Entropy entrainment and dissipation in superfluid Helium

N. Andersson\textsuperscript{1} and G. L. Comer\textsuperscript{2}
\textsuperscript{1} School of Mathematics, University of Southampton, UK
\textsuperscript{2} Department of Physics and Center for Fluids at All Scales, Saint Louis University, St. Louis, MO, USA

Abstract. We discuss finite temperature effects in superfluid Helium, providing a direct demonstration that a model based on treating the excitations in the system as a massless “entropy” fluid is equivalent to the traditional two-fluid approach. In particular, we demonstrate how the entropy entrainment is related to the “normal fluid density”. We also show how the superfluid constraint of irrotationality reduces the number of dissipation coefficients in the system. This analysis provides insight into the more general problem when vortices are present in the superfluid, and we discuss how the so-called mutual friction force can be accounted for. The end product is a formalism for finite temperature effects in a single condensate that can be applied to Helium in the low temperature regime. It can also be used to describe (to a certain extent) colour-flavour locked quark superconductors that may be present at the extreme densities in a neutron star core.
1. Introduction

Low temperature physics continues to be a vibrant area of research, providing a number of interesting and exciting challenges. Many of these are associated with the properties of superfluids/superconductors. Basically, matter appears to have two options when the temperature decreases towards absolute zero. According to classical physics one would expect the atoms in a liquid to slow down and come to rest, forming a crystalline structure. It is, however, possible that quantum effects become relevant before the liquid solidifies leading to the formation of a superfluid condensate (a quantum liquid). This will only happen if the interaction between the atoms is attractive and relatively weak. The archetypal superfluid system is Helium. It is well established that He$_4$ exhibits superfluidity below $T = 2.17$ K. Above this temperature liquid Helium is accurately described by the Navier-Stokes equations. Below the critical temperature the modelling of superfluid He$_4$ requires a “two-fluid” description, see e.g. [1, 2, 3]. Two fluid degrees of freedom are required to explain, in particular, the presence of a second sound associated with thermal waves in the system.

Phenomenologically, the basic behaviour of superfluid Helium is easy to understand if one first considers a system at absolute zero temperature. Then the dynamics is entirely due to the quantum condensate. There exists a single quantum wavefunction, and the momentum of the flow follows directly from the gradient of its phase. This immediately implies that the flow is irrotational. At finite temperatures, one must also account for thermal excitations. That is, not all atoms remain in the ground state. A second dynamical degree of freedom arises since the excitation gas may drift relative to the atoms. In the standard two-fluid model, one makes a distinction between a “normal” fluid component and a superfluid part. The associated densities are to a large extent statistical concepts, as one cannot separate the “normal fluid” from the “superfluid” [4]. It is important to keep this in mind. In fact, there exists a number of alternative approaches to the problem. It has, for example, been demonstrated that the two-fluid equations [1, 2, 3] can be obtained from a “single fluid” model, provided that a distinction is made between a background flow and the sound waves (phonons) in the system [5]. Another interesting strategy is to apply results from extended irreversible thermodynamics [6, 7] to the Helium problem. Extended thermodynamics was developed to deal with a number of unattractive features of the classical results, e.g. the infinite propagation speed of thermal signals associated with Fourier’s law. In order to arrive at a causal description, one introduces additional variables (motivated by the kinetic theory of gases) leading to a system with richer dynamics. One of these variables is the heat flux. In the limit of rapid thermal relaxation one retains Fourier’s law, while a slow relaxation leads to the presence of a second sound. It is easy to show that the latter limit can be used to describe superfluid systems [8, 9].

We now know that many low temperature systems exhibit superfluid properties. The different phases of He$_3$ have been well studied, both theoretically and experimentally [10], and there is considerable current interest in atomic Bose-Einstein condensates [11].
In fact, the relevance of superfluid dynamics reaches beyond systems that are accessible in the laboratory. It is generally expected that neutron stars, which are formed when massive stars run out of nuclear fuel and collapse following a supernova explosion, will contain a number of superfluid phases [12]. This expectation is natural given the extreme core density (reaching several times the nuclear saturation density) and low temperature (compared to the nuclear scale of the Fermi temperatures of the different constituents, about $10^{12}$ K) of these stars. The outer regions of a typical neutron star will contain a, more or less solid, crust where the nuclei form a crystalline lattice. As the density increases the so-called neutron drip is reached. Beyond this point the crust lattice will coexist with degenerate neutrons. If the temperature is sufficiently low, below around $10^9$ K, these neutrons will be superfluid and may flow through the lattice. When the density reaches nuclear saturation, the crust lattice will have given way to the fluid core of the star. In the outer parts of this core, the neutron fluid will co-exist with protons, electrons and perhaps muons. At low temperatures, both neutrons and protons are expected to form condensates. Hence one is forced to consider the dynamics of a neutron superfluid coexisting with a proton superconductor and a relativistic gas of electrons/muons. Modelling this system is a serious challenge. Yet, the nature of the outer core is relatively well understood. The physics of the deep core is much less certain [12]. One alternative is that the composition continues to change as the presence of more massive baryons (hyperons) becomes energetically favourable. Alternatively, the ground state of matter at high density corresponds to a plasma of deconfined quarks and gluons. The different phases of matter provide a number of different channels for Cooper pairing, leading to many potential “superfluid” components. In order to develop a moderately realistic model for a neutron star core we need to improve our understanding of tricky issues concerning hyperon superfluidity and quark colour superconductors [13]. Neutron star observations may provide the only way to constrain our models for this extreme sector of physics.

