Anomalous dynamical scaling from nematic and U(1)-gauge field fluctuations in two dimensional metals

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We analyze the scaling theory of two-dimensional metallic electron systems in the presence of critical bosonic fluctuations with small wave vectors, which are either due to a U(1) gauge field, or generated by an Ising nematic quantum critical point. The one-loop dynamical exponent $z = 3$ of these critical systems was shown previously to be robust up to three-loop order. We show that the cancellations preventing anomalous contributions to $z$ at three-loop order have special reasons, such that anomalous dynamical scaling emerges at four-loop order.

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A well-known mechanism for non-Fermi liquid behavior in two-dimensional metals is provided by scattering at quantum critical bosonic degrees of freedom. The latter can be order parameter fluctuations at a quantum critical point [1], or emergent gauge fields arising from strong interactions [2]. The gauge field propagator diverges at small momenta and thus leads to singular forward scattering. Order parameter fluctuations can be singular at small or large momenta, depending on the nature of the phase transition. Critical order parameter fluctuations with small momenta are present at the onset of nematic order driven by a Pomeranchuk instability in metallic electron systems [3, 4]. In fact, the problem of two-dimensional fermions coupled to a U(1)-gauge field emerging in doped Mott insulators is closely related to the problem posed by an Ising nematic quantum critical point (QCP).

Both problems have a fairly long history. One-loop results for the bosonic propagator and the fermion self-energy were first derived in the context of the U(1)-gauge theory [5], and later for the case of a nematic QCP [3, 4]. The bosonic propagator is substantially modified by Landau damping, and the fermion self-energy scales as $|\omega|^{2/3}$ at low excitation energies, implying a pronounced non-Fermi liquid behavior without Landau quasi-particles. In case of an Ising nematic on a lattice, a momentum dependent form factor in the self-energy leads to a few "cold spots" on the Fermi surface, where quasi-particles survive [4]. The main contributions to the self-energy at Fermi momenta $k_F$ come from particle-hole excitations near $k_F$ and $-k_F$ with a small momentum transfer $q$ that is almost tangential to the Fermi surface, and an excitation energy of the order $|q|^3$. At the one-loop level, both bosonic and fermionic degrees of freedom obey scaling with a dynamical exponent $z = 3$.

Over many years, the one-loop result was expected to be robust. It was believed to be controlled by a $1/N_f$ expansion in the inverse fermion flavor number $N_f$ [6], and at the two-loop level no qualitative modifications were found [7]. Hence, it came as a surprise when Sung-Sik Lee [8] discovered that the naive $1/N_f$-expansion is not valid, and Feynman diagrams of arbitrary loop order contribute even in the limit $N_f \rightarrow \infty$. Shortly afterwards, Metlitski and Sachdev [9] formulated a general scaling theory for the nematic QCP and the related U(1)-gauge field problem. Focusing on dominant processes near $k_F$ and $-k_F$ they derived a low energy field theory which involves only states near those two Fermi points. Symmetry constraints allow for only two independent anomalous scaling exponents, a fermion anomalous dimension $\eta_f$ and an anomalous dynamical exponent $z \neq 3$. A small contribution to $\eta_f$ was found by Metlitski and Sachdev in a three-loop calculation of the fermion self-energy. A renormalization of $z$ might be obtained from a divergence of the boson self-energy, but that quantity was found to be finite up to three-loop order. These results were confirmed in subsequent extensions of the problem which were designed such that the loop expansion corresponds to an expansion in a suitably designed small parameter [10, 11].

Thus, the most important remaining issue is whether anomalous dynamical scaling appears at higher order or not, which has strong implications for observable quantities such as the compressibility [9]. In the following we clarify that question. It turns out that the absence of a renormalization of $z$ at three-loop order is due to cancellations which are specific to that order, and cannot be expected to hold at higher orders. We identify a divergent contribution to the boson self-energy at four-loop order which cannot be cancelled by other contributions, and will thus lead to anomalous dynamical scaling with $z \neq 3$.

