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On the Transition from the Quantum to the Classical Regime for Massive Scalar Particles: A Spatiotemporal Approach

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Abstract

If the classical structure of space-time is assumed to define an a-priori scenario for the formulation of the structure of quantum theory (QT), the coordinate representation of the solutions \( \psi(\vec{x}, t) \) of the Schroedinger equation of a quantum system containing one \((N)\) massive scalar particle has a preferred status. It is then possible to perform a multipolar expansion of the density matrix \( \rho(\vec{x}, t) = |\psi(\vec{x}, t)|^2 \) (and more generally of the Wigner function) around a space-time trajectory \( \vec{x}_c(t) \) to be properly selected. A special set of solutions \( \psi_{EMWF}(\vec{x}, t) \), named Ehrenfest monopole wave functions (EMWF), is characterized by the conditions that: (i) the quantum expectation value of the position operator coincides at any time with the searched classical trajectory, \( \langle \psi_{EMWF}|\hat{\vec{x}}|\psi_{EMWF} \rangle = \vec{x}_c(t) \), and, (ii) Ehrenfest’s theorem holds for the expectation values of the position and momentum operators. The first condition implies the vanishing of the ‘dipole’ term in the multipolar expansion of the density matrix with respect to such trajectory. Ehrenfest’s theorem applied to EMWF leads then to a closed Newton equation of motion for the classical trajectory, where the effective force is the Newton force plus non-Newtonian terms (of order \( \hbar^2 \) or higher) depending on the higher multipoles of the probability distribution \( \rho \). Note that the super-position of two EMWFs is not an EMWF, a result to be strongly hoped for, given the possible unwanted implications concerning classical spatial perception. These results can be extended to \( N \) particle systems and to relativistic quantum mechanics.

Consequently, for the states of a quantum particle which are EMWF, we get the emergence of a corresponding classical ‘effective’ particle following a Newton-like trajectory in space-time. Note that this holds true in the standard framework of quantum mechanics, i.e. by assuming the validity of Born’s rule and the individual system interpretation of the wave function (no ensemble interpretation). These results are valid without any approximation (like \( \hbar \to 0 \), big quantum numbers,...). Moreover, we do not commit ourselves to any ontological interpretation of quantum theory (such as, e.g., the Bohmian one. It will be clear that our trajectories are not Bohm’s trajectories). We will argue that, in substantial agreement with Bohr’s viewpoint, the macroscopic description of the preparation, certain intermediate steps and the detection of the final outcome of experiments involving massive particles are dominated by these ‘classical effective trajectories’.

This approach can be applied to the point of view of de-coherence (in which positions turn out to be selected preferred robust bases) in the case of a diagonal reduced density matrix \( \rho_{\text{red}} \) (an improper mixture) depending on the position variables of a massive particle and of a pointer. When both the particle and the pointer wave functions appearing in \( \rho_{\text{red}} \) are EMWF, the expectation value of the particle and pointer position variables becomes a statistical average on a classical ensemble. In these cases an improper quantum mixture becomes a classical statistical one, thus providing an answer to an open problem of de-coherence about the emergence of classicality.

Our results cast some light on the so-called problem of the classical regime and also provide support to Bohr’s point of view, without adhering to any underlying ontology. Finally, we add some comments on the possible implications of our results for the theory of measurement, emphasizing the relevance of the many-body problem in the description of the macroscopic measuring apparatuses and the fact that the quantum system under investigation should be realistically considered as an open quantum subsystem which does not follow a unitary evolution. With this in view, the ‘collapse of the wave function’ with the eigenvalue-eigenvector link and the standard break of unitarity in the measurement processes would be originated by an idealized instantaneous approximation of a process localized in neither space nor time.

The main open point is whether this description of the emergence of classicality can be extended to experiments with photons, many of which are in the realm of quantum optics but are described
at the macroscopic level in terms of *effective* light-rays of geometrical optics interpreted as *effective* photon trajectories. Finally, our approach should also be extended to massive particles with spin.
I. INTRODUCTION

In the literature of the last twenty years important new developments in different domains of the physical research have been widely accepted as casting some light on the quantum-to-classical transition. More precisely, they configured the so-called problem of the classical regime, i.e. is the question of whether and how the sweeping success of the classical physical description (in particular on the macroscopic scale) can be explained in quantum mechanical terms. There is also agreement on the fact that, even if the problem of the classical regime is relatively independent of the traditional measurement problem, it is equally important in assessing the empirical adequacy of quantum theory (QT) and its interpretations. See Refs.[1–7] as a sample of extensive bibliography on these subjects and the associated issues of the uniqueness of outcomes, of Born’s probability rule, of the acceptance or non-acceptance of the collapse of the wave function as a real process (in connection with the interpretation of the wave function itself) and of the eigenvalue-eigenvector link. In this vast literature, however, the discussion is often characterized by imprecise, purely verbal statements which do not help to characterize the conceptual heart of the problem discussed. Finally, we would like to stress that the essential epistemological difference between Bohr’s traditional viewpoint and the contemporary emphasis on the problem of the classical regime lies in the fact that asserting the latter is tantamount to refusing to take for granted the classical appearances of our macroscopic world. In particular, within the so-called de-coherence approach.

As for the classical limit of quantum theory, at present we have only non conclusive statements about either the $\hbar \to 0$ limit (like the WKB approximation) or the limit of large quantum numbers. Also, Ehrenfest’s standard theorem is essentially a statement about the expectation value of the position and momentum operators in a given quantum state: although they show a quasi-classical behavior, the interpretation of the theorem is still unclear.

Non-relativistic quantum mechanics (QM) is used nowadays in low energy experiments in atomic and molecular physics by taking into account the electro-magnetic field at the order $1/c$ (note that the Galilei group cannot be consequently implemented). If one wishes to take into account relativistic kinematics (i.e., the Poincaré group) one must use relativistic QT (RQT) as an approximation to relativistic quantum field theory (RQFT) valid below the threshold of particle production. On the other hand, in particle and nuclear physics one needs the Fock space of RQFT and its particle interpretation. This framework is unavoidable when one deals with photons: in many cases one can exploit at best a quasi-classical eikonal interpretation with wave fronts replaced by congruences of light rays (interpreted as trajectories of unspecified ‘photons’) or with Gaussian wave-packets of classical light. However, there is no analogue of Ehrenfest’s theorem for the expectation values of the quantum electro-magnetic field between suitable (coherent?) states.

In any actual experiment there is a unique random outcome without quantum superposition of macroscopically distinguishable states and with an outcome probability distribution consistent with the Born rule. The only experimentally relevant quantity is in any case the density matrix associated with the wave function and its standard probabilistic interpretation.

It is important to remark that all of the experiments involving massive particles are planned and interpreted in terms of ‘effective’ particles (see later) localized up to a small
probability cloud and adopting a Newtonian (relativistic when needed) quasi-classical intu-
ition about their motion in space-time. Actually, in the preparation of any actual experiment
(particle physics at CERN, neutron interferometry, atom interferometry, atoms in a reso-
nant cavity,...) one produces a beam of effective particles following a well defined mean trajecto-
y leading to the experimental area, with a well-defined mean value of energy and momentum
(usually defined by the time-of-flight method). In a scattering process one deals
with incoming beams of this type and identifies outgoing beams of other effective particles
with effective trajectories and 4-momentum. In the outcomes of the double slit experiment
one detects macroscopic traces of effective particles on the screen. In atom interferometers
the wave function beyond the beam splitter must be thought of as approximately localizable
in the form of two beams of effective atoms in the two arms, notwithstanding the existing
interference effects. In a radioactive decay an effective nucleus localized in a cavity decays
and the decay products are detectable effective particles. The same happens in the decay of
muons in the atmosphere and in the decay of mesons in EPR experiments. In quantum opt-
ics an effective atom in a cavity emits or absorbs photons by interacting with lasers beams,
while in an atomic fountain a beam of effective atoms feels gravity and interacts with laser
beams. And so on.

It therefore seems that most of the realizable experiments admit a quasi-classical de-
scription in terms of effective massive particles with a mean trajectory and a mean value
of 4-momentum, i.e. effective entities approximately localized in space-time. On the other
hand, in experiments like the double slit, in which interference effects dominate, the de-
scription of the propagation of the massive quantum particles from the slits to the screen is
more appropriately provided by using a ‘wave’ interpretation. This is the starting point of
the pilot wave description [8, 9] (even considered independently of the possible addition of
hidden extra variables describing, e.g., the positions of added ‘real’ particles as in Bohmian
ontological interpretation). Like in the eikonal approximation of classical electro-magnetic
waves, the wave function is parametrized as a modulus (the square root of the density ma-
trix) and a phase (an action variable) while the Schroedinger equation is replaced by two
coupled equations. Then one looks for the effective rays (the world-line of fluid elements
named Bohm trajectories) propagating from the slits to the screen. Avoiding the Bohmian
ontology and exploiting the Feynman path integral representation of the wave function, such
effective rays could be interpreted as effective particles and viewed as an emerging feature
of the quantum reality.

Finally, we should ask ourselves whether there could exist some experimental situations
in which neither the ‘particle’ nor the ‘wave’ aspect dominates the physical description:
could they be detectable with the existing experimental technology?

This state of affairs is well reflected in the following passage of von Weizsäcker.

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1 It is well-known that they are the guidance equation for the velocity field of the tangents to the rays
(implying an equation similar to the Euler equation for a fluid) and an equation of the Hamilton-Jacobi
type for classical massive particles.

2 Let us note that ‘particle’ and ‘wave’ seem to be the only possible spatiotemporal formats to represent
a physical entity. In Ref.[10] a heterodox proposal is advanced concerning the unavoidable epistemic
primacy of this spatiotemporal intuition.
"We ought not to say 'Every experiment that is even possible must be classically described, i.e. localized in space-time', but 'Every actual experiment known to us is classically described in this way, and we do not know how to proceed otherwise' " (p.128 of Ref.[11]).

This citation reflects Bohr’s legacy very well (see Refs.[1, 2]), namely the need for some classical distinction between the quantum system and the apparatus needed by the praxis of experimentalists. While the measuring instrument may of course be described as a quantum-mechanical system, it is only possible to explain its capacity to function as a measure instrument on the assumption that the resulting traces of the interaction between object and instrument can be described in terms of an exchange of energy and momentum in space and time (i.e. action). Actually Bohr stressed:

"The unambiguous interpretation of any measurement must be essentially framed in terms of classical physical theories, and we may say that, in this sense, the language of Newton and Maxwell will remain the language of physics for all time (p.692 of Ref.[12]). Furthermore:

"It is decisive to recognize that, however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms. The argument is simply that by the word 'experiment' we refer to a situation where we can tell others what we have done and learned and that, therefore, the account of the experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics (p. 209 of Ref.[13])."

It is well-known that Bohr regarded QT as the universally correct theory, which would in principle - i.e. given an appropriate experimental arrangement - imply its application to the description of macroscopic measurement apparatuses and observers too. However, Bohr felt that the experimental setup must be described in terms of classical physics if it is to serve as a measuring instrument at all, and that there should be a form of separability between the observed system and the apparatus (the observer). One is then left with the idea that a shifty quantum-classical divide (the so-called Heisenberg cut) is a necessary part of the epistemological structure of QT.

To our knowledge, there has so far been no satisfactory theoretical support to the description of existing experiments in terms of effective quasi-classical particles propagating in space-time. Moreover, this simulation seems to be independent of the choice of any solution of the measurement problem and also of the presence of the environment with its role in de-coherence.

In this paper we intend to throw some additional light on the phenomenology of quantum experimental situations with its apparent dominance of what we have called quasi-classical effective massive particles. More precisely, the term effective intends to stress the nature of the classical structure emerging from the quantum level by means of a mathematical manipulation which allows to recover a Newton-like description of such entities. Here we will not take either the limit $\hbar \to 0$, like for instance in the WKB approximation, or the limit of large quantum numbers, or the $N \to \infty$ limit for multi-particles states.

Lets add a few final remarks on the issue of the so-called collapse of the wave function, even if this does not belong to our main concerns. Actually, a matter-of-fact question about the collapse exists only for an 'ontological' interpretation of the wave function. In that case, however, the very notion of collapse should be replaced by a dynamic explanation of what
goes on during a measurement process, as is the case, e.g., in the GRW theory [7] and other attempts to solve the measurement problem (by means of a unitary, but non-linear, time evolution). If one shares - as is our case - a weak instrumentalist viewpoint about QT, more or less a la Bohr [14], and considers the wave function just as a 'catalogue of probabilities' (as Schroedinger put it), the issue of collapse becomes essentially a question of words. Namely, how to describe the information change following the fact that after a measurement, the 'catalogue of probabilities' must be redefined. In other words, the physical amplification process that leads to recording, which is always highly complicated and irreversible, is such that the 'closure' of a measurement process in QT, though not yet understood, does not imply any practical difficulty. Note, furthermore, that at the experimental level there is no clear, definitive evidence of the validity of the eigenvector-eigenvalue link. In conclusion our approach has nothing to say on the traditional foundational issues of QT, including the unicity of the outcomes. We shall only offer some suggestions about the relevance of the effective Newtonian trajectories in identifying the areas of the detectors where the amplification process, leading to a final pointer position as a result of a measurement, is randomly localized.

A collateral discourse should be made concerning the current views about de-coherence in which the measurement process is approximately described by a unitary transformation involving the quantum system to be measured, the measuring apparatus and the environment. In this way, one gets a 'dynamic' explanation of a sort, of the measuring process in which a collapse phase is not made explicit and interference tends to be suppressed. Of course, this last viewpoint is not the right one to be exploited in the cases in which quantum interference effects represent just what is to be made explicit in a measurement. See, e.g., the insightful analysis of this point in Refs.[1] (see Sect. 3.3). Our contribution to the viewpoint of de-coherence will concern only an instantiation of the important transition from genuinely non-classical quantum improper mixtures (mixing 'ignorance and entanglement') to classical statistical mixture (ignorance only), a transformation that becomes almost automatic with our method, at least in special cases.

This said, we will start from the following assumptions:

1) the wave function describing a quantum system and satisfying the standard time-dependent Schroedinger equation encodes all the probabilistic properties of the given individual system (no ensemble interpretation);

2) space-time structure is presupposed, independently of and before the technical formulation of QT and its interpretations, even if its symbolic structure goes - so speak - beyond space-time. The inertial frames of Galilei and Minkowski space-times define a scenario for describing matter (here we are not considering fields but particles only) whose dynamics satisfies either Newtonian and relativistic classical mechanics (NCM and RCM), or non-relativistic QT and RQT with the transformation rules connecting inertial frames governed by the Galilei and Poincaré groups, respectively. We shall mainly discuss the standard non-relativistic case in inertial frames of Galilei space-time (Newtonian gravity can be present) and we will extend to the inertial frames of Minkowski space-time subsequently (but without relativistic gravity). In Ref.[15] there is a review of the status of understanding of the theory of relativistic massive either classical or quantum particles and what is known about extending the results to non-inertial frames (see also Refs.[16, 17]).
Starting with this space-time oriented point of view, a deterministic point-like classical particle in an inertial frame of Galilei space-time is described in QT by a wave function solution of the Schroedinger equation with a given quantum Hamiltonian and having the full Euclidean 3-space \( \Sigma_t \) at time \( t \) as background. It is accordingly assumed that the coordinate representation has a privileged kinematical and descriptive status among all the possible bases in Hilbert space.

