Boltzmann-Gibbs Distribution of Fortune and Broken Time-Reversible Symmetry in Econodynamics

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Abstract

Within the description of stochastic differential equations it is argued that the existence of Boltzmann-Gibbs type distribution in economy is independent of the time reversal symmetry in econodynamics. Both power law and exponential distributions can be accommodated by it. The demonstration is based on a mathematical structure discovered during a study in gene regulatory network dynamics. Further possible analogy between equilibrium economy and thermodynamics is explored.

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89.65.Gh Economics; econophysics, financial markets, business and management;
05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc)
05.20.-y Classical statistical mechanics.
87.23.Ge Dynamics of social systems
I. INTRODUCTION

Carefully and extensive analysis of real economic and financial data have revealed various exponential and power-law distributions regarding to money, income, wealth, and other economic quantities in ancient and modern social societies [1, 2, 3, 4]. A remarkable analogy between the economic system and a thermodynamical system has been revealed in a recent study [4]. Using a detailed microdynamical model with time reversal symmetry, it was demonstrated that a Boltzmann-Gibbs type distribution exists in economic systems. Indeed, ample empirical analysis support such suggestion [4, 5]. Nevertheless, different microdynamical models lead to apparently different distributions [3, 4]. Those distributions are supported by empirical data, too. The nature of such difference may reveal the difference in corresponding economic structure. For example, this difference has been tentatively attributed to the role played by time reversal symmetry in microdynamical models [4].

Following the tradition of synthesizing interdisciplinary knowledge, such as from biology, physics, and finance [4], in this letter we argue that irrespective of the time reversal symmetry the Boltzmann-Gibbs type distribution always exists. A broader theoretical base is thus provided. The demonstration is performed within the framework of stochastic differential equations and is based on a novel mathematical structure discovered during a recent study in gene regulatory network dynamics [7]. In the light of thermodynamical description, possible explanations for the origin of the difference in various empirical distributions are proposed.

II. BOLTZMANN-GIBBS DISTRIBUTION IN FINANCE

Stochastic differential equations or Langevin equations and their corresponding Fokker-Planck equations have been shown to be a useful modelling tool in economy [1, 2, 8, 9]. One of the best examples is the Black-Scholes formula in option pricing theory [8]. This kind of mathematical formulation provides a direct connections between the microdynamics and the stationary state and has been used to generate various distribution laws [1, 2, 3, 4, 5].

Specifically, the stochastic differential equation may take the following form:

$$\dot{q} = f(q) + N_t(q)\xi(t),$$

(1)

where $f$ and $q$ are $n$-dimensional vectors and $f$ a nonlinear function of the state variable $q$. The state variable is the quantity to specify the economic system. It may be the money,
the income, or other suitable indices. The noise $\xi$ is a standard Gaussian white noise with $l$

independent components: $\langle \xi_i \rangle = 0$, $\langle \xi_i(t)\xi_j(t') \rangle = 2\delta_{ij}\delta(t-t')$, and $i,j = 1,2,...,l$. In Eq.(1) we explicitly factorize out the pure noise source and the state variable dependent part for the convenient of later description.

The specification of the noise in Eq.(1) is through the $n \times n$ diffusion matrix $D(q)$ by the following matrix equation

$$N_I(q)N_I^T(q) = \epsilon D(q),$$  

(2)

where $N_I$ is an $n \times l$ matrix, $N_I^T$ is its the transpose, and $\epsilon$ a nonnegative numerical constant to keep tract of the noise. It plays the role of temperature in thermodynamics. According to Eq.(2) the $n \times n$ diffusion matrix $D$ is both symmetric and nonnegative. For the dynamics of state variable $q$, all what needed from the noise is the diffusion matrix $D$. Hence, it is not necessary to require the dimension of the noise vector $\xi$ be the same as that of the state vector $q$ and to require more specific knowledge of $n \times l$ matrices $\{N_I\}$ beyond Eq.(2).

