Torus invariants of the homogeneous coordinate ring of $G/B$ - connection with Coxeter elements

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Abstract

In this article, we prove that for any indecomposable dominant character $\chi$ of a maximal torus $T$ of a simple adjoint group $G$ over $\mathbb{C}$ such that there is a Coxeter element $w$ in the Weyl group $W$ for which $X(w)_{ss}^T(L_\chi) \neq \emptyset$, the graded algebra $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^\otimes d)^T$ is a polynomial ring if and only if $\dim(H^0(G/B, L_\chi)^T) \leq \text{rank of } G$. We also prove that the co-ordinate ring $\mathbb{C}[h]$ of the cartan subalgebra $h$ of the Lie algebra $\mathfrak{g}$ of $G$ and $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^\otimes d)^T$ are isomorphic if and only if $X(w)_{ss}^T(L_{\alpha_0})$ is non empty for some coxeter element $w$ in $W$, where $\alpha_0$ denotes the highest long root.

Keywords: Indecomposable dominant characters, Coxeter elements.

1 Introduction

In [1], [10], [12] Chevalley, Serre, Shephard-Todd have proved that for any faithful representation $V$ of a finite group $H$ over the field $\mathbb{C}$ of complex numbers, the ring of $H$ invariants $\mathbb{C}[V]^H$ is a polynomial algebra if and only if $H$ is generated by pseudo reflections in $GL(V)$.

Chevalley also proved that for any semisimple algebraic group $G$ over $\mathbb{C}$, the ring $\mathbb{C}[\mathfrak{g}]^G$ of $G$ invariants of the co-ordinate ring $\mathbb{C}[\mathfrak{g}]$ of the adjoint representation $\mathfrak{g}$ of $G$ is a polynomial algebra (refer to page 127 in [3]).

In [13], Steinberg proved that for any semisimple simply connected algebraic group $G$ (over any algebraically closed field $K$) acting on itself by conjugation, the ring of $G$-invariants $K[G]^G$ is a polynomial algebra (refer to theorem in page 41 of [13]).
In [14], D. Wehlau proved a theorem giving a neccessary and sufficient condition for a rational representation \(V\) of a torus \(S\) for which the ring of \(S\) invariants of the co-ordinate ring \(K[V]\) is a polynomial algebra (refer to theorem 5.8 of [14]).

We now set up some notation before proceeding further.

Let \(G\) be a simple adjoint group of rank \(n\) over the field of complex numbers. Let \(T\) be a maximal torus of \(G\), \(B\) be a Borel subgroup of \(G\) containing \(T\). Let \(N_G(T)\) denote the normaliser of \(T\) in \(G\). Let \(W = N_G(T)/T\) denote the Weyl group of \(G\) with respect to \(T\).

We denote by \(\mathfrak{g}\) the Lie algebra of \(G\). We denote by \(\mathfrak{h} \subseteq \mathfrak{g}\) the Lie algebra of \(T\). Let \(R\) denote the roots of \(G\) with respect to \(T\). Let \(R^+ \subset R\) be the set of positive roots with respect to \(B\). Let \(S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset R^+\) denote the set of simple roots with respect to \(B\). Let \(\langle ., . \rangle\) denote the restriction of the Killing form to \(\mathfrak{h}\). Let \(s_i\) denote the simple reflection corresponding to the simple root \(\alpha_i\).

A element \(w\) in \(W\) is said to a Coxeter element if it has a reduced expression of the form \(s_{i_1}s_{i_2}\cdots s_{i_n}\) such that \(i_j \neq i_k\) whenever \(j \neq k\) (refer to [2]).

We denote by \(G/B\), the flag variety of all Borel subgroups of \(G\). For any \(w \in W\), we denote by \(X(w) = BwB/B \subset G/B\) the Schubert Variety corresponding to \(w\). We note that \(X(w)\) is stable for the left action of \(T\) on \(G/B\).

We denote \(L_\chi\), the line bundle associated to a character \(\chi\) of \(T\).

\(X(w)^{ss}_T(L_\chi)\) denote the set of all semistable points with respect to the line bundle \(L_\chi\) and for the action of \(T\) (refer to [5, 9]).

In [5] and [6] some properties of semi stable points \(X(w)^{ss}_T(L_\chi)\) with respect to dominant characters \(\chi\) of \(T\) are studied. For instance in [6], for every simple algebraic group \(G\) all the Coxeter elements \(w\) in the Weyl group \(W\) for which \(X(w)^{ss}_T(L_\chi)\) is non empty for some non trivial dominant character \(\chi\) of \(T\) has been studied.

A further study about the dominant characters \(\chi\) of \(T\) for which there is a Coxeter element \(w\) such that \(X(w)^{ss}_T(L_\chi)\) is non empty, we observed that in the case of \(A_2\), given an indecomposable dominant character \(\chi\) of \(T\) which is in the root lattice (refer to definition 4.1 for the indecomposable dominant character ), \(X(w)^{ss}_T(L_\chi)\) is non empty for some coxeter element \(w\) in \(W\) if and only if \(\chi\) must be one of the following: \(\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\) and \(\alpha_1 + 2\alpha_2\). We also observed that for all these three dominant characters \(\chi\), the ring of \(T\)-invariants of the homogeneous co-ordinate ring \(\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(\mathbb{PGL}_3(\mathbb{C})/B, L_\chi^{\otimes d})\) is a polynomial ring.

In case of \(B_2\) as well, given an indecomposable dominant character \(\chi\) of \(T\) which is in the root lattice, \(X(w)^{ss}_T(L_\chi)\) is non empty for some coxeter element \(w\) in \(W\) if and only if \(\chi\) must be one of the following: \(\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\). We also observed that for all these two dominant characters \(\chi\), the ring of \(T\)-invariants of the homogeneous co-ordinate ring \(\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(SO(5, \mathbb{C})/B, L_\chi^{\otimes d})\) is a polynomial ring.

The computations in the above mentioned special cases tempt us to ask the following question:
Let $G$ be a simple adjoint group over $\mathbb{C}$, the field of complex numbers. Let $T$ be a maximal torus of $G$, $B$ be a Borel subgroup of $G$ containing $T$. Then, for any indecomposable dominant character $\chi$ of $T$ such that there is a Coxeter element $w$ in $W$ such that $X(w)_T^{ss}(\mathcal{L}_\chi)$ is non empty, is the ring of $T$- invariants of the homogeneous co-ordinate ring $\bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_\chi^{\otimes d})$ a polynomial algebra?

In this paper we prove that for any indecomposable dominant character $\chi$ of a maximal torus $T$ of a simple adjoint group $G$ such that there is a Coxeter element $w \in W$ for which $X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$, the graded algebra $\bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring if and only if $\text{dim}(H^0(G/B, \mathcal{L}_\chi)^T) \leq \text{rank of } G$ (refer to theorem 4.8).

Let $\mathfrak{g}$ be the Lie algebra of $G$, let $\mathfrak{h}$ be the Lie algebra of $T$, let $\alpha_0$ be the highest long root. Since $H^0(G/B, \mathcal{L}_{\alpha_0})$ is an irreducible self dual $G$ module with highest weight $\alpha_0$, the $G$ modules $H^0(G/B, \mathcal{L}_{\alpha_0})$ are isomorphic.

On the other hand, the natural $T$-invariant projection from $\mathfrak{g}$ to $\mathfrak{h}$ induces a isomorphism $\text{Hom}(\mathfrak{h}, \mathbb{C}) \to \text{Hom}(\mathfrak{g}, \mathbb{C})^T$. So, we have an isomorphism $\text{Hom}(\mathfrak{h}, \mathbb{C}) \to H^0(G/B, \mathcal{L}_{\alpha_0})^T$.

Thus we have a homomorphism $f : \mathbb{C}[\mathfrak{h}] \longrightarrow \bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ of $\mathbb{C}$ algebras.

In this direction, we prove that the homomorphism $f : \mathbb{C}[\mathfrak{h}] \longrightarrow \bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism if and only if $X(w)_T^{ss}(\mathcal{L}_{\alpha_0})$ is non empty for some coxeter element $w$ in $W$ (refer to theorem 3.3).

The organisation of the paper is as follows:

Section 2 consists of preliminaries and notations.

In section 3, we prove that the homomorphism $f : \mathbb{C}[\mathfrak{h}] \longrightarrow \bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism if and only if $X(w)_T^{ss}(\mathcal{L}_{\alpha_0})$ is non empty for some coxeter element $w$ in $W$.

