On rereading Savage

Yudi Pawitan\textsuperscript{1} and Youngjo Lee\textsuperscript{2}

\textsuperscript{1}Department of Medical Epidemiology and Biostatistics, Karolinska Institutet, Stockholm 17177, Sweden (e-mail: yudi.pawitan@ki.se)

\textsuperscript{2}Department of Statistics, Seoul National University, Gwanak-gu, Seoul 08826, South Korea (e-mail: youngjo@snu.ac.kr).

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Abstract

If we accept Savage’s set of axioms, then all uncertainties must be treated like ordinary probability. Savage espoused subjective probability, allowing, for example, the probability of Donald Trump’s re-election. But Savage’s probability also covers the objective version, such as the probability of heads in a fair toss of a coin. In other words, there is no distinction between objective and subjective probability. Savage’s system has great theoretical implications; for example, prior probabilities can be elicited from subjective preferences, and then get updated by objective evidence, a learning step that forms the basis of Bayesian computations. Non-Bayesians have generally refused to accept the subjective aspect of probability or to allow priors in formal statistical modelling. As demanded, for example, by the late Dennis Lindley, since Bayesian probability is axiomatic, it is the non-Bayesians’ duty to point out which axioms are not acceptable to them. This is not a simple request, since the Bayesian axioms are not commonly covered in our professional training, even in the Bayesian statistics courses. So our aim is to provide a readable exposition the Bayesian axioms from a close rereading Savage’s classic book.

Keywords: Axioms of probability; Bayesian; subjective probability; uncertainty
1 Introduction

With a regular use of the inverse probability method, 19th-century statistics was largely Bayesian. But Bayesian school of statistics as a fully coherent statistical philosophy and methodology emerged during 1950s, perhaps conveniently coinciding with the publication of Savage’s seminal book *The Foundation of Statistics* in 1954. Even the adjective ‘Bayesian’ was only used regularly in its current meaning then after the appearance of many prominent Bayesians that seemed to revolve around Savage (Fienberg, 2006). Fisher (e.g., 1930), following many 19th century mathematicians including George Boole, John Venn and George Chrystal, criticized the use of inverse probability method due to its arbitrary prior probability for unknown parameters. Fisher had used the term Bayesian pejoratively to refer to this method. Savage’s work established Bayesian statistics axiomatically, hence brought legitimacy to prior probability and subjective probability in general.

There is a virtual consensus regarding the use of probability for statistical modeling, but we have yet to reach that for its interpretation and philosophical aspects. Mathematically, Kolmogorov’s axiomatic foundation puts probability as the legitimate child of the more mature measure theory. Kolmogorov’s probability, with its celebrated laws of large numbers, is naturally interpreted as long-run frequency. In orthodox introductory teaching, where most classic textbooks are written by frequentists, probability is said to be meaningless for specific events such as Donald Trump’s re-election in 2024. But people do bet on such specific events, which can only mean that they do have a probability that is not of a long-term variety. The reasoning requires a probability that applies to specific events. Moreover, since different people have different beliefs and temperaments, they may have different probabilities for the same event. This in turn calls for subjective probability.

Savage’s axiomatic development is a culmination of the subjective theory of probability at the heart of Bayesian statistics, so it seems appropriate to put our focus on him. In order not to get too distracted, we shall not discuss Jeffreys (1939) or the development of the so-called objective Bayesian statistics. Rather than ‘subjective’, Savage liked to use the term ‘personalistic,’ but the latter is not in common use, so we shall use ‘subjective’ throughout. Our aim is to provide a brief historical background of the subjective probability, then to go through Savage’s axioms in detail. We discuss two well-known paradoxes: Allais’s and Ellsberg’s. The purpose is not to use them in order to dismiss the axioms, but to illuminate the logical implications of some of them.

2 Historical background

There are two historical strands leading to Savage: (i) Ramsey’s (1931) and de Finetti’s (1937, RDF) definition of subjective probability as a betting quotient, and their fundamental theorem that the bet is coherent if and only if the probability satisfies finitely additive probability axioms. (ii) von Neumann and Morgenstern’s (VNM, 1947) theorem in game theory that a person whose preferences follow certain axioms behave as if he is maximizing a utility function. In effect the RDF theorem assumes objective utility (good ol’ money) and a coherent betting strategy to arrive at subjective probability. This result is not sufficient to deal with the question of Bayesian prior probability. The VNM theorem assumes
objective probability and axioms of rational preferences to arrive at subjective utility. Savage came up with the axioms of rational preferences that unify both subjective probability and subjective utility within the decision framework. This axiomatic approach justifies the use of subjective probability, including subjective priors, in Bayesian statistics.

Let's start with the work of Ramsey in Cambridge and de Finetti in Italy during 1920s. In the Cambridge circle, under the influence of philosophers W. E. Johnson and Bertrand Russel, Keynes (1921) proposed a logical theory of probability as a rational degree of belief of propositions rather than just of events; ‘rational’ is meant to be objective, independent of subjective preferences. But his idea, which still relied on the classical principle of indifference/non-sufficient reason for equiprobable alternatives, never made any impact in statistics or in mathematics. It is of interest historically as it led Ramsey in 1926, who was only 22 at the time, to propose his alternative theory of probability as a subjective degree of belief. During 1920s de Finetti also conceived the same idea independently of Ramsey, though his book was only published in 1937. In their construction, a person’s probability is revealed as a ‘coherent betting quotient’, a value that avoids an external agent to run a Dutch Book – a risk-free bet – against him.

Setting a ‘coherent bet’ is reminiscent to ‘I-cut-you-choose’ method of splitting a cake fairly between two greedy individuals. Even children will immediately see its fairness: John cuts the cake and Patrick chooses first. It is in John’s self-interest to cut it as equally as possible. To cut into unequal pieces is not coherent, as he would knowingly guarantee himself to lose as Patrick will choose the bigger piece.

How do we extend this trick to probability of an arbitrary event, such as re-election of Donald Trump? Say John has a subjective probability 0.25 for such an event. It is not important that he knows it himself, but only that he is prepared to act – to bet – on it. It is the action, speaking louder than any verbal pronouncement, that is supposed to reveal the probability. We set up the following betting game. John (as player) has the chance to win $100 from Patrick (as bookie) if the event occurs, but he has to pay Patrick a price to play the game. How much would he pay? Paying more than $25 would violate his sense of the probability of the event, so he will not do that, but he would be happy to pay, say, $10. But that will of course not reveal his probability.

Here is the trick: Patrick has the right to reverse their roles as player and bookie depending on the amount John decides to pay. If John decides to pay $10, then (he worries) he runs the risk that Patrick would reverse their roles and instead make him pay $100 in return for $10 if the event happens. So, to neutralize this risk, John should pay as large as possible, but as he would not be willing to pay more than $25, so he will arrive at $25 as his fair price. At this fair price John should be indifferent to the role reversal. Choosing any other number runs the risk of a loss: if his subjective probability is \( p \) and he sets the price at \( q \) per winning dollar, he runs the risk of losing \( |p - q| \). So the subjective probability \( p \) might also be called the risk-neutral (not risk-free) probability.

