Superfluidity and Anderson localisation for a weakly interacting Bose gas in a quasiperiodic potential

Xiaolong Deng,1 Roberta Citro,2 Edmond Orignac,3 and Anna Minguzzi1

1Université Joseph Fourier, Laboratoire de Physique et Modélisation des Milieux Condensés, C.N.R.S. B.P. 166, 38042 Grenoble, France
2Department of Physics "E.R. Caianiello" and C.N.I.S.M., Università di Salerno, via S. Allende, 84081 Baronissi, Salerno, Italy
3Université de Lyon, Laboratoire de Physique de l’École Normale Supérieure de Lyon, CNRS UMR5672, 46 Allée d’Italie, 69364 Lyon Cedex 07, France

Using exact diagonalisation and Density Matrix Renormalisation group (DMRG) approach we analyse the transition to a localised state of a weakly interacting quasi-1D Bose gas subjected to a quasiperiodic potential. The analysis is performed by calculating the superfluid fraction, density profile, momentum distribution and visibility for different periodicities of the second lattice and in the presence (or not) of a weak repulsive interaction. It is shown that the transition is sharper towards the maximally incommensurate ratio between the two lattice periodicities, and shifted to higher values of the second lattice strength by weak repulsive interactions. We also relate our results to recent experiments.

PACS numbers:

I. INTRODUCTION

Ultracold atoms in optical lattices1 represent a powerful and robust tool for simulating simple quantum systems with a broad tunability of the parameters, thus serving as quantum simulators2 to reproduce the fundamental phenomena of condensed matter systems. One of the most fascinating and still long-standing problem in condensed matter physics is represented by the localisation of waves in disordered media3. Actually, evidence of the Anderson localisation for light waves in disordered media has been provided by the observation of the modification of the diffusive regime, indicating a transition from a conductor to an insulator transition4,5. However, a careful analysis is necessary in order to separate localisation from absorption effects6. Anderson localisation in three-dimensions has also been recently observed with ultrasound waves7. In the case of electronic waves in solids, the situation is made more complicated by the long range coulombic repulsion between electrons8. Localisation problems can also be considered in the case of bosonic systems in a disordered medium such as 4He in porous material9 or disordered Josephson Junction arrays10. In both of these systems, the bosons are again strongly interacting. Thus, the interplay between disorder and interaction is an important issue in theoretical condensed matter physics. The combination of ultracold atoms and optical potentials is offering a novel platform to study disorder and interaction related phenomena, in which most of the relevant physical parameters, including interaction, are tunable11. Moreover, a disordered optical potential can be realized either with laser speckles12 or with quasi-periodic optical lattices13. For both types of disorder, Anderson localisation has been recently experimentally observed14,15.

In this work we focus on quasiperiodic potentials, which exhibit properties common to both periodic and disordered systems16. As in the case of periodic lattices, their spectrum shows reminiscences of energy bands. On the other hand, owing to the lack of any translational invariance, they support the existence of localised states, which behave very similarly to the ones generated by truly random potentials17. Localisation of noninteracting particles in quasiperiodic potentials has been intensively studied in relation with electronic structures of incommensurate solids18. The case of a bichromatic lattice of Refs.12,13 is described by the so called Harper model19 or 1D tight-binding Aubry-André model20,21,22 the Hamiltonian of which is:

\begin{equation}
H = -t \sum_n (|n\rangle \langle n+1| + |n+1\rangle \langle n|) + \Delta \sum_n \cos(2\pi n\beta + \phi) |n\rangle \langle n|.
\end{equation}

Here \(t\) is the transfer integral between nearest neighbour sites of the main lattice, \(\Delta\) is the amplitude of the quasiperiodic modulation of the potential energy, while \(\beta\) is an irrational number and \(\phi\) is a phase. The Aubry-André model can be experimentally realized by superimposing two optical lattices of incommensurate wavelengths, the primary lattice height, \(s_1\), being much larger than the secondary lattice height \(s_2\):

\begin{equation}
V(x) = s_1 E_{R1} \cos^2(k_1 x) + s_2 E_{R2} \cos^2(k_2 x)
\end{equation}

where \(k_{1,2}\) is the lattice wavevector and \(E_{Ri} = \hbar^2 k_i^2 / 2m\) the respective recoil energy. At variance with the completely random potential which localises all the states as soon as it is switched on23, in the Aubry-André case the strength
of the potential must exceed the critical value $\Delta / t = 2$ to induce a localised ground state with a localisation length $1 / \ln(\Delta / 2t)$ in units of lattice spacing. Moreover, the localised state at $\Delta / t > 2$ is related to the delocalised state at $\Delta / t < 2$ by a duality transformation. The transition point is self-dual, and its spectrum is singular continuous, with wavefunctions having a fractal-like structure.