Given the Schroedinger equation and the space of its solutions, we exploit Ehrenfest’s theorem [18], which states that the equation of motion for the expectation value of the position and momentum operators have some resemblance with Newton-Hamilton equations of motion. If Hilbert space is a suitable function space, the recent mathematical developments of Ref.[19] allow to prove Ehrenfest’s theorem for a system of N particles with mutual Coulomb interaction (i.e. for the main system behind molecular physics). Moreover, both the wave function and the density matrix of a quantum system have the whole 3-space at a fixed time as a support, even when they are strongly peaked like the Gaussian wave packets with minimal position and momentum indeterminacy. Furthermore, we note that even if, in general, Ehrenfest expectation values for well peaked wave functions have non-classical behavior on long times, it has been found that in many cases the spread of wave packets stops and leads to a revival of the original quasi-classical behavior [20]. Finally, let us stress that the crucial obstacle for exploiting Ehrenfest’s theorem is due to the fact that the time-dependent mean values of position operators are not trajectories of classical Newton (or relativistic) particles [4].

The technical input to face the problem of the classical regime comes from the remark that in classical physics (matter or fluids interacting with the electro-magnetic and/or gravitational field) nobody is able to describe extended objects: actually they are always described by making a multi-polar expansion of their energy-momentum tensor (when known) around a classical world-line together with postulated equations of motion for the multi-poles. See Ref.[21] for a review given by one of the fathers of this method and for the analyticity properties required for the validity of the multi-polar expansion. See also Ref.[22] for applications in special relativity.

We will accordingly study the multipolar expansion around a given space-time trajectory of the density matrix \( \rho(\vec{x},t) = \psi^*(\vec{x},t) \psi(\vec{x},t) \) (and of other bilinears like \(-\psi^*(\vec{x},t) \hbar \frac{d}{dx} \psi(\vec{x},t)\), needed for the expectation value of the momentum).

We then search for a special set of solutions \( \psi_{EMWF}(\vec{x},t) \) of the Schroedinger equation for the quantum system with a given Hamiltonian, to be named Ehrenfest monopole wave functions (EMWF), having the following properties:

a) The quantum expectation value of the position operator of the massive particle, at any time, coincides with the searched classical space-time trajectory, \( < \psi_{EMWF} | \vec{x} | \psi_{EMWF} > = \vec{x}_c(t) \), (the ’monopole’ as a classical point-like effective massive particle). This implies the vanishing of the ’dipole’ of the density matrix with respect to the trajectory. Note that the classical ’coherent states’ satisfy the dipole condition.

b) Due to the first half of Ehrenfest equations, the expectation value of the momentum operator gives the classical momentum.

c) The second half of Ehrenfest equations provides a deterministic Newtonian equation of motion for the space-time trajectory, where the effective force is the Newton force plus non-
Newtonian terms (of order $\hbar^2$ or higher) depending upon higher multi-poles of the probability distribution $\rho$ which characterize the probabilistic properties of the quantum entity.

d) The emergent classical space-time trajectories will clearly depend on the chosen EMWF of the given quantum system. Different EMWFs, e.g. eigenstates of non-commuting operators, will have different associated classical trajectories reflecting the properties of the observables associated to such operators. If, in particular, two EMWFs are eigenfunctions corresponding to different eigenvalues of the same operator so that their associated classical trajectories carry information about the properties of the observables of such operator, all of the superpositions of such wave functions which are also EMWFs will still have classical trajectories unrelated to these properties. They will describe, so to speak, different and non-intrinsic or state dependent, properties. One could even say that our emergent Newtonian world is contextual. Finally note that, as is highly desirable, the notion of superposition of classical trajectories is structurally meaningless.

Finally, the multipolar expansion of the associated Wigner function

$$W(\vec{x}, \vec{p}, t) = \int d^3\xi e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{\xi}} \psi^*(\vec{x} - \frac{1}{2} \vec{\xi}, t) \psi(\vec{x} + \frac{1}{2} \vec{\xi}, t)$$

allows to study the expectation value of any operator.

It is important to stress that our method turns out to be applicable to N particle systems too: the density matrix associated to a wave function $\psi(\vec{x}_1, ..., \vec{x}_N, t)$, living at each instant $t$ in a 3N-dimensional configuration space, gives rise to a set of N effective trajectories inside the Euclidean 3-space when all of the dipolar moments of its multiple multipolar expansion vanish around these trajectories. This is a natural solution of a long standing interpretational problem of the N-particle wave functions.

Unlike all other non-relativistic treatments which have problems with their relativistic extension, our approach to the problem of classical regime works exactly in the same way within the new consistent RQM developed in Ref.[16]. The main feature is a new kind of non-locality\(^3\) and spatial non-separability of the description of N-particle system, induced by the Lorentz signature of space-time. It requires a clock synchronization convention for the definition of instantaneous 3-spaces (reduction of the problem of relative times among the particles), and entails non-local definitions of the collective variables (as is well known, there is no unique notion of center of mass in relativistic theory). Consequently, a consistent description of the N particles can be obtained only in terms of relative 3-coordinates (spatial non-separability of subsystems) and entails - unlike the Galilean case - a dependence of the Lorentz boosts upon the inter-particle interactions. Then, the Hilbert space of a relativistic composite system is not the tensor product of the Hilbert spaces of the subsystems. It can only be defined as the tensor product of the Hilbert space of the decoupled non-covariant (non-local, i.e. non measurable) center of mass times the Hilbert space of relative motions.

Another relevant feature of our approach is its applicability within the framework of decoherence. Namely, given a reduced density matrix \textit{(improper quantum mixture based on both ignorance and entanglement)} in the selected preferred robust positional basis for both a massive particle and the pointer of a macroscopic apparatus, we shall show that, if all the particle and pointer wave functions appearing in this description are EMWF, the \textit{improper}
quantum mixture can be replaced by a classical statistical proper mixture based only on our ignorance about the localization of the particle and the pointer effective trajectories in the region allowed by the experimental apparatus. This should contribute to a partial clarification of one of the main open problems of de-coherence.

Some concluding remarks are advanced on experiments involving photons instead of massive particles by taking into account recent weak measurements of the average trajectory of single photons in the double slit experiment [23]. The adaptation of our approach to the non trivial cases of Pauli and Dirac spinors will be dealt with in a separate paper.

While Section II is dedicated to the main technical results, in Section III we give the multipolar expansion of the Wigner function. The results are extended to two particle systems in Section IV.

In Section V we study relativistic quantum particles in the inertial rest frame.

Section VI is devoted to a comparison of experiments in which the ‘particle’ aspects of the wave function dominate, versus cases in which the ‘wave’ aspects dominate.

In Section VII we review the support given by de-coherence to the preferred status of position measurements and we show how EMWFs suggest a natural transition from quantum improper mixtures to classical statistical ones.

A concluding Section contains some general remarks on the problem of classical regime, its extension to the relativistic level and the potentialities of this approach for foundational problems of QM.

In Appendix A we review the existing mathematical problems concerning spatial localization.
II. THE MONOPOLE EHRENFEST WAVE FUNCTIONS

Let us consider a quantum point-particle in non-relativistic QT whose normalized wave function \( \psi(\vec{x}, t) = < \vec{x} | \psi(t) > \in L^2(R^3) \) is a solution of the Schroedinger equation \( i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) = H \psi(\vec{x}, t) \) with classical Hamiltonian \( H = \frac{\vec{p}^2}{2m} + V(\vec{x}) \). We work in the coordinate representation, where \( \hat{x} = \vec{x} \) and \( \hat{p} = -i\hbar \frac{\partial}{\partial x} \) and where the density matrix operator \( \hat{\rho}(t) = |\psi(t)\rangle < \psi(t)| \) takes the following form (the second line is implied by the Schroedinger equation)

\[
\rho_{(o)}(\vec{x}, t) \equiv < \vec{x} | \hat{\rho}(t) | \vec{x} > = |\psi(\vec{x}, t)|^2, \quad \int d^3x \rho_{(o)}(\vec{x}, t) = 1,
\]

\[
\partial_t \rho_{(o)}(\vec{x}, t) = \vec{\partial} \cdot \left( \psi^* \frac{\hat{p}}{2m} \psi - \frac{\hat{p}}{2m} \psi^* \psi \right)(\vec{x}, t) \equiv - \frac{1}{m} \vec{\partial} \cdot \vec{\rho}_{(1)}(\vec{x}, t),
\]

\[
\vec{\rho}_{(1)}(\vec{x}, t) = -\frac{i\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial \vec{x}} - \frac{\partial \psi^*}{\partial \vec{x}} \right)(\vec{x}, t).
\]

The expectation values of the position \( \hat{\vec{x}} = \vec{x} \) and momentum \( \hat{\vec{p}} = -i\hbar \frac{\partial}{\partial x} \) operators in the state \( \psi(\vec{x}, t) \) are

\[
< \hat{\vec{x}} >_{\psi(t)} = \int d^3x \vec{x} \rho_{(o)}(\vec{x}, t), \quad < \hat{\vec{p}} >_{\psi(t)} = -i\hbar \int d^3x \left( \psi^* \frac{\partial \psi}{\partial \vec{x}} \right)(\vec{x}, t) = \int d^3x \vec{\rho}_{(1)}(\vec{x}, t),
\]

and the Ehrenfest theorem implies

\[
\frac{d}{dt} < \hat{\vec{x}} >_{\psi(t)} = \frac{1}{m} < \hat{\vec{p}} >_{\psi(t)}, \quad \frac{d}{dt} < \hat{\vec{p}} >_{\psi(t)} = -< \frac{\partial V(\vec{x})}{\partial \vec{x}} >_{\psi(t)}.
\]

Then, following Refs.[5, 6], we perform a multipolar expansion of the distribution functions \( \rho_{(o)}(\vec{x}, t) \) and \( \vec{\rho}_{(1)}(\vec{x}, t) \) around a given classical trajectory \( \vec{x}_c(t) \) in Galilei space-time

\[
\rho_{(o)}(\vec{x}, t) = \rho_{(o)0}(\vec{x}_c(t), t) \delta^3(\vec{x} - \vec{x}_c(t)) + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, ..., r_n} \rho_{(o)n}^{r_1, ..., r_n}(\vec{x}_c(t), t) \frac{\partial^n}{\partial x^{r_1} ... \partial x^{r_n}} \delta^3(\vec{x} - \vec{x}_c(t)),
\]

\[
\rho_{(o)n}^{r_1, ..., r_n}(\vec{x}_c(t), t) = \int d^3x \left( x^{r_1} - x_{c}^{r_1}(t) \right) ... \left( x^{r_n} - x_{c}^{r_n}(t) \right) \rho_{(o)}(\vec{x}, t),
\]

\[
\rho_{(o)0}(\vec{x}_c(t), t) = \int d^3x \rho_{(o)}(\vec{x}, t) = \rho_{(o)0}(t) = 1,
\]

\[\text{If } A \text{ is a classical quantity, } \hat{A} \text{ denotes the associated quantum operator.}\]
\[ \tilde{\rho}(1)(\vec{x}, t) = \tilde{\rho}(1)_{(o)}(\vec{x}_c(t), t) \delta^3(\vec{x} - \vec{x}_c(t)) + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, \ldots, r_n} \tilde{\rho}^{1..r_n}(\vec{x}_c(t), t) \frac{\partial^n}{\partial x^{r_1} \ldots \partial x^{r_n}} \delta^3(\vec{x} - \vec{x}_c(t)), \]

\[ \tilde{\rho}^{1..r_n}(\vec{x}_c(t), t) = \int d^3x \left( x^{r_1} - x^{r_1}_c(t) \right) \ldots \left( x^{r_n} - x^{r_n}_c(t) \right) \tilde{\rho}(1)(\vec{x}, t) = \]

\[ = \frac{i\hbar}{2} \int d^3x \left( x^{r_1} - x^{r_1}_c(t) \right) \ldots \left( x^{r_n} - x^{r_n}_c(t) \right) \left( \psi^* \frac{\partial \psi}{\partial \vec{x}} - \frac{\partial \psi^*}{\partial \vec{x}} \psi \right)(\vec{x}, t). \]

(2.5)

If the density \(|\psi(\vec{x}, t)|^2\) is concentrated in a spatial region of the order of the de Broglie wavelength, the classical trajectory \(\vec{x}_c(t)\) should be located within this region. Accordingly we shall adopt the notation \(\vec{x}_c(t) = \vec{x}_\psi(t)\).

The higher multipoles are symmetrical in the indices \(r_1, \ldots, r_n\): \(\tilde{\rho}^{r_1..r_n}_{(o)}(\vec{x}_\psi(t), t) = \tilde{\rho}^{r_1..r_n}_{(1)n}(\vec{x}_\psi(t), t)\) and \(\tilde{\rho}^{r_1..r_n}_{(1)n}(\vec{x}_\psi(t), t) = \tilde{\rho}^{r_1..r_n}_{(1)n}(\vec{x}_\psi(t), t)\). The multipoles \(\tilde{\rho}^{r_1..r_n}_{(o)n}(\vec{x}_\psi(t), t)\) are of order \(\hbar^n\) when their spatial extent is concentrated in such a region; otherwise their order is larger than \(\hbar^n\).

By using Eqs.(2.4) and (2.5), the expectations values of position and momentum operators become

\[ < \hat{x} >_{\psi(t)} = \int d^3x \tilde{x} \rho_{(o)}(\vec{x}, t) = \vec{x}_\psi(t) + \tilde{\rho}_{(o)1}(\vec{x}_\psi(t), t), \]

\[ < \hat{p} >_{\psi(t)} = \int d^3x \tilde{p}(1)(\vec{x}, t) = \tilde{\rho}_{(1)o}(\vec{x}_\psi(t), t) = \tilde{\rho}_{(1)o}(t), \]

(2.6)

where \(\tilde{\rho}_{(o)1}(\vec{x}_\psi(t), t)\) is the dipole term of the distribution \(\rho_{(o)}(\vec{x}, t)\).

Note that the trajectory still needs to be specified. We know, for example, that when the multipolar expansion is used to describe extended objects like the Earth, the trajectory is fixed by requiring that it be coincident with some relevant collective variable (as the center of mass in non-relativistic physics). Since our aim here is to extract some classical features from the quantum probability distribution \(|\psi(\vec{x}, t)|^2\), it seems natural impose that the quantum expectation value of the position operator coincides for any \(t\) with the corresponding point \(\vec{x}_\psi(t)\) of the classical spatial trajectory to be chosen. However, due to the first of Eqs.(2.6), this is only possible for those wave functions whose density matrix has a vanishing dipole with respect to the very trajectory itself, i.e., if

\[ < \hat{x} >_{\psi(t)} = \vec{x}_\psi(t), \]

\[ \Rightarrow \tilde{\rho}_{(o)1}(\vec{x}_\psi(t), t) = \int d^3y \tilde{y} \rho_{(o)}(\vec{x}_\psi(t) + \vec{y}, t) = 0, \]

(2.7)

The wave functions satisfying Eq.(2.7) will be named null dipole wave functions (NDWF).

