Super-Renormalizablity of Yang-Mills Models in the Third Order of Perturbation Theory

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Abstract

We continue the investigation from a previous paper concerning the super-renormalizablity of gauge models going to the third order of the perturbation theory. Here we consider only the Yang-Mills case and we prove that this property is true iff some supplementary restrictions are imposed on the constants appearing in the interaction Lagrangian. The usual standard model does not verify these restrictions, but there is hope that such models do exist and they are in agreement with the phenomenology. We consider here only the even-parity contributions.

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1 Introduction

The general framework of perturbation theory consists in the construction of some distribution-valued operators called chronological products [1]. We prefer the framework from [2]: for every set of Wick monomials $W_1(x_1), \ldots, W_n(x_n)$ acting in some Fock space $\mathcal{H}$ one associates the distribution-valued operator $T(W_1(x_1), \ldots, W_n(x_n)) \equiv T^{W_1, \ldots, W_n}(x_1, \ldots, x_n)$ such that a set of axioms, essentially proposed by Bogoliubov, are verified. The modern construction of the chronological products can be done recursively according to Epstein-Glaser prescription [3], [4] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [10] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined but there are some natural limitation on the arbitrariness. If the arbitrariness does not grow with $n$ we have a renormalizable theory. An equivalent point of view uses retarded products [14].

The description of higher spins in the perturbation theory can be problematic. If we describe them by fields carrying only physical degrees of freedom, then the theories are usually not renormalizable. However, one can save renormalizability using ghost fields. Such theories are defined in a Fock space $\mathcal{H}$ with indefinite metric, generated by physical and un-physical fields (called ghost fields). One selects the physical states assuming the existence of an operator $Q$ called gauge charge which verifies $Q^2 = 0$ and such that the physical Hilbert space is by definition $\mathcal{H}_{\text{phys}} \equiv \text{Ker}(Q)/\text{Im}(Q)$. The fact that two distinct mathematical states from $\mathcal{H}$ can be associated to the same physical context is called gauge freedom and the corresponding theories are called gauge theories. The graded commutator $d_Q$ of the gauge charge with any operator $A$ of fixed ghost number

$$d_Q A = [Q, A] \quad (1.1)$$

(where $[\cdot, \cdot]$ denotes the graded commutator) verifies

$$d_Q^2 = 0 \quad (1.2)$$

so $d_Q$ is a co-chain operator in the space of Wick polynomials.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number $T(x)$ called the interaction Lagrangian such that

$$d_Q T = i\partial_\mu T^\mu \quad (1.3)$$

for some other Wick polynomials $T^\mu$. This relation means that the expression $T$ leaves invariant the physical states, at least in the adiabatic limit. Indeed, we have:

$$T(f) \mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{phys}} \quad (1.4)$$

up to terms which can be made as small as desired (making the test function $f$ flatter and flatter). In all known models one finds out that there exist a chain of Wick polynomials $T^\mu, T^{[\mu\nu]}, T^{[\mu\nu\rho]}, \ldots$ such that:

$$d_Q T = i\partial_\mu T^\mu, \quad d_Q T^\mu = i\partial_\nu T^{[\mu\nu]}, \quad d_Q T^{[\mu\nu]} = i\partial_\rho T^{[\mu\nu\rho]}, \ldots \quad (1.5)$$
where the brackets emphasize completely antisymmetric in all indexes; it follows that the chain of relation stops after a finite number of steps. We can also use a compact notation $T^I$ where $I$ is a collection of indexes $I = [\nu_1, \ldots, \nu_p]$ ($p = 0, 1, \ldots$) and one can write compactly the relations (1.3) as follows:

$$d_Q T^I = i \partial_\mu T^I_{\mu}.$$  (1.6)

All these polynomials have the same canonical dimension

$$\omega(T^I) = \omega_0, \ \forall I$$  (1.7)

and the ghost number:

$$gh(T^I) = |I|.$$  (1.8)

If the interaction Lagrangian $T$ is Lorentz invariant, then one can prove that the expressions $T^I$, $|I| > 0$ can be taken Lorentz covariant.

Now we can construct the chronological products

$$T^{I_1, \ldots, I_n}(x_1, \ldots, x_n) \equiv T(T^{I_1}(x_1), \ldots, T^{I_n}(x_n))$$  (1.9)

according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (1.6):

$$d_Q T^{I_1, \ldots, I_n} = i \sum_{l=1}^{n} (-1)^{s_l} \frac{\partial}{\partial x^\mu_l} T^{I_1, \ldots, I_l \mu, \ldots, I_n}$$  (1.10)

are true for all $n \in \mathbb{N}$ and all $I_1, \ldots, I_n$. Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1} |I_{l,j}|.$$  (1.11)

Such identities can be usually broken by anomalies i.e. expressions of the type $A^{I_1, \ldots, I_n}$ which are quasi-local and might appear in the right-hand side of the relation (1.10). It still an unsolved problem to prove, at least in the causal formalism, that the anomalies can be eliminated by convenient redefinitions of the chronological products.

If one can choose the chronological products such that gauge invariance is true then there is still some freedom left for redefining them. To be able to decide if the theory is renormalizable one needs the general form of such arbitrariness.

In a recent paper [9] we have proved that we have some super-renormalizability properties for loop contributions, in the second order of the perturbation theory. We remind the relevant cohomology terminology. We consider a cochains to be an ensemble of distribution-valued operators of the form $C^{I_1, \ldots, I_n}(x_1, \ldots, x_n)$, $n = 1, 2, \cdots$ (usually we impose some supplementary symmetry properties) and define the derivative operator $\delta$ according to

$$(\delta C)^{I_1, \ldots, I_n} = \sum_{l=1}^{n} (-1)^{s_l} \frac{\partial}{\partial x^\mu_l} C^{I_1, \ldots, I_l \mu, \ldots, I_n}.$$  (1.12)
We can prove that
\[
\delta^2 = 0. \tag{1.13}
\]
Next we define
\[
s = d_Q - i\delta, \quad \bar{s} = d_Q + i\delta \tag{1.14}
\]
and note that
\[
ss = \bar{s}s = 0. \tag{1.15}
\]
We call relative cocycles the expressions \(C\) verifying
\[
sC = 0 \tag{1.16}
\]
and a relative coboundary an expression \(C\) of the form
\[
C = \bar{s}B. \tag{1.17}
\]
The relation (1.10) is simply the cocycle condition
\[
sT = 0 \tag{1.18}
\]
and we have showed that the loop contributions of the second order of the perturbation theory are coboundaries, up to super-renormalizable contributions.

In this paper we consider only Yang-Mills models and extend the result to the third order of the perturbation theory. We will prove that this can be done if we impose some supplementary restrictions on the various constants appearing in the interaction Lagrangian. It seems that the usual standard model of electro-weak and strong interactions does not verify these supplementary restrictions, but we hope that one can find an alternative model, verifying these restrictions and compatible with the phenomenology.

In the next Section we will briefly present the Yang-Mills model in our preferred compact notations. In Section 3 we give the basic ideas of causal perturbation theory. In Section 4 we present our main result concerning super-renormalizability in the third order of the perturbation theory for the Yang-Mills model.
2 Yang-Mills Models

We give some results from [8].

2.1 Massless Particles of Spin 1 (Photons)

We consider a vector space $H$ of Fock type generated (in the sense of Borchers theorem) by the vector field $v_\mu$ (with Bose statistics) and the scalar fields $u, \bar{u}$ (with Fermi statistics). The Fermi fields are usually called ghost fields. We suppose that all these (quantum) fields are of null mass. Let $\Omega$ be the vacuum state in $H$. In this vector space we can define a sesquilinear form $\langle \cdot, \cdot \rangle$ in the following way: the (non-zero) 2-point functions are by definition:

$$
\langle \Omega, v_\mu(x_1)v_\mu(x_2)\Omega \rangle = i \eta_{\mu\nu} D_0^{(+)}(x_1 - x_2),
$$

$$
\langle \Omega, u(x_1)\bar{u}(x_2)\Omega \rangle = -i D_0^{(+)}(x_1 - x_2)
$$

and the $n$-point functions are generated according to Wick theorem. Here $\eta_{\mu\nu}$ is the Minkowski metrics (with diagonal 1, $-1, -1, -1$) and $D_0^{(+)}$ is the positive frequency part of the Pauli-Jordan distribution $D_0$ of null mass. To extend the sesquilinear form to $H$ we define the conjugation by

$$
v_\mu^\dagger = v_\mu, \quad u^\dagger = u, \quad \bar{u}^\dagger = -\bar{u}.
$$

Now we can define in $H$ the operator $Q$ according to the following formulas:

$$
[Q, v_\mu] = i \partial_\mu u, \quad [Q, u] = 0, \quad [Q, \bar{u}] = -i \partial_\mu u^\mu
$$

$$
Q\Omega = 0
$$

where by $[\cdot, \cdot]$ we mean the graded commutator. One can prove that $Q$ is well defined: basically it leaves invariant the causal commutation relations. The usefulness of this construction follows from:

Theorem 2.1 The operator $Q$ verifies $Q^2 = 0$. The factor space $\text{Ker}(Q)/\text{Ran}(Q)$ is isomorphic to the Fock space of particles of zero mass and helicity 1 (photons).

2.2 Massive Particles of Spin 1 (Heavy Bosons)

We repeat the whole argument for the case of massive photons i.e. particles of spin 1 and positive mass.

We consider a vector space $H$ of Fock type generated by the vector field $v_\mu$, the scalar field $\Phi$ (with Bose statistics) and the scalar fields $u, \bar{u}$ (with Fermi statistics). We suppose that all these (quantum) fields are of mass $m > 0$. In this vector space we can define a sesquilinear form $\langle \cdot, \cdot \rangle$ in the following way: the (non-zero) 2-point functions are by definition:

$$
\langle \Omega, v_\mu(x_1)v_\mu(x_2)\Omega \rangle = i \eta_{\mu\nu} D_m^{(+)}(x_1 - x_2), \quad \langle \Omega, \Phi(x_1)\Phi(x_2)\Omega \rangle = -i D_m^{(+)}(x_1 - x_2)
$$

$$
\langle \Omega, u(x_1)\bar{u}(x_2)\Omega \rangle = -i D_m^{(+)}(x_1 - x_2), \quad \langle \Omega, \bar{u}(x_1)u(x_2)\Omega \rangle = i D_m^{(+)}(x_1 - x_2)
$$

(2.4)
and the \( n \)-point functions are generated according to Wick theorem. Here \( \mathcal{D}_m^{(+)} \) is the positive frequency part of the Pauli-Jordan distribution \( \mathcal{D}_m \) of mass \( m \). To extend the sesquilinear form to \( \mathcal{H} \) we define the conjugation by

\[
v^\dagger_\mu = v_\mu, \quad u^\dagger = u, \quad \tilde{u}^\dagger = -\tilde{u}, \quad \Phi^\dagger = \Phi. \tag{2.5}
\]

Now we can define in \( \mathcal{H} \) the operator \( Q \) according to the following formulas:

\[
\begin{align*}
[Q, v_\mu] &= i \partial_\mu u, & [Q, u] &= 0, & [Q, \tilde{u}] &= -i (\partial_\mu v^\mu + m \Phi), & [Q, \Phi] &= i m u, & Q\Omega &= 0. \tag{2.6}
\end{align*}
\]

One can prove that \( Q \) is well defined. We have a result similar to the first theorem of this Section:

**Theorem 2.2** The operator \( Q \) verifies \( Q^2 = 0 \). The factor space \( \text{Ker}(Q)/\text{Ran}(Q) \) is isomorphic to the Fock space of particles of mass \( m \) and spin 1 (massive photons).

### 2.3 The Generic Yang-Mills Case

The situations described above (of massless and massive photons) are susceptible of the following generalizations. We can consider a system of \( r_1 \) species of particles of null mass and helicity 1 if we use in the first part of this Section \( r_1 \) triplets \( (v^\mu_a, u_a, \tilde{u}_a, \Phi_a) \), \( a \in I_1 \) of massless fields; here \( I_1 \) is a set of indexes of cardinal \( r_1 \). All the relations have to be modified by appending an index \( a \) to all these fields.

In the massive case we have to consider \( r_2 \) quadruples \( (v^\mu_a, u_a, \tilde{u}_a, \Phi_a) \), \( a \in I_2 \) of fields of mass \( m_a \); here \( I_2 \) is a set of indexes of cardinal \( r_2 \).

