QUANTUM RIEMANNIAN GEOMETRY OF THE DISCRETE INTERVAL AND $q$-DEFORMATION

J. N. ARGOTA-QUIROZ AND S. MAJID

Abstract. We solve for quantum Riemannian geometries on the finite lattice interval $\bullet - \bullet - \cdots - \bullet$ with $n$ nodes (the Dynkin graph of type $A_n$) and find that they are necessarily $q$-deformed with $q = e^{i\frac{\pi}{n+1}}$. This comes out of the intrinsic geometry and not by assuming any quantum group in the picture. Specifically, we discover a novel ‘boundary effect’ whereby, in order to admit a quantum-Levi Civita connection, the ‘metric weight’ at any edge is forced to be greater pointing towards the bulk compared to towards the boundary, with ratio given by $(i+1)_q/(i)_q$ at node $i$, where $(i)_q$ is a $q$-integer. The Christoffel symbols are also $q$-deformed. The limit $q \to 1$ is the quantum Riemannian geometry of the natural numbers $\mathbb{N}$ with rational metric multiples $(i+1)/i$ in the direction of increasing $i$. In both cases there is a unique metric up to normalisation with zero Ricci scalar curvature. Elements of QFT and quantum gravity are exhibited for $n=3$ and for the continuum limit of the geometry of $\mathbb{N}$. The Laplacian the scaler-flat metric becomes the Airy equation operator $\frac{1}{2} \frac{d^2}{dx^2}$, in so far as a limit exists. Scaling this metric by a conformal factor $e^{\psi(x)}$ gives a limiting Ricci scalar curvature proportional to $\frac{1}{2} \frac{d}{dx} \left( \frac{d\psi}{dx} \right)^2$.

1. Introduction

The idea that spacetime coordinates are better modelled as a noncommutative algebra due to quantum gravity effects has gained traction in recent years as the ‘quantum spacetime hypothesis’. While speculated on since the early days of quantum mechanics, the proper study of this idea only became possible with the arrival of the mathematics of noncommutative geometry and related quantum group symmetries in the 1980s and 1990s. Models connecting these with the Planck scale (actually at the deformed phase space level) appeared in [15], while deformed Minkowski space itself with quantum Poincaré group symmetry appeared in [23], for the quantum group which had been proposed in [16]. Other proposals from other contexts included [15] [10]. These various models were flat but over the last 30 years there emerged a systematic and constructive formalism of quantum Riemannian geometry (QRG), see [6] and references therein, which now allows the systematic construction of curved examples. This approach starts with an algebra $\Omega$ of differential forms over the coordinate algebra $A$, formulates a metric as $g \in \Omega^1 \otimes_A \Omega^1$ and a quantum Levi-Civita connection $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ obeying certain properties[5].

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Remarkably, this theory produces nontrivial results even when $A$ is finite dimensional, for example functions on a finite graph \[19\]. This provides a systematic route to geometry on a finite lattice not as an approximation but as an exact quantum geometry within a single formalism that includes such models at one end and classical GR at the other. In the graph case, functions commute amongst themselves but do not commute with differentials. First quantum gravity models on finite graphs appeared in \[21, 2\] and the present paper is now third in that particular sequence. Quantum gravity on fuzzy spheres is another effectively finite model \[17\]. The QRG approach to quantum gravity cannot directly be compared with other approaches such as loop quantum cosmology \[3\], dynamical triangulations \[1\] and causal sets \[11\] due to different methods, but some aspects could eventually connect up. Also note that our conception of QRG is very different from Connes’ spectral triple approach which encodes the noncommutative geometry in an axiomatically defined ‘Dirac operator’ \[8\], but the two approaches are not incompatible. Finite models in Connes approach were applied to quantum gravity in \[14\] and have also been applied to the standard model of particle physics \[9\].

Specifically, in this paper we explore the quantum Riemannian geometry of the finite line graph $\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet$ with $n$ nodes (the Dynkin graph of type $A_n$) as well as the half-line with nodes the natural numbers $\mathbb{N}$. This turns out to be an order of magnitude harder than the case a closed $n$-gon $\mathbb{Z}_n$ or the integers $\mathbb{Z}$ solved in \[2, 20\] respectively. The complication comes from the boundary, i.e. the endpoints, and we discover a remarkable and unexpected new effect. Namely, it has been pointed out from the start \[19\] that there is nothing in the formalism that says that the length of an edge has to be the direction-independent. A graph metric in QRG is just a nonzero real number assigned to every arrow with one arrow in each direction for every ‘link’ or graph-edge. To keep things simple and to conform to

**Figure 1.** The direction coefficient $\phi(i) = \frac{i+1}{i}$ at node $i$ on the half-line $\mathbb{N}$. Metrics that admit a QRG have an arbitrary real number at each edge but in the ratio shown for the inbound direction / outbound direction. Eg at the first link the inbound length is twice the outbound, at the second the ratio is 3:2, etc. The ratio tends rapidly to 1 as we enter the bulk showing that this is an effect due to the endpoint boundary.
physical intuition, one usually insists on the metric being edge-symmetric so that these two directions have the same length. The more general case can certainly be considered eg the polygon case with asymmetric metrics was recently solved in [7] as a generalisation of [2], but there is no particular reason to do so. Our new and rather surprising result is that for the $A_n$ graph with $n > 2$ and for $N$ there is no edge-symmetric QRG. We are forced to introduce a 'direction coefficient' $\phi$ on edges to measure the ratio of the inbound arrow (towards the bulk) length compared to the outbound arrow length and find for $N$ that these have to be a specific rational numbers as shown in Figure 1 in order to admit a quantum-Levi Civita connection. These decays rapidly from 2 at the endpoint down to 1 in the bulk. As long as we keep these ratios, we are free to vary the actual metric coefficients or ‘square-lengths’ as we please, so the moduli of QRGs is the same as classically – a single ‘square-length’ on every link – but the new effect is that if we consider this as the outbound one then the inbound one is a multiple $\phi$ of it, namely twice at the first link from the end, $3/2$ at the link which is one in from the end, etc. The metric coefficients have units of length squared as explained in [21, 6].

The situation for $A_n$ is similar and we again find a canonical choice of quantum-Levi Civita connection provided $\phi_i$ labelling at edges $i = 0, \cdots, n-1$ are now given by $q$-integers

$$\phi_i = \frac{(i+1)_q}{(i)_q}, \quad q = e^{\frac{i\pi}{n+1}}, \quad (i)_q = \frac{q^i - q^{-i}}{q - q^{-1}}$$

defining the canonical QRG for $N$. This is the second and equally unexpected discovery of our analysis, that a finite-lattice interval is intrinsically $q$-deformed in its quantum Riemannian geometry, without a quantum group in sight. The fact that $A_n$ is also the Dynkin graph for $SU_{n+1}$ suggests that there could be a role for $u_q(su_{n+1})$ at the specified root of unity, possibly as some kind of diffeomorphism group, but this remains to be established.

An outline of the paper is as follows. We start with a recap of the formalism in Section 2 including and choice of differential forms on the graph, in Section 2.3. In fact, the canonical choice of $\Omega$ for us will be $\Omega_{min}$ defined on any graph [6]. In our case, it is a certain quotient of the preprojective algebra of type $A_n$, an algebra itself of considerable interest in representation theory. Moreover, we will find that 3-forms and above vanish, so the geometry is in that limited sense 2-dimensional. We take the same form of calculus for $N$ also, but now with $\Omega_1$ infinite-dimensional. The analysis of the QRG is obtained in Section 4 building on a preliminary exploration of $A_2$–$A_5$ by hand (and by Mathematica) in Section 3. The actual moduli of QLCs is rather rich and includes a parameter $s$ which to be $*$-preserving needs to be $|s| = 1$. The allowed metrics also admit a minus sign $\epsilon$ if we keep track of which coefficients are positive. Then, in Section 4, we extrapolate from this to general formulae for $A_n$ and $N$ with a uniform solution for the QRG with the freely chosen metric weights $\{h_i\}$, sign parameter $\epsilon$ in the metric and modulus 1 parameter $s$ in the quantum Levi-Civita connection. The physical case for all metric coefficients positive requires $\epsilon = 1$ and if we want the Christoffel symbols to also be real then we are forced to $s = \pm 1$ (they also simplify vastly on this case). We will only use these values for the rest of the paper. These canonical QRGs are in Proposition 4.3 for $N$ and Corollary 4.4 for $A_n$. 
After finding this canonical form for the QRGs, we then study scalar field theory on them in Section 5. The effect of the direction dependence $\phi_i$ on $N$ translates to a derivative term correction to the Laplacian which alternates with a $(-1)^i$ factor preventing a straightforward continuum limit. However, this is suppressed as $1/i$ so that as the lattice spacing tends to zero, this complication is pushed to the boundary at 0. A secondary effect of the $\phi_i$ factor is that the overall metric factor $\beta^{-1}$ in front of the Laplacian has a correction compared to the same choices of $h_i$ on $\mathbb{Z}$. We analyse this for the case of constant $h_i$ as something like a $1/x$ force towards the origin. Some correlation functions for scalar field theory on the $A_3$ graph are also computed as proof of concept with respect to a measure for integration on the $A_3$ graph.

In Section 6 we study the Riemann curvature of our QRGs on $\mathbb{N}$ and $A_n$ and some aspects of quantum gravity. In both cases we find a unique set of metric coefficients $\{h_i^{\text{flat}}\}$ such that the scalar curvature vanishes (one can similarly solve for any prescribed curvature). For this background, the Laplacian to leading order (again ignoring the suppressed $(-1)^i$ term) is given by the Airy operator $\frac{1}{x} \frac{d^2}{dx^2}$. We also look at metrics modifying the flat one, $h_i = h_i^{\text{flat}} e^{\psi_i}$ and find in the continuum limit that the leading order scalar curvature is $\frac{1}{x} \frac{d}{dx} \left( \frac{d}{dx} \psi \right)^2$. For a natural choice of measure of integration given by the metric itself, the resulting Einstein-Hilbert action in the continuum limit is topological. For $A_3$ we briefly look at quantum gravity defined by the discrete Einstein-Hilbert action. The paper concludes with some final remarks about further work.

2. Recap of quantum Riemannian geometry on graphs

It is quite important that our geometric constructions are not ad-hoc but part of a general framework which applies to most unital algebras and is then restricted to the algebra of functions on the vertices of a graph as explained in [19].

2.1. Quantum Riemannian geometry. We will not need the full generality of the theory and give only the bare bones at this general level, for orientation. Details are in [6].

We work with a unital possibly noncommutative algebra $A$ viewed as a ‘coordinate algebra’. We replace the notion of differential structure on a space by specifying a bimodule $\Omega^1$ of differential forms over $A$. A bimodule means we can multiply a ‘1-form’ $\omega \in \Omega^1$ by ‘functions’ $a, b \in A$ either from the left or the right and the two should associate according to

$$ (a \omega)b = a(\omega b). $$

(2.1)

We also need $d : A \rightarrow \Omega^1$ an ‘exterior derivative’ obeying reasonable axioms, the most important of which is the Leibniz rule

$$ d(ab) = (da)b + a(db) $$

(2.2)

for all $a, b \in A$. We usually require $\Omega^1$ to extend to forms of higher degree to give a graded algebra $\Omega = \oplus \Omega^i$ (where associativity extends the bimodule identity (2.1) to higher degree). We also require $d$ to extend to $d : \Omega^i \rightarrow \Omega^{i+1}$ obeying a graded-Leibniz rule with respect to the graded product $\wedge$ and $d^2 = 0$. This much structure is common to most forms of noncommutative geometry, including [8] albeit there it
is not a starting point. In our constructive approach this ‘differential structure’ is
the first choice we have to make in model building once we fixed the algebra $A$. We
require that $\Omega$ is then generated by $A, dA$ as it would be classically. A first order
calculus is inner if there is $\theta \in \Omega^1$ such that $d = [\theta, \cdot]$ and similarly with graded
commutator for the exterior algebra to be inner.

Next, on an algebra with differential we define a metric as an element $g \in \Omega^1 \otimes_A \Omega^1$
which is invertible in the sense of a map $(\cdot, \cdot) : \Omega^1 \otimes_A \Omega^1 \to A$ which commutes
with the product by $A$ from the left or right and inverts $g$ in the sense
\[(\omega, \cdot) \otimes_A id)g = \omega = (id \otimes_A (\cdot, \omega))g\] (2.3)
for all 1-forms $\omega$. In the general theory one can require quantum symmetry in
the form $\wedge (g) = 0$, where we consider the wedge product on 1-forms as a map
$\wedge : \Omega^1 \otimes_A \Omega^1 \to A$ and apply this to $g$.

Finally, we need the notion of a connection. A left connection on $\Omega^1$ is a linear
map $\nabla : \Omega^1 \to \Omega^1 \otimes A \Omega^1$ obeying a left-Leibniz rule
\[\nabla(a\omega) = da \otimes_A \omega + a\nabla\omega\] (2.4)
for all $a \in A, \omega \in \Omega^1$. This might seem mysterious but if we think of a map
$X : \Omega^1 \to A$ that commutes with the right action by $A$ as a ‘vector field’ then
we can evaluate $\nabla$ as a covariant derivative $\nabla \omega = (X \otimes_A id)\nabla : \Omega^1 \to \Omega^1$ which classically is then a usual covariant derivative on $\Omega^1$. There is a similar notion for
a connection on a general ‘vector bundle’ expressed algebraically but we only need
the $\Omega^1$ case. Moreover, when we have both left and right actions of $A$ forming a
bimodule as we do here, we say that a left connection is a bimodule connection if
there also exists a bimodule map $\sigma$ such that
\[\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes A \Omega^1, \quad \nabla(\omega a) = (\nabla \omega)a + \sigma(\omega \otimes A da)\] (2.5)
for all $a \in A, \omega \in \Omega^1$. The map $\sigma$ if it exists is unique, so this is not additional
data but a property that some connections have. The key thing is that bimodule
connections extend automatically to tensor products as
\[\nabla(\omega \otimes_A \eta) = \nabla \omega \otimes_A \eta + (\sigma(\omega \otimes A (\cdot)) \otimes_A id)\nabla \eta\] (2.6)
for all $\omega, \eta \in \Omega^1$, so that metric compatibility now makes sense as $\nabla g = 0$. A
connection is called QLC or ‘quantum Levi-Civita’ if it is metric compatible and
the torsion also vanishes, which in our language amounts to $\wedge \nabla = d$ as equality of
maps $\Omega^1 \to \Omega^2$. We also have a Riemannian curvature for any connection,
\[R_V = (d \otimes_A id - id \wedge \nabla)\nabla : \Omega^1 \to \Omega^2 \otimes_A \Omega^1,\] (2.7)
where classically one would interior product the first factor against a pair of vector
fields to get an operator on 1-forms. Ricci requires more data and the current state
of the art (but probably not the only way) is to introduce a lifting bimodule map
$i : \Omega^2 \to \Omega^3 \otimes_A \Omega^1$. Applying this to the left output of $R_V$; we are then free to
‘contract’ by using the metric and inverse metric to define $\text{Ricci} \in \Omega^1 \otimes_A \Omega^1$. The associated Ricci scalar and the geometric quantum Laplacian are
\[S = (\cdot, \cdot)\text{Ricci} \in A, \quad \Delta = (\cdot, \cdot)\nabla d : A \to A\] (2.8)
defined again along lines that generalise these classical concepts to any algebra with
differential structure, metric and connection.
Finally, and critical for physics, are unitarity or ‘reality’ properties. We work over \( \mathbb{C} \) but assume that \( A \) is a \( * \)-algebra (real functions, classically, would be the self-adjoint elements). We require this to extend to \( \Omega \) as a graded-anti-involution (so reversing order with an extra sign when odd degree differential forms are involved) and to commute with \( d \). ‘Reality’ of the metric and of the connection in the sense of being \( * \)-preserving are imposed as \[ (2.9) \]

\[
g^\dagger = g, \quad \nabla \circ * = \sigma \circ \dagger \circ \nabla; \quad (\omega \otimes_A \eta)^\dagger = \eta^* \otimes_A \omega^* \]

where \( \dagger \) is the natural \( * \)-operation on \( \Omega^1 \otimes_A \Omega^1 \). These ‘reality’ conditions in a self-adjoint basis (if one exists) and in the classical case would ensure that the metric and connection coefficients are real.

