Zero-sum $K_m$ over $\mathbb{Z}$ and the story of $K_4$

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Abstract

We prove the following results solving a problem raised in Caro-Yuster [10]. For a positive integer $m \geq 2$, $m \neq 4$, there are infinitely many values of $n$ such that there is a weighting function $f : E(K_n) \to \{-1, 1\}$ (and hence a weighting function $f : E(K_n) \to \{-1, 0, 1\}$), such that $\sum_{e \in E(K_n)} f(e) = 0$ but, for every copy $H$ of $K_m$ in $K_n$, $\sum_{e \in E(H)} f(e) \neq 0$. On the other hand, for every integer $n \geq 5$ and every weighting function $f : E(K_n) \to \{-1, 1\}$ such that $|\sum_{e \in E(K_n)} f(e)| \leq (n^2) - h(n)$, where $h(n) = 2(n+1)$ if $n \equiv 0 \pmod{4}$ and $h(n) = 2n$ if $n \equiv 0 \pmod{4}$, there is always a copy $H$ of $K_4$ in $K_n$ for which $\sum_{e \in E(H)} f(e) = 0$, and the value of $h(n)$ is sharp.

1 Introduction

Our main source of motivation is a recent paper of Caro and Yuster [10], extending classical zero-sum Ramsey theory to weighting functions $f : E(K_n) \to \{-r, -r+1, \ldots, 0, \ldots, r-1, r\}$ seeking zero-sum copies of a given graph $H$ subject to the obviously necessary condition that $|\sum_{e \in E(K_n)} f(e)|$ is bounded away from $r\binom{n}{2}$, or even in the extreme case where $|\sum_{e \in E(K_n)} f(e)| = 0$.

In zero-sum Ramsey theory, one studies functions $f : E(K_n) \to X$, where $X$ is usually the cyclic group $\mathbb{Z}_k$ or (less often) an arbitrary finite abelian group. The goal is to show that, under some necessary divisibility conditions imposed on the number of the edges $e(H)$ of a graph $H$ and for sufficiently large $n$, there is always a zero-sum copy of $H$, where by a zero-sum copy of $H$ we mean a copy of $H$ in $K_n$ for which $\sum_{e \in E(H)} f(e) = 0$ (where 0 is the neutral element of $X$). For several results concerning zero-sum Ramsey theory for graphs
see \[1, 3, 4, 5, 6, 8, 11, 12\], for zero-sum Ramsey problems concerning matrices and linear algebraic techniques see \[2, 9, 13, 14\].

The following notation was introduced in \[10\] and the following zero-sum problems over \(\mathbb{Z}\) were considered. For positive integers \(r\) and \(q\), an \((r, q)\)-weighting of the edges of the complete graph \(K_n\) is a function \(f : E(K_n) \to \{-r, \cdots, r\}\) such that \(|\sum_{e \in E(K_n)} f(e)| < q\). The general problem considered in \[10\] is to find nontrivial conditions on the \((r, q)\)-weightings that guarantee the existence of certain bounded-weight subgraphs and even zero-weighted subgraphs (also called zero-sum subgraphs). So, given a subgraph \(H\) of \(K_n\), and a weighting \(f : E(K_n) \to \{-r, \cdots, r\}\), the weight of \(H\) is defined as \(w(H) = \sum_{e \in E(H)} f(e)\), and we say that \(H\) is a zero-sum graph if \(w(H) = 0\). Finally, we say that a weighting function \(f : E(K_n) \to \{-1, 1\}\) is zero-sum-\(H\) free if it contains no zero-sum copy of \(H\).

Among the many results proved in \[10\], the following theorem and open problem are the main motivation of this paper.

**Theorem 1.1** (Caro and Yuster, \[10\]). For a real \(\epsilon > 0\) the following holds. For \(n\) sufficiently large, any weighting \(f : E(K_n) \to \{-1, 0, 1\}\) with \(\sum_{e \in E(K_n)} f(e)\) not of the form \(m = 4d^2\), there are infinitely many integers \(n\) for which there is a weighting \(f : E(K_n) \to \{-1, 0, 1\}\) with \(\sum_{e \in E(K_n)} f(e) = 0\) without a zero-sum copy of \(K_m\).

