Preference Games and Personalized Equilibria, with Applications to Fractional BGP

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Abstract

We study the complexity of computing equilibria in two classes of network games based on flows - fractional BGP (Border Gateway Protocol) games and fractional BBC (Bounded Budget Connection) games. BGP is the glue that holds the Internet together and hence its stability, i.e. the equilibria of fractional BGP games [15], is a matter of practical importance. BBC games [22] follow in the tradition of the large body of work on network formation games and capture a variety of applications ranging from social networks and overlay networks to peer-to-peer networks.

The central result of this paper is that there are no fully polynomial-time approximation schemes (unless PPAD is in FP) for computing equilibria in both fractional BGP games and fractional BBC games. We obtain this result by proving the hardness for a new and surprisingly simple game, the preference game, which is reducible to both fractional BGP and BBC games.

We define a new flow-based notion of equilibrium for matrix games - personalized equilibria - which generalizes both fractional BBC games and fractional BGP games. We prove not just the existence, but the existence of rational personalized equilibria for all matrix games, which implies the existence of rational equilibria for fractional BGP and BBC games. In particular, this provides an alternative proof and strengthening of the main result in [15]. For k-player matrix games, where k = 2, we provide a combinatorial characterization leading to a polynomial-time algorithm for computing all personalized equilibria. For k ≥ 5, we prove that personalized equilibria are PPAD-hard to approximate in fully polynomial time. We believe that the concept of personalized equilibria has potential for real-world significance.
1 Introduction

This paper concerns two classes of games on networks involving fractional flows — fractional BGP games and fractional BBC games. These games model important practical systems such as the Internet and social networks. The stable operating points of these systems have real-world significance and hence there is interest in their pure Nash equilibria. In order to understand the structure and computational complexity of these equilibria, we define two new concepts — personalized equilibria for matrix games and preference games — which are of independent interest. Below we briefly describe and motivate each of the four different kinds of games.

**Fractional BGP games.** The Border Gateway Protocol is the core routing protocol of the Internet. BGP can be viewed as a distributed mechanism for solving the stable paths problem [13]. In this paper, we refer to the fractional version of the stable paths problem introduced in [15] as the fractional BGP game. Intuitively, the fractional BGP game is a game played between Autonomous Systems that assign fractional capacities to the different paths leading to the destination in such a way that they maximize their utility without violating the capacity constraints of downstream nodes. Clearly, the equilibria of this game have significant implications for the stability of the Internet.

**Fractional BBC games.** Consider a social network where people have to spend time and (cognitive) resources to build connections to people. This situation naturally lends itself to being modeled by the Bounded Budget Connection game [22, 23]. In a BBC game, strategic nodes acting under a cost budget form connections with a view to optimizing their proximity to nodes of interest, which in turn depends on other nodes’ strategic actions. BBC games belong to the much studied class of network formation games. These games have applications to a variety of problems ranging from “how to monetize a social network” to “how to structure incentives in a peer-to-peer network to reduce congestion.” Fractional BBC games were defined in [23], which left unresolved the complexity of finding their Nash equilibria.

**Personalized equilibria for matrix games - a generalization.** Imagine a business selling outfits consisting of a pant (solid or striped) and a shirt (cotton or wool). The manager of one location decides on the ratio of striped to solid pants while the manager at the other decides on the ratio of cotton to wool shirts. Each manager is given the same number of shirts and pants (in the proportions decided) and has to assemble and sell the outfits at her own location in such a way as to maximize her individual profits. Personalized equilibria for matrix games capture exactly this situation: each player chooses a distribution over her own actions, but then each player independently customizes the matching of her own actions to the actions of other players in such a way as to maximize individual payoff. The concept of personalized equilibria for matrix games generalizes both fractional BGP and fractional BBC games.

**Preference games - a specialization.** It is New Year’s Eve. You and each of your friends is hosting a party. Each of you has a preference order over the others’ parties and has to determine the fraction of the evening that you will spend at each party. Naturally, one cannot spend more time at a party than the person hosting that particular party. Your optimal action – how long to host your party and which other parties to attend for how long – depends on your preference and other players’ actions. Such preference games arise whenever each player has a preference among her actions and her distribution over her actions is somewhat constrained by others’ distributions. Preference games are reducible in polynomial-time to both fractional BGP and BBC games.

1.1 Our Contributions

Our paper centers on the study of two classes of flow-based network games — fractional BGP games and fractional BBC games, formally defined in Section 2. We address the following two questions:
1. Does a Nash equilibrium always exist and if so what does the set of equilibria look like?

2. From a computational standpoint how difficult is it to find a Nash equilibrium?

To answer the first question, we define a new flow-based notion of equilibrium for matrix games – personalized equilibria. Personalized equilibria for matrix games constitute a novel and useful generalization of the concept of Nash equilibria for both fractional BGP as well as fractional BBC games. By employing this generalization, we are able to show the following results for multi-player matrix games.

- We show that the set of personalized equilibria for any multi-player matrix game is always nonempty and contains a rational point, though it may be nonconvex (Section 4).

It follows that a rational equilibrium always exists for both fractional BGP and BBC games. We provide an alternate characterization for BGP games that enables a simpler existence proof and strengthens the main result in [15]. We expect that personalized equilibria will be applicable elsewhere, since they capture real-world situations in which players have the opportunity to customize their use of others’ actions.

To answer the second question, we create a new combinatorial $k$-player abstract game – the preference game (see Section 3). Preference games are extremely elementary games that are a simultaneous simplification of both fractional BGP and fractional BBC games. By employing this simplified abstraction, we are able to obtain the following by reduction from a Brouwer fixed point problem [5, 6, 7].

- There are no fully polynomial-time approximation schemes (unless $\text{PPAD}$ is in $\text{FP}$) for computing equilibria in preference games (Section 5).

It follows that there are no fully polynomial-time approximation schemes (unless $\text{PPAD}$ is in $\text{FP}$) for computing equilibria in fractional BGP and fractional BBC games, as well as personalized equilibria in multi-player matrix games. Our result for fractional BGP games settles a question left open in [15], while our result for fractional BBC games settles an open question from [22]. Finally, we study the complexity of personalized equilibria in $k$-player matrix games, for fixed $k$.

- For $k = 2$, we provide a combinatorial characterization which implies a polynomial-time algorithm for computing the personalized equilibrium (Section 4).

- For $k \geq 4$, it is $\text{PPAD}$-hard to find personalized equilibria. Furthermore, for $k \geq 5$, there is no fully polynomial time approximation scheme (unless $\text{PPAD}$ is in $\text{FP}$) for finding personalized equilibria (Section 4).

1.2 Related Work

Nash equilibrium [25, 26] is arguably the most influential solution concept in game theory. Decades after Nash, Papadimitriou defined a complexity class $\text{PPAD}$ [29] to characterize proofs that rely on parity arguments. Recently, an exciting breakthrough made in [7] and strengthened in [5] showed the hardness of approximating Nash equilibria. Since then, there has been a flurry of work on the complexity of finding equilibria in a variety of games and markets [27]. All of our hardness proofs build on the framework established by [5, 6, 7] and heavily use their techniques.

BGP has been the focus of much attention since its inception [31, 32]. The integral stable paths problem was introduced [13] to explain the nonconvergence of BGP [33]. The fractional relaxation of the stable paths problem, or the fractional BGP game, was defined in [15] where they proved the existence of an equilibrium but left open the complexity of finding it. [20] gives a distributed algorithm for finding an $\epsilon$-approximation for the fractional BGP game that is guaranteed to converge, although no bounds are given on the time-to-convergence (the main result in this paper implies a polynomial upper bound is unlikely). Other related works include a multicommodity version [30, 24] and mechanism design [11].
The BBC game, introduced in [22, 23], builds on a large body of work in network formation games [17, 4]. A direct precursor to BBC games was introduced in [10], which together with subsequent works [1, 8], focuses on obtaining price of anarchy results [21]. In [22], it is shown that it is \text{NP}-hard [12] to determine whether an equilibrium exists in integral BBC games. Fractional BBC games were also introduced in [23], but the problem of finding an equilibrium was left open. Other related works include a stochastic small-world version [9] and use of contracts [2, 18].

2 Definitions

In this section, we define fractional BGP and fractional BBC games. Our definitions lead to the existence of Nash equilibria for these games using standard fixed-point techniques. We defer the formal proofs of existence of equilibria, however, to Section 4, where we establish the existence of a more general class of equilibria that includes the equilibria for both fractional BGP and BBC games.

2.1 Fractional BGP

The fractional BGP game is based on a new model of [15], introducing the notion of a fractional stable paths solution in the context of BGP. We first present the model of [15], and then define the fractional BGP game, whose Nash equilibria are equivalent to fractional stable paths solutions.

Let $G$ be a graph with a distinguished node $d$, called the destination. Each node $v \neq d$ has a list $\pi(v)$ of simple paths from $v$ to $d$ and a preference relation $\geq_v$ among the paths in $\pi(v)$. For paths $P$ and $P'$ in $\pi(v)$, $P \geq_v P'$ indicates that $v$ prefers $P$ at least as much as $P'$. We say that $P \succ_v P'$ if $P \geq_v P'$ is true but $P' \geq_v P$ is not true. When it is clear from context that we are talking about the preferences for node $v$, we will write $P \geq P'$ instead of $P \geq_v P'$. For a path $S$, we also define $\pi(v, S)$ to be the set of paths in $\pi(v)$ that have $S$ as a suffix. A proper suffix $S$ of $P$ is a suffix of $P$ such that $S \neq P$ and $S \neq \emptyset$.

A feasible fractional paths solution is a set $w = \{w_v : v \neq d\}$ of assignments $w_v : \pi(v) \rightarrow [0, 1]$ satisfying the following:

1. **Unity condition:** for each node $v$, $\sum_{P \in \pi(v)} w_v(P) \leq 1$

2. **Tree condition:** for each node $v$, and each path $S$ with start node $u$, $\sum_{P \in \pi(v, S)} w_v(P) = w_u(S)$.

In other words, a feasible solution is one in which each node chooses at most 1 unit of flow to $d$ such that no suffix is filled by more than the amount of flow placed on that suffix by its starting node. A feasible solution $w$ is *stable* if for any node $v$ and path $Q$ starting at $v$, one of the following holds:

(S1) $\sum_{P \in \pi(v)} w_v(P) = 1$, and for each $P$ in $\pi(v)$ with $w_v(P) > 0$, $P \geq_v Q$; or

(S2) There exists a proper suffix $S$ of $Q$ such that $\sum_{P \in \pi(v, S)} w_v(P) = w_u(S)$, where $u$ is the start node of $S$, and for each $P \in \pi(v, S)$ with $w_v(P) > 0$, $P \geq_v Q$.

In other words, in a stable solution: if node $v$ has not fully chosen paths that it prefers at least as much as $Q$, then it has completely filled path $Q$ by filling some suffix with paths it prefers at least as much as $Q$.

We now define the fractional BGP game. For convenience, let $w_{-v}$ denote $\{w_u : u \neq d, v\}$. Given assignments $w_v$, $w_v'$, and $w_{-v}$ such that $(w_v, w_{-v})$ and $(w_v', w_{-v})$ are both feasible, we say $w_v$ is *lexicographically at least* $w_v'$ (implied: with respect to $w_{-v}$) if the following holds for every path $P$ in $\pi(v)$:

$\sum_{P' \geq P} w_v(P') \geq \sum_{P' \geq P} w_v'(P')$. We say that $w_v$ is *lexicographically maximal* (implied: with respect to $w_{-v}$) if $(w_v, w_{-v})$ is feasible and $w_v$ is lexicographically at least every assignment $w_v'$ such that $(w_v', w_{-v})$ is feasible.

\footnote{A preference relation is a binary relation that is transitive and complete.}
In the fractional BGP Game, a strategy for a node \( v \neq d \) is a weight function \( w_v : \pi(v) \to [0, 1] \) that satisfies the unity and tree conditions, and the preference relation among the strategies of a node \( v \) is defined by the lexicographically at least relation. (Thus, a node’s best response is a lexicographically maximal flow.)

We can now show that fractional stable paths solutions are equivalent to (pure) Nash equilibria in the fractional BGP game. We note that [20] has also independently shown that a fractional stable paths solution is a Nash equilibrium of a suitably defined game.

