MORSE THEORY FOR GROUP PRESENTATIONS

XIMENA FERNÁNDEZ

Abstract. We introduce a novel combinatorial method to study $Q^{**}$-transformations of group presentations or, equivalently, 3-deformations of CW-complexes of dimension 2. Our procedure is based on a refinement of discrete Morse theory that gives a Whitehead simple homotopy equivalence from a regular CW-complex to the simplified Morse CW-complex, with an explicit description of the attaching maps and bounds on the dimension of the complexes involved in the deformation. We apply this technique to show that some known potential counterexamples to the Andrews–Curtis conjecture do satisfy the conjecture.

Any finite presentation $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ of a group $G$ can be transformed into any other presentation of the same group by a finite sequence of the following operations [37, 28]:

1. replace some relator $r_i$ by $r_i^{-1}$;
2. replace some relator $r_i$ by $r_i r_j$ for some $j \neq i$;
3. replace some relator $r_i$ by a conjugate $w r_i w^{-1}$ for some $w$ in the free group $F(x_1, x_2, \ldots, x_n)$;
4. add a generator $x_{n+1}$ and a relator $r_{m+1}$ that coincides with $x_{n+1}$, or the inverse of this operation;
5. replace each relator $r_i$ by $\phi(r_i)$ where $\phi$ is an automorphism of $F(x_1, x_2, \ldots, x_n)$;
6. add a relator 1, or the inverse of this operation.

For balanced presentations (i.e., $m = n$), J. Andrews and M. Curtis conjectured in 1965 [2] that any presentation of the trivial group $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ can be transformed into the empty presentation $\langle \mid \rangle$ by a finite sequence of the operations (1) to (5), called $Q^{**}$-transformations [29, 41]. This question still remains open and it has become one of the most notorious problems in group theory, as well as in low-dimensional topology due to its topological consequences. Indeed, although the original conjecture is in the area of combinatorial group theory, it has an equivalent formulation – first noticed by the anonymous referee of the foundational article [2] – in terms of Whitehead’s simple homotopy theory [39, 10]. It states that any (finite) contractible 2-dimensional CW-complex $K$ 3-deforms to a point. That is, $K$ can be transformed into a point by a sequence of elementary collapses and expansions in which the dimension of the complexes involved is not greater than 3 (see [42] and [23] Sect. 2.3, Ch. 1 for a detailed proof of this equivalence). This conjecture is closely related to other relevant problems in algebraic topology, such as Whitehead asphericity conjecture [38], Zeeman conjecture [43] and the smooth 4-dimensional Poincaré conjecture [24] (see also [23]). The Andrews–Curtis conjecture is known to be true for some classes of complexes (such as the standard spines [20] and the quasi-constructible...
complexes \( [3] \), but the problem still remains unsolved for general 2-complexes or, equivalently, balanced presentations of the trivial group. Moreover, there is a list of balanced presentations of the trivial group for which no \( Q^* \)-trivialization is known. They serve as potential counterexamples to disprove the conjecture (see [23, Sect. 1.1 Ch. XII] and [35]).

Computational approaches to this problem have proven to be limited by the exponential complexity of the algorithms \([5, 8, 22, 27, 31, 32, 34]\). Most of them are based on the exploration and exhibition of possible transformations of type (1)–(3) from a given presentation. However, in \([8]\) M. Bridson exhibited examples of rather small balanced presentations of the trivial group for which the minimum length of any \( Q^* \)-simplification sequence to \( \langle | \rangle \) is super-exponential in the total length of the relators (and hence, computationally intractable with standard methods). In this article, we present a method which combines topological and combinatorial tools, that allows the computational exploration of presentations which are \( Q^* \)-equivalent from a given one without the need of exhibiting the actual list of transformations. This alternative technique to find \( Q^* \)-transformations, based on a refinement of discrete Morse theory, enables us to show that some well known potential counterexamples to the conjecture can be easily \( Q^* \)-trivialized. Moreover, we also apply this theory to presentations of non-trivial groups, proving that some potential counterexamples to the generalized Andrews-Curtis conjecture \([4]\) do satisfy it.

Forman’s discrete Morse theory \([15, 16]\) introduces a combinatorial tool to simplify the cell decomposition of a given (regular) CW-complex up to homotopy equivalence, in terms of the critical cells of discrete Morse functions. Although this theory results in an efficient way to compute the homology of a regular CW-complex, it does not provide sufficient information to recover its homotopy type due to the lack of a combinatorial description of the simplified CW-complex (the Morse CW-complex). Moreover, the procedure does not ensure the preservation of the simple homotopy type. We extend the scope of Forman’s theory to Whitehead deformations and simple homotopy classes. Concretely, given a regular \( n \)-dimensional complex \( K \) and a discrete Morse function on it, we construct an explicit and algorithmically computable cell decomposition of the Morse CW-complex and we prove that it \((n + 1)\)-deforms to \( K \). We deduce a computational method for handling \( 3 \)-deformations of 2-complexes and thus study the Andrews–Curtis conjecture from a new point of view. We present an algorithm to obtain new presentations \( Q^* \)-equivalent to a given one without requiring to specify the exhaustive list of movements to transform one into the other.

Independently, in \([6, 7]\) the authors also used discrete Morse theory to present an algorithm to describe a presentation of the fundamental group of a regular CW-complex. They applied it in a classification problem of prime knots, and also to compute the fundamental group of point clouds.

The article is organized as follows. In Section 1 we present the refinement of discrete Morse theory in terms of internal collapses and Whitehead deformations with bounds in the dimension of the complexes involved. In Section 2 we deduce a method that associates to each presentation \( P \) and acyclic matching in a poset obtained from \( P \), a presentation \( Q \) such that \( P \sim_{Q^*} Q \). In Section 3 we exhibit applications of our method to investigate potential counterexamples to the Andrews–Curtis conjecture. Appendix 1 contains the proof of an intermediate result that is necessary in Section 3. The implementation in SAGE \([36]\) of the algorithms of Section 2 as well as its application to the potential counterexamples at Section 3 can be found at \([13]\) as part of the package Finite Topological Spaces. An
implementation in GAP \cite{18} is also available at the package Posets \cite{13}. An outline of the computational procedure is described in Appendix A.

Note. Most of the results of this article appeared originally in the author’s PhD Thesis \cite{12}.

1. DISCRETE MORSE THEORY AND WHITEHEAD DEFORMATIONS

Discrete Morse theory was introduced by R. Forman \cite{15, 16} as a discrete approach to classical Morse theory for smooth manifolds. It is based on a combinatorial notion of Morse functions for regular CW-complexes. Critical cells of Morse functions on a CW-complex $K$ are linked to the number of cells in each dimension of a new CW-complex, the Morse CW-complex, which is homotopy equivalent to $K$ \cite[Cor. 3.5]{15}. Although discrete Morse theory is a relevant tool to obtain information about the homotopy type of a CW-complex, it does not give information about its Whitehead simple homotopy type. In this section we present a refinement of the theory, showing that discrete Morse theory actually provides a method to simplify the cell structure of an $n$-dimensional complex through an $(n + 1)$-deformation.

We briefly recall here the main concepts in simple homotopy theory and refer the reader to \cite{11, 23} for a more complete exposition. All the CW-complexes in this article will be finite and connected. Given a CW-complex $K$ and a subcomplex $L \subseteq K$, we say that $K$ elementary collapses to $L$ (or $L$ elementary expands to $K$) and denote it by $K \searrow L$ (resp. $L \nearrow K$) if $K = L \cup e^{n-1} \cup e^n$ with $e^{n-1}, e^n \notin L$ and there exists a map $\psi : D^n \to K$ such that $\psi$ is the characteristic map of $e^n$, $\psi|_{D^n - \bar{D}^{n-1}}$ is the characteristic map of $e^{n-1}$ and $\psi(D^{n-1}) \subseteq L^{(n-1)}$, where $L^{(n-1)}$ is the $(n-1)$-skeleton of $L$. In general, $K$ collapses to $L$ (or $L$ expands to $K$) if there is a finite sequence of elementary collapses from $K$ to $L$. We denote it by $K \searrow L$ (resp. $L \nearrow K$). We say that a CW-complex $K$ $n$-deforms to $L$, and denote it by $K \searrow L^n$, if there is a sequence of CW-complexes $K = K_0, K_1, \ldots, K_r = L$ such that $K_i \searrow K_{i+1}$ or $K_i \nearrow K_{i+1}$ for each $0 \leq i \leq r - 1$, and $\dim(K_i) \leq n$ for all $1 \leq i \leq r$. For every $0 \leq i \leq r - 1$, there is a homotopy equivalence $f_i : K_i \to K_{i+1}$ which is an inclusion or a retraction depending on whether $K_i \nearrow K_{i+1}$ or $K_{i+1} \searrow K_i$, respectively. If $K$ $n$-deforms to $L$, then $K$ and $L$ are then related by a deformation $f : K \to L$ defined as the composition of the retractions and inclusions as above, i.e., $f = f_{r-1} \ldots f_1 f_0$. Notice that if $K \searrow L$, then $K$ and $L$ are homotopy equivalent. The converse is not true in general and this obstruction is measured by the Whitehead group \cite{11}.

We now outline the main definitions and results in discrete Morse theory. We refer the reader to \cite{17, 25} for more details. A CW-complex $K$ is said to be regular if for every open cell $e^n$, the characteristic map $D^n \to \bar{e}^n$ is a homeomorphism. A cell $e$ of a regular complex $K$ is a face of a cell $e'$ if $e \subseteq e'$. We denote the face relation by $e \leq e'$. Given a regular CW-complex $K$, a map $f : K \to \mathbb{R}$ is a discrete Morse function if for every cell $e^n$ in $K$, the number of faces and cofaces of $e^n$ for which the value of $f$ does not increase with dimension is at most one. An $n$-cell $e^n \in K^{(n)}$ is a critical cell of index $n$ if the values of $f$ in every face and coface of $e^n$ increase with dimension. A discrete Morse function induces an ordering in the cells, which determines level subcomplexes of $K$. For every $c \in \mathbb{R}$, the level subcomplex $K(c)$ of $K$ is the subcomplex of closed cells $\bar{e}$ of $K$ such that $f(e) \leq c$ in $\mathbb{R}$. Discrete Morse functions serve as a tool to study the homotopy type of $K$. The following theorem summarizes the main results in discrete Morse theory.
Theorem 1.1. \[16, 17\] Let $K$ be a regular CW-complex and let $f : K \to \mathbb{R}$ be a discrete Morse function. Let $a < b$ be real numbers.

