Classical stochastic dynamics and extended $N = 4$ supersymmetric quantum mechanics.

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This work is aimed at demonstrating the possibility to construct new exactly-solvable stochastic systems by use of the extended supersymmetric quantum mechanics ($N = 4$ SUSY QM) formalism. A feature of the proposed approach consists in the fact that probability densities and so obtained new potentials, which enter the Langevin equation, have a parametric freedom. The latter allows one to change the potentials form without changing the temporal behavior of the probability density.

1 $N = 4$ SUSY QM

Let us get started with a brief discussion of the $N = 4$ SUSY QM structure and constructing within its framework isospectral Hamiltonians [1, 2]. The $N = 4$ SUSY QM Hamiltonian has the following form:

$$H_{\sigma_1, \sigma_2} = \frac{1}{2} (p^2 + V_2(x) + \sigma_3^{(1)} V_1^1(x)) \equiv \frac{1}{2} (p^2 + V_1^2(x) + \sigma_3^{(2)} V_1^1(x))$$ (1)

where we have introduced ($\hbar = m = 1$):

$$V_i(x) = W_i'(x) + \frac{1}{2} \sigma_3^{(i)} \frac{W''(x)}{W'(x)}$$ (2)

$W(x)$ is a superpotential, $\sigma_3^{(i)}$ are matrices which commute to each other and have the eigenvalues $\pm 1$

$$\sigma_3^{(1)} = \sigma_3 \otimes 1, \sigma_3^{(2)} = 1 \otimes \sigma_3$$ (3)

Supercharges $Q_i$ of the extended supersymmetric mechanics form the algebra:

$$\{Q_i, \bar{Q}_k\} = 2 \delta_{ik} H; \{Q_i, Q_k\} = 0; i, k = 1, 2$$ (4)

And admit the form:

$$Q_i = \sigma_3^{(i)} (p + i V^{(i+1)}(x)), \quad \bar{Q}_i = \sigma_3^{(i)} (p - i V^{(i+1)}(x))$$ (5)

Constructing isospectral Hamiltonians within the $N = 4$ SUSY QM is based on the fact that four Hamiltonians cast into one supermultiplet. Let us take as an initial one of the Hamiltonians

$$H_{\sigma_1, \sigma_2} = \frac{1}{2} (p - i \sigma_1 V_{\sigma_2}(x))(p + i \sigma_1 V_{\sigma_2}(x)) + \varepsilon \equiv \frac{1}{2} (p - i \sigma_2 V_{\sigma_1}(x))(p + i \sigma_2 V_{\sigma_1}(x)) + \varepsilon$$ (6)
where
\[ V_{\sigma_2}(x) = W'(x) + \frac{1}{2} \sigma_2 W''(x)/W'(x), \] (7)
and \( \varepsilon \) is the so-called factorization energy. In what follows the energy level will be counted from \( \varepsilon \).

Consider an auxiliary equation:
\[ H_{\sigma_2}^2 \varphi(x) = \varepsilon \varphi(x) \] (8)
Its general solution \( \varphi(x, \varepsilon, c) \) is the linear combination of two independent solutions \( \varphi^{(1)}(x, \varepsilon) \) and \( \varphi^{(2)}(x, \varepsilon) \). The expression for \( W(x) \) in terms of \( \varphi(x, \varepsilon, c) \) has the form:
\[ W(x) = \sigma_1^2 \ln\left(1 + \lambda \int_{x_i}^{x} dx' [\varphi(x', \varepsilon, c)]^{2 \sigma_1 \sigma_2}\right) \] (9)
where \( \lambda, x_i \) are two new parameters.

For definiteness we will consider \( \sigma_1 = \sigma_2 = -1 \). Then, relations in the above can be presented in the form of:
\[ H_+ = H_- - \frac{d^2}{dx^2} \ln \varphi(x, \varepsilon, c), \quad \psi_+(x, E) = \frac{1}{\sqrt{2(E - \varepsilon)}} \frac{W\{\varphi(x, \varepsilon, c)\psi_-(x, E)\}}{\varphi(x, \varepsilon, c)} \] (10)
\[ H_+ = H_- - \frac{d^2}{dx^2} \ln(1 + \lambda \int_{x_i}^{x} dx' [\varphi(x', \varepsilon, c)]^2) \]
\[ \psi_+(x, E) = \psi_-(x, E) - \frac{\lambda \varphi(x, \varepsilon, c)}{1 + \lambda \int_{x_i}^{x} [dx'[\varphi(x', \varepsilon, c)]^2} \int_{x_i}^{x} dx' [\varphi(x', \varepsilon, c)] \psi_-(x', E) \] (11)

