On the Role of Quantization of Soft Information in GRAND

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Abstract—In this work, we investigate guessing random additive noise decoding (GRAND) with quantized soft input. First, we analyze the achievable rate of ordered reliability bits GRAND (ORBGRAND), which uses the rank order of the reliability as quantized soft information. We show that multi-line ORBGRAND can approach capacity for any signal-to-noise ratio (SNR). We then introduce discretized soft GRAND (DSGRAND), which uses information from a conventional quantizer. Simulation results show that DSGRAND well approximates maximum-likelihood (ML) decoding with a number of quantization bits that is in line with current soft decoding implementations. For a (128, 106) CRC-concatenated polar code, the basic ORBGRAND is able to match or outperform CRC-aided successive cancellation list (CA-SCL) decoding with codeword list size of 64 and 3 bits of quantized soft information, while DSGRAND outperforms CA-SCL decoding with a list size of 128 codewords. Both ORBGRAND and DSGRAND exhibit approximately an order of magnitude less average complexity and two orders of magnitude smaller memory requirements than CA-SCL.

Index Terms—GRAND, Soft Decision, Quantization, Statistical Model

I. INTRODUCTION

As maximum-likelihood (ML) error correcting decoding of linear codes is NP-complete [1], the engineering paradigm has been to co-design restricted classes of linear codebooks with code-specific decoding methods that exploit the code structure to enable computationally efficient approximate-ML decoding. For example, Bose-Chaudhuri-Hocquenghem (BCH) codes with hard detection Berlekamp-Massey decoding [2], [3], low-density parity-check (LDPC) codes [4] and belief propagation (BP) [5], polar codes with cyclic redundancy check (CRC) codes, which have been selected for all control channel communications in the 5th-generation wireless system (5G) new radio (NR), and CRC-assisted successive cancellation list (CA-SCL) decoding [6], [7], [8], [9].

Modern applications, including augmented and virtual reality, vehicle-to-vehicle communications, the Internet of Things, and machine-type communications, have driven demand for reliable, low-latency communications [10], [11], [12], [13], [14]. These technologies benefit from short (for latency) and high-rate (for efficiency) codes, reviving the possibility of creating high-accuracy universal decoders that are suitable for hardware implementation. Accurate, practically realizable universal decoders offer the possibility of reduced hardware footprint, the provision of hard or soft detection decoding for codes that currently only have one class of decoder, and future-proofing devices against the introduction of new codes.

Guessing random additive noise decoding (GRAND) is a recently established universal decoder that was originally introduced for hard decision demodulation systems [15], [16]. GRAND algorithms operate by sequentially removing putative noise effects from the demodulated received sequence and querying if what remains is in the codebook. The first instance where a codebook member is found is the decoding. If GRAND queries noise effects from most likely to least likely based on available hard or soft information, it identifies a ML decoding. Amongst other results, in the hard detection setting, mathematical analysis of GRAND determines error exponents for a broad class of additive noise models [16].

For an \((n, k)\) code, where \(k\) information bits are transformed into \(n\) coded bits for communication, all GRAND algorithms identify an erroneous decoding after an approximately geometrically distributed number of codebook queries with mean \(2^{n-k}\) [16, Theorem 2] and correctly decode if they identify a codeword beforehand. As a result, an upper bound on the complexity of all GRAND algorithms is determined by the number of redundant bits rather than the code length or rate directly, making them suitable for decoding any moderate redundancy code of any length. The performance difference between GRAND variants stems from their utilisation of statistical noise models or soft information to improve the targeting of their queries.

The evident parallelizability of hard detection GRAND’s codebook queries have already resulted in the proposal [17] and realization [18], [19] of efficient circuit implementations for binary symmetric channels (BSCs). An algorithm has also been introduced for channels subject to bursty noise whose statistical characteristics are known to the receiver [20], [21], which has also resulted in proposed circuit implementations [22], [23].

A natural question is how to make use of soft detection information, when it is available, in order to improve GRAND’s query order and several proposals have been made. Soft GRAND (SGRAND) [24] uses real-valued soft information to build a bespoke noise-effect query order for each received signal and provides a benchmark for optimal decoding accuracy performance. It is possible to create a semi-parallelizable implementation in software using dynamic max-
heap structures, but the requirement for substantial dynamic memory does not lend itself to efficient implementation in hardware.

Quantization of the soft information provides a trade-off between performance and implementation complexity. Symbol reliability GRAND (SRGRAND) \[25, 26\] avails of the most limited quantized soft information where one additional bit tags each symbol as being reliably or unreliably received. SRGRAND is mathematically analysable, implementable in hardware, and provides a 0.5 – 0.75 dB gain over hard decision GRAND \[26\].

Ordered reliability bits GRAND (ORBGRAND) \[27\] aims to bridge the gap between SRGRAND and SGRAND by obtaining the decoding accuracy close to the latter in an algorithm that is suitable for implementation in circuits. For a block code of length \(n\), it uses \([\log_2(n)]\) bits of codebook-independent quantized soft detection information per received bit, the rank order of each received bit’s reliability, to determine an accurate decoding. It retains the hard decision algorithm’s suitability for a highly parallelized implementation in hardware and high throughput VLSI designs have been proposed in \[28, 29, 30, 31\]. Moreover, theoretical results indicate that ORBGRAND is almost capacity achieving in certain signal-to-noise ratio (SNR) regimes \[32\]. It has been observed that ORBGRAND provides near-ML decoding for block error rates (BLERs) greater than 10\(^{-4}\), but is less precise at higher SNR. The latter effect can be explained by the fact that ORBGRAND’s noise effect query order diverges from the optimal rank order in terms of decreasing likelihood. To rectify this, a list decoding approach to improve ORBGRAND’s performance at higher SNR has been suggested \[33\]. A more accurate statistical model based on multi-line fitting of rank orders from which an enhanced query order can be determined has also been introduced \[34\].