The authors posed the following complementary problem:

**Problem 1** (Caro and Yuster, \[10\]). For an integer \(m = 4d^2\), is it true that, for \(n\) sufficiently large, any weighting \(f : E(K_n) \to \{-1, 0, 1\}\) with \(\sum_{e \in E(K_n)} f(e) = 0\) contains a zero-sum copy of \(K_m\)?

The main result in this paper is a negative answer to the above problem for any \(m \geq 2\) except for \(m = 4\) already for weightings of the form \(f : E(K_n) \to \{-1, 1\}\) with \(\sum_{e \in E(K_n)} f(e) = 0\). On the other hand, concerning the study of the existence of zero-sum copies of \(K_4\), we prove a result analogous to Theorem 1.1 where the range \(\{-1, 1\}\) instead of \(\{-1, 0, 1\}\) is considered. Finally, we show that Theorem 1.1 can neither be extended to wider ranges. To be more precise, we gather our results in the following theorem.

**Theorem 1.2.**

1. For any positive integer \(m \geq 2, m \neq 4\), there are infinitely many values of \(n\) such that the following holds: There is a weighting function \(f : E(K_n) \to \{-1, 1\}\) with \(\sum_{e \in E(K_n)} f(e) = 0\) which is zero-sum-\(K_m\) free.

2. Let \(n\) be an integer such that \(n \geq 5\). Define \(g(n) = 2(n + 1)\) if \(n \equiv 0\) (mod 4) and \(g(n) = 2n\) if \(n \not\equiv 0\) (mod 4). Then, for any weighting \(f : E(K_n) \to \{-1, 1\}\) such that \(|\sum_{e \in E(K_n)} f(e)| \leq \binom{n}{2} - g(n)\), there is a zero-sum copy of \(K_4\).

3. There are infinitely many values of \(n\) such that the following holds: There is a weighting function \(f : E(K_n) \to \{-2, -1, 0, 1, 2\}\) with \(\sum_{e \in E(K_n)} f(e) = 0\) which is zero-sum-\(K_4\) free.

Theorem 1.2 together with the above Theorem 1.1 do not only solve Problem 1, they also supply a good understanding of the situation concerning \(K_4\) as the value of \(g(n)\) is
sharp and the upper bound \((1 - \epsilon)n^2/6\) in Theorem 1.1 is nearly sharp, as already observed in [10].

We will use the following notation. Given a weighting \(f : E(K_n) \to \{-r, \cdots, r\}\) and \(i \in \{-r, \cdots, r\}\), denote by \(E(i)\) the set of the \(i\)-weighted edges, that is, \(E(i) = f^{-1}(i)\) and define \(e(i) = |E(i)|\). Given a vertex \(x \in V(K_n)\) we use \(deg_i(x)\) to denote the number of \(i\)-weighted edges incident to \(x\), that is, \(deg_i(x) = |\{u : f(xu) = i\}|\).

In Section 2 we will prove instances 2 and 3 of Theorem 1.2 corresponding to the study of the existence of zero-sum copies of \(K_4\). In order to prove instance 2, we will use an equivalent formulation consequence of the following remark.

### Remark 1.3

A weighting \(f : E(K_n) \to \{-1, 1\}\) satisfies \(|\sum_{e \in E(K_n)} f(e) - g(n)\) if and only if \(\min\{e(-1), e(1)\} \geq \frac{1}{2}g(n)\).

The remark follows from the fact that \(e(1) + e(-1) = \binom{n}{2}\), which implies \(|\sum_{e \in E(K_n)} f(e)| = |e(1) - e(-1)| = \max\{e(-1), e(1)\} - \min\{e(-1), e(1)\} = \binom{n}{2} - 2\min\{e(-1), e(1)\}\).

In Section 2 we will also prove that instance 2 of Theorem 1.2 is best possible by exhibiting, for each \(n \geq 5\), a weighting function \(f : E(K_n) \to \{-1, 1\}\) with \(\min\{e(-1), e(1)\} = \frac{1}{2}g(n) - 1\) and no zero-sum copies of \(K_4\). Moreover, we will characterize the extremal functions.

Finally, relying heavily on Pell equations and some classical biquadratic Diophantine equations, in Section 3 we will prove instance 1 of Theorem 1.2 corresponding to the study of the existence of zero-sum copies of \(K_m\) in 0-weighted weightings, where \(m \neq 4\).