Several assumptions are implicit in this idealized betting game. One is precision: in real life John the player will not have a precise price in mind, so precision is a mathematical convenience. At his fair price, it is assumed that John would be indifferent between being a player or a bookie. In practice he may have different preferences, i.e. the chance of winning $100 feels different from the chance of losing $100, even if they are balanced by the same
price or compensation. A real-life Patrick the bookie will have his own probability of the event and will then set a price higher than that. For example, he may accept even $10 if that is higher than his probability. So, the revelation of John’s probability requires that Patrick does not have his own price, while John must think introspectively and reflexively that Patrick is thinking like himself so that the risk of reversal occurs even when John is willing to pay $20. This condition is satisfied, for example, if both presume that they have access to the same background information.

Furthermore, it must be assumed that the amount of money involved is small relative to their wealth, so that they can still think linearly. In large amounts the utility of money becomes non-linear, so the reasoning must account for risk aversion. Finally, the game might feel contrived with its potential role reversal and various assumptions. Role reversal means you can be either a buyer or a seller of a bet. This is actually what happens in online betting exchanges, where you can choose to buy or sell depending on the difference between your subjective probability and the current ‘market price’ of an event. More generally, it corresponds to buying and shorting an asset in the financial market.

Under those same assumptions, this setup can be extended to an arbitrary but finite number of complementary events, where John specifies the price for each event according to his subjective probabilities. Ramsey and de Finetti proved the fundamental theorem that a betting strategy is coherent if and only if the probabilities follow the axioms of finitely additive probability. Much has been written about the finite additivity of subjective probability, as opposed to the countable additivity of Kolmogorov’s probability. In fact, this is not a serious issue. Williamson (1999) provided a proof of countably additive subjective probability under one extra condition that only a finite amount of money is used, a condition that is already tacitly assumed in order to avoid non-linear utility effect. The assumption of fixed linear utility, however, renders the RDF theorem insufficient as the basis of Bayesian statistics, where a general loss function – equal to minus utility – would be needed.

3 Savage’s axioms

3.1 The elements

Savage conveniently listed his seven axioms as the ‘Postulates of a Personalistic Theory of Decision’ at the front end of his book. The ‘formal subject matter’ includes (i) $S$ the set of ‘states of the world’ $s_1, s'_1, \ldots$; (ii) $F$ the set of ‘consequences’ $f_1, f'_1, \ldots$, (iii) the set of ‘acts’ $f_1, f_2, \ldots$. Formally, an act $f$ is a function from $S$ to $F$. We might write $f(s) = f(s)$ as a consequence at state $s$, but we have preserved Savage’s explicit preference for these notations. To facilitate understanding – and to tie up more easily with Ramsey, de Finetti, and von Neumann and Morgenstern – think of an ‘act’ as a bet on the state of the world, and the ‘consequence’ is the payoff. (In general the payoff can be expressed in utility scale, but for now there is no need to do that yet.)

The ‘state of the world’ is a description sufficient to compute the consequences of each act. Perhaps useful to envision the usual random experiment with $S$ as the sample space, but, unlike in von Neumann and Morgenstern’s lottery, don’t assume that the random generating mechanism is known. For example, imagine putting a bet in a horse race: the
bet and the payoff are obvious, and the sample space is the list of horses. The process of producing a winner is a complex random process; the theory expresses a person’s probability model with his betting behavior.

An abstract term ‘state space’ – rather than ‘sample space’ – is used as the theory makes no distinction between the uncertainty involving unknown fixed parameters, where no random experiment is involved, and the uncertainty associated with truly random outcomes. (In the case of fixed parameters, pedagogically, it is usually presented as if the true parameters are sampled from a subjective probability distribution, but technically no sampling is needed.) A state is said to ‘obtain’, or to ‘be true’, if it is the true parameter in the parameter space $S$, and can be interpreted as ‘realized’ if it is a randomly generated state in the sample space $S$. For example, the winning horse is the ‘true state’ among the participating horses. Subsets of $S$ are called events. An event is said to ‘obtain’ if it contains the true state; in standard probability terms, we say the event ‘occurs.’ We shall use the latter term, because ‘obtain’ is still not in common usage.

$$
\begin{array}{|c|c|c|c|}
\hline
\text{Bet} & \text{Horse A} & \text{Horse B} & \text{Horse C} \\
\hline
f_1 & $100 & $0 & $0 \\
f_2 & $0 & $100 & $0 \\
f_3 & $0 & $25 & $25 \\
\hline
\end{array}
$$

Table 1: A race of three horses A, B and C, and the corresponding payoffs associated with the bets $f_1$, $f_2$ and $f_3$.

Table 1 shows a simple example that captures the elements of the theory. Choosing the bet $f_1$ means you believe Horse A would win, or more realistically, you believe Horse A has a higher chance to win than the other horses. Similarly, choosing $f_2$ means you believe in Horse B; choosing $f_3$ means you believe either B or C would win. As in Ramsey’s and de Finetti’s theory, a betting choice reveals a subjective probability. In one key difference with standard probability model: a horse race may already be completed. As will be clear from the axioms, the whole theory applies unchanged to the bets made on the specific result of a race, even a race that has concluded, of course as long as the result remains unknown to the bettor. In contrast, the standard probability theory does not apply to the result of a specific race.

The final element of the theory is a preference ordering between the acts: ‘$f_1 \leq f_2$’ means ‘$f_1$ is not preferred to $f_2$,’ or more cleanly ‘$f_2$ is preferred to $f_1$;’ this is equivalent to $f_1 < f_2$ or $f_1 = f_2$, meaning either $f_2$ is strictly preferred over $f_1$ or they are equivalent.

Now we are ready for Savage’s seven postulates.

**Axiom 1**

P1 (Weak ordering) The preference order is complete and transitive, meaning that (i) every pair $f_1$ and $f_2$ can be compared and ordered, (ii) for every $f_1$, $f_2$ and $f_3$, if $f_1 \leq f_2$ and $f_2 \leq f_3$, then $f_1 \leq f_3$.

This postulate is similar to von Neumann and Morgenstern’s (1947) Axioms 1 and 2 for subjective utility. Transitivity is a crucial element of any axiomatic system of rational
behavior. A typical concern is on the completeness: we can imagine situations where we put little thoughts, hence have no explicit preferences, over some acts. In statistics this is not likely to be a concern, and Savage advised against considering a system that allows partial ordering. As a simple consequence of the axiom, if $F$ is finite, then there must be an optimal act, i.e. there is no other act strictly preferred over it.

