Cosmological Fluctuations in Delta Gravity

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Abstract. In this work we present the theory of perturbation of Delta Gravity, we discuss the gauge transformations for metric and a perfect fluid in order to present the equations of the evolution of cosmological fluctuations using the hydrodynamic approximation. Then we compute the temperature fluctuations for photons coming from the time of last scattering $t_L$. Finally we present a formula for temperature multi-pole coefficients for scalar modes, which can be used to compare the theory with astronomical observations.

1. Introduction

Delta Gravity (DG), proposed in \cite{1}, is an extension of General Relativity (GR), in this model new fields are added to the Lagrangian by a new symmetry (for details see \cite{1,2,3}). This theory predicts an accelerating Universe without a cosmological constant $\Lambda$. There are a couple of works which have fitted SN-Ia Data in order to obtain some parameters of the theory. Besides, with this fits DG gives new values for different cosmological parameters, among them the Hubble constant\cite{2,4}.

Despite DG gives good results for these measurements, there is still a very important subject to study, which is matter and energy fluctuations in Cosmology. In particular the information given by the anisotropies of matter and energy fluctuations in the Cosmic Microwave Background (CMB) allow us to know about the structure of our Universe such as primordial gravitational waves(measuring polarization of the CMB) and homogeneous temperature of the Universe.

In particular, the correlations of temperature gives us information about the constituents of the Universe, such as baryonic and dark matter. For this task, we have to study the evolutions of scalar fluctuations of the CMB from the moment of last scattering at time $t_L$ to the present. These computations are usually done by computational codes such as CMBFast\cite{5,6} or CAMB\cite{7,8}, where Boltzmann equations for the fluids and its interactions provide us the well known results that are in agreement with the measurements of Planck\textsuperscript{9}.

Nevertheless, one can get a good approximation to this complex problem\cite{10}. Instead of study the evolution of the scalar perturbations using Boltzmann equations, one can divide this problem in two steps: first, by using a hydrodynamic approximation, which consists in taking photons and baryonic plasma as a fluid in thermal equilibrium at the time of recombination, this is due to the high rate of collisions between free electrons and photons at that time. And second, studying the propagation of photons by radial geodesics from the moment when the
Universe switch from opaque to transparent at time $t_L$ until us. Because of the simplicity of this approximation, and the fact that in DG photons still follow geodesics but of an effective metric, we could give an overview of the study of scalar perturbations for the temperature correlation of the CMB. In this work we will give the first steps of this important task, developing the theory of perturbations at first order of DG, where we will discuss the gauge transformations in an extended Friedmann-Lemaître-Robertson-Walker (FRLW) Universe. Then we will show how using the trajectory of photons we can get an expression for temperature fluctuations and we will show that they are gauge invariant, which is a very important theoretical test of the theory. With this result we have found a formula for the scalar contribution to temperature multipole coefficients, this formula can be used to test the theory and also to give a sign of the physical consequence of the “delta matter”, introduced in this theory.

The CMB Power Spectrum is a very important observational measurement for any cosmological model because it provides cosmological parameters that can constrain the model. Many parameters can be obtained directly from the CMB Power Spectrum, for example $h^2\Omega_b, h^2\Omega_c, 100\theta, \tau, A_s$ and $n_s$ [11], but other parameters can be derived from constraining CMB observation with SNe-Ia or BAOs. With the study of the CMB anisotropies in the DG context, we could study two important aspects: if the CMB Power Spectrum is compatible with DG fluctuations equations and if the DG picture is compatible with SNe-Ia or other observational constraints.

In Section 2 we will present the Lagrangian of DG and its equations of motion, then we study the gauge transformation for small perturbations of the geometrical and matter fields. After we chose a gauge, we present the gauge invariant equations of motion for small perturbations. In section 3 we will study the evolution of cosmological perturbations where we guide how to solve the system when the Universe was dominated by radiation and when it was dominated by matter, non numerical results are presented. Then in Section 4 we derive the formula for temperature fluctuation, here we find that this fluctuation can be splitted in three independend and gauge invariants terms. In Section 5 we derive a formula for temperature multipole coefficients for scalar modes, allowing us to test the theory with astronomical observations in future works. Finally we give some conclusions and remarks.

2. Delta Gravity and its perturbation theory
The action for DG is given by Alfaro et all[1] as:

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa} + L_M - \frac{1}{2\kappa} \left( G^{\alpha\beta} - \kappa T^{\alpha\beta} \right) \tilde{g}_{\alpha\beta} + \tilde{L}_M \right),$$

(1)

where $\kappa = \frac{8\pi G}{c^4}$, $\tilde{g}_{\mu\nu} = \delta g_{\mu\nu}$, $L_M$ is the matter Lagrangian and:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[ \sqrt{-g}L_M \right]$$

(2)

$$\tilde{L}_M = \tilde{\phi} \frac{\delta L_M}{\delta \tilde{\phi}} + \left( \partial_\mu \tilde{\phi} \right) \frac{\delta L_M}{\delta \partial_\mu \tilde{\phi}},$$

(3)

with $\tilde{\phi} = \tilde{\delta} \phi$ are the $\tilde{\delta}$ matter fields or “delta matter”. The equations of motion are given by the variation of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, it is easy to see that we get the usual Einstein’s equations varying the action (1) with respect to $\tilde{g}_{\mu\nu}$. By the other hand, variations with respect $g_{\mu\nu}$ gives us the
equation for $\tilde{g}_{\mu\nu}$ as is shown
\begin{equation}
F(\mu\nu)(\alpha\beta)\rho\lambda D_\rho D_\lambda \tilde{g}_{\alpha\beta} + \frac{1}{2} R^{\alpha\beta} \tilde{g}_{\alpha\beta} - R^{\mu\nu} \tilde{g}_{\mu\nu} + \frac{1}{2} \tilde{g}_{\alpha\beta} G^{\mu\nu} = \frac{\kappa}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left[ \sqrt{-\tilde{g}} \left( T^{\alpha\beta} \tilde{g}_{\alpha\beta} + 2 \tilde{L}_M \right) \right],
\end{equation}
with:
\begin{equation}
F(\mu\nu)(\alpha\beta) = P((\rho\nu)(\alpha\beta)) g^{\mu\nu} + P((\rho\nu)(\alpha\beta)) g^{\mu\nu} - P((\rho\nu)(\alpha\beta)) g^{\mu\nu} + P((\alpha\beta)(\mu\nu)) = \frac{1}{4} \left( g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\beta} g^{\mu\nu} \right).
\end{equation}

Here $(\mu\nu)$ denotes the totally symmetric combination of $\mu$ and $\nu$. It is possible to simplify Eq. (4) (see [1]) to get the final system of equation of DG:
\begin{equation}
G^{\mu\nu} = \kappa T^{\mu\nu}
\end{equation}
and
\begin{equation}
F(\mu\nu)(\alpha\beta) D_\rho D_\lambda \tilde{g}_{\alpha\beta} + \frac{1}{2} g^{\mu\nu} R^{\alpha\beta} \tilde{g}_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu}.
\end{equation}

While, the energy momentum conservation now reads as
\begin{equation}
D_\nu T^{\mu\nu} = 0
\end{equation}
and
\begin{equation}
D_\nu \tilde{T}^{\mu\nu} = \frac{1}{2} T^{\alpha\beta} D_\rho \tilde{g}_{\alpha\beta} - \frac{1}{2} T^{\mu\beta} D_\rho \tilde{g}_{\alpha\beta} + D_\beta (\tilde{g}^\beta T^{\alpha\mu}).
\end{equation}

Then, we will work with Eqs. (6), (7), (8) and (9). However, as perturbation theory in the standard sector is well known (see for example [10]), here we will focus on DG sector.

2.1. Perturbation theory of FRLW metric

One important result of DG is that photons follow geodesic trajectories given by the effective metric $\bar{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \bar{g}_{\mu\nu}$, and for a FRLW universe this metrics take the form
\begin{equation}
\tilde{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + R(t)^2 (dx^2 + dy^2 + dz^2),
\end{equation}
and
\begin{equation}
\bar{g}_{\mu\nu} dx^\mu dx^\nu = -3F(t)dt^2 + \tilde{g}_{\mu\nu} dx^\mu dx^\nu = R(t)^2 (dx^2 + dy^2 + dz^2).
\end{equation}

where $F(t)$ is a time dependent function which is determined by the solution of the unperturbed equations system, $R(t)$ is the standard scale factor, which in Section (4) we will show that it is no longer the physical scale factor of the universe. Now, lets consider perturbations of this metrics given by
\begin{equation}
\bar{g}_{\mu\nu} = g_{\mu\nu} + \bar{g}_{\mu\nu}
\end{equation}
and
\begin{equation}
\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \tilde{h}_{\mu\nu}
\end{equation}

It is standard to decompose this perturbation in the Scalar-Vector-Tensor decomposition[12], this way allow us to study these sectors independently. The perturbations then are
\begin{equation}
h_{00} = -E \quad h_{0i} = R \left[ \frac{\partial H}{\partial x^i} + G_i \right] \quad h_{ij} = R^2 \left[ A \delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij} \right],
\end{equation}

where $\delta_{ij}$ is the Kronecker delta, $A$, $B$, and $C_i$ are functions of the unperturbed space-time $g_{\mu\nu}$, and $D_{ij}$ are the components of the unperturbed metric $\bar{g}_{\mu\nu}$.

With these perturbations, the equations of motion for the perturbations can be written in a compact form.

\begin{align}
\nabla_\mu D_\nu T^{\mu\nu} &\equiv \nabla_\nu D_\mu T^{\mu\nu} = 0
\end{align}

\begin{align}
\nabla_\mu D_\nu \tilde{T}^{\mu\nu} &\equiv \nabla_\nu D_\mu \tilde{T}^{\mu\nu} = \frac{1}{2} \nabla_\rho \tilde{g}^{\alpha\beta} \tilde{g}_{\alpha\beta} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{L}_M + D_\beta (\tilde{g}^\beta T^{\alpha\mu}).
\end{align}
Following the standard procedure, we decompose the spatial part of
the metric perturbation into a spatial scalar plus a divergenceless vector:
\[ \tilde{h}_{\mu\nu}(x) = R \left[ \frac{\partial \tilde{H}}{\partial x^i} + \tilde{G}_i \right] \]
(15)
with
\[ \frac{\partial \tilde{G}_i}{\partial x^i} = 0 \quad \frac{\partial D_{ij}}{\partial x^i} = 0 \quad D_{ii} = 0. \]
(15)

This decomposition must be equivalent for \( \tilde{h}_{\mu\nu} \) by Group Theory, so
\[ \tilde{h}_{00} = -\dot{E} \quad \tilde{h}_{i0} = R \left[ \frac{\partial \dot{H}}{\partial x^i} + \dot{G}_i \right] \quad \tilde{h}_{ij} = R^2 \left[ \dot{\delta}_{ij} + \frac{\partial^2 \tilde{B}}{\partial x^i \partial x^j} + \frac{\partial \tilde{C}_i}{\partial x^j} + \frac{\partial \tilde{C}_j}{\partial x^i} + \tilde{D}_{ij} \right], \]
(16)
with
\[ \frac{\partial \tilde{C}_i}{\partial x^i} = 0 \quad \frac{\partial \tilde{D}_{ij}}{\partial x^j} = 0 \quad \tilde{D}_{ii} = 0. \]
(17)

If we replace perturbations in Eqs (6), (7), (8) and (9) we get the equations for perturbations,
however, there are degrees of freedoms which we have to take in account in order to have
physical solutions. In the next subsection we show how to choose a gauge that will eliminate
this unphysical solutions.