Finally, in practical terms, if the exterior algebra is inner then a connection has \( \theta \otimes g + (\text{id} \otimes \alpha) g + \sigma_{12} (\text{id} \otimes (\alpha - \sigma (\otimes \theta))) g = 0 \).

In the \( * \)-algebra case if \( \theta^* = -\theta \) then we need \((\dagger \circ \sigma)^2 = \text{id} \) and \( \sigma \circ \dagger \circ \alpha = \alpha \circ \ast \) for a \( * \)-preserving connection.

### 2.2. Canonical exterior algebras on a graph.

Let \( X \) be a discrete set and \( A = \mathbb{C}(X) \) the usual commutative algebra of complex functions. It can be shown (basically by considering the action of \( \delta \)-functions) that for such an algebra the possible differential structures \( (\Omega^1, d) \) are in 1-1 correspondence with directed graphs with \( X \) as the set of vertices, cf \[8, 19, 6\]. A directed graph just means to draw at most one arrow between some of the vertices, and no self-arrows are allowed. In fact for the calculus to admit a quantum metric the graph needs to be bidirected, i.e. whenever there is an arrow \( x \rightarrow y \) there is also an arrow \( y \rightarrow x \), in other words our data will just be an undirected graph where \( x=y \) means an arrow in both directions. The reason this graph language is useful is that \( \Omega^1 \) has a basis \( \{\omega_{x \rightarrow y}\} \) over \( \mathbb{C} \) exactly labelled by the arrows of the graph. We then define the bimodule structure and differential

\[
f \omega_{x \rightarrow y} = f(x) \omega_{x \rightarrow y}, \quad \omega_{x \rightarrow y} f = \omega_{x \rightarrow y} f(y), \quad df = \sum_{x \rightarrow y} (f(y) - f(x)) \omega_{x \rightarrow y}
\]

and in the bidirected case a quantum metric has the form \[19\]

\[
g = \sum_{x \rightarrow y} g_{x \rightarrow y} \omega_{x \rightarrow y} \otimes_A \omega_{y \rightarrow x}
\]

with weights \( g_{x \rightarrow y} \in \mathbb{R} \setminus \{0\} \) for every arrow. The calculus over \( \mathbb{C} \) is compatible with complex conjugation on functions \( f^*(x) = \overline{f(x)} \) and \( \omega^*_{x \rightarrow y} = -\omega_{y \rightarrow x} \), from which we see that ‘reality’ of the metric in \[2.9\] indeed amounts to real metric weights. It is not required mathematically, but reasonable from the point of view of the physical interpretation, to restrict attention to the edge symmetric case where \( g_{x \rightarrow y} = g_{y \rightarrow x} \) is independent of the direction. First order calculi on sets are always inner with \( \theta = \sum_{x \rightarrow y} \omega_{x \rightarrow y} \), the sum of all arrows.

Finding a QLC for a metric depends on how \( \Omega^2 \) and here there are four canonical choices for \( \Omega \) in the sense that they are defined for any graph. They are all quotients of the path algebra which in degree \( d \) consists of the \( d \)-step paths \( \omega_{x_0 \rightarrow x_1} \otimes \cdots \otimes \omega_{x_{d-1} \rightarrow x_d} \).
\[ \sum_{y:y \to y} \omega_{y \to y} \wedge \omega_{y \to q} = 0 \]

for all fixed \( p, q \) that obey one of four conditions. This leads to the four exterior algebras forming a diamond:

\[ \Omega_{\text{max}} \quad \Omega_{\text{med}} \quad \Omega_{\text{med}'} \quad \Omega_{\text{min}} \]

where the conditions are all \( p, q \) such that

- \( \Omega_{\text{min}} \): all \( p, q \)
- \( \Omega_{\text{med}} \): \( p \neq q \)
- \( \Omega_{\text{med}'} \): \( p \to q \)
- \( \Omega_{\text{max}} \): \( p \neq q, \ p \to q \)

Three of these were explicitly discussed in [6, 22] while \( \Omega_{\text{med}'} \) was recently used in [13]. The exterior derivative \( d \) is given in [6] with only the first two necessarily inner at the level of the exterior algebra.

2.3. Exterior algebras on \( A_n \) and preprojective algebras. The preprojective algebra of a graph is just the path algebra of the graph viewed as bidirected (each edge is viewed as a pair of arrows, one in each direction). For the Dynkin graph of type \( A_n \) with nodes numbered in order 1, 2, \( \cdots \), \( n \) we denote the edges

\[ a_i = \omega_{i \to i+1}, \quad a'_i = \omega_{i+1 \to i} = -a'^*_i, \quad i = 1, \cdots, n-1. \]

The minus sign is not used here in the maths literature (we call it \( a'_i \)), but the sign just amounts to a different normalisation and is needed for our exterior algebras to become \( \ast \)-exterior algebras when working over \( \mathbb{C} \). We also denote by \( \delta_i \) the Kronecker \( \delta \)-functions at the nodes. The path algebra then has the relations that all products of these generators are zero except

\[ \delta^2_i = \delta_i, \quad \delta_i a_i = a_i, \delta_i a_i = 0, \quad \delta_{i+1} a'_{i+1} = a'_{i+1}, \quad a_i a_{i+1} = a'_{i+1} a'_{i}, \quad a_i a'_i = a'_i a_i \]

The dimension of the path algebra in degree 0 is \( n \) with basis \( \delta_i \). In degree 1 it is \( 2(n-1) \)-dimensional with basis \( a_i, a'_i \) and in degree 2 it is \( 2(2n-3) \)-dimensional with basis \( a_i a_{i+1}, a'_{i+1} a'_{i} \) for \( i = 1, \cdots, n-2 \) and \( a_i a'_i, a'_i a_i \) for \( i = 1, \cdots, n-1 \).

**Proposition 2.1.** For a Dynkin graph of type \( A_n \), \( \Omega_{\text{max}} = \Omega_{\text{med}'} \) is a quotient of the path algebra by the relations

\[ a_i a_{i+1} = 0, \quad a'_{i+1} a'_i = 0, \quad i = 1, \cdots, n-2 \quad (2.11) \]

and \( \Omega_{\text{min}} = \Omega_{\text{med}} \) is the further quotient by the relations

\[ a_1 a'_1 = 0, \quad a'_{n-1} a_{n-1} = 0, \quad a_{i+1} a'_{i+1} + a'_i a_i = 0, \quad i = 1, \cdots, n-2 \quad (2.12) \]

and in this latter case \( d = [\theta, \cdot] \) where \( \theta = \sum_i a_i + a'_i \). Here \( \Omega^2_{\text{min}} \) is \( n-2 \)-dimensional and \( \Omega^i_{\text{min}} = 0 \) for \( i \geq 3 \).
Proof. The dimensions up to $\Omega^2$ are clear from the stated bases and quadratic nature of the relations. In degree 3 we consider all 3-step paths and their image in $\Omega_{\text{min}}^1$. Since any 2-steps in same direction vanish by (2.11), the only possible images on the quotient are for zig-zag paths such as $a_{i+1}a_i a'_{i+1} = -a_i' a_i a'_{i+1} = 0$ using (2.12) and then (2.11). Similarly for zig-zag the other way $a_i a'_{i} a_i = -a_i a_{i+1} a'_{i+1} = 0$.

The preprojective $\Pi_n$ has just the (2.12) relations and dimensions $n, 2(n-1), 3(n-2), \ldots, (n-1)2, n$.

We see that $\Omega_{\text{min}}$ is a quotient of this by (2.11). Also, later, we will need a bimodule ‘lifting’ map $i : \Omega_{\text{min}}^2 \to \Omega^1 \otimes A \Omega^1$ such that following this by $\wedge$ is the identity. Given the description above, the natural choice is

$$i(a_i a'_i) = -i(a_{i-1} a_{i-1}) = \frac{1}{2} (a_i \otimes a_i' - a_i' \otimes a_i), \quad i = 2, 3, \ldots, n-1$$

(2.13)

where the product denotes wedge product. We take these same form of exterior algebra relations for the half-line $\mathbb{N}$ just without the upper bound on the indices $i$.

3. Explicit calculations for $A_2, A_3, A_4, A_5$

In this section we give explicit solutions for small $A_n$. For $n \leq 4$ these are manageable by hand and we show the details of the calculation. For $n = 5$, we used Mathematica and Python (independently) and just list the final result.

3.1. $A_2$ geometry. These case is treated in [21], we just cover here for completeness. We work over the directed graph $G(V, E)$ with vertices $V = \{1, 2\}$ and directed edges or ‘arrows’ $E = \{a_1, a'_1\}$ as in Figure 2. The products distinct from zero in $\Omega^1 \otimes A \Omega^1$ are $a_1 \otimes a'_1, a'_1 \otimes a_1$. The exterior algebra $\Omega^2_{\text{max}}$ is 2-dimensional with basis $a_1 \wedge a'_1$ and $a'_1 \wedge a_1$. We work with $\Omega_{\text{min}}$ where these are set to zero.

Using the graded commutator for the external derivatives considering that the calculus have the inner form $\theta = a_1 + a'_1$

$$da = [\theta, a_1] = a'_1 \otimes a_1 + a_1 \otimes a'_1 = da'_1$$

The bimodule map $\alpha = 0$ and

$$\sigma(a_1 \otimes a'_1) = \tau a_1 \otimes a'_1, \quad \sigma(a'_1 \otimes a_1) = \tau' a'_1 \otimes a_1$$

The general form of the central metric is

$$g = f_1 a_1 \otimes a'_1 + f'_1 a'_1 \otimes a_1$$

where $f_1, f'_1$ are in the field, real if we work over $\mathbb{C}$ and impose the reality condition for the metric.
The general form of the connection considering that the calculus is inner
\[ \nabla a_1 = a'_1 \otimes a_1 - \tau_1 a_1 \otimes a'_1, \quad \nabla a'_1 = a_1 \otimes a'_1 - \tau'_1 a'_1 \otimes a_1 \]
Given that we are working in \( \Omega_{\text{min}} \) there is no elements in \( \Omega^2 \) for this case. Then there are not conditions for torsion freeness.

The metric compatible conditions are
\[ a_1 \otimes a'_1 \otimes a_1 : f'_1 + f_1 \tau_1 \tau'_1 = 0 \]
\[ a'_1 \otimes a_1 \otimes a'_1 : f_1 + f'_1 \tau_1 \tau'_1 = 0 \]
These conditions imply that there is a sign \( \epsilon = \tau_1 \tau'_1 = \pm 1 \) with \( f'_1 = \epsilon f_1 \). The \( \ast \)-preserving conditions
\[ |\tau_1| = 1 \]
with \( \tau'_1 = \epsilon \tau_1^{-1} \). We see that there is one sign and one overall normalisation in the metric
\[ g = h_1(a_1 \otimes a'_1 + \epsilon a'_1 \otimes a_1) \]
which allows a QLC with one parameter \( \tau_1 = s \) in characteristic zero
\[ \nabla a_1 = a'_1 \otimes a_1 - s a_1 \otimes a'_1, \quad \nabla a'_1 = a_1 \otimes a'_1 - \epsilon s^{-1} a'_1 \otimes a_1. \]
and the further condition that \( h_1 \) is real and \( |s| = 1 \) for the reality property of the metric and for the connection to be \( \ast \)-preserving in the case over \( \mathbb{C} \). All the connections are flat since \( \Omega^2 = 0 \).

### 3.2. \( A_3 \) geometry
We work over the directed graph \( G(V, E) \) with vertices \( V = \{1, 2, 3\} \) and directed edges \( E = \{a_1, a'_1, a_2, a'_2\} \) as in Figure 3. The products in \( \Omega^1 \otimes \Omega^1 \) different from zero are those where the head of the first arrow connects to the tail of the second arrow, giving the six non-zero elements \( a_1 \otimes a'_1, a_1 \otimes a_2, a'_1 \otimes a_1, a_2 \otimes a'_2, a'_2 \otimes a'_1, a'_2 \otimes a_2 \).

The exterior algebra \( \Omega_{\text{max}} \) for the maximum prolongation has the relations
\[ a_1 \land a_2 = a'_2 \land a'_1 = 0 \]
and we work with the quotient \( \Omega_{\text{min}} \) of this where we add the further relations
\[ a_1 \land a'_1 = a'_2 \land a_2 = 0, \quad a'_1 \land a_1 + a_2 \land a'_2 = 0. \]
The exterior derivative is given for the graded commutator \( d = [\theta, \cdot] \) with the inner element \( \theta = a + a' + a_2 + a'_2 \)
\[ da_1 = a'_1 \land a_1, \quad da'_1 = a'_1 \land a_1, \quad da_2 = -a'_1 \land a_1, \quad da'_2 = -a'_1 \land a_1. \]
The dimensions of the vector spaces of \( \Omega^i \) are 3:4:1.
The metric to be central has to have the form
\[ g = f_1 a_1 \otimes a'_1 + f'_1 a'_1 \otimes a_1 + f_2 a_2 \otimes a'_2 + f'_2 a'_2 \otimes a_2 \]
where \( f_1, f'_1, f_2, f'_2 \) are in the field, and should be real if we work over \( \mathbb{C} \) and want to have the reality condition \( \dagger \circ g = g \).

