co-Hopfian Modules

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Abstract

If $R$ is a ring with 1, we call a unital left $R$-module $M$ co-Hopfian (Hopfian) in the category of left $R$-modules if any monic (epic) $R$-module endomorphism of $M$ is an automorphism. In the case that $R$ is commutative Noetherian, we use results of Matlis to show that, in a particular setting, every submodule of a co-Hopfian injective module is co-Hopfian. We characterize when a finitely generated co-Hopfian module over a commutative Noetherian ring has finite length. We describe the structure of Hopfian and co-Hopfian abelian groups whose torsion subgroup is cotorsion.

If $R$ is a ring with 1, we call a unital left $R$-module co-Hopfian in the category of left $R$-modules if every monic endomorphism of $M$ is an automorphism (equivalently, $M$ is not isomorphic to any of its proper submodules), and, dually, Hopfian if any surjective endomorphism of $M$ is an automorphism (equivalently, if $M$ is not isomorphic to any of its proper quotients). So, at the level of morphisms, Hopfian and co-Hopfian modules have properties in common with finite sets and finite dimensional vector spaces. If $M$ is both Hopfian and co-Hopfian, then for an endomorphism $f : M \to M$ the following conditions are equivalent: $f$ is an injection; $f$ is a surjection; and $f$ is an automorphism. We refer to such modules as bi-Hopfian. In this paper we will concentrate on co-Hopfian modules, pointing out results for Hopfian modules when appropriate (and, by inference, bi-Hopfian modules in certain obvious cases).

We noted in [12, 13] the well-known facts that co-Hopfian $R$-modules need not be finite, or even finitely generated, and that finitely generated modules need not be co-Hopfian. Also, we saw that the class of co-Hopfian $R$-modules fails, in general, to be closed under submodules and quotients.

Vasconcelos [26] and, independently, Strooker [25], showed that a finitely generated module over a commutative ring is Hopfian. Vasconcelos [27] also showed that all finitely generated modules over a commutative ring are co-Hopfian if and only if the ring is 0-dimensional. Note that all finitely generated modules over a commutative ring are bi-Hopfian if and only if the ring is 0-dimensional. In [14], we showed that over a commutative Artinian principal ideal ring, a module is Hopfian (co-Hopfian) if and only if it is finitely generated (among other equivalent conditions).

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Of late, there has been renewed interest in Hopfian and co-Hopfian abelian groups because of their connection with the notions of algebraic entropy, and its dual concept adjoint entropy, of abelian groups (see, e.g., [2] and [6]). The papers [5], [8] list some well-known facts about Hopfian and co-Hopfian groups and modules. Further sources for proofs and so on, can be found in the references of those papers. The paper by Varadarajan [28] contains many results on Hopfian and co-Hopfian objects among rings, Boolean rings, function algebras, and compact manifolds.

Our first section recalls Matlis’ notion of a module with maximal orders [16], which generalizes the descending chain condition. We can then obtain a closure result for a co-Hopfian injective that has maximal orders: all its submodules are also co-Hopfian. In the second section we examine what information we may obtain from the scalar multiplication on an $R$-module $M$. If $R$ is commutative Noetherian and $M$ finitely generated faithful, then $M$ being co-Hopfian places significant restriction on the ring $R$.

Section three provides an example of an abelian group $G$ which is co-Hopfian, but whose injective envelope is not. We indicate that this example may be extended to modules over certain integral domains. The example helps to show that a module and its injective envelope are independent vis-à-vis co-Hopficity. In the next section we present an example of a finitely generated co-Hopfian module (over a commutative Noetherian $R$) which does not have finite length. We characterize when a finitely generated co-Hopfian $M$ does have finite length. In section five we characterize the co-Hopfian (Hopfian) abelian groups whose torsion group is cotorsion, and hence a summand of $G$.

Throughout, absent explicit mention to the contrary, $R$ will be a commutative Noetherian ring with $1 \neq 0$ and $M$ a unital $R$-module. In the special case of abelian groups, we write $G$ in place of $M$. We use $E$ to denote an arbitrary injective $R$-module, $E(M)$ represents the injective envelope of the $R$-module $M$, $\mathbb{Z}(n)$ the cyclic group of order $n$, and $\mathbb{Z}(p^\infty)$ the Prüfer $p$-group. We say a module $M$ is $P$-primary if $\text{Ass}(M) = \{P\}$ (some authors, e.g., [9], would prefer to say that $M$ is $P$-coprimary). We use this terminology for consistency with Matlis’ terminology and the usual terminology for abelian groups. For convenience, we now include a few results from previous work that will be useful.