In the case when \( \varphi(x, \varepsilon, c) \) is a ground state eigenfunction of the initial Hamiltonian, wave function of ground state \( H_+ \) has a form:
\[ \psi_+(x, E_0) = N_0 \frac{\varphi(x, \varepsilon)}{1 + \lambda \int_{x_i}^{x} [dx'[\varphi(x', \varepsilon)]^2} \] (12)

2 Construction of stochastic models associated with \( N = 4 \) SUSY QM.

The Fokker-Planck equation is equivalent to the Langevin equation, however the Fokker-Planck is used more widely in physics, since it is formulated in more common for the probability density \( m_t^\pm(x, x_0) \) language. The Fokker-Planck equation takes the form [3, 4]:
\[ \frac{\partial}{\partial t} m_t^\pm(x, x_0) = \frac{D}{2} \frac{\partial^2}{dx^2} m_t^\pm(x, x_0) \pm \frac{\partial}{\partial x} \Phi(x)m_t^\pm(x, x_0) \]
\[ m_{t=0}(x, x_0) = \{\delta(x - x_0)\}; \quad U_+(x) = \mp \int_0^x dz \Phi(z) \] (13)
with \( U_\pm(x) \) to be the potential entering the Langevin equation. The Fokker-Planck equation describes the stochastic dynamics of particles in potentials \( U_+ \) and \( U_- = -U_+ \). Substituting
\[ m_t^\pm(x, x_0) = \exp \left\{ - [U_+(x) - U_-(x_0)] / D \right\} K_\pm(x, t) \] (14)
the Fokker-Planck equation transforms into the Schrödinger equation with imagine time:

\[
-D \frac{\partial}{\partial t} K_{\pm}(x, t) = \left\{-\frac{D^2}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \left[\Phi^2(x) \pm D\Phi'(x)\right]\right\} K_{\pm}(x, t) = H_{\pm} K_{\pm}(x, t)
\]

in which the diffusion constant can be treated as the "Planck" constant. As in the previous case, let us consider \( H_{-} \) to the initial Hamiltonian.

\[
H_{-} = \frac{1}{2} \left[ p^2 + \left(V_2^{(-)}(x)\right)^2 - DV_2^{(-)\prime}(x) \right]
\]

\[
V_2^{(-)}(x) = W'(x) - \frac{D}{2} \frac{d}{dx} \ln |W'(x)| \equiv -D \frac{d}{dx} \ln \varphi(x, \varepsilon, c)
\]

The force entering the corresponding Langevin equation:

\[
F(x) = -\frac{dU_{\pm}(x)}{dx} = D \frac{d}{dx} \ln \varphi(x, \varepsilon, c), \quad U_{\pm}(x) = -D \ln \varphi(x, \varepsilon, c)
\]

Then, in the same to the \( N = 2 \) SUSY QM way, there is the relation \( U_{+} = -U_{-} \). It directly follows from the fact that \( H_{+} \) is obtained from \( H_{-} \) with changing \( V_2^{(-)}(x) \) to \(-V_2^{(-)}(x)\).

Hence, there is the relation between the stochastic dynamics in a potential and that of in an inverse potential.

Further consideration is based on an important property of \( N = 4 \) SUSY QM, which consists in having the symmetry of \( H_{\sigma_1} \) under \( \sigma_1 \leftrightarrow \sigma_2 \). It leads to:

\[
H_{+}(x, p) = \frac{1}{2} Q_1^{(-)} Q_1^{(-)} \equiv \frac{1}{2} Q_2^{(+)} Q_2^{(+)} = H_{-}(x, p)
\]