We can consider now the most general case involving fields of spin not greater than 1. We take \( I = I_1 \cup I_2 \cup I_3 \) a set of indexes and for any index we take a quadruple \( (v^\mu_a, u_a, \tilde{u}_a, \Phi_a) \), \( a \in I \) of fields with the following conventions: (a) For \( a \in I_1 \) we impose \( \Phi_a = 0 \) and we take the masses to be null \( m_a = 0 \); (b) For \( a \in I_2 \) we take the all the masses strictly positive: \( m_a > 0 \); (c) For \( a \in I_3 \) we take \( v^\mu_a, u_a, \tilde{u}_a \) to be null and the fields \( \Phi_a \equiv \phi_a^H \) of mass \( m_a^H \geq 0 \). The fields \( \phi_a^H \) are called Higgs fields.

If we define \( m_a = 0, \forall a \in I_3 \) then we can define in \( \mathcal{H} \) the operator \( Q \) according to the following formulas for all indexes \( a \in I \) :

\[
\begin{align*}
[Q, v^\mu_a] &= i \partial_\mu u_a, & [Q, u_a] &= 0, & [Q, \tilde{u}_a] &= -i (\partial_\mu v^\mu + m_a \Phi_a), & [Q, \Phi_a] &= i m_a u_a, & Q\Omega &= 0. \tag{2.7}
\end{align*}
\]

If we consider matter fields also i.e some set of Dirac fields with Fermi statistics: \( \psi_A, A \in I_4 \) then we impose

\[
d_Q\psi_A = 0. \tag{2.8}
\]
2.4 The Yang-Mills Interaction

In the framework and notations from the end of the preceding Section we have the following result which describes the most general form of the Yang-Mills interaction [5], [6], [7]. Summation over the dummy indexes is used everywhere.

**Theorem 2.3** Let $T$ be a relative cocycle for $d_Q$ which is at least tri-linear in the fields and is of canonical dimension $\omega(T) \leq 4$ and ghost number $gh(T) = 0$. Then: (i) $T$ is (relatively) cohomologous to a non-trivial co-cycle of the form:

$$T = f_{abc} \left( \frac{1}{2} v_{a\mu} v_{b\nu} F_{c\mu}^{\nu} + u_a v_b^\mu \partial_\mu \tilde{u}_c \right) + f'_{abc} (\Phi_a \phi_b^\mu v_{a\mu} + m_b \Phi_a \tilde{u}_b u_c) + \frac{1}{3!} f''_{abc} \Phi_a \Phi_b \Phi_c + j_a^\mu v_{a\mu} + j_a \Phi_a;$$

where we can take the constants $f_{abc} = 0$ if one of the indexes is in $I_3$; also $f'_{abc} = 0$ if $c \in I_3$ or one of the indexes $a$ and $b$ are from $I_1$; and $j_a^\mu = 0$ if $a \in I_3$; $j_a = 0$ if $a \in I_1$. By definition

$$\phi_a^\mu \equiv \partial^\mu \Phi_a - v_a^\mu$$

Moreover we have:

(a) The constants $f_{abc}$ are completely antisymmetric

$$f_{abc} = f_{[abc]}.$$  \hspace{1cm} (2.11)

(b) The expressions $f'_{abc}$ are antisymmetric in the indexes $a$ and $b$:

$$f'_{abc} = - f'_{bac}.$$ \hspace{1cm} (2.12)

and are connected to $f_{abc}$ by:

$$f_{abc} \, m_c = f'_{cab} m_a - f'_{cba} m_b.$$ \hspace{1cm} (2.13)

(c) The (completely symmetric) expressions $f''_{abc} = f''_{[abc]}$ verify

$$f''_{abc} \, m_c = \begin{cases} \frac{1}{m_c} f'_{abc} (m_a^2 - m_b^2) & \text{for } a, b \in I_3, c \in I_2 \\ - \frac{1}{m_c} f'_{abc} m_b^2 & \text{for } a, c \in I_2, b \in I_3. \end{cases}$$ \hspace{1cm} (2.14)

(d) the expressions $j_a^\mu$ and $j_a$ are bilinear in the Fermi matter fields: in tensor notations;

$$j_a^\mu = \sum_\epsilon \overline{\psi} t_\epsilon^a \otimes \gamma^\mu \gamma_\epsilon \psi \hspace{1cm} j_a = \sum_\epsilon \overline{\psi} s_\epsilon^a \otimes \gamma_\epsilon \psi$$ \hspace{1cm} (2.15)

where for every $\epsilon = \pm$ we have defined the chiral projectors of the algebra of Dirac matrices

$\gamma_\epsilon \equiv \frac{1}{2} (I + \epsilon \gamma_5)$ and $t_\epsilon^a$, $s_\epsilon^a$ are $|I_4| \times |I_4|$ matrices. If $M$ is the mass matrix $M_{AB} = \delta_{AB} M_A$ then we must have

$$\partial_\mu j_a^\mu = m_a \, j_a \quad \Leftrightarrow \quad m_a \, s_\epsilon^a = i(M \, t_\epsilon^a - t_{-\epsilon}^a M).$$ \hspace{1cm} (2.16)
(ii) The relation \( d_Q T = i \partial_\mu T^\mu \) is verified by:

\[
T^\mu = f_{abc} \left( u_a \, v_b \, F^\mu_c - \frac{1}{2} u_a \, u_b \, \partial^\mu \tilde{u}_c \right) + f'_{abc} \Phi_a \, \phi^\mu_b \, u_c + j^\mu_a \, u_a
\]  

(2.17)

(iii) The relation \( d_Q T^\mu = i \partial_\nu T^{\mu \nu} \) is verified by:

\[
T^{\mu \nu} \equiv \frac{1}{2} f_{abc} \, u_a \, u_b \, F^{\mu \nu}_c.
\]  

(2.18)
3 Causal Perturbation Theory

We give here the essential ingredients of perturbation theory. We consider that the canonical dimension of the vector and scalar fields $v^a_{\mu}, u_a, \bar{u}_a, \Phi_a$ is equal to 1 and the canonical dimension of the Dirac fields is $3/2$. A derivative applied to a field raises the canonical dimension by 1. The ghost number of the ghost fields is 1 and for the rest of the fields is null. The Fermi parity of a Fermi (Bose) field is 1 (resp. 0). The canonical dimension of a Wick monomial is additive with respect to the factors and the same is true for the ghost number and the Fermi parity.

3.1 Bogoliubov Axioms

Suppose that the Wick monomials $W_1, \ldots, W_n$ are self-adjoint: $W_j^\dagger = W_j$, $\forall j = 1, \ldots, n$. The chronological products $T(W_1(x_1), \ldots, W_n(x_n))$ $n = 1, 2, \ldots$ are verifying the following set of axioms:

- Skew-symmetry in all arguments $W_1(x_1), \ldots, W_n(x_n)$:
  \[ T(\ldots, W_i(x_i), W_{i+1}(x_{i+1}), \ldots) = (-1)^{f_i f_{i+1}} T(\ldots, W_{i+1}(x_{i+1}), W_i(x_i), \ldots) \]  
  (3.1)
  where $f_i$ is the number of Fermi fields appearing in the Wick monomial $W_i$.

- Poincaré invariance: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $(a, A) \in inSL(2, \mathbb{C})$ we have:
  \[ U_{a,A} T(W_1(x_1), \ldots, W_n(x_n)) U_{a,A}^{-1} = T(A \cdot W_1(A \cdot x_1 + a), \ldots, A \cdot W_n(A \cdot x_n + a)); \]  
  (3.2)
  Sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- Causality: if $x_i \geq x_j$, $\forall i \leq k$, $j \geq k + 1$ then we have:
  \[ T(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_k(x_k)) T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n)); \]  
  (3.3)

- Unitarity: We define the anti-chronological products according to
  \[ (-1)^n \bar{T}(W_1(x_1), \ldots, W_n(x_n)) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1, \ldots, I_r \in \text{Part}(\{1, \ldots, n\})} \epsilon T_{I_1}(X_1) \cdots T_{I_r}(X_r) \]  
  (3.4)
  where the we have used the notation:
  \[ T_{\{i_1, \ldots, i_k\}}(x_{i_1}, \ldots, x_{i_k}) \equiv T(W_{i_1}(x_{i_1}), \ldots, W_{i_k}(x_{i_k})) \]  
  (3.5)
  and the sign $\epsilon$ counts the permutations of the Fermi factors. Then the unitarity axiom is:
  \[ \bar{T}(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_n(x_n))^{\dagger} \]  
  (3.6)
The “initial condition”

\[ T(W(x)) = W(x). \] (3.7)

It can be proved that this system of axioms can be supplemented with

\[
T(W_1(x_1), \ldots, W_n(x_n)) = \sum < \Omega, T(W'_1(x_1), \ldots, W'_n(x_n)) \Omega > : W''_1(x_1), \ldots, W''_n(x_n) :
\] (3.8)

where \( W'_i \) and \( W''_i \) are Wick submonomials of \( W_i \) such that \( W_i =: W'_i W''_i : \) and we have supposed that only Bose fields are present; if Fermi fields are present then some appropriate signs should be inserted. This is called the Wick expansion property.

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials \( W_1, \ldots, W_n \); explicitly:

\[
\omega(< \Omega, T^{W_1, \ldots, W_n} (X) \Omega >) \leq \sum_{i=1}^{n} \omega(W_i) - 4(n - 1)
\] (3.9)

where by \( \omega(d) \) we mean the order of singularity of the (numerical) distribution \( d \) and by \( \omega(W) \) we mean the canonical dimension of the Wick monomial \( W \); in particular this means that we have

\[
T(W_1(x_1), \ldots, W_n(x_n)) = \sum_g t_g(x_1, \ldots, x_n) W_g(x_1, \ldots, x_n)
\] (3.10)

where \( W_g \) are Wick polynomials of fixed canonical dimension and \( t_g \) are distributions in \( n - 1 \) variables (because of translation invariance) with the order of singularity bounded by the power counting theorem [3]:

\[
\omega(t_g) + \omega(W_g) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n - 1)
\] (3.11)

and the sum over \( g \) is essentially a sum over Feynman graphs. Up to now, we have defined the chronological products only for self-adjoint Wick monomials \( W_1, \ldots, W_n \) but we can extend the definition for Wick polynomials by linearity.

The basic construction of Epstein and Glaser is the construction of the causal commutator. In the second order of the perturbation theory this is simply

\[
D(A(x), B(y)) = [A(x), B(y)]
\] (3.12)

where \( A(x), B(y) \) are arbitrary Wick monomials and \([\cdot, \cdot]\) the graded commutator. This distribution is translation invariant and with causal support i.e. it depends only on \( x - y \) and the support is inside the light cones:

\[
supp(D) \subset V^+ \cup V^-.
\] (3.13)

The simple formula (3.12) and support property is the justification of the terminology of causal commutator.
In higher orders of the perturbation theory the generalization of (3.12) is more complicated but we need only the third-order formula which is for even A, B, C:

\[
D(A(x), B(y), C(z)) = -\left[\tilde{T}(A(x), B(y)), C(z)\right] - \left[T(A(x), C(z)), B(y)\right] - \left[T(B(y), C(z)), A(y)\right] \tag{3.14}
\]

and in the general case all commutators are graded and we insert appropriate signs. Again one can prove that this distribution is translation invariant and with causal support i.e. it depends only on the variables \(x - z, y - z\) and has the support in the causal cone

\[
V^{\text{causal}} \equiv \{(x, y, z)|x - z \in V^+, y - z \in V^+\} \cup \{(x, y, z)|x - z \in V^-, y - z \in V^-\}. \tag{3.15}
\]

### 3.2 Third Order Causal Distributions

We remind the fact that the Pauli-Villars distribution is defined by

\[
D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x) \tag{3.16}
\]

where

\[
D_m^{(\pm)}(x) \sim \int dpe^{ip \cdot x} \theta(\pm p_0) \delta(p^2 - m^2) \tag{3.17}
\]

such that

\[
D^{(-)}(x) = -D^{(+)}(-x). \tag{3.18}
\]

This distribution has causal support. In fact, it can be causally split into an advanced and a retarded part:

\[
D = D^{\text{adv}} - D^{\text{ret}} \tag{3.19}
\]

and then we can define the Feynman propagator and antipropagator

\[
D^F = D^{\text{ret}} + D^{(+)}; \quad \bar{D}^F = D^{(+)} - D^{\text{adv}}. \tag{3.20}
\]

All these distributions have singularity order \(\omega(D) = -2\).

