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On the Asymptotics of Nielsen-Olesen Vortices

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Abstract

We investigate analytically and numerically the asymptotic behavior of the Nielsen-Olesen vortex solutions and show that they approach their asymptotic values exponentially but with exponents that differ from the ones quoted in the literature. In particular, it is shown that for values of the Higgs self-coupling that are larger than a critical value, both the Higgs field and the gauge field approach exponentially their asymptotic values with equal exponents that are independent of the Higgs self-coupling.

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The classic paper of Nielsen-Olesen [1] on the existence of vortex solutions in the Abelian-Higgs model has generated significant activity on both particle physics and cosmology during the past two decades. One of the main results of that paper was the proof of the existence of vortex solutions in the Abelian-Higgs model and the derivation of their asymptotic behavior as a function of the parameters in the Lagrangian. This asymptotic behavior has been taken for granted in subsequent papers [2] and quoted to be the same as that obtained in the Nielsen-Olesen paper. However, as we show here, for values of the Higgs self coupling larger than a critical value the asymptotic behavior of the vortex solutions is not the one quoted by Nielsen-Olesen.

The Abelian-Higgs model Lagrangian may be written as:

\[
L = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} |D_\mu \Phi|^2 - \frac{\lambda}{4}(|\Phi|^2 - \eta^2)^2
\] (1)

where \( \Phi \) is a complex scalar (the Higgs field), \( D_\mu = \partial_\mu - i e A_\mu \) and \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

Using the Nielsen-Olesen anzatz:

\[
\Phi = \eta f(r) e^{i \theta}
\] (2)

\[
A_\mu = \hat{e}_\theta \frac{v(r)}{e} r
\] (3)

and rescaling the coordinate \( r \), we obtain the rescaled Nielsen-Olesen equations for the dimensionless functions \( f(r) \) and \( v(r) \):

\[
f''(r) + \frac{1}{r} f'(r) - \left(1 - \frac{v(r)^2}{r^2}\right) f(r) - \beta (f(r)^2 - 1) f(r) = 0
\] (4)

\[
v''(r) - \frac{1}{r} v'(r) + 2(1 - v(r)) f(r)^2 = 0
\] (5)

where \( \beta = \frac{\lambda}{e^2} = \left(\frac{\mu_H}{\mu_A}\right)^2 \) where \( \mu_H \) and \( \mu_A \) are the masses of the Higgs and the gauge field respectively. The boundary conditions are

\[
f(r) \rightarrow 0
\] (6)

\[
v(r) \rightarrow 0
\] (7)

for \( r \rightarrow 0 \) and

\[
f(r) \rightarrow 1
\] (8)

\[
v(r) \rightarrow 1
\] (9)
for $r \to \infty$.

The rate by which $f(r)$ and $v(r)$ are approaching their asymptotic value at $r \to \infty$ may be obtained by using the ansatz:

$$f \to 1 + \delta f \quad (10)$$

$$v \to 1 + \delta v \quad (11)$$

and keeping only lowest order terms in $\delta f$ while keeping all terms in $\delta v$. The resulting equations are:

$$\delta f'' + \frac{1}{r} \delta f' - \frac{(\delta v)^2}{r^2} - 2\beta \delta f = 0 \quad (12)$$

$$\delta v'' - \frac{1}{r} \delta v' - 2\delta v = 0 \quad (13)$$

Using an ansatz of the form

$$\delta v = e^{-\gamma r} r^\alpha (c_1^v + c_2^v) \quad \alpha \quad (14)$$

in (13) and equating the coefficients of $e^{-\gamma r} r^\alpha$ and $e^{-\gamma r} r^{\alpha-1}$ to 0, we obtain

$$\delta v \to c_1^v e^{-\sqrt{2(\beta-4)} r^2} \quad (15)$$

where $c_1^v, c_2^v$ are constants of $O(1)$. Substituting now (15) into (12) and using an ansatz similar to (14) for $\delta f$ we may obtain both the particular and the homogeneous solution of (12) valid for $r \gg 1$. The result is:

$$\delta f \to \delta f_h + \delta f_p \equiv c^f \frac{e^{-\sqrt{2(\beta-4)r}}}{\sqrt{r}} - \frac{(c_1^v)^2 e^{-2\sqrt{2}r}}{2(\beta - 4)r} \quad (16)$$

or

$$\delta f \to c^f \frac{e^{-\sqrt{2(\beta-4)r}}}{\sqrt{r}} \quad \beta \lesssim 4 \quad (17)$$

$$\delta f \to -\frac{(c_1^v)^2 e^{-2\sqrt{2}r}}{2(\beta - 4)r} \quad \beta > 4 \quad (18)$$

where $c^f$ is a constant of $O(1)$. The case $\beta < 4$ is in agreement with the asymptotic behavior obtained by Nielsen-Olesen and subsequent papers up
to a factor of $\frac{1}{\sqrt{r}}$. This factor is usually not quoted in the literature (with the exception of Ref. [3]) even though it can be of some significance.

For $\beta = 4$, (16) implies that $c_1' \to 0$ for finiteness, which justifies the ‘$\beta \sim 4$’ in (17).

The case $\beta > 4$ indicates a significant modification that should be imposed in the results of Nielsen-Olesen. In this case, the exponent in (18) is fixed and is independent of $\beta$ while according to Nielsen-Olesen and later papers the exponent should be $\sqrt{2\beta}$ (same as for $\beta < 4$). The reason that this point was missed is that the term $\frac{(\delta v)^2}{r^2}$, being second order in $\delta v$ was ignored for all values of $\beta$. Clearly however, this term can not be ignored for $\beta > 4$. The condition $\beta > 4$ physically means that the Higgs mass $\mu_H$ is beyond the two vector meson threshold ($\mu_H > 2\mu_A$).

We have checked the validity of (17), (18) by numerically solving the system (4), (5) using collocation at gaussian points (a variation of the relaxation scheme). Fig. 1 shows a plot of $\frac{\ln(\delta f(r))}{r}$ vs $r$ for

$$\beta = \left(\frac{1}{2}, 2, 8, 50\right)$$

Clearly, $\frac{\ln(\delta f(r))}{r}$ is approaching an asymptotic value for all $\beta$ which is in agreement with the exponents in (17), (18). In fact, the asymptotic values corresponding to (19) according to Fig. 1 are

$$(\sim -1.1, \sim -2.05, \sim -3.1, \sim -3.3)$$

In comparison, Refs [1], [2] would predict the asymptotic values $(-1, -2, -4, -10)$ while our analysis predicts $(-1, -2, -2.82, -2.82)$ (since $2\sqrt{2} \simeq 2.82$) which is in much better agreement with the numerical result shown in (20).

In conclusion, we have shown that the width of the scalar part of the Nielsen-Olesen vortex is independent of the Higgs self-coupling $\beta$ for $\beta > 4$ and that for these values of $\beta$ it is equal to the width of the gauge part of the vortex. This result, which was missed in previous studies, may have interesting consequences in applications of the Nielsen-Olesen vortices. Such applications include the stability of embedded strings [4], the existence of vortex solutions in two-Higgs systems, the interactions of vortices etc.. Investigation of these effects is currently in progress.
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Figure Captions

Figure 1: The dependence of \( \frac{\ln(\delta f(r))}{r} \) on \( r \) for \( \beta = (\frac{1}{2}, 2, 8, 50) \). The
asymptotic values should be compared with the predicted exponents in (17),
(18).

References

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