Martingale Hardy spaces with variable exponents

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Aug. 29, 2014

Abstract

In this paper, we introduce Hardy spaces with variable exponents defined on a probability space and try to develop the martingale theory of variable Hardy spaces. Analogous to the classical theory, we prove some inequalities on Doob’s maximal operator and get a $(1, p(\cdot), \infty)$-atomic decomposition for Hardy martingale spaces associated with conditional square functions. As applications, we obtain the dual theorem and John-Nirenberg inequalities in the frame of variable exponents. The key ingredient is that we employ a condition without metric characterization of $p(\cdot)$ to replace the so-called log-Hölder continuity condition in $\mathbb{R}^n$.

1 Introduction

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function such that $0 < \inf_{x \in \mathbb{R}^n} p(x) \leq \sup_{x \in \mathbb{R}^n} p(x) < \infty$. The space $L^{p(\cdot)}(\mathbb{R}^n)$, the Lebesgue space with variable exponent $p(\cdot)$, is defined as the set of all measurable functions $f$ such that for some $\lambda > 0$

$$
\int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx < \infty,
$$

with

$$
\|f\|_{p(\cdot)} \equiv \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.
$$

Then $(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ is a quasi-normed space. Such Lebesgue spaces were introduced by Orlicz [23] in 1931 and studied by O. Kováčik and J. Rákosník [17], X. Fan and D. Zhao [12] and others. We refer to two new monographs [4] and [9] for the recent progress on Lebesgue spaces with variable exponents and some applications in PDEs and variational integrals with nonstandard growth conditions. We also note that in the recent years more attention was turned to the study of function spaces with variable exponent in harmonic analysis; see for instance [3, 6, 7, 10, 22, 25].

Yong Jiao is supported by the National Natural Science Foundation of China(11001273), the Research Fund for the Doctoral Program of Higher Education of China (20100162120035); Wei Chen is supported by the National Natural Science Foundation of China(11101353).

2000 Mathematics subject classification: Primary 60G46; Secondary 60G42.

Key words and phrases: martingales Hardy spaces, variable exponents, atomic decomposition, duality, John-Nirenberg inequalities.
Let $\Omega \subset \mathbb{R}^n$. We say that a function $p(\cdot) : \Omega \to \mathbb{R}$ is locally log-Hölder continuous on $\Omega$ if there exists $c_1 > 0$ such that

$$|p(x) - p(y)| \leq \frac{c_1}{\log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$. Heavily relying on the so-called log-Hölder continuity conditions on the variable exponent functions, in the pioneering work [8], Diening proved that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. An example in [21] showed that log-Hölder continuity of $p(\cdot)$ is essentially the optimal condition when the maximal operator is bounded on variable exponent Lebesgue spaces defined on Euclidean spaces (even in the doubling metric measure spaces; see [14]). On some more questions related to the maximal operator in variable $L^{p(\cdot)}$, we refer to [11] and [18].

Although variable exponent Lebesgue spaces on Euclidean space have attracted a steadily increasing interest over the last couple of years, the variable exponent framework has not yet been applied to the probability space setting. The purpose of the present paper is to introduce Hardy martingale spaces with variable exponent, and try to develop the martingale theory of variable Hardy spaces. For a convenience, we first fix some notations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{P} = \mathcal{P}(\Omega)$ denote the collection of all measurable functions $p(\cdot) : \Omega \to (0, \infty)$ which is called a variable exponent. For a measurable set $A \subset \Omega$, we denote

$$p_+(A) = \sup_{x \in A} p(x), \quad p_-(A) = \inf_{x \in A} p(x)$$

and

$$p_+ = p_+(\Omega), \quad p_- = p_-(\Omega).$$

Compared with Euclidean space $\mathbb{R}^n$, the probability space $\Omega$ has no natural metric structure. The main difficulty is how to overcome the log-Hölder continuity of $p(\cdot)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

The first aim of this paper is to discuss the weak type and strong type inequalities about Doob’s maximal operator. To the best of our knowledge, Aoyama [1] proved that Doob’s maximal inequality is true under some conditions. Namely, if $1 \leq p(\cdot) < \infty$ and there exists a constant $C$ such that

$$\frac{1}{p(\cdot)} \leq C \mathbb{E}\left(\frac{1}{p(\cdot)} |\mathcal{F}_n|\right),$$

then

$$\mathbb{P}\left(\sup_n |f_n| > \lambda\right) \leq C p(\cdot) \int_{\Omega} \left(\frac{|f_\infty|}{\lambda}\right)^{p(\cdot)} d\mathbb{P}, \quad \forall \lambda > 0.$$  \hspace{1cm} (1.2)

And if $1 < p_- \leq p_+ < \infty$ and $p(\cdot)$ is $\mathcal{F}_n$-measurable for all $n \geq 0$, then

$$\|\sup_n |f_n|\|_{p(\cdot)} \leq C p(\cdot) \|f\|_{p(\cdot)}.$$  \hspace{1cm} (1.3)

Obviously, the condition that $p(\cdot)$ is $\mathcal{F}_n$-measurable for all $n \geq 0$ is quite strict. In 2013, Nakai [21] pointed out that there exists a variable exponent $p(\cdot)$ such that $p(\cdot)$ is not $\mathcal{F}_0$-measurable, but (1.3) still holds. In this paper, we removing the condition (1.1), prove the weak type inequality (1.2). Unfortunately we cannot obtain (1.3) directly by the weak type inequality as the classical case in [20]. This is because that the space $L^{p(\cdot)}$ is no longer a rearrangement invariant space, and the formula

$$\int_{\Omega} |f(x)|^p d\mathbb{P} = p \int_0^\infty t^{p-1} \mathbb{P}(x \in \Omega : |f(x)| > t) dt$$
has no variable exponent analogue (see [9]). In order to describe the strong type Doob maximal inequality, we find the following condition without metric characterization of \( p(x) \) to replace log-Hölder continuity in some sense. That is, there exists an absolute constant \( K_{p(\cdot)} \geq 1 \) depending only on \( p(\cdot) \) such that

\[
P(A)^{p_-(A) - p_+(A)} \leq K_{p(\cdot)}, \quad \forall A \in \mathcal{F}.
\]

We often denote \( K_{p(\cdot)} \) simply by \( K \) if there is nothing confused. Under the condition of (1.4), we prove (1.3) is true for any martingale with respect to the atom \( \sigma \)-algebra filtrations. Let \( \mathcal{T} \) be the set of all stopping times with respect to \( \{ \mathcal{F}_n \}_{n \geq 0} \). For a martingale \( f = (f_n)_{n \geq 0} \) and \( \tau \in \mathcal{T} \), we denote the stopped martingale by \( f^\tau = (f_n^\tau)_{n \geq 0} = (f_n \wedge \tau)_{n \geq 0} \).

**Definition 1.1** Given \( p(\cdot) \in \mathcal{P} \). A measurable function \( a \) is called a \((1, p(\cdot), \infty)\)-atom if there exists a stopping time \( \tau \in \mathcal{T} \) such that

- (a1) \( \mathbb{E}(a|\mathcal{F}_n) = 0 \), \( \forall n \leq \tau \),
- (a2) \( \|a\|_\infty \leq \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}^{-1} \).

Denote by \( A(s, p(\cdot), \infty) \) be the set of all sequences of pair \((\mu_k, a^k, \tau_k)\), where \( \mu_k \) are nonnegative numbers, \( a^k \) are \((1, p(\cdot), \infty)\)-atoms satisfying (a1), (a2).

