Fractional revival on non-cospectral vertices

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Abstract

Perfect state transfer and fractional revival can be used to move information between pairs of vertices in a quantum network. While perfect state transfer has received a lot of attention, fractional revival is newer and less studied. One problem is to determine the differences between perfect state transfer and fractional revival. If perfect state transfer occurs between two vertices in a graph, the vertices must be cospectral. Further if there is perfect state transfer between vertices \( a \) and \( b \) in a graph, there cannot be perfect state transfer from \( a \) to any other vertex. No examples of unweighted graphs with fractional revival between non-cospectral vertices were known; here we give an infinite family of such graphs. No examples of unweighted graphs where the pairs involved in fractional revival overlapped were known; we give examples of such graphs as well.

Keywords: Quantum walks, Spectral graph theory, Fractional revival, Subset transfer

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1 Introduction

One important problem in quantum computing is to provide ways to transfer information from one part of network of qubits to another. The issue is complicated by the fact that it is not possible to copy a quantum state.

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One approach to this is based on continuous quantum walks, as we now explain. Here we are given a network of \( n \) qubits, specified by a graph \( X \) with adjacency matrix \( A \). Given \( A \) we define the transition matrix \( U(t) \) at time \( t \) by

\[
U(t) = \exp(i t A).
\]

These matrices are unitary and act on a quantum system whose states are represented by positive semidefinite matrices of size \( n \times n \) and trace 1 (so-called density matrices). If the initial state is given by \( D \), the state of the system at time \( t \) is \( U(t)DU(-t) \). This system is known as continuous quantum walk. (For more information, see [8, 9, 13–15].) We use \( e_a \) to denote the standard basis vector indexed by the vertex \( a \) in \( X \) and set \( D_a = e_a e^T_a \). We have perfect state transfer from vertices \( a \) to vertex \( b \) at time \( t \) if \( D_b = U(t)D_aU(-t) \); if this happens then the state of the \( b \)-qubit at time \( t \) is equal to the state of the \( a \)-qubit at time zero.

Perfect state transfer takes place between vertices at distance \( d \) in the \( d \)-cube, as shown by Christandl et al. [7]. Subsequent work yielded further examples [1, 6, 9], but it also revealed that perfect state transfer is rare: there are only finitely many connected graphs with maximum valency \( k \) that admit perfect state transfer [15]. Because of this, various extensions of the concept have been studied. Our concern in this paper is with fractional revival. We say that fractional revival occurs on the pair of vertices \{\( a, b \)\} at time \( t \), if \( U(t)(D_a + D_b)U(-t) = D_a + D_b \). (1.1) That is equivalent to say that \( U(t)_{a,c} = U(t)_{b,c} = 0 \) for any \( c \neq a, b \) (or \( U(t) \) is permutation-similar to a \( 2 \times 2 \) block diagonal matrix, with one diagonal block indexed by \( a, b \)). If fractional revival occurs on \{\( a, b \)\} and \( U(t)D_aU(-t) \neq D_a \) (equivalently, \( U(t)_{a,b} \neq 0 \)), then we say that proper fractional revival occurs on \{\( a, b \)\}. It includes \( U(t)D_aU(-t) = D_b \), that is, when there is perfect state transfer from \( a \) to \( b \), as a special case.

A number of papers [2, 4, 5, 10, 11] have investigated fractional revival, providing examples and developing the theory. We discuss the contributions of this paper. There are two important properties of perfect state transfer:

(a) If there is perfect state transfer on \( X \) from \( a \) to \( b \), then the vertex-deleted subgraphs \( X \setminus a \) and \( X \setminus b \) are cospectral (we say that \( a \) and \( b \) are cospectral vertices in \( X \)) [15].

(b) If there is perfect state transfer from \( a \) to a second vertex, this vertex is unique [17]. (This is known as monogamy of perfect state transfer.)
Prior to our work, in all examples of proper fractional revival the two vertices involved were cospectral. We provide an infinite family of examples where this is not the case. Similarly, monogamy held in all known examples. We provide examples of graphs with vertices $a$, $b$, $c$ such that proper fractional revival occurs on $\{a, b\}$ and $\{a, c\}$. (Chan et al. [4] constructed examples of weighted graphs where monogamy fails.)

Some of the tools we use are new. In particular we introduce the concept of support graphs, graphs on eigenvalues of an underlying graph, and we also make use of matrix algebras associated to induced subgraphs (induced algebras).

2 Preliminaries

In this section, we associate a graph to a density matrix with respect to an underlying graph $X$, introduce a matrix algebra, and review some necessary conditions and a characterization for proper fractional revival to occur.

2.1 Support graph

Let $X$ be a graph on $n$ vertices with adjacency matrix $A$ and consider quantum walks on $X$. Assume $A$ has exactly $m$ distinct eigenvalues $\theta_1, \ldots, \theta_m$, with $E_1, \ldots, E_m$ being the corresponding projection matrices to the eigenspaces. Then $A = \sum_{r=1}^m \theta_r E_r$ is the spectral decomposition of $A$, and $U(t) = \sum_{r=1}^m e^{it\theta_r} E_r$. Let $a, b \in V(X)$. If $(E_r)_{a,a} = (E_r)_{b,b}$ for all $r$, then $a$ and $b$ are cospectral (this is equivalent to the subgraphs $X\setminus a$ and $X\setminus b$ being cospectral). If $E_r a$ and $E_r b$ are parallel vectors for each $r$, then we say $a$ and $b$ are parallel.

Let $D$ be a density matrix of size $n$. The eigenvalue support of $D$ is the set $\Phi_D = \{(\theta_r, \theta_s) : E_r DE_s \neq 0\}$. As $E_s DE_r = (E_r DE_s)^*$, we know that $(\theta_s, \theta_r) \in \Phi_D$ if and only if $(\theta_r, \theta_s) \in \Phi_D$. We define the support graph of $D$ (with respect to $X$) to be the graph that has the distinct eigenvalues $(\theta_1, \ldots, \theta_m)$ of $A$ as its vertices, with vertices $\theta_r$ and $\theta_s$ adjacent if $(\theta_r, \theta_s) \in \Phi_D$. If $E_r DE_s \neq 0$, then neither $E_r D = 0$ nor $E_s D = 0$, and so $E_r DE_r \neq 0$ and $E_s DE_s \neq 0$, i.e., $(\theta_r, \theta_r) \in \Phi_D$ and $(\theta_s, \theta_s) \in \Phi_D$. Therefore this graph will normally have loops (any vertex on an edge is on a loop).

Let $F$ be the splitting field of the characteristic polynomial of $A$ over $\mathbb{Q}$, that is, $F = \mathbb{Q}(\theta_1, \ldots, \theta_m)$. Denote the Galois group of $F$ over $\mathbb{Q}$ by $\Gamma$. For any matrix $M \in \text{Mat}_{n \times n}(F)$ and $\sigma \in \Gamma$, let $M^\sigma$ denote the matrix obtained by applying $\sigma$ to
We conclude that if \( \theta \) we represent a state with a positive semidefinite matrix instead of normalizing it to a density matrix (scale the matrix to one of trace 1), which does not change the analysis.

Let \( D_1 \) and \( D_2 \) be two density matrices of size \( n \). If \( U(t)D_1U(-t) = D_2 \) for some \( t > 0 \), then we say there is perfect state transfer from \( D_1 \) to \( D_2 \) at time \( t \). If \( U(t)D_1U(-t) = D_1 \) for some \( t > 0 \), then we say \( D_1 \) is periodic at time \( t \). Note that \( D_1 \) is periodic at time \( t \) if and only if \( U(t) \) and \( D_1 \) commute and that fractional revival on \( \{a, b\} \) at time \( t \) is equivalent to the state \( \frac{1}{2}(D_a + D_b) \) being periodic at time \( t \). Since unitarily similar matrices have the same trace, sometimes we represent a state with a positive semidefinite matrix instead of normalizing it to a density matrix (scale the matrix to one of trace 1), which does not change the analysis.

