Centrality of odd unitary $K_2$-functor

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May 7, 2020

Abstract

Let $(R, \Delta)$ be an odd form algebra. We show that the unitary Steinberg group $StU(R, \Delta)$ is a crossed module over the odd unitary group $U(R, \Delta)$ in two major cases: if the odd form algebra has a free orthogonal hyperbolic family satisfying local stable rank condition and if the odd form algebra is sufficiently isotropic and quasi-finite. The proof uses only elementary localization techniques and stability results for $KU_1$ and $KU_2$.

1 Introduction

Centrality of $K_2$ for the general linear group over arbitrary commutative ring $K$ is proved in [14] by W. van der Kallen. His proof actually shows that $St(n, K) \to GL(n, K)$ is a crossed module for sufficiently large $n$. In [13] M. S. Tulenbaev generalized this result for all almost commutative $K$.

Later in [7, 8, 11] S. Sinchuk and A. Lavrenov proved similar result for the Chevalley groups of type $C_l$, $D_l$, and $E_l$.

These proofs use the so-called “another presentation” of the Steinberg group in terms of non-elementary transvections or reduce to the linear case. In [15] we proved that $St(R) \to GL(R)$ is a crossed module for any almost commutative ring $R$ with a complete family of full orthogonal idempotents, i.e. for isotropic linear groups. Our proof also works for matrix algebras over rings with small local stable rank. In the isotropic case there is no natural notion of unimodular vectors, hence we cannot even formulate van der Kallen’s approach in such generality.

For even unitary matrix groups in the sense of A. Bak centrality was announced in 1998 by Bak and G. Tang, but this result is unpublished. In the local case A. Stavrova proved centrality in [12] for all sufficiently isotropic reductive groups, including the classical groups. Also centrality easily follows from

*Research is supported by the Russian Science Foundation grant 19-71-30002.
surjective stability of $KU_2$. See [17] and [18] for injective stability of $KU_1$ in the odd unitary case, the proofs from these papers also give surjective stability of $KU_2$.

In [9] V. Petrov defined odd unitary groups. These groups generalize even unitary groups and most classical groups. They are defined in terms of odd form parameters on modules with hermitian forms, so they have geometric nature but it is hard to apply algebraic constructions such as Stein’s relativization or faithfully flat descent.

We discovered odd form algebras in [17]. This objects are more flexible than Petrov’s module with odd form parameters. Moreover, it is possible to define the unitary group of an odd form algebra and its elementary subgroup. In [16] we proved that all twisted forms of the most important classical reductive group schemes arise as parts of unitary and projective unitary groups of odd form algebras.

Isotropic elementary subgroups of reductive groups are defined in [10] by V. Petrov and A. Stavrova, where they also proved normality. In the present paper we show how parabolic subgroups of classical reductive groups may be described purely algebraically, using orthogonal hyperbolic families from [17]. For non-twisted linear groups this objects reduce to families of full orthogonal idempotents.

Our goal is to generalize the main result from [15] to the case of odd unitary groups. In sections 2–4 we give all necessary definitions concerning odd form algebras, hyperbolic pairs, and orthogonal hyperbolic families. In section 5 we prove the connection between parabolic subgroups of classical reductive groups and parabolic subgroups of odd unitary groups defined in terms of orthogonal hyperbolic families. Sections 6–9 contain the technical part of our work: construction of odd unitary Steinberg pro-group and results about decreasing the orthogonal hyperbolic family. In section 8 we prove that the local unitary group $U(S^{-1}R, S^{-1}\Delta)$ acts on the global Steinberg pro-group $\text{StU}(R, \Delta)(\infty)$ by automorphisms. Finally, in section 9 we prove the main result:

**Theorem.** Let $K$ be a commutative ring, $(R, \Delta, D)$ be an augmented odd form $K$-algebra with an orthogonal hyperbolic family of rank $n$. Suppose that $n \geq 4$ or $n \geq 3$ and the orthogonal hyperbolic family is strong. Suppose also that $R$ is quasi-finite over $K$ or $\text{Asr}(\eta_1, R, \Delta) \leq n - 1$ and $\text{lsr}(R_{11}) \leq n - 2$. Then there is unique action of $U(R, \Delta)$ on $\text{StU}(R, \Delta)$ making it a crossed module, it is consistent with the action of $\text{StU}(R, \Delta) \rtimes \Delta(R, \Delta)$.

Together with Lavrenov’s result [6] that the odd unitary Steinberg group is universally closed out result give an explicit presentation of the universal central extension of the elementary unitary group.

## 2 Odd form algebras

For group operations we use the conventions $^x y = xyx^{-1}$, $x^y = y^{-1}xy$, and $[x, y] = xyx^{-1}y^{-1}$. We often implicitly use the commutator identities $[xy, z] =$
A hermitian form on a module $M$ is a bilinear map $\langle \cdot, \cdot \rangle : M \times M \to \mathbb{C}$ such that $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for all $a, b \in \mathbb{C}$ and $x, y, z \in M$. A complex vector space $V$ equipped with a hermitian form $\langle \cdot, \cdot \rangle$ is called a hermitian space.

A crossed module is a precrossed module with the property $\lambda$ where $\lambda$ from $\Delta$ to $\mathbb{C}$ is an odd form parameter. Such a pair $(\Delta, \lambda)$ is called a quadratic form. A unitary group of $\Delta$ is unital. Fix a ring $\mathcal{S}$ with an action of $\Sigma^*$, where $\Sigma^*$ is the multiplicative semi-group of invertible elements by $x \mapsto x + 1$. By $\Sigma^*$ we denote the multiplicative semi-group of $\mathcal{S}$.

Recall the definitions from \cite{9, 16, 17}. Let $\mathcal{S}$ be arbitrary unital ring and $\lambda \in \mathcal{S}^*$. A map $(\cdot, \cdot) : \mathcal{S} \to \mathcal{S}$ is called a $\lambda$-involution if it is an anti-automorphism of $\mathcal{S}$ with $(s, s) = \lambda s \lambda^{-1}$ and $\overline{\lambda} = \lambda^{-1}$. An involution is the same as a 1-involution, in this case we do not need that $\mathcal{S}$ is unital. Fix a ring $\mathcal{S}$ with a $\lambda$-involution. A hermitian form on a module $M_{\mathcal{S}}$ is a biadditive map $B: M \times M \to \mathcal{S}$ such that $B(m, m') = B(m, m') s$ and $B(m', m) = B(m, m') \lambda$.

Fix a module $M_{\mathcal{S}}$ with a hermitian form $B$. The Heisenberg group is the set $\text{Heis}(B) = M \times \mathcal{S}$ with the group operation $(m, x) + (m', x') = (m + m', x - B(m, m') + x')$. This group has a right $\Sigma^*$-action, $(m, x) \cdot s = (ms, x s)$. An odd form parameter is an $\Sigma^*$-subgroup $\mathcal{L} \leq \text{Heis}(B)$ such that $\mathcal{L}_{\min} \leq \mathcal{L} \leq \mathcal{L}_{\max}$, where $\mathcal{L}_{\min} = \{(0, x - x \lambda) \mid x \in \mathcal{S} \}$ and $\mathcal{L}_{\max} = \{(m, x) \mid B(m, m) + x + x \lambda = 0 \}$.

For any odd form parameter the map $q : M \to \text{Heis}(B)\Lambda, m \mapsto (m, 0) + \lambda$ is called a quadratic form. A unitary group of $M$ consists of the automorphisms of $M$ stabilizing both hermitian and quadratic forms. An $\mathcal{S}$-module with fixed hermitian and quadratic forms is called a quadratic $\mathcal{S}$-module.

It turns out that for any quadratic $\mathcal{S}$-module there is a ring $R$ with an involution and an odd form parameter $\Delta \leq \text{Heis}(B_R)$ (where $B_R(a, b) = \overline{a} b$) such that its unitary group is isomorphic to the unitary group of $M$, see \cite{16, 17}. Such a pair $(R, \Delta)$ is the same as a special unital odd form ring according to the definition below if we consider $\pi$ and $\rho$ are the first and the second projections from $\Delta$ to $R$, and take $\phi(a) = (0, a - \overline{a})$.

An odd form ring is a pair $(R, \Delta)$, where $R$ is a non-unital ring with an involution $x \mapsto \overline{x}$, $\Delta$ is a group (with the group operation $+$), the semi-group $\Delta^*$ acts on $\Delta$ from the right by endomorphisms (the action is denoted by $u \cdot a$), and there are fixed maps $\phi : R \to \Delta, \pi : \Delta \to R, \rho : \Delta \to R$ such that for all $u, v \in \Delta$ and $a, b \in R$

- $\pi(u + v) = \pi(u) + \pi(v), \pi(u \cdot a) = \pi(u) a$.
\[
\begin{align*}
\phi(a + b) &= \phi(a) + \phi(b), \quad \phi(b) \cdot a = \phi(aba); \\
\rho(u + v) &= \rho(u) - \pi(u)\pi(v) + \rho(v), \quad \rho(u \cdot a) = \rho(u)a; \\
\rho(u) + \rho(u) + \pi(u)\pi(u) &= 0; \\
\pi(\phi(a)) &= 0, \quad \rho(\phi(a)) = a - \tilde{a}; \\
[u, v] &= \phi(-\pi(u)\pi(v)); \\
\phi(a) &= 0 \text{ if } a = \tilde{a}; \\
u \cdot (a + b) &= u \cdot a + \phi(b)\rho(u)a + u \cdot b.
\end{align*}
\]

It follows that \(\Delta\) is 2-step nilpotent \((\phi(R) \unlhd \Delta\) is a central subgroup\), \(\rho(\tilde{0}) = 0, \rho(\tilde{-u}) = \rho(u),\) and \(u \cdot 0 = 0\). An odd form ring is called unital if \(R\) is unital and \(u \cdot 1 = u\) for all \(u \in \Delta\). An odd form ring is called special if \((\pi, \rho) : \Delta \to R \times R\) is injective.

A morphism \(f : (R, \Delta) \to (S, \Theta)\) of odd form rings consists of maps \(f : R \to S\) and \(f : \Delta \to \Theta\) preserving all operations. An odd form ideal in an odd form ring \((R, \Delta)\) is a pair \((I, \Gamma)\) such that \(I = \bar{T} \subseteq R\) is an ideal, \(\Gamma \subseteq \Delta\) is an \(R^\circ\)-subgroup, and \(\Delta - I + \phi(\{a \in R \mid a - \tilde{a} \in I\}) \leq \Gamma \leq \{u \in \Delta \mid \pi(u) = \rho(u) = 0\}\). If \((I, \Gamma) \leq (R, \Delta)\), then \((R/I, \Delta/\Gamma)\) is also an odd form ring. We say that the sequence \((I, \Gamma) \to (R, \Delta) \to (R/I, \Delta/\Gamma)\) is short exact.

An odd form ring \((R, \Delta)\) acts on an odd form ring \((S, \Theta)\) if there are multiplication maps \(R \times S \to S, S \times R \to S, \Theta \times R \to \Theta,\) and \(\Delta \times S \to \Theta\) such that for all \(a, a' \in R; b, b' \in S; u, u' \in \Delta;\) and \(v, v' \in \Theta\)

\[
\begin{align*}
ab &= \bar{ba}; \\
(a + a')b &= ab + a'b, \quad a(b + b') = ab + ab'; \\
(aa')b &= a(a'b), \quad (ab)b' = a(bb'), \quad (a)b'a = a(ab'), \quad (ba)b' = b(ab'); \\
(u + u') \cdot b &= u \cdot b + u' \cdot b, \quad u \cdot (b + b') = u \cdot b + \phi(b)\rho(u)b + u \cdot b'; \\
(v + v') \cdot a &= v \cdot a + v' \cdot a, \quad v \cdot (a + a') = v \cdot a + \phi(a)\rho(v)a + v \cdot a'; \\
(u \cdot a) \cdot b &= u \cdot ab, \quad (u \cdot b) \cdot a = u \cdot ba, \quad (u \cdot b) \cdot b' = u \cdot bb'; \\
(v \cdot a) \cdot b &= v \cdot ab, \quad (v \cdot b) \cdot a = v \cdot ba, \quad (v \cdot a) \cdot a' = v \cdot aa'; \\
\phi(a) \cdot b &= \phi(b)\rho(ab), \quad \phi(b) \cdot a = \phi(aba); \\
\pi(u \cdot b) &= \pi(u)b, \quad \pi(v \cdot a) = \pi(v)a; \\
\rho(u \cdot b) &= \bar{b}\rho(u)b, \quad \rho(v \cdot a) = \bar{a}\rho(v)a.
\end{align*}
\]

An action of a unital odd form ring \((R, \Delta)\) on an odd form ring \((S, \Theta)\) is called unital if \(b1 = b = 1b\) for all \(b \in S\) and \(v \cdot 1 = v\) for all \(v \in \Theta\). Actions of \((T, \Xi)\) on \((R, \Delta)\), of \((T, \Xi)\) on \((S, \Theta)\), and of \((R, \Delta)\) on \((S, \Theta)\) are called coherent if in addition for all \(a \in T, b \in R, c \in S, u \in \Xi, v \in \Delta,\) and \(w \in \Theta\)
\( (ab)c = a(bc), (ac)b = a(cb), (ba)c = b(ac); \)
\( (u \cdot b) \cdot c = u \cdot bc, (u \cdot c) \cdot b = u \cdot cb; \)
\( (v \cdot a) \cdot c = v \cdot ac, (v \cdot c) \cdot a = v \cdot ca; \)
\( (w \cdot a) \cdot b = w \cdot ab, (w \cdot b) \cdot a = w \cdot ba. \)

**Lemma 1.** If an odd form ring \((R, \Delta)\) acts on \((S, \Theta)\), then \((S \times R, \Theta \times \Delta)\) is short exact. The actions of \((T, \Xi)\) on \((R, \Delta)\), of \((T, \Xi)\) on \((S, \Theta)\), and of \((R, \Delta)\) on \((S, \Theta)\) are coherent if and only if they induce the action of \((T, \Xi)\) on \((S \times R, \Theta \times \Delta)\) if and only if they induce the action of \((R \times T, \Delta \times \Xi)\) on \((S, \Theta)\), in this case \((S \times R \times T, \Theta \times \Delta \times \Xi)\) is well-defined. A short exact sequence \((I, \Gamma) \rightarrow (R, \Delta) \rightarrow (S, \Theta)\) is well-defined. Moreover, if \((R, \Delta)\) is unital and its actions are unital, then the iterated semi-direct product is also unital.

*Proof.* Note that if \((R, \Delta)\) acts on \((S, \Theta)\), then the group \(\Delta\) acts on the group \(\Theta\) by \(v \cdot u = \phi(\pi(u) \pi(v))\), so \(\Theta \times \Delta\) is well-defined (also \(S \times R = S \oplus R\) as an abelian group). The operations on the semi-direct product are uniquely determined by the operations on the factors and by the action. All axioms and all other claims follow from direct computation using the notion of quadratic maps from [5] and lemma 1 from [16].

Fix a commutative ring \(K\). Odd form \(K\)-algebras from [16] are precisely odd form rings \((R, \Delta)\) with a unital action of the odd form ring \((K, 0)\) with the trivial involution on \(K\) such that \(ak = ka\) for all \(a \in R, k \in K\). Every odd form ring is an odd form \(\mathbb{Z}\)-algebra with uniquely determined multiplication \(\Delta \times \mathbb{Z} \rightarrow \Delta\).

Recall from [16] that a 2-step \(K\)-module is a pair \((M, M_0)\), where \(M\) is a group with a right \(K^*\)-action (the group operation and the action are denoted by \(m + m'\) and \(m \cdot k\)); \(M_0\) is a subgroup of \(M\) and a left \(K\)-module, and there is a map \(\tau: M \rightarrow M_0\) such that

- \([M, M] \leq M_0, [M, M_0] = 0;\)
- \([m \cdot k, m' \cdot k'] = kk'[m, m];\)
- \(m \cdot (k + k') = m \cdot k + kk' \tau(m) + m \cdot k';\)
- \(m \cdot k = k^2m\) for \(m \in M_0\).

In every 2-step \(K\)-module \((M, M_0)\) there are identities \(\tau(m) = m + m \cdot (-1)\) (in particular, \(\tau(m) = 2m\) for \(m \in M_0\)), \(\tau(m \cdot k) = k^2 \tau(m), \tau(m + m') = \tau(m) + [m, m'] + \tau(m')\). Also, \(M/M_0\) is a right \(K\)-module.
An augmented odd form \(K\)-algebra is a triple \((R, \Delta, D)\) such that \((R, \Delta)\) is an odd form \(K\)-algebra and \(D \leq \Delta\) is an \(R^*\)-subgroup and a left \(K\)-module such that for all \(a \in R, k \in K, v \in D\):

- \(\phi(a) \in D, \phi(ka) = k\phi(a);\)
- \(\pi(v) = 0, v \cdot k = k^2v, \rho(kv) = k\rho(v), kv \cdot a = k(v \cdot a).\)

Then \((\Delta, D)\) is a 2-step \(K\)-module. Every odd form \(K\)-algebra \((R, \Delta)\) has an augmentation \(D = \phi(R)\).

A unitary group of an odd form \(K\)-algebra is a certain subset of \(R \times \Delta\). We denote the componts of its elements \(g\) by \(\beta(g) \in R\) and \(\gamma(g) \in \Delta\), so \(g = (\beta(g), \gamma(g))\). Also let \(\alpha(g) = \beta(g) + 1 \in R \times K\). A unitary group is

\[
U(R, \Delta) = \{ g \in R \times \Delta \mid \alpha(g)^{-1} = \overline{\alpha(g)}, \pi(\gamma(g)) = \beta(g), \rho(\gamma(g)) = \overline{\beta(g)} \}.
\]

The group operation is given by \(\alpha(\gamma g') = \alpha(g) \alpha(g')\) and \(\gamma(gg') = \gamma(g) \cdot \alpha(g') + \gamma(g')\). In the special unital case \(U(R, \Delta) \cong \{ g \in R^* \mid g^{-1} = \tilde{g}, (g-1, \tilde{g}-1) \in \Delta \}\), so this definition is consistent with unitary groups of quadratic modules. The unitary group acts on \((R, \Delta)\) by automorphisms in the following way:

\[
\eta_u = \alpha(g) a \overline{\alpha(g)}, \quad \eta_k = (\gamma(g) \cdot \pi(u) + u) \cdot \overline{\alpha(g)}.
\]

It is easy to see that if \((R, \Delta, D)\) is an augmented odd form algebra, then the action of \(U(R, \Delta)\) preserves the augmentation.

In \([10]\) all classical unitary groups are explicitly constructed from augmented odd form algebras. The linear odd form algebra \((R, \Delta, D) = \text{AL}(n, K)\) has \(R = M(n, K)^op \times M(n, K)\) with \((aop, b) = (bop, a)\), its unitary group is \(\text{GL}(n, K)\). The symplectic odd form algebra \((R, \Delta, D) = \text{ASp}(2n, K)\) has \(R = M(2n, K)\) with \(\overline{e_{ij}} = e_{-j,-i}\) \text{sign}(i) \text{sign}(j)\) (the indices are from the set \(\{-n, \ldots, -1, 1, \ldots, n\}\)), its unitary group is \(\text{Sp}(2n, K)\). Similarly, the orthogonal odd form algebra of even rank \((R, \Delta, D) = \text{AO}(2n, K)\) has \(R = M(2n, K)\) with \(\overline{e_{ij}} = e_{-j,-i}\), its unitary group is \(\text{O}(2n, K)\). Finally, the orthogonal odd form algebra of odd rank \((R, \Delta, D) = \text{AO}(2n+1, K)\) has \(R = \bigoplus_{-n \leq i,j \leq n} \mathbb{R}e_{ij}\) with \(e_{ij}e_{kl} = 0\) for \(j \neq k\), \(e_{ij}e_{jk} = e_{ik}\) for \(j \neq 0\), \(e_{00}e_{0k} = 2e_{ik}\), and \(e_{ij} = e_{-j,-i}\), its unitary group is \(\text{SO}(2n+1, K) \times \mathbb{Z}/2\mathbb{Z})(K)\). All these odd form algebras are special with the augmentation \(D = \text{Ker}(\pi)\).