If the first equation points to the expression of \( H^{(+)} \) in terms of \( Q_1(\bar{Q}_1) \), the second one is \( H^{(-)} \) in terms of supercharges \( Q_2(\bar{Q}_2) \). Though the supercharges of \( H^{(+)} \) and \( H^{(-)} \) are essentially different:

\[
H_{+}(x, p) = \frac{1}{2} Q_2^{(+)} \bar{Q}_2^{(+)} = \frac{1}{2} \left[p^2 + \left(V_1^{(+)}(x)\right)^2 - DV_1^{(+)}\prime(x)\right]
\]

\[
V_1^{(+)}(x) = \frac{d}{dx} \left(W + \frac{D}{2} \ln |W'(x)|\right) \equiv D \frac{d}{dx} \ln \left[\varphi(x, \varepsilon, c)\right]_{x_0}^{x} \left[\varphi(x', \varepsilon, c)\right]^{2}
\]

From equations (19) follows that at the quantum level the Hamiltonians \( H_{-} \) and \( H_{+} \) possesses the same spectrum and wave functions, thus the corresponding to them \( K(x, t) \) are the same. At the same time the corresponding to these Hamiltonians stochastic models are described by essentially different potentials, so their \( m_{i}(x, x_0) \) are different. Further, the potential \( U_{+}(x) \) has a nontrivial parametric dependence on \( \lambda \), that corresponds to having the family of stochastic models, the probability densities of which have the same temporal dependence. Having the parametric freedom allows one to change the potential form that is unexpected in the case. In particular, as it has been noticed in literature, physical quantities such as the time of passing a peak of potentials depend essentially on local changes of the potential barrier. Looking at \( H_{+} \), we note that it follows from \( H_{+} \) with changing \( V_1^{(+)}(x) \) to \(-V_1^{(+)}(x)\), as it also takes place in \( N = 2 \) supersymmetry. In its turn the potential of this stochastic system is \( U_{+}(x) = -U_{+}(x) \). The wave functions which are used to calculate \( K_{+}(x, t) \) have the form of (11) in the case.

Getting back to the parametric dependence of \( m_{+,i}(x, x_0) \) and \( m_{+,i}(x, x_0) \), it should be noted that the normalization condition could lead to fixing the \( \lambda \). The same does
not happen in the case of \( m^+_{+\downarrow}(x, x_0) \), if instead of \( \varphi(x, \varepsilon, c) \) we will use the normed wave function (say, the ground state wave function) of the initial Hamiltonian \( H^- \). From definition:

\[
m^+_{+\downarrow}(x, x_0) = \frac{\varphi(x, \varepsilon, c)}{1 + \lambda \int_{-\infty}^{x_0} dx' |\varphi(x', \varepsilon, c)|^2} \times^n \sum_{n=0}^\infty e^{-E_n / D} \psi^+_n(x) \psi^+_n(x_0, E_n)
\]

(20)

in view of eq. (12) it follows that

\[
\int_{-\infty}^{\infty} dx m^+_{+\downarrow}(x, x_0) = \int_{-\infty}^{\infty} dx m^+_{+\downarrow}=0(x, x_0) =
\]

\[
= (\lambda + 1) \int_{-\infty}^{\infty} dx \left[ \frac{\varphi(x, \varepsilon, c)}{1 + \lambda \int_{-\infty}^{x_0} dx' |\varphi(x', \varepsilon, c)|^2} \right]^2 = 1
\]

(21)

It means that the unique restriction to \( \lambda \) is \( \lambda \neq -1 \), that as it was early noticed corresponds to the absence of singularities at the quantum-mechanical level. It is hard to make an analogous study for \( m^+_{-\downarrow}(x, x_0) \) in a general case. However, for the Ornstein-Uhlenbeck process the normalization condition does not remove the parametric ambiguity at the definite choice of \( x_0 \).

### 3 The Ornstein-Uhlenbeck process

Let us demonstrate the proposed scheme of getting new stochastic models with the well-known Ornstein-Uhlenbeck process. The Fokker-Planck equation which describes the Ornstein-Uhlenbeck leads to a quantum-mechanical potential with the harmonic oscillator potential. The factorization energy coincides with the ground state energy in the case with the harmonic oscillator potential.

\[
\left( \frac{p^2}{2} + \frac{\omega^2 x^2}{2} \right) \varphi(x, \varepsilon) = \varepsilon \varphi(x, \varepsilon)
\]

