A Meshless-based Local Reanalysis Method for Structural Analysis

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Abstract This study presents a meshless-based local reanalysis (MLR) method. The purpose of this study is to extend reanalysis methods to the Kriging interpolation meshless method due to its high efficiency. In this study, two reanalysis methods: combined approximations CA) and indirect factorization updating (IFU) methods are utilized. Considering the computational cost of meshless methods, the reanalysis method improves the efficiency of the full meshless method significantly. Compared with finite element method (FEM)-based reanalysis methods, the main superiority of meshless-based reanalysis method is to break the limitation of mesh connection. The meshless-based reanalysis is much easier to obtain the stiffness matrix \( K_m \) even for solving the mesh distortion problems. However, compared with the FEM-based reanalysis method, the critical challenge is to use much more nodes in the influence domain due to high order interpolation. Therefore, a local reanalysis method which only needs to calculate the local stiffness matrix in the influence domain is suggested to improve the efficiency further. Several typical numerical examples are tested and the performance of the suggested method is verified.

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1 Introduction

Analysis, as an important issue of structural design and optimization, always must be repeated because design is a continuously modified process. For most of the complicated engineering problems, the computational cost is commonly high due to repeated full analysis. Therefore, a method, called reanalysis, has been presented to predict the response of modified structures efficiently without full analysis. In the past several decades, there are many reanalysis methods have been developed. Generally, they can be mainly divided into two categories: direct methods (DMs) and approximate methods.

DMs can obtain the exact response of the modified structure, but it can only solve the problems of local or low-rank modifications. There are mainly two DM strategies: Sherman-Morrison-Woodbury (SMW) formula and matrix factorization updating method. The earliest Sherman-Morrison formula can only solve the problems of rank-one modifications [1], then Woodbury developed it for multiple rank-one changes [2]. At this point, the SWM formula had become to a representative DM for low-rank change problems. Another popular DM is based on the matrix factorization updating strategy. As similar as SWM formula, it proposed to solve rank-one modifications by Davis et al. [3], and then extended for multiple-rank modifications [4]. Liu et al. applied this method for the structure reanalysis with added DOFs [5, 6]. Meanwhile, Song et al. presented a direct reanalysis algorithm by updating the triangular factorization in sparse matrix solution [7]. Moreover, many other direct methods have been developed recently. A Moore Penrose generalized inverse has been used to develop DM [8]. Huang and Wang presented a static reanalysis method called Independent Coefficient (IC) method.
to solve large-scale problems [9], and suggested an Indirect Factorization Updating (IFU) method for local modifications [10]. Gao and Wang presented an exact block-based reanalysis method for local modifications [11].

Compared with DMs, approximate methods can solve the high-rank problems, but the exact response usually cannot be obtained. CA developed by Kirsch might be the most popular approximate reanalysis method due to its generality [12]. By combining local and global approximation methods, the CA method inherited the efficiency of local approximation and the accuracy of global approximation. Initially, the classical CA method is commonly used to solve the problem with small modifications [13, 14]. Therefore, Kirsch and Papalambros presented a method to break the bottleneck by introducing a modified initial design (MID) [13]. Chen proposed an approximate two-step method to solve problems of adding DOFs [15]. Sequentially, the CA method has been applied to multiple disciplines, such as structural static reanalysis [12, 16], eigenvalue reanalysis [17-20], topological optimization [13, 21-24], vibration reanalysis [25], linear dynamic reanalysis [26, 27], nonlinear dynamic reanalysis [28, 29], sensitivity reanalysis [30, 31] and others. Moreover, the CA method has been employed to some optimization approaches. For example, Zuo et al. used reanalysis method to improve the efficiency of genetic algorithm (GA) for structural optimization [32]. Recently, some alternative versions have been suggested for some complicated engineering problems. For example, Sun et al. proposed an adaptive reanalysis method for structural optimization [33]. Zuo et al. presented a sensitivity reanalysis using Taylor series expansion and CA method [34]. Zheng and Wu suggested a block CA method for vibration reanalysis [35]. To improve the efficiency of reanalysis method, a parallel CA method using single GPU to improve the computational efficiency was first developed by Wang et al. [36]. Then He et al. suggested a multiple-GPU based parallel IC reanalysis method which breaks through the memory bottleneck of GPU [37]. To extend the applications of reanalysis, Gao and Wang suggested an adaptive time-based
global reanalysis (ATGR) method which is based on Newmark-β method [38]. Materna et al. suggested a residual increment approximations for both linear and nonlinear reanalysis [39]. Based on these techniques, Wang et al. developed a CAD/CAE integrated parallel reanalysis design system [40].

Although the reanalysis method has been made important progress, it is still difficult to apply in complicated models. In our opinion, the most critical bottleneck is to obtain the change of stiffness matrix because the meshes before and after modifications should be identified. Compared with FEM, the meshless method is independent of meshes and the change of the stiffness matrix is easy to obtain. Therefore, we hope to extend the reanalysis for the meshless method and apply to more widely applications.

The earliest meshless method is smoothed particle hydrodynamics (SPH) presented by Lucy and Gingold in 1977 [41, 42], and then thoroughly developed by Monaghan [43]. After that, there are many kinds of meshless methods have been proposed, such as diffuse element method (DEM) [44], material point method (MPM) [45], element free Galerkin (EFG) method [46], reproducing kernel particle method (RKPM) [47, 48], partition of unity method (PUM) [49], HP clouds [50], natural element method (NEM) [51], finite sphere method (FSM) [52], meshless local Petrov-Galerkin (MLPG) method [53], least-squares collocation meshless (LSCM) method [54], local point interpolation method (LPIM) [55], stabilized conforming nodal integration meshfree method (SCNI-MM) [56], reproducing kernel element method (RKEM) [57], local max-entropy (LME) meshfree method [58], reproducing kernel enhanced local RBF method (RKLRBF) [59], generalized meshfree method (GMF) [60], Quasi-Convex meshfree method (QCMF) [61], meshfree peridynamics [62] and smoothed particle Galerkin method [63].

