Whittaker modules for the twisted affine Nappi-Witten Lie algebra $\hat{H}_4[\tau]$

Xue Chen$^1$
School of Applied Mathematics, Xiamen University of Technology
Xiamen 361024, China
E-mail: xuechen@xmut.edu.cn

Cuipo Jiang$^2$
School of Mathematical Science, Shanghai Jiao Tong University
Shanghai, 200240, China
E-mail: cpjiang@sjtu.edu.cn

Abstract

The Whittaker module $M_\psi$ and its quotient Whittaker module $L_{\psi,\xi}$ for the twisted affine Nappi-Witten Lie algebra $\hat{H}_4[\tau]$ are studied. For nonsingular type, it is proved that if $\xi \neq 0$, then $L_{\psi,\xi}$ is irreducible and any irreducible Whittaker $\hat{H}_4[\tau]$-module of type $\psi$ with $k$ acting as a non-zero scalar $\xi$ is isomorphic to $L_{\psi,\xi}$. Furthermore, for $\xi = 0$, all Whittaker vectors of $L_{\psi,0}$ are completely determined.

For singular type, the Whittaker vectors of $L_{\psi,\xi}$ with $\xi \neq 0$ are fully characterized.

Key words: Twisted affine Nappi-Witten Lie algebra; Whittaker vectors; Whittaker modules

2000MSC: 17B10, 17B65

1 Introduction

It is well known that the conformal field theory (CFT) has many applications in various aspects of mathematics and physics. Wess-Zumino-Novikov-Witten (WZNW) models [19] form one of the typical examples of CFTS. WZNW models were considered originally within the framework of semisimple groups. Later it was found that the WZNW models based on non-abelian non-semisimple Lie groups are closely relevant to the construction of exact string backgrounds. For this reason, WZNW models of the special types have drawn much research interest [9, 13, 15, 19]. In 1993, it was observed by Nappi and Witten [15] that the WZNW model based on a central extension of the two-dimensional Euclidean group describes the homogeneous four-dimensional space-time corresponding to a gravitational plane wave. The related Lie algebra, which is called Nappi-Witten Lie algebra, is neither abelian nor semisimple.

The Nappi-Witten Lie algebra $H_4$ is a four-dimensional vector space with $\mathbb{C}$-basis \{a, b, c, d\} subject to the following relations

$[a, b] = c, \quad [d, a] = a, \quad [d, b] = -b, \quad [c, d] = [c, a] = [c, b] = 0.$

---

$^1$Supported in part by China NSF grant 11771281 and Fujian Province NSF Grant 2017J05016.

$^2$Supported by China NSF grants 11531004, 11771281.
Let $(\cdot,\cdot)$ be a symmetric bilinear form on $H_4$ defined by

$$(a,b) = 1, \quad (c,d) = 1, \quad \text{otherwise, } (\cdot,\cdot) = 0.$$ 

One can easily check that $(\cdot,\cdot)$ is a non-degenerate symmetric invariant bilinear form on $H_4$. Just as the non-twisted affine Kac-Moody Lie algebras given in [12], the non-twisted affine Nappi-Witten Lie algebra $\widehat{H}_4$ is defined by

$$H_4 \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}k,$$

with the bracket relations

$$[x \otimes t^m, y \otimes t^n] = [x,y] \otimes t^{m+n} + m(x,y)\delta_{m+n,0}k, \quad [\widehat{H}_4,k] = 0,$$

for $x, y \in H_4$ and $m,n \in \mathbb{Z}$.

Define a linear map $\tau$ of $H_4$ by

$$\tau a = b, \quad \tau b = a, \quad \tau c = -c, \quad \tau d = -d.$$ 

Clearly, $\tau^2 = \text{id}_{H_4}$ and $\tau \in \text{Aut}(H_4)$. It follows that $H_4$ becomes a $\mathbb{Z}_2$-graded Lie algebra:

$$H_4 = (H_4)_0 \oplus (H_4)_1,$$

where

$$(H_4)_0 = \mathbb{C}(a+b), \quad (H_4)_1 = \mathbb{C}(a-b) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$ 

Define a linear transformation $\hat{\tau}$ of $\widehat{H}_4$ by

$$\hat{\tau}(h \otimes t^m + sk) = (-1)^m \tau(h) \otimes t^m + sk$$

for $m \in \mathbb{Z}, h \in H_4,$ and $s \in \mathbb{C}$. It is easy to check that $\hat{\tau}$ is a Lie automorphism of $\widehat{H}_4$. The twisted affine Nappi-Witten Lie algebra $\widehat{H}_4[\tau]$ is

$$\widehat{H}_4[\tau] = \{ u \in \widehat{H}_4 \mid \hat{\tau}(u) = u \}$$

$$= \sum_{m \in \mathbb{Z}} \mathbb{C}(a+b) \otimes t^{2m} \oplus (\sum_{m \in \mathbb{Z}} (\mathbb{C}(a-b) + \mathbb{C}c + \mathbb{C}d) \otimes t^{2m+1}) \oplus \mathbb{C}k,$$

which is a subalgebra of $\widehat{H}_4$ fixed by $\hat{\tau}$. In the following, for convenience, we shall denote $x \otimes t^n$ by $x(n)$.

The representation theory for the non-twisted affine Nappi-Witten Lie algebra $\widehat{H}_4$ has been well studied in [4]. The irreducible restricted modules for $\widehat{H}_4$ with some natural conditions have been classified and the extension of the vertex operator algebra $V_{\widehat{H}_4}(l,0)$ by the even lattice $L$ has been considered in [10]. Verma modules and vertex operator representations for the twisted affine Nappi-Witten Lie algebra $\widehat{H}_4[\tau]$ have also been investigated in [8].

Whittaker modules were originally introduced by D. Arnal and G. Pinczon [2] in the construction of a very vast family of representations for $sl(2)$. A class of Whittaker modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ were defined
by B. Kostant in [11]. Whittaker modules play a critical role in the classification of all irreducible \( sl(2) \)-modules. It was illustrated in [3] that the irreducible modules for \( sl(2) \) consist of highest (lowest) weight modules, Whittaker modules, and a third family obtained by localization. Since the construction of Whittaker modules depends on the triangular decomposition of finite-dimensional complex semisimple Lie algebras, it is reasonable to consider Whittaker modules for other algebras with a triangular decomposition. In the context of quantum groups, Whittaker modules for \( U_q(g) \) and \( U_q(sl_2) \) were investigated in [18] and [16]. Recently, Whittaker modules for the affine Lie algebra \( A_1^{(1)} \), the Virasoro algebra, generalized Weyl algebras, non-twisted affine Lie algebras, and the Schrödinger-Witt algebra as well as the twisted Heisenberg-Virasoro algebra were also considered in [1, 17, 6, 7, 20, 14].

Inspired by the works mentioned above, P. Batra and V. Mazorchuk later generalized the ideas of both Whittaker modules and the underlying categories to a broad class of Lie algebras in [5]. Their framework allowed for a unified explanation of some important results (such as Lemma 2.1 in this paper). Meanwhile, they formulated some conjectures on the form of Whittaker vectors and the classification of irreducible Whittaker modules for Lie algebras with triangular decompositions. The aim of the present paper is to study Whittaker modules for the twisted affine Nappi-Witten Lie algebra \( \widehat{H}_4[\tau] \). Some ideas we use come from [1, 5, 17, 20].

Let \( M \) be an \( \widehat{H}_4[\tau] \)-module and let \( \psi : \widehat{H}_4[\tau]^{(+)} \to \mathbb{C} \) be a Lie algebra homomorphism. Recall that a non-zero vector \( v \in M \) is called a Whittaker vector of type \( \psi \) if \( xv = \psi(x)v \) for every \( x \in \widehat{H}_4[\tau]^{(+)} \). An \( \widehat{H}_4[\tau] \)-module \( M \) is said to be a Whittaker module of type \( \psi \) if there is a type \( \psi \) Whittaker vector \( \omega \in M \) which generates \( M \). In this case \( \omega \) is called a cyclic Whittaker vector. For \( \xi \in \mathbb{C} \), denote by \( L_{\psi,\xi} \) the quotient Whittaker module by the submodule generated by \( (k - \xi)\omega \).

For the non-singular type, that is, \( \psi(c(1)) \neq 0 \), we prove in Theorem 3.2 that \( L_{\psi,\xi} \) is irreducible if and only if \( \xi \neq 0 \), and any irreducible Whittaker \( \widehat{H}_4[\tau] \)-module of type \( \psi \) for which \( k \) acts by a non-zero scalar \( \xi \) is isomorphic to \( L_{\psi,\xi} \). This result together with Corollary 3.3 confirms the Conjecture 33 and Conjecture 34 proposed in [5], in the setting of the twisted affine Nappi-Witten Lie algebra \( \widehat{H}_4[\tau] \). Furthermore, for the more challenging and interesting case that \( \psi(c(1)) \neq 0 \) and \( \xi = 0 \), we determine in Theorem 4.1 all Whittaker vectors of \( L_{\psi,0} \). It turns out that the proof of Theorem 4.1 is quite non-trivial.

For the singular type \( \psi(c(1)) = 0 \), which is also quite interesting, we give in Theorem 5.2 a full characterization of the Whittaker vectors of \( L_{\psi,\xi} \) for \( \xi \neq 0 \). Finally, for the identically zero case \( \psi = 0 \) and \( \xi \neq 0 \), by Theorem 5.5 we give a filtration of the Whittaker module \( L_{0,\xi} \) with the simple sections given by the Verma module \( M(\psi,\xi) \) with multiplicity infinity.