If \( D \) is periodic at time \( \tau \), then

\[
D = U(\tau)DU(-\tau) = \sum_{r,s} e^{i\tau(\theta_r - \theta_s)} E_r DE_s
\]

and if we multiply on the left by \( E_r \) and on the right by \( E_s \), we get

\[
E_r DE_s = e^{i\tau(\theta_r - \theta_s)} E_s DE_r.
\]

We conclude that if \( (\theta_r, \theta_s) \in \Phi_D \), then \( e^{i\tau \theta_r} = e^{i\tau \theta_s} \). Therefore if \( D \) is periodic at time \( \tau \), the function \( e^{i\tau z} \) is constant on the vertices in a connected component of the support graph.

For \( S \subseteq V(X) \), define \( D_S \) to be the diagonal matrix with \( (a, a) \)-entry equal to 1 if \( a \in S \) and equal to 0 otherwise. Note that \( D_S \) (again, we ignore the factor \( \frac{1}{|S|} \)) is a rational state. If \( D_S \) is periodic (sometimes we say \( S \) is periodic for simplicity), then its eigenvalue support has a special form:

**2.1 Theorem.** Let \( X \) be a simple graph and \( D \) be a rational state. If \( D \) is periodic relative to the continuous quantum walk on \( X \) at time \( \tau \), then there is a square-free integer \( \Delta \) such that \( \frac{\theta_r - \theta_s}{\sqrt{\Delta}} \in \mathbb{Z} \) for all \( (\theta_r, \theta_s) \) in the support of \( D \).
Let $D$ be a periodic rational state. The above theorem tells us that there exists a square-free integer $\Delta$ such that for any two vertices $\theta_r, \theta_s$ in the same connected component of the support graph of $D$, $\theta_r - \theta_s$ is an integer multiple of $\sqrt{\Delta}$. In particular, when $D = D_a + D_b$ is periodic, we have a necessary condition on $\Phi_D$ for fractional revival on $\{a, b\}$ to occur, as shown in Theorem 2.8.

### 2.2 Induced algebra

Now assume without loss of generality that $S = \{1, 2, \ldots, k\}$ is periodic at time $\tau$: $U(\tau)D_S = D_S U(\tau)$. Then $U(\tau)$ has form

$$
\begin{pmatrix}
U_0 & 0 \\
0 & U_1
\end{pmatrix},
$$

where $U_0$ is $k \times k$ and

$$
D_S U(\tau) D_S = \begin{pmatrix} U_0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

Now

$$
(D_S U(\tau) D_S)(D_S E_r D_S) = D_S U(\tau) E_r D_S = e^{i\tau \theta_r} D_S E_r D_S
$$

and similarly

$$
(D_S E_r D_S)(D_S U(\tau) D_S) = e^{i\tau \theta_r} D_S E_r D_S;
$$

it follows that $D_S U(\tau) D_S$ and $D_S E_r D_S$ commute.

As before, let $X$ be a connected graph on $n$ vertices with adjacency matrix $A = \sum_{r=1}^{m} \theta_r E_r$. Let $M$ be a matrix indexed by $V(X)$. For a subset $S$ of $V(X)$, denote the submatrix of $M$ with rows and columns indexed by elements of $S$ by $M_{S,S}$. The induced adjacency algebra of $X$ on $S$ is the matrix algebra generated by the matrices

$$
(E_r)_{S,S}, \quad r = 1, \ldots, m.
$$

Denote it by $\mathcal{A}(S)$ and note that it is isomorphic to the algebra generated by the $n \times n$ matrices $D_S E_r D_S$, $r = 1, \ldots, m$, and (2.1) is equivalent to

$$
U_0(E_r)_{S,S} = e^{i\tau \theta_r} (E_r)_{S,S}.
$$

Therefore (2.2) implies the following:

#### 2.2 Lemma

If the subset $S$ of $V(X)$ is periodic relative to the continuous quantum walk on $X$ at time $\tau$, then $U(\tau)_{S,S}$ belongs to the center of $\mathcal{A}(S)$, the induced algebra generated by $(E_r)_{S,S}$.
If $M$ is a matrix with all eigenvalues being simple, then any matrix that commutes with $M$ is a polynomial in $M$.

2.3 Corollary. Assume $S$ is a subset of $V(X)$ that is periodic relative to the continuous quantum walk on $X$ at time $\tau$. If the eigenvalues of $U(\tau)_{S,S}$ are distinct, then the induced algebra $\mathcal{A}(S)$ is commutative.

Now consider the case when $D_S$ is periodic for $S = \{1, 2\}$. If $U_0$ is a diagonal but not a scalar matrix, then the eigenvalues of $U_0$ are distinct and therefore $X$ is not connected, a contradiction. Hence if non-proper fractional revival occurs on $\{1, 2\}$ at time $\tau$, then $U_0$ is a scalar matrix and so we have simultaneous periodicity at vertices 1 and 2 with the same phase factor. When proper fractional revival occurs, the eigenvalues of $U_0$ are distinct (the only $2 \times 2$ diagonalizable matrix with repeated eigenvalues is a scalar matrix), and we obtain another necessary condition for fractional revival.

2.4 Lemma. [4] If proper fractional revival occurs on $S = \{a, b\}$, then the induced algebra $\mathcal{A}(S)$ is commutative.

As mentioned, prior to this work, in all examples of fractional revival the two vertices involved, say $a$ and $b$, were cospectral, a condition stronger than requiring the induced adjacency algebra $\mathcal{A}(\{a, b\})$ to be commutative. Similarly to cospectrality of vertices $a, b$ [15], commutativity of the induced algebra $\mathcal{A}(\{a, b\})$ has some equivalent combinatorial descriptions.

2.5 Theorem. [4] Let $X$ be a connected graph and let $A = \sum_r \theta_r E_r$ be the spectral decomposition of $A$. For vertices $a, b$ in $X$, the following are equivalent:

(a) The induced algebra $\mathcal{A}(\{a, b\})$ is commutative,

(b) there exists some $\gamma$ such that $(E_r)_{a,a} - (E_r)_{b,b} = \gamma (E_r)_{a,b}$ for all $r$,

(c) There exists some $\gamma$ such that $(A^k)_{a,a} - (A^k)_{b,b} = \gamma (A^k)_{a,b}$ for all positive integer $k$,

(d) For some $\gamma$ 

$$\phi(X \setminus a, t) - \phi(X \setminus b, t) = \gamma \sqrt{\phi(X \setminus a, t)\phi(X \setminus b, t) - \phi(X, t)\phi(X \setminus \{a, b\}, t)},$$

6
where for \( S \subseteq V(X) \), \( X \setminus S \) denotes the induced subgraph of \( X \) on \( S \) and \( \phi(X \setminus S) \) denotes the characteristic polynomial of the adjacency matrix \( A(X \setminus S) \). The value of \( \gamma \) is the same in each case, and is rational. We see that \( \gamma = 0 \) if and only if \( a \) and \( b \) are cospectral. If any of the above condition holds, vertices \( a \) and \( b \) are said to be fractionally cospectral \([4]\).

We use \( \psi_{a,b}(X,t) \) to denote \( \sqrt{\phi(X \setminus a,t)\phi(X \setminus b,t) - \phi(X,t)\phi(X \setminus \{a,b\},t)} \) (even if \( \mathcal{A}(\{a,b\}) \) is not commutative).

Theorem 2.5 implies that if \( a \) and \( b \) satisfy certain relations or if \( X \) is of certain type, then the commutativity of the induced algebra \( \mathcal{A}(\{a,b\}) \) implies that \( a \) and \( b \) are cospectral.

