ON THE REGULARITY OF A GRAPH RELATED TO CONJUGACY CLASS SIZES OF A NORMAL SUBGROUP

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Abstract. Given a finite group $G$ with a normal subgroup $N$, the simple graph $\Gamma_G(N)$ is a graph, whose vertices are of the form $|x^G|$ where $x \in N \setminus Z(G)$, and $x^G$ is the $G$-conjugacy class of $N$ containing the element $x$, and two vertices $|x^G|$ and $|y^G|$ are adjacent if they are not co-prime. In this article we prove that, if $\Gamma_G(N)$ is a connected incomplete regular graph, then $N = P \times A$ where $P$ is a $p$-group, for some prime $p$ and $A \leq Z(G)$, and $Z(N) \neq N \cap Z(G)$.

1. Introduction

Given a finite group $G$, by $\text{cs}(G)$ we mean the set of conjugacy class sizes of $G$. It is well known that strong results can be obtained from $\text{cs}(G)$ about the structure of $G$ (see [5] for example).

Some certain graphs are introduced in order to study specific properties of the given finite group $G$. The ones that we are going to use for the rest of this paper, are constructed upon the set $\text{cs}(G)$. The common divisor graph on conjugacy classes, that we denote by $\Gamma(G)$ (see [2]), is a graph whose vertex set is $\text{cs}(G) \setminus \{1\}$ and vertices $v$ and $w$ are adjacent if $\gcd(v, w) > 1$. And the prime graph $\Delta(G)$ whose vertex set is the set of prime divisors of integers in $\text{cs}(G)$, and an edge joins two vertices if there exists some vertex in $\Gamma(G)$ which is divisible by both of them.

The properties of these two graphs, regarding their association to the algebraic structure of the group $G$, has been vastly investigated in the last few decades. We refer to [8] for a survey on this topic.

Let $N$ be a normal subgroup of the finite group $G$, the set $\text{cs}_G(N)$ denotes $G$-conjugacy class sizes of $N$. Discussing the structure of $N$ based on $\text{cs}_G(N)$, could potentially extend the results made regarding $\text{cs}(G)$, and thus these properties has been studied actively in the recent years as well. It seems to be natural to define analogous graphs based on $\text{cs}_G(N)$. Denote by $\Gamma_G(N)$, the graph whose vertex set contains elements $|x^G|$, where $x \in N \setminus Z(G)$ and vertices $v$ and $w$ are adjacent in $\Gamma_G(N)$ if and only if they are adjacent in $\Gamma(G)$ (see [1]).

In [4] it is proved that if $\Gamma(G)$ is $k$-regular then it must be a complete graph of order $k + 1$ for $k = 2, 3$; and the result is extended for any $k \in \mathbb{N}$ in [3].

It is natural to ask, whether the same result holds for the regular graph $\Gamma_G(N)$. Note that, if $\Gamma_G(N)$ is regular disconnected graph, then by [11] Theorem E, $\Gamma_G(N)$ has two complete connected components and the structure of $N$ in that case is determined. So the connected case is left to discuss. In this paper, we aim to prove the following theorem as the main result:
Main Theorem. Let $G$ be a finite group and $N$ be a normal subgroup of $G$, such that $\Gamma_G(N)$ is a connected incomplete regular graph. Then $N/(N \cap Z(G))$ is a $p$-group, for some prime $p$, and $Z(N) \neq N \cap Z(G)$.

We should mention that the remaining case is still open and we could not find any example showing that there exists a case in which $\Gamma_G(N)$ is not complete. But in Theorem 3.4, we prove that if $\Gamma_G(N)$ is a connected 3-regular graph, then $\Gamma_G(N)$ is complete. Checking so many examples for the remaining case has made us to come to believe $\Gamma_G(N)$ is most probably complete for the rest of these cases as well.

2. Preliminary

Definition 2.1. For a given vertex $v$ of the graph $\Gamma$, define the neighborhood of $v$, the set of all the vertices adjacent to $v$, including $v$ itself and denote it by $N_{\Gamma}(v)$.

Definition 2.2. Two distinct vertices $v_1, v_2$ of the graph $\Gamma$ are said to be partners, if:

$$N_{\Gamma}(v_1) = N_{\Gamma}(v_2)$$

Observe that partnership provides an equivalence relation on the set of vertices of the graph.

Lemma 2.3. Let $|x^G|$ be a vertex of $\Gamma_G(N)$, for some $x \in N$ and $\Gamma_G(N)$ is regular. If $y = x^a$ is non-central for some integer $a$, then either $|y^G| = |x^G|$ or $|x^G|$ and $|y^G|$ are partners.

Proof. Assume $|x^G| \neq |y^G|$, then $C_G(x) \subset C_G(y)$. Therefore $|y^G| \mid |x^G|$. As $|y^G|$ and $|x^G|$ have the same degree in $\Gamma_G(N)$, we conclude that they are partners. ■

Lemma 2.4. (see [7, 2]) A finite group $G$ satisfies $n(\Gamma(G)) = 2$ if and only if $G$ is quasi-Frobenius and $G/Z(G)$ has abelian kernel and complement, and both components of $\Gamma(G)$ are single vertices.

