Abstract

In this note we present an application of the Schwarzian derivative. By exploiting some properties of the Schwarzian derivative, we solve the equation appearing in the gravity-dilaton-antisymmetric tensor system. We also mention that this method can also be used to solve some other equations.

The Schwarzian derivative appears naturally in the transformation law of the two-dimensional stress-energy tensor [1]:

\[(\partial_z z')^2 T'(z') = T(z) - \frac{c}{12} \{z', z\}, \tag{1}\]

where \(\{z', z\}\) denotes the Schwarzian derivative:

\[S(g) \equiv \left( \frac{g''}{g'} \right)' - \frac{1}{2} \left( \frac{g''}{g'} \right)^2. \tag{2}\]

It comes as a surprise that it also appears in a differential equation which we obtained during the course of solving the equations of motion of the coupled
system of gravity-dilaton-antisymmetric tensor system. The last equation to be solved [2, 3] is:

\[ Y'' - \frac{\Delta}{2} (Y')^2 + Q(r)Y' = R(r), \quad (3) \]

where \( Q(r) \) and \( R(r) \) are two specific functions of \( r \) (see [3] for exact formulas) and \( \Delta \) is a (non-vanishing) constant.

To solve this equation completely, we follows [2, 3]. Let

\[ g = \int dr \, e^{\Delta Y - \int Q(r)dr}, \quad (4) \]

or,

\[ Y = \frac{1}{\Delta} \left( \ln(g') + \int Q(r)dr \right), \quad (5) \]

eq. (3) becomes

\[ \left( \frac{g''}{g'} \right)' - \frac{1}{2} \left( \frac{g''}{g'} \right)^2 = \tilde{R}(r). \quad (6) \]

Here

\[ \tilde{R}(r) = \Delta R(r) - Q'(r) - \frac{1}{2} Q^2(r) = - \frac{d^2 - 1}{2r^2} + \frac{\tilde{\Lambda} r^{2d-2}}{2 (C_2 + C_1 r^{2d})^2} \quad (7) \]

with

\[ \tilde{\Lambda} = \Delta K - 4 \tilde{d}^2 C_1 C_2 \quad (8) \]

a constant.

The left-hand side of eq. (6) is the Schwarzian derivative of the function \( g \) as we defined in (3). Thus, in order to solve eq. (6), one must find a function \( g \) such that

\[ S(g) = - \frac{d^2 - 1}{2r^2} + \frac{\tilde{\Lambda}}{2} \frac{r^{2d-2}}{(C_2 + C_1 r^{2d})^2}. \quad (9) \]

We will do this in the following sections [2, 3].

1 Notations and Conventions

Let \( f \) be an arbitrary function of \( r \) and \( f' \) be its derivative. The notation \( f(r) \) always represents the value of the function \( f \) at \( r \). Similarly, \( f'(r) \) is the value of the function \( f' \) at \( r \).
For two functions $f$ and $g$, $f g$ is their product\[1\], and $\frac{f}{g}$ is their division, while the composition of two functions $f$ and $g$ is denoted by $f \circ g$. These functions are defined by

\[
(fg)(r) = f(r)g(r),
\]

\[
\left(\frac{f}{g}\right)(r) = \frac{f(r)}{g(r)},
\]

\[
(f \circ g)(r) = f(g(r)).
\]

For product of three or more functions we assume that the composition of two functions has a higher rank of precedence of associativity than the products of functions. Nevertheless the symbol $\frac{f}{g}$ is considered to be a pure entity and can’t be broken. To understand these conventions we have the following examples:

\[
f g \circ h = g \circ h f = f(g \circ h),
\]

\[
\frac{f}{g} \circ h = \left(\frac{f}{g}\right)\circ h = \frac{f \circ h}{g \circ h}.
\]

If $k \in \mathbb{C}$ or $\mathbb{R}$, we define a function $l_k$ as the multiplication of its variable with the number $k$, i.e.,

\[l_k(r) = kr.\]

For convenience we also consider $k$ itself as a constant function taking the value $k$:

\[k(r) = k.\]

Other functions such as the power function $r^s$ with real number $s$ and the exponential function $e^r$ are denoted by $p_s$ and $\text{exp}$ and other elementary functions are denoted by their standard mathematical symbols.

