Pentavalent arc-transitive Cayley graphs on Frobenius groups with soluble vertex stabilizer

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Abstract: A Cayley graph $\Gamma$ is said to be arc-transitive if its full automorphism group $\text{Aut}\Gamma$ is transitive on the arc set of $\Gamma$. In this paper we give a characterization of pentavalent arc-transitive Cayley graphs on a class of Frobenius groups with soluble vertex stabilizer.

Keywords: Arc-transitive graph, Frobenius group, Cayley graph, Soluble vertex stabilizer

MSC: 20B25, 05C25

1 Introduction

Throughout the paper, graphs considered are simple, connected and undirected. For a graph $\Gamma$, we denote the vertex set, edge set, arc set, valency and full automorphism group of $\Gamma$ by $V\Gamma$, $E\Gamma$, $A\Gamma$, $\text{val}(\Gamma)$ and $\text{Aut}\Gamma$, respectively. $\Gamma$ is said to be $G$-vertex-transitive, $G$-edge-transitive or $G$-arc-transitive if $G \leq \text{Aut}\Gamma$ is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$; in particular, if $G = \text{Aut}\Gamma$, then $\Gamma$ is simply called vertex-transitive, edge-transitive or arc-transitive. An $s$-arc of $\Gamma$ is a sequence of vertices $(u_0, u_1, \ldots, u_s)$ such that $u_i$ is adjacent to $u_{i+1}$ and $u_{i-1} \neq u_{i+1}$ for all possible $i$. For a subgroup $G \leq \text{Aut}\Gamma$, $\Gamma$ is said to be $(G, s)$-arc-transitive if $G$ is transitive on the set of $s$-arcs in $\Gamma$. In particular, a 0-arc is called a vertex, and a 1-arc is called an arc for short. As we all know, a graph $\Gamma$ is $G$-arc-transitive for some $G \leq \text{Aut}\Gamma$ if and only if $G$ is transitive on $V\Gamma$ and the vertex stabilizer $G_v$ of $v \in V\Gamma$ in $G$ is transitive on the neighborhood $\Gamma(v)$ of $v$.

A graph $\Gamma$ is called a Cayley graph if there exist a group $G$ and a subset $S \subset G \setminus \{1\}$ with $S = S^{-1} : = \{g^{-1} \mid g \in S\}$ such that the vertices of $\Gamma$ may be identified with the elements of $G$ in such a way that $x$ is adjacent to $y$ if and only if $yx^{-1} \in S$. The Cayley graph $\Gamma$ is denoted by $\text{Cay}(G, S)$. Throughout this paper, we denote the vertex of $\text{Cay}(G, S)$ corresponding to the identity of $G$ by 1.

A graph $\Gamma$ is a Cayley graph on $G$ if and only if $\text{Aut}\Gamma$ contains a subgroup which is regular on vertices and isomorphic to $G$. It is well-known that a Cayley graph is vertex-transitive. However, a Cayley graph is of course not necessarily arc-transitive. Thus much excellent work has dealt with arc-transitive Cayley graphs. In particular, there are many works about cubic and pentavalent Cayley graphs. For the cubic case, see [1–3] for cubic symmetric Cayley graphs on finite nonabelian simple groups, which are normal except for $A_{47}$, see [4] for a characterisation of connected cubic $s$-transitive Cayley graphs, see [5] for a classification of the connected arc-transitive cubic Cayley graphs on $\text{PSL}(2, p)$ where $p \geq 5$ is a prime, and see [6] for a classification of cubic arc-transitive Cayley graphs on a class of Frobenius groups. For the pentavalent case, see [7] for a classification of arc-transitive pentavalent Cayley graphs on finite nonabelian simple groups, see

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The soluble vertex stabilizer for arc-transitive graphs of valency

We give two basic results for pentavalent graphs.

Lemma 2.2.\[14\]

In this section we give some preliminary results, which will be used in the subsequent sections.

After this introductory section, some preliminary results are given in Section 2 and a few examples are given in Section 3. Then we complete the proof of Theorem 1.1 in Section 4.

2 Preliminary Results

In this section we give some preliminary results, which will be used in the subsequent sections.