For the triangle loop contributions in the third order we need some basic distributions. First, we take \(D_j = D_{mj}, j = 1, 2, 3\) and define

\[
d_{D_1, D_2, D_3}(x, y, z) \equiv D^F_3(x - y)[D^{(-)}_2(z - x)D^{(+)}_1(y - z) - D^{(+)}_2(z - x)D^{(-)}_1(y - z)] + D^F_1(y - z)[D^{(-)}_3(x - y)D^{(+)}_2(z - x) - D^{(+)}_3(x - y)D^{(-)}_2(z - x)] + D^F_2(z - x)[D^{(-)}_1(y - z)D^{(+)}_3(x - y) - D^{(+)}_1(y - z)D^{(-)}_3(x - y)] \tag{3.21}
\]

which also with causal support; indeed we have the alternative forms

\[
d_{D_1, D_2, D_3}(x, y, z) = -D^F_{\text{ret}}(x - y)[D^{(-)}_2(z - x)D^{(+)}_1(y - z) - D^{(+)}_2(z - x)D^{(-)}_1(y - z)] + D^\text{adv}_1(y - z)[D^{(-)}_3(x - y)D^{(+)}_2(z - x) - D^{(+)}_3(x - y)D^{(-)}_2(z - x)] + D^\text{adv}_2(z - x)[D^{(-)}_1(y - z)D^{(+)}_3(x - y) - D^{(+)}_1(y - z)D^{(-)}_3(x - y)] \tag{3.22}
\]
\[
d_{D_1, D_2, D_3} (x, y, z) = -D_3^{\alpha\beta\gamma}(x-y)[D_3^{(-)}(z-x)D_1^{(+)}(y-z) - D_2^{(+)}(z-x)D_3^{(-)}(y-z)] \\
+ D_1^{\alpha\beta\gamma}(y-z)[D_3^{(-)}(x-y)D_2^{(+)}(z-x) - D_3^{(+)}(x-y)D_2^{(-)}(z-x)] \\
+ D_2^{\alpha\beta\gamma}(z-x)[D_3^{(-)}(y-z)D_1^{(+)}(x-y) - D_3^{(+)}(y-z)D_1^{(-)}(x-y)] \tag{3.23}
\]

from which it follows that \(d_{D_1, D_2, D_3}(x, y, z)\) is null outside the causal cone \([3.13]\). These distributions have the singularity order \(\omega(d_{D_1, D_2, D_3}) = -2\).

There are some associated distributions obtained from \(d_{D_1, D_2, D_3}(x, y, z)\) applying derivatives on the factors \(D_j = D_{m_j}, j = 1, 2, 3\). For instance we denote
\[
\begin{align*}
D_1^\alpha d_{D_1, D_2, D_3} &\equiv d_{\partial_\alpha D_1, D_2, D_3} \\
D_2^\alpha d_{D_1, D_2, D_3} &\equiv d_{D_1, \partial_\alpha D_2, D_3} \\
D_3^\alpha d_{D_1, D_2, D_3} &\equiv d_{D_1, D_2, \partial_\alpha D_3},
\end{align*}
\tag{3.24}
\]

and so on for more derivatives \(\partial_\alpha\) distributed in an arbitrary way on the factors \(D_j = D_{m_j}, j = 1, 2, 3\). We mention the fact that the operators \(D_\alpha^j, j = 1, 2, 3\) are commutative but they are not derivation operators: they do not verify Leibnitz rule.

When it possible we skip the dependence on \(D_j = D_{m_j}, j = 1, 2, 3\) i.e. we simply write \(d = d_{D_1, D_2, D_3}\). We note the formulas
\[
\begin{align*}
\frac{\partial}{\partial x^\alpha} d &= (D_\alpha^3 - D_\alpha^2)d \\
\frac{\partial}{\partial y^\alpha} d &= (D_\alpha^1 - D_\alpha^3)d \\
\frac{\partial}{\partial z^\alpha} d &= (D_\alpha^2 - D_\alpha^1)d \tag{3.25}
\end{align*}
\]

Apparently, these distributions do not have nice symmetry properties in all the three variables. However, this is not true. Let \(A(x), B(y), C(z)\) be three Wick monomials. Then the triangle one-loop contribution of the causal commutator is of the form:
\[
D_{\text{triangle}}(A(x), B(y), C(z)) = \sum p_j(D_1^\alpha, D_2^\beta, D_3^\gamma)d_j(x, y, z) W_j(x, y, z) \tag{3.26}
\]

where \(d_j\) are distributions of the type \(d_{D_1, D_2, D_3}\), \(p_j\) are polynomials in the operators \(D_\alpha^j, j = 1, 2, 3\) and \(W_j(x, y, z)\) are Wick monomials. For simplicity, let us suppose that the monomials \(A(x), B(y), C(z)\) are even so they causally commute. Then we have
\[
\begin{align*}
D_{\text{triangle}}(B(x), A(y), C(z)) &= \sum p_j(-D_2^\alpha, -D_1^\alpha, -D_3^\alpha)d_j(x, y, z) W_j(y, x, z) \\
D_{\text{triangle}}(A(x), C(y), B(z)) &= \sum p_j(-D_1^\alpha, -D_3^\alpha, -D_2^\alpha)d_j(x, y, z) W_j(x, z, y) \\
D_{\text{triangle}}(C(x), B(y), A(z)) &= \sum p_j(-D_3^\alpha, -D_2^\alpha, -D_1^\alpha)d_j(x, y, z) W_j(z, y, x) \tag{3.27}
\end{align*}
\]
i.e. the exchange of the factors \(A, B, C\) can be accounted for in a natural way. If some of the monomials \(A(x), B(y), C(z)\) are odd, the some signes must be inserted in the preceding sums. For instance, is \(A\) and \(B\) are causally anticommuting, then we have an extra \(-\) sign in the first and the third line above.
In the third order of perturbation theory other causal distributions can appear. They are associated with the so-called one-particle reducible Feynman graphs.

\[ d^{(1)}_{D_1, D_2}(x, y, z) \equiv \bar{D}_1^F(x - y)D_2(z - x) - D_1(x - y)D_2^F(z - x) \]
\[ + D_1^{(-)}(x - y)D_2^{(+)}(z - x) - D_1^{(+)}(x - y)D_2^{(-)}(z - x) \]
\[ d^{(2)}_{D_1, D_2}(x, y, z) \equiv -\bar{D}_1^F(x - y)D_2(y - z) + D_1(x - y)D_2^F(y - z) \]
\[ + D_1^{(+)}(x - y)D_2^{(-)}(y - z) - D_1^{(-)}(x - y)D_2^{(+)}(y - z) \]
\[ d^{(3)}_{D_1, D_2}(x, y, z) \equiv \bar{D}_1^F(z - x)D_2(y - z) - D_1(z - x)D_2^F(y - z) \]
\[ + D_1^{(-)}(z - x)D_2^{(+)}(y - z) - D_1^{(+)}(z - x)D_2^{(-)}(y - z) \] (3.28)

The causal support properties follow from the alternative formulas

\[ d^{(1)}_{D_1, D_2}(x, y, z) = D_1^{\text{ret}}(x - y)D_2^{\text{ret}}(z - x) - D_1^{\text{adv}}(x - y)D_2^{\text{adv}}(z - x) \]
\[ d^{(2)}_{D_1, D_2}(x, y, z) = D_1^{\text{ret}}(y - x)D_2^{\text{ret}}(z - y) - D_1^{\text{adv}}(y - x)D_2^{\text{adv}}(z - y) \]
\[ d^{(3)}_{D_1, D_2}(x, y, z) = D_1^{\text{ret}}(z - x)D_2^{\text{ret}}(y - z) - D_1^{\text{adv}}(z - x)D_2^{\text{adv}}(y - z) \] (3.29)

and this leads to the following Feynman propagators

\[ d^{(1)F}_{D_1, D_2}(x, y, z) = \bar{D}_1^F(x - y)D_2^F(z - x) \]
\[ d^{(2)F}_{D_1, D_2}(x, y, z) = \bar{D}_1^F(y - x)D_2^F(z - y) \]
\[ d^{(3)F}_{D_1, D_2}(x, y, z) = \bar{D}_1^F(z - x)D_2^F(y - z) \] (3.30)

The order of singularity of these distributions is again \( \omega = -2 \). We can define associated distributions as before if we replace \( D_1 \mapsto \partial_\alpha D_1 \), etc. We need to consider the case when one of the distribution \( D_1, D_2 \) is of the type \( D_m \) and the other is of the type \( D_2 \) where

\[ d_2(x) \equiv \frac{1}{2} [D_m^{(+)}(x)^2 - D_m^{(+)}(-x)^2] \] (3.31)

Let us notice that some associated distributions can have some \( \delta \) factors. We denote

\[ \mathcal{K}_j = D_j^{\mu}D_{j\mu}, \quad j = 1, 2, 3 \] (3.32)

and we have for instance

\[ \mathcal{K}_1 d(x, y, z) = 2\delta(y - z)d_2(x - y) \]
\[ \mathcal{K}_2 d(x, y, z) = 2\delta(z - x)d_2(y - z) \]
\[ \mathcal{K}_3 d(x, y, z) = -2\delta(x - y)d_2(z - x) \] (3.33)

and similar relations for the distributions \( d^{(j)}, j = 1, 2, 3 \).
4 Super-Renormalizability in the Third Order

We need the explicit form of the causal commutators $D^{IJK}$. From (3.26) and (3.27) we can obtain some symmetry properties. We also have the ghost number restrictions

$$gh(D^{IJK}) = |I| + |J| + |K|. \quad (4.1)$$

If we use for them the generic form (3.10) then (3.11) gets the form

$$\omega(t_g) + \omega(W_g) \leq 4. \quad (4.2)$$

We suppose that we have establish gauge invariance up to the second order of the perturbation theory i.e.

$$(sT)^I(x) = 0, (sT)^IJ(x,y) = 0 \quad (4.3)$$

and we obtain from the definition the cocycle property

$$(sD)^{IJK}(x,y,z) = 0. \quad (4.4)$$

We determine under what conditions the expression $D^{IJK}$ are coboundaries, up to super-renormalizable terms i.e.

$$D^{IJK}(x,y,z) = (sB)^{IJK}(x,y,z) + \text{super-renormalizable terms} \quad (4.5)$$

and we will need the generic form for the coboundaries $B^{IJK}$.

Like in [9] we replace everywhere for every mass $m$ in the game

$$D_m = D_0 + (D_m - D_0) \quad (4.6)$$

In this way we split $D^{IJK}_{(1)}(x,y,z)$ into a contribution $D^{IJK}_{(1)0}(x,y,z)$ where everywhere $D_m \mapsto D_0$ and a contribution where at least one factor $D_m$ is replaced by the difference $D_m - D_0$. Because we have

$$\omega(D_m - D_0) = -4 \quad (4.7)$$

the second contribution will be super-renormalizable. We need to consider only the first contribution $D^{IJK}_{(1)0}(x,y,z)$ and investigate if it can be written as coboundary. In the preceding expression we have two type of terms: ones associated to the triangle graphs and the other associated to the one-particle reducible graphs.

$$D^{IJK}_{(1)0}(x,y,z) = D^{IJK}_{\text{triangle}}(x,y,z) + D^{IJK}_{\text{PR}}(x,y,z) \quad (4.8)$$

From (3.26) and (3.27) we can obtain that the triangle contribution $D^{IJK}_{\text{triangle}}(x,y,z)$ is invariant with respect to the following transformation, up to some signs; for instance if $I, J, K$ are even, we have invariance with respect to

$$x \leftrightarrow y, I \leftrightarrow J, D_1 \rightarrow -D_2, D_2 \rightarrow -D_1, D_3 \rightarrow -D_3 \quad (4.9)$$
and in the general case we have invariance up to the sign \((-1)^{|I||J|}\) in the first case, etc. Similar symmetry properties are valid for the one-particle irreducible contribution.

In both terms from the righthand side of (4.8) we have delta-contributions i.e. contributions where the operators \(K_j\) \(j = 1, 2, 3\) are present and non-delta-contributions i.e. contributions where the operators \(K_j\) \(j = 1, 2, 3\) are absent; this splitting is unique.

\[
D^{IJK}_{\text{triangle}}(x, y, z) = D^{IJK}_{\text{triangle}}(x, y, z)_0 + D^{IJK}_{\text{triangle}}(x, y, z)_\delta
\]
\[
D^{IJK}_{\text{1PR}}(x, y, z) = D^{IJK}_{\text{1PR}}(x, y, z)_0 + D^{IJK}_{\text{1PR}}(x, y, z)_\delta
\] (4.10)

If \(D^{IJK}_{(1)0}(x, y, z)\) is a coboundary, then the expressions \(D^{IJK}_{\text{triangle}}(x, y, z)_0\) and \(D^{IJK}_{\text{1PR}}(x, y, z)_0\) should also be coboundaries, up to delta-contributions. If this is establish we still have to check that the sum of the delta-contributions from the triangle and the 1PR contributions can be also written as a coboundary.