2.2. Choosing a gauge
Under a spacetime coordinate transformation, metrics perturbations transform as
\[ \Delta h_{\mu\nu}(x) = -\bar{g}_{\lambda\nu}(x) \frac{\partial \epsilon^\lambda}{\partial x^\nu} - \bar{g}_{\mu\lambda}(x) \frac{\partial \bar{\epsilon}^\lambda}{\partial x^\nu} - \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\lambda} \epsilon^\lambda, \]
(18)
in more detail
\[ \Delta h_{ij} = -\frac{\partial \epsilon_i}{\partial x^j} - \frac{\partial \epsilon_j}{\partial x^i} + 2R\bar{R} \delta_{ij} \epsilon_0, \]
(19)
\[ \Delta h_{i0} = -\frac{\partial \epsilon_i}{\partial t} - \frac{\partial \epsilon_0}{\partial x^i} + 2\frac{\dot{R}}{R} \epsilon_i, \]
(20)
\[ \Delta h_{00} = -\epsilon_0^2 \frac{\partial \epsilon_0}{\partial t}. \]
(21)

And for delta perturbations we get
\[ \Delta \tilde{h}_{\mu\nu} = -\bar{g}_{\mu\lambda}(x) \frac{\partial \epsilon^\lambda}{\partial x^\nu} - \bar{g}_{\nu\lambda}(x) \frac{\partial \bar{\epsilon}^\lambda}{\partial x^\nu} - \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\lambda} \epsilon^\lambda, \]
(22)
in detail
\[ \Delta \tilde{h}_{ij} = -F \frac{\partial \epsilon_i}{\partial x^j} - F \frac{\partial \epsilon_j}{\partial x^i} - F \frac{\partial \epsilon_i}{\partial x^j} - 2F \frac{\partial \epsilon_0}{\partial x^i} \epsilon_0 + \left[ 2F \dot{R} + \dot{F} R^2 \right] \delta_{ij}, \]
(23)
\[ \Delta \tilde{h}_{i0} = -3F \frac{\partial \epsilon_i}{\partial t} - 3F \frac{\partial \epsilon_0}{\partial x^i} \frac{\partial \epsilon_i}{\partial x^0} - 2F \frac{\partial \epsilon_0}{\partial t} \epsilon_i + 2 \frac{\dot{R}}{R} \epsilon_i, \]
(24)
\[ \Delta \tilde{h}_{00} = -3 \epsilon_0 \dot{F} - 6F \frac{\partial \epsilon_0}{\partial t} - 2 \frac{\partial \epsilon_0}{\partial t}. \]
(25)

Here \( \epsilon \) and \( \tilde{\epsilon} = \dot{\epsilon} \) defines the coordinates transformation. Also we raised and lowered index
using \( \bar{g}_{\mu\nu} \), so \( \epsilon^0 = -\epsilon_0, \tilde{\epsilon}^0 = -\tilde{\epsilon}_0, \epsilon^i = R^{-2} \epsilon_i \) and \( \tilde{\epsilon}^i = R^{-2} \tilde{\epsilon}_i \).

Following the standard procedure, we decompose the spatial part of \( \epsilon^\mu \) and \( \tilde{\epsilon}^\mu \) into the gradient
of a spatial scalar plus a divergenceless vector:
\[ \epsilon_i = \partial_i \epsilon^S + \epsilon_i^V, \quad \partial_i \epsilon^V = 0, \]
(26)
\[ \tilde{\epsilon}_i = \partial_i \tilde{\epsilon}^S + \tilde{\epsilon}_i^V, \quad \partial_i \tilde{\epsilon}^V = 0 \]
(27)
So, comparison between transformation of Eqs. (14) and (16) with Eqs. (19)-(21) and (23)-(25) give us the gauge transformations of the metric perturbation components:

\[ \Delta A = \frac{2 \dot{R}}{R} \epsilon_0, \quad \Delta B = -\frac{2}{R^2} \epsilon^S, \]
\[ \Delta C_i = -\frac{1}{R^2} \epsilon_i^V, \quad \Delta D_{ij} = 0, \quad \Delta E = 2 \dot{\epsilon}_0, \]
\[ \Delta H = \frac{1}{R} \left( -\epsilon_0 - \dot{\epsilon}^S + \frac{2 \dot{R}}{R} \epsilon^S \right), \quad \Delta G_i = \frac{1}{R} \left( -\dot{\epsilon}_i^V + \frac{2 \dot{R}}{R} \epsilon_i^V \right), \]

(28)

while

\[ \Delta \tilde{A} = \left( \frac{2 \dot{R} F}{R} + \dot{F} \right) \epsilon_0 + \frac{2 \dot{R}}{R} \tilde{\epsilon}_0, \quad \Delta \tilde{B} = -\frac{2}{R^2} (F \epsilon^S + \tilde{\epsilon}^S), \]
\[ \Delta \tilde{C}_i = -\frac{1}{R^2} (F \epsilon_i^V + \tilde{\epsilon}_i^V), \quad \Delta \tilde{D}_{ij} = 0, \quad \Delta \tilde{E} = 6 F \dot{\epsilon}_0 + 3 \dot{F} \epsilon_0 + 2 \ddot{\epsilon}_0, \]
\[ \Delta \tilde{H} = \frac{1}{R} \left( -3 F \epsilon_0 - \tilde{\epsilon}_0 - F \epsilon^S - \tilde{\epsilon}^S + \frac{2 \dot{F}}{R} \tilde{\epsilon}^S + \frac{2 \dot{R}}{R^2} \epsilon^S \right), \quad \Delta \tilde{G}_i = \frac{1}{R} \left( -F \ddot{\epsilon}_i^V - \dot{\epsilon}_i^V + \frac{2 \dot{F}}{R} \epsilon_i^V + \frac{2 \dot{R}}{R^2} \tilde{\epsilon}_i^V \right). \]

(29)

Imposing conditions on the parameters \( \epsilon_\mu \) and \( \dot{\epsilon}_\mu \), there are different scenarios in which we can follow the calculations, however before discuss about this, let study the gauge transformation of energy momentum tensors \( T_{\mu\nu} \) and \( \tilde{T}_{\mu\nu} \).

2.3. \( T_{\mu\nu} \) and \( \tilde{T}_{\mu\nu} \)

Now we will decompose the energy-momentum tensors \( T_{\mu\nu} \) and \( \tilde{T}_{\mu\nu} \) in the same way. For a perfect fluid we would have (for details see [2])

\[ T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu, \]  

(30)

while for \( \tilde{T}_{\mu\nu} \) [1, 2]

\[ \tilde{T}_{\mu\nu} = \tilde{p} \tilde{g}_{\mu\nu} + (\tilde{\rho} + \tilde{p}) u_\mu u_\nu + (\rho + p) \left( \frac{1}{2} (\tilde{g}_{\mu\alpha} u_\nu u^\alpha + \tilde{g}_{\nu\alpha} u_\mu u^\alpha) + u^T_{\mu} u_\nu + u_\mu u^T_{\nu} \right). \]  

(31)

with

\[ g^{\mu\nu} u_\mu u_\nu = -1, \]  

(32)

and

\[ g^{\mu\nu} u_\mu u^T_{\nu} = 0. \]  

(33)

Where \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \) are defined in Eqs. (10) and (11) respectively, besides we considered

\[ p = \tilde{p} + \delta p \]
\[ \rho = \tilde{\rho} + \delta \rho \]
\[ u_\mu = \tilde{u}_\mu + \delta u_\mu \]
\[ \tilde{p} = \tilde{\tilde{p}} + \delta \tilde{p} \]
\[ \tilde{\rho} = \tilde{\tilde{\rho}} + \delta \tilde{\tilde{\rho}} \]
\[ u^T_{\mu} = \tilde{u}^T_{\mu} + \delta u^T_{\mu}. \]  

(34)
Usually we will have an equation of state $p(\rho)$ so we could reduce this system, but for now we will work in the generic case. When we work in the frame $\bar{u}_\mu = (-1, 0, 0, 0)$ we have $\bar{u}_\mu^T = 0$, and the normalization conditions Eqs. \((32)\) and \((33)\) gives

\[
\begin{align*}
\delta u^0 &= \frac{\delta u_0}{2} = h_{00} = 2 \\
\delta u_\mu^T &= \frac{\delta u_\mu^T}{2} = 0
\end{align*}
\]

while $\delta u_i$ and $\delta u_i^T$ are independent dynamical variables (Note that $\delta u^\mu \equiv \delta (g^{\mu\nu} u_\nu)$ is not given by $\tilde{g}^{\mu\nu} \delta u_\nu$, the same happens for $\delta u_i^\mu$). Then the first-order perturbation to both energy-momentum tensors for a perfect fluid are

\[
\delta T_{\mu\nu} = \tilde{p} h_{\mu\nu} + \delta \tilde{g} g_{\mu\nu} + (\bar{p} + \bar{\rho})(\bar{u}_\mu \delta u_\nu + \delta u_\mu u_\nu) + (\delta p + \delta \rho)\bar{u}_\mu \bar{u}_\nu,
\]

or in more detail

\[
\delta T_{\mu\nu} = \tilde{p} h_{\mu\nu} + \delta \tilde{p} g_{\mu\nu} + \tilde{p} h_{\mu\nu} + (\bar{p} + \bar{\rho})(\bar{u}_\mu \delta u_\nu + \delta u_\mu u_\nu) + (\delta \rho + \delta \rho)\bar{u}_\mu \bar{u}_\nu.
\]

While

\[
\delta \tilde{T}_{\mu\nu} = \tilde{p} h_{\mu\nu} + \delta \tilde{p} g_{\mu\nu} + \tilde{p} h_{\mu\nu} + (\bar{p} + \bar{\rho})(\bar{u}_\mu \delta u_\nu + \delta u_\mu u_\nu) + (\delta \rho + \delta \rho)\bar{u}_\mu \bar{u}_\nu
\]

in detail

\[
\begin{align*}
\delta \tilde{T}_{00} &= -\tilde{p} h_{00} - \tilde{p} h_{00} + 3F \delta \rho + \delta \rho \\
\delta \tilde{T}_{0i} &= \tilde{p} h_{0i} + \tilde{p} h_{0i} - (\bar{p} + \bar{\rho})\delta u_i + (\bar{p} + \bar{\rho}) \left\{ \frac{1}{2} [F h_{i0} - \tilde{h}_{i0} - 4F \delta u_i] - \frac{1}{2} \delta u_i^2 \right\} \\
\delta \tilde{T}_{ij} &= \tilde{p} h_{ij} + \delta \tilde{p} R^2 \delta_{ij} + \tilde{p} h_{ij} + \delta \tilde{p} R^2 \delta_{ij} \frac{1}{2} \delta \tilde{p} R^2 \delta_{ij}
\end{align*}
\]

where we used $\delta u^\mu = \delta (g^{\alpha\beta} u_\beta) = \tilde{g}^{\alpha\beta} \delta u_\beta + h^{\mu\nu} \bar{u}_\beta$.

