Quantum Radiation of Uniformly Accelerated Spherical Mirrors

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Abstract

We study quantum radiation generated by a uniformly accelerated motion of small spherical mirrors. To obtain Green’s function for a scalar massless field we use Wick’s rotation. In the Euclidean domain the problem is reduced to finding an electric potential in 4D flat space in the presence of a metallic toroidal boundary. The latter problem is solved by a separation of variables. After performing an inverse Wick’s rotation we obtain the Hadamard function in the wave-zone regime and use it to calculate the vacuum fluctuations and the vacuum expectation for the energy density flux in the wave zone.

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1 Introduction

In this paper we study quantum radiation generated by a uniformly accelerated motion of a small spherical mirror. This is a special case of the dynamical Casimir effect [1]–[2].

There are several reasons that make studies of the dynamical Casimir effect, that is the effect of vacuum polarization and particle creation in the presence of moving boundaries, important. For a special case when boundaries are moving with constant acceleration, there is some similarity of this problem with problems in a constant gravitational field and Hawking effect. Schwinger [3, 4] suggestion that the photon production associated with changes in the quantum vacuum state in a system with collapsing dielectric bubble could be responsible for sonoluminescence has generated a lot of publications (see e.g. a review by Milton [5]). System including mirrors moving in an external gravitational fields were often used in different gedanken experiments in studies of black holes. In particular, Unruh and Wald in their work devoted to the generalized second law of black hole physics [3], considered a gedanken experiment in which a box with mirrored walls filled with radiation is lowered adiabatically toward a black hole. At any particular moment of time a box is practically at rest in a static gravitational fields and hence its boundaries have a non-vanishing constant acceleration. Unruh and Wald demonstrated that the quantum radiation emitted by the mirror boundaries during the adiabatic transition from initial to final position of the box plays a key role in the understanding of the mechanism providing the fulfillment of the generalized second law for such processes. Moving mirrors may also have an application in studies of quantum radiation created by bubble formation during first order phase transitions in the Early Universe.

In a 2 dimensional spacetime the dynamical Casimir effect is studied quite well [7]–[20]. Much less results have been obtained for the relativistic motion of mirrors in physical 4 dimensional spacetime. Quantum radiation from a uniformly accelerated plane mirror was considered in [21]. Quantum effects in the presence of relativistic spherical mirrors which are expanding with uniform acceleration were studied in [22]–[23]. Ford and Vilenkin [24] have extended the plane mirror result to include non-constant acceleration for the case when the acceleration and its derivatives are small.

Recently the interest to the “old” problem of quantum effects in the presence of accelerated mirrors increased. Partially this was stimulated by attempts to obtain a more detailed quantitative description of gedanken experiments with black holes for a more realistic situation, when a 4D mirror has a finite size, is semi-transparent and moves with an acceleration which not constant in time. Since such a problem, especially in an arbitrary external gravitational field is technically very complicated several simplified models were considered. Anderson and Israel [25] calculated quantum fluxes from a spherical 4D mirror which is expanding or contracting with nearly uniform acceleration. Particle creation for adiabatic expansion or contraction of a spherical mirror and for an oscillating spherical cavity were discussed by Setare and Saharian [26, 27]. Quantum effects in a presence of
an expanding or contracting (with constant acceleration) spherical semitransparent 4D mirrors were studied in [28]. Two-dimensional analogue of this problem for an arbitrary acceleration was discussed by Nicolaevici [29]. Similar problem in a more general set-up was considered by Obadia and Parentani [30]. Energy flux and vacuum polarization created by a polarizable body of small size and moving with constant acceleration was calculated in [31].

Very recently systems with mirrors were used for study the problem of the black hole entropy. Using a 2D moving mirror model Mukohyama and Israel [32] introduced a notion of moving-mirror entropy associated with temporarily inaccessible information about the future. Page [33] argued that by surrounding a black hole by a spherical reflecting shell (mirror) one can make the black hole entropy less that its Bekenstein-Hawking value.

In this paper we continue studying the quantum effects in the presence of a 4D moving mirror. We assume that a mirror has a spherical form and it is uniformly accelerated. As a model of such a system we consider a scalar massless field obeying the Dirichlet boundary conditions on the surface of a tube generated by a uniformly accelerated motion (with the acceleration $w$) of a sphere of radius $b$. The complete information about quantum properties of the system is contained in the Feynman Green’s function. To obtain the latter we use Wick’s rotation. In the Euclidean domain the problem is reduced to finding an electric potential in 4D flat space in the presence of a metallic toroidal boundary. The problem we study is greatly simplified by the presence of high symmetry. As a result of this symmetry we can solve the problem by a separation of variables by using the toroidal coordinates and obtain a series representation for the Euclidean Green’s function. After performing an inverse Wick’s rotation we obtain the Hadamard function in the wave-zone regime and use it to calculate the vacuum fluctuations and the vacuum expectation for the energy density flux in the wave zone.