**2.6 Corollary.** Let \( X \) be a simple graph and \( a,b \in V(X) \). Assume the induced adjacency algebra \( \mathcal{A}(\{a,b\}) \) is commutative. If one of the following conditions holds, then \( a \) and \( b \) are cospectral:

- \( a \) is adjacent to \( b \)
- \( X \) is bipartite, and the distance between \( a \) and \( b \) is odd
- \( a \) and \( b \) are of the same degree and are at distance two in \( X \)
- \( X \) is a connected regular graph

Now we derive one more necessary condition for proper fractional revival to occur. Recall that when proper fractional revival occurs on \( S = \{a,b\} \) at time \( \tau \), the two eigenvalues of \( U_{0} \) are both simple. Equation (2.2) implies that for each \( r \), the non-zero columns of \( (E_{r})_{S,S} \) are eigenvectors of \( U(\tau)_{S,S} \) associated to eigenvalue \( e^{i\theta_{r}} \), therefore \( (E_{r})_{S,S} \) has rank at most 1 and vertices \( a \) and \( b \) are parallel. In fact, from \( \det((E_{r})_{S,S}) = (E_{r})_{a,a}(E_{r})_{b,b} - (E_{r})_{a,b}(E_{r})_{b,a} \) and \( (E_{r})_{a,b} = \left\langle E_{r} e_{a}, E_{r} e_{b} \right\rangle \) (where \( \left\langle \cdot, \cdot \right\rangle \) denotes the inner product of vectors), we know that \( \det(E_{r})_{S,S} = 0 \) if and only if

\[
\left\langle E_{r} e_{a}, E_{r} e_{a} \right\rangle \left\langle E_{r} e_{b}, E_{r} e_{b} \right\rangle = |\left\langle E_{r} e_{a}, E_{r} e_{b} \right\rangle|^{2},
\]

which holds if and only if \( E_{r} e_{a} \) and \( E_{r} e_{b} \) are parallel. Since it holds for all \( r \), vertices \( a \) and \( b \) are parallel.

We have obtained three necessary conditions for proper fractional revival to occur: \( \theta_{r} - \theta_{s} = m_{r,s}\sqrt{2} \) for \( (\theta_{r}, \theta_{s}) \in \Phi_{D_{S}} \) (Theorem 2.1), commutativity of \( \mathcal{A}(S) \) (Corollary 2.3), and parallelity of vertices \( a \) and \( b \). To get a deeper description of the eigenvalue support of \( D_{S} \), we explore the above information further.
Assume that \( \lambda_1 \) and \( \lambda_2 \) are the two distinct eigenvalues of \( U_0 \), \( x \) is a unit eigenvector associated to eigenvalue \( \lambda_1 \), and that \( y \) is a unit eigenvector associated to \( \lambda_2 \). Then for each \( r \), either \((E_r)_{S,S} = \alpha_r xx^* \) if \( e^{i\tau \theta_r} = \lambda_1 \) or \((E_r)_{S,S} = \alpha_r yy^* \) if \( e^{i\tau \theta_r} = \lambda_2 \), for some nonnegative number \( \alpha_r \), as \((E_r)_{S,S}\) is positive semidefinite of rank at most 1. Since \((E_r)_{S,S}\) is a real matrix, the two eigenvectors \( x \) and \( y \) can be chosen to be real. Therefore \( \langle x, y \rangle = 0 \), as real eigenvectors associated to distinct eigenvalues of a complex symmetric matrix are orthogonal. Therefore for any two eigenvalues \( \theta_r \) and \( \theta_s \) of \( A \), either \( e^{i\tau \theta_r} = e^{i\tau \theta_s} \) and \((E_r)_{S,S} \) and \((E_s)_{S,S}\) are scalar multiple of each other, or \( e^{i\tau \theta_r} \neq e^{i\tau \theta_s} \) and \((E_r)_{S,S}(E_s)_{S,S} = 0 \). Together with the fact \((\theta_r, \theta_s) \in \Phi_{DS} \) if and only if \((E_r)_{S,S}(E_s)_{S,S} \neq 0 \), this implies:

2.7 Lemma. If proper fractional revival occurs on \([a,b]\) in a graph \( X \), then the support graph of \( D_{[a,b]} \) has two components isomorphic to complete graphs with loops, and the remaining components are isolated vertices (loopless).

In fact, the two nontrivial components of the support graph can be determined by the sign of \((E_r)_{a,b}\). Note that neither \( x \) nor \( y \) has zero entries. For otherwise \( x^Ty = 0 \) implies that \( \{x, y\} = \{[1 0]^T, [0 1]^T\} \), contradicting proper fractional revival occurs on \([a,b]\). Therefore either \( x \) or \( y \) has its two components both positive or both negative. Without loss of generality, assume \( x = \begin{bmatrix} p \\ q \end{bmatrix} \) with \( pq > 0 \) and \( p^2 + q^2 = 1 \), then \( y = \pm \begin{bmatrix} -q \\ p \end{bmatrix} \). Therefore for some non-negative integer \( \alpha_r \),

\[
(E_r)_{S,S} = \alpha_r \begin{bmatrix} p^2 & pq \\ pq & q^2 \end{bmatrix}, \quad \text{or} \quad (E_r)_{S,S} = \alpha_r \begin{bmatrix} q^2 & -pq \\ -pq & p^2 \end{bmatrix},
\]

depending on whether \( e^{i\tau \theta_r} = \lambda_1 \) or \( e^{i\tau \theta_r} = \lambda_2 \). Furthermore, the scalar \( \gamma \) in Theorem 2.5 satisfies \( \gamma = \frac{p}{q} - \frac{q}{p} \).

Let \( C^+ \) be the set of indices \( r \) such that \((E_r)_{a,b} > 0 \), and let \( C^- \) be the set of indices \( r \) such that \((E_r)_{a,b} < 0 \). Then \( C^+ \) and \( C^- \) correspond to the two nontrivial components of the support graph of \( D_{[a,b]} \). Furthermore,

\[
U_0 = \sum_{r \in C^+} \alpha_r e^{i\theta_r} xx^T + \sum_{r \in C^-} \alpha_r e^{i\theta_r} yy^T
\]

implies that \( \sum_{r \in C^+} \alpha_r = \sum_{r \in C^-} \alpha_r = 1 \), as the eigenvalues of a unitary matrix are numbers on the unit circle. Therefore

\[
U_0 = e^{i\tau \theta_r} \begin{bmatrix} p^2 & pq \\ pq & q^2 \end{bmatrix} + e^{i\tau \theta_r} \begin{bmatrix} q^2 & -pq \\ -pq & p^2 \end{bmatrix}.
\]
These three conditions \( (\theta_r - \theta_s = m_r,s \sqrt{\Delta}) \) for \( (\theta_r, \theta_s) \in \Phi_{D_s} \), commutativity of \( \mathcal{A}(S) \), and parallelity of vertices \( a \) and \( b \) plus one more condition (that ensures the fractional revival is proper, that is, \( e^{i\tau \theta_r} \neq e^{i\tau \theta_s} \)) in fact form a characterization of when proper fractional revival occurs on a graph.

2.8 Theorem. Let \( X \) be a connected graph with adjacency matrix \( \Lambda = \sum_r \theta_r E_r \).
Proper fractional revival occurs on \( \{a, b\} \) in \( X \) if and only if the following conditions hold.