Savage already from the beginning pointed out two aspects of an axiomatic system: empirical and normative. The former is as a descriptive and predictive theory of people’s behavior in decision making: the axioms will be considered good or bad according to the closeness of the predictions to actual behavior of people. The normative aspect attempts to guide people’s behavior, particularly behavior that is not consistent with the theory. He brought analogy to logic, where once we agree to certain propositions, then we should follow their logical consequences. Axiomatic systems such as Savage’s or von Neumann and Morgenstern’s carry their supposed normative values from this iron law of logic. Savage’s correction of his own reaction of Allais’s paradox below shows that he considered his system normatively. So in this normative sense, the axioms are axioms of rational behavior.

**Axiom 2**

P2 (Sure-thing principle) For every act $f, g$ and event $E$, $f \leq g$ given $E$ or $g \leq f$ given $E$.

This axiom at the front-end of the book looks deceptively simple, raising the question why it is called the sure-thing principle and why it is controversial. The idea of conditional preference as used in the axiom requires care, because different pairs of acts might agree when $E$ occurs, but do not agree when $E$ does not occur. Conditional preference is defined as follows:

$f \leq g$ given $E$, if and only if $f' \leq g'$ for every $f'$ and $g'$ that agree with $f$ and $g$, respectively, on $E$, and with each other on $E^C$ and $g' \leq f'$ either for all such pairs or for none.

Now that is not so simple. Within the text, the postulate gets an alternative longer version, not requiring a definition:

P2 (Sure-thing principle) If $f, g$, and $f', g'$ are such that:

1. in $E^C$, $f$ agrees with $g$, and $f'$ agrees with $g'$,
2. in $E$, $f$ agrees with $f'$, and $g$ agrees with $g'$;
3. $f \leq g$;

then $f' \leq g'$.

Using a definition in one place and an axiom in another is an interesting stylistic choice; in the latter the conditional preference is an undefined primitive concept characterized in the axiom. In case you think this longer version still looks challenging, Savage illustrated it with an example (Savage, 1972, page 21):
A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, to clarify the matter for himself, he asks whether he would buy if he knew the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew the Democratic candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy.

He then added a famous comment that ‘I know of no other extralogical principle governing decisions that finds such ready acceptance,’ which is why he called the postulate ‘the sure-thing principle.’ Yet, mapping the example to the postulate is not obvious, because the postulate involves four distinct acts, while the businessman contemplates only two. In fact, the example is a more appropriate illustration of the so-called ‘dominance principle,’ which involves consistent ordering of two acts across the states of the world.

| Bet | Horse A | Horse B | Horse C |
|-----|---------|---------|---------|
| f   | $100    | $0      | $0      |
| g   | $0      | $100    | $0      |
| f'  | $100    | $0      | $100    |
| g'  | $0      | $100    | $100    |

Table 2: A race of three horses A, B and C, and the corresponding payoffs associated with four bets labeled according to the acts in postulate P2. The pair (f, g) is a choice between A vs B, while the pair (f', g') is between not-B vs not-A. So the two pairs are logically distinct choices; P2 is a consistency requirement that they have the same preference ordering.

Let’s consider instead the example in Table 2 involving four bets in a horse race. The bet f' is a bet on horses A and C, while g' is on B and C. Define the event \( E \equiv \{A,B\} \). The pairs of bets (f, g) and (f', g') agree on E, while on the complement event \( E^C \) the bets agree within each pair. In words, if E does not occur, the two bets within each pair are equivalent; while if E occurs, the two pairs are indistinguishable in their payoffs. Thus P2 specifies that if, given E, you decide f \( \leq \) g then you must also decide f' \( \leq \) g', and vice versa. This sounds reasonable: in the within-pair comparisons of the bets, only horses A and B contribute preferential information, C is irrelevant because it produces the same payoff within each pair.

Versions of postulate P2 appear in other axiomatic systems. It is closely related to Rubin’s postulate or Milnor’s column linearity postulate (Luce and Raiffa, 1957), which states that adding a constant amount to a column of payoffs – associated with one state – should not change the ordering between acts. If the probabilities of the events are known, as in a lotto, P2 is equivalent to von Neumann and Morgenstern’s (1947) fourth axiom, called the independence axiom, stipulating that the decision maker compares alternatives based on their distinct consequences, and ignores aspects that are the same. The postulate also appears as Samuelson’s special independence assumption (Samuelson, 1952). In social choice theory (e.g., Arrow, 1950), it is related to the axiom of independence of irrelevant alternatives.
Axiom 3

P3 (State-independence) If \( f \equiv g \), \( f' \equiv g' \); and event \( E \) is not null, then \( f \leq f' \) given \( E \), if and only if \( g \leq g' \).

First define the preference ordering of consequences \( g \) and \( g' \) in terms of constant acts \( f \equiv g \) and \( f' \equiv g' \); think of these acts as receiving gifts instead of betting. The axiom states that the ordering of consequences remains the same under any event, or that knowledge of the underlying event does not change the preferential ordering of consequences. P3 is followed by a significant theorem: assuming axioms P1 and P2, axiom P3 is equivalent to the admissibility or the dominance principle, a key principle in Abraham Wald’s non-Bayesian statistical decision theory. In brief, an act is not admissible if there is another act that dominates it in terms of its consequences. The principle states that we must reject non-admissible acts; e.g. we should reject bets whose payoffs at each state are worse than those of another bet.

Leading to P3, Savage also defined a null event, which is an event under which all acts are equivalent; for example, when all bets have the same payoff, so the decision maker could ignore it from further preferential considerations. According to this definition, a null event is not an impossible event in the usual probabilistic sense, but one that the decision maker does not care enough that it makes no impact on his preferences. For example, for someone who does not care to write his will and has no preference what happens to his wealth after he dies, all the life-threatening events are null events. It also means that for the states that matter, i.e. all the non-null states, there must be at least one consequence that is distinct from the others. This will appear in postulate P5.

Since in the limit an event can contain a single state, the axiom can be called ordinal state independence of payoffs. Thus the preferential value of $100 payoff means the same regardless of the underlying state that produces it. This seems appropriate for statistical applications, but there are situations where it may not be the case. In general the axiom leads to state-independent and chance-neutral utility. In health economic or insurance applications, a person’s utility of money changes depending on his underlying state of health. Chance neutrality means, the value one puts to, say, $1000 is the same regardless whether it comes from the regular salary or from winning a lotto. Recently Stefánsson and Bradley (2015) discussed this aspect in rationality and utility theory.

