Parallel algorithms for maximizing monotone one-sided $\sigma$-smooth functions

Hongxiang Zhang$^1$, Yukun Cheng$^2$, Chenchen Wu$^3$, Dachuan Xu$^1$ and Dingzhu Du$^4$

$^1$Beijing Institute for Scientific and Engineering Computing, Beijing University of Technology, Beijing, 100124, P.R. China.
$^2$School of Business, Suzhou University of Science and Technology, Suzhou, 215009, P.R. China.
$^3$College of Science, Tianjin University of Technology, Tianjin, 300384, P.R. China.
$^4$Department of Computer Science, University of Texas, Dallas, Dallas, 75083, USA.

Contributing authors: zanghx010@emails.bjut.edu.cn; ykcheng@amss.ac.cn; wuchenchen_tjut@163.com; xudc@bjut.edu.cn; dzdu@utdallas.edu;

Abstract

In this paper, we study the problem of maximizing a monotone normalized one-sided $\sigma$-smooth ($OSS$ for short) function $F(x)$, subject to a convex polytope (no need to downward-closed [1]). A function $F(x)$ is one-sided $\sigma$-smooth if $\frac{1}{2}u^T\nabla^2 F(x)u \leq \sigma \cdot \frac{\|u\|_1}{\|x\|_1} u^T \nabla F(x)$, for all $x, u \geq 0, x \neq 0$. This problem was first introduced by Mehrdad et al. [1] to characterize the multilinear extension of some set functions. Different with the serial algorithm with name Jump-Start Continuous Greedy Algorithm by Mehrdad et al. [1], we propose Jump-Start Parallel Greedy (JSPG for short) algorithm, the first parallel algorithm, for this problem. The approximation ratio of JSPG algorithm is proved to be $(1 - e^{-\left(\frac{\alpha^2}{\alpha+1}\right)^\sigma}) - \epsilon$ for any any number $\alpha \in (0, 1]$ and $\epsilon > 0$, which improves the approximation ratio of JSCG algorithm in [1]. We also prove that our JSPG algorithm runs in $(O(\log n/\epsilon^2))$ adaptive rounds and consumes $O(n \log n/\epsilon^2)$ queries, where the number of adaptive rounds and function evaluation queries approximately matches the known results for parallel submodular maximization. In addition,
we study the stochastic version of maximizing monotone normalized OSS function, in which the objective function $F(x)$ is defined as $F(x) = \mathbb{E}_{y \sim T}[f(x, y)]$. Here $f$ is a stochastic function with respect to the random variable $Y$, and $y$ is the realization of $Y$ drawn from a probability distribution $T$. For this stochastic version, we design Stochastic Parallel-Greedy (SPG) algorithm, which achieves a result of

$$F(x) \geq (1 - e^{-\left(\frac{\alpha}{\alpha + 1}\right)^2})OPT - O(\kappa^{1/2}),$$

with the same time complexity of JSPG algorithm. Here $\kappa = \max\{5\|\nabla F(x_0) - d_0\|^2, 16\sigma^2 + 2L^2D^2\}$ is related to the preset parameters $\sigma, L, D$ and time $t$.

**Keywords:** OSS function, Parallel algorithm, Down-closed, Monotone

### 1 Introduction

This paper studies the problem of maximizing a monotone normalized one-sided $\sigma$-smooth (OSS) function subject to a convex polytope (no need to downwards-closed [1]). A function $F$ is one-sided $\sigma$-smooth if $\frac{1}{2}u^T \nabla^2 F(x) u \leq \sigma \cdot \frac{1}{\|x\|_1} u^T \nabla F(x)$, for all $x, u \geq 0, x \neq 0$ (See Appendix for examples of OSS functions). The problem to maximize OSS functions plays an important role in many fields, including machine learning [2, 3], document aggregation [4], web search [5], recommender systems [6, 7]. In this paper, we consider the deterministic and stochastic settings of the OSS maximization problems given the basis of the polyhedron constraint, and design two parallel algorithms for them, respectively. Formally the deterministic OSS maximization problem is defined as follows:

$$\max F(x) \quad s.t. \quad x \in [0, 1]^n, x \in P,$$

where $P$ is a convex polytope, and $F : [0, 1]^n \to \mathbb{R}_+$ is a monotone normalized OSS function. For the stochastic OSS problem, let $x \in X \subset [0, 1]^n$ be an optimization variable and $Y$ be a random variable, which both determine the choice of a stochastic function $f : X \times Y \to \mathbb{R}$. The stochastic OSS maximization problem is formally defined as

$$\max F(x) := \max \mathbb{E}_{y \sim T}[f(x, y)] \quad s.t. \quad x \in [0, 1]^n, x \in P,$$

where $P$ is a convex polytope and $0 \in P$, $F$ is the expected value of the stochastic function $f$ with respect to the random variable $Y$, and $y$ is the realization of the random variable $Y$ drawn from a distribution $T$. $\mathbb{E}_{y \sim T}[f(x, y)]$ is a monotone normalized OSS function.

The concept of OSS was first introduced by Mehrdad et al. [1] to describe the properties of multilinear extension [8] of submodular set functions or diversity functions. Similar to Lipschitz smoothness [9], the property of OSS can control the approximation ratio and the complexity of related algorithms. The
main method to maximize the OSS problem is the continuous greedy, whose core is to maximize the multilinear extension function. Submodularity ensures some nice properties for the multilinear extension. For instance, concavity along a direction $d \geq 0$ is used to bound a Taylor series expansion in the continuous greedy analysis. Since nonsubmodular multilinear extensions will not have this concavity property, Mehrdad et al proposed a "one-sided $\sigma$-smoothness" condition which guarantees an alternative bound based on Taylor series. Many well studied functions such as continuous DR-submodular functions are OSS functions when $\sigma = 0$. Chandra et al. [10, 11] stated that the multilinear extension of submodular set function is concave in the non-negative direction. They gave a $(1 - 1/e - \epsilon)$-approximation for the maximization problem of submodular set function subject to a cardinality constraint. When $\sigma > 0$, OSS function includes the multilinear extension of a diversity function, proved by Mehrdad et al. [1]. They provided a tight $(1 - 1/e^{(1-\alpha)/2\sigma})$-approximation for the maximization problem of monotone normalized OSS function, but their polyhedral constraint must have a downward closed property.

Mehrdad et al. [1] first adopted the Frank-Wolfe algorithm to solve OSS problems. Our parallel algorithm is inspired by the Frank-Wolfe Algorithm, but is different from the parallelism of the traditional discrete Greedy algorithms. To be specific, our parallel algorithm improves the computational efficiency by avoiding solving the optimization problem $\max_{\nu \in \mathcal{P}} \nu^T \nabla F(x)$ in each iteration.

Recently, many researchers focused on the study of parallel algorithms [12, 13] to solve optimization problems under big data. Eric and Singer [14] first proposed a parallelism for submodular set function maximization problem with cardinality constraints. For this problem, Eric et al. and Alina et al. [15, 16] respectively designed a near-optimal $(1 - 1/e - \epsilon)$-approximation algorithm with $O(\log n/ \epsilon^2)$ rounds of adaptivity. In [10], Chandra and Quanrud devised an adaptive algorithm to approximately maximize the multilinear relaxation of a submodular set function subject to packing constraints. The algorithm in [10] achieves a near-optimal $(1 - 1/e - \epsilon)$-approximation in $O(\log^2 m \log n/ \epsilon^4)$ rounds, where $n$ is the cardinality of the ground set and $m$ is the number of packing constraints. All the above algorithms are parallel versions of traditional greedy algorithms and respective variants for the submodular function maximization problems. However, there is no specific parallel algorithm to solve the OSS problems.

