u_k^H \mathbb{1}_{[\frac{r}{N}]} \right)^{\otimes (2q-1)} \right) dr
$$

$$
= C_{q,k} u_k^{H(q-1)+1} \sqrt{N} \int_0^{\frac{2q}{N u_k}} H_{2q-1} \left( N^H (B_{r+\frac{1}{N}} - B_r) \right) dr.
$$

(4.37)
where $H_{2q-1}$ denotes the Hermite polynomial of degree $2q - 1$. The convergence in law of the right-hand side of (4.37) to a centered Gaussian distribution as $N \to \infty$ is proven in [2], equation (1.3). As a consequence, the components of $Z^{(N)}$ converge to a Gaussian random variable as $N \to \infty$. Therefore, by the Peccati-Tudor criterion, $Z^{(N)}$ converges in law to a centered Gaussian distribution with covariance $\Sigma$. Hence,

$$\tilde{J}_{2q-1,M,N} = \frac{1}{2^{2M}} \sum_{k=2}^{M^2} Z_k^{(N)} \xrightarrow{\text{Law}} N \left( 0, \frac{1}{2^{2M}} \sum_{j,k=2}^{M^2} \Sigma_{k,j} \right) \quad \text{as } N \to \infty. \quad (4.38)$$

The convergence (4.31) follows from (4.38) by using the fact that

$$\frac{1}{2^{2M}} \sum_{k,j=2}^{M^2} \Sigma_{k,j} = T(2q - 1)! \beta_q 2^{-2M} \sum_{j,k=2}^{M^2} \int_{\mathbb{R}} G_{1,q}(u(k), u(j)) dx = \tilde{\sigma}_{q,M}.$$ 

The proof is now complete. \qed

5 Technical lemmas

In this section we prove several technical results that were used to determine the asymptotic behavior of the variance of $I_{2q-1}(f_{2q-1}, \varepsilon)$ and $\alpha_\varepsilon$. In Lemma 5.1 we provide an alternative expression for the terms $V_i(\varepsilon), i = 1, 2, 3$ defined in (2.14). In Lemma 5.2 we prove some useful bounds that we will use later to estimate the covariance of $p_\varepsilon(B_{t_1} - B_{s_1})$ and $p_\varepsilon(B_{t_2} - B_{s_2}), s_1 \leq t_1, s_2 \leq t_2$ and $s_1 \leq s_2$. In Lemmas 5.3 and 5.4 we estimate the order of $V_1(\varepsilon)$ and $V_3(\varepsilon)$ when $\varepsilon$ is small, while in Lemmas 5.6 and 5.7 we determine the exact behavior of $V_3(\varepsilon)$ and $V_3^{(1)}(\varepsilon)$ as $\varepsilon \to 0$. Finally, we prove Lemmas 5.9 and 5.8, which were used in Lemma 3.3 to determine the behavior of the variance of $I_{2q-1}(f_{2q-1}, \varepsilon)$ for $q \geq 2$.

In what follows, $I$ will denote the identity matrix of dimension 2. In addition, for every square matrix $A$ of dimension 2, we will denote by $|A|$ its determinant.

**Lemma 5.1.** Let $\varepsilon > 0$ be fixed. Define $S_1, S_2, S_3$ by (2.10), (2.11), (2.12) respectively, and $V_1(\varepsilon), V_2(\varepsilon), V_3(\varepsilon)$ by (2.14). Then, for $i = 1, 2, 3$, we have

$$V_i(\varepsilon) = \frac{1}{\pi} \int_{S_i} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} ds_1 ds_2 dt_1 dt_2, \quad (5.1)$$

where $\Sigma = (\Sigma_{i,j})_{i,j=1,2}$ is the covariance matrix of $(B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2})$.

**Proof.** Let $(X, Y)$ be a jointly Gaussian vector with mean zero, covariance $\Sigma = (\Sigma_{i,j})_{i,j=1,2}$, and density $f_\Sigma(x, y)$. First we prove that for every $\theta > 0$,

$$\mathbb{E} [XY p_\theta(X)p_\theta(Y)] = (2\pi)^{-1} \theta^2 |\theta I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2}. \quad (5.2)$$
To prove this, notice that

\[ \mathbb{E} \left[ XY \rho_0(X) \rho_0(Y) \right] = \int_{\mathbb{R}^2} xy \rho_0(x) \rho_0(y) f_{\Sigma}(x, y) \, dx \, dy \]

\[ = (2\pi)^{-2} \theta^{-1} |\Sigma|^{-\frac{1}{2}} \int_{\mathbb{R}^2} xy \exp \left\{ -\frac{1}{2} (x, y) \left( \theta^{-1} I + \Sigma^{-1} \right) (x, y)^T \right\} \, dx \, dy \]

\[ = (2\pi)^{-1} \theta^{-1} |\Sigma|^{-\frac{1}{2}} |\theta^{-1} I + \Sigma^{-1}|^{-\frac{1}{2}} \int_{\mathbb{R}^2} xy \, f_{\tilde{\Sigma}}(x, y) \, dx \, dy, \quad (5.3) \]

where \( \tilde{\Sigma} := (\theta^{-1} I + \Sigma^{-1})^{-1} \) and \( f_{\tilde{\Sigma}}(x, y) \) denotes the density of a Gaussian vector with mean zero and covariance \( \tilde{\Sigma} \). Clearly, \( \theta^{-1} |\Sigma|^{-\frac{1}{2}} |\theta^{-1} I + \tilde{\Sigma}^{-1}|^{-\frac{1}{2}} = |\theta I + \Sigma|^{-\frac{1}{2}} \). Then, substituting this identity in (5.3), we get

\[ \mathbb{E} \left[ XY \rho_0(X) \rho_0(Y) \right] = (2\pi)^{-1} |\theta I + \Sigma|^{-\frac{1}{2}} \int_{\mathbb{R}^2} xy \, f_{\tilde{\Sigma}}(x, y) \, dx \, dy \]

\[ = (2\pi)^{-1} |\theta I + \Sigma|^{-\frac{1}{2}} \tilde{\Sigma}_{1,2}. \]

