U-duality as General Coordinate Transformations, and Spacetime Geometry

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ABSTRACT

We show that the full global symmetry groups of all the D-dimensional maximal supergravities can be described in terms of the closure of the internal general coordinate transformations of the toroidal compactifications of D = 11 supergravity and of type IIB supergravity, with type IIA/IIB T-duality providing an intertwining between the two pictures. At the quantum level, the part of the U-duality group that corresponds to the surviving discretised internal general coordinate transformations in a given picture leaves the internal torus invariant, while the part that is not described by internal general coordinate transformations can have the effect of altering the size or shape of the internal torus. For example, M-theory compactified on a large torus $T^n$ can be related by duality to a compactification on a small torus, if and only if $n \geq 3$. We also discuss related issues in the toroidal compactification of the self-dual string to $D = 4$. An appendix includes the complete results for the toroidal reduction of the bosonic sector of type IIB supergravity to arbitrary dimensions $D \geq 3$.
1 Introduction

U-dualities are discrete subgroups of the global Cremmer-Julia (CJ) symmetries of supergravities [1], which are conjectured to be exact symmetries of string theory or M-theory [2,3]. The origin of the CJ symmetries is in general rather deep, involving a subtle interplay between the geometry of spacetime and the structure of the gauge fields and their interactions. The subtleties increase as the dimension of the spacetime reduces. For example, in the case of the maximal supergravities obtained by toroidal dimensional reduction from $D = 11$, the $GL(1, \mathbb{R}) \sim O(1, 1)$ and $GL(2, \mathbb{R})$ CJ symmetries in $D = 10$ and $D = 9$ can be simply understood as the lower-dimensional interpretation of the general coordinate transformations in the internal directions, corresponding to constant linear transformations $\delta z^i = -\Lambda^i_j z^j$ of the coordinates $z^i$ on the torus\(^3\) However, in $D \leq 8$ the analogous residual coordinate transformation symmetries in the toroidal reduction from $D = 11$ account for only a part of the full CJ symmetry group, and its full structure involves in addition transformations on the “matter” gauge fields coming from the 3-form potential of $D = 11$. On the other hand, from the viewpoint of a toroidal reduction of type IIB supergravity there are always parts of the CJ groups that cannot be understood as general coordinate transformations.

It was observed in [5] that lower-dimensional CJ groups could be used to perform transformations on $p$-brane solutions in higher dimensions that alter their asymptotic or singularity structures. This was done by diagonally dimensionally reducing the $p$-brane to an instanton, and acting on this with an $SL(2, \mathbb{R})$ subgroup of the CJ group in this lower-dimensional Euclidean-signature theory. This transformation is capable of shifting the constant term in the harmonic function describing the original $p$-brane solution, and it is this shift that is responsible for altering the asymptotic or singularity structure of the solution. It is manifest that the symmetry that is achieving such a change of the structure of the solution must be more than merely a general coordinate transformation. Indeed it is only by considering such instanton transformations in $D \leq 8$ that the associated solutions can be mapped into ones with altered singularity structures, and, as we observed previously, it is in $D \leq 8$ that the CJ groups go beyond general coordinate transformations.

In this paper, we shall show that by combining the two descriptions of the $D$-dimensional theory, corresponding to a dimensional reduction either from $D = 11$ or from the type IIB theory, it is in fact possible to interpret any CJ symmetry transformation as coming from

\[^3\]To be precise, the $\mathbb{R}$ factor of the $GL(11 - D, \mathbb{R})$ transformations parameterised by $\Lambda^i_j$ does not, as it stands, leave the $D$-dimensional Einstein-frame metric invariant, and one needs to include a compensating transformation under the homogeneous rescaling transformation of the Lagrangian in order to achieve this [6].
a general coordinate transformation in the internal space of the toroidal compactification in one or the other of the two pictures. Thus one can construct the entire CJ groups in all dimensions as the closure of the internal coordinate transformations of $D = 11$ supergravity and type IIB supergravity, together with the T-duality that relates the IIA and IIB theories. In fact if a given CJ symmetry transformation does not come from the internal general coordinate transformations in one of the reduction pictures, then it necessarily will in the other reduction picture. An example of this was already known in $D = 9$, where it was shown that the $SL(2, \mathbb{R})$ symmetry of type IIB, enlarged to $GL(2, \mathbb{R})$ upon compactification to $D = 9$, could alternatively be understood, by virtue of IIA/IIB T-duality, as the internal coordinate transformations of the 2-torus compactification of $D = 11$ supergravity. An application of this approach is that any U-duality transformation on a particular $p$-brane can be mapped, via a sequence of T-duality transformations, into a general coordinate transformation in either the type IIA or the type IIB theory. Ideas along these lines were indeed used in [4, 5] for the purpose of effecting the same changes to the harmonic functions in $p$-brane solutions that can also be achieved by U-duality transformations [5].

If maximal supergravity in $D \leq 9$ is obtained by toroidal reduction from $D = 11$ then there will be a manifest $GL(11 - D, \mathbb{R})$ global symmetry coming from the internal linear transformations of the $(11 - D)$-torus. On the other hand, if the same $D$-dimensional theory is obtained by toroidal reduction from ten-dimensional type IIB supergravity then there will instead be a manifest $GL(10 - D, \mathbb{R})$ global symmetry (at the level of the equations of motion in even dimensions). This $GL(10 - D, \mathbb{R})$ group is not, of course, simply a subgroup of the previous $GL(11 - D, \mathbb{R})$ of the $D = 11$ reduction. In fact, as we shall show, the closure of the two global symmetry groups gives the full CJ symmetry group in dimension $D$. A simple non-trivial example is provided by the case of maximal supergravity in $D = 8$, for which the full CJ group is $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$. The $SL(3, \mathbb{R})$ factor is easily understood from the point of view of a $T^3$ reduction from $D = 11$, as a subgroup of the $GL(3, \mathbb{R})$ general-coordinate symmetry. From this viewpoint, however, the origin of the $SL(2, \mathbb{R})$ factor is more obscure; it is generated by an interaction between gravity and the 3-form potential $A_{(3)}$ in $D = 11$, giving rise to a dilatonic scalar and an axion in $D = 8$. This dilatonic scalar parameterises the volume of the 3-torus, and, together with the axion from $A_{(3)}$, forms the $SL(2, \mathbb{R})$-invariant factor in the scalar Lagrangian. If, on the other hand, we look at the $D = 8$ theory from the viewpoint of a $T^2$ reduction from the type IIB theory, then the $SL(2, \mathbb{R})$ factor in the CJ group is now easily understood as a subgroup of the $GL(2, \mathbb{R})$ symmetry from internal coordinate transformations on $T^2$. It is now instead the
The $SL(3, \mathbb{R})$ factor whose origin is more obscure; it is an extension of the $SL(2, \mathbb{R})$ symmetry of the ten-dimensional type IIB theory. By combining the two viewpoints, together with the knowledge, coming from T-duality, that the two eight-dimensional theories are really the same, we can therefore understand both the $SL(3, \mathbb{R})$ and the $SL(2, \mathbb{R})$ factors of the CJ symmetry as coming from certain internal coordinate transformations.

It should be emphasised that it is in $D \leq 8$ that the interpretation of the CJ symmetries as internal general coordinate transformations becomes non-trivial, since only then is there a necessity of combining the pictures from the $D = 11$ and the type IIB compacifications in order to be able to give such an interpretation for the entire CJ groups. In dimensions lower than eight the story is a little more complicated than the one described above in $D = 8$, since we now have CJ groups that are simple, rather than products of two factors, and the way in which the two sets of internal coordinate transformations close on the full group is more involved. In the rest of this paper we shall develop a general formalism for showing how the general coordinate symmetries fit together in the two pictures. In section 2, we show by considering the Dynkin diagrams for the CJ algebras, that we can indeed put the $GL(10 - D, \mathbb{R})$ symmetry from the type IIB reduction route together with the $GL(11 - D, \mathbb{R})$ from the $D = 11$ reduction route to give the full $D$-dimensional CJ group. We also give, in section 3, a detailed decomposition of any CJ group transformation in terms of a sequence of general-coordinate transformations of $D = 11$ and type IIB supergravities, intertwined by type IIA/IIB T-duality transformations.

Within a given picture, the CJ transformations that can be described as internal general coordinate transformations are rather trivial, in the sense that they are really just reparameterisation symmetries. We show in section 4 that the remainder of the CJ transformations in the given picture have a more interesting geometrical interpretation, in that they correspond to symmetries that can change the size or the shape of the internal compactifying torus. These kinds of transformation therefore arise in $D \leq 8$. For example, in these dimensions there is a duality symmetry relating M-theory compactified on a given torus to M-theory compactified on tori of different sizes or shapes. However this symmetry of M-theory becomes less mysterious from the type IIB point of view, since it then corresponds to internal general coordinate transformations of the compactifying torus of the type IIB picture.

In section 5, we give analogous discussions for the toroidal reductions to $D = 4$ of six-dimensional pure gravity and $N = 1$ self-dual supergravity, and we obtain their global symmetry groups. After presenting our conclusions in section 6, we include an appendix in
which the complete results for the toroidal dimensional reduction of the bosonic sector of type IIB supergravity are derived, in all dimensions \( D \geq 3 \). This extends previous results for the toroidal reduction of \( D = 11 \) supergravity [8, 9]. The type IIB reduction has a new subtlety concerned with the handling of the self-duality constraint on the 5-form field strength \( H(5) \). We show that in the reduced \( D \)-dimensional theory the \( GL(10 - D, \mathbb{R}) \) symmetry associated with the internal general coordinate transformations is a symmetry only at the level of the equations of motion, when the spacetime dimension is even, but it is a symmetry of the Lagrangian in odd dimensions. Our analysis shows that the entire \( GL(10 - D, \mathbb{R}) \) internal general-coordinate symmetry is preserved and is manifest.

2 U-duality as general coordinate transformations

In appendix A, we derive the \( D \)-dimensional theory following from the toroidal dimensional reduction of the bosonic sector of type IIB supergravity. We do this by starting from the enlarged ten-dimensional theory introduced in [10], where the five-form field strength \( H(5) \) is not constrained to be self-dual. This allows a Lagrangian formulation, given in (A.1), whose equations of motion reduce, after imposing \( H(5) = *H(5) \), to those of type IIB supergravity. In our dimensionally-reduced Lagrangians (A.11) the \( D \)-dimensional consequences of this self-duality condition have yet to be imposed; they are given, dimension by dimension, in (A.17). This formulation has the advantage of allowing the full \( GL(10 - D, \mathbb{R}) \) global symmetry coming from the general coordinate transformations on the compactifying \((10 - D)\)-torus to be manifest. Unlike in \( D = 10 \) itself, it is possible to write a spacetime-covariant \( D \)-dimensional Lagrangian with the consequences of the ten-dimensional self-duality condition already imposed. One way to do this is by using (A.17) to eliminate higher-degree fields in favour of their lower-degree duals. Of course when \( D \) is even this cannot be done in a fully \( GL(10 - D, \mathbb{R}) \)-covariant way, since half of the fields of degree \( D/2 \) must be eliminated in favour of the other half. This can be done in a \( GL(9 - D, \mathbb{R}) \)-covariant way [4]. Thus in odd dimensions, the Lagrangian for \( D \)-dimensional supergravity has the full \( GL(10 - D, \mathbb{R}) \) symmetry manifest, while in even dimensions the \( GL(10 - D, \mathbb{R}) \) is manifest only at the level of the equations of motion.