### 3 Hyperbolic pairs

From now on fix an augmented odd form \(K\)-algebra \((R, \Delta, D)\). A tuple \(\eta = (e_{-\eta}, e_\eta, q_{-\eta}, q_\eta)\) is called a hyperbolic pair if \(e_{\eta}, e_{-\eta}\) are orthogonal idempotents in \(R\) with the property \(e_{-\eta} = e_\eta\) and \(q_{-\eta}, q_\eta\) are elements of \(\Delta\) such that \(\pi(q_{-\eta}) = e_\eta, \pi(q_\eta) = e_{-\eta}, \rho(q_\eta) = \rho(q_{-\eta}) = 0, q_{-\eta} = q_\eta \cdot e_{\eta}, q_\eta = q_{-\eta} \cdot e_{-\eta}\).

We denote the idempotent \(e_\eta + e_{-\eta}\) by \(e_{[\eta]}\). Hyperbolic pairs \(\eta, \tilde{\eta}\) are called orthogonal if \(e_{[\eta]}\) and \(e_{[\tilde{\eta}]}\) are orthogonal idempotents, in this case

\[
\eta \oplus \tilde{\eta} = (e_{-\eta} + e_{-\tilde{\eta}}, e_\eta + e_{\tilde{\eta}}, q_{-\eta} + q_{-\tilde{\eta}}, q_\eta + q_{\tilde{\eta}})
\]
is a hyperbolic pair (note that \( q_\eta \) and \( q_{-\eta} \) commute with \( q_\eta \) and \( q_{-\eta} \)). Clearly, for any hyperbolic pair \( \eta \) the tuple \( -\eta = (e_\eta, e_{-\eta}, q_\eta, q_{-\eta}) \) is also a hyperbolic pair. If \((R, \Delta)\) is constructed by a quadratic module \( M \), then hyperbolic pairs correspond to the so-called hyperbolic subspaces of \( M \) and orthogonal hyperbolic pairs correspond to orthogonal hyperbolic subspaces, see [17] for details.

Let \( e \) and \( \tilde{e} \) be idempotents in any associative ring \( S \). We say that \( e \) Morita dominates \( \tilde{e} \) if \( e \in \mathcal{S}\mathcal{S}\tilde{e} \); this is a pre-order relation. Idempotents \( e \) and \( \tilde{e} \) are called Morita equivalent if they Morita dominate each other (then the rings \( e\mathcal{S}e \) and \( \tilde{e}\mathcal{S}\tilde{e} \) are actually Morita equivalent). Finally, we say that a hyperbolic pair \( \eta \) in an odd form algebra Morita dominates a hyperbolic pair \( \tilde{\eta} \) if \( e_{|\eta|} \) Morita dominates \( e_{|\tilde{\eta}|} \); and similarly for Morita equivalence. For example, \( \eta \) and \( -\eta \) are Morita equivalent. If \( \eta \) and \( \tilde{\eta} \) are orthogonal, then \( \eta \oplus \tilde{\eta} \) Morita dominates both of them (if \( \eta \) and \( \tilde{\eta} \) are Morita equivalent, then they both are Morita equivalent to \( \eta \oplus \tilde{\eta} \)).

We use the notation \( R_{\eta\tilde{\eta}} = e_{\eta}Re_{\tilde{\eta}}, \Delta_{\eta} = \Delta \cdot e_{\eta}, \Delta_{|\eta|'} = \{ u \in \Delta \mid e_{\eta}u = 0 \} \), and \( D_{\eta} = D \cdot e_{\eta} \). Note that \( (\Delta_{|\eta|'}, D_{\eta}) \) is a 2-step \( K \)-module. There are useful formulas

- \( R_{\eta\oplus\tilde{\eta},\tilde{\eta}} = R_{\eta\tilde{\eta}} \oplus R_{\eta,\tilde{\eta}}, \overline{R_{\eta\tilde{\eta}}} = R_{\eta\tilde{\eta}}, R_{\eta\tilde{\eta}}R_{\eta,\tilde{\eta}} \subseteq R_{\eta\tilde{\eta}}; \)
- \( \pi(\Delta_{\eta}) \subseteq R_{\eta\tilde{\eta}}, \pi(\Delta_{|\eta|'}) \subseteq R_{\eta\tilde{\eta}} \oplus R_{-\eta,\tilde{\eta}}, \rho(\Delta_{\eta}) \subseteq R_{-\eta,\tilde{\eta}}, \phi(R_{-\eta,\tilde{\eta}}) \subseteq \Delta_{|\tilde{\eta}|'}; \)
- \( \Delta_{|\eta\oplus\tilde{\eta}|'} = \Delta_{|\eta|'} \oplus \Delta_{|\tilde{\eta}|'} \oplus \phi(R_{-\eta,\tilde{\eta}}); \)
- \( D_{\eta\oplus\tilde{\eta}} = D_{\eta} \oplus D_{\tilde{\eta}} \oplus \phi(R_{-\eta,\tilde{\eta}}). \)

Let \( \eta \) be a hyperbolic pair. We associate to \( \eta \) several subgroups of \( U(R, \Delta) \). For any element \( u \in \Delta_{|\eta|'} \) a transvection \( T^\eta(u) \in U(R, \Delta) \) is given by

\[
\beta(T^\eta(u)) = \rho(u) + \pi(u) - \pi(u), \quad \gamma(T^\eta(u)) = u + q_{-\eta}(\rho(u) - \pi(u)) - \phi(\rho(u) + \pi(u)).
\]

It is easy to see that \( T^\eta : \Delta_{|\eta|'} \rightarrow U(R, \Delta) \) is a well-defined injective group homomorphism. Its image is denoted by \( T^\eta(*) \). The elements \( T^\eta(u) \) are similar to the ESD-transvections from [9].

For any element \( a \in (R_{\eta\tilde{\eta}})^* \) an elementary dilation \( D^\eta(a) \in U(R, \Delta) \) is given by

\[
\beta(D^\eta(a)) = a + \bar{a} - e_{|\eta|}, \quad \gamma(D^\eta(a)) = q_{-\eta}(\bar{a} - e_{-\eta}) + q_{\eta}(a - e_{\eta}) - \phi(a - e_{\eta}),
\]

where \( a^{-1} \) is the inverse of \( a \) in \( R_{\eta\tilde{\eta}} \) and \( \bar{a}^{-1} = \bar{a}^{-T} \) is the inverse of \( \bar{a} \) in \( R_{-\eta,\tilde{\eta}} \). Again, the map \( D^\eta : (R_{\eta\tilde{\eta}})^* \rightarrow U(R, \Delta) \) is a well-defined injective group homomorphism. Its image is denoted by \( D^\eta(*) \). Note that \( D^{-\eta}(a) = D^\eta(\bar{a}^{-1}) \).

Note that \( (R_{|\eta|'}|\eta|', \Delta_{|\eta|'}, D_{|\eta|'}) \) is an odd form \( K \)-subalgebra of \((R, \Delta)\), where \( R_{|\eta|'}|\eta|' = \{ a \in R \mid e_{|\eta|'}a = 0 = ae_{|\eta|'} \}, \Delta_{|\eta|'} = \{ u \in \Delta \mid u \cdot e_{|\eta|} = 0, e_{|\eta|}u = 0 \} \), and \( D_{|\eta|'} = \{ u \in D \mid u \cdot e_{|\eta|} = 0 \} \). Let

\[ U_{|\eta|'} = U(R_{|\eta|'}|\eta|', \Delta_{|\eta|'}) = \{ g \in U(R, \Delta) \mid \beta(g)e_{|\eta|} = 0 = e_{|\eta|} \beta(g), \gamma(g) \cdot e_{|\eta|} = 0 \}. \]
it is a smaller unitary group.

A maximal parabolic subgroup of \( \eta \) is the group
\[
P_\eta = \{ g \in U(R, \Delta) \mid \beta(g)e_{-\eta} = e_{-\eta}\beta(g)e_{-\eta}, e_\eta\beta(g) = e_\eta\beta(g)e_\eta, \gamma(g)e_{-\eta} = q_{-\eta}\beta(g)e_{-\eta} \}.
\]

Finally, a Levi subgroup of \( P_\eta \) is \( L_\eta = P_\eta \cap P_{-\eta} \).

**Lemma 2.** There are decompositions \( L_\eta = D^\eta(*) \times U_{|\eta|'} \) and \( P_\eta = T^\eta(*) \times L_\eta \) (so \( T^\eta(*) \) is called the unipotent radical of \( P_\eta \)). There are identities
\[
\eta(T^\eta(u)) = T^\eta(\gamma(g) \cdot \pi(u) + u) \text{ for } g \in U_{|\eta|'}
\]
and
\[
T^\eta(u)^{D^\eta(a)} = T^\eta(u \cdot a).
\]

**Proof.** The first decomposition and the identities are easy to check. Since \( T^\eta(*) \cap L_\eta = 1 \), there is an inclusion \( T^\eta(*) \times L_\eta \leq P_\eta \). There are well-defined group homomorphisms
\[
p_1 : P_\eta \to (R_{|\eta|'})^*, g \mapsto e_\eta + e_\eta\beta(g)e_\eta
\]
and
\[
p_2 : P_\eta \to U_{|\eta|'}, \beta(p_2(g)) = (1-e_{|\eta|})\beta(g)(1-e_{|\eta|}), \gamma(p_2(g)) = (\gamma(g) - q_{-\eta}\beta(g))(1-e_{|\eta|}).
\]

From a direct computation it follows that \( (p_1, p_2) : P_\eta \to L_\eta \) is a retraction of the inclusion \( L_\eta \leq P_\eta \) and its kernel equals to \( T^\eta(*) \).

Now consider transvections with some particular arguments. If \( \eta \) is a hyperbolic pair and \( u \in D_\eta \), then
\[
\beta(T^\eta(u)) = \rho(u), \quad \gamma(T^\eta(u)) = q_{-\eta} \cdot \rho(u) \cdot u.
\]

Let \( \eta \) and \( \tilde{\eta} \) be orthogonal hyperbolic pairs. If \( a \in R_{\tilde{\eta}, \eta} \), then
\[
\beta(T^\eta(q_{\tilde{\eta}} \cdot a)) = a - \tilde{a}, \quad \gamma(T^\eta(q_{\tilde{\eta}} \cdot a)) = q_{\tilde{\eta}} \cdot a \cdot q_{-\eta} \cdot \tilde{a} - \phi(a).
\]

Clearly, \( T^\eta(q_{\tilde{\eta}} \cdot a) = T^{-\tilde{\eta}}(q_{-\eta} \cdot \tilde{a}) \) for all \( a \in R_{\tilde{\eta}, \eta} \) and all elements from the intersection \( T^\eta(*) \cap T^{-\tilde{\eta}}(*) \) are of this form.

An orthogonal hyperbolic family of rank \( n \) in \( (R, \Delta) \) is a family \( \eta_1, \ldots, \eta_n \) of orthogonal Morita equivalent hyperbolic pairs. In this case we write \( i \) and \(-i\) as indices instead of \( \eta_i \) and \(-\eta_i\). For example, \( R_{-2,1} \) means \( R_{-q_2, q_1} \). An orthogonal hyperbolic family is called free if there are elements \( e_{ij} \in R \) for \( 1 \leq i, j \leq n \) such that \( e_{ij}e_{jk} = e_{ik} \) and \( e_{ii} = e_i \). For example, the odd form algebras \( A_{1}(n, K), A_{2n}(2n, K), O(2n, K), \) and \( A_{2n+1}(2n+1, K) \) have canonical free orthogonal hyperbolic families of rank \( n \), see [16].

From now on fix an orthogonal hyperbolic family \( \eta_1, \ldots, \eta_n \). Elementary transvections are \( T_{ij}(a) = T^\eta(q_{ij} \cdot a) \) for \( a \in R_{ij} \) and \( T_i(u) = T^\eta(u) \) for \( u \in \Delta^0_i = \{ u \in \Delta_i \mid e_{|i|} + \ldots + e_{|n|} \pi(u) = 0 \} \). It is easy to see that
\[
T^\eta(*) = T_i(*) \bigoplus_{j \neq \pm i} T_{ji}(*),
\]
where $T_i(*)$ and $T_j(*)$ are the subgroups of all elementary transvections with given indices. An elementary unitary group $EU(R, \Delta)$ is the subgroup of $U(R, \Delta)$ generated by all elementary transvections, i.e.

$$EU(R, \Delta) = \langle T_{ij}(*) \rangle.$$

Elementary dilations are $D_i(a) = D^j(a)$ for $a \in (R_i)^*$ and $D_0(g) = g$ for $g \in U_{|\eta_1| \oplus \cdots \oplus |\eta_n|}$. Clearly, they satisfy

(D0) $D_i(a) = D_{-i}(a^{-1})$ for $i \neq 0$;
(D1) $D_i(a) D_i(b) = D_i(ab)$ for $i \neq 0$;
(D2) $D_0(g) D_0(h) = D_0(gh)$;
(D3) $[D_i(a), D_j(b)] = 1$ for $0 \neq i \neq \pm j \neq 0$;
(D4) $[D_i(a), D_0(g)] = 1$ for $i \neq 0$.

The product $D(R, \Delta) = \prod_{i=0}^{n-1} D_i(*)$ is the diagonal subgroup of $U(R, \Delta)$, its is the abstract group with the generators $D_i(a)$ for $0 \leq i \leq n$ and the relations (D0) – (D4).

**Lemma 3.** Let $\eta$ be a hyperbolic pair. Suppose that $g \in U(R, \Delta)$ satisfies $a = e_\eta + e_\eta \beta(g) e_\eta \in R^*_{\eta_\eta}$. Then there is unique $u \in \Delta_{\eta}^{[\eta']} \setminus \Delta_{\eta}^{[\eta]}$ such that $T^u(\tilde{-u}) g \in P_{-\eta}$.

**Proof.** Indeed, direct calculation shows existence and uniqueness for $u$. More explicitly,

$$\pi(u) = (1 - e_{|\eta|}) \beta(g) a^{-1},$$
$$\rho(u) = e_{-\eta} \beta(g) a^{-1},$$
$$u = (\gamma(g) \cdot q_{-\eta} \cdot \beta(g) \cdot q_{\eta} \cdot \beta(g) + \phi(\beta(g)) \cdot a^{-1}.$$

\[\square\]

## 4 Steinberg groups

An odd unitary Steinberg group $StU(R, \Delta)$ is generated by symbols $X_{ij}(a)$ for $0 < |i|, |j| \leq n$, $i \neq \pm j$, $a \in R_{ij}$ and by symbols $X_i(u)$ for $0 < |i| \leq n$, $u \in \Delta_0^0$. The relations on these symbols are the following:

(St0) $X_{ij}(a) = X_{-j,-i}(-a)$;
(St1) $X_{ij}(a) X_{ij}(b) = X_{ij}(a + b)$;
(St2) $X_i(u) X_i(v) = X_i(u + v)$;
(St3) $[X_{ij}(a), X_{kl}(b)] = 1$ for $i \neq l \neq j \neq -k \neq i$;
(St4) $[X_{ij}(a), X_{jk}(b)] = X_{ik}(ab)$ for $i \neq \pm k$;
(St5) \([X_{ij}(a), X_{j,-i}(b)] = X_{-i}(\phi(ab))\);

(St6) \([X_i(u), X_j(v)] = X_{-i,j}(-\pi(u)\pi(v))\) for \(i \neq \pm j\);

(St7) \([X_i(u), X_{jk}(a)] = 1\) for \(j \neq i \neq -k\);

(St8) \([X_i(u), X_{ij}(a)] = X_{-i,j}(\rho(u)a) X_j(-u \cdot (-a))\).

There is a canonical map \(st\) : \(StU(R, \Delta) \to U(R, \Delta)\) given by \(X_i(u) \to T_i(u)\) and \(X_{ij}(a) \to T_{ij}(a)\). Indeed, the elementary transvections satisfy the relations (St0)–(St8) by lemma 3 from [17] or by direct computations using our lemma 2. The image of \(st\) is the elementary subgroup \(EU(R, \Delta)\), its kernel is denoted by \(KU_2(R, \Delta)\), and its cokernel is denoted by \(KU_1(R, \Delta)\) (it is the set of cosets in general).

The subgroups \(X_{ij}(R_{ij})\), \(X_j(\Delta^0_i)\), and \(X_j(D_j)\) of \(StU(R, \Delta)\) are called root subgroups. Since \(st\) maps Steinberg generators to elementary transvections, it follows that \(X_{ij}\) and \(X_j\) are injective group homomorphisms. Note that \(X_j(D_j)\) depends on the augmentation, unlike the whole Steinberg group.

Let \(\Phi = \{\pm e_i \pm e_j, \pm 2e_k | 1 \leq i < j \leq n, 1 \leq k \leq n\} \subseteq \mathbb{R}^n\), it is a non-reduced crystallographic root system of type \(BC_n\). Its elements are called long roots, short roots and ultrashort roots depending on their length (2, \(\sqrt{2}\), or 1). Let also \(e_{-i} = -e_i\) for \(1 \leq i \leq n\). For any root \(\alpha\) we assign a root subgroup of \(StU(R, \Delta)\) in the following way:

\[
X_\alpha(*) = \begin{cases} 
X_{ij}(R_{ij}), & \alpha = e_j - e_i, i \neq \pm j; \\
X_i(\Delta^0_i), & \alpha = e_i; \\
X_i(D_i), & \alpha = 2e_i.
\end{cases}
\]

The Steinberg relations imply that this definition is correct for short roots and

\[
[X_\alpha(*), X_\beta(*)] \leq \prod_{i,j \in \Phi, i,j > 0} X_{i\alpha+j\beta(*)}
\]

for all non-antiparallel roots \(\alpha, \beta \in \Phi\), where the right hand side is a nilpotent subgroup of \(StU(R, \Delta)\). The diagonal group \(D(R, \Delta)\) acts on \(StU(R, \Delta)\) (and on every root subgroup) by

(Ad1) \(D_{\alpha(\delta)}X_{jk}(b) = X_{jk}(b)\) for \(j \neq \pm i \neq k\);

(Ad2) \(D_{\delta(\gamma)}X_{ij}(a) = X_{ij}(a)\);

(Ad3) \(D_{\alpha(\delta)}X_{ij}(b) = X_{ij}(ab)\);

(Ad4) \(D_{\alpha(\delta)}X_j(u) = X_j(u)\) for \(i \neq \pm j\);

(Ad5) \(D_{\delta(\gamma)}X_i(u) = X_i(\gamma(\delta) \cdot \pi(u) + u)\);

(Ad6) \(D_{\alpha(\delta)}X_i(u) = X_i(u \cdot a^{-1})\).
where $a^{-1}$ is the inverse in the ring $R_{i_0}$.

Note that the Weyl group $W = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ of $\Phi$ acts on the set of root subgroups by permutations of roots (and on the set of $D_i$ by permutations of indices). It is easy to see that $W$ acts transitively on all non-zero indices, on all pairs $(i,j)$ of indices with $0 \neq i \neq j \neq 0$, and on all roots of given length. Also, $W$ acts transitively on all pairs of non-collinear roots $(\alpha,\beta)$ with given lengths, given angle between them, and given Dynkin diagram of $\Phi \cap (\mathbb{R}\alpha + \mathbb{R}\beta)$. This Dynkin diagram may be $A_1 \times A_1$, $A_1 \times BC_1$, $A_2$, or $BC_2$, it is needed only to distinguish orthogonal short roots in $A_1 \times A_1$ and $BC_2$.

Similarly to [13], it is possible to define root systems $\Phi/\alpha$ and corresponding orthogonal hyperbolic families for all short and ultrashort $\alpha \in \Phi$. Namely, fix such a root $\alpha \in \Phi$. The set $\Phi/\alpha$ is the image of $\Phi \setminus \mathbb{R} \alpha$ in $\mathbb{R}^n/\mathbb{R} \alpha$, it consists of classes $[\beta]$ for all roots $\beta \in \Phi \setminus \mathbb{R} \alpha$. Let $\text{StU}(R,\Delta;\Phi) = \text{StU}(R,\Delta)$ and $D(R,\Delta;\Phi) = D(R,\Delta)$. We construct the new orthogonal hyperbolic family case by case up to the action of the new Weyl group $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_{n-1}$.