The rapid spin-up and subsequent relaxation associated with radio pulsar glitches [14] provides strong, albeit indirect, evidence for neutron star superfluidity. The standard model for these events is based on, in the first instance, the pinning of superfluid vortices (e.g. to the crust lattice) which allows a rotational lag to build up between the superfluid and the part of the star that spins down electromagnetically, and secondly the sudden unpinning which transfers angular momentum from one component to the other leading to the observed spin-change. Key to the modelling of these events is the vortex pinning and the mutual friction (see [15] for a recent discussion) between the two components in the star.

The modelling of neutron star oscillations has also received considerable attention. It is known that different classes of pulsation modes can be, more or less clearly, associated with different aspects of the neutron star model. As an example, a superfluid star has a set of oscillation modes that arise because of the existence of the second sound [16, 17, 18, 19, 20]. The hope is that one will be able to use future observations, e.g. via gravitational waves, to learn more about the interior composition of the star.
Particularly interesting in this respect is the possibility that various oscillation modes may be unstable. The most promising such instability is (according to current thinking) associated with the inertial r-modes, see [22, 23]. The r-mode instability is expected to be active provided that the gravitation radiation reaction, which provides the driving of the instability, is more efficient than the different damping mechanisms that suppress the growth of the mode. On the one hand, this is interesting because it makes the instability window sensitive to the detailed composition of the star. On the other hand, it makes relativistic modelling of the instability exceedingly difficult. Having said that, progress has been made on understanding the nature of the r-modes in a superfluid neutron star [24, 25, 26], in particular concerning the role of the vortex mediated mutual friction damping. We also know that the bulk viscosity associated with hyperons and deconfined quarks can affect the results significantly [27, 28]. In all cases the effects of superfluidity will be considerable.

So far, studies of the dynamics of superfluid neutron stars have almost exclusively considered the zero temperature problem. This is an obvious starting point since i) it simplifies the analysis and ii) mature neutron stars tend to be “cold”, with core temperature below $10^8$ K. However, this logic has an obvious flaw. The critical temperature at which the different phases of matter become superfluid is density dependent [29]. For instance, singlet state pairing of neutrons is expected to be present from just beyond the neutron drip to some point in the fluid core. For any given stellar temperature there should then exist transition regions where thermal effects play a dominant role. A detailed model ought to account for these regions. This involves understanding the role of the thermal excitations. The aim of the present paper is to take some steps towards such an understanding. We will demonstrate the close connection between the variational multifluid framework [31, 30] that we have previously used to model the outer neutron star core, see for example [26, 32, 33, 34], and the classic two-fluid model for He$_4$ at finite temperatures [1, 2, 3].

A similar comparison between the corresponding non-dissipative relativistic formulations has already been carried out by Carter and Khalatnikov [35]. They demonstrate how the convective variational multifluid formalism developed by Carter — see [36, 37] for detailed discussions — on which our multifluid formalism is based, can be translated into the model developed by Khalatnikov and Lebedev [38]. Our analysis provides additional insight into how thermal excitations should be accounted for, as well as an idea of the dissipation coefficients that are needed to complete a finite temperature model. Even though we do not aim to advance the understanding of Helium superfluidity, we believe that our discussion provides some useful insights. The most relevant contribution may be the analysis of the role of the superfluid irrotationality constraint. It should also be noted that our formalism is spiritually close to the extended thermodynamics approach (this point will be discussed in detail elsewhere [40]). This is an interesting reflection of the universality of conducting multifluid models.

Finally, it is worth noting that even though the single particle species model we consider here is not relevant for the conditions in the outer neutron star core it may
of direct astrophysical use. It could be relevant for a low (but finite) temperature quark core in the colour-flavour-locked phase (where a single condensate co-exists with a phonon gas [39]). Studies of the oscillations and instabilities of such a model would be very interesting.

2. Flux-conservative two-fluid model

We take as our starting point the flux-conservative multifluid framework developed in [30]. We consider the simplest conductive system corresponding to a single particle species exhibiting superfluidity. In the canonical framework, such systems have two degrees of freedom — the atoms are distinguished from the massless “entropy”. In the following, the former will be identified by a constituent index $n$, while the latter is represented by $s$. This description is different (in spirit) from the standard two-fluid model for Helium, and it is relevant to investigate how the two models are related. In particular, we want to understand better the various dissipative terms that arise when the system is out of equilibrium. That is, we want to be able to compare our dissipative formalism to the results in the standard literature, e.g. [1, 3]. Our hope is that this will improve our understanding of the role of the thermal excitations. This would be an important step toward more realistic modelling of the various condensates that are expected to be present in a neutron star core.

Our flux-conservative model [30] combines the usual conservation laws for mass, energy and momentum with the results from a variational analysis [31]. The latter is based on using the particle fluxes $n_i^x$ as the main variables and deducing the associated chemical potentials $\mu_x$ and the conjugate momenta $p^x_i$. Because of the so-called entrainment effect, each momentum does not have to be parallel to the associated flux. In the case of a two-component system, with a single species of particle flowing with $n_i^n = n v_i^n$ and a massless entropy with flux $n_i^s = s v_i^s$, the momentum densities are

$$\pi_i^n = np_i^n = mnv_i^n - 2\alpha w_i^{ns},$$

and

$$\pi_i^s = 2\alpha w_i^{ns},$$

where $w_i^{ns} = v_n - v_i^n$ and $\alpha$ represents what is known as the entrainment.