The paper is organized as follows. In Section 2, Whittaker modules \( M_{\psi} \) and \( L_{\psi,\xi} \) for \( \widehat{H}_4[\tau] \) are constructed. In Section 3, the Whittaker vectors of \( M_{\psi} \) and \( L_{\psi,\xi} \) for \( \psi(c(1)) \neq 0 \) and \( \xi \neq 0 \) are studied. In Section 4, the Whittaker vectors of \( L_{\psi,0} \) for \( \psi(c(1)) \neq 0 \) and \( \xi = 0 \) are completely discussed. In the last section, the Whittaker vectors of \( M_{\psi} \) and \( L_{\psi,\xi} \) for \( \psi(c(1)) = 0 \) and \( \xi \neq 0 \), and relations to Verma modules are studied.
Throughout the paper, we use $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}_+$ to denote the sets of the complex numbers, the non-zero complex numbers, the non-negative integers, the integers and the positive integers respectively.

2 Preliminaries

We first recall some general results on Whittaker modules of complex Lie algebras with a quasi-nilpotent Lie subalgebra in [1, 5]. A Lie algebra $n$ is said to be quasi-nilpotent provided that $\bigcap_{k=0}^{\infty} n^k = 0$, where $n^0 := n$, $n^{k+1} := [n^k, n]$ for any $k \in \mathbb{N}$. Let $g$ be a complex Lie algebra and $n$ be a quasi-nilpotent Lie subalgebra of $g$. Let $\psi : n \to \mathbb{C}$ be a Lie algebra homomorphism and $V$ be a $g$-module. A non-zero vector $v \in V$ is called a Whittaker vector of type $\psi$ if $xv = \psi(x)v$ for all $x \in n$. The module $V$ is said to be a type $\psi$ Whittaker module for $g$ if it is generated by a type $\psi$ Whittaker vector. We say that $n$ acts on $V$ locally nilpotently [1] if for any $v \in V$ there is $s \in \mathbb{N}$ depending on $v$ such that $x_1x_2\cdots x_sv = 0$ for any $x_1, x_2, \cdots, x_s \in n$. Let $n^{(\psi)} = \{x - \psi(x) \mid x \in n\}$.

The following result comes from Lemma 3.1 in [1] and Proposition 32 in [5].

Lemma 2.1. ([1, 5]) Let $V$ be a type $\psi$ Whittaker module for $g$. Suppose that the adjoint action of $n$ on $g/n$ is locally nilpotent. Then

(i) $n^{(\psi)}$ acts locally nilpotently on $V$. In particular, $x - \psi(x)$ acts locally nilpotently on $V$ for any $x \in n$;

(ii) any nonzero submodule of $V$ contains a Whittaker vector of type $\psi$;

(iii) if the vector space of Whittaker vectors of $V$ is one-dimensional, then $V$ is simple.

Furthermore, we have

Lemma 2.2. Let $V$ be a type $\psi$ Whittaker module for $g$. Suppose that the adjoint action of $n$ on $g/n$ is locally nilpotent. Then Whittaker vectors in $V$ are all of type $\psi$.

Proof. Let $\psi' : n \to \mathbb{C}$ be a Lie algebra homomorphism and $\psi' \neq \psi$. Assume $v \in V$ is a Whittaker vector of type $\psi'$. Then $(y - \psi'(y))v = 0$, for any $y \in n$. From (i) we know that there is $s \in \mathbb{Z}_+$ such that

$$(x_1 - \psi(x_1))(x_2 - \psi(x_2))\cdots(x_s - \psi(x_s))v = 0, \text{ for any } x_1, x_2, \cdots, x_s \in n.$$ 

Take $s$ minimal. Then there exist $x_2, x_3, \cdots, x_s \in n$ such that

$$v' = (x_2 - \psi(x_2))\cdots(x_s - \psi(x_s))v \neq 0$$

and

$$(x - \psi(x))v' = 0, \text{ for all } x \in n.$$
So \( \nu' \) is a Whittaker vector of type \( \psi \). On the other hand, note that \( \psi'(\mathbf{n}, \mathbf{n}) = 0 \). By inductive assumption, we deduce that
\[
(y - \psi'(y))\nu' = (y - \psi'(y))(x_2 - \psi(x_2))\cdots(x_s - \psi(x_s))v
\]
\[
= \sum_{i=2}^{s} (x_2 - \psi(x_2))\cdots(x_{i-1} - \psi(x_{i-1}))ady(x_i)(x_{i+1} - \psi(x_{i+1}))\cdots(x_s - \psi(x_s))v
\]
\[
= 0
\]
for any \( y \in \mathbf{n} \). That is \( \nu' \) is also a Whittaker vector of type \( \psi' \). Then \( \psi' = \psi \), a contradiction to our assumption. Thus the lemma holds. \( \square \)

The twisted affine Nappi-Witten Lie algebra \( \hat{H}_4[\tau] \) has a natural decomposition:
\[
\hat{H}_4[\tau] = \hat{H}_4[\tau]^+ \bigoplus \hat{H}_4[\tau]^0 \bigoplus \hat{H}_4[\tau]^-,\n\]
where
\[
\hat{H}_4[\tau]^+ = \text{Span}_\mathbb{C}\{(a + b)(n), (a - b)(m), c(m), d(m) \mid n \in 2\mathbb{Z}_+, m \in 2\mathbb{N} + 1\},
\]
\[
\hat{H}_4[\tau]^0 = \text{Span}_\mathbb{C}\{(a + b)(0), k\}.
\]
Denote
\[
b^- = \hat{H}_4[\tau]^- \bigoplus \hat{H}_4[\tau]^0.
\]
Let \( \mathbb{C}[k] \) be the polynomial algebra generated by \( k \). It is easy to see that \( \mathbb{C}[k] \) lies in the center of the universal enveloping algebra \( U(\hat{H}_4[\tau]) \).

The following notation for \emph{odd partitions} and \emph{even pseudopartitions} will be used to describe bases for \( U(\hat{H}_4[\tau]) \) and Whittaker modules.

Just as in [17, 20], for a non-decreasing sequence of positive odd numbers:
\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_t, \quad \lambda_i \in 2\mathbb{N} + 1, \quad 1 \leq i \leq t,
\]
we call \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \) an \emph{odd partition}. For a non-decreasing sequence of non-negative even numbers:
\[
0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_s, \quad \mu_i \in 2\mathbb{N}, \quad 1 \leq i \leq s,
\]
we call \( \bar{\mu} = (\mu_1, \mu_2, \ldots, \mu_s) \) an \emph{even pseudopartition}. Let \( \mathcal{P}_{\text{odd}} \) and \( \mathcal{P}_{\text{even}} \) denote the set of \emph{odd partitions} and \emph{even pseudopartitions}, respectively. For \( \lambda \in \mathcal{P}_{\text{odd}}, \bar{\mu} \in \mathcal{P}_{\text{even}}, \) we also write
\[
\lambda = (1^{\lambda(1)}, 3^{\lambda(3)}, \ldots), \quad \bar{\mu} = (0^{\mu(0)}, 2^{\mu(2)}, \ldots),
\]
where \( \lambda(k) \) and \( \mu(l) \) are the number of times of \( k \) and \( l \) appear respectively in the \emph{odd partition} and \emph{even pseudopartition}, and \( \lambda(k) = \mu(l) = 0 \) for \( k \) and \( l \) sufficiently large.
Let $\bar{\mu} = (\mu_1, \mu_2, \cdots, \mu_s) \in \tilde{P}_{\text{even}}$. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$, $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$, $\eta = (\eta_1, \eta_2, \cdots, \eta_l)$ and $\lambda, \nu, \eta \in P_{\text{odd}}$. We define

$$|\bar{\mu}| = \mu_1 + \mu_2 + \cdots + \mu_s, \quad |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_t,$$

$$\#(\bar{\mu}) = \mu(0) + \mu(2) + \cdots, \quad \#(\lambda) = \lambda(1) + \lambda(3) + \cdots,$$

$$\#(\bar{\mu}, \nu, \lambda, \eta) = \#(\bar{\mu}) + \#(\nu) + \#(\lambda) + \#(\eta),$$

$$(a + b)(-\bar{\mu}) = (a + b)(-\mu_1)(a + b)(-\mu_2) \cdots (a + b)(-\mu_s) = (a + b)(0)^{\mu(0)}(a + b)(-2)^{\mu(2)} \cdots,$$

$$(a - b)(-\nu) = (a - b)(-\nu_1)(a - b)(-\nu_2) \cdots (a - b)(-\nu_r) = (a - b)(-1)^{\nu(1)}(a - b)(-3)^{\nu(3)} \cdots,$$

$$d(-\lambda) = d(-\lambda_1)d(-\lambda_2) \cdots d(-\lambda_t) = d(-1)^{\lambda(1)}d(-3)^{\lambda(3)} \cdots,$$

$$c(-\eta) = c(-\eta_1)c(-\eta_2) \cdots c(-\eta_l) = c(-1)^{\eta(1)}c(-3)^{\eta(3)} \cdots.$$.

For convenience, we define $\bar{0} = (0^0, 1^0, 2^0, \cdots)$ and set $(a + b)(\bar{0}) = (a - b)(\bar{0}) = d(\bar{0}) = c(\bar{0}) = 1 \in U(\hat{H}_4[\tau])$. In what follows, we regard $\bar{0}$ as an element of $P_{\text{odd}}$ and $\tilde{P}_{\text{even}}$.

**Definition 2.3.** ([1, 17, 20]) Let $M$ be an $\hat{H}_4[\tau]$-module and let $\psi : \hat{H}_4[\tau]^{(+)} \rightarrow \mathbb{C}$ be a Lie algebra homomorphism. A non-zero vector $v \in M$ is called a Whittaker vector if $xv = \psi(x)v$ for every $x \in \hat{H}_4[\tau]^{(+)}$. An $\hat{H}_4[\tau]$-module $M$ is said to be a Whittaker module of type $\psi$ if there is a type $\psi$ Whittaker vector $\omega \in M$ which generates $M$. In this case we call $\omega$ a cyclic Whittaker vector.