In this work, we start with the achievable rate of ORBGRAND. We construct a mismatched demapping function to approximate the reliability sorting operation, and then calculate the achievable rate of the corresponding mismatched decoding. We show that multi-line ORBGRAND can be capacity achieving at all SNRs as long as a sufficiently rich multi-line fitting is employed, expanding upon the results of \[32\] that show that the basic version of ORBGRAND \[27\] is near-capacity achieving in certain SNR regimes. The number of lines in practice we show to be quite moderate to achieve near-optimal performance.

We also propose discretized soft GRAND (DSGRAND), an alternative efficient algorithm based on dynamic programming (DP) for soft detection error correction decoding with GRAND that is an approximation to SGRAND. In contrast to ORBGRAND, DSGRAND uses conventional quantizers and does not require received bits to be rank-ordered by their reliability, which can be energy expensive when implemented in circuits. DSGRAND can make use of any level of quantized soft detection information, with increasingly accurate performance as the number of soft information bits per received bit increases. Quantizer optimization for DSGRAND yields to theoretical analysis and heuristic designs, both of which we discuss. Empirical results show DSGRAND outperforms the original basic ORBGRAND at high SNR and performs within 0.25 dB and 0.1 dB close to ML decoding with 2 and 3 bits quantizers, respectively. We also provide the comparison of performance and complexity between GRAND variants and CA-SCL decoding for a (128, 106) 5G CRC-concatenated polar code. With a 3 bits quantizer, proposed DSGRAND performs approximately 0.2 dB better than a CA-SCL decoder \((L = 128)\) with significantly lower average complexity and memory requirements.

This paper is organized as follows. Section II gives background on the problem. The achievable rate of ORBGRAND is discussed in Section III. Section IV presents the proposed DSGRAND algorithm with pseudo code and the quantizer optimization. Simulation results are shown in Section V. Section VI concludes the paper.

II. Preliminaries

In this paper, vectors are written as \(x^n = (x_1, x_2, \ldots, x_n)\). The all-zeros vector of dimension \(n\) is \(0^n\). The \(i\)-th entry of \(x^n\) is \(x_i\). A random variable (RV) is written with an uppercase letter such as \(X\). A realization of \(X\) is written with the corresponding lowercase letter \(x\). A vector of RVs is written as \(X^n = (X_1, X_2, \ldots, X_n)\). Both the probability density function (PDF) of a continuous RV and the probability mass function (PMF) of a discrete RV evaluated at \(x\) are written as \(f_X(x)\). \(F_X(x)\) denotes the cumulative distribution function (CDF) of RV \(X\). The bit-wise “exclusive or” operation is written as \(⊕\). For functions and operations defined with scalar inputs, we use them with vector inputs as their element-wise version, i.e.,

\[
\begin{align*}
    f(x^n) &= (f(x_1), f(x_2), \ldots, f(x_n)), \\
    x^n ⊕ y^n &= (x_1 ⊕ y_1, x_2 ⊕ y_2, \ldots, x_n ⊕ y_n).
\end{align*}
\]

A. Channel Model and ML Decoding

Consider a communication system using a binary \((n, k)\) linear block code. The codeword \(c^n\) is binary phase-shift keying (BPSK) modulated with unit power via

\[
    f_{\text{BPSK}}(c_i) = 1 - 2c_i, \quad i = 1, \ldots, n
\]

and transmitted over a memoryless additive white Gaussian noise (AWGN) channel, i.e.,

\[
    Y = f_{\text{BPSK}}(C) + \sigma N
\]

where \(N\) is an independent zero mean Gaussian noise with variance one. The SNR is given by

\[
    \frac{E_s}{N_0} = \frac{1}{\sigma^2}, \quad \frac{E_b}{N_0} = \frac{2k}{N_0} \cdot \frac{1}{n}.
\]

At the receiver, we have the symbol-wise log-likelihood ratio (LLR) of \(c_i\) based on the observation \(y_i\), \(i = 1, \ldots, n\)

\[
    \tau(y_i) \triangleq \log \frac{f_Y|C(y_i|0)}{f_Y|C(y_i|1)} = \frac{2y_i}{\sigma^2}, \quad i = 1, \ldots, n.
\]

We also have the hard decision \(\hat{c}_i\) and the symbol-wise reliabilities \(\ell_i\) of the coded bit \(c_i\),

\[
    \hat{c}_i \triangleq \frac{1 - \text{sign}(y_i)}{2}, \quad \ell_i \triangleq |\tau(y_i)| = \frac{2|y_i|}{\sigma^2}.
\]
A block-wise ML decoder puts out the codeword

\[
\hat{c}^n = \arg\max_{c^n \in C} f_{Y^n|C^n}(y^n, c^n)
\]

where the score of an error pattern codeword is deduced. GRAND creates binary error patterns

C. GRAND variants

\[\] Fig. 1. Input-output relation of a quantizer for positive real numbers.

\[
\mu: [0, b_1] \rightarrow [0, b_2)
\]

\[
\mu(x) = \begin{cases} 
  v_1, & \text{if } x \in [0, b_1) \\
  v_2, & \text{if } x \in [b_1, b_2) \\
  \vdots \\
  v_Q, & \text{if } x \in [b_{Q-1}, +\infty). 
\end{cases}
\]