In \( D \leq 9 \), the maximal supergravities can be obtained from the dimensional reduction of either \( D = 11 \) supergravity or of ten-dimensional type IIB supergravity, with the two formalisms being related by field redefinitions that are the field-theoretic precursors of type IIA/IIB string T-duality. The full global symmetry of \( D \)-dimensional maximal supergrav-
ity is the maximally non-compact form of $E_{(11-D)}$. The scalar sectors of the Lagrangians describe the cosets $E_{(11-D)}/H_{(11-D)}$, where $H_n$ is the maximal compact subgroup of $E_n$. In the dimensional reduction scheme of [9, 4], which we are also using in this paper, the scalar coset naturally arises in a Borel gauge, with the axionic scalars (those that do not appear in exponentials) having associated dilaton vectors that form the positive roots of $E_{(11-D)}$ [4, 11]. This means that the $E_{(11-D)}$ group can easily be identified, by recognising the subset of dilaton vectors associated with the simple roots.

In the dimensional reduction route from $D = 11$, the metric tensor yields $D$-dimensional potentials $A^i_1$ and $A^i_{0; j}$, with associated dilaton vectors $\tilde{b}'_i$ and $\tilde{b}'_{ij}$; while the 3-form potential yields gauge potentials $A_{(3)}, A_{(2)i}, A_{(1)ij}$ and $A_{(0)ijk}$ with dilaton vectors $\tilde{a}, \tilde{a}_i, \tilde{a}_{ij}$ and $\tilde{a}_{ijk}$ [4, 11]. Here, the indices $i, j, \ldots$ label the directions on the compactifying $(11-D)$-torus, running from $i = 1$ to $i = 11 - D$. (Note that we use primes on the dilaton vectors $\tilde{b}'_i$ and $\tilde{b}'_{ij}$ here, to distinguish them from the dilaton vectors $\tilde{b}_\alpha$ and $\tilde{b}_\alpha\beta$ in the type IIB reduction in appendix A.) The dilaton vectors $\tilde{b}'_{ij}$ and $\tilde{a}_{ijk}$ are the positive roots of $E_{(11-D)}$ in $D \geq 6$. In $D = 5, 4$ and 3 they are augmented by $-\tilde{a}_i, -\tilde{a}_i$, and $\{-\tilde{a}_{ij}, -\tilde{b}'_i\}$ respectively, corresponding to the fact that in these dimensions additional axions arise from the dualisation of higher-degree fields $A_{(3)}, A_{(2)i}$ and $\{A_{(1)ij}, A^i_{1; j}\}$. In all cases, one finds that the simple roots, in terms of which all the others can be written as sums with non-negative integer coefficients, are $\tilde{b}'_{i,i+1}$ and $\tilde{a}_{123}$, where $1 \leq i \leq 10 - D$ [4]. It is easily shown from their dot products that they give the $E_{(11-D)}$ Dynkin diagram:

\[ \begin{array}{cccccccc}
\tilde{b}'_{12} & \tilde{b}'_{23} & \tilde{b}'_{34} & \tilde{b}'_{45} & \tilde{b}'_{56} & \tilde{b}'_{67} & \tilde{b}'_{78} \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array} \]

\[ \begin{array}{c}
| \\
\circ \\
\tilde{a}_{123} \\
\end{array} \]

Diag. 1: The $D = 11$ dilaton vectors $\tilde{b}'_{i,i+1}$ and $\tilde{a}_{123}$ generate the $E_n$ Dynkin diagram.

In using this diagram, it is to be understood that in a given dimension $D$, only those simple roots whose indices are less than or equal to $(11 - D)$ are present. Note that the simple roots $\tilde{b}'_{i,i+1}$ are those of the $SL(11-D, \mathbb{R})$ subalgebra of the $GL(11-D, \mathbb{R})$ general coordinate transformations on the compactifying $(11-D)$-torus.
We can present the analogous discussion for the type IIB reductions given in appendix A. In this case the axions are $\chi, A^\alpha_{(0),\beta}, A^{NS}_{(0)\alpha\beta}, A^{RR}_{(0)\alpha\beta}$ with dilaton vectors $\vec{d}, \vec{b}_{\alpha\beta}, \vec{c}_{\alpha\beta\gamma\delta}, \vec{a}_{\alpha\beta}^{NS}, \vec{a}_{\alpha\beta}^{RR}$. In $D = 4$ and $D = 3$ these dilaton vectors will be augmented by $\{-\vec{a}_{\alpha}^{NS}, -\vec{a}_{\alpha}^{RR}\}$ and $\{-\vec{a}_{\alpha}^{NS}, -\vec{a}_{\alpha}^{RR}\}$ respectively, corresponding to the extra axions coming from the dualisation of the associated higher-degree fields. Again, we find that the dilaton vectors form the positive roots of $E_{(11-D)}$, and here the simple roots turn out to be $\vec{d}, \vec{a}_{23}^{NS}$ and $\vec{b}_{\alpha,\alpha+1}$, for $1 \leq \alpha \leq 9 - D$. From their dot products we find that they generate the following Dynkin diagram:

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\begin{array}{cccccccc}
\vec{d} & \vec{a}_{23}^{NS} & \vec{b}_{34} & \vec{b}_{45} & \vec{b}_{56} & \vec{b}_{67} & \vec{b}_{78} \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
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Diag. 2: The type IIB dilaton vectors $\vec{d}, \vec{a}_{23}^{NS}$ and $\vec{b}_{\alpha,\alpha+1}$ generate the $E_n$ Dynkin diagram.

Here, in a given dimension $D$ only those simple roots with indices less than or equal to $(10 - D)$ are present. Note that the simple roots $\vec{b}_{\alpha,\alpha+1}$ are those of the $SL(10 - D, \mathbb{R})$ subalgebra of the $GL(10 - D, \mathbb{R})$ general coordinate transformations on the compactifying $(10 - D)$-torus.

By comparing the two Dynkin diagrams, we see that each circle representing a simple root is associated with a general-coordinate transformation dilaton vector $\vec{b}_{i,i+1}$ or $\vec{b}_{\alpha,\alpha+1}$ in at least one of the two diagrams. This means that by putting together the $GL(11 - D, \mathbb{R})$ from the $D = 11$ picture, and the $GL(10 - D, \mathbb{R})$ from the type IIB picture, one is able to generate the entire Dynkin diagram for $E_{(11-D)}$. In other words, the entire CJ symmetry of $D$-dimensional maximal supergravity can be generated from general coordinate transformations on the compactifying tori of the two pictures, which are related by T-duality.

This way of patching together the $E_{(11-D)}$ algebra naturally explains the fact that $E_{(11-D)}$ is a symmetry of the Lagrangian in odd dimensions, but only of the equations of motion in even dimensions. This stems from the fact that the $GL(10 - D, \mathbb{R})$ general coordinate symmetry of the type IIB picture is itself a symmetry only at the level of the equations of
motion in even dimensions (as we discussed earlier), owing to the self-duality constraint on
the ten-dimensional 5-form field strength. Thus if $D$ is even, then if a given transformation
within $E_{(11-D)}$ is a symmetry of the Lagrangian, then it must have its origin, in the $D = 11$
picture, as a general coordinate transformation of the $(11 - D)$-torus. On the other hand,
if it is a symmetry only of the equations of motion, then in the type IIB picture it must
originate from a general coordinate transformation on the $(10 - D)$-torus. For example,
in $D = 8$ the $SL(3, \mathbb{R})$ factor in the CJ group, which is a symmetry of the Lagrangian,
is indeed a subgroup of the $GL(3, \mathbb{R})$ coming from general coordinate transformations on
the $T^3$ compactifying $D = 11$. On the other hand the $SL(2, \mathbb{R})$ factor in the CJ group,
which corresponds to an electric/magnetic duality for membranes, is a symmetry only of the
equations of motion, and indeed it originates from the general coordinate transformations
of the $T^2$ compactifying the type IIB theory.

The procedures described above can also be applied to the $O(10 - D, 10 - D)$ symmetry
of the $N = 1$ theory. In this case, corresponding to truncating out the R-R sectors of
the type IIA and IIB theories, we observe that the Dynkin diagrams given in Diag. 1 and
Diag. 2 lose their left-hand-most circles, so that they are now the Dynkin diagrams for
$O(10 - D, 10 - D)$. The T-duality transformation is now a symmetry of the theory, which
interchanges the two “ears” of the Dynkin diagram. Thus the $O(10 - D, 10 - D)$ symmetry
of the $N = 1$ theory can also be viewed as the general coordinate symmetry $GL(10 - D, \mathbb{R})$
intertwined with the discrete T-duality symmetry.

3 Details of the decomposition

In the previous section we have shown how the general coordinate transformations on the
internal compactifying torus in the $D = 11$ picture and in the type IIB picture patch
together to make the full CJ group of $D$-dimensional maximal supergravity. Here, we
give the detailed decomposition of the global symmetry transformations as a sequence of
general coordinate transformations in the two tori, intertwined with type IIA/IIB T-duality
transformations.

To do this, we first make a correspondence between the various fields of the dimensionally-
reduced $D = 11$ and type IIB theories in $D$ dimensions. The simplest way to do this is to
make the correspondence first in $D = 9$; this is summarised in Table 1 below.
|           | IIA         | IIB         |
|-----------|-------------|-------------|
| $D = 10$  | $D = 9$     | T-duality   |
| R-R fields| $A_{(3)}$   | $B_{(3)2}$  |
|           | $A_{(2)2}$  | $A_{RR}^{(2)}$ |
|           | $A_{(1)}^1$ | $A_{(1)}^1$  |
|           | $A_{(0)2}$  | $\chi$      |
| NS-NS fields| $G_{\mu\nu}$ | $A_{NS}^{(1)2}$ |
|           | $A_{(2)1}$  | $A_{NS}^{(1)}$ |
|           | $A_{(1)12}$ | $A_{2}^{(1)}$ |

Table 1: Gauge potentials of type II theories in $D = 10$ and $D = 9$

From these correspondences in $D = 9$, those in all $D \leq 8$ can be obtained by simply performing standard dimensional reductions from $D = 9$. We now therefore introduce indices $a, b, \ldots$, running from 3 to $11 - D$ (so that $i = (1, \alpha) = (1, 2, a)$, etc.). In $D$ dimensions we therefore have for the NS-NS fields

IIA: $A_{(2)1} A_{(1)12} A_{(1)1a} A_{(0)12a} A_{(0)1ab}$

IIB: $A_{NS}^{(2)} A_{(1)}^2 A_{NS}^{(1)a} A_{(0)a}^{NS} A_{(0)ab}^{NS}$

IIA: $A_{(2)}^2 A_{(1)}^a A_{(0)a}^2 A_{(0)b}^a$

IIB: $A_{NS}^a_{(1)} A_{NS}^{(0)2a} A_{NS}^{(0)b} a$

and for the R-R fields

IIA: $A_{(3)} A_{(2)2} A_{(2)a} A_{(1)2a} A_{(1)ab}$

IIB: $B_{NS}^{(3)2} A_{RR}^{(2)} B_{(2)a} A_{RR}^{(1)a} B_{(1)2ab}$

IIA: $A_{(0)2ab} A_{(0)abc} A_{(1)}^1 A_{(0)2} A_{(0)a}$

IIB: $A_{RR}^{(0)ab} B_{(0)2abc} A_{RR}^{(1)2} \chi A_{RR}^{(0)2a}$

Since the internal directions labelled by 1 and by 2 have both been singled out in this correspondence, we see only $GL(9 - D, \mathbb{R})$, corresponding to the $a, b, \ldots$ indices, as a manifest symmetry. Furthermore, we have made a specific choice in keeping the particular components of the $B$ potentials indicated above. The remaining ones, namely $B_{(3)a}, B_{(2)ab}, B_{(1)abc}$ and $B_{(0)abcd}$, have field strengths that are related by Hodge dualisation to the field
strengths of those that we are keeping. This way of truncating the supernumerary fields was discussed in [4]. As we discussed earlier, the theory actually has the full $GL(11-D, \mathbb{R})$ general coordinate symmetry in the $D = 11$ compactification, and the full $GL(10-D, \mathbb{R})$ in the type IIB compactification.