In the sequel we always denote \( p = \min\{p_-, 1\} \).

**Definition 1.2** Given \( p(\cdot) \in \mathcal{P} \). Let us denote by \( H^{s, at}_{p(\cdot)} \) the space of those martingales for which there exists a sequence \((a^k)_{k \in \mathbb{Z}}\) of \((1, p(\cdot), \infty)\)-atoms and a sequence \((\mu_k)_{k \in \mathbb{Z}}\) of nonnegative real numbers such that

\[
f = \sum_{k \in \mathbb{Z}} \mu_k a^k, \quad \text{a.e.,}
\]

and

\[
\left\| \left\{ \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right)^{\frac{p}{2}} \right\|_{p(\cdot)} \right\| < \infty.
\]

Let

\[
A(\{\mu_k\}, \{a^k\}, \{\tau_k\}) = \left\| \left\{ \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right)^{\frac{p}{2}} \right\|_{p(\cdot)} \right\|.
\]

We define

\[
\|f\|_{H^{s, at}_{p(\cdot)}} = \inf A(\{\mu_k\}, \{a^k\}, \{\tau_k\}), \quad (\mu_k, a^k, \tau_k) \in A(s, p(\cdot), \infty),
\]

where the infimum is taken over all decompositions of the form (1.5). In Section 4, we prove that

\[
H^s_{p(\cdot)} = H^{s, at}_{p(\cdot)}, \quad p(\cdot) \in \mathcal{P},
\]
with equivalent norms. See section 2 for the notation $H^s_{p(\cdot)}$. We give some applications of atomic decomposition in Section 5. Recall that the Lipschitz space $\Lambda_q(\alpha)(\alpha \geq 0, q \geq 1)$, is defined as the space of all functions $f \in L^q$ for which

$$\|f\|_{\Lambda_q(\alpha)} = \sup_{\tau} |\{ \tau < \infty \}|^{-\frac{1}{q} - \alpha} \|f - f_\tau\|_q < \infty.$$  

It was proved by Weisz in [28] that the dual space of $H^s_{p(0 < p \leq 1)}$ is equivalent to $\Lambda_2(\alpha)(\alpha = 1/p - 1)$. The new Lipschitz space $\Lambda_q(\alpha(\cdot))$ is introduced in section 5. Let $p(\cdot)$ satisfy (1.4), we obtain that

$$\left(H^s_{p(\cdot)}\right)^* = \Lambda_2(\alpha(\cdot)), \quad 0 < p_- \leq p_+ \leq 1,$$

where $\alpha(\cdot) = 1/p(\cdot) - 1$.

Finally we get the John-Nirenberg inequality in the frame of variable exponents. If $p(\cdot)$ satisfies (1.4), then

$$\|f\|_{BMO_1} \lesssim \|f\|_{BMO_{p(\cdot)}} \lesssim \|f\|_{BMO_1}, \quad 1 \leq p_- \leq p_+ < \infty,$$

which can be regarded as the probability versions of Theorem 3.1 in [15] and Theorem 1.2 or Theorem 5.1 in [16]. See section 5 for the definition of $BMO_{p(\cdot)}$. Furthermore, we also obtain the exponential integrability form of the John-Nirenberg inequality for $BMO_{p(\cdot)}$. We note that the generalized John-Nirenberg inequalities were proved in the frame of rearrangement invariant spaces in [29], but the variable $L^p(\cdot)$ spaces are not rearrangement invariant spaces generally. Again, the condition (1.4) plays an important role in the above results, which admits us to use small trick to estimate the $p(\cdot)$-norm of characterization function and makes inverse Holder’s inequalities available.

Throughout this paper, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ denote the integer set, nonnegative integer set and complex numbers set, respectively. We denote by $C$ the absolute positive constant, which can vary from line to line, and denote by $C_{p(\cdot)}$ the constant dependently only on $p(\cdot)$. The symbol $A \lesssim B$ stands for the inequality $A \leq CB$ or $A \leq C_{p(\cdot)}B$. If we write $A \approx B$, then it stands for $A \lesssim B \lesssim A$.

## 2 Preliminaries and lemmas

In this section, we give some preliminaries necessary to the whole paper. Given $p(\cdot) \in \mathcal{P}$, we always assume that $0 < p_- \leq p_+ < \infty$ if no special statement. The space $L^{p(\cdot)} = L^{p(\cdot)}(\Omega)$ is the collection of all measurable functions $f$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for some $\lambda > 0$,

$$\rho(f/\lambda) = \int_{\Omega} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \, d\mathbb{P} < \infty.$$  

This becomes a quasi-Banach function space when equipped with the quasi-norm

$$\|f\|_{p(\cdot)} \equiv \inf\{\lambda > 0 : \rho(f/\lambda) \leq 1\}.$$  

The following facts are well known; see for example [22].

1. (Positivity) $\|f\|_{p(\cdot)} \geq 0$; $\|f\|_{p(\cdot)} = 0 \iff f \equiv 0$. 

Lemma 2.2 (see [4], page 24) Given Lemma 2.1 (see [7], page 5) Let \( \Omega \), \( F \) for the classical martingale theory. Let \( (\Omega, F, P) \) relative to \( (\Omega, F, P) \) and \( p_+ > 1 \), we define the conjugate exponent \( p'(\cdot) \) by the equation
\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.
\]

We collect some useful lemmas as follows, which will be used in the paper.

**Lemma 2.1** (see [7], page 5) Let \( p(\cdot) \in \mathcal{P} \), and \( p_\geq 1 \) then for all \( r > 0 \), we have
\[
\|f^r\|_{p(\cdot)} = \|f\|_{r_{p(\cdot)}}.
\]

**Lemma 2.2** (see [4], page 24) Given \( p(\cdot) \in \mathcal{P} \), then for all \( f \in L^{p(\cdot)} \) and \( \|f\|_{p(\cdot)} \neq 0 \), we have
\[
\int_{\Omega} \left| \frac{f(x)}{\|f\|_{p(\cdot)}} \right|^{p(x)} dP = 1.
\]

**Lemma 2.3** (see [12], Theorem 1.3 or [4], page 22) Given \( p(\cdot) \in \mathcal{P} \) and \( f \in L^{p(\cdot)} \), then we have
\[
\begin{align*}
(1) & \quad \|f\|_{p(\cdot)} < 1(= 1, > 1) \text{ if and only if } \rho(f) < 1(= 1, > 1); \\
(2) & \quad \|f\|_{p(\cdot)} > 1, \text{ then } \rho(f)^{1/p^+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}; \\
(3) & \quad \|f\|_{p(\cdot)} \leq 1, \text{ then } \rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p^+}.
\end{align*}
\]

**Lemma 2.4** (see [4], H"{o}lder’s inequality) Given \( p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P} \), such that
\[
\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.
\]
Then there exists a constant \( C_{p(\cdot)} \) such that for all \( f \in L^{q(\cdot)}, g \in L^{r(\cdot)} \), and \( fg \in L^{p(\cdot)} \)
\[
\|fg\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{q(\cdot)} \|g\|_{r(\cdot)}.
\]