In this study, a moving kriging (MK) meshless method is utilized. The MK meshless method is a kind of weak-form meshless methods which was developed on the EFG method, and the MK interpolation procedure is employed instead of the moving least
squared (MLS) procedure to construct shape function [64]. This method has great
development potential behaves good stability and excellent accuracy because the weak-
form method can control the error level [65]. Kriging interpolation procedure is an
optimal interpolation algorithm founded by Matheron and Krige [66]. Then it has been
widely applied to computer experiments and engineering design optimization [67]. The
mathematical theory of Kriging interpolation has been described by Stein [68]. For
meshless methods, Gu firstly introduced MK interpolation procedure in element free
Galerkin (EFG) method [64], then Tongsuk and Kanok-Nukulchai applied this method
to one and two-dimensional elasticity problems [69, 70]. Sayakoummane and Kanok-
Nukulchai extended this method to shell structures [71]. As well plate structures were
analyzed with moving kriging-based EFG method by Bui et al. [72-74]. Moreover, the
moving Kriging interpolation was combined with the boundary integral equation (BIE)
method to construct a boundary-type meshfree method for two-dimensional potential
problems [75]. Shaw et al. applied the Kriging interpolation with an error-reproduction
kernel method to solve linear and nonlinear boundary value problems [76]. The Kriging
interpolation also has been extended to the fracture analysis by Fan and Luan [77].
Instead of global formulations described previously, Lam et al. proposed a local weak-
form meshless formulation incorporating Kriging-based shape functions to form a
novel local Kriging meshless method for two-dimensional structural analysis [78] and
it had been applied to dynamic nonlinear problems [79, 80]. Recently, the local kriging
method was employed to solve two-dimensional and three-dimensional transient heat
conduction problems [81], elastodynamic analysis for two-dimensional solids [82], free
vibration [83] and thermal buckling [84] analysis of functionally graded plates.

Compared with FEM, the pre-processing of meshless method is more convenient
because the meshless method only needs the nodal information while information of
meshes is needless. Furthermore, the meshless method behaves strong adaptability and
it’s easy to model, but usually the computational cost of meshless method is higher than
FEM. By using the reanalysis method, the computational cost of meshless method will be significantly reduced.

Therefore, a reanalysis method named meshless-based local reanalysis (MLR) method is suggested in this study. In this method, a structure should be analyzed by meshless method initially. When the structure is modified, the structural response should be predicated by the MLR method. Compared with the FEM-based reanalysis method, it is easier to be implemented because only nodes should be added or removed while a structure is modified. Moreover, the suggested method is based on the MK meshless method which the Kriging interpolation is used to construct the shape function due to satisfying the property of Kronecker’s delta function. Furthermore, a local search strategy for updating change of stiffness matrix is employed in this study to improve the efficiency.

The rest of this paper is represented as follows. Basic theories of the MK meshless method is introduced in Section 2. The MLR method and the local strategy are introduced in Section 3. In Section 4, four typical numerical examples are shown to test the performance of the MLR method. Finally, the conclusions are summarized in Section 5.

2 Basic theories

2.1 Framework of the MLR

The MLR method extended reanalysis methods to the MK meshless method due to its high efficiency, and a local reanalysis algorithm is suggested to improve the efficiency much more. The framework of the MLR method is presented in Fig. 1.
It can be found that the framework of the MLR method is divided into two parts: global and local updating strategy. The global updating strategy needs to calculate the stiffness matrix of all nodes in solution domain while the local updating strategy only needs to calculate the stiffness matrix of the nodes in the influence domain. Nevertheless, the global updating strategy can be utilized to calculate the overall modifications while the local updating strategy is suitable for local modification.

Generally, structural modifications can be divided into three parts: the number of DOFs is unchanged, decreased or increased [12]. It’s important to note that, the local search strategy for updating stiffness matrix of MLR method is unavailable while solving the problems that number of DOFs is fixed, such as change of material parameters (Young’s modulus or Poisson’s ratio). However, the global stiffness matrix can be obtained by the MK meshless method directly.

Fig. 1 Framework of the MLR method
Thus, the MLR method is suitable for local modifications. However, for large or overall modifications, a global stiffness matrix updating strategy was suggested to solve such problems by calculating global stiffness matrix directly.

2.2 Moving kriging meshless method

The MK meshless method has already been studied by several scholars, and the mathematical formulas can be found in their literature [34, 42]. Assuming the distribution functions \( u(x_i) \) in a sub-domain \( \Omega_x \), so that \( \Omega_x \subseteq \Omega \). Supposing that \( u(x_i) \) can be interpolated by nodal values \( x_i (i \in [1, n]) \), \( n \) is the number of the nodes in \( \Omega_x \). Define the MK interpolation \( u^h(x) \) as

\[
   u^h(x) = [p^T(x)S_a + r^T(x)S_b]u(x)
\]

or

\[
   u^h(x) = \sum_{j=1}^{n} \varphi_j(x)u_j,
\]

where \( \varphi_j(x) \) means the MK shape function and it can be rewritten as

\[
   \varphi_j(x) = \sum_{j=1}^{m} p_j(x)S_{a_{ji}} + \sum_{k=1}^{n} r_k(x)S_{b_{ki}},
\]

where \( S_{a_{ji}} \) denotes the element of matrix \( S_a \) at row \( j \) and column \( i \), \( S_{b_{ki}} \) denotes the element of matrix \( S_b \) at row \( k \) and column \( i \). Matrix \( S_a \) and matrix \( S_b \) are defined by the following equations:

\[
   S_a = (P^T R^{-1} P)^{-1} P^T R^{-1},
\]
\[ S_b = R^{-1}(I - PS_a), \]  

(5)

where \( I \) is a unit matrix and the matrix \( P \) is defined as:

\[
P = \begin{bmatrix}
  p_1(x_1) & p_2(x_1) & \cdots & 1 \\
  p_1(x_2) & p_2(x_2) & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  p_1(x_m) & p_2(x_m) & \cdots & 1
\end{bmatrix}.
\]

(6)

