Two-proton radioactivity and three-body decay. IV. Connection to quasiclassical formulation.

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We derive quasiclassical expressions for the three-body decay width and define the “preexponential” coefficients for them. The derivation is based on the integral formulae for the three-body width obtained in the semiclassical approach with simplified three-body Hamiltonian [L.V. Grigorenko and M.V. Zhukov, arXiv:0704.0920v1]. The model is applied to the decays of the first excited 3/2− state of 17Ne and 3/2− ground state of 45Fe. Various qualitative aspects of the model and relations with the other simplified approaches to the three-body decays are discussed.

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I. INTRODUCTION

The “true” two-proton decay \( \text{[1]} \) is the decay mode which is expected to be an ordinary phenomenon in the vicinity of the proton drip line \( \text{[2]} \). This mode corresponds to a specific situation when one-proton emission is energetically (due to the proton separation energy in the daughter system) prohibited and only the simultaneous emission of two protons is possible. The energy conditions are illustrated in Fig. 1 more detailed discussion of the three-body decay modes could be found in \[2, 3\]. From theoretical point of view this situation is a subset of the three-body Coulomb problem in the continuum. A consistent quantum mechanical three-cluster model of the phenomenon was developed in Refs. \[2, 3, 4\] and applied to a range of the nuclear systems from \( ^6\text{Be} \) to \( ^{66}\text{Kr} \) in papers \[5, 6, 7, 8\]. In the works \[9, 10\] possible importance of the “true” two-proton decay phenomenon was demonstrated for astrophysical applications (in the form of the reverse two-proton radiative capture process). Having in mind these applications, which could require precise result in certain cases, the semiclassical model with simplified three-body Hamiltonian was developed in our recent work \[11\]. The model allowed precise calibrations of three-body calculations for decay widths.

On the other hand the semiclassical model \[11\] can be used for derivation of the quasiclassical formulae for the three-body decay width. Formulae of this class can be found in early papers on two-proton radioactivity \[1, 12\]. They were used for qualitative estimates in our works \[2, 3, 7, 8, 13, 14\]. We characterize them as quasiclassical as they are based on a certain factorization of the decay amplitude, which, in reality, is equivalent to existence of classical trajectories for system propagation in the process of the decay. A quasiclassical model for the width was introduced in the papers \[15, 16\] for \( ^{12}\text{O} \) and then utilized in the series of works by Barker and Brown \[17, 18, 19, 20, 21, 22\] in somewhat modified form. It was characterized as “R-matrix approach”, due to formal similarity with two-body R-matrix formalism.

The name of “R-matrix approach” can be, in certain sense, misleading as a derivation procedure is actually not based on the construction of the R-matrix on the “nuclear surface”. The latter has a very complicated shape in the three-body system due to pairwise final state interactions (FSI). Fig. 2 shows schematically the surface limiting the region of classically allowed motion in the nuclear interior space for pairwise distances between clusters \( r_{ij} \). Close to the origin this surface is approximated by the surface of constant hyperradius (ellipsoidal in this space). In the regions of final state interactions \( r_{ij} \ll r_{ik}, r_{kj} \) it is just \( r_{ij} = \text{const} \). Possible values of radii due to triangle conditions \( |r_{ik} - r_{kj}| \leq r_{ij} \leq r_{ik} + r_{kj} \) are limited by solid angle represented by gray tetrahedron in Fig. 2.

When particles penetrate through the Coulomb barrier they propagate predominantly along the “tunnels” of classically allowed regions (though the motion entirely in the classically allowed region is not possible due to the energy condition of Fig. 1).

In this work we derive the quasiclassical expression for the three-body decay width in the case of the existence of narrow states in the subsystems and define the “preexponential” coefficients for the expression. The obtained expression has an important advantage compared to those used previously. We also make a comprehensive overview of quasiclassical three-body formulae used

FIG. 1: Energy conditions for different modes of two-nucleon emission (three-body decay): true three-body decay (a), sequential decay (b).
The only restriction on the auxiliary Hamiltonian form where the WF \( \tilde{\psi}_l(k,r) \) is the “quasibound” solution for Hamiltonian

\[
(H - E_r)\tilde{\psi}_l = (T + V - E_r)\tilde{\psi}_l = 0
\]

at the resonant energy \( E_r \). “Quasibound” means that the solution is matching the irregular (at the origin) Coulomb function \( G_l \) in the subbarrier region and it is normalized to unity in the finite region of radius \( R \). The function \( \varphi_l(kr) \) is the continuum WF of the auxiliary Hamiltonian \( \tilde{H} \)

\[
(\tilde{H} - E)\varphi_l = (T + \tilde{V} - E)\varphi_l = 0
\]

in the S-matrix representation having the asymptotic form

\[
\varphi_l(kr) = \exp(i\delta_l) \left[ F_l(kr) \cos(\delta_l) + G_l(kr) \sin(\delta_l) \right].
\]

The only restriction on the auxiliary Hamiltonian \( \tilde{H} \) is that its eigenfunction should be sufficiently far from resonance at \( E = E_r \).

