Functions tiling simultaneously with two arithmetic progressions

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Funding information
ISF, Grant/Award Number: 1044/21; ERC, Grant/Award Number: 713927

Abstract
We consider measurable functions $f$ on $\mathbb{R}$ that tile simultaneously by two arithmetic progressions $\alpha \mathbb{Z}$ and $\beta \mathbb{Z}$ at respective tiling levels $p$ and $q$. We are interested in two main questions: what are the possible values of the tiling levels $p, q$, and what is the least possible measure of the support of $f$? We obtain sharp results which show that the answers depend on arithmetic properties of $\alpha, \beta$ and $p, q$, and in particular, on whether the numbers $\alpha, \beta$ are rationally independent or not.

MSC 2020
05B45, 15B51 (primary)

1 | INTRODUCTION

1.1

Let $f$ be a measurable function on $\mathbb{R}$, and $\Lambda \subset \mathbb{R}$ be a countable set. We say that the function $f$ tiles $\mathbb{R}$ at level $w$ with the translation set $\Lambda$, or that $f + \Lambda$ is a tiling of $\mathbb{R}$ at level $w$ (where $w$ is a constant), if we have

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = w \quad \text{a.e.} \quad (1.1)$$

and the series in (1.1) converges absolutely a.e.

In the same way, one can define tiling by translates of a measurable function $f$ on $\mathbb{R}^d$, or more generally, on any locally compact abelian group.

If $f = 1_\Omega$ is the indicator function of a set $\Omega$, then $f + \Lambda$ is a tiling at level one if and only if the translated copies $\Omega + \lambda, \lambda \in \Lambda$, fill the whole space without overlaps up to measure zero. To
the contrary, for tilings by a general real or complex-valued function \( f \), the translated copies may have overlapping supports.

Tilings by translates of a function have been studied by various authors, see, for example, [3–12].

1.2

By the support of a function \( f \), we shall mean the set

\[
\text{supp } f := \{ x : f(x) \neq 0 \}.
\]  

(1.2)

In [7], inspired by the Steinhaus tiling problem, the authors studied the following question: how “small” can be the support of a function \( f \) that tiles \( \mathbb{R}^d \) simultaneously by a finite number of lattices \( \Lambda_1, \ldots, \Lambda_N \)? In particular, they posed the question as to what is the least possible measure of the support of such a function \( f \).

The problem is nontrivial even in dimension one and for two lattices only. This case will be studied in the present paper. We thus consider a measurable function \( f \) on \( \mathbb{R} \) that simultaneously tiles by two arithmetic progressions \( \alpha \mathbb{Z} \) and \( \beta \mathbb{Z} \), that is,

\[
\sum_{k \in \mathbb{Z}} f(x - k\alpha) = p, \quad \sum_{k \in \mathbb{Z}} f(x - k\beta) = q \quad \text{a.e.,}
\]  

(1.3)

where \( \alpha, \beta \) are positive real numbers, the tiling levels \( p, q \) are complex numbers, and both series in (1.3) converge absolutely a.e.

It is obvious that if \( p, q \) are both nonzero, then the simultaneous tiling condition (1.3) implies that \( \text{mes}(\text{supp } f) \) can be no smaller than \( \max\{\alpha, \beta\} \). This estimate was improved for nonnegative functions \( f \) in [7, Theorem 2.6], where the authors proved that if \( 0 < \alpha < \beta \) then the tiling condition (1.3) implies that \( \text{mes}(\text{supp } f) \geq \lceil \beta / \alpha \rceil \alpha \). The authors asked in [7, Question 4] what is the least possible measure of the support of a function \( f \) satisfying (1.3). In this paper, we obtain sharp results which improve on the lower bound from [7] and provide a complete answer to this question.

1.3

Notice that if \( f \) is nonnegative, then integrating the first equality in (1.3) over the interval \([0, \alpha)\) yields \( \int_0^\alpha f = p\alpha \), so \( f \) must in fact be integrable. The same holds if \( f \) is complex valued but assumed a priori to be in \( L^1(\mathbb{R}) \). Moreover, in this case we can also integrate the second equality in (1.3) over \([0, \beta)\) and get \( \int_0^\beta f = q\beta \), hence \( p\alpha = q\beta \). This proves the following basic fact:

**Proposition 1.1.** Let \( f \) be a measurable function on \( \mathbb{R} \) assumed to be either nonnegative or in \( L^1(\mathbb{R}) \). If \( f \) satisfies (1.3) then the vector \((p, q)\) is proportional to \((\beta, \alpha)\).

The convolution \( 1_{[0,\alpha)} * 1_{[0,\beta)} \) provides a basic example of a nonnegative function \( f \) satisfying (1.3) with \((p, q) = (\beta, \alpha)\), and such that \( \text{supp } f \) is an interval of length \( \alpha + \beta \).

We are interested in the following two main questions.
(i) Do there exist tilings (1.3) such that the tiling level vector \((p, q)\) is not proportional to \((\beta, \alpha)\)? (In such a case, \(f\) can be neither nonnegative nor integrable.)

(ii) What is the least possible value of \(\text{mes}(\text{supp} \, f)\) for a function \(f\) satisfying (1.3) with a given tiling level vector \((p, q)\)?

In this paper, we answer these questions in full generality. The answers turn out to depend on arithmetic properties of \(\alpha, \beta\) and \(p, q\), and in particular, on whether the numbers \(\alpha, \beta\) are rationally independent or not. Moreover, we will see that the results differ substantially between these two cases.

2 | RESULTS

2.1

First, we consider the case where \(\alpha, \beta\) are rationally independent. In this case, our first result establishes the existence of tilings (1.3) such that the levels \(p, q\) are arbitrary complex numbers, that is, the vector \((p, q)\) is not necessarily proportional to \((\beta, \alpha)\). Moreover, we can construct such tilings with \(\text{mes}(\text{supp} \, f)\) never exceeding \(\alpha + \beta\).

Theorem 2.1. Let \(\alpha, \beta\) be rationally independent. For any two complex numbers \(p, q\) there is a measurable function \(f\) on \(\mathbb{R}\) satisfying (1.3) with \(\text{mes}(\text{supp} \, f) \leq \alpha + \beta\).

We will also prove that while the function \(f\) in Theorem 2.1 has support of finite measure, \(f\) cannot in general be supported on any bounded subset of \(\mathbb{R}\).

Theorem 2.2. Let \(f\) be a measurable function on \(\mathbb{R}\) satisfying (1.3) where \(\alpha, \beta\) are rationally independent. If the vector \((p, q)\) is not proportional to \((\beta, \alpha)\), then \(\text{supp} \, f\) must be an unbounded set.

It is obvious that the result does not hold if \((p, q) = \lambda(\beta, \alpha)\) where \(\lambda\) is a scalar, since in this case the function \(f = \lambda 1_{[0, \alpha)} \ast 1_{[0, \beta)}\) satisfies (1.3) and has bounded support.

The next result clarifies the role of the value \(\alpha + \beta\) in Theorem 2.1. It turns out that for most level vectors \((p, q)\) it is in fact the least possible value of \(\text{mes}(\text{supp} \, f)\).

Theorem 2.3. Let \(\alpha, \beta\) be rationally independent, and suppose that \((p, q)\) is not proportional to any vector of the form \((n, m)\) where \(n, m\) are nonnegative integers. If a measurable function \(f\) on \(\mathbb{R}\) satisfies (1.3) then \(\text{mes}(\text{supp} \, f) \geq \alpha + \beta\).

In particular, this result applies if \(f\) is nonnegative, or is in \(L^1(\mathbb{R})\), or has bounded support. It follows from Proposition 1.1 and Theorem 2.2 that in any one of these cases the tiling level vector \((p, q)\) must be proportional to \((\beta, \alpha)\), and as \(\alpha, \beta\) are rationally independent, \((p, q)\) cannot therefore be proportional to any integer vector \((n, m)\) unless \(p, q\) are both zero. So, we obtain:

Corollary 2.4. Assume that a measurable function \(f\) on \(\mathbb{R}\) is nonnegative, or is in \(L^1(\mathbb{R})\), or has bounded support. If \(\alpha, \beta\) are rationally independent and (1.3) holds for some nonzero vector \((p, q)\), then \((p, q)\) is proportional to \((\beta, \alpha)\) and \(\text{mes}(\text{supp} \, f) \geq \alpha + \beta\).
We thus obtain that for rationally independent $\alpha, \beta$, the convolution $1_{[0,\alpha)} * 1_{[0,\beta)}$ is a function minimizing the value of $\text{mes}(\text{supp } f)$ among all nonnegative, or all integrable, or all boundedly supported, functions $f$ satisfying (1.3) for some nonzero tiling level vector $(p, q)$.

2.2

We now consider the remaining case not covered by Theorem 2.3, namely, the case where the tiling level vector $(p, q)$ is proportional to some vector $(n, m)$ such that $n, m$ are nonnegative integers. By multiplying the vector $(p, q)$ on an appropriate scalar we may suppose that $p, q$ are by themselves nonnegative integers, and by factoring out their greatest common divisor we may also assume $p, q$ to be coprime.

Interestingly, it turns out that in this case the measure of $\text{supp } f$ can drop below $\alpha + \beta$, in a magnitude that depends on the specific values of the tiling levels $p$ and $q$.

**Theorem 2.5.** Let $\alpha, \beta$ be rationally independent, and let $p, q$ be two positive coprime integers. For any $\varepsilon > 0$ there is a measurable function $f$ on $\mathbb{R}$ satisfying (1.3) such that

$$\text{mes} (\text{supp } f) < \alpha + \beta - \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\} + \varepsilon. \quad (2.1)$$

The next result shows that the upper estimate (2.1) is actually sharp.

**Theorem 2.6.** Let $f$ be a measurable function on $\mathbb{R}$ satisfying (1.3) where $\alpha, \beta$ are rationally independent and $p, q$ are positive, coprime integers. Then

$$\text{mes} (\text{supp } f) > \alpha + \beta - \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}. \quad (2.2)$$

The last two results yield that if the tiling levels $p, q$ are positive, coprime integers, then the right-hand side of (2.2) is the infimum of the values of $\text{mes}(\text{supp } f)$ over all measurable functions $f$ satisfying (1.3), but this infimum cannot be attained.

In Theorems 2.5 and 2.6, the tiling levels $p, q$ are assumed to be both nonzero, which does not cover the case where $(p, q) = (1, 0)$ or $(0,1)$. The following result provides the sharp answer in this last case. By symmetry, it is enough to consider $(p, q) = (1, 0)$.

**Theorem 2.7.** Let $\alpha, \beta$ be rationally independent, and let $(p, q) = (1, 0)$. For any $\varepsilon > 0$ there is a measurable function $f$ on $\mathbb{R}$ satisfying (1.3) such that $\text{mes}(\text{supp } f) < \alpha + \varepsilon$. Conversely, any measurable $f$ satisfying (1.3) must have $\text{mes}(\text{supp } f) > \alpha$.

The results above thus fully resolve the problem for rationally independent $\alpha, \beta$.

2.3

We now move on to deal with the other case where $\alpha, \beta$ are linearly dependent over the rationals. Then the vector $(\alpha, \beta)$ is proportional to some vector $(n, m)$ such that $n, m$ are positive integers. By rescaling, it is enough to consider the case $(\alpha, \beta) = (n, m)$ where $n, m$ are positive integers.
The tiling condition (1.3) thus takes the form

\[
\sum_{k \in \mathbb{Z}} f(x - kn) = p, \quad \sum_{k \in \mathbb{Z}} f(x - km) = q \quad \text{a.e.,}
\]

where \( n, m \) are positive integers, \( p, q \) are complex numbers, and both series in (2.3) converge absolutely a.e.

In this case, our first result shows that the tiling levels \( p, q \) cannot be arbitrary.

**Theorem 2.8.** Let \( n, m \) be positive integers, and let \( f \) be a measurable function on \( \mathbb{R} \) satisfying (2.3). Then the vector \( (p, q) \) must be proportional to \( (m, n) \).

This is not quite obvious since \( f \) is neither assumed to be nonnegative nor in \( L^1(\mathbb{R}) \), so the conclusion does not follow from Proposition 1.1. Moreover, Theorem 2.8 is in sharp contrast to Theorem 2.1 which states that for rationally independent \( \alpha, \beta \) there exist tilings (1.3) such that the levels \( p, q \) are arbitrary complex numbers.

The next result gives a lower bound for the support size of a function \( f \) that satisfies the simultaneous tiling condition (2.3) with a nonzero tiling level vector \( (p, q) \).

**Theorem 2.9.** Let \( f \) be a measurable function on \( \mathbb{R} \) satisfying (2.3) where \( n, m \) are positive integers and the vector \( (p, q) \) is nonzero. Then

\[
\text{mes}(\text{supp} f) \geq n + m - \gcd(n, m).
\]

We will also establish that in fact the lower bound in Theorem 2.9 is sharp. Due to Theorem 2.8, it suffices to prove this for the tiling level vector \( (p, q) = (m, n) \).

**Theorem 2.10.** Let \( n, m \) be positive integers, and let \( (p, q) = (m, n) \). Then there is a nonnegative, measurable function \( f \) on \( \mathbb{R} \) satisfying (2.3) and such that \( \text{supp} f \) is an interval of length \( n + m - \gcd(n, m) \).

It follows that \( n + m - \gcd(n, m) \) is the least possible value of \( \text{mes}(\text{supp} f) \) among all measurable functions \( f \) satisfying (2.3) with a nonzero tiling level vector \( (p, q) \). In particular, the convolution \( 1_{[0,n]} \ast 1_{[0,m]} \) (whose support is an interval of length \( n + m \)) does not attain the least possible value of \( \text{mes}(\text{supp} f) \).

The results obtained thus answer the questions above in full generality.

**Remark 2.11.** We note that the case where the tiling levels \( p, q \) are both zero is trivial, since then the zero function \( f \) satisfies (1.3). It is also easy to construct examples where \( \text{supp} f \) has positive but arbitrarily small measure. For example, let \( h \) be any function with \( \text{supp} h = (0, \varepsilon) \), then the function \( f(x) = h(x) - h(x + \alpha) - h(x + \beta) + h(x + \alpha + \beta) \) satisfies (1.3) with \( p, q \) both zero and \( \text{supp} f \) has positive measure not exceeding \( 4\varepsilon \).

### 2.4

The rest of the paper is organized as follows.
In Section 3, we give a short preliminary background and fix notation that will be used throughout the paper.

In Section 4, we prove Theorems 2.1, 2.5, and 2.7, that is, for any two rationally independent \( \alpha, \beta \) and for any tiling level vector \( (p, q) \), we construct a simultaneous tiling (1.3) such that \( \text{mes}(\text{supp} f) \) is minimal, or is arbitrarily close to the infimum.

In Section 5, we prove that if a measurable function \( f \) satisfies the simultaneous tiling condition (1.3) with a tiling level vector \( (p, q) \) that is not proportional to \( (\beta, \alpha) \), then \( \text{supp} f \) must be an unbounded set (Theorem 2.2).

In Section 6, we solve a problem posed to us by Kolountzakis, asking whether there exists a bounded measurable function \( f \) on \( \mathbb{R} \) that tiles simultaneously with rationally independent \( \alpha, \beta \) and with arbitrary tiling levels \( p, q \). We prove that the answer is affirmative, and moreover, \( f \) can be chosen continuous and vanishing at infinity.

In Section 7, we prove Theorems 2.3 and 2.6 that give sharp lower bounds for the measure of \( \text{supp} f \), where \( f \) is any measurable function satisfying the simultaneous tiling condition (1.3) with rationally independent \( \alpha, \beta \).

In the last Section 8, we consider the case where the two numbers \( \alpha, \beta \) are linearly dependent over the rationals. By rescaling, we may assume that \( \alpha, \beta \) are two positive integers \( n, m \). We prove Theorems 2.8, 2.9, and 2.10 using a reduction of the simultaneous tiling problem from the real line \( \mathbb{R} \) to the set of integers \( \mathbb{Z} \).

### 3 Preliminaries and Notation

In this section, we give a short preliminary background and fix notation that will be used throughout the paper.

If \( \alpha \) is a positive real number, then we use \( \mathbb{T}_\alpha \) to denote the circle group \( \mathbb{R}/\alpha\mathbb{Z} \). We let \( \pi_\alpha \) denote the canonical projection map \( \mathbb{R} \to \mathbb{T}_\alpha \). The Lebesgue measure on the group \( \mathbb{T}_\alpha \) is normalized such that \( \text{mes}(\mathbb{T}_\alpha) = \alpha \).

We use \( m(E) \), or \( \text{mes}(E) \), to denote the Lebesgue measure of a set \( E \) in either the real line \( \mathbb{R} \) or the circle \( \mathbb{T}_\alpha \).

If \( \alpha, \beta \) are two positive real numbers, then they are said to be rationally independent if the condition \( n\alpha + m\beta = 0, \, n, m \in \mathbb{Z} \), implies that \( n = m = 0 \). This is the case if and only if the ratio \( \alpha/\beta \) is an irrational number.

By the classical Kronecker’s theorem, if two positive real numbers \( \alpha, \beta \) are rationally independent then the sequence \( \{\pi_\alpha(n\beta)\}, \, n = 1, 2, 3, \ldots \), is dense in \( \mathbb{T}_\alpha \).