Our analysis is based on the effective field theory for the low energy behavior derived by Lee [8, 12] and Metlitski and Sachdev [9]. Focusing on the dominant excitation processes near a Fermi point $k_F$ and its antipode $-k_F$, and discarding irrelevant terms, one obtains an effective low-energy theory described by the Lagrangian [9]

\begin{equation}
L = \sum_{s = \pm} \psi_s^\dagger \left( \eta \partial_x - is \partial_x - \partial_y^2 \right) \psi_s - \sum_{s = \pm} g_s \phi \psi_s^\dagger \phi \psi_s - \frac{N_f}{2e^2} (\partial \phi)^2 + \frac{N_f}{2} r \phi^2.
\end{equation}
Here $\phi$ is a bosonic scalar field, while $\psi_x$, $\psi_y$ are Grassmann fields with $N_f$ flavor components corresponding to fermionic excitations in the two "patches" near $\pm k_F$. In the U(1)-gauge field problem, $\phi$ is the transverse gauge field and $g_+ = -g_-$. For the Ising nematic, $\phi$ is the order parameter field and $g_+ = g_-$. In both cases the physical flavor number is $N_f = 2$. The derivatives are with respect to real space and imaginary time variables. The spatial coordinates have been chosen such that the corresponding momentum variables $k_x$ and $k_y$ are normal and tangential to the Fermi surface at $\pm k_F$, respectively. Several numerical prefactors have been absorbed by a rescaling of fields and space coordinates. In particular $|g_s| = 1$. The U(1)-gauge field is always massless so that $r = 0$. For the Ising nematic case, $r$ is generally finite, but vanishes at the QCP.

In random phase approximation (RPA), which corresponds to a one-loop calculation of the boson and fermion self-energies, the boson and fermion propagators at criticality ($r = 0$) have the form [5]

$$D^{-1}(q) = N_f \left( \frac{4\pi}{k^2} + \frac{|q_0|}{4\pi |q_y|} \right),$$

$$G_s^{-1}(k) = sk_x + k_y^2 - i \frac{k_0}{N_f} \frac{k_0}{|k_0|^{1/3}},$$

with $\kappa = 2 \times 4^{2/3}/(\sqrt{3} (4\pi)^{2/3})$. These propagators solve the RPA equations also self-consistently [6]. Note that the linear frequency term (proportional to $\eta$) in the fermion propagator is subleading compared to the self-energy and has therefore been discarded. The RPA solution describes a non-Fermi liquid with fermions which are strongly scattered at overdamped bosons. Both propagators are homogeneous under the scaling

$$q_y \to \lambda q_y, \quad q_x \to \lambda^2 q_x, \quad q_0 \to \lambda^z q_0$$

with a dynamical exponent $z = 3$. Scaling the fields accordingly by a factor $\lambda^2$, the time derivative in the Lagrangian (1) is irrelevant, the boson mass term is relevant, while all other terms are marginal.

Metlitski and Sachdev [9] have derived a general scaling ansatz for $D(q)$ and $G_s(k)$. Ward identities following from symmetries of the low-energy theory constrain the renormalization of the marginal terms in the Lagrangian such that only two independent renormalizations are possible: a rescaling of the fermion field $\psi = \psi^{1/2}_r$ and a renormalization of the coupling constant $\epsilon^2 = Z_r \epsilon^2$. The former yields an anomalous fermionic scaling dimension, the latter an anomalous dynamical exponent. In the framework of the field theoretical renormalization group, anomalous infrared scaling can be linked to ultraviolet (UV) divergences of the theory [13]. The anomalous scaling dimensions are thus given by [9]

$$\eta_\psi = -\Lambda \frac{\partial}{\partial \Lambda} \log Z_\psi,$$

$$\eta_e = 3 - z = \Lambda \frac{\partial}{\partial \Lambda} \log Z_e,$$

where $\Lambda$ is a UV cutoff restricting momenta and frequencies to $|q_y| \leq \Lambda$, $|q_x| \leq \Lambda^2$, and $|q_0| \leq \Lambda^3$. The anomalous dimensions determine the scaling behavior of physical quantities. For example, the fermion self-energy on the Fermi surface scales as $|\omega|^{(2-\eta_\psi)/z}$, and the fermionic density of states as $\omega^{\eta_\psi/z}$. In one-loop approximation one has $\eta_\psi = 0$ and $\eta_e = 0$, that is, $z = 3$.