**Theorem 1.** A fractional paths solution is stable iff it is lexicographically maximal for every node.

**Proof.** A stable paths solution is a lexicographically maximal flow. Let \( w \) be a fractional stable paths solution. Assume, for the sake of contradiction, that \( w_v \) is not lexicographically maximal with respect to \( w_{-v} \). Then there exists an assignment \( w_v' \) such that \((w_v', w_{-v})\) is feasible and \( w_v' \) is lexicographically greater than \( w_v \) with respect to \( w_{-v} \). Among all such assignments, we set \( w_v^* \) to be an assignment such that the preference of the highest preference path at which \( w_v \) and \( w_v^* \) differ is smallest. Let \( P \) be the highest preference path at which they differ and let \( \mathcal{P} \) denote the set of all paths with the same preference as \( P \).

By the definition of stability, at least one of the two stability conditions must hold for \( P \) in \( w_v \). First, assume (S1) is satisfied. Then we have \( \sum_{P' \in \pi(v)} w_v(P') = 1 \) and each \( P' \in \pi(v) \) with \( w_v(P') > 0 \) is such that \( P' \geq_P P \). This implies that \( \sum_{P' \geq_P} w_v(P') = 1 \). But since \( w_v^* \) satisfies the unity condition, we have \( \sum_{P' \geq_P} w_v^*(P') = 1 \). However, this means that \( w_v \) is lexicographically at least \( w_v^* \) (by definition of “lexicographically at least”), so \( w_v^* \) is not lexicographically greater than \( w_v \), a contradiction.

If (S1) does not hold for \( P \), then condition (S2) must be satisfied for each path in \( \mathcal{P} \). For each \( Q \in \mathcal{P} \), there exists a proper suffix \( S_Q \) (say with start node \( u \)) of \( Q \) such that \( \sum_{P' \in \pi(v,S_Q)} w_v(P') = w_v(S_Q) \), and each \( P' \in \pi(v,S_Q) \) with \( w_v(P') > 0 \) is such that \( P' \geq_P P \); for each \( Q \), we set \( S_Q \) to be the smallest such suffix. By our choice of \( S_Q \) for each \( Q \), we obtain that for \( Q, Q' \in \mathcal{P}, \pi(v,S_Q) \) and \( \pi(v,S_{Q'}) \) are disjoint if \( Q \neq Q' \). Since \((w_v^*, w_{-v})\) satisfies the tree condition, we have \( \sum_{P' \in \pi(v,S_Q)} w_v^*(P') \leq w_v(S_Q) \). Therefore, using the fact that \( \pi(v,S_Q)'s \) are all disjoint, \( \sum_{Q \in \mathcal{P}} \sum_{P' \in \pi(v,S_Q): P' \geq_P} w_v^*(P') \leq \sum_{Q \in \mathcal{P}} w_v(S_Q) = \sum_{Q \in \mathcal{P}} \sum_{P' \in \pi(v,S_Q): P' \geq_P} w_v(P') \). Furthermore, since \( w_v^* \) is identical to \( w_v \) on all paths more preferred than the paths in \( \mathcal{P} \), we obtain that \( \sum_{Q \in \mathcal{P}} w_v^*(Q) \leq \sum_{Q \in \mathcal{P}} w_v(Q) \).

We now consider two cases. If \( \sum_{Q \in \mathcal{P}} w_v^*(Q) < \sum_{Q \in \mathcal{P}} w_v(Q) \), then \( w_v^* \) is lexicographically greater than \( w_v \), leading to a contradiction. Otherwise, we derive a new assignment \( w'_v \) that is identical to \( w_v^* \), except on paths in \( \mathcal{P} \), where it is identical to \( w_v \). This new assignment \( w'_v \) is lexicographically greater than \( w_v \), since \( w_v^* \) was lexicographically greater; the highest preference path at which it differs from \( w_v \), however, has lower preference than that for \( w_v^* \), contradicting our choice of \( w_v^* \).

A lexicographically maximal flow is a stable paths solution. Let \( w_v \) be a lexicographically maximal flow with respect to \( w_{-v} \). Consider any path \( Q \) that starts at a node \( v \). Suppose, for the sake of contradiction, \( Q \) does not satisfy either of the two stability conditions. That is, we have (i) \( \sum_{P \geq Q} w_v(P) < 1 \), and (ii) for each proper suffix \( S \) of \( Q \) with start node \( u \), we have \( \sum_{P \in \pi(v,S), P \geq Q} w_v(P) < w_v(S) \). We derive a new assignment \( w'_v \) which is identical to \( w_v \) except for the following: \( w'_v(Q) = w_v(Q) + \varepsilon \), for a suitably small \( \varepsilon > 0 \); for each proper suffix \( S \) of \( Q \), if there exists a path that is less preferred than \( Q \), shares \( S \), and has positive weight, then we select one such path \( P \) and set \( w'_v(P) = w_v(P) - \varepsilon \). It is easy to see that \( w'_v \) satisfies the unity and tree conditions. However, \( w'_v \) is lexicographically greater than \( w_v \), a contradiction.

\[ \square \]

2.2 Fractional BBC

We define a fractional variant of the Bounded Budget Connection game, as in [23]. A fractional Bounded Budget Connection game (henceforth, a fractional BBC game) is specified by a tuple \( \langle V, d, c, b \rangle \), and a
3 Hardness of Finding Equilibria

In this section, we define a very simple game, the \emph{preference game}, which is a special case of both fractional BGP and fractional BBC games. In Section 3.2 we show that the set of all equilibria in a preference game is not convex, implying that we cannot hope to find an equilibrium for fractional BGP or BBC games using convex programming. We next present, in Section 3.3, our main result: it is PPAD-hard to find an equilibrium in the preference game. Finally, in Section 3.4 we define an \(\epsilon\)-approximate equilibrium for the preference game, which encompasses two previously-defined notions of approximation for fractional BGP. We extend our PPAD-hardness result to approximate equilibria, thereby proving that there are no fully polynomial-time approximation schemes (unless PPAD is in FP) for computing equilibria in both fractional BGP games and fractional BBC games.

3.1 Preference Games

We begin by defining preference games. In a preference game with a set \(S\) of players, each player’s strategy set is \(S\). Each player \(i \in S\) has a preference relation \(\geq_i\) among the strategies. Each player \(i\) chooses a \emph{weight distribution}, which is an assignment \(w_i : S \rightarrow [0, 1]\) satisfying two conditions: (a) the weights add up to 1: \(\sum_{j \in S} w_i(j) = 1\); and (b) the weight placed by \(i\) on \(j\) is no more than the weight placed by \(j\) on \(i\): \(w_i(j) \leq w_j(j)\) for all \(i, j \in S\). As in the case of fractional BGP, the preference relations \(\geq_i\) induce a preference relation among the weight distributions as follows: \(w_i\) is \emph{lexicographically at least} \(w'_i\) if for all \(j \in S\), \(\sum_{k \geq j} w_i(k) \geq \sum_{k \geq j} w'_i(k)\). An equilibrium in a preference game is an assignment \(w = \{w_i : i \in S\}\) such that \(w_i\) is lexicographically maximal for all \(i \in S\).

We now show that the preference game is a special case of both fractional BGP and fractional BBC.

\textbf{Lemma 1.} There is a polynomial-time reduction from the preference game to the fractional BGP game.

\textbf{Proof.} Consider any instance \(P\) of the preference game, consisting of a set of players \(S\) and a preference relation \(\geq_i\) for each \(i \in S\). We will create an instance \(B\) of fractional BGP. For each player \(i \in S\), create a node \(i\) in \(B\). Also create a universal destination node \(d\). For all \(i \neq d\), define \(P_{ii} = \text{the path } (i, d)\). For all \(i, j \neq d\), define \(P_{ij} = \text{the path } (i, j, d)\). For all \(i\): define \(\pi(i)\) (the set of \(i\’s\) preferred paths in \(B\)) as the set \(\{P_{ij} : j \geq i\}\). If \(k \geq j \in P\), then \(P_{ik} \geq P_{kj}\) in \(B\).

Consider any feasible solution \(w = \{w_i\}\) for \(P\), and define weights \(w' = \{w'_i\}\) for \(B\) where \(\forall i, j, w'_i(P_{ij}) = w_i(j)\), and \(w'_i(P) = 0\) for all other paths \(P\). Because \(w\) is feasible, for any \(i \in S\), \(\sum_{j \in S} w_i(j) = \sum_{j \in S} w'_i(j)\).
1. Therefore, for all $i$, $\sum_{\text{paths } P} w'_i(P) = \sum_{P_{ij} \in S} w'_i(P_{ij}) = \sum_{j \in S} w_i(j) = 1$, and the unit condition for $B$ is satisfied. Also, for all $i, j \in S$, $w_i(j) \leq w_{j'}(j)$. Therefore, for all paths $i$ and all paths $P$ starting at $j$, $\sum_{P_{ij} \in \pi(i, P)} w'_i(P_{ij}) = w'_i(P_{ij}) = w_i(j) \leq w_{j'}(j) = w'_j(P_{jj})$, and the tree condition for $B$ is satisfied, and $w'$ is feasible.

Consider an equilibrium $w' = \{w'_i\}$ of $B$, and define weights $w = \{w_i\}$ for $P$ where $\forall i, j, w_i(j) = w'_i(P_{ij})$. Since this is an equilibrium of $B$, it must be feasible and lexicographically maximal. Because it is feasible, for each node $i$, $\sum_{P_{ij} \in \pi(i, P)} w'_i(P_{ij}) \leq 1$. This implies that in our new solution for $P$, $\sum_{j \in S} w_j \leq 1$. Because it is lexicographically maximal, $P_{ii} \geq_i P_{ij} \Rightarrow w'_i(P_{ij}) = 0$, so $\sum_{j \in S} w_i(j) = \sum_{j \geq_i j} w_i(j) \leq 1$. Furthermore, since $P_{ii} \geq_i$ the empty path, $w'_i(P_{ii}) = 1 - \sum_{P_{ij} \in \pi(i, P) \neq i} w'_i(P_{ij})$, and $\sum_{j \in S} w_j(j) = 1$, as required in the preference game. $w'$ is feasible also implies for each node $i$, and each path $P$ with start node $j$, we have $\sum_{P_{ij} \in \pi(i, P)} w'_i(P_{ij}) \leq w'_j(P_{jj})$. However, $\{P \in \pi(j)\} \cap \{P : \pi(i, P) \neq \emptyset\} = \{P_{jj}\}$, by definition of the preference sets. So $\sum_{P_{ij} \in \pi(i, P)} w'_i(P_{ij}) = 0$ unless $P = P_{jj}$, and $\{P \in \pi(i, P_{jj})\} = \{P_{jj}\}$, so if $P = P_{jj}$ then $\sum_{P_{ij} \in \pi(i, P_{jj})} w'_i(P_{ij}) = w'_i(P_{jj})$. Therefore, $w_i(j) \leq w'_j(P_{jj})$, as required for feasibility in $P$.

Now, consider any other feasible assignment $\bar{w} = \{\bar{w}_i\} \cup \{w_j : j \neq i\}$ for $P$. Define $\bar{w}'(P_{ij}) = \bar{w}_i(j)$. Then $\bar{w}' = \{\bar{w}'_i\} \cup \{w'_j : j \neq i\}$ is feasible for $B$, as shown above, and lexicographic maximality of $w'$ says that for every path $P_{ij}$ in $\pi(i)$, $\sum_{P_{ik} \geq P_{ij}} w'_i(P_{ik}) \geq \sum_{P_{ik} \geq P_{ij}} \bar{w}_i(P_{ik})$. Therefore, for every $j \in S$, $\sum_{k \geq j} w_k(k) \geq \sum_{k \geq j} \bar{w}_k(k)$, so $w$ is also lexicographically maximal, and $w$ is an equilibrium for $P$.

