André-Quillen homology via functor homology

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We obtain André-Quillen homology for commutative algebras using relative homological algebra in the category of functors on finite pointed sets.

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1 Introduction.

Let \( \Gamma \) be the small category of finite pointed sets. For any \( n \geq 0 \), let \([n]\) be the set \( \{0, 1, ..., n\} \) with basepoint 0. We assume that the objects of \( \Gamma \) are the sets \([n]\). A left \( \Gamma \)-module is a covariant functor \( \Gamma \to \text{Vect} \) to the category of vector spaces over a field \( K \). For a left \( \Gamma \)-module \( F \) we put

\[
\pi_0(F) := \text{Coker}(d_0 - d_1 + d_2 : F([2]) \to F([1])),
\]

where \( d_1 \) is induced by the folding map \( [2] \to [1] \), \( 1, 2 \mapsto 1 \) while \( d_0 \) and \( d_2 \) are induced by the projection maps \( [2] \to [1] \) given respectively by \( 1 \mapsto 1, 2 \mapsto 0 \) and \( 1 \mapsto 0, 2 \mapsto 1 \). The category \( \Gamma\text{-mod} \) of left \( \Gamma \)-modules is an abelian category with enough projective and injective objects. Therefore one can form the left derived functors of the functor \( \pi_0 : \Gamma\text{-mod} \to \text{Vect} \), which we will denote by \( \pi_* \). Thanks to \cite{4} and \cite{5} we know that \( \pi_* F \) is isomorphic to the homotopy of the spectrum corresponding to the \( \Gamma \)-space \( F \) according to Segal (see \cite{4} and \cite{5}).

Let \( A \) be a commutative algebra over a ground field \( K \) and let \( M \) be an \( A \)-module. There exists a functor \( \mathcal{L}(A,M) : \Gamma \to \text{Vect} \), which assigns \( M \otimes A^\otimes[n] \) to \([n]\) (see \cite{4} or section 3). Here all tensor products are taken over \( K \). It was proved in \cite{4} that \( \pi_*(\mathcal{L}(A,M)) \) is isomorphic to a brave new algebra version of André-Quillen homology \( H^\Gamma(A,M) \) constructed by Alan Robinson and Sarah Whitehouse \cite{10}. The main result of this paper shows that a similar isomorphism exists also for André-Quillen homology if one takes an appropriate relative derived functors of the same functor \( \pi_0 : \Gamma\text{-mod} \to \text{Vect} \).

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A class of proper exact sequences

Thanks to the Yoneda lemma, \( \Gamma^n, n \geq 0 \), are projective generators of the category \( \Gamma\text{-mod} \). Here

\[ \Gamma^n : = K[\text{Hom}_\Gamma ([n], -)]. \]

and \( K[S] \) denotes the free vector space generated by a set \( S \). For left \( \Gamma \)-modules \( F \) and \( T \) one defines the pointwise tensor product \( F \otimes T \) to be the left \( \Gamma \)-module given by \( (F \otimes T)([n]) = F([n]) \otimes T([n]). \) Since \( \Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m} \) one sees that the tensor product of two projective left \( \Gamma \)-modules is still projective. We also have \( \Gamma^n \cong (\Gamma^1)^{\otimes n}. \)

A partition \( \lambda = (\lambda_1, \cdots, \lambda_k) \) is a sequence of natural numbers \( \lambda_1 \geq \cdots \geq \lambda_k \geq 1 \). The sum of partition is given by \( s(\lambda) := \lambda_1 + \cdots + \lambda_k \), while the group \( \Sigma(\lambda) \) is a product of the corresponding symmetric groups

\[ \Sigma(\lambda) := \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}. \]

which is identified with the Young subgroup of \( \Sigma_{s(\lambda)} \). Let us observe that \( \Sigma_n = \text{Aut}_\Gamma ([n]) \) and therefore \( \Sigma_n \) acts on \( \Gamma^n \cong (\Gamma^1)^{\otimes n} \). For a partition \( \lambda \) with \( s(\lambda) = n \) we let \( \Gamma(\lambda) \) be the coinvariants of \( \Gamma^n \) under the action of \( \Sigma(\lambda) \subset \Sigma_n \).

