Quasiprobabilistic Interpretation of Weak measurements in Mesoscopic Junctions

Adam Bednorz and Wolfgang Belzig

1Fachbereich Physik, Universität Konstanz, D-78457 Konstanz, Germany
2Institute of Theoretical Physics, University of Warsaw, Hoża 69, PL-00681 Warsaw, Poland

(Dated: September 6, 2010)

The impossibility of measuring noncommuting quantum mechanical observables is one of the most fascinating consequences of the quantum mechanical postulates. Hence, to date the investigation of quantum measurement and projection is a fundamentally interesting topic. We propose to test the concept of weak measurement of noncommuting observables in mesoscopic transport experiments, using a quasiprobabilistic description. We derive an inequality for current correlators, which is satisfied by every classical probability but violated by high-frequency fourth-order cumulants in the quantum regime for experimentally feasible parameters.

Every measurement in quantum mechanics is in principle described by the projection postulate [1]. However, in practice perfect projective detectors often do not exist and the measurement encounters a finite error. This can be resolved by replacing the projection by Kraus operators defining a positive operator-valued measure (POVM) [2, 3]. The Kraus operators can be continuously changed from projection – strong measurement (exact) – to almost identity operator – weak measurement (with huge random error). Effectively, a POVM means that we take detector’s degrees of freedom as part of the considered Hilbert space and make a projective measurement on the detector. Obviously, in that case the detector-system coupling defines the strength of the measurement of the system. The equivalence of a POVM and the projective measurement follows from Naimark theorem [4]. The actual modeling of a detection scheme by POVM is a long-standing problem [5, 6].

The interpretation of the results of a weak measurement can lead to paradoxes. For instance, if a weak measurement of $A$ performed on the state $\hat{\rho}$ is followed by a projection $B$ then the weak value can be defined $\beta(A) = \text{Tr}(B\hat{\rho})/\text{Tr}(\hat{\rho})$ [5]. The unusual feature of the weak value is that it can exceed the spectrum of $A$, which obviously contradicts our classical intuition. The strange properties of weak measurements have been confirmed experimentally in quantum optics [5], while experiments in solid state physics are proposed [10]. The interpretation of recent experiments on current fluctuations in mesoscopic junctions in the quantum regime [11] is impossible in terms of a usual probability [12]. Instead, we proposed to consider a weak current measurement, which implies a large background noise, but avoids the paradox of a certain average square to become negative. The necessity of a weak measurement lead to corrections in the observed finite-frequency noise, however, a direct experimental proof of this scheme was not feasible.

In this Letter, we first construct a general formula for a quantum quasiprobability, which does not depend on the details of the measurement apparatus and confirms the previously used formulas [12]. Second, we propose a scheme to test experimentally its negativity in frequency domain. To this end, we will derive a classical inequality for high-frequency current correlators of the form

$$C_{\omega'}^{2} \leq (C_{\omega} + 2\pi S_{\omega}/\delta\omega) (C_{\omega} + 2\pi S'_{\omega}/\delta\omega') \tag{1}$$

Here $C_{\omega'} = \langle(I(\omega)I(-\omega))\rangle/t_{0}$ is a fourth-order correlator and $S_{\omega} = \langle(I(\omega)I(\omega'))\rangle/t_{0}$ the frequency-dependent current noise, where $t_{0}$ is the total (long) measurement time and $\delta\omega$ is the bandwidth of the detector ($S$ and $C$ are independent of $t_{0}$). Inequality (1) is satisfied by every classical stochastic process, but can be violated by high-frequency correlators in the quantum regime of a mesoscopic junction for experimentally accessible parameters. We believe our proposed violation of the classical inequality (1) can be realized with the existing techniques [11]. This violation will be a proof of negative values of the quasiprobability. Although it is not necessary to explain the strange features of weak values [3], it offers an alternative test of nonclassicality similar to the Wigner function [12]. Moreover, the quasiprobabilistic interpretation can be easily generalized to an arbitrary sequence of measurements. This interpretation facilitates the transfer to mesoscopic junctions and we present an example, how the negativity of the quasiprobability can be proven in a tunnel contact and discuss the experimental feasibility.