In the recent experiment in Florence by Roati et al., Anderson localisation has been observed for a noninteracting $^{39}$K BEC in an incommensurate bichromatic lattice where the effect of interactions have been cancelled by tuning a static magnetic field in proximity of a Feshbach resonance to set the scattering length to zero. The crossover between extended to localised states has been studied in detail by looking at the expansion of the BEC and by studying spatial and momentum distribution of the atoms, with results in qualitative agreement with the Aubry-André predictions. In particular, they have verified the scaling behaviour of the critical disorder strength and shown that while the transition to localisation is expected to be sharp in the case of a maximally incommensurate ratio $\beta = (\sqrt{5} - 1)/2$, it is broadened for the experimental parameter $\beta = 1.1972$.

So far, the incommensurate lattice supports a localisation transition for noninteracting particles, but a richer physics arises when interactions are present. A first interesting question is how the critical disorder strength would be modified in the presence of a weak interaction. Another question is the nature of the localised phase of interacting bosons. Indeed, the repulsion between bosons disfavours the condensation of all particles in the lowest localised eigenstate, and should instead favour a phases in which although the bosons are localised, their density is homogeneous. This problem has been discussed in the case of a weakly interacting system in Ref. [24], where the ground state of the system has been calculated for a three-colour optical lattice. Introducing repulsive interactions between the atoms, the numerical integration of the 1D time dependent Gross-Pitaevskii equation shows that the ground state wavefunction becomes a superposition of many single-particle localised states, which add up to form an overall extended state. Similar results for a bichromatic lattice have been presented in the work by Lye et al. [13], and Quantum-Monte Carlo calculations for the case of large interaction strengths have been performed by Roscilde [23, 24].

Besides the modification of the critical disorder strength, the combination of lattice and interaction can lead to new phases that compete with the localised state. In absence of disorder, bosons on a lattice with repulsive interactions display, for commensurate filling, a superfluid (SF) to Mott insulator (MI) transition as the repulsion is increased [27], with the superfluid phase displaying large density fluctuations and a gapless excitation spectrum, while the Mott phase is incompressible and has a gap in the excitation spectrum. If one considers both repulsive interactions and disorder, these two effects will compete: while disorder makes the bosons localised, short-range repulsive interaction energy increases as the square of boson density and hence the total energy of the system is minimised by depleting the localised condensate towards a more uniform density distribution. As a result, in a lattice Bose gas with short-range interactions a novel Bose-glass (BG) phase, non superfluid yet compressible, emerges between the superfluid and the Mott-insulator [27]. For the specific case of a one-dimensional Bose gas subjected to an uncorrelated disorder (in absence of a lattice), the phase diagram has been obtained by Giamarchi and Schulz [28], showing that while for zero interactions the system is always localised, for nonzero values of the repulsive interactions a superfluid phase is possible at small values of disorder. Ref. [28] also predicted that the non-superfluid (Bose-glass) phase of an interacting Bose gas is expected to differ markedly from the non-interacting Anderson-localised (AG) phase, e.g. the density profile of a Bose glass phase is rather uniform, in contrast to the highly inhomogeneous density profile of an ideal Bose gas in a disordered potential where all the particles occupy the lowest single-particle localised orbital. The case of lattice hard core bosons with nearest neighbour repulsion was considered in [29]. It was found that a superfluid dome could be present for strong enough nearest neighbour attraction and weak enough disorder. The phase diagram of a disordered, interacting lattice Bose gas in one-dimension has been the subject of several numerical investigations by exact diagonalisation [30], quantum Monte Carlo methods [31, 32], strong coupling expansions [33], and Density-Matrix renormalization group approaches (DMRG) [34], that have established the existence of a Mott insulating phase separated from the superfluid phase by a Bose glass phase for disorder not excessively strong. For stronger disorder, these numerical studies have established that only the Bose glass and the superfluid are present. Also, the existence of a superfluid dome in the phase diagram for a quasiperiodic lattice potential has been obtained in the commensurate case [34, 35, 36].

Here we focus on the quasi-1D Bose gas subjected to a quasiperiodic potential in the presence of interactions and try to analyse the scaling behaviour of the critical interaction strength by exact diagonalisation or DMRG. In particular, we characterise the transition by means of the behaviour of the superfluid density, the density profile and the visibility. In the case of open boundary conditions, where the superfluid density is undefined, another physical quantity is also introduced for the non-interacting case, i.e. the inverse participation ratio and its results are discussed.
II. THE MODEL

A. The Hamiltonian

We consider a one-dimensional Bose gas at zero temperature subjected to a bichromatic lattice potential as the one of Eq. (2)

\[ H = \int_{-\infty}^{\infty} dx \psi_b^\dagger(x) \left( -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi_b(x) \]
\[ + \frac{g}{2} \int_{-\infty}^{\infty} dx \psi_b^\dagger(x) \psi_b^\dagger(x) \psi_b(x) \psi_b(x), \]

