DET-NORM ON FUZZY MATRICES

A. Nagoor Gani\textsuperscript{1,}\textsuperscript{§}, A.R. Manikandan\textsuperscript{2}

\textsuperscript{1,2}PG and Research Department of Mathematics
Jamal Mohamed College
Tiruchirappalli, 620 020, INDIA

Abstract: In this paper we introduce fuzzy det-norm ordering with fuzzy matrices using the structure of $M_n(F)$, the set of $(n \times n)$ fuzzy det-norm ordering with fuzzy matrices is introduced. From this row and column, determinant of the fuzzy norm has been obtained by imposing an equivalence relation on $M_n(F)$. We know that the comparability relation on fuzzy matrices is a partial ordering. We prove that det-norm ordering is a partitions ordering on the set of all idempotent matrices in $M_n(F)$. We begin with the det-norm ordering on fuzzy matrices as analogue of the ordering on real matrices. Several properties of these orderings are derived. Discuss their relationship between these ordering with det-norm ordering and Also we introduce the concept of Fuzzy norm and partitions of $M_n(F)$, Properties of fuzzy det-norm ordering.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh\cite{8} in 1965. Jian Miao

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\textsuperscript{§}Correspondence author
Chen. [2] introduced the Fuzzy matrix partial ordering and generalized inverse. Bertoiuzza [1] introduced the distributivity of t-norm and t-conorms. In 1995, Ragab M. Z. and Emam E. G [7] introduced the determinant and adjoint of a square fuzzy matrix. Meenakshi A.R. and Cokilavany R. [3] introduced the concept of fuzzy 2-normed linear spaces. Nagoorgani A. and Kalyani G. [5] Introduced the Binormed sequences in fuzzy matrices. Nagoorgani A. and Kalyani G. [6] Introduced the Fuzzy matrix m-ordering. ZHOU Min - na [9] Introduced the Characterizations of the Minus Ordering in Fuzzy Matrix Set. Nagoorgani A. and Manikandan A. R. [4] introduced the properties of fuzzy det-norm matrices.

In this paper, we introduce the concept of fuzzy det-norm ordering with fuzzy matrices. The purpose of the introduction is to explain det-norm ordering with fuzzy matrices and partitions of \( M_n(F) \). In Section 2, fuzzy det-norm ordering with fuzzy matrices is introduced in \( M_n(F) \). In Section 3, Properties of det-norm ordering with fuzzy matrices.

2. Preliminaries

We consider \( F = [0, 1] \) the fuzzy algebra with operation \([+,,\cdot]\) and the standard order ” \( \leq \) ” where \( a + b = \max a, b, a \cdot b = \min a, b \) for all \( a, b \) in \( F \). \( F \) is a commutative semi-ring with additive and multiplicative identities 0 and 1 respectively. Let \( M_{mn}(F) \) denote the set of all \( m \times n \) fuzzy matrices over \( F \). In short \( M_n(F) \) is the set of all fuzzy matrices of order \( n \), define ‘+’ and scalar multiplication in \( M_n(F) \) as \( A + B = [a_{ij} + b_{ij}] \), where \( A = [a_{ij}] \) and \( B = [b_{ij}] \) and \( cA = [ca_{ij}] \), where \( c \) is in \([0, 1]\), with these operations \( M_n(F) \) forms a linear space.

3. Fuzzy Det-Norm and Partitions of \( M_n(F) \)

To analysis more properties of \( M_n(F) \) we introduce the concept of norm in \( M_n(F) \) and thus we have defined for every \( A \) in \( M_n(F) \) a non-negative quantity say det-norm is defined in the following way.

**Definition 1.** An \( m \times n \) matrix \( A = [a_{ij}] \) whose components are in the unit interval \([0, 1]\) is called a fuzzy matrix.

**Definition 2.** The determinant \( |A| \) of an \( m \times n \) fuzzy matrix \( A \) is defined as follows; \( |A| = \Sigma_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \) Where \( S_n \) denotes the symmetric group of all permutations of the indices \((1, 2, \cdots, n)\)
**Definition 3.** For every $A$ in $M_n(F)$ the det-norm of $A$ is defined as $\|A\| = det[A]$, where $A = [a_{ij}]$.

