Absolutely convex sets of large Szlenk index

Philip A.H. Brooker

September 27, 2018

Abstract

Let $X$ be a Banach space and $K$ an absolutely convex, weak$^*$-compact subset of $X$. We study consequences of $K$ having a large or undefined Szlenk index, and subsequently derive a number of related results concerning basic sequences and universal operators. We show that if $X$ has a countable Szlenk index then $X$ admits a subspace with a basis and with Szlenk indices comparable to the Szlenk indices of $X$. If $X$ is separable, then $X$ also admits a quotient with these same properties. We also show that for a given ordinal $\xi$ the class of operators whose Szlenk index is not an ordinal less than or equal to $\xi$ admits a universal element if and only if $\xi < \omega_1$; W.B. Johnson’s theorem that the formal identity map from $\ell_1$ to $\ell_\infty$ is a universal non-compact operator is then obtained as a corollary. Stronger results are obtained for operators having separable codomain.

1 Introduction

The Szlenk index is an ordinal index that measures the difference between the norm and weak$^*$ topologies on subsets of dual Banach spaces. It was introduced by Szlenk in [38] to solve (in the negative) the problem of whether there exists a separable, reflexive Banach space whose subspaces exhaust the class of separable, reflexive Banach spaces up to isomorphism. Since then the Szlenk index has found many uses in the study of Banach spaces and their operators, as surveyed in [25] and [33].

In the current paper we study the Szlenk index in two main contexts, the first of these being the theory of basic sequences in Banach spaces. Our work to this end is based fundamentally on the classical method of Mazur for producing subspaces with a basis and, in turn, on the more recent method of Johnson and Rosenthal [21] (a dual version of Mazur’s techniques) for producing quotients with a basis. Our work on the Szlenk index and basic sequences extends previous work in this area by Lancien [24] and Dilworth-Kutzarova-Lancien-Randrianarivony [11].
The initial motivation for writing the current paper was a desire to study the Szlenk index in the context of the problem of finding universal elements for certain subclasses of the class \( L \) of all (bounded, linear) operators between Banach spaces. For operators \( T \in L(X, Y) \) and \( S \in L(W, Z) \), where \( W, X, Y \) and \( Z \) are Banach spaces, we say that \( S \) factors through \( T \) (or, equivalently, that \( T \) factors \( S \)) if there exist \( U \in L(W, X) \) and \( V \in L(Y, Z) \) such that \( VTU = S \). With this terminology, for a given subclass \( C \) of \( L \) we say that an operator \( \Upsilon \in C \) is universal for \( C \) if \( \Upsilon \) factors through every element of \( C \). Typically \( C \) will be the complement \( \mathcal{C} = \mathcal{C} \) of an operator ideal \( \mathcal{I} \) in the sense of Pietsch [32] (that is, \( \mathcal{C} \mathcal{I} \) consists of all elements of \( L \) that do not belong to \( \mathcal{I} \)), or perhaps the restriction \( \mathcal{J} \cap \mathcal{C} \mathcal{I} \) of \( \mathcal{C} \mathcal{I} \) to a large subclass \( \mathcal{J} \) of \( L \); e.g., \( \mathcal{J} \) might denote a large operator ideal or the class of all operators having a specified domain or codomain. One may think of a universal element of the class \( C \) as a minimal element of \( C \) that is ‘fixed’ or ‘preserved’ by each element of \( C \).

The notion of universality for a class of operators goes back to the work of Lindenstrauss and Pełczyński, who obtained the following result.

**Theorem 1.1** (Theorem 8.1 of [26]). Let \( X \) and \( Y \) be Banach spaces and suppose \( T : X \to Y \) is a non-weakly compact operator. Then \( T \) factors the (non-weakly compact) summation operator \( \Sigma : (a_n)_{n=1}^{\infty} \mapsto (\sum_{i=1}^{n} a_i)_{n=1}^{\infty} \) from \( \ell_1 \) to \( \ell_\infty \). In particular, \( \Sigma \) is universal for the class of non-weakly compact operators.

Since the publication of [26] a number of results in a similar spirit to Theorem 1.1 have appeared in the literature. Perhaps the most well-known is the following result of W.B. Johnson [20], which is a special case of Theorem 5.1 of the current paper.

**Theorem 1.2.** Let \( X \) and \( Y \) be Banach spaces and suppose \( T : X \to Y \) is a non-compact operator. Then \( T \) factors the (non-compact) formal identity operator from \( \ell_1 \) to \( \ell_\infty \). In particular, the formal identity operator from \( \ell_1 \) to \( \ell_\infty \) is universal for the class of non-compact operators.

Another universality result of note, due to C. Stegall, is the existence of a universal non-Asplund operator. The Asplund operators have several equivalent definitions in the literature; in the current paper we say that an operator \( T : X \to Y \) is Asplund if \( (T|_Z)^*(Y^*) \) is separable for any separable subspace \( Z \subseteq X \). We refer the reader to Stegall’s paper [37] for further properties and characterisations of Asplund operators. Stegall’s universal operator is defined in terms of the Haar system \( (h_n)_{n=0}^{\infty} \subseteq C(\{0,1\}^\omega) \), where each factor \( \{0,1\} \) is discrete and \( \{0,1\}^\omega \) is equipped with its compact Hausdorff product topology. For the purpose of stating Stegall’s result, we let \( \mu \) denote the product measure on \( \{0,1\}^\omega \) obtained by equipping each factor \( \{0,1\} \) with its discrete uniform probability measure and
let $H : \ell_1 \to L_\infty(\{0, 1\}^\omega, \mu)$ be defined by setting $Hx = \sum_{m=1}^{\infty} x(m)h_{m-1}$ for each $x = (x(m))_{m=1}^{\infty} \in \ell_1$.

**Theorem 1.3 (Theorem 4 of [36])**. Let $X$ and $Y$ be Banach spaces such that $X$ is separable and suppose $T \in \mathcal{L}(X, Y)$ is such that $T^*(Y^*)$ is nonseparable. Then $H$ factors through $T$.

Since the domain of $H$, namely $\ell_1$, is separable, and since $H^\ast$ has non-separable range, $H$ is a non-Asplund operator. It therefore follows from Theorem 1.3 that $H$ is a universal non-Asplund operator. We note that a different universal non-Asplund operator is obtained by the author of the current paper in [4], using the techniques used for studying the Szlenk index in the current paper.

Other universality results besides those mentioned above can be found in the work of Brooker [4], Cilia and Gutiérrez [7], Dilworth [10], Girardi and Johnson [14], Hinrichs and Pietsch [19], Oikhberg [31], and the Handbook survey on operator ideals by Diestel, Jarchow and Pietsch [9]. Theorem 1.2 above has been applied in the study of information-based complexity by Hinrichs, Novak and Woźniakowski [18].

We shall now outline the content of subsequent sections the current paper and the standard notation used throughout. In Section 2 we provide background knowledge and basic results on Szlenk indices, trees, operators acting on Banach spaces over trees, and connections between these topics.

In Section 3 we prove our key result for the Szlenk index, Theorem 3.3, from which a number of subsequent results in Sections 4, 5 and 6 follow. Theorem 3.3 establishes consequences of a dual Banach space containing a weak*-compact subset with Szlenk index larger than a given ordinal. Roughly speaking, Theorem 3.3 asserts that if a dual Banach space $X^\ast$ contains a weak*-compact subset of large Szlenk index, then $X$ and $X^\ast$ contain large tree structures having properties that bear witness to the phenomenon of having a large Szlenk index. Theorem 3.3 is subsequently applied in Section 4 to show if $X$ is an infinite dimensional Banach space with countable Szlenk index, then $X$ admits a subspace with a basis and with Szlenk indices comparable to the Szlenk indices of $X$ (Theorem 4.1). Under the additional assumption that $X$ is separable, Theorem 3.3 also yields the existence of a quotient of $X$ having a basis and with Szlenk indices comparable to the Szlenk indices of $X$ (Theorem 4.2). We then turn our attention to applying Theorem 3.3 to the study of universal operators for classes defined in terms of a strict lower bound for the Szlenk index. We show in particular that a universal operator exists in this setting if and only if the lower bound is countable (Theorem 5.1). Stronger universality results are achieved in Section 6 for operators having separable codomain. In Section 7 we investigate whether the techniques developed in Section 5 and Section 6 can be used to show the existence of universal operators


for classes of Asplund operators having an uncountable strict lower bound for the Szlenk index. Though we do not completely answer this question, we show that the techniques developed in the earlier sections of the paper cannot decide the existence of universal operators in this setting in ZFC.

We now outline the notation and terminology used in the current paper. We work with Banach spaces over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Typical Banach spaces are denoted by the letters $W$, $X$, $Y$ and $Z$, with the identity operator of $X$ denoted $Id_X$. We write $X^*$ for the dual space of $X$ and denote by $\iota_X$ the canonical embedding of $X$ into $X^{**}$. We define $B_X := \{ x \in X \mid \|x\| \leq 1 \}$ and $B_X^0 := \{ x \in X \mid \|x\| < 1 \}$. By a subspace of a Banach space $X$ we mean a linear subspace of $X$ that is closed in the norm topology. For a Banach space $X$, $C \subseteq X$ and $D \subseteq X^*$ we define $C^\perp := \{ x^* \in X^* \mid \forall x \in C, x^*(x) = 0 \}$ and $D^\perp = \{ x \in X \mid \forall x^* \in D, x^*(x) = 0 \}$. We denote by $[C]$ the norm closed linear hull of $C$ in $X$, with a typical variation on this notation being that for an indexed set $\{ x_i \mid i \in I \} \subseteq X$ we may write $[x_i]_{i \in I}$ or $[x_i \mid i \in I]$ in place of $\{ x_i \mid i \in I \}$. By $\overline{D}$ we denote the weak$^*$ closed linear hull of $D$ in $X^*$. We shall make use of the well-known fact that, for a Banach space $X$ and a sequence $(x^*_m)_{m=1}^\infty \subseteq X^*$, the quotient map $Q : X \rightarrow X/\bigcap_{m=1}^\infty \ker(x^*_m)$ has the property that $Q^*$ is an isometric weak$^*$-isomorphism of $(X/\bigcap_{m=1}^\infty \ker(x^*_m))^*$ onto $(x^*_m)_{m=1}^\infty$.

Operator ideals are denoted by script letters such as $\mathcal{I}$. Operator ideals of particular interest in the current paper are:

- $\mathcal{K}$, the compact operators;
- $\mathcal{W}$, the weakly compact operators;
- $\mathcal{X}$, the operators having separable range;
- $\mathcal{X}^*$, the operators whose adjoint has separable range;
- $\mathcal{D}$, the Asplund operators (also known as the decomposing operators); and,
- $\mathcal{I}_\alpha$, the $\alpha$-Szlenk operators for a given ordinal $\alpha$.

All of the operator ideals in the list above are closed, and most of them are well known. For a given ordinal $\alpha$, the class $\mathcal{I}_\alpha$ consists of all operators whose Szlenk index is an ordinal not exceeding $\omega^\alpha$. These classes were studied by the current author in [5], and important relationships between the operator ideals $\mathcal{I}_\alpha$ and other ideals in the list above shall be given in §2.1 below. We note that $\mathcal{X}^*$ is a subclass of $\mathcal{X}$ [32, Proposition 4.4.8].

By $\text{Ord}$ we denote the class of all ordinals, so that by $\alpha \in \text{Ord}$ we mean that $\alpha$ is an ordinal. We write $\text{cof}(\alpha)$ for the cofinality of the ordinal $\alpha$. If $\alpha$ is a successor ordinal, we write $\alpha - 1$ to mean the unique ordinal whose successor is $\alpha$. 

4
For a set $S$ and a subset $R \subseteq S$ we write $\chi^S_R$ for the indicator function of $R$ in $S$, or simply $\chi_R$ if no confusion can result. When discussing a Banach space $\ell_1(S)$ for some set $S$, for $s \in S$ we typically denote by $e_s$ the element of $\ell_1(S)$ satisfying $e_s(s') = 1$ if $s' = s$ and $e_s(s') = 0$ if $s' \neq s$ ($s' \in S$). We thus denote by $(\epsilon_n)_{n=1}^{\infty}$ the standard unit vector basis of $\ell_1 = \ell_1(\mathbb{N})$. Where confusion may otherwise result, we may write $e_s^\ell$ in place of $e_s$ to specify the space $\ell_1(S)$ to which $e_s$ belongs.

We shall repeatedly use the fact that for a set $I$, Banach space $X$ and family $\{x_i \mid i \in I\} \subseteq X$ with $\sup_{i \in I} \|x_i\| < \infty$, there exists a unique element of $\mathcal{L}(\ell_1, X)$ satisfying $e_i \mapsto x_i$, $i \in I$.

For a Banach space $X$, a subset $A \subseteq X$, and $\epsilon > 0$, we say that $A$ is $\epsilon$-separated if $\|x - y\| > \epsilon$ for any distinct $x, y \in A$. For $B \subseteq C \subseteq X$ and $\delta > 0$ we say that $B$ is a $\delta$-net in $C$ if for every $w \in C$ there exists $z \in B$ such that $\|w - z\| \leq \delta$.

## 2 Szlenk indices, trees, and operators on Banach spaces over trees

### 2.1 The Szlenk index

Let $X$ be a Banach space. For each $\epsilon > 0$ define a derivation $s_\epsilon$ on weak$^*$-compact subsets of $X^*$ as follows: for weak$^*$-compact $K \subseteq X^*$ let

$$s_\epsilon(K) := \{x^* \in K \mid \text{diam}(U \cap K) > \epsilon \text{ for every weak$^*$-open } U \ni x^*\}.$$  

Iterate $s_\epsilon$ transfinitely by setting $s_\epsilon^0(K) = K$, $s_\epsilon^{\xi + 1}(K) = s_\epsilon(s_\epsilon^\xi(K))$ for every ordinal $\xi$, and $s_\epsilon^\xi(K) = \bigcap_{\zeta < \xi} s_\epsilon^\zeta(K)$ whenever $\xi$ is a limit ordinal. The $\epsilon$-Szlenk index of $K$, denoted $Sz(K, \epsilon)$, is defined as the smallest ordinal $\xi$ such that $s_\epsilon^\xi(K) \neq \emptyset$, if such an ordinal exists; if no such ordinal exists then $Sz(K, \epsilon)$ is undefined. (Note that, by weak$^*$-compactness, $Sz(K, \epsilon)$ is a successor ordinal when it exists.) Notationally, we write $Sz(K, \epsilon) < \infty$ to mean that $Sz(K, \epsilon)$ is defined, and $Sz(K, \epsilon) = \infty$ to mean that $Sz(K, \epsilon)$ is undefined. If $Sz(K, \epsilon)$ is defined for all $\epsilon > 0$ then the Szlenk index of $K$, denoted $Sz(K)$, is the ordinal $\sup_{\epsilon > 0} Sz(K, \epsilon)$. If $Sz(K, \epsilon)$ is undefined for some $\epsilon > 0$, then $Sz(K)$ is undefined; we write $Sz(K) < \infty$ to mean that $Sz(K)$ is defined, and $Sz(K) = \infty$ to mean that $Sz(K)$ is undefined. Note that while $Sz(K, \epsilon) \leq \xi$ means that $Sz(K, \epsilon)$ is defined and equal to an ordinal not exceeding $\xi$, the statement $Sz(K, \epsilon) \not< \xi$ means either that $Sz(K, \epsilon)$ is undefined or that $Sz(K, \epsilon)$ is defined and exceeds $\xi$; similarly, $Sz(K) \not< \xi$ means either that $Sz(K)$ is undefined or that $Sz(K)$ is defined and equal to an ordinal exceeding $\xi$.

Define the $\epsilon$-Szlenk index of $X$ and the Szlenk index of $X$ to be the indices $Sz(X, \epsilon) := Sz(B_{X^*}, \epsilon)$ and $Sz(X) := Sz(B_{X^*})$, respectively. If $Y$ is a Banach space and $T : X \to Y$ an operator, define the $\epsilon$-Szlenk index of $T$ and the
Szlenk index of $T$ to be the indices $Sz(T, \epsilon) := Sz(T^*(B_{X^*}), \epsilon)$ and $Sz(T) := Sz(T^*(B_{X^*}))$, respectively.

A survey of the Szlenk index and its applications in the context of Banach spaces can be found in [25]. For facts regarding Szlenk indices of operators we refer the reader to [5]. The following proposition collects some well-known facts concerning Szlenk indices of Banach spaces and operators.

**Proposition 2.1.** Let $X$ be a Banach space.

(i) For $K_1 \subseteq K_2 \subseteq X^*$, $\epsilon_1 \geq \epsilon_2 > 0$ and ordinals $\xi_1 \geq \xi_2$ we have $s^{\xi_1}_{\epsilon_1}(K_1) \subseteq s^{\xi_2}_{\epsilon_2}(K_2)$.

(ii) For a subspace $Z \subseteq X$ we have $Sz(Z) \leq Sz(X)$ and $Sz(X/Z) \leq Sz(X)$.

(iii) The following are equivalent:

(a) $Sz(X) < \infty$ (that is, the Szlenk index is defined).

(b) $X$ is an Asplund space.

(c) $X^*$ has the Radon-Nikodým property.

(d) Every separable subspace of $X$ has separable dual.

An argument due to G. Lancien (see Proposition 2 of [25]) shows that if the Szlenk index of a Banach space or an operator is defined, then it is of the form $\omega^\alpha$ for some ordinal $\alpha$; this observation leads to the following definition.

**Definition 2.2.** For $\alpha$ an ordinal define the class

$$SZ_{\alpha} := \{ T \in L \mid Sz(T) \leq \omega^\alpha \}.$$ 

If $T \in SZ_{\alpha}$, we say that $T$ is an $\alpha$-Szlenk.

It is shown in [5] that the classes $SZ_{\alpha}$ are distinct for different values of $\alpha$ and that each such class is a closed operator ideal. Moreover, $SZ_0$ coincides with the class $\mathcal{K}$ of compact operators, whilst the class $\bigcup_{\alpha \in \text{Ord}} SZ_{\alpha}$ of all operators whose Szlenk index is defined coincides with the class $\mathcal{D}$ of Asplund operators. For operators with separable range, the following result from [5] provides information regarding the relationship between the classes $\mathcal{K}^*$, $\mathcal{D}$ and $SZ_{\alpha}$ for $\alpha \in \text{Ord}$.

**Proposition 2.3.** The following chain of equalities holds:

$$\mathcal{K}^* = \mathcal{K} \cap \mathcal{D} = \mathcal{K} \cap \bigcup_{\alpha \in \text{Ord}} SZ_{\alpha} = \mathcal{K} \cap \bigcup_{\alpha < \omega_1} SZ_{\alpha} = \mathcal{K} \cap SZ_{\omega_1}.$$
2.2 Trees

A tree is a partially ordered set \((\mathcal{T}, \preceq)\) for which the set \(\{s \in \mathcal{T} \mid s \preceq t\}\) is well-ordered for every \(t \in \mathcal{T}\). We shall frequently suppress the partial order \(\preceq\) and refer to the underlying set \(\mathcal{T}\) as the tree. An element of a tree is called a node. For \(S \subseteq \mathcal{T}\) we denote by MIN\((S)\) (resp., MAX\((S)\)) the set of all minimal (resp., maximal) elements of \(S\). A subtree of \(\mathcal{T}\) is a subset of \(\mathcal{T}\) equipped with the partial order induced by the partial order of \(\mathcal{T}\), which we also denote \(\preceq\). A chain in \(\mathcal{T}\) is a totally ordered subset of \(\mathcal{T}\). A branch of \(\mathcal{T}\) is a maximal (with respect to set inclusion) totally ordered subset of \(\mathcal{T}\). We say that \(\mathcal{T}\) is well-founded if it contains no infinite branches, and chain-complete if every chain \(\mathcal{C}\) in \(\mathcal{T}\) admits a unique least upper bound. Clearly, every well-founded tree is chain-complete. A subset \(S \subseteq \mathcal{T}\) is said to be downwards closed in \(\mathcal{T}\) if \(S = \bigcup_{s \in S} \{t \in \mathcal{T} \mid s \preceq t\}\). Following Todorčević [39], a path in \(\mathcal{T}\) is a downwards closed, totally ordered subset of \(\mathcal{T}\). An interval in \(\mathcal{T}\) is a subset of \(\mathcal{T}\) of the form \((t', t''), [t', t'']\), \([t', t'')\) or \((t', t'')\), where, for \(t', t'' \in \mathcal{T}\), \((t', t'') := \{t \in \mathcal{T} \mid t' < t < t''\}\) and the other types of intervals are defined analogously. (For a tree \((\mathcal{T}, \preceq)\) and \(s, t \in \mathcal{T}\) we write \(s < t\) to mean that \(s \preceq t\) and \(s \neq t\).) For \(t \in \mathcal{T}\) we define the following sets:

\[
\begin{align*}
\mathcal{T}[\preceq t] &= \{s \in \mathcal{T} \mid s \preceq t\} \\
\mathcal{T}[< t] &= \{s \in \mathcal{T} \mid s < t\} \\
\mathcal{T}[t \preceq] &= \{s \in \mathcal{T} \mid t \preceq s\} \\
\mathcal{T}[t <] &= \{s \in \mathcal{T} \mid t < s\} \\
\mathcal{T}[t+] &= \text{MIN}(\mathcal{T}[< t])
\end{align*}
\]

By \(t^-\) we denote the maximal element of \(\mathcal{T}[< t]\), if it exists (that is, if the order type of \(\mathcal{T}[< t]\) is a successor). If \(s, t \in \mathcal{T}\) are such that \(s \not\preceq t\) and \(t \not\preceq s\), then we write \(s \perp t\). Following terminology introduced in [13], a subtree \(S\) of \(\mathcal{T}\) is said to be a full subtree of \(\mathcal{T}\) if it is downwards closed, \(|S \cap \text{MIN}(\mathcal{T})| = |\text{MIN}(\mathcal{T})|\), and for every \(t \in S\) we have \(|S[t+]| = |\mathcal{T}[t+]|\). A tree is said to be rooted if \(|\text{MIN}(\mathcal{T})| \leq 1\). In particular, a nonempty tree is rooted if and only if it admits a unique minimal element, which we call the root of \(\mathcal{T}\). We denote by \(\mathcal{T}^*\) the subtree \(\mathcal{T} \setminus \text{MIN}(\mathcal{T})\) of \(\mathcal{T}\). For \(t \in \mathcal{T}\) the height of \(t\), denoted \(ht_T(t)\), is the order type of \(\mathcal{T}[< t]\). The height of \(\mathcal{T}\) is the ordinal \(ht(\mathcal{T}) = \sup\{ht_T(t) + 1 \mid t \in \mathcal{T}\}\). Note that \(ht(\mathcal{T}) \leq \omega\) if and only if \(\mathcal{T}[< t]\) is finite for every \(t \in \mathcal{T}\).