**Theorem 0.1.** [13, Thms 1.2-1.4] Let $R$ be a ring.

1. Let $M$ be a co-Hopfian (Hopfian) $R$-module. If $M$ decomposes as a direct sum of a family $\{M_i\}$ of nontrivial $R$-modules, then each $M_i$ is co-Hopfian (Hopfian).

2. Let $\{M_i\}$ be a family of nontrivial $R$-modules for which $\text{Hom}(M_i, M_j) = \text{Hom}(M_j, M_i) = 0$ whenever $i \neq j$. If each $M_i$ is co-Hopfian (Hopfian), then so is $\oplus M_i$.

3. Let $M$ be a co-Hopfian (Hopfian) $R$-module. If $M$ decomposes as a direct sum of a family $\{M_i\}$ of nontrivial $R$-modules, then there are only finitely many summands isomorphic to a given $R$-module $N$. □
Goldsmith and Gong [7] have a relaxed version of part (2) of the theorem: “If 
A, B are (co-)Hopfian and either \( \text{Hom}(A, B) = 0 \) or \( \text{Hom}(B, A) = 0 \), then \( A \oplus B \) 
is (co-)Hopfian.” This is a useful result in the case in which the torsion subgroup of an abelian group is a summand, and is a special case of a more general result they established in [5]. This latter result extends easily to modules.

Certain changes of ring preserve the co-Hopfian (Hopfian) property.

**Theorem 0.2.** [14, Thm 1.3] Let \( R \to R' \) be an epimorphism in the category of rings. Let \( M, N \) be \( R' \)-modules. Then \( \text{Hom}_{R'}(M, N) = \text{Hom}_R(M, N) \).

This result is contained in Proposition XI.1.2 in [24]. It can also be found at the Stacks Project (stacks.math.columbia.edu/tag/04VM) in the form as written. We will use this theorem in the particular cases in which \( R \) is commutative and \( R' \) is a quotient of \( R \), a localization of \( R \), or, if \( R \) is a domain, the fraction field of \( R \).

1 Maximal orders

In [16], Matlis introduced the concept of a module with maximal orders, an \( R \)-module \( M \) such that for each nonzero \( x \in M \), the only prime ideals containing \( \text{ann}_R(x) \) are maximal ideals. For example, any torsion abelian group has maximal orders, or any module over an Artinian \( \text{PIR} \). Matlis offered these modules as a natural generalization of modules with descending chain condition (DCC). Henceforth, let \( \Omega \) represent the maximal spectrum of \( R \). For a given \( R \)-module \( M \) with maximal orders, there is, for each \( m \in \Omega \), an \( m \)-primary component, \( X_m(M) \) in Matlis’ notation, defined by

\[
X_m(M) = \{ x \in M : m^nx = 0, \text{ for some positive integer } n \},
\]

so that \( \text{Ass}(X_m(M)) = \{ m \} \).

The following theorems give the portions of two of Matlis’ theorems most relevant to us. They exhibit striking similarities between modules with maximal orders and torsion abelian groups.

**Theorem 1.1** ([16], Prop 3). Let \( M \) be an \( R \)-module. Then the following are equivalent:

1. \( M \) has DCC.
2. \( M \) is a submodule of \( E_1 \oplus \ldots \oplus E_n \), where \( E_i = E(R/m_i) \), \( m_i \) a maximal ideal of \( R \).
3. \( M \) has maximal orders and finitely generated socle.

**Theorem 1.2** ([16], Thm 1). Let \( M \) be an \( R \)-module. Then the following are equivalent:

1. \( M \) has maximal orders.
2. $M$ is an essential extension of its socle.

3. $M$ is a submodule of $\bigoplus_{m \in \Omega} (\oplus_{n_m} E(R/m))$, for cardinal numbers $n_m$.

4. $M = \bigoplus_{m \in \Omega} X_m(M)$.

5. Every finitely generated submodule of $M$ has finite length.

We are led immediately to the following closure result for co-Hopfian injective modules having maximal orders.