(3)

where \( \psi_b(x) \) is the bosonic field operator, \( m \) is the atomic mass and \( g = 4\pi\hbar^2a_s/m \) is the contact interaction expressed in terms of the scattering length \( a_s \). When the primary lattice height \( s_1 \) is large compared to the atomic recoil energy \( E_{R,1} \) and to the height of the secondary lattice \( s_2 \), we can map the Hamiltonian on a Bose-Hubbard model [37]:

\[ H = -t \sum_{i=1}^{N-1} (b_i^\dagger b_{i+1} + h.c.) + U \sum_{i=1}^{N} n_i(n_i - 1) \]
\[ - \mu \sum_{i=1}^{N} n_i + \sum_{i=1}^{N} \Delta_i n_i, \]

(4)

where \( b_i^\dagger, b_i \) are bosonic field operators on the site \( i \), \( t \) is the hopping amplitude, \( U \) is the on-site interaction, \( \mu \) is the chemical potential, \( N \) is the total number of lattice sites; the parameters \( U, t \) are related to those of the continuum model [3] (e.g. see Refs. [37, 38]), and their respective dependence upon the recoil energy, lattice depth and scattering length can be calculated both analytically and numerically [39]. The effect of the second lattice is to induce a modulation of the on-site energies according to \( \Delta_i = \Delta \cos(2\pi \beta i + \phi) \), with \( \Delta \propto s_2 \) and \( \phi \) being a phase shift while in the experiments \( \beta = k_2/k_1 \). By taking the parameters of [40], one can estimate \( s_2/t = 2.5 - - - 53 \).

In order to characterise the localisation transition, we evaluate the following observables in the case of periodic boundary conditions: (i) the superfluid fraction as the response to twisted boundary conditions, \( f_s = \frac{N^2 P_t \theta^2}{\partial^2 E_{NP}} \), which on the lattice model reads

\[ f_s = \frac{N^2}{N_P t \theta^2} \left( E_{NP}^{PBC} - E_{NP}^{\theta} \right), \]

(5)

where \( N_P \) is the particle number, \( E_{NP}^{PBC} \) is the ground state energy and \( E_{NP}^{\theta} \) is the ground state energy with boundary conditions twisted by an angle \( \theta \) [31], (ii) the spatial density profile \( \langle n_j \rangle = \langle b_j^\dagger b_j \rangle \), and (iii) the momentum distribution, given by the Fourier transform of the one-body density matrix

\[ n(q) = N \sum_{lm} e^{i(l-m)\alpha} \langle b_l^\dagger b_m \rangle, \]

(6)

with \( \alpha = \pi/k_1 \) being the primary lattice spacing and \( N \) a normalisation constant.

B. Numerical methods

The ground state properties of the interacting Bose gas in the bichromatic lattice have been determined by the Density Matrix Renormalization Group (DMRG) method [42, 43] and exact diagonalisation (ED) method (in the noninteracting case).

Concerning DMRG, it is a quasi-exact numerical technique widely employed for studying strongly correlated systems in low dimensions. Based on the renormalization, it finds efficiently the ground state of a relatively large system with quite high precision. We have been considering a system with periodic boundary conditions and used first the infinite-size algorithm to build the Hamiltonian up to the length \( L \), then we have been resorting to the finite-size algorithm to increase the precision within many sweeps. Since the Hilbert space of bosons is infinite, to keep it finite we have
been choosing the maximal number $n_{\text{max}}$ of boson states approximately of the order $5 \langle n \rangle$, where $\langle n \rangle$ is the average number of bosons per site, varying $n_{\text{max}}$ between $n_{\text{max}} = 10$ and $n_{\text{max}} = 25$. The number of eigenstates of the reduced density matrix are chosen in the range $150 - 250$. The number of sites in the DMRG is $N = 50$. The initial phase of the quasi-periodic potential is chosen $\phi = \pi$.

In the noninteracting case, $U = 0$, we have been employing the exact diagonalisation method on a chain of 50 up to 1000 sites. The superfluid fraction depends on the energy change under twisted-phase boundary conditions. We checked that the superfluid fraction computed from $[3]$ does not vary strongly at critical points $\Delta_c / t$ when small angles (say $\theta = 0.1 \text{ rad}$) are replaced with large angles (say $\theta = \pi$). Thus, the location of the critical point is not sensitive to the choice of the twist angle in Eq. $[5]$. This allows us to use anti-periodic boundary conditions (i.e. a twist angle $\theta = \pi$) to calculate the superfluid fraction (both in ED and DMRG). We have also analysed the variation of the superfluid fraction as a function of the shift $\phi$ of the quasi-periodic potential in the Hamiltonian $[4]$. Under some initial phases with fixed length of chain, we found that the critical points are shifted to a value much smaller than $\Delta_c / t = 2$. This behaviour is associated with the formation of an isolated state below the continuum of extended states which is localised near the edges. The origin of this localised state is simply the fact that the on site potential $\Delta_i$ does not have the periodicity of $N$, thus causing strong fluctuations of the potential near the edges in the system with periodic or aperiodic boundary conditions. In our calculations we choose the initial phase such that no spurious localised state is formed near the edges, ensuring that the critical points are close to $\Delta_c / t = 2$. In addition, when we perform finite size scaling we consider a sequence of periodic approximants to the potential $\Delta_i$ associated with larger and larger ring sizes. In the noninteracting limit we consider also open boundary conditions (OBC). In this case the superfluid density $[5]$ fails to be a good indicator of the localisation transition, but the inverse participation ratio $[41]$ can be employed.