12
Due to Eq.(2.7), the dipole certainly vanishes for all the wave functions that are eigenfunctions of the parity operator (a conserved quantity in QT): \( \psi_\pm(-\vec{x}, t) = \pm \psi_\pm(\vec{x}, t) \) if we choose the time axis \( \vec{x}_\psi(t) = 0 \) as the classical trajectory. This case corresponds to an isolated quantum system ‘at rest’ in the origin (e.g., an atom in a cavity). For instance one can build a complete set of NDWFs by considering a basis of energy eigenfunctions of the given quantum system with definite parity (any other basis of eigenfunctions of a maximal set of commuting operators including parity will contain NDWFs only). Any other wave function, including the NDWFs, can be decomposed on this basis, but the multipolar expansion of the NDWF ones will identify trajectories different from those associated to the basis.

Another class of wave functions with vanishing dipole are the ones whose density matrix has a spatial distribution such that for any \( t \), it holds \( \rho_{(o)}(\vec{x}_\psi(t) - \vec{y}, t) = \rho_{(o)}(\vec{x}_\psi(t) + \vec{y}, t) \). Note that wave functions having this symmetry exist for every quantum system describing e.g. collimated beams of massive particles.

Finally, more general wave functions should possibly exist, satisfying Eq.(2.7), without any explicit symmetry at all.

Let us stress, however, that not all of the NDWF satisfy the Ehrenfest theorem. A simple counterexample is the following: consider an energy wave function \( e^{-iE_n t} \psi_n(\vec{x}) \) of a given quantum system, which is also a parity eigenfunction with \( \psi_n(-\vec{x}) = \pm \psi_n(\vec{x}) \). This is a NDWF with a vanishing expectation value of the position operator \( \langle \vec{x} \rangle_{\psi_n} = 0 \) but with a non-vanishing time-independent expectation value \( \langle \hat{p} \rangle_{\psi_n} \) of the momentum operator. Consequently, Eqs.(2.3) cannot hold for them.

We shall then call Ehrenfest monopole wave functions (EMWF) those NDWFs for which both Eqs.(2.3) hold. The important issue here being the implications of the Ehrenfest theorem for the associated classical trajectories \( \vec{x}_\psi(t) \).

For EMWF the first half of Eqs.(2.3) implies that the expectation value (2.6) of the momentum operator coincides with the expression of the momentum of the classical trajectory. Actually

\[
\langle \hat{p} \rangle_{\psi(t)} = \vec{p}_0(\vec{x}_\psi(t), t) = \vec{p}_0(t) = m \dot{\vec{x}}_\psi(t) \overset{def}{=} \vec{p}_\psi(t).
\] (2.8)

Here \( \vec{p}_\psi(t) \) is the classical momentum of a Newtonian particle, which is a quantity that can be measured by means of, e.g., the time-of-flight method.

Then, the second half of Ehrenfest equations (2.3) implies

\[
m \ddot{\vec{x}}_\psi(t) = \langle -\frac{\partial V(\vec{x})}{\partial \vec{x}} \rangle_{\psi(t)} = \vec{F}_\psi(t) = -\frac{\partial V(\vec{x}_\psi(t))}{\partial \vec{x}_\psi} - \\
- \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{r_1..r_n} \rho_{(o)}^{r_1..r_n}(\vec{x}_\psi(t), t) \frac{\partial^{n+1} V(\vec{x}_\psi(t))}{\partial \vec{x}_\psi \partial x_{\psi}^{r_1} \ldots \partial x_{\psi}^{r_n}},
\] (2.9)

namely a deterministic Newton-like equation, in which the standard Newton force is complemented with \( h \)-dependent non-Newtonian forces determined by the higher multi-poles (of order \( h^2 \) or higher) of the probability distribution function \( \rho_{(o)}(\vec{x}, t) \). Note that these higher multi-poles play the role of a ‘quantum guidance’ of the classical effective trajectory, although
in a sense different from the case of Bohmian mechanics. Finally, the quadrupole term \( \rho_{(o)2} \) carries the information on the standard deviation \( \Delta x^r \) of the \( \vec{x} \)-distribution, appearing in the uncertainty relations (see after, Eq.(3.13)).

The Cauchy data at the initial time \( t_o \) for these Newton-like equations of motion are
\[
\vec{x}_{\psi}(t_o) = \langle \vec{x} \rangle_{\psi(t_o)} \quad \text{and} \quad \hat{p}_{\psi}(t_o) = \frac{1}{m} \langle \hat{p} \rangle_{\psi(t_o)}, \quad \text{i.e. the mean value of position and momentum operator at the initial time.}
\]

In conclusion, the EMWF share the following dual nature:

a) on the one hand, they are probability waves describing the standard QT probability distribution for the localization of the particle;

b) on the other hand, they also characterize a deterministic Newton-like trajectory in Galilei space-time with deviations of order \( O(\hbar^2) \) from a true classical Newtonian trajectory. It seems that in such cases the efficacy of the Newtonian intuition of experimentalists in visualizing atoms and molecules is strongly supported.

Furthermore, if \( \psi_1(\vec{x},t) \) and \( \psi_2(\vec{x},t) \) are two generic EMWFs for a given system with 'classical' trajectories \( \vec{x}_{\psi_1}(t) \) and \( \vec{x}_{\psi_2}(t) \) respectively, their linear superposition \( \psi_{12}(\vec{x},t) = \alpha \psi_1(\vec{x},t) + \beta \psi_2(\vec{x},t) \) may or may not be an EMWF. In the case that it is an EMWF centered on a trajectory \( \vec{x}_{\psi_{12}}(t) \) we have
\[
\langle \hat{\vec{x}} \rangle_{\psi_{12}(t)} = \vec{x}_{\psi_{12}}(t) + \rho_{(o)1}(\vec{x}_{\psi_{12}}(t),t) = \vec{x}_{\psi_1}(t) + \vec{x}_{\psi_2}(t) + \int d^3 \vec{x} \vec{x} \left( \alpha^* \beta \psi_1^* \psi_2 + \alpha \beta^* \psi_2^* \psi_1 \right)(\vec{x},t) \neq \vec{x}_{\psi_1}(t) + \vec{x}_{\psi_2}(t). \tag{2.10}
\]

The interference term is real but not definite positive and does not admit a multipolar expansion, because it was defined only for the density function \( |\psi(\vec{x},t)|^2 \). Consequently, the superposition principle cannot hold in general in the set of EMWFs, and superposition of classical trajectories is, as desirable, a meaningless notion. This is not in contradiction with the possibility of expressing an EMWF on a complete basis of NDWFs with definite parity. While NDWFs not in the EMWF class do not have classical trajectories satisfying Newton-like equations of motion, an EMWF superposition of NDWFs will have a classical trajectory satisfying Newton-like equations of motion. Coherent states are a relevant example of EMWFs.

Note, therefore, that the existence of the multipolar expansion around a well defined classical trajectory, identifying NDWF, and the satisfaction of the Ehrenfest theorem, identifying EMWF with Newton-like classical trajectories, express two distinct mathematical properties.

The expectation value of the Hamiltonian operator between NDWF (not EMWF) which are also energy eigenfunctions, i.e. \( \Psi_n(\vec{x},t) = e^{i E_n t} \psi_n(\vec{x}) \), \( \hat{H} \psi_n(\vec{x}) = E_n \psi_n(\vec{x}) \), is discussed at the end of next Section.

Let us consider instead the expectation value of the angular momentum operator \( \hat{L}^i = \frac{1}{2} \epsilon^{ijk} \hat{x}^j \hat{p}^k \) in a state EMWF. By using Eqs.(2.5) and (2.8) we get
\[ <\hat{L}^i>_\psi(t) = \epsilon^{ijk} \int d^3x \, x^i \rho^k(\bar{x}, t) = \]
\[ = \epsilon^{ijk} \left[ x^j(t) \rho^k(\bar{x}_\psi(t), t) + \rho^k(x^j(t), t) \right] = \]
\[ = L^i_\psi + \epsilon^{ijk} \rho^{k_j}(\bar{x}_\psi(t), t), \quad (2.11) \]

where \( L^i_\psi = \epsilon^{ijk} x^j(t) m \dot{x}^k(t) \) is the angular momentum of the deterministic Newtonian particle. Therefore we have \(<\hat{L}^i>_\psi(t) = L^i_\psi\) only for those EMWFs such that the antisymmetric part of the dipole \(\rho^{k_j}(\bar{x}_\psi(t), t)\) of the distribution function \(\bar{\rho}(\bar{x}, t)\) vanishes. In this case, if the EMWF is a normalizable eigenfunction \(\psi_{lm}(\bar{x}, t)\) of \(\hat{L}^2\) and of \(\hat{L}^3\) with eigenvalues \(l\) and \(m\) \((\hat{L}^2_{\psi_{lm}} = \hbar l (l + 1) \psi_{lm}, \hat{L}^3_{\psi_{lm}} = \hbar m \psi_{lm})\), from Eq.(2.8) we get \(\hbar m = <\hat{L}^3>_{\psi_{lm}(t)} = x^1_{\psi_{lm}} \rho^{2}_{(1)0}(t) - x^2_{\psi_{lm}} \rho^{1}_{(1)0}(t) = \hat{L}^3_{\psi_{lm}}\). With the higher distribution functions of the next Section one could see which higher multipoles contribute to \(\hbar l (l + 1) = <\hat{L}^2>_{\psi_{lm}(t)}\) and which connection may be established with \(\hat{L}^2_{\psi_{lm}}\).

Let us remark that in many-particle systems with mutual interactions one must perform the separation of variables between the free decoupled center of mass and relative motions: an Ehrenfest theorem holds true for such relative variables only, but not for the center of mass if it is in a momentum eigenstate. Indeed for a plane wave function \(\psi(\bar{x}, t) = e^{i(\bar{E}t - \bar{p} \cdot \bar{x})}\), which is not normalizable, we get \(\rho(\bar{x}, t) = 1, <\hat{L}^i>_\psi = 0, <\hat{p} >_\psi = \bar{p}\). Therefore the Ehrenfest theorem does not hold and there is no semi-classical visualization for this improper description of a free particle (commonly used for the center-of-mass free motion in scattering theory). This is not true for realistic scattering events, in which there are Gaussian-like wave packets of effective particles collimated towards the scattering region: they are not eigenfunctions of the momentum operator and the effective mean momentum of the beams is evaluated with time-of-flight methods (so that strictly speaking it is not a momentum eigenvalue).

Also the improper non-normalizable localized states \(\psi(\bar{x}, t) = \delta^3(\bar{x} - \bar{x}_o)\) have no semi-classical visualization \(^5\).

Moreover, see Appendix A for comments about the mathematical problems concerning spatial localization.

\(^5\) See Section 4.1 of Ref.[1] for discretized position measurements: obviously, only localization in finite intervals can be described with normalizable wave functions.
III. THE MULTIPOLAR EXPANSION OF THE WIGNER FUNCTION

As shown in Refs. [24], [4], the expectation value of an operator $\hat{\Omega}(\hat{x}, \hat{p})$ function of the position and momentum operators, in the state $\rho(\vec{x}, t)$, can be written

$$<\hat{\Omega}(\hat{x}, \hat{p})>_{\psi(t)} = \int d^3x d^3p (2\pi \hbar)^3 \Omega_w(\vec{x}, \vec{p}) W(\vec{x}, \vec{p}, t). \tag{3.1}$$

This expression depends on the following two quantities:

i) the Weyl symbol of the operator $\hat{\Omega}(\hat{x}, \hat{p})$, which is defined as (see the second of Refs.[16] for more details)

$$\Omega_w(\vec{x}, \vec{p}) = \int d^3\xi e^{i\vec{p} \cdot \vec{\xi}} \langle \vec{x} - \frac{1}{2} \vec{\xi} | \hat{\Omega}(\hat{x}, \hat{p}) | \vec{x} + \frac{1}{2} \vec{\xi} \rangle; \tag{3.2}$$

the Weyl symbols of the position and momentum operators are $\vec{x}_W(\vec{x}, \vec{p}) = \vec{x}$ and $\vec{p}_W(\vec{x}, \vec{p}) = \vec{p}$;

ii) the Wigner function (a non-definite-positive distribution function on phase space; it is definite-positive only for Gaussian coherent states)

$$W(\vec{x}, \vec{p}, t) = \int d^3\xi e^{i\vec{p} \cdot \vec{\xi}} \langle \vec{x} - \frac{1}{2} \vec{\xi} | \hat{\rho}(t) | \vec{x} + \frac{1}{2} \vec{\xi} \rangle = \int d^3\xi e^{i\vec{p} \cdot \vec{\xi}} \psi(\vec{x} - \frac{1}{2} \vec{\xi}, t) \psi^*(\vec{x} + \frac{1}{2} \vec{\xi}, t). \tag{3.3}$$

Let us make the following Cauchy expansions

$$\psi(\vec{x} - \frac{1}{2} \vec{\xi}, t) = \psi(\vec{x}, t) + \sum_{a=1}^{\infty} \frac{(-1)^a}{2^a} \sum_{a_1..a_a} \xi^{a_1}..\xi^{a_a} \frac{\partial^a \psi(\vec{x}, t)}{\partial x^{a_1}..\partial x^{a_a}},$$

$$\psi^*(\vec{x} + \frac{1}{2} \vec{\xi}, t) = \psi^*(\vec{x}, t) + \sum_{b=1}^{\infty} \frac{1}{2^b} \sum_{b_1..b_b} \xi^{b_1}..\xi^{b_b} \frac{\partial^b \psi^*(\vec{x}, t)}{\partial x^{b_1}..\partial x^{b_b}}. \tag{3.4}$$

Then, Eq.(3.1) becomes ($c = a + b$)
\[ W(\vec{x},\vec{p},t) = \int d^3\xi \ e^{\frac{\vec{p} \cdot \vec{x}}{\hbar}} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left( \frac{-\hbar}{2a+b} \right) \sum_{d_1, \ldots, d_{a+b}} \xi^{d_1+\ldots+d_a+d_{a+1}+\ldots+d_{a+b}} \]

\[ \frac{\partial^b \psi^*(\vec{x},t)}{\partial x^{d_{a+1}} \cdots \partial x^{d_{a+b}}} \frac{\partial^a \psi(\vec{x},t)}{\partial x^{d_1} \cdots \partial x^{d_a}} = \]

\[ = \int d^3\xi \ e^{\frac{\vec{p} \cdot \vec{x}}{\hbar}} \sum_{a=0}^{\infty} \sum_{c=0}^{1} \frac{1}{2^c} \sum_{d_1, \ldots, d_c} \xi^{d_1+\ldots+d_a+d_{a+1}+\ldots+d_c} \]

\[ \frac{\partial^{c-a} \psi^*(\vec{x},t)}{\partial x^{d_{a+1}} \cdots \partial x^{d_c}} \frac{\partial^a \psi(\vec{x},t)}{\partial x^{d_1} \cdots \partial x^{d_a}} = \]

\[ = \int d^3\xi \ e^{\frac{\vec{p} \cdot \vec{x}}{\hbar}} \sum_{c=0}^{\infty} \sum_{d_1, \ldots, d_c} \xi^{d_1+\ldots+d_c} \sum_{a=0}^{c} (-)^a \rho(a)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)(\vec{x},t), \]

\[ \rho(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)(\vec{x},t) = \frac{\partial^{c-a} \psi^*(\vec{x},t)}{\partial x^{d_{a+1}} \cdots \partial x^{d_c}} \frac{\partial^a \psi(\vec{x},t)}{\partial x^{d_1} \cdots \partial x^{d_a}}, \quad a > 0, \]

\[ \rho(c)(d_1 \ldots d_c)(-)(\vec{x},t) = \frac{\partial^c \psi^*(\vec{x},t)}{\partial x^{d_1} \cdots \partial x^{d_c}} \psi(\vec{x},t), \quad a = 0, \]

\[ \rho(c)(-)(d_1 \ldots d_c)(\vec{x},t) = \psi^*(\vec{x},t) \frac{\partial^a \psi(\vec{x},t)}{\partial x^{d_1} \cdots \partial x^{d_c}}, \quad a = c. \]