Given the calculus is inner, the connection for an arbitrary \( x \in \Omega \) have the form
\[
\nabla a_1 = a'_1 \otimes a_1 - \tau_1 a_1 \otimes a'_1 - \sigma_1 a_1 \otimes a_2 \\
\nabla a'_1 = a_1 \otimes a'_1 + a'_2 \otimes a'_1 - \tau'_1 a'_1 \otimes a_1 - (\tau'_1 + 1)a_2 \otimes a'_2 \\
\n\nabla a_2 = a_1 \otimes a_2 + a'_2 \otimes a_2 - \tau_2 a_2 \otimes a'_2 - (\tau_2 + 1)a'_1 \otimes a_1 \\
\n\nabla a'_2 = a_2 \otimes a'_2 - \sigma_2 a'_2 \otimes a'_1 - \tau'_2 a'_2 \otimes a_2
\]
where the braiding map is given by
\[
\sigma(a_1 \otimes a'_1) = \tau_1 a_1 \otimes a'_1 \\
\sigma(a_1 \otimes a_2) = \tau_1 a_1 \otimes a_2 \\
\sigma(a'_1 \otimes a_1) = \tau'_1 a'_1 \otimes a_1 + (\tau'_1 + 1)a_2 \otimes a'_2 \\
\sigma(a_2 \otimes a'_2) = \tau_2 a_2 \otimes a'_2 + (\tau_2 + 1)a'_1 \otimes a_1 \\
\sigma(a'_2 \otimes a'_1) = \sigma_2 a'_2 \otimes a'_1 \\
\sigma(a'_2 \otimes a_2) = \tau'_2 a'_2 \otimes a_2
\]

Metric compatibility (2.10) together with the torsion-free conditions produces
\[
\begin{align*}
a_1 \otimes a'_1 \otimes a_1 : & -f_1 \tau_1 \tau'_1 + f'_1 = 0 \\
a_1 \otimes a_2 \otimes a'_2 : & -f_1 \sigma_1 (\tau'_1 + 1) + f_2 = 0 \\
a'_1 \otimes a_1 \otimes a'_1 : & -f'_1 \tau_1 \tau'_1 - f_2 (\tau_2 + 1)\sigma'_2 + f_1 = 0 \\
a_2 \otimes a'_2 \otimes a'_2 : & -f'_1 \sigma_1 (\tau'_1 + 1) - f_2 \tau_2 \tau'_2 + f'_2 = 0 \\
a'_2 \otimes a'_1 \otimes a_1 : & -f'_2 (\tau_2 + 1)\sigma'_2 + f'_1 = 0 \\
a_2 \otimes a_2 \otimes a'_2 : & -f'_2 \tau_2 \tau'_2 + f_2 = 0 \\
a_2 \otimes a'_2 \otimes a'_1 : & -f_1 \tau_1 (\tau'_1 + 1) - f_2 \tau_2 \sigma'_2 = 0 \\
a'_1 \otimes a_1 \otimes a_2 : & -f'_1 \sigma_1 \tau'_1 - f_2 (\tau_2 + 1)\tau'_2 = 0
\end{align*}
\]
Under these conditions, we have two parameters and one sign in the metric as
\[ g = h_1 (\phi a_1 \otimes a'_1 + \epsilon a'_1 \otimes a_1) + h_2 (\frac{1}{\phi} a_2 \otimes a'_2 + \epsilon a'_2 \otimes a_2), \quad \phi = \sqrt{2} \]
where the connection is
\[
\begin{align*}
\tau_1 & = s, \quad \sigma_1 = \frac{h_2 s}{h_1 \epsilon \phi (\epsilon \phi s + 1)}, \quad \tau'_1 = \frac{1}{\epsilon \phi s}, \\
\tau_2 & = -1 + \frac{1}{2 + \epsilon \phi s}, \quad \sigma'_2 = \frac{h_1 \epsilon \phi}{h_2} (s + \epsilon \phi), \quad \tau'_2 = -\frac{1}{\epsilon \phi} \left(1 + \frac{1}{1 + \epsilon \phi s}\right)
\end{align*}
\]
for a free parameter $s$. Notice that only the combination $s\phi$ enters. There is no point considering $\phi = -\sqrt{2}$ in the metric as this would be equivalent to a redefinition of $\epsilon, h_1, h_2$ by a change of sign, so would be an equivalent solution. Finally the *-preserving condition $\sigma \circ \dagger \circ \nabla = \nabla \circ \ast$ under the connection just require

$$|s| = 1 \quad (3.1)$$

with no further constraints on $h_i$ other than to be real.

3.3. $A_4$ geometry. We again work with $\Omega_{\min}$ which now has vector space dimensions $4 : 6 : 2$ with $\Omega^i = 0$ for $i \geq 3$. The relations for $\Omega^2$ are

$$a_1' \wedge a_1 + a_2 \wedge a_2' = 0, \quad a_2' \wedge a_2 + a_3 \wedge a_3' = 0$$

with the rest being equal to zero. The metric in order to be central has to have the form

$$g = f_1 a_1 \otimes a_1' + f_1' a_1' \otimes a_1 + f_2 a_2 \otimes a_2' + f_2' a_2' \otimes a_2 + f_3 a_3 \otimes a_3' + f_3' a_3' \otimes a_3$$

where $f_1, f_1', f_2, f_2', f_3, f_3'$ are in the field, and real for the reality condition $\dagger \circ g = g$ when working over $\mathbb{C}$. The torsion free connection and bimodule braiding map have to have the form

$$\nabla a_1 = a_1' \otimes a_1 - \tau_1 a_1 \otimes a_1' - \sigma_1 a_1 \otimes a_2$$

$$\nabla a_1' = a_1 \otimes a_1' + a_2' \otimes a_1' - \tau_1' a_1' \otimes a_1 - (\tau_1' + 1) a_2 \otimes a_2'$$

$$\nabla a_2 = a_1 \otimes a_2 + a_2' \otimes a_2 - \tau_2 a_2 \otimes a_2' - (\tau_2 + 1) a_1' \otimes a_1 - \sigma_2 a_2 \otimes a_3$$

$$\nabla a_2' = a_2 \otimes a_2' + a_3' \otimes a_2' - \sigma_2' a_2' \otimes a_2 - (\tau_2' + 1) a_3 \otimes a_3'$$

$$\nabla a_3 = a_2 \otimes a_3 + a_3' \otimes a_3 - \tau_3 a_3 \otimes a_3' - (\tau_3 + 1) a_2 \otimes a_2$$

$$\nabla a_3' = a_3 \otimes a_3' - \sigma_3 a_3' \otimes a_2' - \tau_3' a_3' \otimes a_3$$

$$\sigma(a_1 \otimes a_1') = \tau_1 a_1 \otimes a_1'$$

$$\sigma(a_1 \otimes a_2) = \sigma_1 a_1 \otimes a_2$$

$$\sigma(a_1' \otimes a_1) = \tau_1' a_1' \otimes a_1 + (\tau_1' + 1) a_2 \otimes a_2'$$

$$\sigma(a_2 \otimes a_2') = \tau_2 a_2 \otimes a_2' + (\tau_2 + 1) a_1' \otimes a_1$$

$$\sigma(a_2 \otimes a_3) = \sigma_2 a_2 \otimes a_3$$

$$\sigma(a_2' \otimes a_1') = \sigma_2' a_2' \otimes a_1'$$

$$\sigma(a_2' \otimes a_2) = \tau_2' a_2' \otimes a_2 + (\tau_2' + 1) a_3 \otimes a_3'$$

$$\sigma(a_3 \otimes a_3') = \tau_3 a_3 \otimes a_3' + (\tau_3 + 1) a_2 \otimes a_2$$

$$\sigma(a_3' \otimes a_2') = \sigma_3' a_3' \otimes a_2'$$

$$\sigma(a_3' \otimes a_3) = \tau_3' a_3' \otimes a_3$$
The metric preserving conditions are

\[ a_1 \otimes a_1' \otimes a_1 : -f_1 \tau_1 + f'_1 = 0, \]  
\[ a_1 \otimes a_2 \otimes a_2' : -f_1 \sigma_1 (\tau_1' + 1) + f_2 = 0, \]  
\[ a_1' \otimes a_1 \otimes a_1' : -f_1' \tau_1' - f_2' \tau_2 + f_1 = 0, \]  
\[ a_2' \otimes a_2' \otimes a_2 : -f_1' \sigma_1 (\tau_1' + 1) - f_2' \tau_2 + f_2' = 0, \]  
\[ a_2 \otimes a_3 \otimes a_3' = -f_2' \sigma_2 (\tau_2' + 1) + f_3 = 0, \]  
\[ a_2' \otimes a_1' \otimes a_1 : -f_2' (\tau_2 + 1) \sigma_2' + f_1' = 0, \]  
\[ a_2' \otimes a_2' \otimes a_2 : -f_3' \tau_3 + f_3 = 0, \]  
\[ a_2' \otimes a_2' \otimes a_2' : -f_3' \tau_3 - f_3' \tau_2 + f_2 = 0, \]  
\[ a_3' \otimes a_3' \otimes a_3 : -f_3 \tau_3 \tau'_3 - f_3' \sigma_2 (\tau_2' + 1) + f_3 = 0, \]  
\[ a_3' \otimes a_2' \otimes a_2 : -f_3' \tau_3 + f_3' + f_2 = 0, \]  
\[ a_3' \otimes a_3 \otimes a_3' : -f_3' \tau_3 + f_3 = 0, \]  
\[ a_2' \otimes a_2' \otimes a_1' : -f_3' \tau_1 (\tau_1' + 1) - f_2 \tau_2 \sigma_2 = 0, \]  
\[ a_1' \otimes a_2' \otimes a_1 : -f_3' \sigma_1 (\tau_1 + 1) - f_3 (\tau_2 + 1) \tau_2 = 0, \]  
\[ a_3' \otimes a_3' \otimes a_2' : -f_3' \tau_3 + f_3' - f_3' \tau_2 = 0, \]  
\[ a_2' \otimes a_2' \otimes a_3 : -f_3 (\tau_3 + 1) \tau_3 = 0, \]  
\[ a_2' \otimes a_2' \otimes a_3 : -f_3 (\tau_3 + 1) \tau_3 - f_3' \sigma_2 \tau'_2 = 0 \]

There are for metrics that allow one QLC each. Each connection depend in just one free parameter \( s \). Each metric is then to be found to be of the form

\[ g = h_1 (\sigma_1 a_1 \otimes a_1' + \epsilon a_1' \otimes a_1) + h_2 (a_2 \otimes a_2' + \epsilon a_2' \otimes a_2) + h_3 \left( \frac{1}{\phi} a_3 \otimes a_3' + \epsilon a_3' \otimes a_3 \right) \]

where \( \epsilon = \pm 1 \) and

\[ \phi = \frac{1 \pm \sqrt{5}}{2}. \]

We can chose either in what follows (so we have four metrics according to \( \epsilon \) and the sign of the \( \sqrt{5} \)). For any choices of these we now solve for the connection and find for any value \( s \) of \( \tau_1, \)

\[ \tau_1 = s, \quad \tau_2 = -1 + \frac{1}{\phi + \epsilon s}, \quad \tau_3 = -1 + \frac{1}{\phi (1 - \epsilon) (\phi + s) + \epsilon}. \]

\[ \tau_1' = \frac{1 + \phi}{s}, \quad \tau_2' = \frac{\epsilon (\phi + s) (1 - 1/\phi)}{-\phi (\phi + s) + 1}, \quad \tau_3' = \frac{\epsilon}{\phi s}, \quad \sigma_1 = \frac{h_2}{\h_3 \phi}, \]

\[ \sigma_2 = \frac{h_2}{\h_3 (\phi + s)}, \quad \sigma_3 = \frac{h_2}{\h_3} \phi \left( 1 - \epsilon + \frac{\epsilon}{\phi + \epsilon s} \right). \]

Thus, for each metric we have a 1-parameter family of connections with parameter \( s \). In the *-algebra case, reality of the metric demands \( h_i \) real and *-preserving for the connection is equivalent to

\[ |s| = 1 \]

with no further constraints on \( h_i \). So, there is a still a 1-parameter moduli of connections for each of our four metrics, now with \( |s| = 1 \).
3.4. $A_5$ geometry. Now $\Omega_{\text{min}}$ has vector space dimensions $5 : 8 : 3$ again with $\Omega^2_i = 0$ for $i \geq 3$. Without showing the derivation, form of the braiding and connection, and the metric are

$$
\sigma(a_1 \otimes a_1') = \tau_1 a_1 \otimes a_1' \\
\sigma(a_1 \otimes a_2) = \sigma_1 a_1 \otimes a_2 \\
\sigma(a_1' \otimes a_1) = \tau_1' a_1' \otimes a_1 + (\tau_1' + 1) a_2 \otimes a_2' \\
\sigma(a_2 \otimes a_2') = \tau_2 a_2 \otimes a_2' + (\tau_2 + 1) a_1 \otimes a_1' \\
\sigma(a_2 \otimes a_3) = \sigma_2 a_2 \otimes a_3 \\
\sigma(a_2' \otimes a_2) = \tau_2' a_2' \otimes a_2 + (\tau_2' + 1) a_3 \otimes a_3' \\
\sigma(a_2' \otimes a_1) = \sigma_2' a_2' \otimes a_1 \\
\sigma(a_3 \otimes a_3') = \tau_3 a_3 \otimes a_3' + (\tau_3 + 1) a_2' \otimes a_2 \\
\sigma(a_3 \otimes a_4) = \sigma_3 a_3 \otimes a_4 \\
\sigma(a_3' \otimes a_3) = \tau_3' a_3' \otimes a_3 + (\tau_3' + 1) a_4 \otimes a_4' \\
\sigma(a_3' \otimes a_2) = \sigma_3' a_3' \otimes a_2 \\
\sigma(a_4 \otimes a_4') = \tau_4 a_4 \otimes a_4' + (\tau_4 + 1) a_3 \otimes a_3' \\
\sigma(a_4' \otimes a_3) = \sigma_4' a_4' \otimes a_3 \\
\sigma(a_4' \otimes a_4) = \tau_4' a_4' \otimes a_4
$$