More generally, we decompose $\delta u_i (\delta u_i^T)$ into the gradient of a scalar velocity potential $\delta u (\delta \tilde{u})$ and a divergenceless vector $\delta u_i^I (\delta \tilde{u}_i^I)$. And we add the dissipative corrections to the inertia tensor. That is, we write

\[
\begin{align*}
\delta T_{ij} &= \tilde{p} h_{ij} + R^2 \left[ \delta_{ij} \delta p + \partial_i \partial_j \pi^S + \partial_i \pi_i^V + \partial_j \pi_j^V + \pi_i^T \right], \\
\delta T_{00} &= \tilde{p} h_{00} - (\bar{p} + \bar{\rho}) (\partial_i \delta u + \delta u_i^V), \\
\delta T_{0i} &= \tilde{p} h_{0i} - (\bar{p} + \bar{\rho}) (\partial_i \delta u + \delta u_i^V),
\end{align*}
\]

And

\[
\begin{align*}
\delta \tilde{T}_{ij} &= \tilde{p} h_{ij} + R^2 \left[ \delta_{ij} \delta p + \partial_i \partial_j \pi^S + \partial_i \pi_i^V + \partial_j \pi_j^V + \pi_i^T \right] + \tilde{p} h_{ij} \\
&+ FR^2 \left[ \partial_i \delta p + \partial_i \partial j \pi^S + \partial_i \pi_i^V + \partial_j \pi_j^V + \pi_i^T \right], \\
\delta \tilde{T}_{00} &= \tilde{p} h_{00} + \tilde{p} h_{00} - (\bar{p} + \bar{\rho}) (\partial_i \partial u + \partial u_i^V) + (\bar{p} + \bar{\rho}) \left\{ \frac{1}{2} [F h_{i0} - \tilde{h}_{i0} - 4F (\partial_i \delta u + \delta u_i^V)] \right\} \\
&- \partial_i \delta \tilde{u}^V + \partial \tilde{u}_i^V \} \\
\delta \tilde{T}_{0i} &= -\tilde{p} h_{0i} - \tilde{p} h_{0i} + 3F \delta \rho + \delta \rho,
\end{align*}
\]

where $\pi_i^V (\tilde{\pi}_i^V)$ and $\delta u_i^V (\delta \tilde{u}_i^V)$ satisfy conditions analogous to the conditions \((15)\) and \((17)\) satisfied by $C_i (\tilde{C}_i), D_{ij} (\tilde{D}_{ij})$ and $G_i (\tilde{G}_i)$:

\[
\begin{align*}
\partial_i \pi_i^V = \partial_i \tilde{\pi}_i^V = \partial_i \delta u_i^V = \partial_i \delta \tilde{u}_i^V = 0 & \quad \partial_i \pi_{ij}^T = \partial_i \tilde{\pi}_{ij}^T = \partial_i \pi_{ij}^V + \partial_j \pi_i^V + \pi_i^T = \tilde{\pi}_{ij}^T = \tilde{\pi}_{ij}^V = 0.
\end{align*}
\]
2.4. Gauge Transformations for the Energy-Momentum tensors

Again following Weinberg the gauge transformation for $T_{\mu\nu}$ is given by

$$\Delta \delta T_{\mu\nu}(x) = -\tilde{T}_{\lambda\nu}(x) \frac{\partial \epsilon^\lambda}{\partial x^\mu} - \tilde{T}_{\mu\lambda}(x) \frac{\partial \epsilon^\lambda}{\partial x^\nu} - \frac{\partial \tilde{T}_{\nu\mu}}{\partial x^\lambda} \epsilon^\lambda,$$

or in more detail

$$\Delta \delta T_{ij} = -\tilde{p} \left( \frac{\partial \epsilon_i}{\partial x^j} + \frac{\partial \epsilon_j}{\partial x^i} \right) + \frac{\partial}{\partial t}(R^2 \tilde{p}) \delta_{ij} \epsilon_0$$

$$\Delta \delta T_{i0} = -\tilde{p} \frac{\partial \epsilon_i}{\partial t} + \tilde{p} \frac{\partial \epsilon_0}{\partial x^i} + 2\tilde{R} \tilde{p} \epsilon_i$$

$$\Delta \delta T_{00} = 2\tilde{p} \frac{\partial \epsilon_0}{\partial t} + \dot{\tilde{p}} \epsilon_0.$$  

While the gauge transformation of $\delta \tilde{T}_{\mu\nu}$ is given by

$$\Delta \delta \tilde{T}_{\mu\nu} = -\tilde{T}_{\mu\lambda} \frac{\partial \epsilon^\lambda}{\partial x^\nu} - \tilde{T}_{\nu\lambda} \frac{\partial \epsilon^\lambda}{\partial x^\mu} - \frac{\partial \tilde{T}_{\nu\mu}}{\partial x^\lambda} \epsilon^\lambda - \tilde{T}_{\mu\lambda} \frac{\partial \epsilon^\lambda}{\partial x^\nu} - \tilde{T}_{\nu\lambda} \frac{\partial \epsilon^\lambda}{\partial x^\mu} - \frac{\partial \tilde{T}_{\nu\mu}}{\partial x^\lambda} \epsilon^\lambda,$$

in detail

$$\Delta \delta \tilde{T}_{ij} = -\left( \tilde{p} + 3\tilde{F} \right) \frac{\partial \epsilon_i}{\partial x^j} - (\tilde{p} + \tilde{F}) \frac{\partial \epsilon_j}{\partial x^i} - \tilde{p} \frac{\partial \epsilon_i}{\partial x^j} - \tilde{p} \frac{\partial \epsilon_j}{\partial x^i} + \left[ \epsilon_0 \frac{\partial}{\partial t}(R^2 (\tilde{p} + \tilde{F})) + \frac{\partial}{\partial t}(R^2 \tilde{p}) \epsilon_0 \right] \delta_{ij}$$

$$\Delta \delta \tilde{T}_{i0} = -\left( \tilde{p} + 3\tilde{F} \right) \frac{\partial \epsilon_i}{\partial t} + (\tilde{p} + 3\tilde{F}) \frac{\partial \epsilon_0}{\partial x^i} - \tilde{p} \frac{\partial \epsilon_i}{\partial t} + \tilde{p} \frac{\partial \epsilon_0}{\partial x^i} + 2(\tilde{p} + \tilde{F}) \frac{\tilde{R}}{R} \epsilon_i$$

$$\Delta \delta \tilde{T}_{00} = \epsilon_0 \frac{\partial}{\partial t} (\tilde{p} + 3\tilde{F}) + 2(\tilde{p} + 3\tilde{F}) \frac{\partial \epsilon_0}{\partial t} + \tilde{p} \epsilon_0 + 2\tilde{p} \frac{\partial \epsilon_0}{\partial t}.$$  

Using the decomposition for $\epsilon_i$ and $\tilde{\epsilon}_i$ given by Eqs. (26) in order to write these gauge transformations in terms of the scalar, vector and tensor components written above, the transformations (19)-(21) and (23)-(25) with (18)-(19) and (52)-(54) gives the gauge transformation to the pressure, energy density and velocity potential

$$\Delta \delta p = \dot{\tilde{p}} \epsilon_0, \quad \Delta \delta \rho = \dot{\tilde{p}} \epsilon_0, \quad \Delta \delta u = -\epsilon_0.$$  

The other ingredients of the energy-momentum tensor are gauge invariants:

$$\Delta \pi^S = \Delta \pi^V = \Delta \pi^T = \Delta \delta u^V = 0.$$  

While for the other fields we have:

$$\Delta \delta \tilde{\rho} = \frac{\partial}{\partial t}(\tilde{p} + 3\tilde{F}) \epsilon_0 + 2(\tilde{p} + 3\tilde{F}) \tilde{\epsilon}_0 + \tilde{\delta} \tilde{\epsilon}_0 + 2\tilde{p} \tilde{\epsilon}_0 - 3\tilde{F} \tilde{\rho} \Delta \tilde{E} - 3\tilde{F} \tilde{\delta} \rho,$$

$$\Delta \delta \tilde{\mu} = \frac{1}{R^2} \frac{\partial}{\partial t} \left[ R^2 (\tilde{p} + \tilde{F}) \epsilon_0 \right] + \frac{1}{R^2} \frac{\partial}{\partial t} \left( R^2 \tilde{p} \epsilon_0 \right) - \tilde{\delta} \tilde{A} - 3\tilde{F} \tilde{\mu} \Delta \tilde{A} - \tilde{F} \Delta \delta \rho,$$

$$\Delta \delta \tilde{\nu} = \frac{1}{(\tilde{\rho} + \tilde{p})} \left\{ (\tilde{p} + \tilde{F}) \epsilon^S - (\tilde{\rho} + 3\tilde{F}) \epsilon_0 + \tilde{\rho} \epsilon_0 - 2(\tilde{p} + \tilde{F}) \frac{\tilde{R}}{R} \epsilon^S - 2\tilde{p} \frac{\tilde{R}}{R} \epsilon^S \right. \right.$$  

$$\left. + \tilde{\rho} \Delta \tilde{H} + \tilde{\rho} \tilde{R} \Delta \tilde{H} - (\tilde{\rho} + \tilde{p}) \left[ \frac{1}{2} (1 - F) \tilde{R} \Delta \tilde{H} + 2F \Delta \delta \nu \right] \right\},$$

$$\Delta \delta \tilde{\nu} = \frac{1}{(\tilde{\rho} + \tilde{p})} \left\{ (\tilde{p} + \tilde{F}) \epsilon^S + \tilde{\rho} \epsilon^S - 2(\tilde{p} + \tilde{F}) \frac{\tilde{R}}{R} \epsilon^S - \tilde{\rho} \Delta \tilde{G} - \tilde{\rho} \tilde{R} \Delta \tilde{G} - \frac{1}{2} (\tilde{\rho} + \tilde{p})(1 - F) \tilde{R} \Delta \tilde{G} \right\},$$

$$\Delta \delta \tilde{\pi}^S = -2 \frac{\tilde{R}}{R^2} (\tilde{p} + \tilde{F}) \epsilon^S - 2\tilde{p} \frac{\tilde{R}}{R^2} \epsilon^S - \tilde{\rho} \Delta \tilde{B} - \tilde{\rho} \tilde{F} \Delta \tilde{B},$$

$$\Delta \delta \tilde{\pi}^V = -\frac{1}{R^2} (\tilde{p} + \tilde{F}) \epsilon^V + \frac{\tilde{R}}{R^2} \epsilon^V - \tilde{\rho} \Delta \tilde{C} - \tilde{\rho} \tilde{F} \Delta \tilde{C},$$

$$\Delta \delta \tilde{\pi}_{ij} = 0.$$  

Using the results of Eqs. (28), (29) and (55) we get

$$\Delta \tilde{\delta} \tilde{\rho} = \dot{\tilde{\rho}}_{\epsilon} + (\dot{\tilde{\rho}} - 3F \tilde{\rho}) \epsilon_{0}, \tag{64}$$

$$\Delta \tilde{\delta} \tilde{\pi} = \dot{\tilde{\pi}}_{\epsilon} + (\dot{\tilde{\rho}} - 3F \tilde{\rho}) \epsilon_{0}, \tag{65}$$

$$\Delta \tilde{\delta} \tilde{\rho} = \left[(1 - 3F)\frac{\tilde{F}}{2} - \frac{\tilde{\rho}_{0} + \tilde{\rho}^{'}}{\tilde{\rho} + \tilde{\rho}^{'}}\right] \epsilon_{0} - \frac{1}{2}(1 + F)\epsilon_{0} - \frac{1}{2}(1 - F)\left(F \epsilon^{S} + \epsilon^{\tilde{S}}\right), \tag{66}$$

$$\Delta \tilde{\delta} \tilde{\pi}^{' S} = 0, \tag{67}$$

$$\Delta \tilde{\delta} \tilde{\pi}^{' V} = 0, \tag{68}$$

$$\Delta \tilde{\delta} \tilde{\pi}_{ij} = 0. \tag{69}$$

As we said before, there are different elections for $\epsilon$ and $\tilde{\epsilon}$ parameter in order to fix all gauge freedoms, the most commonds and well known are the Newtonian gauge and Synchronous gauge, the first one consists in to choose in Eq. (28) $\epsilon_{S}$ so that $B = 0$, and choose $\epsilon_{0}$ so that $H = 0$. In DG, this election is extended imposing similar conditions in Eq. (29) to $\tilde{\epsilon}^{S}$ and $\tilde{\epsilon}_{0}$ so that $\tilde{B} = \tilde{H} = 0$. There is no remaining freedom to make a gauge transformation in this scheme. Nevertheless in this work we will use the Synchronous scheme, where we choose $\epsilon_{0}$ so that $E = 0$, and the choose $\epsilon^{S}$ so that again $H = 0$, (similar conditions for $\tilde{\epsilon}_{0}$ and $\tilde{\epsilon}^{S}$). In the next section we present the perturbed equations of motions in this frame, and we discuss about the convenience of this election for our purposes.

