General relativistic rotational energy extraction from black holes-accretion disk systems

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The determination of mass and spin parameters of the black holes (BHs) is crucial in the analysis of the merger of BHs and BHs formation and evolution, including accretion. Here we constrain the BH spin with the evaluation of the dimensionless parameter ξ representing the total rotational energy extracted versus the mass of the BH, following procedure introduced in [1] that is independent from the details of the specific extraction process. The energy extraction can power an outflow which can be then observed. We relate the energy extraction to the accreting configurations and the accretion processes occurring in a cluster of agglomerate corotating and counter-rotating tori orbiting one central Kerr SMBH, associating ξ to the characteristics of the accretion processes. We relate the regions of tori parameters to features of the energy extraction processes, binding ξ to properties of light surfaces by using the bundles developed in [2], relating measures in different regions of the spacetimes. We evaluate properties of the BH accretions disks, and correlate spacetimes before and after their transition due to the energy extraction. Light surfaces are related to the generators of Killing horizons, proving limiting frequency of the stationary observers of the geometries. We consider the photon limiting curves of the stationary observers as constraints for various processes regulated by these frequencies, to relate different BH states, prior and after the energy extraction, investigating regions close to the BH horizons and rotational axis. From methodological viewpoint we used a naked singularity -BH correspondence defined with metric bundles to predict observational characteristics of the BH-accretion disk system. The analysis points relevant BH spins a ≈ 0.94M, a ≈ 0.7M and a ≈ 0.3M. We show the relation between the rotational law of the tori, the characteristic frequency of the bundle and the relativistic velocity defining the von Zeipel surfaces. The inferior limit on the formation of corotating tori is ℓ/a ≥ 2, for counter-rotating tori ℓ/a ≤ −22/5 (ℓ is the fluids specific angular momentum).

Keywords: Black holes– Accretion disks–Accretion; Hydrodynamics –Galaxies: active – Galaxies: jets

I. INTRODUCTION

We examine the extraction of the rotational energy from a Kerr black hole (BH) in the presence of accretion disks. The orbiting accretion matter is considered in the form of aggregates of pressure supported geometrically thick disks of perfect fluids, analyzing purely hydrodynamic models of both tori corotating and counter-rotating with respect to the central Kerr BH, and centered on the Kerr BH equatorial plane. This model of tori aggregate is known as eRAD, or RAD when the toroidal components are also misaligned with respect to each others and the rotational axis of BH[4–7]. Our investigation is centered on the exploitation of a connection between supermassive black holes (SMBHs) and their accretion tori and the states of BHs before and after the process of energy extraction.

The BH rotational energy is supposed to be source of many processes of the high energy astrophysics as the active galactic nuclei (AGN) large-scale outflows [1, 8–10]. The relation between BH spin–disk rotational law, accretion and rotational energy extraction is manifold. The luminosity intensity of the AGNs for example is generally attributed to the fall of gas from the accretion disk on the SMBH, located at the center of the galaxy, therefore en-powering the observed jet emission (see for example recent analysis of the galaxy in the X-band NGC2992 [11], and [12] for a discussion of high-resolution image of the interaction between gas clouds and jets of material ejected from a SMBH at the center of the quasar galaxy MG J0414+0534). However, in different phases of SMBHs evolution cocoons of matter may envelop the hole [13]–consequently SMBHs can also be darkened by surrounding material which can be observed on the X-ray band produced by the accreting fluids1. These plasma concentrations, can show

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1 For example mission to explore the X-ray emission sector: XMM-Newton X-ray Multi-Mirror Mission)http://sci.esa.int/science-e/www/area/index.cfm?fareaid=23, RXTE (Rossi X-ray Timing Explorer) http://heasarc.gsfc.nasa.gov/docs/xte/xtegof.html or ATHENA http://the-athena-x-ray-observatory.eu/
a remarkable variation in their luminosity following the phases of BHs growing – [14, 15]. A model of a cluster of misaligned (inclined) tori orbiting a central static SMBH has been developed in the RAD frame in [16, 17] as an orbiting aggregate composed by tori with different inclination angles relative to a central SMBH. The RAD accreting agglomeration can be seen, depending on the tori thickness, as a (multipole) gobules of orbiting matter, with different toroidal spin orientations, covering the embedded central SMBH. BH spin and Galactic morphology have been argued to be strongly connected, for example SMBHs in elliptical galaxies possess higher spins than those in spiral galaxies [18]. The spins seem to be related also to the redshift parameter.

In this regard the investigation is increasingly directed to find correlations between different phenomena framed in the BH-disk system: for example in the BH populations and galaxy age correlation and BH–spin shift–accretion disk correlation and in the jet-accretion correlation–see for example [19–31]. The BH spin variation follows BH interaction with its environment, therefore many approaches in the evaluation of the BHs parameters and estimation of BH accretion history are connected to the accretion disk physics, based for example on the evaluation of the disk luminosity or accretion rates. It has been shown that the dimensionless spin of a central BH \( (a/M) \) and the morphological and equilibrium properties of its accreting disks, are strongly related [32]. SMBHs spin depends on the angular momentum of the infalling materials tracing back a story of BH accretions in diverse epochs of the BH history [33]. However, an issue of these methods consists also in the fact that the SMBHs spin (with mass of \( 10^6 – 10^9 M_\odot \), \( M_\odot \) being solar mass) is strictly correlated with the “mass-problem” and connected with the evaluation of the main features of the accretion processes as the BH accretion rate or the location of the inner edge of the accretion disk. These methods are clearly model dependent [34–36], constituting a relevant issue considering that even the definition of the inner edge of an accreting disk is controversial -see for example [1, 37–42]. On the other hand gravitational waves detection from coalescence of BHs in a binary system may serve, in future, as a further possible method to fix a BH spin parameter [43–47].

In this work we consider the approach introduced in [1], focused on the definition of SMBH (irreducible) mass function and the definition of rotational energy, and the BH classical thermodynamical law adapted for the analysis of the accretion history of SMBHs in presence of episodic accretion and spin shifts. This approach bridges the BH classical thermodynamic laws with the physical processes, leading to a BH transition through interaction with the surrounding environment.

SMBHs are powerful engines of ejections of matter and energy where the radiative efficiency of energy extraction from rotational energy depends on the spin. In numerous examinations of the energy extraction through the accretion process, the rotational energy converted into radiation corresponds to the binding energy of the fluid, connecting the angular momentum of the accreting material with the radiative efficiency. Furthermore, the SMBHs mass growth rate can be connected with a luminosity function of the host galaxy. Here we consider BHs accreting at (super) Eddington luminosity, and super Eddington accretion disks which are geometrically thick and opaque, pressure supported and cooled by advection–[48–51].

From methodological view-point, we use geometrically thick, "stationary" disks as largely adopted as the initial conditions in the set up for simulations of the GRMHD (magnetohydrodynamic) accretion structures for the numerical analysis–[52–55]. The RAD and eRAD models are essentially "constraining-models", providing initial data for dynamical situations. The location of the inner edge of an accreting torus is indeed a key element of the BHs spin estimation, and it is usually identified with marginally stable orbit or with a radius in the region bounded by marginally bounded circular orbit and marginally stable circular orbit according to the accretion disk model, geometrically thick or thin, and the mass of the central attractor. Thick (stationary) disks give a striking good approximation of several aspects of accretion also for more complex dynamical models, estimating the tori elongation on their symmetry plane, the location of the inner edge of quiescent and accreting disks, the location of critical pressure points, an evaluation of the tori thickness, their maximum height.

The inner ringed structure of the eRAD offers a set of interesting scenarios that include its unstable states. More points of accretion can be present in the eRAD inside the ringed structure, which may also include inner shells of jets. Jet in the BH-accretion disk systems can also change the accretion disk inner edge [56–66]. In this article we discuss also the correlation between the dimensionless spin of the central BH and the specific angular momentum of the fluid in the orbiting tori, characterizing the system with the rationalized specific angular momenta of the fluid as \( \ell/a \) or \( \ell/a \sin \theta \), and functions of these variables, supported by considerations related to the general relativistic features typical of the Kerr geometry and describing the rotational law of the eRAD.

The approach introduced in [1] for the determination of SMBHs spin, is based on the evaluation of the dimensionless ratio, \( \xi \), of the released energy versus the BH mass, considering the definition of irreducible mass of BH and the BHs thermodynamical laws connecting the BH spin \( a \) as function of the spin energy and the BH mass. This
approach is quite independent from the details of the specific process of energy extraction\(^2\). Therefore the quantity \(\xi\), which could be measured eventually as energy released from jet ejection for example, can provide an indication of the lower limit on the BH. The BH spin can be evaluated by the observation of the BH mass \(M\) and the energy outflow\([72]\). In the non-isolated SMBHs the outflows, related to the orbiting accretion matter, provide an indication of features of the BH-accretion disks systems. In this scenario a relevant topic is represented by the assumption of counter-rotating material which is an intrinsic feature of the eRAD model, as well as the presence of a magnetosphere. This estimation implies the black hole spin changes only due to the BH mass using definition of irreducible mass, grounded on assumption that energy outflow would be powered by the BH rotational energy only \([36]\).

We constrain the BH rotational energy estimating parameters of accretion features in the eRAD frame and the frequencies of light surfaces related to the analysis of many aspects of BHs physics, limits of stationary observers frequencies, connecting measures in different points of spacetime accessible to the observer and points of different space-times before and after the transition induced by the energy release. We use the concept of metric Killing bundles (MBs) introduced in \([2]\), which are conformal invariant and can be easy read in terms of the light surfaces. Metric Killing bundles define also properties of the local causal structure and thermodynamical properties of BHs as the surface gravity, temperature and luminosity\([73]\). Metric bundles can be used to investigate properties of the geometries close to the horizons, connecting the different metrics of the bundles. The observer could extract information (locally) of the region close to the horizons \(r_+\) and connecting different geometries of the bundle curves, i.e. there is a Killing horizon \(\mathcal{H}\) satisfying the condition \(\mathcal{L}_{\mathcal{H}} \equiv \mathcal{L} \cdot \mathcal{L} = 0\), where \(\mathcal{L}\) is a Killing vector of the geometry \(\mathcal{L} \equiv \partial_t + \omega \partial_\phi\)\([2, 74–77, 79]\). The event horizons of a spinning BH are Killing horizons with respect to the Killing field \(\mathcal{L}_H \equiv \partial_t + \omega_H \partial_\phi\), where \(\omega_H\) is the angular velocity of the horizons. Conditions on \(\omega_H = \text{constant}\) represents the BH rigid rotation. Characteristic frequencies \(\omega\) of the MBs are also the horizons frequencies\(^3\). Metric bundles are defined as curves in an extended plane, \(\mathcal{P} - r\), where \(\mathcal{P}\) is the metrics family parameter and \(r\) is a radial distance. In the extended plane we can consider the horizons confinement and the horizons replicas. For a property \(Q_\pm\) of the horizon as distinguished in the extended plane, as the horizon frequency \(\omega\), there is a replica of the horizon, in the same spacetime when there is a orbit (radius) \(r_+ > r\), such that \(Q(r_+) \equiv Q_+ = Q(r_)\) where \(r_\) is a point of the horizon curve in the extended plane. There are horizon replicas in different geometries along the bundle curves, i.e. there is a \(p \neq p_\) and a \(r_+ > r_\), where \(p\) and \(p_\) are values of the extended plane parameter \(\mathcal{P}\), corresponding to two different geometries (distinguished with two horizontal lines of the extended plane) such that: \(Q(r_+(p), p) \equiv Q_+ = Q(r_+(p_\), p_\). In both points, \((r_+, r_\), there is equal light-like orbital frequency. The horizon confinement is vice versa interpreted as the existence of a region of the extended plane \(\mathcal{P} - r\) where MBs are entirely confined i.e., there are no horizon replicas in any other region of the extended plane, particularly in any other geometry. However this definition is often specified to the confinement of the \(Q\) property in the same geometry. In the Kerr spacetime, this region is upper bounded in the extended plane by the a portion of the horizon curve corresponding to the set of the inner horizon BHs\([2, 76, 77, 79]\).

In this article we consider phenomenology and processes constrained and regulated by the characteristic frequencies of the bundles. We investigate the extraction of rotational energy in dependence of the accretion features and in relation to the light surfaces through MBs by extrapolating information of the physics close to the BH rotational axis and BHs horizons with the concept of replicas. We predict the possibility of observational evidence of the presence of replicas of the horizon, and indications of the replicas along the bundle curves evident in the thermodynamical transitions of the BH after energy release. We claim that it should be possible to observe the replicas along the axis of the black hole. MBs connect different BH states, prior and after the energy extraction. We consider also the rotational law of the fluid in dependence on the central BH spin.

We note that in more general contexts there can be a different setup on the initial and final state of the BH accretion system, a different outflow contribution and the establishment of special instabilities which can change the BH disks system or even change the BH spin orientation due to Bardeen–Petterson effect, where the BH spin changes under...

\(^2\) Note that the rotation energy extraction process is usually connected with the magnetic fields around the BHs, e.g. the jets are usually connected with the Blandford-Znajek process\([67]\) that could be considered as specific form of the magnetic Penrose process\([68, 69]\) that can demonstrate such extraordinary phenomena as acceleration of protons and ions to high energy\([88]\) due to the so called chaotic scattering studied in\([70]\). Quite recently a new variation of the Penrose process has been presented in\([71]\), that represents new efficient way of extraction of rotational energy related to radiation reaction of charged particles in magnetic fields.

\(^3\) In general, a Killing horizon is a light-like hypersurface, generated by the flow of a Killing vector, where the norm of a Killing vector is null. The hypersurface null generators coincide with the orbits of an one-parameter group of isometries, thus there exists a Killing field \(\mathcal{L}\), which is normal to the null surface.
the action of the disk torques-\cite{97}. We base our analysis on the hypothesis that the energy outflow to be measured is powered by the BH spin energy only, however more generally the extracted energy can be determined considering the ratio between the outflow energy versus the spin energy while we consider here the two quantities be coincident. The establishment of runaway instability and the tori self-gravity are also factors neglected in this model that may be relevant, requiring a different characterization of the energy outflow. Furthermore we assume that, from an initial Kerr BH, the final stage of extraction process is a static spherically symmetric Schwarzschild spacetime and we do not consider the contribution of the mass and momentum to the BH subsequent to accretion while geometrically thick disks have large accretion rates (with super Eddington luminosity).

The article plan is as follows: In Sec. (II) we introduce the eRAD model. Sec. (III) focuses on the energy extraction processes: discussion on the energy-spin relations is in Sec. (III A). Metric bundles, horizon replicas and photon frequencies are the focus of Sec. (III B). Tori energetics and accretion is discussed in Sec. (III C). Sec. (IV) contains notes on the RAD and BH-accretion disk spin correlation: the discussion is deepened in Sec. (IV A), while special sets of tori are introduced in Sec. (IV B). Discussion and concluding remarks follow in Sec. (V). Four Appendix sections close this article: in Sec. (A) and Sec. (B) are some explicit solutions, while in Sec. (C) the relation between Von Zeipel surfaces, metric Killing bundles and tori is investigated and in Sec. (D) an adapted solution parametrization is discussed.

