PARTIALLY ADJOINT DISCRETIZATIONS OF ADJOINT OPERATORS

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Abstract. This paper concerns the discretizations in pair of adjoint operators between Hilbert spaces so that the adjoint properties can be preserved. Due to the finite-dimensional essence of discretized operators, a new framework, theory of partially adjoint operators, is motivated and presented in this paper, so that adjoint properties can be figured out for finite-dimensional operators which can not be non-trivially densely defined in other background spaces. A formal methodology is presented under the framework to construct partially adjoint discretizations by a conforming discretization (CD) and an accompanied-by-conforming discretization (ABCD) for each of the operators. Moreover, the methodology leads to an asymptotic uniformity of an infinite family of finite-dimensional operators. The validities of the theoretical framework and the formal construction of discretizations are illustrated by a systematic family of in-pair discretizations of the adjoint exterior differential operators.

The adjoint properties concerned in the paper are the closed range theorem and the strong dualities, whose preservations have not been well studied yet. Quantified versions of the closed range theorem are established for both densely defined operators and partially adjoint discretizations. The notion Poincaré-Alexander-Lefschetz (P-A-L for short) type duality is borrowed for operator theory, and horizontal and vertical P-A-L dualities are figured out for adjoint operators and their analogues are established for partially adjoint discretizations. Particularly by partially adjoint discretizations of exterior differential operators, the Poincaré-Lefschetz duality is preserved as an identity, which was not yet obtained before. The ABCD, a new kind of discretization method, is motivated by and plays a crucial role in the construction of partially adjoint discretizations. Besides, it can be used for the discretization of single operators; for example, in this paper, de Rham complexes that start with the Crouzeix-Raviart finite element spaces are constructed by series of ABCDs, including both the discretized operators and the domain spaces; commutative diagrams with appropriate regularities are constructed thereon. Equivalences are established between primal and dual discretizations for the elliptic source and eigenvalue problems and the primal and dual mixed discretizations of the Hodge Laplacian problems of exterior differential operators based on the partially adjoint discretizations of adjoint operators. This shows how structure-preserving schemes for equations with linear operators can be lead to by partially adjoint discretizations under the new theoretical framework.

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1. Introduction

Let $X$ and $Y$ be two Hilbert spaces with respective inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, and let $(T, D) : X \to Y$ be an unbounded linear operator, $D$ being the domain dense in $X$. The adjoint operator of $(T, D)$, denoted by $(T^*, D^*)$, is defined by

$$\langle T^*w^*, v \rangle_X = \langle w^*, Tv \rangle_Y, \quad \forall v \in D,$$

and the domain $D^*$ consists of such $w^*$s that there exists an element in $X$ taken as $T^*w^*$ to satisfy (1.1). Then $(T^*, D^*)$ is closed, and if $(T, D)$ is closed, the adjoint operator of $(T^*, D^*)$ is $(T, D)$. For $(T, D)$ closed densely defined from $X$ to $Y$, the pair of adjoint operators bear many important adjoint properties, which can be found in, e.g., [13, 30, 44] and other textbooks. These properties play important roles in many fields and particularly the theory and numerical methods of partial differential equations. The preservation of these properties by discretizations will be important for practical applications, but is generally not naturally or easily done. Indeed, adjoint operators may each have kinds of discretizations, and we can not expect any two discretizations for each of them can compose a pair with expected adjoint properties. This paper concerns the discretizations of a pair of adjoint operators so that adjoint properties can be preserved by the discretized operators. Schemes of relevant problems can be constructed with structural properties preserved.

1.1. Contents of the paper. In this paper, we will mainly study two types of adjoint properties. As a first and, to our knowledge, a most fundamental type of adjoint property, Banach’s closed range theorem ([6]) asserts, if a closed densely defined operator has closed range, then its adjoint operator has closed range. This is not only an important issue in functional analysis, but also a fundamental in the theory and numerical analysis of partial differential equations. Though, its preservation or reconstruction by discretizations has not been discussed. The second type is about...
the Poincaré-Alexander-Lefschetz (P-A-L for short) type dualities of adjoint operators. By the notion P-A-L type duality borrowed from algebraic topology, we mean

\[ C \ominus A = B \ominus D, \]

if \( A \oplus B = C \oplus D \) with \( A \subset C \) and \( D \subset B \) being all closed. Here \( \ominus \) and \( \oplus \) denote orthogonal summation and difference, respectively, whose precise meanings are given later. We will specifically discuss the reconstructions of horizontal and vertical P-A-L dualities, cf. Theorems 2.3 and 2.4, which are not yet well studied. The horizontal type duality was occasionally studied in the formulation of isomorphisms between trace spaces for some specific Sobolev spaces, but the metric properties of the trace spaces are generally very subtle to construct and further to preserve. The Poincaré-Lefschetz duality in the form of (1.2) for exterior differential operators is a typical vertical P-A-L type duality; cf. (4.1) and Lemma 4.3. Its reconstruction can be seen discussed in literature but in a weak formulation; namely, only the isomorphism between the two sides of (1.2) can be obtained, cf. (4.7) and Remark 4.14; no identity has been seen reconstructed by discretizations.

For the discretization of a pair of adjoint operators, we are constructing a family of pairs of discrete operators with certain domains, to provide consistent approximation to each of the original operators and thus discretization schemes to relevant problems. The significant aspect is that, any of these discrete operators is of finite-dimensional essence, and the whole infinite family has to bear some asymptotic properties so that the original infinite-dimensional operator can be approximated. Due to the finite-dimensional essence, these operators can not be non-trivially closed densely defined with respect to other background spaces; the standard theory can not apply directly for them for adjoint properties such as P-A-L type dualities. On the other hand, for any single pair of discrete operators, adjoint properties may be established; note for example that operators of finite-dimensional essence all have closed ranges; however, to describe and establish the adjoint properties of the infinite family of finite-dimensional operators in a uniform way is not that straightforward. The main purpose of this paper is thus to construct some new theory serving as a framework to clear up the adjoint properties for finite-dimensional operators, and then to construct discretizations for adjoint operators, so that adjoint properties may be constructed uniformly for an infinite family of finite-dimensional operators. Our concern is focused on the construction of discretizations in pair; this also includes, given a discretization for one operator, to design an accompanied discretization for the other; we may see the adjoint properties bring in convenience in the design and analysis. Also, the structural properties of relevant model problems may this way be preserved by the corresponding discretizations.

The overall approach of this paper is to generalize the relationship “adjoint” between operators to some new relationship like “generalized adjoint”, and then construct pairs of generalized adjoint operators for in-pair discretizations of adjoint operators. There are two key ingredients which allow us to carry out the approach.

First, we introduce a new concept partially adjoint for the relation between operators, and it bears some adjoint properties. Roughly speaking, two operators \( (T_X, D_X) : X \rightarrow Y \) and \( (T_Y, D_Y) : Y \rightarrow X \) are said to be partially adjoint based on \( [T_X, B_X], (T_Y, B_Y)] \), if

\[ D_X = \{ v \in B_X : \langle v, T_Y q \rangle_X - \langle T_X v, q \rangle_Y = 0, \quad \forall q \in D_Y \}, \]

(1.3)
and

$$(1.4) \quad D_Y = \{ q \in B_Y : \langle v, T_Y q \rangle_X - \langle T_X v, q \rangle_Y = 0, \quad \forall \ v \in D_X \},$$

where $[(T_X, B_X), (T_Y, B_Y)]$ is a **base operator pair** formed by $(T_X, B_X) \supset (T_X, D_X)$ and $(T_Y, B_Y) \supset (T_Y, D_Y)$ which satisfy some assumptions. Adjoint properties of the pair $[(T_X, D_X), (T_Y, D_Y)]$ are constructed, and these properties rely on the properties of $[(T_X, B_X), (T_Y, B_Y)]$ only. The criterion $(1.3)-(1.4)$ looks quite the same as $(1.1)$. Though, the partially adjoint operators are not necessarily densely defined in the base pair, and the criterion only influences the domains $D_X$ and $D_Y$, rather than the operations.

Second, we present a general methodology to construct **partially adjoint discretizations** of adjoint operators. Our 2-step construction is quite formal. Step 1, construct a tensorial base pair using tensor products of cell-wise operators and cell-wise shape functions. The tensorial way brings in the advantage that the features of the global base pair can be completely controlled by the features of cell-wise shape function spaces, and can thus be asymptotically uniform for the infinite family of finite-dimensional operators; see Remark 3.24 and Section 4.2.3. Step 2, use a conforming discretization (**CD**) for one operator and an accompanied-by-conforming discretization (**ABCD**) for the other to form a discrete pair. **ABCD** consists of an operator with its domain, and is a new kind of discretization of an original operator, developed directed by the theory of partially adjoint operators. A cell-wise-defined projective interpolator can be defined to the tensorial base, and it maps the domain space of the original operator into the **ABCFES**, the domain space of **ABCD**. Therefore, those pairs that consists of a **CD** and an **ABCD** are called partially adjoint discretizations of the adjoint operators, though various pairs of partially adjoint operators can be constructed based on a same base pair. Adjoint properties including quantified closed range theorem, slice dualities and dual complexes can be established. Further, the **ABCDs** can themselves be a novel approach as single discretizations. In this paper, though, we focus ourselves on the establishment and analysis of adjoint properties of a pair of operators or problems; general features of discretization schemes such as the accuracy can be analyzed in standard ways and will not be discussed in details.

Technically, two nonstandard treatments help us circumvent the major obstacles which arise due to the finite-dimensional essence.

The first one is to choose proper formulation of the adjoint properties to be constructed. As aforementioned, for example, any finite-dimensional operator possesses a closed range, and the reconstruction of the closed range theorem for a family of operator pairs needs to find a proper formulation. We introduce the **index of closed range** ($\text{icr}$) for closed operators, and a closed operator has closed range if and only if its $\text{icr}$ is finite. Then a quantified (and preciser) version of the closed range theorem is established, which reads a closed operator and its adjoint operator have the **same** index of closed range. This quantified property can then be studied for partially adjoint operators and as an asymptotic property for an infinite family of finite-dimensional operators.

Second, we illustrate how the criterion $(1.3)-(1.4)$ can be used to construct a practical discretization of a single original operator. Logically, if a discretization of one operator is established in any way, a discretization of another operator can be simply established in order that $(1.3)$ and $(1.4)$ hold, and a partially adjoint pair follows. This approach is different from classical ones which
construct discrete function spaces by matching the cell-wise shape functions and the requirement of continuity across the interfaces of neighbored cells, particularly the standard finite element triple presented in [18] and followed by numerous textbooks of finite element methods. Instead, our approach utilizes a continuity requirement not necessarily restricted on the interface but possibly on a group of cells, and our approach does not generally correspond to a triple as defined by [18]. Beyond the logical definition, we show that the discretization established this way is one practically implementable by figuring out its basis functions. Examples of ABCD are given. The construction of ABCFES is similar to some nonconforming spline functions ([25, 38, 45, 46]) and with a new way to impose continuities: different from most existing finite element methods, the ABCFES uses a dual way to impose proper continuities. Though, it can be interesting to clarify, these are not mixed element methods. Note that the ABCFESes admit compact-supported basis functions so that they are each a space that can be singly implemented for a discretization scheme for the one-field formulation of the problem, and they themselves can be embedded into exact sequences, such as the complexes which start from the Crouzeix-Raviart element spaces. The ABCD may be further generalized to a wider category, simply called dually consistent schemes, which does not necessarily rely on a base operator pair or be related to partially adjoint discretizations.

To illustrate the validities of the theory and the construction of partially adjoint discretizations, a family of nonconforming Whitney forms are also given for the discretization of exterior differential forms. The conforming and nonconforming Whitney forms form well-coupled partially adjoint operator pairs which bear adjoint properties including the Poincaré-Lefschetz duality preserved as identities (cf. (4.11)) rather than just isomorphisms. The equivalence between the discretization schemes of primal and dual formulations of the elliptic source and eigenvalue problems and the “primal” and “dual” mixed formulations of the Hodge-Laplacian problems are constructed. We also remark that, the conforming and nonconforming Whitney forms provide an alternative way to reconstruct the duality between the trace spaces of the differential forms corresponding to a horizontal P-A-L type duality between adjoint exterior differential operators. Indeed, for $H^{k}(\Omega)$ on $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, the precise metric of the trace space is subtle for $1 \leq k \leq n-1$ and $n \geq 3$. Further, even for the duality between the trace spaces of $H^{0}(\Omega)$ and $H^{2}(\Omega)$ on $\Omega \subset \mathbb{R}^3$, the preservation is not that direct; for example, when we use conforming linear element space to discretize $H^{0}(\Omega)$ and the lowest-degree Raviart-Thomas space to discretize $H^{2}(\Omega)$, their trace spaces have different dimensions, and no direct isomorphism can be established between them.

Finally, we may conclude that, in this paper, motivated by the needs for constructing structure-preserving discretizations, a theoretical framework is presented so that adjoint properties can be constructed for finite-dimensional operators which can hardly be done in the classical theory of adjoint operators. This may allow us to see things in a more appropriate way, particularly for problems with discrete essence. Directed by the theory, formal methodologies are given for the construction of pairs of partially adjoint discretizations, including the construction of an accompanied discretization with respect to a given discretization. The validities of the theory and the formal methodology are illustrated by adjoint-structure-preserving discretizations of exterior differential operators by conforming and nonconforming Whitney forms. The constructions of discrete Helmholtz decompositions illustrate that nonconforming finite element spaces are necessary.
and the internal continuity of the ABCFESes are proper. Further, the Poincaré-Lefschetz duality is reconstructed as identities, for which we know better than the isomorphisms in earlier theory. The discussions in the present paper may be generalized to more general settings in the future.

1.2. Bibliographic review and remarks. The discussions in the present paper fall into the category of functional analysis and structure-preserving discretization methods for partially differential equations. For functional analysis, we refer to briefly the monographs [6, 13, 30, 44]; for relevant discussions on discretization methods, as well as structure-preserving issues, we refer to briefly the monographs and survey papers [1, 3, 5, 11, 12, 17, 18, 27]. For the construction of discretized adjoint operators which we are focused on in this paper, we refer to, e.g., the monograph [1] and the references therein for relevant discussions. Within these discussions, a discrete analogue of (1.1) is generally constructed, though, as remarked before, the reconstruction of the closed range theorem and the P-A-L type dualities, which are the main focuses of the present paper, has not drawn enough attention. We note that kinds of dualities used to be studied in, e.g., [4, 10, 15, 22, 29, 35, 36, 40, 43]. These works mainly focus on the dual representations for finite element spaces. Some of them discussed dual complexes in the manner that, for a given complex by finite element spaces, they consider how dual representations of these spaces can be constructed and connected to form a complex. The technique of dual/auxiliary grids have usually been used for the construction of discretized dual complexes. Different from them, in this paper, for a given complex composed by primal operators, we study the dual complex composed by adjoint operators (like defined in Section 4.1.2 of [1]). This pair of complexes are discretized by pairs of partially adjoint operators, and a pair of discretized complexes consists of a sequence of complex dualities (see (2.6) and Theorem 2.28 for the so-called complex duality). Note that for here, both complexes are constructed on a same grid, and the reconstruction of Poincaré-Lefschetz type identity becomes possible.

As a fundamental model problem, the discretizations of the exterior differential operators has been drawing very broad interests during the past decades. It is the central topic of, e.g., the discrete exterior calculus (DEC, [21, 28]) and finite element exterior calculus (FEEC, [1, 3, 5]). These existing works mainly use the adjointness between the differential operator $d$ and the boundary operator $\partial$, say, Stokes theorem, to construct discretizations for an operator. In the present paper, we study the connections between discretizations based on the adjointness between differential and codifferential operators $d$ and $\delta$, and provide a systematic family of discretizations both for a pair of adjoint operators and for a single operator.

The ABCDs by Whitney forms for exterior differential operators are generalizations of the Crouzeix-Raviart (C-R) finite element ( [20]) for $H^1$ problems. The C-R element is one of the most important finite elements for partially differential equations. It possesses a cell-wise defined interpolator for functions with minimal regularities and this characteristic plays crucial roles in many applications, such as the correct computation of convex variational problems ( [37]). It is also the starting finite element of the Morley-Wang-Xu family [42], the family of minimal consistent finite elements for $H^m$ problems in arbitrary dimension. It has been far well known that, the C-R finite element space and the Raviart-Thomas (R-T) element space for $H(div)$ problem can
form a discrete version of (1.1), cf. (3.23); in this paper, it is easily shown that the criterion (1.3)-(1.4) holds for C-R element space and R-T element space with respect to a tensorial base pair, and a new interpretation of the C-R element interpolator is established. The pair by the C-R element and the R-T element bears some adjoint properties: connections are established in [33] between the C-R element discretization and the R-T element discretization for $H^1$ problem provided the input data is piecewise constants; a discrete Helmholtz decomposition of piecewise constant space is established in [2], corresponding to $L^2 \Lambda^1$, by C-R element space and the kernel of R-T element space in two dimension, and generalized to three dimension by [34]. In this paper, the C-R element space is generalized to a family for the spaces $H\Lambda^k$ with $0 \leq k \leq n$, the connections between primal and dual schemes for $H\Lambda^k$ problems are established for non-constant input data, and the discrete Helmholtz decompositions are established corresponding to $L^2 \Lambda^k$, $0 \leq k \leq n$, $n \geq 2$. For the first time, to our knowledge, discretized de Rham complexes and commutative diagrams that start with the C-R element spaces are constructed.

There have been kinds of interpolators to finite element spaces which work for functions with minimal regularity requirements, such as [16, 19, 23, 24, 26, 31, 32, 41]. For these interpolators, the regularization, smoothing or averaging techniques are usually used based on macroelements consisting of patches of elements. Though, the adjoint projection (see Definition 3.28 and (3.20)) given in this paper is one which can be strictly cell-wise defined and fit for general boundary conditions. It can be quite flexible with respect to the shapes of the cells of the subdivisions.

The examples of ABCDs given in the present paper are all by piecewise polynomials. We remark that the shape functions and the subdivisions for ABCD can be flexible. For example, conforming virtual element space (VEM, cf., e.g., [7-9]) can be adopted for CD for one operator on complicated subdivision, and ABCD with piecewise polynomials can be constructed for the other; also, the utilization of serendipity polynomials and spectral functions for piecewise shape function spaces will be discussed. We emphasize again that, though motivated by the in-pair discretizations of a pair of adjoint operators, the methodology of ABCD, or rather more general as dually consistent schemes, can be fairly used for a single discretization, not necessarily as part of partially adjoint discretizations. This can be illustrated by the Hodge Laplacian problem and other complicated problems. New algorithms and analysis can also be stimulated for eigenvalue computation, Dirichlet-Neumann operator, multigrid methods and so forth.

1.3. Organization of the paper. At the remaining of this section, we introduce some notations and conventions. In Section 2, we present the definition and basic properties of partially adjoint operators. We will firstly revisit some basics of adjoint operators, and then construct the analogue theory for partially adjoint operators. The P-A-L type dualities are presented as Theorems 2.3 and 2.4, and the quantified closed range theorem is established as Theorem 2.10. The definitions of base operator pair and partially adjoint are given as Definition 2.13 and Definition 2.15. Table 1 is given in Section 2.2 for a list and comparison of adjoint properties for adjoint operators and partially adjoint operators. In Section 3, we present a general methodology for constructing partially adjoint discretizations of adjoint operators by CD and ABCD. The definitions of accompanied-by-conforming discretization and partially adjoint discretizations are given
as Definition 3.12 and Definition 3.14. Compact supported basis functions are constructed in Section 3.2, cell-wise-defined interpolations are constructed in Section 3.3, and examples of ABCD are given in Section 3.4. In Section 4, using the methodology of ABCD, we construct a unified family of nonconforming Whitney forms for the exterior differential operators. The conforming and nonconforming Whitney forms compose partially adjoint discretizations for adjoint exterior differential operators, and structural properties are preserved by discretizations.

1.4. Notations and conventions. In this paper, we usually use three capital letters with different mathematical fonts, for example X, Y and Z, for three Hilbert spaces, with inner products \( \langle \cdot, \cdot \rangle_X \), \( \langle \cdot, \cdot \rangle_Y \) and \( \langle \cdot, \cdot \rangle_Z \), and norms \( \| \cdot \|_X \), \( \| \cdot \|_Y \) and \( \| \cdot \|_Z \), respectively. We use, e.g., v, w and Ω to denote elements in X, Y and Z, use P, P and P to denote subsets of X, Y and Z, and use T, T and T to denote operators defined from X, Y and Z, respectively. The fonts of the letters indicate their relations to certain Hilbert spaces.

We use \( \mathcal{N} \) and \( \mathcal{R} \) to denote the null space and range of certain operators. Namely, for example, \( \mathcal{N}(T, D) := \{ v \in D : T v = 0 \} \), and \( \mathcal{R}(T, D) := \{ T v : v \in D \} \). We use the overline \( \bar{\cdot} \) for the closure of a set. A capital letter with under tilde \( \tilde{\cdot} \) is generally a subset of the same letter with tilde \( \hat{\cdot} \).

For a Hilbert space H, we use the notations \( \oplus^H \) and \( \ominus^H \) to denote the orthogonal summation and orthogonal difference; namely, for two spaces A and B in H, the presentation \( A \ominus^H B \) implies that A and B are orthogonal in H, and evaluates as the direct summation of A and B; for \( A \subset B \subset H \), \( B \ominus^H A \) evaluates as the orthogonal complementation of A in B. We use the supscripts \( \neg \) and \( \succ \) to denote the “outer” and “inner” orthogonal complement. For example, \( A^{\neg H} \) to denote the orthogonal complement of a space in H; \( A^{\succ H} \) denotes the orthogonal complement of \( \mathcal{N}(T, A) \) in A. The subscripts H and T can occasionally be dropped. The ring \( \odot \) generally indicates kernel, and the subscript \( \odot_0 \) generally indicates subspace that adjoint-orthogonal to another space. The meanings of \( \ominus, \ominus^\neg, \ominus^\succ \) and \( \odot_0 \) will be made clear once they are used unless there is no ambiguity.

For an operator \( T : X \to Y \), we usually use \( H(T) \) for short to denote the space \( \{ v \in X : T v \in Y \} \). We use \( L^2 \) for the Lebesgue space that consists of square integrable functions.

We usually use a square bracket for a pair of operators, such as \( [(T, D) : X \to Y, (T', D') : X \to Y] \), and \( [(T, D), (T', D')] \) for short when there is no ambiguity.

2. Theory of partially adjoint operators

In this section, we introduce the theory of partially adjoint operators. We first revisit the basics of adjoint operators. Then we present the basics of partially adjoint operators. Some technical proofs are given in a separate subsection. We use X, Y and Z for three Hilbert spaces.

2.1. Basics of adjoint operators revisited. Let \( [(T, D) : X \to Y, (T', D') : Y \to X] \) be a pair of adjoint closed densely defined operators. In this part, we study and present some fundamental properties, known or unknown, of the pair.

2.1.1. Poincaré-Alexander-Lefschetz type dualities. Below is a most fundamental property of adjoint operators, and can be found in every textbook. We call them orthogonal dualities.
Lemma 2.1 (Orthogonal dualities).
\[
\mathcal{N}(T, \mathbb{D}) = \overline{\mathcal{R}(T, \mathbb{D})}^{\perp}, \quad \overline{\mathcal{N}(T, \mathbb{D})} = \mathcal{N}(T, \mathbb{D})^{\perp},
\]
(2.1)
\[
\mathcal{N}(T, \mathbb{D}) = \mathcal{R}(T, \mathbb{D})^{\perp}, \quad \text{and,} \quad \overline{\mathcal{R}(T, \mathbb{D})} = \mathcal{N}(T, \mathbb{D})^{\perp}.
\]

The Helmholtz decomposition below follows immediately.

Lemma 2.2 (Helmholtz decomposition).
\[
X = \mathcal{N}(T, \mathbb{D}) \oplus_{X} \overline{\mathcal{R}(T, \mathbb{D})}, \quad \text{and} \quad Y = \mathcal{N}(T, \mathbb{D}) \oplus_{Y} \overline{\mathcal{R}(T, \mathbb{D})}.
\]
(2.2)

Theorem 2.3 (Horizontal Poincaré-Alexander-Lefschetz type duality). For two pairs of adjoint closed operators \((T, \overline{D}) : X \to Y, (T, \overline{D}) : Y \to X\) and \((T, \overline{D}) : X \to Y, (T, \mathbb{D}) : Y \to X\) such that \(\overline{D} \supseteq \overline{D}\) and, equivalently, \(\mathbb{D} \subseteq \overline{\mathbb{D}}\),
\[
\mathcal{R}(T, \overline{D}) \oplus_{X} \overline{\mathcal{R}(T, \overline{D})} = \mathcal{N}(T, \overline{D}) \oplus_{X} \mathcal{N}(T, \overline{D}),
\]
(2.3)
\[
\mathcal{R}(T, \overline{D}) \oplus_{Y} \overline{\mathcal{R}(T, \overline{D})} = \mathcal{N}(T, \overline{D}) \oplus_{Y} \mathcal{N}(T, \overline{D}).
\]
(2.4)

Proof. The assertion (2.3) follows immediately from that
\[
\begin{cases}
\mathcal{R}(T, \overline{D}) \oplus_{X} \mathcal{N}(T, \mathbb{D}) = \overline{\mathcal{R}(T, \overline{D})} \oplus_{X} \mathcal{N}(T, \overline{D}), \\
\mathcal{R}(T, \overline{D}) \supseteq \mathcal{R}(T, \overline{D}), \quad \text{and} \quad \mathcal{N}(T, \overline{D}) \supseteq \mathcal{N}(T, \overline{D}).
\end{cases}
\]
(2.5)

Similarly can the assertion (2.4) be proved.

Due to the structural properties (2.5), we borrow the notion from algebraic topology and call the two equalities (2.3) and (2.4) Poincaré-Alexander-Lefschetz (P-A-L for short) type dualities, particularly of horizontal type, whereas below is a vertical P-A-L type duality.

Let \([S, \mathbb{E}] : \mathcal{Y} \to \mathcal{Z}, (S, \mathbb{E}) : \mathcal{Z} \to \mathcal{Y}\) be a pair of adjoint closed densely defined operators. Then \(\mathcal{Y} = \mathcal{N}(S, \mathbb{E}) \oplus_{Y} \overline{\mathcal{R}(S, \mathbb{E})}\), and \(\mathcal{Z} = \mathcal{N}(S, \mathbb{E}) \oplus_{Y} \overline{\mathcal{R}(S, \mathbb{E})}\). Further,
\[
\mathcal{R}(T, \overline{D}) \subset \mathcal{N}(S, \mathbb{E}) \iff \overline{\mathcal{R}(S, \mathbb{E})} \subset \mathcal{N}(T, \overline{D}).
\]
(2.6)

The two sides of (2.6) are the dual complex of each other (cf. [1]). In the sequel, we call the relation (2.6) a complex duality.

Theorem 2.4 (Vertical P-A-L type duality). Assume \(\mathcal{R}(T, \overline{D}) \subset \mathcal{N}(S, \mathbb{E})\).
\[
\mathcal{N}(S, \mathbb{E}) \oplus_{Y} \overline{\mathcal{R}(T, \overline{D})} = \mathcal{N}(T, \overline{D}) \oplus_{X} \overline{\mathcal{R}(S, \mathbb{E})}.
\]
(2.7)

Proof. The assertion follows by the Helmholtz decompositions (by two ways) of \(\mathcal{Y}\).

Remark 2.5. In the context of exterior differential calculus, Theorem 2.4 corresponds to the Poincaré-Lefschetz duality; see also Lemma 4.3.

Denote
\[
\mathcal{H}_{\mathcal{Y}} := \mathcal{N}(S, \mathbb{E}) \oplus_{Y} \overline{\mathcal{R}(T, \overline{D})} = (\text{by Theorem 2.4}) \mathcal{N}(T, \overline{D}) \oplus_{X} \overline{\mathcal{R}(S, \mathbb{E})}.
\]
The Hodge decomposition below is well known.
Lemma 2.6 (Hodge decomposition).

\[ (2.8) \quad \mathcal{H}_{\mathcal{Y}} = \overline{\mathcal{R}(T, D)}^\perp \mathcal{H}_{\mathcal{X}}^\perp \oplus \overline{\mathcal{R}(S, E)}. \]

2.1.2. Index of closed range and quantified closed range theorem. The Banach closed range theorem (Theorem 2.7) is another fundamental property. We refer to, e.g., Section VII.5 (“The closed range theorem”) of [44], for a proof in the context of Banach spaces, and, e.g., Section 3.3 of [1] for a proof in the context of Hilbert spaces.

**Theorem 2.7.** For \([\mathcal{T}, \mathcal{D}], (\mathcal{T}, \mathcal{D}) \] a pair of closed densely defined adjoint operators, \(\mathcal{R}(\mathcal{T}, \mathcal{D})\) is closed in \(\mathcal{Y}\) if and only if \(\mathcal{R}(\mathcal{T}, \mathcal{D})\) is closed in \(\mathcal{X}\).

**Definition 2.8.** For \((\mathcal{T}, \mathcal{D}) : \mathcal{X} \to \mathcal{Y}\) a closed operator, denote

\[ \mathcal{D}^\perp := \{ v \in \mathcal{D} : \langle v, w \rangle_{\mathcal{X}} = 0, \forall w \in \mathcal{N}(\mathcal{T}, \mathcal{D}) \}. \]

Define the index of closed range of \((\mathcal{T}, \mathcal{D})\) as

\[ (2.9) \quad \text{icr}(\mathcal{T}, \mathcal{D}) := \begin{cases} \sup_{0 \neq v \in \mathcal{D}^\perp} \frac{||v||_{\mathcal{X}}}{||Tv||_{\mathcal{Y}}}, & \text{if } \mathcal{D}^\perp \neq \{0\}; \\ 0, & \text{if } \mathcal{D}^\perp = \{0\}. \end{cases} \]

Note that \(\text{icr}(\mathcal{T}, \mathcal{D})\) evaluates in \([0, +\infty)\). Evidently, \(\mathcal{R}(\mathcal{T}, \mathcal{D}) = \mathcal{R}(\mathcal{T}, \mathcal{D}^\perp), \mathcal{T}\) is a bijection from \(\mathcal{D}^\perp\) onto \(\mathcal{R}(\mathcal{T}, \mathcal{D})\), and \(\text{icr}(\mathcal{T}, \mathcal{D}) = \text{icr}(\mathcal{T}, \mathcal{D}^\perp)\). Further, \(\text{icr}(\mathcal{T}, \mathcal{D}^\perp)\) plays like the constant for Poincaré inequality in the sense that \(||v||_{\mathcal{X}} \leq \text{icr}(\mathcal{T}, \mathcal{D}^\perp)||Tv||_{\mathcal{Y}}\) for \(v \in \mathcal{D}^\perp\).

Lemma 2.9 below is actually known. We refer to, e.g., Lemma 3.6 of [1] for a proof.