For a vector space \( V \) we let \( S^* (V) \), \( \Lambda^* (V) \) and \( D^* (V) \) be respectively the symmetric, exterior and divided power algebra generated by \( V \). Let us recall that \( S^n (V) = (V^{\otimes n})/\Sigma_n \) is the space of coinvariants of \( V^{\otimes n} \) under the action of the symmetric group \( \Sigma_n \), while \( D^n (V) = (V^{\otimes n})^{\Sigma_n} \) is the space of invariants. Moreover for a partition \( \lambda = (\lambda_1, \cdots, \lambda_k) \) we put

\[ S^\lambda := S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_k}. \]

A similar meaning has also \( \Lambda^\lambda \) and \( D^\lambda \). It follows from the definition that

\[ \Gamma(\lambda) \cong S^\lambda \circ \Gamma^1. \]

In particular \( \Gamma(1, \cdots, 1) \cong \Gamma^n \) and \( \Gamma(n) \cong S^n \circ \Gamma^1 \).

Let

\[ 0 \to T_1 \to T \to T_2 \to 0 \]

be an exact sequence of left \( \Gamma \)-modules. It is called a \( \mathcal{Y} \)-exact sequence if for any partition \( \lambda \) with \( s(\lambda) = n \) the induced map

\[ T([n])^{\Sigma(\lambda)} \to T_2([n])^{\Sigma(\lambda)} \]

is surjective. Here and elsewhere, \( M^G \) denotes the subspace of \( G \)-fixed elements of a \( G \)-module \( M \). For a \( \mathcal{Y} \)-exact sequence \( 0 \to T_1 \to T \to T_2 \to 0 \) the sequence

\[ 0 \to T_1([n])^{\Sigma(\lambda)} \to T([n])^{\Sigma(\lambda)} \to T_2([n])^{\Sigma(\lambda)} \to 0 \]
is also exact. Following to Section XII.4 of [1] we introduce the related notions. An epimorphism $f : F \to T$ is called $\mathcal{Y}$-epimorphism if

$$0 \to \text{Ker}(f) \to F \to T \to 0$$

is a $\mathcal{Y}$-exact sequence. Similarly, a monomorphism $f : F \to T$ is called $\mathcal{Y}$-monomorphism if

$$0 \to F \to T \to \text{Coker}(f) \to 0$$

is a $\mathcal{Y}$-exact sequence. A morphism $f : F \to T$ is called $\mathcal{Y}$-morphism if $F \to \text{Im}(f)$ is a $\mathcal{Y}$-epimorphism and $\text{Im}(f) \to T$ is a $\mathcal{Y}$-monomorphism. A left $\Gamma$-module $Z$ is called $\mathcal{Y}$-projective if for any $\mathcal{Y}$-epimorphism $f : F \to T$ and any morphism $g : Z \to T$ there exist a morphism $h : Z \to F$ such that $g = fh$.

**Lemma 2.1** i) If a short exact sequence is isomorphic to a $\mathcal{Y}$-exact sequence, then it is also a $\mathcal{Y}$-exact sequence.

ii) A split short exact sequence is $\mathcal{Y}$-exact.

iii) A composition of two $\mathcal{Y}$-epimorphisms is still a $\mathcal{Y}$-epimorphism.

iv) If $f$ and $g$ are two composable epimorphism and $fg$ is a $\mathcal{Y}$-epimorphism, then $f$ is also a $\mathcal{Y}$-epimorphism.

v) A composition of two $\mathcal{Y}$-monomorphisms is still a $\mathcal{Y}$-monomorphism.

vi) If $f$ and $g$ are two composable monomorphism and $fg$ is a $\mathcal{Y}$-monomorphism, then $g$ is also a $\mathcal{Y}$-monomorphism.