Moreover, \( r(x) \) in Eq. (1) can be defined by

\[
r(x) = \{R(x_i, x) \ R(x_2, x) \ \cdots \ x\}^T.
\]

(7)

\( R(x_i, x_j) \) is the correlation function between any pair of nodes \( x_i \) and \( x_j \), it can be calculated by

\[
R(x_i, x_j) = \text{cov}[u(x_i)u(x_j)].
\]

(8)

There are many functions that could be chosen as a correlation function [34]. A widely used Gaussian function is employed, and it can be calculated by

\[
R(x_i, x_j) = e^{-\theta r^2},
\]

(9)

where \( r = \|x_i - x_j\| \), in which \( \theta \) is a correlation parameter. It has been studied thoroughly by Bui [73]. In this study, a suitable value of the correlation parameter is given as \( \theta = 1 \). The linear basic \( p^T(x) = [1 \ x \ y] \) is using for numerical analysis. In addition, matrix \( R[R(x_i, x_j)]_{n \times n} \) can be defined by

9
In many problems, the first-order derivative is required, and it can be obtained from
\[ \varphi_{ij}(x) = \sum_{j} p_{ij}(x) S_{ij} + \sum_{k} r_{ij}(x) S_{ij} . \] (11)

For almost all static problems, usually can be simplified as an equilibrium equation:
\[ K_m U = F \] (12)
where \( K_m, F, U \) mean the stiffness matrix, the load vector, the unknown displacement vector respectively, and the stiffness matrix \( K_m \) can be defined as
\[
K_m = 
\begin{bmatrix}
  K_{11} & K_{12} & \cdots \\
  K_{21} & K_{22} & \cdots \\
  \vdots & \vdots & \ddots & \vdots \\
  K_{n1} & K_{n2} & \cdots
\end{bmatrix}
\] (13)
where
\[
K_{ij} = \int_{\Omega} B^T_i D B_j d\Omega \quad (i, j = 1, 2, \cdots) . \] (14)

In the Eq.(14), \( D \) is the constitutive matrix of material and \( B_i, B_j \) can be calculated by
\[
B_i = 
\begin{bmatrix}
  \varphi_{i,x} & 0 \\
  0 & \varphi_{i,y} \\
\end{bmatrix}
\] (2D formula) (15)
or

\[
\mathbf{B}_i = \begin{bmatrix}
\varphi_{lx} & 0 & 0 \\
0 & \varphi_{ly} & 0 \\
0 & 0 & \varphi_{lz} \\
0 & \varphi_{lz} & \varphi_{ky} \\
\varphi_{ly} & 0 & \varphi_{lx} \\
\varphi_{ky} & \varphi_{lx} & 0
\end{bmatrix} \quad (3D \text{ formula}), \quad (16)
\]

where \( \varphi_{lx} \), \( \varphi_{ly} \), \( \varphi_{lz} \) can obtained from Eq.(11). Then the stiffness sub-matrix can be calculated by Eq.(14), then assemble the sub-matrix into a global stiffness matrix.

3 Meshless-based Local Reanalysis method

In this study, two reanalysis algorithms, CA and IFU, have been used to improve the efficiency of computation. In order to improve the efficiency much more, a local reanalysis algorithm is suggested. The framework of MLR is shown in Fig. 1. In this section, the detail of the MLR is expounded.

3.1 Local search strategy for updating changed stiffness matrix

The MLR method integrates the reanalysis and MK meshless methods seamlessly. The MK meshless method is used to calculate the initial response \( \mathbf{U}^* \) by solving Eq.(20) and obtain the modified stiffness matrix \( \mathbf{K}_m \). Then the modified response \( \mathbf{U} \) can be predicted by reanalysis methods.

A local search strategy for updating changed stiffness matrix is suggested to improve the efficiency. By this strategy, the nodes inside the influence domain are used to construct the local stiffness matrix. The influence domain can be defined by background cells, usually the influence domain includes the background cells and their contiguous
background cells whose number of nodes has changed as shown in Fig. 2, where the circular part will be removed after modification. Obviously, there are four background cells whose number of nodes has changed. Therefore, the influence domain is composed of four background cells and their contiguous background cells. Similarly, the strategy is also suitable for problems of adding nodes, as shown in Fig. 3. When some nodes are added to the solution domain, there are four background cells whose number of nodes has changed. Then, the influence domain is also composed of four background cells and their contiguous background cells.

Fig. 2 The influence domain caused by removing nodes

Fig. 3 The influence domain caused by adding nodes
While the initial stiffness matrix $K^m_*$ and the initial displacement vector $U^*$ have been obtained by the MK meshless method, the key issue of reanalysis method is how to obtain the modified stiffness matrix $K_m$. Usually, the modified stiffness matrix $K_m$ can be calculated by the MK meshless method directly, then $\Delta K_m$ can be obtained by Eq.(22). In the FEM-based reanalysis, only the nodes related the modified elements need to be considered. However, due to high order interpolation and influence domain, all nodes in the influence domain of changed nodes should be found and involved in building. Fortunately, for most of the structural design and optimization, the change is local.

Therefore, a local search strategy has been suggested to obtain the change of stiffness matrix $\Delta K_m$, and it only needs to calculate the stiffness of nodes inside the influence domain rather than all nodes. The local search strategy for updating changed stiffness matrix can save much computational cost and improve the efficiency in the problems of small changes because it only needs to calculate a small part of the entire structure.

As shown in Fig. 1, the steps using to obtain $\Delta K_m$ and $K_m$ by the MLR method can be summarized as following:

(1) Searching the nodes which have been added or removed and recording the serial number of nodes. For example, as shown in Fig. 2, the circular part should be removed after modification, so the nodes inside circle should be searched in this step.

(2) Finding the background cells which adding or removing nodes are located on, and constructing the influence domain.