Finally, consider an equilibrium $w = \{w_i\}$ for $P$ and the weights $w' = \{w'_i\}$ for $B$ as defined above. From above, $w'$ is feasible. Consider any other feasible assignment $\bar{w}' = \{\bar{w}'_i\} \cup \{w'_j : j \neq i\}$ for $B$. Since $w$ is an equilibrium, it is lexicographically maximal, so for $\bar{w}' = \{\bar{w}'_i\} \cup \{w'_j : j \neq i\}$ (where $\bar{w}_i(j) = \bar{w}'_i(P_{ij})$, $\forall i, j \in S$, $\sum_{k \geq j} w_k(k) \geq \sum_{k \geq j} \bar{w}_k(k)$). Therefore, $\forall i, j \in S$, $\sum_{\pi_{ik} \geq \pi_{ij} \pi_{ik}} w'_i(P_{ik}) \geq \sum_{\pi_{ik} \geq \pi_{ij} \pi_{ik}} \bar{w}'_i(P_{ik})$, and $w'$ is also lexicographically maximal and an equilibrium for $B$.

**Lemma 2.** There is a polynomial-time reduction from the preference game to the fractional BBC game.

**Proof.** We use a similar reduction from a preference game to fractional BBC. Given any instance $P$ of the preference game. We will create an instance $B$ of fractional BBC = $(V, d, c, b)$, where $V = S$, $d$ is an additional node, $\forall i, j \in V: c(i, j) = 1$, $\forall i: b(i) = 1$, plus length function $l_i$ for each $i \in V$, defined as follows. Let $l_i(k) = \text{the number of } j \text{ such that } j \geq_i k$. $\forall j \neq i, l_i(j, d) = 1, l_i(i, j) = p_i(j)$. $\forall j \neq i, k \neq i, l_i(j, k) = l_i(k, j) = |S| + 1$. $l_i(i, d) = 1 + p_i(i)$. Given a solution to $B$, define a solution to $P$: set $w_i(j)$ = the weight placed on edge $(i, j)$ (for $j \neq i$), and $w_i(i) = \text{the weight placed on edge } (i, d)$.

Consider any instance $P$ of the preference game, consisting of a set of players $S$ and a preference relation $\geq_i$ for each $i \in S$. We will create an instance $B$ of fractional BBC = $(V, d, c, b)$, where $V = S$, $d$ is an additional node, $\forall i, j \in V: c(i, j) = 1$, $\forall i: b(i) = 1$, plus length function $l_i$ for each $i \in V$, defined as follows. Let $l_i(k) = \text{the number of } j \text{ such that } j \geq_i k$. $\forall j \neq i, l_i(j, d) = 1, l_i(i, j) = p_i(j)$. $\forall j \neq i, k \neq i, l_i(j, k) = l_i(k, j) = |S| + 1$. $l_i(i, d) = 1 + p_i(i)$. Given a solution to $B$, define a solution to $P$ by setting $w_i(j)$ = the weight placed on edge $(i, j)$ (for $j \neq i$), and $w_i(i) = \text{the weight placed on edge } (i, d)$.

Since the total cost for all edges is 1, and the total budget for a node is 1, each node in $B$ will place total weight 1 on edges adjacent to it. This exactly corresponds to the requirement that $\sum_j w_i(j) = 1$ in $P$. The possible paths for a one-unit flow from $i$ to $d$ in $B$ are: (1) the path consisting of only edge $(i, d)$, which has cost $p_i(i) + 1 \leq |S| + 1$, (2) a path of the form $(i, j, d)$ through some other node $j$, which has cost $p_i(j) + 1 \leq |S| + 1$, or (3) a path including some edge $(j, k)$ for $j \neq i, k \neq i$, which has cost $|S| + 1$. Therefore, a minimum cost flow will only use paths of the form $(i, d)$ and $(i, j, d)$, so the requirement in $P$ that $w_i(j) \leq w_{j'}(j)$ corresponds to using the weight $j$ places on edge $(j, d)$ as a capacity on that edge when
focusing on the min-cost flow. Now, we only need to show that a node’s best response in B exactly corresponds to a lexicographically maximal weight assignment in P.

Suppose we have a best response for node i in B that corresponds to a weight assignment w in P that is not lexicographically maximal for i. Then, there is some assignment \( w' = w'_i \cup \{ w_j : j \neq i \} \) such that for some \( j \in S, \sum_{k \geq j} w_i(k) < \sum_{k \geq j} w'_i(k) \). There must be some \( k^+ \in S \) such that \( k^+ \geq i, j \) and \( w'_i(k^+) > w_i(k^+) \), and there must be some \( k^- \in S \) such that \( -i \geq k^- \) and \( w'_i(k^-) < w_i(k^-) \). Suppose we move \( \epsilon \) weight in the best response in B from \( P_{ik^-} \) to \( P_{ik^+} \). \( p_i(k^-) > p_i(k^+) \), so moving this weight will decrease the cost of a minimum cost flow, contradicting the fact that this was a best response.

Suppose we have a lexicographically maximal weight assignment w in P that does not correspond to a best response for node i in B. Then, in B, i could move weight from some path \( P_{ij} \) to a different path \( P_{ik} \) to decrease the cost of its min-cost flow. This means that \( p_i(k) < p_i(j) \), or the number of nodes preferred by i over k is smaller than the number of nodes preferred by i over j. Since preference relations are transitive, this implies that \( k \geq i, j \). However, since \( P_{ik} \) had space left, \( w_k(k) < w_k(k) \), so w is not lexicographically maximal.

\[ \square \]

3.2 Non-Convexity

Theorem 2. There exists an instance of the preference game for which the set of equilibria is not convex.

![Figure 1: Example of an instance of the preference game for which the equilibrium set is not convex.](image)

(a) The a players assign weights (b) The a players assign weights 1/2 – (c) Combining half of each equilibrium, 1/2, 1/2, the b players both use b1, the c 1/2, the b players both use b2, the c players both use c2. x assign 1/2 to a1, 1/4 to each of b1 and c1. x could improve by assigning weight only to a1 and b1.

Proof: Consider the following instance of the preference game. We have 3 sets of 2 players each, \( a_1, a_2, b_1, b_2, c_1, c_2 \), and one additional player, x. The preference lists for these nodes are: \( a_1: (a_2, a_1); a_2: (a_1, a_2); b_1: (b_2, b_1); b_2: (b_1, b_2); c_1: (c_2, c_1); c_2: (c_1, c_2); x: (a_1, b_1, c_1, x) \). (Each list gives strategies in order from most preferred to least preferred.) We now show two equilibria whose linear combination is not an equilibrium. In equilibrium \( w \) (figure 1(a)): \( w_{a_1}(a_1) = \frac{1}{2}, w_{a_1}(a_2) = \frac{1}{2}, w_{a_2}(a_2) = \frac{1}{2}, w_{a_2}(a_1) = \frac{1}{2}, w_{b_1}(b_1) = 1, w_{b_2}(b_1) = 1, w_{c_1}(c_2) = 1, w_{c_2}(c_2) = 1, w_x(a_1) = \frac{1}{2}, w_x(b_1) = \frac{1}{2} \). In equilibrium \( w' \) (figure 1(b)): \( w'_{a_1}(a_1) = \frac{1}{2}, w'_{a_1}(a_2) = \frac{1}{2}, w'_{a_2}(a_2) = \frac{1}{2}, w'_{a_2}(a_1) = \frac{1}{2}, w'_{b_1}(b_2) = 1, w_{b_2}(b_2) = 1, w'_{c_1}(c_1) = 1, w'_{c_2}(c_1) = 1, w'_x(a_1) = \frac{1}{2}, w'_x(c_1) = \frac{1}{2} \). It is easy to verify that \( w \) and \( w' \) are both equilibria, and in a solution \( \lambda \cdot w + (1 - \lambda) \cdot w' \) (for any \( \lambda > \frac{1}{2} \)) (figure 1(c)) shows \( \lambda = \frac{1}{2} \), player x would do better by moving more weight to its second preference. Therefore, the convex combination of \( w \) and \( w' \) is not an equilibrium.

\[ \square \]

3.3 PPAD Hardness

We show that finding an equilibrium in preference games is PPAD-hard. By our reductions of Lemmas[1] and [2] this immediately implies that finding equilibria in fractional BGP and fractional BBC games is also PPAD-hard. We will follow the framework of [7], which shows that finding a Nash equilibrium in a degree-3 graphical game is PPAD-hard, using a reduction from the PPAD-complete problem 3-D BROUWER. In
this problem, we are given a 3-D cube in which each dimension is broken down into $2^{-n}$ segments – thereby dividing the cube into $2^{3n}$ cubelets. We are also given a circuit that takes as input the 3 coordinates of the center of a cubelet (each as an $n$-bit number) and returns a 2-bit number that represents one of four 3-D vectors: either $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, or $(-1, -1, -1)$. A solution to the 3-D BROUWER instance is a cubelet vertex such that the set of 8 results obtained by running the circuit on each of the 8 cubelets surrounding the vertex contains each of the four vectors at least once.

As in [7], we will construct a set of gadgets to simulate various arithmetic operators, logical operators, arithmetic comparisons and other operators. We then follow their framework to systematically combine these gadgets to simulate the input boolean circuit and to encode the geometric condition of discrete fixed points in the 3-D BROUWER instance. In the preference game we construct, we specify the preference relation of any player $P$ by an ordered list of a subset of the players, with the last element being $P$, also referred to as the “self” strategy. When we say that a player $P$ plays itself with weight $v$, we mean that $P$ assigns a weight of $v$ to strategy $P$. We’ll engineer the payoffs such that the game is only in equilibrium if the weights assigned by certain players to themselves successfully echo the inputs and outputs of 8 copies of the circuit that surround a solution vertex of the 3-D BROUWER instance.

For this reduction, we require the following sets of players.

1. One player for each of the 3 coordinates (the coordinate players). If the graph is an equilibrium, each coordinate player plays itself with weight equal to its coordinate of the 3-D BROUWER solution vertex.

2. One player for each of the bits of each of the 3 coordinates (the bit players). In order to force these players to correctly represent the bits, we need some additional players. Assuming we’ve correctly calculated the first $i - 1$ bits of coordinate $x$ (call them $x_0, \ldots, x_{i-1}$), we can create the $i^{th}$ bit as follows. One player will play itself with weight $p_i = x - \sum_{j=0}^{i-1} x_j 2^j$. The bit player will play itself with weight equal to the $i^{th}$ bit. If $p_i \geq \frac{1}{2}$, then this bit should be 1. Otherwise, it should be 0. Therefore, in order to properly extract the bits, we create the following four types of players.

   (a) HALF player: In any equilibrium in which a given player plays itself with weight $a$, the HALF player will play itself with weight $\frac{a}{2}$.

   (b) DIFF player: In any equilibrium in which two given players play themselves with weights $a$ and $b$, the DIFF player will play itself with weight $a - b$.

   (c) VALUE player: In any equilibrium, the VALUE player plays itself with weight $\frac{1}{2}$. This can be easily created by combining a player whose first preference is itself with a HALF player.

   (d) LESS player: In any equilibrium in which two given players play themselves with weights $a$ and $b$, respectively, the LESS player plays itself with weight 1 iff $a \geq b$, and plays itself with weight 0 otherwise. (Actually, the LESS player we create will be inaccurate if $a$ and $b$ are very close, which we discuss more below.)

3. One player simulates each type of gate used in the circuit of the 3-D BROUWER instance. For this, we create 3 more types of players.

   (e) AND player: In any equilibrium in which two given players play themselves with weights $a$ and $b$, the AND player will play itself with weight $a \land b$.

   (f) OR player: In any equilibrium in which two given players play themselves with weights $a$ and $b$, the OR player will play itself with weight $a \lor b$. 
(g) NOT player: In any equilibrium in which a given player plays itself with weight $a$, the NOT player will play itself with weight $\neg a$.

4. Finally, we need to ensure that the graph is in equilibrium if and only if all four vectors are represented in the results of the 8 circuits. As in [7], we will represent the output of each circuit using 6-bits, one each for $+x, -x, +y, -y, +z, -z$. Now, the 4 possible result vectors are represented as 100000, 001000, 000010, and 010101. We can use these circuit results with only two additional types of players to feed back into the original coordinate players. First, we will create an OR player for each of the 6 bits (over the 8 vertices), which yields a result of six 1’s if and only if this is a solution vertex. Therefore, an AND player for each coordinate will all return 1 if and only if this is a solution vertex; at least one of the coordinates will be 0 otherwise. We can turn this around using a NOT player for each coordinate, so that we get all 0’s if and only if this is a solution vertex. Finally, we need the last two new player types, which we’ll use to add these results back to a copy of the original coordinates (the result will be the original coordinate player).