We are now in a position to show how the full set of CJ transformations can be expressed purely in terms of general coordinate transformations on the compactifying torus of one or other of the two pictures. Without loss of generality, we shall choose to express the CJ transformations of the $D = 11$ picture in terms of these general coordinate transformations. As usual, we can concentrate on the scalar manifolds, since these realise the entire $E_{11-D}$ CJ groups. The transformations associated with the Kaluza-Klein axions $A^i_{(0)j}$ originate from the $GL(11-D, \mathbb{R})$ of the general coordinate transformations of the $(11-D)$-torus compactifying $D = 11$ supergravity, and so these are already of the required general-coordinate form. The transformations associated with the remaining axions are not of the general-coordinate type, as seen from the $D = 11$ reduction viewpoint. However, we shall now show that they can be expressed as general coordinate transformations when seen from the type IIB reduction viewpoint. For $D \geq 6$, the extra axions are just the $A_{(0)ijk}$ potentials. They can be mapped into Kaluza-Klein axions in the type IIB picture as follows:

$$
\text{IIA } A_{(0)ijk} \xrightarrow{SL(11-D, \mathbb{R})_M} \text{IIA } A_{(0)12a} \xrightarrow{T-\text{dual}} \text{IIB } A^2_{(0)a}
$$

In $D \leq 5$, in addition to the axions described above, which are handled analogously, there are further ones that come from the dualisations of higher-degree fields. These can be discussed dimension by dimension. In $D = 5$, the extra axion is obtained from dualising the 3-form potential $A_{(3)}$. Since the process of dualisation is non-local on the potentials, we shall indicate the sequence of correspondences on the field strengths here.

$$
\text{IIA } *F_{(4)} \xrightarrow{T-\text{dual}} IIB \xrightarrow{*H_{(4)}2} \text{IIB } H_{(1)abcd} \xrightarrow{SL(5, \mathbb{R})_{\text{IIB}}} IIB \xrightarrow{H_{(1)2abc}}
$$

The step indicated by “self-duality constraint” makes use of the lower-dimensional self-duality relations given in (A.17).

In $D = 4$, there are seven extra axions coming from the dualisation of $A_{(2)i}$. The sequence of their correspondences to the Kaluza-Klein axions of the type IIB reduction is
given by

\[
\begin{align*}
\text{IIA} & \quad \ast F_{(3)i} \xrightarrow{SL(7, \mathbb{R})_M} \ast F_{(3)a} \xrightarrow{\text{T-dual}} \ast F_{(2)ab} \xrightarrow{\text{constraint}} \ast H_{(1)2a} \xrightarrow{\text{T-dual}} \ast H_{(3)12a} \\
\text{IIA} & \quad \ast F_{(2)ij} \xrightarrow{SL(8, \mathbb{R})_M} \ast F_{(2)ab} \xrightarrow{\text{T-dual}} \ast F_{(2)12a} \xrightarrow{\text{constraint}} \ast F_{(3)12a} \\
\text{IIA} & \quad \ast F_{(2)} \xrightarrow{SL(8, \mathbb{R})_M} \ast F_{(2)a} \xrightarrow{\text{T-dual}} \ast F_{(2)2a} \xrightarrow{\text{constraint}} \ast F_{(3)2a} \\
\text{IIA} & \quad \ast F_{(2)} \xrightarrow{\text{T-dual}} \ast F_{(2)a} \xrightarrow{SL(8, \mathbb{R})_M} \ast F_{(2)i} \xrightarrow{\text{constraint}} \ast H_{(1)2i} \xrightarrow{\text{T-dual}} \ast F_{(3)2i} \\
\text{IIB} & \quad \ast F_{(2)} \xrightarrow{\text{T-dual}} \ast F_{(2)i} \xrightarrow{\text{constraint}} \ast F_{(3)i} \\
\text{IIB} & \quad \ast F_{(2)} \xrightarrow{\text{T-dual}} \ast F_{(2)i} \xrightarrow{\text{constraint}} \ast F_{(3)i} \\
\text{IIB} & \quad \ast F_{(2)} \xrightarrow{\text{T-dual}} \ast F_{(2)i} \xrightarrow{\text{constraint}} \ast F_{(3)i} \\
\end{align*}
\]

In \( D = 3 \), there are twenty-eight additional axions coming from the dualisation of \( A_{(1)ij} \) and eight from the dualisation of \( A^i_{(1)} \). We consider the twenty-eight axions first, for which we find that the sequence of their correspondence is given by

\[
\begin{align*}
\text{IIA} & \quad \ast F_{(2)ij} \xrightarrow{SL(8, \mathbb{R})_M} \ast F_{(2)ab} \xrightarrow{\text{T-dual}} \ast F_{(2)12a} \xrightarrow{\text{constraint}} \ast F_{(3)12a} \\
\text{IIA} & \quad \ast F_{(2)} \xrightarrow{SL(8, \mathbb{R})_M} \ast F_{(2)a} \xrightarrow{\text{T-dual}} \ast F_{(2)2a} \xrightarrow{\text{constraint}} \ast F_{(3)2a} \\
\text{IIA} & \quad \ast F_{(2)} \xrightarrow{\text{T-dual}} \ast F_{(2)a} \xrightarrow{\text{T-dual}} \ast F_{(2)i} \xrightarrow{\text{constraint}} \ast F_{(3)i} \\
\end{align*}
\]

The sequence for the other eight axions is given by

\[
\begin{align*}
\text{IIA} & \quad \ast F_{(2)} \xrightarrow{SL(8, \mathbb{R})_M} \ast F_{(2)i} \xrightarrow{\text{T-dual}} \ast F_{(2)2a} \xrightarrow{\text{T-dual}} \ast F_{(2)12a} \\
\text{IIA} & \quad \ast F_{(2)} \xrightarrow{\text{T-dual}} \ast F_{(2)i} \xrightarrow{\text{constraint}} \ast F_{(3)i} \\
\end{align*}
\]

where the dots indicate that the sequence then follows (3.4), joining it at the second stage of the first line.

To summarise, in this section we have shown that all the axions that do not directly arise as Kaluza-Klein axions from \( D = 11 \) supergravity can be mapped, by a sequence of internal general-coordinate symmetry transformations and type IIA/IIB T-duality transformations, into Kaluza-Klein axions from type IIB supergravity. A completely analogous converse discussion would allow one to interpret all the axions of the type IIB reduction as Kaluza-Klein axions.
4 M-theory interpretation of U-duality

So far in the paper, we have been focussing on how the full CJ groups may be interpreted in terms of internal general coordinate transformations, either directly on the compactifying torus in the picture \((D = 11\) supergravity or type IIB supergravity) under consideration, or else indirectly, in the sense that we first make a T-duality transformation to the other picture, and interpret the CJ transformation as an internal general coordinate transformation there.

It is of interest also, however, to take a different approach, and to study how the full set of CJ transformations act when we stay entirely within one picture or the other. Some of the CJ transformations will simply be internal general coordinate transformations, but the others, which are more interesting from this point of view, do not have this geometrical interpretation.

To study this, let us consider in more detail the \(D = 8\) example that we discussed previously. We shall consider it as arising from the compactification of M-theory on a 3-torus. As we discussed previously, the \(SL(3, \mathbb{R})\) factor in the \(SL(3, \mathbb{R}) \times SL(2, \mathbb{R})\) Cremmer-Julia group comes from internal general coordinate transformations on \(T^3\), while the \(SL(2, \mathbb{R})\) factor is electric/magnetic symmetry of the membrane [12]. The scalar and gauge fields in the theory, in the notation of [9, 4], are as follows:

Scalars: \(\vec{\phi}, A_{(0)j}^i, \chi \equiv A_{(0)123}\),
Vectors: \(A_{(1)i}, A_{(1)ij}\),
Higher-rank: \(A_{(2)i}, A_{(3)}\),

(4.1)

where \(1 \leq i \leq 3\). The Lagrangian is given by

\[
\mathcal{L} = R*1 - \frac{1}{2}d\vec{\phi} \wedge d\phi - \frac{1}{2}e^{\vec{\phi}}*F_4 \wedge F_4 - \frac{1}{2} \sum_i e^{\vec{a}_i} * F_{(3)i} \wedge F_{(3)i} \\
- \frac{1}{2} \sum_{i<j} e^{\vec{a}_{ij}} * F_{(2)ij} \wedge F_{(2)ij} - \frac{1}{2} \sum_i e^{\vec{b}_i} * F_{(2)i} \wedge F_{(2)i} - \frac{1}{2} e^{\vec{a}_{123}} * d\chi \wedge d\chi \\
- \frac{1}{2} \sum_{i<j} e^{\vec{b}_{ij}} * F_{(1)ij} \wedge F_{(1)ij} + \frac{1}{2} \chi dA_{(3)} \wedge dA_{(3)} \\
- \left( \frac{1}{6} dA_{(2)i} \wedge dA_{(2)j} \wedge A_{(1)k} + \frac{1}{2} dA_{(3)} \wedge dA_{(2)i} \wedge A_{(1)jk} \right) \epsilon^{ijk},
\]

(4.2)

where the “dilaton vectors” \(\vec{a}, \vec{a}_i, \vec{a}_{ij}, \vec{a}_{ijk}, \vec{b}_i, \vec{b}_{ij}\) are constants that characterise the couplings of the dilatonic scalars \(\vec{\phi}\) to the various gauge fields; they are given in [9, 4].

The \(i, j, \ldots\) indices are in the fundamental representation of \(SL(3, \mathbb{R})\), and so in particular the Kaluza-Klein vectors \(A_{(1)i}\) form a triplet of \(SL(3, \mathbb{R})\). Similarly, the “membrane
wrapping mode” vectors $A_{(1)ij}$ form another triplet of $SL(3, \mathbb{R})$. The $SL(3, \mathbb{R})/O(3)$ part of the scalar manifold is parameterised by the Kaluza-Klein axions $\mathcal{A}_{(0)ij}^i$, together with the two linear combinations of the three dilatons $\vec{\phi}$ that do not couple to $A_{(0)123}$. It is convenient to express these two linear combinations in terms of the three linearly-dependent vectors $\vec{\beta}_i = \vec{b}_i + \frac{1}{2} \vec{a}_{123}$, which have the property that $\vec{\beta}_i \cdot \vec{a}_{123} = 0$. In terms of these, we may write the dilaton vectors for the Kaluza-Klein and winding-mode vectors as

$$
\vec{b}_i = \vec{\beta}_i - \frac{1}{2} \vec{a}_{123},
\vec{a}_{ij} = \vec{\beta}_k + \frac{1}{2} \vec{a}_{123} \quad i \neq j \neq k \neq i.
$$

(4.3)

In fact $A_{(0)123}$, which we are calling $\chi$ for convenience, together with the remaining orthogonal linear combination $\varphi \equiv \frac{1}{2} \vec{a}_{123} \cdot \vec{\phi}$ of the three dilatons $\vec{\phi}$, parameterises the $SL(2, \mathbb{R})/O(2)$ part of the scalar manifold. Under $SL(2, \mathbb{R})$ the Kaluza-Klein vectors $\{A_{(1)1}, A_{(1)2}, A_{(1)3}\}$ pair off with the membrane-wrapping vectors $\{A_{(1)23}, A_{(1)13}, A_{(1)12}\}$ respectively, to form three doublets. The field strength $F_{(4)}$ and its dual form another doublet under $SL(2, \mathbb{R})$, and the remaining fields are invariant.