We have to compute explicitly the contributions \(D^{IJK}_{\text{triangle}}(x, y, z)_0\) and \(D^{IJK}_{\text{1PR}}(x, y, z)_0\). It is easy to start a descent procedure and to consider the case of maximal ghost number \(|I| + |J| + |K| = 3\). It is useful to give the generic form compatible with the symmetry property in all three variables (given in the preceding Section) for \(D^{IJK}_{\text{triangle}}, |I| + |J| + |K| = 3\). Having the generic form one can prove that the cohomology is not trivial i.e. from the cocycle identity we cannot obtain the coboundary property. So in the end we will have to compute explicitly the preceding commutator. The computation are straightforward but rather long so we will present only some of them.

We consider the equation

\[
D^{IJK}_{\text{triangle}}(x, y, z)_0 = (\bar{s}B)^{IJK}(x, y, z) + D^{IJK}_\delta(x, y, z)
\] (4.11)

where the expressions \(B^{IJK}\) are restricted by skew-symmetry properties of the type (4.9), ghost number restrictions

\[
gh(B^{IJK}(x, y, z)) = |I| + |J| + |K| - 1
\] (4.12)

and power counting

\[
\omega(t_g) + \omega(W_g) \leq 3
\] (4.13)

similar to (4.1) and (4.2) respectively.
4.1 The Causal Commutators of Ghost Number

We first consider the causal commutator

\[ D^{[\mu}[^{\nu}[^{\rho}] (x, y, z) = D(T^{\mu}(x), T^{\nu}(y); T^{\rho}(z)) \]
\[ - [\tilde{T}(T^{\mu}(x), T^{\nu}(y)), T^{\rho}(z)] + [T(T^{\mu}(x), T^{\rho}(z)), T^{\nu}(y)] - [T(T^{\nu}(y), T^{\rho}(z)), T^{\mu}(x)]. \] (4.14)

The generic form of the triangle contribution is:

\[
D_{\text{triangle}}^{[\mu}[^{\nu}[^{\rho}] (x, y, z) = i[2A_{1,abc}(D_{1}^{\mu}D_{1}^{\nu}D_{1}^{\rho} + D_{2}^{\mu}D_{2}^{\nu}D_{2}^{\rho} + D_{3}^{\mu}D_{3}^{\nu}D_{3}^{\rho})
+ A_{2,abc}(D_{1}^{\mu}D_{2}^{\nu}D_{2}^{\rho} + D_{2}^{\mu}D_{1}^{\nu}D_{2}^{\rho} + D_{1}^{\mu}D_{3}^{\nu}D_{3}^{\rho} + D_{2}^{\mu}D_{3}^{\nu}D_{3}^{\rho} + D_{3}^{\mu}D_{2}^{\nu}D_{3}^{\rho})
+ A_{3,abc}(D_{1}^{\mu}D_{3}^{\nu}D_{3}^{\rho} + D_{2}^{\mu}D_{2}^{\nu}D_{2}^{\rho} + D_{1}^{\mu}D_{1}^{\nu}D_{3}^{\rho} + D_{1}^{\mu}D_{3}^{\nu}D_{2}^{\rho} + D_{2}^{\mu}D_{3}^{\nu}D_{3}^{\rho} + D_{2}^{\mu}D_{1}^{\nu}D_{3}^{\rho} + D_{2}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{2}^{\mu}D_{3}^{\nu}D_{2}^{\rho})
+ A_{4,abc}(D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{1}^{\mu}D_{3}^{\nu}D_{3}^{\rho} + D_{2}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{2}^{\mu}D_{3}^{\nu}D_{3}^{\rho} + D_{3}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{3}^{\mu}D_{3}^{\nu}D_{2}^{\rho})
+ 6A_{5,abc}(D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho})
+ 2A_{6,abc}(D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho})
+ 3A_{7,abc}(D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho} + D_{1}^{\mu}D_{2}^{\nu}D_{3}^{\rho})
+ A_{8,abc}(\eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3} + \eta^{\mu\nu}D_{2}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{2}^{\rho}D_{1} \cdot D_{3} + \eta^{\mu\nu}D_{3}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{3}^{\rho}D_{1} \cdot D_{3})
+ A_{9,abc}(\eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3})
+ A_{10,abc}(\eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3})
+ A_{11,abc}(\eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3})
+ A_{12,abc}(\eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{2} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3} + \eta^{\mu\nu}D_{1}^{\rho}D_{1} \cdot D_{3})
\]
\[= d(x, y, z) u_a(x) u_b(y) u_c(z). \] (4.15)

This expression is invariant (up to a sign) to the transformations

\[ x \leftrightarrow y, \mu \leftrightarrow \nu, D_1 \rightarrow -D_2, D_2 \rightarrow -D_3, D_3 \rightarrow -D_3 \] (4.16)

and

\[ y \leftrightarrow z, \nu \leftrightarrow \rho, D_1 \rightarrow -D_1, D_2 \rightarrow -D_3, D_3 \rightarrow -D_2 \] (4.17)

as we have explained in the preceding Section - see formulas \(3.26\) and \(3.27\).

By direct computation we have the non-zero expressions

\[ A_{2,abc} = f^{(0)}_{abc}, \quad A_{4,abc} = f^{(0)}_{abc} + f^{(3)}_{abc}, \quad A_{5,abc} = \frac{1}{3}(-f^{(0)}_{abc} + f^{(4)}_{abc}), \]
\[ A_{6,abc} = f^{(0)}_{abc} - f^{(4)}_{abc}, \quad A_{7,abc} = \frac{1}{3}(f^{(3)}_{abc} - 2f^{(4)}_{abc}), \quad A_{8,abc} = f^{(0)}_{abc}, \]
\[ A_{10,abc} = -f^{(0)}_{abc}, \quad A_{11,abc} = 2f^{(4)}_{abc}, \quad A_{12,abc} = -f^{(4)}_{abc}. \] (4.18)
where

\[ f_{[abc]}^{(0)} \equiv f_{ca} f_{eb} f_{cpq} \quad (4.19) \]
\[ f_{[abc]}^{(3)} \equiv f'_{ca} f'_{eb} f'_{pqc} \quad (4.20) \]
\[ f_{[abc]}^{(4)} \equiv -i\, T_r([t'_{\alpha}, t'_{\beta}]t'_{\gamma}) \quad (4.21) \]

If we define

\[ g_{ab} = f_{apq} f_{bpq} \quad (4.22) \]
\[ g_{ab}^{(1)} = f'_{pqa} f'_{pqa} \quad (4.23) \]
\[ g_{ab}^{(2)} = \sum_\epsilon T_r(t'_{\alpha}) \quad (4.24) \]

then we have the alternative expressions:

\[ f_{[abc]}^{(0)} = \frac{1}{2} f_{abcd} g_{cd} \quad (4.25) \]
\[ f_{[abc]}^{(3)} = \frac{1}{2} f_{abcd} g_{cd}^{(1)} \quad (4.26) \]
\[ f_{[abc]}^{(4)} = f_{abcd} g_{cd}^{(2)}. \quad (4.27) \]

In the same way we derive consider the causal commutator

\[ D^{[\mu\nu][\rho0]}(x, y, z) = D(T^{\mu\nu}(x), T^\rho(y); T(z)) \]
\[ -[\bar{T}(T^{\mu\nu}(x), T^\rho(y)), T(z)] - [T(T^{\mu\nu}(x), T^\rho(y)), T^\rho(y)] - [T^\rho(y), T(z)], T^{\mu\nu}(x)]. \quad (4.28) \]

The triangle contribution has the form

\[ D_{\text{triangle}}^{[\mu\nu][\rho0]}(x, y, z) = i(B_{1,abc} D_1^\mu D_2^\nu D_3^\rho + B_{2,abc} D_1^\mu D_1^\rho D_3^\nu + B_{3,abc} D_2^\mu D_2^\rho D_1^\nu + B_{4,abc} D_1^\mu D_3^\nu D_1^\rho + B_{5,abc} D_3^\mu D_1^\rho D_1^\nu + B_{6,abc} D_3^\mu D_3^\nu D_1^\rho + B_{7,abc} D_1^\mu D_1^\nu D_2^\rho + B_{8,abc} D_1^\mu D_2^\nu D_2^\rho + B_{9,abc} D_1^\mu D_1^\nu D_2^\rho) \]
\[ + B_{10,abc} \eta^{\mu\rho} D_1^\nu D_2^\nu D_1^\mu \cdot D_2 + B_{11,abc} \eta^{\mu\rho} D_2^\nu D_1^\nu D_1^\mu \cdot D_3 + B_{12,abc} \eta^{\mu\rho} D_2^\nu D_2^\nu D_1^\mu \cdot D_2 \]
\[ + B_{13,abc} \eta^{\mu\rho} D_2^\nu D_1^\nu D_2^\nu \cdot D_3 + B_{14,abc} \eta^{\mu\rho} D_2^\nu D_3^\nu D_1^\nu \cdot D_3 + B_{15,abc} \eta^{\mu\rho} D_3^\nu D_2^\nu \cdot D_3 \]
\[ + B_{16,abc} \eta^{\mu\rho} D_1^\nu D_2^\nu D_3^\nu \cdot D_3 + B_{17,abc} \eta^{\mu\rho} D_2^\nu D_3^\nu D_1^\nu \cdot D_2 + B_{18,abc} \eta^{\mu\rho} D_3^\nu D_1^\nu \cdot D_2) d(x, y, z) \]
\[ u_a(x) u_b(y) u_c(z) - (\mu \leftrightarrow \nu) \quad (4.29) \]

where the non-zero \( B_j \) are:

\[ B_{6,abc} = -f_{[abc]}^{(0)}, \quad B_{7,abc} = f_{[abc]}^{(0)}, \quad B_{8,abc} = -f_{[abc]}^{(0)}, \quad B_{9,abc} = f_{[abc]}^{(0)}, \quad B_{15,abc} = -f_{[abc]}^{(0)}. \quad (4.30) \]
4.2 The Generic Form of the Coboundaries

The generic expression $B_1^{[\mu][\nu][\rho\sigma]}(x, y, z)$ compatible with the restrictions is:

$$B_1^{[\mu][\nu][\rho\sigma]}(x, y, z) = \sum_{j=1}^{13} a_{j,abc} b_j^{[\mu][\nu][\rho\sigma]}(x, y, z) u_a(x) u_b(y) u_c(z)$$  (4.31)

where

$$b_1^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu gal} - \eta^{\mu\sigma} D_1^{\nu \rho} - \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu gal} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_2^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_3^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_4^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_5^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_6^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_7^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_8^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_9^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_{10}^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_{11}^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_{12}^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$

$$b_{13}^{[\mu][\nu][\rho\sigma]}(x, y, z) = (\eta^{\mu\rho} D_1^{\nu \rho} - \eta^{\mu\sigma} D_1^{\nu \rho} + \eta^{\nu\rho} D_2^{\mu \rho} + \eta^{\nu\sigma} D_2^{\mu \rho} + \eta^\nu_\mu D_3^{\rho} d(x, y, z)$$  (4.32)
The equation
\[ D^{[\mu][\nu][\rho]}_{\text{triangle}}(x, y, z)_0 = (\bar{s}B)^{[\mu][\nu][\rho]}(x, y, z) + D^{[\mu][\nu][\rho]}_{\delta}(x, y, z) \] (4.33)
gives the following equations in the sector \( u_a(x) \ u_b(y) \ u_c(z)\):

\[ \begin{align*}
    a_1 + a_2 &= A_1 \\
    a_3 - a_4 + a_8 - a_{13} &= A_2 \\
    -a_2 + a_4 + a_7 + a_9 + a_{13} &= A_3 \\
    -a_1 - a_3 + a_{10} &= A_4 \\
    -a_9 &= A_5 \\
    -a_7 - a_{10} &= A_6 \\
    -a_8 &= A_7 \\
    -a_5 - a_7 - a_{12} - a_{13} &= A_8 \\
    a_1 - a_3 - a_6 - a_{11} + a_{13} &= A_9 \\
    a_2 + a_4 + a_6 - a_9 + a_{12} &= A_{10} \\
    a_7 + a_8 + a_{10} + a_{11} - a_{12} &= A_{11} \\
    a_9 - a_{11} + a_{12} &= A_{12}
\end{align*} \] (4.34)

If we solve the system we obtain the following consistency equations

\[ \begin{align*}
    A_5 + A_6 + A_7 + A_{11} + A_{12} &= 0 \\
    A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 &= 0 \\
    A_1 - A_2 + 2A_5 - A_7 - A_9 - A_{10} + A_{12} &= 0
\end{align*} \] (4.35)