**Definition 4.** A matrix $A$ in $M_n(F)$ is called idempotent if $A^2 = A$ or $\|A^2\| = det[A]$, where $A = [a_{ij}]$.

**Example 1.** If $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$, then

$$A = [0.5] \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} + 0.4 \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} + 0.6 \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{bmatrix},$$

$A = 0.5[0.1 + 0.2] + 0.4[0.3 + 0.4] + 0.6[0.2 + 0.1] = 0.2 + 0.4 + 0.2, A = 0.4$.

If $A^2 = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$, then

$$A^2 = [0.5] \begin{bmatrix} 0.3 & 0.4 \\ 0.4 & 0.5 \end{bmatrix} + 0.4 \begin{bmatrix} 0.4 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} + 0.5 \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}$$

$A^2 = 0.5[0.3 + 0.4] + 0.4[0.4 + 0.4] + 0.5[0.4 + 0.3] = 0.4 + 0.4 + 0.4, A^2 = 0.4$.

Therefore $A^2 = A = 0.4$.

**Definition 5.** For all $A$ in $M_n(F)$ define:

$A\{1\} = \{x \in M_n(F)/\|x\| > \|A\|\}$,

$A\{2\} = \{x \in M_n(F)/\|x\| < \|A\|\}$,

$A\{3\} = \{x \in M_n(F)/\|x\| = \|A\|\}$,

$A\{4\} = \{x \in M_n(F)/AXA = A\}$,

$A\{5\} = \{x \in M_n(F)/XAX = X\}$.

Clearly $M_n(F) = A\{1\} \cup A\{2\} \cup A\{3\}$. The set $A\{1\}$ is called as det-superior to $A$ and $A\{2\}$ det-inferior to $A$. Clearly $A\{3\}$ is det-equivalent to $A$. $A\{4\}$ and $A\{5\}$ are known as the sets of inner and outer inverses of $A$.

**Example 2.** If $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$ and $x = \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$, then

$$\|A\| = [0.5] \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} + 0.4 \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} + 0.6 \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{bmatrix}$$
\[ \|A\| = 0.5[0.1 + 0.2] + 0.4[0.3 + 0.4] + 0.6[0.2 + 0.1] = 0.2 + 0.4 + 0.2, \|A\| = 0.4, \]
\[ \|x\| = [0.7] \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.7 \end{bmatrix} + 0.4 \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} + 0.5 \begin{bmatrix} 0.4 & 0.8 \\ 0.1 & 0.6 \end{bmatrix} \]
\[ \|x\| = 0.7[0.7 + 0.2] + 0.4[0.4 + 0.1] + 0.5[0.4 + 0.1] = 0.7 + 0.4 + 0.4, \|x\| = 0.7. \]
Therefore \( \|x\| > \|A\| \);
\[ \|x\| = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix}, \]
\[ \|x\| = 0.3[0.3 + 0.2] + 0.5[0.3 + 0.2] + 0.1[0.4 + 0.4] = 0.3 + 0.3 + 0.1, \|x\| = 0.3. \]
Therefore \( \|x\| < \|A\| \);
\[ \|x\| = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.3 \\ 0.5 & 0.3 & 0.4 \end{bmatrix}, \]
\[ \|x\| = 0.6[0.4 + 0.3] + 0.4[0.4 + 0.3] + 0.2[0.3 + 0.5] = 0.4 + 0.4 + 0.2, \|x\| = 0.4. \]
Therefore \( \|x\| = \|A\| \).

**Theorem 1.** For each \( A \) in \( M_n(F) \) the following results hold true:

(i) If \( X \in A\{i\} \) then \( X^T \) is also in \( A\{i\} \) for \( i = 1, 2, 3 \) where \( X^T \) is the transpose of \( X \).

(ii) If \( A_1 \in A\{1\} \), \( A_2 \in A\{2\} \), \( A_3 \in A\{3\} \) then \( \|A_1 + A_2 + A_3\| = det[A_1] \).