Now we introduce some standard notations from martingale theory. We refer to three books [13] [19] [27] for the classical martingale theory. Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space. Recall that the conditional expectation operator relative to \( \mathcal{F}_n \) is denoted by \( \mathbb{E}_{\mathcal{F}_n} \), i.e. \( \mathbb{E}(f|\mathcal{F}_n) = \mathbb{E}_{\mathcal{F}_n}(f) \). A sequence of measurable functions \( f = (f_n)_{n \geq 0} \subset L^1(\Omega) \) is called a martingale with respect to \( (\mathcal{F}_n) \) if \( \mathbb{E}_{\mathcal{F}_n}(f_{n+1}) = f_n \) for every \( n \geq 0 \). If in addition \( f_n \in L^{p(\cdot)} \), \( f \) is called an \( L^{p(\cdot)} \)-martingale with respect to \( (\mathcal{F}_n) \). In this case we set
\[
\|f\|_{p(\cdot)} = \sup_{n \geq 0} \|f_n\|_{p(\cdot)}.
\]
If \( \|f\|_{p(\cdot)} < \infty \), \( f \) is called a bounded \( L^{p(\cdot)} \)-martingale and denoted by \( f \in L^{p(\cdot)} \). For a martingale relative to \( (\Omega, \mathcal{F}, P; (\mathcal{F}_n)_{n \geq 0}) \), define the maximal function and the conditional square function of \( f \) respectively as follows \( (f_1 = f_0) \),
\[
M_m f = \sup_{n \leq m} |f_n|, \quad M f = \sup_{n \geq 1} |f_n|,
\]
Then there exists a constant $C$

Adding the above inequalities with $j$

We choose a sequence of simple functions

Proof. Lemma 3.1 Given $p(\cdot) \in \mathcal{P}$ and $1 \leq p_- \leq p_+ < \infty$. $f = (f_n)_{0 \leq n \leq \infty}$ is an $L^{p(\cdot)}$-martingale. Suppose that for any stopping time $\tau$

Then there exists a constant $C_{p(\cdot)}$ such that

\[
\mathbb{P}(\tau < \infty) \leq C_{p(\cdot)} \int_{\{\tau < \infty\}} \left( \frac{|f_\infty|}{\lambda} \right)^{p(x)} d\mathbb{P}, \quad \forall \lambda > 0.
\]

Proof. We choose a sequence of simple functions $\{s_n\}_{n \geq 1}$ such that $s_n \geq p_-\{\{\tau < \infty\}\}$ for any $n$ and the sequence $\{s_n\}_{n \geq 1}$ increases monotonically to $p(x)$ on $\{\tau < \infty\}$. Then for each $n$

\[s_n(x) = \sum_{j=1}^{k_n} \alpha_{n,j} \chi_{A_{n,j}}(x),\]

where the sets $\{A_{n,j}\}$ are disjoint and $\bigcup_{j} A_{n,j} = \{\tau < \infty\}$.

By Hölder’s inequality and Young’s inequality we have

\[
\int_{A_{n,j}} \frac{|f_\infty(x)|}{\lambda} d\mathbb{P} \leq \left( \int_{A_{n,j}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{\alpha_{n,j}} d\mathbb{P} \right)^{\frac{1}{\alpha_{n,j}}} \mathbb{P}(A_{n,j})^{\frac{1}{\alpha'_{n,j}}}
\]

\[
\leq \frac{1}{\alpha_{n,j}} \int_{A_{n,j}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{\alpha_{n,j}} d\mathbb{P} + \frac{\mathbb{P}(A_{n,j})}{\alpha'_{n,j}}
\]

\[
\leq \frac{1}{p_-\{\{\tau < \infty\}\}} \int_{A_{n,j}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{s_n(x)} d\mathbb{P} + \frac{\mathbb{P}(A_{n,j})}{p_+\{\{\tau < \infty\}\}^{p}}.
\]

Adding the above inequalities with $j$ from 1 to $k_n$, we have

\[
\int_{\{\tau < \infty\}} \frac{|f_\infty(x)|}{\lambda} d\mathbb{P} \leq \frac{1}{p_-\{\{\tau < \infty\}\}} \int_{\{\tau < \infty\}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{s_n(x)} d\mathbb{P} + \frac{\mathbb{P}(\tau < \infty)}{p_+\{\{\tau < \infty\}\}^{p}}.
\]
This inequality holds for all \( n \), hence the monotone convergence theorem implies that
\[
P(\tau < \infty) < \int_{\{\tau < \infty\}} \frac{|f_\infty|}{\lambda} \, d\mathbb{P} \leq \frac{1}{p_-(\{\tau < \infty\})} \int_{\{\tau < \infty\}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{p(x)} \, d\mathbb{P} + \frac{P(\tau < \infty)}{(p_+(\{\tau < \infty\}))'}.
\]

Since \( p_+ < \infty \), then \( (p_+(\{\tau < \infty\}))' > 1 \). It follows that
\[
P(\tau < \infty) \left(1 - \frac{1}{(p_+(\{\tau < \infty\}))'}\right) \leq \frac{1}{p_-(\{\tau < \infty\})} \int_{\{\tau < \infty\}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{p(x)} \, d\mathbb{P}.
\]

Therefore, by a simple calculation, we have
\[
P(\tau < \infty) \leq C_{p(\cdot)} \int_{\{\tau < \infty\}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{p(x)} \, d\mathbb{P}.
\]

The following theorem corresponds to Proposition 4 in [1].

**Theorem 3.2** Given \( p(\cdot) \in \mathbb{P} \) and \( 1 \leq p_- \leq p_+ < \infty \). Suppose that \( f = (f_n)_{0 \leq n \leq \infty} \) is a bounded \( L^{p(\cdot)} \)-martingale, then
\[
P(Mf > \lambda) \leq C_{p(\cdot)} \int_{\Omega} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{p(x)} d\mathbb{P}, \quad \forall \lambda > 0.
\]

**Proof.** For any \( \lambda > 0 \), we define a stopping time \( \tau = \inf\{n > 0 : |f_n| > \lambda\} \) (with the convention that \( \inf\emptyset = \infty \)). It is obvious that
\[
\{Mf > \lambda\} = \{\tau < \infty\},
\]
and
\[
\{\tau < \infty\} \subset \{|f_\tau| > \lambda\}.
\]

Note that \( \mathbb{E}_{\mathcal{F}_\tau} \left( \frac{|f_\infty|}{\lambda} \right) > 1 \) a.e. on the set \( \{\tau < \infty\} \). We get that
\[
P(\tau < \infty) = \int_{\{\tau < \infty\}} 1 \, d\mathbb{P} \leq \int_{\{\tau < \infty\}} \mathbb{E}_{\mathcal{F}_\tau} \left( \frac{|f_\infty|}{\lambda} \right) \, d\mathbb{P} = \int_{\{\tau < \infty\}} \frac{|f_\infty(x)|}{\lambda} \, d\mathbb{P}.
\]

It follows immediately from Lemma 3.1 that
\[
P(Mf > \lambda) = P(\tau < \infty) \leq C_{p(\cdot)} \int_{\{\tau < \infty\}} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{p(x)} \, d\mathbb{P} \leq C_{p(\cdot)} \int_{\Omega} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{p(x)} \, d\mathbb{P}.
\]
Lemma 3.3 Given \( p(\cdot) \in \mathcal{P} \). Then
\[
\left( \sup_{n \geq 0} |f_n| \right)^{p(\cdot)} = \sup_{n \geq 0} \left( |f_n|^{p(\cdot)} \right).
\]
This lemma is very obvious, however it will be used frequently below.