$$
\nabla a_1 = a_1' \otimes a_1 - \tau_1 a_1 \otimes a_1' - \sigma_1 a_1 \otimes a_2 \\
\nabla a_1' = a_1 \otimes a_1' + a_2 \otimes a_1' - \tau_1' a_1' \otimes a_1 - (\tau_1' + 1) a_2 \otimes a_2' \\
\nabla a_2 = a_2' \otimes a_2 + a_2 \otimes a_2 + \tau_2 a_2 \otimes a_2' - (\tau_2 + 1) a_1 \otimes a_1' - \sigma_2 a_2 \otimes a_3 \\
\nabla a_2' = a_2 \otimes a_2' + a_2' \otimes a_2 - \tau_2' a_2' \otimes a_2 - (\tau_2' + 1) a_3 \otimes a_3' - \sigma_2' a_2 \otimes a_4 \\
\nabla a_3 = a_3' \otimes a_3 + a_2 \otimes a_3 + \tau_3 a_3 \otimes a_3' - (\tau_3 + 1) a_2 \otimes a_2' - \sigma_3 a_3 \otimes a_4 \\
\nabla a_3' = a_3 \otimes a_3' + a_3' \otimes a_3 - \tau_3' a_3' \otimes a_3 - (\tau_3' + 1) a_4 \otimes a_4' - \sigma_3' a_3' \otimes a_2 \\
\nabla a_4 = a_4' \otimes a_4 + a_3 \otimes a_4 + \tau_4 a_4 \otimes a_4' - (\tau_4 + 1) a_3 \otimes a_3' \\
\nabla a_4' = a_4 \otimes a_4' - \sigma_4 a_4' \otimes a_3 - \tau_4' a_4' \otimes a_4
$$
\[a'_2 \otimes a_2 \otimes a'_2 : -f_3(1 + \tau_3)\sigma'_3 - f'_3\tau_2\tau'_2 + f_2 = 0,
\]
\[a_3 \otimes a'_3 \otimes a_3 : -f_3\tau_3\rho'_3 - f'_2\sigma'_2(\tau'_2 + 1) + f'_3 = 0,
\]
\[a_3 \otimes a_4 \otimes a'_4 : -f_3\tau_3\sigma'_3 + f_4 = 0,
\]
\[a'_3 \otimes a'_2 \otimes a_2 : -f'_3(1 + \tau_3)\sigma'_3 + f'_2 = 0,
\]
\[a'_3 \otimes a'_3 \otimes a'_3 : -f'_3\tau_3\rho'_3 - f'_4(\tau_4 + 1)\sigma'_4 + f_3 = 0,
\]
\[a_4 \otimes a'_4 \otimes a'_4 : -f'_3\tau_3\sigma'_3 - f_4\tau_4\tau'_4 + f'_4 = 0,
\]
\[a'_4 \otimes a'_3 \otimes a'_3 : -f'_3((\tau_4 + 1)\sigma'_4 + f'_3 = 0,
\]
\[a'_4 \otimes a_4 \otimes a'_4 : -f'_4\tau_4\tau'_4 + f_4 = 0,
\]
\[a_2 \otimes a'_2 \otimes a'_1 : -f'_2\tau_1(\tau'_1 + 1) - f_2\tau_2\sigma'_2 = 0,
\]
\[a'_1 \otimes a_1 \otimes a_2 : -f'_1\sigma_1\tau'_1 - f_2(\tau_2 + 1)\tau'_2 = 0,
\]
\[a_3 \otimes a'_3 \otimes a'_2 : -f_3\sigma_3\tau'_3 - f'_2\tau_2(\tau'_2 + 1) = 0,
\]
\[a'_2 \otimes a_2 \otimes a_3 : -f_3(\tau_3 + 1)\rho'_3 - f'_2\sigma_2\tau'_2 = 0,
\]
\[a_4 \otimes a'_4 \otimes a'_3 : -f'_3(\tau_3 + 1)\rho'_3 - f_4\tau_4\sigma'_4 = 0,
\]
\[a'_3 \otimes a_3 \otimes a'_4 : -f'_3(\tau_3 + 1) - f_4(\tau_4 + 1)\tau'_4 = 0
\]

\[g = h_1(\phi_1 a_1 \otimes a'_1 + ca'_1 \otimes a_1) + h_2(\phi_2 a_2 \otimes a'_2 + ca'_2 \otimes a_2) + h_3\left(\frac{1}{\phi_2} a_3 \otimes a'_3 + ca'_3 \otimes a_3\right) + h_4\left(\frac{1}{\phi_1} a_4 \otimes a'_4 + ca'_4 \otimes a_4\right)
\]

where

\[\phi_1 = \sqrt{3}, \quad \phi_2 = \frac{2}{\sqrt{3}}
\]

and \(\epsilon\) is a sign. In the \(*\)-algebra case over \(\mathbb{C}\) we need \(h_i\) real and the \(|s| = 1\) for the reality of the metric and for the connection to be \(*\)-preserving.

The solutions are as follows, some of which look similar to the \(A_4\) case are:

\[\tau_1 = s, \quad \sigma_1 = \frac{h_2\epsilon_\phi_2}{h_1(\epsilon_\phi_1 + \frac{1}{s})}, \quad \tau'_1 = \frac{1}{\epsilon_\phi_1 s}, \quad \tau_2 = -1 + \frac{\epsilon_\phi_2}{\epsilon_\phi_1 s}, \quad \sigma_2 = \frac{h_1}{h_2\epsilon_\phi_2}(\epsilon_\phi_1 + s)
\]

and the remainder explicitly are

\[\tau_3 = \frac{-2\epsilon_\phi_1 s + \epsilon_\phi_1 s - s - 2}{2\epsilon_\phi_1(s - 2) - 4s + 2}, \quad \tau_4 = \frac{\epsilon_\phi_1(s - 2) - 4s + 2}{\epsilon_\phi_1(4 - 2s) + 6s - 3}
\]

\[\tau'_2 = -\frac{\epsilon_\phi_1 + s}{2(\epsilon_\phi_1 s + 1)}, \quad \tau'_3 = \frac{\epsilon_\phi_1(s - 2) - 2s + 1}{\epsilon_\phi_1(s - 2) - 6s + 3}, \quad \tau'_4 = \frac{2(\epsilon_\phi_1(2s - 1) - s - \epsilon_\phi_1 + 4}{\epsilon_\phi_1(s - 2) - 4s + 2}
\]

\[\sigma_2 = \frac{3h_3(\epsilon_\phi_1 s + 1)}{2h_2(\epsilon_\phi_1 s - 1) - s + 2}, \quad \sigma_3 = \frac{h_4(\epsilon_\phi_1(2s - 1) - s + 2)}{h_3(\epsilon_\phi_1(s - 2) - 3s + 6)}
\]

\[\sigma'_2 = -\frac{2h_2(\epsilon_\phi_1 s - 2s + 1)}{3h_3(\epsilon_\phi_1 s + 1)}, \quad \sigma'_3 = \frac{h_3(2\epsilon_\phi_1(s - 2) - 6s + 3)}{h_4(\epsilon_\phi_1(s - 2) - 2s + 1)}
\]

4. Canonical metrics and QLC for \(A_n\) and the Half-line \(\mathbb{N}\)

Here we solve the system of equations for a quantum Riemannian geometry in general, building on our experience for small \(n\).
4.1. Summary of computer results for \( n \leq 8 \). We summarise the results so far as the first entries in the following table.

| \( n \) | \( \dim_3 \Omega^1 \) | metrics with QLC | QLC | \( \pm \)-QLC |
|---|---|---|---|---|
| 2 | 2:2:0 | \( \epsilon, h_1 \) | \( s \) | \( |s| = 1 \) |
| 3 | 3:4:1 | \( \epsilon, h_1, h_2 \) | \( s \) | \( |s| = 1 \) |
| 4 | 4:6:2:0 | \( \epsilon, \pm, h_1, h_2, h_3 \) | \( s \) | \( |s| = 1 \) |
| 5 | 5:8:3:0 | \( \epsilon, h_1, h_2, h_3, h_4 \) | \( s \) | \( |s| = 1 \) |
| \( n \) : \( (n-1) : n - 2 : 0 \) | \( \epsilon, \epsilon, h_1, ..., h_{n-1} \) | \( s \) | \( |s| = 1 \) |

where \( h_i \) are real variables for the reality conditions if we work over \( \mathbb{C} \), \( \epsilon \) a sign and \( \epsilon' \) is a discrete parameter (not necessarily a binary choice) indicating a discrete moduli for certain numerical ‘direction coefficients’ \( \{ \phi_i \} \). The results found so far then fit the general format

\[
g = \sum_{i=1}^{n-1} h_i (\phi_i a_i \otimes a_i' + \epsilon a_i' \otimes a_i); \quad \phi_{n-1} = \frac{1}{\phi_1}, \quad \phi_{n-2} = \frac{1}{\phi_2} \quad (4.1)
\]

as depicted in Figure 4. This means that only the first \( \phi_1, \cdots, \phi_{\lfloor n/2 \rfloor} \) have to be specified, the rest are inverse, and that in the even case \( \phi_2 = 1 \). We also note

\[
h_i \mapsto -h_i, \quad \phi_i \mapsto -\phi_i, \quad \epsilon \mapsto -\epsilon
\]

is a symmetry of the metric in the odd case. Hence without loss of generality, we may assume that \( \phi_2 = 1 \) in the even case and in the odd case we do not need to list both a solution for \( \{ \phi_i \} \) and their negation. We then solved by computer for all remaining \( n \leq 8 \) and all solutions fit this general format with \( \{ \phi_i \} \) summarised in the following table. Due to the above symmetry, for \( A_2, A_3, A_5 \) we do not have a discrete moduli of \( \{ \phi_i \} \) as it can be absorbed in a change of sign of the \( h_i \) and \( \epsilon \), while in the other cases we list them as separate rows.

From these ‘experimental’ results in the table, we make the following observations:

**Remark 4.1.** (1) In all cases in the table we find

\[
\phi_{i+1} = \phi_1 - \frac{1}{\phi_i}, \quad i = 1, 2, 3, 4, 5, 6, 7
\]

except for \( 8(2) \) where this holds for \( i = 1 \) but not for \( i = 2 \), for example.
(2) For each $n$ in the table, there is a unique solution with $\phi_i > 0$, shown in the first row. These are reproduced by the single formula

$$\phi_i = \frac{\sin \left( \frac{(i+1)\pi}{n+1} \right)}{\sin \left( \frac{i\pi}{n+1} \right)} = \frac{(i+1)q}{(i)q}; \quad (i)q := \frac{q^i - q^{-i}}{q - q^{-1}}; \quad q = e^{\pi i / n},$$

in terms of symmetric $q$-integers. Here $i = 1, 2, \cdots, n - 1$ and the values of $\phi_i$ obey

$$2 > \phi_1 > \phi_2 > \phi_3 > \cdots > \phi_{\lfloor \frac{n}{2} \rfloor} \geq 1$$

with equality in the even case, i.e. $\phi_i$ decreases from $\phi_1$ at the endpoint towards $1$ as we approach the midpoint (and equals 1 at the midpoints in the even case). After that, the $1/\phi_i$ follows the same pattern going back up to the other endpoint.

This, along with $\epsilon = 1$, is the unique physical forms of the metric for each $n$, with real $h_i > 0$ the remaining free parameters at each link. The metric coefficient or ‘square-length’ is $h_i \phi_i$ going from $i \rightarrow i+1$ and $h_i$ going from $i+1 \rightarrow i$, and the ‘direction coefficients’ $\phi_i$ is the ratio of these. The above says that there is a longer ‘square length’ travelling into the bulk compared to travelling outward and that this ratio is most at the endpoints and tends to or is 1 in the middle.

(3) In the limit $n \rightarrow \infty$, the physical choice in (2) tends to $\phi(i) = \frac{i+1}{i}$ as in Figure 1. The values of $\phi_i$ for finite $n$ approach these from below and we see that the finite $n$ QRG is a $q$-deformation (for $q$ a root of unity) of the $n \rightarrow \infty$ theory.

(4) For each $n$, the unique positive value of $\phi_1$ are roots of a certain polynomial and $\phi_2$ determined by (1) are roots of a similar polynomial of the same degree. The other allowed pattern of $\phi_1, \phi_2$ are then all joint solutions of (1) and these two polynomials, modulo the global symmetry mentioned above.

| $n$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 2   | 1        | $\sqrt{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{3}{\sqrt{2}}$ | $\frac{3}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ |
| 3   | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(1 - \sqrt{5})$ | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(1 - \sqrt{5})$ | $\frac{1}{2}(1 + \sqrt{5})$ | $\frac{1}{2}(1 - \sqrt{5})$ |
| 4+  | $\sqrt{3}$ | $\frac{2}{\sqrt{3}}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ |
| 4−  | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ |
| 5   | $\sqrt{2} + \frac{1}{\sqrt{2}}$ | $\sqrt{2} - \frac{1}{\sqrt{2}}$ | $\sqrt{2} + \frac{1}{\sqrt{2}}$ | $\sqrt{2} - \frac{1}{\sqrt{2}}$ | $\sqrt{2} + \frac{1}{\sqrt{2}}$ | $\sqrt{2} - \frac{1}{\sqrt{2}}$ |
| 6+  | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ |
| 6−  | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ | $\frac{2}{3}(1 + \sqrt{5})$ | $\frac{2}{3}(1 - \sqrt{5})$ |

**Figure 5.** Table of allowed direction coefficients $\phi_i$ for $n \leq 8$
(5) For every allowed quantum metric for $n \leq 8$, i.e. for every row in the table, there is a unique form of QLC up to the value of $\tau_1 = s$, which is a free parameter required in the $*$-algebra case to obey $|s| = 1$ for the connection to be $*$-preserving.

It is expected that the above patterns hold up for all $n$. In particular, it is clear that over $\mathbb{C}$ and with the required ‘reality’ structures, there is a unique allowed form of quantum metric with positive square-lengths with free parameters $h_1, \cdots, h_{n-1} > 0$ and ‘direction coefficients’ prescribed by \[4.2\].

4.2. General solution for the $A_n$ and $\mathbb{N}$. We now solve for the quantum Riemannian geometry in general $A_n$, motivated by our experience for $n \leq 8$, which also serves as a check. We now consider a general metric with coefficients

$$f_i = \phi_i h_i, \quad f_i' = \epsilon h_i, \quad h_i, \phi_i \neq 0, \epsilon = \pm 1$$

for the weights of increasing and decreasing arrows respectively.

Next, the general form of $\sigma$ with $4(n-2) + 2$ parameters is forced by the bimodule map property and after including the torsion equation, but not yet solving for metric compatibility, is forced to be of the form (call them (*)

$$\sigma(a_i \otimes a_{i+1}) = \sigma_i a_i \otimes a_{i+1}, \quad i = 1, 2, \cdots, n-2$$

$$\sigma(a_i' \otimes a_{i-1}') = \sigma_i' a_i' \otimes a_{i-1}', \quad i = 2, 3, \cdots, n-1$$

$$\sigma(a_1 \otimes a_i') = \tau_i a_1 \otimes a_i'$$

$$\sigma(a_i \otimes a_1') = \tau_{i+1} a_i \otimes a_1', \quad i = 2, \cdots, n-1$$

$$\sigma(a_i' \otimes a_i') = \tau_i' a_i' \otimes a_i', \quad i = 1, \cdots, n-2$$

$$\sigma(a_{n-1}' \otimes a_{n-1}) = \tau_{n-1}' a_{n-1}' \otimes a_{n-1}$$

where the parameters are organised into four families with $n-2$ values each for $\sigma_i, \sigma_i'$ and $n-1$ values each for $\tau_i, \tau_i'$ as shown. The pattern here is that 2-steps in the same line just have one constant as do the back-and-forth steps at the ends where one can only step one way, but when one can go back-and-forth both to the left and to the right, $\sigma$ of one of these is a mixture of both possibilities.

We provide a formal proof now that this is true for all $n$, considering the metric as background data and solving for the connection coefficients. We limit attention to metrics of the form \[4.1\] with $\phi_1, \cdots, \phi_{n-1}$ unknown and $\epsilon = \pm 1$ arbitrary. Such an analysis for a 1-way graph line (which is not our case and cannot admit an invertible quantum metric) was in \[R\] Exercise 8.1. Indeed, it pays to consider the equations of ‘$A_\infty$’ i.e. the natural numbers $\mathbb{N}$ as a discrete half-line and obtain any $A_n$ of interest by truncation.

