CRITICAL POINT COUNTS IN KNOT COBORDISMS: ABELIAN AND METACYCLIC INVARIANTS

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ABSTRACT. For a pair of knots $K_1$ and $K_0$, we consider the set of four-tuples of integers $(g, c_0, c_1, c_2)$ for which there is a cobordism from $K_1$ to $K_0$ of genus $g$ having $c_i$ critical points of each index $i$. We describe basic properties that such sets must satisfy and then build homological obstructions to membership in the set. These obstructions are determined by knot invariants arising from cyclic and metacyclic covering spaces.

1. Introduction

Given a pair of knots $K_1$ and $K_0$ in $S^3$, let $\mathcal{G}(K_1, K_0)$ denote the set of all four-tuples $(g, c_0, c_1, c_2)$ of nonnegative integers for which there is a smooth orientable cobordism from $K_1$ to $K_0$ of genus $g$ having $c_i$ critical points of each index $i$. Our goal is to identify ways in which classical knot theory can provide constraints on this set. The value of $c_1$ is determined by those of $g, c_0$ and $c_2$, so our investigation is reduced studying the sets $\mathcal{G}_g(K_1, K_0)$ consisting of nonnegative pairs $(c_0, c_2)$ for which there is a genus $g$ cobordism from $K_1$ to $K_0$ having $c_0$ and $c_2$ critical points of index 0 and 2, respectively.

A number of well-studied problems can be formulated in terms of $\mathcal{G}(K_1, U)$, where $U$ is the unknot: related topics include the knot four-genus, the slice-ribbon conjecture, problems related to the ribbon-number of ribbon knots, and general unknotted operations. The set $\mathcal{G}_0(K_1, K_0)$ is related to knot concordances and in particular to the existence and properties of ribbon concordances. Papers that touch on aspects of these topics include [1, 2, 4, 6, 11, 13, 16–19, 22–24, 30, 32]. Through the use of cyclic branched covers, this study is related to the study of the handlebody structure of cobordisms between three-manifold, as presented, for instance, in [2].

We have several goals. The first is simply to present this perspective on knot cobordism. Next, we describe how homological invariants of cyclic branched covers of knots provide constraints on the sets $\mathcal{G}(K_1, K_0)$; this work consists of extensions of known results concerning ribbon disks and concordances to the setting of cobordisms. Our use of equivariant homology groups lets us further refine our results. After this, we consider the use of metacyclic invariants; these arise from cyclic covers of cyclic branched covers. Finally, we list some problems that arise from this perspective.

**Summary of results.** In seeking invariants from $M_n(K)$, the $n$–fold branched cover of a knot $K$, or from a $q$–fold cyclic cover of $M_n(K)$, one faces a series of choices: the values of $n$ and $q$; the coefficients $F$ for the homology groups; and the choice of which $q$–fold cover to consider. There is also a decision as to whether to take into account the

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module structure of the homology, viewed as an $F$–module or $\mathbb{F} \mathbb{Z}_n$–module. As has been done in the past, we will follow a path that is sufficiently complicated to illustrate the techniques but is simple enough to avoid technicalities. For instance, we will work with knots for which the associated $\mathbb{F} \mathbb{Z}_n$–modules are of a simple form.

Our main result that is based on cyclic branched covers is the following.

**Theorem 5.1** Suppose that $\Sigma$ is a cobordism from $K_1$ to $K_0$. Then for all $n$, for all prime powers $p$ satisfying $p - 1 \equiv 0 \; \text{mod} \; n$, and for all $\zeta \in \mathbb{F}_p$ satisfying $\zeta^n = 1$, we have

$$c_0(\Sigma) \geq \frac{\beta_1^c(M_n(K_1), \mathbb{F}_p) - \beta_1^c(M_n(K_0), \mathbb{F}_p)}{2} - g(\Sigma).$$

In this statement, $\beta_1^c(M_n(K_1), \mathbb{F}_p)$ is the dimension of the $\zeta$–eigenspace of the $\mathbb{Z}_n$–action on $H_1(M_n(K), \mathbb{F}_p)$, where $\zeta \in \mathbb{F}_p$ is an $n$–root of unity in the finite field with $p$ elements. Averaging over the set of $n$–roots of unity yields the following simplier, but often weaker, result.

**Corollary 5.4** Under the conditions of Theorem 5.1

$$c_0(\Sigma) \geq \frac{\beta_1(M_n(K_1), \mathbb{F}_p) - \beta_1(M_n(K_0), \mathbb{F}_p)}{2(n - 1)} - g(\Sigma).$$

A simple application of Corollary 5.4 concerns 3–stranded pretzel knots: $P_k = P(2k + 1, -2k - 1, 2k + 1)$. These are ribbon knots. It follows from Corollary 5.4 that that if $2i + 1$ and $2j + 1$ are distinct primes, then there is a genus $g$ cobordism from $\alpha P_i$ to $\beta P_j$ having with $c_0 \geq 0$ and $c_2 \geq 0$ critical points of index 0 and 2, respectively, if and only if $c_0 \geq \alpha - g$ and $c_2 \geq \beta - g$. This is proved using 2-fold branched covers.

We will also present an example that depends on the full strength of Theorem 5.1 using higher-fold covers and the eigenspace splitting. The example is built from the knot $10_{153}$, which is a ribbon knot with ribbon number 1 (see [20]). We show that there exists a genus $g$ surface in $B^4$ bounded by $\alpha 10_{153}$ having $c_0$ and $c_2$ index 0 and index 2 critical points, respectively, if and only if $c_0 \geq \alpha + 1 - g$.

Examples in which metacyclic covers yield stronger results will be built from knots $K(k, J)$ illustrated in Figure 1. In the figure, the right band is tied in the knot $J$. The left band has $-k$ full twists and the right band had $k + 1$ full twists. If $J$ is a ribbon knot, then this knot is ribbon: the simple closed curve that goes over each band once in opposite directions has framing 0 and has knot type $J$. This family is of interest because the Seifert form of $K(k, J)$ is independent of the choice of $J$, and thus no homological invariants arising from cyclic branched covers can be used to distinguish a pair $K(k, J_1)$ and $K(k, J_2)$. However, the branched cyclic covers $M_n(K(k, J))$ themselves have cyclic covers, and the homology of these iterated covers does depend on $J$. In Section 8 we will explore these examples in detail, focusing on the case of $k = 1$ and $J$ is a multiple of either $K(1, U) = 6_1$ or $K(2, U) = 10_3$. The obstructions we develop are determined from 3–fold cyclic covers of the 2–fold branched cover of $5^3$, but the proofs of the results require that we consider covers of order $3^b$ for some unknown value of $b$. This is a reflection of an underlying issue that first appeared in [5].
2. The set $G_g(K_1, K_0)$.

In this section, we present in detail the knot invariants of interest and describe some of their basic properties.

2.1. The definition of $G_g(K_1, K_0)$. We view knots as smooth oriented diffeomorphism classes of pairs $(S, K)$ where $S$ is diffeomorphic to $S^3$ and $K$ is diffeomorphic to $S^1$. We will be using the shorthand notation $K \subset S^3$ or simply $K$ for such a pair; $-K$ denotes the pair $(-S, -K)$. A cobordism from a knot $K_1$ to a knot $K_0$ consists of a smooth oriented surface $\Sigma \subset S^3 \times [0, 1]$ for which $\partial (S^3 \times [0, 1], \Sigma) = -(S^3, K_0) \sqcup (S^3, K_1)$. (In particular, $\Sigma \cap (S^3 \times \{1\}) = K_1$.) We will assume that $\Sigma$ is connected. We will also restrict our attention to Morse cobordisms, those for which the projection $\Sigma \to [0, 1]$ is a Morse function.

Viewing $\Sigma$ as a twice punctured surface of genus $g$, we have that $\beta_1(\Sigma) = 2g + 1$; alternatively, $g = (\beta_1(\Sigma) - 1)/2$. Will write $g(\Sigma)$ for the value of $g$.

We let $c_0(\Sigma), c_1(\Sigma)$, and $c_2(\Sigma)$ denote the number of local minima, saddle points, and local maxima of the projection of $\Sigma$ to $[0, 1]$, respectively. The height function on $\Sigma$ determines a handlebody structure on $(\Sigma, K_0)$ having $c_0, c_1$, and $c_2$ handles of dimensions 0, 1, and 2, respectively. We will move between the Morse function and the handlebody decomposition without further comment.

If $g(\Sigma) = 0$, then $\Sigma$ is called a concordance. If $c_2(\Sigma) = 0$, then $\Sigma$ is called a ribbon cobordism.

An Euler characteristic argument shows that for a genus $g$ cobordism with $c_0$, $c_1$, and $c_2$ critical points of each index, we have $c_1 = c_2 + c_0 + 2g$. Thus, to understand the counts of critical points of possible cobordisms, or equivalently the number of handles in the corresponding handlebody structure, we do not need to keep track of the value of $c_1$. (Many past papers focus on $c_1$, for instance in studying the ribbon number of ribbon knots, but notice that if there is a cobordism from $K_1$ to $K_0$ with $c_1$ saddle points, there is also a cobordism from $K_0$ to $K_1$ with $c_1$ saddle points; we can more readily highlight the asymmetry of the general problem by using $c_0$ and $c_2$.)
Definition 2.1. For knots $K_1$ and $K_0$, set

- $G_g(K_1, K_0) = \{(c_0(\Sigma), c_2(\Sigma)) \mid \Sigma$ is a cobordism from $K_1$ to $K_0$ with $g(\Sigma) = g\} \subset (\mathbb{Z}_{\geq 0})^2$.
- $G(K_1, K_0) = \{(g, c_0, c_1, c_2) \mid (c_0, c_2) \in G_g(K_1, K_0)$ and $c_1 = c_2 + c_0 + 2g\} \subset (\mathbb{Z}_{\geq 0})^4$.

2.2. Elementary properties of $G_g(K_1, K_0)$. We begin with the following proposition, which is no more than a restatement of the definition of ribbon cobordism.

Proposition 2.2. There exists a $c_0 \geq 0$ such that $(c_0, 0) \in G_g(K_1, K_0)$ if and only if there exists a genus $g$ ribbon cobordism from $K_1$ to $K_0$.

A cobordism can be modified by adding a pair of critical points of indices 0 and 1, or of indices 1 and 2, without altering the genus. Thus we have the next result.

Proposition 2.3. For a pair of knots $K_1$ and $K_0$, if $(c_0, c_2) \in G_g(K_1, K_0)$, then $(c_0 + i, c_2 + j) \in G_g(K_1, K_0)$ for all $i, j \geq 0$.

It follows that each $G_g(K_1, K_0)$ is a finite union of quadrants, $\bigcup_{(a, b)} Q(a, b)$, where $Q(a, b) := \{(i, j) \mid i \geq a \text{ and } j \geq b\}$.

Figure 2 illustrates the union of quadrants $Q(2, 3) \cup Q(5, 1)$.