(a) \( a \) and \( b \) are parallel

(b) The induced algebra on \( \{a, b\} \) is commutative

(c) Let \( C^+ \) be the set of indices \( r \) such that \( (E_r)_{a,b} > 0 \), and let \( C^- \) be the set of indices \( r \) such that \( (E_r)_{a,b} < 0 \). Then there exist a square-free integer \( \Delta \) such that
\[
\theta_j = \rho_j \sqrt{\Delta}, \text{ for } j \in C^+ \text{, and } \theta_\ell = \omega_\ell \sqrt{\Delta}, \text{ for } \ell \in C^-,
\]
where \( \rho_j \)'s and \( \omega_\ell \)'s are real numbers satisfying
\[
\rho_j - \rho_{j'} \in \mathbb{Z} \text{ and } \omega_\ell - \omega_{\ell'} \in \mathbb{Z}
\]
for all \( j, j' \in C^+ \) and \( \ell, \ell' \in C^- \).

(d) Let \( g = \gcd \left\{ \frac{\theta_r - \theta_s}{\sqrt{\Delta}} : r, s \in C^+, \text{ or } r, s \in C^- \right\} \). There exists \( j \in C^+ \) and \( \ell \in C^- \) such that
\[
\frac{\rho_j - \omega_\ell}{g} \in \mathbb{Z}.
\]
Moreover, if these conditions hold, then \( \frac{2\pi}{g \sqrt{\Delta}} \) is the minimum time for proper fractional revival to occur on \( \{a, b\} \) in \( X \). Note that fractional revival on \( \{a, b\} \) occurs if and only if conditions (a), (b) and (c) hold.

2.3 Equitable partition

We review about equitable partitions of a graph [12]. Let \( X \) be a graph on \( n \) vertices. Let \( \mathcal{P} = \{C_1, \ldots, C_k\} \) be a partition of \( V(X) = \{1, \ldots, n\} \). The characteristic
matrix $P$ of $\mathcal{P}$ is an $n \times k$ matrix with $p_{j\ell} = 1$ if $j \in C_\ell$ and $p_{j\ell} = 0$ otherwise. The normalized characteristic matrix of the partition $\mathcal{P}$ is

$$\hat{P} = P \text{diag}(\frac{1}{\sqrt{|C_1|}}, \ldots, \frac{1}{\sqrt{|C_k|}}).$$

Note that $\hat{P}^T \hat{P} = I_k$ and $\hat{P} \hat{P}^T$ is the orthogonal projection onto the subspace of $\mathbb{R}^n$ consisting of vectors that are constant on cells of $\mathcal{P}$. A partition $\mathcal{P} = \{C_1, \ldots, C_k\}$ is equitable if for any $j, \ell$, the number of neighbors which a vertex in $C_j$ has in $C_\ell$ is independent of the choice of vertex in $C_j$. Denote this fixed number of neighbors as $c_{j\ell}$. Given an equitable partition $\mathcal{P} = \{C_1, \ldots, C_k\}$ of a graph $X$, a weighted graph can be associated to it. Let $\tilde{X}/\mathcal{P}$ be the weighted graph with the cells of $\mathcal{P}$ as its vertices, with the edge between $C_j$ and $C_\ell$ of weight $\sqrt{c_{j\ell}|C_j|}$ (if $C_j \cap C_\ell = 0$ then $C_j$ and $C_\ell$ are not adjacent). $\tilde{X}/\mathcal{P}$ is called the symmetrized quotient graph of $X$ with respect to $\mathcal{P}$.

Eigenvectors of $A(\tilde{X}/\mathcal{P})$ and of $A(\tilde{X}/\mathcal{P})$ are related.

2.9 Lemma. [12] Let $\mathcal{P}$ be an equitable partition of the graph $X$ with $k$ cells. Assume $\hat{P}$ is the normalized characteristic matrix of $\mathcal{P}$. Then

(a) If $A(\tilde{X}/\mathcal{P})x = \theta x$, then $A(X)\hat{P}x = \theta \hat{P}x$.

(b) If $A(X)y = \theta y$, then $A(\tilde{X}/\mathcal{P})\hat{P}^Ty = \theta \hat{P}^Ty$

Given an orthonormal basis $x_1, \ldots, x_k$ of $\mathbb{R}^k$ that consists of eigenvectors of $A(\tilde{X}/\mathcal{P})$, part (a) of the above lemma tells us that $\hat{P}x_1, \ldots, \hat{P}x_k$ is a list of orthonormal eigenvectors of $A(X)$:

$$\langle \hat{P}x_j, \hat{P}x_\ell \rangle = \langle x_j, \hat{P}^T \hat{P}x_\ell \rangle = \langle x_j, I_kx_\ell \rangle = \delta_{j,\ell}.$$

We can further extend the list to an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A(X)$, say $\hat{P}x_1, \ldots, \hat{P}x_k, y_{k+1}, \ldots, y_n$. Then for any $j \in \{k+1, \ldots, n\}$, we have $\hat{P}^Ty_j = 0$. In fact $\langle \hat{P}^T y_j, x_\ell \rangle = \langle y_j, \hat{P}x_\ell \rangle = 0$ for all $\ell \in \{1, \ldots, k\}$ implies that $\hat{P}^T y_j$ lies in the orthogonal complement of $\text{Span}\{x_1, \ldots, x_k\}$, which is $\{0\}$.

3 Stellar Fusion

Let $a, k$ and $c$ be positive integers. Let $X = X(a, k, c)$ denote the graph obtained by taking a copy of the star $K_{1,a+k}$ and a copy of the star $K_{1,c+k}$ and merging $k$
Let $\mathcal{P}$ be a partition of $V(X)$, where the two centers of the two stars, denoted by 0 and 1, respectively, are both singletons of $\mathcal{P}$, the $k$ common neighbours of 0 and 1 form a cell, the remaining $a$ neighbours of 0 form a cell, and the $c$ remaining neighbours of vertex 1 form a cell, as shown in Figure 1. This is an equitable partition, and the corresponding symmetrized quotient graph is a weighted path on 5 vertices, as shown in Figure 2, which we denote by $\hat{X}$.

**3.1 Lemma.** For any positive integers $a, k, c$, vertices 0 and 1 are parallel and the induced algebra of $X = X(a, k, c)$ on $\{0, 1\}$ is commutative. Moreover, if $C^+$ and $C^-$ are defined as in Theorem 2.8 and $\mu = 2k + a + c$ and $\sigma = 4k^2 + (a - c)^2$, then $C^+ = \left\{ -\sqrt{\frac{\mu + \sqrt{\sigma}}{2}}, \sqrt{\frac{\mu + \sqrt{\sigma}}{2}} \right\}$ and $C^- = \left\{ -\sqrt{\frac{\mu - \sqrt{\sigma}}{2}}, \sqrt{\frac{\mu - \sqrt{\sigma}}{2}} \right\}$.

**Proof.** Let $\hat{X}$ be the symmetrized quotient graph of $X$ with respect to the above equitable partition $\mathcal{P}$. Let $\mu = 2k + a + c$ and $\sigma = 4k^2 + (a - c)^2$. Then the eigen-
values of $\hat{X}$ are

$$\begin{align*}
\theta_1 &= 0, \\
\theta_2 &= -\sqrt{\frac{\mu - \sqrt{\sigma}}{2}}, \\
\theta_3 &= \sqrt{\frac{\mu - \sqrt{\sigma}}{2}}, \\
\theta_4 &= -\sqrt{\frac{\mu + \sqrt{\sigma}}{2}}, \\
\theta_5 &= \sqrt{\frac{\mu + \sqrt{\sigma}}{2}}.
\end{align*}$$