These relations highlight one of the main questions considered in this work. We need to understand the role and physical nature of the entrainment between particles and thermal excitations represented by the entropy fluid. This is very different from the entrainment usually considered in neutron star models. In the most commonly studied context, the entrainment between neutrons and protons arises because of the strong nuclear interaction. Each neutron (say) is endowed with a virtual cloud of protons, leading to an effective mass different from the bare neutron mass. In the dynamical description, this effect is represented by the entrainment.
As discussed in [30], the associated momentum equations can be written
\[ f^n_i = \partial_t \pi^n_i + \nabla_j (v^n_i \pi^n_j + D^{nj}_i) + n \nabla_i \left( \mu_n - \frac{1}{2} m_n v^n_i \right) + \pi^n_j \nabla_i v^n_j, \]
(3)
and
\[ f^s_i = \partial_t \pi^s_i + \nabla_j (v^s_i \pi^s_j + D^{sj}_i) + s \nabla_i T + \pi^s_j \nabla_i v^s_j, \]
(4)
where we have used the fact that the temperature follows from \( \mu_s = T \). In these expressions, \( D_{ij} \) represent the viscous stresses while the “forces” \( f^n_i \) allow for momentum transfer between the two components. In the following we will assume that the system is isolated, which means that \( f^n_i + f^s_i = 0 \).

We want to deduce the general form for the dissipative terms in the equations. To do this we follow the procedure discussed in [30], i.e. we combine the standard conservation laws with the Onsager symmetry principle. In the present context, when there is no particle creation, mass conservation leads to
\[ \partial_t n + \nabla_j (n v^n_j) = \Gamma_n = 0. \]
(5)
At the same time entropy can increase, so we have
\[ \partial_t s + \nabla_j (s v^s_j) = \Gamma_s. \]
(6)

From general principles one can show that the energy loss or gain due to external influences follows from (cf. Eq. (33) in [30])
\[ \varepsilon^\text{ext} = \sum_x \left[ v^n_i f^n_i + D^{nj}_i \nabla_j v^n_i + \left( \mu^n - \frac{1}{2} m^n v^n_i \right) \Gamma_x \right]. \]
(7)
In the case of an isolated system \( \varepsilon^\text{ext} = 0 \) so the above relation can be recast as
\[ TT_s = -f^n_i w^n_i - D^{nj}_i \nabla_j v^n_i - D^{nj}_i \nabla_j w^n_i, \]
(8)
where
\[ D_{ij} = D^n_{ij} + D^s_{ij}. \]
(9)

The above results are taken, more or less directly, from [30]. At this point we recognize a conceptual mistake in our previous analysis. When identifying the thermodynamical forces and the associated fluxes that are needed to complete the dissipative model from (8), we omitted a number of terms related to \( \nabla_j v^n_i \). As a result,

\‡ Throughout this paper we use a coordinate basis to represent tensorial relations. This means that we distinguish between co- and contra-variant objects, \( v_i \) and \( v^i \), respectively. Indices, which range from 1 to 3, can be raised and lowered with the (flat space) metric \( g_{ij} \), i.e., \( v^i = g_{ij} v^j \). Derivatives are expressed in terms of the covariant derivative \( \nabla_i \), which is consistent with the metric in the sense that \( \nabla_i g_{kl} = 0 \). This formulation of what is, essentially, a fluid dynamics problem may seem somewhat unfamiliar to some readers, but it has great advantage when we want to discuss the geometric nature of the different dissipation coefficients. We will then also use the volume form \( \epsilon_{ijk} \) which is completely antisymmetric, and has only one independent component (equal to \( \sqrt{g} \) in the present context).
the models discussed in [30] are not as general as they could have been. In fact, if we were to compare our original formulation to the standard dissipative model for superfluid Helium [1, 3] several bulk viscosity terms would be missing.

Let us rework, and correct, the analysis from [30] in the particular case of two fluids. From (8) we identify the three thermodynamic forces $w^{ns}_i$, $\nabla_j v^i_s$ and $\nabla_j w^{ns}_i$. The associated fluxes are $-f^n_i$, $-D^n_{ij}$ and $-D^{nj}_i$. Following the strategy set out in [30], the fluxes will be formed from linear combinations of the fluxes in such a way that (the notation here may seem somewhat elaborate, but it is chosen in order to make the inclusion of additional fluids in the framework straightforward)

$$-f^n_i = L^{nn}_{ij} w^{k}_j + \tilde{L}^{nn}_{ijk} \nabla^k w^{l}_s + \tilde{L}^{nn}_{ijkl} \nabla^k v^l_s,$$

$$-D^n_{ij} = \tilde{L}^{nn}_{ijk} w^{k}_j + L^{nn}_{ijkl} \nabla^k w^{l}_s + \tilde{L}^{nn}_{ijkl} \nabla^k v^l_s,$$

and

$$-D^n_{ij} = \tilde{L}^{nn}_{ijk} w^{k}_j + \tilde{L}^{nn}_{ijkl} \nabla^k w^{l}_s + L^{nn}_{ijkl} \nabla^k v^l_s.$$

In these expressions we have made use of the Onsager symmetry principle. Limiting the model to the inclusion of quadratic terms in the forces in (8), we find that

$$L^{nn}_{ij} = 2R^{nn} g_{ij},$$

$$\tilde{L}^{nn}_{ijk} = S^n \epsilon_{ijk},$$

$$\tilde{L}^{nn}_{ijkl} = S^{nn} \epsilon_{ijkl},$$

$$L^{nn}_{ijkl} = \zeta^n g_{ij} g_{kl} + \eta^n \left( g_{ik} g_{jl} + g_{il} g_{jk} - \frac{2}{3} g_{ij} g_{kl} \right) + \frac{1}{2} \sigma^n \epsilon_{ijm} \epsilon^m_{kl},$$

$$L^{nn}_{ijkl} = \zeta^n g_{ij} g_{kl} + \eta^n \left( g_{ik} g_{jl} + g_{il} g_{jk} - \frac{2}{3} g_{ij} g_{kl} \right) + \frac{1}{2} \sigma^n \epsilon_{ijm} \epsilon^m_{kl},$$

and

$$L^{nn}_{ijkl} = \zeta g_{ij} g_{kl} + \eta \left( g_{ik} g_{jl} + g_{il} g_{jk} - \frac{2}{3} g_{ij} g_{kl} \right) + \frac{1}{2} \sigma \epsilon_{ijm} \epsilon^m_{kl}.$$

We can reduce the number of unspecified dissipation coefficients by noting that the conservation of total angular momentum requires $D_{ij}$ to be symmetric, cf. eq (22) in [30]. This means that we must have

$$S^n = \sigma^n = \sigma = 0.$$

We are left with a system that has 9 dissipation coefficients.