**Lemma 2.9.** For \((\mathcal{T}, \mathcal{D}) : \mathcal{X} \to \mathcal{Y}\) a closed operator, its range \(\mathcal{R}(\mathcal{T}, \mathcal{D})\) is closed if and only if \(\text{icr}(\mathcal{T}, \mathcal{D}) < +\infty\).

**Theorem 2.10.** For \([\mathcal{T}, \mathcal{D}], (\mathcal{T}, \mathcal{D}) \] a pair of closed densely defined adjoint operators,

\[ (2.10) \quad \text{icr}(\mathcal{T}, \mathcal{D}) = \text{icr}(\mathcal{T}, \mathcal{D}). \]

**Proof.** Recalling (2.1), we have

\[ (2.11) \quad \mathcal{D}^\perp = \mathcal{D} \cap (\mathcal{N}(\mathcal{T}, \mathcal{D}))^\perp = \mathcal{D} \cap \overline{\mathcal{R}(\mathcal{T}, \mathcal{D})}. \]

Therefore, provided that \(0 < \text{icr}(\mathcal{T}, \mathcal{D}) < +\infty\) and thus \(\mathcal{R}(\mathcal{T}, \mathcal{D}) = \mathcal{R}(\mathcal{T}, \mathcal{D})\), given \(v \in \mathcal{D}^\perp\), there exists a \(w \in \mathcal{D}^\perp\), such that \(v = T^\perp w\), and

\[ ||v||_\mathcal{X} = \langle v, v \rangle_\mathcal{X} = \langle v, T^\perp w \rangle_\mathcal{X} = \langle Tv, w \rangle_\mathcal{Y} \leq ||Tv||_\mathcal{Y}||w||_\mathcal{Y} \leq \text{icr}(\mathcal{T}, \mathcal{D})||Tv||_\mathcal{Y}||w||_\mathcal{Y}. \]

Therefore, \(||v||_\mathcal{X} \leq \text{icr}(\mathcal{T}, \mathcal{D})||Tv||_\mathcal{Y}\) for any \(v \in \mathcal{D}^\perp\) and \(\text{icr}(\mathcal{T}, \mathcal{D}) \leq \text{icr}(\mathcal{T}, \mathcal{D}) < +\infty\). Similarly, \(\text{icr}(\mathcal{T}, \mathcal{D}) \geq \text{icr}(\mathcal{T}, \mathcal{D})\); note that \((\mathcal{T}, \mathcal{D})\) is the adjoint operator of \((\mathcal{T}, \mathcal{D})\). Namely, if one of \(\text{icr}(\mathcal{T}, \mathcal{D})\) and \(\text{icr}(\mathcal{T}, \mathcal{D})\) is finitely positive, then \(\text{icr}(\mathcal{T}, \mathcal{D}) \neq \text{icr}(\mathcal{T}, \mathcal{D})\).

If \(\text{icr}(\mathcal{T}, \mathcal{D}) = 0\), then \(\mathcal{R}(\mathcal{T}, \mathcal{D}) = \{0\}\) and \(\mathcal{D}^\perp = \{0\}\). It follows then \(\text{icr}(\mathcal{T}, \mathcal{D}) = 0\). Further, if one of \(\text{icr}(\mathcal{T}, \mathcal{D})\) and \(\text{icr}(\mathcal{T}, \mathcal{D})\) is \(+\infty\), then so is the other.

The proof is completed. \(\square\)

**Remark 2.11.** **Theorem 2.10 is a quantified version of the closed range theorem.**

2.2. **Theory of partially adjoint operators.**
2.2.1. Base operator pair and partially adjoint operators. Let \((T, \tilde{M}) : X \to Y\) and \((T, \tilde{N}) : Y \to X\) be two closed operators, not necessarily densely defined. Denote, for \(v \in \tilde{M}\), \(\|v\|_T := (\|v\|_X^2 + \|Tv\|_Y^2)^{1/2}\), and for \(w \in \tilde{N}\), \(\|w\|_T := (\|w\|_Y^2 + \|Tw\|_X^2)^{1/2}\). Denote

\[
M := \left\{ v \in \tilde{M} : \langle v, Tw \rangle_X - \langle Tv, w \rangle_Y = 0, \forall w \in \tilde{N} \right\},
\]

\[
\mathcal{N} := \left\{ v \in \tilde{N} : \langle v, Tw \rangle_X - \langle Tv, w \rangle_Y = 0, \forall w \in \tilde{M} \right\},
\]

\[
M_B := \left\{ v \in \tilde{M} : \langle v, w \rangle_X = 0, \forall w \in \mathcal{N}(T, M); \langle Tv, w \rangle_Y = 0, \forall w \in M \right\},
\]

and

\[
\mathcal{N}_B := \left\{ v \in \tilde{N} : \langle v, w \rangle_Y = 0, \forall w \in \mathcal{N}(T, \mathcal{N}); \langle Tv, w \rangle_X = 0, \forall w \in \mathcal{N}_B \right\}.
\]

Lemma 2.12. \((T, M)\), \((T, N)\), \((T, M_B)\) and \((T, \mathcal{N}_B)\) are all closed.

Proof. Let \(\{v_k\}_{k=1,2,\ldots} \subset M\) be such that \(\|v_i - v_j\|_T \to 0\), as \(i\) and \(j\) tend to infinity. Then, there exists a \(v_\infty \in \tilde{M}\), such that \(\lim_{k \to \infty} \|v_k - v_\infty\|_T = 0\). For any \(w \in \tilde{N}\), \(\langle v_\infty, Tw \rangle_X - \langle Tv_\infty, w \rangle_Y = \lim_{k \to \infty} \langle v_k, Tw \rangle_X - \langle Tv_k, w \rangle_Y = 0\); namely, \(v_\infty \in \tilde{M}\). Therefore, \((T, \tilde{M})\) is closed. The others are proved the same way. \(\Box\)

Definition 2.13. A pair of closed operators \(\left[(T, \tilde{M}) : X \to Y, (T, \tilde{N}) : Y \to X\right]\) is called a base operator pair, if, with notations (2.12), (2.13), (2.14) and (2.15),

1. \(\mathcal{R}(T, \tilde{M}), \mathcal{R}(T, \tilde{N}), \mathcal{R}(T, M)\) and \(\mathcal{R}(T, \mathcal{N})\) are all closed;
2. \(\mathcal{N}(T, M_B)\) and \(\mathcal{R}(T, \mathcal{N}_B)\) are isomorphic, and \(\mathcal{N}(T, \mathcal{N}_B)\) and \(\mathcal{R}(T, M_B)\) are isomorphic.

For \(\left[(T, \tilde{M}), (T, \tilde{N})\right]\) a base operator pair, for nontrivial \(\mathcal{R}(T, \mathcal{N}_B)\) and \(\mathcal{N}(T, M_B)\), denote

\[
\alpha := \inf_{0 \neq v \in \mathcal{N}(T, M_B)} \sup_{w \in \mathcal{R}(T, \mathcal{N}_B)} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X},
\]

and for nontrivial \(\mathcal{N}(T, \mathcal{N}_B)\) and \(\mathcal{R}(T, M_B)\), denote

\[
\beta := \inf_{0 \neq v \in \mathcal{R}(T, \mathcal{N}_B)} \sup_{w \in \mathcal{N}(T, M_B)} \frac{\langle v, w \rangle_Y}{\|v\|_Y \|w\|_Y}.
\]

Then \(\alpha > 0\) and \(\beta > 0\). We further make a convention that,

\[
\begin{cases}
\alpha = 1, & \text{if } \mathcal{N}(T, M_B) = \mathcal{R}(T, \mathcal{N}_B) = \{0\}; \\
\beta = 1, & \text{if } \mathcal{N}(T, \mathcal{N}_B) = \mathcal{R}(T, M_B) = \{0\}.
\end{cases}
\]

Further,

\[\mathcal{R}(T, \tilde{M}) = \mathcal{R}(T, M) \oplus^\perp \mathcal{R}(T, M_B), \quad \mathcal{R}(T, \tilde{N}) = \mathcal{R}(T, \mathcal{N}) \oplus^\perp \mathcal{R}(T, \mathcal{N}_B),\]

and

\[\mathcal{N}(T, \tilde{M}) = \mathcal{N}(T, M) \oplus^\perp \mathcal{N}(T, M_B), \quad \mathcal{N}(T, \tilde{N}) = \mathcal{N}(T, \mathcal{N}) \oplus^\perp \mathcal{N}(T, \mathcal{N}_B).\]
Lemma 2.14. For \([\mathbf{T}, \mathbf{\tilde{M}}), (\mathbf{T}, \mathbf{\tilde{N}})]\) a base operator pair, \(M_B = \{0\}\) if and only if \(N_B = \{0\}\). For \(M_B \neq \{0\}\),

\[
\inf_{0 \neq v \in M_B} \sup_{\nu \in N_B} \frac{\langle v, T\nu \rangle - \langle Tv, \nu \rangle}{\|v\|_T \|T\nu\|} = \inf_{0 \neq \nu \in N_B} \sup_{v \in M_B} \frac{\langle v, T\nu \rangle - \langle Tv, \nu \rangle}{\|v\|_T \|T\nu\|} > 0.
\]

Proof. If \(N(T, M_B) = \{0\}\) (or \(R(T, M_B) = \{0\}\)), then \(R(T, N_B) = \{0\}\) (respectively \(N(T, N_B) = \{0\}\)), and the assertion follows by Definition 2.13 immediately. We only consider the nontrivial case that \(N(T, M_B) \neq \{0\}\) and \(R(T, M_B) \neq \{0\}\).

Denote \(M_B^0 := \{v \in M_B : \langle v, w \rangle_X = 0, \forall w \in N(T, M_B)\}\) and \(N_B^0 := \{w \in N_B : \langle w, v \rangle_Y = 0, \forall v \in N(T, N_B)\}\). Decompose \(M_B = N(T, M_B) \oplus M_B^0\), and for any \(v = \hat{v} + v^\perp\) with \(\hat{v} \in N(T, M_B)\) and \(v^\perp \in M_B^0\), \(\|\hat{v}\|_X + \|T\hat{v}\|_Y\) is a norm of \(v\) equivalent to \(\|v\|_T\) on \(M_B\).

Given \(v \in M_B\), \(v = \hat{v} + v^\perp\), by (2.16) and (2.18), there exists a \(w^\perp \in N_B^0\), such that \(\langle \hat{v}, T\nu \rangle_X = \|\hat{v}\|_X^2\), and \(\|w^\perp\|_T \leq C\|\hat{v}\|_X\). Let \(w_0 \in N(T, N_B)\) be such that

\[
\langle \hat{v}, T\nu \rangle_X - \langle T\hat{v}, w_0 \rangle_Y = (\langle \hat{v}, T\nu \rangle - \langle T\hat{v}, w_0 \rangle)_Y, \forall \nu \in M_B^0.
\]

Then \(\|w_0\|_Y \leq C(\|T\nu\|_T + \|T\hat{v}\|_Y) \leq C(\|\hat{v}\|_X + \|\hat{v}\|_T)\), and

\[
\langle v^\perp, T\nu \rangle_X - \langle T v^\perp, w_0 \rangle_Y = (\langle v^\perp, T\nu \rangle - \langle T v^\perp, w_0 \rangle)_Y,
\]

namely

\[
\langle v, T\nu \rangle_X - \langle Tv, w_0 + w^\perp \rangle_Y = (\langle v, T\nu \rangle - \langle Tv, w_0 \rangle)_Y.
\]

This shows that \(\inf_{0 \neq v \in M_B} \sup_{\nu \in N_B} \frac{\langle v, T\nu \rangle - \langle Tv, \nu \rangle}{\|v\|_T \|T\nu\|} \geq C > 0\). Here we use “C” for a generic constant dependent on \(a, \beta\) and indices of closed range of \(T\) and \(\mathbf{T}\), which may differ at different places.

On the other hand, for the same reason, given \(\nu \in N_B\), there exists \(v \in M_B\), such that \(\langle v, T\nu \rangle_X - \langle Tv, \nu \rangle_Y \neq 0\). Therefore, noting that \((T, M_B)\) and \((\mathbf{T}, \mathbf{N})\) are closed, we have

\[
\inf_{0 \neq \nu \in N_B} \sup_{v \in M_B} \frac{\langle v, T\nu \rangle_X - \langle T v, \nu \rangle_Y}{\|v\|_T \|T\nu\|} = \inf_{0 \neq v \in M_B} \sup_{\nu \in N_B} \frac{\langle v, T\nu \rangle_X - \langle T v, \nu \rangle_Y}{\|v\|_T \|T\nu\|}.
\]

The proof is completed. \(\square\)

Definition 2.15. For \([\mathbf{T}, \mathbf{\tilde{M}}) : \mathbf{X} \to \mathbf{Y}, (\mathbf{T}, \mathbf{\tilde{N}}) : \mathbf{Y} \to \mathbf{X}\] a base operator pair, two operators \((\mathbf{T}, \mathbf{D}) \subset (\mathbf{T}, \mathbf{\tilde{M}})\) and \((\mathbf{T}, \mathbf{\tilde{D}}) \subset (\mathbf{T}, \mathbf{\tilde{N}})\) are called partially adjoint based on \([\mathbf{T}, \mathbf{\tilde{M}}), (\mathbf{T}, \mathbf{\tilde{N}})]\), if

(2.19) \(D = \{v \in \tilde{M} : \langle v, T\nu \rangle_X - \langle Tv, \nu \rangle_Y = 0, \forall \nu \in \tilde{D}\}\),

and (2.20) \(\tilde{D} = \{v \in \tilde{N} : \langle v, T\nu \rangle_X - \langle Tv, \nu \rangle_Y = 0, \forall v \in \tilde{D}\}\).

In the sequel, we may occasionally omit the base operator pair for partially adjoint operators when there is no ambiguity.

Lemma 2.16. Let \((\mathbf{T}, \mathbf{D})\) and \((\mathbf{T}, \mathbf{\tilde{D}})\) be partially adjoint based on \([\mathbf{T}, \mathbf{\tilde{M}}), (\mathbf{T}, \mathbf{\tilde{N}})]\). Then both \((\mathbf{T}, \mathbf{D})\) and \((\mathbf{T}, \mathbf{\tilde{D}})\) are closed.
Proof. Let \( \{v_n\}_{n=1}^\infty \subset D \) be such that \( \|v_m - v_n\|_T \to 0 \) as \( m \) and \( n \) tend to \( \infty \). Then there exists a \( v_\infty \in \widehat{N} \), such that \( \lim_{k \to \infty} \|v_k - v_\infty\|_T = 0 \). It is easy to verify that \( \langle v, Tv_\infty \rangle_X - \langle Tv, v_\infty \rangle_Y = 0 \) for any \( v \in D \), and \( v_\infty \in D \). Namely \( (T, D) \) is closed. So is \( (T, D) \). The proof is completed. □

**Theorem 2.17.** For \( [(T, M), (\overline{T}, \overline{N})] \) a base operator pair, an operator \( (T, D) \) admits a partially adjoint operator \( (T, D) \) based on \( [(T, M), (\overline{T}, \overline{N})] \), if and only if \( (T, D) \) is closed and \( M \subset D \subset M \).

**Proof.** If \( (T, D) \) and \( (T, D) \) are partially adjoint based on \( [(T, M), (\overline{T}, \overline{N})] \), then, by definition, \( M \subset D \) and \( N \subset \overline{D} \), and by Lemma 2.16, the two operators are closed. On the other hand, if \( (T, D) \) is closed and \( D \supset M \), denote

\[
(D) := \{v \in \overline{N} : \langle v, Tv \rangle_X - \langle Tv, v \rangle_Y = 0, \ \forall v \in D \}.
\]

Then \( N \subset D \), \( D = N \oplus (D \cap N_B) \) and \( (T, D) \) is closed. It now suffices to show that

\[
(D) = \{v \in M : \langle v, Tv \rangle_X - \langle Tv, v \rangle_Y = 0, \ \forall v \in D \},
\]

which follows from, as \( D = M \oplus (D \cap N_B) \) and \( \{v \in M : \langle v, Tv \rangle_X - \langle Tv, v \rangle_Y = 0, \ \forall v \in D \} = M \oplus \{v \in M_B : \langle v, Tv \rangle_X - \langle Tv, v \rangle_Y = 0, \ \forall v \in D \cap N_B \}, \)

\[
D \cap M_B = \{v \in M_B : \langle v, Tv \rangle_X - \langle Tv, v \rangle_Y = 0, \ \forall v \in D \cap N_B \} \tag{2.23}
\]

By (2.21),

\[
(D) \cap N_B = \{v \in N_B : \langle v, Tv \rangle_X - \langle Tv, v \rangle_Y = 0, \ \forall v \in D \}
\]

Since \( (T, N_B) \) is closed, \( (T, D \cap N_B) \) is closed. Further, since both \( (T, D \cap M_B) \) and \( (T, D \cap N_B) \) are closed, (2.23) holds by (2.24) and Lemma 2.14. Then (2.22) follows and the proof is completed. □

**Lemma 2.18.** A pair of closed densely defined adjoint operators \( [(T, D), (\overline{T}, \overline{D})] \) are partially adjoint based on any pair of closed operators \( [(T, D), (\overline{T}, \overline{D})] \) such that \( (T, D) \supset (T, D), (\overline{T}, \overline{D}) \supset (T, D), \) and both \( R(T, D) \) and \( R(T, D) \) are closed.

**Proof.** As \( (T, D) \) and \( (T, D) \) are adjoint operators, we can show by definition that \( D = \{v \in \overline{D} : \langle Tv, v \rangle_Y = \langle v, Tv \rangle_X, \ \forall v \in D \} \) and \( \overline{D} = \{v \in \overline{D} : \langle Tv, v \rangle_Y = \langle v, Tv \rangle_X, \ \forall v \in D \} \). It remains to show that \( [(T, \overline{D}), (\overline{T}, \overline{D})] \) is a base operator pair.

Let \( (T, D) \) and \( (T, D) \) be the adjoint operators of \( (T, \overline{D}) \) and \( (T, \overline{D}) \), respectively. Then \( D = \{v \in \overline{D} : \langle Tv, v \rangle_Y = \langle v, Tv \rangle_X, \ \forall v \in D \} \), and \( \overline{D} = \{v \in \overline{D} : \langle Tv, v \rangle_Y = \langle v, Tv \rangle_X, \ \forall v \in \overline{D} \} \). By closed range theorem, both \( R(T, D) \) and \( R(T, D) \) are closed. By Theorem 2.3, \( R(T, D) \oplus R(T, D) = N(T, \overline{D}) \oplus N(T, D) \), and \( R(T, \overline{D}) \oplus R(T, D) = N(T, \overline{D}) \oplus N(T, D) \). This shows that \( [(T, \overline{D}), (\overline{T}, \overline{D})] \) is a base operator pair. The proof is completed. □
In the remaining of this subsection, we present some adjoint properties of partially adjoint operators. They are respective analogues of the properties of adjoint operators; cf. Table 1. Some of the technical proofs are postponed to Section 2.3.

| adjoint properties                              | adjoint operators | partially adjoint operators |
|------------------------------------------------|-------------------|-----------------------------|
| orthogonal duality                             | Lemma 2.1         | Lemmas 2.20 and 2.19        |
| complex duality                                | Equation (2.6)    | Theorem 2.28                |
| quantified closed range theorem                | Theorem 2.10      | Theorem 2.21                |
| Helmholtz decomposition                         | Theorem 2.2       | Theorem 2.23                |
| horizontal P-A-L duality                       | Theorem 2.3       | Theorems 2.24, and, 2.22 (slice isomorphism) |
| Hodge decomposition                             | Lemma 2.6         | Theorem 2.31                |
| vertical P-A-L duality                         | Theorem 2.7       | Theorems 2.30, and, 2.29 (slice isomorphism) |

Table 1. Correspondence between properties of adjoint operators and partially adjoint operators.

2.2.2. Orthogonal dualities. With $[(T, \overline{M}), (\overline{T}, \overline{N})]$ being a base operator pair, denote, for $M \subset D \subset \overline{M},$

$$D_B := \left\{ v \in D : \langle v, w \rangle_X = 0, \forall w \in N(T, M) ; \langle Tv, Tw \rangle_Y = 0, \forall w \in M \right\}$$

and, for $N \subset D \subset \overline{N},$

$$D_B := \left\{ v \in D : \langle v, w \rangle_Y = 0, \forall w \in N(\overline{T}, \overline{N}) ; \langle T\overline{v}, \overline{T}w \rangle_X = 0, \forall w \in \overline{N} \right\} .$$

Then $D_B = D \cap M_B$, and $D_B = D \cap N_B$; $D = D_B \oplus M$, and $D = D_B \oplus N$. Further, $N(T, D_B) = N(T, M_B) \cap D$, $N(\overline{T}, D_B) = N(\overline{T}, N_B) \cap \overline{D}$, $N(T, D) = N(T, M) \oplus N(T, D_B)$, and $\overline{R(T, D)} = \overline{R(T, D_B)} \oplus \overline{R(T, M)}$.

Lemma 2.19. For $[(T, D), (\overline{T}, \overline{D})]$ partially adjoint based on $[(T, \overline{M}), (\overline{T}, \overline{N})],$

1. $N(T, D_B) = \{ v \in N(T, N_B) : \langle v, Tw \rangle_Y = 0, \forall w \in D_B \}$;

2. $\overline{R(T, D_B)} = \{ p \in \overline{R(T, M_B)} : \langle p, v \rangle_Y = 0, \forall v \in N(T, D_B) \}$.

The same orthogonalities hold for $N(\overline{T}, D_B)$ and $\overline{R(\overline{T}, D_B)}$.

Proof. Given $v \in N(T, N_B), \overline{v} \in N(\overline{T}, D_B)$ if and only if $\langle v, Tw \rangle_Y = 0$ for any $w \in D_B$, which implies that $\overline{v} \in D$. This proves the item (1).

Since $\overline{R(T, M_B)}$ is closed, we decompose it to $\overline{R(T, M_B)} = \overline{R(T, D_B)} \oplus \overline{R(T, D_B)^c}$, where the subscript $^c$ denotes the orthogonal complement in $\overline{R(T, M_B)}$. Denote by $J$ the isomorphism from $N(T, N_B)$ onto $R(T, M_B)$, such that $\langle v, Jw \rangle_Y = \langle v, w \rangle_Y$, $w \in N(T, N_B), \forall v \in R(T, M_B)$. Then given $v \in N(T, N_B), \overline{v} \in N(\overline{T}, D_B)$ if and only if $J\overline{v} \in \overline{R(T, D_B)}$; thus $\overline{R(J, N(T, D_B))} = \overline{R(T, D_B)}$. Hence, $\overline{R(T, D_B)} = \overline{R(J, N(T, D_B))^c} = \{ p \in \overline{R(T, M_B)} : \langle p, Jv \rangle_Y = \langle p, v \rangle_Y = 0, \forall v \in N(T, D_B) \}$. This proves the item (2).
The proof is completed. □

**Lemma 2.20.** For \([T, D], (T, D)\) partially adjoint based on \([\hat{T}, \hat{M}], (T, N)\],

1. \(N(T, D) = \{v \in N(T, \hat{M}) : \langle p, v \rangle_X = 0, \forall p \in \text{R}(T, D)\}\);
2. \(\overline{\text{R}(T, D)} = \left\{ p \in \text{R}(T, \hat{M}) : \langle p, v \rangle_Y = 0, \forall v \in N(T, D) \right\}\).

**Proof.** Given \(v \in N(T, \hat{M}) \supset N(T, D)\), \(v \in N(T, D)\) if and only if \(v \in D\), namely for any \(w \in D\),
\[
\langle v, Tw \rangle_X = \langle v, Tw \rangle_Y = 0.
\]
This proves the item (1).

For the item (2),
\[
\overline{\text{R}(T, D)} = \text{R}(T, M) \oplus \text{R}(T, D_B)
\]
\[
= \text{R}(T, M) \oplus \left\{ p \in \text{R}(T, M_B) : \langle p, v \rangle_Y = 0, \forall v \in \text{N}(T, D_B) \right\}
\]
\[
= \text{R}(T, M) \oplus \left\{ p \in \text{R}(T, M_B) : \langle p, v \rangle_Y = 0, \forall v \in \text{N}(T, D) \right\}
\]
\[
= \left\{ p \in \text{R}(T, \hat{M}) : \langle p, v \rangle_Y = 0, \forall v \in \text{N}(T, D) \right\}.
\]

The proof is completed. □

2.2.3. **Quantified closed range theorem for partially adjoint operators.**

**Theorem 2.21.** For \([T, D], (T, D)\) partially adjoint based on \([\hat{T}, \hat{M}], (T, N)\], with notations given in (2.12), (2.13), (2.16), (2.17) and (2.18), if \(\text{icr}(T, D) < \infty\),

\[
\text{icr}(T, D) \leq (1 + \alpha^{-1}) \cdot \text{icr}(T, \hat{M}) + \alpha^{-1} \text{icr}(T, D) + \text{icr}(T, M);
\]

if \(\text{icr}(T, D) < \infty\),

\[
\text{icr}(T, D) \leq (1 + \beta^{-1}) \cdot \text{icr}(T, \hat{M}) + \beta^{-1} \text{icr}(T, D) + \text{icr}(T, N).
\]

We postpone the technical proof of Theorem 2.21 to Section 2.3.2.

2.2.4. **Horizontal slice isomorphisms for partially adjoint operators.**

**Theorem 2.22 (Horizontal slice isomorphism).** For \([T, D], (T, D)\) and \([\hat{T}, \hat{D}], (T, D)\] partially adjoint based on \([\hat{T}, \hat{M}], (T, N)\] such that \(D \subset \hat{D}\) and \(D \subset \hat{D}\), denote \(dR := \overline{\text{R}(T, D)} \ominus \text{R}(T, D)\),
\[
\overline{\text{R}(T, D)} = \overline{\text{R}(T, \hat{D})} \ominus \text{R}(T, D), \quad dN := \text{N}(T, \hat{D}) \ominus \text{N}(T, \hat{D}).
\]
Then \(dR\) and \(dN\) are isomorphic, and \(dR\) and \(dN\) are isomorphic.

When they are nontrivial, with \(\alpha\) and \(\beta\) given in (2.16), (2.17), and (2.18),

\[
\inf \sup_{0 \neq w \in R, v \in N} \frac{\langle w, v \rangle_Y}{\|w\|_Y \|v\|_Y} = \inf \sup_{0 \neq v \in N} \frac{\langle w, v \rangle_Y}{\|w\|_Y \|v\|_Y} \geq \beta,
\]
and

\[
\inf \sup_{0 \neq w \in R, v \in N} \frac{\langle w, v \rangle_X}{\|w\|_X \|v\|_X} = \inf \sup_{0 \neq v \in N} \frac{\langle w, v \rangle_X}{\|w\|_X \|v\|_X} \geq \alpha.
\]
This slice isomorphism is a weak version of the horizontal \textbf{P-A-L} type duality. We postpone the technical proof of Theorem 2.22 to Section 2.3.3.

**Theorem 2.23.** (Helmholtz decomposition) Assume $\mathcal{R}(T, M_B) = \mathcal{N}(T, N_B)$ and $\mathcal{R}(T, N_B) = \mathcal{N}(T, M_B)$. For $[(T, D), (T, \bar{D})]$ any pair of partially adjoint operators based on $[(T, M), (T, \bar{M})]$, (2.29) \[ \mathcal{R}(T, \bar{N}) \oplus^\bot \mathcal{N}(T, M) = \overline{\mathcal{R}(T, \bar{D})} \oplus^\bot \mathcal{N}(T, D), \] and (2.30) \[ \mathcal{R}(T, \bar{M}) \oplus^\bot \mathcal{N}(T, N) = \overline{\mathcal{R}(T, D)} \oplus^\bot \mathcal{N}(T, D). \]

**Proof.** For any pair $[(T, D), (T, \bar{D})]$ partially adjoint based on $[(T, M), (T, \bar{M})]$, since $\mathcal{R}(T, M_B) = \mathcal{N}(T, N_B)$ and $\mathcal{N}(T, D_B) \subset \mathcal{R}(T, M_B)$, by Lemma 2.19, we have $\overline{\mathcal{R}(T, D_B)} = \mathcal{R}(T, M_B) \oplus^\bot \mathcal{N}(T, D_B)$; namely $\mathcal{R}(T, M_B) = \overline{\mathcal{R}(T, D_B)} \oplus^\bot \mathcal{N}(T, D_B)$. Therefore
\[ \overline{\mathcal{R}(T, D)} \oplus^\bot \mathcal{N}(T, D) = \mathcal{R}(T, M) \oplus^\bot \overline{\mathcal{R}(T, D_B)} \oplus^\bot \mathcal{N}(T, D_B) \oplus^\bot \mathcal{N}(T, N) = \mathcal{R}(T, \bar{M}) \oplus^\bot \mathcal{N}(T, N). \]

Similarly, (2.29) can be proved. The proof is completed. \hfill \square

**Theorem 2.24.** (Horizontal \textbf{P-A-L} duality) Assume $\mathcal{R}(T, M_B) = \mathcal{N}(T, N_B)$ and $\mathcal{R}(T, N_B) = \mathcal{N}(T, M_B)$. Then $d\mathcal{R} = d\mathcal{N}$ and $d\mathcal{R} = d\mathcal{N}$.

**Proof.** By Theorem 2.23, note that $\overline{\mathcal{R}(T, D)} \oplus^\bot \mathcal{N}(T, D) = \overline{\mathcal{R}(T, D)} \oplus^\bot \mathcal{N}(T, \bar{D})$, we obtain $d\mathcal{R} = d\mathcal{N}$. That $d\mathcal{R} = d\mathcal{N}$ follows similarly. The proof is completed. \hfill \square

2.2.5. Vertical slice isomorphism for partially adjoint operators. According to Definition 2.13, let $[(S, \tilde{U}) : Y \to Z, (S, \tilde{V}) : Z \to Y]$ be a base operator pair with the notations below:

(2.31) \[ \tilde{U} := \{ p \in \tilde{U} : \langle p, Sq \rangle_Y - \langle S_p, q \rangle_Z = 0, \forall q \in \tilde{V} \}; \]

(2.32) \[ \tilde{V} := \{ q \in \tilde{V} : \langle p, Sq \rangle_Y - \langle S_p, q \rangle_Z = 0, \forall p \in \tilde{U} \}; \]

(2.33) \[ \tilde{U}_B := \{ p \in \tilde{U} : \langle p, q \rangle_Y = 0, \forall q \in \mathcal{N}(S, \tilde{U}); \langle S_p, \tilde{q} \rangle_Z = 0, \forall \tilde{q} \in \tilde{U} \}; \]

(2.34) \[ \tilde{V}_B := \{ q \in \tilde{V} : \langle q, p \rangle_Z = 0, \forall p \in \mathcal{N}(S, \tilde{V}); \langle S_p, \tilde{q} \rangle_Y = 0, \forall \tilde{q} \in \tilde{V} \}; \]

(2.35) \[ \kappa := \inf_{0 \neq \tilde{q} \in \mathcal{N}(S, \tilde{U}_B)} \sup_{w \in \mathcal{R}(S, \tilde{V}_B)} \frac{\langle \tilde{w}, w \rangle_Y}{\| w \|_Y \| \tilde{w} \|_Y} = \inf_{0 \neq w \in \mathcal{R}(S, \tilde{V}_B)} \sup_{\tilde{q} \in \mathcal{N}(S, \tilde{U}_B)} \frac{\langle \tilde{w}, w \rangle_Y}{\| \tilde{w} \|_Y \| w \|_Y}; \]

(2.36) \[ \omega := \inf_{0 \neq w \in \mathcal{R}(S, \tilde{V}_B)} \sup_{\tilde{q} \in \mathcal{N}(S, \tilde{U}_B)} \frac{\langle \tilde{v}, w \rangle_Z}{\| \tilde{v} \|_Z \| w \|_Z} = \inf_{0 \neq \tilde{v} \in \mathcal{N}(S, \tilde{U}_B)} \sup_{w \in \mathcal{R}(S, \tilde{V}_B)} \frac{\langle \tilde{v}, w \rangle_Z}{\| \tilde{v} \|_Z \| w \|_Z}. \]
Then $\kappa > 0$ and $\varpi > 0$. Again, we make the convention that,

\begin{equation}
\begin{cases}
\kappa = 1, & \text{if } \mathcal{N}(\mathcal{S}, \mathcal{U}_B) = \mathcal{R}(\mathcal{S}, \mathcal{V}_B) = \{0\}, \\
\varpi = 1, & \text{if } \mathcal{N}(\mathcal{S}, \mathcal{V}_B) = \mathcal{R}(\mathcal{S}, \mathcal{U}_B) = \{0\}.
\end{cases}
\end{equation}

\textbf{Assumption 2.25.} $\mathcal{R}(T, \tilde{M}) \subset \mathcal{N}(\mathcal{S}, \tilde{U})$ and $\mathcal{R}(S, \tilde{V}) \subset \mathcal{N}(T, \tilde{N})$.