**Proposition 4.2.** For any quantum metric on the natural numbers $\mathbb{N}$ described by $h_i, \phi_i$, there is a 1-parameter moduli of allowed direction coefficients and a 1-parameter family of QLCs as defined iteratively by

$$\phi_{i+1} = \phi_i - \frac{1}{\phi_i}, \quad \tau_{i+1} = -1 + \frac{\phi_i + \epsilon \tau_i}{\phi_i + \epsilon \tau_i}$$

for arbitrary $\phi_1, \tau_1 \neq 0$. The other connection coefficients are then given by

$$\sigma_i = \frac{h_i+1 \phi_i+1}{h_i \phi_i (1 + \tau_i')}, \quad \sigma_i' = \frac{h_{i-1}(\phi_{i-1} + \epsilon \tau_{i-1})}{h_i \phi_i}, \quad \tau_i' = \frac{\phi_i - \phi_{i+1}}{\tau_i}.$$
Moreover, for $h_i, \phi_1$ real, the connection is $*$-preserving iff $|\tau_1| = 1$.

Proof. Writing out the equations for metric compatibility, these break down into groups of increasing vertex. The first group is

$$-f_1 \tau_1 \tau_1' + f_1' = 0$$
$$-f_1' \tau_1 \tau_1' + f_1 - [f_2(\tau_2 + 1)\sigma_2'] = 0$$

(this is the same as we saw for $A_2$ except there we do not have the square bracket term because we do not have $f_2, \tau_2, \sigma_2'$). The next group are 6 more equations for the 4 new variables $\sigma_1, \sigma_2', \tau_2, \tau_2'$ and 2 new parameters $f_2, f_2'$

$$-f_1 \sigma_1(\tau_1' + 1) + f_2 = 0$$
$$-f_1' \sigma_1(\tau_1' + 1) - f_2 \tau_2 + f_2' = 0$$
$$-f_2'(\tau_2 + 1)\sigma_2' + f_2' = 0$$
$$-f_2' \tau_2 \sigma_2' + f_2 - [f_3(\tau_3 + 1)\sigma_3'] = 0$$
$$-f_1' \tau_1(\tau_1' + 1) - f_2 \tau_2 \sigma_2' = 0$$
$$-f_1' \tau_1 - f_2(\tau_2 + 1)\tau_2' = 0$$

(the equations so far as same as we saw for $A_3$ except that there we not have the square bracket term since no variables $f_3, \tau_3, \sigma_3'$). Similarly we have 6 more equations for the 4 new variables $\sigma_2, \sigma_3', \tau_3, \tau_3'$ and 2 new parameters $f_3, f_3'$

$$-f_2 \sigma_2(\tau_2' + 1) + f_3 = 0$$
$$-f_2' \sigma_2(\tau_2' + 1) - f_3 \tau_3 \tau_3' + f_3' = 0$$
$$-f_3'(\tau_3 + 1)\sigma_3' + f_3' = 0$$
$$-f_3' \tau_3 \tau_3' + f_3 - [f_4(\tau_4 + 1)\sigma_4'] = 0$$
$$-f_2' \tau_2(\tau_2' + 1) - f_3 \tau_3 \sigma_3' = 0$$
$$-f_2' \tau_2 \tau_2' - f_3(\tau_3 + 1)\tau_3' = 0$$

(the equations so far are the same as we saw for $A_4$ except that there we not have the square bracket term since no variables $f_4, \tau_4, \sigma_4'$).

The general induction step here is to add a set of six equations for the 4 new variables $\sigma_{i-1}, \sigma_i', \tau_i, \tau_i'$ and 2 new parameters $f_i, f_i'$,

$$-f_{i-1} \sigma_{i-1}(\tau_{i-1}' + 1) + f_i = 0$$
$$-f_{i-1}' \sigma_{i-1}(\tau_{i-1}' + 1) - f_i \tau_i \tau_i' + f_i' = 0$$
$$-f_i'(\tau_i + 1)\sigma_i' + f_i' = 0$$
$$-f_i' \tau_i \sigma_i' + f_i - [f_{i+1}(\tau_{i+1} + 1)\sigma_{i+1}'] = 0$$
$$-f_{i-1}' \tau_{i-1}(\tau_{i-1}' + 1) - f_i \tau_i \sigma_i' = 0$$
$$-f_{i-1}' \sigma_{i-1} \tau_{i-1}' - f_i(\tau_i + 1)\tau_i' = 0$$

(which to this point would be the equations for $A_{i+1}$, except for $A_{i+1}$ itself we would not have the square bracket equations due to no $f_{i+1}, \tau_{i+1}, \sigma_{i+1}'$). This solves the half-line case, and we also noted for later how to extract the $A_n$ solutions from the same analysis.
To solve the system, we rewrite the above $i$th step equations as

\[
(\tau_i + 1)\sigma'_i = \frac{f_{i+1}'}{f_i'}, \quad \tau_i\tau'_i = \frac{f_i}{f_{i+1}} - \frac{f_{i+1}}{f_i'_{i+1}}, \quad \sigma_{i-1}(\tau'_i + 1) = \frac{f_i}{f_{i-1}}
\]

\[
f_i\sigma_i' - f_{i-1}'\tau_i = f_i, \quad f_i'\tau_i + f_{i-1}' = f_i', \quad f_i - f_{i+1}' = \frac{f_i'}{f_i} - \frac{f_{i-1}'}{f_{i-1}}
\]

If we write $f_i = h_i\phi_i$ and $f_i' = \epsilon h_i$ for some unknown $\phi_i$ and $\epsilon = \pm 1$ then the last equation is

\[
\phi_{i+1} = \phi_i - \frac{\epsilon}{\phi_i} + \frac{1}{\phi_{i-1}}
\]

which if we start off without the $\phi_{i-1}$ terms gives the iteration equation for $\phi_i$ as stated.

Also, from the first and 4th we get

\[
\tau_{i+1} = -1 + \frac{f_{i+1}'}{f_{i+1}'}\sigma_{i+1} = -1 + \frac{f_{i+1}'}{f_{i+1}'}(f_i + f_i'\tau_i) = -1 + \frac{\phi_i + \epsilon \tau_i}{\phi_i}
\]

as stated. Similarly, from the 3rd and the 5th we get

\[
\tau'_i = \frac{f_{i-1}'\sigma_{i-1} - f_i'}{f_i} = \frac{f_{i-1}'}{f_{i-1}(1 + \tau_{i-1}'')} = \frac{\epsilon}{\phi_{i-1}(1 + \tau_{i-1}') - \epsilon \phi_{i-1}}
\]

as another (redundant) recursion relation. The 3rd, 4th and 2nd moreover give

\[
\sigma_{i-1} = \frac{f_i}{f_{i-1}(1 + \tau_{i-1}')}, \quad \sigma_i' = \frac{f_{i-1} + f_{i-1}'\tau_{i-1}}{f_i}, \quad \tau_i' = \frac{\epsilon \phi_i - \phi_{i+1}}{\tau_i}
\]

which can be used to determine $\sigma_{i-1}, \sigma_i'$ and obtain $\tau_i'$ from $\tau_i$ as stated (the two sequences are compatible with this relation).

The stated iterative equations have a unique solution given initial values of $\phi_1, \tau_1$. The $\tau_1'$ is determined as

\[
\tau_1' = \frac{1}{\tau_1}(\phi_1 - \phi_1 + 1) = \frac{1}{\tau_1 \phi_1}.
\]

For the last part, *-compatibility of the connection, we apply $\sigma^{-1} \circ \dagger = \dagger \circ \sigma$ to the relations (4.4) and (4.6) obtaining the conditions $|\tau_i| = |\tau_i'| = 1$. Applying the *-compatible conditions to (4.3), we require the condition $\sigma_i' = 1/\tau_{i+1}'$ which from their form in the proposition holds if

\[
|\tau_i| = 1 - \phi_i/\phi_{i-1},
\]

where we drop the last term for $i = 1$, giving $|\tau_i| = 1$. We prove this by induction. Using the recurrence relation for $\tau_{i+1}$ we then obtain $|\tau_{i+1}| = 1 - \phi_{i+1}/\phi_i$ as expected. Finally, the case for relations (4.5) requires the relations $|\Delta_i|_i^2 = 1$, where $\Delta_i = (1 + \tau_{i+1})(1 + \tau_i') - \tau_{i+1}\tau_i'$. Using the form of $\tau_i'$ this again reduces to $|\tau_i| = 1 - \phi_i/\phi_{i-1}$ as proven.

The iteration for $\phi_i$ here can be done in closed form. If $\phi_1 = x$, then

\[
\phi_i = \frac{1}{2} \left(\sqrt{x^2 - 4} \frac{1}{\frac{1}{2} - \frac{1}{2} (-\sqrt{x^2 - 4} - x)^{\frac{3}{2}}} \left(\sqrt{x^2 - 4} - x\right)^{\frac{1}{2}} - 1\right) + x
\]
Figure 6. (a) $\phi_1 \geq 2$ leads to $\phi_i$ asymptotically constant while $\phi_1 < 2$ leads to $\phi_i$ oscillatory. Here $\phi_1 = 2 \cos \left( \frac{\pi x}{2} \right)$ is suitable for $A_n$ and its $\phi_i$ blows up at $i$ a multiple of $n+1$. (b) Smaller $\phi_1$ including $\phi_1 = 2 \cos \left( \frac{\pi x}{3} \right) = 1$ and perturbations of it.

\[
\phi_2 = x - \frac{1}{x}, \quad \phi_3 = \frac{x(x^2 - 2)}{x^2 - 1}, \quad \phi_4 = \frac{(x^4 - 3x^2 + 1)}{x(x^2 - 2)}, \quad \phi_5 = \frac{x(x^4 - 4x^2 + 3)}{x^4 - 3x^2 + 1},
\]

etc., and we then use this solution to determine the recursion relations for $\tau_i$. We see from $\phi_3$ that demanding edge-symmetry where $\phi_i = 1$ at all $i$, is not an option. Qualitatively, we see from Figure 6 that there are two phases for the system:

1. $\phi_1 \geq 2$ (Open phase): Here $\phi_i$ decays rapidly to an asymptote $\frac{1}{2} \left( \sqrt{\phi_1^2 - 4} + \phi_1 \right)$.
   This case leads to solutions on the half-line graph $\mathbb{N}$.
2. $0 < x < 2$ (Finite phase): Here $\phi_i$ is oscillatory, could have zeros and singularities and be periodic for certain $\phi_1$ (as illustrated). This case leads to solutions on $A_n$ as a subgraph of $\mathbb{N}$.

The critical line started by $\phi_1 = 2$ between these two regions is particularly simple. It can be approached from either side but more naturally from above.

**Proposition 4.3.** (Canonical quantum Riemannian geometry of $\mathbb{N}$). For the half-line graph $\mathbb{N}$, the canonical direction coefficient is given by $\phi_1 = 2$. If the metric $h_i$ and initial $\tau_1$ are rational then all coefficients of the quantum geometry are rational.

In particular for initial $\tau_1 = s = \pm 1$ and $\epsilon = 1$,

\[
\phi_i = \frac{i + 1}{i}, \quad \tau_i = \frac{s(-1)^{i-1}}{i}; \quad \tau'_i = -\tau_{i+1}, \quad \sigma_i = \frac{h_{i+1}}{h_i} (1 + \tau_{i+1}), \quad \sigma'_i = \frac{h_{i-1}}{h_i} \frac{1}{1 + \tau_i}.
\]

**Proof.** Here $\phi_1 = 2$ is best approached for the analytic solution from above but one can see directly that the $\phi_i$ stated has this initial value and solves the required recursion relation. For $\tau_i, \tau'_i$ (which are independent of the metric) the recursion has an analytic solution using Pochhammer functions. For example

\[
\tau_1 = s, \quad \tau_2 = -\frac{16s + 8}{2(8s + 16)}, \quad \tau_3 = \frac{120s + 24}{18(4s + 20)}, \quad \tau_4 = -\frac{2016s + 864}{288(12s + 28)}.
\]
etc. This simplifies greatly when $|s| = 1$, namely if $s = e^{i\theta}$, then
\[
\tau_i = -\frac{(6i + (-1)^i + 3) e^{i\theta} + 2i + 3(-1)^i + 1}{i ((-1)^i(2i + 1) + 3) e^{i\theta} + 3(-1)^i(2i + 1) + 1}.
\]
We show the result when $s = \pm 1$ and also for this case the resulting $\tau'_i, \sigma_i, \sigma'_i$. □

Although we typically work over $\mathbb{C}$ in mathematical physics, it is striking that this quantum geometry on $\mathbb{N}$ works over the rational numbers $\mathbb{Q}$. Finally, we turn to the finite case as promised.

**Corollary 4.4.** (Canonical quantum Riemannian geometry of $A_n$). For the finite interval graph $A_n$ with metric defined by $h_i, \phi_i$, there is a quantum Riemannian geometry of the form in Proposition 4.2 provided $\phi_1$ is such that the stated iteration leads to $\phi_n = 0$ and all preceding $\phi_i \neq 0$. In this case
\[
\phi_{n-i} = \frac{1}{\phi_i}.
\]
The canonical physical choice is provided by $\phi_1 = 2 \cos\left(\frac{\pi}{n+1}\right)$ and results in $\phi_i > 0$ for $i = 1, \ldots, n-1$ according to (4.2). If $\tau_1 = s = \pm 1$ and $\epsilon = 1$ then
\[
\tau_i = \frac{s(-1)^{i-1}}{(i)_q}, \quad q = e^{\frac{i\pi}{n+1}}
\]
as a $q$-deformation of the solution on $\mathbb{N}$ in Proposition 4.3 and $\tau'_i, \sigma_i, \sigma'_i$ given in terms of this by the same formulae as there.

**Proof.** We solve the iterative system as before but there is no $f_n$ etc for our truncated graph so we need
\[
f_{n-1} = f'_{n-1} \tau_{n-1} \tau'_{n-1}
\]
in order that the relevant equation in the last group of 6 holds without that square bracketed term. This is $\tau_{n-1} \tau'_{n-1} = \phi_{n-1}$ which comparing with the general formula $\tau_{n-1} \tau'_{n-1}$ needs $\phi_n = 0$. As the preceding $\phi_i$ should all be nonzero, this is the first time this should happen. Also, this happens in the inductive formula for $\phi_i$ when $\phi_1 = \frac{1}{\phi_{n-1}}$ etc. We similarly apply the inductive formula to prove the general relationship shown. For $\epsilon = 1$ as here, one can check that (4.2) has the required property.

In order to prove the assertion $\phi_1 = 1/\phi_{n-i}$, we proceed by induction on $i$. For $i = 1$ the recursive relation for $\phi_n = 0$ as assumed gives $\phi_1 = 1/\phi_{n-1}$ as required. Next, assuming $\phi_i = 1/\phi_{n-i}$ as induction hypothesis and using the recursive relation for $\phi_{i+1}$ and $\phi_{n-i}$ gives
\[
\phi_{i+1} = \phi_1 - \frac{1}{\phi_i} = \phi_1 - \phi_{n-i} = 1/\phi_{n-i-1}
\]
as required.