The Lie algebra homomorphism $\psi$ is called nonsingular if $\psi(c(1)) \neq 0$, otherwise $\psi$ is called singular. The commutator relation in the definition of $\hat{H}_4[\tau]$ forces that

$$\psi((a + b)(n)) = \psi((a - b)(m)) = \psi(c(m)) = 0,$$

for $n \in 2\mathbb{Z}_+, m \in 2\mathbb{Z}_+ + 1$.

Fix a Lie algebra homomorphism $\psi : \hat{H}_4[\tau]^{(+)} \rightarrow \mathbb{C}$, and let $\mathbb{C}_\psi$ be the one-dimensional $\hat{H}_4[\tau]^{(+)}$-module for which $x\alpha = \psi(x)\alpha$ for $x \in \hat{H}_4[\tau]^{(+)}$ and $\alpha \in \mathbb{C}^*$. Then an induced $\hat{H}_4[\tau]$-module is defined by

$$M_\psi = U(\hat{H}_4[\tau]) \otimes_{U(\hat{H}_4[\tau]^{(+)} \mathbb{C}_\psi).}$$

(2.1)

For $\xi \in \mathbb{C}$, since $k$ is in the center of $\hat{H}_4[\tau]$, $(k - \xi)M_\psi$ is a submodule of $M_\psi$. Define

$$L_{\psi, \xi} = M_\psi / (k - \xi)M_\psi,$$

(2.2)

and let $\overline{\tau} : M_\psi \rightarrow L_{\psi, \xi}$ be the canonical homomorphism. Then $L_{\psi, \xi}$ is a quotient module of $M_\psi$ for $\hat{H}_4[\tau]$. We have the following results for $M_\psi$ and $L_{\psi, \xi}$.
Proposition 2.4. (i) $M_\psi$ and $L_{\psi,\xi}$ are both Whittaker modules of type $\psi$, with cyclic Whittaker vectors $\omega := 1 \otimes 1$ and $\tilde{\omega} := \tau \otimes 1$, respectively;

(ii) The set
\[
\{ \mathbf{k}^t (a + b) (-\tilde{\mu}) (a - b) (-\nu) d (-\lambda) c (-\eta) \omega \mid (\tilde{\mu}, \nu, \lambda, \eta) \in \mathcal{P}_{\text{even}} \times \mathcal{P}_{\text{odd}} \times \mathcal{P}_{\text{odd}} \times \mathcal{P}_{\text{odd}}, \ t \in \mathbb{N} \} \tag{2.3}
\]
forms a basis of $M_\psi$. $L_{\psi,\xi}$ has a basis
\[
\{ (a + b) (-\tilde{\mu}) (a - b) (-\nu) d (-\lambda) c (-\eta) \omega \mid (\tilde{\mu}, \nu, \lambda, \eta) \in \mathcal{P}_{\text{even}} \times \mathcal{P}_{\text{odd}} \times \mathcal{P}_{\text{odd}} \times \mathcal{P}_{\text{odd}} \}; \tag{2.4}
\]

(iii) $M_\psi$ has the universal property in the sense that for any Whittaker module $M$ of type $\psi$ generated by $\omega'$, there is a surjective module homomorphism $\varphi : M_\psi \to M$ such that $u \omega \mapsto u \omega'$, for $u \in U(b^-)$. $M_\psi$ is called the universal Whittaker module of type $\psi$;

(iv) $L_{\psi,\xi}$ has the universal property in the sense that for any Whittaker module $\tilde{M}$ of $\hat{H}_4[\tau]$ of type $\psi$ generated by a Whittaker vector $\tilde{\omega}'$, such that $\mathbf{k}$ acts as the scalar $\xi$, there is a surjective module homomorphism $\bar{\varphi} : L_{\psi,\xi} \to \tilde{M}$ such that $u \tilde{\omega} \mapsto u \tilde{\omega}'$, for $u \in U(\hat{H}_4[\tau][-] \oplus \mathbb{C}(a + b)(0))$.

Proof. (i)-(iii) is obvious. We only prove (iv). Let $\tilde{M}$ be a Whittaker module of $\hat{H}_4[\tau]$ of type $\psi$ generated by a cyclic Whittaker vector $\tilde{\omega}'$, such that $\mathbf{k}$ acts as the scalar $\xi$. By (iii), there is a surjective module homomorphism $\varphi$ from $M_\psi$ to $\tilde{M}$ such that $\varphi(\omega) = \tilde{\omega}'$. Since on $\tilde{M}$, $\mathbf{k}$ acts as the scalar $\xi$, we know that $\varphi(M_\psi(\mathbf{k} - \xi) \omega) = 0$. So $\varphi$ induce a surjective module homomorphism $\bar{\varphi}$ from $L_{\psi,\xi}$ to $\tilde{M}$ such that $\bar{\varphi}(\tilde{\omega}) = \tilde{\omega}'$. \hfill \Box

Remark 2.5. For any $x \in \hat{H}_4[\tau]^+[\omega']$, $u \omega = u \omega$, $u \in U(\hat{H}_4[\tau][-] \oplus \mathbb{C}(a + b)(0))$, we have
\[
(x - \psi(x))\omega' = [x, u] \tilde{\omega}.
\]

Lemma 2.6. For $n \in 2\mathbb{Z}^+, m \in 2\mathbb{N} + 1$, $\tilde{\mu} = (\mu_1, \mu_2, \ldots, \mu_s) \in \mathcal{P}_{\text{even}}$, $\nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathcal{P}_{\text{odd}}$, the following formulas hold.
\[
(a + b)(n)(a + b)(-\tilde{\mu}) = \sum_{i=1}^s 2n\delta_{n,\mu_i} \mathbf{k}(a + b)(-\tilde{\mu}^{(i)}) + (a + b)(-\tilde{\mu})(a + b)(n), \tag{2.5}
\]
where $|\tilde{\mu}^{(i)}| = |\tilde{\mu}| - n$ and $\#(\tilde{\mu}^{(i)}) < \#(\tilde{\mu})$;
\[
(a + b)(n)(a - b)(-\nu) = -2 \sum_{i=1}^r (a - b)(-\nu^{(i)}) c(n_i) + (a - b)(-\nu)(a + b)(n), \tag{2.6}
\]
where $|\nu^{(i)}| - n_i = |\nu| - n$, $n_i < n$ and $n_i \in 2\mathbb{Z} + 1$, $\#(\nu^{(i)}) < \#(\nu)$;
(a - b)(m)(a + b)(-\mu) = 2 \sum_{i=1}^{s} (a + b)(-\tilde{\mu}^{(i)})c(m_i) + (a + b)(-\mu)(a - b)(m), \quad (2.7)

where \(|\tilde{\mu}^{(i)}| - m_i = |\mu| - m, m_i \leq m (m_i \in 2\mathbb{Z} + 1), and \mu^{(i)}(0) < \mu(0) if m_i = m, 
#(\tilde{\mu}^{(i)}) < #(\tilde{\mu});

(a - b)(m)(a - b)(-\nu) = -\sum_{i=1}^{r} 2m\delta_{m,\nu} k(a - b)(-\nu^{(i)}) + (a - b)(-\nu)(a - b)(m), \quad (2.8)

where |\nu^{(i)}| = |\nu| - m and #(\nu^{(i)}) < #(\nu).

Proof. We give a proof of (2.5) by induction on #(\tilde{\mu}). The proof for the rest formulas are similar. Let #(\tilde{\mu}) = 1 and \tilde{\mu} = (\mu_1). Then

(a + b)(n)(a + b)(-\tilde{\mu}) = 2n\delta_{n,\tilde{\mu}}k + (a + b)(-\tilde{\mu})(a + b)(n).

In this case #(\tilde{\mu}^{(i)}) = 0 and (a + b)(-\tilde{\mu}^{(i)}) = 1 in (2.5).

Suppose that (2.5) holds for #(\tilde{\mu}) < s. Let #(\tilde{\mu}) = s and \tilde{\mu} = (\mu_1, \mu_2, \cdots, \mu_s). We denote (a + b)(-\tilde{\mu}) = (a + b)(-\tilde{\mu}')(a + b)(-\mu_s), where \tilde{\mu}' = (\mu_1, \mu_2, \cdots, \mu_{s-1}). Then

\[
(a + b)(n)(a + b)(-\tilde{\mu}) = \sum_{i=1}^{s-1} 2n\delta_{n,\mu_i}k(a + b)(-\tilde{\mu}'^{(i)}) + (a + b)(-\tilde{\mu}')(a + b)(n)(a + b)(-\mu_s)
\]

\[
= \sum_{i=1}^{s-1} 2n\delta_{n,\mu_i}k(a + b)(-\tilde{\mu}'^{(i)})(a + b)(-\mu_s) + 2n\delta_{n,\mu_s}k(a + b)(-\tilde{\mu}') + (a + b)(-\tilde{\mu})(a + b)(n)
\]

\[
= \sum_{i=1}^{s} 2n\delta_{n,\mu_i}k(a + b)(-\tilde{\mu}^{(i)}) + (a + b)(-\tilde{\mu})(a + b)(n),
\]

where (a + b)(-\tilde{\mu}'^{(i)}) = (a + b)(-\mu_1) \cdots (a + b)(-\mu_i) \cdots (a + b)(-\mu_{s-1}), (a + b)(-\tilde{\mu}^{(i)}) = (a + b)(-\mu_1) \cdots (a + b)(-\mu_i) \cdots (a + b)(-\mu_{s-1})(a + b)(-\mu_s) and (a + b)(-\mu_i) means this factor is deleted. (2.5) is proved. \square

3 \hspace{1em} \textbf{Whittaker vectors in } L_{\psi, \xi} \text{ and } M_\psi \text{ for } \psi(c(1)) \neq 0 \text{ and } \xi \neq 0

In this section, it is always assumed that \psi(c(1)) \neq 0, meaning that \psi is nonsingular. Let \(M_\psi \) and \(L_{\psi, \xi} \) defined by (2.1) and (2.2) be the Whittaker modules for the twisted affine Nappi-Witten Lie algebra \(\hat{H}_4[\tau] \). The key results of this section are shown in Theorem 3.2 and Corollary 3.3 in which the Whittaker vectors in \(L_{\psi, \xi} \) and \(M_\psi \) are characterized.