III. ANALYSIS OF THE LOCALISATION TRANSITION

We proceed here to present the results for the localisation transition occurring at increasing height of the secondary lattice, with the purpose of analysing the localisation transition at varying the parameter $\beta$, i.e. the periodicity of the secondary lattice. Although mathematically the transition is expected only for irrational values of $\beta$ $[22]$, in the numerical calculations $\beta$ is a rational number. In the experiment, $\beta$ is fixed by the ratio of the wavelengths of the two laser beams used for creating the optical lattices, which are only known up to a finite number of significant digits, so that the rational or irrational character of $\beta$ cannot be decided. In principle, if $\beta$ is rational, the system is in a periodic potential, and it thus possesses only extended states forming a finite number of bands for any strength of the potential. However, if $\beta$ is given by the ratio of two mutually prime integers, with a large denominator, the periodical behaviour can only be observed on a very large length scale. Since, in contrast to the ideal case studied in Mathematics $[22]$, both in simulations and in experiments the number of sites of the lattice is finite, the very large lengthscale periodicity mentioned above may not be accessible in practice, and the question arises on the occurrence and the nature of the Anderson localisation transition (or crossover). More precisely, in the case of $\Delta \gg t$, if the periodicity of the potential is $N$, degenerate perturbation theory provides an estimate of $\sim t(t/\Delta)^{N-1}$ for the bandwidth of the extended states. On timescales small with respect to the inverse of the bandwidth, particles will exhibit the appearance of localisation. We thus investigate here the Anderson localisation behaviour of a finite lattice and rational $\beta$ parameter, which is relevant for the ongoing experiments.

A. Noninteracting case

We begin our analysis by considering first the noninteracting limit $U = 0$, In this case, the ground state of the Hamiltonian Eq.$[41]$ can be obtained by exact diagonalisation.

1. The inverse participation ratio

In the noninteracting regime it is possible to adopt several definition of localisation. One which turns out to be particularly convenient for our problem is the inverse participation ratio in the ground state $[41]$. This can be expressed as

$$P_0 = \sum_i |\psi_i^{(0)}|^4$$

(7)
FIG. 1: (Color online) Inverse participation ratio $IPR = 1 - \delta w/N$ (adimensional) for a noninteracting Bose gas in a quasiperiodic potential with $\beta = (\sqrt{5} - 1)/2$ with open boundary conditions as a function of the number of sites in the chain for various values of the height of the second lattice $\Delta$ (in units of $t$), from bottom the values are $\Delta/t = 1.9, 1.95, 2, 2.05,$ and $2.1$. The inset shows the same quantity in logarithmic scale.

where $\psi_i^{(0)}$ is the normalised ground state wavefunction of the noninteracting Hamiltonian (4) on the basis $|i\rangle$ of the lattice site occupation number. The inverse participation ratio is related to the probability of return to the initial point [41] and allows to distinguish between extended and localised states: it decays as $1/N$ where $N$ is the number of lattice sites in the extended regime and goes to a constant value in the localised regime.

The inverse participation ratio can be extracted from the numerical calculations for a system with open boundary conditions by computing the average number fluctuations,

$$\delta w = \frac{1}{N} \sum_i \left[ \langle n_i^2 \rangle - \langle n_i \rangle^2 \right]. \quad (8)$$

Indeed, using the expansion of the boson site operators $\hat{b}_i$ on the operators $\hat{b}_\eta$ corresponding to the eigenvectors $\psi_i^{(\eta)}$ of the Hamiltonian, i.e. $\hat{b}_i = \sum_\eta \psi_i^{(\eta)} \hat{b}_\eta$, and the fact that in the ground state all the $N$ particles occupy the state $\psi_i^{(0)}$ we obtain a relation between the number fluctuations and the inverse participation ratio

$$\delta w = N (1 - \sum_i |\psi_i^{(0)}|^4). \quad (9)$$

Since the inverse participation ratio has two distinct large-$N$ behaviour in the extended and localised regimes, it is a good indicator to identify the transition from the extended regime, where $\delta w/N = 1 + O(1/N)$, to the localised regime where $\delta w/N = 1 - \text{const}$. Fig[4] shows the change in behaviour of the inverse participation ratio across the localisation threshold.