If we use the explicit expressions for \( A_j, \ j = 1, \ldots, 12 \) we obtain the equation

\[ 2f^{(0)}_{[abc]} + f^{(3)}_{[abc]} - f^{(4)}_{[abc]} = 0 \] (4.36)

and this is one of the restrictions on the constants from the Lagrangian which are necessary to have the super-renormalizability property.
In the same way we have the generic form

\[ B_{j}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = \sum_{j=1}^{13} b_{j,abc} b_{j}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) u_{a}(x)u_{b}(y)u_{c}(z) \]  

where

\[ b_{1}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho}D_{1}^{\nu}\eta_{1}^{\sigma} + \eta^{\mu\nu}D_{1}^{\rho}\eta_{1}^{\sigma}) - \eta^{\mu\rho}D_{2}^{\nu}\eta_{2}^{\sigma} + \eta^{\mu\nu}D_{2}^{\rho}\eta_{2}^{\sigma} + \eta^{\mu\sigma}D_{1}^{\nu}D_{1}^{\rho} + \eta^{\nu\sigma}D_{1}^{\mu}D_{1}^{\rho} \]

\[ + \eta^{\mu\rho}D_{2}^{\nu}D_{2}^{\rho} - \eta^{\mu\nu}D_{2}^{\rho}D_{2}^{\rho} - \eta^{\mu\sigma}D_{1}^{\nu}D_{2}^{\rho} + \eta^{\nu\sigma}D_{1}^{\mu}D_{2}^{\rho})d(x, y, z) \]

\[ b_{2}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho}D_{1}^{\nu}\eta_{1}^{\sigma} + \eta^{\mu\nu}D_{1}^{\rho}\eta_{1}^{\sigma}) - \eta^{\mu\rho}D_{2}^{\nu}\eta_{2}^{\sigma} + \eta^{\mu\nu}D_{2}^{\rho}\eta_{2}^{\sigma} + \eta^{\mu\sigma}D_{1}^{\nu}D_{1}^{\rho} + \eta^{\nu\sigma}D_{1}^{\mu}D_{1}^{\rho}d(x, y, z) \]

\[ b_{3}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho}D_{1}^{\nu}\eta_{1}^{\sigma} + \eta^{\mu\nu}D_{1}^{\rho}\eta_{1}^{\sigma}) - \eta^{\mu\rho}D_{2}^{\nu}\eta_{2}^{\sigma} + \eta^{\mu\nu}D_{2}^{\rho}\eta_{2}^{\sigma} + \eta^{\mu\sigma}D_{1}^{\nu}D_{1}^{\rho} + \eta^{\nu\sigma}D_{1}^{\mu}D_{1}^{\rho}d(x, y, z) \]

\[ b_{4}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho})D_{1}^{\mu} \cdot D_{2}^{\nu}d(x, y, z) \]

\[ b_{5}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho} D_{1}^{\nu}D_{3}^{\sigma} - \eta^{\mu\nu} D_{1}^{\rho}D_{3}^{\sigma} - \eta^{\mu\sigma} D_{1}^{\rho}D_{3}^{\rho} + \eta^{\rho\sigma} D_{1}^{\mu}D_{3}^{\rho}) + \eta^{\mu\rho} D_{2}^{\nu}D_{3}^{\sigma} - \eta^{\mu\nu} D_{2}^{\rho}D_{3}^{\sigma} - \eta^{\mu\sigma} D_{2}^{\rho}D_{3}^{\rho} + \eta^{\rho\sigma} D_{2}^{\mu}D_{3}^{\rho})d(x, y, z) \]

\[ b_{6}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho} D_{1}^{\nu}D_{3}^{\sigma} - \eta^{\mu\nu} D_{1}^{\rho}D_{3}^{\sigma} - \eta^{\mu\sigma} D_{1}^{\rho}D_{3}^{\rho} + \eta^{\rho\sigma} D_{1}^{\mu}D_{3}^{\rho}) + \eta^{\mu\rho} D_{2}^{\nu}D_{3}^{\sigma} - \eta^{\mu\nu} D_{2}^{\rho}D_{3}^{\sigma} - \eta^{\mu\sigma} D_{2}^{\rho}D_{3}^{\rho} + \eta^{\rho\sigma} D_{2}^{\mu}D_{3}^{\rho})d(x, y, z) \]

\[ b_{7}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho})(D_{1}^{\mu} \cdot D_{3}^{\nu} + D_{2}^{\mu} \cdot D_{3}^{\nu})d(x, y, z) \]

\[ b_{8}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\eta^{\mu\rho} D_{3}^{\nu}D_{3}^{\sigma} - \eta^{\mu\nu} D_{3}^{\rho}D_{3}^{\sigma} - \eta^{\mu\sigma} D_{3}^{\rho}D_{3}^{\rho} + \eta^{\rho\sigma} D_{3}^{\mu}D_{3}^{\rho})d(x, y, z) \]  

and the equation

\[ D_{\text{triangle}}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) = (\bar{s} B_{1}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) + D_{3}^{[\mu\nu][\rho\sigma]\emptyset}(x, y, z) \]

gives the following equations in the sector \( u_{a}(x) u_{b}(y) u_{c}(z) \):

\[
\begin{align*}
a_{10} - a_{11} - a_{13} - b_{3} &= B_{1} \\
-a_{8} + a_{11} - b_{1} - b_{6} &= B_{2} \\
-a_{2} - a_{9} - a_{11} + b_{1} &= B_{3} \\
a_{4} - a_{5} - b_{1} &= B_{4} \\
a_{1} + b_{5} + b_{8} &= B_{5} \\
-a_{1} + b_{6} &= B_{6} \\
a_{3} - a_{7} - b_{6} &= B_{7} \\
a_{4} - a_{11} - b_{2} - b_{5} &= B_{8} \\
a_{5} + a_{8} - b_{3} &= B_{9} \\
a_{7} + a_{12} - a_{13} + b_{2} + b_{4} &= B_{10} \\
-a_{9} + a_{12} - b_{1} + b_{5} + b_{7} &= B_{11} \\
a_{1} - a_{8} - a_{12} + b_{1} &= B_{12} \\
-a_{4} - a_{6} - b_{1} &= B_{13} \\
a_{2} - b_{6} - b_{7} + b_{8} &= B_{14}
\end{align*}
\]
\begin{align*}
  a_2 - b_5 - b_7 &= B_{15} \\
  a_6 + a_9 - b_2 + b_7 &= B_{16} \\
  a_4 - a_{12} - b_3 + b_6 &= B_{17} \\
  a_3 + a_{10} - b_4 + b_5 &= B_{18} \\
\end{align*}

The systems (4.34) and (4.40) can be used to obtain the parameters $a$ and $b$ but they do not produce new restrictions on $A$ and $B$. So if we impose the restriction (4.36) we have the super-renormalizability property in the top ghost number. We can use a descent procedure to show that this property stays true for all triangle contributions of $D^{IJK}$ without derivatives on the fields. So, if we consider the expression

$$
D^{IJK} - (\bar{s}B_1)^{IJK}
$$

(4.41)

we eliminate all the terms without derivatives on the fields.
4.3 The Causal Commutators for Ghost Number 2

There are two relevant causal commutators of this type:

\[
D^{[0][\mu\nu]}(x, y, z) = D(T(x), T(y), T^{\mu\nu}(z))
\]

\[-[\bar{T}(x, y), T^{\mu\nu}(y)], T(z)] - \left[ T^{[0][\nu\mu]}(x, z), T(z) \right] - \left[ T^{[0][\mu\nu]}(y, z), T(x) \right]
\]

(4.42)

and

\[
D^{[\mu][\nu\emptyset]}(x, y, z) = D(T^\mu(x), T^\nu(y), T(z))
\]

\[-[\bar{T}^{[\mu][\nu]}(x, y), T(z)] - \left[ T^{[\mu][\nu\emptyset]}(x, z), T^{\nu}(y) \right] + \left[ T^{[\mu][\nu\emptyset]}(y, z), T^{\mu}(x) \right].
\]

(4.43)

Both expressions have a contribution \( D_1 \sim uuF \) and a contribution \( D_2 \sim uuF \). The contributions of the second type are by explicit computation:

\[
D^{[0][\mu\nu]}_{2}(x, y, z) = i f^{(0)}_{abc}
\]

\[\left[ \mathcal{D}_1 \partial_\nu d(x, y, z) W^{1\mu\nu}(x, y, z) - \mathcal{D}_1 \partial_\nu d(x, y, z) W^{2\mu\nu}(x, y, z) \right] - (\mu \leftrightarrow \nu)
\]

\[+ i \mathcal{D}_1 \cdot \mathcal{D}_2 d(x, y, z) W^{3\mu\nu}(x, y, z)
\]

(4.44)

and

\[
D^{[\mu][\nu\emptyset]}_{2}(x, y, z) = i f^{(0)}_{abc}
\]

\[\left[ -\mathcal{D}_2 \cdot \mathcal{D}_3 d(x, y, z) W^{1\mu\nu}(x, y, z) + \mathcal{D}_1 \cdot \mathcal{D}_3 d(x, y, z) W^{2\mu\nu}(x, y, z) \right]
\]

\[+ \left( -\mathcal{D}_1 \partial_\nu d_2 + \mathcal{D}_1 \partial_\nu d_2 + \mathcal{D}_2 \partial_\nu d_3 - \mathcal{D}_1 \partial_\nu d_3 \right) d(x, y, z) W^{1\mu\nu}(x, y, z)
\]

\[+ \left( -\mathcal{D}_1 \partial_\nu d_2 + \mathcal{D}_1 \partial_\nu d_2 + \mathcal{D}_2 \partial_\nu d_3 - \mathcal{D}_2 \partial_\nu d_3 \right) d(x, y, z) W^{2\mu\nu}(x, y, z)
\]

\[- \mathcal{D}_1 \partial_\nu d(x, y, z) W^{3\mu\nu}(x, y, z)
\]

\[+ (-\mathcal{D}_1 \partial_\nu d_2 - \mathcal{D}_1 \partial_\nu d_2 + \mathcal{D}_2 \partial_\nu d_3 + \mathcal{D}_2 \partial_\nu d_3) d(x, y, z) W^{1\mu\nu}(x, y, z)
\]

\[+ (-\mathcal{D}_1 \partial_\nu d_2 - \mathcal{D}_1 \partial_\nu d_2 + \mathcal{D}_2 \partial_\nu d_3 - \mathcal{D}_2 \partial_\nu d_3) d(x, y, z) W^{2\mu\nu}(x, y, z)
\]

\[+ (-\mathcal{D}_1 \partial_\nu d_2 - \mathcal{D}_1 \partial_\nu d_2 + \mathcal{D}_2 \partial_\nu d_3 - \mathcal{D}_2 \partial_\nu d_3) d(x, y, z) W^{3\mu\nu}(x, y, z)
\]

(4.45)

where

\[
W^{1\mu\nu}_1 \equiv F^{\mu\nu}_a(x) u_b(y) u_c(z), \quad W^{2\mu\nu}_2 \equiv u_a(x) F^{\mu\nu}_b(y) u_c(z), \quad W^{3\mu\nu}_3 \equiv u_a(x) u_b(y) F^{\mu\nu}_c(z)
\]

(4.46)

In the expressions (4.41) we have some supplementary terms of the type \( uuF \) coming from \(- (sB_1)^{1JK} \). The coboundary equations are in this case:

\[
D_2^{[0][\mu\nu]} + \frac{i}{2} \sum b_{j,abc} j^{[\mu][\sigma][0][\nu]}_{j} W_{1,\rho\sigma} + (x \leftrightarrow y) = (sB_2)^{[0][\mu\nu]} + D_8^{[0][\mu\nu]}
\]

(4.47)

and

\[
D^{[\mu][\nu\emptyset]} + \frac{i}{2} \sum a_{j,abc} j^{[\mu][\nu][\rho]}_{j} W_{3,\rho\sigma} = (sB_2)^{[\mu][\nu\emptyset]} + D_8^{[\mu][\nu\emptyset]}
\]

(4.48)
4.4 The Generic Form of the Coboundaries

The various restrictions lead to the following generic forms:

\[ B_2^{[\rho [\mu \nu]}(x, y, z) = \sum_{j=1}^{30} c_{j,abc} B_j^{[\rho [\mu \nu]}(x, y, z) \]  