Generally, one key step to optimize OSS problems [3] or continuous submodular problems [17, 18] is to compute a linear optimization problem: $\max_{\nu \in \mathcal{P}} \nu^T \nabla F(x)$, where $x, \nu$ belongs to a bounded constraint. However, solving this linear optimization problem requires the exact value of gradient $\nabla F(x)$, which is not feasible for stochastic OSS problem. In this paper, we focus on the case where $F$ is a monotone normalized OSS function. For the stochastic OSS problem, an approach to avoid computing $\nabla F(x)$ is modifying the existing algorithms by replacing gradients $\nabla F(x)$ with their stochastic estimation $\nabla f(x, y)$. However, this modification may lead to arbitrarily poor solutions.
Recently, Mokhtari et al. [20] confirmed that if the feasible region is uniformly bounded, the gradient function $\nabla F(x)$ is $L$-Lipschitz continuous and the variance of the unbiased stochastic gradients $\nabla f(x, y)$ is bounded by the constant $\theta^2$, then $\mathbb{E}[\|\nabla F(x_t) - d_t\|^2] \leq \kappa$, where $d_t$ is the Linear replacement of stochastic gradient $\nabla f(x, y)$, $(d_t = (1 - \rho_t)d_{t-1} + \rho_t\nabla f(x_t, y_t))$, where $\rho_t = \frac{4}{t+8}^{2/3}$ is a positive step size dependent on time $t$ and the initial vector $d_0$ is 0) and $\kappa = \max\{5\|\nabla F(x_0) - d_0\|^2, \frac{16\sigma^2 + 2L^2D^2}{(t+9)^{2/3}}\}$ ($L, D$ are constants). This conclusion is very helpful for our design of parallel algorithms for stochastic OSS problems.

1.1 Main Contributions

In this paper, we propose two parallel algorithms to maximize the deterministic OSS problem and the stochastic OSS problem, subject to a convex polytope constraint (no need to downwards-closed), respectively.

- For the deterministic monotone normalized OSS maximization problem, we design the Jump-Start Parallel-Greedy (JSPG) algorithm by combining different techniques, including Jump-Start, Frank-Wolfe, Continuous Greedy and parallel computing. JSPG algorithm iterates by constantly accessing the relationship between the $\text{OPT}$ and the gradient function $\nabla F(x)$, and finally outputs the solution. We theoretically prove that JSPG algorithm has an approximation ratio of $\left(1 - e^{-\left(\frac{\alpha}{\alpha+1}\right)^{2\sigma}} - \epsilon\right)$ for any number $\alpha \in [0, 1]$ and $\epsilon > 0$, and it runs in $O(\log n/\epsilon^2)$ adaptive rounds and consumes $O(n \log n/\epsilon^2)$ queries. Furthermore, JSPG algorithm can be applied to compute the continuous submodular function, which can output a $(1 - 1/e - \epsilon)$ approximately optimal solution.

- For the stochastic monotone normalized OSS maximization problem, we design the stochastic Parallel-Greedy (SPG) algorithm, by combining different techniques, including the Linear replacement of stochastic gradient, Frank-Wolfe, Continuous Greedy and parallel computing. SPG algorithm outputs the final solution through constantly accessing the relationship between the $\text{OPT}$ and $d_t$ in each iteration. Furthermore, SPG algorithm is proved that the final output solution $F(x)$ satisfies $F(x) \geq (1 - e^{-\left(\frac{\alpha}{\alpha+1}\right)^{2\sigma}} - \epsilon)\text{OPT} - O(\kappa^{1/2})$, and its time complexity is the same with the one of JSPG algorithm.

1.2 Organization

The remainder of this paper is organized as follows. Section 2 introduced some definitions and necessary lemmas for the algorithms design. In Section 3 and 4, two parallel algorithms are proposed for the deterministic and stochastic OSS problem, respectively. The theoretical analysis for algorithms’ efficiency are also provided. The last section concludes this work.
2 Preliminaries

In this section, we would introduce some definitions and notations in advance, which are used throughout the whole paper. Let $F$ be a monotone normalized OSS function. For any vectors $x, y \in [0, 1]^N$, we say $x \leq y$, if and only if $x_i \leq y_i$ holds. Let $x \lor y$ be the coordinate-wise maximum of $x$ and $y$, and $x \land y$ be the coordinate-wise minimum.

**Definition 1 OSS function:** Given a continuously twice differentiable function $F : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$. $F$ is defined to be is one-sided $\sigma$-smooth (OSS for short) at point $x \geq 0$, if $F$ satisfies
\[
u T \nabla^2 F(x) u \leq \sigma \cdot \frac{2\|u\|_1}{\|x\|_1} u^T \nabla F(x),
\]
for all $u \geq 0$. Function $F$ is OSS, if (3) holds at all points of its domain.

By Definition 1, we have that a OSS function $F$ is OSS at any non-zero point of its domain. This property of OSS captures the concavity in the forward direction when $\sigma = 0$. Furthermore, inequality (3) means that the second derivative of $F(x)$ is bounded by the linear form of its gradient function. In fact, this inequality is one of the necessary conditions for the approximation algorithm design for the non-convex maximization problem.

**Definition 2 Monotone & Normalized:** A OSS function $F : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is monotone if $F(x) \leq F(y)$, for all $x, y \in \mathbb{R}^n_{\geq 0}$, and $x \leq y$; and $F(x)$ is normalized if it satisfies $F(0) = 0$.

Generally, when a continuous Frank-Wolfe technique is used to solve OSS problem, a local condition, i.e., $\eta$-local, can help to analyze the approximation ratio and the complexity of the algorithm.

**Definition 3 $\eta$-local:** For any $\eta \geq 0$, $\epsilon \in [0, 1]$, $x + \epsilon u \in P$, we say $F \in \mathbb{C}^2$ is $\eta$-local at $x, u$ if
\[
u T \nabla F(x + \epsilon u) \geq (1 - \eta \epsilon) u^T \nabla F(x)
\]
holds for all $u, x \in P$, where $\mathbb{C}^2$ is the twice continuous differentiable space.

The core of a parallel algorithm is the technique of ”threshold loop call”. Generally, a proper threshold can be set by using an estimation of the optimal value $OPT$. Following, we propose some trivial bounds of $OPT$ before our parallel algorithms design.

**Lemma 1 OPT BOUNDS:** Let $OPT$ be the optimal value of the problem to maximize the monotone normalized OSS function $F(x)$, subject to a convex polytope
Denote $r$ to be the rank of the polytope $P$. Then
\[ \max \{ F(x) \mid \|x\|_1 = r, x \in P \cap [0, 1]^n \} \leq \text{OPT} \leq \min \{ F(\max_{x \in P} \|x\|_1), F(1) \}, \]
or
\[ F \left( r \frac{\arg \max_z \|z\|_1, z \in P}{\|z\|_2} \right) \leq \text{OPT} \leq \min \{ F(\max_{x \in P} \|x\|_1), F(1) \}. \]

The lower bound in (5) is obvious. As $r$ is the rank of polytope $P$, we have $x = r \frac{\arg \max_z \|z\|_1, z \in P}{\|z\|_2} \in P \cap [0, 1]^n$. Thus $\text{OPT} \geq F(x)$, leading to the lower bound of $\text{OPT}$ in (5). In addition, the monotone property of $F(x)$ promises the upper bound, since $x \leq 1$.