(22)

\[
E_n = n D \omega, \psi^-_n(x, E_n) = (\omega / D)^{1/2} H_n(x \sqrt{\omega / D}) e^{-\sqrt{\omega / D} x^2}, n = 0, 1, ...
\]

(23)

where \( \varepsilon = \frac{D \omega}{2} \) (the energy counts from \( \varepsilon \)). The potential entering the Langevin equation:

\[
U^-_n = \omega x^2 / 2 = D \xi^2 / 2, \quad \xi = \sqrt{\omega / D} x.
\]

(24)

For the Ornstein-Uhlenbeck process it is easy to see that, at least for the case of \( x_0 = 0 \), the normalized condition does also not fix the value of \( \lambda \). For an arbitrary stochastic process the same is hard to prove. Calculating \( m^+_{+\downarrow}(x, x_0) \) we use the wave functions (11). First of all it has to be pointed out of having the equilibrium value of \( m^+_{+\downarrow}=\infty(x, x_0) \):

\[
m^+_{+\downarrow}=\infty(x, x_0) = (\lambda + 1) \left( \frac{\omega}{\pi D} \right)^{1/2} \frac{e^{-\xi^2}}{(1 + \lambda / 2 + \lambda / 2 \Phi(\xi))^2}, \quad \Phi(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-t^2} dt
\]

(25)

As it was mentioned in the above, the normalized condition of the probability density does not remove the \( \lambda \) - parametric freedom. Using the wave functions \( \Psi^+_n(x, E_n) \), which
are expressed in terms of $\Psi_-(x, E_n)$, after simple but tedious calculations we get the expression for the probability density:

$$m^+_{\tau,t}(\xi, \xi_0) = \lambda \left( \frac{\omega}{\pi D} \right)^{1/2} e^{-\xi^2} + \frac{1}{2} \left( \frac{\omega}{D} \right)^{1/2} \frac{d}{d\xi} \left[ 1 + \frac{1}{1 + \lambda/2 + \lambda/2\Phi(\xi)} \times \left( 1 + \frac{\xi^2}{\sqrt{1 - z^2}} \right) \right]$$

The form of $m^+_{\tau,t}(\xi, \xi_0)$ can be viewed from the following plots under the following parametrization $\xi_0 = 0$, $z = e^{-\omega t}$.

Fig. [14] show the results of numerical evaluation of the dependence of $m^+_{\tau,t}(x, 0)$ on $x$ and $\lambda$ at different $z$. Thus, when $z = 1$ $m^+_{\tau,t}(x, 0)$ has well pronounced $\delta$-like shape in concordance with initial conditions. With decreasing of $z$ $m^+_{\tau,t}(x, 0)$ changes its shape and this changes are especially significant at $\lambda \rightarrow -1$. In this case the shift of the maximum of probability density occurs, as well as emergence of substantial asymmetry.
4 Conclusions.

The described procedure of the obtaining of exactly-solved stochastic models allows to use the results of numerous works concerning exactly-solved as well as quasi-exactly-solved quantum mechanical problems. The distinctive feature of this approach is an existence of parametric freedom in potentials, that enter the in the Langevin equation, as well as in transitional probability densities. This situation takes place at any modification in the spectrum of initial Hamiltonian. It’s important to mention, that normalization condition for $m_{+,t}^\pm(x, x_0)$, most probably, could be fulfilled only when certain relation between $x_0$ and $\lambda$ exists. This means that stochastic model with $U^\pm_\sigma(x)$ is "bad". At the same time $m_{+,t}^\pm(x, x_0)$ is a "good" transitional probability density, which preserve the parametric freedom.

As is well known, [7, 8], local modifications of the shape of the potential in Langevin equation could lead to substantial changes of such quantities as times of the passing through barrier and time of live in metastable state. Although this quantities are mainly determined by first non-zero energy level in spectrum of $H_{\sigma_1}^\sigma_2$, the $\lambda$-freedom allows for significant variations of their values. This especially important for potentials with several local minima, which could emerge when constructing isospectral Hamiltonians with factorization energy $\varepsilon < E_0$. In this case potentials $U_{\sigma_1}^{\sigma_2}$ with two wells emerge, symmetric, as well as asymmetric and $\lambda$-freedom reveals in substantial modification of the shape and height of the barriers, what leads to changes in the rate of the inter-well transitions.

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