### 2.5. Fields equations and energy momentum conservations in synchronous gauge

Under this gauge fixing, perturbed Einstein equations Eq. (6) reads (at first order):

$$- 4\pi G(\delta \rho + 3\delta p + \nabla^{2} \pi^{S}) = \frac{1}{2} \left[3 \dot{A} + \nabla^{2} \dot{B}\right] + \frac{\dot{R}}{2R} \left[3 \dot{A} + \nabla^{2} \dot{B}\right] \tag{71}$$

While the energy-momentum conservation gives

$$\delta \dot{p} + \nabla^{2} \pi^{S} + \partial_{0}[\tilde{\rho} + \tilde{p}] \delta u] + \frac{3\dot{R}}{R} (\tilde{\rho} + \tilde{p}) \delta u = 0 \tag{72}$$

$$\delta \dot{\tilde{\rho}} + \frac{3\dot{\tilde{R}}}{R} (\delta \rho + \delta p) + \nabla^{2} \left[R^{-2}(\tilde{\rho} + \tilde{\rho}^{'}) \delta u + \frac{\dot{\tilde{R}}}{R} \pi^{S}\right]$$

$$+ \frac{1}{2}(\tilde{\rho} + \tilde{\rho}^{'}) \partial_{0} \left[3 A + \nabla^{2} B\right] = 0. \tag{73}$$

if we define

$$\Psi \equiv \frac{1}{2} \left[3 A + \nabla^{2} B\right] \tag{74}$$

we have

$$- 4\pi G R^{2}(\delta \rho + 3\delta p + \nabla^{2} \pi^{S}) = \frac{\partial}{\partial t} \left(R^{2} \dot{\Psi}\right) \tag{75}$$

and

$$\delta \dot{\tilde{\rho}} + \frac{3\dot{\tilde{R}}}{R} (\delta \rho + \delta p) + \nabla^{2} \left[R^{-2}(\tilde{\rho} + \tilde{\rho}^{'}) \delta u + \frac{\dot{\tilde{R}}}{R} \pi^{S}\right]$$

$$+ \frac{1}{2}(\tilde{\rho} + \tilde{\rho}^{'}) \dot{\Psi} = 0. \tag{76}$$

The unperturbed Einstein equations correspond to the Friedmann equations. In the Delta sector, computations gives the nonperturbative equations:

$$3 \dot{F} \frac{\dot{R}}{R} = \kappa (3 F \tilde{\rho} + \tilde{\rho}) \tag{77}$$
and
\[ 12F \frac{\dot{R}}{R} + 6F \left( \frac{\dot{R}}{R} \right)^2 + 3F \frac{\dot{R}}{R} - 3\dot{F} = \kappa (\ddot{\rho} + 3\dot{\rho} + 3F\dot{\rho} + 3F\ddot{\rho}) \] (78)

and the perturbative contribution (at first order) is
\[ \left[ 2F \frac{\dot{R}}{R} + \ddot{F} \right] \left[ 3A + \nabla^2 B \right] + \left[ 6F \frac{\dot{R}}{R} + \frac{5}{2} F \right] \left[ 3\dot{A} + \nabla^2 \dot{B} \right] - \left[ 2 \frac{\dot{R}}{R} \right] \left[ 3\ddot{A} + \nabla^2 \ddot{B} \right] + 3F \left[ 3\ddot{A} + \nabla^2 \ddot{B} \right] = \kappa (3\dot{\rho} + \ddot{\rho} + F\delta\rho + 3F\delta p + \nabla^2 \ddot{\pi} + F\nabla^2 \pi) \] (79)

Besides, Eq. (31) gives the equation for 00 component
\[ \delta\dot{\rho} + \frac{3\dot{R}}{R} (\ddot{\rho} + \ddot{\rho}) + \frac{3\dot{F}}{2} (\delta\rho + \delta p) + \nabla^2 \left[ \left( \frac{\ddot{\rho} + \ddot{\rho}}{R^2} \right) \delta u + \left( \frac{\dot{\rho} + \dot{\rho}}{R^2} \right) \dot{\delta u} \right] + \frac{3F}{2} \delta u \left[ (\dot{A} + \dot{B}) - \dot{\partial}_0 (F[3A + \nabla^2 B]) \right] = 0, \] (80)

while the $i0$ component gives
\[ \delta\dot{\rho} + \partial_0 [(\dot{\rho} + \dot{\rho}) \delta u] + \partial_0 [(\ddot{\rho} + \ddot{\rho}) \delta u] = \partial_0 [(\dot{\rho} + \dot{\rho}) F\delta u] + 3(\dot{\rho} + \dot{\rho}) F\delta u + \frac{3\dot{R}}{R} (\ddot{\rho} + \ddot{\rho}) \delta u + \frac{3\dot{F}}{R} (\dot{\rho} + \dot{\rho}) \delta u = 0. \] (81)

Analogous to the standard sector, lets define
\[ \dot{\Psi} \equiv \frac{1}{2} \left[ 3\dot{A} + \nabla^2 \dot{B} \right], \] (82)

then the gravitational equation becomes
\[ \left[ 2F \frac{\dot{R}}{R} + \ddot{F} \right] R^2 \dot{\Psi} + \left[ 6F \frac{\dot{R}}{R} + \frac{5}{2} F \right] R^2 \dot{\Psi} + 3F \nabla^2 \dot{\Psi} - \frac{d}{dt} \left( R^2 \dot{\Psi} \right) = \kappa \left( \frac{3}{2} \delta\dot{\rho} + \frac{\dot{\rho} + \ddot{\rho} + 3F\delta p + \nabla^2 \ddot{\pi} + F\nabla^2 \pi} \right) \] (83)

while, the delta energy conservation now reads:
\[ \delta\dot{\rho} + \frac{3\dot{R}}{R} (\ddot{\rho} + \ddot{\rho}) + \frac{3\dot{F}}{2} (\delta\rho + \delta p) + \nabla^2 \left[ \left( \frac{\ddot{\rho} + \ddot{\rho}}{R^2} \right) \delta u + \left( \frac{\dot{\rho} + \dot{\rho}}{R^2} \right) \dot{\delta u} \right] + (\dot{\rho} + \dot{\rho}) \dot{\Psi} + (\ddot{\rho} + \ddot{\rho}) \dot{\Psi} = 0. \] (84)

The study of the non perturbative sector was already treated in Alfaro et all (1, 2, 4) and applied on the observation of supernovas. We will refer to those results when it will be needed.

For now, we only need the expresion for the time dependend function $F(t)$, which is
\[ F(Y) = \frac{3}{2} (2C_2 - C_1) Y \frac{1}{C} \left( \frac{\sqrt{\frac{Y}{C}} + 1}{\sqrt{\frac{Y}{C} + 1}} \right) - 2C_2 + C_3 \frac{Y}{C} \frac{1}{\sqrt{\frac{Y}{C} + 1}}, \] (85)

where $Y \equiv Y(t) = R(t)/R_0$ is the quotient between the scale factor at a time $t$ over the scale factor in the actuality (which for our purposes we will consider equal to one). $L_2$ and $C$ are constants that we can use to fit our results, or take the values obtained in previous works [2, 4].

Also, we have to remark that our definition of $\Psi$ is not the usual since the standard definition is with the time derivative of fields $A$ and $B$, respectively. Our election is given because in the delta sector, appears explicitly the combinations of this fields without a time derivative, so if the reader wants to compare results with other works, he or she should take in consideration this detail. In the next section we will discuss about the evolution of the cosmological fluctuations which will help us to compute the scalar contribution to the CMB.
3. Evolution of cosmological fluctuations

Until now he have developed the perturbation theory in DG, now we are interested on study the evolution of the cosmological fluctuations in order to have a physical interpretation of the delta matter fields which this theory naturally introduces. Even in the standard cosmology, the system of equations which describes this perturbations are way so complicated to allow an analytic solutions and there are comprehensive computer programs to this task, such as CMBfast\[5, 6\] and CAMB\[7, 8\], however such computer programs can not give a clear understanding of the physical phenomena involved. Nevertheless, there are some good approximations that allow to compute the spectrum of the CMB fluctuations with a rather good agreement with this computer programs\[10, 13\]. In particular we are going to extend Weinberg approach for this task, which consist in two main points: the first one is the so-called hydrodynamic limit, which consist on that near recombination time photons were in local thermal equilibrium with the baryonic plasma, then photons could be treated hydrodynamically, like plasma and cold dark matter. And second, a sharp transition from thermal equilibrium to complete transparency at a moment $t_L$ of last scattering.

Since we are going to reproduce this approach, we consider standard components of the universe, that means photons, neutrinos, baryons and cold dark matter. Then the task is to understand the role of their respective delta-counter part. Also we have neglected anisotropic both inertia tensor and took usual state quation for pressures and energy densities and its perturbations. Besides, as we will treat photons and delta photons hydrodynamically, we will put $\delta u_\gamma = \delta u_B$ and $\delta \tilde{u}_\gamma = \delta \tilde{u}_B$. And finally, as synchronous scheme does not fix completely the gauge freedom, one can use the remaining freedom to put $\delta u_D = 0$, which means that cold dark matter evolves at rest with respect to universe expansion. In our theory, extended synchronous scheme also has an extra freedom which we will use to put $\delta \tilde{u}_D = 0$ as its standard part. Now we will present the equations for both sectors, however we will provide a more detail in the delta sector because the solution of the Einstein’s equations are already calculated by Weinberg.

Einstein’s equations and its energy momentum conservation gives

$$\frac{d}{dt} \left( a^2 \dot{\Psi}_q \right) = -4\pi Ga^2 \left( \delta \rho_{Dq} + \delta \rho_{Bq} + 2\delta \rho_{\gamma q} + 2\delta \rho_{\nu q} \right), \quad \text{(86)}$$

$$\delta \dot{\rho}_{\gamma q} + 4H\delta \rho_{\gamma q} - (4q/3a)\bar{\rho}_\gamma \delta u_{\gamma q} = -(4/3)\bar{\rho}_\gamma \dot{\Psi}_q, \quad \text{(87)}$$

$$\delta \dot{\rho}_{Dq} + 3H\delta \rho_{Dq} = -\bar{\rho}_D \dot{\Psi}_q, \quad \text{(88)}$$

$$\delta \dot{\rho}_{Bq} + 3H\delta \rho_{Bq} - (q/a)\bar{\rho}_B \delta u_{\gamma q} = -\bar{\rho}_B \dot{\Psi}_q, \quad \text{(89)}$$

$$\delta \dot{\rho}_{\nu q} + 4H\delta \rho_{\nu q} - (4q/3a)\bar{\rho}_\nu \delta u_{\nu q} = -(4/3)\bar{\rho}_\nu \dot{\Psi}_q, \quad \text{(90)}$$

it is useful to rewrite these equations in term of the dimensionless fractional perturbation

$$\delta_{aq} = \frac{\delta \rho_{aq}}{\bar{\rho}_a + \bar{\rho}_\alpha}, \quad \text{(91)}$$
where $\alpha$ runs over $\gamma$, $\nu$, $B$ and $D$. Considering that $R^4 \ddot{\rho}_\gamma$, $R^4 \ddot{\rho}_\nu$, $R^3 \ddot{\rho}_D$, $R^3 \ddot{\rho}_B$ are time independent, Eqs (80-90) now read

\[
\frac{d}{dt} \left( a^2 \dot{\Psi} \right) = -4\pi Ga^2 \left( \dot{\rho}_D \delta_{Dq} + \dot{\rho}_B \delta_{Bq} + \frac{8}{3} \dot{\rho}_\gamma \delta_{\gamma q} + \frac{8}{3} \dot{\rho}_\nu \delta_{\nu q} \right),
\]

(92)

\[
\dot{\gamma} - \frac{q^2}{a^2} \delta_{u\gamma} = -\dot{\Psi} \quad (93)
\]

\[
\dot{\delta}_{Dq} = -\dot{\Psi} \quad (94)
\]

\[
\dot{\delta}_{Bq} - \frac{q^2}{a^2} \delta_{u\gamma} = -\dot{\Psi} \quad (95)
\]

\[
\dot{\delta}_{uv} - \frac{q^2}{a^2} \delta_{u\nu} = -\dot{\Psi} \quad (96)
\]

\[
\frac{d}{dt} \left( \frac{(1+S) \delta_{u\gamma}}{a} \right) = -\frac{1}{3} \delta_{\gamma q} \quad (97)
\]

\[
\frac{d}{dt} \left( \frac{\delta_{uv}}{a} \right) = -\frac{1}{3} \delta_{\nu q} \quad (98)
\]

where $S = 3\ddot{\rho}_B/4\dot{\rho}_\gamma$. By the other side, in the delta sector we will also use a dimensionless fractional perturbation, however this perturbation is defined as the delta transformation of Eq. (101)

\[
\tilde{\delta}_\alpha \equiv \ddot{\delta}_\alpha = -\frac{\ddot{\rho}_\alpha}{\dot{\rho}_\alpha + \ddot{\rho}_\alpha} \frac{\ddot{\rho}_\alpha}{\dot{\rho}_\alpha} \delta_\alpha.
\]