Taking into account that \( \tilde{\Sigma}_{1,2} \) is given by

\[ \tilde{\Sigma}_{1,2} = \theta^2 |\theta I + \Sigma|^{-1} \Sigma_{1,2}, \]

we conclude that

\[ \mathbb{E} \left[ XY \rho_0(X) \rho_0(Y) \right] = (2\pi)^{-1} \theta^2 |\theta I d + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2}, \]

as required. From (5.2), we can write

\[ V_t(\varepsilon) = 2 \int_{S_i} \mathbb{E} \left[ p_\varepsilon(B_t_1 - B_s_1) p_\varepsilon(B_t_2 - B_s_2) \right] \, ds_1 ds_2 dt_1 dt_2 \]

\[ = 2 \int_{S_i} \mathbb{E} \left[ (B_t_1 - B_s_1)(B_t_2 - B_s_2) p_\varepsilon(B_t_1 - B_s_1) p_\varepsilon(B_t_2 - B_s_2) \right] \, ds_1 ds_2 dt_1 dt_2 \]

\[ = \frac{1}{\pi} \int_{S_i} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} \, ds_1 ds_2 dt_1 dt_2. \]

This finishes the proof of (5.1). \( \square \)

**Lemma 5.2.** Let \( s_1, s_2, t_1, t_2 \in \mathbb{R}_+ \) be such that \( s_1 \leq s_2 \), and \( s_i \leq t_i \) for \( i = 1, 2 \). Denote by \( \Sigma \) the covariance matrix of \( (B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2}) \). Then, if \( s_1 < s_2 < t_2 < t_1 \), or \( s_1 < t_1 < s_2 < t_2 \), there exists \( 0 < \delta < 1 \) such that

\[ |\Sigma| \geq \delta (t_1 - s_1)^{2H} (t_2 - s_2)^{2H}. \quad (5.4) \]

In addition, if \( s_1 < s_2 < t_2 < t_1 \), then there exists \( 0 < \delta < 1 \) such that

\[ |\Sigma| \geq \delta ((a + b)^{2H} c^{2H} + (b + c)^{2H} a^{2H}), \quad (5.5) \]

where \( a := s_2 - s_1, \ b := t_1 - s_2, \) and \( c := t_2 - t_1. \)
Proof. The result follows from the local non-determinism property of the fractional Brownian motion (see [5], Lemma 9).

Lemma 5.3. Let \( \varepsilon > 0 \) and define \( V_1(\varepsilon) \) by (2.14). Then, for every \( \frac{2}{3} < H < 1 \) we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{3 - \frac{2}{H}} V_1(\varepsilon) = 0. \tag{5.6}
\]

Proof. Changing the coordinates \((s_1, s_2, t_1, t_2)\) by \((s_1, a := s_2 - s_1, b := t_1 - s_2, c := t_2 - t_1)\) in (5.1), we get

\[
V_1(\varepsilon) \leq \frac{1}{\pi} \int_{[0,T]^4} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} ds_1 da db dc, \tag{5.7}
\]

where \( \Sigma \) denotes the covariance matrix of \((B_{a+b}, B_{a+b+c} - B_a)\), namely,

\[
\Sigma_{1,1} = (a + b)^{2H}, \tag{5.8}
\]

\[
\Sigma_{2,2} = (c + b)^{2H}, \tag{5.9}
\]

\[
\Sigma_{1,2} = \frac{1}{2} ((a + b + c)^{2H} + b^{2H} - c^{2H} - a^{2H}). \tag{5.10}
\]

Integrating the \( s_1 \) variable in (5.7) we obtain

\[
V_1(\varepsilon) \leq T \cdot \frac{1}{\pi} \int_{[0,T]^3} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} da db dc. \tag{5.11}
\]

Next we bound the right-hand side of (5.11). Applying (5.5), (5.8), (5.9) and (5.10), we get

\[
|\varepsilon I + \Sigma| = (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 = \varepsilon^2 + \varepsilon \Sigma_{1,1} + \varepsilon \Sigma_{2,2} + |\Sigma| \\
\geq \delta (\varepsilon^2 + \varepsilon(a + b)^{2H} + \varepsilon(b + c)^{2H} + (a + b)^{2H}e^{2H} + (b + c)^{2H}a^{2H}), \tag{5.12}
\]

for some \( \delta > 0 \) only depending on \( H \). Using the inequality \( \Sigma_{1,2} \leq (a + b)^H(b + c)^H \), as well as (5.11) and (5.12), we deduce that there exists a constant \( K \) only depending on \( T, H \) such that

\[
V_1(\varepsilon) \leq K \int_{[0,T]^3} \frac{(a + b)^H(b + c)^H}{\Theta_\varepsilon(a, b, c)^{\frac{3}{2}}} da db dc, \tag{5.13}
\]

where the function \( \Theta_\varepsilon \) is defined by

\[
\Theta_\varepsilon(a, b, c) := \varepsilon^2 + \varepsilon(a + b)^{2H} + \varepsilon(b + c)^{2H} + c^{2H}(a + b)^{2H} + a^{2H}(b + c)^{2H}. \tag{5.14}
\]

By the arithmetic mean-geometric mean inequality, we have

\[
\frac{1}{2} ((a + b)^{2H} + (b + c)^{2H}) \geq (a + b)^H(b + c)^H,
\]

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and
\[ \frac{1}{2} (c^2 H (a + b)^2 + a^2 H (b + c)^2 H) \geq (a + b)^H (b + c)^H (ac)^H. \]

Consequently,
\[ \Theta \geq 2(a + b)^H (b + c)^H (\varepsilon + (ac)^H). \]

Therefore, by (5.13) there exists a constant \( K > 0 \) only depending on \( T \) and \( H \) such that
\[ V_1(\varepsilon) \leq K \int_{[0,T]^3} (a + b)^{-H/2} (b + c)^{-H/2} (\varepsilon + (ac)^H)^{-3/2} dadbdc \]
\[ \leq K \int_{[0,T]^3} b^{-H} (\varepsilon + (ac)^H)^{-3/2} dadbdc. \]