The integral in Eq. (1) can be rewritten in terms of Wronskian, see Ref. [11]:

\[
2M \int_0^R \varphi_l^*(V - \tilde{V})\psi_l dr = \left[ \varphi_l^* \left( \frac{d}{dr}\psi_l \right) - \left( \frac{d}{dr}\varphi_l^* \right)\psi_l \right] \bigg|_{r=R} = -i \exp(-i\delta_l) k_r W(F_l,G_l).
\]

The function \( \psi_l \) on the left from \( R \) is obtained as a solution of the Schrödinger equation [normalized only by asymptotic condition \( \Psi_l = G_l \)]. To the right of the matching point it can be written explicitly in the form correctly normalized both asymptotically (to \( G_l \)) and in the internal region (to unity):

\[
\tilde{\psi}_l(k_r,r) \geq R \left( \int_0^R |\psi_l(k_r,r)|^2 dr \right)^{-1/2} \psi_l(k_r) G_l(k_r). \]

Neglecting the terms of the order \( \delta_l \) we obtain

\[
\Gamma = \frac{1}{MR^2} \int_0^1 |\psi_l(k_r,x)|^2 dx G_l^2(k_r). \]

For small widths the relation \( G_l \gg F_l \) is true and the penetrability can be identified as

\[
P_l(k_r,R) = \frac{k_r R}{F_l^2(k_r) + G_l^2(k_r)} \approx \frac{k_r R}{G_l^2(k_r)}. \]

Then we obtain for the width in Eq. (1) the standard R-matrix expression

\[
\Gamma = 2\gamma^2 P_l(k_r,R) = 2\gamma^2 W_L \theta^2 P_l(k_r,R), \tag{3}
\]

where the Wigner limit for the reduced width \( \gamma^2 W_L \) and the dimensionless reduced width (DRW) \( \theta^2 \) are identified as

\[
\gamma^2 W_L = \frac{1}{2MR^2}, \quad \theta^2 = \frac{|\psi_l(k_r,R)|^2}{\int_0^1 |\psi_l(k_r,x)|^2 dx}. \tag{4}
\]

The DRW is expected to be of the order of unity for \( R \) chosen in the subbarrier region. However, the DRW \( \theta^2 \) decreases relatively strongly with radius \( R \) for two reasons: (i) the normalization integral in the denominator is growing with \( R \) [27]; (ii) the numerator of Eq. (3) is decreasing as \( G_l^2 \) with \( R \) under the barrier. This decrease is compensated in Eq. (3) by growth of the penetrability as \( 1/G_l^2 \). Thus for values of \( R \) under the barrier the width should decrease slowly with radius \( R \) up to some value of the radius (close to the outer classical turning point of the barrier), where it is practically stabilized.

### A. Integration region in calculations of widths

The role of the integration limit in the calculations of the internal normalization can be understood using the R-matrix example. Similar calculations can be found in the book [24], where they illustrate a somewhat different issue. In the case of a square well potential and absence of Coulomb interaction the normalization integral for the internal region is easily evaluated

\[
N_1 = \int_0^1 [(kRx)j_l(kRx)]^2 dx = \frac{(kR)^2}{2} [j_l(kR)^2 - j_{l+1}(kR)j_{l-1}(kR)] \approx \frac{(kR)^2}{2} [j_l(kR)^2]. \tag{5}
\]
The approximate equality is good at resonance and becomes exact if \(E_r \to 0\). The dimensionless reduced width is in that case

\[
\theta_1^2 = 2 \left( 1 - \frac{j_{l+1}(kR)j_{l-1}(kR)}{[j_l(kR)]^2} \right)^{-1}.
\]

We can redefine the DRW in such a way that penetrability could be evaluated at the nuclear surface but the effect of the contribution to normalization from the subbarrier region is already taken into account. The normalization integral is reasonable to calculate up to the first zero of the irregular function \(G_l(k_r, r)\). The WF comes from under the barrier approximately at this point and calculation of the normalization integral beyond this point loose a sense (the particle can not anymore be considered to be in the “internal region”).

Beyond the matching point \(R\) the function can be approximated as

\[
G_l(k_r, r) = (k_r r) n_l(k_r, r) \sim r^{-l},
\]

with a good precision and integration in Eq. (4) can be extended to infinity. The integral of such approximated function converges for integration up to infinity for \(l > 0\). Thus the normalization for the whole subbarrier domain is

\[
N_2 = N_1 + [(kR)r_j(kR)]^2 \int_1^\infty x^{-2l} dx \approx \frac{2l + 1}{2l - 1} \frac{(kR)^2}{2} [j_l(kR)]^2.
\]

The “redefined” dimensionless reduced width is exactly written as

\[
\theta_2^2 = 2 \left( \frac{2l + 1}{2l - 1} - \frac{j_{l+1}(kR)j_{l-1}(kR)}{[j_l(kR)]^2} \right)^{-1}.
\]

The ratio of the reduced widths calculated at square well boundary by Eq. (5) and for “effective infinity” by Eq. (6) is approximately

\[
\frac{\theta_2^2}{\theta_1^2} \approx \frac{(2l - 1)}{(2l + 1)},
\]

again the expression becomes an identity for \(E_r \to 0\).

In Table I the mentioned values are provided for the simple model employed here. We can find out that (i) the effect of the subbarrier region is very important for weak barriers (e.g. \(l = 1\) in our example) and it gradually diminish as the barrier grows; (ii) the effect is not absolutely negligible even for quite high barriers; (iii) the direct numerical comparison shows that the DRW value \(\theta_2^2\) calculated for “effective infinity” is exactly consistent with the width obtained in scattering calculations, and should be considered as a correct definition.

