On the probability of staying above a wall for the (2 + 1)-dimensional SOS model at low temperature

Pietro Caputo1 · Fabio Martinelli1 · Fabio Lucio Toninelli2

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Abstract We obtain sharp asymptotics for the probability that the (2 + 1)-dimensional discrete SOS interface at low temperature is positive in a large region. For a square region \( \Lambda \), both under the infinite volume measure and under the measure with zero boundary conditions around \( \Lambda \), this probability turns out to behave like \( \exp(-\tau_\beta(0) L \log L) \), with \( \tau_\beta(0) \) the surface tension at zero tilt, also called step free energy, and \( L \) the box side. This behavior is qualitatively different from the one found for continuous height massless gradient interface models (Bolthausen et al., Commun Math Phys 170(2):417–443, 1995; Deuschel et al., Stochastic Process Appl 89(2):333–354, 2000).

Keywords SOS model · Loop ensembles · Random surface models · Entropic repulsion · Large deviations

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1 Dipartimento di Matematica e Fisica, Università Roma Tre, Largo S. Murialdo 1, 00146 Rome, Italy
2 Université de Lyon, CNRS and Institut Camille Jordan, Université Lyon 1, 43 bd du 11 novembre 1918, 69622 Villeurbanne, France
1 Introduction

Let $\mathbb{P}_\Lambda$ denote the Gibbs measure of the $(2 + 1)$-dimensional SOS model on a box $\Lambda \subset \mathbb{Z}^2$ with zero boundary condition. The configurations are discrete height functions $\eta : \Lambda \mapsto \mathbb{Z}$ whereas $\eta(x) = 0$ for $x \notin \Lambda$. The probability measure is given by

$$\mathbb{P}_\Lambda(\eta) = \frac{\exp \left( -\beta \sum_{|x-y|=1} |\eta(x) - \eta(y)| \right)}{Z_\Lambda},$$

where $\beta > 0$ is the inverse temperature, and $Z_\Lambda$ denotes the associated normalizing factor, called partition function. We will mostly consider the case where $\Lambda = \Lambda_L = [-L, L]^2 \cap \mathbb{Z}^2$ is the square of side $2L + 1$ in $\mathbb{Z}^2$ centered at the origin.

It is well known that, if $\beta$ is sufficiently large (as we assume from here on), the limit of $\mathbb{P}_\Lambda_L$ as $L \to \infty$ exists (in the sense that the probability of any local event converges), and is denoted $\mathbb{P}$, the infinite-volume Gibbs measure; see e.g. [3].

The infinite volume measure is characterized by the fact that heights have finite variance and exponentially decaying tails: the interface is globally very rigid and flat, the height is exactly zero on a set of sites of density $1 - O(\exp(-4\beta))$ and typical fluctuations are isolated spikes; see [3,4,7]. The question we investigate here is that of large fluctuations of the interface, namely, the asymptotics of the probability that the interface is positive in a fixed large region. In order to formulate our main result, let us recall the definition of the surface tension at zero tilt, often referred to as step free energy:

**Definition 1.1** Let $\xi$ be the height function on $\Lambda_L^n$ such that $\xi(x) = 1$ if $x = (x_1, x_2)$ with $x_2 \geq 0$, and $\xi(x) = 0$ otherwise. Let $Z_{\Lambda_L}^\xi$ be the partition function on $\Lambda_L$ with boundary condition $\xi$ (see Sect. 2.1 below for more details). Then, the surface tension at zero tilt is defined as

$$\tau_\beta(0) = -\lim_{L \to \infty} \frac{1}{2\beta L} \log \frac{Z_{\Lambda_L}^\xi}{Z_{\Lambda_L}}.$$  (1.1)

It is a known fact that $\tau_\beta(0)$ is well defined and that, for $\beta$ sufficiently large, one has $\tau_\beta(0) > 0$, see Lemma 2.4 below for more details. We have then:

**Theorem 1.2** There exists $\beta_0 > 0$ such that for any $\beta \geq \beta_0$ one has

$$\lim_{L \to \infty} \frac{1}{L \log L} \log \mathbb{P}_\Lambda (\eta(x) \geq 0 \text{ for every } x \in \Lambda_L) = -2\tau_\beta(0).$$  (1.2)

The same limit holds if we replace $\mathbb{P}_\Lambda$ by $\mathbb{P}$.

Actually, it will be clear from the proof that the result still holds if we replace the inequality $\eta(x) \geq 0$ with $\eta(x) \geq n$, for any fixed $n > 0$.

We now describe the heuristics behind Theorem 1.2. In [7] (see also [6] for a summary of the main results) the scaling limit of the shape of the SOS surface in the box $\Lambda_L$ with zero boundary conditions and conditioned to be non-negative was
established in full detail. The SOS interface lifts rigidly to a height $H(L) = \lfloor \frac{1}{4\beta} \log L \rfloor$, in order to create room for downward spike-like fluctuations (entropic repulsion). As a consequence there are $H(L)$ macroscopic level lines, following approximately $\partial \Lambda_L$, where the height of the surface jumps (roughly) by one. A fraction $1 - o(1)$ of the level lines is at distance $o(L)$ from $\partial \Lambda_L$ while the rest has a non trivial scaling limit as $L \to \infty$, with flat and curved parts and $1/3$ fluctuation exponent along the flat part. Roughly each of the level lines at distance $o(L)$ from $\partial \Lambda_L$ entails a surface energy cost $|\partial \Lambda_L| \beta \tau(0) = 8\beta L \log L$, which explains (1.2). The difficulty that arises in substantiating this heuristics is that the $H(L)$ contours have mutual interactions. If these are naively estimated, they produce an additive term, of apriori indefinite sign, of order $O(\beta \log L)$ in the energy cost. Here $\epsilon_\beta = c_\beta/\beta > 0$ is a constant tending to zero as $\beta \to \infty$, but non-zero for any finite $\beta$. While this problem can be avoided when looking for a lower bound on the l.h.s. of (1.2), simply by imposing that the contours stay sufficiently far one from the other to neglect the interaction, as an upper bound we would get nothing better than $-2\tau(0) + \epsilon_\beta$.

The solution we find is an iterative monotonicity argument (Theorem 4.1), based on the FKG properties of the SOS model, which we believe is of interest by itself. This allows us to conclude that the possibly attractive effect of the mutual interaction potential is more than compensated by the loss of entropy due to the fact that the contours cannot mutually cross. As a consequence, the surface tension associated to $n$ SOS contours is at least the sum of the individual surface tensions (Corollary 4.2).

1.1 Discussion

Since the early work of Lebowitz and Maes [14], the problem of computing the sharp large deviation behavior of the positivity event $\eta(x) \geq 0$, $x \in \Lambda_L$, has attracted much attention. Refined estimates have been obtained for continuous height models such as the Gaussian free field on $\mathbb{Z}^d$, see [1,2,8], as well as for more general lattice massless free fields [9]. A large deviation theory for such models was further developed in [10]. The problem is of particular relevance in the study of the entropic repulsion phenomenon [4], see e.g. [16] for a survey. Considerable progress has been recently made for the SOS model [5–7] and for the discrete Gaussian model [15] for which the SOS gradient term $|\eta(x) - \eta(y)|$ in the energy function is replaced by $(\eta(x) - \eta(y))^2$, but the question of computing the limit in (1.2) remained unaddressed.

As a matter of comparison, let us briefly recall the known results for the two-dimensional continuous Gaussian case. If $\mathbb{P}_L$ denotes the 2D Gaussian free field on $\Lambda_L$ with zero boundary condition, then for any $\delta \in (0, 1)$ one has

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1 After completing this work we realized that a conceptually similar argument was put forward by Bricmont et al. [4, Appendix 1] to compare the step free energy to the free energy associated to a single macroscopic step in the boundary condition.
\[
\lim_{L \to \infty} \frac{1}{(\log L)^2} \log P_L \left( \eta(x) \geq 0 \text{ for every } x \in \Lambda_{(1-\delta)L} \right) = -\kappa(\delta),
\]

where \( \kappa(\delta) > 0 \) is a constant related to the relative capacity of the set \( \Lambda_{(1-\delta)L} \) with respect to \( \Lambda_L \) which satisfies \( \kappa(\delta) \to \infty \) as \( \delta \to 0 \); see \cite[Theorem 3]{1}. On the other hand, boundary effects dominate if all heights in \( \Lambda_L \) are required to be nonnegative, and one expects \cite[Section 3]{9} that

\[
\lim_{L \to \infty} \frac{1}{L} \log P_L \left( \eta(x) \geq 0 \text{ for every } x \in \Lambda_L \right) = -\chi,
\]

for some \( \chi > 0 \). Because of its discrete nature, the SOS interface considered in our work presents a very different behavior. First, the rigidity of the interface allows one to consider the infinite volume limit—whereas the 2D massless free field does not admit such a limit. Second, while the typical height in the bulk under the positivity constraint is of order \( \log L \) just as in the case of the 2D massless free field, the cost of such a shift is much higher due to the unavoidable presence of as many as \( H(L) \) macroscopic level lines each of which has a definite cost proportional to the length. In particular, boundary terms do not dominate here and the estimate of Theorem 1.2 holds for \( P \) as well as for \( P_{\Lambda_L} \).

2 Contours, surface tension, etc.

Here we define the model, and the notion of contours of the SOS interface. To express the law of contours we shall use a cluster expansion for partition functions of the SOS model. Finally we recall the definition of surface tension for a general tilt, and some of its properties.

2.1 SOS model: basic definitions and notation

We call a bond (resp. dual bond) any straight line segment joining two neighboring sites in \( \mathbb{Z}^2 \) (resp. of \( \mathbb{Z}^2^* \), the dual lattice of \( \mathbb{Z}^2 \)). Here \( \mathbb{Z}^2 \) and \( \mathbb{Z}^2^* \equiv \mathbb{Z}^2 + (1/2, 1/2) \) are thought of as embedded in \( \mathbb{R}^2 \). For any finite \( \Lambda \subset \mathbb{Z}^2 \), let \( B_\Lambda \subset \mathbb{Z}^2 \) denote the set of bonds of the form \( e = xy \) with \( x \in \Lambda \) and \( y \in \Lambda \cup \partial \Lambda \), where \( \partial \Lambda \) is the external boundary of \( \Lambda \), i.e. the set of \( y \in \Lambda^c \) such that \( xy \) is a bond for some \( x \in \Lambda \). A height configuration \( \tau : \Lambda^c \mapsto \mathbb{Z} \) is called a boundary condition. We define \( \Omega_\Lambda^\tau \) as the set of height functions \( \eta : \mathbb{Z}^2 \mapsto \mathbb{Z} \) such that \( \eta(x) = \tau(x) \) for all \( x \notin \Lambda \). The SOS Hamiltonian in \( \Lambda \) with boundary condition \( \tau \) is the function defined by

\[
\mathcal{H}_\Lambda^\tau(\eta) = \sum_{xy \in B_\Lambda} |\eta(x) - \eta(y)|, \quad \eta \in \Omega_\Lambda^\tau.
\]

The SOS Gibbs measure in \( \Lambda \) with boundary condition \( \tau \) at inverse temperature \( \beta \) is the probability measure \( \mathbb{P}_\Lambda^\tau \) on \( \Omega_\Lambda^\tau \) given by

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\[ \mathbb{P}_\Lambda^\tau(\eta) = \frac{1}{Z^\tau_\Lambda} \exp\left(-\beta \mathcal{H}_\Lambda^\tau(\eta)\right), \]

where \( Z^\tau_\Lambda \) is the partition function

\[ Z^\tau_\Lambda = \sum_{\eta \in \Omega^\tau_{\Lambda}} \exp\left(-\beta \mathcal{H}_\Lambda^\tau(\eta)\right). \]

When \( \tau = 0 \) we simply write \( Z_\Lambda \) for \( Z^0_\Lambda \) and \( \mathbb{P}_\Lambda \) for \( \mathbb{P}^0_\Lambda \). We often consider boxes \( \Lambda \) of rectangular shape, and write \( \Lambda_{L,M} \), with \( L, M \in \mathbb{N} \), for the rectangle \( \Lambda_{L,M} = ([−L, L] \times [−M, M]) \cap \mathbb{Z}^2 \) centered at the origin. When \( L = M \) we write \( \Lambda_L \) for the square of side \( 2L + 1 \).

