Fields and forms on $\rho$-algebras

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Abstract. In this paper we introduce non-commutative fields and forms on a new kind of non-commutative algebras: $\rho$-algebras. We also define the Frölicher–Nijenhuis bracket in the non-commutative geometry on $\rho$-algebras.

Keywords. Non-commutative geometry; $\rho$-algebras; Frölicher–Nijenhuis bracket.

1. Introduction

There are some ways to define the Frölicher–Nijenhuis bracket in non-commutative differential geometry. The Frölicher–Nijenhuis bracket on the algebra of universal differential forms of a non-commutative algebra, is presented in [2], the Frölicher–Nijenhuis bracket in several kinds of differential graded algebras are defined in [6] and the Frölicher–Nijenhuis bracket on colour commutative algebras is defined in [7]. But this notion is not defined on $\rho$-algebras in the context of non-commutative geometry. In this paper we introduce the Frölicher–Nijenhuis bracket on a $\rho$-algebra $A$ using the algebra of universal differential forms $\Omega(A)$.

A $\rho$-algebra $A$ over the field $k$ (C or R) is a $G$-graded algebra ($G$ is a commutative group) together with a twisted cocycle $\rho: G \times G \to k$. These algebras were defined for the first time in the paper [1] and are generalizations of usual algebras (the case when $G$ is trivial) and of $Z$ ($Z_2$)-superalgebras (the case when $G$ is $Z$ resp. $Z_2$). Our construction of the Frölicher–Nijenhuis bracket for $\rho$-algebras, in this paper, is a generalization of this bracket from [2].

In $\mathfrak{x}$ we present a class of non-commutative algebras which are $\rho$-algebras, derivations and bimodules. In $\mathfrak{x}$ we define the algebra of (non-commutative) universal differential forms $\Omega(A)$ of a $\rho$-algebra $A$. In $\mathfrak{x}$ we present the Frölicher–Nijenhuis calculus on $A$, the Nijenhuis algebra of $A$, and the Frölicher–Nijenhuis bracket on $A$. We also show the naturality of the Frölicher–Nijenhuis bracket.

2. $\rho$-Algebras

In this section we present a class of non-commutative algebras that are $\rho$-algebras. For more details see [1].

Let $G$ be an abelian group, additively written, and let $A$ be a $G$-graded algebra. This implies that the vector space $A$ has a $G$-grading $A = \bigoplus_{a \in G} A_a$, and that $A_a A_b = A_{a+b}$.
(1) The \( \rho \) is a \( \rho \)-algebra with the trivial group \( G \):

\[
\rho (a; b) = \rho (\psi_a) \bigg( a b 2 G; \bigg)
\]

(2) Let \( G = \mathbb{Z} \) (\( \mathbb{Z}_2 \)) be the group and the cocycle \( \rho (a; b) = (a b)^{\epsilon_b} \), for any \( a b 2 G \). In this case any \( \rho \)-algebra is a super(commutative) algebra.

Examples.

1) Any usual (commutative) algebra is a \( \rho \)-algebra with the trivial group \( G \).

2) Let \( G = \mathbb{Z} \) (\( \mathbb{Z}_2 \)) be the group and the cocycle \( \rho (a; b) = (a b)^{\epsilon_b} \), for any \( a b 2 G \). In this case any \( \rho \)-algebra is a super(commutative) algebra.

3) The \( N \)-dimensional quantum hyperplane \( S_N \), is the algebra generated by the unit element and \( N \) linearly independent elements \( x_1, \ldots, x_N \) satisfying the relations:

\[
x_i x_j = qx_j x_i; \quad i < j
\]

for some fixed \( q 2 \mathbb{K}; q \neq 0 \). \( S_N \) is a \( \mathbb{Z}^N \)-graded algebra, i.e.,

\[
S_N = \bigoplus_{n_1, \ldots, n_N} S_{n_1, \ldots, n_N};
\]

with \( S_{n_1, \ldots, n_N} \) the one-dimensional subspace spanned by products \( x^{n_1} \ldots x^{n_N} \). The \( \mathbb{Z}^N \)-degree of these elements is denoted by

\[
x^{n_1} \ldots x^{n_N} = n = n_1 + \ldots + n_N.
\]

Define the function \( \rho : \mathbb{Z}^N \rightarrow \mathbb{Z}^N \) as

\[
\rho (\alpha_1^0) = q^{\sum_{k=1}^{N} n_1 \alpha_{k}^1};
\]

with \( \alpha_{j_k} = 1 \) for \( j < k \), \( 0 \) for \( j = k \) and \( 1 \) for \( j > k \). It is obvious that \( S_N \) is a \( \rho \)-commutative algebra.