(h) COPY player: In any equilibrium in which a given player plays itself with weight $a$, the COPY player will also play itself with weight $a$.

(i) SUM player: In any equilibrium in which two given players play themselves with weights $a$ and $b$, the SUM player will play itself with weight $\min(a + b, 1)$.

If the coordinates represented a solution vertex to the 3-D BROUWER instance, then all the values we’ve added back in will be zero; so the coordinate players cannot do better by changing their strategies. On the other hand, if the coordinates do not form a solution vertex, then at least one of the values is 1, so that the coordinate player will have incentive to change strategies and play more weight on itself.

We now describe how to create the new types of players (gadgets) required for the reduction. For each of these gadget definitions, we assume we are given a preference game such that in any equilibrium, node $X$ plays itself with weight $v_1$ and node $Y$ plays itself with weight $v_2$. For the first three gadgets, we assume $v_1, v_2 \in \{0, 1\}$. For the rest of the gadgets, we assume $v_1, v_2 \in [0, 1]$.

**OR**($X, Y$)
We can add a new node $R = \text{OR}(X, Y)$ that will play itself with weight $v_1 \lor v_2$ in any equilibrium. Create a node $R_1$ with preference list $(X, Y, R_1)$. Let node $R$’s preference list be $(R_1, R)$. Now, if $v_1$ and/or $v_2$ is 1, then $R_1$ will play $R_1$ with weight 0, so $R$ will play itself with weight 1. If both $v_1$ and $v_2$ is 0, then $R_1$ will play itself with weight 1, so $R$ will play $R_1$ with weight 1 and $R$ with weight 0.

**NOT**($X$)
We can add a new node $N = \text{NOT}(X)$ that will play itself with weight $\neg v_1$ in any equilibrium. Let node $N$’s preference list be $(X, N)$. Clearly, $N$ will play $X$ as much as $v_1$ and will play $N$ with the remainder.

**AND**($X, Y$)
We can add a new node $A = \text{AND}(X, Y)$ that will play itself with weight $v_1 \land v_2$ in any equilibrium. Assemble the OR and NOT gadgets $\text{NOT}(\text{OR(\text{NOT}(X), \text{NOT}(Y)))}$. 

9
**SUM**(X, Y)

We can add a new node \( S = \text{SUM}(X, Y) \) that will play itself with weight \( \max(1, v_1 + v_2) \) in any equilibrium. Create a node \( S_1 \) with preference list \( (X, Y, S_1) \). Let node \( S \)'s preference list be \((S_1, S)\). Now, clearly node \( S_1 \) will play \( S_1 \) with weight \( \max(0, 1 - v_1 - v_2) \), and node \( S \) will play \( S_1 \) that same amount. So node \( S \) will play itself with weight \( 1 - \max(0, 1 - v_1 - v_2) \). In other words, if \( v_1 + v_2 \geq 1 \), then \( S \) will play itself with weight 1. Otherwise, \( S \) will play itself with weight \( 1 - 1 + v_1 + v_2 = v_1 + v_2 \), as desired.

**DIFF**(X, Y)

We can add a new node \( D = \text{DIFF}(X, Y) \) that will play itself with weight \( v_1 - v_2 \) if \( v_1 > v_2 \), or 0 otherwise in any equilibrium. Create a node \( D_1 \) with preference list \( (X, D_1) \). \( D_1 \) will play itself with weight \( 1 - v_1 \). Now set the preference list for \( D \) to \( (D_1, Y, D) \). \( D \) will play itself with weight \( \min(0, 1 - (1 - v_1) - v_2) = \min(0, v_1 - v_2) \), as desired.

**COPY**(X)

We can add a new node \( C = \text{COPY}(X) \) that will play itself with weight \( v_1 \) in any equilibrium. Create a node \( C_1 \) with preference list \( (X, C_1) \). \( C_1 \) will play itself with weight \( 1 - v_1 \). Set the preference list for node \( C \) to \( (C_1, C) \). \( C \) will play \( C_1 \) with weight \( 1 - v_1 \), leaving weight \( v_1 \) on \( C \).

**DOUBLE**(X)

We can add a new node \( M = \text{DOUBLE}(X) \) that will play itself with weight \( \min(1, v_1 + 2) \) in any equilibrium. Create player \( M_1 = \text{COPY}(X) \) and set \( M \) as \( \text{SUM}(X, M_1) \).

**LESS**(X, Y)

Given \( \epsilon_i \) \((0 < \epsilon_i \leq \frac{1}{2})\). We can add a new node \( L = \text{LESS}(X, Y) \) to the game that in any equilibrium will play only itself if \( v_1 - v_2 \geq \epsilon_i \), and will play \( L_1 \) (for a new node \( L_1 \)) if \( v_1 \leq v_2 \). First create \( D = \text{DIFF}(X, Y) \). Then create \( M_1 = \text{DOUBLE}(D) \). For \( i = 1 \) to \(-\log \epsilon_i \), create player \( M_{i+1} = \text{DOUBLE}(M_i) \). Call the last \( \text{DOUBLE} \) player node \( L \) and the extra player for the sum player of the last \( \text{DOUBLE} \) player node \( L_1 \). If \( v_1 \leq v_2 \), the \( \text{DIFF} \) player will return 0, so player \( L \) will play the result of multiplying 0 by 2 many times, or 0. If \( v_1 - v_2 \geq \epsilon_i \), player \( L \) will play the max of 1 and \( (v_1 - v_2) \star 2^{-\log \epsilon_i} = (v_1 - v_2) \star \frac{1}{\epsilon_i} \geq \frac{\epsilon_i}{\epsilon_i} = 1 \).

**HALF**(X)

We can add a new node \( H = \text{HALF}(X) \) that will play itself with weight \( v_1 / 2 \) in any equilibrium. Create a node \( H_1 \) with preference list \((X, H_1)\). \( H_1 \) will play itself with weight \( 1 - v_1 \). Then create two more nodes: \( H_2 \) and \( H_3 \). Node \( H_2 \) has preference list \((H_1, H_3, H_2)\). Node \( H_3 \) has preference list \((H_1, H, H_3)\). Set the preference list for node \( H \) to be \((H_1, H_2, H)\). Each of \( H, H_2, \) and \( H_3 \) will use its first choice with weight \( 1 - v_1 \), leaving \( v_1 \) for its other two choices. Then, we have \( w_H(H) + w_H(H_2) = v_1, w_H(H_2) + w_H(H_3) = v_1, \) and \( w_{H_3}(H_3) + w_{H_2}(H) = v_1 \). In any equilibrium, it must be true that \( w_H(H_2) = w_{H_2}(H_2) \), \( w_{H_3}(H_3) = w_{H_3}(H_3) \), and \( w_{H_3}(H) = w_{H_3}(H) \). Solving this gives \( w_H(H) = w_H(H_2) = w_{H_2}(H_2) = w_{H_3}(H_3) = w_{H_3}(H) = \frac{v_1}{3} \).

As in [7], our \( \text{LESS} \) player plays the specified action (itself, in our case) with weight 1 if \( v_1 \geq v_2 + \epsilon_i \), and plays itself with weight 0 if \( v_1 \leq v_2 \), but will play some unspecified fraction on itself if \( v_2 < v_1 < v_2 + \epsilon_i \). We use the \( \text{LESS} \) player to extract the bits representing the coordinates of a cubelet to be passed into the circuit. This procedure is identical to that of [7]. Let \( X \) denote the \( x \)-coordinate player, and let \( X_1 = \text{COPY}(X) \).
For $i$ from 1 through $n$, we create players $B_i = \text{LESS}(2^{-i}, X_i)$ and $X_{i+1} = \text{DIFF}(X_i, \text{HALF}^i(B_i))$, where $\text{HALF}^i$ indicates applying the $\text{HALF}$ gadget $i$ times. It can be shown that as long as $x$ is not too close to a multiple of $2^{-n}$, we will extract its $n$ bits correctly. If this is not the case, however, we will not properly extract the bits, and our circuit simulation may return an arbitrary value. We resolve this problem using the same technique as in [7]: we compute the circuit for a large constant number of points surrounding the vertex and take the average of the resulting vectors. Since these details are almost identical to that of [7], Lemma 4], we omit them. From this reduction, we get:

**Theorem 3.** It is $\text{PPAD}$-hard to find an equilibrium in a given preference game.

### 3.4 Approximate equilibria

Given the hardness of finding exact equilibria in preference games (and fractional BGP and BBC games), a natural next question is whether it is easier to find approximate equilibria. We define an $\epsilon$-equilibrium of a $k$-player preference game to be a set of weight distributions $w_1, \ldots, w_k$ that satisfy the following conditions for every player $i$: (a) $\sum_j w_i(j) = 1$; (b) for each $j$, $w_i(j) \leq w_j(j) + \epsilon$; and (c) for each $j$, either $\sum_{\ell \geq j} w_i(\ell) \geq 1 - \epsilon$ or $|w_i(\ell) - w_j(\ell)| \leq \epsilon$. In other words, the weight assigned by a player $i$ on another player $j$ is at most $\epsilon$ more than the weight assigned by $j$ on itself; and for any $i$ and $j$, either $i$ plays a total weight of at least $1 - \epsilon$ on players it prefers at least as much as $j$ or the weight assigned by $i$ on $j$ differs from that assigned by $j$ to itself by at most $\epsilon$. Note that there exists some threshold preference such that any player preferred strictly more than that must be “filled” to within $\epsilon$ of the allowed weight. The rest of at least $1 - \epsilon$ weight must be placed on players at the threshold preference. At most $\epsilon$ weight is left for players with preference lower than the threshold.

Two notions of approximation have been defined for fractional BGP: an $\epsilon$-solution by [15] and $\epsilon$-stable solution by [20]. The (polynomial-time) reduction of Lemma 1 mapping a given preference game instance $P$ to a fractional BGP game instance $B$ has the property that any $\epsilon$-solution or $\epsilon$-stable solution for $B$ is, in fact, an $\epsilon$-equilibrium for $P$. This implies that any $\text{PPAD}$-hardness on finding $\epsilon$-equilibrium for preference games immediately yields an equivalent result for both notions of approximation for fractional BGP.

**Theorem 4.** It is $\text{PPAD}$-hard to find an $\epsilon$-equilibrium for preference games, for some $\epsilon$ inverse polynomial in $n$.

**Proof.** Our proof follows the framework of [5, 6] for proving the hardness of approximating Nash equilibria in 2-player games. This framework starts with a high-dimensional discrete fixed point problem, BROUWER, which is also $\text{PPAD}$-complete. The input to the problem is a Boolean circuit that assigns a color from $\{1, \ldots, n, n + 1\}$ to each interior node of an $n$-dimensional grid $\{0, 1, \ldots, 8\}^n$. This grid has about $2^{3n}$ cells, each of which is an $n$-dimensional hypercube. The discrete fixed point is defined to be a panchromatic simplex inside a hypercube. This framework of [5, 6] uses a new geometric condition for discrete fixed points, which requires that the average of $n^3$ sampled points in the interior of the targeted panchromatic simplex is inverse-polynomially close to the zero vector. The rest of the proof follows the framework of [7].

Our broad definition of an $\epsilon$-equilibrium poses additional technical challenges which did not occur in the reductions of [5, 6]. In particular, in the presence of errors, our Boolean gadgets only approximately simulate the Boolean operations, while in previous reductions, the Boolean gadgets are precise. Therefore, most of our technical effort is to prevent the magnification of errors in Boolean simulation. In our proof, we have designed a CORRECTION gadget to accomplish this.

We focus on the necessary changes for the gadgets of Theorem 3 to account for errors, and the description and use of the new CORRECTION gadget. Other details closely match those of [5, 6, 7].
Let $\epsilon_1$ (the measure of the fragility of our LESS gadget) be a real number such that $\epsilon \leq \epsilon_1^3$. Then, we have the following error bounds.

**Lemma 3.** Assuming node $X$ plays itself with weight $v_1'$, $v_1 - 2\epsilon_1 \leq v_1' \leq v_1 + 2\epsilon_1$, and node $Y$ plays itself with weight $v_2'$, $v_2 - 2\epsilon_1 \leq v_2' \leq v_2 + 2\epsilon_1$, each of the boolean gadgets defined in the proof of Theorem 3 plays itself within $\pm (4\epsilon_1 + 6\epsilon)$ of the correct value for the correct $v_1$ and $v_2$ inputs.