Applying Lemma 2.2 to \(\hat{H}_4[\tau] \), we have...
Lemma 3.1. Let $\psi(c(1)) \neq 0$ and $L_{\psi, \xi}$ be a Whittaker module for $\widehat{H}_4[\tau]$ defined by (2.2). Then the Whittaker vectors in $L_{\psi, \xi}$ are all of type $\psi$.

We are now in a position to give the main result of this section.

Theorem 3.2. Assume that $\psi(c(1)) \neq 0$ and $\xi \in \mathbb{C}^*$. Let $\tilde{\omega} = \overline{1} \otimes 1 \in L_{\psi, \xi}$. Then

1. $\omega' \in L_{\psi, \xi}$ is a Whittaker vector if and only if $\omega' = u\tilde{\omega}$ for some $u \in \mathbb{C}^*$;
2. $L_{\psi, \xi}$ is irreducible;
3. any Whittaker $\widehat{H}_4[\tau]$-module of type $\psi$ for which $\psi(c(1)) \neq 0$ and $k$ acts by a non-zero scalar $\xi \in \mathbb{C}$ is irreducible and isomorphic to $L_{\psi, \xi}$.

Proof. (1) It is clear that if $\omega' = u\tilde{\omega}$ for some $u \in \mathbb{C}^*$, then $\omega'$ is a Whittaker vector. We now prove the necessity. Note that for $m, n \in \mathbb{Z}_+$,

$$[c(2m + 1), d(-2n - 1)] = (2m + 1)\delta_{m,n}.$$ 

Then

$$s = \bigoplus_{m \in \mathbb{Z}_+} (\mathbb{C}c(2m + 1) \oplus \mathbb{C}d(-2m - 1)) \oplus \mathbb{C}k$$

is a Heisenberg algebra and $L_{\psi, \xi}$ can be viewed as an $s$-module on which $k$ acts as $\xi \neq 0$. Since every highest weight $s$-module generated by one element with $k$ acting as a non-zero scalar is irreducible, it follows that $L_{\psi, \xi}$ can be decomposed into a direct sum of irreducible highest weight modules of $s$ with the highest weight vectors

$$\{ (a + b)(-\bar{\mu})(a - b)(-\nu)c(-\eta)d(-1)^{t}\tilde{\omega} \mid (\bar{\mu}, \nu, \eta) \in \bar{\mathcal{P}}_{\text{even}} \times \mathcal{P}_{\text{odd}} \times \mathcal{P}_{\text{odd}}, t \in \mathbb{N} \}.$$ 

Thus if $\omega \neq \omega' \in L_{\psi, \xi}$ is a Whittaker vector, then $\omega'$ can be written as

$$\omega' = \sum_{i \in I} x_i (a + b)(-\bar{\mu}^{(i)})(a - b)(-\nu^{(i)})c(-\eta^{(i)})d(-1)^{t_i}\tilde{\omega},$$

where $I$ is a finite index set, $x_i \in \mathbb{C}^*$. If there exists $i \in I$ such that $t_i > 0$, by considering the action of $c(1)$ on $\omega'$, we obtain

$$(c(1) - \psi(c(1)))\omega' = \sum_{i \in I} x_i t_i \xi (a + b)(-\bar{\mu}^{(i)})(a - b)(-\nu^{(i)})c(-\eta^{(i)})d(-1)^{t_i-1}\tilde{\omega} \neq 0,$$

a contradiction. Then

$$\omega' = \sum_{i \in I} x_i (a + b)(-\bar{\mu}^{(i)})(a - b)(-\nu^{(i)})c(-\eta^{(i)})\tilde{\omega},$$

where $I$ is a finite index set, $x_i \in \mathbb{C}^*$. For $i \in I$, let

$$\bar{\mu}^{(i)} = (0^{a_{i0}}, 2^{a_{i1}}, \ldots, (2n)^{a_{in}}),$$

$$\nu^{(i)} = (1^{b_{i0}}, 3^{b_{i1}}, \ldots, (2m + 1)^{b_{im}}),$$

$$\eta^{(i)} = (1^{c_{i0}}, 3^{c_{i1}}, \ldots, (2l + 1)^{c_{il}}),$$

and

$$\omega = \sum_{i \in I} x_i (a + b)(-\bar{\mu}^{(i)})(a - b)(-\nu^{(i)})c(-\eta^{(i)})\tilde{\omega},$$

$$\tilde{\omega} = \left\{ (a + b)(-\bar{\mu})(a - b)(-\nu)c(-\eta)d(-1)^{t}\tilde{\omega} \mid (\bar{\mu}, \nu, \eta) \in \bar{\mathcal{P}}_{\text{even}} \times \mathcal{P}_{\text{odd}} \times \mathcal{P}_{\text{odd}}, t \in \mathbb{N} \}.$$
where \( n, m, l, a_{ij}, b_{ij}, c_{ij} \in \mathbb{N} \). Then \( \omega' \) can be written as

\[
\omega' = \sum_{i \in I} x_i(a+b)(0)^{a_0}(a+b)(-2)^{a_1} \cdots (a+b)(-2n)^{a_n} \\
(a-b)(-1)^{b_0}(a-b)(-3)^{b_1} \cdots (a-b)(-2m-1)^{b_m} c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l-1)^{c_l} \bar{\omega},
\]

where \( I \) is a finite index set, \( n, m, l, a_{ij}, b_{ij}, c_{ij} \in \mathbb{N} \).

**Case I** Suppose \( n \geq 0 \) and \( \{a_{in} \mid i \in I\} \neq \{0\} \).

If \( m \geq n \) and there exists \( i \in I \) such that \( b_{im} \neq 0 \), by (2.5) and (2.6), we have

\[
((a+b)(2m+2) - \psi((a+b)(2m+2)))\omega' = \sum_{i \in I} x_i(-2\psi(c(1))b_{im}(a+b)(0)^{a_0}(a+b)(-2)^{a_1} \cdots (a+b)(-2n)^{a_n} \\
(a-b)(-1)^{b_0}(a-b)(-3)^{b_1} \cdots (a-b)(-2m-1)^{b_m} c(-\eta^{(i)})\bar{\omega} \neq 0,
\]
a contradiction.

If \( m < n \) and there exists \( i \in I \) such that \( b_{im} \neq 0 \), by (2.7) and (2.8), we obtain

\[
((a-b)(2n+1) - \psi((a-b)(2n+1)))\omega' = \sum_{i \in I} x_i 2\psi(c(1))a_{im}(a+b)(0)^{a_0}(a+b)(-2)^{a_1} \cdots (a+b)(-2n)^{a_n-1} \\
(a-b)(-1)^{b_0}(a-b)(-3)^{b_1} \cdots (a-b)(-2m-1)^{b_m} c(-\eta^{(i)})\bar{\omega} \neq 0,
\]
a contradiction.

**Case II** Suppose \( m \geq 0 \) and \( \{b_{im} \mid i \in I\} \neq \{0\} \) and \( \{a_{in} \mid i \in I\} = \{0\} \) for any \( n \geq 0 \).

Then \( v \) has the following form

\[
\omega' = \sum_{i \in I} x_i(a-b)(-1)^{b_0}(a-b)(-3)^{b_1} \cdots (a-b)(-2m-1)^{b_m} c(-\eta^{(i)})\bar{\omega}.
\]

By considering the action of \((a+b)(2m+2)\) on \( \omega' \). We also get a contradiction.

**Case III** Suppose \( l \geq 0 \) and \( \{c_{il} \mid i \in I\} \neq \{0\} \) and \( \{a_{in} \mid i \in I\} = \{b_{im} \mid i \in I\} = \{0\} \) for any \( n, m \geq 0 \).

In this case,

\[
\omega' = \sum_{i \in I} x_i c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l-1)^{c_l} \bar{\omega}.
\]

We have

\[
(d(2l+1) - \psi(d(2l+1)))\omega' = \sum_{i \in I} x_i c_{il}(2l+1)\xi c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l-1)^{c_l-1} \bar{\omega} \neq 0,
\]
a contradiction. So we deduce that \( \omega' = u \bar{\omega} \) for some \( u \in \mathbb{C}^* \), with which we prove (1).

(2) follows from (1) and (iii) of Lemma 2.1. (3) follows from (2) and (iv) of Proposition 2.4. \( \square \)
Corollary 3.3. Let $\psi(c(1)) \neq 0$ and $M_\psi$ be a universal Whittaker module for $\hat{H}_4[\tau]$, generated by the Whittaker vector $\omega = 1 \otimes 1$. Then $v \in M_\psi$ is a Whittaker vector if and only if $v = u\omega$ for some $u \in \mathbb{C}[k]$.