If $\alpha$ is short, then without loss of generality $\alpha = e_n - e_{n-1}$. Let $\eta_\infty = \eta_{n-1} \oplus \eta_n$ be the new hyperbolic pair, $\text{StU}(R,\Delta;\Phi/\alpha)$ and $D(R,\Delta;\Phi/\alpha)$ be the groups with respect to the orthogonal hyperbolic family $\eta_1,\ldots,\eta_{n-2},\eta_\infty$. There is a well-defined homomorphism $F_\alpha : \text{StU}(R,\Delta;\Phi/\alpha) \to \text{StU}(R,\Delta;\Phi)$ given by

- $X_{ij}(a) \mapsto X_{ij}(a)$ for $i,j \neq \pm \infty$;
- $X_{in}(a) \mapsto X_{i,n-1}(ae_n - e_{n-1}) X_{i,n}(ae_n)$ for $i \neq \pm \infty$;
- $X_{n,j}(a) \mapsto X_{n-1,j}(ae_n) X_{n,j}(e_na)$ for $j \neq \pm \infty$;
- $X_{j}(u) \mapsto X_{j}(u)$ for $j \neq \pm \infty$;
- $X_{\infty}(u) \mapsto X_{\infty}(u \cdot e_n) X_{\infty}(u \cdot e_{n-1}) X_{-\infty}(u \cdot e_n) X_{-\infty}(u \cdot e_{n-1})$;
- $X_{-\infty}(u) \mapsto X_{-\infty}(u \cdot e_n) X_{-\infty}(u \cdot e_{n-1}) X_{n,n}(e_n \rho(u)e_{n-1})$.

On $X_{i,-\infty}$ and $X_{-\infty,i}$ this homomorphism is defined by (St0).

If $\alpha$ is ultrashort, then without loss of generality $\alpha = e_1$. Let $\text{StU}(R,\Delta;\Phi/\alpha)$ and $D(R,\Delta;\Phi/\alpha)$ be the groups with respect to the orthogonal hyperbolic family $\eta_2,\ldots,\eta_n$. There is a well-defined homomorphism $F_\alpha : \text{StU}(R,\Delta;\Phi/\alpha) \to \text{StU}(R,\Delta;\Phi)$ given by

- $X_{ij}(a) \mapsto X_{ij}(a)$;
- $X_{i}(u) \mapsto X_{i}(u - q_1 \cdot \pi(u) - \eta_{-1} \cdot \pi(u)) X_{-i}(e_{-1} \pi(u)) X_{1,i}(e_1 \pi(u))$.

Note that in this case $X_i$ are defined on different sets in $\text{StU}(R,\Delta;\Phi/\alpha)$ and $\text{StU}(R,\Delta;\Phi)$.

In all cases $\text{st}_\Phi \circ F_\alpha = \text{st}_\Phi/\alpha$. The set $\Phi/\alpha$ is a root system of type $BC_{n-1}$, it parametrizes the root subgroups of $\text{StU}(R,\Delta;\Phi/\alpha)$. We have $F_\alpha(X_{\beta}(\ast)) = \prod_{\beta + \alpha \in \Phi} X_{\beta + \alpha}(\ast) \ast \Delta(R,\Delta;\Phi), \text{T}_{\alpha}(\ast)) \ast \Delta(R,\Delta;\Phi,\ast) \leq \Delta(R,\Delta;\Phi/\alpha)$, where $\text{T}_{\alpha}(\ast) = \text{st}(X_{\alpha}(\ast))$. Clearly, $\Phi/\alpha = \Phi/-\alpha$ (i.e. there is a canonical bijection between these sets, the corresponding Steinberg and diagonal groups are also canonically
isomorphic) and \((\Phi/\alpha)/[\beta] \cong (\Phi/\beta)/[\alpha]\) for non-collinear \(\alpha\) and \(\beta\) (where both sides are defined or not simultaneously). We denote \((\Phi/\alpha)/[\beta]\) by \(\Phi/\{\alpha, \beta\}\), it is defined if and only if neither \(\alpha\) nor \(\beta\) is long, \(\alpha\) and \(\beta\) are non-collinear, and they are not orthogonal short roots inside some root subsystem of type \(BC_2\). Note that \(\Phi/\{\alpha, \beta\}\) depends only on the span of \(\alpha\) and \(\beta\). We say that \(\Phi/\alpha\) is obtained from \(\Phi\) by elimination of \(\alpha\), similarly for the orthogonal hyperbolic family and the Steinberg group.

We also need the groups

\[
U^+(R, \Delta; \Phi) = (X_{ij}(R, \Delta; \Phi), T_k(\Delta^\alpha_i) \mid i < j, k > 0)
\]

\[
U^-(R, \Delta; \Phi) = (X_{ij}(R, \Delta; \Phi), T_k(\Delta^\alpha_i) \mid i > j, k < 0).
\]

Clearly, these groups are nilpotent and \(st\) is injective on them. If \(\alpha\) lies in the basis of \(BC_n\) (i.e., it is \(e_i\) or \(e_{i+1} - e_i\) for \(1 \leq i \leq n - 1\)), then we may similarly define \(U^\pm(R, \Delta; \Phi/\alpha)\). It is easy to see that \(U^\pm(R, \Delta; \Phi) = U^\pm(R, \Delta; \Phi/\alpha) \times X_{\pm\alpha}^*(\cdot)\) and \(U^\pm(R, \Delta; \Phi/\alpha)\) are normalized by both \(X_{\pm\alpha}(\cdot)\) and \(X_{\alpha}(\cdot)\).

We say that \(P^\pm(R, \Delta; \Phi) = st(U^\pm(R, \Delta; \Phi)) \times D(R, \Delta; \Phi)\) are opposite parabolic subgroups of \(U(R, \Delta)\), \(D(R, \Delta; \Phi)\) is their common Levi subgroup, and \(U^\pm(R, \Delta; \Phi)\) are their unipotent radicals. If the orthogonal hyperbolic family has rank 1, then we get the maximal parabolic subgroups from lemma 2. Clearly, \(StU(R, \Delta; \Phi)\) is generated by \(U^+(R, \Delta; \Phi)\) and \(U^-(R, \Delta; \Phi)\).

Note that all constructions from [15] may be stated in terms of odd form algebra. If \(S\) is a \(K\)-algebra with a complete family of orthogonal idempotents \(\varepsilon_1, \ldots, \varepsilon_n\), then \(R = S^{op} \times S\) is a \(K\)-algebra with the involution \((a^{op}, b) = (b^{op}, a)\), \((R, \Delta_{\text{max}})\) is a special unital odd form \(K\)-algebra, and \(U(R, \Delta_{\text{max}}) \cong S^*\).

Moreover, \(e_{-i} = (\varepsilon_i^{op}, 0)\) and \(e_i = (0, \varepsilon_i)\) are orthogonal idempotents for \(0 < i \leq n\), they determine orthogonal hyperbolic pairs \(\eta_i = (e_{-i}, e_i, q_{-i}, q_i)\) (here \(q_{-i}\) and \(q_i\) are uniquely determined since the odd form algebra is special). Also \(\varepsilon_i\) are Morita equivalent if and only if \(\eta_i\) are Morita equivalent. The linear Steinberg group constructed by \(\varepsilon_i\) is canonically isomorphic to the unitary Steinberg group constructed by \(\eta_i\).

5 Parabolic subgroups of classical groups

In this section we show that our parabolic groups coincide with the usual parabolic subgroups of reductive groups in the classical case. Definition and basic properties of parabolic subgroups of reductive groups may be found in [3] and in Exp. XXVI, [4].

**Proposition 1.** Let \(K\) be a local commutative ring with the maximal ideal \(m\) and \((R, \Delta, D)\) be one of \(\text{AL}(n, K)\), \(\text{ASp}(2n, K)\), \(\text{AO}(n, K)\). Then every hyperbolic pair \(\eta = (e_{-\eta}, e_\eta, q_{-\eta}, q_\eta)\) is conjugate to an orthogonal sum of some \(\eta_{\pm i}\) for \(0 < i \leq n\) under the action of \(U(R, \Delta)\).

**Proof.** In the linear case \(R = M(n, K)^{op} \times M(n, K)\) and without loss of generality \(e_\eta \in M(n, K)\). We claim that \(e_\eta R e_\eta\) is isomorphic to a matrix algebra over \(K\). It
follows from freeness of all finite projective $K$-modules in the linear case and if $R$ is itself a matrix algebra. The only remaining case is $(R, \Delta, D) = AO(2n+1, K)$ and $2 \not\in K^*$. In this case there is a ring map: $R \to M(2n+1, K), e_{ij} \mapsto e_{ij}$ for $j \neq 0$ and $e_{i0} \mapsto 2e_{i0}$ (the indices are from $-n$ to $n$). It remains to show that rep is injective on $e_\eta R e_\eta$. Indeed, if $a = \sum a_i e_{i0} \in e_\eta R e_\eta \cap \text{Ker}(\text{rep})$ for some $a_i \in K$, then $2a_0 = 0$ and $a = ae_\eta$, hence $a = 0$.

Decomposing $\eta$ into an orthogonal sum and applying induction, we may assume that $e_\eta R e_\eta \cong K$ (and $e_\eta \in M(n, K)$ in the linear case). Then there are $a \in e_\eta R e_\eta$ and $b \in e_\eta R e_\eta$ such that $ab = e_\eta$ and $ba = e_\eta$ again by freeness of finite projective modules and applying rep in the odd orthogonal case if $2 \not\in K^*$ (since rep is injective on $e_\eta R e_\eta$ and $e_\eta R e_\eta$). We claim that there is $g \in U(R, \Delta)$ such that $(\beta(g) + e_\eta)a = e_\eta$ and $b(\beta(g) + e_\eta) = e_\eta$. We are going to make changes $a \mapsto \alpha(g) a$ and $b \mapsto b\alpha(g)$ for $g \in U(R, \Delta)$ until we get $a = b = e_\eta$.

If $e_i a \in \text{m}_{e_i n}$ for all $i \neq 0$, then $(R, \Delta) = AO(2n+1, K)$, $2 \in K^*$, and $e_\eta a \notin \text{m}_{e_\eta n}$. Hence $e_n a(T_{-n}(u_{-n})) a e_n \notin \text{m}_{e_n n}$ (see [10] for a description of $AO(2n+1, K)$) and from now we may assume that there is $i \neq 0$ with $e_i a \notin \text{m}_{e_i n}$. If $e_i a \in \text{m}_{e_i n}$ for all $i > 0$, then there is $j > 0$ such that $e_{-j} a \notin \text{m}_{e_{-j} n}$ and we are not in the linear case. In the symplectic case $e_j \alpha(T_{-j}((-)) a e_n \notin \text{m}_{e_{jn}}$, and in the orthogonal case $e_j \alpha(g) a e_n \notin \text{m}_{e_{jn}}$ for $g \in U(R, \Delta)$ with $\beta(g) = e_{-j} e_j + e_{-j} - e_{-j} a e_j$ and $\gamma(g) = q_{-j} \cdot (e_{-j} - e_{-j} a e_j)$ (i.e. $g$ is a reflection). Hence from now we may assume that there is $i > 0$ such that $e_i a \notin \text{m}_{e_i n}$.

Clearly, there is $g \in (D_1(*), T_{n1}(*)) \mid 0 < i < n$ such that $e_n \alpha(g) a = e_n$. Hence we may assume that $e_n a = e_n$. Now it is easy to see that $\alpha(T^n(q_n + q_{-n} a - q_n a)) a = e_n$ (where $T^n$ parametrizes the unipotent radical of a maximal parabolic subgroup) because $a^2 = a$ and $\bar{a} a = 0$. Hence without loss of generality $a = e_n$, so $be_n = e_n$. Finally, $b\alpha(g) = e_n = \alpha(g) a$ for $g = T^{-n}(q_{-n} + q_{-n} \cdot \bar{b} - q_{-n} a \cdot \bar{b})$ because $b^2 = b$ and $b\bar{b} = 0$.

Let $K$ be any commutative ring and $(R, \Delta, D)$ be a classical odd form $K$-algebra. Let $L$ be the corresponding twisted form of one of the group schemes $SL(n, -)^m$, $\text{Sp}(2n, -)^m$, $\text{SO}(n, -)^m$. Fix an orthogonal hyperbolic family in $(R, \Delta)$. Clearly, $E \mapsto P^\pm(R \otimes K E, \Delta \otimes K E) \cap G(E)$ are group schemes over $K$, where $E/K$ is arbitrary extension of commutative rings. We denote these group schemes by $P^\pm((R, \Delta) \cap G$, and similarly for the Levi subgroup and the unipotent radicals. By proposition [1] fppf locally and up to an inner isomorphism the orthogonal hyperbolic family is obtained from the standard one by factoring out roots from the base on every direct factor of the split classical odd form algebra in the symplectic and the orthogonal cases.

In the linear case the same is true if we allow changing the signs of the elements of the standard orthogonal hyperbolic family. Note that if $\eta = \bar{\eta} \oplus \bar{\eta}$ in the orthogonal hyperbolic family, where $\bar{\eta}$ is a sum of standard hyperbolic pairs of $AL(n, K)$ and $\bar{\eta}$ is a sum of opposite standard hyperbolic pairs, then we may change $\eta$ to $\bar{\eta}$ and $\bar{\eta}$ (in any order) in the orthogonal hyperbolic family preserving the parabolic subgroups. If in the orthogonal hyperbolic family there are con-
secutive hyperbolic pairs $\tilde{\eta}$ and $\check{\eta}$, where $\tilde{\eta}$ is a sum of standard hyperbolic pairs and $\check{\eta}$ is a sum of opposite standard hyperbolic pairs, then we may swap them preserving the parabolic subgroups. Finally, if the orthogonal hyperbolic family is $\tilde{\eta}_1, \ldots, \tilde{\eta}_k, \check{\eta}_k+1, \ldots, \check{\eta}_l$, where $\tilde{\eta}_i$ are sums of standard hyperbolic pairs and $\check{\eta}_i$ are sums of opposite standard hyperbolic pairs, then $-\tilde{\eta}_k, \ldots, -\tilde{\eta}_1, \check{\eta}_k+1, \ldots, \check{\eta}_l$ has the same parabolic subgroups and is obtained from the standard one by factoring out roots from the base (up to an inner automorphism).

Hence (and by descent) $P^\pm(R, \Delta) \cap G$ are opposite parabolic subgroups of $G$ with the common Levi subgroup $D(R, \Delta) \cap G$ and the unipotent radicals $st(U^\pm(R, \Delta))$ in the sense of reductive groups.

**Proposition 2.** Let $K$ be a commutative ring, $G$ is a twisted form of $\text{SL}(n, -)^m$ for $n \geq 3$ and $m \geq 1$, $\text{Sp}(2n, -)^m$ for $n \geq 1$ and $m \geq 1$, or $\text{SO}(n, -)^m$ for $n \geq 3$ and $m \geq 1$. Fix a parabolic subgroup $P \leq G$ and a Levi subgroup $L \leq P$ in the sense of reductive groups. Let $(R, \Delta, D)$ be the corresponding classical augmented odd form algebra. Then there is an orthogonal hyperbolic family $\eta_1, \ldots, \eta_k$ in $(R, \Delta)$ such that $L = D(R, \Delta) \cap G$ and $P = st(U^+(R, \Delta)) \times L$.

**Proof.** Note that $(R, \Delta, D)$ is well-defined by [10], theorem 1. By Exp. XXVI, corollary 1.8 and lemma 1.14 in [3] or proposition 5.2.3 and corollary 5.4.6 in [3], for every point of Spec($K$) there is an fppf extension $E/K$ covering this point such that the base change $G_E$ is isomorphic to the split group, under this isomorphism $P_E$ maps to a standard parabolic subgroup, and $L_E$ maps to its standard Levi subgroup. In other words, there is a subset $J$ of the Dynkin diagram (a power of one of $A_l$, $B_l$, $C_l$, or $D_l$) such that $P$ is generated by the standard torus, all root subgroups with positive roots, and all root subgroups with negative roots not involving $J$ in their decomposition over the basis (as an fppf sheaf of groups). Decomposing $K$ into a finite product of rings and decomposing $(R, \Delta, D)$ into a finite product of smaller classical odd form algebras, we may assume that $E/K$ is an fppf covering and $J$ has isomorphic intersections with all connected components of the Dynkin diagram of $G_E$ (not necessarily equal because of the outer automorphisms).

It follows that the intersection of $P_E$ with the $i$-th direct factor of $(R, \Delta, D)_E$ corresponds to an orthogonal hyperbolic family $\eta_{h,1}, \ldots, \eta_{h,k}$ obtained from the standard one by elimination the base roots not lying in $J$. If this orthogonal hyperbolic family is not invariant under the outer automorphism, then we may choose these families in all factors simultaneously in a coherent way and descend the orthogonal hyperbolic family $\bigoplus_i \eta_{h,1}, \ldots, \bigoplus_i \eta_{h,k}$ to $(R, \Delta, D)$ (because the only automorphisms of $G_E$ preserving $L$ are the inner automorphisms by $L$ and some outer automorphisms).

If it is invariant, then we have three cases. In the even orthogonal case the outer automorphism changes the sign of $\eta_{h,1}$ and this hyperbolic pair is actually the standard one. We may eliminate this hyperbolic pair (passing from $D_k$ to $B_{k-1}$) and then apply descent. The parabolic subgroup is preserved since $\phi(R_{l,-1}) = \phi(R_{-1,1}) = 0$. In the linear case if $k$ is even, then we may change the orthogonal hyperbolic family to $\eta_{\frac{k}{2}+1} \oplus -\eta_{\frac{k}{2}}, \ldots, \eta_{h,k} \oplus -\eta_{h,1}$ (so instead of
We use the notation from [15]. Let \( C \) be an object of \( \text{Alg}_{A_{2k-1}} \) we have \( C_k \) and then apply descent. Finally, in the linear case and if \( k \) is odd, we change the orthogonal hyperbolic family to \( \eta_i, \eta_{i+1} \oplus -\eta_i, \eta_{i+1}, \ldots, \eta_i, k \oplus -\eta_i, 1 \) and then apply descent. In this case instead of \( A_{2k} \) we get \( BC_k \). □

Note that if the orthogonal hyperbolic family in a split odd form algebra is obtained by elimination from the standard one and it is invariant under an outer automorphism, the our proof decreases the rank of the family in order to apply faithfully flat descent. On the other hand, our main results require that the rank is not too small.

Actually, we may now describe Steinberg groups of all classical isotropic reductive group. Let \( G \) be a classical reductive group over \( K \) (i.e. its Dynkin diagram consists only of copies of \( A_t, B_t, C_t, \) and \( D_t \)), \( P \leq G \) is a parabolic subgroup, and \( L \leq P \) is a Levi subgroup. Suppose that \( P \) is sufficiently reductive in the sense of [10]. Then we may defined the Steinberg group as the group generated by symbols \( X_A(a) \) for a relative root \( A \) and a parameter \( a \) from the corresponding finite projective \( K \)-module. The relations on them are \( X_A(a)X_A(b) = X_A(a+b) \prod_{i>1} X_A(q_i^A(a,b)) \), where \( q_i^A \) are polynomial maps homogeneous of degree \( i \), and the Chevalley commutator formulas (see [10] for details). Moreover, \( L \) acts on this Steinberg group. This construction is preserved if we take the factor of \( G \) by its center, hence it suffices to consider only semi-simple group schemes of adjoint type. Up to a decomposition of \( K \) and \( G \) into direct products, \( G \) is a twisted form of \( \text{PGL}(n,K)^m, \text{PSp}(2n,K)^m, \) or \( \text{PSO}(n,K)^m \). If \( G \) is not a twisted form of \( \text{PSO}(8,K)^m \), then it has a central extension \( G' \) such that \( G' \) it a twisted form of \( \text{GL}(n,K)^m, \text{Sp}(2n,K)^m, \) or \( \text{SO}(n,K)^m \), i.e. the Steinberg group of \( G' \) is described in terms of odd form algebras and orthogonal hyperbolic families. This is true even if \( G \) is a twisted form of \( \text{PSO}(8,K)^m \) and the descent data does not involve the triality. Twisted forms of \( \text{PSO}(8,K)^m \) involving the triality have only parabolic subgroups with isotropic rank at most 2; it is too small for our purposes.