**Proof.** The properties i)- iv) are clear. Let $f : B \to C$ and $g : A \to B$ be monomorphisms. One can form the following diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
A & \xrightarrow{g} & B \\
\downarrow_{1_A} & \searrow_{f} & \downarrow \\
0 & \to & X \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & C \\
\downarrow & \searrow & \downarrow \\
0 & \to & Z \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{1_Y} & Y \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

Assume $f$ and $g$ are are $\mathcal{Y}$-monomorphisms, then for any partition $\lambda$ with $s(\lambda) = n$
one has a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & B([n])^{\Sigma(\lambda)} & \longrightarrow & X([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
\downarrow^{1_A} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & C([n])^{\Sigma(\lambda)} & \longrightarrow & Z([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
\downarrow & & \downarrow^{h} & & \downarrow & & \downarrow & & \\
Y([n])^{\Sigma(\lambda)} & \longrightarrow & Y([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
\end{array}
\]

The diagram chasing shows that \( h \) is an epimorphism and therefore \( fg \) is a \( \mathcal{Y} \)-monomorphism and \( v) \) is proved. Assume now \( fg \) is a \( \mathcal{Y} \)-monomorphism. Then we have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & B([n])^{\Sigma(\lambda)} & \longrightarrow & X([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
\downarrow^{1_A} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & C([n])^{\Sigma(\lambda)} & \longrightarrow & Z([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Y([n])^{\Sigma(\lambda)} & \longrightarrow & Y([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
\end{array}
\]

The diagram chasing shows that \( l \) is an epimorphism and therefore \( f \) is a \( \mathcal{Y} \)-monomorphism and therefore we get \( vi) \). \( \square \)

As an immediate corollary we obtain that the class of all \( \mathcal{Y} \)-exact sequences is proper in the sense of Mac Lane [4]. We now show that there are enough \( \mathcal{Y} \)-projective objects.

**Lemma 2.2** i) For any partition \( \lambda \) the left \( \Gamma \)-module \( \Gamma(\lambda) \) is a \( \mathcal{Y} \)-projective object.

ii) A morphism \( f : F \rightarrow T \) of left \( \Gamma \)-modules is \( \mathcal{Y} \)-epimorphism iff for any partition \( \lambda \) the induced morphism

\[ \text{Hom}_{\Gamma-\text{mod}}(\Gamma(\lambda), F) \rightarrow \text{Hom}_{\Gamma-\text{mod}}(\Gamma(\lambda), T) \]

is an epimorphism.
iii) For any left $\Gamma$-module $F$ there is a $\mathcal{Y}$-projective object $Z$ and $\mathcal{Y}$-epimorphism $f : Z \to F$.

iv) Any projective $\mathcal{Y}$-module is a direct summand of the sum of objects of the form $\Gamma(\lambda)$.

v) The tensor product of two $\mathcal{Y}$-projective left $\Gamma$-modules is still $\mathcal{Y}$-projective.

Proof. Let $\lambda$ be a partition with $s(\lambda) = n$. By definition $\Gamma(\lambda) = H_0(\Sigma(\lambda), \Gamma^n)$. Hence for any left $\Gamma$-module $F$ one has

$$\text{Hom}_{\Gamma-\text{mod}}(\Gamma(\lambda), F) \cong H^0(\Sigma(\lambda), \text{Hom}_{\Gamma-\text{mod}}(\Gamma^n, F) \cong F(n)^{\Sigma(\lambda)}.$$ 

The assertions i) and ii) are immediate consequence of this isomorphisms. To proof iii) we set

$$X(\lambda) := \text{Hom}_{\Gamma-\text{mod}}(\Gamma(\lambda), F).$$

Moreover, for each $x \in X(\lambda)$ we let $f_x : \Gamma(\lambda) \to F$ be the corresponding morphism. Take $Z = \bigoplus_\lambda \bigoplus_{x \in X(\lambda)} \Gamma(\lambda)$. Then the collection $f_x$, $x \in X(\lambda)$ yields the morphism $f : Z \to F$. We have to show that it is a $\mathcal{Y}$-epimorphism. Let $g : \Gamma(\lambda) \to F$ be a morphism of left $\Gamma$-modules. By ii) we need to lift $g$ to $Z$. By our construction $g \in X(\lambda)$ and therefore the inclusion $\Gamma(\lambda) \to Z$ corresponding to the summand $g \in X(\lambda)$ is an expected lifting and iii) is proved. The proof of iii) shows that one can assume $P$ to be a sum of $\Gamma^\lambda$ and iv) follows. To proof the last statement one observes that, for any partitions $\lambda$ and $\mu$ one has

$$\Gamma(\lambda) \otimes \Gamma(\mu) \cong (\Gamma^{s(\lambda)} \otimes \Gamma^{s(\mu)})^{\Sigma(\lambda) \times \Sigma(\mu)} = (\Gamma^{s(\lambda) + s(\mu)})^{\Sigma(\lambda) \times \Sigma(\mu)}$$

and therefore $\Gamma(\lambda) \otimes \Gamma(\mu)$ is $\mathcal{Y}$-projective. 

