A concise, general proof of the Multiplicative Ergodic Theorem

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October 27, 2021

Abstract

The semi-invertible version of Oseledeč’s multiplicative ergodic theorem providing a decomposition of the underlying state space of a random linear dynamical system into fast and slow spaces is obtained for a strongly measurable cocycle on a separable Banach space. Measurable growth estimates on subspaces for linear operators are identified and combined with a modified version of Kingman’s subadditive ergodic theorem to obtain this main result for an ergodic invertible system on a Lebesgue probability space.

1 Introduction

The multiplicative ergodic theorem is a fundamental tool in the study of linear dynamical systems. Given a Banach space $X$ write $B(X)$ for the space of bounded linear operators on $X$. Given an ergodic system $\sigma : \Omega \to \Omega$ and function $L : \Omega \to B(X)$, one may compose copies of $L$ along orbits and investigate long term behaviour of any $x \in X$. Such an $L$ is called a cocycle said to be forward-integrable if $\log^+ \|L\| \in L^1(\Omega)$. Iteratively set $L^{(0)} = 1_X : X \to X$, and write $L^{(n)} := L_{\sigma^n} \circ \cdots \circ L_\omega$ for $n \in \mathbb{N}$. The identity $L^{(n+m)} = L^{(n)} \circ L^{(m)}$ then immediately follows. If $L$ is required to be invertible, the choice $L^{(n)} = L_{\sigma^{-n}}^{-1} \circ \cdots \circ L^{-1}_\omega$ for $n < 0$ is necessary in order to retain this relation for all $n \in \mathbb{Z}$. If $L^{-1}$ is forward-integrable then $L$ is said to be backward-integrable. One may seek ways of describing $X$ in terms of long term behaviour of vectors under $L^{(n)}$ as $n \to \infty$. Given any normed space $V$ write $S_V = \{x \in V : \|x\| = 1\}$ and $B_V = \{x \in V : \|x\| < 1\}$. Oseledeč [11] proved the following in 1965:

**Theorem 1.** Let $\Omega$ be a Lebesgue probability space and $\sigma : \Omega \to \Omega$ be an invertible measure preserving transformation. Let $L : \Omega \to GL_d(\mathbb{R})$ satisfy $\log^+ \|L\|, \log^+ \|L^{-1}\| \in L^1(\Omega)$ where $GL_d$ denotes the invertible $d$-dimensional matrices. Then there are measurable numbers $\lambda_i(\omega), i \in \{1, \cdots, r\}$ and a direct sum decomposition of invariant measurable subspaces $\mathbb{R}^d = \bigoplus E_i(\omega)$ such that for $x \in S_{E_i(\omega)}$,

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \|L^{(n)}x\| = \pm \lambda_i(\omega).$$

This limit converges uniformly in $x$.

In a setting where all invertibility assumptions are dropped the conclusions are weaker. The following flag decomposition is proven in the work of Raghunathan [13]:

**Theorem 2.** Let $\Omega$ be a Lebesgue probability space and $\sigma : \Omega \to \Omega$ be a (not necessarily invertible) measure preserving transformation. Let $L : \Omega \to Mat_d(\mathbb{R})$ satisfy $\log^+ \|L\| \in L^1(\Omega)$ where $Mat_d$ denotes the $d$-dimensional matrices. Then there are measurable numbers $\lambda_i(\omega), i \in \{1, \cdots, r\}$ and a flag of invariant measurable subspaces $\mathbb{R}^d = V_1(\omega) > V_2(\omega) > \cdots > V_r(\omega)$ such that for $x \in V_i(\omega) \setminus V_{i+1}(\omega)$,

$$\lim_{k \to \infty} \frac{1}{k} \log \|L^{(k)}x\| = \lambda_i(\omega).$$

These results have been successively extended to more general settings, by Ruelle to compact operators on Hilbert spaces [15], by Mañé to compact operators on Banach spaces with additional continuity assumptions [14] and by Thieullen to quasicompact operators [16].

The case where the underlying vector space is a separable Banach space was first presented by Lian and Lu [8]. Their monograph concludes with a decomposition result assuming only almost-everywhere injectivity of the cocycle in a separable Banach space.

The injectivity condition isn’t necessary to obtain a decomposition: Froyland, Lloyd and Quas [9] demonstrated that if $\sigma$ is invertible, but no such condition is made of $L(\omega)$ then the space may still be written as a sum of fast spaces. This first paper dealt with the finite dimensional case, but in order to apply this result to more useful settings, González-Tokman and Quas [3] generalised this statement to a separable Banach space. However, there is some ambiguity to the statement and consequentially the proof - it is claimed that a measurable composition of $X$ exists without a
full discussion of what this means, which is crucial given that in the infinite dimensional setting no statement may be made of the slow spaces. Their subsequent shorter proof in the case where the dual $X^* = B(X, \mathbb{R})$ is separable uses intuitive notions of volume growth that current work partially builds upon, but relies on less than fully rigorous references to cocycles whose domain varies depending on $\omega$. Another example of this kind of emphasis on volume growth may be seen in the noninvertible result of Blumenthal in [1], where a flag is obtained with no separability assumptions but with a much stronger uniform measurability condition for the cocycle.

Given measurable spaces $A$ and $B$ write $\mathcal{M}(A \to B)$ for the space of measurable functions from $A$ to $B$. Let $\mathcal{F}$ be the Borel sigma algebra induced by the strong operator topology on $X$. The space $\mathcal{SM}$ of strongly measurable functions consists of measurable maps with respect to this choice of sigma algebra:

$$\mathcal{SM}(A \to \mathcal{B}(X)) = \mathcal{M}(A \to (\mathcal{B}(X), \mathcal{F})).$$

Write $\mathcal{G}_k X = \{ V \leq X : \text{dim } V = k \}$ for the Grassmannian of subspaces of dimension $k$. Given $T \in \mathcal{B}(X)$ define the slowest growth of vectors in a given subspace under $T$:

$$g(T, V) = gr(V) = \inf_{x \in V} \|Tx\|, \rho_k T = \sup_{V \in \mathcal{G}_k} g(T, V).$$

The $\rho_k$ are called Bernstein numbers in the work of Pietsch [12], which gives an overview of similar kinds of statistics in Banach spaces. Given a strongly measurable forward-integrable cocycle $\mathcal{L}$ on Lebesgue probability space there are decreasing sequences $(\mu_i)_{i \in \mathbb{N}}$ and $\lambda_i$ of invariant functions

$$\mu_k = \lim_{n \to \infty} \frac{1}{n} \rho_k \mathcal{L}^{(n)}_\omega, \lambda_1 = \mu_1, \lambda_{i+1} = \mu_{inf(t, \mu_i < \lambda_i)}.$$

While there are countably many $\mu_k$, there may only be finitely many $\lambda_i$, referred to henceforth as the Lyapunov exponents. The main result, an extension of the semi-invertible result to separable Banach spaces, may now be stated:

**Theorem 3.** Let $(\Omega, \sigma, X, \mathcal{L})$ be a strongly measurable forward-integrable random linear dynamical system with $(\Omega, \sigma)$ an ergodic invertible map on a Lebesgue probability space and having Lyapunov exponents $(\mu_i)_{i=1}^\infty$ and $(\lambda_i)_{i=1}^\infty$, where $1 \leq L \leq \infty$. Then each $0 \leq l < L$ there is a direct sum decomposition into equivariant spaces $X = (\bigoplus_{i\leq L} E_i(\omega)) \oplus V_{l+1}(\omega)$ with the $E_i : \Omega \to \mathcal{G}_{m_i} X$ measurable, $m_i \in \mathbb{N}$ and having

$$\frac{1}{n} \log \| \mathcal{L}^{(n)}_\omega \|_{E_i(\omega)} \|, \frac{1}{n} \log \inf \{ \| \mathcal{L}^{(n)}_\omega x \| : x \in \mathcal{S}_{E_i(\omega)} \} \to \lambda_i,$$

and

$$V_l(\omega) = \{ x \in X : \lim_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}^{(n)}_\omega x \| \leq \lambda_l \}.$$

The projection $\Pi$ onto $\bigoplus_{j<l} E_j$ parallel to $V_l$ is strongly measurable and tempered, that is to say, $\lim_{n \to \infty} \frac{1}{n} \log \| \Pi_{l\nu} \| = 0$ almost surely. $L \geq 1$ and there is a nontrivial decomposition exactly when $\nu = \lim_{n \to \infty} \mu_n < \lambda_1$.

The choice of construction method for the Lyapunov exponents is a fundamental aspect to each proof: in [8] the first Lyapunov exponent is defined according to the asymptotic growth rate of $\| \mathcal{L}^{(n)}_\omega \|$ followed by the Lyapunov exponents being first described as the asymptotic growth rate for nonzero elements of the fast spaces in the statement of the theorem. While singular value decomposition is no longer available in these contexts, some construction must be written down that may be viewed as having echoes of a proof of the singular value decomposition of a transformation. The work presented here explicitly relies on a notion of singular values for arbitrary elements of $\mathcal{B}(X)$. Given a $T \in \mathcal{B}(X)$, sufficient conditions on these singular values for contracting fast growing regions of $\mathcal{G}X$ under the action $V \to TV$ are derived. A modified version of Kingman’s subadditive theorem is established in section 3 and used to guarantee the asymptotic growth rates of the singular values of $\mathcal{L}^{(n)}_\omega$.

The results from past papers stated thus far have not required ergodicity. Throughout the rest of this paper, ergodicity will be assumed, simplifying the classification of invariant functions, although all results as is typical may be formulated with this assumption dropped. The end result of this paper may be found in [3]. In this Banach space setting, the main theorem yields in a trichotomous classification of forward-integrable cocycles on separable Banach spaces: one of

- $\mathcal{L}$ fails to be quasicompact, with $\mu_i = \nu$ for all $i \in \mathbb{N}$ - no fast spaces may be detected
- $\mathcal{L}$ is quasicompact, with finitely many finite dimensional fast spaces growing at rates $\lambda_1 > \cdots > \lambda_L > \lambda_{L+1} = \nu$
- $\mathcal{L}$ is quasicompact, with a countable sequence of finite dimensional fast spaces growing at rates $\lambda_1 > \lambda_2 > \cdots \to \nu$. 


These three possibilities just correspond to situations where there are no, finitely many or countably many fast spaces. The number \( \nu \) is an alternative choice to the typical index of compactness \( \kappa = \lim_{n \to \infty} \frac{1}{n} \| \mathcal{L}^{(n)} \| \), that proves simpler to work with in this proof. In the appendix it is verified that these quantities are equal, so that the final result represents an extension of the result of Lian and Lu with the injectivity assumption dropped, avoiding measurability or domain concerns present \( \text{[3]} \) and \( \text{[4]} \).