\textbf{Lemma 2.26.} Provided Assumption 2.25,

\[ \mathcal{R}(T, \tilde{M}) \subset \mathcal{N}(\mathcal{S}, \tilde{U}), \quad \text{and} \quad \mathcal{R}(S, \tilde{V}) \subset \mathcal{N}(\tilde{T}, \tilde{N}). \]

\textit{Proof.} Given $p \in \mathcal{M}$, $T_p \in \mathcal{N}(\mathcal{S}, \tilde{U})$. For any $q \in \tilde{V}$, since $Sq \in \mathcal{N}(\mathcal{S}, \tilde{U})$,

\[ \langle T_p, Sq \rangle_Y - \langle STp, q \rangle_Z = \langle T_p, Sq \rangle_Y - \langle Tp, Sq \rangle_Y = 0. \]

Thus $T_p \in \mathcal{U}$. This proves $\mathcal{R}(T, \tilde{M}) \subset \mathcal{N}(\mathcal{S}, \tilde{U})$. The other part is proved the same way. \hfill \Box

Provided Assumption 2.25, denote

\[ \tilde{\mathcal{H}} := \mathcal{N}(\mathcal{S}, \tilde{U}) \oplus^+ \mathcal{R}(T, \tilde{M}), \quad \mathcal{H} := \mathcal{N}(\mathcal{S}, \tilde{U}) \oplus^+ \mathcal{R}(T, M), \]

and

\[ \tilde{\mathfrak{S}} := \mathcal{N}(\tilde{T}, \tilde{N}) \oplus^+ \mathcal{R}(S, \tilde{V}), \quad \mathfrak{S} := \mathcal{N}(\tilde{T}, \tilde{N}) \oplus^+ \mathcal{R}(S, V). \]

\textbf{Assumption 2.27.} $\tilde{\mathcal{H}}$ and $\tilde{\mathfrak{S}}$ are isomorphic, and $\mathcal{H}$ and $\mathfrak{S}$ are isomorphic.

Provide Assumption 2.27, denote

\begin{equation}
\chi := \inf_0 \sup_{p \in \tilde{\mathcal{H}} \in \tilde{\mathfrak{S}}} \frac{\langle p, v \rangle_Y}{\|p\|_Y \|v\|_Y}, \quad \epsilon := \inf_0 \sup_{p \in \mathcal{H} \in \mathfrak{S}} \frac{\langle p, v \rangle_Y}{\|p\|_Y \|v\|_Y},
\end{equation}

and

\begin{equation}
\chi := \inf_0 \sup_{p \in \tilde{\mathcal{H}} \in \tilde{\mathfrak{S}}} \frac{\langle p, v \rangle_Y}{\|p\|_Y \|v\|_Y}, \quad \epsilon := \inf_0 \sup_{p \in \mathcal{H} \in \mathfrak{S}} \frac{\langle p, v \rangle_Y}{\|p\|_Y \|v\|_Y}.
\end{equation}

Then $\chi > 0$ and $\epsilon > 0$. Particularly, we make the convention that

\begin{equation}
\begin{cases}
\chi = 1, & \text{if } \tilde{\mathcal{H}} = \tilde{\mathfrak{S}} = \{0\}, \\
\epsilon = 1, & \text{if } \mathcal{H} = \mathfrak{S} = \{0\}.
\end{cases}
\end{equation}

\textbf{Theorem 2.28 (Complex duality).} Let $[(T, D), (T, \mathbb{I})]$ and $[(S, E), (S, \mathbb{E})]$ be partially adjoint based on $[(T, \tilde{M}), (\tilde{T}, \tilde{N})]$ and $[(\mathcal{S}, \tilde{U}), (\mathcal{S}, \tilde{V})]$, respectively. Provided Assumption 2.25,

\[ \mathcal{R}(T, D) \subset \mathcal{N}(\mathcal{S}, \mathbb{E}) \iff \mathcal{R}(S, E) \subset \mathcal{N}(T, \mathbb{I}). \]

\textit{Proof.} Assuming $\mathcal{R}(T, D) \subset \mathcal{N}(\mathcal{S}, \mathbb{E})$, given $q \in E$, we obtain $Sq \in \mathcal{N}(\mathcal{T}, \tilde{N})$, and $\langle Sq, q \rangle_Y = 0$ for any $q \in \mathcal{R}(T, D) \subset \mathcal{N}(\mathcal{S}, \mathbb{E})$. Therefore $Sq \in \mathcal{N}(\mathcal{T}, \mathbb{I})$; namely $\mathcal{R}(S, E) \subset \mathcal{N}(T, \mathbb{I})$. The other direction can be obtained the same way. The proof is completed. \hfill \Box
For \([\langle T, D \rangle, \langle T, \bar{D} \rangle] \) and \([\langle S, E \rangle, \langle S, \bar{E} \rangle] \) which are partially adjoint based on \([\langle T, \bar{M} \rangle, \langle T, \bar{N} \rangle] \) and \([\langle S, \bar{U} \rangle, \langle S, \bar{V} \rangle] \), respectively, assuming \(\mathcal{R}(S, E) \subset N(T, D) \) and \(\mathcal{R}(T, D) \subset N(S, E) \) (by Theorem 2.28), denote
\[
\mathcal{H} := \mathcal{N}(S, E) \ominus^\perp \mathcal{R}(T, D), \quad \text{and} \quad \mathfrak{H} := \mathcal{N}(T, D) \ominus^\perp \mathcal{R}(S, E).
\]

**Theorem 2.29** (Vertical slice isomorphism). Assume \(\mathcal{R}(S, E) \subset N(T, D) \). Provided Assumptions 2.25 and 2.27, \(\mathcal{H} \) and \(\mathfrak{H} \) are isomorphic. Further, if \(\mathcal{H} \neq \{0\} \), with notations as (2.17), (2.36), (2.37), (2.38), (2.39), and (2.40),
\[
(2.41) \quad \inf_{0 \neq \eta \in \mathcal{H}} \sup_{u \in \mathfrak{H}} \frac{\langle \eta, u \rangle_Y}{\|\eta\|_Y \|u\|_Y} = \inf_{0 \neq \eta \in \mathfrak{H}} \sup_{u \in \mathcal{H}} \frac{\langle \eta, u \rangle_Y}{\|\eta\|_Y \|u\|_Y} \geq \max \left[ \min(\beta, \chi), \min(\sigma, \epsilon) \right].
\]

This is a weak version of the vertical P-A-L type duality. We postpone the proof to Section 2.3.4.

**Theorem 2.30** (Vertical P-A-L duality). With the same assumptions as Theorem 2.28 and provided that \(\mathcal{R}(T, \bar{M}_B) = N(T, \bar{N}_B) \), that \(\mathcal{N}(S, \bar{U}_B) = \mathcal{R}(S, \bar{V}_B) \), and that \(\mathcal{H} = \mathfrak{H} \), the vertical P-A-L type duality holds that
\[
(2.42) \quad \mathcal{H} = \mathfrak{H}.
\]

**Proof.** By Theorem 2.23,
\[
\mathcal{R}(T, D) \ominus^\perp \mathcal{N}(T, D) = \mathcal{R}(T, \bar{M}) \ominus^\perp \mathcal{N}(T, \bar{N}).
\]

Similar to (2.29), we have
\[
\mathcal{R}(S, E) \ominus^\perp \mathcal{N}(S, E) = \mathcal{R}(S, \bar{N}) \ominus^\perp \mathcal{N}(S, \bar{U}).
\]

Note that
\[
\mathcal{R}(T, \bar{M}) \ominus^\perp \mathcal{N}(T, \bar{N}) = \mathcal{R}(T, \bar{M}) \ominus^\perp \mathfrak{H} \ominus^\perp \mathcal{R}(S, \bar{V}) = \mathcal{R}(T, \bar{M}) \ominus^\perp \mathfrak{H} \ominus^\perp \mathcal{R}(S, \bar{V})
\]
\[
= \mathcal{N}(S, \bar{U}) \ominus^\perp \mathfrak{H} \ominus^\perp \mathcal{R}(S, \bar{V}) = \mathcal{N}(S, \bar{U}) \ominus^\perp \mathcal{N}(S, \bar{U}_B) \ominus^\perp \mathcal{R}(S, \bar{V})
\]
\[
= \mathcal{N}(S, \bar{U}) \ominus^\perp \mathcal{R}(S, \bar{V}_B) \ominus^\perp \mathcal{R}(S, \bar{V}) = \mathcal{N}(S, \bar{U}) \ominus^\perp \mathcal{R}(S, \bar{V}).
\]

Thus
\[
\mathcal{R}(T, D) \ominus^\perp \mathfrak{H} \ominus^\perp \mathcal{R}(S, E) = \mathcal{R}(T, D) \ominus^\perp \mathcal{N}(T, D) = \mathcal{R}(T, \bar{M}) \ominus^\perp \mathcal{N}(T, \bar{N})
\]
\[
= \mathcal{R}(S, \bar{N}) \ominus^\perp \mathcal{N}(S, \bar{U}) = \mathcal{R}(S, E) \ominus^\perp \mathcal{N}(S, E) = \mathcal{R}(S, E) \ominus^\perp \mathcal{H} \ominus^\perp \mathcal{R}(T, D).
\]

The proof of Theorem 2.30 is completed. \(\square\)

**Theorem 2.31** (Hodge decomposition). With the assumptions of Theorem 2.30, there holds
\[
(2.43) \quad \mathcal{R}(T, \bar{M}) \ominus^\perp \mathcal{N}(T, \bar{N}) \mathcal{R}(T, D) \ominus^\perp \mathcal{H} = \mathfrak{H} \ominus^\perp \mathcal{R}(S, E).
\]
Remark 2.32. With the assumptions of Theorem 2.30, we can prove
\[ \mathcal{R}(T, \overline{M}) \oplus N(T, N) = R(T, M) \oplus \mathcal{R}(T, \overline{N}) = R(S, V) \oplus \mathcal{R}(S, \overline{V}) = R(S, U) \oplus N(S, \overline{U}). \]
Hence the Hodge decomposition (2.43) has various equivalent presentations.

According to the proof of Theorem 2.30, provided that \( \mathcal{R}(T, \overline{M}_B) = N(T, \overline{N}_B) \) and that \( N(S, \overline{U}_B) = R(S, \overline{V}_B) \), we have
\[ \mathcal{H} = \overline{\mathcal{H}} \iff \mathcal{H} = \overline{\mathcal{H}}. \]
Indeed, \( \mathcal{R}(T, \overline{M}) \oplus N(T, N) = \mathcal{R}(T, M) \oplus N(T, \overline{N}) = \mathcal{R}(T, M) \oplus \overline{\mathcal{H}} \oplus \mathcal{R}(S, V) \), and \( N(S, \overline{U}) \oplus \mathcal{R}(S, V) = N(S, \overline{U}) \oplus \mathcal{R}(S, \overline{V}) \). That \( \mathcal{H} = \overline{\mathcal{H}} \) implies \( \mathcal{R}(T, \overline{M}) \oplus \mathcal{R}(S, V) = N(T, \overline{N}) \) and that \( \mathcal{H} = \overline{\mathcal{H}} \) follows. The reverse direction can be shown the same way.

2.3. Some technical proofs.

2.3.1. An elementary lemma for Hilbert spaces.

Lemma 2.33. Let \( A, B \) and \( B_0 \subset B \) be closed subspaces of a Hilbert space \( H \). Assume
\[ \inf_{0 \neq b \in B} \sup_{a \in A} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \gamma > 0. \]

(1) Denote \( A_{\perp B_0} := \{ a \in A : \langle a, b \rangle_H = 0, \ \forall b \in B_0 \} \) the polar set of \( B_0 \) in \( A \), and \( A_{\parallel B_0} := \{ a \in A : \langle a, a' \rangle_H = 0, \ \forall a' \in A_{\perp B_0} \} \). Then
\[ (2.44) \quad \inf_{0 \neq b \in B_0} \sup_{a \in A_{\parallel B_0}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A_{\parallel B_0}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma. \]

(2) Let \( B_0^c \) be the orthogonal complement of \( B_0 \) in \( B \), and \( A_{\perp B_0^c} := \{ a \in A : \langle a, b \rangle_H = 0, \ \forall b \in B_0^c \} \). If further
\[ \inf_{0 \neq a \in A} \sup_{b \in B} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \gamma, \]
then
\[ (2.45) \quad \inf_{0 \neq b \in B_0^c} \sup_{a \in A_{\perp B_0^c}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A_{\perp B_0^c}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma. \]

Proof. Given \( b \in B_0 \), there exists an \( a' \in A \), such that \( \langle a', b \rangle_H \geq \gamma \|a'\|_H \|b\|_H \). Let \( a \) be the projection of \( a' \) into \( A_{\parallel B_0} \), then \( \langle a, b \rangle_H = \langle a', b \rangle_H \), and \( \|a\|_H \leq \|a'\|_H \). Therefore, \( \langle a, b \rangle_H \geq \gamma \|a\|_H \|b\|_H \), and
\[ \inf_{0 \neq b \in B_0} \sup_{a \in A_{\parallel B_0}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma. \]
On the other hand, given \( 0 \neq a \in A_{\parallel B_0} \), there exists a \( b \in B_0 \), such that
\[ \langle a, b \rangle_H \neq 0. \]
Thus
\[ \inf_{0 \neq b \in B_0} \sup_{a \in A_{\parallel B_0}} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A_{\parallel B_0}} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma. \] This proves (2.44).
Now we turn to (2.45). Denote $I : A \to B$ such that $\langle I a, b' \rangle_H = \langle a, b' \rangle_H, \forall b' \in B$. Then $I$ is an isomorphism and $\mathcal{R}(I, A \perp B) = B_0$. Thus $\inf_{0 \neq b \in B_0} \sup_{a \in A \perp B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} = \inf_{0 \neq a \in A \perp B_0} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} > 0$. Given $a \in A \perp B_0$, there exists a $b' \in B$, such that $\langle a, b' \rangle_H \geq \gamma \|a\|_H \|b'\|_H$. We further set $b$ the projection of $b'$ into $B_0$, then $\langle a, b \rangle_H \geq \gamma \|a\|_H \|b\|_H$. Thus $\inf_{0 \neq a \in A \perp B_0} \sup_{b \in B_0} \frac{\langle a, b \rangle_H}{\|a\|_H \|b\|_H} \geq \gamma$. This proves (2.45), and completes the proof of the lemma.

2.3.2. Proof of Theorem 2.21. Recall that $D^* := \{v \in D : \langle v, w \rangle_X = 0, \forall w \in N(T, D)\}$. Denote $D_B^* := \{v \in D_B : \langle v, w \rangle_X = 0, \forall w \in N(T, D_B)\}$. Then $D_B^* = D^* \cap D_B$. Given $v \in D_B^*$, there exists a unique decomposition $v = v^* + \tilde{v}$, such that $v^* \in M_B^*$ and $\tilde{v} \in N(T, M_B)$. Note that both $v$ and $v^*$ are orthogonal to $N(T, D_B) \subset N(T, M_B)$ in $X$, and thus $\tilde{v} \in N(T, M_B) \perp N(T, D_B)$. Further, $\langle v, Tv \rangle_X = \langle T \tilde{v}, v \rangle_Y = 0, \forall v \in D_B^*$, and thus

$$
\langle \tilde{v}, T \tilde{v} \rangle_X = -(\langle v^*, T v \rangle_X - \langle T v^*, v \rangle_Y), \forall v \in D_B^*.
$$

By Lemma 2.33(2), with $\mathcal{R}(T, \overline{D_B}) = \{p \in \mathcal{R}(T, N_B), \langle p, w \rangle_X = 0, \forall w \in N(T, D_B)\}$, we have

$$
\inf_{0 \neq w \in N(T, M_B) \perp N(T, D_B)} \sup_{p \in \mathcal{R}(T, \overline{D_B})} \frac{\langle v, p \rangle_X}{\|v\|_X \|p\|_X} \geq \inf_{0 \neq w \in N(T, M_B) \perp N(T, D_B)} \sup_{w \in \mathcal{R}(T, \overline{D_B})} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X} = \alpha.
$$

Therefore, as $\mathcal{R}(T, \overline{D_B})$ is closed,

$$
\|\tilde{v}\|_X \leq \alpha^{-1} \sup_{v \in D_B^*} \frac{\langle v^*, T \tilde{v} \rangle_X - \langle T v^*, v \rangle_Y}{\|T \tilde{v}\|_X} \leq \alpha^{-1} \|v^*\|_X + \alpha^{-1} \sup_{\tilde{v} \in D_B^*} \frac{\|T v^*\|_Y}{\|T \tilde{v}\|_X} = \alpha^{-1} \|v^*\|_X + \alpha^{-1} \|\tilde{v}\|_X = \alpha^{-1} \|v\|_X + \alpha^{-1} \|\tilde{v}\|_X,
$$

Then $\|v\|_X \leq \|v^*\|_X + \|\tilde{v}\|_X \leq ((1 + \alpha^{-1}) \cdot \text{icr}(T, M) + \alpha^{-1} \text{icr}(T, \overline{D}) ) \|Tv\|_Y$. Thus

$$
\text{icr}(T, D_B) \leq (1 + \alpha^{-1}) \cdot \text{icr}(T, \overline{M}) + \alpha^{-1} \text{icr}(T, \overline{D}).
$$

Now, given $w \in D^*$, there exists a unique decomposition $w = w^*_B + w_0$, such that $w^*_B \in D_B^* \subset D^*$, $Tw_0 \in \mathcal{R}(T, M)$, and $\langle Tw_0, Tw_0 \rangle_Y = 0$. Then $\|w^*_B\|_X \leq \text{icr}(T, D_B) \|Tw_0\|_Y$; further, as $w_0 \in D^*$ and $Tw_0 \in \mathcal{R}(T, M)$, $\|w_0\|_X \leq \text{icr}(T, M) \|Tw_0\|_Y$. Thus $\|w\|_X \leq \|w^*_B\|_X + \|w_0\|_X \leq \text{icr}(T, D_B) \|Tw_0\|_Y + \text{icr}(T, M) \|Tw_0\|_Y \leq \text{icr}(T, D_B) + \text{icr}(T, M) \|Tw_0\|_Y$. Namely,

$$
\text{icr}(T, D) \leq (1 + \alpha^{-1}) \cdot \text{icr}(T, \overline{M}) + \alpha^{-1} \text{icr}(T, \overline{D}) + \text{icr}(T, M).
$$

Similarly, can (2.26) be proved. The proof is completed.

2.3.3. Proof of Theorem 2.22. If $N(T, M_B) = \{0\} = \mathcal{R}(T, N_B)$ or $N(T, N_B) = \{0\} = \mathcal{R}(T, M_B)$, the assertion is straightforward. We consider the case that $N(T, M_B), \mathcal{R}(T, N_B), N(T, N_B)$ and $\mathcal{R}(T, M_B)$ are all nontrivial. Note that $N(T, D) = N(T, M) \oplus (N(T, M_B) \cap D)$, and $N(T, \overline{D}) = N(T, N) \oplus (N(T, N_B) \cap \overline{D})$. Then

$$
N(T, N_B) \cap \overline{D} = (N(T, N_B) \cap \overline{D}) \oplus^\perp (N(T, \overline{D}) \oplus^\perp N(T, \overline{D})) = (N(T, N_B) \cap \overline{D}) \oplus^\perp \mathcal{I} N.
$$
Note also that $\mathcal{R}(T, M) \perp \mathcal{N}(T, N)$ in $Y$, “$\perp$” for orthogonal. By Theorem 2.20,

$$\mathcal{R}(T, \bar{D}) = \mathcal{R}(T, M) \oplus \{ \nu \in \mathcal{R}(T, M_B) : \nu \perp (\mathcal{N}(T, N_B) \cap \bar{D}) \} \subset \mathcal{Y} := \mathcal{R}(T, M) \oplus A_b.$$  

Similarly

$$\mathcal{R}(T, \bar{D}) = \mathcal{R}(T, M) \oplus \{ \nu \in \mathcal{R}(T, M_B) : \nu \perp (\mathcal{N}(T, N_B) \cap \bar{D}) \} \subset \mathcal{Y} := \mathcal{R}(T, M) \oplus B.$$  

By Lemma 2.33(2), we can show

$$\inf_{0 \neq \nu \in \mathcal{N}(T, N_B) \cap \bar{D}} \sup_{\nu \in \mathcal{R}(T, M_B)} \frac{\langle \nu, \nu \rangle_Y}{\|\nu\|_Y \|\nu\|_Y} = \beta.$$  

Therefore, as $\bar{D} = (\mathcal{N}(T, N_B) \cap \bar{D}) \oplus (\mathcal{N}(T, N_B) \cap \bar{D})$, $\inf_{0 \neq \nu \in \bar{D}} \sup_{\nu \in \mathcal{R}(T, M_B)} \frac{\langle \nu, \nu \rangle_Y}{\|\nu\|_Y \|\nu\|_Y} \geq \beta.$  

Then by Lemma 2.33(1), with $\bar{D} = A_b \oplus B$ and $A_b = \{ \nu \in A_b : \nu$ is orthogonal to $\bar{D} \}$,  

$$\inf_{0 \neq \nu \in \bar{D}} \sup_{\nu \in \mathcal{R}(T, M_B)} \frac{\langle \nu, \nu \rangle_Y}{\|\nu\|_Y \|\nu\|_Y} = \inf_{0 \neq \nu \in \bar{D}} \sup_{\nu \in \mathcal{R}(T, M_B)} \frac{\langle \nu, \nu \rangle_Y}{\|\nu\|_Y \|\nu\|_Y} \geq \beta.$$  

This proves (2.27). If $\bar{D} = \{0\}$, then easily we know $\bar{D} = \{0\}$, and vice versa. (2.28) can be proved the same way. The proof is completed.  

2.3.4. Proof of Theorem 2.29.

**Lemma 2.34.** Provided Assumptions 2.25 and 2.27,

$$\inf_{0 \neq \nu \in \mathcal{N}(S, \bar{U}) \cap \bar{R}(T, N_B) \cap \bar{R}(S, \vec{V})} \frac{\langle \nu, \nu \rangle_Y}{\|\nu\|_Y \|\nu\|_Y} = \inf_{0 \neq \nu \in \mathcal{N}(S, \bar{U}) \cap \bar{R}(T, N_B) \cap \bar{R}(S, \vec{V})} \frac{\langle \nu, \nu \rangle_Y}{\|\nu\|_Y \|\nu\|_Y} \geq \max \left[ \min(\beta, \chi), \min(\sigma, \epsilon) \right].$$  

**Proof.** Direct calculation leads to that

$$\mathcal{N}(T, \bar{N}) \perp \mathcal{R}(S, \vec{V}) = \mathcal{N}(T, N_b) \perp \mathcal{S} = \mathcal{S} \oplus \mathcal{R}(S, V_B)$$  

and

$$\mathcal{N}(S, \bar{U}) \perp \mathcal{R}(T, M) = \mathcal{H} \perp \mathcal{R}(T, M_B) = \mathcal{N}(S, U_B) \perp \mathcal{H}.$$  

If $\mathcal{H} = \mathcal{S} = \{0\}$, then $\chi = 1$, and

$$\inf_{0 \neq \nu \in \mathcal{N}(S, \bar{U}) \cap \mathcal{R}(T, M_B)} \frac{\langle \nu, \nu \rangle_Y}{\|\nu\|_Y \|\nu\|_Y} = \beta = \min(\chi, \beta).$$  

Otherwise, given $\nu \in \mathcal{N}(S, \bar{U}) \cap \mathcal{R}(T, M)$, there exists a unique decomposition $\nu = \nu_1 + \nu_2$, such that $\nu_1 \in \mathcal{H}$ and $\nu_1 \in \mathcal{R}(T, M_B)$; $\|\nu_1\|^2_Y = \|\nu_1\|^2_Y + \|\nu_2\|^2_Y$. Then, by Assumption 2.27, there exist $\nu_2 \in \mathcal{S}$, such that $\langle \nu_2, \nu_2 \rangle_Y = \|\nu_2\|^2_Y$, and $\|\nu_2\|^2_Y \leq \chi^{-1}\|\nu_2\|^2_Y$; further, with $\beta$ given in (2.17), let $\nu_1 \in \mathcal{N}(T, N_B)$ be such that $\langle \nu_1, \nu_1 \rangle_Y = \|\nu_1\|^2_Y$, and $\|\nu_1\|^2_Y \leq \beta^{-1}\|\nu_1\|^2_Y$. Set
\( \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{N}(\mathbb{T}, \Bar{\mathbf{N}}) \oplus^\perp \mathcal{R}(\mathbf{S}, \mathbf{V}) \), then \( \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = \langle \mathbf{q}, \mathbf{q} \rangle_\mathbf{V} \), and \( \|\mathbf{v}\|_\mathbf{V}^2 \leq \max(\chi^{-1}, \beta^{-1})^2 \|\mathbf{q}\|_\mathbf{V}^2 \). Hence,

\[
\inf_{0 \neq \mathbf{q} \in \mathcal{N}(\mathbb{T}, \bar{\mathbf{N}}) \oplus^\perp \mathcal{R}(\mathbf{T}, \mathbf{M})} \sup_{\mathbf{v} \in \mathcal{N}(\mathbb{T}, \bar{\mathbf{N}}) \oplus^\perp \mathcal{R}(\mathbf{S}, \mathbf{V})} \frac{\langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V}}{\|\mathbf{q}\|_\mathbf{V} \|\mathbf{v}\|_\mathbf{V}} \geq \min(\chi, \beta).
\]

Similarly, we can prove

\[
\inf_{0 \neq \mathbf{q} \in \mathcal{N}(\mathbb{T}, \bar{\mathbf{N}}) \oplus^\perp \mathcal{R}(\mathbf{T}, \mathbf{M})} \sup_{\mathbf{v} \in \mathcal{N}(\mathbb{T}, \bar{\mathbf{N}}) \oplus^\perp \mathcal{R}(\mathbf{S}, \mathbf{V})} \frac{\langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V}}{\|\mathbf{q}\|_\mathbf{V} \|\mathbf{v}\|_\mathbf{V}} \geq \min(\epsilon, \omega).
\]

The proof is completed. \( \square \)

**Proof of Theorem 2.29.** With the structure of \( \mathbf{V} \),

\[
\mathcal{R}(\mathbf{S}, \mathbf{E}) = \mathcal{R}(\mathbf{S}, \mathbf{V}) \oplus^\perp \mathcal{R}(\mathbf{S}, \mathbf{V}_B \cap \mathbf{E})
\]

and

\[
\mathcal{N}(\mathbb{T}, \mathbb{D}) = \mathcal{N}(\mathbb{T}, \mathbb{N}) \oplus^\perp [\mathcal{N}(\mathbb{T}, \mathbb{N}_B) \cap \mathbb{D}] = \mathcal{R}(\mathbf{S}, \mathbf{V}) \oplus^\perp \mathbf{\mathcal{H}} \oplus^\perp (\mathcal{N}(\mathbb{T}, \mathbb{N}_B) \cap \mathbb{D}).
\]

Therefore,

\[
\mathbf{\mathcal{H}} = \mathcal{N}(\mathbb{T}, \mathbb{N}) \oplus^\perp \mathcal{R}(\mathbf{S}, \mathbf{E}) := \mathbf{A}_1 \oplus^\perp \mathbf{A}_2,
\]

where

\[
\mathbf{A}_1 = \mathbf{\mathcal{H}} \oplus^\perp [\mathcal{N}(\mathbb{T}, \mathbb{N}_B) \cap \mathbb{D}], \text{ and } \mathbf{A}_2 = \mathcal{R}(\mathbf{S}, \mathbf{V}_B \cap \mathbf{E}).
\]

By Lemma 2.20, noting that \( \mathcal{N}(\mathbf{S}, \Bar{\mathbf{U}}) = \mathcal{R}(\mathbf{T}, \mathbf{M}) \oplus^\perp \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus^\perp \mathbf{\mathcal{H}} \subset \mathbf{V} \), and that \( \mathcal{R}(\mathbf{T}, \mathbf{M}) \) is orthogonal to \( \mathcal{N}(\mathbb{T}, \Bar{\mathbf{N}}) \supset \mathcal{R}(\mathbf{S}, \Bar{\mathbf{V}}) \), we have

\[
\mathcal{N}(\mathbf{S}, \mathbf{E}) = \left\{ \mathbf{q} \in \mathcal{N}(\mathbf{S}, \Bar{\mathbf{U}}) : \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = 0, \forall \mathbf{v} \in \mathcal{R}(\mathbf{S}, \mathbf{E}) = \mathcal{R}(\mathbf{S}, \mathbf{V}) \oplus^\perp \mathbf{A}_2 \right\}
\]

\[
= \mathcal{R}(\mathbf{T}, \mathbf{M}) \oplus^\perp \left\{ \mathbf{q} \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus^\perp \mathbf{\mathcal{H}} : \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = 0, \forall \mathbf{v} \in \mathbf{A}_2 \right\}.
\]

Meanwhile, noting that \( \mathcal{N}(\mathbb{T}, \mathbb{N}) = \mathcal{R}(\mathbf{S}, \mathbf{V}) \oplus^\perp \mathbf{\mathcal{H}} \),

\[
\mathcal{R}(\mathbf{T}, \mathbf{D}) = \left\{ \mathbf{q} \in \mathcal{R}(\mathbf{T}, \Bar{\mathbf{M}}) : \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = 0, \forall \mathbf{v} \in \mathcal{N}(\mathbb{T}, \mathbb{D}) = \mathcal{R}(\mathbf{S}, \mathbf{V}) \oplus^\perp \mathbf{A}_1 \right\}
\]

\[
= \mathcal{R}(\mathbf{T}, \mathbf{M}) \oplus^\perp \left\{ \mathbf{q} \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = 0, \forall \mathbf{v} \in \mathbf{A}_1 \right\}.
\]

Therefore,

\[
\mathbf{\mathcal{H}} = \mathcal{N}(\mathbf{S}, \mathbf{E}) \oplus^\perp \mathcal{R}(\mathbf{T}, \mathbf{D}) = \mathbf{B}_1 \oplus^\perp \mathbf{B}_2,
\]

where

\[
\mathbf{B}_1 = \left\{ \mathbf{q} \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus^\perp \mathbf{\mathcal{H}} : \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = 0, \forall \mathbf{v} \in \mathbf{A}_2 \right\}
\]

and

\[
\mathbf{B}_2 = \left\{ \mathbf{q} \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) : \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = 0, \forall \mathbf{v} \in \mathbf{A}_1 \right\}.
\]

Denote

\[
\mathbf{B}_{1,2} := \left\{ \mathbf{q} \in \mathcal{R}(\mathbf{T}, \mathbf{M}_B) \oplus^\perp \mathbf{\mathcal{H}} : \langle \mathbf{q}, \mathbf{v} \rangle_\mathbf{V} = 0, \forall \mathbf{v} \in \mathbf{A}_1 \right\}.
\]
and we use the same notation.