The notations used in this paper are listed in Table 1.

| Table 1 Symbol Description |
|-----------------------------|
| $\delta, \sigma, \epsilon, \eta$ | The preset parameters which are positive; |
| $\lambda$ | The estimation value of $\text{OPT}$; |
| $P$ | Polyhedron: such as linear function polyhedron $Ax \leq b$, matrix polyhedron, etc.; |
| $\{e_P\}_r$ | The set of basis vectors of polytope $P$, where $r$ is the rank of polytope $P$. Generally, the base of a polytope is the linear maximum independent group of the polytope, and the rank of a polyhedron is the number of elements in its linear maximum independent group; |
| $\nu$ | Basis Vectors in $\{e_P\}_r$; |
| $u$ | A vector belongs to $[0, 1]^n$; |
| $d_t$ | Estimated value of the gradient $f(x_t, y_t)$; |
| $D$ | A constant parameter (In Assumption 12) given in advance, used to bound the Euclidean norm of a vector $x \in P$; |
| $\theta$ | A constant parameter given in advance, which is used to bound $\nabla f(x, y)$; |
| $L$ | The Lipschitz parameter. |

3 Jump-Start Parallel-Greedy (JSPG) Algorithm for Deterministic Setting

This section concentrates on the deterministic setting, in which the monotone normalized OSS function $F(x)$ is determined by the decision variable $x \in P \cap [0, 1]^n$. The Jump-Start Parallel-Greedy (JSPG) algorithm is designed for the deterministic OSS problem.

\footnote{The basis of the polyhedron $P$ is required to exist and to be solvable in polynomial time.}
3.1 JSPG Algorithm Design

JSPG algorithm is inspired by the work \cite{1}, in which Mehrdad et al. proposed a Jump-Start Continuous Greedy (JSCG) algorithm to solve the monotone normalized OSS maximization problem. In each iteration of JSCG algorithm, a solution is dynamically improved by adding a direction vector $\nu$ to current solution $x$ with a fixed and conservative step size $\delta > 0$. The key idea of the JSCG algorithm actually originated from the continuous-greedy algorithm, where the value of a submodular set function is gradually increased by adding only one element $x$ per iteration. Such an operation leads to a slow speed and does not work well for big data applications.

Our JSPG algorithm also applies the Frank-Wolfe skill, which approximates the objective function as a linear function, at each iteration point $x_k$.

$$F(x) \approx F(x_k) + (x - x_k)^T \nabla F(x_k).$$

Hence

$$\max_{x \in P} F(x) \approx \max F(x_k) + (x - x_k)^T \nabla F(x_k), x \in P.$$ 

Since $F(x_k)$ and $(x_k)^T \nabla F(x_k)$ are constants. The above optimization problem is thus equivalent to the following problem

$$\max x^T \nabla F(x_k), x \in P.$$ 

Before describing the detailed layout of JSPG algorithm in Algorithm 1, let us introduce it briefly as follows. Given a threshold $\lambda$, JSPG algorithm firstly selects an initial feasible solution

$$x \leftarrow \alpha \arg \max_{x \in P} \|x\|_1, \ \alpha \to 0, \ \alpha \in (0, 1].$$

Based on this initial feasible solution, JSPG algorithm goes to identify all good directions from the base set $\{e_P\}_r$ by Frank-Wolfe skill, that is choosing all $\{\nu_i \in \{e_P\}_r \mid \nu_i^T \nabla F(x) \geq (1 - \epsilon)\mu \lambda\}$ (If the base set of the polyhedron constraint does not exist, we can replace it with the coordinates of the solution vector, and the algorithm still works). JSPG algorithm continues to increase along all these directions uniformly by a dynamical increment $\delta$. The value of $\delta$ depends on $\mu, \eta$, where $\mu = \left(\frac{\alpha}{\alpha + 1}\right)^{2\sigma}$. At last, the threshold $\lambda$ is updated when the inner loop has no direction to choose.

JSPG algorithm improves JSCG algorithm from two following perspectives. In each iteration,

- Instead of increasing the current solution $x$ only along a single best direction in JSCG algorithm, our JSPG algorithm identifies all good directions from the base set by Frank-Wolfe skill, and increases along all of these directions uniformly.
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- Instead of increasing the current solution $x$ along a direction by a fixed step size in JSCG algorithm, our JSPG algorithm dynamically adjusts the step size $\delta$.

**Algorithm 1** JSPG: Jump-Start Parallel-Greedy($F, \lambda, P, \epsilon$)

**Input**: OSS function $F : [0, 1]^n \cap P \rightarrow \mathbb{R}_+^*$; $P$: down-closed convex polytope with rank $r$; $\{e_P\}_r$: the set of standard vector bases of $P$.

**Parameter**: $\lambda$: the upper bound of $OPT$, $\alpha \in (0, 1]$, $\alpha, \epsilon, \eta > 0$, $\sigma \geq 0$.

**Output**: A fractional solution $x$.

1. $x \leftarrow \alpha \arg \max_{x \in P} \|x\|_1$.
2. $\mu = \left(\frac{\alpha}{\alpha+1}\right)^{2\sigma}$.
3. $t \leftarrow \alpha$.
4. **while** $t < 1$ and $\lambda \geq e^{-\mu}OPT$, **do**
5.   Let $M_\lambda = \{v_i \in \{e_P\}_r : v_i^T \nabla F(x) \geq (1 - \epsilon)\mu \lambda\}$.
6.   $S_x \leftarrow M_\lambda$.
7. **while** $S_x$ is not empty and $t < 1$, **do**
8.   **A.** Choose maximal $\delta$ s.t.
9.   \begin{align}
10.   F\left(x + \frac{\delta}{\|S_x\|} \sum_{v_i \in S_x} v_i\right) - F(x) & \geq \mu(1 - \epsilon)^2 \frac{\delta}{\|S_x\|} \sum_{v_i \in S_x} \|v_i\|_\lambda, \\
11.   \delta & \leq \min\left\{\frac{1}{\eta^{\frac{1}{\mu(1 - \epsilon)}}}, \frac{1}{\mu(1 - \epsilon)}\right\}, \\
12.   t & + \delta \leq 1.
13.   \end{align}
14. **B.** $x \leftarrow x + \frac{\delta}{\|S_x\|} \sum_{v_i \in S_x} v_i$,
15. **C.** Update $S_x = \{v_i \in \{e_P\}_r : v_i^T \nabla F(x) \geq (1 - \epsilon)\mu \lambda\}$.
16. $t \leftarrow t + \delta$.
17. **end while**
18. $\lambda \leftarrow (1 - \epsilon)\lambda$.
19. **end while**

Our algorithm’s parallel technique is different from traditional discrete parallel greedy (DPG) algorithm [15, 21]. Firstly, the DPG algorithm is actually a traditional discrete algorithm. Our JSPG algorithm is a parallel one based on Frank-Wolfe skill, which can be used to solve the continuous problem. Secondly, the DPG algorithm is only suitable for the discrete submodular optimization problem, as the property of set submodularity ensures that the DPG algorithm easily addresses the relationship among the $OPT$, the iterative function value $F(x)$ and the threshold $\lambda$. This relationship is critical to obtain the approximation ratio. Our parallel algorithm is not restricted to the property of submodularity and multilinearity, and thus can be applied to all
other non-submodular OSS problems. To overcome the missing of submodularity and multilinearity, we use the quadratic differentiability and the OSS property of objective function to analyze JSPG algorithm.