(iii) \( \|A_1 A_2 A_3\| = det[A_2] \).

(iv) \( A^T \in A\{3\} \) for all \( A \) in \( M_n(F) \).

**Proof.** (i) \( \|X\| = \|X^T\| \), since \( \|X\| = det[A] \) for all \( X \) in \( A\{3\} \).

(ii) \( \|A_1\| > \|A\|, \|A_2\| < \|A\|, \|A_3\| = \|A\| \). Therefore \( \|A_1 + A_2 + A_3\| = det[A_1 + A_2 + A_3] = det[A_1] + det[A_2] + det[A_3] = det[A_3] = A_3 \).

(iii) \( \|A_1 A_2 A_3\| = det[A_1 A_2 A_3] = det[A_1] + det[A_2] + det[A_3] = det[A_2] = \|A_2\| \).

(iv) \( \|A\| = \|A^T\| \) or \( det[A] = det[A^T] \). Therefore for all \( A \) in \( M_n(F) \), \( A^T \in A\{3\} \). \( \square \)
Example 3. (i) \( \|X\| = \|X^T\| \) for all \( X \) in \( A\{i\} \), where \( i = 1, 2, 3 \).

Case (i): \( A\{1\} \), \( x = \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix} \), \( \|x\| = 0.7 \), \( x^T = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.4 & 0.8 & 0.6 \\ 0.5 & 0.2 & 0.7 \end{bmatrix} \),
\( \|x^T\| = 0.7[0.7 + 0.2] + 0.4[0.4 + 0.5] + 0.1[0.2 + 0.5] = 0.7 + 0.4 + 0.1 \), \( \|x^T\| = 0.7 \).

Therefore \( \|x\| = \|x^T\| = 0.7 \).

Case (i): \( A\{2\} \), \( \|x\| = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \end{bmatrix} \), \( \|x\| = 0.3 \), \( x^T = \begin{bmatrix} 0.3 & 0.4 & 0.4 \\ 0.5 & 0.7 & 0.5 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \),
\( \|x^T\| = 0.3[0.3 + 0.2] + 0.4[0.3 + 0.1] + 0.4[0.2 + 0.1] = 0.3 + 0.3 + 0.2 \), \( \|x^T\| = 0.3 \).

Therefore \( \|x\| = \|x^T\| = 0.3 \).

Case (i): \( A\{3\} \), \( \|x\| = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.3 \\ 0.5 & 0.3 & 0.4 \end{bmatrix} \), \( \|x\| = 0.4 \), \( x^T = \begin{bmatrix} 0.6 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.3 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} \),
\( \|x^T\| = 0.6[0.4 + 0.3] + 0.4[0.4 + 0.2] + 0.5[0.3 + 0.2] = 0.4 + 0.4 + 0.3 \), \( \|x^T\| = 0.4 \).

Therefore \( \|x\| = \|x^T\| = 0.4 \).

Example 4. Let \( A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \), \( A_1 = \begin{bmatrix} 0.2 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix} \), \( A_2 = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 \\ 0.2 & 0.6 & 0.3 \end{bmatrix} \) and \( A_3 = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 0.2 & 0.6 & 0.4 \\ 0.3 & 0.5 & 0.4 \end{bmatrix} \).

\( \|A\| = 0.4 \)
\( \|A_1\| = 0.2[0.5 + 0.6] + 0.3[0.7 + 0.6] + 0.6[0.6 + 0.5] 
\quad = 0.2 + 0.3 + 0.6 \)
\( \|A_1\| = 0.6 \)
\( \|A_2\| = 0.3[0.3 + 0.2] + 0.6[0.1 + 0.2] + 0.7[0.1 + 0.2] 
\quad = 0.3 + 0.2 + 0.2 \)
\( \|A_2\| = 0.3 \)
\( \|A_3\| = 0.4[0.4 + 0.4] + 0.1[0.2 + 0.3] + 0.3[0.2 + 0.3] 
\quad = 0.4 + 0.1 + 0.3 \)
\( \|A_3\| = 0.4 \)
\[ A_1 + A_2 + A_3 = \begin{bmatrix} 0.4 & 0.6 & 0.7 \\ 0.7 & 0.6 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix} \]