We now turn to consider the strong type inequality (1.3). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let
\[
\mathcal{D}_n = \{ A^n_j \}_{j \geq 1}, \quad \text{for each } n \geq 0,
\]
be decompositions of \( \Omega \) such that \((B_n)_{n \geq 0} = (\sigma(\mathcal{D}_n))_{n \geq 0}\) is increasing and \( \mathcal{F} = \sigma \left( \bigcup_{n \geq 0} B_n \right) \). It follows from [26] that
\[
\mathbb{E}_{B_n}(f) = \sum_{j=1}^{\infty} \left( \frac{1}{\mathbb{P}(A^n_j)} \int_{A^n_j} f(x) d\mathbb{P} \right) \chi_{A^n_j}.
\]
Then
\[
\int_{\Omega} (Mf)^{p(x)} d\mathbb{P} \leq \int_{\Omega} \sup_n \left\{ \sum_{j=1}^{\infty} \left( \frac{1}{\mathbb{P}(A^n_j)} \int_{A^n_j} |f(x)| d\mathbb{P} \right)^{p(x)} \right\} d\mathbb{P} (3.1)
\]
\[
= \int_{\Omega} \left\{ \sup_n \sum_{j=1}^{\infty} \left( \frac{1}{\mathbb{P}(A^n_j)} \int_{A^n_j} |f(x)| d\mathbb{P} \right)^{\frac{p(x)}{p_+}} \chi_{A^n_j} \right\} d\mathbb{P} (3.2)
\]

Lemma 3.4 Let \( p(\cdot) \in \mathcal{P}, 1 < p_- \leq p_+ < \infty \) and satisfy (1.4). Suppose that \( f \in L^{p(\cdot)} \) and \( \|f\|_{p(\cdot)} \leq 1/2 \). Then for all measurable sets \( B \),
\[
\left( \frac{1}{\mathbb{P}(B)} \int_B |f(x)| d\mathbb{P} \right)^{\frac{p(x)}{p_-}} \leq K \left( \frac{1}{\mathbb{P}(B)} \int_B |f(x)|^{\frac{p(x)}{p_-}} d\mathbb{P} + 1 \right).
\]

Proof. Let \( q(x) = p(x)/p_- \), then for any \( x \in B \),
\[
q(x) \leq p(x), \quad \text{and} \quad q_-(B) \leq p(x).
\]
Let \( f|_B(x) = f(x), x \in B \). Then \( \|f|_B\|_{p(\cdot)} \leq 1/2 \). Let \( g = f/\|f|_B\|_{p(\cdot)} \). It follows from Lemma 2.2 that
\[
\int_B \left| \frac{f(x)}{\|f|_B\|_{p(\cdot)}} \right|^{q_-(B)} d\mathbb{P} = \int_{B \cap \{|g| \geq 1\}} |g(x)|^{q_-(B)} d\mathbb{P} + \int_{B \cap \{|g| < 1\}} |g(x)|^{q_-(B)} d\mathbb{P}
\]
\[
\leq 1 + \mathbb{P}(\Omega).
\]
Then
\[
\|f|_B\|_{q_-(B)} \leq \left( 1 + \mathbb{P}(\Omega) \right)^{\frac{1}{q_-(B)}} \|f|_B\|_{p(\cdot)} \leq (1 + \mathbb{P}(\Omega)) \|f\|_{p(\cdot)} \leq 1.
\]
Using Hölder’s inequality and (1.4), we find that
\[
\left( \frac{1}{\mathbb{P}(B)} \int_B |f(x)| d\mathbb{P} \right)^{q(x)} \leq \left( \frac{1}{\mathbb{P}(B)} \int_B |f(x)|^{q(B)} d\mathbb{P} \right)^{q(x)/q(B)}.
\]

\[
= \mathbb{P}(B)^{-q(x)/q(B)} \|f\|_{q(B)}^{q(x)}.
\]

\[
\leq \mathbb{P}(B)^{-q(x)/q(B)} \|f\|_{q(B)}^{q(x)}.
\]

\[
= \mathbb{P}(B)^{-q(x)/q(B)} \|f\|_{q(B)}^{q(x)} \int_B |f(x)|^{q(B)} d\mathbb{P}
\]

\[
\leq \mathbb{P}(B)^{-q(x)/q(B)} \|f\|_{q(B)}^{q(x)} \int_B |f(x)|^{q(B)} d\mathbb{P}
\]

\[
= \mathbb{P}(B)^{-q(x)/q(B)} \|f\|_{q(B)}^{q(x)} \int_B |f(x)|^{q(B)} d\mathbb{P}
\]

\[
\leq K^{p(B)} \left( \frac{1}{\mathbb{P}(B)} \int_B (|f(x)|^{q(x)} + 1) d\mathbb{P} \right)
\]

\[
\leq K \left( \frac{1}{\mathbb{P}(B)} \int_B (|f(x)|^{q(x)} + 1) d\mathbb{P} \right).
\]

**Theorem 3.5** Let $D_n = \{A^n_j\}_{j \geq 1}$, for each $n \geq 0$, be decompositions of $\Omega$ such that $(B_n)_{n \geq 0} = (\sigma(D_n))_{n \geq 0}$ is increasing and $\mathcal{F} = \sigma \left( \bigcup_{n \geq 0} B_n \right)$. Let $p(\cdot)$ satisfy (1.4) and $1 < p_- \leq p_+ < \infty$. Then for any martingale $f \in L^p(\cdot)$ with respect to $(B_n)_{n \geq 0}$,

\[
\| \sup_n |f_n| \|_{p(\cdot)} \leq C_{p(\cdot)} \| f \|_{p(\cdot)}.
\]

**Proof.** We assume that $\|f\|_{p(\cdot)} \leq 1/2$ by homogeneity and let $q(x) = p(x)/p_-$. Then by Lemma 3.4 and the classical Doob maximal inequality

\[
\int_\Omega \left\{ \sup_n \sum_{j=1}^\infty \left( \frac{1}{\mathbb{P}(A^n_j)} \int_{A^n_j} |f(x)| d\mathbb{P} \right)^{q(x)/p}_- \right\} d\mathbb{P}
\]

\[
\leq \int_\Omega \left\{ \sup_n \sum_{j=1}^\infty K \left( \frac{1}{\mathbb{P}(A^n_j)} \int_{A^n_j} (|f(x)|^{p(x)} + 1) d\mathbb{P} \right)^{p(\cdot)} \right\} d\mathbb{P}
\]

\[
= K^{p_-} \left\| \sup_n \mathbb{E}_{B_n} (|f|^{q(\cdot)} + 1) \right\|_{p_-}
\]

\[
\leq C_{p_-} K^{p_-} \left\| |f|^{q(\cdot)} + 1 \right\|_{p_-} \leq C
\]

By (3.2), we have $\int_\Omega (Mf)^{p(\cdot)} d\mathbb{P} \leq C$. Now the proof is complete.

In the time of this writing, we do not know if the condition (1.4) is sufficient for the Doob maximal inequality in generalized probability spaces.