Let \( f \) be a measurable function on \( \mathbb{R} \), and suppose that the series

\[
\sum_{k \in \mathbb{Z}} f(x - k\alpha)
\]

converges absolutely for every \( x \in \mathbb{R} \). Then the sum (3.1) is an \( \alpha \)-periodic function of \( x \), so it can be viewed as a function on \( \mathbb{T}_\alpha \). We denote this function by \( \pi_\alpha(f) \). If the sum (3.1) converges absolutely not everywhere but almost everywhere, then the function \( \pi_\alpha(f) \) is defined in a similar way on a full measure subset of \( \mathbb{T}_\alpha \).

We observe that the simultaneous tiling condition (1.3) can be equivalently stated as the requirement that \( \pi_\alpha(f) = p \) a.e. on \( \mathbb{T}_\alpha \), and that \( \pi_\beta(f) = q \) a.e. on \( \mathbb{T}_\beta \).
If \( f \) is in \( L^1(\mathbb{R}) \), then the sum (3.1) converges absolutely almost everywhere, and moreover, the function \( \pi_\alpha(f) \) is in \( L^1(\mathbb{T}_\alpha) \) and satisfies \( \int_{\mathbb{T}_\alpha} \pi_\alpha(f) = \int_{\mathbb{R}} f \).

The set \( \text{supp } f := \{x : f(x) \neq 0\} \) will be called the support of the function \( f \). If we have \( \text{supp } f \subset \Omega \) then we will say that \( f \) is supported on \( \Omega \).

We observe that if \( \text{supp } f \) is a set of finite measure in \( \mathbb{R} \), then in the sum (3.1) there are only finitely many nonzero terms for almost every \( x \in \mathbb{R} \), which implies that the function \( \pi_\alpha(f) \) is well-defined on a full measure subset of \( \mathbb{T}_\alpha \).

4 | INCOMMENSURABLE ARITHMETIC PROGRESSIONS: CONSTRUCTING SIMULTANEOUSLY TILING FUNCTIONS WITH SMALL SUPPORT

In this section we prove Theorems 2.1, 2.5, and 2.7, that is, for any two rationally independent \( \alpha, \beta \) and for any tiling level vector \( (p, q) \), we construct a simultaneous tiling (1.3) such that \( \text{mes}(\text{supp } f) \) is minimal, or is arbitrarily close to the infimum.

Throughout this section, we shall suppose that \( \alpha, \beta > 0 \) are two fixed, rationally independent real numbers.

4.1

It will be convenient to introduce the following terminology:

**Definition 4.1.** By an elementary set (in either \( \mathbb{R} \), \( \mathbb{T}_\alpha \) or \( \mathbb{T}_\beta \)) we mean a set that can be represented as the union of finitely many disjoint closed intervals of finite length.

We will use \( \text{int}(U) \) to denote the interior of an elementary set \( U \).

**Lemma 4.2.** Let \( A \) be an elementary set in \( \mathbb{T}_\alpha \). Then given any nonempty open interval \( J \subset \mathbb{T}_\beta \), no matter how small, one can find an elementary set \( U \subset \mathbb{R} \) such that

(i) \( \pi_\alpha(U) = A \);
(ii) \( \pi_\alpha \) is one-to-one on \( \text{int}(U) \);
(iii) \( \pi_\beta(U) \subset J \).

Moreover, \( U \) can be chosen inside the half-line \((r, +\infty)\) for any given number \( r \).

**Proof.** We choose \( \delta > 0 \) smaller than both the length of \( J \) and \( \alpha \), and we decompose the elementary set \( A \) as a union \( A = A_1 \cup \cdots \cup A_n \), where each \( A_j \) is a closed interval in \( \mathbb{T}_\alpha \) of length smaller than \( \delta \), and \( A_1, \ldots, A_n \) have disjoint interiors. Let \( U_j \) be a closed interval in \( \mathbb{R} \) such that \( A_j \) is a one-to-one image of \( U_j \) under \( \pi_\alpha \). By translating the sets \( U_j \) by appropriate integer multiples of \( \alpha \) we can ensure that \( \pi_\beta(U_j) \subset J \) (due to Kronecker’s theorem, as \( \alpha, \beta \) are rationally independent), and that the sets \( U_1, \ldots, U_n \) are pairwise disjoint and all of them are contained in a given half-line \((r, +\infty)\). Then the set \( U := U_1 \cup \cdots \cup U_n \) is an elementary set contained in \((r, +\infty)\) and satisfying the properties (i), (ii), and (iii) above. \( \square \)
Lemma 4.3. Let $A \subset \mathbb{T}_\alpha$ be an elementary set, and $\varphi$ be a measurable function on $A$. Given any nonempty open interval $J \subset \mathbb{T}_\beta$, one can find an elementary set $U \subset \mathbb{R}$ and a measurable function $f$ on $\mathbb{R}$, such that

(i) $\pi_\alpha(U) = A$;
(ii) $\pi_\beta(U) \subset J$;
(iii) $m(U) = m(A)$;
(iv) $f$ is supported on $U$;
(v) $\pi_\alpha(f) = \varphi$ a.e. on $A$.

Moreover, $U$ can be chosen inside the half-line $(r, +\infty)$ for any given number $r$.

Proof. Use Lemma 4.2 to find an elementary set $U \subset \mathbb{R}$ such that $\pi_\alpha(U) = A$, $\pi_\alpha$ is one-to-one on $\text{int}(U)$, and $\pi_\beta(U) \subset J$. Notice that the first two properties imply that $m(U) = m(A)$. Recall also that Lemma 4.2 allows us to choose the set $U$ inside any given half-line $(r, +\infty)$. We define a function $f$ on $\mathbb{R}$ by $f(x) := \varphi(\pi_\alpha(x))$ for $x \in \text{int}(U)$, and $f(x) = 0$ outside of $\text{int}(U)$. Then $f$ is a measurable function supported on $U$. As $\pi_\alpha$ is one-to-one on $\text{int}(U)$, we have $\pi_\alpha(f) = \varphi$ on the set $\pi_\alpha(\text{int}(U))$, a full measure subset of $A$. The properties (i)–(v) are thus satisfied and the claim is proved.

4.2

The next lemma incorporates a central idea of our tiling construction technique. Roughly speaking, the lemma asserts that one can find a function $f$ on $\mathbb{R}$ with prescribed projections $\pi_\alpha(f)$ and $\pi_\beta(f)$, and that, moreover, $\text{mes}(\text{supp } f)$ need never exceed the total measure of the supports of the projections.

Lemma 4.4. Suppose that we are given two elementary sets $A \subset \mathbb{T}_\alpha$, $B \subset \mathbb{T}_\beta$, both of positive measure, as well as two measurable functions $\varphi$ on $A$, and $\psi$ on $B$. Then there is a closed set $\Omega \subset \mathbb{R}$ (a union of countably many disjoint closed intervals accumulating at $+\infty$) and a measurable function $f$ supported on $\Omega$, such that

(i) $m(\Omega) = m(A) + m(B)$ (in particular, $\Omega$ has finite measure);
(ii) $\pi_\alpha(\Omega) = A$, $\pi_\alpha(f) = \varphi$ a.e. on $A$;
(iii) $\pi_\beta(\Omega) = B$, $\pi_\beta(f) = \psi$ a.e. on $B$.

Moreover, $\Omega$ can be chosen inside the half-line $(r, +\infty)$ for any given number $r$.

Proof. We choose an arbitrary decomposition of the set $A$ as a union $A = \bigcup_{k=1}^{\infty} A_k$, where each $A_k \subset \mathbb{T}_\alpha$ is an elementary set and the sets $A_1, A_2, ...$ have nonempty and disjoint interiors. We do the same also for the set $B$, that is, we let $B = \bigcup_{k=1}^{\infty} B_k$, where the $B_k$ are elementary sets in $\mathbb{T}_\beta$ with nonempty, disjoint interiors.

Now, we apply Lemma 4.3 to the elementary set $A_1$, the function $\varphi$, and an arbitrary nonempty open interval $J \subset B$. We obtain from the lemma an elementary set $U_1 \subset \mathbb{R}$ and a measurable function $g_1$ on $\mathbb{R}$, satisfying the conditions $\pi_\alpha(U_1) = A_1$, $\pi_\beta(U_1) \subset B$, $m(U_1) = m(A_1)$, the function $g_1$ is supported on $U_1$, and $\pi_\alpha(g_1) = \varphi$ a.e. on $A_1$.

Next, we apply Lemma 4.3 again but with the roles of $\alpha, \beta$ interchanged, to the elementary set $B_1$, the function $\psi - \pi_\beta(g_1)$, and an arbitrary nonempty open interval $J \subset A \setminus A_1$. The lemma
yields an elementary set $V_1 \subset \mathbb{R}$ and a measurable function $h_1$ on $\mathbb{R}$, such that $\pi_\alpha(V_1) = B_1$, $\pi_\alpha(V_1) \subset A \setminus A_1$, $m(V_1) = m(B_1)$, the function $h_1$ is supported on $V_1$, and $\pi_\beta(h_1) = \psi - \pi_\beta(g_1)$ a.e. on $B_1$.

We continue the construction in a similar fashion. Suppose that we have already constructed the sets $U_k, V_k$ and the functions $g_k, h_k$ for $1 \leq k \leq n - 1$. Using Lemma 4.3, we find an elementary set $U_n \subset \mathbb{R}$ and a measurable function $g_n$ on $\mathbb{R}$, such that $\pi_\alpha(U_n) = A_n$, $\pi_\beta(U_n) \subset B \setminus \bigcup_{k=1}^{n-1} B_k$, $m(U_n) = m(A_n)$, $g_n$ is supported on $U_n$, and

$$\pi_\alpha(g_n) = \varphi - \sum_{k=1}^{n-1} \pi_\alpha(h_k) \quad \text{a.e. on } A_n. \quad (4.1)$$

Then, we use again Lemma 4.3 to find an elementary set $V_n \subset \mathbb{R}$ and a measurable function $h_n$ on $\mathbb{R}$, such that $\pi_\beta(V_n) = B_n$, $\pi_\alpha(V_n) \subset A \setminus \bigcup_{k=1}^{n} A_k$, $m(V_n) = m(B_n)$, $h_n$ is supported on $V_n$, and

$$\pi_\beta(h_n) = \psi - \sum_{k=1}^{n} \pi_\beta(g_k) \quad \text{a.e. on } B_n. \quad (4.2)$$

We may assume that the sets $U_1, V_1, U_2, V_2, \ldots$ are pairwise disjoint, that all of them lie inside a given half-line $(r, +\infty)$, and that they accumulate at $+\infty$. Indeed, Lemma 4.3 allows us to choose the sets such that they satisfy these properties. (In fact, one can check that by their construction the sets necessarily have disjoint interiors.)

Finally, we define

$$\Omega := \bigcup_{n=1}^{\infty} (U_n \cup V_n), \quad f := \sum_{n=1}^{\infty} (g_n + h_n). \quad (4.3)$$

The sum on the right-hand side of $(4.3)$ is well-defined as the terms in the series have disjoint supports. We will show that the properties (i), (ii), and (iii) are satisfied.

We begin by verifying that (i) holds. Indeed, $m(U_n) = m(A_n)$, $m(V_n) = m(B_n)$, and the sets $U_1, V_1, U_2, V_2, \ldots$ are disjoint. It follows that

$$m(\Omega) = \sum_{n=1}^{\infty} (m(U_n) + m(V_n)) = \sum_{n=1}^{\infty} (m(A_n) + m(B_n)) = m(A) + m(B). \quad (4.4)$$

Next we verify that (ii) is satisfied. Indeed, we have $\pi_\alpha(U_n) = A_n$ and $\pi_\alpha(V_n) \subset A$ for every $n$, hence $\pi_\alpha(\Omega) = A$. We must show that also $\pi_\alpha(f) = \varphi$ a.e. on $A$. It would suffice to verify that this holds on each $A_n$. Notice that $\pi_\alpha(\text{int}(U_k))$ is disjoint from $A_n$ for $k \neq n$, and $\pi_\alpha(V_k)$ is disjoint from $A_n$ for $k \geq n$. Hence, using $(4.1)$ this implies that

$$\pi_\alpha(f) = \pi_\alpha(g_n) + \sum_{k=1}^{n-1} \pi_\alpha(h_k) = \varphi \quad \text{a.e. on } A_n. \quad (4.5)$$

In a similar way, we can show that (iii) holds as well. We have $\pi_\beta(V_n) = B_n$ and $\pi_\beta(U_n) \subset B$ for every $n$, hence $\pi_\beta(\Omega) = B$. To see that $\pi_\beta(f) = \psi$ a.e. on $B$, we verify that this is the case on each $B_n$. But $\pi_\beta(\text{int}(V_k))$ is disjoint from $B_n$ for $k \neq n$, and $\pi_\beta(U_k)$ is disjoint from $B_n$ for $k \geq n + 1$. 

Hence, (4.2) implies that
\[ \pi_\beta(f) = \pi_\beta(h_n) + \sum_{k=1}^{n} \pi_\beta(g_k) = \psi \quad \text{a.e. on } B_n. \]  
(4.6)

Thus, all the properties (i), (ii), and (iii) are satisfied and Lemma 4.4 is proved.

\[ \square \]

4.3

We can now use Lemma 4.4 in order to prove Theorems 2.1 and 2.7.

**Proof of Theorem 2.1.** Let \( p, q \) be any two complex numbers. Apply Lemma 4.4 to the sets \( A = \mathbb{T}_\alpha \), \( B = \mathbb{T}_\beta \), and to the constant functions \( \varphi = p, \psi = q \). The lemma yields a measurable function \( f \) on \( \mathbb{R} \), supported on a set \( \Omega \subset \mathbb{R} \) of measure \( \alpha + \beta \), and such that \( \pi_\alpha(f) = p \) a.e. on \( \mathbb{T}_\alpha \), while \( \pi_\beta(f) = q \) a.e. on \( \mathbb{T}_\beta \), that is, \( f \) satisfies the tiling condition (1.3). The theorem is thus proved.

\[ \square \]

**Proof of Theorem 2.7.** Let \( (p, q) = (1, 0) \). Given \( \varepsilon > 0 \), we apply Lemma 4.4 to the sets \( A = \mathbb{T}_\alpha \) and \( B = [0, \varepsilon] \subset \mathbb{T}_\beta \), and to the functions \( \varphi = 1, \psi = 0 \). The lemma yields a set \( \Omega \subset \mathbb{R} \) satisfying \( m(\Omega) = \alpha + \varepsilon, \pi_\beta(\Omega) = B \), as well as a measurable function \( f \) supported on \( \Omega \) and such that \( \pi_\alpha(f) = 1 \) a.e. on \( \mathbb{T}_\alpha \), and \( \pi_\beta(f) = 0 \) a.e. on \( \mathbb{T}_\beta \). Notice though that the condition \( \pi_\beta(\Omega) = B \) ensures that \( \pi_\beta(f) = 0 \) a.e. also on \( \mathbb{T}_\beta \setminus B \). Hence, the tiling condition (1.3) is satisfied. This proves one part of the theorem.

To prove the converse part, we suppose that \( f \) is a measurable function on \( \mathbb{R} \) satisfying (1.3) with \( (p, q) = (1, 0) \), that is, \( \pi_\alpha(f) = 1 \) a.e. on \( \mathbb{T}_\alpha \) and \( \pi_\beta(f) = 0 \) a.e. on \( \mathbb{T}_\beta \). It follows from the first assumption that the set \( \Omega := \text{supp} f \) has measure at least \( \alpha \), as \( \pi_\alpha(\Omega) \) is a set of full measure in \( \mathbb{T}_\alpha \). We must show that actually \( m(\Omega) > \alpha \). Suppose to the contrary that \( m(\Omega) = \alpha \). Then \( \pi_\alpha(\Omega) \) cannot be a set of full measure in \( \mathbb{T}_\alpha \) unless \( \pi_\alpha \) is one-to-one on a full measure subset of \( \Omega \). But then the assumption that \( \pi_\alpha(f) = 1 \) a.e. on \( \mathbb{T}_\alpha \) implies that \( f = 1 \) a.e. on its support \( \Omega \), which in turn contradicts our second assumption that \( \pi_\beta(f) = 0 \) a.e. on \( \mathbb{T}_\beta \). Hence, we must have \( m(\Omega) > \alpha \), and so the second part of the theorem is also proved.

\[ \square \]

4.4

Next we turn to prove Theorem 2.5. The proof will require the following notion:

**Definition 4.5.** An \( n \times m \) matrix \( M = (c_{ij}) \) is called a **doubly stochastic array** (with uniform marginals) if the entries \( c_{ij} \) are nonnegative, and
\[ \sum_{j=1}^{m} c_{ij} = m, \quad 1 \leq i \leq n, \]
(4.7)
\[ \sum_{i=1}^{n} c_{ij} = n, \quad 1 \leq j \leq m, \]
(4.8)
that is, the sum of the entries at each row is \( m \) and at each column is \( n \).
By the support of the matrix $M = (c_{ij})$, we refer to the set

$$\text{supp } M = \{(i, j) : c_{ij} \neq 0\}.$$ 

In [7, Question 7], the authors posed the following question, which arose in connection with the simultaneous tiling problem in finite abelian groups: what is the least possible size of the support of a doubly stochastic $n \times m$ array?