By using

\[ \int d^3\xi \ e^{\frac{\vec{p} \cdot \vec{x}}{\hbar}} \xi^{d_1+\ldots+d_c} = (-i\hbar)^c \frac{\partial^c \delta^3(\vec{p})}{\partial p^{d_1} \cdots \partial p^{d_c}}, \]

and by performing a multipolar expansion of the distribution functions \( \rho(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)(\vec{x},t), \)

\[ \rho(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)(\vec{x},t) = \rho(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)\alpha(\vec{x}\psi(t),t) \delta^3(\vec{x} - \vec{x}_\psi(t)) + \]

\[ + \sum_{n=1}^{\infty} \frac{(-\hbar)^n}{n!} \sum_{r_1, \ldots, r_n} \rho^{r_1 \ldots r_n}(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)n(\vec{x}\psi(t),t) \frac{\partial^n \delta^3(\vec{x} - \vec{x}_\psi(t))}{\partial x^{r_1} \cdots \partial x^{r_n}}, \]

\[ \rho^{(o)}(\vec{x}_\psi(t),t) = \int d^3x \rho^{(o)}(\vec{x},t) = \rho^{(o)}(t) = 1, \]

\[ \rho(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)\alpha(\vec{x}\psi(t),t) = \int d^3x \frac{\partial^{c-a} \psi^*(\vec{x},t)}{\partial x^{d_{a+1}} \cdots \partial x^{d_c}} \frac{\partial^a \psi(\vec{x},t)}{\partial x^{d_1} \cdots \partial x^{d_a}} = \]

\[ = \rho(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)\alpha(t), \quad (\rho^{(c)}(-)(d_1 \ldots d_a)\alpha(t)) \text{ for } a = c, \]

\[ \rho^{r_1 \ldots r_n}(c)(d_{a+1} \ldots d_c)(d_1 \ldots d_a)n(\vec{x}\psi(t),t) = \int d^3x \left(x^{r_1} - x^{r_1}_\psi(t)\right) \cdots \left(x^{r_n} - x^{r_n}_\psi(t)\right) \frac{\partial^{c-a} \psi^*(\vec{x},t)}{\partial x^{d_{a+1}} \cdots \partial x^{d_c}} \frac{\partial^a \psi(\vec{x},t)}{\partial x^{d_1} \cdots \partial x^{d_a}}, \]

\[ n = 1, a = c = 0, \quad \rho^{(o)1}(\vec{x}\psi(t),t) = 0, \text{ for EMWF} \]

\[ (3.7) \]
we get the following multipolar expansion for the Wigner function:

\[
W(\vec{x}, \vec{p}, t) = W_{(o)}(\vec{x}_\psi(t), \vec{p}_\psi(t))\delta^3(\vec{x} - \vec{x}_\psi(t)) + \\
+ \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum_{r_1\ldots r_n} W_{(o)}^{r_1\ldots r_n}(\vec{x}_\psi(t), \vec{p}, t) \frac{\partial^n \delta^3(\vec{x} - \vec{x}_\psi(t))}{\partial x_1 \ldots \partial x_n} = \\
= \sum_{c=0}^{\infty} \sum_{d_1\ldots d_c} W_{(o)(c)}^{d_1\ldots d_c}(\vec{x}_\psi(t), t) \frac{\partial^c \delta^3(\vec{p})}{\partial p_{d_1} \ldots \partial p_{d_c}} \delta^3(\vec{x} - \vec{x}_\psi(t)) + \\
+ \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \sum_{c=0}^{\infty} \sum_{d_1\ldots d_c} W_{(n)(c)}^{r_1\ldots r_n d_1\ldots d_c}(\vec{x}_\psi(t), t) \frac{\partial^c \delta^3(\vec{p})}{\partial p_{d_1} \ldots \partial p_{d_c}} \delta^3(\vec{x} - \vec{x}_\psi(t))
\]

\[
W_{(o)(o)}(\vec{x}_\psi(t), t) = \rho_{(o)}(\vec{x}_\psi(t), t) = 1,
\]

\[
W_{(1)(o)}(\vec{x}_\psi(t), t) = \rho_{(o)1}(\vec{x}_\psi(t), t) = 0,
\]

\[
W_{(o)(1)}^{d_1}(\vec{x}_\psi(t), t) = \frac{-i\hbar}{2} \sum_{a=0}^{c} (-)^a \rho_{(1)(d_{a+1})d_a}(\vec{x}_\psi(t), t) = \\
= -\frac{i\hbar}{2} \int d^3x \left( \psi^* \frac{\partial \psi}{\partial x_1} - \frac{\partial \psi^*}{\partial x_1} \psi \right)(\vec{x}, t) = \rho_{(o)1}^{d_1}(t) = p_{\psi}^{d_1}(t) = m \dot{\vec{x}}^{d_1}(t),
\]

\[
W_{(o)(c)}^{d_1\ldots d_c}(\vec{x}_\psi(t), t) = (-i\hbar)^c \sum_{a=0}^{c} (-)^a \rho_{(c)(d_{a+1}\ldots d_c)(d_1\ldots d_a)}(\vec{x}_\psi(t), t),
\]

\textbf{The monopole part of the Wigner function is}

\[
W_{(o)}(\vec{x}, \vec{p}, t) = \sum_{c=0}^{\infty} \sum_{d_1\ldots d_c} W_{(o)(c)}^{d_1\ldots d_c}(\vec{x}_\psi(t), t) \frac{\partial^c \delta^3(\vec{p})}{\partial p_{d_1} \ldots \partial p_{d_c}} \delta^3(\vec{x} - \vec{x}_\psi(t)) = \\
= \sum_{c=0}^{\infty} \frac{(-i\hbar)^c}{2^c} \sum_{a=0}^{c} (-)^a \sum_{d_1\ldots d_c} \rho_{(c)(d_{a+1}\ldots d_c)(d_1\ldots d_a)}(\vec{x}_\psi(t), t) \frac{\partial^c \delta^3(\vec{p})}{\partial p_{d_1} \ldots \partial p_{d_c}} \delta^3(\vec{x} - \vec{x}_\psi(t)) = \\
= \left( \delta^3(\vec{p}) + \vec{p}_\psi(t) \cdot \frac{\partial \delta^3(\vec{p})}{\partial \vec{p}} \right) + \\
+ \sum_{c=2}^{\infty} \frac{(-i\hbar)^c}{2^c} \sum_{a=0}^{c} (-)^a \sum_{d_1\ldots d_c} \rho_{(c)(d_{a+1}\ldots d_c)(d_1\ldots d_a)}(\vec{x}_\psi(t), t) \frac{\partial^c \delta^3(\vec{p})}{\partial p_{d_1} \ldots \partial p_{d_c}} \delta^3(\vec{x} - \vec{x}_\psi(t)).
\]
This distribution is concentrated in $\vec{p} = 0$ and $\vec{x} = \vec{x}_\psi(t)$ so that the problem of not being definite positive disappears at the monopole level.

The *marginals*, i.e. the distribution functions for operators depending on either the position or the momentum operators only, are

$$W(\vec{x}, t) = \int d^3 p \, W(\vec{x}, \vec{p}, t) = \rho_{(o)}(\vec{x}, t) = \delta^3(\vec{x} - \vec{x}_\psi(t)) + \sum_{n=2}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, \ldots, r_n} \rho_{(o)n}^{r_1, \ldots, r_n}(\vec{x}_\psi(t), t) \frac{\partial^n \delta^3(\vec{x} - \vec{x}_\psi(t))}{\partial x^{r_1} \cdots \partial x^{r_n}} = \delta^3(\vec{x} - \vec{x}_\psi(t)) + \sum_{n=2}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, \ldots, r_n} W_{(o)(n)}^{r_1, \ldots, r_n}(\vec{x}_\psi(t), t) \frac{\partial^n \delta^3(\vec{x} - \vec{x}_\psi(t))}{\partial x^{r_1} \cdots \partial x^{r_n}},$$

$$\rho_{(o)n}^{r_1, \ldots, r_n}(\vec{x}_\psi(t), t) = \rho_{(c)(d_1, \ldots, d_n)}^{r_1, \ldots, r_n}(\vec{x}_\psi(t), t) |_{c = a = 0}.$$ (3.10)

$$W(\vec{p}, t) = \int d^3 x \, W(\vec{x}, \vec{p}, t) = \frac{1}{(2\pi \hbar)^3} \int d^3 x_1 d^3 x_2 e^{i\vec{p}(\vec{x}_1 - \vec{x}_2) \psi^*(\vec{x}_1, t) \psi(\vec{x}_2, t)} = \sum_{c=0}^{\infty} \sum_{d_1, \ldots, d_c} W_{(o)c}^{d_1, \ldots, d_c}(\vec{x}_\psi(t), t) \frac{\partial^c \delta^3(\vec{p})}{\partial p^{d_1} \cdots \partial p^{d_c}}.$$ (3.11)

For EMWF, we recover $<\hat{x}^i \hat{p}^j >_{\psi(t)} = \int \frac{d^3 x \, d^3 p}{(2\pi \hbar)^3} \vec{x} \vec{W}(\vec{x}, \vec{p}, t) = \vec{x}_\psi(t)$ and $<\hat{p}^i \hat{x}^j >_{\psi(t)} = \int \frac{d^3 x \, d^3 p}{(2\pi \hbar)^3} \vec{p} \vec{W}(\vec{x}, \vec{p}, t) = \vec{p}_\psi(t) = m \vec{x}_\psi(t)$.

It follows that the monopole and dipole of the momentum distribution $\vec{\rho}_{(1)}$ determine the quantities

$$<\hat{x}^i \hat{p}^j >_{\psi(t)} = \frac{i\hbar}{2} \delta^{ij} - x^i_{\psi}(t) \rho_{(1)o}^{ij}(\vec{x}_\psi(t), t) + \rho_{(1)(1)}^{ij}(\vec{x}_\psi(t), t),$$

$$<\hat{p}^i \hat{x}^j >_{\psi(t)} = -\frac{i\hbar}{2} - x^i_{\psi}(t) \rho_{(1)o}^{ij}(\vec{x}_\psi(t), t) + \rho_{(1)(1)}^{ij}(\vec{x}_\psi(t), t),$$ (3.12)
in agreement with the commutation relations $[\hat{x}^i, \hat{p}^j] = i\hbar \delta^{ij}$.

Moreover, for EMWF, it follows from Eqs.(3.7)

$$<\hat{x}^i \hat{x}^j >_{\psi(t)} = x^i_{\psi}(t) x^j_{\psi}(t) + \frac{1}{2} \rho_{(o)2}^{ij}(\vec{x}_\psi(t), t),$$

$$<\hat{p}^i \hat{p}^j >_{\psi(t)} = -\hbar^2 \rho_{(2)(-)(ij)o}(t).$$ (3.13)
Therefore, for the EMWF class, the standard deviations $\Delta x^r = \sqrt{\langle (\hat{x}^r)^2 \rangle_{\psi(t)} - \langle \hat{x}^r \rangle_{\psi(t)}^2}$ of the $\vec{x}$-distribution and $\Delta p^r = \sqrt{\langle (\hat{p}^r)^2 \rangle_{\psi(t)} - \langle \hat{p}^r \rangle_{\psi(t)}^2}$ of the $\vec{p}$-distribution, appearing in the uncertainty relations $\Delta x^r \Delta p^r \geq \hbar / 2$, are determined by the following multipoles:

\[
(\Delta x^r)^2 = \frac{1}{2} \rho_{(o)2}^{rr} (\vec{x}_\psi(t), t) \quad \text{(the quadrupole of $\rho_{(o)}$)};
\]

\[
(\Delta p^r)^2 = -\frac{\hbar^2}{2m} \rho_{(2)(-)(rr)_{(o)}}(t) = \langle (\hat{p}^r)^2 \rangle_{\psi} - \langle \hat{p}^r(t) \rangle^2 \quad \text{(the monopoles of $\rho_{(2)(-)(ij)}$ and $\rho_{(1)(o)}$ of Eqs.(3.7) and (2.8)).}
\]

As previously said, an energy eigenfunction, i.e. $\Psi_n(\vec{x}, t) = e^{\frac{i}{\hbar} E_n t} \psi_n(\vec{x})$ with $\hat{H} \psi_n(\vec{x}) = E_n \psi_n(\vec{x})$, $\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{x})$, is a NDWF not of the EMWF type with a classical trajectory $< \hat{x} >_{\psi_n(t)} = 0$, if it is a parity eigenstate. For it we get

\[
E_n = \langle \hat{H} \rangle_{\psi_n(t)} = \frac{\langle \hat{p}^2 \rangle_{\psi_n} + \langle V(\vec{x}) \rangle_{\psi_n}}{2m},
\]

\[
\frac{\langle \hat{p}^2 \rangle_{\psi_n}}{2m} = -\frac{\hbar^2}{2m} \sum_r \rho_{(2)(-)(rr)_{(o)}}(t),
\]

\[
\langle V(\vec{x}) \rangle_{\psi_n} = V(\vec{x}_{\psi_n}(t)) + \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{r_1..r_m} \rho_{(o)m}^{r_1..r_m} (\vec{x}_{\psi_n}(t), t) \frac{\partial^m V(\vec{x}_{\psi_n}(t))}{\partial x_{\psi_n}^{r_1} .. \partial x_{\psi_n}^{r_m}}.
\]

(3.14)

For this NDWF the mean kinetic energy does not coincide with the vanishing classical kinetic energy $\frac{1}{2} m \dot{\vec{x}}_{\psi_n}^2(t) = 0$.

Note, finally, that the force $\vec{F}_{\psi_n}(t)$ appearing in Eq.(2.9) is not minus the gradient of the expectation value of the potential, $\vec{F}_{\psi_n}(t) \neq - \frac{\partial}{\partial \vec{x}_{\psi_n}} \langle V(\vec{x}) \rangle_{\psi_n}$, except when all the higher multipoles for $n \geq 2$ are independent of the classical trajectory, i.e. $\rho_{(o)n}^{r_1..r_n}(\vec{x}_\psi(t), t) = \rho_{(o)n}^{r_1..r_n}(t)$. 

