Abstract Binary linear codes with good parameters have important applications in secret sharing schemes, authentication codes, association schemes, and consumer electronics and communications. In this paper, we construct several classes of binary linear codes from vectorial Boolean functions and determine their parameters, by further studying a generic construction developed by Ding et al. recently. First, by employing perfect nonlinear functions and almost bent functions, we obtain several classes of six-weight linear codes which contains the all-one codeword. Second, we investigate a subcode of any linear code mentioned above and consider its parameters. When the vectorial Boolean function is a perfect nonlinear function or a Gold function in odd dimension, we can completely determine the weight distribution of this subcode. Besides, our linear codes have larger dimensions than the ones by Ding et al.’s generic construction.

Keywords: Vectorial Boolean function, linear code, extended Walsh spectrum, secret sharing scheme, authentication code.

1 Introduction

Boolean functions are the building blocks of symmetric-key cryptography and coding theory. Symmetric-key cryptography includes block ciphers and stream ciphers. Boolean functions are used to create substitution boxes (S-boxes) with good cryptographic properties (low differential uniformity, high algebraic degree, large nonlinearity and so on) in block ciphers, and utilized as nonlinear filters and combiners in stream ciphers when they satisfy the main cryptographic criteria simultaneously. Error-correcting codes have long been known to have many applications in CD players, high speed communications, cellular phones, and (massive) data storage devices. They have been widely studied by many researchers in the past four decades and a lot of progresses on coding theory have been made. Particularly, Boolean functions can be used to construct binary linear and nonlinear codes, and the two related classes of
binary codes which are the Reed-Muller [1, 2] codes and Kerdock codes [3, 4, 5] are well-known.

Binary linear codes with good parameters have wide applications in secret sharing schemes [6, 7, 8, 9], authentication codes [10, 11], and association schemes [12], in addition to their applications in consumer electronics and communications. In the past two decades, the design of binary linear codes derived from (vectorial) Boolean functions has been a research topic of increasing importance. Many classes of binary codes with good parameters have been obtained, for instance see [13, 14] and the references therein. Generally speaking, there are two generic constructions of binary linear from (vectorial) Boolean functions. The first one is based on highly nonlinear vectorial Boolean functions such as perfect nonlinear (PN) functions and almost bent functions. For an \((m, s)\)-function, this construction can generate a linear code with length \(2^m\) and dimension at most \(m + s + 1\). The second construction is based on the support set of some highly nonlinear Boolean functions such as bent and semi-bent functions. For an \(m\)-variable Boolean function, such construction can provide a linear code such that the length of this code is equal to the size of the support of this function and the dimension of this code is at most \(m + 1\) [13, 14]. In the present paper, we extend the second construction to vectorial Boolean functions. In general, for an \((m, s)\)-function \(F\) and an arbitrary component function \(f\), we can construct a linear code such that the length of this code is equal to the size of the support of \(f\), the dimension of this code is at most \(m + s\), and the minimum Hamming distance can be expressed by means of the nonlinearity of \(F\). When \(F\) is an arbitrary PN function or AB function, we can determine the length, dimension, and weight distribution of the linear codes generated by our construction. All of these codes are six-weight linear codes and contain the all-one codeword; When \(m\) is small, some examples of our linear codes are optimal or at least have the same parameters as the best known codes [15] (see some examples in Remark 1). Further, we define a subcode of any linear code given above and consider its weight distribution. When \(F\) is a PN function or Gold function in odd dimension, we can completely determine the weight distribution of such subcode.

The remainder of this paper is organized as follows. In Section 2, the notation and the necessary preliminaries required for the subsequent sections are reviewed. In Section 3, we present the construction of linear codes from vectorial Boolean functions and provide the parameters of those codes. In Section 4, we focus on calculating the weight distributions of some subcodes of the codes given in Section 3. Finally, Section 5 concludes the paper.
2 Preliminaries

For any positive integer \( m \), we denote by \( \mathbb{F}_2^m \) the vector space of \( m \)-tuples over the finite field \( \mathbb{F}_2 = \{0, 1\} \), and by \( \mathbb{F}_{2^m} \) the finite field of order \( 2^m \). For simplicity, we denote by \( \mathbb{F}_{2^m}^* \) the set \( \mathbb{F}_2^m \setminus \{(0, 0, \cdots, 0)\} \), and \( \mathbb{F}_{2^m}^* \) denotes the set \( \mathbb{F}_{2^m} \setminus \{0\} \). We use \(+\) (resp. \(\sum\)) to denote the addition (resp. a multiple sum) in \( \mathbb{Z} \) or in the finite field \( \mathbb{F}_{2^m} \), and \(\oplus\) (resp. \(\bigoplus\)) to denote the addition (resp. a multiple sum) in \( \mathbb{F}_2 \). For simplicity, when there will be no ambiguity, we shall allow us to use \(+\) instead of \(\oplus\). The vector space \( \mathbb{F}_2^m \) is isomorphic to the finite field \( \mathbb{F}_{2^m} \) through the choice of some basis of \( \mathbb{F}_{2^m} \) over \( \mathbb{F}_2 \). Indeed, let \( (\lambda_1, \lambda_2, \cdots, \lambda_m) \) be a basis of \( \mathbb{F}_{2^m} \) over \( \mathbb{F}_2 \), then every vector \( x = (x_1, \cdots, x_m) \) of \( \mathbb{F}_2^m \) can be identified with the element \( x_1\lambda_1 + x_2\lambda_2 + \cdots + x_m\lambda_m \in \mathbb{F}_{2^m} \). The finite field \( \mathbb{F}_{2^m} \) can then be viewed as an \( m \)-dimensional vector space over \( \mathbb{F}_2 \); each of its elements can be identified with a binary vector of length \( m \), the element \( 0 \in \mathbb{F}_{2^m} \) is identified with the all-zero vector. We shall use \( x \) to denote indifferently elements in \( \mathbb{F}_{2^m} \) and in \( \mathbb{F}_2^m \).

A Boolean function of \( m \) variables is a function from \( \mathbb{F}_2^m \) into \( \mathbb{F}_2 \). We denote by \( \mathcal{B}_m \) the set of Boolean functions of \( m \) variables. Any Boolean function \( f \in \mathcal{B}_m \) can be expressed by its truth table, i.e.,

\[
 f = [f(0, \cdots, 0, 0), f(0, \cdots, 0, 1), \cdots, f(1, \cdots, 1, 0), f(1, \cdots, 1, 1)].
\]

We say that a Boolean function \( f \in \mathcal{B}_m \) is balanced if its truth table contains an equal number of ones and zeros, that is, if its Hamming weight equals \( 2^{m-1} \), where the Hamming weight of \( f \), denoted by \( n_f \), is defined as the size of the support of \( f \) in which the support of \( f \) is defined as \( D_f = \{ x \in \mathbb{F}_2^m | f(x) \neq 0 \} \). Given two Boolean functions \( f \) and \( g \) in \( m \) variables, the Hamming distance between \( f \) and \( g \) is defined as \( d_H(f, g) = |\{ x \in \mathbb{F}_2^m | f(x) \neq g(x) \}| \). Any Boolean function \( f \) of \( m \) variables can also be expressed in terms of a polynomial in \( \mathbb{F}_2[x_1, \cdots, x_m]/(x_1^2 \oplus x_1, \cdots, x_m^2 \oplus x_m) \):

\[
 f(x_1, \cdots, x_m) = \bigoplus_{u \in \mathbb{F}_2^m} a_u \left( \prod_{j=1}^n x_j^{u_j} \right) = \bigoplus_{u \in \mathbb{F}_2^m} a_u x^u,
\]

where \( a_u \in \mathbb{F}_2 \). This representation is called the algebraic normal form (ANF). The algebraic degree, denoted by \( \text{deg}(f) \), is the maximal value of \( \text{wt}(u) \) such that \( a_u \neq 0 \). A Boolean function is an affine function if its algebraic degree is at most 1. The set of all affine functions of \( m \) variables is denoted by \( \mathcal{A}_m \). Recall that \( \mathbb{F}_{2^m} \) is isomorphic as a \( \mathbb{F}_2 \)-vector space to \( \mathbb{F}_2^m \) through the choice of some basis of \( \mathbb{F}_{2^m} \) over \( \mathbb{F}_2 \). The Boolean functions over \( \mathbb{F}_{2^m} \) can also be uniquely expressed by a univariate polynomial

\[
 f(x) = \sum_{i=0}^{2^m-1} a_i x^i,
\]

3
where \( a_0, a_{2m-1} \in \mathbb{F}_2, a_i \in \mathbb{F}_{2^m} \) for \( 1 \leq i < 2^m - 1 \) such that \( a_i = a_{2i \mod {2^m-1}} \), and the addition is modulo 2. The algebraic degree \( \text{deg}(f) \) under this representation is equal to \( \max \{ \text{wt}(\tilde{i}) : a_i \neq 0, 0 \leq i < 2^m \} \), where \( \tilde{i} \) is the binary expansion of \( i \) (see e.g. [16]). The nonlinearity \( \text{nl}(f) \) of a Boolean function \( f \in \mathcal{B}_m \) is defined as the minimum Hamming distance between \( f \) and all the affine functions:

\[
\text{nl}(f) = \min_{g \in \mathcal{A}_m} (d_H(f, g)).
\]

In other words, the nonlinearity of a Boolean function \( f \) in \( m \) variables equals the minimum Hamming distance between the binary vector of length \( 2^m \) listing the values of the function to the Reed-Muller code \( \text{RM}(1, m) \) of length \( 2^m \). The maximal nonlinearity of all Boolean functions in \( m \) variables equals by definition the covering radius of \( \text{RM}(1, m) \) [17]. Let \( x = (x_1, x_2, \cdots , x_m) \) and \( a = (a_1, a_2, \cdots , a_m) \) both belong to \( \mathbb{F}_2^m \) and let \( a \cdot x \) be any inner product; for instance the usual inner product \( a \cdot x = a_1x_1 \oplus a_2x_2 \oplus \cdots \oplus a_mx_m \), then the Walsh transform of a Boolean function \( f \in \mathcal{B}_m \) at point \( a \) is defined by

\[
\hat{f}(a) = \sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + a \cdot x}.
\]

Note that changing the inner product changes the order of the values of the Walsh transform but not the multiset of these values which is called the Walsh spectrum. Over \( \mathbb{F}_{2^m} \), the Walsh transform of the Boolean function \( f \) at \( \alpha \in \mathbb{F}_{2^m} \) can be defined by

\[
\hat{f}(\alpha) = \sum_{x \in \mathbb{F}_{2^m}} (-1)^{f(x) + \text{Tr}_t^m(\alpha x)},
\]

where \( \text{Tr}_t^m(x) = \sum_{i=0}^{m/t-1} x^{2^i} \) is the trace function from \( \mathbb{F}_{2^m} \) to \( \mathbb{F}_{2^t} \) in which \( t \) is a positive divisor of \( m \). The well-known Parseval relation [18] states that: for any \( m \)-variable Boolean function, we have \( \sum_{u \in \mathbb{F}_{2^m}} \hat{f}^2(u) = 2^{2m} \). Parseval’s relation implies that, for a Boolean function of \( m \) variables, the mean of square of Walsh spectrum equals \( 2^m \). Then the maximum of the square of Walsh spectrum is greater than or equal to \( 2^m \) and therefore \( \max_{u \in \mathbb{F}_{2^m}} |\hat{f}(u)| \geq 2^{m/2} \). This implies that the nonlinearity of Boolean functions in \( m \) variables is upper-bounded by \( 2^{m-1} - 2^{m/2-1} \), which is tight for even \( m \).