\[\square\]

3 Definition of Andrè-Quillen homology and the functor $\mathcal{L}$

The definition Andrè-Quillen homology is based on the framework of homotopical algebra [8] and it is given as follows. We let $C_*(V_*)$ be the chain complex associated to a simplicial vector space $V_*$. Let $A$ be a commutative algebra over a ground field $K$ and let $M$ be an $A$-module. A simplicial resolution of $A$ is an augmented simplicial object $P_* \to A$ in the category of commutative algebras, which is a weak equivalence (in other words $C_*(P_*) \to A$ is a weak equivalence). A simplicial resolution is called free if $P_n$ is a polynomial algebra over $K$ for all $n \geq 0$. Any commutative algebra posses a free simplicial resolution which is unique up to homotopy. Then the Andrè-Quillen homology is defined by

$$D_*(A, M) := H_*(C_*(\Omega^1 P_* \otimes P_* M)),$$
where $\Omega^1$ is the Kähler 1-differentials and $P_* \to A$ is a free simplicial resolution. In the dimension 0 we have $D_0(A, M) \cong \Omega^1_A \otimes_A M$.

As we mentioned the functor $L(A, M) : \Gamma \to \text{Vect}$ is given on objects by $[n] \mapsto M \otimes A^{\otimes n}$. For a pointed map $f : [n] \to [m]$, the action of $f$ on $L(A, M)$ is given by

$$f_*(a_0 \otimes \cdots \otimes a_n) = b_0 \otimes \cdots \otimes b_m,$$

where $b_j = \prod_{f(i) = j} a_i$, $j = 0, \cdots, n$.

Example 3.1 Let $M = A = K[t]$. In this case one has an isomorphism

$$L(K[t], K[t]) \cong S^* \circ \Gamma^1.$$

To see this isomorphism, one observes that $\Gamma^1$ assigns the free vector space on a set $[n]$ to $[n]$ and therefore both functors in the question assign the ring $K[t_0, \cdots, t_n]$ to $[n]$. An important consequence of this isomorphism is the fact that the functor $L(K[t], K[t])$ is $\mathcal{Y}$-projective.

Lemma 3.2 For any commutative algebra $A$ and any $A$-module $M$, one has a natural isomorphism $\pi_0(L(A, M)) \cong \Omega^1_A \otimes_A M$.

Proof. This is a consequence of Proposition 1.15 and Proposition 2.2 of [5]. $\square$

Lemma 3.3 i) Let $A$ be a commutative algebras and let

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of $A$-modules. Then

$$0 \to L(A, M_1) \to L(A, M) \to L(A, M_2) \to 0$$

is a $\mathcal{Y}$-exact sequence.

ii) Let $f : B \to A$ be a surjective homomorphism of commutative algebras, then for any $A$-module $M$ the induced morphism of left $\Gamma$-modules

$$L(B, M) \to L(A, M)$$

is a $\mathcal{Y}$-epimorphism.

Proof. One observes that for any partition $\lambda$ with $s(\lambda) = n$ one has

$$(L(A, M)([n]))^{\Sigma(\lambda)} = (M \otimes A^{\otimes n})^{\Sigma(\lambda)} \cong M \otimes D^\lambda(A).$$

Since we are over field the tensor product is exact and we obtain i). By the same reason $f$ has a linear section, which yields also a linear section of $D^\lambda(B) \to D^\lambda(A)$, because $D^\lambda$ is a functor defined on the category of vector spaces. $\square$
4 Relative derived functors