20
IV. NON-RELATIVISTIC TWO-PARTICLE SYSTEMS

Let $\psi(\vec{x}_1, \vec{x}_2, t)$ be a two-particle wave function, $\vec{x}_1(t)$ and $\vec{x}_2(t)$ two classical trajectories. Let us remark that even if $\psi(\vec{x}_1, \vec{x}_2, t)$ lives in the Cartesian product of two Euclidean spaces, the trajectories $\vec{x}_1(t)$ and $\vec{x}_2(t)$ live in the standard Galilei space-time.

The mean values of the position and momentum operators are expressed in terms of the density matrix $\rho(o)(\vec{x}_1, \vec{x}_2, t) = \vert \psi(\vec{x}_1, \vec{x}_2, t) \vert^2$ and of two other probability distributions $\rho_{(1)(i)}(\vec{x}_1, \vec{x}_2, t)$

$$< \hat{x}_i >_{\psi(t)} = \int d^3x_1 d^3x_2 \hat{x}_i \rho(o)(\vec{x}_1, \vec{x}_2, t),$$
$$< \hat{p}_i >_{\psi(t)} = \int d^3x_1 d^3x_2 \rho_{(1)(i)}(\vec{x}_1, \vec{x}_2, t).$$

and the Ehrenfest theorem implies

$$\frac{d}{dt} < \hat{x}_i >_{\psi(t)} = \frac{1}{m_i} < \hat{p}_i >_{\psi(t)}, \quad i = 1, 2,$$
$$\frac{d}{dt} < \hat{p}_i >_{\psi(t)} = - \frac{\partial V_i(\vec{x}_i)}{\partial \vec{x}_i} >_{\psi(t)}.$$

The double multipolar expansion of the density matrix is

$$\rho(o)(\vec{x}_1, \vec{x}_2, t) = \vert \psi(\vec{x}_1, \vec{x}_2, t) \vert^2 =$$
$$= \rho(o)_{10}(\vec{x}_1(t), \vec{x}_2(t)) \delta^3(\vec{x}_1 - \vec{x}_1(t)) +$$
$$+ \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, ..., r_n} \rho_{(o)1n}^{r_1...r_n}(\vec{x}_1(t), \vec{x}_2(t)) \frac{\partial^n \delta^3(\vec{x}_1 - \vec{x}_1(t))}{\partial x_1^{r_1} ... \partial x_1^{r_n}},$$

$$n = 0, 1, ... \quad \rho_{(o)1n}^{r_1...r_n}(\vec{x}_1(t), \vec{x}_2(t)) =$$
$$= \rho(o)_{1020}(\vec{x}_1(t), \vec{x}_2(t), t) \delta^3(\vec{x}_2 - \vec{x}_2(t)) +$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{m!} \sum_{s_1, ..., s_m} \rho_{(o)12m}^{r_1s_1...s_m}(\vec{x}_1(t), \vec{x}_2(t), t) \frac{\partial^m \delta^3(\vec{x}_2 - \vec{x}_2(t))}{\partial x_2^{s_1} ... \partial x_2^{s_m}}.$$

21
\[ \rho_{(o)}(\vec{x}_1, \vec{x}_2, t) = |\psi(\vec{x}_1, \vec{x}_2, t)|^2 = \]
\[ = \sum_{n,m} \frac{(-)^n (-)^m}{n! m!} \sum_{r_1 \ldots r_n s_1 \ldots s_m} \rho_{(o)1n2m}^{r_1 \cdots r_n s_1 \cdots s_m}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) \]
\[ = \partial^n \delta^3(\vec{x}_1 - \vec{x}_1\psi(t)) (\vec{x}_2 - \vec{x}_2\psi(t)) \]
\[ = \rho_{(o)120}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) \delta^3(\vec{x}_1 - \vec{x}_1\psi(t)) \delta^3(\vec{x}_2 - \vec{x}_2\psi(t)) - \]
\[ - \sum_{s_1} \rho_{(o)1s_121}^{r_1}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) \delta^3(\vec{x}_1 - \vec{x}_1\psi(t)) \frac{\partial \delta^3(\vec{x}_2 - \vec{x}_2\psi(t))}{\partial x_2^{s_1}} - \]
\[ - \sum_{r_1 s_1} \rho_{(o)121}^{r_1 s_1}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) \frac{\partial \delta^3(\vec{x}_1 - \vec{x}_1\psi(t))}{\partial x_1^{r_1}} \frac{\partial \delta^3(\vec{x}_2 - \vec{x}_2\psi(t))}{\partial x_2^{s_1}} + \ldots. \]
\[ (4.3) \]

The condition \( < \hat{\mathbf{p}}_1 >_{\psi(t)} = \vec{x}_1\psi(t) \) requires the vanishing of the dipole \( \rho_{(o)1120}^{r_1}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) = 0 \), while \( < \hat{\mathbf{p}}_2 >_{\psi(t)} = \vec{x}_2\psi(t) \) requires the vanishing of the dipole \( \rho_{(o)1021}^{r_1}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) = 0 \). If the third dipole vanishes too, i.e. \( \rho_{(o)1121}^{r_1 s_1}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) = 0 \), it follows \( < \hat{\mathbf{p}}_1 \hat{\mathbf{p}}_2 >_{\psi(t)} = \partial_{\psi(t)}(\vec{x}_1\psi(t), \vec{x}_2\psi(t)) \).

For EMWF, for which the Ehrenfest theorem holds, by making a multipolar expansion of \( \rho_{(1)(i)}(\vec{x}_1, \vec{x}_2, t) = \left( \psi^* i \hbar \frac{\partial \psi}{\partial x_1} \right)(\vec{x}_1, \vec{x}_2, t) \), from the monopole terms \( \rho_{(1)(i)0}^{r}(t) \), due to the first of Eqs.(4.2), we get
\[ < \hat{\mathbf{p}}_1 >_{\psi(t)} = m_i \vec{x}_1\psi(t). \]  
\[ (4.4) \]

With potentials \( V_i(\vec{x}_1, \vec{x}_2) = V_{12}(\vec{x}_1 - \vec{x}_2) + V_{ext}(\vec{x}_i) \), if all dipoles vanish, Eqs (4.2) and (4.3) imply the following coupled deterministic equations of motion for the two trajectories
\[ m_i \ddot{x}_1\psi(t) = - \frac{\partial V_i(\vec{x}_1\psi(t), \vec{x}_2\psi(t))}{\partial \vec{x}_1\psi} - \]
\[ - \sum_{n,m} \frac{(-)^n (-)^m}{n! m!} \sum_{r_1 \ldots r_n s_1 \ldots s_m} \rho_{(o)1n2m}^{r_1 \cdots r_n s_1 \cdots s_m}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) \]
\[ \frac{\partial^{n+m} V_i(\vec{x}_1\psi(t), \vec{x}_2\psi(t))}{\partial x_1^{r_1} \cdots \partial x_1^{s_n} \partial x_2^{r_2} \cdots \partial x_2^{s_m}}. \]
\[ (4.5) \]

On the other hand, if only one dipole vanishes, for instance \( \rho_{(o)1120}^{r_1}(\vec{x}_1\psi(t), \vec{x}_2\psi(t), t) = 0 \), we have only \( < \hat{\mathbf{p}}_1 >_{\psi(t)} = \vec{x}_1\psi(t) \) and we can write a deterministic equation of motion for particle 1 but not for particle 2.
Therefore, by means of a double multipolar expansion, it is possible to find solutions of the Schrödinger equation with two effective classical particles, solutions with only one effective classical particle and, in general, solutions without effective classical particles.

Let us stress that for a two-particle quantum system there is a further ambiguity in the position basis of Hilbert space. Actually, according to the standard notion of separability, the Hilbert space is described as the tensor product $H = H_1 \otimes H_2$, so that a wave function is separable if $\psi(\vec{x}_1, \vec{x}_2) = \psi_1(\vec{x}_1) \psi_2(\vec{x}_2)$ and entangled if $\psi(\vec{x}_1, \vec{x}_2) \neq \psi_1(\vec{x}_1) \psi_2(\vec{x}_2)$.

However, in the case of an isolated system of two interacting particles, the only way of solving the Schrödinger equation is to transform to center-of-mass and relative variables. Then, by means of a unitary transformation, the Hilbert space is represented as $H = H_{c.m} \otimes H_{rel}$. Now, as always happens in scattering theory, a wave function is separable if $\Psi(\vec{x}, \vec{r}) = \Psi_{c.m}(\vec{x}) \Psi_{rel}(\vec{r})$ and entangled (note that this is a totally different notion of entanglement compared to one mentioned above) if $\Psi(\vec{x}, \vec{r}) \neq \Psi_{c.m}(\vec{x}) \Psi_{rel}(\vec{r})$. In scattering theory, however, one never considers wave packets with superpositions of the center of mass. The total momentum is a constant of motion and most likely there is some kind of super-selection rule forbidding these superpositions.

Now the center of mass is a free quantum particle, while the relative quantum motion can be either effectively ‘classical’ (zero dipole) or not. The free center of mass is not associated with an effective ‘classical’ particle, because it is described by a plane wave (non-zero dipole) having the momentum of the isolated two-particle system.

N particle systems can be treated in the same way.
V. RELATIVISTIC QUANTUM MECHANICS IN THE REST FRAME INSTANT FORM OF DYNAMICS

The extension of our approach to special relativity needs a consistent formulation of RQT in inertial frames of Minkowski space-time, a corresponding Ehrenfest theorem, and a well-defined non-relativistic limit to QT in the inertial frames of Galilei space-time.

In non-relativistic physics, a classical theory of particles in Galilean space-time (where both time and 3-space are absolute notions) is usually worked out with the position of particles as basic preferred observables.

Newtonian classical mechanics is 'local' in a strict sense if external potentials acting on the particles are admitted but inter-particle (instantaneous) potentials \( V(\vec{x}_i - \vec{x}_j) \) are not ('non-local' case). Both in the 'local' and 'non-local' cases there is no problem with the definition of the Galilean center of mass. The Galilei generators (constants of motion for isolated systems) are 'local' and only the Galilei energy possibly depends on 'non-local' potentials \( V(\vec{x}_i - \vec{x}_j) \). The Galilei boosts do not depend on the potentials and can be used to define the Newtonian center of mass (unique canonical and covariant collective position variable), which is assumed to be a 'measurable' quantity. The above kinds of 'non-locality' are harmless in the non-relativistic theory. One can safely define a sub-system of an isolated system at least in absence of non-local potentials.

This viewpoint becomes problematic in special relativity and we will see that although relativistic classical mechanics of isolated systems in Minkowski space-time hatches the same two sources of 'non-locality' as the non-relativistic case, in a relativistic framework these give way to significant consequences.

A. Problems of the Relativistic Framework

The lack of a well-defined quantum-relativistic framework, independent of quantum field theory yet compatible with it (e.g. such to allow a description of charged particles interacting with the electro-magnetic field) is at the origin of many problems for the formulation of relativistic extensions of various existing proposals for the solution of the quantum measurement problem.

The basic problems affecting this attempt, essentially residing in the Lorentz signature of Minkowski space-time, gave rise in the last century to a vast literature, starting from Fokker (see Ref.[25–27] and the bibliography therein). They are:

A) The lack of an absolute notion of 3-space and the ensuing necessity to choose a convention for the synchronization of distant clocks.

B) The structure of the Poincaré group. While in the realizations of the (mass-)extended Galilei algebra the energy generator only depends on the mutual particle interaction (sum of the kinetic and the potential energy), in the (Dirac) instant form of relativistic dynamics both the energy and the Lorentz boost generators are interaction dependent. Since the actual structure of the Poincaré generators follows from the energy-momentum tensor of an isolated system, such generators are consequently 'non-local' quantities in the sense of requiring information about the whole 3-space \( \Sigma_\tau \) identified by the clock synchronization convention.
C) The lack of a collective variable for an isolated system having all the properties of the Newton center of mass (see Refs. [15–17, 27]). As is well known, the only intrinsic (i.e. containing information about the isolated system alone) relativistic collective variables (replacing the Newtonian center of mass and tending to it in the non-relativistic limit) are the canonical non-covariant Newton-Wigner center of mass (or center of spin), the non-canonical covariant Fokker-Pryce center of inertia and the non-canonical non-covariant Møller center of energy. All these can be built in terms of the Poincare’ generators alone. They are all 'non-local', non measurable quantities due to the 'non-locality' of the generators.

D) In isolated systems 'non-local' potentials must depend on space-like relative variables only. This is a consequence of the elimination of relative times in the theory of relativistic bound states to avoid time-like levels not present in spectroscopic data. Consequently, even in the free case, sub-systems of the isolated system are kinematically interconnected in the physical Hilbert space ('spatial non-separability'; 'weak relationism'; no intrinsic definition of subsystems). This kinematical 'non-locality' allows the presence of 'non-local' potentials of the Coulomb or Darwin type, e.g. in the treatment of systems of charged particles plus the electro-magnetic field (after the gauge fixation of the un-physical gauge variables of the electro-magnetic field). Such 'non-locality' is allowed by special relativity so that, given the Cauchy data, no problem arises concerning time evolution.

Usually, QT is considered 'local' in the sense of the no-signalling theorem and 'non-local' for the EPR correlations among sub-systems. 6.

Let us stress that the 'kinematical non-locality' and 'spatial non-separability' connected with Lorentz signature imply that the relativistic description is intrinsically 'non-local in the previous sense before taking into account the specific sense of the quantum EPR 'non-locality'. We shall refer to these two kinds of 'non-locality' appearing in relativistic quantum formulations as 'kinematical' and 'causal', respectively.

Let us note, incidentally, that the 'kinematical non-locality' must be taken into account at the non-relativistic level if the 1/c coupling to the electro-magnetic field must be taken into account, as in atomic and molecular physics.

B. Relativistic Quantum Mechanics

In Ref.[16] a consistent formulation of RQT is expounded which takes into account all the known problems implied by the Lorentz signature. This formulation is obtained as the restriction to inertial frames of the parametrized Minkowski theories of Ref.[25], [17], describing isolated systems in arbitrary admissible non-inertial frames (see Ref.[26]). This inertial formulation, the rest-frame instant form of dynamics, is originally developed in the rest-frame of the isolated system (and then re-expressed in arbitrary inertial frames) after the decoupling of the external, non-covariant, non-local, non-measurable canonical center of mass. The clock synchronization convention corresponds to a 3+1 splitting of space-time centered on the Fokker-Planck center of inertia of the system (as an observer) and exploits radar 4-coordinates: a proper time \( \tau \) and 3-coordinates having the observer as origin. In the resulting Euclidean so-called Wigner 3-spaces of the rest frame, N positive-energy particles

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6 This issue is still debated, as can be seen in the last Section of Ref.[28]
are described by N Wigner spin-1 position 3-vectors $\vec{\eta}_i(\tau)$, $i=1,..,N$, and by 3-momenta $\vec{\kappa}_i(\tau)$. $\sum_i \vec{\kappa}_i \approx 0$ is the rest-frame condition and the 3-coordinates of the internal 3-c.m. are eliminated by the vanishing of the internal (interaction-dependent) Lorentz boosts (in the free case we have $\sum_i \vec{\eta}_i \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} \approx 0$). Therefore, here too, only relative variables survive.