Proof. Since $k$ is the center of $\hat{H}_4[\tau]$, it is easy to see that $v = u\omega$ is a Whittaker vector if $u \in \mathbb{C}[k]$. Conversely, let $v \in M_\psi$ be a Whittaker vector. By (2.3), we can write

$$v = \sum_{\mu,\nu,\lambda,\eta} p_{\mu,\nu,\lambda,\eta}(k)(a + b)(-\mu)(a - b)(-\nu)d(-\lambda)c(-\eta)\omega,$$

where $p_{\mu,\nu,\lambda,\eta}(k) \in \mathbb{C}[k]$. There exists a module homomorphism $\varphi : M_\psi \rightarrow L_{\psi,\xi}$ with $\omega \mapsto \bar{\omega}$. Clearly

$$\varphi(v) = \sum_{\mu,\nu,\lambda,\eta} p_{\mu,\nu,\lambda,\eta}(\xi)(a + b)(-\mu)(a - b)(-\nu)d(-\lambda)c(-\eta)\bar{\omega}$$

is a Whittaker vector of $L_{\psi,\xi}$. Then by Theorem 3.2, $\varphi(v) \in \mathbb{C}\bar{\omega}$. Thus $(\bar{\mu}, \bar{\nu}, \bar{\lambda}, \bar{\eta}) = (0, 0, 0, 0)$ for any $p_{\mu,\nu,\lambda,\eta}(k) \neq 0$, then $v = u\omega$ for $u \in \mathbb{C}[k]$. \qed

4 Whittaker vectors for the case that $\psi(c(1)) \neq 0$ and $\xi = 0$

In this section, we will give all the Whittaker vectors of $L_{\psi,0}$ for which $\psi(c(1)) \neq 0$. The following is the main result of this section.

Theorem 4.1. Let $\psi : \hat{H}_4[\tau]^{(+)} \rightarrow \mathbb{C}$ be a Lie algebra homomorphism such that $\psi(c(1)) \neq 0$. Then any Whittaker vector of the Whittaker module $L_{\psi,0}$ for $\hat{H}_4[\tau]$ is a non-zero linear combination of elements in $\{\bar{\omega}, c(\eta)\bar{\omega} \mid \eta \in \mathcal{P}_{odd}\}$.

Proof. Since $\xi = 0$, it is easy to see that $\{c(\eta)\bar{\omega} \mid \eta \in \mathcal{P}_{odd}\}$ are Whittaker vectors. Denote by $N$ the submodule of $L_{\psi,0}$ generated by $\{c(\eta)\bar{\omega} \mid \eta \in \mathcal{P}_{odd}\}$.

It suffices to prove that $\bar{\omega}$ is the only linear independent Whittaker vector of $L_{\psi,0}/N$. For convenience, we still denote by $u$ the image of $u \in L_{\psi,0}$ in $L_{\psi,0}/N$. Assume that $u \in L_{\psi,0}/N$ is a Whittaker vector. Suppose that $u \neq \bar{\omega}$. Then we may assume that

$$u = \sum_{i \in I} x_i d(-1)^{c_{i0}} \cdots d(-2m - 1)^{c_{im}}(a + b)(0)^{a_{i0}} \cdots (a + b)(-2n)^{a_{in}} \cdot (a - b)(-1)^{b_{i0}} \cdots (a - b)(-2l - 1)^{b_{il}}\bar{\omega},$$

where $I$ is a finite set of index, $m, n, l, c_{ij}, a_{ij}, b_{ij} \in \mathbb{N}$, $x_i \in \mathbb{C}^*$. For $i \in I$, set

$$u_i = d(-1)^{c_{i0}} \cdots d(-2m - 1)^{c_{im}}(a + b)(0)^{a_{i0}} \cdots (a + b)(-2n)^{a_{in}}(a - b)(-1)^{b_{i0}} \cdots (a - b)(-2l - 1)^{b_{il}}\bar{\omega}.$$

Suppose that there exists $i \in I$ such that $c_{im} > 0$. Let $J \in I$ be such that for $i \in J$,

$$\sum_{k=0}^{m} c_{ik} = \max_{j \in I} \left\{ \sum_{k=0}^{m} c_{jk} \right\}.$$
By considering the action of \((a - b)(2k + 1)\) on \(u\), if for some \(i \in J\) there exists \(0 \leq k \leq n\) such that \(a_{ik} \neq 0\), we can get

\[
2\psi(c(1))x_i a_{ik} d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{im}(a + b)(0)^{a_{10}} \cdots (a + b)(-2k)^{a_{ik} a_{ik} - 1} \cdots (a + b)(-2n)^{a_{im}}(a - b)(-1)^{b_{im}} \cdots (a - b)(-2l - 1)^{b_{il} \omega} = 0,
\]
a contradiction. So we have for \(i \in J\),

\[
\sum_{k=0}^{n} a_{ik} = 0. \tag{4.1}
\]

Similarly for \(i \in J\),

\[
\sum_{k=0}^{l} b_{ik} = 0. \tag{4.2}
\]

Let \(J_1 \subset J\) be such that for \(i \in J_1\), \(c_{im} \neq 0\). Let \(J_2 \subset J_1\) be such that

\[
c_{im} = \max_{j \in J_1} \{c_{jm}\}.
\]

We may assume that \(1 \in J_2\) and \(x_1 = 2\psi(c(1))\). By considering the action of \((a + b)(2k + 2)\) and \((a - b)(2k + 1)\), for \(0 \leq k \leq m\), and noticing that

\[
(a + b)(2k + 2)d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{im} \omega}
\]

\[
= -c_{1m}d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{im} - 1}(a - b)(-2m + 2k + 1)\omega + \cdots. \tag{4.3}
\]

\[
(a - b)(2k + 1)d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{im} \omega}
\]

\[
= -c_{1m}d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{im} - 1}(a + b)(-2m + 2k)\omega + \cdots. \tag{4.4}
\]

We can deduce that there exists \(i \in I\) such that

\[
c_{i0} = c_{10}, c_{i1} = c_{11}, \ldots, c_{im-1} = c_{1m-1}, c_{im} = c_{1m} - 1, \quad \sum_{k=0}^{n} a_{ik} + \sum_{k=0}^{l} b_{ik} \neq 0. \tag{4.5}
\]

Let \(I_1 \subset I\) be such that for \(i \in I_1\),

\[
\{c_{ij} : 0 \leq j \leq m\} \text{ satisfies (4.5)}. \tag{4.5}
\]

Let \(I_2 \subset I_1\) be such that for \(i \in I_2\),

\[
n_1 = \sum_{k=0}^{n} a_{ik} = \max_{i \in I_1} \left\{ \sum_{k=0}^{n} a_{ik} \right\}.
\]

Let \(I'_2 \subset I_1\) be such that for \(i \in I'_2\),

\[
l_1 = \sum_{k=0}^{l} b_{ik} = \max_{i \in I_1} \left\{ \sum_{k=0}^{l} b_{ik} \right\}.
\]

\[12\]
By (4.3), we have \( n > 0 \), and by (4.4), we have \( n_1 > 0 \). Then

\[
\begin{align*}
    u &= 2\psi(c(1))d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m}} \bar{\omega} \\
    &\quad + \sum_{i \in I_2} x_i d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1} (a + b)(0)^{a_0} \cdots (a + b)(-2n)^{a_{1m}}. \\
    &\quad (a - b)(-1)^{b_0} \cdots (a - b)(-2l - 1)^{b_{1l}} \bar{\omega} \\
    &\quad + \sum_{i \in I_1 \setminus (I'_2 \cup I_2)} x_i d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1} (a + b)(0)^{a_0} \cdots (a + b)(-2n)^{a_{1m}}. \\
    &\quad (a - b)(-1)^{b_0} \cdots (a - b)(-2l - 1)^{b_{1l}} \bar{\omega} + \sum_{i \in I_1 \setminus I_2} x_i u_i.
\end{align*}
\]

If \( n_1 > 2 \), let \( j \in I_2, 0 \leq k \leq n \) be such that \( a_{jk} \neq 0 \). Considering the action of \(((a - b)(2k + 1) - \psi((a - b)(2k + 1)))\) on \( u \), then the term

\[
2x_j a_{jk} \psi(c(1))d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1} (a + b)(0)^{a_0} \cdots (a + b)(-2k)^{a_{1k} - 1} \\
\cdots (a + b)(-2n)^{a_{jn}} (a - b)(-1)^{b_0} \cdots (a - b)(-2l - 1)^{b_{1l}} \bar{\omega}
\]

only appears in

\[
(a - b)(2k + 1) x_j d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1}. \\
(a + b)(0)^{a_0} \cdots (a + b)(-2k)^{a_{jk}} \cdots (a + b)(-2n)^{a_{jn}} (a - b)(-1)^{b_0} \cdots (a - b)(-2l - 1)^{b_{1l}} \bar{\omega}
\]

with coefficient \( 2x_j a_{jk} \psi(c(1)) \). This means that \( x_j = 0 \), a contradiction. So \( n_1 \leq 2 \). Similarly, we have \( l_1 \leq 2 \).

If for \( j \in I_2 \), there exists \( 0 \leq k \leq l \) such that \( b_{jk} \neq 0 \), considering the action of \((a + b)(2k + 2)\) on \( u \), we can deduce that \( x_j = 0 \), a contradiction. So we have

\[
\sum_{k=0}^{l} b_{jk} = 0, \text{ for } i \in I_2.
\]

Similarly, \( \sum_{k=0}^{n} a_{ik} = 0, \text{ for } i \in I'_2 \). So we have for \( i \in I_1 \),

\[
0 \leq \sum_{k=0}^{n} a_{ik} + \sum_{k=0}^{l} b_{ik} \leq 2.
\]

Suppose that there exists \( j \in I_1 \) such that

\[
u_j = d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1}(a - b)(-2k - 1) \bar{\omega},\]
for some $0 \leq k \leq l$. If $k \neq m$, consider the action of $(a + b)(2k + 2)$ on $u$. Since
\[
(a + b)(2k + 2)d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{1m}}\bar{\omega}
\]
\[
= -c_{1m}d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{1m-1}}(a - b)(2k - 2m + 1)\bar{\omega} + \cdots,
\]

it follows that the term
\[
d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{1m-1}}\bar{\omega}
\]
appears only in $x_j(a + b)(2k + 2)u_j$ with coefficient $-2\psi(c(1))x_j \neq 0$, a contradiction. So $k = m$ and
\[
x_j = -c_{1m}\psi((a - b)(1)).
\]

Suppose there exists $j \in I_1$ such that
\[
\sum_{k=0}^{n} a_{jk} = 1 \quad \text{and} \quad \sum_{k=0}^{l} b_{jk} = 1.
\]