3.2 Approximation Ratio of JSPG Algorithm

The property of OSS enables us to bound the first order Taylor’s polynomial of the objective function, which is helpful to compute the approximation ratio of JSPG algorithm.

Lemma 2 [1] Let \( x \in [0, 1]^n \backslash \{0\}, u \in [0, 1]^n \). There exits \( \epsilon > 0 \) such that \( (x + \epsilon u) \in [0, 1]^n \). Let \( F : [0, 1]^n \rightarrow \mathbb{R} \) be a monotone normalized function which is OSS on \( [x, x + \epsilon u] \), then we have

\[
    u^T \nabla F(x + \epsilon u) \leq \left( \frac{\|x + \epsilon u\|_1}{\|x\|_1} \right)^{2\sigma} u^T \nabla F(x).
\]

(5)

Because

\[
    \frac{\|x + \epsilon u\|_1}{\|x\|_1} \leq \frac{\|x + u\|_1}{\|x\|_1} = 1 + \frac{\|u\|_1}{\|x\|_1} \leq 1 + \frac{1}{\alpha},
\]

we continue to explore (5) to be

\[
    u^T \nabla F(x + \epsilon u) \leq \left( \frac{\alpha + 1}{\alpha} \right)^{2\sigma} (u^T \nabla F(x)).
\]

(6)

Lemma 3 For any solution \( x \) obtained in the intermediate stage of JSPG algorithm, we have \( OPT - F(x) \leq \lambda \).

Proof JSPG algorithm starts from a non-zero solution, such that the initial value of objective function \( F \) is greater than zero. Moreover, due to the monotone property, \( F(x) \) increases, and thus \( OPT - F(x) \) monotonically decreases with \( x \) increasing. Suppose \( z \) is an optimal solution, meaning \( OPT = F(z) \). Then \( z \vee x \geq z \), ensuring that \( F(z \vee x) \geq OPT \). It’s a contradiction, and thus \( z \vee x \notin P \cap [0, 1]^n \). Let \( u = x \vee z - x \), then

\[
    OPT - F(x) \leq F(x \vee z) - F(x) = (i) \langle \nabla F(x + \epsilon u), u \rangle \\
    \leq (ii) \langle \mu^{-1} \nabla F(x), u \rangle \leq (iii) \langle \mu^{-1} \nabla F(x), z \rangle \leq (iv) \lambda,
\]

(7)

where \( \mu = \left( \frac{\alpha + 1}{\alpha} \right)^{2\sigma} \). Specifically, equality (i) is from the Median Theorem of continuous function \( F \). Lemma 5 guarantees inequality (ii). Combining the monotonicity of \( F \), which implies \( \nabla F(x) \geq 0 \), and the fact of \( u \leq z \), inequality (iii) is achieved. The last inequality (iv) is correct, since \( z^T \nabla F(x) \leq (1 - \epsilon)\mu \lambda \). This lemma holds. \( \square \)
Lemma 4  For any solution $x$ obtained in the intermediate stage of JSPG algorithm, we have
\[ \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i^T \nabla F(x) \geq (1 - \epsilon) \left( \frac{\alpha}{\alpha + 1} \right)^{2\sigma} (OPT - F(x)). \] (8)

Proof  By the choices of $\delta$ and $\mu$ in JSPG algorithm, we have
\[ \nu_i^T \nabla F(x) \geq (1 - \epsilon) \mu \lambda = (1 - \epsilon) \left( \frac{\alpha}{\alpha + 1} \right)^{2\sigma} \lambda. \] (9)
Combining Lemma 3 and (9), it is easy to deduce
\[ \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i^T \nabla F(x) \geq (1 - \epsilon) \left( \frac{\alpha}{\alpha + 1} \right)^{2\sigma} (OPT - F(x)). \]
This lemma holds. \qed

Theorem 1  When JSPG algorithm terminates, the output solution $x$ satisfies $F(x) \geq (1 - O(\epsilon))(1 - e^{-\mu})OPT$.

Proof  By Lemma 3 and Line 9-10 in Algorithm 1, we have
\begin{align*}
F \left( x + \delta \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i \right) &- F(x) \\
\geq \mu (1 - \epsilon)^2 \delta \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i \| \lambda \\
\geq \mu (1 - \epsilon)^2 \delta \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i \| (OPT - F(x)),
\end{align*}
(10)

By JSPG algorithm, solution $x$ increases during the whole operation, which leads to the increase of $F(x)$ by the monotone property of $F(x)$. Thus we analyze the approximation ratio of JSPG algorithm by reducing the dimensionality of variables. Given a solution $x$, let $l_x = \sum x_i$ be the sum of all elements of $x$. Define a function $G : \mathbb{R} \rightarrow \mathbb{R}$, such that for any $x$, $G(l_x) = F(x)$. So by (10), we have
\begin{align*}
G \left( l_x + \delta \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i \right) - G(l_x) &\geq \mu (1 - \epsilon)^2 \delta \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i \| (OPT - G(l_x)).
\end{align*}
Based on the choice of $\delta$ in the Algorithm 1, we have
\[ \frac{dG(l_x)}{dl_x} = \lim_{\delta \rightarrow 0} \frac{G \left( l_x + \delta \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i \right) - G(l_x)}{\delta \frac{1}{|S_x|} \sum_{\nu_i \in S_x} \nu_i} \]
\[ \geq \mu (1 - \epsilon)^2 (OPT - G(l_x)). \] (11)

By Line 7-16 in Algorithm 1, we know the intermediate solution $x$ depends on parameter $t$, thus it can be denoted by $x(t)$. Clearly, because the initial value of $t$ is $\alpha$, the initial solution is $x(\alpha)$. As the step size $\delta$ is adjusted iteratively, we denote
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the step size as $\delta_i$ in the i-th iteration for convenience. By the update of $t$ in Line 12 of Algorithm 1, the final value of $t$, denoted by $\hat{t}$, is $t = \sum_i \delta_i + \alpha \geq 1$, and thus the final output of Algorithm 1 is $x(\hat{t})$.