\[ \|A_1 + A_2 + A_3\| = 0.4[0.6 + 0.6] + 0.6[0.7 + 0.6] + 0.7[0.6 + 0.6] = 0.4 + 0.6 + 0.6 \]

\[ \|A_1 + A_2 + A_3\| = 0.6 \]

\[ \|A_1 + A_2 + A_3\| = \text{det}[A_1] + \text{det}[A_2] + \text{det}[A_3] = 0.6 + 0.3 + 0.4 = 0.6 = \|A_2\| \]

\[ A_1 A_2 A_3 = \begin{bmatrix} 0.2 & 0.6 & 0.4 \\ 0.3 & 0.6 & 0.4 \\ 0.3 & 0.6 & 0.4 \end{bmatrix} \]

\[ \|A_1 A_2 A_3\| = 0.2[0.4 + 0.4] + 0.6[0.3 + 0.3] + 0.4[0.3 + 0.3] = 0.2 + 0.3 + 0.3 \]

\[ \|A_1 A_2 A_3\| = 0.3 \]

\[ \|A_1 A_2 A_3\| = \text{det}[A_1]\text{det}[A_2]\text{det}[A_3] = [0.6][0.3][0.4] = 0.3 = \|A_2\| \]

(iv) \[\|A\| = \|A_T\|.\]

Therefore \[\|A\| = \|A_T\| = 0.4\] or \[\text{det}[A] = \text{det}[A^T]\] for all \(A \in M_n(F), A^T \in A(3)\).

**Theorem 2.**

(i) For all \(X \in A\{4\}\), \(\|A\| \leq \|X\|\).

(ii) For all \(X \in A\{5\}\), \(\|X\| \leq \|A\|\).

Further for all \(X\) in \(A\{4\} \cap A\{5\}\) the matrices \(AX\) and \(XA\) are idempotent.

**Proof.** If \(X \in A\{4\}\), then \(AXA = A\). Therefore

\[\|AXA\| = \text{det}[A] \Rightarrow \text{det}[A]\text{det}[X]\text{det}[A] = \text{det}[A] = \|A\| \Rightarrow \|A\| \leq \|X\|.\]

(i) If \(X \in A\{5\}\), then \(XAX = X\). Therefore

\[\|XAX\| = \text{det}[X] \Rightarrow \text{det}[X]\text{det}[A]\text{det}[X] = \text{det}[X] = \|X\| \Rightarrow \|X\| \leq \|A\|.\]

(ii) If \(X\) in \(A\{4\} \cap A\{5\}\), then

\[AXA = A,\] (1)
\[ XAX = X, \tag{2} \]

\[ XAXA = XA \Rightarrow (XA)^2 = XA \text{ from (1) and } AXAX = Ax \Rightarrow (AX)^2 = AX \text{ from (2)}. \]

Therefore \(XA\) and \(AX\) are idempotent. \(\square\)

**Example 5.** (i) If \(X\) in \(A\{4\}\), then \(\|AXA\| = \|A\| \leq \|X\|\)

\[
A = \begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\text{ and } x = \begin{bmatrix}
0.4 & 0.8 & 0.2 \\
0.1 & 0.6 & 0.7
\end{bmatrix}
\]

\[
\|AXA\| = \begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.4 & 0.5 \\
0.1 & 0.6 & 0.7
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\]

\[
\|AXA\| = 0.4
\]

\[
\Rightarrow \|A\| \leq \|X\|
\]

If \(X\) in \(A\{5\}\), then \(\|XAX\| = \|X\| \leq \|A\|\)

\[
A = \begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\text{ and } \|x\| = \begin{bmatrix}
0.4 & 0.7 & 0.2 \\
0.1 & 0.5 & 0.3
\end{bmatrix}
\]

\[
\|XAX\| = \begin{bmatrix}
0.3 & 0.5 & 0.1 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
0.3 & 0.5 & 0.1 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\]

\[
\|XAX\| = 0.4 \Rightarrow \|X\| \leq \|A\|.\]