(3.1)

with corresponding eigenvectors (with vertices ordered as $u_1, 0, u_2, 1, u_3$)

$$v_1 = \left[ \frac{\sqrt{c}}{\sqrt{a}}, 0, -\frac{\sqrt{c}}{\sqrt{k}}, 0, 1 \right]^T,$$

$$v_2 = \begin{bmatrix}
-\frac{c \sqrt{\alpha-a^{3/2}+\sqrt{\alpha}}}{2k\sqrt{c}} \\
\frac{\sqrt{\alpha-\sqrt{\alpha}(c-a+\sqrt{\alpha})}}{2\sqrt{2k\sqrt{c}}} \\
-\frac{2k+c-a+\sqrt{\alpha}}{2\sqrt{c}} \\
\frac{\sqrt{\mu-\sqrt{\alpha}}}{\sqrt{2c}} \\
1
\end{bmatrix},$$

$$v_3 = \begin{bmatrix}
-\frac{c \sqrt{\alpha-a^{3/2}+\sqrt{\alpha}}}{2k\sqrt{c}} \\
\frac{\sqrt{\alpha-\sqrt{\alpha}(c-a+\sqrt{\alpha})}}{2\sqrt{2k\sqrt{c}}} \\
-\frac{2k+c-a+\sqrt{\alpha}}{2\sqrt{c}} \\
\frac{\sqrt{\mu-\sqrt{\alpha}}}{\sqrt{2c}} \\
1
\end{bmatrix},$$

(3.2)

$$v_4 = \begin{bmatrix}
-\frac{c \sqrt{\alpha-a^{3/2}+\sqrt{\alpha}}}{2k\sqrt{c}} \\
\frac{\sqrt{\alpha-\sqrt{\alpha}(c-a+\sqrt{\alpha})}}{2\sqrt{2k\sqrt{c}}} \\
-\frac{2k+c-a+\sqrt{\alpha}}{2\sqrt{c}} \\
\frac{\sqrt{\mu+\sqrt{\alpha}}}{\sqrt{2c}} \\
1
\end{bmatrix},$$

$$v_5 = \begin{bmatrix}
-\frac{c \sqrt{\alpha-a^{3/2}+\sqrt{\alpha}}}{2k\sqrt{c}} \\
\frac{\sqrt{\alpha-\sqrt{\alpha}(c-a+\sqrt{\alpha})}}{2\sqrt{2k\sqrt{c}}} \\
-\frac{2k+c-a+\sqrt{\alpha}}{2\sqrt{c}} \\
\frac{\sqrt{\mu+\sqrt{\alpha}}}{\sqrt{2c}} \\
1
\end{bmatrix}.$$

As the characterization of proper fractional revival in Theorem 2.8 is for unweighted graphs (in fact it works for integer weighted graphs but not for all weighted graphs), we now use Lemma 2.9 and the comments below it to further get eigenvector information of $A(X)$. First let $\hat{P}$ be the normalized characteristic matrix of the equitable partition $\mathcal{P}$ and for $j = 1, \ldots, 5$ let $(\theta_j, v_j)$ be an eigenpair of $A(\hat{X})$ as above with $v_j$ scaled to have norm 1. Then $u_j = \hat{P}v_j$ is an eigenvector of $A(X)$ associated to $\theta_j$, $j = 1, \ldots, 5$. Now assume that $u_1, \ldots, u_5, u_6, \ldots, u_{a+k+c+2}$ is an orthonormal basis of $\mathbb{R}^{a+k+c+2}$ consisting of eigenvectors of $A(X)$. For $j = 6, \ldots, a + k + c + 2$, the eigenvector $u_j$ is associated to eigenvalue 0 (as $\text{rk}(A(X)) = 4$), and its components on each cell of $\mathcal{P}$ sum up
to 0 (as \( \hat{P}^T u_j = 0 \)). Therefore the components of \( u_j \) corresponding to vertices 0 and 1 are both 0, as the two vertices are singletons of the partition. Since \( u_1 = P v_1 \) also has the two corresponding components 0, vertex 0 and vertex 1 have the same eigenvalue support, and the support is \( \Phi = \{ \theta_2, \theta_3, \theta_4, \theta_5 \} \). Furthermore, as \( \theta_2, \theta_3, \theta_4 \) and \( \theta_5 \) are all simple eigenvalues of \( A(X) \), we know that vertex 0 and vertex 1 are parallel and orthogonal projection onto the eigenspace associated to \( \theta_j \) of \( A(X) \) is \( E_j = u_j u_j^T = \hat{P} v_j v_j^T \hat{P}^T \) for \( j = 2, 3, 4, 5 \). Now observe that \( (E_r)_{0,0} - (E_r)_{1,1} \) is a fixed number for \( r = 2, 3, 4, 5 \); by Theorem 2.5, the induced algebra \( \mathcal{A}((0,1)) \) is commutative. Since \( (E_j)_{0,1} > 0 \) for \( j = 4, 5 \) and \( (E_j)_{0,1} < 0 \) for \( j = 2, 3 \), we have \( C^+ = \{ \theta_4, \theta_5 \} \) and \( C^- = \{ \theta_2, \theta_3 \} \).

For a rational number \( x \), let \( |x|_2 \) denote the 2-adic norm of \( x \).

3.2 Lemma. Let \( X = X(a, k, c) \) and let \( \theta_3 \) and \( \theta_5 \) be the two eigenvalues of \( X \) as given in (3.1). Then \( X \) admits fractional revival on \( \{0, 1\} \) if and only if \( \theta_3^2 \) and \( \theta_5^2 \) are both integers with the same square-free part \( \Delta \) and further proper fractional revival occurs if and only if \( \frac{\theta_3}{\sqrt{\Delta}} \notin \mathbb{Z}, \frac{\theta_5}{\sqrt{\Delta}} \notin \mathbb{Z} \).

Proof. From Lemma 3.1 we know that the induced algebra \( \mathcal{A}((0,1)) \) is commutative and vertices 0 and 1 are parallel with the same eigenvalue support \( C^+ \cup C^- \), where \( C^+ = \{ \theta_3, \theta_4 = -\theta_5 \} \) and \( C^- = \{ \theta_2, \theta_3 = -\theta_3 \} \). Condition (c) in Theorem 2.8 is equivalent to both \( 2\theta_3 \) and \( 2\theta_5 \) being integer multiples of \( \sqrt{\Delta} \) for some square-free integer \( \Delta \), which holds if and only if

\[
\frac{\theta_3}{\sqrt{\Delta}}, \frac{\theta_5}{\sqrt{\Delta}} \in \mathbb{Z}.
\]

In fact, for \( j = 3, 5 \), there exists an integer \( m_j \) such that \( 2\theta_j = m_j \sqrt{\Delta} \) if and only if \( 4\theta_j^2 = m_j^2 \Delta \), if and only if \( \theta_j^2 = m_j^2 \Delta/4 \). As \( \theta_j \) is an algebraic integer and \( m_j^2 \Delta/4 \) is a rational number, we conclude that \( \theta_j^2 = m_j^2 \Delta/4 \) is an integer. Since \( \Delta \) is square-free, we know \( m_j^2 \) is an even integer, hence \( m_j \) is an even integer and \( \theta_j \) is an integer multiple of \( \sqrt{\Delta} \). Now assume that

\[
\theta_3 = \alpha \sqrt{\Delta}, \theta_5 = \beta \sqrt{\Delta}
\]

for some integers \( \alpha \) and \( \beta \). Since fractional revival occurs if and only if condition (a), (b) and (c) of Theorem 2.8 hold, we have proved the first claim. For proper fractional revival to occur, we just need to show that condition (d) in
Theorem 2.8 holds if and only if the multiplicities of 2 as a factor of $\alpha$ and $\beta$ are different. With the notation of Theorem 2.8

$$g = \gcd\left\{\frac{\theta_r - \theta_s}{\sqrt{\Delta}} : r, s \in C^+, \text{ or } r, s \in C^-\right\}$$

$$= \gcd\left\{\frac{\theta_3 - \theta_2}{\sqrt{\Delta}}, \frac{\theta_5 - \theta_4}{\sqrt{\Delta}}\right\}$$

$$= 2 \gcd\{\alpha, \beta\}$$

and $\rho_{a_1-a_2} = \frac{a_1-a_2}{2 \gcd(a, pr)}$, which is not an integer if and only if the multiplicities of 2 as a factor in $\alpha$ and $\beta$ are distinct, that is, $|\alpha|_2 \neq |\beta|_2$. In this case, the minimum time for proper fractional revival to occur is $\tau = \frac{\pi}{\gcd(a, \beta) \sqrt{\Delta}}$.