The world-lines $x_i^\mu(\tau)$ of the particles are derived (interaction-dependent) quantities and in general do not satisfy vanishing Poisson brackets: note then that at the classical level a kind of non-commutative structure already emerges due to the Lorentz signature of space-time. The world-lines are built in terms of the Fokker-Pryce center-of-inertia 4-vector $Y^\mu(\tau)$, the conserved 4-momentum $P^\mu$ of the isolated system and the 3-positions $\vec{\eta}_i(\tau)$, as shown in Ref.[16] $x_i^\mu(\tau) = Y^\mu(\tau) + \epsilon_\mu^\nu(\vec{h}) \eta_i^\nu(\tau) \footnote{Where $\vec{h} = \vec{P}/\sqrt{\epsilon_\mu^\nu P^\mu P^\nu}$ and $\epsilon_\mu^\nu(\vec{h})$ are the spatial columns of the standard Wigner boost bringing the time-like 4-vector $P^\mu$ at rest.}$. The standard 4-momenta $p_i^\mu(\tau)$, satisfying $p_i^2 = m_i^2 c^2$ in the free case, are derived quantities too.

As shown in Ref.[16], taking into account the elimination of the relative times needed for a consistent treatment of relativistic bound states [26, 29, 30] requires the description of RQT in a Hilbert space of the type $H = H_{c.m} \otimes H_{rel}$ where $H_{rel}$ is the Hilbert space of the relative variables. $H_{c.m}$ is the Hilbert space of the decoupled non-covariant relativistic center of mass, which must be described in terms of frozen Jacobi data to avoid Hegerfeldt theorem (instantaneous spreading of c.m. wave packets). These Jacobi data are described by a self-adjoint velocity operator $\hat{h}$ and by an operator $\hat{z}$, whose classical expression is the product of the rest mass of the isolated system for the initial value of the classical analogue of the Newton-Wigner position operator. As explained in Appendix A, this operator and therefore even $\hat{z}$, cannot be taken as self-adjoint. This implies bad localization of this non-measurable quantity. The operator $\hat{Y}^\mu$, corresponding to the classical Fokker-Pryce center of inertia, is defined in terms of the quantum Jacobi data and of the quantum internal mass and rest spin after the choice of a suitable operator ordering.

Because of the elimination of relative times, this Hilbert space is not unitarily equivalent to the standard separable Hilbert space $H_1 \otimes H_2 \otimes ...$ built in terms of the single particle Hilbert spaces $H_i$, like in the non-relativistic case. It is seen that, unlike the standard non-relativistic situation, the Lorentz signature implies not only kinematical non-locality but also non separability.

The non-relativistic limit of the RQT given in Ref.[16] reproduces the standard QT in the center-of-mass and relative variable configuration representation after a unitary transformation to the Hamilton-Jacobi description of the Newton center of mass (frozen center of mass limit of the relativistic Jacobi data). Since atomic and molecular physics require the presence of the electromagnetic field at the order $1/c$ (at this order neither the Galilei nor the Poincaré group are implemented), in such experimental situations, even non-relativistic QT, being the $c \rightarrow \infty$ limit of RQT, should be presented in this way$^8$.

While in QT one can consider one particle with external interaction (with non-conserved Galilei generators), in RQT if we expect to maintain the Poincaré generators and the kind

$^7$ For a review of the results obtained so far along these lines, see Ref.[15], where the treatment of the gravitational field (at the classical level) is also given.
of interactions compatible with Poincaré algebra [26, 29, 30] under control, a one particle system must be treated as an open subsystem of an isolated N-particle system.

Here, we will therefore consider an isolated system of two free particles as the simplest non trivial model for studying the transition from the quantum to the classical relativistic world. Interactions can be introduced along the lines of Refs.[26, 29, 30]. After separating the external free decoupled center of mass and having satisfied the rest-frame conditions, we get a Schroedinger equation for a Lorentz scalar wave function \( \psi(\vec{\rho}, \tau) \) on the Wigner 3-space, depending only on the relative variable \( \vec{\rho} = \vec{n}_1 - \vec{n}_2 \) in the configuration representation

\[
i\hbar \frac{\partial}{\partial \tau} \psi(\vec{\rho}, \tau) = \left( \sqrt{m_1^2 c^2 + \hat{\vec{\rho}}^2} + \sqrt{m_2^2 c^2 + \hat{\vec{\pi}}^2} \right) \psi(\vec{\rho}, \tau) = \hat{H} \psi(\vec{\rho}, \tau),
\]

(5.1)

where \( \hat{\vec{\pi}} = i\hbar \frac{\partial}{\partial \vec{\rho}} \) is the relative momentum operator conjugate to the relative position operator \( \hat{\vec{\rho}} \).

Modulo ordering problems, the operators \( \hat{\vec{n}}_i, i = 1,2 \), associated with the localization of the two particles inside the Wigner 3-spaces, can be expressed as well defined functions of \( \hat{\vec{\rho}} \) and \( \hat{\vec{\pi}} \) starting from the classical expressions given in Ref.[26].

A first technical problem with Eq.(5.1) is the necessity of exploiting pseudo-differential operators [31] in order to give a meaning to quantities like \( \sqrt{m_i^2 c^2 + \hat{\vec{\pi}}^2} \).

With this technique we can study the space of solutions of the relativistic Schroedinger equation. We can associate a density function \( \rho(\vec{o})(\vec{\rho}, \tau) = |\psi(\vec{\rho}, \tau)|^2 \) and the general Wigner distribution function to each normalizable solution, like in the non-relativistic case. We can then define their multipolar expansions around a classical trajectory \( \vec{\rho}_\psi(\tau) \), which at any instant of proper time \( \tau \) defines a relative position 3-vector in the Wigner 3-space \( \Sigma_\tau \).

A problem with such pseudo-differential operators in the quantum Hamiltonian is the necessity of characterizing the function space needed as Hilbert space in such a way that a relativistic version of the Ehrenfest theorem holds for the expectation values of the relative position and relative momentum operators, as a consequence of the Heisenberg equations

\[
\frac{d}{d\tau} < \vec{\rho} >_{\psi(\tau)} = < \vec{\rho} >_{\psi(\tau)} \frac{d}{d\tau} \psi(\tau) = 0,
\]

\[
\frac{d}{d\tau} < \vec{\pi} >_{\psi(\tau)} = 0.
\]

(5.2)

In the interacting case there are two types of interacting classes with a complete control of the Poincaré algebra [26, 29, 30]: A) \( \hat{H} = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + V(\vec{\rho}_i^2)} + \hat{\vec{\pi}}^2 \) with arbitrary potential \( V \); B) \( \hat{H} = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \hat{\vec{\rho}}^2} + V(\vec{\rho}_i^2) \), but only for special potentials like Coulomb plus Darwin.

\[9\] In the interacting case there are two types of interacting classes with a complete control of the Poincaré algebra [26, 29, 30]: A) \( \hat{H} = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + V(\vec{\rho}_i^2)} + \hat{\vec{\pi}}^2 \) with arbitrary potential \( V \); B) \( \hat{H} = \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \hat{\vec{\rho}}^2} + V(\vec{\rho}_i^2) \), but only for special potentials like Coulomb plus Darwin.
where \( \hat{v} \) is the 3-velocity operator. Thus, in this case, we can invert the relation between \( \hat{v} \) and \( \hat{\pi} \) to get \( \hat{\pi} = \frac{m \hat{v}}{\sqrt{4 - \hat{v}^2}} \) in the equal mass case, recovering the relativistic standard connection between 3-momentum and 3-velocity.

If the dipole of the density matrix vanishes, we get \(< \hat{\rho} >_{\psi(\tau)} = \check{\rho}_\psi(\tau) \). Then for EMWFs from the first half of Ehrenfest theorem we obtain

\[
\check{\rho}_\psi(\tau) = < \hat{v} >_{\psi(\tau)},
\]

\[
\Rightarrow \quad < \hat{\pi} >_{\psi(\tau)} = \frac{m \check{\rho}_\psi(\tau)}{4 - \hat{\rho}_\psi^2(\tau)}, \quad (equal\ mass\ case).
\]

while the second part of Ehrenfest theorem provides the deterministic second order relativistic equation \( \check{\rho}_\psi(\tau) = 0 \).

In presence of interactions, the calculations become very complex. We see, however, that the emergence of 'classical' results for EMWF follows the same pattern of the non-relativistic case, in the center-of-mass - relative configuration representation for the RQT of Ref.[16].

In conclusion, for a 2-particle system only the relative variable in the Wigner 3-spaces can sustain a classical behavior in the sense that, if the dipole of the density matrix vanishes and the wave function is of the EMWF type, the system can described by a 'classical' relative variable.

On the other hand, the decoupled external ('non-local', 'non-measurable') center of mass is described by a plane wave corresponding to the conserved momentum of the isolated system and, like in the non-relativistic scattering cases, it does not admit a classical visualization.

The reconstruction of effective classical world-lines requires the study of the expectation value of the operators \( \hat{x}_i^\mu = \hat{Y}^\mu + \epsilon_\mu (\hat{h}) \hat{\eta}_i^\mu \) among wave functions whose part describing the relative motion is an EMWF. This is a difficult task which we intend to deal with in a future paper, trying to take into account the problems of localization described in Appendix A. In any case, the experimental determination of the world-line will lead to the localization of effective relativistic massive particles in a world-tube with a radius of the order of nanometers, i.e. much larger than the Compton wavelength of the isolated system.

The extension to N-particle systems is straightforward.
VI. ON THE POSSIBLE RELEVANCE OF CLASSICAL 'EFFECTIVE' Trajectories in Actual Experiments Involving Massive Particles

As previously stated, we believe that the proposed notion of classical 'emergence'- specifically the effective monopole massive particles - justifies the actual efficacy of the quasi-classical Newtonian intuition implicitly used by atomic and molecular physicists in devising experiments and describing the properties of complex systems. This is true, in particular, for the experimental description of the preparation, the intermediate steps and the final outcome of experiments in particle, nuclear, atomic and molecular physics, where the 'particle' aspect dominates. Actually in the preparation of all the existing experiments there are classical macroscopic apparatuses creating a beam of quantum particles with a mean classical trajectory and a mean momentum, described in terms of Newtonian or relativistic beams of effective quasi-classical particles. This is true from particle accelerators to atoms localized in a cavity. We argue that the wave functions describing the input beams should have small dipoles approximating EMWFs. It seems that no conceivable experiment can avoid, at a certain level of approximation, this quasi-classical description. Note that the preparation stage should often be considered independently of the type of measurement to be made in the subsequent stages of the experiment.

Let us remark that in all the experiments in which the relevant system is a 'quantum massive particle', notwithstanding the criticism of the POVM approach [32], we meet the so-called problem of the 'wave-particle' dualism: a single electron in QT is, strictly speaking, neither a particle nor a wave, but sometimes its 'particle aspects' dominate while in other cases the 'wave aspects' do so. Both the 'particle aspects' and the 'wave aspects' are summarized by the wave function of the electron and become dominant in different experiments.

All this is of course an idealization. As already observed, the preparation of both scattering experiments (particles at CERN, atoms, molecules) and the double slit experiment require collimated beams of effective particles and the effective mean momentum (and mean energy) of the beam is defined with time-of-travel methods (in Refs. [32] it is said that this quantity does not derive from the momentum operator of a quantum particle, which instead is needed for the indetermination relations). It is important to stress that only when the mean energy of a collimated beam is below a certain threshold we can speak of an 'effective one particle' in the beam.

In what follows we discuss atom interferometers and the double slit experiment as two relevant instantiations of the 'particle' and 'wave' aspects.

A. Effective Particles in the Arms of Atom Interferometers from Feynman Path Integral

Atom interferometry is a rapidly developing area of research in which interference effects \( \psi = \psi_1 + \psi_2 \) with \( \psi_i \) describing the atom beam in the arm \( i \) compatibly coexist with a quasi-classical description of the beams in the two arms, in terms of effective atoms, based on Feynman path integral approach. As shown in Refs.[33] this is possible because in each arm the dominating classical trajectory generates a classical action \( S_{cl}(\vec{x}_f, t_f; \vec{x}_i, t_i) \gg \hbar \). This classical action is the main tool for determining the phases inside the atom interferometer.
and for generating the classical trajectories that, in presence of Newtonian gravity, allow for instance to perform measures of $g$ and test the universality of free fall.$^{10}$

Our results are compatible with this picture, suggesting that the wave functions in the two arms have associated density matrices $|\psi_i|^2$ with small dipole and almost EMWF wave functions: this is a condition on the modulus of the path integral and not on its phase. The non-Newtonian forces we get for non-quadratic potentials should be connected with higher order terms beyond the classical action.

### B. Wave and Particle Aspects in the Double Slit Experiment

In the double slit experiment with massive particles one traditionally emphasizes their wave aspect mainly due to the importance of the interference effects involved. However, particle aspects must be considered too for a satisfactory account of the whole phenomenon. In order to clarify this point we shall make recourse to some mathematical aspects of Bohmian mechanics, disregarding its ontological interpretation. Note that our approach helps to clarify aspects of the problem of the classical regime without any implication concerning interpretational problems of QT.

The starting point of the ‘pilot wave’ description of QT $[8,9]$ is a rewriting of the Schrödinger equation in a form reminiscent of the eikonal approximation of classical electromagnetic waves, where a wave is represented as a wave front and a congruence of rays (the light rays, trajectories of unspecified massless photons) orthogonal to it$^{11}$. The wave function of the quantum massive particle is written in the form

$$\psi(\vec{x},t) = \sqrt{\rho_{(o)}(\vec{x},t)} e^{i\frac{\hbar}{m} S(\vec{x},t)}, \quad \rho_{(o)} = |\psi|^2,$$

where $\rho_{(o)}$ is the density matrix and $S$ an action variable.

Then the Schrödinger equation with Hamiltonian $\frac{\vec{p}^2}{2m} + V$ is rewritten as the following pair of coupled equations:

1) A guidance equation that is the continuity equation (2.1) $\partial_t \rho_{(o)}(\vec{x},t) = \vec{\partial} \cdot \vec{\rho}_{(1)}(\vec{x},t)$.

The Bohm velocity field, i.e. the field of the tangent vectors to the effective rays, is

$$\vec{v}_B(\vec{x},t) = \frac{\vec{\rho}_{(1)}(\vec{x},t)}{\rho_{(o)}(\vec{x},t)} = \frac{\hbar}{m} m \frac{\vec{\partial} \psi(\vec{x},t)}{\psi(\vec{x},t)} = \frac{1}{m} \vec{\partial} S(\vec{x},t).$$

The Bohm velocity field is the velocity field of a fluid of hydrodynamic type whose fluid elements follow the Bohmian trajectories. In the double slit experiment, the initial points can be any of the points inside the two slits.

2) An equation of the Hamilton-Jacobi type for the action $S(\vec{x},t)$

$$-\partial_t S(\vec{x},t) = \frac{1}{2m} \left( \vec{\partial}^2 S(\vec{x},t) \right)^2 + V(\vec{x},t) + Q(\vec{x},t),$$

$^{10}$ See the discussion on the possibility of testing the gravitational redshift with quantum atoms in Refs.[34].