The paper is organized as follows. Section 2 discusses the set up of the problem. We formulate the equation for a massless scalar field in a presence of a uniformly accelerated spherical mirror. We also discuss here Wick’s rotation. In Section 3 we obtain a series representation for the Euclidean Green’s function. An analytical continuation to the Minkowski spacetime and wave-zone regime are discussed in Section 4. In Sections 5 and 6 we calculate $\langle \phi^2(x) \rangle_{\text{ren}}$ and the energy density flux in the wave zone.

## 2 Model and Geometry

Our purpose is to study quantum radiation from a uniformly accelerated small spherical body with a mirror-like boundary. In order to describe such a motion it is convenient to introduce in the Minkowski spacetime with a metric

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2,$$ (2.1)
new (Rindler) coordinates

\[ T = \rho \sinh \tau, \quad Z = \rho \cosh \tau, \quad X = X, \quad Y = Y. \]  \hfill (2.2)

These coordinates cover the right wedge \( R^+ \) of the spacetime where \( Z > |T| \). A line \( \rho = \rho_0, X = 0, Y = 0 \) represent a world line of a uniformly accelerated observer, moving with a constant acceleration \( w = \rho_0^{-1} \) (as measured in his reference frame) in the \( Z \)-direction, while the 3-dimensional plane \( \tau = \text{const} \) is a set of events which are simultaneous from the point of view of the observer. In these coordinates metric (2.1) is

\[ ds^2 = -\rho^2 d\tau^2 + d\rho^2 + dX^2 + dY^2, \]  \hfill (2.3)

and the boundary \( \Sigma_+ \) of a uniformly accelerated spherical mirror is described by the equation

\[ X^2 + Y^2 + (\rho - \rho_0)^2 = b^2, \]  \hfill (2.4)

where \( b \) is the radius of the mirror which is smaller than the distance to the horizon \( \rho_0 \).

This equation in the original Cartesian coordinates is

\[ X^2 + Y^2 + (\sqrt{Z^2 - T^2 - w^{-1}})^2 = b^2. \]  \hfill (2.5)

In fact, since (2.5) is invariant under the reflection \( Z \to -Z \), it describes two surfaces, \( \Sigma_+ \) and \( \Sigma_- \) which correspond to the solution of (2.5) with \( Z > 0 \) and \( Z < 0 \), respectively.

It is evident that for this type of motion, the mirror \( \Sigma_+ \) can affect the state of the field only in the region where \( T - Z > 0 \), while the region lying under the null plane \( N_- : T - Z = 0 \) is causally disconnected from the region of influence of \( \Sigma_+ \).

Our aim is to study the influence of an accelerated motion of a spherical mirror on a scalar massless quantum field \( \varphi \). In our consideration the background geometry is flat. Nevertheless, it is convenient to consider at first a general action for a scalar massless field in curved geometry

\[ W[\varphi] = -\frac{1}{2} \int dx^4 \ g^{1/2} \left( g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + \xi R \varphi^2 \right). \]  \hfill (2.6)

The field equation is

\[ \Box \varphi - \xi R \varphi = 0, \]  \hfill (2.7)

where \( \Box = g^{-1/2} \partial_{\mu} \left( g^{1/2} g^{\mu\nu} \partial_{\nu} \right) \). The stress-energy tensor, \( T_{\mu\nu} \), for the field has the form

\[ T_{\mu\nu} = (1 - 2\xi) \varphi_{,\mu} \varphi_{,\nu} + \left( 2\xi - \frac{1}{2} \right) g_{\mu\nu} g^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} \]

\[ - 2\xi \left( \varphi \varphi_{,\mu\nu} - g_{\mu\nu} \varphi_{,\alpha} \varphi^{,\alpha} \right) + \xi \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \varphi^2. \]  \hfill (2.8)

When the parameter \( \xi \) of non-minimal coupling vanishes, \( T_{\mu\nu} \) reduces to the canonical stress-energy tensor. The case

\[ \xi = 1/6 \]  \hfill (2.9)
corresponds to the conformally invariant theory with $T^\mu_\mu = 0$.

For our problem, the curvature $R$ vanishes, the field $\varphi$ propagates in the exterior of the surface $\Sigma$, and obeys the Dirichlet boundary condition on $\Sigma$

$$\varphi|_{\Sigma} = 0. \quad (2.10)$$

We denote by $G^{(1)}(x, x')$ the corresponding Hadamard function

$$G^{(1)}(x, x') = \langle 0 | \hat{\varphi}(x) \hat{\varphi}(x') + \hat{\varphi}(x') \hat{\varphi}(x) | 0 \rangle. \quad (2.11)$$

This is symmetric function of its arguments $x$ and $x'$ obeying the boundary conditions

$$G^{(1)}(x, x')|_{x \in \Sigma} = G^{(1)}(x, x')|_{x' \in \Sigma} = 0. \quad (2.12)$$

Hadamard function depends on the choice of the state of the quantum field. Since our problem possesses the invariance with respect to a reflection $T \rightarrow -T$, we use the state which also obeys the same symmetry. This is evidently a preferable state for which the calculations are greatly simplified. For this state, the Hadamard function $G^{(1)}(x, x')$ can be obtained by the standard Wick's rotation prescription.