The ansatz for the reduction of the metric, following the notation of [4], is

$$
\begin{align*}
\text{ds}^2_{11} &= e^{\frac{1}{2} \vec{g} \cdot \vec{\phi}} \text{ds}^2_8 + \sum_{i=1}^3 e^{2 \vec{\gamma}_i \cdot \vec{\phi}} (dz^i + A_{(1)}^i + A_{(0)j}^i dz^j)^2,
\end{align*}
$$

(4.4)

where the vectors $\vec{g}$ and $\vec{f}_i$ can be found in [4]. The volume form of the physical 3-torus is therefore given by $e^{(\vec{\gamma}_1 + \vec{\gamma}_2 + \vec{\gamma}_3) \cdot \vec{\phi}} dz^1 \wedge dz^2 \wedge dz^3$. The exponential factor turns out to be $e^{-\frac{1}{2} \vec{a}_{123} \cdot \vec{\phi}}$, and hence the volume form is $e^{-\varphi} dz^1 \wedge dz^2 \wedge dz^3$. Since $\varphi$ is the dilaton of the $SL(2, \mathbb{R})/O(2)$ scalar manifold, and since $dz^1 \wedge dz^2 \wedge dz^3$ is invariant under the $SL(3, \mathbb{R})$ transformations $\delta z^i = A^i_j z^j$, it follows that the volume of the 3-torus, namely

$$
V_3 = e^{-\varphi},
$$

(4.5)

is $SL(3, \mathbb{R})$ invariant, and furthermore is given in terms of the dilaton $\varphi$ of the $SL(2, \mathbb{R})$ factor in the $D = 8$ CJ group.

In the Einstein frame, the mass per unit $p$-volume for an electric $p$-brane supported by a field strength with kinetic term of the form $e^{\vec{c} \cdot \vec{\phi}} F^2$ is $M \sim e^{-\frac{1}{2} \vec{c} \cdot \vec{\phi}} [13, 14]$. From (4.2) and (4.3), we therefore see that the masses of the particles supported by the Kaluza-Klein vectors and the winding-mode vectors depend on the $SL(2, \mathbb{R})$ dilaton as follows:

$$
M_{KK} \sim e^{\frac{1}{2} \varphi}, \quad M_{WM} \sim e^{-\frac{1}{2} \varphi}.
$$

(4.6)
We are assuming here, for simplicity, that the axion $\chi$ is zero. We shall discuss the more general case below.) Thus, as expected, we see that the Kaluza-Klein particles become massless in the limit when the volume of the 3-torus goes to infinity.\footnote{To be precise, in order to see that the BPS black hole solitons supported by the Kaluza-Klein vectors indeed describe the Kaluza-Klein particle states coming from $D = 11$, we should calculate their masses in the $D = 11$ M-metric and not the $D = 8$ Einstein metric. In general, if two metrics are related by $g_{\mu\nu}^{(A)} = \Omega^2 g_{\mu\nu}^{(B)}$, where $\Omega$ is a constant, then the masses per unit $p$-volume of a $p$-brane are related by $m^{(A)}_p = \Omega^{-p-1} m^{(B)}_p$.\footnote{Note that the field equations in $D = 11$ M-theory are related to those in $D = 8$ supergravity by a Hodge duality.} Thus the mass of the Kaluza-Klein particles in the M-metric is of the form $e^{\frac{1}{4}\phi} e^{-\frac{1}{8}\delta} \sim e^{\frac{1}{4}\phi} = V^{-1/4}$. This is the expected scale dependence for the Kaluza-Klein masses arising from compactification on a 3-torus.} (This is the same phenomenon as the one discussed by Witten in \cite{Witten:1995im}, where the Kaluza-Klein particles in the type IIA string become massless as the dilaton grows large, leading to the opening out of the eleventh dimension.) On the other hand, if the volume of the 3-torus goes to zero then the Kaluza-Klein particles become heavy, while the winding-mode particles now become massless. Thus the particle spectrum of M-theory compactified on a small $T^3$ is identical to the particle spectrum of M-theory compactified on a large $T^3$ of reciprocal volume. This implies that M-theory compactified on a large $T^3$ is dual to M-theory compactified on a small $T^3$ of reciprocal volume. (Recall that we are taking $\chi = 0$ here for simplicity.)

It is not enough just to study the particle spectrum in order to justify this duality conjecture. Further supporting evidence can be found from the spectra of the other $p$-branes in the theory. For example, the masses of the fundamental string and two D-strings, measured in the Einstein metric, are dependent only on the shape of the $T^3$, but not on the size. Actually, the conjecture about the equivalence of the compactification on a large $T^3$ and a small $T^3$ is really just equivalent to the conjecture of U-duality, and the supporting evidence for the one conjecture is equivalent to that for the other. This is a consequence of the fact that the conjectured $T^3$ duality symmetry is implied by the $Z_2$ Weyl group symmetry of $SL(2, \mathbb{Z})$ factor of the U-duality group in $D = 8$.

In the CJ group $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, the $SL(3, \mathbb{R})$ subgroup comes from internal general coordinate transformations. Upon quantisation, it is discretised to $SL(3, \mathbb{Z})$. This discrete subgroup has the feature that it leaves the size and shape of the compactifying 3-torus unchanged. For this reason, different moduli which are related by this $SL(3, \mathbb{Z})$ symmetry
transformation should be identified. On the other hand, the $SL(2, \mathbb{Z})$ factor of the U-duality group cannot all be regarded as general coordinate transformations of the internal space from the 11-dimensional point of view. (In fact an $\mathbb{R}$ subgroup of $SL(2, \mathbb{R})$ does originate from the internal general coordinate transformations, as we shall discuss below.) It is generated by the axion $A_{(0)123}$ (which does not parameterise any modulus of the 3-torus) and the dilatonic scalar $\varphi$ that measures the volume of the compactifying 3-torus. This implies that under this $SL(2, \mathbb{Z})$ transformation, the volume of the 3-torus ranges from infinity to zero. The $D = 8$ U-duality symmetry implies that M-theory compactified on $M_8 \times T^3$ for various sizes of the volume that are related each other by $SL(2, \mathbb{Z})$ transformations are dual to each other. To be precise, if we start with M-theory compactified on a 3-torus of volume $V_0$, and a value $\chi_0$ for the axion, then it is dual to M-theory compactified on a 3-torus of the same shape but with volume $V$, and an axion $\chi$, given by

$$V = \frac{1}{c^2 V_0 + (c \chi_0 + d)^2 V_0^{-1}}$$

$$\chi = \frac{(a \chi_0 + b)(c \chi_0 + d) + a c V_0^2}{c^2 V_0^2 + (c \chi_0 + d)^2}$$

where $a$, $b$, $c$ and $d$ are integers satisfying $a d - b c = 1$.

Classically, the $GL(3, \mathbb{R}) \sim \mathbb{R} \times SL(3, \mathbb{R})$ of internal general coordinate transformations (which also utilises a compensating “trombone” [15] rescaling symmetry so as to leave the lower-dimensional metric invariant [4]) can change the volume as well as the shape of the 3-torus. In particular, it is the $\mathbb{R}$ factor that rescales the volume, while the $SL(3, \mathbb{R})$ factor changes the shape. However, the scalar corresponding to this rescaling symmetry has been “reassigned” to the $SL(2, \mathbb{R})$ factor in the full $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ CJ group, and so upon quantisation, where it is conjectured that the U-duality group $SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$ survives, the volume-changing transformations (4.7) survive within the $SL(2, \mathbb{Z})$ factor, and not the $SL(3, \mathbb{Z})$ factor. This should be contrasted with the type IIA $D = 10$ and $D = 9$ cases, where the volume-changing factors in the classical $GL(1, \mathbb{R})$ and $GL(2, \mathbb{R})$ CJ groups are completely broken at the quantum level. This explains why in these two cases the physics is changed by changing the radius of the $S^1$ or the volume of the $T^2$. (In the $D = 10$ case the weakly-coupled type IIA string is mapped to a $D = 11$ theory as the circle gets

---

3 The $GL(11 - D, \mathbb{R})$ symmetry of the $D$-dimensional supergravity, which acts on the lower dimensional fields, can be derived from the covariance of the eleven-dimensional theory under constant linear coordinate transformations in the internal $T^{11-D}$. However, if the $GL(11 - D, \mathbb{R})$ symmetry is viewed just as a transformation on the $D$-dimensional fields, it will now imply an active transformation on the internal space, and hence will have the effect of changing the size and shape of the internal torus, since in the dimensional reduction procedure a specific configuration for the internal coordinates is chosen. Nevertheless, we shall still refer to this $D$-dimensional $GL(11 - D, \mathbb{R})$ global symmetry as internal general coordinate transformations.
large $R$, whilst in the $D = 9$ case the type IIB theory emerges as the volume of the $T^2$ shrinks to zero \[^3\]. In $D \leq 8$, on the other hand, the volume of the compactifying torus can be changed by surviving U-duality symmetries, and so in such cases one has duality symmetries relating M-theory compactified on tori of different radii.

In the above discussion, we said that M-theory on large a 3-torus is dual to M-theory on a small 3-torus with reciprocal volume, provided that the Kaluza-Klein modes and membrane wrapping modes are interchanged. At first sight, this might seem to be in contradiction with a fundamental membrane interpretation, where the Kaluza-Klein modes are those of the membrane itself. If we consider a fundamental membrane in $D = 11$, then its particle modes compactified on $T^3$ might be expected to have masses of the form
\[
m^2 \sim x V^{-\frac{2}{3}} + y V^\frac{4}{3},
\]
(4.8)
where $x$ and $y$ are related to the mode numbers for the Kaluza-Klein and winding modes respectively, and they would also depend on the parameters specifying the fixed shape of the 3-torus. This mass formula is obviously not invariant under the $V \to 1/V$ transformation. However the analogy between strings and membranes is not so close as to make the mass formula (4.8) justifiable, since there is no sensible perturbative spectrum analysis for membranes that are not wrapped around the torus. Thus the formula (4.8) really only arises in an approximate analysis where the Kaluza-Klein contributions are treated as a small perturbation to a large winding-mode contribution \[^{\text{[13]}}\]. It would therefore be inappropriate to expect such an approximate formula to be invariant under the interchange of Kaluza-Klein and winding modes.