Let \( 0 < y < \frac{3H}{2} - 1 \) be fixed, and define \( \gamma := \frac{2y}{3H} + 1 - \frac{2}{3H} \). By the weighted arithmetic mean-geometric mean inequality, we have
\[ \gamma \varepsilon + (1 - \gamma)(ac)^H \geq \varepsilon \gamma (ac)^{(1 - \gamma)H}. \]

Hence, by (5.15), we get
\[ \varepsilon^3 - \frac{2}{3} V_1(\varepsilon) \leq K \varepsilon^3 - \frac{2}{3} \int_{[0,T]^3} b^{-H} (ac)^{-\frac{3}{2}(1 - \gamma)H} dadbdc \]
\[ = K \varepsilon^3 - \frac{1}{3} \pi \left( \int_{0}^{T} b^{-H} db \right) \left( \int_{[0,T]^2} (ac)^{-1 + y} dadbdc \right). \]

This implies that (5.16) holds and the proof of the lemma is complete. \( \square \)

**Lemma 5.4.** Let \( \varepsilon > 0 \) be fixed. Define \( V_2(\varepsilon) \) by (2.14). Then, for every \( \frac{2}{3} < H < 1 \),
\[ \lim_{\varepsilon \to 0} \varepsilon^3 - \frac{2}{3} V_2(\varepsilon) = 0. \]

**Proof.** Changing the coordinates \((s_1, s_2, t_1, t_2)\) by \((s_1, a := s_2 - s_1, b := t_2 - s_2, t_1 - t_2)\) in (5.1) for \( i = 2 \), and integrating \( s_1 \), we obtain, as before
\[ V_2(\varepsilon) \leq \frac{T}{\pi} \int_{[0,T]^3} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} dadbdc, \]

where the matrix \( \Sigma \) is given by
\[ \Sigma_{1,1} = (a + b + c)^{2H}, \]
\[ \Sigma_{2,2} = b^{2H}, \]
\[ \Sigma_{1,2} = \frac{1}{2}((a + b)^{2H} + (b + c)^{2H} - c^{2H} - a^{2H}). \]
Using relation (5.11) in Lemma 5.2 as well as (5.18), (5.19) and (5.20), we get
\[ |\varepsilon I + \Sigma| = (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2} = \varepsilon^2 + \varepsilon(\Sigma_{1,1} + \Sigma_{2,2}) + |\Sigma| \geq \varepsilon^2 + \varepsilon((a + b + c)^{2H} + b^{2H}) + \delta(a + b + c)^{2H}b^{2H}. \]  
(5.21)

From (5.17) and (5.21) we deduce that there exists a constant \( K > 0 \), only depending on \( T \) and \( H \), such that
\[ V_2(\varepsilon) \leq K \int_{[0,T]^3} \frac{\Sigma_{1,2}}{(\varepsilon^2 + \varepsilon(b^{2H} + (a + b + c)^{2H}) + b^{2H}(a + b + c)^{2H})^{\frac{3}{2}}} dadbdc. \]  
(5.22)

The term \( \Sigma_{1,2} \) can be written as
\[ \Sigma_{1,2} = \frac{1}{2}((a + b)^{2H} + (b + c)^{2H} - a^{2H} - c^{2H}) = Hb \int_0^1 ((a + bv)^{2H-1} + (c + bv)^{2H-1}) dv, \]
which implies
\[ \Sigma_{1,2} \leq 2Hb(a + b + c)^{2H-1}. \]  
(5.23)

From (5.22) and (5.23), we deduce that there exists a constant \( K > 0 \) only depending on \( T \) and \( H \), such that
\[ V_2(\varepsilon) \leq K \int_{[0,T]^3} \frac{b(a + b + c)^{2H-1}}{(\varepsilon^2 + \varepsilon(b^{2H} + (a + b + c)^{2H}) + b^{2H}(a + b + c)^{2H})^{\frac{3}{2}}} dadbdc. \]  
(5.24)

Therefore, using the inequality
\[ (\varepsilon^2 + \varepsilon(b^{2H} + (a + b + c)^{2H}) + b^{2H}(a + b + c)^{2H})^{\frac{3}{2}} \geq (\varepsilon(a + b + c)^{2H} + b^H(a + b + c)^{2H})^\frac{3}{2}, \]
we get
\[ V_2(\varepsilon) \leq K \int_{[0,T]^3} (a + b + c)^{-(H+1)} b(\varepsilon + b^H)^{-\frac{3}{2}} dadbdc \leq K \left( \int_{[0,T]^2} (a + c)^{-(H+1)} dadc \right) \left( \int_0^T b(\varepsilon + b^H)^{-\frac{3}{2}} db \right). \]  
(5.25)

The term \((a + c)^{-(H+1)}\) is clearly integrable over the region \( 0 \leq a, c \leq T \). To bound the integral over \( 0 \leq b \leq T \) of \( b(\varepsilon + b^H)^{-\frac{3}{2}} \) we proceed as follows. Define \( y := H - \frac{2}{3} \). Notice that \( 0 < y < 1 \) due to the condition \( \frac{2}{3} < H < 1 \). Therefore, by the weighted arithmetic mean-geometric mean inequality, we have
\[ y\varepsilon + (1 - y) b^{2H} \geq \varepsilon^y b^{2H(1-y)}. \]  
(5.26)
From (5.25) and (5.26), it follows that there exists a constant $K > 0$, only depending on $H$ and $T$, such that
\[
\varepsilon^3 \frac{2}{\pi} V_2(\varepsilon) \leq K \varepsilon^3 \frac{2}{\pi} \int_0^T b^{1-3H(1-y)} db \\
= K \varepsilon^3 \frac{2}{\pi} \int_0^T b^{3H-2} db.
\] (5.27)

The integral in the right-hand side of (5.27) is finite thanks to the condition $H > \frac{2}{3}$. Relation (5.16) then follows by taking limit as $\varepsilon \to 0$ in (5.27).