![Figure 2. Graph of $Q(2, 3) \cup Q(5, 1)$.](image)

If for some pair of knots $K_1$ and $K_0$ and $g \geq 0$, the graphic in Figure 2 represents $G_g(K_1, K_0)$, then the fact that there are no point on either axis implies that there does not exist a genus $g$ ribbon cobordism from $K_1$ to $K_0$ or from $K_0$ to $K_1$.

Next, we observe the most basic ways in which points in $G_g(K_1, K_0)$ determine points in $G_{g+1}(K_1, K_0)$

Proposition 2.4. For a pair of knots $K_1$ and $K_0$, suppose that $(c_0, c_2) \in G_g(K_1, K_0)$.

1. If $c_0 > 0$, then $(c_0 - 1, c_2) \in G_{g+1}(K_1, K_0)$.
2. If $c_2 > 0$, then $(c_0, c_2 - 1) \in G_{g+1}(K_1, K_0)$.

Proof. In terms of cross-sections of the cobordism, an index 0 critical point at height $t$ corresponds to the addition of an unknotted, unlinked component to the cross-section of $\Sigma$ as the height increases past $t$. The same addition can be realized by performing a trivial band move to the cross-section at height just below $t$. This corresponds to adding a critical point of index 1 in exchange for eliminating the index 0 critical point. It increases the genus by 1. A similar construction eliminates index 2 critical points. □
Example 2.5. Figure 3 illustrates how a point in \( G_g \) generates points in \( G_{g+1} \). In this example, the point \((4, 2) \in G_0\). Using Proposition 2.4 we see that \( \{(3, 2), (4, 1)\} \subset G_1 \). This in turn implies that \( \{(2, 2), (3, 1), (4, 0)\} \subset G_2 \). It next follows that \( \{(1, 2), (2, 1), (3, 0)\} \subset G_3 \). As a consequence, we have \( \{(0, 2), (1, 1), (2, 0)\} \subset G_4 \) and then that \( \{(0, 1), (1, 0)\} \subset G_5 \). Finally, \((0, 0) \in G_g \) for all \( g \geq 6 \).

In this example, if the first figure represents \( G_0 \) for some pair of knots, we are not asserting the remaining diagrams illustrate the \( G_g \), but only that they represent subsets of the \( G_g \). Example 5.6 in Section 5 we will show that \( G(K_1, K_0) \) can be strictly larger than the set guaranteed by Proposition 2.4.

![Figure 3. Possible sets \( G_g(K_1, K_2) \).](image)

2.3. The set of \( G(K_1, K_2) \) and the associated sequence. It is apparent that each \( G_g \) is determined by a unique finite set of points and that for large \( g \), \( G_g \) consists of the entire quadrant. This is summarized in the following theorem.

**Theorem 2.6.** Each set \( G(K_1, K_0) \) is determined by a finite sequence

\[ S(K_1, K_0) = ((g_1, a_1, b_1), (g_2, a_2, b_2), (g_3, a_3, b_3), (g_4, a_4, b_4), \ldots, (g_k, 0, 0)) \]

of elements in \((\mathbb{Z}_{\geq 0})^3\) which is lexicographically ordered. There is a unique minimal length such sequence.

As an example, some of the terms of the lexicographically ordered sequence corresponding to the regions in Figure 3 are

\[ \left( (0, 4, 2), (1, 3, 2), (1, 4, 1), (2, 2, 2), (2, 3, 1), (2, 4, 0), \ldots, (5, 0, 1), (5, 1, 0), (6, 0, 0) \right). \]

A general problem that seems to be beyond currently available techniques is to determine if there are any constraints on the sequences that can arise from a pair of knots other than those that are a consequence of Propositions 2.3 and 2.4. For instance, the ribbon conjecture can be stated as the following: if \((0, c_0, c_2) \in S(K, U)\) for some \(c_0\) and \(c_2\), then \((0, c'_0, 0) \in S(K, U)\) for some \(c'_0\). The generalized ribbon conjecture states that if \((g, c_0, c_1) \in S(K, U)\) for some \(g, c_0\) and \(c_2\), then \((g, c'_0, 0) \in S(K, U)\) for some \(c'_0\). See Section 10 for a further discussion.
2.4. The case of $K_0$ is unknotted. Understanding $G_g(K, U)$ is equivalent to analyzing surfaces bounded by $K$ in $B^4$. Given a knot $K \subset S^3$, we let $\Sigma \subset B^4$ with $\partial \Sigma = K$. We will assume the radial function is Morse on $\Sigma$; hence, we can define the count of critical points as before.

Definition 2.7. For knot a knot $K$, set
\[ B_g(K) = \{(c_0(\Sigma), c_2(\Sigma)) \mid \Sigma \subset B^4, \partial \Sigma = K, \text{ and } g(\Sigma) = g\}. \]

The following is clear.

Proposition 2.8. For any knot $K$, $(c_0, c_2) \in B_g(K)$ if and only if $(c_0 - 1, c_2) \in G_g(K, U)$.

The sets $G_g(K_1, K_0)$ and $B_g(K_1 \# -K_0)$ are related, but note that in considering $G_g(K_1 \# -K_0)$ we have lost the asymmetry of the general problem. Let $b(K)$ denote the minimum number of index 0 critical points in a ribbon disk for $K \# -K$. This invariant is related to classical three-dimensional knot invariants. For instance, let $br(K)$ denote the bridge index of $K$. A ribbon disk for $K \# -K$ with $c_0 = br(K)$ and $c_1 = br(K) - 1$ is easily constructed; thus $b(K) \leq br(K)$. Results concerning the interplay between these invariants appears in [18, Section 1]. See also Problem 5 in Section 10.

Given a cobordism from $K_1$ to $K_0$, we can start with the ribbon surface for $K_0 \# -K_0$ to build a slicing surface for $K_1 \# -K_0$; use the cobordism to change $K_1 \# -K_0$ into $K_0 \# -K_0$ and then attach a slice disk. This leads to the next result.

Theorem 2.9. If $(c_0, c_2) \in G_g(K_1, K_0)$, then $(c_0 + b(K_0), c_2) \in B_g(K_1 \# -K_0)$.

In the reverse direction, given a surface bounded by $K_1 \# -K_0$, we can build a cobordism from $K_1$ to $K_0$: build a cobordism from $K_1$ to $K_1 \# -K_0 \# -K_0$ and then cap it off with the surface bounded by $K_1 \# -K_0$. This yields the following.

Theorem 2.10. If $(c_0, c_2) \in B_g(K_1 \# -K_0)$, then $(c_0 - 1, c_2 + b(K_0)) \in G_g(K_1, K_0)$.

3. Covering spaces and equivariant knot theory

In this section, we set up the notation for covering spaces and the general theory of the associated equivariant homology theory. We then consider a technical issue that arises from the following situation. A homomorphism $\rho: \pi_1(X) \to \mathbb{Z}_m$ determines a homomorphism $\rho: \pi_1(X) \to \mathbb{Z}_{km}$ for any $k$ via inclusion; we will need to understand relationships between the equivariant homology groups of the associated $m$–fold and $km$–fold cyclic covers.

3.1. Cyclic covers of knots. Let $K \subset S^3$ be a knot and let $\Sigma \subset S^3 \times [0, 1]$ be a cobordism between knots.

Definition 3.1.

- $M_n(K)$ will denote $n$–fold cyclic cover of $S^3$ branched over $K$.
- $\tilde{K}$ denotes the preimage of $K$ in $M_n(K)$.
- $W_n(\Sigma)$ and $\tilde{\Sigma}$ denote the $n$–fold cyclic cover of $S^3 \times [0, 1]$ branched over $\Sigma$ and the preimage of $\Sigma$. 
• $M_\infty(K)$ and $W_\infty(\Sigma)$ will denote the infinite cyclic covers of $S^3 \setminus K$ and $(S^3 \times [0,1]) \setminus \Sigma$.

3.2. Covering space theory. For any group $\Gamma$, let $K_\Gamma$ denote an Eilenberg-MacLane space for $\Gamma$ and let $E_\Gamma$ denote its universal cover. All spaces $X$ considered here will be connected manifolds and covering spaces will be abelian, so we need not discuss details about the underlying point set topology and basepoint issues.

If $X$ is connected and $\rho: \pi_1(X) \to \Gamma$ is a homomorphism, then it induces a map $X \to K_\Gamma$. The pullback of $E_\Gamma \to K_\Gamma$ to $X$ is a covering space $\tilde{X}_\rho$. Points in the preimage of a basepoint in $\tilde{X}_\rho$ correspond to elements of $\Gamma$ and components of $\tilde{X}_\rho$ corresponds to cosets of $\rho(\pi_1(X)) \subset \Gamma$.

3.3. Equivariant homology and Betti numbers. For any CW–complex $X$, suppose that $T$ is an order $m$ homeomorphism that preserves the CW–structure. Let $F$ be a separable field, for instance $\mathbb{Q}$, $\mathbb{R}$, or a finite field $F_p$. Let $F$ have algebraic closure $\overline{F}$. The homology group $H_i(X,\overline{F})$ splits into $m$ eigenspaces, $H^\omega_i(X,\overline{F})$ for the $m$ distinct $m$–roots of unity $\zeta \in \overline{F}$. These eigenspaces are isomorphic to the homology groups associated to an eigenspace splitting of the CW–chain complex. We define $\beta^\omega_i(X,T,F)$ to be the dimension of $H^\omega_i(X,\overline{F})$. If $\zeta_1$ and $\zeta_2$ are Galois conjugate, then $\beta^\omega_i(X,T,F) = \beta^\omega_i(X,T,F)$.

If $\rho: \pi_1(Y) \to \mathbb{Z}_m$ is a homomorphism, then there is an induced $m$–fold covering space $\tilde{Y}_\rho$ with canonical deck transformation $T_\rho$. We will sometimes highlight the role of $\rho$ in our notation by writing $\beta^\omega_i(\tilde{Y}_\rho,\rho,F)$ for $\beta^\omega_i(\tilde{Y}_\rho,T_\rho,F)$. In our applications, the space $Y$ will be either $M_n(K)$ or $W_n(\Sigma)$. The corresponding covers $\tilde{Y}_\rho$ are called metacyclic branched covers of $K$. In the case that $n = 2$, the cover $\tilde{M}_2(K)$ is what is called a regular $2m$–fold dihedral cover of $S^3$ branched over $K$.

3.4. Relations between equivariant Betti numbers. For a connected manifold $X$, suppose that $\rho: \pi_1(X) \to \mathbb{Z}_m$ is a homomorphism. Let $\rho': \pi_1(X) \to \mathbb{Z}_{km}$ be induced by the inclusion $\mathbb{Z}_m \subset \mathbb{Z}_{km}$.

**Theorem 3.2.** With the conditions given above, the induced $km$–fold cover of $X$ is the disjoint union of $k$ copies of the $m$–fold cover of $X$:

$$\tilde{X}_{\rho'} \cong \tilde{X}_{\rho} \sqcup \tilde{X}_{\rho} \sqcup \cdots \sqcup \tilde{X}_{\rho}.$$  

The order $km$ deck transformation shifts each summand to the next. The last summand is mapped to the first via the order $m$ deck transformation of $\tilde{X}_{\rho}$.