(ii) If \(X\) in \(A\{5\}\) \(XAXA = XA \Rightarrow (XA)^2 = XA\)

\[
XAXA = \begin{bmatrix}
0.3 & 0.5 & 0.1 \\
0.4 & 0.7 & 0.2 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
0.3 & 0.5 & 0.1 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\]

\[
XAXA = 0.4 \text{ } XA = \begin{bmatrix}
0.4 & 0.7 & 0.2 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
0.3 & 0.5 & 0.1 \\
0.4 & 0.5 & 0.3
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.2 & 0.5
\end{bmatrix}
\]

\[
XA = 0.4, \ (XA)^2 = XA = 0.4.
\]

If \(X \in A\{4\}\)

\[
AXAX = \begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.4 & 0.5 \\
0.4 & 0.8 & 0.2 \\
0.1 & 0.6 & 0.7
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\]

\[
AXAX = 0.4
\]

\[
AX = \begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.4 & 0.5 \\
0.4 & 0.8 & 0.2 \\
0.1 & 0.6 & 0.7
\end{bmatrix}
\]
\[(AX)^2 = AX = 0.4.\]
Therefore \(XA\) and \(AX\) are idempotent.

4. Properties of Det-Norm Ordering with Fuzzy Matrices

Definition 6. The det-norm ordering \(A \leq B\) in \(M_n(F)\) is defined as \(A \leq B \iff \|A\| \leq \|B\|\) Or \(A \leq B \iff \text{det}[A] \leq \text{det}[B]\)

Example 6. \(A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}\) and \(B = \begin{bmatrix} 0.7 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}\), \(\|A\| = 0.4, \|B\| = 0.6\)
\(\|A\| = 0.7[0.5 + 0.6] + 0.6[0.4 + 0.6] + 0.8[0.4 + 0.6] = 0.6 + 0.6 + 0.6, \|B\| = 0.6.\)
Therefore \(A \leq B \iff \|A\| \leq \|B\|\).

Theorem 3. The det-ordering is not a partial ordering.

Proof. (i) \(\text{det}[A] \leq \text{det}[B]\) for all \(A \in M_n(F)\). Hence \(A \leq B\).
Therefore reflexivity is true.

(ii) \(A \leq B \Rightarrow \|A\| \leq \|B\|, B \leq A \Rightarrow \|B\| \leq \|A\|, A \leq B\) and \(B \leq A \Rightarrow \|A\| = \|B\|\).
But \(\|A\| = \|B\|\) does not imply \(A = B\).
Therefore anti symmetry is not true.

(iii) \(A \leq B, B \leq C \Rightarrow A \leq C\) for all \(A, B, C \in M_n(F)\). For \(A \leq B = \|A\| \leq \|B\|\)
\[ B \leq C = \|B\| \leq \|C\|, \]
\[ A \leq C = \|A\| \leq \|C\|. \]
Therefore transitivity is true.
Thus the det-ordering is not a partial ordering in \(M_n(F)\). \(\square\)

Example 7. \(A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}\), \(B = \begin{bmatrix} 0.7 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}\) and \(C = \begin{bmatrix} 0.8 & 0.6 & 0.8 \\ 0.5 & 0.9 & 0.7 \\ 0.8 & 0.6 & 0.8 \end{bmatrix}\), \(\|A\| = 0.4, \|B\| = 0.6, \|C\| = 0.8[0.8 + 0.6] + 0.6[0.5 + 0.7] + 0.8[0.5 + 0.8] = 0.8 + 0.6 + 0.8, \|C\| = 0.8.\)
\(\|A\| \leq \|A\|\) for all \(A \in M_n(F), A \leq A\).
Therefore reflexivity is true.
(ii) \( A \leq B = \|A\| \leq \|B\| = 0.4 \leq 0.6, \ B \leq A = \|B\| \leq \|A\| = 0.6 \preceq 0.4, \)
\( A \leq B \) and \( B \leq A \) \( \Rightarrow \|A\| = \|B\|. \)
But \( \|A\| = \|B\| \) does not imply \( A = B. \)
Therefore anti symmetry is not true.