Namely, let us make a rotation of time $T$ in the complex plane

$$T \rightarrow iT. \quad (2.13)$$

Under this rotation the metric becomes Euclidean

$$ds_E^2 = dT^2 + dX^2 + dY^2 + dZ^2, \quad (2.14)$$

and the equation of the mirror surface takes the form

$$X^2 + Y^2 + (\sqrt{Z^2 + T^2} - w^{-1})^2 = b^2. \quad (2.15)$$

This surface $\Sigma_E$ is a 4-dimensional torus $S^1 \times S^2$, which is obtained by the rotation of a sphere $S^2$ of the radius $b$ around a circle $S^1$ of the radius $w^{-1}$ ($b < w^{-1}$). Denote by $G_E(x, x')$ the Euclidean Green function, that is a symmetric (with respect $x$ and $x'$) solution of the equation

$$\Box_E G_E(x, x') = -\delta(x'x'), \quad (2.16)$$

defined in the exterior of the torus $\Sigma_E$ and obeying the boundary conditions

$$G_E(x, x')|_{x \in \Sigma} = G_E(x, x')|_{x' \in \Sigma} = 0. \quad (2.17)$$

For space-like separation of the arguments, the Hadamard function $G^{(1)}$ can be obtained from $G_E$ by the Wick’s rotation

$$G^{(1)}(x, x') = 2G_E(x, x')|_{T \rightarrow iT, T' \rightarrow iT'}. \quad (2.18)$$
It should be emphasized that using the Euclidean approach greatly simplifies calculations. On the other hand this method has restrictions. In particular, the Green’s functions obtained by the analytical continuation of $G_E(x,x')$ correspond to the averages of the $\hat{\varphi}(x)\hat{\varphi}(x')$ for a special quantum state singled out by its $T \rightarrow -T$ invariance. Moreover, because of the symmetry $Z \rightarrow -Z$, the corresponding Green’s function obeys the Dirichlet boundary conditions on the both surfaces, $\Sigma_+$ and $\Sigma_-$, and hence always gives us a result when two accelerated mirror are present.

In the next sections we shall obtain an expression for the Euclidean Green function and discuss the problem of its analytical continuation to the physical Minkowski spacetime later.

3 Euclidean Green Function

The Euclidean Green function $G_E(x,x')$ coincides with an electric potential at a point $x$ created by a point charge located at a point $x'$ of the 4-dimensional space in the presence of a conducting surface $\Sigma_E$. This problem can be solved by using the toroidal coordinates, for which the 4-dimensional Laplace operator $\Box_E$ allows separation of variables. The toroidal coordinates $(\eta, \psi, \gamma, \phi)$ are related to the Cartesian coordinates as follows

$$X = \frac{a \sin \gamma}{B} \cos \phi, \quad Y = \frac{a \sin \gamma}{B} \sin \phi,$$

$$Z = \frac{a \sinh \eta}{B} \cos \psi, \quad T = \frac{a \sinh \eta}{B} \sin \psi,$$

where $B = \cosh \eta - \cos \gamma$. The metric (2.14) in these coordinates takes the form

$$ds^2_E = \Omega^2 \, ds^2, \quad \Omega = \frac{a}{B},$$

$$d\bar{s}^2 = dH^2 + dS^2,$$

where

$$dH^2 = d\eta^2 + \sinh^2 \eta \, d\psi^2$$

is a metric on a hyperboloid $H^2$ and

$$dS^2 = d\gamma^2 + \sin^2 \gamma \, d\phi^2$$

is a metric on the unit sphere $S^2$.

In these coordinates surfaces $\eta =$-const are torii. By substituting (3.1)–(3.2) into (2.13) one can easily find that

$$a = \sqrt{\frac{1}{w^2} - b^2}, \quad \tanh \eta_0 = \sqrt{1 - w^2 b^2},$$

where $\eta_0$ is the value of $\eta$ corresponding to $\Sigma_E$. Points with $\eta < \eta_0$ lie in the exterior of $\Sigma_E$. 

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Later we shall use the following expressions in the toroidal coordinates for the square of the distance $R^2$ from the origin to the point $x = (X, Y, Z, T)$ and for the square of the distance $R^2(x, x')$ between points $x$ and $x'$

\[ R^2 = a^2 \frac{\cosh \eta + \cos \gamma}{\cosh \eta - \cos \gamma}, \]  
(3.8)

\[ R^2(x, x') = \frac{2a^2 (\cosh \Lambda - \cos \lambda)}{(\cosh \eta - \cos \gamma)(\cosh \eta' - \cos \gamma')}, \]  
(3.9)

where $\lambda$ and $\Lambda$ are geodesic distances on a unit sphere $S^2$ and a unit hyperboloid $H^2$, respectively. They are defined as

\[ \cos \lambda = \cos \gamma \cos \gamma' + \cos(\phi - \phi') \sin \gamma \sin \gamma', \]  
(3.10)

and

\[ \cosh \Lambda = \cosh \eta \cosh \eta' - \cos(\psi - \psi') \sinh \eta \sinh \eta'. \]  
(3.11)