**Proof.** This result follows from standard covering space theory.  

**Theorem 3.3.** Suppose that $\rho: \pi_1(X) \to \mathbb{Z}_m$ is a homomorphism and $\rho': \pi_1(X) \to \mathbb{Z}_{km}$ is the composition of $\rho$ with the inclusion $\mathbb{Z}_m \subset \mathbb{Z}_{km}$. Let $T_\rho$ be the order $m$ deck transformation of $\tilde{X}_\rho$ and let $T_{\rho'}$ be the order $km$ deck transformation of $\tilde{X}_{\rho'}$. Then the $k$ power of $T_{\rho'}$ is a transformation of order $m$ and

$$\beta^\omega_i(\tilde{X}_{\rho'},T_{\rho'}^k,F) = k\beta^\omega_i(\tilde{X}_\rho,T_\rho,F).$$
3.5. The equivariant CW–chain complex. Suppose that \( X \) has the structure of a CW–complex. That structure lifts to give a compatible CW–structure on each covering space; there is also a lifted CW–structure on branched covers, assuming that the branch set in \( X \) is a subcomplex. In particular, there are CW–chain complexes \( C_*(\tilde{X},\mathbb{F}) \) for any covering space or branched covering spaces we consider.

We will also be working with pairs \((X,M)\) that have a relative CW–structure, and these structures also lift to covering spaces.

If \( \rho: \pi_1(X) \rightarrow \mathbb{Z}_m \) is a homomorphism and \( \tilde{X}_\rho \) is the induced cover with deck transformation \( T_\rho \), then for each \( m\)–root of unity \( \zeta \in \mathbb{F} \) there is a subcomplex \( C_*(\tilde{X}_\rho,T_\rho,\mathbb{F}) \).

The homology of this complex is the equivariant homology discussed earlier.

**Theorem 3.4.** Suppose that \( \mathbb{F} \) contains a primitive \( m\)–root of unity \( \zeta \). Let \( X \) be a space or a pair of spaces. Suppose that \( \rho: \pi_1(X) \rightarrow \mathbb{Z}_m \) induces a cover \( \tilde{X}_\rho \) with deck transformation \( T_\rho \). Let \( \rho': \pi_1(X) \rightarrow \mathbb{Z}_{km} \) be the composition of \( \rho \) and the inclusion map \( \mathbb{Z}_m \subset \mathbb{Z}_{km} \), and let \( \tilde{X}_{\rho'} \) be the associated cover with deck transformation \( T_{\rho'} \).

- For all \( i \), \( \dim C_i(\tilde{X}_\rho,T_\rho,\mathbb{F}) = \dim C_i(X) \).
- For all \( i \), \( \dim C_i(\tilde{X}_{\rho'},(T_{\rho'})^k,\mathbb{F}) = k \dim C_i(X) \).

**Proof.** Let \( t \) generate \( \mathbb{Z}_m \). As a \( \mathbb{F}[\mathbb{Z}_m] \)–module, \( C_i(\tilde{X}_\rho,\mathbb{F}) \) splits as a direct sum of modules isomorphic to \( \mathbb{F}[\mathbb{Z}_m] \). There is one summand for each \( i\)–cell of \( X \). We then have the decomposition \( \mathbb{F}[\mathbb{Z}_m] \cong \bigoplus_{i=0}^{n-1} \mathbb{F}[\mathbb{Z}_m]/(t-\zeta^i) \). The summand \( \mathbb{F}[\mathbb{Z}_m]/(t-\zeta^i) \) is a \( \zeta^i \)–eigenspace of the action. Thus, each \( i\)–cell of \( X \) provides an eigenvector in \( C_i(\tilde{X}_\rho,T_\rho,\mathbb{F}) \). The second statement then follows from Theorem 3.3. \( \square \)

3.6. Pairs of spaces. Let \((X,Y)\) be a CW–pair and let \( \rho: H_1(X) \rightarrow \mathbb{Z}_m \). Then there is an associated covering space pair \((\tilde{X},\tilde{Y})\) and we can consider the equivariant relative homology groups of this cover. All the statements in the Section 3.5 above carry over to this relative setting.

3.7. Computing the equivariant homology for spaces associated to knots. For any given knot, the computation of \( \beta^*(M_n(K),\mathbb{F}_\rho) \) is fairly straightforward, using little more that what is covered in, say, Rolfsen’s text \[29\]. The computation of the metacyclic invariants can be technically challenging; in particular, they are not determined by a Seifert matrix. For this reason, we will restrict our examples to those for which for which the computation is quickly accessible.

4. Handlebody structure

**Theorem 4.1.** The pair \((W_n(\Sigma) \setminus \bar{\Sigma}, M_n(K_0) \setminus \bar{K})\) has a relative handlebody decomposition with:

- \( nc_0(\Sigma) \) 1–handles.
• \( \text{nc}_1(\Sigma) \) 2–handles.
• \( \text{nc}_2(\Sigma) \) 3–handles.

**Proof.** See, for instance, [10, Proposition 6.2.1] for a description of the handlebody structure on \((S^3 \times [0,1]) \setminus \Sigma\). That structure lifts to the covering space. \( \square \)

**Theorem 4.2.** The pair \((W_n(\Sigma), M_n(K_0))\) has a relative handlebody decomposition with:
• \( \text{nc}_0(\Sigma) \) 1–handles.
• \( \text{nc}_1(\Sigma) \) 2–handles.
• \( \text{nc}_2(\Sigma) + 2g(\Sigma) \) 3–handles.

**Proof.** We have that \((W_n(\Sigma), M_n(K_0))\) is built from \((W_n(\Sigma) \setminus \sim, M_n(K_0) \setminus \tilde{K}_0)\) via handle additions. For each \( i \)–handle in \( \Sigma \) there is an \((i + 2)\)–handle added.

The surface \( \Sigma \) can be built with one 0–cell and \( \beta_1(\Sigma) \) 1–cells. The 0–cell and the first 1–cell comprise \( K_0 \). Hence, in building \((W_n(\Sigma), M_n(K_0))\) the added 2–handle and the first 3–handle complete the construction of a product neighborhood of \( M_n(K_0) \). There remain \((\beta_1(\Sigma) - 1) \) 3–handles to add. Finally, \( \beta_1(\Sigma) - 1 = 2g \). \( \square \)

5. Homological constraints arising from cyclic branched covers

5.1. Homological constraints. In this section, we will denote the order \( n \) deck transformation of \( M_n(K) \) by \( T \). That is, no confusion should result by using the symbol \( T \) without notating its dependence on \( K \) and \( n \). We will work with finite fields of prime order, \( \mathbb{F}_p \), that contain primitive \( n \)–roots of unity; that is, \( p - 1 \equiv 0 \pmod{n} \). Unless specified, we will not assume that a given \( n \)–root of unity \( \zeta \) is primitive. The main result of this section is the following theorem.

**Theorem 5.1.** Suppose that \( \Sigma \) is a cobordism from \( K_1 \) to \( K_0 \). Then for all \( n \), for all prime powers \( p \) satisfying \( p - 1 \equiv 0 \pmod{n} \), and for all \( \zeta \in \mathbb{F}_p \) satisfying \( \zeta^n = 1 \), we have

\[
\text{c}_0(\Sigma) \geq \frac{\beta_1^1(M_n(K_1), T, \mathbb{F}_p) - \beta_1^2(M_n(K_0), T, \mathbb{F}_p)}{2} - g(\Sigma).
\]

Before proving this, we isolate the case \( \zeta = 1 \) in a lemma and then prove another lemma that will simplify our exposition.

**Lemma 5.2.** Let \( K \) be a knot and let \( \{n,p\} \) be a relatively prime pair. Then the 1–eigenspace of the deck transformation acting on \( H_1(M_n(K), \mathbb{F}_p) \) is trivial.

**Proof.** This is a fairly standard result, the proof of which we outline. Let \( f \colon M_n(K) \to S^3 \) be the branched cover. Given any cell \( d \) in a compatible CW–structure on \( S^3 \), we can choose a lift \( \tilde{d} \) in \( M_n(K) \) and define the transfer to be \( \tau(d) = \sum_{i=0}^{n-1} T^i(\tilde{d}) \). The choice of coefficients ensures that \( \tau \) induces an isomorphism to the 1–eigenspace, \( H_i(S^3, \mathbb{F}_p) \to H_i^1(M_n(K), \mathbb{F}_p) \). The target is thus trivial for \( i = 1 \). \( \square \)

**Lemma 5.3.** Let \((W,M)\) be a CW–pair supporting an action \( T \) of \( \mathbb{Z}_n \). Suppose that \( \mathbb{F} \) is a field containing an element \( \zeta \neq 1 \) for which \( \zeta^n = 1 \). Finally, assume that \( T \)
preserves the components of $M$; that is, $T_*$ acts trivially on $H_0(M, \mathbb{F})$. Then with \( \mathbb{F} \)-coefficients,
\[
\beta_1(W) \leq \beta_1^\xi(M) + \dim(C_1^\xi(W, M))
\]
and
\[
\beta_1(W) \geq \beta_1^\xi(M) + \dim(C_1^\xi(W, M)) - \dim(C_2^\xi(W, M)).
\]

Proof. Removing cells of dimension 3 or higher does not affect any of the terms in the statement, so we can assume that $W$ is a 2–complex. In the proof, to simplify the presentation we suppress the “\( \mathbb{F} \)” in notation for chain complexes, homology groups, and Betti numbers.

The group $H_0^\xi(M) = 0$. Thus, from the long exact sequence, we have
\[
H_2^\xi(W, M) \to H_1^\xi(M) \to H_1^\xi(W) \to H_1^\xi(W, M) \to 0.
\]
From this it follows that
\[
\beta_i^\xi(W) = \beta_i^\xi(W, M) + \beta_i^\xi(M) - \dim(\text{Image}(H_2^\xi(W, M) \to H_1^\xi(M))).
\]
Since $\beta_i^\xi(W, M) \leq \dim(C_i^\xi(W, M))$, the first inequality in the statement of the lemma is immediate.

We have $\dim(\text{Image}(H_2^\xi(W, M) \to H_1^\xi(M))) \leq \beta_2^\xi(W, M)$; substituting into Equation (1) yields
\[
\beta_i^\xi(W) \geq \beta_i^\xi(M) + \beta_i^\xi(W, M) - \beta_i^\xi(W).
\]
We have that $c_0^\xi(W, M) = 0$, so a standard Euler characteristic argument implies that
\[
\beta_1^\xi(W, M) - \beta_2^\xi(W) = \dim(C_1^\xi(W, M)) - \dim(C_2^\xi(W, M)).
\]
Hence,
\[
\beta_1^\xi(W) \geq \beta_1^\xi(M) + \dim(C_1^\xi(W, M)) - \dim(C_2^\xi(W, M)),
\]
as desired. \( \square \)

Proof of Theorem 5.1. To simplify notation, we let $W = W_\alpha(\Sigma) \setminus \tilde{\Sigma}$, $\partial_0 W = M_\alpha(K_0) \setminus \tilde{K}_0$ and $\partial_1 W = M_\alpha(K_1) \setminus \tilde{K}_1$.