This problem was solved recently in [13] and independently in [2].

**Theorem 4.6 ([2, 13]).** For all $n, m$ the minimal size of the support of an $n \times m$ doubly stochastic array is equal to $n + m - \gcd(n, m)$.

In particular, there exists an $n \times m$ doubly stochastic array whose support size is as small as $n + m - \gcd(n, m)$. We will use this fact in the proof of Lemma 4.8 below.

### 4.5

**Lemma 4.7.** Let $p, q$ be two positive integers, and $0 < \gamma < \min\{\alpha q^{-1}, \beta p^{-1}\}$. Then there is a system $\{L_{ij}\}$, $1 \leq i \leq p$, $1 \leq j \leq q$, of disjoint closed intervals in $\mathbb{R}$, with the following properties.

1. (i) Each interval $L_{ij}$ has length $\gamma$.
2. (ii) $\pi_{\alpha}(L_{ij})$ is an interval $I_j \subset \mathbb{T}_{\alpha}$ that does not depend on $i$.
3. (iii) $\pi_{\beta}(L_{ij})$ is an interval $J_i \subset \mathbb{T}_{\beta}$ that does not depend on $j$.
4. (iv) $I_1, ..., I_q$ are disjoint closed intervals in $\mathbb{T}_{\alpha}$.
5. (v) $J_1, ..., J_p$ are disjoint closed intervals in $\mathbb{T}_{\beta}$.

**Proof.** We choose integers $m_1, ..., m_q$ such that the intervals $I_j := [m_j \beta, m_j \beta + \gamma]$, $1 \leq j \leq q$, are disjoint in $\mathbb{T}_{\alpha}$. This is possible due to Kronecker’s theorem, as $\alpha, \beta$ are rationally independent and $q \gamma < \alpha$. As we also have $p \gamma < \beta$, we can find in a similar way integers $n_1, ..., n_p$ such that the intervals $J_i := [n_i \alpha, n_i \alpha + \gamma]$, $1 \leq i \leq p$, are disjoint in $\mathbb{T}_{\beta}$. We then define the intervals $L_{ij} \subset \mathbb{R}$ by

$$L_{ij} := [0, \gamma] + n_i \alpha + m_j \beta, \quad 1 \leq i \leq p, \ 1 \leq j \leq q.$$  

Then each interval $L_{ij}$ has length $\gamma$, and we have $\pi_{\alpha}(L_{ij}) = I_j$, $\pi_{\beta}(L_{ij}) = J_i$, so all the required properties (i)–(v) are satisfied.

Finally, we show that the intervals $L_{ij}$ must be disjoint. Indeed, suppose that two intervals $L_{i_1,j_1}$ and $L_{i_2,j_2}$ have a point $x$ in common. Then, on one hand, from (ii) we obtain $\pi_{\alpha}(x) \in I_{j_1} \cap I_{j_2}$, which in turn using (iv) implies that $j_1 = j_2$. On the other hand, by (iii) we also have $\pi_{\beta}(x) \in J_{i_1} \cap J_{i_2}$, and hence $i_1 = i_2$, which now follows from (v). Hence, the intervals $L_{i_1,j_1}$ and $L_{i_2,j_2}$ cannot intersect unless $(i_1, j_1) = (i_2, j_2)$.

**Lemma 4.8.** Let $p, q$ be two positive coprime integers, and $0 < \gamma < \min\{\alpha q^{-1}, \beta p^{-1}\}$. Then there is an elementary set $\Omega \subset \mathbb{R}$ and a measurable function $f$ supported on $\Omega$, such that

1. (i) $m(\Omega) = (p + q - 1)\gamma$;
2. (ii) the set $A = \pi_{\alpha}(\Omega)$ in $\mathbb{T}_{\alpha}$ has measure $q \gamma$;
3. (iii) the set $B = \pi_{\beta}(\Omega)$ in $\mathbb{T}_{\beta}$ has measure $p \gamma$;

---

`FUNCTIONSTILINGSIMULTANEOUSLYWITHTWOARITHMETICPROGRESSIONS`
(iv) \( \pi_\alpha(f) = p \) a.e. on \( A \);
(v) \( \pi_\beta(f) = q \) a.e. on \( B \).

It is instructive to compare this result with Lemma 4.4. Recall that the sets \( A, B \) in Lemma 4.4 can be any two elementary subsets of \( \mathbb{T}_\alpha \) and \( \mathbb{T}_\beta \), respectively, and that the projections \( \pi_\alpha(f), \pi_\beta(f) \) can be any two measurable functions on \( A \) and \( B \) respectively, but the measure of the support \( \Omega \) must in general be as large as the sum of \( m(A) \) and \( m(B) \). To the contrary, in Lemma 4.8 we are able to reduce the measure of the support \( \Omega \) to be strictly smaller than the sum of \( m(A) \) and \( m(B) \).

Proof of Lemma 4.8. Let \( \{L_{ij}\}, 1 \leq i \leq p, 1 \leq j \leq q \), be the system of disjoint closed intervals given by Lemma 4.7. We use Theorem 4.6 to find a \( p \times q \) doubly stochastic array \( M = (c_{ij}) \), whose support is of size \( p + q - 1 \) (which is the smallest possible size as \( p, q \) are coprime). We define a function \( f \) on \( \mathbb{R} \) by \( f(x) = c_{ij} \) for \( x \in L_{ij}, 1 \leq i \leq p, 1 \leq j \leq q \), and \( f(x) = 0 \) if \( x \) does not lie in any one of the intervals \( L_{ij} \).

Let \( \Omega \) be the support of the function \( f \). Then \( \Omega \) is the union of those intervals \( L_{ij} \) for which \( (i, j) \in \text{supp} M \). As \( |\text{supp} M| = p + q - 1 \), and as the intervals \( L_{ij} \) are disjoint and have length \( \gamma \), it follows that \( m(\Omega) = (p + q - 1)\gamma \).

Recall that \( \pi_\alpha(L_{ij}) \) is a closed interval \( I_j \subset \mathbb{T}_\alpha \) not depending on \( i \), and the intervals \( I_1, \ldots, I_q \) are disjoint. Let \( A = I_1 \cup \cdots \cup I_q \), then \( A \) has measure \( q\gamma \). We show that \( \pi_\alpha(f) = p \) a.e. on \( A \). It would suffice to verify that this holds on each one of the intervals \( I_j \). Indeed, as the sum of the entries of the matrix \( M \) at the \( j \)th column is \( p \), we have

\[
\pi_\alpha(f) = \sum_{i=1}^{p} c_{ij} = p \quad \text{on } I_j. \tag{4.10}
\]

Next, in a similar way, we let \( B = J_1 \cup \cdots \cup J_p \), then \( B \) has measure \( p\gamma \). We show that \( \pi_\beta(f) = q \) a.e. on \( B \), by checking that this holds on each \( J_i \). And indeed, this time due to the sum of the entries of \( M \) at the \( i \)th row being \( q \), we get

\[
\pi_\beta(f) = \sum_{j=1}^{q} c_{ij} = q \quad \text{on } J_i. \tag{4.11}
\]

Finally, as \( p \) is nonzero, it follows from (4.10) that \( \pi_\alpha(f) \) does not vanish on \( A \), hence \( \pi_\alpha(\Omega) \) must cover \( A \). But \( \pi_\alpha(\Omega) \) is a subset of \( A \), so we get \( \pi_\alpha(\Omega) = A \). Similarly, also \( q \) is nonzero, so (4.11) implies that \( \pi_\beta(\Omega) \) covers \( B \), and as \( \pi_\beta(\Omega) \) is also a subset of \( B \) we conclude that \( \pi_\beta(\Omega) = B \). The lemma is thus proved.

4.6

Now we are able to prove Theorem 2.5 based on the results above.

Proof of Theorem 2.5. Let \( p, q \) be two positive coprime integers, and denote

\[
\sigma := \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}. \tag{4.12}
\]
Given $\varepsilon > 0$, we choose a number $\gamma$ such that $\sigma - \varepsilon < \gamma < \sigma$ (we can assume that $\varepsilon$ is smaller than $\sigma$). We then use Lemma 4.8 to obtain an elementary set $\Omega_1 \subset \mathbb{R}$ of measure $(p + q - 1)\gamma$, and a measurable function $f_1$ supported on $\Omega_1$, such that the elementary set $A_1 := \pi_\alpha(\Omega_1)$ has measure $q\gamma$, the elementary set $B_1 := \pi_\beta(\Omega_1)$ has measure $p\gamma$, and such that $\pi_\alpha(f_1) = p$ a.e. on $A_1$, while $\pi_\beta(f_1) = q$ a.e. on $B_1$.

Next, we apply Lemma 4.4 to the two elementary sets $A_2 := \mathbb{T}_\alpha \setminus \text{int}(A_1)$, $B_2 := \mathbb{T}_\beta \setminus \text{int}(B_1)$, and to the constant functions $\varphi = p$, $\psi = q$. The lemma allows us to find a set $\Omega_2 \subset \mathbb{R}$ and a measurable function $f_2$ supported on $\Omega_2$, such that $\pi_\alpha(\Omega_2) = A_2$, $\pi_\alpha(f_2) = p$ a.e. on $A_2$, $\pi_\beta(\Omega_2) = B_2$, $\pi_\beta(f_2) = q$ a.e. on $B_2$, and

$$m(\Omega_2) = m(A_2) + m(B_2) = (\alpha - q\gamma) + (\beta - p\gamma) = \alpha + \beta - (p + q)\gamma. \quad (4.13)$$

The lemma also allows us to choose $\Omega_2$ to be disjoint from $\Omega_1$.

We now define

$$\Omega := \Omega_1 \cup \Omega_2, \quad f := f_1 + f_2. \quad (4.14)$$

Then $f$ is supported by $\Omega$. As $\Omega_1$ and $\Omega_2$ are disjoint, we have

$$m(\Omega) = m(\Omega_1) + m(\Omega_2) = \alpha + \beta - \gamma. \quad (4.15)$$

But recall that we have chosen $\gamma$ such that $\gamma > \sigma - \varepsilon$, hence (4.15) yields that $\text{mes}(\text{supp } f) < \alpha + \beta - \sigma + \varepsilon$. That is, the condition (2.1) is satisfied.

We must verify that $f$ satisfies the tiling condition (1.3). We first show that $\pi_\alpha(f) = p$ a.e. on $\mathbb{T}_\alpha$. Indeed, we have $\pi_\alpha(\Omega_1) = A_1$, $\pi_\alpha(\Omega_2) = A_2$, where $A_1, A_2$ have disjoint interiors and their union is the whole $\mathbb{T}_\alpha$. Moreover, $\pi_\alpha(f) = \pi_\alpha(f_1) = p$ a.e. on $A_1$, and $\pi_\alpha(f) = \pi_\alpha(f_2) = p$ a.e. on $A_2$, which proves the claim.

Finally, we show that also $\pi_\beta(f) = q$ a.e. on $\mathbb{T}_\beta$. In a similar way, we have $\pi_\beta(\Omega_1) = B_1$, $\pi_\beta(\Omega_2) = B_2$, and $B_1, B_2$ have disjoint interiors and their union is $\mathbb{T}_\beta$. As before, we have $\pi_\beta(f) = \pi_\beta(f_1) = q$ a.e. on $B_1$, and $\pi_\beta(f) = \pi_\beta(f_2) = q$ a.e. on $B_2$. This shows that the tiling condition (1.3) indeed holds and thus the theorem is proved.

\[\Box\]

4.7 Remarks

(1) Let us say that a measurable function $f$ on $\mathbb{R}$ is piecewise constant if there is a strictly increasing real sequence $\{\lambda_n\}, n \in \mathbb{Z}$, with no finite accumulation points, such that $f$ is constant a.e. on each one of the intervals $[\lambda_n, \lambda_{n+1})$ (note that these intervals constitute a partition of $\mathbb{R}$). One can verify that our proof of Theorems 2.1, 2.5, and 2.7 yields a function $f$ that is not only measurable, but in fact is piecewise constant on $\mathbb{R}$.

(2) Our construction method allows us to choose the function $f$ in Theorems 2.1, 2.5, and 2.7 to have “dispersed” support, that is, $f$ can be supported on the union of (countably many) small intervals that are located far apart from each other.
5 | SIMULTANEOUS TILING BY FUNCTIONS OF BOUNDED SUPPORT

5.1

One can easily notice that our proof of Theorems 2.1, 2.5, and 2.7 above yields a function $f$ whose support lies inside any given half-line $(r, +\infty)$, so that $\text{supp } f$ is bounded from below. One may ask whether the function $f$ can be chosen such that the support is bounded from both above and below at the same time.

The answer is obviously “yes” if we have $(p, q) = \lambda(\beta, \alpha)$ where $\lambda$ is a scalar, since in this case the function $f = \lambda 1_{[0,\alpha)} \ast 1_{[0,\beta)}$ satisfies the simultaneous tiling condition (1.3) and has bounded support.

To the contrary, if the tiling level vector $(p, q)$ is not proportional to $(\beta, \alpha)$, then Theorem 2.2 provides the question above with a negative answer: $f$ cannot be supported on any bounded set. This theorem will be proved in the present section.

We note that our proof in fact does not use the assumption that $\alpha, \beta$ are rationally independent. However if $\alpha, \beta$ are linearly dependent over the rationals, then we know from Theorem 2.8 that there do not exist any simultaneous tilings (1.3) with a level vector $(p, q)$ that is not proportional to $(\beta, \alpha)$, so the result is vacuous in this case.

5.2

We now turn to prove Theorem 2.2. To this end, we shall use a result due to Anosov [1, Theorem 1] that we state here as a lemma:

**Lemma 5.1 ([1]).** Let $\varphi \in L^1(\mathbb{T}_\alpha)$. If the equation

$$\psi(x) - \psi(x + \beta) = \varphi(x) \quad \text{a.e.} \quad (5.1)$$

has a measurable solution $\psi : \mathbb{T}_\alpha \to \mathbb{C}$, then $\int_{\mathbb{T}_\alpha} \varphi = 0$.

In fact, in [1, Theorem 1] a more general version of this result was stated and proved, in the context of a measure-preserving transformation acting on a finite measure space. Here we only state the result in the special case where the transformation is a rotation of the circle $\mathbb{T}_\alpha$.

**Proof of Theorem 2.2.** Assume that $f$ is a measurable function on $\mathbb{R}$ satisfying (1.3). We shall suppose that $f$ has bounded support, and prove that this implies that the vector $(p, q)$ must be proportional to $(\beta, \alpha)$.

By translating $f$ we may assume that $\text{supp } f \subset [0, n\beta)$, where $n$ is a positive, large enough integer. We can then write

$$f = \sum_{k=0}^{n-1} f_k, \quad f_k := f \cdot 1_{[k\beta,(k+1)\beta)}.$$  

(5.2)
By the first condition in (1.3), we have
\[ \sum_{k=0}^{n-1} \pi_\alpha(f_k) = \pi_\alpha(f) = p \quad \text{a.e.} \quad (5.3) \]

The second condition in (1.3) can be equivalently stated as
\[ \sum_{k=0}^{n-1} f_k(x + k\beta) = q \cdot 1_{[0,\beta)}(x) \quad \text{a.e.} \quad (5.4) \]

It follows from (5.4) that
\[ \sum_{k=0}^{n-1} \pi_\alpha(f_k)(x + k\beta) = q \cdot \pi_\alpha(1_{[0,\beta)})(x) \quad \text{a.e.} \quad (5.5) \]

Let us define
\[ \varphi := p - q \cdot \pi_\alpha(1_{[0,\beta)}), \quad \psi_k := \pi_\alpha(f_k), \quad 0 \leq k \leq n - 1, \quad (5.6) \]

then \( \varphi \in L^1(\mathbb{T}_\alpha) \), while the \( \psi_k \) are measurable functions on \( \mathbb{T}_\alpha \). If we now subtract the equality (5.5) from (5.3), this yields
\[ \sum_{k=1}^{n-1} (\psi_k(x) - \psi_k(x + k\beta)) = \varphi(x) \quad \text{a.e.} \quad (5.7) \]

Lastly, we introduce a measurable function \( \psi \) on \( \mathbb{T}_\alpha \) defined by
\[ \psi(x) := \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \psi_k(x + j\beta), \quad (5.8) \]

and observe that (5.7) can be reformulated as
\[ \psi(x) - \psi(x + \beta) = \varphi(x) \quad \text{a.e.} \quad (5.9) \]

This allows us to apply Lemma 5.1, which yields \( \int_{\mathbb{T}_\alpha} \varphi = 0 \). But using (5.6) we have
\[ \int_{\mathbb{T}_\alpha} \varphi = \int_{\mathbb{T}_\alpha} \left( p - q \cdot \pi_\alpha(1_{[0,\beta)}) \right) = p\alpha - q \int_{\mathbb{R}} 1_{[0,\beta)} = p\alpha - q\beta. \quad (5.10) \]

We conclude that \( p\alpha - q\beta = 0 \), so the vector \( (p, q) \) is proportional to \( (\beta, \alpha) \). \( \square \)

6 | SIMULTANEOUS TILING BY A BOUNDED FUNCTION

6.1

The following question was posed to us by Kolountzakis: Let \( \alpha, \beta \) be rationally independent. Given two arbitrary complex numbers \( p, q \), does there exist a bounded measurable function \( f \) on \( \mathbb{R} \), satisfying the simultaneous tiling condition (1.3)?
The answer is once again “yes” if we have $(p, q) = \lambda(\beta, \alpha)$, $\lambda \in \mathbb{C}$, since in this case the continuous, compactly supported function $f = \lambda 1_{(0, \alpha)} \ast 1_{(0, \beta)}$ satisfies (1.3).