The author is indebted to Anthony Quas for his enthusiastic encouragement and guidance throughout.

2  The Grassmannian

When discussing subspaces of \( X \), we may also consider the space of subspaces of \( X \) they lie in:

**Definition 4.** Given vector spaces \( U, V \leq X \) with \( U \oplus V = X \), write \( \Pi_{U|V} \) for the projection defined by \( \Pi_{U|V}(u + v) = u \) for any \( u \in U \) and \( v \in V \). The Grassmannian is defined as the set

\[
\mathcal{G}X = \{ V \text{ closed } \leq X : \text{ there exists a projection } X \xrightarrow{\Pi} V \text{ with } \| \Pi \| < \infty \}
\]

\[
= \{ V \text{ closed } \leq X : \text{ there exists } W \leq X \text{ with } \| \Pi_{V|W} \| < \infty \}.
\]

\( \mathcal{G}X \) may be metrised by any of a few equivalent choices of distance between spaces, such as the Hausdorff distance between unit spheres. \( \mathcal{G}_k X \subset \mathcal{G}X \) has already been defined. Write \( \mathcal{G}^k X = \{ V \leq X : \text{codim } V = k \} \subset \mathcal{G}X \).

**Lemma 5.** Suppose that \( X \) is a Banach space. Then

- If \( X \) is separable then \( \mathcal{G}_k X \) is separable and complete.
- If \( \Omega \) is a measurable space, \( E \in \mathcal{M}(\Omega \rightarrow \mathcal{G}_k X) \) and \( A \in \mathcal{SM}(\Omega \rightarrow \mathcal{B}(X)) \) then the pointwise pushforward \( \omega \rightarrow A(\omega)E(\omega) \) is measurable.

*Proof.* This result is established in the appendix on Grassmannians in \( \text{[3]} \).

**Lemma 6.** If we take \( B = \mathcal{B}(X) \) then we have the following characterisation of the space of strongly measurable functions:

\[
\mathcal{SM}(\Omega \rightarrow \mathcal{B}(X)) = \left\{ \mathcal{L} : \Omega \rightarrow \mathcal{B}(X) : \begin{array}{l}
\text{for each } x \in X, \\
(\omega \rightarrow L_\omega x) \in \mathcal{M}(\Omega \rightarrow X)
\end{array} \right\}.
\]

*Proof.* An account is given in appendix A of \( \text{[3]} \).

The following hold:

**Lemma 7.** Let \( T \in \mathcal{B}(X,Y) \) and \( S \in \mathcal{B}(Y,Z) \).

- \( g_T(V)g_S(TV) \leq g_{S\circ T}(V) \leq \min\{ ||TV||g_S(TV), g_T(V)||S|| \} \)
- \( \rho_T(T)\rho_S(S) \leq \rho_T(S \circ T) \leq \rho_T(T)||S|| \)

We make use of the following straightforward construction:

**Lemma 8.** Let \( \Omega \) be a measurable space. Suppose that \( \{ x_n \}_{n \in \mathbb{N}} \subseteq X \) for some measurable space \( X \). Then if we can write down a measurable set \( S_n \subseteq \Omega \) for each \( n \in \mathbb{N} \) such that \( \bigcup_n S_n = \Omega \) then there is an associated measurable \( N : \Omega \rightarrow \mathbb{N} \) given by \( N_\omega = \inf \{ n : \omega \in S_n \} \). Further, the map \( \omega \rightarrow x_{N_\omega} \) is measurable.

This principle is applied to check measurability in the context of spaces which are separable.

3  Facts about linear maps

**Lemma 9.** Let \( X \) be a normed space. Then \( g : \mathcal{B}(X) \times \mathcal{G}_k X \rightarrow [0, \infty) \) is continuous.

*Proof.* Let \( \epsilon > 0 \). Here use the following metric inducing the product topology:

\[
d((S,V),(T,W)) = ||S - T|| + d(V,W).
\]

First note that \( g \) is continuous at \( (0,V) \) for any \( V \) since \( g(0,V) = 0 \) and if \( d(V,W) + ||T - 0|| < \epsilon \) then \( g(T,W) < \epsilon \). Otherwise, the projection onto the sphere \( p_T : T = \frac{1}{||T||} T \) is continuous. Let \( \epsilon > 0 \). Since \( g(T,V) = ||T||g(p_T T, V) \) for nonzero \( T \), it suffices to check \( g \) is continuous on \( \mathcal{S}_{\mathcal{B}(X)} \times \mathcal{G}_k X \). Let \( (S,V),(T,W) \in \mathcal{S}_{\mathcal{B}(X)} \times \mathcal{G}_k X \)
be such that
\[ d((S, V), (T, W)) = \|S - T\| + d(V, W) < \epsilon. \]
Since \( V \) and \( W \) are finite dimensional their spheres are compact and we may choose an \( x \in S_V \) with \( \|Sx\| = g(S, V) \). Choose a \( y \in S_W \) with \( \|x - y\| < \epsilon \). Then
\[ g(T, W) \leq \|T y\| \leq \|T - S\| + \|S\|(y - x) + \|Sx\| < \epsilon + \epsilon + g(S, V). \]

By the same argument swapping \((S, V)\) and \((T, W)\) then, \( g(S, V) < g(T, W) + 2\epsilon \) and continuity (in fact, uniform continuity) on \( S_B(X) \times G_k X \) is clear. Thus \( g \) is continuous on \( B(X) \times G_k X \) as required.

**Corollary 10.** The quantities \( \rho_k : B(X) \to \mathbb{R} \) are measurable functions.

**Proof.** The \( \rho_k \) may now be written as the supremum of the functions \( \{T \mapsto g(T, V) : V \in G_k X\} \) which may be written as the supremum of a countable family since \( G_k X \) is separable.

**Lemma 11.** Grassmannian contraction estimates: Let \( T \in B(X) \) and \( \Theta \in (\rho_{k+1} T, \rho_k T) \). Suppose that \( V, W \in G_k X \) are choices of fast growing spaces:
\[ g_T(V), g_T(W) > \Theta. \]

Then
\[ d(TV, TW) < \frac{2}{1 - \frac{\rho_{k+1} T}{\Theta}} \cdot \rho_{k+1} T. \]
In particular, if \( \Theta > 2\rho_{k+1} T \) then
\[ d(TV, TW) < \frac{4\rho_{k+1} T}{\Theta}. \]

**Proof.** If \( V = W \) there is nothing to prove so assume otherwise. Let \( a \in V \setminus W \). Then \( W \oplus \text{span}\{a\} \in G_{k+1} X \) so that there is some \( a - b \in W \oplus \text{span}\{a\} \) with \( b \in W \) and \( g_T(a - b) \leq \rho_{k+1} T \). Set \( c = a - b \). By fastness \( \|a\| \leq \frac{2\|a\|}{\Theta} \) and \( \|b\| \leq \frac{2\|a\|}{\Theta} \).

\[ \|Tc\| \leq \|c\| \rho_{k+1} T \leq (\|a\| + \|b\|) \rho_{k+1} T \leq \|Ta\| + \|Tb\| + \rho_{k+1} T \leq 2\|Ta\| + \|Tc\| \rho_{k+1} T, \]

So that
\[ \|Tc\| < \frac{2}{1 - \frac{\rho_{k+1} T}{\Theta}} \|Ta\|. \]
Then since \( a \) isn’t in the kernel then \( d\left(\frac{T a}{\|T a\|}, TW\right) \leq \frac{\|T c\|}{\|T a\|} < \frac{2}{1 - \frac{\rho_{k+1} T}{\Theta}} \rho_{k+1} T \). Since the choice of \( a \)
was arbitrary,
\[ d(TV, TW) < \frac{2}{1 - \frac{\rho_{k+1} T}{\Theta}} \rho_{k+1} T. \]

**Lemma 12.** Let \( X \) be a separable Banach space. Then for every \( k > 0 \) there is a measurable map \( b : G_k X \to G_k X \) such that for each \( V \in G_k X \),
\[ V = \text{span}\{b_i(V) : i = 1\}^k. \]

**Proof.** Stated in [3].

**Lemma 13.** Let \( V \in G_k X \) with bounded projection \( \Pi : X \to V \). Let \( T \in B(X) \). Then \( \rho_{k+1}(T) \leq \rho(T \circ \Pi) \).

**Proof.**
\[ \rho_{k+1}(T) = \sup_{U \in G_{k+1} X} g_T(U) \leq \sup_{U \in G_{k+1} X} g_T(U \cap V) = \sup_{U \in G_{k+1} \geq |V|} g_T(U) = \sup_{U \in G_{k+1} V} g_T(U \cap \Pi) \leq \rho(T \circ \Pi). \]
Lemma 14. Let $X = E \oplus V = E' \oplus V'$ with corresponding projections $\Pi : X \to V \in \mathcal{G}^k X$ and $\Pi' : X \to V' \in \mathcal{G}^k X$. Let $T \in \mathcal{B}(X)$ such that $\Pi' \circ T = T \circ \Pi$, and suppose further that $g_T(E) > \|T\|\|\|$. Then $\rho(T \circ \Pi) \leq 4\|\Pi\|\|T\|\|\Pi\|$.

**Proof.** Make use of the inequality $\rho(T \circ \Pi) \leq \|\Pi\|\rho(T|V)\|$. Let $U \in \mathcal{G}V$ with $g_{T^1}(U) > e^{-\epsilon} \rho(\Pi \circ T|V) = e^{-\epsilon} \rho(T|V)$. Let $x \in \mathcal{S}_{U \circ E}$. Observe that $\|\Pi' \circ T \circ x\| \leq \|\Pi\|\|T\|\|x\|$ and similar for $1 - \Pi'$ so that

\[
\|Tx\| \geq \max\{\|\Pi' \circ T\|, \|(1 - \Pi') \circ T\|\} \geq \max\{\rho(T)\|\Pi\|, \rho(T)\|1 - \Pi\|\} \\
\geq \max\{\|\Pi\|, \|(1 - \Pi)\|\} \geq \epsilon^{-1} \rho(T|V) \geq \epsilon^{-1} \rho(T \circ \Pi) \geq \epsilon^{-1} \rho(T - \Pi) \geq \epsilon^{-1} \rho(T \circ \Pi).
\]

We may then conclude that $g_T(U) \geq \epsilon^{-1} \rho(T \circ \Pi)$ with $\epsilon > 0$ arbitrary, so that $\rho(T \circ \Pi) \leq 4\rho_{k+1}(T \circ \Pi)\|\Pi\|\|\Pi\|$ as required.