### B. Evaluation of the integral in Eq. (1)

If the width of the resonance in the auxiliary Hamiltonian is sufficiently small, then, in proximity of the auxiliary resonance energy \(E_a\), we can write confidently that in the internal region

\[
\varphi_l(kr) = \frac{\sqrt{\pi \nu}}{2} \frac{\Gamma_a(E)/2\pi}{(E - E_a)^2 + 1}\hat{\psi}_l(k_a, r),
\]

where function \(\hat{\psi}_l(k_a, r)\) is also the quasibound WF as \(\hat{\psi}_l(k_r, r)\) (namely with resonant boundary condition and normalized to unity in the internal domain) but taken at the resonant energy \(E_a\) of the auxiliary Hamiltonian

\[
\hat{\psi}_l(k_a, r > R) \propto G_l(k_a, r), \int_0^R dr \left| \hat{\psi}_l(k_a, r) \right|^2 = 1.
\]

The coefficient \(D_l\) provides the normalization of the Breit-Wigner profile, which is already practically normalized

\[
\int_{-\infty}^\infty dE \frac{\Gamma_a(E)/2\pi}{(E - E_a)^2 + \Gamma_a^2(E_a)/4} = 1,
\]

within the energy domain of interest:

\[
1 = D_l \int_0^{E_a + \Delta E} dE \frac{\Gamma_a(E)/2\pi}{(E - E_a)^2 + \Gamma_a^2(E_a)/4}.
\]

For reasonably narrow states \(D_l \equiv 1\) means that \(\Delta E = (3 - 5)\Gamma_a(E)\). So this coefficient is always reasonably close to 1. The coefficient \(\sqrt{\pi \nu}/2\) in the Eq. (7) is partly connected with normalization of the partial radial functions \(\varphi_l(kr)\)

\[
\int_0^\infty \varphi_l^*(k'r)\varphi_l(kr) dr = \frac{\pi}{2}\delta(k' - k),
\]

and partly with conversion from integration over \(dk\) to integration over \(dE\). Now substituting (7) into (1) and using (3) the identity

\[
\Gamma(E_r) = \Gamma(E_a) \frac{\theta_2^2 D_l}{\theta_1^2} \left| \int_0^R dr \hat{\psi}_l(k_a, r) (\hat{V} - V) \hat{\psi}_l(k_r, r) \right|^2 (E_r - E_a)^2 + \Gamma_a(E_r)^2/4
\]

is obtained.

If the difference between energies \(E_r - E_a\) is small compared to the height of the barrier, the quasibound WFs \(\hat{\psi}_l\) and \(\hat{\psi}_l\) should be quite close to each other and provide close DRW values

\[
E_r - E_a \ll E_{bar}, \hat{\psi}_l(k_a, r) \approx \hat{\psi}_l(k_r, r) \rightarrow \theta_2^2/\theta_1^2 \approx 1.
\]

Also the variation of the kinetic energy should be small in such a case and the whole variation in energy should
be due to the potential energy change and it should be true that
\[
\frac{\partial^2}{\partial^2} \int_0^R dr \hat{\psi}_l^*(k_a, r) \left( \hat{V} - V \right) \hat{\psi}_l(k_r, r) \right|^2 = (E_r - E_a)^2 B_l ,
\]
with coefficient $B_l \approx 1$. The examples of actual calculated values of these coefficients are provided in Table I for the model of the previous Section. They indicate that the auxiliary resonance $E_a$ should be sufficiently narrow and sufficiently close to the physical resonance $E_r$ to make the approximation of Eq. (1) really precise. In the opposite case the value $D_l$ needs to be correspondingly renormalised to preserve the identity (5). Anyhow we keep in mind that for states broader than $10^{-2}$ MeV the substitution (2) provides results which are valid only within a factor of 2.

The issues discussed above are not of importance for the standard R-matrix phenomenology, but they should be clearly understood before we turn to the three-body case.

III. THREE-BODY CASE

In paper [11] we used the simplified three-body Hamiltonian in which the proton-proton FSI was neglected [28]. In this Hamiltonian we used also a simplified expression for the three-body Coulomb interaction, which allows an isolation of the degrees of freedom. The three-body width in this simplified model was obtained as

\[
\Gamma(E_{3r}) = \frac{8E_{3r}}{\pi} \int_0^1 d\varepsilon \frac{M_x M_y}{k_x(\varepsilon)k_y(\varepsilon)} |A(\varepsilon)|^2 ,
\]

where energy and momenta of the subsystems are

\[E_x = \varepsilon E_{3r} , \quad E_y = (1 - \varepsilon) E_{3r} , \quad k_i = \sqrt{2M_i E_i} .\]