Suppose we have two conformally related 4 dimensional spaces with metrics $d\tilde{s}^2$ and $ds^2 = \Omega^2 d\tilde{s}^2$. Then the box-operators in these spaces are related as follows

\[ \Box - \frac{1}{6} \tilde{R} = \Omega^{-3} (\tilde{\Box} - \frac{1}{6} \tilde{R}) \Omega. \]  
(3.12)

In our case curvatures $R_E$ and $\tilde{R}$ for metrics $ds^2_E$ and $d\tilde{s}^2$ given by (3.3) – (3.4) vanish and using (3.12) one gets

\[ \tilde{\Box} \tilde{G}(x, x') = -\tilde{\delta}(x, x'), \]  
(3.13)

where

\[ G_E(x, x') = \Omega^{-1}(x) \Omega^{-1}(x') \tilde{G}(x, x'), \]  
(3.14)

and

\[ \tilde{\delta}(x, x') = \frac{\delta(x - x')}{\sqrt{g}}. \]  
(3.15)

The operator $\tilde{\Box}$ is of the form

\[ \tilde{\Box} = \Delta_H + \Delta_S, \]  
(3.16)

where

\[ \Delta_H = \partial^2_\eta + \coth \eta \partial_\eta + \frac{1}{\sinh^2 \eta} \partial^2_\psi, \]  
(3.17)

and

\[ \Delta_S = \partial^2_\gamma + \cot \gamma \partial_\gamma + \frac{1}{\sin^2 \gamma} \partial^2_\phi, \]  
(3.18)

are the Laplace operators on the unit hyperboloid and unit sphere, respectively.

Expanding the Green function $\tilde{G}$ over spherical harmonics $Y_{\ell m}$ which form a complete set on the unit sphere we can write

\[ \tilde{G}(x, x') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{G}_\ell(p, p') Y_{\ell m}(q) Y_{\ell m}(q') = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) \tilde{G}_\ell(p, p') P_\ell(\cos \lambda), \]  
(3.19)
where \( x = (p,q), \; x' = (p',q') \), and \( p, p' \) are points on \( H \) and \( q, q' \) are points on \( S \). Here \( P_\ell(z) \) is the Legendre polynomial and \( \lambda \) is a geodesic distance (angle) between \( q = (\gamma, \phi) \) and \( q' = (\gamma', \phi') \) on \( S \) defined by relation (3.10).

The functions \( \tilde{G}_\ell(p, p') \) are 2-dimensional Green functions of the operator \( \Delta_H - \ell(\ell+1) \) which are regular inside the disc \( 0 \leq \eta < \eta_0 \) and obey the Dirichlet boundary conditions at the boundary of the disc.

Using the Fourier decomposition with respect to the angle variable \( \psi \) one get the following representation for \( \tilde{G}_\ell(p, p') \)

\[
\tilde{G}_\ell(p, p') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{-im(\psi-\psi')} G_{\ell m}(\eta, \eta'),
\]

where \( G_{\ell m}(\eta, \eta') \) obeys the equation

\[
\left( \frac{d^2}{d\eta^2} + \coth \eta \frac{d}{d\eta} - \frac{m^2}{\sinh^2 \eta} - \ell(\ell+1) \right) G_{\ell m}(\eta, \eta') = -\frac{\delta(\eta - \eta')}{\sinh \eta},
\]

and the boundary condition

\[
G_{\ell m}(\eta_0, \eta') = G_{\ell m}(\eta, \eta_0) = 0.
\]

The required Green functions \( G_{\ell m} \) must be also regular at \( \eta = 0 \).

Linear independent solutions of the homogeneous version of the equation (3.21) are the associated Legendre functions \( P_{m}^{\ell} \) and \( Q_{m}^{\ell} \). For complex values of their argument \( z \) and parameters \( \nu, \mu \) these functions are defined as (see [34], eq. 3.2.3 and 3.2.5)

\[
P_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left( \frac{z+1}{z-1} \right)^{\mu/2} F(-\nu, \nu+1; 1-\mu; (1-z)/2),
\]

\[
Q_{\nu}^{\mu}(z) = e^{i\mu \pi} 2^{-\nu-1} \sqrt{\pi} \Gamma(\nu+1) \Gamma(\nu+3/2) z^{\nu-\mu-1} (z^2-1)^{\mu/2} \\
\times F(1+\nu, 1+\mu, 1, 1+\nu+\mu; z^2/z^2).
\]

The function \( F \) which enters these relations is the hypergeometric function.