$^{11}$ As well-known, according to Bohmian ontology, such effective rays are interpreted as Bohmian trajectories of added ‘real’ particles.
where $Q(\vec{x}, t)$ is the so-called quantum potential and has the following expression

$$Q(\vec{x}, t) = -\frac{\hbar^2}{2m} \frac{\partial^2 \sqrt{\rho_o(\vec{x}, t)}}{\sqrt{\rho_o(\vec{x}, t)}}.$$  \(6.4\)

Note that Eq.(6.3) resembles the Hamilton-Jacobi equation for a classical material particle.

Eqs.(6.2) and (6.3) imply a quasi-Newtonian equation of motion for the fluid similar to the classical Euler equation

$$m \frac{d}{dt} \vec{v}_B(\vec{x}, t) = m \left( \partial_t + \vec{v}_B(\vec{x}, t) \cdot \vec{\partial} \right) \vec{v}_B(\vec{x}, t) = -\vec{\partial} \left( V(\vec{x}, t) + Q(\vec{x}, t) \right). \tag{6.5}$$

Independently of Bohm’s ontological interpretation, if $\vec{x}_B(t)$ is formally a Bohmian trajectory, we can use the multipolar expansions (2.4) and (2.5) around it in Eq.(6.2) to get the following expression

$$\vec{\rho}(1)(\vec{x}, t) = \vec{\rho}_o(\vec{x}_\psi(t), t) \delta^3(\vec{x} - \vec{x}_\psi(t)) + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, \ldots, r_n} \vec{\rho}^{r_1, \ldots, r_n}_{(1)\rho}(\vec{x}_\psi(t), t) \frac{\partial^n \delta^3(\vec{x} - \vec{x}_\psi(t))}{\partial x^{r_1} \ldots \partial x^{r_n}} =$$

$$= \delta \left( S(\vec{x}, t) \rho_o(\vec{x}, t) = \delta \left( \rho_o(\vec{x}_\psi(t), t) \delta^3(\vec{x} - \vec{x}_\psi(t)) + \right. \right. + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, \ldots, r_n} \vec{\rho}^{r_1, \ldots, r_n}_{(o)\rho}(\vec{x}_\psi(t), t) \frac{\partial^n \delta^3(\vec{x} - \vec{x}_\psi(t))}{\partial x^{r_1} \ldots \partial x^{r_n}} \right), \tag{6.6}$$

whose monopole term is (Eq.(2.6) is used)

$$\vec{\rho}(1)_{\rho}(\vec{x}_B(t), t) = \langle \hat{\vec{p}}_\psi(t) >= m \vec{v}_B(\vec{x}_B(t), t) + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \sum_{r_1, \ldots, r_n} \vec{\rho}^{r_1, \ldots, r_n}_{(o)\rho}(\vec{x}_B(t), t) \frac{\partial^n \delta S(\vec{x}_B(t), t)}{\partial x^{r_1} \ldots \partial x^{r_n}}. \tag{6.7}$$

No statement can be made, at this stage, on the vanishing of the dipole term $\vec{\rho}_{(o)1}(\vec{x}_B(t), t)$ of the density matrix.

According to Bohm’s ontology, the effective rays are real particles hitting on the detector so that - in our interpretation - near that point of the detector, the Bohmian trajectory should be reinterpreted as an effective monopole particle with $\vec{\rho}_{(o)1}(\vec{x}_B(t), t) = 0$, i.e. having vanishing dipole and being an EMWF. Now, when is this picture valid?

Consider the double slit experiment. The Bohmian trajectories emanating from the two slits correspond to a fluid whose fluid trajectories rarefy in the spatial regions corresponding
to the dark fringes on the screen, where the wave function is nearly zero or of order $O(\hbar)$ (if it is represented as a Feynman path integral this means that there is destructive interference between the Feynman paths).

On the other hand, in the regions corresponding to the bright fringes of the screen there is a concentration of fluid trajectories. In each point of this region the density matrix is near a maximum (where we have $\rho_{ij}(\vec{x},t) \approx \text{const.}$ with all spatial derivatives small), around which the quantum potential $Q(\vec{x}_B(t))$ of Eq. (6.7) is very small (of order $O(\hbar^2)$), so that Eq. (6.3) is approximately the Hamilton equation of a classical particle. Then $S(\vec{x}_B(t),t) \approx S_{\text{clas}}(\vec{x}_{\text{clas}}(t),t)$, where $S_{\text{clas}}$ is a classical action evaluated along a classical trajectory $\vec{x}_{\text{clas}}(t) \approx \vec{x}_B(t)$. By exploiting the stationary phase approximation, one finds that in such points of the screen, the Feynman path integral representation of the double slit wave function $\Psi$ is dominated by the path $\vec{x}_{\text{clas}}(t)$ and has the form $\psi(\vec{x}_{\text{clas}}(t),t) \approx \sqrt{\rho_{ij}(\vec{x}_{\text{clas}}(t),t)} e^{iS_{\text{clas}}(\vec{x}_{\text{clas}}(t),t)} \approx \text{const.} e^{iS_{\text{clas}}(\vec{x}_{\text{clas}}(t),t)}$. This means, however, that like in the arms of the atom interferometers, in the points of the screen corresponding to the bright fringes the wave function is well approximated by an EMWF (modulo corrections of order $O(\hbar^2)$) corresponding to an effective monopole particle with an effective trajectory $\vec{x}_{\text{clas}}(t)$ hitting the screen. Since $S_{\text{clas}}$ is a stationary action, its variations are small so that Eq. (6.7) implies $<\hat{p}_i(t)> \approx m \vec{v}_B(\vec{x}_B(t),t)|_{\vec{x}_B(t)=\vec{x}_{\text{clas}}(t)}$ and Eqs. (2.7) and (2.8) imply $<\vec{x}>_{\psi(t)} = \vec{x}_{\text{clas}}(t) + O(\hbar^2) \approx \vec{x}_B(t)$ and $<\hat{p}>_{\psi(t)} = \hat{p}_\psi(t) = m \vec{x}_{\text{clas}}(t) \approx m \vec{v}_B(\vec{x}_B(t) \approx \vec{x}_{\text{clas}}(t),t)$.

Let us stress that the non-Newtonian forces of the second order Newton-like equations of motion of Eq. (2.9), acting on an effective monopole particle, are not derivable from a potential (like in general happens with viscous forces). In the chain of approximations described above, they replace the effect of the quantum potential in the Hamilton equation (6.3) and this enlightens the difference between the Feynman path $\vec{x}_{\text{clas}}(t)$ and the Bohmian trajectory $\vec{x}(t)$ near the points of the bright fringes.

If $\chi_i^{(\text{bright})}$, $-\infty < i < \infty$, are the characteristic functions identifying the regions of the bright fringes on the screen and $\chi_j^{(\text{dark})}$, $-\infty < j < \infty$, the characteristic functions identifying the regions of the dark fringes, the wave function on the screen is well approximated by an expression of the type $\psi(\vec{x}\text{screen},t) \approx \sum_{i=-\infty}^{\infty} \chi_i^{(\text{bright})} \psi_i^{(\text{bright})}(\vec{x}\text{screen},t) + \sum_{j=-\infty}^{\infty} \chi_j^{(\text{dark})} \psi_j^{(\text{dark})}(\vec{x}\text{screen},t)$, where $\chi_i^{(\text{bright})} \psi_i^{(\text{bright})}(\vec{x}\text{screen},t)$ is an EMWF, modulo corrections of order $O(\hbar)$ and $\chi_j^{(\text{dark})} |\psi_j^{(\text{dark})}(\vec{x}\text{screen},t)|^2 \approx O(\hbar^2)$.

We can conclude, therefore, that even in the case of the double slit, the experiments with a beam of massive particles could be interpreted, with all the interference effects taken into account, as outcomes of a beam of effective monopole particles hitting the screen, rather than as real Bohmian particles.

Let us recall that Bohmian quantum trajectories have recently become a computational tool to obtain approximations in many-body problems connected with molecular dynamics, whenever it is possible to have a subset of variables of the molecule to be treated quantum mechanically under the influence of the rest of the system treated classically. This type of hybrid quantum/classical description exploits the wave packet of the subsystem to define

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12 See for instance Ref. [35] for the double slit experiment.
Bohmian quantum trajectories which, in turn, are used to calculate the force acting on the classical variables. See for instance Refs.[36–38] and the associated hydrodynamic formulation of QT [39]. Our approach suggests that it should be possible to recover the main results of these approaches by suitably reinterpreting Bohmian trajectories as *effective* trajectories associated with EMWFs.
VII. ON DE-COHERENCE AND THE TRANSITION FROM IMPROPER QUANTUM MIXTURES TO CLASSICAL STATISTICAL ONES

A. Preferred Role of Positions and Aspects of Classicality from De-coherence

A remarkable contribution to the clarification of the dynamics of QT and the problem of classical regime is due to the de-coherence-approach. The pervading presence of the environment in every realistic description of a macroscopic apparatus interacting with a microscopic system has led to a deeper understanding of the properties of entanglement. On the one hand, the de-coherence approach pointed to the generic existence of a selection of robust preferred positional bases of pointer states for the apparatuses (the only relevant collective variables in the many-body description of an apparatus that reveal the integer number resulting from the measurement). On the other hand, it showed the tendency to an extremely rapid decay of the off-diagonal interference terms in the reduced density matrix describing the relevant quantum degrees of freedom of the microscopic system and of the pointer.\(^{13}\)

However, it is now commonly agreed that de-coherence is not a solution to the quantum measurement problem (see, e.g., Refs.[1, 40, 41]). In particular, the reduced density matrix reflects a quantum improper mixture mixing 'ignorance and entanglement' in a non-separable way and not a classical statistical proper mixture based on 'ignorance' only. Consequently, the probabilistic Born rule describes the right experimental results but cannot be read in terms of classical probability.

Anyway, de-coherence supports a preferred role of position measurements: this fact, together with the assumption we made about the methodologically autonomous role of space-time vis-a-vis quantum theory, once again leads more to a preferred role of the coordinate representation in Hilbert space.

B. From Improper Quantum Mixtures to Classical Statistical Ones

Let us now exploit the results previously obtained in the two-particle case in order to show that multipolar expansions can be instrumental in describing the transition from quantum improper mixtures to classical statistical ones.

Consider a reduced density matrix (improper mixture), obtained for a 'system + apparatus + environment' 3-universe, by means of de-coherence (no off-diagonal terms after a short time) after having summed over all the irrelevant variables, except for a pointer position \(\vec{x}_1\) and a particle \(\vec{x}_2\)

\[\rho_{(\text{red,dec})}(\vec{x}_1, \vec{x}_2, t) = \sum_w p_w \rho_{(o)(w)}(\vec{x}_1, \vec{x}_2, t), \quad \sum_w p_w = 1,\]

\[\rho_{(o)w}(\vec{x}_1, \vec{x}_2, t) = <\vec{x}_1, \vec{x}_2|\hat{\rho}_{(o)(w)}|\vec{x}_1, \vec{x}_2> = |\psi_{w}(\vec{x}_1, \vec{x}_2, t)|^2,\]

\[\hat{\rho}_{(o)(w)} = |1, 2, w> <1, 2, w|,\] \hspace{1cm} (7.1)

\(^{13}\) After summation over the environment and the other many-body degrees of freedom of the apparatus.
and perform a multipolar expansion around the two trajectories $\vec{x}_1(t)$ and $\vec{x}_2(t)$, imposing the vanishing of all the three dipole terms, for each value of $w$. Moreover let us assume that all the wave functions be EMWF.

Then, we get

$$<\hat{x}_1^i \hat{x}_2^j>_{(red)} = \sum_w p_w <\hat{x}_1^i \hat{x}_2^j>_{(w)} = \sum_w p_w \int d^3x_1 d^3x_2 <1,2,w|\hat{x}_1^i \hat{x}_2^j|1,2,w> = \sum_w p_w \int d^3x_1 d^3x_2 x_1^i x_2^j \rho_{(o)(w)}(\vec{x}_1, \vec{x}_2, t) = \sum_w p_w x_{1w}^i(t) x_{2w}^j(t).$$

(7.2)

It is seen therefore that, after de-coherence has suppressed all the interference terms, the monopole term describes a classical statistical mixture (ignorance due to unspecified Cauchy data inside the volume of the apparatus, with probabilities $p_w$) obtained from an improper mixture (ignorance + entanglement with probability $p_w$).

The monopole provides a value $\vec{x}_{1w}(t)$ of the pointer and $\vec{x}_{2w}(t)$ of the particle position with probability $p_w$, in a standard measurement. The probabilities are the same as in the reduced matrix (namely in accord with the Born rule in the framework of de-coherence). Note that no statement is necessary here about the collapse of wave functions since it is implied that the measurement process is ruled by a unitary transformation up to the end. The underlying strong hypothesis is, of course, that the robust preferred basis selected by de-coherence must be not only positional but also such that all the dipoles (and also the interference terms) are negligible for every value of $w$.

In the relativistic case, in order to have the Poincaré group under control, we must consider isolated systems system + apparatus + environment, i.e. a 3-universe, where system + apparatus is an open subsystem. Another relevant open subsystem is formed by system + a collective variable of the apparatus (simulating the pointer). In any case, after the separation of the external non-local non-measurable center of mass described by the Jacobi data and the elimination of the internal center of mass inside the Wigner 3-spaces, one is left with a description of the 3-universe in terms of relative variables only. Relativistic de-coherence must be described in terms of these. The strategy then is: (i) identifying the important relative variables to be measured in order to obtain information on the system (a crucial relative variable is the one connecting the pointer to the system); (ii) understanding their dynamics by treating the remaining relative variables perturbatively \(^{14}\); (iii) finding the reduced density matrix depending on the relevant relative variable by summing over all the other relative variables of the 3-universe; (iv) verifying whether the wave functions appearing in the reduced density are EMWFs so that even at the relativistic level one may get a transition from an improper quantum mixture to a classical statistical one, for measurements of the relevant relative variable.

\(^{14}\) Like for the Jacobi coordinates in the description of molecules: choose the Jacobi coordinates describing the dominating chemical bonds and treat the other chemical bonds perturbatively.
C. Beyond De-coherence: Open Quantum Systems

De-coherence describes relevant properties of a microscopic system plus a macroscopic apparatus plus environment, in which only a collective variable of the apparatus, the pointer, is selected and described. The difficulty of treating the quantum dynamics of the many-body degrees of freedom of the macroscopic apparatus is not really faced, being out of reach, and the final description is often similar to a standard instantaneous strong impulsive measurement. At best, there is an estimate of the time in which the off-diagonal terms of the reduced density matrix decay.