To find $\tau_i$ for the canonical solution with $\phi_1 = 2 \cos\left(\frac{\pi}{n+1}\right)$ we iterated the recursion relation in Proposition 4.2 to fill out a table of $\tau_i$ values for small $n$ values, see Figure 7. Note that standard recursion methods eg on Mathematica do not yield a general answer in closed form. We then ‘recognised’ the general formula as stated.
Once found, it is easy enough to check that \( \tau_i \) obeys the recursion relation in Proposition 4.2 for \( \phi_i = (i + 1)q/iq \) and compute the other values. □

The canonical choice of \( \phi_1 \) is illustrated in Figure 6. It is also worth noting that for \( s = \pm 1 \), the \( \tau_i \) in this case enjoy symmetries similar to those of \( \phi_1 \), namely for odd \( n \):

\[
\tau_1 = \pm 1, \quad \tau_{n-1} = \tau_2, \quad \tau_{n-2} = \tau_3, \quad \tau_{n-3} = \tau_4, \quad \tau_{n-4} = (-1)^{n-1} \sin \left( \frac{\pi}{n+1} \right)
\]

and for even \( n \):

\[
\tau_1 = \pm 1, \quad \tau_{n-1} = -\tau_2, \quad \tau_{n-2} = -\tau_3, \quad \tau_{n-4} = -\tau_4.
\]

Similarly iterating for small \( n \) but general \( \tau_1 = s \) and computing the associated \( \tau_i, \sigma_i, \sigma_i' \) recovers for \( A_2, \cdots, A_5 \) the explicit values reported in Section 3 for \( \epsilon = 1 \).

More generally, one can start with \( \phi_1 = 2 \cos \left( \frac{j\pi}{n+1} \right) \) for \( j = 1, 2, \cdots, n-1 \), but some of these differ only by a sign so the solution they generate can be absorbed in \( \epsilon \), and others can be excluded as some of the \( \phi_i \) they generate vanish. Also, for specific \( n \) there can be further ‘irregular’ solutions not generated by our method. For example, if we set \( n = 8 \) then all the solutions for \( \phi_1 \) such that \( \phi_8 = 0 \) are given by

\[
\phi_1 : \quad \pm 2 \cos \left( \frac{\pi}{9} \right), \quad \pm 2 \cos \left( \frac{2\pi}{9} \right), \quad \pm 2 \cos \left( \frac{3\pi}{9} \right), \quad \pm 2 \cos \left( \frac{4\pi}{9} \right)
\]

of which the 3rd solution is just \( \pm 1 \) and can be excluded as not all the \( \phi_2, \cdots, \phi_7 \) are nonzero. The other three feature in the table in Figure 5 (we only listed one of the signs since the other can be absorbed in the choice of \( \epsilon \)). This explains the ‘regular’ part of the table but we also see from row 8(2) that for specific \( n \), we do not generate all the solutions by our method. Indeed, row 8(2) has the same initial \( \phi_1 \) as row 8(3) while our method only gives one set of \( \phi_i \) for an initial \( \phi_1 \).
5. LAPLACIAN AND ELEMENTS OF QFT ON $A_3$

Here we compute the Laplacian $\Box = (., \nabla d\psi)$ for a field $\psi$ on $A_n$ and $N$ with their canonical QRGs in Proposition 4.3 and Corollary 4.4 with $h_i$ real, $s = \pm 1$ and $\epsilon = 1$ there. Henceforth, we repurpose $\epsilon > 0$ as a lattice spacing for the continuum limit of the geometry on $N$.

**Lemma 5.1.** The Laplacian operator $\Box f$ of a function $f = \sum_i f(i)\delta_i$ on the vertices of the $A_n$ graph with $n \geq 3$ has the form

$$\Box f = \sum_{i=1}^{n-1} (f(i+1) - f(i))(\nabla a_i - \nabla a'_i)$$

$$= \left( (f(1) - f(2)) \frac{\tau_1 + 1}{h_1 \phi_1} \right) \delta_1$$

$$+ \sum_{i=2}^{n-1} \left( (f(i) - f(i-1))(\tau'_i + 1) + (f(i) - f(i+1))(\tau_i + 1) \right) \left( \frac{1}{h_{i-1}} + \frac{1}{h_i \phi_i} \right) \delta_i$$

$$+ (f(n) - f(n-1)) \left( \frac{\tau'_n + 1}{h_{n-1}} \right) \delta_n$$

**Proof.** Because the calculus is inner, we have $df = [\theta, f] = \sum_{i=1}^{n-1} (f(i+1) - f(i))(a_i - a'_i)$ where $\theta = \sum_{i=1}^{n-1} a_i + a'_i$. Using this in the definition of the Laplacian $\Box f = (., \nabla df)$ for an arbitrary function $f$ we gives the first expression for the Laplacian.

Next, the general form for the connection corresponding to $a_1$, $a_{n-1}$ and $a_i$ (valid only for $1 < i < n - 1$) using again that the calculus is inner

$$\nabla a_1 = a'_1 \otimes a_1 - \tau_1 a_1 \otimes a'_1 - \sigma_1 a_1 \otimes a_2$$

$$\nabla a'_1 = a_1 \otimes a'_1 + a'_2 \otimes a'_1 - \tau'_1 a'_1 \otimes a_1 - (\tau'_1 + 1)a_2 \otimes a'_2$$

$$\nabla a_{n-1} = a_{n-2} \otimes a_{n-1} + a_{n-1} \otimes a_{n-1} - \tau_{n-1} a_{n-1} \otimes a_{n-1} - (\tau_{n-1} + 1)a_{n-2} \otimes a_{n-2}$$

$$\nabla a'_{n-1} = a_{n-1} \otimes a'_{n-1} - \sigma_{n-1} a'_{n-1} \otimes a_{n-1} - \tau'_{n-1} a'_{n-1} \otimes a_{n-1}$$

$$\nabla a_i = a'_i \otimes a_i + a_i \otimes a_i - \tau_i a_i \otimes a'_i - (\tau_i + 1)a_i \otimes a_{i+1} - \sigma_i a_i \otimes a_{i+1}$$

$$\nabla a'_i = a_i \otimes a'_i + a'_{i+1} \otimes a'_i - \tau'_i a'_i \otimes a_i - (\tau'_i + 1)a_{i+1} \otimes a'_{i+1} - \sigma'_i a'_i \otimes a'_{i+1}$$

Arranging the terms and applying the inverse metric, we recover the explicit form stated.

We also use this for the case of $N$ without the final values.

### 5.1. $\mathbb{N}$ and its continuum limit.

The Laplacian for the connection of the Proposition 4.3 with general values of the metric $h_i$ comes out as

$$(\Box f)(1) = (f(1) - f(2)) \frac{1 + s}{2h_1}$$

$$(\Box f)(i) = \left( -(\Delta_z f)(i) + \frac{(-1)^{s} s}{i} (f(i+1) - f(i-1)) \right) \left( \frac{1}{h_{i-1}} + \frac{1}{h_i (1 + \frac{1}{\epsilon})} \right) ; \quad i > 1$$

in terms of the usual discrete Laplacian $\Delta_z f(i) = f(i+1) + f(i-1) - 2f(i)$. For the sake of discussion we take $s = 1$ in order to avoid $(\Box f)(1) = 0$ for all $f$ but an
alternative, which amounts to ignoring the \((-1)^i\) term, is to average the Laplacian between \(s = \pm 1\). For reference,

\[
(\Box h) f(i) = -(f(i+1) + f(i-1) - 2f(i)) \left( \frac{1}{h_{i-1}} + \frac{1}{h_i} \right)
\]

was the Laplacian for an infinite line graph \(Z\) with metric weights \(h_i\) as found in [2]. Compared to this, we see two effects of the truncation to \(N\) both going as \(1/i\) so that they are not visible far from the boundary at \(i = 1\).

1. The metric-dependent factor \(\frac{1}{h_{i-1}} + \frac{1}{h_i}\) for \(Z\) increases slightly as \(i \to 1\).
2. There is a derivative correction but with an alternating sign \((-1)^i\).

We now look at both of these in the context of a lattice approximation of \((0, \infty) \subset \mathbb{R}\) sampled at \(x = \epsilon i\) where \(i \in \mathbb{N}\), \(\epsilon > 0\) now denotes the lattice spacing, and we consider a function \(f\) as either \(f(x)\) or \(f(i)\) with this correspondence. We implement the lattice spacing by a constant value \(h_i = \epsilon^2\) in the inbound (increasing \(i\) direction), but one also has similar results for the more symmetrical \(h_i = \epsilon^2 \sqrt{i+(i+1)}\).

Then the metric-dependent factor is

\[
\frac{1}{h_{i-1}} + \frac{1}{h_i(1 + \frac{1}{i})} = \frac{1}{\epsilon^2 \beta^{-1}}, \quad \beta^{-1}(i) = 1 + \frac{1}{1 + \frac{1}{i}} = \frac{2i+1}{i+1}
\]

(5.1)

1. We first ignore the term with \((-1)^i\) as this clearly has no continuum limit and we will argue that its effects are minimal. In this case, we have as \(\epsilon \to 0\),

\[
\beta(x) = 1 - \frac{1}{2 + \frac{x}{\epsilon}} = 1 + \frac{\epsilon}{4x} + O(\epsilon^2), \quad \frac{1}{\epsilon^2} \Delta Z f = \frac{d^2 f}{dx^2} + O(\epsilon^2)
\]

To interpret what happens at order \(\epsilon\) we consider solving the time-independent Schroedinger equation \(\Box f = 4mEf\) for a mass \(m\) and energy \(E\) in our normalisation of \(\Box\). We set \(h = 1\) for present purposes. Then to \(O(\epsilon^2)\), the equation we are solving is

\[
\left( -\frac{1}{2m} \frac{d^2}{dx^2} - \frac{E\epsilon^2}{2x} \right) f = Ef.
\]

This does not have an immediate parallel with quantum mechanics as the ‘potential’ term is energy-dependent but we can see that energy \(E\) shifts by an amount which is \(E\epsilon\) times a ‘potential’ \(-\frac{1}{2x}\), with eigenfunctions to \(O(\epsilon^2)\) obtained by solving this. One could therefore think of this as like a \(\frac{1}{x}\) force driving solutions towards the boundary as \(x \to 0\). We illustrate this in Figure 8. This shows the solution to

\[
\Box f = 4mEf, \quad f(1) = 1 - \alpha, \quad f(2) = 1 - 2\alpha, \quad \alpha = \frac{4mE\epsilon^2}{1 + 4mE\epsilon^2}
\]

where the initial conditions are such that the linearly extrapolated value at the origin is \(f(0) = 1\) and \(4mEf(1) = (\Box f)(1) = (f(1) - f(2))/\epsilon^2\) as required. The effect of the \((-1)^i\) is to produce ripples in the solution which are less pronounced as larger \(x\) and which get faster and smaller as \(\epsilon \to 0\) (since there are more steps in the range \((0, x)\) for any finite \(x\)).

2. For a theoretical picture of the term with the \((-1)^i\) factor, we discuss two ways to think about this, at least intuitively. One is to live with the lack of continuity
Figure 8. Numerical solutions of $\Box f = 4mEf$ at $mE = 15$ and different values of $h_i = \epsilon^2$, converging to a smooth solution as $\epsilon \to 0$.

and just keep the $(-1)^i$ factor which in the limit of $\epsilon \to 0$ stands for an infinitely-rapidly alternating function of $x = i/\epsilon$, but which makes sense for any finite $\epsilon > 0$. In this case, the other parts of the expression has a limit and we obtain

$$\Box f = -2 \frac{d^2 f}{dx^2} + (-1)^i \frac{4}{x} \frac{df}{dx} + O(\epsilon)$$

in so far as this makes sense. The other approach is to sample our functions only at even $i$ and replace $(-1)^i$ by its average value at $i$ and $i + 1$, i.e. by $\frac{1}{2} \left( \frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{2\epsilon(i+1)}$, which tends to $\frac{x^2}{2\epsilon}$ plus higher order in $\epsilon$. In this case, one can say, again intuitively, that

$$\Box f = -\frac{d^2 f}{dx^2} + \frac{\epsilon}{x^2} \frac{df}{dx} \beta^{-1}(x) + O(\epsilon^2) = -\left( \frac{d^2 f}{dx^2} + 4 \beta' \frac{df}{dx} \right) \beta^{-1} + O(\epsilon^2)$$

where we recognise $\beta'(x) = -\frac{2}{\epsilon^2}$, $\beta^{-1} = \frac{1}{2\epsilon}$, and $\beta' = \frac{1}{2\epsilon^2}$ are the classical Laplacian for metric $g = \beta dx \otimes dx$ and connection $\nabla dx = \frac{\beta'}{\epsilon} dx \otimes dx$ in our notations. (Namely, it differs by a factor $-4$ in the $\beta'$ coefficient given that $\beta \approx 1/2$.) We also recall from that the limit of the quantum geometry is more precisely a 2-dimensional noncommutative differential calculus rather than a classical calculus.

The overall picture is that the direction-dependent quantum metric on $\mathbb{N}$ cannot be avoided as we approach the $i = 1$ boundary and cannot be mapped to an effective continuum limit, but we can begin to get a feel for its physical significance as something like an effective force pushing solutions towards the boundary and possibly a further velocity dependent force. This analysis was for constant metrics $h_i$ or similar. We will mention another natural choice in Section 6.
5.2. QFT on a finite lattice interval \(A_n\). The Laplacian from Lemma 5.1 for the \(A_n\) geometry given for the Corollary 4.4 is

\[
\Box f(1) = (f(1) - f(2)) \frac{s + 1}{(2)_q h_1}; \quad \Box f(n) = (f(n) - f(n - 1)) \frac{1 + (-1)^n s}{h_{n-1}},
\]

\[
\Box f(i) = \left(-\Delta_z f(i) + \frac{(-1)^i s}{i_q} (f(i + 1) - f(i - 1))\right) \left(1 + \frac{h_{i-1}(i)_q}{h_i(i + 1)_q}\right) \frac{1}{h_{i-1}},
\]

for \(i = 2, \ldots, n - 1\). We used that \((n)_q = 1\). It is convenient to write the Laplacian in the form

\[
\Box f = (L f) \beta^{-1}; \quad \beta^{-1}(1) = \frac{1}{h_1}, \quad \beta^{-1}(n) = \frac{1}{h_{n-1}}, \quad \beta^{-1}(i) = \left(1 + \frac{h_{i-1}(i)_q}{h_i(i + 1)_q}\right) \frac{1}{h_{i-1}},
\]

for \(i = 2, \ldots, n - 1\). Then for the free field QFT partition functions etc we are interested in

\[
Z = \int \prod_{i=1}^n d\psi(i)d\bar{\psi}(i)e^{\frac{i}{2} \sum_{i=1}^n \mu(i)\bar{\psi}(i)(L\psi(i))\beta^{-1}(i) - m^2\psi(i)}
\]

with \(\alpha\) a real coupling constant. This is quadratic and hence can be evaluated as a determinant. The key part in the massless case is \(\det(L)\) which does not depend on the metric \(h_i\) coefficients. This also applies in the real scalar field case where we have \(\prod_i d\psi(i)\). One can find that \(\det(L) = 0\) if \(s = -1\) and if \(s = 1\) then

\[
\det(L) = \frac{4}{(2)_q(n-1)_q} \prod_{i=1}^{n-2} (i + 1)_q + (-1)^i
\]

This is then modified by the mass and by the \(\mu_i\) and \(\beta^{-1}(i)\) factors.