## 2 The case of \(K_4\)

We will use standard graph theoretical notation to denote particular graphs. Having said this, \(K_{1,3}\) will stand for the star with three leaves, \(K_3 \cup K_1\) for the disjoint union of a triangle and a vertex, \(P_k\) for a path with \(k\) edges, and \(C_k\) for a cycle with \(k\) edges.

A weighting function \(f : E(K_n) \to \{-1, 1\}\) is zero-sum-\(K_4\) free if and only if the graph induced by \(E(-1)\) (or equivalently \(E(1)\)) is \(\{K_{1,3}, K_3 \cup K_1, P_3\}\)-free (in the induced sense). The following lemma, which characterizes the \(K_3\)-free subclass of the family of \(\{K_{1,3}, K_3 \cup K_1, P_3\}\)-free graphs, will be useful in proving the forthcoming results. We define

\[
h(n) = \begin{cases} 
  n + 1, & \text{if } n \equiv 0 \pmod{4}, \\
  n, & \text{otherwise}.
\end{cases}
\]

### Lemma 2.1

Let \(G\) be a \(\{K_{1,3}, K_3, P_3\}\)-free graph on \(n\) vertices. Then each component of \(G\) is isomorphic to one of \(C_4, K_1, K_2\) or \(P_2\). Moreover, \(e(G) \leq h(n) - 1\), and equality holds if and only if \(G \cong J \cup \bigcup_{i=1}^{q} C_4\), where \(J \in \{\emptyset, K_1, K_2, P_2\}\) and \(q = \lfloor \frac{n}{4} \rfloor\).

**Proof.** Let \(J\) be a connected component of \(G\). If \(J\) has at most 3 vertices, then it is easy to see that \(J \in \{K_1, K_2, P_2\}\). So assume that \(J\) has at least 4 vertices. Then, since \(J\) is \(\{K_3, P_3\}\)-free, we infer that \(J\) has no vertex of degree larger than 2 and so we can deduce that \(J \cong C_4\). Further, we note that \(e(J) = |J|\) if \(J \cong C_4\), and \(e(J) = |J| - 1\) otherwise. This implies that, among all \(\{K_{1,3}, K_3, P_3\}\)-free graphs on \(n\) vertices, \(G\) has maximum number of edges if and only if \(G \cong J \cup \bigcup_{i=1}^{q} C_4\), where \(J \in \{\emptyset, K_1, K_2, P_2\}\) and \(q = \lfloor \frac{n}{4} \rfloor\). Since, clearly, \(e(J \cup \bigcup_{i=1}^{q} C_4) = h(n) - 1\), the proof is complete. \(\square\)
Lemma 2.2. Let $n \geq 5$ and $f : E(K_n) \to \{−1, 1\}$ be a zero-sum-$K_4$ free coloring. Let $G_{−1}$ and $G_1$ be the graphs induced by $E(−1)$ and $E(1)$, respectively. Then at least one of $G_{−1}$ or $G_1$ is triangle-free.

Proof. Suppose for contradiction that both $G_{−1}$ and $G_1$ have a triangle. Let $abc$ be a triangle in $G_{−1}$ and $uvw$ a triangle in $G_1$. Suppose first that $abc$ and $uvw$ have a vertex in common, say $a = u$. Consider the graph $J$ induced by the $−1$-edges among vertices in $\{a, b, c, v, w\}$. If a vertex $x \in \{b, c\}$ is neighbor of both $v$ and $w$, then $\{x, v, w, a\}$ would induce a $K_{1,3}$ in $G_{−1}$, which is not possible. If no vertex $x \in \{b, c\}$ is adjacent to some $y \in \{v, w\}$, then $\{a, b, c, y\}$ would induce a $K_3 \cup K_1$ in $G_{−1}$, which again is not possible. Hence, $\{b, c, v, w\}$ induces two independent edges. But then $\{b, c, v, w\}$ induces a $P_3$ in $G_{−1}$, a contradiction. Hence, we can assume that any pair of triangles such that one has only $−1$-edges and the other only $1$-edges are vertex disjoint. This implies that from any vertex in $\{u, v, w\}$ there is at most one $−1$-edge to vertices from $\{a, b, c\}$. Analogously, from any vertex in $\{a, b, c\}$ there is at most one $1$-edge to vertices from $\{u, v, w\}$. But this implies that there are at most 6 edges between $\{a, b, c\}$ and $\{u, v, w\}$, which is false. Since in all cases we obtain a contradiction, we conclude that at least one of $G_{−1}$ or $G_1$ is triangle-free. 