(3) Searching the nodes in the influence domain. Then the local initial stiffness matrix $K^*_{m_{-z}}$ can be isolated from the global initial stiffness matrix $K^*_m$. 
(4) Finding the Gauss points which are associated with all the recorded nodes, then the local modified stiffness matrix $K_{m_l}$ can be obtained by the MK method, and the change of local stiffness matrix $\Delta K_{m_l}$ can be obtained from

$$\Delta K_{m_l} = K_{m_l} - K_{m_l}^*.$$  \hfill (17)

(5) Obviously, because the change of stiffness matrix of outside influence domain is zero, the change of global stiffness matrix is equal to the change of local stiffness matrix.

$$\Delta K_m = \Delta K_{m_l}.$$  \hfill (18)

(6) The global modified stiffness matrix can be obtained from

$$K_m = K_m^* + \Delta K_m.$$  \hfill (19)

3.2 Meshless-based CA reanalysis

Assuming the equilibrium equation of initial structure is given by the following equation:

$$K_m^* U^* = F,$$  \hfill (20)

where $K_m^*$ means the stiffness matrix which can be obtained by the MK meshless method from Eq.(13), $U^*$, $F$ denote the displacement vector and the load vector respectively. After redesigning the structure, the modified equilibrium equation changed as

$$K_m U = F,$$  \hfill (21)

where
\[
K_m = K_m^* + \Delta K_m .
\] (22)

Obviously, the displacement vector \( \mathbf{U} \) will be predicted by reanalysis method rather than full analysis. Assuming that the displacements \( \mathbf{U} \) of a new design can be estimated by a linear combination of \( s \) independent basis vectors, \( \mathbf{U}_1, \mathbf{U}_2, \ldots \):

\[
\mathbf{U} = y_1\mathbf{U}_1 + y_2\mathbf{U}_2 + \cdots = \mathbf{U}_B\mathbf{y} .
\] (23)

Assuming that \( s \ll n \), \( \mathbf{U}_B \) is the \( n \times s \) matrix of the basis vectors and \( \mathbf{y} \) is a vector of coefficients to be determined:

\[
\mathbf{U}_B = [\mathbf{U}_1, \mathbf{U}_2, \ldots] ,
\] (24)

\[
\mathbf{y}^T = [y_1, y_2, \ldots] .
\] (25)

The response \( \mathbf{U} \) can be predicted by the following steps:

(1) Constructing the matrix of the basis vectors \( \mathbf{U}_B \) by Eq. (24). The initial value and the following series of basis vectors can be obtained from:

\[
\mathbf{U}_i = (K_m^*)^{-1}\mathbf{F} = \mathbf{U}^* ,
\] (26)

\[
\mathbf{U}_{i+1} = -\mathbf{B}\mathbf{U}_i \quad (i = 1, 2, \ldots)
\] (27)

where the matrix \( \mathbf{B} \) is defined as

\[
\mathbf{B} = (K_m^*)^{-1}\Delta K_m .
\] (28)

(2) Constructing the reduced stiffness matrix \( \mathbf{K}_R \) and load vector \( \mathbf{F}_R \) by:
\[ K_R = U_B^T K_m U_B, \quad F_R = U_B^T F. \]  

(29)

(3) Calculating the vector of coefficients \( y \) by solving the following equation:

\[ K_R y = F_R. \]  

(30)

(4) Updating the modified displacement by Eq.(23).

(5) In order to calculate the accuracy of stress-strain results, the modified strain and stress should be calculated by the following equations:

\[
\varepsilon = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix} U_B K^{-1}_R F_R,
\]  

(31)

\[ \sigma = c \varepsilon, \]  

(32)

where \( c \) is the matrix of material constants obtained by experiments.

3.3 Meshless-based IFU reanalysis

Compared with the CA method, the IFU method only calculates the displacement of the influenced DOFs when solving Eq.(21). Assuming the modified displacement is

\[ U = U^* + \Delta U, \]  

(33)

then Eq.(21) becomes

\[ K_m (U^* + \Delta U) = F \]  

(34)

or
\[
K_m \Delta U = F - K_m U^*.
\]  
(35)

Defining the residual value of the initial displacement \( \delta \) as

\[
\delta = F - K_m U^*.
\]  
(36)

It worth mentioning that only some members of \( \delta \) are non-zero while the modification is local. Basing on this property, the modified displacement \( U \) can be predicted by the following steps:

(1) Calculating the Cholesky factorization of initial stiffness matrix by Eq.(37);

\[
K_m^* = L_{m}L_{m}^T
\]  
(37)

(2) Calculating measurement vector: \( \delta \) and \( \Delta \) by Eq.(36) and Eq.(38) respectively;

\[
\Delta = sum \left[ \left| K_m - K_m^* \right| \right] + \left| \delta \right|
\]  
(38)

(3) Recording the unbalanced DOFs: If \( |\Delta(i)| > 0 \), the \( i \)-th DOF is unbalanced, and \( i \) should be recorded in \( S_d \) by ascending order, and the number of unbalanced DOFs is \( n_d \).

(4) Extracting unbalanced equations:

For \( i = 1 \) to \( n_d \)

\[
K_n(i,:) = K_m(S_d(i,:),:); \quad \delta_n(i) = \delta(S_d(i));
\]

End for.

(5) Applying extra constrains on \( K_m^* \):
For $i=n_d$ to 1

$$V(:, i) = L_0(:, S_d(i)); V(S_d(i), i) = 0;$$

$$L_0(S_d(i), :) = 0; L_0(:, S_d(i)) = 0; L_0(S_d(i), S_d(i)) = 1;$$

End for.

(6) Calculating the right-hand vectors or extra constraints $R$:

For $i=1$ to $n_d$

$$R(:, i) = K_m(:, S_d(i)); R(S_d(i), :) = 0; R(S_d(i), i) = 1;$$

End for.

(7) Calculating the fundamental solution system $B$ of the balanced equations by solving the Eq.(39) using SWM formula [2].

$$(L_0L_0^T + VV^T)B = R$$

(8) Reducing the unbalanced equations by $K_R = K_uB$.

(9) Solving the reduced equation by $K_Ry = \delta_u$.

(10) Calculating the increment of displacements by $\Delta U = By$.

(11) Calculating modified displacement $U$ by Eq.(33).