To conclude the general analysis, let us write down the final expressions for the dissipative fluxes. To do this we use the decomposition

$$\nabla_i v^i_j = \Theta^s_{ij} + \frac{1}{3} g_{ij} \Theta_s + \epsilon_{ijk} W^k_s,$$

where we have introduced the expansion

$$\Theta_s = \nabla_j v^j_s,$$
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the trace-free shear

$$\Theta_{ij}^s = \frac{1}{2} \left( \nabla_i v_j^s + \nabla_j v_i^s - \frac{2}{3} g_{ij} \Theta_s \right),$$ (22)

and the “vorticity”

$$W_s^i = \frac{1}{4} \epsilon^{ijk} (\nabla_j v_k^s - \nabla_k v_j^s).$$ (23)

We will use analogous expressions for gradients of the relative velocities. The definition of the various quantities should be obvious.

We arrive at

$$- f_n^i = 2 R_m^{mn} w_{ns}^i + 2 S_m^{mn} W_{ns}^i,$$ (24)

$$- D_{ij}^n = S_{klm}^{mn} \epsilon_{ij}^{km} v_{ns}^k + g_{ij} (\zeta^{mn} \Theta_{ns} + \zeta^n \Theta_s) + 2 \eta^{mn} \Theta_{ij}^s + 2 \eta^n \Theta_{ij}^s + \sigma^{mn} \epsilon_{ijk} W_{ns}^k,$$ (25)

and

$$- D_{ij} = g_{ij} (\zeta^n \Theta_{ns} + \zeta \Theta_s) + 2 \eta^n \Theta_{ij}^s.$$ (26)

3. The superfluid constraint

Let us now assume that we are considering a superfluid system. For low temperatures and velocities the fluid described by (3) should then be irrotational. To impose this constraint we need to appreciate that it is the momentum that is quantised in a rotating superfluid, not the particle velocity [31]. This means that we require

$$\epsilon^{klm} \nabla_i p_m^i = 0.$$ (27)

To see how this affects the equations of motion, we rewrite (3) as

$$n \partial_t p_n^i + n \nabla_i \left[ \mu_n - \frac{m}{2} v_n^2 + v_n^i p_n^i \right] - n \epsilon_{ijk} v_n^j (\epsilon^{klm} \nabla_i p_m^i) = f_n^i - \nabla_j D_{ij}^n.$$ (28)

That is, we have

$$\partial_t p_n^i + \nabla_i \left[ \mu_n - \frac{m}{2} v_n^2 + v_n^i p_n^i \right] = \frac{1}{n} \left[ f_n^i - \nabla_j D_{ij}^n \right].$$ (29)

If we take the curl of this equation we see that the dissipative fluxes must satisfy

$$\epsilon^{ijk} \nabla_j \left[ \frac{1}{n} \left( f_k^n - \nabla_l D_{lk}^n \right) \right] = 0.$$ (30)

In other words, we should have

$$\nabla_i \Phi = \frac{1}{n} \left( f_k^n - \nabla_l D_{lk}^n \right)$$ (31)

for some scalar $\Phi$. This constraint ensures that the superfluid remains irrotational, i.e., there is no generation of turbulence or vorticity.

In order to satisfy this constraint, it is useful to express (8) in terms of the variables $j_{ns}^i = n w_{ns}^i$ and $v_s^i$ rather than the variables used in the previous section. This means that we have

$$T \Gamma_s = - F_i^{kn} j_{ns}^i - D_{ij}^n \nabla^l j_{ns}^l - D_{ij} \nabla^i v_s^j$$ (32)
where we have defined
\[ \mathcal{F}_n^i = \frac{1}{n} \left[ f_n^i - \left( \frac{\nabla^j n}{n} \right) D_{ji}^n \right] , \tag{33} \]
and
\[ \mathcal{D}_{ij}^n = \frac{1}{n} D_{ij}^n . \tag{34} \]

It follows that (31) becomes
\[ \nabla^i \Phi = \mathcal{F}_n^i - \nabla^j \mathcal{D}_{ji}^n . \tag{35} \]

Repeating the analysis from the previous section in terms of the new variables, we see that the thermodynamic fluxes will now be formed from
\[ - \mathcal{F}_n^i = \mathcal{L}_{ij}^n j_{ns} + \tilde{\mathcal{L}}_{ijk}^n j_{ns} + \tilde{\mathcal{L}}_{ijkl}^n j_{ns} + \tilde{\mathcal{L}}_{ijkl}^n j_{ns} + \tilde{\mathcal{L}}_{ijkl}^n j_{ns} , \tag{36} \]
\[ - \mathcal{D}_{ij}^n = \tilde{\mathcal{L}}_{ij}^n j_{ns} + \tilde{\mathcal{L}}_{ij}^n j_{ns} + \tilde{\mathcal{L}}_{ij}^n j_{ns} + \tilde{\mathcal{L}}_{ij}^n j_{ns} , \tag{37} \]
and
\[ - D_{ij} = \tilde{\mathcal{L}}_{ij}^n j_{ns} + \tilde{\mathcal{L}}_{ij}^n j_{ns} + \tilde{\mathcal{L}}_{ij}^n j_{ns} + \tilde{\mathcal{L}}_{ij}^n j_{ns} . \tag{38} \]

Recall that the conservation of total angular momentum requires \( D_{ij} \) to be symmetric.