By Remark 1.3 the next result is equivalent to instance 2 of Theorem 1.2.

Theorem 2.3. Let $n \geq 5$ and $f : E(K_n) \to \{−1, 1\}$ such that $\min\{e(−1), e(1)\} \geq h(n)$. Then there is a zero-sum $K_4$.

Proof. Let $f : E(K_n) \to \{−1, 1\}$ be such that $\min\{e(−1), e(1)\} \geq h(n)$ and suppose for contradiction that it has no zero-sum $K_4$. Let $G_{−1}$ and $G_1$ be the graphs induced by $E(−1)$ and $E(1)$, respectively. Then both $G_{−1}$ and $G_1$ are $\{K_{1,3}, K_3 \cup K_1, P_3\}$-free graphs. By Lemma 2.2, $G_{−1}$ or $G_1$ is $K_3$-free. So we may assume, without loss of generality, that $G_{−1}$ is triangle-free. It follows by Lemma 2.1 that $e(−1) = |E(G_{−1})| \leq h(n) − 1$, which is a contradiction to the hypothesis.

The following theorem shows that Theorem 2.3 is best possible and characterizes the extremal zero-sum-$K_4$ free weightings. We will use Mantel’s Theorem, that any graph on $n$ vertices and at least $\frac{n^2}{4} + 1$ edges contains a copy of $K_3$.

Theorem 2.4. Let $n \geq 5$ and $f : E(K_n) \to \{−1, 1\}$ such that $e(1) = h(n) − 1$. Then $f$ is zero-sum-$K_4$ free if and only if the graph induced by $E(1)$ is isomorphic to $J \cup \bigcup_{i=1}^{q} C_4$, where $J \in \{\emptyset, K_1, K_2, P_2\}$ and $q = \lfloor \frac{n}{4} \rfloor$.

Proof. If the graph induced by $E(1)$ is isomorphic to $J \cup \bigcup_{i=1}^{q} C_4$, where $J \in \{\emptyset, K_1, K_2, P_2\}$ and $q = \lfloor \frac{n}{4} \rfloor$, it is easy to check that $f$ is zero-sum-$K_4$ free. Conversely, let $f$ be zero-sum-$K_4$ free. Then the graphs $G_{−1}$ and $G_1$ induced by $E(−1)$ and $E(1)$, respectively, are both $\{K_{1,3}, K_3 \cup K_1, P_3\}$-free. If $n = 5$, it is easy to check that the only $\{K_{1,3}, K_3 \cup K_1, P_3\}$-free graph with $h(5) − 1 = 4$ edges is isomorphic to $C_4 \cup K_1$, and so we are done. Hence, we may assume that $n \geq 6$. Observe that

$$e(−1) = \frac{n(n−1)}{2} − h(n) + 1 \geq \frac{n(n−1)}{2} − n = \frac{n(n−3)}{2},$$

whose right-hand side is at least $\frac{n^2}{4}$ for $n \geq 6$. Hence, by Mantel’s Theorem, $G_{−1}$ has a triangle, and, by Lemma 2.2, this implies that $G_1$ is triangle-free. It follows that $G_1$ is a
\{K_{1,3}, K_3, P_3\}\text{-free graph on } n \text{ vertices and with } h(n) - 1 \text{ edges. Thus, with Lemma 2.1, we obtain that } G_1 \text{ is isomorphic to } J \cup \bigcup_{i=1}^{q} C_4, \text{ where } J \in \{\emptyset, K_1, K_2, P_2\} \text{ and } q = \lceil \frac{n}{4} \rceil, \text{ and we are done.}\]

The following theorem is instance 2 from Theorem 1.2. It shows that, whenever we take a wider range for the weighting function \( f \), we cannot hope for a result as in Theorem 2.3 anymore.

**Theorem 2.5.** There are infinitely many values of \( n \) such that the following holds: There is a weighting function \( f : E(K_n) \to \{-2, -1, 0, 1, 2\} \) with \( \sum_{e \in E(K_n)} f(e) = 0 \) which is zero-sum-\( K_4 \) free.