We may assume that
\[
u_j = d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{1m-1}}(a + b)(-2r^{(j)})(a - b)(-2s^{(j)} - 1)\bar{\omega}.
\]

We consider the action of $(a + b)(2s^{(j)} + 2)$ on $u$, then the term
\[
d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{1m-1}}(a + b)(-2r^{(j)})\bar{\omega}
\]
appears only in
\[
x_j(a + b)(2s^{(j)} + 2)u_j
\]
with coefficient $-2x_j\psi(c(1)) \neq 0$, a contradiction. So there exists no $j \in I_1$ such that
\[
\sum_{k=0}^{n} a_{jk} = 1, \quad \sum_{k=0}^{l} b_{jk} = 1.
\]

Let $j \in I_1$ be such that
\[
\sum_{k=0}^{n} a_{jk} = 1 \quad \text{and} \quad \sum_{k=0}^{l} b_{jk} = 0.
\]

We may assume that
\[
u_j = d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{1m-1}}(a + b)(-2m^{(j)})\bar{\omega}.
\]

Consider the action of $(a - b)(2m^{(j)} + 1)$ on $u$, then the term
\[
d(-1)^{c_{10}} \cdots d(-2m - 1)^{c_{1m-1}}\bar{\omega}
\]
appears only in
\[ x_j(a - b)(2m^{(j)} + 1)u_j \]
with coefficient \(2\psi(c(1))x_j \neq 0\), a contradiction. So for \(j \in I_1\), if \(\sum_{k=0}^n a_{jk} \neq 0\), then
\[ \sum_{k=0}^n a_{jk} = 2 \text{ and } \sum_{k=0}^l b_{jk} = 0. \]

Let \(I_4 \subseteq I_1\) be such that for \(j \in I_4\), \(\sum_{k=0}^n a_{jk} = 2 \text{ and } \sum_{k=0}^l b_{jk} = 0\). By (4.4), \(I_4 \neq \emptyset\). We may assume that for \(j \in I_4\),
\[ u_j = d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1}(a + b)(-2m^{(j)}_1)(a + b)(-2m^{(j)}_2)\omega, \]
and \(0 \leq m^{(j)}_1 \leq m^{(j)}_2\). If \(m^{(j)}_1 + m^{(j)}_2 \neq m\), we consider the action of \((a - b)(2m^{(j)}_2 + 1) - \psi((a - b)(2m^{(j)}_2))\) on \(u\), then the term
\[ d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1}(a + b)(-2m^{(j)}_1)\omega \]
appears only in \(x_j(a - b)(2m^{(j)}_2 + 1)u_j\) with coefficient
\[ 2x_j\psi(c(1)) \neq 0, \text{ if } m^{(j)}_1 \neq m^{(j)}_2, \text{ or } 4x_j\psi(c(1)) \neq 0, \text{ if } m^{(j)}_1 = m^{(j)}_2, \]
a contradiction. So \(m^{(j)}_1 + m^{(j)}_2 = m\), that is,
\[ u_j = d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1}(a + b)(-2m^{(j)}_1)(a + b)(-2m + 2m^{(j)}_1)\omega. \quad (4.6) \]
By (4.4), we have for \(j \in I_4\),
\[ x_j = c_{1m}, \text{ if } m^{(j)}_1 \neq m/2, \]
or
\[ x_j = \frac{1}{2} c_{1m}, \text{ if } m^{(j)}_1 = m/2. \]
Let \(I_5 \subseteq I_1\) be such that for \(j \in I_5\), \(\sum_{k=0}^n a_{jk} = 0 \text{ and } \sum_{k=0}^l b_{jk} = 2\). By (4.3), \(I_5 \neq \emptyset\), if \(m \geq 1\). We may assume that
\[ u_j = d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1}(a - b)(-2r^{(j)}_1 - 1)(a - b)(-2r^{(j)}_2 - 1)\omega \]
such that \(r^{(j)}_2 \geq r^{(j)}_1 \geq 0\). If \(r^{(j)}_1 + r^{(j)}_2 \neq m - 1\), we consider the action of \((a + b)(2r^{(j)}_2 + 2)\) on \(u\), then the term
\[ d(-1)^{c_1} \cdots d(-2m - 1)^{c_{1m} - 1}(a - b)(-2r^{(j)}_1 - 1)\omega \]
appears only in $x_j(a + b)(2r_j^{(j)} + 2)u_j$ with coefficient
\[
-2x_j \psi(c(1)) \neq 0, \text{ if } r_1^{(j)} \neq r_2^{(j)}, \text{ or } \\
-4x_j \psi(c(1)) \neq 0, \text{ if } r_1^{(j)} = r_2^{(j)},
\]
a contradiction. Then together with (4.3), we have for $j \in I_5$,
\[
u_j = d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}(a - b)(-2r_1^{(j)} - 1)(a - b)(-2m + 2r_1^{(j)} + 1)\bar{\omega},
\]
for some $0 \leq r_1^{(j)} \leq \frac{m-1}{2}$ with coefficient $-c_{1m}$ if $r_1^{(j)} \neq \frac{m-1}{2}$, or $-\frac{1}{2}c_{1m}$ if $r_1^{(j)} = \frac{m-1}{2}$.

By the above proof, we have
\[
u = 2\psi(c(1))d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}\bar{\omega} \\
-c_{1m}\psi((a - b)(1))d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}(a - b)(-2m - 1)\bar{\omega} \\
+ \sum_{i \in I_4} x_i d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}(a + b)(-2m + 2m_1^{(i)})\bar{\omega} \\
+ \sum_{i \in I_5} x_i d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}(a - b)(-2r_1^{(j)} - 1)(a - b)(-2m + 2r_1^{(j)} + 1)\bar{\omega} \\
+ \sum_{i \in I_4 \setminus (I_4 \cup I_5)} x_i u_i + \sum_{i \in I_4 \cap I_5} x_i u_i,
\]
where for $j \in I_4$, $0 \leq m_1^{(j)} \leq m/2$, and $x_j = c_{1m}$ or $\frac{1}{2}c_{1m}$, depending on $m_1^{(j)} \neq m/2$ or $m_1^{(j)} = m/2$, and for $j \in I_5$, $0 \leq r_1^{(j)} \leq \frac{m-1}{2}$, $x_j = -c_{1m}$ or $-\frac{1}{2}c_{1m}$, depending on $r_1^{(j)} \neq \frac{m-1}{2}$ or $r_1^{(j)} = \frac{m-1}{2}$.

We now consider the action of $d(2m + 1)$ on $u$, then we have
\[
(d(2m + 1) - \psi(d(2m + 1)))u \\
= -c_{1m}\psi((a - b)(1))d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}(a + b)(0)\bar{\omega} \\
+ \sum_{i \in I_4} x_i d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}(2\psi(c(1)) + (a + b)(-2m + 2m_1^{(i)}) \bar{\omega}) \\
(a - b)(2m - 2m_1^{(i)} + 1) + (a + b)(-2m_1^{(i)})(a - b)(2m_1^{(i)} + 1)\bar{\omega} \\
+ \sum_{i \in I_5} x_i d(-1)^{c_{i_0}} \cdots d(-2m - 1)^{c_{m-1}}(-2\psi(c(1)) + (a - b)(-2m + 2r_1^{(i)} + 1) \bar{\omega}) \\
(a + b)(2m - 2r_1^{(i)}) + (a - b)(-2r_1^{(i)} - 1)(a + b)(2r_1^{(i)} + 2)\bar{\omega} \\
+ \sum_{i \in I_4 \setminus I_1} (d(2m + 1) - \psi(d(2m + 1)))x_i u_i = 0.
\]
So we have
\[
\left(\sum_{i \in I_4} x_i - \sum_{j \in I_5} x_j\right) \psi(c(1)) = 0.
\]
Since for $i \in I_4, j \in I_5$, $x_i = c_{1m}$ or $\frac{1}{2}c_{1m}$, $x_j = -c_{1m}$ or $-\frac{1}{2}c_{1m}$, and $\psi(c(1)) \neq 0$, it follows that $c_{1m} = 0$, a contradiction. This proves that for all $i \in I$,
\[
c_{i_0} = \cdots = c_{1m} = 0.
\]
Then
\[ u = \sum_{i \in I} x_i(a+b)(0)^{a_0} \cdots (a+b)(-2n)^{a_\infty}(a-b)(-1)^{b_0} \cdots (a-b)(-2l-1)^{b_d} \bar{\omega} \]
is a Whittaker vector of type \( \psi \) of the Weyl algebra, which is linearly spanned by
\[ \{(a+b)(2r), (a-b)(2s+1), c(1) \mid r, s \in \mathbb{Z} \}, \]
with relations:
\[ [(a+b)(2r), (a-b)(2s+1)] = -2\delta_{r+s,0}c(1). \]
Then we immediately have \( u = \bar{\omega} \).

5 Whittaker vectors for \( \psi(c(1)) = 0 \)

In this section, we assume that \( \psi(c(1)) = 0 \), that is, \( \psi \) is singular. We shall continue to investigate the form of Whittaker vectors and the connections between Whittaker modules and Verma modules for the twisted affine Nappi-Witten Lie algebra \( \tilde{H}_4[\tau] \).

We still follow the notations in Theorem 3.2 and Corollary 3.3. In particular, \( \bar{\omega} \) (resp. \( \omega \)) denotes the cyclic Whittaker vector \( \bar{1} \otimes 1 \) (resp. \( 1 \otimes 1 \)) for \( L_{\psi,\xi} \) (resp. \( M_{\psi} \)).

Note that \( \psi((a+b)(n)) = \psi((a-b)(m)) = \psi(c(l)) = 0 \) for \( n \in 2\mathbb{Z}_+, m \in 2\mathbb{Z}_+ + 1 \), \( l \in 2\mathbb{N} + 1 \). Denote \( \psi((a-b)(1)) \) by \( \sigma_1 \), we have the following lemma by straightforward computations.

