Markovian embedding of non-Markovian quantum collisional models

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(Dated: May 11, 2014)

A wide class of non-Markovian completely positive master equations can be formulated on the basis of quantum collisional models. In this phenomenological approach the dynamics of an open quantum system is modeled through an ensemble of stochastic realizations that consist in the application at random times of a (collisional) completely positive transformation over the system state. In this paper, we demonstrate that these kinds of models can be embedded in bipartite Markovian Lindblad dynamics consisting of the system of interest and an auxiliary one. In contrast with phenomenological formulations, here the stochastic ensemble dynamics and the inter-event time interval statistics are obtained from a quantum measurement theory after assuming that the auxiliary system is continuously monitored in time. Models where the system inter-collisional dynamics is non-Markovian [B. Vacchini, Phys. Rev. A 87, 030101(R) (2013)] are also obtained from the present approach. The formalism is exemplified through bipartite dynamics that leads to non-Markovian system effects such as an environment-to-system back flow of information.

PACS numbers: 03.65.Yz, 42.50.Lc, 03.65.Ta, 02.50.Ga

I. INTRODUCTION

The description of open quantum systems through local in time Markovian evolutions is well understood from both mathematical [1] and physical [2] point of views. As is well know, under a completely positive condition, Lindblad equations provide the more general evolution structure of the system density matrix [1, 2]. On the other hand, in the last years an ever increasing interest have been paid to establishing a non-Markovian generalization of the open quantum system theory formulated in terms of non-local in time evolutions [3]. There exist diverse formalisms for describing memory effects. One leading program consist in generalizing Lindblad equations by replacing the rates of each dissipative channel by a time-convoluted kernel function. A wide class of both phenomenological [4, 5] and theoretical approaches [6, 22] were formulated for building and characterizing master equations of that kind, which in turn lead to a completely positive solution map.

In the category of phenomenological approaches, quantum collisional models (QCMs) provided a fundamental tool for establishing a non-Markovian generalization of Lindblad equations [4, 5]. In this formalism, the evolution of an open quantum system follows from an average performed over an ensemble of stochastic realizations of the system state. Each realization consists in the application, at random times, of a completely positive transformation. The events can be read as a “collision” or interaction with the environment. Depending on the statistics of the collision times and the system inter-event dynamics different non-Markovian master equations were established [5, 8]. Over that basis, the emergence of non-Markovian effects such as a system-to-environment back flow of information [23, 24] were also analyzed in the recent literature [25, 26].

The collisional superoperator, the inter-event system dynamics, and the collision time statistics are the main ingredients of the approach. They must be defined, in an arbitrary way, from the beginning. Therefore, besides its usefulness, the QCM model does not have associated a microscopic description, neither it is completely understood which kind of underlying mechanism may induce the structure of the stochastic dynamics. The main goal of this paper is to provide a rigorous physical frame to answering these issues.

The basic idea consists in embedding the non-Markovian system evolution in a Markovian bipartite dynamics. It is defined by the system of interest and an auxiliary (ancilla) system. We demonstrate that there exist bipartite Markovian interactions that induce the same system non-Markovian dynamics. In this way, “microscopic interactions” that lead to the master equations associated to the QCM are found. On the other hand, by assuming that the auxiliary system in continuously monitored in time, over the basis of a (Markovian) quantum jump approach [27, 29], we find that the realizations of the QCM can be put in one to one correspondence with the realizations of the measurement apparatus. In this way, the stochastic dynamics of the QCM is established from a quantum measurement theory. In addition, this modeling allows to characterizing the inter-event statistics from the Markovian Lindblad description.

In Ref. [8] Vacchini introduced a generalized QCM where, in contrast to previous approaches [3, 8], the system inter-event dynamics is defined by a non-Markovian propagator. On the basis of an underlying tripartite Markovian dynamics we show that this generalization can also be described with the present frame. Even when the stochastic realizations consist of successive collisional events with a non-Markovian inter-event dynamics [8],
they cannot be read as the result of a continuous measurement action performed over the system of interest. In fact, in contrast with the results of Ref. [22], here we demonstrate that QCMs can consistently be recovered when measuring the auxiliary ancilla system. The non-Markovian quantum jump approach developed in [22] relies on more general bipartite interactions. Additionally, the monitoring action is performed over the system of interest.

It is interesting to note that collisional models were also proposed as a phenomenological tool for deriving Markovian irreversible dynamics [11, 30]. Furthermore, from a quantum information perspective [31], similar approaches were introduced by considering collisions with a string of auxiliary qubits systems [32, 33]. When the system-string interaction is defined by partial swap and controlled-not qubits operations, specific Markovian master equations describe the system dynamics [33]. Generalization of these ideas to non-Markovian dynamics were considered recently in Refs. [35, 39]. The stretched relation of these results with the present formalism is also investigated.

The paper is outlined as follows. In Sect. II we present the Markovian embedding, where the system density matrix is obtained by using projector techniques [3]. In Sect. III, from a standard quantum measurement theory, we obtain the stochastic ensemble dynamic after assuming that the auxiliary system is subjected to a measurement process. These results rely in the standard quantum jump approach [27, 29] applied to bipartite dynamics. In Sec. IV, we analyze some examples that exhibits the main features of the present approach. A back flow of information from the system to the environment is explicitly shown. In Sec. V, some generalizations of the standard collisional approach are provided. The dynamics presented in Ref. [8] is recovered from a tripartite Markovian dynamics. The formalisms of Refs. [34, 37] are analyzed in this context. In Sec. VI we present the conclusions.

II. MARKOVIAN EMBEDDING

In this section, it is demonstrated that non-Markovian QCMs can be obtained by tracing out a bipartite Markovian dynamics. We deal the case of stationary renewal statistics.

A. Phenomenological renewal collisional models

The superoperator $E_s$ that define each collisional events is written as

$$E_s[\rho] = \sum_\alpha V_\alpha \rho V_\alpha^\dagger, \quad \sum_\alpha V_\alpha^\dagger V_\alpha = I_s,$$  \hspace{1cm} (1)

where the set of operators $\{V_\alpha\}$ act on the system Hilbert space. $I_s$ is the identity matrix. Between collision events the system dynamics is defined by an arbitrary Lindblad generator $L_s$ [unitary plus dissipative contributions]. Thus, given that the last event happened at time $t'$, the inter-event evolution follows from the propagator $\exp[(t-t')L_s]$. By assuming that the collision times define a renewal process, with waiting time distribution $w(t)$ [3], it is possible to demonstrate that the average system density matrix $\rho_s^a$ is governed by the equation

$$\frac{d}{dt} \rho_s^a = L_s[\rho_s^a] + \int_0^t dt' k(t-t')C_s[\exp[(t-t')L_s]\rho_s^u].$$  \hspace{1cm} (2)

The superoperator $C_s$ and the kernel function read

$$C_s = E_s - I_s, \quad k(u) = \frac{uw(u)}{1 - w(u)},$$ \hspace{1cm} (3)

where $u$ is a Laplace variable [$f(u) \equiv \int_0^\infty dt e^{-ut} f(t)$]. Notice that here, due to the assumed (stationary) renewal property, the kernel does not depend separately on the time variables $t$ and $t'$. On the other hand, if $[C_s, L_s] = 0$, in an interaction representation with respect to $L_s$ Eq. (2) (under the replacement $L_s \rightarrow 0$) recovers the evolution introduced in Ref. [8].

B. Bipartite Markovian dynamics

We introduce a bipartite arrangement defined by the system of interest $S$ and an auxiliary (ancilla) system $A$. Their joint density matrix is $\rho^{sa}$. Therefore, their marginal density matrices follow from a partial trace,

$$\rho_s^a = Tr_a[\rho^{sa}], \quad \rho_t^s = Tr_s[\rho^{sa}].$$ \hspace{1cm} (4)

The bipartite dynamics is defined by a Markovian Lindblad equation

$$\frac{d}{dt} \rho_t^{sa} = L_t\rho_t^{sa} = (L_s + L_a + C_{sa})\rho_t^{sa},$$ \hspace{1cm} (5)

where the (arbitrary) Lindblad generators $L_s$ and $L_a$ define the system and ancilla dynamics respectively. The contribution $C_{sa}$ introduces their mutual interaction.