Figure 2 shows the average number fluctuation as a function of the height of the secondary lattice $\Delta$ for different choices of the quasiperiodic potential. In the first panel, we choose for $\beta$ subsequent approximants of the irrational number $(1 + \sqrt{5})/2$, according to the Fibonacci sequence: $F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$, as $\lim_{n \to \infty} F_{n+1}/F_n = (1 + \sqrt{5})/2$. The figure shows that the transition at $\Delta/t = 2$ becomes more and more pronounced as the order of the approximation increases. In the second panel, we show the behaviour of the inverse participation ratio for three choices of $\beta$ which we will consider throughout the paper: the maximally irrational value $\beta = (\sqrt{5} - 1)/2$, the value
FIG. 2: (Color online) Average number fluctuation $\delta w$ (adimensional) for a noninteracting Bose gas in a quasiperiodic potential with open boundary conditions as a function of the height of the second lattice $\Delta$ (in units of $t$) for a number of lattice sites $N = 200$, at varying choices of $\beta$. First panel, subsequent approximants of the number $\beta = (1 + \sqrt{5})/2$ according to the Fibonacci sequence. Second panel $\beta = (\sqrt{5} - 1)/2$ (blue solid line), $\beta = 830/1076$ (green dotted line), and $\beta = 1032/862$ (red dash-dotted line).

adopted in the first Florence experiment [13] $\beta = 830/1076$, and the value used in the second Florence experiment [15] $\beta = 1032/862$.

2. The superfluid fraction

Since the inverse participation ratio is defined only in the noninteracting limit, we further characterise the transition by monitoring the superfluid fraction, which is nonzero in the extended phase and vanishes in the localised phase. We obtain the superfluid fraction from the change of the ground state energy of the system to twisted boundary conditions, Eq. (5). Figure 3 shows the behaviour of the superfluid fraction as a function of the height of the secondary lattice $\Delta$, for various choices of the quasiperiodicity parameter $\beta$ and for a number of lattice sites equal to 50, which is close to the experimental conditions. We have considered three possible values of $\beta$: the irrational number $\beta = (\sqrt{5} - 1)/2$, and the values $\beta = 830/1076$, $\beta = 1032/862$ adopted in the Florence experiments [13, 15]. From the data we see that the superfluid fraction vanishes around the value $\Delta/t = 2$, as predicted by the Aubry-André model [20], the details however depend on the choice of $\beta$. We remark that the transition is particularly smooth in the case of the second Florence experiment, where the potential has five minima very close in energy.

The Anderson localisation transition can be also seen by looking at the spatial density profiles, illustrated in Fig. 4 at increasing values of $\Delta$. In correspondence to the transition the profile displays a clear change of behaviour as all the particles tend to cluster around the localised state. The position of the Anderson-localised peak corresponds to the position of the absolute minimum of the quasiperiodic potential.

3. Finite size scaling

All the results of the previous section were obtained for a system of finite length. In such a system, there is no actual localisation transition as the localisation length can never exceed the length of the system and the spectrum always remains a pure point one. In order to interpolate the results to the case of an infinite system which possesses a true localisation transition between localised and extended state, we make a finite-size scaling Ansatz. [44] We thus assume that the superfluid fraction is of the form:

$$f_s(N, \Delta) = N^{-a} \Phi \left( \frac{N}{\xi(\Delta)} \right), \tag{10}$$

where $\xi(\Delta)$ is a characteristic length that diverges at the localisation transition as $|\Delta - \Delta_c|^{-\nu}$. The function $\Phi(x)$ is assumed to be regular for $x \ll 1$ and to behave as $\Phi(x) \sim x^a$ for $x \gg 1$ on the extended side, so that the superfluid fraction becomes independent of system size in the thermodynamic limit.
FIG. 3: Superfluid fraction for a noninteracting Bose gas in a quasiperiodic potential as a function of the height of the second lattice \( \Delta \) in units of the hopping strength \( t \) for three choices of the quasiperiodicity period \( \beta \): maximally irrational value \( \beta = (\sqrt{5} - 1)/2 \) (solid lines), first Florence experiment \[13\] \( \beta = 830/1076 \) (dashed lines), and second Florence experiment \[15\] \( \beta = 1032/862 \) (dot-dashed lines). All the curves are drawn for \( \phi = \pi \) and \( N = 50 \).

FIG. 4: (Colour online) Contour plot of the density profile for a noninteracting Bose gas in a quasiperiodic potential with periodic boundary conditions at increasing height of the second lattice \( \Delta \) for a number of lattice sites \( N = 50 \), at varying choices of \( \beta \): \( \beta = (\sqrt{5} - 1)/2 \) (a), \( \beta = 830/1076 \) (b), and \( \beta = 1032/862 \) (c). In the three cases localisation is clearly visible for \( \Delta > 2t \). The value of the phase shift has been chosen \( \phi = 3\pi/10 \) for (a) and (b), and \( \phi = -2\pi/5 \) for (c).