In section 4, we prove that for any indecomposable dominant character $\chi$ of a maximal torus $T$ of a simple adjoint group $G$ over $\mathbb{C}$ such that there is a Coxeter element $w \in W$ for which $X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$, the graded algebra $\bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring if and only if $\text{dim}(H^0(G/B, \mathcal{L}_\chi)^T) \leq \text{rank of } G$.

2 Preliminaries and Notations

Let $G$ be a simple adjoint group of rank $n$ over the field of complex numbers. Let $T$ be a maximal torus of $G$, $B$ be a Borel subgroup of $G$ containing $T$. Let $N_G(T)$ denote the normaliser of $T$ in $G$. Let $W = N_G(T)/T$ denote the Weyl group of $G$ with respect to $T$.

We denote by $\mathfrak{g}$ the Lie algebra of $G$. We denote by $\mathfrak{h} \subseteq \mathfrak{g}$ the Lie algebra of $T$. Let $R$ denote the roots of $G$ with respect to $T$. Let $R^+ \subset R$ be the set of positive roots with respect to $B$. Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset R^+$ denote the set of simple roots with respect to $B$. Let $\langle.,.\rangle$ denote the restriction of the Killing form to $\mathfrak{h}$. Let $\hat{\alpha}_i$ denote the co-root corresponding
to \(\alpha_i\). Let \(\varpi_1, \varpi_2, \cdots, \varpi_n\) fundamental weights corresponding to \(S\). Let \(s_i\) denote the simple reflection corresponding to the simple root \(\alpha_i\). For any subset \(J\) of \(\{1, 2, \cdots, n\}\), we denote by \(W_J\) the subgroup of \(W\) generated by \(s_j, j \in J\). We denote \(J^c\) is the complement of \(J\) in \(\{1, 2, \cdots, n\}\). We denote \(P_J\) the parabolic subgroup of \(G\) containing \(B\) and \(W_J\). In particular, we denote the maximal parabolic subgroup of \(G\) generated by \(B\) and \(\{s_j; j \neq i\}\) by \(P_i\) (refer to \([4]\)).

Let \(w_0\) denote the longest element of \(W\) corresponding to \(B\). Let \(B^- = w_0 B w_0^{-1}\) denote the Borel subgroup of \(G\) opposite to \(B\) with respect to \(T\).

Let \(\{E_\beta : \beta \in R\} \cup \{H_\beta : \beta \in S\}\) be the Chevalley basis for \(\mathfrak{g}\) (refer to \([3]\)).

A element \(w\) in \(W\) is said to a Coxeter element if it has a reduced expression of the form \(s_{i_1} s_{i_2} \cdots s_{i_n}\) such that \(i_j \neq i_k\) whenever \(j \neq k\) (refer to \([2]\)).

We denote by \(G/B\), the flag variety of all Borel subgroups of \(G\). For any \(w \in W\), we denote by \(X(w) = B w B \subset G/B\) the Schubert Variety corresponding to \(w\). We note that \(X(w)\) is stable for the left action of \(T\) on \(G/B\).

Let \(\chi \in X(B)\), we have an action of \(B\) on \(\mathbb{C}\), namely \(b.k = \lambda(b^{-1})k, b \in B, k \in \mathbb{C}\). \(L := G \times \mathbb{C}/\sim\), where \(\sim\) is the equivalence relation defined by \((gb, b.k) \sim (g, k), g \in G, b \in B, k \in \mathbb{C}\). Then \(L\) is the total space of a line bundle over \(G/B\). We denote by \(\mathcal{L}_\chi\), the line bundle associated to \(\chi\).

\[X(w)_{T}^d(\mathcal{L}_\chi)\] denote the set of all semistable points with respect to the line bundle \(\mathcal{L}_\chi\) and for the action of \(T\) (for precise definition, refer to \([8],[9]\)).

Let \(\mathfrak{g} = \text{Lie}(G)\) be the adjoint representation of \(G\) and \(\mathfrak{h} = \text{Lie}(T)\). Let \(\alpha_0\) be the highest long root.

Since \(G\) is simple, the adjoint representation \(\mathfrak{g}\) of \(G\) is an irreducible representation with highest weight \(\alpha_0\).

Let \(\phi_i : \mathfrak{g} \to \mathfrak{h}\) be the \(T\)-invariant projection. Then, \(\phi_1\) induces a natural isomorphism \(\text{Hom}(\mathfrak{h}, \mathbb{C}) \to \text{Hom}(\mathfrak{g}, \mathbb{C})^T\).

Since \(H^0(G/B, \mathcal{L}_{\alpha_0})\) is an irreducible self dual \(G\) module with highest weight \(\alpha_0\), the \(G\) modules \(H^0(G/B, \mathcal{L}_{\alpha_0})\), \(\text{Hom}(\mathfrak{g}, \mathbb{C})\) are isomorphic.

So, we have an isomorphism \(\text{Hom}(\mathfrak{h}, \mathbb{C}) \to H^0(G/B, \mathcal{L}_{\alpha_0})^T\).

Thus we have a homomorphism \(f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d \in \mathbb{Z}_{>0}} H^0(G/B, \mathcal{L}_{\alpha_0}^d)^T\) of \(\mathbb{C}\) algebras. (1)

### 3 Relationship between \(\mathbb{C}[\mathfrak{h}]\) and homogeneous co-ordinate ring of \(G/B\) associated to highest long root

In this section, we show that the homomorphism \(f : \mathbb{C}[\mathfrak{h}] \to \bigoplus_{d \in \mathbb{Z}_{>0}} H^0(G/B, \mathcal{L}_{\alpha_0}^d)^T\) as in (1) is injective.
Further, we also prove that \( f : \mathbb{C}[\mathfrak{h}] \to \oplus_{\alpha \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}^{w_0}_\alpha) \) is an isomorphism if and only if \( X(w)^{\mathfrak{g}^\phi}(L_{\alpha_0}) \) is non-empty for some coxeter element \( w \) in \( W \).

We first set up some notation.

For each positive root \( \alpha \in R^+ \), we denote by \( U_\alpha \), the \( T \)-stable root subgroup of \( B \) corresponding to \( -\alpha \).

Let \( U^- \) be the unipotent radical of the opposite Borel subgroup \( w_0Bw_0^{-1} \).

Then, we have \( U^- = \prod_{\alpha \in \Phi^+} U_\alpha \).

Let \( v^+ = E_{\alpha_0} \) be a highest weight vector of \( \mathfrak{g} \). Consider \( U^- v^+ \subset \mathfrak{g} \) be the \( U^- \) orbit of \( v^+ \).

**Lemma 3.1.** The restriction map \( \phi := \phi_1|_{U^- v^+} : U^- v^+ \to \mathfrak{h} \) is onto.

**Proof.** Since \( \alpha_0 \) is dominant, we can choose a simple root \( \gamma_1 \) such that \( \langle \alpha_0, \gamma_1 \rangle \geq 1 \). Choose distinct simple roots \( \gamma_2, \gamma_3, \ldots, \gamma_n-1 \) such that for all \( r = 1, 2, \ldots, n-1 \), \( \sum_{j=1}^{n-1} \gamma_j \) is a root.

Denote \( \theta_r = \sum_{j=1}^{r} \gamma_j, \ r = 1, 2, \ldots, n-1 \). Again since \( \langle \alpha_0, \theta_r \rangle \geq 1 \) for \( 1 \leq r \leq n-1 \), each \( \beta_r := \alpha_0 - \theta_r \) is a root.

For every choices of \( c_0, c_r, c'_r \in \mathbb{C}, \ 1 \leq r \leq n-1 \) we claim that
\[
\phi(\exp(c_0E_{-\alpha_0}))(\exp(c_1E_{-\beta_1}))(\exp(c'_1E_{-\theta_1}))(\exp(c_2E_{-\beta_2}))(\exp(c'_2E_{-\theta_2})) \cdots (\exp(c_{n-1}E_{-\beta_{n-1}}))(\exp(c'_nE_{-\theta_{n-1}}))(E_{\alpha_0}) = -c_0 H_{\alpha_0} - \sum_{r=1}^{n-1} c_r c'_r H_{\beta_r}.
\]

Take a typical monomial \( M = \frac{c_0}{m_0}E_{-\alpha_0} c_{a_1}^{a_1} E_{-\beta_1} \cdots c_{a_{n-1}}^{a_{n-1}} E_{-\beta_{n-1}} c_{b_1}^{b_1} E_{-\theta_1} \cdots c_{b_{n-1}}^{b_{n-1}} E_{-\theta_{n-1}} \) occurring in the expansion of
\[
(\exp(c_0E_{-\alpha_0}))(\exp(c_1E_{-\beta_1}))(\exp(c'_1E_{-\theta_1}))(\exp(c_2E_{-\beta_2}))(\exp(c'_2E_{-\theta_2})) \cdots (\exp(c_{n-1}E_{-\beta_{n-1}}))(\exp(c'_nE_{-\theta_{n-1}}))
\]

Then \( M v^+ \) has weight zero if and only if \( (1 - m_0) \alpha_0 = \sum_{j=1}^{n-1} a_j \beta_j + \sum_{k=1}^{n-1} b_k \theta_k \).