Let \(\mathcal{T} = (\mathcal{T}, \preceq)\) be a tree, \(\alpha\) an ordinal and \(\psi : \alpha \rightarrow \mathcal{T}\) a surjection. Then \(\psi\) induces a well-ordering of \(\mathcal{T}\) that extends \(\preceq\). Indeed, define \(A_0 = \mathcal{T}[\preceq \psi(0)]\) and, if \(\beta > 0\) is an ordinal such that \(A_\gamma\) has been defined for all \(\gamma < \beta\), define \(A_\beta = \mathcal{T}[\preceq \psi(\beta)] \setminus \bigcup_{\gamma < \beta} \mathcal{T}[\preceq \psi(\gamma)]\). The induced well-order \(\leq\) of \(\mathcal{T}\) is defined by declaring \(s \leq t\), where \(s \in A_\beta\) and \(t \in A_{\beta'}\), if \(\beta < \beta'\) or if \(\beta = \beta'\) and \(s \preceq t\). Note that if \(\mathcal{T}\) is countable and \(ht(\mathcal{T}) \leq \omega\) then the well-ordering of \(\mathcal{T}\) induced as above
by a surjection of \( \omega \) onto \( T \) is of order type \( \omega \). In fact, the following statements are equivalent:

(i) \( T \) is countable and \( T[\prec t] \) is finite for every \( t \in T \);

(ii) \( T \) is countable and \( ht(T) \leq \omega \); and,

(iii) There exists a bijection \( \tau \) of \( \omega \) onto \( T \) such that \( \tau(l) \preceq \tau(m) \) implies \( l \preceq m \) for \( l, m < \omega \).

Example 2.4. Let \( \Omega := \bigcup_{n<\omega} \prod_n \omega \). That is, \( \Omega \) is the set of all finite (including possibly empty) sequences of finite ordinals. We define an order \( \sqsubseteq \) on \( \Omega \) by saying that \( s \sqsubseteq t \) if and only if \( s \) is an initial segment of \( t \). Note that \( \Omega \) is a rooted tree, with its root being the empty sequence \( \emptyset \). For \( n < \omega \) and \( t \in \Omega \) we denote by \( n \cdot t \) the concatenation of \( (n) \) with \( t \); that is, \( n \cdot t = (n) \) if \( t = \emptyset \) and \( n \cdot t = (n, n_1, \ldots, n_k) \) if \( t = (n_1, \ldots, n_k) \). It is straightforward to show that for an arbitrary tree \((T, \preceq)\) the following statements are equivalent to statements (i)-(iv) above and to each other:

(iv) \( T \) is order-isomorphic to a subtree of \( \Omega \); and,

(v) \( T \) is order-isomorphic to a downwards-closed subtree of \( \Omega^* \).

Moreover, if \( T \) is rooted then (i)-(iv) are equivalent to:

(vi) \( T \) is order-isomorphic to a downwards-closed subtree of \( \Omega \).

For a tree \((T, \preceq)\) we inductively define a decreasing (with respect to set inclusion) family of downwards closed subtrees of \( T \), indexed by the ordinals, by setting

\[
T^{(0)} = T;
\]

\[
T^{(\xi+1)} = T^{(\xi)} \setminus \text{MAX}(T^{(\xi)}) \quad \text{for every ordinal } \xi; \quad \text{and},
\]

\[
T^{(\xi)} = \bigcap_{\xi' < \xi} T^{(\xi')} \quad \text{if } \xi \text{ is a limit ordinal}.
\]

The fact that \( T^{(\xi)} \) is downwards closed for all ordinals \( \xi \) is a straightforward transfinite induction: for the inductive step, note that the property of being downwards closed passes from \( T^{(\xi)} \) to \( T^{(\xi+1)} \) immediately from the definition of \( T^{(\xi+1)} \), and passes to \( T^{(\xi)} \) when \( \xi \) is a limit ordinal by the elementary fact that the intersection of a family of downwards closed subtrees of a tree is itself downwards closed.

The rank of a node \( t \in T \) is defined to be the unique ordinal \( \rho_T(t) \) such that \( t \in T^{(\rho_T(t))} \setminus T^{(\rho_T(t)+1)} \), if it exists. For an ordinal \( \xi \) we henceforth denote by
Thus, if $t_0 \in T$ is such that $\rho_T(t_0)$ does not exist, there exists $t_1 \in T[t_0+]$ such that $\rho_T(t_1)$ does not exist; similarly, there exists $t_2 \in T[t_1+]$ such that $\rho_T(t_2)$ does not exist, and in this way we inductively define an infinite chain $(t_0)_n<\omega$ in $T$, hence $T$ is not well-founded. Conversely, if $T$ contains an infinite branch, $B$ say, then, with $s_n$ denoting the element of $B$ of height $n$ in $T$, we have by induction that $\{s_n \mid n<\omega\} \subseteq T^{(\xi)}$ for all ordinals $\xi$, hence $\rho_T(s_n)$ is undefined for all $n$. We deduce that $\rho_T(t)$ exists for all $t \in T$ if and only if $T$ is well-founded, if and only if $T^{(\xi)} = \emptyset$ for some ordinal $\xi$. If $T$ is well-founded, the rank of $T$ is the ordinal

$$\rho(T) := \min \{\xi \mid T^{(\xi)} = \emptyset\} = \sup \{\rho_T(t) + 1 \mid t \in T\}.$$  

Notice that if $T$ is rooted, with root $t_0$, say, then $T^{(\rho_T(t_0))} = \{t_0\}$, hence $\rho(T) = \rho_T(t_0) + 1$ is a successor ordinal.

We now give the definition of a blossomed tree, due to Gasparis [13].

**Definition 2.5.** We say that a countable tree $(T, \preceq)$ is blossomed if it is rooted, well-founded, and for each $t \in T \setminus \text{MAX}(T)$ there exists a bijection $\psi : \omega \rightarrow T[t+]$ such that $m \leq n < \omega$ implies $\rho_T(\psi(m)) \leq \rho_T(\psi(n))$.

Blossomed trees are used in [13] to study fixing properties of operators of large Szlenk index acting on $C(K)$ spaces. The important property of a blossomed tree $T$ in studying the Szlenk index is that for every $t \in T \setminus \text{MAX}(T)$ and cofinite subset $Q \subseteq T[t+]$ we have

$$\sup \{\rho_T(t') \mid t' \in Q\} = \sup \{\rho_T(t'') \mid t'' \in T[t+]\};$$  

(2.2)

this condition clearly holds for any blossomed tree and, moreover, any countable, rooted, well-founded tree $T$ satisfying the property stated at (2.2) admits a subtree $S$ such that $\rho(S) = \rho(T)$ and $S$ is blossomed. Thus, blossomed trees can be thought of as the ‘minimal’ trees satisfying these conditions and, moreover, the formally stronger definition of a blossomed tree is typically more convenient to work with than the property stated at (2.2) for the purposes of proving results concerning the Szlenk index.

The following example guarantees a rich supply of blossomed trees. Note that other examples of blossomed trees, namely the Schreier families of finite subsets of $\mathbb{N}$, are used by Gasparis in [13]. The construction of trees in Example 2.6 below is essentially the same as that given by Bourgain on p.91 of [2].
Example 2.6. We construct, via transfinite induction on $\xi < \omega_1$, a family $(T_\xi)_{\xi < \omega_1}$ consisting of blossomed subtrees of $\Omega$ that satisfy $\rho(T_\xi) = \xi + 1$ for each $\xi < \omega_1$. Set $T_0 = \{\emptyset\}$. Suppose $\xi > 0$ is an ordinal such that the $T_\zeta$ has been defined for all $\zeta < \xi$; we define $T_\xi$ as follows. Let $(\xi_n)_{n=0}^\infty$ be a non-decreasing, cofinal sequence in $\xi$, and set

$$T_\xi = \{\emptyset\} \cup \{ n^- t \mid n < \omega, t \in T_{\xi_n} \}.$$  

A straightforward transfinite induction on $\zeta \leq \xi$ shows that

$$\forall \zeta \leq \xi \quad T_\zeta^{(C)} = \{\emptyset\} \cup \bigcup_{n<\omega} \{ n^- t \mid t \in T_\xi^{(C)} \},$$

hence

$$\forall \zeta < \xi \quad \text{MAX}(T_\zeta^{(C)}) = \bigcup_{n<\omega} \{ n^- t \mid t \in \text{MAX}(T_{\xi_n}^{(C)}) \}.$$  

Taking $\zeta = \xi$ in (2.3) yields $T_\xi^{(C)} = \{\emptyset\}$, hence $\rho(T_\emptyset) = \xi$ and $\rho(T_\xi) = \xi + 1$.

For $n < \omega$ let $i_n : T_{\xi_n} \rightarrow T_\xi$ be the map $t \mapsto n^- t$. From (2.4) we have $\rho(T_\xi(n^- t)) = \rho(T_{\xi_n}(t))$ every $n < \omega$ and $t \in T_{\xi_n} \setminus \text{MAX}(T_{\xi_n})$ the map $\psi : \omega \rightarrow T_{\xi_n}[t+]$ is a bijection such that $(\rho(T_n \psi(m)))_{m=0}^\infty$ is non-decreasing (as per Definition 2.6), then $i_n \circ \psi : \omega \rightarrow T_\xi[(n^- t)+]$ is a bijection with

$$(\rho(T_\xi(i_n \circ \psi(m))))_{m=0}^\infty = (\rho(T_{\xi_n}(n^- \psi(m))))_{m=0}^\infty = (\rho(T_{\xi_n}(\psi(m))))_{m=0}^\infty$$

non-decreasing. Similarly, $n \mapsto (n)$ is a bijection of $\omega$ onto $T_\xi[\emptyset+]$ and

$$(\rho(T_\xi((n))))_{n=0}^\infty = (\rho(T_{\xi_n}(\emptyset)))_{n=0}^\infty = (\xi_n)_{n=0}^\infty$$

is non-decreasing. We have now shown that $T_\xi$ is blossomed, as required.

The following proposition collects properties of blossomed trees that we shall need in subsequent sections of the current paper.

Proposition 2.7. Let $(S, \preceq')$ and $(T, \preceq)$ be countable, rooted, well-founded trees.

(i) If $S$ is blossomed and $\rho(T) \leq \rho(S)$ then $T$ is order isomorphic to a downwards closed subtree of $S$.

(ii) If $S$ is blossomed and $S'$ is a full subtree of $S$ then $S'$ is blossomed and $\rho(S') = \rho(S)$.

Assertion (i) of Proposition 2.7 is a trivial generalisation of Lemma 2.7 of [13], requiring little change in the proof. Assertion (ii) is Lemma 2.8 of [13]. We refer the reader to [13] for the proofs.
There are various natural topologies for trees, many of which are described in [30]. The tree topology of interest to us is the coarse wedge topology, which is compact and Hausdorff for many trees. The coarse wedge topology of \((T, \preceq)\) is that topology on \(T\) formed by taking as a subbase all sets of the form \(T[t \preceq]\) and \(T \setminus T[t \preceq]\), where the order type of \(T[\prec t]\) is either 0 or a successor ordinal. For a tree \((T, \preceq)\), \(t \in T\) and \(F \subseteq T\), define
\[
W_T(t, F) := T[t \preceq] \setminus \bigcup_{s \in F} T[s \preceq].
\]
The following proposition is clear.

**Proposition 2.8.** Let \((T, \preceq)\) be a tree and let \(t \in T\) be such that the order type of \(T[\prec t]\) is 0 or a successor ordinal. Then the coarse wedge topology of \(T\) admits a local base of clopen sets at \(t\) consisting of all sets of the form \(W_T(t, F)\), where \(F \subseteq T[t+]\) is finite.

The following result is proved in the aforementioned paper of Nyikos.

**Theorem 2.9.** ([30, Corollary 3.5]) Let \(T\) be a tree. The following are equivalent.

(i) \(T\) is chain-complete and \(\text{MIN}(T)\) is finite.

(ii) The coarse wedge topology of \(T\) is compact and Hausdorff.

We conclude the current subsection on trees with the following proposition.

**Proposition 2.10.** Let \((T, \preceq)\) be a tree with \(\text{ht}(T) \leq \omega\).

(i) Let \(S \subseteq T\) be a downwards closed subset of \(T\). Then \(S\) is closed in the coarse-wedge topology of \(T\).

(ii) Let \((S, \preceq')\) be a tree and suppose \(\phi : S \to T\) is an order-isomorphism of \(S\) onto a downwards closed subset of \(T\). Then \(\phi\) is coarse wedge continuous.

**Proof.** We first prove (i). Suppose \(t \in T \setminus S\). Then \(T[t \preceq]\) is open in the coarse wedge topology of \(T\) since \(\text{ht}_T(t) < \omega\). Moreover \(S \cap T[t \preceq] = \emptyset\) since \(t \notin S\) and \(S\) is downwards closed. Since \(t \in T \setminus S\) was arbitrary we conclude that \(T \setminus S\) is open, hence \(S\) is closed in the coarse wedge topology of \(T\).

To prove (ii), first note that the sets \(T[t \preceq]\) and \(T \setminus T[t \preceq]\), where \(t\) varies over all of \(T\), form a subbasis of clopen sets for the coarse wedge topology of \(T\). To establish the continuity of \(\phi\) it therefore suffices to show that \(\phi^{-1}(T[t \preceq])\) is clopen in \(S\) for every \(t \in T\). To this end suppose \(t \in T\). If \(t \notin \phi(S)\) then \(\phi^{-1}(T[t \preceq]) = \emptyset\) since \(\phi\) is an order isomorphism and \(\phi(S)\) is downwards closed in \(T\). On the other hand if \(t \in \phi(S)\), say \(t = \phi(s)\), then \(\phi^{-1}(T[t \preceq]) = S[s \preceq']\) since \(\phi\) is an order isomorphism. \(\square\)
2.3 Operators on Banach spaces over trees

Let \((\mathcal{T}, \preceq)\) be a tree. Define \(\Sigma_{\mathcal{T}} : \ell_1(\mathcal{T}) \to \ell_\infty(\mathcal{T})\) by setting

\[
\Sigma_{\mathcal{T}} w = \left( \sum_{s \preceq t} w(s) \right)_{t \in \mathcal{T}}, \quad w \in \ell_1(\mathcal{T}).
\]

Equivalently, \(\Sigma_{\mathcal{T}}\) is the unique element of \(\mathcal{L}(\ell_1(\mathcal{T}), \ell_\infty(\mathcal{T}))\) satisfying \(\Sigma_{\mathcal{T}}e_t = \chi_{\mathcal{T}[t \preceq]}\) for each \(t \in \mathcal{T}\), with \(\|\Sigma_{\mathcal{T}}\| = 1\) for nonempty \(\mathcal{T}\).

Notice that we can state some existing universality results in terms of operators of the form \(\Sigma_{\mathcal{T}}\). For instance, taking \(\mathcal{T}\) to be the set of natural numbers \(\mathbb{N}\) equipped with its usual order \(\leq\), the operator \(\Sigma_{\mathcal{T}}\) is the aforementioned universal non-weakly compact operator of Lindenstrauss and Pełczyński [26]. Moreover, taking \(\mathcal{T}\) to instead be the set of natural numbers \(\mathbb{N}\) equipped with the trivial order \(=\) yields \(\Sigma_{\mathcal{T}}\) as the formal identity operator from \(\ell_1\) to \(\ell_\infty\), shown by Johnson in [20] to be universal for the class of non-compact operators. Amongst the outcomes of the current paper is that we add to the collection of trees \((\mathcal{T}, \preceq)\) for which the corresponding operator \(\Sigma_{\mathcal{T}}\) is universal for the complement of some operator ideal.

We shall use the following proposition to determine whether \(\Sigma_{\mathcal{T}}\) factors through \(T\), for certain trees \((\mathcal{T}, \preceq)\) and operators \(T\).

**Proposition 2.11.** Let \((\mathcal{T}, \preceq)\) be a tree, \(X\) and \(Y\) Banach spaces and \(T \in \mathcal{L}(X, Y)\). The following are equivalent:

(i) \(\Sigma_{\mathcal{T}}\) factors through \(T\).

(ii) There exist \(\delta > 0\) and families \((x_t)_{t \in \mathcal{T}} \subseteq B_X\) and \((x^*_t)_{t \in \mathcal{T}} \subseteq T^*B_Y^*\) such that

\[
\langle x^*_t, x_s \rangle = \begin{cases} 
\langle x^*_s, x_s \rangle \geq \delta & \text{if } s \preceq t \\
0 & \text{if } s \not\preceq t 
\end{cases} \quad s, t \in \mathcal{T}. \tag{2.5}
\]

**Proof.** First suppose that (i) holds and that \(U \in \mathcal{L}(\ell_1(\mathcal{T}), X)\) and \(V \in \mathcal{L}(Y, \ell_\infty(\mathcal{T}))\) are such that \(VTU = \Sigma_{\mathcal{T}}\). For each \(t \in \mathcal{T}\) let \(f^*_t\) be the element of \(\ell_\infty(\mathcal{T})^*\) satisfying \(\langle f^*_t, z \rangle = \|V\|^{-1}z(t)\) for every \(z \in \ell_\infty(\mathcal{T})\). For each \(s, t \in \mathcal{T}\) let \(x_s = \|Ue_s\|^{-1}Ue_s \in B_X\) and \(x^*_t = T^*V^*f^*_t \in T^*B_Y^*\). Then for \(s, t \in \mathcal{T}\) we have

\[
\langle x^*_t, x_s \rangle = \langle T^*V^*f^*_t, \|Ue_s\|^{-1}Ue_s \rangle \\
= \|Ue_s\|^{-1}\langle f^*_t, VUe_s \rangle \\
= \|Ue_s\|^{-1}\langle f^*_t, \Sigma_{\mathcal{T}}e_s \rangle \\
= \begin{cases} 
\|Ue_s\|^{-1}\|V\|^{-1} \geq (\|U\|\|V\|)^{-1} & \text{if } s \preceq t \\
0 & \text{if } s \not\preceq t 
\end{cases}
\]

12
hence (i) implies (ii).

Now suppose that (ii) holds and let \( \delta > 0 \), \((x_t)_{t \in T} \subseteq B_X \) and \((x_t^*)_{t \in T} \subseteq T^* B_{Y^*} \) be such that (2.5) holds. Define \( U : \ell_1(T) \to X \) by setting

\[
U w = \sum_{t \in T} \frac{w(t)}{\langle x_t^*, x_t \rangle} x_t
\]

for each \( w \in \ell_1(T) \). Then \( U \) is well-defined, linear and continuous with \( \|U\| \leq \delta^{-1} \).

For each \( t \in T \) choose \( v_t^* \in B_{Y^*} \) such that \( T^* v_t^* = x_t^* \). The map \( V : Y \to \ell_\infty(T) \)

given by setting \( V y = (\langle v_t^*, y \rangle)_{t \in T} \) for each \( y \in Y \) is well-defined, linear and continuous with \( \|V\| \leq 1 \). To complete the proof we show that \( VTU = \Sigma_T \). To this end note that for \( s \in T \) we have

\[
VTU e_s = VT(\langle x_s^*, x_s \rangle^{-1} x_s) = (\langle x_s^*, x_s \rangle^{-1} \langle v_t^*, T x_s \rangle)_{t \in T} = (\langle x_s^*, x_s \rangle^{-1} \langle x_t^*, x_s \rangle)_{t \in T} = \chi_T[s \preceq s],
\]

hence \( VTU = \Sigma_T \).