Assumptions are imposed for $\mathcal{A}$ that $B_{1.5} = B_2$. Indeed, given $q_1 \in B_{1.5}$, there exists a unique decomposition $q_1 = q_1 + q_2$ with $q_1 \in R(T, M_B)$ and $q_2 \in \mathcal{H}$. Then, for any $\psi \in \mathcal{Y}$, $\langle q_1, \psi \rangle_Y = \langle q_2, \psi \rangle_Y = 0$ and thus $q_1 \in R(T, M_B)$. This way $B_{1.5} \subset B_2$ and $B_{1.5} = B_2$. Hence

$$\mathcal{H} = \left\{ q_1 \in R(T, M_B) \oplus^+ \mathcal{H} : \langle q_1, \psi \rangle_Y = 0, \ \forall \psi \in A_1 \right\}$$

$$\oplus^+ \left\{ q_1 \in R(T, M_B) \oplus^+ \mathcal{H} : \langle q_1, \psi \rangle_Y = 0, \ \forall \psi \in A_1 \right\} = C_1 \oplus^+ C_2.$$

Therefore, that $\mathcal{Y} = \{0\}$ implies that $\mathcal{H} = \{0\}$. Similarly, that $\mathcal{H} = \{0\}$ implies that $\mathcal{Y} = \{0\}$. By Lemma 2.33(2),

$$\inf_{0 \neq \psi \in N(T, T_B) \oplus \mathcal{Y}\oplus A_2} \sup_{\psi \in C} \frac{\langle \psi, \psi \rangle_Y}{\|\psi\|_Y \|\psi\|_Y} \geq \inf_{0 \neq \psi \in N(T, T_B) \oplus \mathcal{Y}\oplus A_2} \sup_{\psi \in C} \frac{\langle \psi, \psi \rangle_Y}{\|\psi\|_Y \|\psi\|_Y}.$$

As $C_2 = \{ q_1 \in C_1 : \langle q_1, \psi \rangle_Y = 0, \ \forall \psi \in \mathcal{Y}\}$ and $\mathcal{Y} \subset N(T, T_B) \oplus \mathcal{Y} \oplus A_2$, by Lemma 2.33(1),

$$\inf_{0 \neq \psi \in N(T, T_B) \oplus \mathcal{Y}\oplus A_2} \sup_{\psi \in \mathcal{Y}} \frac{\langle q_1, \psi \rangle_Y}{\|q_1\|_Y \|\psi\|_Y} = \inf_{0 \neq \psi \in N(T, T_B) \oplus \mathcal{Y}\oplus A_2} \sup_{\psi \in C_1} \frac{\langle q_1, \psi \rangle_Y}{\|q_1\|_Y \|\psi\|_Y} \geq \inf_{0 \neq \psi \in N(T, T_B) \oplus \mathcal{Y}\oplus A_2} \sup_{\psi \in \mathcal{Y}} \frac{\langle q_1, \psi \rangle_Y}{\|q_1\|_Y \|\psi\|_Y}.$$

The proof is then completed by Lemma 2.34.

3. Partially adjoint discretizations of adjoint operators

Let $\Omega$ be a domain. For $\Xi$ a subdomain of $\Omega$, we denote by $R^\Xi$ and $E^\Xi$ the restriction to and extension from $L_{loc}^1(\Xi)$ from and to $L_{loc}^1(\Omega)$, the spaces of locally integrable functions, respectively. Namely,

$$R^\Xi : L_{loc}^1(\Omega) \rightarrow L_{loc}^1(\Xi), \quad R^\Xi v = v|_{\Xi}, \text{ for } v \in L_{loc}^1(\Omega),$$

and

$$E^\Xi : L_{loc}^1(\Xi) \rightarrow L_{loc}^1(\Omega), \quad E^\Xi v = \begin{cases} v, & \text{on } \Xi, \\ 0, & \text{else}, \end{cases} \text{ for } v \in L_{loc}^1(\Xi).$$

We use the same notation $L_{loc}^1$ for both scalar and non-scalar locally integrable functions, and use the same notation $R^\Xi$ and $E^\Xi$ for both scalar and non-scalar functions.

Let $X, Y$ and $Z$ be three Hilbert spaces that each consists of locally integrable functions defined on $\Omega$. Assume,

$$E^\Xi R^\Xi v \in X, \quad \forall v \in X, \text{ for any subdomain of } \Xi \subset \Omega.$$

Denote $X_\Xi := R(R^\Xi, X)$. For $v, w \in X_\Xi$, define an inner product $\langle v, w \rangle_{X_\Xi} := \langle E^\Xi v, E^\Xi w \rangle_X$. Similar assumptions are imposed for $Y$ and $Z$, and spaces $Y_\Xi$ and $Z_\Xi$ and inner products thereon are defined the same way.
Let \( T \) be a linear operator that maps locally integrable functions compatible to \( X \) to locally integrable functions compatible to \( Y \). We assume the operation of \( T \) is local, namely, for \( v, v' \) locally integrable,

\[
R_{\Omega}^a v = R_{\Omega}^a v' \implies R_{\Omega}^a (Tv) = R_{\Omega}^a (Tv'), \quad \text{for any subdomain } \Xi \subset \Omega.
\]

We call (3.1) and (3.2) the localization assumptions. Similarly, \( T \) is a linear operator that maps locally integrable functions compatible to \( Y \) to locally integrable functions compatible to \( X \), and its operation is local. Let \( \{(T, D) : X \to Y, (T, D) : Y \to X\} \) be a pair of closed densely defined adjoint operators. This section considers the in-pair discretizations of \( [(T, D), (T, D)] \). Particularly, we use a conforming discretization (CD) for one operator and an accompanied-by-conforming discretization (ABCD), constructed below, for the other to construct a pair of partially adjoint discretizations for the adjoint operators.

3.1. Partially adjoint discretizations by CD and ABCD.

3.1.1. Tensorial base pair and partially adjoint discretized operators. Let \( G \) be a subdivision of \( \Omega \) to cells. Define formally the product of a set of function spaces \( \{\Upsilon(K)\}_{K \in G} \) defined cell by cell such that \( E^{\Omega}_{K} \Upsilon(K) := R(E^{\Omega}_{K}, \Upsilon(K)) \) for all \( K \in G \) are compatible,

\[
\prod_{K \in G} \Upsilon(K) := \sum_{K \in G} E^{\Omega}_{K} \Upsilon(K),
\]

and the summation is direct. Then \( v_{\theta} \in \prod_{K \in G} \Upsilon(K) \) if and only if there exist (uniquely) a set of functions \( \{v_{K}\}_{K \in G} \), with \( v_{K} \in \Upsilon(K) \) for any \( K \), such that \( v_{\theta} = \sum_{K \in G} E^{\Omega}_{K} v_{K} \). Further, it holds that

\[
X = \prod_{K \in G} X_{K}, \quad Y = \prod_{K \in G} Y_{K}, \quad \text{and} \quad Z = \prod_{K \in G} Z_{K}.
\]

On every \( K \in G \), let \( T_{K} \) and \( T_{K} \) be two operators that coincide with the local operation of \( T \) and \( T \) on \( K \), respectively, in the sense that

\[
T_{K} R_{\Omega}^a v = R_{\Omega}^a T_{K} v, \quad \text{and} \quad T_{K} R_{\Omega}^a v = R_{\Omega}^a T_{K} v.
\]

Note again that the same notation \( R_{\Omega}^a \) can take different meanings at different places.

Define formally the product of operators \( T_{\theta} = \prod_{K \in G} T_{K} \) by

\[
T_{\theta} : X \to Y, \quad T_{\theta} \left( \sum_{K \in G} E^{\Omega}_{K} v_{K} \right) = \sum_{K \in G} \left( E^{\Omega}_{K} T_{K} v_{K} \right), \quad \text{with } v_{K} \in X_{K}, \forall K \in G.
\]

Then we call \( T_{\theta} \) a discretization of \( T \). Evidently, \( T_{\theta} v = T v \), for \( v \in H(T : X \to Y) \).

We similarly define \( T_{\theta} = \prod_{K \in G} T_{K} \) and \( T_{\theta} \) is a discretization of \( T \) on \( G \). Discretizations of \( S \) and \( S \), say \( S_{\theta} \) and \( S_{\theta} \), are constructed the same way.

On every \( K \in G \), let there be two finite-dimensional shape function spaces \( P(K) \subset H(T_{K} : X_{K} \to Y_{K}) \) and \( \dot{P}(K) \subset H(T_{K} : Y_{K} \to X_{K}) \) constructed. Define

\[
\overline{P}(G) := \prod_{K \in G} P(K), \quad \text{and} \quad \overline{P}(G) := \prod_{K \in G} \dot{P}(K).
\]
Denote
\[ \mathbf{P}(G) := \{ v \in \tilde{\mathbf{P}}(G) : \langle v, T_G w \rangle_X - \langle T_B v, w \rangle_Y = 0, \forall w \in \tilde{\mathbf{P}}(G) \}, \]
\[ \mathbf{P}(G) := \{ v \in \tilde{\mathbf{P}}(G) : \langle v, T_B w \rangle_X - \langle T_B v, w \rangle_Y = 0, \forall w \in \tilde{\mathbf{P}}(G) \}, \]
\[ \mathbf{P}_B(G) := \{ v \in \tilde{\mathbf{P}}(G) : \langle v, w \rangle_X = 0, \forall w \in \mathcal{N}(T_G, \mathbf{P}(G)); \langle T_B v, T_B w \rangle_Y = 0, \forall w \in \mathbf{P}(G) \}, \]
and
\[ \mathbf{P}_B(G) := \{ v \in \tilde{\mathbf{P}}(G) : \langle v, w \rangle_Y = 0, \forall w \in \mathcal{N}(T_B, \mathbf{P}(G)); \langle T_G v, T_G w \rangle_X = 0, \forall w \in \mathbf{P}(G) \}. \]

**Assumption 3.1.** \( \mathcal{N}(T_G, \mathbf{P}_B(G)) \) and \( \mathcal{R}(T_G, \mathbf{P}_B(G)) \) are isomorphic; \( \mathcal{N}(T_G, \mathbf{P}_B(G)) \) and \( \mathcal{R}(T_G, \mathbf{P}_B(G)) \) are isomorphic.

Provided Assumption 3.1, denote
\[ \alpha_G := \inf_{0 \neq v \in \mathcal{N}(T_G, \mathbf{P}_B(G))} \sup_{w \in \mathcal{R}(T_G, \mathbf{P}_B(G))} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X} = \inf_{0 \neq v \in \mathcal{R}(T_G, \mathbf{P}_B(G))} \sup_{v \in \mathcal{N}(T_G, \mathbf{P}_B(G))} \frac{\langle v, w \rangle_X}{\|v\|_X \|w\|_X} \]
and
\[ \beta_G := \inf_{0 \neq v \in \mathcal{N}(T_G, \mathbf{P}_B(G))} \sup_{w \in \mathcal{R}(T_G, \mathbf{P}_B(G))} \frac{\langle v, w \rangle_Y}{\|v\|_Y \|w\|_Y} = \inf_{0 \neq v \in \mathcal{R}(T_G, \mathbf{P}_B(G))} \sup_{v \in \mathcal{N}(T_G, \mathbf{P}_B(G))} \frac{\langle v, w \rangle_Y}{\|v\|_Y \|w\|_Y}. \]

Then \( \alpha_G > 0 \) and \( \beta_G > 0 \).

**Remark 3.2.** In this section, we will not discuss in detail the trivial cases such as \( \mathcal{N}(T_G, \mathbf{P}_B(G)) = \{0\} \) and \( \mathcal{R}(T_G, \mathbf{P}_B(G)) = \{0\} \), as the relevant discussions are straightforward. Similar trivial cases will be omitted for Assumptions 3.4 and 3.6.

**Lemma 3.3.** Provided Assumption 3.1, \([T_G, \tilde{\mathbf{P}}(G)] : X \to Y, (T_G, \tilde{\mathbf{P}}(G)) : Y \to X\) is a base operator pair.

All spaces discussed in this section are of finite-dimension, and the closedness can easily be verified. The base operator pair constructed this way is called a tensorial base pair.

Also, on every \( K \), let there be \( \mathbf{Q}(K) \subset H(S_K : Y_K \to Z_K) \) and \( \mathbf{Q}(K) \subset H(S_K : Z_K \to Y_K) \) constructed. Define
\[ \tilde{\mathbf{Q}}(G) := \prod_{K \in G} \mathbf{Q}(K), \text{ and } \quad \mathbf{Q}(G) := \prod_{K \in G} \mathbf{Q}(K). \]
We similarly introduce the notation \( \mathbf{Q}(G), \mathbf{Q}(G), \mathbf{Q}_B(G) \) and \( \mathbf{Q}_B(G) \) with respect to \( S_G \) and \( S_G \).

**Assumption 3.4.** \( \mathcal{N}(S_G, \mathbf{Q}_B(G)) \) and \( \mathcal{R}(S_G, \mathbf{Q}_B(G)) \) are isomorphic; \( \mathcal{N}(S_G, \mathbf{Q}_B(G)) \) and \( \mathcal{R}(S_G, \mathbf{Q}_B(G)) \) are isomorphic.

Provided Assumption 3.4, denote
\[ \omega_G := \inf_{0 \neq p \in \mathcal{N}(S_G, \mathbf{Q}_B(G))} \sup_{q \in \mathcal{R}(S_G, \mathbf{Q}_B(G))} \frac{\langle p, q \rangle_Y}{\|p\|_Y \|q\|_Y} = \inf_{0 \neq q \in \mathcal{R}(S_G, \mathbf{Q}_B(G))} \sup_{p \in \mathcal{N}(S_G, \mathbf{Q}_B(G))} \frac{\langle p, q \rangle_Y}{\|p\|_Y \|q\|_Y} > 0, \]
and

\[ \kappa_G := \inf_{0 \neq p \in N(S_G, Q_G)} \sup_{q \in R(S_G, Q_G)} \frac{\langle p, q \rangle_{Z}}{\|p\|_{Z} \|q\|_{Z}} = \inf_{0 \neq q \in R(S_G, Q_G)} \sup_{p \in N(S_G, Q_G)} \frac{\langle p, q \rangle_{Z}}{\|p\|_{Z} \|q\|_{Z}} > 0. \]

Provided Assumption 3.4, \([ (S_G, \overline{Q}(G)) : Y \rightarrow Z, (S_G, \overline{Q}(G)) : Z \rightarrow Y \] is a base operator pair.

Assumption 3.5. \( R(T_G, \overline{P}(G)) \subset N(S_G, \overline{Q}(G)), \) and \( R(S_G, \overline{Q}(G)) \subset N(T_G, \overline{P}(G)) \).

Provided Assumption 3.5, by Lemma 2.26, \( R(T_G, P(G)) \subset N(S_G, Q(G)) \), and, \( R(S_G, Q(G)) \subset N(T_G, P(G)) \). Denote,

\[ \overline{H}_G := N(S_G, Q(G)) \ominus R(T_G, \overline{P}(G)), \quad \overline{H}_G := N(S_G, Q(G)) \ominus R(T_G, P(G)), \]

and

\[ \overline{S}_G := N(T_G, P(G)) \ominus R(S_G, \overline{Q}(G)), \quad \overline{S}_G := N(T_G, P(G)) \ominus R(S_G, Q(G)). \]

Assumption 3.6. \( \overline{H}_G \) and \( \overline{S}_G \) are isomorphic; \( \overline{H}_G \) and \( \overline{S}_G \) are isomorphic.

Particularly, denote

\[ \chi_G := \inf_{0 \neq p \in H_G} \sup_{v \in \overline{S}_G} \frac{\langle p, v \rangle_{Y}}{\|p\|_{Y} \|v\|_{Y}} = \inf_{0 \neq v \in \overline{S}_G} \sup_{p \in H_G} \frac{\langle p, v \rangle_{Y}}{\|p\|_{Y} \|v\|_{Y}} > 0 \]

and

\[ \epsilon_G := \inf_{0 \neq p \in H_G} \sup_{v \in \overline{S}_G} \frac{\langle p, v \rangle_{Y}}{\|p\|_{Y} \|v\|_{Y}} = \inf_{0 \neq v \in \overline{S}_G} \sup_{p \in H_G} \frac{\langle p, v \rangle_{Y}}{\|p\|_{Y} \|v\|_{Y}} > 0. \]

3.1.2. Adjoint properties of partially adjoint operators. We have immediately the properties below. They are each a correspondence of the results in Section 2.2.

Theorem 3.7. For \([ (T_G, D_G), (T_G, I_G) ] \) partially adjoint based on \([ (T_G, \overline{P}(G)), (T_G, \overline{P}(G)) ] \),

\[ \text{icr}(T_G, D_G) \leq (1 + \alpha_{G}^{-1}) \cdot \text{icr}(T_G, \overline{P}(G)) + \alpha_{G}^{-1} \text{icr}(T_G, I_G) + \text{icr}(T_G, P(G)). \]

and

\[ \text{icr}(T_G, I_G) \leq (1 + \beta_{G}^{-1}) \cdot \text{icr}(T_G, \overline{P}(G)) + \beta_{G}^{-1} \text{icr}(T_G, D_G) + \text{icr}(T_G, P(G)). \]

Theorem 3.8 (Horizontal slice isomorphism). Let \([ (T_G, D_G), (T_G, \overline{D}(G)) ] \) and \([ (T_G, \overline{D}(G), (T_G, \overline{D}(G)) ] \) be partially adjoint based on \([ (T_G, \overline{P}(G)), (T_G, \overline{P}(G)) ] \), such that \( D_G \subset \overline{D}_G \) and \( \overline{D}_G \subset \overline{D}_G \). Denote

\[ dR_G := R(T_G, \overline{D}_G) \ominus R(T_G, D_G), \quad dR_G := R(T_G, \overline{D}_G) \ominus R(T_G, \overline{D}_G), \quad dN_G := N(T_G, \overline{D}_G) \ominus N(T_G, \overline{D}_G) \] and \( dN_G := N(T_G, \overline{D}_G) \ominus N(T_G, \overline{D}_G) \). Then

\[ \inf_{0 \neq w \in dR_G} \sup_{v \in dN_G} \frac{\langle w, v \rangle_{Y}}{\|w\|_{Y} \|v\|_{Y}} = \inf_{0 \neq v \in dN_G} \sup_{w \in dR_G} \frac{\langle w, v \rangle_{Y}}{\|w\|_{Y} \|v\|_{Y}} \geq \beta_{G} > 0, \]
and

\[ \inf_{0 \leq v \leq \|v\|} \sup_{x \in N_G} \frac{\langle w, v \rangle_x}{\|w\|} = \inf_{0 \leq v \leq \|v\|} \sup_{x \in N_G} \frac{\langle w, v \rangle_x}{\|w\|} \geq \alpha > 0. \]

If further, \( N(T, P_B(\mathcal{G})) = \mathcal{R}(\mathbb{T}_G, P_B(\mathcal{G})) \) and \( N(T, P_B(\mathcal{G})) = \mathcal{R}(T, P_B(\mathcal{G})) \), then,

1. (Horizontal P-A-L type duality)

\[ dR_G = dN_G, \text{ and } d\mathcal{R}_G = dN_G; \]

2. (Helmholtz decomposition) for \( (T, D_G) \) and \( (\mathbb{T}, D_G) \) any pair of partially adjoint operators based on \( [(T, P_G(\mathcal{G})), (\mathbb{T}, P_G(\mathcal{G}))] \).

\[ \mathcal{R}(\mathbb{T}_G, P_G(\mathcal{G})) \oplus N(T, P_G(\mathcal{G})) = \mathcal{R}(T, D_G) \oplus N(T, D_G), \]

and

\[ \mathcal{R}(T, P_G(\mathcal{G})) \oplus N(T, P_G(\mathcal{G})) = \mathcal{R}(T, D_G) \oplus N(T, D_G). \]

For \( [(S_G, E_G), (S_G, E_G)] \) partially adjoint based on \( [(S_G, Q_B(\mathcal{G})), (S_G, \bar{Q}_B(\mathcal{G}))] \), assuming \( \mathcal{R}(S_G, E_G) \subset N(\mathbb{T}_G, D_G) \), which implies \( \mathcal{R}(T, D_G) \subset N(S_G, E_G) \) (cf. Theorem 2.28), define

\[ \mathcal{H}_G := N(S_G, E_G) \oplus \mathcal{R}(T, D_G), \text{ and } \mathfrak{H}_G := N(T, D_G) \oplus \mathcal{R}(S_G, E_G). \]

**Theorem 3.9** (Vertical slice isomorphism). Provided Assumptions 3.5 and 3.6, if \( \mathcal{R}(S_G, E_G) \subset N(\mathbb{T}_G, D_G) \).

\[ \inf_{0 \leq u \leq \|u\|} \sup_{\mathcal{H}_G} \frac{\langle p, u \rangle_{\mathcal{H}_G}}{\|p\|_{\mathcal{H}_G} \|u\|_{\mathcal{H}_G}} = \inf_{0 \leq u \leq \|u\|} \sup_{\mathcal{H}_G} \frac{\langle p, u \rangle_{\mathcal{H}_G}}{\|p\|_{\mathcal{H}_G} \|u\|_{\mathcal{H}_G}} \geq \max \left[ \min(\beta_G, \chi_G), \min(\sigma_G, \epsilon_G) \right]. \]

If further \( \mathcal{R}(T, P_B(\mathcal{G})) = N(\mathbb{T}_G, P_B(\mathcal{G})), N(S_G, Q_B(\mathcal{G})) = \mathcal{R}(S_G, Q_B(\mathcal{G})), \) and \( \mathcal{H}_G = \mathfrak{H}_G \), then

1. (Vertical P-A-L type duality)

\[ \mathcal{H}_G = \mathfrak{H}_G; \]

2. (Hodge decomposition)

\[ \mathcal{R}(T, P_G(\mathcal{G})) \oplus N(\mathbb{T}_G, P_G(\mathcal{G})) = \mathcal{R}(T, D_G) \oplus \mathcal{H}_G = \mathfrak{H}_G \oplus \mathcal{R}(S_G, E_G). \]

**Remark 3.10.** The “Hodge decomposition” here is different from the “discrete Hodge decomposition” given as Equation (5.6) of [1], which reads, using the same notations as defined in the present paper,

\[ E_G = \mathcal{R}(T, D_G) \oplus \mathcal{H}_G \oplus (\mathcal{R}(T, D_G) \oplus \mathcal{H}_G)^{\perp}, \]

where \( (\mathcal{R}(T, D_G) \oplus \mathcal{H}_G)^{\perp} \) is the orthogonal complementation of \( \mathcal{R}(T, D_G) \oplus \mathcal{H}_G \) in \( E_G \) with respect to the inner product of \( \mathcal{Y} \).
3.1.3. Accompanied-by-conforming discretization and discretized spaces. On \( K \in \mathcal{G} \), denote 
\[ H(T_K) := \{ v \in X_K : T_K v \in Y_K \}. \]
The assumption below, which is mild, is useful.

**Assumption 3.11.** \( \langle v, T_K w_K \rangle_{X_K} - \langle T_K v, w_K \rangle_{Y_K} = 0 \), for any \( w_K \in \mathbb{P}_0(K) \) and any \( v \in H(T_K) \) on every \( K \in \mathcal{G} \).

By Assumption 3.11, for any \( w_K \in \mathbb{P}_0(K) \),
\[ \langle v, T_G E_{K}^{\Omega} w_K \rangle_{X} - \langle T v, E_{K}^{\Omega} w_K \rangle_{Y} = 0, \forall v \in D. \]
Therefore, \( E_{K}^{\Omega} w_K \in D \) and \( T_G E_{K}^{\Omega} w_K = T E_{K}^{\Omega} w_K \). Moreover, \( \prod_{K \in \mathcal{G}} \mathbb{P}_0(K) \subset D \).

**Definition 3.12.** \( (T_G, D_{G}^{abc}) \) is called an accompanied-by-conforming discretization, ABCD for short, on \( \mathcal{G} \) of \( (T, D) \), if there is a \( \mathbb{D}_{G}^{c} \subset D \) such that \( \left( (T_G, D_{G}^{abc}), (T, \mathbb{D}_{G}^{c}) \right) \) is partially adjoint based on some tensorial base pair \( \left( (T_G, \tilde{P}(\mathcal{G})), (T_G, \tilde{P}(\mathcal{G})) \right) \) on \( \mathcal{G} \). \( D_{G}^{abc} \) is called an accompanied-by-conforming finite element space, ABCFES for short, on \( \mathcal{G} \) of \( D \).

**Lemma 3.13.** Provided Assumption 3.11, there exists a pair of operators \( [(T_G, D_{G}^{abc}), (T, D_{G})] \) partially adjoint based on \( [(T_G, \tilde{P}(\mathcal{G})), (T_G, \tilde{P}(\mathcal{G}))] \), such that \( D_{G} \subset D \).

**Proof.** We simply set \( D_{G} := D \cap \tilde{P}(\mathcal{G}) \), and \( D_{G} := \{ v_{G} \in P_{G} : \langle v_{G}, T_{G} w_{G} \rangle_{X} - \langle T_{G} v_{G}, w_{G} \rangle_{Y} = 0, \forall w_{G} \in D_{G}^{c} \} \). Then \( \prod_{K \in \mathcal{G}} \mathbb{P}_0(K) \subset D_{G} \), and \( (T_G, D_{G}) \) and \( (T, D_{G}) \) are partially adjoint based on \( [(T_G, \tilde{P}(\mathcal{G})), (T_G, \tilde{P}(\mathcal{G}))] \). The proof is completed. \( \square \)

**Definition 3.14.** Let \( [(T_G, D_{G}), (T, D_{G})] \) be partially adjoint based on \( [(T_G, \tilde{P}(\mathcal{G})), (T_G, \tilde{P}(\mathcal{G}))] \) on the subdivision \( \mathcal{G} \). If \( D_{G} \subset D \) or \( D_{G} \subset D \), then \( [(T_G, D_{G}), (T, D_{G})] \) is called a partially adjoint discretization of \( [(T, D), (T, D)] \).

3.1.4. ABCD used for variational problems. Here we consider a typical elliptic variational problem: given \( f \in X \), find \( u \in D \) such that
\[ \langle Tu, v \rangle_{Y} + \langle u, v \rangle_{X} = \langle f, v \rangle_{X}, \forall v \in D. \]

It follows that \( Tu \in D \), and \( f = T(Tu) + u \).

Let \( [(T_G, D_{G}^{abc}), (T_G, \mathbb{D}_{G}^{c})], \mathbb{D}_{G}^{c} \subset D \), be a partially adjoint discretization of \( [(T, D), (T, D)] \).

We here use \( (T_G, D_{G}^{abc}) \) as a discretization of \( (T, D) \), and consider the discretization of (3.3): to find \( u_{G} \in D_{G}^{abc} \), such that
\[ \langle T_{G} u_{G}, v_{G} \rangle_{Y} + \langle u_{G}, v_{G} \rangle_{X} = \langle f, v_{G} \rangle_{X}, \forall v_{G} \in D_{G}^{abc}. \]

Immediately (3.3) and (3.4) are well-posed. Denote \( \| v_{G} \|_{T_{G}} := (\| v_{G} \|_{X}^{2} + \| T_{G} v_{G} \|_{Y}^{2})^{1/2}. \)

**Theorem 3.15.** Let \( u \) and \( u_{G} \) be the solutions of (3.3) and (3.4), respectively.
\[ \| u - u_{G} \|_{T_{G}} \leq \inf_{w_{G} \in D_{G}^{abc}} \| u - w_{G} \|_{T_{G}} + \inf_{w_{G} \in D_{G}} \| T_{G} u - w_{G} \|_{T_{G}} \cdot \sup_{v_{G} \in D_{G}^{abc}} \inf_{w_{G} \in D_{G}^{abc} \cap D} \| v_{G} - w_{G} \|_{T_{G}}. \]

\[ \| v_{G} \|_{T_{G}} = 1. \]
Lemma 3.17. Therefore

It can be easily proved by definition that the global spaces are tensor products of certain local spaces. The proof is completed.

3.2. Characteristics of localization.

3.2.1. Structure localization of tensorial pairs. Using the notations defined in Section 3.1, on every \( K \in \mathcal{G} \), denote with respect to \( T_K \) and \( \Pi_K \),

- \( \hat{P}(K) := \{ v \in P(K) : T_K v = 0 \}, P^*(K) := \{ v \in P(K) : \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}(K) \}; \)
- \( P_0(K) := \{ v \in P(K) : \langle v, \Pi_K w \rangle_{X_K} - \langle T_K v, w \rangle_{Y_K} = 0, \forall w \in P(K) \}; \)
- \( \hat{P}_0(K) := \{ v \in P_0(K) : T_K v = 0 \}, P^*_0(K) := \{ v \in P_0(K) : \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_0(K) \}; \)
- \( P_B(K) := \{ v \in P(B) : \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_0(K); \langle T_K v, T_K w \rangle_{Y_K} = 0, \forall w \in P_0(K) \}; \)
- \( \hat{P}_B(K) := \{ v \in P_B(K) : T_K v = 0 \}, P^*_B(K) := \{ v \in P_B(K), \langle v, w \rangle_{X_K} = 0, \forall w \in \hat{P}_B(K) \}. \)

Similarly, denote

- \( \hat{P}(K) := \{ v \in \hat{P}(K) : T_K v = 0 \}, P^*(K) := \{ v \in \hat{P}(K) : \langle v, w \rangle_{Y_K} = 0, \forall w \in \hat{P}(K) \}; \)
- \( P_0(K) := \{ v \in \hat{P}(K) : \langle v, T_K w \rangle_{Y_K} - \langle T_K v, w \rangle_{X_K} = 0, \forall w \in P(K) \}; \)
- \( \hat{P}_0(K) := \{ v \in \hat{P}_0(K) : T_K v = 0 \}, P^*_0(K) := \{ v \in \hat{P}_0(K) : \langle v, w \rangle_{Y_K} = 0, \forall w \in \hat{P}_0(K) \}; \)
- \( P_B(K) := \{ v \in P_B(K) : \langle v, w \rangle_{Y_K} = 0, \forall w \in \hat{P}_0(K); \langle T_K v, T_K w \rangle_{X_K} = 0, \forall w \in P_B(K) \}; \)
- \( \hat{P}_B(K) := \{ v \in P_B(K) : T_K v = 0 \}, P^*_B(K) := \{ v \in P_B(K), \langle v, w \rangle_{Y_K} = 0, \forall w \in \hat{P}_B(K) \}. \)

It can be easily proved by definition that the global spaces are tensor products of certain local spaces. We survey them in the two lemmas below.