Recall that a graph $X$ is said to be periodic if there is a time $t$ such that $D_a$ is periodic at time $t$ for all $a \in V(X)$.

3.3 Corollary. If $X(a, k, c)$ admits fractional revival between vertices 0 and 1, then $X$ is periodic with minimum period $t = \frac{2\pi}{g \sqrt{\Delta}}$, where $g = \gcd\{\theta_3, \theta_5 / \sqrt{\Delta}\}$.

Proof. From Corollary 3.3 of [13], we know that a graph $X$ with adjacency matrix $A = \sum_{r=1}^{m} \theta_r E_r$ is periodic if and only if all its eigenvalues are integer multiples of $\sqrt{\Delta}$ for a square-free integer $\Delta$ (if $\Delta = 1$ if all the eigenvalues are integers); in this case the minimum period for $X$ is $t = (2\pi) / (g \sqrt{\Delta})$, where $g$ is the greatest common divisor of $\{\theta_r / \sqrt{\Delta}\}_{r=1,...,m}$.

The graph $X(a, k, c)$ has exactly 5 distinct eigenvalues as shown in (3.1), with $\theta_2, \theta_3 = -\theta_2, \theta_4, \theta_5 = -\theta_4$ all simple, and $\theta_1 = 0$. Now the result follows from Lemma 3.2.

3.4 Corollary. If proper fractional revival occurs on $\{0, 1\}$ in $X = X(a, k, c)$ at time $\tau$, then it also occurs at $(2j + 1)\tau$ for all positive integer $j$, and $X$ is periodic at $2j \tau$ for all positive integer $j$.

Proof. From Lemma 3.2 and Corollary 3.3, we know that the minimum time for fractional revival on $\{0, 1\}$ to occur in $X(a, k, c)$ is $\tau = (\pi) / (g \sqrt{\Delta})$ and the minimum period is $t = 2\tau = (2\pi) / (g \sqrt{\Delta})$, where $g = \gcd\{\theta_3 / \sqrt{\Delta}, \theta_5 / \sqrt{\Delta}\} = \gcd\{\alpha, \beta\}$. Both $U(\tau)$ and $U(2\tau)$ are block diagonal, with $U(2\tau)|_{\{0, 1\}, \{0, 1\}}$ being a scalar matrix, therefore $U(2j \tau)$ is block diagonal with the $2 \times 2$ block being a scalar matrix, and $U((2j + 1)\tau)$ is block diagonal with the $2 \times 2$ block not a scalar matrix.
Note that there exist parameters $a, k, c$ such that the graph $X(a, k, c)$ is periodic, but does not admit proper fractional revival.

3.5 Example. Let $X_1 = X(1, 4, 1)$, then $\theta_3 = 1$ and $\theta_5 = 5$. Therefore $|\theta_3|_2 = |\theta_5|_2 = 1$. So $X$ is periodic but does not admit proper fractional revival, by Lemma 3.2 and Corollary 3.3. Similarly, $X_2 = X(1, 16, 25)$ is also such a graph, with $\theta_3 = 3$ and $\theta_5 = 7$.

3.6 Corollary. $X(a, k, c)$ admits proper fractional revival if and only if there exist integers $\alpha, \beta$ with $|\alpha|^2 \neq |\beta|^2$ and a square-free integer $\Delta$, such that

$$\Delta(\beta^2 - \alpha^2) = \sqrt{4k^2 + (a - c)^2},$$

(3.3)

$$\Delta(\alpha^2 + \beta^2) = 2k + a + c.$$  
(3.4)

Proof. From Lemma 3.2 we know that $X$ admits proper fractional revival on $\{0, 1\}$ if and only if

$$\theta_3^2 = \frac{1}{2}(2k + a + c - \sqrt{4k^2 + (a - c)^2}) = \alpha^2 \Delta$$  
(3.5)

and

$$\theta_5^2 = \frac{1}{2}(2k + a + c + \sqrt{4k^2 + (a - c)^2}) = \beta^2 \Delta$$  
(3.6)

for some integers $\alpha, \beta$ and $\Delta$, with $\Delta$ square-free, and $|\alpha|^2 \neq |\beta|^2$.

The result follows as equations (3.5) and (3.6) are equivalent to (3.3) and (3.4). \qed

Now we show that there are triples $(a, k, c)$ such that $X(a, k, c)$ admits proper fractional revival.

3.7 Theorem. There exist graphs which admit proper fractional revival between non-cospectral vertices, and there are infinitely many such graphs.

Proof. We prove the result by showing that there are infinitely many positive integer triples $(a, k, c)$ such that proper fractional revival on $\{0, 1\}$ occurs in $X(a, k, c)$ with vertices 0 and 1 non-cospectral. Note that vertices 0 and 1 are non-cospectral in $X(a, k, c)$ if and only if $a \neq c$. Without loss of generality, assume $a < c$. Now we make use of Corollary 3.6. Let $p$ be a prime such that $p \equiv 1 \pmod{4}$, then by Fermat’s Theorem on sums of two squares, $p = f^2 + g^2$ for some positive integers $f > g$. Now we pick integers
\(\Delta\) (square-free) and \(\alpha < \beta\), such that \(|\alpha|_2 \neq |\beta|_2\) and \(p | \Delta (\beta^2 - \alpha^2)\). Let \(d\) be the integer satisfying \(\Delta (\beta^2 - \alpha^2) = pd = (f^2 + g^2)d\). Then the triple \((a, k, c)\) with
\[
\begin{align*}
k &= fgd \\
c &= \frac{\Delta (\beta^2 + \alpha^2) + (f^2 - g^2)d}{2} - fgd = \Delta \alpha^2 + f(d - g) \\
a &= \frac{\Delta (\beta^2 + \alpha^2) - (f^2 - g^2)d}{2} - fgd = \Delta \alpha^2 - g(d - g)
\end{align*}
\]
is a set of integer solutions of equations (3.5) and (3.6). If further
\[
\frac{\beta}{\alpha} < \sqrt{2} + 1
\]
then \(a, k, c\) are all positive integers. Let \(\tau = \frac{\pi}{\sqrt{\Delta \gcd(\alpha, \beta)}}\), then there is proper fractional revival on \([0, 1]\) at time \(\tau\). There are infinitely many such graphs, as we have infinitely many choices for \(p, \Delta, \alpha, \beta\).

In particular, let \(p = 5\), then \(f = 2\) and \(g = 1\). If we let

- let \(\Delta = 5\), then we can pick any positive integer \(\alpha\) and \(\beta = 2\alpha\). In this case, \(a = 2\alpha^2, k = 6\alpha^2\) and \(c = 11\alpha^2\).

- let \(\beta = 3\) and \(\alpha = 2\), then we can pick any square free integer \(\Delta\). In this case, \(a = 3\Delta, k = 2\Delta\) and \(c = 6\Delta\).

- let \(\beta = 6\) and \(\alpha = 4\), then we can pick any square free integer \(\Delta\). In this case, \(a = 12\Delta, k = 8\Delta\) and \(c = 24\Delta\).

- let \(\beta = 8\) and \(\alpha = 7\), then we can pick any square free integer \(\Delta\). In this case, \(a = 46\Delta, k = 6\Delta\) and \(c = 55\Delta\).