With these conventions, we can derive the following derivative rules:

\[
l_k' = k, \quad p_s' = sp_{s-1}, \quad \text{arctan}' = \frac{1}{1+p^2}, \quad \ln' = p_{-1},
\]

\[k' = 0, \quad \exp' = \exp, \quad l_n' = p_{-1},\]

and the derivative rule for composition of function is

\[
(f \circ g)' = g'f' \circ g
\]

\(^1\)We denote $f f$ by $f^2$, $f f f$ by $f^3$, and so on.
2 Some Properties of the Schwarzian Derivative

Now we list some elementary properties of the Schwarzian derivative.

(1) If \( f \) and \( g \) are two functions, we have

\[
S(f \circ g) = (g')^2 \ S(f) \circ g + S(g).
\] (17)

(2) If \( f = \frac{la + b}{lc + d} \) for some numbers \( a, b, c \) and \( d \), namely, \( f(r) = \frac{ar + b}{cr + d} \), then

\[
S \left( \frac{la + b}{lc + d} \right) = 0.
\] (18)

This is the well-know fact that the fractional linear transformation is a global conformal transformation of the complex sphere. By using this result we have

\[
S \left( \frac{la + b}{lc + d} \circ g \right) = S(g).
\] (19)

That is to say, if \( g \) is a special solution of the equation \( S(f) = R \), the general solution will be

\[
f = \frac{la + b}{lc + d} \circ g = \frac{ag + b}{cg + d}
\] (20)

with constants \( a, b, c, d \) such that \( ad - bc = 1 \).

(3) For \( s \in \mathbb{R} \), we have

\[
S(l_k) = 0,
\] (21)

\[
S(p_s) = - \frac{s^2 - 1}{2p_2},
\] (22)

\[
S(\exp) = - \frac{1}{2},
\] (23)

\[
S(\ln) = \frac{1}{2p_2},
\] (24)

\[
S(\tan) = 2,
\] (25)

\[
S(\arctan) = - \frac{2}{(1 + p_2)^2}.
\] (26)
3 The Complete Solution of Eq. (9)

The function $\tilde{R}$ in Eq. (9) is

$$\tilde{R} = \tilde{d}^2 - 1 + \pi p_{2d} - 2 \frac{1}{(C_2 + C_1 p_2)^2} \circ p_{\tilde{d}}. \tag{27}$$

For $C_1 C_2 > 0$, we have

$$\tilde{R} = S(p_{\tilde{d}}) + (p_{\tilde{d}}')^2 S(h_1) \circ p_{\tilde{d}}$$

$$= S(h_1 \circ p_{\tilde{d}}) \tag{28}$$

where $h_1$ is a yet-unknown function such that

$$S(h_1) = \frac{\Lambda}{2d^2 C_2^2} \frac{1}{(1 + \frac{C_2}{C_1} p_2)^2}$$

$$= S(l \sqrt{C_1/C_2}) + (l' \sqrt{C_1/C_2})^2 \frac{\Lambda}{2d^2 C_1 C_2} \frac{1}{(1 + p_2)^2} \circ l \sqrt{C_1/C_2}$$

$$= S(h_2 \circ l \sqrt{C_1/C_2}). \tag{29}$$

Now we want to find a function $h_2$ such that

$$S(h_2) = \frac{\Lambda}{2d^2 C_1 C_2} \frac{1}{(1 + p_2)^2}$$

$$= S(\arctan) + \frac{\Delta K}{2d^2 C_1 C_2} \frac{1}{(1 + p_2)^2} \circ \arctan$$

$$= S(\arctan) + (\arctan')^2 \frac{\Delta K}{2d^2 C_1 C_2} \circ \arctan$$

$$= S(h_3 \circ \arctan). \tag{30}$$

Note that in the third line we have considered $\frac{\Delta K}{2d^2 C_1 C_2}$ to be a constant function. The function $h_3$ in the above is easy to find to be $\tan \circ l_k$ with