**Proof.** Let \( X \cup Y \) be a partition of the vertex set of \( K_n \) and consider the weighting function \( f : E(K_n) \to \{-2, -1, 0, 1, 2\} \) such that

\[
f(uv) = \begin{cases} 
-2, & \text{if } u, v \in X \\
1, & \text{if } u, v \in Y \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly, \( f \) is zero-sum-\( K_4 \) free. On the other hand, \( \sum_{e \in E(K_n)} f(e) = 0 \) if and only if

\[
-2 \frac{|X|(|X| - 1)}{2} + |Y|(|Y| - 1) = 0,
\]

which is equivalent to \( (2|Y| - 1)^2 - 2(2|X| - 1)^2 = -1 \). Hence, solving the latter equation is equivalent to solve the following Pell’s equation

\[
y^2 - 2x^2 = -1, \tag{1}
\]

for (odd) integers \( x = 2|X| - 1 \) and \( y = 2|Y| - 1 \). It is well-known that the Diophantine equation \( y^2 - 2x^2 = \pm 1 \) has infinitely many solutions given by

\[
x_k = \frac{a^k - b^k}{a - b} = \frac{a^k - b^k}{2\sqrt{2}}, \quad y_k = \frac{a^k + b^k}{2},
\]

where \( a = 1 + \sqrt{2}, b = 1 - \sqrt{2} \) and \( k \in \mathbb{N} \). Moreover, since \( y_k^2 - 2x_k^2 = (-1)^k \), the solutions for equation (1) are the pairs \((x_k, y_k)\) where \( k \) is odd. Observe also that, for odd \( k \), \( x_k \) and \( y_k \) are odd, too. Hence, each odd \( k \) gives us a solution \((\frac{a^k + 1}{2}, \frac{a^k + 1}{2})\) for \((|X|, |Y|)\) and thus for \( n = \frac{x_k + y_k}{2} + 1 \) and we are done.

For the sake of comprehension, let us compute small values of \( n = \frac{x_k + y_k}{2} + 1 \) and exhibit how the partition \((|X|, |Y|) = (\frac{x_k + 1}{2}, \frac{y_k + 1}{2})\) gives a zero-sum weighting function \( f \) as described in the theorem. Recall that we only want to consider solutions for odd values of \( k \). So we have \((x_1, x_3, x_5, \ldots) = (1, 5, 29, \ldots)\) and \((y_1, y_3, y_5, \ldots) = (1, 7, 41, \ldots)\), and the corresponding sequence of \( n \)'s is \((2, 7, 36, \ldots)\). The case of \( n = 2 \) is not interesting for vaquity reasons. For \( n = 7 \), the partition is \((|X|, |Y|) = (3, 4)\), thus there will be \((\frac{3}{2})\) edges weighted with \(-2\), \((\frac{3}{2})\) edges weighted with \(1\) and the rest of edges weighted with \(0\), adding up to zero. For \( n = 36 \), the partition is \((|X|, |Y|) = (15, 21)\), and the sum of weighted edges is \(-2\binom{15}{2} + 1\binom{21}{2} = (-2) \cdot 105 + 1 \cdot 210 = 0.\)
3 The case of $K_m, m \neq 4$

A balanced $\{-1, 1\}$-weighting function $f : E(K_n) \to \{-1, 1\}$ is a function for which $e(-1) = e(1)$. In Section 2, we prove that, for $n \geq 5$, any function $f : E(K_n) \to \{-1, 1\}$ with sufficiently many edges assigned to each type contains a zero-sum $K_4$. In this section, we prove that this is not true for $K_m$ with $m \in \mathbb{N} \setminus \{1, 4\}$. In other words, we exhibit, for infinitely many values of $n$, the existence of a balanced weighting function $f : E(K_n) \to \{-1, 1\}$ without a zero-sum copies of $K_m$, where $m \neq 1, 4$. In order to define those functions, consider first the following Pell equation:

$$8x^2 - 8x + 1 = y^2.$$  \hfill (2)

It is well known that such a Diophantine equation has infinitely many solutions given by the recursion

$$(x_1, y_1) = (1, 1),
$$

$$(x_2, y_2) = (3, 7),
$$

$$y_k = 6y_{k-1} - y_{k-2}, \quad x_k = \frac{y_k + x_{k-1} + 1}{3}.$$

**Lemma 3.1.** Let $n$ be a positive integer and consider the complete graph $K_n$ and a partition $V(K_n) = A \cup B$ of its vertex set. Then the function $f : E(K_n) \to \{-1, 1\}$ defined as

$$f(e) = \begin{cases} -1, & \text{if } e \subset A \\ 1, & \text{otherwise,} \end{cases}$$

is balanced if and only if $n = \frac{1 + y_k}{2}$ and $|A| = x_k$ for some $k \in \mathbb{N}$.