II. FLUID CONFIGURATIONS ON THE KERR SPACETIME

We consider geometrically thick tori of perfect fluids orbiting a central Kerr SMBH. The metric tensor of the spacetime background can be written in Boyer-Lindquist (BL) coordinates \( \{t, r, \theta, \phi\} \) as follows

\[
ds^2 = -dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + \frac{2M}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 ,
\]

\[
\rho^2 \equiv r^2 + a^2 \cos \theta^2 , \quad \Delta \equiv r^2 - 2Mr + a^2 , \quad r_{\pm} \equiv M \pm \sqrt{M^2 - a^2} ; \quad r_+ \equiv M + \sqrt{M^2 - a^2} ,
\]

where \( r_{\pm} \) are the outer and inner Killing horizons respectively, \( r_+ \) is the outer ergosurface. Here \( M \) is the (ADM and Komar) mass parameter and the specific angular momentum is given as \( a = J/M \), where \( J \) is the total angular momentum of the gravitational source. There is \( r_+ < r_+ \) on the planes \( \theta \neq 0 \) and is \( r_+ = 2M \) on the equatorial plane \( \theta = \pi/2 \). In the region \( r \in [r_+, r_+] \) (outer ergoregion or simply ergoregion) there is \( g_{tt} > 0 \) and \( t \)-Boyer-Lindquist coordinate becomes spacelike. This fact implies that static observers cannot exist inside the ergoregion. For convenience we introduce quantities

\[
\alpha = \sqrt{(\Delta \rho^2/A)} ; \quad \omega_z = 2aMr/A ,
\]

where

\[
A \equiv (r^2 + a^2)^2 - a^2 \Delta \sigma , \quad \sigma \equiv \sin^2 \theta ,
\]

the lapse function and the frequency of the zero angular momentum fiducial observer (or ZAMOS) \cite{74}, whose four velocity is \( u^\alpha = (1/\alpha, 0, 0, \omega_z/\alpha) \) which is orthogonal to the surface of constant \( t \). Since the metric is independent of \( \phi \) and \( t \), the covariant components \( p_{\phi} \) and \( p_t \) of a fluid four–momentum are conserved along its geodesic. Consequently \( E \equiv -g_{ab}c_t^a p_b \) and \( L \equiv g_{ab}c_\phi^a p_b \), are constants of motion for test particle orbits, where \( \xi_t = \partial_t \) is the Killing field representing the stationarity of the Kerr geometry and \( \xi_\phi = \partial_\phi \) is the rotational Killing field (the vector \( \xi_t \) becomes spacelike in the ergoregion). It is convenient to introduce also the fluid angular frequency \( \Omega \) and the fluid specific angular momentum \( \ell \) as follows

\[
\Omega \equiv \frac{u^\phi}{u^t} = - \frac{E g_{\phi t} + g_{tt} L}{E g_{\phi \phi} + g_{\phi t} L} = - \frac{g_{t \phi} + g_{\phi t} \ell}{g_{\phi \phi} + g_{\phi t} } \quad \ell \equiv \frac{L}{E} = - \frac{g_{\phi t} u^t + g_{\phi \phi} u^\phi}{g_{tt} u^t + g_{\phi \phi} } = - \frac{g_{\phi \phi} + \Omega g_{\phi \phi}}{g_{tt} + g_{\phi \phi} \Omega} .
\]

The case of an orbiting circular or toroidal configuration is defined by the constraint \( u^\phi = 0 \), as we assume the motion on the fixed equatorial plane which is also the plane of symmetry of each toroid of the configurations (\( \sigma = 1 \)), besides no motion is in the \( \theta \) angular direction and it is \( u^\theta = 0 \). We mainly consider in our analysis the case of positive values of \( a \) for corotating (\( L > 0 \)) and counter-rotating (\( L < 0 \)) orbits, or for corotating (\( \ell > 0 \)) and counter-rotating (\( \ell < 0 \)) fluids. Couple of orbiting toroids, \( (i, o) \), with \( \ell_i \ell_o > 0 \) are said corecorotating while toroids having \( \ell_i \ell_o < 0 \) are said corecounterrotating. The geodesic structures, composed by the radii regulating the particle dynamics, regulate also large part of the disks dynamics especially in the case of geometrically thick tori, namely the marginally circular orbit for
is the outer ergosurface on the equatorial plane, and maximum density profile points, in [3] there is a discussion on the wavelike solution for a torus. The sequence of tori \( C_{-}^{1} < C_{-}^{2} < C_{+}^{3} \) is therefore counter-rotating. Values of fluid specific angular momentum \( \ell_{-}^{1}, \ell_{-}^{2}, \ell_{+}^{3} \) for the three tori are signed on the panel as the BH dimensionless spin \( a/M \). The values of the \( K \) parameters, levels of the tori effective potential, \( K_{-}^{1}, K_{-}^{2}, K_{+}^{3} \) are in legends. Right panels: pictorial representations of a view from above of the ringed structure of eRAD, the center is noted the presence of a BH. Right panels show structures of differently thick tori composing the ringed structure. For this representation we used a wave model for the ringed disk, adapted to the need for representation it follows the considerations of [3]. (The dimensionless spin \( a/M = 0.994298M : r_{+} = 2M \) is the outer ergosurface on the equatorial plane, \( r_{-}^{(\ell)} = \ell^{(r_{-})} \) see Eqs (6). Left panel from above emphasizing the inner eRAD structure and right panel in a front-view to emphasize the thickness of the toroidal components. Black center region is the Kerr BH, light gray surface is the outer ergosurface, the parameters for the inner central and outer torus of the eRAD are respectively: \( (\ell = 2.10728, K = 0.74385), (\ell = 2.5, K = 0.867797), (\ell = 3, K = 0.927362) \).

FIG. 1: Upper line. Left panel: eRAD–Ringed accretion disk of the order 3, composed by three tori, inner corotating \( C_{-}^{1} \) with cusp, middle quiescent corotating \( C_{-}^{2} \) and the outer counter-rotating \( C_{+}^{3} \) torus. The sequence of tori \( C_{-}^{1} < C_{-}^{2} < C_{+}^{3} \) is therefore counter-rotating. Values of fluid specific angular momentum \( \ell_{-}^{1}, \ell_{-}^{2}, \ell_{+}^{3} \) for the three tori are signed on the panel as the BH dimensionless spin \( a/M \). The values of the \( K \) parameters, levels of the tori effective potential, \( K_{-}^{1}, K_{-}^{2}, K_{+}^{3} \) are in legends. Right panels: pictorial representations of a view from above of the ringed structure of eRAD, the center is noted the presence of a BH. Right panels show structures of differently thick tori composing the ringed structure. For this representation we used a wave model for the ringed disk, adapted to the need for representation it follows the considerations of [3]. (The dimensionless spin \( a/M = 0.994298M : r_{+} = 2M \) is the outer ergosurface on the equatorial plane, \( r_{-}^{(\ell)} = \ell^{(r_{-})} \) see Eqs (6). Left panel from above emphasizing the inner eRAD structure and right panel in a front-view to emphasize the thickness of the toroidal components. Black center region is the Kerr BH, light gray surface is the outer ergosurface, the parameters for the inner central and outer torus of the eRAD are respectively: \( (\ell = 2.10728, K = 0.74385), (\ell = 2.5, K = 0.867797), (\ell = 3, K = 0.927362) \).

timelike particles \( r_{+}^{\pm} \), which is also the photon orbit, the marginally bounded orbit is \( r_{mbo}^{\pm} \), and the marginally stable circular orbit is \( r_{msa}^{\pm} \) for corotating (–) and counter-rotating (+) motion. Radii of geodesic structures are relevant to the accretion physics. We consider also the radius \( r_{M}^{\pm} \) as solution of \( \partial_{\ell}^{2} = 0 \), and the set of radii \( r_{(\ell) mbo}^{\pm} \) and \( r_{(\ell) s}^{\pm} \) or more generally \( \left( r_{(\ell) mbo}^{\pm}, r_{(\ell) s}^{\pm}, r_{(\ell) M}^{\pm} \right) \) where

\[
\begin{align*}
r_{(\ell) mbo}^{\pm} : & \quad \ell_{\pm} \left( r_{(\ell) mbo}^{\pm} \right) = \ell_{\pm} \left( r_{mbo}^{\pm} \right) = \ell_{\pm} \left( r_{msa}^{\pm} \right) = \ell_{\pm}^{\pm}, \\
r_{(\ell) s}^{\pm} : & \quad \ell_{\pm} \left( r_{(\ell) s}^{\pm} \right) = \ell_{\pm} \left( r_{(\ell) s}^{+} \right) = \ell_{\pm}^{\pm}, \quad \text{and} \quad r_{(\ell) M}^{\pm} : \quad \ell_{\pm} \left( r_{(\ell) M}^{\pm} \right) = \ell_{\pm}^{\pm}
\end{align*}
\]

where there is \( r_{+}^{\pm} < r_{mbo}^{\pm} < r_{msa}^{\pm} < r_{(\ell) s}^{\pm} < r_{(\ell) M}^{\pm} < r_{+}^{\pm} \).

Each toroid is described by a one-species particle perfect fluid (simple fluid) energy momentum tensor where

\[
T_{ab} = (p + \varrho)u_{a}u_{b} + pg_{ab},
\]

where from the conservation of the energy momentum tensor \( \nabla^{a}T_{ab} = 0 \) projected along each fluid four velocity \( u^{b} \) and the related 3-sheet spatial metric tensor projector \( h^{ab} = g^{ab} + u^{a}u^{b} \) we obtain the continuity (density \( \varrho \) evolution) equation and Euler equation for the pressure respectively

\[
\begin{align*}
\varrho u^{a}\nabla_{a} \varrho + (p + \varrho)\nabla^{a}u_{a} = 0 \\
(p + \varrho)u^{a}\nabla_{a}u^{b} + h^{bc}\nabla_{b}p = 0,
\end{align*}
\]
where \( g \) and \( p \) are the total energy density and pressure as measured by an observer moving with the fluid. Properly chosen boundary conditions determine the inner ringed structure of the tori agglomerate. Then model constructed in \([3-6, 16, 17, 32, 42, 80]\) is known as ringed accretion disk (RAD) for tori with a generic inclination with respect the central attractor (aggregates of tilted tori). For the case we consider here, where all tori are centered on the equatorial plane of the attractor the model is distinguished as eRAD, Figs (1). For the symmetries of the problem, we always assume \( \partial_t Q = 0 \) and \( \partial_i Q = 0 \), being \( Q \) a generic spacetime tensor (we can refer to this assumption as the condition of ideal hydrodynamics of equilibrium). As consequences of this choice, the notion of the fluid is described by the Euler equation only. We assume moreover a barotropic equation of state \( p = p(\rho) \). From the Euler equation (8) we obtain

\[
\frac{\partial_\rho p}{\varrho} = -\partial_p W + \frac{\Omega_\rho \ell}{1 - \Omega_\ell}, \quad W \equiv \ln V_{eff}(\ell), \quad V_{eff}(\ell)^2 = \sqrt{\frac{g_{\phi\phi} - g_{rt}g_{\phi\phi}}{g_{\phi\phi} + 2g_{\rho\phi} + \ell^2 g_{tt}}} \tag{9}
\]

where \( V_{eff}(\ell) \) is the effective potential and the function \( W \) is Paczynski-Wiita (P-W) potential. Assuming the fluid is characterized by the specific angular momentum \( \ell \) constant (see also \([81]\)), we consider the equation for \( V_{eff} = K = \text{constant} \) obtaining the toroidal configurations as surfaces of constant pressure or \( \Sigma_i = \text{constant} \) for \( i \in (p, \varrho, \ell, \Omega) \), where it is indeed \( \Omega = \Omega(\ell) \) and \( \Sigma_i = \Sigma_j \) for \( i, j \in (p, \varrho, \ell, \Omega) \). The toroidal surfaces are obtained from the equipotential surfaces, critical points of the effective potential \( V_{eff}(\ell) \). This set of results is known as von Zeipel solutions.

The function \( V_{eff}(\ell) \) is related to the energy \( E \) of the test particle as it is \( V_{eff}(\ell)^2 = L^2/\ell^2 = E^2 \). Clearly there is \( \lim_{\ell \to \infty} L(\ell) = \sqrt{E^2} \) and it is \( \partial_\ell L(\ell) \neq 0 \). Models of geometrically thick tori allow the determination of a wide number of aspects of disks morphology, dynamics and stability. The inner and outer edges of an equilibrium torus are also strongly constrained. For each torus, the extrema of the effective potential functions fix the center \( r_{\text{cent}} \), as minimum point of the effective potential and the maximum point for the hydrostatic pressure. Cusped \( C_x \) equipotential surfaces are associated to tori accreting onto the central BH, due to the Paczynski-Wiita (P-W) hydro-gravitational instability mechanism at the cusp \( r_x \). More specifically, the matter outflows because of a violation of mechanical equilibrium of the tori, due to an instability in the balance of the gravitational and inertial forces and the pressure gradients in the fluid. The cusp \( r_x \), inner edge of the accreting torus, corresponds to the maximum point of the effective potential and also the zero point for the hydrostatic pressure. The distribution of these critical points, fixing the distribution of tori in the eRAD, as well as features of stability properties, are governed by the RAD rotational law \( \ell(r, \sigma; a) = \partial_r V_{eff}(r, \sigma; a) = 0 \), regulating the distribution of tori in the RAD. Function \( K(r) \equiv V_{eff}(\ell(r)) \) on the other hand provides the values of \( K \) at the critical pressure points inside each torus of the agglomerate. Thus the equipressure surfaces at \( K = \text{constant} \), could be closed, determining equilibrium or quiescent \( C \) configurations, cusped \( C_x \) for the "accreting" tori, or also open \( O_x \) for proto-jet configurations related to jets \([4, 80, 82-84]\), constituting funnels of matter with a cusp on the equatorial plane, characterizing the possible inter-disk activity in the agglomerate, such as the arising of proto-jets shells, collision or double accretion \([4, 42]\). Each orbiting toroid is governed by the general relativistic hydrodynamic Boyer condition of equilibrium configurations of rotating perfect fluid. Within the so called "Boyer’s condition", we can determine the boundary of stationary, barotrope, perfect fluid body as the surfaces of constant pressure (also equipotential surfaces) which are also called Boyer surfaces\([86]\).

We can define the ranges of fluids specific angular momentum \( (L_1, L_2, L_3) \) governing the fluids and tori topology as follows (we adopt the notation \( Q_* \equiv \mathcal{Q}(r_*) \) for any quantity \( Q \) evaluated on a radius \( r_* \)):

- \( \ell \in L_1 \): for \( \ell \in L_1 \) there are quiescent and cusped tori—where \( \mp L_1^\pm \equiv [\mp l_{mso}^\pm, \pm l_{mbo}^\pm] \). Therefore tori have topologies \((C_1, C_x)\): the accretion point is in \( r_x \in [r_{mbo}, r_{mso}] \) and the center with maximum pressure in \( r_{\text{cent}} \in [r_{mso}, r_{mbo}] \) (for tori with \( K < 1 \)).

- \( \ell \in L_2 \): for \( \ell \in L_2 \) there are quiescent tori and proto-jets—\( \mp L_2^\pm \equiv [\pm l_{mbo}^\pm, \pm l_{mbo}^\pm] \). Topologies \((C_2, O_x)\) are possible; the unstable point is \( r_j \in [r_{\gamma}, r_{mbo}] \) and center with maximum pressure \( r_{\text{cent}} \in [r_{mbo}, r_{\gamma}] \); For these configurations there is \( K_j > 1 \).

- \( \ell \in L_3 \): for \( \ell \in L_3 \) there are only quiescent tori \( C_3 - d \equiv L_3^\pm \equiv \ell \geq \pm l_{\gamma}^\pm \) and center \( r_{\text{cent}} > r_{\gamma} \).

For illustration of the presented definition see Figs (2): left panel shows the open surfaces and the limiting cusped surfaces with cusps at \( r_{\gamma}^\pm \) for counterrotating and corotating fluids respectively, enlightening the spatial separations between the two sets of fluids, and the vertical direction parallel to the BH rotational axis. Center and right panels show radii \( (r_s, r) \) solutions of \( \ell(r) = \ell(r_s) \) for corotating and counterrotating tori used in Eq. (6), the separation \( r_s - r > 0 \) is the distance between the center of maximum pressure and maximum pressure in the disk (cusp) when the minimum is defined. The distance depends on the BH spin and the rotation orientation of the fluid respect to the central BH (there is \( r > r_+ \) and \( r_s > r \)).
FIG. 2: Left panel: open surfaces, solutions of Euler equations for corotating fluids (−) plain and counter-rotating fluids (+) dashed. Black region is the BH, gray region is the outer ergoregion. We can note the presence of cusp for the photon orbit location $r_{γ}^{±}$, $M$ is the BH mass, $r_{mbo}$ is for marginally bounded orbit, curves are open equipotential surfaces associated to proto-jets, limiting almost collimated funnels at the horizons are present as limiting surfaces at various constant values of specific fluid angular momenta: black curve is $ℓ = ± 0$, 7, purple $ℓ = ℓ^{±}(r_{mso})$, cyan $ℓ = ℓ^{±}(r_{mbo})$, light-gray curve is $ℓ = ± 6.5$, green curve is $ℓ = ±3.8$, blue curve is $ℓ = ±8$. Center and right panels: solution of condition $ℓ^{±}(r) = ℓ^{±}(r_{s})$ for corotating (−) and counter-rotating (+) fluids respectively. The pair $(r,r_{s})$ relates centers and cusps (in general critical points) of family of tori for different black holes.

III. ENERGY EXTRACTION

In order to look for global characteristics of rotational energy extraction from black holes we have to introduce specific functions of the black hole spin, which will be addressed in Sec. (III A). It is also convenient to introduce here some notable BH spins:

$$a^{e}_{mso}/M \equiv 2\sqrt{2}/3 \approx 0.942809 : r_{mso}^{−} = r_{e}^{+} , \quad a^{e}_{mbo}/M \equiv 2\left(\sqrt{2} − 1\right) \approx 0.828427 : r_{mbo}^{−} = r_{e}^{+} , \quad (10)$$

$$a^{e}_{γ}/M \equiv 1/\sqrt{2} \approx 0.707107 : r_{γ}^{−} = r_{e}^{+} , \quad (11)$$

related to the cross of the radii $\{r_{mso}, r_{mbo}, r_{γ}\}$ with the $r_{e}^{+} = 2M$, outer ergosurface on the equatorial plane. We proceed in Sec. (III A) with the analysis of the BH rotational energy in the eRAD context, while we expand the discussion with the inclusion of the metric bundles, and the concept of horizon replicas in Sec. (III B). This section closes in Sec. (III C) with an estimation of the mass-flux, the enthalpy-flux and the flux thickness, determined by the geometric properties of spacetime via the torus effective potential.