Since \( \ell \) and \( m \) are independent parameters in the equation, we shall need the Legendre functions both for \( |m| \leq \ell \) and for \( |m| > \ell \). For the latter case and for the standard definition of \( P_{\ell}^{m}(z) \) these functions vanish, while \( \Gamma(\ell - m + 1)P_{\ell}^{m} \) remains finite in the limit of integer \( \ell \) and \( m \). Thus instead of \( P_{\nu}^{\mu}(z) \) it is more convenient to use the following functions

\[
\mathcal{P}_{\nu}^{\mu}(z) = \Gamma(\nu - \mu + 1)P_{\nu}^{\mu}(z).
\]

It is understood, that for integer \( \nu \) and \( \mu \) these functions are defined by continuity. Functions \( \mathcal{P}_{\ell}^{m} \) are regular at \( z = 1 \). We also define

\[
\mathcal{Q}_{\nu}^{\mu}(z) = \frac{e^{-\mu \pi}}{\Gamma(\nu + \mu + 1)} Q_{\nu}^{\mu}(z).
\]
Functions $P_{\nu}^\mu(z)$ and $Q_{\nu}^\mu(z)$ are analytic functions of their complex arguments $\nu$, $\mu$, and $z$ defined in the complex plane $z$ with a cut along the real axis lying to the left of $z = 1$. For integer value $\mu = m$ these functions obey the following symmetry relations

$$P_{\nu}^{-m}(z) = P_{\nu}^m(z), \quad Q_{\nu}^{-m}(z) = Q_{\nu}^m(z). \tag{3.27}$$

Using relation (3.2.13) of [34] we obtain the following expression for the Wronskian of the functions $P_{\nu}^\mu(z)$ and $Q_{\nu}^\mu(z)$

$$W[P_{\nu}^\mu(z), Q_{\nu}^\mu(z)] \equiv P_{\nu}^\mu(z) \frac{d}{dz} Q_{\nu}^\mu(z) - Q_{\nu}^\mu(z) \frac{d}{dz} P_{\nu}^\mu(z) = \frac{1}{1 - z^2}. \tag{3.28}$$

The following functions are vanishing at $z = z_0$

$$O_{\nu}^m(z|z_0) = Q_{\nu}^m(z) - \frac{Q_{\nu}^m(z_0)}{P_{\nu}^m(z_0)} P_{\nu}^m(z). \tag{3.29}$$

The Green function $G_{\ell m}(\eta, \eta')$ is

$$G_{\ell m}(\eta, \eta') = P_{\ell}^m(cosh \eta_<) O_{\ell}^m(cosh \eta_> | z_0), \tag{3.30}$$

where $z_0 = \cosh \eta_0$. Combining the obtained results we get the following representation for the Euclidean Green function $\tilde{G}$

$$\tilde{G}(x, x') = \frac{1}{8\pi^2} \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(\cos \lambda) \sum_{m=0}^{\infty} \beta_m \cos[m(\psi - \psi')] P_{\ell}^m(cosh \eta_<) O_{\ell}^m(cosh \eta_>|z_0), \tag{3.31}$$

where $\beta_0 = 1$ and $\beta_{m \geq 1} = 2$. In order to get this relation we used the symmetry properties (3.27).

Function $Q_{\nu}^m(z)$ which enter the definition (3.29) of $O_{\nu}^m(z|z_0)$ evidently does not depend on on the position of the boundary $z_0$. Let us calculate now the corresponding boundary independent part of $\tilde{G}(x, x')$. Namely, we denote by $\tilde{G}^0(x, x')$ the expression (3.31) where instead of $O_{\ell}^m$ we substitute its boundary independent part $Q_{\ell}^m$. Using relation 3.11.4 from [34] we have

$$\sum_{m=0}^{\infty} \beta_m \cos[m(\psi - \psi')] P_{\ell}^m(z_<) Q_{\ell}^m(z_> = Q_{\ell}(z_< z_> - \cos(\psi - \psi')\sqrt{(z_>^2 - 1)(z_<^2 - 1)}). \tag{3.32}$$

Using this relation together with Heine formula (see equation 3.11.10 of Ref.[34])

$$\sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(t') Q_{\ell}(t) = \frac{1}{t - t'}, \tag{3.33}$$

we obtain

$$\tilde{G}^0(x, x') = \frac{(\cosh \eta - \cosh \gamma)(\cosh \eta' - \cosh \gamma')}{8\pi^2 a^2 (\cosh \Lambda - \cos \lambda)}, \tag{3.34}$$
where $\lambda$ and $\Lambda$ are defined by (3.10) and (3.11). This result implies that $G_E^0$ related to $\tilde{G}^0$ by (3.14) coincides with the vacuum Green function

$$G_E^0(x, x') = \frac{1}{4\pi^2 R^2(x, x')}.$$  

(3.35)

Thus the renormalized Euclidean Green defined as

$$G_E^{\text{ren}}(x, x') = G_E(x, x') - G_E^0(x, x')$$  

(3.36)

has the following series representation

$$G_E^{\text{ren}}(x, x') = -\frac{BB'}{8\pi^2 a^2} \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos \gamma) \sum_{m=0}^{\infty} \beta_m \cos[m(\psi - \psi')]$$  

$$\times \mathcal{P}_m^m(\cosh \eta) \mathcal{P}_m^m(\cosh \eta') \frac{Q_m^m((wb)^{-1})}{P_m^m((wb)^{-1})},$$  

(3.37)

where $B = \cosh \eta \cos \gamma$ and $B' = \cosh \eta' \cos \gamma'$.

