Non-Abelian Energy Loss at Finite Opacity

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A systematic expansion in opacity, $L/\lambda$, is used to clarify the non-linear behavior of induced gluon radiation in quark-gluon plasmas. The inclusive differential gluon distribution is calculated up to second order in opacity and compared to the zeroth order (factorization) limit. The opacity expansion makes it possible to take finite kinematic constraints into account that suppress jet quenching in nuclear collisions below RHIC ($\sqrt{s} = 200$ AGeV) energies.

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Introduction. The production of high transverse momentum jets in QCD is always accompanied by gluon showers. For jets produced inside nuclei via $e^+ A$ or quark-gluon plasmas via $A + A$, final state interactions of jet and radiated gluons induce further radiation and also broaden the gluon shower. The non-abelian radiative energy loss in a medium is expected to be observable as “jet quenching”. The observable consequences in nuclear collisions should be seen as a suppression of the high $p_T$ tails of single hadron distributions [1,2] and a broadening of the jet cone [3].

Non-abelian energy loss in pQCD can be calculated analytically in two limits. In both cases, the energy of the leading parton is assumed to be large enough that its angular deflection can be neglected. One analytic limit applies to thin plasmas where the mean number, $\bar{n} = L/\lambda$, of jet scatterings is small [4-6]. The other limit is the thick plasma one [7,8] where $\bar{n} \gg 1$. This mean number is a measure of the opacity or geometrical thickness of the medium:

$$\bar{n} = \frac{N \sigma_{el}}{A_e} = \int dz \langle \sigma_{el} \rho \rangle \approx \frac{dN}{dy} \frac{\sigma_{el}}{2\pi R_G^2} \log \frac{R_G}{\tau_0}. \quad (1)$$

For a constant (box) density $\rho = N/(LA_e)$, $L$ is the target thickness and $\lambda = 1/(\rho \sigma_{el})$ is the average mean free path. For a sharp geometry cylinder we can interpret $L = 1.2 A^{1/3} \equiv R_s$ as the nuclear radius. For a more realistic 3+1D expanding Gaussian cylinder with $\rho(x_{\perp}, \tau) = (\tau_0/\tau) \rho_0 \exp\left(-\left(x_{\perp}^2 + \Delta^2 \tau^2\right)/R_G^2\right)$, $I_0(2|x_{\perp}|\Delta \tau/R_G^2)$, $L$ is replaced by the equivalent rms Gaussian transverse radius $R_G = 0.75 A^{1/3}$ fm. The rightmost form in (1) is obtained by averaging over the expanding Gaussian cylinder. Here $\tau_0$ is the formation time of the plasma.

At RHIC energies ($\sqrt{s} \approx 200$ AGeV) the expected rapidity density of the gluons is $dN/dy \sim 1000$ for $A = 200$. With an elastic cross section $\sigma_{el} \sim 2$ mb and a plasma formation time $\sim 0.5$ fm/c, we obtain $\bar{n} \sim 4$. This suggests that neither analytic approaches may be strictly applicable in practice. However, because of the non-abelian analog [9,10] of the Landau-Pomeranchuk-Migdal (LPM) effect, the radiation intensity angular distribution and total energy loss are controlled by the combined effect of the number of scatterings $\bar{n} = L/\lambda$ and a formation probability $p_f = L/\tau_f = L \mu^2/(2sE)$ of the gluon in the medium. In [11] the induced energy loss was shown to be nonlinear in the nuclear thickness, $\Delta E \propto \mu^2 L^2/\lambda$ assuming the $\bar{n} \gg 1$ limit. In [12], we showed that the angular pattern was even more strongly nonlinear in the exclusive (tagged target) case. We show below that in the inclusive case, important features of the radiation pattern are in fact governed by the product $\bar{n}p_f = L^2(\mu^2/k^+\lambda)$. Because of this, even the first order in opacity reproduces the $L^2$ dependence of the energy loss and the range of applicability of the finite opacity expansion derived below may extend to realistic targets.

Another important motivation for our approach is the apparent absence of jet quenching observed at SPS energies [11,12]. As we find below, the opacity expansion shows that finite kinematic constraints suppress greatly the non-abelian energy loss at SPS energies. The work reported in this Letter extends Ref. [11] by including virtual corrections [13] necessary to compute inclusive (vs. tagged) jet quenching. Detailed derivation of these results is given in Ref. [14].