On the other hand, the problem is nontrivial if the vector $(p, q)$ is not proportional to $(\beta, \alpha)$. We note that in this case, a bounded function $f$ satisfying (1.3) cannot be supported on any set of finite measure. Indeed, if mes(supp $f$) is finite then $f$ must be in $L^1(\mathbb{R})$, which is not possible due to Proposition 1.1.

We will nevertheless prove that the question above admits an affirmative answer. Moreover, one can always choose $f$ to be continuous and vanishing at infinity:

**Theorem 6.1.** Let $\alpha, \beta$ be rationally independent. For any two complex numbers $p, q$ one can find a continuous function $f$ on $\mathbb{R}$ vanishing at infinity, and satisfying (1.3).

### 6.2

In what follows, we assume $\alpha, \beta$ to be rationally independent. Our proof of Theorem 6.1 is based on the technique used to prove Lemma 4.4, but this time we will use the following variant of Lemma 4.3.

**Lemma 6.2.** Let $A$ be an elementary set in $\mathbb{T}_\alpha$, and $\varphi$ be a continuous function on $A$. Then given any $\delta > 0$ and any nonempty open interval $J \subseteq \mathbb{T}_\beta$, one can find an elementary set $U \subseteq \mathbb{R}$ and a continuous function $f$ on $\mathbb{R}$ such that

1. $\pi_\alpha(U) = A$;
2. $\pi_\beta(U) \subseteq J$;
3. $f$ is supported on $U$, $|f(x)| \leq \delta$ for all $x \in U$;
4. $\pi_\alpha(f) = \varphi$ on some elementary set $A' \subseteq A$, $m(A \setminus A') < \delta$.

Moreover, $U$ can be chosen inside the half-line $(r, +\infty)$ for any given number $r$.

**Proof.** We apply Lemma 4.2 to the elementary set $A$ and to the open interval $J$. The lemma yields an elementary set $U_0 \subseteq \mathbb{R}$ such that $\pi_\alpha(U_0) = A$, $\pi_\alpha$ is one-to-one on $\text{int}(U_0)$, and $\pi_\beta(U_0) \subseteq J$. Let $M := \sup |\varphi(t)|$, $t \in A$, and choose an integer $N$ sufficiently large so that $N\delta > M$. We then find integers $m_1, \ldots, m_N$ such that, if we denote $U_j := U_0 + m_j \alpha$, $1 \leq j \leq N$, then $\pi_\beta(U_j) \subseteq J$ for every $j$. This is possible due to Kronecker’s theorem, as $\alpha, \beta$ are rationally independent and $\pi_\beta(U_0)$ is a compact subset of the open interval $J$. We can also choose the integers $m_1, \ldots, m_N$ such that the sets $U_1, \ldots, U_N$ are disjoint and all of them are contained in a given half-line $(r, +\infty)$.

We now find an elementary set $U_0' \subseteq \text{int}(U_0)$, such that the (also elementary) set $A' := \pi_\alpha(U_0')$ satisfies $m(A \setminus A') < \delta$. Let $f_0$ be a continuous function on $\mathbb{R}$, supported on $U_0$, and satisfying $f_0(x) = \varphi(\pi_\alpha(x))$ for $x \in U_0'$, and $|f_0(x)| \leq M$ for every $x \in \mathbb{R}$. As $\pi_\alpha$ is one-to-one on $\text{int}(U_0)$, we have $\pi_\alpha(f_0) = \varphi$ on the set $A'$.

Finally, we define the continuous function

$$f(x) := \frac{1}{N} \sum_{j=1}^{N} f_j(x), \quad f_j(x) := f_0(x - m_j \alpha). \quad (6.1)$$

Then $f_j$ is supported on $U_j$, $1 \leq j \leq N$, and hence $f$ is supported on the union

$$U := U_1 \cup U_2 \cup \cdots \cup U_N. \quad (6.2)$$
Recall that $U_1, \ldots, U_N$ are disjoint sets, and that $|f_j| \leq M$ for each $j$. It thus follows from (6.1) that $|f(x)| \leq MN^{-1} \leq \delta$ for every $x \in \mathbb{R}$. So, property (iii) is satisfied.

Notice that $\pi_\alpha(U_j) = \pi_\alpha(U_0) = A$ for every $j$. In particular, this implies (i).

As $f_j$ is a translate of $f_0$ by an integer multiple of $\alpha$, we have $\pi_\alpha(f_j) = \pi_\alpha(f_0)$ for each $1 \leq j \leq N$. It follows that $\pi_\alpha(f) = \pi_\alpha(f_0) = \varphi$ on $A'$. So, (iv) is established.

Lastly, $\pi_\beta(U_j) \subset J$ for every $j$, hence by (6.2) we have $\pi_\beta(U) \subset J$ as well. We conclude that also the condition (ii) holds and the lemma is proved.

\[ \square \]

**Proof of Theorem 6.1.** The approach is similar to the proof of Lemma 4.4, so we shall be brief. We construct by induction a sequence $A_1, A_2, \ldots$ of pairwise disjoint elementary sets in $\mathbb{T}_\alpha$, a sequence $B_1, B_2, \ldots$ of pairwise disjoint elementary sets in $\mathbb{T}_\beta$, a sequence $U_1, V_1, U_2, V_2, \ldots$ of pairwise disjoint elementary sets in $\mathbb{R}$ accumulating at infinity, and a sequence $g_1, h_1, g_2, h_2, \ldots$ of continuous functions on $\mathbb{R}$, in the following way.

Suppose that we have already constructed the sets $A_k, B_k, U_k, V_k$ and the functions $g_k, h_k$ for $1 \leq k \leq n - 1$. We use Lemma 6.2 to find an elementary set $U_n \subset \mathbb{R}$, and a continuous function $g_n$ on $\mathbb{R}$, such that $\pi_\alpha(U_n)$ is disjoint from the sets $A_1, \ldots, A_{n-1}$, $\pi_\beta(U_n)$ is disjoint from the sets $B_1, \ldots, B_{n-1}$, $g_n$ is supported on $U_n$, $|g_n(x)| \leq 2^{-n}$ for all $x \in U_n$, and $\pi_\alpha(g_n) = p - \sum_{k=1}^{n-1} \pi_\alpha(h_k)$ on some elementary set $A_n \subset \mathbb{T}_\alpha$, which is disjoint from $A_1, \ldots, A_{n-1}$, and such that

\[
(1 - 2^{-n})\alpha < m(A_1 \cup \cdots \cup A_n) < \alpha. \tag{6.3}
\]

Then, we use again Lemma 6.2 but with the roles of $\alpha, \beta$ interchanged, to find an elementary set $V_n \subset \mathbb{R}$, and a continuous function $h_n$ on $\mathbb{R}$, such that $\pi_\beta(V_n)$ is disjoint from the sets $A_1, \ldots, A_{n-1}$, $\pi_\alpha(V_n)$ is disjoint from the sets $B_1, \ldots, B_{n-1}$, $h_n$ is supported on $V_n$, $|h_n(x)| \leq 2^{-n}$ for all $x \in V_n$, and $\pi_\beta(h_n) = q - \sum_{k=1}^{n} \pi_\beta(g_k)$ on some elementary set $B_n \subset \mathbb{T}_\beta$, which is disjoint from $B_1, \ldots, B_{n-1}$, and such that

\[
(1 - 2^{-n})\beta < m(B_1 \cup \cdots \cup B_n) < \beta. \tag{6.4}
\]

We observe that Lemma 6.2 allows us to choose the sets $U_1, V_1, U_2, V_2, \ldots$ to be pairwise disjoint and accumulating at $+\infty$. So, we may assume this to be case.

Finally, we define $f := \sum_{n=1}^{\infty} (g_n + h_n)$, which is a continuous function on $\mathbb{R}$ vanishing at infinity. Similarly to the proof of Lemma 4.4, one can check that $\pi_\alpha(f) = p$ on the union $\bigcup_{n=1}^{\infty} A_n$, a set of full measure in $\mathbb{T}_\alpha$, while $\pi_\beta(f) = q$ on $\bigcup_{n=1}^{\infty} B_n$, a set of full measure in $\mathbb{T}_\beta$. Thus, $f$ satisfies the simultaneous tiling condition (1.3). (We note that both sums in (1.3) have only finitely many nonzero terms for almost every $x \in \mathbb{R}$.)

\[ \square \]

6.3 Remarks

(1) One can choose the function $f$ in Theorem 6.1 to be not only continuous but in fact smooth. To this end it suffices to replace Lemma 6.2 with a similar version, where $\varphi$ and $f$ are smooth functions.

(2) If the tiling level vector $(p, q)$ is not proportional to $(\beta, \alpha)$, then the function $f$ in Theorem 6.1 can only have slow decay at infinity. In fact, $f$ cannot be in $L^1(\mathbb{R})$ due to Proposition 1.1.
In this section, we prove Theorems 2.3 and 2.6. These theorems give a sharp lower bound for the measure of \( \text{supp } f \), where \( f \) is an arbitrary measurable function satisfying the simultaneous tiling condition (1.3).

Our proof is based on a graph-theoretic approach. We will show that any simultaneously tiling function \( f \) induces a weighted bipartite graph, whose vertices and edges are also endowed with a measure space structure. The main method of the proof is an iterative leaves removal process that we apply to this graph.

Throughout this section, we again suppose that \( \alpha, \beta > 0 \) are two fixed, rationally independent real numbers.

### 7.1 Bipartite graphs and iterative leaves removal

We start by introducing some purely graph-theoretic concepts and notation.

A bipartite graph is a triple \( G = (A, B, E) \), consisting of two disjoint sets \( A, B \) of vertices, and a set \( E \subseteq A \times B \) of edges. The sets \( A, B \) and \( E \) may be infinite, and may even be uncountable. However, we will assume that each vertex in the graph \( G \) has finite degree.

For any set of vertices \( S \subseteq A \cup B \), we denote by \( E(S) \) the set of all edges that are incident to a vertex from \( S \).

For each \( k \geq 0 \) we let \( A_k \) be the set of vertices of degree \( k \) in \( A \), and \( B_k \) be the set of vertices of degree \( k \) in \( B \). In particular, \( A_1 \) and \( B_1 \) are the sets of leaves in \( A \) and \( B \), respectively. Note that the sets \( A_k, B_k \) form a partition of \( A \cup B \).

A vertex \( v \in A \cup B \) will be called a star vertex if all the neighbors of \( v \) in the graph are leaves. We denote by \( A_* \) the set of star vertices that belong to \( A \), and by \( B_* \) the set of star vertices that belong to \( B \).

**Definition 7.1 (Leaves removal).** Given a bipartite graph \( G = (A, B, E) \) with no isolated vertices, we define its \( A \)-leaves-removed-graph to be the graph

\[
G_A = (A \setminus A_1, B \setminus B_*, E \setminus E(A_1)),
\]

that is, \( G_A \) is the graph obtained from \( G \) by removing all the leaves on the \( A \)-side (including the edges incident to those leaves) and then dropping the star vertices in \( B \), which are the vertices on the \( B \)-side that became isolated due to the removal of all their neighbors. Similarly, we define the \( B \)-leaves-removed graph to be

\[
G_B = (A \setminus A_*, B \setminus B_1, E \setminus E(B_1)).
\]

**Remark 7.2.** Notice that assuming \( G \) to have no isolated vertices implies that the new graph \( G_A \) must have no isolated vertices either. Indeed, when we remove the leaves from \( A \), the only vertices that become isolated are those in \( B_* \), and we make sure to remove these vertices from \( B \). Similarly, the graph \( G_B \) has no isolated vertices.
**Definition 7.3** (Iterative leaves removal). Given a bipartite graph $G = (A, B, E)$ with no isolated vertices, we define its leaves-removal-graph-sequence $G^{(n)} = (A^{(n)}, B^{(n)}, E^{(n)})$ as follows. We let $G^{(0)} = G$, and for each $n$, if $n$ is even we let $G^{(n+1)} = (G^{(n)})_A$, while if $n$ is odd then $G^{(n+1)} = (G^{(n)})_B$.

In other words, the sequence is obtained by consecutive removal of leaves alternately from each side of the graph. First we remove all the leaves from the $A$-side, as well as the star vertices on the $B$-side. By doing so, we may have created some new leaves on the $B$-side, as some vertices in $B$ may have lost all their neighbors in $A$ but one. In the second step, we remove all the leaves from the $B$-side and the star vertices on the $A$-side. Then again some vertices on the $A$-side may become leaves. The process continues in a similar fashion.

Notice that if at the $n$th step of the iterative process there are no leaves to be removed on the relevant side of the graph, then we simply obtain $G^{(n+1)} = G^{(n)}$.

**Definition 7.4** (Weighted bipartite graph). We say that a bipartite graph $G = (A, B, E)$ is weighted if it is endowed with an edge-weight function $w : E \to \mathbb{C}$ that assigns a complex-valued weight to each edge of the graph.

For each vertex $v \in A \cup B$, the sum of the weights of all the (finitely many) edges incident to $v$ will be called the weight of the vertex $v$.

### 7.2 The graph induced by a subset of the real line

We now turn our attention to a specific construction of a bipartite graph.

**Definition 7.5** (The induced graph $G(\Omega)$). Let $\Omega$ be an arbitrary subset of $\mathbb{R}$. We associate to $\Omega$ a bipartite graph $G(\Omega)$ defined as follows. The set of vertices of the graph is the union of the two disjoint sets $A = \pi^\alpha(\Omega)$ and $B = \pi^\beta(\Omega)$, which form the bipartition of the graph. The set of edges $E$ of the graph consists of all edges of the form $e(x) := (\pi^\alpha(x), \pi^\beta(x))$ where $x$ goes through the elements of $\Omega$.

**Remark 7.6**. We note that distinct points $x, y \in \Omega$ correspond to distinct edges $e(x), e(y)$ in $E$. Indeed, if $e(x) = e(y)$ then we must have $x - y \in \alpha\mathbb{Z} \cap \beta\mathbb{Z}$, which in turn implies that $x = y$ since $\alpha, \beta$ are rationally independent. Thus, the representation of the elements of $\Omega$ as edges in the graph is one-to-one. In the sequel, we will often identify edges of the graph with elements of the set $\Omega$.

**Definition 7.7** (Finite degrees assumption). We say that a set $\Omega \subset \mathbb{R}$ satisfies the finite degrees assumption if each vertex in the graph $G(\Omega)$ has finite degree. This is the case if and only if for every $x \in \mathbb{R}$, the sets $\Omega \cap (x + \alpha\mathbb{Z})$ and $\Omega \cap (x + \beta\mathbb{Z})$ have both finitely many elements.

In what follows, we shall assume that the given set $\Omega \subset \mathbb{R}$ satisfies the finite degrees assumption.

**Remark 7.8**. Notice that the graph $G(\Omega) = (A, B, E)$ has no isolated vertices. Indeed, if $a$ is a vertex in $A$ then $a = \pi^\alpha(x)$ for some $x \in \Omega$, so $a$ is incident to the edge $e(x) = (\pi^\alpha(x), \pi^\beta(x))$. Similarly, any vertex $b \in B$ is incident to at least one edge in $E$. 


Remark 7.9. Let $G_A(\Omega)$ be the $A$-leaves-removed-graph of $G(\Omega)$. Notice that $G_A(\Omega)$ is the graph induced by the set $\Omega_A = \Omega \setminus E(A_1)$, where here we identify edges of the graph with elements of the set $\Omega$ (see Remark 7.6). Thus, we have $G_A(\Omega) = G(\Omega_A)$. Similarly, the $B$-leaves-removed-graph $G_B(\Omega)$ of $G(\Omega)$ is the graph induced by the set $\Omega_B = \Omega \setminus E(B_1)$ (where again edges of the graph are identified with elements of $\Omega$). Hence, the iterative leaves removal process applied to the graph $G(\Omega)$ induces a sequence of sets $\Omega(n) \subset \mathbb{R}$, satisfying $\Omega(n+1) \subset \Omega(n) \subset \Omega$ for all $n$, and such that the leaves-removal-graph-sequence $G(n)(\Omega)$ is given by $G(n)(\Omega) = G(\Omega(n))$.

7.3 Vertices and edges as measure spaces

Assume now that $\Omega$ is a measurable subset of $\mathbb{R}$, satisfying the finite degrees assumption. In this case the induced graph $G(\Omega)$ can be endowed with an additional measure space structure, as follows.

Recall that we have endowed $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ with the Lebesgue measure, normalized such that $m(\mathbb{T}_\alpha) = \alpha$ and $m(\mathbb{T}_\beta) = \beta$. We notice that the two vertex sets $A = \pi_\alpha(\Omega)$ and $B = \pi_\beta(\Omega)$ of the graph $G(\Omega) = (A, B, E)$ are measurable subsets of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, respectively. We may therefore consider $A$ and $B$ as measure spaces, with the measure space structure induced from the embedding of $A$ and $B$ into $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, respectively.