Axiom 4

P4 (Consequence independence) If consequences \( f, f', g, g' \); events \( A, B \); and acts \( f_A, f_B, g_A, g_B \) are such that:

1. \( f' < f \), \( g' < g \)
2a. \( f_A(s) = f \), \( g_A(s) = g \) for \( s \in A \)
   \( f_A(s) = f' \), \( g_A(s) = g' \) for \( s \notin A \)
2b. \( f_B(s) = f \), \( g_B(s) = g \) for \( s \in B \)
   \( f_B(s) = f' \), \( g_B(s) = g' \) for \( s \notin B \)
3. \( f_A \leq f_B \)

then \( g_A \leq g_B \).
The act $f_A$ can be interpreted as a bet on event $A$, as it offers a strictly better payoff if $A$ occurs; similarly for the other three acts. So $f_A \leq f_B$, i.e. preferring $B$ over $A$ even though they have the same payoffs, must mean you believe $B$ is more likely than $A$. The axiom states that the choice of events to bet on is independent of the size of the payoff. In the horse race example (Table 1), the choice of Horse B over A should remain the same if the payoff is changed to $200 or any other value. Axioms 3 and 4 are the key pillars that support a great weight: the separation between subjective probability and utility, or between belief and value, which are normally tangled in ordinary preferences. This means (i) utility of payoffs can be defined independently of the underlying state, including its probability, and (ii) the subjective probability of an event is meaningful independently of the value of the event. The former can be interpreted as chance neutrality: when you win say $1M in a lottery, you should feel the same regardless whether the winning probability is close to impossible $10^{-9}$ or just small $10^{-6}$. Recently Stefánsson and Bradley (2015) propose the idea of chance itself having a utility.

**Axiom 5**

P5 (Non-triviality) There is at least one pair of consequences $f, f'$ such that $f' < f$.

This must the case under one state of the world. In view of the discussion of null events in Axiom 3 above, this axiom is needed to avoid a trivial null state space. In other words, there is at least one event, containing one state, that you are willing to bet on.

### 3.2 Qualitative probability and Axiom 6

Axioms 1-5 form a natural set, as they are sufficient to construct a qualitative subjective probability. It is similar to the logical probability concept that appeared in Keynes’s proposal, which he considered necessary when there is no sufficient knowledge to set up quantitative probability. In this framework we can say, for example, that ‘event $A$ is less probable than $B’ without specifying their numerical probabilities. In his 1941 paper ‘Heuristic Reasoning and the Theory of Probability’ and delightful 1954 book ‘Mathematics and Plausible Reasoning,’ the mathematician G. Polya also described a coherent setup of qualitative logical probability as a tool for characterizing the experimental stage of mathematical proofs.

Defining 0 as null event, $S$ as the state space, a relational operator $\preceq$ is a qualitative probability if, for all events $B, C$ and $D$,

1. $\preceq$ is complete and transitive,
2. $B \preceq C$ if and only if $B \cup D \preceq C \cup D$, provided $B \cap D = C \cap D = 0$,
3. $0 \preceq B, 0 \preceq C$.

It is, however, more convenient to express the relation in probability notation. We first define a finitely additive probability measure as a function of subsets $B$ of $S$, such that
1. $P(B) \geq 0$ for every $B$.

2. If $B \cap C = 0$, $P(B \cup C) = P(B) + P(C)$.

3. $P(S) = 1$.

The probability measure $P$ and the qualitative probability $\preceq$ are said to agree if for every $B$ and $C$, $P(B) \leq P(C)$ is equivalent to $B \preceq C$. Unfortunately Axioms 1-5 are not sufficient to guarantee the existence of a probability measure that agrees with $\preceq$. Savage proved a series of theorems leading to postulate P6', which provides the guarantee.

$$P6'$$ If $B \prec C$, there exists a partition $\cup_i E_i$ of $S$, such that $B \cup E_i \prec C$ for any $i$.

The key idea here is that $S$ is rich enough so that it can be divided into a fine partition, such that any piece in the partition is too small to change the strict ordering of $B$ and $C$. Savage viewed this postulate as a precursor to postulate P6 needed for a full-blown quantitative subjective probability.

$$P6$$ (Small-event continuity) If $f < g$, and $h$ is any consequence; then there exists a partition $\cup_i E_i$ of $S$ such that, if $f$ is so modified to take value $h$ on $E_i$ for any $i$, other values being undisturbed, then the modified $f < g$. The same ordering is also preserved when $g$ is modified instead.

The postulate requires $S$ to be rich enough, at least infinitely countable, to allow a fine partition, and each act is a smooth function over $S$ that a modification on a small event does not upset the preference of the act. Typically this is interpreted to mean there is no consequence either infinitely better or infinitely worse than the others that its inclusion in a modification of an act upsets its ordering. Another consequence is that the probability is non-atomic in the sense that for any $\rho$, $0 \leq \rho \leq 1$, and event $E$, there is $B \subseteq E$, such that $P(B) = \rho P(E)$.

Allais’s paradox below shows that a small-event discontinuity can occur, even with bounded consequences, if the small event turns an act into a sure bet. The continuity axiom is closely related to von Neumann and Morgenstern’s third axiom, sometimes called Archimedian property for no obvious reasons. It is a continuity requirement in the set of acts – lotteries – in that no act is infinitely more or infinitely less preferable than any other act. Further technical work is needed to deal with finite state spaces as in the horse race example. Wakker (1989), for example, proposed a model where the set of consequences is a connected separable topological space.

Axioms 1–6 are sufficient to construct quantitative/numerical subjective probability, which conveniently reduces the calculation of preferences between acts in terms of arithmetic comparisons of numbers. However, the acts must be of special form, i.e. they are bets on events. So what we have is numerical probability of any event. At this point the theory is difficult to navigate, since there is no explicit theorem stating exactly what has been achieved by the time we reach Axiom 6. Comparisons of arbitrary acts with arbitrary consequences will have to wait until the introduction of utility and Axiom 7.
Finite additivity

Similarly to Ramsey-de Finetti’s coherent betting strategy, Savage’s axioms 1-6 are necessary and sufficient to construct finitely additive probability. This is in contrast to Kolmogorov’s countably additive probability, which is the basis of most results in modern probability theory. However, agreeing with de Finetti, Savage knew ‘of no argument leading to the requirement of countable additivity... therefore seems better not to assume countable additivity outright as a postulate ...’. Kolmogorov (1933) himself argued that the countable additivity axiom was essential theoretically, but ‘it is almost impossible to elucidate its empirical meaning.’

Assuming countably additive probability, we cannot have a uniform distribution over, say, positive integers. De Finetti (1970, vol 1, p-122) considered that counter-intuitive. However, his reasoning was not clear. Countable additivity ‘forces me to choose some finite subset of them ... to which I attribute a total probability of at least 99%....’ Savage (1972, page 43) also wrote ‘many of us have a strong intuitive tendency to regard as natural probability problems about the necessarily only finitely additive uniform probability densities on the integers, on the line and on the plane.’ Under finite additivity, we can of course have a uniform distribution on an arbitrarily large collection of integers, but the collection must be finite, so it is not clear how finite additivity solves the problem.

The loss of countable additivity is a serious loss. For example, Schervish et al (1984) showed that we lose conglomerability. Under countable additivity, for any event $E$ and partition $\cup_i B_i$ of $S$, $P(E)$ must be in the interval between $\inf_i P(E|B_i)$ and $\sup_i P(E|B_i)$. This is intuitively compelling, but it is not true if we only have additivity. As with the Ramsey-deFinetti’s subjective probability, there is a known remedy to Savage’s version also: Villegas (1964) added a monotone continuity condition to make the probability countably additive.