**Claim:** For all \( j = 1, 2, \ldots, n-1 \) and \( k = 1, 2, \ldots, n-1 \), there exist unique \( r \) in \( \{1, 2, \ldots, n-1\} \) such that \( a_r = b_r = 1 \) and \( a_j = b_j = 0 \) for all \( j \neq r \).

Now \( M v^+ \) has weight 0 implies \( (1 - m_0) \alpha_0 - \sum_{j=1}^{n-1} a_j \beta_j - \sum_{k=1}^{n-1} b_k \theta_k = 0 \).

\[
\Rightarrow (m_0 - 1) \alpha_0 + \sum_{j=1}^{n-1} a_j \beta_j + \sum_{k=1}^{n-1} b_k \theta_k = 0
\]
\[
\Rightarrow (m_0 - 1) \alpha_0 + \sum_{j=1}^{n-1} a_j \beta_j + \sum_{k=1}^{n-1} b_k (\alpha_0 - \beta_k) = 0
\]
\[
\Rightarrow (m_0 - 1) \alpha_0 + (\sum_{k=1}^{n-1} b_k) \alpha_0 + \sum_{j=1}^{n-1} a_j \beta_j - \sum_{k=1}^{n-1} b_k \beta_k = 0.
\]
\[
\Rightarrow ((m_0 - 1) + \sum_{k=1}^{n-1} b_k) \alpha_0 + \sum_{j=1}^{n-1} (a_j - b_j) \beta_j = 0.
\]

Since \( \{\alpha_0, \beta_j, j = 1, 2, \ldots, n-1\} \) linearly independent, we have
\( (m_0 - 1) + \sum_{k=1}^{n-1} b_k = 0 \) and \( a_j = b_j \) for all \( j = 1, 2, \ldots, n-1 \).

Since \( m_0 \) and \( b_j \)s are non-negative integers, we have either \( m_0 = 1 \) and \( a_j = b_j \) for all \( j \) or \( m_0 = 0 \) and there exist unique \( k \) such that \( b_k = 1 \) and \( b_j = 0 \) for all \( j \neq k \).

Again \( H_{\alpha_0}, H_{\beta_r} \) are linearly independent since \( \alpha_0 \) and \( \beta_r \) are linearly independent. So, we have a surjective map \( U^- v^+ \cong (U_{-\alpha_0} \prod_{j=1}^{n-1} U_{-\beta_j} \prod_{k=1}^{n-1} U_{-\theta_k})(U^- v^+ \to \mathfrak{h} \) given by
\[
(\prod_{j=1}^{n-1} u_{-\beta_j}(c_j) \prod_{k=1}^{n-1} u_{-\theta_k}(c'_k)) v^+ \mapsto -c_0 H_{\alpha_0} - \sum_{r=1}^{n-1} c_r c'_r H_{\beta_r}.
\]
where $u_\alpha(c) = \exp(cE_\alpha)$, $\alpha \in R, c \in \mathbb{C}$.

Hence $\phi : U^{-v^+} \rightarrow \mathfrak{h}$ is onto. This completes the proof of the lemma. \hfill \Box

We have

**Corollary 3.2.** The homomorphism $f : \mathbb{C}[\mathfrak{h}] \rightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T$ as in (1) is injective.

**Proof.** By the lemma 3.1, we have $\phi : U^{-v^+} \rightarrow \mathfrak{h}$ is onto

So, $\phi^* : \mathbb{C}[\mathfrak{h}] \rightarrow \mathbb{C}[U^{-v^+}]$ is injective.

Since $U^{-[v^+]}$ is affine open subset of $G/B$, we have restriction map $H^0(G/B, \mathcal{L}_\alpha^{\otimes d}) \rightarrow \mathbb{C}[U^{-v^+}]$ for all $d \in \mathbb{Z}_{\geq 0}$.

So we get a map $H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T \rightarrow \mathbb{C}[U^{-v^+}]$ for all $d \in \mathbb{Z}_{\geq 0}$

Hence we have a homomorphism $g : \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T \rightarrow \mathbb{C}[U^{-v^+}]$ of $\mathbb{C}$ algebras.

Now we have the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}[\mathfrak{h}] & \xrightarrow{f} & \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T \\
\phi^* \downarrow & & \downarrow g \\
\mathbb{C}[U^{-v^+}] & \xrightarrow{=} & \mathbb{C}[U^{-v^+}]
\end{array}
$$

So, we have $g \circ f = \phi^*$.

Hence the homomorphism $f : \mathbb{C}[\mathfrak{h}] \rightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T$ is injective. \hfill \Box

We now prove the following theorem.

**Theorem 3.3.** The homomorphism $f : \mathbb{C}[\mathfrak{h}] \rightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T$ is an isomorphism if and only if $X(w)^w(T)(\mathcal{L}_\alpha)$ is non empty for some coxeter element $w$ in $W$.

**Proof.** By the theorem in 4.2 in \cite{6}, $X(w)^w(T)(\mathcal{L}_\alpha)$ is non empty for some coxeter element $w$ if and only if $G$ is of type $A_n, B_2$ and $C_n$.

Now we prove that the homomorphism $f : \mathbb{C}[\mathfrak{h}] \rightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T$ is an isomorphism when $G$ is of type $A_n, B_2$ and $C_n$.

By the corollary 3.2, the graded homomorphism $f : \mathbb{C}[\mathfrak{h}] \rightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T$ is injective. Hence we have $\text{sym}^d(h) \subset H^0(G/B, \mathcal{L}_\alpha^{\otimes d})^T$.

Let $\alpha_0 = \sum m_i \omega_i$, $J := \{ i \in \{1, 2, \cdots, n\}; m_i \geq 1 \}$.

Let $P = F_J$. Let $U_P$ be unipotent radical of the opposite parabolic subgroup of $P$ determind by $T$ and $B$.

Take line bundle $\mathcal{L}_0^{\otimes d}$ on $G/P$ and restric to $U_P$.

Since $U_P$ is affine space, $\mathcal{L}_0^{\otimes d}$ is trivial on $U_P$.

So, we have $H^0(U_P, \mathcal{L}_0^{\otimes d}) = \mathbb{C}[U_P]$ , regular fuctions on $U_P$.

So, $H^0(G/P, \mathcal{L}_0^{\otimes d})$ is a $T$ sub module of $\mathbb{C}[U_P]$.

Now considering weights,
the weight zero in $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})$ corresponding to weight $d\alpha_0$ in $\mathbb{C}[U^-_P]$. 

$X_{-\alpha_0}^{-a_0} X_{-\beta_1}^{-a_1} X_{-\beta_2}^{-a_2} \cdots X_{-\beta_{n-1}}^{-a_{n-1}} X_{-\theta_1}^{b_1} \cdots X_{-\theta_n}^{b_n-1}$ has weight $d\alpha_0$ in $\mathbb{C}[U^-_P]$ if and only if 

$a_0a_0 + \sum_{j=1}^{n-1} a_j \beta_j + \sum_{k=1}^{n-1} b_k \theta_k - d\alpha_0 = 0$

$\Rightarrow (a_0 - d)\alpha_0 + \sum_{j=1}^{n-1} a_j \beta_j + \sum_{k=1}^{n-1} b_k (\alpha_0 - \beta_k) = 0$

$\Rightarrow ((a_0 - d)\alpha_0 + \sum_{j=1}^{n-1} a_j \beta_j - \sum_{k=1}^{n-1} b_k \beta_k) = 0$

$\Rightarrow (a_0 - d)\alpha_0 + \sum_{j=1}^{n-1} (a_j - b_j)\beta_j = 0$.

Since $\{\alpha_0, \beta_j, j = 1, 2, \ldots, n-1\}$ linearly independent, we have $a_j = b_j$ for all $j = 1, 2, \ldots, n-1$.