When approaching the transition from the localised side, this characteristic length is obviously the localisation length. On the delocalised state, the identification of the characteristic length is less straightforward. Using the Aubry-André duality, there is now a dimensionless localisation length \( \ell \) defined in momentum space, which measures the number of plane waves of quasi-momentum \( k + n\beta \) that must be combined in the repeated zone scheme to form an extended wavefunction. When these quasi-momenta are folded back in the first Brillouin zone, we end up with a wavefunction formed by linear combination of \( \ell \) distinct plane waves, whose momenta are in the interval \([-\pi/\alpha, \pi/\alpha]\]. The typical distance between these different momenta can be estimated to be of the order of \( 2\pi/\ell\alpha \). Thus, in order to resolve the difference between these momenta, it is necessary to be in a system of size \( N \) such that \( 2\pi/(N\alpha) \ll 2\pi/\ell\alpha \) i.e. \( N \gg \ell \). Thus, the length \( \ell \) has the interpretation of the minimal system length necessary to distinguish a extended state from a localised state. We thus identify the length \( \ell\alpha \) to \( \xi(\Delta)\alpha \) in the extended regime. Using the exact result on the localisation length in the Aubry-André model, we have \( \xi(\Delta) = 1/|\ln(\Delta/2t)| \sim 1/|\Delta - 2t| \). Thus, the exponent \( \nu = 1 \) on both sides of the transition. Using this expression for \( \xi(\Delta) \), we have been able to collapse our numerical data...
FIG. 5: Double logarithmic scaling plot of the superfluid fraction in the case of non-interacting bosons ($\beta = (\sqrt{5} - 1)/2$). The data for different system sizes are collapsing on a single curve once properly scaled. Deviations from scaling far away from the transition point are visible on the graph, indicating that the corresponding points are already outside of the scaling region.

on a single curve, as can be seen on Fig. 5 using an exponent $a = 1/2$. This leads to a superfluid fraction vanishing as $|2t - \Delta|^{1/2}$ as the transition is approached from the extended side in the thermodynamic limit.

B. Weakly interacting case

Using DMRG we can access the regime of arbitrary interaction strength \cite{36}. In particular, we consider here the effect of weak repulsive interactions. In this regime, interaction reduces localisation, as repulsion keeps particles from coming on top of each other. Figure 6 shows the superfluid fraction as a function of the height of the secondary lattice $\Delta$. The comparison with the noninteracting case shows that in the interacting case the localisation transition is shifted to higher values of the secondary lattice $\Delta$. The delocalised phase, having both non-vanishing compressibility and non-vanishing superfluid fraction is in a Luttinger liquid phase. \cite{36, 46} In particular, it displays a quasi-long-range superfluid order, with the single particle density matrix decaying as a power law as a function of distance. This also gives a power law divergence of the momentum distribution at low momentum in the infinite-system limit \cite{36}. The observation of the shift of the transition could provide a sensitive measure of the interaction strength $U$, hence of the scattering length $a_s$.

1. Finite size scaling

In the interacting case, we can also develop a finite size scaling Ansatz analogous to \cite{10}. We now assume that $\xi(\Delta) \sim |\Delta - \Delta_c(U)|^{-\nu}$, where $\Delta_c(U)$ is the transition point in the presence of interaction $U$. The main difficulty in applying finite size scaling to the transition in the interacting case is that in contrast to the non-interacting case of section IIIA3 we do not have a priori knowledge of $\Delta_c(U)$. However, we note that in the localized phase, $f_s$ is decreasing exponentially with the system size, and at the critical point, according to finite-size scaling is decreasing as a power law. By plotting $f_s$ as a function of $\Delta$ with $U = 0.02t$, we find that $f_s$ is a non-decreasing function of $N$ for $\Delta/t < 3.2$. Thus, the localized phase is obtained for $\Delta > 3.2t$. We have assumed that $\Delta_c = 3.2t$ was the location...
FIG. 6: Comparison of the superfluid fraction for a noninteracting (straight lines, \( U = 0 \)) and weakly interacting Bose gas (lines with circles, \( U/t = 0.02 \)) in a quasiperiodic potential as a function of the height of the secondary lattice \( \Delta/t \) at varying choices of the periodicity parameter: solid lines, \( \beta = (\sqrt{5} - 1)/2 \); dot-dashed lines, \( \beta = 1032/862 \). Note the enhancement of the superfluid fraction by the interaction for both cases. For all the curves we have used a number of lattice sites \( N = 50 \) and a filling factor \( 1/2 \).

of the critical point, and fitted the \( N \) dependence of \( f_s(\Delta_c, N) \) to obtain the exponent \( a = 0.38 \). We have been able to collapse the curves (see Fig. 7) using \( \nu = 1.25 \). The collapse of the data points is a verification of the scaling hypothesis.

