Faster FAST
(Feedback Arc Set in Tournaments)

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Abstract
We present an algorithm that finds a feedback arc set of size $k$ in a tournament in time $n^{O(1)}2^{O(\sqrt{k})}$. This is asymptotically faster than the running time of previously known algorithms for this problem.

1 Introduction
A tournament is a directed graph in which every pair of vertices is connected by exactly one arc. A feedback arc set is a set of arcs whose removal makes the remaining digraph acyclic. Given a tournament, the Feedback Arc Set in Tournaments (FAST) problem asks for the smallest feedback arc set in the tournament. This problem is NP-hard [1, 2]. Hence we shall consider a parameterized version of the problem, $k$-FAST, in which one is given a tournament and a parameter $k$, and one has to find a feedback arc set of size $k$ if one exists. In [3] it was shown (among other things) that this problem can be solved in time $n^{O(1)}+k^{O(\sqrt{k})}$.

(Here and elsewhere $n$ denotes the number of vertices in the tournament.) The interesting aspect of this running time is the subexponential dependence on $k$, as the fact that the problem is fixed parameter tractable and moreover has a polynomial kernel was established earlier [5]. Given that [3] is titled Fast FAST, there is a big temptation to publish a paper titled Faster FAST. Not being able to resist this temptation, we present here a different algorithm that offers a mild improvement to the running time.

Theorem 1 There is an algorithm that solves $k$-FAST in time $2^{O(\sqrt{k})}n^{O(1)}$.

Observe that equivalently, this running time can be written as $n^{O(1)}+2^{O(\sqrt{k})}$ (this only changes the constants in the $O$ notation). The running time of the

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The algorithm of [3] remains polynomial in \( n \) for \( k = O((\log n / \log \log n)^2) \), whereas the running time of our algorithm remains polynomial for \( k = O((\log n)^2) \).

The algorithm presented in [3] is based on the color coding technique (introduced in [4]), and specifically on a certain random coloring lemma: for every graph with \( k \) edges, if one colors its vertices at random by \( O(\sqrt{k}) \) colors, then with probability at least \( 2^{-O(\sqrt{k})} \) the coloring is proper (see [3] for an exact statement of this lemma). In [3] this lemma is used in combination with dynamic programming to design an algorithm for \( k \)-FAST. Moreover, this lemma may be of interest beyond the specific application to the \( k \)-FAST problem. The algorithm presented in the current paper is also based on dynamic programming. However, it does not use the random coloring lemma.

2 The algorithm

The feedback arc set problem is equivalent to finding a linear ordering of the vertices (numbering them from 0 to \( n - 1 \)) that minimizes the number of arcs pointing backwards. Had the tournament been acyclic, there would have been a simple local test that would tell us where to place a vertex \( v \) in this linear ordering. We call it the indegree test. Under this test, the proposed location for vertex \( v \) is \( i \) if and only if \( v \) has \( i \) incoming arcs (and \( n - i - 1 \) outgoing arcs).

What happens if the minimum feedback arc set is of size \( k > 0 \)? In this case, the degree test might be incorrect. Let \( \pi \) be an optimal linear ordering (one with only \( k \) backward arcs). Let the error of the indegree test for vertex \( v \) be the absolute value of the difference between its location under \( \pi \) and the number of incoming arcs that \( v \) has. Then the sum of errors over all vertices is at most \( 2k \) (since each feedback arc contributes at most 2 to the error). It follows that for every value of \( d \) (we shall later choose \( d = \Theta(\sqrt{k}) \)) there are at most \( 2k/d \) vertices for which the error is more than \( d \). Let \( D_\pi \) denote the set of all such vertices.

As we shall see shortly, given \( D_\pi \), a minimum feedback arc set can be computed in time \( n^{O(1)}2^{O(|D_\pi|+d)} \). For \( d = \Theta(\sqrt{k}) \) the running time becomes \( n^{O(1)}2^{O(\sqrt{k})} \) as desired. The difficulty is that \( D_\pi \) is not given to us, and handling this issue is the purpose of the following discussion.

We say that three vertices in the tournament form a triangle if their corresponding arcs form a directed cycle. At least one of the arcs in a triangle is a feedback arc. It is known that a tournament is acyclic if and only if it does not contain triangles. Call an arc suspect if it belongs to some triangle. Call an arc a major suspect if it belongs to at least \( t \) triangles, where \( t = \Theta(\sqrt{k}) \) is a parameter to be chosen later. Call a vertex bad if at least \( t \) arcs incident with it are major suspects. Let \( B \) denote the set of all bad vertices in a tournament. Clearly, given the tournament, the set \( B \) can be computed in polynomial time. We shall now show that for appropriate choices of \( d \) and \( t \), \( D_\pi \subset B \), and moreover, that like \( D_\pi \), the size of \( B \) is \( O(\sqrt{k}) \).