**Problem 3.6** Let $p(\cdot)$ satisfy (1.4) with $1 < p_- \leq p_+ < \infty$. Then for any martingale $f \in L^p(\cdot)$ with respect to $(F_n)_{n \geq 0}$,

\[
\| \sup_n |f_n| \|_{p(\cdot)} \leq C_{p(\cdot)} \| f \|_{p(\cdot)}^?.
\]
Remark 3.7 It is well known that $|\mathcal{E}_{\mathcal{F}_n}(f)|^p \leq E_{\mathcal{F}_n}(|f|^p)$ for $1 \leq p < \infty$. However, it is easy to give inverse examples to show that one can never expect a variable exponent version, namely,

$$
|\mathcal{E}_{\mathcal{F}_n}(f)|^{p(\cdot)} \leq C_{p(\cdot)E_{\mathcal{F}_n}(|f|^{p(\cdot)})}, \quad 1 \leq p(\cdot) < \infty.
$$

Hence the essential difficulty to deal with Problem 3.6 is how to overcome or avoid the use of the inequality (3.3).

4 Atomic characterization of variable Hardy martingale space

In this section we construct the atomic decomposition of martingale Hardy space with variable exponents. Here we use Definition 1.1 and 1.2.

Proposition 4.1 Given $p(\cdot) \in \mathcal{P}$. Let $f \in H_{s,at}^{p(\cdot)}$, i.e., $f = \sum \mu_k a^k$.

1. We have

$$
\left( \sum_{k \in \mathbb{Z}} \mu_k^{p_+} \right)^{\frac{1}{p_+}} \leq A(\{\mu_k\}, \{a^k\}, \{\tau_k\}).
$$

2. If $p_+ \leq 1$, then

$$
\sum_{k \in \mathbb{Z}} \mu_k \leq A(\{\mu_k\}, \{a^k\}, \{\tau_k\}).
$$

3. For any $k \in \mathbb{Z}$ we have

$$
\|a^k\|_{H_{s,at}^{p(\cdot)}} \leq 1.
$$

Proof. (1) The convexity implies that

$$
\int_{\Omega} \left( \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\lambda \||\chi_{\{\tau_k < \infty\}}||_{p(\cdot)}} \right)^{p(x)} \right)^{\frac{1}{p(x)}} d\mathbb{P} \geq \int_{\Omega} \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\lambda \||\chi_{\{\tau_k < \infty\}}||_{p(\cdot)}} \right)^{p(x)} d\mathbb{P} = \sum_{k \in \mathbb{Z}} \int_{\tau_k < \infty} \left( \frac{\mu_k}{\lambda \||\chi_{\{\tau_k < \infty\}}||_{p(\cdot)}} \right)^{p(x)} d\mathbb{P}.
$$

Now if we set $\lambda = \left( \sum_{k \in \mathbb{Z}} \mu_k^{p_+} \right)^{\frac{1}{p_+}}$, and then we obtain

$$
\int_{\Omega} \left( \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\lambda \||\chi_{\{\tau_k < \infty\}}||_{p(\cdot)}} \right)^{p(x)} \right)^{\frac{1}{p(x)}} d\mathbb{P} \geq \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k}{\lambda} \right)^{p_+} \int_{\Omega} \left( \frac{\chi_{\{\tau_k < \infty\}}}{\||\chi_{\{\tau_k < \infty\}}||_{p(\cdot)}} \right)^{p(x)} d\mathbb{P} = 1.
$$

By the definition of $A(\{\mu_k\}, \{a^k\}, \{\tau_k\})$, we get the desired result.

(2) and (3) are obvious.

Theorem 4.2 Let $p(\cdot) \in \mathcal{P}$. If the martingale $f \in H_{s,at}^{p(\cdot)}$, then there exist a sequence $(a^k)_{k \in \mathbb{Z}}$ of $(1, p(\cdot), \infty)$-atoms and a sequence $(\mu_k)_{k \in \mathbb{Z}}$ of nonnegative real numbers such that for all $n \geq 0$,

$$
\sum_{k \in \mathbb{Z}} \mu_k \mathcal{E}_{\mathcal{F}_n} a^k = f_n, \quad a.e
$$

(4.1)
and

\[ A(\{\mu_k\}, \{a^k\}, \{\tau_k\}) \lesssim H_{p(\cdot)}^s. \]

Moreover the sum \( \sum_{k \in \mathbb{Z}} \mu_k a^k \) converges to \( f \) in \( H_{p(\cdot)}^s \). Conversely, if the martingale \( f \) has a decomposition of \( (4.1) \), then

\[ \|f\|_{H_{p(\cdot)}^s} \lesssim \inf A(\{\mu_k\}, \{a^k\}, \{\tau_k\}), \]

where the infimum is taken over all the decompositions of the form \( (4.1) \).

**Proof.** Assume that \( f \in H_{p(\cdot)}^s \). Let us consider the following stopping times for all \( k \in \mathbb{Z} \)

\[ \tau_k = \inf \{ n \in \mathbb{N} : s_{n+1}(f) > 2^k \}. \]

The sequence of these stopping times is obviously non-decreasing. For each stopping time \( \tau \), denote \( f_n^\tau = f_{n \wedge \tau} \). It is easy to see that

\[ f_n = \sum_{k \in \mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}). \]

Let

\[ \mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}, \quad \text{and} \quad a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}. \]

If \( \mu_k = 0 \) then let \( a_n^k = 0 \) for all \( k \in \mathbb{Z}, n \in \mathbb{N} \). Then \( (a_n^k)_{n \geq 0} \) is a martingale for each fixed \( k \in \mathbb{Z} \). Since \( s(f^{\tau_k}) = s_{\tau_k}(f) \leq 2^k \), we get

\[ s \left( (a_n^k)_{n \geq 0} \right) \leq \frac{s(f^{\tau_{k+1}}) + s(f^{\tau_k})}{\mu_k} \leq \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1}. \]

Hence it is easy to check that \( (a_n^k)_{n \geq 0} \) is a bounded \( L_2 \)-martingale. Consequently, there exists an element \( a^k \in L_2 \) such that \( \mathbb{E}_{\mathcal{F}_n} a^k = a_n^k \). If \( n \leq \tau_k \), then \( a_n^k = 0 \), and \( s(a^k) \leq \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1} \).

Thus we conclude that \( a^k \) is really a \((1, p(\cdot), \infty)\)-atom.

Denote \( \mathcal{O}_k = \{\tau_k < \infty\} = \{s(f) > 2^k\} \). Recalling that \( \tau_k \) is non-decreasing for each \( k \in \mathbb{Z} \), we have \( \mathcal{O}_k \supset \mathcal{O}_{k+1} \). Then

\[ \sum_{k \in \mathbb{Z}} \left( 3 \cdot 2^k \chi_{\mathcal{O}_k}(x) \right)^p \]

is the sum of the geometric sequence \( \{(3 \cdot 2^k \chi_{\mathcal{O}_k}(x))^p\}_{k \in \mathbb{Z}} \). Thus, we can claim that

\[ \sum_{k \in \mathbb{Z}} (3 \cdot 2^k \chi_{\mathcal{O}_k}(x))^p \approx \left( \sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{O}_k}(x) \right)^p \approx \left( \sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}}(x) \right)^p. \]
Indeed, for each fixed \( x_0 \in \Omega \), there is \( k_0 \in \mathbb{Z} \) such that \( x_0 \in O_{k_0} \) but \( \not\in O_{k_0+1} \), then

\[
\sum_{k=-\infty}^{k_0} \left( 3 \cdot 2^k \chi_{O_k}(x_0) \right)^p = \sum_{k=-\infty}^{k_0} \left( 3 \cdot 2^k \right)^p
= \left( 3 \cdot 2^{k_0} \right)^p \frac{1}{1 - 2^{-p}}
\leq \left( 3 \cdot 2^{k_0} \right)^p \left( \frac{1}{1 - \frac{1}{2}} \right)^p
= \left( \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{O_k}(x_0) \right)^p
\leq \left( \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{O_k \setminus O_{k+1}}(x_0) \right)^p.
\]