3.3 Conditional probability

Conditional probability can be constructed starting with conditional preference as described in Axiom 2. If $\preceq$ is a qualitative probability relation, then for events $B$ and $C$, and non-null event $B$, we can define the conditional relation $B \preceq C$ given $D$ to mean $B \cap D \preceq C \cap D$. It can then be shown that the conditional relation is also a qualitative probability. If $\preceq$ is numerically represented by (‘almost agrees with’) the probability measure $P(\cdot)$, then the conditional relation is represented by

$$P(B|D) = \frac{P(B \cap D)}{P(D)}.$$  

This is of course the key formula that allows update of subjective probability, the crucial step of learning from experience, which forms the basis of Bayesian inference. Savage then moved imperceptibly from the qualitative to a full quantitative conditional probability; presumably all follows from Axioms 1–6.

Then came (i) a familiar introduction of the Bayesian method with updating of prior probability $P(B_i)$ of event $B_i$, using data $x = (x_1, \ldots, x_n)$, to produce posterior probability

$$P(B_i|x) \propto P(x|B_i)P(B_i).$$
And (ii) an exposition of exchangeability based on de Finetti’s example on the inference of the success probability \( p \) from a sequence of Bernoulli trials. The classical objective view is that \( p \) is a fixed unknown parameter. For de Finetti, ‘objective’ is simply an agreement of subjective opinions. Different people may start with different opinion – hence different priors – but their posterior probability will eventually agree, so there is no need for the objective view. In both expositions there is an emphasis on consistency as \( n \) tends to infinity, and there is no discussion on how to get the prior distribution.

### 3.4 Utility, Axiom 7 and the theorem

Up to this point, acts that are bets on events can be compared by numerical probabilities. But we cannot compare acts with arbitrary consequences, which is of course necessary for general statistical applications. To do that we need the concept of utility \( U \), such that

\[
f \leq g \text{ if and only if } U(f) \leq U(g).
\]  

(1)

In this case we say that the preference ordering agrees with the utility. Given a probability measure \( P(\cdot) \) on \( S \), the utility \( U(f) \) is the expected value

\[
U(f) \equiv \int_S U(f(s))dP(s),
\]

where \( U(\cdot) \) is the utility function applied to consequences \( f(s) \). The utility concept allows a convenient arithmetic representation of acts and their preference ordering. If we consider the axioms normatively as axioms of rational decisions, then the utility function is a calculator of rationality, so it has special role in any discussion of rationality and economic behavior.

Savage’s utility theory is largely influenced by von Neumann and Morgenstern’s (1947, VNM) utility. In the final foundational chapter (Chapter 5), Savage discussed the utility concept at great length, providing the historical background from the time of Daniel Bernoulli in the 18th century, and defending it from the then orthodox economic view that had dismissed any meaningful value of cardinal – as opposed to ordinal – utility. He was siding strongly with von Neumann and Morgenstern, and perhaps even going beyond them by interpreting the theory normatively as a guide of rational behavior.

As we have seen above, three of the six axioms are in fact closely related to VNM’s three axioms to establish the existence of utility as the basis of preferential ordering of lotteries. By first defining a gamble as an act with a finite set of consequences, Savage first proved that Axioms 1–6 are sufficient to guarantee the existence of a utility function that agrees with the preferences between gambles. So Axiom 7 is introduced to allow theoretically infinite number of consequences:

**P7 (Strong dominance)** If \( f \leq g(s) \) for every \( s \) in \( B \), then \( f \leq g \) given \( B \).

This means, if every consequence of \( g \) is preferable to \( f \) as a whole, then \( g \) must be preferable to \( f \). Savage considered this as a stronger version of the sure-thing principle. P7 is clearly a stronger version of the dominance principle, which only compares the consequences
\[ f(s) \leq g(s) \] at each state \( s \). As we mentioned above, Savage proved that, assuming P1 and P2, postulate P3 is equivalent to the dominance principle. This means that P7 makes P3 redundant in the whole set of axioms (cf. Hartmann, 2020).

The theorem

The utility concept completes Savage’s axiomatic development, and the overall theorem is stated as follows: The preference ordering of acts satisfies the postulates P1–P7 if and only if there exists a unique finitely additive non-atomic probability measure \( P \) on \( S \) and a real-value bounded utility function \( U(\cdot) \) on \( F \), such that the preference ordering agrees with expected utility in the sense of [1].

This means someone who acts rationally, in the sense that his preferences satisfy the axioms, behaves as if he has a probability measure and a utility function, and he maximizes the expected utility. In principle he does not have to be even aware of his subjective probability and utility function. However, in practice it is much easier to first assume explicitly the probability and the utility function, then work out the implied preferences. For example, in Bayesian statistics, one typically starts with a prior distribution and a minus squared-error loss as utility. According of the theorem, maximizing the expected utility corresponds to choosing an optimal act. By staying within the convenient utility framework, in other words ‘by following the expected utility theory,’ your preferences are guaranteed ‘rational.’ Any decision that violates the expected utility can be shown to violate at least one of the axioms.

Savage’s theorem can be seen as an amalgamation of Ramsey-de Finetti’s subjective probability and von Neumann-Morgenstern’s utility. Strictly speaking, when the probability is objective – as in a lotto – the utility theory is covered in von Neumann-Morgenstern’s framework, otherwise it is in Savage’s.

4 Paradoxes

We now discuss two well-known paradoxes, not in order to dismiss the axioms, but to illuminate the logical content and implications of some of them.

4.1 Allais’s paradox

The French economist and nobel laureate M. Allais (1953) described two betting situations where people tend to prefer alternatives that contradict the expected utility theory, hence violating at least one of the axioms. It is an important paradox, because among the violators, when Allais first presented it in a 1952 meeting on the economic theory of risk, were future Nobel prize winners in economics (Paul Samuelson, Milton Friedman and Kenneth Arrow). Savage also happened to be there; his reaction will be presented below.

Situation 1. Choose between these two bets (it’s important to use these large amounts of money, because utility is magnitude-dependent, though one can modify $2,500,000 with smaller amounts such as $1,000,000 and preserve the paradox):
I. Win $500,000 with probability 1,
II. Win $2,500,000 with probability 0.1,
   or $500,000 with probability 0.89
   or nothing with probability 0.01.

Situation 2. Choose between these two bets:

III. Win $500,000 with probability 0.11,
     or nothing with probability 0.89.
IV. Win $2,500,000 with probability 0.1,
    or nothing with probability 0.9.

Assuming a linear utility function, the expected utility of bet II is

\[ U(\text{II}) = 0.1 \times 2,500,000 + 0.89 \times 500,000 = 695,000, \]

which is greater than \( U(\text{I}) = 500,000 \), so you should prefer II. But in reality, most people would be happy with the smaller but sure amount of money in option I. The small potential of getting nothing in bet II is not compensated by the another small potential of a bigger jackpot. So people tend to prefer I over II. This is a well-known risk aversion, but, wait, it is not yet the paradox in question.