We also endow the edge set $E$ with a measure space structure, induced from the identification of $E$ with $\Omega$ as a (measurable) subset of $\mathbb{R}$ as in Remark 7.6. (Notice that we do not endow $E$ with the measure space structure induced from the embedding of $E$ into the product space $A \times B$.)

In the sequel, we will also consider the entire vertex set $V := A \cup B$ as a single measure space, formed by the disjoint union of the two measure spaces $A$ and $B$.

Lemma 7.10 (Measurability lemma).

(i) For each $k$ the set $A_k$ of vertices of degree $k$ in $A$ is a measurable subset of $A$.
(ii) The set $A_\ast$ of star vertices in $A$ (that is, the vertices in $A$ all of whose neighbors are leaves) is a measurable subset of $A$.
(iii) If $S \subset A$ is a measurable set of vertices, then $E(S)$ (the set of edges incident to a vertex in $S$) is a measurable subset of $E$.

Similar assertions hold for the sets $B_k$, $B_\ast$ and $S \subset B$.

Proof. If $a$ is a vertex in $A$, then the degree of $a$ in the graph $G(\Omega)$ is equal to $\pi_\alpha(1_\Omega)(a)$. Hence, $\pi_\alpha(1_\Omega)$ is an everywhere finite, measurable function on $\mathbb{T}_\alpha$. As for each $k$ the set $A_k$ is the preimage of $\{k\}$ under this function, it follows that $A_k$ is measurable.

By a similar argument, also the set $B_k$ is measurable for each $k$.

Next we observe that

$$A_\ast = A \setminus \pi_\alpha(\Omega \cap \pi_\beta^{-1}(B \setminus B_1)), \quad B_\ast = B \setminus \pi_\beta(\Omega \cap \pi_\alpha^{-1}(A \setminus A_1)), \quad (7.3)$$

hence both sets $A_\ast$ and $B_\ast$ are measurable.

Finally, let $S$ be a measurable subset of $A$. Identifying the edges of the graph with elements of $\Omega$, we have $E(S) = \pi_\alpha^{-1}(S) \cap \Omega$, hence $E(S)$ is measurable. Similarly, for any measurable set $S \subset B$, the set $E(S) = \pi_\beta^{-1}(S) \cap \Omega$ is measurable. □
Remark 7.11. Recall from Remark 7.9 that the iterative leaves removal process induces a sequence of sets $\Omega^{(n)} \subset \mathbb{R}$, satisfying $\Omega^{(n+1)} \subset \Omega^{(n)} \subset \Omega$ for all $n$, and such that the leaves-removal-graph-sequence $G^{(n)}(\Omega)$ is given by $G^{(n)}(\Omega) = G(\Omega^{(n)})$. It follows from Lemma 7.10 that if $\Omega$ is a measurable subset of $\mathbb{R}$, then all the sets $\Omega^{(n)}$ are measurable too, as the set of edges removed at each step of the iterative process is measurable.

For a vertex $a \in A$, we denote by $\deg_A(a)$ the degree of $a$ in the graph $G(\Omega)$. Similarly, we denote by $\deg_B(b)$ the degree of a vertex $b \in B$. Then $\deg_A$ and $\deg_B$ are nonnegative, integer-valued functions on $A$ and $B$, respectively.

**Lemma 7.12** (Edge counting lemma). $\deg_A$ is a measurable function on $A$. Moreover, for any measurable set of vertices $S \subset A$ we have

$$m(E(S)) = \int_S \deg_A. \quad (7.4)$$

In particular,

$$m(E(A_k)) = k \cdot m(A_k), \quad k = 1, 2, 3, \ldots \quad (7.5)$$

Similar assertions hold for $\deg_B$ and $B_k$.

Notice that the integral in (7.4) may be finite or infinite, but in any case it has a well-defined value, as $\deg_A$ is a nonnegative function.

**Proof of Lemma 7.12.** Let $S \subset A$ be a measurable set. By identifying the edges of the graph $G(\Omega)$ with elements of $\Omega$, we have $E(S) = \pi_\alpha^{-1}(S) \cap \Omega$. Then

$$m(E(S)) = \int_{\mathbb{R}} 1_{E(S)} = \int_{\mathbb{R}} \pi_\alpha(1_{E(S)}) \quad (7.6)$$

(theses equalities hold both if $E(S)$ has finite or infinite measure). But notice that for a vertex $a \in A$, we have

$$\pi_\alpha(1_{E(S)})(a) = \begin{cases} \deg_A(a), & a \in S, \\ 0, & a \notin S. \end{cases} \quad (7.7)$$

Thus, (7.6) and (7.7) imply (7.4). Finally, (7.5) is a consequence of (7.4), as the function $\deg_A$ attains the constant value $k$ on the set $A_k$. \qed

**Remark 7.13.** Let $\mu_A$ be the measure on $A$ obtained as the image under the map $\pi_\alpha$ of the Lebesgue measure restricted to $\Omega$. The assertion of Lemma 7.12 may be equivalently stated by saying that $\deg_A$ is the Radon–Nikodym derivative of $\mu_A$ with respect to the Lebesgue measure on $A$. 
7.4 | A brief digression: Measure preserving graphs (graphings)

The graph $G(\Omega)$ endowed with its measure space structure is closely related to the notion of a measure preserving graph, or a graphing, so we will discuss this relation briefly here. For a detailed exposition we refer to the book by Lovász [14, chapter 18].

A Borel graph is a graph $(V, E)$ where the vertex set $V$ is a standard Borel space (i.e., the measure space associated to a separable, complete metric space), and the edge set $E$ is a Borel set in $V \times V$. One can show that if $\Omega \subset \mathbb{R}$ is a Borel set, then the induced graph $G(\Omega)$ is a Borel graph.

A measure preserving graph, or a graphing, is a Borel graph $(V, E)$ whose vertex set $V$ is endowed with a probability measure $\lambda$, such that for any two measurable sets $U, W \subset V$ we have

$$\int_U n_W(x) d\lambda(x) = \int_W n_U(x) d\lambda(x), \quad (7.8)$$

where $n_U(x)$ and $n_W(x)$ denote the number of neighbors of $x$ within the sets $U$ and $W$, respectively. The last condition relates the graph structure to the measure space structure by requiring that “counting” the edges between $U$ and $W$ from $U$, yields the same result as counting them from $W$. One can show based on Lemma 7.12 that the graph $G(\Omega)$ satisfies the condition (7.8).

We point out however that in [14] the notion of a graphing includes the additional assumption that the degrees of the vertices in the graph are bounded by a certain constant. To the contrary, for the graph $G(\Omega)$ we only assume that each vertex has finite degree, allowing the existence of vertices with arbitrarily large degrees.

7.5 | The graph induced by a simultaneously tiling function

Let now $f$ be a measurable function on $\mathbb{R}$, and consider the graph $G(\Omega) = (A, B, E)$ induced by the set $\Omega := \text{supp}(f)$. By identifying the edges of the graph with elements of the set $\Omega$ (as in Remark 7.6), we may view $f$ as a function on the set of edges of the graph. Thus, $G(\Omega)$ becomes a weighted graph, with the weight function $f$.

Lemma 7.14. Let $f$ be a measurable function on $\mathbb{R}$, $\text{mes}(\text{supp} f) < +\infty$, and assume that $f$ satisfies the simultaneous tiling condition (1.3). Then $f$ can be redefined on a set of measure zero so as to satisfy also the following two additional properties.

(i) The set $\Omega := \text{supp} f$ satisfies the finite degrees assumption.
(ii) If the induced graph $G(\Omega) = (A, B, E)$ is weighted by the function $f$, then each vertex from $A$ has weight $p$, while each vertex from $B$ has weight $q$.

Proof. Denote the given function by $f_0$, and let $\Omega_0 := \text{supp} f_0$. Let $X_0$ be the set of all points $x \in \mathbb{R}$ satisfying the conditions

$$\sum_{k \in \mathbb{Z}} 1_{\Omega_0}(x - k\alpha) < +\infty, \quad \sum_{k \in \mathbb{Z}} f_0(x - k\alpha) = p, \quad (7.9)$$

as well as the conditions

$$\sum_{k \in \mathbb{Z}} 1_{\Omega_0}(x - k\beta) < +\infty, \quad \sum_{k \in \mathbb{Z}} f_0(x - k\beta) = q. \quad (7.10)$$
The assumptions imply that $X_0$ is a set of full measure in $\mathbb{R}$. Then also the set

$$X := \bigcap_{(n,m) \in \mathbb{Z}^2} (X_0 + n\alpha + m\beta)$$

(7.11)

has full measure in $\mathbb{R}$. We define $f := f_0 \cdot 1_X$, then $f$ coincides with $f_0$ a.e. We will show that the new function $f$ satisfies the two additional conditions (i) and (ii).

Let $G(\Omega) = (A, B, E)$ be the graph induced by the set $\Omega := \text{supp } f = \Omega_0 \cap X$. We first verify the condition (i), namely, we show that each vertex of $G(\Omega)$ has finite degree. Indeed, let $a \in A$, then $a = \pi_\alpha(x)$ for some $x \in \Omega$, and the degree of $a$ is the number of elements in the set $\Omega \cap (x + \alpha \mathbb{Z})$. But this set has finitely many elements, which follows from the first condition in (7.9) using the fact that $\Omega \subset \Omega_0$ and $x \in \Omega \subset X_0$. Hence, each vertex $a \in A$ has finite degree in the graph $G(\Omega)$. Similarly, each vertex $b \in B$ also has finite degree.

Now let the graph $G(\Omega)$ be weighted by the function $f$. We show that condition (ii) holds. Indeed, let $a \in A$, then again $a = \pi_\alpha(x)$ for some $x \in \Omega$. As $\Omega \subset X$ and the set $X$ is invariant under translations by elements from $\alpha \mathbb{Z}$, we have $x + \alpha \mathbb{Z} \subset X$ and thus $f$ coincides with $f_0$ on the set $x + \alpha \mathbb{Z}$. This implies that $\pi_\alpha(f)(a) = \pi_\alpha(f_0)(a) = p$, where the last equality follows from the second condition in (7.9) using the fact that $x \in \Omega \subset X_0$. But $\pi_\alpha(f)(a)$ is exactly the weight of the vertex $a$ in the graph $G(\Omega)$, hence the vertex $a$ has weight $p$. The proof that each vertex $b \in B$ has weight $q$ is similar.

In what follows, we assume that $f$ is a measurable function on $\mathbb{R}$ satisfying the simultaneous tiling condition (1.3). As our goal is to obtain a lower bound for the measure of the support of $f$, we assume that $\Omega := \text{supp } f$ is a set of finite measure.

We endow the graph $G(\Omega) = (A, B, E)$ with the weight function $f$. By redefining the values of $f$ on a set of measure zero (using Lemma 7.14) we can assume with no loss of generality that every vertex in the graph has finite degree, and that the vertices from $A$ have weight $p$, while the vertices from $B$ have weight $q$.

We will also assume that the tiling levels $p$ and $q$ in (1.3) are both nonzero (the case where one of $p, q$ is zero is covered by Theorem 2.7). This implies that the supports of the functions $\pi_\alpha(f)$ and $\pi_\beta(f)$ coincide with $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, respectively, up to a set of measure zero. Hence,

$$m(A) = \alpha, \quad m(B) = \beta.$$  

(7.12)

## 7.6 The Euler characteristic

Recall that the set $E$ of edges of the graph $G(\Omega)$ is endowed with a measure space structure, induced from the identification of $E$ with $\Omega$ as a measurable subset of $\mathbb{R}$ (Remark 7.6). In particular, $m(E) = m(\Omega) < +\infty$.

**Definition 7.15 (Euler characteristic).** The quantity

$$\chi = m(A) + m(B) - m(E)$$

(7.13)

will be called the *Euler characteristic* of the graph $G(\Omega) = (A, B, E)$. 

We call this quantity the “Euler characteristic” since it is the difference between the total measure of the vertices in the graph and the total measure of the edges.

Similarly, we let
\[ \chi^{(n)} = m(A^{(n)}) + m(B^{(n)}) - m(E^{(n)}) \]  
(7.14)
denote the Euler characteristics of the leaves-removal-graph-sequence \( G^{(n)}(\Omega) \).

Let \( L^{(n)} \) be the set of leaves removed at the \( n \)th step of the iterative leaves removal process, that is, if \( n \) is even then \( L^{(n)} = A_1^{(n)} \) (the set of leaves in \( A^{(n)} \)), and if \( n \) is odd then \( L^{(n)} = B_1^{(n)} \) (the set of leaves in \( B^{(n)} \)). The next lemma gives a lower bound for the measure of the set \( L^{(n)} \) in terms of the Euler characteristic \( \chi^{(n)} \).

**Lemma 7.16** (Removed leaves estimate). Assume that \( \alpha > \beta \). Then
\[ m(L^{(0)}) > \chi^{(0)}, \]  
(7.15)
and for all \( n \geq 1 \) we have
\[ m(L^{(n)}) \geq 2\chi^{(n)}. \]  
(7.16)

The assumption that \( \alpha > \beta \) can be made with no loss of generality, for otherwise we may simply interchange the roles of \( \alpha \) and \( \beta \). The reason we need to make this assumption is that we have chosen to begin the iterative leaves removal process by removing leaves from the \( A \)-side. (If we had \( \beta > \alpha \) then the process would have to begin by removing leaves from the \( B \)-side.)

To prove Lemma 7.16, we will first establish two additional lemmas. The first one gives a lower bound for the measures of the sets of leaves \( A_1 \) and \( B_1 \).

**Lemma 7.17.** We have
\[ m(A_1) \geq 2m(A) - m(\Omega), \]  
(7.17)
and similarly,
\[ m(B_1) \geq 2m(B) - m(\Omega). \]  
(7.18)

**Proof.** Recall that we denote by \( A_k \) the set of vertices in \( A \) of degree \( k \). As the sets \( A_k \) form a partition of \( A \), we have
\[ m(A) = \sum_{k=1}^{\infty} m(A_k). \]  
(7.19)

In turn, the sets \( E(A_k) = \pi^{-1}(A_k) \cap \Omega \) form a partition of \( \Omega \), and by Lemma 7.12 we have \( m(E(A_k)) = km(A_k) \). Hence,
\[ m(\Omega) = \sum_{k=1}^{\infty} m(E(A_k)) = \sum_{k=1}^{\infty} km(A_k). \]  
(7.20)
Using (7.19) and (7.20), we conclude that

\[2m(A) - m(A_1) = m(A_1) + 2 \sum_{k=2}^{\infty} m(A_k) \leq \sum_{k=1}^{\infty} km(A_k) = m(\Omega),\]  

(7.21)

which proves (7.17). The inequality (7.18) can be proved in a similar way. □

The next lemma is a more symmetric version of the previous one.

**Lemma 7.18.** We have

\[m(A_1) + m(B_1) \geq 2\chi.\]  

(7.22)

In other words, the measure of the set of leaves in the graph \(G(\Omega)\), both from \(A\) and from \(B\), is at least \(2\chi\). This is an immediate consequence of Lemma 7.17. Indeed, taking the sum of (7.17) and (7.18) yields

\[m(A_1) + m(B_1) \geq 2m(A) - m(\Omega) + 2m(B) - m(\Omega) = 2\chi.\]  

(7.23)

Now we can prove Lemma 7.16 based on the previous two lemmas.

**Proof of Lemma 7.16.** Recall from (7.12) that we have \(m(A) = \alpha\), \(m(B) = \beta\), and that we have assumed \(\alpha > \beta\). Hence, using Lemma 7.17 we obtain

\[m(L^{(0)}) \geq 2\alpha - m(\Omega) > \alpha + \beta - m(\Omega) = \chi^{(0)},\]  

(7.24)

and so (7.15) is proved. Next, for \(n \geq 1\) we apply Lemma 7.18 to the graph \(G^{(n)}(\Omega)\). The lemma gives

\[m(A_1^{(n)}) + m(B_1^{(n)}) \geq 2\chi^{(n)}.\]  

(7.25)

However, we observe that for \(n \geq 1\), the set of leaves \(B_1^{(n)}\) is empty if \(n\) is even, and \(A_1^{(n)}\) is empty if \(n\) is odd, due to the removal of the leaves in the previous step of the iterative leaves removal process. Hence, (7.16) follows from (7.25).