A. The energy-spin relations

We start by considering the spin function $A(ξ)$:

$$A(ξ) \equiv 2\sqrt{−(ξ − 2)(ξ − 1)^{2}}ξ , \quad (12)$$

relating the dimensionless BH spin $a/M$ to the dimensionless ratio $ξ = M_{rot}/M$ representing the total released rotational energy versus BH mass (measured by an observer at infinity, that is $M_{rot} ≡ M − M_{irr}$ where $M_{irr}$ is the irreducible BH mass), assuming a process ending with the total extraction of the rotational energy of the central Kerr BH—Figs (3). (Here and in the following where we do not intend differently, we shall use dimensionless quantities, viceversa where necessary we will make explicit the dependence on the mass.). We can express Eq. (12) in the form

$$ξ_{±}^{±} = 1 ± \sqrt{r_{±}^{2}} , \quad (13)$$
relating directly the energy parameter $\xi$ to the horizons–see Figs (3). However, as there is $\xi \propto M - M_{\text{irr}}$ (here $\xi$ has units of mass $M$), only solution $\xi^\pm$ has to be considered. Considering $A(\xi) \equiv a_s^{(\pm)}$, solving for $\xi$ we find the eight functions $\xi_s^{(\pm)}$

$$\xi_s^{(\pm)} = 1b \sqrt{1 - \frac{(a_s^{(\pm)})^2}{2}}, \quad \text{where} \quad b = \pm; \quad \xi = \pm. \quad (14)$$

Note the general solution $A(\xi) = a_s^{(\pm)}$, for a couple of spins $a_s^{(\pm)}$, provides eight functions $\xi_s^{(\pm)}$, four for each spin $a_s^{(+)}$ and $a_s^{(-)}$ according to the four (not related) signs $b$ and $\xi$–see Figs (6). The surface area $A_{\text{BH}} = 16\pi M_{\text{irr}}^2$ of the event horizon identifies the BH irreducible (or rest) mass. The total BH mass $M$ can be decomposed into the mass $M_{\text{irr}}$, the rotational energy and, eventually, the electromagnetic energy contribution in the Kerr-Newman solution. Therefore there is $M^2 = M_{\text{irr}}^2 + J^2/4M_{\text{irr}}^2$ (here $J$, black hole’s angular momentum, has units of mass $M$ as measured in the asymptotical flat region). The maximum rotational energy which can be extracted from the black hole is $(M - M_{\text{irr}})$. A result of Christodolou and Ruffini gives an upper limit on the energy extraction from the Kerr BH, to the rotational energy extraction, assuming a final stage of the process resulting a static Schwarzschild BH [89]. The bottom limit of $M$, at the end of extraction process, has to be $M_{\text{irr}}$. More precisely, considering $M(0)$ and $J(0)$ the mass and angular momentum of the initial state of the BH, the upper limit for the energy extracted during the stationary process bringing the BH to the state (1) is $M(0) - M_{\text{irr}}(0)$. All the quantities are evaluated at the state (0) prior the process, therefore all the quantities evaluated here inform on the status of the BH-accretion disk system at its initial state (0), by evaluating the related $\xi$ parameter. The BH angular momentum in the final state (1) is zero. Obtaining the limit of $\xi = \sqrt{2} - 2\sqrt{2}$ of energy extraction (hence $\xi \in [0, \xi_1]$) at the state (0) (prior the extraction) there is an extreme Kerr spacetime (having spin $a = M$). Eq. (12) has been found considering the rotational (spin) energy $E_{\text{rot}} \approx (M - M_{\text{irr}})^2$, where $c$ is the light velocity and $M_{\text{irr}} = \frac{1}{2} \sqrt{a^2 + r^2}$ is the BH irreducible mass. The rotational energy is equal to the extracted total energy $E$ of the outflow $\xi \equiv E_{\text{rot}}/Mc^2$.

Consider the state (0) prior the extraction, there is

$$M_{\text{irr}}^2 = \frac{1}{2} \left(\sqrt{M^4 - J^2} + M^2\right), \quad \text{therefore} \quad M(0)^2 - M_{\text{irr}}(0)^2 = \left(\frac{J(0)M(0)}{M(0)(2M_{\text{irr}}(0))}\right)^2 \quad (15)$$

the variation is thus

$$\frac{\delta M_{\text{irr}}}{M_{\text{irr}}} = \frac{\delta M - \delta J(0)\omega_H}{\sqrt{M(0)^2 - \frac{J(0)^2}{M(0)^2}}}, \quad \text{where} \quad \delta M_{\text{irr}} \geq 0, \quad \text{thus} \quad (\delta M - \delta J(0)\omega_H) \geq 0. \quad (16)$$

($\omega_H$ is the frequency of for the outer Killing horizon.). From the first law of thermodynamic $M^2 = \frac{J(0)^2}{4M_{\text{irr}}^2} + M_{\text{irr}}^2$, and (extracted rotational) energy emission is essentially\

$$M_{\text{rot}} = M(0) - \sqrt{\frac{1}{2} \left(M(0)^2 - \sqrt{M(0)^4 - J(0)^2}\right)}, \quad \text{and} \quad \frac{M_{\text{rot}}}{M(0)} = \left(1 - \sqrt{1 + \sqrt{1 - \frac{J(0)^2}{M(0)^2}}}\right). \quad (17)$$

In general, there is $A(\xi) \in [0, 1]$ (dimensionless) and $\xi \in [0, 2]$. However, an immediate calculus from definition of rotational energy, $\xi = 1 - M_{\text{irr}}/M$, with quantities evaluated at the initial state prior the process leads to the restricted range $\xi \in [0, \xi_1]$ where $\xi_1 \equiv \frac{1}{2} (2 - \sqrt{2})$, limiting therefore the energy extracted to a superior of $\approx 29\%$ of the mass $M$. It is convenient to view $\xi$ in its extended range $[0, 2]$, and to focus on an extended range corresponding to the outer and inner horizons in the sense explained below. The extension is intended as a reparametrization of the

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4 We note that analogue reasoning guides a first evaluation of the energy radiated by gravitational waves following two black holes collisions resulting in a merger. (Particularly considering three Schwarzschild BHs the maximum limit is coincidentally $\xi_1$ otherwise in case of rotating BHs the maximum values is $\xi = 1/2$)

5 Note that the Smarr formulas can be modified considered an embedded black hole actually an astrophysical not isolated BH immersed in material environment, where there is an additional contribution of the non-vanishing energy-stress tensor due to the matter–see Bardeen in [90].
spin \( a(\xi) \in [0, 1] \), since \( A(\xi) \) shows remarkable properties of symmetries. We shall see that this extension does not preserve the symmetry when applied to certain properties of the disk.

As clear from Figs (7), functions \( A(\xi) \) and \( \xi^A \) of Eq. (13) are symmetric in \( \xi \in [0, 2] \). For spin value corresponding to the Schwarzschild spacetime, \( A(\xi) = 0 \), there is \( \xi = 0 \) (note in the extended range \( \xi \in [0, 2] \), \( A(\xi) = 0 \) corresponds also to the values \( \xi = 1 \) and \( \xi = 2 \)). Each curve \( A = A = \) constant corresponds generally to four values \( \xi_1 < \xi_2 < \xi_3 < \xi_4 \), excluding some notable cases as the value \( A = 1 \) and \( A = 0 \). There is a maximum as function of the spin, \( \xi \equiv \frac{1}{2}(2 - \sqrt{2}) \) (and \( \xi_m \equiv \frac{1}{2}(2 + \sqrt{2}) \)) where \( A(\xi) = 1 \) and \( r_+ = M \) (the extreme Kerr BH). Measuring \( \xi \) therefore will provide indication of the BH spin. This method, independent of the specific model of energy extraction, was introduced first in [1], and applied in other analysis in [1, 8–10]. Here we connect the quantities \( \{A(\xi), \xi\} \) with the tori parameters \( \{\ell, K\} \) and fluid angular velocity \( \omega \). The analysis using function \( A(\xi) \) in Eq. (12) distinguishes continuum curves of classes of attractors in a plane \( a/M - \xi \).

The study of the relativistic velocity of the fluid, constrained by the characteristic frequencies of the metric bundle introduced below, allows to alternatively constrain the tori fluids and to constrain structures dependent on the light surfaces defined by the boundary of stationary observers orbits. The relativistic velocity of the fluid \( \omega \equiv u^\theta/u^t \) is related to the function \( 1/\ell \) of the fluid specific angular momentum and the conditions of von Zeipel, governing the relation between \( \omega \) and \( \ell \) (see Eqs (5)–(91)). Von Zeipel theorem characterizes any stationary, axisymmetric, non-self-gravitating perfect fluids in circular motion in the gravitational field of a central compact object. It guarantees that the angular velocity \( \omega \) depends only on the specific angular momentum \( \ell \) and the metric components, therefore velocity \( \omega \) and parameter \( \ell \) have common iso-surfaces (assuming a barotropic equation of state). The von Zeipel theorem eventually represents a set of integrability conditions to compute the equilibrium toroidal solutions. Tori models defined here have four velocity \( u^\theta \) sharing symmetries with the stationary observers, therefore bounded by the limiting \( \omega_\pm \) relativistic velocities (light-like orbital angular frequencies) of photons defining the light-surfaces regulating many aspects of the BH physics and accretion physics and jet launching or the magnetosphere structure around a BH accretion disk system. Light surfaces are given explicitly in Appendix (B). The connection with the von Zeipel surfaces is explored in Appendix (C). The fluid specific angular momentum is therefore delimited by the the like orbital frequencies \( \omega_\pm \) defining the light surfaces and, as particular case the horizon angular velocity \( \omega_H^+ = \omega_\pm(r_+) \).

The concept of metric bundles introduced here is based on the classes of all the geometries having the same limit frequency \( \omega \) in a point; these classes can be represented as curves in a plane called the extended plane, where they are all tangent to the curve defining the Killing horizons of the Kerr geometries. The angular momentum of the fluid and its relativistic velocity are constrained in a point by the frequency values of the intersection of two bundle curves, at fixed plane. There are situations where these frequencies are replicated in other portions of the spacetime (the replicas) thus giving the same constraints. For this reason it is useful to use the concept of "extended plane" introduced in [2] in the definition of metric Killing bundles. Some bundles on the equatorial plane of the Kerr spacetimes are represented in Figs 4. The characteristic frequencies \( \omega \) are the limiting photon-like frequencies of the stationary observers \( \omega_\pm \), and also the horizons frequency of the BH identified in the extended plane by the tangent point of the bundle with the horizon.

Metric bundles \( B_\omega \) are collections of black holes or black holes and naked singularities (NSs) defined generally for any axially symmetric spacetime with Killing horizons.

Each geometry of the bundles has equal "limiting photon frequency", \( \omega_\pm \), which is the characteristic frequency of
the metric bundle, and it is also demonstrated be the frequency of a BH Killing horizon as defined in the extended plane \( r/M - a/M \). Metric bundles are represented by curves in the extended plane tangent to the horizon curve, which therefore emerges as the envelope surface of all the bundles. The metric bundles \( \mathcal{B}_0 \) introduced in [2] satisfy the conditions \( \mathcal{L} \cdot \mathcal{L} = 0 \) defining the null particles frequencies \( \omega \) for the vector \( \mathcal{L} \equiv \xi_\mu + \omega \xi_\phi \) which is also an horizon frequency for a certain spacetime with spin \( a \) on the curve \( a_\pm \equiv \sqrt{r(2 - r)} \) in units of mass. (We note that the curves would be \( \pm \sqrt{r(2 - r)} \) featuring counter-rotating orbits.). We deepen this definition below.

### B. Metric bundles, horizon replicas and photon frequencies

Metric bundles (MBs) of the Kerr geometry were fully characterized in [79]—see also [2, 73, 76, 77]. This definition concentrates on the null orbits frequencies of the stationary observers on the orbits \( r \), and introduce the concept of replicas for light-surfaces with equal photon frequencies, which is useful to connect different points of the spacetime, and different geometries following the BH transition for the rotational energy release. More precisely we define the replicas in the extended plane \( a/M - r/M \), as a special set of points \( \{p_i\}_{i=1}^n \) of the extended plane corresponding to equal (positive) limiting frequency \( \omega > 0 \) [76, 77]. It is proved that there is a maximum of \( \kappa = 2 \) on the section of the extended plane \( a > 0 \) for fixed plane. In the case considered here, replicas are a couple of orbits \( (r, r_1) \) (and planes \( (\sigma, \sigma_1) \)) corresponding to the same value of the limiting frequency \( \omega \) which is also the horizon frequency of the BH defined by the tangency condition of the bundle with the horizon: all the geometries of the bundles have at an point \( r \) and \( \sigma \), one equal limiting frequency (the characteristic bundle frequency). More generally, we can extend the definition of bundle to group geometries on the basis of an equal value of a property \( Q \), but the convenience of the choice of the orbital limiting frequencies for photon is manifold: this definition is naturally related to the geometry symmetries, it has a numerous of relevant astrophysical applications; eventually, in the extended plane, the BHs horizons emerge as the envelope surface of all the bundle curves, providing therefore also an horizon definition.

The concept of horizon confinement can be introduced by considering that there is a replica when in the same spacetime it is possible to find at least a couple of points having the same value of properties \( Q \). There is a confinement viceversa, when that value is not replicated. In the Kerr spacetime, part of the inner horizons frequencies are "confined". The confinement analysis, which is a study of curves \( \mathcal{B}_{\pm} \) topology in the extended plane and particularly the self-crossing of the bundle curves in the extended plane, provides the possibility to extract information of determined local properties of the spacetime in some regions, by their possible replicas in other regions more accessible to the observers. Conversely, it provides a way to connect measures in different spacetimes, bundles connect different points in different spacetimes, all characterized by equal value of the property \( Q \). Therefore MBs connect two or more points in the same spacetime or different spacetimes and one point of the horizon in the extended plane to other points by equal values of the quantity \( Q \).

In the Kerr geometry we restrict the definition to the BH case, therefore the section of the extended plane \( a/M \in [0, 1] \) or \( \mathcal{A} \equiv a\sqrt{\sigma} \in [0, M] \) for \( \mathcal{A} - r/M \) (the quantity \( \mathcal{A} \) should not be confused with spin function Eq. (12)), and applying the symmetries for corotating and counterrotating fluids, related to the poloidal axis, connecting a point \( p_\mu \), typically a point of the outer horizon of the BH geometry, to a set of points \( \{p_i\} \) all characterized by same value of photon orbital frequency \( \omega_\star \) of the horizon. It can be proven that in the extended plane \( a/M - r/M \), for fixed plane \( \theta = \text{constant} \), it is possible to connect two points at most through the frequency of the outer horizon (existence of at most a replica of the outer horizon in the same geometry). In the case of the inner horizon the situation is more complex and it is not always possible to find a replica of the horizon[2]. The dependence of the orbit on the poloidal angle \( \theta \), allows a more accurate study of the regions close to the BH rotational axis, revealing interesting observational implications[2]. Similarly accurate transformation from the bundles in \( a/M - r/M \) to \( \mathcal{A} - r/M \) has been discussed in [76, 77]. In this context it is clear that \( \omega \) is tied to the rotational energy extraction (we also add further notes on this issue in Sec. (IV A)). The process of energy extraction brings the point on the horizon from \( p_\mu \) to a point \( p_1 \) on the horizon curve, \( a_\pm \) in the plane \( a/M - r/M \), and then a rigid rotation in the sense of [2], a shift from an horizontal line of the extended plane to another. Specifying these arguments, we consider the BHs horizons frequencies \( \omega_\mu^\pm \) and the horizons curves in the extended plane \( \mathcal{E}_\mu - r/M \) respectively on the equatorial plane

\[
\omega_\mu^\pm = \frac{a}{2r_\pm}, \quad \xi_\mu^\pm = 1 \mp \sqrt{1 - \frac{r}{2}}, \quad \xi_\nu^\pm = 1 \mp \sqrt{\frac{r}{2}}, \tag{18}
\]

see Figs 5,6,19 and Eqs (13) for functions \( \xi_\mu^\pm \)—Figs (3). It is shown the frequency \( \omega = 1/\sqrt{27} \) relevant to the bundle structure for each value of \( a/M \). On the equatorial plane, the zero of the bundles define the Schwarzschild static case, in this spacetime the frequency is related to photon orbit \( r = 3M \).