## 4 Wave Zone Regime

We discuss now the problem of analytical continuation of the results obtained in the Euclidean space to the “physical” spacetime with the Minkowski metric. In order to do this it is sufficient to express coordinates $(\eta, \psi, \gamma, \phi)$ in terms of Euclidean coordinates $(T, X, Y, Z)$ and after make the Wick’s rotation $T \rightarrow -iT$. To simplify calculations we assume that the arguments of $G_E^{\text{ren}}$ are split in time direction. Namely this object is of interest for our calculations. Thus we have

$$G_E^{\text{ren}}(\mathcal{X}, \mathcal{X}') = -\frac{BB'}{8\pi^2 a^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sum_{m=0}^{\infty} \beta_m \mathcal{P}_m^m(\cosh \eta) \mathcal{P}_m^m(\cosh \eta') \frac{Q_m^m((wb)^{-1})}{P_m^m((wb)^{-1})},$$  

(4.1)

where $\mathcal{X} = (T, X, Y, Z)$ and $\mathcal{X}' = (T', X, Y, Z)$. We used also that $\cosh \eta_0 = (wb)^{-1}$.

Simple calculations also give

$$\frac{B^2}{a^2} = \frac{4a^2}{(R^2 - a^2)^2 + 4a^2(X^2 + Y^2)},$$  

(4.2)

$$\cosh^2 \eta = \frac{(R^2 + a^2)^2}{(R^2 - a^2)^2 + 4a^2(X^2 + Y^2)},$$  

(4.3)

where $R^2 = T^2 + R^2$ and $R^2 = X^2 + Y^2 + Z^2$.

If we are interested in the wave-zone limit of $G_E^{\text{ren}}$ it can be further simplified. This limit can be found if one makes the Wick’s rotation $T \rightarrow -iT$ first, then puts

$$T = U + R, \quad X = R \sin \Theta \cos \Phi, \quad Y = R \sin \Theta \sin \Phi, \quad Z = R \cos \Theta$$  

(4.4)
and after this takes the limit $R \to \infty$ with $(U, \Theta, \Phi)$ fixed. Keeping the leading terms one gets
\[ \frac{B^2}{a^2} = \frac{a^2 z^2}{R^2 U^2}, \quad (4.5) \]
\[ \cosh \eta = z, \quad z = \frac{1}{\sqrt{1 + a^2 \sin^2 \Theta/U^2}}. \quad (4.6) \]

The right-hand side of the expression for $\cosh^2 \eta$ is less than 1. This means that as a result of Wick’s rotation $\eta$ takes pure imaginary value. In the wave zone we have
\[ G_{E}^{\text{ren}}(U, U'; R, \Theta) \sim \frac{G(U, U'; \Theta)}{R \, R'}, \quad (4.7) \]
\[ G(U, U'; \Theta) = -\frac{a^2 z z'}{8\pi^2 U U'} \sum_{\ell=0}^{\infty} (2\ell + 1) \sum_{m=0}^{\infty} \beta_m \, P_{\ell}^m(z) P_{\ell}^m(z') \frac{Q_{\ell}^m(z_0)}{P_{\ell}^m(z_0)}, \quad (4.8) \]
where $z_0 = (wb)^{-1}$.

### 5 \langle \varphi^2 \rangle^{\text{ren}} in the Wave Zone

In the coincidence limit the function $G_{E}^{\text{ren}}$ is finite and gives $\langle \varphi^2 \rangle^{\text{ren}}$. Thus we have the following representation for $\langle \varphi^2 \rangle^{\text{ren}}$ in the wave zone
\[ \langle \varphi^2(x) \rangle^{\text{ren}} = -\frac{a^2}{8\pi^2 R^2 U^2} \mathcal{F}(z, z_0), \quad (5.1) \]
where
\[ \mathcal{F}(z, z_0) = z^2 \sum_{\ell=0}^{\infty} (2\ell + 1) \sum_{m=0}^{\infty} \beta_m \, [P_{\ell}^m(z)]^2 \frac{Q_{\ell}^m(z_0)}{P_{\ell}^m(z_0)}, \quad (5.2) \]
and
\[ z = \frac{1}{\sqrt{1 + (a \sin \Theta/U)^2}}, \quad a = \sqrt{\frac{1}{w^2} - b^2}. \quad (5.3) \]

The function $\mathcal{F}(z, z_0)$ depends on constants $w$ and $b$ which specify the problem. Besides this it depends on the coordinates $U$ and $\Theta$ on $\mathcal{J}^+$ which enter only through a combination $U/\sin \Theta$. The independence of the result of $\Phi$ is a consequence of the invariance of the mirror-like boundary $\Sigma$, (2.3), under rotation in the $X$–$Y$-plane. Moreover, the equation (2.3) for $\Sigma$ is also invariant under a boost transformation in the $T$–$Z$-plane. It can be shown (see e.g. Ref. [31]) that as the result of this symmetry $\langle \varphi^2 \rangle^{\text{ren}}$ near $\mathcal{J}^+$ must have the form $\sim R^{-2} U^{-2} f(\sin \Theta/U)$. The fact that our result (5.1) does have the form dictated by the symmetry of the problem gives an independent test of the correctness of the calculations. It is also easy to see that all the dependence on $U$ enters through the dimensionless time parameter $u = U/a$.