6 Localization and homotopes

We use the notation from [15]. Let \( C \) be a category. Objects of its pro-completion \( \text{Pro}(C) \) are contravariant functors \( X(\infty) : \mathcal{I}_X \to C, i \mapsto X(i) \), where the category \( \mathcal{I}_X \) is small and filtered. Elements of \( \mathcal{I}_X \) are called indices of \( X \). We omit applications of \( X(\infty) \) to morphisms \( i \to j \) in \( \mathcal{I}_X \) in our formulas if this morphism is clear from the context (say, if \( j \) is sufficiently large).

A pre-morphism \( f : X(\infty) \to Y(\infty) \) consists of a function \( f^* \) from the set of indices of \( X(\infty) \) to the set of indices of \( Y(\infty) \) (not necessarily functorial), and of morphisms \( f(i) : X(f^*(i)) \to Y(i) \) for all \( i \in \mathcal{I}_Y \) such that for any morphism \( i \to j \) in \( \mathcal{I}_Y \) there is an index \( k \in \mathcal{I}_X \) making the composition \( X(k) \to X(f^*(i)) \to Y(i) \) equal to \( X(k) \to X(f^*(j)) \to Y(j) \to Y(i) \). A composition of pre-morphisms \( f : X(\infty) \to Y(\infty) \) and \( g : Y(\infty) \to Z(\infty) \) is given by \( (g \circ f)^*(i) = f^*(g^*(i)) \) and \( (g \circ f)(i) = g(i) \circ f^*(i) \). Two pre-morphisms \( f, g : X(\infty) \to Y(\infty) \) are called equivalent if for every \( i \in \mathcal{I}_Y \) there is \( j \in \mathcal{I}_X \) making the composition
There is a fully faithful functor \( \mathcal{C} \to \text{Pro}(\mathcal{C}) \) if we consider every object from \( \mathcal{C} \) as a functor from the index category with only one object and only one morphism. It follows that \( X^{(\infty)} \) is the projective limit of \( X^{(i)} \) in \( \text{Pro}(\mathcal{C}) \).

The category of pro-sets \( \text{Pro}(\text{Set}) \) has all finite limits, so we may consider algebraic objects in \( \text{Pro}(\text{Set}) \) such as groups and odd form algebras. Every algebraic formula (say, the commutator) defines a morphism in \( \text{Pro}(\text{Set}) \) from a product of such algebraic objects to an algebraic objects. If \( a \) is a variable of such a formula, then \( a \in X^{(\infty)} \) means that \( X^{(\infty)} \) is the domain of \( a \). Let \( X^{(\infty)} \) and \( Y^{(\infty)} \) be pro-sets with the same index category \( \mathcal{I} \). The pro-set \( Z^{(\infty)} \) with the index category \( \mathcal{I} \) and with \( Z^{(i)} = X^{(i)} \times Y^{(i)} \) is their product in \( \text{Pro}(\text{Set}) \), the first projection is given by the pre-morphism \( \pi_X^{(i)}(i) = i, \pi_X^{(i)}(x, y) = x \), and similarly for the second projection. If \( X^{(\infty)} = Y^{(\infty)} \), then the pre-morphism \( \Delta^{(i)}(i) = i, \Delta^{(i)}(x) = (x, x) \) gives the diagonal morphism \( X^{(\infty)} \to X^{(\infty)} \times X^{(\infty)} \).

Now let \( f, g : X^{(\infty)} \to Y^{(\infty)} \) be two pre-morphisms with \( f^{(i)}(i) = g^{(i)}(i) = i \). Suppose that \( f^{(i)}(x) = f^{(j)}(x) \in Y^{(i)} \) and \( g^{(i)}(x) = g^{(j)}(x) \in Y^{(i)} \) for all \( x \in X^{(i)} \) and all morphisms \( i \to j \) in \( \mathcal{I} \). Then the pro-set \( W^{(\infty)} \) with the index category \( \mathcal{I} \) and with \( W^{(i)} = \{x \in X^{(i)} \mid f^{(i)}(x) = g^{(i)}(x)\} \) is the equalizer of \( f \) and \( g \) in \( \text{Pro}(\text{Set}) \). The equalizer morphism \( W^{(\infty)} \to X^{(\infty)} \) is given by the pre-morphism \( e^{(i)}(i) = i, e^{(i)}(x) = x \).

There is a faithful functor \( \text{Pro}(\text{Grp}) \to \text{Pro}(\text{Set}) \), hence every pro-group may be considered as a group object in \( \text{Pro}(\text{Set}) \) (but not conversely). A morphism \( f \in \text{Pro}(\text{Set})(G^{(\infty)}, H^{(\infty)}) \) for pro-groups \( G^{(\infty)} \) and \( H^{(\infty)} \) is a morphism of pro-groups if and only if \( f \) is a morphism of group objects, i.e. if \( f(gg') = f(g)f(g') \) for \( g, g' \in G^{(\infty)} \). If \( G^{(\infty)} \) is a group object in \( \text{Pro}(\text{Set}) \) and a group \( H \) acts on \( G^{(\infty)} \) by automorphisms, then their semi-direct product \( G^{(\infty)} \rtimes H \) is also a group object, it is the formal projective limit of \( G^{(i)} \rtimes H \). The same is true for actions of abstract odd form \( K \)-algebras on odd form \( K \)-algebras in \( \text{Pro}(\text{Set}) \) if the action is coherent with the actions of \( (K, 0) \).

Recall that a morphism \( f \in \mathcal{C}(X, Y) \) is a split epimorphism if it admits a section. Split epimorphisms are universal epimorphisms, i.e. they are preserved under pullbacks.

From now on fix a commutative ring \( K \) and a multiplicative subset \( S \subseteq K^\bullet \). We usually use the index category \( S \), its objects are elements of \( S \), and \( S(s, s') = \{ s'' \in S \mid s' = ss'' \} \) (the composition and identity morphisms are obvious). For every \( K \)-module \( M \) let \( M^{(s)} = \{ m(s) \mid m \in M \} \), it is isomorphic to \( M \) as a \( K \)-module. Let also \( M^{(\infty)} \) be their formal projective limit under the maps \( m^{ss'} \to m^{ss''} \). Clearly, \( M^{(\infty)} \) is a \( K \)-module in \( \text{Pro}(\text{Set}) \) and a pro-group under addition. Also \( (\bigoplus_{i=1}^n M_i)^{(\infty)} \simeq \bigoplus_{i=1}^n M_i^{(\infty)} \). Every bilinear map \( f : M \times N \to L \) gives a bilinear morphism of pro-sets \( f : M^{(\infty)} \times N^{(\infty)} \to L^{(\infty)}, (m^{(s)}, n^{(t)}) \mapsto f(m^{(s)}, n^{(t)}) \).
The relations are flatness of $S$. For example, if the group $\iota$ the definition of $M$, $M_0$ are denoted by $M(M_0)$, the operations on $S^{-1}M$ are given by

- $m \cdot \frac{1}{s} + m' \cdot \frac{1}{s'} = (m \cdot s' + m' \cdot s) \cdot \frac{1}{ss'}$,  
- $(m \cdot \frac{1}{s}) \cdot (m' \cdot \frac{1}{s'}) = (m \cdot k) \cdot \frac{1}{ss'}$.

Clearly, $(M, M_0) \rightarrow (S^{-1}M, S^{-1}M_0)$, $m \mapsto m \cdot \frac{1}{s}$ is a morphism of 2-step $K$-modules and $S^{-1}M/S^{-1}M_0 \cong S^{-1}(M/M_0)$ as $S^{-1}K$-modules. It follows that $S^{-1}M \cong \prod_{K} S^{-1}K$ (see the definition in [16]), though we do not require flatness of $M/M_0$.

Now we define homotopies of $(M, M_0)$. Let $M^{(s)}$ be the group generated by the group $M_0^{(s)}$ and the elements $m^{(s)}$ for all $m \in M$, the map $M_0^{(s)} \rightarrow M^{(s)}$ is denoted by $\iota$ (so we distinguish the generators $\iota(m^{(s)})$ and $m^{(s)}$ for $m \in M_0$). The relations are $m^{(s)} + m'^{(s)} = (m + m')^{(s)}$ for all $m, m'$ and $m^{(s)} = \iota(sm^{(s)})$ for $m \in M_0$. It is easy to see that the sequence $M_0^{(s)} \rightarrow M^{(s)} \rightarrow (M/M_0)^{(s)}$ is short exact, where the left map is $\iota$ and the right map is $m^{(s)} \mapsto (m + M_0)^{(s)}(s), \iota(m^{(s)}) \rightarrow 0$. The pro-group $M^{(\infty)}$ is the formal projective limit of $M^{(s)}$ under the maps $M^{(ss')} \rightarrow M^{(s)}, m^{(ss')} \rightarrow (m \cdot s')^{(s)}, \iota(m^{(ss')}) \rightarrow \iota((s'm)^{(s)})$.

Hence $M^{(\infty)} \rightarrow (M/M_0)^{(\infty)}$ is a short exact sequence of pro-group in the following sense: the left morphism is the equalizer of 0 and the right morphism in Pro(Set), the right morphism is an epimorphism in Pro(Set), and any morphism $f \in \text{Pro}(\text{Set})(M^{(\infty)}, X^{(\infty)})$ factors through $(M/M_0)^{(\infty)}$ if and only if $f(m + m') = f(m')$ for $m \in M^{(\infty)}, m' \in M^{(\infty)}$.

The pair $(M^{(\infty)}, M_0^{(\infty)})$ is a 2-step $K$-module in Pro(Set). The action $M^{(\infty)} \times K \rightarrow M^{(\infty)}$ and the operations $[-,-] : (M/M_0)^{(\infty)} \times (M/M_0)^{(\infty)} \rightarrow M_0^{(\infty)}$, $\tau : M^{(\infty)} \rightarrow M_0^{(\infty)}$ are given by

- $m^{(s)} \cdot k = (m \cdot k)^{(s)}$, $\iota(m'^{(s)}) \cdot k = \iota((k^2 m')^{(s)})$ for $k \in K$, $m \in M$, $m' \in M_0$;
- $\tau(m^{(s)}) = (s \tau(m))^{(s)}$, $\iota(m'^{(s)}) = (2m')^{(s)}$ for $m \in M$, $m' \in M_0$;
- $[m^{(s)}, m'^{(s)}] = (s[m, m'])^{(s)}$ for $m, m' \in M/M_0$.

Moreover, $(M^{(s)}, M_0^{(s)})$ are also 2-step $K$-modules and the structure maps from the definition of $M^{(\infty)}$ are morphisms of 2-step $K$-modules. Note that $(M^{(1)}, M_0^{(1)}) \cong (M, M_0)$.

Let $(M_i, M_0)$ be 2-step $K$-modules for $1 \leq i \leq n$. Fix some bilinear maps $[-,-] : M_i/M_0 \times M_j/M_0 \rightarrow \bigoplus_k M_{0k}$ for $i < j$. Then $(N, N_0) = \bigoplus_i (M_i, M_0)$ is also 2-step $K$-module, where $N_0 = \bigoplus_i M_0$ and $N = \bigoplus_i M_i$, the operations are defined in the obvious way. Moreover, $N^{(\infty)} \cong \bigoplus_i N_i^{(\infty)}$. 

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From now on fix an augmented odd form $K$-algebra $(R, \Delta, D)$ with an orthogonal hyperbolic family $\eta_1, \ldots, \eta_n$. There is a decomposition $R = \bigoplus_{-n \leq i,j \leq n} R_{ij}$, where $R_{ij} = e_i e_j$ and $e_0 = 1 - \sum_{i \neq 0} e_i \in R \times K$. Also $(\Delta, D) = \bigoplus_i (\Delta_i, D_i) \oplus \bigoplus_{i+j > 0} (\phi(R_{ij}), \rho(R_{ij}))$, where $D_i = D \cdot e_i$ and $\Delta_i = \Delta \cdot e_i$. Note that $\phi(R_{ij}) \cong R_{ij}$ for $i + j > 0$. Every 2-step $K$-module $(\Delta_i, D_i)$ further decomposes as $\bigoplus_{j \neq 0} (q_{ij}, R_{ij}, 0) \oplus (\Delta_{ij}^0, D_i)$, where $\Delta_{ij}^0 = \{ u \in \Delta_i \mid \pi(u) = e_0 \pi(u) \}$. Then $(\Delta^{(\infty)}, D^{(\infty)}) \cong \bigoplus_i (\Delta^{(\infty)}_i, D_i^{(\infty)}) \oplus \bigoplus_{i+j > 0} (\phi(R_{ij}^{(\infty)}), \rho(R_{ij}^{(\infty)}))$. All $(R^{(s)}, \Delta^{(s)}, D^{(s)})$ are augmented odd form $K$-algebras and $(R^{(\infty)}, \Delta^{(\infty)})$ is an odd form $K$-algebra in $\text{Pro}(\text{Set})$. The operations are given for $a \in R$, $u \in \Delta$, $v \in D$ by

- $a^{(s)} = \tilde{a}^{(s)}$;
- $u^{(s)} \cdot a^{(s)} = (u \cdot sa)^{(s)} \in \Delta^{(s)}$, $v^{(s)} \cdot a^{(s)} = (v \cdot sa)^{(s)} \in D^{(s)}$;
- $\phi(a^{(s)}) = \phi(a)^{(s)} \in D^{(s)}$, $\pi(u^{(s)}) = \pi(u)^{(s)}$;
- $\rho(u^{(s)}) = (sp(u)^{(s)})$, $\rho(v^{(s)}) = \rho(v)^{(s)}$.

Note that the localization $(S^{-1}R, S^{-1}\Delta, S^{-1}D)$ is an odd form $S^{-1}K$-algebra with $(u \cdot \frac{1}{s}) \cdot \frac{1}{s} = (u \cdot a) \cdot \frac{1}{s}$, $\phi(\frac{1}{s}) = \frac{\phi(a)}{s}$, $\pi(u \cdot \frac{1}{s}) = \frac{\pi(u)}{s}$, and $\rho(u \cdot \frac{1}{s}) = \frac{\rho(u)}{s}$.

**Lemma 4.** There is a well-defined action of the abstract odd form $K$-algebra $(S^{-1}R, S^{-1}\Delta)$ on the odd form $K$-algebra $(R^{(\infty)}, \Delta^{(\infty)})$ in $\text{Pro}(\text{Set})$ (coherent with the actions of $(K, 0)$) given by the pre-morphisms

- $a^{(ss')} \cdot \frac{1}{s} = (ab)^{(s)}$, $\frac{1}{s^2} a^{(ss')} = (ba)^{(s)}$;
- $u^{(ss')} \cdot \frac{1}{s} = (u \cdot s'b)^{(s)} \in \Delta^{(s)}$, $v^{(ss')} \cdot \frac{1}{s} = (v \cdot b)^{(s)} \in D^{(s)}$;
- $(w \cdot \frac{1}{s^2}) \cdot a^{(ss')} = (w \cdot a)^{(s)}$.

for $a, b \in R$, $u, w \in \Delta$, and $v \in D$. Moreover, $(R, \Delta)$ acts on each $(R^{(s)}, \Delta^{(s)})$ and the multiplications on $e_i$ and $q_i$ are the projections on the corresponding semi-direct factors.

**Proof.** This follows from direct computation. \hfill \square

Clearly, the localization $(S^{-1}R, S^{-1}\Delta)$ depends on $D$ only up to an isomorphism. Actually, $(R^{(\infty)}, \Delta^{(\infty)})$ also depends on $D$ only up to an isomorphism. Let $\Delta^{(s)} = \{ u^{(s)} \mid u \in \Delta \}$ with the structure maps $u^{(ss')} \mapsto u \cdot s'^{(s)}$, then $\Delta^{(\infty)} \rightarrow \Delta^{(\infty)}$, $u^{(s)} \rightarrow u^{(s)}$ is an isomorphism with the inverse $u^{(s^2)} \mapsto (u \cdot s)^{(s)}$, $v^{(s^2)} \rightarrow v^{(s)}$ for $u \in \Delta$, $v \in D$. The operations on $(R^{(\infty)}, \Delta^{(\infty)})$ are given in the same way as on $(R^{(\infty)}, \Delta^{(\infty)})$ with the exception $\phi(a^{(s^2)}) = \phi(a)^{(s)}$ (since we do not have the embedding $i$ for $\Delta^{(s)}$). The multiplication $\Delta^{(\infty)} \times S^{-1}R \rightarrow \Delta^{(\infty)}$ is given by a nicer formula $u^{(ss')}$, $\frac{1}{s} = (u \cdot b)^{(s)}$. But we need the construction via an augmentation in order to define a Steinberg pro-group.
7 Presentations of pro-groups

Now we are ready to define a Steinberg pro-group. Let \( \text{StU}(R, \Delta)(s) \) be the group generated by symbols \( X_{ij}(a) \) and \( X_{j}(u) \) for \( a \in R^{(s)}_{ij} \) and \( u \in (\Delta^{(s)})_{j}^{0} \). The relations on these symbols are (St0)–(St8). There are obvious structure homomorphisms \( \text{StU}(R, \Delta)^{(ss')} \to \text{StU}(R, \Delta)^{(s)}, \) and a Steinberg pro-group \( \text{StU}(R, \Delta)^{()}(\infty) \) is the formal projective limit of all \( \text{StU}(R, \Delta)^{(s)} \). The generators may be considered as the pre-morphisms \( X_{ij}: R^{(\infty)}_{ij} \to \text{StU}(R, \Delta)^{(\infty)} \) and \( X_{j}: (\Delta^{(\infty)})_{j}^{0} \to \text{StU}(R, \Delta)^{(\infty)} \) of pro-groups. Also there is a pre-morphism \( \text{StU}(R, \Delta)^{(\infty)} \to U(R^{(\infty)}_{ij}, (\Delta^{(\infty)})_{j}) \) of pro-groups, where every \( s: \text{StU}(R, \Delta)^{(s)} \to U(R^{(s)}_{ij}, (\Delta^{(s)})_{j}) \) is the restriction of the homomorphism corresponding to the odd form algebra \( (R^{(s)} \times R, (\Delta^{(s)}) \times \Delta) \) (it exists by lemma [3]).

**Lemma 5.** Let \( G^{(\infty)} \) be a pro-group. Then every morphism \( \text{StU}(R, \Delta)^{(\infty)} \to G^{(\infty)} \) of pro-groups is uniquely determined by its compositions with the generators \( X_{ij} \) and \( X_{j} \). Morphisms \( f_{ij}: R^{(\infty)}_{ij} \to G^{(\infty)} \) and \( f_{k}: (\Delta^{(\infty)})_{k}^{0} \to G^{(\infty)} \) of pro-sets are restrictions of a morphism \( \text{StU}(R, \Delta)^{(\infty)} \to G^{(\infty)} \) of pro-groups if and only if they satisfy (St0)–(St8) in Pro\( (\text{Set}) \).

**Proof.** This follows directly from the definition of \( \text{StU}(R, \Delta)^{(\infty)} \). \( \square \)

We say that the orthogonal hyperbolic family is strong if all \( e_{i} \) are Morita equivalent. Sometimes we write the variables from \( R^{(\infty)} \times R^{(\infty)} \) as \( a \otimes b \) and the variables from \( \Delta^{(\infty)} \times R^{(\infty)} \) as \( u \boxtimes a \).

