A novel family of 1-D robust chaotic maps

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Abstract

Chaotic dynamics of various continuous and discrete-time mathematical models are used frequently in many practical applications. Many of these applications demand the chaotic behavior of the model to be robust. Therefore, it has been always a challenge to find mathematical models which exhibit robust chaotic dynamics. In the existing literature there exist a very few studies of robust chaos generators based on simple 1-D mathematical models. In this paper, we have proposed an infinite family consisting of simple one-dimensional piecewise smooth maps which can be effectively used to generate robust chaotic signals over a wide range of the parameter values.

Keywords: Robust Chaos; Piecewise-smooth Map; Lyapunov exponent; Logistic Map
AMS 2010 codes: 34H10

1 Introduction

Application of chaos in various fields of engineering and technology has increased rapidly over the past few decades. Some of the most important such applications include secured communication using chaos, generation of random number in cryptography based on chaos etc. [3, 9, 11–17, 19]. These applications of chaos have entirely changed the trend of research in many engineering fields. Due to these emerging important applications of chaos, the research related to chaotic systems and chaos generators have become one of the important fields of research these days. Design of chaos generators is one the important topic of interest in this field. Several studies have already been reported on this topic in last few years [1, 4, 6, 7, 10, 20, 21].

Many chaos generators have been designed based on different continuous-time mathematical models. Chua circuit [4], chaotic circuit based on Lorentz and Rossler systems [1] are some of such examples existing in the literature. The advanced technology in these days requires the integrability of any such chaos generator on chip. However, most of the chaos generators based on continuous-time mathematical models are difficult to be integrated on chip. Because of this drawback, studies of chaos generators based on discrete-time mathematical

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ISSN 2444-8656
doi:10.2478/AMNS.2020.2.00073
models are necessary. As a result of that, various discrete-time chaos generators have been studied [7, 10, 21], which have attracted much attention for having simple design and robust chaotic behaviour. These chaos generators are mainly implemented by imitating the existing 1-D chaotic maps (e.g. Logistic Map, Tent Map etc.) and have been described to be effective techniques of generating chaotic signals [20, 21]. However, many practical applications demands these chaotic signals to be robust [5, 8].

Let us discuss the phenomenon robust chaos in brief. If there exist a neighbourhood in the parameter space such that a system has a unique chaotic attractor throughout that neighbourhood, i.e., if the system has no periodic windows or coexisting attractors inside that neighbourhood, then such type of dynamics is called robust chaos. In general, dynamics of a chaotic map is not always robust chaotic. As an example, we know that the dynamics of the Logistic map shows a period-doubling root to chaos. However, if we continuously tune the parameter value of the Logistic map inside any neighbourhood entirely lying within its chaotic range, the stable chaotic dynamics is destroyed very often due to the appearance of periodic windows. Therefore the Logistic map does not exhibit robust chaotic dynamics. In fact, mathematical models which exhibit robust chaotic dynamics are rare. Therefore, it is in general difficult to guarantee the robustness of the chaotic behaviour of any chaos generator. Hence, it is necessary to study simple discrete-time mathematical models which exhibit robust chaos.

In the literature there are very few studies on mathematical models exhibiting robust chaos and their circuit implementation [2, 5, 18].

In this paper, we have proposed a family of one-dimensional piecewise smooth maps which exhibit robust chaotic behaviour in a wide range of system parameter values. This family of maps have been constructed via a minor modification of the Logistic map, which is one of the very simple discrete-time mathematical model used to generate chaotic signals. Although, due to the lack of robust chaotic behaviour the circuits based on the Logistic map is unable to produce robust chaotic signal. However, we have shown that a minor modification of the Logistic model changes the entire scenario nicely. This modification allows to produce a family of 1-D maps displaying not only chaotic but also robust chaotic behaviour over a sufficiently wide range of the parameter values.

2 Mathematical model and its analysis

The Logistic map is defined on the closed and bounded interval $[0, 1]$ of the real line by

$$x_{n+1} = rx_n(1-x_n) \tag{1}$$

where $0 \leq r \leq 4$.

Now consider the one-dimensional piecewise smooth discontinuous map

$$x_{n+1} = \begin{cases} \lambda x_n(1-x_n) : 0 \leq x_n < x_b \\ x_n - \lambda^* : x_b \leq x_n \leq 1 \end{cases} \tag{2}$$

Where $x_b$ denotes the border, $\lambda$ is any value of the parameter $r$, at which the map (1) is chaotic and $0 < \lambda^* \leq x_b$.