Since $a_0$ and $b_j$'s are non-negative integers, we have either $a_0 = d$, $\sum_{k=1}^{n-1} b_k = 0$ or $a_0 = 0$ and $\sum_{k=1}^{n-1} b_k = d$.

Let $V_d := \{X_{-\alpha_0}^{-a_0} (X_{-\beta_1}^{-a_1} X_{-\theta_1}^{a_1}) (X_{-\beta_2}^{-a_2} X_{-\theta_2}^{a_2}) \cdots (X_{-\beta_{n-1}}^{-a_{n-1}} X_{-\theta_{n-1}}^{a_{n-1}}); \sum_{i=0}^{n-1} a_i = d\}$.

In type $A_n, B_2$ and $C_n$, $\dim(G/P) = 2n - 1$.

So, we can identify $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ with $V_d$.

And also we can identify $V_d$ as a subset of $\text{sym}^d(h)$.

Therefore we have $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T \subset \text{sym}^d(h)$.

So, $H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T = \text{sym}^d(h)$.

Hence the homomorphism $f : \mathbb{C}[h] \to \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism.

Since $H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d}) = H^0(G/P, \mathcal{L}_{\alpha_0}^{\otimes d})$, the homomorphism $f : \mathbb{C}[h] \to \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^{\otimes d})^T$ is an isomorphism.

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**When $G$ is not of type $A_n, B_2$ or $C_n$, we prove that $\dim(G/P) \geq 2n$.**

**Type $B_n$, $n \neq 2$:**

In this case, the highest long root $\alpha_0$ is $\varpi_2$. So we have $P = P_2$.

The dimension of $U^-_{P_2}$ is $\# \{ \alpha \in R^+ / \alpha \geq \alpha_2 \} = 4n - 5$.

Since $U^-_{P_2}$ affine open subset of $G/P$, $\dim(G/P) = 4n - 5$.

Hence we have $\dim(G/P) \geq 2n$ for $n \neq 2$.

**Type $D_n$:**

In this case, the highest long root $\alpha_0$ is $\varpi_2$. So we have $P = P_2$.

The dimension of $U^-_{P_2}$ is $\# \{ \alpha \in R^+ / \alpha \geq \alpha_2 \} = 4n - 7$.

Since $U^-_{P_2}$ affine open subset of $G/P$, $\dim(G/P) = 4n - 7$.

Hence we have $\dim(G/P) \geq 2n$ for $n \geq 4$.

**Type $E_6$:**

The highest long root $\alpha_0 = \varpi_2$. Hence we have $P = P_2$.

The dimension of $U^-_{P_2}$ is $\# \{ \alpha \in R^+ / \alpha \geq \alpha_2 \} = 21$.

Then $\dim(G/P) = 21$.

Hence we have $\dim(G/P) > 12$.

**Type $E_7$:**

The highest long root $\alpha_0 = \varpi_1$. Hence we have $P = P_1$. 


The dimension of $U_{P_1} = \#\{\alpha \in R^+/\alpha \geq \alpha_1\} = 33$.
Then $\dim(G/P) = 33$.
Hence we have $\dim(G/P) > 14$.

**Type $E_8$:**
The highest long root $\alpha_0 = \varpi_8$. Hence we have $P = P_8$.
The dimension of $U_{P_8} = \#\{\alpha \in R^+/\alpha \geq \alpha_8\} = 57$.
Then $\dim(G/P) = 57$.
Hence we have $\dim(G/P) > 16$.

**Type $F_4$:**
The highest long root $\alpha_0 = \varpi_1$. Hence we have $P = P_1$.
The dimension of $U_{P_1} = \#\{\alpha \in R^+/\alpha \geq \alpha_1\} \geq 8$.
Hence we have $\dim(G/P) > 8$.

**Type $G_2$:**
The highest long root $\alpha_0 = \varpi_2$. Hence we have $P = P_2$.
The dimension of $U_{P_2} = \#\{\alpha \in R^+/\alpha \geq \alpha_2\} = 5$.
Hence we have $\dim(G/P) > 4$.
Since $\dim(G/P) \geq 2n$, the krull dimension of $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{\alpha_0}^\otimes d) > 2n$.
Hence $\dim(\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{\alpha_0}^\otimes d)^T) > n$.
Therefore the homomorphism $f : \mathbb{C}[\mathfrak{h}] \to \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^\otimes d)^T$ is not an isomorphism if $G$ is not of the type $A_n, B_2$ or $C_n$.
This completes the proof of the theorem. \qed

Let $\mathbb{P}(\mathfrak{g})$ be the projective space corresponding to the affine space $\mathfrak{g}$.

**Corollary 3.4.** $\mathbb{P}(\mathfrak{g})//G \simeq (G/B(\mathcal{L}_{\alpha_0}))^*/N_G(T)$, when $G$ is of type $A_n, B_2$ or $C_n$.

**Proof.** By the Chevalley restriction theorem we have the restriction map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$ is an isomorphism.
So we have $\mathbb{P}(\mathfrak{g})//G = \mathbb{P}(\mathfrak{h})//=W$.
Since $G$ is of type $A_n, B_2$ or $C_n$, by the theorem 3.3 we have $\mathbb{C}[\mathfrak{h}] \simeq \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^\otimes d)^T$.

Then $\mathbb{C}[\mathfrak{h}]^W \simeq \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\alpha_0}^\otimes d)^{N_G(T)}$.
Hence $\mathbb{P}(\mathfrak{h})//=G \simeq (G/B(\mathcal{L}_{\alpha_0}))^*/N_G(T)$.
Therefore $\mathbb{P}(\mathfrak{g})//G \simeq (G/B(\mathcal{L}_{\alpha_0}))^*/N_G(T)$. \qed
4 A description of line bundles \( \mathcal{L}_\chi \) on \( G/B \) for which 
\( \bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_\chi^\otimes d)_T \) is a polynomial ring

In this section, We prove that for any indecomposable dominant character \( \chi \) of a maximal torus \( T \) of a simple adjoint group \( G \) such that there is a Coxeter element \( w \in W \) for which 
\( X(w)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset \), the graded algebra \( \bigoplus_{d \in \mathbb{Z} \geq 0} H^0(G/B, \mathcal{L}_\chi^\otimes d)_T \) is a polynomial ring if and only if 
\( \text{dim}(H^0(G/B, \mathcal{L}_\chi)_T) \leq \text{rank of } G \).

Notation: we use additive notation for the group \( X(T) \) of characters of \( T \).

\( X(T)_i^+ \) denotes the set of all dominant characters of \( T \).

**Definition 4.1.** A non trivial dominant character \( \chi \) of \( T \) is said to be decomposable if there is a pair of non trivial dominant characters \( \chi_1, \chi_2 \) of \( T \) such that \( \chi = \chi_1 + \chi_2 \). Otherwise we will call it indecomposable.

\( X(T)_i^+ \) denotes the set of all indecomposable elements of \( X(T)_i^+ \).

**Lemma 4.2.** Let \( G \) be a simple adjoint group of type \( A_{n-1} \). Let \( \chi = \sum_{i=1}^{n-1} a_i \alpha_i \), where \( a_i \in \mathbb{N} \) for each \( i = 1, 2, ..., n-1 \) be an element of \( X(T)_i^+ \) such that \( \langle \chi, \alpha_{n-1} \rangle = 0 \). Suppose that \( X(s_{n-1}...s_1)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset \) then

(i) The coefficients \( a_i, i = 1, 2, ..., n-1 \) satisfy the following inequality:

\[
a_1 \geq a_2 \geq a_3 \geq ... \geq a_{n-2} = 2 \text{ and } a_{n-1} = 1.
\]

(ii) \( \chi \) must be of the form \( i\varpi_1 + \varpi_{n-i} \) for some \( 2 \leq i \leq n-1 \).

**Proof.** Since \( X(s_{n-1}...s_1)^{ss}_T(\mathcal{L}_\chi) \neq \emptyset \), we have \( s_{n-1}...s_1(\chi) \leq 0 \). and also given \( \chi \) dominant  
So, \( a_i \geq a_{i+1} \) for each \( i = 1, 2, ..., n-2 \).

Now we prove that \( a_{n-1} = 1 \).

If \( a_{n-1} \geq 2 \), let \( i \) be the largest positive integer such that \( a_{n-i} = ia_{n-1} \).

Since \( 2a_{n-1} - a_{n-2} = \langle \chi, \alpha_{n-1} \rangle = 0 \), we must have \( i \geq 2 \). So \( a_{n-(i+1)} \neq (i+1)a_{n-1} \)

\( \Rightarrow a_{n-(i+1)} = ia_{n-1} + c \), where \( 0 \leq c \leq a_{n-1} - 1 \).