Lemma 3.16. (1) \( P(G) = \prod_{K \in \mathcal{G}} P_0(K), P_B(G) = \prod_{K \in \mathcal{G}} P_B(K) \);

(2) \( N(T_G, \hat{P}(G)) = \prod_{K \in \mathcal{G}} P_0(K), N(T_G, P(G)) = \prod_{K \in \mathcal{G}} P_0(K), N(T_G, P_B(G)) = \prod_{K \in \mathcal{G}} \hat{P}_B(K) \);

(3) \( R(T_G, \hat{P}(G)) = \prod_{K \in \mathcal{G}} R(T_K, P(K)), R(T_G, P(G)) = \prod_{K \in \mathcal{G}} R(T_K, P_0(K)), R(T_G, P_B(G)) = \prod_{K \in \mathcal{G}} R(T_K, P_B(K)). \)

Lemma 3.17. (1) \( P(G) = \prod_{K \in \mathcal{G}} P_0(K), P_B(G) = \prod_{K \in \mathcal{G}} P_B(K) \);
Moreover, Proof.

Lemma 3.20. \(N(T_G, \hat{P}(G)) = \prod_{K \in \mathcal{G}} \hat{P}(K), N(T_G, P(G)) = \prod_{K \in \mathcal{G}} P_0(K), N(T_G, \hat{P}_B(G)) = \prod_{K \in \mathcal{G}} \hat{P}_B(K)\); (3) \(R(T_G, \hat{P}(G)) = \prod_{K \in \mathcal{G}} R(T_K, \hat{P}(K)), R(T_G, P(G)) = \prod_{K \in \mathcal{G}} R(T_K, P_0(K)), R(T_G, \hat{P}_B(G)) = \prod_{K \in \mathcal{G}} R(T_K, \hat{P}_B(K))\).

Based on the tensor products, the global structures can be realized locally. We present several lemmas below. They can be actually proved the same way, and we will only present the proof of Lemma 3.18 in details.

Lemma 3.18. Assumption 3.1 holds, if and only if, on every \(K \in \mathcal{G}\), there exists an \(\alpha_K > 0\) such that

\[
(3.5) \quad \inf_{0 \neq v \in P_B(K)} \sup_{w \in R(T_K, P_B(K))} \frac{\langle v, w \rangle_{x_K}}{\|v\|_{x_K} \|w\|_{x_K}} = \inf_{0 \neq w \in R(T_K, P_B(K))} \sup_{v \in P_B(K)} \frac{\langle v, w \rangle_{x_K}}{\|v\|_{x_K} \|w\|_{x_K}} = \alpha_K,
\]

and on every \(K \in \mathcal{G}\), there exists a \(\beta_K > 0\) such that

\[
(3.6) \quad \inf_{0 \neq w \in P_B(K)} \sup_{v \in R(T_K, P_B(K))} \frac{\langle v, w \rangle_{x_K}}{\|v\|_{x_K} \|w\|_{x_K}} = \inf_{0 \neq v \in P_B(K)} \sup_{w \in R(T_K, P_B(K))} \frac{\langle v, w \rangle_{x_K}}{\|v\|_{x_K} \|w\|_{x_K}} = \beta_K.
\]

Moreover,

\[\alpha_G = \inf_{K \in \mathcal{G}} \alpha_K, \quad \text{and} \quad \beta_G = \inf_{K \in \mathcal{G}} \beta_K.\]

Proof. Provided \(0 \neq v \in N(T_G, P_B(G))\), there exists a \(w \in \prod_{K \in \mathcal{G}} R(T_K, P_B(K))\), such that \(E_K^\alpha v_K, w)_{x_K} \geq \alpha_G \|v_K\|_{x_K} \|w\|_{x_K}.\) Then \(\langle v_K, R^K w \rangle_{x_K} \geq \alpha_G \|v_K\|_{x_K} \|R^K w\|_{x_K}\) with \(R^K w \in R(T_K, P_B(K))\). This proves

\[\alpha_K = \inf_{0 \neq v \in P_B(K)} \sup_{w \in R(T_K, P_B(K))} \frac{\langle v, w \rangle_{x_K}}{\|v\|_{x_K} \|w\|_{x_K}} \geq \alpha_G, \quad \forall K \in \mathcal{G}.\]

On the other hand, provided that \(0 \neq v \in N(T_G, P_B(G))\), there exists a \(w \in R(T_G, P_B(G))\), such that \(\langle R^K v, R^K w \rangle_{x_K} \geq \alpha_K \|R^K v\|_{x_K} \|R^K w\|_{x_K}\) and \(\|R^K v\|_{x_K} = \|R^K w\|_{x_K}\) on every \(K \in \mathcal{G}\). Further, \(\langle v, w \rangle_{x_K} = \sum_{K \in \mathcal{G}} \langle R^K v, R^K w \rangle_{x_K} \geq \sum_{K \in \mathcal{G}} \alpha_K \|R^K v\|_{x_K} \|R^K w\|_{x_K} \geq \inf_{K \in \mathcal{G}} \alpha_K \|v\|_{x_K} \|w\|_{x_K}\) on every \(K \in \mathcal{G}\). This proves

\[\inf_{0 \neq v \in N(T_G, P_B(G))} \sup_{w \in R(T_G, P_B(G))} \frac{\langle v, w \rangle_{x_K}}{\|v\|_{x_K} \|w\|_{x_K}} = \alpha_G \geq \inf_{K \in \mathcal{G}} \alpha_K.\]

The proof is completed. \(\square\)

Lemma 3.19. \(\text{icr}(T_G, \hat{P}(G)) = \sup_{K \in \mathcal{G}} \text{icr}(T_K, P(K)). \text{icr}(T_G, P(G)) = \sup_{K \in \mathcal{G}} \text{icr}(T_K, P_0(K)).\)

Lemma 3.20. \(N(T_G, P_B(G)) = R(T_G, P_B(G))\) if and only if on every \(K \in \mathcal{G}\), \(\hat{P}_B(K) = R(T_K, P_B(K))\).
Lemma 3.21. Assumption 3.4 holds, if and only if, on every $K \in \mathcal{G}$, there exists a $\sigma_K > 0$ such that

$$
\inf_{0 \neq p \in Q_0(K)} \sup_{q \in R(S_K, Q_0(K))} \frac{\langle p, q \rangle_{Y_K}}{\|p\|_{Y_K} \|q\|_{Y_K}} = \inf_{0 \neq q \in R(S_K, Q_0(K))} \sup_{p \in Q_0(K)} \frac{\langle p, q \rangle_{Y_K}}{\|p\|_{Y_K} \|q\|_{Y_K}} = \sigma_K,
$$

and on every $K \in \mathcal{G}$, there exists a $\kappa_K > 0$ such that

$$
\inf_{0 \neq p \in Q_0(K)} \sup_{q \in R(S_K, Q_0(K))} \frac{\langle p, q \rangle_{Z_K}}{\|p\|_{Z_K} \|q\|_{Z_K}} = \inf_{0 \neq q \in R(S_K, Q_0(K))} \sup_{p \in Q_0(K)} \frac{\langle p, q \rangle_{Z_K}}{\|p\|_{Z_K} \|q\|_{Z_K}} = \kappa_K.
$$

Further

$$
\sigma_{\mathcal{G}} = \inf_{K \in \mathcal{G}} \sigma_K, \text{ and } \kappa_{\mathcal{G}} = \inf_{K \in \mathcal{G}} \kappa_K.
$$

Lemma 3.22. Assumption 3.5 holds, if and only if on every $K \in \mathcal{G}$,

$$
\mathcal{R}(T_K, P(K)) \subset \mathcal{N}(S_K, Q(K)), \text{ and } \mathcal{R}(S_K, Q(K)) \subset \mathcal{N}(T_K, P(K)).
$$

Provided Assumption 3.5, denote, on $K \in \mathcal{G}$,

$$
\widetilde{H}_K := \mathcal{N}(S_K, Q(K)) \ominus \mathcal{R}(T_K, P(K)), \quad H_K := \mathcal{N}(S_K, Q_0(K)) \ominus \mathcal{R}(T_K, P_0(K)),
$$

and

$$
\overline{H}_K := \mathcal{N}(T_K, P(K)) \ominus \mathcal{R}(S_K, Q(K)), \quad \overline{H}_K := \mathcal{N}(T_K, P_0(K)) \ominus \mathcal{R}(S_K, Q_0(K)).
$$

Then

$$
\overline{H}_{\mathcal{G}} = \bigsqcup_{K \in \mathcal{G}} \overline{H}_K, \quad H_{\mathcal{G}} = \bigsqcup_{K \in \mathcal{G}} H_K, \quad \overline{H}_{\mathcal{G}} = \bigsqcup_{K \in \mathcal{G}} \overline{H}_K, \text{ and } \overline{H}_{\mathcal{G}} = \bigsqcup_{K \in \mathcal{G}} \overline{H}_K.
$$

Lemma 3.23. Assumption 3.6 holds, if and only if, on every $K \in \mathcal{G}$, there exist constants $\chi_K, \epsilon_K > 0$, such that

$$
\inf_{0 \neq p \in H_K} \sup_{v \in H_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \inf_{0 \neq v \in H_K} \sup_{p \in H_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \chi_K,
$$

and

$$
\inf_{0 \neq p \in H_K} \sup_{v \in H_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \inf_{0 \neq v \in H_K} \sup_{p \in H_K} \frac{\langle p, v \rangle_{Y_K}}{\|p\|_{Y_K} \|v\|_{Y_K}} = \epsilon_K.
$$

Moreover,

$$
\chi_{\mathcal{G}} = \inf_{K \in \mathcal{G}} \chi_K, \text{ and } \epsilon_{\mathcal{G}} = \inf_{K \in \mathcal{G}} \epsilon_K.
$$

Remark 3.24 (Asymptotics of an infinite family of finite-dimensional operators). Let $\mathcal{F}^\mathcal{G}$ be a family of subdivisions of $\Omega$. Denote

$$
\alpha_{\mathcal{F}} := \inf_{G \in \mathcal{F}} \inf_{K \in \mathcal{G}} \alpha_K, \text{ and } \beta_{\mathcal{F}} := \inf_{G \in \mathcal{F}} \inf_{K \in \mathcal{G}} \beta_K;
$$

$$
\rho_{\mathcal{F}} := \sup_{G \in \mathcal{F}} \sup_{K \in \mathcal{G}} \text{icr}(T_K, P(K)) \text{ and } \rho_{\mathcal{F}} := \sup_{G \in \mathcal{F}} \sup_{K \in \mathcal{G}} \text{icr}(T_K, P_0(K));
$$
though, we now show that the supports of the basis functions of $$\text{Theorem 3.26.}$$

On any cell $$K$$, the set of functions $$\{v_i : i \in I_1, \hat{K} \cap \text{supp}(b_i) \neq \emptyset\}$$ are linearly independent.

**Theorem 3.26.** Provided Assumption 3.25, $$D_\mathcal{G}$$ admits a set of locally supported basis functions.

*Proof.* We prove this theorem by constructing a set of basis functions.

By Assumption 3.25, given $$K \in \mathcal{G}$$, set $$I_2^K := \{i \in I_2 : \hat{K} \cap \text{supp}(b_i) \neq \emptyset\}$$, and there exist a set of functions $$\{v_i^K : i \in I_2^K\} \subset P_B(K)$$, such that $$\langle v_i^K, T_K R_{12} K b_j \rangle_{X_K} - \langle T_K v_i^K, R_{12} K b_j \rangle_{Y_K} = 0, \forall j \in I_2^K$$. Denote further $$P_b^K(K) := \{v \in P_B(K) : \langle v, T_K R_{12} K b_j \rangle_{X_K} - \langle T_K v, R_{12} K b_j \rangle_{Y_K} = 0, \forall j \in I_2^K\}$$. Note that $$P_B(K) \supset \text{span} \{b_i^K : i \in I_2^K\}$$, and they two are equal if and only if $$P_b^K(K) = \{0\}$$. Therefore, the space $$P(K)$$ can be decomposed to $$P(K) = P_0(K) \oplus \text{span} \{v_i^K : i \in I_2^K\} \oplus P_b^K(K)$$. 

Let on every $$G \in \mathcal{F}^G$$ be constructed a pair of partially adjoint discretizations of $$[(T, D), (\mathbb{T}, \mathbb{D})]$$, so that a family of partially adjoint discretizations is formed as $$\{[(T_G, D_G), (\mathbb{T}_G, \mathbb{D}_G)]\}_{G \in \mathcal{F}^G}$$. Then

1. the adjoint properties of the operator pairs in $$\{[(T_G, D_G), (\mathbb{T}_G, \mathbb{D}_G)]\}_{G \in \mathcal{F}^G}$$ can be controlled by the parameters defined by (3.9), (3.10), (3.11), and (3.12):

   (2) all these parameters can be controlled cell by cell, and subdivision by subdivision.

This way, we can have an infinite family of finite-dimensional operators which bears uniform asymptotic properties based on the cell-wise properties. See Lemma 4.2.1 for a precise example.

3.2.2. Locality of supports of basis functions. Let $$[(T_G, D_G), (\mathbb{T}_G, \mathbb{D}_G)]$$ be a pair of partially ad-

joint discretizations of $$[(T, D), (\mathbb{T}, \mathbb{D})]$$. An explicit set of basis functions are usually used in the implementation of the discretization schemes based on $$D_G$$ and $$\mathbb{D}_G$$. Generally, the basis functions of $$\mathbb{D}_G$$ are determined by solving a linear system, and it is possible that the supports may be vast. Though, we now show that, the supports of the basis functions of $$D_G$$ can be controlled by the supports of the basis functions of $$\mathbb{D}_G$$. Here, $$[(T_G, D_G), (\mathbb{T}_G, \mathbb{D}_G)]$$ is not necessarily a partially adjoint discretization of $$[(T, D), (\mathbb{T}, \mathbb{D})]$$.

We use $$\text{LSB}(\mathbb{D}_G) = \{b_i\}_{i=1}^{\dim(\mathbb{D}_G)}$$ to denote a set of linearly independent locally supported basis functions of $$\mathbb{D}_G$$. By Theorem 2.17, since $$(T_G, D_G)$$ and $$(\mathbb{T}_G, \mathbb{D}_G)$$ are partially adjoint, $$\prod_{K \in \mathcal{G}} P_0(K) = \mathbb{P}(G) \subset \mathbb{D}_G$$. Therefore, we can particularly set

\[
(3.13) \quad \text{LSB}(\mathbb{D}_G) = \left[\prod_{K \in \mathcal{G}} P_0(K)\right] \oplus \left[\text{LSB}(\mathbb{D}_G) \cap \prod_{K \in \mathcal{G}} P_B(K)\right] = \text{span} \{b_i\}_{i \in I_1} \oplus \text{span} \{b_i\}_{i \in I_2},
\]

where $$I_1$$ and $$I_2$$ are two distinct sets of indices, and $$I_1 \cup I_2 = \{1, \ldots, \dim(\mathbb{D}_G)\}$$.

**Assumption 3.25.** On any cell $$K$$, the set of functions $$\{b_i|_K : i \in I_2, \hat{K} \cap \text{supp}(b_i) \neq \emptyset\}$$ are linearly independent.

**Theorem 3.26.** Provided Assumption 3.25, $$D_G$$ admits a set of locally supported basis functions.

The precise meaning of “locally supported” is given in the end of the proof.
Now, by definition,

\[
D_G = \left\{ v \in \overline{P} : \langle v, T_G w \rangle_X - \langle T_G v, w \rangle_Y = 0, \ \forall \ w \in D_G \right\}
\]

\[
= P(G) \oplus \bigcap_{K \in G} P_B(K) \oplus \left\{ v \in \sum_{K \in G} \sum_{j \in I_2} \text{span} \left\{ E^K_j \right\} : \langle v, T_G b_j \rangle_X - \langle T_G v, b_j \rangle_Y = 0, \ j \in I_2 \right\}
\]

\[
= P(G) \oplus \bigcap_{K \in G} P_B(K) \oplus \sum_{j \in I_2} \left\{ v \in \sum_{K \cap \text{supp}(b_j) \neq \emptyset} \text{span} \left\{ E^K_j \right\} : \langle v, T_G b_j \rangle_X - \langle T_G v, b_j \rangle_Y = 0 \right\}
\]

Therefore, \( D_G \) admits two types of basis functions, namely,

**Type-I:** \( E^K_0 P_0(K) \) and \( E^K_{\log} P_B(K) \) for all \( K \in G \);

**Type-II:** for \( j \in I_2 \), such \( v \in \sum_{K \cap \text{supp}(b_j) \neq \emptyset} \text{span} \left\{ E^K_j \right\} \) that

\[
\sum_{K \in G} \langle v, T_K R^K_{ij} b_j \rangle_X - \langle T_K v, R^K_{ij} b_j \rangle_Y = 0.
\]

(3.14)

The supports of basis functions of Type-I are each one cell. The supports of basis functions of Type-II are all contained in the support of \( b_j \). The proof is completed.

**Remark 3.27.** Given any \( b_i \), \( i \in I_2 \), there are \([\# \{ K \in G : \hat{K} \cap \text{supp}(b_i) \neq \emptyset \} - 1] \) basis functions of Type-II corresponding to \( b_i \). They are each a combination of those \( v^K_i \), determined up to one degree of freedom by (3.14).

3.3. **Cell-wise-defined interpolation of ABCFES and optimal ABCD.**

3.3.1. **Interpolator on one cell.** We here use the notations defined in the previous subsections.

**Definition 3.28** (Adjoint projection). Define \( T^K : H(T_K) \to \overline{P}(K) \) by

\[
(1) \quad \langle T^K v, T_K w \rangle_X = \langle v, T_K w \rangle_X - \langle T_K v, w \rangle_Y, \ \forall \ w \in \overline{P}(K);
\]

\[
(2) \quad \langle T^K v, w \rangle_X = \langle v, w \rangle_X, \ \forall \ w \in P_0(K);
\]

\[
(3) \quad \langle T^K v, T_K w \rangle_Y = \langle T_K v, T_K w \rangle_Y, \ \forall \ w \in P_0(K).
\]

**Remark 3.29.** By Assumption 3.11,

\[
\langle T^K v, T_K w \rangle_X = \langle T_K T^K v, w \rangle_Y = 0 = \langle v, T_K w \rangle_X - \langle T_K v, w \rangle_Y, \ \forall \ w \in \overline{P}(K).
\]

Therefore, for \( v \in H(T_K) \),

\[
\langle T^K v, T_K w \rangle_X = \langle T_K T^K v, w \rangle_Y = \langle v, T_K w \rangle_X - \langle T_K v, w \rangle_Y, \ \forall \ w \in \overline{P}(K).
\]

This is why the interpolator is called "adjoint projection".

**Lemma 3.30.** Provided Assumption 3.1, the interpolator \( T^K \) is well-defined, and \( T^K v = v \) for \( v \in P(K) \). Further,
(a) with $\beta_K = \inf_{0 \neq w \in \mathbb{R}^n \setminus \mathbb{P}_B(K)} \sup_{v \in \mathbb{P}_B(K)} \frac{\langle \nu, w \rangle_{\mathbb{Y}_K}}{\|v\|_{\mathbb{Y}_K} \cdot \|w\|_{\mathbb{Y}_K}}$ as given in (3.6),

$$||T_K \nu\|_{\mathbb{Y}_K} \leq (1 + \beta_K^{-1}) ||T_K \nu||_{\mathbb{Y}_K}, \text{ and } ||T_K(v - T_K^\nu)\|_{\mathbb{Y}_K} \leq (2 + \beta_K^{-1}) \inf_{w \in \mathbb{P}(K)} ||T_K(v - w)||_{\mathbb{Y}_K};$$

(b) with $\gamma_K := \inf_{0 \neq v \in \mathbb{P}(K)} \sup_{v \in \mathbb{P}(K)} \frac{\langle \nu, T_K v \rangle_{\mathbb{X}_K} - \langle T_K v, \nu \rangle_{\mathbb{Y}_K}}{||v||_{T_K} \cdot ||w||_{T_K}}$, and $\rho_K := \text{icr}(T_K, P_0(K))$,

$$||T_K^\nu||_{T_K} \leq (2 + \rho_K + \gamma_K^{-1} + \beta_K^{-1}) ||v||_{T_K}, \text{ and } ||v - ||T_K^\nu||_{T_K} \leq (3 + \rho_K + \gamma_K^{-1} + \beta_K^{-1}) \inf_{w \in \mathbb{P}(K)} ||v - w||_{T_K}.$$ 

**Proof.** Provided Assumption 3.1, by Lemma 3.18 and Lemma 2.14, $\gamma_K > 0$. By the decomposition of $P(K)$ and $P(K)$, there exist uniquely $v_B \in P_B(K)$, $\bar{v}_0 \in P_0(K)$ and $v_0' \in P_0^+(K)$, such that

1. $\langle v_B, T_K v \rangle_{\mathbb{X}_K} - \langle T_K v, v \rangle_{\mathbb{Y}_K} = \langle v, T_K v \rangle_{\mathbb{X}_K} - \langle T_K v, v \rangle_{\mathbb{Y}_K}, \forall v \in \mathbb{P}_B(K);$ 
2. $\langle \bar{v}_0, w \rangle_{\mathbb{X}_K} = \langle v, w \rangle_{\mathbb{X}_K}, \forall w \in \mathbb{P}_0(K);$ 
3. $\langle T_K v_0', T_K w \rangle_{\mathbb{Y}_K} = \langle T_K v, T_K w \rangle_{\mathbb{Y}_K}, \forall w \in \mathbb{P}_0^+(K).$

Besides, $||\bar{v}_0||_{\mathbb{X}_K} \leq ||v||_{\mathbb{X}_K}$ and $||T_K v_0'||_{\mathbb{X}_K} \leq ||T_K v||_{\mathbb{Y}_K}$. Moreover,

1. $\langle \bar{v}, T_K v \rangle_{\mathbb{X}_K} - \langle T_K \bar{v}, v \rangle_{\mathbb{Y}_K} = 0, \forall v \in \mathbb{P}_B(K)$, if $\bar{v} \in \{\bar{v}_0, v_0\};$
2. $\langle \bar{v}, w \rangle_{\mathbb{X}_K} = 0, \forall w \in \mathbb{P}_0(K)$, if $\bar{v} \in \{v_B, v_0'\};$
3. $\langle T_K \bar{v}, T_K w \rangle_{\mathbb{Y}_K} = 0, \forall w \in \mathbb{P}_0^+(K)$, if $\bar{v} \in \{v_B, \bar{v}_0\}.$

Therefore, $T_K^\nu$ is uniquely determined as $v_B + \bar{v}_0 + v_0'$, and $T_K$ is well defined. Evidently $T_K^\nu = v$ for $v \in P(K)$.

By (3.6), $||T_K v_B||_{\mathbb{X}_K} \leq \sup_{v \in \mathbb{P}_B(K)} \frac{1}{\beta_K} \frac{\langle T_K v, v \rangle_{\mathbb{Y}_K}}{||v||_{\mathbb{Y}_K}} \leq ||T_K v||_{\mathbb{Y}_K} / \beta_K$. Therefore,

$$||T_K^\nu||_{\mathbb{Y}_K} \leq ||T_K v||_{\mathbb{Y}_K} / \beta_K + ||T_K v||_{\mathbb{Y}_K} = (1 + \beta_K^{-1}) ||T_K v||_{\mathbb{Y}_K}.$$ 

Similarly, for any $w \in P(K)$, $||T_K(v - T_K^\nu)\|_{\mathbb{Y}_K} \leq ||T_K(v - w)||_{\mathbb{Y}_K} + ||T_K(T_K^\nu - T_K(v - w))||_{\mathbb{Y}_K} \leq ||T_K(v - w)||_{\mathbb{Y}_K} + (1 + \beta_K^{-1}) ||T_K(v - w)||_{\mathbb{Y}_K}$, and therefore,

$$||T_K(v - T_K^\nu)||_{\mathbb{Y}_K} \leq (2 + \beta_K^{-1}) \inf_{w \in \mathbb{P}(K)} ||T_K(v - w)||_{\mathbb{Y}_K}.$$ 

On the other hand, $\gamma_K \|v_B\|_{T_K} \leq \sup_{v \in \mathbb{P}_B(K)} \frac{\langle v_B, T_K v \rangle_{\mathbb{X}_K} - \langle T_K v_B, v \rangle_{\mathbb{Y}_K}}{||v||_{T_K}} \leq ||v||_{T_K}$. Therefore,

$$||T_K^\nu||_{T_K} \leq (2 + \rho_K + \gamma_K^{-1} + \beta_K^{-1}) ||v||_{T_K}.$$ 

Finally, it holds with any $w \in P(K)$ that

$$||v - T_K^\nu||_{T_K} \leq ||v - w||_{T_K} + ||T_K^\nu(v - w)||_{T_K} \leq (3 + \rho_K + \gamma_K^{-1} + \beta_K^{-1}) ||v - w||_{T,K}.$$ 

The proof is completed. 

\[\square\]
**Discrete commutative diagram.** With respect to the spaces $\mathbb{Q}(K)$ and $\mathbb{Q}(K)$, we can define interpolation $I^s_K : H(S_K) \to \mathbb{Q}(K)$ by

1. $(I^s_K q, S_K q)_{\mathbb{Y}_K} - (S_K I^s_K q, q)_{\mathbb{Z}_K} = (q, S_K q)_{\mathbb{Y}_K} - (S_K q, q)_{\mathbb{Z}_K}$, \( \forall q \in \mathbb{Q}_B(K) \);
2. $(I^s_K q, w)_{\mathbb{Y}_K} = (q, w)_{\mathbb{Y}_K}$, \( \forall w \in \hat{\mathbb{Q}}_0(K) \);
3. $(S_K I^s_K q, S_K w)_{\mathbb{Z}_K} = (S_K q, S_K w)_{\mathbb{Z}_K}$, \( \forall w \in \mathbb{Q}_B^0(K) \).

The interpolators $I^t_K$ and $I^s_K$ commute in the sense below.

**Lemma 3.31.** Provided Assumption 3.5, that $\mathcal{H}_K = \overline{\mathcal{S}}_K$, and that $\mathcal{R}(T_K, H(T_K)) \subset N(S_K, H(S_K))$, $I^s_K T_K^T v = T_K I^t_K v$ for $v \in H(T_K)$.

**Proof.** As by Assumption 3.5 and Lemma 3.22, $T_K I^t_K v \in \hat{\mathbb{Q}}(K)$, we will verify $I^s_K T_K^T v = T_K I^t_K v$ by verifying

\[ (T_K I^t_K v, q)_{\mathbb{Y}_K} = (T_K v, q)_{\mathbb{Y}_K}, \quad \forall q \in \hat{\mathbb{Q}}_0(K) \]

and

\[ (T_K I^t_K v, S_K v)_{\mathbb{Y}_K} - (S_K I^t_K v, v)_{\mathbb{Z}_K} = (T_K v, S_K v)_{\mathbb{Y}_K} - (S_K T_K v, v)_{\mathbb{Z}_K}, \quad \forall v \in \mathbb{Q}_B(K). \]

Firstly, by Assumption 3.11,

\[ (T_K v, w)_{\mathbb{X}_K} - (v, T_K w)_{\mathbb{X}_K} = 0, \quad \forall v \in H(T_K), \quad w \in \mathbb{P}_0(K). \]

By the definition of $I^t_K$,

\[ (T_K I^t_K v, w)_{\mathbb{Y}_K} - (I^t_K v, T_K w)_{\mathbb{X}_K} = (T_K v, w)_{\mathbb{Y}_K} - (v, T_K w)_{\mathbb{X}_K}, \quad \forall v \in H(T_K), \quad w \in \mathbb{P}_B(K). \]

Given $\hat{v} \in \overline{\mathcal{S}}_K \subset \hat{\mathbb{P}}_B(K) \oplus \hat{\mathbb{P}}_0(K)$, decompose $\hat{v} = \hat{v}_0 + \hat{v}_B$, with $\hat{v}_0 \in \hat{\mathbb{P}}_0(K)$ and $\hat{v}_B \in \hat{\mathbb{P}}_B(K)$. Then, for any $v \in H(T_K)$,

\[ (T_K I^t_K v, \hat{v})_{\mathbb{Y}_K} = (T_K I^t_K v, \hat{v}_0 + \hat{v}_B)_{\mathbb{Y}_K} = (by \ (3.17)) (T_K I^t_K v, \hat{v}_B)_{\mathbb{Y}_K} = (T_K v, \hat{v}_B)_{\mathbb{Y}_K} = (T_K v, \hat{v}_B + \hat{v}_0)_{\mathbb{Y}_K} = (by \ (3.17)) (T_K v, \hat{v}_0 + \hat{v}_B)_{\mathbb{Y}_K} = (T_K v, \hat{v})_{\mathbb{Y}_K}. \]

Therefore, for any $q \in \hat{\mathbb{Q}}_0(K) = T_K \mathbb{P}_0(K) + \mathcal{H}_K = T_K \mathbb{P}_0(K) + \overline{\mathcal{S}}_K$, we have by (3.18) and the definition of $I^t_K$, that $\langle T_K I^t_K v, q \rangle_{\mathbb{Y}_K} = \langle T_K v, q \rangle_{\mathbb{Y}_K}$. This verifies (3.15).