4) The algebra of matrix \( M_n (\mathbb{C}) \) \( [5] \) is \( \rho \)-commutative as follows:

Let

\[
p = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 1
\end{pmatrix}, \quad \text{and} \quad q = \begin{pmatrix}
e^0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Then \( pq = \epsilon qp \) and \( M_n (\mathbb{C}) \) is generated by the set \( B = \{0; 1; \cdots; n\} \).
Fields and forms on \( \rho \)-algebras

It is easy to see that \( p^ap^b = e^{ab}p^ap^b \) and \( q^bp^a = e^{ab}p^a q^b \) for any \( a, b = 0; 1; \ldots; n \). Let \( G := Z_G \) and \( x_{\alpha} = (\alpha; 0\alpha_2) \in G \) and \( x_{\alpha} := p^{0\alpha_2}q_{0\alpha_2} \in M_n(\mathbb{C}) \). If we denote \( \rho(\alpha; \beta) = e^{\alpha\beta_1} \alpha_2 \beta_2 \) then \( x_{\alpha}x_{\beta} = \rho(\alpha; \beta)x_{\beta}x_{\alpha} \), for any \( \alpha; \beta \in G \), \( x_{\alpha}x_{\beta} \in B \).

It is obvious that the map \( \rho: G \rightarrow G \otimes \mathbb{C} \), \( \rho(\alpha; \beta) = e^{\alpha\beta} \alpha_2 \beta_2 \) is a cocycle and that \( M_n(\mathbb{C}) \) is a \( \rho \)-commutative algebra.

Let \( \alpha \) be an element of the group \( G \). A \( \rho \)-derivation \( X \) of \( A \), of degree \( \alpha \) is a bilinear map \( X: A \rightarrow A \) of \( G \)-degree \( \mathbb{K} \) i.e. \( X: A \rightarrow A \otimes A \) such that one has for all elements \( f, g \in A \) and \( g \in A \),

\[
X(fg) = (Xf)g + \rho(\alpha; \beta) f (Xg) \quad (4)
\]

Without any difficulties it can be obtained that if algebra \( A \) is \( \rho \)-commutative, \( f, g \in A \), and \( X \) is a \( \rho \)-derivation of degree \( \alpha \), then \( fX \) is a \( \rho \)-derivation of degree \( \mathbb{K} \) i.e. \( \mathbb{K} \) and the \( G \)-degree \( \mathbb{K} \).

\[
(fX)(gh) = ((fX)g)h + \rho(\alpha; \beta) f (Xg)h
\]

and \( fX: A \rightarrow A \), \( A \rightarrow A \).

We say that \( X: A \rightarrow A \) is a \( \rho \)-derivation if it has degree equal to \( G \)-degree \( \mathbb{K} \) i.e. \( X: A \rightarrow A \) \( A \rightarrow A \).

\[
X(fg) = (Xf)g + \rho(\alpha; \beta) f (Xg) \quad (5)
\]

It is known that the \( \rho \)-commutator of two \( \rho \)-derivations is again a \( \rho \)-derivation and the linear space of all \( \rho \)-derivations is a \( \rho \)-Lie algebra, denoted by \( \rho \)-Der \( A \).