Our claim is that for each fixed value of the border $x_b$ where $x_b < 0.375$, the above map (2) exhibits robust chaotic dynamics. As we know that chaos is a deterministic bounded dynamical phenomena of a system, therefore we initiate the proof of our claim by showing that any orbit of the map (2) remains bounded in the phase space. In other words, the map (2) is a selfmap of $[0, 1]$. Let $f_L$ and $f_R$ denote the left hand and right hand side maps of (2) respectively. Whether $x_n < x_b$ or $x_n \geq x_b$, $x_{n+1} \in [0, 1]$ because $f_L$ is the Logistic map for $\lambda$, therefore it is a selfmap of $[0, 1]$ in the first case and $f_R(x_n) = x_n - \lambda^*$, which belongs to $[0, 1]$, for $0 < \lambda^* \leq x_b$ in the second case. Therefore it is assured that the dynamics of the considered map (2), remains confined in the bounded interval $[0, 1]$ of the phase space.

Next we move on to show that the map (2) satisfies the property ‘sensitive dependence on initial condition’. We shall present a brief idea in support of our claim. Suppose $x_0$ and $(x_0 + \delta_0)$ are two nearby points of the phase
space, separated by a distance \( \delta_0 \) with \( |\delta_0| \ll 1 \). A map \( F : X \to X \) (where \( X \) is a closed and bounded interval of real line) defined by

\[
x_{n+1} = F(x_n)
\]

is said to satisfy the property ‘sensitive dependence on initial condition’ if

\[
|\delta_n| = |F^n(x_0 + \delta_0) - F^n(x_0)| = |\delta_0|e^{n\beta}
\]

where \( \beta > 0 \). This \( \beta \) is known as the ‘Lyapunov exponent’ of the map given by the formula

\[
\beta = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \ln|F'(x_i)|
\]

for almost all the initial conditions \( x_0 \), where \( F' \) denotes the first derivative of \( F \). The positiveness of the Lyapunov exponent \( \beta \) is a key signature of chaos for any system.

As the map (2) is not differentiable at \( x = x_b \) therefore first we assume that the orbit \{ \( x_0, x_1, x_2, \ldots \) \} does not contain the point \( x_b \). We already know that Logistic map given by (1) is chaotic in a specific range of the parameter \( r \) and hence it satisfies ‘sensitive dependence on initial condition’ in that range of the parameter \( r \). Now notice that the distance between any two points of the phase space remains unaltered under the application of the right hand side map \( f_R \) of (2). In other words if \( x_0, y_0 \) be any two points in \([x_b, 1]\), then

\[
|f_R(x_0) - f_R(y_0)| = |x_0 - y_0|
\]

where \( f_R \) denote the right hand side linear map of (2) defined on the interval \([x_b, 1]\). Therefore the distance between two different points of the phase space is not at all affected by the right hand side map of (2). On the other hand the left hand side map \( f_L \) has the property ‘sensitive dependence on initial condition’ as it is nothing but the Logistic map restricted in a shorter domain, which exhibits chaotic dynamics. Again any orbit must enter the interval \([0, x_b]\) repeatedly as the left hand map continuously pushes any point inside \([x_b, 1]\), towards the interval \([0, x_b]\). Therefore if two different orbits start from two different points then the orbits enter the interval \([0, x_b]\) again and again, as a result of which the two orbits separates exponentially fast from each other.

Moreover it is desired that more number of iterates of (2) should fall in the left hand side compartment as the left hand side map \( f_L \) is responsible for the chaotic nature of the map (2). Therefore we shall choose the value of the parameter \( \lambda \) very close to \( x_b \). The reason is that it will increase the stretching strength of the right hand side map and due to which any iterate lying on the right hand compartment will enter into the left hand compartment in less time and hence it will ensure that more number of iterates of an orbit will lie on the left hand side compartment of the phase space.

Another point we want to specify here is that the border must lie to the left of the point \( x = 0.375 \), i.e. \( x_b < 0.375 \). Otherwise we may not have a robust chaos. We simply show that the possibility of periodic attractor can be avoided by such a choice for the position of the border and therefore it ensures the occurrence of robust chaos. If a stable periodic orbit of period \( n \) exist for some parameter value for the map (3) then we must have,

\[
\prod_{i=0}^{n-1} |F'(x_i)| < 1
\]

where \( F' \) denotes the first derivative of \( F \) and \{ \( x_0, x_1, \cdots, x_{n-1} \) \} denotes the stable periodic orbit of period \( n \). Here first we assume that \( x_b \) is not a point on this periodic orbit.