4 A balanced subadditive ergodic theorem

Consider a decomposition of the interval $[a, b] = [a, c] \cup [c, b] \subset \mathbb{Z}$. In discrete time, where these intervals represent different intervals in which dynamics may occur. Given a cocycle $\mathcal{L} : \Omega \to \mathcal{B}(X)$, for fixed $\omega$ and each $b \geq a \in \mathbb{Z}$ we may define the map $\mathcal{L}_{a \to b}(\omega) := \mathcal{L}_{a \to b}^{[\nu, \omega]}$, which may be thought of as the evolution rule for $X$ from time $a$ to time $b$. Under this notation it is easy to see that for any $a < c < b \in \mathbb{Z}$, $\mathcal{L}_{a \to c} \circ \mathcal{L}_{c \to b} = \mathcal{L}_{a \to b}$. The inequality $\|\mathcal{L}_{a \to b}\| \leq \|\mathcal{L}_{a \to c}\| \|\mathcal{L}_{c \to b}\|$ holds. Taking log of each side, one obtains the following triangle inequality-like bound on the growth of points in $X$ from time $a$ to $b$:

\[
f_{a \to b}(\omega) := \log \|\mathcal{L}_{a \to b}\| \leq \log \|\mathcal{L}_{a \to c}\| + \log \|\mathcal{L}_{c \to b}\| = f_{a \to c}(\omega) + f_{c \to b}(\omega).
\]

**Definition 15.** Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}(\Omega \to \mathbb{R})$ satisfy

\[
f_{m,n}(\omega) \leq f_m(\sigma^n \omega) + f_n(\omega).
\]

Then $\mathcal{F}$ is referred to as a subadditive family of measurable functions.

The following view is useful:

**Definition 16.** A subadditive family $\{f_n\}_{n \in \mathbb{N}}$ generates a stationary subadditive process $\{f_{a \to b} : a < b, a, b \in \mathbb{Z}\}$ and vice versa via the relation

\[
f_{a \to b} := f_{b-a} \circ \sigma^a.
\]

A subadditive process is a collection $\{f_{a \to b}\}_{a < b \in \mathbb{Z}} \subseteq \mathcal{M}(\omega \to \mathbb{R})$ such that for all $a < c < b \in \mathbb{Z}$,

\[
f_{a \to b} \leq f_{a \to c} + f_{c \to b},
\]

and the stationarity condition is

\[
f_{a+1 \to b+1} = f_{a \to b} \circ \sigma^1.
\]

As such both notations may be interchanged as appropriate. A similar formalism is outlined in [2].

The Kingman theorem concerns subadditive families of functions, here a slight refinement when the underlying transformation is invertible will be required.

**Theorem 17 (Kingman [6]).** Let $(\Omega, \sigma, \mathcal{F})$ be an ergodic system and let $\{f_n\}_{n \in \mathbb{N}} \subset L^1\Omega$ be subadditive. Then there is a constant $C \in [-\infty, \infty]$ such that

\[
\frac{1}{n} f_n(\omega) \to C = \lim_{n \to \infty} \frac{1}{n} \int f_n
\]

pointwise almost everywhere.

In the case where $\sigma$ is invertible, one easily obtains the following useful corollary:

**Corollary 18.** Let $(\Omega, \sigma, \mathcal{F})$ be an invertible ergodic system on a Lebesgue probability space and let $\{f_n\}_{n \in \mathbb{N}} \subset L^1\Omega$ be subadditive. Then

\[
\lim_{n \to \infty} \frac{1}{n} f_n(\sigma^{-n} \omega) = \lim_{n \to \infty} \frac{1}{n} f_n(\omega)
\]

pointwise almost everywhere.
Proof. Simply set \( g_n = f_n \circ \sigma^{-n} \). Then \( g_n \) is subadditive with respect to \( \sigma^{-1} \):

\[
g_{m+n}(\omega) = f_{m+n}(\sigma^{-(m+n)}\omega) \leq f_m(\sigma^{-m-n}\omega) + f_n(\sigma^{m-n}\omega)
\]

\[= g_{m}(\sigma^{-n}\omega) + g_{n}(\omega),\]

so that certainly there exists some \( L \) such that \( \frac{1}{n} f_n \circ \sigma^{-n} \rightarrow L \). It remains to check that this limit and \( \frac{1}{n} f_n(\omega) \rightarrow L' \) coincide. Let \( \epsilon > 0 \). Then there exists an \( N \) such that

\[
\mathbb{P}(\omega : \frac{1}{n} f_n(\omega) \in (L' - \epsilon, L' + \epsilon)) > \frac{1}{2}
\]

and

\[
\mathbb{P}(\omega : \frac{1}{n} g_n(\omega) = \frac{1}{n} f_n(\sigma^{-n}\omega) \in (L - \epsilon, L + \epsilon)) > \frac{1}{2},
\]

so that the set

\[
\{\omega : \frac{1}{n} f_n(\sigma^{-n}\omega) \in (L - \epsilon, L + \epsilon)\} \cap \{\omega : \frac{1}{n} g_n(\omega) \in (L' - \epsilon, L' + \epsilon)\}
\]

has positive measure, whence \( |L - L'| < 2\epsilon \). \( \epsilon \) was arbitrary though, so that \( L = L' \).

Kingman’s original result may be modified in a few directions. One may very rapidly obtain the following, which is also shown in [2]:

**Lemma 19.** Let \( \{f_n\}_{n \in \mathbb{N}} \subseteq L^1\Omega \) be a subadditive family of functions. Then

\[
\frac{1}{n} f_{n\rightarrow 2n} \rightarrow C = \lim_{n \to \infty} \frac{1}{n} \int f_n.
\]

**Proof.** The lower bound is immediately clear, since

\[
\frac{1}{n} f_{n\rightarrow 2n}(\omega) \geq \frac{1}{n} (f_{0\rightarrow 2n} - f_{0\rightarrow n}) \rightarrow 2C - C = C \text{ as } n \to \infty.
\]

On the other hand, let \( \epsilon > 0 \), then since \( \frac{1}{n} \int f_n \rightarrow C \) there exists some \( N \) such that for all \( n \geq N \),

\[
\frac{1}{n} \int f_n \leq C + \epsilon.
\]

Given some fixed \( n \geq N \) and some \( j < N \), set \( j = \lfloor \frac{n}{2} \rfloor \) and break up the interval \([0, n) = [n, n + j) \sqcup [n + j, n + j + N), \ldots \sqcup [n + j + (t-1)N, n + j + \lfloor \frac{n}{2} \rfloor N] \sqcup [n + j + \lfloor \frac{n}{2} \rfloor N, 2n)\).

Applying subadditivity in this manner

\[
f_{n\rightarrow 2n} \leq f_{n\rightarrow n+j} + \sum_{t=0}^{t} f_{n+j+iN\rightarrow n+j+(i+1)N}(\omega) + f_{n+j+iN\rightarrow n}
\]

\[
\leq S_N[f_1 \circ \sigma^n(\omega) + \sum_{t=0}^{t} f_{n} \circ \sigma^{n+j+iN}(\omega) + S_N[f_1 \circ \sigma^{2n-N}(\omega)]
\]

The above is valid for all \( j \in [0, N) \), so averaging and dividing by \( nN \) we obtain

\[
\frac{1}{n} f_{n\rightarrow 2n} \leq \frac{1}{n}(S_N[f_1 \circ \sigma^n + S_N[f_1 \circ \sigma^{2n-N}] + \frac{1}{nN} S_n f_n \circ \sigma^n)
\]

\[
\leq 2\epsilon + \frac{1}{nN} n(N(C + \epsilon))
\]

for sufficiently large \( n \). Since \( \epsilon \) was arbitrary, the result is proven.

A similar result for \( f_{n\rightarrow n} \) will be crucial to the main result. The proof here uses the following lemma, which is a simplified version of result 3.9 in [14]:

**Lemma 20** (Backward Vitali). Let \( (\Omega, \mathcal{F}, \sigma) \) be an invertible ergodic system on a Lebesgue probability space. Suppose that there is a sequence of integer valued functions \( j_k \in \mathcal{M}(\Omega \rightarrow \mathbb{N}) \) such that \( j_k(\omega) \rightarrow \infty \) as \( k \rightarrow \infty \). Then there is a measurable \( A \subseteq \Omega \) and a measurable \( j \in \mathcal{M}(A \rightarrow \mathbb{N}) \) such that writing \( I_\omega = \{\sigma^i \omega : i \in \{-j, -j+1, \ldots, j\}\} \), the \( I_\omega \) are disjoint and

\[
\mathbb{P}(\bigcup_{\omega \in A} I_\omega) > 1 - \epsilon.
\]

**Theorem 21** (Kingman’s theorem for balanced intervals). Let \( (\Omega, \mathcal{F}, \sigma) \) be an ergodic system with invertible base, and let \( \{f_n\}_{n \in \mathbb{N}} \) be subadditive sequence of measurable functions on \( \Omega \). Then

\[
\frac{1}{n} f_{2n}(\sigma^{-n}\omega) \rightarrow C = \lim_{n \to \infty} \frac{1}{n} \int f_n
\]

pointwise almost everywhere.
Proof. Kingman’s theorem immediately yields an upper bound: since
\[
\frac{1}{2n}f_{2n}(\sigma^{-n}\omega) \leq \frac{1}{2n}((f_n(\sigma^{-n}\omega) + f_n(\omega))) \to \frac{1}{2}C + \frac{1}{2}C
\]
it is clear that
\[\tilde{C}_\omega = \limsup_{n \to \infty} \frac{1}{2n}f_{2n}(\sigma^{-n}\omega) \leq C.\]