The following result may be proved by an appeal to Proposition 2.11 but we give the equally easy direct proof.

**Proposition 2.12.** Let \((S, \preceq')\) and \((T, \preceq)\) be trees and suppose that \( S \) is order isomorphic to a subtree of \( T \). Then \( \Sigma_S \) factors through \( \Sigma_T \).

**Proof.** Let \( \phi \) be an order-embedding of \( S \) into \( T \). Let \( A \in \mathcal{L}(\ell_1(S), \ell_1(\phi(S))) \) be the map satisfying \( Aw = w \circ \phi^{-1} \) for each \( w \in \ell_1(S) \), let \( U \in \mathcal{L}(\ell_1(\phi(S)), \ell_1(T)) \) be defined by setting \( (U u)|_{\phi(S)} = u \) and \( (U u)|_{T \setminus \phi(S)} \equiv 0 \) for each \( u \in \ell_1(\phi(S)) \), let \( V \in \mathcal{L}(\ell_\infty(T), \ell_\infty(\phi(S))) \) be the map satisfying \( V v = v \circ \phi \) for each \( v \in \ell_\infty(T) \), and let \( B \in \mathcal{L}(\ell_\infty(\phi(S)), \ell_\infty(S)) \) be the map satisfying \( B z = z \circ \phi \) for each \( z \in \ell_\infty(\phi(S)) \). For each \( s \in S \) we have

\[
BV \Sigma_T U A e_s^S = BV \Sigma_T U e_{\phi(s)}^S = BV \Sigma_T e_{\phi(s)}^T = BV \chi_T[\phi(s) \preceq s] = B \chi_{\phi(S)}[\phi(s) \preceq s] = \chi_S[s \preceq s],
\]

hence \( BV \Sigma_T U A = \Sigma_S \).

Notice that if \( \mathcal{I} \) and \( \mathcal{J} \) are operator ideals and \( T \in \mathcal{J} \cap \mathbb{C} \mathcal{I} \) is universal for \( \mathbb{C} \mathcal{I} \), then every universal element of \( \mathbb{C} \mathcal{I} \) belongs to \( \mathcal{J} \). In particular, it is a consequence of the following proposition that if \( \mathcal{I} \) is an operator ideal and \( T \) is a tree such that \( \Sigma_T \) is universal for \( \mathbb{C} \mathcal{I} \), then any operator universal for \( \mathbb{C} \mathcal{I} \) is strictly singular.

**Proposition 2.13.** Let \((T, \preceq)\) be a tree. Then:

(i) \( \Sigma_T \) is strictly singular.

13
(ii) $\Sigma_T$ is weakly compact if and only if $T$ is well-founded.

(iii) $\Sigma_T$ is compact if and only if $T$ is finite, if and only if $\Sigma_T$ is finite rank.

Suppose $(T, \preceq)$ be a tree. For the purposes of proving Proposition 2.13 we now recall the definition of the James tree space of $T$, denoted $J(T)$, which is the completion of $c_{00}(T)$ with respect to the norm $\| \cdot \|_{J(T)}$ on $c_{00}(T)$ that is defined by setting

$$\| x \|_{J(T)} = \sup \left\{ \left( \sum_{i=1}^{k} \left| \sum_{t \in S_i} x(t) \right|^2 \right)^{1/2} \mid S_1, \ldots, S_k \subseteq T \text{ pairwise disjoint intervals} \right\}$$

for each $x \in c_{00}(T)$. Notice that the formal identity map $(c_{00}(T), \| \cdot \|_{\ell_1(T)}) \rightarrow (c_{00}(T), \| \cdot \|_{J(T)})$ is continuous with norm 1, and therefore admits a (unique) continuous linear extension $A_T \in \mathcal{L}(\ell_1(T), J_2(T))$. Moreover, the linear map $x \mapsto (\sum_{s \preceq t} x(s))_{t \in T}$ from $c_{00}(T)$ to $\ell_\infty(T)$ is continuous with norm 1 with respect to the norm $\| \cdot \|_{J(T)}$ on $c_{00}(T)$, and thus extends (uniquely) to some $B_T \in \mathcal{L}(J(T), \ell_\infty(T))$. Since $\Sigma_T = B_T A_T$ we have that $\Sigma_T$ factors through the James tree space $J(T)$.

**Proof of Proposition 2.13**. Assertion (i) of the proposition follows from the fact that the codomain of $\Sigma_T$, namely $\ell_1(T)$, is $\ell_1$-saturated, whilst $\Sigma_T$ has already been seen above to factor through the $\ell_2$-saturated space $J(T)$. (See Lemma 2 and the final remark of [17] for details of the proof that $J(T)$ is $\ell_2$-saturated.)

For (ii), the assertion that $\Sigma_T$ is weakly compact whenever $T$ is well-founded follows from the aforementioned fact that $\Sigma_T$ factors through the James tree space $J(T)$ of $T$ and the fact that $J(T)$ is reflexive if and only if $T$ is well-founded. The proof of this latter fact is obtained via a straightforward transfinite induction on $\rho(T)$, using the following facts: an $\ell_2$-direct sum of a family of reflexive spaces is reflexive; and, for a rooted tree $T$, the Banach space $(\bigoplus_{t \in \text{MIN}(T)} J(T|t \preceq t)) \ell_2$ is isometrically isomorphic to a codimension 1 subspace of $J(T)$; the remaining details are omitted. On the other hand, if $T$ is not well-founded then $T$ contains a path order-isomorphic to $\mathbb{N}$ equipped with its usual order $\leq$. It follows then by Proposition 2.12 that $\Sigma_T$ factors the universal non-weakly compact operator of Lindenstrauss and Pełczyński (Theorem 1.1), hence $\Sigma_T$ fails to be weakly compact whenever $T$ is not well founded.

To prove (iii), first note that if $T$ is finite then the codomain is the finite dimensional space $\ell_\infty(T)$, hence $\Sigma_T$ is finite rank and therefore compact. Conversely, if $T$ is infinite then the set $\{ \Sigma_T e_t \mid t \in T \}$ is an infinite 1-separated subset of $\Sigma_T B_{\ell_1(T)}$, hence in this case $\Sigma_T$ is non-compact, hence non-finite rank.

$\square$
We now establish a connection between the rank $\rho(\mathcal{T})$ and the Szlenk indices of $\Sigma_\mathcal{T}$ in the particular case that $\mathcal{T}$ is blossomed.

**Proposition 2.14.** Let $(\mathcal{T}, \preceq)$ be a blossomed tree and $\epsilon \in (0, 1)$. Then $Sz(\Sigma_\mathcal{T}^\epsilon, \epsilon) \geq \rho(\mathcal{T})$.

**Proof.** Fix $\epsilon \in (0, 1)$, let $t_0$ denote the root of $\mathcal{T}$ and let $B = \Sigma_\mathcal{T}^\epsilon B_{\ell_\infty(\mathcal{T})^\star}$. For each $t \in \mathcal{T}^\star$ let $f_t \in \ell_\infty(\mathcal{T}^\star)^\star$ be the evaluation functional of $\ell_\infty(\mathcal{T}^\star)$ at $t$ and let $g_t := \Sigma_\mathcal{T}^\star f_t \in B$. Let $g_t := 0 \in B$. Note that $\{g_t^* | t \in \mathcal{T}\}$ is $\epsilon$-separated, for if $s, t \in \mathcal{T}$ are such that $s \notin \overline{T}[t \preceq t]$ then $\|g_s^* - g_t^*\| \geq \langle g_t^* - g_s^*, e_t \rangle > 1 > \epsilon$. For each $t \in \mathcal{T} \setminus \max(\mathcal{T})$ let $(t_m)_{m=0}^\infty$ be an enumeration of $\mathcal{T}[t^+]$ with $(\rho_T(t_m))_{m=0}^\infty$ non-decreasing. For every $x \in \ell_1(\mathcal{T}^\star)$ and $t \in \mathcal{T} \setminus \max(\mathcal{T})$ we have $\langle g_{t_m}, x \rangle = \sum_{s \preceq t_m} x(s) \rightarrow \sum_{s \preceq t} x(s) = \langle g_t^*, x \rangle$ as $m \rightarrow \infty$, so that $(g_{t_m})_{m=0}^\infty$ is weak* convergent to $g_t^*$.

We will show by transfinite induction that

$$\{g_t^* | t \in \mathcal{T}^{[\xi]}\} \subseteq s_\xi^\epsilon(B). \quad (2.6)$$

for every $\xi < \rho(\mathcal{T})$. (2.6) is trivially true for $\xi = 0$. Suppose $\zeta \in (0, \rho(\mathcal{T}))$ is such that (2.6) holds for every $\xi < \zeta$; to complete the induction we show that $\{g_t^* | t \in \mathcal{T}^{[\zeta]}\} \subseteq s_\zeta^\epsilon(B)$. To this end suppose $t \in \mathcal{T}^{[\zeta]}$. For each $m < \omega$ the weak* closed set $s_\epsilon^{\rho_T(t_m)}(B)$ contains the weak* limit of the sequence $(g_{t_m}^*)_{m=0}^\infty \subseteq s_\epsilon^{\rho_T(t_m)}(B)$, namely $g_t^*$, hence $g_t^* \in s_\epsilon(s_\epsilon^{\rho_T(t_m)}(B)) = s_\epsilon^{\rho_T(t_m)+1}(B)$ since $\|g_t^* - g_s^*\| > \epsilon$ for all $t \in [m, \omega)$. It follows that $g_t^* \in \bigcap_{m<\omega} s_\epsilon^{\rho_T(t_m)+1}(B) = s_\epsilon^{\rho_T(t)}(B) = s_\zeta^\epsilon(B)$, as required. With the induction now complete, taking $\xi = \rho(\mathcal{T}) - 1$ in (2.6) yields $g_{t_0}^* \in s_\epsilon^{\rho_T(t)}(B)$, from which the proposition follows. \qed

### 3 Absolutely convex sets of large Szlenk index

This section is devoted to proving our key result, Theorem 3.3 from which a number of results in subsequent sections of the paper are derived. The work presented here follows after the work of several other authors who have studied the structure of subspaces and quotients of Banach spaces having Szlenk index larger than a given ordinal. We shall now sketch some previous results and then briefly explain the contributions of the current paper to this topic. The first result of interest to us is the following result due to G. Lancien.

**Proposition 3.1** (Proposition 3.1 of [24]). Let $X$ be a Banach space and $\xi < \omega_1$. If $Sz(X) > \xi$ then there exists a separable subspace $Y$ of $X$ such that $Sz(Y) > \xi$.

It follows easily from Proposition 3.1 that if $X$ is a Banach space with $Sz(X) < \omega_1$, then there exists a separable subspace $Y$ of $X$ such that $Sz(Y) = Sz(X)$. For
instance, take $Y$ to be the closed linear hull of $\bigcup_{n=1}^{\infty} Y_n$, where, for each $n \in \mathbb{N}$, $Y_n$ is a separable subspace of $X$ with $Sz(Y_n) > Sz(X, 1/n) - 1$.

To prove Proposition 3.1, Lancien showed (c.f. Lemma 3.4 of [24]) that for a suitable tree $T$ of rank $\xi + 1$, the estimate $Sz(X) > \xi$ implies the existence of families of vectors $(x_t)_{t \in T} \subseteq B_X$ and $(x^*_t)_{t \in T} \subseteq B_{X^*}$ satisfying certain properties that bear witness to the fact that $Sz(X) > \xi$. Without giving the precise details of Lancien’s construction here, we mention that the subspace $Y$ is taken to be the closed linear span of the family $(x_t)_{t \in T}$. Recently, Dilworth, Kutzarova, Lancien and Randrianarivony have adapted Lancien’s construction from [24] to show that, under the additional hypothesis that $X$ is reflexive, $(x_t)_{t \in T}$ may be assumed to be a basic sequence for a suitable enumeration of $T$ (Proposition 3.1(i) of [11]). It follows that if $X$ is reflexive and $Sz(X) = \omega^\alpha$ for some $\alpha < \omega_1$, then $X$ admits a subspace $Y$ with a basis and satisfying $Sz(Y) = Sz(X)$ (to see this, consider the estimate $Sz(X) > \omega^\alpha$).

In Proposition 3.5 of [24], Lancien combined the techniques developed in the proof Proposition 3.4 of [24] (mentioned in the preceding paragraph) with the techniques developed by Johnson and Rosenthal in [21] for constructing weak$^*$-basic sequences in dual Banach spaces. In particular, Lancien showed the following:

**Proposition 3.2** (Proposition 3.5 of [24]). Let $X$ be a separable Banach space and $\xi < \omega_1$. If $Sz(X) > \xi$ then there exists a subspace $Z$ of $X$ such that $X/Z$ has a basis and $Sz(X/Z) > \xi$.

It follows from Proposition 3.2 that if $X$ is a separable Banach space with $Sz(X) = \omega^{\alpha+1}$ for some $\alpha < \omega_1$, then $X$ has a subspace $Z$ such that $X/Z$ has a basis and $Sz(X/Z) = Sz(X)$ (as above, consider the estimate $Sz(X) > \omega^\alpha$).

We extend these earlier results in several ways, one of which is to study the consequences for the quotient and subspace structure of a Banach space $X$ arising from a sequence of estimates of the form $Sz(X, \epsilon_n) > \xi_n$, $n < \omega$, rather than just a single estimate of the form $Sz(X, \epsilon) > \xi$. This more general approach shall later yield dividends by taking $(\epsilon_n)_{n<\omega}$ to be dense in $(0, \infty)$, allowing us to show in particular that for a Banach space $X$ of countable Szlenk index the Szlenk index of $X$ is attained by some subspace of $X$ with a basis (Theorem 4.1). We show moreover that if such $X$ is separable, then the Szlenk index of $X$ is attained by a quotient of $X$ with a basis. Another important consequence of our approach is the equation (3.1) below, which will be crucial for proving universality results for the classes $SZ_\alpha$ of non-$\alpha$-Szlenk operators in Section 5.

Part (i) of Theorem 3.3 below may be viewed as a refinement and extension of the aforementioned constructions given in Lemma 3.4 of [24] and Proposition 3.1(i) of [11], while part (ii) of Theorem 3.3 builds on the ideas developed in the proof of Proposition 3.5 of [24].
In order to state Theorem 3.3 we introduce the following notation. For a family $T = ((T_n, \preceq_n))_{n<\omega}$, where each $(T_n, \preceq_n)$ is a rooted tree, define $[T] := \{\emptyset\} \cup \bigcup_{n<\omega} (\{n\} \times T_n^*)$, so that $[T]$ is a rooted tree when equipped with the order $\preceq_T$ on $[T]$ defined by setting $\emptyset \preceq_T t$ for all $t \in [T]$ and $(n_1, t_1) \preceq_T (n_2, t_2)$ if and only if $n_1 = n_2$ and $t_1 \preceq_{n_1} t_2$.

**Theorem 3.3.** Let $X$ be a Banach space, $K \subseteq X^*$ a non-empty, absolutely convex, weak*-compact set, $\delta, \theta > 0$ positive real numbers, $(\epsilon_n)_{n<\omega}$ a family of positive real numbers, $(\xi_n)_{n<\omega}$ a family of countable ordinals such that $s^{\xi_n}_\epsilon(K) \neq \emptyset$ for all $n < \omega$, and $\mathcal{T} = ((T_n, \preceq_n))_{n<\omega}$ a family of countable, well-founded, rooted trees such that $\rho(T_n) \leq \xi_n + 1$ for all $n < \omega$.

(i) There exist families $(x^*_i)_{i \in [\mathcal{T}]} \subseteq K$ and $(x_i)_{i \in [\mathcal{T}]} \subseteq S_X$ such that

$$\langle x^*_i, x_s \rangle = \begin{cases} \langle x^*_i, x_s \rangle > \frac{\epsilon_n}{s+\theta} & \text{if } s \preceq_T t \in \{n\} \times T_n^*, s, t \in [\mathcal{T}]^*, n < \omega. \\ 0 & \text{if } s \not\preceq_T t \end{cases}$$

Moreover, for any bijection $\tau: \omega \rightarrow [\mathcal{T}]$ such that $\tau(l) \preceq_T \tau(m)$ implies $l \leq m$ we may choose $(x_i)_{i \in [\mathcal{T}]}$ so that $(x_{\tau(m)})^\omega_{m=1}$ is a basic sequence with basis constant not exceeding $1 + \delta$.

(ii) Let $Z = \bigcap_{i \in [\mathcal{T}]} \ker(x^*_i)$ and let $Q: X \longrightarrow X/Z$ be the quotient map. If $X^*$ is norm separable then the families $(x^*_i)_{i \in [\mathcal{T}]}$ and $(x_i)_{i \in [\mathcal{T}]}$ in (i) may be chosen so that $(x_{\tau(m)})^\omega_{m=1}$ is shrinking and $(Qx_{\tau(m)})^\omega_{m=1}$ is a shrinking basis for $X/Z$ with basis constant not exceeding $1 + \delta$.

The proof Theorem 3.3 shall invoke the following lemma due to G. Lancien [24], who established the result for the special case $K = B_{X^*}$ and $\zeta$ of the form $\omega^\alpha$ for some ordinal $\alpha$; the same argument gives the more general statement presented below.

**Lemma 3.4.** ([24, p.67]) Let $X$ be a Banach space, $K \subseteq X^*$ an absolutely convex, weak*-compact set, $\zeta$ an ordinal and $\epsilon > 0$. If $s^{\epsilon}_\zeta(K) \neq \emptyset$ then

$$\forall n < \omega \quad 0 \in s^{\epsilon 2^n}_{\zeta/2^n+1}(K).$$

We require the following result, which is Lemma 2.2 of [11].

**Lemma 3.5.** Let $X$ be a Banach space, $\nu > 0$ a real number, $F$ a finite dimensional subspace of $X^*$, $A$ a $\frac{2^n}{1+2^n}$-net in $S_F$ and $\{y^*_f \mid f^* \in A\} \subseteq S_X$ a family such that $\inf\{|f^*(y^*_f)| \mid f^* \in A\} \geq \frac{4\nu}{1+2^n}$. Then for every $x^* \in \{y^*_f \mid f^* \in A\}$ we have $\sup\{|x^*(y)| \mid y \in B_{F^*}\} \geq \frac{1}{2^\nu} \|x^*\|$. 17
We do not know a reference for the following result, so we provide the straightforward proof.

**Lemma 3.6.** Let $X$ be a Banach space and $x^* \in X^*$. Then

$$\{ x^* + \epsilon B_{X^*} + C^\perp \mid \epsilon > 0, C \subseteq X, |C| < \infty \}$$

is a local base for the weak* topology of $X^*$ at $x^*$.

**Proof.** We assume $x^* = 0$, from which the general case follows easily. For $\epsilon > 0$ and finite $C \subseteq X$ define $N(C, \epsilon) := \bigcap_{x \in C}\{ y^* \in X^* \mid |y^*(x)| < \epsilon \}$. Since $\epsilon B_{X^*} + C^\perp \subseteq N(C, \epsilon)$ it suffices to show that $\epsilon B_{X^*} + C^\perp$ is weak*-open for $\epsilon > 0$ and $C \subseteq X$ finite.

Fix $\epsilon > 0$ and $C$ a finite subset of $X$. Set $Y = \text{span}(C)$, let $P \in \mathcal{L}(X)$ be a projection with range $Y$, and let $\lambda > 0$ be small enough that

$$\lambda B_Y \subseteq \left\{ \sum_{x \in C} \lambda_x x \mid \lambda_x \in \mathbb{K}, \sum_{x \in C} |\lambda_x| \leq 1 \right\}.$$

Since $P(B_X) \subseteq \|P\|^{-1}(\lambda B_Y)$, for $y^* \in N(C, \|P\|^{-1}\lambda\epsilon)$ we have

$$\|P^*y^*\| = \sup\{ |\langle y^*, Py \rangle| \mid y \in B_X \}$$

$$\leq \sup\{ |\langle y^*, z \rangle| \mid z \in \|P\|^{-1}(\lambda B_Y) \}$$

$$\leq \|P\|\lambda^{-1}\sup\left\{ |\langle y^*, \sum_{x \in C} \lambda_x x \rangle| \mid \sum_{x \in C} |\lambda_x| \leq 1 \right\}$$

$$< \epsilon,$$

hence $y^* = P^*y^* + (Id_{X^*} - P^*)y^* \in \epsilon B_{X^*} + C^\perp$. It follows that for $u^* \in B_{X^*}^\circ$, $v^* \in C^\perp$ and $y^* \in N(C, \|P\|^{-1}\lambda\epsilon)$ we have

$$\epsilon u^* + v^* + (1 - \|u^*\|)y^* = \epsilon (u^* + (1 - \|u^*\|)\epsilon^{-1}P^*y^*) + (v^* + (1 - \|u^*\|)(Id_{X^*} - P^*)y^*)$$

$$\in \epsilon B_{X^*} + C^\perp.$$

In particular, for any $u^* \in B_{X^*}^\circ$ and $v^* \in C^\perp$ we have that $\epsilon u^* + v^* + (1 - \|u^*\|)N(C, \|P\|^{-1}\lambda\epsilon)$ is a weak* neighbourhood of $\epsilon u^* + v^*$ contained in $\epsilon B_{X^*} + C^\perp$, hence $\epsilon B_{X^*} + C^\perp$ is weak*-open.