□

**Lemma 7.19** (Monotonicity). For every \(n\), we have

\[\chi^{(n+1)} \leq \chi^{(n)}.\]  

(7.26)

**Proof.** Suppose first that \(n\) is even. By the definitions of \(\chi^{(n+1)}\) and \(G^{(n+1)}(\Omega)\), we have

\[\chi^{(n+1)} = m(A^{(n+1)}) + m(B^{(n+1)}) - m(E^{(n+1)})\]

\[= (m(A^{(n)}) - m(A_1^{(n)})) + (m(B^{(n)}) - m(B_1^{(n)})) - (m(E^{(n)}) - m(E^{(n)}(A_1^{(n)})))\]

\[= \chi^{(n)} - m(A_1^{(n)}) - m(B_1^{(n)}) + m(E^{(n)}(A_1^{(n)})) = \chi^{(n)} - m(B_1^{(n)}),\]  

(7.27)
where in the last equality we used \( m(E^{(n)}(A^{(n)}_1)) = m(A^{(n)}_1) \) (Lemma 7.12). Hence, for even \( n \) we have

\[
\chi^{(n+1)} = \chi^{(n)} - m(B^{(n)}_s),
\]

(7.28)

Similarly, for odd \( n \) we have

\[
\chi^{(n+1)} = \chi^{(n)} - m(A^{(n)}_s),
\]

(7.29)

and the inequality (7.26) follows.

\[\square\]

### 7.7 Jump sets and measure jumps

Let us denote by \( J^{(n)} \) the set \( B^{(n)}_s \) if \( n \) is even, or the set \( A^{(n)}_s \) if \( n \) is odd. The set \( J^{(n)} \) will be called a jump set. The equalities (7.28) and (7.29), established in the proof of Lemma 7.19, say that for every \( n \) we have

\[
m(J^{(n)}) = \chi^{(n)} - \chi^{(n+1)}.
\]

(7.30)

**Definition 7.20** (Measure jump). Whenever it happens that \( \chi^{(n)} > \chi^{(n+1)} \), or equivalently, whenever we have \( m(J^{(n)}) > 0 \), we will say that a measure jump has occurred.

**Lemma 7.21** (Finite subtree lemma). Assume that for some \( n \) the set \( J^{(n)} \) is nonempty (in particular, this is the case if \( \chi^{(n)} > \chi^{(n+1)} \)). Then for each vertex \( v \in J^{(n)} \), the connected component of \( v \) in the graph \( G(\Omega) \) is a finite tree. Moreover, if \( v, w \) are two distinct vertices in \( J^{(n)} \) then their respective connected components in \( G(\Omega) \) are disjoint.

**Proof.** By definition, \( J^{(n)} \) is either \( B^{(n)}_s \) or \( A^{(n)}_s \) depending on whether \( n \) is even or odd. We consider the case where \( n \) is even (the case where \( n \) is odd is similar). Then \( J^{(n)} = B^{(n)}_s \) is the set of star vertices in \( B^{(n)} \), that is, the vertices in \( B^{(n)} \) all of whose neighbors in the graph \( G^{(n)} \) are leaves. Recalling that all the degrees of vertices in \( G(\Omega) \) are finite, it follows that the connected component of a vertex \( v \in J^{(n)} \) in the graph \( G^{(n)} \) is a finite tree (of height one, if we view \( v \) as the root of the tree). As \( G^{(n)} \) was obtained from \( G^{(n-1)} \) by leaves removal, the connected component of \( v \) in the graph \( G^{(n-1)} \) is again a finite tree (of height at most two, when counted from the root \( v \)). Continuing in the same way, we conclude that the connected component of \( v \) in the graph \( G^{(n-j)} \) is a finite tree (of height at most \( j+1 \) from the root \( v \)), for each \( j = 1, 2, \ldots, n \). In particular, for \( j = n \) we obtain the first assertion of the lemma.

Next we turn to prove the second assertion of the lemma. Consider two distinct vertices \( v, w \in J^{(n)} \). As all the neighbors of both \( v \) and \( w \) within \( G^{(n)} \) are leaves, \( v \) and \( w \) cannot have any common neighbor in \( G^{(n)} \). Hence, the connected components of \( v \) and \( w \) in the graph \( G^{(n)} \) are disjoint. As \( G^{(n)} \) was obtained from \( G^{(n-1)} \) by leaves removal, the connected components of \( v \) and \( w \) in the graph \( G^{(n-1)} \) are also disjoint. Continuing in the same way, we conclude that the connected components of \( v \) and \( w \) in the graph \( G^{(n-j)} \) are disjoint for each \( j = 1, 2, \ldots, n \). In particular, this is the case for \( j = n \) and so the second assertion of the lemma follows. \[\square\]
7.8 Proof of Theorem 2.3

We now turn to prove Theorem 2.3. Recall that the theorem asserts that if the tiling level vector \((p, q)\) is not proportional to any vector of the form \((n, m)\) where \(n, m\) are nonnegative integers, then \(m(Ω) \geq α + β\). To prove this result, we will assume that the tiling levels \(p\) and \(q\) are both nonzero and that

\[
m(Ω) < α + β,
\]

and we will show that this implies that \((p, q)\) must be proportional to some vector of the form \((n, m)\) where \(n, m\) are two positive integers.

Recall from (7.12) that we have \(m(A) = α\), \(m(B) = β\), hence (7.31) is equivalent to the assumption that

\[
m(E) < m(A) + m(B),
\]

that is, the total measure of the edges in the graph \(G(Ω)\) is strictly smaller than the total measure of the vertices.

The following lemma shows that to prove Theorem 2.3, it would be enough to establish the existence of a finite connected component in the graph \(G(Ω)\).

**Lemma 7.22 (Total weight equality).** Assume that the graph \(G(Ω)\) has a finite connected component \(H\), and suppose that \(H\) has \(m\) vertices in \(A\) and \(n\) vertices in \(B\). Then

\[
mp = nq.
\]

**Proof.** Recall that we have assumed (using Lemma 7.14) that the weight of each vertex in \(G(Ω)\) is either \(p\) or \(q\), depending on whether this vertex lies in \(A\) or in \(B\). Consider the total weight of the connected component \(H\), that is, the sum of the weights of all the edges in \(H\). On one hand, this sum is the same as the sum of the weights of the vertices of \(H\) that belong to \(A\), and therefore it is equal to \(mp\). On the other hand, this sum is also the same as the sum of the weights of the vertices of \(H\) that belong to \(B\), so it must also be equal to \(nq\). Hence, the equality in (7.33) must hold. □

We can now complete the proof of Theorem 2.3.

**Proof of Theorem 2.3.** We may assume with no loss of generality that \(α > β\). Consider the graph \(G(Ω)\) and its leaves-removal-graph-sequence \(G^{(n)}(Ω)\). Suppose that (7.31) holds, then equivalently we have (7.32) and thus \(χ(0) > 0\). We claim that after at most \(r = \lceil \frac{α + β}{χ(0)} \rceil\) steps of the iterative leaves removal process, a measure jump must occur. Indeed, if not then \(χ(n) = χ(0)\) for each \(0 \leq n \leq r\). But then Lemma 7.16 implies that the measure of the set \(L^{(n)}\) of the leaves removed at the \(n\)th step of the iterative process, is at least \(χ(0)\) for each \(0 \leq n \leq r\). Thus, the total measure of the removed leaves must be at least \((r + 1)χ(0)\). But \((r + 1)χ(0)\) is greater than \(m(A) + m(B)\) which is the total measure of the set of vertices in the entire graph \(G(Ω)\), so we arrive at a contradiction. Hence, a measure jump must occur.

We thus conclude that there exists at least one jump set \(J^{(n)}\) of positive measure, that is, there is \(n\) such that \(m(J^{(n)}) = χ^{(n)} - χ^{(n+1)} > 0\). In particular the jump set \(J^{(n)}\) is nonempty. Then by Lemma 7.21, any vertex \(v \in J^{(n)}\) belongs to a finite connected component of the graph \(G(Ω)\). Thus,
$G(\Omega)$ has a finite connected component. By Lemma 7.22, there exist two positive integers $n, m$ such that $mp = nq$. We conclude that the vector $(p, q)$ is proportional to $(n, m)$. Theorem 2.3 is thus proved.

7.9 The total jump set

We now move on toward our next goal, which is to prove Theorem 2.6. This will require a more detailed analysis of the jump sets that occur in our iterative leaves removal process. We start with the following lemma.

Lemma 7.23 (Euler characteristics limit). We have

$$\lim_{n \to \infty} \chi(n) \leq 0.$$  \hfill (7.34)

Notice that the existence of the limit in (7.34) is guaranteed due to the monotonicity of the sequence $\chi(n)$ (Lemma 7.19).

Proof of Lemma 7.23. Let $G^{(n)}(\Omega) = (A^{(n)}, B^{(n)}, E^{(n)})$ be the leaves-removal-graph-sequence of $G(\Omega) = (A, B, E)$, and recall that

$$A^{(n+1)} \subset A^{(n)}, \quad B^{(n+1)} \subset B^{(n)}, \quad E^{(n+1)} \subset E^{(n)};$$  \hfill (7.35)

as the graph $G^{(n+1)}$ is obtained from $G^{(n)}$ by the removal of vertices and edges. We define the graph limit $G^{(\omega)}(\Omega) = (A^{(\omega)}, B^{(\omega)}, E^{(\omega)})$ by

$$A^{(\omega)} = \bigcap_{n} A^{(n)}, \quad B^{(\omega)} = \bigcap_{n} B^{(n)}, \quad E^{(\omega)} = \bigcap_{n} E^{(n)}.$$  \hfill (7.36)

Equivalently, $G^{(\omega)}(\Omega)$ is the graph induced by the set $\Omega^{(\omega)} = \bigcap_{n} \Omega^{(n)}$ (the equivalence can be verified in a straightforward way using the finite degrees assumption).

It follows from (7.35) and (7.36) that

$$m(A^{(n)}) \to m(A^{(\omega)}), \quad m(B^{(n)}) \to m(B^{(\omega)}), \quad m(E^{(n)}) \to m(E^{(\omega)}),$$  \hfill (7.37)

and consequently,

$$\lim_{n \to \infty} \chi^{(n)} = \lim_{n \to \infty} (m(A^{(n)}) + m(B^{(n)}) - m(E^{(n)})) \quad = m(A^{(\omega)}) + m(B^{(\omega)}) - m(E^{(\omega)}) = \chi^{(\omega)},$$  \hfill (7.38)

where $\chi^{(\omega)}$ is the Euler characteristic of the graph $G^{(\omega)}(\Omega)$.

Now, suppose to the contrary that $\chi^{(\omega)} > 0$. Then we may apply Lemma 7.18 to the graph limit $G^{(\omega)}(\Omega)$ and obtain that there must be a set of leaves with positive measure in $G^{(\omega)}(\Omega)$. Let $v$ be any leaf of $G^{(\omega)}(\Omega)$, then $v$ has exactly one neighbor $w_0$ in $G^{(\omega)}(\Omega)$. Notice that the vertex $v$ must have at least one more neighbor in the original graph $G(\Omega)$, for otherwise $v$ is a leaf in $G(\Omega)$ and should have been removed in either the first or second step of the leaves removal process. Let $w_0, w_1, \ldots, w_k$ be all the neighbors of $v$ in $G(\Omega)$ (there can be only finitely many neighbors due to the finite degrees assumption). As the vertices $w_1, \ldots, w_k$ are no longer in the graph limit $G^{(\omega)}(\Omega)$,
for each $1 \leq j \leq k$ there is $n_j$ such that $w_j$ is not in $G^{(n_j)}(\Omega)$. Hence, if we let $N := \max\{n_1, \ldots, n_k\}$ then $G^{(N)}(\Omega)$ does not contain any one of the vertices $w_1, \ldots, w_k$. Thus, $v$ is a leaf already in the graph $G^{(N)}(\Omega)$. But then $v$ should have been removed at the $N$th step of the leaves removal process, so $v$ cannot belong to the graph limit $G^{(\omega)}(\Omega)$. We thus arrive at a contradiction. This shows that $\chi^{(\omega)}$ cannot be positive and the lemma is proved.

**Definition 7.24** (The total jump set). The set

$$J = \bigcup_{n=0}^{\infty} J^{(n)}$$

will be called the **total jump set** of the graph $G(\Omega)$.

Recall that $J^{(n)}$ is a subset of $B$ if $n$ is even, and $J^{(n)}$ is a subset of $A$ if $n$ is odd. Hence, $J$ is a subset of the entire vertex set $V = A \cup B$ of the graph $G(\Omega) = (A, B, E)$. Moreover, if we consider $V$ as a measure space, formed by the disjoint union of the two measure spaces $A$ and $B$, then $J$ is a measurable subset of $V$ (Lemma 7.10).

We also notice that the sets $J^{(n)}$ form a partition of $J$ (being disjoint sets) and hence the measure $m(J)$ of the total jump set is equal to the sum of all the measure jumps. By the proof of Theorem 2.3, we know that at least one measure jump must occur, which implies that the set $J$ has positive measure. Now we prove a stronger result:

**Lemma 7.25** (Lower bound for the total jump measure). We have

$$m(J) \geq m(A) + m(B) - m(\Omega).$$  \hfill (7.40)

**Proof.** Using (7.38), we have

$$\chi^{(0)} - \chi^{(\omega)} = \lim_{n \to \infty} (\chi^{(0)} - \chi^{(n)}) = \lim_{n \to \infty} \sum_{k=0}^{n-1} (\chi^{(k)} - \chi^{(k+1)}) =$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n-1} m(J^{(k)}) = m(J).$$  \hfill (7.42)

But due to Lemma 7.23, we know that $\chi^{(\omega)}$ is nonpositive, thus

$$m(J) = \chi^{(0)} - \chi^{(\omega)} \geq \chi^{(0)}$$

which establishes (7.40).

**Lemma 7.26** (Total jump set as a set of representatives). Every connected component of the graph $G(\Omega)$, which is a finite tree, intersects the total jump set $J$ at exactly one vertex. Conversely, each vertex $w \in J$ lies in a connected component of the graph $G(\Omega)$, which is a finite tree.

Thus, we may consider the total jump set $J$ as a set of representatives, containing a unique representative vertex for each connected component of the graph $G(\Omega)$, which is a finite tree.
Proof of Lemma 7.26. Recall that $G^{(n+1)}(\Omega)$ is obtained from $G^{(n)}(\Omega)$ by (i) the removal of the leaves in $A^{(n)}$ if $n$ is even, or the leaves in $B^{(n)}$ if $n$ is odd; (ii) the removal of the edges incident to the leaves removed; and (iii) the removal of the set $J^{(n)}$ of vertices that become isolated (which is the set $B^s_n$ if $n$ is even, or the set $A^s_n$ if $n$ is odd).

Now let $H$ be a connected component of the graph $G(\Omega)$, and assume that $H$ is a finite tree. Then the iterative leaves removal process necessarily exhausts the tree $H$ after a finite number of steps (this can be easily proved by induction on the size of the tree). Moreover, the tree $H$ gets exhausted at the unique step $n$ for which $J^{(n)} \cap H$ is nonempty, and $J^{(n)} \cap H$ must then consist of exactly one vertex.

(It is worth mentioning that at the last step $n$ when the tree gets exhausted, it may happen that there is only one edge of the tree left to be removed. In this case, one of the vertices of this edge will be considered as a leaf, while the other vertex will be considered as an element of the set $J^{(n)}$.)

Thus, each connected component of the graph $G(\Omega)$, which is a finite tree, contributes exactly one vertex to the total jump set $J$.

Conversely, consider a vertex $w \in J$. Then $w$ belongs to the set $J^{(n)}$ for some $n$, so by Lemma 7.21 the connected component of $w$ in the graph $G(\Omega)$ is a finite tree. □

Lemma 7.27 (Upper bound for the total jump measure). Assume that the tiling levels $p$, $q$ are two positive coprime integers. Then

$$m(J) \leq \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}. \quad (7.44)$$

Remark 7.28. Let us explain our intuition behind Lemma 7.27. Recall that each vertex $w \in J$ is a representative of a connected component of $G(\Omega)$, which is a finite tree (Lemma 7.26). Let $H$ be one of these connected components, and suppose that $H$ has $m$ vertices in $A$ and $n$ vertices in $B$. Using Lemma 7.22, it follows that $mp = nq$. But as $p$, $q$ are now assumed to be positive coprime integers, this implies that $q$ must divide $m$, and $p$ must divide $n$. In particular, we have $m \geq q$ and $n \geq p$. Hence, the connected component of each vertex $w \in J$ contributes at least $q$ vertices to $A$, and at least $p$ vertices to $B$. So, intuitively we may expect to have

$$m(A) \geq q \cdot m(J), \quad (7.45)$$

and

$$m(B) \geq p \cdot m(J). \quad (7.46)$$

But notice that according to (7.12) we have $m(A) = \alpha$, $m(B) = \beta$, so that the two inequalities (7.45) and (7.46) together imply (7.44). This explains why intuitively one may expect that Lemma 7.27 should be true.

We now turn to the formal proof of Lemma 7.27. Let $V = A \cup B$ be the vertex set of the graph $G(\Omega) = (A, B, E)$, and let $V' \subset V$ be the set of those vertices whose connected component in the graph $G(\Omega)$ is a finite tree. Let

$$A' = V' \cap A, \quad B' = V' \cap B. \quad (7.47)$$

Lemma 7.29. The sets $A'$, $B'$ (and hence also $V' = A' \cup B'$) are measurable.
Proof. Consider the sets $J_A := J \cap A$ and $J_B := J \cap B$. The sets $J_A, J_B$ are measurable since $A, B$, and $J$ are measurable sets.

Recall that every connected component in the graph $G(\Omega)$, which is a finite tree, has a representative vertex $v \in J$ (Lemma 7.26). Hence, $A'$ is the set of all vertices $a \in A$ such that there is a finite path connecting $a$ to some element $v \in J = J_A \cup J_B$.