(a) If every cell $e \in K$ such that $f(e) \in (a, b]$ is not critical, then $K(b) \setminus K(a)$.

(b) If $e^n \in K$ is the only critical cell with $f(e^n) \in (a, b]$, then there is a continuous map $\varphi : \partial D^n \to K(a)$ such that $K(b)$ is homotopy equivalent to $K(a) \cup_{\varphi} D^n$.

(c) $K$ is homotopy equivalent to a CW-complex with exactly one cell of dimension $k$ for every critical cell of index $k$.

Given $K$ a regular CW-complex, denote by $\mathcal{X}(K)$ its face poset, that is, the poset of cells of $K$ ordered by the face relation $\leq$. Let $\mathcal{H}(\mathcal{X}(K))$ be the Hasse diagram of the poset $\mathcal{X}(K)$, a digraph whose vertices are the cells of $K$ and whose edges are the ordered pairs $(e, e')$ such that $e < e'$ and there exists no $e'' \in K$ such that $e < e'' < e'$ (in that case, we say that $e'$ covers $e$ and we denote this by $e \prec e'$).

Every discrete Morse function $f : K \to \mathbb{R}$ has an associated set $M_f$ of pairings of cells of $K$, where $(e, e') \in M_f$ if and only if $e < e'$ and $f(e) \geq f(e')$. Moreover, $M_f$ is an acyclic matching in $\mathcal{X}(K)$. Recall that a pairing $M$ of cells in $K$ is said to be an acyclic matching if each cell of $K$ is involved in at most one pair of $M$ and the directed graph $\mathcal{H}_M(\mathcal{X}(K))$ obtained by reversing the orientation of the edges of $\mathcal{H}(\mathcal{X}(K))$ associated to matched pairs of cells is acyclic. In \[10\], M. Chari proved that a subset $C$ of cells of $K$ is the set of critical cells of a discrete Morse function $f$ on $K$ if and only if there is an acyclic matching $M$ in $\mathcal{X}(K)$ such that $C$ is the set of nodes of $\mathcal{X}(K)$ not incident to any edge in $M$.

We next present a series of results in simple homotopy theory as preliminary steps to our refinement of Theorem 1.1. Our theory is built over the idea of internal collapses, a generalization of the standard collapses (see \[25\], Ch. 11 and \[26\], Ch. 11). We will show that internal collapses can be thought of as a way of performing $(n+1)$-deformations from a CW-complex of dimension less than or equal to $n$ to another one with fewer number of cells. This general notion of collapse will be the key to a better understanding of the close connection between discrete Morse theory and Whitehead deformations of bounded dimension.

The cornerstone of the concept of internal collapses is the following fact. If $K \setminus L$ and we attach a cell $e$ to $K$, then we still have a deformation from $K \cup e$ to $L \cup \tilde{e}$, where $\tilde{e}$ is attached with an inherited attaching map (cf. \[11\], Prop. 7.1). When needed, we may emphasize the attaching map $\varphi : \partial D^n \to K$ of an $n$-cell by writing $K \cup e^n$ as $K \cup_{\varphi} D^n$.

Lemma 1.2. Let $K$ be a CW-complex of dimension less than or equal to $n$. Let $\varphi : \partial D^n \to K$ be the attaching map of an $n$-cell $e^n$. If $K \setminus L$, then $K \cup e^n \setminus L \cup \tilde{e}^n$, where the attaching map $\tilde{\varphi} : \partial D^n \to L$ of $\tilde{e}^n$ is defined as $\tilde{\varphi} = r \varphi$ with $r : K \to L$ the canonical strong deformation retract induced by the collapse $K \setminus L$.

Proof. Let $i : L \to K$ be the inclusion map and let $r : K \to L$ be the strong deformation retract induced by the collapse $K \setminus L$. There is a homotopy $H : \partial D^n \times I \to K$, $ir\varphi \equiv_H \varphi$ that allows to perform the following elementary moves

$$K \cup_{\varphi} D^n \looparrowright (K \cup_{\varphi} D^n) \cup_{ir\varphi} D^n \cup_H D^n \times I \setminus \varphi K \cup_{ir\varphi} D^n.$$ 

Finally, the collapse $K \setminus L$ induces a collapse $K \cup_{ir\varphi} D^n \setminus L \cup_{r\varphi} D^n$, since the image of the attaching map $ir\varphi$ is included in $L$. Given that the dimension of the complex
By definition, there are elementary moves the homotopy inclusion, then the maps \( \varphi \) are homotopic. Let \( \varphi : \partial D_j \to K \cup \bigcup_{i<j} e_i \) be the attaching map of \( e_j \). If \( K \subseteq L \), then there exist CW-complexes \( Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_d \) of dimension less than or equal to \( n+1 \) such that for every \( j = 1, 2, \ldots, d \),

\[
K \cup \bigcup_{i=1}^{j} e_i \not\rightarrow Z_j \not\rightarrow L \cup \bigcup_{i=1}^{j} \tilde{e}_i
\]

where the attaching map \( \tilde{\varphi}_j : \partial D_j \to \bigcup_{i<j} \tilde{e}_i \) of the cell \( \tilde{e}_j \) is defined inductively as follows: \( \tilde{\varphi}_1 = r_0 \varphi_1 \) with \( r_0 : K \to L \) the canonical strong deformation retract and for \( j > 1 \), \( \tilde{\varphi}_j = \tilde{r}_{j-1} t_{j-1} \varphi_j \) where \( \tilde{r}_{j-1} : Z_{j-1} \to L \cup \bigcup_{i<j} \tilde{e}_i \) is the strong deformation retract and \( t_{j-1} : K \cup \bigcup_{i<j} e_i \to Z_{j-1} \) is the inclusion.

**Proof.** We proceed by induction on the number of cells \( d \). If \( d = 1 \), the assertion follows by Lemma 1.2. Suppose that the statement holds for \( d \geq 1 \). Thus, we have the following sequence of expansions and collapses

\[
K \cup \bigcup_{i=1}^{d} e_i \not\rightarrow Z_d \not\rightarrow L \cup \bigcup_{i=1}^{d} \tilde{e}_i.
\]

There is a strong deformation retract \( \tilde{r}_d : Z_d \to \bigcup_{i=1}^{d} \tilde{e}_i \) and an inclusion \( i_d : K \cup \bigcup_{i=1}^{d} e_i \to Z_d \). Let \( \varphi_{d+1} : \partial D_{d+1} \to K \cup \bigcup_{i=1}^{d} e_i \) be the attaching map of the cell \( e_{d+1} \). Define

\[
\tilde{\varphi}_{d+1} : \partial D_{d+1} \to L \cup \bigcup_{i=1}^{d} \tilde{e}_i \text{ as } \tilde{\varphi}_{d+1} = \tilde{r}_{d+1} \tilde{\varphi}_{d+1}.
\]

If \( \tilde{i}_d : L \cup \bigcup_{i=1}^{d} \tilde{e}_i \to Z_d \) denotes the inclusion, then the maps \( \tilde{i}_d \varphi_{d+1} \) and \( \tilde{i}_d \tilde{\varphi}_{d+1} \) are homotopic. Let \( H : \partial D_{d+1} \times I \to Z_d \) be the homotopy \( H = \tilde{i}_d \tilde{\varphi}_{d+1} \). Define

\[
Z_{d+1} := Z_d \cup \bigcup_{i=1}^{d} e_{d+1} D_{d+1} \cup i_{d+1} \varphi_{d+1} D_{d+1} \cup H (D_{d+1} \times I).
\]

By definition, there are elementary moves

\[
Z_d \cup i_{d+1} \varphi_{d+1} D_{d+1} \not\rightarrow Z_{d+1} \not\rightarrow Z_d \cup i_{d+1} \tilde{\varphi}_{d+1} D_{d+1}.
\]
Since \( \text{Im}(\varphi_{d+1}) \subseteq K \cup \bigcup_{i=1}^{d} e_i \), the collapse \( Z_d \setminus K \cup \bigcup_{i=1}^{d} e_i \) induces a collapse \( Z_d \cup \varphi_{d+1} \). Analogously, \( Z_d \cup \tilde{\varphi}_{d+1} \) induces a collapse \( Z_d \cup \tilde{\varphi}_{d+1} \) and the result follows. \( \square \)

**Corollary 1.4.** Let \( K \) be an \( n \)-dimensional CW-complex and let \( L \leq K \) be a subcomplex of \( K \) such that \( L \searrow \ast \). Then \( K \setminus^ {n+1} K/L \).

**Definition 1.5.** Under the conditions of Proposition 1.3, we say that there is an internal collapse from \( K \cup \bigcup_{i=1}^{d} e_i \) to \( L \cup \bigcup_{i=1}^{d} \tilde{e}_i \).

We next prove that the composition of a sequence of internal collapses in \( n \)-dimensional complexes is an \( (n+1) \)-deformation.

**Theorem 1.6.** Let \( L \) be a CW-complex on dimension \( n \). Let \( L_0 \leq K_0 \leq L_1 \leq K_1 \leq \ldots \leq L_N \leq K_N \leq L_{N+1} = L \) be a sequence of CW-subcomplexes of \( L \) such that \( K_j \searrow L_j \) for all \( j = 0, 1, \ldots N \). If \( L_{j+1} = K_j \cup \bigcup_{i=1}^{d_j} e_j^i \), then there are internal collapses from \( L_{j+1} \) to \( L_j \cup \bigcup_{i=1}^{d_j} \tilde{e}_j^i \) for all \( j = 0, 1, \ldots, N \) that induce an \( (n+1) \)-deformation

\[
L \searrow^{n+1} L_0 \cup \bigcup_{j=0}^{N} \bigcup_{i=1}^{d_j} \tilde{e}_j^i \]

**Proof.** We proceed by induction on \( N \). For \( N = 0 \), the statement follows from Proposition 1.3. For \( N \geq 1 \), by inductive hypothesis there is an \( (n+1) \)-deformation

\[
L_0 \cup \bigcup_{j=0}^{N-1} \bigcup_{i=1}^{d_j} \tilde{e}_j^i \searrow L_N .
\]

Moreover, there exist a CW-complex \( Z \) of dimension \( n+1 \) such that

\[
L_0 \cup \bigcup_{j=0}^{N-1} \bigcup_{i=1}^{d_j} \tilde{e}_j^i \searrow Z \searrow L_N
\]

(see for instance [11 Ch.II]). Suppose that \( L_N \searrow K_N \subseteq L_{N+1} \), where \( L_{N+1} = K_N \cup \bigcup_{j=1}^{d_N} e_j^N \). By Proposition 1.3, the internal collapse from \( L_{N+1} \) to \( L_N \cup \bigcup_{j=1}^{d_N} e_j^N \) induces an \( (n+1) \)-deformation (where the attaching maps of \( e_j^N \) are induced from the attaching maps of \( e_j^N \)). On the other hand, an argument analogous to the proof of Proposition 1.3 shows that it is

\footnote{Here, the attaching maps of the cells \( \tilde{e}_j^i \) are induced by the internal collapses (see Proposition 1.3).}
possible to construct a sequence of CW-complexes $Z \leq Z_1 \leq Z_2 \leq \ldots \leq Z_{d_N}$ of dimension less than or equal to $n + 1$ such that for every $d = 1, 2, \ldots, d_N$

$$\left( L_0 \cup \bigcup_{j=0}^{N-1} d_j \right) \bigcup \bigcup_{i=1}^d \tilde{e}_j^N \cup Z_J \cup \left( L_N \cup \bigcup_{i=1}^d \tilde{e}_i^N \right)$$

where the attaching map $\tilde{e}_j^N : \partial D_j \to L \cup \bigcup_{i<d} \tilde{e}_i$ of the cell $\tilde{e}_d^N$ is induced by the attaching map of $\tilde{e}_d^N$.