**Proof.** Suppose first that $f$ is balanced and let $|A| = x$. Then

$$e(-1) = \frac{x(x - 1)}{2} = \frac{1}{2} \binom{n}{2},$$

which yields

$$n^2 - n - (2x^2 - 2x) = 0,$$

and therefore $n = \frac{1 + \sqrt{8x^2 - 8x + 1}}{2}$. But this is only an integer if $8x^2 - 8x + 1 = y^2$ for some integer $y$, and we obtain equation (2). Hence, $|A| = x = x_k$ and $n = \frac{1 + y_k}{2}$ for some $k \in \mathbb{N}$. Conversely, suppose that $n = \frac{1 + y_k}{2}$ and $|A| = x_k$ for some $k \in \mathbb{N}$. Then

$$n = \frac{1 + y_k}{2} = \frac{1 + \sqrt{y_k^2}}{2} = \frac{1 + \sqrt{8x_k^2 - 8x_k + 1}}{2}.$$

Thus $n$ is the positive root of

$$n^2 - n - (2x_k^2 - 2x_k) = 0,$$  \hfill (3)

which is equivalent to

$$x_k(x_k - 1) = \frac{1}{2} \binom{n}{2}.$$  

Since the left hand side of this equation is precisely $e(-1)$ and the right hand side is half the number of the edges of $K_n$, it follows that $f$ is balanced. \hfill $\square$
Lemma 3.2. Let \( n \) be a positive integer and consider the complete graph \( K_n \) and a partition \( V(K_n) = A \cup B \) of its vertex set. Then the function

\[
f(e) = \begin{cases} 
-1, & \text{if } e \subset A \text{ or } e \subset B \\
1, & \text{otherwise},
\end{cases}
\]

is balanced if and only if \( n = k^2 \) and \( \{|A|, |B|\} = \{\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\} \) for some \( k \in \mathbb{N} \).

Proof. Suppose first that \( f \) is balanced and let \( |A| = w \). Then

\[
e(1) = w(n - w) = \frac{1}{2} \binom{n}{2},
\]

which is equivalent to

\[
w^2 - nw + \frac{1}{4}n(n - 1) = 0.
\]

Hence,

\[
w = \frac{n \pm \sqrt{n}}{2},
\]

which is an integer if and only if \( n = k^2 \) for some \( k \in \mathbb{N} \). So we obtain \( n = k^2 \) and \( w \in \{\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\} \). Since \( |B| = n - |A| = k^2 - w \), it follows easily that \( \{|A|, |B|\} = \{\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\} \) and we are done.

Conversely, suppose that \( n = k^2 \) and \( \{|A|, |B|\} = \{\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\} \) for some \( k \in \mathbb{N} \). Without loss of generality, assume that \( |A| = \frac{1}{2}k(k+1) \). Then

\[
e(1) = |A|(n - |A|) = \frac{1}{2}k(k+1) \left( k^2 - \frac{1}{2}k(k+1) \right) = \frac{1}{4}k^2(k^2 - 1) = \frac{1}{2} \binom{n}{2},
\]

implying that \( f \) is balanced. \( \square \)

We define the set \( S_1 \) as the set of all integers \( n_k = \frac{1+y_k}{2} \), \( k \in \mathbb{N} \), where \( (x_k, y_k) \) is the \( k \)-th solution of \( (2) \), that is,

\[
S_1 = \left\{ \frac{1+y_k}{2} \mid k \in \mathbb{N} \right\}.
\]

Further, let \( S_2 \) be the set of all integer squares, that is,

\[
S_2 = \{k^2 \mid k \in \mathbb{N}\}.
\]

Lemmas 3.1 and 3.2 yield the following corollary.

Corollary 3.3. For any integer \( m \in \mathbb{N} \setminus (S_1 \cap S_2) \), there are infinitely many positive integers \( n \) such that there exists a balanced function \( f : E(K_n) \to \{-1, 1\} \) without zero-sum \( K_m \).