(99)

in [2], they have found that

\[
\frac{\ddot{\rho}_R}{\dot{\rho}_R} = 2(C_2 - F(R)) \quad \text{and} \quad \frac{\ddot{\rho}_M}{\dot{\rho}_M} = \frac{3}{2} \frac{C_1 - F(R)}{F(R)}
\]

(100)

We will assume that this quotient holds for every species. Also using the result that $R^4 \ddot{\rho}_\gamma/(C_2 - F)$, $R^4 \ddot{\rho}_\nu/(C_2 - F)$, $R^3 \ddot{\rho}_D/(C_1 - F)$, $R^3 \ddot{\rho}_B/(C_1 - F)$ are time independent, the equations for the delta sector are

\[
\frac{d}{dt} \left( \frac{(1+S) \delta_{u\gamma}}{R} \right) - 2(C_2 - F) \frac{d}{dt} \left( \frac{(S - \tilde{S}) \delta_{u\gamma}}{R} \right) - F \frac{d}{dt} \left( \frac{(1+S) \delta_{u\gamma}}{R} \right) - 2\tilde{F} \frac{d}{dt} \left( \frac{(S - \tilde{S}) \delta_{u\gamma}}{R} \right) = 0 \quad (101)
\]

\[
\tilde{\gamma} = \frac{q^2}{R^2} (\delta_{u\gamma} + F \delta_{u\gamma}) + \tilde{\dot{\Psi}} - \dot{\theta}_0 (F \dot{\Psi}) = 0 \quad (102)
\]

\[
\tilde{\delta}_D + \tilde{\dot{\Psi}} - \dot{\theta}_0 (F \dot{\Psi}) = 0 \quad (103)
\]

\[
\tilde{\delta}_B - \frac{q^2}{R^2} (\delta_{u\gamma} + F \delta_{u\gamma}) + \tilde{\dot{\Psi}} - \dot{\theta}_0 (F \dot{\Psi}) = 0 \quad (104)
\]

\[
\tilde{\delta}_\nu - \frac{q^2}{R^2} (\delta_{u\gamma} + F \delta_{u\gamma}) + \tilde{\dot{\Psi}} - \dot{\theta}_0 (F \dot{\Psi}) = 0 \quad (105)
\]

with $\tilde{S} = 3\ddot{\rho}_D/4\dot{\rho}_\gamma$. Before to trying to find solutions validates up to recombination time, we need to find initial conditions that will be imposed. At sufficiently early times the universe was
dominated by radiation, and as Friedmann equations holds we can use the good aproximation
\[ R \propto \sqrt{t} \] and \[ 8\pi G \bar{\rho}_R/3 = 1/4t^2 \], while \( S \) and \( \tilde{S} \ll 1 \). Here
\[ \bar{\rho}_M \equiv \bar{\rho}_D + \bar{\rho}_B, \quad \bar{\rho}_R \equiv \bar{\rho}_\gamma + \bar{\rho}_\nu. \] (108)

Besides, we are interested on adiabatic solutions, in the sense that all the \( \delta_{aq} \) and \( \tilde{\delta}_{aq} \) become equal at very early times. So, if we make the ansatz,
\[ \delta_{\gamma q} = \delta_{\nu q} = \delta_{B q} = \delta_{D q} = \delta_q \quad \delta u_{\gamma q} = \delta u_{\nu q} = \delta u_q, \] (109)
\[ \tilde{\delta}_{\gamma q} = \tilde{\delta}_{\nu q} = \tilde{\delta}_{B q} = \tilde{\delta}_{D q} = \tilde{\delta}_q \quad \tilde{\delta} u_{\gamma q} = \tilde{\delta} u_{\nu q} = \tilde{\delta} u_q, \] (110)
Finally, we drop the term \( q^2/R^2 \) because we are considering very early times. Then Eqs. (92)-(98) becomes
\[ \frac{d}{dt} (t \Psi_q) = -\frac{1}{t} \delta_q, \] (111)
\[ \dot{\delta}_q = -\Psi_q, \] (112)
and
\[ \frac{d}{dt} \left( \frac{\delta u_q}{\sqrt{t}} \right) = -\frac{1}{t} \delta_q. \] (113)

While Eqs. (101)-(107) becomes
\[ \left[ 2\dot{F} \frac{\dot{R}}{R} + \ddot{F} \right] R^2 \Psi + \left[ 6F \frac{\dot{R}}{R} + \frac{5}{2} \ddot{F} \right] R^2 \dot{\Psi} + 3FR^2 \ddot{\Psi} \]
\[ -\frac{d}{dt} \left( R^2 \ddot{\Psi} \right) = \frac{R^2}{t^2} \left( \ddot{\delta} + (2C_2 - F)\delta \right), \] (114)
\[ \ddot{\delta} + \ddot{\Psi} - \partial_0 (F\Psi) = 0, \] (115)
\[ \frac{\ddot{\delta}}{3R} + \frac{d}{dt} \left( \frac{\delta u}{R} \right) - F \frac{d}{dt} \left( \frac{\delta u}{R} \right) = 0. \] (116)

Inspection of Eq. (85) show that at this era that for \( R \ll C \) we have \( F \propto -L_2 R \sqrt{\Omega}/3 \) (if \( C_1 = C_2 = 0 \)), also \( F \propto -2C_2 \). Also, in DG time is related with \( R(t) \) as
\[ t(Y) = \frac{2\sqrt{C}}{3H_0 \sqrt{\Omega}_R} \left( \sqrt{Y + C(Y - 2C)} + 2C^\frac{3}{2} \right), \] (117)
where \( Y = R/R_0 = R \) assuming \( R_0 = 1 \), \( H_0 = \dot{R}_0/R_0 \) is the usual Hubble parameter which we recall this is not longer the physical Hubble parameter observed by Planck\(^9\) and Riess\(^14\). \( \Omega_R \) is the density for radiation. So in radiation era time and \( R(t) \) was related by \( R(t) = (3H_0 \sqrt{\Omega_R})^{1/2} t^{1/2} \). This complete system Eqs. (111)-(113) and Eqs. (114)-(116) has analytical solution \( (C_1 = C_2 = 0) \):
\[ \delta_{\gamma q} = \delta_{B q} = \delta_{D q} = \delta_{\nu q} = \frac{q^2 t^2 R_q}{R^2}, \] (118)
\[ \dot{\Psi}_q = -\frac{tq^2 R_q}{R^2}, \] (119)
\[ \delta u_{\gamma q} = \delta u_{\nu q} = -\frac{2t^3 q^2 R_q}{9 R^2}. \]  

(120)

Where

\[ q^2 R_q \equiv -R^2 H_q + 4\pi G R^2 \delta \rho_q + q^2 H \delta u_q, \]  

(121)

is a gauge invariant quantity, which take a time independent value for \( q/R \ll H \). In the other side, we get

\[ \tilde{\delta} = -\frac{L^2 \sqrt{C} q^2 R_q t^2}{3 R}, \]  

(122)

\[ \dot{\Psi} = \frac{L^2 \sqrt{C} q^2 R_q t}{R}, \]  

(123)

\[ \delta \tilde{u} = \frac{L^2 \sqrt{C} q^2 R_q t^3}{R}. \]  

(124)

Note that Eq. \( \text{(93)} \)–ec. \( \text{(95)} \) gives

\[ \frac{d}{dt} (\delta_{\gamma} - \delta_B) = 0, \]  

(125)

this imply that if we start from adiabatic solutions \( \delta_{\gamma} = \delta_B \) holds over all the universe evolution (the same happen for its delta version, from Eq. \( \text{(102)} \)–Eq \( \text{(104)} \)).

3.1. Matter era

In this era we have \( R \propto t^{2/3} \), so (still using \( S = \tilde{S} = 0 \)) we have

\[ \frac{d}{dt} (a^2 \Psi_q) = -4\pi G \rho_D a^2 \delta D_q \]  

(126)

\[ \delta D_q = -\Psi_q \]  

(127)

\[ \frac{d}{dt} \left( \frac{(1 + R) \delta u_{\gamma q}}{a} \right) = -\frac{1}{3a} \delta_{\gamma q} \]  

(128)

\[ \frac{d}{dt} \left( \frac{\delta u_{\nu q}}{a} \right) = -\frac{1}{3a} \delta_{\nu q} \]  

(129)

while the delta sector

\[ \left[ \frac{2F \dot{R}}{R} + \ddot{F} \right] R^2 \Psi + \left[ 6F \frac{\dot{R}}{R} + \frac{5}{2} \ddot{F} \right] R^2 \dot{\Psi} + 3FR^2 \ddot{\Psi} \]  

\[ -\frac{d}{dt} \left( R^2 \dot{\Psi} \right) = \frac{2R^2}{3t^2} \left( \delta_D + (3C_1 - F) \frac{\delta D}{2} \right) \]  

(130)

\[ \dot{\delta}_\gamma - \frac{q^2}{R^2} (\delta \tilde{u}_\gamma + F \delta u_\gamma) + \dot{\Psi} - \delta_0 (F \Psi) = 0 \]  

(131)

\[ \dot{\delta}_D + \dot{\Psi} - \delta_0 (F \Psi) = 0 \]  

(132)

\[ \frac{\dot{\delta}_\nu}{3R} + \frac{d}{dt} \left( \frac{\delta \tilde{u}_\nu}{R} \right) - F \frac{d}{dt} \left( \frac{\delta u_\gamma}{R} \right) = \]  

(133)
in this era

\[ R(t) = \left( \frac{3H_0\sqrt{\Omega_R}}{2\sqrt{C}} \right)^{2/3} t^{2/3} \]  

(134)

and (using \( C_3 = -C^{3/2}L_2/3 \))

\[ F(t) = \frac{3}{2} (2C_2 - C_1) \frac{Y}{C} \left( \sqrt{\frac{Y}{C}} + 1 \ln \left( \frac{\sqrt{\frac{Y}{C}} + 1}{\sqrt{\frac{Y}{C}} - 1} \right) - 2 \right) - 2C_2 + C_3 \frac{Y}{C} \sqrt{\frac{Y}{C} + 1} \]

\[ \sim -\frac{L_2}{3} R(t)^{3/2} \]  

(135)

In order to get all transfer functions we have to compare solutions with the full equation system (with \( \rho_B = \tilde{\rho}_B = 0 \)). To do this task lets make the change of variable

\[ \frac{d}{dt} = \frac{H_{EQ} \sqrt{1 + y}}{\sqrt{2}} \frac{d}{dy} \]  

(136)

Also, we will use the following parametrization for all perturbations

\[ \delta_{Dq} = k^2 R^0_d(y)/4, \quad \delta_{\gamma q} = \delta_{\nu q} = k^2 R^0_d \gamma(y)/4, \]

\[ \dot{\psi}_q = (k^2 H_{EQ}/4\sqrt{2}) R^0_q f(y), \quad \delta u_{\nu q} = \delta u_{\nu q} = (k^2 \sqrt{2}/4 H_{EQ}) R^0_q \gamma(y), \]

and

\[ \delta_{Dq} = k^2 R^0_d \tilde{d}(y)/4, \quad \delta_{\gamma q} = \delta_{\nu q} = k^2 R^0_d \tilde{\gamma}(y)/4, \]

\[ \dot{\tilde{\psi}}_q = (k^2 H_{EQ}/4\sqrt{2}) R^0_q \tilde{f}(y), \quad \delta \tilde{u}_{\nu q} = \delta \tilde{u}_{\nu q} = (k^2 \sqrt{2}/4 H_{EQ}) R^0_q \tilde{\gamma}(y). \]

Then Eqs. (126)-(129) and Eqs. (130)-(133) become

\[ \sqrt{1 + y} \frac{d}{dy} \left( y^2 f(y) \right) = -\frac{3}{2} d(y) - \frac{4r(y)}{y} \]  