Hard Gluon Distribution. At zeroth order in opacity, the gluon emission from the hard production vertex is approximately given by

$$\frac{dN^{(0)}}{dx dk_{\perp}} = C_R \frac{\pi}{x} \left(1 - x + x^2\right) \frac{1}{k_{\perp}^2}, \quad (2)$$

where $x = k^+/E^+ \approx \omega/E$, and $C_R$ is the Casimir of the (spin 1/2) jet in the $dR$ dimensional color representation. For a spin 1 jet the gluon splitting function must be substituted above. The differential energy distribution outside a cone defined by $k_{\perp}^2 > \mu^2$ is given by

$$\frac{df^{(0)}}{dx} = \frac{2C_R \sigma_s}{\pi} \left(1 - x + \frac{x^2}{2}\right) E \log \frac{k_{\perp}^2_{\text{max}}}{\mu}, \quad (3)$$

where the upper kinematic limit is

$$k_{\perp}^2_{\text{max}} = \min\{4E^2 x^2, 4E^2 x(1-x)\}. \quad (4)$$

The energy loss outside the cone is then given by

$$\Delta E^{(0)} = \frac{4C_R \sigma_s}{3\pi} E \log \frac{E}{\mu} \quad (5)$$

in the leading log($E/\mu$) approximation. While this overestimates the radiative energy loss in the vacuum...
(self-quenching), it is important to note that \( \Delta E^{(0)}/E \sim 50\% \) is typically much larger than the medium induced energy loss.

**Opacity Expansion.** To compute the induced radiation, we assume as in \([6,12]\) that the quark gluon plasma can be modeled by nine well-separated, i.e. \( \lambda > 1/\mu_c \), color-screened Yukawa potentials. In contrast to \([5]\), we consider here the inclusive gluon distribution induced by a finite medium which remains **unobserved**. Without target tagging, we must add double Born (virtual) amplitudes \( V_j \) to the real (Direct) amplitudes \( D_i \) derived in \([5]\). We denote by \( G_0 \) the basic eikonal hard emission amplitude

\[
G_0 = -2ig_{\alpha} e_{\perp} \cdot H e^{i\omega_0 z_0 c} , \tag{6}
\]

where we use the shorthand notation of \([5]\), \( H \equiv k_{\perp}/k_{\perp}^2 \), \( \omega_0 \equiv k_{\perp}^2/2xeE \). The hard parton originates at \( z_0 \), and \( c \) denotes the color matrix \( T_c \) in its \( d_f \) dimensional representation. In Ref. \([11]\) we developed an iterative procedure to generate from this amplitude the sum of all amplitude terms to **any** order in opacity including both direct and virtual terms.

To first order in opacity it is sufficient to consider the classes of amplitudes denoted \( G_0D_1 \) (sum of three graphs) and \( G_0V_j \) (sum of four graphs) resulting from direct and virtual terms due to a scattering at position \( z_1 \). The cross section for the induced radiation consists of \( 3^2 \) real and \( 2 \times 4 \) double Born contributions that sum to a simple result

\[
\text{Tr} \left( G_0D_1D_1^\dagger G_0^\dagger + (G_0V_j G_0^\dagger + \text{h.c.}) \right) = 4g_{\alpha}^2 C_RC_Ad_R \left[ -2(C_1 \cdot B_1) \left( 1 - \cos(\omega_1 \Delta z_1) \right) \right] , \tag{7}
\]

with \( C_m \) and \( \omega_m \) obtained from \( H \) and \( \omega_0 \) through the substitution \( k_{\perp} \rightarrow k_{\perp} - q_{\perp}m \) and \( B_m \equiv \overline{H} - C_m \). Note that the interaction of the radiated gluon in the medium brought in a factor \( C_A = N_c \). Unlike the tagged case \([5]\), the above first order correction to the hard (factorization) distribution in Eq. \([3]\) has no simple classical cascade limit. In the tagged case, the virtual corrections lead to long range non-local interference effects that have no classical analog. On the other hand, a remarkable **"color triviality"** \([4]\) is found at the inclusive level where all jet scattering effects cancel \([5]\). As show in detail in Ref. \([11]\), the color factor at order \( n = 2 \) is simply \( C_RC_A^2d_R \), in contrast to the vastly more complicated color structure of the tagged case \([5]\).

The non-abelian LPM effect is seen in Eq. \([5]\) as arising from the gluon formation factor

\[
\Phi(\Delta z_1) = 1 - \cos(\omega_1 \Delta z_1) , \tag{8}
\]

where \( \Delta z_1 = z_1 - z_0 \). This must also be averaged over the longitudinal target profile, \( n(z) \). For a box density \([5]\) of thickness \( L = 1.2A^{1/3} \), \( n = \theta(L - z)/L \). For analytic simplicity we take instead an exponential form \( n_c(z) = e^{-z/L_c}/L_c \). In order to compare results for the two cases, we must require identical mean target depths, i.e. \( L_c = L/2 \). With \( n_c(z) \) the ensemble averaged formation factor is

\[
\int_0^\infty \frac{dL}{L} e^{-z/L} \Phi(\Delta z_1) = \frac{(k - q_{\perp})^2 L^2}{16\pi^2 E^2} \left( 1 + \frac{4g_{\alpha}^2}{\pi^2} \right) \frac{\pi^2}{(q_{\perp}^2 + \mu_c^2)^2} \frac{4k_{\perp}q_{\perp}L^2}{16\pi^2 E^2} \left( 1 + \frac{4g_{\alpha}^2}{\pi^2} \right) \tag{9}
\]

The formation probability in this case is controlled by simple Lorentzian factors.