As an example, we let \(n = 3\) and \(s = 1\). We have \((2)_q = \sqrt{2}\) and \((3)_q = 1\) and with weights \(\mu_i\), we have an action for a complex (or real) 3-vector \(\psi = (\psi_1, \psi_2, \psi_3)\). In these terms, the action is quadratic,

\[
S[\psi] = \bar{\psi}_1 \mu_1 \left(\frac{2}{h_1\sqrt{2}} \psi_1 - m^2 \psi_1\right) + \bar{\psi}_3 \mu_3 \left(\frac{2}{h_2} \psi_3 - m^2 \psi_3\right)
\]

\[
+ \bar{\psi}_2 \mu_2 \left(\left(\frac{\sqrt{2}}{h_2} + \frac{1}{h_1}\right) \left(-\frac{1}{\sqrt{2}} - 1\right) \psi_1 + \left(\frac{1}{\sqrt{2}} - 1\right) \psi_3 + 2\psi_2\right) - m^2 \psi_2
\]

\[
= \bar{\psi}.B.\psi
\]

for a matrix

\[
B = \begin{pmatrix}
\mu_1 \left(\frac{\sqrt{2}}{h_1} - m^2\right) & -\mu_1 \frac{\sqrt{2}}{h_1} & 0 \\
-\mu_2 \left(1 + \frac{1}{\sqrt{2}}\right) \left(\frac{\sqrt{2}}{h_2} + \frac{1}{h_1}\right) & \mu_2 \left(\frac{\sqrt{2}}{h_2} + \frac{1}{h_1}\right) - m^2 & \mu_2 \left(\frac{\sqrt{2}}{h_2} - 1\right) \left(\sqrt{2} + 1\right) \\
0 & \frac{2\mu_2}{h_2} & \mu_3 \left(-m^2\right)
\end{pmatrix}
\]

now including the mass term. Hence, the partition function \(Z\) is again given as usual for a Gaussian via the determinant

\[
\det(B) = \frac{\mu_1 \mu_2 \mu_3}{h_1 h_2} \left(h_1^2 m^2 \left(-h_2^2 m^4 + 2\sqrt{2} h_2 m^2 - 2\sqrt{2} + 2\right)
\right.
\]

\[
+ h_1 \left(\left(\sqrt{2} + 2\right) h_2^2 m^4 + 2 \left(\sqrt{2} - 2\right) h_2 m^2 - 2\sqrt{2} + 4\right)
\]

\[
- h_2 \left(\sqrt{2} - 1\right) \left(h_2 m^2 - 2\right).
\]
For the constant case $h_1 = h_2$, this simplifies to
\[
\det(B) = -\frac{\hbar_1^{\mu_2 \mu_3}}{h_1^3} \left( h_1 m^2 \left( h_1 m^2 - 3\sqrt{2} - 2 \right) + \sqrt{2} + 1 \right) - 2.
\]

One can similarly compute correlation functions.

6. Curvatures and elements of quantum gravity on $n = 3$

In this section we compute the curvatures in terms of the $h_i$ real parameters and $s = \pm 1$ and $\epsilon = 1$, i.e. for the standard real solutions on $\mathbb{N}$ in Proposition 4.3 and on $\mathbb{A}_n$ and Corollary 4.4 with $\phi_1 = 2 \cos \left( \frac{1}{n+1} \pi \right)$. We then use the lifting map (2.13) to define the Ricci tensor as in [6] and the Ricci scalar by $S$ which we use throughout the section. We also repurpose $\epsilon$ as the lattice spacing in the case of $\mathbb{N}$.

In general, the Riemann curvature (2.7) for our form of connection reduces to
\[
R_{\psi} a_1 = 0
\]
\[
R_{\psi} a_i' = (\tau_i (\sigma_i^2 - \tau_i) + \sigma_i^2) a_i' \wedge a_i \otimes a_i' + (\tau_i (\tau_i - \sigma_i) + \tau_i^2) a_i' \wedge a_1 \otimes a_2
\]
\[
R_{\psi} a_i = (\tau_i (\sigma_i - \tau_i) + \sigma_i^2 - \sigma_i (\tau_i + 1) + \tau_i (\tau_i - 1) a_i \wedge a_i' \otimes a_i + (\tau_i (\tau_i - 1) - a_i')
\]
\[
R_{\psi} a_i' = (\tau_i (\sigma_i^2 - \tau_i) + \sigma_i^2 - \sigma_i (\tau_i^2 - 1) + \tau_i (\tau_i^2 - 1 - a_i')
\]
\[
R_{\psi} a_{n-1} = (\tau_{n-1} (\sigma_{n-2} - \tau_{n-2}) + \sigma_{n-2} a_{n-1} \wedge a_{n-1} \otimes a_{n-1} + (\tau_{n-1} (\tau_{n-2} - \tau_{n-1}) + \tau_{n-2} a_{n-1} \wedge a_{n-1} \otimes a_{n-2}
\]

for $i = 2, \cdots, n - 2$ on $\mathbb{A}_n$ and the same without the final cases on $\mathbb{N}$.

6.1. Curvatures for $\mathbb{N}$. The results for the canonical solution in Proposition 4.3 with $s = \pm 1$ are as follows. For the Riemann curvature, we find
\[
R_{\psi} a_1 = 0
\]
\[
R_{\psi} a_i' = \left( \frac{(1 - 2s)}{4} - \frac{(1 + 2s)}{6} \right) a_i' \wedge a_1 \otimes a_2 - \left( \frac{(s + 2)}{\rho_1 (s - 2)} \right) a_i' \wedge a_1 \otimes a_1'
\]
\[
R_{\psi} a_i = \left( \frac{(i - (-1)^i s)}{i^2} + \frac{a_i (i + (1 - (-1)^i s))}{(i + 1)^2} \right) a_i \wedge a_i' \otimes a_i
+ \frac{(-1)^i s}{(i - 1)} + \frac{1}{\rho_{i-1} (i - (-1)^i s)} a_i \wedge a_i' \otimes a_{i-1}
\]
\[
R^\nabla a_i' = \left( -\frac{(i+1)(s+2)}{\rho_i} \right) a_1 \otimes a_2 - \left( \frac{s(s+2)}{\rho_i} \right) a_1 \otimes a_i' \\
+ \frac{(-1)^i s}{i+1} \left( \frac{(i+1)(s+2)}{\rho_i} \right) a_1 \otimes a_i' \\
+ \frac{(-1)^i s}{i+1} \left( \frac{(i+1)(s+2)}{\rho_i} \right) a_1 \otimes a_i' \\
+ \frac{(-1)^i s}{i+1} \left( \frac{(i+1)(s+2)}{\rho_i} \right) a_1 \otimes a_i'
\]

for \( i \geq 2 \). The Ricci tensor for the lift (2.13) is then
\[
\text{Ricci} = \left( \frac{s(s-2)}{4} - \frac{s(s+2)}{6} \right) a_1 \otimes a_2 - \left( \frac{s(s+2)}{\rho_i} \right) a_1 \otimes a_i' \\
+ \sum_{i \geq 2} \left\{ \left( \frac{s(s-2)}{4} - \frac{s(s+2)}{6} \right) a_1 \otimes a_i' \\
+ \frac{(-1)^i s}{i+1} \left( \frac{(i+1)(s+2)}{\rho_i} \right) a_1 \otimes a_i' \\
+ \frac{(-1)^i s}{i+1} \left( \frac{(i+1)(s+2)}{\rho_i} \right) a_1 \otimes a_i' \right\}
\]

There are no Ricci flat solutions but note that the only coefficient that does not decay \( O(\frac{1}{x^2}) \) for generic \( \rho_i \) is the coefficient of \( a_i \otimes a_i' \) which asymptotes to \( \frac{1}{\rho_i} - \frac{1}{\rho_{i-1}} \).

Contracting with ( , ), the Ricci scalar is then
\[
S(1) = \frac{1}{8h_1} \left( 1 + \frac{2(s+2)}{\rho_1(s-2)} \right) \\
S(2) = \frac{1}{18h_2} \left( 1 - \frac{6(s+2)}{\rho_1(s-2)} + \frac{6(s+3)}{\rho_2(s+3)} \right) \\
S(i) = \frac{1}{2h_i} \left( \frac{1}{i+1} - \frac{\rho_i}{i} + \frac{\rho_{i-1} \rho_{i-2}}{(i-1)(i)} \right)
\]

with
\[
S(i) = \frac{1}{2h_i} \left( \rho_{i-1} \rho_{i-2} - \rho_{i-1} + \frac{1}{\rho_i} \right) + O(\frac{1}{i})
\]

for generic \( h_i \). In the constant \( h_i \) case, however, we have to look to the next order and then one finds
\[
S(i) = \frac{1}{2h_i(i+1)^2} \left( 1 + \frac{2}{i} + i(9 + i + 3i^2) + (-1)^i s(4 + 2i^2(1 + i)(-7 + 2i + 4i^2)) \right) \\
= (-1)^i \frac{s}{h_i^2} \left( 4 - \frac{2}{i} + O(\frac{1}{i^2}) \right)
\]

which has a non-continuum alternating term suppressed for large \( i \), in line with such a term term in the Laplacian in Section 5. One can ignore it and consider that the curvature should be taken as zero.
Alternatively, we can land on $S = 0$, in fact on any prescribed function for the curvature, provided we use an oscillatory $h_i$ which will then not have a classical limit itself. We explore this option next.

**Proposition 6.1.** On $\mathbb{N}$, there is a unique metric \( \{ h_i \} \) up to normalisation such that $S = 0$, given by

\[
\begin{align*}
  &s = 1 : \quad h^\text{flat}_i = 2h_1 \frac{(\frac{i}{2})^2}{(i + 1)} = 2h_1 \left\{ \begin{array}{ll}
  i + 1 & \text{if } i \text{ even} \\
  \frac{i^2}{i+1} & \text{if } i \text{ odd}
  \end{array} \right.
  \\
  &s = -1 : \quad h^\text{flat}_i = 2h_1 \frac{(\frac{i}{2})^2}{(i + 1)} = h_1 \frac{\frac{i^2}{i+1}}{i+1} \left\{ \begin{array}{ll}
  i & \text{if } i \text{ even} \\
  1 & \text{if } i \text{ odd}
  \end{array} \right.
\end{align*}
\]

for any initial value $h_1$.

**Proof.** From the form of $S(i)$ it is clear that we can solve iteratively to find $h_i$ for any initial $h_1$. Doing this for $s = \pm 1$ gives the solution shown. \( \square \)

For the rest of the section, we focus on $s = 1$ but there is a similar story for $s = -1$. First note that setting $s = 1$ and $h_1 = \epsilon^3$ for a small number $\epsilon > 0$ and repeating the analysis in Section 5, the metric-dependent factor in the Laplacian becomes

\[
\frac{1}{h_{i-1}} + \frac{1}{h_i(1 + \frac{1}{i})} = \frac{1}{\epsilon^2} \beta^{-1}(i); \quad \beta^{-1}(i) = \left\{ \begin{array}{ll}
  \frac{i(i^2 + 1)}{(i^2 - 1)^2} & \text{if } i \text{ even} \\
  \frac{i}{i+1} & \text{if } i \text{ odd}
  \end{array} \right.
\]

in place of (5.1). We can take the continuum limit with the leading order $\beta^{-1}(i) = \frac{i}{i+1}$ and the parallel analysis (1) in Section 5. Ignoring the $(-1)^i$ differential term as we did before, gives that $\Box f = 4mEf$ becomes the Airy equation

\[
\frac{d^2 f}{dx^2} + 4mExf = 0 \quad (6.1)
\]

with a real decaying cosine-wave-like solution if $4mE > 0$ and, say, $f(0) = 1, f'(0) = 0$. In addition, we can expect ripples in the discrete solution visible for small $i$ due to the even values of $\beta^{-1}(i)$ and due to the $(-1)^i$ differential term as discussed in Section 5. Meanwhile, the QFT action depends on the measure $\mu_i$ and if we take the obvious choice $\mu_i = h_i$ then this cancels the $1/x$ in the continuum limit and we obtain a multiple of the free field action (again ignoring the suppressed $(-1)^i$ term in the Laplacian), which is perhaps reasonable as the curvature is zero.

Next, we consider metrics near to the flat one in a conformal sense,

\[
h_i = h^\text{flat}_i g_i; \quad \rho_i = \rho^\text{flat}_i \eta_i; \quad \eta_i = \frac{g_{i+1}}{g_i}
\]

with $h^\text{flat}_i$ from Proposition 6.1 for $s = 1$. Then a calculation with $h_1 = \epsilon^3$ and $i = x/\epsilon$ as above and working to leading order in $\epsilon$, we have

\[
S(x) = \frac{1}{2h^\text{flat}_i g_i} \left( \eta_{i-1}(\eta_{i-2} - \eta_{i-1}) + \frac{1}{\eta_{i-1}} - \frac{1}{\eta_i} \right) = \frac{1}{4\epsilon x g} \left( \frac{1}{\eta^3} - 1 \right) \frac{d\eta}{dx}
\]

where

\[
\eta_i = 1 + \frac{g_{i+1}}{g_i} = 1 + \epsilon g^{-1} \frac{dg}{dx} + O(\epsilon^2).
\]
Putting this in, we have to leading order in $\epsilon$,

$$S(x) = \frac{3\epsilon}{4x^2} g^{-2} \frac{dg}{dx} \left( g^{-1} \frac{dg}{dx} \right) = \frac{3\epsilon}{8x} e^{-\psi} \left( \frac{d\psi}{dx} \right)^2$$

if we write $g(x) = e^{\psi(x)}$ for a real scalar field $\psi$.

We briefly consider the Einstein-Hilbert action for such metric fluctuations near the scalar-flat metric, expressed in $\psi(x)$. We need to fix the measure $\mu_i$ in

$$S[h] = \sum_i \mu_i S(i)$$

and based on experience in [20] for $\mathbb{Z}$, we take $\mu_i = h_i = h^{flat}_i g_i$. The theory behind how to choose this measure is not clearly understood, but we expect some power of the metric. Classically, one would have $\sqrt{\text{det}(g)}$ for the measure but in [20] it gave more reasonable answers and related to $\Omega^1$ there being 2-dimensional. Our $\Omega^1$ is not exactly a free module but is more like this far from the boundary. In this case,

$$S[\psi] = \sum_{i=1}^{\infty} h^{flat}_i g_i S(i) \to \frac{3\epsilon^3}{4} \int_0^\infty dx \frac{d\psi}{dx} \left( \frac{d\psi}{dx} \right)^2 = -\frac{3\epsilon^3}{4} \left( \frac{d\psi}{dx} \right)(0^+)$$

to leading order as $\epsilon \to 0$, given that $h^{flat}_i = 2\epsilon^2 x$ to leading order and assuming our fields decay at $\infty$. The $\epsilon^3$ can be absorbed in $\mu$ or in a coupling constant in front of the action. The action here is topological and appears to amount to a trivial theory on the boundary at $x = 0^+$ (approaching from the bulk) but could be more interesting before we take the continuum limit and if we look more closely at the boundary.