**Case 1:** If \( c = 0 \).

Since \( \chi \) dominant and \( \langle \chi, \alpha_{n-1} \rangle = 0 \) we have \( a_{n-j} \geq ja_{n-1}, j = 1, 2, ..., n-2 \) and also we have \( a_j \geq a_{j+1} \) for each \( j = 1, 2, ..., n-2 \).

**Claim:** \( \chi \) must be of the form \( ia_{n-1}(\sum_{j=1}^{n-1} \alpha_j) + a_{n-1}(\sum_{j=n+1-i}^{n-1}(n-j)\alpha_j) \).

Since \( c = 0 \), \( a_{n-(i+1)} = ia_{n-1} \).

Now we prove \( a_{n-(i+2)} = ia_{n-1} \).

Since \( \chi \) is dominant, \( 2a_{n-(i+1)} - a_{n-(i+2)} - a_{n-i} \geq 0 \)

\( \Rightarrow 2ia_{n-1} - a_{n-(i+2)} - ia_{n-i} \geq 0 \)

\( \Rightarrow ia_{n-i} \geq a_{n-(i+2)} \).

and also we have \( a_{n-(i+2)} \geq a_{n-(i+1)} = ia_{n-i} \)
So, \( a_{n-(i+2)} = ia_{n-i} \).

Similarly, we can prove \( a_{n-j} = ia_{n-1} \) for \( j = i + 3, \ldots, n - 1 \).

Now we prove \( a_{n-(i-1)} = (i - 1)a_{n-1} \).

Since \( \chi \) is dominant, \( a_{n-(i-1)} + a_{n-3} \geq a_{n-2} + a_{n-i} = 2a_{n-1} + ia_{n-1} = (i + 2)a_{n-1} \).

Since \( a_{n-j} \leq ja_{n-1} \), \( a_{n-(i-1)} + a_{n-3} \leq (i + 2)a_{n-1} \).

So, we conclude that \( a_{n-(i-1)} = (i - 1)a_{n-1} \) and \( a_{n-3} = 3a_{n-1} \).

Similarly, we can prove that \( a_{n-j} = ja_{n-1} \) for \( j = i - 2, i - 3, \ldots, 4 \).

Since \( \chi \) is of the form \( ia_{n-1}(\sum_{j=1}^{n-i} \alpha_j) + a_{n-1}(\sum_{j=n+1-i}^{n-1} (n-j)\alpha_j) = a_{n-1}(i\varpi_1 + \varpi_{n-i}) \).

This forces that \( \chi \) is decomposable, since \( a_{n-1} \geq 2 \). This is a contradiction to the indecomposability of \( \chi \).

This proves that if \( c = 0 \) then \( a_{n-1} = 1 \).

Case 2: If \( c > 0 \).

\[ \langle \chi, \hat{a}_1 \rangle = 2a_1 - a_2 \geq a_1 \geq a_{n-(i+1)} = ia_{n-1} + c. \]

Similarly, \( \langle \chi, \hat{a}_{n-1} \rangle \geq 2ia_{n-1} - ia_{n-1} - c - (i - 1)a_{n-1} = a_{n-1} - c \geq 1. \)

Thus, \( \chi - (i\varpi_1 + \varpi_{n-i}) \) is still a non zero dominant weight which is in the root lattice. This is a contradiction to the indecomposability of \( \chi \).

So, \( c = 0 \) and \( a_{n-1} = 1 \) and this proves (i).

(ii). Using above argument we also see that \( \chi \) is of the form \( i\varpi_1 + \varpi_{n-i} \) where \( i \) is the largest positive integer such that \( a_{n-i} = ia_{n-1} \).

\[ \square \]

**Lemma 4.3.** Let \( G \) be a simple adjoint group of type \( A_{n-1} \). Let \( \chi = \sum_{i=1}^{n-1} a_i \alpha_i \), where \( a_i \in \mathbb{N}, i = 1, \ldots, n - 1 \) be an element of \( X(T)_i^+ \) such that \( \langle \chi, \hat{a}_1 \rangle = 0 \). Suppose that \( X(s_1 \cdots s_{n-1})_{\mathbb{T}}^s(\mathcal{L}_\chi) \neq \emptyset \), then

(i) The coefficient \( a_i, i = 1, \ldots, n - 1 \) satisfy the following inequality:

\[ 1 = a_1 \leq a_2 \leq \cdots \leq a_{n-1} \]

(ii) \( \chi \) must be of the form \( \varpi_1 + i\varpi_{n-1} \) for some \( 2 \leq i \leq n - 1 \).

**Proof.** Similar to lemma 4.2. \[ \square \]

**Lemma 4.4.** Let \( G \) be a simple adjoint group of type \( A_{n-1} \). Let \( \chi = \sum_{i=1}^{n-1} a_i \alpha_i \), where \( a_i \in \mathbb{N}, i = 1, \ldots, n - 1 \) be an element of \( X(T)_i^+ \). If \( X(s_{i+1} \cdots s_{n-1}s_1s_{i+1})_{\mathbb{T}}^s(\mathcal{L}_\chi) \neq \emptyset \) for some \( 2 \leq i \leq n - 3 \), then \( \chi = \alpha_1 + \cdots + \alpha_{n-1} \).

**Proof.** Since \( X(s_{i+1} \cdots s_{n-1}s_1s_{i+1})_{\mathbb{T}}^s(\mathcal{L}_\chi) \neq \emptyset \) for some \( 2 \leq i \leq n - 3 \), \( s_{i+1} \cdots s_{n-1}s_1s_{i+1}(\chi) \leq 0 \).

So, we have \( \sum_{j=1}^{n-i}(a_{j+1} - a_j)\alpha_j + (a_{i+1} - a_1 - a_{n-i})\alpha_{i+1} + \sum_{k=i+2}^{n} (a_{i+2} - a_{n-1})\alpha_k \leq 0. \)

Since \( \chi \) is dominant, we have \( a_{i+1} \leq a_i \leq \cdots \leq a_2 \leq a_1, a_{i+1} \leq a_{i+2} \leq \cdots \leq a_{n-1} \) and \( 2a_{i+1} - a_i - a_{i+1} \geq 0 \) then \( a_{i+1} = a_i = a_{i+2} \).

Similarly, we can prove that \( a_1 = a_2 = \cdots = a_{n-1} \).

Therefore \( \chi = a_1(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}). \)
Since \(\chi\) is indecomposable and \(a_i \in \mathbb{N}\), we have \(a_1 = 1\).
Hence \(\chi = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\).

\[\square\]

**Lemma 4.5.** Let \(G\) be a simple adjoint group of type \(A_{n-1}\). Let \(\chi = i\varpi_1 + \varpi_{n-i} \in X(T)_i^+\) for some \(2 \leq i \leq n - 3\) then \(\dim(H^0(G/B, L_{\chi})^T) > n - 1\).

**Proof.** Since \(\chi = i\varpi_1 + \varpi_{n-i}\), each integer in \(\{1, 2, \cdots, n\}\) occurs in the standard young tableau corresponding to \(T\) - invariant standard monomial of shape \(\chi\) (refer to [11] for standard monomial) exactly once.
Hence \(\dim(H^0(G/B, L_{\chi})^T) = \binom{n-1}{i-1}\).

Since \(2 \leq i \leq n - 3\), \(i < n - 2\) and so \(i - j < n - (j + 2)\) for \(j = 1, 2, \cdots, i - 2\).
So we have \((n - 2)(n - 3)\cdots(n - i) > i\).\[\square\]

**Lemma 4.6.** Let \(G\) be a simple adjoint group of type \(A_{n-1}\). Let \(\chi = \varpi_i + i\varpi_{n-1} \in X(T)_i^+\) for some \(2 \leq i \leq n - 3\) then \(\dim(H^0(G/B, L_{\chi})^T) > n - 1\).

**Proof.** Let \(w_0\) be longest weyl group element.
Since \(\chi = \varpi_i + i\varpi_{n-1} \in X(T)_i^+, -w_0\chi = i\varpi_1 + \varpi_{n-i} \in X(T)_i^+\).