3.8 Example. Graph \(X(3, 2, 6)\) in Figure 3 admits proper fractional revival between (non-cospectral) vertices 0 and 1 at time \(\pi\). Other examples include:
\(X(6, 3, 14)\) at \(t = \pi / \sqrt{2}\), \(X(6, 4, 12)\) at \(t = \pi / \sqrt{3}\), \(X(9, 6, 18)\) at \(t = \pi / \sqrt{5}\), \(X(2, 6, 11)\) at \(t = \pi / \sqrt{5}\), \(X(2, 6, 28)\) at \(t = \pi / 2\).

3.9 Remark. When \(k = 1\), (3.3) implies that 4 is the difference of two nonzero integer squares (if \(a \neq c\)) or that \(2 = \Delta (\beta^2 - \alpha^2)\) with \(\Delta, \alpha, \beta \in \mathbb{Z}\) (if \(a = c\)). But there are no such integer pairs. Therefore, tree graphs in this family do not admit proper fractional revival.
Nevertheless, there is an infinite family of trees that admits proper fractional revival between cospectral vertices: Let $X$ be the graph obtained by connecting the two centers of two copies of the star $K_{1,a}$. Label the two centers vertices 0 and 1, respectively. Then $X$ admits proper fractional revival on \{0, 1\} at time $t = \frac{2\pi}{\sqrt{4a+1}}$.

3.1 Commutative algebra

In Lemma [3.1], we show that for the graph $X = X(a, k, c)$ the induced algebra on \{0, 1\} is commutative by using the spectral decomposition of $A(X)$ and verifying that $(E_r)_{0,0} - (E_r)_{1,1}$ is independent of $r$ (given that $\theta_r$ lies in the eigenvalue support of the vertex 0). As a computational technique, this has serious difficulties because the entries of $E_r$ will be floating point numbers. We show how to carry out this computation in exact arithmetic, using the polynomial condition in Theorem [2.5](d).

The characteristic polynomial of $X(a, k, c)$ is:

$$t^{a+k+c-2}(t^4 - (a + 2k + c)t^2 + ak + ck + ac).$$

Recall that the vertex of degree $a + k$ is denoted by 0, and the vertex of degree $c + k$ is denoted by 1. The characteristic polynomials of $X \setminus 0$ and $X \setminus 1$ are, respectively,

$$t^{a+k+c-2}(t^2 - c - k), \quad t^{a+k+c-2}(t^2 - a - k).$$

Finally

$$\phi(X \setminus \{0, 1\}, t) = t^{a+k+c}, \quad \psi_{0,1}(X, t) = kt^{a+k+c-1}.$$  

We see that $\phi(X \setminus 0, t) - \phi(X \setminus 1, t) = \frac{a-c}{k}\psi_{0,1}(X, t)$, therefore by Theorem [2.5], the induced algebra of $X$ on \{0, 1\} is commutative.
From
\[(tI - A)^{-1})_{a,b} = \frac{(adj(tI - A))_{b,a}}{\phi(X,t)}\]
and
\[(tI - A)^{-1})_{a,b} = \sum_r \frac{(E_r)_{a,b}}{t - \theta_r},\]
we get that for each \(r\),
\[(E_r)_{0,1,0,1} = \lim_{t \to \theta_r} \frac{t - \theta_r}{\phi(X,t)} \begin{bmatrix} \phi(X \setminus 0, t) & \psi_{0,1}(X, t) \\ \psi_{0,1}(X, t) & \phi(X \setminus 1, t) \end{bmatrix} \]
Note that \((E_r)_{1,1}\) is the coefficient of \(1/(t - \theta_r)\) in the partial fraction expansion of \(\frac{\phi(X \setminus 1, t)}{\phi(X,t)}\), and the other entries have similar expressions.

By way of example, for the triple \((3, 2, 6)\),
\[
\phi(X \setminus 0, t) = \frac{1}{10} \left( \frac{1}{t-3} + \frac{4}{t-2} + \frac{4}{t+2} + \frac{1}{t+3} \right),
\]
\[
\psi_{0,1}(X, t) = \frac{2}{10} \left( \frac{1}{t-3} - \frac{1}{t-2} - \frac{1}{t+2} + \frac{1}{t+3} \right),
\]
\[
\phi(X \setminus 1, t) = \frac{1}{10} \left( \frac{4}{t-3} + \frac{1}{t-2} + \frac{1}{t+2} + \frac{4}{t+3} \right).
\]
The eigenvalues of \(X\), as denoted in (3.1), are \(\theta_1 = 0, \theta_2 = -2, \theta_3 = 2, \theta_4 = -3, \theta_5 = 3, \) and
\[
(E_2)_{0,1,0,1} = (E_3)_{0,1,0,1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix},
(E_4)_{0,1,0,1} = (E_5)_{0,1,0,1} = \frac{1}{10} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.
\]
For any positive integer \(m\), we get exactly the same matrices for the triples \((3m, 2m, 6m)\), and so all graphs in this family admit proper fractional revival.

### 4 No balanced fractional revival on \(\{0, 1\}\) in \(X(a, k, c)\)

Again, let \(a \leq c\), here we consider whether balanced fractional revival on \(\{0, 1\}\) could occur in \(X(a, k, c)\), that is, whether there is a time \(\tau\) such that \(|U(\tau)_{0,0}| = |U(\tau)_{0,1}| = 1/\sqrt{2}\). Let \(X\) be a simple connected graph. Assume proper fractional revival occurs on \(\{1, 2\}\) at time \(\tau\) in \(X\). Then \(U(\tau)_{1,2,1,2}\) is unitary. Take \(\theta_1 \in C^+, \theta_2 \in C^-\), then by equation (2.3)
\[
U(\tau)_{1,2,1,2} = e^{i\tau \theta_1} \begin{bmatrix} p^2 + e^{i\tau \theta} q^2 & pq(1 - e^{i\tau \theta}) \\ pq(1 - e^{i\tau \theta}) & q^2 + e^{i\tau \theta} p^2 \end{bmatrix},
\]

for some positive numbers $p \geq q$ with $p^2 + q^2 = 1$, where $\theta = \theta_2 - \theta_1$.

Now, balanced fractional revival occurs at time $\tau$ if and only if $\frac{1}{2} = |pq(1 - e^{r\tau\theta})|^2 = 2p^2q^2(1 - \cos(\tau\theta))$, which implies that
\[
\cos(\tau\theta) = 1 - \frac{1}{4p^2q^2}
= \frac{4p^2q^2 - (p^2 + q^2)^2}{4p^2q^2} \quad (\text{as } p^2 + q^2 = 1)
= -\frac{(p^2 - q^2)^2}{4p^2q^2}
= -\frac{1}{4}\left(\frac{p}{q} - \frac{q}{p}\right)^2 \in \mathbb{Q}, \tag{4.1}
\]
where the rationality comes from the fact that $(E_r)_{1,1} - (E_r)_{2,2} = (\frac{p}{q} - \frac{q}{p})(E_r)_{1,2}$ for each $r$, and therefore $\frac{p}{q} - \frac{q}{p}$ is the rational scalar $\gamma$ in Theorem 2.5.

Now assume that $X$ is also periodic at vertices 1,2, say at time $\tau_1$. Then $e^{i\tau_1\theta_1} = e^{i\tau_1\theta_2}$, that is $e^{i\tau_1\theta} = 1$, or equivalently, $\tau_1\theta = 2k\pi$ for some integer $k$. Assume that the minimum time when proper fractional revival occurs on $\{1,2\}$ is $\tau_0$, then $\tau = r\tau_0$ and $\tau_1 = r_1\tau_0$ for some $r, r_1 \in \mathbb{Z}$. Therefore $\frac{r}{r_1} \in \mathbb{Q}$, and $\tau\theta$ is a rational multiple of $\pi$, with $\cos(\tau\theta) \in \mathbb{Q}$, which implies that $\cos(\tau\theta) \in [0, \frac{1}{2}, -1]$ as $\cos(\tau\theta) \leq 0$. But equation (4.1) in fact tells us that $\sqrt{-\cos(\tau\theta)} \in \mathbb{Q}$. Therefore $\cos(\tau\theta) \in [0, -1]$.