The “energy distribution” coefficient is defined

\[A(\varepsilon) = \int_0^\infty dX dY \varphi_{\varepsilon, l_x}(k_x(\varepsilon)X) \varphi_{\varepsilon, l_y}(k_y(\varepsilon)Y) \times V_3(X, Y) \varphi_{\varepsilon, l_x, l_y, S}(k_r, X, Y) ,\]

where the function $\varphi_{\varepsilon, l_x, l_y, S}(k_r, X, Y)$ is the radial part of the solution with the outgoing asymptotic behaviour for a simplified three-body Hamiltonian

\[
\left( H - \tilde{E}_{3r} \right) \varphi = \left( H_x + H_y + V_3(\rho) - \tilde{E}_{3r} \right) \varphi = \left( T_x + T_y + V_x(X) + V_y(Y) + V_3(\rho) - \tilde{E}_{3r} \right) \varphi = 0
\]

at the complex pole energy $\tilde{E}_{3r} = E_{3r} + i\Gamma/2$. The three-body potential $V_3(\rho)$, which is depending only on the hyperradius $\rho$, is used to form the three-body resonance and control the resonant energy. The functions $\varphi_{\varepsilon, l_x}, \varphi_{\varepsilon, l_y}$ are the continuum eigenfunctions of the separable auxiliary Hamiltonian $H = H_x + H_y$ and they are normalized by the asymptotic condition (2) where phase shifts are defined for the subhamiltonians

\[
\begin{align*}
(T_x + V_x(X) - E_x)\varphi_{\varepsilon, l_x}(k_x X) &= 0 , \\
(T_y + V_y(Y) - E_y)\varphi_{\varepsilon, l_y}(k_y Y) &= 0 .
\end{align*}
\]

In the case when there are narrow resonant states in both “X” (at energy $E_{3xa}$) and “Y” (at energy $E_{3ya}$) subsystems, the substitution (2) can be used for both $\varphi_{\varepsilon, l_x}$ and $\varphi_{\varepsilon, l_y}$. It is clear that this approximation is physical only when the so-called “Y” Jacobi system is chosen. In such a Jacobi system $X$ is a coordinate between the core and a proton, $Y$ is a distance between the “X” subsystem center of mass and the second proton. The hyperradius is then defined as

\[
\rho^2 = \frac{A_c}{A_c + 1} X^2 + \frac{A_c + 1}{A_c + 2} Y^2 ,
\]

where $A_c$ is the core mass. This approximation is good enough only in relatively heavy nuclei where the Y Jacobi coordinate is close to the single-particle proton coordinate due to a large core mass. The total auxiliary Hamiltonian $\tilde{H}$ has a resonant energy

\[E_a = E_{3xa} + E_{3ya} , \quad k_{ia} = \sqrt{2M_i E_{ia}} .\]

Denoting the following integral as

\[\langle V_3 \rangle = \int_0^\infty dX dY \varphi_{\varepsilon, l_x}(k_x X) \varphi_{\varepsilon, l_y}(k_y Y) V_3(\rho) \times \varphi_{\varepsilon, l_x, l_y, S}(k_r, X, Y) ,\]

where $\varphi$ are normalized quasibound WFs for the subsystems, the width is obtained as

\[
\Gamma(E_{3r}) = \frac{E_{3r} \langle V_3 \rangle^2}{2\pi} \int_0^1 d\varepsilon \frac{D_x}{(E_x - E_{3xa})^2 + \Gamma_{x}(E_x)^2/4} \times \frac{D_y}{(E_y - E_{3ya})^2 + \Gamma_{y}(E_y)^2/4} .
\]

The expression for the two-proton width was, for the first time, obtained in a similar form in Ref. [12]. It is difficult to say why this work did not attract attention and why its results have never been used. A possible reason could be the relatively complicated procedure used in Ref. [12] and a lack of qualitative investigation of the model properties and the nature of approximations involved.

Using the following notations

\[
\Gamma_i(E_i) = 2\gamma_i^2 P_i(\varepsilon, r_{ch_i}, Z_i) = \frac{\theta_i^2}{4\pi} \frac{Z_i}{M_{r_{ch_i}}} \frac{P_i(E_i, r_{ch_i}, Z_i)}{P_i(1 - \varepsilon) E_{r_{ch_i}} Z_Y ,}
\]

\[
\Delta_i(\varepsilon, E, E_{ia}) = \left( 1 - \frac{\varepsilon E}{E_{ia}} \right)^2 - \frac{\Gamma_{ia}(E)^2}{4\varepsilon^2} \right)^{-1} ,
\]

we can rewrite Eq. (13) in the “dimensionless” form for the penetration part of the expression:

\[
\Gamma(E_{3r}) = \frac{D_x D_y}{2\pi} \frac{\gamma_x^2 \gamma_y^2 E_{3r} \langle V_3 \rangle^2}{E_{3xa}^2 E_{3ya}^2} \int_0^1 d\varepsilon \frac{P_{3r} \Delta_i(\varepsilon, E_{3r}, E_{3xa}) \Delta_i(1 - \varepsilon, E_{3r}, E_{3ya})}{E_{ia}^2} .
\]
The expression for width is now explicitly factorized into “preexponent” (which has the dimension of energy) and dimensionless “exponential” part

\[ \int_0^1 d\varepsilon \mathcal{P}_{I_x} \Delta I_{y} \sim \exp \left[ -\sqrt{M/E_{3r}} 2Z_{\text{core}} C \right], \]

where \( C \) is coefficient of the order of unity.