We recall that the SOS model satisfies the so-called FKG inequality [12] with respect to the natural partial order defined by \( \eta \leq \eta' \iff \eta(x) \leq \eta'(x) \) for every \( x \). That is, if \( f \) and \( g \) are two increasing (w.r.t. the above partial order) functions, then

\[ \mathbb{E}_\Lambda^n( fg) \geq \mathbb{E}_\Lambda^n(f) \mathbb{E}_\Lambda^n(g) \]

for any region \( \Lambda \) and any boundary condition \( \tau \), where \( \mathbb{E}_\Lambda^n \) denotes expectation w.r.t. \( \mathbb{P}_\Lambda^n \). To prove the FKG inequality one can establish directly the validity of the FKG lattice condition

\[ \mathbb{P}_\Lambda^n(\eta \lor \eta') \mathbb{P}_\Lambda^n(\eta \land \eta') \geq \mathbb{P}_\Lambda^n(\eta) \mathbb{P}_\Lambda^n(\eta'). \] (2.1)

2.2 Geometric contours, \( h \)-contours etc.

We use the following notion of contours.

**Definition 2.1** Two sites \( x, y \) in \( \mathbb{Z}^2 \) are said to be separated by a dual bond \( e \) if their distance (in \( \mathbb{R}^2 \)) from \( e \) is \( \frac{1}{2} \). A pair of orthogonal dual bonds which meet in a site \( x^* \in \mathbb{Z}^{2*} \) is said to be a linked pair of bonds if both are on the same side of the forty-five degrees line (w.r.t. to the horizontal axis) across \( x^* \). A geometric contour (for short a contour in the sequel) is a sequence \( e_0, \ldots, e_n \) of dual bonds such that:

1. \( e_i \neq e_j \) for \( i \neq j \), except for \( i = 0 \) and \( j = n \) where \( e_0 = e_n \).
2. For every \( i \), \( e_i \) and \( e_{i+1} \) have a common vertex in \( \mathbb{Z}^{2*} \).
3. If \( e_i, e_{i+1}, e_j, e_{j+1} \) all have a common vertex \( x^* \in \mathbb{Z}^{2*} \), then \( e_i, e_{i+1} \) and \( e_j, e_{j+1} \) are linked pairs of bonds.

We denote the length of a contour \( \gamma \), i.e. the number of distinct bonds in \( \gamma \), by \( |\gamma| \), its interior (the sites in \( \mathbb{Z}^2 \) it surrounds) by \( \Lambda_\gamma \) and its interior area (the number of such sites) by \( |\Lambda_\gamma| \). Moreover we let \( \Delta_\gamma \) be the set of sites in \( \mathbb{Z}^2 \) such that either their distance (in \( \mathbb{R}^2 \)) from \( \gamma \) is \( \frac{1}{2} \), or their distance from the set of vertices in \( \mathbb{Z}^{2*} \) where two non-linked bonds of \( \gamma \) meet equals \( 1/\sqrt{2} \). Finally we let \( \Delta_\gamma^+ = \Delta_\gamma \cap \Lambda_\gamma \) and \( \Delta_\gamma^- = \Delta_\gamma \setminus \Delta_\gamma^+ \). Given a contour \( \gamma \) we say that \( \gamma \) is an \( (h \)-contour) for the configuration \( \eta \) if

\[ \eta|_{\Delta_\gamma^-} \leq h - 1, \quad \eta|_{\Delta_\gamma^+} \geq h. \]

Finally \( C_{\gamma,h} \) will denote the event that \( \gamma \) is an \( h \)-contour.
Fig. 1 Example of a SOS configuration above the wall in the $7 \times 7$ box $\Lambda_3$ with zero boundary conditions: white sites have height $0$, shaded sites have height $1$ and darker sites have height $2$. Notice that according to Definition 2.1 there are three $1$-contours and two $2$-contours.

To illustrate the above definitions with a simple example, consider the elementary contour given by the square of side $1$ surrounding a site $x \in \mathbb{Z}^2$. In this case, $\gamma$ is an $h$-contour iff $\eta(x) \geq h$ and $\eta(y) \leq h-1$ for all $y \in \{x \pm e_1, x \pm e_2, x+e_1+e_2, x-e_1-e_2\}$. We observe that a geometric contour $\gamma$ could be at the same time an $h$-contour and an $h'$-contour with $h \neq h'$. More generally two geometric contours $\gamma, \gamma'$ could be contours for the same surface with different height parameters even if $\gamma \cap \gamma' \neq \emptyset$, but then the interior of one of them must be contained in the interior of the other; see Fig. 1 for an example.

2.3 Cluster expansion

So called cluster expansions are a well established tool for the analysis of random interfaces at low-temperature; see e.g. [3] where both the SOS model and the discrete gaussian model are considered. Here we shall need a particular expansion that allows us to take into account the extra constraints that appear naturally in the partition function of the SOS model on a region $\Lambda$ delimited by two contours; see (2.5) below.

Given a finite connected set $\Lambda \subset \mathbb{Z}^2$, let $\partial_\Lambda$ denote the set of $y \in \Lambda$ either at distance $1$ from $\partial \Lambda$ or at distance $\sqrt{2}$ from $\partial \Lambda$ in the south-west or north-east direction. Fix $U_+, U_- \subset \partial_\Lambda$, and let $Z_{\Lambda,U_+,U_-}$ denote the SOS partition function in $\Lambda$ with the sum over $\eta$ restricted to those $\eta \in \Omega_\Lambda^0$ such that $\eta(x) \geq 0$ for all $x \in U_+$ and $\eta(x) \leq 0$ for all $x \in U_-$. Clearly, if $U_- \cap U_+ \neq \emptyset$, then $\eta(x) = 0$ is fixed for all $x \in U_- \cap U_+$. If $\Lambda = \emptyset$ then $Z_{\Lambda,U_+,U_-} := 1$. We refer the reader to [5, App. A] for a proof of the following statement based on the general cluster expansion from [13].

Lemma 2.2 There exists $\beta_0 > 0$ independent of $\Lambda$ such that for all $\beta \geq \beta_0$, for all finite connected $\Lambda \subset \mathbb{Z}^2$ and $U_+, U_- \subset \partial_\Lambda$:

$$\log Z_{\Lambda,U_+,U_-} = \sum_{V \subset \Lambda} \varphi_{U_+,U_-}(V),$$

where the potentials $\varphi_{U_+,U_-}(V)$ satisfy

(i) $\varphi_{U_+,U_-}(V) = 0$ if $V$ is not connected.
(ii) \( \varphi_{U_+, U_-}(V) = \varphi_0(V) \) if \( V \cap (U_+ \cup U_-) = \emptyset \), for some shift invariant potential \( V \mapsto \varphi_0(V) \), that is

\[ \varphi_0(V) = \varphi_0(V + x) \quad \forall \, x \in \mathbb{Z}^2, \]

where \( \varphi_0 \) is independent of \( U_+, U_-, \Lambda \).

(iii) For all \( V \subset \Lambda \):

\[ \sup_{U_+, U_- \subset \partial \Lambda} |\varphi_{U_+, U_-}(V)| \leq \exp(- (\beta - \beta_0) d(V)) \]

where \( d(V) \) is the cardinality of the smallest connected set of all dual bonds separating points of \( V \) from points of its complement (a dual bond separates \( V \) from \( V^c \) iff it is orthogonal to a bond connecting \( V \) to \( V^c \)).

### 2.4 Nested contours

Consider the rectangle \( \Lambda_{L,M} \), for some \( L, M \in \mathbb{N} \), and let \( P_{\Lambda} \) denote the \( SOS \) Gibbs measure in \( \Lambda := \Lambda_{L,M} \) with zero boundary conditions. Given two contours \( \gamma, \gamma' \), we write \( \gamma \subset \gamma' \) if \( \Lambda_\gamma \subset \Lambda_{\gamma'} \). Fix \( n \in \mathbb{N} \) and pick \( n \) geometric contours \( \gamma_1, \ldots, \gamma_n \) such that \( \gamma_{i+1} \subset \gamma_i \), for every \( i = 1, \ldots, n - 1 \). Consider the event \( \cap_{i=1}^n \mathcal{E}_{\gamma_i,i} \) that \( \gamma_i \) is an \( i \)-contour for all \( i = 1, \ldots, n \). The probability of this event under \( P_{\Lambda} \) can be expressed as

\[ P_{\Lambda} (\cap_{i=1}^n \mathcal{E}_{\gamma_i,i}) = \frac{Z(\gamma_1, \ldots, \gamma_n; L, M)}{Z_{\Lambda}}, \quad (2.3) \]

where \( Z_{\Lambda} \) denotes the partition function of the \( SOS \) model in \( \Lambda = \Lambda_{L,M} \) with zero boundary conditions and \( Z(\gamma_1, \ldots, \gamma_n; L, M) \) stands for the same summation restricted to the configurations \( \eta \in \mathcal{O}^0_{\Lambda} \) such that \( \gamma_i \) is an \( i \)-contour for each \( i = 1, \ldots, n \). By applying the cluster expansion in Lemma 2.2, with \( \Lambda = \Lambda_{L,M} \) and \( U_\pm = \emptyset \), we can write

\[ Z_{\Lambda} = \exp \left( \sum_{V \subset \Lambda} \varphi_0(V) \right), \quad (2.4) \]

To expand the partition function \( Z(\gamma_1, \ldots, \gamma_n; L, M) \), define \( S_i := \Lambda_{\gamma_i-1} \setminus \Lambda_{\gamma_i} \), for \( i = 1, \ldots, n + 1 \), where \( \Lambda_{\gamma_0} = \Lambda \) and \( \Lambda_{\gamma_{n+1}} = \emptyset \), and set \( \Delta^+_i = S_i \cap \Delta^+_{\gamma_{i-1}} \) and \( \Delta^-_i = S_i \cap \Delta^-_{\gamma_{i}} \), with the understanding that \( \Delta^+_1 = \Delta^-_{n+1} = \emptyset \). Notice that \( \Delta^\pm_i \subset \partial S_i \). Using the notation of Lemma 2.2 a direct computation proves that

\[ Z(\gamma_1, \ldots, \gamma_n; L, M) = \exp \left( -\beta \sum_{i=1}^n |\gamma_i| \right) \prod_{i=1}^{n+1} Z_{S_i, \Delta^+_i, \Delta^-_i}. \quad (2.5) \]
The term $\sum_{i=1}^{n} |\gamma_i|$ accounts for the minimal energy associated to the given contours. The fact that the surface gradient across a contour $\gamma_i$ must be at least 1 is encoded by the constraints on $\Delta_i^+, \Delta_i^-$ appearing in $Z_{S_i, \Delta_i^+, \Delta_i^-}$.