Thus

\[
A(\{\mu_k\}, \{a^k\}, \{\tau_k\}) = \left\| \left\{ \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(.)}} \right\} \right\|_{L_p(\cdot)}^p = \left\| \left\{ \sum_{k \in \mathbb{Z}} \left( 3 \cdot 2^k \chi_{O_k \setminus O_{k+1}}(x) \right) \right\} \right\|_{L_p(\cdot)}^p.
\]

\[
\leq \left\| \sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{O_k \setminus O_{k+1}} \right\|_{L_p(\cdot)}^p.
\]

\[
= \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \sum_{k \in \mathbb{Z}} \frac{3 \cdot 2^k \chi_{O_k \setminus O_{k+1}}(x)}{\lambda} \right)^{p(x)} \frac{d\mathbb{P}}{\lambda} \leq 1 \right\}
= \inf \left\{ \lambda > 0 : \sum_{k \in \mathbb{Z}} \int_{O_k \setminus O_{k+1}} \left( \frac{3 \cdot 2^k}{\lambda} \right)^{p(x)} \frac{d\mathbb{P}}{\lambda} \leq 1 \right\}
\approx \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{s(f)}{\lambda} \right)^{p(x)} \frac{d\mathbb{P}}{\lambda} \leq 1 \right\}.
\]

Therefore, we obtain

\[
A(\{\mu_k\}, \{a^k\}, \{\tau_k\}) \leq \|s(f)\|_{p(.)} = \|f\|_{H^s_{p(.)}}.
\]

We now verify the sum \( \sum_{k \in \mathbb{Z}} \mu_k a^k \) converges in \( H^s_{p(.)} \). By the equality \( s(f - f^{\tau_k})^2 = s(f)^2 - s(f \tau_k)^2 \) we have

\[
s(f - f^{\tau_k}), s(f^{\tau_k}) \leq s(f) \quad \text{and} \quad s(f - f^{\tau_k}), s(f^{\tau_k - k}) \to 0 \quad \text{a.e., as } k \to \infty.
\]

Consequently, by the dominated convergence theorem in variable \( L^p(\cdot) \) (Theorem 2.62 in [4])

\[
\left\| f - \sum_{k=-M}^{N} \mu_k a^k \right\|_{H^s_{p(.)}}^p \leq \left\| f - f^{\tau_N+1} \right\|_{H^s_{p(.)}}^p + \left\| f^{\tau_{-M}} \right\|_{H^s_{p(.)}}^p,
\]

converges to 0 a.e. as \( M, N \to \infty \).
Conversely, by the definition of $(1,p(\cdot),\infty)$-atom, we have almost everywhere

$$s(a) = s(a)\chi_{\{\tau < \infty\}} \leq \|s(a)\|_{\infty} \chi_{\{\tau < \infty\}} \leq \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}^{-1} \chi_{\{\tau < \infty\}},$$

where $a$ is a $(1,p(\cdot),\infty)$-atom. By the subadditivity of the conditional quadratic variation operator, we obtain that

$$s(f) \leq \sum_{k \in \mathbb{Z}} \mu_k s(a^k) \leq \sum_{k \in \mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}}.$$

Thus

$$\|f\|_{H^s_{p(\cdot)}} = \|s(f)\|_{p(\cdot)} \leq \left\| \sum_{k \in \mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right\|_{p(\cdot)} \leq \left\| \left\{ \sum_{k \in \mathbb{Z}} \left( \mu_k \frac{\chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right)^p \right\}^{\frac{1}{p}} \right\|_{p(\cdot)} \leq \mathcal{A}(\{\mu_k\}, \{a^k\}, \{\tau_k\}).$$

Hence we can conclude that $\|f\|_{H^s_{p(\cdot)}} \approx \|f\|_{H^s_{p(\cdot),\infty}}$ and the proof is complete now.

5 The duality and John-Nirenberg theorem

In this section we prove the dual space of $H^s_{p(\cdot)}$ by the atomic decomposition established in section 4 and the John-Nirenberg inequalities.

**Proposition 5.1** Let $p(\cdot) \in \mathcal{P}$ and satisfy (1.4).

1. If $q(\cdot) \in \mathcal{P}$ satisfies (1.4), then $p(\cdot) + q(\cdot)$ also satisfies (1.4);
2. $\frac{1}{p(\cdot)}$ satisfy (1.4);
3. If assume $1 < p(\cdot) < \infty$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then $q(\cdot)$ satisfies (1.4);
4. If assume $q(\cdot) \in \mathcal{P}$ satisfies (1.4) and $\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)}$, then $r(\cdot)$ satisfies (1.4).

**Proof.** (1) If $q(\cdot) = C$ is a constant, then the result is trivial. Set $h(\cdot) = p(\cdot) + q(\cdot)$, then $h_-(A) - h_+(A) \geq p_-(A) + q_-(A) - p_+(A) - q_+(A)$. Hence

$$\mathbb{P}(A)^{h_-(A) - h_+(A)} \leq \mathbb{P}(A)^{p_-(A) - p_+(A) + q_-(A) - q_+(A)} \leq \mathcal{K}_p \mathcal{K}_q \triangleq K.$$

(2) We have

$$\mathbb{P}(A)^{1/p_+(A) - 1/p_-(A)} = \mathbb{P}(A)^{p_-(A) - p_+(A) \cdot p_+(A) - p_-(A)} \leq \mathcal{K}_{p_+(A) p_-(A)}^{1/p(\cdot)}.$$

If $p_-(\Omega) \geq 1$, then $\mathcal{K}_{p_+(A) p_-(A)}^{1/p(\cdot)} \leq \mathcal{K}_p$. If $0 < p_-(\Omega) < 1$, then $\mathcal{K}_{p_+(A) p_-(A)}^{1/p(\cdot)} \leq \mathcal{K}_{1/p_+(A) p_-(A)}^{1/p(\cdot)} \triangleq K$.

(3) Set $h(\cdot) = 1 - \frac{1}{p(\cdot)}$. We get

$$\mathbb{P}(A)^{h_-(A) - h_+(A)} = \mathbb{P}(A)^{1 - 1/p_-(A) - 1 + 1/p_+(A)} \leq \mathcal{K}_{p_+(A) p_-(A)}^{1/p(\cdot)} \triangleq K.$$
Hence we have $1 - \frac{1}{p} \cdot q$ satisfies (1.4). Using (2), we get desired result.

(4) This can be easily proved by (1) and (2) similarly to the proof of (3). The proof is complete.

It is easy to prove that for all $B \in \mathcal{F}$
\[
\mathbb{P}(B)^{p_-(B)-p(x)} \quad \text{and} \quad \mathbb{P}(B)^{p(x)-p_+(B)} \leq K \quad \forall x \in B,
\]
if $p(\cdot)$ satisfies (1.4). Using this result, we have the following lemma.