Following the discussion of Eq. (10) we can write

\[ \langle V_3 \rangle^2 = (E_{3r} - E_{xa} - E_{ya})^2 D_3, \quad (15) \]

where the dimensionless coefficient \( D_3 \) is presumed to be close to unity. Finally we get for the width:

\[ \Gamma(E_{3r}) = \frac{D_x D_y D_3 \theta^2 \theta^2}{2\pi} \frac{E_{3r}(E_{3r} - E_{xa} - E_{ya})^2}{M_y M_{3r} r_{ch}^{2} E_{xa} E_{ya}} \int_0^1 d\varepsilon \times \mathcal{P}_{I_x I_{y}}(\varepsilon, E_{3r}) \Delta I_{y} (1 - \varepsilon, E_{3r}, E_{ya}) \quad (16) \]

The \( D_3 \) values calculated in the simplified three-body model are given in Table II. It can be shown that the ratio \( \langle V_3^2 \rangle / \langle V_3 \rangle^2 \) is a measure of the WF to be outside the interaction region (this statement is trivial to check for the square well potential). The values of \( \langle V_3^2 \rangle \) and \( \langle V_3 \rangle^2 \) calculated in the three-body model for \(^{17}\text{Ne}\) and \(^{45}\text{Fe}\) are also provided in Table III. The ratios \( \langle V_3^2 \rangle / \langle V_3 \rangle^2 \) are reasonably consistent with the \( D_3 \) values and are quite close to unity. This indicates that the WFs are predominantly localized in the “internal” regions.

IV. DISCUSSION

Table III demonstrates a sensitivity of three-body widths estimated by quasiclassical expressions to different ingredients of the models. However, before discussing these effects we should make some overview of the models.

A. Our previous quasiclassical model

In the pioneering work of Goldansky the differential probability of the two-proton decay was estimated as

\[ w(E_{3r}, \varepsilon) \sim \exp \left[ -\frac{2\pi Z_{\text{core}} \sqrt{M}}{\sqrt{E_{3r}}} \left( \frac{1}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{1-\varepsilon}} \right) \right]. \]

In our works the quasiclassical expression for the two-proton width was used in the form

\[ \Gamma_s(E_{3r}) = \frac{6E_{3r}^{1/2}}{\pi(r_{ch x} r_{ch y})^{3/2}(M_y M_x)^{3/4}} \int_0^1 d\varepsilon \mathcal{P}_{I_x I_{y}}(\varepsilon, E_{3r}). \quad (17) \]

The “exponential” coefficient \( \mathcal{P}_{I_x I_{y}}(\varepsilon, E_{3r}) \) is closely related to the function \( w(E_{3r}, \varepsilon) \) above. The motivation for the preexponent choice was like follows. Let’s consider the two-proton width in the situation of no barrier (no Coulomb interaction and zero angular momentum \( l = 0 \)):

\[ \Gamma = 2\gamma^2 P_l(k, r_{ch}, Z) = 2 \frac{\theta^2 (kr_{ch}) \theta^2}{2Mr_{ch}^2} = 2v \frac{\gamma^2}{r_{ch}} = \frac{1}{\tau(r_{ch}/2)}. \]

The width in that case is formally just the inverse flight time for distance \( r_{ch}/2 \) [denoted as \( \tau(r_{ch}/2) \) above]. Using the same assumption (no Coulomb and zero angular momentum \( l_x = l_y = 0 \) we obtain from Eq. (17)

\[ \Gamma_s(E_{3r}) = \frac{6r_{ch x} r_{ch y} E_{3r}^{1/2}}{\pi(r_{ch x} r_{ch y})^{3/2}(M_y M_x)^{3/4}} \int_0^1 d\varepsilon k_x k_y. \]

Using the integral value

\[ \int_0^1 d\varepsilon \sqrt{\varepsilon(1-\varepsilon)} = \frac{\pi}{8}, \]

the estimate for the width is obtained as

\[ \Gamma_s(E_{3r}) = \frac{3E_{3r}^{1/2}}{2(2r_{ch x} r_{ch y})^{1/2}(M_y M_x)^{1/4}} \approx \frac{3}{8} \frac{v}{r_{ch}} \approx \frac{1}{\tau(r_{ch})}. \]

Thus, the width in such a “no barrier” case is normalized to the inverse flight time to some “typical internal distance” \( r_{ch} \).

The above idea of the preexponent derivation can be found too simplistic. However, calculations show that for the considered cases Eq. (17) demonstrates a good consistency with Eq. (16) when the channel radii are chosen in a sound way (namely they should be close to the inner classical turning point of the Coulomb barrier [27]).

B. Special cases of the present model

Let’s consider the some special cases of Eq. (16).

1. True three-body decay, \( E_{xa} \gg E_{3r}, E_{ya} \gg E_{3r} \)

In that case the main contribution to the energy integral in (16) is connected with \( \varepsilon = 1 \). Replacing the slowly varying functions \( \Delta \) by their constant values at \( \varepsilon = 1/2 \) we obtain

\[ \Gamma_m(E_{3r}) = \frac{E_{3r}(E_{3r} - E_{xa} - E_{ya})^2}{(E_{xa} - E_{3r} - E_{ya})^2 (E_{ya} - E_{3r} - E_{ya})^2} \times \frac{2\pi}{D_x D_y D_3 \gamma^2} \int_0^1 d\varepsilon \mathcal{P}_{I_x I_{y}}(\varepsilon, E_{3r}). \quad (18) \]

Thus we obtain formula analogous to (17), used in our previous works, but with different preexponent. It should be noted:

(i) The preexponent in this form is explicitly depending on the resonance properties of the subsystems.
(ii) It can be seen in Table III that agreement is very good between the value $\Gamma$ in Eq. (15) and the approximation $\Gamma_m$ in Eq. (15), calculated neglecting the variation of functions $\Delta_l$ within the decay window.