Let us consider now the tangent curves to the horizons in the extended planes. There are the functions of the
FIG. 4: Metric Killing bundles of the Kerr spacetimes, on the equatorial plane, in the extended plane $a/M - r/M$ see also Figs (5). $\omega$ is the bundle characteristic frequency which is a limiting photon frequency of the stationary observers, constant values of $\omega$ are signed on the curves. All the bundles are tangent to the horizon curves (inner and outer horizons) in the extended plane. The central black region is the set of Kerr BHs. Inner and outer Killing horizons are represented. The extreme Kerr black hole (EBH) spacetime $a = \pm 1$ is also shown. $r^+_{\rho}$ is the outer ergosurface, counter-rotating orbits are also shown ($\omega a < 0$). The limit of the Schwarzschild spacetimes is on point $(a = 0, r = 0)$ and point $(a = 0, r = 2M)$. for $|a| > M$ there is a naked singularity (NS). Details on the structure of the metric bundles are in [2, 74, 76, 77].

frequency

$$a_g = \frac{4\omega}{4\omega^2 + 1}, \text{ and } \xi_\pm \equiv 1 \mp \frac{1}{\sqrt{4\omega^2 + 1}}, \quad \xi_{\tau\tau} \equiv 1 \mp \frac{2\omega}{\sqrt{4\omega^2 + 1}}$$

where $a_g$ is the curve of tangent points of the bundles with the horizons, from now on the tangent curve to the horizon, in the extended plane $a/M - r/M$. In this way we represent the extracted energy in terms of characteristic frequency of the bundle and through the tangency condition of the bundle in the extended plane–Figs (6). Tangent $a_g$ can be written in terms of extracted rotational energy as $\xi_\pm$, represented in Figs. (6); we note the limiting value $a = M$. We introduce the curves $\xi_{\tau\tau}$ and $\xi_\pm$ in the extension of the plane for extended values of $\xi$. In Figs. (6) we represented the horizons curves of Figs (7) parametrized as $\xi_{\tau\tau}$ in the extended plane, the inner and outer horizons are shown. The limiting null-like particle frequencies of the stationary observes, horizons frequencies as characteristic frequencies of the bundles $B$ are on the equatorial plane given by

$$\omega_\xi \equiv \frac{4r\sqrt{-(\xi - 2)(\xi - 1)^2\xi \mp \sqrt{r^4[\xi - 2(\xi - 1)^2]2(\xi - 2)\xi + r}}}{r[-8(\xi - 2)(\xi - 1)^2\xi + r^3 - 4(\xi - 2)(\xi - 1)^2\xi r]}$$

see– Figs (4). On the equatorial plane, the bundles are expressed in the form

$$a_{\pm} = \frac{2\omega \pm \sqrt{r^4\omega^2[1 - r(r + 2)\omega^2]}}{(r + 2)\omega^2}.$$ 

In our case, the bundles in the plane $\xi - r/M$ are evaluated, i.e. bundles describe the Schwarzschild BH, the Kerr BHs as well as Kerr NSs. Considering the counter-rotating orbits, as in [77], we take into account also $a < 0$. Accordingly, bundles can be given in the extended plane $\xi - r/M$, providing the eight solutions for $a_{\pm}^{(\pm)} \equiv a_g^{\pm}$–see Eq. (13). Horizons replicas on the equatorial plane are

$$r^+_{\rho} \equiv \frac{1}{2} \left( \sqrt{\frac{32r_x}{a^2} - a^2 \pm 6\sqrt{1 - a^2 - 22 - r_x}} \right);$$
FIG. 5: Metric Killing Bundles of the Kerr spacetimes (the equatorial plane) in the extended plane $\xi - r/M$. $\xi$ is the rotational energy extraction parameter—see also Figs (4). The bundle characteristic frequency $\omega$, constant on each bundle is a limiting photon frequency of the stationary observers. The special frequency $\omega = 1/\sqrt{27}$, related to the zeros of the metric bundles (which is the Schwarzschild geometry) and the photon orbit is signed on the curve. All the bundles are tangent to the horizon curves (inner and outer horizons) in the extended plane. Inner and outer Killing horizons are represented as well as the horizons replicas (dashed curves) $r^\pm_{\rho}$: for fixed spacetimes there are two orbits with equal horizons frequencies $\omega^\pm_{\mu}$ one on the horizons $r^\pm$, the second on the orbit replica—see also Figs (7). The extreme Kerr black hole (EBH) spacetime (for $\xi = 0$) is also shown. $r^+$ is the outer ergosurface, counter-rotating orbits ($\omega_a<0$) are also shown. The limit represented by the Schwarzschild spacetime in on points $(a = 0, r = 0)$ and $(a = 0, r = 2M)$. For $|a| > M$ there are naked singularities (NS). Arrows set the increasing values of the frequencies magnitude. We note the special role of the photon orbits $r^{\gamma} = 3M$ and marginally bounded orbit $r^{mbo} = 4M$ for the Schwarzschild spacetime (the limiting case $a = 0$ in the extended plane) for each BH spin. Below–left and center panels show a zoom on the region $\xi \in [0, \xi_{\ell}]$. Below–right panel shows a selection of curves. Some frequencies, for $r^{\gamma} = 3$ are shown—see also Figs (8).

FIG. 6: Spin energy extraction. Left panel: curves $\xi_\mu^{\pm}$, $\xi_\tau^{\pm}$, defined in Eqs (19), of tangent curves of the horizons in the extended plane as functions of the horizons frequencies $\omega$. Value $\omega = 1/2$ is the frequency of the extreme BH horizon. Spin function $a_g$ is the tangent curve to the horizon as function of the horizon frequency—see Eq. (25). Black lines are $\xi_{\ell}$ and $\xi_{m}$, maxima of $\xi$ as functions of the BH spin $a/M$. Values $\xi_{\ell}$ and $\xi_{m}$, are maxima of $\mathcal{A}(\xi)$ (dimensionless spin of the BH) where $\mathcal{A}(\xi) = 1$ and $r_+ = M$ (the extreme Kerr BH). Solution $\mathcal{A}(\xi) = \xi$, dotted line, is the crossing point $a_g = \xi_{g}$ and $\xi_{g} = a_g$ for the outer and inner horizon of the spacetime with spin $a = a_g = \xi_{g} = \xi_{g}$. Spin $\mathcal{A}(\xi)$ is defined in Eq. (12) as function of dimensionless emission energy $\xi$. Center panel: plot of the functions $\xi_{\mu}^{\pm}$ and $\xi_{\tau}^{\pm}$, rotational energy, defined in Eqs (18) as function of $r$, horizons points in the extended plane: $r/M = 1$ corresponds to the extreme Kerr BH spacetime, $r = 0$ is a limiting value correspondent to the Schwarzschild singularity with $r = 2M$, which corresponds also to radius $r^+_a$ outer ergosurface on the equatorial plane. $r_{\gamma} = 3M$ is the photon orbit on the equatorial plane for the Schwarzachild spacetime. The region $r \in [0, M]$ refers to the inner horizons, while $r \in [M, 2M]$ corresponds to the outer horizons. Right panel: 3D solutions $\xi_{\mu}^{\pm}$ in Eq. (13), of $\omega = \omega_\pm(a, r)$, for $\sigma = 1$ (equatorial plane) and $a = \mathcal{A}(\xi)$ on the metric bundles Eq. (21) as function of the horizon frequency.
where there is $\omega_H^+ = \omega_+ (r_+^p)$ respectively—see [2] and Figs (7). In Figs (5) we note the special role of the photon orbits $r_+ = 3M$ and marginally bounded orbit $r_{mbo} = 4M$ for the Schwarzschild spacetime (the limiting case $a = 0$ in the extended plane) for each BH spin. Right panel shows a selection of curves, where frequencies for $r_\gamma = 3M$ are shown. Considering the notion of replicas, an observer on a point $p$ in the BH spacetime with (dimensionless) spin $a/M$ has orbital frequencies limited by the photon orbital frequencies $\omega_{\pm}$, here $\omega_{\pm}^E$ of Eqs (20), reachable only in the case of null-particles. On the outer horizon of the BH spacetime there is $\omega_\omega = \omega_H^+$ and the frequency "window" characterizing the stationary observes reduces to circle defining the BH outer horizon. On the other hand, in the case of weak naked singularities as studied in [2, 76–78], there is a region (of $a/M$ and $A = \sqrt{a/M}$) proximate to the singularity where the frequency range bounded by $\omega_{\pm}$ is not null (in this case interpretable as absence of the horizons) but reduced to a minimum. These singularities have peculiar characteristics emerging very clearly as properties of their bundles $B_\omega$ in the extended plane. Bundles corresponding to these NSs are tangent to a portion of the inner horizon curve which is not confined. This is related to a region in the plane $r - \omega$ that can be further reduced to a minimal range of frequencies and spins around the central limiting value $a = M$ (a bottleneck) [75]. The issue of naked singularity characterization goes beyond the targets of this analysis, however we can say that such NSs are characterized by this bottleneck regions connected to the horizons as their "remnants" or "memory" of the BHs horizons in the extended plane[2]. This region coincides with the region where repulsive gravity effects appear. NSs region of the extended plane containing parts of the MBs tangent to the horizons have a role in the MBs analysis of BHs as the origins of the bundles, which are points of the axis $r = 0$ also in NSs.

Replicas connect the null vectors $L(r_+, a, \sigma)$ and $L(r_p, a, \sigma_p)$ (we also consider the special case $\sigma = \sigma_p$), i.e., $\omega_H^+(a) = \omega_\omega$ defining the bundles $B_\omega$ and connecting points belonging to $B_\omega$ in the same spacetime or different spacetimes. Here we consider the observers registering the presence of a replica at the point $p$ of the BH spacetime with spin $\sigma_p$, belonging to the Killing bundle $B_\omega$. The observer will find the replica of the BH horizon frequency $\omega_H^+(a_p)$ at point $p$, therefore her/his orbital stationary frequency will be $\omega_p \in [\omega_*, \omega_\omega]$ where one of $(\omega_*, \omega_\omega)$ is the horizon frequency, i.e. the horizons frequency $\omega_H^+$ is replicated on a pair of orbits $(r_+, r_p)$, the second light-like frequency $\omega_\omega$ is the frequency of a horizon in a BH spacetime which is correlated by the bundles to $B_\omega$ in the considered spacetime. The relation between the two frequencies $(\omega_*, \omega_\omega)$ is determined by a characteristic ratio studied in details in [2].

In the extended plane, $\xi - r/M$, the extracted rotational energy is $\xi = 1 - (\sqrt{1 - (-2 - r)^2} + 1)/(\sqrt{2})$ see Figs (3).

Clearly, the extended plane $\xi - r/M$ does not describe bundles, or portions of bundles $B_\omega$, in the NSs region similarly to the plane $a/M - r/M$, particularly in the case fully contained in NS region which is defined by the frequency $\omega = 1/2$, corresponding to the Kerr extreme BH as tangent point bundle-horizon\textsuperscript{6}. Thus it cannot describe energy extraction from a NS although we use a NS to construct the bundles on the BH region.

\textsuperscript{6} Exploring the role of the NSs as considered in the extended plane, we note that the three solutions $\xi(r)/M$ not verifying the condition $1 - M_\omega r/M \in [0, \xi]$ could provide an indication of the portion of the bundles contained in the NSs region. We note that considering the characteristic frequency $\omega = 0.43 < 0.5$ of the bundle on the equatorial plane tangent to the curve of the outer horizon in the extended plane, thus, $\omega = 0.43 = \omega_H^+$, there is a replica in the spacetime at $a = 2M$, not shown however in the plane $\xi - r/M$—as clear from Figs (8). It is also evident that these bundle structures are very similar to those obtained for the Schwarzschild case.
Sequence of diagrams illustrating the orbital frequencies $\omega_\xi^\pm$ in Eqs (20) of null particles, limits of stationary observers as functions of the extracted energy $\xi$. Left upper panel: 3D plots of $\omega_\xi^\pm$ as functions of $\xi$, rotational energy and the radius $r/M$, $M$ is the BH mass (as measured at infinity). Planes $\omega = 0$ related to the Schwarzschild solution and $\omega = 1/2$ frequency of the extreme Kerr BH horizon are represented. Note the presence of negative frequencies at $r > 2M$, outer stationary limit and horizon of the limiting Schwarzschild spacetime. Center panel: curves for different $\xi$. The limiting case of the Schwarzschild solution $a = 0$ is shown. The extreme Kerr BH case $a = M$ is also shown. (Note that curves do not represent NSs solutions.) Curves provide the inner and outer Killing horizons. Right upper panel: curves $\omega_\xi^\pm$ in a portion of the extended plane $\xi - r/M$ Bottom left panel: bundle at fixed frequency. Black region is the BH in the extended plane, spin $a_\gamma$ is the tangent point between the bundle and the horizon, $r_\pm^\gamma$ are points of replicas the NS at spin $a = 2M$. Center panel: bundle in the plane $\xi - r/M$, replicas $r_\pm^\gamma$ for fixed frequency and spin $a_\gamma$ and the radius $r_\pm^\gamma$ are shown. Right panel: different bundles in the plane $\xi - r/M$ at fixed frequency, the limit case, $r = M$, for the extreme BH is also shown. See also Figs (5).

Function $A(\xi)$ links the former state spin $a_0$ to the rotational energy extraction in the subsequent phase where the BH is settled in a Schwarzschild BH. Therefore analysis of a quantity $Q(\xi)$ relates quantities $Q(0)$, before the transition, to the energy extraction measured by $\xi$ – Figs (8)–[92–94].

In the processes of accretion and classical interaction between BHs and the surrounding matter and fields, the horizon area, $A_{BH}$, can remain constant or can increase. Assuming that all the (corotating) fluid provides an initial contribution through the cusp $r_\xi$, from its specific angular momentum and a specific internal energy, the mass and angular momentum of the BH grow by the corresponding amount given by $dM_{BH} = E_x dM_0$, and $dJ_{BH} = L_x dM_0$ ($[J] = M^2$). Next section focuses on the analysis of quantities relative to the disks as functions of the extracted energy measured by the observers. In Figs (9) we represent the geodesic structure $\{r_{mso}, r_{mbo}\}$ as functions of the rotational energy parameter, including the curves $\ell$ constant which define each torus of the RAD agglomeration as functions of the extracted rotational energy in the range $\xi \in [0, 2]$ and for corotating and counter-rotating fluids. The ranges defined by values $\ell_{mso}$, $\ell_{mbo}$ and $\ell_\gamma$ define the topology of the critical configurations. In Figs (10) the curves $K(\ell) \equiv K_{crit}(\ell)$ are shown as functions of $\xi$ providing the values of the $K$ parameter of each toroidal configuration (at $\ell =$ constant) at the critical points of pressure inside the configurations. (Note that the functions in Figs (10) are not symmetric in the left and right range of the extreme value $\xi_\ell$) Therefore, in this frame we solve equation $\ell^\pm = \ell$ obtaining the curves:

$$\tilde{a}^\pm = \frac{1}{2} \left( \pm \sqrt{\ell^2 - 4\ell(\ell - 1)\sqrt{\ell} - 4(\ell - 1)r + \ell + 2\sqrt{\ell}} \right),$$

representing the families of tori-BHs systems. Note that solutions of $\ell^- = \ell$ or $\ell^+ = \ell$ where $\ell$ is positive or negative, if fluids are corotating or counter-rotating. (Alternatively solutions are given in Sec. (A).) Considering the geodesic structure in the extended plane, we can find classes of spin-radius

$$a_\gamma^- = -\frac{1}{2}(r - 3)\sqrt{\ell}, \quad a_\gamma^+ = \frac{1}{2}(r - 3)\sqrt{\ell}, \quad \frac{\ell^\pm(a_\gamma)}{a_\pm} = -\frac{2}{r - 2}$$

connecting the geodesic structures of different spins of the Kerr BHs, where $a_\gamma^\pm$ refers to the photon orbits $r_\gamma^\pm$, and
FIG. 9: Plane $r/M - \xi$, extracted rotational energy parameter of Eqs (12). Upper line panels and bottom right panel: Black lines are the marginal bounded orbits, $r_{mbo}$, marginal stable orbit $r_{mso}$ and $r_{mco} = r_\gamma$ marginal circular orbit and photon orbit for the Schwarzschild spacetime ($a = 0$). Red curves are for counter-rotating fluids, blue curves for corotating fluids. Right bottom panel: Curves $\ell(\xi; r) =$constant evaluated on $r_{mso}$ (plain curve), $r_{mbo}$ (dashed curve), and $r_{mco}$ (dotted curves). Therefore each curve represents a torus. Upper panels represent curves $\ell^\pm(a\xi)$ in the plane $r/M-\xi$ for corotating and counter-rotating curves respectively. The maximum $\xi_\ell$ is the vertical black line. Bottom right panel shows the geodesic structure as function of $\xi_\ell$, black region is the central BH.

$\ell^\pm(a\pm)$ refers to the horizons. Similar definitions can be used for $a_{\pm mso}^\pm$, showed in Figs (9). Explicit solutions in different parameterizations are in Sec. (A). Solving the problem $\ell^\pm(\xi, r) = \ell$, we find functions $\xi(r, \ell)$ connecting the tori, defined by values $\xi = 0$, and locations of maximum and minimum pressure points ($r$), with the rotational energy $\xi$, which now can be read in the extended plane. Figs (13) and Figs (14) show the results of $\ell(\xi, r) = \ell$ for corotating and counter-rotating tori and the location of the inner edge of accretion tori as function of the rotational energy $\xi$.

