Equivalence Principle in the New General Relativity

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We study the problem of whether the active gravitational mass of an isolated system is equal to the total energy in the tetrad theory of gravitation. The superpotential is derived using the gravitational Lagrangian which is invariant under parity operation, and applied to an exact spherically symmetric solution. Its associated energy is found equal to the gravitational mass. The field equation in vacuum is also solved at far distances under the assumption of spherical symmetry. Using the most general expression for parallel vector fields with spherical symmetry, we find that the equality between the gravitational mass and the energy is always true if the parameters of the theory $a_1$, $a_2$ and $a_3$ satisfy the condition, $(a_1 + a_2)(a_1 - 4a_3/9) \neq 0$. In the two special cases where either $(a_1 + a_2)$ or $(a_1 - 4a_3/9)$ is vanishing, however, this equality is not satisfied for the solutions when some components of the parallel vector fields tend to zero as $1/\sqrt{r}$ for large $r$.

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§1. Introduction

It was shown by Møller\(^1\) that a tetrad description of gravitational field allows a more satisfactory treatment of the energy-momentum complex than general relativity (GR). The Lagrangian formulation of the theory was given by Pellegrini and Plebanski\(^2\). Hayashi and Nakano\(^3\) independently formulated the tetrad theory of gravitation as a gauge theory of the spacetime translation group. In the earlier attempts, admissible Lagrangians were limited by the assumption that the field equation has the Schwarzschild solution. Møller\(^4\) later suggested to abandon this assumption and to look for a wider class of Lagrangians. Meyer\(^5\) formulated the tetrad theory as a special case of Poincaré gauge theory\(^6,7\). Sáez\(^8\) generalized the theory into a scalar-tetrad theory.

Hayashi and Shirafuji\(^9\) studied the geometrical and observational basis of the tetrad theory of gravitation. Geometrically the tetrad fields are identified with the parallel vector fields defined by the underlying absolute parallelism. Incidentally they gave the name, new general relativity (NGR), to the theory of gravitation based on absolute parallelism, since Einstein\(^10\) was the first to introduce the notion of absolute parallelism in physics. They assumed the Lagrangian with three unknown parameters, denoted by \(a_1\), \(a_2\) and \(a_3\). In order to reproduce the correct Newtonian limit, the first two parameters should satisfy a condition called the Newtonian approximation condition, which allows us to express these two parameters in terms of an unknown dimensionless parameter \(\epsilon\). An exact, vacuum solution of the gravitational field equation was found for static, spherically symmetric case, where the parallel vector fields take a diagonal form. This solution describes the spacetime around a mass point located at the origin, and will be referred to as the "exact spherically symmetric solution" hereafter. Then comparison with solar-system experiments showed that the value of \(\epsilon\) should be very small. By contrast only an upper bound has been estimated for the remaining parameter \(a_3\).\(^9,11\) The singularity problem of the exact solution has been studied.\(^12,13\) If \((a_1 + a_2) = 0\), then the theory reduces to the one studied by Hayashi and Nakano\(^3\) and Møller.\(^4\)

The equation of motion for a test particle was discussed based on the response equation of the energy-momentum tensor of matter fields.\(^9\) In particular, when the intrinsic spin of the fundamental particles constituting the test particle can be ignored, the response equation reduces to the covariant conservation law of GR, and hence the world line of the test particle is a geodesics of the metric defined by the parallel vector fields. Accordingly, for spinless test particles the passive gravitational mass is equal to the inertial mass.

In GR the equality between passive gravitational mass and inertial mass is predicted to be valid also for massive bodies like planets which contains appreciable fraction (about \(5 \times 10^{-10}\) for the Earth) of gravitational self-energy. This equality means that gravity pulls on gravitational binding energy of a massive body just as it does on other forms of mass-energy, and is referred to as the strong equivalence principle.\(^14\) Nordtvedt\(^15\) developed the parametrized post-Newtonian (PPN) formalism and calculated the deviation from the unity of the ratio of gravitational mass to inertial mass in terms of the PPN parameters. Analysis of the lunar laser ranging (LLR) over past 24 years has confirmed that the Earth and the Moon accelerate equally to the sun within fractional difference less than \(5 \times 10^{-13}\), which yields the Nordtvedt parameter \(\eta = -0.0010 \pm 0.0010.\(^{16,17,18}\)

This parameter \(\eta\) (\(\equiv 0\) in GR) can be expressed
as \( \eta = 4\beta - \gamma - 3 \), in fully conservative theory which possesses a full complement of the post-Newtonian conservation laws: energy, momentum, angular momentum and center-of-mass motion.\[^{14}\] Here \( \beta \) and \( \gamma \) are the so-called Eddington-Robertson parameters.\[^{19,20}\]

Thus, if NGR is a fully conservative theory in the above sense, then the Nordtvedt parameter \( \eta \) vanishes, since the Eddington-Robertson parameters are expressed as \( \beta = 1 - \epsilon/2 \) and \( \gamma = 1 - 2\epsilon \) in terms of the dimensionless parameter \( \epsilon \) mentioned above. It can be shown that energy, momentum and angular momentum are conserved in NGR. We must be careful not to conclude that \( \eta = 0 \), however, since NGR is not a metric theory: Accordingly we are now trying to develop its PPN formalism.