IV. MOMENTUM DISTRIBUTION

The momentum distribution is one of the experimentally accessible observables\[15\], and allows to distinguish among the extended and localised regimes. We have calculated the momentum distribution for various choices of the quasiperiodicity parameter \( \beta \), both in the noninteracting and in the weakly interacting case.

Our results are reported in Fig. 8. In the extended regime, in addition to the central peak at \( q = 0 \) side peaks in the momentum distribution appear, at a position related to the value of the periodicity of the secondary lattice and corresponding to the beating of the two lattices. The position of the secondary peaks is at \( Q = \pm \frac{2\pi}{\alpha}(1 - \beta) \). In the localised regime, the central peak of the momentum distribution is broadened till becomes of the order of the Brillouin zone (Fig. 8(a) and (b)), while the secondary peaks are strongly suppressed. This is in agreement the experimental findings\[15\]. The broadening of the distribution in momentum space indicates that the states in real space are more and more localised. As interactions are turned on, we see a restoration of the peaks of the momentum distribution as a signature of the enhanced superfluid behaviour found by the analysis of the superfluid fraction.

The transition from extended to localised phase has been characterised in the experiment\[15\] by studying the visibility of the secondary peaks in the momentum distribution. In Fig. 9 we show the visibility of the momentum distribution, defined as \( V = [n(0) - n(k_1)]/[n(0) + n(k_1)] \), both for the noninteracting and for the interacting case. The loss of visibility starts to become important at values of the height of the secondary lattice \( \Delta/t \) which depend on the choice of \( \beta \) and of the interaction strength. In the case of an irrational \( \beta = (\sqrt{5} - 1)/2 \) and in the noninteracting limit the visibility starts to decrease right at the transition point \( \Delta/t = 2 \), while for other choices of \( \beta \) as in the experiments the visibility remains close to one to larger values of \( \Delta/t \), indicating a smoother crossover from extended to localised states. Similarly, in the interacting case the visibility starts to decrease from its value one at larger value of \( \Delta/t \), reflecting the shift of the transition point.
In summary, we have analysed the superfluid to Anderson-localisation transition for an interacting Bose gas subjected to a quasiperiodic potential, focussing on the case of a finite-size lattice. We have investigated the effects of changing the values of the quasiperiodicity parameter $\beta$ and of the interactions. In the case of periodic boundary conditions, by studying the superfluid fraction we have found that the transition is sharper for irrational values of $\beta$ and smoother when several minima in the potential have almost the same energy. We have also found that the transition is shifted to higher values of the secondary lattice height when the repulsive interactions are taken into account. Implementation of this idea could lead to an accurate method to measure of the interaction coupling strength, which is directly related to the s-wave scattering length. In the case of open boundary conditions, the inverse participation ratio is introduced as a good indicator of the transition, and its scaling analysis reproduces again the change of character of the transition from irrational values of $\beta$ to rational ones (with a large denominator). We have furthermore shown that also in the interacting case the change of the ground state of the system might be probed by analysing the momentum distribution of the gas, which displays side peaks in the superfluid phase but not in the deeply localised phase, and the visibility of the momentum distribution. In perspective, a detailed analysis of the shift of the transition at increasing the interaction strength, as well as the inclusion of a harmonically trapping potential, seem to us valuable extensions of the present work.
FIG. 8: (Colour online) Momentum distribution (in units of the primary lattice spacing $\alpha$) of a Bose gas in the quasiperiodic lattice, for various choices of the height of the secondary lattice $\Delta/t = 1$ (solid lines), $\Delta/t = 2$ (dashed lines) and $\Delta/t = 3$ (dot-dashed lines); of the interaction strength $U/t = 0$ (left panels) and $U/t = 0.02$ (right panels) and of the periodicity parameter $\beta = (\sqrt{5} - 1)/2$ (top panels) and $1032/862$ (bottom panels). For all the curves we have used a number of lattice sites $N = 50$ and a filling factor $1/2$.

FIG. 9: Visibility of the momentum distribution (adimensional) of a Bose gas in the quasiperiodic lattice, as a function of the height of the secondary lattice $\Delta/t$ for three choices of the quasiperiodicity parameter $\beta$ (solid line: $\beta = (\sqrt{5} - 1)/2$, dotted line $\beta = 830/1076$, dash-dotted line $\beta = 1032/862$); left panel, noninteracting case; right panel interacting case with $U = 0.02t$. For all the curves we have used a number of lattice sites $N = 50$ and a filling factor $1/2$. 
Acknowledgements

R.C. acknowledges funding from Marie-Curie Intra-European fellowship, X.D. funding from CNRS and A.M. from the MIDAS-STREP project.