Now we ask about the possibility of finding specific system-ancilla interactions such that the marginal system density matrix $\rho_t^s$ [Eq. (4)] fulfill the evolution (2). With this goal in mind, the superoperator $C_{sa}$ is defined as

$$C_{sa}[\rho] = \sum_{\alpha, l} \gamma_l([V_{wl}\rho V_{wl}^\dagger] + [V_{wl}\rho, V_{wl}^\dagger])]$$ \hspace{1cm} (6)

where $\gamma_l$ are dissipative rates and the operator $V_{wl}$ is

$$V_{wl} = V_{al} \otimes |a_l\rangle \langle a_l|.$$  \hspace{1cm} (7)

The set of operators $\{V_{wl}\}$ are the same as in Eq. (1). The states $\{|a_0\rangle, |a_l\rangle\}$, $l = 1, 2, \ldots, \dim\{\mathcal{H}_a\} - 1$, form a complete orthogonal normalized basis in the Hilbert space $\mathcal{H}_a$ of the ancilla system. Hence, excepting the state $|a_0\rangle$, the index $l$ runs over all available states. Notice that operators (7) introduce irreversible ancilla transitions between the state $|a_0\rangle$ and any of the remaining possible states $|a_l\rangle$, that is, $|a_0\rangle \not\sim |a_l\rangle$. 

1. Ancilla dynamics

With the previous choice of operators [Eq. (7)], it is simple to write down a closed Markovian evolution for the ancilla state $\rho_t^a$. From Eqs. (5) and (10) we get

$$\frac{d}{dt}\rho_t^a = L_a\rho_t^a = (L_a + C_a)\rho_t^a. \quad (8)$$

The extra Lindblad term reads

$$C_a\rho_t^a = \sum_l \gamma_l \{[A_l, \rho_t^a A_l^\dagger] + [A_l^\dagger, \rho_t^a A_l]\}, \quad (9)$$

where $A_l = |a_l\rangle\langle a_l|$. Straightforwardly, this superoperator can be rewritten as

$$C_a\rho_t^a = -\frac{1}{2}\gamma \{\rho_t^a, \rho_t^a\} + \gamma \langle a_l | a_l \rangle \rho_t^a, \quad (10)$$

Here, $\{\cdots\}$ denotes an anticommutation operation, and the ancilla state $\bar{\rho}_a$ is

$$\bar{\rho}_a = \sum_l \frac{\gamma_l}{\gamma} |a_l\rangle\langle a_l|, \quad \gamma = \sum_l \gamma_l, \quad (11)$$

which in fact satisfies $\text{Tr}_a[\bar{\rho}_a] = 1$.

2. Non-Markovian system dynamics

In contrast to Eq. (8), the evolution of the system state $\rho_t^s$ is non-Markovian. Its calculation is a little more involved, which here is obtained by using a projector technique [2, 3]. Let introduce the projectors $\mathcal{P}$ and $\mathcal{Q}$,

$$\mathcal{P}\rho_t^a = \text{Tr}_a[\rho_t^a] \otimes \bar{\rho}_a, \quad \mathcal{P} + \mathcal{Q} = I_{sa}, \quad (12)$$

where $I_{sa}$ is the identity matrix in the bipartite system-ancilla Hilbert space. $\bar{\rho}_a$ is the ancilla state (11). The election of this projector definition will becomes clear in the next section.

The bipartite evolution [13] can be projected in a relevant and irrelevant contributions [3]

$$\frac{d}{dt}\mathcal{P}\rho_t^a = \mathcal{P}\mathcal{L}(\mathcal{P} + \mathcal{Q})\rho_t^a, \quad (13)$$

$$\frac{d}{dt}\mathcal{Q}\rho_t^a = \mathcal{Q}\mathcal{L}(\mathcal{P} + \mathcal{Q})\rho_t^a. \quad (14)$$

On the other hand, consistently with the projectors definition (12), a separable state defines the bipartite initial condition

$$\rho_0^a = \rho_0^s \otimes \bar{\rho}_a, \quad (15)$$

where $\rho_0^s$ is an arbitrary system state. With this initial state, it follows that $\mathcal{Q}\rho_0^a = 0$. Therefore, Eq. (14) can be integrated [3] as $\mathcal{Q}\rho_t^a = \int_0^t dt' \exp[\mathcal{Q}\mathcal{L}(t-t')]\mathcal{Q}\mathcal{L}\rho_t^a$, which in turn, after replacing in Eq. (13) leads to the convolent evolution

$$\frac{d}{dt}\mathcal{P}\rho_t^a = \mathcal{P}\mathcal{L}(\mathcal{P} + \mathcal{Q})\rho_t^a + \mathcal{P}\mathcal{L}\int_0^t dt' \exp[\mathcal{Q}\mathcal{L}(t-t')]\mathcal{Q}\mathcal{L}\rho_t^a. \quad (16)$$

The superoperator $\mathcal{L}$ is defined by Eq. (19). From Eqs. (6) and (7), it can be rewritten as

$$\mathcal{L}[\bullet] = (\mathcal{L}_s + \mathcal{L}_a)[\bullet] - \frac{1}{2}\gamma \langle \langle a_0 | a_0 | \bullet \rangle \rangle + \gamma \mathfrak{E}_a\langle \langle a_0 | a_0 | \bullet \rangle \rangle \otimes \bar{\rho}_a, \quad (17)$$

where the collision superoperator $\mathfrak{E}_a$ and the ancilla state $\bar{\rho}_a$ are defined by Eqs. (11) and (11) respectively. Eqs. (12) and (17) lead to

$$\mathcal{P}\mathcal{L}[\bullet] = \mathcal{P}\mathcal{L}_a[\bullet] + \mathcal{P}\mathfrak{E}_a\langle \langle a_0 | a_0 | \bullet \rangle \rangle \otimes \bar{\rho}_a, \quad (18)$$

where $\mathfrak{E}_a$ follows from Eq. (3). With these last two expressions it is possible to evaluate all contributions in Eq. (16). By using that $\langle \langle a_0 | \bar{\rho}_a | a_0 \rangle \rangle = 0$, we get

$$\mathcal{P}\mathcal{L}\rho_t^a = \mathcal{L}_s[\rho_t^a] \otimes \bar{\rho}_a, \quad (19)$$

where the ancilla superoperator $\mathcal{L}_a$ follows from Eq. (3). We have also used that $\mathcal{C}_a[\bar{\rho}_a] = 0$ [see Eqs. (10) and (11)]. Similarly, it is possible to demonstrate that $\mathcal{Q}\mathcal{L}\rho_t^a = (\mathcal{L}_s + \mathcal{L}_a)(\rho_t^a \otimes \mathbb{L}_a[\bar{\rho}_a])$, which by induction implies the expression

$$\exp[\mathcal{Q}\mathcal{L}t]\mathcal{Q}\mathcal{L}\rho_t^a = \exp[(\mathcal{L}_s + \mathcal{L}_a)t](\rho_t^a \otimes \mathbb{L}_a[\bar{\rho}_a]). \quad (20)$$

By introducing the previous results in Eq. (16), using that $\text{Tr}_a[\mathcal{L}_a(\bullet)] = 0$, straightforwardly we recover the convoluted evolution (2) with the kernel function

$$k(t) = \gamma \langle \langle a_0 | \exp(t\mathbb{L}_a)[\bar{\rho}_a] | a_0 \rangle \rangle, \quad (20a)$$

$$= \gamma \frac{d}{dt} \langle \langle a_0 | \exp(t\mathbb{L}_a)[\bar{\rho}_a] | a_0 \rangle \rangle. \quad (20b)$$

This is the main result of this section. It demonstrate that the non-Markovian evolution (2) also arises as the marginal dynamics of a Markovian bipartite dynamics. In addition, here the kernel function is not arbitrary. In fact, it is completely determined from the ancilla dynamics [see Eqs. (3) and (20)]. Notice that the solution map $\rho_0^a \rightarrow \rho_t^a$ associated to the evolution (2) with the kernel (20) is, by construction, completely positive.