How about between III and IV? A small reduction in the probability of winning $500,000 in III is rewarded by a large increase in the jackpot in IV. As expected, most people would indeed prefer IV over III. Now we have a paradox: that choice turns out to violate the expected utility theory. Kahneman and Tversky (1979) conducted many surveys of people’s preferences in bets similar to Allais’s and confirmed the offending preferences.

The usual explanation of the paradox is nothing to do with non-linear utility nor with risk aversion. We can first put the four bets in terms of their expected utilities for a general non-linear utility function \( U(\cdot) \) (for convenience, let’s set \( U(0) \equiv 0 \), and drop 000s from the numbers):

\[
\begin{align*}
U(\text{I}) & = U(500) \\
U(\text{II}) & = 0.1U(2,500) + 0.89U(500) \\
U(\text{III}) & = 0.11U(500) \\
U(\text{IV}) & = 0.1U(2,500)
\end{align*}
\]

Preferring I>II implies \( U(500) > 0.1U(2,500) + 0.89U(500) \), or \( 0.11U(500) > 0.1U(2,500) \). But the last preference ordering means you should prefer III over IV, which contradicts people’s common preference of IV over III.

The common preference is usually cited as a violation of the sure-thing principle. Imagine 100 lottery tickets numbered 1 to 100, where you pick one ticket, and a winning number is chosen randomly. The payoffs are given in Table \( \text{5} \). Then the four bets in the table are exactly equivalent as above, and we can see clearly the set-up of the sure-thing principle. Under the principle, the last column (tickets 12-100) represents the irrelevant event, so the preference I over II must be consistent with III over IV.

Savage initially chose like most people do (i.e., preferring I over II, and IV over III), but after realizing that he violated the sure-thing principle, he changed his preference to
Table 3: Allais’s four bets are equivalent to the bets in this table, where the columns indicate ticket numbers and the entries are the payoffs. You pick one number between 1 and 100, and a winning number is selected randomly. So, the probability of winning is 0.01, 0.10, and 0.89 for the three columns, respectively.

III > IV. He wrote (Savage, 1972, page 103) ‘in reversing my preference between Gambles 3 and 4 I have corrected an error.’ But, why did he reverse III and IV, not I and II? The sure-thing principle only states that the two pairs of bets must be consistent, but does not say which one is the right choice. In any case, Savage’s correction highlights the normative value of the axioms as a guide for rational decision, with power to correct our subjective preferences. It raises an interesting question whenever there is a conflict: which have the primacy? Do we modify our preferences or modify our axioms? We discuss this further in the Discussion Section.

| Bet | 1   | 2–11 | 12–100 |
|-----|-----|------|--------|
| I   | 500 | 500  | 500    |
| II  | 0   | 2500 | 500    |
| III | 500 | 500  | 0      |
| IV  | 0   | 2500 | 0      |

Table 4: Modified bets $I_0$ and $III_0$ in Allais’s paradox, following Table 3, where the payoff of some tickets are set to zero. For clarity, the original bets are also listed.

The explanation of Allais’s paradox as a violation of the sure-thing principle is well-known, but we can also argue that the paradox violates the small-event continuity axiom (Axiom 6). Viewed in this perspective, the paradox occurs because of a conflict between the axiom and risk aversion. Let’s modify the payoff of some bets on events with small probability, i.e. on ticket 1 with probability 0.01, to get $I_0$ and $III_0$. These modifications are shown in Table 4. To be clear, the reasoning steps are itemized as follows:

- By the strong-dominance principle (Axiom 7), we must have $II > I_0$ and $IV > III_0$.

- $III$ is a small-event modification of $III_0$ using a comparable-sized payoff, so by Axiom 6, we should have $IV > III$. This is the commonly observed preference, so in fact it is justified by Axioms 6 and 7.

- Now, the modification $I_0 \rightarrow I$ is exactly the same as $III_0 \rightarrow III$, so again by Axiom 6, we should have $II > I$. However, the small-event modification from $I_0$ to $I$ generates a
sure gain, so for most people the ordering II > I₀ gets reversed to I > II, thus violating Axiom 6.

In this analysis, the preference IV > III is seen not as an error, but a choice that agrees – at least in spirit if not formally – with Axioms 6 and 7. So, as we remarked above, there is actually no rational basis for Savage to reverse his initial preference of IV > III. The reason he reversed it was because he first preferred I > II, which was justified by risk aversion. But risk aversion is an empirical phenomenon that is not implied by any of the axioms and plays no role in Savage’s theory. So, normatively, reversing the risk-averse preference I > II is perhaps a more consistent choice. But you would sacrifice risk aversion in order to follow Axiom 6.

Instead of using Axiom 6, we could also invoke the sure-thing principle to justify II > I, but here we want to highlight the violation of Axiom 6. We could also say: Axiom 6 and the sure-thing principle together contradict risk aversion. Or, risk aversion and the sure-thing principle together contradict Axiom 6. So, to the extent that the small event in Axiom 6 is an approximation of real events with small probability, the axiom is in conflict with risk aversion. It is an important conflict, because risk aversion is closely connected to the universally accepted law of diminishing returns. In view of the discussion following Axiom 6, the reversal of preference occurs even when there is no event with infinitely better or infinitely worse consequence. The reversal – marking a discontinuity – occurs when even a small-event modification creates certainty, as people behave differently when dealing with sure events.

Such a violation is not a rare or exotic phenomenon; as described by Kahneman and Tversky (1979) it is commonly seen, for example, in gambling and insurance decisions. They took the paradox seriously and developed an alternative axiomatic system called the prospect theory, in part to allow for discontinuity when the probability is near 0 or near 1. Instead of maximizing expected utility, in prospect theory a decision maker maximizes the score

\[ V \equiv \sum_i \pi(p_i)U(f_i), \]  

where \( U(f_i) \) is a function of the payoff \( f_i \) (they called it ‘value function,’ but in principle it works like a utility function), \( p_i \) is the probability of state \( i \), and \( \pi(\cdot) \) is a decision weight function with \( \pi(0) \equiv 0 \) and \( \pi(1) \equiv 1 \). In the expected utility theory \( \pi(p_i) = p_i \), but in general \( \pi(p) \neq p \). An important feature of the theory is that the decision weight function has a discontinuity property: \( \pi(0^+) > 0 \) and \( \pi(1^-) < 1 \), which implies a sub-certainty property \( \pi(p) + \pi(1-p) < 1 \) when \( 0 < p < 1 \). Furthermore, the prospect score \([2]\) can be re-written as

\[ V = \sum_i p_i \{ \pi(p_i)U(f_i)/p_i \} \equiv \sum_i p_i U^*(f_i, p_i), \]

where \( U^*(f_i, p_i) \equiv \pi(p_i)U(f_i)/p_i \) becomes a chance-dependent or state-dependent utility function, thus violating Axiom 3.