The 1–handles and 2–handles in the relative handlebody structure on $(W, \partial_0 W)$ are each freely permuted by the action of the generating deck transformation $T$. That is, for $i = 1$ and $i = 2$ we have that the CW–chain complex $C_i(W, \partial_0 W, \mathbb{F}_p)$ splits as a $\mathbb{F}_p[\mathbb{Z}_n]$–module into $c_i$ copies of $\mathbb{F}_p[T]/\langle 1 - T^n \rangle$. Each of these splits into $n$ eigenspaces; letting $\xi$ be a primitive $n$–root of unity,
\[
\mathbb{F}_p[T]/\langle 1 - T^n \rangle \cong \bigoplus_{i=0}^{n-1} \mathbb{F}_p[T]/\langle \xi^i - T \rangle.
\]
We have that $\zeta = \xi^i$ for some $i$, so the $\zeta$–eigenspace of the relative CW–chain complex of $(W, \partial_0 W)$ has $c_0$ generators in dimension 1 and $c_1$ generators in dimension 2. That is, $\dim(C_1^\xi(W, \partial_0 W, \mathbb{F}_p)) = c_i$. The first inequality of Lemma 5.3 gives
\[
\beta_1^\xi(W, T, \mathbb{F}_p) \leq \beta_1(\partial_0 W, T, \mathbb{F}_p) + c_0(\Sigma).
\]

We have a similar construction of $W$ starting with $\partial_1 W = M_\alpha(K_1) \setminus \tilde{K}_1$. In this case, we have $\dim(C_1^\xi(W, \partial_1 W)) = c_2$ and $\dim(C_2^\xi(W, \partial_1 W)) = c_1$. Using the second
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inequality in Lemma \[5.3\]

\[\beta_1^p(W, T, F_p) \geq \beta_1^p(\partial_1 W, T, F_p) + c_2(\Sigma) - c_1(\Sigma).\]

Combining these, we see that

\[\beta_1^p(\partial_0 W, F_p) + c_0(\Sigma) \geq \beta_1^p(\partial_1 W, F_p) + c_2(\Sigma) - c_1(\Sigma).\]

Recall that \(c_1(\Sigma) = c_0(\Sigma) + c_2(\Sigma) + 2g(\Sigma)\). The previous inequality can be rewritten as

\[\beta_1^p(\partial_0 W, T, F_p) + c_0(\Sigma) \geq \beta_1(\partial_1 W, T, F_p) + c_2(\Sigma) - \left(c_0(\Sigma) + c_2(\Sigma) + 2g(\Sigma)\right).\]

This inequality simplifies to give

\[c_0(\Sigma) \geq \frac{\beta_1^p(\partial_1 W, T, F_p) - \beta_1^p(\partial_0 W, T, F_p)}{2} - g(\Sigma).\]

The proof is finished by noting that completing the covers to form the branched cyclic covers adds generators to the CW–complex that are all in the 1–eigenspace and thus do not change the computation. \(\Box\)

Early work \[24\] studying ribbon knots provided homological constraints on the structure of the number the minimum number of index 1 critical points in a ribbon disk based on the homology of the 2–fold branched covers. The next theorem is a fairly simple generalization of such results. Notice that we do not restrict to the ribbon situation, 2–fold covers, or the case of \(g = 0\).

**Corollary 5.4.** Under the conditions of Theorem \[5.1\]

\[c_0(\Sigma) \geq \frac{\beta_1(M_n(K_1), F_p) - \beta_1(M_n(K_0), F_p)}{2(n - 1)} - g(\Sigma).\]

**Proof.** The proof consists of summing over the \(n - 1\) eigenspaces. \(\Box\)

**Example 5.5.** Let \(P_k\) denote the pretzel knot \(P(2k + 1, -2k - 1, 2k + 1)\). These are ribbon knots having Seifert form

\[
\begin{pmatrix}
0 & k \\
 k + 1 & 0
\end{pmatrix}.
\]

Each bounds a ribbon disk with one saddle point and two minimum. We have \(H_1(M_2(P_k)) \cong \mathbb{Z}_{2k+1} \oplus \mathbb{Z}_{2k+1}\).

We want to consider the sets \(G_g(nP_1, mP_2)\) and for convenience assume that \(n \geq m\). This example presents the case of \(g = 0\) and the next considers \(g > 0\).

Our first observation is that \(aJ_k\) bounds a ribbon disk with \(a\) saddle points and \(a + 1\) minima. From this it is easily seen that there is a concordance \(\Sigma\) from \(nP_1\) to \(mP_2\) with \(c_0(\Sigma) = n, c_1(\Sigma) = n + m,\) and \(c_2(\Sigma) = m\). That is, \((n, m) \in G_0(nP_3, mP_3)\).

Using \(\mathbb{Z}_3\)–coefficients in Theorem \[5.1\], we see that

\[c_0(\Sigma) \geq \frac{2n - 0}{2} - 0 = n.\]

Similar, working with \(\mathbb{Z}_5\)–coefficients we have \(c_2(\Sigma) \geq m\). Thus, \(G_0(nP_3, mP_3)\) is precisely the quadrant with vertex \((n, m)\), that is \(Q(n, m)\).
Example 5.6. If $m > 0$ and $n > 0$, then by Proposition 2.4 we have $Q(n - 1, m) \cup Q(n, m - 1) \subset \mathcal{G}_1(nP_1, mP_2)$. Here we show that this is a proper containment, that in fact, $\mathcal{G}_1(nP_1, mP_2) = Q(n - 1, m - 1)$.

The construction of a cobordism is simple. In the initial concordance that we built, the local maxima were at levels below the local minima. Because of this, the concordance can be modified by replacing disk neighborhoods of a maximum point and a minimum point by an annulus near an increasing path from the maximum to the minimum. The effect is to decrease both $c_0$ and $c_2$ by 1 in exchange for increasing the genus by 1; that is $\mathcal{G}_1(nP_3, mP_5) \subset Q(n - 1, m - 1)$. Theorem 5.1 immediately implies that this inclusion must be an equality.

The process can be repeated to prove that for $g \leq m$ we have $\mathcal{G}_g(nP_1, mP_2) = Q(n - g, m - g)$. Finally, Proposition 2.4 implies that for $m \leq g \leq n$ we have $\mathcal{G}_g(nP_1, mP_2) = Q(n - g, 0)$. For $g \geq n$ we have $\mathcal{G}_g(nP_1, mPJ_2) = Q(0, 0)$.

Figure 4 illustrates the sets $\mathcal{G}_g(4P_1, 2P_2)$.

Example 5.7. Let $K = 10_{153}$. We consider cobordisms to the unknot. For this knot $H_1(M_2(K)) = 0$ and $H_1(M_5(K)) \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$. Clearly Theorem 5.1 and Corollary 5.4 provide no information in the case of 2–fold covers. Using 5–fold covers does. We first observe that there are precisely two nontrivial 5–eigenspaces in $\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$, each 1–dimensional, as can be seen as follows. Clearly there are at most two nontrivial eigenspaces. Poincaré duality implies that if there is a $\zeta$–eigenvector, there is also a $\zeta^{-1}$–eigenvector; we present a proof in this in the appendix as Lemma B.1.

Using either eigenvalue, Theorem 5.1 implies that for any cobordism from $nK$ to the unknot, we have $c_0(\Sigma) \geq n - g$. Using Corollary 5.4 yields the weaker result that $c_0(\Sigma) \geq n/2 - g$. The improvement by a factor of two is expected, since two of the eigenspaces are trivial and two have dimension 1.

6. The infinite cyclic cover and the Alexander module

It has been known that the rank of the Alexander module of a knot has an upper bound that is determined by the genus of a surface bounded by the knot in $B^4$ and the critical point structure of that surface. We now generalize that observation, focusing on cobordisms.

Recall that $M_\infty(K_i)$ and $W_\infty(\Sigma)$ represent the infinite cyclic covers of the complements of the $K_i$ and $\Sigma$. In general, suppose we have a finite CW–complex $X$ and a
homomorphism $\rho: H_1(X) \to \mathbb{Z}$; then $\rho$ induces an infinite cyclic cover $\tilde{X}_\rho$. The group $H_1(\tilde{X}_\rho, \mathbb{Q})$ is a finitely generated module over the PID $\mathbb{Q}[t, t^{-1}]$. We denote this module by $A(X, \rho, \mathbb{Q}[t, t^{-1}])$. There is splitting

$$A(X, \rho, \mathbb{Q}[t, t^{-1}]) \cong \bigoplus_{j=1}^n \mathbb{Q}[t, t^{-1}]/(f_j(t)),$$

where $f_j$ divides $f_{j+1}$ for $j < n$. This splitting is unique and the value of $n$ is called the rank of the module.

**Definition 6.1.** Let $X$ be a space (or pair of spaces) supporting a map $\rho: H_1(X) \to \mathbb{Z}$ with associated infinite cyclic cover $\tilde{X}_\rho$. We denote by $\beta_i(\tilde{X}_\rho, \mathbb{Q}[t, t^{-1}])$ the $\mathbb{Q}[t, t^{-1}]$–rank of $A(X, \rho, \mathbb{Q}[t, t^{-1}])$. When $\rho$ is implicit, it is dropped from the notation.

For the complements of the $K_i$ and of $\Sigma$ there are canonical maps of the first homology to $\mathbb{Z}$, and thus we can suppress the “$\rho$” in our notation. The infinite cyclic cover $W_\infty(\Sigma)$ is built from the infinite cyclic cover $M_\infty(K_0)$ by adding the lifts of $c_0$ 1–handles, followed by $c_1$ 2–handles, and then 3–handles. There is a similar decomposition arising for $M_\infty(K_1)$. The proof of Theorem 5.1 carries over to this setting, yielding the following result.

**Theorem 6.2.** Suppose that $\Sigma$ is a cobordism from $K_1$ to $K_0$. Then

$$c_0(\Sigma) \geq \frac{\beta_1(M_\infty(K_0), \mathbb{Q}[t, t^{-1}]) - \beta_1(M_\infty(K_0), \mathbb{Q}[t, t^{-1}])}{2} - g(\Sigma).$$

This result can be strengthened by focusing on the direct sum decomposition of the module $A(X, \rho, \mathbb{Q}[t, t^{-1}])$ that corresponds to irreducible elements in $\mathbb{Q}[t, t^{-1}]$. For any irreducible polynomial $f$ we can set $\beta_f^i(\tilde{X}, \rho, \mathbb{Q}[t, t^{-1}])$ to be the rank of the $f$–primary summand of $A(X, \rho, \mathbb{Q}[t, t^{-1}])$. The proof of the following result is much the same as that for the previous theorem. (As an alternative, one can switch to the ring $\mathbb{Q}[t, t^{-1}]_{(f)}$, which denotes the localization at $f$, that is, the ring formed from $\mathbb{Q}[t, t^{-1}]$ by adding a multiplicative inverse to all nontrivial elements $g$ that are relatively prime to $f$. This is a PID with a unique prime, represented by $f$.)