We next observe that a vertex $a \in A$ is connected to some vertex $v \in J_A$ if and only if $a$ belongs to the set $(\pi_\alpha \circ \pi_\beta^{-1} \circ \pi_\alpha \circ \pi_\beta^{-1})^n(J_A)$ for some $n$. This is because when moving from a vertex in $A$ to a neighbor vertex in $B$, we first go from the vertex to some of its incident edges (which corresponds to picking an element of $\Omega$ belonging to the preimage of the vertex under the map $\pi_\alpha$), and then go from this edge to its other endpoint vertex (which corresponds to taking the image of the edge under $\pi_\beta$). Similarly, when moving from a vertex in $B$ to a neighbor vertex in $A$, we first pick an edge in the preimage under the map $\pi_\beta$ and then take the image of the edge under $\pi_\alpha$.

For a similar reason, a vertex $a \in A$ is connected to some vertex $v \in J_B$ if and only if $a$ belongs to the set $(\pi_\alpha \circ \pi_\beta^{-1} \circ \pi_\alpha \circ \pi_\beta^{-1})^n(J_B)$ for some $n$.

(Wave note that here we consider $\pi_\alpha$ and $\pi_\beta$ as maps defined on the set $\Omega$, thus inverse images under these maps are understood to be subsets of $\Omega$.)

We have thus shown that

$$A' = \bigcup_{n=0}^{\infty} \left[ (\pi_\alpha \circ \pi_\beta^{-1} \circ \pi_\alpha \circ \pi_\beta^{-1})^n(J_A) \cup (\pi_\alpha \circ \pi_\beta^{-1} \circ \pi_\alpha \circ \pi_\beta^{-1})^n(\pi_\alpha \circ \pi_\beta^{-1})(J_B) \right],$$

so the measurability of $A'$ follows from the measurability of the sets $J_A, J_B$ and the fact that the measurability of a set is preserved under both images and preimages with respect to the maps $\pi_\alpha$ and $\pi_\beta$.

The proof that the set $B'$ is also measurable is similar. □

Proof of Lemma 7.27. Assume that the tiling levels $p$ and $q$ are two positive coprime integers. We must prove that (7.44) holds, or equivalently, that (7.45) and (7.46) are both satisfied. We will prove (7.45) only. The proof of (7.46) is similar.

Recall that the connected component of any vertex $v \in J$ in the graph $G(\Omega)$ is a finite tree (Lemma 7.26). For each $v \in J$, we let $h_A(v)$ be the number of vertices of the connected component of $v$ that lie in the set $A$. Then $h_A$ is a nonnegative, integer-valued function on $J$. As each finite connected component of $G(\Omega)$ must contain at least $q$ vertices in $A$ (Remark 7.28), we have $h_A(v) \geq q$ for every $v \in J$.

We will show that the function $h_A$ is measurable and satisfies

$$\int_J h_A \leq m(A').$$

(7.49)

Notice that once (7.49) is established, we can conclude (using $A' \subset A$) that

$$m(A) \geq m(A') \geq \int_J h_A \geq q \cdot m(J),$$

(7.50)

and (7.45) follows. So, it remains to show that $h_A$ is measurable and satisfies (7.49).

Recall that we denote by $L^{(n)}$ the set of leaves removed at the $n$th step of the iterative leaves removal process, that is, if $n$ is even then $L^{(n)} = A_1^{(n)}$ (the set of leaves in $A^{(n)}$), and if $n$ is odd then $L^{(n)} = B_1^{(n)}$ (the set of leaves in $B^{(n)}$). We construct by induction two sequences $\{\phi_A^{(n)}\}$ and $\{\psi_A^{(n)}\}$ of
functions on $V' = A' \cup B'$, as follows. We define

$$\psi_A^{(0)} = 1_{A'}, \quad \psi_A^{(n+1)} = \psi_A^{(n)} + \phi_A^{(n)}, \quad (7.51)$$

and

$$\phi_A^{(n)}(v) = \begin{cases} 
-\psi_A^{(n)}(v), & v \in V' \cap L^{(n)} \\
\sum_w \psi_A^{(n)}(w), & v \in V' \setminus L^{(n)}
\end{cases} \quad (7.52)$$

$$\quad \sum_w \psi_A^{(n)}(w), & v \in V' \setminus L^{(n)} \quad (7.53)$$

where $w$ goes through all the vertices in $V' \cap L^{(n)}$ who are neighbors of $v$ in the graph $G^{(n)}(\Omega)$ (if there are no such neighbors then the sum is understood to be zero).

The motivation behind this construction is that we view the function $\psi_A^{(n)}$ as assigning a certain mass to each vertex in $V'$. We start with the function $\psi_A^{(0)}$ that assigns a unit mass to each vertex in $A'$, and zero mass to each vertex in $B'$. At the $n$th step of the leaves removal process, by adding $\phi_A^{(n)}$ to $\psi_A^{(n)}$ we subtract the mass from each removed leaf in $V'$ and add the mass back to the neighbor of the leaf, so that at each step the mass is transferred from the removed leaves to their neighbors.

In particular, each $\psi_A^{(n)}$ is a nonnegative, integer-valued function on $V'$.

Notice that whenever the leaves removal process exhausts a connected component $H$ of the graph $G(\Omega)$ (so that $H$ is a finite tree by Lemma 7.21), then the total mass accumulated from all the $A'$-vertices of $H$ is transferred into the unique representative vertex $v$ of $H$ in the total jump set $J$ (Lemma 7.26), and the value of $\psi_A^{(n)}(v)$ will remain fixed from that point on. This implies that the sequence $\psi_A^{(n)}$ converges pointwise to the function $h_A$ on $J$, and to zero on $V' \setminus J$.

We now give an equivalent way of constructing the function $\phi_A^{(n)}$.

Let $g_n$ be a function on the set $E$ of edges of the graph $G(\Omega) = (A, B, E)$, defined as follows. We let $g_n(e) = \psi_A^{(n)}(v)$ if $e$ is a leaf edge in the graph $G^{(n)}(\Omega)$, incident to a vertex $v \in V' \cap L^{(n)}$. If this is not the case, then we let $g_n(e) = 0$. Then $g_n$ is supported on the subset $E^{(n)}(V' \cap L^{(n)})$ of the edge set $E$. By identifying the edges of $G(\Omega)$ with elements of $\Omega$ (Remark 7.6) we may view $g_n$ as a function on $\Omega$. If $n$ is even, then $g_n(x) = \psi_A^{(n)}(\pi_\alpha(x))$ if $x$ is in the set $\pi^{-1}_\alpha(V' \cap L^{(n)}) \cap \Omega^{(n)}$, and $g_n(x) = 0$ otherwise. Similarly, if $n$ is odd then $g_n(x) = \psi_A^{(n)}(\pi_\beta(x))$ if $x$ is in the set $\pi^{-1}_\beta(V' \cap L^{(n)}) \cap \Omega^{(n)}$, and $g_n(x) = 0$ otherwise.

Now consider again the definition (7.52), (7.53) of the function $\phi_A^{(n)}$. Notice that up to a sign, both of (7.52) and (7.53) involve summation of the values $g_n(e)$ over all the edges $e$ incident to the vertex $v$, with the only difference that in (7.52) there is just one such edge $e$ (as $v$ is a leaf in the graph $G^{(n)}(\Omega)$) while in (7.53) the vertex $v$ might have several neighbors. Observe also that the identification of edges in the graph with elements of $\Omega$, enables us to express summation over neighbors in terms of the projections $\pi_\alpha$ and $\pi_\beta$. Thus, if we extend the function $g_n$ to the whole $\mathbb{R}$ by setting $g_n(x) = 0$ for $x \notin \Omega$, then the function $\phi_A^{(n)}$ can be equivalently defined by $\phi_A^{(n)} = (-1)^{n+1} \pi_\alpha(g_n)$ on $A'$, and $\phi_A^{(n)} = (-1)^n \pi_\beta(g_n)$ on $B'$.

It follows from this equivalent definition, using induction on $n$, that $\phi_A^{(n)}$ and $\psi_A^{(n)}$ are both measurable functions on $V'$. Moreover, these functions are both in $L^1(V')$, as we have $\int_\Omega g_n = \int_{V' \cap L^{(n)}} \psi_A^{(n)}$ and the projections $\pi_\alpha$ and $\pi_\beta$ map $L^1(\Omega)$ into $L^1(A)$ and $L^1(B)$, respectively.
As a consequence, the function \( h_A \) (the pointwise limit of \( \psi^{(n)}_A \) on \( J \)) is a measurable function on \( J \).

Now, notice that we have \( \int_{V'} \psi^{(0)}_A = m(A') \). We claim that in fact \( \int_{V'} \psi^{(n)}_A = m(A') \) for every \( n \), that is, when the mass is transferred from leaves to neighbors along the leaves removal process, the total mass remains constant. This is equivalent to the assertion that \( \int_{V'} \phi^{(n)}_A = 0 \) for every \( n \). Indeed, we have \( \phi^{(n)}_A = (-1)^{n+1} \pi_\alpha(g_n) \) on \( A' \), and \( \phi^{(n)}_A = (-1)^n \pi_\beta(g_n) \) on \( B' \). Hence,

\[
(-1)^n \int_{V'} \phi^{(n)}_A = - \int_{A'} \pi_\alpha(g_n) + \int_{B'} \pi_\beta(g_n) = - \int_{\Omega} g_n + \int_{\Omega} g_n = 0. \tag{7.54}
\]

We have thus proved that \( \int_{V'} \psi^{(n)}_A = m(A') \) for every \( n \). Moreover, the functions \( \psi^{(n)}_A \) are non-negative and converge pointwise to \( h_A \) on \( J \), and to zero on \( V' \setminus J \). Hence, we may apply Fatou’s lemma, which yields

\[
m(A') = \lim_{n \to \infty} \int_{V'} \psi^{(n)}_A \geq \int_{V'} \lim_{n \to \infty} \psi^{(n)}_A = \int_J h_A, \tag{7.55}
\]

and we arrive at (7.49). Together with (7.50), this implies (7.45). The proof of (7.46) can be done in a similar way. Lemma 7.27 is thus proved.

### 7.10 Proof of Theorem 2.6

We can now turn to prove Theorem 2.6. Recall that the theorem asserts that if the tiling levels \( p, q \) are two positive, coprime integers, then we must have

\[
m(\Omega) > \alpha + \beta - \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}. \tag{7.56}
\]

Let us combine the lower bound (Lemma 7.25) and the upper bound (Lemma 7.27) for the total jump measure. As by (7.12), we have \( m(A) = \alpha, m(B) = \beta \), this yields

\[
\alpha + \beta - m(\Omega) \leq m(J) \leq \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}, \tag{7.57}
\]

and as a consequence,

\[
m(\Omega) \geq \alpha + \beta - \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}. \tag{7.58}
\]

We thus almost arrive at (7.56). It only remains to show that an equality cannot occur. We will need the following lemma.

**Lemma 7.30.** Suppose that we are given a full measure subset \( J \) of the total jump set \( J \). Let \( V \) be the set of all vertices \( v \in V \) whose connected component in the graph \( G(\Omega) \) intersects \( J \). Then \( V \) is a full measure subset of \( V' \).
Proof. By definition, $\mathcal{V}$ is the set of those vertices $v \in V$ such that there is a finite path connecting $v$ to some vertex in $\mathcal{J}$. Let

$$\mathcal{J}_A = \mathcal{J} \cap A, \quad \mathcal{J}_B = \mathcal{J} \cap B, \quad \mathcal{V}_A = \mathcal{V} \cap A, \quad \mathcal{V}_B = \mathcal{V} \cap B.$$  \hfill (7.59)

By the same argument as in the proof of Lemma 7.29, we have

$$\mathcal{V}_A = \bigcup_{n=0}^{\infty} \left[ (\pi_\alpha \circ \pi_\beta^{-1} \circ \pi_\alpha^{-1})^n(\mathcal{J}_A) \cup (\pi_\alpha \circ \pi_\beta^{-1} \circ \pi_\alpha^{-1})^n(\pi_\alpha \circ \pi_\beta^{-1})(\mathcal{J}_B) \right].$$  \hfill (7.60)

But the right-hand sides of (7.48) and (7.60) coincide up to a set of measure zero, as both the image and the preimage of a null set under the maps $\pi_\alpha$ and $\pi_\beta$ is again a null set. Hence, $\mathcal{V}_A$ is a full measure subset of $A'$. In a similar way, $\mathcal{V}_B$ is a full measure subset of $B'$. Thus, $\mathcal{V} = \mathcal{V}_A \cup \mathcal{V}_B$ is a full measure subset of $V' = A' \cup B'$.

Now suppose to the contrary that there is equality in (7.58). Then the two inequalities in (7.57) also become equalities. In particular, we obtain

$$m(J) = \min \left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}. \quad (7.61)$$

This means that we must have $m(A) = q \cdot m(J)$ or $m(B) = p \cdot m(J)$. We shall suppose that $m(A) = q \cdot m(J)$ (the other case is similar). In turn, this implies that all the inequalities in (7.50) become equalities, that is,

$$m(A) = m(A') = \int_J h_A = q \cdot m(J).$$  \hfill (7.62)

As we know that $h_A(v) \geq q$ for every $v \in J$, it follows from $\int_J h_A = q \cdot m(J)$ that $h_A(v) = q$ for every $v$ in some full measure subset $J$ of $J$. In other words, the connected component of every $v \in J$ has exactly $q$ vertices in $A$. In turn, using Lemma 7.22 it follows that any such a connected component must have exactly $p$ vertices in $B$. For each $v \in J$, let $h_B(v)$ be the number of vertices of the connected component of $v$ that lie in $B$. Then we have $h_B(v) = p$ for every $v \in J$, and hence a.e. on $J$.

Consider functions $\phi^{(n)}_B, \psi^{(n)}_B$ on the set $V' = A' \cup B'$, defined analogously to the functions $\phi^{(n)}_A, \psi^{(n)}_A$ from the proof of Lemma 7.27. Then the sequence $\psi^{(n)}_B$ converges pointwise to $h_B$ on $J$, and to zero on $V' \setminus J$. But as the connected component of every $v \in J$ has exactly $p + q$ vertices ($q$ of them in $A$, and $p$ of them in $B$) then after at most $p + q$ steps of the iterative leaves removal process, all the mass $\psi^{(n)}_B$ on such a connected component is concentrated on the representative vertex of the connected component in $J$. Using Lemma 7.30, we know that if $\mathcal{V}$ is the set of all vertices whose connected component intersects $J$, then $\mathcal{V}$ is a full measure subset of $V'$. We conclude that for all $n \geq p + q$ we have $\psi^{(n)}_B = p$ on $J = \mathcal{V} \cap J$ (and hence, a.e. on $J$), and $\psi^{(n)}_B = 0$ on $V \setminus J$ (and hence, a.e. on $V' \setminus J$). In turn, this implies that $\int_{V'} \psi^{(n)}_B = p \cdot m(J)$ for all $n \geq p + q$. But on the other hand, observe (as in the proof of Lemma 7.27) that we have $\int_{V'} \psi^{(n)}_B = m(B')$ for every $n$. We conclude that

$$m(B') = p \cdot m(J).$$  \hfill (7.63)
Next we claim that $m(B') = m(B)$. Indeed, if not, then $B \setminus B'$ is a positive measure subset of $B$. It follows that the set $\pi_\alpha(\pi_\beta^{-1}(B \setminus B') \cap \Omega)$, consisting of those vertices in $A$ that have a neighbor belonging to $B \setminus B'$ in the graph $G(\Omega)$, is a positive measure subset of $A \setminus A'$. But this implies that $m(A') < m(A)$, contradicting (7.62). Hence, we must have $m(B') = m(B)$.

We conclude that

$$m(B) = m(B') = p \cdot m(J). \quad (7.64)$$

Finally, we combine (7.62) and (7.64) to obtain

$$\frac{\alpha}{q} = m(J) = \frac{\beta}{p}, \quad (7.65)$$

and it follows that $p\alpha = q\beta$. But this contradicts the assumption that $\alpha, \beta$ are rationally independent. This establishes that equality in (7.58) cannot occur, and completes the proof of Theorem 2.6.

### 8 SIMULTANEOUS TILING BY INTEGER TRANSLATES

In this section, we turn to deal with the case where the numbers $\alpha, \beta$ are linearly dependent over the rationals. By rescaling, it would be enough to consider the case $(\alpha, \beta) = (n, m)$ where $n, m$ are positive integers.

We will prove Theorems 2.8, 2.9, and 2.10 by showing that if a measurable function $f$ on $\mathbb{R}$ satisfies the simultaneous tiling condition (2.3) then the tiling level vector $(p, q)$ must be proportional to $(n, m)$, and if the level vector $(p, q)$ is nonzero then the least possible measure of the support of $f$ is $n + m - \gcd(n, m)$.

The approach is based on a reduction of the simultaneous tiling problem from the real line $\mathbb{R}$ to the set of integers $\mathbb{Z}$. In particular, we will prove that $n + m - \gcd(n, m)$ is also the least possible size of the support of a function $g$ on $\mathbb{Z}$ that tiles the integers simultaneously (with a nonzero level vector) by two arithmetic progressions $n\mathbb{Z}$ and $m\mathbb{Z}$.