We have proved that there is an $(n+1)$-deformation from $L_{N+1}$ to

$$\left( L_0 \cup \bigcup_{j=0}^{N-1} d_j \right) \cup \bigcup_{i=1}^{d_N} \tilde{e}_i^N$$

(equivalently, to $L_0 \cup \bigcup_{j=0}^{N-1} d_j \tilde{e}_j^N$ by abuse of notation). \qed

In what follows, we interpret discrete Morse theory in terms of internal collapses. In particular, we deduce a combinatorial method to simplify the cell structure of a regular CW-complex preserving its simple homotopy type, with bounds in the deformation. Every discrete Morse function $f$ on a regular CW-complex $K$ can be described as a sequence of internal collapses in its CW-structure as follows.

**Lemma 1.7.** Let $K$ be a regular CW-complex. Then, $M$ is an acyclic matching in $\mathcal{X}(K)$ with unmatched set of cells $C$ if and only if there exist a sequence of subcomplexes of $K$

$$(*)_1 \quad K_0 \leq L_1 \leq K_1 \ldots \leq K_{N-1} \leq L_{N-1} \leq K_N = K$$

such that $K_j \cap L_j$ for all $1 \leq j \leq N$ and the set of cells of $K$ that was not collapsed in any of the collapses $K_j \cap L_j$ is equal to $C$.

**Proof.** Given an acyclic matching $M$ in $\mathcal{X}(K)$, there is a linear extension $\mathcal{L}$ of the (order induced by the) directed acyclic graph $\mathcal{H}_M(\mathcal{X}(K))$ such that if $(e, e') \in M$, the cells $e, e'$ follow consecutively in total order $\mathcal{L}$ (see [23], Thm. 11.2]). Let $\mathcal{L}_M$ be the ordering induced by $\mathcal{L}$ with the additional relations $e = \mathcal{L}_M e'$ if $(e, e') \in M$. It is easy to see that $f_M : K \to \mathbb{R}$ defined by $f_M(e) = \left| e' \in K : e' \leq \mathcal{L}_M e \right|$ is a discrete Morse function with critical cells the set $C$ of unmatched cells of $M$. If $c_1 \leq c_2 \leq \ldots \leq c_N \in \mathbb{N}_0$ are the images under $f_M$ of the critical cells of $K$, define $L_j = K(c_j)$ for $1 \leq j \leq N$, $K_{j-1} = K(c_j - 1)$ for $1 \leq j \leq N - 1$ and $K_N = K$. Notice that $L_1 \leq K_1 \leq \ldots \leq L_N \leq K_N$ is a sequence of CW-subcomplexes of $K$ and by Theorem 1.1, $K_j \cap L_j$ for all $1 \leq j \leq N$. The set of cells of $K$ that are not involved in any of the collapses $K_j \cap L_j$ is $\{ e \in K(c_j) \setminus K(c_j - 1) : 1 \leq j \leq N \} = C$, the set of critical cells of $f_M$.

Conversely, given a sequence $(*_1)$ of subcomplexes of a regular CW-complex $K$, define $M$ as the set of pairings of cells $(e^{k+1}_i, e^k_i)$ of $K$ associated to each elementary collapse in $K_j \cap L_j$ for all $1 \leq j \leq N$. Since each cell is involved in at most one elementary collapse, $M$ is a matching in $\mathcal{X}(K)$. Suppose that there is a simple cycle

$$(*)_2 \quad e^{k-1}_i < e^k_i < \ldots < e^2_i < e^1_i$$

in $\mathcal{H}_M(\mathcal{X}(K))$ where $(e^{k-1}_i, e^k_i) \in M$ for all $1 \leq i \leq l$. Let $K_j$ be the minimal subcomplex of $K$ in the sequence $(*_1)$ that contains all the cells of the cycle $(*_2)$. For every pair of cells $(e^{k-1}_i, e^k_i)$ in $(*_2)$, $e^{k-1}_i$ is also a face of $e^{k-1}_{i-1}$, which in turn is not part of any other
elementary collapse along with a higher dimensional cell. So none of the pairs \((e_{i-1}^k, e_i^k)\) determines an elementary collapse in \(K_j \setminus L_j\). This contradicts the minimality of \(K_j\). Hence, \(M\) is an acyclic matching in \(\mathcal{X}(K)\).

**Remark 1.8.** Given a regular CW-complex \(K\) and an acyclic matching in \(\mathcal{X}(K)\), the associated sequence of internal collapses \((\ast)\) in \(K\) of Lemma 1.7 can be performed in a way that all the collapses involved are performed in decreasing dimension. That is, if \((e_{k-1}^k, e_k^k)\) and \((e_{k'}^{k'}, e_{k'}^{k'})\) are pair of cells involved in collapses \(K_j \setminus L_j\) and \(K_j' \setminus L_j'\) respectively with \(j' \leq j\), then the dimensions \(k, k'\) satisfy \(k' \leq k\). This construction can be achieved by considering in the proof of Lemma 1.7 a linear extension \(L\) of the order induced by \(\mathcal{H}_M(\mathcal{X}(K))\) such that it also respects the order given by the dimension (the existence of such \(L\) follows by a slight modification of proof of [25 Thm. 11.2]).

We next state the simple homotopy version of Theorem 1.1 that provides an explicit construction of the Morse CW-complex and establishes bounds on the dimension of the deformation.

**Theorem 1.9.** Let \(K\) be a regular CW-complex of dimension \(n\) and let \(M\) be an acyclic matching in \(\mathcal{X}(K)\). Then \(M\) induces a sequence of subcomplexes of \(K\)

\[
K_0 \leq L_1 \leq K_1 \ldots \leq K_{N-1} \leq L_{N-1} \leq K_N = K
\]

such that \(K_j \setminus L_j\) for all \(1 \leq j \leq N\) and the set of cells of \(K\) non-collapsed in \(K_j \setminus L_j\) \(\forall j\) corresponds to the unmatched cells in \(M\). Moreover, if \(K_M\) is the CW-complex obtained after performing in \(K\) the sequence of internal collapses induced by \(M\), then \(K \setminus \bigcup_{j=0}^{n+1} K_M\) (where \(K_M\) is as defined in the proof of Lemma 1.7 can be performed in a way that all the collapses involved are performed in decreasing dimension. Then, \(K_M\) has exactly one cell of dimension \(k\) for every unmatched cell of dimension \(k\) in \(M\).

**Proof.** It is consequence of Lemma 1.7 and Theorem 1.6.

We focus now in the explicit combinatorial description of the deformation provided by Theorem 1.9 in the case of 2-complexes.

**Remark 1.10.** Notice that internal collapses transform 2-dimensional regular CW-complexes into combinatorial complexes. Recall that a CW-complex \(K\) of dimension 2 is called **combinatorial** if for each 2-cell \(e^2\), its attaching map \(\varphi : S^1 \rightarrow K(1)\), seen as a cellular map by assigning a CW-structure on \(S^1\), is a combinatorial map (that is, \(\varphi\) sends each open 1-cell of \(S^1\) either homeomorphically onto an open 1-cell of \(K\) or it collapses it to a 0-cell of \(K\), see [23 Ch.II]). Thus, one can think of the attaching map of a 2-cell in a combinatorial complex of dimension 2 just as the ordered list of oriented 1-cells. Suppose that there is an elementary internal collapse from the combinatorial 2-complex \(K \cup \bigcup_{i=1}^{n} e_i\) to \(L \cup \bigcup_{i=1}^{n} \tilde{e}_i\), in which \(K \subseteq L\). We endow the CW-complex \(L \cup \bigcup_{i=1}^{n} \tilde{e}_i\) with a combinatorial structure as follows. If \(K = L \cup \{x, v\}\), where \(v\) is a 0-cell and \(x\) a 1-cell, then for every 2-cell in \(\bigcup_{i=1}^{n} e_i\) containing an edge \(x^\epsilon\) with \(\epsilon = 1\) or \(-1\) (where by \(x^{-1}\) we mean the 1-cell \(x\) traversed with the opposite orientation), modify its attaching map by replacing each occurrence of \(x^\epsilon\) by 1. Similarly, if \(K = L \cup \{x, e\}\) with \(x\) a 1-cell and \(e\) a 2-cell with attaching map \(xx_1 \ldots x_r\), then for every 2-cell in \(\bigcup_{i=1}^{n} e_i\) containing edge \(x^\epsilon\) modify its attaching map by replacing each occurrence of \(x^\epsilon\) by \((x_1 \ldots x_r)^{-\epsilon}\) and leaving the remaining cells unchanged. As a result, after performing a sequence of internal collapses to a regular CW-complex \(K \cup \bigcup_{i=1}^{n} e_i\) we obtain a CW-complex \(L \cup \bigcup_{i=1}^{n} \tilde{e}_i\) with a natural combinatorial structure.
Example 1.11 (Torus). Let $K$ be the oriented regular CW-complex of Figure 1. Let $M$ be the acyclic matching in $\mathcal{X}(K)$ described in Figure 2. By Theorem 1.1, the Morse CW-complex $K_M$ has one cell of dimension 0, two cells of dimension 1 and one cell of dimension 2. There is also a formula to compute the incidence number of critical cells of consecutive dimension by means of $M$ (see [13] and [25, Thm. 11.13]). However, this information does not determine the homotopy type of $K_M$. In fact, the incidence number of the critical 2-cell into each of the critical 1-cells is 0. But the same number of cells in each dimension and incidence numbers can be observed in the minimal CW-structure of $S^1 \vee S^1 \vee S^2$. We give next an explicit description of the attaching map of the 2-cell using Theorem 1.9.