$$k = \frac{1}{2d} \sqrt{\frac{\Delta K}{C_1 C_2}}. \tag{31}$$

because

$$S(h_3) = \frac{\Delta K}{2d^2 C_1 C_2}$$

$$= S(l_k) + (l_k')^2 S(\tan) \circ l_k$$

$$= S(\tan \circ l_k). \tag{32}$$
Combining all the above steps, one finds that
\[ \tilde{R} = S(\tan \circ l_k \circ \arctan \circ \sqrt{C_1/C_2} \circ p_d). \] (33)
and a special solution of eq. (9) is found to be:
\[ g_0 = \tan \circ l_k \circ \arctan \circ \sqrt{C_1/C_2} \circ p_d, \] (34)
namely,
\[ g_0(r) = \tan \left( k \arctan \sqrt{\frac{C_1}{C_2}} r \right). \] (35)

The special case considered in [4] corresponds to \( k = 1 \). By using Mathematica, one can easily check that the above function is indeed a solution of eq. (9).

The general solution of eq. (9) is obtained from the above special solution, eq. (35), by an arbitrary \( SL(2, R) \) transformation:
\[ g(r) = a_0 g_0(r) + b_0 c_0 g_0(r) + d_0, \] (36)
where \( a_0, b_0, c_0 \) and \( d_0 \) consist of an \( SL(2, R) \) matrix:
\[ a_0 d_0 - b_0 c_0 = 1. \] (37)

Here we have three independent constants. This is the right number for a third order ordinary differential equation. So we obtain the complete solution to eq. (9).

With this general solution in hand one can proceed to obtain the general solution of (3). For details please see [3].

4 More examples

There are also other examples which can be solved by the above method. One example comes from the special case \( \tilde{d} = 0 \) and the equation is as follows \( (\Delta = a^2) \):
\[ Y'' - \frac{a^2}{2} (Y')^2 + Q(r) Y' = R(r) \] (38)
where
\[ Q(r) = \frac{1 + a^2}{r} + \frac{1 + aC_3}{r \ln \frac{r_0}{r}}, \] (39)
\[ R(r) = \frac{a^2}{2r^2} + \frac{1 + aC_3}{r^2 \ln \frac{r_0}{r}} + \frac{C_3^2}{2r^2 \left( \ln \frac{r_0}{r} \right)^2} + \frac{(D - 3)C_4^2 - 4C_4 - 4(D - 1)}{4(D - 2) r^2 \left( \ln \frac{r_0}{r} \right)^2}. \] (40)
In terms of
\[ g(r) = \int dr e^{a^2 Y - \int Q(r) dr}, \]
(41)
or, equivalently,
\[ Y = \frac{1}{a^2} (\ln g') + \int Q(r) dr, \]
(42)
the equation for \( Y \) can be written as
\[ S(g) = \tilde{R}(r), \]
(43)
where \( S(g) \) is the Schwarzian derivative of \( g \) and
\[
\tilde{R}(r) &= a^2 R(r) - Q'(r) - \frac{1}{2} Q^2(r) \\
&= \frac{1}{2r^2} + \frac{1 - K}{2r^2 (\ln \frac{r_0}{r})^2} \]
(44)
with
\[ K = 4(1 + aC_3) + \frac{4(D - 1) + 4C_4 - (D - 3)C_4^2}{2(D - 2)} a^2. \]
(45)
For different choices of \( K \) the special solution is as follows:
(1) For \( K = 0 \),
\[ g_0 = \ln \left| \ln \frac{r_0}{r} \right|; \]
(46)
(2) For \( K > 0 \) (\( k = \sqrt{K} \)),
\[ g_0 = \left| \ln \frac{r_0}{r} \right|^k; \]
(47)
(3) For \( K < 0 \) (\( k = \sqrt{-K} \)),
\[ g_0 = \tan \left( \frac{k}{2} \ln \left| \ln \frac{r_0}{r} \right| \right). \]
(48)

For more examples and extensive discussions, we refer the readers to [5].

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