(4.49)

where

\[ B_1^{[\rho [\mu \nu]}(x, y, z) = D_1^\rho d(x, y, z) W_1^{\mu \nu} \]
\[ B_2^{[\rho [\mu \nu]}(x, y, z) = D_2^\rho d(x, y, z) W_2^{\mu \nu} \]
\[ B_3^{[\rho [\mu \nu]}(x, y, z) = D_3^\rho d(x, y, z) W_3^{\mu \nu} \]
\[ B_4^{[\rho [\mu \nu]}(x, y, z) = D_4^\rho d(x, y, z) W_4^{\mu \nu} \]
\[ B_5^{[\rho [\mu \nu]}(x, y, z) = D_5^\rho d(x, y, z) W_5^{\mu \nu} \]
\[ B_6^{[\rho [\mu \nu]}(x, y, z) = D_6^\rho d(x, y, z) W_6^{\mu \nu} \]
\[ B_7^{[\rho [\mu \nu]}(x, y, z) = D_7^\rho d(x, y, z) W_7^{\mu \nu} \]
\[ B_8^{[\rho [\mu \nu]}(x, y, z) = D_8^\rho d(x, y, z) W_8^{\mu \nu} \]
\[ B_9^{[\rho [\mu \nu]}(x, y, z) = D_9^\rho d(x, y, z) W_9^{\mu \nu} \]

\[ B_{10}^{[\rho [\mu \nu]}(x, y, z) = D_1^\rho d(x, y, z) W_{10}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{11}^{[\rho [\mu \nu]}(x, y, z) = D_2^\rho d(x, y, z) W_{11}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{12}^{[\rho [\mu \nu]}(x, y, z) = D_3^\rho d(x, y, z) W_{12}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{13}^{[\rho [\mu \nu]}(x, y, z) = D_4^\rho d(x, y, z) W_{13}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{14}^{[\rho [\mu \nu]}(x, y, z) = D_5^\rho d(x, y, z) W_{14}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{15}^{[\rho [\mu \nu]}(x, y, z) = D_6^\rho d(x, y, z) W_{15}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{16}^{[\rho [\mu \nu]}(x, y, z) = D_7^\rho d(x, y, z) W_{16}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{17}^{[\rho [\mu \nu]}(x, y, z) = D_8^\rho d(x, y, z) W_{17}^{\mu \nu} - (\mu \leftrightarrow \nu) \]
\[ B_{18}^{[\rho [\mu \nu]}(x, y, z) = D_9^\rho d(x, y, z) W_{18}^{\mu \nu} - (\mu \leftrightarrow \nu) \]

(4.50)
\[ B_{28}^{[\rho][\eta][\nu]}(x, y, z) = d(x, y, z) \partial^\rho F_{\alpha}^{\nu}(x) u_b(y) u_c(z) \]
\[ B_{29}^{[\rho][\eta][\nu]}(x, y, z) = d(x, y, z) u_a(x) \partial^\rho F_{\beta}^{\nu}(y) u_c(z) \]
\[ B_{30}^{[\rho][\eta][\nu]}(x, y, z) = d(x, y, z) u_a(x) u_b(y) \partial^\rho F_{\gamma}^{\nu}(z) \]

(4.51)

and

\[ B_{2}^{[\mu][\nu][\rho]}(x, y, z) = \sum_{j=1}^{10} d_{j,abc} B_{j}^{[\mu][\nu][\rho]}(x, y, z) \]

(4.52)

where

\[ B_{1}^{[\mu][\nu][\rho]}(x, y, z) = D^\mu_1 d(x, y, z) W_{1}^{\mu\nu} + D^\rho_2 d(x, y, z) W_{2}^{\mu\nu} + D^\nu_1 d(x, y, z) W_{1}^{\mu\rho} - D^\rho_3 d(x, y, z) W_{3}^{\mu\rho} - D^\nu_2 d(x, y, z) W_{2}^{\mu\rho} - D^\mu_2 d(x, y, z) W_{2}^{\nu\rho} \]
\[ B_{2}^{[\mu][\nu][\rho]}(x, y, z) = D^\rho_5 d(x, y, z) W_{3}^{\mu\rho} - D^\mu_3 d(x, y, z) W_{3}^{\nu\rho} - D^\nu_3 d(x, y, z) W_{3}^{\mu\rho} - D^\rho_5 d(x, y, z) W_{1}^{\mu\rho} + D^\nu_4 d(x, y, z) W_{1}^{\nu\rho} - D^\mu_4 d(x, y, z) W_{1}^{\nu\rho} \]
\[ B_{3}^{[\mu][\nu][\rho]}(x, y, z) = D^\mu_7 d(x, y, z) W_{2}^{\mu\rho} + D^\nu_3 d(x, y, z) W_{2}^{\mu\rho} + D^\rho_6 d(x, y, z) W_{2}^{\nu\rho} - D^\rho_6 d(x, y, z) W_{3}^{\mu\rho} - D^\nu_4 d(x, y, z) W_{3}^{\nu\rho} + D^\rho_5 d(x, y, z) W_{3}^{\nu\rho} \]
\[ B_{4}^{[\mu][\nu][\rho]}(x, y, z) = D^\rho_5 d(x, y, z) W_{3}^{\mu\rho} - D^\mu_1 d(x, y, z) W_{1}^{\mu\rho} - D^\nu_2 d(x, y, z) W_{2}^{\mu\rho} - D^\mu_2 d(x, y, z) W_{2}^{\nu\rho} - D^\mu_2 d(x, y, z) W_{3}^{\nu\rho} - D^\nu_4 d(x, y, z) W_{3}^{\nu\rho} \]
\[ B_{5}^{[\mu][\nu][\rho]}(x, y, z) = \eta^\rho \partial_1 d(x, y, z) W_{1}^{\mu\nu} - \eta^\mu \partial_2 d(x, y, z) W_{2}^{\mu\rho} + \eta^\nu \partial_1 d(x, y, z) W_{1}^{\mu\rho} + \eta^\rho \partial_3 d(x, y, z) W_{3}^{\mu\rho} \]
\[ B_{6}^{[\mu][\nu][\rho]}(x, y, z) = \eta^\nu \partial_1 d(x, y, z) W_{1}^{\mu\rho} + \eta^\mu \partial_2 d(x, y, z) W_{2}^{\mu\rho} + \eta^\rho \partial_3 d(x, y, z) W_{3}^{\mu\rho} \]
\[ B_{7}^{[\mu][\nu][\rho]}(x, y, z) = \eta^\rho \partial_1 d(x, y, z) W_{1}^{\mu\sigma} - \eta^\sigma \partial_2 d(x, y, z) W_{2}^{\mu\sigma} - \eta^\sigma \partial_3 d(x, y, z) W_{3}^{\mu\sigma} \]
\[ B_{8}^{[\mu][\nu][\rho]}(x, y, z) = \eta^\mu \partial_1 d(x, y, z) W_{1}^{\mu\rho} - \eta^\rho \partial_2 d(x, y, z) W_{2}^{\mu\rho} - \eta^\rho \partial_3 d(x, y, z) W_{3}^{\mu\rho} \]
\[ B_{9}^{[\mu][\nu][\rho]}(x, y, z) = \eta^\sigma \partial_1 d(x, y, z) W_{1}^{\mu\rho} + \eta^\rho \partial_2 d(x, y, z) W_{2}^{\mu\rho} + \eta^\rho \partial_3 d(x, y, z) W_{3}^{\mu\rho} \]
\[ B_{10}^{[\mu][\nu][\rho]}(x, y, z) = d(x, y, z) \left[ \partial^\mu F_{\alpha}^{\nu}(x) u_b(y) u_c(z) + u_a(x) \partial^\nu F_{\beta}^{\rho}(y) u_c(z) + u_a(x) u_b(y) \partial^\rho F_{\gamma}^{\mu}(z) \right] \]

(4.53)

(4.54)
From the equation (4.47)

\[ D_2^{[\emptyset][\mu\nu]} + \frac{i}{2} \sum [b_{j,abc}b_{j}^{[\rho\sigma][\mu\nu]}W_{1,\rho\sigma} + (x \leftrightarrow y)] = (\bar{s}B_2)^{[\emptyset][\mu\nu]} + D_3^{[\emptyset][\mu\nu]} \]

we get in the sector \( \mathcal{D}\mathcal{D}d(x, y, z)uu\partial F \)

\[
\begin{align*}
  c_1 + c_{19} - c_{29} &= 0 \\
  c_4 + c_{22} - c_{28} &= 0 \\
  c_7 + c_{25} + c_{28} + c_{29} &= 0 
\end{align*}
\]

(4.55)

and in the sector \( \mathcal{D}\mathcal{D}d(x, y, z)uuF \)

\[
\begin{align*}
  c_{14} + c_{23} &= b_1 \\
  c_{11} - c_{19} &= b_5 + f_0 \\
  c_{17} + c_{19} - c_{23} &= b_2 \\
  -c_{10} + c_{20} &= b_6 \\
  -c_{13} - c_{22} &= b_8 \\
  -c_{16} - c_{20} + c_{22} &= b_5 \\
  c_{10} - c_{14} + c_{26} &= b_3 \\
  -c_{11} + c_{13} - c_{25} &= b_6 \\
  c_{16} - c_{17} + c_{25} - c_{26} &= b_1 \\
  -c_{15} + c_{24} &= 0 \\
  -c_{12} - c_{21} &= 0 \\
  -c_{18} + c_{21} + c_{24} &= 0 \\
  c_{12} + c_{15} - c_{27} &= 0 \\
  c_{18} + c_{27} &= 0 \\
  -c_{1} + c_{2} &= b_7 \\
  c_{1} - c_{5} + c_{8} &= b_4 \\
  -c_{2} + c_{4} - c_{7} &= b_7 \\
  -2c_3 &= f_0 \\
  c_3 + c_6 - c_9 &= 0 
\end{align*}
\]

(4.56)

From the equation (4.48)

\[ D^{[\mu][\nu]\emptyset} + \frac{i}{2} \sum_j a_{j,abc}b_{j}^{[\rho][\mu\nu]}[\emptyset]W_{3,\rho\sigma} = (\bar{s}B_2)^{[\mu][\nu]\emptyset} + D_3^{[\mu][\nu]\emptyset} \]

(4.57)

it follows in the sector \( \mathcal{D}\mathcal{D}d(x, y, z)uu\partial F \)

\[- c_{12} + d_{10} = 0\]
\[
\begin{align*}
    c_{12} + c_{21} - c_{30} &= 0 \\
    -c_{15} + c_{28} &= 0 \\
    c_{15} + c_{24} + c_{30} &= 0 \\
    -c_{18} - c_{28} - d_{10} &= 0 \\
    c_{18} + c_{27} &= 0 \\
    -c_{29} + d_5 &= 0 \\
    c_{29} - d_3 + d_9 + d_{10} &= 0 \\
    d_3 - d_8 - d_{10} &= 0 \\
\end{align*}
\]

and in the sector \( \mathcal{D}\mathcal{D}_d(x, y, z) uuF \)

\[
\begin{align*}
    c_1 + c_{19} - d_5 &= 0 \\
    c_{10} - d_1 - d_6 &= 0 \\
    c_7 - d_3 + d_5 &= 0 \\
    c_{16} - c_{27} - d_4 + d_6 &= -f_0 \\
    c_{13} + c_{27} - d_2 &= -f_0 \\
    c_4 - c_{19} + d_3 &= 0 \\
    -c_{18} + c_{25} + d_8 &= 0 \\
    -c_9 + d_1 + d_7 &= f_0 \\
    -c_{12} + d_3 - d_8 &= 0 \\
    -c_3 - c_{21} + d_4 - d_7 &= 0 \\
    -c_{15} - c_{25} - d_3 &= 0 \\
    -c_6 + c_{21} + d_2 &= f_0 \\
    -c_1 + c_{18} + c_{22} - d_9 &= 0 \\
    c_9 - c_{10} + d_7 &= 0 \\
    -c_7 + c_{12} + d_9 &= 0 \\
    c_3 - c_{16} - c_{24} - d_7 &= 0 \\
    -c_4 + c_{15} - c_{22} &= 0 \\
    c_6 - c_{13} + c_{24} &= 0 \\
    -c_2 - c_{20} + d_4 - d_9 &= a_2 \\
    -c_{11} + d_2 + d_8 &= a_1 \\
    -c_8 + d_2 + d_9 &= -a_3 - f_0 \\
    -c_{17} + c_{26} + d_4 - d_8 &= a_4 \\
    -c_5 + c_{20} + d_1 &= a_{10} - f_0 \\
    -c_{14} - c_{26} + d_1 &= a_8 \\
    c_2 - c_{17} - c_{23} - d_5 &= a_9 + f_0 \\
    -c_8 + c_{11} - d_5 &= a_7 - f_0
\end{align*}
\]
\[ c_5 - c_{14} + c_{23} = -a_{13} + f_0 \]
\[ -c_{16} - c_{18} + d_1 - d_2 = 0 \]
\[ c_{10} - c_{13} + c_{18} - d_4 = 0 \]
\[ c_{12} - c_{15} + c_{16} + d_4 = -f_0 \]
\[ 2c_{17} + 2d_3 = a_6 - f_0 \]
\[ -c_{11} + c_{14} - c_{17} = a_{12} \]
\[ -c_{25} - c_{27} - d_5 - d_9 = 0 \]
\[ -c_{19} - c_{22} + c_{27} + d_8 = 0 \]
\[ c_{21} + c_{24} - c_{25} - d_8 = 0 \]
\[ c_{20} + c_{23} - c_{26} - d_6 = a_{11} \]
\[ 2c_{26} + 2d_7 = a_5 \]  
\[(4.59)\]

The systems (4.55) + (4.56) + (4.58) + (4.59) can be used to obtain the parameters \(c\) and \(d\). No constraints are necessary on the physical parameters of the system.