(iii) The dimensionless coefficient $D_x D_y D_3$ is not that different from unity (at the level of 10–50%). The $D_x$ and $D_y$ coefficients should be very close to unity for sufficiently narrow states, so we need to know only the properties of the two-body subsystem to fix this component of the model. However, the $D_3$ coefficient is beyond the R-matrix ideology and requires a validation within the three-body model.

(iv) If both subsystems (on $X$ and $Y$ coordinates) are equivalent, then $E_{xa} \equiv E_{ya}$ and the three-body width dependence on the two-body resonance position should have the following typical dependence:

$$\Gamma(E_{3r}) \sim (E_{xa} - E_{3r}/2)^{-2}.$$  \hspace{1cm} (19)

This dependence we have observed in the calculations within the simplified three-body model for $^{45}$Fe (two equivalent final state interactions), see Fig. 14 of Ref. [11].

2. Sequential decay, $E_{xa} < E_{3r}, E_{ya} \gg E_{3r}$

It is quite simple to integrate over $d\varepsilon$ in the Eq. (19) for these energy conditions:

$$\Gamma(E_{3r}) \approx \frac{D_y (V_3)^2}{(E_{3r} - E_{xa} - E_{ya})^2} \Gamma_{ya}(E_{3r} - E_{xa})$$

$$\Gamma(E_{3r}) \approx D_y D_3 \Gamma_{ya}(E_{3r} - E_{xa}).$$

Thus the width is reduced to a two-body expression with some modifications, which takes into account the three-body character of the model considered for the resonant state.

3. True three-body decay, $E_{xa} > E_{3r}, E_{ya} \gg E_{3r}$

In this case we can approximate the $(E_{3r} - E_{xa} - E_{ya})^2$ by $E_{ya}^2$ in the numerator and $(E_{ya} - E_{3r}/2)^2$ by $E_{ya}^2$ in the denominator and obtain

$$\Gamma(E_{3r}) = \frac{2}{\pi} \frac{D_x D_y D_3 \gamma_x^2 \gamma_y^2 E_{3r}}{(E_{xa} - E_{3r}/2)^2} \int_0^1 d\varepsilon P_{l_{ia}}(\varepsilon, E_{3r}).$$  \hspace{1cm} (20)

This situation is close to the one final state interaction (OFSI) model considered in Ref. [11] for methodological purposes. The three-body width dependence on the two-body resonance position is again

$$\Gamma(E_{3r}) \sim (E_{xa} - E_{3r}/2)^{-2}.$$  \hspace{1cm} (19)

as in Eq. (19) where two equivalent FSIs are considered. This behaviour is reasonably well reproduced for $^{17}$Ne within the simplified three-body OFSI model, see Fig. 5 in Ref. [11].

It should, however, be kept in mind that the considered approximation is valid and precise if the energy of the higher resonance $E_{ya}$ is kept significantly below the barrier. It is necessary for two reasons. (i) Geometric proximity of the resonance WFs $\psi_{i_{ia}}^*(X,Y)$ and $\psi_{l_{ia}}(X,Y)$ is required for Eq. (15) which is based on the smallness of kinetic energy contribution to the variation of total energy. (ii) The width of the upper resonance should be sufficiently small as demonstrated in Section 1.2. All the mentioned conditions for this special case are difficult to meet in practice. It can be seen in Fig. 5 of Ref. [11] that the width calculated for the $^{17}$Ne 3/2$^+$ state in OFSI model (one subsystem is nonresonant) follows the analytical dependence Eq. (19) with significant deviations, while in the case of calculations for $^{45}$Fe shown in Fig. 13 (both subsystems have resonances) the agreement is practically perfect.

A quasiclassical expression for the width was introduced in the papers [15, 16] for $^{12}$O in the form

$$\Gamma_b(E_{3r}) = \frac{2D_x \gamma_x^2 \gamma_y^2 E_{3r}}{\pi E_{xa}} \int_0^1 d\varepsilon P_{l_{ia}}(\varepsilon, E_{3r}, E_{xa}).$$  \hspace{1cm} (21)

(given here in our notations), which is practically equivalent to Eq. (20). It was introduced without derivation and analysis of the involved approximations. The expression was utilized in the series of works [17, 18, 19, 20, 21, 22] by Barker and Brown in somewhat modified form. We can see in Table III that for the systems under consideration the disagreement between “correct” value $\Gamma$ Eq. (16) and “special case” $\Gamma_b$ Eq. (21) is as large as a factor 2–4. It is also easy to show analytically that for systems with $E_{xa} \equiv E_{ya}$ (that means for all the ground state decays) the difference should be about a factor 4.

C. Diproton model

The nature of approximations used for derivation of the quasiclassical formula (19) is such that they can not be used for derivation of a corresponding formula for a “diproton” model. The formula (21) [actually analogous formula] was used in the papers [18, 19, 21, 22] in the “diproton” form by formally making the “X” subsystem to be subsystem of the $p-p$ motion. We find serious problems in such an approach.