In Ref.[42] there is, instead, a full dynamic treatment of a spin 1/2 micro-system interacting with an apparatus of $N \gg 1$ spins (the pointer being simulated by a quantum dot described by a Curie-Weiss magnet) in equilibrium with a thermal bath of phonons (the environment). The total Hamiltonian is explicitly given. Before the interaction, the pointer is in a meta-stable paramagnetic state. The Hamiltonian evolution can be studied in detail and several time scales emerge in the basis determined by the interaction Hamiltonian. There is a very short reduction time during which the off-diagonal terms of the system-pointer density matrix decay: re-appearance after a long recurrence time is avoided by either de-coherence or a dynamic mechanism. The pointer plus the bath then goes out of equilibrium and the open quantum macro-system is described via methods similar to the ones of irreversible quantum statistical mechanics (though with finite $N \gg 1$ rather than with the thermodynamic limit) because the interaction with the micro-system leads to a break of ergodicity. In a so-called registration time, the pointer relaxes in a random way to one of the two possible ferromagnetic states. It should be noted, however, that the authors of Ref.[42] suggest reading the unique random outcome of single experiments, but we discarded their proposal for a solution of the quantum measurement problem in the framework of a statistical interpretation of QT from the beginning.

This example shows that, when one is able to face the dynamics in some way, more structure emerges with respect to the ordinary de-coherence approaches. See also Ref.[43] for other advances in the treatment of open quantum system which seems to be the main existing technique for dealing with quantum macro-systems. In any case, an open quantum subsystem generically does not evolve in a unitary way [44].

Unfortunately, when the relevant variables of the micro-system and of the pointer are positional, we lack analogous dynamic results in order to deal with the quantum measurement problem.
VIII. CONCLUSIONS

In this paper we have shown that, given the Hilbert space of the solutions of the Schrödinger equation in the preferred coordinate representation for a given quantum system involving massive scalar particles (either isolated or interacting with an external potential), it is possible to make a multipolar expansion of the associated density matrix (and the Wigner function) around a classical trajectory. By requiring that the quantum expectation value of the position operator coincides, at any time, with such classical trajectory, it follows that a sufficient condition for this coincidence is that the spatiotemporal multipolar expansion has a null dipole. The subset of wave functions giving rise to expectation values of position and momentum operators satisfying the Ehrenfest theorem have been named EMWF. The relevance of EMWFs is the emergence of a notion of effective classical particles in the sense that the trajectories chosen for the multipolar expansion turn out to be solution of second order equations of motion with forces determined by the quantum Hamiltonian and deviating from Newtonian (or relativistic) forces by terms in $\hbar^n$ ($n \geq 2$). Since these wave functions are solutions of the Schrödinger equation, their whole expansion contains all the information about the interference effects present in the given system in any experiment under study. These classical effective trajectories have nothing to do with Bohmian trajectories. Moreover, as highly desirable, the EMWFs do not satisfy a superposition principle.

We have also shown that the extension of these results to N-particle states allows us to make a transition from a 3N-dimensional configuration space to N effective classical trajectories in the Euclidean 3-space for a certain class of wave functions.

Our approach suggests the association of such classical effective particles with the 'particles' mentioned in the terminology of experimentalists and phenomenologists when they describe the preparation, the intermediate steps and the final outcome of all the actual experiments.

Also, the EMWFs provide a direction of solution to an open problem of de-coherence, since they allow to characterize the transition from improper quantum mixtures to classical statistical ones in special cases.

Our results have been extended to RQM by using the formulation given in Ref.[14].

In conclusion, our approach accepts the validity of QT and of its Born rule, with the associated randomness of the unique outcomes as a result of measurements, assuming only that the wave function describes a single quantum system and not an ensemble and that the mathematical structure of QT is defined in a given spatiotemporal background. This last assumption implies a preferred role of the position variables in accord with the privileged role of pointer positions, giving us a position signal as a result of the unique outcome of the measurements, in the theory of de-coherence. The same preferred role is present in Bohmian mechanics in a hidden variable framework. However, our classical Newtonian trajectories are not Bohmian trajectories and only play the role of effective emerging physical properties without ontological status. On the other hand, position variables are very complex mathematical quantities in QT, as shown in Appendix A. In recent developments they turn out to be at best unsharp observables described by POVM [45–47]. The implied bad localization is relevant for particle positions (tracks in bubble chambers, atoms fixed in a cavity...) but it is only a noise in the case of macroscopic pointers.

The emerging classical trajectories are relevant in supporting Bohr’s point of view for the description of the technically possible (not merely gedanken) experiments, but do not
imply any explicit choice among the existing interpretations of the theory of measurement. Their construction, however, allows to make some suggestions about the measurement issue if we observe that every measurement apparatus is a macroscopic quantum object, which should be described as a many-body system in a QFT for condensed matter, and that any measurement lasts a finite period of time (which can contain different physical time scales for the various stages of the measurement). Some position collective variable of the apparatus will describe the pointer. Since the system under measurement interacts with the apparatus and with a persisting environment, it is an open subsystem of the isolated system which is better described as 'quantum system + quantum apparatus + quantum environment'. The crucial point of this issue is that the mutual potentials acting on the subsystems of the global isolated system cannot in general be simulated as external potential acting on the subsystem to be investigated and this forbids time evolution of the latter from being represented by a unitary transformation, [44] even if the total isolated system evolves unitarily. In other words, a break of unitarity at the level of the subsystem is natural with or without the collapse hypothesis. Moreover, in many-body physics, stochastic processes are natural for both open quantum systems and thermodynamical reasons. All this means that the standard presentation of quantum systems as unitarily evolving under the action of an external potential up to measurement is not a realistic description of the real process but only an idealization implying a discontinuous break of unitarity in the evolution of a subsystem in an external potential when the standard description of an instantaneous measurement is given.

Actually, this idealized description of the standard collapse postulate together with the eigenvalue-eigenvector link makes use of the almost instantaneous von Neumann measurements, which are a very poor approximation to the complexity of the above many-body situation. New types of measurements (weak [48], protective [49],...) tend to extend the time of measurement. Probably the main support to the standard point of view indirectly comes from the success in interpreting atom spectroscopy. The spontaneous localization of GRW theory [7] is another way to avoid facing the many-body issue by simulating it with stochastic processes acting on the investigated system and suitable time scales. Ref.[42] is the most advanced attempt in treating a many-body problem in the only area (the spin systems) where we have enough results to allow to perform explicit calculations. In this way one gets various time scales for the measurement process which are compatible with (and even able to extend) the de-coherence results.

Our results on the emergence of classical Newtonian trajectories and on their role in the description of experiments in agreement with Bohr’s point of view suggest that in the idealized framework of an evolving wave function of a quantum system (the one in which the postulate of the ”wave function collapse” is introduced) the following scheme should be instrumental. Namely: decomposing the wave function on a suitable preferred position basis defined near the surface of the measuring apparatus (i.e., compatible with the geometry and structure of the detector; maybe in terms of an over-complete basis like coherent states), as was done in Subsection B of Section VI for the double slit experiment. In this way the wave function will appear as a superposition of many EMWFs, whose associated classical Newtonian trajectories identify different points on the surface of the apparatus. The suggestion is that, in each measurement, one of these EMWF components is randomly selected as the relevant initiator of a many-body amplification process leading to a unique reading of the pointer consistently with Born’s rule (using the coefficients of the decomposition on the preferred position basis). Since the readings of the pointer of the detector are interpretable
at the macroscopic level (in Bohr’s spirit) as a classical **effective** particle hitting the detector (tracks of particles, points on the screen of the double-slit experiment,...) the given suggestion seems reasonable. The other components of the wave function, not of the EMWF type, will not produce a reading on the pointer (think of the dark fringes in the double slit experiment).

In conclusion we obtained a kind of two-level structure. A probabilistic quantum world and a deterministic classical one, in a mathematically connected and consistent way. With this in view, the so-called Heisenberg cut between the two worlds becomes an even more evanescent divide.

A further clarification of this topic will require new experimental data in the mesoscopic energy region using the best modern technology and a mathematical effort to find suitable approximations to the many-body physics relevant to the theory of measurement.

An open problem is the extension of our approach from massive quantum particles to photons, in the sense of checking whether all the recent experimental developments in quantum optics allow a re-interpretation in terms of **effective monopole photons** moving along emergent classical **effective rays** of light. In this case, however, we have no Schroedinger equation and no Ehrenfest theorem for photons. On the other hand, e.g. in the case of the double-slit experiment with photons there are *weak measurements of effective average trajectories of single photons*[21] and proposals of Bohmian trajectories for photons obtained by studying spin 1 wave equations [50]. Furthermore, laser beams are visualized as Gaussian beams of classical light at the macroscopic level and the focalization of photons in the arms of interferometers can be visualized in terms of light rays of geometrical optics. Finally, other examples are given by single atoms trapped in cavities and interacting with laser beams.

In order to deal with this problem, one should start from the free quantum electromagnetic field in the radiation gauge (where there is a transverse vector potential $\vec{A}_\perp$ representing the magnetic field and its conjugate momentum, i.e. the transverse electric field $\vec{E}_\perp$) and describe it in the Fock space. One should then consider the expectation value of the two quantum fields between coherent states and compare the time derivatives of these expectation values with the Hamilton equations of a classical electro-magnetic field in the radiation gauge. Should an analogue of the Ehrenfest theorem emerge, we could imagine obtaining emerging Gaussian packets of classical light. Within an eikonal approximation, they should give rise to rays of light simulating the trajectories of **effective** photons.

Another non trivial open problem is the extension of our approach to massive particles with spin, described by Pauli and Dirac spinors.
Appendix A: Mathematical Problems with Localization of Positions

Our initial assumption was a space-time scenario in which the structure of QT must be formulated and discussed, even if its symbolic structure goes - so to speak - beyond space-time. This presupposition from the very beginning entails a privileged role of the position observables and position measurements. We know that even de-coherence ends up with a privileged role of position observables. However, the discussion of position measurements has not come to an agreed upon solution and many problems have not even reached an accepted formulation yet.

In this Appendix we recall the mathematical problems connected with the localization of massive particles in QT and RQT.

The position measurement for scalar massive particles forces us to face the problems of unbound position operators with a continuous spectrum. It is well-known that this is a source of highly non-trivial problems.

a) In the standard approach, the instantaneous precise measurement of self-adjoint bounded operators (physical observables) with a discrete spectrum is formalized by using projection operators (or projection valued measures, PVM) using the spectral representation. In this way one formulates the pragmatic notion of the 'non-unitary collapse' of the wave function (the projection postulate) and the eigenvalue-eigenvector link.

On the other hand, in the case of position measurements we have unbounded self-adjoint operators and no normalized position eigenstates (the improper Dirac kets |⃗x>, sharp eigenstates of the position operator, satisfying <⃗x|⃗y> = δ3(⃗x − ⃗y)).

As noted in Ref.[1], the notion of 'collapse' of the wave functions requires position wave functions with a finite support (inside the apparatus). Physically, this is connected to the fact that an arbitrarily precise measurement of position requires arbitrarily strong coupling to the system and an arbitrarily large amount of energy. Therefore one must generalize the standard formalism to include unsharp positions with bad localization. This formalism, also named projection-operator-valued-measures (POVM), was introduced in Refs.[45] and then developed in Refs. [46].

The results of measurements of a POVM give imprecise information of stochastic type on

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15 If n and |n> are the eigenvalues and eigenvectors of a self-adjoint bounded operator with discrete non-degenerate spectrum, the associated PVM is defined by the orthogonal projectors \( P_n = |n><n| \), \( P_n P_m = \delta_{nm} P_n \). \( \sum_n P_n = I_H \), where \( I_H \) is the identity operator in Hilbert space. If \( \rho \) is the density matrix of the measured system, the probability of the outcome n is \( P_n = Tr(\rho P_n) \).

16 A POVM is a set of n (n can be bigger than the dimension of the Hilbert space) self-adjoint non-orthogonal positive semi-definite operators \( E_i \) (the 'effects') satisfying \( \sum_{i=1}^{n} E_i = I_H \). If \( \rho \) is the density matrix of the measured system, the probability of the outcome associated with the (non repeatable) measurement of the operator \( E_i \) is \( P_i = Tr(\rho E_i) \). Neumark's dilation theorem guarantees that every POVM can be implemented as a PVM in a bigger Hilbert space \( H \otimes H_{ancilla} \), where \( H_{ancilla} \) can be imagined as describing a measuring apparatus: in this case one can write \( E_i = M_i \dagger M_i \) (\( M_i \) may be non-positive operators). Given the \( E_i \)'s, there are infinite solutions for the operators \( M_i \). This means that the POVM can arise by infinite different experimental apparatuses (each one described by a different \( H_{ancilla} \)).
the localization of the particle (see Refs.[47] for the status of continuous quantum position weak measurements\(^{17}\).)

b) The un-sharp POVM of positions analyzed in Ref.[46] have been recently used to extend the implications of the Wigner-Araki-Yanase (WAY) theorem\(^{18}\) [51] (see also Ref.[14]) to unbound self-adjoint operators. In Ref.[52] there is the analysis of the possibility of momentum-conserving measurements of the position of a quantum particle, as well as a discussion of the limitations imposed by the conservation law to a position measurement of the pointer of an apparatus. It turns out that, in repeatable measurements, the position to be measured (either the particle or the pointer) must commute with the (particle part) of the conserved total momentum. It is also shown that the WAY theorem precludes accurate and repeatable measurements of the pointer position (given the momentum conservation) unless the Yanase condition (i.e., the assumption that the pointer observable commutes with the conserved quantity) is fulfilled.

c) As shown in Ref.[53], the relativistic Newton-Wigner operator cannot be self-adjoint (again bad localization: macroscopic tracks of particles in bubble chambers) but only symmetric (see Ref. [54] for the bad localization implied by symmetric position operators).

d) Concerning the experimental and theoretical limits put on the measurements of position (center of mass) of an atom, see Refs. [55, 56]. In this case, the uncertainty of the localization (of the order of nanometers) is much bigger than the Compton wavelength of the atom itself!

Therefore, we still lack a well defined framework to be used in the area of the foundational problems of QT in all of the approaches, like de-coherence, where position measurements are crucial.

\(^{17}\) A continuous weak measurement is one in which information is continuously extracted from the system: one divides time into a sequence of intervals \(\Delta t\), considers a weak measurement in each interval, puts the strength of each measurement proportional to \(\Delta t\) and finally takes the limit \(\Delta t \to 0\). A possible POVM for the time interval \(\Delta t\) are the operators \(A(\alpha) = \left(\frac{4k\Delta t}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} dx e^{2k\Delta t(x-\alpha)^2} |x><x|\) with \(\alpha\) a continuous index. If the initial state is \(|\psi> = \int dx \psi(x)|x>\), the probability density of measuring the outcome \(\alpha\), when \(\Delta t\) is small, is \(P(\alpha) = Tr[A(\alpha)^\dagger A(\alpha)|\psi><\psi|]\). Consequently one gets the following result for the expectation values of the position \(<\alpha> = \int_{-\infty}^{\infty} d\alpha \alpha P(\alpha) = \int_{-\infty}^{\infty} dx x|\psi(x)|^2 = \langle x \rangle\).

A continuous measurement results if one makes a sequence of these measurements and takes the limit \(\Delta \to 0\). In this way more and more measurements are made in any finite time interval, but each of them is increasingly weak. Then, a stochastic Schroedinger equation emerges which describes the evolution of the original state \(|\psi>\) under the effect of the measurements.

\(^{18}\) The total angular momentum of the object plus apparatus cannot be conserved in an accurate and repeatable measurement of a particular component.
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