**Lemma 5.5.** Let $c, \beta, \alpha$ and $\gamma$ be real numbers such that $c, \beta > 0$, $\alpha > -1$ and $1 + \alpha + \gamma \beta < 0$. Then we have
\[
\int_0^\infty a^\alpha (c + a^\beta)^\gamma da = \beta^{-1} c^{\frac{\alpha + 1 + \beta \gamma}{\beta}} B \left( \frac{\alpha + 1}{\beta}, -\frac{1 + \alpha + \gamma \beta}{\beta} \right),
\] (5.28)
where $B(\cdot, \cdot)$ denotes the Beta function.

**Proof.** Making the change of variables $x = a^\beta$ in the left-hand side of (5.28) we obtain
\[
\int_0^\infty a^\alpha (c + a^\beta)^\gamma da = \beta^{-1} \int_0^\infty x^{\frac{\alpha + 1 - \beta}{\beta}} (c + x)^\gamma dx.
\] (5.29)

Hence, making the change of variables $a = \frac{x^\beta}{c}$ in the right hand side of (5.29) we get
\[
\int_0^\infty a^\alpha (c + a^\beta)^\gamma da = \beta^{-1} c^{\frac{\alpha + 1 + \beta \gamma}{\beta}} \int_0^\infty a^{\frac{\alpha + 1 - \beta}{\beta}} (1 + a)^\gamma da.
\] (5.30)

Finally, the change of variables $x = \frac{a^\beta}{1 + a}$ in the right hand side of (5.30) leads to
\[
\int_0^\infty a^\alpha (c + a^\beta)^\gamma da = \beta^{-1} c^{\frac{\alpha + 1 + \beta \gamma}{\beta}} \int_0^1 x^{\frac{\alpha + 1 - \beta}{\beta}} (1 - x)^{\frac{\beta + 1 + \alpha + \gamma \beta}{\beta}} dx,
\] (5.31)
which implies the desired result.

**Lemma 5.6.** Let $\varepsilon, T > 0$, and define $V_3(\varepsilon)$ by (2.14). Then, for every $\frac{2}{3} < H < 1$ we have
\[
\lim_{\varepsilon \to 0} \varepsilon^3 \frac{2}{\pi} V_3(\varepsilon) = \sigma^2,
\] (5.32)
where $\sigma^2$ is given by (3.2).

**Proof.** Changing the coordinates $(x, u_1, u_2)$ by $(a := u_1, b := x - u_1, c := u_2)$ in (5.1) for $i = 3$, we obtain
\[
V_3(\varepsilon) = \frac{1}{\pi} \int_{[0,T]^3} 1_{(0,T)}(a + b + c)(T - (a + b + c)) |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} dadbdc,
\] (5.33)
where the matrix $\Sigma$ is given by
\[
\Sigma_{1,1} = a^{2H}, \\
\Sigma_{2,2} = c^{2H}, \\
\Sigma_{1,2} = \frac{1}{2}((a + b + c)^{2H} + b^{2H} - (b + c)^{2H} - (a + b)^{2H}).
\]
We can easily check, as before, that
\[
\mu = \frac{\varepsilon I + \Sigma}{\sqrt{B}D}
\]
We show that
\[
\lim_{\varepsilon \to 0} \frac{\varepsilon^{3-\frac{2}{H}} V_3(\varepsilon)}{\pi} = \int_{\mathbb{R}_+^3} 1_{(0,T)}(b) \frac{H(2H - 1)(T - b) ac b^{2H - 2}}{(1 + a^{2H} + c^{2H} + a^{2H} c^{2H})^2} dv_1 dv_2 
\]
which implies
\[
\Psi_\varepsilon(a, b, c) = \frac{H(2H - 1)(T - b) ac b^{2H - 2}}{(1 + a^{2H} + c^{2H} + a^{2H} c^{2H})^2} \\
= (T - b) b^{2H - 2} ac (1 + a^{2H})^{-\frac{3}{2}} (1 + c^{2H})^{-\frac{3}{2}}.
\]
Therefore, provided we show that $1_{(0,T)}(\varepsilon^{\frac{1}{2H}}(a + c) + b)\Psi_\varepsilon(a, b, c)$ is dominated by a function integrable in $\mathbb{R}_+^3$, we obtain the following identity by applying the dominated convergence theorem in (5.35)
\[
\lim_{\varepsilon \to 0} \frac{\varepsilon^{3-\frac{2}{H}} V_3(\varepsilon)}{\pi} = \frac{H(2H - 1)}{\pi} \int_{\mathbb{R}_+^3} 1_{(0,T)}(b) \frac{H(2H - 1)(T - b) b^{2H - 2} ac ((1 + a^{2H})(1 + c^{2H}))^{-\frac{3}{2}}}{\pi} dv_1 dv_2.
\]
Making the change of variables $x = \frac{b}{\varepsilon}$, and using Lemma 5.5 we obtain (5.32). Next we show that $1_{(0,T)}(\varepsilon^{\frac{1}{2H}}(a + c) + b)\Psi_{3,0,\varepsilon}(a, b, c)$ is dominated by a function integrable
Lemma 5.7. Let $T, \varepsilon > 0$ be fixed. Define $V_3^{(1)}(\varepsilon)$ by (2.16). Then, for every $\frac{2}{3} < H < 1$ it holds
\[ \lim_{\varepsilon \to 0} \varepsilon^{3-\frac{2}{H}} V_3^{(1)}(\varepsilon) = \sigma^2, \]
where $\sigma^2$ is given by (5.2).