**Lemma 2** For \( D_\pi \) and \( B \) as defined above, if \( d \geq 4t \) and \( t \geq \sqrt{k} \), then \( D_\pi \subset B \).
Proof. We need to show that every vertex with error $d' > d$ is incident with at least $t$ arcs that are major suspects. Consider such a vertex $v$, let $i$ be its location in $\pi$ and let $i + d'$ be the number of $v$’s incoming arcs. (The case in which $i - d'$ is the number of incoming arcs is handled in a similar way and is omitted.) Let $F$ be the set of vertices that come after $v$ in $\pi$ and yet have an arc directed from them to $v$. Clearly, $|F| \geq d'$. Let us denote the vertices in $F$ by $v_1, v_2, \ldots$ in the order of their appearance after $v$. A crucial observation is that for every $j$, the location of $v_j$ in $\pi$ has to be no sooner than $i + 2j$. Otherwise, $\pi$ is not an optimal linear arrangement, because the size of the feedback arc set can be decreased by doing one cyclic shift on the block of vertices that starts at $v_i$ and ends at $v_j$ (where all vertices in the block move one location down except $v_i$ that moves to the original location of $v_j$). Doing this cyclic shift, $j$ arcs are removed from the feedback arc set and less than $j$ arcs join the feedback arc set (only arcs incident with $v_i$ are affected by the cyclic shift).

Now consider the arc $(v_j, v_i)$, which is a feedback arc in $\pi$. Let us consider only triples of vertices $(v_i, v_j, u)$ where vertex $u$ has to lie in $\pi$ between $v_i$ and $v_j$, and moreover, the arc $(v_i, u)$ is directed towards $u$ (in agreement with the linear order $\pi$). By the observation above, there are at least $j$ possibilities for the choice of $u$. The triple $(v_i, v_j, u)$ forms a triangle unless the arc $(v_j, u)$ is directed towards $u$, which makes it too a feedback arc in $\pi$.

Now we use the fact that the feedback arc set of $\pi$ has size $k$. Consider only values of $j$ between $2t$ and $4t$ (here we used the assumptions the $d' \geq d \geq 4t$). For each such vertex $v_j$, the arc $(v_j, v_i)$ is not a major suspect only if at least $t$ arcs $(v_j, u)$ are feedback arcs in $\pi$. Hence for $v_i$ not to be incident with $t$ major suspects, there must be more than $t^2$ feedback arcs in $\pi$, which is a contradiction for $t \geq \sqrt{k}$.

Proposition 3 For $B$ as defined above and $t = \sqrt{6k}$, $|B| < t$.

Proof. Each vertex of $B$ is incident with at least $t$ arcs which are major suspects (we shall think of this as being exactly $t$, by possibly ignoring some of the major suspects), and each such arc is in at least $t$ triangles (again, we shall think of this as being exactly $t$, by possibly ignoring some of the triangles). This gives a count of $t^2|B|$ triangles (though possibly the same triangle might be counted more than once). Each such triangle has at least one feedback arc, and hence we have counted $t^2|B|$ feedback arcs. The problem is that the same arc might have been counted several times. We bound now the number of times that a single arc $(u, v)$ might be counted.

The arc $(u, v)$ may serve as a major suspect twice, once for $u$ and once for $v$. This gives $2t$ triangles in which $(u, v)$ might have been counted. Also for every other major suspect incident with $u$ (or with $v$), the arc $(u, v)$ might appear in one of its triangles. This gives another $2t$ triangles in which $(u, v)$ might appear. Finally, for every vertex $w \notin \{u, v\}$ in $B$, if either $(w, u)$ or $(w, v)$ is a major suspect, then the arc $(u, v)$ might again be counted in a triangle. This gives at most $2|B|$ additional triangles. Altogether an arc might be counted at most $4t + 2|B|$ times.
As the total number of feedback arcs is $k$, we obtain the inequality $t^2|B|/(4t + 2|B|) \leq k$, which implies the proposition.

We can now describe our algorithm. We assume without loss of generality that the value of $k$ is known (the algorithm may try all values of $k$ in increasing order until the first one that succeeds). Given a value of $k$, let us fix $t = 3\sqrt{k}$ and $d = 12\sqrt{k}$. The main steps of the algorithm are as follows.

1. Compute the set $B$ of bad vertices.

2. For each location $i$ in the linear order, compute a candidate set $C(i)$ that contains those vertices whose indegree is between $i - d$ and $i + d$, plus the vertices of $B$. In addition compute a prefix set $P(i)$ that contains those vertices not in $B$ with indegree less than $i - d$.

3. Using these candidate sets and prefix sets, compute a minimum feedback arc set using dynamic programming.

We now elaborate on these main steps, proving the correctness of the algorithm and bounding its running time.