If \(C = -\infty\) then there is nothing more to show. Otherwise,
\[
C_{\sigma^{-1}\omega} = \liminf_{n \to \infty} \frac{1}{2n}f_{2n} \circ \sigma^{-n}(\sigma^{-1}\omega)
\geq \liminf_{n \to \infty} \left(\frac{1}{2n}f_{2(n+1)} \circ \sigma^{-(n+1)}(\omega) - f_{2} \circ \sigma^{-1}(\omega)\right)
\geq \liminf_{n \to \infty} \frac{1}{2n}f_{2n} \circ \sigma^{-n}(\omega) - \limsup f_{2} \circ \sigma^{-1}(\omega)
= C_{\omega} - 0,
\]
whence \(C \leq \tilde{C}\) is constant. Fix \(\epsilon > 0\). It is sufficient to show that \(C - \epsilon \leq \Theta(C, \epsilon)\) where \(\Theta(C, \epsilon) \to C\) as \(\epsilon \to 0\). The set of \(j\) such that \(f_{-j-j}\) is small is infinite, so denote the ordered elements
\[
\{j : f_{-j-j}(\omega) < 2j(C + \epsilon)\} = \{j_k(\omega) : k \in \mathbb{N}\}.
\]
Since \(f\) is integrable, it is uniformly integrable and so there exists a \(\delta > 0\) such that if \(\mathbb{P}(A) < \delta\) then \(\int_A f < \epsilon\). Apply Backward Vitali to the \(j_k\)s and the \(-j_k\)s with parameter \(\epsilon' := \min\{\epsilon, \delta\}\) to obtain an \(A \subseteq \Omega\) and a \(j : A \to \mathbb{N}\) according to theorem 20 such that writing \(\mathcal{B} = \Omega \setminus \bigcup_{\omega \in A} I_\omega\), we have
\[
\mathbb{P}(\mathcal{B}) < \delta
\]
and so
\[
\int_{\Omega \setminus \bigcup_{\omega \in A} I_\omega} f_{j_1} < \epsilon.
\]
While every \(j(\omega) = j_i(\omega)\), for some \(i\) so that for every \(\omega \in A\),
\[
f_{-j(\omega)-j(\omega)}(\omega) \leq 2j(C + \epsilon).
\]
Write \(A_{\pm} = \{\sigma^\pm j_i(\omega) : \omega \in A\}\). In addition we may define measurable maps \(T_{\pm} \in \mathcal{M}(\Omega \to \mathbb{N})\) by
\[
T_+(\omega) = \inf\{t \in \mathbb{N}_0 : \sigma^t\omega \in A_+\}, \quad
T_-(\omega) = \inf\{t > T_+(\omega) : \sigma^t\omega \in A_-\}.
\]
\(T_+\) is the earliest nonnegative time such that \(\sigma^{T_+}\omega\) is the start of a period at growth rate guaranteed close to \(C\). \(T_-\) by contrast seeks the first time in the past which was the end of such an interval. Each are almost surely finite since \(\mathbb{P}(A_{\pm}) = \mathbb{P}(A) > 0\).

The idea of the next step is that \([0, n]\) may be measurably broken up into intervals of slow growth and with small gaps inbetween. Set \(b_0 = 0\) and recursively iterate along the orbit for \(i > 0\):
\[
a_i(\omega) = T_{-i}(\sigma^{a_i-1}(\omega)), \\
b_i(\omega) = T_{+i}(\sigma^{b_i-1}(\omega)).
\]
For any \(m \in \mathbb{N}_0\) we may set \(t_m(\omega) = \max\{t : b_i(\omega) \leq m\}\). Since \(j\) and \(T_{\pm}\) are measurable and the \(t_m(\omega) \to \infty\) as \(m \to \infty\), it is possible to pick an \(N\) such that for all \(n \geq N\), \(\mathbb{P}(j \geq N) < \frac{1}{4}\epsilon'\), \(\mathbb{P}(t_n = 0) < \frac{1}{4}\epsilon'\) and \(\mathbb{P}(T_+ \geq N) < \frac{1}{4}\epsilon'\). Let \(n \geq N\). The orbit of \(\omega\) is considered over the following intervals of times:
\[
[0, n] = [0, a_1(\omega)) \cup [a_1(\omega), b_1(\omega)) \cup [b_1(\omega), a_2(\omega)) \cup [a_2(\omega), b_2(\omega)) \cup \\
\ldots \cup [b_{i_n}(\omega)-1(\omega), a_{i_n}(\omega)) \cup [a_{i_n}(\omega), b_{i_n}(\omega)) \cup [b_{i_n}(\omega), n).
\]
Time \(a_i \to b_i\) is guaranteed to have low growth:
\[
f_{a_i(\omega) \to b_i(\omega)}(\omega) \leq (b_i(\omega) - a_i(\omega))(\tilde{C}_\omega + \epsilon),
\]
and in addition there is a good chance that the gaps between the \(a_i\)s and the \(b_i\)s won’t be too large:
\[
\mathbb{P}(a_i(\omega) \geq N) = \mathbb{P}(T(\omega) \geq N) < \frac{1}{4}\epsilon'.
\]
Since $T_\epsilon$ seeks the endpoints of the periods of guaranteed growth, it holds that either $T_\epsilon(\sigma^n\omega) = n - b_{t_n}(\omega)$ or $t_n(\omega) = 0$. Therefore,

$$\mathbb{P}(n - b_{t_n}(\omega) \geq N) \leq \mathbb{P}(T_\epsilon(\sigma^n\omega) \geq N) + \mathbb{P}(t_n(\omega) = 0) < \mathbb{P}(T_\epsilon \geq N) + \frac{1}{2}\epsilon' < \frac{1}{2}\epsilon'. $$

Set

$$\Lambda = \{a_1 \geq N \text{ or } n - b_{t_n}(\omega) \geq N\},$$

so that it is immediate from above that $\mathbb{P}(\Lambda) < \epsilon'$. Given some measurable $g$ write $S_p g = \sum_{i=0}^{p-1} g \circ \sigma^i$.

We then repeatedly apply subadditivity to $f_0 \to n$:

$$f_n(\omega) = f_0 \to n(\omega)$$

$$\leq (f_0 \to a_1(\omega) + f_{a_1(\omega)} \to b_1(\omega) + \cdots + f_{a_n(\omega)} \to b_{t_n}(\omega) + f_{b_{t_n}(\omega)} \to n(\omega))$$

$$\leq \sum_{i=1}^{t_n(\omega)} f_{a_i(\omega)} \to b_i(\omega) + \sum_{i=1}^{t_n(\omega)} f_{b_{i-1}(\omega)} \to a_i(\omega) + f_{b_{t_n}(\omega)} \to n(\omega)$$

$$\leq \sum_{i=1}^{b_{t_n}(\omega)(\omega)} (b_i(\omega) - a_i(\omega))(C + \epsilon) +$$

$$\sum_{i=1}^{b_{t_n}(\omega)(\omega)} 1_B(\sigma^i\omega)f_1 \circ \sigma^i + f_{b_{t_n}(\omega)} \to n(\omega).$$

In the case where $t_n(\omega) = 0$ the first two terms are empty sums. Outside $\Lambda$ it is guaranteed that $[0, a_1(\omega)) \subseteq [0, M]$ and $[b_{t_n}(\omega)(\omega), n) \subseteq [n - M, n)$. Therefore

$$f_n(\omega) \leq \sum_{i=0}^{n-1} (C + \epsilon) 1_B(\sigma^i\omega) + \sum_{i=0}^{n-1} 1_B(\sigma^i\omega)f_1(\sigma^i\omega) +$$

$$\sum_{i=0}^{M} (|f_1|)(\sigma^i\omega) + |C| + \epsilon + \sum_{i=n-M}^{n-1} (|f_1|)(\sigma^i\omega) + |C| + \epsilon +$$

$$1_{\Lambda(\omega)}|S_n||f_1|$$

$$= (C + \epsilon)S_n 1_B(\omega) + S_n(f_1 1_B)(\omega) +$$

$$S_M|f_1| + S_M|f_1(1 - \sigma^n - M)| + 1_{\Lambda(\omega)}|S_n||f_1|.$$ 

It remains to check that each of these terms are small enough to yield the result when we divide through by $n$ and integrate:

$$\frac{1}{n} \int (C + \epsilon)S_n 1_B(\omega) + S_n(f_1 1_B) = \int ((C + \epsilon) 1_B + f_1 1_B)$$

$$= (C + \epsilon)\mathbb{P}(B^c) + \int_B |f_1|$$

$$\leq \max\{(C + \epsilon)(1 - \epsilon), C + \epsilon\} + \epsilon$$

$$\leq (C + \epsilon) + \epsilon(|C| + \epsilon) + \epsilon,$$

and

$$\frac{1}{n} \int (S_M|f_1| + S_M|f_1(1 - \sigma^n - M)| + 1_{\Lambda(\omega)}|S_n||f_1|)$$

$$= \frac{1}{n} \int (M|f_1| + M|f_1|) + \frac{1}{n} \int \Lambda|S_n||f_1|$$

$$\leq \frac{2M\epsilon}{n} + \frac{1}{n} \sum_{i=0}^{n-1} \int_{\sigma^i\Lambda} |f_1|$$

$$\leq \frac{2M\epsilon}{n} + \epsilon.$$

Putting these together we obtain

$$(C - \epsilon) \leq \frac{1}{n} \int \Omega f_n \leq (C + \epsilon + \epsilon(|C| + \epsilon) + \epsilon + \frac{2M\epsilon}{n} + \epsilon$$

$$\to C + (3 + |C|)\epsilon$$ as $n \to \infty$.

Letting $\epsilon \to 0$ we obtain $C \leq C$ as required.
Linearity of the cocycles yields subadditivity of these families, which allow us to apply Kingman.

**Definition 22.** Given a random linear dynamical system $\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})$, write

$$\lambda_\mu(x) = \limsup_n \frac{1}{n} \log \| \mathcal{L}_\omega^{(n)} x \| \in [-\infty, \infty).$$

Write $m_i$ for the multiplicity of $\lambda_i$, i.e., $m_i$ is defined as the largest $m \in \mathbb{N}$ such that

$$\mu_{m_1 + \cdots + m_{i-1} + m} = \lambda_i.$$

**Lemma 23.** The quantities $\lambda_i, \mu_i$, and $\nu$ are almost everywhere constants. Further, $\lambda_2$ exists if and only if $\lambda_1 > \nu$.

**Proof.** The existence of the limits are guaranteed by applying Kingman to particular choices of subadditive sequences:

$$f_k^n(\omega) = \log \rho_k(\mathcal{L}_\omega^{(n)}),$$

$$g_n(\omega) = \log \| \mathcal{L}_\omega^{(n)} \|_c.$$ 

The second statement is a trivial consequence of the definitions $\nu = \lim_{n \to \infty} \mu_n$ and $\lambda_2 = \mu_{\inf\{t : \mu_t < \mu_1\}}$.

\[ \square \]

## 5 Decomposing a cocycle

The tools obtained thus far are now used to decompose quasicompact cocycles.