By Lemma 2.2 the class of $\mathcal{V}$-exact sequences has enough projective objects. Thanks to [4] this allows us to construct the relative derived functors. Let us recall that an augmented chain complex $X_* \to F$ is called a $\mathcal{V}$-resolution of $F$ if it is exact (that is $H_i(X_*) = 0$, for $i > 0$ and $H_0(X_*) \cong F$) and all boundary maps $X_{n+1} \to X_n$ are $\mathcal{V}$-morphisms, $n \geq 0$. It follows from Lemma 2.2 that $X_* \to F$ is a $\mathcal{V}$-resolution iff for any partition $\lambda$ the augmented complex

$$\text{Hom}_{\Gamma\text{-mod}}(\Gamma(\lambda), X_*) \to \text{Hom}_{\Gamma\text{-mod}}(\Gamma(\lambda), F)$$

is exact. A $\mathcal{V}$-resolution $Z_* \to F$ is called $\mathcal{V}$-projective resolution if for all $n \geq 0$ the left $\Gamma$-module $Z_n$ is a $\mathcal{V}$-projective object. We define $\pi_*^V(F)$ using relative derived functors of the functor $\pi_0 : \Gamma\text{-mod} \to \text{Vect}$. In other words we put

$$\pi_n^V(F) := H_n(\pi_0(Z_*)), \ n \geq 0,$$

where $Z_* \to F$ is a $\mathcal{V}$-projective resolution. By [4] this gives the well-defined functors $\pi_n^V : \Gamma\text{-mod} \to \text{Vect}, \ n \geq 0$.

Lemma 4.1 If $K$ is a field of characteristic zero, then $\pi_*(F) \cong \pi_*^V(F)$.

Proof. In this case all exact sequences are $\mathcal{V}$-exact, because for any finite group $G$, the functor $M \mapsto M^G$ is exact. □

Lemma 4.2 For left $\Gamma$-modules $F, T$ one has an isomorphism

$$\pi_*^V(F \otimes T) \cong \pi_*^V(F) \otimes T([0]) \oplus F([0]) \otimes \pi_*^V(T).$$

Proof. The result in the dimension 0 is known (see Lemma 4.2 of [4]). Let $Z_* \to F$ and $R_* \to T$ be $\mathcal{V}$-projective resolutions. By Lemma 2.2 $Z_* \otimes R_* \to F \otimes T$ is also $\mathcal{V}$-projective resolution. Thus

$$\pi_*^V(F \otimes T) = H_* (\pi_0(Z_* \otimes R_*)) \cong$$

$$H_* (\pi_0^V(Z_*) \otimes R_*([0]) \oplus Z_*([0]) \otimes \pi_0^V(R_*)) \cong$$

$$\pi_*^V(F) \otimes T([0]) \oplus F([0]) \otimes \pi_*^V(T),$$

where the last isomorphism follows from the Eilenberg-Zilber theorem and Künneth theorem. □

Lemma 4.3 Let $\epsilon : X_* \to A$ be a simplicial resolution in the category of commutative algebras and let $M$ be an $A$-module. Then the associated chain complex of the simplicial $\Gamma$-module $C_*(\mathcal{L}(X_*, M)) \to \mathcal{L}(A, M)$ is a $\mathcal{V}$-resolution.

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Proof. Since \( \epsilon \) is weak equivalence of simplicial algebras it is a homotopy equivalence in the category of simplicial vector spaces. Thus \( M \otimes D^\lambda(X_\ast) \to M \otimes D^\lambda(A_\ast) \) is also a homotopy equivalence, for any partition \( \lambda \). It follows that

\[
\mathcal{L}(X_\ast, M)([n])^{\Sigma(\lambda)} \to \mathcal{L}(A, M)([n])^{\Sigma(\lambda)}
\]

is also a homotopy equivalence of simplicial vector spaces. \( \Box \)

The following is our main result.

**Theorem 4.4** For any commutative ring \( A \) and any \( A \)-module \( M \), there is a canonical isomorphism

\[ D_i(A, M) \cong \pi_i^Y(\mathcal{L}(A, M)), \; i \geq 0 \]

between the André-Quillen homology and relative derived functors of \( \pi_0 \) applied on the functor \( \mathcal{L}(A, M) \).