One verifies immediately that for such an algebra \( A \), \( \rho \)-Der \( A \) is not only a \( \rho \)-Lie algebra but also a left \( A \)-module with the action of \( A \) on \( \rho \)-Der \( A \) defined by

\[
(fX)g = f (Xg) \quad (5)
\]

Let \( M \) be a \( \rho \)-bimodule over a \( \rho \)-commutative algebra \( A \), with the usual properties, in particular \( \rho(\alpha; \beta) \rho(\gamma; \delta) \rho(\beta; \gamma) = \rho(\alpha; \gamma) \rho(\gamma; \delta) \rho(\beta; \gamma) \) for any \( f, g, h \in A \), \( \alpha, \beta, \gamma, \delta \in A \). Then \( M \) is also a right \( A \)-module with the right action on \( M \) defined by

\[
\psi f = \rho(\alpha; \beta) \rho(\gamma; \delta) \rho(\beta; \gamma) \psi f
\]

In fact \( M \) is a bimodule over \( A \), i.e.

\[
f(\psi g) = (f\psi)g \quad (5)
\]

Let \( M \) and \( N \) be two \( G \)-graded \( \rho \)-bimodules over the \( \rho \)-algebra \( A \). Let \( f: M \rightarrow N \) be an \( A \)-bimodule homomorphism of degree \( \alpha \in G \) if \( f: M_{\beta} \rightarrow N_{\alpha + \beta} \) such that \( f(um) = \rho(\alpha; \beta) f(m) \) for any \( \alpha, \beta \in G \) and \( m \in M \). We denote by \( \text{Hom}_A(M; N) \) the space of \( A \)-bimodule homomorphisms of degree \( \alpha \) and by \( \text{Hom}^A_{\alpha}(M; N) = \text{Hom}_A(M; N) \) the space of all \( A \)-bimodule homomorphisms.

3. Differential forms on a \( \rho \)-algebra

\( A \) is a \( \rho \)-algebra as in the previous section. We denote by \( \Omega^1_A(\mathbb{A}) \) the space generated by the elements: \( x_{\alpha}x_{\beta} = \alpha \) with the usual relations:

\[
d(\alpha + \beta) = d(\alpha) + d(\beta); \quad d(\alpha x_{\beta}) = d(\alpha)x_{\beta} + \alpha d(\beta) \quad \text{and} \quad d1 = 0;
\]

where \( 1 \) is the unit of the algebra \( A \).
If we denote by $\Omega^1(A) = \sum_{n} \Omega^n_\alpha(A)$ then $\Omega^1(A)$ is an $A$-bimodule and satisfies the following theorem of universality.

**Theorem 1.** For any $A$-bimodule $M$ and for any derivation $X : A \to M$ of degree $\mathfrak{K}$ there is an $A$-bimodule homomorphism $f : \Omega^1(A) \to M$ of degree $\mathfrak{K}$ such that $X = f \circ d$. The homomorphism is uniquely determined and the corresponding $X \subseteq f$ establishes an isomorphism between $\rho$-$\text{Der}_{\mathfrak{K}}(A;M)$ and $\text{Hom}_{\mathfrak{K}}(\Omega^1(A);M)$.

**Proof.** We define the map $f : \Omega^1(A) \to M$ by $f(\mathfrak{K}aX\phi) = \rho(\mathfrak{K}aX\phi)M$ which transform the usual Leibniz rule for the operator $d$ into the $\rho$-Leibniz rule for the derivation $X$.

Starting from the $A$-bimodule $\Omega^1(A)$ and the $\rho$-algebra $\Omega^0(A) = A$ we build up the algebra of differential forms over $A$.

This algebra will be a new $\overline{\rho}$-algebra

$$\Omega(A) = \sum_{n \geq 0, \mu \geq G} \Omega^n_\alpha(A)$$

graded by the group $G = \mathbb{Z}$ and generated by elements $a \otimes A \otimes \Omega^0_{\mathfrak{K}j}(A)$ of degree $0; \mathfrak{K}j$ and their differentials $da \otimes \Omega^1_{\mathfrak{K}j}(A)$ of degree $(1; \mathfrak{K}j)$.