Proof. By (2.16) and (2.19),
\[ V_3^{(1)}(\varepsilon) = (2q - 1)! \beta_3^2 \int_{S_3} G_3^{(q)}(t_1 - s_1, t_2 - s_2), \]
where $S_3$ is defined by (2.12). Changing the coordinates $(s_1, s_2, t_1, t_2)$ by $(a := t_1 - s_1, b := s_2 - t_1, c := t_2 - s_2)$ in (5.39), and using (2.20), we obtain
\[ V_3^{(1)}(\varepsilon) = \frac{1}{\pi} \int_{\mathbb{R}^3_+} \int_0^T (a + b + c) \left( \varepsilon + a^{2H} \right)^{-\frac{2}{H}} \left( \varepsilon + c^{2H} \right)^{-\frac{2}{H}} \mu(a + b, a, c) da db dc. \]

Next, using the identity
\[ \mu(x + y, x, z) = H(2H - 1) x z \int_{[0, 1]^2} (y + x v_1 + z v_2)^{2H-2} dv_1 dv_2, \]
we get
\[ \varepsilon^{3-\frac{2}{H}} V_3^{(1)}(\varepsilon) = \frac{H(2H - 1)}{\pi} \int_0^T \int_{\mathbb{R}^3_+} \int_{[0, 1]^2} \left| \varepsilon + a^{2H} \right|^{\frac{2}{H}} \left| \varepsilon + c^{2H} \right|^{\frac{2}{H}} \mu(a + b, a, c) (a + c)(T - b - \varepsilon \sqrt{\frac{1}{\pi}} (a + c)) \]
\[ \times (1 + a^{2H})^{-\frac{2}{H}} (1 + c^{2H})^{-\frac{2}{H}} ac(b + \varepsilon \sqrt{\frac{1}{\pi}} (av_1 + cv_2))^{2H-2} dv_1 dv_2 db dc. \]
Notice that the argument of the integral in the right-hand side of (5.41) is dominated by the function
\[
\Theta(a, b, c, v_1, v_2) := \frac{TH(2H - 1)}{\pi} (1 + a^{2H})^{-\frac{3}{2}} (1 + c^{2H})^{-\frac{3}{2}} abc^{2H - 2}.
\]
The integral \(\int_0^T \int_{[0,1]^2} \Theta(a, b, c, v_1, v_2) dv_1 dv_2 dacdb\) is finite thanks to condition \(H > \frac{2}{3}\). Therefore, applying the dominated convergence theorem to (5.41), we get
\[
\lim_{\varepsilon \to 0} \varepsilon^3 \pi V_3^{(1)}(\varepsilon) = \frac{H(2H - 1)}{\pi} \int_0^T \int_{\mathbb{R}_+^3} (T - b)(1 + a^{2H})^{-\frac{3}{2}} (1 + c^{2H})^{-\frac{3}{2}} abc^{2H - 2} dacdb.
\]
Making the change of variables \(x = \frac{b}{T}\), and using Lemma 5.5 we obtain (5.38).

**Lemma 5.8.** Let \(T, \varepsilon > 0\) and \(q \in \mathbb{N}\), \(q \geq 2\) be fixed. Define \(G_{1,x}^{(q)}(u_1, u_2)\) by (2.20). Then, for every \(\frac{3}{q} < H < \frac{4q - 3}{4q - 2}\), it holds that
\[
\int_{\mathbb{R}_+^3} G_{1,x}^{(q)}(u_1, u_2) dx du_1 du_2 < \infty. \tag{5.42}
\]

**Proof.** Let \(T, \varepsilon > 0\), and \(q \in \mathbb{N}\) be fixed, and define the sets
\[
\mathcal{T}_1 := \{(x, u_1, u_2) \in \mathbb{R}_+^3 \mid u_1 - x \geq 0, \ x + u_2 - u_1 \geq 0\},
\]
\[
\mathcal{T}_2 := \{(x, u_1, u_2) \in \mathbb{R}_+^3 \mid u_1 - u_2 \geq 0\},
\]
\[
\mathcal{T}_3 := \{(x, u_1, u_2) \in \mathbb{R}_+^3 \mid x - u_1 \geq 0\}.
\]
Since \(\mathbb{R}_+^3 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3\), it suffices to prove that \(G_{1,x}^{(q)}(u_1, u_2)\) is integrable in \(\mathcal{T}_i\), for \(i = 1, 2, 3\).

To prove the integrability of \(G_{1,x}^{(q)}(u_1, u_2)\) in \(\mathcal{T}_1\) we change the coordinates \((x, u_1, u_2)\) by \((a := x, b := u_1 - x, c := x + u_2 - u_1)\). Then,
\[
\int_{\mathcal{T}_1} G_{1,x}^{(q)}(u_1, u_2) dx du_1 du_2 = \int_{\mathbb{R}_+^3} G_{1,a}^{(q)}(a + b, b + c) dacdbdc. \tag{5.43}
\]
Next we prove that the right hand of (5.43) is finite. Notice that
\[
G_{1,a}^{(q)}(a + b, b + c) = (1 + (a + b)^{2H})^{-\frac{1}{2}q}(1 + (b + c)^{2H})^{-\frac{1}{2}q}\mu(a, a + b, b + c)^{2q-1}.
\]
By the Cauchy-Schwarz inequality, we get \(\mu(a, a + b, b + c) \leq (a + b)^H(b + c)^H\), and consequently,
\[
G_{1,a}^{(q)}(a + b, b + c) \leq (1 + (a + b)^{2H})^{-1}(1 + (b + c)^{2H})^{-1}.
\]
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Hence, using the inequalities $\frac{2}{3}a + \frac{1}{3}b \geq a^\frac{2}{3}b^\frac{1}{3}$ and $\frac{2}{3}c + \frac{1}{3}b \geq c^\frac{2}{3}b^\frac{1}{3}$, we deduce that there exists a constant $K$ only depending on $T$ and $H$ such that the following bounds hold

$$G^{(q)}_{1,a}(a + b, b + c) \leq K(abc)^{-\frac{2H}{3}}$$

if $a, b, c \geq 1$,

$$G^{(q)}_{1,a}(a + b, b + c) \leq K(1 + b^{2H})^{-1}(1 + c^{2H})^{-1}$$

if $a \leq 1$,

$$G^{(q)}_{1,a}(a + b, b + c) \leq K(1 + b^{2H})^{-1}(1 + a^{2H})^{-1}$$

if $c \leq 1$,

$$G^{(q)}_{1,a}(a + b, b + c) \leq K(1 + a^{2H})^{-1}(1 + c^{2H})^{-1}$$

if $b \leq 1$.

Using the previous bounds, as well as condition $H > \frac{3}{4}$, we deduce that $G^{(q)}_{1,a}(a + b, b + c)$ is integrable in the variables $a, b, c \geq 0$.