Since \(H^0(G/B, L_{\chi})^* = H^0(G/B, L_{-w_0\chi})\), \(\dim(H^0(G/B, L_{\chi})^T) = \dim(H^0(G/B, L_{-w_0\chi})^T)\).
By the previous lemma we have \(\dim(H^0(G/B, L_{-w_0\chi})^T) > n - 1\).
Hence \(\dim(H^0(G/B, L_{\chi})^T) > n - 1\).\[\square\]

**Lemma 4.7.** Let \(G\) be a simple adjoint group of type \(A_{n-1}\), \(n \neq 4\). Let \(\chi \in X(T)^+\). If \(w \in W\) is a Coxeter element such that \(X(w)^+_T(L_{\chi}) \neq \emptyset\) then \(\{i \in \{1, \cdots, n\} : l(ws_i) = l(w) - 1\} \subseteq \{1, n - 1\}\).

**Proof.** Let \(\chi = \sum_{i=1}^{n-1} a_i \alpha_i\), where \(a_i \in \mathbb{N}\). Suppose there is a \(2 \leq i \leq n - 2\) such that \(l(ws_i) = l(w) - 1\). Since \(w\chi \leq 0\) we have \(a_{i-1} + a_{i+1} \leq a_i\). Since \(\langle \chi, \alpha_{i-1} \rangle \geq 0\) and \(\langle \chi, \alpha_{i+1} \rangle \geq 0\) we have \(2a_{i-1} \geq a_{i-2} + a_i\) and \(2a_{i+1} \geq a_i + a_{i+2}\).
So, we have \(2a_i \geq 2(a_{i-1} + a_{i+1}) \geq 2a_i + a_{i-2} + a_{i+2}\)
\[\Rightarrow a_{i-2} + a_{i+2} = 0\]
\[\Rightarrow a_{i-2} = a_{i+2} = 0\]
\[\Rightarrow i - 2 \leq 0\) and \(i + 2 \geq n\)
\[\Rightarrow i = 2\) and \(i = n - 2\)
\[\Rightarrow i = 2\) and \(n = 4\) which is contradiction to assumption \(n \neq 4\). This completes the proof of the lemma.\[\square\]
We now prove the following theorem.

**Theorem 4.8.** Let $G$ be a simple adjoint group over $\mathbb{C}$. Let $\chi \in X(T)^+_1$ be such that there is a Coxeter element $w \in W$ for which $X(w)_{\chi}^*(\Lambda_{\chi}) \neq \emptyset$, the graded algebra $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is a polynomial ring if and only if $\dim(H^0(G/B, \mathcal{L}_{\chi})^T) \leq \text{rank of } G$.

**Proof.** We prove the theorem by using case by case analysis.

For a given simple adjoint group $G$ and for any indecomposable dominant character $\chi$ of $T$ such that $X(w)_{\chi}^*(\Lambda_{\chi}) \neq \emptyset$ for some Coxeter element $w$ in $W$, we prove that either the graded algebra $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is a polynomial ring and $\dim(H^0(G/B, \mathcal{L}_{\chi})^T) \leq \text{rank of } G$ or the graded algebra $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is a not polynomial ring and $\dim(H^0(G/B, \mathcal{L}_{\chi})^T) > \text{rank of } G$.

**Type $A_{n-1}, n \neq 4$:**

From lemma 4.7, if $w \in W$ a Coxeter elements $w$ such that $X(w)_{\chi}^*(\Lambda_{\chi})$ is non empty then $w = s_{i+1}. \ldots s_{n-1}s_i \ldots s_1$ for some $1 \leq i \leq n-2$ or $w = s_1 \ldots s_{n-1}$.

When $w = s_{n-1} \ldots s_1$ using lemma 4.2, the indecomposable dominant character $\chi$ of $T$ for which $X(w)_{\chi}^*(\Lambda_{\chi})$ is non empty are $\chi = i\varpi_1 + \varpi_{n-i}$ for $1 \leq i \leq n-1$.

If $i = n-1$: $\chi = n\varpi_1$, in this case there is only one $T$ invariant monomial. Hence $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is a polynomial ring in one variable.

If $i = n-2$:

we have $\chi = (n-2)\varpi_1 + \varpi_2$.

$\dim(H^0(G/B, \mathcal{L}_{\chi})^T) = n-1$.

Consider the map $\phi : \mathbb{C}[X_1, \ldots, X_{n-1}] \longrightarrow \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is given by $X_i \mapsto p_{i1}p_{2i-1} \ldots p_{(n-1)i}$ where $p_{ii}, p_{2i-1} \ldots p_{(n-1)i}$ Plücker co-ordinates and $i$ denotes omiting of $i$.

Using the standard monomial of shape $d\chi$ we can see that $\phi$ is surjective. So we have

(1) the surjective map $\phi : \mathbb{C}[X_1, \ldots, X_{n-1}] \longrightarrow \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$.

Let $P = P_1 \cap P_2$. Since $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T = \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P, \mathcal{L}_{\chi}^d)^T$, we have

(2) the krull dimension of $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T = n-1$.

From (1) and (2) we conclude that the map $\mathbb{C}[X_1, \ldots, X_{n-1}] \longrightarrow \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is an isomorphism.

Hence $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is a polynomial ring.

If $2 \leq i \leq n-3$:

$\chi = i\varpi_1 + \varpi_{n-i}$, by the lemma 4.5 we have $\dim(H^0(G/B, \mathcal{L}_{\chi})^T) > n-1$.

**Claim:** $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_{\chi}^d)^T$ is not a polynomial ring.
Let $P = P_i \cap P_{n-i}$. Since $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P, L_{\chi}^{\otimes d})_T$, we have the krull dimension of $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T = 1 + i(n - 1 - i)$. On the other hand we have $\dim(H^0(G/B, L_{\chi})_T) = \binom{n}{i}$.

With out loss of generality, we may assume that $i \leq (n - 1)/2$.

So, we have $i(n - 1 - i) < (n - 1)(n - i)/2$.

Since $i \leq n - 3, i - j < n - (j + 2)$ for $j = 1, \ldots, i - 3$.

So $(n - 2)(n - 3) \cdots (n - i + 1) > 3.4 \cdots i$

$\Rightarrow \binom{n}{i} > 1 + i(n - 1 - i)$.

Then we have $\dim(H^0(G/B, L_{\chi})_T) > \text{krull dimension of } \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$.

Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is not a polynomial ring.

If $i = 1$:

we have $\chi = \varpi_1 + \varpi_{n-1} = \alpha_1 + \cdots + \alpha_{n-1}$. By the theorem 3.3, we have the ring $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is a polynomial ring.

When $w = s_1 \cdots s_{n-1}$, By the lemma 2.7, the indecomposable dominant character $\chi$ of $T$ for which $X(w)^{ss}_T(L_{\chi})$ is non empty are $\chi = \varpi_i + i\varpi_{n-1}$ for $1 \leq i \leq n - 1$.

If $i = n - 1$:

we have $\chi = n\varpi_{n-1}$.

Since $-w_0\chi = n\varpi_1 + \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ and $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ are isomorphic. We proved that $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is a polynomial ring.

Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is a polynomial ring.

If $i = n - 2$:

In this case $\chi = \varpi_{n-2} + (n - 2)\varpi_{n-1}$.

Since $-w_0\chi = \varpi_2 + (n - 2)\varpi_1 + \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ and $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ are isomorphic.

We proved that $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is a polynomial ring.

Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is a polynomial ring.

If $2 \leq i \leq n - 3$:

$\chi = \varpi_i + i\varpi_{n-1}$, by the lemma 4.6, we have $\dim(H^0(G/B, L_{\chi})_T) > n - 1$.

Claim: $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is not a polynomial ring.

Since $-w_0\chi = \varpi_{n-i} + i\varpi_1 + \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ and $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ are isomorphic.

We proved that $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is not a polynomial ring.

Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is not a polynomial ring.

If $i = 1$:

we have $\chi = \varpi_1 + \varpi_{n-1} = \alpha_1 + \cdots + \alpha_{n-1}$. By the theorem 3.3, we have the ring $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is a polynomial ring.

When $w = s_{i+1} \cdots s_{n-1}s_i \cdots s_1$, where $2 \leq i \leq n - 3$, By the lemma 4.4, the indecomposable dominant character $\chi$ such that $X(w)^{ss}_T(L_{\chi})$ is non empty is $\alpha_1 + \cdots + \alpha_{n-1}$. By the theorem 2.3, the ring $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L_{\chi}^{\otimes d})_T$ is a polynomial ring.
Type $A_3$:

The indecomposable dominant characters for which there is a coxeter $w$ such that $X(w)^{ss}_{\chi}(L_\chi)$ is non empty are $\alpha_1 + \alpha_2 + \alpha_3$, $3\alpha_1 + 2\alpha_2 + \alpha_3$, $\alpha_1 + 2\alpha_2 + \alpha_3$ and $\alpha_1 + 2\alpha_2 + 3\alpha_3$.