**Lemma 5.2** Let $p(\cdot) \in \mathcal{P}$, $0 < p_- \leq p_+ < \infty$ and satisfy (1.4). Then for all set $B \in \mathcal{F}$, we have
\[
\mathbb{P}(B)^{1/p_-(B)} \approx \mathbb{P}(B)^{1/p(x)} \approx \mathbb{P}(B)^{1/p_+(B)} \approx \|\chi_B\|_{p(\cdot)} \quad \forall x \in B.
\]

**Proof.** Obviously, we have $\mathbb{P}(B)^{1/p_-(B)} \leq \mathbb{P}(B)^{1/p(x)} \leq \mathbb{P}(B)^{1/p_+(B)}$, for all $x \in B$. Since (1.4), we have
\[
\mathbb{P}(B)^{1/p(x)} \leq \mathbb{P}(B)^{p(x)-p_+(B)} \leq K \frac{1}{p(\cdot)} \leq K.
\]
This implies $\mathbb{P}(B)^{1/p(x)} \leq K \mathbb{P}(B)^{1/p_-(B)}$.

Then it is easy to check that $\mathbb{P}(B)^{1/p_-(B)} \approx \mathbb{P}(B)^{1/p(x)} \approx \mathbb{P}(B)^{1/p_+(B)}$. And we have
\[
\frac{\chi_B(x)}{\mathbb{P}(B)^{1/p_-(B)}} \approx \frac{\chi_B(x)}{\mathbb{P}(B)^{1/p(x)}},
\]
that is
\[
\left(\frac{\chi_B(x)}{\mathbb{P}(B)^{1/p_-(B)}}\right)^{p(x)} \geq \frac{\chi_B(x)}{\mathbb{P}(B)} \geq \left(\frac{\chi_B(x)}{K \mathbb{P}(B)^{1/p_-(B)}}\right)^{p(x)}.
\]
So
\[
\int_{\Omega} \left(\frac{\chi_B(x)}{\mathbb{P}(B)^{1/p_-(B)}}\right)^{p(x)} d\mathbb{P} \approx \int_{\Omega} \frac{\chi_B(x)}{\mathbb{P}(B)} d\mathbb{P} = 1.
\]
Consequently, we get $\|\chi_B\|_{p(\cdot)} \approx \mathbb{P}(B)^{1/p_-(B)}$ and we get the desired result.

**Corollary 5.3** Let $p(\cdot) \in \mathcal{P}$ and satisfy (1.4).

(1) Then for all set $B \in \mathcal{F}$, we have
\[
\|\chi_B\|_1 \approx \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{q(\cdot)},
\]
where
\[
1 = \frac{1}{p(x)} + \frac{1}{q(x)}.
\]

(2) Let $q(\cdot) \in \mathcal{P}$ and satisfies (1.4). Then for all set $B \in \mathcal{F}$, we have
\[
\|\chi_B\|_{r(\cdot)} \approx \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{q(\cdot)},
\]
where
\[
\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.
\]
We shall show that for all \( f \in L^p \) we first claim that

\[
L^p \quad \text{Definition 5.4}
\]

Given 1 \( \leq q < \infty \). Define \( \Lambda_q(\alpha(\cdot)) \) as the space of functions \( f \in L^q \) for which

\[
\| f \|_{\Lambda_q(\alpha(\cdot))} = \sup_{\tau \in T} \| \chi_{\{\tau < \infty\}} \|^1_{\alpha(\cdot)} \| \chi_{\{\tau < \infty\}} \|^{-1}_q \| f - f^\tau \|_q
\]

is finite.

**Theorem 5.5** Given \( p(\cdot) \in \mathcal{P} \), \( 0 < p_- \leq p_+ \leq 1 \) and \( p(\cdot) \) satisfies (1.4). Then

\[
\left( H^s_{p(\cdot)} \right)^* = \Lambda_2(\alpha(\cdot)), \quad \alpha(x) = 1/p(x) - 1.
\]

**Proof.** We first claim that \( \alpha(\cdot) \) satisfies (1.4) by Proposition 5.1(1). Let \( \varphi \in \Lambda_2(\alpha(\cdot)) \subset L^2 \) and for all \( f \in L^2 \), define

\[
l_\varphi(f) = \mathbb{E}(f \varphi).
\]

We shall show that \( l_\varphi \) is a bounded linear functional on \( H^s_{p(\cdot)} \). By Theorem 4.2, we know that \( L^2 \) is dense in \( H^s_{p(\cdot)} \). Take the same stopping times \( \tau_k \), atoms \( a^k \) and nonnegative numbers \( \mu_k \) as we did in Theorem 4.2. It follows from Theorem 4.2 that \( f = \sum_{k \in \mathbb{Z}} \mu_k a^k \) (\( \forall f \in L^2 \)). Hence

\[
l_\varphi(f) = \mathbb{E}(f \varphi) = \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k \varphi).
\]

By the definition of the atom \( a^k \), \( \mathbb{E}(a^k \varphi) = \mathbb{E}(a^k(\varphi - \varphi^{\tau_k})) \) always holds. It follows from Corollary 5.3 that

\[
\| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)} \approx \| \chi_{\{\tau_k < \infty\}} \|_{\alpha(\cdot)} \| \chi_{\{\tau_k < \infty\}} \|_2 \| \chi_{\{\tau_k < \infty\}} \|_2.
\]

Thus, using Hölder’s inequality we can conclude that

\[
|l_\varphi(f)| \leq \sum_{k \in \mathbb{Z}} \mu_k \int_\Omega |a^k| |\varphi - \varphi^{\tau_k}| d\mathbb{P}
\]

\[
\leq \sum_{k \in \mathbb{Z}} \mu_k \|a^k\|_2 \|\varphi - \varphi^{\tau_k}\|_2
\]

\[
\leq \sum_{k \in \mathbb{Z}} \mu_k \|\{\tau_k < \infty\}\|^{\frac{1}{2}} \|\varphi - \varphi^{\tau_k}\|_2
\]

\[
\leq \sum_{k \in \mathbb{Z}} \mu_k \|\varphi\|_{\Lambda_2(\alpha(\cdot))}.
\]

Then, we obtain from Proposition 4.1 and Theorem 4.2 that

\[
|l_\varphi(f)| \leq \|f\|_{H^s_{p(\cdot)}} \|\varphi\|_{\Lambda_2(\alpha(\cdot))}.
\]
Consequently, \( l_\varphi \) can be extended to \( H_{p(\cdot)}^s \) uniquely.

On the other hand, let \( l \) be an arbitrary bounded linear functional on \( H_{p(\cdot)}^s \). We shall show that there exists \( \varphi \in \Lambda_2(\alpha(\cdot)) \) such that \( l = l_\varphi \) and

\[
\|\varphi\|_{\Lambda_2(\alpha(\cdot))} \lesssim \|l\|.
\]

Since \( 0 < p_- \leq p_+ \leq 1 \), thus it follows from Theorem 2.8 in \cite{17} that

\[
\|f\|_{H_{p(\cdot)}^s} \leq \|s(f)\|_2 = \|f\|_2, \quad \forall f \in L^2.
\]

Then the space \( L^2 \) can be embedded continuously in \( H_{p(\cdot)}^s \). Consequently, there exists \( \varphi \in L^2 \) such that

\[
l(f) = E(f\varphi), \quad \forall f \in L^2.
\]

Let \( \tau \) be an arbitrary stopping time and

\[
g = \frac{\varphi - \varphi^\tau}{\|\varphi - \varphi^\tau\|_2 \|\chi_{\{\tau<\infty\}}\|_2}.
\]

Then \( g \) is not necessarily a \((1, p(\cdot), \infty)\)-atom but it satisfies (1) in Definition 1.1, thus we have

\[
s(g) = s(g)\chi_{\{\tau<\infty\}}.
\]