Note that \(q\) bits reliability quantization equivalently quantizes the LLR \(\tau(y)\) and the channel output \(y\) with \(q+1\) bits, i.e., one extra bit is required for the sign (or equivalently the hard decision \(\hat{c}\)). To simplify explanations in future sections, we define the equivalent quantizer \(\bar{\mu}(\cdot)\) for LLRs,

\[
\bar{\mu}(\tau(y)) = \begin{cases} 
  \mu(\tau(y)), & \text{if } \tau(y) \geq 0 \\
  -\mu(-\tau(y)), & \text{otherwise} 
\end{cases}
\]

and \(\bar{\mu}(\cdot)\) for channel output \(y\),

\[
\bar{\mu}(y) = \bar{\mu}(\frac{2y}{\sigma^2}) - \frac{\sigma^2}{2}.
\]

B. Reliability quantization

A conventional \(q\) bits (\(Q = 2^q\)-level) reliability quantizer \(\mu(\cdot): [0, \infty) \rightarrow [0, \infty)\) maps a positive input value in range \([b_{i-1}, b_i)\) to a output value \(v_i\) for \(i = 1, \ldots, Q\), where \(b_0 = 0\) and \(b_Q = +\infty\), i.e.,

\[
\mu(x) = \begin{cases} 
  v_1, & \text{if } x \in [0, b_1) \\
  v_2, & \text{if } x \in [b_1, b_2) \\
  \vdots \\
  v_Q, & \text{if } x \in [b_{Q-1}, +\infty). 
\end{cases}
\]

\[
\mu: [0, \infty) \rightarrow [0, \infty)
\]

\[
\mu(x) = \begin{cases} 
  v_1, & \text{if } x \in [0, b_1) \\
  v_2, & \text{if } x \in [b_1, b_2) \\
  \vdots \\
  v_Q, & \text{if } x \in [b_{Q-1}, +\infty). 
\end{cases}
\]

In a hard decision BSC, GRAND [15], [16] doesn’t have any reliability information and thus uses weight \(w_i = 1\) for \(i = 1, \ldots, n\), i.e., the score of error pattern \(e^n\) is equal to its Hamming weight. If a bursty statistical channel characterization is available at the receiver, a Markov-informed order can be used to generate error patterns [20], [21].

SGRAND uses the non-quantized reliability as the weight [23], i.e.,

\[
\bar{w}_i = \ell_i, \quad i = 1, \ldots, n.
\]

Theorem 1. SGRAND provides the ML decision by

\[
\hat{c}^n = \hat{e}^n \oplus \hat{z}^n, \quad \text{where } \hat{z}^n = \arg\min_{e^n: \hat{e}^n \oplus e^n \in C} \sum_{i=1}^{n} e_i \cdot \ell_i.
\]

Proof. We check both possibilities \(\hat{c}_i = 0, 1\) for eq. (9).

- If \(\tau(y_i) \geq 0\) and \(\hat{c}_i = 0\),

\[
(1 - 2c_i) \cdot \tau(y_i) = (1 - 2(\hat{c}_i \oplus e_i)) \cdot \tau(y_i) = (1 - 2e_i) \cdot \ell_i.
\]

- If \(\tau(y_i) < 0\) and \(\hat{c}_i = 1\),

\[
(1 - 2c_i) \cdot \tau(y_i) = (1 - 2(\hat{c}_i \oplus e_i)) \cdot \tau(y_i) = (1 - 2(1 - e_i)) \cdot \ell_i = (1 - 2e_i) \cdot \ell_i.
\]

\[
S(e^n) \hat{=} \sum_{i=1}^{n} e_i \cdot w_i
\]
We have an ML decision by identify the noise effect $z^n$ via
\[
\hat{z}^n = \arg \max_{e^n:z^n \in e^n \in \mathbb{C}} \sum_{i=1}^{n} (1 - 2(\hat{c}_i \oplus e_i)) \cdot \tau (y_i) \tag{24}
\]
\[
= \arg \max_{e^n:z^n \in e^n \in \mathbb{C}} \sum_{i=1}^{n} (1 - 2e_i) \cdot \ell_i \tag{25}
\]
\[
= \arg \min_{e^n:z^n \in e^n \in \mathbb{C}} \sum_{i=1}^{n} e_i \cdot \ell_i. \tag{26}
\]

By the definition of noise effect eq. (13), we have the ML decision via $\hat{z}^n = \hat{c}^n \oplus \hat{z}^n$. Note that this proof is equivalent to [24] Theorem III.1.

Theorem [1] shows that SGRAND provides ML decisions. However, SGRAND requires a dynamic data structure to generate the error patterns rank-ordered by the score depending upon the real-valued reliabilities $\ell^n$.

SGRAND uses a one bit quantizer for the reliabilities, i.e., SGRAND sets $w_i = +\infty$ for bits with reliability $\ell_i$ higher than a threshold, tagging them as being perfectly reliable, and $w_i = 1$ for those below the threshold, resulting in error patterns following increasing Hamming weight within the region of unreliable bits [26], i.e.,
\[
w_i = \begin{cases} 1, & \text{if } \ell_i < \delta \\ +\infty, & \text{otherwise}. \end{cases} \tag{27}
\]

D. ORBGRAND

Figure 2 shows that the rank-ordered reliabilities are increasing almost linearly at low to moderate SNR regime. Based on this observation, the basic version of ORBGRAND [27] considers the received bits rank-ordered in increasing reliability and their weights are increasing linearly, i.e.,
\[
w_i = r, \quad \ell_i \text{ is the } r\text{-th smallest element in } \ell^n. \tag{28}
\]

ORBGRAND sorts the reliabilities $\ell^n$ and set the weights $w^n$ to its rank orders $r \in \{1, 2, \ldots, n\}$, i.e., ORBGRAND uses a $\lceil \log_2(n) \rceil$ bits input-related statistical model-based quantizer. Then error pattern generation could be solved by determining distinct integer partitions. ORBGRAND’s advantage is that, once ranking is complete, pattern generation can be done on the fly with essentially no memory. ORBGRAND provides near-ML decoding for BLERs greater than $10^{-4}$, but it is less precise at high SNR. To overcome this problem, a multi-line ORBGRAND with a more sophisticated statistical model is proposed in [34].