(12) The modified strain and stress can be calculated by Eq.(40) and Eq.(41) to test the accuracy of stress-strain results.
\[
\varepsilon = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
(U^* + \left(L_0 L_0^T + VV^T\right)^{-1} RK R^T) \delta_u ,
\]

\[
\sigma = c \varepsilon ,
\]

where \( c \) is the matrix of material constants obtained by experiments.

### 4 Numerical examples

In order to test the accuracy and efficiency of the MLR method, four examples will be solved by the proposed methods. The first two cases are 2D problems, and the number of DOFs is decreased in Section 4.1 while it is increased in Section 4.2. Another two cases are 3D problems, and also involve the problems of decreased and increased DOFs which have been shown in Section 4.3 and Section 4.4. Moreover, the first two cases are tested in concentrated load and another two cases are simulated in uniformed load. These four cases involve 2D and 3D, decreased and increased DOFs, concentrated and uniformed load problems, thus the performance of the MLR method could be verified thoroughly. In this study, the comparison has been made between the MLR and full analysis, and the errors of displacement, Von Mises stress, Von Mises strain are defined by the following formulas:

\[
E_u = \frac{\left\| U_{MLR} - U_{FA} \right\|}{\left\| U_{FA} \right\|} \times 100\%
\]

\[
E_\varepsilon = \frac{\left\| \varepsilon_{MLR} - \varepsilon_{FA} \right\|}{\left\| \varepsilon_{FA} \right\|} \times 100\%
\]
\[ E_\sigma = \left( \frac{\sigma_{MLR} - \sigma_{FA}}{\sigma_{FA}} \right) \times 100\% \]  

(44)

where \( U_{MLR} \), \( \varepsilon_{MLR} \), \( \sigma_{MLR} \) mean the results of MLR method, and \( U_{FA} \), \( \varepsilon_{FA} \), \( \sigma_{FA} \) mean the results of full analysis. All the simulations in this study was performed on an Intel® Core(TM) i5-3450 3.10GHz CPU with 8FB of memory within MATLAB R2014a in x64 Windows 7.

4.1 The rectangular plate optimization

As shown in Fig. 4, a rectangular plate is considered. The dimension of the plate is \( L \times D \) and the state of plane stress is considered, where the \( L=100\text{mm} \), \( D=50\text{mm} \), the Young’s modulus \( E=200\text{GPa} \), the Poisson’s ratio \( \nu=0.3 \), and the vertical concentrated load \( F=1000\text{mN} \).

![Fig. 4 The initial design and the modified structure of the rectangular plate](image)

![Fig. 5 The initial and the modified nodal distribution of the rectangular plate](image)
As shown in the left of Fig. 5, 5083 irregularly distributed field nodes are used in the initial structure. To investigate the performance of the MLR, the structure demonstrated in the right of Fig. 5 assumed to be the modified shape. The modified structure is only composed of 3341 nodes, and the percent of reduced DOFs is up to 34.3% of the original meshless model. Then the structural response will be predicted by the MLR method.

In general, the accuracy of the CA method relies on the number of basis vectors. Fig. 6 shows the variation in displacement of x-axis and y-axis respectively when the number of basis vectors is increased, and No. 2 node has been used which is the loading point. A comparison is made with respect to the result of the full analysis method. Obviously, the accuracy of results is improved when the number of basis vectors is increased.

According to Fig.9, it can be found that the accuracy of reanalysis solution converges to stability when the number of basis vectors grows up to 10. Sequentially, with more basis vectors, the accuracy of predicted response can’t be significantly improved. Moreover, analysis results comparisons between the MLR and the full analysis are illustrated in Fig. 7, Fig. 8 and Fig. 9. It is obvious that the result of the MLR is very
close to the result of full analysis, and the displacements of some selected DOFs are listed in
Tab. 1. The displacement, strain and stress errors of the CA method with 10 basis vectors calculated by Eq.(42) are 0.075%, 0.69%, 0.61% respectively. However, all the errors of the IFU method are 0, it obtained the exact solutions.

Fig. 7 The displacement results of the MLR and the full analysis methods

Fig. 8 The von Mises strain results of the MLR and the full analysis methods

Fig. 9 The von Mises stress results of the MLR and the full analysis methods
Tab. 1 Displacement error analysis of the MLR method

| DOF ID | CA method     | IFU method     | Full analysis | Displacement error |
|--------|---------------|---------------|---------------|--------------------|
|        |               |               |               | CA                | IFU               |
| 521    | -1.48859E-5   | -1.48728E-5   | -1.48728E-5   | 8.77844E-4        | 0                 |
| 522    | -1.08025E-5   | -1.07985E-5   | -1.07985E-5   | 3.67652E-4        | 0                 |
| 1111   | -9.52E-8      | -9.66E-8      | -9.66E-8      | 0.01395           | 0                 |
| 1112   | -2.4733E-6    | -2.4784E-6    | -2.4784E-6    | 0.00203           | 0                 |
| 2221   | 2.2155E-6     | 2.2152E-6     | 2.2152E-6     | 1.36715E-4        | 0                 |
| 2222   | -1.0379E-6    | -1.037E-6     | -1.037E-6     | 8.98309E-4        | 0                 |
| 5101   | 2.68836E-5    | 2.69254E-5    | 2.69254E-5    | 0.00155           | 0                 |
| 5102   | -9.43354E-5   | -9.42683E-5   | -9.42683E-5   | 7.11762E-4        | 0                 |

Tab. 2 Von Mises strain error analysis of the MLR method

| Node ID | CA method     | IFU method     | Full analysis | Von Mises strain error |
|---------|---------------|---------------|---------------|-----------------------|
|         |               |               |               | CA                | IFU               |
| 100     | 1.8655E-6     | 1.8726E-6     | 1.8726E-6     | 0.00378              | 0                 |
| 115     | 7.072E-7      | 6.995E-7      | 6.995E-7      | 0.01111              | 0                 |
| 197     | 1.141E-6      | 1.1415E-6     | 1.1415E-6     | 4.45611E-4           | 0                 |
| 369     | 1.1053E-6     | 1.1161E-6     | 1.1161E-6     | 0.00968              | 0                 |
| 442     | 5.564E-7      | 5.587E-7      | 5.587E-7      | 0.00421              | 0                 |
| 884     | 5.942E-7      | 5.919E-7      | 5.919E-7      | 0.00387              | 0                 |
| 1057    | 6.34E-7       | 6.369E-7      | 6.369E-7      | 0.00457              | 0                 |
| 2141    | 1.743E-7      | 1.754E-7      | 1.754E-7      | 0.00653              | 0                 |