It is the purpose of this paper to study a different aspect of the equivalence principle, namely the problem of whether or not the active gravitational mass (or simply the gravitational mass) of an isolated system is equal to its inertial mass, i.e. the total energy divided by the square of the velocity of light.\[^{21}\] As is well known, this problem is settled affrmatively in GR.\[^{21}\] In the case of \( (a_1 + a_2) = 0 \) Mikhail et al.\[^{22}\] calculated the energy of two spherically symmetric solutions, and found that the energy in one of the two solutions does not coincide with the gravitational mass. Shirafuji et al.\[^{23}\] extended the calculation to all the stationary asymptotically flat solutions with spherical symmetry, dividing them into two classes: The one in which the components, \( (b^a_0) \) and \( (b^{(0)}_\alpha) \), of the parallel vector fields \( (b^k_\mu) \) tend to zero faster than \( 1/\sqrt{r} \) for large \( r \) and the other in which those components go to zero like \( 1/\sqrt{r} \).\[^{†}\] It was found that the equality of the gravitational and inertial masses holds true only in the first class.

In this paper we study the problem for stationary, spherically symmetric systems without making any assumption on the parameters, \( a_1, a_2 \) and \( a_3 \) other than the Newtonian approximation condition.

We organize this paper as follows. In §2 we give a brief review of the NGR. In §3 we derive the superpotential of the total energy-momentum complex, and apply it to the exact spherically symmetric solution. In §4 we study stationary spherically symmetric solutions at far distances in the linear approximation. In §5 we calculate the energy in the special case of \( (a_1 - 4a_3/9) = 0 \), taking all the leading terms into account beyond the linear approximation. The final section is devoted to conclusion and discussion.

§2. A brief review of the NGR

We assume spacetime to admit absolute parallelism, i.e. to have a quadruplet of linearly independent parallel vector fields \( (b^k_\mu) \) satisfying

\[
D_\nu b^k_\mu = b^k_\mu,\nu - \Gamma^\lambda_{\mu\nu} b^k_\lambda = 0
\]

\[^{*}\]Since we use the unit \( c = 1 \), we need not draw a distinction between the inertial mass and the total energy

\[^{†}\]In this paper Latin indices \((i, j, k, ...)\) represent the vector number, which runs from \((0)\) to \((3)\), while Greek indices \((\mu, \nu, \rho, ...)\) represent the world-vector components running from \(0\) to \(3\). The spatial part of Latin indices are denoted by \((a, b, c, ...)\), while that of Greek indices by \((\alpha, \beta, \gamma, ...)\).
with \( b_{\mu,\nu}^k = \partial_\nu b_{\mu}^k \). Solving (1), we get the nonsymmetric connection

\[
\Gamma^\lambda_{\mu\nu} = b^k_\nu b^\lambda_{\mu,\nu},
\]

(2)

which defines the torsion tensor as

\[
T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = b^k_\nu (b^\mu_{\mu,\nu} - b^\mu_{\nu,\mu}).
\]

(3)

The curvature tensor defined by \( \Gamma^\lambda_{\mu\nu} \) is identically vanishing, however. The metric tensor is given by the parallel vector fields as

\[
g_{\mu\nu} = b_{k\mu}^k b_{k\nu}^k,
\]

(4)

where we raise or lower Latin indices by the Minkowski metric \( \eta_{ij} = \eta^{ij} = \text{diag}(-1,+1,+1,+1) \).

Assuming the invariance under

a) the group of general coordinate transformations,

b) the group of global Lorentz transformations, and

c) the parity operation,

we write the gravitational Lagrangian density in the form

\[
\mathcal{L}_G = \frac{\sqrt{-g}}{\kappa} \left[ a_1 (t^\mu_{\lambda\nu} t_{\mu\lambda}) + a_2 (v^\mu v_\mu) + a_3 (a^\mu a_\mu) \right],
\]

(5)

where \( a_1, a_2 \) and \( a_3 \) are dimensionless parameters of the theory, \( \kappa \) and

\[
t_{\mu\nu\lambda} = \frac{1}{2} (T_{\mu\nu\lambda} + T_{\nu\mu\lambda}) + \frac{1}{6} (g_{\lambda\mu} v_\nu + g_{\lambda\nu} v_\mu) - \frac{1}{3} g_{\mu\nu} v_\lambda,
\]

(6)

\[
v_\mu = T^\lambda_{\lambda\mu},
\]

(7)

\[
a_\mu = \frac{1}{6} \epsilon_{\mu\rho\sigma} T^{\nu\rho\sigma}
\]

(8)

with \( \epsilon_{\mu\rho\sigma} \) being the completely antisymmetric tensor normalized as \( \epsilon_{0123} = \sqrt{-g} \). By applying variational principle to the above Lagrangian, we get the field equation:

\[
I^{\mu\nu} = \kappa T^{\mu\nu}
\]

(9)

with

\[
I^{\mu\nu} = 2\kappa [D_\lambda F^{\mu\nu\lambda} + v_\lambda F^{\mu\nu\lambda} + H^{\mu\nu} - \frac{1}{2} g^{\mu\nu} L_G],
\]

(10)

* Throughout this paper we use the relativistic units, \( c = G = 1 \). The Einstein constant \( \kappa \) is then equal to \( 8\pi \). We will denote the symmetric part by the parenthesis ( ) and the antisymmetric part by the square bracket [ ].