(iii) \( A \leq B = \|A\| \leq \|B\| = 0.4 \leq 0.6, \ B \leq C = \|B\| \leq \|C\| = 0.6 \preceq 0.8, \)
\( A \leq C = \|A\| \leq \|C\| = 0.4 \preceq 0.8. \)
Therefore transitivity is true.
Thus the det-ordering is not a partial ordering in \( M_n(F) \).

**Theorem 4.** If \( A \leq B \), then:

(i) \( A^T \leq B^T \);

(ii) \( AB^T \leq BB^T, B^TA \leq B^TB \);

(iii) \( A^TA \leq B^TB, A^TB \leq BB^T, A^n \leq B^n \) for any positive integer \( n \).

**Proof.** (i) \( \|A\| = \det[A^T], \|B\| = \det[B^T]. \)
Therefore \( \|A\| \leq \|B\| \Rightarrow \det[A^T] \leq \det[B^T], \) i.e. \( A \leq B \Rightarrow A^T \leq B^T. \)

(ii) \( \det[AB^T] \leq \det[A]\det[B^T] = \det[A]\det[B] = \det[A]. \) Since \( A \leq B, \)
\[ \det[BB^T] = \det[B]\det[B^T] = \det[B]\det[B] = \det[B], \]
\( A \leq B \Rightarrow \det[A] \leq \det[B] \Rightarrow \det[AB^T] \leq \det[BB^T]. \)
Similarly \( A \leq B \Rightarrow B^TA \leq B^T B. \)

(iii) \( \det[A^TA] = \det[A^T]\det[A] = \det[A]\det[A] = \det[A], \)
\[ \det[B^TB] = \det[B^T]\det[B] = \det[B]\det[B] = \det[B], \]
\( A \leq B \Rightarrow \det[A] \leq \det[B] \Rightarrow \det[A^TA] \leq \det[B^TB] \Rightarrow A^T A \leq B^T B. \)
Similarly \( A \leq B \Rightarrow A^T A \leq B^T B. \)

(iv) \( \det[A^n] = \det[A..n\times] = \det[A] \)
\( \det[A^n] = \det[A..n\times] = \det[A] \)
\( \det[B^n] = \det[B..n\times] = \det[B] \)
\( \det[B^n] = \det[B..n\times] = \det[B] \)
\( A \leq B \Rightarrow \det[A] \leq \det[B] \Rightarrow \det[A^n] \leq \det[B^n]. \)
Therefore \( A^n \preceq B^n \) for any positive integer \( n. \)

**Example 8.** \( A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5 \end{bmatrix}, \) and \( B = \begin{bmatrix} 0.7 & 0.6 & 0.8 \\
0.4 & 0.5 & 0.6 \\
0.8 & 0.6 & 0.7 \end{bmatrix} \)
\( \|A\| = 0.4 \) and \( \|B\| = 0.6 \)
\[
A^T = \begin{bmatrix}
0.5 & 0.3 & 0.5 \\
0.4 & 0.1 & 0.2 \\
0.6 & 0.4 & 0.5
\end{bmatrix}
\]
\[
\|A^T\| = 0.5[0.1 + 0.2] + 0.3[0.4 + 0.2] + 0.5[0.4 + 0.1] = 0.2 + 0.3 + 0.4
\]
\[
\|B^T\| = 0.6 + 0.4 + 0.6
\]
\[
\|A^T\| = 0.6
\]
(i) \(\|A\| = det[A^T] = 0.4, \|B\| = det[B^T] = 0.6\).