To prove the integrability of $G^{(q)}_{1,x}(u_1, u_2)$ in $T_2$ we change the coordinates $(x, u_1, u_2)$ by $(a := x, b := u_2, c := u_1 - x - u_2)$. Then,

$$\int_{T_2} G^{(q)}_{1,x}(u_1, u_2) dx du_1 du_2 = \int_{\mathbb{R}^3_+} G^{(q)}_{1,a}(b, a + b + c) dadbdc.$$

Next we prove that $G^{(q)}_{1,a}(a + b, b + c)$ is integrable in the variables $a, b, c \geq 0$. Using inequality $\mu(a, a + b + c, b) \leq (a + b + c)^H b^H$, as well as the condition $q \geq 2$, we obtain

$$G^{(q)}_{1,a}(a + b, b + c) = (1 + (a + b + c)^{2H})^{-\frac{q}{2}}(1 + b^{2H})^{-\frac{q}{2}}(1 + c^{2H})^{-\frac{q}{2}} \mu(x, a + b + c, b)^3$$

$$\times \left(\frac{\mu(a, a + b + c, b)}{\sqrt{(1 + b^{2H})(1 + (a + b + c)^{2H})}}\right)^{2(a-2)}$$

$$\leq (1 \vee a \vee b \vee c)^{-5H}(1 \vee b)^{-5H} \mu(a, a + b + c, b)^3.$$  \hspace{1cm} (5.44)

Similarly, using $q \geq 1$ we can prove that

$$G^{(q)}_{1,a}(b, a + b + c) \leq (1 \vee a \vee b \vee c)^{-2H}(1 \vee b)^{-2H}.$$ \hspace{1cm} (5.45)

In addition, using the representation

$$\mu(a, a + b + c, b) = \frac{1}{2}((a + b)^{2H} + (b + c)^{2H} - a^{2H} - c^{2H})$$

$$= Hb \int_0^1 ((a + bu)^{2H-1} + (c + bu)^{2H-1}) du,$$

we deduce that there exist constants $K, K'$ only depending on $H$ such that

$$\mu(a, a + b + c, b) 1_{(0,a \land c)}(b) \leq K 1_{(0,a \land c)}(b)b((a + b)^{2H-1} + (c + b)^{2H-1})$$

$$\leq K' 1_{(0,a \land c)}(b)b(a \lor c)^{2H-1}$$

$$\leq K'(1 \lor b)(1 \lor a \lor c)^{2H-1}.$$  \hspace{1cm} (5.46)
Using (5.49), we deduce that there exists a constant $K > 0$ such that

$$G_{1,a}^{(q)}(b, a + b + c) \mathbb{1}_{(0,a \land c)}(b) \leq K \mathbb{1}_{(0,a \land c)}(b)(1 \lor a \lor b \lor c)^{-5H}(1 \lor b)^{-5H+3}(1 \lor a \lor c)^{6H-3} \leq K(1 \lor a \lor c)^{H-3}(1 \lor b)^{-5H+3}.$$

Using the previous inequality, as well as the condition $H > \frac{5}{4}$, we deduce that $G_{1,a}^{(q)}(b, a + b + c)$ is integrable in $\{(a, b, c) \in \mathbb{R}^3_+ \mid b \leq a \land c\}$. In addition, from (5.45) we obtain

$$G_{1,a}^{(q)}(b, a + b + c) \mathbb{1}_{(0,a \land c)}(a) \leq (1 \lor b)^{-2H}(1 \lor b \lor c)^{-2H}.$$

Therefore, using condition $H > \frac{3}{4}$, we deduce that $G_{1,a}^{(q)}(b, a + b + c)$ is integrable in $\{(a, b, c) \in \mathbb{R}^3_+ \mid a \leq b \land c\}$. By symmetry $G_{1,a}^{(q)}(b, a + b + c)$ is integrable in $\{(a, b, c) \in \mathbb{R}^3_+ \mid c \leq a \land b\}$. From the previous analysis we conclude that $G_{1,x}^{(q)}(u_1, u_2)$ is integrable in $\mathcal{T}_2$.

To prove the integrability of $G_{1,x}^{(q)}(u_1, u_2)$ in $\mathcal{T}_3$, we change the coordinates $(x, u_1, u_2)$ by $(a := u_1, b := x - u_1, c := u_2)$. Then,

$$\int_{\mathcal{T}_3} G_{1,x}^{(q)}(u_1, u_2)dxdu_1du_2 = \int_{\mathbb{R}^3_+} G_{1,a+b}^{(q)}(a, c)dadbc.$$

To bound $G_{1,a+b}^{(q)}(a, c)$ we proceed as follows. Using inequality $\mu(a + b, a, c) \leq a^H e^H$, we deduce that

$$G_{1,a+b}^{(q)}(a, c) \leq (1 + a^{2H})^{-1}(1 + e^{2H})^{-1} \leq (1 \lor a)^{-2H}(1 \lor c)^{-2H}.$$

As a consequence, $G_{1,a+b}^{(q)}(a, c)$ is integrable in $\{(a, b, c) \in \mathbb{R}^3_+ \mid b \leq 1\}$. In addition, from relation

$$\mu(x + y, x, z) = H(2H - 1)xyz \int_{[0,1]^2} (y + xv_1 + zv_2)^{2H-2}dv_1dv_2,$$

we can prove that

$$\mu(x + y, x, z) \leq H(2H - 1)xyz^{2H-2}.$$

Using (5.49), we deduce that there exists a constant $K > 0$, only depending on $H$ and $q$, such that

$$G_{1,a+b}^{(q)}(a, c) \leq K ((1 + a^{2H}) (1 + e^{2H}))^{-\frac{1}{2}-q} (ac)^{2q-1} (2q-1)(H-1) \leq K ((1 \lor a) (1 \lor c))^{-H-2qH+2q-1} b^2(2q-1)(H-1).$$
Taking into account that $H < \frac{4q-3}{4q-2}$, we get $2(2q-1)(H-1) < -1$, and hence

$$\int_{1 \vee a \vee c}^{\infty} G^{(q)}_{1,a+b}(a,c)db \leq K \left( ((1 \lor a) (1 \lor c))^{-H-2qH+2q-1}(1 \lor a \lor c)^{2(2q-1)(H-1)+1} \right)$$

$$\leq K (1 \lor a)^{-2H+\frac{1}{2}} (1 \lor c)^{-2H+\frac{1}{2}}, \quad (5.50)$$

where in the last inequality we used the relation

$$(1 \lor a \lor c)^{2(2q-1)(H-1)+1} \leq (1 \lor a)^{(2q-1)(H-1)+\frac{1}{2}} (1 \lor c)^{(2q-1)(H-1)+\frac{1}{2}}.$$