3. Main results

Lemma 3.1. Let $G$ be a finite group and $N$ be a normal subgroup of $G$ such that $N/(N \cap Z(G))$ is divisible by two distinct prime divisors $p_1$ and $p_2$. Let $x_0, y_0 \in N$ be non-central $p_1$ and $p_2$-elements (respectively) such that $x_0y_0 = y_0x_0$. Also, assume that $\Gamma_G(N)$ is a connected incomplete regular graph. Then denoting by $v_0, w_0$ and $z_0$ the sizes of conjugacy classes of $G$, containing $x_0, y_0$ and $x_0y_0$, respectively, the followings hold:

(a) There exists a non-central $p_1$-element $x_1 \in N$ and a non-central $p_2$-element $y_1 \in N$, such that $v_1 = |x_1^G|$, $w_1 = |y_1^G| \in N_{\Gamma_G(N)}(z_0)$, where $v_1$ and $w_1$ are not adjacent in $\Gamma_G(N)$, $(v_1, p_1p_2) = p_2$ and $(w_1, p_1p_2) = p_1$.

(b) $v_0$ is divisible by $p_2$, $w_0$ is divisible by $p_1$. In particular $z_0$ is divisible by $p_1p_2$.

Proof. Since $\Gamma_G(N)$ is a connected incomplete regular graph, then for every vertex $v$ of $\Gamma_G(N)$, there exist two distinct non-adjacent vertices in $N_{\Gamma_G(N)}(v)$. Therefore non-central elements $x_1$ and $y_1$ exist in $N$ such that $v_1 = |x_1^G|$, $w_1 = |y_1^G|$ and $(v_1, w_1) = 1$, while they are both connected to $z_0 = (x_0y_0)^G$. By Lemma 2.3 we may assume that $o(x_1)$ and $o(y_1)$ are both power of some prime, but distinct primes.
Recall that $p_1$ can not divide both $|x_1^G|$ and $|y_1^G|$, assuming that $p_1 \not| |x_1^G|$, thus $C_G(x_1)$ must contain some Sylow $p_1$-subgroup. Without loss of generality we may suppose that $x_1x_0 = x_0x_1$. If $\alpha(x_1)$ is not a power of $p_1$, by Lemma 3.2, $v_1$ and $v_2$ should be partners. But that yields a contradiction, otherwise $v_1$ and $w_1$ must be adjacent, therefore $x_1$ has to be a $p_1$-element. Same argument holds for $y_1$ as well.

Now, we prove (b). On the contrary, assume that $p_2 \not| v_0$ consequently, $C_G(x_0)$ contains a Sylow $p_2$-subgroup of $G$. Similar to the given arguments in the previous paragraph, we come to conclusion that $v_0$ and $w_1$ should be partners, and that contradicts the assumption, stating the non-adjacency of $v_1$ and $w_1$. Similarly we get that $w_2$ is divisible by $p_1$. Therefore, since we know $v_0|z_0$ and $w_0|z_0$ is divisible by $p_1p_2$, as desired.

**Theorem 3.2.** Let $G$ be a finite group and $N$ be a normal subgroup of $G$ such that $N/(N \cap Z(G))$ is divisible by two distinct primes $p_1$ and $p_2$ and $x_0, y_0 \in N$ be non-central $p_1$ and $p_2$-elements(respectively) such that $x_0y_0 = y_0x_0$. Also, assume $\Gamma(G(N))$ is a connected regular graph. Then $\Gamma(G(N))$ is complete.

**Proof.** On the contrary assume that $\Gamma(G(N))$ is not complete, so we can apply Lemma 3.1. Accordingly there exist such vertices $z_0 := [(x_0y_0)^G], v_0 := |x_0^G|$ and $w_0 := |y_0^G|$ as described in the statement of Lemma 3.1 also let $v_1$ and $w_1$ be the elements such as those in the statement of Lemma 3.1(a); and define $A = N/r_1(N)(v_1) \setminus \{z_0, v_1\}$. On the contrary, assume that $x_0s = sx_0$ then the conclusion by applying Lemma 2.3 is that vertices $v_0$ and $|s^G|$ must be partners, and so should $v_1$ and $|s^G|$, which implies that $v_1$ and $w_1$ are adjacent. So $s$ is a $p_1$-element.

On the other hand, $C_G(s)$ contains a Sylow $p_2$-subgroup of $G$ (it is mentioned above that $(p_1p_2, |s^G|) = 1$), hence we may assume that $sy_1 = y_1s$ and $sy_0 = y_0s$. Therefore, applying Lemma 2.3 results in partnership of $|s^G|, w_0, v_1,$ and $w_1$ which implies adjacency of $v_1$ and $w_1$, a contradiction. Accordingly, every vertex in $A$ is divisible by both $p_1$ and $p_2$.

Observe, that $d(v_1) = |A| + 1$. By regularity of $\Gamma(G(N))$ we have $d(z_0) = |A| + 1$. By Lemma 3.1(b), $z_0$ is adjacent to all vertices in $\{v_1, w_1\} \cup A$, which implies that $w_1 \in A$, that again is a contradiction.