† The dimensionless parameters \( \kappa a_i \) of Ref.9) are here denoted by \( a_i \) for convenience.
where

\[ F^{\mu\nu\lambda} = \frac{1}{2} b^{k\mu} \frac{\partial L_G}{\partial b^k_{\nu,\lambda}} = \frac{1}{\kappa} \left[ a_1 \left( t^{\mu\nu\lambda} - t^{\mu\lambda\nu} \right) + a_2 \left( g^{\mu\nu} v^\lambda - g^{\mu\lambda} v^\nu \right) - \frac{a_3}{3} \epsilon^{\mu\nu\lambda\rho} a_\rho \right] = -F^{\mu\nu}, \]  

(11)

\[ H^{\mu
u} = T^{\rho\sigma\mu} F_{\rho\sigma}^{\nu} - \frac{1}{2} T^{\nu\rho\sigma} F_{\rho\sigma}^{\mu} = H^{\nu\mu}, \]  

(12)

\[ L_G = \frac{L_M}{\sqrt{-g}}, \]  

(13)

\[ T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta L_M}{\delta b^{k\mu}}. \]  

(14)

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Here \( L_M \) denotes the Lagrangian density of material fields and \( T^{\mu\nu} \) is the material energy-momentum tensor which is nonsymmetric in general. In order to reproduce the correct Newtonian limit, we require the parameters \( a_1 \) and \( a_2 \) to satisfy the condition

\[ a_1 + 4a_2 + 9a_1 a_2 = 0 \]  

(15)

called the Newtonian approximation condition,\(^3\) which can be solved to give

\[ a_1 = -\frac{1}{3(1 - \epsilon)}, \quad a_2 = \frac{1}{3(1 - 4\epsilon)} \]  

(16)

with \( \epsilon \) being a dimensionless parameter. The comparison with solar-system experiments shows that \( \epsilon \) should be given by\(^3\)

\[ \epsilon = -0.004 \pm 0.004, \]  

(17)

which we assume throughout this paper.

It is well known that the conservation law in GR is given by

\[ T_{GR}^{\mu\nu} = 0, \]  

(18)

where \( T_{GR}^{\mu\nu} \) is the symmetric material energy-momentum tensor of GR and the semicolon denotes covariant derivative with respect to the Christoffel symbol. This law does not follow from (9), however. Instead, we can derive the response equation

\[ T^{\mu\nu}_{;\nu} = K^{\nu\lambda\mu} T_{[\nu\lambda]}, \]  

(19)

where \( K^{\nu\lambda\mu} \) is the contortion tensor given by

\[ K^{\nu\lambda\mu} = \frac{1}{2} \left( T^{\nu\lambda\mu} - T^{\lambda\nu\mu} - T^{\mu\nu\lambda} \right) = -K^{\lambda\nu\mu}. \]  

(20)
The antisymmetric part $T_{[\mu\nu]}$ is due to the contribution from the intrinsic spin of fundamental spin-1/2 particles. For macroscopic test particles for which the effects due to intrinsic spin can be ignored, their energy-momentum tensor can be supposed to be symmetric and satisfy (18). The equation of motion for such macroscopic test particles is then the geodesic equation of the metric.

In static, spherically symmetric spacetime the parallel vector fields take a diagonal form, and the field equation (9) can be exactly solved in vacuum to give

$$\left( b^k_{\mu} \right) = \begin{pmatrix} \left( \frac{1 - \frac{m}{pr}}{\left(1 + \frac{m}{qr}\right)^{q/2}} \right)^{p/2} & 0 \\ 0 & \left( \frac{1 - \frac{m}{pr}}{\left(1 + \frac{m}{qr}\right)^{q/2}} \right)^{(2-p)/2} \left( 1 + \frac{m}{qr} \right)^{(2+q)/2} \delta^a_{\alpha} \end{pmatrix}, \tag{21}$$

where $m$ is a constant of integration, and the constants $p$ and $q$ are given by

$$p = \frac{2}{(1-5\epsilon)}[(1-5\epsilon + 4\epsilon^2)^{1/2} - 2\epsilon], \quad q = \frac{2}{(1-5\epsilon)}[(1-5\epsilon + 4\epsilon^2)^{1/2} + 2\epsilon]. \tag{22}$$

In spherical polar coordinates (21) gives the line-element

$$ds^2 = -\left(1 - \frac{m}{pr}\right)^{p} + \left(1 - \frac{m}{pr}\right)^{2-p} \left( 1 + \frac{m}{qr} \right)^{2+q} \left[ dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \tag{23}$$

From the asymptotic behavior of the component $g_{00}$ of the metric tensor, the constant $m$ can be identified with the gravitational mass of the central gravitating system. It is clear that if the parameter $\epsilon$ vanishes, the line-element (23) coincides with the Schwarzschild solution written in the isotropic coordinates.