Let us now consider the constraint (35). We need
\[ \nabla^i \Phi = \nabla^j \left( \mathcal{L}_{ijkl}^n j_{ns} + \tilde{\mathcal{L}}_{ijkl}^n j_{ns} \right) - \left( \mathcal{L}_{ij}^n - \nabla^k \tilde{\mathcal{L}}_{ikj}^n j_{ns} - \tilde{\mathcal{L}}_{ij}^n j_{ns} \right) . \tag{39} \]

That is we must have
\[ \tilde{\mathcal{L}}_{ij}^n = 0 , \tag{40} \]
and
\[ \mathcal{L}_{ij}^n = \nabla^k \tilde{\mathcal{L}}_{ikj}^n , \tag{41} \]
which leaves us with
\[ \nabla^i \Phi = \nabla^j \left( \mathcal{L}_{ijkl}^n j_{ns} + \tilde{\mathcal{L}}_{ijkl}^n j_{ns} \right) . \tag{42} \]

In other words, we must have
\[ \mathcal{L}_{ijkl}^n = \hat{\mathcal{L}}_{ijkl} g_{ijkl} , \tag{43} \]
and
\[ \mathcal{L}_{ijkl}^n = \hat{\mathcal{L}}_{ijkl} g_{ijkl} , \tag{44} \]
which means that
\[ \Phi = \hat{\mathcal{L}}_{ijkl}^n j_{ns} + \hat{\mathcal{L}}_{ijkl}^n j_{ns} . \tag{45} \]

Finally, it is straightforward (given the results in the previous section) to show that
\[ - D_{ij} = g_{ij} (\hat{\mathcal{L}}_{ijkl}^n j_{ns} + \hat{\mathcal{L}}_{ijkl}^n j_{ns} + 2\eta \Theta s) . \tag{46} \]

That is, only four dissipation coefficients remain once we have imposed the superfluid constraint.
We want to compare this result to the standard two-fluid model for Helium, e.g. the results discussed in chapter 9 of [1]. In order to do this, we need to translate our variables into those that are usually considered. In addition to providing a useful “sanity check” on our analysis, this will also give us a direct translation between the various coefficients. This should be useful for future modelling of superfluid neutron stars. After all, the Helium dissipation coefficients have been studied in detail both experimentally and theoretically (mainly through kinetic theory models).

4. Translation to the orthodox framework

The relationship between our framework and the traditional non-dissipative two-fluid model for Helium has already been discussed by Prix [31]. To extend the discussion to the dissipative problem is, as we will now demonstrate, straightforward.

4.1. Non-dissipative case

It is natural to begin by identifying the drift velocity of the quasiparticle excitations in the two models. After all, this is the variable that leads to the “two-fluid” dynamics. Moreover, since it distinguishes the flow that is affected by friction it has a natural physical interpretation. In the standard two-fluid model this velocity, $v_i^N$, is associated with the “normal fluid” component. In our framework, the excitations are directly associated with the entropy of the system, which flows with $v_i^s$. These two quantities should be the same, and hence we have

$$v_i^N = v_i^s.$$  
(47)

The second fluid component, the “superfluid”, is usually associated with a “velocity” $v_i^S$. This quantity is directly linked to the gradient of the phase of the superfluid condensate wave function. This means that it is, in fact, a momentum. As discussed by Prix [31] we should identify

$$v_i^S = \frac{\pi^i_n}{\rho} = \frac{p^i_n}{m}.$$  
(48)

These identifications lead to

$$\rho v_i^S = \rho \left[ (1 - \varepsilon) v_i^n + \varepsilon v_i^N \right],$$  
(49)

where $\varepsilon = 2\alpha/\rho$. We see that the total mass current is

$$\rho v_i^S = \frac{\rho}{1 - \varepsilon} v_i^S - \frac{\varepsilon \rho}{1 - \varepsilon} v_i^N.$$  
(50)

If we introduce the superfluid and normal fluid densities,

$$\rho_S = \frac{\rho}{1 - \varepsilon}, \quad \text{and} \quad \rho_N = -\frac{\varepsilon \rho}{1 - \varepsilon},$$  
(51)
we have the usual result;
\[ \rho v^i_n = \rho_S v^i_S + \rho_N v^i_N. \] (52)
Obviously, it is the case that \( \rho = \rho_S + \rho_N \). This completes the translation between
the two formalisms. Comparing the two descriptions, it is clear that the variational
approach has identified the natural physical variables; the average drift velocity of the
excitations and the total momentum flux. Since the system can be “weighed” the total
density \( \rho \) also has a clear interpretation. In constrast, the standard model uses quantities
that only have a statistical meaning [4]. The density \( \rho_N \) is inferred from the mean drift
momentum of the excitations. That is, there is no “group” of excitations that can be
identified with this density. Since the superfluid density \( \rho_S \) is inferred from \( \rho_S = \rho - \rho_N \),
it is a statistical concept as well. At the end of the day, this is not a problem. The
various quantities can be calculated from microscopic theory and the results compare
well to experiments. The main point we want to make is that the variables in the
variational formulation have a more immediate physical interpretation.

The above results show that the entropy entrainment coefficient follows from the
“normal fluid” density according to
\[ \alpha = -\frac{\rho_N}{2} \left( 1 - \frac{\rho_N}{\rho} \right)^{-1}. \] (53)
This shows that the entrainment coefficient diverges as the temperature increases
towards the superfluid transition and \( \rho_N \to \rho \). At first, this may seem an unpleasant
feature of the model. However, it is simply a manifestation of the fact that the two
fluids must lock together as one passes through the phase transition. The model remains
non-singular as long as \( v^i_n \) approaches \( v^i_s \) sufficiently fast as the critical temperature is
approached.