Therefore, the expansion (2.2) implies

$$Z(\gamma_1, \ldots, \gamma_n; L, M) = \exp \left( -\beta \sum_{i=1}^{n} |\gamma_i| + \sum_{V \subseteq S_i} \varphi_{\Delta_i^+, \Delta_i^-}(V) \right).$$

The ratio (2.3) then becomes

$$P_{\Lambda} \left( \bigcap_{i=1}^{n} C_{\gamma_i} \right) = \exp \left( -\beta \sum_{i=1}^{n} |\gamma_i| + \Psi_{\Lambda}(\gamma_1, \ldots, \gamma_n) \right),$$

where

$$\Psi_{\Lambda}(\gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n+1} \sum_{V \subseteq S_i: \text{ or } V \cap (\Delta_i^+ \cup \Delta_i^-) \neq \emptyset} (\varphi_{\Delta_i^+, \Delta_i^-}(V) - \varphi_0(V)) - \sum_{V \subseteq \Lambda: V \cap (\bigcup_{i=1}^{n} \gamma_i) \neq \emptyset} \varphi_0(V).$$

(2.7)

where the condition $V \cap (\bigcup_{i=1}^{n} \gamma_i) \neq \emptyset$ means that $V$ intersects more than just one $S_i$. When $n = 1$, we have only one contour $\gamma_1 = \gamma$ and we define

$$\psi_{\Lambda}(\gamma) := \Psi_{\Lambda}(\gamma_1).$$

(2.8)

Observe that property (iii) of the potentials $\varphi_{U_+, U_-}$ in Lemma 2.2 implies in particular that

$$|\psi_{\Lambda}(\gamma)| \leq 3 \sum_{V \subseteq \Lambda: V \cap (\Delta_i^+ \cup \Delta_i^-) \neq \emptyset} e^{-2(\beta - \beta_0)d(V)} \leq \varepsilon(\beta)|\gamma|,$$

(2.9)

where $\lim_{\beta \to \infty} \varepsilon(\beta) = 0$. Later on it will be convenient to introduce the quantity $\psi_{\infty}(\gamma)$ defined as $\psi_{\Lambda}(\gamma)$ but without the restriction $V \subseteq \Lambda$, i.e. now $S_1 = \mathbb{Z}^d \setminus \Lambda_{\gamma}$.

### 2.5 The staircase ensemble

Consider the rectangle $\Lambda_{L, M}$, for some $L, M \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and integers $-M \leq a_1 \leq \cdots \leq a_n \leq M$, and $-M \leq b_1 \leq \cdots \leq b_n \leq M$. (2.10)

and set $a_0 = b_0 = -(M + 1)$ and $a_{n+1} = b_{n+1} = M + 1$. We define a “staircase” height $\tau$ at the external boundary $\partial \Lambda_{L, M}$ of our rectangle which, starting from height zero at the base of the rectangle (i.e. the set $(x, -(M + 1)), x = -L, \ldots, L$) jumps by one at the locations specified by the two $n$-tuples $\{a_i, b_i\}$ until it reaches height $n$:
Fig. 2 A sketch of the staircase boundary condition (2.11) in the rectangle $\Lambda_{L,M}$ for $n = 2$. The points $z_i, z'_i$ have coordinates $z_i = (-L-1, a_i)$, and $z'_i = (L+1, b_i)$.

\[
\tau(u, v) = \begin{cases} 
  i & \text{if } u = -L-1 \text{ and } a_i \leq v < a_{i+1} \text{ or } u = L+1 \text{ and } b_i \leq v < b_{i+1}, \\
  0 & \text{if } u \in [-L, L] \text{ and } v = -M - 1 \\
  n & \text{if } u \in [-L, L] \text{ and } v = M + 1, 
\end{cases}
\]  

(2.11)

where $i \in \{0, \ldots, n\}$, see Figure 2. Note that if two or more values of the $a_i$ or $b_i$ coincide then the boundary height $\tau$ takes jumps higher than 1 at those points.

Next, let $Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L, M)$ denote the partition function of the SOS model in $\Lambda_{L,M}$ with boundary condition $\tau$ as in (2.11).

Let also $Z_\Lambda$ denote as above the partition function of the SOS model in $\Lambda = \Lambda_{L,M}$ with zero boundary condition everywhere. We want to compute the ratio

\[
\frac{Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L, M)}{Z_\Lambda}. 
\]

(2.12)

To expand the partition function in the numerator of (2.12), we need the notion of an open contour. This is defined as in Definition 2.1 except that $e_0 \neq e_n$. Since the boundary conditions force the surface height to grow from height zero at the bottom base of $\Lambda_{L,M}$ to height $n$ at the top base, necessarily any configuration $\eta$ of the SOS interface compatible with $\tau$ must satisfy the following property.

Given $\eta$ there exist uniquely defined non-crossing open contours $\gamma_i$, $i = 1, \ldots, n$, joining the dual lattice points $x_i := (-L-1/2, a_i-1/2)$ and $y_i := (L+1/2, b_i-1/2)$ such that $\eta(x) \leq i - 1$ for all $x \in \Delta_i^-$ and $\eta(x) \geq i - 1$ for all $x \in \Delta_i^+$ where $\Delta_i^\pm$ are now the sets defined as follows. Let $S_i \subset \Lambda_{L,M}$ denote the region bounded by $\gamma_i$ and $\gamma_{i-1}$, where $\gamma_{n+1}$ is the top boundary of $\Lambda_{L,M}$ and $\gamma_0$ is the bottom boundary of $\Lambda_{L,M}$. Then $\Delta_i^-$ is defined as the set of $x \in S_i$ such that either their distance from $\gamma_i$...
is $\frac{1}{2}$, or their distance from the set of vertices in $\mathbb{Z}^2$ where two non-linked bonds of $\gamma_i$ meet equals $1/\sqrt{2}$. Similarly, $\Delta_i^+$ is the set of $x \in S_i$ such that either their distance from $\gamma_{i-1}$ is $\frac{1}{2}$, or their distance from the set of vertices in $\mathbb{Z}^2$ where two non-linked bonds of $\gamma_{i-1}$ meet equals $1/\sqrt{2}$. Lemma 2.2 here implies

$$Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L, M) = \sum_{\gamma_1, \ldots, \gamma_n} \exp \left( -\beta \sum_{i=1}^{n} |\gamma_i| + \sum_{i=1}^{n+1} \sum_{V \subset S_i} \varphi_{\Delta_i^+, \Delta_i^-}(V) \right),$$

where the sum ranges over all possible values of the open contours $\gamma_i : x_i \to y_i$ inside $\Lambda_{L,M}$ with the non-crossing constraints. Recalling that $Z_{\Lambda}$ can be expanded as in (2.4), one finds that

$$\frac{Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L, M)}{Z_{\Lambda}} = \sum_{\gamma_1, \ldots, \gamma_n} \exp \left( -\beta \sum_{i=1}^{n} |\gamma_i| + \Phi_{L,M}(\gamma_1, \ldots, \gamma_n) \right), \quad (2.13)$$

where

$$\Phi_{L,M}(\gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n+1} \sum_{V \subset S_i} \sum_{V \cap (\Delta_i^+ \cup \Delta_i^-) \neq \emptyset} (\varphi_{\Delta_i^+}(V) - \varphi_0(V)) - \sum_{V \subset \Lambda_{L,M}} \varphi_0(V), \quad (2.14)$$

where the condition $V \cap (\cup_{i=1}^{N} \gamma_i) \neq \emptyset$ means that $V$ intersects more than just one $S_i$. Equation (2.13) expresses the ratio (2.12) as the partition function of a gas of $n$ interacting non-crossing open contours $\gamma_1, \ldots, \gamma_n$ within $\Lambda_{L,M}$ such that $\gamma_i : x_i \to y_i$, $i = 1, \ldots, n$. Using (iii) in Lemma 2.2 the limit as $M \to \infty$ of the above expression is well defined, so that the following holds.

**Lemma 2.3** For any integers $n$, $\{a_i, b_i\}_{i=1}^{n}$ satisfying (2.10), the limit

$$\mathcal{Z}(a_1, \ldots, a_n; b_1, \ldots, b_n; L) := \lim_{M \to \infty} \frac{Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L, M)}{Z_{\Lambda}}$$

exists and it satisfies

$$\mathcal{Z}(a_1, \ldots, a_n; b_1, \ldots, b_n; L) = \sum_{\gamma_1, \ldots, \gamma_n} \exp \left( -\beta \sum_{i=1}^{n} |\gamma_i| + \Phi_{L,\infty}(\gamma_1, \ldots, \gamma_n) \right), \quad (2.15)$$

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where the sum ranges over all possible values of the open contours in the strip $\Lambda_{L,\infty} := [-L, L] \times \mathbb{Z}$ and $\Phi_{L,\infty}$ is defined as in (2.14) with $\Lambda_{L,\infty}$ replaced by $\Lambda_{L,\infty}$.

**Proof** Using (iii) in Lemma 2.2 it is immediate to check that for any family of contours $(\gamma_1, \ldots, \gamma_n)$

$$
\lim_{M \to \infty} \Phi_{L,M}(\gamma_1, \ldots, \gamma_n) = \Phi_{L,\infty}(\gamma_1, \ldots, \gamma_n).
$$

As in (2.9) we have

$$
|\Phi_{L,M}(\gamma_1, \ldots, \gamma_n)| \leq \varepsilon(\beta) \sum_{i=1}^{n} |\gamma_i|.
$$

(2.16)

Hence, for $\beta$ large enough, the conclusion follows by dominated convergence. \Box

### 2.6 Surface tension

Here we recall the definition and some properties of the surface tension corresponding to arbitrary tilt. Let us rewrite (2.15) in the case $n = 1$ as

$$
\mathcal{Z}(a_1; b_1; L) = \sum_{\gamma} \exp \left( -\beta |\gamma| + \Phi_{L,\infty}(\gamma) \right),
$$

(2.17)

where the sum ranges over all open contours in the strip $\Lambda_{L,\infty}$ joining the dual lattice points $x_1 := (-L - 1/2, a_1 - 1/2)$ and $y_1 := (L + 1/2, b_1 - 1/2)$.

**Lemma 2.4** There exists $\beta_0 > 0$ such that the following holds for all $\beta \geq \beta_0$. Let $\mathcal{Z}(a_1; b_1; L)$, denote the partition function (2.17). Assume that as $L \to \infty$ one has $(b_1 - a_1)/(2L) \to \lambda \in \mathbb{R}$ and set $\theta = \tan^{-1}(\lambda)$. Then the function

$$
\tau_\beta(\theta) = -\lim_{L \to \infty} \frac{\cos(\theta)}{2\beta L} \log \mathcal{Z}(a_1; b_1; L),
$$

is well defined and positive in $(-\pi/2, \pi/2)$. It is convex in the following sense: defining, for $x \in \mathbb{R}^2$, $\tau_\beta(x) = \|x\| \tau_\beta(\theta_x)$ with $\theta_x$ the angle formed by the vector $x$ with the horizontal axis, $\tau_\beta$ is a convex function on $\mathbb{R}^2$. Moreover,

$$
\limsup_{L \to \infty} \frac{1}{2\beta L} \sup_{a_1,b_1} \log \mathcal{Z}(a_1; b_1; L) \leq -\tau_\beta(0),
$$

(2.18)

**Proof** Existence and the stated properties of the surface tension are known facts [11, Section 4.16]. It is also known, see [11, Section 4.20], that $\tau_\beta(\theta)$ tends to $|\cos \theta| + |\sin \theta|$, as $\beta \to \infty$. In particular, $\tau_\beta(0) \to 1$, $\beta \to \infty$. Strictly speaking the proofs in [11] are carried out for the contour ensemble associated to the 2D Ising model, which has the form (2.17) but with slightly different potentials in the “decoration
term” $\Phi_{L,\infty}(\gamma)$. However, thanks to the properties listed in Lemma 2.2, the same proofs actually apply to our model in (2.17) as well.