So far we have considered the U-duality in $D = 8$. The generalisation to lower dimensional U-duality is straightforward. The $SL(11 - D, \mathbb{Z})$ leaves the size and shape of the $(11 - D)$-torus invariant, whilst the parts of the U-duality transformations that are not internal general coordinate transformations can change the size or shape of the torus. This implies that there is a duality symmetry of M-theory compactified on various sizes and shapes of internal torus. To summarise, when M-theory is compactified on $M_{10} \times S^1$, the small circle limit gives rise to type IIA perturbative string theory. If $M_{11} = M_9 \times T^2$, the small volume limit of the $T^2$ give rise to the type IIB string. If $M_{11} = M_8 \times T^3$, there is an $SL(2, \mathbb{Z})$ multiplet of sizes for the volume that should be identified (as given in \([4.7]\)), but different shapes of the $T^3$ give rise to inequivalent moduli. When the dimension of the internal torus is bigger than 3, then tori even of different shapes should be identified. It would be of interest to study the identifications in the internal spaces in the lower-dimensional examples.
5 $T^2$ reductions of $D = 6$ gravity and supergravity

In sections 2 and 3, we showed that the full CJ groups for all $D$-dimensional maximal supergravities can be described in terms of the closure of the internal general coordinate transformations of the toroidal compactifications of $D = 11$ supergravity and of type IIB supergravity, with type IIA/IIB T-duality linking the two pictures. The T-duality and the general coordinate transformations $GL(10 - D, \mathbb{R})$ of the type IIB theory are both perturbative transformations of string theory. The non-perturbative aspect arises as the internal general coordinate transformations of the toroidal reduction of 11-dimensional supergravity. The fact that all the non-perturbative transformations within the CJ symmetries arise from internal general coordinate transformations in M-theory may shed some light on the non-perturbative aspects of string theory.

As we mentioned in the introduction, the CJ symmetry transformations can have the effect of altering the asymptotic structure and also the curvature structure of a $p$-brane soliton. Obviously, a general coordinate transformation cannot do this. What we have shown is that it is really the type IIA/IIB T-duality transformation that is responsible for this change of the geometrical structure of the $p$-brane, and hence the nature of this change may be understandable within the perturbative framework.

It is well known that the $R \leftrightarrow 1/R$ transformation of type IIA/IIB T-duality is a stringy phenomenon, which becomes a symmetry in the heterotic string, i.e. when the R-R fields are truncated out. At the level of supergravity this symmetry amounts to an orthogonal transformation of the dilatons, accompanied by an interchange of the Kaluza-Klein vector and the winding-mode vector (i.e. the vector coming from the dimensional reduction of the antisymmetric tensor that couples to the string). In fact this type of symmetry, at the field theory level, can also arise in a pure gravity theory.

To illustrate this, let us consider pure gravity in $D = 6$, and then perform a dimensional reduction to $D = 4$ on a 2-torus. It is straightforward to see that the four-dimensional Lagrangian has a manifest $GL(2, \mathbb{R})$ global symmetry, which is the residual symmetry of the general coordinate transformations of the internal 2-torus. The scalar coset contains two dilatons $\phi_1$ and $\phi_2$, and one axion $\chi = A^1_{0,2}$. The two Kaluza-Klein vectors, $A^1_{(1)}$ and $A^2_{(1)}$, form a doublet. We may now show that at the level of equations of motion, there is an additional $R \leftrightarrow 1/R$ discrete symmetry, where $R$ is the radius of the circle compactifying from $D = 5$ to $D = 4$. To this see, we first dimensionally reduce $D = 6$ gravity to $D = 5$, where we will obtain a dilaton $\phi_1$ and a Kaluza-Klein vector $A^1_{(1)}$, together the metric. We
now dualise the Kaluza-Klein vector to a 2-form potential $A_2$. The Lagrangian for this dualised system is given by

$$L_5 = R*1 - \frac{1}{2}*d\phi_1 \wedge d\phi_1 - \frac{4}{2} \sqrt{6} \phi_1^* F_3 \wedge F_3,$$  \hspace{1cm} (5.1)$$

where $F_3 = dA_2$. This theory admits an electric string soliton solution. The form of the Lagrangian is analogous to that for the bosonic sector of $N = 1$ supergravity in $D = 10$. Thus we expect that upon dimensional reduction to $D = 4$, there should be a discrete symmetry that interchanges the new Kaluza-Klein vector $A_2^{(1)}$ with the winding-mode vector $A_{(1)}$ that comes from the dimensional reduction of $A_2$. Indeed, performing the reduction we find

$$L_4 = R*1 - \frac{1}{2}*d\phi_1 \wedge d\phi_1 - \frac{4}{2} \sqrt{6} \phi_1^* F_2 \wedge F_2 - \frac{1}{2} e^{-\phi_1/\sqrt{6} - \phi_2/\sqrt{3}} *dF_2 \wedge dF_2,$$  \hspace{1cm} (5.2)$$

where $F_3 = dA_2 - A_2^{(1)} \wedge dA_{(1)}$, $F_2 = dA_{(1)}$, and $F_2^2 = dA_2^{(1)}$. It is straightforward to see that (5.2) is invariant under the following discrete transformation

$$\phi_1 \quad \phi_2 \rightarrow \Lambda \phi_1 \quad \phi_2 = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) \rightarrow \left( \begin{array}{cc} 1/3 & -2\sqrt{2}/3 \\ -2\sqrt{2}/3 & -1/3 \end{array} \right) \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right)$$  \hspace{1cm} (5.3)$$

$$A_{(1)} \rightarrow A_2^{(1)} \quad A_2^{(1)} \rightarrow A_{(1)} \quad A_2 \rightarrow A_{(2)} + A_{(1)} \wedge A_2^{(1)}.$$  \hspace{1cm} (5.4)$$

Note that we have $\Lambda = \Lambda^{-1}$. If we measure the radius of the compactifying circle from $D = 5$ to $D = 4$ in the five-dimensional “string” metric \[ i.e. the metric in which the mass of the 5-dimensional string soliton is independent of the $\phi_1$ dilaton modulus, then it is given by $R = e^{-\phi_1/\sqrt{6} - \phi_2/\sqrt{3}}$. This transforms as $R \rightarrow 1/R$ under the dilaton transformation (5.3). Note that in the Lagrangian (5.2), the $SL(2, \mathbb{R})$ factor in the $GL(2, \mathbb{R})$ coming from the internal general coordinate transformations is broken, and instead, we gain an $R \rightarrow 1/R$ discrete symmetry. If instead we had not dualised the Kaluza-Klein vector in $D = 5$, the resulting 4-dimensional Lagrangian would have been invariant under $GL(2, \mathbb{R})$, but then it would not have been invariant under the discrete symmetry. At the level of the equations of motion, where the choice of whether or not to dualise $A_{(1)}$ is immaterial, the theory is invariant under both the $GL(2, \mathbb{R})$ and the discrete symmetries.

\footnote{If the dilaton coupling of the 3-form field strength that couples to the fundamental string is of the form $e^{c \phi} F_3^2$ in the Einstein-frame metric $ds_5^2$ in $D$-dimensions, then the $D$-dimensional string-frame metric is $ds_{2tr}^2 = e^{-\frac{1}{2} c \phi} ds_5^2$. The $D$-dimensional string coupling constant is given by $g_D = e^{-(D-2)/8 c \phi}$.}
The example we have just been considering illustrates that an $R \leftrightarrow 1/R$ symmetry can arise in the context of a pure gravity theory. From the six-dimensional point of view, it interchanges one Kaluza-Klein vector with the dual of the other Kaluza-Klein vector, by contrast with the situation in string theory where Kaluza-Klein and winding modes are interchanged. Of course this theory is just a toy model, and it is not supersymmetric. We may, however, embed it in a supersymmetric theory, in which case the $R \leftrightarrow 1/R$ symmetry will be embedded as part of a larger symmetry of the supersymmetric theory. The minimal supergravity in $D = 6$ is $N = 1$, which contains the metric and a self-dual 3-form field strength $G_{(3)}$. If this is directly reduced on a 2-torus to $D = 4$, the internal $GL(2, \mathbb{R})$ general coordinate transformations are symmetries only at the level of the equations of motion, owing to the self-duality condition on $G_{(3)}$. (This is analogous to the situation for the reduction of type IIB supergravity, which is discussed in appendix A.) Now, we shall follow the same strategy that we did previously for pure gravity, and first dualise the Kaluza-Klein vector $A_1^{(1)}$ in $D = 5$. This gives the Lagrangian

$$\mathcal{L}_5 = R^* \mathbf{1} - \frac{1}{2} \ast d \phi_1 \wedge d \phi_1 - \frac{1}{2} e^{\frac{4}{\sqrt{6}} \phi_1} \ast F_{(3)} \wedge F_{(3)} - \frac{1}{2} e^{\frac{2}{\sqrt{6}} \phi_1} \ast G_{(2)} \wedge G_{(2)} , \quad (5.5)$$

where $G_{(2)} = dB_{(1)}$ comes from the dimensional reduction of $G_{(3)}$, and $F_{(3)} = dA_{(2)} - \frac{1}{2} B_{(1)} \wedge dB_{(1)}$ is the field strength coming from the dualisation of $F_{(2)}^1$. After a further reduction to $D = 4$, followed by a dualisation of $A_{(2)}$ to give an axion $\chi$, we obtain the Lagrangian

$$\mathcal{L}_4 = R^* \mathbf{1} - \frac{1}{2} \ast d \phi \wedge d \phi - \frac{1}{2} e^{2 \phi} \ast d \chi \wedge d \chi - \frac{1}{2} e^{\sqrt{2} \phi} \ast d \sigma \wedge d \sigma
- \frac{1}{2} e^{\sqrt{2} \phi} \ast F_{(2)}^{+} \wedge F_{(2)}^{+} - \frac{1}{2} e^{- \phi} \ast F_{(2)}^{0} \wedge F_{(2)}^{0} - \frac{1}{2} e^{- \sqrt{2} \phi} \ast F_{(2)}^{-} \wedge F_{(2)}^{-}
- \chi (F_{(2)}^{+} \wedge F_{(2)}^{+} + \frac{1}{2} F_{(2)}^{0} \wedge F_{(2)}^{0}) , \quad (5.6)$$

where

$$F_{(2)}^{+} = dA_{(1)}^{+} - \sigma dA_{(1)}^{0} - \frac{1}{2} \sigma^2 dA_{(1)}^{-} , \quad F_{(2)}^{0} = dA_{(1)}^{0} + \sigma dA_{(1)}^{-} , \quad F_{(2)}^{-} = dA_{(1)}^{-} . \quad (5.7)$$

The vector potential $A_{(1)}^{+}$ arises from the reduction of $A_{(2)}$ in $D = 5$; it is in fact dual to the Kaluza-Klein vector $A_{(1)}^{1}$. The potential $A_{(1)}^{0}$ comes from the reduction of $B_{(1)}$ in $D = 5$, and $A_{(1)}^{-}$ is the new Kaluza-Klein vector $A_{(1)}^{2}$. Finally, $\sigma$ is the axion coming from the reduction of $B_{(1)}$. The dilatons $\phi$ and $\varphi$ are related to the standard $\phi_1$ and $\phi_2$ by

$$\phi = -\sqrt{\frac{2}{3}} \phi_1 + \frac{1}{\sqrt{3}} \phi_2 , \quad \varphi = \frac{1}{\sqrt{3}} \phi_1 + \sqrt{\frac{2}{3}} \phi_2 . \quad (5.8)$$