**Theorem 1.3.** Let $E = \bigoplus_{m \in \Omega} (\oplus_{n_m} E(R/m))$, with each $n_m < \infty$. Any submodule $M$ of $E$ is co-Hopfian.

**Proof.** We know $E$ is co-Hopfian [14, Thm 2.1]. There is no loss of generality in assuming $E = E(M)$. By Theorem 1.2, $X_m(M) \subseteq \oplus_{n_m} E(R/m)$ for each $m \in \Omega$. Since $n_m$ is finite, $X_m(M)$ has DCC (Theorem 1.1) and so is co-Hopfian. Hence, by Theorem 0.1, $M = \oplus X_m(M)$ is co-Hopfian, because $\text{Hom}(X_m(M), X_n(M)) = 0$ if $m \neq n$. \qed

This theorem generalizes the co-Hopfian part of an observation made in [14, Rem 3.6] for the case that $R$ is Artinian. If $R$ is not Artinian, then the module $E$ in the theorem may not be Hopfian, since $E(R/m)$ may not be Hopfian (see the discussion in [14, §2]).

**Corollary 1.4.** Let $M$ be an $R$-module having maximal orders. If $E(M)$ is co-Hopfian, then so is $M$. \qed

**Remark 1.5.** Theorem 1.3 has a Hopfian counterpart for abelian groups: any reduced subgroup of $E = \oplus_p \mathbb{Z}(p^\infty)$, the sum over the positive primes, is Hopfian. If $M \subseteq E$, then $M$ is torsion and so $M = \oplus M_p$ is the direct sum of its $p$-primary components. Furthermore, $M_p \subseteq \mathbb{Z}(p^\infty)$ for each $p$. If $M$ is reduced, then $M_p \neq \mathbb{Z}(p^\infty)$ for any $p$ since $M$ has no divisible subgroup. So, $M_p$ is a cyclic group for all $p$. Since $\mathbb{Z}$ is Noetherian, each $M_p$ is Hopfian. But $\text{Hom}(M_p, M_q) = 0$ if $p \neq q$, so $M$ is Hopfian. The remark remains valid if we replace the ring of integers by any principal ideal domain $R$. \qed

It takes only a little more effort to generalize to the case that $R$ is a Dedekind domain. Such a domain is a particular example of an $h$-local domain, an integral domain characterized by two conditions: (1) every nonzero prime ideal is contained in only one maximal ideal; (2) every nonzero element is contained in only finitely many maximal ideals.

**Theorem 1.6.** Let $R$ be a Dedekind domain and $M \subseteq \oplus_{p} E(R/P)$. If $M$ is reduced, then $M$ is Hopfian.

**Proof.** It is clear that $M$ is torsion. Matlis has shown that primary decomposition holds over any $h$-local ring (see any of [18 Thm 8.5], [19 Thm3.1], [20 Thm 22]). Hence, $M = \oplus_{p} M_P$, where $M_P = X_P(M)$, the $P$-primary component of $M$. Conveniently, $M_P \cong R_P \otimes M$ coincides with the localization of $M$ at $P$ [19].
Prop 2], so the notation should cause no confusion. Of course, $M_P \subseteq E(R/P)$ for all $P$ in $\Omega$.

If $M$ is reduced, then so is $M_P$ ($M_P$ is also an $R_P$-module; if $M_P$ had an $R_P$-injective summand $I$, then $I$ would be an $R$-injective summand of $M_P$, hence of $M$. Let $R$ Dedekind implies $R_P$ is a PID, so, as an $R_P$-module, $M_P$ is a reduced submodule of $R_P(\pi^\infty)$, where $\pi$ is a generator of the maximal ideal $PR_P$ of $R_P$. Hence, $M_P$ is a cyclic $R_P$-module and, since $R_P$ is Noetherian, a Hopfian $R_P$-module.

Let $f : M \to M$ be an $R$-module epimorphism. Then the localized map $f_P : M_P \to M_P$ is an $R_P$-module epimorphism for each $P$. Since each $M_P$ is Hopfian, each $f_P$ is a monomorphism and so $f$ is a monomorphism and $M$ is Hopfian.