We will construct the quasiprobability by a deconvolution from a suitable POVM. The real parts of weak values can then be expressed as averages with respect to the quasiprobability. Let us begin with the basic properties of a POVM. The Kraus operators $\hat{K}(A)$ for an observable described by $A$ with continuous outcome $A$ need only to satisfy $\int d\hat{A}\hat{K}^\dagger(A)\hat{K}(A) = 1$. The act of measurement on the state defined by the density matrix $\hat{\rho}$ results in the new state $\hat{\rho}(A) = \hat{K}(A)\hat{\rho}\hat{K}^\dagger(A)$. The new state yields a normalized and positive definite probability density $\rho(A) = \text{Tr}\hat{\rho}(A)$. The procedure can be repeated recursively for an arbitrary sequence of (not necessarily commuting) operators $A_{1}, \ldots, A_{n}$ [14],

$$\hat{\rho}(A_{1}, \ldots, A_{n}) = \hat{K}(A_{n})\hat{\rho}(A_{1}, \ldots, A_{n-1})\hat{K}^\dagger(A_{n}) \tag{2}$$
The corresponding probability density is given by 
\[ \rho(A_1, \ldots, A_n) = \text{Tr} \hat{\rho}(A_1, \ldots, A_n). \] 

The corresponding probability density is given by 
\[ \rho(A_1, \ldots, A_n) = \text{Tr} \hat{\rho}(A_1, \ldots, A_n). \] 

We now define a family of Kraus operators, namely \( \hat{K}_\lambda(A) = (2\lambda/\pi)^{1/4} \exp(-\lambda\hat{A} - A)^2 \). It is clear that \( \lambda \to \infty \) should correspond to exact, strong, projective measurement, while \( \lambda \to 0 \) is a weak measurement and gives a large error. We also see that strong projection changes the state (by collapse), while \( \lambda \to 0 \) gives \( \hat{\rho}(A) \sim \hat{\rho} \), and hence this case corresponds to the weak measure-

The corresponding probability density is given by 
\[ \text{Tr} \text{ } \hat{\rho}(A) \] 

should correspond to exact, strong, projective measure-

The corresponding probability density is given by 
\[ \text{Tr} \text{ } \hat{\rho}(A) \] 

We shall apply the above scheme to the measurement of current \( I(t) \) through a mesoscopic junction in a sta-

We shall apply the above scheme to the measurement of current \( I(t) \) through a mesoscopic junction in a sta-

The complex weak values require a different interpretation [8], which can also be generalized to sequential mea-

The complex weak values require a different interpretation [8], which can also be generalized to sequential mea-

Here and throughout the text we use Latin arguments in time domain and Greek ones in frequency domain, related by \( a(\omega) = \int dt \ e^{i\omega t} a(t) \). Note, that the delta function of the frequency sum has a cut-off of the order of the measuring time \( t_0 \) (larger than all relevant timescales of the system), which in some following expressions is a simple prefactor and does not enter final conclusions.

Let us define the fluctuating noise spectral density 
\[ X_\omega = \int_{-\infty}^{\infty} \delta I(\alpha) \delta I(-\alpha) d\omega \] 

with the initial density matrix \( \hat{\rho} = \hat{\rho} \) for \( n = 0 \). We can interpret \( g \) in (3) as some internal noise of the detectors which, in the limit \( \lambda \to 0 \), should not influence the system. We define the quasiprobability \( g_\lambda = \text{Tr} \hat{\rho}_\lambda \) and abbreviate \( \rho \equiv \rho_0 \). In this limit (4) reduces to 
\[ \hat{\rho}(A) = \int \frac{d\chi}{2\pi} e^{-i\chi A_n} \hat{\rho}(A_1, \ldots, A_{n-1}) e^{i\chi A_n/2}. \] 