**Lemma 24.** Let $\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})$ be a strongly measurable random linear dynamical system with ergodic base on a separable Banach space. Suppose that $\mathcal{L}$ satisfies the quasicompactness condition $\nu < \lambda_1$. Then there is a unique measurable choice of fast space $E : \Omega \to \mathcal{G}_{m_1} X$ for which the following hold almost surely:

- Equivariance: $\mathcal{L}_\omega E(\omega) = E(\sigma \omega)$
- $\lambda(x) = \lambda_1$ for every $x \in E(\omega) \setminus \{0\}$
- $\lim_{n \to \infty} \frac{1}{n} \log g(\mathcal{L}_\omega^{(n)}, E(\omega)) = \lambda_1.$

**Proof.** Since $X$ is separable, by lemma 24, $\mathcal{G}_{m_1} X$ is separable: choose some dense $\{E_i\}_{i \in \mathbb{N}} \subseteq \mathcal{G}_{m_1} X$. Let $\epsilon \in (0, \frac{1}{2}(\lambda_1 - \lambda_2))$. The sets

$$A_i = \{\omega \in \Omega : g(\mathcal{L}_{\sigma \omega}^{(2n)}, E_i) > e^{-\epsilon} \rho_k(\mathcal{L}_{\sigma \omega}^{(2n)})\}$$

are measurable and cover $\Omega$, since for fixed $\omega$, density of the $E_i$s and continuity of $g(\mathcal{L}_{\sigma \omega}^{(2n)})$ means that we may find an $i$ with $g(\mathcal{L}_{\sigma \omega}^{(2n)}, E_i)$ as close to $\rho_k(\mathcal{L}_{\sigma \omega}^{(2n)})$ as we please. Then lemma 23 above provides measurable functions

$$\iota^{(n)}(\omega) = \inf\{i \in \mathbb{N} : g(\mathcal{L}_{\sigma \omega}^{(2n)}, E_i) > e^{-\epsilon} \rho_k(\mathcal{L}_{\sigma \omega}^{(2n)})\},$$

$$\bar{E}^{(n)}(\omega) := E_{\iota^{(n)}(\omega)},$$

and the pushforward

$$E^{(n)}(\omega) = \mathcal{L}_{\sigma \omega}^{(n)} \bar{E}^{(n)}(\omega)$$

is also then measurable by lemma 23.

First, we establish that this sequence is Cauchy, and therefore convergent, to a family of spaces $E \in \mathcal{M}(\Omega \to \mathcal{G}_{m_1} X)$.

For almost every $\omega \in \Omega$ the fastest $m_1$ dimensional growth rate

$$\frac{1}{n} \log \rho_k(\mathcal{L}_{\sigma \omega}^{(n)}) \to \lambda_1,$$

$$\frac{1}{n} \log \rho_k(\mathcal{L}_{\omega}^{(n)}) \to \lambda_1,$$

$$\frac{1}{kn} \log \rho_k(\mathcal{L}_{\sigma \omega}^{(2n)}) \to \lambda_1,$$

and

$$\rho_k(\mathcal{L}_{\sigma \omega}^{(n)}) \geq g_{E_{\sigma \omega}^{(n)}}(\bar{E}^{(n)}(\omega)) > e^{-\epsilon} \rho_k(\mathcal{L}_{\sigma \omega}^{(n)}).$$
Thus for each $\omega$ in this full measure set we may choose $M_\omega$ such that for $n \geq M_\omega$ we can usefully estimate growth under $\mathcal{L}$ in a few cases:

\[
\|\mathcal{L}^{(n)}_{\sigma-n_\omega}\|, \|\mathcal{L}^{(n)}_{\omega}\| \in (e^{n(\lambda_1-\epsilon)}, e^{n(\lambda_1+\epsilon)}),
\]

\[
g(\mathcal{L}^{(2n)}_{\sigma-n_\omega}, \mathcal{E}^{(n)}(\omega)) \in (e^{2n(\lambda_1-\epsilon)}, e^{2n(\lambda_1+\epsilon)}),
\]

and

\[
\|\mathcal{L}^{-(n+1)}_{\sigma-n_\omega}\| < e^{n\epsilon}.
\]

Applying the inequalities in lemma 7 then,

\[
g(\mathcal{L}^{(n)}_{\sigma-n_\omega}, \mathcal{E}^{(n)}(\omega)) \geq \frac{g(\mathcal{L}^{(2n)}_{\sigma-n_\omega}, \mathcal{E}^{(n)}(\omega))}{\|\mathcal{L}^{(n)}_{\omega}\|} \geq e^{n(\lambda_1-3\epsilon)}.
\]

and

\[
g(\mathcal{L}^{(n)}_{\sigma-n_\omega}, \mathcal{E}^{(n)}(\omega)) = g(\mathcal{L}^{(n)}_{\sigma-n_\omega}, \mathcal{E}^{(n)}(\omega)) \geq \frac{g(\mathcal{L}^{(2n)}_{\sigma-n_\omega}, \mathcal{E}^{(n)}(\omega))}{\|\mathcal{L}^{(n)}_{\sigma-n_\omega}\|} \geq e^{n(\lambda_1-3\epsilon)}.
\]

In addition, $\mathcal{L}^{-(n+1)}_{\sigma-n_\omega}\mathcal{E}^{(n+1)}(\omega)$ is also guaranteed to have fast growth under $\mathcal{L}^{(n)}_{\sigma-n_\omega}$:

\[
g(\mathcal{L}^{(n)}_{\sigma-n_\omega}, \mathcal{L}^{-(n+1)}_{\sigma-n_\omega} \mathcal{E}^{(n+1)}(\omega)) \geq \frac{g(\mathcal{L}^{(n+1)}_{\sigma-(n+1)_\omega}, \mathcal{E}^{(n+1)}(\omega))}{\|\mathcal{L}^{(n)}_{\sigma-(n+1)_\omega}\|} \geq e^{(n+1)(\lambda_1-3\epsilon)} \geq e^{n(\lambda_1-4\epsilon)}.
\]

$E^{(n)}(\omega)$ consists then of the image of vectors that were fast from time $-n$ to 0, and will grow fast from 0 to $n$. Then by lemma 11 with $\Theta = e^{n(\lambda_1-4\epsilon)}$ we have

\[
d(E^{(n)}(\omega), E^{(n+1)}(\omega)) = d\left(\mathcal{E}^{(n)}(\sigma\omega), \mathcal{L}^{(n)}_{\sigma-n_\omega}(\mathcal{E}^{(n)}(\sigma\omega))\right) < e^{n(\lambda_1+\epsilon)} < e^{-n(\lambda_1-\lambda_2-3\epsilon)}.
\]

Thus $E^{(n)}(\omega)$ is Cauchy and convergent since $\mathcal{G}_{\mathcal{M}_1}, X$ is complete, say to $E(\omega)$.

To prove equivariance, observe that for $n \geq \max\{M_\mathcal{L}, M_{\sigma_\omega}\}$; we find $E^{(n+1)}(\sigma\omega)$ is fast under $\mathcal{L}^{(n)}_{\sigma-n_\omega}$:

\[
g(\mathcal{L}^{(n)}_{\sigma-n_\omega}, \mathcal{E}^{(n+1)}(\sigma\omega)) \geq \frac{g(\mathcal{L}^{(n+1)}_{\sigma-n_\omega}, \mathcal{E}^{(n+1)}(\sigma\omega))}{\|\mathcal{L}_{\omega}\|} \geq e^{(n+1)(\lambda_1-\epsilon)} e^{-n\epsilon} \geq e^{n(\lambda_1-3\epsilon)}.
\]

Then, once again, by closeness of images of fast spaces,

\[
d(\mathcal{L}^{(n)}_{\sigma-n_\omega}, E^{(n+1)}(\sigma\omega), E^{(n)}(\omega)) < e^{-n(\lambda_1-\lambda_2-3\epsilon)},
\]

whence

\[
d(E^{(n+1)}(\sigma\omega), \mathcal{L}_{\omega} E^{(n)}(\omega)) = d(\mathcal{L}^{(n+1)}_{\sigma-n_\omega}, \mathcal{E}^{(n+1)}(\sigma\omega), \mathcal{L}_{\omega} E^{(n)}(\omega)) \leq e^{n\epsilon} d(\mathcal{L}^{(n)}_{\sigma-n_\omega}, \mathcal{E}^{(n)}(\sigma\omega), E^{(n)}(\omega)) < e^{-n(\lambda_1-\lambda_2-4\epsilon)},
\]

so that $\mathcal{L}_{\omega} E(\omega) = \lim_{n \to \infty} \mathcal{L}^{(n)}_{\omega} E^{(n)}(\omega) = E(\sigma\omega)$.

To check that $E(\omega)$ is fast, choose $x \in \mathcal{S}_{x}(\omega)$. Then since for $n \geq M_\mathcal{L}$ we have $d(E^{(n)}(\omega), E(\omega)) < e^{-n(\lambda_1-\lambda_2-\epsilon)}$, for each such $n$ we may choose an $x_n \in \mathcal{S}_{E^{(n)}(\omega)}$ with $\|\mathcal{L}^{(n)}_{\sigma-n_\omega}(x-x_n)\| < e^{-n(\lambda_1-\lambda_2-\epsilon)}$. Since $x_n \in E^{(n)}(\omega)$ and $g(\mathcal{L}^{(n)}_{\sigma-n_\omega}, E^{(n)}(\omega)) \geq e^{n(\lambda_1-3\epsilon)}$ it then follows that,

\[
\|\mathcal{L}^{(n)}_{\omega} x\| \geq \|\mathcal{L}^{(n)}_{\omega} x_n - \mathcal{L}^{(n)}_{\omega} (x-x_n)\| \geq e^{n(\lambda_1-3\epsilon)} - e^{-n(\lambda_1-\lambda_2-\epsilon)} \|\mathcal{L}^{(n)}_{\omega}\| \Rightarrow \|\mathcal{L}^{(n)}_{\omega} x\| \geq \frac{1}{e} e^{n(\lambda_1-\epsilon)}.
\]

Thus we may conclude that as well as being equivariant, $E(\omega)$ is fast for all sufficiently large $n$; since the choice of $x$ was arbitrary the growth is uniform:

\[
g(\mathcal{L}^{(n)}_{\omega}, E(\omega)) > e^{n(\lambda_1-\epsilon)}.
\]
On the other hand, for $n$ sufficiently large we also have
\[ g(\mathcal{L}_{\omega}^{(n)}, E(\omega)) \leq \rho_k(\mathcal{L}_{\omega}^{(m)}) < e^{\lambda_2}, \]
whence $\frac{1}{n} \log g(\mathcal{L}_{\omega}^{(n)}, E(\omega)) \to \lambda_1$. Finally, we check uniqueness: Suppose that $E(\omega)$ and $E'(\omega)$ are both equivariant and fast, so that for every $\omega$ there is some $N$ such that for $n \geq N$,
\[ g_{\mathcal{L}_{\omega}^{(n)}}(E(\omega)), g_{\mathcal{L}_{\omega}^{(n)}}(E'(\omega)) > e^{\lambda_1 - \epsilon}. \]
Define $\varphi \in \mathcal{M}(\Omega \to [0,1])$ by $\varphi(\omega) = d(E(\omega), E'(\omega))$. Applying lemma 11 for almost every $\omega$ we have
\[ \varphi(\sigma^n \omega) = d(E(\sigma^n \omega), E'(\sigma^n \omega)) = d(\mathcal{L}_{\omega}^{(n)}, E(\omega), \mathcal{L}_{\omega}^{(n)} E'(\omega)) < 4 \rho_{k+1}(\mathcal{L}_{\omega}^{(n)}) \to 0. \]
$\varphi$ tends to zero along all orbits. Therefore the sets $\{ \varphi(\omega) > \epsilon \}$ all have measure zero, whence $\varphi$ vanishes almost everywhere and $E = E'$. □

The following lemma provides the top fast space for a general quasicompact cocycle:

**Lemma 25**. Let $\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})$ be a quasicompact, semi-invertible random linear dynamical system. Then there exists a forward-equivariant decomposition $X = E(\omega) \oplus V(\omega)$, where $V : \Omega \to \mathbb{G}^k X$ and the corresponding projection is a strongly measurable $\Pi : \Omega \to B(X)$. $V$ is a slow growing space: $\lim_{n \to \infty} \frac{1}{n} \log \| \mathcal{L}_{\omega}^{(n)} |V(\omega)| \| \to \lambda_2$ almost surely. Finally, $\Pi_{\omega}$ is tempered.