### III. Quantum Measurement Theory

In the previous section we have found an underlying bipartite Markovian dynamics that leads to the non-Markovian system dynamics. Here, on the same basis we find a clear physical interpretation to the ensemble of realizations (3) associated to the master equation (2).

#### A. Quantum jumps in the bipartite dynamics

The realizations of the collision model do not rely on a quantum measurement theory. Nevertheless, this link
can be established by studying the bipartite dynamics when a measurement process is performed over the ancilla system. Specifically, we assume that the apparatus is sensitive to all ancilla transitions $|a_0\rangle \rightarrow |a_i\rangle$. As the bipartite dynamics is Markovian, from a standard quantum jump approach it is possible to associate each realization of the monitoring process with a realization in the system-ancilla Hilbert space such that

$$\rho_{t_s}^{sa} = \overline{\rho_{t_s}^{sa}(t)}.$$ 

(21)

Here, $\rho_{t_s}^{sa}(t)$ is a stochastic density matrix and the overbar denotes an ensemble average. The time evolution of $\rho_t^{sa}$ is defined by Eq. (5). As usual, the stochastic dynamics of $\rho_{t_s}^{sa}(t)$ consists of disruptive transformations associated to each recording event, while in the intermediate time intervals it is smooth and non-unitary.

Consistently with a quantum measurement theory, in each detection event the bipartite state suffer the (measurement) transformation

$$\rho \rightarrow \mathcal{M}\rho = \frac{\mathcal{J}\rho}{\text{Tr}_s[\mathcal{J}\rho]}.$$ 

(22)

where the superoperator $\mathcal{J}$ takes into account all possible transitions $|a_0\rangle \rightarrow |a_i\rangle$ that lead to a detection event. Assuming that $\mathcal{L}$ does not induce this kind of transitions, from Eq. (6) we write

$$\mathcal{M}\rho = \frac{\sum_{a_i} \gamma_i V_{ai}\rho V_{ai}^\dagger}{\text{Tr}_s[\sum_{a_i} \gamma_i V_{ai}\rho V_{ai}^\dagger]},$$ 

(23a)

$$= \frac{\mathcal{E}_s \langle a_0 | \rho | a_0 \rangle}{\text{Tr}_s[\langle a_0 | \rho | a_0 \rangle]} \odot \overline{\rho_a}.$$ 

(23b)

This last expression follows from the definition of the operators $V_{ai}$, Eq. (7). On the other hand, the conditional evolution of $\rho_{t_s}^{sa}(t)$ between detection events is given by the normalized propagator

$$\mathcal{T}_c(t)\rho = \frac{\mathcal{T}(t)\rho}{\text{Tr}_s[\mathcal{T}(t)\rho]},$$ 

(24)

where the unnormalized propagator $\mathcal{T}(t)$ is

$$\mathcal{T}(t)\rho = \exp[t\mathcal{D}]\rho.$$ 

(25)

Here, the exponential superoperator is defined by the generator $\mathcal{D}$, which is the complement of $\mathcal{J}$, that is, $\mathcal{L} = \mathcal{D} + \mathcal{J}$. Hence, from Eq. (6) it reads

$$\mathcal{D}\rho = (\mathcal{L}_a + \mathcal{L}_s)\rho - \frac{\gamma}{2} \{ |a_0\rangle \langle a_0 |, \rho \}.$$ 

(26)

The measurement transformation $\mathcal{M}$ and the propagator $\mathcal{T}_c(t)$ completely define the structure of the realizations of $\rho_{t_s}^{sa}(t)$. It only remains to define the algorithm that allows to obtain the random detection times.

Here they are characterized through a survival probability function $P_0(t|\rho)$ [27]. Given that at time $t$ the bipartite system state is $\rho$, the probability of not happening any detection up to time $t$ is

$$P_0(t - \tau|\rho) = \text{Tr}_s[\mathcal{T}(t - \tau)\rho] = \text{Tr}_s[\mathcal{e}^{t\mathcal{D}}\rho].$$ 

(27)

With this function the realizations can be obtained as follows. Given the initial state $\rho_0^{sa}$, the time $t_1$ of the first detection event follows by solving the equation $P_0(t_1 - 0|\rho_0^{sa}) = r$, where $r$ is a random number in the interval $(0, 1)$. The dynamic of $\rho_{t_1}^{sa}(t)$ in the interval $(0, t_1)$ is defined by Eq. (24). At $t = t_1$ the disruptive transformation [Eq. (23)] $\rho_{t_1}^{sa}(t_1) = \mathcal{M}\rho_{t_1}^{sa}(t_1)$ is applied. The subsequent dynamics is the same. In fact, after the $n_{th}$-measurement event at time $t_n$, $\rho_{t_n}^{sa}(t_n) = \mathcal{M}\rho_{t_n}^{sa}(t_n)$, the time $t_n+1$ for the next detection event follows from $P_0(t_{n+1} - t_n|\mathcal{M}\rho_{t_n}^{sa}(t_n)) = r$, where again $r$ is a random number in the interval $(0, 1)$. The dynamic in the interval $(t_n, t_{n+1})$ is defined by the conditional propagator.

The realizations generated with this algorithm fulfill Eq. (21) (see for example Appendix A of Ref. [22]).

### B. Stochastic realizations

The standard quantum jump approach allows to defining the realizations of $\rho_{t_s}^{sa}(t)$. Straightforwardly from this object it is possible to obtain the partial stochastic dynamics of each system,

$$\rho_{t_s}^{st}(t) = \text{Tr}_a[\rho_{t_s}^{sa}(t)], \quad \rho_{t_s}^{st}(t) = \text{Tr}_a[\rho_{t_s}^{sa}(t)].$$ 

(28)

Furthermore, from Eq. (21), the relations $\rho_t^{a} = \overline{\rho_t^{a}}$, and $\rho_t^{s} = \overline{\rho_t^{s}}$ are also valid. Given the separable initial condition (15), from Eqs. (23) and (26) it is simple to realize that $\rho_{t_s}^{st}(t)$ becomes separable at all times

$$\rho_{t_s}^{st}(t) = \rho_{t_s}^{st}(t) \odot \rho_t^{a}(t).$$ 

(29)

In fact, given the absence of initial correlations, the conditional dynamic remains separable [see Eq. (26)]. Furthermore, in each detection event, given a separable input, the post measurement state also becomes separable. Nevertheless, notice that $\rho_{t_s}^{st}(t)$ and $\rho_t^{a}(t)$ are statistically correlated. Below, we describe their dynamics.

#### 1. Ancilla realizations

After taking a partial trace over Eq. (23), from Eq. (29) we deduce that in each measurement event the ancilla state suffer the transformation

$$\rho_{t_s}^{st}(t) \rightarrow \text{Tr}_s[\mathcal{M}\rho_{t_s}^{st}(t)] = \frac{\mathcal{J}_a \rho_{t_s}^{st}(t)}{\text{Tr}_a[\mathcal{J}_a \rho_{t_s}^{st}(t)]} = \overline{\rho_a},$$ 

(30)

where the ancilla superoperator $\mathcal{J}_a$ is

$$\mathcal{J}_a[\rho] = \gamma \langle a_0 | \rho | a_0 \rangle \overline{\rho_a}. $$

(31)
Hence, the collapsed ancilla state is always the same [Eq. (11)]. Similarly, from Eqs. (24) and (20), we deduce that between detection events the conditional ancilla dynamics is defined by the (unnormalized) superoperator \( T_a(t) \rho = \exp[t \mathcal{D}_a] \rho \), where
\[
\mathcal{D}_a \rho = \mathcal{L}_a \rho - \frac{\gamma}{2} \{|a_0 \rangle \langle a_0|, \rho\} + .
\] (32)

For separable initial conditions, this propagator also applies at the initial time. This simplification explain the chosen initial state (15) and the projectors (12).