§3. Superpotential of the NGR and calculation of the energy

The following identity can be derived from the invariance of $\mathcal{L}_G$ under general coordinate transformations: \(^{1,24}\)

$$-\frac{\delta \mathcal{L}_G}{\delta b^k_{\nu}} b^k_{\mu} - \frac{\partial \mathcal{L}_G}{\partial b^k_{\lambda,\nu}} b^k_{\lambda,\mu} + \delta^\nu_\mu \mathcal{L}_G - \partial_\lambda \left( \frac{\partial \mathcal{L}_G}{\partial b^k_{\mu,\lambda}} b^k_{\mu} \right) \equiv 0. \tag{24}$$

When $(b^k_{\mu})$ satisfies the gravitational field equation (9), this implies

$$\sqrt{-g} \left( T^\nu_\mu + t^\nu_\mu \right) = \partial_\lambda \left( 2\sqrt{-g} F^\nu_\mu \right), \tag{25}$$
where \( t_{\mu}^{\nu} \) is the canonical energy-momentum complex of the gravitational field

\[
\sqrt{-g} t_{\mu}^{\nu} = -\frac{\partial L_G}{\partial b_{\lambda,\nu}^k} b^k_{\lambda,\mu} + \delta_{\mu}^{\nu} L_G.
\]  

(26)

The total energy-momentum complex is then defined by

\[
\mathcal{M}_{\mu}^{\nu} = \sqrt{-g} (T_{\mu}^{\nu} + t_{\mu}^{\nu}) = U_{\mu}^{\nu\lambda,\lambda}
\]  

(27)

with \( U_{\mu}^{\nu\lambda} \) being the superpotential.

\[
U_{\mu}^{\nu\lambda} = 2\sqrt{-g} F_{\mu}^{\nu\lambda}.
\]  

(28)

Since the tensor \( F_{\mu}^{\nu\lambda} \) is antisymmetric with respect to \( \nu \) and \( \lambda \), the \( \mathcal{M}_{\mu}^{\nu} \) of (27) satisfies ordinary conservation law,

\[
\partial_{\nu} \mathcal{M}_{\mu}^{\nu} = 0.
\]  

(29)

The total energy is now given by

\[
E = -\int \mathcal{M}_{0}^{0} d^3x = -\lim_{r\to\infty} \int_{r=\text{constant}} U_{0}^{0\alpha} n_\alpha dS,
\]  

(30)

where \( n_\alpha \) is the outward unit 3-vector normal to the surface element \( dS \).

Let us calculate the superpotential by writing the Lagrangian (5) in the form

\[
\mathcal{L}_G = \frac{\sqrt{-g}}{\kappa} \left[ a_1 L^{(1)} + a_2 L^{(2)} + a_3 L^{(3)} \right],
\]  

(31)

where \( L^{(1)} = t_{\lambda\mu\nu} t_{\lambda\mu\nu} \), \( L^{(2)} = v^\mu v_\mu \) and \( L^{(3)} = a^\mu a_\mu \). Writing (11) in the form

\[
F_{\mu}^{\nu\lambda} = \frac{1}{\kappa} \left[ a_1 F_{\mu}^{(1)} + a_2 F_{\mu}^{(2)} + a_3 F_{\mu}^{(3)} \right],
\]  

(32)

we get

\[
F_{\mu}^{(1)} = \frac{1}{2} \left[ 2T_{\mu}^{\nu\lambda} + T_{\lambda\mu}^{\nu\lambda} - T_{\nu\mu}^{\nu\lambda} - (\delta_{\mu}^{\nu} \nu\lambda - \delta_{\mu}^{\nu} \nu\lambda) \right],
\]  

(33)

\[
F_{\mu}^{(2)} = \left( \delta_{\mu}^{\nu} \nu\lambda - \delta_{\mu}^{\nu} \nu\lambda \right),
\]  

(34)

\[
F_{\mu}^{(3)} = -\frac{1}{9} \left[ T_{\mu}^{\nu\lambda} - T_{\lambda\mu}^{\nu\lambda} + T_{\nu\mu}^{\nu\lambda} \right],
\]  

(35)

*The Lagrangian used by Möller is different from that used by Hayashi and Shirafuji by a factor (-2). Accordingly, the definition (28) is different from that of Möller by a factor (-2).
where \( F^{(1)}_{\mu} \), \( F^{(2)}_{\mu} \) and \( F^{(3)}_{\mu} \) correspond to \( L^{(1)} \), \( L^{(2)} \) and \( L^{(3)} \), respectively. So with the help of (28) the superpotential of the NGR can be written as

\[
U_{\mu}^{\nu\lambda} = \frac{2\sqrt{-g}}{\kappa} \left[ \left( a_{1} - \frac{a_{3}}{9} \right) T^{\nu\lambda}_{\mu} + \left( \frac{a_{1}}{2} + \frac{a_{3}}{9} \right) \left( T^{\lambda\nu}_{\mu} - T^{\nu\lambda}_{\mu} \right) - \left( \frac{a_{1}}{2} - a_{2} \right) \left( \delta_{\mu}^{\nu} v^{\lambda} - \delta_{\mu}^{\lambda} v^{\nu} \right) \right].
\] (36)

As an example, let us apply (36) to the exact solution (21). The appropriate components \( U_{00}^{\alpha} \) are given by

\[
U_{00}^{\alpha} = -\frac{m}{2\kappa pq r^2} \left[ (p - q - 8)(a_{1} - 2a_{2}) \frac{m}{r} + 4(p - q)(a_{1} - 2a_{2}) + 4pq(2a_{1} - a_{2}) + 3(p - q) \frac{a_{1} m}{r} \right],
\] (37)

with \( p \) and \( q \) given by (22). Using (37) in (30), we get

\[
E = \frac{8\pi m}{\kappa pq} [(p - q)(a_{1} - 2a_{2}) + pq(2a_{1} - a_{2})]
= m \left[ (2a_{1} - a_{2}) - 2\epsilon(a_{1} - 2a_{2}) \right] = m,
\] (38)

where the relation \((p - q)/pq = -2\epsilon\), which follows from (22), is used in the second equality and (16) is employed in the last one. This means that the total energy of the exact solution is just the same as the gravitational mass of the central gravitating system.