**Theorem 6.3.** Suppose that $\Sigma$ is a genus $g$ cobordism from $K_1$ to $K_0$. Then for any irreducible polynomial $f \in \mathbb{Q}[t, t^{-1}]$,

$$c_0(\Sigma) \geq \frac{\beta_f^1(M_\infty(K_1), \mathbb{Q}[t, t^{-1}]) - \beta_f^1(M_\infty(K_0), \mathbb{Q}[t, t^{-1}])}{2} - g(\Sigma).$$

The following corollary is immediate.

**Corollary 6.4.** If knots $K$ and $J$ have nontrivial Alexander polynomials with a pair of distinct irreducible factors, then for any cobordism $\Sigma$ from from $nK$ to $mJ$ we have

$$c_0(\Sigma) \geq n/2 - g$$

and

$$c_2(\Sigma) \geq m/2 - g.$$

For related results in the case of ribbon concordances, see [7].
7. Knots $K(1, \alpha 6_1)$, $K(1, \beta 10_3)$, and their associated metacycle covers.

A metacyclic invariant of a knot $K$, or of a surface $\Sigma \subset S^3 \times [0,1]$, is one that is derived from a cyclic cover of the branched cover of $K$ or $\Sigma$. The use of such invariants in knot theory already appears in early work, such as Reidemeiser’s 1932 book \cite{27,28}. The role of such invariants in concordance first appeared in the work of Casson and Gordon \cite{5}. That paper, which introduced what is now called Casson-Gordon theory, was restricted to 2–bridge knots $B((2k+1)^2, 2)$. We will build our examples using the 2–bridge knots $B((2k+1)^2, 2k)$. The reason for the different choice is that Casson and Gordon were interested in showing that particular knots are not slice; we want to start with knots that are slice and explore their slice disks and concordances between them.

Our examples are built from two knots from this family: $K(1, U) = B(9, 2) = 6_1$ and $K(2, U) = B(25, 4) = 10_3$, but further examples are easily constructed.

Figure 1 gave an illustration of a knot $K(k, J)$. For $J$ unknotted, this is $B((2k+1)^2, 2k)$. We can think of $K(1, J)$ as being built from $B((2k+1)^2, 2k)$ by removing a neighborhood of a circle linking the right band in the Seifert surface shown in Figure 1 (for which the right band unknotted) and replacing that neighborhood with the complement of the knot $J$ in $S^3$. The identification of the boundaries interchanges the meridian and longitude. This creates a new knot in $S^3$, formed from $K_1(U)$ by tying the knot $J$ in a band on the Seifert surface, as desired. We will focus on two specific examples: $K(1, \alpha 6_1)$ and $K(1, \beta 10_3)$, where $\alpha$ and $\beta$ are nonnegative integers.

7.1. Ribbon disks for $K(k, J)$.

**Theorem 7.1.** If $J$ is ribbon and bounds a ribbon disk with $n$ minima, then $K(k, J)$ is ribbon, bounding a ribbon disk with $2n$ minima.

**Proof.** The knot $B((2k+1)^2, 2k)$ is ribbon: a simple closed curve $\gamma$ that passes over both bands of the Seifert surface once is unknotted and has framing zero. The ribbon disk has one index one critical point and two minima. A ribbon disk for $K(1, J)$ is built by removing an annular neighborhood of $\gamma$ (on the Seifert surface for $K(k, J)$) and replacing it with a pair of ribbon disks for $J$. \qed

7.2. The 2–fold branched cover of $K(1, J)$. An algorithm of Akbulut-Kirby \cite{3} provides a surgery diagram of the 2–fold branched cover of $K(1, J)$, as shown on the left in Figure 5. $M_2(K(1, J))$ is given as surgery on a two-component link, with one of the components unknotted and the other representing $J \# J^r$, where $J^r$ denotes $J$ with its string orientation reversed. Since all the knots $J$ we consider are reversible, we have left out the superscript “$r$” and do not orient the circles labeled with $J$. Also, we can write $2J$ rather then $J \# J^r$ when needed. As describe in, for instance, \cite{29}, that surgery diagram can be modified to appears the diagram on the right. This illustrates the 2–fold branched cover as formed from the lens space $L(9, 2)$ by removing two parallel copies of a core circle and replacing each with a copy of the complement of $J$.

As an immediate consequence, we have the following.

**Theorem 7.2.** For all $J$, $H_1(M_2(K(1, J))) \cong \mathbb{Z}_9$. 

7.3. The homology of the metacyclic cover of $K(1, J)$. It is evident that the 3–fold cyclic cover of $M_2(K(1, J))$ is built from the 3–fold cyclic cover of $L(9, 2)$, which is the lens space $L(3, 2)$, by removing a pair of parallel core circles and replacing them with copies of $M_3(J) \setminus \tilde{J}$. This is illustrated in Figure 6. We will thus need the following. Let $\tilde{M}_2^3(K(1, J))$ denote the nontrivial 3–fold cyclic cover of $M_2(K(1, J))$.

**Theorem 7.3.** There is an isomorphism $H_1(\tilde{M}_2^3(K(1, J))) \cong \mathbb{Z}_3 \oplus H_1(M_3(J))^2$.

**Proof.** For any knot $J$, let $X_1$ and $X_2$ be copies of the 3–fold cyclic cover of $S^3 \setminus J$. We have $H_1(X_i) \cong \mathbb{Z} \oplus H_1(M_3(J))$.

The torus boundary of $X_i$ has natural boundary curves, $m_i$, and $l_i$, lifts of the meridian and longitude of $J$. The curve $m_i$ represents an element of infinite order in $\mathbb{Z} \oplus H_1(M_3(J))$, and after a change of basis represents $1 \oplus 0$. The curve $l_i$ is null-homologous in $X_i$, bounding a lift of a Seifert surface.

In Figure 6 the curves $m_i$ and $l_i$ are attached to the longitude and meridian, respectively, of the curves labeled $\tilde{J}_3$. (Notice that there is an interchange of meridian and longitude.)

One can now undertake a Mayer-Vietoris argument. The covering space is split into four components by the three evident tori in Figure 6, that is, the peripheral tori to the three curves illustrated. As just described, two are related to the 3–fold covers of $J$, one is a solid torus with core $\gamma$ (corresponding to the 3/2–surgery), and one is the compliment of the three component link that is illustrated, having homology generated by three meridians, which we denote $\alpha_0$, $\beta_1$ and $\beta_2$, corresponding to the 3/2–surgery curves and the two $\tilde{J}_3$. We let $T = H_1(M_3(J))$. Via the Mayer-Vietoris sequence, we
see the homology is a quotient of
\[
(\mathbb{Z}(\alpha) \oplus \mathbb{Z}(\beta_1) \oplus \mathbb{Z}(\beta_2)) \oplus (\mathbb{Z}(m_1) \oplus T) \oplus (\mathbb{Z}(m_2) \oplus T) \oplus \mathbb{Z}(\gamma).
\]

The identification along the three tori, each with rank two first homology, introduces six relations. Taking them in order, meridian first and initially along the surgery torus, yields the following, where we write \(l_1\) despite it equaling 0, to make the gluing maps more evident:

- \(\alpha = -2\gamma\)
- \(\beta_1 + \beta_2 = 3\gamma\)
- \(\beta_1 = l_1\)
- \(\alpha = m_1\)
- \(\beta_2 = l_2\)
- \(\alpha = m_2\).

None of these involve the summand \(T \oplus T\) and so, in effect, they are relations defining a quotient of \(\mathbb{Z}^6 \cong \langle \alpha, \beta_1, \beta_2, m_1, m_2, \gamma \rangle\). A simple exercise shows the quotient is isomorphic to \(\mathbb{Z}_3\), as desired. In our case, we have either \(T \cong (\mathbb{Z}_7)^2\) or \(T \cong (\mathbb{Z}_{19})^2\).

To apply this result, we will use the following and its immediate corollary.

**Lemma 7.4.** \(H_1(M_3(61)) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_7\) and \(H_1(M_3(103)) \cong \mathbb{Z}_{19} \oplus \mathbb{Z}_{19}\).

**Proof.** This is a standard knot theoretic computation; see, for instance [29]. More generally, as described in the appendix, one can readily show that for \(q\) odd, \(H_1(M_q(B((2k + 1)^2, 2k))) \cong \mathbb{Z}_{(k+1)^q-k^q} \oplus \mathbb{Z}_{(k+1)^q-k^q}\). \(\square\)

**Corollary 7.5.** \(H_1(\widehat{M}_2^3(K(1, 61))) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_7)^4\) and \(H_1(\widehat{M}_2^3(K(1, 103))) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}_{19})^4\).

### 7.4. The eigenvalue decomposition of \(H_1(\widehat{M}_2^3(K(1, J)))\).

For any field \(\mathbb{F}\), there is an action of \(\mathbb{Z}_3\) on \(H_1(\widehat{M}_2^3(K(1, J)), \mathbb{F})\). In the case that \(\mathbb{F}\) contains a primitive 3–root of unity \(\zeta\), the homology \(H_1(\widehat{M}(K(1, J)), \mathbb{F})\) splits into eigenspaces, as described in Section 3. Note that \(\mathbb{F}_7\) and \(\mathbb{F}_{19}\) both contain such roots of unity. When no confusion can result, we will use the same symbol \(\zeta\) to denote a primitive cube roots of unity in \(\mathbb{F}_7\) and in \(\mathbb{F}_{19}\).

**Theorem 7.6.** With the set-up described above:

- \(\beta_1^c(\widehat{M}_2^3(K(1, 61)), \mathbb{F}_7) = 2\alpha\).
- \(\beta_1^c(\widehat{M}_2^3(K(1, 61)), \mathbb{F}_{19}) = 0\).
- \(\beta_1^c(\widehat{M}_2^3(K(1, 103)), \mathbb{F}_7) = 0\).
- \(\beta_1^c(\widehat{M}_2^3(K(1, 103)), \mathbb{F}_{19}) = 2\beta\).

**Proof.** Considering the \(\mathbb{F}_7\)–homology, we have \(H_1(\widehat{M}_2^3(K(1, 61)), \mathbb{F}_7) \cong (\mathbb{F}_7)^{2\alpha}\) arises entirely from the \(2\alpha\) copies of \(M_3(J) \setminus \bar{J}\) that appear in the covering space. Thus the proof of the first statement comes down to analyzing the eigenspace splitting of the \(\mathbb{Z}_3\)–action on \(M_3(J) \cong \mathbb{F}_7 \oplus \mathbb{F}_7\). We claim that the 1–eigenspace is trivial and the
\( \zeta \)-eigenspaces and \( \zeta^{-1} \)-eigenspaces are both 1-dimensional. This can be shown with an explicit computation, or one can argue abstractly, as follows. A transfer argument, using the branched covering map \( M_3(J) \to S^3 \) shows that the 1-eigenspace is trivial. Poincaré duality implies that the \( \zeta \)-eigenspace and \( \zeta^{-1} \)-eigenspace are isomorphic (see Lemma B.1 for a proof).