We can consider this issue from a different side: the analysis can be started from the original expressions (11), (12) [not from approximation (13)] and the situation studied with one resonant and one nonresonant subsystem. The direct integration in (12) for the case of one resonant and one nonresonant continuum in the subsystems leads in the general case to the expression analogous to (21). However it contains a complicated additional coefficient that is strongly dependent on fine details of the system geometry and the behaviour of the nonresonant continuum. Thus we can conclude that in this case Eq.
should be correct only within an order of the magnitude. The analysis provided in Ref. [11] for the case physically corresponding to the “diproton” model (the OFSI model in the “T” Jacobi system) showed that the situation in that particular case is even worse. The semianalytical three-body model of Ref. [11] provides the width values which are very small (compared to those typically evaluated in diproton model). They could be matched to those obtained from Eq. (21) only if very small channel radii \( r_{chy} \) \((1−2\, \text{fm})\) are chosen for emission of the “diproton”. This requirement is certainly not consistent with the practice of the “diproton” model application, where the \( r_{chy} \) is typically chosen as a nucleus radius plus some “radius of diproton”.

D. Relation to the three-body calculations

Some important points could be understood using the information listed in Table III. Treatment of the Coulomb interaction in the present model follows the approximations discussed in detail in Ref. [11]: “no p-p” means that Coulomb interaction between protons is just neglected (product of charges are \( Z_x = Z_y = Z_{\text{core}} \), both in \( X \) and \( Y \) subsystems); effective treatment of Coulomb “Eff.” means that Coulomb interaction in \( Y \) coordinate is formed by proton and “effective particle” \( \text{core} + \) proton (product of charges are \( Z_x = Z_{\text{core}} \) in \( X \) and \( Z_y = Z_{\text{core}} + 1 \) in \( Y \) subsystem). Sensitivity to the choice of the Coulomb treatment is relatively high (factor 1.5 – 5, depending on particular model).

The values of \( \Gamma \) derived from Eq. (16) reasonably well agree with the corresponding results of the three-body calculations with the simplified Hamiltonian from Ref. [11]. The disagreement can be reduced if (i) the widths of the subsystems (\( \Gamma_x \), \( \Gamma_y \)) are taken the same as in the three-body model, (ii) corrections for coefficient \( D_3 \) (see Table III) are taken into account and (iii) radii of channels in Eq. (13) are close to the inner classical turning points \( \Gamma_{\text{cha}} \) for corresponding potentials in the three-body model (see Table III).

The sensitivity of the results to the “unphysical” (not leading to modification of observable values \( \Gamma_{\text{xa}}, \Gamma_{\text{ya}} \)) variation of channel radii is moderate (about factor of 1.5) except for the \( \Gamma_s \) model Eq. (17) from our previous works. However, even this model is providing results consistent with Eq. (16) if the channel radii are chosen close to classical turning point.

From comparison of the simplified three-body model with the realistic three-body model in Ref. [11] we can see that the calculations using the effective Coulomb interaction are reasonably close to realistic results and should be a preferable choice. However, even calculations with the effective Coulomb interaction differs from the realistic results typically by a factor 1.3 – 3. Thus the calculations in the quasiclassical model presented here (which itself is an approximation to the three-body model with simplified Hamiltonian of Ref. [11]) are not a replacement for the realistic three-body calculations if a better than mentioned precision is requested.

V. Conclusion.

In this work we derived the quasiclassical (“R-matrix type”) formula for two-proton decay widths. The preexponent coefficients are defined and evaluated numerically using the simplified three-body model. The following aspects of the obtained results should be emphasized. (i) The derivation is based on the three-body model with a simplified Hamiltonian \([11]\), which omits one FSI (p-p) and treats another in an approximate way (one of core-p interactions). The first approximation can be justified by the weakness of the p-p interaction compared to core-p interaction and should be good in heavy systems. The second approximation is connected with the finite mass of the core and also becomes well justified in heavy systems. (ii) The derivation of the quasiclassical formula requires existence of narrow states in both core-p subsystems. This condition is also well satisfied for the ground states of relatively heavy systems. (iii) The derived formula is basically the same as that obtained by Galitsky and Cheltsov in Ref. \([12]\) by a somewhat different procedure. As far as we started from construction and validation of the simplified three-body model \([11]\) we were able to define a precision of approximations used for transition to the quasiclassical model and define ingredients of the model which remained undefined in \([12]\). (iv) The most important dependencies of the three-body width \([10]\) are fixed by observable properties of the subsystems (positions and widths of the lowest resonances in the subsystems). However, there is “unphysical” sensitivity to channel radii when the observables for the subsystems are fixed. The quasiclassical formula provides a good agreement with the simplified three-body model when the radii of channels in the subsystems are chosen to be close and outside of the inner classical turning points of the barriers. (v) The formula (21), used in papers \([17, 18, 19, 20, 21, 22]\), is a special case of Eq. (17). It can be obtained by formally assuming the energy of one of the states in the subsystems to go to positive infinity. Such an assumption is unphysical when both valence protons populate states with the same quantum numbers. Thus formula (21) is valid within a factor which can be as large as 4. (vi) The derived expressions can not be used in the form of the “diproton” model without introducing a large (above an order of magnitude) uncertainty. Having in mind the origin and scale of the uncertainties introduced by reducing the realistic three-body model to a simplified three-body model (discussed in details in \([11]\)) and by reducing the simplified three-body model to the quasiclassical three-body model (considered in this work) we now get a basis for the appropriate (within
the limits of its reliability) application of this model for estimates of the two-proton widths.

VI. ACKNOWLEDGEMENTS

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Γ obtained by two-body scattering calculations [it exactly coincides with Γ defined by Eq. (3) for reduced widths

θ 

sense of Eq. (6) (Γ 

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2

8

Table I, II.

Ref. [11], Tables I, II.