We denote by $v_i, x_i, e_i$ the 0-cells, 1-cells and 2-cells of $K$ respectively. Initially, the attaching map of the critical 2-cell $e_{10}$ is

$$x_3^{-1} x_5 x_2^{-1} x_9.$$  

After the internal collapse associated to the pair $(x_9, e_9)$ (or 9 for short), the attaching map of $\tilde{e}_{10}$ becomes

$$x_3^{-1} x_5 x_2^{-1} x_8 x_5^{-1} x_4^{-1}.$$
(see Figure 3 (a)). After performing the internal collapses associated to the pairs 8 and 7, the induced attaching map of the 2-cell $\tilde{e}_{10}$ is
\[x_3^{-1}x_5x_2^{-1}x_2x_6x_3x_4x_6^{-1}x_5^{-1}x_4^{-1}\]
(see Figure 3 (b) and (c)). Now, the internal collapses 5, 3, 2 associated to pairs of 0-cells and 1-cells mean that the corresponding edges are collapsed into a point, and they translate into the elimination of each occurrence of $x_2^\epsilon, x_3^\epsilon, x_5^\epsilon$ (with $\epsilon = \pm 1$) from the description of the attaching map of $\tilde{e}_{10}$, that ends up being equal to
\[x_6x_4^{-1}x_4^{-1}\].

See Figure 3 for a picture of the complete procedure of deformation.

![Figure 3](image-url)

**Figure 3.** Internal collapses in the torus $T$. The CW-complex obtained after performing successively the internal collapses associated to the matched pairs 9 (a), 8 (b), 7 (c), 5 (d), 3 (e) and 2 (f).

**Remark 1.12.** Given a regular CW-complex $K$ of dimension 2 and an acyclic matching $M$ in $\mathcal{X}(K)$, the associated sequence of internal collapses ([x_1]) can be performed in any order in the process of construction of $K_M$. Concretely, if $(v, x) \in M$ is a pair of matched cells of dimension 0 and 1, the associated internal collapse induces the replacement of $x^\epsilon$ by 1 in the combinatorial description of the attaching map of every 2-cell. This operation commutes with the transformation induced by the rest of the internal collapses, since $x$ is not part of any other element of $M$. Let $(x_1, e_1), (x_2, e_2) \in M$ be two pairs of matched

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2By abuse of notation, we denote by $\tilde{e}$ to all the intermediate stages of the cell obtained from a critical cell $e$ after performing a subsequence of internal collapses.
cells of dimension 1 and 2. The attaching maps of $e_i$ in $K$ can be described as $x_i w_i$ where $w_i$ does not contain $x_i$. It is impossible that both $w_1$ contains an occurrence of $x_2$ and $w_2$ an occurrence of $x_1$, since this would contradict the acyclicity of $H_M(X(K))$. In particular, after performing any of the transformations induced by these internal collapses, the modified attaching map $\tilde{e}_i$ will still contain a single occurrence of $x_i$, $i = 1, 2$. So it is possible to perform these two transformations in any order, and they yield the same result. An inductive application of this step completes the proof for an arbitrary number of transpositions.

Example 1.13 (Higher dimensions). For dimension $n > 2$, internal collapses also preserve some of the rigid combinatorial structure of regular CW-complexes. Indeed, they transform regular complexes into higher dimensional combinatorial complexes, characterized by a discrete description of the attaching maps (see [23, Ch.II] for the definitions). A general algorithmic description of the Morse CW-complex in high dimensions is, however, limited by the lack of a simple combinatorial description of the attaching map of $n$-cells for $n > 2$. Notwithstanding, the Morse CW-complex can be still described in particular examples.

For instance, let $K$ be the 3-dimensional regular CW-complex constructed as two solid pyramids $K_1$ and $K_2$ whose boundary is jointly identified as in Figure 4. Each pyramid can be seen as the cone on a regular CW-structure $P$ of the projective plane $\mathbb{R}P^2$.

![Figure 4](image-url)

**Figure 4.** Top: Two subdivided solid pyramids $K_1$ and $K_2$ whose boundaries are jointly identified. Each pyramid is the cone on a regular CW-structure $P$ of the projective plane $\mathbb{R}P^2$. Bottom: The top view of $K_1$ and $K_2$ showing the identification of its ‘top’ cells.
Let $M$ be the acyclic matching in $X(K)$ described at Figure 5. The associated internal collapses, depicted in Figure 6, simplify the 3-skeleton of $K$ to obtain a CW-complex with a single (critical) 3-cell whose attaching map can be concretely described as follows. Let $S$ be the triangulation of $S^2$ induced by the construction of $K$ illustrated at Figure 7. The attaching map of the critical 3-cell in the Morse CW-complex $K_M$ is the quotient of the canonical homeomorphism $\varphi : S^2 \to S$ under the identification in $S$ of antipodal points (via the homeomorphism). This construction shows a simple CW-structure of the 3-dimensional real projective space $\mathbb{R}P^3$. A minimal CW-structure of $\mathbb{R}P^3$ (with a single 0-cell $v_2$, a 1-cell $x_{10}$, a 2-cell $e_5$ and a 3-cell $b_8$) is obtained after further performing the internal collapses associated to the matched cells in solid black at Figure 5.

Figure 5. Acyclic matching $M$ in $X(K)$ in solid red (and a maximal acyclic matching $M_{\text{max}}$ in solid red and black). In dashed lines, the sub-posets $X(K_1)$, $X(K_2)$ and $X(P)$ with $P$ the regular CW-structure of the projective plane $\mathbb{R}P^2$ in the basis of $K_1$ and $K_2$. Here, $e_9, \ldots, e_{12}$ and $e_{13}, \ldots, e_{16}$ denote the ‘internal’ 2-cells in $K_1$ and $K_2$ respectively with vertices $\{v_1, v_4, v_i\}$ for $i = 2, 3$. The 3-cells in $K_1$ and $K_2$ are denoted by $b_1, \ldots, b_4$ and $b_5, \ldots, b_8$ respectively. The 1-cells $x_{11}$ and $x_{12}$ join the vertices $v_1$ and $v_4$ in $K_1$ and $K_2$ respectively. The ‘internal’ 1-cells in the CW-structure $P$ of the projective plane in the basis of $K_1$ and $K_2$ are labelled as $x_7, \ldots, x_{10}$.

2. $Q^{**}$-transformations of group presentations

In this section, we present an algorithm based in discrete Morse theory to obtain finite presentations that are $Q^{**}$-equivalent to a given one without tracking down explicitly the $Q^{**}$-transformations involved. The method has a geometric core based in the connection between group presentations and 2-dimensional CW-complexes [23, Ch.I].
Figure 6. Internal collapses in $K$ determined by the acyclic matching $M$. Here $P$ is the CW-structure of projective plane $\mathbb{R}P^2$ at the basis of the pyramids.

Figure 7. Left: A triangulation $S$ of $S^2$ (the boundary of $D^3$). Right: The Morse CW-complex $K_M$ obtained as the identification of antipodal points in $S \cup \varphi \, D^3$, with $\varphi : \partial D^3 \to S$ the canonical homeomorphism.
In the same way as the collapses and expansions determine simple homotopy classes of CW-complexes, the transformations (1)–(5) define classes of group presentations, the \(Q^{\ast\ast}\)-classes. There is a correspondence between 3-deformation classes of CW-complexes of dimension 2 and \(Q^{\ast\ast}\)-classes of group presentations. Such correspondence is based on a standard way to map a given complex \(K\) to an associated presentation \(\mathcal{P}_K\), and a presentation \(\mathcal{P}\) to a 2-complex \(K_\mathcal{P}\). Concretely, if \(\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle\) is a finite group presentation, its associated \emph{standard complex} \(K_\mathcal{P}\) is a CW-complex with a single vertex and an oriented 1-cell \(e_i^1\) for each generator \(x_i\). The 2-cells \(e_j^2\) of \(K_\mathcal{P}\) correspond to the relators \(r_j\), which determine closed edge paths on the 1-skeleton as the attaching maps. Conversely, if \(K\) is a finite CW-complex of dimension 2 and \(T\) is a spanning tree of \(K^{(1)}\) (the 1-skeleton of \(K\)), then \(K \wedge_3 T\) and the \emph{standard presentation} \(\mathcal{P}_K\) is defined as a presentation of the fundamental group of \(K/T\). Different choices of spanning trees result in \(Q^{\ast\ast}\)-equivalent presentations. It is important to note that \(\mathcal{P}_K\) contains not only information about the fundamental group of \(K\), but also about its 3-deformation class.

Given group presentations \(\mathcal{P}, \mathcal{Q}\), if \(\mathcal{P}\) can be transformed into \(\mathcal{Q}\) by a finite sequence of \(Q^{\ast\ast}\)-transformations, we say that \(\mathcal{P}\) is \(Q^{\ast\ast}\)-equivalent to \(\mathcal{Q}\) and we denote \(\mathcal{P} \sim_{Q^{\ast\ast}} \mathcal{Q}\). It can be shown that \(\mathcal{P} \sim_{Q^{\ast\ast}} \mathcal{Q}\) if and only \(K_\mathcal{P} \wedge_3 K_\mathcal{Q}\). Similarly, if \(K, L\) are CW-complexes of dimension 2, \(K \wedge_3 L\) if and only if \(\mathcal{P}_K \sim_{Q^{\ast\ast}} \mathcal{P}_L\). Moreover, \(\mathcal{P} \sim_{Q^{\ast\ast}} \mathcal{P}_K\) and \(K \wedge_3 K_\mathcal{P}\). Balanced presentations of the trivial group correspond to contractible complexes.

Theorem \([13]\) implies the following result in terms of group presentations when the dimension of the CW-complex is 2.

**Corollary 2.1.** Let \(K\) be a regular CW-complex of dimension 2 and let \(M\) be an acyclic matching in \(K\). Then, \(\mathcal{P}_K \sim_{Q^{\ast\ast}} \mathcal{P}_{K_M}\).

We will show next an algorithm to construct \(\mathcal{P}_{K_M}\) for any matching \(M\) in \(\mathcal{X}(K)\). We start with the case when the matching \(M\) only pairs cells of dimension 0 and 1.