Lemma 5.1. (i) If \( \sigma_1 = 0 \), then \( (a+b)(0)\bar{\omega} \) is a Whittaker vector of \( L_{\psi,\xi} \);
(ii) If \( \sigma_1 \neq 0 \), then \( (\xi(a+b)(0) - \sigma_1 c(-1))\bar{\omega} \) is a Whittaker vector of \( L_{\psi,\xi} \).

The following theorem gives a full characterization of Whittaker vectors of \( L_{\psi,\xi} \) for \( \psi(c(1)) = 0 \), \( \xi \neq 0 \).

Theorem 5.2. Let \( \psi(c(1)) = 0 \), \( \xi \neq 0 \). Set
\[ z = \begin{cases} 
(a+b)(0), & \text{if } \sigma_1 = 0, \\
\xi(a+b)(0) - \sigma_1 c(-1), & \text{if } \sigma_1 \neq 0,
\end{cases} \]
then \( v \) is a Whittaker vector of \( L_{\psi,\xi} \) if and only if \( v = uz \) for some \( u \in \mathbb{C}[z] \), where \( \mathbb{C}[z] \) is the polynomial algebra generated by \( z \).

Proof. By Lemma 5.1 and induction on the degree of \( u \) as a polynomial, it is easy to check that \( \mathbb{C}[z]\bar{\omega} \) are Whittaker vectors of \( L_{\psi,\xi} \). Conversely, let \( v \) be a Whittaker vector of \( L_{\psi,\xi} \). Note that for \( m, n \in 2\mathbb{N} + 1 \),
\[ [c(m), d(-n)] = m\delta_m,nk, \]
then
\[ s = \bigoplus_{m \in 2\mathbb{N} + 1} (\mathbb{C}c(m) \oplus \mathbb{C}d(-m)) \oplus \mathbb{C}k \]
is a Heisenberg algebra and \( L_{\psi, \xi} \) can be viewed as an \( \mathfrak{s} \)-module such that \( k \) acts as \( \xi \neq 0 \). Since every highest weight \( \mathfrak{s} \)-module generated by one element with \( k \) acting as a non-zero scalar is irreducible, it follows that \( L_{\psi, \xi} \) can be decomposed into a direct sum of irreducible highest weight modules of \( \mathfrak{s} \) with the highest weight vectors

\[
\{(a + b)(-\mu)(a - b)(-\nu)c(-\eta)\bar{\omega} \mid (\mu, \nu, \eta) \in \mathcal{P}_{even} \times \mathcal{P}_{odd} \times \mathcal{P}_{odd}\}.
\]

So if \( \bar{\omega} \neq v \in L_{\psi, \xi} \) is a Whittaker vector, then \( v \) is a linear combination of elements of the form

\[
(a + b)(-\mu)(a - b)(-\nu)c(-\eta)\bar{\omega}, \quad (\mu, \nu, \eta) \in \mathcal{P}_{even} \times \mathcal{P}_{odd} \times \mathcal{P}_{odd}.
\]

We may assume that

\[
v = \sum_{i \in I} x_i(a + b)(0)^{a_0}(a + b)(-2)^{a_1} \cdots (a + b)(-2n)^{a_{im}}.
\]

\[
(a - b)(-1)^{b_0}(a - b)(-3)^{b_1} \cdots (a - b)(-2m - 1)^{b_{im}} c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l - 1)^{c_i} \bar{\omega},
\]
352

where \( I \) is a finite index set, \( n, m, l, a_{ij}, b_{ij}, c_{ij} \in \mathbb{N} \).

**Case I** Suppose \( n > 0 \) and \( \{a_{im} \mid i \in I\} \neq \{0\} \).

If \( m > n \) and there exists \( i \in I \) such that \( b_{im} \neq 0 \), we obtain

\[
((a - b)(2m + 1) - \psi((a - b)(2m + 1)))v
\]

\[
= \sum_{i \in I} x_i(2(2m + 1)\xi b_{im}(a + b)(0)^{a_0}(a + b)(-2)^{a_1} \cdots (a + b)(-2n)^{a_{im}}).
\]

\[
(a - b)(-1)^{b_0}(a - b)(-3)^{b_1} \cdots (a - b)(-2m - 1)^{b_{im} - 1} c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l - 1)^{c_i} \bar{\omega}
\]

\[
\neq 0,
\]

a contradiction. Now assume \( m < n \) and there exists \( i \in I \) such that \( b_{im} \neq 0 \), we deduce

\[
((a + b)(2n) - \psi((a + b)(2n)))v
\]

\[
= \sum_{i \in I} x_i(2(2n)\xi a_{im}(a + b)(0)^{a_0}(a + b)(-2)^{a_1} \cdots (a + b)(-2n)^{a_{im} - 1}).
\]

\[
(a - b)(-1)^{b_0}(a - b)(-3)^{b_1} \cdots (a - b)(-2m - 1)^{b_{im}} c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l - 1)^{c_i} \bar{\omega}
\]

\[
\neq 0,
\]

a contradiction again.

**Case II** Suppose \( m \geq 0 \) and \( \{b_{im} \mid i \in I\} \neq \{0\} \) and \( \{a_{im} \mid i \in I\} = \{0\} \) for any \( n > 0 \).

Then \( v \) has the following form

\[
v = \sum_{i \in I} x_i(a + b)(0)^{a_0}(a - b)(-1)^{b_0}(a - b)(-3)^{b_1} \cdots (a - b)(-2m - 1)^{b_{im}}.
\]

\[
c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l - 1)^{c_i} \bar{\omega}.
\]

It is easy to check that \((a - b)(2m + 1) - \psi((a - b)(2m + 1)))v \neq 0, \) a contradiction.
Case III Suppose \( l \geq 1 \) and \( \{c_{il} \mid i \in I\} \neq \{0\} \) and \( \{a_{in} \mid i \in I\} = \{b_{in} \mid i \in I\} = \{0\} \) for any \( n > 0, m \geq 0 \).

In this case,

\[
v = \sum_{i \in I} x_i (a + b)(0)^{a_0} c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l - 1)^{c_l} \bar{\omega}.
\]

We have

\[
(d(2l + 1) - \psi(d(2l + 1)))v = \sum_{i \in I} x_i (2l + 1) \xi c_{il}(a + b)(0)^{a_0} c(-1)^{c_0} c(-3)^{c_1} \cdots c(-2l - 1)^{c_l - 1} \bar{\omega} \neq 0,
\]
a contradiction.

If \( \sigma_1 = 0 \), by Case III, it also leads to a contradiction for \( l \geq 0 \). Then

\[
v = \sum_{i \in I} x_i (a + b)(0)^{a_0} \bar{\omega},
\]

and \( v \in \mathbb{C}[(a + b)(0)] \bar{\omega} \). If \( \sigma_1 \neq 0 \), then

\[
v = \sum_{i \in I} x_i (a + b)(0)^{a_0} c(-1)^{c_0} \bar{\omega}.
\]

For any \( n \in 2\mathbb{Z}_+, m \in 2\mathbb{N} + 1 \), it is easy to see that

\[
((a + b)(n) - \psi((a + b)(m)))v = ((a - b)(m) - \psi((a - b)(m)))v = (c(m) - \psi(c(m)))v = 0.
\]

We consider \((d(m) - \psi(d(m)))v \) for \( m \in 2\mathbb{N} + 1 \). Then we have

\[
(d(m) - \psi(d(m)))v = \sum_{i \in I} x_i ([d(m), (a + b)(0)^{a_0}] c(-1)^{c_0} \bar{\omega} + (a + b)(0)^{a_0} [d(m), c(-1)^{c_0}] \bar{\omega})
\]

\[
= \sum_{i \in I} x_i (a_{i0}(a - b)(m)(a + b)(0)^{a_0 - 1} c(-1)^{c_0} \bar{\omega} + m \delta_{m,1} \xi c_{i0}(a + b)(0)^{a_0} c(-1)^{c_0 - 1} \bar{\omega}).
\]

By the above equation, if \( m > 1 \), \((d(m) - \psi(d(m)))v = 0 \). If \( m = 1 \), we have

\[
(d(1) - \psi(d(1)))v = \sum_{i \in I} x_i (a_{i0}(a - b)(1)(a + b)(0)^{a_0 - 1} c(-1)^{c_0} + \xi c_{i0}(a + b)(0)^{a_0} c(-1)^{c_0 - 1} \bar{\omega}) = 0,
\]

as \( v \) is a Whittaker vector.
In the following we prove that \( v = \sum_{i \in I} x_i(a+b)(0)^{a_0} c(-1)^{c_0} \bar{\omega} \in \mathbb{C}[z] \bar{\omega} \), where \( z = \xi(a+b)(0) - \sigma_1 c(-1) \). We may assume that 

\[ c_{10} = \max_{i \in I} \{ c_i \}. \]

By (5.1), \( a_{10} = 0 \). Then we have

\[
v = \sum_{i \in I} x_i(a+b)(0)^{a_0} c(-1)^{c_0} \bar{\omega} \\
= x'_1 c(-1)^{c_{10}} \bar{\omega} + \sum_{i \in I, c_{10} < c_{10}} x_i(a+b)(0)^{a_0} c(-1)^{c_0} \bar{\omega} \\
= x'_1 \sigma_1^{c_0} \sigma_1 c(-1) - \xi(a+b)(0)^{c_{10}} \bar{\omega} + \sum_{i \in I, g_{10} < c_{10}} y_i(a+b)(0)^{h_0} c(-1)^{g_0} \bar{\omega}.
\]

Denote \( \sum_{i \in I, g_{10} < c_{10}} y_i(a+b)(0)^{h_0} c(-1)^{g_0} \bar{\omega} \) by \( v' \). Then \( v' \) is also a Whittaker vector. By inductive assumption, \( v' \in \mathbb{C}[z] \bar{\omega} \). Thus \( v \in \mathbb{C}[z] \bar{\omega} \).