In Figs (2) we give the spread between the rotational law curves which demonstrates the ranges of parameters for the existence of tori, tori extension on the equatorial plane and maximum possible distance between tori in the eRAD. We relate in this way centers and cusps (in general critical points) of family of tori for different black holes, determined as solutions of the problem $\ell(r) = \ell(r_\gamma)$ on the rotational curve of the eRAD. The distance relates the point of maximum and minimum pressure inside each torus, enlightening the outstretching of the torus on its equatorial plane and the location of the inner edge (cusp).

The parameter $\xi$ on the horizon at the extreme case $a = M$ where the frequency is $\omega = 1/2$ and $\xi = 1 \pm 1/\sqrt{2}$. The characteristic frequencies of the bundles, seen as horizons frequencies in the extended plane, are

$$\omega^\pm_g \equiv \frac{2 \pm \sqrt{16(\xi - 2)\xi(\xi - 1)^2 + 4}}{8\sqrt{-(\xi - 2)(\xi - 1)^2} \xi};$$

(25)

$\omega^-$ has a saddle point for the extraction parameter $\xi = (3 \pm \sqrt{6})/3$ correspondent to the spin $a = a_{mso}^\prime$ and frequency
Our analysis focuses on a state (0) prior the extraction process, considering the RAD tori of polytropic fluids, with pressure given by polytropic equation of state $p = \rho\Gamma + \frac{\Gamma}{\Gamma-1}P$, where $\Gamma$ is a polytropic constant, $\gamma \equiv (1 + 1/n)$ is the polytropic index. The limiting case of polytropic index $\gamma = 0$ would correspond to the case of zero pressure represented by gravitating dust of test particles. We estimate the mass-flux, the enthalpy-flux (which is related to the temperature parameter), and the flux thickness, which are important for the evaluation of the energy release from RADs—see [84, 85]. In details, definitions of these quantities are listed in Table (I). All these quantities can be written in general form $\mathcal{O}(r_x, r_s, n) = q(n, \kappa)(W_s - W_x)d(n)$, and $\mathcal{P} = \mathcal{O}(r_x, r_s, n)/r_s\Omega_K(r_x)$, considering that $\Omega_K(r_x)$ is the Keplerian frequency of the accreting tori cusp $r_x$ (the inner edge of accreting disk), where $\{q(n, \kappa), d(n)\}$ are functions of the polytropic index for each torus. Parameters $(\kappa, n)$ within the constraints $q(n, \kappa) = \tilde{q} = \text{constant}$, fix a polytropic-family, while $r_s < r_x$ is related to thickness of the accreting matter flow and the potential $W = \ln V_{\text{eff}}$, thus $W_s(W_x)$ denotes, for a torus with fixed specific angular momentum $\ell$, the (constant) value of the potential of the $p = \text{constant}$ surface corresponding to radius $r_s\ (r_x)$. We consider also the quantity $W - W(r_j)$, where $r_j \approx r_x \approx r_{\text{mbo}}$ is a limiting case corresponding to a large centrifugal component of the disk tending to balance the gravity and pressure force components in the torus. In this case the inner edge of the accreting tori $r_x \approx r_{\text{mbo}}$; there is $W_x \equiv W(r_x) = \ln K_x \approx 0$. In our analysis, for this first evaluation, we define the accretion point $\tilde{r}_x(a)$ and $\tilde{r}_s(a)$ as in Figs (13) (there is therefore $1 > K_s > K_x$). As the cusp approaches the limiting radius $r_{\text{mbo}}$, the potential $W_x \approx 0$, which is also the limiting asymptotic value for very large $r$ as well as for the emergence of the proto-jets for $\ell \in \mathbb{L}_2$. The couple of parameters $\{r_s(a), r_x(a)\}$ has been fixed, to simplify the comparison of the $\mathcal{O}^\pm$ and $\mathcal{P}^\pm$ quantities in the corotating and counter-rotating tori, and to characterize the dependence of these quantities on the SMBH spin-to mass ratio $a/M$.

We examine the fraction of energy produced inside the flow and not radiated through the surface but swallowed by central SMBH, the efficiency $\eta = \mathcal{L}/\dot{M}c^2$, together with the total luminosity $\mathcal{L}$, the total accretion rate, $\dot{M}$, and accretion for a stationary flow, $\dot{M} = \dot{M}_s$. We examine also $\mathcal{P}$-quantities—Figs (10,12,13) , for $\kappa \equiv n + 1 = 4(n = 3)$, with the new variables independent from details of the selected specific polytrope, $\Psi^\pm = \mathcal{O}(r_x, r_s, n)/q(n, K)\Omega_K(r_x)$ for $\mathcal{O}$-quantities and $N^\pm = \mathcal{O}(r_x, r_s, n)/q(n, K)\Omega_K(r_x)$ for $\mathcal{P}$-quantities. Considering therefore the rate of the thermal-energy carried at the cusp and the disk accretion rate $\dot{m} = \dot{M}/\dot{M}_{\text{Edd}}$, as well as the mass flow rate through the cusp (i.e., mass loss accretion rate). The eRAD is in fact a geometrically thin disk with a complex inner ringed structure composed by both corotating and counter-rotating geometrically thick accretion disks having many features common with the thick and opaque disks as the expected super Eddington luminosity. This evaluation, leading to the results shown in Figs (12), connects diverse states of the BH and its RAD system, more specifically, different initial states (0), prior the rotational energy extraction. The procedure is based on the geometric considerations derived from the geometrically thick torus model considered here: each RAD component is pressure supported and subjected to conditions of the von Zeipel theorem, ensuring the integrability of the Euler equation and assumptions on the boundary conditions (which are assumptions on the pressure at the center, point of the maximum pressure, and torus edge, equipressure surface). We limited the description of the situation of BH, neglecting the contribution of mass feeding processes, leading likely to a multi-stage evaluation, as an interactive process may be engaged between geometry and disk similar to the runaway instability. For corotating and counter-rotating fluid characteristics, fluid momenta and location of cusps have different role for energy extraction parameter $\xi$. For corotating tori the variation

\[ \omega = (1/2\sqrt{2}, 1/\sqrt{2}) - \text{see Figs (11) and Sec. (IV A).} \]
FIG. 10: $K^\pm_{\text{crit}}$ is $K^\pm(r; a) \equiv V_\text{eff}(\ell(r), a)$ on the equatorial plane, there is spin $a = A(\xi)$, see Eq. (12), $\xi$ is the rotational energy parameter. $\ell(r)$ is the fluid specific angular momentum. Panels show $K^\pm(r; \xi) = \text{constant}$ and $\ell^\pm(\mathcal{A}_\xi) = \text{constant}$ in $(\xi, r/M)$ plane for corotating ($-$) counter-rotating tori ($+$) tori, $M$ is the central BH mass. It is clear the role of $\xi_\ell$, maximum for $A(\xi)$ (black vertical line).

FIG. 11: Left panel: metric Killing bundles on the equatorial plane for selected values of the characteristic bundle frequencies $\omega$ signed on the panel, $\omega = 1/2$ is the bundle tangent to the extreme Kerr spacetime. Black region is the BH in the extended plane, $r = 2M$ is the schwarzschild horizon and the outer engosurface. Brown and purple curves are correspondent bundles to one tangent spin $a_g/M = a_{\text{maxo}}/M = 2\sqrt{2}/3$ see discussion in Sec. (III A) and Sec. (IVA). Right panel: $\omega$ is the bundle characteristic frequency and light particle orbital frequency, $\xi$ is the parameter for the maximum rotational energy extracted. $r_{\rho}^\pm$ are horizon replicas on the equatorial plane see Eq. (22). $\omega_{\rho}^\pm$ are defined in Eq. (25) and are the horizons frequencies. Quantities $a_{\omega_g}^\pm$ refer to the limiting condition on the energy process and $\ell = L/E = \omega_{\ell}^\pm$ it refers to quantity $\ell_{\text{max}}$ from analysis of Sec. (IVA). $\xi_\ell$ is the maximum energy extracted.
of the location of the minimum points of the pressure, the instability points of the toroidal configurations, changes minimally with respect to counter-rotating tori with the parameter $\xi$. A relevant aspect concerning accretion is whether the presence of double accretion phase due to the doubled inner ringed structure of the RAD, enhances BH accretion, considering also the possibility of interrupted phase of accretion due to the inner screening torus of the couple. There is the maximum of two accreting tori in the eRAD, with the outer torus being counter-rotating, and inner torus, being corotating. This scenario is then enriched by the possibility of inter disk emission jets and obscuring tori, which are typical aspects of the RAD presence around the central SMBH, and shells of jets. The processes related to counter-rotating fluids may however be drastically different from the corotating case especially due to the influence of the ergosurface. There are indeed corotating solutions in the ergoregion for these tori stretching with the inner quiescent or cusped tori down to very close vicinity of the horizons for large BH spins. This is essentially due to the fact that the fluid in the tori could be considered stationary in the sense that the fluid four velocity is $u = \xi_t + \omega \xi_\phi$, where $\omega$ is the relativistic angular momentum. On the other hand, a cusped (corotating) torus can stretch towards the horizon enhancing the so called runaway instability – see Figs (18). We conclude from Figs (13) that the functions are not symmetric in the left and right range of the limiting value $\xi_c$, corresponding to the extreme Kerr BH.

IV. ON THE RAD AND BH SPIN–ACCRETION DISK CORRELATION

We conclude this analysis exploring tori characteristics in terms of dimensionless BH spin. Here we investigate in detail the dependence of the RAD rotational law and the BH spin. It was proved in [5, 6, 32] that there is a relation between the specific angular momentum of the fluid ($\ell_i, \ell_o$) of the inner ($i$) and outer ($o$) torus of an eRAD couple
FIG. 13: Analysis of the energetics of BH-accretion disks systems in Sec. (III C). Upper left panel: Solutions for the extracted rotational energy parameter $\xi$: $\ell^\pm(r, \xi) = \ell$ for corotating (−) (cyan) and counter-rotating (+) (red) tori respectively, of Sec (A) at different angle view, where the BH spin is $a = A(\xi)$ of Eq. (12), function of $\xi$. Functions $\xi(\ell)$ are represented for corotating and counter-rotating fluids as functions of $(\ell, r/M)$, $\ell$ is a constant value of the distribution law of the eRAD and specific angular momentum of the torus. Exact expression of solutions $\ell^\pm(r, \xi) = \ell$ are also in Sec. (A). Upper right panel: $(V_{eff}(r_{mbo}^+)) - (V_{eff}(r_{mbo}^-)) = constant$ respectively where $a = A(\xi)$ in the plane $(\xi, \ell)$, $\xi$ is the extracted rotational energy parameter and $\ell$ is the fluid specific angular momentum, $(-)$ is for counter-rotating fluids, $(\ell)$ is for corotating fluids. Right and left bottom panel: evaluation of $P$- and $O$-quantities of Table (I) for corotating and counter-rotating tori, versus $\xi$. Plots of $N_{\pm}^\pm \equiv r_x(W_{\pm}^\pm(r_x) - W_{\pm}^\pm)^{\gamma}(\Omega_{K}(r_x)^{\pm})^{-1}$ for $P$-quantities analysis and $R_{\pm}^\pm \equiv (W_{\pm}^\pm(r_x) - W_{\pm}^\pm)^{\gamma}$ for $O$-quantities analysis for corotating ([−]continuum curves) and counterrotating ([+]dashed curves) tori at different $r_x = r^\pm_x \in \{\bullet, \blacksquare, \blacklozenge, K, O_K\}$ and $r_x \in \{\bullet, K, \blacksquare, K, O_K\}$ where $\gamma = n + 1$, with $\gamma = 1/n + 1$ is the polytropic index. Radii $(r_x, r_s)$ and the associated angular momentum $\ell$ and $K$ parameters are shown with $\{\bullet, \blacksquare, \blacklozenge, K, O_K\}$. $\Omega_K$ is the Keplerian angular velocity, $r_x$ is the accreting torus inner edge (inner edge of accreting torus), $r_s$ is related to thickness of the accreting matter flow. $r_{mbo}$ is the marginally bounded orbit. $\Omega_K^\pm$ has been considered for the counter-rotating fluids. The maximum location of inner edge is $r_x \lesssim r_{mbo}$—see Figs (12). Vertical black line is the maximum value of the rotational energy parameter $\xi_x$, corresponding to the extreme Kerr BH.

and the central BH spin $a$, different for corotating and counter-rotating couples. We investigate a simple correlation within the limits of the range of values of the dimensionless spin $a/M$ of the BH ($M$ is the BH ADM mass) and the fluid specific angular momentum $\ell$, here used to parameterize the torus of the orbiting aggregate, through the value assumed along the distribution curve $\ell(r)$ (the eRAD rotational law) at the point of maximum density and pressure inside each toroidal configuration of the agglomeration. Arguments in support of the existence of such correlation in the form of superior and inferior boundaries of the range, have been discussed in [32, 81], while in [75] discussion on the possible geometrical origin of this limit has been highlighted. In this section we resume this topic considering the ergoregion as stationary (corotating) disk solution can also be considered for sufficiently large values of the central BH spin. Therefore, the model is also studied in the regions close to the static limit

Function $\ell(r)$ is considered as a possible reference distribution on the fluid angular momentum in an extended region of any ("fast" rotating) accretion disk. The condition of almost spherical accretion provides a first limit on the rotational law of the fluid in the accretion disk. Sometimes this limit is know as "Bondi regime". Thick disks considered in this article are regulated by a significant centrifugal force which is more generally assumed superior or equal the Keplerian specific angular momentum $\ell_K = L/E$. In this sense the "slow rotation" case is referred to as "Bondi flows" (being the limit of free fall accretion disks) [87]. In these (quasi)spherical "Bondi" accretion conditions, the angular momentum is not relevant in the dynamical forces balance, i.e., the (specific) angular momentum in the
FIG. 14: Inner edge \( r_x = \text{constant} \) of the cusped tori or the cusp location of a proto-jet in the plane \((\ell, \xi)\), \(\ell\) is the fluid specific angular momentum and \(\xi\) is the energy parameter, for corotating \((-\)) and counter-rotating \((+\)) tori. \(\lambda\) is the elongation \(r_{\text{out}} - r_{\text{in}}\), evaluated considering \(K = V_{\text{eff}}(a, r, K, \ell)\); \(r_x(a, r, K, \ell)\) is evaluated considering functions \(K = V_{\text{eff}}(A\xi, \ell, r)\), (where \(a = A(\xi)\)). Central curve is the fluid specific angular momentum \(\ell^\pm(\xi, r)\) as function of \(r/M > r_+(\xi)\) for selected values of \(\xi\), \(r_+\) is the BH horizon as function of extracted rotational energy.

disk is smaller than the Keplerian one. On the other hand, an accretion disk must have an extended region where matter has a large centrifugal component \((\ell \geq \ell_K)\). It will be convenient to analyze the accretion disk properties in terms of the ratio \(\ell/a\) or \(\ell - a\) as an important parameter for these models.  

A. On the RAD rotational law \(\ell(r)\), torus specific angular momentum \(\ell\) and BH spin \(a/M\).

In [80, 81], variables \((\ell \pm a)\) and \((\ell \pm a\sigma)\) have been considered to construct the accreting tori around a Kerr SMBH. Many properties of the fluid effective potential are determined by the quantity \(\bar{\ell} \equiv \ell/(a\bar{\sigma}^2)\) where there are the limiting values \(\ell = -a\bar{\sigma}^2\) and \(\ell = a\bar{\sigma}^2\) (\(\bar{\sigma} = \sin \theta\) being off function of the polar angle \(\theta\)). The origin of this quantity can be understood by considering the following fact. For simplicity we use here all dimensionless quantities, we introduce the rotational version of the Killing vectors \(\xi_t\) and \(\xi_\phi\), i.e., the canonical vector fields \(\tilde{V} \equiv (r^2 + a^2)\partial_t + a\partial_\phi\) and \(\tilde{W} \equiv \partial_\phi + (a\bar{\sigma}^2)\partial_t\).

Then the contraction of the geodesic four-velocity with \(\tilde{W}\) leads to the (non-conserved) quantity \(L - E(a\bar{\sigma}^2)\), function of the conserved quantities \((E, L)\), the spacetime parameter \(a\) and the polar coordinate \(\theta\); on the equatorial plane it then reduces on \(L - Ea\). We note the existence of two limiting values related to two bundles frequencies and limiting momenta: \(\omega = a\sigma\) and \(\omega = 1/(a\sqrt{\bar{\sigma}})\). When we consider the principal null congruence, \(\gamma_{\pm} \equiv \pm \partial_t + \Delta^{-1}V\),

7 As discussed in Sec. (II) there are closed toroidal configurations for \(\pm \ell^\pm \geq \pm \ell^\pm_{\text{ino}}\) and \(K^\pm \in [K^\pm_{\text{ino}}, 1]\). However in [81] orbiting stationary configurations for different values of \(K\) and \(\ell\) have been constructed. At lower momentum (in magnitude) there are closed surfaces very close to the outer horizon, and related to the inner Roche lobe of the closed tori. The stability of these configurations which have not cusp has however still to be assessed [61].
the angular momentum $L = (a\tilde{\sigma})^2$ that is $\tilde{\ell} = 1$ (and $E = +1$, in proper unit), every principal null geodesic is then characterized by $^8 \tilde{\ell} = 1$.