Let $\Gamma$ be a $G$-vertex-transitive graph. Then, for $a \in V\Gamma$, the stabilizer $G_a$ is a core-free subgroup in $G$, that is, $\cap_{g \in G} G_a^g = 1$. Set $H = G_a$ and $D = \{x \mid \alpha^x \in \Gamma(\alpha)\}$. Then $D$ is a union of several double cosets $HxH$. Moreover, $\Gamma$ is isomorphic to the coset graph $\text{Cos}(G, H, D)$ defined over $\{Hx \mid x \in G\}$ with edge set $\{(Hg_1, Hg_2) \mid g_2 g_1^{-1} \in D\}$.

The following statements for coset graphs are well known.

(a) $\Gamma$ is undirected if and only if $D = D^{-1} = \{x^{-1} \mid x \in D\}$.

(b) $\Gamma$ is connected if and only if $(H, D) = G$.

(c) $\Gamma$ is $G$-arc-transitive if and only if $D = HgH$ for $g \in G$ with $g^2 \in H$; moreover, $g$ can be chosen as a 2-element such that $g \in N_G(H \cap H^g)$.

For a Cayley graph $\Gamma = \text{Cay}(G, S)$, Let $\text{Aut}(G, S) = \{a \in \text{Aut}(G) \mid S^a = S\}$. Then we have the following basic result.

Lemma 2.1. ([12, Lemma 2.1]) Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph. Then the normalizer $N_{\text{Aut}}(G) = G: \text{Aut}(G, S)$.

The soluble vertex stabilizer for arc-transitive graphs of valency 5 is known.

Lemma 2.2.\[14\] Let $\Gamma$ be a pentavalent $(G, s)$-transitive graph for some $G \leq \text{Aut}\Gamma$ and $s \geq 1$. Let $v \in V\Gamma$. If $G_v$ is soluble, then $|G_v| > 80$ and $s \leq 3$. Furthermore, one of the following holds:

(1) $s = 1$, $G_v \cong \mathbb{Z}_5$, $D_{10}$ or $D_{20}$;

(2) $s = 2$, $G_v \cong F_{20}$, or $F_{20} \times \mathbb{Z}_2$;

(3) $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$.

We give two basic results for pentavalent graphs.
Lemma 2.3. Let $\Gamma = \text{Cay}(G, S)$ be a connected pentavalent graph with soluble stabilizer. Assume that $\text{Aut}\Gamma$ contains a subgroup $X$ such that $\Gamma$ is $X$-arc-transitive and $G \leq X$. Then $X_1 \cong \mathbb{Z}_5, D_{10}$, or $F_{20}$.

Proof. Since $\Gamma$ is connected, $G = \langle S \rangle$, and thus $\text{Aut}(G, S)$ acts faithfully on $S$. So $\text{Aut}(G, S) \leq S_5$. By Lemma 2.1, $X \leq \text{Aut}_1(G) = G: \text{Aut}(G, S)$. Thus $X_1 \leq \text{Aut}(G, S) \leq S_5$. Note that $X_1$ is transitive on $S$, so $X_1 \cong \mathbb{Z}_5, D_{10}$, or $F_{20}$. We say a vertex-transitive graph $\Gamma$ is a normal cover of its quotient graph $\Gamma_N$ if $\Gamma$ and $\Gamma_N$ have the same valency, where $N < \text{Aut}\Gamma$ is not transitive on $V\Gamma$.

Lemma 2.4. Let $\Gamma$ be a connected pentavalent $X$-arc-transitive graph with soluble stabilizer, and let $N < X$ such that $X/N$ is insoluble, where $X \leq \text{Aut}\Gamma$. Then $\Gamma$ is a normal cover of $\Gamma_N$.

Proof. Let $u \in V\Gamma$, and let $B = u^N$ be an orbit of $N$ acting on $V\Gamma$. Then $K$ is the kernel of $X$ acting on $V\Gamma$. Then $K_u$ is soluble as $K_u \leq X_u$. By the Frattini argument, we have that $K = NK_u$. Note that $K/N \cong NK_u/N \cong K_u/(N \cap K_u)$, so $K/N$ is soluble. Since $X/N$ is insoluble, $X/K \cong (X/N)/(K/N)$ is insoluble. Thus $\Gamma$ is a normal cover of $\Gamma_N$.

The next lemma gives a classification of locally primitive Cayley graphs on abelian groups.