**Lemma 6.** Let \( i, j, k \) be non-zero indices from \( \{-n, \ldots, n\} \). Then there is \( N \geq 0 \) such that the morphism

\[
(R^{(\infty)}_{ij} \times R^{(\infty)}_{jk})^{N} \times (R_{i,-j}^{(\infty)} \times R_{-j,k}^{(\infty)})^{N} \to R_{ik}, (a_{p} \otimes b_{p})_{l=\pm j}^{1 \leq p \leq N} \to \sum_{l,p} a_{ip}b_{lp}
\]

is a split epimorphism in Pro\( (\text{Set}) \). If the orthogonal hyperbolic family is strong, then the same is true for

\[
(R^{(\infty)}_{ij} \times R^{(\infty)}_{jk})^{N} \to R_{ik}, (a_{p} \otimes b_{p})_{p=1}^{N} \to \sum_{p} a_{p}b_{p}.
\]

**Proof.** Let \( e_{i} = \sum_{l=\pm j} \sum_{p} x_{ip}y_{tp} \) for some \( x_{ip} \in R_{ii} \) and \( y_{tp} \in R_{ti} \). Then \( c^{(s)} \mapsto (x_{ip}^{(s)} \otimes (y_{tp}c^{(s)}))_{i,p} \) is a required section. The second claim may be proved similarly. \( \square \)

**Lemma 7.** Let \( i, j \) be non-zero indices from \( \{-n, \ldots, n\} \). Then there is \( N \geq 0 \) such that the morphism

\[
((\Delta^{(\infty)})_{j}^{0} \times R^{(\infty)}_{ji})^{N} \times ((\Delta^{(\infty)})_{-j}^{0} \times R^{(\infty)}_{-j,i})^{N} \times R_{-i,i} \to (\Delta^{(\infty)})_{i}^{0},
\]

\[
((u_{ip} \boxtimes a_{ip})_{l=\pm j}^{1 \leq p \leq N}, b) \mapsto \sum_{l,p} a_{ip} \cdot a_{lp} + \phi(b)
\]

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is a split epimorphism in Pro(\text{Set}). If the orthogonal hyperbolic family is strong, then the same is true for

\[(\Delta(\infty)_{ij}^0 \times R_{ij}^{(\infty)} \times R_{-\text{i},\text{i}}) \rightarrow (\Delta(\infty)_{ij}^0, (u_p \boxtimes a_p)_{p=1}^N, b) \rightarrow \sum_p u_p \cdot a_p + \phi(b).\]

**Proof.** Let \(e_i = \sum_{l=\pm j} \sum_p x_{ip}y_{ip}\) for some \(x_{ip} \in R_{il}\) and \(y_{ip} \in R_{li}\). Then

\[
u(s^2) + \iota(v(s^2)) \rightarrow \left((u(s), s x_{ip} + i(v(s)) \cdot x_{ip}) \boxtimes y_{ip}^{(s)}\right)_{l,p},
\]

\[
\sum_{(l,p) < (p',p')} y_{lp}' x_{lp}' (s^2 \rho(u) + s^2 \rho(v)(s)) x_{lp} y_{ip}
\]

is a required section. The strong case is similar. \(\square\)

**Proposition 3.** If \(n \geq 3\), then the Steinberg group \(\text{StU}(R, \Delta)\) is perfect.

**Proof.** Indeed, let \(0 \neq i \neq k \neq 0\) and choose an index \(j\) different from \(0, \pm i, \pm j\). Then \(X_{ik}(*)\) lies in the subgroup generated by \([X_{i,\pm k}(*), X_{\pm k,j}(*)]\) by (St1), (St4), and lemma \(\Box\) applied to \(S = \{1\}\). Similarly, let \(0 \neq i\) and choose an index \(j\) different from \(0\) and \(\pm i\). Then \(X_{i}(*)\) lies in the subgroup generated by \([T_{\pm j}(*), T_{\pm j,i}(*)], T_{\pm j,i}(*)\) by (St2), (St5), (St8), and lemmas \(\Box\) \(\Box\) applied to \(S = \{1\}\). \(\square\)

Now we describe \(R_{ij}^{(\infty)}\) and \((\Delta(\infty))_{ij}^0\) as in lemmas \(\Box\) \(\Box\) but with explicit relations.

**Lemma 8.** Let \(i, j, k\) be non-zero indices from \([-n, \ldots, n]\), \(G^{(\infty)}\) be a pro-group, \(f_{\pm j}: R_{i,\pm j}^{(\infty)} \times R_{\pm j,k}^{(\infty)} \rightarrow G^{(\infty)}\) be morphisms of pro-sets. Then \(f_{\pm j}\) factor through a morphism \(g: R_{ik}^{(\infty)} \rightarrow G^{(\infty)}\) of pro-groups if and only if they satisfy

- \([f_l(a \otimes b), f_{l'}(a' \otimes b') = 1\) for \(l, l' = \pm j\);
- \(f_l((a + a') \otimes b) = f_l(a \otimes b) f_l(a' \otimes b)\) for \(l = \pm j\);
- \(f_l(a \otimes (b + b')) = f_l(a \otimes b) f_l(a \otimes b')\) for \(l = \pm j\);
- \(f_l(a \otimes bc) = f_{l'}(ab \otimes c)\) for \(l, l' = \pm j, a \in R_{il}^{(\infty)}, b \in R_{il'}^{(\infty)}, c \in R_{il'}^{(\infty)}\).

If the orthogonal hyperbolic family is strong, then the same is true for \(f_j\) without \(f_{-j}\) and for \(l = l' = j\) in the formulas.

**Proof.** Suppose that \(f_{\pm j}\) satisfy the formulas (in the other direction the claim is obvious). Note that \(g\) is unique by lemma \(\Box\) hence we may assume that \(G\) is a group. Let \(e_i = \sum_{l=\pm j} \sum_p x_{ip}y_{ip}\) for some \(x_{ip} \in R_{il}\) and \(y_{ip} \in R_{li}\). Define \(g\) by

\[
g(c(s^2)) = \prod_{l=\pm j} \prod_p f_l(x_{ip}^{(s)} \otimes (y_{ip} c)^{(s)})
\]
for sufficiently large $s$. It is easy to see that $g$ is a homomorphism for large $s$. Moreover,

$$f_{\pm j}(a^{(s^2)} \otimes b^{(s^2)}) = \prod_{l,p} f_{\pm j}((sx_lpy_l a)^{(s)} \otimes (sb)^{(s)})$$

$$= \prod_{l,p} f_{\pm j}(x_l^{(s)} \otimes (s^2 y_l p a b)^{(s)}) = g((s^2 ab)^{(s^2)})$$

for large $s$. In the strong case the proof is similar.

Lemma 9. Let $i, j$ be non-zero indices from $\{-n, \ldots, n\}$, $G^{(\infty)}$ be a pro-group, $f_{\pm j}: (\Delta^{(\infty)}_i)^0_{\pm j} \times R^{(\infty)}_{i\pm j} \rightarrow G^{(\infty)}$ and $g: R^{(\infty)}_{-l_i} \rightarrow G^{(\infty)}$ be morphisms of pro-sets. Then $f_{\pm j}$ and $g$ factor through a morphism $h: (\Delta^{(\infty)}_i)^0 \rightarrow G^{(\infty)}$ of pro-groups if and only if they satisfy

- $[f_1(u \boxtimes a), g(b)] = 1$ for $l = \pm j$;
- $[f_1(u \boxtimes a), f_{l'}(v \boxtimes b)] = g(-\overline{a\pi(u)\pi(v)b})$ for $l, l' = \pm j$;
- $f_1((u + u') \boxtimes a) = f_1(u \boxtimes a) f_1(u' \boxtimes a)$ for $l = \pm j$;
- $f_1(u \boxtimes (a + a')) = f_1(u \boxtimes a) g(\overline{a'}\rho(a) f_1(u \boxtimes a')$ for $l = \pm j$;
- $f_1(u \boxtimes ab) = f_{l'}(u \cdot a \boxtimes b)$ for $l, l' = \pm j, u \in (\Delta^{(\infty)}_i)^0, a \in R^{(\infty)}_{ii}, b \in R^{(\infty)}_{ii'}$;
- $g(a + a') = g(a) g(a')$;
- $g(\overline{a}) = g(a)^{-1}$;
- $f_1(\overline{a}(a \boxtimes b) = g(\overline{b \cdot ab})$ for $l = \pm j, a \in R^{(\infty)}_{ij}, b \in R^{(\infty)}_{ii}$.

Proof. Suppose that $f_{\pm j}$ and $g$ satisfy the formulas (in the other direction the claim is obvious). Note that $h$ is uniquely determined by lemma 7; hence we may assume that $G$ is a group. Let $e_i = \sum_{l = \pm j} \sum_p x_l p y_l p$ for some $x_l p \in R_{ii}$ and $y_l p \in R_{ii'}$. Define $h$ by

$$h(u^{(s^2)} + \ell(v^{(s^2)})) = \prod_{l,p} f_1((u^{(s)} \cdot sx_l p + \ell(v^{(s)}) \cdot x_l p) \boxtimes y_l^{(s)})$$

$$g\left(\sum_{(l,p) < (l', p')} y_l^{(s')} x_l^{(s')} (s^5 \rho(u) + s^2 \rho(v))^{(s)} x_l p y_l p\right)$$

for sufficiently large $s$, where $(l, p)$ runs over $\{j, -j\} \times \{1, \ldots, N\}$ with the lexicographic order. It is easy to see that $h$ is a homomorphism for large $s$. 

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Moreover, 
\[
f_{\pm j}(u(s^3) + \iota(v(s^3))) \boxtimes a(s^3) = \prod_{l,p} f_{\pm j}(u(s^3) + \iota(v(s^3))) \boxtimes (ax_{lp}y_{lp}(s^3))
\]
\[
g(\sum_{(l,p) < (l',p')} (ax_{l'p'}y_{l'p'}(s^3)) \rho(u(s^3) + \iota(v(s^3)))) (ax_{lp}y_{lp}(s^3))
\]
\[
= \prod_{l,p} f_1((u(s^3) + \iota(v(s^3))) \cdot (sax_{lp}(s^3) \boxtimes y_{lp}^{(s)})
\]
\[
g(\sum_{(l,p) < (l',p')} (ax_{l'p'}y_{l'p'}(s^3)) \rho(u(s^3) + \iota(v(s^3)))) (ax_{lp}y_{lp}(s^3))
\]
\[
= h((u \cdot s^3 a)(s^3) + \iota((v \cdot s^3 a)(s^3)))
\]
and
\[
g(a(s^3)) \equiv g(\sum_{(l,p) < (l',p')} (x_{l'p'}y_{l'p'}(s^3)) a(s^3) (x_{lp}y_{lp}(s^3))
\]
\[
= \prod_{l,p} f_1(\iota(\phi(ax_{lp}(s^3)) \boxtimes y_{lp}^{(s)})
\]
\[
g(\sum_{(l,p) < (l',p')} (x_{l'p'}y_{l'p'}(s^3)) (a - \bar{a})(s^3) (x_{lp}y_{lp}(s^3))
\]
\[
= h(\iota(\phi(a)(s^3)))
\]
for large $s$. The strong case is similar. \qed

8 Elimination of an ultrashort root

We are ready to prove that the morphism $F_n : StU(R, \Delta; \Phi/\alpha)^{\infty} \rightarrow StU(R, \Delta; \Phi)^{\infty}$ of pro-groups is an isomorphism for any $\alpha$ if $n$ is sufficiently large.

Lemma 10. Suppose that $n \geq 3$ and $\alpha \in \Phi$ is a non-long root. Then $F_n : StU(R, \Delta; \Phi/\alpha)^{\infty} \rightarrow StU(R, \Delta; \Phi)^{\infty}$ is an epimorphism of pro-groups. If $\Phi/\{\alpha, \beta\}$ is defined and in addition the orthogonal hyperbolic family is strong or $\beta$ is ultrashort, then $F_{\{\alpha, \beta\}} : StU(R, \Delta; \Phi/\{\alpha, \beta\})^{\infty} \rightarrow StU(R, \Delta; \Phi)^{\infty}$ is also an epimorphism of pro-groups.

Proof. To prove the first statement, note that any generator $X_3$ of $StU(R, \Delta; \Phi)^{\infty}$ factors through $F_n$ unless $\gamma$ and $\alpha$ are collinear. By symmetry, it suffices to consider only the case $\gamma = \alpha$. If $\alpha$ is short, then without loss of generality $\alpha = e_n - e_{n-1}$. Then $X_{n-1,n}(ab) = X_{n,\pm 1}(a) X_{\pm 1,n}(b)$ for $a \in R_{\pm 1,n}^{(\infty)}$ and $b \in R_{\pm 1,n}^{(\infty)}$, hence we are done by lemmas \ref{short1} and \ref{short2}. If $\alpha$ is ultrashort, then without loss of generality $\alpha = e_1$. Then $X_1(u \cdot b) = X_{\pm 2,2}(\rho(u) b) [X_{\pm 2,2}(\pm u), X_{\pm 2,2}(\pm u)]$ and $X_1(\phi(ab)) = [X_{\pm 2,2}(a), X_{\pm 2,2}(b)]$ for $u \in (\Delta^{(\infty)})_{\pm 2}^{0}, a \in R_{\pm 1,\pm 2}^{(\infty)}$, and $b \in R_{\pm 2,2}^{(\infty)}$. Hence we are done by lemmas \ref{ultrashort1}, \ref{ultrashort2}, and \ref{ultrashort3}.
Now suppose that $\Phi_1 (\alpha, \beta)$ is defined. Any generator $X$, of $\text{StU}(R, \Delta; \Phi)$ factors through $\text{StU}(R, \Delta; \Phi_1 (\alpha, \beta))$ unless $\gamma$ lies in the span of $\alpha$ and $\beta$. We again apply lemmas 5, 6, 7. If $\Phi_1 (\alpha, \beta)$ is of type $A_2$, then without loss of generality $\alpha = e_n - e_{n-1}$ and $\beta = e_n - e_{n-1}$. We may use the identity $X_{n-1,n}(ab) = X_{n-1,1}(a)X_{n-1,n}(b)$ for $a \in \mathcal{R}_{n-1,1}^\infty$ and $b \in \mathcal{R}_{1,n}^\infty$ (here we need that the orthogonal hyperbolic family is strong). If $\Phi_1 (\alpha, \beta)$ is of type $A_1 \times A_1$, then $n \geq 4$ and without loss of generality $\alpha = e_n - e_{n-3}$, $\beta = e_n - e_{n-1}$, so we use the identity $X_{n-1,n}(ab) = [X_{n-1,\pm(n-2)}(a), X_{\pm(n-2),n}(b)]$ for $a \in \mathcal{R}_{n-1,\pm(n-2)}^\infty$ and $b \in \mathcal{R}_{\pm(n-2),n}^\infty$.

Suppose that $\Phi_1 (\alpha, \beta)$ is of type $A_1 \times BC_1$. Then without loss of generality $\alpha = e_n - e_{n-1}$, $\beta = e_1$, and $\gamma$ coincide with one of them. If $\gamma = \alpha$, then we have the identity $X_{n-1,n}(ab) = [X_{n-1,\pm1}(a), X_{\pm1,n}(b)]$ for $\alpha \in \mathcal{R}_{n-1,\pm1}^\infty$, $b \in \mathcal{R}_{\pm1,n}^\infty$. If $\gamma = \beta$, then there are the identities $X_{1}(u \cdot b) = X_{1}(\rho(u) b) [X_{\pm1}(\pm u), X_{\pm1}(\pm b)]$ and $X_{1}(\phi(ab)) = [X_{-1,\pm2}(a), X_{\pm2,1}(b)]$ for $\rho(u) = e, 0$.

Finally, suppose that $\Phi_1 (\alpha, \beta)$ is of type $BC_2$. Without loss of generality, $\alpha = e_n - e_{n-1}$, $\beta = e_{n-1}$, and $\gamma$ coincides with one of them. If $\gamma = \alpha$, then there is an identity $X_{n-1,n}(ab) = [X_{n-1,\pm1}(a), X_{\pm1,n}(b)]$ for $a \in \mathcal{R}_{n-1,\pm1}^\infty$ and $b \in \mathcal{R}_{\pm1,n}^\infty$. If $\gamma = \beta$, then we apply the identities $X_{n-1}(u \cdot b) = X_{n-1}(\rho(u) b) [X_{\pm1}(\pm u), X_{\pm1,n-1}(\pm b)]$ and $X_{n-1}(\phi(ab)) = [X_{-1,\pm1}(a), X_{\pm1,n-1}(b)]$ for $u \in (\Delta^\infty)_0, a \in R_{n-1,\pm1}^\infty$, $b \in R_{\pm1,n}^\infty$.

Proposition 4. Suppose that $n \geq 3$. Then for every ultrashort root $\alpha$ the morphism $F_\alpha : \text{StU}(R, \Delta; \Phi_\alpha) \rightarrow \text{StU}(R, \Delta; \Phi)$ is an isomorphism of pro-groups.

Proof. Without loss of generality, $\alpha = e_1$. We denote the generators of $\text{StU}(R, \Delta; \Phi_\alpha)$ by $X_{ij}$ and $X_j$, in the first case $i, j \in \{\pm 2, \ldots, \pm n\}$ and in the second case $i, j \in \{\pm 1, \ldots, \pm n\}$. We have to construct morphisms $\bar{X}_{ij} : \mathcal{R}_{ij}^\infty \rightarrow \text{StU}(R, \Delta; \Phi_\alpha)^\infty$, $\bar{X}_j : (\Delta^\infty)_0 \rightarrow \text{StU}(R, \Delta; \Phi)^\infty$ for $i, j \in \{\pm 1, \ldots, \pm n\}$ and to show that they satisfy (St0)–(St8).

Let

- $\bar{X}_{ij}(a) = X_{ij}(a)$ and $\bar{X}_j(u) = X_j(u)$ for $i, j \neq \pm 1$;
- $\bar{X}_{\pm 1,j}(a) = \bar{X}_{-j,\mp 1}(\pm a) = X_{j}(q_{\pm 1} \cdot a)$ for $j \neq \pm 1$;
- $\bar{X}_{i,1}(a \otimes b) = [\bar{X}_{-i,1}(a), \bar{X}_{i,1}(b)]$ for $i \neq \pm 1$, $a \in \mathcal{R}_{-1,1}^\infty$, $b \in \mathcal{R}_{1,1}^\infty$;
- $\bar{X}_{i}(u \boxtimes b) = \bar{X}_{i,1}(\rho(u) b) [X_i(\pm u), \bar{X}_{i,1}(\pm b)]$ for $i \neq \pm 1$, $u \in (\Delta^\infty)^0_{i,1}$, $b \in \mathcal{R}_{i,1}^\infty$.

Now we have $\bar{X}_\beta$ for all $\beta \notin \alpha$, and these generators satisfy the Steinberg relations not involving $R\alpha$. Since no Steinberg relation involves $\alpha$ and $-\alpha$ simultaneously, it suffices to construct $\bar{X}_1$ and prove the Steinberg relations for it.
At first we show (St7) for \( \bar{X}_{-1,1}(a \otimes b) \) and \( \bar{X}_{1}(u \boxtimes b) \). Clearly, \( \bar{X}_{ik}(c) \) commutes with \( \bar{X}_{-1,1}(a \otimes b) \) and \( \bar{X}_{1}(u \boxtimes b) \) if \( j \) and \( k \) are different from \( \pm i \) and \( j \neq -1 \neq -k \). If \( k \neq \pm 1, \pm i \), then it is easy to see that

\[
\bar{X}_{ik}(c)\bar{X}_{-1,1}(a \otimes b) = \bar{X}_{-1,1}(a \otimes b), \quad \bar{X}_{ik}(c)\bar{X}_{1}(u \boxtimes b) = \bar{X}_{1}(u \boxtimes b),
\]

\[
\bar{X}_{-ik}(c)\bar{X}_{-1,1}(a \otimes b) = \bar{X}_{-1,1}(a \otimes b), \quad \bar{X}_{-ik}(c)\bar{X}_{1}(u \boxtimes b) = \bar{X}_{1}(u \boxtimes b).
\]

Since \( \bar{X}_{\pm 1,1}(cd) = [\bar{X}_{\pm 1,j}(c), \bar{X}_{j}(d)] \) for \( j \notin \{ -1, 1, -i, i \} \), \( c \in R_{\pm 1,i}^{(\infty)} \) and \( d \in R_{j}^{(\infty)} \), it follows that \( \bar{X}_{\pm 1,1}(a \otimes b) \) and \( \bar{X}_{1}(u \boxtimes b) \) by lemma \( \square \) By (St0), \( \bar{X}_{j, \pm 1}(c) \) also commutes with \( \bar{X}_{-1,1}(a \otimes b) \) and \( \bar{X}_{1}(u \boxtimes b) \) if \( j \neq \{ -i, i, 1 \} \).