#### 8.1

We begin by introducing the notion of tiling by translates of a function on the set of integers $\mathbb{Z}$. Let $g$ be a function on $\mathbb{Z}$, and $\Lambda$ be a subset of $\mathbb{Z}$. We say that $g + \Lambda$ is a tiling of $\mathbb{Z}$ at level $w$ if we have

$$\sum_{\lambda \in \Lambda} g(t - \lambda) = w, \quad t \in \mathbb{Z}, \quad (8.1)$$

and the series (8.1) converges absolutely for every $t \in \mathbb{Z}$.

We are interested in simultaneous tiling of the integers by two arithmetic progressions $n\mathbb{Z}$ and $m\mathbb{Z}$. We thus consider a function $g$ on $\mathbb{Z}$ satisfying

$$\sum_{k \in \mathbb{Z}} g(t - kn) = p, \quad \sum_{k \in \mathbb{Z}} g(t - km) = q, \quad t \in \mathbb{Z}, \quad (8.2)$$

where \( n, m \) are positive integers, \( p, q \) are complex numbers, and both series in (8.2) converge absolutely for every \( t \in \mathbb{Z} \).

### 8.1.1

We begin with the following basic result.

**Proposition 8.1.** Let \( g \) be a function on \( \mathbb{Z} \) satisfying (8.2), where \( n, m \) are positive integers. Then \( g \in \ell^1(\mathbb{Z}) \), and the vector \((p, q)\) must be proportional to \((m, n)\).

**Proof.** First we observe that

\[
\sum_{t \in \mathbb{Z}} |g(t)| = \sum_{t=0}^{n-1} \sum_{k \in \mathbb{Z}} |g(t - kn)|. \tag{8.3}
\]

By assumption, the inner sum on the right-hand side of (8.3) converges for every \( t \). Hence, the sum on the left-hand side converges as well, which shows that the function \( g \) must be in \( \ell^1(\mathbb{Z}) \). Next, we have

\[
\sum_{t \in \mathbb{Z}} g(t) = \sum_{t=0}^{n-1} \sum_{k \in \mathbb{Z}} g(t - kn) = np, \tag{8.4}
\]

where the last equality follows from condition (8.2). In a similar way, we also have

\[
\sum_{t \in \mathbb{Z}} g(t) = \sum_{t=0}^{m-1} \sum_{k \in \mathbb{Z}} g(t - km) = mq, \tag{8.5}
\]

again using (8.2). Hence, \( np = mq \), that is, the vector \((p, q)\) is proportional to \((m, n)\). \(\square\)

### 8.1.2

Let us recall (see Definition 4.5) that an \( n \times m \) matrix \( M = (c_{ij}) \) is called a *doubly stochastic array* if its entries \( c_{ij} \) are nonnegative, and the sum of the entries at each row is \( m \) and at each column is \( n \). We have seen that the minimal size of the support of an \( n \times m \) doubly stochastic array is \( n + m - \gcd(n, m) \) (Theorem 4.6). In the proof of Lemma 4.8, we used one part of this result, namely, the part which states that there exists an \( n \times m \) doubly stochastic array whose support size is as small as \( n + m - \gcd(n, m) \).

In what follows, we will use the other part of the result, that is, the part which states that \( n + m - \gcd(n, m) \) constitutes a lower bound for the support size of any \( n \times m \) doubly stochastic array. Actually, we will need a stronger version of this result, proved in [2], which establishes that the same lower bound holds also for complex-valued matrices, that is, even without assuming that the matrix entries are nonnegative.
Theorem 8.2 (see [2, Theorem 3.1]). Let $M = (c_{ij})$ be an $n \times m$ complex-valued matrix satisfying (4.7) and (4.8), that is, the sum of the entries at each row is $m$ and at each column is $n$. Then the support of $M$ has size at least $n + m - \gcd(n, m)$.

8.1.3

By the support of a function $g$ on $\mathbb{Z}$ we mean the set
\[
\text{supp } g = \{ t \in \mathbb{Z} : g(t) \neq 0 \}. \tag{8.6}
\]

In the next result, we use Theorem 8.2 to give a lower bound for the support size of any function $g$ on $\mathbb{Z}$ that tiles simultaneously by the two arithmetic progressions $n\mathbb{Z}$ and $m\mathbb{Z}$ with a nonzero tiling level vector $(p, q)$.

Theorem 8.3. Let $g$ be a function on $\mathbb{Z}$ satisfying (8.2) where $n, m$ are positive integers and the vector $(p, q)$ is nonzero. Then $\text{supp } g$ has size at least $n + m - \gcd(n, m)$.

Proof. By Proposition 8.1, the function $g$ is in $\ell^1(\mathbb{Z})$, and the tiling level vector $(p, q)$ is proportional to $(m, n)$. By multiplying the function $g$ on an appropriate scalar, we may suppose that $(p, q) = (m, n)$.

We will first prove the result in the special case where $n, m$ are coprime. Let $\mathbb{Z}_{nm}$ be the additive group of residue classes modulo $nm$. Define a function
\[
h(t) := \sum_{k \in \mathbb{Z}} g(t - km), \quad t \in \mathbb{Z}. \tag{8.7}
\]
Then $h$ is periodic with period $nm$, so it may be viewed as a function on $\mathbb{Z}_{nm}$.

Let $H_k$ denote the subgroup of $\mathbb{Z}_{nm}$ generated by the element $k$. One can verify using (8.2) and (8.7) that the function $h$ tiles the group $\mathbb{Z}_{nm}$ by translations along each one of the two subgroups $H_n$ and $H_m$, that is to say,
\[
\sum_{s \in H_n} h(t - s) = m, \quad \sum_{s \in H_m} h(t - s) = n, \quad t \in \mathbb{Z}_{nm}. \tag{8.8}
\]

Next, we denote by $\mathbb{Z}_n$ and $\mathbb{Z}_m$ the additive groups of residue classes modulo $n$ and $m$, respectively. As $n, m$ are coprime, then by the Chinese remainder theorem there is a group isomorphism $\varphi : \mathbb{Z}_{nm} \to \mathbb{Z}_n \times \mathbb{Z}_m$ given by $\varphi(t) = (t \mod n, t \mod m)$. This isomorphism allows us to lift the function $h$ to a new function
\[
M : \mathbb{Z}_n \times \mathbb{Z}_m \to \mathbb{R} \tag{8.9}
\]
defined by $M(\varphi(t)) = h(t), t \in \mathbb{Z}_{nm}$. We use (8.9) as an alternative way to represent a complex-valued $n \times m$ matrix $M$, in which the rows of the matrix are indexed by residue classes modulo $n$, while the columns indexed by residue classes modulo $m$.

We now claim that the sum of the entries of the matrix $M$ at each row is equal to $m$ and at each column is equal to $n$. To see this, we observe that the isomorphism $\varphi$ maps the subgroup $H_n$ of $\mathbb{Z}_{nm}$ onto the subgroup $\{0\} \times \mathbb{Z}_m$ of $\mathbb{Z}_n \times \mathbb{Z}_m$. Hence, for each $i \in \mathbb{Z}_n$, the set $\{(i, j) : j \in \mathbb{Z}_m\}$ is
the image under \( \varphi \) of a certain coset of \( H_n \) in \( \mathbb{Z}_{nm} \), say, the coset \( a_i - H_n \). It follows that

\[
\sum_{j \in \mathbb{Z}_m} M(i, j) = \sum_{s \in H_n} h(a_i - s) = m, \tag{8.10}
\]

where in the last equality we used (8.8). In a similar way, \( \varphi \) maps the subgroup \( H_m \) onto \( \mathbb{Z}_n \times \{0\} \), so for each \( j \in \mathbb{Z}_m \) the set \( \{ (i, j) : i \in \mathbb{Z}_n \} \) is the image under \( \varphi \) of a coset \( b_j - H_m \), and we obtain

\[
\sum_{i \in \mathbb{Z}_n} M(i, j) = \sum_{s \in H_m} h(b_j - s) = n, \tag{8.11}
\]

again using (8.8). We thus see that the sum of the entries of \( M \) at each row is \( m \) and at each column is \( n \).

Notice that we cannot say that \( M \) is a doubly stochastic array, as the entries of \( M \) are not guaranteed to be nonnegative (see Definition 4.5). Nevertheless, we can now invoke Theorem 8.2 that is valid also for complex-valued matrices. As \( n, m \) are coprime, it follows from the theorem that the support of \( M \) has size at least \( n + m - 1 \). As \( \text{supp} \ h \) and \( \text{supp} \ M \) are of the same size, we conclude that

\[
| \text{supp} \ h | \geq n + m - 1. \tag{8.12}
\]

Lastly, we observe that if \( h(t) \neq 0 \) for some \( t \in \mathbb{Z} \), then \( g \) does not vanish on at least one element of the arithmetic progression \( \{ t - knm : k \in \mathbb{Z} \} \) due to (8.7). But these arithmetic progressions are pairwise disjoint as \( t \) goes through a complete set of residues modulo \( nm \). This shows that \( \text{supp} \ g \) has size at least as large as the size of \( \text{supp} \ h \). So, combined with (8.12) this implies that \( \text{supp} \ g \) is of size at least \( n + m - 1 \).

We have thus proved the result in the special case where \( n, m \) are coprime. To prove the result in the general case, we now let \( n, m \) be two arbitrary positive integers. We then write \( n = dn', \ m = dm' \), where \( d = \gcd(n, m) \) and \( n', m' \) are coprime. For each \( 0 \leq j \leq d - 1 \), we consider the function

\[
g_j(t) := g(j + dt), \quad t \in \mathbb{Z}. \tag{8.13}
\]

It follows from (8.2) that each \( g_j \) tiles the integers simultaneously by the two arithmetic progression \( n' \mathbb{Z} \) and \( m' \mathbb{Z} \) at levels \( p \) and \( q \), respectively. As \( n', m' \) are coprime (and the tiling level vector is nonzero) then, by what we have proved above, the size of \( \text{supp} \ g_j \) must be at least \( n' + m' - 1 \). It follows that

\[
| \text{supp} \ g | = \sum_{j=0}^{d-1} | \text{supp} \ g_j | \geq d(n' + m' - 1) = n + m - d, \tag{8.14}
\]

and we arrive at the desired conclusion. \( \square \)

We note that the correspondence between the \( n \times m \) doubly stochastic arrays and the nonnegative functions that tile the group \( \mathbb{Z}_{nm} \) by translations along each one of the two subgroups \( H_n \) and \( H_m \) (where \( n, m \) are coprime) was pointed out in an earlier version of [7].
8.1.4

Our next result shows that the lower bound in Theorem 8.3 is in fact sharp.

**Theorem 8.4.** For any two positive integers \( n, m \) there exists a nonnegative function \( g \) on \( \mathbb{Z} \), supported on a set of \( n + m - \gcd(n, m) \) consecutive integers, and satisfying (8.2) with \((p, q) = (m, n)\).

**Proof.** Let \( \chi_k \) denote the indicator function of the subset \{0, 1, ..., \( k - 1 \)\} of \( \mathbb{Z} \). We consider a function \( g \) on \( \mathbb{Z} \) defined as the convolution

\[
g(t) = (\chi_n * \chi_m)(t) = \sum_{s \in \mathbb{Z}} \chi_n(t - s) \chi_m(s), \quad t \in \mathbb{Z}.
\] (8.15)

Then \( g \) is supported on the set \{0, 1, ..., \( n + m - 2 \)\} of size \( n + m - 1 \). As the function \( \chi_n \) tiles at level one by translation with \( n\mathbb{Z} \), and \( \chi_m \) also at level one by translation with \( m\mathbb{Z} \), we can deduce from (8.15) that \( g \) satisfies the simultaneous tiling condition (8.2) with \((p, q) = (m, n)\). This proves the result in the case where \( n, m \) are coprime.

To prove the result in the general case, we write as before \( n = d n' \), \( m = d m' \), where \( d = \gcd(n, m) \) and \( n', m' \) are coprime. Let \( h \) be a nonnegative function on \( \mathbb{Z} \), supported on a set of \( n' + m' - 1 \) consecutive integers, which tiles simultaneously by the two arithmetic progressions \( n'\mathbb{Z} \) and \( m'\mathbb{Z} \) at levels \( m \) and \( n \), respectively (such a function \( h \) exists, by what we have proved above). We then define a function \( g \) on \( \mathbb{Z} \) by

\[
g(j + dt) := h(t), \quad 0 \leq j \leq d - 1, \quad t \in \mathbb{Z}.
\] (8.16)

Then \( g \) satisfies the simultaneous tiling condition (8.2), and \( g \) is supported on a set of \( d(n' + m' - 1) = n + m - d \) consecutive integers, as required. \( \square \)

8.2

Given a measurable function \( f \) on \( \mathbb{R} \), we define for each \( x \in \mathbb{R} \) a function \( f_x \) on the set of integers \( \mathbb{Z} \), given by

\[
f_x(t) := f(x + t), \quad t \in \mathbb{Z}.
\] (8.17)

The following lemma gives a connection between tilings of \( \mathbb{R} \) and tilings of \( \mathbb{Z} \).

**Lemma 8.5.** Let \( f \) be a measurable function on \( \mathbb{R} \), and \( \Lambda \) be a subset of \( \mathbb{Z} \). Then \( f + \Lambda \) is a tiling of \( \mathbb{R} \) at level \( w \) if and only if \( f_x + \Lambda \) is a tiling of \( \mathbb{Z} \) at the same level \( w \) for almost every \( x \in \mathbb{R} \).

**Proof.** Let \( f + \Lambda \) be a tiling of \( \mathbb{R} \) at level \( w \), then we have

\[
\sum_{\lambda \in \Lambda} f(x - \lambda) = w
\] (8.18)
for all \( x \) in some set \( E \subset \mathbb{R} \) of full measure. It follows that for \( t \in \mathbb{Z} \) we have

\[
\sum_{\lambda \in \Lambda} f_x(t - \lambda) = \sum_{\lambda \in \Lambda} f(x + t - \lambda) = w \tag{8.19}
\]

provided that \( x \in E - t \). (The series in both (8.18) and (8.19) are understood to converge absolutely.) Hence, \( f_x + \Lambda \) is a tiling of \( \mathbb{Z} \) at level \( w \) for every \( x \) belonging to the set \( \bigcap_{t \in \mathbb{Z}} (E - t) \), which is also a set of full measure in \( \mathbb{R} \).

Conversely, let \( f_x + \Lambda \) be a tiling of \( \mathbb{Z} \) at level \( w \) for almost every \( x \in \mathbb{R} \). Then

\[
\sum_{\lambda \in \Lambda} f(x - \lambda) = \sum_{\lambda \in \Lambda} f_x(-\lambda) = w \quad \text{a.e.} \tag{8.20}
\]

(with absolute convergence of the series) and so \( f + \Lambda \) is a tiling of \( \mathbb{R} \) at level \( w \).

**Lemma 8.6.** Let \( f \) be a measurable function on \( \mathbb{R} \). Then

\[
\text{mes}(\text{supp } f) = \int_{0}^{1} | \text{supp } f_x | \, dx. \tag{8.21}
\]

The proof of this lemma is standard and so we omit the details.

### 8.3

Now we can prove Theorems 2.8, 2.9, and 2.10.

**Proof of Theorem 2.8.** Let \( n, m \) be positive integers, and \( f \) be a measurable function on \( \mathbb{R} \) satisfying (2.3). By Lemma 8.5, the function \( g = f_x \) then satisfies the simultaneous tiling condition (8.2) for almost every \( x \in \mathbb{R} \). Using Proposition 8.1, we conclude that the vector \((p, q)\) must be proportional to \((m, n)\).

**Proof of Theorem 2.9.** Let \( f \) be a measurable function on \( \mathbb{R} \) satisfying (2.3) where \( n, m \) are positive integers and the vector \((p, q)\) is nonzero. By Lemma 8.5, the function \( g = f_x \) then satisfies the simultaneous tiling condition (8.2) for almost every \( x \in \mathbb{R} \). By applying Theorem 8.3 to the function \( g = f_x \), we obtain that \( | \text{supp } f_x | \geq n + m - \gcd(n, m) \) for almost every \( x \in \mathbb{R} \). Finally, combining this with Lemma 8.6 we conclude that

\[
\text{mes}(\text{supp } f) = \int_{0}^{1} | \text{supp } f_x | \, dx \geq n + m - \gcd(n, m), \tag{8.22}
\]

and so the theorem is proved.

**Proof of Theorem 2.10.** Let \( n, m \) be positive integers, and \((p, q) = (m, n)\). Let \( g \) be the function given by Theorem 8.4, that is, \( g \) is a nonnegative function on \( \mathbb{Z} \), supported on a set of \( n + m - \gcd(n, m) \) consecutive integers, and satisfying (8.2). We then construct a measurable (in fact, piecewise constant) nonnegative function \( f \) on \( \mathbb{R} \) given by \( f(x + t) = g(t) \) for every \( t \in \mathbb{Z} \) and \( x \in [0, 1) \). Then \( f \) is supported on an interval of length \( n + m - \gcd(n, m) \), and \( f \) satisfies the tiling condition (2.3) by Lemma 8.5.
ACKNOWLEDGMENTS
We thank Mihalis Kolountzakis for posing to us the problem discussed in Section 6. This research was supported by ISF Grant Number: 1044/21 and ERC Starting Grant Number: 713927.

JOURNAL INFORMATION
The Proceedings of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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