**Proof.** By lemma 12 we may choose a measurable family of bases
\[ (b_i)_{i=1}^k : G_{m_1} X \to S^k. \]
Write $v_i(\omega) = b_i(E(\omega))$ which is itself then measurable. Let $\{(q_j)_{j=1}^k \}$ be dense in $\mathbb{R}^k$. Let $T_j \in \mathcal{M}(\Omega \times X \to X)$ be defined by
\[ T_j(\omega, x) = x - \sum_{i=1}^k q_j v_i(\omega), \]
so that for each $\omega \in \Omega$, the collection $\{T(\omega, \cdot)\}$ is dense in translations of $X$ by elements of $E(\omega)$. Let $\epsilon < \frac{1}{2} (\lambda_1 - \lambda_2)$. Define $t_n \in \mathcal{M}(X \setminus E(\omega) \to \mathbb{N})$ for $n \in \mathbb{N}$ by
\[ t_n(\omega, x) = \inf \{ j \in \mathbb{N} : \| \mathcal{L}_{\omega}^{(n)} T_j x \| \leq e^{\epsilon} \rho_{k+1}(\mathcal{L}_{\omega}^{(n)}) \| T_j x \| \}. \]
$\Pi^{(n)} \in \mathcal{M}(X \to X)$ may then be defined piecewise by
\[ \Pi^{(n)}(x) = \begin{cases} 0, & \text{if } x \in E(\omega), \\ T_n(\omega, x) \text{ otherwise,} \end{cases} \]
and
\[ P^{(n)}(x) = x - \Pi^{(n)}(x). \]
To see that $t$ is finite on $X \setminus E(\omega)$, first note that the set
\[ \{ T_j x : j \in \mathbb{N} \} \]
is dense in the set $S = E(\omega) + x$. The set
\[ U = \{ y \in E(\omega) \oplus \text{span} \{ x \} : \| \mathcal{L}_{\omega}^{(n)} y \| < e^{\frac{1}{2} \epsilon} \rho_{k+1}(\mathcal{L}_{\omega}^{(n)}) \| y \| \} \]
is open and scale invariant - for every $\theta \neq 0$ we have $\theta U = U$, so
\[ U' = (E(\omega) + x) \cap U \]
is open and nonempty in $S$, whence
\[ U' \cap \{ T_j x \} \neq \emptyset, \]
so that $t_n(\omega, x)$ may be found in finite time. To see that $t_n$ is measurable, note that
\[ \tau_n^{-1}(1, \cdots, m) = \bigcup_{i=1}^m \{ (\omega, x) : \| \mathcal{L}_{\omega}^{(n)} T_i x \| \leq e^{\epsilon} \rho_{k+1}(\mathcal{L}_{\omega}^{(n)}) \| T_i x \| \} \]
Let $x \in \mathbb{S}_X$. Immediately from the definition, $P^{(n)}_\omega x \in E(\omega)$. By convergence of the $\frac{1}{n} \log \rho_b$, for all $n$ greater than or equal to some $N_\omega$,

$$\|L^{(n+1)}_\omega \Pi^{(n)}_\omega (x)\| \leq e^{(n+1)(\lambda_2+\epsilon)} \|\Pi^{(n)}_\omega x\|,$$
$$\|L^{(n+1)}_\omega \Pi^{(n+1)}_\omega (x)\| \leq e^{(n+1)(\lambda_2+\epsilon)} \|\Pi^{(n)}_\omega x\|,$$
$$\|L^{(n+1)}_\omega \Pi^{(n+1)}_\omega (x)\| \leq e^{(n+1)(\lambda_2+\epsilon)} \|\Pi^{(n)}_\omega x\|,$$
$$\|L^{(n+1)}_\omega \Pi^{(n+1)}_\omega (x)\| \leq e^{(n+1)(\lambda_2+\epsilon)} \|\Pi^{(n)}_\omega x\|,$$
$$
\|L^{(n)}_\omega \Pi^{(n)}_\omega (x)\| \leq e^{n(\lambda_1+\epsilon)} + e^{n(\lambda_2+\epsilon)} \|\Pi^{(n)}_\omega x\|,
$$

which rearranged yields
$$\|\Pi^{(n)}_\omega (x)\| \leq \frac{1 + e^{2n_\epsilon}}{1 - e^{-n(\lambda_1-\lambda_2-2\epsilon)}} \leq e^{3n_\epsilon}.$$

Consider differences between successive approximative slow components:
$$\|\Pi^{(n+1)}_\omega (x) - \Pi^{(n)}_\omega (x)\| = \|P^{(n+1)}_\omega (x) - P^{(n)}_\omega (x)\|$$
$$\leq \frac{\|L^{(n+1)}_\omega (P^{(n+1)}_\omega (x) - P^{(n)}_\omega (x))\|}{\|L^{(n+1)}_\omega \Pi^{(n+1)}_\omega (x)\|}$$
$$\leq e^{-(n+1)(\lambda_1-\epsilon)} \left(\|L^{(n+1)}_\omega \Pi^{(n)}_\omega (x)\| + \|L^{(n+1)}_\omega \Pi^{(n+1)}_\omega (x)\|\right)$$
$$\leq e^{-(n+1)(\lambda_1-\epsilon)} \left(\|L^{(n+1)}_\omega \Pi^{(n)}_\omega (x)\| + e^\epsilon \|\Pi^{(n+1)}_\omega (x)\|\right)$$
$$\leq e^{-(n+1)(\lambda_1-\lambda_2-2\epsilon)} (e^{3n_\epsilon} + e^{3(\lambda_2+\epsilon)})$$
$$\leq e^{-(n+1)(\lambda_1-\lambda_2-7\epsilon)},$$

whence gaps between subsequent points decay exponentially, so that $(\Pi^{(n)}_\omega (x))_{n \in \mathbb{N}}$ forms a Cauchy and thus convergent sequence. Set
$$\Pi_\omega (x) = \lim_{n \to \infty} \Pi^{(n)}_\omega (x).$$

Note that $\Pi_\omega \mathbb{S}_X$ is then bounded, since
$$\|\Pi_\omega x\| = \lim_{n \to \infty} \|\Pi^{(n)}_\omega x\|$$
$$\leq \|\Pi^{(n)}_\omega x\| + \lim_{n \to \infty} \|\Pi^{(n)}_\omega x - \Pi^{(n)}_\omega (x)\|$$
$$\leq e^{3n_\epsilon} + e^{-N(\lambda_1-\lambda_2-2\epsilon)} \frac{1}{1 - e^{-N(\lambda_1-\lambda_2-2\epsilon)}},$$

the final line being independent of choice of $x$. Not only then is $\Pi^{(n)}_\omega x$ convergent, but we have the estimate
$$\|\Pi^{(n)}_\omega x - \Pi_\omega x\| \leq \sum_{i=1}^{\infty} \|\Pi^{(n+i)}_\omega x - \Pi^{(n+i-1)}_\omega x\| \leq e^{-(\lambda_1-\lambda_2-\epsilon)} \frac{1}{1 - e^{-(\lambda_1-\lambda_2-\epsilon)}}.$$

$\Pi_\omega$ is linear: to see this let $b, c \in X$ and $t \in \mathbb{R}$, and set
$$d := \Pi^{(n)}_\omega (b + tc) - \Pi^{(n)}_\omega b - t \Pi^{(n)}_\omega c.$$

Certainly $d \in E(\omega)$, since
$$d = \Pi^{(n)}_\omega (b + tc) - (b + tc) + b - \Pi^{(n)}_\omega b + tc - t \Pi^{(n)}_\omega c$$
$$= -P^{(n)}_\omega (b + tc) + P^{(n)}_\omega b + t P^{(n)}_\omega c \in E(\omega).$$
Then apply the estimate for $\Pi^{(n)}$

$$\|d\| \leq e^{-n(1-\epsilon)} \|L^{(n)}d\|$$

$$\leq e^{-n(1-\epsilon)/2} \left( \|L^{(n)}(\Pi^{(n)}(b + tc))\| + \|L^{(n)}(b)\| + \|L^{(n)}(c)\| \right)$$

$$< e^{-n(1-\epsilon)/2} \left( \|\Pi^{(n)}(b + tc)\| + \|\Pi^{(n)}b\| + \|\Pi^{(n)}c\| \right)$$

$$< e^{-n(1-\epsilon)/2} \left( \|\Pi^{(n)}(b + tc)\| + \|\Pi^{(n)}b\| + \|\Pi^{(n)}c\| + \epsilon \right)$$

$$\leq e^{-n(1-\epsilon)/2} \left( \frac{1}{1 - e^{-n(1-\epsilon)/2}} \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$ 