Now we prove (St6). If \( j \notin \{ -1, 1, -i, i \} \), then clearly

\[
\bar{X}_{j}(c)\bar{X}_{-1,1}(a \otimes b) = \bar{X}_{-1,1}(a \otimes b), \quad \bar{X}_{j}(c)\bar{X}_{1}(u \boxtimes b) = \bar{X}_{1}(u \boxtimes b)\bar{X}_{j}(c).
\]

Since \( \bar{X}_{\pm 1,i}(\phi(c)) \) commutes with \( \bar{X}_{-1,1}(a \otimes b) \) and \( \bar{X}_{1}(u \boxtimes b) \) for \( j \notin \{ -1, 1, -i, i \} \), \( c \in R_{\pm 1,i}^{(\infty)} \), \( d \in R_{j}^{(\infty)} \), it follows by lemma \( \square \) that \( \bar{X}_{\pm 1,i}(\phi(c)) \) commutes with \( \bar{X}_{-1,1}(a \otimes b) \) and \( \bar{X}_{1}(u \boxtimes b) \). If \( j \notin \{ -1, 1, -i, i \} \), then for \( c \in R_{\pm 1,i}^{(\infty)} \), \( d \in R_{j}^{(\infty)} \), \( v \in (\Delta^{(\infty)})_{j} \) we have

\[
\bar{X}_{-1,1}(a \otimes b, \bar{X}_{1}(u \boxtimes b)) = \bar{X}_{1}(\phi(c)) = \bar{X}_{1}(u \boxtimes b, \bar{X}_{-1,1}(c \otimes d), \bar{X}_{1}(u \otimes d)) \quad \text{for all possible } i, j.
\]

Also,

\[
\bar{X}_{-1,1}(a \otimes (b + b')) = \bar{X}_{i, -1}(a \otimes b) \bar{X}_{i, 1}(a \otimes b'),
\]

\[
\bar{X}_{-1,1}(a \otimes b) = [\bar{X}_{-1, -i}(- \bar{a}), \bar{X}_{-1,1}(- \bar{a})]^{-1} = \bar{X}_{-1,1}(- \bar{a} \otimes \bar{a}).
\]

If \( i \neq \pm j \), the for \( a \in R_{-1, i}^{(\infty)} \), \( b \in R_{j}^{(\infty)} \), \( c \in R_{j}^{(\infty)} \) we have

\[
\bar{X}_{-1,1}(a \otimes bc) = [\bar{X}_{-1, -i}(a \bar{X}_{ij}(b), \bar{X}_{j}(c)) \bar{X}_{i}(b) \bar{X}_{j}(c)] = \bar{X}_{-1,1}(a \otimes c).
\]

Let \( i \notin \{ -1, 1, -2, 2 \} \). Since \( \bar{X}_{l, -1}(a \otimes bc) = \bar{X}_{l, -1}(abc \otimes d) \) for \( l, l' \in \{ -2, 2 \} \), \( a \in R_{-1, i}^{(\infty)} \), \( b \in R_{j}^{(\infty)} \), \( c \in R_{j}^{(\infty)} \), \( d \in R_{l}^{(\infty)} \), it follows by lemma \( \square \) that \( \bar{X}_{l, -1}(a \otimes bc) = \bar{X}_{l, -1}(abc \otimes d) \) for \( l, l' \in \{ -2, 2 \} \), \( a \in R_{-1, i}^{(\infty)} \), \( b \in R_{j}^{(\infty)} \), \( c \in R_{j}^{(\infty)} \). By the same lemma, there is a morphism \( \tilde{X}_{-1, 1} : R_{-1, i}^{(\infty)} \rightarrow \text{StU}(R, \Delta, \Phi(\alpha))^{(\infty)} \) of pro-groups such that \( \tilde{X}_{-2}(a \otimes b) = \tilde{X}_{-1,1}(ab) \). Moreover, \( \tilde{X}_{-1,1}(a) = \tilde{X}_{-1,1}(- \bar{a}) \) and \( \tilde{X}_{-1,1}(a \otimes b) = \tilde{X}_{-1,1}(ab) \) for all \( i \neq \pm 1 \).
We are ready to construct \( \tilde{X}_1(u) \). For all \( i, j \in \{-1, 1\} \) there is a commutation relation
\[
\tilde{X}_1^i((u \otimes b)X c) = \tilde{X}_1^i(u \otimes c)\tilde{X}_{-1,1}(-b\pi(u)\pi(v)c).
\]
Also,
\[
\tilde{X}_1^i((u + u') \otimes b) = \tilde{X}_1^i(u \otimes b) \tilde{X}_1^i(u' \otimes b),
\]
\[
\tilde{X}_1^i(u \otimes (b + b')) = \tilde{X}_1^i(u \otimes b) \tilde{X}_{-1,1}(b\rho(u)b) \tilde{X}_1^i(u \otimes b').
\]
If \( i \neq \pm j \), then for \( u \in (\Delta(\infty))^0, a \in R_{-j,i}^{(\infty)}, b \in R_{ij}^{(\infty)}, c \in R_{ji}^{(\infty)} \)
\[
\tilde{X}_1^i(u \otimes bc) = \tilde{X}_{-1,i}(\rho(u)bc)\tilde{X}_1^i(-c)\tilde{X}_{11}(bc) = \tilde{X}_1^i(u \cdot b \otimes c),
\]
\[
\tilde{X}_1^i((\phi(ab) \otimes c) = \tilde{X}_{-j,i}((-ab)c)\tilde{X}_1^i(-a)[\tilde{X}_1((\phi(ab))] = \tilde{X}_{-1,1}(\phi(ab)).
\]

Let \( i \neq \pm 2 \). Since \( \tilde{X}_1^i(u \otimes bc d) = \tilde{X}_{11}(u \cdot bc d) \) for \( l, l' \in \{-2, 2\}, u \in (\Delta(\infty))^0, b \in R_{ij}^{(\infty)}, d \in R_{i1}^{(\infty)}, c \in R_{ji}^{(\infty)} \), it follows by lemma \( \Box \) that \( \tilde{X}_1^i(u \otimes bc) = \tilde{X}_1^i(u \cdot b \otimes c) \) for \( l, l' \in \{-2, 2\}, u \in (\Delta(\infty))^0, b \in R_{ij}^{(\infty)}, c \in R_{ji}^{(\infty)} \).

Similarly, \( \tilde{X}_1^{\pm 2}(\phi(ab) \otimes bc) = \tilde{X}_1^{\pm 2}(u \otimes bc) \) and \( \tilde{X}_1^{\pm 1}(a) = \tilde{X}_1(a) \). Moreover, \( \tilde{X}_1^i(u \otimes b) = \tilde{X}_1^i(u \cdot b) \) for all \( i \neq 1 \). Clearly, \( \tilde{X}_1(1) \) satisfies (St2), (St5), (St6), (St7), and the case of (St8) where \( \tilde{X}_1(a) \) appears in the right hand side.

For any \( i \neq \pm 1 \) let \( j \neq \{-1, 1, -i, i\} \). We have
\[
\tilde{X}_1^i(\phi(ab) \otimes c)\tilde{X}_1^i(u \otimes b) = \tilde{X}_1^i((\phi(ab) \otimes c)\tilde{X}_1^i(u \otimes b)) = \tilde{X}_1^i(\phi(ab) \otimes bc)\tilde{X}_1^i(u \otimes b).
\]

From lemma \( \Box \), it follows that there is unique morphism \( G_{\alpha} : \text{StU}(R, \Delta; \Phi)^{(\infty)} \to \text{StU}(R, \Delta; \Phi/\alpha)^{(\infty)} \) of pro-groups such that \( G_{\alpha}(X_{ij}(a)) = \tilde{X}_{ij}(a) \) and \( G_{\alpha}(X_{ij}(u)) = \tilde{X}_{ij}(u) \). Moreover, \( G_{\alpha} \circ F_{\alpha} = \text{id} \). Since \( F_{\alpha} \) is an epimorphism of pro-groups, \( F_{\alpha}^{-1} \) is an isomorphism with \( F_{\alpha}^{-1} = G_{\alpha} \). \( \square \)

9 Elimination of a short root

We need a technical lemma to handle the case \( n = 3 \).

Lemma 11. Suppose that \( n \geq 3 \) and the orthogonal hyperbolic family is strong. Let \( \{-,-,\varepsilon\}_{ij} : R_{ij}^{(\infty)} \times R_{ij}^{(\infty)} \times R_{ij}^{(\infty)} \to G^{(\infty)} \) be morphisms of pro-sets for \( i, j \in \{-1, 1\}, G^{(\infty)} \) be a pro-group with additive notation. Moreover, suppose that these morphisms satisfy
\[
\{a, b, c\}_{ij} + \{a', b', c'\}_{kl} = \{a', b', c'\}_{kl} + \{a, b, c\}_{ij}, \{a, b, c\}_{ij} \text{ are triadditive};
\]
\[ \{a, bc, d\}_{ij} = \{ab, c, d\}_{ij} + \{a, b, cd\}_{jk}; \]
\[ \{a, b, c\}_{-i,i} = \{a, b\}, c, d\}_{-i,i}, \{a, b, ac\}_{-i,i} = 0; \]
\[ \{a, bc, d\}_{ij} = \{\bar{b}, ac, d\}_{-i,j}, \{a, b, cd\}_{ij} = -\{\bar{c}, \bar{b}, \bar{d}\}_{-j,-i}. \]

if both sides are defined. Then \( \{a, b, c\}_{ij} = 0 \) for all \( i, j \in \{-1, 1\} \).

**Proof.** First of all, these axioms imply that \( \{ab, \bar{c}d, e\}_{ij} = \{ac, \bar{b}d, e\}_{-i,j} \) for \( b, c \in R_{2,i}^{(\infty)} \) and \( \{ab, \bar{c}d, e\} = \{ac, \bar{b}d, \bar{e}\} \) for \( c, d \in R_{2,i}^{(\infty)} \). From this and lemma 6 we get

\[
\begin{align*}
\{axyb, c, d\}_{ij} &= \{axyb, c, d\}_{ij} \text{ for } x, y \in R_{\pm 2, \pm 2}, \\
\{a, bxy, d\}_{ij} &= \{a, bxy, d\}_{ij} \text{ for } x, y \in R_{\pm 2, \pm 2}, \\
\{a, b, cxyd\}_{ij} &= \{a, b, cxyd\}_{ij} \text{ for } x, y \in R_{\pm 2, \pm 2}, \\
\{ab, cd, e\}_{ij} &= \{ab, cd, e\}_{ij} \text{ for } x \in R_{\pm 2, \pm 2}, \\
\{a, b, cxe\}_{ij} &= \{a, b, cxe\}_{ij} \text{ for } x \in R_{\pm 2, \pm 2}, \\
\{a, bxyd, e\}_{ij} &= \{a, bxyd, e\}_{ij} \text{ for } x \in R_{\pm 2, \pm 2} \text{ and } y \in R_{\pm 2, \pm 2}.
\end{align*}
\]

Let \( \bar{R}_{22} \) be the factor of \( R_{22} \) by the ideal \( I \) generated by all additive commutators \( [x, y] = xy - yx \). Then \( \bar{R}_{22} \) is a unital commutative ring and \( \bar{R}_{21} = R_{21}/IR_{21} \) is a finite projective \( \bar{R}_{22} \)-module with the dual \( \bar{R}_{12} = R_{12}/R_{12}I \). Locally in the Zarisky topology on \( \bar{R}_{22} \) the modules \( \bar{R}_{31} \) is free. Hence there are elements \( f_t \in \bar{R}_{22}, g_t \in \bar{R}_{22}, u_{tp} \in \bar{R}_{21}, v_{tp} \in \bar{R}_{12} \) for \( 1 \leq t \leq M, 1 \leq p \leq N_t \) such that

\[
\begin{align*}
\sum_{p} v_{tp}u_{tp} &= f_t \pmod{I} \quad \text{for } p \neq q, \\
u_{tp}v_{tq} &= f_t \pmod{I}, \\
\sum_{t} f^2_t g_t &= 1 \pmod{I},
\end{align*}
\]

where \( f_t \in \bar{R}_{11} \) means \( \sum_r x_r f_t y_r \) for some fixed elements \( x_r \in R_{12} \) and \( y_r \in R_{21} \) with \( \sum_r x_r y_r = 1 \).

Hence it remains to prove that \( \{au_{tp}, v_{tp}, bu_{tp}, q, q\} = 0 \) for all \( t, p, p', q, q' \) (here we again use lemma 6). If \( p \neq p' \) and \( q \neq q' \), then this follows from lemma 6 and the identity

\[
\{au_{tp}, v_{tp}, bu_{tp}, c, d\} = \{au_{tp}v_{tp}, b, c, d\} + \{au_{tp}, v_{tp}, b, cu_{tp}, q, d\}.
\]

The case \( p \neq q' \) and \( q \neq q' \) reduces to the case \( p \neq p' \) and \( q \neq q' \) using the properties of \( \{a, b, c\}_{ij} \). Up to symmetry, the only remaining case is \( p = q \). But this follows from \( \{a, b, ac\}_{-i,i} = 0 \) and other properties. \( \square \)

**Proposition 5.** Suppose that \( n \geq 4 \) or \( n \geq 3 \) and the orthogonal hyperbolic family is strong. Then for every short root \( \alpha \) the morphism \( F\alpha: \text{StU}(R, \Delta; \Phi/\alpha)^{(\infty)} \to \text{StU}(R, \Delta; \Phi)^{(\infty)} \) is an isomorphism of pro-groups.
Proof. Without loss of generality, \( \alpha = e_n - e_{n-1} \). We denote the generators of \( \text{StU}(R, \Delta; \Phi/\alpha) \) and \( \text{StU}(R, \Delta; \Phi) \) by \( X_{ij} \) and \( \tilde{X}_{ij} \), in the first case \( i, j \in \{ \pm 1, \ldots, \pm (n-2), \pm \infty \} \) and in the second case \( i, j \in \{ \pm 1, \ldots, \pm n \} \) (where \( \eta_{\infty} = \eta_{n-1} \oplus \eta_n \)). We have to construct morphisms \( \tilde{X}_{ij} : R^{(\infty)}_{ij} \to \text{StU}(R, \Delta; \Phi/\alpha) \) and \( \tilde{X}_j : (\Delta^{(\infty)})_0 \to \text{StU}(R, \Delta; \Phi/\alpha) \) for \( i, j \in \{ \pm 1, \ldots, \pm n \} \) and to show that they satisfy (St0)–(St8). Let

- \( \tilde{X}_{ij}(a) = X_{ij}(a) \) and \( \tilde{X}_j(u) = X_j(u) \) for \( |i|, |j| < n - 1 \);
- \( \tilde{X}_{ij}(a) = \tilde{X}_{j,-i}(-a) = X_{i\infty}(a) \) for \( |i| < n - 1, j \in \{ n-1, n \} \);
- \( \tilde{X}_j(u) = \tilde{X}_{j,-i}(-a) = X_{\infty j}(a) \) for \( i \in \{ n-1, n \}, |j| < n - 1 \);
- \( \tilde{X}_{1-n,n}(a) = \tilde{X}_{n,n-1}(-a) = X_{\infty}(\phi(a)) \) and \( \tilde{X}_{n-1,n}(b) = \tilde{X}_{n-1,n}(-\overline{b}) = X_{\infty}(\phi(b)) \);
- \( \tilde{X}_{n-1,n}(u \otimes v) = [\tilde{X}_{1-n}(-u), \tilde{X}_n(v)] \) for \( u \in (\Delta^{(\infty)})_{1-n, n}^0, v \in (\Delta^{(\infty)})_{n}^0, b \in R_{1-n,n}^{(\infty)} \);
- \( \tilde{X}_n^1(a \otimes b) = [\tilde{X}_{1-n}(u), \tilde{X}_{n-1,n}(b)] \tilde{X}_n(u \cdot (-b)) \) for \( u \in (\Delta^{(\infty)})_{1-n, n}^0 \);
- \( \tilde{X}_{n-1,n}^i(a \otimes b) = [\tilde{X}_{n-1,i}(a), \tilde{X}_{n-1,n}(b)] \) for \( |i| < n - 1, a \in R_{n-1,i}^{(\infty)}, b \in R_{n-1,n}^{(\infty)} \).

Now we have \( \tilde{X}_\beta \) for all \( \beta \notin \mathbb{R}\alpha \), and these generators satisfy the Steinberg relations not involving \( \pm \alpha \). Since no Steinberg relation involves \( \alpha \) and \(-\alpha \) simultaneously, it suffices to construct \( \tilde{X}_{n-1,n} \) and prove the Steinberg relations for it. Note that there is an element \( \sigma \) of the Weyl group with the properties \( \sigma^2 = 1 \) and \( \eta_{n-1} = -\eta_n \), stabilizes \( \alpha \). At first we show most cases of (St3) and (St7). Obviously, \( \tilde{X}_{n-1,n}^i(a \otimes b) \) commutes with \( \tilde{X}_{jk}(c) \) and \( \tilde{X}_k(u) \) for \( |i| < n - 1 \) and \( \{ j, k \} \cap \{ n-1, n \} = \emptyset \) if in addition \( j, k \neq \pm i \). If \( j \notin \{ -i, i, n-1, n \} \), then it is easy to see that

\[
\tilde{X}_{i}(c) \tilde{X}_{n-1,n}(a \otimes b) = \tilde{X}_{n-1,n}(a \otimes b), \quad \tilde{X}_{i}(c) \tilde{X}_{n-1,n}(a \otimes b) = \tilde{X}_{n-1,n}(a \otimes b),
\]
\[
\tilde{X}_{i-1}(c) \tilde{X}_{n-1,n}(a \otimes b) = \tilde{X}_{n-1,n}(a \otimes b), \quad \tilde{X}_{i-1}(c) \tilde{X}_{n-1,n}(a \otimes b) = \tilde{X}_{n-1,n}(a \otimes b).
\]

Also, \( \tilde{X}_{n-1,n}(u \otimes v) \) and \( \tilde{X}_{n-1,n}(u \otimes v) \) commute with \( \tilde{X}_{ij}(c) \) for \( |i|, |j| < n - 1 \). Moreover, for \( |i| < n - 1 \) we have

\[
\tilde{X}_{n-1,i}(d) \tilde{X}_{n-1,n}(u \otimes v) = \tilde{X}_{n-1,n}(u \otimes v),
\]
\[
\tilde{X}_{n-1,i}(d) \tilde{X}_{n-1,n}(u \otimes v) = \tilde{X}_{n-1,n}(u \otimes v),
\]
\[
\tilde{X}_{n-1,i}(d) \tilde{X}_{n-1,n}(u \otimes v) = \tilde{X}_{n-1,n}(u \otimes v).
\]
Using these cases of (St3) and (St7), we may prove easy cases of (St5) as follows:

\[ \tilde{X}_{n,-1,n}(a \otimes b) = \tilde{X}_{n,-1,n}(a \otimes b) \tilde{X}_n(\phi(cab)), \]
\[ \tilde{X}_{n,1,n}(a \otimes b) = \tilde{X}_{1,-n}(\phi(-abc)) \tilde{X}_{n,-1,n}(a \otimes b), \]
\[ \tilde{X}_{n,-1,n}(u \otimes v) = X_{n,-1,n}(u \otimes v) X_{n}(\phi(-c\pi(u)\pi(v))), \]
\[ \tilde{X}_{n,1,n}(u \otimes v) = X_{1,-n}(\phi(\pi(u)\pi(v))) X_{n,-1,n}(u \otimes v). \]

We are ready to prove the remaining instances of (St3). If \( n \geq 4 \), then \( \tilde{X}_{n,1,1}(a \otimes b) \) commutes with \( \tilde{X}_{n,1,1}(a \otimes b) \) for \( c \in R_{i,\pm j}^{(\infty)} \), \( d \in R_{\pm j,\pm j}^{(\infty)} \), \( c' \in R_{-1,\pm j}^{(\infty)} \), and \( d' \in R_{-1,\pm j}^{(\infty)} \), where \( j \) is a non-zero index different from \( \pm i \), \( \pm (n-1) \), and \( \pm n \). If \( n = 3 \) and the orthogonal hyperbolic family is strong, then

\[ \tilde{X}_{1,1,1}(a \otimes b) \tilde{X}_{1,1,1}(a \otimes b) = \tilde{X}_{1,1,1}(a \otimes b) \tilde{X}_{1,1,1}(a \otimes b) \]
for \( c \in R_{1,1,1}^{(\infty)} \), \( d \in R_{1,1,1}^{(\infty)} \), \( c' \in R_{-1,1,1}^{(\infty)} \), and \( d' \in R_{-1,1,1}^{(\infty)} \). Hence in any case \( \tilde{X}_{n,1,1}(a \otimes b) \) commutes with \( \tilde{X}_{1,1,1}(a \otimes b) \) and \( \tilde{X}_{1,1,1}(a \otimes b) \) by lemma [lemma]..