With reference to Figs (15) we considered the "bundles-like" solutions $B_{a\sigma}$ with Killing vector $L$ with $\omega = a\sigma^2$. We note that these particular bundles solutions appear tangent to the horizon curve in the extended plane $a/M - r/M$. The study of these special solutions is relevant considering the following relations hold: there is $a_0 \equiv 1/(\omega\sqrt{\sigma})$ where $a_0$ is the origin of the bundles $B_{a\sigma}$ that is at $\omega =$constant in the extended plane $a/M - r/M$, the origin is at $r = 0[2, 76, 77]$. It is worth noting that in the extended plane at fixed spacetime (an horizontal line), on a fixed plane these curves are bundles at constant frequency. Eventually, we can evaluate the gradient of $\omega - a\sigma^2 =$constant. The following relations also hold: $E - \omega_H L > 0$, thus $\ell < 1/\omega_H$, $\omega_H$ is the outer horizon frequency in the extended plane. Therefore, featuring an adapted process in terms of the variation of the moment of the BH, there is $\delta J/\delta M < 1/\omega_H$, which is expression of the condition $\delta M_{\ell\ell} > 0$ (here $J$ has using of mass square). Note that there is the limit $\ell_{\text{max}} = 1/\omega$ independently from $\sigma$ in the spherically symmetric Schwarzschild spacetime. Condition $1/\omega_\pm = a\sigma$ has no solution, but there is the singular surface of the ergospheres, $a^{+}_\sigma \equiv \sqrt{(r-2)r/(\sigma-1)}$, in the extended plane as reported in Figs (15). The condition $1/\omega_\pm = a\sqrt{\sigma}$ is realized only at the origin $r = 0$, particularly for $\sigma = 1$ there is $\omega_\pm = 1/a$ (then $\ell = a$).

On the other hand, considering the coincidence condition with the horizon curves, the condition $\omega_\pm (r_\pm) = a\sigma$ for the bundles $B_{a\sigma}$ is solved for $a_{g\sigma} \equiv (\sqrt{4\sigma -1})/2\sigma$ which, evaluated on the horizons $r_\pm$, gives respectively

---

$^8$ Here we consider the quantity $\tilde{\ell} = \ell/(a\sigma^2) = L/(Ea\sigma^2)$; then if $L = a\sigma^2$ we have $\tilde{\ell} = 1/E$ and therefore for $E = +1$ there is $\tilde{\ell} = 1$.  

*FIG. 15*: Black region is the BH in the extended plane, the boundary is the horizons curve in the extended plane. There is $\sigma = \sin^2 \theta$. Numbers of the left panels are values of constant plane $\sigma$. Upper line. Left panel shows metric Killing bundles $B_{a\sigma}$ at constant $\sigma$, defined by condition $\omega - a\sigma = 0$. Central panel shows the ergospheres curves $a^{+}_\sigma$ in the extended plane which is a spin function, for different $\sigma$s following colors relative to the left panel. Full characterization is in [2, 76, 77]. Bottom line. Curves of constant $\omega - a\sigma$ for different $\sigma$s according to colors in upper line panels. Bottom right panel shows the curves on the equatorial plane $\sigma = 1$. It can be noted the horizon curves. Upper right panel shows notable spins $(a/M)$, radii $(r/M)$ and characteristic frequencies of the bundles $\omega$ as function of the plane $\sigma = \sin^2 \theta \in [0, 1]$. $a_{g\sigma}$ is the tangent curve to the horizons of the bundles $B_{a\sigma}$ $r_{\pm}(a_{g\sigma})$ are the horizon curve for tangent spin, depending on the plane $\sigma$ from definition of $a_{g\sigma}$. $\omega_{\pm}$ are limiting frequencies for stationary observers.
\[
\begin{align*}
\omega_{g0}^\pm &= (2\omega \mp \sqrt{(2\omega - 1)^2}/(2\sqrt{4\omega - 1}) + 1.
\end{align*}
\]

The condition implies \( \ell(r_{g0}) = 1/(a\sigma) \). The tangency condition of the metric bundles are given in [2] with curve \( a_g \) for \( B_\sigma \). The solution of the condition \( a_g = a_{g0} \) for the frequency \( \omega \) is \( \omega_{g0}^\pm (2\omega \mp \sqrt{(2\omega - 1)^2}/(2\sqrt{4\omega - 1}) + 1. \) The horizon curve is clearly delineated as an asymptotic limit of the bundles.

The characteristic frequencies of the bundles, seen as horizons frequencies, are the functions \( \omega_{g0}^\pm \) of Eq. (25). \( \omega_g^- \) has a saddle point for \( \xi = (3 + \sqrt{6})/3 \), corresponded to the spin \( A(\xi) = a_{mso}^1 \) and frequencies \( \omega_{g0}^+ = (1/2\sqrt{2}, 1/\sqrt{2}) \) - see Figs (11). The maximum extractable energy \( \epsilon_\ell \equiv (2 - \sqrt{\omega})/2 = 1/\ell_{\max} = 2 + \sqrt{2} \), confirming the relation \( \ell_{\max} = 1/\omega \) - see Figs (11). The two bundles whose tangent spin on the equatorial plane is \( a_g/M = a_{mso}^1 \), are tangent to the horizon curve in the extended plane on \( r = 2/3, r = 4/3 \), with characteristic frequencies \( \omega = \{1/(2\sqrt{2}), 1/\sqrt{2}\} \) and origin dimensionless spins in the naked singularity regions, \( a_0/M = (2\sqrt{2}/\sqrt{2}) \) respectively on the equatorial plane. Obtaining the limit of \( \ell \ell = 1/2 \) (2 - \sqrt{\omega}) in the static case \( a = 0 \), the bundle frequencies have an extreme as function of \( r/M \) for the orbit of photon \( r = 3M \) where \( \omega_{\pm} = 1/\sqrt{27}\sigma \).

Spin \( a_{g0}^1 \equiv M/\sqrt{2} \) is solution of \( \partial_{a} a \ln s = 0 \) where \( s = \omega_{H0}^+ / \omega_{H0}^- \). Another relevant spin is \( a_g = \sqrt{3}/2 \) which solves the problem for \( \partial_{a} a \equiv 0 \) and where \( s = 1/3, a_0 = 1/\sqrt{2} \), where \( \omega_{H0}^+ = 1/2 \pm 1/\sqrt{2} \) [76].

A different adapted solution parameterizations is discussed in Appendix (D).

B. Sets of tori

An example of distributions of momenta and radius as functions of the spin in the considered parametrization is shown in Figs (17,18). In Fig. (2) we show the dispersion in the corotating and counter-rotating distribution of tori in the RAD.

Considering the situation on the equatorial plane \( (\sigma = 1) \), there are lower bounds in magnitude of the rationalized momentum \( \ell/a \). It is easy to see that this limit is always given as inferior limit (in magnitude) of the extreme rotating BH case. Therefore, corotating fluid configurations are formed for \( \ell/a \geq 2 \) where the limiting value increases with decreasing spin. In the counter-rotating case, fluid configurations can form with \( \ell/a < -22/5 \). The detailed situation is shown in Figs (16), which considers also the limits for the formation of stationary configurations in the ergoregion. For the formation of counter-rotating accreting tori and proto-jets there are the limiting ratios \( \ell/a = -2(1 + \sqrt{2}) \), and \( \ell/a = -7 \). In [81] it was shown that the limiting cases \( \ell = \pm 1 \) do not admit any toroidal Boyer configurations, \( \ell = \ell_{g0} +/- = \pm a \).

A more accurate description is therefore shown in Figs (18), curves of \( \ell = \text{constant} \) show one torus evolution. The analysis points out two relevant spins: the first at \( a = 0.7M \), clear also from the Figure (17), and a second relevant spin is \( a \approx 0.3M \). The crossing of areas and curves show the regions of RAD tori parameter where collision is more probable to occur. This behavior has also a role in the density seeds formation. Considering different regions of the parameters, the separation (corotating-counter-rotating) is larger for the accretion than the proto-jets parameter space, indicating that at high fluid specific angular momentum the corotating and counter-rotating fluids may have not been differentiation in the formation of the early phases of jets, and the separation remains in the range from \( 2M \) to \( 4M \) independently from the spin. The model reveals also the possibility of jet shells of corotating or counter-rotating fluids.

A detailed analysis points out the following spins :

\[
\begin{align*}
a_0 &= 0.638285M : r_{mso}^- = r_{m+}, \quad a_0 = 0.372583M : r_{mso}^- = r_{m+}, \quad a_0 = 0.313708M : r_{mbo}^- = r_{\gamma}^+ \\
a_1 &= 0.172564M : \ell_{mso}^- = \ell_{m+}, \quad a_2 = 0.390781M : \ell_{\gamma}^- = \ell_{mbo}^+, \quad a_3 = 0.508865M : \ell_{\gamma}^- = \ell_{mso}^+
\end{align*}
\]

where \( a_1 < a_2 < a_3 < a_5 \).

There is a discriminating value in two spin classes, determined by the value \( a \approx 0.3M \), see Figs (17) [5, 32]. We distinguish the set of spins \( \{a_\alpha, a_\beta, a_\gamma\} \) defined by the cross of the radii of the corotating and counter-rotating geodetic structures, and the spins \( \{a_h1, a_h2, a_h3\} \) defined by the cross of the corotating and counterrotating fluid angular momentum curves as functions of the BH dimensionless spin. Spins \( \{a_\alpha, a_\beta, a_\gamma\} \) regulate the eRAD inner structure associated to classes of SMBHs defined according to their spins. This can be seen by considering the role of the geodetic structures in the torus model construction, fixing location of maximum and minimum of pressure in the torus, respectively the center and inner edge of the torus, and the related orbital regions shown in Figs (17). The inner eRAD structure is defined by the number of tori, their locations and relative rotation, if corotating or counterrotating with the central BH, and their topology. Spins \( \{a_h1, a_h2, a_h3\} \) define regions of the rationalized specific angular momentum, \( \ell/a = L/\sigma(aE) \), for corotating and counterrotating fluids, regulating also condition on the colliding tori [32, 42]. Furthermore, in Figs (17) a quasilinear relation \( \ell/a \) as function of \( a \) is shown, in logarithm scale: \( \ell \sim a^{b_1+1} \) + \( e^{b0} \) where \( b_1 \) and \( b_0 \) are constant, and it can be explained by an analysis for small dimensionless
FIG. 16: Corotating fluids (−) and counter-rotating fluids (+). The photon orbit location is \( r^\pm \), \( M \) is the BH mass, \( r_{\text{mbo}}^\pm \) is for marginally bounded orbit, \( r_{\text{mso}}^\pm \) is the marginally stable orbits. Left: increments versus spin of the BH for marginally stable orbits, marginally bounded orbits and photon orbits. Central panel: rate of increments versus spin of the BH. These are dimensionless quantities, the increment of gradients is larger for the parameter range determining formation of quiescent tori with limiting unstable proto-jets. Right panel: \( \ell \) fluid specific angular momentum and spin \( a/M \) of the BH, \( \ell^\pm = \ell - \epsilon \) where \( r^\pm = 2M \) is the outer ergosurface. In right panel role of spins \( \{a_{\beta}^\pm, a_{mbo}^\pm, a_\gamma^\pm\} \) are shown. No tori are defined for fluid specific angular momentum in the region \( \ell < \ell_{mbo} \) (gray).

FIG. 17: Plot of the geodesic structure \( r/a \) (left upper panel) and \( r/M \) (center upper panel) as function of the BH dimensionless spin \( a/M \), where \( r \) is the marginally stable orbit (mso), marginally bounded orbit (mbo), and the photon orbit (or marginally circular orbit) \( r_\gamma \) for corotating (−) and counter-rotating (+) fluids. Black region is the BH \( r < r_+ \) where \( r_+ \) is the outer horizon. Gray strip is the outer ergoregion. Purple region is region where cusps of accreting corotating tori are located. Note that some cross the outer ergosurface. Similarly gray region refers to counterrotating fluids. Upper right panel and bottom left panel show the specific angular momentum \( \mp \ell/a \) for counterrotating and corotating fluids according to the notation used in other panels. Here there is \( \ell_* \equiv \ell(r_\bullet) \) for any radius \( r_\bullet \). Spins \( \{a_{h1}, a_{h2}, a_{h3}\} \) (from the crossing of \( \ell^\pm \) curves) and \( \{a_\beta, a_\alpha, a_\gamma\} \) (from the crossing of the geodetic radii curves). Bottom center panel and right panel describe corotating and counter-rotating fluids respectively. Curves of specific angular momentum versus spin \( a \) constant in the plane \( r/a \), gray curves, and evaluated on the geodesic curves as signed in the panel, curves lines on the geodesic radii are also plotted according to the notation on the panel. Each curve is a class of torus/attractor.

spin \( a/M \). It is clear from Figs (9) how the functions of the rotational energy parameter \( \xi \) determining the inner structure of the eRAD in terms of the relative location of corotating or counterrotating tori, is more articulated for small \( \xi \in [0, \xi_0] \), that is for geometries \( a < a_\beta \approx 0.57M \).

On the equatorial plane, the frequency has an extreme for \( a = \sqrt{2} \) where \( \omega = 1/(2\sqrt{2}) \). In Figs (16) we represent the ratio \( \delta r^\pm_\bullet = (r^\pm_\bullet - r^\pm_\star)/a \) and the increment \( \partial_a \delta r^\pm_\bullet \) for \( \star \) and \( \bullet \) denoting geodetic limiting orbits. The differences \( a\delta r^\pm_\bullet \) provide a first indication on the construction of the couples of (counter-rotating orbits and therefore the possibility of formation and evolution under spin transition, remaining almost constant for \( a \in [0, M] \). Curve representing classes
of torus/attractor according to the ratios $r/a$ and $\ell/a$ are shown in Figs (17). We mention here the extreme case of “steady” fluid with respect to the central object, in other words the cases $\ell = 1$, or the counter-rotating case $\ell = -1$. With respect to these limiting conditions, the corotating and counter-rotating configurations are constrained by the range $\ell < 1$ as shown in [81]. The critical points for the pressure can be only at $\ell < -1$, therefore only counter-rotating tori configurations are allowed with $\ell < 1$. As the ergosurface, independently from the spin, is located at $r^+ = 2M$, the ratio $\ell^−/a$ has a maximum on the stationary limit on $a = 2M$, correspondent to a NS where there is limting $L = E = 2E$ (see analysis [74, 75]). The bundle with origin in $a_0 = 2M$ on the equatorial plane is tangent to the extreme Kerr BH on the extended plane, where the characteristic frequency is $\omega = 1/2$, corresponding to the maximum rotational energy $\xi = \xi_\ell$, while maximum for ratio $\ell/a$ is at $\ell/a = 2\sqrt{2}$. 

V. DISCUSSION AND CONCLUDING REMARKS 

Black holes are the powerful engines of the active galactic nuclei. A very relevant question of the BH accretion physics in this context is determination of the characteristic BH parameters, its spin $a/M$ and mass $M$. This issue is certainly a preponderant aspect of the physics of accretion and jet emission, such that the BH accretion history can be traced in the evolution of its spin. 

Variation of BH mass and spin parameters can be the results of combination of phases of mass growth from accretion and spins-ups processes, or rotational energy extraction, eventually constituting a spin-down mechanism. The BH spin evolution may consist in several periods of episodic accretions or of a continuum prolonged accretion regime. The spin of SMBHs traces the angular momentum of the accreted material, tracing consequently the SMBH feeding. The accretion processes may be evident also by the presence of accretion disk misalignment. In these processes, the presence of fluid viscosity and the Lense-Thirring effect enhanced by the BH rotation affect the disks alignment evolution, empowering the BH spin evolution (see for example role of the Bardeen–Pettersson effect), and constituting also a possible mechanism for the origin of the inner discrete inhomogeneous eRAD structure. 

Methods to provide an evaluation of mass and spin of a BH are usually highly model (and process) dependent. The establishment of the location of inner edge of accretion disks, or the analysis of the jet/radio power of radio-loud AGNs[1, 8–10], based on a correlation between the beam power and spin are reliable methods. As discussed in Sec. (III C), we can detect the BH spin evolution through the efficiency of the energy extraction. 

The approach we consider here relates the BH spin “states” prior and after a (complete) rotational energy extraction [36]. A mayor characteristic of this approach is the fact that it is essentially model-independent, in the sense that it does not depend on the details of the process of the energy extraction, considering exclusively the final and initial BH states where its spin is only affected by the extraction of the reducible BH mass, according to the BH thermodynamical laws. The rotational energy extraction parameter $\xi$ is defined by the ratio between the total energy $(M-$ADM mass) and the BH irreducible mass $M_{irr}$. Therefore, we proceed to study the BH thermodynamical processes related to a series of high energy phenomena affecting the BH accretion system, combined in the RAD model of tori aggregates. 