Having related the main variables, let us consider the form of the equations of
motion. We start with the inviscid problem. It is common to work with the total
momentum. Thus we combine (3) and (4) to get
\[ 0 = f^n_i + f^s_i = \partial_t (\pi^n_i + \pi^s_i) + \nabla_i \left( v^n_l \pi^n_i + v^s_l \pi^s_i \right) + n \nabla_i \mu_n + s \nabla_i T \]
\[ - n \nabla_i \left( \frac{1}{2} m v_n^2 \right) + \pi^n_l v^n_i + \pi^s_l v^s_l. \] (54)
Here we have
\[ \pi^n_i + \pi^s_i = \rho v^n_i \equiv j_i \] (55)
which defines the total momentum density. From the continuity equations we see that
\[ \partial_t \rho + \nabla_i j^i = 0. \] (56)
The pressure \( \Psi \) follows from [30]
\[ \nabla_i \Psi = n \nabla_i \mu_n + s \nabla_i T - \alpha \nabla_i w_{ns}^2. \] (57)
We also need
\[ v^n_l \pi^n_i + v^s_l \pi^s_i = v^S_l j^l + v^N_l j^0 \] (58)
where we have defined
\[ j^0_i = \rho_N(v_N^i - v_S^i) = \pi^s_i \] (59)
and
\[ \pi^s_i \nabla_i v_n^l + \pi^s_i \nabla_i v_s^l = n \nabla_i \left( \frac{1}{2} m v_n^2 \right) - 2 \alpha w_{ns}^l \nabla_i w_{ns}^l . \] (60)

Putting all the pieces together we have
\[ \partial_t j_i + \nabla_i \left( v_S^j j_l^l + v_L^l j^0_i \right) + \nabla_i \Psi = 0 . \] (61)

The second equation of motion follows directly from (29);
\[ \partial_t v^S_i + \nabla_i \left( \tilde{\mu}_S + \frac{1}{2} v_S^2 \right) = 0 \] (62)
where we have defined
\[ \tilde{\mu}_S = \frac{1}{m} \mu_n - \frac{1}{2} (v_n^i - v_S^i)^2 . \] (63)

The above relations show that our inviscid equations of motion are identical to the standard ones, cf. [1, 3]. The identified relations between the different variables also provide a direct way to translate the quantities in the two descriptions. In particular, we have demonstrated how the “normal fluid density” corresponds to the entropy entrainment in our model.

4.2. The dissipative case

Let us now move on to the dissipative problem. From (46) we immediately see that in the dissipative case we need to augment (61) by the divergence of
\[ D_{ij} = -g_{ij} \left[ \zeta \Theta_s + \hat{\zeta}^n \nabla_i \left( n w_{ns}^l \right) \right] - 2 \eta \Theta^s_{ij} . \] (64)

This result should be compared to the dissipative equations in, for example, [1]. In that description, the dissipation in the total momentum flux follows from the divergence of
\[ \tau_{ij} = -g_{ij} \left[ \zeta_1 \nabla_l \left( j^l - \rho v_N^l \right) + \zeta_2 \nabla_l v_N^l \right] - 2 \eta \Theta^s_{ij} . \] (65)
That is,
\[ \tau_{ij} = -g_{ij} \left[ \zeta_1 \nabla_l \left( \rho w_{ns}^l \right) + \zeta_2 \Theta_s \right] - 2 \eta \Theta^s_{ij} . \] (66)
First of all we see that the two shear viscosity coefficients are the same. Secondly, we identify
\[ \zeta = \zeta_2 , \quad \hat{\zeta}^n = m \zeta_1 . \] (67)

Moving on to the second momentum equation we need the gradient of, cf. (45),
\[ \frac{1}{m} \Phi = \frac{1}{m} \left[ \hat{\zeta}^n \nabla_l \left( n w_{ns}^l \right) + \hat{\zeta}^n \Theta_s \right] . \] (68)
From [I] we see that we should compare this to $h$ where
\[ h = -\zeta_3 \nabla l \left( j^l - \rho v_N^l \right) - \zeta_4 \nabla l v_N^l \] (69)
or
\[ h = -\zeta_3 \nabla l \left( \rho w_{ns}^l \right) - \zeta_4 \Theta_s . \] (70)

Once we identify
\[ \hat{\zeta}_{nn} = m^2 \zeta_3 , \quad \hat{\zeta}^n = m \zeta_4 \] (71)
we see that the two formulations agree perfectly. Moreover, it is obvious that $\zeta_1 = \zeta_4$ as required by the Onsager symmetry.

In order to complete the comparison of the two models, we need to comment on the (perhaps surprising) absence of dissipative heat flux terms in our model. At first sight, this would seem to be at odds with the traditional expressions [I] which contain Fourier’s law for the heat conductivity, $q_i = \kappa \nabla_i T$. For consistency, our model requires $\kappa = 0$, i.e. the thermal conductivity must vanish. Is this an unattractive feature of our model? In fact, it is not. First of all, it should be noted that the heat flux is intimately related to the entropy flow. In a two-component model one does not have the freedom to introduce an “independent” heat flux in addition to the massless entropy flux $n_{s}^l$, without at the same time introducing a new dynamical degree of freedom. That this makes sense physically is clear from the fact that the thermal conductivity in Helium arises from the interaction between phonons and rotons [I], which can drift at different rates. Our model is therefore a valid representation of the cold regime where the condensate coexists with thermal phonons. It is well-known that the thermal conductivity $\kappa$ vanishes in that case.

It is also relevant to comment on the well-known problems associated with Fourier’s law, i.e. the fact that it leads to a non-causal behaviour of thermal signals. This issue was one of the main motivations for the development of extended irreversible thermodynamics [6, 7]. A truly sound model for superfluid Helium ought to reflect these developments. Even though such a model is yet to be formulated, it is clear that our approach will allow us to make progress in this direction. This follows naturally from the discussion in [10] where we demonstrate that the relaxation time associated with the entropy flux in heat conductivity problems is intimately related to the entrainment.