Remark 1.7. (1) There is a common thread running through Remark 1.5 and Theorem 1.6. If $R$ is an integral domain, then it has a fraction field $Q$, and we may form the quotient module $K = Q/R$. The module $K$ is a divisible $R$-module since $Q$ is. However, despite the fact that $Q$ is an injective $R$-module, $K$ need not be, since a quotient of an injective is not necessarily injective. But, if $R$ is hereditary, for example if $R$ is a Dedekind domain, then $K$ will be injective [1] §1.5. Since $K$ is a torsion module, it is the direct sum of its $P$-torsion components $K_P$, $P \in \Omega$. Once again, it is convenient that the $P$-component $K_P$ is, in fact, the localization of $K$ at $P$, and is isomorphic to $E(R/P)$ [17, Thm 5, p 580]. The importance of the module $K$ can be appreciated by consulting [13, 17, 18, 19, 20].

(2) Theorem 1.2.4, in conjunction with Theorem 0.1, allows us to reduce questions of the of co-Hopficity (Hopficity) of modules $M$ with maximal orders to the question of the co-Hopficity (Hopficity) of the $m$ primary components $X_m(M) : M$ is co-Hopfian (Hopfian) if and only if $X_m(M)$ is co-Hopfian (Hopfian) for each $m \in \Omega$. This observation allows us to produce many co-Hopfian or Hopfian modules. For example, at the level of abelian groups, $G = \oplus G_p$, the sum over the positive primes, is both co-Hopfian and Hopfian if each $G_p$ is a finite abelian $p$-group.

(3) Theorem 1.3 applies to torsion modules over a 1-dimensional domain, in particular to torsion abelian groups (as well as to modules over Artinian rings). It would be nice to know what conditions allow a converse, that is, when $M$ co-Hopfian implies $E(M)$ is. We saw in [14] that the converse holds if $R$ is an Artinian $PIR$. We will say more in this regard in Section 5.

Recall that an $R$-module is finitely cogenerated if its injective envelope is a finite sum of injective envelopes of simple modules [14, Def 3.4]. Using the two theorems of Matlis, we can rephrase Theorem 1.3 as follows.

**Theorem 1.8.** If $M$ is an $R$-module having maximal orders, and $X_m(M)$ is finitely cogenerated for each $m \in \Omega$, then $M$ is co-Hopfian (note that since $\Omega$ may be infinite, we are not assuming that $M$ itself is finitely cogenerated).
2 The role of scalar multiplication

Co-Hopfian modules over a commutative Noetherian ring $R$ are naturally modules over a ring of quotients. This fact is particularly useful if the module is finitely generated.

If $M$ is an $R$-module, then left multiplication by $r \in R$ defines an endomorphism of $M$ which is well-known to be an injection if and only if $r$ belongs to no associated prime of $M$. The set $S$ which is the complement in $R$ of the union of the associated primes of $M$ is multiplicative. If $M$ is co-Hopfian, then each element of $S$ acts as an automorphism on $M$ and $M$ is an $S^{-1}R$-module [21, Prop 6.5] (Matlis [15, Lemma 3.2] used this approach, at least implicitly, to show that $E(R/P)$ is an $R_P$-module for $P$ a prime ideal of $R$).

Since the canonical map $R \rightarrow S^{-1}R$ is an epimorphism in the category of rings, $M$ is co-Hopfian (Hopfian) as an $R$-module if and only if it is co-Hopfian (Hopfian) as an $S^{-1}R$-module (Theorem 0.2). This approach gives an easy proof of the fact that a torsion-free divisible abelian group is co-Hopfian (Hopfian) if and only if it is a finite dimensional vector space over the rationals. However, we get no help in the case of torsion abelian $p$-groups, since such a group is always a $\mathbb{Z}(p)$-module. We also get no help for groups such as $G = \bigoplus \mathbb{Z}(p)$, the sum over the positive primes, for which every nonzero prime ideal of $\mathbb{Z}$ is an associated prime of $G$, so that $S = \{\pm 1\}$ and the associated ring of quotients is $\mathbb{Z}$.