**Lemma 3.7.** Let $X$ be a Banach space, $K$ and $L$ weak*-compact subsets of $X^*$, $\xi$ an ordinal and $\epsilon > 0$. If $x^* \in s^\xi(K)$ and $y^* \in L$ then $x^* + y^* \in s^\xi(K + L)$.

**Proof.** We proceed by transfinite induction on $\xi$. The assertion of the lemma is true for $\xi = 0$. Suppose that $\xi > 0$ is an ordinal such that the assertion of
the lemma is true for $\xi = \zeta$; we will show that it is true then for $\xi = \zeta + 1$. Let $x^* \in s_\xi(1)(K)$ and $y^* \in L$. Since $x^* \in s_\xi(K)$ it follows from the induction hypothesis that $x^* + y^* \in s_\xi(K + L)$. Since $x^* \in s_\xi(K)$ we have that for any weak$^*$-neighbourhood $U$ of $x^* + y^*$ there exist $x_1^*, x_2^* \in s_\xi(K) \cap (-y^* + U)$ such that $\|x_1^* - x_2^*\| > \epsilon$. Since $x_1^* + y^*, x_2^* + y^* \in U \cap (K + L)$ and $\|(x_1^* + y^*) - (x_2^* + y^*)\| > \epsilon$, we deduce that $x^* + y^* \in s_{\xi + 1}(K + L)$.

Now suppose $\zeta$ is a limit ordinal and the assertion of the lemma is true for all $\xi < \zeta$. For $x^* \in s_\xi = \bigcap_{\xi < \zeta} s_\xi(K)$ and $y^* \in L$ we have

$$x^* + y^* \in \bigcap_{\xi < \zeta} s_\xi(K + L) = s_\zeta(K + L),$$

which completes the induction.

The final preliminary result before proving Theorem 3.3 is the following theorem due to Kadets [22] and Klee [23]. A short proof due to Davis and Johnson (sketched in [8]) can be found on p.13 of Lindenstrauss and Tzafriri’s book [27].

**Theorem 3.8.** Let $(X, \| \cdot \|)$ be a Banach space and $c > 1$ a real number. If $X^*$ is norm separable then $X$ admits a norm $\| \cdot \|$ such that the following properties hold:

(i) For every $x^* \in X^*$ we have $\|x^*\| \leq \|x^*\| \leq c\|x^*\|$.

(ii) If $(x_n^*) \subseteq X^*$ and $x^* \in X^*$ are such that $x_n^* \rightharpoonup x^*$ and $\|x_n^*\| \to \|x^*\|$, then $\|x_n^* - x^*\| \to 0$.

We briefly indicate the main idea of Davis and Johnson’s proof of Theorem 3.8. For $X$ as in the statement of Theorem 3.8 one defines a norm $\| \cdot \|$ on $X^*$ as follows: Let $c > 1$ be a real number, let $B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots$ be a sequence of finite-dimensional subspaces of $X^*$ whose union is dense in $X^*$, and for each $n \in \mathbb{N}$ let $Q_n : X^* \to X^*/B_n$ denote the quotient map. For $x^* \in X^*$ define

$$\|x^*\| := \|x^*\| + (c - 1)\sum_{n=1}^\infty 2^{-n}\|Q_n x^*\|.$$

Following the argument in [27], $\| \cdot \|$ is a norm on $X^*$ that is dual to some norm on $X$ which we also denote $\| \cdot \|$. Clearly (i) holds for $\| \cdot \|$. That (ii) holds for $\| \cdot \|$ is verified in [27].

As is implicit in the statement of Theorem 3.8 when we apply the renorming result Theorem 3.8, we shall use $\| \cdot \|$ to denote also the corresponding induced norm on duals, subspaces, quotients and operators on $X$. 

19
Proof of Theorem 3.3. We shall first prove (i) without any assumption on the norm density of \( X^* \), then show how to modify the arguments in the proof of (i) to obtain also the assertions of (ii) when \( X^* \) is assumed to be norm separable.

Fix \( \delta, \theta > 0 \) and let \( \nu = \theta/(38 + 3\theta) \), so that

\[
\frac{1 - 3\nu}{4(2 + \nu)(1 + \nu)} \geq \frac{1}{8 + \theta} \tag{3.3}
\]

Let \( \tau : \omega \rightarrow \mathbb{S} \) be a bijection such that \( \tau(l) \leq \tau(m) \) implies \( l \leq m \) (c.f. the paragraph preceding Example 2.4), noting that we necessarily have \( \tau(0) = \emptyset \), the root of \( \mathbb{S} \). For \( 0 < m < \omega \) we may write \( \tau(m) = (n_m, t_m) \), where \( n_m < \omega \) and \( t_m \in T_{n_m} \). Fix a sequence \( (\delta_m)_{m=1}^{\infty} \subseteq (0, 1) \) of positive real numbers such that \( \sum_{m=1}^{\infty} \delta_m < \infty \) and \( \prod_{m=1}^{\infty}(1 - \delta_m) \geq (1 + \delta)^{-1} \). Proceeding via an induction over \( m \in [1, \omega) \), we shall construct families \( (f_{\tau(m)}^*)_{m<\omega} \subseteq X^* \) and \( (x_{\tau(m)})_{1\leq m<\omega} \subseteq S_X \) satisfying the following conditions for all \( m \in [1, \omega) \):

(I) \( f_{\tau(m)}^* \in S_{\tau(m)/2}^{\rho_{n_m}(t_m)}((1 + \nu \sum_{j=1}^{m}2^{-j})K) \);  

(II) For all \( i, j \in [1, m] \),

\[
\langle f_{\tau(j)}^*, x_{\tau(i)} \rangle = \begin{cases} 
\langle f_{\tau(i)}^*, x_{\tau(i)} \rangle > \frac{(1-3\nu)\delta_{m}}{4(2+\nu)} & \text{if } \tau(i) \leq \tau(j) \\
0 & \text{if } \tau(i) \nleq \tau(j) \end{cases} \tag{3.4}
\]

(III) For all \( x \in \text{span}\{x_{\tau(l)} \mid 1 \leq l < m \} \) and scalars \( a \) we have \( \|x + ax_{\tau(m)}\| \geq (1 - \delta_m)\|x\| \).

Since for \( 0 < m < \omega \) we have \( S_{\tau(m)/2}^{\rho_{n_m}(t_m)}((1 + \nu \sum_{j=1}^{m}2^{-j})K) \subseteq (1 + \nu)K \), once the induction is complete the first assertion of (i) then follows from (3.3), (I) and (II) by taking \( x_t^* = \frac{1}{1+p}f_t^* \) for each \( t \in \mathbb{S}^* \).

The second assertion of (i) follows from the Grunblum criterion (see, e.g., Proposition 1.1.9 of [II]) and the fact that, by (III), for \( 1 \leq l \leq m < \omega \) and scalars \( a_1, \ldots, a_m \) we have

\[
\left\| \sum_{q=1}^{m} a_q x_{\tau(q)} \right\| \leq \frac{1}{\prod_{q=l+1}^{m} (1 - \delta_q)} \left\| \sum_{q=1}^{m} a_q x_{\tau(q)} \right\| \leq (1 + \delta) \left\| \sum_{q=1}^{m} a_q x_{\tau(q)} \right\|.
\]

For each \( n < \omega \) let \( o_n \) denote the root of \( T_n \). It follows from (3.2) that

\[
0 \in \bigcap_{n<\omega} S_{\tau_n/2}^{\xi_n}(K) \subseteq \bigcap_{n<\omega} S_{\rho(T_n)}^{-1}(K) = \bigcap_{n<\omega} S_{\tau_n/2}^{\rho(T_n)(o_n)}(K). \tag{3.5}
\]
Define \( f^*_\tau(0) := 0 \in \bigcap_{n<\omega} s^{\xi_n}_{\epsilon_n/2}(K) \). Since 

\[
f^*_\tau(0) \in s^{\rho_{\tau_n}(\omega_n)}_{\epsilon_n/2}(K) \subseteq s^{\rho_{\tau_n}(t_1)+1}_{\epsilon_n/2}(K) = s_{\epsilon_n/2}(s^{\rho_{\tau_n}(t_1)}_{\epsilon_n/2}(K)),
\]

it follows from the definition of the derivation \( s_{\epsilon_n/2} \) that there exists \( f^*_\tau(1) \in s^{\rho_{\tau_n}(t_1)}_{\epsilon_n/2}(K) \) such that \( \|f^*_\tau(1)\| = \|f^*_\tau(1) - f^*_\tau(0)\| > \epsilon_n/4 \). Choose \( x_{\tau(1)} \in S_X \) so that 

\[
\langle f^*_\tau(1), x_{\tau(1)} \rangle > \frac{\epsilon_n}{4}.
\]

It is readily checked that (I)-(III) hold for \( m = 1 \).

Fix \( k \in [1, \omega) \) and suppose that the points \( f^*_\tau(m) \in s^{\rho_{\tau_n}(t_m)}_{\epsilon_n/2}((1+\nu \sum_m 2^{-j})K) \) and \( x_{\tau(m)} \in S_X \) have been defined for \( 1 \leq m \leq k \) in such a way that properties (I)-(III) are satisfied for \( 1 \leq m \leq k \). Let \( (k+1)^- \) denote the unique ordinal less than \( k+1 \) and such that \( \tau((k+1)^-) = \tau(k+1)^- \). To carry out the inductive step of the proof we show how to construct \( f^*_\tau(k+1) \in s^{\rho_{\tau_n}(k+1)}_{\epsilon_n+1/2}((1+\nu \sum_{j=1}^{k+1} 2^{-j})K) \) and \( x_{\tau(k+1)} \in S_X \) so that (I)-(III) are satisfied for \( 1 \leq m \leq k+1 \). Our first task will be to define \( f^*_\tau(k+1)^- \) as a point inside a certain \( \text{weak}^* \)-neighbourhood of \( f^*_\tau(k+1)^- \) and then show that (I) holds for \( m = k+1 \). To this end let \( G \) be a finite \( \delta_{k+1} \)-net in \( S_{\text{span}\{x_{\tau(i)}\}_{1 \leq i \leq k}} \) and for each \( g \in G \) let \( h_g \in X^* \) be such that \( \langle h_g^*, g \rangle = 1 \). Set \( F = \text{span}\{f^*_\tau(i) \mid 1 \leq i \leq k\} \cup \{h_g^* \mid g \in G\} \subseteq X^* \), let \( A \) be a finite \( \frac{\nu}{4+2\nu} \)-net in \( S_F \) and let \( \{y_{f^*} \mid f^* \in A\} \subseteq S_X \) be such that \( f^*(y_{f^*}) \geq \frac{4+2\nu}{4+2
u} \) for each \( f^* \in A \). Let 

\[
\mathcal{U}_1 = \bigcap_{i=1}^k \left\{ x^* \in X^* \mid |\langle x^* - f^*_\tau(k+1)^-, x_{\tau(i)} \rangle| < \frac{2^{-k-4\nu}(1-3\nu)\epsilon_{n_k}}{k(2+\nu)(1+\nu)} \right\}; \quad \text{and,} \quad \\
\mathcal{U}_2 = f^*_\tau(k+1)^- + \frac{\nu \epsilon_{n_k+1}}{4(2+\nu)} B^o_X \oplus \left\{ x_{\tau(i)} \mid 1 \leq i \leq k \right\} \cup \{y_{f^*} \mid f^* \in A\} \bigg]^\perp.
\]

Note that \( \mathcal{U}_2 \) is a \( \text{weak}^* \)-open neighbourhood of \( f^*_\tau(k+1)^- \) by Lemma 3.6 hence \( \mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2 \) is a \( \text{weak}^* \)-open neighbourhood of \( f^*_\tau(k+1)^- \). On the one hand, if \( (k+1)^- = 0 \) then, by (3.5) and the hypothesis that (I) holds for \( 1 \leq m \leq k \), we have 

\[
f^*_\tau(k+1)^- \in s^{\rho_{\tau_n}(k+1)}_{\epsilon_{n_k+1}/2}(K) \subseteq s^{\rho_{\tau_n}(k+1)}_{\epsilon_{n_k+1}/2}((1+\nu \sum_{j=1}^k 2^{-j})K) \subseteq s^{\rho_{\tau_n}(k+1)}_{\epsilon_{n_k+1}/2}((1+\nu \sum_{j=1}^k 2^{-j})K).
\]

\[21\]
On the other hand, if \((k + 1)\) is not equal to 0 then, by the hypothesis that (I) holds for \(1 \leq m \leq k\), we have

\[
\begin{align*}
 f^*_\tau(k+1) \in S^{\rho_{\tau k+1}(t_{k+1})}_{\epsilon_{\tau k+1}/2}((1 + \nu \sum_{j=1}^{k} 2^{-j})K) 
 \subseteq S^{\rho_{\tau k+1}(t_{k+1})}_{\epsilon_{\tau k+1}/2}((1 + \nu \sum_{j=1}^{k} 2^{-j})K) 
 \subseteq S^{\rho_{\tau k+1}(t_{k+1})}_{\epsilon_{\tau k+1}/2}((1 + \nu \sum_{j=1}^{k} 2^{-j})K).
\end{align*}
\]

It follows from (3.6), (3.7), and the definition of the derivation \(s^i\) for \(i > 0\) and \(\xi \in \text{Ord}\) that there exists \(u^* \in U \cap s^{\rho_{\tau k+1}(t_{k+1})}_{\epsilon_{\tau k+1}/2}((1 + \nu \sum_{j=1}^{k} 2^{-j})K)\) such that \(\|f^*_\tau(k+1) - u^*\| > \epsilon_{\tau k+1}/4\). Define

\[
 f^*_\tau(k+1) := u^* - \sum_{l=1}^{k} \frac{\langle u^* - f^*_\tau(k+1)^{-}, x^*_{\tau(l)} \rangle}{\langle f^*_\tau(l), x^*_{\tau(l)} \rangle} (f^*_\tau(l) - f^*_\tau(l)^{-}).
\]

Since \(u^* \in U_1\) and since \(f^*_\tau(l) - f^*_\tau(l)^{-} \in (1 + \nu)K\) for \(1 \leq l \leq k\) (by the assumption that (I) holds for \(1 \leq m \leq k\)), it follows from the definition of \(f^*_\tau(k+1)\) that we have \(f^*_\tau(k+1) - u^* \in cK\), where \(c > 0\) is a scalar that may be taken to satisfy

\[
\begin{align*}
 c &\leq \sum_{l=1}^{k} \frac{\|u^*_{\tau(k+1)} - f^*_\tau(k+1)^{-}, x^*_{\tau(l)} \|}{\|f^*_\tau(l), x^*_{\tau(l)} \|} 2(1 + \nu) \\
 &\leq \sum_{l=1}^{k} \frac{2^{-k-1} \nu}{k} \frac{\|u^*_{\tau(k+1)} - f^*_\tau(k+1)^{-}, x^*_{\tau(l)} \|}{(1 + \nu)} (1 - 3\nu) \epsilon_{\tau l} \\
 &= \sum_{l=1}^{k} \frac{2^{-k-1} \nu}{k} \\
 &= \nu 2^{-k-1}. \quad (3.8)
\end{align*}
\]

An appeal to Lemma 3.7 yields

\[
 f^*_\tau(k+1) = u^* + (f^*_\tau(k+1) - u^*) \in S^{\rho_{\tau k+1}(t_{k+1})}_{\epsilon_{\tau k+1}/2}((1 + \nu \sum_{j=1}^{k+1} 2^{-j})K)
\]

hence (I) holds for \(m = k + 1\).

We now show how to define \(x^*_{\tau(k+1)}\) and then verify that (II) and (III) hold for \(m = k + 1\). Since \(u^* \in U_2\) we may write \(u^* = f^*_\tau(k+1)^{-} + y^* + x^*\), where \(\|y^*\| < \frac{\epsilon_{\tau k+1}}{4(2 + \nu)}\)
and \( x^* \in (\{x_{\tau(i)} \mid 1 \leq i \leq k\} \cup \{y_\ast \mid f^* \in A\})^\perp \). Since

\[
\|x^*\| \geq \|u^* - f^*_{\tau(k+1)}\| - \|y^*\| > \frac{\epsilon_{n_{k+1}}}{4} - \frac{\nu \epsilon_{n_{k+1}}}{4(2 + \nu)} > \frac{(1 - \nu)\epsilon_{n_{k+1}}}{4},
\]

an application of Lemma 3.5 with \( x \) and \( y \) in the current proof yields we need to prove the case where at least one of the conditions \( \tau \) holds for \( 1 \leq j \leq k \). Since \((II)\) holds for \( 1 \leq j \leq k \), it is clear from the definition of \( f \) that.

\[
\tau(\perp) \subseteq \langle \tau \rangle |(\perp) \rangle \ni 1 = \langle \tau \rangle |(\perp) \rangle.
\]

We now show that \((II)\) holds for \( m = k + 1 \). By the induction hypothesis, we need to prove the case where at least one of \( i \) and \( j \) is equal to \( k + 1 \). Since \( x_{\tau(k+1)} \in S_{F_k} \subseteq S_{X} \) and

\[
\langle u^*, x_{\tau(k+1)} \rangle = \langle u^*, y \rangle - \langle u^*, f^*_{\tau(k+1)} \rangle = \langle u^* + y^*, y \rangle
\]

\[
> \frac{(1 - 2\nu)\epsilon_{n_{k+1}}}{4(2 + \nu)} - \frac{\nu \epsilon_{n_{k+1}}}{4(2 + \nu)}
\]

\[
= \frac{(1 - 2\nu)\epsilon_{n_{k+1}}}{4(2 + \nu)}.
\]

We now show that \((II)\) holds for \( m = k + 1 \). By the induction hypothesis, we need to prove the case where at least one of \( i \) and \( j \) is equal to \( k + 1 \). Since \( x_{\tau(k+1)} \in S_{F_k} \subseteq \bigcap_{j=1}^{k} \ker(f^*_{\tau(j)}) \) we have \( \langle f^*_{\tau(j)}, x_{\tau(k+1)} \rangle = 0 \) for \( 1 \leq j \leq k \). Moreover, it is clear from the definition of \( f^*_{\tau(k+1)} \) and the fact that \( x_{\tau(k+1)} \in \bigcap_{j=1}^{k} \ker(f^*_{\tau(j)}) \) that

\[
\langle f^*_{\tau(k+1)}, x_{\tau(k+1)} \rangle = \langle u^*, x_{\tau(k+1)} \rangle - 0 = \langle u^*, x_{\tau(k+1)} \rangle > \frac{(1 - 3\nu)\epsilon_{n_{k+1}}}{4(2 + \nu)}.
\]

Since \((II)\) holds for \( 1 \leq m \leq k \), if \( l, i \in [1, k] \) then

\[
\langle f^*_l - f^*_i, x_{\tau(i)} \rangle = \begin{cases} \langle f^*_i, x_{\tau(i)} \rangle, & \text{if } i = l \\ 0, & \text{if } i \neq l. \end{cases}
\]

(3.9)

It follows that if \( 1 \leq i \leq k \) then

\[
\langle f^*_{\tau(k+1)}, x_{\tau(i)} \rangle = \langle u^*, x_{\tau(i)} \rangle - \frac{\langle u^* - f^*_{\tau(k+1)} - x_{\tau(i)} \rangle}{f^*_i, x_{\tau(i)}}(f^*_{\tau(i)}, x_{\tau(i)})
\]

\[
= \langle f^*_{\tau(k+1)}, x_{\tau(i)} \rangle
\]

\[
= \begin{cases} \langle f^*_{\tau(i)}, x_{\tau(i)} \rangle > \frac{(1 - 3\nu)\epsilon_{n_{k}}}{4(2 + \nu)} & \text{if } \tau(i) \not\perp \tau(k + 1) \\ 0 & \text{if } \tau(i) \perp \tau(k + 1) \end{cases},
\]

23
hence (II) holds for $m = k + 1$.

Finally, we show that (III) holds for $m = k + 1$. Let $a$ be a scalar and, to avoid triviality, let $x \in \text{span}\{x_{\tau(i)} \mid 1 \leq i \leq k\}$ be nonzero. Let $g_x \in G$ be such that $\|\|x\|^{-1}x - g_x\| \leq \delta_{k+1}$. Since $\langle h^*_x, g_x \rangle = 1$ and $x_{\tau(k+1)} \in \ker(h^*_x)$ we have

\[
\|x + ax_{\tau(k+1)}\| \geq |\langle h^*_x, x + ax_{\tau(k+1)} \rangle| \\
= |\langle h^*_x, \|x\|^{-1}x \rangle|\|x\| \\
\geq (|\langle h^*_x, \|x\|^{-1}x - g_x \rangle| - |\langle h^*_x, \|x\|^{-1}x - g_x \rangle|)\|x\| \\
\geq (1 - \delta_{k+1})\|x\|,
\]

which completes the proof of part (i) of the theorem.