**Definition 1.** ([19]) Let \( m \) be an even integer and \( f \) be a Boolean function of \( m \) variables. If \( \text{nl}(f) = 2^{m-1} - 2^{m/2-1} \), then we say that \( f \) is bent.

For odd number of variables \( m \), the maximum nonlinearity of an \( m \)-variable Boolean functions for \( m \) odd is \( 2^{m-1} - 2^{m-2} \) when \( m = 1, 3, 5, 7 \) and the question for the maximum nonlinearity of functions in odd \( m \geq 9 \) variables is still completely open.
Definition 2. Let $m$ be an odd integer and $f$ be a Boolean function of $m$ variables. If the set formed by the Walsh spectrum of $f$ equals $\{0, \pm 2^{(m+1)/2}\}$, then we say that $f$ is semi-bent (or near-bent).

The nonlinearity of a Boolean function $f \in \mathcal{B}_m$ can be calculated as

$$nl(f) = 2^{m-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^m} |W_f(a)|$$

$$= 2^{m-1} - \frac{1}{2} \max_{\omega \in \mathbb{F}_{2m}} |W_f(\omega)|.$$ 

Given two integers $m$ and $s$, a mapping from $\mathbb{F}_2^m$ to $\mathbb{F}_2^s$ which is often called an $(m, s)$-function or a vectorial Boolean function if the values $m$ and $s$ are omitted, can be viewed (and vice versa) as a function $G$ from the vectorial space $\mathbb{F}_2^m$ to the vectorial space $\mathbb{F}_2^s$. Particularly, $G$ is called a Boolean function when $s = 1$. Let $G$ be an $(m, s)$-function, the Boolean functions $g_1(x), \cdots, g_s(x)$ of $m$ variables defined by $G(x) = (g_1(x), \cdots, g_s(x))$ are called the coordinate functions of $G$. Further, the Boolean functions, which are the linear combinations, with non all-zero coefficients of the coordinate functions of $G$, are called component functions of $G$. The component functions of $G$ can be expressed as $a \cdot G$ where $a \in \mathbb{F}_2^s$. If we identify every element of $\mathbb{F}_2^s$ with an element of finite field $\mathbb{F}_2$, then the component functions $g_\alpha$ of $G$ can be expressed as $\text{Tr}^s_1(\alpha G)$, where $\alpha \in \mathbb{F}_2^*$. For any $(a, b) \in \mathbb{F}_2^s \times \mathbb{F}_2^m$, the Walsh transform of $G$ at $(a, b)$ is defined as

$$\hat{G}(a, b) = \sum_{x \in \mathbb{F}_2^m} (-1)^{a \cdot G(x) + b \cdot x}.$$

If $(\alpha, \beta) \in \mathbb{F}_2^s \times \mathbb{F}_2^m$, the Walsh transform of $G$ at $(\alpha, \beta)$ is defined as

$$\hat{G}(\alpha, \beta) = \sum_{x \in \mathbb{F}_2^m} (-1)^{\text{Tr}^s_1(\alpha G(x)) + \text{Tr}^m_1(\beta x)}.$$

We call extended Walsh spectrum of $G$ (and we shall denote by $\text{EW}_G$) the multi-set of the absolute values of all the Walsh transform of $G$. The nonlinearity $nl(G)$ of an $(m, s)$-function $G$ is the minimum Hamming distance between all the component functions of $G$ and all affine functions in $m$ variables. According to the definition of Walsh transform, we have

$$nl(G) = 2^{m-1} - \frac{1}{2} \max_{(a, b) \in \mathbb{F}_2^s \times \mathbb{F}_2^m} |\hat{G}(a, b)|$$

$$= 2^{m-1} - \frac{1}{2} \max_{(\alpha, \beta) \in \mathbb{F}_2^s \times \mathbb{F}_2^m} |\hat{G}(\alpha, \beta)|.$$ 

The nonlinearity $nl(G)$ is upper-bounded by $2^{m-1} - \frac{2^{m-1}}{2}$ when $m = s$. This upper bound is tight for odd $m = s$. For even $n = m$, the best known value of the nonlinearity of $(n, m)$-functions is $2^{n-1} - 2^\frac{n}{2}$.
Definition 3. Let $m$ be an odd integer and $G$ be an $(m, m)$-function. If $\text{nl}(G) = 2^{m-1} - 2^{\frac{m-1}{2}}$, then $G$ is called almost bent (AB).

It is well-known that the extended Walsh spectrum values of an almost bent $(m, m)$-functions $G$ are 0 and $2^{(m+1)/2}$ and thus an $(m, m)$-function is almost bent if and only if all of its component functions are semi-bent.

Definition 4. For two integers $m$ and $s$, an $(m,s)$-function is called bent vectorial if it nonlinearity is equal to $2^{m-1} - 2^{m/2-1}$

Clearly, an $(m,s)$-function is bent vectorial if and only if all of its component functions are bent. The bent vectorial functions exist only for even $m$ and $s \leq m/2$ [20]. They are characterized by the fact that all their derivatives $D_a F(x) = F(x) + F(x + a)$, $a \in \mathbb{F}_{2^m}$, are balanced (i.e. take each value of $\mathbb{F}_2$ the same number of times $2^{m-s}$) and are then also called perfect nonlinear (PN).

3 Linear codes from vectorial Boolean functions

In this section, we will give a method for obtaining linear codes from vectorial Boolean functions and get the parameters of those codes.

Let us first recall some basic definitions related to linear codes. Let $p$ be a prime, $m$ a positive integer, $r$ a positive divisor of $m$ and $q = p^r$. An $[n, k, d]_q$ linear code $C$ over $\mathbb{F}_q$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming distance $d$. Recall that $d = \min_{a,b \in C} d_H(a,b)$ where $d_H$ denotes the Hamming distance between vectors (called codewords) $a = (a_1, a_2, \cdots, a_n) \in C$ and $b = (b_1, b_2, \cdots, b_n) \in C$, i.e., $d_H(a,b) = |\{1 \leq i \leq n : a_i \neq b_i\}|$. For a given codeword $a = (a_1, a_2, \cdots, a_n) \in C$, the Hamming weight $\text{wt}(a)$ is defined as the number of nonzero coordinates. A generator matrix $G$ of a linear $[n, k, d]$ code $C$ is a $k \times n$ matrix whose rows form a basis of $C$. The dual code $C^\perp$ is the orthogonal subspace under the usual inner product in $\mathbb{F}_q^n$. Usually, if the context is clear we omit the subscript $q$ by convention in the sequel. Highly nonlinear Boolean and vectorial Boolean functions (or more generally functions valued in $\mathbb{F}_q$) have important applications in cryptography and coding theory. In coding theory, such functions have been used to construct linear codes with good parameters, for instance in papers [13, 14, 21, 22, 23, 24, 25, 26].

Let $m, s$ be two integers and $F$ be a vectorial Boolean function from $\mathbb{F}_{2^m}$ to $\mathbb{F}_{2^s}$. For any $\lambda \in \mathbb{F}_{2^s}$, we denote by $f_{\lambda}$ the Boolean function $Tr_1^n(\lambda F)$ which is a component function of $F$. Recall from Section 2 that $n_{f_{\lambda}}$ denotes the Hamming weight of $f_{\lambda}$ and $D_{f_{\lambda}}$ denotes the support of $f_{\lambda}$. Let $D_{f_{\lambda}} = \{d_1, d_2, \cdots, d_{n_{f_{\lambda}}}\}$, we define a linear code of length $n_{f_{\lambda}}$ over $\mathbb{F}_2$ as follows:

$$C_{D_{f_{\lambda}}} = \{c_{x,y} : x \in \mathbb{F}_{2^m}, y \in \mathbb{F}_{2^s}\},$$ (1)
where \( c_{x,y} = (Tr_1^m(xd_1) + Tr_1^s(yF(d_1)), \ldots, Tr_1^m(xd_n) + Tr_1^s(yF(d_n))) \).

We can easily see that the code \( C_{D_{f_\lambda}} \) is linear. For any \((m,s)\)-function \( G \), its graph is defined as the set \( \{(x,y) \in \mathbb{F}_2^m \times \mathbb{F}_2^s : y = G(x)\} \). Then we can see that the codewords of the linear code \( C_{D_{f_\lambda}} \) are the evaluations of all linear functions at those elements of the graph of \( F \) whose abscissa belong to the support of some fixed component function of \( F \). Hence the codewords are the restrictions of all codewords of the extended simplex code of length \( 2^m \) to such elements of the graph of \( F \).

In the rest of this section, we give the parameters of the linear code \( C_{D_{f_\lambda}} \), which are heavily relied on the extended Walsh spectrum of \( F \).

**Proposition 1.** Let \( F \) be an \((m,s)\)-function. For any \( \lambda \in \mathbb{F}_2^s \), let \( f_\lambda = Tr_1^s(\lambda F) \) and let \( n_{f_\lambda} \) be the Hamming weight of \( f_\lambda \), equal to the size of the support \( D_{f_\lambda} \) of \( f_\lambda \). If \( 2^m - 2nl(F) < n_{f_\lambda} \), then the linear code \( C_{D_{f_\lambda}} \) defined by (1) has length \( n_{f_\lambda} \), dimension \( m + s \) and minimum Hamming weight no less than \( nl(F) - \frac{2^m - n_{f_\lambda}}{2} \).

**Proof.** By (1), it is clear that every codeword in \( C_{D_{f_\lambda}} \) has length \( n_{f_\lambda} \). We now prove that the linear code \( C_{D_{f_\lambda}} \) has dimension \( m + s \) if \( 2^m - 2nl(F) < n_{f_\lambda} \). For doing this, we only need to prove that for any two distinct pairs \((x_1, y_1), (x_2, y_2) \in \mathbb{F}_2^m \times \mathbb{F}_2^s \), the Hamming distance between codewords \( c_{x_1,y_1} \) and \( c_{x_2,y_2} \) is not equal to zero, i.e., \( d_H(c_{x_1,y_1}, c_{x_2,y_2}) \neq 0 \), where \( c_{x,y} \) is defined by (1). Define

\[
A = \sum_{d \in D_{f_\lambda}} (-1)(Tr_1^m(x_1d) + Tr_1^s(y_1F(d)) + (Tr_1^m(x_2d) + Tr_1^s(y_2F(d))
\]

Let us use \( t(d) \) to denote \( Tr_1^m((x_1 + x_2)d) + Tr_1^s((y_1 + y_2)F(d)) \), we have

\[
\left\{ \begin{array}{l}
|\{d \in D_{f_\lambda} : t(d) = 0\}| - |\{d \in D_{f_\lambda} : t(d) = 1\}| = A \\
|\{d \in D_{f_\lambda} : t(d) = 0\}| + |\{d \in D_{f_\lambda} : t(d) = 1\}| = n_{f_\lambda} \\
|\{d \in D_{f_\lambda} : t(d) = 1\}| = d_H(c_{x_1,y_1}, c_{x_2,y_2})
\end{array} \right.
\]

Thus we get

\[
d_H(c_{x_1,y_1}, c_{x_2,y_2}) = \frac{1}{2}(n_{f_\lambda} - A).
\]

We distinguish the following three cases to calculate the values of \( d_H(c_{x_1,y_1}, c_{x_2,y_2}) \).