(137)

\[ \sqrt{1 + y} \frac{d}{dy} r(y) = \frac{k^2 g(y)}{y} - k^2 g(y) \]  

(138)

\[ \sqrt{1 + y} \frac{d}{dy} d(y) = -y f(y) \]  

(139)

\[ \sqrt{1 + y} \frac{d}{dy} \left( \frac{g(y)}{y} \right) = -\frac{r(y)}{3} \]  

(140)

and

\[ - \left[ (1 + 2y) y F'(y) + y(1 + y) F''(y) \right] d(y) + \left[ 6F(y) + \frac{5}{2} y F'(y) \right] y \sqrt{1 + y} f(y) + 3 F(y)y^2 \sqrt{1 + y} f'(y) \]

\[ - \sqrt{1 + y} \frac{d}{dy} \left( y^2 \tilde{f}(y) \right) = \frac{3d(y)}{2} + \frac{4\tilde{r}(y)}{y} + \left( \frac{3}{2} C_1 - \frac{F(y)}{2} \right) \frac{3d(y)}{2} + (2C_2 - F) \frac{4r(y)}{y} \]  

(141)

\[ \sqrt{1 + y} \frac{d}{dy} \tilde{d}(y) = -y \tilde{f}(y) - \sqrt{1 + y} \frac{d}{dy} d(y) \]  

(142)

\[ \sqrt{1 + y} \frac{d}{dy} \tilde{r}(y) = \frac{k^2}{y} \left[ \tilde{g}(y) + F(y) \gamma(y) \right] - y \tilde{f}(y) - \sqrt{1 + y} \frac{d}{dy} d(y) \]  

(143)

\[ \sqrt{1 + y} \frac{d}{dy} \left( \frac{\tilde{g}(y)}{y} \right) = -\frac{\tilde{r}(y)}{3} + \sqrt{1 + y} F(y) \frac{d}{dy} \left( \frac{g(y)}{y} \right) \]  

(144)
In this notation, initial conditions are
\[ d(y) \rightarrow r(y) \rightarrow y^2 \]
\[ f(y) \rightarrow -2 \quad g(y) \rightarrow -\frac{y^4}{9} \]
while for delta sector (with \( C_1 = C_2 = 0 \))
\[ \tilde{d}(y) = \tilde{r}(y) = -\frac{L_2 C^{3/2}}{3} y^3 \]
\[ \tilde{f}(y) = \sqrt{2} L_2 C^{3/2} y \quad \tilde{g}(y) = \frac{L_2 C^{3/2}}{2} y^5 \]

We do not present numerical solutions here because the aim of this work is to trace a guide for future works, in particular in the numeric computation of multipole coefficients for temperature fluctuations in the CMB.

4. Derivation of temperature fluctuations
It is possible to find expressions analogous to temperature fluctuations usually obtained by Boltzmann equations by studying photons propagation in FLRW perturbed coordinates, with the condition \( \bar{g}_{i0} = 0 \). For DG the metric which photons follow is given by
\[ g_{00} = -((1+3F(t))c^2 + E(x, t) + \tilde{E}(x, t)) \quad g_{0j} = R^2(t)(1+3F(t))\delta_{ij} + h_{ij}(x, t) + \tilde{h}_{ij}(x, t) \quad (145) \]
A raylight that is propagating in direction to the center of FLRW coordinate system from a direction \( \hat{n} \) will have a comovil radial coordinate \( r \) related with \( t \) by
\[ 0 = g_{\mu
u}dx^\mu dx^\nu = -((1+3F(t))c^2 + E(r\hat{n}, t) + \tilde{E}(r\hat{n}, t))dt^2 + (R^2(t)(1+3F(t)))h_{rr}(r\hat{n}, t) + \tilde{h}_{rr}(r\hat{n}, t))dr^2, \]
in other words
\[ \frac{dr}{dt} = -\left( \frac{(1+3F(t))c^2 + E + \tilde{E}}{R^2(t)(1+3F(t)) + h_{rr} + \tilde{h}_{rr}} \right)^{1/2} \simeq -\frac{c}{R(t)} + \frac{c(h_{rr} + \tilde{h}_{rr})}{2(1+3F(t))R^3(t)} - \frac{E + \tilde{E}}{2(1+3F(t))cR(t)} \quad (146) \]
where \( \tilde{R}(t) \) is the modified scale factor given by
\[ \tilde{R}(t) = R(t) \sqrt{\frac{1 + F(t)}{1 + 3F(t)}} \quad (147) \]
Now we will use the approximation of a sharp transition between opaque and transparent universe at a moment \( t_L \) of last scattering, at red shift \( z \simeq 1090 \). With this approximation, the relevant term at first order in Eq. (147) is
\[ r(t) = c \left[ s(t) + \int_{t_L}^t \frac{dt'}{R(t')} N(cs(t')\hat{n}, t') \right], \quad (149) \]
where
\[ N(x, t) \equiv \frac{1}{2(1+3F)} \left[ \frac{h_{rr}(x, t) + \tilde{h}_{rr}(x, t)}{R^2} - \frac{E(x, t)}{c^2} - \frac{\tilde{E}(x, t)}{c^2} \right], \quad (150) \]
and \( s(t) \) is the zero order solution for the radial coordinate which has a value of \( r_L \) when \( t = t_L \):

\[
s(t) = r_L - \int_{t_L}^{t} \frac{dt'}{R(t')} = \int_{t}^{t_0} \frac{dt'}{R(t')}. \tag{151}
\]

In particular, if a raylight arrives to \( r = 0 \) at a time \( t_0 \), then Eq. (149) gives

\[
0 = s(t_0) + \int_{t_L}^{t} \frac{dt'}{R(t')} N(cs(t')\hat{n}, t') = r_L + \int_{t_L}^{t_0} \frac{dt}{R(t)} (N(cs(t)\hat{n}, t) - 1). \tag{152}
\]

In an interval of time \( \delta t_L \) between departure of successive raylights at a time \( t_L \) of last scattering produces an interval of time \( \delta t_0 \) between the arrival of the raylights at \( t_0 \) given by the variation of Eq. (152):

\[
0 = \frac{\delta t_L}{R(t_L)} \left[ 1 - N(csL\hat{n}, t_L) + c \int_{t_L}^{t_0} \frac{dt}{R(t)} \left( \frac{\partial N(r(t)\hat{n}, t)}{\partial r} \right) \right] + \delta t_L \left( \delta u_L'(crL\hat{n}, t_L) + \delta u_L''(crL\hat{n}, t_L) \right) + \frac{\delta t_0}{R(t_0)} [-1 + N(0, t_0)]. \tag{153}
\]

The terms of velocities of the photon gas or photon-electron-nucleon arise for the variation respect to the time of the radial coordinate \( r_L \) since the emission of light in Eq. (152). The exchange rate of \( N(s(t)\hat{n}, t) \)

\[
\frac{d}{dt} N(s(t)\hat{n}, t) = \left( \frac{\partial}{\partial t} N(r\hat{n}, t) \right)_{r = cs(t)} - c \frac{1}{R(t)} \left( \frac{\partial N(r\hat{n}, t)}{\partial r} \right)_{r = cs(t)},
\]

then in Eq. (153) we get

\[
0 = \frac{\delta t_L}{R(t_L)} \left[ 1 - N(0, t_L) + \int_{t_L}^{t_0} dt \left( \frac{\partial N(r\hat{n}, t)}{\partial t} \right) \right] + \delta t_L (\delta u_L'(rL\hat{n}, t_L) + \delta u_L''(rL\hat{n}, t_L)) + \frac{\delta t_0}{R(t_0)} [-1 + N(0, t_0)]. \tag{154}
\]

This result gives the ratio between the intervals of time coordinate between raylights are emitted and received, however we are interested in this ratio but for proper time, which in DG is defined with the original metric \( g_{\mu\nu} \)

\[
\delta \tau_L = \sqrt{1 + \frac{E(rL, t_L)}{c^2}} \delta t_L, \quad \delta \tau_0 = \sqrt{1 + \frac{E(0, t_0)}{c^2}} \delta t_0, \tag{155}
\]

which at first order gives the ratio of an received frequency over an emitted one as

\[
\frac{\nu_0}{\nu_L} = \frac{\delta \tau_L}{\delta \tau_0} = \frac{\tilde{R}(t_L)}{R(t_0)} \left[ 1 + \frac{1}{2c^2} \left( E(rL\hat{n}, t) - E(0, t_0) \right) \right] - \int_{t_L}^{t_0} \left( \frac{\partial}{\partial t} N(r\hat{n}, t) \right)_{r = cs(t)} dt - \tilde{R}(t)(\delta u_L'(rL\hat{n}, t) + \delta u_L''(rL\hat{n}, t)) \tag{156}
\]

The observed temperature at the present time \( t_0 \) from direction \( \hat{n} \) is

\[
T(\hat{n}) = \left( \frac{\nu_0}{\nu_L} \right) (\tilde{T}(t_L) + \delta T(csL\hat{n}, t_l)), \tag{157}
\]
In absence of perturbations the observed temperature in all direction should be

$$T_0 = \left( \frac{\tilde{R}(t_L)}{R(t_0)} \right) T(t_L),$$  \hspace{1cm} (158)

so the ratio of the temperature shift observed which comes from direction \( \hat{n} \) over its unperturbed value is

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{T(\hat{n}) - T_0}{T_0} = \frac{\nu_0 \tilde{R}(t_0)}{\nu_L R(t_L)} - 1 + \frac{\delta T(c r_L \hat{n}, t_L)}{T(t_L)}$$

$$= \frac{1}{2c^2} (E(r_L \hat{n}, t) - E(0, t_0)) - \int_{t_L}^{t_0} dt \left( \frac{\partial}{\partial t} N(r \hat{n}, t) \right)_{r=cs(t)}$$

$$- \frac{\dot{R}(t)(\delta u^r \gamma(r_L, \hat{n}, t) + \delta \tilde{u}^r \gamma(r_L, \hat{n}, t)) + \frac{\delta T(c r_L \hat{n}, t_L)}{T(t_L)}}{T(t_L)}. \hspace{1cm} (159)$$

For scalar perturbations in any gauge with \( h_{00} = 0 \), metric perturbations are given by

$$h_{00} = -E, \quad h_{ij} = (1 + F) R^2 \left[ A \delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} \right]$$

$$\tilde{h}_{00} = -\tilde{E}, \quad \tilde{h}_{ij} = (1 + F) R^2 \left[ \tilde{A} \delta_{ij} + \frac{\partial^2 \tilde{B}}{\partial x^i \partial x^j} \right] \hspace{1cm} (160)$$

Besides for scalar perturbations radial velocity of the photon fluid and its delta version are given in terms of the potencial of velocities \( \delta u_\gamma \) and \( \delta \tilde{u}_\gamma \) respectively

$$\delta u^r \gamma = (\bar{g} + \tilde{g})^r \nu \frac{\partial \delta u_{\nu}}{\partial x^\nu} = \frac{1}{(1 + F(t)) R^2} \frac{\partial \delta u_{\nu}}{\partial r}$$

$$\delta \tilde{u}^r \gamma = (\bar{g} + \tilde{g})^r \frac{\partial \delta \tilde{u}_{\nu}}{\partial x^\nu} = \frac{1}{(1 + F(t)) R^2} \frac{\partial \delta \tilde{u}_{\nu}}{\partial r}. \hspace{1cm} (161)$$

Then Eq. (159) gives the scalar contribution to temperature fluctuations

$$\left( \frac{\Delta T(\hat{n})}{T_0} \right)^S = \frac{1}{2c^2} (E(r_L \hat{n}, t) - E(0, t_0)) - \int_{t_L}^{t_0} dt \left( \frac{\partial}{\partial t} N(r \hat{n}, t) \right)_{r=cs(t)}$$

$$- \frac{1}{(1 + 3F(t)) R} \left( \frac{\partial \delta u_{\nu} (c r_L \hat{n}, t)}{\partial r} + \frac{\partial \delta \tilde{u}_{\nu} (c r_L \hat{n}, t)}{\partial t} \right) + \frac{\delta T(c r_L \hat{n}, t_L)}{T(t_L)}. \hspace{1cm} (162)$$

where now

$$N = \frac{1}{2} \left[ A + \frac{\partial^2 B}{\partial r^2} \right] + \left( \frac{\tilde{A} + \frac{\partial^2 \tilde{B}}{\partial r^2}}{1 + 3F} \right) - \frac{E}{1 + 3F} - \frac{\tilde{E}}{1 + 3F} \hspace{1cm} (163)$$