To prove (2.18) we distinguish two cases. If $|b_1 - a_1| > 4L$ we use again (2.9) to obtain

$$\sup_{a_1, b_1: |b_1 - a_1| > 4L} Z(a_1; b_1; L) \leq \sum_{\gamma_1: |\gamma_1| \geq 5L} e^{-(\beta - \varepsilon(\beta))|\gamma_1|},$$

where $\gamma_1$ is a contour from $(-(L + 1, a_1)$ to $((L + 1), b_1)$. Clearly the above sum is negligible w.r.t. $\exp(-2\beta L \tau_{\beta}(0))$ as $L \to \infty$ for $\beta$ large enough. If instead $|b_1 - a_1| \leq 4L$, then the estimate [11, Eq. (4.12.3)] together with convexity of the surface tension allows one to conclude (2.18).

It is not hard to check that the special case $\theta = 0$ coincides with the quantity in Definition 1.1. Indeed, using the notation from Definition 1.1 together with (2.13) (with $n = 1$ and $a_1 = b_1 = 0$),

$$\frac{Z_{\Lambda_L}^k}{Z_{\Lambda_L}} = \frac{Z(0; 0; L, L)}{Z_{\Lambda_L}} = \sum_{\gamma_1} \exp(-\beta|\gamma_1| + \Phi_{L,\infty}(\gamma_1)).$$

The same arguments of Lemma 2.3 can be used to check that

$$\lim_{L \to \infty} \frac{1}{L} \log \frac{Z_{\Lambda_L}^k}{Z_{\Lambda_L}} = \lim_{L \to \infty} \frac{1}{L} \log Z(0; 0; L).$$

### 3 Lower bound

Here we prove the lower bound in Theorem 1.2. We first establish a lower bound on the probability of having zero height at the boundary of a square.

**Lemma 3.1** For $\beta \geq \beta_0$ there exists $c_\beta > 0$ such that for any $L \in \mathbb{N}$:

$$\mathbb{P}(\eta_{\partial \Lambda_L} = 0) \geq e^{-c_\beta L}.$$

**Proof** Recall that $\mathbb{P}(\cdot) = \lim_{K \to \infty} \mathbb{P}_{\Lambda_K}(\cdot)$. Expanding as in (2.4), we see that

$$\mathbb{P}_{\Lambda_K}(\eta_{\partial \Lambda_L} = 0) = \frac{Z_{\Lambda_K \setminus \partial \Lambda_L}}{Z_{\Lambda_K}} = \exp \left( - \sum_{V \subset \Lambda_K, V \cap \partial \Lambda_L \neq \emptyset} \varphi_0(V) \right),$$

where $V \cap \partial \Lambda_L \neq \emptyset$ is equivalent to $V$ not contained in $\Lambda_K \setminus \partial \Lambda_L$. From the decay properties of the potentials $\varphi_0$ stated in Lemma 2.2, the desired result follows. 

$\square$ Springer
3.1 Proof of the lower bound in Theorem 1.2

If we prove the lower bound for \( P \) in (1.2) we also have the same lower bound for \( \mathbb{P} \) by using Lemma 3.1 and

\[
P(\eta_{\Lambda} \geq 0) \geq \mathbb{P}(\eta_{\Lambda} = 0) \mathbb{P}_\Lambda(\eta_{\Lambda} \geq 0). \tag{3.1}
\]

To prove the lower bound for \( \mathbb{P}_\Lambda \), we proceed by restricting the set of configurations to an event \( E \) defined as follows. Fix \( N := H(L) = \lfloor \frac{1}{4\beta} \log L \rfloor \). Define the nested annular regions \( \hat{U}_i := \Lambda_{-3e_i} - \Lambda_{-e_i}, i = 1, \ldots, N \), where \( e_0 = 0 \) and \( e_i = i(i+1)/2 \). Notice that each \( \hat{U}_i \) consists of 3 nested disjoint annuli each of width \( i \). We define \( \mathcal{U}_i \) as the middle one, i.e., \( \mathcal{U}_i = \Lambda_{-(3e_i - 1)} - \Lambda_{-(3e_i + 1)} \). These sets are such that \( d(\mathcal{U}_i, \mathcal{U}_{i+1}) \geq 2i + 1 \), where \( d(\cdot, \cdot) \) stands for the euclidean distance.

For each \( i \), define the set \( C_i \) of all contours \( \gamma \) such that \( \gamma \subset \mathcal{U}_i \) and \( \gamma_i \) surrounds \( \Lambda_{-(3e_i + 1)} \). We consider the event \( E \) that for each \( i = 1, \ldots, N \) there exists an \( i \)-contour \( \gamma_i \in C_i \):

\[
E = \bigcup_{\gamma_1 \in C_1, \ldots, \gamma_N \in C_N} \mathcal{C}_{\gamma_1, 1} \cap \cdots \cap \mathcal{C}_{\gamma_N, N}.
\]

For a fixed choice of \( \gamma_i \subset C_i, i = 1, \ldots, N \) we write \( S_i = \Lambda_{\gamma_i - 1} - \Lambda_{\gamma_i} \), and \( \Delta_i^+ = S_i \cap \Delta_{\gamma_i}^+ \), and \( \Delta_i^- = S_i \cap \Delta_{\gamma_i}^- \) as in Sect. 2.4. We define \( \mathcal{Z}_i \) as the partition function in \( S_i \) with boundary conditions \( i - 1 \) in \( \partial S_i \), and with the following constraints: \( \eta(x) \leq i - 1 \) for \( x \in \Delta_i^+ \), \( \eta(x) \geq i - 1 \) for \( x \in \Delta_i^- \) and \( \eta(\ell) \geq 0 \) for all \( x \in S_i \). Then

\[
\mathbb{P}_\Lambda(\eta_{\Lambda} \geq 0; \mathcal{C}_{\gamma_1, 1} \cap \cdots \cap \mathcal{C}_{\gamma_N, N}) = \frac{e^{-\beta \sum_{i=1}^{N} |\gamma_i|} \prod_{i=1}^{N+1} \mathcal{Z}_{\Delta_i^+, \Delta_i^-}}{\mathcal{Z}_\Lambda}. \tag{3.2}
\]

Below, we shall take \( n := \lfloor \log \log L \rfloor \) and fix arbitrary contours \( \gamma_1^* \in C_1, \ldots, \gamma_n^* \in C_n \), and sum over the remaining contours \( \gamma_i, i = n + 1, \ldots, N \).

**Lemma 3.2** Fix \( \beta \geq \beta_0 \) and fix \( \gamma_i^* \in C_1, \ldots, \gamma_n^* \in C_n \), where \( n = \lfloor \log \log L \rfloor \). Then

\[
\mathbb{P}_\Lambda(\eta_{\Lambda} \geq 0; E) \geq \frac{1}{2} \sum_{\gamma_{n+1} \in C_{n+1}, \ldots, \gamma_N \in C_N} \mathbb{P}_\Lambda(\eta_{\Lambda} \geq 0; \cap_{k=1}^{n} \mathcal{C}_{\gamma_k^*, k} \cap_{j=n+1}^{N} \mathcal{C}_{\gamma_j^*, j}).
\]

**Proof** Let \( F_i \) denote the event that there is more than one \( i \)-contour in \( C_i \). Then

\[
\mathbb{P}_\Lambda(\eta_{\Lambda} \geq 0; E) \geq \sum_{\gamma_{n+1} \in C_{n+1}, \ldots, \gamma_N \in C_N} \mathbb{P}_\Lambda(\eta_{\Lambda} \geq 0; \cap_{k=1}^{n} \mathcal{C}_{\gamma_k^*, k} \cap_{j=n+1}^{N} \mathcal{C}_{\gamma_j^*, j} \cap_{i=n+1}^{N} F_i^c).
\]
Thus, it suffices to show that for any fixed choice of \( \gamma^*_k \in C_k, k = 1, \ldots, n \) and \( \gamma_j \subset C_j, j = n + 1, \ldots, N \):

\[
\mathbb{P}_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \bigcap_{k=1}^n \mathcal{E}^*_{\gamma^*_k,k} ; \bigcap_{j=n+1}^N \mathcal{E}_{\gamma_j,j} ; \bigcup_{i=n+1}^N F_i \right) \\
\leq \frac{1}{2} \mathbb{P}_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \bigcap_{k=1}^n \mathcal{E}^*_{\gamma^*_k,k} ; \bigcap_{j=n+1}^N \mathcal{E}_{\gamma_j,j} \right).
\]

Suppose the \( j \)-contour \( \gamma_j \in C_j \) is given for each \( j = n + 1, \ldots, N \). If \( F_i \) occurs then there must be a \( i \)-contour \( \gamma \in C_i, \gamma \neq \gamma_i \), such that either \( \gamma \subset S_{i+1} \) or \( \gamma \subset S_i \). In particular, if \( \bigcup_{i=n+1}^N F_i \) occurs, then, for some \( i \in [n + 1, N + 1] \), there exists either an \((i - 1)\)-contour or an \( i \)-contour \( \gamma \) inside \( S_i \) and surrounding \( \Lambda L_{-(3Li+i)} \). Let \( \pi_{S_i,\Delta^+_i,\Delta^-_i} \) denote the probability measure corresponding to the partition function \( Z_{S_i,\Delta^+_i,\Delta^-_i} \). From [7, Proposition 2.7] one has that for any fixed contour \( \gamma \) inside \( S_i \), for any \( h \in \mathbb{N} \):

\[
\pi_{S_i,\Delta^+_i,\Delta^-_i}(\mathcal{E}_{\gamma,h}) \leq \exp \left( -\beta |\gamma| + Ce^{-4\beta h} |S_i| + Ce^{-4\beta h} |\gamma| \log |\gamma| \right).
\]

Here and below, by \( C \) we mean a positive constant that does not depend on \( \beta \) and \( L \), whose value may change at each occurrence. Since \( |S_i| \leq Cl \leq L \log L \), and \( \log |\gamma| \leq 2 \log L \), taking either \( h = i \) or \( h = i - 1 \), with \( i \geq n + 1 \) one has that \( e^{-4\beta h} |S_i| \leq L \log L \) and \( e^{-4\beta h} \log |\gamma| \leq 2 \log L \) and therefore

\[
\pi_{S_i,\Delta^+_i,\Delta^-_i}(\mathcal{E}_{\gamma,h}) \leq \exp (- (\beta - 1) |\gamma| + L), \tag{3.3}
\]

as soon as \( \beta \) and \( L \) are large enough. If \( \gamma \) is required to surround \( \Lambda L_{-3Li+i} \), then necessarily \( |\gamma| \geq 2L \). Let \( p_i \) denote the \( \pi_{S_i,\Delta^+_i,\Delta^-_i} \)-probability that there exists either an \((i - 1)\)-contour or an \( i \)-contour \( \gamma \) inside \( S_i \) and surrounding \( \Lambda L_{-3Li+i} \). Summing over \( \gamma \subset S_i \) with \( |\gamma| \geq 2L \) in (3.3), one finds that for \( \beta \) large enough, \( p_i \leq e^{-L} \). From (3.2), using a union bound and the fact that \( Ne^{-L} \leq 1/2 \), it follows that

\[
\mathbb{P}_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \bigcap_{k=1}^n \mathcal{E}^*_{\gamma^*_k,k} ; \bigcap_{j=n+1}^N \mathcal{E}_{\gamma_j,j} ; \bigcup_{i=n+1}^N F_i \right) \\
\leq \sum_{i=n+1}^N p_i \mathbb{P}_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \bigcap_{k=1}^n \mathcal{E}^*_{\gamma^*_k,k} ; \bigcap_{j=n+1}^N \mathcal{E}_{\gamma_j,j} \right) \\
\leq \frac{1}{2} \mathbb{P}_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \bigcap_{k=1}^n \mathcal{E}^*_{\gamma^*_k,k} ; \bigcap_{j=n+1}^N \mathcal{E}_{\gamma_j,j} \right).
\]