Proof. Let \( m \in \mathbb{N} \setminus (S_1 \cap S_2) \). By Lemmas 3.1 and 3.2 there is a balanced function \( f : E(K_n) \to \{-1, 1\} \) for each \( n \in S_1 \cup S_2 \). Suppose that there is a zero-sum \( K_m \) in such a weighting \( f \) for a given \( n \in S_1 \cup S_2 \). Then, the function \( f \) restricted to the edges of \( K_m \) is a balanced function on \( E(K_m) \), which is not possible by Lemmas 3.1 and 3.2 since \( m \in \mathbb{N} \setminus (S_1 \cap S_2) \). Since \( S_1 \cup S_2 \) has infinitely many elements, it follows that there are infinitely many positive integers \( n \) such that there exists a balanced function \( f : E(K_n) \to \{-1, 1\} \) without zero-sum \( K_m \). \( \square \)
Now we can state the main result of this section, which is equivalent to instance 3 of Theorem 1.2.

**Theorem 3.4.** For any integer $m \in \mathbb{N} \setminus \{1, 4\}$, there are infinitely many positive integers $n$ such that there exists a balanced weighting $f : E(K_n) \to \{-1, 1\}$ which is zero-sum-$K_m$ free.

**Proof.** By Corollary 3.3, for any $m \in \mathbb{N} \setminus (S_1 \cap S_2)$, there are infinitely many positive integers $n$, such that there exists a balanced weighting function $f : E(K_n) \to \{-1, 1\}$ without a zero-sum $K_m$. We will show that $S_1 \cap S_2 = \{1, 4\}$. Let $q$ be an integer such that $q^2 \in S_1$ (and thus $q^2 \in S_1 \cap S_2$). Then $q^2$ must be the positive root of equation (3) for some $x_k$. Thus we need to know for which positive integers $q$ and $x$ the following is possible:

$$q^4 - q^2 - (2x^2 - 2x) = 0. \quad (5)$$

Note that equation (5) can be written as:

$$Q^2 - 2X^2 = -1. \quad (6)$$

where $Q = 2q^2 - 1$ and $X = 2x - 1$. Again (as in the proof of Theorem 2.5), we have to deal with the Diophantine equation $Q^2 - 2X^2 = \pm 1$, which has infinitely many solutions given by

$$Q_k = \frac{a^k + b^k}{2}, \quad X_k = \frac{a^k - b^k}{a - b} = \frac{a^k - b^k}{2\sqrt{2}},$$

where $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$. Since we need to solve equation (6) (that is, with $-1$ on the right side), we know that $k$ must be odd. Therefore, according to the definition of $Q$, we need to determine all odd $k$’s such that

$$Q_k + 1 = 2q^2,$$

or equivalently,

$$2Q_k + 2 = 4q^2,$$

and so,

$$a^k + b^k + a + b = (2q)^2. \quad (7)$$

We consider two cases:

**Case 1.** If $k \equiv 1 \pmod{4}$, we will prove that the left side of equation (7) is $4Q_{\frac{k-1}{2}}Q_{\frac{k+1}{2}}$. Note that $ab = -1$ and, since in this case $\frac{k-1}{2}$ is even, we have $(ab)^{\frac{k-1}{2}} = (-1)^{\frac{k-1}{2}} = 1$. Hence,

$$a^k + b^k + a + b = a^k + b^k + (ab)^{\frac{k-1}{2}}(a + b)$$

$$= a^{\frac{k-1}{2}}a^{\frac{k+1}{2}} + b^{\frac{k-1}{2}}b^{\frac{k+1}{2}} + a^{\frac{k-1}{2}}b^{\frac{k+1}{2}}(a + b)$$

$$= a^{\frac{k-1}{2}}a^{\frac{k+1}{2}} + b^{\frac{k-1}{2}}b^{\frac{k+1}{2}} + a^{\frac{k+1}{2}}b^{\frac{k-1}{2}} + a^{\frac{k-1}{2}}b^{\frac{k-1}{2}}$$

$$= (a^{\frac{k-1}{2}} + b^{\frac{k-1}{2}})(a^{\frac{k+1}{2}} + b^{\frac{k+1}{2}})$$

$$= 4Q_{\frac{k-1}{2}}Q_{\frac{k+1}{2}}.$$

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Thus, by (7), we conclude that \( Q_{k-1}Q_{k+1} \) is a perfect square. We know that, for all \( i \), \( Q_i \) and \( Q_{i+1} \) are coprimes. Thus, it follows that both \( Q_{k-1} \) and \( Q_{k+1} \) are perfect squares. Coming back to equation (6), the following must be satisfied
\[
Y^4 - 2X^2 = -1
\]
where \( Q_{k+1} = Y^2 \). But, the only possible solution for the Diophantine equation (8) is \((Y, X) = (\pm 1, 1)\). Hence, \( Q_{k+1} = 1 \), which means that \( k = 1 \), and so \( Q = Q_1 = 1 \). Since \( Q = 2q^2 - 1 \) and \( q > 0 \), we conclude that \( q = 1 \).