Remark 5.3. Suppose \( \psi(c(1)) = 0 \). Set

\[
z = \begin{cases} 
(a+b)(0), & \text{if } \sigma_1 = 0, \\
(a+b)(0)k - \sigma_1 c(-1), & \text{if } \sigma_1 \neq 0.
\end{cases}
\]

The same argument as in Theorem 5.2, by replacing \( \xi \) with \( k \) and \( \bar{\omega} \) with \( \omega \) whenever necessary, proves that \( v \) is a Whittaker vector of \( M_\psi \) if and only if \( v = u \omega \) for some \( u \in \mathbb{C}[z,k] \), where \( \mathbb{C}[z,k] \) is the polynomial algebra generated by \( z \) and \( k \).

In the following we consider the case that \( \psi \) is identically zero. We first recall the definition of Verma module for the twisted affine Nappi-Witten Lie algebra \( \hat{H}_4[\tau] \) given in [8]. If \( \psi \) is identically zero, for \( l \in \mathbb{C} \), let

\[ M_l = U(\hat{H}_4[\tau])((a+b)(0) - l) \bar{\omega}, \]

then the quotient module 

\[ M(\xi,l) := L_{\psi,\xi}/M_l \]

is a Verma module for \( \hat{H}_4[\tau] \). By Theorem 2.1 of [8], we immediately obtain the following lemma.

Lemma 5.4. ([8]) For \( \xi, l \in \mathbb{C} \), the Verma module \( M(\xi,l) \) of \( \hat{H}_4[\tau] \) is irreducible if and only if \( \xi \neq 0 \).

Theorem 5.5. If \( \psi \) is identically zero and \( \xi \neq 0 \), for each \( l \in \mathbb{C} \) and \( i \in \mathbb{Z}_{\geq 1} \), define

\[ M^i = U(\hat{H}_4[\tau])((a+b)(0) - l)^i \bar{\omega}. \]

Then}

20
(i) $M^i$ is a Whittaker submodule of $L_{0,\xi}$, with a cyclic Whittaker vector $\omega_i = ((a + b)(0) - l)^i\bar{\omega}$, and $M^{i+1}$ is a maximal submodule of $M^i$;

(ii) $L_{0,\xi} \cong M^i$ for any $i \in \mathbb{N}$;

(iii) $M^i/M^{i+1} \cong M(\xi,l)$ with the multiplicity space consisting of polynomials in one variable;

(iv) $L_{0,\xi}$ has a filtration

$$L_{0,\xi} = M^0 \supseteq M^1 \supseteq \cdots \supseteq M^i \supseteq \cdots$$

with the simple sections given by the Verma module $M(\xi,l)$ with multiplicity infinity.

Proof. For (i), by Theorem 5.2, $((a + b)(0) - l)^i\bar{\omega}$ is a Whittaker vector of $L_{0,\xi}$, thus $M^i$ is a Whittaker module.

For (ii), we define the linear map for $i \in \mathbb{N}$,

$$\phi : L_{0,\xi} \to M^i$$

$$u\bar{\omega} \mapsto u((a + b)(0) - l)^i\bar{\omega},$$

where $u \in U(\hat{H}_4[\tau](-) \oplus \mathbb{C}(a + b)(0))$. It is easy to check that $\phi$ is an isomorphism of modules. Thus $L_{0,\xi} \cong M^i$.

For (iii), it is immediate that the linear map

$$\phi : L_{0,\xi} \to M^i/M^{i+1}$$

$$u\bar{\omega} \mapsto u((a + b)(0) - l)^i\bar{\omega},$$

is an epimorphism of modules and $\text{Ker} \phi = M^1$. Thus $M^i/M^{i+1} \cong L_{0,\xi}/M^1 = M(\xi,l)$. Then $M^{i+1}$ is a maximal submodule of $M^i$ by Lemma 5.4.

(iv) follows from (iii) and Lemma 5.4. □

Theorem 5.6. If $\psi$ is identically zero and $\xi = 0$, then $\{d(-\lambda)\bar{\omega} \mid \lambda \in \mathcal{P}_{\text{odd}}\}$ generates a maximal non-trivial submodule $U$ of $L_{0,0}$. Furthermore, $L_{0,0}/U$ is isomorphic to the one-dimensional $\hat{H}_4[\tau]$-module $\mathbb{C}\bar{\omega}$.

Proof. By the fact that $\psi \equiv 0$ and the definition of the submodule $U$, we see that

$$(a + b)(-n)\bar{\omega}, \ (a - b)(-m)\bar{\omega}, \ c(-m)\bar{\omega}, \ d(-m)\bar{\omega} \in U$$

for all $n \in 2\mathbb{N}$ and $m \in 2\mathbb{N} + 1$. Thus

$$(a + b)(-\bar{\mu})(a - b)(-\nu)d(-\lambda)c(-\eta)\bar{\omega} \in U$$

for all $(\bar{\mu}, \nu, \lambda, \eta) \in \tilde{P}_{\text{even}} \times P_{\text{odd}} \times P_{\text{odd}} \times P_{\text{odd}}$ with $\#(\bar{\mu}, \nu, \lambda, \eta) > 0$. Since $k\bar{\omega} = \hat{H}_4[\tau]^+\bar{\omega} = 0$, it follows that each element of $U$ can be written as a linear combination of elements of the form $(a + b)(-\bar{\mu})(a - b)(-\nu)d(-\lambda)c(-\eta)\bar{\omega}$ with $\#(\bar{\mu}, \nu, \lambda, \eta) > 0$. Moreover, $\bar{\omega} \notin U$ as $\xi = 0$. Therefore, $L_{0,0}/U$ is isomorphic to the one-dimensional $\hat{H}_4[\tau]$-module $\mathbb{C}\bar{\omega}$ and $U$ is a maximal non-trivial submodule of $L_{0,0}$. □
Remark 5.7. If $\psi$ is identically zero and $\xi = 0$, then by Lemma 5.4

$$M(0, l) = L_{0,0}/M_l$$

is reducible, where $M_l = U(\hat{H}_4[[\tau]])((a + b)(0) - l)\bar{\omega}$. Furthermore, denote by $\bar{\omega}$ the image of $\bar{\omega} \in L_{0,0}$ in $L_{0,0}/M_l$. By Theorem 2.1 of [8], we have

(i) if $l = 0$, $\bar{U} = \langle \{d(-\lambda)\bar{\omega} \mid \lambda \in P_{\text{odd}}\} \rangle$ is a maximal non-trivial submodule of $M(0, l)$;

(ii) if $l \neq 0$, $\bar{U}' = \langle \{c(-\eta)\bar{\omega} \mid \eta \in P_{\text{odd}}\} \rangle$ is a maximal non-trivial submodule of $M(0, l)$.

References

[1] D. Adamović, R. Lu, K. Zhao, Whittaker modules for the affine Lie algebra $A_1^{(1)}$, Adv. Math. 289 (2016) 438-479.

[2] D. Arnal, G. Pinczon, On algebraically irreducible representations of the Lie algebra $sl(2)$, J. Math. Phys. 15 (1974) 350-359.

[3] R. Block, The irreducible representations of the Lie algebra $sl(2)$ and of the Weyl algebra, Adv. Math. 39 (1) (1981) 69-110.

[4] Y. Bao, C. Jiang, Y. Pei, Representations of affine Nappi-Witten algebras, J. Algebra 342 (2011) 111-133.

[5] P. Batra, V. Mazorchuk, Blocks and modules for Whittaker pairs, J. Pure Appl. Algebra 215 (2011) 1552-1568.

[6] G. Benkart, M. Ondrus, Whittaker modules for generalized Weyl algebras, Represent. Theory 13 (2009) 141-164.

[7] K. Christodouloupolou, Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras, J. Algebra 339 (2008) 2871-2890.

[8] X. Chen, C. Jiang, Q. Jiang, Representations of the twisted affine Nappi-Witten algebras, J. Math. Phys. 54 (5) (2013) 051703, 20 pp.

[9] J. Distler, talk at the “Strings 93” Conference, Berkeley, May 1993.

[10] C. Jiang, S. Wang, Extension of vertex operator algebra $V_{\hat{H}_4}(l, 0)$, Algebra Colloq. 21 (3) (2014) 361-380.

[11] B. Kostant, On Whittaker vectors and representation theory, Invent. Math. 48 (1978) 101-184.

[12] V.G. Kac, Infinite-Dimensional Lie Algebras, third ed., Cambridge University Press, Cambridge, UK, 1990.
[13] E. Kiritsis, C. Kounnas, String propagation in gravitational wave backgrounds, Phys. Lett. B 594 (1994) 368-374.

[14] D. Liu, Y. Wu, L. Zhu, Whittaker modules for the twisted Heisenberg-Virasoro algebra, J. Math. Phys. 51 (2) (2010) 023524, 12 pp.

[15] C. Nappi, E. Witten, Wess-Zumino-Witten model based on a nonsemisimple group, Phys. Rev. Lett. 23 (1993) 3751-3753.

[16] M. Ondrus, Whittaker modules for $U_q(sl_2)$, J. Algebra 289 (2005) 192-213.

[17] M. Ondrus, E. Wiesner, Whittaker modules for the Virasoro algebra, J. Algebra Appl. 8 (3) (2009) 363-377.

[18] A. Sevostyanov, Quantum deformation of Whittaker modules and Toda lattice, Duke Math. J. 105 (2000) 211-238.

[19] E. Witten, Non-abelian bosonization in two-dimensions, Commun. Math. Phys. 92 (1984) 455-472.

[20] X. Zhang, S. Tan, H. Lian, Whittaker modules for the Schrödinger-Witt algebra, J. Math. Phys. 51 (8) (2010) 083524, 17 pp.