6.2. Curvatures for $A_n$. We proceed from the general expression for the curvature and put in the connection in Corollary 4.4. We define

$$c_i := (i)_q (-1)^i s + (i)^2_q - (i - 1)_q (i + 1)_q, \quad d_i := (i)^2_q - (i - 1)_q (i + 1)_q$$

Then the curvature for $s = \pm 1$ is

$$\text{R}_\nabla a_1 = 0$$

$$\text{R}_\nabla a'_1 = \left( \rho_1 (3)_q \left( (3)_q (2)_q^2 c_2 - (2)_q^2 (3)_q \right) a'_1 \wedge a_1 \otimes a_2 \right.$$

$$\left. + \frac{(2)_q + s}{\rho_1 (3)_q s} - (2)_q + \frac{(3)_q}{(2)_q} \right) a'_1 \wedge a_1 \otimes a'_1$$

$$\text{R}_\nabla a_i = \left( -\rho_{i-1} \left( \frac{(i - 1)_q (i + 1)_q}{(i)_q} - \frac{(-1)^i s (i - 1)_q}{c_i} \right) - \frac{(i + 1)_q}{(i)_q} + \frac{(i + 2)_q}{(i + 1)_q} \right.$$

$$\left. + \rho_i \frac{(-1)^i s (i)_q}{(i + 1)_q} \left( \frac{(i + 1)_q}{(i)_q} + (1)^i s \right) \right) a_i \wedge a'_i \otimes a_i$$

$$\left. + (-1)^i s \left( \frac{(-1)^i s + (i)_q}{(i - 1)_q (i)_q} + \frac{1}{\rho_{i-1}} \frac{(-1)^i s + (i)_q}{(i + 1)_q (i - 1)_q} \right) a_i \wedge a'_i \otimes a'_{i-1} \right.$$

$$\left. \right)$$

$$\text{R}_\nabla a'_i = -\frac{1}{(i + 1)_q} \left( \frac{1}{\rho_{i-1}} \frac{((-1)^i s (i)_q + 1)c_i}{(i - 1)_q} + \frac{(i + 1)_q^2}{(i)_q} - (i + 2)_q \right)$$
\[ 2\text{Ricci} = \frac{1}{\rho_l} \left( (i + 1)_q \frac{((1 - s(-1)^i(i + 1)_q) c_{i+1}}{(i)_q(i + 2)_q} \right) a'_i \otimes a_i' + \frac{1}{n - 1} \left( a'_{n-1} \right) \otimes a_{n-1}' \]

\[ R^\nu a'_{n-1} = \frac{1}{2h_{n-2}} \left( \rho_{n-2} \frac{(n - 1)_q(s(n - 1)_q)}{(n - 2)_q a_{n-2}'} + (n - 3)_q a_{n-2} \right) + \rho_{n-2} \frac{(n - 2)_q(1 + (-1)^n s(n - 1)_q)}{(n - 1)_q a_{n-1}'} + \frac{1}{h_{n-1}} \left( \frac{(n - 1)_q d_{n-1} - d_n a_{n-1}'}{2(n - 1)_q a_{n-1}'} \right) \]

\[ R^\nu a'_{n-1} = 0 \]

for \( i = 2, \ldots, n - 2 \). The Ricci tensor is then
for \( n > 3, i = 3, \cdots, n - 2 \). There are no Ricci flat solutions. The Ricci scalar is then

\[
S(1) = -\frac{1}{2h_1(2)q} \left( \frac{(2)q + s}{\rho_1(3)q} - (2)q + (3)q \right)
\]

\[
S(2) = \frac{1}{2(3)q^2\rho_2} \left( \frac{1}{(4)q} \frac{(3)q}{(3)q} - (3)q \right) + d_3 s + \frac{(2)q ((2)q + s) c_2}{\rho_1}
\]

\[
S(i) = \frac{1}{2\rho_i} \left( \rho_{i-1} \frac{(i + 1)q}{(i)q} - \frac{(i)q}{(i - 1)q} \right) + \rho_{i-2}\rho_{i-1} \left( \frac{(i-1)s(i-2)q(i-1)q}{c_{i-1}(i-1)^2} - \frac{(i-2)q(i)q}{c_{i-1}(i-1)^2} \right)
\]

\[
+ \rho_{i-1} \frac{(i+1)q(i-1)q(1 + (i-1)s(i)q)}{c_{i}(i)} + 1 - \frac{(i+2)q(i)q}{(i+1)q}
\]

\[
S(n - 1) = \frac{1}{2h_{n-2}} \left( \frac{1}{(n - 1)q} - \frac{(n - 1)q}{(n - 2)q} \right)
\]

\[
+ \rho_{n-2} \frac{(-1)^n s(n - 2)q((n - 1)q + (-1)^n s)}{(n - 1)q^2 c_{n-1}}
\]

\[
+ \rho_{n-3} \frac{(-1)^n s(n - 3)q(n - 1)q}{(n - 2)q^2 c_{n-2}} - \frac{(n - 3)q(n - 1)q}{(n - 2)q^2 c_{n-2}}
\]

\[
S(n) = \frac{1}{2h_{n-1}} \left( \frac{(-1)^n s(n - 1)q d_n - d_n d_{n-1}}{(n - 1)q c_{n-1}} - \rho_{n-2} \frac{(n - 2)q(1 + (-1)^n s(n - 1)q)}{(n - 1)q c_{n-1}} \right)
\]

These formulae show how the geometry of \( A_n \) \( q \)-deforms that of \( \mathbb{N} \). As with Proposition 6.1, there is again a unique metric \( h^{flat} \) up to overall scale such that \( S = 0 \).

We conclude with a small example for \( n = 3, s = 1 \). Then \((2)q = \sqrt{2}, (3)q = 1\) and we obtain

\[
S = \left\{ 1, 4 \left( \frac{1}{h_1} - \frac{3\sqrt{2} + 4}{h_2} \right), 0, 4 \left( \frac{3 - 2\sqrt{2}}{h_1} - \frac{\sqrt{2}}{h_2} \right) \right\}
\]

at the three points. This vanishes at \( h_2 = (4 + 3\sqrt{2})h_1 \). If we reconsider QFT of a scalar field for this value, the determinant of the operator in the action becomes

\[
\det(B) = -\frac{\mu_1 \mu_2 \mu_3}{h_1^4} \left( h_1 m^2 \left( h_1 m^2 + 3\sqrt{2} - 8 \right) + 8 \left( 5\sqrt{2} - 7 \right) \right) + 48\sqrt{2} - 68
\]

in place of our previous result in Section 5.2, i.e., of the same form but more complicated.

Next, if we write \( \mu_1 = h_1, \mu_3 = h_2 \) then get for the Einstein-Hilbert action

\[
S[\rho] := \sum_i \mu_i S(i) = \frac{1}{4} \left( (3 - 2\sqrt{2}) \rho - \frac{(3\sqrt{2} + 4)}{\rho} - \sqrt{2} + 1 \right); \quad \rho = \frac{h_2}{h_1}
\]

but note that we can get any coefficients for the two powers of \( \rho \) by scaling \( \mu_i \). Sticking with the obvious values, if we ignore the constant then

\[
Z = \int dh_1 dh_2 e^{\frac{1}{c}(\tilde{\phi} - \rho)} = \int_0^\infty h_1 dh_1 \int_{-\infty}^\infty d\rho e^{\frac{1}{c}(\tilde{\phi} - \rho)}; \quad c = 24 + 17\sqrt{2}
\]

for a real positive coupling constant \( G \). The first integral is an infinite volume which we ignore and the second has a divergence at \( \rho = 0 \) so we cut it off at \( \epsilon > 0 \).
Using l'Hopital’s rule, we then find
\[
\langle \rho^m \rangle = \int_0^\infty d\rho e^{\frac{1}{2}(\frac{2}{\rho} - \rho)} \rho^m = \begin{cases} 
0 & m > 0 \\
1 & m = 0 \\
\infty & m < 0 
\end{cases}
\]
which is not too interesting.

Letting \( \mu_1, \mu_3 \) be a convex linear combination of the \( h_i \) does not allow one to change the sign of \( c \) here, leading to the same behaviour. But if we allow ourselves more flexibility in the choice of \( \mu_i \) then we can change the sign. For example,
\[
\mu_1 = h_1 - h_2, \quad \mu_3 = h_1 + h_2; \quad S[\rho] = 8 - 2(\sqrt{2} - 1)(\frac{2}{\rho} + \rho)
\]
which for suitable \( G \) gives the same as above but with \( c = -2 \). This time, the integrals converge,
\[
\int_0^\infty d\rho e^{-\frac{1}{2}(\frac{2}{\rho} + \rho)} \rho^m = 2^{\frac{m+3}{2}} K_{m+1} \left( \frac{2\sqrt{2}}{G} \right)
\]
for BesselK functions. These diverge as \( G \to 0 \) and as \( G \to \infty \), but the expectation behave like
\[
\langle \rho^m \rangle \to \begin{cases} 
2^{\frac{m}{2}} & G \to 0 \\
\infty & G \to \infty
\end{cases}
\]
while, for example, the relative uncertainty has a limit
\[
\frac{\Delta \rho}{\langle \rho \rangle} = \frac{\sqrt{\langle \rho^2 \rangle} - \langle \rho \rangle^2}{\langle \rho \rangle} \to 1; \quad \frac{\langle \rho^3 \rangle}{\langle \rho \rangle^2} \to 2
\]
as \( G \to \infty \) (the ‘strong gravity’ limit). This looks more reasonable for a theory of quantum gravity on 3 points in the sense that it follows the same pattern as other models\cite{21, 2, 17}, but the choice of \( \mu \) here requires to be understood much better in order to justify such a change. In particular, since we introduced the \( \mu_i \) as a measure of ‘integration’ for the action, we might expect \( \mu_i > 0 \) which for the above would need \( \rho < 1 \). In can truncate the integrals to this region; they still converge but are no longer given in closed form by Bessel functions.

7. Concluding remarks

We have found that the quantum geometry of a finite discrete lattice in 1 dimensions is intrinsically \( q \)-deformed, but from solving for a QLC and not from assuming a quantum group. Also note that the structure of \( \Omega \) is rather nontrivial as \( \Omega^1, \Omega^2 \) are not free modules, i.e. the differential structure is not parallelisable with a global basis. In fact, the exterior algebra \( \Omega \) is a quotient of the preprojective algebra of Dynkin type \( A_n \), which is intimately connected to the representation theory and structure of the Lie algebra \( sl_{n+1} \). This suggests that there should be a reduced quantum group \( u_q(sl_{n+1}) \) (in some conventions) at play with \( q \) an even \( 2(n + 1) \)-th root of unity at play. This should be explored further as well as implied links to the physics of spin-chains as quantum integrable systems. Finally, the mathematical structure suggests that it should also be interesting to look at quantum Riemannian
geometry (and quantum spin ‘chains’, reduced quantum groups) for the other types of Dynkin graphs.

On the physics side, our emergence of \( q \)-deformation out of discretisation together with the belief that \( q \)-deformation in 2+1 quantum gravity with points sources corresponds to a cosmological constant (see [24] for an overview) suggests that switching on the cosmological constant could be equivalent to discretisation of quantum geometry. The models are very different, but if we naively equate \( q = e^{\frac{i\pi}{\lambda_c}} \) with \( q = e^{-\frac{1}{2\pi^2}} \), where \( \lambda_c \) is the cosmological constant length scale and \( \lambda_p \) the Planck length, then we get \( n \sim \pi \lambda_c / \lambda_p \). Here, \( \lambda_c = 1/\sqrt{\Lambda} \) for positive cosmological constant \( \Lambda \). If a similar ideas were to hold for our observed 3+1 universe, where it appears that \( \lambda_c \sim 10^{26} \text{m} \), this would give \( n \sim 3 \times 10^{61} \). This is merely for comparison and we not suggesting any particular role for the QRG of the \( A_n \) graph with this many nodes in 2+1 quantum gravity. Morally speaking, however, the idea that we discretise spacetime as a way to regularise quantum gravity, seems to match up with switching on the cosmological constant and could explain why the latter is very small compared to the Planck scales and yet nonzero. This would agree with other evidence [25] that it could arise as a consequence of quantum gravity effects that render spacetime noncommutative. Looking more specifically at the role of \( A_n \) in quantum gravity, we speculate that this could perhaps be relevant to the geometry of an open string of \( n \) Planck lengths.

There is also a lot to be done on the QRG of \( \mathbb{N} \). This we found to be rational, i.e. if the metric coefficients are rational then so are all Christoffel symbols etc. We found that there were non-continuum \((-1)^i\) terms which have no continuum limit but which would be pushed to the origin at \( x = 0 \) in the limit. In that case the unique (up to scale) flat metric on \( \mathbb{N} \) led for the Laplacian to the Airy equation in the bulk, but the behaviour around \( x = 0 \) needs much more attention. This would also affect our conclusion for the field theory of a conformal factor \( e^{\psi} \) on this flat metric, which we found to be topological and hence reducing to the value at the origin, but again on the assumption of a particular choice \( \mu_i = h_i \) of the measure for ‘integration’ prior to taking the continuum limit. There is as of yet no general theory for this measure as well as a lack of a variational calculus in general. Also, we only examined the limit of scalars but the limit of the Ricci tensor etc can also be studied provided we limit the differential structure correctly. For \( \mathbb{Z} \), this is 2-dimensional and limits to a certain noncommutative 2-dimensional calculus on the line [2] with the classical calculus as a quotient. This is the reason why we have any curvature in the first place. For \( \mathbb{N} \), the calculus is more complicated particularly around the origin, but in the bulk we would expect a similar limit. It also remains to look at particle creation and other possible quantum gravity effects [26], such as found for \( \mathbb{Z} \) in [20] but now adapted to \( \mathbb{N} \), as well as use both \( A_n, \mathbb{N} \) as parts of higher-dimensional models as in [3] for \( \mathbb{Z}_n \).

Finally, we only took a first look at quantum theory and quantum gravity in a functional integral approach on \( A_n \) for \( n = 3 \), which so far did not appear too interesting for the ‘obvious’ choice of \( \mu_i \), but the quantum gravity model looked more like other models [21, 2, 17] if we allowed more ourselves more flexibility in the choice of \( \mu_i \). This phenomenon as well as general \( n \) should also be examined.
Because of the $q$-deformation, these models also have a rich structure in the numerics which, however, manages to stay real for the particular root of unity in the picture. These are some directions for further work.

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Queen Mary University of London, School of Mathematical Sciences, Mile End Rd, London E1 4NS, UK

Email address: j.n.argotaquiroz@qmul.ac.uk, s.majid@qmul.ac.uk