Calculations beyond the one-loop approximation are generally performed by expanding around the RPA solution, that is, by inserting RPA propagators for the internal lines in Feynman diagrams representing higher order contributions [7–9]. This corresponds to a resummation of terms of arbitrary order. It reduces the infrared divergences and the number of diagrams contributing in a given loop order. However, the integrations are complicated by the non-rational frequency dependence of the fermionic RPA propagator. Since the one-loop fermion self-energy depends only on frequency, not momentum, Ward identities are still valid order by order at least in the zero frequency limit [9]. Metlitski and Sachdev computed the anomalous dimensions from the fermion and boson self-energies at zero frequency up to three-loop order [9]. At two-loop order, no contributions were found. At three-loop order (see Fig. 1), a small contribution to the fermion anomalous dimension, $\eta_\psi \approx \pm 0.065$ for $N_f = 2$, was discovered, with a plus (minus) sign for the nematic (gauge field) system. No contribution to $\eta_e$ was found up to three-loop order. Divergent contributions to the boson self-energy obtained from individual Feynman diagrams (of Aslamasov-Larkin type, see Fig. 1) cancel each other such that the sum is finite. It remained open whether similar cancellations occur at higher orders.

We now discuss the general structure of contributions to the anomalous scaling dimensions. The fermionic anomalous dimension $\eta_\psi$ is obtained from the logarithmic UV divergence of $Z_\psi = 1 - \partial \Sigma_e(k)/\partial k_s$, where $\Sigma_e(k)$ is the fermion self-energy and $k_s = sk_x + k_y^2$. The anomalous dynamical scaling dimension is determined by a logarithmic UV divergence of $Z_e = 1 + \frac{\lambda^2}{2N_f} \partial^2 H(q)/\partial q_y^2$, where

![Figure 1. Three-loop contributions to the boson (left) and fermion (right) self-energies computed by Metlitski and Sachdev [9]. The solid lines represent fermions, the wiggly lines bosons.](image-url)
The gain of a power in $\Lambda$ is a consequence of the symmetrization, while the 3-point loop decays as $\Lambda^{-1}$ for large $p$ and fixed $q$. Moreover, $p_y$ is constrained to the size of $q_y$. As a consequence, the symmetrized Aslamasov-Larkin diagram is finite in the UV limit [9]. The decay of the symmetrized 3-point loop with $\Lambda^{-1}$ is a consequence of the symmetrization, while the kinematic constraint of $p_y$ is present already in the unsymmetrized function $\Pi_{3,s}(q,p,-q-p)$.

The gain of a power in $\Lambda$ for a symmetrized loop with a fixed external momentum holds generally. In the Supplementary Materials we show that $\Pi_{N,s}^{ym}(q_1,\ldots,q_N)$ vanishes if one of the momenta $q_i$ vanishes. This implies that the symmetrized loop decays at least as $\Lambda^{-2(N-3)-1}$.
if one momentum stays fixed while the others tend to infinity. If a fixed momentum \( q \) is injected at a vertex and extracted at another vertex of the same loop, while all other external momenta become large (this is possible only for \( N \geq 4 \)), there is even a suppression of order \( \Lambda^{-2} \). This gain in power-counting from symmetrization guarantees that contributions to the boson self-energy diverge at most logarithmically, with a prefactor of the order \( q_y^2 \) for \( q_y = 0 \). Quadratic and linear UV divergences of the boson self-energy indicated by power-counting must cancel. The same conclusion can be drawn from a Ward identity following from current conservation [9]. On the other hand, the kinematic constraint suppressing the 3-point loop in the Aslamasov-Larkin diagram is more special. It does hold for \( N \)-point loops with general \( N \), too, but only in case that only one large bosonic momentum passes through the loop, that is, when a large momentum is injected into the loop at one of the vertices and extracted again at another, while the remaining \( N - 2 \) momenta \( q_i \) remain finite [17]. However, this restriction does not apply if a fermion loop is connected to a fermion line (open or closed) by three or more boson propagators, as in Fig. 3. For such diagrams, there is no suppression of the UV divergence from kinematic constraints.

![Diagram](image)

Figure 3. The necessary building block for a diagram to be singular.

the fermionic self-energy, a fermion loop connected to another fermion line by at least three boson propagators appears already at three-loop order (see Fig. 1). This is the logarithmically divergent contribution to \( Z_\psi \) identified by Metlitski and Sachdev [9]. For the boson self-energy, fermion lines connected by three boson lines as in Fig. 3 appear only at four-loop order. Examples are the diagrams (c), (d) and (e) in Fig. 2. Those diagrams can thus be expected to contribute to a logarithmic divergence of \( Z_\psi \), and thus to an anomalous dynamical exponent.