Now we show most cases of (St4). If \( j \) is different from \( \pm i \), \( \pm n \), and \( \pm (n-1) \), then

\[ \tilde{X}_{n,1,i}(a \otimes b) = \tilde{X}_{n,1,i}(a \otimes b) \tilde{X}_{n,1,i}(a \otimes b), \]
\[ \tilde{X}_{n,1,n}(a \otimes b) = \tilde{X}_{n,1,n}(a \otimes b) \tilde{X}_{n,1,n}(a \otimes b). \]

Also,

\[ \tilde{X}_{n,1,i}(a \otimes b) = \tilde{X}_{n,1,i}(a \otimes b) \tilde{X}_{n,1,i}(a \otimes b), \]
\[ \tilde{X}_{n,1,n}(a \otimes b) = \tilde{X}_{n,1,n}(a \otimes b) \tilde{X}_{n,1,n}(a \otimes b). \]

For other generators and for \( |i| < n-1 \) we have

\[ \tilde{X}_{n,1,i}(a \otimes b) = \tilde{X}_{n,1,i}(a \otimes b) \tilde{X}_{n,1,i}(a \otimes b), \]
\[ \tilde{X}_{n,1,n}(a \otimes b) = \tilde{X}_{n,1,n}(a \otimes b) \tilde{X}_{n,1,n}(a \otimes b). \]

The remaining cases of (St5) may be proved in the following way. For \( |i| < n-1, x \in R_{n,i}^{(\infty)}, y \in R_{1,n-1,i}^{(\infty)}, z \in R_{n,i}^{(\infty)}, w \in R_{i,n-1}^{(\infty)} \) there are identities

\[ \tilde{X}_{n,1,n}(x \otimes b) \tilde{X}_{n,1,n}(y) = \tilde{X}_{n,1,n}(x \otimes b) \tilde{X}_{n,1,n}(y), \]
\[ \tilde{X}_{n,1,n}(u \otimes b) \tilde{X}_{n,1,n}(v) = \tilde{X}_{n,1,n}(u \otimes b) \tilde{X}_{n,1,n}(v). \]
Hence by lemma 6 we have

\[ [\tilde{X}_{n-1,n}^1(u \otimes b), \tilde{X}_{n-1,n}(x)] = \tilde{X}_{1-n}(\phi(\rho(u)b)), \quad [\tilde{X}_{n-1,n}(x), \tilde{X}_{n-1,n}^1(u \otimes b)] = \tilde{X}_n(\phi(\rho(u)b)), \]

It follows that all \( \tilde{X}_{n-1,n}^1(u \otimes b) \), \( \tilde{X}_{n-1,n}^n(u \otimes v) \), and \( \tilde{X}_{n-1,n}^1(u \otimes b) \) commute with each other. Also these morphisms are biadditive:

\[
\begin{align*}
\tilde{X}_{n-1,n}^1((a + a') \otimes b) &= \tilde{X}_{n-1,n}^1(a \otimes b) \tilde{X}_{n-1,n}^1(a' \otimes b), \\
\tilde{X}_{n-1,n}^1(a \otimes (b + b')) &= \tilde{X}_{n-1,n}^1(a \otimes b) \tilde{X}_{n-1,n}^1(a \otimes b'), \\
\tilde{X}_{n-1,n}^n((u + u') \otimes v) &= \tilde{X}_{n-1,n}^n(u \otimes v) \tilde{X}_{n-1,n}^n(u' \otimes v), \\
\tilde{X}_{n-1,n}^n((u \otimes (v + v')) &= \tilde{X}_{n-1,n}^n(u \otimes v) \tilde{X}_{n-1,n}^n(u \otimes v'), \\
\tilde{X}_{n-1,n}^1(u \otimes (b + b')) &= \tilde{X}_{n-1,n}^1(u \otimes b) \tilde{X}_{n-1,n}^1(u \otimes b'), \\
\tilde{X}_{n-1,n}^1((u + u') \otimes b) &= \tilde{X}_{n-1,n}^1(u' \otimes b) \tilde{X}_{n-1,n}^1(u \otimes b). 
\end{align*}
\]

Biadditivity implies that \( [\tilde{X}_{n,i}(a), \tilde{X}_{i,1-n}(b)] = [\tilde{X}_{n-1,i}(-a), \tilde{X}_{i,n}(-b)]^{-1} = \tilde{X}_{n-1,i}(-\tilde{a} \otimes \tilde{b}) \), hence we may apply the element \( \sigma \) from the Weyl group to every identity involving \( \tilde{X}_{n,i}^1(a \otimes b) \) and \( \tilde{X}_{n-1,n}^n(u \otimes v) \).

Now we show that the morphisms \( \tilde{X}_{n-1,n}^1(a \otimes b), \tilde{X}_{n-1,n}^n(u \otimes v) \) and \( \tilde{X}_{n-1,n}^1(u \otimes b) \) are balanced. If \(|i|, |j| < n - 1, i \neq \pm j, a \in R_{n-1,i}, b \in R_{ij}^\infty, c \in R_{jn}^\infty \), then it is easy to see that

\[
\tilde{X}_{n-1,n}^1(a \otimes bc) = \tilde{X}_{n-1,n}^j(ab \otimes c).
\]

For \(|i| < n - 1, u \in (\Delta^{(\infty)})_{1-n}^0, v \in (\Delta^{(\infty)})_n^0, b \in R_{1-n,i}^\infty, c \in R_{in}^\infty \) we have

\[
\tilde{X}_{n-1,n}^1(u \otimes bc) = \tilde{X}_{n-1,n}^i(\rho(u)b \otimes c).
\]

For \(|i| < n - 1, a \in R_{n-1,i}^\infty, u \in (\Delta^{(\infty)})_i^0, c \in R_{in}^\infty \) we also have

\[
\tilde{X}_{n-1,n}^i(a \rho(u) \otimes c) = \tilde{X}_{n-1,n}^{-i}(a \otimes \rho(u)c).
\]

If \(|i| < n - 1, u \in (\Delta^{(\infty)})_i^0, v \in (\Delta^{(\infty)})_i, b \in R_{1-n}^\infty \), then

\[
\tilde{X}_{n-1,n}^i(u \otimes v \cdot b) = \tilde{X}_{n-1,n}^i(\pi(u) \pi(v) \otimes b).
\]

Moreover, if \(|i| < n - 1, u \in (\Delta^{(\infty)})_i^0, a \in R_{n-1,i}^\infty, b \in R_{in}^\infty \), then \( \tilde{X}_{n-1,n}^i(u \otimes \phi(ab)) = 1 \) Hence by lemma 6 it follows that

\[
\tilde{X}_{n-1,n}^i(u \otimes \phi(a)) = 1.
\]

Using that \( \tilde{X}_{n-1,n}^i(u \otimes v) \) is balanced, it is easy to prove that for \(|i| < n - 1, u \in (\Delta^{(\infty)})_i^0, a \in R_{1-i,n}^\infty, b \in R_{1-n,n}^\infty \) there is an identity

\[
\tilde{X}_{n-1,n}^i(u \cdot a \otimes b) = \tilde{X}_{n-1,n}^i(a \rho(u) \otimes ab).
\]
Finally, for $|i| < n - 1$, $a \in R_{n-1,i}^{(\infty)}$, $b \in R_{i-1,n}^{(\infty)}$, $c \in R_{i-1,n}^{(\infty)}$, we have

$$\tilde{X}_{n-1,n}^{-i}(\phi(ab) \otimes c) = \tilde{X}_{n-1,n}^{-i}(a \otimes bc) \tilde{X}_{n-1,n}^{-i}(-b \otimes ac).$$

If $n \geq 4$, then $\tilde{X}_{n-1,n}^{-i}(abc \otimes d) = \tilde{X}_{n-1,n}^{-i}(a \otimes bc) \tilde{X}_{n-1,n}^{-i}(b \otimes cd)$ for $i, j \in \{-1,1\}$, $b \in R_{i-1,n}^{(\infty)}$, $c \in R_{i-1,n}^{(\infty)}$. Hence by lemmas 6 and 8 there is unique morphism $\tilde{X}_{n-1,n}^{-i}: R_{n-1,n}^{(\infty)} \to \text{StU}(R, \Delta; \Phi/\alpha)$ of pro-groups such that $\tilde{X}_{n-1,n}^{-i}(\alpha \otimes b) = \tilde{X}_{n-1,n}(\alpha \otimes b)$.

Now suppose that $n = 3$ and the orthogonal hyperbolic family is strong. Let $(a, b, c)_{ij} = \tilde{X}_{n-1,n}^{-i}(ab \otimes c) \tilde{X}_{n-1,n}^{-i}(a \otimes bc)$ for $i, j \in \{-1,1\}$. This morphisms satisfy all axioms from lemma 11 (where $\tilde{X}_{n-1,n}^{-i}(\alpha \otimes ab) = \tilde{X}_{n-1,n}^{-i}(\alpha \otimes xab)$) follows from $\tilde{X}_{n-1,n}^{-i}(\phi(\tilde{a} \tilde{x} a) \otimes b) = \tilde{X}_{n-1,n}^{-i}(\tilde{a} \otimes ab) \tilde{X}_{n-1,n}^{-i}(-\tilde{a} \otimes \tilde{b} \otimes \tilde{c})$). Hence $(a, b, c)_{ij} = 0$ and by lemmas 6 and 7 there is unique morphism $\tilde{X}_{n-1,n}: R_{n-1,n}^{(\infty)} \to \text{StU}(R, \Delta; \Phi/\alpha)$ of pro-groups such that $\tilde{X}_{n-1,n}^{-i}(\alpha \otimes b) = \tilde{X}_{n-1,n}(\alpha \otimes b)$.

In any case, lemmas 6 and 7 with balancing properties imply that $\tilde{X}_{n-1,n}^{-i}(\alpha \otimes b) = \tilde{X}_{n-1,n}(\alpha \otimes b)$, $\tilde{X}_{n-1,n}^{-i}(u \otimes v) = \tilde{X}_{n-1,n}(\pi(u) \pi(v))$, $\tilde{X}_{n-1,n}^{-i}(u \otimes b) = \tilde{X}_{n-1,n}(\rho(u)b)$. Hence $\tilde{X}_{n-1,n}(\alpha)$ satisfies (St0)–(St7). Also (St8) holds if $\alpha$ appears in the right ([$\tilde{X}_{n-1,n}^{-i}(\alpha \otimes b)$]) follows using the symmetry $\sigma$). The remaining case of (St8) follows from

$$\tilde{X}_{n-1,n}(\alpha \otimes b) = \tilde{X}_{n-1,n}(\alpha \otimes b) \tilde{X}_{n-1,n}(\alpha \otimes b).$$

By lemma 8 there is a morphism $G_{\alpha}: \text{StU}(R, \Delta; \Phi) \to \text{StU}(R, \Delta; \Phi/\alpha)$ of pro-groups such that $G_{\alpha}(X_{ij}(\alpha)) = \tilde{X}_{ij}(\alpha)$ and $G_{\alpha}(X_{j}(\alpha)) = \tilde{X}_{j}(\alpha)$, $G_{\alpha} \circ F_{\alpha} = \text{id}$. Hence by lemma 10 also $F_{\alpha} \circ G_{\alpha} = \text{id}$. \hfill \qed

## 10 Local – global principle

By lemma 11 the group $D(S^{-1}R, S^{-1}\Delta; \Phi)$ acts on $\text{StU}(R, \Delta; \Phi)^{(\infty)}$ and on $(R^{(\infty)}, \Delta^{(\infty)})$ making $\text{St}$ equivariant. By propositions 4 and 5, $D(S^{-1}R, S^{-1}\Delta; \Phi/\alpha)$ also acts on $\text{StU}(R, \Delta; \Phi)^{(\infty)}$ making $\text{St}$ equivariant if $n \geq 4$ or $n \geq 3$ and the orthogonal hyperbolic family is strong. Moreover, under this assumption every element from $D(S^{-1}R, S^{-1}\Delta; \Phi/\alpha) \ast D(S^{-1}R, S^{-1}\Delta; \Phi/\beta)$ with trivial image in $U(S^{-1}R, S^{-1}\Delta)$ acts trivially by lemma 10 (so $U(S^{-1}R, S^{-1}\Delta)$ acts on $\text{StU}(R, \Delta)^{(\infty)}$). In particular, the Weyl group acts on $\text{StU}(R, \Delta)$ if the orthogonal hyperbolic family is free (if we take $S = \{1\}$).

Recall that a sequence $a_1, \ldots, a_k$ in a unital ring $A$ is called left unimodular if there is a sequence $b_1, \ldots, b_k$ in $A$ such that $\sum_i b_i a_i = 1$. The ring $A$ satisfies $\text{sr}(A) \leq k - 1$ if for every left unimodular sequence $a_1, \ldots, a_k$ there are elements $c_1, \ldots, c_{k-1} \in A$ such that $a_1 + c_1 a_2, \ldots, a_{k-1} + c_{k-1} a_k$ is also unimodular. More generally, let $M_i$ be right $A$-modules for $1 \leq i \leq k$. A sequence $m_1, \ldots, m_k$ for $m_i \in M_i$ is called left unimodular if there are $f_i \in M_i$ such that $\sum_i f_i m_i = 1$.\hfill \hfill 30
Now let \((R, \Delta)\) be an odd form \(K\)-algebra with a free orthogonal hyperbolic family \(\eta_1, \ldots, \eta_k\). Let \(\Lambda = \{ \rho(u) \mid u \in \Delta_{1-1}^0, \pi(u) = 0 \}\), this is an even form parameter in the sense that \(\{a - \overline{a} \mid a \in R_{1-1}\} \leq \Lambda \leq \{a \in R_{1-1} \mid a + \overline{a} = 0\}\) and \(\Lambda a \leq \Lambda\) for any \(a \in R_{11}\). We say that \(\text{Asr}(\eta_1; R, \Delta) \leq k - 1\) if \(\text{sr}(R_{11}) \leq k - 1\) and for every unimodular sequence \(a_{-k}, \ldots, a_{-1}, b_1, \ldots, b_k\) with \(a_k \in R_{-1,1}\) and \(b_k \in R_{11}\) there is a matrix \(\{e_{ij} \in R_{1-1}\}_{i,j=1}^k\) such that \(e_{ii} = e_{-i,-i}\), \(e_{i,-i} \in \Lambda\), and the sequence \(a_1 + \sum_i c_{1i}b_i, \ldots, a_k + \sum_i c_{ki}b_i\) is left unimodular in \(R_{11}\). For example, if there are elements \(e_{1,-1} \in R_{1-1}\) and \(e_{-1,1} \in R_{11}\) such that \(e_{1,-1}e_{-1,1} = e_1\) and \(e_{-1,1}e_{1,-1} = e_{-1}\), then \(\text{Asr}(\eta_1; R, \Delta) \leq k\) is equivalent to the condition \(\text{Asr}(R_{11}, \Lambda e_{-1,1}) \leq k\) from [1] [2].

We say that \(\text{Asr}(\eta_1; R, \Delta) \leq k\) if for every maximal ideal \(m \leq K\) the inequality \(\text{Asr}(\eta_1; R, \Delta, \Lambda_m) \leq k\) holds. For example, it is easy to prove that \(\text{Asr}(R, \Delta) \leq 1\) if \(R\) is quasi-finite over \(K\) (i.e., it is a direct limit of finite \(K\)-algebras).

The next proposition shows surjectivity for \(KU_1(R, \Delta)\). If there are \(e_{ij} \in R_{ij}\) for all \(i, j \neq 0\) such that \(e_{ij}e_{jk} = e_{ik}\) and \(e_{ii} = i\), then this result already appears in [9].

**Proposition 6.** Suppose that the orthogonal hyperbolic family is free and \(\text{Asr}(\eta_1; R, \Delta) \leq n - 1\). Then \(U(R, \Delta)\) is generated by \(U(R, \Delta)\) and \(U(\eta_1)\).

**Proof.** Recall that \(U(\eta_1)\) normalizes \(EU(R, \Delta)\). Let \(g \in U(R, \Delta)\) be any element. First of all, suppose that \(e_n\alpha(g)e_n = e_n\) (we may consider \(e_i\alpha(g)e_i + e_i \in R_{ii}\)). In this case there is unique \(u \in \Delta_{[n]}^n\) such that \(T^n(\pm u) g \in P_{\eta_{-n}}\) by lemma [3]. For elements of the parabolic subgroup \(P_{\eta_{-n}}\) the claim is clear.

In the general case multiply \(g\) by elementary transvections from the left until \(e_n\beta(g)e_n \) becomes 0. Note that the sequence \(\{e_n\alpha(g)e_n\}_{i=-\infty}^0 \cup \{e_i\alpha(g)e_n\}_{i=\infty}^0\) is left unimodular (where \(e_0 = 1 - \sum_{i \neq 0} e_i \in R \times K\)). By \(\text{sr}(R_{11}) \leq n - 1\) and properties of the stable rank there are \(b_i \in R_{ii}\) for \(1 \leq i \leq n\) such that \(e_i\alpha(g)e_n + b_i\alpha(g)e_0\alpha(g)e_n\) is left unimodular. Hence there is \(h \in EU(R, \Delta)\) such that \(e_i\alpha(h)g\) is left unimodular. By \(\text{Asr}(R, \Delta) \leq n - 1\) there is \(h' \in EU(R, \Delta)\) such that \(e_n\alpha(h'h')g\) is invertible in \(R_{nn}\). Then clearly there is \(h'' \in EU(R, \Delta)\) such that \(e_n\alpha(h'h'h')g\) is invertible in \(R_{nn}\), but \(n \geq 2\) (if \(n = 1\), then the stable rank condition imply that \(R_{nn} = 0\) and there is nothing to prove).

By the main results of [17] [18], \(\text{StU}(R, \Delta) \cap U(\eta_1)\) is generated by \(T_i a\) and \(T_j(u)\) for \(|i|, |j| < n\) if \(\text{sr}(R_{11}) \leq n - 2\). The proof actually implies surjectivity for \(KU_2(R, \Delta)\).

**Proposition 7.** Suppose that the orthogonal hyperbolic family is free, \(n \geq 4\) or \(n \geq 3\) and the orthogonal hyperbolic family is strong, and \(\text{sr}(\eta_1; R, \Delta) \leq n - 2\). Then every element from \(KU_2(R, \Delta)\) may be generated by \(X_{ij}(a)\) and \(X_{ij}(u)\) for \(|i|, |j| < n\).

**Proof.** The proofs of lemmas 7 and 8 from [17] actually works in the Steinberg group. This means that \(\text{StU}(R, \Delta) = PLQ\), where \(P = \langle X_{ij}(\ast), X_{ij}(\ast) | -i, j \neq \ast \rangle\)
 decomposition $U(\cdot)$ is a Steinberg parabolic subgroup, $L = \langle X_{i,-n}(\cdot), X_{-n}(\cdot) \mid i \geq -1 \rangle$, and $Q = \langle X_{ij}(\cdot), X_{j}(\cdot) \mid 2 \leq j \leq n \rangle$. Now let $plq \in KU_2(R, \Delta)$ for some $p \in P$, $l \in L$, and $q \in Q$. It follows that $l = 1$, hence we may assume that $p \in \langle X_{ij}(\cdot), X_{j}(\cdot) \mid |i|, |j| < n \rangle$ (the factor from $\langle X_{in}(\cdot), X_{n}(\cdot) \rangle$ may be pushed into $q$). Let $w = X_{1n}(e_{1n})X_{n1}(-e_{1n})X_{1n}(e_{1n}) \in StU(R, \Delta)$ be an element swapping $\eta_i$ with $\eta_n$, $q = q'd$ for $q' \in \langle X_{ij}(\cdot), X_{j}(\cdot) \mid i \leq 1$ and $2 \leq j \rangle$ and $d \in \langle X_{ij}(\cdot) \mid 2 \leq i, j \leq n \rangle$. Obviously, $q' \in \langle X_{ij}(\cdot), X_{j}(\cdot) \mid -n < i \leq 1$ and $2 \leq j < n \rangle$. It remains to prove that $d \in \langle X_{ij}(\cdot) \mid 1 \leq i, j \leq n \rangle$. But $d$ commutes with $w$ since $d$ trivially acts on $X_{1n}(\cdot)$ and $X_{n1}(\cdot)$ (recall that $st(d) \in U_n(\cdot)(R, \Delta)$ and $^w d \in \langle X_{ij}(\cdot) \mid 1 \leq i, j \leq n \rangle$).