We have now achieved the main objective of this work. We have demonstrated that our dissipative two-fluid formulation, with one of the fluids being associated with the massless entropy flow, reproduces the orthodox model for superfluid Helium. This comparison is valuable since it enables us to use available results for the various dissipation coefficients. It also demonstrates that any suggestions that it is “difficult” to relate the variational formulation to microphysical calculations [41] are based on a lack of understanding of the problem.
5. Vortices and mutual friction

The analysis in the previous two sections provides useful insights into the dynamics of a single component superfluid at finite temperatures. From a conceptual point of view, it is important to understand how the superfluid irrotationally constraint simplifies the dynamics of the two fluid system. In particular, we have shown that the number of dissipation coefficients is reduced from nine to four. However, the final model may be of limited practical use.

The superfluid constraint is, in general, too severe. In reality a superfluid can rotate by forming an array of vortices. To describe such a system, we must revert to the dissipative fluxes (24)-(26). However, this more general description still fails to account for all the dissipative channels in the problem. In particular, it does not easily accommodate the vortex mediated mutual friction force. In the simplest description [15, 42] we expect a force

$$f_{i}^{mf} = B' n_{v} \epsilon_{ijk} \kappa_{j} u_{ns} + B n_{v} \epsilon_{ijk} \epsilon^{klm} \hat{\kappa}_{j} \kappa_{i} u_{ns}^m,$$

(72)

to act on the particles (with a balancing force affecting the excitations). Here $n_{v}$ is the vortex area density and $\kappa^{i}$ is associated with the rotation of the system (the hat represents a unit vector) [13]. This force follows after averaging over the, locally straight, vortex array.

In [30] we discussed how this force could be accounted for in our dissipative model. This analysis was not entirely successful. The main reason for this is that the variational description assumes that the system is isotropic. This is obviously no longer the case when one introduces an array of vortices with a preferred direction. This problem can be resolved in different ways. One can either add an additional “fluid” degree of freedom, representing the averaged vorticity, to the variational discussion (see [44, 45] for an interesting discussion and [46] for a relativistic account). Formally, this may be the most natural approach. In particular since the mutual friction then arises as a linear friction associated with the drift of vortices relative to the excitations.

A more direct alternative would be to augment the analysis of the dissipative fluxes with the preferred direction $\hat{\kappa}^{j}$. This leads to quite a large number of possible extra dissipative terms. To see this, let us briefly return to (13)-(18). These relations followed the assumption that the dissipative fluxes must be linear in the thermodynamical forces. As a result, a two index coefficient like $L_{ij}^{nn}$ can only be constructed out of the metric $g_{ij}$. If we have an additional vector in the problem, then a number of additional two-index objects can be written down. We can then have

$$L_{ij}^{nn} = 2 \mathcal{R}_{ij}^{nn} g_{ij} + \mathcal{R}_1 \hat{\kappa}_{i} \hat{\kappa}_{j} + \mathcal{R}_2 \epsilon_{ijk} \hat{\kappa}_{k}.$$

(73)

The force resulting from this expression can be written

$$- f_{i}^{n} = L_{ij}^{nn} w_{ns}^j = 2 \mathcal{R}_{ij}^{nn} w_{ns}^i + \mathcal{R}_1 \hat{\kappa}_{i} (\hat{\kappa}_{j} w_{ns}^j) + \mathcal{R}_2 \epsilon_{ijk} \hat{\kappa}_{k} w_{ns}^j.$$

(74)
In order to compare this to (72) we rewrite the latter as
\[
- f_i^{\text{mf}} = B \rho_n \kappa \left[ w_i^{\text{ns}} - (\hat{\kappa}_j w_j^{\text{ns}}) \hat{\kappa}_i \right] - B' \rho_n n_v \kappa \epsilon_{ijk} \hat{\kappa}_j w_k^{\text{ns}}, \tag{75}
\]
and we see that we should identify
\[
2 R_{\text{im}} = - R_1 = B \rho_n \kappa \quad \text{and} \quad R_2 = B' \rho_n n_v \kappa. \tag{76}
\]

This provides a simple and natural generalisation of the dissipative framework discussed in this paper. Of course, it was designed only to account for the standard form of the vortex mutual friction. It does not in any way provide a completely general description of a system with vortices. Such a model would allow a number of additional dissipative terms, and would be much more complicated. This would nevertheless be an interesting problem to consider. After all, we have not yet accounted for the self-induced flow of (curved) vortices, the vortex tension etcetera [47, 48, 49].

6. Discussion

In this paper we have developed a dissipative two-fluid model, based on distinguishing the particle flux from a massless entropy flow. Correcting a conceptual mistake in a previous analysis [30] we have formulated a general model for an isotropic system, which requires the determination of nine dissipation coefficients. We then demonstrated how imposing the constraint of irrotationality, which is expected for a (pure) superfluid, reduces the complexity of the problem. The final model is in one-to-one correspondence with the classic two-fluid model for He\textsubscript{4} and we have provided a translation between the different variables. This comparison highlights the link between the entropy entrainment in our model and the “normal fluid density” in the standard description (see also [31]). This analysis makes it possible to use available results from kinetic theory [1] in our formulation. Finally, we discussed how the presence of vortices in the superfluid affects the model. In particular, we indicated how one may account for the vortex mediated mutual friction force. Our final model should be directly applicable to low temperature, single “particle species” systems, ranging from laboratory systems to astrophysical objects.