**Case 1.** \( y_1 + y_2 = 0 \).

Obviously, in this case \( x_1 + x_2 \neq 0 \) and \( y_1 + y_2 \neq \lambda \). Since \( Tr_1^s(\lambda F(d)) = 1 \) if \( d \in D_{f_\lambda} \) and \( Tr_1^s(\lambda F(d)) = 0 \) otherwise, we have:

\[
\left\{ \begin{array}{l}
\sum_{d \in D_{f_\lambda}} (-1)Tr_1^m((x_1 + x_2)d) + \sum_{d \in \mathbb{F}_2^m \setminus D_{f_\lambda}} (-1)Tr_1^m((x_1 + x_2)d) = 0 \\
\sum_{d \in D_{f_\lambda}} (-1)Tr_1^m((x_1 + x_2)d) + Tr_1^s(\lambda F(d)) + \sum_{d \in \mathbb{F}_2^m \setminus D_{f_\lambda}} (-1)Tr_1^m((x_1 + x_2)d) + Tr_1^s(\lambda F(d)) = \widehat{F}(\lambda, x_1 + x_2)
\end{array} \right.
\]

7
which is equivalent to
\[
\begin{cases}
A + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d)} = 0 \\
-A + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d)} = \hat{F}(\lambda, x_1 + x_2).
\end{cases}
\]

This implies that \( A = -\frac{1}{2} \hat{F}(\lambda, x_1 + x_2) \). By (2), we have
\[
d_H(c_{x_1, y_1}, c_{x_2, y_2}) = \frac{1}{4} (2n_{f_{\lambda}} + \hat{F}(\lambda, x_1 + x_2)).
\]
\( \text{(3)} \)

**Case 2.** \( y_1 + y_2 = \lambda \).

If \( x_1 + x_2 \neq 0 \), similarly to Case 1, we have
\[
\begin{cases}
\sum_{d \in D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d)} + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d)} = 0 \\
\sum_{d \in D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d) + \text{Tr}^1_1(\lambda F(d))} + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d) + \text{Tr}^1_1(\lambda F(d))} = \hat{F}(\lambda, x_1 + x_2),
\end{cases}
\]
which is equivalent to
\[
\begin{cases}
-A + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d)} = 0 \\
A + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d)} = \hat{F}(\lambda, x_1 + x_2).
\end{cases}
\]
and then \( A = \frac{1}{2} \hat{F}(\lambda, x_1 + x_2) \). It follows from (2) that
\[
d_H(c_{x_1, y_1}, c_{x_2, y_2}) = \frac{1}{4} (2n_{f_{\lambda}} - \hat{F}(\lambda, x_1 + x_2)).
\]
\( \text{(4)} \)

If \( x_1 + x_2 = 0 \), we have \( A = \sum_{d \in D_{f_{\lambda}}} (-1)^1 = -n_{f_{\lambda}} \). This implies that
\[
d_H(c_{x_1, y_1}, c_{x_2, y_2}) = n_{f_{\lambda}} \text{ when } x_1 + x_2 = 0,
\]
\( \text{(5)} \)
according to (2).

**Case 3.** \( y_1 + y_2 \in \mathbb{F}_2^{*} \setminus \{0, \lambda\} \).

Note that \( \lambda + y_1 + y_2 \neq 0 \) in this case. We have
\[
\begin{cases}
\sum_{d \in D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d) + \text{Tr}^1_1((y_1 + y_2)F(d))} + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d) + \text{Tr}^1_1((y_1 + y_2)F(d))} = \hat{F}(y_1 + y_2, x_1 + x_2) \\
\sum_{d \in D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d) + \text{Tr}^1_1((y_1 + y_2 + \lambda)F(d))} + \sum_{d \in \mathbb{F}_2^{m} \setminus D_{f_{\lambda}}} (-1)^{\text{Tr}^m_1((x_1 + x_2)d) + \text{Tr}^1_1((y_1 + y_2 + \lambda)F(d))} = \hat{F}(y_1 + y_2 + \lambda, x_1 + x_2).
\end{cases}
\]
Note that $\text{Tr}_1^*(\lambda F(d)) = 1$ if $d \in D_{f_h}$ and $\text{Tr}_1^*(\lambda F(d)) = 0$ otherwise. Then we have

$$
\begin{align*}
A + \sum_{d \in \mathbb{F}_2^m \setminus D_{f_h}} (-1)^{\text{Tr}_1^m((x_1+x_2)d) + \text{Tr}_1^*((y_1+y_2)F(d))} &= \hat{F}(y_1 + y_2, x_1 + x_2) \\
-A + \sum_{d \in \mathbb{F}_2^m \setminus D_{f_h}} (-1)^{\text{Tr}_1^m((x_1+x_2)d) + \text{Tr}_1^*((y_1+y_2)F(d))} &= \hat{F}(y_1 + y_2 + \lambda, x_1 + x_2).
\end{align*}
$$

Thus, we have $A = \frac{1}{2}(\hat{F}(y_1 + y_2, x_1 + x_2) - \hat{F}(y_1 + y_2 + \lambda, x_1 + x_2))$ and hence, by (2),

$$
d_H(c_{x_1,y_1}, c_{x_2,y_2}) = \frac{1}{4}(2n_{f_h} - \hat{F}(y_1 + y_2, x_1 + x_2) + \hat{F}(y_1 + y_2 + \lambda, x_1 + x_2)). \quad (6)
$$

Combining (3), (4) and (5), we have

$$
d_H(c_{x_1,y_1}, c_{x_2,y_2}) \in \left\{ \frac{1}{4}(2n_{f_h} + \hat{F}(\lambda, \alpha)), \frac{1}{4}(2n_{f_h} - \hat{F}(\lambda, \alpha)), n_{f_h} \right\} \quad (7)
$$

for $y_1 + y_2 \in \{0, \lambda\}$, and by (6) we have

$$
d_H(c_{x_1,y_1}, c_{x_2,y_2}) = \frac{1}{4}(2n_{f_h} - \hat{F}(\gamma, \beta) + \hat{F}(\gamma + \lambda, \beta)), \quad (8)
$$

for $y_1 + y_2 \in \mathbb{F}_2^* \setminus \{0, \lambda\}$, in which $\alpha \in \mathbb{F}_2^m, \gamma \in \mathbb{F}_2^*, \{0, \lambda\}, \beta \in \mathbb{F}_2^m$. So we have

$$
d_H(c_{x_1,y_1}, c_{x_2,y_2}) \geq \frac{1}{2}(n_{f_h} - \max_{(\mu,\nu) \in \mathbb{F}_2^m \times \mathbb{F}_2^m} |\hat{F}(\mu, \nu)|) > 0. \quad (9)
$$

The last inequality follows from the condition $2^m - 2nl(F) < n_{f_h}$ which means that $\max_{(\mu,\nu) \in \mathbb{F}_2^m \times \mathbb{F}_2^m} |\hat{F}(\mu, \nu)| < n_{f_h}$ according to (1). Therefore, if $2^m - 2nl(F) < n_{f_h}$ we have $c_{x_1,y_1} \neq c_{x_2,y_2}$ for any two distinct pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{F}_2^m \times \mathbb{F}_2^*$ and hence $\mathcal{C}_{D_{f_h}}$ has dimension $m + s$. Furthermore, for any two distinct pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{F}_2^m \times \mathbb{F}_2^*$, we have $d_H(c_{x_1,y_1}, c_{x_2,y_2}) \geq \frac{1}{2}(n_{f_h} - \max_{(\mu,\nu) \in \mathbb{F}_2^m \times \mathbb{F}_2^m} |\hat{F}(\mu, \nu)|) = \frac{1}{2}(n_{f_h} - 2^m + 2nl(F)) = nl(F) - \frac{1}{2}(2^m - n_{f_h})$. This implies that the minimum Hamming distance of the linear code $\mathcal{C}_{D_{f_h}}$ is no less than $n_{f_h} - \frac{1}{2}(2^m - n_{f_h})$.

This completes the proof. \hfill $\square$

**Theorem 1.** Let $F$ be an $(m, s)$-function with $nl(F) > 0$. For any integer $w \in EW_F$, Rel. (1) allows designing two linear codes with parameters $[2^{m-1} - w/2, m + s, nl(F) - 2^{m-2} - w/4]$ and $[2^{m-1} + w/2, m + s, nl(F) - 2^{m-2} + w/4]$ respectively.

**Proof.** Suppose that the component functions of $F$ are $f_1, f_2, \ldots, f_{2^s-1}$. If $w \in EW_F$, there exists a function $f_i$ ($1 \leq i \leq 2^s - 1$) and an element $a \in \mathbb{F}_2^*$ such that $|W_{f_i}(a)| = w$. Clearly, there exist $s$ component functions $f_i, f_{j_1}, f_{j_2}, \ldots, f_{j_{s-1}}$ such that $l f_i + \sum_{i=1}^{s-1} l_i f_{j_i} \notin A_m$ for any $(l, l_1, \ldots, l_{s-1}) \in \mathbb{F}_2^s$, since any function in $A_m$ (the set of $m$-variable Boolean functions with algebraic degree no more than 1) has
nonlinearity 0 and the \((m, s)\)-function \(F\) has nonlinearity greater than 0. Therefore, \(l(f_i + Tr^m_i(ax) + c) + \sum_{i=1}^{s-1} l_i f_j = l f_i + \sum_{i=1}^{s-1} l_i f_j + Tr^m_i(ax) + c \notin A_m\), where \(c \in \mathbb{F}_2\), for any \((l, l_1, \cdots, l_{s-1}) \in \mathbb{F}_2^s\). Define an \((m, s)\)-function \(F' = (f_i + Tr^m_i(ax) + c, f_{j1}, f_{j2}, \cdots, f_{js-1})\). It can be easily checked that \(nl(F') = nl(F)\).

**Case A.** If \(W_{f_i}(a) = w\), by taking \(F = F'\) and \(f_\lambda = f_i + Tr^m_i(ax)\) in (1), we have \(n_{f_\lambda} = |D_{f_\lambda}| = 2^m - |\{x \in \mathbb{F}_{2^m} : f_\lambda(x) = 0\}| \leq |D_{f_\lambda}| = \sum_{x \in \mathbb{F}_{2^m}} (-1)^{f_\lambda(x)} = \sum_{x \in \mathbb{F}_{2^m}} (-1)^{f_i(x)} + Tr^m_i(ax) = w\). This implies that \(n_{f_\lambda} = 2^{m-1} - w/2\) and hence we can get a \([2^{m-1} - w/2, m + s, nl(F') - 2^{m-2} - w/4]\)-code by Proposition 1. Similarly, by taking \(F = F'\) and \(f_\lambda = f_i + Tr^m_i(ax) + 1\) in (1), we can get a \([2^{m-1} + w/2, m + s, nl(F') - 2^{m-2} + w/4]\)-code by Proposition 1.

**Case B.** If \(W_{f_i}(a) = -w\), we can get the same codes as Case A with similar discussion.

This completes the proof. \[\square\]

Let \(F\) be a perfect nonlinear function from \(\mathbb{F}_{2^m}\) to \(\mathbb{F}_{2^m/2}\), where \(m\) is even. It is well-known that all values in the extended Walsh spectrum of \(F\) are equal to \(2^{m/2}\). Thus, by Theorem 1, we can immediately get the following two corollaries.