Similar arguments give the remaining statements. \( \square \)

7.5. 

**Metacyclic covers of** \( nK(1, \alpha 61) \) **and** \( mK(1, \beta 103) \). Let \( \rho: H_1(M(nK(1, J))) \to \mathbb{Z}_3 \) be nonzero on \( a \) of the natural \( \mathbb{Z}_3 \)-summands and be 0 on \( (n-a) \) of the summands. We wish to understand the eigenspace decomposition of the homology of the associated cover. This will be clarified by the following result concerning the lens space \( L(9, 2) \).

We can prove with a simple construction and should make the subsequent results evident.

**Lemma 7.7.** Let \( X = L(9, 2) \). Suppose that \( \rho: H_1(nX) \to \mathbb{Z}_3 \) is nonzero on \( a \geq 1 \) of the natural \( \mathbb{Z}_3 \)-summands. Then the associated 3-fold cover satisfies

\[
\tilde{nX}_3 \cong aL(3, 2) \# 3(n-a)L(9, 2) \# 2(a-1)S^1 \times S^2.
\]

**Theorem 7.8.**

**A.** Suppose that \( \rho: H_1(nM_2(K(1, \alpha 61))) \to \mathbb{Z}_3 \) is nonzero on \( a \geq 0 \) of the natural \( \mathbb{Z}_3 \)-summands. Then

- If \( a \geq 1 \), then \( \beta_1^{(\tilde{M}_2)(nK(1, \alpha 61)), \rho, \mathbb{F}_7} = 2a + a - 1 \).
- If \( a \geq 1 \), then \( \beta_1^{(\tilde{M}_2)(nK(1, \alpha 61)), \rho, \mathbb{F}_{19}} = a - 1 \).
- If \( a = 0 \), then \( \beta_1^{(\tilde{M}_2)(nK(1, \alpha 61)), \rho, \mathbb{F}_7} = \beta_1^{(M_2(nK(1, \alpha 61)), \rho, \mathbb{F}_{19})} = 0 \).

Similarly,

**B.** Suppose that \( \rho: H_1(nM_2(K(1, \beta 103))) \to \mathbb{Z}_3 \) is nonzero on \( a' \geq 0 \) of the natural \( \mathbb{Z}_3 \)-summands. Then

- If \( a' \geq 1 \), then \( \beta_1^{(\tilde{M}_2)(nK(1, \beta 103)), \rho, \mathbb{F}_{19}} = 2a' + a' - 1 \).
- If \( a' \geq 1 \), then \( \beta_1^{(\tilde{M}_2)(nK(1, \beta 103)), \rho, \mathbb{F}_7} = a' - 1 \).
- If \( a' = 0 \), then \( \beta_1^{(\tilde{M}_2)(nK(1, \beta 103)), \rho, \mathbb{F}_{19}} = \beta_1^{(M_2(nK(1, \beta 103)), \rho, \mathbb{F}_7)} = 0 \).

**Proof.** Most of the terms that appear in the statements are evident from the construction, with perhaps one exception. In the first formula there is the term \( a - 1 \) which arises from the \( S^1 \times S^2 \) summands. To clarify this, we will consider the case of \( a = 2 \) and the more general situation of \( L(mn, q) \) with the homomorphism \( \rho \) mapping onto \( \mathbb{Z}_n \) on both factors. Then the \( n \)-fold cyclic cover is \( L(m, q) \# (n-1)S^1 \times S^2 \). The homology with \( \mathbb{F} \) coefficients has a summand \( \mathbb{F}^{n-1} \). As a \( \mathbb{Z}_n \)-module this is \( \mathbb{F}[\mathbb{Z}_n]/(1-t) \). In the case that \( \mathbb{F} \) contains a primitive \( n \)-root of unity, this splits into \( (n-1) \) eigenspaces of dimension 1. \( \square \)

8. 

**Cobordisms between** \( nK(1, \alpha 61) \) **and** \( mK(1, \beta 103) \).

To simplify the discussion, we will assume that \( n \geq m > 0 \). Let \( \Sigma \) be a genus \( g \) cobordism from \( nK(1, \alpha 61) \) and \( mK(1, \beta 103) \). We continue to denote the 2-fold cover
of $S^3 \times [0,1]$ branched over $\Sigma$ by $W_2(\Sigma)$; this is a cobordism from $M_2(nK(1, \alpha 6_1))$ to $M_2(mK(1, \beta 10_3))$.

8.1. Extending homomorphisms from $H_1(M_2(nK(1, \alpha 6_1)))$ to $H_1(W_2(\Sigma))$. We have that the $M_2(nK(1, \alpha 6_1))$ and $M_2(nK(1, \beta 10_3))$ are $\mathbb{Q}$–homology spheres and thus there are $\mathbb{Q}/\mathbb{Z}$–valued non-singular symmetric linking forms on each one. For any $\mathbb{Q}$–homology sphere $M$, the linking form provides an identification of $H_1(M)$ with $\text{hom}(H_1(M), \mathbb{Q}/\mathbb{Z})$. We remind the reader that a metabolizer for such a linking form on an abelian group of order $l^2$ is a subgroup of order $l$ on which the linking form is identically 0.

The following result is an immediate consequence of a theorem of Gilmer in [9]. In summary, suppose that $K$ bounds a genus $g$ surface in $B^4$. Then according to [9, Lemma 1], the homology group $H_1(M_2(K))$ splits as a direct sum $A \oplus B$, where $B$ has a presentation of size $2g \times 2g$ and the linking form on $A$ is metabolic; notice that this implies that if $B \cong \mathbb{Z}_k^b$, then $k \leq 2g$. Denote by $\phi$ the restriction map
\[ \phi: \text{hom}(H_1(W_2(\Sigma)), \mathbb{Q}/\mathbb{Z}) \to \text{hom}(H_1(nM_2(K_1)) \oplus H_1(-M_2(mM_2(K_0))), \mathbb{Q}/\mathbb{Z}). \]

**Theorem 8.1.** Suppose that $n+m \geq 2g$ and recall there is an isomorphism $H_1(nM_2(K_1)) \oplus H_1(-mM_2(K_0)) \cong (\mathbb{Z}_9)^n \oplus (\mathbb{Z}_9)^m$. For some $\epsilon \geq 0$, the linking form on this group splits off a summand that is isomorphic to $(\mathbb{Z}_9)^{n+m-2g+\epsilon}$ which contains a metabolizer $\mathcal{M} \subset H$, all elements of which are in the image of $\phi$. In particular, the order of $\mathcal{M}$ is at least $3^{n+m-2g}$.

To apply this result, we clearly need to have $n + m > 2g$. To simplify our considerations, we will assume that $n > 2g$.

**Corollary 8.2.** Suppose that $\Sigma$ is a genus $g$ cobordism from $nK(1, \alpha 6_1)$ to $mK(1, \beta 10_3)$ and assume that $n > 2g$. Then there is a surjective homomorphism $\rho: H_1(M_2(nK(1, \alpha 6_1))) \to \mathbb{Z}_3^b$ for some $b$.

**Proof.** Theorem 8.1 provides a set $\mathcal{M}$ of homomorphisms $H_1(nM_2(K_1)) \oplus H_1(-M_2(mM_2(K_0))) \to \mathbb{Q}/\mathbb{Z}$ that extend to homomorphisms $\rho'$ on $H_1(W_2(\Sigma))$. The order of $\mathcal{M}$ is $3^{n+m-2g}$ and the order of a metabolizer for $H_1(-M_2(mM_2(K_0)))$ is $3^m$. It follows that if $3^{n+m-2g} > 3^m$, then some element in $\mathcal{M}$ is not contained in $0 \oplus H_1(-M_2(mM_2(K_0))$ and thus must be nontrivial on $H_1(M_2(mM_2(K_0)))$. This will occur as long as $n > 2g$. Call one such element $\rho$ and let $\rho'$ denote an extension of $\rho$ to $H_1(W_2(\Sigma))$.

The image of $\rho'$ is a finite cyclic subgroup $G \subset \mathbb{Q}/\mathbb{Z}$. Projecting $G$ to its 3–primary summand does not change its restriction to the boundary, so we can assume that $\rho'$ takes values in $\mathbb{Z}_{3^b}$ for some $b$. If $\rho$ is not of order 3, then it can be multiplied by 3 so that it does have order 3.

8.2. The $3^b$–fold cyclic cover of $W(\Sigma)$. Let $\pi: \tilde{W}_2^3(\Sigma) \to W_2(\Sigma)$ denote the $3^b$–fold cyclic cover of $W_2(\Sigma)$ associated to the homomorphism $\rho'$ defined above. We let $\partial_1(\tilde{W}_2^3) = \pi^{-1}(M_2(nK(1, \alpha 6_1)))$ and $\partial_0(\tilde{W}_2^3) = \pi^{-1}(M_2(nK(1, \alpha 10_3)))$.

We can now apply Theorem 3.3. Let $\zeta$ be a primitive 3–root of unity and consider the $\mathbb{Z}_3^b$–action on $\partial_1(\tilde{W}_2^3)$, the $3^{b-1}$ power of order $3^b$ deck transformation, which we denote by $T = S^{3^{b-1}}$. 
Theorem 8.3. Assume that the restriction \( \rho: M_2(nK(1, \alpha 6_1)) \to \mathbb{Z}_3 \) is nonzero on \(a \geq 1\) of the \(n\) summands. Also suppose that the restriction is nonzero on \(a' \geq 0\) of the \(m\) summands of \(H_1(M_2(nK(\beta 10_3)))\).

\[
\beta_1^c(\partial_t(\widetilde{W}_2^3), T, \mathbb{F}_7) = 3^{b-1}(2a\alpha + a - 1).
\]

\[
\beta_1^c(\partial_0(\widetilde{W}_2^3), T, \mathbb{F}_7) = 3^{b-1}(a' - 1) \text{ if } a' \geq 1 \text{ and } \beta_1^c(\partial_0(\widetilde{W}_2^3), \mathbb{F}_7) = 0 \text{ if } a' = 0.
\]

Applying a relative version of Theorem 3.4 along with Theorem 4.2 gives the next result.