We will also require the universal derivation $d : \Omega^1(A)$ which can be extended to a $\overline{\rho}$-derivation of the algebra $\Omega(A)$ of degree $(1;0)$ in such a way that $d^2 = 0$ and $\overline{\rho} \cdot G = \rho : \Omega^1(A) \otimes \Omega^1(A) \to \Omega^1(A)$ denote by $\omega \wedge \theta \in \Omega^{n+m}(A)$ the product of forms $\omega \in \Omega^n(A)$ and $\theta \in \Omega^m(A)$ in the algebra $\Omega(A)$. Then

$$d(\omega \wedge \theta) = d\omega \wedge \theta + \overline{\rho}((1;0); \psi; \alpha))d\omega \wedge d\theta;$$

and

$$d^2(\omega \wedge \theta) = \overline{\rho}((1;0); \psi + 1; \alpha))d\omega \wedge d\theta + \overline{\rho}((1;0); \psi; \alpha))d\omega \wedge d\theta = 0 : (8)$$

Hence

$$\overline{\rho}((1;0); \psi + 1; \alpha)) + \overline{\rho}((1;0); \psi; \alpha)) = 0: (9)$$

From these relations it follows that

$$\overline{\rho}((1;0); \psi; \alpha)) = (1)^n \varphi(\alpha);$$

where $\varphi : G \to U(\mathfrak{k})$ is the group homomorphism $\varphi(\alpha) = \overline{\rho}((1;0); \theta; \alpha))$. From the properties of the cocycle $\rho$,

$$\overline{\rho}(\psi; \alpha); \psi; \beta)) = (1)^m \varphi^m(\alpha) \varphi^n(\beta) \rho(\alpha; \beta)$$

for any $n; m \in \mathbb{Z}$ and $\alpha; \beta \in G$.

**PROPOSITION 1.**

Let $A$ be a $\rho$-algebra with the cocycle $\rho$. Then any cocycle $\overline{\rho}$ on the group $G$ with the conditions $\overline{\rho} \cdot G = \rho$ and $\overline{\rho}$ are given by $\varphi$ for some homomorphism $\varphi : G \to U(\mathfrak{k})$. 

We will denote below \( \Omega^\bullet(\mathfrak{g};\varphi) \) or simply \( \Omega^\bullet(\mathfrak{g}) \) the \( G \)-graded algebra of forms with the cocycle \( \varphi \) and the derivation \( d = d_\varphi \) of degree \( (1,0) \).

Therefore for any \( \rho \)-algebra \( A \), a group homomorphism \( \varphi : G \rightarrow U(\mathbb{k}) \) and an element \( \alpha \in G \), we have the complex:

\[
0 \to A_\alpha^0 \to \Omega_\alpha^1(\mathfrak{g};\varphi) \to \Omega_\alpha^2(\mathfrak{g};\varphi) \to \cdots
\]

The cohomology of this complex term \( \Omega_\alpha^n(\mathfrak{g};\varphi) \) is denoted by \( H^\alpha_\varphi(\mathfrak{g};\varphi) \) and will be called as the de Rham cohomology of the \( \rho \)-algebra \( A \).

**PROPOSITION 2.**

Let \( f : A \to B \) be a homomorphism of degree \( \alpha \in G \) between the \( G \)-graded \( \rho \)-algebras. There is a natural homomorphism \( \Omega(f) : \Omega^\bullet(A) \to \Omega^\bullet(B) \) which in degree \( n \) is \( \Omega(f)_n : \Omega^n(A) \to \Omega^n(B) \) and has the \( G^\alpha \)-degree \( \alpha \cdot (n+1) \alpha \) given by

\[

\Omega(f)_n(\theta) = f(\theta)
\]

**4. Frölicher–Nijenhuis bracket of \( \rho \)-algebras**

**4.1 Derivations**

Here we present the Frölicher–Nijenhuis calculus over the algebra of forms defined in the previous section.

Denote by \( \Der_{\mathfrak{g};\alpha}(\mathfrak{g};\varphi)(\mathfrak{g}) \) the space of derivations of degree \( \mathfrak{g} \cdot \alpha \) i.e. an element \( D \in \Der_{\mathfrak{g};\alpha}(\mathfrak{g};\varphi)(\mathfrak{g}) \) satisfies the relations:

1. \( D \) is linear,
2. the \( G^\alpha \)-degree of \( D \) is \( \mathfrak{g} \cdot \alpha = \mathfrak{g} \cdot \alpha \), and
3. \( D(\rho \cdot \theta) = D \rho \cdot \theta + [D, \rho](\mathfrak{g} \cdot \alpha) \cdot \theta \) for any \( \rho \in \mathfrak{g} \), \( \theta \in \Omega^n(\mathfrak{g};\varphi) \).