**Corollary 1.** Let \(F\) be a perfect nonlinear function from \(\mathbb{F}_{2^m}\) to \(\mathbb{F}_{2^m/2}\) where \(m\) is even. Then there exist two linear codes with parameters \([2^{m-1} - 2^{m/2} - 1, 3m/2, 2^{m-2} - 3 \cdot 2^{m/2-2}]\) and \([2^{m-1} + 2^{m/2}, 3m/2 - 1, 2^{m-2} - 2^{m/2-2}]\) respectively.

We have mentioned that Nyberg proved in [20] that an \((m, s)\)-function is perfect nonlinear (or equivalently, bent) only if \(m\) is even and \(s \leq m/2\). In fact, for any even integer \(m \geq 4\), bent vectorial \((m, m/2)\)-functions do exist. For examples, we list below some primary constructions of bent vectorial \((m, m/2)\)-functions in the form of \(F(x, y)\), where \((x, y) \in \mathbb{F}_{2^m/2} \times \mathbb{F}_{2^m/2}\).

1. \(F(x, y) = L(x \pi(y)) + H(y) [20]\), where the product \(x \pi(y)\) is calculated in \(\mathbb{F}_{2^m/2}\), \(L\) is any linear or affine mapping from \(\mathbb{F}_{2^m/2}\) onto itself, \(\pi\) is any permutation of \(\mathbb{F}_{2^m/2}\) and \(H\) is any \((m/2, m/2)\)-function. This class of functions are called strict Maiorana-McFarland class. Maiorana-McFarland class of bent vectorial functions can be extended to a more general class which is called general Maiorana-McFarland class, see [27, 28].

2. \(F(x, y) = G(xy^{2^{m/2}-2}) [28]\), where \(G\) is a balanced \((m/2, m/2)\)-function. The component functions of \(F\) belongs to the class of \(\mathcal{PS}_{ap}\) functions [29].

3. \(F(x, y) = xG(yx^{2^{m-2}}) [30]\), where \(G\) is an \(o\)-polynomial on \(\mathbb{F}_{2^m/2}\). This class bent vectorial functions are called \(\mathcal{H}\) class in [30].

There are also some primary constructions of bent vectorial \((m, m/2)\)-functions from single term or multiple terms trace functions, see for examples in [31, 32].
Remark 1. When \( m \) is small, some linear codes introduced in Corollary 1 are optimal or at least have the same parameters as the best known codes listed in [15]. For examples:

- For \( m = 6 \), we define \( F(x, y) = (\text{Tr}_3^3(xy), \text{Tr}_1^3(ax), \text{Tr}_1^3(a^2xy)) \), where \( x, y \in \mathbb{F}_{2^6} \) and \( \alpha \) is the default primitive element of \( \mathbb{F}_{2^6} \) in Magma version 2.12-16. Let \( f_\lambda = \text{Tr}_3^3(xy) \). Then \( C_{D_\lambda} \) defined by (1) is a \([28,9,10]\)-code with weight enumerator \( 1 + 84z^{10} + 63z^{12} + 216z^{14} + 63z^{16} + 84z^{18} + z^{28} \) from our Magma program, which is an optimal code [15] and confirms the result of Corollary 1. If we define \( F(x, y) = (\text{Tr}_3^3(xy) + 1, \text{Tr}_1^3(ax), \text{Tr}_1^3(a^2xy)) \) and \( f_\lambda = \text{Tr}_3^3(xy) + 1 \). Then \( C_{D_\lambda} \) defined by (1) is a \([36,9,14]\)-code with weight enumerator \( 1 + 108z^{14} + 63z^{16} + 168z^{18} + 63z^{20} + 108z^{22} + z^{36} \) by Magma programm, which is an optimal code and confirms the result of Corollary 1.

- For \( m = 8 \), we define \( F(x, y) = (\text{Tr}_4^4(xy), \text{Tr}_1^4(ax), \text{Tr}_1^4(a^2xy), \text{Tr}_1^4(a^3xy)) \), where \( x, y \in \mathbb{F}_{2^8} \) and \( \alpha \) is the default primitive element of \( \mathbb{F}_{2^8} \) in Magma version 2.12-16. Let \( f_\lambda = \text{Tr}_1^4(xy) \). Then the linear code \( C_{D_\lambda} \) defined by (1) is a \([120,12,52]\)-code with weight enumerator \( 1 + 840z^{52} + 255z^{56} + 1904z^{60} + 255z^{64} + 840z^{68} + z^{120} \) according to our Magma program, which confirms the result of Corollary 1. This code has the same parameters as a best known linear code given in [15]. If we define \( F(x, y) = (\text{Tr}_1^4(xy)+1, \text{Tr}_1^4(ax), \text{Tr}_1^4(a^2xy), \text{Tr}_1^4(a^3xy)) \) and \( f_\lambda = \text{Tr}_1^4(xy)+1 \). Then the linear code \( C_{D_\lambda} \) defined by (1) is a \([136,12,60]\)-code with weight enumerator \( 1 + 952z^{60} + 255z^{64} + 1680z^{68} + 255z^{72} + 952z^{76} + z^{136} \) by our Magma program, which confirms the result of Corollary 1. This code has the same parameters as a best known linear code given in [15].

If \( F \) is an almost bent function from \( \mathbb{F}_{2^m} \) to itself, then by definition, all values in the extended Walsh spectrum of \( F \) belong to the set \( \{0, 2^{(m+1)/2}\} \).

Corollary 2. Let \( F \) be an almost bent function from \( \mathbb{F}_{2^m} \) to itself. Then there exists three linear codes with parameters \([2^m-1-2^{(m-1)/2}, 2m, 2^{m-2} - 3 \cdot 2^{(m-3)/2}], [2^{m-1} + 2^{(m-1)/2}, 2m, 2^{m-2} - 2^{(m-3)/2}] \) and \([2^{m-1}, 2m, 2^{m-2} - 2^{(m-1)/2}] \) respectively.

We list the known power almost bent functions \( F(x) = x^d \) on \( \mathbb{F}_{2^m} \) in the following:

1. \( d = 2^i + 1 \), where \( \gcd(m, i) = 1 \) is odd [33]. These power functions are called Gold functions.

2. \( d = 2^{2i} + 2^i + 1 \), where \( i \geq 2 \leq (m-1)/2 \) and \( \gcd(m, i) = 1 \). The AB property of this function is equivalent to a result given by Kasami [34]; Welch also obtained this result but never published it. These power functions are called Kasami functions or Kasami-Welch functions.
3. $d = 2^{(m-1)/2} + 3$. These power functions were conjectured AB by Welch and this was proved by Canteaut, charcoal and Dobbertin in [35].

4. $d = 2^{(m-1)/2} + 2^{(m-1)/4} - 1$, where $m \equiv 1 \mod 4$. These power functions were conjectured AB by Niho, and this was proved by Hollman and Xiang in [36].

5. $d = 2^{(m-1)/2} + 2^{(3m-1)/4} - 1$, where $m \equiv 3 \mod 4$. These power functions were also conjectured AB by Niho, and this was proved by Hollman and Xiang in [36]. The power functions in these two last cases are called Niho functions.

Remark 2. For small number of $m$, some linear codes introduced in Corollary 2 are optimal or the same as the best known codes [15]. For examples:

1. For $m = 5$, we define $F(x) = x^3$ on $\mathbb{F}_{2^5}$, in which the finite field $\mathbb{F}_{2^5}$ generated by the default primitive polynomial $x^5 + x^2 + 1$ in Magma version 2.12-16. Let $f_\lambda$ be the function $Tr_1^7(x^3 + \alpha^2 x), Tr_1^7(x^3)$, $Tr_1^7(x^3 + \alpha^15 x)$ respectively, where $\alpha$ is a root of the equation $x^5 + x^2 + 1 = 0$. Then the linear codes $C_{D_{f_\lambda}}$ defined by (1) are code [12, 10, 2]-code with weight enumerator $1 + 30z^2 + 255z^4 + 452z^6 + 255z^8 + 30z^{10} + z^{12}$, [16, 10, 4]-code with weight enumerator $1 + 60z^4 + 256z^6 + 390z^8 + 256z^{10} + 60z^{12} + z^{16}$ and [20, 10, 6]-code with weight enumerator $1 + 90z^6 + 255z^8 + 332z^{10} + 255z^{12} + 90z^{14} + z^{20}$, respectively, by Magma programs. This confirms the results of Corollary 2. These three codes are optimal [15].

2. For $m = 7$, we define $F(x) = x^3$ on $\mathbb{F}_{2^7}$, in which the finite field $\mathbb{F}_{2^7}$ generated by the default primitive polynomial $x^7 + x + 1$ in Magma version 2.12-16. Let $f_\lambda$ be the function $Tr_1^7(x^3 + \alpha^7 x), Tr_1^7(x^3)$, $Tr_1^7(x^3 + \alpha^19 x)$ respectively, where $\alpha$ is a root of the equation $x^7 + x + 1 = 0$. Thus, the linear codes $C_{D_{f_\lambda}}$ defined by (1) are code [56, 14, 20]-code with weight enumerator $1 + 756z^{20} + 4095z^{24} + 6680z^{28} + 4095z^{32} + 756z^{36} + z^{56}$, [64, 14, 24]-code with weight enumerator $1 + 1008z^{24} + 4096z^{28} + 6174z^{32} + 4096z^{36} + 1008z^{40} + z^{64}$ and [72, 14, 28]-code with weight enumerator $1 + 1260z^{28} + 4095z^{32} + 5672z^{36} + 4095z^{40} + 1260z^{44} + z^{72}$, respectively, by Magma programs. This confirms the results of Corollary 2. These codes are the same as the best known codes with such parameters and are almost optimal because the upper bounds on the minimum Hamming weight of length 56, 72, 64 with dimension 14 are 21, 29, 25 respectively [15].

3. For $m = 9$, we define $F(x) = x^3$ on $\mathbb{F}_{2^9}$, in which the finite field $\mathbb{F}_{2^9}$ generated by the default primitive polynomial $x^9 + x + 1$ in Magma version 2.12-16. Let $f_\lambda$ be the function $Tr_1^9(x^3 + \alpha^9 x), Tr_1^9(x^3)$, $Tr_1^9(x^3 + \alpha^{10} x)$ respectively, where $\alpha$ is a root of the equation $x^9 + x^4 + 1 = 0$. Thus, the linear codes $C_{D_{f_\lambda}}$ defined by (1) are code [240, 18, 104]-code with weight enumerator $1 + 14280z^{104} + 65535z^{112} + 102512z^{120} + 65535z^{128} + 14280z^{136} + z^{240}$, [256, 18, 112]-code with
weight enumerator $1 + 16320z^{112} + 65536z^{120} + 98430z^{128} + 65536z^{136} + 16320z^{144} + z^{256}$ and [272, 18, 120]-code with weight enumerator $1 + 18360z^{120} + 65535z^{128} + 94352z^{136} + 65535z^{144} + 18360z^{152} + z^{272}$, respectively, by Magma programs. This confirms the results of Corollary 2. The fist two codes are the same as the best known codes.

3.1 The weight distribution of $C_{D_{f\lambda}}$ when $F$ is a perfect nonlinear function

Let $C$ be a binary linear $[n, k, d]$-code including the all-one codeword. Then the number $A_{w_1}$ of codewords with Hamming weight $w_1$ is equal to the number $A_{w_2}$ of codewords with Hamming weight $w_2 = n - w_1$.