Next, let us consider the differential equation:

$$\frac{d}{dl_x} \left[ e^{\mu(1-\epsilon)^2 l_x} G(l_x) \right]$$

$$= \mu(1-\epsilon)^2 e^{\mu(1-\epsilon)^2 l_x} G(l_x) + \epsilon e^{\mu(1-\epsilon)^2 l_x} \frac{d}{dl_x} G(l_x)$$

$$\geq \mu(1-\epsilon)^2 e^{\mu(1-\epsilon)^2 l_x} G(l_x) + \mu(1-\epsilon)^2 e^{\mu(1-\epsilon)^2 l_x} (OPT - G(l_x))$$

$$\geq \mu(1-\epsilon)^2 e^{\mu(1-\epsilon)^2 l_x} OPT.$$ (12)

Integrating the LHS and RHS of (12) between $l_{x(\alpha)}$ and $l_{x(t)}$, we get

$$e^{\mu(1-\epsilon)^2 l_{x(t)}} \cdot G(l_{x(t)}) - e^{\mu l_{x(\alpha)}} \cdot G(l_{x(\alpha)})$$

$$\geq \int_{l_{x(\alpha)}}^{l_{x(t)}} \left( \mu(1-\epsilon)^2 e^{\mu(1-\epsilon)^2 l_x} \cdot OPT \right) dl_x$$

$$= \mu(1-\epsilon)^2 \cdot OPT \cdot \left[ e^{(1-\epsilon)^2 \mu l_{x(t)}} - e^{(1-\epsilon)^2 \mu l_{x(\alpha)}} \right]$$

$$= OPT \cdot \left[ e^{(1-\epsilon)^2 \mu l_{x(t)}} \right] - e^{(1-\epsilon)^2 \mu l_{x(\alpha)}}] .$$ (13)

Due to $\alpha \to 0$, then $l_{x(\alpha)} \to 0$. So

$$G(l_{x(t)}) \geq OPT \cdot \left[ e^{(1-\epsilon)^2 \mu l_{x(t)}} \right] + G(l_{x(\alpha)}) \cdot e^{(1-\epsilon)^2 \mu l_{x(\alpha)}}$$

$$\geq OPT \cdot \left[ 1 - e^{(1-\epsilon)^2 \mu l_{x(t)}} \right] .$$ (14)

According to the termination conditions of JSPG algorithm, following two cases shall be distinguished.

**Case 1.** If $t = \hat{t} \geq 1$ at the end of the algorithm, then $l_{x(t)} = \sum x_i(\hat{t}) \geq \|x_i(\hat{t})\|_2 = t$. Hence

$$F(x(\hat{t})) = G(l_{x(\hat{t})}) \geq (1 - O(\epsilon)) \left( 1 - e^{-\mu} \right) \cdot OPT.$$ (15)

**Case 2.** If $\lambda \leq e^{-\mu}OPT$, then

$$F(x) = G(l_x) \geq \left( 1 - e^{-\mu} \right) \cdot OPT.$$ (16)

Combining (15) and (16) as above, we eventually have

$$F(x) \geq (1 - O(\epsilon)) \left( 1 - e^{-\left( \frac{\lambda}{\alpha+1} \right)^{2\sigma}} \right) \cdot OPT$$

This theorem is obtained. \(\square\)

### 3.3 Number of Iterations & Number of Oracle Queries

According to JSPG algorithm, one of A. (1), A. (2), and A. (3) (in Algorithm 1) must tightly hold along the update of $\delta$. Furthermore, when A. (3) is tight, the algorithm terminates. So we only need to analyze A. (1) and A. (2).
Theorem 2 For JSPG algorithm, if the step size $\delta$ is determined by $A.(2)$ in each iteration, then the inner loop iterates at most $O(n)$ times, and the total loop iterates at most $O(n/\epsilon)$ times. If the step size $\delta$ is determined by $A.(1)$ in each iteration, then the inner loop iterates at most $O(\log n/\epsilon)$ times, and the number of iterations of total loop is at most $O(\log n/\epsilon^2)$.

Proof Assuming the step size $\delta$ is determined by $A.(2)$ in each iteration, the result on the number of iterations is not hard to obtain. Next, we need to analyze the case for $A.(1)$.

Define $\Delta = \frac{\delta}{\eta} \| \sum_{i \in S} \nu_i \|$. Given the current step size $\delta$. Let $\Delta'$ denote the value of $\Delta$ before updating current $\delta$ and $\Delta''$ denote the value of $\Delta$ after updating. Recall that $l_x = \sum x_i$ for $x$ and $G$ is a function $G: \mathbb{R} \to \mathbb{R}$, satisfying $G(l_x) = F(x)$ for any $x$. Hence

$$\lambda(1 - \epsilon)^2 \Delta' = |G(x + \Delta') - G(x)| = |\nabla G(x + \epsilon \Delta'), \Delta'| \geq (ii) |\mu^{-1}(G'(x + \Delta'), \Delta')| \geq (iii) |\mu^{-1}(G'(x + \Delta'), \Delta'')| \geq (iv) |\mu^{-1}(1 - \eta \epsilon)G'(x + \Delta''), \Delta''| \geq (v) |\mu^{-1}\lambda(1 - \epsilon)\Delta''| \geq |\lambda(1 - \epsilon)\Delta''|.$$ \hspace{1cm} (17)

Equality (i) is from the Median Theorem of continuous function $F$. Inequality (ii) holds, due to the OSS smoothness and the monotonic decreasing property of $\Delta$. The inequality (iii) is correct since $\delta$ decreases during the whole operation of JSPG algorithm and $\|\sum_{i \in S_x} \nu_i \| = 1$. Since $F(x)$ is $\eta$-local, we get inequality (iv). The inequality (v) holds, due to the choice of $S_x$. So from (17), we have $\Delta'' \leq (1 - \epsilon)\Delta'$. Not that $\Delta' - \Delta''$ is just the increment of $x$ between two adjacent iterations. Let $k$ be the total number of iterations. Based on the the choice of $\delta$ (suppose the value of $\delta$ is $O(n^{-c})$, $c$ represents any finite number) and due to the fact of $\max_{x \in P \cap [0,1]^n} \|x\| \leq r$ (where $r$ is the rank of $P$), $k$ can be computed through $r(1 - \epsilon)^k = n^{-c}$. So the number of the iterations of inner is at most $O\left(\frac{\log n}{\epsilon}\right)$.

In addition, $\lambda$ is the upper bound of $OPT$ at the beginning of JSPG algorithm. It updates as $\lambda \leftarrow (1 - \epsilon)\lambda$ during the operation. So one of terminate conditions $\lambda < e^{-\mu OPT}$ makes us get that the outer loop iterates at most $O\left(\frac{1}{\epsilon}\right)$ times, through the equation $OPT(1 - \epsilon)^k = e^{-\mu OPT}$.

Therefore, we conclude that the total loop iterate at most $O\left(\frac{\log n}{\epsilon^2}\right)$ times. This theorem holds. $\square$

By Algorithm 1, the number of oracle queries to $F$ is at most $O(\log n/\epsilon^2)$, and the number of oracle queries to $\nabla F(x)$ is at most $O(n \log n/\epsilon^2)$. Thus, we obtain the following result.

Theorem 3 The number of oracle queries to $F$ is at most $O(\log n/\epsilon^2)$, and the number of oracle queries to $\nabla F(x)$ is at most $O(n \log n/\epsilon^2)$. 

Traditionally continuous greedy algorithms mainly appear in solving multilinear extension problems of the submodular set function. The key idea is that the partial derivative of the function is equal to the marginal return of the corresponding component, which is guaranteed by the property of submodularity. To avoid using the property of submodularity, our algorithm applies the idea of Frank-Wolfe Algorithm with the help of the quadratic differentiability and the OSS property of objective function. This makes our algorithm be applied more widely. Compared with the algorithm in [1], our algorithm can ensure the complexity of the sub-optimization problem \( \max_{\nu \in \mathcal{P}} \nu^T \nabla F(x) \) to be quantifiable. The OSS problems are mainly continuous problems, which are also applicable to the continuous expansion of some combination problems. The traditional rounding techniques mentioned in the paper [1] are all available.