**Case 2.** If \( k \equiv 3 \pmod{4} \), then we will prove that the left side of equation (7) is \( 8X_{k-1}X_{k+1} \).

Recall that \( ab = -1 \) and, since in this case \( \frac{k+1}{2} \) is even, we have \( (ab)^{\frac{k+1}{2}} = (-1)^{\frac{k+1}{2}} = 1 \). Hence,
\[
a^k + b^k + a + b = a^k + b^k + (ab)^{\frac{k+1}{2}}(a + b)
= a^k + b^k - (ab)^{\frac{k+1}{2}}(a + b)
= a^{\frac{k-1}{2}}a^{\frac{k+1}{2}} + b^{\frac{k-1}{2}}b^{\frac{k+1}{2}} - a^{\frac{k-1}{2}}b^{\frac{k+1}{2}}(a + b)
= a^{\frac{k-1}{2}}a^{\frac{k+1}{2}} + b^{\frac{k-1}{2}}b^{\frac{k+1}{2}} - a^{\frac{k-1}{2}}b^{\frac{k+1}{2}} - a^{\frac{k+1}{2}}b^{\frac{k+1}{2}}
= (a^{\frac{k-1}{2}} - b^{\frac{k-1}{2}})(a^{\frac{k+1}{2}} - b^{\frac{k+1}{2}})
= 8X_{k-1}X_{k+1}.
\]

Thus, by (7), we conclude that \( 2X_{k-1}X_{k+1} \) is a perfect square. We know that \( X_{k-1} \) and \( X_{k+1} \) have different parity. Observe that, for \( k \equiv 3 \pmod{4} \), \( X_{\frac{k-1}{2}} \) is odd and \( X_{\frac{k+1}{2}} \) is even. Since for all \( i \), \( X_i \) and \( X_{i+1} \) are coprimes, also \( X_{\frac{k-1}{2}} \) and \( 2X_{\frac{k+1}{2}} \) are coprimes, from which it follows that both \( X_{\frac{k-1}{2}} \) and \( 2X_{\frac{k+1}{2}} \) are perfect squares. Particularly, coming back to equation (6), we obtain
\[
Q^2 - 2W^4 = -1
\]
where \( X_{\frac{k-1}{2}} = W^2 \). Note that equation (9) is the well known Ljunggren Equation 1 + \( Q^2 = 2W^4 \). Such a Diophantine equation has solutions only for \( W = 1 \) and \( W = 13 \), which correspond respectively to \( X_1 \) and \( X_7 \) (because \( X_1 = 1 = 1^2 \) and \( X_7 = 169 = 13^2 \)). Therefore, we have two possibilities, either \( k = 3 \) (that is \( X_{\frac{3-1}{2}} = X_1 \)), or \( k = 15 \) (that is \( X_{\frac{15-1}{2}} = X_7 \)). The second case is disclaimed since \( X_{\frac{15-1}{2}} = X_8 = 408 = 2 \cdot (204) \) and 204 is not a perfect square. The first case, corresponding to \( k = 3 \), leads to \( X_{\frac{3-1}{2}} = X_1 = 1 \) and \( X_{\frac{3+1}{2}} = X_2 = 2 \). The solution \((Q_1, X_1) = (1, 1)\) gives \( q = 1 \) as we saw in Case 1. The solution \((Q_2, X_2) = (3, 2)\) gives \( q = 2 \) (since \( Q = 2q^2 - 1 \) an \( q > 0 \)).

From both cases we conclude that, if \( q^2 \in S_1 \cap S_2 \) then either \( q = 1 \) or \( q = 2 \). Hence, \( S_1 \cap S_2 = \{1, 4\} \) which concludes the proof.

**4 Conclusions**

While the situation about zero-sum copies of \( K_m \) over \( \mathbb{Z} \)-weightings is fairly clear now, a lot of interesting results can be proved when the graphs in question are not complete graphs.
Several examples are given in [10] (for example, certain complete bipartite graphs and many more), and in a forthcoming paper [7] which is under preparation.

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