We now prove (ii). To this end suppose $X^*$ is norm separable and let $(z^*_m)_{m=1}^\infty$ a norm dense sequence in $X^*$. To see that $(x_{\tau(m)})_{m=1}^\infty$ may be chosen to be a shrinking basis, we modify the proof of (i) by extending the list of conditions (I)-(III) to include the following fourth condition:

(IV) $x_{\tau(m)} \in \bigcap_{j=1}^{m-1} \ker(z^*_j)$.

In the inductive construction involving the verification of properties (I)-(III), we amend the argument to ensure that (IV) holds for all $m \in [1, \omega)$ as follows. For the basis step we require no change in the argument since $\bigcap_{j=1}^0 \ker(z^*_j) = X$. For the inductive step we change the definition of $F$ so that

\[
F = \text{span}\left( \{f_{\tau(i)}^* \mid 1 \leq i \leq k\} \cup \{h^*_g \mid g \in G\} \cup \{z^*_j \mid 1 \leq j \leq k\} \right).
\]

Since $x_{\tau(k+1)}$ is defined so that $x_{\tau(k+1)} \in S_{F_+}$, the induction yields that (IV) holds for all $m \in [1, \omega)$. With this modification it is now easy to see that $(x_{\tau(m)})_{m=1}^\infty$ is shrinking. Indeed, let $f^* \in [(x_{\tau(m)})_{m=1}^\infty]^*$ and fix $\epsilon > 0$. Let $\hat{f}^* \in X^*$ be an extension of $f^*$ to $X$ and let $N < \omega$ be such that $\|\hat{f}^*-z^*_N\| < \epsilon$. By (IV) we have $\langle z^*_N, x_{\tau(m)} \rangle = 0$ for all $m > N$, hence $m > N$ implies

\[
\|f^*|_{x_{\tau(m)}|_{m>N}}\| = \|\hat{f}^*|_{x_{\tau(m)}|_{m>N}}\| \leq \|(\hat{f}^* - z^*_N)|_{x_{\tau(m)}|_{m>N}}\| + \|z^*_N|_{x_{\tau(m)}|_{m>N}}\| < \epsilon.
\]

In particular, $\lim_{M \to \infty} \|f^*|_{x_{\tau(m)}|_{m>M}}\| = 0$. As $f^* \in [(x_{\tau(m)})_{m=1}^\infty]^*$ was arbitrary, $(x_{\tau(m)})_{m=1}^\infty$ is shrinking by Proposition 3.2.6 of [1].

We now show how to modify the proof of (i) further so that $(Qx_{\tau(m)})_{m=1}^\infty$ is a shrinking basis for $X/Z$ with basis constant not exceeding $1 + \delta$. In a similar spirit to the proof of Proposition 3.5 of [24], the main idea is to modify the proof of (i) to incorporate the arguments from Johnson and Rosenthal’s proof of Theorem III.1 of [21]. To this end let $\|\cdot\|$ be an equivalent norm on $X$ such that properties (i) and (ii) of Theorem 3.8 hold with $c = (1 + \delta)^{1/2}$ and let $(z_p)_{p=1}^\infty$ be a norm dense sequence
\[ v^* \in [(v_j)_{j=1}^m]^* \] with \[ \|v^*\| = 1 \] there is a natural number \( p \leq p_m \) such that \[ |\langle v, z_p \rangle - \langle v^*, v \rangle| \leq \delta_m'/3 \] for all \( v \in [(v_j)_{j=1}^m] \).

(VI) \[ |\langle v_m, z_p \rangle| < \delta_m'/3 \] for all \( z_p \) with \( p \leq p_{m-1} \).

At the basis step of the induction we set \( p_0 = 0 \) and use Helly's theorem (or Goldstine's theorem), the density of \( (z_p)_{p=1}^\infty \) in \( S(X,\|\|) \) and the total boundedness of \( S((v_1),\|\|) \) and \( S((v_1)^*,\|\|) \) to obtain also \( p_1 > p_0 \) large enough that (V) holds for \( m = 1 \) (we leave the straightforward details to the reader). Since \( p_0 \) is defined to be 0, which is not in the index set of the sequence \( (z_p)_{p=1}^\infty \), (VI) is true for \( m = 1 \).

At the inductive step of the modified construction we assume that for some \( k \in [1,\omega) \) the properties (I)-(VI) hold for \( m = 1, \ldots, k \). We again use Helly's theorem to obtain \( p_{k+1} > p_k \) so that (V) holds for \( m = k + 1 \). To obtain that (VI) is true for \( m = k + 1 \) we modify the argument in the proof of (i) as follows. Let \( e \in \mathbb{R} \) be such that

\[
ke \frac{4(2 + \nu)}{(1 - 3\nu)e_{n+1}}2(1 + \nu) \sup \{\|y^*\| \mid y^* \in K\} = \frac{1}{2} \frac{(1 - 3\nu)e_{n+1}}{4(2 + \nu)} \frac{1}{(1 + \delta)^{1/2}}
\]

and define

\[
\mathcal{U}_3 = \bigcap_{i=1}^k \left\{ x^* \in X^* \mid |\langle x^* - f^*_{\tau(k+1)\ominus i}, x_{\tau(j)} \rangle| < e \right\}; \text{ and,}
\]

\[
\mathcal{U}_4 = \bigcap_{p=1}^{p_k} \left\{ x^* \in X^* \mid |\langle x^* - f^*_{\tau(k+1)\ominus p}, z_j \rangle| < \frac{1}{2} + \frac{(1 - 3\nu)e_{n+1}}{4(2 + \nu)} \right\}.
\]

We modify the definition of \( \mathcal{U} \) in the proof of (i) so that \( \mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4 \) and, as in the proof of (i), choose \( u^* \in \mathcal{U} \cap S_{\epsilon_{n+1}/2} f^*_{\tau(k+1)\ominus 1} (1 + \nu \sum_{j=1}^k 2^{-j}K) \) such that \( \|f^*_{\tau(k+1)\ominus j} - u^*\| > e_{n+1}/4 \). Since (II) holds for \( 1 \leq m \leq k + 1 \) and since \( u^* \in \mathcal{U}_3 \), it follows from the definition of \( f^*_{\tau(k+1)} \) that

\[
\|f^*_{\tau(k+1)\ominus j} - u^*\| \leq \frac{1}{2} + \frac{(1 - 3\nu)e_{n+1}}{4(2 + \nu)} \frac{1}{(1 + \delta)^{1/2}}.
\]
Moreover, since $u^* \in \mathcal{U}_1$ we deduce that for all $p \leq p_k$ we have

$$ |\langle f^*_\tau(k+1) - f^*_\tau(k+1)^-, z_p \rangle| \leq \|f^*_\tau(k+1) - u^*\| \|z_p\| + |\langle u^* - f^*_\tau(k+1)^-, z_p \rangle| $$

$$ \leq \frac{1}{2} \frac{\delta'_{k+1} (1 - 3\nu) \epsilon_{n_{k+1}}}{3} \leq \frac{1}{(1 + \delta)^{1/2}} \frac{1}{4(2 + \nu)} + \frac{1}{2} \frac{\delta'_{k+1} (1 - 3\nu) \epsilon_{n_{k+1}}}{3} \leq \frac{1}{(1 + \delta)^{1/2}} \frac{1}{4(2 + \nu)} + \frac{1}{2} \frac{\delta'_{k+1} (1 - 3\nu) \epsilon_{n_{k+1}}}{3}. $$

(3.10)

Since

$$ \|f^*_\tau(k+1) - f^*_\tau(k+1)^-\| \geq \|f^*_\tau(k+1) - f^*_\tau(k+1)^-\| \geq \langle f^*_\tau(k+1) - f^*_\tau(k+1)^-, x_{\tau(k+1)} \rangle > \frac{(1 - 3\nu) \epsilon_{n_{k+1}}}{4(2 + \nu)}, $$

it follows from (3.10) that for

$$ v_{k+1} := \|f^*_\tau(k+1) - f^*_\tau(k+1)^-\|^{-1}(f^*_\tau(k+1) - f^*_\tau(k+1)^-) $$

we have $|\langle v_{k+1}, z_p \rangle| \leq \delta'_{k+1}/3$ for all $p \leq p_k$. Thus (I)-(VI) hold for all $m \in [1, \omega)$ with these modifications to the proof of (i).

Still following the argument in [21], our next step is to show that $(v_m)_{m=1}^\infty$ is a basic sequence whose basis constant with respect to $\|\cdot\|$ is no larger than $(1 + \delta)^{1/2}$. To this end fix $m \in [1, \omega)$ and let $v \in [(v_\ell)_{\ell=1}^m]$ be such that $\|v\| = 1$. Choose $v^* \in [v_\ell]_{1 \leq \ell \leq m}$ such that $\langle v^*, v \rangle = 1 = \|v^*\|$ and choose $p \leq p_m$ so that (V) holds for $v^*$. Then $|\langle v, z_p \rangle| \geq 1 - \delta'_{m}/3$, hence for any scalar $a$ we have

$$ \|v + av_{m+1}\| \left\{ \begin{array}{lcl} > 1 & \text{if } |a| > 2 \\ \geq \|v, z_p\| + \langle av_{m+1}, z_p \rangle \geq (1 - \frac{\delta'_{m}}{3}) - \frac{2\delta'_{m}}{3} & \text{if } |a| \leq 2 \\ \geq 1 - \delta'_{m}. \end{array} \right.$$ 

It follows that $\|\sum_{q=1}^m a_q v_q\| \leq \frac{1}{1 - \delta'_{m}} \|\sum_{q=1}^{m+1} a_q v_q\|$ for any scalars $a_1, \ldots, a_m, a_{m+1}$. Thus for $1 \leq l \leq m < \omega$ and any scalars $a_1, \ldots, a_m$ we have

$$ \left\| \sum_{q=1}^l a_q v_q \right\| \leq \frac{1}{\prod_{q=l+1}^m (1 - \delta'_{q})} \left\| \sum_{q=1}^m a_q v_q \right\| \leq (1 + \delta)^{1/2} \left\| \sum_{q=1}^m a_q v_q \right\|. $$

(3.11)

By Grunblum’s criterion, $(v_m)_{m=1}^\infty$ is a basic sequence whose basis constant with respect to $\|\cdot\|$ is no larger than $(1 + \delta)^{1/2}$.

Let $(v_m)_{m=1}^\infty$ be the sequence of functionals in $[(v_m)_{m=1}^\infty]^*$ biorthogonal to $(v_m)_{m=1}^\infty$ and define $T : X \rightarrow [(v_m)_{m=1}^\infty]^*$ by $\langle Tx, v \rangle = \langle v, x \rangle$ for $x \in X$ and $v \in [(v_m)_{m=1}^\infty]^*$. 26
That is, \( Tx = (v_x)x \) for each \( x \in X \). Note that \( \ker(T) = \bigcap_{m=1}^{\infty} \ker(v_m) \).
Moreover since \( f^*_\tau(0) = 0 \) we have
\[
f^*_t = \sum_{\emptyset \leq s \leq t} (f^*_s - f^*_{s-})
\]
for each \( t \in \mathbb{T} \), hence
\[
\ker(T) = \bigcap_{m=1}^{\infty} \ker(v_m) = \bigcap_{t \in \mathbb{T}} \ker(f^*_t - f^*_{t-}) = \bigcap_{t \in \mathbb{T}} \ker(f^*_t) = \bigcap_{t \in \mathbb{T}} \ker(x^*_t) = Z.
\]

Following the argument in the proof of Theorem III.1 of [12] yields the equality
\[
T(X) = \{(v^*_m)_{m=1}^{\infty} \} \text{ and the existence of a linear isometry } T : (X/Z, \| \cdot \|) \to \{(v^*_m)_{m=1}^{\infty}, \| \cdot \| \} \text{ such that } TQx = Tx \text{ for every } x \in X.
\]
By Fact 6.6 of [12], for \( m \in [1, \omega] \) we have
\[
TQx \tau(m) = Tx \tau(m) = \sum_{m'=1}^{\infty} \langle Tx \tau(m), v_{m'} \rangle v^*_{m'} = \sum_{m'=1}^{\infty} \langle v_{m'}, x_{\tau(m)} \rangle v^* = \frac{\langle f^*_\tau(m), x_{\tau(m)} \rangle}{\| f^*_\tau(m) - f^*_{\tau(m)} \|} v^*_{m'}.
\]
For each \( m \in [1, \omega] \) let \( a_m = \langle f^*_{\tau(m)}, x_{\tau(m)} \rangle / \| f^*_\tau(m) - f^*_{\tau(m)} \|^{-1} \), so that \( TQx \tau(m) = a_m v^*_m \) for each such \( m \). As \( T \) is an isometry with respect to \( \| \cdot \| \), then, with respect to \( \| \cdot \| \), \( (Qx_{\tau(m)})_{m=1}^{\infty} \) is a basis for \( X/Z \) isometrically equivalent to \( (a_m v^*_m) \), whose basis constant coincides with the basis constant of \( (v^*_m)_{m=1}^{\infty} \), which coincides with the basis constant of \( (v_m)_{m=1}^{\infty} \), which is no larger than \( (1+\delta)^{1/2} \) (as shown above).
It follows that for \( 1 \leq l \leq m < \omega \) and any scalars \( a_1, \ldots, a_m \) we have
\[
\left\| \sum_{q=1}^{l} a_q Qx_{\tau(q)} \right\| \leq (1+\delta)^{1/2} \left\| \sum_{q=1}^{l} a_q Qx_{\tau(q)} \right\| \leq (1+\delta) \left\| \sum_{q=1}^{m} a_q Qx_{\tau(q)} \right\| \leq (1+\delta) \left\| \sum_{q=1}^{m} a_q Qx_{\tau(q)} \right\|.
\]
In particular, \( (Qx_{\tau(m)})_{m=1}^{\infty} \) is a basis for \( X/Z \) whose basis constant with respect to \( \| \cdot \| \) is no larger than \( 1 + \delta \). It remains then to show that \( (Qx_{\tau(m)})_{m=1}^{\infty} \) is shrinking. As \( (Qx_{\tau(m)})_{m=1}^{\infty} \) is equivalent to \( (a_m v^*_m)_{m=1}^{\infty} \), which is shrinking if and only if \( (v^*_m)_{m=1}^{\infty} \) is shrinking, it follows from the duality between shrinking and boundedly complete bases (see, e.g., Corollary 6.1 of [35]) that, to complete the proof, it suffices to show that \( (v^*_m)_{m=1}^{\infty} \) is boundedly complete. To this end we recall the following definition from [21]:

**Definition 3.9.** Let \( X \) be a Banach space. A sequence \( (y^*_m)_{m=1}^{\infty} \subseteq X^* \) is said to be **weak*-basic** if there is a sequence \( (y_m)_{m=1}^{\infty} \subseteq X \) so that \( (y_m, y^*_m)_{m=1}^{\infty} \) is biorthogonal and for each \( y^* \in (y^*_m)_{m=1}^{\infty} \) we have \( \sum_{q=1}^{m} \langle y^*, y_q \rangle y^*_q \) as \( m \to \infty \).

The following result is proved in [21].
Proposition 3.10 (Proposition II.1 of [21]). Let \( X \) be a Banach space, \( (y_m^*)_{m=1}^{\infty} \subseteq X^* \) and \( Q : X \rightarrow X/\bigcap_{m=1}^{\infty} \ker(y_m^*) \) the quotient map. Then

(a) \( (y_m^*)_{m=1}^{\infty} \) is weak*–basic if and only if \( X/\bigcap_{m=1}^{\infty} \ker(y_m^*) \) has a basis \( (e_m^*)_{m=1}^{\infty} \) with associated biorthogonal functionals \( (e_m^*)_{m=1}^{\infty} \) such that \( Q^*e_m^* = y_m^* \) for all \( m \in \mathbb{N} \). It follows that if \( (y_m^*)_{m=1}^{\infty} \) is weak*–basic, then \( (y_m^*)_{m=1}^{\infty} \) is basic.

(b) The following are equivalent:

(i) \( (y_m^*)_{m=1}^{\infty} \) is a boundedly complete weak*–basic sequence; and,

(ii) \( (y_m^*)_{m=1}^{\infty} \) is weak*–basic and \( (y_m^*)_{m=1}^{\infty} = (y_m^*)_{m=1}^{\infty} \).

We shall first apply (a) of Proposition 3.10 to show that \( (v_m)_{m=1}^{\infty} \) is weak*–basic, then apply (b) of Proposition 3.10 to deduce that \( (v_m)_{m=1}^{\infty} \) is boundedly complete, as desired. For \( m \in [1, \omega) \) let

\[
e_m := \frac{\| f_{\tau(m)}^* - f_{\tau(m)}^* \|}{\langle f_{\tau(m)}^*, x_{\tau(m)} \rangle} Q_{\tau(m)},
\]

so that \( T e_m = v_m^* \). For \( m \in [1, \omega) \) let

\[
v_m^{**} := (\langle v_q \rangle_{q=1}^\infty | v_m) | \langle v_q \rangle_{q=1}^\infty
\]

and \( e_m^* := T^* v_m^{**} \), so that \( (v_m^{**})_{m=1}^{\infty} \) and \( (e_m^*)_{m=1}^{\infty} \) are the sequences of biorthogonal functionals associated to the basic sequences \( (v_m)_{m=1}^{\infty} \) and \( (e_m)_{m=1}^{\infty} \), respectively. For \( 1 \leq m < \omega \) and \( x \in X \) we have

\[
\langle Q^* e_m^*, x \rangle = \langle e_m^*, Q x \rangle = \langle v_m^{**}, T Q x \rangle = \langle T x, v_m \rangle = \langle v_m, x \rangle,
\]

hence \( Q^* e_m^* = v_m \) for each \( m \in [1, \omega) \). By Proposition 3.10 (a), \( (v_m)_{m=1}^{\infty} \) is weak*–basic. By Proposition 3.10 (b), to complete the proof of Theorem 3.3 it now suffices to show that \( \left( (v_m)_{m=1}^{\infty} \right) = (v_m)_{m=1}^{\infty} \).

For each \( m \in [1, \omega) \) the operator \( S_M : (v_m)_{m=1}^{\infty} \rightarrow (v_m)_{m=1}^{\infty} \) given by setting

\[
S_M y^* = \sum_{m=1}^{M} \langle y^*, y_m \rangle v_m
\]

for each \( y^* \in (v_m)_{m=1}^{\infty} \), satisfies \( \| S_M \| \leq \prod_{m=M+1}^{\infty} \frac{1}{1 - s_m} \).

Suppose \( y^* \in (v_m)_{m=1}^{\infty} \). Then \( (S_M y^*)_{M=1}^{\infty} \) converges weak* to \( y^* \) since \( (v_m)_{m=1}^{\infty} \) is weak*–basic, hence \( \lim \inf_M \| S_M y^* \| \geq \| y^* \| \). On the other hand, since \( \| S_M \| \rightarrow 1 \) we have \( \lim \sup_M \| S_M y^* \| \leq \| y^* \| \). It follows that \( \| S_M y^* \| \rightarrow \| y^* \| \) as \( M \rightarrow \infty \), hence \( \| S_M y^* - y^* \| \rightarrow 0 \) since \( \| \cdot \| \) satisfies property (ii) of Theorem 3.3. As \( S_M y^* \in ((v_m)_{m=1}^{\infty} \) for all \( M \), we conclude that \( y^* \in ((v_m)_{m=1}^{\infty} \), which completes the proof of Theorem 3.3.
The following corollary of Theorem 3.3 may be useful in situation where one considers the $\epsilon$-Szlenk index for only a single $\epsilon > 0$ (rather than a for a sequence $(\epsilon_n)_{n<\omega}$), such as the work in the current paper on universal operators.

**Corollary 3.11.** Let $X$ be a Banach space, $K \subseteq X^*$ an absolutely convex, weak$^*$-compact set, $\epsilon > 0$, $\xi > 0$ a countable ordinal, and $(T, \preceq)$ a countable, well-founded, rooted tree such that $\rho(T) \leq \xi + 1$. If $s^\xi_n(K) \neq \emptyset$ then there exist families $(x^*_t)_{t \in T} \subseteq K$ and $(x_t)_{t \in T} \subseteq S_X$ such that

$$\langle x^*_t, x_s \rangle = \begin{cases} \frac{\epsilon_n}{17} & \text{if } s \preceq t \land s \in T, \\ 0 & \text{if } s \not\preceq t \lor s \notin T. \end{cases}$$

(3.13)

**Proof.** Suppose $s^\xi_n(K) \neq \emptyset$ so that, by Lemma 3.4, $s^{\xi+1}_n(K) \supseteq s^{\xi/2}_n(K) \neq \emptyset$. Let $t_0$ be a set such that $t_0 \notin T$ and let $(T_0, \preceq_0)$ be the tree obtained by setting $T_0 = T \cup \{t_0\}$ and extending $\preceq$ to $T_0$ by making $t_0$ the unique minimal element of $T_0$. Let $\xi_0 = \xi + 1$, so that $\rho(T_0) \leq \xi_0 + 1$ and $s^{\xi_0}_n(K) \neq \emptyset$. The conclusion of the corollary follows from an application of Theorem 3.3(i) with $\theta = 1/2$, $\epsilon_n = \epsilon/2$ for all $n < \omega$, $\xi_n = 0$ for $0 < n < \omega$, and $(T_n, \preceq_n)$ a tree consisting of a single node for $0 < n < \omega$ (since $([T]*, \preceq_T) = (\{0\} \times T, \preceq_T)$ is, in this case, naturally order isomorphic to $T$). \hfill \Box

4 Basic sequences of large Szlenk index

In this section we continue with the notation introduced in Section 3. Our first result concerns basic sequences in Banach spaces of countable Szlenk index.