If we consider the expression

\[ D^{IJK} - (\bar{s}(B_1 + B_2)^{IJK} \]
\[(4.60)\]

we remain only with terms \(\sim uFF\) and \(FFF\).
4.5 The Causal Commutators for Ghost Number 1

We have the contribution $D_1 \sim \omega v d, D_2 \sim \omega \pi F$ and $D_3 \sim \omega F F u$:

$$
D_3^{[0][\mu]}(x, y, z) = i \phi(x, y, z) F_\omega \pi F \omega \pi (y) u_c(z) \\
- \phi(x, y, z) F_\omega \pi F \omega \pi (y) u_c(z) + \phi(x, y, z) F_\omega \pi F \omega \pi (y) F_c \omega \pi (z)
$$

(4.61)

4.6 The Generic Form of the Coboundaries

$$
B_3^{[0][\mu]}(x, y, z) = \sum c_{j, abc} \hat{B}_j^{[0][\mu]}(x, y, z) \\
B_3^{[\mu][0]}(x, y, z) = \sum d_{j, abc} \hat{B}_j^{[\mu][0]}(x, y, z)
$$

(4.62)

where

$$
\hat{B}_1^{[0][\mu]}(x, y, z) = d(x, y, z) \eta_{\mu \sigma} F_a^{\mu \sigma}(x) F_b^{\mu \sigma}(y) u_c(z) - (\mu \leftrightarrow \nu) \\
\hat{B}_2^{[0][\mu]}(x, y, z) = d(x, y, z) \eta_{\mu \sigma} [u_a(x) F_b^{\mu \sigma}(y) F_c^{\mu \sigma}(z) - F_a^{\mu \sigma}(x) u_b(y) F_c^{\mu \sigma}(z)] - (\mu \leftrightarrow \nu)
$$

(4.63)

and

$$
\hat{B}_1^{[\mu][0]}(x, y, z) = d(x, y, z) \eta_{\mu \sigma} F_a^{\mu \sigma}(x) F_b^{\mu \sigma}(y) u_c(z) \\
\hat{B}_2^{[\mu][0]}(x, y, z) = d(x, y, z) \eta_{\mu \sigma} [u_a(x) F_b^{\mu \sigma}(y) F_c^{\mu \sigma}(z) + F_a^{\mu \sigma}(x) u_b(y) F_c^{\mu \sigma}(z)] \\
\hat{B}_3^{[\mu][0]}(x, y, z) = d(x, y, z) \eta_{\mu \sigma} [u_a(x) F_b^{\mu \sigma}(y) F_c^{\mu \sigma}(z) + F_a^{\mu \sigma}(x) u_b(y) F_c^{\mu \sigma}(z)] \\
\hat{B}_4^{[\mu][0]}(x, y, z) = d(x, y, z) \eta_{\mu \sigma} F_a^{\mu \sigma}(y) F_b^{\mu \sigma}(z) u_c(z) \\
\hat{B}_5^{[\mu][0]}(x, y, z) = d(x, y, z) \eta_{\mu \sigma} F_a^{\mu \sigma}(y) F_b^{\mu \sigma}(z) u_c(z) + F_a^{\mu \sigma}(x) u_b(y) F_c^{\mu \sigma}(z)]
$$

(4.64)

We consider the equation

$$
D_3^{[0][\mu]} + \hat{D}_3^{[0][\mu]} = (\bar{s} B_3)^{[0][\mu]}
$$

(4.65)

where $\hat{D}_3$ contains the contributions coming from $-(\bar{s}(B_1 + B_2))^{IJK}$; we obtain in the sector $Dd(X)FFu$ the following equations

$$
\hat{c}_1 + \hat{d}_3 = -c_1 - c_2 + f_0 \\
-\hat{c}_1 + \hat{d}_4 = -c_1 + c_2 \\
\hat{c}_2 + \hat{d}_3 = -c_3 \\
-\hat{c}_2 + \hat{d}_2 = -c_5 \\
\hat{c}_3 + \hat{d}_4 = -c_1 + f_0 \\
\hat{c}_2 + \hat{d}_1 = -c_3 \\
-\hat{d}_1 - \hat{d}_3 = -c_9 \\
\hat{d}_2 - \hat{d}_3 = -c_{19}
$$
\begin{align*}
-\hat{d}_3 - \hat{d}_4 &= -c_{11} - c_{20} \\
\hat{d}_5 &= -\frac{1}{2} c_4 \\
\hat{d}_6 &= -\frac{1}{2} c_7 \\
-\hat{d}_5 - \hat{d}_6 &= -\frac{1}{2} c_1 \\
-2\hat{d}_6 &= -c_2 \\
\hat{d}_6 &= -\frac{1}{2} (c_5 + c_8) - f_0 \\
-\frac{1}{2} \hat{c}_1 &= -\frac{1}{2} c_{28} \\
-\frac{1}{2} \hat{d}_2 - \hat{d}_5 &= 0 \\
-\frac{1}{2} \hat{d}_3 - \hat{d}_6 &= -\frac{1}{2} c_{29} \\
\end{align*}

We can solve this system \textit{iff} we impose

\[ f_{abc}^{(3)} = 6 f_{abc}^{(0)} \]  \hspace{1cm} (4.67)

and if we combine with (4.36) we get also

\[ f_{abc}^{(4)} = 8 f_{abc}^{(0)} \]  \hspace{1cm} (4.68)

One can show also by direct computation that the contribution $D_4$ bilinear in the scalar fields $\Phi_a$ does not produces new restriction.
4.7 The Causal Commutators in the Dirac Sector

We define

\[ t_{ae}^{(1)} \equiv \sum_b g_{ab} t^c_b \quad t_{ae}^{(2)} \equiv \sum_b t^b_c t^c_b \]

\[ t_{ae}^{(3)} \equiv \sum_b s_b^{-} t^c_a s_b^c \quad t_{ae}^{(4)} \equiv -i \sum_{b,c} f_{bca} s_b^{-} s_c^c \]

and we have the following Dirac contribution in ghost number 1:

\[
D_5(T(x), T(y); T^\mu(z)) =
\]

\[ \frac{i}{2} D_1^\nu D_2^\rho d(x, y, z) u_a(x) [\bar{\Psi}(y) t_{ae}^{(1)} \otimes \gamma_\rho \gamma_\nu \gamma_\mu \gamma_\epsilon \Psi(z) + \bar{\Psi}(z) t_{ae}^{(1)} \otimes \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\epsilon \Psi(y)]
\]

\[ + 2i D_1^\nu D_2^\rho d(x, y, z) u_a(z) [\bar{\Psi}(x) t_{ae}^{(2)} \otimes \gamma_\nu \gamma_\rho \gamma_\mu \gamma_\epsilon \Psi(y) - 4i D_1^\nu D_2^\rho d(x, y, z) u_a(z) \bar{\Psi}(x) t_{ae}^{(2)} \otimes \gamma_\mu \gamma_\nu \gamma_\epsilon \Psi(y) - i D_1^\nu D_2^\rho d(x, y, z) u_a(z) \bar{\Psi}(x) t_{ae}^{(3)} \otimes \gamma_\rho \gamma_\mu \gamma_\nu \gamma_\epsilon \Psi(y)]
\]

\[ + \frac{i}{2} D_1^\nu D_3^\rho d(x, y, z) u_a(z) [\bar{\Psi}(x) t_{ae}^{(1)} \otimes \gamma_\nu \gamma_\rho \gamma_\mu \gamma_\epsilon \Psi(y) + i D_1^\nu D_3^\rho d(x, y, z) u_a(z) \bar{\Psi}(x) t_{ae}^{(1)} \otimes \gamma_\mu \gamma_\nu \gamma_\epsilon \Psi(y)]
\]

\[ + \frac{i}{2} D_2^\nu D_3^\rho d(x, y, z) u_a(z) [\bar{\Psi}(x) t_{ae}^{(1)} \otimes \gamma_\rho \gamma_\nu \gamma_\mu \gamma_\epsilon \Psi(y) + i D_2^\nu D_3^\rho d(x, y, z) u_a(z) \bar{\Psi}(x) t_{ae}^{(1)} \otimes \gamma_\mu \gamma_\nu \gamma_\epsilon \Psi(y)]
\]

\[ + i D_2^\nu D_3^\rho d(x, y, z) u_a(z) \bar{\Psi}(x) t_{ae}^{(4)} \otimes \gamma_\nu \gamma_\epsilon \Psi(y) + i D_2^\nu D_3^\rho d(x, y, z) u_a(z) \bar{\Psi}(x) t_{ae}^{(4)} \otimes \gamma_\nu \gamma_\epsilon \Psi(y)
\]

(4.70)
4.8 The Generic Form of the Coboundaries

We impose the coboundary condition

$$D_5^{[0][\mu]}(x, y, z)_0 = (\bar{s}B_5)^{[0][\mu]}(x, y, z) + D_5^{[0][\mu]}(x, y, z)$$

(4.71)

and from the various restrictions we have the generic forms:

$$B_5^{[\mu][\nu]}(x, y, z) =$$

$$D_4^\rho d(X) u_a(x) \tilde{\Psi}(y) F_{ac}^{(1)} \otimes \gamma^\rho \gamma_c \Psi(z) + D_2^\rho d(X) u_a(y) \tilde{\Psi}(x) F_{ac}^{(2)} \otimes \gamma^\rho \gamma_c \Psi(z)$$

$$+ D_2^\rho d(X) u_a(x) \tilde{\Psi}(y) F_{ac}^{(3)} \otimes \gamma^\rho \gamma_c \Psi(z) + D_1^\rho d(X) u_a(y) \tilde{\Psi}(x) F_{ac}^{(4)} \otimes \gamma^\rho \gamma_c \Psi(z)$$

$$+ D_3^\rho d(X) u_a(x) \tilde{\Psi}(y) F_{ac}^{(5)} \otimes \gamma^\rho \gamma_c \Psi(z) + D_3^\rho d(X) u_a(y) \tilde{\Psi}(x) F_{ac}^{(6)} \otimes \gamma^\rho \gamma_c \Psi(z)$$

$$+ D_4^\rho d u_a(x) \tilde{\Psi}(z) F_{ac}^{(10)} \otimes \gamma^\rho \gamma_c \Psi(y) + D_2^\rho d u_a(y) \tilde{\Psi}(z) F_{ac}^{(11)} \otimes \gamma^\rho \gamma_c \Psi(y)$$

$$+ D_3^\rho d u_a(z) \tilde{\Psi}(x) F_{ac}^{(15)} \otimes \gamma^\rho \gamma_c \Psi(y) + D_3^\rho d u_a(y) \tilde{\Psi}(x) F_{ac}^{(16)} \otimes \gamma^\rho \gamma_c \Psi(y)$$

$$+ D_2^\rho d u_a(x) \tilde{\Psi}(y) F_{ac}^{(19)} \otimes \gamma^\rho \gamma_c \Psi(y) + D_1^\rho d u_a(z) \tilde{\Psi}(y) F_{ac}^{(20)} \otimes \gamma^\rho \gamma_c \Psi(y)$$

$$+ D_2^\rho d u_a(z) \tilde{\Psi}(x) F_{ac}^{(24)} \otimes \gamma^\rho \gamma_c \Psi(y) + D_1^\rho d u_a(y) \tilde{\Psi}(x) F_{ac}^{(25)} \otimes \gamma^\rho \gamma_c \Psi(y)$$

$$+ D_3^\rho d u_a(z) \tilde{\Psi}(y) F_{ac}^{(29)} \otimes \gamma^\rho \gamma_c \Psi(y) + D_3^\rho d u_a(y) \tilde{\Psi}(x) F_{ac}^{(30)} \otimes \gamma^\rho \gamma_c \Psi(y)$$

$$+ d(X)[u_a(x) \partial^\nu \tilde{\Psi}(y) F_{ac}^{(28)} \otimes \gamma^\rho \gamma_c \Psi(z) - u_a(y) \partial^\rho \tilde{\Psi}(x) F_{ac}^{(29)} \otimes \gamma^\nu \gamma_c \Psi(z)$$