From Eqs. (20) and (27), we notice that the survival probability can be rewritten as \([P_b(t - \tau | \rho) \rightarrow P_b(t - \tau)]\)
\[
P_b(t - \tau) = \text{Tr}_a[\exp[\mathcal{D}_a(t - \tau)] \hat{\rho}_a].
\] (33)

In fact, the ancilla state is always the same after a detection event. Consequently, the measurement statistics correspond to a renewal process, that is, the inter-event probability distribution is always the same. On the other hand, it is simple to realize that the measurement transformation (30), the conditional ancilla dynamics defined by Eq. (32), and the survival probability (33) also arise by formulating the quantum jump approach over the basis of Eq. (3). In fact, \( \mathbb{L}_a = \mathcal{D}_a + \mathcal{J}_a \).

2. System realizations

Given the separability property (29), from Eq. (28) it follows that in each detection event (ancilla measurement apparatus), the system suffer the transformation
\[
\rho^{st}_s(t) \rightarrow \text{Tr}_a[\mathcal{M} \rho^{st}_a(t)] = \mathcal{E}_s[\rho^{st}_a(t)],
\] (34)

that is, the transformation associated to a collision event. On the other hand, given that a measurement event happened at time \( \tau \), from Eqs. (24) and (20) we deduce that the posterior system conditional evolution is given by
\[
\rho^{st}_s(t) = \text{Tr}_a[\mathcal{T}_a(t - \tau) \rho^{st}_a(\tau)] = \exp[\mathcal{L}_s(t - \tau)] \rho^{st}_s(\tau).
\] (35)

This inter-event evolution also correspond to the dynamics of the QCM. Therefore, by assuming that the measurement process is performed over the ancilla system, the realizations of the system of interest have the same structure than in the phenomenological QCM. This is the main result of this section. Notice that each system collisional event happens when the measurement apparatus detects an ancilla transition.

The renewal property of the realizations was proven previously. In fact, form the survival probability (34) we define the waiting time distribution \( w(t) = -(d/dt) P_b(t) \), which delivers
\[
w(t) = -\text{Tr}_a[D_a \exp[t \mathcal{D}_a] \hat{\rho}_a], \quad w(t) = \gamma \langle a_0 | \exp(t \mathcal{D}_a) \hat{\rho}_a | a_0 \rangle.
\] (36a)

In deriving this expression we used Eq. (32) and that \( \text{Tr}_a[\mathcal{L}_a \rho] = 0 \). Hence, in the present modeling the quantum jump approach allows to write the waiting time distribution in terms of the ancilla dynamics. Indeed, from Eqs. (34) and (35), we deduce that the ancilla dynamics mainly determine the statistic of the system realizations.

C. Consistence between master equation and ensemble of realizations

For showing the consistence of the developed results, it remains to demonstrate that the waiting time distribution (33), which determine the realizations statistics, and the kernel (20), which determine the density matrix evolution, fulfill in the Laplace domain the relation (3). The Laplace transform of Eq. (33) reads
\[
w(u) = \gamma \langle a_0 | \frac{1}{u - \mathcal{D}_a} \hat{\rho}_a | a_0 \rangle, \tag{37}
\]
while from Eq. (20) we obtain
\[
k(u) = \gamma \langle a_0 | \frac{1}{u - \mathbb{L}_a} \hat{\rho}_a | a_0 \rangle. \tag{38}
\]

In deriving this expression we used that \( \langle a_0 | \hat{\rho}_a | a_0 \rangle = 0 \). On the other hand, using that \( \mathbb{L}_a = \mathcal{D}_a + \mathcal{J}_a \) it follows the relation
\[
\frac{1}{u - \mathbb{L}_a} = \sum_{n=0}^{\infty} \left[ \frac{1}{u - \mathcal{D}_a} \mathcal{J}_a \right]^n \frac{1}{u - \mathcal{D}_a}. \tag{39}
\]

By introducing this expression in Eq. (38) and by using the definition (31) we get
\[
k(u) = \sum_{n=1}^{\infty} w(u) = \frac{w(u)}{1 - w(u)}, \tag{40}
\]
which recovers the relation (3) associated to the phenomenological approach.

IV. EXAMPLE

In this section, we study the dynamics of a two-level system, which in turn may be read, for example, as a qubit unit. In quantum information arrangements it is expected that decoherence and dissipation are “mediated” by interactions with extra quantum subunits. Therefore, as ancilla we consider another system whose dynamics is able to develops quantum coherent effects. For simplicity it is also taken as a two-level system.

In the approach developed in the previous sections, the collision statistics is completely defined by the ancilla dynamics. Hence, in the next example, it structure depends on underlying quantum coherent effects. We remark that this feature is foreign in phenomenological formulations where the waiting time distribution is usually defined by a linear combination of exponential functions (38). We demonstrate that this kind of statistics arise when the ancilla dynamics is completely incoherent. This property motivate the dynamics studied below. Both dephasing and dissipative channels are formulated.
As system we consider a two-level system whose Hamiltonian reads $H_s = \hbar \omega_s \sigma_z/2$, where $\omega_s$ is the transition frequency between its eigenstates, denoted as $|\pm\rangle$, while $\sigma_z$ is the $z$-Pauli matrix. The ancilla system is also a two-level system. In an interaction representation with respect to $H_s$ the evolution of the bipartite state $\rho_s^{\text{sa}}$ reads

\[
\frac{d\rho_{s}^{\text{sa}}}{dt} = \frac{-i\Delta}{2} [I_s \otimes \sigma_z, \rho_s^{\text{sa}}] + \gamma (|V_s \rho_t^{\text{sa}} V_s^\dagger| + [V_s \rho_t^{\text{sa}}, V_s^\dagger]).
\]  

(41)

The first unitary contribution defines the ancilla Hamiltonian. It is given by the $x$-Pauli matrix $\sigma_x$ written in the basis of $\sigma_z$ eigenstates: $|\pm\rangle$. The Lindblad contribution is written in terms of the operator [see Eq. (7)]

\[
V = \sigma_z \otimes \sigma.
\]  

(42)

Here, $\sigma = \sigma_z \otimes \sigma$. Hence, $V$ leads to a dissipative coupling between both systems. The initial bipartite state [see Eq. (11)] is taken as

\[
\rho_{0}^{\text{sa}} = \rho_s^{\text{sa}} \otimes \rho_t^{\text{sa}} \langle - | - \rangle,
\]  

(43)

where $\rho_t^{\text{sa}}$ is an arbitrary system state. The ancilla begins in its lower state.

Performing the partial trace $\rho_s^{\text{sa}} = \text{Tr}_s [\rho_s^{\text{sa}}]$, the bipartite evolution (11) leads to

\[
\frac{d\rho_{t}^{\text{sa}}}{dt} = -\frac{i\Delta}{2} [\sigma_z, \rho_t^{\text{sa}}] + \gamma (|\sigma_t \rho_t^{\text{sa}} \sigma_t^\dagger| + [\sigma_t \rho_t^{\text{sa}}, \sigma_t^\dagger]).
\]  

(44)

This marginal ancilla dynamics corresponds to a quantum fluorescent system [2, 28], where $\gamma$ defines its natural decay rate while $\Delta$ is the Rabi frequency. On the other hand, the interaction defined by Eq. (42) lead to a decoherence system channel [33]. Hence, only the system coherences are affected by the undesirable interaction.

1. System stochastic realizations

The measurement apparatus record the ancilla transitions $|+\rangle \rightarrow |\rangle$. Therefore, from Eqs. (23) and (24) we deduce that in each measurement event the ancilla collapse to its ground state $\rho_s = |\rangle \langle -|$, while the system suffer the completely positive transformation

\[
E_s[\rho] = \sigma_z \rho \sigma_z.
\]  

(45)

As is well known, this superoperator lead to a change of sign in the system coherences [3]. On the other hand, during the successive measurement events the system dynamics is frozen, that is, it does not evolves. This conclusion follows from Eq. (33) and (41).