[1] O. Morsch and M. Oberthaler, Rev. Mod. Phys. 78, (2006) 179
[2] R. P. Feynman, Int. J. Theor. Phys. 21, (1982) 467
[3] P. W. Anderson, Phys. Rev. 109, (1958) 1492.
[4] M. P. Van Albada and A. Lagendijk, Phys. Rev. Lett. 55, (1985) 2692
[5] D.S. Wiersma, P. Bartolini, A. Lagendijk, and R. Righini, Nature 390, (1997) 671
[6] M. Störzer, P. Gross, C.M. Aegerter and G. Maret Phys. Rev. Lett. 96 (2006) 063904
[7] Hefei Hu, A. Strybulevich, J.H. Page, S.E. Skipetrov and B.A. van Tiggelen, Nature Phys. 4 (2008) 945
[8] P. A. Lee and T. V. Ramakhrishnan Rev. Mod. Phys. 57 (1985) 287
[9] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, and M. Lewenstein, Phys. Rev. Lett. 91, (2003) 080403
[10] J.E. Lye, L. Fallani, C. Fort, V. Guarrera, M. Modugno, D. Wiersma, C. Fort, and M. Inguscio, Phys. Rev. Lett. 95, (2005) 070401
[11] L. Fallani, J.E. Lye, V. Guarrera, C. Fort, and M. Inguscio, Phys. Rev. Lett. 98, (2007) 130404
[12] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lujan, D. Clement, L. Sanchez-Palencia, P. Bouyer and A. Aspect, Nature 453, (2008) 891
[13] L. Fallani, J.E. Lye, V. Guarrera, C. Fort, and M. Inguscio, Phys. Rev. Lett. 98, (2007) 130404
[14] J. E. Lye, L. Fallani, C. Fort, V. Guarrera, M. Modugno, D. S. Wiersma, and M. Inguscio, Phys. Rev. Lett. 95, (2005) 170411; T. Schulte, S. Drenkelforth, J. Kruse, R. Teimeyer, K. Sacha, J. Zakrzewski, M. Lewenstein, W. Ertmer, and J. J. Arlt, New J. Phys. 8, (2006) 230
[15] T. Roscilde, Phys. Rev. A 77 (2008) 063605
[16] T Roscilde, arXiv:0804.2769
[17] M.P. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B 40 (1989) 546.
[18] T. Giamarchi and H. J. Schulz, Phys. Rev. B, 37 (1988) 325
[19] C. Doty and D. S. Fisher, Phys. Rev. B 45 (1992), 2167
[20] K. Runge and G.T. Zimanyi Phys. Rev. B, 49 (1994), 15212.
[21] G.G. Batrouni and R.T. Scalettar, Phys. Rev. B, 46 (1992) 9051
[22] B.V. Svistunov, Phys. Rev. B, 54 (1996) 16131, N. V. Prokof’ev and B. V. Svistunov, Phys. Rev. Lett., 80 (1998) 4355
[23] J. K. Freericks and H. Monien Phys. Rev. B 53, (1996) 2691
[24] P. Schmitteckert, T. Schulze, C. Schuster, P. Schwab and U. Eckern, Phys. Rev. Lett., 80 (1998) 560
[25] G. Roux, T. Barthei, I. P. McCulloch, C. Kollath, U. Schollwoeck, and T. Giamarchi, Phys. Rev. A, 78(2008) 023628
[26] Xialong Deng, R. Citro, A. Minguzzi, and E. Orignac, Phys. Rev. A, 78 (2008) 013625
[27] D. Jaksch et al., Phys. Rev. Lett., 81(1998) 3108
[28] H.P. Buchler, G. Blatter and W. Zwerger, Phys. Rev. Lett., 90(2003) 130401
[29] I. Bloch, J. Dalibard and W. Zwerger, Rev. Mod. Phys. 50, (2008) 885
[30] J.E. Lye, L. Fallani, C. Fort, V. Guarrera, M. Modugno, D.S. Wiersma, and M. Inguscio, Phys. Rev. A 75, (2007) 061603(R).
[31] D. J. Thouless, Phys. Rep. 13 (1974) 93
[32] S.R. White, Phys. Rev. Lett. 69, (1992) 2863; Phys. Rev. B 48, (1993) 10345
[33] U. Schollwoeck, Rev. Mod. Phys. 77, (2005) 259; R.M. Noack and S. Mannara AIP Conf. Proc. 789, (2005) 93; K. Hallberg Adv. Phys. 55, (2006) 477.
[34] M. N. Barber in Phase Transitions and Critical Phenomena vol. 8, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1983).
[35] The presence of a shallow trapping potential would not alter the conclusions of our analysis.
[36] F. D. M. Haldane, Phys. Rev. Lett. 81, (1981) 1840.