Tab. 3 Von Mises stress error analysis of the MLR method

| Node ID | CA method     | IFU method     | Full analysis | Von Mises stress error |
|---------|---------------|---------------|---------------|-----------------------|
|         |               |               |               | CA                | IFU               |
| 100     | 370.25767     | 372.00852     | 372.00852     | 0.00471              | 0                 |
| 115     | 155.78426     | 154.03874     | 154.03874     | 0.01133              | 0                 |
| 197     | 251.40034     | 251.54859     | 251.54859     | 5.89379E-4           | 0                 |
| 369     | 225.92672     | 227.88026     | 227.88026     | 0.00857              | 0                 |
| 442     | 108.27899     | 106.57767     | 106.57767     | 0.01596              | 0                 |
| 884     | 98.82425      | 99.51534      | 99.51534      | 0.00694              | 0                 |
| 1057    | 135.48721     | 137.05081     | 137.05081     | 0.01141              | 0                 |
| 2141    | 19.50026      | 20.57304      | 20.57304      | 0.05215              | 0                 |
4.2 Support bracket redesign

A redesign of support bracket was considered. The idea is to design a support bracket which will act as a cantilever beam to support an end load and will be fixed on two pin holes. Fig. 10 shows the geometry of the initial design where the fillet radius is 7.5mm. Similarly, it also shows the geometry of the modified design where the fillet radius is 2.5mm. A concentrated load $F$ of 1000mN is applied at the free end, and the Young’s modulus $E=200GPa$, the Poisson’s ratio $\nu=0.3$.

![Initial and modified designs](image)

**Fig. 10** The initial design and the modified design of the support bracket

![Nodal distribution](image)

**Fig. 11** The initial and the modified nodal distribution of the support bracket

As shown in the Fig. 11, 2611 irregularly distributed field nodes are used in the initial structure while only 2647 nodes are used in the modified structure, and the percent of
adding DOFs is 1.4% of the original meshless model. Then the structural response will be predicted by the MLR method.

As shown in Fig. 12, it can be found that the accuracy of reanalysis solution converges to stability when the number of basis vectors grows up to 8. Sequentially, with more basis vectors, the accuracy of predicted response can’t be significantly improved. Moreover, analysis results comparisons between the MLR and the full analysis are illustrated in Fig. 13, Fig. 14 and Fig. 15 receptively. It is obvious that the result of the MLR is very close to the result of the full analysis, and the displacements of some selected DOFs are listed in Tab. 4. The displacement, strain and stress errors of the CA method are 1.1%, 6.7%, 5.1% respectively while all the errors of the IFU method are 0.
Fig. 14 The von Mises strain result of the MLR and the full analysis methods

Fig. 15 The von Mises stress result of the MLR and the full analysis methods

Tab. 4 Displacement error analysis of the MLR method

| DOF ID | CA method   | IFU method   | Full analysis | Displacement error |
|--------|-------------|--------------|---------------|--------------------|
|        |             |              |               | CA                |
| 282    | -7.6368E-4  | -7.6032E-4  | -7.6032E-4    | 0.00442            |
| 283    | -6.86184E-5 | -7.09861E-5 | -7.09861E-5   | 0.03335            |
| 897    | 2.61588E-4  | 2.64118E-4  | 2.64118E-4    | 0.00958            |
| 898    | -0.00126    | -0.00127    | -0.00127      | 0.00347            |
| 2797   | 6.1366E-5   | 6.07579E-5  | 6.07579E-5    | 0.01001            |
| 2798   | -6.91991E-4 | -6.87641E-4 | -6.87641E-4   | 0.00633            |
| 3199   | 3.99416E-5  | 3.89738E-5  | 3.89738E-5    | 0.02483            |
| 3200   | -9.057E-4   | -9.04355E-4 | -9.04355E-4   | 0.00149            |
Tab. 5 Von Mises strain error analysis of the MLR method

| Node ID | CA method | IFU method | Full analysis | Von Mises strain error |
|---------|-----------|------------|---------------|------------------------|
|         | CA        | IFU        |               |                        |
| 451     | 2.9325E-6 | 3.0135E-6  | 3.0135E-6     | 0.02689                |
| 452     | 3.253E-6  | 3.3428E-6  | 3.3428E-6     | 0.02689                |
| 494     | 7.005E-6  | 7.2029E-6  | 7.2029E-6     | 0.02747                |
| 495     | 6.9909E-6 | 7.1869E-6  | 7.1869E-6     | 0.02726                |
| 817     | 4.93E-6   | 5.0274E-6  | 5.0274E-6     | 0.01938                |
| 818     | 4.9525E-6 | 5.0377E-6  | 5.0377E-6     | 0.01691                |
| 1788    | 5.1274E-6 | 5.0403E-6  | 5.0403E-6     | 0.01728                |
| 1789    | 4.4536E-6 | 4.3937E-6  | 4.3937E-6     | 0.01363                |

Tab. 6 Von Mises stress error analysis of the MLR method

| Node ID | CA method | IFU method | Full analysis | Von Mises stress error |
|---------|-----------|------------|---------------|------------------------|
|         | CA        | IFU        |               |                        |
| 451     | 623.21646 | 771.96421  | 771.96421     | 0.02689                |
| 452     | 686.68504 | 836.26515  | 836.26515     | 0.02689                |
| 494     | 1544.71825| 1588.29902 | 1588.29902    | 0.02744                |
| 495     | 1541.57486| 1584.6989  | 1584.6989     | 0.02721                |
| 817     | 1071.28757| 1084.77122 | 1084.77122    | 0.01243                |
| 818     | 1072.06909| 1081.96295 | 1081.96295    | 0.00914                |
| 1788    | 674.1351  | 648.47392  | 648.47392     | 0.03957                |
| 1789    | 630.06725 | 603.10841  | 603.10841     | 0.0447                 |