When $\chi = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_0$, by theorem 3.3, we have $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.

When $\chi = 3\alpha_1 + 2\alpha_2 + \alpha_3 = 4\omega_1$, there is only one $T$ invariant monomial. Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring in one variable.

When $\chi = \alpha_1 + 2\alpha_2 + 3\alpha_3 = 4\omega_3$. Since $-w_0\chi = 4\omega_1$, $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ and $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^{T\alpha}$ are isomorphic. Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial ring.

Now we deal the special case of $\chi = 2\omega_2$ in $A_3$. In this case the Coxeter element is $w = s_1s_3s_2$. Let a typical monomial $\prod_{i<j} p_{ij}^{m_{ij}}$ in the Plücker co-ordinates which is $T$-invariant. Then it is easy to see that each of the indices $1, 2, 3, 4$ occur same number of times. So if $p_{12}$ (resp. $p_{13}$) is a factor of $T$ invariant monomial $M$, then $p_{34}$ (resp. $p_{24}$) is also a factor of $M$. Also, if $p_{14}$ is a factor of $T$ invariant monomial $M$, then $p_{23}$ also a factor of $M$. But by the Plücker relation we have

$$p_{14}p_{23} = p_{13}p_{24} - p_{12}p_{34}.$$ 

So, $p_{13}p_{24}$ and $p_{12}p_{34}$ generate the ring of $T$ invariants of $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})$, where $P_2$ is the the maximal parabolic subgroup associated to $\alpha_2$.

(1) Hence we have a surjective map $\mathbb{C}[p_{13}p_{24}, p_{12}p_{34}] \rightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})^T$.

(2) The krull dimension of $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})^T$ is two.

From (1) and (2) we conclude that the map $\mathbb{C}[p_{13}p_{24}, p_{12}p_{34}] \rightarrow \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})^T$ is an isomorphism.

So, $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_2, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial algebra. Hence $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, \mathcal{L}_\chi^{\otimes d})^T$ is a polynomial algebra.

This completes the proof for the type $A_{n-1}$.

Type $B_n$, $n \neq 2$:

By the theorem 4.2 in [6], when $G$ is of type $B_n$, the coxeter elements $w$ for which there is dominant character such that $X(w)^{ss}_{\chi}(L_\chi)$ is non empty is $s_ns_{n-1}\cdots s_2s_1$. The indecomposable dominant character with this property is $\chi = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \overline{\omega}_1$. 

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Now consider the standard representation $\mathbb{C}^{2n+1}$ of $SO_{2n+1}$. Then

$$\dim(Sym^2(SO_{2n+1}^*)) = (n + 1)(2n + 1).$$

By Weyl dimension formula, the dimension of the irreducible representation $V(2\mathfrak{w}_1)$ of $SO_{2n+1}$ is

$$\prod_{\alpha \in \Phi^+, \alpha \geq \alpha_1} \frac{\langle 2\mathfrak{w}_1 + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

Again since $\langle 2\mathfrak{w}_1 + \rho, \alpha \rangle = \langle \rho, \alpha \rangle$ for $\alpha \neq \alpha_1$, we have

$$\dim(V(2\mathfrak{w}_1)) = \prod_{\alpha \in \Phi^+, \alpha \geq \alpha_1} \frac{\langle 2\mathfrak{w}_1 + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

The set of $\alpha \in \Phi^+$ such that $\alpha \geq \alpha_1$ is $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_n, \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + 2\alpha_n, \ldots, \alpha_1 + 2\alpha_2 + \cdots + \alpha_n\}$.

We now calculate $\langle 2\mathfrak{w}_1 + \rho, \alpha \rangle$ for all $\alpha \geq \alpha_1$.

$$\frac{\langle 2\mathfrak{w}_1 + \rho, \alpha_1 + \cdots + \alpha_i \rangle}{\langle \rho, \alpha_1 + \cdots + \alpha_i \rangle} = \frac{i + 2}{i}, \quad 1 \leq i \leq n - 1.$$

$$\frac{\langle 2\mathfrak{w}_1 + \rho, \alpha_1 + \cdots + \alpha_n \rangle}{\langle \rho, \alpha_1 + \cdots + \alpha_n \rangle} = \frac{2n + 3}{2n - 1}.$$

$$\frac{\langle 2\mathfrak{w}_1 + \rho, \alpha_1 + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_n) \rangle}{\langle \rho, \alpha_1 + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_n) \rangle} = \frac{2n - j + 2}{2n - j}, \quad 2 \leq j \leq n.$$

Hence

$$\dim(V(2\mathfrak{w}_1)) = \prod_{\alpha \in \Phi^+, \alpha \geq \alpha_1} \frac{\langle 2\mathfrak{w}_1 + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = n(2n + 3).$$

From (1) and (2) we can conclude that $(Sym^2((\mathbb{C}^{2n+1})^*))^{SO_{2n+1}}$ is one dimensional, namely generated by the quadratic form $q$ which defines the orthogonal group $SO_{2n+1}$. Hence we have

$$Sym^2(\mathbb{C}^{2n+1})^* = V(2\mathfrak{w}_1)^* + Cq.$$

where $q = \sum_{i=1}^{n} X_i X_{2n+2-i}$. Since $q$-vanishes on $SO_{2n+1}(\mathbb{C})/P_1$, where $P_1$ is the maximal parabolic associated to $\alpha_1$, there is a unique quadratic relation among the variables $X_iX_{2n+2-i}$, $i = 1, 2, \ldots, n + 1$ on $SO_{2n+1}(\mathbb{C})/P_1$, namely $aX_{n+1}^2 = \sum_{i} X_i X_{2n+2-i}$ for some non zero $a \in \mathbb{C}$ on $SO_{2n+1}(\mathbb{C})/P_1$ (refer to [7]).
Now we explain all the \( T \)-invariant polynomials restricted to \( \text{SO}_{2n+1}(\mathbb{C})/P_1 \). Take a \( T \)-invariant polynomial \( X_1^{m_1}X_2^{m_2}\cdots X_{2n+1}^{m_{2n+1}} \) with \( m_i = m_{2n+2-i} \). The above relation implies that every \( T \)-invariant polynomial restricted to \( \text{SO}_{2n+1}(\mathbb{C})/P_1 \) is a linear combination of the monomials of the form \( (X_1X_{2n+1})^{r_1}(X_2X_{2n})^{r_2}\cdots(X_{n-1}X_{n+3})^{r_{n-1}}(X_nX_{n+2})^{r_n} \) for some \( r_i \)'s in \( \mathbb{Z}_{\geq 0} \). Thus

\[
\text{(3)} \text{ the map } \mathbb{C}[X_1X_{2n+1}, X_2X_{2n}, \cdots X_{n-1}X_{n+3}, X_nX_{n+2}] \to \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_1, L^{\otimes d}_{2\varpi_1})^T \text{ is onto.}
\]

On the other hand we have \( \dim(U^-_{P_1}) = |\{ \alpha \in R^+ : \alpha \geq \alpha_1 \}| = 2n-1 \), where \( U^-_{P_1} \) be unipotent radical of the opposite parabolic subgroup of \( P_1 \) determind by \( T \) and \( B \).

Since \( U^-_{P_1} \) is open subset of \( G/P_1 \), the dimension of the affine cone over \( G/P_1 \) is of dimension \( 2n \). So we have

\[
\text{(4) the Krull dimension of } \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_1, L^{\otimes d}_{2\varpi_1})^T \text{ is } 2n-n = n.
\]

From (3) and (4) we conclude that \( \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_1, L^{\otimes d}_{2\varpi_1})^T = \mathbb{C}[X_1X_{2n+1}, X_2X_{2n}, \cdots, X_{n-1}X_{n+3}, X_nX_{n+2}] \) is a polynomial ring.

Hence \( \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L^{\otimes d}_{2\varpi_1})^T \) is a polynomial ring.

Now we prove that \( \dim(H^0(G/B, L^{\otimes d}_{2\varpi_1})^T) \leq \text{rank of } G \).

The \( T \) invariant monomials in \( \text{Sym}^2((\mathbb{C}^{2n+1})^*) \) are of the form \( X_iX_j \) for some \( 1 \leq i \leq n+1, j = 2n+2-i \).

Hence we have \( \dim(\text{Sym}^2((\mathbb{C}^{2n+1})^*))^T \) is \( n+1 \).