Thanks to Lemma 3.2 the lower bound in Theorem 1.2 follows if we prove that

\[
\sum_{\gamma_{n+1} \in C_{n+1}, \ldots, \gamma_{N} \in C_N} \mathbb{P}_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \bigcap_{k=1}^n \mathcal{E}^*_{\gamma^*_k,k} ; \bigcap_{j=n+1}^N \mathcal{E}_{\gamma_j,j} \right) \\
\geq \exp \left( -8 \beta \tau_{\beta}(0) NL (1 + o(1)) \right), \tag{3.4}
\]

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for any fixed choice of $\gamma^*_k \in C_k$, $k = 1, \ldots, n$, with $n = \lfloor \log \log L \rfloor$. To prove (3.4) we start by observing that by the FKG inequality one has

$$\frac{Z_{S_i, \Delta_i^+, \Delta_i^-}^+}{Z_{S_i, \Delta_i^+, \Delta_i^-}^-} = \pi_{S_i, \Delta_i^+, \Delta_i^-}^- (\eta(x) \geq 0, \forall x \in S_i) \geq \prod_{x \in S_i} \pi_{S_i, \Delta_i^+, \Delta_i^-}^- (\eta(x) \geq 0),$$

where $Z_{S_i, \Delta_i^+, \Delta_i^-}^+$ is defined above (3.2), $Z_{S_i, \Delta_i^+, \Delta_i^-}^-$ is as in Sect. 2.3, and $\pi_{S_i, \Delta_i^+, \Delta_i^-}^-$ denotes the probability measure associated to the partition function $Z_{S_i, \Delta_i^+, \Delta_i^-}^-$. From [5, Proposition 3.9] one has that $\pi_{S_i, \Delta_i^+, \Delta_i^-}^- (\eta(x) \geq 0) \geq 1 - Ce^{-4\beta(i-1)}$ for any $x \in S_i$. Using $1 - x \geq e^{-2x}$ for $0 \leq x \leq 1/2$, one has

$$\frac{Z_{S_i, \Delta_i^+, \Delta_i^-}^+}{Z_{S_i, \Delta_i^+, \Delta_i^-}^-} \geq \exp \left( -2C |S_i| e^{-4\beta(i-1)} \right).$$

Therefore, in (3.2) we can estimate

$$\mathbb{P}_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0; \bigcap_{k=1}^n \psi_{\gamma^*_k, k}; \bigcap_{j=n+1}^N \psi_{\gamma_j, j} \right) \geq \exp \left( -\beta \sum_{i=1}^n |\gamma^*_i| - \beta \sum_{i=n+1}^N |\gamma_i| - 2C \sum_{i=1}^{N+1} |S_i| e^{-4\beta(i-1)} \right) \prod_{i=1}^{N+1} \frac{Z_{S_i, \Delta_i^+, \Delta_i^-}^+}{Z_{\Lambda}}.$$

Expanding as in (2.6) one obtains

$$\prod_{i=1}^{N+1} \frac{Z_{S_i, \Delta_i^+, \Delta_i^-}^+}{Z_{\Lambda}} = \exp \left( \Psi_\Lambda \left( \gamma^*_1, \ldots, \gamma^*_n, \gamma_{n+1}, \ldots, \gamma_N \right) \right),$$

where $\Psi$ is given in (2.7). Estimating $|S_i| \leq CiL$ one finds

$$\sum_{i=1}^N |S_i| e^{-4\beta(i-1)} \leq CL.$$  \hspace{1cm} (3.6)

On the other hand, the term $|S_{N+1}| e^{-4\beta N} = |\Lambda_{\gamma_N}| e^{-4\beta H(L)}$ satisfies

$$|S_{N+1}| e^{-4\beta N} \leq L^2 e^{-4\beta H(L)} \leq CL,$$  \hspace{1cm} (3.7)

where we use $e^{-4\beta H(L)} \leq C/L$. Note that it is at this point of the argument that it is crucial to have $N$ as large as $H(L)$. From (3.6)-(3.7) one has $\sum_{i=1}^{N+1} |S_i| e^{-4\beta(i-1)} \leq CL$. From this bound and (3.5) we obtain
\[ P_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \cap_{k=1}^n C_{\gamma_i^*, k} ; \cap_{j=n+1}^N C_{\gamma_j, j} \right) \geq \exp \left( \frac{-\beta}{2} \sum_{i=1}^n |\gamma_i^*| - \beta \sum_{i=n+1}^N |\gamma_i| + \psi_{\Lambda}(\gamma_1^*, \ldots, \gamma_n^*, \gamma_{n+1}, \ldots, \gamma_N) - CL \right). \]  

(3.8)

Next, we want to show that the interactions among the contours are negligible in our setting. Let \( \psi_{\Lambda}(\gamma) \) denote the potential associated to a single contour \( \gamma \) as defined in (2.8).

**Lemma 3.3** Take \( \beta \geq \beta_0 \). Uniformly in the choice of \( \gamma_1 \in C_1, \ldots, \gamma_N \in C_N \) one has

\[ \left| \psi_{\Lambda}(\gamma_1, \ldots, \gamma_N) - \sum_{i=1}^N \psi_{\Lambda}(\gamma_i) \right| \leq \sum_{i=1}^N |\gamma_i|e^{-\beta i/2}. \]

**Proof** Notice that any \( V \subset \Lambda \) such that \( d(V, \gamma_i) \leq 1 \) and \( d(V, \gamma_{i+1}) \leq 1 \) must have \( d(V) \geq 2i \). Thus the sum of the potentials associated to \( V \)'s that have \( d(V, \gamma_i) \leq 1 \) and are such that \( d(V, \gamma_j) \leq 1 \) for some \( j \neq i \) contributes at most \( |\gamma_i|e^{-\beta i/2} \) if \( \beta \) is large enough. \( \square \)

From (3.8) and Lemma 3.3 one has

\[ P_{\Lambda_L} \left( \eta_{\Lambda_L} \geq 0 ; \cap_{i=1}^n C_{\gamma_i^*, i} ; \cap_{j=n+1}^N C_{\gamma_j, j} \right) \geq \exp \left( \frac{-\beta}{2} \sum_{i=1}^n |\gamma_i^*| - \beta \sum_{i=n+1}^N |\gamma_i| + \sum_{i=1}^N \psi_{\Lambda}(\gamma_i) - CL \right). \]  

(3.9)

For \( i = 1, \ldots, n \), we can use the rough estimates \( |\gamma_i| \leq CLn \leq CL \log \log L \) and \( |\psi_{\Lambda}(\gamma_i)| \leq C|\gamma_i| \) (cf. (2.9)) to obtain

\[ \exp \left( \frac{-\beta}{2} \sum_{i=1}^n |\gamma_i| + \sum_{i=1}^n \psi_{\Lambda}(\gamma_i) \right) \geq \exp(-o(L \log L)). \]  

(3.10)

For \( n < i \leq N \) we need the following statement.

**Lemma 3.4** Uniformly over \( i \) such that \( n < i \leq N \), one has

\[ \sum_{\gamma \in C_i} \exp \left( -\beta |\gamma| (1 + e^{-\beta i/2}) + \psi_{\Lambda}(\gamma) \right) \geq \exp \left( -8\beta \tau_\beta(0)L(1 + o(1)) \right). \]  

(3.11)

We first conclude the proof of the lower bound in Theorem 1.2, assuming the estimate of Lemma 3.4. From Lemma 3.2 and (3.9)–(3.10) we have

\[ \square \]
On the probability of staying...

\[ \mathbb{P}_{\Lambda}(\eta_{\Lambda} \geq 0) \geq \mathbb{P}_{\Lambda}(\eta_{\Lambda} \geq 0; E) \geq \exp(-o(L \log L)) \times \]

\[ \sum_{\gamma_{n+1} \in \mathcal{C}_{n+1}, \ldots, \gamma_N \in \mathcal{C}_N} \exp \left( -\beta \sum_{i=n+1}^N |\gamma_i|(1 + e^{-\beta i/2}) + \sum_{i=1}^N \psi(\gamma_i) \right). \]

From Lemma 3.4 and using \( NL = 1/(4\beta) L \log L + O(L) \) one has

\[ \mathbb{P}_{\Lambda}(\eta_{\Lambda} \geq 0) \geq \exp \left( -8 \beta \tau \beta(0) NL (1 + o(1)) \right). \]

This concludes the proof.

Proof of Lemma 3.4 First observe that \( \gamma \in \mathcal{C}_i \) implies \( |\gamma| \leq |S_i| \leq L \log L \) and therefore for \( i \geq \log \log L \) and \( \beta \geq \beta_0 \) one has

\[ |\gamma|e^{-\beta i/2} = o(L). \]

Next, observe that we may safely replace \( \psi(\gamma) \) in (3.11) by the quantity \( \psi(\gamma) \) (see the end of Sect. 2.4). Indeed, any connected set \( \mathbb{V} \) that touches both \( \mathcal{U}_i \) and \( \partial \Lambda \) must have \( d(\mathbb{V}) \geq \frac{1}{2} (\log \log L)^2 \). Thus, we have to show that

\[ \sum_{\gamma \in \mathcal{C}_i} \exp(-\beta |\gamma| + \psi(\gamma)) \geq \exp \left( -8 \beta \tau \beta(0)L (1 + o(1)) \right). \quad (3.12) \]

To prove (3.12) we fix \( i \) and partition the set \( \mathcal{U}_i \) into rectangles \( R_j, j = 1, \ldots, m \), with height \( i \) and basis \( i^{2-\varepsilon} \), so that there are \( m \sim 8Li^{-2+\varepsilon} \) such rectangles, see Fig. 3. For simplicity, let us assume that the partitioning is exact so that \( \mathcal{U}_i \) is the union of the \( R_j \)'s plus four squares at the corners as in Fig. 3. The modifications in the general case are straightforward.