**Lemma 2.2.** Let \(K\) be a regular CW-complex of dimension \(n\) and let \(M\) be a matching in the subposet of \(\mathcal{X}(K)\) of cells of dimension 0 and 1 with only one critical cell of dimension 0. Let \(T\) be the subcomplex of \(K\) of the closed matched cells \(T = \bigcup_{e \in M} \bar{e}\). Then, \(M\) is acyclic if and only if \(T\) is a spanning tree in the 1-skeleton \(K^{(1)}\).

**Proof.** Recall that given an acyclic matching \(M\) in the Hasse diagram of \(\mathcal{X}(K)\), the graph \(\mathcal{H}_M(\mathcal{X}(K))\) is the (acyclic) directed graph obtained by reversing in \(\mathcal{H}(X)\) the orientation of the edges corresponding to pairs in \(M\). Since the induced subgraph of \(\mathcal{H}(\mathcal{X}(K))\) whose vertices are the cells of dimension 0 and 1 is the barycentric subdivision of the 1-skeleton of \(K\), the result follows directly from the following fact: any simple cycle

\[ v_1 < x_1 > v_2 < x_2 > \ldots < v_k < x_k > v_1 \]

in \(\mathcal{H}_M(\mathcal{X}(K))\), with \(v_i\) of dimension 0 and \(x_i\) of dimension 1, is in correspondence with a simple cycle

\[ v_1, v_2, \ldots, v_k, v_1 \]

in the 1-dimensional subcomplex of \(K\) of matched pairs of cells of dimensions 0 and 1. \(\square\)

As a consequence of the previous lemma and Theorem \([13]\) we obtain an alternative proof of the fact that an \(n\)-dimensional CW-complex \(K\) \((n + 1)\)-deforms to \(K/T\), where \(T\) is a spanning tree of \(K^{(1)}\). Indeed, it reduces to noting that \(K/T\) is homeomorphic to \(K_M\), where \(M\) is the acyclic matching induced by \(T\).
We now provide a simple description of the group presentation associated to $K_M$ for any acyclic matching $M$, which will be easily tractable through computer assistance. We need first some definitions.

Definition 2.3 (Rewriting rule). Let $P$ be a group presentation and let $r$ be a relator of $P$ given by the word $w_1x^ew_2$, where $w_1$ and $w_2$ are words on the generators, the generator $x$ appears neither in $w_1$ nor in $w_2$ and $e = +1$ or $-1$. The equivalent expression of $x$ induced by $r$ is defined as $(w_1^{-1} w_2^{-1})^e$.

Remark 2.4. If $P$ is a group presentation and $x$ is a generator of $P$ such that it appears only once in a relator $r$, then $P$ is $Q^\ast\ast$-equivalent to the presentation obtained after eliminating the generator $x$ and the relator $r$ and applying the rewriting rule to every occurrence of $x$ in the rest of the relators. Indeed, if $r'$ is another relator containing $x$, by cyclically permuting $r'$ if necessary (this is operation (3)), we can assume $r'$ reads as $x'x$, with $e = 1$ or $-1$. We replace $r'$ by the product $sr'$ where $s$ is a suitable cyclic permutation of $r$ (or its inverse) to eliminate this occurrence of $x$ (here we apply operations (1), (2) and (3)). We iterate this procedure until no occurrence of $x$ is left.

Example 2.5. If $P = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1}, xy^{-2} \rangle$, then the equivalent expression of $x$ induced by the relator $r = xy^{-2}$ is $y^2$. By Remark 2.4 $P$ is $Q^\ast\ast$-equivalent to the presentation $P = \langle y \mid y^2y^2y^{-1}y^{-2}y^{-1} \rangle$, i.e. $\langle y \mid y \rangle$.

Definition 2.6. Let $K$ be a regular CW-complex of dimension 2. Let $M$ be an acyclic matching in $X(K)$ such that there is only one critical cell of dimension 0. Denote by $M_0$ the subset of matched pairs of cells of dimension 0 and 1 and by $M_1 = \{(x_1, e_1), \ldots, (x_m, e_m)\}$ the subset of matched pairs of cells of dimension 1 and 2. The Morse presentation $Q_{K,M}$ is the presentation $Q_m$ defined by the following iterative procedure:

- $Q_0$ is the standard presentation $P_K$ constructed using the spanning tree $T$ induced by $M_0$ (see Lemma 2.2). The generators of $Q_0$ are the unmatched 1-cells of $K$ with respect to the matching $M_0$, and its relators are the words induced by the attaching maps of the 2-cells of $K$. We associate to each relator $r$ in $Q_0$ the 2-cell that produced it.
- For $0 \leq i < m$, let $Q_{i+1}$ be the presentation obtained from $Q_i$ after removing the relator associated to $e_i$ and the generator $x_i$, and applying the rewriting rule on every occurrence of the generator $x_i$ in the rest of the relators.

We will see next that the Morse presentation $Q_{K,M}$ is an algorithmic description of $P_{K_M}$. Concretely, there is a correspondence between the internal collapses in $K$ induced by $M$ and the combinatorial transformations in $P_K$ described in Definition 2.6. The acyclicity of $M$ implies that the construction of $Q_{K,M}$ does not depend on the choice of the order in $M_1$; that is, the internal collapses determined by $M_1$ can be performed in any order (see Remark 1.12).

Proposition 2.7. Let $K$ be a regular 2-dimensional CW-complex, and let $M$ be an acyclic matching in $X(K)$ with only one critical cell of dimension 0. Then, $Q_{K,M} = P_{K_M}$ for a suitable choice of orientations and basepoints in $K_M$.

Proof. Given an acyclic matching $M$ in $X(K)$, by Lemma 1.7 and Remarks 1.8 and 1.12 there is a sequence of internal collapses (1) of decreasing dimension such that the collapses at each dimension can be performed in any order. If $M_0$ and $M_1 = \{(x_1, e_1), \ldots, (x_m, e_m)\}$ are as in Definition 2.6, then they describe an (ordered) sequence of internal collapses to
transform $K$ into $K_M$. For each $1 \leq i < m$, the combinatorial transformation performed to get $Q_{i+1}$ from $Q_i$ parallels exactly the geometric description of the elementary internal collapse indicated by the pair $(x_i, e_i)$ (see Remark 1.10). Denote by $K_M$, the CW-complex obtained from $K$ after performing the internal collapses induced by the pairs in $M_1$. Now, let $T$ be the spanning tree induced by $M_0$. Since $K_M = K_{M_1}/T$, it follows that $Q_m = Q_{K,M}$ is the standard presentation associated to $K_M$ for the right choice of orientations and basepoints.

By Corollary 2.1, Proposition 2.7 implies that $Q_{K,M}$ is another representative of the $Q^{**}$-class of $P_K$.

**Theorem 2.8.** Let $K$ be a regular 2-dimensional CW-complex, and let $M$ be an acyclic matching in $\mathcal{X}(K)$ with only one critical cell of dimension 0. Then, $Q_{K,M} \sim_{Q^{**}} P_K$.

We deduce next a combinatorial technique to study the $Q^{**}$-class of a given presentation $P$, by means of acyclic matchings.

**Theorem 2.9.** Let $P$ be a finite presentation of a finitely presented group and $K'_p$ the barycentric subdivision of $K_p$. Let $M$ be an acyclic matching in $\mathcal{X}(K'_p)$ with only one critical cell of dimension 0. Then $P \sim_{Q^{**}} Q_{K'_p,M}$.

**Proof.** The result follows from the sequence of $Q^{**}$-equivalences

$$P \sim_{Q^{**}} P_{K_p} \sim_{Q^{**}} P_{K'_p} \sim_{Q^{**}} P_{(K'_p)_M} = Q_{K'_p,M},$$

where the equivalence $P \sim_{Q^{**}} P_{K_p}$ is well known (see for instance [12]), $P_{K_p} \sim_{Q^{**}} P_{K'_p}$ holds since every 2-complex 3-deforms to its barycentric subdivision, and $P_{K'_p} \sim_{Q^{**}} P_{(K'_p)_M} = Q_{K'_p,M}$ is consequence of Theorem 2.8 applied to the regular 2-dimensional CW-complex $K'_p$.

**Example 2.10.** Let $P = \langle x, y \mid x^2, xy^{-2} \rangle$. The poset $\mathcal{X}(K'_p)$ associated to $P$ is depicted in Figure 8. This poset admits an explicit algorithmic description from $P$. If $o$ is the minimal point that represents the only vertex of $K_p$, for every generator $g$ of $P$ there is a subposet of $\mathcal{X}(K'_p)$, a model of $S^1$, consisting of four points $o, g, g^{-1}$, and $g$. We refer to these points as generator elements. For every relator $r_i$ of $P$, there is a subposet of $\mathcal{X}(K'_p)$, a model of $D^2$, given by a cone with apex $v_r$ over a model of $S^1$ that represents the subdivision of the 2-cell associated to $r_i$ in $K_p$. We refer to the points of this subposet as relator elements. Finally, there are cover relations between the generator elements and the relator elements. For each $g'$ in $r_i$ there is an edge from the generator elements $g, g^{-1}$ to the homonym relator elements $g, g^{-1}, g^{-1}$. Note that $\mathcal{X}(K'_p)$ has 29 points and, in general, $|\mathcal{X}(K'_p)| = 4(l + m) + 1$ where $l$ is the total relator length and $m$ the number of relators of $P$.

Consider an acyclic matching $M$ in $\mathcal{X}(K'_p)$ with a single critical cell of dimension 0, Figure 9.

First, we compute the presentation $Q_0$ associated to $K'_p/T$, $T$ being the spanning tree in $K'_p^{(1)}$ induced by $M$, see [13]. Notice that, although $Q_0$ is $Q^{**}$-equivalent to $P$, it has a larger number of generators and relators.

\[
Q_0 = \langle x_6, \ldots, x_{15} \mid x_7, x_6^{-1} x_7^{-1} x_8, x_8^{-1} x_9, x_9^{-1} x_10, x_{10}^{-1} x_{11}, x_{11}^{-1} x_{12}, x_{13}, x_{12} x_{13}^{-1} x_{14}, x_{14}^{-1} x_{15}, x_{12} x_{15}^{-1} \rangle
\]

3Here, the subscript of each critical 1-cell corresponds to the label in Figure 9.
Figure 8. Top: The barycenter subdivision of $K_P$. Bottom: The face poset $\mathcal{X}(K'_P)$.