It is known that even in situations that Eq.(1) is not an exact description, and perhaps it would never be in a rigorous sense in economy, it may still serve as the first approximation for further modelling [1, 2]. Indeed, it has been empirically verified to be a rather accurate description in economy [8, 9, 10, 11]. Because the energy function or Hamiltonian has played a dominant role in equilibrium physics processes, the crucial question is whether or not a similar quantity exists in a more general setting. In the following we present an argument leading to the positive answer.

There exists several ways to deal with the stochastic equations equation in the form of Eq.(1) and (2). The most commonly used are those of Ito and Stratonovich methods [1, 2, 12]. However, with those methods the connection between the existence of energy function like quantity in Eq.(1) and the stationary distribution is not clear when the time reversal symmetry is broken [12]. The difficulty for finding such potential function can be illustrated by the fact that usually $D^{-1}(q)f(q)$ cannot be written as the gradient of a scalar function $\phi(q)$ in the absence of detailed balance condition or in the broken time reversal symmetry. This will become precise as we proceed.

During a recent study of the robustness of the genetic switch in a living organism [7], it was discovered that Eq.(1) can be transformed into the following form,

$$[A(q) + C(q)]\dot{q} = \partial_q\phi(q) + N_{II}(q)\xi(t),$$  

(3)
where the noise $\xi(t)$ is from the same source as that in Eq.(1). Here we tentatively name $A(q)$ the adaptation matrix, $C(q)$ the conservation matrix, and the scalar function $\phi(q)$ the fortune function. The gradient operation in state space is denoted by $\partial_q$. The adaptation matrix $A(q)$ is defined through the following matrix equation

$$N_{II}(q)N_{II}^T(q) = \epsilon A(q) ,$$

(4)

which guarantees that $A$ is both symmetric and nonnegative. The $n \times n$ conservation matrix $C$ is antisymmetric. We define

$$A(q) + C(q) = 1/[D(q) + Q(q)] \equiv M(q) .$$

with the $n \times n$ matrix $M$ is the solution of following two matrix equations [13]. The first equation is the potential condition

$$\partial_q \times [M(q)f(q)] = 0 ,$$

(5)

which gives $n(n - 1)/2$ conditions [the wedge product for two arbitrary vectors $v_1$ and $v_2$ in $n$-dimension: $[v_1 \times v_2]_{ij} = v_1_i v_2_j - v_1_j v_2_i , i, j = 1, 2, ..., n$]. The second equation is the generalized Einstein relation between the adaptation and diffusion matrices in the presence of conservation matrix

$$M(q)D(q)M^T(q) = \frac{1}{2}[M(q) + M^T(q)] .$$

(6)

which gives $n(n + 1)/2$ conditions [13]. The fortune function $\phi(q)$ is connected to the deterministic force $f(q)$ by

$$\partial_q \phi(q) = M(q)f(q) .$$

For simplicity we will assume $\det(A) \neq 0$ in the rest of the letter. Hence $\det(M) \neq 0$ [14]. Thus, the adaptation matrix $A$, the conservation matrix $Q$ and the fortune function $\phi$ in Eq.(3) and (4) can be completely determined by Eq.(1) and (2). The breakdown of detailed balance condition or the time reversal symmetry is represented by the finiteness of the conservation matrix

$$C(q) \neq 0 ,$$

(7)

or equivalently $Q \neq 0$. The usefulness of the formulation of Eq.(3) and (4) is already manifested in the successful solution of outstanding stable puzzle in gene regulatory dynamics [7] and in solving two fundamental controversies in population genetics [15].
A few remarks on Eq.(3) are in order. In the light of classical mechanics in physics, Eq.(3) is in precisely the form of Langevin equation. The fortune function $\phi$ corresponds to the potential function but opposite in sign to reflect the fact that in economy there is a tendency to seek the peak or maximum of fortune. The adaptive matrix $A$ plays the role of friction. It represents adaptive dynamics and is the dynamical mechanism to seek the nearby fortune peak. The conservation matrix $C$ plays the role analogous to a magnetic field. Its dynamics is similar to that of the Lorentz force, hence conserves the fortune. As in classical mechanics, the finiteness of the conservation matrix $C$ breaks the time reversal symmetry.