§4. The spherically symmetric parallel vector fields and its energy in the linear approximation

The solution (21) is the only exact spherically symmetric solution in vacuum that we know at present, and it is not clear to us whether there exist any other spherically symmetric solutions in vacuum. In view of this let us consider a wider class of spherically symmetric solutions at far distances from the source. Consider an isolated, gravitating system with spherical symmetry, and restrict attention to the weak field far from the source. The most general form of the parallel vector fields is given in the Cartesian coordinates by

\[
\left( b^{k}_{\mu} \right) = \begin{pmatrix} C(r) & G(r)n^{\alpha} \\ H(r)n^{\alpha} & \delta_{\alpha}^{\beta} D(r) + E(r)n^{\alpha}n^{\alpha} + F(r)\epsilon_{\alpha\beta\gamma}n^{\gamma} \end{pmatrix},
\] (39)
where the two real functions $C(r)$ and $D(r)$ are supposed to approach 1 at infinity, while the remaining four real functions, $E(r)$, $F(r)$, $G(r)$ and $H(r)$, must tend to zero there. Here we define the radial unit vector $n^a$ and $n^\alpha$ by

$$n^\alpha = \frac{x^\alpha}{r} = \delta^\alpha_a n^a,$$  \hspace{1cm} (40)

without making distinction between upper and lower indices. Using the freedom to redefine the radial coordinate $r$, we can eliminate the function $E(r)$ from the components $(b^\mu_a)$. Accordingly we can put $E(r) = 0$ without loss of generality. The metric tensor $g_{\mu\nu}$ is then written as

$$g_{00} = -(C^2 - H^2),$$  \hspace{1cm} (41a)

$$g_{0\alpha} = \{-CG + DH\}n_\alpha,$$  \hspace{1cm} (41b)

$$g_{\alpha\beta} = (D^2 + F^2)\delta_{\alpha\beta} - (F^2 + G^2)n_\alpha n_\beta.$$  \hspace{1cm} (41c)

According to (30), the total energy of an isolated system can be calculated if the superpotential is known up to order $O(1/r^2)$. It is then enough to know the parallel vector fields up to order $O(1/r)$. So let us restrict our attention to the weak field far from the source, and analyze the field equation in vacuum up to order $O(1/r^3)$. This has been performed in GR, showing quite generally that the gravitational mass is equal to the total energy for any stationary isolated system. In the NGR, however, the analysis of the field equation in vacuum has not yet been done for the general case of $(a_1 + a_2) \neq 0$ even in the weak field approximation. This is due to the fact that the gravitational field equation of the NGR is more complicated than the Einstein equation of GR.

Let us suppose that the leading term of the five unknown functions is given by some power of $1/r$, and that $(b^k_\mu)$ can be represented as

$$b^k_\mu = \left( \frac{1 + b}{r^s} \right) \frac{j}{r^t} n^\alpha \left( \frac{h}{r^u} n^a \right) \left( 1 + \frac{d}{r^v} \right) \delta^a_\alpha + \frac{f}{r^w} \epsilon_{\alpha\beta\gamma} n^\beta.$$  \hspace{1cm} (42)

Here the powers, $s$, $t$, $u$, $v$ and $w$, are positive unknown constants at the beginning of the calculation, and their value will be determined by the field equation of NGR. Here the constant coefficients, $b$, $j$, $h$, $d$ and $f$, are also unknown, but they can be assumed to be nonvanishing without loss of generality since the powers of $r$ are left unknown. Use (42) in (41) gives the asymptotic behavior of the metric tensor which involves linear and quadratic terms of the unknown constants, $b$, $j$, $h$, $d$ and $f$: The linear terms are dominant if the powers satisfy the following inequality

$$s < 2u, \quad v < 2t, \quad v < 2w,$$  \hspace{1cm} (43)

* The constant $b$ should not be confused with $\text{det}(b^k_\mu)$, which we denote by $\sqrt{-g}$.  }
which we call the condition of the linear approximation.

Now apply the vacuum field equation (9) to (42), assuming the condition (43). Keeping only the leading terms, which are shown to be linear in the five unknown constants due to (43), we get the nonvanishing components of $I_{\mu\nu}$:

\[
I_{00} = -\frac{2}{r^2} \left\{ \frac{s(s-1)(a_1 + a_2)b}{r^s} - \frac{v(v-1)(a_1 - 2a_2)d}{r^v} \right\},
\]

\[
I_{\alpha0} = 2(u + 1)(u - 2) \frac{(a_1 + a_2)h}{r^{u+2}} n_\alpha,
\]

\[
I_{\alpha\beta} = \frac{n_\alpha n_\beta}{r^2} \left\{ \frac{s(s+2)(a_1 - 2a_2)b}{r^s} - \frac{v(v+2)(a_1 + 4a_2)d}{r^v} \right\} - \frac{\delta_{\alpha\beta}}{r^2} \left\{ \frac{s^2(a_1 - 2a_2)b}{r^s} - \frac{v^2(a_1 + 4a_2)d}{r^v} \right\} + (w + 1)(w - 2) \frac{(a_1 - \frac{4}{9}a_3)f}{r^{w+2}} \epsilon_{\alpha\beta\gamma} n_\gamma.
\]