**Proof.** OR

If $v_1$ and/or $v_2$ is 1, then $v_1'$ and/or $v_2'$ is at least $1 - 2\epsilon_1$, and node $R_1$ will play $R_1$ with weight at most $2\epsilon_1 + \epsilon$, so $R$ will play $R$ with weight at least $1 - 2\epsilon_1 - 2\epsilon$. If both $v_1$ and $v_2$ are 0, then $v_1'$ and $v_2'$ are at most $2\epsilon_1$, and node $R_1$ will play $R_1$ with weight at least $1 - 4\epsilon_1 - 2\epsilon$, so $R$ will play $R$ with weight at most $4\epsilon_1 + 3\epsilon$.

NOT

If $v_1 = 1$, $v_1'$ is at least $1 - 2\epsilon_1$, and node $N$ will play itself with weight at most $2\epsilon_1 + \epsilon$. If $v_1 = 0$, $v_1'$ is at most $2\epsilon_1$, and node $N$ will play $N$ with weight at least $1 - 2\epsilon_1 - \epsilon$.

AND

The AND gadget concatenates other new players to get $\neg (\neg v_1 \lor \neg v_2)$. Each NOT may add at most one additional $\epsilon$ error to the given value, and the OR may add up to $3\epsilon$ error (on top of the sum of the errors from both inputs). So the AND player will return a value within an additive $4\epsilon_1 + 6\epsilon$ of the correct 0 or 1 answer. \qed

**Lemma 4.** Each of the arithmetic gadgets plays itself within $\pm 5\epsilon$ of the correct value for the input it is given.

**Proof.** SUM

Node $S_1$ will play $S_1$ with weight $w(S_1T) \in [\max(0, 1 - v_1' - v_2' - 2\epsilon), \max(0, 1 - v_1' - v_2' + 2\epsilon)]$. So node $S$ will play $S$ with weight $w_S(S) \in [v_1' + v_2' - 3\epsilon, v_1' + v_2' + 3\epsilon]$, unless $w_S(S_1) = 0$, which means $v_1' + v_2' \geq 1 - 2\epsilon$. In this case, node $S$ will play $S$ with weight at least $1 - \epsilon$.

DIFF

Node $D_1$ will play $D_1T$ with weight $w_{D_1}(D_1) \in [\max(0, 1 - v_1' - \epsilon), \max(0, 1 - v_1' + \epsilon)]$. Node $D$ will play $D$ with weight $w_D(D) \in [\max(0, v_1' - v_2' - 3\epsilon), \max(0, v_1' - v_2' + 3\epsilon)]$, unless $w_D(D_1) = 0$ which means $v_1' \geq 1 - \epsilon$. In this case, node $D$ will play $D$ with weight at least $1 - v_2' - 2\epsilon$ and at most $1 - v_2' + \epsilon$ (not $2\epsilon$ because we cannot underfill the strategy with weight 0).

COPY

Node $C_1$ will play $C_1$ with weight at least $1 - v_1' - \epsilon$ and at most $1 - v_1' + \epsilon$. Node $C$ will play $C$ with weight at least $v_1' - 2\epsilon$ and at most $v_1' + 2\epsilon$.

HALF

Node $H_1$ will play $H_1$ with weight $w_{H_1}(H_1) \in [1 - v_1' - \epsilon, 1 - v_1' + \epsilon]$, and each other player will play its second and third preferences with total weight between $1 - w_{H_1}(H_1) - \epsilon$ and $1 - w_{H_1}(H_1) + \epsilon$. Each other player will play itself half of this amount plus or minus $3\epsilon$ (this is easy to verify by writing the system of
inequalities and checking the extreme points). Therefore, node $H$ plays $H$ with weight at least $\frac{v_1'}{2} - 4\varepsilon$ and at most $\frac{v_1'}{2} + 4\varepsilon$.

**DOUBLE**

The DOUBLE gadget consists of a copy player, which adds at most $2\varepsilon$ error, and a sum player, which adds at most $3\varepsilon$ error on top of the sum of the errors in the two inputs. Therefore, node $M$ plays $M$ with weight at least $2v_1' - 5\varepsilon$ and at most $2v_1' + 5\varepsilon$.

**Lemma 5.** The LESS player will play itself with weight $< \varepsilon_I$ if it is given $v_1', v_2'$ such that $v_1' \leq v_2'$, and with weight $> 1 - \varepsilon_I$ if $v_1' - v_2' \geq \varepsilon_I$.

**Proof.** LESS

The LESS gadget inherits its susceptibility to error from its initial DIFF player (which was, in the exact equilibrium case, non-zero if and only if $v_1 < v_2$). For the case where $v_1 < v_2$, we can account for the errors of the DOUBLE players (used to repeatedly amplify the difference) simply by adding extra iterations of DOUBLE. Since we stipulated that $\varepsilon \leq \varepsilon_I^2$, a value that started $\leq 5\varepsilon$ will remain $< \varepsilon_I$, even after doubling enough times to push a value $\geq \varepsilon_I$ to a value over 1 (including extra multiplications to account for the DOUBLE errors). Therefore, the LESS player will play itself with weight less than $\varepsilon_I$ if $v_1' \leq v_2'$ and with weight greater than $1 - \varepsilon_I$ if $v_1' - v_2' \geq \varepsilon_I$.

Next, we generate another gadget that can be used to amplify the results of each boolean logic player before using it, in order to ensure that each input within the circuit is close to the correct value.

**CORRECTION**

After a single gate (if the inputs are within additive $2\varepsilon_I$ of the correct 0 or 1 inputs), a player will play itself at least $1 - 4\varepsilon_I - 6\varepsilon$ if the correct answer is 1, and at most $4\varepsilon_I + 6\varepsilon$ if the correct answer is 0 (based on the analysis in the proof of Lemma [3]). Therefore, we need only to add a LESS player to determine whether or not the result is $< \frac{1}{2}$ and adjust the value in the correct direction using HALF or DOUBLE players.

**Lemma 6.** By using a CORRECTION gadget after each boolean logic gadget, we can ensure that the output from each gate is at most $2\varepsilon_I$ away from the correct output.

**Proof.** The results of a single gate gadget will be at least $1 - 4\varepsilon_I - 6\varepsilon$ if the correct answer is 1, and at most $4\varepsilon_I + 6\varepsilon$ if the correct answer is 0. If the result is $< \frac{1}{2}$, we will add three HALF players: the first reduces any result (at most $5\varepsilon_I + 4\varepsilon$) to at most $\frac{5\varepsilon_I}{4} + 6\varepsilon$, the second reduces it to at most $\frac{5\varepsilon_I}{4} + 7\varepsilon$, and with $\varepsilon \ll \varepsilon_I$. If the result is $> \frac{1}{2}$, we add a single DOUBLE player, which should give us a result of at least $1 - \varepsilon$ (since the input is very close to 1, the extra player in the SUM portion of the gadget has to play 0). However, we do collect a small additional error term because of the LESS used in the CORRECTION player.

We can use the LESS player as an if-statement (as needed above) as follows: LESS will play one of two strategies with weight close to 1, the other with weight close to 0. Say $P_1$ is the strategy that will be played with weight close to 1 ($\geq 1 - \varepsilon_I$) if and only if $v_1 < \frac{1}{2}$, $P_2$ is the strategy played with weight close to 1 ($\geq 1 - \varepsilon_I$) if and only if $v_1 \geq \frac{1}{2}$. We create the necessary players for both the HALF gadget and the DOUBLE gadget, but add $P_2$ as the first choice preference for the three players in the HALF gadget (labeled $H$, $H_2$ and $H_3$ in the gadget description), and add $P_1$ as the first choice preference for the COPY and SUM players in the DOUBLE gadget (players $C$ and $S$, but not players $C_1$ and $S_1$). Add one additional
player SUM$(H, D)$, where $H$ is the HALF player and $D$ is the DOUBLE player (one of the two is playing itself with weight close to 0). To show the correctness of the CORRECTION gadget, consider the following case analysis, assuming the result we are trying to correct is value $v \in \{0, 5\epsilon_l\}, (1 - 5\epsilon_l, 1]\}$. Call the four players that make up the DOUBLE gadget $C_1$ (the extra player for the COPY portion), $C$ (the COPY player), $S_1$ (the extra player for the SUM portion), and $S$ (the SUM player), and the four players that make up the HALF gadget $H_1, H_2$ and $H_3$ (as above):

**Case 1:** $v \leq 5\epsilon_l$. $C_1$ will play itself with weight at least $1 - v - \epsilon \geq 1 - 5\epsilon_l - \epsilon$. $C$ will play $P_1$ with weight at least $1 - \epsilon_l - \epsilon$. It must play the rest of its weight on the heavily-weighted $C_1$. $S_1$ will play some amount on the player that has weight $v$ and some on $C$, but must have at least $1 - 2\epsilon_l - 3\epsilon$ left for itself. $S$ will play at least weight $1 - \epsilon_l - \epsilon$ on $P_1$, and must play the rest of its weight on heavily-weighted player $S_1$, leaving 0 on itself.

Meanwhile, $H_1$ will play at least $1 - v - \epsilon$ on itself, so each of $H, H_2$ and $H_3$ will use up to within $\epsilon$ of the weight of $P_2$ (which may be 0), and of the weight of $H_1$ (at least $1 - v - 2\epsilon$), leaving at most $v + 2\epsilon$ to be divided in half. As stated above, this remaining amount will be split to within $\pm 3\epsilon$ across the strategies, so the result will be at most $\frac{5}{2}\epsilon_l + 4\epsilon$. Since $\epsilon$ is much smaller than $\epsilon_l$, the additive $\epsilon$ values with each iteration of the HALF gadget will be covered by the $\epsilon_l$.

The SUM player in the CORRECTION gadget will return a value at most the correct sum ($\leq \epsilon_l$ from the previous paragraph) plus $3\epsilon$.

**Case 2:** $v \geq 1 - 5\epsilon_l$. $C_1$ will play itself with weight at most $1 - v + \epsilon \leq 5\epsilon_l + \epsilon$. $C$ will play $P_1$ with weight at most $\epsilon_l + \epsilon$, and will play $C_1$ with weight at most $5\epsilon_l + 2\epsilon$, leaving at least $1 - 6\epsilon_l + 3\epsilon$ on itself. $S_1$ will try to play at least $1 - 5\epsilon_l - \epsilon$ on the player that has weight $v$ on itself and at least $1 - 6\epsilon_l - 4\epsilon$ on $C$, which will leave nothing left for itself. $S$ will play at most $\epsilon_l + \epsilon$ on $P_1$, and at most $\epsilon$ on $S_1$, leaving at least $1 - \epsilon_l - 2\epsilon$ for itself.

Any errors in the HALF player for this case will be if our player puts $> 0$ weight on the HALF player. However, this will only help to inflate the final result of the CORRECTION gadget.

The SUM player in the CORRECTION gadget will return a value at least the correct sum ($\geq 1 - \epsilon_l - 2\epsilon$) minus $3\epsilon$, or at least $1 - \epsilon_l - 5\epsilon > 1 - 2\epsilon_l$.

Using this CORRECTION gadget after each gate, we keep our input values to within $2\epsilon_l$ of the correct values, as required.  

After the corrections, we’re left with the following possible errors due to the $\epsilon$-approximation. We have small errors in the bit extraction, which are no larger than the parallel errors in $\ell$ (they verify that these small error values will not affect the final result). We also have small errors (at most $2\epsilon_l$) coming out of the circuit. As in $[5][6]$, we will repeat the circuit a polynomial number of times and take the average in order to override any errors from the LESS gadgets in the bit extraction.

Taking an average of two results requires 3 steps: first we divide each “bit” in half (we cannot take the average of the entire values because we have a max value of 1 for any player, so the average of two 1’s would come out to $\frac{1}{2}$). Here, we may pick up $4\epsilon$ of error for each of the two results. Then, we sum the two. The total error so far is at most $11\epsilon$. Finally, we take half of the sum, which also divides the error in half, but may add up to an additional $4\epsilon$ of error, for a total additional error of at most $9.5\epsilon$ from taking the average of 2 results.