Theorem 2. Let $F$ be a perfect nonlinear function from $\mathbb{F}_2^m$ to $\mathbb{F}_{2^m/2}$, where $m$ is even. For every $\lambda \in \mathbb{F}_2^*_{2^m/2}$, $C_{D_{f\lambda}}$ is an $[n_{f\lambda}, 3m/2, n_{f\lambda}/2 - 2m/2 - 1]$-code with the weight distribution given in Table 1, where $f_\lambda = Tr_1^{m/2}(\lambda F)$ and $n_{f\lambda} \in \{2^{m-1} - 2^{m/2-1}, 2^{m-1} + 2^{m/2-1}\}$.

| Weight $w$ | Multiplicity $A_w$ |
|------------|-------------------|
| $0$        | $1$               |
| $n_{f\lambda} - 2^m$ | $2^{m/2-1}n_{f\lambda} - 2^{-m}n_{f\lambda}^2 - 2^{-m/2} + 1/4$ |
| $n_{f\lambda} - 2^{m/2}$ | $2^{m-1}$ |
| $n_{f\lambda} + 2^{m/2}$ | $(2n_{f\lambda}^2 - 2^{3m/2}n_{f\lambda})2^{-m} + 2^{3m/2} - 3\cdot 2^{m-1} - 1/2$ |
| $n_{f\lambda} + 2^{m}$ | $2^{m-1}$ |
| $n_{f\lambda}$ | $2^{m/2-1}n_{f\lambda} - 2^{-m}n_{f\lambda}^2 - 2^{-m/2} + 1/4$ |
| $n_{f\lambda}$ | $1$               |

Proof. Note that for any $\lambda \in \mathbb{F}_2^*_{2^m/2}$ the Boolean function $f_\lambda$ is a bent function and hence $\hat{f_\lambda}(a) = \pm 2^{m/2}$ for all $a \in \mathbb{F}_2^m$. Thus, we have $n_{f\lambda} = |\{x \in \mathbb{F}_2^m : f_\lambda(x) = 1\}| \in \{2^{m-1} - 2^{m/2-1}, 2^{m-1} + 2^{m/2-1}\}$. Note that $F$ has nonlinearity $2^{m-1} - 2^{m/2-1}$. By Proposition 1, we immediately obtain that $C_{D_{f\lambda}}$ is an $[n_{f\lambda}, 3m/2, n_{f\lambda}/2 - 2m/2 - 1]$-code. In what follows, we discuss the weight distribution of $C_{D_{f\lambda}}$. Since the code $C_{D_{f\lambda}}$ is linear, the Hamming weights of all nonzero codewords of $C_{D_{f\lambda}}$ belong to the set constituted by the values of Hamming distances between all pairs of codewords in $C_{D_{f\lambda}}$. Thus, by (7) and (8) we can see that the Hamming weights of all codewords of $C_{D_{f\lambda}}$ belong to the set $\{1/2n_{f\lambda} \pm 2^{m/2-2}, 1/2n_{f\lambda} \pm 2^{m/2-1}, 1/2n_{f\lambda}, n_{f\lambda}\} \cup \{0\}$. So we
Lemma 2. Let \( \gcd(|\{\lambda \in \text{Table}\text{\ spectrum takes at most three values}\}) \cdot n_a \in \mathbb{Z} \), \( w \) includes the all-one and all-zero vectors. So we have \( d \) determine the number \( A \). We now \( C \). \( C \) is an almost bent functions

\[
\begin{align*}
  w_1 &= 0, w_2 = \frac{n_a - 2^{m/2}}{2}, w_3 = \frac{n_a - 2^{m/2-1}}{2}, \\
  w_4 &= \frac{n_a}{2}, w_5 = \frac{n_a + 2^{m/2-1}}{2}, w_6 = \frac{n_a + 2^{m/2}}{2}, w_7 = n_a,
\end{align*}
\]

since the all values in the extended Walsh spectrum of \( F \) are equal to \( 2^{m/2} \). We now determine the number \( A_{w_i} \) of codewords with weight \( w_i \) in \( C_{D_{f_{\lambda}}} \), where \( i = 1, 2, \ldots, 7 \). Clearly we have \( A_{w_1} = A_{w_7} = 1 \). Furthermore, it follows from (7) and (8) that the Hamming weight of any nonzero codeword \( c_{x,y} \) belongs to the set \( \{1/2n_{f_{\lambda}} \pm 2^{m/2-2}, n_{f_{\lambda}}\} \) if and only if \( y \in \{0, \lambda\} \). Note that the set \( \{c_{x,y} : x \in \mathbb{F}_{2^m}, y \in \{0, \lambda\}\} \) includes the all-one and all-zero vectors. So we have \( w_3 + w_5 = |\{c_{x,y} : x \in \mathbb{F}_{2^m}, y \in \{0, \lambda\}\}| - 2 = 2^{m+1} - 2 \) and then \( A_{w_3} = A_{w_5} = 2^m - 1 \) by the observation made before the present theorem. Note that any two elements \( d_i \) and \( d_j \) of the set \( D_{f_{\lambda}} \) must be distinct if \( i \neq j \). Then by [18, Theorem 10], the minimum weight of the dual code of \( C_{D_{f_{\lambda}}} \) is no less than 3. According to the first three Pless Power Moments [37, p. 260], we have

\[
\begin{align*}
  &\left\{ \begin{array}{l}
    A_{w_1} + A_{w_2} + A_{w_3} + A_{w_4} + A_{w_5} + A_{w_6} + A_{w_7} = 2^{3m/2} \\
    w_1 A_{w_1} + w_2 A_{w_2} + w_3 A_{w_3} + w_4 A_{w_4} + w_5 A_{w_5} + w_6 A_{w_6} + w_7 A_{w_7} = n_f \frac{2^{3m/2-1}}{2} \\
    w_1^2 A_{w_1} + w_2^2 A_{w_2} + w_3^2 A_{w_3} + w_4^2 A_{w_4} + w_5^2 A_{w_5} + w_6^2 A_{w_6} + w_7^2 A_{w_7} = n_f (n_f + 1) \frac{2^{3m/2-2}}{2}
  \end{array} \right.
\end{align*}
\]

Recall that \( A_{w_1} = A_{w_7} = 1 \) and \( A_{w_3} = A_{w_5} = 2^m - 1 \) and note that \( A_{w_2} = A_{w_6} \). Thus

\[
\begin{align*}
  &\left\{ \begin{array}{l}
    2A_{w_2} + A_{w_4} = 2^{3m/2} - 2^{m+1} \\
    (w_2^2 + w_4^2) A_{w_2} + w_4^2 A_{w_4} = n_f (n_f + 1) 2^{3m/2-2} - n_f^2 - (n_f^2/2 + 2^{m-3}) (2^m - 1)
  \end{array} \right.
\end{align*}
\]

By solving this system of equations, we can get the weight distribution of \( C_{D_{f_{\lambda}}} \) given in Table 1. This completes the proof.

\[
\Box
\]

3.2 The weight distribution of \( C_{D_{f_{\lambda}}} \) when \( F \) is an almost bent functions

Lemma 1 ([38]). Let \( f \) be a Boolean function of \( m \) variables such that its Walsh spectrum takes at most three values \( 0, \pm 2^s \). Then \( \{|a \in \mathbb{F}_2^m : \hat{f}(a) = 0\}| = 2^m - 2^{m-2s} \), \( \{|a \in \mathbb{F}_2^m : \hat{f}(a) = 2^s\}| = 2^{2m-2s-1} + (-1)^{f(0)} 2^{m-s-1} \), and \( \{|a \in \mathbb{F}_2^m : \hat{f}(a) = -2^s\}| = 2^{2m-2s-1} - (-1)^{f(0)} 2^{m-s-1} \).

Lemma 2. Let \( F(x) = x^d \) be an almost bent function over \( \mathbb{F}_{2^m} \), where \( m \) is odd and \( \gcd(d, 2^m-1) = 1 \). Define \( N_{f_{\lambda}}(t) = |\{(x,y) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} \setminus \{0, \lambda\} : \hat{f}_{\lambda} \hat{f}_{\lambda+y}(x) - \hat{f}_{\lambda}(x) = t\}| \), where \( \lambda \in \mathbb{F}_{2^m}^* \) and \( f_{\lambda} = Tr_{1}^m (\lambda F) \). Then we have \( N_{f_{\lambda}}(2^{(m+3)/2}) = 2^{2m-4} - 2^{m-3} \).
$N_{f_\lambda}(-2^{(m+3)/2}) = 2^{2m-4} - 2^{m-3}$, $N_{f_\lambda}(2^{(m+1)/2}) = 2^{2m-2} - 2^{m-1}$, $N_{f_\lambda}(-2^{(m+1)/2}) = 2^{2m-2} - 2^{m-1}$, $N_{f_\lambda}(0) = 3 \cdot 2^{2m-3} + 2^{m-2} - 2^m$ and $N_{f_\lambda}(t) = 0$ for every other value of $t$.

**Proof.** Note that

$$\widehat{f}_\lambda(x) - \widehat{f}_\lambda(x) = \sum_{z \in \mathbb{F}_{2^m}} (-1)^{T_{11}(z^d + xz)} = \sum_{z \in \mathbb{F}_{2^m}} (-1)^{T_{11}(yz^d + xz)}$$

$$= \sum_{z \in \mathbb{F}_{2^m}} (-1)^{T_{11}(z^d + x(\lambda+y))} - \sum_{z \in \mathbb{F}_{2^m}} (-1)^{T_{11}(z^d + xy)}$$

$$= \sum_{z \in \mathbb{F}_{2^m}} (-1)^{T_{11}(z^d + wz)} - \sum_{z \in \mathbb{F}_{2^m}} (-1)^{T_{11}(z^d + w(1+\lambda/2))}$$

$$= \widehat{f}_1(w) - \widehat{f}_1\left(w \left(1 + \frac{\lambda}{y}\right)^\frac{1}{2}\right)$$

where $w = x(\lambda + y)^{-\frac{1}{2}}$. Then we can see that $N_{f_\lambda}(t) = |\{(w, y) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} \setminus \{0, \lambda\} : \widehat{f}_\lambda(w) = \widehat{f}(w(1+\lambda/y)^{-1/d}) = t\}|$. Note that for any $w \in \mathbb{F}_{2^m}$, $w(1+\lambda/y)^{-1/d}$ ranges over $\mathbb{F}_{2^m} \setminus \{0, \lambda\}$ when $y$ ranges over $\mathbb{F}_{2^m} \setminus \{0, \lambda\}$. Note also that $|\{(w \in \mathbb{F}_{2^m} : \widehat{f}_\lambda(w) = 0\}| = 2^{m-1}$, $|\{(w \in \mathbb{F}_{2^m} : \widehat{f}_\lambda(w) = 2^{(m+1)/2}\}| = 2^{m-2} + 2^{(m+3)/2}$ and $|\{(w \in \mathbb{F}_{2^m} : \widehat{f}_\lambda(w) = -2^{(m+1)/2}\}| = 2^{m-2} - 2^{(m+3)/2}$, according to Lemma 1 and $f_\lambda(0) = 0$. So we have $N_{f_\lambda}(2^{(m+3)/2}) = (2^{m-2} + 2^{(m-3)/2})(2^{m-2} - 2^{(m-3)/2}) = 2^{2m-4} - 2^{m-3}$, $N_{f_\lambda}(-2^{(m+3)/2}) = (2^{m-2} - 2^{(m-3)/2})(2^{m-2} + 2^{(m-3)/2}) = 2^{2m-4} - 2^{m-3}$, $N_{f_\lambda}(2^{(m+1)/2}) = (2^{m-2} + 2^{(m-3)/2} - 1)(2^{m-1} + 2^{m-2} - 2^{(m-3)/2}) = 2^{2m-2} - 2^{m-1}$, $N_{f_\lambda}(-2^{(m+1)/2}) = (2^{m-2} - 2^{(m-3)/2} - 1)(2^{m-1} + 2^{m-2} + 2^{(m-3)/2}) = 2^{2m-2} - 2^{m-1}$, and $N_{f_\lambda}(0) = 2^m(2^m - 2) - N_{f_\lambda}(2^{(m+3)/2}) - N_{f_\lambda}(-2^{(m+3)/2}) - N_{f_\lambda}(2^{(m+1)/2}) - N_{f_\lambda}(-2^{(m+1)/2}) = 3 \cdot 2^{2m-3} + 2^{m-2} - 2^m$. This completes the proof. 