III. ACHIEVABLE RATE OF ORBGRAND

In this section, the achievable rate of ORBGRAND is discussed. In contrast to the approach taken in [32], our analysis is based on order statistics and mismatched decoding theory.

![Graph showing rank-ordered reliabilities](image-url)

Fig. 2. Random samples of rank-ordered reliability at $E_s/N_0 = 3$ dB for $n = 128$.

A. Order statistics of the reliabilities

Let $\ell(r)$ be the $r$-th smallest element in vector $\ell^n$ and $L(r)$ be the $r$-th order statistic of the reliability samples $\ell^n$. Since $L_i = |\tau(Y_i)|$, the reliabilities $L_i$ are independent and identically folded Gaussian distributed, i.e.,
\[
L_i \overset{i.i.d.}{\sim} f_{FG}(a) = p_{FG} \left( a \left| \frac{2}{\sigma^2}, \frac{4}{\sigma^2} \right. \right) \tag{29}
\]
where $p_{FG}(x | \mu, \sigma^2)$ denotes the PDF of a folded Gaussian distributed RV, i.e.,
\[
p_{FG}(a | \mu, \sigma^2) = \begin{cases} p_G \left( a | \mu, \sigma^2 \right) + p_G \left( -a | \mu, \sigma^2 \right), & \text{if } a \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{30}
\]

where $p_G(x | \mu, \sigma^2)$ denotes the PDF of a Gaussian RV. We have then the distribution of $L(r)$
\[
f_{L(r)}(a) = \frac{n!}{(r-1)!(n-r)!} f_L(a) F_L(a)^{r-1} (1 - F_L(a))^{n-r}. \tag{31}
\]

As mentioned in Section II-C, basic ORBGRAND does not provide ML decisions in general, since ORBGRAND uses an input-related model-based quantizer, which maps the $r$-th smallest reliability $\ell(r)$ to its rank order $r$, i.e.,
\[
\ell(r) \mapsto r. \tag{32}
\]

We separate this mapping in two parts,
\[
\ell(r) \mapsto E[L(r)] \text{ and } E[L(r)] \mapsto r. \tag{33}
\]

We first map the reliability $\ell(r)$ to its expectation $E[L(r)]$. Since the reliabilities $\ell^n$ are independent and identically distributed (i.i.d.) distributed with finite variance, the variance of the order statistics approaches zero asymptotically in the number of samples [35], i.e.,
\[
\lim_{n \to +\infty} \text{Var} \left[L(r)\right] = 0. \tag{34}
\]
An example for finite code length is shown in Figure 3. As a result, \( \ell(r) \rightarrow E[L(r)] \) introduces only very limited performance loss. In [34] Sec. 3.D, a multi-line ORBGRAND is proposed. The performance loss of multi-line ORBGRAND vanishes if a precise enough curve fitting of \( E[L(r)] \) is used as weight \( w_n \).

The expectation \( E[L(r)] \) is then mapped to the rank order \( r \), i.e., ORBGRAND considers the received bits rank-ordered in increasing reliability and their weights are increasing linearly, which is only precise at moderate SNR. An example is shown in Figure 4 the expectations are calculated using eq. (31). To evaluate the loss introduced by \( E[L(r)] \rightarrow r \), we define a continuous curve fitting \( \lambda(r) \) for the discrete function \( E[L(r)] \).

By construction, we have that

\[
\lambda^{-1}(E[L(r)]) = r. \tag{35}
\]

Thus, \( \lambda^{-1}(\ell(r)) \) is an approximation of \( r \), i.e.,

\[
\lambda^{-1}(\ell(r)) \approx r. \tag{36}
\]

An example for a realization \( Y^n \) at 8 dB is shown in Figure 5. We observe that the approximation in eq. (36) is pretty precise, i.e., by using the function \( \lambda^{-1}(\cdot) \), we remove the statistical information of \( E[L(r)] \) and treat the reliabilities increasing almost linearly as basic ORBGRAND.

B. Achievable rates and mismatched decoding

Consider a binary input memoryless symmetric (BMS) channel \( f_{Y|C} \) with codewords \( C^n \) uniformly distributed in a codebook \( \mathcal{C} \). The highest rate that can be supported, in the Shannon sense, is governed by mutual information, \( I(C;Y) \). An ML decoder eq. (9) uses the LLR \( \tau(Y) \) as decoding input and achieves the mutual information.

In contrast to ML decoding, a mismatched decoder [30] Sec. 8.2] uses a different function \( M(Y) \) from \( \tau(Y) \) as the decoding input providing the soft information about bit \( C \), i.e.,

\[
e^n = \arg \max_{c^n \in \mathcal{C}} \sum_{i=1}^{n} (1 - 2c_i) \cdot M(y_i). \tag{37}
\]

Defining

\[
R(Y,C,M(Y),s,r) = \max_{r,s} E \left[ \log_{2} \frac{e^{sM(Y)(1-2c)\tau(C)}}{\sum_{c \in \{0,1\}} P_C(c) e^{sM(Y)(1-2c)\tau(C)}} \right]. \tag{38}
\]

where \( s \geq 0 \), and where \( r : \{0,1\} \rightarrow \mathbb{R} \) is a real-valued function with finite expectation \( E[r(C)] \)\( < \infty \). By [37], it is known that the block error probability of mismatched decoding approaches zero for \( n \) approaching infinity if

\[
nk = k/n < R(Y,C,M(Y),s,r). \tag{40}
\]