Using relation (5.50) as well as condition $H > \frac{3}{4}$, we conclude that $G^{(q)}_{1,a+b}(a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}^3_+ \mid 1 \lor a \lor c \leq b\}$. In addition, from (5.48) we obtain

$$\mu(x + y, x, z) \leq H(2H - 1)xz \int_{[0,1]^2} (xv_1 + zv_2)^{2H-2}dv_1dv_2$$

$$\leq H(2H - 1)xz \int_0^1 ((x \lor z)w)^{2H-2}dw$$

$$= Hxz(x \lor z)^{2H-2} = H(x \land z)(x \lor z)^{2H-1}.$$

Hence, there exist constants $K, \tilde{K} \geq 0$ such that

$$G^{(q)}_{1,a+b}(a,c)1_{(a \land c, a \lor c)}(b)$$

$$= \left( (1 + a^{2H}) (1 + c^{2H}) \right)^{-\frac{3}{2} - q} \mu(a + b, a, c)^{2q-1}$$

$$\leq K \left( ((1 \lor a) (1 \lor c))^{-H-2qH} (a \lor c)^{2q-1} (a \lor c)^{(2q-1)(2H-1)} \right)$$

$$\leq K \left( ((1 \lor a) (1 \lor c))^{-H-2qH} (1 \lor (a \land c))^{2q-1} (1 \lor a \lor c)^{(2q-1)(2H-1)} \right)$$

$$= K (1 \lor (a \land c))^{-H(2q+1)+2q-1} (1 \lor a \lor c)^{-3H-2q+2qH+1}. \quad (5.51)$$

Using relation (5.51) as well as condition $H > \frac{3}{4}$, we obtain that $G^{(q)}_{1,a+b}(a,c)$ is integrable in the region $\{(a,b,c) \in \mathbb{R}^3_+ \mid a \land c \leq b \leq a \lor c\}$. Finally, applying (5.47) we can prove that $G^{(q)}_{1,a+b}(a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}^3_+ \mid b \leq a \land c\}$. From the previous analysis we conclude that $G^{(q)}_{1,a+b}(a,c)$ is integrable in the variables $a, b, c \geq 0$, which in turn implies that $G^{(q)}_{1,a+b}(u_1, u_2)$ is integrable in $T_3$ as required.

\[ \square \]

**Lemma 5.9.** Let $T, \varepsilon > 0$ and $q \in \mathbb{N}$, $q \geq 2$ be fixed, and define $G^{(q)}_{0,x}(u_1, u_2)$ by (2.20). Then, for every $\frac{2}{3} < H < \frac{3}{4}$, we have

$$\int_{[0,T]^3} G^{(q)}_{0,x}(u_1, u_2) dx du_1 du_2 < \infty.$$
Proof. Let $T, \varepsilon > 0$, and $q \in \mathbb{N}$, and define the sets

\[
\tilde{T}_i := \{(x, u_1, u_2) \in [0, T]^3 \mid u_1 - x \geq 0, \ x + u_2 - u_1 \geq 0\},
\tilde{T}_2 := \{(x, u_1, u_2) \in [0, T]^3 \mid u_1 - x - u_2 \geq 0\},
\tilde{T}_3 := \{(x, u_1, u_2) \in [0, T]^3 \mid x - u_1 \geq 0\}.
\]

Since $[0, T]^3 = \tilde{T}_1 \cup \tilde{T}_2 \cup \tilde{T}_3$, it suffices to check the integrability of $G_{0,x}^{(q)}(u_1, u_2)$ in $\tilde{T}_i$, for $i = 1, 2, 3$. To prove integrability in $\tilde{T}_1$ we make change the coordinates $(x, u_1, u_2)$ by $(a := x, b := u_1 - x, c := x + u_2 - u_1)$. Then,

\[
\int_{\tilde{T}_1} G_{0,x}^{(q)}(u_1, u_2) \, dx \, du_1 \, du_2 \leq \int_{[0,T]^3} G_{0,a}^{(q)}(a + b + c) \, da \, db \, dc.
\]

By the inequality $\mu(a, a + b, b + c) \leq (a + b)^H (b + c)^H$, we can write

\[
G_{0,a}^{(q)}(a + b, b + c) \leq (a + b)^{-2H} (b + c)^{-2H}.
\]

Therefore, using $\frac{2a}{3} + \frac{b}{3} \geq \frac{3}{2}b^\frac{1}{H}$ and $\frac{2c}{3} + \frac{b}{3} \geq \frac{9}{4}b^\frac{1}{H}$, as well as (5.52), we deduce that there exists a universal constant $K$ such that

\[
G_{0,a}^{(q)}(a + b, b + c) \leq K(abc)^{-\frac{4H}{3}}.
\]

The right hand side in the previous inequality is integrable in $[0, T]^3$ thanks to the condition $H < \frac{3}{4}$. Therefore, $G_{0,x}^{(q)}(u_1, u_2)$ is integrable in $\tilde{T}_1$.