Prospect theory is considered one of the cornerstones of behavior economics; Kahneman was awarded Nobel prize in economics in 2002 for this work. In particular, the theory provides better explanations of distinct human reactions to gains and losses, and of behavior
near impossibility or certainty. Allais’s paradox can be seen as being due to special behavior associated with sure gain in Bet I. The scores are

\[
\begin{align*}
V(I) &= U(500) \\
V(II) &= \pi(0.1)U(2,500) + \pi(0.89)U(500) \\
V(III) &= \pi(0.11)U(500), \\
V(IV) &= \pi(0.1)U(2,500).
\end{align*}
\]

The preferences I > II and IV > III imply:

\[
\{1 - \pi(0.89)\}U(500) > \pi(0.1)U(2,500) \\
\pi(0.1)U(2,500) > \pi(0.11)U(500),
\]

giving \(\pi(0.89) + \pi(0.11) < 1\), which is the sub-certainty property anticipated (and allowed) in the prospect theory.

We close this section with another explanation of the paradox, highlighting an unstated assumption in the setup of the sure-thing principle. Let’s assume that you have to make two simultaneous bets: (I vs II) and (III vs IV) based on a single draw of the lottery in Table 3. Preferring I > II and IV > III implies (I+IV) > (II+III), which is irrational because they have exactly the same payoffs. The other preference (I+III) vs (II+IV) depends on one’s utility of money, so there is no immediate inconsistency issue. This means that for the sure-thing principle to hold, the bets are presumed to be made simultaneously based on a single realization of the random outcome. Violators of the principle – which include many brilliant economists – are likely thinking of two independent bets, based on two random draws of the lottery. Had the dependence of the single draw of the lottery been made explicitly, a rational person is not likely to violate the principle.

| Bet   | 1   | 2–11 | 12–100 |
|-------|-----|------|--------|
| I+IV  | 500 | 3000 | 500    |
| II+III| 500 | 3000 | 500    |
| I+III | 1000| 1000 | 500    |
| II+IV | 0   | 5000 | 500    |

Table 5: Allais’s paradox explained in terms of two simultaneous bets on the outcome of a single draw of a lottery. Preferring I > II and IV > III implies (I+IV) > (II+III), which is irrational because they have exactly the same payoffs.

### 4.2 Ellsberg’s paradox

Ellsberg (1961) set up a paradox that suggests that people treat objective and subjective uncertainty differently. Briefly, in an urn, there are 90 coloured balls: 30 are red and the other 60 are an unknown mixture of black and yellow. You pick one ball from the urn and are given the option of (I) getting $100 if the ball is red, or (II) getting $100 if the ball is black. Which option would you prefer? While the payoff table is similar to the bets in
Table 6: The first situation from Ellsberg (1961): an urn contains 30 red balls and 60 black balls and yellow balls together, but with unknown proportion. You pick one ball from the urn; the table shows the payoffs based on the color of the ball. Do you prefer option I or II?

|       | Red | Black | Yellow |
|-------|-----|-------|--------|
| I     | 100 | 0     | 0      |
| II    | 0   | 100   | 0      |

Table 2, but the logical content is distinct because of the single information on black and yellow balls here.

Now consider the second scenario: Again, you pick one ball from the urn and are given the option of (III) getting $100 if the ball is red or yellow; or (IV) getting $100 if the ball is black or yellow. Which option would you prefer now?

|       | Red | Black | Yellow |
|-------|-----|-------|--------|
| III   | 100 | 0     | 100    |
| IV    | 0   | 100   | 100    |

Table 7: The second situation from Ellsberg (1961): similar setup as Table 1, but different payoffs. Do you prefer option III or IV?

If you are like most people (e.g., Camerer, 1992), you would prefer I over II, and IV over III. It shows that people tend to prefer the objective probability over the subjective one; behavioral economists call this tendency *ambiguity aversion*. This is clearly in violation of the sure-thing principle, which requires that preferring I over II implies preferring III over IV, and vice versa. According to the principle, the presence of yellow balls should not influence the preference between red and black balls.

In this example, the probability of picking a black or yellow ball is between 0 to 2/3, but this probability is subjective. Savage’s theory implies that prior probability can be determined by self-interrogation of subjective preferences. He supported de Finetti that binomial probability can be interpreted in terms of subjective probability alone. Let assume, as Bayes and Laplace did, the probability of picking a black ball follows the uniform distribution on 0 to 1. Then, Laplace’s law of succession shows that the probability of \( k \) black balls is 1/61 for all \( k = 1, \ldots, 60 \). This distribution satisfies de Finetti’s exchangeability, and was discussed thoroughly in Section 3.7 of Savage’s book. The expected value of the number of black balls is 30, and if we perform the experiment repeatedly the marginal chance of a black ball being selected is 1/3.

Thus, if we treat the subjective probability objectively, then I and II are equivalent; similarly III and IV. However, in a specific bet, the number of black balls could be 15 or 45, or any number between 0 to 60. While the number of red balls is predetermined as 30, the number of black balls is not determined but only has an expectation 30. Thus, the number of black balls has another layer of uncertainty, an ambiguity. Because of this ambiguity people prefer I over II. A Bayesian may say that he believes in his prior, so that I and II are equivalent, but if so he is ambiguity-neutral and ignores the extra uncertainty.
in a specific situation.

This paradox also highlights the descriptive and normative aspects of an axiomatic system. If the system is only descriptive, then we simply say that it does not predict well in this case. But if it is normative, as basic logic or arithmetic, then the violators must reconsider and correct their choices. Ellsberg (1961) reported various reactions among some well-known economists and statisticians. Some did not violate the principle (G. Debreu, R. Schlaifer, P. Samuelson). Some violated the principle ‘cheerfully’ (J. Marschak, N. Dalkey); others ‘sadly and persistently, having looked into their hearts, found conflicts with the axioms and decided to satisfy their preferences and let the axioms satisfy themselves. Still others (H. Raiffa) tend, intuitively, to violate the axioms but feel guilty about it ...’ Some who previously felt ‘first-order commitment’ to the principle were ‘surprised and dismayed to find that they wished, in these situations, to violate the Sure-Thing Principle.’ This special group ‘seems to deserve respectful consideration’ because it included none other than L. J. Savage himself.

As with the Allais’s paradox, there is a setup where ambiguity aversion is clearly irrational. It requires one to play two simultaneous bets I vs II, followed by III vs IV based on a single draw from the balls. First let’s put a price on each bet: since I and IV have objective probabilities, assuming linear utility, the fair prices are $33 and $67, respectively. If you have ambiguity aversion, you prefer I over II, and IV over III. This means you set lower for II (say $30) and III (say $65) relative to I and IV. If so, then I can run a Dutch Book against you: I will buy both bets II and III from you for a total of $95, and with that I am guaranteed to win $100. Now we can explain Ellsberg’s paradox as follows: The sure-thing principle implicitly presumes the preferred acts to be acted together by one abstract decision maker based on the a single realization of the random outcome. But, unless stated explicitly, a real person considers them one at a time as independent decisions based on independent draws from the balls. Explicitly requesting a rational and intelligent person to make two simultaneous bets based on a single draw will remove the aversion in this case. However, if the person is making independent single bets based on independent draws, it is not obvious whether ambiguity aversion violates the sure-thing principle.