Note that \( \rho_0 = \rho_\lambda = \rho \), so the last measurement does not need to be weak (it can be even a projection), and marginal distributions are consistent with absence of a measurement, \( \int dA_k \rho(A) = \rho(\ldots, A_{k-1}, A_{k+1}, \ldots) \). In the case of commuting operators the quasiprobability reduces to the usual probability \( \rho = \rho_\infty \). For \( A_1 = \hat{x} \) and \( A_2 = \hat{p} \) with \( [\hat{x}, \hat{p}] = i\hbar \) we obtain the Wigner function \( g(x, p) = g(p, x) = W(x, p) \). The definition preserves locality – for \( (\hat{A}, \hat{B}) \) acting in two separate Hilbert spaces we have \( \hat{\rho} = \rho_\hat{A} \hat{\rho}_\hat{B} = \rho_0(\hat{A}) \rho_0(\hat{B}) \). The averages with respect to \( \rho \) are easily calculated by means of the generating function [5], e.g. \( \langle A \rangle_\rho = \text{Tr} \hat{A} \hat{\rho} \), \( \langle AB \rangle_\rho = \text{Tr} \{ \hat{A}, \hat{B} \} \hat{\rho}/2 \), \( \langle ABC \rangle_\rho = \text{Tr} \hat{C} \{ \hat{B}, \hat{A}, \hat{\rho} \}/4 \) for \( A = (A, B, C) \). This ordering of operators is called time symmetric [11, 15]. To relate the quasiprobability to weak values, we have to consider two measurements: \( \hat{A} \) and \( \hat{B} \). The real part of the weak value is just the average \( \text{Re} \bar{B}(A)_\rho = \langle A \rangle_\rho \) with respect to the conditional quasiprobability \( \rho(A|B) = \rho(A, B)/\rho(B) \).
correlations in the frequency domain instead, which is more suitable for electric current measurements.

We denote the measured current operator in the Heisenberg picture by $\hat{I}(t)$. Strong, projective measurement can be performed only if we are interested in the long-time limit. The finer the time resolution we want the weaker the measurement must be as $[\hat{I}(t), \hat{I}(t')] \neq 0$. We define a generating functional of the quasiprobability in the weak measurement limit by

$$\hat{\rho} = \int D\chi e^{-i \int \chi(t) \hat{I}(t) dt}$$

with $\mathcal{T} (\hat{T})$ denoting (anti)time ordering. This represents a straightforward generalization of the generating function obtained from the trace of (5), in which the time-variable labels the subsequent measurements. The averages of current powers (noise and third cumulant) have been already calculated [9] with respect to the quasiprobability (without introducing this notion) and measured [11]. In experiments, the large measured offset noise plays the role of $g$ in (3) preventing paradoxical results [12]. We emphasize that in long-time averages the quasiprobability becomes a conventional probability and reproduces the formula for the usual full counting statistics [20], also confirmed experimentally [27, 28].

We consider a quantum point contact described by the Hamiltonian:

$$\hat{H} = \sum_n \int dx \left\{ i\hbar v_n [\hat{\psi}^\dagger_n(x) \partial_x \hat{\psi}_n(x) - L \leftrightarrow R] + \frac{q_0}{\hbar} \delta(x)[\hat{\psi}^\dagger_n(x) \hat{\psi}_R(x) - \hat{\psi}^\dagger_R(x) \hat{\psi}_n(x)] - eV \theta(x)[\hat{\psi}^\dagger_n(x) \hat{\phi}_L(x) + \hat{\psi}^\dagger_R(x) \hat{\phi}_R(x)] \right\}.$$  \hspace{1cm} (10)