**Theorem 2.** The space \( \mathfrak{g} \)-Lie algebra with \( \mathfrak{g} \)-bracket \( [\mathfrak{g} \cdot \alpha, \mathfrak{g} \cdot \alpha] = \mathfrak{g} \cdot \alpha \)

**4.2 Fields**

Let us denote by \( L : \Hom_{\mathfrak{g};\alpha}(\mathfrak{g};\varphi)(\mathfrak{g};\alpha) \to \mathfrak{g} \)-Lie algebra with \( \mathfrak{g} \)-bracket \( [\mathfrak{g} \cdot \alpha, \mathfrak{g} \cdot \alpha] = \mathfrak{g} \cdot \alpha \) which is given by

\[
L([X,Y]) = [L_X, L_Y] = L_X L_Y - \rho([X,Y]) L_Y L_X
\]

and will be referred to as the \( \mathfrak{g} \)-Lie bracket of fields.

**Lemma 1.** Each field \( \alpha \in \mathfrak{g} \cdot \alpha \) is by definition an \( \alpha \)-bimodule homomorphism \( \Omega^\bullet(\mathfrak{g};\varphi)(\mathfrak{g}) \) and it prolongs uniquely to a graded \( \mathfrak{g} \)-derivation \( f(X) = f_X : \Omega^\bullet(\mathfrak{g};\varphi)(\mathfrak{g}) \) of degree \( \mathfrak{g} \cdot \alpha \) by

\[
f_X(\theta) = f(\theta)
\]
\[ j_x (a) = 0 \quad \text{for} \quad a \in A = \Omega^0 (A); \]
\[ j_x (\omega) = X (\omega) \quad \text{for} \quad \omega \in \Omega^1 (A) \]

and

\[ j_x (\omega_1 \wedge \omega_2 \wedge \xi) = \sum_{i=1}^{k-1} \pi (1; \xi) \; i \; \sum_{j=1}^{k-1} \varphi_0 (j) \omega_1 \wedge \omega_2 \wedge \xi \]

for any \( \omega_1 \in \Omega^1 (A_{\alpha \beta}) \). The \( \pi \)-derivation \( j_x \) is called the contraction operator of the field \( X \).

**Proof.** This is an easy computation.

With some abuse of notation we also write \( \omega (X) = X (\omega) = j_x (\omega) \) for \( \omega \in \Omega^1 (A) \) and \( X \times X (A) = \text{Hom}_A^1 \Omega^1 (A) \).

4.2.1 **Algebraic derivations:** A \( \pi \)-derivation \( D : \text{Der}_{\#_{\pi}} \Omega (A) \) is called algebraic if \( D_{\Omega^0 (A)} = 0 \). Then \( D (a \omega) = \pi (\#_{\pi} \alpha) \; a \Omega (\omega) \) and \( D (\omega a) = D (\omega) a \) for any \( a \in A \), \( \#_{\pi} \) and \( \omega \in \Omega (A) \). It results that \( D \) is an \( A \)-bimodule homomorphism. We denote by \( \text{Hom}_{\text{alg}} \Omega (A) \) the space of \( A \)-bimodule homomorphisms from \( \Omega (\#_{\pi} \alpha) \) to \( \Omega (\#_{\pi} \alpha) \) of degree \( \#_{\pi} \alpha \). Then an algebraic derivation \( D \) of degree \( \#_{\pi} \alpha \) is from \( \text{Hom}_{\alpha} \Omega (A) \). We denote by \( \text{Der}_{\#_{\pi}}^{\text{alg}} \Omega (A) \) the space of all \( \pi \)-algebraic derivations of degree \( \#_{\pi} \alpha \) from \( \Omega (A) \). Since \( D \) is a \( \pi \)-derivation, \( D \) has the following expression on the product of 1-forms \( \omega_1 \in \Omega^1 (A) \) and \( \omega_2 \in \Omega^1 (A) \):

\[ D (\omega_1 \wedge \omega_2 \wedge \xi) = \sum_{i=1}^{k-1} \pi (1; \xi) \; i \; \sum_{j=1}^{k-1} \varphi_0 (j) \omega_1 \wedge \omega_2 \wedge \xi \]

and the derivation \( D \) is uniquely determined by its restriction on \( \Omega^1 (A) \),

\[ K := D_{\Omega^1 (A)} \cdot \text{Hom}_A (\Omega^1 (A) \cdot \Omega^1 (A)) \]  \hspace{1cm} (13)