If $R$ is any commutative ring, and $R$ is not co-Hopfian as an $R$-module, then the total quotient ring of $R$, the ring $Q = S^{-1}R$, with $S$ the set of nonzerodivisors of $R$, has the property that if $R \subseteq M \subseteq Q$, with $M$ an $R$-module, then $M$ is not co-Hopfian. In order to justify this claim, we recall some terminology [13, §2]. An $R$-module $M$ is called torsion-free if multiplication by $a$ is an injective endomorphism of $M$ for each regular element $a$ of $R$, and $M$ is divisible if $aM = M$ for each regular element $a$ of $R$. It is clear that $R$ is torsion-free as an $R$-module, and that $Q$ is also torsion-free. Hence, if $M$ is an $R$-module and if $R \subseteq M \subseteq Q$, then $M$ is torsion-free. So, if $M$ is co-Hopfian, then $M$ must be divisible [13 Thm 2.1]. Thus, multiplication by any regular element $a$ must be an automorphism of $M$. Hence, as discussed above, $M$ must be a $Q$-module. This is not the case ($1_R \in M$, so $qM \nsubseteq M$ if $q \in Q \setminus M$, whence $M$ is not a $Q$-module). So, we have established the following result.

**Theorem 2.1.** Let $R$ be a commutative ring, $Q$ its total quotient ring. Then $Q$, thought of as an $R$-module, is a minimal co-Hopfian extension of the $R$-module $R$. 

**Remark 2.2.** Eisenbud [3, §11.3] calls any $R$-submodule of the total quotient ring of $R$ a fractional ideal, a concept most of us are more familiar with in the context of finitely generated submodules of the fraction field of an integral domain. Lambek [10, §4.3 Exercise 13] establishes an even more general context.

If $R$ is commutative, $M$ a nonzero $R$-module, and $I = \text{ann}_R(M)$, then $M$ is a faithful $R/I$-module. Since $M$ is co-Hopfian as an $R$-module if and only if it
is co-Hopfian as an $R/I$-module (Theorem 0.2), there is no loss of generality in assuming that $M$ is a faithful $R$-module.

So, let $M$ be a nonzero faithful co-Hopfian $R$-module. Moreover, let $M$ be finitely generated. This last assumption places significant restriction on $R$. Let $U$ be the set of units of $R$. If $r \in U$, it is clear that left multiplication by $r$ is an automorphism. Conversely, if left multiplication by $r$ is an automorphism, then $JM = M$, where $J = (r)$. Since $M$ is faithful, $J = R$ and so $r \in U$ [9, Lemma p 174]. So, if $M$ is finitely generated, then left multiplication by $r \in R$ is an automorphism of $M$ if and only if $r$ is a unit.

If, in addition, $R$ is Noetherian, then left multiplication by $r$ is a monic endomorphism of $M$ if and only if $r$ is in $S$, the complement in $R$ of the union of the associated primes of $M$. If $M$ is co-Hopfian, this endomorphism must be an isomorphism. By the preceding argument, $S = U$. But $U$ is the complement in $R$ of the union of the maximal ideals of $R$. Since $M$ is finitely generated (and nonzero) and $R$ is Noetherian, $\text{Ass}(M)$ is a finite nonempty set. Hence, the union of the maximal ideals of $R$ is equal to the union of the associated primes of $M$. So, we have the following theorem.

**Theorem 2.3.** Let $M$ be a nonzero finitely generated faithful $R$-module, $R$ commutative Noetherian. If $M$ is co-Hopfian, then $R$ is semi-local, and its maximal ideals are precisely those ideals maximal in $\text{Ass}(M)$. \hfill $\blacksquare$

If the only associated primes of $M$ are maximal, then $R$ must be Artinian.

### 3 An example

Generally speaking, a module and its injective envelop are independent relative to co-Hopficity. This feature is easily exhibited for abelian groups. The group $\mathbb{Z}$ is not co-Hopfian but $\mathbb{Q}$, its injective envelope is. The following example provides a group that is co-Hopfian, but whose injective envelope is not.

**Example 3.1.** In personal correspondence, E. Enochs sketched a proof that the abelian group $G = \prod \mathbb{Z}(p)$, the product over positive primes $p$, is co-Hopfian. In response to a question we posed on math.stackexchange.com [11], a colleague (“moonlight”) provided an argument that $E(G) = \mathbb{Q}^c \bigoplus (\oplus_{p} \mathbb{Z}(p^\infty))$, where $c$ is the cardinality of the continuum, so $E(G)$ is clearly not co-Hopfian (Theorem 0.1). \hfill $\blacksquare$

If $G$ be the countably infinite direct sum of copies of $\mathbb{Z}(p)$ for some prime $p$. Then $G$ is not co-Hopfian (Theorem 0.2). But $E(G)$ is then the countably infinite direct sum of copies of $\mathbb{Z}(p^\infty)$ and is also not co-Hopfian (see section 6). Finally, if $G$ is finitely generated, then $E(G)$ is the sum of finitely many $\mathbb{Z}(p^\infty)$ for various primes $p$ [23 Thm 4.7], and so both $G$ and $E(G)$ are co-Hopfian.