In general, we have \( R(Y,C,M(Y),s,r) \leq I(C;Y) \) with equality if \( M(Y) = \tau(Y) \), \( s = 1 \) and \( r(\cdot) = 1 \). Fixing \( r(\cdot) = 1 [38] \) and maximizing over \( s \) yields the generalized mutual information [49]. Maximizing over \( r,s \) yields the highest achievable rate, the LM-rate [37]. For a BMS channel, we have

\[
R_{LM}(Y,C,M(Y)) = \max_{r,s} R(Y,C,M,s,r) \tag{41}
\]

\[
= \max_{r,s} \left[ \log_{2} \frac{e^{sM(Y)(1-2c)\tau(C)}}{e^{sM(Y)\tau(0)} + e^{sM(Y)\tau(1)}} \right]. \tag{42}
\]

Note that we have \( R_{LM}(Y,C,\tau(Y)) \) vs. code length (number of samples) at \( E_s/N_0 = 6 \) dB.

Fig. 3. Variance of the \( r \)-th order statistic vs. code length (number of samples) at \( E_s/N_0 = 6 \) dB.

The LM-rate of a basic ORBGRAND is given by

\[
R_{LM}(Y,C,\nu(Y)). \tag{44}
\]

An example for \( n = 128 \) is shown in Figure 5 where the loss is defined by

\[
\left[ E_s/N_0 \right]_{dB} \rightarrow \left[ C^{-1}_{\text{bpsk}}(R_{LM}(Y,C,\nu(Y))) \right]_{dB}. \tag{45}
\]

Note that eq. (44) is not an achievable rate for finite code length \( n \). The code length is only related to the statistical model \( E[L(r)] \), and consequently related to \( \nu(\cdot) \).

We have following conclusions about the performance of ORBGRAND: consistent with the results in [32], since the rank-ordered weights are increasing almost linearly at low to moderate SNR regimes, the basic version of ORBGRAND is an effective approach; n the high SNR regime, multi-line ORBGRAND [34] Sec. 3.D] exhibits a negligible loss relatively to capacity-achieving SGRAND if a precise enough curve fitting is used for multi-line ORBGRAND.

IV. DISCRETIZED SOFT GRAND

As discussed in Section II.C error pattern generation rank-ordered by score \( \sum_{i=1}^{n} e_i \cdot \ell_i \) requires dynamic memory and is hard to implement in hardware efficiently. In contrast to ORBGRAND, DSGRAND envisages a conventional quantization of the real-valued reliabilities \( \ell^n \) into a restricted number of categories determined by a quantization level without the need for received bit reliabilities to be rank-ordered.
the DSGRAND may generate (details are discussed in Sec-

possible binary vectors to a subset sum problem [40], i.e., our target is to find all
Algorithm 2. A two-dimensional array

Fig. 5. \( \lambda^{-1} (c_{(r)}) \) and \( r \) for \( n = 128 \) at 3 dB.

A. Detailed algorithm

The reliabilities \( \ell^n \) are quantized to discrete input weight \( w^n \) via

\[
\ell \equiv \mu (\bar{\ell}) , \quad i = 1, \ldots, n.
\]  

(46)

The pseudo code of the proposed DSGRAND is shown in Algorithm 2. A two-dimensional array \( \Lambda \) of size \( n \times s_{\text{max}} \) is required as a DP table to store the boolean type elements, where \( s_{\text{max}} \) denotes the maximum score of the error patterns the DSGRAND may generate (details are discussed in Section V-A).

For a given score \( s \), the error pattern generation is equivalent to a subset sum problem [40], i.e., our target is to find all possible binary vectors \( e^n \) such that

\[
\sum_{i=1}^{n} e_i \cdot w_i = s.
\]  

(47)

We first check the existence of error patterns (line 6–11). The element \( \Lambda[i, s + 1] \) denotes whether we could find a subset in \( w^n \) to achieve sum \( s \). \( \Lambda[i, s + 1] \) is marked as true if one of the following conditions is met,

- \( w_i = s \): \( w_i \) is equal to the sum \( s \).
- \( \Lambda[i - 1, s + 1] \) is true: If there exists a subset in \( w^{i-1} \) to achieve sum \( s \), the sum \( s \) could be achieved if \( w_i \) is also allowed to be considered.
- \( \Lambda[i - 1, s - w_i + 1] \) is true: If there exists a subset in \( w^{i-1} \) to achieve sum \( s - w_i \), the sum \( s \) is achieved by including \( w_i \) in the subset.

By reusing the results of score \( s - 1 \), we need \( \mathcal{O}(n) \) operations to check the existence of error patterns with score \( s \).

If the existence \( \Lambda[n, s + 1] \) is marked as true (line 12–13), the error patterns are generated with help of a stack \( \Xi \) to avoid recursive functions. The stack \( \Xi \) contains pattern structures \( \langle i, j, d, e^n \rangle \), where

- \( i \) denotes the row index in \( \Lambda \).
- \( j \) denotes the target score. Note that \( j \) is not always equal to \( s \).
- \( e^n \) denotes the current error pattern.
- \( d \) denotes the difference between the sum \( s \) and the score of the current error pattern, i.e., \( d = s - \sum_{i=1}^{n} e_i \cdot w_i \).

Firstly, an initial structure \( \langle n, s, s - w_n, (0^{n-1}, 1) \rangle \) is pushed into the stack \( \Xi \) (line 15). Then, we pop pattern structures from the stack and push new pattern structures until it is empty to generate all error patterns with score \( s \). If we get a structure \( \langle i, j, d, e^n \rangle \) from \( \Xi \) (line 17),

- If \( d = 0 \), an error pattern with score \( s \) is found. We check whether \( e^n \oplus e^n \in \mathbb{C} \). If so, then \( e^n \) the estimate of the noise effect \( z^n \) given the discretized soft information (line 18–22).