Lemma 2.5. [15, Theorem 1.1] Let $\Gamma$ be a connected locally primitive Cayley graph of an abelian group of valency at least 3. Then one of the following holds:

1. $\Gamma = K_n, K_{n,n}, K_n \times \cdots \times K_n$;
2. $\Gamma$ is the standard double cover of $K_2^k$;
3. $\Gamma$ is a normal or bi-normal Cayley graph of an elementary abelian 2-group or a meta-abelian 2-group.

3 Examples

In this section we give some examples of pentavalent arc-transitive graphs.

Example 3.1. Let $T = \text{PSL}(2, 4)$. Take a subgroup $H \cong \mathbb{Z}_5$ of $T$. Let $g \in \text{N}_T(L) \setminus L$ be an involution such that $\langle H, g \rangle = T$, where $L = H^g \cap H$. Let $G \cong \mathbb{Z}_5^3 : \mathbb{Z}_3$ be a subgroup of $T$. Then $G$ is transitive on $[T:H]$, and so the coset graph $\mathbb{C}_{12} = \text{Cos}(T, H, HgH)$ is a pentavalent $T$-arc-transitive Cayley graph on $G$.

Example 3.2. Let $T = \text{PSL}(2, 9)$. Take a subgroup $H \cong D_{10}$ of $T$. Let $g \in \text{N}_T(L) \setminus L$ be an involution such that $\langle H, g \rangle = T$, where $L = H^g \cap H$. Let $G \cong \mathbb{Z}_3^2 : \mathbb{Z}_6$ be a subgroup of $T$. Then $G$ is transitive on $[T:H]$, and so the coset graph $\mathbb{C}_{36} = \text{Cos}(T, H, HgH)$ is a pentavalent $T$-arc-transitive graph but not a Cayley graph on $G$.

4 Proof of Theorem 1.1

In this section we will prove Theorem 1.1 by a series of lemmas.

Let $G = W:H \cong \mathbb{Z}_d^n ; \mathbb{Z}_n$ be a primitive Frobenius group, where $p$ is a prime, and $d, n$ are integers. Assume that $\Gamma = \text{Cay}(G, S)$ is a connected pentavalent arc-transitive Cayley graph on $G$ with soluble vertex stabilizer. First of all, we study the case where the full automorphism group $A := \text{Aut}\Gamma$ is soluble.

Suppose that $G \cong D_{2p}$. Then we have the following lemma, see [16, Proposition 2.7].

Lemma 4.1. Let $G$ be a dihedral group of order $2p$, and let $\Gamma$ be a connected pentavalent arc-transitive Cayley graph on $G$. Then $\Gamma \cong G(2p, 5)$ with $p \equiv 1(\text{mod } 5)$ and $p > 11$, $\text{Aut}\Gamma \cong (\mathbb{Z}_p : \mathbb{Z}_5) : \mathbb{Z}_2$. 

Let $F$ be the Fitting subgroup of $A$, that is, $F$ is the largest nilpotent normal subgroup of $A$. Then $F' = 1$, and $C_A(F) \leq F$ as $A$ is soluble.

For a group $H$ and a prime $p$, we denote the Sylow $p$-subgroup of $H$ by $H_p$.

**Lemma 4.2.** If $G \not\cong D_{2p}$, then $W$ is normal in $A$.

*Proof.* We claim that $G \cap F \neq 1$. Suppose that $G \cap F = 1$. Since $F \leq A_u$, $|F| \mid |A_u|$. It follows that $|F| \mid 80$ by Lemma 2.2, where $u \leq V$. Assume that $G$ is transitive on $V$. Then $|G| \mid |F|$. Since $H$ acts irreducibly on $W$, $G \cong \mathbb{Z}_5 \mathbb{Z}_4$ or $\mathbb{Z}_2^2 \mathbb{Z}_5$. Since there exists no connected arc-transitive pentavalent graphs of order $4p$ for each prime $p \geq 5$ by [18, Theorem 1.1], the former case does not occur. By [17], there exists no connected arc-transitive pentavalent graphs of order 80, so the latter case is excluded. Similarly, we also exclude the case where $F_G \cong K_2$.

Thus $\Gamma$ is a normal cover of $\Gamma_F$. Then $|F|$ divides $|G|$. Since $C_G(F) \leq F$, $C_G(F) = 1$. Therefore $G$ acts faithfully on $F$. It follows that $G \leq \text{Aut}(F)$. Thus $2 \mid |F|$.