Here it is important to notice that $I_{\alpha0}$ and $I_{[\alpha\beta]}$ are dominated by the linear term irrespectively of the condition of the linear approximation. Also using (42) in (36), we get the leading terms for the appropriate components of the superpotential:

\[
U_{00}^\alpha = - \frac{2n^\alpha}{kr} \left[ \frac{s(a_1 + a_2)b}{r^s} - \frac{v(a_1 - 2a_2)d}{r^v} \right].
\]

By virtue of (44) and (46) we can show that the equations, $I_{00} = 0$ and $I_{[\alpha\beta]} = 0$, give the following results:

1) if $s < v$ then $b = 0$, because $(a_1 - 2a_2) \neq 0$,

2) if $s > v$ then $d = 0$, because $(a_1 + 4a_2) \neq 0$, and

3) if $s = v \neq 1$ then $b = d = 0$, because

\[
\det \begin{pmatrix}
(a_1 + a_2) & -(a_1 - 2a_2) \\
(a_1 - 2a_2) & -(a_1 + 4a_2)
\end{pmatrix} \neq 0.
\]

All the above three cases contradict our assumption that $b \neq 0$ and $d \neq 0$. Therefore, we find that the powers $s$ and $v$ should be given by

\[
s = v = 1.
\]

Then the equation $I_{00} = 0$ is automatically satisfied and the remaining one, $I_{[\alpha\beta]} = 0$, gives

\[
d = (2\epsilon - 1)b.
\]
We notice that the asymptotic behavior of the diagonal exact solution (21) satisfy (49) and (50) with \( b = -m \). Also this result of (49) is physically very satisfactory in view of the superpotential (47), because we can then get a finite value of the energy.

Next the \((\alpha 0)\)-component of the field equation in vacuum, \( I_{\alpha 0} = 0 \), implies that

\[
(u - 2)(a_1 + a_2) = 0. \tag{51}
\]

When \((a_1 + a_2) \neq 0\), this gives

\[
u = 2, \tag{52}
\]

compatibly with the condition of the linear approximation (43). Using (49) and (51) in (41), we see that the component \( g_{00} \) of the metric tensor behaves asymptotically like

\[
g_{00} = -\left(1 + \frac{2b}{r}\right), \tag{53}
\]

which indicates that the constant \( c \) is related to the gravitational mass \( m \) of the isolated system by

\[
b = -m, \tag{54}
\]

since as shown in §2 the world line of a spinless test particle is a geodesics of the metric. So (50) can now be written as

\[
d = (1 - 2\epsilon)m. \tag{55}
\]

In the special case of \((a_1 + a_2) = 0\), on the other hand, the power \( u \) is not constrained by the field equation \( I_{\alpha 0} = 0 \). However, the exact form of all the spherically symmetric solutions was found in this special case, and the energy of those solutions was calculated as will be explained at the end of this section.

Finally from the skew part, \( I_{[\alpha \beta]} = 0 \), we get

\[
(w - 2)\left(a_1 - \frac{4a_3}{9}\right) = 0. \tag{56}
\]

When \((a_1 - 4a_3/9) \neq 0\), it follows from the condition (56) that

\[
w = 2, \tag{57}
\]

satisfying the condition of the linear approximation (43). In the special case of \((a_1 - 4a_3/9) = 0\), on the other hand, the power \( w \) is not restricted by the field equation. We shall discuss this case separately in the next section.

Collecting the above arguments together, we see that when the parameters satisfy \((a_1 + a_2)(a_1 - 4a_3/9) \neq 0\), the field equation \( I_{\mu \nu} = 0 \) can be solved in the linear approximation to give the following asymptotic form of \((b_k^\mu)\):

\[
\begin{pmatrix}
(1 - \frac{m}{r}) & \frac{j n_\alpha}{r} \\
\frac{h}{r^2} n_\alpha & (1 + \frac{m(1 - 2\epsilon)}{r})\delta_\alpha^\alpha + \frac{f}{r^2} c_\alpha^\beta n^\gamma
\end{pmatrix}
\]

11
\[ (b^k_\mu(\text{exact})) + \begin{pmatrix} 0 & \frac{kn_\alpha}{r^2} \\ 0 & 0 \end{pmatrix} + O \left( \frac{1}{r^2} \right), \]  

(58)

where \( (b^k_\mu(\text{exact})) \) denotes the exact solution (21) in the asymptotic form. Substituting (58) into (36) gives

\[ U_0^\alpha = 2mn^\alpha \left[ (a_1 + a_2) + (1 - 2\epsilon)(a_1 - 2a_2) \right] = 2mn^\alpha, \]  

(59)

where we have used (16) in the last equation. From (30) we then get

\[ E = m. \]  

(60)

Thus, the gravitational mass is equal to the total energy for an isolated spherically symmetric system.