Averaging over the momentum transfer \( q_{\perp} \) via the color Yukawa potential leads finally to the gluon double differential distribution

\[
\frac{dN}{dq_{\perp}^2} = \frac{dN^{(0)}}{dq_{\perp}^2} \frac{L}{\lambda_g} \int_{q_{\perp}^2}^{q_{\perp}^2_{\text{max}}} d^2 q_{\perp} \frac{d^2q_{\perp}^2}{\pi(q_{\perp}^2 + \mu_c^2)^2} \frac{2k_{\perp}q_{\perp}(k - q_{\perp})^2 L^2}{16\pi^2 E^2} \left( 1 + \frac{4g_{\alpha}^2}{\pi^2} \right) \tag{10}
\]

where the opacity factor \( L/\lambda_g = N\sigma^{(g)}/A_{\perp} \) arises from the sum over the \( N \) distinct targets. Note that the radiated gluon mean free path \( \lambda_g = (C_A/C_R)\lambda \) appears rather than the jet mean free path. It is the color triviality of Eq. \([4]\) that allows us to absorb \( C_R \) factor into the hard distribution, Eq. \([2]\), and \( C_A \) factor into \( \lambda_g \).

The upper kinematic bound on the momentum transfer \( q_{\perp}^2_{\text{max}} = s/4 \approx 3E\mu/(1/\mu_c^2 + 1/(\mu_c^2 + q_{\perp}^2_{\text{max}})) \). For SPS and RHIC energies, this finite limit cannot be ignored as we show below.

The second order contribution in opacity requires a more complex calculation involving the sum of \( 7^2 \) direct and \( 2 \times 96 \) virtual terms as discussed in Ref. \([11]\). The result is

\[
\text{Tr} \left( G_0D_1D_2D_2^\dagger D_1^\dagger G_0^\dagger + (G_0V_2 G_0^\dagger + \text{h.c.}) \right) + (G_0V_1 V_2 D_2^\dagger G_0^\dagger + \text{h.c.}) \right) = 4g_{\alpha}^2 C_RC_A^2d_R \left[ 2C_1 \cdot B_1 \left( 1 - \cos(\omega_1 \Delta z_1) \right) \right.

\]

\[
+ 2C_2 \cdot B_2 \left( \cos(\omega_2 \Delta z_2) - \cos(\omega_2 \Delta z_1 + \Delta z_2) \right) \right.

\]

\[
- 2C_{(12)} \cdot B_2 \left( \cos(\omega_2 \Delta z_2) - \cos(\omega_{12} \Delta z_1 + \omega_{12} \Delta z_2) \right) \right.

\]

\[
- 2C_{(12)} \cdot B_{2(12)} \left( 1 - \cos(\omega_{12} \Delta z_1) \right) \right] \tag{11}
\]

where with \( C_{(mn)} \) and \( \omega_{(mn)} \) obtained from \( H \) and \( \omega_0 \) through the substitution \( k_{\perp} = k_{\perp} - q_{\perp}m - q_{\perp}n \) and \( B_m(nl) = C_m - C_{(mn)} \). Note the color triviality of this inclusive result \([5,11]\) in contrast to the tagged target case \([5]\). Both Eqs. \([3,11]\) actually hold for either quark or gluon jets in the small \( x \) approximation.

We note further that in the two important limits \( |k_{\perp}| \rightarrow 0 \) and \( |k_{\perp}| \rightarrow \infty \) the distributions vanish for any fixed \( x \) due to azimuthal angular averaging.
\[
x dN^{(n)} / dx d\mathbf{k}_\perp \propto \int_0^{2\pi} d\phi / \pi \mathbf{k}_\perp \cdot \mathbf{q}_{m\perp} = 0.
\]

This is clearly seen in Fig. 1.

The average over the scattering points, \( z_1, z_2 \) in the second order case is performed with the exponential density profile as follows:

\[
\langle \cdots \rangle = \int_0^{\infty} 3 d\Delta z_1 / L \int_0^{\infty} 3 d\Delta z_2 / L \ e^{-z_1 z_2 / L} \cdots , \quad (13)
\]

where \( \Delta z_i = z_i - z_{i-1} \). We must use \( L_e = L/(n+1) \) for \( n \)th order in opacity in order to insure that the first moment of the \( n \)th scattering center is identical to that in a box distribution \( \langle z_m \rangle_n = mL/(n+1) \) as discussed in [3].