Secondly, $\langle T_K I^t_K v, \hat{v} \rangle_{\mathbb{Y}_K} = \langle T v, \hat{v} \rangle_{\mathbb{Y}_K}$, for any $\hat{v} \in \hat{\mathbb{P}}(K) = \hat{\mathbb{P}}_B(K) \oplus \hat{\mathbb{P}}_0(K)$. Indeed, if $\hat{v}_B \in \hat{\mathbb{P}}_B(K)$, by the definition of $I^t_K$, $\langle T_K I^t_K v, \hat{v}_B \rangle_{\mathbb{Y}_K} = \langle T v, \hat{v}_B \rangle_{\mathbb{Y}_K}$; if $\hat{v}_0 \in \hat{\mathbb{P}}_0(K)$, by (3.17),

\[ \langle T_K I^t_K v, \hat{v}_0 \rangle_{\mathbb{Y}_K} = \langle T_K I^t_K v, \hat{v}_0 \rangle_{\mathbb{Y}_K} - (T_K I^t_K v, T_K \hat{v}_0)_{\mathbb{Y}_K} = 0 = \langle T_K v, \hat{v}_0 \rangle_{\mathbb{Y}_K} - (v, T_K \hat{v}_0)_{\mathbb{Y}_K} = \langle T v, \hat{v}_0 \rangle_{\mathbb{Y}_K}. \]

Therefore, as $\mathcal{R}(S_K, \mathbb{Q}(K)) \subset \hat{\mathbb{P}}(K)$,

\[ (T_K I^t_K v, S_K q)_{\mathbb{Y}_K} - (S_K T_K I^t_K v, q)_{\mathbb{Z}_K} = (T_K I^t_K v, S_K q)_{\mathbb{Y}_K} - (S_K T_K v, q)_{\mathbb{Z}_K} \quad \forall q \in \mathbb{Q}(K). \]

Then (3.16) is verified and the proof is completed. \( \square \)
3.3.2. A cell-wise interpolator that maps $D$ into $D_{G}^{abc}$.

Global interpolator and approximation. Now define

$$
\mathbb{T}_{G}^{T} : \prod_{K \in G} H(T_{K}) \to \prod_{K \in G} P(K), \text{ by } (\mathbb{T}_{G}^{T} v_{G})_{|K} = \mathbb{T}_{K}(v_{G}|K), \forall K \in G.
$$

Then $\mathbb{T}_{G}^{T}$ is projective onto $\mathbb{P}(G)$. Moreover, defined as an interpolator for $D$, $\mathbb{T}_{G}^{T}$ requires the appropriate regularity and of minimal dependence.

**Lemma 3.32.** Let $[(T_{G}, D_{G}^{\text{abc}}), (\mathbb{T}_{G}, D_{G}^{c})]$ be partially adjoint based on $[(T_{G}, \mathbb{P}(G)), (\mathbb{T}_{G}, \mathbb{P}(G))]$, and $D_{G}^{c} \subset D$. If $v \in D$, then $\mathbb{T}_{G}^{T} v \in D_{G}^{abc}$.

**Proof.** It suffices for proving $\mathbb{T}_{G}^{T} v \in D_{G}^{abc}$ to show

$$
\langle \mathbb{T}_{G}^{T} v, T_{G} v \rangle_{X} - \langle T_{G}^{*} v, v \rangle_{Y} = 0, \forall v \in D_{G}^{c} = \prod_{K \in G} P_{0}(K) \oplus (D_{G}^{c} \cap \prod_{K \in G} P_{B}(K)).
$$

Given $v \in \prod_{K \in G} P_{0}(K)$, with $v_{K} := v|_{K}$ and $v_{K} := v|_{K}$,

$$
\sum_{K \in G} \langle \mathbb{T}_{K}^{T} v_{K}, T_{K} v_{K} \rangle_{X_{K}} = \sum_{K \in G} 0 = 0.
$$

Given $v \in D_{G}^{c} \cap \prod_{K \in G} P_{B}(K)$, it holds with $v_{K} := v|_{K}$ and $v_{K} := v|_{K}$ that

$$
\langle \mathbb{T}_{K}^{T} v_{K}, T_{K} v_{K} \rangle_{X_{K}} - \langle T_{K}^{*} v_{K}, v_{K} \rangle_{Y_{K}} = \langle v_{K}, T_{K} v_{K} \rangle_{X_{K}} - \langle T_{K} v_{K}, v_{K} \rangle_{Y_{K}}, \forall K \in G,
$$

therefore

$$
\sum_{K \in G} \langle \mathbb{T}_{K}^{T} v_{K}, T_{K} v_{K} \rangle_{X_{K}} - \langle T_{K}^{*} v_{K}, v_{K} \rangle_{Y_{K}} = \sum_{K \in G} \langle v_{K}, T_{K} v_{K} \rangle_{X_{K}} - \langle T_{K} v_{K}, v_{K} \rangle_{Y_{K}} = 0,
$$

where we have used the fact that $v \in D_{G}^{c} \subset D$ and thus

$$
\sum_{K \in G} \langle v_{K}, T_{K} v_{K} \rangle_{X_{K}} - \langle T_{K} v_{K}, v_{K} \rangle_{Y_{K}} = \langle v, T_{G} v \rangle_{X} - \langle T_{G} v, v \rangle_{Y} = 0.
$$

The proof is then completed. \qed

**Remark 3.33.** The interpolation $\mathbb{T}_{G}^{T} : D \to D_{G}^{abc}$ is both projective and strictly cell-wise defined with the minimal regularity requirement. Here, by “strictly cell-wise defined”, we mean with respect to any $K \in G$, $(\mathbb{T}_{G}^{T} v_{1})_{|K} = (\mathbb{T}_{G}^{T} v_{2})_{|K}$ for any $v_{1}, v_{2} \in D$ such that $v_{1}|_{K} = v_{2}|_{K}$.

The global stability and approximation follows directly from Lemma 3.30.

**Theorem 3.34.** Provided Assumption 3.1,

(a) $\|T_{G}^{T} v\|_{Y} \leq (1 + \beta_{G}^{-1})\|T_{G} v\|_{Y},$ and $\|T_{G}(v - T_{G}^{T} v)\|_{Y} \leq (2 + \beta_{G}^{-1}) \inf_{w \in \mathbb{P}(G)} \|T_{G}(v - w)\|_{Y};$

(b) with $\gamma_{G} := \inf_{0 \neq w \in \mathbb{P}(G)} \sup_{v \in \mathbb{P}(G)} \frac{\langle v, T_{G} v \rangle_{X} - \langle T_{G} v, v \rangle_{Y}}{\|v\|_{T_{G}} \|v\|_{T_{G}}} \text{ and } \rho_{G} := \text{icr}(T_{G}, P(G)),$

$\|\mathbb{T}_{G}^{T} v\|_{T_{G}} \leq (2 + \rho_{G} + \gamma_{G}^{-1} + \beta_{G}^{-1})\|v\|_{T_{G}},$ and $\|v - \mathbb{T}_{G}^{T} v\|_{T_{G}} \leq (3 + \rho_{G} + \gamma_{G}^{-1} + \beta_{G}^{-1})\|v - w\|_{T_{G}}.$
Remark 3.35. The construction of $\mathbb{T}_g^T$ depends on $(T_g, \mathbf{P}(G))$ and $(T_g, \mathbf{P}(G))$, but not on $D$ or $\mathbb{D}$, neither on $D_G$ or $\mathbb{D}_G$. Meanwhile, the constants in Theorem 3.34 depend on $(T_g, \mathbf{P}(G))$ and $(T_g, \mathbf{P}(G))$, but not on $D$ or $\mathbb{D}$, neither on $D_G$ or $\mathbb{D}_G$.

Global commutative diagram. Also, a global interpolation is defined as

$$(3.21) \quad \int_{T_g}^S \left[ \sum_{k \in G} H(S_k) \to \mathbf{Q}(G) \right] \quad \text{by} \quad (\int_{T_g}^S \mathbf{q})|_k = \int_{T_g}^S (\mathbf{q})|_k, \forall K \in G.$$ 

Lemma 3.36. Provided the assumptions of Lemma 3.31, $T_g^T \mathbf{v} = \mathbb{I}^T \mathbf{v}$ for $\mathbf{v} \in \prod_{K \in G} H(T_K)$.

3.3.3. Optimal ABCD. Let $u$ and $u_G$ be the solutions of (3.3) and (3.4), respectively. Then the error estimation reads

$$\|u - u_G\|_{T_g} \leq C \inf_{w_G \in \mathbf{P}(G)} \|u - w_G\|_{T_g} + \inf_{w_G \in D_G} \|T_g u - w_G\|_{T_g} \cdot \sup_{v_G \in D_G} \inf_{w_G \in D_G} \|v_G - w_G\|_{T_g}.$$ 

In general, $D_G^{abc} \subset D$, and

$$\|u - u_G\|_{T_g} \leq C \inf_{w_G \in \mathbf{P}(G)} \|u - w_G\|_{T_g} + \inf_{w_G \in D_G} \|T_g u - w_G\|_{T_g}.$$ 

Namely, up to a constant, the scheme can exploit the best approximation by $\mathbf{P}(G)$ to $D$, and the total accuracy depends on the approximation by $\mathbb{D}_G^c$ to $Tu$. It is interesting to note by Remark 3.35 that the constant $C$ does not rely on the choice of $D_G^c$. We can simply set $\mathbb{D}_G^C := \mathbb{D} \cap \mathbf{P}(G)$; this is the biggest possible conforming subspace, namely an optimal $CD$, and the obtained scheme bears equivalently the best accuracy among all ABCDs. The ABCD constructed this way is called the optimal ABCD.

3.4. Examples of ABCDs. Let $\Omega \subset \mathbb{R}^2$ be a simply-connected polygon. As usual, we use $\nabla$, curl, div and rot to denote the gradient operator, curl operator, divergence operator and rot operator, respectively. We use $H^1(\Omega)$, $H^1_0(\Omega)$, $H(\text{div}, \Omega)$, $H_0(\text{div}, \Omega)$, $H(\text{rot}, \Omega)$, $H_0(\text{rot}, \Omega)$, and $L^2(\Omega)$ to denote certain Sobolev spaces, and specifically, denote $L^2_0(\Omega) := \left\{ w \in L^2(\Omega) : \int_\Omega w dx = 0 \right\}$. For here, we denote vector-valued quantities by “_”. We use $(\cdot, \cdot)$ with subscripts to represent $L^2$ inner product. We here present some examples of partially adjoint discretizations, by $CD$ and $ABCD$, to the adjoint operators $[(\nabla, H^1(\Omega)), (\text{div}, H_0(\text{div}, \Omega))]$ and $[(\nabla, H^1_0(\Omega)), (\text{div}, H(\text{div}, \Omega))]$.

Let $T_h$ be a shape-regular triangular subdivision of $\Omega$ with mesh size $h$, such that $\overline{\Omega} = \bigcup_{T \in T_h} \overline{T}$, and every boundary vertex is connected to at least one interior vertex. Denote by $\mathcal{E}_h$, $\mathcal{E}_h^b$, $\mathcal{X}_h$, $\mathcal{X}_h^b$, and $\chi_h^b$, the set of edges, interior edges, boundary edges, vertices, interior vertices, boundary vertices and corners, respectively. We use the subscript “ $h$ ” to denote the dependence on triangulation. In particular, an operator with the subscript “ $h$ ” indicates that the operation is performed cell by cell.
Denote by $\mathbb{V}_h^1$ and $\mathbb{V}_h^2$ the continuous linear and quadratic element spaces, and by $V_{\text{rt}}^h$ and $V_{\text{bdm}}^h$ the Raviart-Thomas [39] and Brezzi-Douglas-Marini [14] element space of lowest degree on $T_h$. Denote $\mathbb{V}_h^{1,0} := \mathbb{V}_h^1 \cap H_0^1(\Omega)$, $\mathbb{V}_h^{2,0} := \mathbb{V}_h^2 \cap H_0^1(\Omega)$, $V_{\text{rt}}^{0,0} := V_{\text{rt}}^h \cap H_0(\text{div}, \Omega)$ and $V_{\text{bdm}}^{0,0} := V_{\text{bdm}}^h \cap H_0(\text{div}, \Omega)$. We also use $V_{\text{bdm}}^h$ and $V_{\text{bdm}}^{0,0}$ for the rotation of $V_{\text{bdm}}^h$ and $V_{\text{bdm}}^{0,0}$, respectively, which are conforming subspaces of $H(\text{rot}, \Omega)$ and $H_0(\text{rot}, \Omega)$, respectively.

3.4.1. A nonconforming $H(\text{div})$ element space by Raviart-Thomas’ shape functions. On a triangle $T$, denote $\mathbb{R}(T) := \text{span} \{ \alpha + \beta x : \alpha \in \mathbb{R}^2, \beta \in \mathbb{R} \}$. Then

$$
\mathcal{R}(\text{div}, \mathbb{R}(T)) = \mathbb{R} = \mathcal{N}(\nabla, P_1(T)), \quad \mathcal{N}(\text{div}, \mathbb{R}(T)) = \mathbb{R}^2 = \mathcal{R}(\nabla, P_1(T)).
$$

Denote

$$
\mathbb{R}(T_h) := \prod_{T \in T_h} \mathbb{R}(T), \quad \mathbb{P}_1(T_h) := \prod_{T \in T_h} P_1(T).
$$

Define

$$
\mathbb{R}_{h}^{\text{abc}} := \left\{ \mathbf{v} \in \mathbb{R}(T_h) : \sum_{T \in \mathcal{T}_h} (\mathbf{v}_T, \nabla \mathbf{v}_h)_T + (\text{div} \mathbf{v}_T, \mathbf{v}_h)_T = 0, \quad \forall \mathbf{v}_h \in \mathbb{V}_h^{1,0} \right\},
$$

and

$$
\mathbb{R}_{h}^{\text{abc}} := \left\{ \mathbf{v} \in \mathbb{R}(T_h) : \sum_{T \in \mathcal{T}_h} (\mathbf{v}_T, \nabla \mathbf{v}_h)_T + (\text{div} \mathbf{v}_T, \mathbf{v}_h)_T = 0, \quad \forall \mathbf{v}_h \in \mathbb{V}_h^1 \right\}.
$$

The proposition is easy to verify.

**Proposition 3.37.**

1. $[(\nabla, \mathbb{P}_1(T_h)), (\text{div}_h, \mathbb{R}(T_h))]$ is a base operator pair.

2. $[(\nabla, \mathbb{V}_h^{1,0}), (\text{div}_h, \mathbb{R}_{h}^{\text{abc}})]$ and $[(\nabla, \mathbb{V}_h^1), (\text{div}_h, \mathbb{R}_{h}^{\text{abc}})]$ are both partially adjoint operators based on $[(\nabla, \mathbb{P}_1(T_h)), (\text{div}_h, \mathbb{R}(T_h))]$.

3. $(\text{div}_h, \mathbb{R}_{h}^{\text{abc}})$ is an ABCD of $(\nabla, H(\text{div}, \Omega))$ and $(\text{div}_h, \mathbb{R}_{h}^{\text{abc}})$ is an ABCD of $(\nabla, H_0(\text{div}, \Omega))$.

Locally-supported basis functions of $\mathbb{R}_{h}^{\text{abc}}$ and $\mathbb{R}_{h}^{\text{abc}}$. Note that the spaces $\mathbb{V}_h^1$ and $\mathbb{V}_h^{1,0}$ each admits a set of locally support basis functions, the supports of which are each a patch around a vertex (for $\mathbb{V}_h^1$) or around an interior vertex (for $\mathbb{V}_h^{1,0}$). See Figure 1 for a reference. For $a \in \mathcal{X}_h$, denote by $\psi_a$ the basis function of $\mathbb{V}_h^1$ supported on the patch around $a$. By following the procedure of Theorem 3.26, a set of locally supported basis functions can be constructed for $\mathbb{R}_{h}^{\text{abc}}$ as well as $\mathbb{R}_{h}^{\text{abc}}$ by two steps. And we illustrate the supports below as well.

**Step 1.** On a cell $T \in T_h$ with vertices $a_i$ and opposite edges $e_i$, $i = 1:3$, let $\lambda_i$, $i=1:3$, be the barycentric coordinates, and let $S$ denote its area; cf. Figure 2, left. Denote

$$
\mathbf{b}_{T}^{a_i} := \frac{1}{2S}(x + a_j - a_j - a_k), \quad i = 1, 2, 3, \quad \{i, j, k\} = \{1, 2, 3\}.
$$

Then, $\{\mathbf{b}_{T}^{a_i}, i = 1, 2, 3\}$ form a set of linearly independent basis functions of $\mathbb{R}(T)$. Particularly, $\mathbf{b}_{T}^{a_i} \cdot \mathbf{n} \mid_{e_j} = (1 - 2\delta_{ij})/|e_j|$, and

$$
(\mathbf{b}_{T}^{a_i}, \nabla \lambda_j)_T + (\text{div} \mathbf{b}_{T}^{a_i}, \lambda_j)_T = \delta_{ij}, \quad 1 \leq i, j \leq 3.
$$

(3.22)
We note that $\lambda_j = \psi_{a_j}|_T$.

**Step 2.** The basis functions on every cell are combined associated with $\psi_a$ to form a global basis function, with $a \in E^i_h$ for $\mathbb{R}^{abc}_T h_0$ (with $a \in E^0_h$ for $\mathbb{R}^{abc}_{h_0}$, respectively). Finally, the constructed basis functions fall into two categories.

- **Category–I:** Supported on successive cells. $\mathbb{R}^{abc}_{T h_0}$ consists of basis functions of this type.

  On a patch of a vertex (see Figure 1 for patches of interior vertices $A$ and $B$ and a patch of boundary vertex $C$) which consists of $m$ cells, there are $(m-1)$ such basis functions; see Figure 3 for illustrations. Here $A$ is an interior vertex; we recall Figure 1. The support of $\psi_A$ consists of 6 triangles. The related basis functions then take the formulation $\sum_{i=1}^{6} \gamma_i b^A_T$, such that

  \[
  \sum_{i=1}^{6} (\gamma_i b^A_T, \nabla(\psi_A|_T))_T + (\gamma_i \mathrm{div} b^A_T, \psi_A|_T)_T = 0.
  \]

  Then 5 linearly independent basis functions can be solved out, and they are each supported on a two-cell patch contained in the support of $\psi_A$. See Figure 4(b) for illustration on a patch of a boundary vertex $C$ (cf. Figure 1).

- **Category–II:** Supported on a single cell. $\mathbb{R}^{abc}_h$ consists of basis functions of this type added on $\mathbb{R}^{abc}_{h_0}$.
Figure 3. A is an interior vertex; cf. Figure 1. Five basis functions associated with the interior vertex A. The shadowed parts are respectively the supports of the basis functions.

On a patch of a boundary vertex (see Figure 1 for a patch of boundary vertex C) which consists of $m$ cells, there are $m$ such basis functions. See Figure 4(a) for illustration on a patch of a boundary vertex C.

Figure 4. C is boundary vertex with a three-cell patch; cf. Figure 1. The basis functions associated with C are: (a) with no essential boundary condition, supported on a single cell; (b) with essential boundary condition, supported on the shadowed parts.

Remark 3.38. As shown in Figure 1 (illustrated by Figure 5), a same pair of neighbored triangles can be covered in both the supports of $\psi_A$ and $\psi_B$. We have to note that, the basis functions of $RT_h^{abc}$ supported thereon associated with $A$ consists of $b_T^A$ and $b_T^A$, while the basis functions associated with $B$ consists of $b_T^B$ and $b_T^B$. Further, basis functions associated with different vertices are all linearly independent.

Cell-wise-defined global interpolation. By virtue of Definition 3.28, the cell-wise interpolator $I^RT_T$ is defined by

$$(I^RT_T, \nabla v)_T + (\text{div} I^RT_T, v)_T = (\tau, \nabla v)_T + (\text{div} \tau, v)_T, \ \forall v \in P_1(T), \ \text{for} \ \tau \in H(\text{div}, T).$$
By noting the duality (3.22), we have precisely

$$\mathbb{I}_T^{RT} = \sum_{i=1}^{3} \left[ (\tau, \nabla \lambda_i)_T + (\text{div} \tau, \lambda_i)_T \right] b_T^a.$$

On the triangulation $\mathcal{T}_h$, define the global interpolator by

$$\mathbb{I}_T^{RT}_h : \prod_{T \in \mathcal{T}_h} H(\text{div}, T) \to \mathbb{R}^{T}(\mathcal{T}_h), \quad (\mathbb{I}_T^{RT}_h \tau)_T = \mathbb{I}_T^{RT} \tau|_T, \forall T \in \mathcal{T}_h.$$

Then

$$\| \sigma - \mathbb{I}_h^{RT} \sigma \|_{\text{div}_h} \leq C \inf_{\tau_h \in \mathbb{R}^{T}(\mathcal{T}_h)} \| \sigma - \tau_h \|_{\text{div}_h},$$

$C$ depending on the regularity of the triangulation only. Further

$$\mathcal{R}(\mathbb{I}_h^{RT}, H(\text{div}, \Omega)) \subset \mathbb{R}^{T_{abc}}_h, \quad \text{and} \quad \mathcal{R}(\mathbb{I}_h^{RT}, H_0(\text{div}, \Omega)) \subset \mathbb{R}^{T_{abc}}_h.$$

**Discrete Helmholtz decomposition.** By Theorem 3.8, a discrete Helmholtz decomposition holds that

$$\prod_{T \in \mathcal{T}_h} (P_0(T))^2 = \mathcal{R}(\nabla, \mathcal{V}_{h_0}^1) \oplus^+ \mathcal{N}(\text{div}_h, \mathbb{R}^{T_{abc}}_h) = \mathcal{R}(\nabla, \mathcal{V}_{h}^1) \oplus^+ \mathcal{N}(\text{div}_h, \mathbb{R}^{T_{abc}}_h).$$

3.4.2. *The Crouzeix-Raviart element: reinterpretation.* Temporarily in this section, we use the superscript $\perp$ to denote rotation. Namely, $\mathcal{X}_h^\perp = (-y, x)$. On a triangle $T$, denote $\mathbb{R}^{T \perp}(T) := \text{span} \left\{ \alpha + \beta \mathcal{X}_h^\perp : \alpha \in \mathbb{R}^2, \beta \in \mathbb{R} \right\}$. Then,

$$\mathcal{R}(\text{rot}, \mathbb{R}^{T \perp}(T)) = \mathcal{N}(\text{curl}, P_1(T)), \quad \text{and} \quad \mathcal{N}(\text{rot}, \mathbb{R}^{T \perp}(T)) = \mathcal{R}(\text{curl}, P_1(T)).$$

Denote $\mathbb{R}^{T \perp}(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} \mathbb{R}^{T \perp}(T)$, and denote

$$\mathbb{R}^{T \perp}_h := \mathbb{R}^{T \perp}(\mathcal{T}_h) \cap H(\text{rot}, \Omega), \quad \text{and} \quad \mathbb{R}^{T \perp}_{h_0} := \mathbb{R}^{T \perp}(\mathcal{T}_h) \cap H_0(\text{rot}, \Omega).$$
Lemma 3.39. Given $v_h \in \mathbb{P}^1_h$, $v_h \in V^\text{CR}_h$ ($v_h \in V_{h0}^\text{CR}$, respectively) if and only if it holds for $v_h$ and any $T_h \in \mathbb{RT}_{h0}^+$ (any $T_h \in \mathbb{RT}_h^+$, respectively) that

$$ (3.23) \quad \sum_{T \in T_h} \int_T (\text{curl} v_h \cdot T_h - v_h \text{rot} T_h) = 0. $$

Remark 3.40. The identity (3.23) is usually called a discrete Green formula and is far well known for any $v_h \in V_{h0}^\text{CR}$ (or $v_h \in V_h^\text{CR}$) and any $T_h \in \mathbb{RT}_{h0}^+$ ($T_h \in \mathbb{RT}_h^+$, respectively). Though, the "if" part of the lemma, which is also evident, has not been seen discussed in literature.

Proposition 3.41. (1) $[(\text{rot}, \mathbb{RT}_h^+), (\text{curl}_h, V_{h0}^\text{CR})]$ and $[(\text{rot}, \mathbb{RT}_{h0}^+), (\text{curl}_h, V_h^\text{CR})]$ are both partially adjoint based on $[(\text{rot}_h, \mathbb{RT}_h^+(T_h)), (\text{curl}_h, \mathbb{P}_1(T_h))]$.

(2) $(\text{curl}_h, V_{h0}^\text{CR})$ is an ABCD of $(\text{curl}, H_0^1(\Omega))$ and $(\text{curl}_h, V_h^\text{CR})$ is an ABCD of $(\text{curl}, H^1(\Omega))$.

By following the procedure of Theorem 3.26, we can construct a set of locally-supported global basis functions for $V_{h0}^\text{CR}$ and $V_h^\text{CR}$. This set of basis functions are identical to the standard basis functions of the Crouzeix-Raviart element. Similarly, by following Definition 3.28 and (3.20), we can define local and global interpolators $\mathbb{T}_T^\text{CR}$ and $\mathbb{I}_h^\text{CR}$, they coincide to the standard Crouzeix-Raviart interpolators for $H^1(T)$ and $H^1(\Omega)$ functions, respectively.

Again and in the sequel, for a cell $T \in T_h$, we use $a_i$ and $e_i$ for the vertices and opposite edges, $h_i$ be the height on $e_i$, $i = 1:3$. Let $a_i$, be the barycentric coordinates, and let $|e_i|$ and $|h_i|$ denote the length of $e_i$ and $h_i$, respectively, $i=1,2,3$. Denote $\psi_i := \frac{1}{|h_i|} (x^i - x_k^i)$, $i=1:3$. Then $\psi_j$ are the restrictions of the respective global basis functions of $\mathbb{RT}_h^+$, and $\psi_i \cdot t_j = \delta_{ij}$ on $e_j$, $t_j$ being the unit tangential vector along $e_j$. Denote

$$ b_k := (a_i + a_j - a_k)|e_k|, \quad \{i, j, k\} = \{1, 2, 3\}, \quad k = 1, 2, 3. $$

Then

$$ (3.24) \quad (\psi_i, \text{curl} b_j)_T - (\text{rot} \psi_j, b_j)_T = \delta_{ij}, \quad 1 \leq i, j \leq 3. $$

By virtue of Definition 3.28, the cell-wise interpolator $\mathbb{T}_T^\text{CR}$ is defined by

$$ (\mathbb{T}_T^\text{CR} v, \text{rot} \eta)_T - (\text{curl} v, \eta)_T = (v, \text{rot} \eta)_T - (\text{curl} v, \eta)_T, \quad \forall \eta \in \mathbb{RT}_h^+(T), \quad v \in H^1(T). $$

By noting the duality (3.24), we have precisely

$$ \mathbb{T}_T^\text{CR} v = \sum_{i=1}^{3} [(v, \text{rot} \psi_i)_T - (\text{curl} v, \psi_i)_T] b_i = \sum_{i=1}^{3} (\int_{e_i} v)b_i. $$

This is exactly the classical Crouzeix-Raviart interpolators for $H^1(T)$ functions. The nodal parameter of the C-R element has an interpretation that

$$ \int_{e_i} v = \int_{e_i} v \psi_j \cdot t_j = \sum_{j=1}^{3} \int_{e_j} v \psi_j \cdot t_j = (v, \text{rot} \psi_j)_T - (\text{curl} v, \psi_j)_T. $$
The nodal parameter indeed collects the information of the interpolated function along the whole boundary of $T$ or even the whole cell $T$ by the aid of the “dual” basis $\psi_i$.

By Lemma 3.36, the local and global commutative diagrams hold.

**Lemma 3.42.** Denote by $P_T$ the $L^2$ projection to the space of constants on a triangle $T$, and by $P_h$ the $L^2$ projection to the space of piecewise constants on a grid $\mathcal{T}_h$. The below exact complexes and commutative diagrams hold:

\[
\begin{align*}
0 & \rightarrow H^1(\Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{\text{curl}} 0; \\
0 & \rightarrow V^\text{CR}_h \xrightarrow{\text{curl}_h} R^\text{abc}_h \xrightarrow{\text{div}_h} \prod_{T \in \mathcal{T}_h} P_0(T) \xrightarrow{\text{curl}} 0.
\end{align*}
\]

**Proof.** The commutative diagrams follow by Lemmas 3.31 and 3.36. The exactness of the complexes follow from the exactness of the dual complexes. Particularly, for example, the dual complex of

\[
\begin{align*}
0 & \rightarrow H^1(\Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \xrightarrow{\text{curl}} 0; \\
0 & \rightarrow V^\text{CR}_h \xrightarrow{\text{curl}_h} R^\text{abc}_h \xrightarrow{\text{div}_h} \prod_{T \in \mathcal{T}_h} P_0(T) \xrightarrow{\text{curl}} 0.
\end{align*}
\]

is

\[
\begin{align*}
0 & \rightarrow L^2_0(\Omega) \cap \prod_{T \in \mathcal{T}_h} P_0(T) \xrightarrow{\text{curl}} 0.
\end{align*}
\]

As $\Omega$ is simply connected, the latter complex is well known to be exact, and by Theorem 3.9, so is the former one. The proof is completed. \qed

3.4.3. A nonconforming $H^1$ element space by enhanced quadratic polynomials. Let $T \in \mathcal{T}_h$ be a triangle, and denote on $T$

\[
\psi_B := \lambda_1 \lambda_2 (\lambda_1 - \lambda_2) + \lambda_2 \lambda_3 (\lambda_2 - \lambda_3) + \lambda_3 \lambda_1 (\lambda_3 - \lambda_1)
\]

and

\[
\psi_0 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 - \frac{1}{6}.
\]
Note that \((\text{curl} \psi_4) \cdot \mathbf{n}|_{\partial} = \lambda \lambda,\) \(\{i, j, k\} = \{1, 2, 3\},\) and \((\nabla \psi_0, \tau) + (\psi_0, \text{div} \tau) = 0,\) \(\forall \tau \in P_1(T).\)

Therefore, with respect to \(P_1(T),\) set \(P_{0}^{2}(T) := \text{span} \{\psi_0\}.\) Decompose, \(P_{0}^{2}(T) = P_{0}^{2}(T) \oplus P_{B}^{2}(T),\)

where \(P_{B}^{2}(T) := \{v \in P_{0}^{2}(T) : (\nabla v, \nabla \psi_0) = 0\}.

Evidently, \(\mathcal{R}(\text{div}, P_1(T)) = \mathcal{N}(\nabla, P_{B}^{2}(T)).\) On the other hand, \(\mathcal{N}(\text{div}, P_1(T))\) and \(\mathcal{R}(\nabla, P_{B}^{2}(T))\) are isomorphic: it is easy to verify that, given \(0 \neq \tau \in \mathcal{R}(\nabla, P_{B}^{2}(T)),\) there exists an \(\eta \in \mathcal{N}(\text{div}, P_1(T))\) such that \((\tau, \eta)_{\tau} \neq 0;\) given \(\eta \in \mathcal{N}(\text{div}, P_1(T)),\) there exists a \(0 \neq \tau \in \mathcal{R}(\nabla, P_{B}^{2}(T)),\) such that \((\tau, \eta)_{\tau} \neq 0.

Define

\[
V_{e}^{FS} := \left\{ v_h \in \prod_{T \in \mathcal{T}_h} P_{0}^{2}(T) : \sum_{T \in \mathcal{T}_h} (v_h, \text{rot} \tau)_T + (\text{curl} v_h, \tau)_T = 0, \ \forall \tau \in V_{e}^{BDM} \right\},
\]

and

\[
V_{e0}^{FS} := \left\{ v_h \in \prod_{T \in \mathcal{T}_h} P_{0}^{2}(T) : \sum_{T \in \mathcal{T}_h} (v_h, \text{rot} \tau)_T + (\text{curl} v_h, \tau)_T = 0, \ \forall \tau \in V_{e}^{BDM} \right\}.
\]

Note that the finite element spaces possess the same continuity as that of the Fortin-Soulie element space \([25]\), and we thus call them enriched Fortin-Soulie element spaces.