The computation of four-loop diagrams is difficult. To see that anomalous dynamical scaling indeed emerges at four-loop order, we have evaluated the sum of all diagrams in the symmetry class of the diagram (e) in Fig. 2. These are all diagrams where two four-point loops are connected by three boson propagators. There are six topologically distinct such diagrams. Six out of twelve integrations could be done analytically by residues, which leads to a lengthy expression involving several thousand terms. It is more convenient to perform only four integrations by residues, and the remaining eight integrations numerically. The numerical integration is complicated anyway by the singular structure of the integrands. Details of the evaluation are presented in the Supplementary Materials. Summing all contributions, we expect a logarithmic UV divergence of the form

\[
\Pi^{(4e)}(q) \sim C^{(4e)} \frac{q_y^2}{e^2} \log (\Lambda/|q_y|)
\]

for \( q_0 = q_x = 0 \) and \( \Lambda \gg |q_y| \). Our calculation clearly yields a UV divergence with a negative prefactor. A cancellation or kinematic constraint removing the divergence does not occur. Unfortunately, we were not able to obtain stable results for ratios \( \Lambda/|q_y| \) beyond \( 10^2 \). Up to that ratio the divergence seems to be actually stronger than logarithmic (but weaker than any power-law).

The four-loop diagrams of type (e) do not contain any divergent subdiagrams from vertex or self-energy insertions. The vertex corrections contained as subdiagrams in those diagrams include a 4-point loop with a fixed external boson momentum. The symmetrized vertex correction obtained by summing all permutations of boson vertices at the loop is thus finite due to the cancellations under symmetrization discussed above. The divergence in the sum of diagrams of type (e) is thus a primitive divergence which is not a consequence of the divergent fermionic renormalization \( Z_\psi \). By contrast, the diagram (c) in Fig. 2 contains a divergent self-energy insertion, and diagram (d) a vertex correction which is most likely divergent, too. These divergences have to be compensated by counterterms, to disentangle them from UV divergences contributing to \( Z_\psi \).

In summary, we have shown that the quantum field theory describing the Ising nematic QCP and non-relativistic electrons coupled to a \( U(1) \)-gauge field in two dimensions acquires anomalous dynamical scaling at four-loop order. After the unexpected discovery of a fermionic anomalous dimension at three-loop order by Metlitski and Sachdev [9], this establishes another significant deviation from the at first sight robust one-loop result in that theory. The results on the minimal order required for anomalous scaling seem more natural when viewing symmetrized fermion loops as effective bosonic interactions. Then, when counting only loops involving bosons, both anomalous dimensions actually appear at two-loop order.

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ANOMALOUS DYNAMICAL SCALING FROM NEMATIC AND U(1)-GAUGE FIELD FLUCTUATIONS IN TWO DIMENSIONAL METALS: SUPPLEMENTARY MATERIAL

In this Supplementary Material we provide an explicit expression for the fermion loops. We prove that symmetrized loops vanish when one of the external energy-momentum variables vanishes, and we present details on the calculation of the four-loop contribution to the bosonic self-energy.

Explicit expression for N-point loops

The \( N \)-point fermion loop on patch \( s \) is defined by an integrated product of \( N \) fermion propagators as

\[
\Pi_{N,s}(q_1, \ldots, q_N) = I_{N,s}(p_1, \ldots, p_N) = N_f \int \frac{d k_0}{2 \pi} \int \frac{2 \pi}{2 \pi} \prod_{j=1}^{N} G_s(k - p_j). \tag{S1}
\]

The energy-momentum variables \( q_j \) and \( p_j \) are related by \( q_j = p_{j+1} - p_j \) for \( j = 1, \ldots, N - 1 \), and \( q_N = p_1 - p_N \). Note that \( q_1 + \cdots + q_N = 0 \) due to energy and momentum conservation. A Feynman graph representing a fermion loop is shown in Fig. S1.

Since we expand around the one-loop (RPA) solution, the fermion propagators in the loop have the form

\[
G_s^{-1}(k) = s k_x + k_y^2 - i \tilde{\kappa} \frac{k_0}{|k_0|^{1/3}} \tag{S2}
\]

with \( \tilde{\kappa} = \kappa/N_f \).