Suppose that $p$ is even. By the previous paragraph, we have that $n \geq 7$. Assume that $\Phi(F) = 1$. Since $|F|$ divides both 80 and $|G|$, we obtain that $F_2 \cong \mathbb{Z}_4^2$ and $F_2' \cong \mathbb{Z}_5$ or 1, where $k \leq \min(4, d)$. Note that $C_A(F) \leq F$, so $G \leq \text{GL}(k, 2)$. By [19], there exists no $k$ and $G$ satisfying the above relation. Thus $\Phi(F) = 1$. Since $F_5 \cong \mathbb{Z}_5$, we conclude that $\Phi(F_2) = \Phi(F) \leq F_2$. Let $\bar{F}_2 = F_2/\Phi(F)$ and $\bar{H} = H\Phi(F)/\Phi(F)$. Then $\bar{H} \cong H$. If $\bar{C}_P(\bar{F}_2) = 1$, then $\bar{C}_P(\bar{F}_2) = 1$, and so $\bar{H} \leq \text{GL}(k, 2)$, where $k \leq 3$. By [19], $\bar{F}_2 = \mathbb{Z}_4^2$ and $\bar{H} \cong \mathbb{Z}_7$. It follows that $|F_2| = 16 = d$ and $3$, which is a contradiction.

Suppose that $n$ is odd. Assume that $\Phi(F) = 1$. Then $F_2 \cong \mathbb{Z}_2^2$ and $2^k \mid n$, where $k \leq 4$. Note that $G \leq \text{GL}(k, 2)$, it follows that $W \leq \text{GL}(k, 2)$, and $Z_{2^n} \leq \text{GL}(k, 2)$. By [19], there exists no $n$ and $W$ satisfying the above relation. Thus $\Phi(F) = 1$. Since $F_5 \cong \mathbb{Z}_5$, we conclude that $\Phi(F_2) = \Phi(F) \leq F_2$. Let $\bar{W} = W\Phi(F)/\Phi(F_2)$ and $\bar{F}_2 = F_2/\Phi(F_2)$. If $\bar{C}_P(\bar{F}_2) = 1$, then $\bar{C}_P(\bar{F}_2) = 1$, refer to [20, p.174, Theorem 1.4], a contradiction. Thus $\bar{C}_P(\bar{F}_2) = 1$, and so $\bar{W} \leq \text{GL}(k, 2)$, where $k \leq 3$. By [19], this is impossible.

To sum up, $F \cap G = 1$. Since $W$ is minimal in $G$, $W \leq F$. Note that $G \unlhd D_{2p}$, so $\Gamma$ is a cover of $\Gamma_{F_p}$, and thus $W = F_p$. Therefore, $W$ is normal in $A$. This completes the proof.

**Lemma 4.3.** With the hypothesis of Lemma 4.2, then $G$ is normal in $A$, and $A_u \cong \mathbb{Z}_5 \mathbb{Z}_{10}$, $D_{220}$, or $F_{20}$, where $u \leq V$.

*Proof.* By Lemma 4.2, $W$ is normal in $A$. Since $G \not\cong D_{2p}$, $n \geq 2$. Then $G$ is a normal cover of $\Gamma_W$. By Lemma 2.5, either $G/W \leq A/W$ or $\Gamma_W \cong K_g$ or $K_5$. In the former case, $G \leq A$. By Lemma 2.3, $A_u \cong \mathbb{Z}_6 \mathbb{Z}_{10}$, or $F_{20}$, where $u \leq V$. If $\Gamma_W \cong K_g$, then $A/W \leq \text{Aut}(\Gamma_W) \cong S_6$. Note that $5 \cdot 6 \mid |A/W|$, so $A_u$ is insoluble, which is a contradiction. Similarly, we can exclude the case where $\Gamma_W \cong K_5$. This completes the proof of Lemma 4.3.

In the remaining section, we study the case where the full automorphism group $A$ is insoluble. Denote by $R$ the radical of $A$, that is, $R$ is the largest soluble normal subgroup of $A$.

Suppose that $R = 1$. Then we have the following lemma.