The statistics of the time interval between successive detections events define a renewal process. Its probability distribution is given by Eq. (30). Under the associations $|a_0\rangle \rightarrow |+\rangle$, and $\mathcal{D}_a[\rho] = -(i\Delta/2)[\sigma_z, \rho]$

\[
\langle \rho_s^{\text{sa}}(t) | - \rangle = \langle + | \rho_s^{\text{sa}}(t) | + \rangle + \langle - | \rho_s^{\text{sa}}(t) | - \rangle,
\]  

(47)

that is, from the partial trace of $\rho_s^{\text{sa}}(t)$. The states $|ii\rangle$, $i,j = +,-$, provide a complete basis of the bipartite Hilbert space. The realizations of $\rho_s^{\text{sa}}(t)$ follows from a “standard Markovian quantum jump approach” formulated on the basis of Eq. (41). We have taken the initial condition $\rho_s^{\text{sa}}(0) = |x_+\rangle \langle x_+| \otimes |\rangle \langle -|$, where $|x_+\rangle = (1/\sqrt{2})(|+\rangle + |\rangle)$ is an eigenstate of $\sigma_x$. In Fig. 1(a), we see that in each recording event the bipartite coherence $\langle + | \rho_s^{\text{sa}}(t) | + \rangle$ collapse to zero

\[
\langle + | \mathcal{M} \rho_s^{\text{sa}}(t) | + \rangle = 0.
\]  

(48)

This result follows from the action of the operator (12), which induces the ancilla transitions $|+\rangle \sim |-\rangle$. 

FIG. 1: Realizations of matrix elements of the stochastic density matrix $\rho_s^{\text{sa}}(t)$ and $\rho_t^{\text{sa}}(t)$. (a) $\langle + | \rho_s^{\text{sa}}(t) | + \rangle$. (b) $\langle + | \rho_s^{\text{sa}}(t) | + \rangle$. The characteristic parameters of the bipartite evolution (11) satisfy $\Delta/\gamma = 6$. 

Notice that Eqs. (15) and (16) completely define the system realizations. 

In Fig. 1 we show a realization of the system coherence $\rho_s^{\text{sa}}(t)$. In order to show the consistence of the developed approach, it was obtained from the realizations of the underlying bipartite dynamics, 

\[
\langle + | \rho_t^{\text{sa}}(t) | - \rangle = \langle + | \rho_t^{\text{sa}}(t) | + \rangle + \langle - | \rho_t^{\text{sa}}(t) | - \rangle,
\]  

(47)
On the other, the bipartite coherence \((+|\rho_{st}^s(t)|-\rangle\) suffers the disruptive changes \((-|\rho_{st}^s(t)|-\rangle \to -(+|\rho_{st}^s(0)|-\rangle\), Fig. 1(b). By calculating the measurement transformation (43), from Eq. (42) we get
\[
\langle +|\mathcal{M}\rho_{st}^s(t)|-\rangle = \langle +|\rho_{st}^s(0)|-\rangle - \langle +|\rho_{st}^s(t)|-\rangle
\]
By an explicitly calculation of the conditional evolution defined by the operator \(\mathcal{D}\), Eq. (43), it follows that the quotient of the previous bipartite matrix elements is an invariant of the conditional evolution, delivering the observed property
\[
\langle +|\mathcal{M}\rho_{st}^s(t)|-\rangle = \langle +|\rho_{st}^s(0)|-\rangle
\]
where we have used that \(\langle +|\rho_{st}^s(0)|-\rangle = \langle +|\rho_{st}^s(t)|-\rangle\) [Eq. (43)]. Therefore, in each measurement event the coherence \(\langle +|\rho_{st}^s(t)|-\rangle\), beside a change of sign, recovers its initial value.

In Fig. 1(c), we plot the realization of \(\langle +|\rho_{st}^s(t)|-\rangle\) obtained from Eq. (17), that is by adding the two bipartite coherences. As both coherences \(\langle +|\mathcal{M}\rho_{st}^s(t)|-\rangle\) and \(\langle -|\mathcal{M}\rho_{st}^s(t)|-\rangle\) always oscillate in a complementary way, during the inter-event time intervals \(\langle +|\rho_{st}^s(t)|-\rangle\) is constant, while in the measurement events it changes of sign. In this way, we explicitly show that the underlying quantum jump approach lead to the realizations of the phenomenological collision model. In fact, the action of the superoperator (15) only introduce a change of sign in the system coherences. In a similar way, it is possible to show that the system populations are not affected by the dynamics, that is, \(\langle \pm|\rho_{st}^s(t)|\pm\rangle = \langle \pm|\rho_{st}^s(0)|\pm\rangle\).

2. Density matrix evolution

In Fig. 2 we show the average coherence behavior obtained from the ensemble of realizations shown in Fig. 1 (noisy curve). Furthermore, we present the exact solution of the coherence that follows from the master equation (23) (black full line). Taking into account the underlying Lindblad equation (11), it can be written as
\[
\frac{d\rho_s^t}{dt} = \int_0^t dt'k(t-t')\mathcal{C}_s[\rho_s^t].
\]
The superoperator \(\mathcal{C}_s = (\mathcal{E}_s - I_s)\), from Eq. (15) reads
\[
\mathcal{C}_s[\rho] = \frac{1}{2}[(\sigma_z \bullet \sigma_z) + (\sigma_z \bullet \sigma_z)].
\]
On the other hand, the kernel is determined by the general expression (20). From Eq. (11) it follows
\[
k(t) = 2\gamma^2t e^{-(3/4)\gamma t} \left\{ \frac{\sinh(\gamma/4)\sqrt{\gamma^2 - 16\Delta^2}}{\sqrt{\gamma^2 - 16\Delta^2}} \right\}.
\]
This kernel and the waiting time distribution (10) fulfill the Laplace relation (3).

3. Environment-to-system back flow of information

The analysis of Refs. [23, 24] demonstrate that QCMs may lead to non-Markovian effects such as an environment-to-system back flow of information [23]. This property or phenomenon can be defined on the basis of “any measure” that in the Markovian case present a monotonic time decay behavior [2]. One well known example is the relative entropy between two states.

FIG. 2: System coherence. Full line, exact solution Eq. (53). Dotted (noisy) line, average coherence \(\langle +|\rho_{st}^s(t)|-\rangle\) obtained by averaging \(10^3\) realizations. Grey line, relative entropy \(E(\rho_s^t||\rho_s^0)\), Eq. (53). The parameters are the same than in Fig. 1, \(\Delta/\gamma = 6\).

Consistently with the system stochastic realizations, Eq. (53) does not modify the populations, \(\langle \pm|\rho_{st}^s(t)|\pm\rangle = \langle \pm|\rho_{st}^s(0)|\pm\rangle\). On the other hand, working in a Laplace domain, the coherences \(c_{\pm}^t \equiv \langle \pm|\rho_{st}^s(t)|\mp\rangle\) read
\[
c_{\pm}^t = c_{\pm}^0 \left\{ e^{-\gamma t} - \frac{2\Delta^2}{\gamma^2 + 2\Delta^2} + e^{-\gamma t/4} \right\} \times \left[ \frac{\gamma}{\gamma^2 + 2\Delta^2} \cosh(\gamma t) + \frac{\gamma(\gamma^2 + 8\Delta^2)}{4(\gamma^2 + 2\Delta^2)} \sinh(\gamma t) \right],
\]
where for shortening the expression we introduced the “frequency”
\[
\varphi = \sqrt{(\gamma/4)^2 - \Delta^2}.
\]
As we are not interested in quantifying the non-Markovian effects, for simplicity here we consider the relative entropy with respect to the stationary state

\[
E(\rho_s^t||\rho_s^\infty) = \text{Tr}[\rho_s^t (\ln_2 \rho_s^t - \ln_2 \rho_s^\infty)],
\]

where \(\rho_s^\infty = \lim_{t \to \infty} \rho_s^t\). Hence, the back flow of information arises if there exists times \(t_2 > t_1\) such that \(E(\rho_s^{t_2}||\rho_s^\infty) > E(\rho_s^{t_1}||\rho_s^\infty)\). Below we show that this feature arises in the dynamics described previously.