4 Stochastic Parallel-Greedy (SPG) Algorithm for Stochastic Setting

In this section, we continue to study the problem of maximizing a monotone normalized OSS function \( F(x) \) under a stochastic setting. In this stochastic version, the objective function \( F(x) \) is defined as \( F(x) = \mathbb{E}_{y \sim T} f(x, y) \), where \( f \) is a stochastic function with respect to the random variable \( Y \), and \( y \) is the realization of \( Y \) drawn from a probability distribution \( T \). In this section, we design a Stochastic Parallel-Greedy (SPG for short) algorithm for the stochastic OSS maximization problem.

4.1 Stochastic Parallel-Greedy (SPG) Algorithm Design

Stochastic Parallel-Greedy algorithm is a stochastic variant of JSPG algorithm. It starts from zero instead of jumping start. For JSPG algorithm, we need to accurately calculate the gradient value. But it is difficult to get the exact value of \( \nabla F(x) \) if the objective function obeys a unknown or complex probability distribution. To overcome this difficulty, we use the estimated gradient

\[
d_t = (1 - \rho_t) d_{t-1} + \rho_t \nabla f(x_t, y_t),
\]

where \( \rho_t = \left( \frac{4}{t+8} \right)^{\frac{1}{2}} \) is a positive step size, dependent on \( t \) and the initial vector \( d_0 = 0 \).

To design a stochastic parallel algorithm, we need to find a good alternative to the gradient function, but it is not enough to rely on \( d_t \). So three other assumptions are required:

Assumption 1 The Euclidean norm of the elements in the down-closed convex polytope \( P \) are uniformly bounded, i.e., for all \( x \in P \), \( \|x\| \leq D \).

Assumption 2 The gradients of objective function \( F(x) \) are \( L \)-Lipschitz continuous over the sets \( X \). That is for all \( x, x' \in X \), we have

\[
\|\nabla F(x) - \nabla F(x')\| \leq L \|x - x'\|. \tag{18}
\]
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Assumption 3 The variance of the unbiased stochastic gradients \(\nabla f(x, y)\) is bounded above by \(\theta^2\). That is for all \(x \in X\), we have
\[
E\|\nabla F(x, y) - \nabla f(x, y)\| \leq \theta^2.
\] (19)

Under Assumptions 1-3, \(E[\|\nabla F(x_t) - d_t\|^2]\) is bounded by a parameter [20]. This conclusion plays an important role in our design of parallel algorithms for solving stochastic OSS problems.

Our SPG algorithm in Algorithm. 2 includes two phases. The first phase is to find all good directions by solve \(\{\nu_i \in \{e_P\}_r : \nu_i^T d_t \geq (1 - \epsilon) \mu \lambda\}\), and to increase along all of these directions uniformly in Line 6-7; The second phase is to increase \(x\) along these directions by a dynamical increment \(\delta\) in Line 8-18.

**Algorithm 2** SPG: stochastic Parallel-Greedy\((F, \lambda, P, \epsilon)\)

**Input:** OSS function \(F(x) := \mathbb{E}_{y \sim T}[f(x, y)] : X \times Y \rightarrow R_+; P: \) down-closed convex polytope of rank \(r; \{e_P\}_r: \) the set of standard vector bases of \(P, 0 \in P\).

**Parameter:** \(\lambda: \) the upper bound of \(OPT, \epsilon, \eta, \sigma, \kappa, \rho_t \geq 0, \alpha \in (0, 1]\).

**Output:** A fractional solution \(x\).

1: \(x \leftarrow 0\).
2: \(\mu = \left(\frac{\alpha}{\alpha + 1}\right)^{2\sigma}\).
3: \(t \leftarrow 0\).
4: \(d_t = 0\).
5: \textbf{while} \(t < 1 \text{ and } \lambda \geq e^{-\mu}OPT, \text{ do}\)
6: \hspace{1em} Let \(M_{\lambda} = \{\nu_i \in \{e_P\}_r : \nu_i^T d_t \geq (1 - \epsilon) \mu \lambda\}\).
7: \hspace{1em} \(S_x \leftarrow M_{\lambda}\).
8: \hspace{1em} \textbf{while} \(S_x\) is not empty and \(t < 1, \text{ do}\)
9: \hspace{2em} \textbf{A.} Choose \(\delta\) maximal s.t.
10: \hspace{3em} 1) \(F\left(x + \frac{\delta}{|S_x|} \sum_{\nu_i \in S_x} \nu_i\right) - F(x) \geq \mu(1 - \epsilon)^2 \frac{\delta}{|S_x|} \sum_{\nu_i \in S_x} \nu_i(\lambda + \mu^{-1}\kappa^{1/2}r),\)
11: \hspace{3em} 2) \(\delta \leq \min\left\{\frac{1}{n\eta}, \frac{1}{\mu^2(1-\epsilon)}\right\},\)
12: \hspace{3em} 3) \(t + \delta \leq 1\).
13: \hspace{2em} \textbf{B.} \(x \leftarrow x + \frac{\delta}{|S_x|} \sum_{\nu_i \in S_x} \nu_i,\)
14: \hspace{2em} \(d_t = (1 - \rho_t)d_t' + \rho_t \nabla f(x, y),\)
15: \hspace{2em} \(t \leftarrow t + \delta.\)
16: \hspace{1em} \textbf{C.} Update \((Where \ t' \text{ represents the moment before } t \text{ is updated, } \rho_t = \frac{4}{t+8}^{2/3}).\)
17: \hspace{1em} \textbf{end while}
18: \hspace{1em} \(S_x = \{\nu_i \in \{e_P\}_r : \nu_i^T d_t \geq (1 - \epsilon) \mu \lambda\}.\)
19: \hspace{1em} \textbf{end while}
20: \(\lambda \leftarrow (1 - \epsilon)\lambda.\)
21: \textbf{end while}
4.2 Approximation Ratio of SPG Algorithm

**Lemma 5** [20] Under Assumptions 1-3, we have

$$\mathbb{E}[\|\nabla F(x_t) - d_t\|^2] \leq \kappa,$$

(20)

where $$\kappa = \max \left\{ \frac{5\|\nabla F(x_0) - d_0\|^2}{(t+9)^{2/3}}, 16\theta^2 + 2L^2D^2 \right\}.$$

**Lemma 6** For any solution $$x$$ obtained in the intermediate stage of SPG algorithm, we have

$$OPT - F(x) \leq \lambda + \mu^{-1}\kappa^{1/2}r.$$

(21)

**Proof** During the operation of SPG algorithm, the value of $$F(x)$$ starts from a number greater than zero and increases monotonically. Thus $$OPT - F(x)$$ gradually decreases with $$x$$ increasing. Let $$z$$ be an optimal solution, then

$$OPT - F(x) \leq F(x \vee z) - F(x) = \langle \nabla F(x + \epsilon u), u \rangle$$

$$\leq \langle \mu^{-1}\nabla F(x), u \rangle \leq \langle \mu^{-1}\nabla F(x), z \rangle \leq \mu^{-1}\langle d_t + (\nabla F(x) - d_t), z \rangle$$

(22)

$$\leq \lambda + \mu^{-1}\kappa^{1/2}r.$$

The analysis is similar to the proof for Lemma 3, and so we don’t explain it in detail. □

**Theorem 4** Let $$F$$ be a $$\eta$$-local and monotone normalized OSS function and $$P$$ be a convex polytope including 0. By setting $$\delta \leq \min\left\{ \frac{1}{\eta n}, \frac{1}{(1 - \epsilon)^2\mu} \right\}$$, the value of the output $$F(x)$$ from SPG algorithm is larger than

$$F(x) \geq (1 - O(\epsilon)) \left( 1 - e^{-\left(\frac{\delta}{\alpha + 1}\right)^2} \right) (1 - o(1))OPT - O(\kappa^{1/2})$$

for any given parameter $$\alpha \in (0,1]$$, $$\sigma, \epsilon, \kappa$$.