For completeness let us recapitulate the known results\(^{23}\) about the exact, spherically symmetric solutions and their total energy. Referring to the general expression for the parallel vector fields given by (39), the solutions are divided into two classes: (1) the solution with \( F(r) = 0 \) and (2) the solution with \( F(r) \neq 0 \). The general solution of the class (1) involves an arbitrary function of \( r \), and therefore there exist solutions whose components \( (b^a_0) \) asymptotically behave like \( 1/r^u \) for any positive value of \( u \). The calculated energy was shown to coincide with the gravitational mass only when the components \( (b^a_0) \) go to zero faster than \( 1/\sqrt{r} \). When \( (b^a_0) \sim hn^a/\sqrt{r} \) for large \( r \), the quadratic terms of \( h \) must be taken into account in the superpotential, and the energy does not coincide with the gravitational mass, being given by

\[ E = m + \frac{h^2}{2}. \]  

(61)

If \( (b^a_0) \) behave like \( 1/r^u \) with \( 0 < u < 1/2 \), the calculated energy is infinite, implying that the solutions with such an asymptotic behavior is physically unacceptable. Next let us turn to the general solution of the class (2), which was shown to involve an arbitrary constant parameter, and to have the asymptotic behavior of (42) with \( h = 0 \) and \( w = 2 \): The calculated energy was shown to coincide with the gravitational mass.

**§5. The energy in the special case of \((a_1 - 4a_3/9) = 0\)**

Now let us turn to the special case of \((a_1 - 4a_3/9) = 0\), in which we must go beyond the linear approximation discussed in the preceding section since the field equation does not impose any restriction on the power \( w \). We notice that when the parameters, \( a_1 \), \( a_2 \), and \( a_3 \) satisfy the conditions, \( (a_1 + a_2) = 0 \) and \( (a_1 - 4a_3/9) = 0 \), together with the Newtonian approximation condition (15), the gravitational Lagrangian (5) reduces to the Einstein-Hilbert Lagrangian of GR written in terms of the tetrad field, thus
leading to inconsistency of the field equation (9): For example, the $I_{\mu\nu}$ is symmetric, while $T_{\mu\nu}$ of spinor fields is nonsymmetric. Accordingly we assume $(a_1 + a_2) \neq 0$ in this special case.

Now consider the parallel vector fields $(b^k\mu)$ which asymptotically behave like (42) with $u = 2$. Since the power $w$ cannot be fixed by the field equation, we shall discuss the three cases, $w > 1/2$, $w = 1/2$ and $w < 1/2$, separately.

(i) When $w > 1/2$, it can be shown by the same argument as in the previous section that the powers $s$ and $v$ satisfy either $s = v = 1$, or $s > 1$ and $v > 1$. If $s = v = 1$, the linear approximation studied in the preceding section is still valid, and the parallel vector fields behave asymptotically like

$$(b^k\mu) = (b^k\mu\text{(exact)}) + \begin{pmatrix} 0 \\ \frac{n_\alpha}{r} \\ 0 \end{pmatrix},$$

and the energy coincides with the gravitational mass.

If it happens that $s > 1$ and $v > 1$, on the other hand, the superpotential dies out at infinity faster than $1/r^2$, and hence the calculated energy will be vanishing. The gravitational mass is also vanishing since $g_{00}$ tends to 1 for large $r$ faster than $1/r$. Therefore, such a solution, if it exists, is devoid of physical meaning, although we cannot exclude its existence by the present approximate treatment.

(ii) When $w = 1/2$, the field equation $I_{\alpha\beta} = 0$ implies that the powers $s$ and $v$ must satisfy one of the three possible alternatives: (a) $s = v = 1$, (b) $s = 1$ and $v > 1$ or (c) $s > 1$ and $v = 1$.

Let us start with the case (a) of $s = v = 1$. The left-hand side of (9) is given by

$$I_{00} = I_{0\alpha} = I_{\alpha0} = 0,$$

$$I_{\alpha\beta} = -\left[b(a_1 - 2a_2) - d(a_1 + 4a_2) + 2f^2(a_1 + a_2)\right] \left(\frac{\delta_{\alpha\beta} - 3n_\alpha n_\beta}{r^3}\right)$$

up to order $O(1/r^3)$, where we have used $(a_1 - 4a_3/9) = 0$, and the constant $c$ satisfies (54) since $s = 1$ and $u = 2$. The vacuum field equation then provides the condition

$$d = \frac{1}{a_1 + 4a_2} \left[b(a_1 - 2a_2) + 2f^2(a_1 + a_2)\right] = (1 - 2\epsilon)m + 2\epsilon f^2.$$

Similarly the superpotential is expressed by

$$U_{0\alpha} = -\frac{2n_\alpha}{kr^2} \left[(a_1 + a_2)b - (a_1 - 2a_2)d + \frac{1}{2}(a_1 - 2a_2)f^2\right] = -\frac{2mn_\alpha}{kr^2} \left[1 - \frac{1 - 2\epsilon f^2}{1 - \epsilon f^2}\right],$$

(65)
where we have used (16) as well as (64). With the help of (30) we then have

$$E = m - \frac{1 - 2\epsilon}{2(1 - \epsilon)} f^2. \tag{66}$$

In the case (b) of $s = 1$ and $v > 1$, the terms proportional to the constant $d$ do not contribute to the leading term of $I_{\alpha\beta}$ and $U_0^{0\alpha}$. Accordingly, the calculated energy is written as

$$E = \frac{m}{4\epsilon(1 - \epsilon)}, \tag{67}$$

which goes to infinity as $\epsilon$ tends to zero (i.e., $(a_1 + a_2) \to 0$). This means that when the parameter $\epsilon$ is varied, the solution of the case (b), if it exists, will be singular at $\epsilon = 0$.