5 Discussions

Savage dedicated a whole chapter to address criticisms of the (personalistic) subjective probability as well as his own criticisms of other views of probability. First, it is perhaps useful to go through a number of statements that capture his views (page 46, 56, 57). We can recognize here his strong influence on modern Bayesianism; see e.g. Lindley (2000).

...any mathematical problem concerning personal probability is necessarily a problem concerning probability measures – the study of which is currently called by mathematicians mathematical probability – and conversely.

...the concept of personal probability introduced and illustrated in the preceding chapter is... the only probability concept essential to science and other activities that call upon probability.
...the role of mathematical theory of probability is to enable the person using it to detect inconsistencies in his own real or envisaged behavior.

**Ambiguous probability**

Ellsberg’s paradox (1961) highlights an example where people react differently to different types of uncertainty, roughly as ‘sure’ and ‘unsure’ or ambiguous probability. Savage dismissed the notion of second-order probability to deal with such different levels of uncertainty as in ‘the probability that $B$ is more probable than $C$ is greater than the probability that $F$ is more probable than $G$.’ The concept of ambiguous belief or ‘imprecise probability’ was later taken up, for example, by Schmeidler (1989), Gilboa and Schmeidler (1993) and Binmore (2009).

Savage also did not support a model where the probability of $B$ is a random variable $b$ with respect to the second probability. He was concerned with ‘endless hierarchy’ that would be difficult to interpret, but the hierarchical models of this type have become popular in recent years. In fact, it is now common to think of Bayesian models as hierarchical, but Savage did not consider subjective probability as part of a hierarchical model.

**Inexact magnitude**

As with de Finetti’s coherence argument, the postulates imply that one can determine with high accuracy his subjective probability that Donald Trump will be re-elected. In reality we can only expect some rough magnitudes. Savage responded that the subjective theory, as a normative theory, is ‘a code of consistency for the person applying it, not a system of predictions...’ De Finetti’s (1931, page 204) response to the same problem is also relevant here: ‘...to apply mathematics, you must act as though the measured magnitudes have precise values. This fiction is very fruitful... To go, with the help of mathematics, from approximate premises to approximate conclusions, I must go by way to an exact algorithm...’

**Other views of probability**

Savage considered the subjective probability as lying not in between, but beside, the logical and objective views, because these latter two are meant to be free of individual preferences. For him, the strength of his axiomatic system is that it deals explicitly with the problem of decision under uncertainty. He saw a weakness in other subjective views, such as Koopman’s (1940), that did not explicitly deal with the problem of individual decisions. De Finetti’s coherent betting approach was criticized because it needs to assume the utility of money is linear (or the bet is small). In his later life de Finetti (1964) moved in the direction of Savage by putting probability and utility together within a decision theory.

For the objective view – i.e., frequentist – probability is primary, decision secondary. Furthermore, probability is only given to very special events, e.g. repeatable ones such as coin tosses, but not to very specific event such as the unification of Korea. For Savage, the objective view is ‘charged with circularity.’ It relies on the existence of processes that can be represented by infinitely repeatable events, but the degree of approximation is determined...
by the same theory of probability. Savage also rejected the need for a dualistic view of probability in inference, which allows both objective and subjective ones.

Objectivity in science

No one will stop you from betting your own money, but how can we justify the subjective beliefs in science? We may define objectivity as agreement between reasonable minds, excluding the possibility of differing individual preferences. Savage put forward some arguments why there is no reason to exclude subjective probability as part of scientific reasoning. One argument is, by consistency, any two differing opinions will be brought closer by sufficiently large evidence. The objective view presumes a common universally accepted opinion as the goal of science, but in reality there are ‘pairs of opinions, neither of which can be called extreme ... which cannot be expected to brought into close agreement after the observational program’ (Savage, 1972, page 68).

Ryder (1981) pointed out that an external agent can run a Dutch Book against two individuals that hold differing subjective probabilities of an event, i.e., make money from them regardless of what happens. Say John has a subjective probability 0.25 for re-election of Donald Trump, while Patrick has 0.15. Alice comes along to bet $100 against John, and $\text{-}100$ against Patrick (meaning she plays the bookie against John, but lets Patrick be the bookie; or, even more clearly, she buys from Patrick and sells to John). If Trump is re-elected, Alice’s gain is

$$g_1 = 0.25 \times 100 - 100 - 0.15 \times 100 + 100 = 10,$$

while, if he is not, she gains

$$g_2 = 0.25 \times 100 - 0.15 \times 100 = 10.$$

The only way to avoid the Dutch Book is for John and Patrick to have the same probability. (From the financial-market perspective, the two players behave like two mini-markets that offer different prices, which provide an arbitrage opportunity – i.e. risk-free investment – to an external agent.) Note that both John and Patrick are individually coherent, and if there is no communication between them, they are none the wiser about the Dutch Book run against them. But if they are closely related individuals with regular contacts, and especially with a joint economy, we could say they are incoherent, because they have allowed a Dutch Book against them.

If it is not too farfetched of an analogy, a collection of scientists in a scientific area is meant to be a closely communicating group with a common interest. So if the individual scientists have differing subjective probabilities of relevant events, the group can be said to be incoherent. This might explain the reluctance of nonBayesians or science in general to allow subjective preferences in the formal assessment of evidence.

6 Conclusions

An axiomatic system carries the strength of logic: once we agree with a set of propositions, then we must agree with their implications. It is therefore important that we understand
Savage’s axioms in details. Savage clearly viewed Bayesian statistics not simply about attaching a prior distribution to a statistical problem, but a holistic framework for making rational decisions under uncertainty. The framework is strong enough to carry a normative force.

Axioms cannot be proved or disproved mathematically, but they can be checked empirically. For example, the fifth axiom in Euclid’s geometry – the so-called parallel postulate – can be corroborated if the total sum of angles of the triangle in the space is 180 degrees. In a universe where the angles do not sum to 180 degrees its geometry cannot be Euclidian. On the earth surface, large triangles will have angles adding up to more than 180 degree. Thus, the axioms can be checked via experience or observations and we may encounter situations that violate the axioms. Shafer (1986) previously revisited Savage primarily to object the normative aspect of the theory, in light of accumulating empirical evidence that people violate the postulates.

For statistical applications, curiously Savage did not discuss the problem of how to choose a prior distribution, which was already considered by many writers from the 19th century as a weakness in the inverse probability method. Much of this problem seems to be addressed currently by the so-called objective methods, such as Jeffreys’s non-informative prior, but it is a substantial topic in itself and way beyond the scope of this paper.

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