Theorem 8.4. Let \(C_i^c(\widetilde{W}_2^3, \partial_0(\widetilde{W}_2^3), \mathbb{F}_7)\) be the \(\zeta\)-eigenspace of the CW–chain group under the \(\mathbb{Z}_3\)–action given as the \(3^{b-1}\)–power of its deck transformation. Then

\[
\dim C_1^c(\widetilde{W}_2^3, \partial_0(\widetilde{W}_2^3), T, \mathbb{F}_7) = 3^{b-1}(2c_0(\Sigma)).
\]

\[
\dim C_2^c(\widetilde{W}_2^3, \partial_0(\widetilde{W}_2^3), T, \mathbb{F}_7) = 3^{b-1}(2c_1(\Sigma)).
\]

\[
\dim C_3^c(\widetilde{W}_2^3, \partial_0(\widetilde{W}_2^3), T, \mathbb{F}_7) = 3^{b-1}(2c_2(\Sigma) + 2g(\Sigma)).
\]

Theorem 8.5. Let \(\Sigma\) be a genus \(g\) cobordism from \(nK(1, \alpha 6_1)\) to \(mK(1, \beta 10_3)\). Assume that \(n > 2g\). Then

\[
c_0(\Sigma) \geq \frac{2a\alpha + 1 - m}{4} - g.
\]

Proof. The proof is much like the one for Theorem 5.1. We work with the \(\zeta\)-eigenspaces of the \(\mathbb{Z}_3\)-actions. Consider the fact that \(\widetilde{W}_2^3\) is built from \(\partial_0(\widetilde{W}_2^3)\). We have

\[
\beta_1^c(\widetilde{W}_2^3, T, \mathbb{F}_7) \leq 3^{b-1}(m - 1) + 3^{b-1}(2c_0(\Sigma))
\]

The first summand comes from the homology of the boundary, using the fact that in Theorem 8.3, we have \(a' - 1 \leq m\). Turning the bordism upside down and using that fact that \(\widetilde{W}_2^3\) is built from \(\partial_1(\widetilde{W}_2^3)\) by adding 1–handles and 2–handles that correspond to the index two and index one critical points of \(\Sigma\), respectively, we find that

\[
\beta_1^c(\widetilde{W}_2^3, T, \mathbb{F}_7) \geq 3^{b-1}(2a\alpha + a - 1) + 3^{b-1}(2c_2(\Sigma)) - 3^{b-1}(2c_1(\Sigma)).
\]

Together, these inequalities imply

\[
(2a\alpha + a - 1) + 2c_2(\Sigma) - (2c_1(\Sigma)) \leq (m - 1) + (2c_0(\Sigma)).
\]

We have that \(c_1(\Sigma) = c_0(\Sigma) + c_2(\Sigma) + 2g(\Sigma)\). Substituting yields

\[
(2a\alpha + a - 1) + 2c_2(\Sigma) - 2(c_0(\Sigma) + c_2(\Sigma) + 2g(\Sigma)) \leq (m - 1) + 2c_0(\Sigma).
\]

This simplifies to give

\[
c_0(\Sigma) \geq \frac{2a\alpha + 1 - m}{4} - g.
\]

Finally, since \(a \geq 1\), we have the desired result:

\[
c_0(\Sigma) \geq \frac{2a + 1 - m}{4} - g.
\]
8.3. **Strengthening the bounds.** The difference between the lower bound provided by Theorem 8.5 and the best upper bound that we can prove with a realization result is quite large. For instance, we have the following realization result.

**Theorem 8.6.** If \( g \leq \min\{n(2\alpha + 1), m(2\beta + 1)\} \), then there is a genus \( g \) cobordism \( \Sigma \) from \( nK(\alpha_6) \) to \( mK(\beta_{10}) \) satisfying

\[
c_0(\Sigma) = n(2\alpha + 1) - g \quad \text{and} \quad c_0(\Sigma) = m(2\beta + 1) - g
\]

**Proof.** The construction given in Example 5.6 can be easily modified to produce the result. What is essential is that the canonical ribbon disks can be pieced together to form a concordance in which the local maxima are beneath the local minima. □

A limitation in this theorem is the absence of \( n \) in the bound on \( c_0 \) given Theorem 8.5. We want to explore this briefly. We have an inclusion of \( (\mathbb{Z}_3)^{m+n} \) into a group with nonsingular linking form:

\[
(\mathbb{Z}_3)^n \oplus (\mathbb{Z}_3)^m \subset (\mathbb{Z}_9)^n \oplus (\mathbb{Z}_9)^m.
\]

We have assumed that \( n + m > 2g \) and identified a subgroup \( \mathcal{M} \subset (\mathbb{Z}_9)^n \oplus (\mathbb{Z}_9)^m \) of order \( 3^{n+m-2g} \) upon which the linking form is identically 0. In the proof of Theorem 8.5 we used the fact that if \( n > 2g \) then \( \mathcal{M} \cap (\mathbb{Z}_3)^n \oplus 0 \) is nontrivial. But in fact, if \( n \) is large in comparison to \( m \) and \( g \), then the rank of the intersection \( \mathcal{M} \cap (\mathbb{Z}_3)^n \oplus 0 \) must be large as well; in particular, rather than use \( a \geq 1 \) in the argument, we could find metabolizing elements for which \( a \) is much larger. Similar, we used the obvious fact that \( a' \leq m \); with care, we could also show that it is possible to assume that \( a' \) is close to 0. We have opted not to undertake the careful analysis of self-annihilating subgroups of the standard linking form on \((\mathbb{Z}_9)^{n+m}\) that is required to establish these better bounds.

9. **Non-reversible knots**

To conclude our presentation of examples, we consider a subtle family of examples built from knots \( K \) and \( K^r \), where \( K^r \) denotes the reverse of \( K \). Such knots are difficult to distinguish by any means. For instance, all abelian invariants are identical for the two knots. It is not known at the moment whether any invariants that are built from the Heegaard Floer knot complex \( \text{CFK}^\infty(K) \) defined in [25], such as its involutive counterpart, defined in [15], can distinguish them. The successful application of metacyclic invariants to distinguishing knots from their reverses began with the work of Hartley, [14].

Figure 7 illustrates a knot that we will denote \( P = P(J_1, J_2) \). The starting knot is the pretzel knot \( P(3, -3, 3) \), and knots \( J_1 \) and \( J_2 \) are placed in the two bands. Notice that we have indicated the orientation of \( P \). We let \( P^r \) denote reverse of the knot; that is, the knot with the same diagram except with the arrow reversed (the use of \( P^r \) rather than the more standard notation \( P^r \) will simplify some notation later on). These knots have formed the basis of a variety of concordance result related to reversibility; see, for instance, [21]. In past papers that used these knots, the \( J_1 \) were chosen so that the knots could be shown not to be concordant. We will let \( J_1 \) and \( J_2 \) be slice knots, so that
they the $P$ and $P^*$ are themselves slice, and or results apply to consider concordances
between them.

\begin{figure}[ht]
\centering
\includegraphics[width=0.5\textwidth]{knot.png}
\caption{The knot $P(J_1, J_2)$.}
\end{figure}

We will now briefly summarize the results of some calculations related to these, leaving
the details to [21]. First, we have for the 3–fold cover that $H_1(M_3(P)) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_7$. This

\textit{Figure 7. The knot $P(J_1, J_2)$.}

group splits into a 2–eigenspace and a 4–eigenspace for the deck transformation using
$\mathbb{F}_7$–coefficients. If $z$ and $w$ are linking circles to the two bands, with $\tilde{z}$ and $\tilde{w}$ being
chosen lifts to $M_3(P)$, then the 2–eigenspace and 4–eigenspace are spanned by $\tilde{z}$ and $\tilde{w}$, respectively. For $P^*$ we have $\tilde{z}^*$ and $\tilde{w}^*$ as eigenvectors, but because of the reversal,
they are the 4–eigenvectors and 2–eigenvectors, respectively.

Let $\Sigma$ be a cobordism from $P$ to $P^*$.

We let $W_3$ be the 3–fold cover of $S^3 \times [0, 1]$ branched over $\Sigma$. In Section 8 we considered
a metabolizer $\mathcal{M}$ of the linking form on $H_1(M_3(P)) \oplus H_1(-M_3(P^*))$. This metabolizer
must be invariant under the $\mathbb{Z}_3$–action, and thus is spanned by eigenvectors. Here are
the possibilities.

- $\mathcal{M}$ is a 2–eigenspace, spanned by $\{\tilde{z}, \tilde{w}^*\}$.
- $\mathcal{M}$ is a 4–eigenspace, spanned by $\{\tilde{w}, \tilde{z}^*\}$.
- $\mathcal{M}$ contains a nontrivial 2–eigenvector $a\tilde{z} + b\tilde{w}^*$ and a 4–eigenvector $c\tilde{w} + b\tilde{z}^*$.

We now wish to find obstructions based on the 7–fold cyclic covers of the spaces
involved. There are three cases to consider. Here is a summary of what arises.

- **Case 1:** Considering the eigenvector $\tilde{z}$, the corresponding cover of $M_3(P)$ will
  have first homology that depends on the homology of $M_7(J_1)$. For the eigenvector
  $\tilde{w}^*$, the corresponding cover of $M_3(P^*)$ will have first homology that depends on
  the homology of $M_7(J_2)$.

- **Case 2:** This is similar. For the eigenvector $\tilde{w}$, the corresponding cover of
  $M_3(P)$ will have first homology that depends on the homology of $M_7(J_2)$. For
  the eigenvector $\tilde{z}^*$, the corresponding cover of $M_3(P^*)$ will have first homology
  that depends on the homology of $M_7(J_1)$.

- **Case 3:** The last case splits into subcases, depending on whether the coefficients
  $a, b, c,$ and $d$ are zero or not. The most interesting case is when, say $a \neq 0 \neq b$. Then
  the corresponding 7–fold cover of $M_3(P)$ will involve the first homology of
\(M_7(J_1)\) and the corresponding 7–fold cover of \(M_3(P^*)\) will also involve the first homology of \(M_7(J_2)\).

From this it should be clear that by choosing \(J_1\) and \(J_2\) so that the rank of the first homology groups \(H_1(M_7(J_1), \mathbb{F}_p)\) and \(H_1(M_7(J_2), \mathbb{F}_{p'})\) are large for appropriate primes \(p\) and \(p'\), then regardless of which metabolizer arises, there will be obstructions to the values of \(c_0(\Sigma)\) and \(c_2(\Sigma)\) being small. This can be achieved by letting \(J_1\) be a multiple of 61 and letting \(J_2\) be a multiple of 103. A computation as described in the appendix shows \(H_1(M_7(61)) \cong \mathbb{Z}_{127} \oplus \mathbb{Z}_{127}\) and \(H_1(M_7(103)) \cong \mathbb{Z}_{2059} \oplus \mathbb{Z}_{2059}\). The number 2059 has prime factors 29 and 71. All of \(\mathbb{F}_{127}\), \(\mathbb{F}_{29}\) and \(\mathbb{F}_{71}\) contain primitive 7–roots of unity.

To construct examples in Section 7 we used the fact that the 3–fold cover of \(L(9,2)\) is \(L(3,2)\). For carrying out an explicit computation here, we would need to know the homology of the 7–fold cover of \(M_3(P)\) corresponding to each eigenspace of the \(\mathbb{Z}_3\)–action. Regardless of what these groups are, their ranks in comparison to the rank of \(H_1(M_7(\alpha J_1))\) or \(H_1(M_7(\beta J_2))\) will be small if \(\alpha\) and \(\beta\) are large. This permits one to prove the following result.