\footnote{For this reason, this $R \leftrightarrow 1/R$ symmetry lies outside the general-coordinate $GL(2, \mathbb{R})$ symmetry.
The scalar sector in (5.6) contains two independent $SL(2, \mathbb{R})/O(2)$ cosets, one parameterised by $(\phi, \chi)$, and the other by $(\varphi, \sigma)$. We shall denote the corresponding global symmetries by $SL(2, \mathbb{R})_G$ and $SL(2, \mathbb{R})_T$. The $SL(2, \mathbb{R})_G$ is a subgroup of the $GL(2, \mathbb{R})$ internal general-coordinate transformations, and hence it is a symmetry only at the level of the equations of motion (owing to the self-duality condition), where $F^0_{(2)}$ and its dual form a doublet. It is therefore an electric/magnetic duality symmetry of the 2-form field strength that comes from the self-dual 3-form in $D = 6$. The $SL(2, \mathbb{R})_T$ symmetry, on the other hand, leaves the Lagrangian (5.6) itself invariant, with the three potentials $(A^+_1, A^0_1, A^-_1)$ transforming as a triplet. This can be made manifest by defining the matrix

$$
\mathcal{M} = \begin{pmatrix}
e^{-\frac{1}{\sqrt{2}}\varphi} + \frac{i}{2} \sigma e^{\frac{1}{\sqrt{2}}\varphi} & \frac{1}{\sqrt{2}} \sigma e^{\frac{1}{\sqrt{2}}\varphi} \\
\frac{1}{\sqrt{2}} \sigma e^{\frac{1}{\sqrt{2}}\varphi} & \frac{1}{e^{\frac{1}{\sqrt{2}}\varphi}}
\end{pmatrix},
$$

and the quantity

$$
A = \begin{pmatrix} A^0_1 \\
-A^-_1
\end{pmatrix},
$$

in terms of which (5.10) can be written as

$$
\mathcal{L}_4 = R\ast \mathbb{I} - \frac{1}{2} d\ast d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \ast d\chi \wedge d\chi + \frac{1}{2} \text{tr}(d\ast \mathcal{M}^{-1} \wedge d\mathcal{M}) - \frac{1}{2} e^{-\phi} \text{tr}(\ast H \wedge \mathcal{M} H^T \mathcal{M}^{-1}) - \frac{1}{2} \chi \text{tr}(H \wedge H),
$$

(5.11)

where $H = dA$.

The Lagrangian (5.11) is manifestly invariant under $SL(2, \mathbb{R})_T$, whose action on the scalars and gauge potentials is given by

$$
\mathcal{M} \longrightarrow \Lambda \mathcal{M} \Lambda^T, \quad A \longrightarrow \Lambda A \Lambda^{-1}.
$$

(5.12)

It is also evident from the form of the Lagrangian that its equations of motion are invariant under $SL(2, \mathbb{R})_G$, where, as explained earlier, the field strength $F^0_{(2)}$ and its dual form a doublet. In fact under $SL(2, \mathbb{R})_G$ the other 2-form field strengths form two further doublets, namely $(F^+_1, \ast F^-_1)$ and $(F^-_1, \ast F^+_1)$. The $R \leftrightarrow 1/R$ symmetry that we discussed previously in the pure gravity theory is now a part of $SL(2, \mathbb{R})_T$, corresponding to

$$
\Lambda = \begin{pmatrix} 0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

(5.13)

In terms of the complex field $\tau \equiv \sigma/\sqrt{2} + i e^{-\varphi/\sqrt{2}}$, this corresponds to $\tau \leftrightarrow -\tau^{-1}$.

It was shown in [7] that the 2-torus reduction of the six-dimensional $N = 2$ supergravity to $D = 4$ gives a theory with $SL(2, \mathbb{R})_S \times O(2,2)$ global symmetry. The
$O(2,2) \sim SL(2, \mathbb{R})_1 \times SL(2, \mathbb{R})_2$ factor is the expected T-duality symmetry for a string theory reduced on a 2-torus; this is a symmetry of the four-dimensional Lagrangian. The factor $SL(2, \mathbb{R})_1$ is generated by the Kaluza-Klein axion, while $SL(2, \mathbb{R})_2$ is generated by the winding-mode axion. The remaining $SL(2, \mathbb{R})_S$ factor is an electric/magnetic S-duality [18, 19], which is a symmetry only at the level of the equations of motion. In this section, we have seen that the dimensional reduction of $N = 1$ supergravity in $D = 6$ gives a theory in $D = 4$ with the global symmetry $O(2,2) \sim SL(2, \mathbb{R})_G \times SL(2, \mathbb{R})_T$. This, however, is not the same as the $O(2,2)$ of T-duality in [17]. In fact, our $SL(2, \mathbb{R})_G$ is the same as $SL(2, \mathbb{R})_1$, while our $SL(2, \mathbb{R})_T$ is the diagonal subgroup of $SL(2, \mathbb{R})_S$ and $SL(2, \mathbb{R})_2$.

6 Conclusions

In this paper, we have shown explicitly how the full Cremmer-Julia symmetries of the maximal supergravities in all dimensions $3 \leq D \leq 11$ can be interpreted in terms of general coordinate transformations on the internal compact toroidal dimensions. Specifically, we showed how the combination of such general coordinate transformations in the $D = 11$ reduction picture and in the type IIB reduction picture, intertwined by the T-duality that relates the type IIA and type IIB theories, conspire together to generate the entire $E_{(11-D)}$ CJ groups. (In fact in $D = 10$ type IIA, and in $D = 9$, the CJ groups are simply the internal general coordinate groups $GL(1, \mathbb{R})$ and $GL(2, \mathbb{R})$ from the toroidal compactification of $D = 11$ supergravity, and it is not necessary to invoke the intertwining with the type IIB theory in these cases.)

The scheme that we have described allows us to interpret the global symmetry algebras of all the maximal supergravities, with the exception of the $D = 10$ type IIB theory itself, in terms of internal general coordinate transformations. (Of course the global symmetry $GL(2, \mathbb{R})$ of the $S^1$ reduction of the type IIB theory can be understood as internal general-coordinate symmetries of the $T^2$ reduction of $D = 11$ supergravity.)

Up to this point, our discussion was a classical one at the level of the supergravity field theories. The symmetry is conjectured [2] to survive as a discrete U-duality, in the context of string or M-theory. The discrete U-duality group $E_n(\mathbb{Z})$ can then be viewed as the closure of $SL(11-D, \mathbb{Z})_M$ and $SL(10-D, \mathbb{Z})_{IIIB}$, which are discretised subgroups of the internal general coordinate transformations of toroidally compactified M-theory and type IIB theory respectively. These two groups are interwined by the perturbative T-duality transformation of the type IIA/IIB strings. This decomposition gives a better understanding of the origins
of the U-duality groups. For example, it provides a natural understanding for why sets of moduli that are related by U-duality transformations should be identified, since any physical compactifying torus $T^n$ is invariant under $SL(n, \mathbb{Z})$ transformations.

It should be emphasised, however, that in a given picture (i.e. either M-theory or the type IIB string), the U-duality group also involves (in $D \leq 8$) transformations that are not simply internal general coordinate transformations on the torus. The scaling transformation of the torus, which classically is part of the internal general coordinate transformations, is no longer a symmetry at the quantum level in those cases ($D = 10$ and $D = 9$) where it remains as a direct-product $\mathbb{R}$ factor in the CJ group. (This $\mathbb{R}$ factor in $GL(11 - D, \mathbb{R})$ cannot survive the quantisation in $D \geq 9$ since there is no such group as $GL(n, \mathbb{Z})$.) However, in $D \leq 8$ it is absorbed into the full CJ group, and transformations that change the size of the torus do survive in the quantum discretisation to the U-duality symmetry group. A consequence of this is that whereas the compactification of M-theory on $T^n$ gives different physics under inversion of the volume of the torus when $n \leq 2$, it gives equivalent physics when $n \geq 3$.

The perturbative T-duality that relates the type IIA and type IIB strings on circles of reciprocal radii, in the infinite-radius limit, requires further comment. Let us consider compactifying the type IIA and type IIB strings on circles with radii $R_{\text{IIA}}$ and $R_{\text{IIB}} = 1/R_{\text{IIA}}$ respectively. (These are measured in the corresponding ten-dimensional string metrics.) The two theories are equivalent, by virtue of T-duality, with the following relation between the string coupling constants

$$\frac{g_{\text{IIA}}^{10}}{\sqrt{R_{\text{IIA}}}} = g_{\text{IIA}}^9 = g_{\text{IIB}}^9 = \frac{g_{\text{IIB}}^{10}}{\sqrt{R_{\text{IIB}}}}.$$  

(6.1)

In other words, we have

$$g_{\text{IIA}}^{10} = \frac{g_{\text{IIB}}^{10}}{R_{\text{IIB}}}, \quad \text{or equivalently,} \quad g_{\text{IIB}}^{10} = \frac{g_{\text{IIA}}^{10}}{R_{\text{IIA}}}. \quad (6.2)$$

Thus for any finite but non-vanishing value of radius $R_{\text{IIA}}$ or $R_{\text{IIB}}$, both $g_{\text{IIA}}^{10}$ and $g_{\text{IIB}}^{10}$ can be small and non-vanishing, and so both the type IIA and the type IIB string are within their weak-coupling perturbative regimes. However, in the limit where one of the radii approaches infinity (or equivalently, where the other radius becomes infinitesimal), the situation becomes more subtle. It follows from (6.2) that when the compactifying circle of one string theory shrinks until it is comparable in size to its coupling constant, the dual string with reciprocal compactification radius becomes strongly coupled. For example, if $R_{\text{IIA}}$ goes to zero, then the type IIA string coupling $g_{\text{IIA}}^{10}$ must also be taken to zero in order
for the theory to map to the type IIB string at large radius at finite string coupling. In other words, it is the free $D = 9$ type IIA string on a zero-radius circle that is T-dual to the $D = 10$ type IIB string. This free $D = 9$ type IIA string can also be obtained from M-theory on a 2-torus with vanishing volume [6]. If instead we keep the string coupling constant of type IIA (or IIB) to be fixed, and send the corresponding radius to zero, it will map to the type IIB (or IIA) string at infinite string coupling. In particular, it implies that the type IIB theory at fixed coupling on a zero-radius circle is mapped to $D = 11$ M-theory (with an infinite radius for the eleventh, as well as for the tenth, dimension), rather than to the weakly-coupled perturbative type IIA string in $D = 10$. This relation between the type IIA string (or M-theory) and the type IIB string obviously goes beyond the weak-coupling regime.

In this paper, we also discussed the toroidal reduction of six-dimensional pure gravity and $N = 1$ supergravity to $D = 4$. In the former case, we found that there is an $R \leftrightarrow 1/R$ discrete symmetry at the level of the equations of motion, in addition to the internal general coordinate $GL(2, \mathbb{R})$ symmetry. In the case of $N = 1$ supergravity, we saw that the equations of motion have an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry.