Figure 9. Top: The barycenter subdivision of $K_P$ with arrows and labels associated to an acyclic matching $M$. Bottom: The face poset $\mathcal{X}(K'_P)$ with the matching $M$ in red. The labels correspond to an induced discrete Morse function $f_M$. Critical cells are empty bullets.
We perform below the iterative procedure of reduction of $Q_0$ described in Definition 2.6. This is a sequence of $Q^{**}$-transformations induced by $M$ to obtain $Q_{K,M}$, a new presentation derived from $Q_0$ with fewer generators and relators that is also $Q^{**}$-equivalent to $P$ (see Proposition 2.7). At each step, the generator and relator involved in the transformation of the next step is in bold.

- $Q_1 = \langle x_6, \ldots, x_{14} \mid x_7, x_6^{-1} x_7^{-1} x_8, x_8^{-1} x_9, x_6^{-1} x_9^{-1} x_{10}, x_{10} x_{11}, x_{11} x_{12}, x_9, x_{12} x_{13}, x_{13} x_{14} x_{15}, x_{12} x_{14} \rangle$
- $Q_2 = \langle x_6, \ldots, x_{13} \mid x_7, x_6^{-1} x_7^{-1} x_8, x_8^{-1} x_9, x_6^{-1} x_9^{-1} x_{10}, x_{10} x_{11}, x_{11} x_{12}, x_9, x_{12} x_{13}, x_{13} x_{14}, x_{12} x_{14} \rangle$
- $Q_3 = \langle x_6, \ldots, x_{12} \mid x_7, x_6^{-1} x_7^{-1} x_8, x_8^{-1} x_9, x_6^{-1} x_9^{-1} x_{10}, x_{10} x_{11}, x_{11} x_{12}, x_9, x_{12} x_{13}, x_{13} x_{14}, x_{12} x_{14} \rangle$
- $Q_4 = \langle x_6, \ldots, x_{11} \mid x_7, x_6^{-1} x_7^{-1} x_8, x_8^{-1} x_9, x_6^{-1} x_9^{-1} x_{10}, x_{10} x_{11}, x_{11} x_{12} \rangle$
- $Q_5 = \langle x_6, \ldots, x_{10} \mid x_7, x_6^{-1} x_7^{-1} x_8, x_8^{-1} x_9, x_6^{-1} x_9^{-1} x_{10}, x_{10} x_{11}, x_{11} x_{12} \rangle$
- $Q_6 = \langle x_6, \ldots, x_9 \mid x_7, x_6^{-1} x_7^{-1} x_8, x_8^{-1} x_9, x_6^{-1} x_9^{-1} x_{10}, x_{10} x_{11}, x_{11} x_{12} \rangle$
- $Q_7 = \langle x_6, x_7, x_8 \mid x_7, x_6^{-1} x_7^{-1} x_8, (x_8 x_9)^{-1} \rangle$
- $Q_8 = \langle x_6, x_7 \mid x_7, (x_7 x_8)^{-1} \rangle$
- $Q_9 = \langle x_6 \mid x_6^4 \rangle$

Hence, $P \sim_{Q^{**}} \langle x_6 \mid x_6^4 \rangle$.

Remark 2.11. Given a presentation $P$ and a matching $M$ in $K_P'$ with only one critical 0-cell, we showed that $P \sim_{Q^{**}} Q_{K'_P,M}$. We now estimate a sufficient number of $Q^{**}$-transformations to obtain $Q_{K'_P,M}$ from $P$. Let $n$ be the number of generators of $P$, $m$ the number of relators and $k$ the total relator length. We base our reasoning in the proof of Theorem 2.9. The equivalences $P \sim_{Q^{**}} P_{K_P}$ and $P_{K_P} \sim_{Q^{**}} P_{K'_P}$ can be achieved in $O(n + m)$ and $O(k)$ $Q^{**}$-transformations respectively. By Corollary 2.1, $K'_P / \sim^{3} (K'_P)_M$ and the estimated number of $Q^{**}$-transformations required to obtain $P_{(K'_P)_M}$ from $P_{K'_P}$ is proportional to the number of elementary expansions and collapses needed to deform $K'_P$ into $(K'_P)_M$. This is bounded by the square of the number of cells of $K'_P$, which is proportional to $k$.

3. Applications to the Andrews–Curtis Conjecture

In this section, we apply the techniques developed in Section 2 to prove that some of the potential counterexamples to the Andrews–Curtis conjecture do satisfy the conjecture.

There is a common belief that the Andrews–Curtis conjecture is false. However, not a single counterexample could be found yet. Over the last fifty years, a list of examples of balanced presentations of the trivial group which are not known to be trivializable via $Q^{**}$-transformations has been compiled. They serve as potential counterexamples to disprove the Andrews–Curtis conjecture (see [23] Ch. XII for a detailed reference).

For a potential counterexample $P$, the general outline we will adopt to prove that $P \sim_{Q^{**}} \langle \mid \rangle$ is to find an appropriated acyclic matching $M$ in $\mathcal{X}(K_P)$ such that the presentation $Q_{K'_P,M}$, which is in the same $Q^{**}$-class as $P$, is computationally tractable and easily transformed into the trivial presentation $\langle \mid \rangle$. Note that, given $P$ and the acyclic matching $M$, the complexity of the computation of the presentation $Q_{K'_P,M}$, the Morse presentation associated to $P$, is $O(k)$ with $k$ the total length relator of $P$. We emphasize that our method of $Q^{**}$-transformation makes use of the transformation (4), the only transformation in (1)–(5) that increases the total length relator of the presentation. This point is central for the posterior manageable reduction of the $Q^{**}$-equivalent Morse presentation.
We say that a presentation \( P \) is greedy trivializable if the reduction algorithm for simplification of presentations described in [21] can transform \( P \) into \( \langle \ | \rangle \). This procedure was originally developed for Tietze simplification of presentations and it consists of a loop of two phases which is iterated until no further operation is possible. The search phase attempts to reduce the length-relator by replacing long substrings of relators by shorter equivalent ones. That is, if there are relators \( r_1 \) and \( r_2 \) such that a suitable cyclic permutation of \( r_1 \) reads \( uv \) and a cyclic permutation of \( r_2 \), or its inverse reads as \( wv \), and the length of \( u \) is greater than the length of \( w \), then \( r_2 \) is replaced by \( uw^{-1} \). The elimination phase involves the elimination of generators \( x \) which occur only once in some relator \( r \) (in which case the reduction consist of the elimination of the generator \( x \) and the relator \( r \)).

**Remark 3.1.** There is only one situation in which the reduction algorithm [21] performs the transformation (6). Namely, suppose that the presentation \( P \) has a relator \( r_i \) which is equal to another relator \( r_j \). In that case, the algorithm replaces relator \( r_i \) by a 1 and then eliminates the latter 1. This transformation changes the deficiency of the presentation and does not preserve the (simple) homotopy type of the associated CW-complex \( K_P \).

Note that the previous situation is not possible if \( P \) is a balanced presentation of the trivial group, since if \( P \) has (after possibly a sequence of \( Q^{**} \)-transformations) one relator equal to another, then it is in the same \( Q^{**} \)-class as a presentation with a relator equal to 1. Thus, \( K_P \) has non-trivial second homology group, which is not possible. Therefore, the reduction algorithm performs only \( Q^{**} \)-transformations if the input is a balanced presentation of the trivial group.

### 3.1. Experimental results

We next present an application of our method to computationally prove that some potential counterexamples of the Andrews–Curtis conjecture can be \( Q^{**} \)-trivialized. Concretely, given a balanced presentation \( P \), we computationally generate an acyclic matching \( M \) in \( X(K'_P) \) such that the presentation \( Q_{K'_P,M} \) is greedily trivializable. All the computations are made using SAGE [36]. The code to replicate the following examples can be found at [13], whereas the outline of the routine is described in Appendix A. We remark that in all of these examples of balanced presentations \( P \) of the trivial group, the reduction algorithm applied to the presentation \( Q_{K'_P,T} \), with \( T \) the spanning tree induced by the matching \( M \), is not able to trivialize it (cf. Lemma 2.2).

**Example 3.2** (Akbulut & Kirby [1]). The family of balanced presentations of the trivial group \( \mathcal{AK}_n \) was inspired by the case \( n = 4 \), which corresponds to a handle decomposition of the Akbulut-Kirby 4-sphere:

\[ \mathcal{AK}_n = \langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle, \quad n \geq 1. \]

We prove that the presentation \( \mathcal{AK}_2 = \langle x, y \mid xyx = yxy, x^2 = y^3 \rangle \) satisfies the Andrews–Curtis conjecture. Indeed, it is simple to find an acyclic matching \( M \) in \( X(K'_\mathcal{AK}_2) \) such that \( Q_{\mathcal{AK}_2,M} \) is greedily trivializable (see Appendix A). Since \( \mathcal{AK}_2 \sim Q^{**} Q_{\mathcal{AK}_2,M} \), this shows that \( \mathcal{AK}_2 \) satisfies the Andrews–Curtis conjecture. This fact was previously proved in [31] using genetic algorithms (also by S. M. Gersten in an unpublished work [19]). However, their methods were focused on the explicit sequence of transformations to trivialize the presentation, rather than to give a proof of its existence. For \( n = 1 \), \( \mathcal{AK}_1 \) is simply \( Q^{**} \)-trivializable (see Example 2.5). For \( n > 2 \), the question remains open.
Example 3.3 (Miller III & Schupp [33]). Given $w$ a word in $x$ and $y$ with exponent sum 0 on $x$ and $n > 0$,
\[ \mathcal{MS}_n(w) = \langle x, y \mid x^{-1}y^nx = y^{n+1}, x = w \rangle \]
is a balanced presentation of the trivial group. A well-studied subfamily of the presentations $\mathcal{MS}_n(w)$ is that given by $w_3 = y^{-1}xyx^{-1}$ (see [22, 31]). In these works, the authors proved that $\mathcal{MS}_n(w_3)$ is $Q^{**}$-trivializable for $n \leq 2$. The case $n = 3$ was the smallest potential counterexample of this family. With our algorithmic method, we prove that $\mathcal{MS}_3(w_3) = \langle x, y \mid x^{-1}y^3x = y^4, x = y^{-1}xyx^{-1} \rangle$
does satisfy the Andrews–Curtis conjecture. Precisely, we computationally generate an acyclic matching $M$ such that the Morse presentation $Q_{\mathcal{K}_{\mathcal{MS}_3(w_3)}^Q}^M$ is greedily trivializable (see Appendix A).