We can add CORRECTION gadgets periodically during the averaging and during the final OR, AND and NOT of the results to keep our total errors under $2\epsilon_l$. In other words, if this is a solution vertex for
BROUWER, then we will have 6 players, each playing at most $2\epsilon_l$. If this is not a solution vertex, then at least one of the 6 players will play at least $1 - 2\epsilon_l$. Suppose we have an $\epsilon$-equilibrium in this game, and the x-coordinate player is playing value $x$. This is a SUM player, and the extra player from the SUM gadget must be playing between $1 - x - \epsilon$ and $1 - x + \epsilon$. Therefore, the sum of the two values it is adding (a copy of the coordinate player and the feedback NOT player) must be between $x - 3\epsilon$ (if this player overfills each of its top strategies by $\epsilon$) and $x + 3\epsilon$ (if this player underfills each of its top strategies by $\epsilon$). We know that the copy player must be playing the same value as the coordinate player to within $2\epsilon$ (between $x - 2\epsilon$ and $x + 2\epsilon$). Adding this range to a number $\geq 1 - 2\epsilon_l$ cannot possibly give something in the range $[x - 3\epsilon, x + 3\epsilon]$, so the feedback player must be playing a value at most $2\epsilon_l$ on itself (since we know the feedback player will play either a value $\leq 2\epsilon_l$ or a value $\geq 1 - 2\epsilon_l$), and the correct feedback must be 0, so this is a valid fixed point.

Theorem 4 implies that it is PPAD-hard to find an equilibrium in both fractional BGP and fractional BBC games. Since it is PPAD-hard to find a fractional BGP equilibrium, it is natural to next consider special instances when it might be easier to find an equilibrium. For instance, in real world internet routing, BGP path preferences are primarily based on a combination of security considerations and shortest paths. What would happen if we restrict ourselves to path preferences that echo the real world? Unfortunately, using only small adjustments to the above hardness proof, we show that it is PPAD-hard to find an equilibrium even if all preferences are based only on shortest path lengths.

**Theorem 5.** Fractional BGP is PPAD-hard even if each node’s preference list consists of all paths, ordered shortest to longest based on edge length (where each node defines its own edge lengths, which may not obey triangle inequality).

**Proof.** We will implicitly translate the proof of Theorem 4 to a corresponding proof for BGP, by assuming a destination node $T$, each preference by player $V$ for a player $U$ is now a preference for a path $(V \rightarrow U \rightarrow T)$ (abbreviated $(VUT)$), and each preference by player $V$ for “self” is now a preference for path $(V \rightarrow T)$ (abbreviated $(VT)$). We will add a set of edge lengths for each node in the gadgets such that the preferences in the gadget definitions follow shortest path distances according to the specified lengths.

For each node $U$ used in each of the gadgets, the preference list is of the form $(UVT, UT), (UVT, UWT, UT), (UVT, UWT, UZT, UT)$ (the last is only for the HALF player in the CORRECTION gadget). For preferences of the first form, we will assign edge lengths $l(UV) = 1, l(VT) = 1$, and all other lengths are 3. Clearly, to get to $T$ through any node other than $V$, the cost will be greater than 3, so the direct path will be preferred. The distance via $V$ is 2, so this will be preferred over the direct path. For preferences of the second form, we will assign edge lengths $l(UV) = 1, l(VT) = 1, l(UW) = 2, l(WT) = 1$, and all other lengths are 4. Clearly, the preferences for the 3 paths in the list will be correctly ordered based on distance. Any path involving a node other than $V$ or $W$ will have length greater than 4. Edges $VW$ and $WV$ both also have length 4, so any path to $T$ that uses $V$ or $W$ (that is preferred over the direct path) cannot include both $V$ and $W$. This leaves only the paths in the original preference list. For preferences of the third form, we will assign edge lengths $l(UV) = 1, l(VT) = 1, l(UW) = 2, l(WT) = 1, l(UZ) = 3, l(ZT) = 1$, and all other lengths are 5. Similar reasoning shows that this preserves the preference list.

**Theorem 6.** Fractional BGP is PPAD-hard even if all preferred paths are preference-ordered based on the path length (where each node defines its own distances on the edge lengths, and these distances form a metric and obey triangle inequality), assuming we may only use edges from a given template graph.
Proof. As in the proof of [5] we will implicitly translate the proof of Theorem [4] to a corresponding proof for BGP, by assuming a destination node $T$, each preference by player $V$ for a player $U$ is now a preference for a path $(VUT)$, and each preference by player $V$ for “self” is now a preference for path $(VT)$.

We will add a set of edge lengths for each node in the gadgets such that the preferences in the gadget definitions follow shortest path distances according to the specified metrics.

First, we will replace each direct path with a 2-hop path, by adding an extra node (whose only preference is for its own direct path). In other words, we will replace any path of the form $UT$ with a path of the form $UU'T$. We will replace any use of a direct path, such as $VUT$, with a use of the modified path: $VUUT$. We will remove all other edges straight to $T$ from the template graph, and we remove all edges between a new node $U'$ except the edge from $U$. Removing edges straight to $T$ is necessary because a preference list $(VUT, VT)$ does not obey triangle inequality for any metric. However, the list $(VUUT, VVT)$ is a valid preference list if $V$ uses the following edge lengths: $l(VU) = 1, l(UU') = 1, l(U'T) = 1, l(VV') = 2, l(V'T) = 2$ (assuming $VT$ is not allowed). Removing other edges into $U'$ is necessary because otherwise any path $VUT$ would have to be preferred at least as much as $VUUT$.

Now, for each node $U$ used in each of the gadgets, the preference list is of the form $(UVT, UTT), (UVT, UWT, UTT)$, or $(UVT, UWT, UZT, UTT)$. With the new additional nodes, each node now has a preference list of the form $(UVV'T, UU'T), (UVVT, UWW'T, UU'T)$, or $(UVVT, UWW'T, ZZ'T, UTT)$. For the first type of preference list, we will define the length of each leg of the most preferred path to be 1, the length of each leg of the second path to be 2, and any other edge in the graph has length 3. It is easy to verify that these lengths obey triangle inequality and give the required preference order. For the second type of preference list, we will assign edge lengths $l(UV) = 2, l(VV') = 1, l(V'T) = 1, l(UW) = 2, l(WW') = 2, l(W'T) = 1, l(UU') = 3, l(U'T) = 3$. In order to ensure triangle inequality, set $l(VW) = l(WV) = 4$. The rest of the edges in the graph had length 5 (so any path to the root containing any node other than $U, U', V, V', W, W'$ has length at least 10). In the smaller graph containing only $U, U', V, V', W, W'$, the paths to the root that haven’t been included in the preferences list or specifically excluded by restricting the edges are $UWVW'T$ (which has length 9) and $UWVV'T$ (which has length 8) - both are longer than any path in the preference list. For the third type of preference list, we will assign edge lengths $l(UV) = 3, l(VV') = 1, l(V'T) = 1, l(UW) = 3, l(WW') = 2, l(W'T) = 1, l(UZ) = 2, l(ZZ') = 3, l(Z'T) = 2, l(UU') = 4, l(U'T) = 4$. In order to ensure triangle inequality, set $l(VW) = l(WV) = 6$, $l(VZ) = l(ZV) = l(WZ) = l(ZW) = 5$. The rest of the edges in the graph have length 5.

Since we’ve added an additional edge to every path, this construction adds up to $\epsilon$ error for each player in the proof of theorem[4]. However, these errors will still be overpowered by the $\epsilon_1$ errors from our LESS gadget, so the proof could easily be adjusted to compensate. 

Notice, if any edge may be used, and if the preferences are based on shortest path lengths for a metric defined for each node, then there is a trivial algorithm for finding an equilibrium: each node only follows the “direct to destination” path. Since a metric must obey triangle inequality, this path length cannot be strictly longer (cannot be less preferred) than any path including additional nodes.

4 Existence and Rational Solutions via Personalized Equilibria

We introduce a new notion of an equilibrium for matrix games based on min-cost flows. Because the flow-based payoff functions enable each player to individually match her distribution to her opponents’ distributions, we call this a personalized equilibrium. We study the structural properties of personalized equilibria and analyze the complexity of finding such an equilibrium. We show that both the fractional BGP game and the fractional BBC game are special cases of matrix games in which players seek a personalized best response.
We first define personalized equilibria for two player games. We then extend it to multi-player games, including multi-player games with succinct representations. Consider a matrix game \((R, C)\) between two players ROW and COLUMN, in which player ROW has strategies \(r_1, r_2, \ldots, r_m\) and player COLUMN has strategies \(c_1, c_2, \ldots, c_n\). \(R \in \mathbb{R}^{m \times n}\) is the payoff matrix of ROW, and \(C \in \mathbb{R}^{m \times n}\) is the payoff matrix of COLUMN.

Like a standard bimatrix game, if player ROW selects \(r_i\) and player COLUMN selects \(c_j\), the payoff to ROW is \(R[i, j]\) and the payoff to COLUMN is \(C[i, j]\). Suppose ROW selects a distribution \(x\) among the strategies \(\{r_1, r_2, \ldots, r_m\}\), and COLUMN selects a distribution \(y\) among \(\{c_1, c_2, \ldots, c_n\}\). Unlike payoffs defined for mixed strategies, in which the payoff to ROW is \(\sum_{i,j} x[i]y[j]R[i, j]\) and the payoff to COLUMN is \(\sum_{i,j} x[i]y[j]C[i, j]\), we define the payoffs using flows. The payoff to ROW is:

\[
\text{Payoff (ROW)} = \max_{u_{i,j}} \sum_{i,j} u_{i,j} R[i, j] \tag{1}
\]

subject to \(\sum_{j} u_{i,j} = x[i], \ \forall i\) and \(\sum_{i} u_{i,j} = y[j], \ \forall j\);

The payoff to COLUMN is:

\[
\text{Payoff (COLUMN)} = \max_{v_{i,j}} \sum_{i,j} v_{i,j} C[i, j] \tag{2}
\]

subject to \(\sum_{j} v_{i,j} = x[i], \ \forall i\) and \(\sum_{i} v_{i,j} = y[j], \ \forall j\).

In other words, Payoff (ROW) is the cost of a 1-unit min-cost flow from source \(r\) to destination \(c\) in the directed graph \(G_R = (V_R, E_R)\), with

\[
V_R = \{r, c, r_1, r_2, \ldots, r_m, c_1, c_2, \ldots, c_n\} \quad E_R = \{(r \rightarrow r_i), \ \forall i\} \cup \{(r_i \rightarrow c_j), \ \forall i, j\} \cup \{(c_j \rightarrow c), \ \forall j\},
\]

where the capacity of edge \((r \rightarrow r_i)\) is \(x[i]\), the capacity of edge \((c_j \rightarrow c)\) is \(y[j]\), and the capacity of all other edges is \(+\infty\). The cost of edge \((r_i \rightarrow c_j)\) is \(-R[i,j]\), and the cost of all other edges is 0. We note that for any distributions \(x\) and \(y\), a unit-flow from \(r\) to \(c\) always exists, so the above payoff function is well-defined.

Similarly, Payoff (COLUMN) is the cost of a 1-unit minimum-cost flow from source \(c\) to destination \(r\) in the directed graph \(G_C = (V_C, E_C)\), with

\[
V_C = \{r, c, r_1, r_2, \ldots, r_m, c_1, c_2, \ldots, c_n\} \quad E_C = \{(c \rightarrow c_j), \ \forall j\} \cup \{(c_j \rightarrow r_i), \ \forall i, j\} \cup \{(r_i \rightarrow r), \ \forall i\},
\]

where the capacity of edge \((c \rightarrow c_j)\) is \(y[j]\), the capacity of edge \((r_i \rightarrow r)\) is \(x[i]\), and the capacity of all other edges is \(+\infty\). The cost of edge \((c_j \rightarrow r_i)\) is \(-C[i,j]\), and the cost of all other edges is 0.

Because there is no condition such as \(u[i, j] = v[i, j]\) in Eqn. (1), or in the payoff function for COLUMN (UMN) each player can individually choose the best way to match the distributions. We therefore refer to these payoff functions as personalized payoff functions, and we call an equilibrium for the game with these payoffs a personalized equilibrium. Using personalized payoffs, each player plays a distribution across her strategy space and chooses how to combine it with the strategy distributions of the other players.