In the next step we will study the gauge transformations of these fluctuations, for this we will use the following identity for the fields \( B \) and \( \tilde{B} \)

$$\left( \frac{\partial^2 \tilde{B}}{\partial r^2} \right)_{r=s(t)} = -\left( \frac{d}{dt} \left( R \frac{\partial \tilde{B}}{\partial r} + \tilde{R} \frac{\partial \tilde{B}}{\partial r} + \tilde{R}^2 \tilde{B} \right) + \frac{\partial}{\partial t} \left( \tilde{R} \frac{\partial \tilde{B}}{\partial r} + \tilde{R}^2 \tilde{B} \right) \right)_{r=s(t)}. \hspace{1cm} (164)$$

Then temperature fluctuations becomes:

$$\left( \frac{\Delta T(\hat{n})}{T_0} \right)^S = \left( \frac{\Delta T(\hat{n})}{T_0} \right)_{early} + \left( \frac{\Delta T(\hat{n})}{T_0} \right)_{late} + \left( \frac{\Delta T(\hat{n})}{T_0} \right)_{ISW}. \hspace{1cm} (165)$$
where

\[
\left( \frac{\Delta T(\hat{n})}{T_0} \right)_{early} = -\frac{1}{2} \hat{R}(t_L) \hat{B}(r_L \hat{n}, t_L) - \frac{1}{2} \hat{R}^2(t_L) \bar{B}(r_L \hat{n}, t_L) + \frac{1}{2} E(r_L \hat{n}, t_L) + \frac{\delta T(r_L \hat{n})}{T(t_L)} - \hat{R}(t_L) \left[ \frac{\partial}{\partial r} \left( \frac{1}{2} \hat{B}(r \hat{n}, t_L) + \frac{1}{(1 + 3F)\hat{R}^2(t_L)} \hat{\delta u}_r(r \hat{n}, t_L) \right) \right]_{r=r_L}
\]

\[
- \left\{ \left( \frac{1}{2} \hat{R}(t_L) \hat{B}(r_L \hat{n}, t_L) + \frac{1}{2} \hat{R}^2(t_L) \bar{B}(r_L \hat{n}, t_L) \right) + \hat{R}(t_L) \left[ \frac{\partial}{\partial r} \left( \frac{1}{2} \hat{B}(r \hat{n}, t_L) + \frac{1}{(1 + 3F)\hat{R}^2(t_L)} \hat{\delta u}_r(r \hat{n}, t_L) \right) \right] \right\}
\]

\[\text{(167)}\]

\[
\left( \frac{\Delta T(\hat{n})}{T_0} \right)_{late} = \frac{1}{2} \hat{R}(t_0) \hat{R}(0, t_0) + \frac{1}{2} \hat{R}^2(t_0) \bar{B}(0, t_0) - \frac{1}{2} E(0, t_0) + \hat{R}(t_0) \left[ \frac{\partial}{\partial r} \left( \frac{1}{2} \hat{B}(r \hat{n}, t_0) + \frac{1}{(1 + 3F)\hat{R}^2(t_0)} \hat{\delta u}_r(r \hat{n}, t_0) \right) \right]_{r=0}
\]

\[
+ \left\{ \left( \frac{1}{2} \hat{R}(t_0) \hat{B}(0, t_0) + \frac{1}{2} \hat{R}^2(t_0) \bar{B}(0, t_0) \right) + \hat{R}(t_0) \left[ \frac{\partial}{\partial r} \left( \frac{1}{2} \hat{B}(r \hat{n}, t_0) + \frac{1}{(1 + 3F)\hat{R}^2(t_0)} \hat{\delta u}_r(r \hat{n}, t_0) \right) \right] \right\}
\]

\[\text{(168)}\]

The “late” term is the sum of direction independent terms and a term proportional to \( \hat{n} \), which was added in order to represent the local anisotropies of the gravitational field and the local fluid. In GR this terms only contribute in the multipolar expansion for \( l = 0 \) and \( l = 1 \), so we will ignore their contribution even for DG.

### 4.1. Gauge transformations

Let’s study gauge transformations for photons propagating in the metric \( g_{\mu\nu} \) for a parameter \( \epsilon_\mu \), with this metric we get

\[
\Delta A = \frac{2R}{(1 + F)RF^2(1 + 3F)} \epsilon_0, \quad \Delta B = -\frac{2}{1 + F(1 + 3F)} \epsilon_0, \quad \Delta C_i = -\frac{1}{1 + F(1 + 3F)} \epsilon_i, \quad \Delta D_{ij} = 0, \quad \Delta E = 2\frac{\partial}{\partial t} \left( \frac{\epsilon_0}{1 + 3F} \right), \quad \text{(169)}
\]

\[
\Delta H = -\frac{1}{\sqrt{1 + FR}} \left[ R^2 \frac{\partial}{\partial t} \left( \frac{\epsilon_0}{1 + 3F} \right) + \frac{\epsilon_0}{1 + 3F} \right], \quad \Delta G_i = -\frac{R}{\sqrt{1 + FR}} \frac{\partial}{\partial t} \left( \frac{\epsilon_i}{1 + 3F} \right), \quad \text{and}
\]

\[
\Delta \tilde{A} = \frac{1}{(1 + F)RF^2} \left[ \frac{\partial}{\partial t} \left( FR^2 \right) \frac{\epsilon_0}{1 + 3F} \right], \quad \Delta \tilde{B} = -\frac{1}{(1 + F)RF^2} \left[ \frac{2F}{1 + F} \epsilon_0 + \frac{3F}{1 + 3F} \epsilon_0 \right], \quad \Delta \tilde{C}_i = -\frac{F}{1 + F(1 + 3F)} \epsilon_i, \quad \Delta \tilde{D}_{ij} = 0, \quad \Delta \tilde{E} = 6F \frac{\partial}{\partial t} \left( \frac{\epsilon_0}{1 + 3F} \right) + \frac{3F}{1 + 3F} \epsilon_0.
\]

\[
\Delta \tilde{H} = -\frac{1}{\sqrt{1 + FR}} \left[ FR^2 \frac{\partial}{\partial t} \left( \frac{\epsilon_0}{1 + 3F} \right) + \frac{3F \epsilon_0}{1 + 3F} \right], \quad \Delta \tilde{G}_i = -\frac{1}{\sqrt{1 + FR}} \left[ FR^2 \frac{\partial}{\partial t} \left( \frac{\epsilon_i}{1 + 3F} \right) \right]. \quad \text{(170)}
\]
Also, considering the sum of the perturbations
\[ \Delta A + \Delta \tilde{A} = \frac{1}{(1+F)R^2} \frac{\partial}{\partial t}[(1+F)R^2] \frac{\epsilon_0}{1+3F} \] (171)
\[ \Delta B + \Delta \tilde{B} = -\frac{2\epsilon S}{(1+F)R^2} \] (172)
\[ \Delta E + \Delta \tilde{E} = 2(1+3F) \frac{\partial}{\partial t} \left( \frac{\epsilon_0}{1+3F} \right) + \frac{3\dot{F}}{1+3F} \epsilon_0 \] (173)
\[ \Delta H + \Delta \tilde{H} = -\frac{1}{\sqrt{1+FR}} \left[ (1+F)R^2 \frac{\partial}{\partial t} \left( \frac{\epsilon S}{(1+F)R^2} \right) + \epsilon_0 \right] \] (174)
\[ \Delta C_i + \Delta \tilde{C}_i = -\frac{\epsilon_i^V}{(1+F)R^2} \] (175)
\[ \Delta G_i + \Delta \tilde{G}_i = -\frac{1}{\sqrt{1+FR}} \left[ (1+F)R^2 \frac{\partial}{\partial t} \left( \frac{\epsilon_i^V}{(1+F)R^2} \right) \right] \] (176)

Now we will study gauge transformations which preserve the condition \( g_{i0} = g_{i0} + \tilde{g}_{i0} = 0 \), it means \( \Delta H + \Delta \tilde{H} = 0 \), this gives us a solution for \( \epsilon_0 \) given by
\[ \epsilon_0 = -(1+F)R^2 \frac{\partial}{\partial t} \left( \frac{\epsilon S}{(1+F)R^2} \right) \] (177)

When we study how “ISW” term transform under this type of transformations we found that \( \Delta ISW = 0 \). While for the “early” term we should note that temperature perturbations transforms as
\[ \Delta \delta T(r_L \hat{n}, t) = \dot{T}(t) \frac{\epsilon_0}{1+3F}, \] (178)
then, as \( \dot{T} \dot{R} = cte \), we get
\[ \frac{\Delta \delta T(r_L \hat{n}, t)}{T(t_L)} = \frac{\dot{R}}{R} \frac{\epsilon_0}{1+3F} \] (179)

Using this result, we also get that the “early” term is invariant under this gauge transformation.

4.2. Single modes
Now we will assume that since last scattering until now all the scalar contributions are dominated by an unique mode, so that any perturbation \( X(x,t) \) could be written as
\[ X(x,t) = \int d^3q \alpha(q)e^{i\mathbf{q} \cdot \mathbf{x}} X_q(t), \] (180)
with \( \alpha(q) \) an stochastic variable, normalized so that
\[ \langle \alpha(q)\alpha^*(q') \rangle = \delta^3(\mathbf{q} - \mathbf{q'}) \] (181)
Then Eqs. (166) and (168) become
\[
\left( \frac{\Delta T(\hat{n})}{T_0} \right)^S_{early} = \int d^3 \mathbf{q} \alpha(\mathbf{q}) e^{i \mathbf{q} \cdot \hat{n}(t_L)} \left( \mathcal{F}(q) + \dot{\mathcal{F}}(q) + i \dot{q} \cdot \mathcal{G}(q) + \ddot{\mathcal{G}}(q) \right) \tag{182}
\]
\[
\left( \frac{\Delta T(\hat{n})}{T_0} \right)^S_{ISW} = -\frac{1}{2} \int_{t_0}^{t_1} dt \int d^3 \mathbf{q} \alpha(\mathbf{q}) e^{i \mathbf{q} \cdot \hat{n}(t)} \frac{d}{dt} \left[ \ddot{R}^2(t) \dot{B}_q(t) + \ddot{R}(t) \dot{R}(t) \dot{B}_q(t) + A_q(t) - \frac{E_q(t)}{1 + 3F(t)} \right] + \left( \ddot{R}^2(t) \dot{B}_q(t) + \ddot{R}(t) \dot{R}(t) \dot{B}_q(t) + \ddot{A}_q(t) - \frac{E_q(t)}{1 + 3F(t)} \right) \tag{183}
\]
where
\[
\mathcal{F}(q) = -\frac{1}{2} \ddot{R}^2(t) \dot{B}_q(t_L) - \frac{1}{2} \ddot{R}(t) \dot{R}(t_L) \dot{B}_q(t_L) + \frac{1}{2} E_q(t_L) + \frac{\delta T_q(t_L)}{T(t_L)} \tag{184}
\]
\[
\dot{\mathcal{F}}(q) = -\frac{1}{2} \ddot{R}^2(t) \dot{B}_q(t_L) - \frac{1}{2} \ddot{R}(t_L) \dot{R}(t_L) \dot{B}_q(t_L) \tag{185}
\]
\[
\mathcal{G}(q) = -q \left( \frac{1}{2} \ddot{R}(t_L) \dot{B}_q(t_L) + \frac{1}{1 + 3F(t_L)} \dot{R}(t_L) \delta u_r(t_L) \right) \tag{186}
\]
\[
\ddot{\mathcal{G}}(q) = -q \left( \frac{1}{2} \ddot{R}(t_L) \dot{B}_q(t_L) + \frac{1}{1 + 3F(t_L)} \dot{R}(t_L) \delta u_r(t_L) \right) \tag{187}
\]
are the so-called form factors. We emphasize that the combination for the form factors \( \mathcal{F}(q) + \dot{\mathcal{F}}(q) \) and \( \mathcal{G}(q) + \ddot{\mathcal{G}}(q) \) and the expression inside the integral are gauge invariants under gauge transformation which preserve \( g_0 \) equal to zero.