If the size of the ball is small we have $z_0 \to \infty$. In this regime one can use the following asymptotics
\[ P_{\ell}^m(z_0) \sim \frac{(2z_0)^\nu}{\sqrt{\pi}} \Gamma(\nu + \frac{1}{2}), \quad \text{Re}(\nu) > -1/2, \quad (5.4) \]
\[ Q_{\nu}(z_0) \sim \frac{(2z_0)^{-\nu-1}\sqrt{\pi}}{\Gamma(\nu + \frac{3}{2})}, \quad (5.5) \]

and hence
\[ \frac{Q_{\nu}(z_0)}{P_{\nu}(z_0)} \sim \frac{2\pi}{(2z_0)^{2\nu+1}(2\nu + 1) [\Gamma(\nu + \frac{1}{2})]^2}, \quad Re(\nu) > -1/2. \quad (5.6) \]

Since the asymptotic of this ratio does depend on \( m \) we can write the small \( b \) expansion of \( F(z, z_0) \) as follows

\[ F(z, z_0) = 2\pi z^2 \sum_{\ell=0}^{\infty} \frac{1}{(2z_0)^{2\ell+1} [\Gamma(\ell + \frac{1}{2})]^2} F_\ell(z), \quad (5.7) \]

where
\[ F_\ell(z) = \sum_{m=0}^{\infty} \beta_m [P_\ell^m(z)]^2. \quad (5.8) \]

The leading (proportional to \( b \)) contribution for small \( b \) is given by \( \ell = 0 \) term in the series \((5.8)\). Notice that for integer \( m \)
\[ P_0^m(z) = 2^{-m}(z^2 - 1)^{m/2} F(m + 1, m; m + 1; 1 - \frac{z}{2}), \quad (5.9) \]

(see formula 3.6.1.1 of [34]). Using the following property of the hypergeometric function
\[ F(b, a; b; \xi) = (1 - \xi)^{-a}, \quad (5.10) \]

we get
\[ P_0^m(z) = \left( \frac{z - 1}{z + 1} \right)^{m/2}. \quad (5.11) \]

To calculate \( P_\ell^m(z) \) for \( \ell > 1 \) one can use the following relation
\[ P_{\ell+1}^m(z) = (z^2 - 1) \frac{dP_\ell^m(z)}{dz} + (\ell + 1) z P_\ell^m(z). \quad (5.12) \]

In particular, one has
\[ P_1^m(z) = (z + m) \left( \frac{z - 1}{z + 1} \right)^{m/2}, \]
\[ P_2^m(z) = (3z^2 + 3zm + m^2 - 1) \left( \frac{z - 1}{z + 1} \right)^{m/2}. \quad (5.13) \]

Summation of series \((5.12)\) can be performed in the closed form by using Maple program. It can be shown that \( F_\ell(z) \) is a polynomial of \( z \) of the order \( 2\ell + 1 \). The first few harmonics of \( F_\ell(z) \) are
\[ F_0(z) = z, \]
\[ F_1(z) = \frac{1}{2}(5z^3 - 2z^2 - 3z + 2), \quad (5.14) \]
\[ F_2(z) = \frac{1}{2}(63z^4 - 18z^3 - 70z^2 + 12z + 15). \]

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Using these results one can show that the leading contribution to $\langle \varphi^2(x) \rangle_{\text{ren}}^{\text{ren}}$ for the small radius of the mirror $b \ll w^{-1}$ is

$$
\langle \varphi^2(x) \rangle_{\text{ren}}^{\text{ren}} \sim -\frac{bw}{8\pi^2 R^2 w^2 [1 + (\sin \Theta/u)^2]^{3/2}},
$$

(5.15)

where $u = U/a \approx wU$ is the dimensionless retarded time. By using relations (5.1), (5.7), and (5.14) one can also easily obtain higher corrections to $\langle \varphi^2(x) \rangle_{\text{ren}}^{\text{ren}}$ as powers of $bw$.

6 Energy Density Flux in the Wave Zone

The calculation of the energy density flux in the solid angle $d\Omega$ in the wave zone

$$
\frac{dE}{dU d\Omega} = \lim_{R \to \infty} R^2 \langle T_{UU} \rangle_{\text{ren}}^{\text{ren}}
$$

(6.1)

is similar to the calculation of $\langle \varphi^2(x) \rangle_{\text{ren}}^{\text{ren}}$ but more involved.