The width Γ obtained by two-body scattering calculations [it exactly coincides with Γ defined by Eq. (3) for reduced widths θ^2].

| l  | θ_1^2 | θ_2^2 | θ_2^2/θ_1^2 | B_l(E_a = 1.1E_r) | B_l(E_a = 2E_r) | B_l(E_a = 5E_r) | Γ (MeV) |
|----|-------|-------|--------------|-------------------|-----------------|----------------|---------|
| 1  | 2.044 | 0.671 | 0.329        | 0.81              | 0.75            | 0.58          | 3.46 x 10^{-2} |
| 2  | 2.013 | 1.205 | 0.598        | 1.04              | 1.06            | 1.11          | 5.58 x 10^{-4}  |
| 3  | 2.007 | 1.432 | 0.714        | 1.007             | 1.011           | 1.03          | 2.14 x 10^{-6}  |
| 4  | 2.002 | 1.557 | 0.777        | 1.0022            | 1.0046          | 1.013         | 3.54 x 10^{-9}  |

Table II: Parameters for quasiclassical approximation calculated in a three-body (TFSI) model Ref. [11].

| E   | E_θ | l_y | E_ya | E_theta | (V^2)_θ | (V^2)_ya | (V^2)_ya/(V^2)_θ | D_θ |
|-----|-----|-----|------|---------|---------|---------|------------------|-----|
| 17Ne | 0.344 | 0 | 0.535 | 2 | 0.96 | 1.495 | 1.714 | 2.426 | 1.414 | 1.338 |
| 45Fe | 1.154 | 1 | 1.480 | 1 | 1.480 | 2.960 | 3.285 | 3.991 | 1.215 | 1.098 |

Table III: Width sensitivity to the treatment of the charges in the subsystems and to channel radii for fixed properties of the subsystems “X” and “Y”. The three-body widths Γ_s, Eq. (17), Γ_b, Eq. (21), Γ_m, Eq. (18), Γ, Eq. (16) with D_b = 1, are given in 10^{-14} MeV units for 17Ne and 10^{-19} MeV units for 45Fe. The two-proton decay energies E_2p are 0.344 MeV for 17Ne and 1.154 MeV for 45Fe. The recent experimental data on 45Fe [23] provide Γ_2p = 2.85_{-0.65}^{+0.85} x 10^{-19} MeV [T_1/2 (2p) = 1.6_{-0.3}^{+0.5} ms] for E_2p = 1.154(16) MeV and two-proton branching ratio Br(2p) = 0.57. The E_2a, E_ya values are given in Table II. The widths Γ_xa and Γ_ya are chosen to be the same as in the corresponding potential model of Ref. [11], Tables I, II.

| Coulomb | l_y | θ_1^2 | r_ch (fm) | Γ_xa (keV) | l_y | θ_2^2 | r_ch (fm) | Γ_ya (keV) | Γ_s | Γ_b | Γ_m | Γ |
|---------|-----|-------|-----------|------------|-----|-------|-----------|------------|-----|-----|-----|-----|
| 17Ne    | No p-p | 0.986 | 3.53^a   | 17.9       | 2   | 1.85  | 3.53^a    | 3.5        | 2.63 | 1.75 | 3.70 | 3.72 |
|         | No p-p | 0.667 | 5.06^b   | 17.9       | 2   | 1.145 | 4.04^b    | 3.5        | 9.44 | 2.04 | 4.33 | 4.35 |
|         | Eff.   | 0.667 | 5.06^b   | 17.9       | 2   | 1.145 | 4.12      | 2.2        | 1.86 | 0.385 | 0.845 | 0.844 |
|         | Eff.   | 0.667 | 5.06^b   | 17.9       | 2   | 0.162 | 7.0^c     | 2.2        | 2.27 | 0.510 | 1.12 | 1.10 |
|         | Eff.   | 0.363 | 8.0^c    | 17.9       | 2   | 1.145 | 4.12      | 2.2        | 5.84 | 0.522 | 1.15 | 1.13 |
|         | Eff.   | 0.363 | 8.0^c    | 17.9       | 2   | 0.162 | 7.0^c     | 2.2        | 71.2 | 0.691 | 1.52 | 1.50 |
| 45Fe    | No p-p | 1.07  | 4.72^b   | 0.257      | 1   | 1.07  | 4.72^b    | 0.257      | 10.5 | 3.24 | 12.7 | 12.9 |
|         | No p-p | 1.07  | 4.72^b   | 0.257      | 1   | 0.888 | 4.94^a    | 0.257      | 16.6 | 3.36 | 13.1 | 13.3 |
|         | Eff.   | 1.07  | 4.72^b   | 0.257      | 1   | 1.03  | 4.76      | 0.15       | 3.94 | 1.14 | 4.54 | 4.62 |
|         | Eff.   | 1.088 | 4.94^a   | 0.257      | 1   | 0.85  | 4.98^a    | 0.15       | 6.24 | 1.18 | 4.70 | 4.78 |
|         | Eff.   | 1.088 | 7.5^c    | 0.257      | 1   | 0.128 | 7.56      | 0.15       | 667 | 2.02 | 8.02 | 8.16 |

^a| This is radius from systematics r_ch = 1.4(A_core + 1)^{1/3} used in the papers 2, 4, 7, 8.

^b| This is radius of the classical inner turning point for potential Ref. [11], Tables I, II.

^c| At this radius the nuclear potential becomes negligible in potential model Ref. [11], Tables I, II.