In Fig. 2 we also plotted \(E(\rho_s^t||\rho_s^\infty)\) (grey full line) where \(\rho_s^t\) is the solution of the Eq. (51). The stationary state is the diagonal matrix \(\rho_s^\infty = \text{diag}\{\langle +|\rho_0^s|+\rangle, \langle -|\rho_0^s|-\rangle\}\}. Clearly the time behavior is non-monotonous, indicating a back-flow of information. Furthermore, the oscillatory behavior of \(E(\rho_s^t||\rho_s^\infty)\) is correlated with the oscillatory behavior of the coherences, which arise whenever \(\varphi\) is a complex quantity, that is, from Eq. (54), \(\Delta > (\gamma/4)\).

4. Incoherent ancilla dynamics

For the dynamics (41), the ancilla dynamics develops quantum coherent effects, Eq. (44), which in turn determine the waiting time distribution, Eq. (50). Here, we introduce an alternative ancilla dynamics which only induces incoherent transitions. Instead of Eq. (41), for the dynamics (44), which in turn determine the collision statistics [Eq. (46)], in this case the stationary state can be obtained by considering a generalized amplitude damping superoperator [31]. Hence, the ancilla dynamics [Eq. (8)] only leads to the incoherent (classical) transitions \(|+\rangle \xrightarrow{\gamma} |-\rangle\) and \(|-\rangle \xrightarrow{\beta} |+\rangle\). Its statistical behavior is defined by a (two-level) classical rate master equation.

We assume that the recording apparatus is only sensitive to the ancilla transition \(|+\rangle \xrightarrow{\gamma} |-\rangle\), that is, the transition induced by the operator \(V\). In this situation, from Eqs. (23) and (57), we deduce that the collisional superoperator again reads \(\mathcal{E}_c[\rho] = \sigma_z \rho \sigma_z\) [Eq. (15)]. Thus, the system evolution is given by Eq. (51). Nevertheless, the kernel follows from Eq. (4), where the waiting time distribution can be calculated from Eq. (56). We get

\[
w(u) = \left(\frac{\gamma}{u + \gamma}\right) \left(\frac{\beta}{u + \beta}\right).
\]

In the time domain \(w(t)\) is the convolution of two exponential functions. The system coherences become \(c_{u}^{\pm} = \frac{c_{u}^{\pm}(u + \gamma + \beta)}{2u^2 + 2u(u + \gamma + \beta) + \gamma \beta}\), which can be written as a linear combination of exponential functions. Independently of the initial conditions, in this case the dynamics does not present an environment-to-system back flow of information, suggesting that underlying coherent effects may be necessary for the development of this phenomenon.

Taking an ancilla system with higher number of states, all of them coupled via incoherent transitions, the waiting time distribution results defined by more complex expressions which in the time domain are linear combinations of exponential functions. For example, taking an unidirectional coupling \(|a_0\rangle \sim |a_1\rangle \sim \cdots |a_m\rangle \sim |a_0\rangle\), all of them with rate \(\gamma\), the waiting time distribution becomes \(w(u) = [(u + \gamma)/\gamma]^{m+1}\). This kind of distributions, which rely on incoherent ancilla dynamics, were considered, for example, in Ref. [8].

B. Dissipative channels

In the previous example, Eqs. (50) and (51) define a non-Markovian decoherence channel. One may also consider interactions that lead to dissipative channels. For example, maintaining the ancilla dynamics (44), a depolarizing [31] non-Markovian channel arises by introducing two bipartite Lindblad terms \([x, y\text{ in Eq. (6)}]\) defined by the operators \(V_x = \sqrt{p}\sigma_x \otimes \sigma\) and \(V_y = \sqrt{1 - p}\sigma_y \otimes \sigma\), where the parameter \(p\) satisfies \(0 < p < 1\). With the same collision statistics [Eq. (10)], in this case the stationary system state becomes \(\rho_s^\infty = (1/2)I_s\). A thermal stationary state can be obtained by considering a generalized amplitude damping superoperator [31].

V. GENERALIZED COLLISIONAL MODELS

In the previous sections we associated the basic master equation of the collision model [Eq. (2)] with an underlying Markovian microscopic dynamics, Eq. (6). Furthermore, the realizations of the model, given that the ancilla system is continuously monitored in time, were established on the basis of the quantum jump approach. In this section, we show that these results also apply in different possible generalizations of the basic approach.

A. Non-stationary renewal collision dynamics

The basic ingredients of the present approach remain valid when the evolution of the ancilla system, in the bipartite Lindblad dynamics [3], depends explicitly on time, \(\mathcal{L}_a \rightarrow \mathcal{L}_a(t)\). Under this situation, the main change is the measurement statistics. While it remains a renewal process, the waiting time distribution explicitly depends on the observation time. This case can be worked out with the elements introduced in the previous sections.
B. Non-renewal collision statistics

With the same system realizations, the formalism may becomes non-renewal when the measurement process is non-renewal. Basically this situation occurs whenever the ancilla resetting state is not always the same. This case arises, for example, when the operators (7) are generalized as

\[ V_{al} = V_a \otimes |a_i\rangle \langle a_i| \]  

(59)

Hence, instead of a unique state |a_0\rangle, here many of them play the same role. Assuming that the measurement apparatus is sensitive to “all transitions” \( |a_0\rangle \rightarrow |a_i\rangle \), the stochastic ancilla becomes non-renewal. This case may corresponds, for example, to optical cascade systems [28].

While the structure of the systems realizations remains the same, the statistics of the inter-event time intervals can only be determinate by knowing the ancilla state at all times. Therefore, for generating the system realizations unavoids one also must to generates the ancilla realizations.

C. Non-Markovian inter-collision dynamics

Maintaining the renewal property, in Ref. [8] Vaccini introduced an interesting generalization that consists in assuming that the inter-event dynamics is non-Markovian. This situation naturally arises when considering a system interacting successively with a string of qubits systems [33, 37].

Instead of the Markovian evolution defined by Eq. (35), it is taken as

\[ \rho_w^s(t) = \mathcal{G}(t-\tau)[\rho_w^s(\tau)], \]  

(60)

where \( \mathcal{G}(t) \) is an arbitrary (trace preserving) completely positive propagator that cannot be written as a semigroup, \( \mathcal{G}(t) \neq \exp[t\mathcal{L}_s] \) [8]. Here, we demonstrate that this case can also be covered with the present formalism.

The generalized QCM can be embedded in a tripartite stochastic state. The evolution of their joint density matrix \( \rho_{s}^{ab} \) is written as

\[ \frac{d}{dt} \rho_{s}^{ab} = \mathcal{L}_{s} \rho_{s}^{ab} = (\mathcal{L}_{sb} + \mathcal{L}_{a} + \mathcal{C}_{sab}) \rho_{s}^{ab}. \]  

(61)

The first superoperator reads

\[ \mathcal{L}_{sb} = \mathcal{L}_{s} + \mathcal{L}_{b} + \mathcal{C}_{sab}. \]  

(62)

Here, \( \mathcal{L}_s \) and \( \mathcal{L}_b \) are arbitrary Lindblad equations for the systems S and B respectively. \( \mathcal{C}_{sab} \) is an extra Lindblad contribution that introduce an arbitrary interaction (unitary and dissipative) between them. As before, \( \mathcal{L}_a \) defines the dynamics of the ancilla system A. The contribution \( \mathcal{C}_{sab} \) introduces a dissipative interaction between the three systems,

\[ \mathcal{C}_{sab}[\rho] = \sum_{\alpha, l, m} \gamma_l [V_{al}, \rho V_{al}^\dagger] + [V_{al} \rho, V_{al}^\dagger], \]  

(63)

where \( \gamma_l \) are the dissipative rates and the operators are

\[ V_{al} = V_a \otimes |a_0\rangle \langle a_0| \otimes |b_m\rangle \langle b_m|. \]  

(64)

The system operators \( V_a \) and the states \(|a_i\rangle \) are the same than in Eq. (7), where the index \( l = 1, 2, \ldots, \dim(\mathcal{H}_a) - 1 \) does not include the single state \(|a_0\rangle \). On the other hand, the states \(|b_m\rangle \), \( m = 0, 1, \ldots \dim \mathcal{H}_b - 1 \) form a complete orthonormal basis in the Hilbert space \( \mathcal{H}_b \) of \( B \). Notice that here the state \(|b_0\rangle \) must be included in the summation index \( m \). For simplicity, the tripartite initial state is chosen separable

\[ \rho_0^{sab} = \rho_0^s \otimes \rho_a \otimes |b_0\rangle \langle b_0|. \]  

(65)

where \( \rho_0^s \) is an arbitrary system state and \( \rho_a \) follow from Eq. (11).