In addition to the fractional BGP and BBC games, this concept of personalized equilibria is inspired by the correlated equilibrium of Aumann (3). Recall that the correlated payoff function requires \(u_{i,j} = v_{i,j}\) in Eqn. (1) and Eqn. (2), but relaxes Nash’s condition of \(u_{i,j} = v_{i,j} = x[i]y[j]\). We are considering payoff functions (personalized payoff functions) which further relax this by removing \(u_{i,j} = v_{i,j}\).
One can extend the personalized payoff functions to multi-player matrix games. Suppose we are given a $k$-player matrix game $G$, with $S_i$ being a set of $m_i$ strategies for player $i$, $1 \leq i \leq k$, and $u_i : \prod_j S_j \rightarrow \mathbb{R}$ being the utility function for player $i$. As in a mixed strategy, each player $i$ chooses a probability distribution $p_i : S_i \rightarrow [0, 1]$ over the strategies in $S_i$. Given $p_1, \ldots, p_k$, the personalized payoff function for player $i$ is computed as follows. Construct a hypergraph $H_i$ with $V = \cup_j S_j$ as the set of nodes and $E = \prod_j S_j$ as the set of hyperedges. Consider a (fractional hypergraph) matching defined by an assignment $w_i : E \rightarrow \mathbb{R}$ of weights to each hyperedge that satisfies the condition that the sum of weights of all hyperedges adjacent to any strategy $s \in S_j$ (for any $j$) equals $p_j(s)$. Define the weight of matching $w_i$ as $\sum_{e \in E} w_i(e)u_i(e)$. The payoff to player $i$ is then simply the cost of the maximum-weight matching in $H_i$.

The concept of personalized equilibria is extendible to games with succinct representations such as graphical games [19] and multimatrix games [34]. It can also be viewed as a relaxation of correlated equilibrium, as mentioned above.

**Theorem 7.** Finding an equilibrium in the fractional BGP game can be reduced to finding a personalized equilibrium in a matrix game.

**Proof.** Consider any instance $B$ of fractional BGP. We will create a matrix game, $M$, such that a solution to the $M$ is a personalized equilibrium if and only if a corresponding solution to $B$ is an equilibrium.

For each node $v$ in $B$, create a player $v'$ in $M$. Assign $v'$ one strategy $P'$ for each path $P \in \pi(v)$, plus one strategy for "no path." Let $q_v(P) = \text{the number of paths } Q \text{ such that } P \geq Q$. Next, we will define the payoff to $v'$ for a hyperedge in $M$ containing $P'$ (for $P \in \pi(v)$). If this hyperedge contains all proper suffixes of $P$, then the payoff to $v'$ will be $q_v(p) + 1$. Otherwise, the payoff to $v'$ will be 0. All hyperedges including the "no path" strategy for $v'$ will have payoff 0 for $v'$.

Given a set of distributions $\{p_v\}$ and a set of hyperedge weights $w$ in $M$, we can assign path weights $w'$ in $B$: $w'_v(P) = p_v'(P')$. If "no path" has any weight, this weight is not assigned in $B$.

We will show that a solution in $B$ is feasible if and only if the corresponding solution to $M$ is feasible, then show the correspondence of equilibria. The unity condition is clearly preserved: the distribution for a node $u$ in $M$ is a distribution of 1 unit. The weights placed on edges in $B$ also sum to 1. Now let's consider the tree condition. Suppose the tree condition is violated in $B$. Then, there exists a path $S$ starting at some node $u$ such that for another node $v$, $\sum_{P \in \pi(v), S} w'_v(P) > w'_u(S)$. This means that in $M$, we had a strategy $S'$ of node $u'$ such that for node $v'$, $\sum_{e \in E, S \subseteq e} w_{v'}(e) > p_u(S)$, which means the solution to $M$ was also infeasible. Now, suppose we have a solution to $M$ that is infeasible. Then, there is some $S' \in S_{v'}$ such that for some node $v'$, $\sum_{e \in E, S' \subseteq e} w_{v'}(e) > p_u(S')$. If the weight placed on $S'$ was from a path that did not include $s$ as a subpath, $v'$ could move the excess weight from $S'$ onto any strategy of $u'$ without changing the payoff, so all remaining weight on $S'$ much be from paths that contains $S'$ as a suffix. But then, for $B$, we have $\sum_{P \in \pi(u), S} w'_v(P) > w'_u(S)$ - another infeasible solution.

As a first step in showing that the equilibria are equivalent, we will note that fractional BGP preference lists across paths can be replaced with preference weights. Any weights that preserve the $\geq$ relationship will also preserve the set of equilibria. To show this, first define preference weights $u_v(P)$ for all paths $P \in \pi(v)$ such that $u_v(P) \geq u_v(Q)$ if and only if $P \geq v Q$. Now, suppose we have an equilibrium $w$ using weights $u$ which is not lexicographically maximal, plus a lexicographically maximal solution $w'$. Let $P = \text{the set of paths}$ such that for all $P$ and $Q$ in $P$, $P \geq v Q$, $Q \geq v P$, and for all $Q > v P$, $w_v(Q) = w'_v(Q)$. Then, we know that

1. $\sum_{Q : Q \geq v P, P \in P} w_v(Q) = \sum_{Q : Q \geq v P, P \in P}$
2. $u_v(P) = u_v(Q)$ if $P \in P, Q \in P$
3. \( u_v(P) > u_v(Q) \) if \( P \in \mathcal{P}, Q \notin \mathcal{P} \).

Replacing \( w \) with \( w' \) will keep the same weights on all paths with strictly higher preference weight that \( u_v(P), P \in \mathcal{P} \), increase the weight of paths in \( \mathcal{P} \) by some total increase amount \( I \), and decrease the total weight by \( I \) of the paths with preference weight \( < u_v(P), P \in \mathcal{P} \), thereby improving the solution. Similarly, if we have an equilibria using preference weights, it must be lexicographically maximal.

Now, we can write a linear program to find a best response for fractional BGP. In this LP, \( w_v \) is the set of weights assigned by node \( v \).

\[
\max \sum_{P \in \pi(v)} w_v(P)u_v(P) \\
\sum_{P \in \pi(v)} w_v(P) \leq w_u(S) \quad S \in \pi(u), 1 \leq u \leq k \\
w_v(P) \geq 0 \quad P \in \pi(v)
\]

We will also adjust the linear program for finding a best response using personalized payoffs to work for graphical games. In the graphical representation, we write \( e \in E \) to represent a hyperedge, where a hyperedge is a subset of at most one strategy per player (compared to exactly one strategy per player previously). We still use \( w_v(e) \) to mean the weight placed by player \( v \) on hyperedge \( e \), \( u_v(e) \) to mean the payoff to player \( v \) for hyperedge \( e \), and \( p_v(s) \) to mean the weight placed by player \( v \) on his own strategy \( s \). The following linear program defines a best response for player \( v \).

\[
\max \sum_{e \in E} w_v(e)u_v(e) \\
\sum_{e : s \in e} w_v(e) \leq p_u(s) \quad s \in S_u, 1 \leq u \leq k \\
w_v(e) \geq 0 \quad e \in E
\]

Now, if we assign preference weights for \( B: \ u_v(P) = u_v(e) \) where \( e \) is the hyperedge in \( M \) corresponding to \( P \) and all suffixes of \( P \), the two linear programs are exactly equivalent. Therefore, the set of equilibria is exactly equivalent. \( \square \)

**Theorem 8.** Finding an equilibrium in the fractional BBC game can be reduced to finding a personalized equilibrium in a matrix game.

**Proof.** Consider any instance of fractional BBC. Create a player in the matrix game for each node in the BBC instance. Assign the player one action for each available edge in the BBC instance. For any hyperedge in the matrix game, a player’s payoff is negative of the length of the shortest path to the destination made up of a subset of the edges represented by that hyperedge (or negative of the disconnection penalty if there is no such path to the destination). The proof that this preserves the set of equilibria is similar to the above proof for fractional BGP. \( \square \)

### 4.1 Existence and Rational Solutions

**Theorem 9.** For every multi-player matrix game, a personalized equilibrium always exists.
Proof. Given the matrix game $G$, we construct the $k$-player game $G$ in which the $i$th player’s strategy space is the set of all probability distribution functions over $S_i$, and the payoff is given by the personalized payoff function defined above. Then a personalized equilibrium of $G$ is equivalent to a Nash equilibrium of $G$. By [28, Proposition 20.3], a game has a pure Nash equilibrium if the strategy space of each player is a compact, non-empty, convex space, and the payoff function of each player is continuous on the strategy space of all players and quasi-concave in the strategy space of the player. The set of probability distributions over $S_i$ is clearly nonempty, convex, and compact. Furthermore, given probability distributions $p_i$ over $S_i$, $1 \leq i \leq k$, the payoff for any player $i$ is simply the solution to the following linear program with variables $w_i(e)$, over $e \in E$.

$$\max_{e \in E} \sum_{e \in E} w_i(e) u_i(e)$$

$$\sum_{e \in S \in e} w_i(e) = p_j(s) \quad s \in S_j, 1 \leq j \leq k$$

$$w_i(e) \geq 0 \quad e \in E$$

It is easy to see that the payoff function is both continuous in the probability distributions of all players, and quasi-concave in the strategy space of player $i$, thus completing the proof of the theorem. 

Theorem 10. For any matrix game with all rational payoffs, there exists a personalized equilibrium in which the probability assigned by each player to each strategy is a rational number.

Proof. Let $G$ be a $k$-player matrix game. (Please refer to the beginning of Section 4 for relevant notation.) For each player $i$, let $p_i : S_i \rightarrow [0, 1]$ denote a probability distribution over its strategies. If $p = (p_1, \ldots, p_k)$ forms a personalized equilibrium, then it provides a feasible solution to the following linear program over variables $w_i(e)$, where $e \in \prod_j S_j$ and $1 \leq i \leq k$, and $p_i(s)$, where $1 \leq i \leq k$ and $s \in S_i$:

$$\sum_{e \in S \in e} w_i(e) = p_j(s) \quad s \in S_j, 1 \leq j \leq k, 1 \leq i \leq k$$

$$\sum_{s \in S_i} p_i(s) = 1 \quad 1 \leq i \leq k$$

$$w_i(e) \geq 0 \quad 1 \leq i \leq k, e \in E$$

(3)

Furthermore, if $p$ is a personalized equilibrium, then each $(p_i, w_i)$ pair maximizes $\sum_e w_i(e) u_i(e)$ subject to LP (3). Suppose $p$ is not a personalized equilibrium, yet satisfies LP (3). This is so if and only if there exists a player $\ell$ for which $(p_\ell, w_\ell)$ does not maximize $\sum_e w_\ell(e) u_\ell(e)$. Suppose $(p_\ell', w_\ell')$ with $w_\ell' \neq w_\ell$ is an optimal choice for player $\ell$. Then, $\delta = w_\ell' - w_\ell$ is a feasible solution to the following LP:

$$\sum_{e \in E} \delta(e) u_\ell(e) > 0$$

$$\sum_{e \in S \in e} \delta(e) = 0 \quad s \in S_j, 1 \leq j \leq k$$

$$\delta(e) \geq -w_\ell(e) \quad e \in E$$

(4)

If $F$ is the set of hyperedges for which $\delta(e)$ is negative, then $\delta(e)$ satisfies LP (4) only if $w_\ell(e) > 0$ for all
at the start of Section 4:

\[ \text{Proof. Recall the secondary definition of the personalized payoff to player ROW in a two-player game given} \]
\[ \text{graph } G \\text{A 2-player personalized equilibrium can always be found in polynomial time.} \]

Theorem 11.