Now in the map (2), \( f'_R(x_i) = 1 \) for all values of \( \lambda, x_b \) and for any \( x_i \in (x_b, 1) \). Again \( f'_L(x_i) \) may be less than zero for some values of \( \lambda \) and \( x_i \in (0, x_b) \), which we can avoid by a judicious choice of the border \( x_b \). Now \( f'_L(x_i) > 1 \) for any \( x_i \in (0, x_b) \) if \( x_b < 0.375 \). Therefore in that case from (4) we can immediately conclude that there exist no stable periodic orbit of (2) for any value of the parameters. Here it is also worth discussing the following approach. The condition \( F'(x_i) > 1 \) for \( i = 0, 1, \cdots, n-1 \) is a sufficient condition for a periodic orbit
\{x_0, \ldots, x_{n-1}\}$ to be unstable. This condition leads actually to the somewhat weaker condition $x_b < \frac{1}{2}(1 - \frac{1}{\lambda})$, although in that case $x_b < 3/8 = 0.375$ (corresponding to $\lambda = 4$) is more practical.

Here we discuss one thing that what if $x_b$ becomes a point on a periodic orbit $O(x_0) = \{x_0, x_1, \ldots, x_{n-1}\}$ of (2). The essential question is that can this type of periodic orbits be stable? If $x_k = x_b$ for some $k$ then we have $x_{k+1} < 0$. So it is sure that some points of $\{x_0, x_1, \ldots, x_{n-1}\}$ lies inside $[0, x_b)$ i.e. there exist no periodic orbit, entirely contained in any one compartment of the phase space. Now let us consider that we start from a point $(x_b, \beta)$, where $|\beta| << 1$. Then the orbits $O'(x_b)$ and $O'(x_b + \beta)$ visits the left compartment of the phase space infinitely often. Now as $f_L$ is a chaotic map, therefore $d(O'(x_b + \beta), O'(x_b)) \neq 0$ ($d$ here denotes the distance between two sets in the Euclidean norm) and hence $O'(x_b + \beta)$ does not converge to $\{x_0, x_1, \ldots, x_{n-1}\}$. Therefore any periodic orbit of (2) containing $x_b$ is always unstable.

3 Numerical verification

Now we verify our claim numerically. We show the graph of Lyapunov exponents and the bifurcation diagrams of the map (2) for the corresponding values of $x_b = 0.3$, $\lambda^* = 0.28$ and $x_b = 0.2$, $\lambda^* = 0.18$. The value of the parameter $\lambda$ has been varied inside the interval $[3.5, 4]$, which includes the parameter range for which the map (1) is chaotic. To be more precise, the Feigenbaum point (chaos onset) is $0.356995 \ldots$. We draw both the graph of Lyapunov exponent and bifurcation diagrams considering a very small spacing of $0.001$ in between any two consecutive values of the parameter $\lambda$. The figures Fig. (1(a)) and Fig. (2(a)) show that the Lyapunov exponents are positive in both the cases and hence giving an indication of existence of chaos. Moreover the bifurcation diagrams given by Fig. (1(b)) and Fig. (2(b)) confirm that in both the cases we have a robust chaotic dynamics.

![Graph of the Lyapunov exponent of the map (2) with respect to the parameter $\lambda$, where $x_b = 0.2$, $\lambda^* = 0.18$.](image1)

![Bifurcation diagram of the map (2) with respect to the parameter $\lambda$, where $x_b = 0.2$, $\lambda^* = 0.18$.](image2)

![Graph of the Lyapunov exponent of the map (2) with respect to the parameter $\lambda$, where $x_b = 0.3$, $\lambda^* = 0.28$.](image3)

![Bifurcation diagram of the map (2) with respect to the parameter $\lambda$, where $x_b = 0.3$, $\lambda^* = 0.28$.](image4)

Next we vary the parameter $\lambda^*$ inside the interval $(0, x_b)$, for some fixed value of $\lambda$ and $x_b$. The Lyapunov exponents for the corresponding cases have been shown in Fig. (3(a)) and Fig. (4(a)). The existence of positive
Lyapunov exponents give a strong indication of chaos. Fig. (3(b)) and Fig. (4(b)) show the bifurcation diagrams corresponding to \( \lambda = 3.7, x_b = 0.3 \) and \( \lambda = 4, x_b = 0.3 \) respectively. The bifurcation diagrams have been drawn taking a spacing of 0.001 in between any two consecutive values of \( \lambda^* \). These two diagrams are showing the existence of robust chaos.