**Proof** Let $$x(t)$$ represents the variable $$x$$ related to time $$t$$, and $$x(0)$$ and $$x(1)$$ are used to denote the initial and the final solutions, respectively. Since $$F(x)$$ is twice continuous differentiable and $$P$$ is a convex set,

$$F(x(t + \delta)) = F\left( x(t) + \frac{\delta}{|S_{x(t)}|} \sum_{\nu_i \in S_{x(t)}} \nu_i \right)$$

$$= F(x(t)) + \frac{\delta}{|S_{x(t)}|} \sum_{\nu_i \in S_{x(t)}} \nu_i \cdot \nabla F\left( x(t) + \frac{\epsilon \delta}{|S_{x(t)}|} \sum_{\nu_i \in S_{x(t)}} \nu_i \right),$$

where

$$\frac{\delta}{|S_{x(t)}|} \sum_{\nu_i \in S_{x(t)}} \nu_i \cdot \nabla F\left( x(t) + \frac{\epsilon \delta}{|S_{x(t)}|} \sum_{\nu_i \in S_{x(t)}} \nu_i \right)$$
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\[ \geq (1 - \epsilon \delta \eta) \frac{\delta}{S_x(t)} \sum_{\nu_i \in S_x(t)} \nu_i \cdot \nabla F(x(t)) \]

\[ \geq (1 - \epsilon \delta \eta) \frac{\delta}{S_x(t)} \sum_{\nu_i \in S_x(t)} \nu_i \cdot (d_t + (\nabla F(x(t) - d_t))) \]

\[ \geq (1 - \epsilon \delta \eta) \frac{\delta}{S_x(t)} \sum_{\nu_i \in S_x(t)} \nu_i \cdot d_t + (1 - \epsilon \delta \eta) \frac{\delta}{S_x(t)} \sum_{\nu_i \in S_x(t)} \nu_i \cdot (\nabla F(x(t)) - d_t) \]

\[ \geq (1 - \epsilon \delta \eta) \cdot (1 - \epsilon) \mu \lambda + (1 - \epsilon \delta \eta) \frac{\delta}{S_x(t)} \sum_{\nu_i \in S_x(t)} \nu_i \cdot (\nabla F(x(t)) - d_t) \]

\[ \geq (1 - \epsilon \delta \eta) \cdot (1 - \epsilon) \mu \lambda - (1 - \epsilon \delta \eta) \delta \| \nabla F(x(t)) - d_t \| \]

\[ \geq (1 - \epsilon \delta \eta) \cdot (1 - \epsilon) \mu \lambda (OPT - F(x(t))) - \mu \gamma \epsilon^{1/2} r - (1 - \epsilon \delta \eta) \delta \| \nabla F(x(t)) - d_t \| \]

\[ \geq (1 - \epsilon \delta \eta) \cdot (1 - \epsilon) \mu \lambda (OPT - F(x(t))) - (1 - \epsilon \delta \eta) \gamma \epsilon^{1/2} (\mu \lambda + 1) \]

Define \( M = (1 - \epsilon \delta \eta) OPT \). Because \( F \) is nonnegative, \( \delta \leq \frac{1}{\mu \lambda} \), and \( \| \nabla F(x(t)) - d_t \| \leq \gamma \epsilon^{1/2} \), we have

\[ F(x(t + \delta)) \geq F(x(t)) + (1 - \epsilon) \mu \delta (M - F(x(t))) - (1 - \epsilon \delta \eta) \gamma \epsilon^{1/2} (\mu \lambda + 1) \]

which can be deduced as

\[ (1 - (1 - \epsilon) \mu \delta) (M - F(x(t))) \geq (M - F(x(t + \delta))) - (1 - \epsilon \delta \eta) \gamma \epsilon^{1/2} (\mu \lambda + 1) \]

By induction,

\[ (1 - (1 - \epsilon) \mu \delta)^{1/\delta} (M - F(x(0))) \geq (M - F(x(1))) - (1 - \epsilon \delta \eta) \gamma \epsilon^{1/2} (\mu \lambda + 1) \]

Since \( \delta \leq \frac{1}{\mu (1 - \epsilon)} \), we have

\[ (1 - (1 - \epsilon) \mu \delta)^{1/\delta} \leq e^{-\mu (1 - \epsilon)} \]

Therefore,

\[ F(x(1)) \geq (1 - e^{-\mu (1 - \epsilon)}) M + F(x(0)) - (1 - \epsilon \delta \eta) \gamma \epsilon^{1/2} (\mu \lambda + 1) \]

\[ \geq (1 - e^{-\mu (1 - \epsilon)}) M - (1 - \epsilon \delta \eta) \delta \gamma \epsilon^{1/2} (\mu \lambda + 1) \]

\[ = (1 - e^{-\mu (1 - \epsilon)}) (1 - \epsilon \delta \eta) OPT - (1 - \epsilon \delta \eta) \gamma \epsilon^{1/2} (\mu \lambda + 1) \]

\[ = (1 - O(\epsilon)[1 - e^{-\mu}] (1 - o(1)) OPT - O(\delta \gamma^{1/2}) \]

This theorem holds. \( \square \)

5 Conclusion

In this paper, we study the problem of maximizing a monotone normalized OSS function \( F(x) \), subject to a convex polytope constraint. The deterministic and stochastic versions of this problems are both considered and we respectively design two parallel algorithms for them. Specifically, for the deterministic OSS maximization problem, we propose a \( ((1 - e^{-\mu (1 - \epsilon)}) - \epsilon) \)-approximation JSPG algorithm for any any number \( \alpha \in [0, 1] \) and \( \epsilon > 0 \). The time complexity of this deterministic algorithm is \( O(\log n / \epsilon^2) \), in while, the number of oracle queries to \( F \) is at most \( O(\log n / \epsilon^2) \), and the number of
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oracle queries to $\nabla F(x)$ at most $O(n \log n/\epsilon^2)$. For the stochastic OSS maximization problem, the designed SPG algorithm outputs a result of $(F(x) \geq (1-e^{-\left(\frac{d_o}{\kappa + \frac{1}{2}}\right)^2} - \epsilon)OPT - O(\kappa^{1/2}) \ (\kappa = \frac{\max\{5\|\nabla F(x_0) - d_o\|^2, 16\sigma^2 + 2L^2D^2\}}{(t+9)^{2/3}})$ under the same time complexity with the one of JSPG algorithm.