Now we deal with the semi-local case.

**Proposition 8.** Suppose that $R$ is semi-local and $n \geq 1$. Then there is Gauss decomposition

$$U(R, \Delta) = st(U^- (R, \Delta; \Phi/e_1)) st(U^+ (R, \Delta; \Phi/e_1)) st(U^- (R, \Delta; \Phi/e_1)) D(R, \Delta; \Phi/e_1).$$

Moreover, $U(R, \Delta)$ is isomorphic to the group $G$ generated by $T_{ij}(\cdot)$ and $D(R, \Delta; \Phi/\alpha)$ for all ultrashort $\alpha$ with the Steinberg relations and the relations of type $g = 1$

$$g \in T_{ij}(\cdot) * T_{-i,j} * T_{-i,j} * T_{-i,-j}(\cdot) * D(R, \Delta; \Phi/e_i) * D(R, \Delta; \Phi/e_j)$$

with trivial image in $U(R, \Delta)$.

**Proof.** We prove Gauss decomposition for $n = 2$, the general case then follows by elimination of $e_1$ and induction. Let $g \in U(R, \Delta)$ be any element. At first suppose that $e_2\alpha(g)e_2$ is invertible as an element of $R_{22}^\ast$. Then there is $u \in \Delta^{|2|}_2$ such that $T^2(-u)g \in P_{n-2}$ by lemma 3, so we are done.

In the general case we have to find $h \in T^{-2}(\cdot) \times U_{|g|}$ such that $e_2\alpha(hg)e_2 \in R_{22}^\ast$ (then there is $h \in T^{-2}(\cdot)$ with the same property). This may be done modulo the Jacobson radical of $R$, hence we may assume that $R$ is semi-simple and $(\Delta, \Delta)$ is unital special. Decomposing into a direct product, we may further assume that $R_{22}$ is a matrix algebra over a division ring. We denote the rank over this division ring by rk. Choose $h$ in such a way that $rk(e_2\alpha(g)e_2)$ is maximal possible. We need to prove that $e_2\alpha(g)e_2$ is invertible, i.e., $rk(e_2\alpha(g)e_2) = rk(\alpha(g)e_2)$. If $rk((e_{-1} + e_1 + e_2)\alpha(g)e_2) > rk(e_2\alpha(g)e_2)$, then there is $h \in T_{21}(\cdot) T_{2,-1}(\cdot)$ such that $rk(e_2\alpha(hg)e_2) > rk(e_2\alpha(g)e_2)$, a contradiction. If $rk((1-e_0)\alpha(g)e_2) > rk(e_2\alpha(g)e_2)$, then there is $h \in T^{-1,-2}(\cdot) T_{2,-1}(\cdot)$ such that $rk((e_1 + e_2)\alpha(hg)e_2) > rk(e_2\alpha(g)e_2)$, a contradiction. Finally, if $rk(\alpha(g)e_2) > rk(\alpha(g)e_2)$, then recall that the sequence $e_{2}\alpha(g)e_2, e_{-1}\alpha(g)e_2, e_0\alpha(g)e_2, e_1\alpha(g)e_2, e_2\alpha(g)e_2$ is left unimodular. Hence there is $h \in T^{-1}(\cdot) T^{-1}(\cdot)$ such that $rk((1-e_0)\alpha(hg)e_2) = rk(\alpha(g)e_2)$, a contradiction.

Now we prove the second claim. Note that there is a natural map $f: StU(R, \Delta) \to G$. We show that

$$G = f(U^- (R, \Delta; \Phi/e_1)) f(U^+ (R, \Delta; \Phi/e_1)) f(U^- (R, \Delta; \Phi/e_1)) D(R, \Delta; \Phi/e_1).$$

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Denote the right hand side by \( G' \). Clearly, \( G' \) is closed under multiplications on \( f(U^-(R, \Delta; \Phi/e_1)) \) and \( D(R, \Delta; \Phi/e_1) \) from the right. We show that \( G' \) is stable under the action of the Weyl group (i.e. it is preserved under permutations of \( \eta_i \) and sign changes), then obviously \( G = G' \). By definition, \( G' \) is stable under the reflection \( e_1 \mapsto -e_1 \). By Gauss decomposition for \( U(\eta_1 \oplus \ldots \oplus \eta_n) \), \( G' \) is stable under the transposition \( (e_1, e_2) \mapsto (e_2, e_1) \).

Recall that Gauss decomposition holds for isotropic linear groups (for example, see lemma 7 in [15]), and it actually does not need Morita equivalence of the idempotents. Using this decomposition for \( U(\eta_1 \oplus \ldots \oplus \eta_n) \), it is easy to see that \( G' \) is stable under all transpositions \( (e_i, e_{i+1}) \mapsto (e_{i+1}, e_i) \) for \( i > 1 \). The whole Weyl group is generated by transpositions of adjacent indices and the reflection \( e_1 \mapsto -e_1 \).

From lemma 7 it follows by induction that

\[
\text{st}(U^+(R, \Delta)) \cap D(R, \Delta) = \text{st}(U^-(R, \Delta)) \cap (\text{st}(U^+(R, \Delta)) \rtimes D(R, \Delta)) = 1.
\]

Now if \( g \in G \) has trivial image in \( U(R, \Delta) \), then the factor from \( f(U^+(R, \Delta; \Phi/e_1)) \) is trivial. Hence \( g \in f(U^-(R, \Delta; \Phi/e_1)) \rtimes D(R, \Delta; \Phi/e_1) \), and this group is isomorphic to its image in \( U(R, \Delta) \).

We are ready to construct the action of \( U(S^{-1}R, S^{-1}\Delta) \) on \( \text{StU}(R, \Delta)^{(\infty)} \) in the interesting cases.

**Theorem 1.** Let \( K \) be a commutative ring, \((R, \Delta)\) be an odd form \( K\)-algebra with an orthogonal hyperbolic family of rank \( n \), \( S \leq K^* \) be a multiplicative subset. Suppose that \( n \geq 4 \) or \( n \geq 3 \) and the orthogonal hyperbolic family is strong. Suppose also that \( S^{-1}R \) is semi-local or the orthogonal hyperbolic family is free and \( \text{Asr}(\eta_1; S^{-1}R, S^{-1}\Delta) \leq n - 2 \). Then \( U(S^{-1}R, S^{-1}\Delta) \) acts on \( \text{StU}(R, \Delta)^{(\infty)} \) and \( \text{st} \) is equivariant, this action is the usual one on every \( D(S^{-1}R, S^{-1}\Delta; \Phi/\alpha) \).

**Proof.** In the semi-local case this follows from proposition 8 and lemma 10. In the stable rank case note that \( U(S^{-1}R, S^{-1}\Delta) \) is generated by \( U[\eta_{n-1} \oplus \eta_n] \) and \( \text{StU}(S^{-1}R, S^{-1}\Delta) \) by proposition 6. The semi-direct product of these groups acts on \( \text{StU}(R, \Delta; \Phi)^{(\infty)} \) by lemma 10 and propositions 11 14. We have to show that every element \((g, h) \in \text{StU}(S^{-1}R, S^{-1}\Delta) \rtimes U[\eta_{n-1} \oplus \eta_n] \) with trivial image in \( U(R, \Delta) \) acts trivially. By injective stability of \( \text{KU}_1 \) and surjectivity of \( \text{KU}_2 \) we may assume that \( g \in \text{StU}(S^{-1}R[\eta_{n-1}, \eta_n], S^{-1}\Delta[\eta_n]) \). But then \((g, h) \) acts trivially on \( \text{StU}(R, \Delta; \Phi/\{e_1, \ldots, e_{n-1}\}) \), hence on \( \text{StU}(R, \Delta; \Phi) \) by lemma 10.

\[\square\]

11 Unitary Steinberg crossed module

Now we prove the main results. Recall that the Steinberg group is perfect for \( n \geq 3 \) by proposition 8.
**Theorem 2.** Let \((R, \Delta)\) be an odd form ring with an orthogonal hyperbolic family of rank \(n \geq 3\). Suppose that \(R\) is semi-local. Then there is unique action of \(U(R, \Delta)\) on \(StU(R, \Delta)\) making \(st\) equivariant, consistent with the action of \(StU(R, \Delta) \rtimes D(R, \Delta)\).

*Proof.* Clearly, \(StU(R, \Delta)\) and \(D(R, \Delta; \Phi/\alpha)\) for all ultrashort \(\alpha\) act on \(StU(R, \Delta)\) making \(st\) equivariant by proposition [1]. By lemma [10] and proposition [3] these actions glue together to the required action of \(U(S^{-1}R, S^{-1}\Delta)\). Now \(StU(R, \Delta) \rightarrow EU(R, \Delta)\) is a central perfect extension, so the uniqueness follows from abstract group theory.  

**Theorem 3.** Let \(K\) be a commutative ring, \((R, \Delta, D)\) be an augmented odd form \(K\)-algebra with an orthogonal hyperbolic family of rank \(n\). Suppose that \(n \geq 4\) or \(n \geq 3\) and the orthogonal hyperbolic family is strong. Suppose also that \(R\) is quasi-finite over \(K\) or \(\operatorname{lsr}(\eta_i, R, \Delta) \leq n - 1\) and \(\operatorname{lsr}(R_{11}) \leq n - 2\). Then there is unique action of \(U(R, \Delta)\) on \(StU(R, \Delta)\) making \(st\) equivariant, consistent with the action of \(StU(R, \Delta) \rtimes \Delta(R, \Delta)\).

*Proof.* Note that the quasi-finite case easily reduces to the case when \(R\) is finite \(K\)-algebra. First of all, we show that any \(g \in KU_2(R, \Delta)\) lies in the center of \(StU(R, \Delta)\). For any non-zero indices \(i \neq \pm j\) let

\[
\begin{align*}
\alpha_1 &= \{ k \in K | [g, X_{ij}(kR_{ij})] = 1 \}, \\
\alpha_2 &= \{ k \in K | [g, X_j(kD^0_j)] = 1 \}, \\
\alpha_3 &= \{ k \in K | [g, X_j(\Delta_j^0 \cdot k)] = 1 \}.
\end{align*}
\]

Clearly, \(\alpha_1\) and \(\alpha_2\) are ideals. The set \(\alpha_3\) is also an ideal if \(\alpha_2 = K\). By theorem [1] \(g\) acts trivially on \(StU(R, \Delta)^{(\infty)}\), where the multiplicative subset is \(K \setminus m\) for a maximal ideal \(m \leq K\). Hence \(\alpha_i\) are not contained in any maximal ideal. In other words, \(\alpha_i = K\) and \(KU_2(R, \Delta) \leq StU(R, \Delta)\) is central.

Next, we show that \(EU(R, \Delta)\) is normalized by any \(g \in U(R, \Delta)\). For any non-zero indices \(i \neq j\) let

\[
\begin{align*}
\beta_1 &= \{ k \in K | ^gT_{ij}(kR_{ij}) \leq EU(R, \Delta) \}, \\
\beta_2 &= \{ k \in K | ^gT_j(kD^0_j) \leq EU(R, \Delta) \}, \\
\beta_3 &= \{ k \in K | ^gT_j(\Delta_j^0 \cdot k) \leq EU(R, \Delta) \}.
\end{align*}
\]

Again, \(\beta_1\) and \(\beta_2\) are ideals. The set \(\beta_3\) is an ideal if \(\beta_2 = K\). Apply theorem [1] to the multiplicative subset \(K \setminus m\) for any maximal ideal \(m\). It follows that \(\beta_i\) are not contained in any maximal ideal, so \(\beta_i = K\) and \(EU(R, \Delta)\) is normalized by \(U(R, \Delta)\) (and by \(G\) in the classical case).

It remains to prove that for any \(g \in U(R, \Delta)\) (or \(g \in G\) in the classical case) there is an endomorphism \(StU(R, \Delta) \rightarrow StU(R, \Delta), X_{ij}(a) \rightarrow ^gX_{ij}(a), X_j(u) \rightarrow ^gX_j(u)\) making \(st\) equivariant. Indeed, since \(StU(R, \Delta) \rightarrow EU(R, \Delta)\) is a central perfect extension, such endomorphisms are unique for all \(g\) if they exist and are multiplicative on \(g\). Let \(Y_{ij}(a) = st^{-1}(^gT_{ij}(a))\) and \(Y_j(u) = st^{-1}(^gT_j(u))\), they
are certain cosets of $KU_d(R, \Delta)$. Note that all commutators of these cosets are one-element sets, so we may consider them as elements of the Steinberg group.

We are going to choose canonical elements $Y_{ij}(a) \in \mathcal{Y}_{ij}(a)$ and $Y_j(u) \in \mathcal{Y}_j(u)$ satisfying the Steinberg relations with the following property: for any maximal ideal $m \leq K$ there is $s \in K \setminus m$ such that the maps

$$R_{ij}^{(ss')} \xrightarrow{X_{ij}} \text{StU}(R, \Delta)^{(ss')} \xrightarrow{\text{Ad}_g^{(s)}} \text{StU}(R, \Delta)^{(s)} \to \text{StU}(R, \Delta),$$

$$(\Delta^{(s)})_j^{(0)} \xrightarrow{X_j} \text{StU}(R, \Delta)^{(ss')} \xrightarrow{\text{Ad}_g^{(s)}} \text{StU}(R, \Delta)^{(s)} \to \text{StU}(R, \Delta)$$

coincide with $Y_{ij}$ and $Y_j$, where $\text{Ad}_g^{(s)}$ is the action of $g$ from theorem 1 and $s'$ is sufficiently large.

We begin with (St3) and (St7). Fix non-zero indices $i \neq j' \neq -j \neq -i' \neq i$. For any non-zero indices $i \in R_{ij}$ and $u \in \Delta^0_j$ consider the ideals

$$\mathfrak{c}_1 = \{k \in K \mid [\mathcal{Y}_{ij}(a), \mathcal{Y}_{i'}(kR_{i'j'})] = 1\};$$

$$\mathfrak{c}_2 = \{k \in K \mid [\mathcal{Y}_j(u), \mathcal{Y}_{i'}(kR_{i'j'})] = 1\}.$$

By theorem 1 for each maximal ideal $m \leq K$ elements $gT_{ij}(a)$, $gT_j(u)$, and $g$ act in the same way on $X_{i'j'}(sR_{i'j'})$ for some $s \in K \setminus m$, so $s \in \mathfrak{c}_1 \cap \mathfrak{c}_2$ for some $s \in K \setminus m$. Hence $\mathfrak{c}_i = K$ and $[\mathcal{Y}_{ij}(a), \mathcal{Y}_{i'}(R_{i'j'})] = [\mathcal{Y}_j(u), \mathcal{Y}_{i'}(R_{i'j'})] = 1$.

Fix non-zero indices $i \neq \pm j$ and let $Y_{ij}^l(a \otimes b) = [\mathcal{Y}_{ij}(a), \mathcal{Y}_{i'}(b)]$ for all non-zero $l \notin \{-i, i, -j, j\}$. Note that the symbols $Y_{ij}^l(a \otimes b)$ are biadditive and commute with each other. Take two indices $l, l' \notin \{-i, i, -j, j\}$ and elements $a \in R_{il}, b \in R_{il'}$. Consider the ideal

$$\mathfrak{d}_1 = \{k \in K \mid Y_{ij}^l(a \otimes bc) = Y_{ij}^{l'}(ab \otimes c) \text{ for all } c \in kR_{i,j}\}.$$

By theorem 1, $\mathfrak{d}_1$ is not contained in any maximal ideal of $K$, so by lemma 8 there is a unique homomorphism $Y_{ij} : R_{ij} \to \text{StU}(R, \Delta)$ such that (St4) holds (then $Y_{ij}$ is also compatible with the action of $g$ on all pro-groups with respect to the maximal ideals). For any non-zero indices $i \neq \pm j$ and for all $u \in \Delta^0_j$ the set

$$\mathfrak{d}_2 = \{k \in K \mid [\mathcal{Y}_i(u), \mathcal{Y}_j(v)] = Y_{-i,j}(-u)p(v) \text{ for all } v \in \phi(kR_{-j,i}) + \Delta^0_j \cdot k\}$$

is also an ideal. By theorem 1, $\mathfrak{d}_2 = K$. Hence (St6) holds.

Now fix a non-zero index $i$ and let $Y_{-i,i}^l(a \otimes b) = [Y_{-i,i}(a), Y_{i}((b)]$ for all non-zero $l \neq \pm i$. Note that $Y_{-i,i}^l(a \otimes b)$ commute with each other and are biadditive. Take two indices $l, l' \notin \{-i, i\}$ and elements $a \in R_{-i,i}, b \in R_{il'}$. Consider the ideal

$$\mathfrak{e}_1 = \{k \in K \mid Y_{-i,i}^l(a \otimes bc) = Y_{-i,i}^{l'}(ab \otimes c) \text{ for all } c \in kR_{i,i}\}.$$

By theorem 1, $\mathfrak{e}_1$ is not contained in any maximal ideal of $K$, hence by lemma 8 there is a unique homomorphism $Y_{-i,i} : R_{-i,i} \to \text{StU}(R, \Delta)$ such that (St5) holds. Clearly, it also satisfies $Y_{-i,i}(a) = Y_{-i,i}(-a)$. 

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Similarly, fix a non-zero index $i$ and let $Y^i_l(u \boxtimes b) = Y_{-l,i}(\rho(u)b) [Y_l(-u), Y_{li}(-b)]$ for all non-zero $l \neq \pm i$. It is easy to see that $Y^i_l(u \boxtimes b)$ commutes with $Y_{-l,i}^j$. Moreover,

$$\left[ Y^i_l(u \boxtimes b), Y^i_l(u' \boxtimes b') \right] = Y_{-l,i}(-b' \pi(u)\pi(u')b)$$

$$Y^i_l((u + u') \boxtimes b) = Y^i_l(u \boxtimes b) Y^i_l(u' \boxtimes b),$$

$$Y^i_l(u \boxtimes (b + b')) = Y^i_l(u \boxtimes b) Y_{-l,i}(-b' \rho(u)b) Y^i_l(u \boxtimes b').$$

For any $u \in \Delta^0_l$, $b \in R_{li}$, $c \in R_{li'}$ consider the ideals

$$\mathfrak{e}_2 = \{ k \in K \mid Y^i_l(\phi(a) \boxtimes b) = Y_{-l,i}(\overline{\pi(a)b}) \text{ for all } a \in kR_{-l,i} \},$$

$$\mathfrak{e}_3 = \{ k \in K \mid Y^i_l(u \boxtimes cd) = Y^i_l(u \cdot c \boxtimes d) \text{ for all } d \in kR_{li'} \}.$$

By theorem 1, $\mathfrak{e}_2 = \mathfrak{e}_3 = K$. Hence by lemma 9 there is a unique homomorphism $Y_j: \Delta^0_l \to StU(R, \Delta)$ such that $Y_j(\phi(a)) = Y_{-l,j}(a)$ and (St8) holds (then $Y_j$ is also compatible with the action of $g$ on all pro-groups with respect to the maximal ideals).

Clearly, the maps $Y_{ij}$ and $Y_j$ satisfy all Steinberg relations. □

Now by theorem 1 from [9] the Steinberg group $StU(R, \Delta)$ is centrally closed for $n \geq 5$. Hence it is a universal central extension of the elementary group under assumptions of theorem 2 or 3 if in addition $n \geq 5$. By the main result from [10] it follows that if $G$ is a reductive classical group over $K$ and $P \leq G$ is a sufficiently isotropic parabolic subgroup, then the elementary subgroup $E_P(G)$ is normal in $G$. Hence the corresponding Steinberg group is a crossed module over $G$ and a universal central extension of $E_P(G)$ provided that $P$ is sufficiently isotropic.

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