For a given score \( s \), the error pattern generation is equivalent to a subset sum problem [40], i.e., our target is to find all possible binary vectors \( e^n \) such that

\[
\sum_{i=1}^{n} e_i \cdot w_i = s.
\]  

(47)
Algorithm 2: DSGRAND

Input : hard decisions \( \hat{c}^n \), weights \( w^n \), maximum score \( s_{\text{max}} \)
Output: estimates \( \hat{c}^n \), number of guesses \( n_p \), decoding state \( \phi \)
1 \( \phi \leftarrow \text{false}, n_p \leftarrow 1 \)
2 if \( \hat{c}^n \in C \) then
3 \( \hat{c}^n \leftarrow \hat{c}^n, \phi \leftarrow \text{true} \)
4 return
5 for \( s = 0, 1, \ldots, s_{\text{max}} \) do
6 \( \Lambda[1, s + 1] \leftarrow (w_1 = s) \)
7 for \( i = 2, 3, \ldots, n \) do
8 \( \Lambda[i, s + 1] \leftarrow \Lambda[i - 1, s + 1] \text{ or } (w_i = s) \)
9 if \( \Lambda[i, s + 1] = \text{false} \) then
10 if \( s - w_i \geq 0 \) then
11 \( \Lambda[i, s + 1] \leftarrow \Lambda[i - 1, s - w_i + 1] \)
12 if \( \Lambda[n, s + 1] = \text{false} \) then
13 continue
14 \( \Xi \leftarrow \emptyset \)
15 push \( (\Xi, (n, s, s - w_n, (0^{n-1}, 1))) \)
16 while \( \Xi \neq \emptyset \) do
17 \( (i, j, d, e^n) \leftarrow \text{pop}(\Xi) \)
18 if \( d = 0 \) then
19 \( n_p \leftarrow n_p + 1 \)
20 if \( \hat{c}^n + e^n \in C \) then
21 \( \hat{c}^n \leftarrow \hat{c}^n + e^n, \phi \leftarrow \text{true} \)
22 return
23 \( e^n \leftarrow \text{copy}(e^n) \)  // if necessary
24 if \( i > 1, \Lambda(i - 1, j + 1) \) then
25 \( \Xi \leftarrow 0, \Xi_{i-1} \leftarrow 1 \)
26 push \( (\Xi, (i - 1, j, d + w_i - w_{i-1}, e^n)) \)
27 if \( i > 1, j - w_i \geq 0, \Lambda(i - 1, j - w_i + 1) \) then
28 \( e_{i-1} \leftarrow 1 \)
29 push \( (\Xi, (i - 1, j - w_i, d - w_{i-1}, e^n)) \)
30 return

In general, the proposed DSGRAND algorithm requires \( \mathcal{O}(n \cdot S(\hat{z})) \) operations to construct the DP table and \( \mathcal{O}(n_p) \) operations to generate and test the error patterns.

In contrast to the set \( S \) used in SGRAND \[24\], the stack \( \Xi \) in our implementation is not order-sensitive, i.e., for a given score \( s \), every single thread could execute line 17—29 independently to other threads without maintaining the order of the stack. In our experiments, the size of \( \Xi \) never exceeds 12.

We note that a decoder with discrete input may find multiple codewords with a same likelihood, which is a probability zero event for the continuous case. In our implementation, DSGRAND returns the first valid codeword that is found without checking for the existence of other codewords with the same score.

B. Quantizer optimization

For a system with conventional quantizer, the mutual information is given by \( I(C; \hat{\mu}(Y)) \). Replacing the optimal decoder input \( \tau(Y^n) \) with \( \hat{\mu}(\tau(Y^n)) \) can be thought of as using a mismatched decoder \[37\]. Note that \( \hat{\mu}(\cdot) \) and \( \hat{\mu}(\cdot) \) denote the channel output quantizer and LLR quantizer, respectively, which are equivalent to the reliability quantizer \( \mu(\cdot) \) (see Section [I-B]).

In this work, we consider three types of conventional quantizers.

**Heuristic:** An uniform quantizer is used. The step size \( \beta \) of quantization is heuristically chosen via
\[
\beta = \frac{2}{\sigma^2} \frac{1 - \sigma/2}{Q}. \quad (48)
\]
The first term \( 2/\sigma^2 \) normalizes for the increase in reliability with SNR, while the second term, \((1 - \sigma/2)/Q\) ensures that approximately 30% of the least reliable bits are accurately assigned quantized reliabilities, while the 70% most reliable bits are grouped together,
\[
b_i = i\beta, \quad i = 1, \ldots, Q - 1. \quad (49)
\]

**Uniform quantizer:** An uniform quantizer uses fixed quantization step size \( \beta \), while the output values \( v^Q \) are not constrained. Review the mutual information and LM-rate of systems with quantized output introduced in Section [III-B]. We know that the mutual information \( I(C; \hat{\mu}(Y)) \) is only related to the quantization boundaries \( b^{Q-1} \). The LM-rate \( R_{\text{LM}}(Y, C, \hat{\mu}(\tau(Y))) \) is equal to the mutual information only if the output values \( v^Q \) are optimized. We first optimize the step size \( \beta \) for boundaries \( b_i = i\beta, \quad i = 1, \ldots, Q - 1 \) by maximizing the mutual information,
\[
\beta = \arg \max_{\beta} I(C; \hat{\mu}(Y)). \quad (50)
\]