**Proposition 3.43.**

1. \([\text{curl}_h, \prod_{T \in \mathcal{T}_h} P_{0}^{2}(T)), (\text{rot}_h, \prod_{T \in \mathcal{T}_h} P_1(T))\) is a base operator pair.

2. \([\text{curl}_h, V_{e}^{FS}), (\text{rot}_h, V_{e0}^{BDM})\] \(\text{and} \) \([\text{curl}_h, V_{e0}^{FS}), (\text{rot}_h, V_{e0}^{BDM})\] are partially adjoint based on \([\text{curl}_h, \prod_{T \in \mathcal{T}_h} P_{0}^{2}(T)), (\text{rot}_h, \prod_{T \in \mathcal{T}_h} P_1(T))].\)

3. \((\text{curl}_h, V_{e}^{FS})\) is an \(\text{ABCD}\) of \((\text{curl}, H^1(\Omega))\) and \((\text{curl}_h, V_{e0}^{FS})\) is an \(\text{ABCD}\) of \((\text{curl}, H_0^{1}(\Omega)).\)

**3.4.4. A nonconforming \(H(\text{div})\) element space by enhanced Brezzi-Douglas-Marini’s shape functions.** Let \(T\) be a triangle. Denote

\[
P_{1}^{2}(T) := P_1(T) \oplus \text{span} \{\text{curl} \psi_B\}.
\]

Evidently, \(\mathcal{R}(\text{div}, P_{1}^{2}(T)) = \mathcal{N} = \mathcal{N}(\nabla, P_{2}(T)).\) On the other hand, it can be shown by elementary calculus that

\[
\inf_{\tau \in \mathcal{N}(\text{div}, P_{1}^{2}(T))} \sup_{\eta \in \mathcal{R}(\text{div}, P_{2}(T))} \frac{(\tau, \eta)_{\tau}}{||\tau||_{0,T} ||\eta||_{0,T}} = \inf_{\eta \in \mathcal{N}(\nabla, P_{3}(T))} \sup_{\tau \in \mathcal{N}(\text{div}, P_{1}^{2}(T))} \frac{(\tau, \eta)_{\tau}}{||\tau||_{0,T} ||\eta||_{0,T}} \geq C > 0,
\]

with \(C\) depending on the regularity of \(T\) only.

Define

\[
V_{e}^{BDM} := \left\{ \tau_h \in \prod_{T \in \mathcal{T}_h} P_{1}^{2}(T) : \sum_{T \in \mathcal{T}_h} (\tau_h, \nabla v_h)_T + (\text{div} \tau_h, v_h)_T = 0, \ \forall v_h \in V_{e0}^{2} \right\},
\]

and

\[
V_{e}^{BDM} := \left\{ \tau_h \in \prod_{T \in \mathcal{T}_h} P_{1}^{2}(T) : \sum_{T \in \mathcal{T}_h} (\tau_h, \nabla v_h)_T + (\text{div} \tau_h, v_h)_T = 0, \ \forall v_h \in V_{e0}^{2} \right\}.
\]
Proposition 3.44. (1) \( \left( \text{div}, \prod_{T \in \mathcal{T}_h} P_1^+(T), (\nabla, \prod_{T \in \mathcal{T}_h} P_2(T)) \right) \) is a base operator pair.

(2) \( \left( \text{div}, V_{\text{eBDM}}, (\nabla, V^2_{\text{h0}}) \right) \) and \( \left( \text{div}, V_{\text{eBDM}}, (\nabla, V^2_{\text{h0}}) \right) \) are both partially adjoint based on

\[ \left( \text{div}, \prod_{T \in \mathcal{T}_h} P_1^+(T), (\nabla, \prod_{T \in \mathcal{T}_h} P_2(T)) \right). \]

(3) \( \text{div}, V_{\text{eBDM}} \) is an ABCD of \( \text{div}, H(\text{div}, \Omega) \) and \( \text{div}, V_{\text{h0}}^{\text{eBDM}} \) is an ABCD of \( \text{div}, H_0(\text{div}, \Omega) \).

By following the procedure of Theorem 3.26, a set of locally supported basis functions can be constructed for \( V_{\text{eFS}} \) as well as \( V_{\text{h0}}^{\text{eFS}} \), and \( V_{\text{h0}}^{\text{eBDM}} \) as well as \( V_{\text{eBDM}} \). Similarly, by following Definition 3.28 and (3.20), we can define interpolators \( I_{\text{h}}^{\text{eFS}} \) and \( I_{\text{h}}^{\text{eBDM}} \).

Similar to Lemma 3.42, we can prove the local and global exact complexes and commutative diagrams.

Lemma 3.45. Denote by \( P_T \) the \( L^2 \) projection to the space of constants on a triangle \( T \), and by \( P_h \) the \( L^2 \) projection to the space of piecewise constants on a grid \( \mathcal{T}_h \). The below exact complexes and commutative diagrams hold:

\[
\begin{array}{cccccccc}
\mathbb{R} & \rightarrow & H^1(T) & \xrightarrow{\text{curl}} & H(\text{div}, T) & \xrightarrow{\text{div}} & L^2(T) & \xrightarrow{f} & \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R} & \rightarrow & P_2^+(T) & \xrightarrow{\text{curl}} & P_1^+(T) & \xrightarrow{\text{div}} & \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\
\end{array}
\]

(3.30)

\[
\begin{array}{cccccccc}
\mathbb{R} & \rightarrow & H^1(\Omega) & \xrightarrow{\text{curl}} & H(\text{div}, \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \xrightarrow{f} & \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{R} & \rightarrow & V_{\text{eFS}}(\Omega) & \xrightarrow{\text{curl}} & V_{\text{h0}}^{\text{eFS}} & \xrightarrow{\text{div}} & \prod_{T \in \mathcal{T}_h} P_0(T) & \xrightarrow{f} & \mathbb{R} \\
\end{array}
\]

(3.31)

\[
\begin{array}{cccccccc}
\{0\} & \rightarrow & H^1(\Omega) & \xrightarrow{\text{curl}} & H(\text{div}, \Omega) & \xrightarrow{\text{div}} & L^2(\Omega) & \xrightarrow{f} & \{0\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \rightarrow & V_{\text{h0}}^{\text{eFS}} & \xrightarrow{\text{curl}} & V_{\text{h0}}^{\text{eBDM}} & \xrightarrow{\text{div}} & L^2(\Omega) \cap \prod_{T \in \mathcal{T}_h} P_0(T) & \xrightarrow{f} & \{0\} \\
\end{array}
\]

(3.32)

4. Nonconforming Whitney \( k \)-forms for exterior differential operators

4.1. \( L^2 \) theory of exterior calculus revisited. Following [1], we denote by \( \Lambda^k(\Xi) \) the space of differential \( k \)-forms on an \( n \)-dimensional domain \( \Xi \), and \( L^2 \Lambda^k(\Xi) \) consists of differential \( k \)-forms with coefficients in \( L^2(\Xi) \) component by component. \( L^2 \Lambda^k(\Xi) \) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle_{L^2 \Lambda^k} \). The exterior differential operator \( d^k : \Lambda^k(\Xi) \rightarrow \Lambda^{k+1}(\Xi) \) is an unbounded operator from \( L^2 \Lambda^k(\Xi) \) to \( \Lambda^{k+1}(\Xi) \). Denote, for \( 0 \leq k \leq n-1 \),

\[
H \Lambda^k(\Xi) := H(d^k : L^2 \Lambda^k(\Xi) \rightarrow L^2 \Lambda^{k+1}(\Xi)) = \left\{ \omega \in L^2 \Lambda^k(\Xi) : d^k \omega \in L^2 \Lambda^{k+1}(\Xi) \right\},
\]
then $HA^k(\Xi)$ is a Hilbert space with the norm $\|\omega\|_{L^2(\Lambda^k(\Xi))} + \|d^k\omega\|_{L^2(\Lambda^{k+1}(\Xi))}$. Denote by $H_0A^k(\Xi)$ the closure of $C_0^\infty \Lambda^k(\Xi)$ in $HA^k(\Xi)$.

The Hodge star operator $\star$ maps $L^2\Lambda^k(\Xi)$ isomorphically to $L^2\Lambda^{n-k}(\Xi)$ for each $0 \leq k \leq n$. The codifferential operator $\delta_k$ defined such that $\star \delta_k \omega = (-1)^k d^{n-k} \star \omega$ is unbounded from $\Lambda^k(\Xi)$ to $\Lambda^{k-1}(\Xi)$. Denote

$$H^* \Lambda^k(\Xi) := H(\delta_k : L^2\Lambda^k(\Xi) \to L^2\Lambda^{k-1}(\Xi)) = \left\{ \mu \in L^2\Lambda^k(\Xi) : \delta_k \mu \in L^2\Lambda^{k-1}(\Xi) \right\},$$

and $H_0^* \Lambda^k(\Xi)$ the closure of $C_0^\infty \Lambda^k(\Xi)$ in $H^* \Lambda^k(\Xi)$. Then $H^* \Lambda^k(\Xi) = \star HA^{n-k}(\Xi)$, and $H_0^* \Lambda^k(\Xi) = \star H_0A^{n-k}(\Xi)$. Further $N(d^k, HA^k) = \star N(\delta_{n-k}, H^* \Lambda^{n-k})$, and $R(d^{k-1}, HA^{k-1}) = \star R(\delta_{n-k+1}, H^* \Lambda^{n-k+1})$.

**Lemma 4.1.** \textsuperscript{[1]} The adjoint of

(1) $(d^k,HA^k(\Xi))$ is $(\delta_{k+1}, \star H_0A^{n-k-1}(\Xi))$;

(2) $(d^k,H_0A^k(\Xi))$ is $(\delta_{k+1}, \star H^* \Lambda^{n-k-1}(\Xi))$;

(3) $(\delta_{k+1}, \star H^* \Lambda^{n-k-1}(\Xi))$ is $(d^k,H_0A^k(\Xi))$;

(4) $(\delta_{k+1}, \star H_0A^{n-k-1}(\Xi))$ is $(d^k,HA^k(\Xi))$.

It is well known that $R(d^k, HA^k(\Xi))$ is closed in $L^2\Lambda^{k+1}(\Xi)$, $0 \leq k \leq n-1$. Meanwhile, $R(d^k, H_0A^k(\Xi))$, $R(\delta_{k+1}, H_0^* \Lambda^{k+1}(\Xi))$, and $R(\delta_{k+1}, H^* \Lambda^{k+1}(\Xi))$ are all closed. In the sequel, we will drop the domain $\Xi$ when there is no ambiguity brought in.

Lemmas 4.2 and 4.3 follow from Lemma 4.1 immediately.

**Lemma 4.2** (Helmholtz decomposition). For $1 \leq k \leq n$

$$L^2\Lambda^k = R(d^{k-1}, HA^{k-1}) \oplus N(\delta_k, \star H_0A^{n-k}) = R(d^{k-1}, H_0A^{k-1}) \oplus N(\delta_k, \star HA^{n-k}),$$

and for $0 \leq k \leq n-1$

$$L^2\Lambda^k = N(d^k, HA^k) \oplus R(\delta_{k+1}, \star H_0A^{n-k-1}) = N(d^k, H_0A^k) \oplus R(\delta_{k+1}, \star HA^{n-k-1}).$$

The complex structures hold that

$$R(d^{k-1}, HA^{k-1}) \subset N(d^k, HA^k), \text{ and, } R(d^{k-1}, H_0A^{k-1}) \subset N(d^k, H_0A^k).$$

Denote, for $1 \leq k \leq n-1$,

$$\mathcal{H}A^k := N(d^k, HA^k) \oplus R(d^{k-1}, HA^{k-1}), \text{ and } \mathcal{H}_0A^k := N(d^k, H_0A^k) \oplus R(d^{k-1}, H_0A^{k-1}).$$

**Lemma 4.3** (Poincaré-Lefschetz duality). For $1 \leq k \leq n-1$

$$\mathcal{H}A^k = \star \mathcal{H}_0A^{n-k}, \text{ and, } \mathcal{H}_0A^k = \star \mathcal{H}A^{n-k}.$$

Now we pay particular attention on the isomorphisms between border spaces (4.2) which are not well presented yet. Denote $H^A^k := \{ \omega \in HA^k : d^k \omega = 0 \}$, $H_0^A^k := \{ \omega \in H_0A^k : d^k \omega = 0 \}$, and, $H^A^{k+1} := \{ \omega \in HA^{k+1} : \langle \omega, \mu \rangle_{L^2(\Lambda^k)} = 0, \forall \mu \in HA^k \}$. Define the border space by

$$H_B^A^k := \{ \omega \in HA^k : \langle d^k \omega, d^k \mu \rangle_{L^2(\Lambda^{k+1})} = 0, \forall \mu \in H_0A^k, \langle \omega, \mu \rangle_{L^2(\Lambda^k)} = 0, \forall \mu \in H_0^A^k \}.$$

Then by the closedness of $d^k$ and $\delta_k$,

$$R(d^k, H_B^A^k) = R(d^k, HA^k) \oplus R(d^k, H_0A^k), \text{ and } N(d^k, H_B^A^k) = N(d^k, HA^k) \oplus N(d^k, H_0A^k).$$
The lemma below holds as a horizontal $P\cdot A\cdot L$ type duality by Theorem 2.3 immediately.

**Lemma 4.4.** $\mathcal{R}(d^{k-1}, \star H_B \Lambda^{k-1}) = N(\delta_k, \star H_B \Lambda^{n-k}); \ N(d^k, H_B \Lambda^k) = \mathcal{R}(\delta_{k+1}, \star H_B \Lambda^{n-k-1}).$

Further, denote

$$\hat{H}_B \Lambda^k := \left\{ \omega \in \hat{H} \Lambda^k : \langle \omega, \mu \rangle_{L^2 \Lambda^k} = 0, \forall \mu \in \hat{H}_0 \Lambda^k \right\},$$

and

$$H_B^\perp \Lambda^k := \left\{ \omega \in H^\perp \Lambda^k : \langle d^k \omega, d^k \mu \rangle_{L^2 \Lambda^{k+1}} = 0, \forall \mu \in H_0 \Lambda^k \right\}.$$

The following two lemmas hold immediately.

**Lemma 4.5.** $H_B \Lambda^k = \hat{H}_B \Lambda^k \oplus H_B^\perp \Lambda^k$, orthogonal in $L^2 \Lambda^k$.

**Lemma 4.6.** $d^{k-1}$ is an isomorphism from $H_B^\perp \Lambda^{k-1}$ onto $\star \hat{H}_B \Lambda^{n-k}$, and $\delta_k$ is an isomorphism from $\star H_B^\perp \Lambda^{n-k}$ onto $\hat{H}_B \Lambda^{k-1}$.

**Remark 4.7.** The spaces $H_B \Lambda^{k-1}$ and $\star H_B \Lambda^{n-k}$ are isomorphic to the trace spaces of $H \Lambda^{k-1}$ and $\star H \Lambda^{n-k}$, respectively. Though, the intrinsic metric of the trace spaces can be subtle, while the metrics of $H_B \Lambda^{k-1}$ and $\star H_B \Lambda^{n-k}$ are clear. We illustrate Lemmas 4.4 and 4.6 with the decomposition below:

$$H_B \Lambda^{k-1} = \hat{H}_B \Lambda^{k-1} \oplus \delta_k \delta_{k-1} \cdots \delta_2 \delta_1 \delta_0 = \star H_B \Lambda^{n-k}.$$

### 4.2. ABCDs by Whitney $k$-forms of exterior differential operators.

#### 4.2.1. Basics of Whitney $k$-forms.

Following [3], denote by $\mathcal{P} \cdot \Lambda^k$ the space of differential $k$-forms with polynomial coefficients of degree of at most $r$, denote by $\kappa$ the Koszul differential operator which maps a $k$-form $\omega$ to a $(k-1)$-form obtained by contracting $\omega$ with the identity vector field. Particularly, let $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n$, then

$$\kappa(dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k}) = \sum_{1 \leq j < k} (-1)^{j+1} x^{\alpha_j} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_j-1} \wedge dx^{\alpha_{j+1}} \wedge \cdots \wedge dx^{\alpha_k}.$$

The trimmed spaces of polynomial forms of degree $r \geq 1$ are defined as:

$$\mathcal{P}^{-r} \Lambda^{0} = \mathcal{P} \cdot \Lambda^{0}, \ \mathcal{P}^{-r} \Lambda^{k} := \left\{ \omega \in \mathcal{P} \cdot \Lambda^{k} : \kappa \omega \in \mathcal{P}^{-r} \Lambda^{k-1} \right\}, \ 1 \leq k \leq n.$$

**Lemma 4.8.** ([3]) The polynomial de Rham complex

$$\mathcal{P}^{-r} \Lambda^{0} \xrightarrow{d} \mathcal{P}^{-r} \Lambda^{1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}^{-r} \Lambda^{n} \rightarrow 0$$

is exact.

**Lemma 4.9.** $\mathcal{R}(d^k, \mathcal{P}^{-r} \Lambda^k) = N(\delta_{k+1}, \star \mathcal{P}^{-r} \Lambda^{n-(k+1)}); \ \mathcal{R}(\delta_{k+1}, \star \mathcal{P}^{-r} \Lambda^{n-(k+1)}) = N(d^k, \mathcal{P}^{-r} \Lambda^k).$
4.2.3. Particularly, with $\tilde{d}^{-1} \Lambda_n^k - (k+1) \Lambda^k$. Similarly, $\mathcal{R}(d^k, \mathcal{P}_1 \Lambda^k) = N(\delta_{k+1}, \mathcal{P}_1^{-1} \Lambda^{n-(k+1)})$. The proof is completed. \hfill \Box

4.2.2. Whitney k-forms. Now, let $\Omega$ be an $n$-dimensional Lipschitz domain and $\mathcal{F}^\mathcal{G} := \{G_h\}$ be a family of shape-regular and quasi-uniform subdivisions of $\Omega$ by simplexes. Denote

$$P_1 \Lambda^k(\mathcal{G}_h) := \bigcap_{T \in \mathcal{G}_h} P_1 \Lambda^k(T), \quad \text{and} \quad P_0 \Lambda^k(\mathcal{G}_h) := \bigcap_{T \in \mathcal{G}_h} P_0 \Lambda^k(T).$$

The well-known lowest-degree conforming trimmed polynomial finite element spaces, known as the Whitney k-forms, are defined as

$$(4.6) \quad W_h^k \Lambda^k := \bigcap_{T \in \mathcal{G}_h} P_1^{-1} \Lambda^k(T) \cap H \Lambda^k, \quad \text{and} \quad W_{h0}^k \Lambda^k := W_h^k \Lambda^k \cap H_0 \Lambda^k.$$

Lemma 4.10. [3] There exists a constant $C_{k,n}$ uniform for $\mathcal{F}^\mathcal{G}$, such that

$$\text{icr}(d^k, W_h^k \Lambda^k) \leq C_{k,n}, \quad \text{and} \quad \text{icr}(d^k, W_{h0}^k \Lambda^k) \leq C_{k,n}.$$  

Remark 4.11. We can find Lemma 4.10 proved in Section 5 of [3] but in the formulation of discrete Poincaré inequalities. We further note that, for any $k$ and $n$, there is a constant $C_{k,n} > 0$, such that $\text{icr}(d^k, W_h^k \Lambda^k) \geq C_{k,n}$ and $\text{icr}(d^k, W_{h0}^k \Lambda^k) \geq C_{k,n}$ when $h$ is small enough.

Lemma 4.12. [1] $\mathcal{R}(d^k, W_h^k \Lambda^k) \subset N(d^{k+1}, W_h^k \Lambda^k)$ and $\mathcal{R}(d^k, W_{h0}^k \Lambda^k) \subset N(d^{k+1}, W_{h0}^k \Lambda^k)$.

Define

$$(4.7) \quad \mathcal{H}_h^k \Lambda^k := N(d^k, W_h^k \Lambda^k) \ominus \mathcal{R}(d^{k-1}, W_h^k \Lambda^{k-1}), \quad \text{and} \quad \mathcal{H}_{h0}^k \Lambda^k := N(d^k, W_{h0}^k \Lambda^k) \ominus \mathcal{R}(d^{k-1}, W_{h0}^k \Lambda^{k-1}).$$

Lemma 4.13. [1] $\mathcal{H}_h^k \Lambda^k$ is isomorphic to $\mathcal{K} \Lambda^k$ and $\mathcal{H}_{h0}^k \Lambda^k$ is isomorphic to $\mathcal{K}_0 \Lambda^k$.

Remark 4.14. $\mathcal{H}_h^k \Lambda^k$ is isomorphic to $\star \mathcal{H}_{h0}^k \Lambda^{n-k}$ in the space $L^2 \Lambda^k$.

4.2.3. ACM by Whitney k-forms. For a subdivision $\mathcal{G}_h$, $d^k_h$ and $\delta_{k+1}$ denote the cell wise operations by $d^k$ and $\delta_{k+1}$, respectively. We follow the lines in Section 3.1 for adjoint operator pairs. Particularly, with $\widetilde{P}(\mathcal{G}) = P_1 \Lambda^k(\mathcal{G}_h)$ and $\overline{P}(\mathcal{G}) = *P_1^{-1} \Lambda^{n-(k+1)}(\mathcal{G}_h)$, we have $P(\mathcal{G}) = \{0\}$ and $\overline{P}(\mathcal{G}) = \{0\}$, which implies that the respective indices of closed range are zero. By Lemmas 4.8 and 4.9, Assumptions 3.1, 3.4, 3.5, and 3.6 can be verified. The relevant parameters evaluate as $\alpha_\mathcal{G} = \beta_\mathcal{G} = 1$, $\sigma_\mathcal{G} = \kappa_\mathcal{G} = 1$, and $\chi_\mathcal{G} = \varepsilon_\mathcal{G} = 1$ where as $\overline{\mathcal{H}}_\mathcal{G} = \overline{\delta}_\mathcal{G} = \{0\}$ and $\mathcal{H}_\mathcal{G} = \overline{\delta}_\mathcal{G} = \{0\}$. For simplicity and for the ease of readers, we just use here the notations defined in Section 3.1. The results in the remaining of this subsection follow by Section 3.1 directly, except Lemma 4.21, for which we will present a proof.
Lemma 4.15. \[ \left( d_h^k, P_{-1}^{-1}A^k(G_h), (\delta_{k+1,h}, \star P_{-1}^{-1}A^{n-(k+1)}(G_h)) \right) \] is a base operator pair.

Define
\[ W_h^k \Lambda^k := \left\{ \omega_h \in P_{-1}^{-1}A^k(G_h) : \langle \omega_h, \delta_{k+1,h} \mu_h \rangle_{L^2(A^k)} = 0, \forall \mu_h \in \star W_{h0}^k \Lambda^{n-k-1} \right\}, \]
and
\[ W_{h0}^k \Lambda^k := \left\{ \omega_h \in P_{-1}^{-1}A^k(G_h) : \langle \omega_h, \delta_{k+1,h} \mu_h \rangle_{L^2(A^k)} = 0, \forall \mu_h \in \star W_{h0}^k \Lambda^{n-k-1} \right\}. \]

Note that \( W_h^k \Lambda^k \) and \( W_{h0}^k \Lambda^k \) can be recognized as the classical Crouzeix-Raviart element spaces.

Theorem 4.16. Both \[ \left( d_h^k, W_h^k \Lambda^k \right), (\delta_{k+1,h}, \star W_{h0}^k \Lambda^{n-k-1}) \] and \[ \left( d_h^k, W_{h0}^k \Lambda^k \right), (\delta_{k+1,h}, \star W_{h0}^k \Lambda^{n-k-1}) \] are partially adjoint based on \[ \left( d_h^k, P_{-1}^{-1}A^k(G_h) \right), (\delta_{k+1,h}, \star P_{-1}^{-1}A^{n-k-1}(G_h)) \].

Proposition 4.17. \( W_h^k \Lambda^k \) and \( W_{h0}^k \Lambda^k \) each admits a set of locally supported basis functions.

We refer to Section 3.4.1 for an example with precise constructions of basis functions.

Lemma 4.18. \( R(d_h^k, W_h^k \Lambda^k) \subset N(d_h^{k+1}, W_h^k \Lambda^{k+1}) \) and \( R(d_h^k, W_{h0}^k \Lambda^k) \subset N(d_h^{k+1}, W_{h0}^k \Lambda^{k+1}) \).

Denote
\[ \mathcal{H}_{h}^{\text{abc}} \Lambda^k := N(d_h^k, W_h^k \Lambda^k) \otimes I \subset R(d_h^k, W_h^k \Lambda^{k-1}), \] and
\[ \mathcal{H}_{h0}^{\text{abc}} \Lambda^k := N(d_h^k, W_{h0}^k \Lambda^k) \otimes I \subset R(d_h^k, W_{h0}^k \Lambda^{k-1}). \]

Theorem 4.19. Let \( \mathbb{d}_h^k \) be defined by Definition 3.28 and (3.20). Then \( \mathbb{d}_h^k H \Lambda^k \subset W_h^k \Lambda^k \) and \( \mathbb{d}_h^k H_0 \Lambda^k \subset W_{h0}^k \Lambda^k \). Further, the commutative diagrams below hold.

\[
\begin{array}{ccccccccc}
\mathbb{R} & \rightarrow & H \Lambda^0 & \quad \rightarrow & H \Lambda^1 & \rightarrow & \cdots & \rightarrow & H \Lambda^n & \rightarrow & \mathbb{R} \\
\downarrow \mathbb{d}_h^0 & & \downarrow \mathbb{d}_h^1 & & \downarrow \mathbb{d}_h^{k-1} & & \downarrow \mathbb{d}_h^n \\
\mathbb{R} & \rightarrow & W_h^k \Lambda^0 & \rightarrow & W_h^k \Lambda^1 & \rightarrow & \cdots & \rightarrow & W_h^k \Lambda^n & \rightarrow & \mathbb{R}.
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_0 \Lambda^0 & \rightarrow & H_0 \Lambda^1 & \rightarrow & \cdots & \rightarrow & H_0 \Lambda^n & \rightarrow & 0 \\
\downarrow \mathbb{d}_h^0 & & \downarrow \mathbb{d}_h^1 & & \downarrow \mathbb{d}_h^{k-1} & & \downarrow \mathbb{d}_h^n \\
0 & \rightarrow & W_{h0}^k \Lambda^0 & \rightarrow & W_{h0}^k \Lambda^1 & \rightarrow & \cdots & \rightarrow & W_{h0}^k \Lambda^n & \rightarrow & 0.
\end{array}
\]

Moreover, if the de Rham complex below is exact:
\[ \mathbb{R} \rightarrow H \Lambda^0 \rightarrow H \Lambda^1 \rightarrow \cdots \rightarrow H \Lambda^n \rightarrow \mathbb{R}, \]
all the complexes aforementioned are exact.

Lemma 4.20. For \( \omega \in H \Lambda^k \) with \( C_{k,n} \) uniform for \( \mathcal{G} \),
\[ \| \omega - \mathbb{d}_h^k \omega \|_{d_h^k} \leq C_{k,n} \inf_{\mu_h \in \mathbb{W}_{h0}^k \Lambda^k} \| \omega - \mu_h \|_{d_h^k}, \text{ on } \mathcal{G}. \]
Lemma 4.21. For $0 \leq k \leq n - 1$, there exists a constant $C_{k,n}$, such that for $h$, the meshsize of $\mathcal{G}_h$, small enough,

$$(1 - C_{k,n}h) \text{icr}(\delta_{k+1}, \star \mathcal{W}_h \Lambda^{n-(k+1)}) \leq \text{icr}(d_h^k, \mathcal{W}_h \Lambda^k) \leq (1 + C_{k,n}h) \text{icr}(\delta_{k+1}, \star \mathcal{W}_h \Lambda^{n-(k+1)}),$$

and

$$(1 - C_{k,n}h) \text{icr}(\delta_{k+1}, \star \mathcal{W}_h \Lambda^{n-(k+1)}) \leq \text{icr}(d_h^k, \mathcal{W}_h \Lambda^k) \leq (1 + C_{k,n}h) \text{icr}(\delta_{k+1}, \star \mathcal{W}_h \Lambda^{n-(k+1)}).$$

Proof. We firstly prove that, for $T$ a simplex and $0 \leq k \leq n - 1$, by elementary calculus for polynomials, there exists a constant $\bar{C}_{k,n}$, depending on the regularity of $T$ only, such that $\text{icr}(d^k, \mathcal{P}_1 \Lambda^k(T)) \leq \bar{C}_{k,n} \cdot h_T$, $h_T$ the size of $T$. Indeed, note that

$$(4.10) \quad \mathcal{P}_1 \Lambda^k(T) = \mathcal{P}_0 \Lambda^k(T) + \kappa(\mathcal{P}_0 \Lambda^k(T)) = \text{span} \{ dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k} : \alpha \in \mathcal{I}k,n \}$$

$$\oplus^k \text{span} \left\{ \sum_{1 \leq j \leq k+1} (-1)^{j+1} \tilde{x}^{\alpha_j} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_{j+1}} \wedge \cdots \wedge dx^{\alpha_k+1} : \alpha \in \mathcal{I}k+1,n \right\}$$

$$:= \mathcal{P}_0 \Lambda^k(T) \oplus^k \mathcal{P}_0 \Lambda^k(T)^{\perp},$$

where we use $\mathcal{I}k,n$ to denote the set of $k$-indices $\alpha$ such that $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n$, and $\tilde{x}^{\alpha_j} = x^{\alpha_j} - c_{\alpha_j}$, such that $\int_T x^{\alpha_j} = 0$. Then $\text{icr}(d^k, \mathcal{P}_1 \Lambda^k(T)) = \text{icr}(d^k, \mathcal{P}_0 \Lambda^k(T)^{\perp})$. Now given $\mu \in \text{icr}(d^k, \mathcal{P}_0 \Lambda^k(T)^{\perp})$, we write

$$\mu = \sum_{\alpha \in \mathcal{I}k+1,n} C_{\alpha} \sum_{1 \leq j \leq k+1} (-1)^{j+1} \tilde{x}^{\alpha_j} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_{j+1}} \wedge \cdots \wedge dx^{\alpha_k+1},$$

and then

$$||d^k \mu||_{\mathcal{L}^2(\Lambda^{k+1}(T))}^2 = \left|\sum_{\alpha \in \mathcal{I}k+1,n} C_{\alpha} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_k+1} \right|^2_{\mathcal{L}^2(\Lambda^{k+1}(T))} = |T| \sum_{\alpha \in \mathcal{I}k+1,n} C_{\alpha}^2.$$ 

Further, with $C_{k,n}$ depending on the regularity of $T$,

$$||\mu||_{\mathcal{L}^2(\Lambda^k(T))}^2 \leq C_{k,n}h_T^2 \left|\sum_{\alpha \in \mathcal{I}k,n} C_{\alpha} \sum_{1 \leq j \leq k+1} ((-1)^{j+1}(\nabla \tilde{x}^{\alpha_j}) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{j-1}} \wedge dx^{\alpha_{j+1}} \wedge \cdots \wedge dx^{\alpha_k+1}) \right|^2_{\mathcal{L}^2(\Lambda^{k+1}(T))}$$

$$= C_{k,n}h_T^2 (k+1)|T| \sum_{\alpha \in \mathcal{I}k+1,n} C_{\alpha}^2 = C_{k,n}h_T^2 (k+1)||d^k \mu||_{\mathcal{L}^2(\Lambda^{k+1}(T))}^2.$$ 

This way, $\text{icr}(d^k, \mathcal{P}_1 \Lambda^k(T)) \leq \bar{C}_{k,n} \cdot h_T$. It then follows that

$$\text{icr}(d_h^k, \mathcal{P}_1 \Lambda^k(\mathcal{G}_h)) \leq \bar{C}_{k,n} \cdot h, \quad \text{icr}(\delta_{k+1,h}, \star \mathcal{P}_1 \Lambda^{n-(k+1)}(\mathcal{G}_h)) \leq \bar{C}_{n-(k+1),n} \cdot h.$$

Submit all relevant parameters into Theorem 3.7, and we obtain

$$\text{icr}(\delta_{k+1}, \star \mathcal{W}_h \Lambda^{n-(k+1)}) \leq \bar{C}_{k,n} \cdot h \leq \text{icr}(d_h^k, \mathcal{W}_h \Lambda^k) \leq \text{icr}(\delta_{k+1}, \star \mathcal{W}_h \Lambda^{n-(k+1)}) + \bar{C}_{k,n} \cdot h.$$
and
\[ \text{icr}(\delta_{k+1}, \star W_h^t \Lambda^{n-k}) - C_{k,h} \leq \text{icr}(d_h^k, W_{h0}^{abc} \Lambda^k) \leq \text{icr}(\delta_{k+1}, \star W_h^{t0} \Lambda^{n-k-1}) + C_{k,h}. \]

By noting the upper and lower bounds of \( \text{icr}(\delta_{k+1}, \star W_h^{t0} \Lambda^{n-k-1}) \) and \( \text{icr}(\delta_{k+1}, \star W_h^{t0} \Lambda^{n-k-1}) \), we obtain the assertion of Lemma 4.21. The proof is completed. \( \square \)

The lemma reveals that, the partially adjoint operators \( (\delta_{k+1}, \star W_h^{t0} \Lambda^{n-k-1}) \) and \( (d_h^k, W_h^{abc} \Lambda^k) \) have asymptotically the same indices of closed range up to a high order term as \( h \) tends to zero.