Example 3.4 (Barmak [4]). The generalized Andrews–Curtis conjecture states that any two presentations $P$ and $Q$ with simple homotopy equivalent standard complexes $K_P, K_Q$ are $Q^{**}$-equivalent [23, Ch.I]. A strong version of the generalized conjecture asserts that under the previous conditions, $P$ can be obtained from $Q$ by performing only operations (1) to (3) (called $Q^*$-transformations). Recently, Barmak found a counterexample to this strong formulation of the conjecture [4]. He proved that $B_1 = \langle x, y \mid [x, y], 1 \rangle$
is not $Q^*$-equivalent to $B_2 = \langle x, y \mid [x, [x, y^{-1}]]^2y[y^{-1}, x]y^{-1}, [x, [[y^{-1}, x], x]] \rangle$,
but $K_{B_1} \cap K_{B_2}$ (here $[x, y] = xyx^{-1}y^{-1}$). He asked whether these presentations are $Q^{**}$-equivalent or not, being $B_1$ and $B_2$ a potential counterexample to the generalized Andrews–Curtis conjecture.

We produce an acyclic matching $M$ in $\mathcal{A}(K_{B_2}')$ such that the associated Morse presentation is greedily transformed into $B_1$ via the reduction algorithm (see Appendix A). By Theorem 2.9 and Remark 3.1, we conclude that $B_1 \sim_{Q^*} B_2$ and hence, that this counterexample to the strong version of the conjecture does not disprove the generalized Andrews–Curtis conjecture.

3.2. Gordon’s potential counterexamples. In [9, Sect. 3.8–3.10], the author brings to the attention the following family of balanced presentations of the trivial group proposed by Gordon
\[ \mathcal{G}_{n,m,p,q} = \langle x, y \mid x = [x^m, y^n], y = [y^p, x^q] \rangle \quad n, m, p, q \in \mathbb{Z}, \]
where $[x, y] = xyx^{-1}y^{-1}$. Many instances of this family of presentations are potential counterexamples of the Andrews–Curtis conjecture [30]. For instance, the presentation $\mathcal{MS}_3(w_3)$ of Example 3.3 (see also [33]) is a representative of this family for $n = m = q = -1, p = -3$. It is not known in general whether a presentation $\mathcal{G}_{n,m,p,q}$ is $Q^{**}$-trivializable. In [5], the authors proved that any presentation in this sequence with total length-relator up to 14 can be transformed into $\langle x, y \mid \rangle$ by exploring the space of possible sequences of transformations (1)–(3). We focus our attention on the subfamily
\[ \mathcal{G}_q := \mathcal{G}_{-1,-1,-1,-q} = \langle x, y \mid x = [x^{-1}, y^{-1}], y = [y^{-1}, x^{-q}] \rangle, \]
\[ = \langle x, y \mid x = x^{-1}y^{-1}xy, y = y^{-1}x^{-q}yx^q \rangle. \]
We show that \( \mathcal{G}_q \) is \( Q^{**} \)-trivializable for all \( q \in \mathbb{N} \), by an inductive procedure based on our theory from Section 2.

**Theorem 3.5.** The presentation \( \mathcal{G}_q = \langle x, y \mid x = x^{-1}y^{-1}xy, \ y = y^{-1}x^{-q}yx^q \rangle \) satisfies the Andrews–Curtis conjecture for all \( q \in \mathbb{N} \).

**Proof.** If \( q \) is even, \( \mathcal{G}_q \) results greedily trivializable. Indeed, given \( q = 2k \) with \( k \in \mathbb{N} \), the reduction algorithm replaces \( k \) times the string \( x^{-1}y^{2k} \) by its equivalent expression \( y \) in the second relator \( y^{-2}x^{-2k}yx^{2k} \), which results in \( y^{-2}x^{-k}y \) or, equivalently, \( y^{-1}x^{-k} \). Now, the first relator \( x^{-2}y^{-1}xy \) is transformed into \( x \) after replacing the string \( y \) by \( x^{-k} \). The resulting presentation \( \langle x, y \mid x, y^{-1}x^{-k} \rangle \) is clearly \( Q^{**} \)-trivializable. Hence, \( \mathcal{G}_{2k} \sim Q^{**} \langle \ | \ \rangle \).

For \( q \) odd, that is \( q = 2k - 1 \), a similar procedure transforms \( \mathcal{G}_{2k-1} \) into
\[
\tilde{\mathcal{G}}_k = \langle x, y \mid x^{-2}y^{-1}xy, y^{-1}x^{-k+1}yx^{-1}y^{-1} \rangle, \ k \in \mathbb{N}.
\]
For \( k = 1 \), it is easy to check that \( \tilde{\mathcal{G}}_1 \) is trivializable. For \( k = 2 \),
\[
\tilde{\mathcal{G}}_2 = \langle x, y \mid x^{-2}y^{-1}xy, y^{-1}x^{-3}yx^{-1}y^{-1} \rangle.
\]
If we replace the first relator \( r_1 \) by \( r_1 r_2 \), we obtain \( \tilde{r}_1 = x^{-3}y^{-1} \). Now, \( y \) occurs only once at \( \tilde{r}_1 \). After replacing \( y \) by its equivalent expression \( x^{-3} \) in \( r_2 \) and eliminating the generator \( y \) and relator \( \tilde{r}_1 \), the resulting \( Q^{**} \)-equivalent presentation
\[
\langle x \mid x \rangle
\]
is trivializable.

For \( q > 3 \) odd (that is \( q = 2k - 1 \) with \( k > 2 \)), the reduction algorithm is not able to trivialize \( \mathcal{G}_q \) (neither \( \tilde{\mathcal{G}}_k \)). We prove by induction on \( k \) that \( \mathcal{G}_k \sim Q^{**} \langle \ | \ \rangle \). For \( k \geq 3 \), consider the acyclic matching \( M_k \) in the poset \( X_k = \mathcal{X}(K_k') \) of Figure 10.

The Morse presentation \( \mathcal{Q}_{K_k', M_k} \) is \( (Q^{**} \)-equivalent to
\[
\langle a_0, a_1, a_2 \mid a_0a_1^{-1}a_0, a_2a_1^{-1}a_2^{-1}a_2a_1^{-1}a_2a_1^{-1}a_2a_1^{-1} \rangle
\]
(see Appendix 1) and it can be inductively trivialized. In fact, there exists a sequence of \( Q^{**} \)-equivalent presentations
\[
\mathcal{Q}_{K_k', M_k} \sim \mathcal{Q}^{**} \langle a_0, a_2 \mid a_2a_0^{-2}a_2a_0^{-1}, a_2a_0^{-2}a_2a_0^{-1} \rangle
\]
\[
\sim Q^{**} \langle a_0, a_2 \mid a_2a_0^{-2}a_2a_0^{-1}, a_2a_0^{-2}a_2a_0^{-1} \rangle
\]
\[
\sim Q^{**} \langle a_0, a_2 \mid a_2a_0^{-2}a_2a_0^{-1}, a_2a_0^{-2}a_2a_0^{-1} \rangle
\]
\[
\sim Q^{**} \langle a_0, a_2 \mid a_2a_0^{-2}a_2a_0^{-1}, a_2a_0^{-2}a_2a_0^{-1} \rangle
\]
\[
\sim Q^{**} \tilde{\mathcal{G}}_{k-1}.
\]
For the first \( \mathcal{Q}^{**} \)-equivalence, notice that the generator \( a_1 \) appears only once in the first relator of \( \mathcal{Q}_{K_k', M_k} \). Thus, after replacing \( a_1 \) by its equivalent expression \( a_0^2 \) in the rest of the relators, the generator \( a_1 \) and the relator \( a_0a_1^{-1}a_0 \) can be removed. Now, for the second equivalence, the second relator can be rewritten as \( a_2^{-1}a_0(a_2^{-1}a_0^{-k+2}a_2)(a_2^{-1}a_0^{-k}a_2a_0^{-1}) \). It can be deduced from the first relator that the string \( a_2^{-1}a_0^{-k}a_2a_0^{-1} \) is equivalent to \( a_0^{-2}a_2^{-1} \). After replacing \( a_2^{-1}a_0^{-k}a_2a_0^{-1} \) by its equivalent expression in the second relator, the latter is transformed into \( a_2^{-1}a_0(a_2^{-1}a_0^{-k+2}a_2)(a_0^{-2}a_2^{-1}a_2a_0^{-1}) \). By replacing this time \( a_0^{-2}a_2^{-1}a_2a_0^{-1} \) by \( a_2^{-1}a_0^{-1}a_2a_0^{-1} \), it is transformed into \( a_2^{-1}a_0^{-k+2}a_2a_0^{-1}a_2a_0^{-1} \) and, hence, the resulting presentation is \( Q^{**} \)-equivalent to \( \tilde{\mathcal{G}}_{k-1} \). By inductive hypothesis, \( \tilde{\mathcal{G}}_{k-1} \) is \( Q^{**} \)-trivializable and therefore, so is \( \mathcal{Q}_{K_k', M_k} \). \( \square \)
Figure 10. Acyclic matching in $X_k$ (edges in red). Critical points are the empty bullets. Edges between generator elements and relator elements are not depicted unless they belong to the matching. The figure illustrates the subposets of $X_k$ corresponding to the generator elements and the first relator elements (top) and the second relator elements (bottom).

Appendix A. Computational procedure for examples in Section 3.1

The algorithm to compute a presentation $Q^{**}$-equivalent to a given one is implemented in SAGE [36]. The code can be found at the repository [13].

We describe next the main functions. Given as input the lists gens and rels of generators and relators, the function `group_presentation(gens, rels)` computes the induced group presentation $P = \langle\text{gens} \mid \text{rels}\rangle$. The method `simplified()` applies the reduction algorithm described in Section 3 [21]. It can be verified that the method `simplified()` is not able to reduce the number of generators and relators of the original presentation in any of the Examples of Section 3.1.

The function `presentation_poset(gens, rels)` returns $X(K'_P)$, that is, the face poset of the barycentric subdivision of the standard complex induced by $P$.

Given a poset $X$, we randomly generate an acyclic matching with a single critical point at level 0 with the function `spanning_matching(X)`.

Now, given an acyclic matching $M$ in the poset $X(K'_P)$, the function `Morse_presentation(gens, rels, M)` computes the associated Morse presentation $Q_{K'_P,M}$ (see Definition 2.6). By Theorem 2.9, the latter is $Q^{**}$-equivalent to $P$.