The above two algorithms are the first parallel algorithms proposed to solve monotone normalized OSS problem subject to a convex polytope constraint (no need to downward-closed [1]). Most of OSS problems are continuous, and thus our algorithms can be applied directly. However, if the optimization problems are discrete, then we need to round the corresponding non-discrete results. Generally, the traditional rounding techniques mentioned in [1, 10] are all available. JSPG and SPG algorithms are also suitable for nonmonotone situations, but the performance of them under nonmonotone situations may be poor. Therefore, designing an efficient parallel algorithm for nonmonotone OSS problems is an interesting topic.

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Appendix A  Examples of the OSS function

In this section, we propose two examples of the one-sided \( \sigma \)-smooth function.

- \( \sigma = 0 \): Continuous DR-submodular function (e.g. Multilinear extension of set functions [10])

Proof  Firstly, we give the definition of the continuous DR-submodular function

**Definition 4** A continuously twice differentiable function \( F : \mathbb{R}_{\geq 0}^n \to \mathbb{R} \) is DR-submodular if it satisfies

\[
F(k e_i + x) - F(x) \geq F((k + l) e_i + x) - F(le_i + x),
\]

where \( k, l \in \mathbb{R}_{+} \) and \( x, (k e_i + x), ((k + l) e_i + x) \in \mathbb{R}_{\geq 0}^n \).

In [22], Bian et.al. proposed the second-order condition of continuous DR-submodular function: a continuously twice differentiable function \( F : \mathbb{R}_{\geq 0}^n \to \mathbb{R} \) is DR-submodular if and only if

\[
\frac{\partial^2 F}{\partial x_i \partial x_j} \leq 0, \quad \forall i, j \in [n].
\]

Next, let \( A \) denote the second-order Hessian matrix of the DR-submodular function, we get \( A_{ij} \leq 0 \) for any \( i, j \in [n] \). Then for any vector \( u = (u_1, \ldots, u_n)^T \geq 0 \), the following inequality holds

\[
u^T A u = u_1^2 A_{11} + \ldots + u_1 u_n A_{n1} + \ldots + u_n A_{nn} \leq 0.
\]

So the continuous DR-submodular function is one-sided \( 0- \)smooth.

- \( \sigma > 0 \): \([1]\) \( F(x) = \frac{1}{2} x^T M x + b^T x \) is OSS if \( M \) is a \( \sigma \)-semi-metric. Where \( M \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, b \geq 0 \)

Proof  Let \( M \in \mathbb{R}^{n \times n} \) be a non-negative symmetric \( \sigma \)-semi-metric (\( \sigma \)-semi-metric means that \( M_{i,j} \leq \sigma (M_{i,k} + M_{k,j}) \) holds for any \( i, j, k \in [n] \)). Note that \( \nabla^2 F(x) = M, \nabla F(x) = M x + b \). So

\[
\sigma (\nabla_i F(x) + \nabla_j F(x)) \geq \sigma (\sum_{k=1}^{n} M_{i,k} x_k + \sum_{k=1}^{n} M_{j,k} x_k)
\]
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\[
= \sum_{k=1}^{n} \sigma(M_{i,k} + M_{j,k})x_k \geq \sum_{k=1}^{n} M_{i,j}x_k
= \|x\|_1 M_{i,j} = \|x\|_1 \nabla^2_{i,j} F(x)
\]

where \( u = (e_i, e_j)^T \).

\[\square\]

Appendix B Discretization of JSPG Algorithm

The parallel algorithms are generally applied for the situation of big data. The proof for the approximation ratio in Theorem 1 strictly depends on the infinite value of \( n \). However, if \( n \) is a finite number, then parameter \( \delta \) cannot be arbitrarily small, and the process of integration in the analysis may not be realized. So it is necessary for us study the process from the perspective of discretization. In fact, our JSPG algorithm can run in a polynomial time by the bounded locality and smoothness conditions. In the following, we propose a different analysis, that relies on OSS smoothness and \( \eta \)-local boundedness, for the discretization version.

**Theorem 5** Let \( F : [0, 1]^n \rightarrow \mathbb{R}_+ \) be a \( \eta \)-local monotone normalized OSS function and \( P \) be a convex polytope. By setting \( \delta \leq \min\{\frac{1}{\eta n}, \frac{1}{(1-\epsilon)^2\mu}\} \), the approximation ratio of JSPG algorithm is \((1 - e^{-(\frac{n}{\alpha+1} + \sigma)} - \epsilon)(1 - o(1))\), for any given parameter \( \alpha \in (0, 1], \sigma, \epsilon > 0 \).

**Proof** Recall that \( x(t) \) is the intermediate solution \( x \) corresponding to parameter \( t \). So the initial solution is \( x(\alpha) \), and final output is \( x(\hat{t}) \). Because \( F(x) \) is twice continuous differentiable and \( P \) is a convex set, we have

\[
F(x(t + \delta)) = F\left(x(t) + \frac{\delta}{|S_x(t)|} \sum_{\nu_i \in S_x(t)} \nu_i\right)
\]

\[
= F(x(t)) + \frac{\delta}{|S_x(t)|} \sum_{\nu_i \in S_x(t)} \nu_i \cdot \nabla F\left(x(t) + \frac{\epsilon\delta}{|S_x(t)|} \sum_{\nu_i \in S_x(t)} \nu_i\right).
\]

The second part of RHS in (B2) is further deduced as

\[
\frac{\delta}{|S_x(t)|} \sum_{\nu_i \in S_x(t)} \nu_i \cdot \nabla F\left(x(t) + \frac{\epsilon\delta}{|S_x(t)|} \sum_{\nu_i \in S_x(t)} \nu_i\right)
\geq (1 - \epsilon\delta \eta) \frac{\delta}{|S_x(t)|} \sum_{\nu_i \in S_x(t)} \nu_i \cdot \nabla F(x(t))
\geq (1 - \epsilon\delta \eta) \cdot \delta \mu \lambda (1 - \epsilon)
\geq (1 - \epsilon\delta \eta) \cdot \delta \mu (OPT - F(x(t)))(1 - \epsilon),
\]

(B3)
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where the first, the second and the the last inequalities are from the property of \( \eta \)-local, Lemma 4 and Lemma 3, respectively.

Now let us define \( M = (1 - \epsilon \delta \eta)OPT \). Since \( F(x) \) is non-negative and \( \delta \leq \frac{1}{\eta n} \),

\[
F(x(t + \delta)) \geq F(x(t)) + (1 - \epsilon)\mu \delta (M - F(x(t))),
\]

from which we can deduce that

\[
(1 - (1 - \epsilon)\mu \delta)(M - F(x(t))) \geq (M - F(x(t + \delta))).
\]

Applying the fact of \( \hat{t} - \alpha \approx O(1/\delta) \), the following can be obtained by induction

\[
(1 - (1 - \epsilon)\mu \delta)^{1/\delta} (M - F(x(\alpha))) \geq (M - F(x(\hat{t}))). \quad (B4)
\]

Since \( \delta \leq \frac{1}{\mu(1 - \epsilon)} \), we have

\[
(1 - (1 - \epsilon)\mu \delta)^{1/\delta} \leq e^{-\mu(1 - \epsilon)}. \quad (B5)
\]

Substitute (B5) into (B4),

\[
F(x(\hat{t})) \geq (1 - e^{-\mu(1 - \epsilon)})M + F(x(\alpha)) \\
\geq (1 - e^{-\mu(1 - \epsilon)})M \\
= (1 - O(\epsilon)[1 - e^{-\mu}](1 - o(1)))OPT.
\]

This theorem holds. \( \square \)