**Non-uniform quantizer:** A non-uniform quantizer has no constraints on the boundaries and output values. Similar to the uniform quantizer, we optimize the boundaries by
\[
b_1, \ldots, b_{Q-1} = \arg \max_{b_1, \ldots, b_{Q-1}} I(C; \hat{\mu}(Y)). \quad (51)
\]

For given boundaries \( b^{Q-1} \), the optimal output values \( v^Q \) is always given by
\[
v_i = \log \int_{b_{i-1}}^{b_i} f_{\tau(Y)}(x|C) \, dx, \quad i = 1, \ldots, Q. \quad (52)
\]

As mentioned in Section [V-A], our implementation of DSGRAND only accepts positive integers as weights. We now map the optimal output values \( v^Q \) to integers. Because of eq. (16), the output values \( v^Q \) are normalized by \( \alpha = 1/v_1 \), such that \( v_1 = 1 \). The output values \( v^Q \) are then mapped to their nearest integers via round function. The resulting LM-rate is given by
\[
R_{\text{LM}}(Y, C, \hat{\mu}(\tau(Y))). \quad (53)
\]

Note that we have
\[
R_{\text{LM}}(\hat{\mu}(Y), C, \hat{\mu}(\tau(Y))) = R_{\text{LM}}(Y, C, \hat{\mu}(\tau(Y))) \quad (54)
\]
since the LLR quantizer \( \hat{\mu}(\tau(Y)) \) carries the expected boundaries information as channel output quantizer \( \hat{\mu}(Y) \).

Design examples for above mentioned quantizers for BPSK modulated codewords over real-valued AWGN channels at 4 dB and 7 dB are shown in Table I and Table II. Their achievable rates are displayed in Figure 7.

V. SIMULATION RESULTS

For simulated performance evaluation, we consider BPSK modulated codewords over real-valued AWGN channels.

A. Maximum score of DSGRAND

We start with the distribution of the score of the correct noise effect \( S(Z^n) \). Since 1) the codewords are uniformly distributed on the codebook \( \mathcal{C} \); and 2) the codeword is transmitted over a symmetric channel, i.e., \( f_Y(y) = f_Y(-y) \), we could use the all-zero codeword assumption to evaluate the expected score. Assuming that all-zero codeword \( 0^n \) is transmitted, the LLRs are i.i.d. Gaussian RVs with mean \( 2/\sigma^2 \) and variance \( 4/\sigma^2 \)

\[
\tau(Y) \sim f_{\tau(Y)}(a) = \mathcal{G}\left(a,\frac{2}{\sigma^2},\frac{4}{\sigma^2}\right). \tag{55}
\]

We have now the score function

\[
S(Z^n) = \sum_{i=1}^{n} |\tau(Y_i)| \cdot Z_i \tag{56}
\]

\[
= \sum_{i=1}^{n} \tau(Y_i) \cdot 1 \{ \tau(Y_i) > 0 \} \tag{57}
\]

where the indicator function \( 1 \{ \cdot \} \) equals 1 if the proposition is true and 0 otherwise. Define mixed-type RVs \( V_i, i = 1, \ldots, n \)

\[
V_i \triangleq \tau(Y_i) \cdot 1 \{ \tau(Y_i) > 0 \}. \tag{58}
\]

\( V_1, \ldots, V_n \) are i.i.d.

\[
V_i \sim f_V(a) = \begin{cases} 
\delta(a) \int_{-\infty}^{0} f_{\tau(Y)}(x) \, dx, & \text{if } a > 0 \\
0, & \text{otherwise}
\end{cases} \tag{59}
\]

where \( \delta(\cdot) \) denotes the Dirac delta function. Thus, we have the PDF of the score \( S(Z^n) \)

\[
f_{S(Z^n)}(a) = f_V(a)^{\otimes n}, \tag{60}
\]

\( \otimes \) denotes the n-th convolution power.
The GRAND algorithm is now terminated if the score of the current error pattern exceeds a threshold $s_{\text{max}}$. The performance loss on BLER is upper bounded\(^1\) by

$$
\int_{s_{\text{max}}}^{\infty} f_s(z^n)(a) \, da.
$$

(61)

In addition, such a threshold test enables the decoder to reject a decision when it is not reliable enough, which reduces the number of undetected errors if the threshold is carefully optimized (see, e.g., [41]). For a given threshold $s_{\text{max}}$, we define a binary RV $\Phi$ as

$$
\Phi = \mathbb{1}\{ S(z^n) \leq s_{\text{max}} \}.
$$

(62)

The proposition of the indicator function (62) reads as “GRAND finds an estimate $\hat{C}^n$ with a score smaller than $s_{\text{max}}$”. Then, the undetected error probability of the algorithm is given as

$$
\Pr \left( \hat{C}^n \neq C^n, \Phi = 1 \right).
$$

(63)

Observe that the overall error probability is equal to the summation of detected and undetected error probabilities, i.e., we have

$$
\Pr \left( \hat{C}^n \neq U^n \right) = \sum_{\phi \in \{0,1\}} \Pr \left( \hat{C}^n \neq C^n, \Phi = \phi \right)
$$

(64)

which simply follows from the law of total probability. The parameter $s_{\text{max}}$ controls the BLER and undetected block error rate (uBLER) tradeoff [41], [42]. In particular, (63) becomes equal to the left-hand side of (64) if $s_{\text{max}} = \infty$.