---

4The package Posets [14] in GAP [18] also contains an implementation of the functions described in this section.
In Examples 3.2 and 3.3 we find in short runtime an appropriate matching $M$ such that the method `simplified()` applied to `Morse_presentation(gens, rels, M)` produce as output the presentation $\langle \mid \rangle$. In Example 3.4, given $\text{gens}$ and $\text{rels}$ the generators and relators of $B'$, we generate a matching $M$ such that `Morse_presentation(gens, rels, M).simplified()` is the presentation $\langle x, y \mid [x, y] \rangle$. By Remark 3.1, this implies that the original presentation $B' = \langle \text{gens} \mid \text{rels} \rangle$ is $Q^{**}$-equivalent to $B = \langle x, y \mid [x, y], 1 \rangle$.\footnote{The list of transformations involved in the simplification of the Morse presentation $Q_{K'_{G_{2k-1}} M_k}$ can be verified using the function `SimplifyPresentation` in GAP \cite{18} (equivalent to the method `simplified()` in SAGE \cite{36}) and it is included in the repository \cite{13} for completeness.}

The following script summarizes the procedure:

```python
P = group_presentation(gens, rels)
X = presentation_poset(gens, rels)
l = len(P.simplified().generators())
while l>1:
    M = spanning_matching(X)
    Q = Morse_presentation(gens, rels, M)
    S = Q.simplified()
    l = len(S.generators())
print(S)
```

**Appendix B. The Morse presentation $Q_{K'_{G_{2k-1}} M_k}$**

In this appendix we give a proof of the construction of the Morse presentation $Q_{K'_{G_{2k-1}} M_k}$ for all $k \in \mathbb{N}$. Recall that $G_{2k-1} = \tilde{G}_k = \langle x, y \mid x^{-2}y^{-1}xy, y^{-1}x^{-k+1}yx^{-1}y^{-1} \rangle$, $k \in \mathbb{N}$. The poset $X_k$ is the face poset of the regular CW-complex $K'_{\tilde{G}_k}$ associated to $\tilde{G}_k$ (see Figure 11).

![Figure 11. The regular CW-complex $K'_{\tilde{G}_k}$. The 1-cells $x^{1+}$ and $x^{-1}$ in braces are repeated $k-1$ times and correspond to the substring $x^{-k+1}$ of the second relator of $\tilde{G}_k$.](image-url)
We label the 1-cells \( x_1, x_2, \ldots, x_{20+2k} \) as in Figure 12.

![Figure 12. Labeling of the 1-cells of \( K'_m \). The matching \( M_k \) in \( X(K'_m) \) is represented with red arrows.](image)

The presentation \( Q_0 \) has generators that coincide with the set of 1-cells \( x_1, x_2, \ldots, x_{20+2k} \) and relators \( r_1, r_2, \ldots, r_{20+2k} \) that correspond either to a non-critical 1-cell (that is, relators \( x_8, x_{11}, x_{14}, x_{16+2k} \)) or to the attaching map of a 2-cell. Concretely,

\[
Q_0 = \langle x_1, x_2, \ldots, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15+2k}, \ldots, x_{18+2k}, x_{19+2k}, x_{20+2k} | x_8, x_{11}, x_{14}, x_{16+2k},
\]

\[
x_5^2 x_6^{-1}, x_6 x_1^{-1} x_5^{-1}, x_7 x_2^{-1} x_5^{-1}, x_8 x_1^{-1} x_9^{-1}, x_9 x_4^{-1} x_1^{-1} x_{10}^{-1} x_3^{-1} x_1^{-1}, x_{11} x_1 x_2^{-1}, x_{12} x_2 x_3^{-1}, x_{13} x_3 x_{14}^{-1}, x_{14} x_4 x_5^{-1} x_{15} x_6 x_3^{-1} x_{17} x_2^{-1} x_{18} x_4 x_1^{-1} x_{19}^{-1} \ldots, x_{13+2k} x_2^{-1} x_{14+2k} x_1^{-1} x_{15+2k} x_{16+2k}, x_{15+2k} x_4 x_{16+2k}, x_{16+2k} x_4 x_{17+2k} x_{17+2k} x_2^{-1} x_{18+2k} x_{18+2k} x_3^{-1} x_{19+2k} x_4 x_{20+2k} x_{20+2k} x_3^{-1} x_{15}^{-1} \rangle
\]

For each relator, we display in bold either the generator associated to the 1-cell to which the underlying 2-cell is matched or the generator that is not critical.

The critical 1-cells are the ones linked to the generators \( x_2, x_{15+2k}, x_{18+2k} \), whereas critical 2-cells are in correspondence with the relators \( x_{11} x_1 x_2^{-1}, x_{14} x_4 x_5^{-1}, x_{19+2k} x_4^{-1} x_{20+2k} \) (in blue). After performing the reductions associated to the non-critical 1-cells (or equivalently, the collapse to a point of the associated spanning tree on the 1-skeleton), we obtain

\[
Q_0 \sim Q'' = \langle x_1, x_2, \ldots, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15+2k}, x_{17+2k}, x_{18+2k}, x_{19+2k}, x_{20+2k} | x_8, x_{11}, x_{14}, x_{16+2k}, x_{17} x_2^{-1} x_{18}, x_{18} x_1, x_{19}, \ldots, x_{13+2k} x_2^{-1} x_{14+2k} x_1^{-1} x_{15+2k},
\]

\[
x_5 x_6^{-1}, x_6 x_1^{-1} x_5^{-1}, x_7 x_2^{-1} x_5^{-1}, x_8 x_1^{-1} x_9^{-1}, x_9 x_4^{-1} x_1^{-1} x_{10}^{-1} x_3^{-1} x_1^{-1}, x_{11} x_1 x_2^{-1}, x_{12} x_2 x_3^{-1}, x_{13} x_3 x_{14}^{-1}, x_{14} x_4 x_5^{-1} x_{15} x_6 x_3^{-1} x_{17} x_2^{-1} x_{18} x_4 x_1^{-1} x_{19}^{-1} \ldots, x_{13+2k} x_2^{-1} x_{14+2k} x_1^{-1} x_{15+2k} x_{16+2k}, x_{15+2k} x_4 x_{16+2k}, x_{16+2k} x_4 x_{17+2k} x_{17+2k} x_2^{-1} x_{18+2k} x_{18+2k} x_3^{-1} x_{19+2k} x_4 x_{20+2k} x_{20+2k} x_3^{-1} x_{15}^{-1} \rangle
\]

Now, we perform iteratively the reductions associated to the internal collapses involving pairs of 1-cells and 2-cells. At each step, we replace a generator by its equivalent expression induced by the associated matched pair of cells, and then we remove the corresponding generator and relator. The generator and relator involved at each stage of the procedure are marked in bold.
\[ G_k \sim \langle x_1, x_2, \ldots, x_{7}, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15+2k}, x_{17+2k}, x_{18+2k}, x_{19+2k}, x_{20+2k} | x_{5}x_2^{-1} x^{-1}, x_{6}x_1^{-1} x^{-1}, x_{7}x_2^{-1} x^{-1}, x_1 x_9^{-1} x_{10}^{-1} x_{10} x_3^{-1} x_{11} x_{12} x_9^{-1} x_{13} x_3^{-1} x_4 x_5^{-1} x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{14+2k}, x_{15+2k}, x_{16+2k} | x_{5}x_2^{-1} x^{-1}, x_{6}x_1^{-1} x^{-1}, x_{7}x_2^{-1} x^{-1}, x_1 x_9^{-1} x_{10}^{-1} x_{10} x_3^{-1} x_{11} x_{12} x_9^{-1} x_{13} x_3^{-1} x_4 x_5^{-1} x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{17} x_5^{-1} x_{18+2k} x_1^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]

\[ G_{k} \sim \langle x_1, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{15}, \ldots, x_{15} x_2^{-1} x^{-1}, x_{16} x_3^{-1} x_{17} x_2^{-1} x^{-1} x_{18} x_4^{-1} x_{19}^{-1} x_{13+2k} x_2^{-1} x_{14+2k} x_{15+2k} x_3^{-1} x_{17+2k} x_2^{-1} x_{18+2k} x_1^{-1} x_{19+2k} x_3^{-1} x_{15} \rangle \]
\[ Q \sim \left( x_1, x_2, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}, X_{15}, X_{15+2k}, X_{18+2k} \right) \]
\[ X_{18+2k} X_2 x_5^{−1}, X_{15} X_2 x_1 x_9^{−1} (x_2 x_1^{−1} k^{−1}) X_{15+2k}, X_{18+2k} x_1 x_2^{−1} x_2 x_1^{−1} k^{−1} X_{15+2k} \]
\[ \sim Q^{+} \left( x_1, x_2, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}, X_{15}, X_{15+2k}, X_{18+2k} \right) \]
\[ x_5 x_2^{−1} x_6^{−1}, x_5 x_1^{−1} x_7^{−1}, x_7 x_2^{−1}, x_1 x_9^{−1}, x_9 x_2^{−1} x_1 x_9^{−1}, x_9 x_2^{−1} k^{−1} x_{15+2k}, X_{18+2k} x_1 x_2^{−1} x_2 x_1^{−1} k^{−1} X_{15+2k} \]
\[ \sim Q^{+} \left( x_1, x_2, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}, X_{15}, X_{15+2k}, X_{18+2k} \right) \]
\[ x_5 x_2^{−1} x_6^{−1}, x_5 x_1^{−1} x_7^{−1}, x_7 x_2^{−1}, x_1 x_9^{−1}, x_9 x_2^{−1} x_1 x_9^{−1}, x_9 x_2^{−1} k^{−1} x_{15+2k}, X_{18+2k} x_1 x_2^{−1} x_2 x_1^{−1} k^{−1} X_{15+2k} \]
\[ \sim Q^{+} \left( x_1, x_2, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}, X_{15}, X_{15+2k}, X_{18+2k} \right) \]
\[ x_5 x_2^{−1} x_6^{−1}, x_5 x_1^{−1} x_7^{−1}, x_7 x_2^{−1}, x_1 x_9^{−1}, x_9 x_2^{−1} x_1 x_9^{−1}, x_9 x_2^{−1} k^{−1} x_{15+2k}, X_{18+2k} x_1 x_2^{−1} x_2 x_1^{−1} k^{−1} X_{15+2k} \]

Therefore,

\[ Q K_{\phi_h} M_k \sim Q^{+} \left( x_2, x_{15+2k}, X_{18+2k} \right) \]
\[ X_{18+2k} x_2^{−1} x_2 x_{15+2k}, X_{18+2k} x_2^{−1} x_2 x_{15+2k}, X_{18+2k} x_2^{−1} x_2 x_{15+2k}, X_{18+2k} x_2^{−1} x_2 x_{15+2k} \]

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**DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, UK AND DEPARTAMENTO DE MATEMÁTICA, FCEN, UNIVERSIDAD DE BUENOS AIRES, ARGENTINA.**

*Email address*: xfernand@dm.uba.ar