4.3 Bridge

As shown in Fig. 16, a simplified bridge model is considered. The middle of the bridge is subjected to uniformed load $q$ and the dimension of the bridge deck is $L \times D$, where the $L=100mm$, $D=10mm$, the Young’s modulus $E=200GPa$, the Poisson’s ratio $\nu=0.3$, and the vertical uniformed load $q=110mN/mm$. 
As shown in the left of Fig. 17, 12189 irregularly distributed field nodes are used in the initial structure. To investigate the performance of the MLR, the structure demonstrated in the right of Fig. 17 assumed to be the modified shape. The modified structure is only composed of 11019 nodes, and the percent of reduced DOFs is 9.6% of the original meshless model. Then the structural response will be predicted by the MLR method.

In this case, 10 is chosen as the number of basic vectors, and analysis results comparisons between reanalysis and full analysis are illustrated in Fig. 18, Fig. 19, Fig. 20. Moreover, and the displacements of some selected DOFs are listed in Tab. 7. The
displacement, strain, stress errors of the CA method are 0.544%, 2.3%, 1.9% respectively while all the errors of the IFU method are 0.

![Fig. 18 The displacement result of the MLR and the full analysis methods](image1)

![Fig. 19 The von Mises strain result of the MLR and the full analysis methods](image2)

![Fig. 20 The von Mises stress result of the MLR and the full analysis methods](image3)

Tab. 7 Displacement error analysis of the MLR method

| DOF ID | CA method | IFU method | Full analysis | Displacement error |
|--------|-----------|------------|---------------|--------------------|
|        |           |            |               | CA     | IFU     |
| 131    | -2.68E-7  | -2.674E-7  | -2.674E-7     | 0.00226 | 3.14E-8 |
| 132    | -4.96923E-5 | -4.97644E-5 | -4.97644E-5  | 0.00145 | 0       |
| 19133  | 1.892E-7  | 1.895E-7   | 1.895E-7      | 0.00172 | 1.22E-8 |
| 19134  | 2.1113E-6 | 2.1015E-6  | 2.1015E-6     | 0.00468 | 8E-10   |
| 27731  | -2.668E-7 | -2.667E-7  | -2.667E-7     | 6.22801E-4 | 1.75E-8 |
| 27732  | -5.0272E-6 | -5.0074E-6 | -5.0074E-6   | 0.00395 | 1E-9    |
| 33044  | 1.723E-7  | 1.708E-7   | 1.708E-7      | 0.0089  | 3.64E-8 |
| 33045  | -5.5204E-5 | -5.53011E-5 | -5.53011E-5 | 0.00176 | 5E-10   |
Tab. 8 Von Mises strain error analysis of the MLR method

| Node ID | CA method | IFU method | Full analysis | Von Mises strain error |
|---------|-----------|------------|---------------|------------------------|
|         |           | CA         | IFU           | CA         | IFU         |
| 12      | 2.301E-7  | 2.303E-7   | 2.303E-7      | 0.001      | 0           |
| 652     | 2.13E-7   | 2.143E-7   | 2.143E-7      | 0.00609    | 0           |
| 3830    | 1.158E-7  | 1.182E-7   | 1.182E-7      | 0.02027    | 0           |
| 5372    | 3.71E-8   | 3.65E-8    | 3.65E-8       | 0.01737    | 0           |
| 6017    | 1.522E-7  | 1.505E-7   | 1.505E-7      | 0.01131    | 0           |
| 7195    | 4.017E-7  | 3.94E-7    | 3.94E-7       | 0.01962    | 0           |
| 7522    | 4.225E-7  | 4.222E-7   | 4.222E-7      | 8.00375E-4 | 0           |
| 9225    | 4.872E-7  | 4.816E-7   | 4.816E-7      | 0.0116     | 0           |

Tab. 9 Von Mises stress error analysis of the MLR method

| Node ID | CA method | IFU method | Full analysis | Von Mises stress error |
|---------|-----------|------------|---------------|------------------------|
|         |           | CA         | IFU           | CA         | IFU         |
| 12      | 45.40663  | 45.4437    | 45.4437       | 8.15842E-4 | 0           |
| 652     | 41.85062  | 42.24334   | 42.24334      | 0.0093     | 0           |
| 3830    | 22.55975  | 23.12509   | 23.12509      | 0.02445    | 0           |
| 5372    | 7.36588   | 7.28433    | 7.28433       | 0.0112     | 0           |
| 6017    | 18.81806  | 18.67553   | 18.67553      | 0.00763    | 0           |
| 7195    | 46.7825   | 45.65297   | 45.65297      | 0.02474    | 0           |
| 7522    | 61.03802  | 61.03334   | 61.03334      | 7.67271E-5 | 0           |
| 9225    | 63.86115  | 63.33949   | 63.33949      | 0.00824    | 0           |

4.4 L-frame

As shown in Fig. 21, a 3D L-frame under uniform load is considered. The top right edge of the L-frame is subjected to uniform load $q$, and the undersurface is fixed. Where the load $q=100mN/mm$, and the Young’s modulus $E=200GPa$, the Poisson’s ratio $\nu=0.3$. A ribbed plate has been added to the right-angle of the L-frame to improve stiffness as shown in Fig. 21.
As shown in the left of Fig. 22, 2016 irregularly distributed field nodes are used in the initial structure. To investigate the performance of the MLR, a ribbed plate was added to the L-frame as shown in the right of Fig. 22, and there are 2100 nodes left, the percent of adding DOFs is 4.2% of the initial meshless model.