There is considerable scope for future developments in this problem area. First of all, it would be relevant to allow for causal dissipative heat flux terms, building on the discussion in [40]. Secondly, we want to use the experience gained here to develop finite temperature models for the different superfluid components expected to be present in a neutron star core. The final model discussed in this paper, in essence developed to represent superfluid Helium, may be immediately relevant (in a certain temperature regime) for a compact star with a colour superconducting quark core [13]. Further work is required to formulate a model for the coexisting neutron superfluid and proton superconductor expected in the outer core of a neutron star. Other exotic phases, like hyperon superfluids, may be even more complex. However, by demonstrating the intimate link between the entropy entrainment and the thermal excitations, the present analysis has provided a key ingredient for such models.
References

[1] I.M. Khalatnikov, *An introduction to the theory of superfluidity* (W. A. Benjamin, Inc., New York, 1965)
[2] J. Wilks *Liquid and solid Helium* (Clarendon Press, Oxford, 1967)
[3] S.J. Putterman, *Superfluid hydrodynamics* (North-Holland, Amsterdam, 1974)
[4] L. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1959)
[5] S.J. Putterman & P.H. Roberts, Physica 117A 369 (1983)
[6] D. Jou, J. Casas-Vázquez & G. Lebon, *Extended irreversible thermodynamics* (Springer, Berlin, 1993)
[7] I. Müller & T. Ruggeri, *Extended thermodynamics*, (Springer, New York, 1993)
[8] A. Greco & I. Müller, Arch. Rat. Mech. Anal. 85 279 (1984)
[9] M.S. Mongiovi, Phys. Rev. B 48 6276 (1993).
[10] D. Vollhardt & P. Wölfle, *The superfluid phases of Helium 3*, (Taylor & Francis, London 2002)
[11] C.J. Pethick & H. Smith, *Bose-Einstein condensation in dilute gases*, (Cambridge Univ. Press, Cambridge 2002)
[12] P. Haensel, A.Y. Potekhin & D.G. Yakovlev, *Neutron stars 1: Equation of state and structure* (Springer, New York 2007)
[13] M.G. Alford, K. Rajagopal, T. Schaefer & A. Schmitt, *Color superconductivity in dense quark matter* preprint arXiv:0709.4639
[14] A.G. Lyne, S.L. Shemar & F. Graham Smith, MNRAS 315, 534 (2000)
[15] N. Andersson, T. Sidery & G.L. Comer, MNRAS 368 162 (2006)
[16] R. I. Epstein, Ap. J. 333, 880 (1988)
[17] G. Mendell, Ap. J. 380, 515 (1991)
[18] U. Lee, Astron. Astrophys. 303, 586 (1995)
[19] G. L. Comer, D. Langlois, L. M. Lin, Phys. Rev. D 60, 104025 (1999)
[20] N. Andersson and G. L. Comer, Mon. Not. R Astro. Soc. 328, 1129 (2001)
[21] N. Andersson & K.D. Kokkotas, Mon. Not. R Astro. Soc. 299 1059 (1998)
[22] N. Andersson & K.D. Kokkotas, Int. J. Mod. Phys. D 10 381 (2001)
[23] N. Andersson, Class. Quantum Grav. 20 R105 (2003)
[24] L. Lindblom & G. Mendell, Phys. Rev. D 61 104003 (2000)
[25] S. Yoshida & U. Lee, Phys. Rev. D 67 124019 (2003)
[26] R. Prix, G.L. Comer & N. Andersson, Mon. Not. R Astro. Soc. 348 625 (2004)
[27] M. Nayyar & B.J. Owen, Phys. Rev. D 73 084001 (2006)
[28] M.G. Alford, M. Braby & A. Schmitt, preprint arXiv:0806.0285
[29] N. Andersson, G.L. Comer & K. Glampedakis, Nucl. Phys. A 763 212 (2005)
[30] N. Andersson & G.L. Comer, Class. Quantum Grav. 23 5505 (2006)
[31] R. Prix, Phys. Rev. D 69 043001 (2004)
[32] K. Glampedakis, N. Andersson & D.I. Jones, Phys. Rev. Lett 100 081101 (2007)
[33] K. Glampedakis, N. Andersson & D.I. Jones, Mon. Not. R. Astro. Soc. in press, preprint arXiv:0801.4638
[34] K. Glampedakis & N. Andersson, preprint arXiv:0806.3664
[35] B. Carter & I.M. Khalatnikov, Phys. Rev. D 45 4536 (1992)
[36] B. Carter, “Covariant Theory of Conductivity in Ideal Fluid or Solid Media”, in A. Anile and M. Choquet-Bruhat, eds., *Relativistic Fluid Dynamics* (Noto, 1987), pp. 1–64, (Springer-Verlag, Heidelberg, Germany, 1989)
[37] N. Andersson & G.L. Comer, Living Reviews in Relativity, 10 no. 1 (2007)
[38] I. M. Khalatnikov & V. V. Lebedev, Phys. Lett. A 91 70 (1982)
[39] C. Manuel & F.J. Llanes-Estrada, JCAP 8 1 (2007)
[40] N. Andersson & G.L. Comer, *Extracting extended thermodynamics from a variational multifiud approach* in preparation
Entropy entrainment and dissipation in superfluid Helium

[41] M.E. Gusakov, Phys. Rev. D 76 083001 (2007)
[42] H.E. Hall & W.F. Vinen Proc. R. Soc. Lond. A 238 215 (1956)
[43] T. Sidery, N. Andersson & G.L. Comer, MNRAS 385 335 (2008)
[44] J.A. Geurst, Physica B 154 327 (1989)
[45] K. Yamada, K. Miyake & S. Kashiwamura, J. Phys. Soc. Japan 76 014601 (2007)
[46] B. Carter and D. Langlois, Nucl. Phys. B 454, 402 (1995)
[47] I.L. Bekarevich & I.M. Khalatnikov, Sov. Phys. JETP 13 643 (1961)
[48] G. Mendell, Ap. J. 380, 530 (1991)
[49] R.J. Donnelly, Quantized vortices in Helium II (Cambridge Univ. Press 1991)