We write \( D = j (K) = j_K \) to express this dependence. Note that \( j_K (\omega) = K (\omega) \) for \( \omega \in \Omega^1 (A) \). Next we will use the following notations:

\[ \Omega^1_{\#_{\pi} \alpha} (A) := \text{Hom}_A (A; \Omega^1 (A)) \]
\[ \Omega^1_{\#_{\pi} \alpha} (A) := \text{Hom}_A (A; \Omega^1 (A)) \]

Elements of the space \( \Omega^1_{\#_{\pi} \alpha} (A) \) will be called field-valued \( \#_{\pi} \alpha \)-forms.
4.2.2 Nijenhuis bracket:

**Theorem 3.** The map \( j : \Omega^1_{k+1;\mathfrak{K}} (A) \rightarrow \mathcal{D}\text{-Der}^{\text{alg}}_{k;\mathfrak{K}} \Omega (A) \), \( K \ni j_K \) defined by

\[
j_K (\omega_1 \wedge \omega_2 \wedge \ldots \wedge \ell) = \sum_{i=1}^k \mathfrak{K} (k + 1; \alpha_i) \cdot i \sum_{j=1}^1 \mathfrak{D}_j \omega_j
\]

is an isomorphism and satisfies the following properties:

1) \( j_K : \Omega^1_{k;\mathfrak{K}} (A) \rightarrow \Omega^1_{k+1;\mathfrak{K}} (A) \).
2) \( j_K (\omega \wedge \theta) = j_K \omega \wedge \theta + \mathcal{D} (k; \alpha \cdot \mathfrak{K} (1; \beta)) \omega \wedge j_K \theta \) for any \( \theta \in \Omega^1_{\mathfrak{K}} (A) \).
3) \( j_K (\omega) = 0 \) and \( j_K (\omega) = K (\omega) \) for any \( \omega \in \Omega^1_{2;\mathfrak{K}} (A) \).

The module of \( \mathcal{D}\text{-Der}^{\text{alg}}_{k;\mathfrak{K}} \Omega (A) \) is isomorphic to the Frölicher–Nijenhuis bracket. By definition, the Nijenhuis bracket of the elements \( K \in \text{Hom}_\mathfrak{K} (\Omega^1 (A); \Omega^1_{k+1;\mathfrak{K}} (A)) \) and \( \mathcal{D} \in \text{Hom}_\mathfrak{K} (\Omega^1 (A); \Omega^1_{k;\mathfrak{K}} (A)) \) is given by the formula

\[
[K;L]^A = j_K \left( \mathcal{D} (k; \alpha \cdot \mathfrak{K} (1; \beta)) \right) j_L \quad K
\]

or

\[
[K;L]^A (\omega) = j_K (\mathcal{L} (\omega)) \quad (1)^i \phi^j (\alpha) \phi^k (\alpha) \mathfrak{D} (\alpha; \beta) j_L K (\omega)
\]

for all \( \omega \in \Omega^1 (A) \):

### 4.2.3 The Frölicher–Nijenhuis bracket:

The exterior derivative \( d \) is an element of \( \mathcal{D}\text{-Der}^{\text{alg}}_{k;\mathfrak{K}} \Omega (A) \). In the view of the formula \( L_X = [j_X \mu d] \) for fields \( X \) we define \( K \in \Omega^1_{k;\mathfrak{K}} (A) \) and \( \mathcal{D} \in \mathcal{D}\text{-Der}^{\text{alg}}_{k;\mathfrak{K}} \Omega (A) \) by \( L_K := \{ j_K \mu d \} \). Then the mapping \( L : \Omega^1_{\mathfrak{K}} \rightarrow \mathcal{D}\text{-Der}^{\text{alg}}_{k;\mathfrak{K}} \Omega (A) \) is injective by the universal property of \( \Omega^1_{\mathfrak{K}} (A) \), since \( L_K (\omega) = j_K (\mu d) = K (\mu d) \) for all \( \omega \in \Omega^1_{\mathfrak{K}} (A) \).