The \( k_x \) and \( k_y \) integrations in Eq. (S1) can be easily done by residues. The \( N \)-point loop can then be written in the form

\[
I_{N,s} = \frac{N_f}{2} \sum_{i < j} \int_{p_{i0}}^{p_{j0}} \frac{d k_0}{2 \pi} \Theta \left( \frac{p_{i0} - p_{j0}}{p_{i0} - p_{j0}} \right) (p_{i0} - p_{j0})^{N-3} \prod_{k \neq i,j} J_{ijk,s}(k_0), \tag{S3}
\]

for \( N \geq 3 \), where

\[
J_{ijk,s}(k_0) = \frac{1}{s D_{ijk} + F_{ijk} + i \Omega_{ijk}(k_0)}, \tag{S4}
\]
with

\[ D_{ijk} = p_{ix}(p_{ky} - p_{jy}) + \text{cyl}, \]

\[ F_{ijk} = (p_{jy} - p_{iy})(p_{ky} - p_{iy})(p_{iy} - p_{ky}), \]

\[ \Omega_{ijk}(k_0) = k - \frac{k_0 - p_0}{|k_0 - p_0|^{2/3}}(p_{ky} - p_{jy}) + \text{cyl}. \]

Here "cyl" denotes cyclic permutations of the indices \( i, j, k \).

Note that the \( k \)-integration converges both in the infrared and ultraviolet limits, such that the \( N \)-point loop is a cutoff-independent function of the external energy-momentum variables. Under a rescaling of the form \( q_{iy} \rightarrow \lambda q_{iy}, \quad q_{ix} \rightarrow \lambda^2 q_{ix}, \quad q_0 \rightarrow \lambda^3 q_0 \), the \( N \)-point loop scales homogeneously as \( \lambda^{2(3-N)} \).

Constructing the loop with bare instead of RPA propagators, one obtains the same expression (S3), with \( \Omega_{ijk}(k_0) \) replaced by the \( k_0 \)-independent function \( \Omega_{ijk} = -p_{00}(p_{ky} - p_{jy}) + \text{cyl} \). The \( k_0 \)-integration can then be performed analytically. The function \( \Omega_{ijk} \) is subleading in the scaling limit compared to \( D_{ijk} \) and \( F_{ijk} \). Summing over the two patches the result for the \( N \)-point loop with bare propagators agrees with an earlier result derived by performing the scaling limit after the \( k \)-integration [1].

**Reduction for symmetrized \( N \)-point loops**

Here we show that the symmetrized \( N \)-point loop vanishes, if one of the external momenta vanishes. The symmetrized \( N \)-point loop is given by a sum over all permutations of external momenta. For a vanishing external momentum two fermion lines in the loop carry the same internal momentum. By permutations the doubled fermion line is cycled around the loop. Let us consider the symmetrized 4-point loop as an example. Without loss of generality, we can choose \( q_4 \) as vanishing and sum the three permutations which are cyclic in \( q_1, q_2, q_3 \). The sum is given by

\[ \int \frac{dk_0}{2\pi} \int \frac{d^2 k}{(2\pi)^2} \frac{G_s(k - p_1)G_s(k - p_2)G_s(k - p_3) - G_s(k - p_1) + G_s(k - p_2) + G_s(k - p_3)}{G_s(k - p_1)G_s(k - p_2)G_s(k - p_3)}, \]

where we used that \( q_4 = p_1 - p_4 = 0 \). Since the denominator of \( G_s(k - p_i) \) is linear in \( k_x \), the integrand can be written as a \( k_x \)-derivative,

\[ -s \frac{\partial}{\partial k_x} [G_s(k - p_1)G_s(k - p_2)G_s(k - p_3)]. \]

Performing the \( k_x \)-integration one thus finds that the 4-point loop vanishes if one leg has a vanishing momentum. The extension to arbitrary \( N \) is straightforward, the sum of all cyclic permutations of \( N - 1 \) legs, while one leg with zero momentum is kept separate, will cancel exactly. By dimensional analysis, the UV scaling of the symmetrized loop with one fixed external momentum is thus reduced by a factor \( \Lambda^{-1} \).

Using analyticity and the invariance under \( q_i \rightarrow -q_i \) one can conclude that symmetrized loops vanish even quadratically, if a vanishing momentum \( q \) enters and leaves the same loop at two distinct vertices, provided that the other momenta remain finite. Due to momentum conservation this is possible only for \( N \geq 4 \). Hence, the UV scaling of symmetrized loops for \( N - 2 \) large momenta and two fixed external momenta \( q \) and \( -q \) is reduced by a factor \( \Lambda^{-2} \).