**Proof of the Main Theorem.** Let $\pi(N/(N \cap Z(G))) = \{p_1, \ldots, p_n\}$ and $P_i$'s are distinct primes. By Theorem 3.2 for every $p_i$-element $a_i, |a_i^G|$ is divisible by $(\prod_{j=1}^n p_j)/p_i$. If $n > 2$, we get that $\Gamma(G(N))$ is complete, a contradiction. Therefore $n \leq 2$.

First, assume $n = 2$. If $Z(N) \not\subset Z(G)$, then there exist a $p_1$-element $x_0$ and a $p_2$-element $y_0$ in $N \setminus Z(G)$ such that $x_0y_0 = y_0x_0$, and so we get a contradiction, by applying Theorem 3.2. So we may assume $Z(N) \subset Z(G)$ and $|N/Z(N)| = p_1^{n_1}p_2^{n_2}$, for integers $n_1$ and $n_2$.

Note that for every non-central element $x \in N, |x^G|$ is not divisible by $p_1p_2$, otherwise $\Gamma(G(N))$ is complete. Therefore, for every non-central $p_i$-element $x$, we have $|x^N| = p_i^{n_i}$, where $i \in \{1, 2\}$ and $\{i, j\} = \{1, 2\}$. By applying Lemma 2.3 $\Gamma(N)$ is a disconnected graph with two vertices of sizes $p_i^{n_i}$, for $i = 1, 2$. Now, considering Lemma 2.3 $N$ is a quasi-Frobenius group with abelian kernel and complement. Therefore the Frobenius complement of $N/(N \cap Z(G))$ is a cyclic $p_i$-group, for some
\[ i = 1, 2. \] Without loss of generality, we may assume that the Frobenius complement of \( N/(N \cap Z(G)) \) is a cyclic \( p_2 \)-group. Let \( x \) be a non-central \( p_2 \)-element of \( N \), such that \( (xN \cap Z(G)) \) is a Sylow \( p_2 \)-subgroup of \( N/(N \cap Z(G)) \). Therefore, any non-central \( p_2 \)-element of \( N \) is \( x^G \)'s partner, in which case the graph must be complete and that is a contradiction. Therefore \( n = 1 \), as desired.

If \( Z(N) \subseteq Z(G) \), contradiction is obtained since, each vertex is divisible by \( p \). Therefore, \( Z(N) \not\subseteq Z(G) \) must be the case, and the proof is complete. \( \blacksquare \)

**Theorem 3.3.** Let \( G \) be a finite group and \( N \) a normal subgroup of \( G \). The graph \( \Gamma_G(N) \) is 3-regular if and only if \( \Gamma_G(N) \) is either isomorphic to \( K_4 \) or it is the union of two components both of which are isomorphic to \( K_4 \).

**Proof.** First of all according to [1] Theorem A, we know that \( n(\Gamma_G(N)) \leq 2 \) therefore there are two cases for the graph \( \Gamma_G(N) \), it is either disconnected with two components or connected. First, suppose \( \Gamma_G(N) \) is disconnected then according to the second part of [1] Theorem B it consists of two cliques, also due to the regularity condition with \( k = 3 \) these components are both isomorphic to \( K_4 \).

Now let us discuss the former case, where the graph is connected, it either has a triangle or not. Suppose it does have a triangle and does not contain a \( K_4 \). Let \( a, b \) and \( c \) be vertices of the triangle \( T \) if a vertex \( d \) exists in \( \Gamma_G(N) \), adjacent to two vertices of \( T \) then by taking into account [1] Theorem B the condition of [4] Lemma 2.4 is satisfied so \( \Gamma_G(N) \) contains an induced subgraph \( C_n, n \geq 6 \). Also according to [4] Lemma 2.3 there exists an induced subgraph \( C_n \) in the graph \( \Delta(G) \), thus there are three independent vertices in \( \Delta(G) \) which contradicts [6] Theorem A.

If there is no triangle in \( \Gamma_G(N) \), considering that \( \Gamma_G(N) \) cannot be a tree, there definitely exists a cycle in it. Now take the cycle of the minimal length in this graph, suppose it is of length \( m \geq 4 \), for some integer \( m \). For the cases where \( m = 4, 5 \) [4] Lemma 2.5] would be applicable if there exists an edge, with no vertices in common with the cycle. Assuming the minimum length of cycles is 4, exactly 4 edges exit this cycle, considering that it has no triangles, there are at least two vertices outside of this cycle and thus there definitely has to be another edge having no intersections with this cycle. And if \( m = 5 \) each vertex on this cycle has a neighbour outside of the cycle. If any two vertices of the cycle shared a neighbour there will be a cycle of length 4 or less but that can not occur, as a result there exists an edge with no common vertices with this cycle, henceforth the condition of [4] Lemma 2.5] is met. Therefore by applying [6] Theorem A we get a contradiction. And for \( m > 6 \), by applying [4] Lemma 2.3] and [6] Theorem A] we reach the final contradiction. \( \blacksquare \)

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