**Lemma 4.4.** If $R = 1$, then $(G, \Gamma) = (\mathbb{Z}_2^2 \mathbb{Z}_3, C_{12})$.

*Proof.* Let $N$ be a minimal normal subgroup of $A$. Since $R = 1$, $N = T_i \times \cdots \times T_\ell \cong T^\ell$, where $T_i \cong T$ is non-abelian simple. By [21], $T$ is one of the following:

- $\text{PSL}(4, 2)$, $\text{PSU}(3, 8)$, $M_{11}$, $\text{PSp}(4, 3)$, $\text{PSL}(3, q)(q < 9)$, $\text{PSL}(2, q)(q > 3)$.

Let $W_i = T_i \cap W$, where $1 \leq i \leq \ell$, and let $N = N \cap H$. Since $G$ is a Frobenius group, $L$ is a diagonal subgroup of $N$. Write $L = \langle h_1 h_2 \cdots h_\ell \rangle$, where $\langle h_1 \rangle \cong L$. Let $H_1 = \langle h_1 \rangle$ and $G_1 = W_i H_1$, where $1 \leq i \leq \ell$. Then $G_1$ is a Frobenious group. Let $m = \left\lvert \frac{F_1}{[G_1]} \right\rvert \prod_{i=2}^{\ell} \left\lvert \frac{F_i}{[W_i]} \right\rvert$. Since $N \cap G = W: L$, it follows that $\left\lvert N \cap G \right\rvert \mid |A_u|$, and $m \mid 80$.

Suppose that $T = \text{PSL}(4, 2)$. By [19], $G_i \leq A_7$ and $A_7$ has no subgroup with index 2, 5 or 10. So $m$ does not divide $|A_u|$, which is a contradiction. Similarly, we can exclude the cases where $T = \text{PSU}(3, 8)$, $M_{11}$, $\text{PSp}(4, 3)$ and $\text{PSL}(3, q)(q < 9)$. 

Suppose that $T = \text{PSL}(2, q)$ where $q = r^t$ with $r$ a prime. According to [22, Theorem 6.25], $G_t$ is isomorphic to a subgroup of one of the following groups:

$$\mathbb{Z}_r^4:\mathbb{Z}_{(q-1)} \mathbf{, ~ D}_{2(q-1)} \mathbf{, ~ A}_5 \mathbf{, ~ PGL}(2, r^t)$$

where $d = (2, r - 1)$, and $f \mid e$. In the following, we process our analysis by several cases.

**Case 1:** $G_1 \not\leq \mathbb{Z}_r^4:\mathbb{Z}_{(q-1)}$.

If $t > 1$, then $(q + 1)^2$ divides 80 since $m$ divides $|u|$. It follows that $(q + 1)^2$ divides 16, and thus $q = 3$, which is a contradiction. Thus $t = 1$, namely, $N \cong T$. For this case, $q + 1$ divides 80. It implies that $q = 4, 7, 9, 19$ or 79. Since $W \cong \mathbb{Z}_r^4$ with $e_i \leq e$, we conclude that $W \cong \mathbb{Z}_2^2, \mathbb{Z}_7, \mathbb{Z}_3^2, \mathbb{Z}_{19}$ or $\mathbb{Z}_{79}$.

Assume that $W \cong \mathbb{Z}_2^2$. Then $G \cong \mathbb{Z}_3^2: \mathbb{Z}_3$. By Example 3.1, $\Gamma \cong C_{12}$. Assume that $W \cong \mathbb{Z}_2^2$. Then $G \cong \mathbb{Z}_2^2: \mathbb{Z}_8$ or $\mathbb{Z}_2^2: \mathbb{Z}_4$. Since $\text{PSL}(2, 9)$ has no subgroup with order 72, we can exclude the former case. For the latter case, by Example 3.2, we can also exclude this case.

Assume that $W \cong \mathbb{Z}_7, \mathbb{Z}_{19}$ or $\mathbb{Z}_{79}$. Then $G$ is isomorphic to one of the following

$$\mathbf{D}_{14}, \mathbf{Z}_7: \mathbf{Z}_6, \mathbf{D}_{38}, \mathbf{Z}_{19}: \mathbf{Z}_6, \mathbf{Z}_{19}: \mathbf{Z}_3: \mathbf{Z}_2, \mathbf{Z}_{158}, \mathbf{Z}_{79}: \mathbf{Z}_6, \mathbf{Z}_{79}: \mathbf{Z}_2: \mathbf{Z}_6, \mathbf{Z}_{79}: \mathbf{Z}_7: \mathbf{Z}_8$$