4.2.1 Two Player Personalized Equilibria

4.2 Complexity of finding Personalized Equilibria

We have thus argued that \( p \) is a personalized equilibrium if and only if there exists \( w = (w_1, \ldots, w_k) \)

such that \( p \) and \( w \) satisfy LP (3) and, if LP (5) is feasible for some \( \ell \) and \( F \), then \( w_\ell(e) \) should not be positive

for all \( e \) in \( F \). We thus add the following constraints to LP (3):

\[ \min_{e \in F} w_\ell(e) = 0, \text{ for all } F \text{ and } \ell \text{ such that LP (5) is feasible.} \]

By taking all combinations of one hyperedge from each of the above product constraints, we get an exponential number of linear programs (with all rational coefficients), the union of which precisely describes all personalized equilibria. By Theorem 9, at least one of these linear programs is feasible, which implies that

there exists a personalized equilibrium with all rational probabilities. \( \square \)

4.2 Complexity of finding Personalized Equilibria

4.2.1 Two Player Personalized Equilibria

It is not hard to show that the set of all two-player personalized equilibria is convex. In fact, we can give a stronger characterization, which will lead to a polynomial time algorithm.

Theorem 11. A 2-player personalized equilibrium can always be found in polynomial time.

Proof. Recall the secondary definition of the personalized payoff to player ROW in a two-player game given at the start of Section 4.

Payoff (ROW) is the cost of a 1-unit minimum-cost flow from source \( r \) to destination \( c \) in the directed graph \( G_R = (V_R, E_R) \), with

\[ V_R = \{ r, c, r_1, r_2, \ldots, r_m, c_1, c_2, \ldots, c_n \} \]
\[ E_R = \{ (r \rightarrow r_i), \forall i \} \cup \{ (r_i \rightarrow c_j), \forall i, j \} \cup \{ (c_j \rightarrow c), \forall j \}, \]

where the capacity of edge \( (r \rightarrow r_i) \) is \( x[i] \), the capacity of edge \( (c_j \rightarrow c) \) is \( y[j] \), and the capacity of all other edges is \( +\infty \). The cost of edge \( (r_i \rightarrow c_j) \) is \( -R[i,j] \), and the cost of all other edges is 0.

A similar definition of a flow on a graph \( G_C \) gives the payoff function for player COLUMN.

Now, let graph \( G \) be the union of \( G_R \) and \( G_C \). We will now consider a subgraph \( G' = (V', E') \subset G \), such that \( V' = V_R \cap V_C, (r_i \rightarrow c_j) \in E_R \) is in \( E' \) if and only if \( R[i,j] \geq R[i',j] \) for all \( i' \), and \( (c_j \rightarrow r_i) \in E_C \) is in \( E' \) if and only if \( C[i,j] \geq C[i, j'] \) for all \( j' \).

Any directed cycle in \( G' \) corresponds to a personalized equilibria. Consider any cycle \( \{ r_{i_1}, c_{j_1}, r_{i_2}, c_{j_2}, \ldots, r_{i_l}, c_{j_l} \} \) in \( G' \), each node played with weight \( \frac{1}{2} \). Player ROW can match each of his strategies \( r_{i_k} \) with player COLUMN’S strategy \( c_{j_k} \). Since this is a best response for player ROW, ROW cannot do better by changing to another strategy. Similarly, player ROW can match each of his strategies \( c_{j_k} \) with player ROW’S strategy \( r_{i_{(k+1)}} \) for \( k < l, c_{j_l} \) can be matched with \( r_{i_1} \).

Every personalized equilibria is a linear combination of cycles in \( G' \). Starting with any bipartite graph from \( G' \) in which the in-degree equals the out-degree of each node (a characteristic of any personalized equilibria), we can remove any cycle (which is a personalized equilibria) and we are still left with a bipartite graph with the same characteristic. \( \square \)
4.2.2 Multi-player personalized equilibria

**Theorem 12.** For multiplayer games, the set of all personalized equilibria may not be convex.

*Proof.* Consider the following example, with 3 players, 2 strategies per player. Player $i$ has strategies $a_i$ and $b_i$. Let $P_1(a_1, a_2, a_3) = \text{the payoff to player 1 for hyperedge } \{a_1, a_2, a_3\}$. The payoffs to player 1 are: $P_1(a_1, a_2, a_3) = P_1(a_1, b_2, b_3) = 1$, $P_1(b_1, a_2, b_3) = P_1(b_2, b_2, a_3) = 2$, the other 4 payoffs for player 1 are all 0. The payoffs for the other players are 1 for all hyperedges. In this example, the pure strategies $a_1, a_2, a_3$ and $a_1, b_2, b_3$ are both equilibria. However, a combination of these two, $a_1 = 1, a_2 = a_3 = \lambda, b_2 = b_3 = (1 - \lambda)$, is not an equilibrium, since player $p$ would prefer to play hyperedges $\{b_1, a_2, b_3\}$ and $\{b_1, b_2, a_3\}$.

We have shown that the preference game is a special case of fractional BGP, and we have shown that fractional BPG is a special case of finding personalized equilibria. Therefore, the non-convex example from Section 3.2 also proves theorem 12.

**Theorem 13.** It is PPAD-hard to find a personalized equilibrium in general matrix games.

*Proof.* Again, this is shown via the reduction from preference games to fractional BGP to personalized equilibria.

**Theorem 14.** It is PPAD-hard to find 4-player personalized equilibria.

*Proof.* For this theorem, we first note that when reducing from preference games to fractional BGP to personalized equilibria, we keep the same number of players. We also preserve the number of players “depended on” for the payoff of a particular strategy. In preference games, a payoff depends only on the single other player being chosen, and a player will only place weight on other players it prefers over itself. When reduced to fractional BGP, the paths considered by a node are only one or two-hop paths: and the node only considers two-hop paths that it prefers over its “direct” path. When reduced to finding personalized equilibria in a matrix game, we keep the same number of players. The payoff for a hyperedge depends only on a number of players equal to the number of hops in the path represented by that hyperedge; in this case, at most 2. The only strategies that will ever be chosen by a node are the strategies corresponding to players preferred over that node in the original instance of the preference game.

Therefore, if we start with the reduction from 3-DIMENSIONAL BROUWER used in Section 3.3, we have a graphical matrix game in which each node’s strategy “depends on” one or two other nodes, and each node “influences” the strategy for one or two other nodes. (We can easily make this a max of 2, because if a node influences 3 other nodes, we just switch to influencing one of them plus a copy gadget, which can influence the other(s).) So we can say that in+out degree is at most 4. Each node has 2 strategies.

We want to represent this as a 4 player game with more strategies per player.

First we have to slightly transform the graph in order to get two properties:

1. Our graph should have max degree of 3 (in + out)
2. If a node “depends on” 2 other nodes, we want an edge (undirected is fine) between these two nodes.

This edge counts in the degree.

We have a degree 4 graph and we want to change it to degree 3. Any degree 4 node currently has 2 inputs plus 2 outputs. We can change both of the outputs into a single copy gadget with 2 outputs - the copy has one input and 2 outputs, or degree three.
Now we have a graph with max degree 3, but we need the edges from property 2 (an edge between any two nodes X and Y that influence the same third node, Z). Suppose we have two nodes X and Y that both influence a third node Z, and node X has degree 3 already. Just add a new node X′ that copies X, and make this copy influence Z. Now X still has degree 3, X′ has degree 2 + the edge between X′ and Y.

After the above conversions, we have a graph with those 2 properties. Create a 3-coloring (possible because max degree is 3). Create one player per color. Each player has 2 strategies for each node in that color (one for the 0 strategy of that node, one for the 1 strategy).

Add dummy strategies as necessary so that each of the 3 players has the same number of strategies. Also add a fourth player with half the number of strategies as any other player.

This gives us 4 players. Let the strategies for player 1 be \( \{a_{10}, a_{11}, a_{20}, a_{21}, \ldots, a_{k0}, a_{k1}\} \). The strategies for player 2 are \( \{b_{10}, b_{11}, \ldots, b_{k0}, b_{k1}\} \). The strategies for player 3 are \( \{c_{10}, c_{11}, \ldots, c_{k0}, c_{k1}\} \). The strategies for player 4 are \( \{d_1, d_2, \ldots, d_k\} \).

Next we will assign payoffs for each hyperedge. Start by giving each hyperedge the same payoff as in the graphical game (we can do this because no two nodes influencing the same strategy are strategies of the same player). Notice that these payoffs will not depend at all on player 4. All of player 4’s payoffs start at 0. Let \( p_i(w, x, y, z) \) = the payoff to player i if player 1 plays \( w \), 2 plays \( x \), player 3 plays \( y \), player 4 plays \( z \). Now we want to add to these payoffs in order to ensure that each player plays each strategy pair equally.

Let \( M > \) the largest payoff so far. Now, change the following payoffs:

\[
p_1(a_{si}, x, y, d_s) + = M \text{ (player 1 is playing either strategy from the node numbered } s, \text{ player 4 is playing his } s^{th} \text{ strategy).}
\]

\[
p_2(w, b_{si}, y, d_s) + = M \text{ (player 2 is playing either strategy from the node numbered } s, \text{ player 4 is playing his } s^{th} \text{ strategy).}
\]

\[
p_3(w, x, c_{si}, d_s) + = M \text{ (player 3 is playing either strategy from the node numbered } s, \text{ player 4 is playing his } s^{th} \text{ strategy).}
\]

\[
p_4(a_{si}, x, y, d_{(s+1)}) + = M \text{ (player 1 is playing either strategy from the node numbered } s, \text{ player 4 is playing his } s + 1^{st} \text{ strategy).}
\]

If \( f_i(x) \) = the amount player \( i \) plays strategy \( x \) then in any equilibrium we must have (for all \( s \))

\[
f_1(a_{s0}) + f_1(a_{s1}) = f_4(d_s)
\]

\[
f_2(b_{s0}) + f_2(b_{s1}) = f_4(d_s)
\]

\[
f_3(c_{s0}) + f_3(c_{s1}) = f_4(d_s)
\]

\[
f_4(d_s) = f_1(a_{s-1}0) + f_1(a_{(s-1)1}) \text{ for } s > 0
\]

\[
f_4(d_0) = f_1(a_{k0}) + f_1(a_{k1})
\]

These equations imply that:

\[
f_1(a_{s0}) + f_1(a_{s1}) = f_1(a_{s-1}0) + f_1(a_{(s-1)1}) \text{ for } s > 0
\]

\[
f_1(a_{s0}) + f_1(a_{s1}) = f_1(a_{k0}) + f_1(a_{k1}) \text{ for } s > 0
\]

\[
f_2(b_{s0}) + f_2(b_{s1}) = f_1(a_{s0}) + f_1(a_{s1})
\]

\[
f_3(c_{s0}) + f_3(c_{s1}) = f_1(a_{s0}) + f_1(a_{s1})
\]

In other words, given an equilibrium in this game, we can simply multiply by the number of pairs (nodes) per player to get an equilibrium in the graphical game.
With a slight modification, the above proof also establishes PPAD-hardness for approximating personalized equilibria in 5-person games. We simply need to note that in the approximation gadgets created for the preference game hardness proof, the maximum length of any preference list is 4 instead of 3 (one of the players in the CORRECTION gadget “depends on” 3 other players). The rest of the proof remains intact.

5 Concluding Remarks

We note that our PPAD-hardness results from section [3] also apply to two other problems reduced in [16] to fractional BGP. The first of these problems is finding fractional stable matchings in hypergraphic preference systems. The second is finding fractional kernels in directed graphs.

We raise a number of open questions.

• Is finding a personalized equilibrium in general matrix games in PPAD? Although we show PPAD-hardness for general games, we have not settled the question of PPAD-membership. We have shown that a rational solution always exists, so finding an exact equilibrium may be in PPAD.

• We show that it is possible to find a personalized equilibrium for a 2-person game in polynomial time, and it is PPAD-hard to find a personalized equilibrium in a 4-person game. However, the hardness of finding these equilibria in 3-person games remains open.

• In this paper, we concentrate on a version of BBC games in which all nodes want to reach a single universal destination. However, in [22], the utility of a node in a BBC game is defined as an affinity-weighted average of the shortest path length (or minimum cost flow in the fractional case) to all other nodes. They show that an equilibrium always exists with multiple destinations, and our hardness results of course extend to this model, since we can define only one non-zero affinity, but it is unknown whether rational equilibria always exist.

• Our reduction from preference games to BBC games does not apply in the “multiple destinations” model if all affinities must be equal. Is this special instance of BBC games PPAD-hard as well?

Personalized equilibria and preference games both have a number of real world and theoretical applications, and seem to be natural end points in a spectrum of “personalized” fractional games. As more games are added to this hierarchy, we hope to fully understand the relationships and behavior of fractional equilibria.

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