**Theorem 9.1.** For any non-negative integers \(g\), \(c_0\) and \(c_2\), there are positive integers \(\alpha\) and \(\beta\) such that the knot \(P = P(\alpha 61, \beta 103)\) has the following properties.

- \(P(\alpha 61, b 103)\) is a ribbon knot.
- Any genus \(g\) cobordism \(\Sigma\) from \(P\) to \(P^*\) has \(c_0(\Sigma) \geq c_0\) and \(c_2(\Sigma) \geq c_2\).

## 10. Problems

1. Is \(G_g(K_1, K_0)\) always a quadrant, of the form \(Q(a, b)\), for some \(a\) and \(b\)?
2. An affirmative answer to the previous question would be implied by a positive answer to the following: If \((a + 1, b) \in G_g(K_1, K_0)\) and \((a, b + 1) \in G_g(K_1, K_0)\), then is \((a, b) \in G_g(K_1, K_0)\)? This would also imply Gordon’s Conjecture 13: If \(K_1\) is ribbon concordant to \(K_0\) and \(K_0\) is ribbon concordant to \(K_1\), then \(K_1 = K_0\).
3. An even simpler generalization of Gordon’s conjecture is the following statement: if for some \(c_0\) and \(c_2\), \((c_0, 0) \in G_g(K_1, K_0)\) and \((0, c_2) \in G_g(K_1, K_0)\), then \(G_g(K_1, K_2) = Q(0, 0)\).
4. If \((a + 1, b + 1) \in G_g(K_1, K_0)\), then is \((a, b) \in G_{g+1}(K_1, K_0)\)?
5. Recall that the bridge number of \(K\) is denoted \(b(K)\) and we defined \(b(K)\) to be the minimum number of index 0 critical points of a slice disk for \(K \# -K\). It is elementary to show that \(b(K) \leq b(K)\). It is also not difficult to construct ribbon knots \(K\) with large bridge index that bound disks in the four-ball with one saddle point. Using these knots we see that \(b(K) - b(K)\) can be arbitrarily large.

For the torus knot \(T_{2,3}\) we have \(b(T_{2,3}) = 2\) and it is elementary to see that \(b(T_{3,3}) = 2\). In fact, in 18 it is shown that for torus knots, \(b(K) = b(K)\). Yet there are still basic examples that are unresolved: for \(K = nT_{2,3}\) we have \(b(K) = n + 1\); is it true that \(b(nT_{2,3}) = n + 1\)?
Appendix A. The knots $K(k, J)$

Here we summarize the computations required in Section 7 that determine the homology groups of covering spaces associated to $K(k, J)$. Recall that if $J$ is unknotted, this is the two-bridge knot $B((2k + 1)^2, 2k)$. It is the basic building block for the examples in Lemma 7.4.

A.1. A Seifert surface for $K(k, J)$ and its Seifert form. The knot $K(k, J)$ has a genus 1 Seifert surface $F$ built by attaching two bands to a disk, one with framing $k + 1$ and other with framing $-k$. The first band has a knot $J$ tied it it. This was illustrated in Figure 5. The Seifert matrix with respect to the natural basis $\{a, b\}$ of $H_1(F)$ is

$$A_k = \begin{pmatrix} k + 1 & 1 \\ 0 & -k \end{pmatrix}.$$  

The classes $a$ and $b$ are represented by simple closed curves on $F$ representing the unknot and the knot $J$. If we change basis, letting $a' = a - b$ and $b' = b$ then the Seifert matrix becomes

$$B_k = \begin{pmatrix} 0 & k + 1 \\ k & -k \end{pmatrix}.$$  

These generators are still represented by simple closed curves, the first of which is unknotted and the second of which represents $J$.

A.2. The homology of the cyclic branched covers of $K(k, J)$. We next have the computation of the needed homology groups.

**Theorem A.1.** Let $K(k, J)$ be as above. Then $H_1(M_2(K(k, J))) \cong \mathbb{Z}_{(2k + 1)^2}$. For $n$ odd, $H_1(M_n(K(k, J))) \cong \mathbb{Z}_d \oplus \mathbb{Z}_d$, where $d = (k + 1)^n - k^n$.

**Proof.** The homology group $H_1(M_2(K(k, J)))$ is presented by $A_k + A_T^k$, where $A^T$ denotes the transpose. This $2 \times 2$ matrix has one if its entries a 1, so it presents a cyclic group. The order of that group is the absolute value of the determinant of the matrix. As an alternative, the presence of $J$ does not affect the Seifert matrix or the homology of the cover. If $J$ is the unknot, then the 2-fold branched cover is the lens space $L((2k+1)^2, 2k)$.

The homology group $H_1(M_n(K(k, J)))$ can be computed using a formula of Seifert [31]; see [8] for a more recent treatment. In our notation, this result states that for a knot $K$ with Seifert matrix $B$, $H_1(M_n(K))$ is presented by

$$\Gamma^n - (\Gamma - \text{Id})^n,$$

where $\Gamma = (B^T - B)^{-1}B^T$.

In our case, one readily computes that

$$\Gamma = \begin{pmatrix} k + 1 & -k \\ 0 & -k \end{pmatrix},$$  

and thus we are interested in the group presented by

$$A_k = \begin{pmatrix} k + 1 & -k \\ 0 & -k \end{pmatrix}^n - \begin{pmatrix} k & -k \\ 0 & -k - 1 \end{pmatrix}^n.$$
For some \( b \), this is of the form
\[
A_k = \begin{pmatrix}
(k+1)^n - k^n & b \\
0 & (-k)^n - (-k-1)^n
\end{pmatrix}.
\]
Since \( n \) is odd, this can be rewritten as
\[
A_k = \begin{pmatrix}
(k+1)^n - k^n & b \\
0 & (k+1)^n - k^n
\end{pmatrix}.
\]
With a bit more work we could show that \( b = 0 \), but instead we rely on a theorem of Plans [26] (or see [29, Chapter 8D]): the homology of an odd-fold cycle branched cover is a double. \( \square \)

A.3. A number theoretic observation. In our examples, we considered the cases of \( H_1(M_3(K(1, U))) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_7 \) and \( H_1(M_3(K(2, U))) \cong \mathbb{Z}_{19} \oplus \mathbb{Z}_{19} \). We observed that both \( \mathbb{F}_7 \) and \( \mathbb{F}_{19} \) contain primitive 3–roots of unity, since \( 7 \equiv 1 \mod 3 \) and \( 19 \equiv 1 \mod 3 \). This is not a coincidence. Our examples were the cases of \( p = 3 \) and either \( k = 1 \) or \( k = 2 \) in the following theorem, which follows immediately from a standard application of the binomial theorem or from Fermat’s Little Theorem.

**Theorem A.2.** If \( p \) is prime, then for all \( k \), \( (k+1)^p - k^p \equiv 1 \mod p \).

Appendix B. The eigenspace structure of \( H_1(M_n(K), \mathbb{F}_p) \).

In his survey paper on knot theory [12], Gordon used a duality argument to prove that the first homology of the infinite cyclic cover of a knot, viewed as a module over the ring \( \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}] \), is isomorphic to its dual module, in which the action of \( t \) is replaced with the action of \( t^{-1} \). A similar argument can be applied in the setting of \( n \)–fold cyclic branched covers. Here we give a simple proof of a consequence of such a result. Duality is still required to the extent that it implies that the linking form of a three-manifold is nonsingular.

**Theorem B.1.** Assume that \( H_1(M_n(K)) \cong \mathbb{F}_p^k \) for some \( k \). Suppose that \( n \) divides \( p-1 \), so that \( \mathbb{F}_p \) contains a primitive \( n \)–root of unity, \( \xi \). Then \( H_1(M_n(K)) \) splits into a direct sum of \( \xi^i \)–eigenspaces, denoted \( E_i \), under the action of the deck transformation \( T_\ast \). In addition, \( E_0 \) is trivial and \( E_i \cong E_{n-i} \) for all \( i, 0 < i < n \).

**Proof.** Since \( T_\ast \) satisfies \( T_\ast^n = 1 \), the splitting into a direct sum of eigenspaces is an elementary fact from linear algebra.

Let \( \text{lk}(x, y) \in \mathbb{F}_p \) denote the \( \mathbb{F}_p \)-valued linking form on \( H_1(M_n(K)) \). Recall that the linking form is symmetric, nonsingular and equivariant with respect to the action of a homeomorphism, in particular with respect to \( T_\ast \).

**Claim 1:** The eigenspaces \( E_i \) and \( E_j \) are orthogonal with respect to the linking form unless \( i = j = 0 \) or \( i = n - j \).

To see this, suppose that \( x \in E_i \) and \( y \in E_j \). Then
\[
\xi^i \text{lk}(x, y) = \text{lk}(T_\ast x, y) = \text{lk}(x, T_\ast^{-1} y) = \text{lk}(x, \xi^{-j} y) = \xi^{-j} \text{lk}(x, y).
\]
It follows that \((\xi^i - \xi^{-j})\text{lk}(x, y) = 0\). This can be rewritten as \((\xi^i - \xi^{n-j})\text{lk}(x, y) = 0\). If \(i \neq 0\), then \(\xi^i - \xi^{n-j} \neq 0\) unless \(i = n - j\). Thus, if \(i \neq 0\) and \(i \neq n - j\), then \(\text{lk}(x, y) = 0\).

**Claim 2:** \(E_0\) is trivial. We can now write

\[
H_1(M(K)) \cong E_0 \oplus E_{n/2} \bigoplus_{1 \leq i < n/2} (E_i \oplus E_{n-i}).
\]

(The summand \(E_{n/2}\) exists if and only if \(n\) is even, in which case it represents the \(-1\)-eigenspace.)

If \(x \in E_0\), then \(x + T_*x + \cdots + T^{n-1}_*x = nx\) is in the image of the transfer map \(\tau: H_1(S^3) \to H_1(M_n(K))\), and thus equals 0. We can write \(p - 1 = nk\) for some \(k\), and so \((p - 1)x = 0\). But \(p - 1\) is relatively prime to \(p\), and so we have \(x = 0\), as desired.

**Claim 3:** \(E_i \cong E_{n-i}\) for all \(i, 0 < i < n\).

This is automatic for \(E_{n/2}\) in the case the \(n\) is even. We focus on a summand \(E_i \oplus E_{n-i}\) for \(1 \leq i < n/2\).

Suppose that \(E_i\) is of dimension \(a\) and \(E_{n-i}\) is of dimension \(b\). By choosing bases for these eigenspaces, the linking form can be represented by an \((a + b) \times (a + b)\) matrix with entries in \(\mathbb{F}_p\). Both \(E_i\) and \(E_{n-i}\) are self-orthogonal, so there are blocks with all entries 0 of size \(a \times a\) and \(b \times b\). The nonsingularity implies that \(a \leq (a + b)/2\) and \(b \leq (a + b)/2\). This can occur only if \(a = b\).

\[\square\]

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