**Theorem 3.** Let $F(x) = x^d$ be an almost bent function over $\mathbb{F}_{2m}$, where $m$ is odd and $\text{gcd}(d, 2^m - 1) = 1$. For every $\lambda \in \mathbb{F}_{2m}^*$, $\mathcal{C}_{D_{f_\lambda}}$ is a $[2^{m-1}, 2m, 2^{m-2} - 2^{(m-1)/2}]$-code with the weight distribution given in Table 2, where $f_\lambda = Tr_{1}^{m}(\lambda F)$.

**Proof.** It is clear that for any $\lambda \in \mathbb{F}_{2m}^*$ the Boolean function $f_\lambda$ is a semi-bent function and hence $\widehat{f}_\lambda(a) \in \{0, \pm 2^{(m+1)/2}\}$ for all $a \in \mathbb{F}_{2m}$ and $nl(F) = 2^{m-1} - 2^{(m-1)/2}$. Note that $F$ is bijective. We have $f_\lambda$ is balanced for any $\lambda \in \mathbb{F}_{2m}^*$ [28] and hence $n_{f_\lambda} = 2^m - 1$. Hence $\mathcal{C}_{D_{f_\lambda}}$ is an $[n_{f_\lambda}, 3m/2, n_{f_\lambda}/2 - 2^{m/2} - 1]$-code by Proposition 1.

In what follows we determine the weight distribution of $\mathcal{C}_{D_{f_\lambda}}$. Let us consider the Hamming weight of $c_{x, y}$ for any $(x, y) \in \mathbb{F}_{2m} \times \mathbb{F}_{2m}$, where $c_{x, y}$ is defined in (1). Recall that the all values in the extended Walsh spectrum of $F$ are belong to the set $\{0, 2^{(m+1)/2}\}$, then by (7) and (8) we can see that the codewords in $\mathcal{C}_{D_{f_\lambda}}$ have weights: $w_1 = 0, w_2 = 2^{m-1} - 2^{m-1}/2, w_3 = 2^{m-1} - 2^{m-1}/2$. 

15
Table 2: The weight distribution of the code of Theorem 3

| Weight $w$ | Multiplicity $A_w$ |
|------------|------------------|
| 0          | 1                |
| $2^{m-2} - 2^{(m-1)/2}$ | $2^{2m-4} - 2^{m-3}$ |
| $2^{m-2} - 2^{(m-3)/2}$ | $2^{2m-2}$ |
| $2^{m-2}$ | $3 \cdot 2^{2m-3} + 2^{m-2} - 2$ |
| $2^{m-2} + 2^{(m-3)/2}$ | $2^{2m-2}$ |
| $2^{m-2} + 2^{(m-1)/2}$ | $2^{2m-4} - 2^{m-3}$ |
| $2^{m-2}$ | 1                |

For any $\lambda \in \mathbb{F}_2^*$, we denote by $H_\lambda$ a set such that: (1) $H_\lambda$ is a vector subspace of $\mathbb{F}_2^s$ with dimension $s - 1$; and (2) $H_\lambda \cup \{\lambda + H_\lambda\} = \mathbb{F}_{2^s}$, i.e., $\mathbb{F}_{2^s} = \{x + y : x \in \{0, \lambda\}, y \in H_\lambda\}$. Let $m, s$ be two integers and $F$ be a vectorial Boolean function from $\mathbb{F}_{2^m}$ to $\mathbb{F}_{2^s}$. For any $\lambda \in \mathbb{F}_{2^s}^*$, we denote by $f_\lambda$ the Boolean function $Tr_1^s(\lambda F)$ and define a linear code of length $n_{f_\lambda}$ over $\mathbb{F}_2$ as follows:

$$C_{D_{f_\lambda}} = \{c_{x,y} : x \in \mathbb{F}_{2^m}, y \in H_\lambda\},$$

(10)

4 The weight distribution of subcodes

The linear code $C_{D_{f_\lambda}}$ defined by (1) includes the all-one codeword. In this section, we focus on calculating the weight distribution of those subcodes of $C_{D_{f_\lambda}}$ which do not contain the all-one codewords.

For any $\lambda \in \mathbb{F}_{2^s}^*$, we denote by $H_\lambda$ a set such that: (1) $H_\lambda$ is a vector subspace of $\mathbb{F}_2^s$ with dimension $s - 1$; and (2) $H_\lambda \cup \{\lambda + H_\lambda\} = \mathbb{F}_{2^s}$, i.e., $\mathbb{F}_{2^s} = \{x + y : x \in \{0, \lambda\}, y \in H_\lambda\}$. Let $m, s$ be two integers and $F$ be a vectorial Boolean function from $\mathbb{F}_{2^m}$ to $\mathbb{F}_{2^s}$. For any $\lambda \in \mathbb{F}_{2^s}^*$, we denote by $f_\lambda$ the Boolean function $Tr_1^s(\lambda F)$ and define a linear code of length $n_{f_\lambda}$ over $\mathbb{F}_2$ as follows:

$$C_{D_{f_\lambda}, H_\lambda} = \{c_{x,y} : x \in \mathbb{F}_{2^m}, y \in H_\lambda\},$$

(10)
where \( n_{f_L} = |D_{f_L}| = |\{d \in \mathbb{F}_{2^m} : f_L(d) \neq 0\}| = |\{d_1, d_2, \ldots, d_{n_{f_L}}\}| \) and \( c_{x,y} = (\text{Tr}_1^m(xd_1) + \text{Tr}_1^m(yF(d_1)), \ldots, \text{Tr}_1^m(xd_n) + \text{Tr}_1^m(yF(d_{n_{f_L}}))) \).

Let \( F \) be an \((m, s)\)-function, where \( m, s \) be two integers. It can be easily checked that, for any \( \lambda \in \mathbb{F}_{2^s}^* \), the linear code \( C_{D_{f_L}, H_L} \) defined by (10) is a subcode of \( C_{D_{f_L}} \) defined by (1). By Proposition 1, we directly have the following theorem.

**Theorem 4.** Let \( F \) be an \((m, s)\)-function. For any \( \lambda \in \mathbb{F}_{2^m}^* \), if \( 2^m - 2n(F) < n_{f_L} \), where \( f_L = \text{Tr}_1^m(\lambda F) \), denoting by \( n_{f_L} \) the size of the support of \( f_L \), the linear code \( C_{D_{f_L}, H_L} \) defined by (10) has length \( n_{f_L} \), dimension \( m + s - 1 \) and minimum Hamming weight no less than \( n(F) - \frac{2^m - n_{f_L}}{2} \).

### 4.1 The weight distribution of \( C_{D_{f_L}, H_L} \) when \( F \) is a perfect nonlinear function

**Theorem 5.** Let \( F \) be a perfect nonlinear function from \( \mathbb{F}_{2^m} \), where \( m \) is even, to \( \mathbb{F}_{2^{m/2}} \) such that \( F(0) = 0 \). For every \( \lambda \in \mathbb{F}_{2^m}^* \) and any \( H_L, C_{D_{f_L}, H_L} \) is an \([n_{f_L}, 3m/2 - 1, n_{f_L}/2 + 2m/2 - 1]-\)code with the weight distribution given in Table 3, where \( f_L = \text{Tr}_1^m(\lambda F) \) and \( n_{f_L} \in \{2^{m-1} - 2^{m/2-1}, 2^{m-1} + 2^{m/2-1}\} \) is defined as the size of the support of \( f_L \).

| Weight \( w \) | Multiplicity \( A_w \) |
|----------------|------------------|
| 0              | 1                |
| \( \frac{n_{f_L}}{2} - 2\frac{m/2-1}{2} \) | \( 2^{m/2-2}n_{f_L} - 2^{-m-1}n_{f_L}^2 - 2^{-m-2} + \frac{1}{8} \) |
| \( \frac{n_{f_L}}{2} - 2\frac{m/2-2}{2} \) | \( 2^{m-1-n_{f_L}} \frac{2^{1-m}}{2} \) |
| \( \frac{n_{f_L}}{2} \) | \( (n_{f_L}^2 - 2^{3m/2-1}n_{f_L})2^{-m} + 2^{3m/2-1} - 3 \cdot 2^{-m-2} - \frac{1}{4} \) |
| \( \frac{n_{f_L}}{2} + 2\frac{m/2-2}{2} \) | \( 2^{m-1-n_{f_L}} \frac{2^{1-m}}{2} \) |
| \( \frac{n_{f_L}}{2} + 2\frac{m/2-1}{2} \) | \( 2^{m/2-2}n_{f_L} - 2^{-m-1}n_{f_L}^2 - 2^{-m-2} + \frac{1}{8} \) |

**Proof.** Recall that for any \( \lambda \in \mathbb{F}_{2^m}^* \) the Boolean function \( f_L \) is a bent function and so \( \hat{f}_L(a) = \pm 2^{m/2} \) for all \( a \in \mathbb{F}_{2^m} \). This implies that \( n_{f_L} = |\{x \in \mathbb{F}_{2^m} : f_L(x) = 1\}| \in \{2^{m-1} - 2^{m/2-1}, 2^{m-1} + 2^{m/2-1}\} \). Recall also that \( F \) has nonlinearity \( 2^{m-1} - 2^{m/2-1} \). By Theorem 4 we have that \( C_{D_{f_L}, H_L} \) is a linear code with length \( n_{f_L} \) and dimension \( 3m/2 - 1 \).