Appendix

A Type IIB reduction, and $GL(10 - D, \mathbb{R})$

Our starting point is the Lagrangian introduced [10], whose equations of motion, when supplemented by the externally (consistently) imposed self-duality constraint $H_{(5)} = *H_{(5)}$, give the equations of motion for the bosonic fields of type IIB supergravity:

$$\mathcal{L} = R * \mathbb{I} - \frac{1}{2} d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} * d\chi \wedge d\chi - \frac{1}{4} * H_{(5)} \wedge H_{(5)}$$

$$- \frac{1}{2} e^{-\phi} F_{(3)}^{NS} \wedge F_{(3)}^{NS} - \frac{1}{2} e^{\phi} F_{(3)}^{RR} \wedge F_{(3)}^{RR} + \frac{1}{2} B_{(4)} \wedge dA_{(2)}^{NS} \wedge dA_{(2)}^{RR} , \quad \text{(A.1)}$$

where

$$F_{(3)}^{NS} = dA_{(2)}^{NS} , \quad F_{(3)}^{RR} = dA_{(2)}^{RR} - \chi dA_{(2)}^{NS} , \quad H_{(5)} = dB_{(4)} + \frac{1}{2} \epsilon_{pq} A_{(2)}^{p} \wedge dA_{(2)}^{q} . \quad \text{(A.2)}$$

In the expression for $H_{(5)}$, we are using the notation that $A_{(2)}^{NS} = A_{(2)}^{1}$, and $A_{(2)}^{RR} = A_{(2)}^{2}$ here, and $\epsilon_{12} = 1$. Note that $H_{(5)}$ is normalised so that its kinetic term is one half of the “canonical” one. This is because eventually we shall be substituting the solution of the self-duality condition back into the equations, and this will lead to a compensating doubling.
We now reduce the Lagrangian to $D$ dimensions, using procedures and conventions that are analogous to those used for the reduction of $D = 11$ supergravity in [9, 4]. Thus we reduce the metric to $D$ dimensions using the ansatz

$$ds_{10}^2 = e^\bar{s} \bar{\varphi} ds_D^2 + \sum_\alpha e^{2\bar{\gamma}_\alpha \cdot \bar{\varphi}} (h^\alpha)^2,$$

(A.3)

where $h^\alpha = dz^\alpha + A_{(4)\alpha} + A_{(3)\beta} dz^\beta$. Here, we are using indices $\alpha, \beta, \ldots = 2, 3, \ldots, 11 - D$ to label the internal compactified dimensions. Note that we begin labelling these with $\alpha = 2$, corresponding to the reduction step on the coordinate $z_2$ from $D = 10$ to $D = 9$. The coordinate $z_1$ will be reserved for the reduction of $D = 11$ supergravity to $D = 10$. The vectors $\vec{s}$ and $\vec{\gamma}_\alpha$ have $10 - D$ components, and we have $\vec{\gamma}_\alpha = \frac{1}{2}(\vec{s} - \vec{f}_\alpha)$ where

$$\vec{s} = (s_2, s_3, \ldots, s_{11-D}) \ ,$$

$$\vec{f}_\alpha = \left(0, 0, \ldots, 0, (10 - \alpha) s_\alpha, s_{\alpha+1}, s_{\alpha+2}, \ldots, s_{11-D}\right),$$

(A.4)

and $s_\alpha = \sqrt{2/((10 - \alpha)(9 - \alpha))}$. With these definitions, it follows that a pure Einstein-Hilbert action in $D = 10$ reduces to an Einstein-Hilbert action in $D$ dimensions together with canonically-normalised dilatonic scalars and vectors. Note that $\vec{s}$ and $\vec{f}_\alpha$ satisfy the dot-product relations

$$\vec{s} \cdot \vec{s} = \frac{10-D}{4(D-2)} \ , \quad \vec{s} \cdot \vec{f}_\alpha = \frac{2}{D-2} \ , \quad \vec{f}_\alpha \cdot \vec{f}_\beta = 2 \delta_{\alpha\beta} + \frac{2}{D-2} .$$

(A.5)

The various potentials in (A.1) are reduced according to the ansätze

$$B_{(4)} \rightarrow B_{(4)} + B_{(3)\alpha} dz^\alpha + \frac{1}{2} B_{(2)\alpha\beta} dz^\alpha dz^\beta + \frac{1}{6} B_{(1)\alpha\beta\gamma} dz^\alpha dz^\beta dz^\gamma + \frac{1}{24} B_{(0)\alpha\beta\gamma\delta} dz^\alpha dz^\beta dz^\gamma dz^\delta ,$$

$$A_{(2)} \rightarrow A_{(2)} + A_{(1)\alpha} dz^\alpha + \frac{1}{2} A_{(0)\alpha\beta} dz^\alpha dz^\beta ,$$

(A.6)

where the $A$ potentials can carry either NS or RR superscripts. Note that we are in general suppressing the $\wedge$ symbols between differential forms. The field strengths are defined according to the following conventions:

$$H_{(5)} \rightarrow H_{(5)} + H_{(4)\alpha} h^\alpha + \frac{1}{2} H_{(3)\alpha\beta} h^\alpha h^\beta + \frac{1}{6} H_{(2)\alpha\beta\gamma} h^\alpha h^\beta h^\gamma + \frac{1}{24} H_{(1)\alpha\beta\gamma\delta} h^\alpha h^\beta h^\gamma h^\delta ,$$

$$F_{(3)} \rightarrow F_{(3)} + F_{(2)\alpha} h^\alpha + \frac{1}{2} F_{(1)\alpha\beta} h^\alpha h^\beta ,$$

(A.7)

where the fields $F$ can carry NS or RR superscripts. From these, one can read off the expressions for the field strengths in terms of the potentials. For the fields coming from $H_{(5)}$, we
find

\[ H_{(5)} = G_{(5)} - G_{(3)} A_A^\alpha A_A^\alpha + \frac{1}{2} G_{(2)\alpha \beta} A_\alpha^\alpha A_\beta^\beta - \frac{1}{6} G_{(1)\alpha \beta \gamma} A_\alpha^\alpha A_\beta^\beta \hat{A}_\gamma^\gamma \]
\[ + \frac{1}{24} G_{(0)\alpha \beta \gamma \delta} A_\alpha^\alpha A_\beta^\beta \hat{A}_\gamma^\gamma \hat{A}_\delta^\delta, \]
\[ H_{(4)\alpha} = \gamma_\alpha \beta\left[G_{(3)\beta} + G_{(2)\beta} \hat{A}_\beta^\gamma + \frac{1}{2} G_{(1)\beta \delta} \hat{A}_\delta^\gamma + \frac{1}{6} G_{(0)\beta \delta \epsilon} \hat{A}_\delta^\epsilon \hat{A}_\gamma^\gamma \hat{A}_\epsilon^\epsilon \right], \]
\[ H_{(3)\alpha \beta} = \gamma_\alpha \beta_{\gamma\delta} \left[G_{(2)\gamma \delta} - G_{(1)\gamma \delta \epsilon} \hat{A}_\epsilon^\epsilon + \frac{1}{2} G_{(0)\gamma \delta \epsilon \zeta} \hat{A}_\epsilon^\epsilon \hat{A}_\zeta^\zeta \right], \]
\[ H_{(2)\alpha \beta \gamma} = \gamma_\alpha \beta_{\gamma \delta} \gamma_\delta (G_{(1)\beta \gamma \delta} + G_{(0)\beta \gamma \sigma \zeta} \hat{A}_\zeta^\zeta \hat{A}_\sigma^\sigma), \]
\[ H_{(1)\alpha \beta \gamma \delta} = \gamma_\alpha \beta_{\gamma \delta} \gamma_\delta \gamma_\tau G_{(0)\gamma \sigma \rho \lambda \tau}, \] (A.8)

where

\[ G_{(5)} = dB_{(4)} + \frac{1}{2} \epsilon_{pq} A_p{^2} dA_q{^2}, \]
\[ G_{(3)\alpha} = dB_{(3)\alpha} + \frac{1}{2} \epsilon_{pq} (A_p{^2} dA_q{^1} - A_{(1)\alpha} dA_q{^2}), \]
\[ G_{(3)\alpha \beta} = dB_{(2)\alpha \beta} + \frac{1}{2} \epsilon_{pq} (A_p{^2} dA_q{^2} + A_{(0)\alpha \beta} dA_q{^2} + 2 A_{(1)\alpha} dA_q{^0}), \] (A.9)
\[ G_{(2)\alpha \beta \gamma} = dB_{(1)\alpha \beta \gamma} - \frac{3}{2} \epsilon_{pq} (A_p{^2} dA_q{^2} - A_{(0)\alpha \beta} dA_q{^0}), \]
\[ G_{(1)\alpha \beta \gamma \delta} = dB_{(0)\alpha \beta \gamma \delta} + 3 \epsilon_{pq} A_p{^2} dA_q{^2} dA_q{^0}. \]

The fields coming from the NS-NS and R-R 3-forms are given by

\[ F_{NS}^{(3)} = dA_{NS}^{(2)} - dA_{NS}^{(1)\alpha} A_\alpha^\alpha A_\beta^\beta, \]
\[ F_{NS}^{(2)\alpha} = \gamma_\alpha \beta (dA_{NS}^{(1)\beta} + dA_{NS}^{(0)\alpha} A_\alpha^\alpha), \]
\[ F_{NS}^{(1)\alpha \beta} = \gamma_\alpha \beta A_{NS}^{(0)\gamma \delta}, \]
\[ F_{RR}^{(3)} = dA_{RR}^{(2)} - \chi dA_{NS}^{(2)} - (dA_{RR}^{(1)\alpha} - \chi dA_{NS}^{(1)\alpha}) A_\alpha^\alpha A_\beta^\beta + \frac{1}{2} (dA_{RR}^{(0)\alpha \beta} - \chi dA_{NS}^{(0)\alpha \beta}) A_\alpha^\alpha A_\beta^\beta, \]
\[ F_{RR}^{(2)\alpha} = \gamma_\alpha \beta (dA_{RR}^{(1)\beta} - \chi dA_{NS}^{(1)\beta} + (dA_{RR}^{(0)\beta \gamma} - \chi dA_{NS}^{(0)\beta \gamma}) A_\gamma^\gamma), \]
\[ F_{RR}^{(1)\alpha \beta} = \gamma_\alpha \beta (dA_{RR}^{(0)\gamma \delta} - \chi dA_{NS}^{(0)\gamma \delta}). \] (A.10)

In these expressions, the quantity \( \gamma_{\alpha \beta} \) is the inverse of \( \tilde{\gamma}_{\alpha \beta} \equiv \delta_{\alpha \beta} + A_{(0)\beta} \), and \( \hat{A}_{\alpha} \equiv \gamma_{\alpha \beta} A_{\beta} \) (see [3, 4]). Note that \( dz^\alpha = \gamma_{\alpha \beta} h^\beta - \hat{A}_\alpha \).