Studying the BH mass and the outflow energy, we obtain a lower limit on the BH spin. 

In this analysis we related three diverse topics of BH physics: accretion physics, the BH extraction process energy.
and the metric bundles concept grounded on the light surfaces definition. Features of the extraction processes with BH thermodynamics introduced in [1], have been discussed and revisited through metric bundles in Sec. (III A), and related to accretion physics in Sec. (III C). The dimensionless $\xi$ is evaluated for the BH initial ADM mass, relating the spin to the mass evaluation for the initial state of the BH. In this analysis we made use of some simplificative assumptions which we plan to focus on in future analysis. Firstly, we assumed a process leading to total rotational energy extraction; on the other hand, contribution of mass feeding processes has been neglected especially in the evaluation of the accretion process, eventually not considering the contribution of accretion to the BH mass, or the so-called runaway instability that is essentially an iterative process between BHs and disks. We showed the relation between the rotational law of the tori, the characteristic frequency of the bundle and the relativistic velocity defining the von Zeipel surfaces, in Section (IV).

Black hole luminosity is another relevant variable for the BH spin evolution, specifically, if the BH is accreting at near Eddington luminosity, which may imply super Eddington accretion rates for the accreting disks, as it is the case for the thick tori considered here. Luminosity and accretion rates should reflect the BH spin-up evolution: the BH may have an initial state of super-Eddington luminosity, followed by a phase characterized by lower luminosity. The standard accretion disk theory prescribes that the radiative efficiency of energy conversion in such processes is related to the BH spin, because it is related to the location of inner edge, established by the presence of a tori cusp of the accreting configuration. In the eRAD model we discussed the connection between the specific angular momentum of the disk, which is a parameter related to the point of maximum pressure in the torus, and the BH spin.

The binding energy of the orbiting matter is radiated leaving the inner edge, matter freely falling into the hole, assuming the total amount of energy converted into radiation being binding energy. There is therefore a relation between the accreting material angular momentum and the radiative efficiency. We specialize this relation in Sec. (III C).

In Sec. (IV), we discussed for RAD the BH spin-accretion disk correlation. The attractor-ringed-disk system shows various symmetry properties with respect to the quantities $\ell = \ell/(a\sigma)$ and $R = r/a$, where $\sigma = \sin \theta$ [81]. Noticeably, many properties of the RAD depend mainly on the rationalized specific angular momentum $\bar{\ell}$. There are relevant spins $a \approx 0.94M$, $a \approx 0.7M$ and $a \approx 0.3M$. We show from the analysis of ratios $\ell/a$, the formation of corotating tori is bounded to $\ell/a \geq 2$, counter-rotating tori to $\ell/a \leq -22/5$.

At fixed specific angular momentum $\ell$, the zeros of the $\mathcal{L} \cdot L$, as functions of $\ell$, define limiting surfaces of the fluid configurations (light surfaces for solutions in terms of $\omega$). For fluids with specific angular momentum $\ell \in L_3$, the limiting surfaces are the cylinder-like surfaces crossing the equatorial plane on a point which is increasingly far from the attractor with increasing specific angular momentum magnitude. A second surface, embracing the BH, appears, matching the outer surface at the cusp $r_\Lambda$. For $\ell = \ell_\Lambda$, a cusp appears together with an inner surface closed on the BH. A further remarkable aspect of the application of the metric bundles approach to the analysis of the BH energetics, consists in the use a NS-BH correspondence to predict observational characteristics of the BH accretion disk systems, through the extension of the geometry in the naked singularity region in the extended plane, since the metric bundles, tangent to the horizons curve in the extended plane, are in part contained in the NS region. Results of the analysis of metric bundles are in Figs (15,11,8,7,6,5,4). Metric bundles are defined in the extended plane, considered in Figs (3) in terms of the rotational energy. Metric bundles are curves tangent to the horizon curve in the extended plane. The tangent point changes along the horizon curve during the process of rotational energy extraction. The characteristic frequencies of the bundle provide limits to the angular momentum of the fluid, and can be measured with the replicas orbits.

A further question regarding the measurement of the BH spin emerges from the fact that the applied methods are related to phenomena in the very vicinity of the horizon, where the general relativistic effects are predominant. We explored these conditions adapted to the metric bundles concept related to definition of the light surfaces.

This analysis informs of the overall situation before any occurring process involving disks and central BH. All the quantities are evaluated at the state $(0)$ prior the process, therefore all the quantities evaluated here inform on the status of the BH-accretion disk system at its initial state $(0)$, by evaluating the related $\xi$ parameter. We proceed with the characterization of the state $(0)$ in terms of the eRAD structure in relation with the rotational energy extracted, characterized by a determined inner composition of tori, relatively to fluid corotation or counter-rotation, or the associated topological state with the instauration of certain unstable phases or the position of the inner edge and certain characteristics of the energetics of the accretion. However, the analysis describes the state prior the extraction processes, considering the rotational energy of the BH unaltered. Results of the analysis of the tori morphological and topological characteristic have been analysed in Fig (9), (10), and (14). The energetics is considered in Figs (12), Figs (13). The conditions for an eRAD with counterrotating tori is explored in Sec. (IV B), and in Figs (16), Figs (17). This analysis individuates the spins $\{a_s, a_3, a_6\}$ and $\{a_{h1}, a_{h2}, a_{h3}\}$ significant for the conditions on the fluids in the initial configurations of tori. Concerning how the accretion disks and particularly the part of inner disk and inner edge affects this process we refer for example to [92]. The inner edge (cusp) is constrained in Figs (2) in terms of the spin, as the center of maximum pressure in the torus for corotating and counterrotating fluids, in Figs (9) in terms of the parameter $\xi$. The cusp is in the range $r_x \in [r_{\text{mho}}, r_{\text{mso}}]$ and it is constrained in Figs (10)- Figs (17),
in terms of the specific angular momentum. In these processes we should consider for the thick disks the instauration of the runaway process, and in the case of RADs the runaway-runaway process. Runaway instability affecting thick tori and their BH attractors implies large accretion rates typical of these tori—the BH mass increases changing the spacetime properties, and in turn affects the orbiting accreting disk. In this process, the inner edge moves inwards, or the cusp moves inside the disk inducing increase of mass transfer. In the RAD system one may consider the possibility of Runaway-Runaway instability, originating when the Runaway instability affecting the BH and the inner accreting torus of the agglomeration, is accompanied by the consequences of the background modification on the outer tori of the RAD which can collide, accrete or stabilize in dependence on initial conditions [5]. It is the combination of runaway instability, involving the inner edge of the inner accreting torus of the RAD, with the consequent destabilization of the aggregate. The accretion modifies the inner torus morphology and the background geometry having repercussions in the whole RAD structure establishing a sequence of events having different possible outcomes.

More specifically, characteristic features of different accretion disk models outlining the torus interior structure, the mechanism of energy transport in the disk, and more generally the different accretion dynamics might affect the energy extraction processes from the central BH, constraining eventually the assumptions on the tori-BH model adopted here. The magnetic fields and, in particular contexts, the quantum effects of extraction in the BH vicinity are among the most relevant mechanisms that can interact in the extraction process, requiring a different characterization of the energy outflow. Together with these effects however, there can be a different setup on the initial and final state of the BH accretion system, a different outflow contribution and the establishment of special instabilities which can change the BH disks system as the runaway instability or the change in spin orientation due to Bardeen–Petterson effect. More specifically, in the jets emission, characterizing SMBH also in AGNs, the frame dragging for the spinning attractor is thought to play an important role, the Lense–Thirring effect induced by the central attractor can engine also the Bardeen–Petterson effect (a process resulting in the tearing up of the orbiting disk) [97]). The change in the disks structure can induce a spin shift in magnitude and orientation of the BH. The warping and twisting of the disks, the Bardeen–Petterson effect, depends on the frame-dragging and the disk viscosity, the presence of magnetic fields, and the original disk inclination with respect to the orbital plane of a star companion. There can be a final steady state of the initially misaligned torus in a coplanar accreting inner torus with the outer part of the original disk aligning more slowly on the longer timescales. These misaligned disks are expected to be characteristics of the transient periods of the BH–accretion disk evolution. Bardeen-Petterson effect is also proposed as a possible cause of the counter-alignment of BH and disk spins. In fact BH can grow rapidly if they acquire most of accreting mass in a sequence of randomly oriented accretion episodes [98, 99]. The chaotical accretion in AGNs could produce counterrotating accretion disks or strongly misaligned disks with respect to the central SMBH spin, and misaligned disks evolution orbiting a Kerr BH might lead to a tearing up of the disk into several planes with different inclinations with a precession of the BH spin. The BH spin changes under the action of the disk torques, as the disk, being subjected to Lense-Thirring precession, becomes twisted and warped—[100], therefore it is possible to study the aligning of Kerr BHs and accretion disks [101], or the effects of BH spin on misaligned accretion disks in relation to the role of the disk inner edge and the alignment of the angular momentum with the BH spin—[102]. The method used here of measuring BH spin is largely not model dependent, however it is based on the hypothesis that the energy outflow to be measured is powered by the BH spin energy only, therefore the outflow energy is taken as a lower bound on the BH spin energy. More generally the extracted energy can be determined considering the ratio between the outflow energy versus the spin energy. In here we considered the two quantities be coincident, being based on the suggestion that the jets beam power can be directly related to BH spin, linking jet power and accretion tori with jet mechanism supposed to be originated in the accretion disk with magnetic fields, and extraction of the BH spin energy powering the outflow from the hole. More precisely we consider the dimensionless ratio between the released energy (coincident with the rotational energy of the initial BH state) to the BH mass (ADM total mass of the initial BH state). We assume that all the BH rotational energy is extracted, and the spin changes only by extraction of the reducible BH mass. The energy outflow can be empowered however from the the accretion of disk material and separately the rotational energy of the black hole itself. We assume that, from an initial Kerr BH, the final stage of extraction process is a static spherically symmetric Schwarzschild spacetime. The hypothesis is that all the outflow is induced by the rotational energy extraction leaving constant (and not increased) the BH irreducible mass. A divergence from this hypothesis can alter the analysis of Figs (13) which do not consider the contribution of the mass and momentum to the BH subsequent to accretion and in the case of runaway-instability. In fact at the initial and final state the BH-accretion disk system which is stable (at least for few seconds), the runaway instability typical of SMBHs and thick tori could undermine this situation, the runaway-instability affecting the structure of the torus inner part can be considered as a purely dynamical effect, meaning that time-scale is extremely short occurring therefore before the start of processes as the viscous transport of angular momentum in the disk in the accretion model supported by such processes. To consider properly these aspects, having fixed the details of the process, a sequence of steady states should be considered, re-evaluating the disks mass and momentum (consequently its inner edge) and the BH spin and mass altered by the accretion. When an accretion disk is subjected to the runaway instability, a large
portion of its mass fall into the BH within a few dynamical time. Furthermore geometrically thick disks, which are particularly subjected to runaway instability, have large accretion rates (with super Eddington luminosity) and especially for the largest tori (great corotating specific angular momentum or for the counterrotating tori far from the BH) may also have a relevant self gravity. For small BH spin the corotating tori matter inflow may have also an initial specific momentum relatively large. The divergence of the radial mass transfer expected as an outcome of the runaway-instability clearly can lead to the disk destruction. The runaway instability of such systems is based on the following mechanism: accretion from the disk induces an increase of the BH mass, consequently the geometry background is affected and resettle in a new state. As a consequence of this, the accretion disk can also never reach a completely steady state. Considering therefore the case of disks increasing the BH mass two faces for critical and overcritical accretion from thick disk are possible i.e. the disk cusp can move inwardly, slowing the mass transfer and consequently reaching eventually a stable situation or, alteratively, the cusp can move outwardly in the disk, the mass transfer increases (in velocity), leading to the runaway instability. Further factors have been neglected in this model and may be relevant also for the combinations with the runaway instability the establishment of such instability and its outcomes, as the tori self-gravity seeming to favor the instability, the black hole rotation which can have a stabilizing effect on the runaway instability, and finally a different rotational law for the torus with a non-constant distribution of the angular momentum (which might have a stabilizing effect in relation to runaway instability).

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Appendix A: Explicit solutions $\ell^\pm(\xi, r) = \ell$

A convenient parametrization of solutions $\ell^\pm(\xi, r) = \ell$ are as follows. See also Figs (13). For corotating fluids, there is for $\ell^- (\xi, r) = \ell$

$$\xi = \frac{1}{2} \left( 2 \mp \sqrt{2\xi \Psi_\beta^- - 2\Psi_\alpha^- + 4} \right), \quad \xi = \frac{1}{2} \left( 2 \mp \sqrt{2\xi \Psi_\beta^- - 2\Psi_\alpha^- + 4} \right),$$

$$\xi = \frac{1}{2} \left( 2 \mp \sqrt{2\xi \Psi_\beta^+ + 2\Psi_\alpha^+} \right), \quad \xi = \frac{1}{2} \left( 2 \mp \sqrt{2\xi \Psi_\beta^+ + 2\Psi_\alpha^+} \right),$$

where $\xi = \pm$

$$\Psi_\alpha^\pm \equiv \sqrt{\pm 4\ell^3/2 - 16(r - 1)r^2 + 4\ell^2(3r - 2)r + \ell^4}, \quad \Psi_\beta^\pm \equiv \pm 4\ell^{3/2} + 4(r - 2)r - 2\ell^2. \quad (A4)$$

For counter-rotating orbits $\ell^+ (\xi, r) = \ell$, we have

$$\xi = \frac{1}{2} \left( 2 \pm \sqrt{2\xi \Psi_\beta^+ - 2\Psi_\alpha^+ + 4} \right), \quad \xi = \frac{1}{2} \left( 2 \pm \sqrt{2\xi \Psi_\beta^+ - 2\Psi_\alpha^+ + 4} \right),$$

$$\xi = \frac{1}{2} \left( 2 \pm \sqrt{2\xi \Psi_\beta^- + 2\Psi_\alpha^-} \right), \quad \xi = \frac{1}{2} \left( 2 \pm \sqrt{2\xi \Psi_\beta^- + 2\Psi_\alpha^-} \right).$$

(See Figs (13).). Using the approach introduced with the bundle on the extended plane, featuring classes of BH solutions, we found the spin functions for this problem for corotating and counter-rotating fluids,

$$a_1 = \frac{1}{2} \left[ -y_i^+ + \ell - 2\sqrt{\ell} \right], \quad a_2 = \frac{1}{2} \left( y_i^+ + \ell - 2\sqrt{\ell} \right), \quad a_3 = \frac{1}{2} \left( -y_i^+ + \ell + 2\sqrt{\ell} \right), \quad a_3 = \frac{1}{2} \left( y_i^+ + \ell + 2\sqrt{\ell} \right),$$

where there is

$$y_i^+ = \sqrt{\ell^2 \pm 4\ell(r - 1)\sqrt{\ell}} - 4(r - 1)r, \quad y_i^\mp = \sqrt{4\ell^{3/2} \mp 2y_i^- (\ell + 2\sqrt{\ell})} - 2\ell^2 + 4(r - 1)^2; \quad (A8)$$

$$y_{i\ell i}^+ = \sqrt{4(r - 1)^2 - 2 \left( \left( \ell^2 \mp \ell \left[ y_i^+ - 2\ell^{3/2} \right] \right) - 2 \left( \sqrt{\ell} y_i^+ \right) \right)}.$$
These solutions are for \( r = y, y \) being the Cartesian flat coordinate on the equatorial plane of the Kerr BH coincident with the symmetry plane of the torus. Therefore, rearranging the terms to make explicit for corotating \((-\)\) and counterrotating \((+)\) tori we have

\[
\begin{align*}
a_1 & : \; (+) \; \xi = 1 \pm \frac{1}{2} \sqrt{2 - y_{ii}^+}; \quad (-) \; \xi = 1 \pm \frac{1}{2} \sqrt{y_{ii}^+ + 2}, \\
a_2 & : \; (+) \; \xi = 1 \pm \frac{1}{2} \sqrt{2 - y_{ii}^+}; \quad (-) \; \xi = 1 \pm \frac{1}{2} \sqrt{y_{ii}^+ + 2}, \\
a_3 & : \; (+) \; \xi = 1 \pm \frac{1}{2} \sqrt{2 - y_{iii}^+}; \quad (-) \; \xi = 1 \pm \frac{1}{2} \sqrt{y_{iii}^+ + 2}, \\
a_4 & : \; (+) \; \xi = 1 \pm \frac{1}{2} \sqrt{2 - y_{iii}^+}; \quad (-) \; \xi = 1 \pm \frac{1}{2} \sqrt{y_{iii}^+ + 2}.
\end{align*}
\]  

(A11)  

(A12)  

(A13)  

(A14)