In the rest of this proof, we determine the weight distribution of \( C_{D_{f_L}, H_L} \). It can be easily seen that the code \( C_{D_{f_L}, H_L} \) does not include the all-one codeword. Thus, by
recalling the proof of Theorem 2, we can assume that the codewords in $C_{D_{f_\lambda},H_\lambda}$ have weights:

$$w_1 = 0, \quad w_2 = \frac{n_{f_\lambda} - 2^{m/2}}{2}, \quad w_3 = \frac{n_{f_\lambda} - 2^{m/2 - 1}}{2},$$

$$w_4 = \frac{n_{f_\lambda}}{2}, \quad w_5 = \frac{n_{f_\lambda} + 2^{m/2 - 1}}{2}, \quad w_6 = \frac{n_{f_\lambda} + 2^{m/2}}{2}.$$ 

We now determine the number $A_{w_i}$ of codewords with weight $w_i$ in $C_{D_{f_\lambda},H_\lambda}$, where $i = 1, 2, \ldots, 6$. Obviously we have $A_{w_1} = 1$. By (7) and (8), the Hamming weight of any nonzero codeword $c_{x,y}$, where $c_{x,y}$ is defined by 10, belongs to the set $\{1/2n_{f_\lambda} \pm 2^{m/2 - 2}, n_{f_\lambda}\}$ if and only if $y \in \{0, \lambda\}$. Note that $\lambda \notin H_\lambda$; the Hamming weight of any nonzero codeword $c_{x,y}$ belongs to the set $\{1/2n_{f_\lambda} \pm 2^{m/2 - 2}, n_{f_\lambda}\}$ if and only if $y = 0$. Define $C' = \{c_{x,0} \in C_{D_{f_\lambda},H_\lambda} : x \in \mathbb{F}_{2^m}\}$. We can easily see that $C'$ is a subcode of $C_{D_{f_\lambda},H_\lambda}$ with length $n_{f_\lambda}$ and dimension $m$. Moreover, we can see that the codewords in $C'$ have weights $w_1, w_3, w_5$ and the numbers of codewords in $C'$ with weight $w_1, w_3, w_5$ respectively are equal to $A_{w_1}, A_{w_3}, A_{w_5}$ respectively. It follows from [18, Theorem 10] that the minimum weight of the dual code of $C'$ is no less than 3 since any two elements $d_i$ and $d_j$ of the set $D_{f_\lambda}$ must be distinct if $i \neq j$. According to the first two Pless Power Moments [37, p. 260], we have

$$\begin{pmatrix} A_{w_1} + A_{w_3} + A_{w_5} \\ w_1 A_{w_1} + w_3 A_{w_3} + w_5 A_{w_5} \end{pmatrix} = 2^m,$$

$$n_{f_\lambda}w_5 = n_{f_\lambda}2^{m-1}.$$ 

Recall that $A_{w_1} = 1$. By solving this system of equations, we have $A_{w_3} = (2^m - 1 - n_{f_\lambda}2^{-m/2})/2$ and $A_{w_5} = (2^m - 1 + n_{f_\lambda}2^{-m/2})/2$. By [18, Theorem 10], we immediately get that the minimum weight of the dual code of $C'$ is no less than 3 since any two elements $d_i$ and $d_j$ of the set $D_{f_\lambda}$ must be distinct if $i \neq j$. According to the first three Pless Power Moments [37, p. 260], we have

$$\begin{pmatrix} A_{w_1} + A_{w_2} + A_{w_3} + A_{w_4} + A_{w_5} + A_{w_6} \\ w_1 A_{w_1} + w_2 A_{w_2} + w_3 A_{w_3} + w_4 A_{w_4} + w_5 A_{w_5} + w_6 A_{w_6} \\ w_1^2 A_{w_1} + w_2^2 A_{w_2} + w_3^2 A_{w_3} + w_4^2 A_{w_4} + w_5^2 A_{w_5} + w_6^2 A_{w_6} \end{pmatrix} = 2^{3m/2 - 1},$$

$$n_{f_\lambda}w_6 = n_{f_\lambda}2^{3m/2 - 2}.$$ 

Recall that the values $A_{w_1}, A_{w_3}, A_{w_5}$. Then by solving this system of equations, we can get the values of $A_{w_4}, A_{w_2}, A_{w_6}$. This completes the proof.

**Example 1.** For $m = 8$, we define $F(x,y) = (Tr_1^4(x,y), Tr_1^4(\alpha xy), Tr_1^4(\alpha^2 xy), Tr_1^4(\alpha^3 xy))$, where $x, y \in \mathbb{F}_{2^4}$ and $\alpha$ is the default primitive element of $\mathbb{F}_{2^4}$ in Magma version 2.12-16, and $f_\lambda = Tr_1^4(x)$ with $H_\lambda = c_1 \alpha^2 + c_2 \alpha^2 + c_3 \alpha^3$ where $(c_1, c_2, c_3) \in \mathbb{F}_3^3$. With the help of Magma, we can get the linear code $C_{D_{f_\lambda},H_\lambda}$ defined by (10) is a $[120,11,52]$-code with weight enumerator $1 + 420z^{52} + 120z^{56} + 952z^{60} + 135z^{64} + 420z^{68}$, which confirms the result of Theorem 5.
4.2 The weight distribution of $C_{Df, H}$ when $F$ is a Gold function

For odd $m$ and $e = 2^i + 1$ where $\gcd(i, m) = 1$ and $1 \leq i \leq (n - 1)/2$, the $(m, m)$-functions $F(x) = x^e$ are called Gold functions (Gold functions also exist in even dimensions, but we only consider the odd case in this paper). These functions are almost bent and are bijective. We now consider the weight distribution of the linear codes from the Gold functions in odd dimensions. We first need the following lemmas.

**Lemma 3** ([16]). Let $m, k$ be two integers such that $m \geq k \geq 1$. Let $E$ be a subspace of $\mathbb{F}_2^m$ with dimension $k$ and with orthogonal space $E^\perp = \{x \in \mathbb{F}_2^m : \forall y \in E, \text{Tr}_1^m(xy) = 0\}$. Then we have that $\sum_{x \in E} (-1)^{\text{Tr}_1^m(\alpha x)}$ equals $2^k$ if $\alpha \in E^\perp$ and is 0 otherwise.

**Lemma 4.** Let $f$ be a balanced quadratic semi-bent function defined on $\mathbb{F}_{2m}$ and $T$ be the set $\{\alpha \in \mathbb{F}_2^m : \hat{f}(\alpha) = 0\}$. Then $T$ is a subspace of $\mathbb{F}_2^m$ with dimension $m - 1$. If we assume that $T^\perp = \{0, w\}$, then

$$\sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + f(x + b)} = \begin{cases} 2^m, & \text{if } b = 0 \\ -2^m, & \text{if } b = w \\ 0, & \text{otherwise} \end{cases}$$

**Proof.** It is easy to see that $\sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + f(x + b)} \in \{0, \pm 2^m\}$, since $f$ has algebraic degree 2 and so $f(x) + f(x + b)$ has algebraic degree at most 1. Note that $\sum_{\alpha \in \mathbb{F}_2^m} \hat{f}(\alpha)^4 = 2^m \sum_{b \in \mathbb{F}_2^m} \left(\sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + f(x + b)}\right)^2$, see [39]. Then by Lemma 1, there only exist two elements $\{0, w\}$, in $\mathbb{F}_2^m$ such that $|\sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + f(x + b)}| = 2^m$, where $b \in \{0, w\}$. So we have $\sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + f(x + b)} = 0$ if $b \in \mathbb{F}_2^m \setminus \{0, w\}$. On the other hand, we have $\sum_{b \in \mathbb{F}_2^m} \sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + f(x + b)} = \sum_{x \in \mathbb{F}_2^m} (-1)^{f(x)} \sum_{b \in \mathbb{F}_2^m} (-1)^{f(x + b)} = 0$ since $f$ is balanced. This implies that $\sum_{x \in \mathbb{F}_2^m} (-1)^{f(x) + f(x + w')} = -2^m$.

We now prove that $T$ is a subspace with dimension $m - 1$. Clearly, $0 \in T$ since $f$ is balanced. Moreover, $|T| = 2^{m-1}$ according to Lemma 1. Hence, for proving that $T$ is a subspace with dimension $m - 1$, we only need to prove that for any two distinct elements $\alpha, \beta \in T$ such that $\hat{f}(\alpha) = \hat{f}(\beta) = 0$ we have $\hat{f}(\alpha + \beta) = 0$. Note that $0 = \hat{f}^2(\alpha) = \sum_{x, y \in \mathbb{F}_{2m}} (-1)^{f(x) + f(y) + \text{Tr}_1^m(\alpha y)} = \sum_{x, b \in \mathbb{F}_{2m}} (-1)^{f(x) + f(x + b) + \text{Tr}_1^m(ab)} = \sum_{b \in \mathbb{F}_{2m}} (-1)^{\text{Tr}_1^m(ab)} \sum_{x \in \mathbb{F}_{2m}} (-1)^{f(x) + f(x + b)} = 2^n (1 - (-1)^{\text{Tr}_1^m(\alpha w')})$, which implies that $\text{Tr}_1^m(\alpha w') = 0$. Similarly, we have $\text{Tr}_1^m(\beta w') = 0$ from $\hat{f}^2(\beta)$. Furthermore, we have $\hat{f}^2(\alpha + \beta) = 2^n (2^n (1 - (-1)^{\text{Tr}_1^m((\alpha + \beta) w')})$, which is equal to 0 since $\text{Tr}_1^m((\alpha + \beta) w') = \text{Tr}_1^m(\alpha w') + \text{Tr}_1^m(\beta w') = 0$. Therefore, $T$ is a subspace with dimension $m - 1$.

In what follows, we will prove the rest assertion of this lemma. From the above discussion, we only need to prove $w = w'$. Note that $0 = \sum_{\alpha \in T} \hat{f}^2(\alpha) = \sum_{\alpha \in T} \sum_{x, y \in \mathbb{F}_{2m}} (-1)^{f(x) + f(y) + \text{Tr}_1^m(\alpha y)} = \sum_{x, \beta \in \mathbb{F}_{2m}} (-1)^{f(x) + f(x + \beta) + \text{Tr}_1^m(\alpha \beta)}$, which leads to $w = w'$. This completes the proof.
Lemma 5. Let \( f_\lambda = Tr_1^m(\lambda x^d) \) be a balanced quadratic semi-bent function and \( T \) be the set \( \{ \alpha \in \mathbb{F}_2^m : \hat{f}(\alpha) = 0 \} \). Then we have \( T^\perp = \{ 0, \lambda^{-\frac{1}{2}} \} \).

Lemma 6. Let \( f \) be a quadratic semi-bent function defined on \( \mathbb{F}_2^m \), where \( m \geq 5 \). Let \( T \) be the set \( \{ \alpha \in \mathbb{F}_2^m : \hat{f}(\alpha) = 0 \} \) and \( E \) be a subspace of \( \mathbb{F}_2^m \) with dimension \( m-1 \) and \( E^\perp = \{ 0, w \} \). Define \( N_f(E, t) = |\{ \alpha \in E : \hat{f}(\alpha) = t \}| \). Then we have

\[
N_f(E, 0^+) = 2^{m-2}, \quad N_f(E, 2^{(m+1)/2}) = 2^{m-3} + 2^{(m-3)/2}(1 - f(0) - f(w)) \quad \text{and} \quad N_f(E, -2^{(m+1)/2}) = 2^{m-3} - 2^{(m-3)/2}(1 - f(0) - f(w)).
\]

Proof. Note that

\[
\sum_{\alpha \in E} \hat{f}(\alpha) = \sum_{\alpha \in E} \sum_{x, y \in \mathbb{F}_2^m} (-1) f(x) + f(y) + Tr_1^m(\alpha(x+y)) = 2^{m-1} \sum_{\alpha \in E} \sum_{x \in \mathbb{F}_2^m} (-1) f(x) + f(x+\alpha) \quad \text{(by Lemma 3)}
\]

\[
= 2^{m-1} \sum_{\alpha \in E^\perp} \sum_{x \in \mathbb{F}_2^m} (-1) f(x) + f(x+w) = 2^{m-1} + 2^{m-1} \sum_{x \in \mathbb{F}_2^m} (-1) f(x) + f(x+w).
\]

It is easy to see that \( w \not\in T^\perp \), thus we have \( \sum_{x \in \mathbb{F}_2^m} (-1) f(x) + f(x+w) = 0 \) by Lemma 4. So we have \( \sum_{\alpha \in E} \hat{f}(\alpha) = 2^{2m-1} \). Recall that \( f \) is semi-bent. Thus, we have

\[
(2^{(m+1)/2})^2 (N_f(E, 2^{(m+1)/2}) + N_f(E, -2^{(m+1)/2})) = 2^{2m-1}.
\]

This implies that

\[
N_f(E, 2^{(m+1)/2}) + N_f(E, -2^{(m+1)/2}) = 2^{m-2}.
\]

(11)

Then we have \( N_f(E, 0) = 2^{m-2} \). Note also that

\[
\sum_{\alpha \in E} \hat{f}(\alpha) = \sum_{\alpha \in E} \sum_{x \in \mathbb{F}_2^m} (-1) f(x) + Tr_1^m(\alpha x) = \sum_{x \in \mathbb{F}_2^m} (-1) f(x) \sum_{\alpha \in E} (-1) Tr_1^m(\alpha x).
\]

Then by Lemma 3, we have \( \sum_{\alpha \in E} \hat{f}(\alpha) = 2^m(1 - f(0) - f(w)) \), which is equivalent to saying that

\[
2^{(m+1)/2}(N_f(E, 2^{(m+1)/2}) - N_f(E, -2^{(m+1)/2})) = 2^m(1 - f(0) - f(w)).
\]

(12)

Combining Equations (11) and (12), we can deduce that \( N_f(E, 2^{(m+1)/2}) = 2^{m-3} + 2^{(m-3)/2}(1 - f(0) - f(w)) \) and \( N_f(E, -2^{(m+1)/2}) = 2^{m-3} - 2^{(m-3)/2}(1 - f(0) - f(w)) \).