We fix for every rectangle \( R_j \) the points \( x_j \) and \( y_j \) that are the midpoints of the two shorter side. Consider an open contour \( \hat{\gamma}_j \) connecting \( x_j \) to \( y_j \) which is entirely contained in \( R_j \) (see Fig. 3). For technical reasons it is convenient to consider a closed path \( \gamma \) that agrees with \( \hat{\gamma}_j \) on \( R_j \). The latter is defined as follows. Let \( \hat{\gamma} \) be the closed contour contained in \( \mathcal{U}_i \) which coincides with \( \hat{\gamma}_j \) inside \( R_j \), it is given by straight segments in all other rectangles \( R_k, k \neq j \), and by a straight right angle shape at each of the four corner squares. Then we define \( \psi(\hat{\gamma}_j) \) as \( \psi(\hat{\gamma}) \) (see text after (2.9)) but with the restriction to those sets \( \mathbb{V} \) which have distance from \( \hat{\gamma}_j \) at most \( 1 \). It follows

Fig. 3 The partition of \( \mathcal{U}_i \) into rectangles \( R_j, j = 1, \ldots, m \) (left). A single path \( \hat{\gamma}_j : x_j \to y_j \) inside the rectangle \( R_j \) (right).
from [11, Sections 4.12 and 4.15] that for a fixed index $j$ one has, for $i$ large:

$$\sum_{\hat{\gamma}_j \cdot x_j \to y_j \in R_j} \exp \left( -\beta |\hat{\gamma}_j| + \psi_{\infty}(\hat{\gamma}_j) \right) \geq \exp \left( -\beta \tau_{\beta}(0) i^{2-\epsilon} (1 + o(1)) \right). \tag{3.13}$$

The point is that the height $i$ of the rectangles $R_j$ is much larger than the typical vertical fluctuation $i^{1-\epsilon/2}$ of paths $\hat{\gamma}_j$, so the restriction to be in $R_j$ is not modifying the partition function significantly.

Suppose now that $\gamma \in C_i$ is a contour passing through all the points $x_j, y_j$ that can be written as the composition of $\hat{\gamma}_1, \ldots, \hat{\gamma}_m$ where $\hat{\gamma}_j$ is as in the sum above, and assume that it has some prescribed shape at the four corners of $U_i$, e.g. a right angle form as in Fig. 3. Then it is immediate to check that $|\gamma| \leq \sum_{j=1}^m |\hat{\gamma}_j| + O(i)$, and

$$|\gamma| = \sum_{j=1}^m |\hat{\gamma}_j| + O(i).$$

The latter estimate holds thanks to the decay properties of the potentials, so that the mutual interaction between $\hat{\gamma}_j$ and $\hat{\gamma}_{j-1}$ is $O(i)$ uniformly in $j = 1, \ldots, m$. Thus, by restricting the sum in (3.12) to contours as in (3.13) one obtains

$$\sum_{\gamma \in C_i} \exp \left( -\beta |\gamma| + \psi_{\infty}(\gamma) \right) \geq \exp \left( -\beta \tau_{\beta}(0) i^{2-\epsilon} (1 + o(1)) \right).$$

Since $m \sim 8L i^{-2+\epsilon}$, the desired estimate follows. □

4 A monotonicity property of the SOS model

Recall the staircase ensemble defined in Sect. 2.5 with partition function

$$Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L),$$

as defined in Lemma 2.3. In this section we establish the following important monotonicity property.

**Theorem 4.1** There exists $\beta_0 > 0$ such that, for any $\beta > \beta_0$ and any $L \in \mathbb{N}$

$$Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L) \leq \prod_{i=1}^n Z(a_i; b_i; L). \tag{4.1}$$

The above estimate allows one to control the partition function of $n$ interacting open contours by means of the partition functions of $n$ non-interacting open contours. In particular, Theorem 4.1 and Lemma 2.4 yield the following corollary.

**Corollary 4.2** Fix $n \in \mathbb{N}$, and suppose that as $L \to \infty$ one has $(b_i - a_i)/L \to \lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$. Then
\[
\limsup_{L \to \infty} \frac{1}{2L} \log \mathcal{Z}(a_1, \ldots, a_n; b_1, \ldots, b_n; L) \leq -\beta \sum_{i=1}^{n} \frac{\tau_{\beta}(\theta_i)}{\cos(\theta_i)}
\]

where \(\theta_i = \tan^{-1}(\lambda_i)\).

The proof of Theorem 4.1 is based on the following key lemma.

**Lemma 4.3** Given \(\{a_i, b_i\}_{i=1}^{n}\), let \(\{a'_i, b'_i\}_{i=1}^{n}\) be defined by

\[
a'_i = a_i, \quad b'_i = b_i, \quad i = 1, \ldots, n-1; \quad a'_n = a_n + 1, \quad b'_n = b_n + 1.
\]

Then

\[
\mathcal{Z}(a_1, \ldots, a_n; b_1, \ldots, b_n; L) \leq \mathcal{Z}(a'_1, \ldots, a'_n; b'_1, \ldots, b'_n; L).
\]

**Proof of Lemma 4.3** Set \(\Lambda := \Lambda_{L,M}\) for some large fixed \(M > \max\{a_n, b_n, -a_1, -b_1\}\). Let \(\tau, \tau'\) be the SOS boundary conditions associated to \(\{a_i, b_i\}_{i=1}^{n}\) and \(\{a'_i, b'_i\}_{i=1}^{n}\) according to (2.11). Given \(s \in [0, 1]\) consider the auxiliary boundary condition \(\tau_s : \partial \Lambda \mapsto \mathbb{R}\) defined by

\[
\tau_s(x_1, x_2) = \begin{cases} 
    n - 1 + s & \text{if } (x_1, x_2) = (-L - 1, a_n) \text{ or } (x_1, x_2) = (L + 1, b_n); \\
    \tau'(x_1, x_2) & \text{otherwise}.
\end{cases}
\]

Next, we consider the partition function \(Z_{\tau_s}^{\Lambda}\) associated to \(\tau_s\) (strictly speaking we have only defined the model for integer valued boundary condition, but it is straightforward to extend it to the real valued case). Notice that \(\tau_s = s \tau + (1 - s) \tau'\). We shall see that \(Z_{\tau_s}^{\Lambda}\) is differentiable w.r.t. \(s \in [0, 1]\) so that

\[
Z_{\tau_s}^{\Lambda} = Z_{\tau'}^{\Lambda} = \int_0^1 ds \frac{d}{ds} Z_{\tau_s}^{\Lambda}.
\]

In order to compute the above derivative we proceed as follows. Define the points \(z = (-L + 1, a_n), w = (-L, a_n)\) and \(z' = (L + 1, b_n), w' = (L, b_n)\), so that \(w\) (resp. \(w'\)) is the nearest neighbor of \(z\) (resp. \(z'\)) in \(\Lambda\), see Fig. 4.
Let $B^*_\Lambda = B_{\Lambda} \setminus \{wz, w'z'\}$ denote all bonds with at least one vertex in $\Lambda$ with the exception of the two bonds $wz$ and $w'z'$. Define the energy function $H^{\tau,*}_{\Lambda}(\eta), \eta \in \Omega^\tau_{\Lambda}$ by

$$H^{\tau,*}_{\Lambda}(\eta) = \sum_{xy \in B^*_\Lambda} |\eta(x) - \eta(y)| + \phi(\eta(w)) + \phi(\eta(w')),$$

where

$$\phi(h) = (h - n) \mathbb{1}_{\{h \geq n\}} + (n - 1 - h) \mathbb{1}_{\{h \leq n - 1\}}, \ h \in \mathbb{Z}.$$
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the height of \( \hat{\tau} \) by 0, 1 instead of \( n - 1, n \). Finally, since \( G_{s,1} \) is a bounded local function, we can take the limit \( M \to \infty \) in (4.2) and get that

\[
Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L) - Z(a_1, \ldots, a_n + 1; b_1, \ldots, b_n + 1; L) \\
\leq \beta \left( \lim_{M \to \infty} \frac{\mathbb{E}^{\Lambda}_M}{Z_L} \right) \int_0^1 ds \, \pi_{\infty, \tau}^{\hat{\tau}, *}(G_{s,1}),
\]

where \( \pi_{\infty, \tau}^{\hat{\tau}, *}(\cdot) \) denotes the weak limit as \( M \to \infty \) of \( \pi^{\hat{\tau}, *}_\Lambda \), that is the Gibbs measure on \( \Lambda_{L,\infty} = [-L, L] \times \mathbb{Z} \) with boundary condition at height 1 at the vertices \( x = (x_1, x_2) \) with either \( x_1 = -(L + 1) \) and \( x_2 \geq a_n + 1 \) or \( x_1 = L + 1 \) and \( x_2 \geq b_n + 1 \); the boundary height is unspecified at the vertices \( z, z' \) (this simply means that the terms corresponding to bonds \( w_z \) and \( w'_z \) do not appear in the interaction) and otherwise it is equal to zero. The existence of the limits mentioned above can be proved again from the cluster expansion representation as in Lemma 2.3. By symmetry one has that

\[
\pi_{\infty, \tau}^{\hat{\tau}, *}(\eta(w) \geq 1; \eta(w') \geq 1) = \pi_{\infty, \tau}^{\hat{\tau}, *}(\eta(w) \leq 0; \eta(w') \leq 0),
\]

so that

\[
\pi_{\infty, \tau}^{\hat{\tau}, *}(G_{s,1}) = -\pi_{\infty, \tau}^{\hat{\tau}, *}(G_{1-s,1}) \quad \text{and} \quad \int_0^1 ds \, \pi_{\infty, \tau}^{\hat{\tau}, *}(G_{s,1}) = 0.
\]

In conclusion

\[
Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L) \leq Z(a_1', \ldots, a_n'; b_1', \ldots, b_n'; L)
\]

and the lemma is proved. \( \square \)

We can now complete the proof of Theorem 4.1. By iterating Lemma 4.3 arbitrarily many times, we have that

\[
Z(a_1, \ldots, a_n; b_1, \ldots, b_n; L) \leq \lim_{k \to \infty} Z(a_1, \ldots, a_{n-1}, a_n + k; b_1, \ldots, b_{n-1}, b_n + k; L).
\]

On the other hand, using the explicit representation (2.15) together with the rough bound (2.16) to control the large deviations of the \( n \)-th contour \( \gamma_n \), we have that

\[
\lim_{k \to \infty} Z(a_1, \ldots, a_{n-1}, a_n + k; b_1, \ldots, b_{n-1}, b_n + k; L) = Z(a_1, \ldots, a_{n-1}; b_1, \ldots, b_{n-1}; L)Z(a_n; b_n; L).
\]

In conclusion, we have factorized out the contribution of the \( n \)-th contour. By repeating the above reasoning for \((a_{n-1}, b_{n-1}), (a_{n-2}, b_{n-2}), \ldots, (a_2, b_2)\) we finally get (4.1).
5 Upper bound

If we prove the upper bound for $P$ in (1.2), then we can obtain the upper bound for $P_{\Lambda_L}$ by using (3.1) and Lemma 3.1. From now on we concentrate on proving the upper bound for $P$.

For any event $A$, note that

$$P(\eta_{\Lambda_L} \geq 0) \leq \frac{P(A)}{P(A | \eta_{\Lambda_L} \geq 0)}.$$  

(5.1)

Indeed, (5.1) is obtained by multiplying by $P(\eta_{\Lambda_L} \geq 0)$ both sides of the obvious inequality $1 \leq P(A) / P(A, \eta_{\Lambda_L} \geq 0)$.

For any $\delta > 0$ and $K > 0$, define $A(\delta, K)$, as the event that there exists a lattice circuit $C$ surrounding $\Lambda' := \Lambda_{(1-\delta)L}$ such that $\eta(x) \geq H(L) - K$, for all $x \in C$, where as usual $H(L) = \lfloor \frac{1}{4\beta} \log L \rfloor$.