5. Coefficients of multipolar temperature expansion: Scalar modes
Now, as an application of the previous results, we will study the contribution of the scalar modes for temperature-temperature correlation, given by:
\[
C_{TT, l} = \frac{1}{4\pi} \int d^2 \mathbf{n} \int d^2 \mathbf{n'} P_l(\mathbf{n} \cdot \mathbf{n'}) \langle \Delta T(\hat{n}') \Delta T(\hat{n}) \rangle , \tag{188}
\]
where \( \Delta T(\hat{n}) \) is the stochastic variable which gives the deviation of the average of observed temperature in direction \( \hat{n} \), and \( \langle ... \rangle \) denotes the average over the position of the observer. However this quantity is not the observed one, if not
\[
C^{obs}_{TT, l} = \frac{1}{4\pi} \int d^2 \mathbf{n} \int d^2 \mathbf{n'} P_l(\mathbf{n} \cdot \mathbf{n'}) \Delta T(\hat{n}) \Delta T(\hat{n}'), \tag{189}
\]
nevertheless, the mean square fractional difference between this and Eq. (188) is \( 2/(2l + 1) \), and therefore may be neglected for \( l \gg 1 \).
In order to calculate this coefficients we use the following expansion in spherical harmonics
\[
e^{i \hat{n} \cdot \mathbf{q}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \tilde{d}(\rho) Y_l^m(\hat{n}) Y_l^{m*}(\hat{q}) \tag{190}
\]
with \( \tilde{d}(\rho) \) the spherical Bessel’s functions. Using this in Eq. (182), and replacing the factor \( i \dot{q} \cdot \hat{n} \) for time derivatives of Bessel’s functions, the scalar contribution of T-T fluctuations observed in direction \( \hat{n} \) are
\[
\langle \Delta T(\hat{n}) \rangle^S = \sum_{l=1}^S d_{l\alpha} T_{l\alpha} Y_l^m(\hat{n}) , \tag{191}
\]
where
\[
a_{TT,lm}^S = 4\pi l^3 T_0 \int d^3q \alpha(q) Y_l^{i*}(\hat{q}) \left[ j_i(qr_L)(F(q) + \bar{F}(q)) + j'_i(qr_L)(G(q) + \bar{G}(q)) \right].
\]

(192)

Here \( \alpha(q) \) is the stochastic parameter for the dominant scalar mode, normalized so that
\[
\langle \alpha(q)\alpha^*(q') \rangle = \delta^3(q - q').
\]

(193)

Inserting this in Eq. (188) we get
\[
C_{TT,l}^S = 16\pi^2 T_0^2 \int_0^\infty q^2 dq \left[ j_i(qr_L)(F(q) + \bar{F}(q)) + j'_i(qr_L)(G(q) + \bar{G}(q)) \right]^2
\]

(194)

Now we will consider the case \( l \gg 1 \). In this limit we can use the following approximation for Bessel’s functions:[15]:
\[
j_i(\rho) \rightarrow \begin{cases} \cos(b) \cos \left[ (\nu(tan b - b) - \pi/4) / (\nu \sqrt{\sin b}) \right] & \rho > \nu, \\ 0 & \rho < \nu, \end{cases}
\]

(195)

where \( \nu \equiv l + 1/2 \), and \( \cos b \equiv \nu/\rho \), with \( 0 \leq b \leq \pi/2 \). Besides, for \( \rho > \nu \gg 1 \) the phase \( \nu(tan b - b) \) is a function of \( \rho \) that grows very fast, so derivatives on Bessel’s functions only acts in its phase:
\[
j'_i(\rho) \rightarrow \begin{cases} -\cos(b)\sqrt{\sin b} \sin \left[ (\nu(tan b - b) - \pi/4) / \nu \right] & \rho > \nu, \\ 0 & \rho < \nu. \end{cases}
\]

(196)

Using this limits in Eq. (194) and doing the change of integration variable from \( q \) to \( b = \cos^{-1}(\nu/qr_L) \) we get
\[
C_{TT,l}^S = \frac{16\pi^2 T_0^2}{r_L^3} \int_0^{\pi/2} \frac{db}{\cos^2 b} \left[ \mathcal{F} \left( \frac{\nu}{r_L \cos b} \right) + \bar{\mathcal{F}} \left( \frac{\nu}{r_L \cos b} \right) \cos[\nu(tan b - b) - \pi/4] \right.
\]
\[
- \sin b \left( \mathcal{G} \left( \frac{\nu}{r_L \cos b} \right) + \bar{\mathcal{G}} \left( \frac{\nu}{r_L \cos b} \right) \right) \sin[\nu(tan b - b) - \pi/4] \left. \right] \right)^2.
\]

(197)

As we said, for \( \nu \gg 1 \) functions \( \cos[\nu(tan b - b) - \pi/4] \) and \( \sin[\nu(tan b - b) - \pi/4] \) oscillate very rapidly, so the average of is values at square are 1/2, while the cross terms in average are zero. Dropping the distinction between \( l \) and \( \nu = l + 1/2 \), and doing another change of variable of integration, from \( b \) to \( \beta = 1/\cos b \), Eq. (197) then becomes
\[
l(l+1)C_{TT,l}^S = \frac{8\pi^2 T_0^2 \beta^3}{r_L^3} \int_1^\infty \frac{\beta d\beta}{\sqrt{\beta^2 - 1}} \left[ \mathcal{F} \left( \frac{l\beta}{r_L} \right) + \bar{\mathcal{F}} \left( \frac{l\beta}{r_L} \right) \right]^2 + \frac{\beta^2 - 1}{\beta^2} \left( \mathcal{G} \left( \frac{l\beta}{r_L} \right) + \bar{\mathcal{G}} \left( \frac{l\beta}{r_L} \right) \right)^2.
\]

(198)

with \( d_A = r_L \tilde{R}_L \) is the angular diameter distance of the surface of last scattering. In order to follow with the calculation we need to know the value of \( \bar{B}_q \), to find it we use the off diagonal equation from Delta sector which give us
\[
\dot{A}_q = \dot{A}_q F + A_q \dot{F} - 2R^2(\rho + p)\delta u_q - R^2(\dot{\rho} + \ddot{p})\delta u_q - (\rho + p)\delta \bar{u}_q,
\]

(199)
so if we use this with the definition of $\tilde{\Psi}$:

$$\dot{\tilde{\Psi}} = \frac{1}{2}(3\dot{\tilde{A}}_q - q^2 \dot{\tilde{B}}_q)$$  \hspace{1cm} (200)$$

it allow us to find $\dot{\tilde{B}}$. Now we will use the approximation of that perturbations of gravitation field are dominated by perturbations of dark matter density. In this regime we have $\dot{A}_q(t_L) = 0$ and in the synchronic gauge velocity densities for Dark matter are zero, then

$$\dot{\tilde{A}}_q(t_L) = A_q(t_L)\dot{\tilde{F}}(t_L).$$  \hspace{1cm} (201)$$

thus

$$\dot{\tilde{B}}_q(t_L) = \frac{3}{q^2} A_q(t_L)\dot{\tilde{F}}(t_L) - \frac{2\tilde{\Psi}_q(t_L)}{q^2} = \frac{3}{q^2} A_q(t_L)\dot{\tilde{F}}(t_L) - \frac{2\tilde{\Psi}_q(t_L)}{q^2},$$  \hspace{1cm} (202)$$

where

$$q^2 A_q = 8\pi G R^2 \delta \rho_D - 2HR^2 \tilde{\Psi}_q = 3H^2 R^2 \delta Dq - 2HR^2 \tilde{\Psi}_q.$$  \hspace{1cm} (203)$$

In GR $\dot{B}_q = -2\tilde{\Psi}_q/q^2$, and $\tilde{\Psi}_q \propto t^{-1/3}$ implies $\dot{B}_q = 2\tilde{\Psi}_q/3tq^2$, this the usual form factors are:

$$\mathcal{F}(q) = \frac{1}{3}\delta_{\gamma q}(t_L) + \frac{\tilde{\Psi}_q(t_L)}{q^2} \left( \dot{R}(t_L)\ddot{R}(t_L) - \frac{2}{3} \frac{\dot{R}^2(t_L)}{t_L} \right),$$  \hspace{1cm} (204)$$

$$\mathcal{G}(q) = -q \frac{\delta u_{\gamma q}(t_L)}{(1 + 3F(t_L))R(t_L)} + \frac{\dot{R}(t_L)\tilde{\Psi}_q(t_L)}{q}.$$  \hspace{1cm} (205)$$

where we use $\delta T_d/T = \delta \rho_D/4\bar{\rho}_\gamma$ and $\tilde{\delta}_{\gamma q}/3$. While for the “delta” contribution $\dot{\tilde{\Psi}}_q$ and $\ddot{\tilde{\Psi}}_q$ satisfies the same relation than the standard case, due our decomposition, we have

$$\tilde{\mathcal{F}}(q) = -\frac{3}{2} \frac{A_q(t_L)}{q^2} \left( \dot{R}^2(t_L)\ddot{R}(t_L) + \dot{R}(t_L)\dddot{R}(t_L) \dot{\tilde{F}}(t_L) + \frac{\tilde{\Psi}_q(t_L)}{q^2} \left( \dot{R}(t_L)\ddot{R}(t_L) - \frac{2}{3} \frac{\dot{R}^2(t_L)}{t_L} \right) \right),$$  \hspace{1cm} (206)$$

$$\tilde{\mathcal{G}}(q) = -q \frac{\delta u_{\gamma q}(t_L)}{(1 + 3F(t_L))R(t_L)} + \frac{\dot{R}(t_L)\tilde{\Psi}_q(t_L)}{q}.$$  \hspace{1cm} (207)$$

Numerical solutions and some other considerations should be taken in order to compute the solution for the perturbations, however this will be part of a future work. It is remarkable the structure of eq. ([108]), delta sector contributes additively inside the integral, if we set all delta sector equal to zero we recover directly the result for scalar temperature-temperature multipole coefficients in GR given by Weinberg.

6. Conclusions
We had developed the theory of perturbation for Delta Gravity and its gauge transformation, extension of the usual frameworks was obtained with similar conclusions: Newtonian gauge fix completely the gauge freedom of the perturbations, while Synchronous gauge leaves a residual gauge transformation which can be use to set $\delta u_D = 0$ (and also $\tilde{\delta} u_D = 0$). Then we computed the equations for cosmological perturbations using the hidrodynamic approximation, which we solved for radiation era, while for a matter dominated Universe we presented the equations with
its initial conditions, however we did not solve them here because this will be done in a future work.

As in GR, we can find an expression for temperature fluctuations in DG, this was done studying the propagation of photons for an effective metric from the moment of last scattering until us, and the main result is that this expression is invariant under gauge transformation that preserve $g_{00}$ equal to zero. After this, the multipole coefficients for temperature were derived for scalar modes, here we found that DG affects additively, which could have an observational effect that could be compared with Planck results, and give a physical meaning for the so called “delta matter”.

With the full scalar expression for the CMB Power Spectrum coefficients, we can find the shape of the spectrum. In order to achieve it, we have to determine the best cosmological parameters that can describe the observational spectrum given by Planck [11]. The determination of the cosmological parameters could be difficult (from a computational point of view), but if we constraints the cosmological parameters with the SNe-Ia analysis [4] the determination of the CMB Power Spectrum in DG could be easier. In the context of the controversy about the $H_0$ value [14] and other problems as the curvature measurements [16] or the possibility of a universe with less Dark Energy [17], this work could provide an alternative to solve the today cosmological puzzle. A future work in this line is being carried out.

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