Rotman [22] used the group $G$ in the example to show that the torsion subgroup of an abelian group is not necessarily a direct summand. We used it in [13] to show that the class of co-Hopfian abelian groups is not closed under quotients. Via the example, that class is not closed under essential extensions.
either. Furthermore, \( G \) provides a counterexample to the converse of Theorem 3.1 of [13], since \( G \) and \( tG \) are co-Hopfian, but \( G/tG \) is not.

The example can be extended to modules over certain integral domains in exact analogy to the abelian group case. We will discuss the details of the example, its extension, and some generalizations, in a future article.

4 Chain conditions

It is well-known that if an \( R \)-module \( M \) has DCC, then \( M \) is co-Hopfian. This result holds for general \( R \), associative with 1, and unital left \( R \)-modules \( M \). E. Enochs suggested that we examine if there is any relation between co-Hopficity and finite length for finitely generated \( R \)-modules if \( R \) is commutative Noetherian. The following example shows that there are finitely generated modules over such \( R \) which are co-Hopfian but not of finite length.

**Example 4.1.** Any commutative Noetherian ring \( R \) has a commutative Noetherian total quotient ring \( Q = S^{-1}R \), where \( S \) is the set of nonzerodivisors of \( R \). But \( Q \) is an Artinian ring (and so of finite length as a \( Q \)-module) if and only if all prime ideals in \( \text{Ass}(R) \) are minimal [24, p. 287, example 2]. So, let \( R \) be any commutative Noetherian ring for which \( \text{Ass}(R) \) has embedded primes. Then \( Q \) is not an Artinian ring, and so not of finite length as a \( Q \)-module. But \( Q \) is finitely generated (by 1) as a \( Q \)-module, and hence co-Hopfian since it is a quoring [13, Thm. 2.3].

For instance, let \( R = k[x, y] \), \( k \) a field. Let \( I = (x^2, xy) \). The ideal \( I \) has primary decomposition \((x) \cap (x, y)^2\). Hence, the associated primes of \( I \) are \( P_1 = (x) \) and \( P_2 = (x, y) \), and \( P_1 \subset P_2 \). Let \( Q \) be the total quotient ring of \( R/I \). This particular example can be generalized to any finite number of variables, yielding a \( Q \) of any finite Krull dimension.

Let \( R = k[x_1, \ldots, x_n] \). Let \( I = (x_1^2, x_1x_2, \ldots, x_1x_n) \). Then \( I \) has primary decomposition \((x_1) \cap (x_1, \ldots, x_n)^2\). Hence, the associated primes of \( I \) are \( P_1 = (x_1) \) and \( P_2 = (x_1, \ldots, x_n) \) with \( P_1 \subset P_2 \). Let \( Q \) be the total quotient ring of \( R/I \). Then \( Q \) has dimension \( n \).

Generally, co-Hopficity is independent of chain conditions: the abelian group \( \mathbb{Z}(p^\infty) \) has DCC but not ACC, while the \( Q \)-module \( Q \) of the preceding example has ACC but not DCC. Any finite abelian group is co-Hopfian and has both chain conditions, while \( Q \), thought of as an abelian group, has neither.

The next theorem, which is elementary, determines when co-Hopfian implies finite length. Maximal orders are relevant.

**Theorem 4.2.** Let \( R \) be commutative Noetherian, \( M \) a finitely generated co-Hopfian \( R \)-module, and \( I = \text{ann}_R(M) \). The following are equivalent.

1. \( M \) has finite length.
2. \( R' = R/I \) is Artinian.
3. \( M \) has maximal orders.

4. \( M \) is Artinian as an \( R' \)-module.

5. \( M \) is finitely cogenerated as an \( R' \)-module.

**Proof.** By Theorem 1.4, \( R/I \) is semilocal and \( \text{Ass}(M) \supseteq \Omega(R/I) \). For \( M \) to have finite length, we must have \( \text{Spec}(R/I) = \Omega(R/I) \), i.e., \( R/I \) must be Artinian. So, the equivalences (1) \( \iff \) (2) \( \iff \) (4) are clear [3 Cor. 2.17].