Proposition 5.1 For any $\delta > 0$, there exists a constant $K > 0$ such that

$$\lim_{L \to \infty} P(A(\delta, K) | \eta_{\Lambda_L} \geq 0) = 1.$$  

Proof Let $\partial_\star \Lambda_L$ denote the internal boundary of $\Lambda_L$. Observe that $A(\delta, K)$ is monotone increasing so that by the FKG inequality

$$P(A(\delta, K) | \eta_{\Lambda_L} \geq 0) \geq P(A(\delta, K) | \eta_{\Lambda_L} \geq 0, \eta_{\partial_\star \Lambda_L} = 0).$$

Therefore, the proposition follows once we know that for some $K = K(\delta)$ one has

$$\lim_{L \to \infty} P(A(\delta, K) | \eta_{\Lambda_L} \geq 0, \eta_{\partial_\star \Lambda_L} = 0) = 1.$$  

(5.2)

Under the conditioning $\eta_{\Lambda_L} \geq 0, \eta_{\partial_\star \Lambda_L} = 0$, one has an SOS interface in $\Lambda_{L-1}$ with a wall at height zero and zero boundary conditions. The result of [7, Theorem 2] implies that with probability converging to 1, within $\Lambda_{L-1}$, there exists an $h$-contour surrounding $\Lambda' = \Lambda_{(1-\delta)L}$, for all $h \leq H(L) - K$ as soon as $K$ is a sufficiently large constant depending on $\delta$. This implies (5.2).

It follows that to prove the upper bound in (1.2) it is sufficient to establish:

Proposition 5.2 For any $\delta > 0$, for any $K > 0$, one has

$$\limsup_{L \to \infty} \frac{1}{2L \log L} \log P(A(\delta, K)) \leq -\tau(0)(1 - \delta).$$  

(5.3)

5.1 Proof of Proposition 5.2

The first observation is that we may impose zero boundary conditions outside a very large set, e.g. $\Lambda_M$ with $M \gg L^2$, and therefore we may consider $\bar{P} := P_{\Lambda_M}$ instead.
of $\mathbb{P}$ in (5.3). The reason is that the probability that there is a contour surrounding $\Lambda'$ and not contained in, say, $\Lambda_{L_2}$ is a negligible $O(\exp(-L^2))$, as one can check easily using a rough estimate as in (2.9). Then, $A(\delta, K)$ can be considered as a local event (localized in $\Lambda_{L_2}$) and by definition of thermodynamic limit one can approximate arbitrarily well $\mathbb{P}(A(\delta, K))$ by $\mathbb{P}(A(\delta, K))$, if $M$ is sufficiently large.

The event $A(\delta, K)$ implies that for each $h = 1, \ldots, N := H(L) - K$ there exists (at least) one $h$-contour surrounding $\Lambda'$. Therefore, there must exist $\Lambda_M \supset \gamma_1 \supset \cdots \supset \gamma_N \supset \Lambda'$ such that $\gamma_h$ is an $h$-contour:

$$
\mathbb{P}(A(\delta, K)) \leq \sum_{\gamma_1 \supset \cdots \supset \gamma_N \supset \Lambda'} \mathbb{P}(\cap_{i=1}^N \mathcal{C}_{\gamma_i,i}).
$$

(5.4)

Here we use the notation $\Lambda_M \supset \gamma_1 \supset \cdots \supset \gamma_N \supset \Lambda'$ when the contours satisfy $\Lambda_M \supset \Lambda_{\gamma_1} \supset \cdots \supset \Lambda_{\gamma_N} \supset \Lambda'$.

For a fixed choice of $\gamma_1 \supset \cdots \supset \gamma_N$ the above probability is computed in (2.6):

$$
\mathbb{P}(\cap_{i=1}^N \mathcal{C}_{\gamma_i,i}) = \exp\left(-\beta \sum_{i=1}^N |\gamma_i| + \Psi_{\Lambda_M}(\gamma_1, \ldots, \gamma_N)\right).
$$

(5.5)

To deal with the summation in (5.4) we consider a decomposition of each contour into four “irreducible” pieces, which will be responsible for the main contributions, plus some negligible corner terms.

Let $S_v$ and $S_h$ denote, respectively, the vertical and horizontal infinite strips obtained by prolonging the sides of the square $\Lambda'$:

$$
S_v = \left\{ x = (x_1, x_2) \in \mathbb{Z}^2 : |x_1| \leq (1 - \delta)L \right\},
$$

$$
S_h = \left\{ x = (x_1, x_2) \in \mathbb{Z}^2 : |x_2| \leq (1 - \delta)L \right\}.
$$

Let $S^u_v$, resp. $S^b_v$, denote the top, resp. bottom part of $S_v$, i.e. the part that lies above, resp. below, the square $\Lambda'$. Similarly, let $S^\ell_h$, resp. $S^r_h$, denote the portion of $S_h$ to the left, resp. right, of the square $\Lambda'$.

We now define the irreducible components of a fixed contour $\gamma$ containing $\Lambda'$. Consider the portion of $\gamma$ that intersects $S^u_v$. This must contain at least one crossing, defined as an open contour connecting the opposite vertical sides of $S^u_v$ that is fully contained in the interior of $S^u_v$. Let $\gamma'$ denote the most internal crossing, i.e. the one that lies closest to the square $\Lambda'$. We repeat the same construction in the strips $S^b_v$, $S^\ell_h$ and $S^r_h$, to define $\gamma^\ell$, $\gamma^b$ and $\gamma^r$ as the most internal crossings. We say that $\gamma^u$, $u \in \{\ell, \ell, b, r\}$, form the irreducible components of the contour $\gamma$. We call $x^u$, $y^u$ the endpoints of $\gamma^u$, with $x^u$ coming after $y^u$ if $y^u$ is given a counter clockwise orientation. See Fig. 5. It is easy to convince oneself that any contour containing the square $\Lambda'$, such that its irreducible components coincide with the given $\gamma^t$, $\gamma^\ell$, $\gamma^b$, $\gamma^r$, must have the following property: If we travel along $\gamma^t$ in the direction $y^t \to x^t$, and then follow the contour, the irreducible components we meet are, in order: $\gamma^\ell$ in the direction $y^\ell \to x^\ell$, then $\gamma^b$ in the direction $y^b \to x^b$, then $\gamma^r$ in the direction $y^r \to x^r$,
Fig. 5 Example of a contour $\gamma$ surrounding the square $\Lambda'$. The irreducible components of $\gamma$ are the thicker paths

and finally again $\gamma^t$ in the direction $y^t \to x^t$. Thus we can write any $\gamma$ with given irreducible components $\gamma^t, \gamma^\ell, \gamma^b, \gamma^r$ as the composition

$$\gamma = \gamma^t \circ \eta^{t,\ell} \circ \gamma^\ell \circ \eta^{\ell,b} \circ \gamma^b \circ \eta^{b,r} \circ \gamma^r \circ \eta^{r,t},$$

(5.6)

where $\eta^{u,v}$ denotes a path connecting $x^u$ and $y^v$ for $u, v \in \{t, \ell, b, r\}$.

Let $\gamma_1, \ldots, \gamma_N$ denote a collection of nested contours as in (5.4). We write $\gamma_i^u$ for the corresponding irreducible components, and $\eta_i^{u,v}$ for the remaining components. Clearly, by applying the decomposition (5.6) for each $i$, one has

$$|\gamma_i| = |\eta_i^{t,\ell}| + |\eta_i^{\ell,b}| + |\eta_i^{b,r}| + |\eta_i^{r,t}| + \sum_u |\gamma_i^u|,$$

where the sum ranges over $u \in \{t, \ell, b, r\}$.

Next, we want to decouple the four irreducible pieces, by writing $\Psi_{\Lambda_M}(\gamma_1, \ldots, \gamma_N)$ as the sum of a main term $\sum_u \Psi_u(\gamma_1^u, \ldots, \gamma_N^u)$ and a correction term associated to the corner pieces $\eta_i$ and to the interactions between distinct irreducible regions. To this end it will be convenient to enlarge the strips $S_v, S_h$ by an amount of order $(\log L)^2$. This will ensure that the expression (5.5) factorizes (up to lower order terms) into the product of four distinct pieces which, see Lemma 5.4 below, can each be reinterpreted as probabilities from the SOS staircase ensemble defined in Sect. 2.5. To define the potential $\Psi_u(\gamma_1^u, \ldots, \gamma_N^u)$ we proceed as follows.

We start with $u = t$. Let $S_v'$ denote the infinite vertical strip obtained by enlarging the original strip $S_v$ by $(\log L)^2$:

$$S_v' = \left\{ x \in \mathbb{Z}^2 : d(x, S_v) \leq (\log L)^2 \right\}.$$

Let $\hat{x}_i^t$ denote the point on the left boundary of $S_v'$ which has the same vertical coordinate as $x_i^t$ and let $\hat{y}_i^t$ denote the point on the right boundary of $S_v'$ which has the same vertical coordinate as $y_i^t$. Let $\tilde{\gamma}_i^t$ denote the open contour joining $\hat{x}_i^t$ and $\hat{y}_i^t$ obtained by connecting $\hat{x}_i^t$ and $x_i^t$ by a straight line, then using $\gamma_i^t$ from $x_i^t$ to $y_i^t$ and then connecting $y_i^t$ and $\hat{y}_i^t$ by a straight line; see Fig. 6. This defines a set of ordered, non-crossing paths
Lemma 5.3 Let $\Psi_{\Lambda_M}$ denote the potential from (5.5). There exists $\beta_0, C > 0$ such that: for any choice of $\gamma_1, \ldots, \gamma_N$ in (5.4) with $\gamma_1 \subset \Lambda_{L^2/2}$, for any $\beta \geq \beta_0$ one has

$$|\Psi_{\Lambda_M}(\gamma_1, \ldots, \gamma_N) - \sum_u \Psi_u(\gamma_1^u, \ldots, \gamma_N^u)|$$

$$\leq C \sum_{i=1}^{N+1} (|\eta_i^{\ell,}\ell| + |\eta_i^{b,}\ell| + |\eta_i^{r,}\ell| + |\eta_i^{r,}\ell|) + C(\log L)^3 \quad (5.8)$$

Proof We are going to use the properties of the potentials listed in Lemma 2.2. In particular, we use the fact that for $\beta$ large enough, for any $\Gamma \subset \mathbb{Z}^2$, any $\lambda > 0$ one has

$$\sum_{V \subset \mathbb{Z}^2} \sup_{V \cap \Gamma \neq \emptyset, d(V) \geq \lambda} |\varphi_{U_+, U_-}(V)| \leq C |\Gamma| e^{-\lambda} \quad (5.9)$$

for some constant $C > 0$. In the potential $\Psi_{\Lambda_M}$ one has a sum over subsets $V \subset \Lambda_M$, while the potential $\Psi_u$ contains sums over $V$ in the corresponding strips of width $2L'$. Since we assume $\gamma_1 \subset \Lambda_{L^2/2}$, one has that $d(\gamma_1, \Lambda_M) > L^2/4$ and therefore adding all $V$’s which are not contained in $\Lambda_M$ does not change the value of $\Psi_{\Lambda_M}(\gamma_1, \ldots, \gamma_N)$ by more than a constant. Similarly, using the fact that there are $N = O(\log L)$ contours and that $\gamma_i^l$ is at distance at least $\lambda = (\log L)^2$ from the complement of $S_v'$, when we compute $\Psi_1$, we may remove the constraint that $V \subset S_v'$ at the cost of an additive term $O((\log L)^3)$. Indeed, separating the contribution from the straight pieces in $\gamma_i^l$, and observing that $\max_i |\gamma_i^l| \leq CL^2$ (since all contours belong to $\Lambda_M$, with $M = L^2$) one has that the sum over all $V \not\subset S_v'$ at distance less than 1 from $\bigcup_{i=1}^N \gamma_i^l$ contributes at most

$$CNL^2e^{-(\log L)^2} + CN(\log L)^2 \leq C(\log L)^3.$$
The same applies to all $\Psi_u, u \in \{t, \ell, b, r\}$. The same reasoning shows that the sum over all $V$’s such that $V$ intersects both $\gamma_i^u$ and $\gamma_j^v$, for arbitrary $i, j$ is at most a constant if $u \neq v$. It remains to deal with the contribution from all the $V$’s which intersect some corner term $\eta_i^{u,v}$. By the rough bound (5.9) these can be estimated by $C|\eta_i^{u,v}|$. Putting together these facts one arrives at (5.8).