Since

\[
\frac{1}{p(x)} = \frac{1}{2} + \frac{1}{1/\alpha(x)} + \frac{1}{2},
\]

then by Hölder’s inequality we get

\[
g \in H_{p(\cdot)}^s \quad \Rightarrow \quad \|g\|_{H_{p(\cdot)}^s} \lesssim \|s(\varphi - \varphi^\tau)\|_{p(\cdot)}
\]

\[
\lesssim \frac{\|s(\varphi - \varphi^\tau)\|_2 \|\chi_{\{\tau<\infty\}}\|_2 \|\chi_{\{\tau<\infty\}}\|_2}{\|\varphi - \varphi^\tau\|_2 \|\chi_{\{\tau<\infty\}}\|_2 \|\chi_{\{\tau<\infty\}}\|_2}
\]

\[
= 1.
\]

Thus

\[
\|l\| \gtrsim l(g) = E(g(\varphi - \varphi^\tau))
\]

\[
= \|\chi_{\{\tau<\infty\}}\|_2 \|\chi_{\{\tau<\infty\}}\|_2 \|\varphi - \varphi^\tau\|_2
\]

and we get that \( \|\varphi\|_{\Lambda_2(\alpha(\cdot))} \lesssim \|l\| \) and the proof is complete.

We now turn to the John-Nirenberg theorem with variable exponent. Recall that \( BMO_p(1 \leq p < \infty) \) is the space of those functions \( f \) for which

\[
\|f\|_{BMO_p} = \sup_{\tau \in T} \|\chi_{\{\tau<\infty\}}\|_p \|f - f^\tau\|_p < \infty.
\]
**Definition 5.6** Given $p(\cdot) \in \mathcal{P}$ and $\mathcal{T}$ be the sets of all stopping times relative to $\{F_n\}_{n \geq 0}$. Define

$$BMO_{p(\cdot)} = \left\{ f = (f_n)_{n \geq 0} : \|f\|_{BMO_{p(\cdot)}} < \infty \right\},$$

where

$$\|f\|_{BMO_{p(\cdot)}} = \sup_{\tau \in \mathcal{T}} \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}^{-1} \|f - f^{\tau - 1}\|_{p(\cdot)}.$$  

**Lemma 5.7** (see [20]) If $1 \leq p < \infty$, then

$$\|f\|_{BMO_1} \approx \|f\|_{BMO_p}.$$  

**Theorem 5.8** If $p(\cdot) \in \mathcal{P}$, $1 < p_- \leq p_+ < \infty$ satisfies (1.4), then we have that for all $f \in BMO_1$

$$\|f\|_{BMO_1} \lesssim \|f\|_{BMO_{p(\cdot)}} \lesssim \|f\|_{BMO_1}.$$  

**Proof.** By Hölder’s inequality and Corollary 5.3, we have that

$$\|f - f^{\tau - 1}\|_1 \lesssim \frac{\|f - f^{\tau - 1}\|_{p(\cdot)} \|\chi_{\{\tau < \infty\}}\|_{p'(\cdot)}^{p(\cdot)}}{\|\chi_{\{\tau < \infty\}}\|_1} = \frac{\|f - f^{\tau - 1}\|_{p(\cdot)} \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)} \|\chi_{\{\tau < \infty\}}\|_{p'(\cdot)}}{\|\chi_{\{\tau < \infty\}}\|_1} \lesssim C_{p(\cdot)} \|f\|_{BMO_{p(\cdot)}},$$

where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$  

Hence $\|f\|_{BMO_1} \lesssim \|f\|_{BMO_{p(\cdot)}}$.  

Since

$$\|f - f^{\tau - 1}\|_{p(\cdot)} \lesssim \|f - f^{\tau - 1}\|_{p_+} \|\chi_{\{\tau < \infty\}}\|_{p_+} \|\chi_{\{\tau < \infty\}}\|_{p_+},$$

then by Lemma 5.7, we get

$$\|f - f^{\tau - 1}\|_{p(\cdot)} \lesssim \|f\|_{BMO_1} \|\chi_{\{\tau < \infty\}}\|_{p_+} \|\chi_{\{\tau < \infty\}}\|_{p_+} \|\chi_{\{\tau < \infty\}}\|_{p_+}.$$  

Thus by Corollary 5.3

$$\frac{\|f - f^{\tau - 1}\|_{p(\cdot)}}{\|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}} \lesssim \|f\|_{BMO_1} \|\chi_{\{\tau < \infty\}}\|_{p_+} \|\chi_{\{\tau < \infty\}}\|_{p_+} \|\chi_{\{\tau < \infty\}}\|_{p_+}^{-1} \|f\|_{BMO_1}.$$  

This means

$$\|f\|_{BMO_{p(\cdot)}} \lesssim \|f\|_{BMO_1}. $$
Theorem 5.9 Let \( p(\cdot) \in \mathcal{P} \) satisfy 1.1 and 1 < \( p_- \leq p_+ < \infty \), then there exist constants \( C_1, C_2 > 0 \) such that for any \( f \in BMO_1, \tau \in \mathcal{T} \) and \( t > 0 \),

\[
\left\| \chi_{\{\tau < \infty\} \cap \{f - f_{\tau - 1} \geq t\}} \right\|_{p(\cdot)} \leq C_1 e^{-\frac{C_2 t}{\|f\|_{BMO_1}}} \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}.
\]

Proof. Using Lemma 2.1 and the idea before this theorem, we point out that for any \( r \geq 1 \)

\[
\sup_{\tau} \left\| \frac{|f - f_{\tau - 1}|^r}{\|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}}^{1/r} \right\|_{p(\cdot)} \leq \|f\|_{BMO_{rp(\cdot)}} \leq C_0 \|f\|_{BMO_1} \triangleq C_0.
\]

This implies that

\[
\left\| \frac{|f - f_{\tau - 1}|^r}{\|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}} \right\|_{p(\cdot)} \leq C_0 \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}.
\]

Then we get that

\[
\left\| \chi_{\{\tau < \infty\} \cap \{f - f_{\tau - 1} \geq t\}} \right\|_{p(\cdot)} \leq \frac{1}{t^r} \left\| \frac{|f - f_{\tau - 1}|^r}{\|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}} \right\|_{p(\cdot)} \leq \frac{C_0 t}{t^r} \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}.
\]

If \( t \geq 2C_0 \), we take \( r = \frac{t}{2C_0} \geq 1 \), then

\[
\left( \frac{C_0}{t} \right)^r \leq \frac{1}{2^r} = e^{-r \ln 2} = e^{-\frac{r}{2C_0} \ln 2} = e^{-\frac{r}{\|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}} \ln 2} = e^{-\frac{C_2 t}{\|f\|_{BMO_1}}},
\]

where \( C_2 = \frac{1}{2C_0} \ln 2 \).

If \( t < 2C_0 \), take \( C_2 = \frac{1}{2C_0} \ln 2 \). Then \( e^{-\frac{C_2 t}{\|f\|_{BMO_1}}} = \left( \frac{1}{2} \right)^{\frac{t}{2C_0}} > 1/4 \). Since \( \{\tau < \infty\} \cap \{f - f_{\tau - 1} \geq t\} \subset \{\tau < \infty\} \), then

\[
\left\| \chi_{\{\tau < \infty\} \cap \{f - f_{\tau - 1} \geq t\}} \right\|_{p(\cdot)} \leq \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)} \leq 4e^{-\frac{C_2 t}{\|f\|_{BMO_1}}} \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}.
\]

Finally, we obtain the desired result.

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