(1) \( \Rightarrow \) (3) : Finite length implies DCC, implies maximal orders by Theorem 1.1.

(3) \( \Rightarrow \) (5) : By Theorem 1.2, \( M \) is an essential extension of its socle and \( M \) is a submodule of \( \bigoplus_{m \in \Omega} (\oplus_{n_m E(R/m)}) \), for cardinal numbers \( n_m \). But, the sum has only finitely many summands [23 Thm. 4.7]. So, the socle of \( M \) is finitely generated and \( M \) is finitely cogenerated.

(5) \( \Rightarrow \) (1) : \( M \) is an essential extension of its socle, and the socle is finitely generated. So, \( M \) has the form stated in Theorem 1.1.2. Hence, \( M \) has DCC, and so finite length.

**Remark 4.3.** The ring \( Q \) from the example has annihilator 0 as a \( Q \)-module. It does not have finite length. But \( Q \) is not Artinian, so the example is consistent with the theorem.

Generally, if \( M \) has maximal orders, then \( R/\text{ann}_R(M) \) is a 0-dimensional ring. As mentioned in the introduction, Vasconcelos showed that all finitely generated \( R \)-modules are co-Hopfian if and only if \( R \) is 0-dimensional. One might naturally wonder for which 0-dimensional \( R \) is it the case that the only co-Hopfian \( R \)-modules are the finitely generated ones. We discussed this question in [14] and showed that if \( R \) is an Artinian PIR, then an \( R \)-module \( M \) is co-Hopfian if and only if it if finitely generated.

## 5 Abelian groups

Throughout this section, all groups are abelian. Theorem 0.2 shows that if there is a well-defined structure theory for a class of modules, then we may be able to exploit that structure to extract some information about the module relative to the Hopfian or co-Hopfian property. We did this throughout [14]. For certain abelian groups, we can follow the same strategy.

A group \( C \) is **cotorsion** if every extension of \( C \) by a torsion-free group splits. So, if the torsion subgroup \( tG \) of a group \( G \) is cotorsion, then it is a summand of \( G \). Hence, \( G \) is co-Hopfian if and only if both \( tG \) and \( G/tG \) are (cf. the remark following the proof of [13 Thm 3.1]). Thus, \( G/tG \cong \mathbb{Q}^n \) for some positive integer \( n \) (torsion-free and co-Hopfian implies divisible [13 Thm 2.1]). Torsion cotorsion groups have a well-known structure: they are the direct sum of a divisible group and a bounded group [4 Cor 54.4]. A group \( B \) is **bounded** if \( nB = 0 \) for some positive integer \( n \). We will always assume that \( n \) is the least
such integer. This usage of bounded is traditional in (infinite) abelian group theory.

So if \( tG \) is cotorsion and co-Hopfian, then \( tG \cong D \oplus B \), where \( D \) is torsion divisible and co-Hopfian, and \( B \) is bounded and co-Hopfian. The group \( D \) is the direct sum of finitely many (and possibly 0) groups \( \mathbb{Z}(p^\infty) \) for each prime \( p \). The bounded group \( B \) is the direct sum of torsion cyclic groups \([1 \text{ Thm } 17.2]\). But \( nB = 0 \) implies that \( B \) is the direct sum of finitely many \( p \)-components, one for each prime divisor of \( n \). So, we need only know when a bounded \( p \)-group is co-Hopfian (see Remark 1.7 (2)).

**Theorem 5.1.** A bounded \( p \)-group is co-Hopfian (Hopfian) if and only if the group is finite.

**Proof.** Let \( G \) be a bounded \( p \)-group, and suppose \( p^kG = 0 \). Then we can express \( G \) as \( G = \bigoplus_{j=1}^{k-1} (\bigoplus_{n_j} \mathbb{Z}(p^j)) \). By Theorem 0.1, \( G \) co-Hopfian (Hopfian) implies each \( n_j \) must be finite. Hence, \( G \) is a finite sum of finite groups and so is a finite group. Hence, \( G \) is co-Hopfian (Hopfian). The converse is obvious since any finite group is co-Hopfian (Hopfian).

The following corollary is immediate.