From (5.5), if $\gamma_1 \subset \Lambda_{L^2/2}$, then Lemma 5.3 implies for $\beta$ large enough:

$$\bar{\mathbb{P}}(\bigcap_{i=1}^N \mathcal{C}_{\gamma_1,i}) \leq \exp \left( -\frac{1}{2} \beta \sum_{i=1}^N (|\eta_i^{t,\ell}| + |\eta_i^{\ell,b}| + |\eta_i^{b,r}| + |\eta_i^{r,t}|) + C(\log L)^3 \right) \times \prod_u \exp \left( -\beta \sum_{i=1}^N |\gamma_i^u| + \Psi_u(\gamma_1^u, \ldots, \gamma_N^u) \right).$$

(5.10)

Let us now go back to (5.4). Using a very rough bound one can easily obtain

$$\bar{\mathbb{P}}(\gamma_1 \not\subset \Lambda_{L^2/2}) \leq e^{-L^2}.$$  

(5.11)

Indeed, write the expansion (2.6) with only one contour and estimate the decoration term $|\psi/\Lambda_1(\gamma_1)| \leq c_\beta |\gamma_1|$, with a constant $c_\beta > 0$ that vanishes as $\beta \to \infty$, and then use a simple Peierls’ argument together with the fact that $\gamma_1 \not\subset \Lambda_{L^2/2}$ implies $|\gamma_1| \geq L^2/2$.

From (5.11) and (5.10), summing over all choices of the points

$$(x, y) = \{(x_i^u, y_i^u), i = 1, \ldots, N; u = t, \ell, b, r\},$$

one has that up to the additive error term $e^{-L^2}$, $\bar{\mathbb{P}}(A(\delta, K))$ is upper bounded by

$$\sum_{(x, y)} \left( \prod_{i=1}^N \Theta(x_i^t, y_i^t) \Theta(x_i^\ell, y_i^b) \Theta(x_i^b, y_i^r) \Theta(x_i^r, y_i^t) \right) \prod_u \mathcal{Z}_u(x^u, y^u),$$

(5.12)

where

$$\mathcal{Z}_u(x^u, y^u) := \sum_{\gamma_1^u, \ldots, \gamma_N^u} \exp \left( -\beta \sum_{i=1}^N |\gamma_i^u| + \Psi_u(\gamma_1^u, \ldots, \gamma_N^u) \right),$$

(5.13)

and

$$\Theta(x_i^u, y_i^v) := e^{C(\log L)^3} \sum_{\eta: x_i^u \to y_i^v} \exp \left( -\frac{1}{2} \beta |\eta| \right).$$

(5.14)

The sum in (5.13) ranges over all open contours $\gamma_i^u : y_i^v \to x_i^u$ such that $\gamma_i^u, \gamma_j^v$ do not cross for $i \neq j$ and such that $\gamma_i^u$ is more internal (closer to $\Lambda'$) than $\gamma_j^v$ for $i > j$. 

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Since we are doing an upper bound, we may neglect the constraint that $\gamma_i^u$ does not cross the boundary of $\Lambda'$. The sum in (5.14) ranges over all paths from $x_i^u \to y_i^v$. The following lemma summarizes the main estimate we need.

**Lemma 5.4** For any $u$, uniformly in the choice of the points $x^u$, $y^u$, one has

$$Z_u(x^u, y^u) \leq \exp\left(-2\beta \tau_\beta(0) NL(1 - \delta)(1 + o(1))\right). \tag{5.15}$$

Let us conclude the proof by assuming the validity of Lemma 5.4. From (5.14) one has

$$\sum_{x_i^u, y_i^v} \Theta(x_i^u, y_i^v) \leq e^{C\log L^3},$$

for some new constant $C$. Therefore, one has the upper bound

$$\sum_{(x, y)} \left( \prod_{i=1}^N \Theta(x_i^u, y_i^v) \Theta(x_i^x, y_i^x) \Theta(x_i^y, y_i^y) \right) \leq e^{C\log L^4}.$$  

From (5.4)–(5.12), using the uniform bound (5.15) for each $u$, one has

$$\widetilde{P}(A(\delta, K)) \leq e^{C\log L^4} \exp\left(-8\beta \tau_\beta(0) N (1 - \delta)L(1 + o(1))\right).$$

Since $N = \frac{1}{\tau_\beta} \log L(1 + o(1))$ the conclusion (5.3) follows.

### 5.2 Proof of Lemma 5.4

The core of the proof is the monotonicity argument of Theorem 4.1 that allows us to consider each of the $N$ contours separately; see Sect. 4. To be able to apply this we first need to reformulate the problem in terms of $SOS$ contours. Without loss of generality we assume that $u = t$. Let $\hat{x}_i^t$, $\ldots$, $\hat{y}_i^t$ denote the points on the boundary of $S'_{v_i}$ as defined before (5.7), and call $a_{N-i+1}$ the vertical coordinate of $\hat{x}_i^t$ and $b_{N-i+1}$ the vertical coordinate of $\hat{y}_i^t$, $i = 1, \ldots, N$. Let $Z(a_1, \ldots, a_N; b_1, \ldots, b_N; L')$, $L' = (1 - \delta)L + (\log L)^2$, denote the partition function of the $N$ contours in the strip $S'_c$ as defined in Lemma 2.3. We claim that

$$Z_t(x^t, y^t) \leq e^{C\log L^3} Z(a_1, \ldots, a_N; b_1, \ldots, b_N; L'). \tag{5.16}$$

Let us first conclude the proof of Lemma 5.4 assuming the validity of the estimate (5.16). From (5.16) and Theorem 4.1 we can bound $Z_t(x^t, y^t)$ from above by a product of partition functions of a single contour:

$$Z_t(x^t, y^t) \leq e^{C\log L^3} \prod_{i=1}^N Z(a_i; b_i; L').$$
The surface tension bound (2.18) then implies the desired estimate (5.15).

To conclude the proof of Lemma 5.4, it remains to prove (5.16). To this end, observe that by the expansion (2.15), one has

\[ Z(a_1, \ldots, a_N; b_1, \ldots, b_N; L') = \sum_{\hat{\gamma}_1, \ldots, \hat{\gamma}_N} \exp \left( -\beta \sum_{i=1}^N |\hat{\gamma}_i| + \Phi_{L', \infty}(\hat{\gamma}_1, \ldots, \hat{\gamma}_N) \right), \]

where the sum ranges over all collections of non-crossing contours \( \hat{\gamma}_i : \hat{x}_i^t \rightarrow \hat{y}_i^t \). Let us restrict this summation to paths of the form \( \hat{\gamma}_i = \gamma_t^i \), i.e. paths which have a straight line from \( \hat{x}_i^t \) to \( x_i^t \), a regular path \( \gamma_t^i : x_i^t \rightarrow y_i^t \), and a straight line from \( y_i^t \rightarrow \hat{y}_i^t \); see Fig. 6. By summing over the regular parts \( \gamma_t^i \) and using \( |\hat{\gamma}_i^t| = |\gamma_t^i| + 2(\log L)^2 \) one has

\[ Z(a_1, \ldots, a_N; b_1, \ldots, b_N; L') \geq \sum_{\gamma'_1, \ldots, \gamma'_N} \exp \left( -\beta \sum_{i=1}^N |\gamma'_i| + \Phi_{L', \infty}(\gamma'_1, \ldots, \gamma'_N) - 2\beta N (\log L)^2 \right). \]

By the definition (5.7), one has \( \Phi_{L', \infty}(\gamma'_1, \ldots, \gamma'_N) = \Psi_t(\gamma_1^t, \ldots, \gamma_N^t) \). Therefore, using \( N \leq (4\beta)^{-1} \log L \), we conclude

\[ Z(a_1, \ldots, a_N; b_1, \ldots, b_N; L') \geq Z_t(x^t, y^t) e^{-C(\log L)^3}. \]

This ends the proof of (5.16).

References

1. Bolthausen, E., Deuschel, J.-D., Giacomin, G.: Entropic repulsion and the maximum of the two-dimensional harmonic crystal. Ann. Probab. 29(4), 1670–1692 (2001)
2. Bolthausen, E., Deuschel, J.-D., Zeitouni, O.: Entropic repulsion of the lattice free field. Commun. Math. Phys. 170(2), 417–443 (1995)
3. Brandenberger, R., Wayne, C.E.: Decay of correlations in surface models. J. Stat. Phys. 27(3), 425–440 (1982)
4. Bricmont, J., El Mellouki, A., Fröhlich, J.: Random surfaces in statistical mechanics: roughening, rounding, wetting, . . . J. Stat. Phys. 42(5–6), 743–798 (1986)
5. Caputo, P., Lubetzky, E., Martinelli, F., Sly, A., Toninelli, F.L.: Dynamics of 2+1 dimensional sos surfaces above a wall: slow mixing induced by entropic repulsion. Ann. Probab. 42, 1516–1589 (2014)
6. Caputo, P., Lubetzky, E., Martinelli, F., Sly, A., Toninelli, F.L.: The shape of the (2+1)D SOS surface above a wall. C. R. Math. Acad. Sci. Paris 350(13–14), 703–706 (2012)
7. Caputo, P., Lubetzky, E., Martinelli, F., Sly, A., Toninelli, F.L.: Scaling limit and cube-root fluctuations in sos surfaces above a wall. J. Eur. Math. Soc. arXiv:1302.6941 (2013, preprint)
8. Deuschel, J.-D.: Entropic repulsion of the lattice free field. II. The 0-boundary case. Commun. Math. Phys. 181(3), 647–665 (1996)
9. Deuschel, J.-D., Giacomin, G.: Entropic repulsion for massless fields. Stoch. Process. Appl. 89(2), 333–354 (2000)
10. Deuschel, J.-D., Giacomin, G., Ioffe, D.: Large deviations and concentration properties for \( \nabla \phi \) interface models. Probab. Theory Relat. Fields 117(1), 49–111 (2000)
11. Dobrushin, R., Kotecký, R., Shlosman, S.: Wulff construction, Translations of Mathematical Monographs, vol. 104. American Mathematical Society, Providence (1992) (A global shape from local interaction, Translated from the Russian by the authors)
12. Fortuin, C.M., Kasteleyn, P.W., Ginibre, J.: Correlation inequalities on some partially ordered sets. Commun. Math. Phys. 22, 89–103 (1971)
13. Kotecký, R., Preiss, D.: Cluster expansion for abstract polymer models. Commun. Math. Phys. 103(3), 491–498 (1986)
14. Lebowitz, J.L., Maes, C.: The effect of an external field on an interface, entropic repulsion. J. Stat. Phys. 46(1–2), 39–49 (1987)
15. Lubetzky, E., Martinelli F., Sly, A.: Harmonic pinnacles in the Discrete Gaussian model. arXiv:1405.5241 (2014, preprint)
16. Velenik, Y.: Localization and delocalization of random interfaces. Probab. Surv. 3, 112–169 (2006)