The last case (c) of $s > 1$ and $v = 1$ can be examined in the similar manner. Since $s > 1$, the gravitational mass of the central body must be vanishing, and furthermore the energy takes a value

$$E = \frac{1 - 2\epsilon}{2(1 - \epsilon)} f^2, \tag{68}$$

which is negative due to the experimental value of $\epsilon$ given by (17). Thus, the case (c) should be discarded.

Therefore, when the power $w$ is equal to $1/2$, the energy of the spherically symmetric body differs from the gravitational mass as measured from far distances. It must be noticed, however, that we have not yet actually found a solution with such asymptotic behavior with $w = 1/2$. The above result merely implies that the existence of such a solution is not excluded by the field equation which considers only up to order $O(1/r^3)$: Detailed analysis based on exact solutions is desirable.

(iii) When $w < 1/2$, quadratic terms of $f$ contribute to the superpotential with the asymptotic behavior like $1/r^{2w}$, and hence the energy integral will diverge, giving an infinite value. Thus, the solution with $w < 1/2$, if it exists, must be discarded.

§6. Conclusion and discussion

We studied the problem whether or not the equality of the active gravitational mass and the inertial mass (i.e., the total energy) is ensured by the gravitational field equation in NGR, restricting ourselves to isolated spherically symmetric systems. Based on the identity following from the general coordinate invariance, we defined the gravitational energy-momentum complex and derived the explicit expression for the superpotential which allows us to calculate the total energy of isolated systems. We first applied it to the exact spherically symmetric solution of diagonal matrix form, and found that the equality of the active gravitational mass and the total energy is satisfied.
Since we do not know whether the spherically symmetric solution is unique, we then discussed the asymptotic method to calculate the total energy. Namely, we started from a quite general expression for the parallel vector fields with spherical symmetry, and solved the vacuum field equation at far distances from the source. The main results can be summarized as follows:

(a) In the generic case of \((a_1 + a_2)(a_1 - 4a_3/9) \neq 0\), the linear approximation can be applied to solve the field equation. All the components of the parallel vector fields other than \((b^{(0)}_0)\) are asymptotically determined by the field equation, and coincide with those of the exact solution up to order \(O(1/r)\). The components \((b^{(0)}_0)\) do not contribute to the total energy, however, and the calculated value of the energy agrees with the gravitational mass.

(b) In the special case of \((a_1 + a_2) = 0\) the asymptotic behavior of the components \((b^a_0)\) is not fixed at all by the field equation. We know, however, that for any asymptotic behavior of \((b^a_0)\), there exist exact solutions with such asymptotic behavior.\(^{23}\) The total energy is finite only when \((b^a_0) \sim 1/r^u\) with \(u \geq 1/2\), and furthermore if \(u = 1/2\), the quadratic terms of \((b^a_0)\) contribute to the total energy, violating the equality of the total energy and the gravitational mass.

(c) In the special case of \((a_1 - 4a_3/9) = 0\), the gravitational field equation does not impose any condition on the \(\epsilon_{ab3}n^b\)-term of \((b^a_0)\). We showed that if this term tends to zero at infinity like \(1/\sqrt{r}\), the equality between the gravitational mass and the total energy is violated. It is yet to be studied, however, whether exact solutions with such asymptotic behavior really exist.

Thus, in the special cases of (b) and (c) above, there are anomalous solutions which violate the equality of the energy and the gravitational mass. The characteristic feature of these solutions is that specific components of the parallel vector fields tend to zero as \(1/\sqrt{r}\) for large \(r\). The physical meaning of these anomalous solutions is not yet clear, however. Is it reasonable to rule out such anomalous solutions by demanding that all the components of the parallel vector fields should tend to flat-space value faster than \(1/\sqrt{r}\)?

As we noticed, if the parameters satisfy both the conditions, \((a_1 + a_2) = 0\) and \((a_1 - 4a_3/9) = 0\), then the gravitational Lagrangian in the NGR coincides with the Einstein-Hilbert Lagrangian in GR expressed in terms of the tetrad field. The gravitational field equation of the NGR then becomes inconsistent: In fact, the L.H.S. of (9) will be symmetric but the R.H.S. is nonsymmetric. Therefore, we assumed in this paper that the combinations of the parameters, \((a_1 + a_2)\) and \((a_1 - 4a_3/9)\), do not vanish at the same time. The special case of \((a_1 + a_2) = 0\) has been studied in rather detail\(^{3,4,9,22,23}\). Another case of \((a_1 - 4a_3/9) = 0\) deserves more attention in this respect.

We assumed the spherical symmetry to obtain the solution at large distances. In GR, however, it is well known that such assumption of spherical symmetry can be removed. In fact, Einstein equation ensures that the metric tensor at far distances is the same as the Schwarzschild metric up to order \(O(1/r)\) for any isolated stationary system. Is it possible in NGR to investigate the vacuum solution at far distances without assuming spherical symmetry?

We have ignored the parity-violating term, \(v_\mu a^\mu\), throughout this paper. The possibility that this term plays an important role was suggested by Müller-Hoissen and
Nitsch\textsuperscript{26}) in the case of $(a_1 + a_2) = 0$. It seems interesting to take into account this parity-violating term also in the case of $(a_1 - 4a_3/9) = 0$.

We shall address ourselves to these problems in future work.
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