To prove the integrability of $G_{0,x}^{(q)}(u_1, u_2)$ in $\tilde{T}_2$ we change the coordinates $(x, u_1, u_2)$ by $(a := x, b := u_2, c := u_1 - x - u_2)$. Then,

\[
\int_{\tilde{T}_2} G_{0,x}^{(q)}(u_1, u_2) \, dx \, du_1 \, du_2 \leq \int_{[0,T]^3} G_{0,a}^{(q)}(b + a + c) \, da \, db \, dc.
\]

In order to bound the term $G_{0,a}^{(q)}(b + a + c)$ we proceed as follows. Applying the inequality $\mu(a, a + b + c, b) \leq (a + b + c)^H b^H$, as well as the condition $q \geq 2$, we obtain

\[
G_{0,a}^{(q)}(b + a + c) = (a + b + c)^{-5H} b^{-5H} \mu(a, a + b + c, b)^3 \\
\times \left( \frac{\mu(b, a + b + c, b)}{b^H (a + b + b)^H} \right)^{2(q-2)} \\
\leq (a + b + c)^{-5H} b^{-5H} \mu(a, a + b + c, b)^3.
\]

(5.53)

On the other hand, by the relation

\[
\mu(a, a + b + c, b) = \frac{1}{2} \left( (a + b)^{2H} + (b + c)^{2H} - a^{2H} - c^{2H} \right) \\
= Hb \int_{0}^{1} ((a + bw)^{2H-1} + (c + bw)^{2H-1}) \, dw,
\]

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we deduce that there exists a constant \( K > 0 \) such that

\[
\mu(a, a + b + c, b) \leq 1_{(0, a \wedge c)}(b) H b \int_0^1 \left( (a + bw)^{2H-1} + (c + bw)^{2H-1} \right) dw
= Kb(a \vee c)^{2H-1}.
\] (5.54)

Using (5.53) and (5.54) we get

\[
G^{(q)}_{0,a}(b, a + b + c) \leq Kb^{-5H+3}(a + b + c)^{-5H}(a \vee c)^{6H-3}
\leq Kb^{-5H+3}(a \vee c)^{H-3}.
\] (5.55)

From (5.55) as well as the condition \( H < \frac{3}{4} \), we deduce that \( G^{(q)}_{0,a}(b, a + b + c) \) is integrable in \( \{(a, b, c) \in [0, T]^3 \mid b \leq a \wedge c\} \). In addition, using the relation \( \mu(a, a + b + c, b) \leq (a + b + c)^{H} b^{H} \), we can prove that

\[
G^{(q)}_{0,a} (b, a + b + c) \leq b^{-2H} c^{-2H}.
\]

Therefore, by the condition \( H < \frac{3}{4} \), we deduce that \( G^{(q)}_{0,a}(b, a + b + c) \) is integrable in \( \{(a, b, c) \in [0, T]^3 \mid a \leq b \wedge c\} \). Similarly, we can prove that

\[
G^{(q)}_{0,a} (b, a + b + c) \leq b^{-2H} a^{-2H},
\]

and hence, since \( H < \frac{3}{4} \) we conclude that \( G^{(q)}_{0,a}(b, a + b + c) \) is integrable in \( \{(a, b, c) \in [0, T]^3 \mid c \leq b \wedge a\} \). From the analysis we conclude that \( G^{(q)}_{0,a}(b, a + b + c) \) is integrable in \([0, T]^3\).

To prove the integrability of \( G^{(q)}(u_1, u_2) \) in \( \tilde{T}_3 \) we change the coordinates \((x, u_1, u_2)\) by \((a := u_1, b := x - u_1, c := u_2)\) to get

\[
\int_{\tilde{T}_3} G^{(q)}_{0,x}(u_1, u_2) dx du_1 du_2 \leq \int_{[0,T]^3} G^{(q)}_{0,a+b}(a, c) da db dc.
\]

In order to bound the term \( G^{(q)}_{0,a+b}(a, c) \) we proceed as follows. From relation

\[
\mu(x + y, x, z) = H(2H - 1) x z \int_{[0,1]^2} (y + xv_1 + zv_2)^{2H-2} dv_1 dv_2,
\] (5.56)

we can deduce that

\[
\mu(x + y, x, z) \leq H(2H - 1) x y^{2H-2}.
\]

Hence, since

\[
G^{(q)}_{0,a+b}(a, c) = a^{-2H-2qH} c^{-2H-2qH} \mu(a + b, a, c)^{2q-1},
\] (5.57)
we deduce that there exists a constant $K > 0$ only depending on $H$ such that

$$G_{0,a+b}^{(q)}(a,c)1_{(a \land c, a \lor c)}(b) \leq a^{-H-2qH+2q-1}c^{-H-2qH+2q-1}b^{2(2q-1)(H-1)}1_{(a \land c, a \lor c)}(b).$$

(5.58)

Since $q \geq 2$, we have that $H < \frac{3}{2} < \frac{3}{5} \leq \frac{2q}{1+2q}$. As a consequence, from (5.58) we deduce that $G_{0,a+b}^{(q)}(a,c)$ is integrable in $\{(a,b,c) \in \mathbb{R}^3_+ \mid b \geq a, c\}$. In addition, by (5.56) we get

$$\mu(x+y,x,z) \leq H(2H-1)xyz \int_{[0,1]^2} ((x \land z)w_1)^{2H-2}dw_1dw_2$$

$$= Hxyz(x \lor z)^{2H-2} = H(x \land z)(x \lor z)^{2H-1}.$$ (5.59)

Therefore, there exists constant $K \geq 0$ such that

$$G_{0,a+b}^{(q)}(a,c)1_{(a \land c, a \lor c)}(b) \leq K(a \land c)^{-H(2q+1)+2q-1}(a \lor c)^{-3H-2q+2qH+1}1_{(a \land c, a \lor c)}(b).$$

(5.59)

From (5.59), and $H < \frac{3}{4} < \frac{3}{5} \leq \frac{2q}{1+2q}$, it follows that $G_{0,a+b}^{(q)}(a,c)$ is integrable in $\{(a,b,c) \in [0,T]^3 \mid a \land c \leq b \leq a \lor c\}$. Finally, by inequalities $\mu \leq a^{HcH}$ and (5.57), we get

$$G_{0,a+b}^{(q)}(a,c)1_{(0,a \land c)}(b) \leq a^{-2H}c^{-2H}.$$ (5.60)

Using (5.60) as well as condition $H < \frac{3}{4}$, we deduce that $G_{0,a+b}^{(q)}(a,c)$ is integrable in $\{(a,b,c) \in [0,T]^3 \mid b \leq a \land c\}$. From the previous analysis it follows that $G_{0,x}^{(q)}(u_1,u_2)$ is integrable in $\tilde{T}$ as required.

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