Numerical results comparing the first and second order in opacity corrections to the hard distribution Eq. (4) are illustrated in Figs. 1 and 2. We consider a 50 GeV quark jet in a medium with \( \lambda_g = 1 \) fm. A screening scale \( \mu = 0.5 \) GeV and \( \alpha_s = 0.3 \) were assumed. The “angular” distribution in Fig. 1 shows that the first order angular distribution is wider than the medium independent hard \((1/k_\perp^2)\) distribution. The second order corrections redistribute the gluons further. In Fig. 2 the \( k_\perp \) integrated contributions to the gluon intensity, \( dI / dx \), are shown for the same conditions as in Fig. 1. The induced intensity is concentrated at small \( x \) in contrast to the relatively constant intensity originating from the hard “self-quenching” term [2].

This formula breaks down at both \( x \to 0 \) and \( x \to 1 \) because \( |k_\perp|_{\max} \) cannot be approximated by \( \infty \) and because the small \( x \) approximations used above break down as \( x \to 1 \).

The total radiative energy loss is then given by

\[
\Delta E^{(1)} = C_{R \alpha_s} \frac{L^2 \mu^2}{N(E)} \log \frac{E}{\mu}, \quad (17)
\]
asymptotic value 4. We find that $N$ constraints cause with $E$ for $E_{\text{jet}} = 5, 50, 500$ GeV (at SPS, RHIC, LHC) is plotted as a function of the opacity $\Delta E/L \lambda_g$. ($\lambda_g = 1$ fm, $\mu = 0.5$ GeV). Solid curves show first order, while dashed curves show results up to second order in opacity. The energy loss (solid triangles) from Ref. [6] (with $\bar{v} = 2.5$) is shown for comparison.

$N(E) = 4$ if the kinematic bounds are ignored as in Eq. (10). In practice, we emphasize that finite kinematic constraints cause $N(E)$ to deviate considerably from the asymptotic value 4. We find that $N(E) = 7.3, 10.1, 24.4$ for $E = 500, 50, 5$ GeV. Together with the logarithmic dependence of energy, these kinematic effects suppress greatly the energy loss at lower (SPS) energies as seen in Fig. 3. This is in contrast to the approximately energy independent result in Ref. [8] where the finite kinematic bounds were neglected.

**Conclusions.** We calculated the effect of final state interactions on the induced gluon differential distribution up to second order in opacity for hard jets produced in nuclear reactions. This work generalizes Ref. [8] by taking into account virtual corrections to calculate inclusive rates and provides an complementary analytic approach to clarify nonlinear jet quenching effects predicted in Refs. [3,7]. The inclusive $xdN/dxdk_1^2$, $dI/dx$, and $\Delta E$, was studied as a function of nuclear thickness for jet energies in the SPS, RHIC and LHC range. One of our main results is the demonstration that the second order contribution to the integrated energy loss remains surprisingly small up to realistic nuclear opacities $L/\lambda_g \sim 5$ (except at low SPS energies). The rapid convergence of the opacity expansion even for realistic opacities results from the fact that the effective expansion parameter is actually the product of the opacity and the gluon formation probability $L\mu^2/e_e$ (see (15)). The leading quadratic dependence of the energy loss on nuclear thickness therefore arises from the simple first order term in this approach. The detailed pattern of angular broadening and the $x$ dependence is also dominated by the first order contribution. We note that our first order (effectively power-law) $|k_1|$ and $x$ distributions, however, differ considerably from the Gaussian form obtained in the eikonal resummation approach of [8] that applies to thick targets. Surprisingly, on the other hand, the magnitude of the integrated $\Delta E(L)$ is rather similar at RHIC energies.

At SPS energies kinematic effects suppress greatly the energy loss relative to [9]. Our estimates provide a natural explanation for the absence of jet quenching in $Pb+Pb$ at 160 AGeV that has been a puzzle up to now [10]. The short duration of the dense phase further limits the effective opacity at the SPS. A duration of $L/\lambda_g \sim 2$ for example leads to a total energy loss only $\sim 100$ MeV, which is much too small to be observable in soft multiple scattering background [11]. At RHIC energies, on the other hand, a significant nonlinear (in $A$) pattern of suppression [12] of high $p_{\perp}$ hadrons relative to scaled $pp$ data should be observable to enable a direct test of non-abelian energy loss mechanisms in dense matter. We note finally that the simplicity of the first and second order results [13] will make it possible to improve significantly Monte Carlo event simulations of jet quenching [11].

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[Note added in proof: U. Wiedemann informed us of an opacity expansion derived independently in [3].]

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