Step (1) can be done in time $O(n^3)$ by checking for each triple of vertices whether it forms a triangle, then identifying those arcs that are major suspects (members of at least $t$ triangles), and putting in $B$ those vertices that are incident with at least $t$ major suspects. By Proposition 3 we have that $|B| \leq t$ and by Lemma 2 we have that $D_\pi \subset B$.

Given $B$, Step (2) can also be performed in polynomial time, since the indegree of vertices can be computed in polynomial time. The properties required from Step (2) are summarized in the following proposition.

**Proposition 4** In the optimal linear arrangement $\pi$, for every location $i$, the vertex in location $i$ is one of the vertices of the candidate set $C(i)$ (as computed in Step (2) of the algorithm). Moreover, every vertex of the prefix set $P(i)$ is placed in $\pi$ prior to location $i$.

**Proof.** Let $v$ be the vertex placed by $\pi$ in location $i$. We need to show that $v \in C(i)$. If $v \in D_\pi$ then this follows from Lemma 2 because $D_\pi \subset B$. Hence it remains to consider the case that $v \not\in D_\pi$. In this case the indegree of $v$ is in the range $[i - d, i + d]$, again implying that $v \in B$, as desired. A similar argument shows that all vertices of the prefix set $P(i)$ are placed in $\pi$ before location $i$.

Now we can use dynamic programming to find which linear order among those that respect the candidate sets has the smallest feedback arc set. We scan the locations from 0 to $n - 1$. On reaching location $i$ we need only know two things:

1. Which vertices of $C(i)$ have been placed up to location $i$. There are at most $2^{|C(i)|}$ possibilities for such subsets.
2. How many backward arcs we have placed so far. For each choice of subset $C'(i)$ as in item (1), we need to remember just the smallest number of backward arcs that can be attained in a linear arrangement that up to $i$ placed $C'(i)$ and did not place $C(i) - C'(i)$.

At step $i$, one can place at location $i$ any one of the vertices $v$ of $C(i) - C'(i)$. (If $C(i) - C'(i)$ is empty, the corresponding branch of the dynamic programming dies off.) Thereafter, $C'(i + 1)$ can be computed in a straightforward way as $C(i + 1) \cap (C'(i) \cup \{v\})$. Likewise, the number of backward arcs can be updated by adding to the previous total those arcs going from $v$ to $C'(i)$ and from $v$ to $P(i)$.

Let $C = \max_i |C_i|$. Then the size of the dynamic programming table constructed by this dynamic programming algorithm is at most $n2^{3C}$, and the running time of the algorithm is polynomial in the size of the table. Hence to prove Theorem 1 it remains to prove the following proposition.

**Proposition 5** For a choice of $t$ and $d$ as above, for every $i$, the size of the candidate set $C(i)$ is at most $52 \sqrt{k}$.

**Proof.** The candidate set $C(i)$ contains all of $B$, which by our choice of $t = 3\sqrt{k}$ and Proposition 3 contains at most $3\sqrt{k}$ vertices. In addition it contains those vertices not in $D_x$ whose indegree is between $i - d$ and $i + d$. There are at most $4d + 1$ such vertices (because any such vertex has to be in a location between $i - 2d$ and $i + 2d$ in $\pi$), and by our choice of $d = 12 \sqrt{k}$ this contributes at most $48 \sqrt{k} + 1$ additional vertices to $C(i)$.

In summary, the algorithm presented above runs in time $n^{O(1)}2^{O(\sqrt{k})}$ and finds a feedback arc set of size $k$ in an $n$-vertex tournament, if the tournament has such a feedback arc set. This proves Theorem 1.

### 3 Conclusions

Are there algorithms for $k$-FAST with running times that are substantially better than $n^{O(1)}2^{O(\sqrt{k})}$? Specifically, can we extend the range of values of $k$ for which the running time is polynomial beyond $k = O((\log n)^2)$? If yes, then this will imply that FAST can be solved in time $2^{o(n)}$ (details omitted). The NP-hardness results of [1, 2] do not show that a running time of $2^{o(n)}$ for FAST is unlikely (e.g., they do not show that this will imply a similar running time for SAT). But still, solving FAST in time $2^{o(n)}$ seems to require substantially new techniques, and hence the author does not anticipate major improvements over the bounds in the current paper in the near future.

If major improvements are not to be expected, what about minor improvements? Here much can be done. Kernelization techniques (such as in [3] and [4]) can offer improvements that are significant when $k$ is small. For larger values of $k$, improvements can come from optimizing the values of parameters (such as of $d$ and $t$) so as to minimize the value of the hidden constant in the exponent of
the $2^{O(\sqrt{n})}$ term. Moreover, some minor modifications to the algorithm can lead to further improvements. We have not attempted any of these optimizations in the current paper, because this surely deserves a separate publication, to be titled *Fastest faster FAST*.

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**References**

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