It was heuristically argued [13] and rigorous demonstrated [16] that the stationary distribution $\rho(q)$ in the state space is, if exists,

$$\rho(q) \propto \exp \left( \frac{\phi(q)}{\epsilon} \right).$$

(8)

Therefore, the fortune function $\phi$ acquires both the dynamical meaning through Eq.(3) and the steady state meaning through Eq.(8). Specifically, in the so-called zero-mass limit to differentiate from Ito and Stratonovich methods, the Fokker-Planck equation for the probability distribution $\rho(q, t)$ takes the form [16]

$$\partial_t \rho(q, t) = \partial_q M^{-1}(q)[\epsilon \partial_q - \partial_q \phi(q)]\rho(q, t).$$

(9)

Here $\partial_t$ is a derivative with respect to time and $\partial_q$ represents the gradient operation in state space. We note that Eq.(8) is a stationary solution to Eq.(9) even it may not be normalizable, that is, even when the partition function $Z = \int d^n q \rho(q)$ is ill-defined. Again, we emphasize that no time reversal symmetry is assumed in reaching this result. This completes our demonstration on the existence of the Boltzmann-Gibbs distribution in economy.

Using $M^{-1}(q) = D(q) + Q(q)$ and $f(q) = [D(q) + Q(q)]\partial_q \phi(q)$, Eq.(9) can be rewritten in a more suggestive form [16]

$$\partial_t \rho(q, t) = \partial_q^T \epsilon D(q) \partial_q + \epsilon (\partial_q^T Q(q)) - f(q) |\rho(q, t).$$

(10)

It is clear that in the presence of time reversal symmetry, i.e. $Q = 0$, one can directly read the fortune function $\phi$ from above form of Fokker-Planck equation as $\partial_q \phi(q) = D^{-1}(q)f(q)$.

For the sake of completeness, we list the Fokker-Placnk equations corresponding to Ito
and Stratonovich treatments of Eq.(1) and (2) 

\[ \partial_t \rho_I(q, t) = \sum_{i=1}^{n} \partial_{q_i} \left[ -f_i(q) + \epsilon \sum_{j=1}^{n} \partial_{q_j} D_{ij}(q) \right] \rho_I(q, t) , \quad (Ito) \]  

and

\[ \partial_t \rho_S(q, t) = \sum_{i=1}^{n} \partial_{q_i} \left[ -f_i(q) + \sum_{j=1}^{n} \sum_{k=1}^{l} N_{i,jk}(q) \partial_{q_j} N_{ik}(q) \right] \rho_S(q, t) . \quad (Str) \]

The connection of fortune function with both dynamical (Eq.(1) and (2)) and stationary state is indeed not clear in above two equations. Nevertheless, it has been shown [16] that there are corresponding fortune functions, adaptation and conservation matrices to Eq.(11) and (12). We point out here that when the matrix $N_I$ is independent of state variable, Eq.(11) and (12) are the same but may still differ from Eq.(9), because the gradient of the antisymmetric matrix $Q(q)$ may not be zero. This last property shows that the time reversal symmetry is indeed important.

### III. TWO EXAMPLES

The Fokker-Planck equation used by Silva and Yakovenko [5] has the form:

\[ \partial_t \rho_{sy}(q, t) = \partial_q \left[ a(q) + \partial_q b(q) \right] \rho_{sy}(q, t) , \quad (13) \]

and Fokker-Planck equation used by Bouchaud and Mezard [3] has the form

\[ \partial_t \rho_{bm}(q, t) = \partial_q \left[ (J(q - 1) + \sigma^2 q + \sigma^2 q \partial_q q) \right] \rho_{bm}(q, t) . \quad (14) \]

They are all in one dimension. We immediately conclude that the conservation matrix $C$, equivalently $Q$, is zero, because there is no conservation matrix in one dimension. In accordance with the definition in the present letter, which is consistent with that in nonequilibrium processes [12], the dynamics described by above two equations can be effectively classified as time reversal symmetric.