The fermionic operators satisfy anticommutation relations $\{\hat{\psi}_a(x), \hat{\psi}_b^\dagger(x')\} = 0$ and $\{\hat{\psi}_a(x), \hat{\psi}_b(x')\} = \delta_{ab} \delta(x-x')$ for $a, b = L, R, \bar{R}, \bar{L}$. The transmission coefficients are $T_n = \cos^2(q_n/\hbar v_n)$. We apply (9) to the current operator $I = \sum_n e\nu_n \hat{\psi}^\dagger_n(0_+) \hat{\psi}_n(0_-) - L \leftrightarrow R$ and the density matrix $\hat{\rho} \propto \exp(-\hat{H}/k_B T)$. To find conditions in which the inequality (11) is violated, it is enough to consider the case $V = 0$. Using Eq. (9) we obtain $S_n = hG \nu(\alpha) + C_{n,\beta} = hF G e^2(w(\alpha) + w(\beta))/2$ [21, 23]. Here we denote $w(\omega) = \omega \coth(\omega/2k_B T)$, conductance $G = \sum_n e^2 T_n/\pi \hbar$ and Fano factor $F = \sum_n R_n T_n/\sum_n T_n$. In the tunneling limit, $T_n \ll 1$, for a finite bias voltage $V$ we only need to replace $w(\omega)$ with $(w(\omega + eV/\hbar) + w(\omega - eV/\hbar))/2$ and $F = 1$. For $T = 35mK$, $\delta \omega = \delta \omega' = 2\omega' = 2\pi \cdot 200MHz$, $\omega = 2\pi \cdot 6GHz$, $G^{-1} = 500k\Omega$ and $V = 0$, we get $B = 1.4$, which contradicts our classical expectation [3] and clearly shows that the quasiprobability $\hat{\rho}$ must take negative values. Generally, the violation occurs for sufficiently small $G < G_{\min}$, as shown in Fig. 1. At $eV = 0$ and $\delta \omega \approx \delta \omega' \gg 2\omega' \geq k_B T/\hbar$, we have $G_{\min} = 3\sqrt{\omega'/\omega'F e^2/2h}$. For larger conductance, one can still find a reasonable range of parameters for the violation at $G^{-1} = 5k\Omega$, as shown in Fig. 1b. The strongest violation occurs at low temperature and voltage but at large bandwidth. Unfortunately, the typical experimental conductance is with $G^{-1} \approx 50\Omega$ even larger and would require either $\omega \sim 2\pi \cdot 1THz$ or a temperature $\sim 1mK$. However, we can make the reasonable assumption that all modes of the junction are independent and replace the inequality (11) – valid for the whole junction – by the same one for a single mode. If we can assume that the modes are independent and have similar transmission coefficients ($T_n \ll 1$) we can simply divide $C$ and $S$ by the number of modes in (11). Note that $C$ enters there linearly while $S$ enters quadratically, so effectively we weaken the contribution from the second cumulant. For a tunnel
junction we can thus replace $G$ by $G/N$ where $N$ is a lower bound of the contributing modes ($T_{\eta} > 0$), which must be larger than $G\pi \hbar/e^2$. The results in Fig. 1 are valid for example for $N_0 = 100 < (\hbar/2e^2)/50 \approx 258$. In this case, it is necessary to ensure that most of the modes contributing to the transport have small transmission eigenvalues.

The cumulants are never measured directly. The second cumulant always contains a large offset noise generated by detector and amplifier and effectively described by detector and amplifier and effectively described by detector’s amplifier. The offset noise may be smaller in the case of a many-mode tunnel junction as it is due to detector’s amplifier. The offset noise may be smaller in the case of a many-mode tunnel junction and one way around is to measure cross correlations by different detectors and amplifiers [12]. The fourth cumulant should also contribute to photon counting statistics [30, 31], but in the limit $\delta \omega \rightarrow 0$ the photon statistics is dominated by the second cumulants in [11].

We have shown that the unusual properties of weak measurements can be interpreted in terms of a real quasi-probability, which can take negative values. Our interpretation agrees well with predictions and measurements of the current fluctuations in mesoscopic junctions. Its direct confirmation would be the measurement of high-frequency fourth-order averages of the current through the junction. By a violation of the inequality [11], the negativity of the quasi-probability could be directly demonstrated. Finally, the separation between detector and system is somewhat arbitrary. One could argue that simply adding some noise can restore the positive probability. This is why an experimental estimate of the detector noise $g$ or the strength of the measurement $\lambda$ is also desirable.

We are grateful for fruitful discussions with and help-ful suggestions by J. Gabelli and B. Reulet. Financial support from the DFG through SFB 767 and SP 1285 is acknowledged.

* Electronic address: [Adam.Bednorz@fuw.edu.pl](mailto:Adam.Bednorz@fuw.edu.pl)

[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton U.P., Princeton, 1932).

[2] K. Kraus, *States, Effects and Operations* (Springer, Berlin, 1983).

[3] See e.g. J. Preskill, *Quantum Information and Computation* (www.theory.caltech.edu/people/preskill/ph229).

[4] M. A. Naimark, Izv. Akad. Nauk. SSSR, Ser. Mat. 4, 277 (1940).

[5] V. B. Braginsky and F. Ya. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).

[6] A. N. Jordan and M. Böttiker, Phys. Rev. B 71, 125333 (2005); A. N. Jordan, A. N. Korotkov, and M. Böttiker, Phys. Rev. Lett. 97, 026805 (2006); A. N. Jordan and A. N. Korotkov, Phys. Rev. B 74, 085307 (2006); A. N. Korotkov in *Quantum Noise in Mesoscopic Physics*, Y. V. Nazarov (Ed.), (Kluwer, Dordrecht, 2003).