$$+ d(X)[u_a(x) \partial^\nu \tilde{\Psi}(y) F_{ac}^{(28)} \otimes \gamma^\rho \gamma_c \Psi(z) - u_a(y) \partial^\rho \tilde{\Psi}(x) F_{ac}^{(29)} \otimes \gamma^\nu \gamma_c \Psi(z)]$$
\[ \frac{d}{dx} \left[ u(x) \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] = \frac{d}{dx} \left[ u(x) \frac{\partial}{\partial x} \right] \frac{\partial}{\partial y} \]
\[ +d(x, y, z)u_{a}(z)[\bar{\Psi}(x)G_{ae}^{(15)} \otimes \gamma^{\mu} \gamma_{\epsilon} \partial^{\mu} \Psi(y) + \bar{\Psi}(y)G_{ae}^{(15)} \otimes \gamma^{\mu} \gamma_{\epsilon} \partial^{\mu} \Psi(x)] \\
+ D_{1}^{a}d(x, y, z)u_{a}(x)\bar{\Psi}(y)G_{ae}^{(16)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(z) \\
- D_{2}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(16)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(z) \\
+ D_{3}^{a}d(x, y, z)u_{a}(x)\bar{\Psi}(y)G_{ae}^{(17)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(z) \\
- D_{4}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(17)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(z) \\
+ D_{5}^{a}d(x, y, z)u_{a}(x)\bar{\Psi}(y)G_{ae}^{(18)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(z) \\
- D_{6}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(18)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(z) \\
+ D_{7}^{a}d(x, y, z)u_{a}(x)\bar{\Psi}(y)G_{ae}^{(19)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
- D_{8}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(19)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
+ D_{9}^{a}d(x, y, z)u_{a}(x)\bar{\Psi}(y)G_{ae}^{(20)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
- D_{10}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(20)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
+ D_{11}^{a}d(x, y, z)u_{a}(x)\bar{\Psi}(y)G_{ae}^{(21)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
- u_{a}(y)\bar{\Psi}(z)G_{ae}^{(21)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(x) \\
+ D_{12}^{a}d(x, y, z)u_{a}(z)\bar{\Psi}(x)G_{ae}^{(22)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
- D_{13}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(22)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
+ D_{14}^{a}d(x, y, z)u_{a}(z)\bar{\Psi}(x)G_{ae}^{(23)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
- D_{15}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(23)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
+ D_{16}^{a}d(x, y, z)u_{a}(z)\bar{\Psi}(x)G_{ae}^{(24)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(y) \\
- D_{17}^{a}d(x, y, z)u_{a}(y)\bar{\Psi}(x)G_{ae}^{(24)} \otimes \gamma^{\mu} \gamma^{\epsilon} \gamma_{\epsilon} \Psi(x) \} \\
- \{ \mu \leftrightarrow \nu \} (4.73) \]
We get from the equation \((4.71)\)

\[ D^{0[0][\mu]}_5(x, y, z)_0 = (\bar{s}B_5)^{0[0][\mu]}(x, y, z) + D^{0[0][\mu]}_5(x, y, z) \]

we obtain the following system

\[
\begin{align*}
-F_{40} + F_{46} + F_{48} - G_{21} &= 0 \\
F_{40} + F_{47} - G_{19} - G_{20} &= \frac{1}{2}t_1 \\
-F_{41} - F_{42} + F_{47} + G_{21} &= 0 \\
F_{40} + F_{42} + F_{44} + G_{24} &= -\frac{1}{2}t_1 \\
F_{41} + F_{43} + F_{45} - G_{24} &= \frac{1}{2}t_1 \\
-F_{41} + F_{44} - G_{22} - G_{23} &= 2t_2 - t_3 \\
-F_{43} - F_{47} - F_{48} - G_{18} &= 0 \\
F_{43} - F_{46} - G_{16} - G_{17} &= -\frac{1}{2}t_1 \\
-F_{44} - F_{45} - F_{46} + G_{18} &= 0 \\
-F_{1} + F_{20} - F_{24} + 2F_{40} - 2F_{48} + G_{6} &= 0 \\
F_{1} + F_{22} - 2F_{40} - G_{4} + G_{5} + 2G_{19} &= 0 \\
-F_{2} - F_{23} + F_{25} + G_{21} &= 0 \\
F_{2} + F_{26} + G_{4} + G_{20} &= 0 \\
F_{3} - F_{5} - F_{22} + 2F_{42} - 2F_{47} - G_{6} &= 0 \\
-F_{4} - F_{8} - F_{19} - F_{25} - 2F_{41} &= 0 \\
F_{4} - F_{9} - F_{26} - 2F_{42} + G_{6} &= 0 \\
-F_{6} + F_{8} - F_{21} + 2F_{41} - G_{21} &= 0 \\
F_{6} + F_{9} + 2F_{42} + G_{5} - G_{20} &= 0 \\
-F_{7} + F_{19} - F_{27} - 2F_{40} - G_{6} &= 0 \\
F_{7} + F_{21} + 2F_{40} - G_{5} - G_{19} &= 0 \\
F_{23} + F_{27} - G_{4} + G_{19} &= 0 \\
-F_{1} - F_{7} - F_{10} - F_{16} &= 0 \\
F_{1} - F_{9} - F_{17} + G_{9} &= 0 \\
F_{2} - F_{6} - F_{13} - 2F_{42} - 2F_{44} - G_{9} &= t_1 \\
-F_{3} - F_{11} + F_{16} + G_{24} &= t_4 \\
F_{3} + F_{17} + G_{7} + G_{23} &= -4t_2 \\
-F_{4} + F_{11} - F_{15} - 2F_{41} - 2F_{45} + G_{9} &= 0 \\
F_{4} + F_{13} + 2F_{41} - G_{7} + G_{8} + 2G_{22} &= 2t_3 \\
-F_{5} + F_{7} - F_{12} - G_{24} &= t_1 + t_4 \\
F_{5} + F_{9} + G_{8} - G_{23} &= 0
\end{align*}
\]
\[-F_8 + F_{10} - F_{18} - G_9 = t_1\]
\[F_8 + F_{12} - G_8 - G_{22} = -2t_3\]
\[F_{14} + F_{18} - G_7 + G_{22} = 0\]
\[-F_{10} + F_{21} - F_{23} + 2F_{43} + 2F_{48} + G_3 = 0\]
\[F_{10} + F_{19} - 2F_{43} - G_1 + G_2 + 2G_{16} = t_1\]
\[-F_{11} - F_{24} + F_{26} + 2F_{47} + G_{18} = 0\]
\[F_{11} + F_{25} + 2F_{46} + G_1 + G_{17} = t_1\]
\[F_{12} - F_{14} - F_{19} + 2F_{45} + 2F_{46} - G_3 = 0\]
\[-F_{13} - F_{17} - F_{22} - F_{26} - 2F_{44} - 2F_{47} = 0\]
\[F_{13} - F_{18} - F_{25} - 2F_{45} - 2F_{46} + G_3 = 0\]
\[-F_{15} + F_{17} - F_{20} + 2F_{44} - G_{18} = 0\]
\[F_{15} + F_{18} + 2F_{45} + G_2 - G_{17} = 0\]
\[-F_{16} + F_{22} - F_{27} - 2F_{43} - 2F_{48} - G_3 = 0\]
\[F_{16} + F_{20} + 2F_{43} - G_2 - G_{16} = -t_1\]
\[F_{24} + F_{27} + 2F_{48} - G_1 + G_{16} = 0\]
\[-F_2 - F_{29} + F_{33} = 0\]
\[-F_4 - F_{33} + G_{14} = 0\]
\[-F_6 + F_{29} - G_{14} = 0\]
\[-F_7 - F_{28} + F_{32} - 2F_{40} = 0\]
\[-F_8 - F_{32} - 2F_{41} - G_{14} = 0\]
\[-F_9 + F_{28} - 2F_{42} + G_{14} = 0\]
\[F_{11} - F_{31} + F_{35} + 2F_{43} = 0\]
\[F_{13} + F_{31} + 2F_{44} - G_{15} = 0\]
\[F_{15} - F_{35} + 2F_{45} + G_{15} = 0\]
\[F_{16} - F_{30} + F_{34} = 0\]
\[F_{17} + F_{30} + G_{15} = 0\]
\[F_{18} - F_{34} - G_{15} = 0\]
\[F_{19} - F_{32} - G_{10} = 0\]
\[F_{21} + F_{32} - F_{36} = 0\]
\[F_{23} + F_{36} + G_{10} = 0\]
\[F_{25} - F_{33} + G_{10} = 0\]
\[F_{26} + F_{33} - F_{37} = 0\]
\[F_{27} + F_{37} - G_{10} = 0\]
\[F_{34} - F_{38} + G_{3} + 2G_{18} = 0\]
\[-F_{34} - G_2 - G_{11} + 2G_{17} = 0\]
\[F_{35} - F_{39} - G_{18} = 0\]
\[-F_{35} + G_{11} - G_{17} = 0\]
\[F_{38} - G_{1} + G_{11} + 2G_{16} = 0\]
\[F_{39} - G_{11} - G_{16} = 0\]
\[F_{20} + F_{30} + F_{39} + 2F_{46} = 0\]
\[F_{22} - F_{30} + 2F_{47} - G_{13} = 0\]
\[F_{24} - F_{39} + 2F_{48} + G_{13} = 0\]
\[F_{25} + F_{31} + F_{38} = 0\]
\[F_{26} - F_{31} + G_{13} = 0\]
\[F_{27} - F_{38} - G_{13} = 0\]
\[F_{28} + F_{37} - G_{6} = 0\]
\[-F_{28} - G_{5} - G_{12} = 0\]
\[F_{29} + F_{36} + G_{21} = 0\]
\[-F_{29} + G_{12} + G_{20} = 0\]
\[-F_{36} - G_{12} + G_{19} = 0\]
\[-F_{37} - G_{4} + G_{12} = 0\]
\[-F_{31} - F_{33} = 0\]
\[-F_{37} - G_{13} = 0\]
\[F_{38} - G_{10} = 0\]

(4.74)

We can solve this system \textit{iff} we impose

\[t_1 - 2t_2 - t_3 + t_4 = 0.\]

(4.75)
4.9 The Final Result

Collecting the results from the preceding Subsections we obtain the following result:

**Theorem 4.1** The equation

\[ D_{\text{triangle}}^{IJK}(x, y, z)_0 = (\bar{s}B)^{IJK}(x, y, z) + D_{\text{triangle}}^{IJK}(x, y, z)_\delta \] (4.76)

is true iff the following restrictions are true:

\[
\begin{align*}
    f_{abc}^{(3)} &= 6f_{abc}^{(0)} \\
    f_{abc}^{(4)} &= 8f_{abc}^{(0)} \\
    t_1 - 2t_2 - t_3 + t_4 &= 0.
\end{align*}
\] (4.77)

To complete the proof we have to prove first a similar result for the 1PR contributions, namely:

\[ D_{\text{1PR}}^{IJK}(x, y, z)_0 = (\bar{s}b)^{IJK}(x, y, z) + D_{\text{1PR}}^{IJK}(x, y, z)_\delta \] (4.78)

is true without other restrictions.

It can be seen that both delta-contributions (obtained from the triangle and 1PR contributions) are non-trivial, i.e. they cannot be made null. If we apply the relations of the type (3.33) we will get two type of terms: (a) terms with derivatives on the delta distribution and (b) terms without derivatives on the delta distribution. We can write contribution (a) as a coboundary plus a contribution (b). So in the end we have to check that the remaining contribution (b) is null. This follows by direct computations. This means that we have

**Theorem 4.2** The equation

\[ D_{\text{IJK}}^{IJK}(x, y, z)_{(1)} = (\bar{s}B)^{IJK}(x, y, z) + \text{super - renormalizable terms} \] (4.79)

is true iff the following restrictions are true:

\[
\begin{align*}
    f_{abc}^{(3)} &= 6f_{abc}^{(0)} \\
    f_{abc}^{(4)} &= 8f_{abc}^{(0)} \\
    t_1 - 2t_2 - t_3 + t_4 &= 0.
\end{align*}
\] (4.80)

The two-loop contribution can be analysed in the same way and does not bring new constraints. This is our final result.

5 Conclusions

We have proved that the super-renormalizability property is true for Yang-Mills models in the third order of the perturbation theory if we have the three relations from above. We have checked that the electro-weak sector does not fulfill them so we must look for another gauge group having two properties: it should lead to a solution of the preceding equations and it should be in agreement with the phenomenology (not very “far” from the standard model). This problem will be addressed in further publications. However, let us mention that the second relation (4.80) gives in the QCD sector that the number of colors must be 3. We must also investigate if the super-renormalizability property can be implemented in arbitrary orders of the perturbation theory and try to extend the result for gravity also.
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