In this case, 10 is chosen as the number of basic vectors for CA method, and analysis results comparisons between reanalysis and full analysis are illustrated in Fig. 23, Fig. 24, Fig. 25. Moreover, and the displacements of some selected DOFs are listed in Tab. 10. The displacement, strain, stress errors of the CA method are 1.87%, 5.6%, 5.2% respectively while all the errors of the IFU method are 0.
Fig. 23 The displacement result of the MLR and the full analysis methods

Fig. 24 The von Mises strain result of the MLR and the full analysis methods

Fig. 25 The von Mises stress result of the MLR and the full analysis methods
Tab. 10 Displacement error analysis of the MLR method

| DOF ID | CA method   | IFU method   | Full analysis | Displacement error |
|--------|-------------|--------------|---------------|--------------------|
|        |             |              |               | CA | IFU |
| 29     | 9.2201E-6  | 9.1687E-6   | 9.1687E-6     | 0.00561 | 0   |
| 30     | 1.16594E-5 | 1.17816E-5  | 1.17816E-5    | 0.01037 | 0   |
| 2903   | 1.56476E-5 | 1.56234E-5  | 1.56234E-5    | 0.00155 | 0   |
| 2904   | 1.46668E-5 | 1.48395E-5  | 1.48395E-5    | 0.01164 | 0   |
| 3711   | -5.9957E-6 | -5.9831E-6  | -5.9831E-6    | 0.00211 | 0   |
| 3712   | -9.288E-7  | -9.284E-7   | -9.284E-7     | 3.45795 | 0   |
| 6176   | 3.57369E-4 | 3.64841E-4  | 3.64841E-4    | 0.02048 | 0   |
| 6177   | -9.53544E-5| -9.57467E-5| -9.57467E-5   | 0.0041  | 0   |

Tab. 11 Von Mises strain error analysis of the MLR method

| Node ID | CA method   | IFU method   | Full analysis | Von Mises strain error |
|---------|-------------|--------------|---------------|------------------------|
|         |             |              |               | CA | IFU |
| 6       | 1.1546E-6  | 1.1613E-6   | 1.1613E-6     | 0.00579 | 0   |
| 126     | 5.149E-7   | 5.225E-7    | 5.225E-7      | 0.01463 | 0   |
| 418     | 1.3054E-6  | 1.3194E-6   | 1.3194E-6     | 0.01064 | 0   |
| 621     | 3.677E-7   | 3.707E-7    | 3.707E-7      | 0.00822 | 0   |
| 1065    | 3.747E-7   | 3.811E-7    | 3.811E-7      | 0.01666 | 0   |
| 1843    | 1.3226E-6  | 1.3288E-6   | 1.3288E-6     | 0.00464 | 0   |
| 1962    | 8.539E-7   | 8.834E-7    | 8.834E-7      | 0.03341 | 0   |
| 1989    | 1.2031E-6  | 1.2413E-6   | 1.2413E-6     | 0.03071 | 0   |

Tab. 12 Von Mises stress error analysis of the MLR method

| Node ID | CA method   | IFU method   | Full analysis | Von Mises stress error |
|---------|-------------|--------------|---------------|------------------------|
|         |             |              |               | CA | IFU |
| 6       | 204.60153  | 205.98867    | 205.98867     | 0.00673 | 0   |
| 126     | 74.60447   | 77.23895     | 77.23895      | 0.03411 | 0   |
| 418     | 261.05919  | 263.87043    | 263.87043     | 0.01065 | 0   |
| 621     | 73.44396   | 74.12374     | 74.12374      | 0.00917 | 0   |
| 1065    | 67.5001    | 67.48854     | 67.48854      | 1.71261 | 0   |
| 1843    | 264.46262  | 265.71453    | 265.71453     | 0.00471 | 0   |
| 1962    | 170.68602  | 176.60388    | 176.60388     | 0.03351 | 0   |
| 1989    | 240.61834  | 248.24881    | 248.24881     | 0.03074 | 0   |
4.5 **Accuracy and efficiency comparison**

Four numerical examples have been tested in this section, these cases include large or small modification, reduce or increase nodes, 2D or 3D problems. It can be found that the accuracy of CA method is high for both 2D and 3D problems, even for large modification. Meanwhile, the IFU method can obtain the exact response of the modified structure, there are almost no errors for both four cases. Moreover, the accuracy of reducing nodes modifications is much higher than adding nodes modifications by using CA method and the accuracy of stress-strain results is lower than displacement results.

To investigate the performance of the MLR, the running time of the above four examples which cost by the full analysis and the MLR methods was recorded respectively as shown in Fig. 26. It can be found that the efficiency of the MLR is much higher than the full analysis for local modifications, and for large modifications, the CA method behaves much better than the IFU method. In addition, although the accuracy of CA method is lower than IFU method, but the CA method is more efficient, especially for large modifications.

![Fig. 26 The comparison of computational cost of the MLR and the full analysis methods](image-url)
5 Conclusions

Compared with other FE-based reanalysis methods, the meshless-based reanalysis method is easier to be implemented because only points should be added or removed while a structure is modified without considering the connection of nodes. By using the MK method to construct stiffness matrix, the specified essential boundary conditions can be easily implemented due to the property of Kronecker’s delta function that Kriging interpolation procedure possesses. However, because more nodes in the influence domain are involved in constructing shape function, more relative nodes should be considered in building the modified stiffness matrix. Therefore, a local strategy is suggested. By this strategy, only the nodes inside the influence domain are used to construct local stiffness matrix rather than all nodes. Furthermore, considering the expensive computational cost of meshless methods, the advantage in term of efficiency of the meshless-based reanalysis is more obvious.

The effect of the number of basis vectors on analysis results has been discussed in this study, and the strain and stress formulations based on MLR are also given to make a comparison of accuracy between reanalysis and full analysis. Four numerical examples have shown that the accuracy of the MLR method is available even for large modification problems and this method can save much computational cost. Moreover, this study not only made comparisons of displacement, but also made comparisons of strain and stress, and the result shows that the accuracy of stress-strain results is available. In summary, the MLR method is a high-efficiency method with nice accuracy both in displacement, strain and stress.

However, further research is still needed to improve the accuracy of the meshless-based reanalysis method for the large deformation problems. It should also be extended to dynamic problems to fully show the advantages, such as crack propagation.
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