We determine the system realizations over the basis of a standard quantum jump approach formulated on the basis of Eq. (61). As before, the measurement apparatus is only sensitive to transitions of the auxiliary system \( A \). Therefore, the transformation associated to each detection event, instead of Eq. (23), here reads

\[ \mathcal{M}_\rho = \frac{\mathcal{E}_s[\sum m a_0 b_m \rho |a_0 b_m\rangle \langle a_0 b_m|]}{\operatorname{Tr}_s[\sum m a_0 b_m \rho |a_0 b_m\rangle \langle a_0 b_m|]} \otimes \rho_a \otimes |b_0\rangle \langle b_0|. \]  

(66)

The collisional superoperator \( \mathcal{E}_s \) is given by Eq. (1). On the other hand, the (tripartite) conditional dynamics can be written as in Eqs. (24) and (25). Nevertheless, here the superoperator \( \mathcal{D} \) reads

\[ \mathcal{D}_{\rho} = (\mathcal{L}_{sb} + \mathcal{L}_{a}) \rho - \frac{\gamma}{2} (|a_0\rangle \langle a_0|, \rho) +. \]  

(67)

In deriving this result we used that \( \sum a V_a V_a = \mathcal{I}_s \), and \( \sum_{b=0}^{\dim \mathcal{H}_b-1} |b_m\rangle \langle b_m| = \mathcal{I}_b \). With the previous definition of \( \mathcal{D} \), the expression for the survival probability, Eq. (27), remains almost the same, \( P_0(t-\tau|\rho) = \operatorname{Tr}_{sab}[e^{\mathcal{D}t}\rho] \).

Over the basis of the previous two equations and the initial condition (17), it is simple to conclude that the tripartite stochastic state \( \rho_{s}^{st}(t) = \rho_{s}^{st}(t) \) can be written at all times as

\[ \rho_{s}^{st}(t) = \rho_{s}^{st}(t) \otimes \rho_{a}^{st}(t). \]  

(68)

The dynamics for the ancilla state \( \rho_{a}^{st}(t) \) remains the same as before, that is, Eqs. (30) to (32) are not modified by the introduction of system \( B \). In consequence, the measurement statistics, defined by the survival probability (33), or equivalently the waiting time distribution (50), is also the same.
The induced stochastic system dynamics follows from 
\( \rho_{st}^{a}(t) = \text{Tr}_{ab}[\rho_{sab}^{a}(t)] \). Hence, in each recording event the state suffer the disruptive transformation

\[
\rho_{s}^{a}(t) \to \text{Tr}_{ab}[M\rho_{sab}^{a}(t)] = \mathcal{E}_{s}[\rho_{s}^{a}(t)]. \tag{69}
\]

This expression follows straightforwardly from Eq. (60), after using Eq. (63) and noting that \( \sum_{m}(b_{m} \bullet |m\rangle) = \text{Tr}_{b} \bullet \). On the other hand, the inter-collision dynamic [Eq. (53)], here is \( \rho_{s}^{a}(t) = \text{Tr}_{ab}[\mathcal{T}_{s}(t - \tau)\rho_{sab}^{a}(\tau)] \). Given that \( \mathcal{T}_{s}(t) \) follows from Eqs. (24) and (26), the operator \( D \) [Eq. (57)] and the separability property defined by Eqs. (65) and (68) lead to

\[
\rho_{s}^{a}(t) = \text{Tr}_{b}[\exp(t\mathcal{L}_{ab}) |b_{0}\rangle \langle b_{0}|] \rho_{s}^{a}(\tau). \tag{70}
\]

This conditional dynamics recovers the phenomenological proposal Eq. (60). Hence, the non-Markovian propagator \( \mathcal{G}(t) \) reads

\[
\mathcal{G}(t) = \text{Tr}_{b}[\exp(t\mathcal{L}_{ab}) |b_{0}\rangle \langle b_{0}|]. \tag{71}
\]

This is the main result of this section. It implies that the generalized phenomenological approach of Ref. [8] can be described over the basis of a tripartite Markovian evolution. If \( \mathcal{L}_{ab} = \mathcal{L}_{s} + \mathcal{L}_{b} \), that is, when the system \( S \) and the ancilla \( B \) do not interact, the formalism of the previous section, \( \mathcal{G}(t) = \exp(t\mathcal{L}_{s}) \), is recovered. Hence, given the structure of the operators [21], it becomes clear that the main role of system \( B \) is to modify the inter-collision system dynamics.

The realizations defined by the measurement transformation [60] and the inter-event dynamics [71] are similar to that found in Ref. [22], where a non-Markovian generalization of the quantum jump approach was defined over a similar basis by assuming that the system of interest is submitted to a measurement process. Nevertheless, the present treatment explicitly demonstrate that collisional dynamics can only be linked with a quantum measurement theory if the monitoring action is performed over the auxiliary ancilla system.

The non-local character of the propagator \( \mathcal{G}(t) \) can be showed by writing Eq. (71) in the Laplace domain as \( \mathcal{G}(u) = \text{Tr}_{b}[\{u - \mathcal{L}_{ab}\}^{-1}|b_{0}\rangle \langle b_{0}|]. \) This expression can be rewritten as \( \mathcal{G}(u) = \{\text{Tr}_{a}[\{u - \mathcal{L}_{ab}\}^{-1}(u - \mathcal{L}_{ab})|b_{0}\rangle \langle b_{0}|]\}^{-1} \times \{\mathcal{G}(u)^{-1}\}^{-1}. \) Using in the curly brackets that \( X^{-1} \times Y^{-1} = (Y \times X)^{-1} \), where \( X \) and \( Y \) are arbitrary matrices, it follows \( \mathcal{G}(u) = \{\mathcal{G}(u)^{-1}[\text{Tr}_{a}[\{u - \mathcal{L}_{ab}\}^{-1}|b_{0}\rangle \langle b_{0}|] - \text{Tr}_{a}[\{u - \mathcal{L}_{ab}\}^{-1}\mathcal{L}_{ab}|b_{0}\rangle \langle b_{0}|]\}^{-1} \), which in turn leads to

\[
\mathcal{G}(u) = \frac{1}{u + \mathcal{K}(u)}, \tag{72}
\]

where the system superoperator \( \mathcal{K}(u) \) is

\[
\mathcal{K}(u) = \left\{ \text{Tr}_{a}\left[\frac{1}{u - \mathcal{L}_{ab}}|b_{0}\rangle \langle b_{0}|\right]\right\} \text{Tr}_{b}\left[\frac{1}{u - \mathcal{L}_{ab}} - \mathcal{L}_{ab}|b_{0}\rangle \langle b_{0}|\right].
\]

Hence, in the time domain we get

\[
\frac{d}{dt}\mathcal{G}(t) = \int_{0}^{t} dt' \mathcal{K}(t - t')\mathcal{G}(t'). \tag{73}
\]

where \( \mathcal{K}(t - t') \) is defined by its Laplace transform \( \mathcal{K}(u) \).

The evolution of \( \rho_{st}^{a} \) can be obtained from Eq. (61) by using projector techniques. A simpler way is to calculate the average behavior of the ensemble of stochastic realizations (see Ref. [8]). On the other hand, the QCM introduced by Ciccarello, Palma, and Giovannetti in Ref. [32], which relies on interaction with a qubits-string, can also be recovered from the present approach. In fact, as demonstrated in Ref. [8] it arises by taking \( \mathcal{E}_{s} \to I_{s} \). Hence, each collision only resets the evolution induced by \( \mathcal{G}(t) \). The results presented by Rybar et. al. in Ref. [35] rely on a similar approach. All non-Markovian effects arise because the ancilla string begin in a correlated state [37]. Nevertheless, in our approach that formalism seems to be equivalent to a system-ancilla dynamics coupled via a unitary evolution, which in turn leads to a random-like superposition of Hamiltonian system propagators. Therefore, extra analysis are necessary for establishing a full mapping between both approaches.