Thus $d = 0$ and $\Pi\omega \in B(X)$. By construction, $\Pi\omega E(\omega) = 0$. On the other hand, $P\omega X = E(\omega)$ since for any $x \in X$ and $n \in N$ we have $x - \Pi^{(n)}x \in E(\omega)$. $\Pi\omega$ is then idempotent, since for any $x \in X$

$$\Pi^2\omega x - \Pi\omega x = \Pi\omega \circ P\omega x \in \Pi\omega E(\omega) = \{0\},$$

and is thus a projection. Set $V(\omega) = \Pi\omega X$ so that $X = V(\omega) \oplus E(\omega)$. For all $n \geq N\omega$, and any $x \in V\omega(x)$, because of the exponential rate of convergence there is a sequence of approximants

$$\|x_n - x\| < C\omega e^{-n(1-\epsilon)}$$

with

$$\|L^{(n)}x_n\| \in [0, e^{n(1-\epsilon)}].$$

Therefore,

$$\|L^{(n)}x\| < \|L^{(n)}\| \|x - x_n\| + \|L^{(n)}x_n\| < e^{n(1+8\epsilon)},$$

whence

$$\|L^{(n)}\|_{V(\omega)} < e^{n(1+8\epsilon)}$$

for $n \geq N\omega$. By the definition of $\lambda_2$, it is possible to choose $Y_n \in \mathcal{G}_{m+1}(\omega)$ with $\frac{1}{n}g^{(n)}(Y_n) \rightarrow \lambda_2$, which means by dimension counting that there is always some $x_n \in S_{V(\omega)\cap Y} \omega$ with $\frac{1}{n}\|L^{(n)}x_n\| \rightarrow \lambda_2$ and so

$$\frac{1}{n}\log \|L^{(n)}\|_{V(\omega)} \rightarrow \lambda_2.$$ 

Further, the map $\omega \mapsto \Pi\omega$ is strongly measurable, since for each $x \in X$ the map $\omega \mapsto \Pi\omega x$ is the limit of a sequence of measurable functions.

To see that $V(\omega)$ is equivariant it is sufficient to show that $P\sigma\omega \circ L\omega x = 0$ for all $x \in V(\omega)$. If this were not the case, then $L\omega x$ would have a nonzero component in $E(\sigma\omega)$. From this it would follow that $\lambda_\omega(x) = \lambda_1$ which would contradict the fact that $\|L^{(n)}\|_{V(\omega)} < e^{n(1+8\epsilon)}$ for sufficiently large $n$.

As for temperedness of the projections: Since $\Pi\omega$ is bounded pointwise we may choose an $A \subseteq \Omega$ of positive measure on which $\|\Pi\omega\|$ is at most some $M > 0$. Then define a new cocycle $L'$ by

$$L'\omega = \begin{cases} L\omega \circ \Pi\omega & \text{if } \omega \in A, \\ L & \text{otherwise.} \end{cases}$$

Then $L'\omega$ is forward-integrable since

$$\int_{\Omega} \log^+ \|L'\omega\| \leq \int_{\Omega} \log^+ \|L\omega\| + \int_{A} \log^+ M < \infty.$$ 

Since $A$ has positive measure, there is almost surely an $n$ such that $L^{(n)}\omega E(\omega) = 0$, and so

$$L^{(n)}\omega X \subseteq V(\sigma^n\omega).$$

We may then conclude that for each $x \in X, \lambda'_\omega(x) \leq \lambda_2$ and $\lambda'_1 \leq \lambda_2$. Applying lemma 19 to the subadditive families $g_n = \log \|L^{(n)}\|$ and $g'_n = \log \|L^{(n)}\|$ there is an $N_1$ such that for $n \geq N_1$,

$$\|L^{(n)}\|_{E(\omega)} < e^{n(1+\epsilon)} \text{ and } \|L^{(n)}\|_{E(\omega)} > e^{n(1-\epsilon)}.$$ 

Clearly $L^{(n)}_{n \to 2n} \neq L^{(n)}_{n \to 2n}$ since the latter has a greater norm. Therefore there must be some $j$ such that

$$L^{(n)}_{n \to 2n} = L^{(n)}_{n \to j} \circ P_{\sigma^n\omega} \circ L^{(n)}_{j \to 2n} = L^{(n)}_{n \to 2n} \circ P_{\sigma^n\omega}.$$ 

In addition, there exists an $N_2$ such that for $n \geq N_2$,

$$g^{(n)}_{\sigma^n\omega}(E(\sigma^n\omega)) \in (e^{n(1-\epsilon)}, e^{n(1+\epsilon)}).$$
As a final condition, there exists some \( N_3 \in \mathbb{N} \) such that for all \( n \geq N_3 \), 
\[ g_{\omega}(E(\omega)) \in (e^{n(\lambda_1 - \epsilon)}, e^{n(\lambda_1 + \epsilon)}). \]

Let \( x \in \mathcal{S}_X \). Putting these together, for all \( n \geq \max\{N_1, N_2, N_3, \frac{1}{\epsilon}\} \),
\[
\|P_{\sigma^n x}\| \leq e^{-n(\lambda_1 - \epsilon)}\|L_{\omega}^{(n)} P_{\sigma^n x}\|
\]
\[
\leq e^{-n(\lambda_1 - \epsilon)} (\|L_{\omega}^{(n)} x\| + \|L_{\omega}^{(n)} \Pi_{x} x\|)
\]
\[
\leq e^{-n(\lambda_1 - \epsilon)} (e^{n(\lambda_1 + \epsilon)} ||x|| + \|L_{\omega}^{n-2} x\|)
\]
\[
\leq e^{-n(\lambda_1 - \epsilon)} (e^{n(\lambda_1 + \epsilon)} + \|L_{\omega}^{n-2} x\|)||x||
\]
\[
\leq e^{2n \epsilon} + e^{-n(\lambda_1 - \lambda_2 - 2\epsilon)} \leq e^{3n \epsilon}.
\]

\( \epsilon \) was arbitrary and the norms of \( \Pi \) and \( P \) differ by at most 1 so \( \frac{1}{n} \log \|P_{\sigma^n x}\|, \frac{1}{n} \log \|\Pi_{x} x\| \to 0 \) as required.

**Corollary 26.** \( V(\omega) = \tilde{V}(\omega) = \{x \in X : \lambda_\omega(x) \leq \lambda_2\} \).

**Proof.** \( \frac{1}{n} \log \|L_{\omega}^{(n)} V(\omega)\| \to 0 \) establishes the fact that \( V(\omega) \subseteq \tilde{V}(\omega) \). Conversely, any \( x \in \tilde{V}(\omega) \setminus V(\omega) \) would have \( P_{\omega}(x) \neq 0 \) so that \( \lambda_\omega(x) = \lambda_1 \), contradicting the definition of \( \tilde{V} \).

**6 Proof of main result**

Finally, we may conclude with the main result, a well behaved decomposition of the space acted on by a random linear dynamical system:

**Proof of theorem[3]**. The decomposition is obtained inductively. At each stage it is shown that if \( \lambda_{i+1} \) exists, there exists an equivariant decomposition
\[
X = E_{\leq 1}(\omega) \oplus V_{i+1}(\omega)
\]
with \( E_{\leq 1} \in \mathcal{S}M(\omega \to G_M, X) \) and boudned projections
\[
\Pi_{i+1} : X \to V_{i+1}(\omega) \text{ and } P_{\omega} : X \to E_{\leq 1}(\omega).
\]
The existence of the top fast space has already been established - here denote this \( E_{\leq 2}(\omega) \oplus V_{2}(\omega) \) with measurable projections \( \Pi_{\omega} \) and \( P_{\omega} \).

Suppose that the statement is true up to \( i = 1 \). If \( \lambda_i = \nu \) then we are done, so suppose otherwise - that there exists \( \lambda_{i+1} \geq \nu \). The projection \( \Pi_{\omega} \) is pointwise bounded, so that there exists some \( M > 0 \) such that \( A = \{\|\Pi_{\omega}\| < M\} \) has positive measure.

\[
\mathcal{L}_{\omega}^{(n)} = \mathcal{L}_{\omega}^{(n)} \circ \Pi_{\omega} = \Pi_{\sigma^n \omega} \circ \mathcal{L}_{\omega}^{(n)}.
\]
\[
\|\Pi_{x} \mathcal{L}_{\omega}^{(n)} \| < e^{n \epsilon}.
\]
\[
g_{\omega}(E(\omega)) > \|L_{\omega}^{(n)} V_{i+1}(\omega)\|.
\]

Therefore applying lemma[13] and [14] to \( \mathcal{L}_{\omega}^{(n)} = \mathcal{L}_{\omega}^{(n)} \circ \Pi_{\omega} \), for each \( k \) the following holds:
\[
\rho_{k+M_1}(\mathcal{L}_{\omega}^{(n)}) \leq \rho_{k}(\mathcal{L}_{\omega}^{(n)}) \leq 4\|\Pi_{x} \mathcal{L}_{\omega}^{(n)} \| \|\Pi_{\omega}\| \rho_{k+M_1}(\mathcal{L}_{\omega}^{(n)}) \leq 4e^{n \epsilon} \|\Pi_{\omega}\| \|\Pi_{k+M_1}(\mathcal{L}_{\omega}^{(n)})\|,
\]
whence \( \mu_{k+M_1} = \mu_{k} \) and \( \nu = \nu' \). Further then, \( m_{k} = m_{k+M_1} \) and \( \lambda + \lambda' = \lambda_{k+1} \). Set \( V_{i+1}(\omega) = V_{i}(\omega) \cap V'(\omega) \) and \( \Pi_{i+1} = \Pi_{\omega} \circ \Pi_{\omega}' \). Write \( E_{\leq i}(\omega) = E_{\leq i}(\omega) + E_{i}(\omega) \). The equality \( L_{\omega} = L_{\omega}' \) holds on \( V_{i}(\omega) \), whence \( \lambda_\omega = \lambda_\omega \) on \( V_{i}(\omega) \). Since \( A \) has positive measure, there is almost surely an \( N \) such that for all \( n \geq N \), \( L_{\omega}^{(n)} E_{\leq i}(\omega) = 0 \), \( L_{\omega}^{(n)} X \subseteq V_{i}(\omega) \), and \( L_{\omega}' X \subseteq V_{i}(\omega) \). We may then conclude that for each \( x \in X \), \( \lambda_\omega(x) \leq \lambda_i \) and \( \lambda_i \leq \lambda \). As such, for each \( i < l \), have \( E_{i}(\omega) \subseteq V'(\omega) \). As for \( E_{l}(\omega) \), by equivariance \( E_{l}(\omega) = L_{\omega} \mathcal{L}_{\sigma^{-n} \omega} E(\sigma^{-n} \omega) \subseteq V_{l}(\omega) \). Thus \( X = E_{\leq l}(\omega) \oplus E_{l}(\omega) \oplus (V_{l}(\omega) \cap V'(\omega)) = E_{\leq l}(\omega) \oplus E_{l}(\omega) \oplus V_{i+1}(\omega) \).
\(\Pi_{i+1}\) is then also tempered:
\[
0 \leq \frac{1}{n} \log \|\Pi_{i+1} \sigma_{n}\omega\| \leq \frac{1}{n} \log \|\Pi_{i} \sigma_{n}\omega\| + \frac{1}{n} \log \|\Pi_{i} \sigma_{n}\omega\| \to 0.
\]

Let \(\epsilon > 0\). There exists an \(N\) such that for \(n \geq N\), \(\|P_{n} \sigma_{n}\omega\|, \|\Pi_{i} \sigma_{n}\omega\| < e^{-\epsilon n}\) and so for all \(x \in X\) we have \(\max\{\|P_{n} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|, \|\Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|\} \leq \|\mathcal{L}_{n}^{(n)} x\|\).