**Theorem 4.1.** Let $X$ be an infinite dimensional Banach space such that $SZ(X) < \omega_1$ and let $\delta > 0$. Then there exists a subspace $Y \subseteq X$ such that $Y$ has a shrinking basis with basis constant not exceeding $1 + \delta$ and such that

$$\forall \epsilon > 0 \quad SZ\left(Y, \frac{\epsilon}{66}\right) \geq SZ(X, \epsilon),$$

(4.1)

hence $SZ(Y) = SZ(X)$.

**Proof.** Fix $\theta \in (0, \sqrt{65} - 8)$ and fix $\{\epsilon_n \mid n < \omega\}$, a countable dense subset of $(0, \infty) \subseteq \mathbb{R}$. We apply Theorem 3.3 with $K = B_X^*$, $\xi_n = SZ(X, \epsilon_n) - 1$ for each $n < \omega$, and $T = ((T_n, \preceq_n))_{n<\omega}$ a family of blossomed trees with $\rho(T_n) = SZ(X, \epsilon_n)$ for each $n < \omega$, to obtain families $(x^*_t)_{t \in T} \subseteq S_X$ and $(x_t)_{t \in T} \subseteq B_X$ such that

$$\langle x^*_t, x_s \rangle = \begin{cases} \frac{\epsilon_n}{8 + \theta} > \frac{\epsilon_n}{17} & \text{if } s \preceq t \in \{n\} \times T^*_n, \\ 0 & \text{if } s \not\preceq t \lor s \notin T^*_n. \end{cases}$$

(4.2)
Let \( Y_0 = [x_{(m)}]_{m=1}^\infty \). It follows from (4.2) and an argument similar to that used to prove (2.6) that
\[
\forall n < \omega \quad \forall (n, t) \in \{n\} \times \mathcal{T}_n^* \quad x^*_{(n, t)}|_{Y_0} \in s_{n/(8+\theta)}^{\rho_{n}(t)}(B_{Y_0}^*)
\]
and, subsequently, that \( 0 \in s_{n/(8+\theta)}^{\rho_{n}(t)}(B_{Y_0}^*) = s_{n/(8+\theta)}^{\rho_{n}(t)}(B_{Y_0}^*) \) for each \( n < \omega \). Thus,
\[
\forall n < \omega \quad S\zeta\left(Y_0, \frac{\epsilon_n}{8+\theta}\right) \geq S\zeta(X, \epsilon_n). \tag{4.3}
\]
Let \( \tau : \omega \rightarrow [\mathcal{X}] \) be a bijection such that \( \tau(m) \lesssim \tau(m') \) implies \( m \leq m' \). By (4.3) and Theorem 3.3 there exist families \((y_t)_{t \in [\mathcal{X}]^*} \subseteq S_{Y_0} \) and \((y_t^\prime)_{t \in [\mathcal{X}]^*} \subseteq B_{Y_0}^* \) such that
\[
\langle y_t^*, y_s \rangle = \begin{cases} 
\frac{\epsilon_n}{65} & \text{if } s \lesssim t \in \{n\} \times \mathcal{T}_n^*, \\
0 & \text{if } s \not\lesssim t
\end{cases} \quad s, t \in [\mathcal{X}]^*, \ n < \omega. \tag{4.4}
\]
and \((y_t(m))_{m=1}^\infty \) is a shrinking basic sequence with basis constant not exceeding \( 1 + \delta \). Let \( Y = [y_t(m)]_{m=1}^\infty \). It follows from (4.4) that
\[
\forall n < \omega \quad \forall (n, t) \in \{n\} \times \mathcal{T}_n^* \quad y^*_{(n, t)}|_{Y} \in s_{\epsilon_n/65}^{\rho_{n}^*(t)}(B_{Y^*})
\]
and, subsequently, that \( 0 \in s_{\epsilon_n/65}^{\rho_{n}^*(t)}(B_{Y^*}) = s_{\epsilon_n/65}^{\rho_{n}^*(t)}(B_{Y^*}) \) for each \( n < \omega \). Thus,
\[
\forall n < \omega \quad S\zeta\left(Y, \frac{\epsilon_n}{65}\right) \geq S\zeta(X, \epsilon_n). \tag{4.5}
\]
For each \( \epsilon > 0 \) choose \( N(\epsilon) < \omega \) such that \( \epsilon_{N(\epsilon)} \in \left[\frac{65\epsilon}{66}, \epsilon\right] \). From (4.5) we obtain
\[
\forall \epsilon > 0 \quad S\zeta\left(Y, \frac{\epsilon}{66}\right) \geq S\zeta\left(Y, \frac{\epsilon_{N(\epsilon)}}{65}\right) \geq S\zeta(X, \epsilon_{N(\epsilon)}) \geq S\zeta(X, \epsilon),
\]
which completes the proof of the theorem. \( \square \)

Two applications of Theorem 3.3 were used in the proof of Theorem 4.1 - the first to achieve separable reduction and the second to obtain a shrinking basic sequence. Clearly, if \( X \) is assumed norm separable then only one application of Theorem 3.3 is required, in which case the number 65 in (4.1) may be replaced by \( 8 + \theta \) for any \( \theta > 0 \). Moreover, in the general case we may replace 65 by \( 16 + \theta \) for any \( \theta > 0 \); this is achieved by proving a version of Lemma 3.4 of [24] for families \((\epsilon_n)_{n<\omega} \subseteq (0, \infty)\) and blossomed trees \((T_n, \lesssim_n)_{n<\omega}\) (as in the proof Theorem 3.3), then applying this generalisation of Lemma 3.4 of [24] to achieve separable reduction in the proof of Theorem 4.1 with \( \epsilon/2 \) (rather than \( \epsilon/(8 + \theta) \)) replacing \( \epsilon \).

We now turn our attention to quotients. Our main result in this direction is the following:
Theorem 4.2. Let $X$ be an infinite dimensional Banach space with separable dual and let $\delta > 0$. Then there exists a subspace $Z \subseteq X$ such that $X/Z$ has a shrinking basis with basis constant not exceeding $1 + \delta$ and such that
\[
\forall \epsilon > 0 \quad Sz\left(\frac{X}{Z}, \frac{\epsilon}{\delta}\right) \geq Sz(X, \epsilon),
\] hence $Sz(X/Z) = Sz(X)$.

Proof. Fix a countable, dense subset $\{\epsilon_n \mid n < \omega\}$ of $(0, \infty)$. We apply Theorem 3.3 with $K = B_{\infty}$, $\xi_n = Sz(X, \epsilon_n) - 1$ for each $n < \omega$, and $\mathfrak{T} = \{(T_n, \preceq_n)\}_{n<\omega}$ a family of blossomed trees with $\rho(T_n) = Sz(X, \epsilon_n)$ for each $n < \omega$. Let $\tau : \omega \rightarrow [[\mathfrak{T}]]$ be a bijection such that $\tau(m) \preceq_\mathfrak{T} \tau(m')$ implies $m \leq m'$. By Theorem 3.3 there exist families $(x_t)_{t \in [\mathfrak{T}]} \subseteq S_X$ and $(x_t^*)_{t \in [\mathfrak{T}]} \subseteq B_X^*$, such that
\[
\langle x_t^*, x_s \rangle = \begin{cases} 
\frac{2\epsilon_n}{17} & \text{if } s \preceq_\mathfrak{T} t \in \{n\} \times T_n^* \\
0 & \text{if } s \not\preceq_\mathfrak{T} t
\end{cases},
\] s, t \in [[\mathfrak{T}]]^*, n < \omega. (4.7)
and $(Q x_{\tau(m)})_{m=1}^\infty$ is a shrinking basis for $X/\bigcap_{t \in [\mathfrak{T}]} \ker(x_t^*)$ with basis constant not exceeding $1 + \delta$, where $Q : X \rightarrow X/\bigcap_{t \in [\mathfrak{T}]} \ker(x_t^*)$ is the quotient map. Let $Z = \bigcap_{t \in [\mathfrak{T}]} \ker(x_t^*)$. To complete the proof we will show that (4.6) holds.

We may assume that the families $(x_t)_{t \in [\mathfrak{T}]}$ and $(x_t^*)_{t \in [\mathfrak{T}]}$ above are those constructed in the proof of Theorem 3.3. Let $(f_t^*)_{t \in [\mathfrak{T}]}$ and $(v_m)_{m=1}^\infty$ also be as in the proof of Theorem 3.3. We have
\[
\text{span}\{x_t^* \mid t \in [[\mathfrak{T}]]^*\} = \text{span}\{f_t^* \mid t \in [[\mathfrak{T}]]^*\} = \text{span}\{v_m \mid 1 \leq m < \omega\} \subseteq Q^*((X/Z)^*),
\] where the first equality is immediate from the definitions, the second equality follows from the inductively verified fact that
\[
\forall k < \omega \quad \text{span}\{f_{\tau(m)} \mid 1 \leq m \leq k\} = \text{span}\{v_m \mid 1 \leq m \leq k\},
\] and the final inclusion follows from (3.12). Since $\|Q\| = 1$ and since $Q^*$ is an isometric embedding it follows respectively that $Q x_t \in B_{X/Z}$ and that $(Q^*)^{-1}(x_t^*)$ is a well-defined element of $B_{(X/Z)^*}$ for every $t \in [[\mathfrak{T}]]^*$. Since for $s, t \in [[\mathfrak{T}]]^*$ and $n < \omega$ we have
\[
\langle (Q^*)^{-1}(x_t^*), Q x_s \rangle = \langle x_t^*, x_s \rangle = \begin{cases} 
\frac{2\epsilon_n}{17} & \text{if } s \preceq_\mathfrak{T} t \in \{n\} \times T_n^* \\
0 & \text{if } s \not\preceq_\mathfrak{T} t
\end{cases},
\] and since span$\{Q x_t \mid t \in [[\mathfrak{T}]]^*\}$ is norm dense in $X/Z$, an argument similar to that used to prove (2.6) yields
\[
\forall n < \omega \quad \forall (n, t) \in \{n\} \times T_n^* \quad (Q^*)^{-1}(x_{(n,t)}^*) \in s^{\rho_{\mathfrak{T}}(t)}_{2\epsilon_n/17}(B_{(X/Z)^*})
\]
and, subsequently, that $0 \in s_{2n/17}(B_{(X/Z)^*}) = s_{2n/17}(B_{(X/Z)^*})$ for each $n < \omega$. Thus,

$$\forall n < \omega \ S\sigma(X, 2\epsilon_n/17) \geq S\sigma(X, \epsilon_n).$$

(4.9)

For each $\epsilon > 0$ choose $N(\epsilon) < \omega$ such that $\epsilon_{N(\epsilon)} \in [17\epsilon/18, \epsilon]$. From (4.9) we obtain

$$\forall \epsilon > 0 \ S\sigma(X/Z, \epsilon) \geq S\sigma(X/Z, 2\epsilon_{N(\epsilon)}/17) \geq S\sigma(X, \epsilon_{N(\epsilon)}) \geq S\sigma(X, \epsilon),$$

which completes the proof of the theorem.

5 Universal operators of large Szlenk index

In this section we classify the ordinals $\beta$ for which the class $\mathcal{C}\mathcal{Z}_\beta$ admits a universal element. The following result provides this classification via a consideration of operators of the form $\Sigma_T$.

Theorem 5.1. Let $X$ and $Y$ be Banach spaces, $\alpha < \omega_1$, $T \in \mathcal{L}(X,Y) \setminus \mathcal{Z}_\alpha(X,Y)$ and $(T, \preceq)$ a countably infinite, rooted, well-founded tree with $\rho(T) < \omega^\alpha + 1$. Then $\Sigma_T$ factors through $T$. Moreover if $T$ is blossomed and $\rho(T) \geq \omega^\alpha$, then $\Sigma_T$ is universal for $\mathcal{C}\mathcal{Z}_\alpha$. It follows that, for an ordinal $\beta$, the class $\mathcal{C}\mathcal{Z}_\beta$ admits a universal element if and only if $\beta < \omega_1$.

Proof. Let $\epsilon' > 0$ be small enough that $s_{\epsilon'}(T^*(B_Y^*)) \neq \emptyset$ and let $N < \omega$ be large enough that $\rho(T) \leq \omega^\alpha 2^N + 1$. Set $\epsilon = 2^{-N-1} \epsilon'$, so that $s_{\epsilon}(T^*(B_Y^*)) \neq \emptyset$ by (3.2). An application of Corollary 3.11 yields families $(x_t)_{t \in T^*} \subseteq S_X$ and $(x^*_t)_{t \in T^*} \subseteq T^* B_Y^*$ such that

$$\langle x^*_s, x_t \rangle = \begin{cases} 
\langle x^*_s, x_t \rangle > \frac{\epsilon}{17} & \text{if } s \preceq t, \\
0 & \text{if } s \not\preceq t,
\end{cases} \quad s, t \in T^*.$$

By Proposition 2.11, $\Sigma_T$ factors through $T$.

We now suppose that $T$ is blossomed and $\rho(T) \geq \omega^\alpha$. Since $T$ is infinite and rooted we have $\rho(T) \geq 2$ and that $\rho(T)$ is a successor ordinal. It follows that $\rho(T) > \omega^\alpha$, hence by Proposition 2.14 we have

$$S\sigma(\Sigma_T^*) \geq S\sigma(\Sigma_T^*, \epsilon) \geq \rho(T) > \omega^\alpha,$$

so that $\Sigma_T \in \mathcal{C}\mathcal{Z}_\alpha$. It follows that $\Sigma_T$ is universal for $\mathcal{C}\mathcal{Z}_\alpha$.

Finally, let $\beta$ be an arbitrary ordinal. If $\beta < \omega_1$ then, by the second assertion of Theorem 5.1, $\Sigma_{\omega_1^\beta}$ is universal for $\mathcal{C}\mathcal{Z}_\beta$, where $\omega_1^\beta$ is as constructed in Example 2.6. Now suppose on the other hand that $\beta \geq \omega_1$; to complete the proof we
show that $\mathcal{C} \mathcal{L} \beta$ does not admit a universal element. Suppose by way of contra-
position that $\mathcal{C} \mathcal{L} \beta$ does admit a universal element, $\Upsilon$ say. By Theorem 2.6 of [6] we have $Sz(C(\omega^{\beta} + 1)) = \omega^{\beta+1}$, where $C(\omega^{\beta} + 1)$ denotes the Banach space of continuous scalar-valued functions on the compact ordinal $\omega^{\beta} + 1$. It follows that $\Upsilon$ factors through the identity operator of $C(\omega^{\beta} + 1)$, hence $Sz(\Upsilon)$ is defined and satisfies $Sz(\Upsilon) \leq Sz(C(\omega^{\beta} + 1)) = \omega^{\beta+1}$. Moreover, the identity operator of $\ell_1$ belongs to $\mathcal{C} \mathcal{L} \beta$ since $\ell_1$ is not an Asplund space, hence $\Upsilon$ factors through $\ell_1$ and, in particular, $\Upsilon$ has separable range and its Szlenk index is defined. It thus follows by Proposition 2.3 that $Sz(\Upsilon) <\omega_1$, hence $\Upsilon \in \mathcal{L} \omega_1 \subseteq \mathcal{L} \beta$ - a contra-
diction. Thus $\mathcal{C} \mathcal{L} \beta$ does not admit a universal element whenever $\beta \geq \omega_1$.

Remark 5.2. It is straightforward to observe that we may replace $\Sigma T$ by $\Sigma T$ in
the statement of Theorem 5.1. However, the reason for our choic e of $\Sigma T$ over $\Sigma T$ is that universal operators may be thought of as ‘minimal’ elements of the class for which they are universal, and the operator $\Sigma T$ can be thought of as naturally ‘smaller’ than $\Sigma T$ since $T^*$ is a subtree of $T$ and $\Sigma T$, therefore factors through $\Sigma T$ by Proposition 2.12. Moreover, $T$ is not order isomorphic to a subtree of $T^*$ since $T$ is assumed to be well-founded.

The following result can be proved directly using techniques develop-
ed else-
where, for example in [6], but since we shall refer to this result later we provide a quick proof here using Theorem 5.1.

Corollary 5.3. Let $(T, \preceq)$ be an infinite blossomed tree. Then $Sz(\Sigma T^*) = \rho(T)\omega$.

Proof. Let $\alpha$ be the ordinal satisfying $\omega^{\alpha} \leq \rho(T) < \omega^{\alpha+1}$. Since $Sz(C(\omega^{\alpha} + 1)) = \omega^{\alpha+1} > \omega^{\alpha}$ by Samuel’s computation in [34] of $Sz(C(K))$ for countable compact Hausdorff $K$, by Theorem 5.1 we have that $\Sigma T^*$ factors through $C(\omega^{\alpha+1})$, hence $Sz(\Sigma T^*) \leq Sz(C(\omega^{\alpha+1})) = \omega^{\alpha+1} = \rho(T)\omega$.

As $T$ is infinite and rooted we have $\rho(T) \geq 2$. Moreover, as noted in Section 2.2 $\rho(T)$ is a successor ordinal. It follows that $\rho(T) > \omega^{\alpha}$, hence $Sz(\Sigma T^*) \geq Sz(\Sigma T^*, \epsilon) \geq \rho(T) > \omega^{\alpha}$ by Proposition 2.14. As $Sz(\Sigma T^*)$ is a power of $\omega$, we deduce that $Sz(\Sigma T^*) \geq \omega^{\alpha+1} = \rho(T)\omega$, which completes the proof.

Remark 5.4. Theorem 1.2 above, due to W.B. Johnson, may be obtained as an immediate consequence of Theorem 5.1 and the fact that $\mathcal{K} = \mathcal{L} 0$ by Proposition 2.3 of [5].

The following proposition relates some of the factorisation results of the current paper to known relationships between various closed operator ideals.

Proposition 5.5. Let $I$ be a cofinal subset of $\omega_1$ and for each ordinal $\xi \in I$ let $T_\xi$ be a blossomed tree with $\rho(T_\xi) = \xi + 1$. For $T \in \mathcal{X}$, the following are equivalent:
(i) $T$ factors $\Sigma_{T^*}$ for every $\xi \in I$.

(ii) $T$ factors $\Sigma_{\Omega}$.

(iii) $T$ factors $\Sigma_T$ for every countable tree $T$ with $ht(T) \leq \omega$.

(iv) $T \in C(\Omega)$.

(v) $T \in C(\mathcal{D})$.

(vi) $T \in C(\mathcal{D}_{\omega_1})$.

(vii) $T \in C(\bigcup_{\alpha \in \text{Ord}} \mathcal{D}_\alpha)$.

(viii) $T \in C(\bigcup_{\alpha < \omega_1} \mathcal{D}_\alpha)$.

The proof of Proposition 5.5 relies on the following result from [4].

**Theorem 5.6.** Let $X$ and $Y$ be Banach spaces and $T \in \mathcal{L}(X,Y)$. Suppose that at least one of $X$ and $Y$ is separable and that $T \notin \mathcal{K}(X,Y)$. Then $\Sigma_{\Omega}$ factors through $T$.

**Proof of Proposition 5.5.** The equivalence of (iv) to (viii) is Proposition 2.11 of [5]. To complete the proof it suffices to show that (v) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (viii).

To see that (v) $\Rightarrow$ (ii), let $X$ and $Y$ be Banach spaces and $T \in \mathcal{L}(X,Y) \setminus \mathcal{K}(X,Y)$. Let $\bar{T} : X \rightarrow Y$ be the separable codomain operator given by setting $\bar{T}x = T_x$ for all $x \in X$. Since $\bar{T} \notin \mathcal{K}$, by Theorem 5.6 there exist $U \in \mathcal{L}(\ell_1(\Omega),X)$ and $V \in \mathcal{L}(\bar{T}(X),\ell_\infty(\Omega))$ such that $V \bar{T}U = \Sigma_{\Omega}$. By the injectivity of $\ell_\infty(\Omega)$ [27, p.105], $V$ admits a continuous linear extension $\tilde{V} \in \mathcal{L}(Y,\ell_\infty(\Omega))$, and for such $\tilde{V}$ we have $\tilde{V}T = \Sigma_{\Omega}$. Thus (v) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) follows from Proposition 2.12 and the fact that every countable tree $T$ with $ht(T) \leq \omega$ is order isomorphic to a subtree of $\Omega$, whilst (iii) $\Rightarrow$ (i) is immediate from the fact that blossomed trees are by definition countable and well-founded.