Therefore \(\|A\| \leq \|B\| \Rightarrow det[A^T] \leq det[B^T] = 0.4 \leq 0.6\)

ie., \(A \leq B \Rightarrow A^T \leq B^T\)

\[
\|AB^T\| = \begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.4 & 0.8 \\
0.6 & 0.5 & 0.6 \\
0.8 & 0.6 & 0.7
\end{bmatrix}
\]
\[
\|AB^T\| = 0.6[0.4 + 0.4] + 0.6[0.4 + 0.4] + 0.6[0.4 + 0.4] = 0.4 + 0.4 + 0.4
\]
\[
\|AB^T\| = 0.4
\]
\[
\|A\||\|B^T\| = [0.4][0.6] = 0.4
\]
\[
\|BB^T\| = \begin{bmatrix}
0.7 & 0.6 & 0.8 \\
0.4 & 0.5 & 0.6 \\
0.8 & 0.6 & 0.7
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.4 & 0.8 \\
0.6 & 0.5 & 0.6 \\
0.8 & 0.6 & 0.7
\end{bmatrix}
\]
\[
\|BB^T\| = 0.8[0.6 + 0.6] + 0.6[0.6 + 0.6] + 0.7[0.6 + 0.5] = 0.6 + 0.6 + 0.6
\]
\[
\|BB^T\| = 0.6
\]
\[
\|A^TA\| = \begin{bmatrix}
0.5 & 0.3 & 0.5 \\
0.4 & 0.1 & 0.2 \\
0.6 & 0.4 & 0.5
\end{bmatrix}
\begin{bmatrix}
0.5 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.4 \\
0.5 & 0.2 & 0.5
\end{bmatrix}
\]
\[
\|A^TA\| = 0.5[0.4 + 0.4] + 0.4[0.4 + 0.4] + 0.5[0.4 + 0.4] = 0.4 + 0.4 + 0.4
\]
\[
\|A^TA\| = 0.4
\]
\[
\|A^T||A\| = [0.4][0.4] = 0.4
\]
\[
\|B^TB\| = \begin{bmatrix}
0.7 & 0.4 & 0.8 \\
0.6 & 0.5 & 0.6 \\
0.8 & 0.6 & 0.7
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.6 & 0.8 \\
0.4 & 0.5 & 0.6 \\
0.8 & 0.6 & 0.7
\end{bmatrix}
\]
\[
\|B^TB\| = 0.8[0.6 + 0.6] + 0.6[0.6 + 0.6] + 0.7[0.6 + 0.6]
\]
\[ \|B^T B\| = 0.6 \]
\[ \|AA^T\| = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \]
\[ \|AA^T\| = 0.4 + 0.4 + 0.4 = 0.4 \]
\[ \|A\|\|A^T\| = [0.4][0.4] = 0.4 \]

Therefore \( A \leq B \Rightarrow AA^T \leq BB^T \).

5. Conclusion

In this paper, a new definition for the det-norm ordering and its properties are suggested in fuzzy environment. A numerical example is given to clarify the developed theory and the proposed det-norm ordering with fuzzy matrix.

References

[1] Bertoiuzza, On the distributivity of t-norm and t-conorms, pro, 2nd IEEE internal. On fuzzy system (IEEE press, Piscataway, NJ, 1993) 140-147

[2] Jian Miao Chen. (1982). "Fuzzy matrix partial ordering and generalized inverse." Fuzzy sets system. 105:453-458

[3] Meenakshi A.R. and Cokilavany R., On fuzzy 2-normed linear spaces, The Journal of fuzzy mathematics, volume 9(No.2) 2001 (345-351)

[4] Nagoorgani A. and Manikandan A. R. On Fuzzy det-norm matrices. J. Math. Comput. Sci. 3 (2013), No. 1, 233-241, ISSN: 1927-5307

[5] Nagoorgani A. and Kalyani G. Binormed sequences in fuzzy matrices. Bulletin of Pure and Applied Science. Vol. 22E(No.2) 2003; P. 445-451

[6] Nagoorgani A. and Kalyani G. "Fuzzy matrix m-ordering." Science Letters, vol.27, No.3 and 4,2004
[7] Ragab M. Z. and Emam E. G. The determinant and adjoint of a square fuzzy matrix, An international journal of Information Sciences-Intelligent systems, Vol 84, 1995, 209-220.

[8] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338{353}.

[9] ZHOU Min - na "Characterizations of the Minus Ordering in Fuzzy Matrix Set" journal of ningbo university ( nsee ) Vol.21 No.4 Dec. 2008