Proof. Thanks to Lemma 3.2 the result is true for \( i = 0 \). First consider the case, when \( M = A = K[t] \). In this case André-Quillen homology vanishes in positive dimensions by definition. On the other hand \( \mathcal{L}(K[t], K[t]) \) is \( \mathcal{Y} \)-projective thanks to Example 3.1 and therefore \( \pi_i^Y(\mathcal{L}(A, M)) \) vanishes for all \( i > 0 \). One can use Lemma 4.2 to conclude that \( \pi_i^Y(\mathcal{L}(A, A)) \) vanishes for all \( i > 0 \) provided \( A \) is a polynomial algebra. For the next step, we proof that the result is true if \( A \) is a polynomial algebra and \( M \) is any \( A \)-module. We have to prove that \( \pi_i^Y(\mathcal{L}(A, M)) \) also vanishes for \( i > 0 \). We already proved this fact if \( M = A \). By additivity the functor \( \pi_i^Y(\mathcal{L}(A, -)) \) vanishes on free \( A \)-modules. By Lemma 3.3 the functor \( \pi_*^Y(\mathcal{L}(A, -)) \) assigns the long exact sequence to a short exact sequence of \( A \)-modules. Therefore we can consider such an exact sequence associated to a short exact sequence of \( A \)-modules

\[ 0 \to N \to F \to M \to 0 \]

with free \( F \). Since the result is true if \( i = 0 \) one obtains by induction on \( i \), that \( \pi_i^Y(\mathcal{L}(A, M)) = 0 \) provided \( i > 0 \). Now consider the general case. Let \( P_\ast \to A \) be a free simplicial resolution in the category of commutative algebras. Then we have

\[
\Omega^1_{P_\ast} \otimes_{P_\ast} M \cong \pi_0^Y(\mathcal{L}(P_\ast, M))
\]

Thanks to Lemma 4.3 \( C_\ast(\mathcal{L}(P_\ast, M)) \to \mathcal{L}(A, M) \) is a \( \mathcal{Y} \)-resolution consisting with \( \pi_*^Y \)-acyclic objects and the result follows. \( \Box \)

The main theorem together with the main result of [7] yields:

**Corollary 4.5** If \( \text{Char}(K) = 0 \), then for any commutative algebra \( A \) and any \( A \)-module \( M \) one has a natural isomorphism

\[ D_\ast(A, M) \cong H_\ast^Y(A, M). \]
This fact was also proved in [10] based on the combinatorical and homotopical analysis of the space of fully grown trees.

**Remarks.** i) We let \( t : \Gamma^{op} \to \text{Vect} \) be the functor which assigns the vector space of all maps \( f : [n] \to K \), \( f(0) = 0 \) to \([n]\). Then \( t \otimes_\Gamma F \cong \pi_0(F) \) (see Proposition 2.2 [3]). Hence \( \pi^Y \) can be also defined as the relative derived functors of the functor \( t \otimes_\Gamma (-) : \Gamma\text{-nod} \to \text{Vect} \). More generally one can take any functor \( T : \Gamma^{op} \to \text{Vect} \) and define \( \text{Tor}^Y(T, F) \) as the value of the relative derived functors (with respect of \( \mathcal{Y} \)-exact sequences) of the functor \( T \otimes_\Gamma (-) : \Gamma\text{-nod} \to \text{Vect} \). Then our result claims that

\[
D_\ast(A, M) \cong \text{Tor}^Y_\ast(t, \mathcal{L}(A, M)).
\]

Based on the Proposition 1.15 of [5] the argument given above shows that

\[
D^{(n)}_\ast(A, M) \cong \text{Tor}^Y_\ast(\Lambda^n \circ t, \mathcal{L}(A, M)),
\]

where \( D^{(n)}_\ast(A, M) \) are defined using Kähler \( n \)-differentials:

\[
D^{(n)}_\ast(A, M) := H_\ast(C_\ast(\Omega^n_p \otimes P, M))
\]

and for \( n = 1 \) one recovers the main theorem.

ii) All results remains true if \( K \) is any commutative ring and \( A \) and \( M \) are projective as \( K \)-module.

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