Since $\text{PSL}(2, 19)$ has no subgroup with order 114 and 342 by MAGMA [23], we can exclude the cases where $G \cong \mathbb{Z}_{19}: \mathbb{Z}_6$, and $\mathbb{Z}_{19}: \mathbb{Z}_3$. Besides, we can also exclude the cases where $G \cong \mathbf{D}_{14}, \mathbf{D}_{38}$ and $\mathbf{D}_{158}$ by [16, Proposition 2.7], and where $G \cong \mathbf{Z}_7: \mathbf{Z}_6, \mathbf{Z}_{79}: \mathbf{Z}_6, \mathbf{Z}_{79}: \mathbf{Z}_2: \mathbf{Z}_6$ and $\mathbf{Z}_{79}: \mathbf{Z}_7: \mathbf{Z}_8$ by [16, Theorem 4.2].

**Case 2:** $G_1 \not\leq \mathbb{D}_{2(q-1)}$.

By the discussion as above, $N \cong T$. If $G_1 \not\leq \mathbb{D}_{2(q-1)}$, then $q(q + 1)$ divides 80. It follows that $q = 4$. Thus $G \cong \mathbb{D}_6$. By [16, Proposition 2.7], $\Gamma \cong K_6, \text{Aut}\Gamma \cong S_6$, and so $|u| \mid 80$, which is a contradiction. Thus $G_1 \not\leq \mathbb{D}_{2(q-1)}$. For this case, $q(q - 1)$ divides 80. It follows that $q = 5$, and then $G \cong \mathbb{D}_6$. By above discussion, this case is also excluded.

**Case 3:** $G_1 \not\leq \mathbb{A}_5$.

Then $G_1 \cong \mathbb{D}_6$ or $\mathbb{D}_{10}$, and $N \cong T$. Since $|A| \mid 80$, we can exclude the case where $G \cong \mathbb{D}_{10}$. Arguing as Case 2, we can exclude the case where $G \cong \mathbb{D}_6$.

**Case 4:** $G_1 \not\leq \mathbb{PGL}(2, r^t)$.

Let $r = \frac{r^t - 1}{r^t - 1}$. Then $r$ divides 80. If $e - f = 1$, then $e = 2, f = 1$ and $r(r^t + 1) \mid 80$. It follows that $r = 2$ and $N \cong T$. Hence $G \cong \mathbb{D}_6$ or $\mathbb{D}_{10}$. However, the case does not occur by Case 3. Thus $e > f + 1$, and so $r = 2$. Then $\frac{r^t - 1}{r^t - 1} = 5$, which is impossible. This completes the proof of Lemma 4.4.

In what follows, we will prove that $R = 1$.

**Lemma 4.5. The radical $R = 1$.**

**Proof.** Suppose that $R = 1$. Set $L = R \cap G$. Let $L \neq 1$. Since $W$ is minimal and normal in $G$, $W \leq L$. Note that $A/R$ is insoluble, so $\Gamma$ is a normal cover of $\Gamma_R$ by Lemma 2.4.

Let $\mathcal{U} = GR/R$. Then $\mathcal{U} \cong G/(G \cap R)$ is cyclic. Since $\Gamma$ is a Cayley graph of $G$, $\Gamma_R$ is a Cayley graph of $\mathcal{U}$. By Lemma 2.5, we obtain that $\Gamma_R \cong K_6, K_{5,5}$. Thus $\text{Aut}\Gamma_R \cong S_6$, or $S_5 \times \mathbb{Z}_2$. It follows that $|u| \mid 80$, a contradiction.

Suppose that $L = 1$. Let $\mathcal{U} = GR/R$. Then $\mathcal{U} \cong G$. Let $A = A/R$. Then $\mathcal{A} = \mathcal{U} \mathcal{A}$, where $\mathcal{U} \in VT_R$. Arguing as the proof of Lemma 4.4 with $\mathcal{A} = \mathcal{U} \mathcal{A}$ in place of $A = GA_u$, $\mathcal{A}$ is almost simple, and $\mathcal{U} \cong \mathbb{Z}_2^2: \mathbb{Z}_3$. Therefore, $A$ is almost simple, which is a contradiction.

The assertion of Theorem 1.1 follows from Lemmas 4.1 and 4.3-4.5.

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