Finally, the implication (i) $\Rightarrow$ (viii) is a consequence of Proposition 2.14. 

6 When the codomain is separable

It is evident from the definition of the operator $\Sigma_T$ associated to a tree $(T, \preceq)$ that the range of $\Sigma_T$ is contained in the closed linear span in $\ell_\infty(T)$ of the indicator functions $\chi_{T_{[t,\preceq]}}$, for $t \in T$. For example, the range of the universal non-compact operator $\ell_1 \rightarrow \ell_\infty$ of Johnson [20] is contained in the subspace $c_0$ of $\ell_\infty$, whilst the range of the Lindenstrauss-Pełczyński universal non-weakly compact summation
operator from \( \ell_1 \) to \( \ell_\infty \)\[^{[20]}\] is contained in the subspace \( c \) of \( \ell_\infty \) consisting of all convergent scalar sequences. In both these cases, the range is contained (up to isometric isomorphism) in a separable \( C(K) \) space. In the papers of Johnson\[^{[20]}\] and Lindenstrauss-Pelczyński\[^{[26]}\] it is noted that stronger versions of the universal operator theorems presented there hold under restriction to the class of operators having separable codomain. More precisely, it is noted in\[^{[20]}\] that if \( T : X \rightarrow Y \) is noncompact and \( Y \) is separable, then \( T \) factors the formal identity operator from \( \ell_1 \) to \( c_0 \). Moreover, in\[^{[26]}\] it is noted that if \( T : X \rightarrow Y \) is non-weakly compact and \( Y \) is separable, then \( T \) factors the summation operator from \( \ell_1 \) to \( c \) defined by \((a_n)_{n=1}^\infty \mapsto (\sum_{i=1}^n a_i)_{n=1}^\infty \). In a similar vein, we show in the current section that for every \( \alpha < \omega_1 \) there exists an operator \( \Upsilon_\alpha \) from \( \ell_1 \) into a separable \( C(K) \) space with \( Sz(\Upsilon_\alpha) > \omega^\alpha \) and such that \( \Upsilon_\alpha \) factors through any \( T : X \rightarrow Y \) with \( Y \) separable and \( Sz(T) \not\subseteq \omega^\alpha \).

Let \((T, \preceq)\) be a rooted and chain-complete tree and let \( t_\emptyset \) denote the root of \( T \). By Theorem 2.9\[^{[20]}\] the coarse wedge topology of \( T \) is compact Hausdorff, thus for such \( T \) we shall denote by \( C(T) \) the Banach space (with the sup norm) of coarse-wedge-continuous scalar-valued functions on \( T \). We denote by \( C_0(T) \) the codimension-1 subspace \( \{ f \in C(T) \mid f(t_\emptyset) = 0 \} \) of \( C(T) \). Since for all \( t \in T \) with \( ht_T(t) \) either 0 or a successor ordinal we have that \( T[t \preceq] \) is clopen with respect to the coarse wedge topology of \( T \), the set

\[
\{ \chi_{T[t \preceq]} \mid t \in T, \, ht_T(t) \text{ is 0 or a successor} \} \cup \{ 0 \} \quad (6.1)
\]

is a subset of \( C(T) \) that is closed under taking products and separates points of \( T \). It follows by the Stone-Weierstrass theorem that if \( T \) is countable then, since \( C(T) \) is the closed linear span of \( \{ \chi_{T[t \preceq]} \mid t \in T \} \), \( C(T) \) is separable.

The following definition establishes the class of operators from which we shall draw our examples of universal non-\( \alpha \)-Szlenk operators with separable codomain. We note that although the definition can be adapted to trees of arbitrarily large height, such generality is unnecessary for our purposes.

**Definition 6.1.** Let \((T, \preceq)\) be a rooted, well-founded tree. Define \( \sigma_T : \ell_1(T) \rightarrow C(T) \) by

\[
(\sigma_T x)(t) = \sum_{s \preceq t} x(s), \quad x \in \ell_1(T), \, t \in T.
\]

That is, \( \sigma_T \) is the unique element of \( \mathcal{L}(\ell_1(T), C(T)) \) that maps each \( e_t \in \ell_1(T) \) to \( \chi_{T[t \preceq]} \in C(T) \). Similarly, define \( \hat{\sigma}_T \) to be the unique element of \( \mathcal{L}(\ell_1(T^*), C_0(T)) \) that maps each \( e_t \in \ell_1(T^*) \) to \( \chi_{T[t \preceq]} \in C_0(T) \).

Notice that Proposition 2.14 holds true with \( \hat{\sigma}_T \) in place of \( \Sigma_T \). Indeed, since \( C_0(T) \) naturally embeds linearly and isometrically into \( \ell_\infty(T^*) \) via the restriction map \( R \in \mathcal{L}(C_0(T), \ell_\infty(T^*)) \), defined by setting \( R(f) = f|_T \), for each \( f \in C_0(T) \),
and since $\Sigma_{T^*} = R\hat{\sigma}_T$, we have $\hat{\sigma}_T^*(B_{C_0(T^*)}) = \Sigma_{T^*}(B_{l_\infty(T^*)})$. We thus deduce that $Sz(\hat{\sigma}_T) = Sz(\Sigma_{T^*})$ since the Szlenk indices of $\hat{\sigma}_T$ and $\Sigma_{T^*}$ are determined by the same subset of $l_1(T^*)^*$.

The following theorem is the main result of the current section.

**Theorem 6.2.** Let $X$ and $Y$ be Banach spaces, $\alpha < \omega_1$, $T \in \mathcal{L}(X,Y) \setminus \mathcal{B}_\alpha(X,Y)$ and $(T, \preceq)$ a countably infinite, rooted, well-founded tree with $\rho(T) < \omega^{\omega+1}$. If $Y$ is separable then $\hat{\sigma}_T$ factors through $T$. Moreover if $T$ is blossomed and $\rho(T) \geq \omega^\alpha$ then $\hat{\sigma}_T$ is universal for the class of non-$\alpha$-Szlenk operators having separable codomain.

Similarly to the comments in Remark 5.2 regarding Theorem 5.1, we note that although $\hat{\sigma}_T$ may be replaced by $\sigma_T$ in the statement of Theorem 6.2, we present Theorem 6.2 as stated since $\hat{\sigma}_T$ may be viewed as being naturally ‘smaller’ than $\sigma_T$.

A smallness condition of some kind on $Y$ is necessary for the first assertion of Theorem 6.2 to hold in general. To see this, by Corollary 5.3 it is enough to observe that for a countably infinite, rooted, well-founded tree $(T, \preceq)$, $\hat{\sigma}_T$ does not factor through $\Sigma_{T^*}$. Firstly, the fact that such $T$ is infinite and well-founded implies that $\mathrm{MAX}(T)$ contains an infinite anti-chain $\{t_n \mid n < \omega\}$, so that $\hat{\sigma}_T$ is non-compact since the set $\{\hat{\sigma}_Te_{t_n} \mid n < \omega\}$ is an infinite $\epsilon$-separated subset of $\hat{\sigma}_T(B_{l_1(T^*)})$ for any $\epsilon \in (0, 1)$. Secondly, the norm separability of $C_0(T)$ and the fact that $l_\infty(T^*)$ is a Grothendieck space implies that $\mathcal{L}(l_\infty(T^*), C_0(T)) = \mathcal{W}(l_\infty(T^*), C_0(T))$ [16]. Thirdly, the fact that $T$ is well-founded implies that $\Sigma_{T^*}$ is weakly compact by Proposition 2.13, hence for any $V \in \mathcal{L}(l_\infty(T^*), C_0(T)) = \mathcal{W}(l_\infty(T^*), C_0(T))$ we have that $V\Sigma_{T^*}$ is compact since $l_\infty(T^*)$ is isomorphic to a $C(K)$ space and therefore has the Dunford-Pettis Property [16]. Finally, since $V\Sigma_{T^*}$ therefore cannot factor $\hat{\sigma}_T$ for any $V \in \mathcal{L}(l_\infty(T^*), C_0(T))$, we conclude that $\Sigma_{T^*}$ does not factor $\hat{\sigma}_T$.

To prove Theorem 6.2 we first establish the following continuous analogue of Proposition 2.11.

**Proposition 6.3.** Let $K$ be a compact Hausdorff space, $I$ an index set and $\{K_i\}_{i \in I}$ a family of clopen subsets of $K$. For Banach spaces $X$ and $Y$ and $T \in \mathcal{L}(X,Y)$ the following are equivalent:

(i) $T$ factors the unique element of $\mathcal{L}(l_1(I), C(K))$ satisfying $e_i \mapsto \chi_{K_i}$, $i \in I$.

(ii) There exists $(\delta_i)_{i \in I} \subseteq \mathbb{R}$ with $\inf_{i \in I} \delta_i > 0$, a family $(x_i)_{i \in I} \subseteq X$ with $\sup_{i \in I} \|x_i\| < \infty$ and a weak*-continuous map $\Xi : K \rightarrow Y^*$ such that

$$
\forall i \in I \quad \forall k \in K_i \quad \langle \Xi(k), Tx_i \rangle = \begin{cases} 
\delta_i, & k \in K_i \\
0, & k \notin K_i 
\end{cases}.
$$
Proof. For each $k \in K$ let $g_k^*$ denote the evaluation of functional of $C(K)$ at $k$; that is, $g_k^*(f) = f(k)$ for each $f \in C(K)$.

Suppose (i) holds. Let $S$ denote the unique element of $\mathcal{L}(\ell_1(I), C(K))$ satisfying $e_i \mapsto \chi_{K_i}, i \in I$, and let $U \in \mathcal{L}(\ell_1(I), X)$ and $V \in \mathcal{L}(Y, C(K))$ be such that $S = VTU$. The map $k \mapsto g_k^*$ is a homeomorphic embedding of $K$ into $C(K)^*$ with respect to the weak-$\ast$-topology of $C(K)^*$, hence the map $\Xi : K \to Y^\ast$ defined by setting $\Xi(k) = V^*g_k^*$ for each $k \in K$ is weak-$\ast$-continuous. For each $i \in I$ set $x_i = Ue_i$, so that $\sup_{i \in I} \|x_i\| \leq \|U\| < \infty$. Then for $i \in I$ and $k \in K$ we have

$\langle \Xi(k), Tx_i \rangle = \langle V^*g_k^*, Tu e_i \rangle = \langle g_k^*, VTUe_i \rangle = \langle g_k^*, Se_i \rangle = \langle g_k^*, \chi_{K_i} \rangle = \begin{cases} 1, & k \in K_i \\ 0, & k \notin K_i \end{cases}$.

By taking $\delta_i = 1$ for each $i \in I$ we see that (ii) holds, as desired.

Now suppose (ii) holds. Let $U$ be the element of $\mathcal{L}(\ell_1(I), X)$ defined by setting $Ue_i = \delta_i^{-1}x_i$ for each $i \in I$, noting that $U$ is well-defined with $\|U\| \leq (\inf_{i \in I} \delta_i)^{-1} \sup_{i \in I} \|x_i\|$. Let $V$ be the element of $\mathcal{L}(Y, C(K))$ satisfying $(Vy)(k) = \langle \Xi(k), y \rangle$ for $y \in Y$ and $k \in K$, noting that $V$ is well-defined with $\|V\| = \sup_{k \in K} \|\Xi(k)\| < \infty$. For $i \in I$ and $k \in K$ we have

$\langle VTUe_i \rangle(k) = \langle g_k^*, VTUe_i \rangle = \delta_i^{-1} \langle V^*g_k^*, Tx_i \rangle = \delta_i^{-1} \langle \Xi(k), Tx_i \rangle = \begin{cases} 1, & k \in K_i \\ 0, & k \notin K_i \end{cases}$,

hence $VTUe_i = Se_i$ for every $i \in I$, hence $VTU = S$. \hfill \Box

The following lemma is another key ingredient required for the proof of Theorem 6.2.

Lemma 6.4. Let $X$ and $Y$ be Banach spaces such that $Y$ is separable. Let $T \in \mathcal{L}(X,Y), \delta > 0, (\mathcal{R}, \preceq')$ a blossomed tree and $(x_r)_{r \in \mathcal{R}} \subseteq S_X$ and $(x'_r)_{r \in \mathcal{R}} \subseteq T^\ast(B_{Y^\ast})$ families such that

$\langle x'_r, x_r \rangle = \begin{cases} \langle x'_r, x_r \rangle > \delta & \text{if } r \preceq' s \\ 0 & \text{if } r \not\preceq' s \end{cases}, \quad r, s \in \mathcal{R}.$ \hfill (6.2)

Then for any $\xi < \rho(\mathcal{R})$ and any $r_0 \in \mathcal{R}^{[k]}$ there exists a full subtree $\mathcal{S}$ of $\mathcal{R}[r_0 \preceq']$ and a family $(y'_s)_{s \in \mathcal{S}} \subseteq B_{Y^\ast}$ such that

$\langle y'_s, Tx_r \rangle = \langle x'_s, x_r \rangle, \quad s \in \mathcal{S}, r \in \mathcal{R}$ \hfill (6.3)

and the map $s \mapsto y'_s$ from $\mathcal{S}$ to $Y^\ast$ is coarse-wedge-to-weak$^\ast$ continuous.
Proof. We proceed by induction on \( \xi \). For the base case, namely \( \xi = 0 \), fix \( r_0 \in R[0] \), let \( S = \{r_0\} \) and choose \( y_{r_0}^* \in B_Y^* \) such that \( T^*y_{r_0}^* = x_{r_0} \). In this way we see that the assertion of the lemma is true in the case \( \xi = 0 \).

We now address the inductive step. Suppose \( 0 < \zeta < \rho(R) \) and that the assertion of the lemma holds for all \( \xi < \zeta \); we will now show it is then true for \( \xi = \zeta \). To this end fix \( r_0 \in R[0] \) and for each \( t \in R[r_0+] \) let \( S_t \) be a full subtree of \( R[t \preceq \cdot] \) and \( (y_s^*)_{s \in S_t} \subseteq B_Y^* \) a family such that

\[
\langle y_s^*, Tx_r \rangle = \langle x_s^*, x_r \rangle, \quad s \in S_t, \ r \in R
\]  

(6.4)

and the map \( \Xi : s \mapsto y_s^* \) from \( S_t \) to \( Y^* \) is coarse-wedge-to-weak* continuous. Let \( d \) be a metric on \( B_Y^* \) that is compatible with the weak* topology on \( B_Y^* \) and let \( (t_n)_{n=0}^\infty \) be an injective sequence in \( R[r_0+] \) such that \( (y_{t_n}^*)_{n=0}^\infty \) is weak* convergent. Let \( y_{r_0}^* \) denote the weak*-limit of \( (y_{t_n}^*)_{n=0}^\infty \). Passing to a subsequence we may assume that \( d(y_{t_n}^*, y_{r_0}^*) < 1/n \) for each \( n < \omega \). By Proposition 2.8 and the continuity of the maps \( \Xi_{t_n}, n < \omega \), for each \( n < \omega \) we may choose a finite set \( F_n \subseteq S_{t_n}[t_n+] \) such that

\[
\forall n < \omega \ \forall t \in W_{S_{t_n}}(t_n, F_n) \quad d(\Xi_{t_n}(t), \Xi_{t_n}(t_n)) < \frac{1}{n}.
\]

Define \( S := \{r_0\} \cup \bigcup_{n<\omega} W_{S_{t_n}}(t_n, F_n) \) and \( \Xi : S \rightarrow Y^* \) by

\[
\Xi(s) = \begin{cases} y_{r_0}^*, & s = r_0 \\ \Xi_{t_n}(s), & s \in W_{S_{t_n}}(t_n, F_n), \ n < \omega \end{cases}
\]

It is straightforward to check that \( S \) is a full subtree of \( R[r_0 \preceq \cdot] \) since \( W_{S_{t_n}}(t_n, F_n) \) is a full subtree of \( S_{t_n} \) for every \( n < \omega \). To see that (6.3) holds for this \( S \), note that since (6.1) holds for \( t = t_n \), for all \( n < \omega \), we need only check that (6.3) holds in the case where \( s = r_0 \). To this end note that for all \( r \in R \) we have

\[
\langle y_{r_0}^*, Tx_r \rangle = \lim_{n \rightarrow \omega} \langle y_{t_n}^*, Tx_r \rangle = \lim_{n \rightarrow \omega} \langle x_{t_n}^*, x_r \rangle = \langle x_{r_0}^*, x_r \rangle,
\]

where the final equality follows from (6.2). Thus (6.3) holds for all \( s \in S \) and \( r \in R \).

To complete the proof it remains only to establish the continuity of \( \Xi \). Since each \( \Xi_{t_n} \) is continuous, for \( n < \omega \), the only nontrivial case to check is whether \( \Xi \) is continuous at \( r_0 \). Fix \( \lambda > 0 \). Let \( N < \omega \) be large enough that \( N\lambda > 2 \) and let \( \mathcal{F} = \{t_0, \ldots, t_{N-1}\} \). For each \( s \in W_S(r_0, \mathcal{F}) \setminus \{r_0\} \) there exists a unique \( n_s \geq N \) such that \( t_{n_s} \preceq s \). So for \( s \in W_S(r_0, \mathcal{F}) \setminus \{r_0\} \) we have

\[
d(\Xi(s), \Xi(r_0)) \leq d(\Xi(s), \Xi(t_{n_s})) + d(\Xi(t_{n_s}), \Xi(r_0)) < \frac{1}{n_s} + \frac{1}{n_s} \leq \frac{2}{N} < \lambda.
\]

It follows that \( d(\Xi(s), \Xi(r_0)) < \lambda \) for all \( s \in W_S(r_0, \mathcal{F}) \), hence \( \Xi \) is continuous at \( r_0 \) since \( W_S(r_0, \mathcal{F}) \) is open in \( S \) by Proposition 2.8. \( \square \)
Proof of Theorem 6.2. Suppose $Y$ is separable. Let $(R, \preceq')$ be a blossomed tree with $\rho(R) = \rho(T)$ (c.f. Example 2.6), let $\epsilon' > 0$ be small enough that $s_{e'}^{\epsilon'}(T^*(B_{Y^*})) \neq \emptyset$, and let $N < \omega$ be large enough $\rho(T) \leq \omega^\omega + 1$. Set $\epsilon = 2^{-N-1}\epsilon'$, so that $s_{\epsilon}^\rho(T)^{-1}(T^*(B_{Y^*})) \neq \emptyset$ by (5.2). An application of Corollary 3.11 yields families $(x_s)_{s \in R} \subseteq S_X$ and $(x_s^*)_{s \in R} \subseteq T^*(B_{Y^*})$ such that

$$
\langle x_s^*, x_r \rangle = \begin{cases} 
\frac{r}{17} & \text{if } r \preceq' s, \ r, s \in R, \\
0 & \text{if } r \not\preceq' s.
\end{cases}
$$

(6.5)

We apply Lemma 6.4 with $r_0$ the root of $R$ and $\xi = \rho(T) - 1$ to obtain a full subtree $S$ of $R$ and a family $(y_s^*)_{s \in S} \subseteq B_{Y^*}$ such that

$$
\langle y_s^*, T x_r \rangle = \langle x_s^*, x_r \rangle, \quad s, r \in S,
$$

(6.6)

and the map $\Xi : s \mapsto y_s^*$ from $S$ to $Y^*$ is coarse-wedge-to-weak* continuous. From (6.5) and (6.6) we deduce that

$$
\langle y_s^*, T x_r \rangle = \begin{cases} 
\frac{r}{17} & \text{if } r \preceq' s, \ r, s \in S, \\
0 & \text{if } r \not\preceq' s.
\end{cases}
$$

(6.7)

By an application of Proposition 6.3 with $K = S$, index set $I = S$, clopen sets $K_s = S[s \preceq']$ for $s \in S$, and $\delta_s = \epsilon/17$ for all $s \in S$, we obtain that $\sigma_S$ factors through $T$. So to prove the first assertion of Theorem 6.2 it now suffices to show that $\dot{\sigma}_T$ factors through $\sigma_S$. To this end we now define three operators, $S$, $R$, and $P$, so that $\dot{\sigma}_T = PS\sigma_S R$. Let $\phi : T \rightarrow S$ be an order-isomorphism of $T$ onto a downward-closed subtree of $S$, noting that such an embedding exists by Proposition 2.7(i). Since $\phi$ is coarse wedge continuous by Proposition 2.10(ii), the operator $S \in \mathcal{L}(C(S), C(T))$ given by setting $Sf = f \circ \phi$ for each $f \in C(S)$ is well-defined. Let $R \in \mathcal{L}(\ell_1(T^*), \ell_1(S))$ be operator defined by

$$(Rx)(s) = \begin{cases} 
x(\phi^{-1}(s)), & s \in T^*, x \in \ell_1(T^*), s \in S, \\
0, & s \notin T^*.
\end{cases}$$

Let $t_0$ denote the root of $T$ and define $P \in \mathcal{L}(C(T), C_0(T))$ by setting $Pf = f - f(t_0)\chi_T$ for each $f \in C(T)$. Since for $t \in T^*$ we have

$$PS\sigma_S e_t = PS\sigma_S e_{\phi(t)} = PS\chi_{S[\phi(t) \leq']} = P \chi_{T[0 \leq]} = \chi_{T[0 \leq]} = \dot{\sigma}_Te_t,$$

we conclude that $\dot{\sigma}_T = PS\sigma_S R$, which completes the proof of the first assertion of the theorem.