In what follows we analyze how different underlying dynamics lead to dephasing and dissipative inter-collision dynamics [8, 30].

### 1. Dephasing inter-collision dynamics

In this example, both the system an the ancillas are two-level systems. Their tripartite Markovian evolution is given by Eq. (61). In an interaction representation with respect to the system Hamiltonian, we write

\[
\mathcal{L}_{ab}[^{\rho} \rho] = \frac{-i}{\hbar}[H_{s}, \rho] = \frac{-i\lambda}{2}[\sigma_{z} \otimes I_{a} \otimes \sigma_{z}, \rho], \tag{74}
\]

where \( \sigma_{j}, j = x, y, z, \) are the Pauli matrices defined in each Hilbert space. Hence, the system of interest \( S \) and the auxiliary system \( B \) are coupled via a Hamiltonian interaction. The isolated dynamics of ancilla \( A \) is unitary

\[
\mathcal{L}_{a}[\rho] = \frac{-i}{\hbar}[H_{a}, \rho] = \frac{-i\Delta}{2}[I_{a} \otimes \sigma_{z} \otimes I_{s}, \rho]. \tag{75}
\]

The dissipative tripartite interaction [Eq. (53)] reads

\[
\mathcal{L}_{sab}[\rho] = \gamma \sum_{m=0,1} [(V_{m} \rho V_{m}^{\dagger}) + [V_{m} \rho, V_{m}^{\dagger}]]. \tag{76}
\]

The index \( m = 0,1 \), runs over the basis \( \{ |b_{0}\rangle , |b_{1}\rangle \} \) of system \( B \). The two operators \( V_{m} \) are

\[
V_{m} = \sigma_{x} \otimes \sigma \otimes |b_{m}\rangle \langle b_{m}|, \tag{77}
\]

where as before \( \sigma \) is the lowering operator, here defined in the Hilbert space of system \( A \). Consistently with Eq. (65), the initial tripartite state is

\[
\rho_{0}^{sab} = \rho_{0} \otimes |\cdot\rangle \langle \cdot| \otimes |b_{0}\rangle \langle b_{0}|. \tag{78}
\]

From the previous definitions, Eqs. (60) and (69) lead to the collision system superoperator

\[
\mathcal{E}_{s}[\rho] = \sigma_{z} \rho \sigma_{z}. \tag{79}
\]
Notice that $\sigma_x$ arise from the first (system) operator contribution in Eq. (77). On the other hand, the dynamics of system $A$ again is defined by Eq. (11). Consequently, the waiting time distribution is given by Eq. (10). The inter-collision dynamic follows from Eq. (71) and (74). By an explicit calculation, we get the completely positive (non-Markovian) dephasing superoperator

$$
\mathcal{G}(t)\rho = \frac{1}{2}[1 + d(t)]\rho + \frac{1}{2}[1 - d(t)]\sigma_z\rho\sigma_z,
$$

(80)

which in turn can be rewritten as

$$
\mathcal{G}(t)\rho = \begin{pmatrix}
+|\rho|+&d(t)\ (+|\rho|+)
+|\rho|-&d(t)\ (-|\rho|-)
\end{pmatrix}.
$$

(81)

The function $d(t)$ defines the system coherences behavior. It reads $d(t) = \cos(\lambda t)$.

In the first example worked out in Ref. [8], the superoperator is given by Eq. (79), while the propagator $\mathcal{G}(t)$ is given by Eq. (80) (see supplemental material of [8]). Hence, our results provides a clear microscopic description for that phenomenological model. The waiting time distribution, instead of Eq. (10), is a classical one like Eq. (85). That case can be recovered replacing the ancilla dynamics (79) by

$$
\mathcal{L}_a[\rho] = \beta((A, \rho a^a A^\dagger) + [A\rho a^a, A^\dagger]),
$$

(82)

with the operator

$$
A = I_b \otimes \sigma^\dagger \otimes I_b.
$$

(83)

As explained previously, diverse “underlying classical” waiting time distributions can be obtained by adding extra ancilla states, all of them coupled by incoherent transitions.

2. Dissipative inter-collision dynamics

Instead of the dephasing evolution (80), the inter-collision dynamics may also lead to dissipative effects. This property is defined by the superoperator $\mathcal{L}_b$ [Eq. (74) in the previous example]. For example $\mathcal{L}_{ab}$ may correspond to a Jaynes-Cumming interaction, which couples the system to a set of Bosonic field modes initially in the vacuum state $|\tilde{0}\rangle$. This case, which has been studied in Refs. [8, 31] can be analyzed over the basis developed previously.

VI. SUMMARY AND CONCLUSIONS

Phenomenological QCMs provided an important theoretical tool for establishing and describing non-Markovian completely positive dynamics. In this paper we have developed a solid physics basis for understanding this approach. It relies on a Markovian embedding of the non-Markovian system density matrix evolution, which in turn allows to derive the phenomenological trajectories from a quantum measurement theory.

First, we focused our analysis on the leading case in which the collision statistics is defined by a renewal process, while the inter-event dynamics is defined by a Markovian quantum semigroup. By using projector techniques we demonstrated that the non-Markovian density matrix evolution [Eq. (2)] can be obtained, without involving any approximation, from a bipartite Markovian dynamics where the system of interest interact with an auxiliary ancilla system, Eq. (14). The memory kernel that determines the system evolution becomes defined by the ancilla dynamics, Eq. (20). The proposed Markovian embedding allows to associate a clear microscopic dynamics to the QCM. In fact, Lindblad equations are linked with well defined microscopic dynamics.

In a second step, we assumed that the ancilla system is continuously monitored in time. Hence, over the basis of the quantum jump approach formulated for the bipartite dynamics, we find that the realizations of the QCM are recovered from the marginal conditional stochastic system dynamics, Eq. (21). In fact, each recording event of the ancilla measurement apparatus lead to the collisional transformations of the phenomenological approach, Eq. (41). The inter-collision system dynamics follows from the conditional bipartite dynamics between detection events, Eq. (85). The waiting time distribution of the inter-event time interval also becomes defined by the ancilla dynamics, Eq. (85). In this way, the phenomenological realizations of the collisional approach were derived from a quantum measurement theory.

The Markovian embedding and the link with the quantum jump approach were explicitly shown through an example where the dynamics of both the system of interest and the auxiliary one develop in two dimensional Hilbert spaces (Figs. 1 and 2). In contrast to phenomenological formulations, here the collision statistics arises from quantum coherent effects developing in the ancilla Hilbert space. A system-to-environment back flow of information characterize the dynamics. In contrast, when the ancilla dynamics is completely incoherent, this feature is absent.

The previous finding provide a solid basis for proposing different generalizations of the QCM. For example, non-stationary renewal collision dynamics can be obtained by introducing an explicit time dependence in the ancilla dynamics. Non-renewal collision statistics can be related to a non-renewal ancilla measurement process. On the other hand, we showed that by introducing a second auxiliary system the inter-collision dynamics becomes defined by a non-Markovian propagator, Eq. (11). This finding allowed us to recover a recent proposed generalization of the QCM [8], which in fact can also be embedded in a Markovian evolution and their realizations derived from a quantum measurement theory. From this result, we also concluded that some non-Markovian collisional models formulated in terms of qubits logical operations [31] can also be recovered from our formalism.
The present analysis allows us to read the phenomenological QCMs from a novel perspective. Besides a solid physical basis of the corresponding non-Markovian dynamics, the developed approach provides an alternative and power tool for describing non-Markovian memory effects in open quantum systems.

Acknowledgments

This work was supported by CONICET, Argentina, under Grant No. PIP 1142009010211.

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