**Theorem 4.** For any \( \mathcal{D}\text{-Der}^{\text{alg}}_{k;\mathfrak{K}} \Omega (A) \), there are unique homomorphisms \( \Omega^1_{k;\mathfrak{K}} (A) \) and \( L \in \Omega^1_{k+1;\mathfrak{K}} (A) \) such that

\[
D = L_K + j_L
\]

We have \( L = 0 \) if and only if \( \mathcal{D} \mu d \) = 0. \( D \) is algebraic if and only if \( K = 0 \).
Proof. The map $D : a \mapsto D (a)$ is a $\rho$-derivation of degree $\alpha$ so $D \cdot : A ! \Omega (A) \Omega (A)$ has the form $K \cdot d$ for an unique $K \in \Omega (A)$. The defining equation for $K$ is $D (a) = j_k d a = L (K (a))$ for $a \in A$. Thus $D \cdot L (K)$ is an algebraic derivation, so $D \cdot L = j_L$ for an unique $L \in \Omega (A)$.

By the Jacobi identity, we have

$$0 = [j_K ; [L d]] = [[j_K ; L d] + [L (j_k) d]) ; (l (j_0)) ; [l ; [j_K ; L d]]$$

so $L = 0$. It follows that $D (L) = [j_L ; L d] = L (L)$ and using the injectivity of $L$ results that $L = 0$.

Let $K \in \Omega (A)$ and $L \in \Omega (A)$. Definition of the $\rho$-Lie derivation results in $[L (K) ; [L_2 ; d]] = 0$ and using the previous theorem results that is a unique element which is denoted by $K \cdot L \in \Omega (A)$ such that

$$[L (K) ; [L_2 ; d]] = L (K (L)) \tag{16}$$

and this element will be the denoted by the abstract Frölicher–Nijenhuis of $K$ and $L$.

**Theorem 5.** The space $\Omega (A) = \Omega (A)$ with the usual grading and the Frölicher–Nijenhuis is a $\rho$-graded Lie algebra. $L : \Omega (A) ; \rho ; \rho$ $\rho$-Der $\Omega (A)$ is an injective homomorphism of $\rho$-graded Lie algebras. For fields in $\text{Hom} (A, \Omega (A) ; A)$ the Frölicher–Nijenhuis coincides with the bracket defined in [2].

4.3 Naturality of the Frölicher–Nijenhuis bracket

Let $f : A \mapsto B$ be an homomorphism of degree 0 between the $\rho$-graded $\rho$-algebras $A$ and $B$. Two forms $K \in \Omega (A)$ and $L \in \Omega (A)$ are $f$-related if we have

$$K \cdot L (f) = L (K (f))$$

where $\Omega (f) : \Omega (A) \mapsto \Omega (B)$ is the homomorphism from $\Omega (A)$ induced by $f$.

**Theorem 6.**

1. If $K$ and $K^0$ are $f$-related as above then $j_k \cdot K^0 \Omega (f) = \Omega (f) ; [j_k \cdot K^0 \Omega (A)]$.
2. If $j_k \cdot K (f) = \Omega (f) ; j_k \cdot K (A)$, then $K$ and $K^0$ are $f$-related, where $d (A) \Omega (A)$ is the space of exact 1-forms.
3. If $K_j$ and $K_j^0$ are $f$-related for $j = 1 ; 2$ then $j_k \cdot K_j$ and $j_k \cdot K_j^0$ are $f$-related and also $\Omega (K_j ; K_j^0)$ and $\Omega (K_j ; K_j^0)$ are $f$-related.
4. If $K$ and $K^0$ are $f$-related then $L \cdot K^0 \Omega (f) = \Omega (f) ; L (K (A)) = \Omega (B)$.
5. If $L \cdot K^0 \Omega (f) ; [L \cdot K (A)] = \Omega (f)$, then $K \cdot K^0$ and $K \cdot K^0$ are $f$-related.
6. If $K_j$ and $K_j^0$ are $f$-related for $j = 1 ; 2$ then their Frölicher–Nijenhuis brackets $[K_j ; K_j^0]$ and $[K_j^0 ; K_j^0]$ are $f$-related.

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