Appendix B: Explicit solutions for light-surfaces

The zeros of \( \mathcal{L} \cdot \mathcal{L} = 0 \), for the Killing field \( \mathcal{L} = \xi_t + \omega \xi_\phi \), defines the light surfaces \( r(\sigma; a, \omega) \) when solved for the radius \( r \) and the metric bundles \( a(\omega; r, \sigma) \). Explicit solutions for the light surface at any plane \( \sigma \equiv \sin^2 \theta \) are the four radii:

\[
\begin{align*}
r_{u}^\pm & = \frac{1}{2} \left( u_* \pm \sqrt{-\frac{4(a\omega - 1)^2}{\sigma u_*^2} - bb_\pi - c_*} \right), \quad u_* = \{+u_0, -u_0\}, \quad uu_* = \sqrt{c_* - bb_\pi}, \\
c_* & = \frac{1}{3} \left[ 2\sqrt{bb_\pi^2 - 3cc_\pi \cos \left(\frac{1}{3} \cos^{-1} (dd_\pi)\right)} + bb_\pi \right], \\
dd_\pi & = \frac{36a^2bb_\pi(-1 + \omega^4)(a^2\omega^2 - 1) + 54(a\omega - 1)^4 + bb_\pi^3\omega^2}{\sigma^2}, \\
cc_\pi & = \frac{-4a^2(-1 + \omega^4)a^2\omega^2 - 1}{\sigma^2}, \quad bb_\pi = \frac{-a^2(-1 + \omega^4)a^2\omega^2 - 1}{\sigma^2}.
\end{align*}
\]

(B1)  

(B2)  

(B3)  

(B4)

where \( a_0 \equiv 1/\omega \sqrt{\sigma} \) is the bundle origin spin–Figs (19). On the equatorial plane, \( \sigma = 1 \), light surfaces are the radii

\[
\begin{align*}
r_1 & = \frac{2u_k \cos \left(\frac{1}{3} \cos^{-1} (-u_x)\right)}{\sqrt{3}}, \quad r_2 = \frac{2u_k \sin \left(\frac{1}{3} \sin^{-1} (u_x)\right)}{\sqrt{3}}, \quad u_k = \sqrt{\frac{1}{\omega^2} - a^2}, \quad u_x = \frac{3\sqrt{3}\omega^2 u_k}{(a\omega + 1)^2}.
\end{align*}
\]

(B5)

The analysis through bundles connects classes of spacetimes with equal limiting photon frequencies in some points, replicas in the same spacetimes, with the classes of RAD disks which we can expect in different phases of their evolution. Any point of the light surface is related to an horizon in the extended plane. Excluding the counterrotating orbits, for any point \((r, \sigma)\) there is a maximum of two frequencies. One point is the outer horizon of the BH geometry, the second frequency is also a frequency of the horizon on a geometry connected by the bundle curve in the plane, crossing this point, according to the analysis of Sec. (III A). A detailed study of the bundles \( \mathcal{B}_\omega \) at fixed frequency is in [2, 76, 77]; for fixed frequency and plane \( \mathcal{B}_{a\omega} \) prevalently studied in Sec. (III A) on the equatorial plane \( \sigma = 1 \) relevant for the eRAD case. There is a detailed discussion of the class of bundles \( \{\mathcal{B}_\omega\}_\sigma \) with frequencies \( \omega \in [\omega_H, \omega_H^+] \) and \( \sigma \in [0, 1] \) (note there is also the range \( \sigma = [-1, 0] \) considered within the axial symmetry of the metric in [2, 76, 77]). We considered \( \omega \) as the horizons frequencies for \( a > 0 \), including limiting points of the extended plane on line \( a = 0 \) then \((r = 0, a = 0)\) and \((r = 2M, a = 0)\). Transformations from \( \mathcal{B}_{a\omega, \sigma} \) to a \( \mathcal{B}_{a\omega, \sigma, 1} \) relate different (couples of) points of the same spacetime (horizontal line of the extended plane) and different spacetimes related by transformations along the bundle curve, at fixed initial state 0 (with parameters \( a_0, M_0 \)) of final of a BH transition (with parameter \( a_1, M_1 \)). In general, at fixed \( a = \bar{a} \), the light surfaces are given by the collections of points crossing the horizontal line \( a = \bar{a} \) in the classes \( \mathcal{B}_{a\omega, \sigma} \) for all values of \( \sigma \) and \( \omega \). Not all the bundles intersect a fixed horizontal line (there are frequencies not accessible in a given spacetime, depending on the polar angle), some are confined in a region of the extended plane as it occurs for a portion of the inner horizon tangent bundles. Other bundles are partially contained in the naked singularity region of the extended plane, the bundle tangent to the extreme Kerr BH spacetime for \( \sigma = 1 \) is entirely contained in the NS sector apart for the tangent point. Some bundles cross each other, for corotating photons there is a maximum of two bundles at fixed plane. The observation of the BH-accretion disks and RAD associated phenomenology should show traces of the presence of replicas, Eq. (22), and the relation with the states before and after the rotational energy extraction, described as a shift on the horizon curve. Main
characteristic quantities can be considered as function of the energy parameter \( \xi \) which can then be detected by the measure of energy outflow.

Under spin transition this light cylinder is deformed. There is a fixed point \( p \) on the external horizon in the extended plane which shifts rigidly (in the sense of [2]) on the curve \( a_\perp \). The BHs horizon is independent of \( \theta \) and thus it rotates rigidly. The analysis of bundles in the external plane shows the existence of a collection of points with the same frequency of the horizon \( \omega_H^I \) for \( r_+ \) and \( \{(r_s, \sigma_s)\} \) which could serve to relate magnetospheres in the two consequential states\(^9\).

Appendix C: Von Zeipel surfaces, metric Killing bundles and tori

Here we explore the connection between the von Zeipel surfaces, defined as \( \Omega = \text{constant}, \) metric bundles, which are curves with light-like orbital frequency \( \omega = \text{constant on the plane } a/M - r/M, \) and eRAD tori defined by condition \( \ell = \text{constant}. \) From Eq. (5) we have \( \Omega(\ell) = \Omega, \) or viceversa the explicit expression \( \ell(\Omega) = \ell. \) It is convenient to introduce the rotational law of the eRAD with explicit dependence on the polar parameter \( \sigma: \)

\[
\ell_\sigma^2 \equiv \frac{Cr \mp C r_1}{Cr_2}, \quad \text{where} \quad C r \equiv a^3 r^2 (\sigma - 2) \sigma + a^5 [-(\sigma - 1)] \sigma + ar^3 (4 - 3r) \sigma; \tag{C1}
\]

\[
C r_1 \equiv \sqrt{r} [a^2 + (r - 2)r]^2 [r^2 - a^2 (\sigma - 1)] [a^2 (\sigma - 1) + r^2]; \tag{C2}
\]

\[
C r_2 \equiv a^2 r^2 [-2r(\sigma - 1) + 3\sigma - 4] + a^4 (\sigma - 1)[r(\sigma - 1) - \sigma] + (r - 2)^2 r^3; \tag{C3}
\]

with \( \ell_\sigma^2 = \ell_\perp^2 \) on the equatorial plane (\( \sigma = 1 \)). Important is the special case \( a = 0 \) where \( \Omega = \pm \frac{1}{\sqrt{r^3 \sin^2(\theta)}} \) (therefore the limit \( \Omega \sqrt{\sigma} = 2/r^{3/2} \)) using Eq. (5). In the Kerr (BH) spacetime, for any spin \( a/M \) on the equatorial plane \( \sigma = 1, \)

\(^9\) Concerning the proto-jets emission, in [4, 14, 89] the set of RAD open-cusped configurations and open configurations which are not related to the critical points of the effective potential were studied. These correspond to the solutions \( \Pi = 0 \) of: (a) \( \Pi(\ell) = g_{\phi \phi} + 2g_{\phi t} + \ell^2 g_{tt}, \) (b) \( \Pi(u^i, u^\phi) = g_{tt}(u^t)^2 + 2g_{\phi t}u^tu^\phi + g_{\phi \phi}(u^\phi)^2 = (u^t)^2 (g_{tt} + 2g_{\phi t} \Omega + g_{\phi \phi} \Omega^2) \approx (u^t)^2 \mathcal{L} \cdot \mathcal{L}, \) and (c) \( \Pi(L, E) = \mathcal{E}^2 g_{\phi \phi} + 2Eg_{\phi t} L(t) + g_{tt} L(t)^2, \) These quantities are of course related to the equation \( \mathcal{L} \cdot \mathcal{L} = 0 \) providing solutions of stationary observers. Thus, \( \Pi \) is related to the normalization factor \( \gamma \) for the stationary observers, establishing thereby the light-surfaces. The effective potential, related to the four-velocity component \( u_\ell = g_{\phi t} u^\phi + g_{tt} u^t, \) is not well defined on the zeros of \( \Pi(\ell). \)
FIG. 20: Surfaces $\ell = \text{constant}$ in the extended plane. Each curve define an eRAD torus. Black region is the BHs. $a/M$ is the dimensionless BH spin, $\ell = \text{constant}$ is the fluid specific angular momentum and $\ell(r)$ is the eRAD rotational law, $\ell^+$ is for corotating fluids and $\ell^-$ for counter-rotating fluids. Marginally stable circular orbits radii are respectively $r_{\text{ms}}$. 

FIG. 21: On the equatorial plane $\sigma = 1$, $r_{\mp}$ are the outer and inner horizons of the BH, $r_+^*$ is the outer ergosurface. $r_{\pm}^*$ are the photon orbits for the counter-rotating and corotating motion respectively. $a = 0$ is the Schwarzschild static geometry. $a = M$ is the extreme Kerr spacetime. $\omega_{\pm}$ are the limiting photon stationary orbit frequencies and metric bundles frequencies (for $\omega_{\pm} = \text{constant}$ for varying $a/M$). Quantity $\Omega_{\pm}$ is the relativistic velocity in Eq. (C5) there is

$$\Omega_{\pm}^2 \equiv \mp \frac{1}{r^{3/2} + a}, \quad \Omega_{\mp} = \Omega(\ell_{\pm}), \quad (\sigma = 1), \quad (C5)$$

(+ is for counter-rotating and − is for corotating). Example of curves $\ell = \text{constant}$ (tori) in the extended plane are in Fig. (20). Figs (21) show the relation between $\omega_{\pm}$ (bundle characteristic frequencies and limiting photon stationary orbits) and $\Omega$. Surfaces of constant $\Omega^2$ are the von Zeipel surfaces, surfaces of constant $\ell$ are the matter configurations considered here, surfaces of constant $\omega_{\pm}$ are light surfaces defined by stationary observers that define metric bundles—see Appendix (B) and [91]. Clearly $\partial_r \omega_{\mp} = 0$ is solved for $r_{\pm}^*$, where there is also $\Omega_{\pm} = \omega_{\pm}$ respectively, a further solution is for the horizons $r_{\pm}$. We solve the problem $\Omega(\ell) = \omega_{\pm}$ for $\ell$, obtaining the solution

$$\ell_{\pm}^* \equiv -g_{\phi\phi} \mp \sqrt{g_{\phi\phi}^2 - g_{\phi\phi}g_{tt}} \frac{g_{\phi\phi} \omega_{\pm}^2}{g_{tt}} \quad (\omega_{\pm} = \Omega(\ell)), \quad (C6)$$

$$\omega_{\pm}^\pm \equiv -g_{\phi\phi} \mp \sqrt{g_{\phi\phi}^2 - g_{\phi\phi}g_{tt}} \frac{g_{\phi\phi}}{g_{\phi\phi}}, \quad \text{where} \quad \omega_{\pm}^\pm = \omega_{\pm} \quad \text{on} \quad \sigma = 1. \quad (C7)$$

From Fig. (22) it can be deduced that $\ell_{\pm}^* = 1/(\Omega_{\pm}(r_{\pm}^*)) = 1/\omega_{\pm} = a_0$ respectively, where $a_0$ is the bundle origin, $\omega_{\pm}$ is the bundle characteristic frequency, and orbital frequency of the photon on $r_{\pm}^*$ for the spacetime $a \neq a_0$ (on the equatorial plane).
In [80] the following variables were introduced:

\[ \Delta_\pm \equiv \ell \pm a, \quad \text{where} \quad \Delta_+ \Delta_- > 0, \]  

where

\[ \ell = \frac{\Delta_- + \Delta_+}{2}, \quad a = \frac{(\Delta_+ - \Delta_-)}{2}, \quad A_\pm \equiv \frac{\Delta_\pm + \Delta_\mp}{2} : \quad A_+ = a > 0, \quad A_- = \ell. \]  

Then we can solve the Boyer problem to find out the tori in terms of the \( \Delta_\pm \) and not on the rotational law. The critical points for the pressures where solutions of the Boyer problem exist are given by:

\[ \Delta_- = \frac{a(r-1)r^2 - \sqrt{r^3\Delta^2}}{\Delta + X}, \quad \text{and} \quad \Delta_+ = -\frac{2a^3 - ar[8 + (r-7)r]}{\Delta + X} + \sqrt{r^3\Delta^2}, \]  

\[ \Delta_- = \frac{a(r-1)r^2 + \sqrt{r^3\Delta^2}}{\Delta + X}, \quad \text{and} \quad \Delta_+ = \frac{2a^3 - ar[8 + r(r-7)] + \sqrt{r^3\Delta^2}}{\Delta + X}, \]  

where \( X = -r(r-2)(r-1) \) and \( \Delta_+ - \Delta_- = 2a \).

We consider here a different problem, re-phrasing the analysis above re-considering the solutions for the Euler equation for the tori, by using new variables. We focus on the potential on the equatorial plane:

\[ V_{\text{eff}} \equiv \sqrt{\frac{r \left[ \left( \frac{\Delta_+ + \Delta_-}{2} \right)^2 + (r-2)r \right]}{2\Delta_+^2 + r^3 + \Delta_+ \Delta_- r}}, \quad \text{where} \quad \ell = \frac{\Delta_+ - \Delta_-}{2}, \quad a = \frac{\Delta_+ + \Delta_-}{2}, \quad \{\Delta_{\pm} = a \mp \ell\} \]  

We consider now the tori constrained by the condition \( \partial_{\pm} V_{\text{eff}} \equiv \partial_{\Delta_{\pm}} V_{\text{eff}} = 0 \) where there is:

\[ \partial_- V = 0; \quad \text{for} : \quad (1) \Delta_- = \frac{\Delta_+ r}{r - 4} \quad \text{or} \quad (2) \Delta_- = -\Delta_+ - \frac{2(r-4)r}{\Delta_+}; \]  

Appendix D: Adapted solution parameterization

FIG. 22: Von Zeipel surfaces for the static Schwarzschild spacetime \( a = 0 \), for BH spin \( a = 0.9 M \), for the extreme Kerr spacetime. Black region is the central BH. \( r_+ \) is a photon orbit for corotating (−) and counter-rotating orbit (+). \( \ell \)-constant is the fluid specific angular momentum, \( \Omega(\ell_{\pm}) \) is the relativistic frequencies in Eqs (5).
or equivalently solving for $\Delta_+$ we find

$$\ell(1) = \frac{\Delta_-(r - 4)}{r}, \quad \ell(2) = \frac{\Delta_+}{a} \equiv \frac{1}{2} \left[ -\Delta_+ \pm \sqrt{\Delta_+^2 - 8(r - 4)r} \right],$$

(D7)

where in terms of specific angular momentum there is

$$\ell(1) = -\frac{2a}{r - 2}, \quad \ell(2) = \frac{4r - a^2 - r^2}{a},$$

(D8)

there is (1) = (2) for $r = r_+$, and $\ell = \sqrt{(a^4 + 4\sqrt{1 - a^2}a^2 + 8\sqrt{1 - a^2} + 8)/a^2 - a}$.

On the other hand, there is

$$\partial_+ V = 0, \quad \Delta_+ = \frac{\Delta_-(r - 4)}{r}, \quad \Delta_+ = -\Delta_+ - 2\frac{r^2}{\Delta_+}.$$

(D9)

The last case is $\Delta_- = \Delta_+^\pm \equiv \left[ -\Delta_+ \pm \sqrt{\Delta_+^2 - 8(r - 4)r} \right]/2$, in terms of specific angular momenta we find respectively

$$\ell(1) = -\frac{2a}{r - 2}, \quad \ell(2) = \frac{a^2 + r^2}{a}.$$

(D10)

Note solution (1) solved as well as the derivative for $(a - \ell)$; there is (1) = (3) for $r = r_+$. On the other hand, the problem $\ell^\pm = \ell$ is solved for both corotating and counter-rotating fluids giving the solutions

$$\Delta_+ = \Delta_- + \frac{2}{r^3 - \Delta_-^2} \Delta_- \frac{r^2}{\Delta_-^2} \left[ \frac{r^2 + \Delta_-^2}{(r^2 - \Delta_-^2)^2} \right]^2,$$

(D11)

$$\Delta_+ = \Delta_- + \frac{2}{r^3 - \Delta_-^2} \Delta_- \frac{r^2}{\Delta_-^2} \left[ \frac{r^2 + \Delta_-^2}{(r^2 - \Delta_-^2)^2} \right]^2.$$

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