**Corollary 5.2.** A bounded group is co-Hopfian (Hopfian) if and only if each of its \( p \)-primary components is a finite group, whence it is a finite group.

**Remark 5.3.** Theorem 5.1 applies more generally to \( p \)-groups which are direct sums of cyclics. As in the proof of Theorem 5.1 if \( G = \bigoplus_{j=1}^{\infty} (\bigoplus_{n_j} \mathbb{Z}(p^j)) \) is co-Hopfian (Hopfian), then each \( n_j \) must be finite. It remains to show that \( n_j \neq 0 \) for only finitely many \( j \). In the contrary case, we can use “shift operators” as in \([28 \text{ p } 300]\) to show that \( G \) is not co-Hopfian (Hopfian). The corollary follows immediately (for torsion \( G \) which are direct sums of cyclics).

The structure of co-Hopfian torsion cotorsion groups is now obvious. Let \( D = \bigoplus_p (\bigoplus_{n_p} \mathbb{Z}(p^\infty)) \), the exterior sum over the positive primes with the \( n_p \) nonnegative integers (to insure \( D \) is co-Hopfian). Then \( D \) is a co-Hopfian divisible (=injective) abelian group.

**Theorem 5.4.** A torsion cotorsion group \( G \) is co-Hopfian if and only if \( G = D \oplus B \), where \( B \) is a finite group.

With \( D \) and \( B \) as in the theorem, we have the following.

**Corollary 5.5.** Let \( G \) be a group for which \( tG \) is cotorsion. Then \( G \) is co-Hopfian if and only if \( G \cong \mathbb{Q}^n \oplus D \oplus B \).

**Remark 5.6.** It is well-known that the group \( \mathbb{Z}(p^\infty) \) is not Hopfian, but that the group \( \mathbb{Q} \) is. Hence, a torsion cotorsion group is Hopfian if and only if it is a finite group. Similarly, if \( G \) is a group for which \( tG \) is cotorsion, \( G \) is Hopfian if and only if \( G \cong \mathbb{Q}^n \oplus B \).
Now, let $E \cong \mathbb{Q}^n \oplus [\oplus_p (\oplus_{n_p} \mathbb{Z}(p^\infty))]$ be an arbitrary injective (=divisible) co-Hopfian abelian group. What are the co-Hopfian subgroups of $E$? Based on what we have done so far, there are some obvious examples:

1. The torsion-free co-Hopfian subgroups of $E$ are isomorphic to $\mathbb{Q}^m$ with $m \leq n$.
2. Any torsion subgroup of $E$ is co-Hopfian (Theorem 1.3).
3. Any subgroup $G = \mathbb{Q}^m \oplus T$, where $T$ is a torsion subgroup of $\bigoplus (\oplus_{n_p} \mathbb{Z}(p^\infty))$ and $m \leq n$, is co-Hopfian.
4. If the expression for $E$ involves only a finite number of primes, then the only subgroups of $E$ are the $\mathbb{Q}^m \oplus T$ described in item (3), since for any subgroup $G$ of $E$, $tG$ will be cotorsion. We will examine the case involving infinitely many primes in the future.

Remark 5.7. Note that in the third item each $p$-component of $T$ is cotorsion, even though $T$ itself may not be (say $T = \oplus_p \mathbb{Z}(p)$, the sum over the positive primes).

The discussion generalizes to an arbitrary integral domain $R$ if we restrict $E$ to injectives of the form $E \cong K^n \oplus \bigoplus_{m \in \Omega} (\oplus_{m_p} E(R/m))$, $K$ the field of fractions of $R$, and the $m_p$ nonnegative integers.

6 A correction

The direct product portion of [13, Thm. 1.3] is incorrect. H. Storrer was kind enough to provide a counterexample. Let $G = \prod \mathbb{Z}(p^\infty)$, the product being over the positive primes. The torsion subgroup of $G$ is $\oplus \mathbb{Z}(p^\infty)$, and is a direct summand of $G$. The complementary summand is isomorphic to a direct sum of uncountably many copies of $\mathbb{Q}$ [22, Exercise 9.36], and so is not co-Hopfian [13, Theorem 1.4]. Therefore, $G$ is not co-Hopfian by the same theorem. Nonetheless, [13, Example 3] is unaffected, since the group $\prod \mathbb{Z}(p)$ is co-Hopfian by the argument alluded to in the example of Section 3.

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