[7] K. V. Bayandin, A. V. Lebedev, G. B. Lesovik, JETP 106, 117 (2008).

[8] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988); Y. Aharonov and L. Vaidman, Phys. Rev. A 41, 11 (1990); Y. Aharonov and D. Rohrlich, *Quantum Paradoxes* (Wiley-VCH, Weinheim, Germany, 2005).

[9] G. J. Pryde et al., Phys. Rev. Lett. 94, 220405 (2005); A. M. Steinberg, Phys. Rev. Lett. 74, 2405 (1995); H. M. Wiseman, Phys. Rev. A 65, 032111 (2002).

[10] N. S. Williams and A. N. Jordan, Phys. Rev. Lett. 100, 026804 (2008); A. Romito, Y. Gefen, and Y. M. Blanter, 056801 (2008); V. Shpitalkin, Y. Gefen and A. Romito, 101, 226802 (2008).

[11] E. Zakka-Bajjani et al., Phys. Rev. Lett. 99, 236803 (2007); 104, 206802 (2010); J. Gabelli and B. Reulet, 100, 026601 (2008); J. Stat. Mech. P01049 (2009).

[12] A. Bednorz and W. Belzig, Phys. Rev. Lett. 101, 206803 (2008).

[13] E. P. Wigner, Phys. Rev. 40 749 (1932); M. Hillery et al., Phys. Rep. 106, 121 (1984).

[14] A. Schmidt, Ann. of Phys. 173, 103 (1987); C. Anastopoulos and N. Savvidou, J. Math. Phys. 47, 122106 (2006); 48, 032106 (2007).

[15] B. Berg et al., Phys. Rev. A 80, 033624 (2009).

[16] H. F. Hofmann, Phys. Rev. A 81, 012103 (2010).

[17] G. Mitchison, R. Jozsa, S. Popescu, Phys. Rev. A 76, 062105 (2007).

[18] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 2007).

[19] B. Misra and E. C. G. Sudarshan, J. Math. Phys. (N.Y.) 18, 756 (1977).

[20] A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857 (1985); A. J. Leggett, J. Phys. Condens. Matter 14, R415 (2002).

[21] Y. M. Blanter and M. Büttiker, Phys. Rep. 336, 1 (2000).

[22] A. V. Galaktionov, D. S. Golubev, and A. D. Zaikin, Phys. Rev. B 68, 235333 (2003); 72, 205417 (2005).

[23] A. Bednorz and W. Belzig Phys. Rev. B 81, 125112 (2010).

[24] J. Salo, F. W. J. Hekking, and J. P. Pekola, Phys. Rev. B 74, 125427 (2006); T. T. Heikkilä and T. Ojanen, Phys. Rev. B 75, 035335 (2007).

[25] S. Bachmann, G. M. Graf, and G. B. Lesovik, J. Stat. Phys. 138, 333 (2010).

[26] L.S. Levitov and G.B. Lesovik, JETP Lett. 58, 230 (1993); L.S. Levitov, H.W. Lee, G.B. Lesovik, J. Math. Phys. 37, 4345 (1996); W. Belzig and Y.V. Nazarov, Phys. Rev. Lett. 87, 197006 (2001); Y.V. Nazarov and M. Kindermann, Eur. J. Phys. B 35, 413 (2003); M. Kindermann and Y.V. Nazarov in Quantum Noise in Mesoscopic Physics, Y.V. Nazarov (Ed.), (Kluwer, Dordrecht, 2003).

[27] M. I. Reznikov et al., Phys. Rev. Lett. 75 3340 (1995); A. Kumar et al., 76, 2778 (1996); R.J. Schoelkopf et al., 78, 3370 (1997).

[28] B. Reulet et al., Phys. Rev. Lett. 91, 196601 (2003); Y. Bonzom et al., 95, 176601 (2005); 101, 016803 (2008).

[29] B. Reulet et al., in: *Perspectives of Mesoscopic Physics*, A. Aharony, O. Entin-Wohlman (Eds.). (World Scientific, Singapore, 2010)

[30] C.W.J. Beenakker and H. Schomerus, Phys.Rev.Lett. 86, 700 (2001); 93, 096801 (2004); A.V. Lebedev, G.B. Lesovik, and G. Blatter, Phys. Rev. B 81, 155421 (2010).

[31] J. Gabelli et al., Phys. Rev. Lett. 93 056801 (2004).