Rewriting them in symmetric form with respect to the derivative of state variable $q$ as in Eq.(10), we have

\[ \partial_t \rho_{sy}(q, t) = \partial_q \left[ a(q) + \partial_q b(q) \right] \rho_{sy}(q, t) , \quad (15) \]

and

\[ \partial_t \rho_{bm}(q, t) = \partial_q \left[ (J(q - 1) + \sigma^2 q + \sigma^2 q \partial_q q) \right] \rho_{bm}(q, t) . \quad (16) \]
The corresponding wealth functions can be immediate read out as

\[ \phi_{sy}(q) = -\int_{q_0}^{q} dq' \frac{a(q') + (\partial_q b(q'))}{b(q')} \]
\[ = -\frac{a}{b}(q - q_0), \text{ if } a, b = \text{constant} \quad (17) \]

\[ \phi_{bm}(q) = -\int_{q_0}^{q} dq' \left( \frac{J(q' - 1) + 2\sigma^2 q'}{\sigma^2 q'^2} \right) \]
\[ = -\frac{J}{\sigma^2 q} \left( 2 + \frac{J}{\sigma^2} \right) \ln q + \frac{J}{\sigma^2 q_0} + \left( 2 + \frac{J}{\sigma^2} \right) \ln q_0 \quad (18) \]

They are exactly what found in Ref.[5] and [3]: the first one corresponds to an exponential distribution and the second one a power law distribution according to the Boltzmann-Gibbs distribution Eq.(8).

IV. ENSEMBLES AND STATE VARIABLES

Having defined a precise meaning of time reversal symmetric and have demonstrated that the Boltzmann-Gibbs distribution even in the absence of time reversal symmetry, we explore further connection between econodynamics and statistical physics.

There are two general types of situations which would generate different distributions in statistical physics and thermodynamics. The first one is to link to constraints on the system under various conditions. In statistical physics such constraints are described by various ensembles and free energies. For example, there are canonical and grand-canonical ensembles. There are Gibbs and Helmholtz free energies, entropy, enthalpy, etc. Those ensembles have their characteristic distributions. It would be interesting to know the corresponding situations in economy.

Even with a given constraint, the form of distribution depends on the choice of state variable. For example, for ideal gas model, the distributions are different if views from the kinetic energy and from velocity. Hence, there is a question of appropriate state variable for a given situation, with which the physics becomes particular transparent. It would be interesting to know what be the appropriate variables to describe an economic system.

Within this context, the difference between what discovered by Dragulecu and Yakovenko [4] and Bouchaud and Mezard [3] is perhaps more due to the difference in choices of state variables, because it seems they are describing the same situation of same system under same constraints.
A remark on terminology is in order. It was demonstrated above that regardless of the time reversal symmetry the Boltzmann-Gibbs distribution, Eq. (8), exists. The fortune function $\phi$ has an additional dynamical meaning defined in Eq. (3). Both exponential and power law distribution can be represented by Eq. (8). In fact, it is well known that power law distributions exist in statistical physics. A nontrivial example is the Kosterlitz-Thouless transition [17]. Thus it does not appear appropriate to call the power law distribution non-Boltzmann-Gibbs distribution. Such a terminology confusion was already noticed before [18].

In the view of the dominant role of entropy in Kosterlitz-Thouless transition [17], the ubiquitous existence of power law distribution in economy may suggest that the entropy effect is rather important in econodynamics. This may corroborate with the suggestion of “superthermal” in economy [5].

V. CONCLUSIONS

In this letter we demonstrate that the existence of Boltzmann-Gibbs distribution in finance is independent of time reversal symmetry. Both power law and exponential distributions are within its description. In analogous to similar situation in statistical physics, the differences among those distributions discovered empirically in economy are likely the result of different choices of state variables to describe the same system in econodynamics.

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