If $\cos(\tau\theta) = 0$, then $\gamma = \frac{p}{q} - \frac{q}{p} = 0$ and vertices 1 and 2 are cospectral. It can be shown that $X$ admits perfect state transfer between 1 and 2 at time $2\tau$ in this case [5].

If $\cos(\tau\theta) = -1$, then $e^{i\tau\theta} = -1$, $e^{i2\tau\theta} = 1$ and $X$ is periodic at vertices 1,2 at time $2\tau$. In this case, $\gamma = \frac{p}{q} - \frac{q}{p} = 2$, that is, $(A^k_{1,1} - A^k_{2,2}) = 2A^k_{1,2}$ for any positive integer $k$. In fact, $\frac{p}{q} = 1 + \sqrt{2}$ in this case. The achieved results are summarized in the following lemma.

4.1 Lemma. Assume that graph $X$ admits balanced fractional revival on $\{a, b\}$ at time $\tau$ and non-proper fractional revival on $\{a, b\}$ at time $t$. Then either vertices $a$ and $b$ are cospectral and $X$ admits perfect state transfer between them at time $2\tau$, or $a$ and $b$ are non-cospectral and $X$ is periodic at them at time $2\tau$. In the second case, $(A^k)_{a,a} - A^k_{b,b} = 2A^k_{a,b}$, for all nonnegative integer $k$, that is, the scalar $\gamma$ in Theorem 2.5 satisfies $\gamma = 2$.

4.2 Corollary. $X(a, k, c)$ does not admit balanced fractional revival on $\{0, 1\}$ for any positive integers $a, k, c$. 
Proof. From Lemma 4.1 we know that if balanced fractional revival occurs between vertices 0 and 1, then \( \frac{p}{q} = 1 + \sqrt{2} \notin \mathbb{Q} \). From equation (3.2) we know \( \frac{p}{q} = -c + a - \sqrt{4k^2 + (a-c)^2} \), which is a rational number by Corollary 3.6, a contradiction.

5 Fractional revival is polygamous

Let \( X \) be a graph. If there is perfect state transfer from \( a \) to \( b \) and from \( a \) to \( c \), then \( b = c \). For fractional revival, weighted graphs where there is proper fractional revival on \( \{a, b\} \) and on \( \{a, c\} \) with \( a, b, c \) distinct vertices of \( X \) were constructed [4], but no unweighted graph examples were known. Here we construct unweighted graphs where there is proper fractional revival on \( \{a, b\} \) and on \( \{a, c\} \) for \( b \neq c \).

Assume that \( X \) is a graph on \( m \) vertices with adjacency matrix \( A(X) \), and \( Y \) is a graph on \( n \) vertices with adjacency matrix \( A(Y) \). Their Cartesian product \( X \Box Y \) is the graph with adjacency matrix \( A(X \Box Y) = I_m \otimes A(Y) + A(X) \otimes I_n \), and the transition matrix of continuous time quantum walk on \( X \Box Y \) at time \( \tau \) is

\[
U_{X \Box Y}(\tau) = U_X(\tau) \otimes U_Y(\tau).
\]

Therefore if there is a time \( t \) such that \( X \) is periodic at vertex \( x_1 \) and graph \( Y \) admits proper fractional revival on \( \{y_1, y_2\} \), then \( X \Box Y \) admits proper fractional revival on \( \{(x_1, y_1), (x_1, y_2)\} \) at time \( t \). Now we take \( X \) to be \( K_2 \), which is periodic at integer multiples of \( \pi \), admits perfect state transfer at odd integer multiples of \( \pi/2 \), and admits proper fractional revival at all other time. By (5.1) we know that if for some positive integer \( \ell \) we can find a graph \( Y \) that admits proper fractional revival on \( \{a, b\} \) at time \( \tau = \frac{\pi}{2\ell+1} \) and is periodic at vertices \( a, b \) at time \( 2\tau \), then \( K_2 \Box Y \) admits fractional revival between vertices \( \{0, a\} \) and \( \{1, a\} \) at time \( 2\tau \), and admits proper fractional revival on \( \{0, a\} \) and \( \{0, b\} \) at time \( t = \pi = (2\ell + 1)\tau \).

Corollary 3.4 tells us that if proper fractional revival occurs on \( \{0, 1\} \) at time \( \tau \) in \( X(a, k, c) \), then it also occurs at time \( (2j + 1)\tau \), and \( X(a, k, c) \) is periodic at vertices 0 and 1 at time \( 2j\tau \) for any positive integer \( j \). The graph \( X_1 = X(16, 36, 37) \) admits fractional revival between vertex 0 and 1 at \( \frac{\pi}{3} \), the graph \( X_2 = X(10, 30, 55) \) admits fractional revival at \( \frac{\pi}{5} \), and the graph \( X_3 = X(27, 18, 54) \) admits fractional revival at time \( \frac{\pi}{7} \). In fact there are infinitely many such graphs.

5.1 Theorem. There are infinitely many unweighted graphs where the fractional revival pairs overlap (fractional revival is polygamous).
Proof. We show that there are infinitely many graphs in the family $X = X(a, k, c)$ such that $X$ admits fractional revival at time $\frac{\pi}{2\ell+1}$ for some integer $\ell$. Let $p$ be a prime such that $p \equiv 1 \pmod{4}$. Then $p = f^2 + g^2$ for some $f > g$ by Fermat’s theorem. Let $r$ be any positive integer. Then the graph $X(a, k, c)$ with

$$a = p^2 r^2 - gp(2r + 1)(f - g), k = fg p(2r + 1), c = p^2 r^2 + fp(2r + 1)(f - g)$$

admits proper fractional revival at time $\tau = \frac{\pi}{p}$. By the argument before the theorem, we know that $K_2 \square X$ admits proper fractional revival with overlapping pairs.

\[\square\]

6 Further questions

We showed in Corollary 4.2 that balanced fractional revival does not occur in $X(a, k, c)$, the only class of graphs known where proper fractional revival occur between non-cospectral vertices. Apart from looking for other families, we wonder:

6.1 Question. Is there a tree $T$ where there is proper fractional revival between two non-cospectral vertices $a$ and $b$?

6.2 Question. Can balanced fractional revival occur between non-cospectral vertices?

Assume proper fractional revival occurs on $\{a, b\}$. Then there exists a square-free integer $\Delta$ such that $\frac{\theta_r - \theta_s}{\sqrt{\Delta}} \in \mathbb{Z}$ for any $\theta_r, \theta_s \in C^+$ or any $\theta_r, \theta_s \in C^-$. If further $a$ and $b$ are cospectral, then the two sets $C^+$ and $C^-$ are both closed under taking algebraic conjugates, and therefore there exists a square-free integer $\Delta$ and integers $a^+, a^-, b_r$ such that $\theta_r = \frac{a^+ b_r + \sqrt{\Delta}}{2}$ for any $\theta_r \in C^+$, and $\theta_s = \frac{a^- b_r + \sqrt{\Delta}}{2}$ for any $\theta_s \in C^-$. When proper fractional revival occurs on $X(a, k, c)$, the eigenvalues in the support are quadratic integers as in the cospectral case. Are there unweighted graphs with proper fractional revival such that $C^+$ and $C^-$ are not closed under taking algebraic conjugates (then it occurs between non-cospectral vertices), or

6.3 Question. Is there a graph $X$ with proper fractional revival on $\{a, b\}$ such that eigenvalues in the support of $a$ are not integers or quadratic integers? Vertices $a$ and $b$ must be noncospectral in $X$ if it happens.
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