Rearranging this last inequality and letting \(x \in E_{\leq 1}\),
\[
\|\mathcal{L}_{n}^{(n)} x\| \geq e^{-\epsilon \max\{\|P_{n} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|, \|\Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|\}}
\]
\[
\geq e^{-\epsilon \max\{\|\mathcal{L}_{n}^{(n)} \Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|, \|\mathcal{L}_{n}^{(n)} \Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|\}}
\]
\[
\geq e^{-\epsilon \max\{e^{(\lambda_{n})^{-1}} \|P_{n} \mathcal{L}_{n}^{(n)} \Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|, e^{(\lambda_{n})^{-1}} \|\Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|\}}
\]
\[
\geq e^{-\epsilon} e^{(\lambda_{n})^{-1}} \max\{\|\Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|, \|P_{n} \mathcal{L}_{n}^{(n)} \Pi_{i} \sigma_{n}\omega \mathcal{L}_{n}^{(n)} x\|\} \geq \frac{1}{e^{\epsilon}} e^{(\lambda_{n})^{-2} \epsilon}.
\]

For \(n \geq \max\{N, \frac{1}{\epsilon} \log 2\}\) it follows that \(g_{\mathcal{L}_{n}^{(n)}}(E_{\leq 1}(\omega)) \geq e^{(\lambda_{n})^{-3} \epsilon}\). The characterisation
\[
V_{i+1}(\omega) = \{x \in X : \limsup_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}_{n}^{(n)} x\| \leq \lambda_{i+1}\}
\]
holds. \(\square\)

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Appendix: Equivalence of growth statistics

The Gelfand numbers may be defined by \( s_k(T) = \inf_{V \in \mathcal{G}^{k-1}X} \|T|_V\| \). Throughout this section we assume \( T \in \mathcal{B}(X) \).

**Lemma 27.** \( \rho_k \leq s_k \).

**Proof.** Let \( \epsilon > 0 \). Choose \( V \in \mathcal{G}_kX \) with \( g_T(V) \geq \rho_k(T) - \epsilon \). Then since any \( W \in \mathcal{G}^{k-1}X \) intersects \( V \) nontrivially, suppose \( x \) is a unit vector in the intersection. Then \( \rho_k(T) \leq \|Tx\| \leq \|Tw\| \). \( W \) was arbitrary so \( \rho_k(T) \leq s_k(T) \).

A better bound than the following may be found in the work of Pietsch on \( s \)-numbers \([12]\).

**Lemma 28.** For all \( k, s_k \leq 4^{-k-1}((k-1)!)^{\frac{1}{2}} \rho_k \).

**Proof.** For \( k = 1 \) each quantity is just the norm of \( T \), so equality holds. Write \( c_i = 4^{1-i}((i-1)!)^{-\frac{1}{2}} \). Suppose that the proposition holds for each \( l < k \) with \( k \geq 2 \). Choose \( U \in \mathcal{G}_{k-1}X \) with \( g_T(U) \geq (1 - \epsilon) \rho_{k-1}(T) \), so that in particular

\[
g_T(U) \geq (1 - \epsilon)c_{k-1}s_{k-1}(T) \geq (1 - \epsilon)c_{k-1}s_k(T).
\]

Choose a complement \( TU \oplus V = X \) with \( \Pi = \Pi_{TU}V \) such that \( \|\Pi\| \leq (k-1)^{\frac{1}{2}} + \epsilon \). Since \( T^{-1}V \in \mathcal{G}^{k-1}X \), we may choose \( x \in S_{T^{-1}V} \) with \( \|Tx\| \geq s_k(T) - \epsilon \). Set \( W = U \oplus \text{span}\{x\} \in \mathcal{G}_kX \) and let \( a = b + tx \in SW \). Applying \( T \),

\[
\|Ta\| \geq \max\{\frac{\|Tb\|}{\|\Pi\|}, \|tT\|\frac{1}{\|\Pi\|}\}
\geq (1 + \sqrt{k-1} + \epsilon)^{-1}\max\{(1 - \epsilon)c_{k-1}s_k(T)\|b\|, s_k(T)(1 - \epsilon)\|t\|\}
\geq (2\sqrt{k-1} + \epsilon)^{-1}(1 - \epsilon)c_{k-1}\max\{\|b\|, \|t\|\}
\geq \frac{1 - \epsilon}{4(\sqrt{k-1} + \epsilon)}s_k(T).
\]

\( a \) was arbitrary so the final line is a lower bound for \( g_T(W) \). \( \epsilon \) was also arbitrary, so that \( \rho_k(T) \geq \frac{1}{4c_{k-1}\sqrt{k-1}}s_k(T) = c_k s_k(T) \).

**Lemma 29.** The Gelfand numbers satisfy

\[ s_k(T) \geq k^{\frac{1}{2}}\|T\|_e. \]

**Proof.** Let \( V \in \mathcal{G}^{k-1}X \). We may by the theorem of Kadets \([17]\) choose a complement \( V \oplus W = X \) with \( \|\Pi_VW\| \leq \sqrt{k-1} + \epsilon \). Then \( T\Pi_VW \) is finite rank, so \( \|T\|_e \leq \|T - T\Pi_VW\| \leq \|T\Pi_VW\| \leq \|T\Pi_VW\| \leq \|T\Pi_VW\| \leq \|T\|_e \). Taking the inf over such \( V \) and letting \( \epsilon \to 0 \) we obtain the bound.

**Lemma 30.** The index defined at the start of this article agrees with the usual index of compactness

\[ \lim_{n \to \infty} \mu_n = \nu = \kappa = \lim_{n \to \infty} \frac{1}{n} \log \|L^{(n)}\|_e. \]

**Proof.** Since \( \rho \) dominates the compactness seminorm up to a multiplicative constant, \( \kappa \leq \mu_k \) for every \( k \in \mathbb{N} \), so certainly \( \kappa \leq \nu \). It remains to verify that \( \kappa \geq \nu \). There exists a \( \delta > 0 \) such that for all \( P(A) < \delta \), \( \int_A \log \|\mathcal{L}\| < \frac{1}{2} \). Choose \( N \) sufficiently large that

\[ P(\|L^{(N)}\|_e \geq e^{N(\kappa + \epsilon)}) < \frac{1}{2} \delta. \]

Choose \( r \) sufficiently large that

\[ P(G) = P(L^{(N)}) X \] may be covered by at most \( e^{rn} e^{N(\kappa + \frac{1}{2} + \epsilon)} \)-balls.

Set \( f_n(\omega) = \log \log \{t > 0 : L^{(N)} X \} \). In this case

\[ f_{m+n}(\omega) \leq \log \log \{a : L^{(m)} X \} \] is covered by \( e^{rn} a \)-balls, and \( L^{(m)} X \) by \( e^{rn} b \)-balls

so that the family is subadditive, \( \int_G f \leq \int_G \log \|L^{(N)}\|_e < \kappa + \frac{1}{2} \epsilon \). Thus we may apply Kingman again to obtain that \( \frac{1}{2} f_n(\omega) \to C \leq \kappa + \epsilon \). Almost surely, for sufficiently large \( n \) we may guarantee the following:
• $\|L^{(n)}\|_c < e^{n(k+\epsilon)}$

• There is a $V \in G_k X$ with $g_{c_{\omega}}(V) > e^{n(\mu_k - \epsilon)}$.

• $L^{(n)}B_X$ is covered by at most $e^{rn}$ $e^{n(k+\epsilon)}$-balls

• $e^{rn} > (2k)^k$.

Choose a basis of unit vectors $x_i$ for $V$ with $d(x_i, \text{span}\{x_j\}_{j<i}) = 1$. Write

$$\Lambda = \left\{ \sum_{i=1}^k a_i x_i : a_i \in \{0, \pm 2e^{n(k+\epsilon)} \|L^{(n)}\|_{c_{\omega}}/\|L^{(n)}\|_{c_{\omega}}, \ldots \}, |a_i| < \frac{1}{2} \right\} \subseteq B_X$$

If two members $a = \sum_{i} a_i x_i$ and $b = \sum_{i} b_i x_i$ are distinct then there is a maximal $j \leq k$ with $a_j \neq b_j$. Then

$$\|a - b\| \geq |a_j - b_j| d(L^{(n)}x_j, L^{(n)} \text{span}\{x_i : i \leq j\}) > e^{n(k\epsilon - \epsilon)} > \|L^{(n)}\|_c.$$

The points in $L^{(n)} \Lambda$ are then of distance at least $2e^{n(k+\epsilon)} > \|L^{(n)}\|_c$ apart, and there are at least

$$|\Lambda| = \prod_{i=1}^k \frac{1}{2} \left[ \frac{e^{n(k+\epsilon)}}{\|L^{(n)}\|_{c_{\omega}} \|L^{(n)}\|_{c_{\omega}}} \right] \geq (2k)^{-k} \prod_{i=1}^k e^{n(\mu_k - \epsilon - (n+k+\epsilon))} = e^{nk(\mu_k - n - 3\epsilon)}$$

of them. Each member of a cover of $L^{(n)}B_X$ by $e^{n(k+\epsilon)}$-balls contains at most one element of $\Lambda$, so the cover has cardinality at least $e^{nk(\mu_k - n - 3\epsilon)}$. On the other hand, this quantity is bounded by $e^{rn}$;

$$e^{rn} \geq e^{nk(\mu_k - n - 3\epsilon)}.$$

Taking log and rearranging we obtain

$$k \leq \frac{r}{\mu_k - n - 4\epsilon}.$$ 

In the case that $\mu_k > n + 5\epsilon$, it must hold that $k < \frac{r}{\mu_k - n - 4\epsilon}$. This bound on $k$ shows the number of $\mu$s greater than $n + 5\epsilon$ is finite, whence $\mu_k \downarrow n = \nu$ as $k \to \infty$. \qed