Using (4.8) it is easy to show that when one calculates $\langle T_{UU} \rangle_{\text{ren}}^{\text{ren}}$ it is sufficient to keep only derivatives with respect to $U$ and $U'$. All other derivatives effectively introduce extra power of $R^{-1}$ and hence do not contribute to the flux of the energy density at infinity. Hence in the wave zone we have

$$
\frac{dE}{dU} = [(1 - 2\xi) \frac{\partial U}{\partial U'} - \xi (\partial^2_U + \partial^2_{U'})] \mathcal{G}(U, U'; \Theta),
$$

(6.2)

where

$$
\mathcal{G}(U, U'; \Theta) = -\frac{a^2}{8\pi^2} \frac{\mathcal{F}(z, z'|z_0)}{UU'},
$$

(6.3)

$$
\mathcal{F}(z, z'|z_0) = zz' \sum_{\ell=0}^{\infty} (2\ell + 1) \mathcal{F}_\ell(z, z'|z_0),
$$

(6.4)

$$
\mathcal{F}_\ell(z, z'|z_0) = \sum_{m=0}^{\infty} \beta_m \mathcal{P}^m_\ell(\frac{z}{z_0}) \mathcal{Q}^m_\ell(\frac{z'}{z_0}).
$$

(6.5)

In the lowest order in $bw$ one has

$$
\mathcal{F}(z, z'|z_0) \sim \frac{zz'}{z_0} \sqrt{\frac{(1 + z)(1 + z') + \sqrt{(1 - z)(1 - z')}}{\sqrt{(1 + z)(1 + z') - \sqrt{(1 - z)(1 - z')}}}.
$$

(6.6)

To obtain this result we took into account that only $\ell = 0$ contributes in this order, and used expression (5.11).

It is easy to check that

$$
\frac{\partial z}{\partial U} = \frac{z(1 - z^2)}{U}. \quad (6.7)
$$

Thus we have

$$
\partial_U = \partial_U|_z + \frac{z(1 - z^2)}{U} \partial_z|_U.
$$

(6.8)
We use notation $\partial_a |_b$ to indicate that the partial derivative with respect to $a$ is taken by assuming that $b$ is fixed. Direct calculations by using Maple give

$$\frac{dE}{dU} \sim -\frac{a^2}{8\pi^2 z_0 U^4} f(z, \xi), \quad (6.9)$$

where

$$f(z, \xi) = z \left[ (1 + z^2) - 2\xi (2 + z^2 + 3z^4) \right]. \quad (6.10)$$

Since $a \sim w^{-1}$, $z_0 \sim (wb)^{-1}$ we have

$$\frac{dE}{dU} \sim -\frac{bw^3}{8\pi^2 u^4} f(z, \xi). \quad (6.11)$$

For small values of $u$ and $\Theta \neq 0, \pi$, $z \sim u/|\sin \Theta|$ and hence

$$\frac{dE}{dU} \sim -\frac{bw^3}{8\pi^2 |\sin \Theta| u^3}. \quad (6.12)$$

For large $u$ one has

$$\frac{dE}{dU} \sim -\frac{bw^3 (1 - 6\xi)}{8\pi^2 u^4}. \quad (6.13)$$

### 7 Conclusion

In this paper we considered the vacuum polarization effects in the presence of uniformly accelerated spherical mirrors. We considered a scalar field model. Under assumption that a size of the mirror, $b$, is much smaller than the inverse acceleration, $w^{-1}$, we calculated the field fluctuation and the energy density flux created by the accelerated body at infinity. This flux is given by (6.11), and for the canonical energy (i.e. for $\xi = 0$) it is always negative. Its divergence $\sim U^{-3}$ near $U = 0$ is connected with the idealization of the problem: it is assumed that the motion remains uniformly accelerated for an infinite interval of time. The boost-invariance property connected with this assumption significantly simplifies calculations. In particular, the quantity $dE/(dUd\Omega)$, which describes the angular distribution of the energy density flux at $\mathcal{J}^+$ at given moment of the retarded time $U$, besides common scale dependence, $U^{-4}$, depends on one invariant variable, $\sin \Theta/aU$.

It should be emphasized again that due to the symmetry of the problem the Euclidean approach used in this paper gives a result for two-mirror system, one of them moving in $R_+$ domain and the other moving in $R_-$ domain. Thus what we obtain as the result of calculations is joint radiation of such two mirrors. Moreover, the analytical continuation of the Euclidean expressions gives the result for the "vacuum" expectation values for a very special choice of the "vacuum" state, namely the state which is invariant under boost transformations. It would be interesting to study a similar effect of quantum radiation by a single accelerated mirror and for other quantum states, e.g. for a state obtained from
a Minkowski vacuum by switching-on procedure. This kind of problems must be investigated directly in the Minkowski spacetime. Nevertheless, some elements of a construction used in this paper and connected with the boost symmetry may be useful.

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