Denote
\[ W_{h0}^{t0} \Lambda^k := \{ \omega_h \in W_{h0}^{t0} \Lambda^k : d_h^k \omega_h = 0 \}, \quad W_{h0}^{t0} \Lambda^k := \{ \omega_h \in W_{h0}^{t0} \Lambda^k : \langle \omega_h, \mu_h \rangle_{L^2} = 0, \forall \mu_h \in W_{h0}^{t0} \Lambda^k \}, \]
\[ W_h^t \Lambda^k := \{ \omega_h \in W_h^t \Lambda^k : d_h^k \omega_h = 0 \}, \quad \text{and,} \quad W_h^{t0} \Lambda^k := \{ \omega_h \in W_h^{t0} \Lambda^k : \langle \omega_h, \mu_h \rangle_{L^2} = 0, \forall \mu_h \in W_h^{t0} \Lambda^k \}. \]

Denote accordingly
\[ W_{hB}^t \Lambda^k := \{ \omega_h \in W_{hB}^t \Lambda^k : \langle \omega_h, \mu_h \rangle_{L^2} = 0, \forall \mu_h \in W_{hB}^{t0} \Lambda^k, \langle d_h^k \omega_h, d_h^k \mu_h \rangle_{L^2} = 0, \forall \mu_h \in W_{hB}^{t0} \Lambda^k \}, \]
and
\[ W_{hB}^{abc} \Lambda^k := \{ \omega_h \in W_{hB}^{abc} \Lambda^k : \langle \omega_h, \mu_h \rangle_{L^2} = 0, \forall \mu_h \in W_{hB}^{abc} \Lambda^k, \langle d_h^k \omega_h, d_h^k \mu_h \rangle_{L^2} = 0, \forall \mu_h \in W_{hB}^{abc} \Lambda^k \}. \]

**Theorem 4.22.** The P-A-L type dualities hold.

(1) (Horizontal duality)
\[ \mathcal{R}(d_h^k, W_{hB}^{abc} \Lambda^k) = N(\delta_{k+1}, \star W_{hB}^{t0} \Lambda^{n-k-1}), \quad \text{and,} \quad \mathcal{R}(\delta_{k+1}, \star W_{hB}^{t0} \Lambda^{n-k-1}) = N(d_h^k, W_{hB}^{abc} \Lambda^k). \]

(2) (Discrete Poincaré-Lefschetz duality)
\[ \mathcal{H}_{hB}^{abc} \Lambda^k = \star \mathcal{H}_{h0}^{t0} \Lambda^{n-k}, \quad \text{and,} \quad \mathcal{H}_{hB}^{abc} \Lambda^k = \star \mathcal{H}_{h0}^{t0} \Lambda^{n-k}. \]

**Theorem 4.23.** The orthogonal decompositions of \( \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) \) hold:

(1) (Discrete Helmholtz decomposition.) For \( 1 \leq k \leq n \),
\[ \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) = \mathcal{R}(d_h^{k-1}, W_{hB}^{abc} \Lambda^{k-1}) \oplus^\perp N(\delta_{k+1}, \star W_{hB}^{t0} \Lambda^{n-k-1}) = \mathcal{R}(d_h^{k-1}, W_{h0}^{abc} \Lambda^{k-1}) \oplus^\perp N(\delta_k, \star W_h^{t0} \Lambda^{n-k}); \]
\( \text{for } 0 \leq k \leq n - 1, \)
\[ \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) = N(d_h^k, W_{hB}^{abc} \Lambda^k) \oplus^\perp \mathcal{R}(\delta_{k+1}, \star W_{hB}^{t0} \Lambda^{n-k-1}) = N(d_h^k, W_{h0}^{abc} \Lambda^k) \oplus^\perp \mathcal{R}(\delta_k, \star W_h^{t0} \Lambda^{n-k-1}). \]

(2) (Discrete Hodge decomposition.) For \( 1 \leq k \leq n - 1 \),
\[ \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) = \mathcal{R}(d_h^{k-1}, W_{hB}^{abc} \Lambda^{k-1}) \oplus^\perp \mathcal{H}_{h0}^{abc} \Lambda^k = \star \mathcal{H}_{h0}^{t0} \Lambda^{n-k} \oplus^\perp \mathcal{R}(\delta_{k+1}, \star W_{hB}^{t0} \Lambda^{n-k-1}) \]
\[ = \mathcal{R}(d_h^{k-1}, W_{h0}^{abc} \Lambda^{k-1}) \oplus^\perp \mathcal{H}_{h0}^{abc} \Lambda^k \oplus^\perp \mathcal{R}(\delta_{k+1}, \star W_{h0}^{abc} \Lambda^{n-k}). \]

**Remark 4.24.** The constructions of discrete Helmholtz decomposition illustrate that nonconforming finite element spaces are necessary and the internal continuity of the ABCDFES are proper.

4.3. Primal and dual discretizations of variational problems.
4.3.1. Primal and dual discretizations of the elliptic source problem. We consider the source problem: given \( f \in L^2 \Lambda^k \), find \( \omega \in H^A \), such that

\[
\langle d^k \omega, d^k \mu \rangle_{L^2 \Lambda^{k+1}} + \langle \omega, \mu \rangle_{L^2 \Lambda^k} = \langle f, \mu \rangle_{L^2 \Lambda^k}, \quad \forall \mu \in H^A.
\]

(Equation 4.13)

Evidently, \( d^k \omega \in H^A_n \Lambda^{k+1} = \ast H^A_n \Lambda^{n-(k+1)} \).

The dual formulation of the problem is then to find \((\sigma, \overline{\omega}) \in H^A_n \Lambda^{k+1} \times L^2 \Lambda^k \), such that

\[
\left\{ \begin{array}{l}
\langle \sigma, \tau \rangle_{L^2 \Lambda^{k+1}} - \langle \overline{\omega}, \delta_{k+1} \tau \rangle_{L^2 \Lambda^k} = 0, \\
\langle \delta_{k+1} \sigma, \overline{\mu} \rangle_{L^2 \Lambda^k} + \langle \overline{\omega}, \overline{\mu} \rangle_{L^2 \Lambda^k} = \langle f, \overline{\mu} \rangle_{L^2 \Lambda^k},
\end{array} \right.
\forall \tau \in \ast H^A_n \Lambda^{n-(k+1)}, \quad \forall \mu \in L^2 \Lambda^k
\]

(Equation 4.14)

Then \( \overline{\omega} = \omega \), and \( \sigma = d^k \omega = d^k \omega \).

Let \( P^k_h \) denote the \( L^2 \) projection onto \( P_0 \Lambda^k(\mathcal{G}_h) \). We consider respective discretizations:

- for (4.13): find \( \omega_h \in W^{abc}_h \Lambda^k \), such that

\[
\langle d^k \omega_h, d^k \mu_h \rangle_{L^2 \Lambda^{k+1}} + \langle \omega_h, \mu_h \rangle_{L^2 \Lambda^k} = \langle f, \mu_h \rangle_{L^2 \Lambda^k}, \quad \forall \mu_h \in W^{abc}_h \Lambda^k;
\]

- for (4.14): find \((\sigma_h, \overline{\omega}_h) \in \ast W^{t}_h \Lambda^{n-(k+1)} \times P_0 \Lambda^k(\mathcal{G}_h) \), such that

\[
\left\{ \begin{array}{l}
\langle \sigma_{h+1}, \overline{\omega}_h \rangle_{L^2 \Lambda^{k+1}} - \langle \overline{\omega}_h, \delta_{k+1} \tau_h \rangle_{L^2 \Lambda^k} = 0, \\
\langle \delta_{k+1} \sigma_h, \overline{\mu}_h \rangle_{L^2 \Lambda^k} + \langle \overline{\omega}_h, \overline{\mu}_h \rangle_{L^2 \Lambda^k} = \langle f, \overline{\mu}_h \rangle_{L^2 \Lambda^k},
\end{array} \right.
\forall \tau_h \in \ast W^{t}_h \Lambda^{n-(k+1)}, \quad \forall \mu_h \in P_0 \Lambda^k(\mathcal{G}_h)
\]

(Equation 4.16)

Note that we here use first-order numerical quadrature for the inner products denoted by a projection \( P^k_h \) or \( P^{k+1}_h \). We will not emphasize this again in the sequel. By discrete Poincaré inequalities, (4.16) is wellposed by standard way.

**Lemma 4.25.** Let \( \omega_h \) and \((\sigma_h, \overline{\omega}_h)\) be the solutions of (4.15) and (4.16), respectively. Then

\[
\overline{\omega}_h = P^k_h \omega_h, \quad P^k_h \omega_h + \delta_{k+1} \sigma_h = P^k_h f, \quad \text{and} \quad d^k \omega_h = P^{k+1}_h \sigma_h
\]

Proof. Let \( \omega_h \) be the solution of (4.15). For any \( \mu_h \in P_1^{-}\Lambda^k(\mathcal{G}_h) \) and \( \tau_h \in \ast W^{t}_h \Lambda^{n-(k+1)} \), with \( \overline{\sigma}_h \in \ast W^{t}_h \Lambda^{n-(k+1)} \), \( \overline{\omega}_h \in \ast W^{t}_h \Lambda^{n-(k+1)} \),

\[
\left\{ \begin{array}{l}
\langle d^k_h \omega_h, d^k_h \mu_h \rangle_{L^2 \Lambda^{k+1}} + \langle \omega_h, \mu_h \rangle_{L^2 \Lambda^k} + \langle \delta_{k+1} \sigma_h, \mu_h \rangle_{L^2 \Lambda^k} - \langle \overline{\omega}_h, d^k_h \mu_h \rangle_{L^2 \Lambda^{k+1}} = \langle f, \mu_h \rangle_{L^2 \Lambda^k}, \\
\langle \delta_{k+1} \tau_h, \omega_h \rangle_{L^2 \Lambda^k} - \langle \tau_h, d^k_h \omega_h \rangle_{L^2 \Lambda^{k+1}} = 0.
\end{array} \right.
\]

Choosing \( \mu_h \) to be any in \( P_0 \Lambda^k(\mathcal{G}_h) \), we obtain

\[
P^k_h \omega_h + \delta_{k+1} \overline{\sigma}_h = P^k_h f.
\]

Thus

\[
\langle d^k_h \omega_h, d^k_h \mu_h \rangle_{L^2 \Lambda^{k+1}} - \langle \overline{\sigma}_h, d^k_h \mu_h \rangle_{L^2 \Lambda^{k+1}} = 0, \quad \forall \mu_h \in P_1^{-}\Lambda^k(\mathcal{G}_h),
\]

and therefore, \( d^k \omega_h = P^{k+1}_h \overline{\sigma}_h \). Further, by the second equation of (4.17),

\[
\langle \tau_h, P^{k+1}_h \overline{\sigma}_h \rangle_{L^2 \Lambda^{k+1}} - \langle \delta_{k+1} \tau_h, P^k_h \omega_h \rangle_{L^2 \Lambda^k} = 0, \quad \forall \tau_h \in \ast W^{t}_h \Lambda^{n-(k+1)}.
\]

We have thus

\[
\left\{ \begin{array}{l}
\langle \overline{\sigma}_h, \overline{\omega}_h \rangle_{L^2 \Lambda^{k+1}} - \langle \overline{\omega}_h, \delta_{k+1} \tau_h \rangle_{L^2 \Lambda^k} = 0, \\
\langle \delta_{k+1} \sigma_h, \overline{\mu}_h \rangle_{L^2 \Lambda^k} + \langle \overline{\omega}_h, \overline{\mu}_h \rangle_{L^2 \Lambda^k} = \langle f, \overline{\mu}_h \rangle_{L^2 \Lambda^k}, \quad \forall \tau_h \in \ast W^{t}_h \Lambda^{n-(k+1)}.
\end{array} \right.
\]
Namely \((\bar{\sigma}_h, P_h^k \omega_h)\) solves (4.16). The proof is completed. \(\square\)

**Remark 4.26.** Lemma 4.25 reconstructs the equivalence between the primal and dual formulations. Moreover, given the solution of one of (4.15) and (4.16), the solution of the other can be computed cell by cell by Lemma 4.25.

**Remark 4.27.** We here focus on the adjoint equivalence between schemes. The accuracies of the schemes, numerical quadratures taken into consideration, can be studied in standard ways.

### 4.3.2. Primal and dual discretizations of elliptic eigenvalue problems.

We consider the eigenvalue problem: find \(\omega \in H\Lambda^k\) and \(\lambda \neq 0\), such that

\[
\langle d^k \omega, d^k \mu \rangle_{L^2 H^k} = \lambda \langle \omega, \mu \rangle_{L^2 H^k}, \quad \forall \mu \in H\Lambda^k.
\]

Provided that \((\omega, \mu)\) is such an eigenpair, denote \(\sigma = d^k \omega\), then \(\sigma \in H_0^* \Lambda^{k+1}\), and

\[
\langle \delta_{k+1} \sigma, \delta_{k+1} \tau \rangle_{L^2 H^k} = \lambda \langle \sigma, \tau \rangle_{L^2 H^k}, \quad \forall \tau \in H_0^* \Lambda^{k+1}.
\]

(4.18) and (4.19) are the primal and dual formulations for the nonzero eigenvalues of \(\delta_{k+1} d^k\), and the two have the same nonzero eigenvalues.

**Lemma 4.28.** Let \(P_h^k\) denote the \(L^2\) projection onto \(P^k \Lambda^k(G_h)\). Let \(\omega_h \in W_h^{abc} \Lambda^k\) and \(\lambda_h \neq 0\) be such that

\[
\langle d_h^k \omega_h, P_h^k \mu_h \rangle_{L^2 H^k} = \lambda_h \langle P_h^k \omega_h, P_h^k \mu_h \rangle_{L^2 H^k}, \quad \forall \mu_h \in W_h^{abc} \Lambda^k.
\]

Then there exists a unique \(\sigma_h \in W_{h,0}^t \Lambda^{n-(k+1)}\), such that

\[
\langle \delta_{k+1} \sigma_h, \delta_{k+1} \tau_h \rangle_{L^2 H^k} = \lambda_h \langle \sigma_h, \tau_h \rangle_{L^2 H^k}, \quad \forall \tau_h \in W_{h,0}^t \Lambda^{n-(k+1)}.
\]

Further, \(P_h^{k+1} \sigma_h = d_h^k \omega_h\) and \(P_h^k \omega_h = \delta_{k+1} \sigma_h\).

The proof is the same as that of Lemma 4.25, and we omit it here.

### 4.3.3. “Primal”, “dual”, and “completely” mixed discretizations of Hodge Laplacian.

Now, given \(f \in L^2 \Lambda^k\), we consider the Hodge Laplacian problem: find \(\omega \in H\Lambda^k \cap H_0^* \Lambda^k\), such that

\[
\langle d^k \omega, d^k \mu \rangle_{L^2 H^k} + \langle \delta_k \omega, \delta_k \mu \rangle_{L^2 H^k} = \langle f, P_{\mathcal{H}} f, \mu \rangle_{L^2 H^k}, \quad \forall \mu \in H\Lambda^k \cap H_0^* \Lambda^k.
\]

Here \(P_{\mathcal{H}}\) denotes the \(L^2\) projection to \(\mathcal{H} \Lambda^k\).

The classical mixed formulations can be based on that \(\omega \in H\Lambda^k\) or \(\omega \in H_0^* \Lambda^k\), and we here simply call them “primal” and “dual” mixed formulations. They are respectively

- “primal” mixed formulation: to find \((\omega^p, \zeta^p, \vartheta^p) \in H\Lambda^k \times H\Lambda^{k-1} \times \mathcal{H} \Lambda^k\), such that, for \((\mu, \eta, \zeta) \in H\Lambda^k \times H\Lambda^{k-1} \times \mathcal{H} \Lambda^k\),

\[
\begin{cases}
\langle \zeta^p, \eta \rangle_{L^2 H^{k-1}} = 0 \\
\langle \vartheta^p, \mu \rangle_{L^2 H^k} + \langle d^{k-1} \vartheta^p, \mu \rangle_{L^2 H^k} = 0 \\
\langle \zeta^p, \mu \rangle_{L^2 H^{k-1}} + \langle d^{k-1} \zeta^p, \mu \rangle_{L^2 H^{k-1}} + \langle d^k \omega^p, d^k \mu \rangle_{L^2 H^k} = \langle f, \mu \rangle_{L^2 H^k}.
\end{cases}
\]

(4.23)
“dual” mixed formulation: to find \((\omega^d, \sigma^d, \vartheta^d) \in H^0_0 \Lambda^k \times H^0_0 \Lambda^{k+1} \times \mathcal{H} \Lambda^k\), such that, for \((\mu, \tau, \varsigma) \in H^0_0 \Lambda^k \times H^0_0 \Lambda^{k+1} \times \mathcal{H} \Lambda^k\),

\[
\begin{align*}
\langle \sigma^d, \tau \rangle_{L^2 \Lambda^{k+1}} & \hspace{1cm} \langle \omega^d, \varsigma \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta^d, \mu \rangle_{L^2 \Lambda^k} & \hspace{1cm} -\langle \omega^d, \delta_{k+1} \tau \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta^d, \mu \rangle_{L^2 \Lambda^k} & \hspace{1cm} +\langle \delta_{k+1} \omega^d, \delta_{k+1} \mu \rangle_{L^2 \Lambda^k} = \langle f, \mu \rangle_{L^2 \Lambda^k}.
\end{align*}
\]

We here consider their respective discretizations:

- for (4.23): to find \((\omega^p_h, \sigma^p_h, \vartheta^p_h) \in W_h^{abc} \Lambda^k \times W_h^{abc} \Lambda^{k-1} \times \mathcal{H}_h^{abc} \Lambda^k\), such that, for \((\mu_h, \eta_h, \varsigma_h) \in W_h^{abc} \Lambda^k \times W_h^{abc} \Lambda^{k-1} \times \mathcal{H}_h^{abc} \Lambda^k\),

\[
\begin{align*}
\langle \sigma^p_h, \tau \rangle_{L^2 \Lambda^{k+1}} & \hspace{1cm} \langle \omega^p_h, \varsigma_h \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta^p_h, \mu_h \rangle_{L^2 \Lambda^k} & \hspace{1cm} -\langle \omega^p_h, \delta_{k+1} \tau \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta^p_h, \mu_h \rangle_{L^2 \Lambda^k} & \hspace{1cm} +\langle \delta_{k+1} \omega^p_h, \delta_{k+1} \mu_h \rangle_{L^2 \Lambda^k} = \langle f, \mu_h \rangle_{L^2 \Lambda^k}.
\end{align*}
\]

- for (4.24): to find \((\omega^d_h, \sigma^d_h, \vartheta^d_h) \in \star W_{h0}^t \Lambda^{n-k} \times \star W_{h0}^t \Lambda^{n-(k+1)} \times \mathcal{H}_{h0}^t \Lambda^k\), such that, for \((\mu_h, \tau_h, \varsigma_h) \in \star W_{h0}^t \Lambda^{n-k} \times \star W_{h0}^t \Lambda^{n-(k+1)} \times \mathcal{H}_{h0}^t \Lambda^k\),

\[
\begin{align*}
\langle \sigma^d_h, \tau \rangle_{L^2 \Lambda^{k+1}} & \hspace{1cm} \langle \omega^d_h, \varsigma \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta^d_h, \mu_h \rangle_{L^2 \Lambda^k} & \hspace{1cm} -\langle \omega^d_h, \delta_{k+1} \tau_h \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta^d_h, \mu_h \rangle_{L^2 \Lambda^k} & \hspace{1cm} +\langle \delta_{k+1} \omega^d_h, \delta_{k+1} \mu_h \rangle_{L^2 \Lambda^k} = \langle f, \mu_h \rangle_{L^2 \Lambda^k}.
\end{align*}
\]

We can further consider a “completely” mixed formulation: to find \((\omega, \sigma, \zeta, \vartheta) \in L^2 \Lambda^k \times H^0 \Lambda^{k+1} \times H \Lambda^{k-1} \times \mathcal{H} \Lambda^k\), such that, for \((\mu, \tau, \eta, \varsigma) \in L^2 \Lambda^k \times H^0 \Lambda^{k+1} \times H \Lambda^{k-1} \times \mathcal{H} \Lambda^k\),

\[
\begin{align*}
\langle \sigma, \tau \rangle_{L^2 \Lambda^{k+1}} & \hspace{1cm} \langle \omega, \varsigma \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta, \mu \rangle_{L^2 \Lambda^k} & \hspace{1cm} -\langle \omega, \delta_{k+1} \tau \rangle_{L^2 \Lambda^k} = 0 \\
\langle \vartheta, \mu \rangle_{L^2 \Lambda^k} & \hspace{1cm} +\langle \delta_{k+1} \omega, \delta_{k+1} \mu \rangle_{L^2 \Lambda^k} = \langle f, \mu \rangle_{L^2 \Lambda^k}.
\end{align*}
\]

**Lemma 4.29.** For \(f \in L^2 \Lambda^k\), the problem (4.27) admits a unique solution \((\omega, \sigma, \zeta, \vartheta)\), and

\[
\|\omega\|_{L^2 \Lambda^k} + \|\sigma\|_{\delta_{k+1}} + \|\varsigma\|_{\delta^{k+1}} + \|\vartheta\|_{L^2 \Lambda^k} \leq C\|f\|_{L^2 \Lambda^k}.
\]

Further, \(\sigma = \delta^k \omega\), \(\zeta = \delta_k \omega\), and \(\omega\) solves (4.22).

**Proof.** For (4.28), we only have to verify Brezzi’s conditions, which hold by the orthogonal Hodge decomposition

\[
L^2 \Lambda^k = R(\delta^{k-1}, H \Lambda^{k-1}) \oplus^+ \mathcal{H} \Lambda^k \oplus^+ R(\delta_{k+1}, H^0 \Lambda^{k+1}),
\]

with the closeness of \(R(\delta^{k-1}, H \Lambda^{k-1})\) and \(R(\delta_{k+1}, H^0 \Lambda^{k+1})\). The remaining assertions are straightforward. The proof is completed. \(\square\)
The discretization of (4.27) is: find \((\omega_h, \sigma_h, \zeta_h, \vartheta_h) \in \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) \times \star \mathcal{W}_h^k \Lambda^{n-(k+1)} \times \mathcal{W}_h^{abc} \Lambda^{k-1} \times \mathcal{K}_h^{abc} \Lambda^k\), such that, for \((\mu_h, \tau_h, \eta_h, \varsigma_h) \in \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) \times \star \mathcal{W}_h^t \Lambda^{n-(k+1)} \times \mathcal{W}_h^{abc} \Lambda^{k-1} \times \mathcal{K}_h^{abc} \Lambda^k\),

\[
\begin{cases}
\langle P_{h}^{k+1} \sigma_h, P_{h}^{k+1} \tau_h \rangle_{L^2 \Lambda^{k+1}} & \langle \omega_h, \varsigma_h \rangle_{L^2 \Lambda^k} = 0 \\
-\langle \omega_h, \delta_{k+1} \tau_h \rangle_{L^2 \Lambda^k} & = 0 \\
\langle P_{h}^{k+1} \zeta_h, P_{h}^{k-1} \eta_h \rangle_{L^2 \Lambda^{k-1}} & -\langle \omega_h, d_{h}^{k-1} \eta_h \rangle_{L^2 \Lambda^{k-1}} = 0 \\
+\langle \delta_{k+1} \sigma_h, \mu_h \rangle_{L^2 \Lambda^k} & +\langle d_{h}^{k-1} \zeta_h, \mu_h \rangle_{L^2 \Lambda^{k-1}} = \langle f, P_{h}^{k} \mu_h \rangle_{L^2 \Lambda^{k}}
\end{cases}
\]  

(4.29)

**Lemma 4.30.** Given \(f \in L^2 \Lambda^k\), the problem (4.29) admits a unique solution \((\omega_h, \sigma_h, \zeta_h, \vartheta_h)\), and

\[
\|\omega_h\|_{L^2 \Lambda^k} + \|\sigma_h\|_{L^2 \Lambda^{k-1}} + \|\zeta_h\|_{L^2 \Lambda^{k-1}} + \|\vartheta_h\|_{L^2 \Lambda^k} \leq C\|f\|_{L^2 \Lambda^k}.
\]

The constant \(C\) depends on \(\text{icr}(\delta_{k+1}, \star \mathcal{W}_h^t \Lambda^{n-(k+1)})\) and \(\text{icr}(d_{h}^{k-1}, \mathcal{W}_h^{abc} \Lambda^{k-1})\).

**Lemma 4.31.** Let \((\omega_h, \sigma_h, \zeta_h, \vartheta_h)\), \((\omega_p^h, \zeta_p^h, \vartheta_p^h)\) and \((\omega_d^h, \sigma_d^h, \vartheta_d^h)\) be the solutions of (4.29), (4.25) and (4.26), respectively. Then

\[
\langle \vartheta_h \rangle = \vartheta_d^h, \quad \langle \vartheta_p^h \rangle = \vartheta_p^h, \quad \langle \vartheta_d^h \rangle = \vartheta_d^h, \quad \langle \omega_h \rangle = \omega_d^h, \quad \langle \sigma_h \rangle = \sigma_d^h, \quad \langle \omega_p^h \rangle = \omega_p^h, \quad \text{and}, \quad \langle d_{h}^{k-1} \zeta_h \rangle = d_{h}^{k-1} \zeta_p^h
\]

(4.30)

**Proof.** Let \((\omega_d^h, \sigma_d^h, \vartheta_d^h)\) be the solution of (4.26). Then, with \(\widehat{\zeta}_h \in \mathcal{W}_h^{abc} \Lambda^{k-1}\),

\[
\langle \vartheta_d^h \rangle_{L^2 \Lambda^k} + \langle \delta_{k+1} \sigma_d^h, \mu_h \rangle_{L^2 \Lambda^k} + \langle \delta_{k} \omega_d^h, \sigma_d^h \rangle_{L^2 \Lambda^{k-1}} + \langle d_{h}^{k-1} \zeta_p^h, \mu_h \rangle_{L^2 \Lambda^{k-1}} = \langle f, P_{h}^{k} \mu_h \rangle_{L^2 \Lambda^{k}}
\]

for any \(\mu_h \in \star \mathcal{P}_1 \Lambda^{n-k}(\mathcal{G}_h)\). Choose particularly \(\mu_h \in \mathcal{P}_0 \Lambda^k(\mathcal{G}_h)\), we have

\[
\langle \vartheta_d^h \rangle + \langle \delta_{k+1} \sigma_d^h \rangle + \langle d_{h}^{k-1} \zeta_p^h \rangle = P_{h}^{k} f,
\]

and

\[
\langle \delta_{k} \omega_d^h, \delta_{k} \mu_h \rangle_{L^2 \Lambda^{k-1}} = 0, \quad \forall \mu_h \in \star \mathcal{P}_1 \Lambda^{n-k}(\mathcal{G}_h),
\]

which leads to that \(\delta_{k} \omega_d^h = P_{h}^{k} \widehat{\zeta}_h\). Further, noting that \(\langle \delta_{k} \omega_d^h, \eta_h \rangle_{L^2 \Lambda^{k-1}} = \langle \omega_d^h, d_{h}^{k-1} \eta_h \rangle_{L^2 \Lambda^{k-1}}\) for \(\eta_h \in \mathcal{W}_h^{abc} \Lambda^{k-1}\), we obtain \(\langle P_{h}^{k} \widehat{\zeta}_h, P_{h}^{k-1} \eta_h \rangle_{L^2 \Lambda^{k-1}} = \langle \omega_d^h, d_{h}^{k-1} \eta_h \rangle_{L^2 \Lambda^{k-1}} = 0\) for \(\eta_h \in \mathcal{W}_h^{abc} \Lambda^{k-1}\).

In all, \((P_{h}^{k} \omega_d^h, P_{h}^{k} \sigma_d^h, P_{h}^{k} \vartheta_d^h) \in \mathcal{P}_0 \Lambda^k(\mathcal{G}_h) \times \star \mathcal{W}_h^t \Lambda^{n-(k+1)} \times \mathcal{W}_h^{abc} \Lambda^{k-1} \times \mathcal{K}_h^{abc} \Lambda^k\) satisfies the system (4.29), and thus \((P_{h}^{k} \omega_d^h, P_{h}^{k} \sigma_d^h, P_{h}^{k} \vartheta_d^h) = (\omega_h, \sigma_h, \zeta_h, \vartheta_h)\). This proves (4.30), and (4.31) can be proved the same way. The proof is completed.

**Remark 4.32.** Given the solution of one of (4.29) (4.25) and (4.26), the solutions of the others can be computed cell by cell, by Lemma 4.31 and by the aid of (4.32).

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