Since \( \text{Sym}^2((\mathbb{C}^{2n+1})^*) = V(2\varpi_1)^* + \mathbb{C}q, \dim(V(2\varpi_1)^T) = n \).

Hence \( \dim(H^0(G/B, L^{\otimes d}_{2\varpi_1})^T) = \text{rank of } G \).

This completes the proof for type \( B_n, n \neq 2 \).

**Type \( B_2 \):**

By the theorem 4.2 in [6], when \( G \) is of type \( B_2 \), the coxeter elements \( w \) for which there is dominant character such that \( X(w)^{\alpha_0}_{\chi}(L_\chi) \) is non empty are \( s_2s_1 \) and \( s_1s_2 \). The indicomposible dominant character with this property for the coxeter element \( s_2s_1 \) is \( \chi = \alpha_1+\alpha_2 \). In this case using similar arguements as in type \( B_n, n \neq 2 \), we can prove that the ring \( \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L^{\otimes d}_{\chi})^T \) is a polynomial ring.

The indicomposible dominant character \( \chi \) for the coxeter element \( w = s_1s_2 \) for which \( X(w)^{\alpha_0}_{\chi}(L_\chi) \) is non empty is \( \chi = \alpha_1+2\alpha_2 = \alpha_0 \). By the theorem 3.3, the ring \( \oplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/B, L^{\otimes d}_{\alpha_0})^T \) is a polynomial ring.
Type $C_n$: When $G$ is of the type $C_n$, by theorem 4.2 of [6], the only coxeter element $w$ for which there is dominant character such that $X(w)^{ss}_{T}(L_\chi)$ is non empty is $w = s_{n} s_{n-1} \cdots s_{2} s_{1}$. Further, the indecomposable dominant character $\chi$ with this property is $2\omega_{1} = 2(\sum_{i \neq n} \alpha_{i}) + \alpha_{n} = \alpha_{0}$. By the theorem 3.3, the ring of $T$ invariants $\oplus_{d \in \mathbb{Z}_{\geq 0}} H^{0}(G/B, L^{\otimes d})^{T}$ is a polynomial ring.

We now prove that $\text{dim}(H^{0}(G/B, L_{2\omega_{1}})^{T}) = \text{rank of } G$. Since $\chi = \alpha_{0}$, $H^{0}(G/B, L_{\chi}) = \mathfrak{g}$. So, we have $H^{0}(G/B, L_{\chi})^{T} = \mathfrak{h}$. Hence $\text{dim}(H^{0}(G/B, L_{2\omega_{1}})^{T}) = \text{rank of } G$.

Type $D_4$: By theorem 4.2 of [6], the only coxeter elements $w \in W$ for which there exist a dominant weight $\chi$ such that $X(w)^{ss}_{T}(L_\chi)$ is non empty are $s_{4} s_{3} s_{2} s_{1}$, $s_{4} s_{1} s_{2} s_{3}$ and $s_{3} s_{1} s_{2} s_{4}$. The indecomposable dominant characters with this property are $2(\alpha_{1} + \alpha_{2}) + \alpha_{3} + \alpha_{4}$, $2(\alpha_{3} + \alpha_{2}) + \alpha_{1} + \alpha_{4}$ and $2(\alpha_{4} + \alpha_{2}) + \alpha_{1} + \alpha_{3}$ to the coxeter elements $s_{4} s_{3} s_{2} s_{1}$, $s_{4} s_{1} s_{2} s_{3}$ and $s_{3} s_{1} s_{2} s_{4}$ respectively. Since there is an automorphism of the Dynkin diagram sending $\alpha_{1}$ to $\alpha_{3}$ and fixing $\alpha_{2}$ and $\alpha_{4}$ and there is also an automorphism that sends $\alpha_{1}$ to $\alpha_{4}$ and fixing $\alpha_{2}$ and $\alpha_{3}$. If $\sigma'$ is an automorphism of dynkin diagram, we get a $\sigma : G \to G$ automorphism of algebraic groups such that $\sigma(B) = B$, $\sigma(T) = T$ and $\sigma(\alpha_{i}) = \sigma'(\alpha_{i})$ for all $i = 1, \cdots, 4$. Further, we have $H^{0}(G/B, L_{\chi})$ and $H^{0}(G/B, L_{\sigma(\chi)})$ are isomorphic as $G$-modules where the action of $G$ on $H^{0}(G/B, L_{\sigma(\chi)})$ via $\sigma$. Thus, $H^{0}(G/B, L_{\chi})^{T} = H^{0}(G/B, L_{\sigma(\chi)})^{T}$. So it is sufficient to consider the case when $\chi = 2\omega_{1} = 2\alpha_{1} + 2\alpha_{2} + \alpha_{3} + \alpha_{4}$.

Now consider the standard representation $\mathbb{C}^{8}$ of $SO_{8}$. Then, we have

\begin{equation}
(1) \quad \text{dim}(\text{Sym}^{2}(\mathbb{C}^{8}^{*})) = 36.
\end{equation}

By using the Weyl dimension formula and by proceeding with similar calculation above we can see that the dimension of the irreducible representation $V(2\omega_{1})$ is 35. \hspace{1cm} (2)

From (1) and (2) we have

\[ \text{Sym}^{2}(\mathbb{C}^{8}^{*}) = V(2\omega_{1})^{*} + \mathbb{C}q. \]

where $q = \sum_{i=1}^{8} X_{i} X_{9-i}$. Since $q$-vanishes on $SO_{8}(\mathbb{C})/P_{1}$, where $P_{1}$ is the maximal parabolic associated to $\alpha_{1}$, there is a unique quadratic relation among the variables $X_{i} X_{9-i}$, $i = 1, 2, 3, 4$ on $SO_{8}(\mathbb{C})/P_{1}$, namely $-X_{1} X_{8} = \sum_{i}^{4} X_{i} X_{9-i}$ on $SO_{8}(\mathbb{C})/P_{1}$ (refer to [7]).

Now we explain all the $T$-invariant polynomials restricted to $SO_{8}(\mathbb{C})/P_{1}$. Take a $T$-invariant polynomial $X_{1}^{m_{1}} X_{2}^{m_{2}} \cdots X_{8}^{m_{8}}$ with $m_{i} = m_{9-i}$. The above relatation implies that...
every $T$-invariant polynomial restricted to $SO_8(\mathbb{C})/P_1$ is a linear combination of the monomials of the form $(X_2X_7)^{m_2}(X_3X_6)^{m_3} \cdots (X_4X_5)^{m_4}$. Thus

(3) the map $\mathbb{C}[X_2X_7, X_3X_6, X_4X_5] \to \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_1, L^{\otimes d}_T)$ is onto.

On the other hand we have $\dim(U_{-P_1}) = |\{\alpha \in R^+ : \alpha \geq \alpha_1\}| = 6$, where $U_{-P_1}$ be unipotent radical of the opposite parabolic subgroup of $P_1$ determined by $T$ and $B$.

Hence the dimension of the affine cone $G/P_1$ is of dimension 7. So we have

(4) the Krull dimension of $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_1, L^{\otimes d}_T)$ is 3.

From (3) and (4) we conclude that $\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^0(G/P_1, L^{\otimes d}_T)$ is a polynomial ring.

Now we prove that $\dim(H^0(G/B, L^{\otimes d}_T)) \leq \text{rank of } G$.

The $T$ invariant monomials in $\text{Sym}^2((\mathbb{C}^8)^*)$ are of the form $X_iX_j$ for $1 \leq i \leq 4, j = 9-i$. Hence we have $\dim(\text{Sym}^2((\mathbb{C}^8)^*)^T)$ is 4.

Since $\text{Sym}^2((\mathbb{C}^8)^*) = V(2\varpi_1)^* + \mathbb{C}q$, $\dim(V(2\varpi_1)^T) = 3$.

Hence $\dim(H^0(G/B, L^{\otimes d}_T)) \leq \text{rank of } G$.

Type $D_n, n \neq 4$:

By theorem 4.2 of [6], the coxeter element $w$ for which there is dominant character such that $X(w)^T(L_\chi)$ is non empty is $s_ns_{n-1} \cdots s_2s_1$. The indicomposable dominant character with this property is $\chi = 2\varpi_1 = 2(\alpha_1 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$. Proof in this case is similar to that of $\chi = 2\varpi_1$ in type $B_n$.

For other types:

By theorem 4.2 of [6], there are no coxeter element $w$ and dominant character $\chi$ such that $X(w)^T(L_\chi)$ is non empty.

□

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