For a DSGRAND, the LLRs are quantized by function $\mu(\cdot)$. The PMF of discrete RV $\mu(V)$ is given by

$$
f_{\mu(V)}(a) = \begin{cases} 
\int_{-\infty}^{0} f_\tau(Y)(x) \, dx, & \text{if } a = 0 \\
\int_{b_{i-1}}^{b_i} f_\tau(Y)(x) \, dx, & \text{if } a = v_1, \ldots, v_Q 
\end{cases}
$$

(65)

where $Q, b_i, v_i$ are the quantization parameters (see Section II-B). Thus, we have the PMF of the score $S(Z^n)$

$$
f_{S(Z^n)}(a) = f_{\mu(V)}(a)^{\oplus n}.
$$

(66)

In our simulations of DSGRAND, we always use a $s_{\text{max}}$ such that the performance loss eq. (61) is close to the target BLER.

### B. Error correction performance

Figures 8-14 present the decoding performance and average complexity (in number of guesses) of several BCH and Reed–Solomon (RS) codes of different length and code rate with DSGRAND, basic ORBGRAND and SGRAND. We have following observations.

- Basic ORBGRAND performs close to SGRAND at moderate SNR, but is less precise at higher SNR, which matches our analysis in Section III

\(^1\)The exact performance loss on BLER is the joint probability of event “ML decoder is capable to find the transmitted codeword for $Y^n$” and “GRAND with a maximum score misdecodes $Y^n$”. Eq. (61), is the probability of event “GRAND with a maximum score misdecodes $Y^n$”, which is a upper bound of the performance loss.
the mutual information maximizing quantizer at high SNR.

- The 3 bits quantizer with heuristic approach provides near-ML performance (within 0.1 dB loss) for all operating points.

Figure 15 shows the performance of the GRANDs and CA-SCL decoders. We consider a \( (128, 106 + 11) \) 5G polar code concatenated with 11 bits CRC. With a 3 bits quantizer, proposed DSGRAND performs \( \approx 0.2 \) dB better than a CA-SCL decoder with list size \( L = 128 \).
C. Computational complexity

The main budget of computational power for a GRAND is the parity check, which requires \( n_p \cdot n \cdot (n - k) \) exclusive or operations in fully parallel implementations. Note that the complexity of parity check is much smaller in serial implementations, since the parity check function would return a false once a unsatisfied equation is found, instead of check all \( n - k \) equations. Additionally, both ORBGRAND and DSGRAND need some real additions and compare operations. ORBGRAND requires a size-\( n \) sorting operation to get the rank information of the reliabilities.

D. Memory requirement

For a DSGRAND, the required memory space is the DP table and the stack \( \Xi \), i.e., \( n \cdot (s_{\text{max}} + |\Xi|) \) bits and \( 3 \cdot |\Xi| \) real numbers. In the our experiments, the size of \( \Xi \) never exceeds 12 for all codes, all quantization levels and all operating points. The selection of maximum score \( s_{\text{max}} \) could be found in Section V.A and is SNR-related. We use \( s_{\text{max}} = 38 \) for the DSGRAND (\( q = 3 \)) at \( E_b/N_0 = 5.5 \) dB shown in Figure 15 and Table III. On the other side, an CA-SCL requires \( n \cdot L \) real numbers and \( 2n \cdot L \) bits memory space.

Table III shows the decoding complexity in (average) number of required operations and memory requirement of three decoders in Figure 15 at \( E_b/N_0 = 5.5 \) dB. For DSGRAND and ORBGRAND, we use the fully parallel implementation, i.e., \( n_p \cdot n \cdot (n - k) \) exclusive or operations are required.

In Table III we also provide the complexity and space score for the decoders, i.e., we assume that a real addition operation is eight times more complex than an exclusive or, a compare operation is six times more complex than an exclusive or, and the complexity of a size-\( n \) sort (and find median) operation is close to \( 6 \cdot n \cdot \log_2 n \) exclusive or operations. Under this assumption, the proposed DSGRAND requires only a few percent of the complexity and space of a CA-SCL decoder at \( E_b/N_0 = 5.5 \) dB.

VI. Conclusions

Universal decoders have many practical benefits, including the ability to support an arbitrary number of distinct codes with one efficient piece of software or hardware, enabling the best choice of code for each application and future proofing devices to the introduction of new codes. Particularly as new applications drive demand for shorter, higher rate error correcting codes, GRAND is a promising approach to realizing this possibility. GRAND with soft input information handles codes for which there are only hard detection decoders, such as BCH codes, or no established error correcting decoder, such as CRCs [43], [44], to soft detection decoding. Consistently with theoretical predictions [45], results from GRAND algorithms show that decoding performance is largely driven by the quality of the decoder rather than that of the code, and that good CRC codes and codes selected at random can offer comparable performance to highly structured ones [20], [43], [27], [13], [46].

In this work, we have analyzed the achievable rate of ORBGRAND, ORBGRAND, which requires a size-\( n \) sorting...
operation on reliabilities, in effect at most \( \lceil \log_2(n) \rceil \) bits of quantization. Our approach is based on order statistics and mismatched decoding. We have shown that multi-line ORBGRAND can be capacity achieving at all SNR points with precise enough curve fitting through multi-line models. Its complexity is also far lower than CA-SCL, both in terms of and computation and, in particular, memory.

We have introduced DSGRAND, that uses a conventional quantizer. DSGRAND inherits all the desirable features of GRAND algorithms, including universality, parallelizability and reduced algorithmic effort as SNR increases. It can avail of any level of quantization, provides improved error correction performance as quantization level increases, and obviates the need for a sorter. The numerical result shows that DSGRAND outperforms basic ORBGRAND at high SNR and performs within 0.25 dB and 0.1 dB close to ML decoding with 2 and 3 bits quantizers, respectively. DSGRAND outperforms CA-SCL with sharply lower complexity and memory requirements. With respect to basic ORBGRAND, it has somewhat lower complexity but somewhat higher memory requirements.

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