**Calculation of the four-loop analogue of the Aslamazov-Larkin diagram**

The four-loop contributions of type (e), where two 4-point fermion loops are connected by three boson propagators, are given by

\[ \Pi^{(4e)}(q) = -2g^4 g_+^4 \int \frac{dl_1}{(2\pi)^3} \int \frac{dl_2}{(2\pi)^3} D(l_1 + \frac{q}{2})D(l_2 - l_1)D(l_2 - l_2) \]

\[ \times \Pi_{4,+}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

\[ \times \Pi_{4,-}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

\[ \times \Pi_{4,+}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

\[ \times \Pi_{4,-}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

\[ \times \Pi_{4,+}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

\[ \times \Pi_{4,-}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

\[ \times \Pi_{4,+}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

\[ \times \Pi_{4,-}^{(4)}(\frac{q}{2} - l_2, l_2 - l_1, l_1 + \frac{q}{2}, -q) \]

The symmetrized 4-point loop \( \Pi_{4,-}^{(4)} \) is defined by the sum of the six distinct permutations (without normalization factors such as \( 1/6 \)). The individual Feynman diagrams corresponding to these permutations are depicted in Fig. S2. Note that \( g^2 g_+^2 = 1 \). We set the external momentum to \( q = (0, 0, q_y) \) and analyze the behavior for small \( q_y \). Since the
Figure S2. The six four-loop contributions with two 4-point fermion loops contributing to the boson self-energy. All diagrams can be grouped into one by symmetrizing one of the 4-point loops.

external frequency is zero, the replacement of all loop frequencies by its negative produces the conjugate and shows that $\Pi^{(4e)}$ is real. In the same way we conclude that the integral is invariant under $q_y \to -q_y$, which means that the respective choice which N-point loop is on the plus and on the minus patch can be accounted for by a factor of 2.

The 4-point loop is given by Eq. (S3). Only a few more simplifications are possible. With some effort it is possible to perform the integrals in $l_{1x}$ and $l_{2x}$ by residues. However, this leads to a proliferation of terms and does not reduce the numerical effort required for the remaining integrations. Hence, we perform the integration over the bosonic variables $l_1$ and $l_2$ fully numerically. The frequency integration within the fermion loops has to be done numerically, too.

We introduce stretched spherical integration variables by substituting $l_{0i} = r \tilde{l}_{0i}$, $l_{ix} = r \tilde{l}_{ix}$, and $l_{iy} = r \tilde{l}_{iy}$, where $r$ runs from 0 to $\Lambda$, and the tilde-variables are confined to a unit sphere. Scaling out the $r$-dependence, the integral can then be written in the form

$$\Pi^{(4e)}(q_y) = \frac{q_y^2}{c^2} \int_0^\Lambda \frac{dr}{r} \int d\Omega F(\Omega, q_y / r),$$

where $\Omega$ denotes the integration over the sphere. A logarithmic divergence is then signaled by a nonzero result of the surface integral over $\Omega$ for $|q_y| \ll r$, and the prefactor of the divergence is given by

$$C^{(4e)} = \lim_{\tilde{q}_y \to 0} \int d\Omega F(\Omega, \tilde{q}_y).$$

$\Pi_{4s}$ itself contains already 5 terms from the sum over $i < j$ (one term vanishes), the product $\Pi_{4s} \Pi_{4s}^{\text{sym}}$ then contains $5 \times 30 = 150$ summands, each of which consists of a product of four $J_{ijk}$. We employed the computer algebra capabilities of Wolfram Mathematica and exported the results for an integration with the adaptive routine “Divonne” from CUBA [2]. The integrand contains a large number of (integrable) poles. For $q_y / r \to 0$ poles coalesce and a substantial sign problem occurs. The integral $\int d\Omega F(\Omega, \tilde{q}_y)$ turns out to be negative for small $|\tilde{q}_y|$ with increasing absolute value for decreasing $|\tilde{q}_y|$. The integration routine converges well down to $|q_y| / r$ ratios of the order $10^{-2}$, where $\int d\Omega F(\Omega, \tilde{q}_y)$ reaches the value $-0.63$. Since $\int d\Omega F(\Omega, \tilde{q}_y)$ still increases for $|\tilde{q}_y| < 10^{-2}$, where the numerical integration becomes unstable, we cannot determine the final (saturated) result for $C^{(4e)}$.

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