For the second assertion of the theorem, we now suppose that $T$ is blossomed and $\rho(T) \geq \omega^\omega$. As $T$ is infinite and rooted, we have that $\rho(T) \geq 2$ and $\rho(T)$
is a successor ordinal, hence \( \rho(T) > \omega^\alpha \). Since \( T \) is blossomed, an application of Proposition 2.14 yields \( \sigma(T) \geq \rho(T) > \omega^\alpha \). Moreover, as noted in the paragraph following Definition 6.1, \( \sigma(T) \) coincides with \( \sigma(\sigma(T)) \), hence \( \sigma(T) \) is non-\( \alpha \)-Szlenk. Note also that the codomain of \( \sigma(T) \), namely \( C_0(T) \), is norm separable since \( T \) is countable (c.f. the discussion at (6.1)). On the other hand, by the first assertion of the theorem we have that \( \sigma(T) \) factors through any non-\( \alpha \)-Szlenk operator with separable codomain, hence we conclude that \( \sigma(T) \) in this case universal for the class of non-\( \alpha \)-Szlenk operators with separable codomain.

\[ \square \]

Remark 6.5. Bourgain [2], in a study of fixing properties of operators of large Szlenk index acting on \( C(K) \) spaces, represented \( C(L) \) spaces with \( L \) countable, compact and Hausdorff as spaces of scalar-valued functions on blossomed trees. Bourgain associates to each tree \( T \) constructed in Example 2.6 of the current paper a Banach space \( X \), isometrically isomorphic to \( C_0(T) \), defined as the completion of \( c_{00}(T) \) (the space of finitely-supported scalar-valued functions on \( T \)) with respect to the norm \( \| \cdot \|_\xi \) defined by setting

\[
\| x \|_\xi = \sup_{t \in \tau_{\xi}} \left| \sum_{s \subseteq t} x(s) \right|, \quad x \in c_{00}(T).
\]

The main assertion of Theorem 6.2 may be recast as follows: \( \text{Suppose } \alpha, \xi < \omega_1 \text{ are such that } \omega^\alpha \leq \xi < \omega^{\alpha+1} \text{ and let } T_{\xi} \in \mathcal{L}(\ell_1(T_{\xi}^*), X_{T_{\xi}^*}) \text{ denote the continuous linear extension of the formal identity map from } (c_{00}(T_{\xi}^*), \| \cdot \|_{\ell_1(T_{\xi}^*)}) \text{ to } X_{T_{\xi}^*}. \text{ Then } T_{\xi} \text{ is universal for the class of non-\( \alpha \)-Szlenk operators with norm separable codomain.} \)

We conclude the current section with some observations regarding the aforementioned universal operator theorems of Johnson [20] and Lindenstrauss-Pelczyński [26]. In particular, we note the following corollaries of Theorem 1.1 and Theorem 1.2 respectively. These results appear in [20] and [26], respectively, under the stronger hypothesis that \( Y \) is norm separable.

Corollary 6.6. Let \( X \) and \( Y \) be Banach spaces such that \( Y \) has weak*-sequentially compact dual ball and let \( T = \mathcal{L}(X, Y) \) be non-weakly compact. Then \( T \) factors the summation operator \( (a_n)_{n=1}^\infty \mapsto (\sum_{i=1}^n a_i)_{n=1}^\infty \) from \( \ell_1 \) to \( c \).

Proof. By Theorem 1.1 there exist \( U \in \mathcal{L}(\ell_1, X) \) and \( V \in \mathcal{L}(Y, \ell_\infty) \) such that \( VTU \) is the summation operator from \( \ell_1 \) to \( \ell_\infty \). For \( n \in \mathbb{N} \) let \( f^n \) denote the \( n \)th coordinate functional on \( \ell_\infty \); that is, \( f^n(f) = f(n) \) for every \( f \in \ell_\infty \). Since \( Y \) has weak*-sequentially compact dual ball there is a weak*-convergent subsequence \( (V^* f_{n_k})_{k=1}^\infty \) of \( (V^* f_n)_{n=1}^\infty \). Define \( A \in \mathcal{L}(\ell_1, X) \) by setting \( A e_k = U e_n \) for each \( k \in \mathbb{N} \) a define \( B \in \mathcal{L}(Y, c) \) by setting \( B y = ((V^* f_{n_k}^*, y))_{k=1}^\infty \) for each \( y \in Y \). For
we have

\[
(BTAe_k)(l) = \langle V^*f_{n_l}^*, T Ae_k \rangle = \langle f_{n_l}^*, V T U e_{n_k} \rangle = \begin{cases} 
1, & l \geq k \\
0, & l < k,
\end{cases}
\]

hence \( BTA \) coincides with the summation operator from \( \ell_1 \) to \( c_0 \).

\[ \text{Corollary 6.7.} \text{ Let } X \text{ and } Y \text{ be Banach spaces such that } Y \text{ has weak}^*\text{-sequentially compact dual ball and let } T \in \mathcal{L}(X,Y) \text{ be non-compact. Then } T \text{ factors the formal identity mapping from } \ell_1 \text{ to } c_0. \]

\[ \text{Proof.} \text{ By Theorem 1.2 there exist } U \in \mathcal{L}(\ell_1, X) \text{ and } V \in \mathcal{L}(Y, \ell_\infty) \text{ such that } VTU \text{ is the formal identity mapping from } \ell_1 \text{ to } \ell_\infty. \text{ For } n \in \mathbb{N} \text{ let } f_n^* \text{ denote the } n\text{th coordinate functional on } \ell_\infty. \text{ Since } Y \text{ has weak}^*\text{-sequentially compact dual ball there is a subsequence } (V^*f_{n_k}^*)_{k=1}^\infty \text{ of } (V^*f_n^*)_{n=1}^\infty \text{ converging weak}^* \text{ to some } y^* \in Y^*. \text{ Define } A \in \mathcal{L}(\ell_1, X) \text{ by setting } Ae_k = U e_{n_k} \text{ for each } k \in \mathbb{N} \text{ and define } B \in \mathcal{L}(Y, c_0) \text{ by setting } By = ((V^*f_{n_k}^* - y^*, y))_{k=1}^\infty \text{ for each } y \in Y. \text{ Since }

\[
\forall k \in \mathbb{N} \; \langle y^*, T U e_{n_k} \rangle = \lim_{l \to \infty} \langle V^*f_{n_l}^*, T U e_{n_k} \rangle = \lim_{l \to \infty} \langle f_{n_l}^*, V T U e_{n_k} \rangle = 0,
\]

it follows that for \( k, l \in \mathbb{N} \) we have

\[
(BTAe_k)(l) = (BTU e_{n_k})(l) = \langle V^*f_{n_l}^* - y^*, T U e_{n_k} \rangle = \langle f_{n_l}^*, V T U e_{n_k} \rangle = \begin{cases} 
1, & l = k \\
0, & l \neq k,
\end{cases}
\]

hence \( BTA \) coincides with the formal identity mapping from \( \ell_1 \) to \( c_0 \).

We do not know if the statement of Theorem 6.2 remains true if the condition that \( Y \) be norm separable is relaxed and \( Y \) is only assumed to have weak\(^*\)-sequentially compact dual ball.

\[ \text{7 Uncountable Szlenk indices and uncountable biorthogonal systems} \]

The proof of Theorem 5.1 makes use of the fact that an operator \( T \) can fail to be \( \alpha \)-Szlenk for a given ordinal \( \alpha \) in essentially two different ways. More precisely, it can be that \( Sz(T) \) is defined and larger than \( \omega^\alpha \), or it can be that \( Sz(T) \) is undefined. In particular, a non \( \alpha \)-Szlenk operator can be either Asplund or non-Asplund. This observation and the first assertion of Theorem 5.1 lead naturally to the following question.
Question 7.1. Let \( \alpha \geq \omega_1 \) be an uncountable ordinal. Does \( \mathcal{D} \cap \mathcal{Z}_\alpha \) admit a universal element?

In light of the approach taken in Theorem 5.1 to prove the existence of universal elements of \( \mathcal{Z}_\alpha \) for \( \alpha < \omega_1 \), it is natural to guess that a first step towards answering Question 7.1 could involve consideration of operators of the form \( \Sigma_T \), where \((T, \preceq)\) is a tree, and extending the definition of a blossomed tree (c.f. Definition 2.5) to the uncountable setting as follows: say that a tree \((T, \preceq)\) is blossomed if it is rooted, well-founded, and for every \( t \in T \setminus \text{MAX}(T) \) there exists a bijection \( \psi_t : \max\{\omega, \text{cof}(\rho_T(t))\} \to T[t+] \) such that \( \zeta \leq \zeta' < \max\{\omega, \text{cof}(\rho_T(t))\} \) implies \( \rho_T(\psi_t(\zeta)) \leq \rho_T(\psi_t(\zeta')) \). Examples of such trees \( T \) with \( \rho(T) = \xi + 1 \) for a given ordinal \( \xi \) may be obtained via a similar construction to that provided in Example 2.6, but with \( T \) consisting of finite sequences of ordinals (ordered by extension, as in Example 2.6). Moreover, under this more general definition a blossomed tree, a blossomed tree \( (T, \preceq) \) satisfies \( \rho(T) < \omega_1 \) if and only if \( T \) is countable and satisfies the usual definition of blossomed tree given in Definition 2.5. The natural candidate for a universal element of \( \mathcal{D} \cap \mathcal{Z}_\omega_1 \) under this approach is \( \Sigma_S \), where for each \( \alpha < \omega_1 \) we let \((T_\alpha, \preceq_\alpha)\) be a blossomed tree with \( \rho(T_\alpha) = \alpha + 1 \) and set \( S = \bigcup_{\alpha < \omega_1} (\{\alpha\} \times T_\alpha) \), with an order \( \preceq \) on \( S \) defined by setting \( (\alpha, t) \preceq (\alpha', t') \) if and only if \( \alpha = \alpha' \) and \( t \preceq_\alpha t' \). We do not know the answer to Question 7.1, even in the case \( \alpha = \omega_1 \). However, as we shall now see, operators of the form \( \Sigma_T \), where \((T, \preceq)\) is a well-founded tree - cannot be expected to provide absolute examples of universal elements of the classes \( \mathcal{D} \cap \mathcal{Z}_\alpha \) for \( \alpha \geq \omega_1 \) in general. In particular, we shall see that it is consistent with ZFC that the operator \( \Sigma_S \) defined above is not universal for \( \mathcal{D} \cap \mathcal{Z}_\omega_1 \).

Let \( Z \) a Banach space and \((T, \preceq)\) a tree. Let \( O_T = \{ t \in T \mid h_T(t) = 0 \text{ or } h_T(t) \text{ is a successor} \} \), noting that \( O_T = T \) if and only if \( h_T(T) \leq \omega \). In particular, \( O_T = T \) whenever \( T \) is well-founded. If \( \Sigma_T \) factors through \( Z \) then, by Proposition 2.11 there exist \( \delta > 0 \), \( (x_t)_{t \in T} \subseteq Z \) and \( (x_t^*)_{t \in T} \subseteq Z^* \) satisfying (2.5). It follows that \( Z \) admits a biorthogonal system of cardinality \( |O_T| \), namely \( (x_t, z_t^*)_{t \in O_T} \), where

\[
z_t^* = \begin{cases} 
  x_t^* - x_t^* & \text{if } h_T(t) > 0 \\
  x_t^* & \text{if } h_T(t) = 0 
\end{cases}, \quad t \in O_T.
\]

The following proposition is now immediate.

**Proposition 7.2.** Let \( Z \) be a Banach space not admitting an uncountable biorthogonal system and let \((T, \preceq)\) be a tree such that \( O_T \) is uncountable. Then \( \Sigma_T \) does not factor through \( Z \).
It is consistent with ZFC that there exists an Asplund space $W$ with $Sz(W) > \omega_1$ and $W$ does not admit an uncountable biorthogonal system. Thus, by Proposition 7.2, it is consistent with ZFC that the operator $\Sigma_S$ defined earlier in the current section is not universal for $D \cap C[\mathcal{P}_{\omega_1}]$. An example of a compact Hausdorff space $K$ such that $C(K)$ is such a space $W$ was constructed in the 1970s by Kunen, though the construction was not published until much later in [29]. (For further historical remarks concerning the existence of uncountable biorthogonal systems, see Remark 4 of [40].) Since a Banach space $C(L)$ is Asplund if and only if $L$ is scattered [28], the space $C(K)$ arising from Kunen’s construction is Asplund. Moreover, the Cantor-Bendixson rank of Kunen’s space $K$ is larger than $\omega_1$. Thus, the $C(K)$ space arising from Kunen’s construction is indeed an example of such a space $W$ once we have observed the following fact: for $L$ a compact Hausdorff space the Szlenk index of $C(L)$ is bounded below by the Cantor-Bendixson rank of $L$. This is an easy consequence of the well-known fact that the mapping that takes $l \in L$ to the evaluation functional of $C(L)$ at $l$ is a homeomorphic embedding with respect to the weak* topology, and the image of $L$ under this embedding is a 1-separated subset of $B_{C(L)^*}$. From this fact it is easy to see that $Sz(C(L), 1)$ is bounded below by the Cantor-Bendixson rank of $L$, hence $Sz(C(L))$ is bounded below by the Cantor-Bendixson rank of $L$.

Finally, we mention a more recent construction of Brech and Koszmider [3], who establish the consistency of a scattered compact Hausdorff space $J$ having Cantor-Bendixson rank equal to $\omega_2 + 1$ and such that $C(J)$ does not admit an uncountable biorthogonal system. If $T$ is a tree that is blossomed in the generalised sense introduced at the beginning of the current section, and if $\rho(T) = \omega_2 + 1$, then $Sz(\Sigma_T) = \omega_2 \omega$. However, by Proposition 7.2, $\Sigma_T$ does not factor through the Brech-Koszmider space $C(J)$ which satisfies $Sz(C(J)) \geq \omega_2 \omega$.

References

[1] Albiac, F., and Kalton, N. J. Topics in Banach space theory, vol. 233 of Graduate Texts in Mathematics. Springer, New York, 2006.

[2] Bourgain, J. The Szlenk index and operators on $C(K)$-spaces. Bull. Soc. Math. Belg. Sér. B 31, 1 (1979), 87–117.

[3] Brech, C., and Koszmider, P. Thin-very tall compact scattered spaces which are hereditarily separable. Trans. Amer. Math. Soc. 363, 1 (2011), 501–519.

[4] Brooker, P. A. H. Banach spaces and operators with non-separable dual. In preparation.
[5] Brooker, P. A. H. Asplund operators and the Szlenk index. *J. Operator Theory* 68 (2012), 405–442.

[6] Brooker, P. A. H. Szlenk and $w^*$-dentability indices of the Banach spaces $C([0,\alpha])$. *J. Math. Anal. Appl.* 399, 2 (2013), 559–564.

[7] Cilia, R., and Gutiérrez, J. M. Nonexistence of certain universal polynomials between Banach spaces. *J. Math. Anal. Appl.* 427, 2 (2015), 962–976.

[8] Davis, W. J., and Johnson, W. B. A renorming of nonreflexive Banach spaces. *Proc. Amer. Math. Soc.* 37 (1973), 486–488.

[9] Diestel, J., Jarchow, H., and Pietsch, A. Operator ideals. In *Handbook of the geometry of Banach spaces*, Vol. I. North-Holland, Amsterdam, 2001, pp. 437–496.

[10] Dilworth, S. J. Universal noncompact operators between super-reflexive Banach spaces and the existence of a complemented copy of Hilbert space. *Israel J. Math.* 52, 1-2 (1985), 15–27.

[11] Dilworth, S. J., Kutzarova, D., Lancien, G., and Randrianarivony, N. L. Equivalent norms with the property (β) of Rolewicz. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* (2016), 1–13.

[12] Fabian, M., Habala, P., Hájek, P., Montesinos Santalucía, V., Pelant, J., and Zizler, V. *Functional analysis and infinite-dimensional geometry*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8. Springer-Verlag, New York, 2001.

[13] Gasparis, I. Operators on $C(K)$ spaces preserving copies of Schreier spaces. *Trans. Amer. Math. Soc.* 357, 1 (2005), 1–30.

[14] Girardi, M., and Johnson, W. B. Universal non-completely-continuous operators. *Israel J. Math.* 99 (1997), 207–219.

[15] Godefroy, G., Kalton, N. J., and Lancien, G. Szlenk indices and uniform homeomorphisms. *Trans. Amer. Math. Soc.* 353, 10 (2001), 3895–3918.

[16] Grothendieck, A. Sur les applications linéaires faiblement compactes d’espaces du type $C(K)$. *Canadian J. Math.* 5 (1953), 129–173.

[17] Hagler, J., and Odell, E. A Banach space not containing $l_1$ whose dual ball is not weak* sequentially compact. *Illinois J. Math.* 22, 2 (1978), 290–294.
[18] Hinrichs, A., Novak, E., and Woźniakowski, H. Discontinuous information in the worst case and randomized settings. *Math. Nachr.* 286, 7 (2013), 679–690.

[19] Hinrichs, A., and Pietsch, A. The closed ideal of \((c_0, p, q)\)-summing operators. *Integral Equations Operator Theory* 38, 3 (2000), 302–316.

[20] Johnson, W. B. A universal non-compact operator. *Colloq. Math.* 23 (1971), 267–268.

[21] Johnson, W. B., and Rosenthal, H. P. On \(\omega^*\)-basic sequences and their applications to the study of Banach spaces. *Studia Math.* 43 (1972), 77–92.

[22] Kadec’, M. Ĭ. On the connection between weak and strong convergence. *Dopovidi Akad. Nauk Ukrain. RSR* 1959 (1959), 949–952.

[23] Klee, V. Mappings into normed linear spaces. *Fund. Math.* 49 (1960), 25–34.

[24] Lancien, G. On the Szlenk index and the weak\(^*\)-dentability index. *Quart. J. Math. Oxford Ser. (2)* 47, 185 (1996), 59–71.

[25] Lancien, G. A survey on the Szlenk index and some of its applications. *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 100, 1-2 (2006), 209–235.

[26] Lindenstrauss, J., and Pełczyński, A. Absolutely summing operators in \(L^p\)-spaces and their applications. *Stud. Math.* 29 (1968), 275–326.

[27] Lindenstrauss, J., and Tzafriri, L. *Classical Banach spaces. I.* Springer-Verlag, Berlin, 1977. Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[28] Namioka, I., and Phelps, R. R. Banach spaces which are Asplund spaces. *Duke Math. J.* 42, 4 (1975), 735–750.

[29] Negrepontis, S. Banach spaces and topology. In *Handbook of set-theoretic topology*. North-Holland, Amsterdam, 1984, pp. 1045–1142.

[30] Nyikos, P. J. Various topologies on trees. In *Proceedings of the Tennessee Topology Conference (Nashville, TN, 1996)* (1997), World Sci. Publ., River Edge, NJ, pp. 167–198.
[31] Oikhberg, T. A note on universal operators. In Ordered structures and applications. Positivity VII (Zaanen centennial conference), Leiden, the Netherlands, July 22–26, 2013. Basel: Birkhäuser/Springer, 2016, pp. 339–347.

[32] Pietsch, A. Operator ideals, vol. 20 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, 1980.

[33] Rosenthal, H. P. The Banach spaces C(K). In Handbook of the geometry of Banach spaces, Vol. 2. North-Holland, Amsterdam, 2003, pp. 1547–1602.

[34] Samuel, C. Indice de Szlenk des C(K) (K espace topologique compact dénombrable). In Seminar on the geometry of Banach spaces, Vol. I, II (Paris, 1983), vol. 18 of Publ. Math. Univ. Paris VII. Univ. Paris VII, Paris, 1984, pp. 81–91.

[35] Singer, I. Bases in Banach spaces. I. Springer-Verlag, New York, 1970. Die Grundlehren der mathematischen Wissenschaften, Band 154.

[36] Stegall, C. The Radon-Nikodým property in conjugate Banach spaces. Trans. Amer. Math. Soc. 206 (1975), 213–223.

[37] Stegall, C. The Radon-Nikodým property in conjugate Banach spaces. II. Trans. Amer. Math. Soc. 264, 2 (1981), 507–519.

[38] Szlenk, W. The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces. Studia Math. 30 (1968), 53–61.

[39] Todorčević, S. Trees and linearly ordered sets. In Handbook of set-theoretic topology. North-Holland, Amsterdam, 1984, pp. 235–293.

[40] Todorčević, S. Biorthogonal systems and quotient spaces via Baire category methods. Math. Ann. 335, 3 (2006), 687–715.

philip.a.h.brooker@gmail.com