Now we give an numerical evidence in support of our mathematical proposition that \( x_b \) must satisfy \( x_b < 0.375 \) in order to have a robust chaos. At the end of Section-2 it is proven that \( x_b < 0.375 \) is a sufficient condition for a chaotic dynamic in the sense that there are no stable periodic orbits. Fig. (5) shows the bifurcation diagram of (2) for \( x_b = 0.48 \) and \( \lambda^* = 0.47 \). It clearly shows the existence of a period two orbit for some range of the parameter \( \lambda \) inside the interval \([3.5, 4]\). We illustrate the reason behind this behaviour. We know that the modulus of the slope of \( f_L \) at any point \( 0.375 < x < 0.5 \) is less than 1 as we have

\[
f'_L(x) = \lambda (1 - 2x)
\]

where \( f_L \) denotes the left hand side map of (2). Starting from any point \( x_0 \), suppose at any \( k \)-th instant, the iterate \( x_k \) falls inside the interval \((0.375, x_b)\), where \( 0.375 < x_b < 0.5 \). Then the next iterate must lie to the right hand side compartment of the map (2), i.e. \( x_{k+1} > x_b \). Now if the stretching of the right hand map \( f_R \) be such that the next iterate \( x_{k+2} \) again falls inside \((0.375, x_b)\) then the resulting orbit settles down into a stable period two orbit.

\[
x_{n+1} = f_R f_L(x_n)
\]

If \( x_f \) is the fixed point of the map (6) and \( x_f \in (0.375, x_b) \), where \( 0.375 < x_b < 0.5 \), then the resulting dynamics settles down in a period two stable orbit, given by \((x_f, f_L(x_f))\) as the slope of the composed map (6) is less than 1. For example suppose we want to detect the period two orbit at \( \lambda = 3.75 \) in the above case. Then \( x_f \) is the fixed point of the map

\[
x_{n+1} = -0.47 + 3.75x_n(1 - x_n)
\]
Therefore here \( x_f = 0.435 \) and the stable period two orbit is given by \((0.435, 0.922)\).

\[
x_{n+1} = f_R f_L(x_n)
\]

(7)

where \( f_L \) and \( f_R \) have been assumed to be the left and right hand maps of (2) respectively. Now if \( 0.375 < x_f < x_b \), then the slope of the map (7) is less than 1 and since in this case \( 0.375 < x_f < x_b \) therefore the period two orbit is a stable periodic orbit.

![Bifurcation diagram](image)

**Fig. 5** Bifurcation diagram of (2) with respect to \( \lambda \) where \( x_b = 0.48, \lambda^* = 0.47 \).

4 Conclusion

Summarizing the above discussions we conclude that the equation in (2) actually represents an infinite family of robust chaotic maps. For each fixed value of \( x_b < 0.375 \), (2) becomes a map with two parameters, \( \lambda \) and \( \lambda^* \). As \( \lambda \) is varied inside the range, for which the Logistic map (1) is chaotic or \( \lambda^* \) is varied inside the interval \((x_b - \delta^*, x_b)(0 < \delta^* \ll 1)\), a family of robust chaotic maps are generated. Therefore the parameter space of (2) is a subspace of \( \mathbb{R}^2 \) (where \( \mathbb{R} \) is the set of all real numbers), i.e. \((\lambda, \lambda^*) \in I_R \times (x_b - \delta^*, x_b) \subset \mathbb{R}^2 \) \((I_R \subset [0,4] \) is that range of \( \lambda \) for which the map (1) is chaotic). Therefore an infinite number of robust chaotic signals can be generated by (2) just by fixing the position of the border and choosing the values of the parameters \( \lambda \) and \( \lambda^* \) judiciously. The main advantage this study is that the model discussed here is a simple 1-D discrete-time mathematical model. Moreover, as specific equations for the maps are known, therefore it makes the circuit implementation of the proposed model easier.

**Acknowledgement:** I thank the anonymous reviewer for careful reading of the manuscript and providing constructive remarks, due to which the quality of the paper has improved to a great extent.

**Conflict of Interest:** The author declare that there is no conflict of interest.

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