It is now straightforward to calculate the D-dimensional Lagrangian following from (A.3). We find

\[ \mathcal{L} = R \ast 1 - \frac{1}{2} \ast d\tilde{d} - \frac{1}{2} \ast d\tilde{d} + \frac{1}{2} \ast d\tilde{d} + \frac{1}{4} \ast d\tilde{d} \ast H_{(5)} \land H_{(5)} - \frac{1}{2} \sum \ast \tilde{e}^\alpha \tilde{d}^\alpha \ast H_{(4)\alpha} \land H_{(4)\alpha} \]
\[ - \frac{1}{4} \sum_{\alpha < \beta} \ast \tilde{e}^\alpha \tilde{d}^\beta \ast \gamma_{H_{(3)\alpha \beta} \land H_{(3)\alpha \beta} - \frac{1}{4} \sum_{\alpha < \beta < \gamma} \ast \tilde{e}^\alpha \tilde{d}^\beta \ast H_{(2)\alpha \beta \gamma} \land H_{(2)\alpha \beta \gamma} \]
\[ - \frac{1}{4} \sum_{\alpha < \beta < \gamma < \delta} \ast \tilde{e}^\alpha \tilde{d}^\beta \ast \gamma_{H_{(1)\alpha \beta \gamma \delta} \land H_{(1)\alpha \beta \gamma \delta}} \] (A.11)
Note that field strengths are given by from the dimensional reduction from $10$ to $D$.

\[
\begin{align*}
D &= 7: \quad \frac{1}{2} e^{\alpha} \phi_{(1)} \wedge F_{(1)\alpha} \wedge F_{(1)\beta} \epsilon^{\alpha\beta}, \\
D &= 6: \quad \frac{\epsilon_{pq}}{2} B_{(2)} \wedge dA_{(2)} + dA_{(2)} - \frac{1}{2} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} - \frac{1}{2} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} \\
D &= 5: \quad \frac{1}{8} B_{(3)} \wedge dA_{(3)} + dA_{(3)} - \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} - \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} \\
D &= 4: \quad \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} - \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} \\
D &= 3: \quad \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} - \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} \\
D &= 2: \quad \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} - \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} \\
D &= 1: \quad \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} - \frac{1}{8} \sum_{\alpha<\beta} e^{\alpha\beta} \phi_{(1)\alpha} \wedge F_{(1)\beta} \\
\end{align*}
\]

Here, $\tilde{\phi} \equiv (\phi, \bar{\phi})$, where $\bar{\phi}$ denotes the set of dilatonic scalars appearing in (A.3) that come from the dimensional reduction from $10$ to $D$ dimensions, and $\phi$ is the original dilaton of the ten-dimensional type IIB theory. The various dilaton vectors for the $D$-dimensional field strengths are given by

\[
\begin{align*}
\tilde{a}^{NS} &= (-1, -4\bar{s}) , \\
\tilde{a}^{NS} &= (-1, -4\bar{s} + \bar{f}_\alpha) , \\
\tilde{a}^{NS} &= (-1, -4\bar{s} + \bar{f}_\alpha + \bar{f}_\beta) , \\
\tilde{a}^{RR} &= (1, -4\bar{s}) , \\
\tilde{a}^{RR} &= (1, -4\bar{s} + \bar{f}_\alpha) , \\
\tilde{a}^{RR} &= (1, -4\bar{s} + \bar{f}_\alpha + \bar{f}_\beta) , \\
\tilde{b}_\alpha &= (0, -\bar{f}_\alpha) , \\
\tilde{b}_{\alpha\beta} &= (0, -\bar{f}_\alpha + \bar{f}_\beta) , \\
\tilde{d} &= (2, 0) , \\
\tilde{c} &= (0, -8\bar{s}) , \\
\tilde{c}_\alpha &= (0, -8\bar{s} + \bar{f}_\alpha) , \\
\tilde{c}_{\alpha\beta} &= (0, -8\bar{s} + \bar{f}_\alpha + \bar{f}_\beta) , \\
\tilde{c}_{\alpha\beta\gamma} &= (0, -8\bar{s} + \bar{f}_\alpha + \bar{f}_\beta + \bar{f}_\gamma) , \\
\tilde{c}_{\alpha\beta\gamma\delta} &= (0, -8\bar{s} + \bar{f}_\alpha + \bar{f}_\beta + \bar{f}_\gamma + \bar{f}_\delta) .
\end{align*}
\]
or

\[ + \frac{1}{48} \varepsilon_{pq} B_{(0)\alpha\beta\gamma\delta} \wedge dA^p_{(2)} \wedge dA^q_{(1)\epsilon} \epsilon^{\alpha\beta\gamma\delta\epsilon}, \]

\[ D = 4 : \quad \left( \frac{1}{10} B_{(2)\alpha\beta} \wedge dA^{NS}_{(0)\gamma\delta} \wedge dA^{RR}_{(0)\zeta} + \frac{1}{90} \varepsilon_{pq} B_{(0)\alpha\beta\gamma\delta} \wedge dA^p_{(2)} \wedge dA^q_{(1)\epsilon} \right. \]

\[ \left. + \frac{1}{45} B_{(0)\alpha\beta\gamma\delta} \wedge dA^{NS}_{(0)\gamma\delta} \wedge dA^{RR}_{(1)\zeta} + \frac{1}{24} \varepsilon_{pq} B_{(1)\alpha\beta\gamma} \wedge dA^p_{(1)\delta} \wedge dA^q_{(0)\zeta} \right) \epsilon^{\alpha\beta\gamma\delta\zeta\epsilon}, \]

\[ D = 3 : \quad \left( \frac{1}{38} B_{(1)\alpha\beta\gamma} \wedge dA^{NS}_{(0)\delta} \wedge dA^{RR}_{(0)\zeta\eta} - \frac{1}{96} \varepsilon_{pq} B_{(0)\alpha\beta\gamma\delta} \wedge dA^p_{(1)\epsilon} \wedge dA^q_{(0)\zeta\eta} \right) \epsilon^{\alpha\beta\gamma\delta\epsilon\zeta\eta}, \]

\[ D = 2 : \quad \frac{1}{102} B_{(0)\alpha\beta\gamma\delta} \wedge dA^{NS}_{(0)\epsilon\zeta} \wedge dA^{RR}_{(0)\eta\theta} \epsilon^{\alpha\beta\gamma\delta\epsilon\zeta\eta\theta}. \]

(Recall that we are sometimes using the notation \( A^p \) with \( p = 1 \) or \( p = 2 \) to indicate \( A^{NS} \) or \( A^{RR} \) respectively.)

The \( D \)-dimensional Lagrangian (A.11) has a global \( GL(10-D, \mathbb{R}) \) invariance. In particular, the various fields with \( \alpha, \beta, \ldots \) indices each transform irreducibly under \( GL(10-D, \mathbb{R}) \), in the corresponding antisymmetric-tensor representations. Furthermore, it is evident from the invariance of the Lagrangian that the duals of the \( H \) fields, defined by

\[ \tilde{H}_{(D-5)} \equiv e^{\tilde{\alpha}\tilde{\beta}} \ast H_{(5)} , \quad \tilde{H}^{\alpha}_{(D-4)} \equiv e^{\tilde{\alpha}\tilde{\beta}} \ast H_{(4)\alpha} , \quad \tilde{H}^{\alpha\beta}_{(D-3)} \equiv e^{\tilde{\alpha}\tilde{\beta}} \ast H_{(3)\alpha\beta} , \]

\[ \tilde{H}^{\alpha\beta\gamma}_{(D-2)} \equiv e^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} \ast H_{(2)\alpha\beta\gamma} , \quad \tilde{H}^{\alpha\beta\gamma\delta}_{(D-1)} \equiv e^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} \ast H_{(1)\alpha\beta\gamma\delta} , \]

(A.14)

transform covariantly as antisymmetric-tensor representations with upstairs \( GL(10-D, \mathbb{R}) \) indices.

We now consider the self-duality condition which must be imposed on the ten-dimensional 5-form field strength. It is useful to do this by postponing the imposition of the self-duality constraint until the lower dimension \( D \) has been reached. At this stage, the original condition \( H_{(5)} = \ast H_{(5)} \) becomes a set of conditions relating the various dimensionally-reduced components of the original \( H_{(5)} \). To carry out this procedure, it is useful to define the following forms in the internal space:

\[ \Sigma = \frac{1}{(10-D)!} \varepsilon_{\beta_1\beta_2\ldots\beta_{(10-D)}} h^{\beta_1} \wedge h^{\beta_2} \wedge \ldots \wedge h^{\beta_{(10-D)}}, \]

\[ \Sigma_{\alpha_1} = \frac{1}{(9-D)!} \varepsilon_{\alpha_1\beta_1\ldots\beta_{(9-D)}} h^{\beta_1} \wedge \ldots \wedge h^{\beta_{(9-D)}}, \]

\[ \Sigma_{\alpha_1\alpha_2} = \frac{1}{(8-D)!} \varepsilon_{\alpha_1\alpha_2\beta_1\ldots\beta_{(8-D)}} h^{\beta_1} \wedge \ldots \wedge h^{\beta_{(8-D)}}, \]

\[ \ldots \]

We then find that the dimensional reduction of the ten-dimensional Hodge dual of \( H_{(5)} \) gives

\[ (-1)^D \ast H_{(5)} \rightarrow \tilde{H}_{(D-5)} \Sigma - \tilde{H}^{\alpha}_{(D-4)} \Sigma_{\alpha} + \frac{1}{2} \tilde{H}^{\alpha\beta}_{(D-3)} \Sigma_{\alpha\beta} - \frac{1}{6} \tilde{H}^{\alpha\beta\gamma}_{(D-2)} \Sigma_{\alpha\beta\gamma} + \frac{1}{24} \tilde{H}^{\alpha\beta\gamma\delta}_{(D-1)} \Sigma_{\alpha\beta\gamma\delta}, \]

(A.16)

where the tilded fields are defined in (A.14). From this and (A.7), we see that the ten-dimensional self-duality condition \( H_{(5)} = \ast H_{(5)} \) becomes, in the various lower dimensions:

\[ D = 9 : \quad \tilde{H}_{(4)} = H_{(4)2}, \]
\[
D = 8 : \quad \tilde{H}_{(3)} = \frac{1}{2} \epsilon^{\alpha\beta} H_{(3)\alpha\beta}, \quad \tilde{H}_{(4)} = -\epsilon^{\alpha\beta} H_{(4)\beta},
\]
\[
D = 7 : \quad \tilde{H}_{(2)} = -\frac{1}{6} \epsilon^{\alpha\beta\gamma} H_{(2)\alpha\beta\gamma}, \quad \tilde{H}_{(3)} = \frac{1}{2} \epsilon^{\alpha\beta\gamma} H_{(3)\beta\gamma},
\]
\[
D = 6 : \quad \tilde{H}_{(1)} = \frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} H_{(1)\alpha\beta\gamma\delta}, \quad \tilde{H}_{(2)} = -\frac{1}{6} \epsilon^{\alpha\beta\gamma\delta} H_{(2)\beta\gamma\delta}, \quad \tilde{H}_{(3)} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} H_{(3)\gamma\delta},
\]
\[
D = 5 : \quad \tilde{H}_{(1)} = -\frac{1}{24} \epsilon^{\alpha\beta\gamma\delta\sigma} H_{(1)\beta\gamma\delta\sigma}, \quad \tilde{H}_{(2)} = \frac{1}{6} \epsilon^{\alpha\beta\gamma\delta\sigma} H_{(2)\gamma\delta\sigma},
\]
\[
D = 4 : \quad \tilde{H}_{(1)} = \frac{1}{24} \epsilon^{\alpha\beta\gamma\delta\sigma\rho} H_{(1)\gamma\delta\sigma\rho}, \quad \tilde{H}_{(2)} = -\frac{1}{6} \epsilon^{\alpha\beta\gamma\delta\sigma\rho} H_{(2)\delta\sigma\rho},
\]
\[
D = 3 : \quad \tilde{H}_{(1)} = -\frac{1}{24} \epsilon^{\alpha\beta\gamma\delta\sigma\rho\lambda} H_{(1)\delta\sigma\rho\lambda}. \tag{A.17}
\]

It should be emphasised that these conditions are all \(GL(10 - D, \mathbb{R})\) covariant, and so at the level of the equations of motion, the full \(GL(10 - D, \mathbb{R})\) general coordinate symmetry from the reduction on the \((10 - D)\)-torus is manifest. It is possible to use these equations to substitute back into (A.11), and thereby eliminate the doubling of the degrees of freedom associated with the original non-self-dual \(H_{(5)}\) field in \(D = 10\). When \(D\) is odd, this can be done in a completely \(GL(10 - D, \mathbb{R})\)-covariant way, by using the individual equations in (A.17) to eliminate sets of either \(H\) or \(\tilde{H}\) fields. When \(D\) is even, the explicit elimination of redundant fields requires a breaking of \(GL(10 - D, \mathbb{R})\) to \(GL(9 - D, \mathbb{R})\), since one has to eliminate half of the set of fields of degree \(D/2\).

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