This completes the proof. □
Lemma 7. Let $f_\nu$, $f_\lambda$ and $f_\mu$ be three Boolean functions of $m$ variables such that $f_\nu = f_\lambda + f_\mu$. Then we have $\sum_{\alpha \in \mathbb{F}_2^m} (f_\lambda(\alpha) - \hat{f}_\mu(\alpha))^2 = 2^{m+2}n_\nu$, $\sum_{\alpha \in \mathbb{F}_2^m} (\hat{f}_\lambda(\alpha) + \hat{f}_\mu(\alpha))^2 = 2^{m+2}n_\nu$, where $n_\nu$ denotes the size of support of $f_\nu$.

Proof. We have $\sum_{\alpha \in \mathbb{F}_2^m} (f_\lambda(\alpha) - \hat{f}_\mu(\alpha))^2 = \sum_{\alpha \in \mathbb{F}_2^m} (\hat{f}_\lambda(\alpha) + \hat{f}_\mu(\alpha) - 2\hat{f}_\lambda(\alpha)\hat{f}_\mu(\alpha)) = \sum_{\alpha \in \mathbb{F}_2^m} \hat{f}_\lambda(\alpha) + \sum_{\alpha \in \mathbb{F}_2^m} \hat{f}_\mu(\alpha) - 2 \sum_{\alpha \in \mathbb{F}_2^m} \hat{f}_\lambda(\alpha)\hat{f}_\mu(\alpha)$. Note that $\sum_{\alpha \in \mathbb{F}_2^m} \hat{f}_\lambda(\alpha)\hat{f}_\mu(\alpha) = \sum_{\alpha \in \mathbb{F}_2^m} \sum_{x \in \mathbb{F}_2^m} (-1)^{f_\lambda(x) + f_\mu(y)} \sum_{y \in \mathbb{F}_2^m} (-1)^{f_\lambda(y) + f_\mu(y)} = \sum_{x,y \in \mathbb{F}_2^m} (-1)^{f_\lambda(x) + f_\mu(y)} = 2^m$. Thus, we have $\sum_{\alpha \in \mathbb{F}_2^m} (f_\lambda(\alpha) - \hat{f}_\mu(\alpha))^2 = 2^{m+2}n_\nu$ and $\sum_{\alpha \in \mathbb{F}_2^m} (\hat{f}_\lambda(\alpha) + \hat{f}_\mu(\alpha))^2 = 2^{m+2}n_\nu$. This completes the proof. \hfill $\Box$

Lemma 8. Let $F(x) = x^e$ be the Gold functions in odd dimension $m \geq 5$, where $e = 2^i + 1$ and $i$ is such that $\gcd(i, n) = 1$ and $1 \leq i \leq (n-1)/2$. Let $f_\lambda = Tr_1^n(\mu F)$ and $f_\mu = Tr_1^n(\mu F)$ be two Boolean functions, where $\lambda, \mu$ are two distinct elements of $\mathbb{F}_2^*$. We define a subset $S_\lambda$ formed by the codewords as follows:

$$S_\lambda = \{ s_x : x \in \mathbb{F}_2^m \},$$

where $s_x = (Tr_1^m(xd_1 + \lambda F(d_1)), \cdots, Tr_1^m(xd_{2^m-1} + \lambda F(d_{2^m-1}))$ and $D_{\lambda + \mu} = \{ d \in \mathbb{F}_2^m : Tr_1^m((\lambda + \mu)F(d)) = 1 \}$. Then the weight distribution of the codewords included in $S_\lambda$ is shown as in Table 4.

| Weight $w$ | Multiplicity $A_w$ |
|------------|---------------------|
| $2^{m-2} - 2^{m-1}/2$ | $2^{m-4} + 2^{m-5}/2 (Tr_1^m(\lambda \mu) - Tr_1^m(\lambda \mu))$ |
| $2^{m-2} - 2^{m-3}/2$ | $2^{m-2} + 2^{m-3}/2 (Tr_1^m(\lambda \mu) - Tr_1^m(\lambda \mu))$ |
| $2^m - 2$ | $3 \cdot 2^{m-3}$ |
| $2^{m-2} + 2^{m-3}/2$ | $2^{m-2} + 2^{m-3}/2 (Tr_1^m(\lambda \mu) - Tr_1^m(\lambda \mu))$ |
| $2^{m-2} + 2^{m-1}/2$ | $2^{m-4} + 2^{m-5}/2 (Tr_1^m(\lambda \mu) - Tr_1^m(\lambda \mu))$ |

Proof. Note that for any $x \in \mathbb{F}_2^m$ we have

$$\begin{cases}
\sum_{d \in \mathbb{F}_2^m \setminus D_{\lambda + \mu}} (-1)^{Tr_1^m(xd + \lambda F(d))} + \sum_{d \in D_{\lambda + \mu}} (-1)^{Tr_1^m(xd + \lambda F(d))} = \hat{f}_\lambda(x) \\
\sum_{d \in \mathbb{F}_2^m \setminus D_{\lambda + \mu}} (-1)^{Tr_1^m(xd + \lambda F(d))} - \sum_{d \in D_{\lambda + \mu}} (-1)^{Tr_1^m(xd + \lambda F(d))} = \hat{f}_\mu(x)
\end{cases}.$$
This implies that, for any $x \in \mathbb{F}_2^n$, we have

$$\text{wt}(s_x) = 2^{m-2} - \frac{1}{4}(\widehat{f}_\lambda(x) - \widehat{f}_\mu(x)).$$

(13)

We can easily see that for any $x \in \mathbb{F}_2^n$, $\widehat{f}_\lambda(x) - \widehat{f}_\mu(x) \in \{0, \pm 2^{(m+1)/2}, \pm 2^{(m+3)/2}\}$. Therefore, for obtaining the weight distribution of the codewords in $S_\lambda$, we need to calculate the distribution of the values $0, \pm 2^{(m+1)/2}, \pm 2^{(m+3)/2}$ in the set $\{\widehat{f}_\lambda(x) - \widehat{f}_\mu(x) : x \in \mathbb{F}_2^n\}$. For doing this, we denote $T_0 = \{\alpha \in \mathbb{F}_2^m : \widehat{f}_\lambda(\alpha) = 0\}$, $T_0^\perp = \{0, \lambda^{-\frac{1}{2}}\}$, $T_1 = \mathbb{F}_2^m \setminus \{T_0\}$, $S_0 = \{\alpha \in \mathbb{F}_2^m : \widehat{f}_\mu(\alpha) = 0\}$, $S_0^\perp = \{0, \mu^{-\frac{1}{2}}\}$, $S_1 = \mathbb{F}_2^m \setminus \{T_0\}$.

We define

$$c_1 = \{\alpha \in T_0 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (0, 0)\},$$

$$c_2 = \{\alpha \in T_0 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (0, 2^{\frac{m+1}{2}})\},$$

$$c_3 = \{\alpha \in T_0 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (0, -2^{\frac{m+1}{2}})\},$$

$$c_4 = \{\alpha \in T_1 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (2^{\frac{m+1}{2}}, 0)\},$$

$$c_5 = \{\alpha \in T_1 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (2^{\frac{m+1}{2}}, 2^{\frac{m+1}{2}})\},$$

$$c_6 = \{\alpha \in T_1 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (2^{\frac{m+1}{2}}, -2^{\frac{m+1}{2}})\},$$

$$c_7 = \{\alpha \in T_1 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (-2^{\frac{m+1}{2}}, 0)\},$$

$$c_8 = \{\alpha \in T_1 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (-2^{\frac{m+1}{2}}, 2^{\frac{m+1}{2}})\},$$

$$c_9 = \{\alpha \in T_1 : (\widehat{f}_\lambda(\alpha), \widehat{f}_\mu(\alpha)) = (-2^{\frac{m+1}{2}}, -2^{\frac{m+1}{2}})\}.$$

If the values of $c_i$ for $1 \leq i \leq 9$ are known, then we can obtain the distribution of the values $0, \pm 2^{(m+1)/2}, \pm 2^{(m+3)/2}$ in the set $\{\widehat{f}_\lambda(x) - \widehat{f}_\mu(x) : x \in \mathbb{F}_2^n\}$ and hence give the weight distribution of the codewords in $S_\lambda$. We now compute the values of $c_i$ for $1 \leq i \leq 9$. By Lemma 4, $T_0$ and $S_0$ are two subspaces of $\mathbb{F}_2^m$ with dimension $m - 1$. Note that $T_0 \neq S_0$ since $\lambda^{-\frac{1}{2}} \neq \mu^{-\frac{1}{2}}$. Then by Lemma 6, we have

$$c_1 = 2^{m-2},$$

(14)

$$c_2 = 2^{m-3} + 2^{(m-3)/2} (1 - T_{1^m} (\mu_\lambda)), $$

(15)

$$c_3 = 2^{m-3} - 2^{(m-3)/2} (1 - T_{1^m} (\lambda_\mu)), $$

(16)

$$c_4 = 2^{m-3} + 2^{(m-3)/2} (1 - T_{1^m} (\lambda_\mu)), $$

(17)

$$c_7 = 2^{m-3} - 2^{(m-3)/2} (1 - T_{1^m} (\mu_\lambda)).$$

(18)

By Lemma 7, we have $\sum_{\alpha \in \mathbb{F}_2^m} (\widehat{f}_\lambda(\alpha) - \widehat{f}_\mu(\alpha))^2 = 2^{2m+1}$. This implies that $2^{m+1} (c_2 + c_3 + c_4 + c_7) + 2^{m+3} (c_6 + c_8) = 2^{2m+1}$. Combining (15)(16)(17)(18), we have

$$c_6 + c_8 = 2^{m-3}. $$

(19)
Similarly, it follows from \( \sum_{\alpha \in \mathbb{F}_{2^m}} (\hat{f}_\lambda(\alpha) + \hat{f}_\mu(\alpha))^2 = 2^{2m+1} \) that
\[ c_5 + c_9 = 2^{m-3}. \] (20)
According to Lemma 1, we have
\[ c_4 + c_5 + c_6 = c_2 + c_5 + c_8 = 2^{m-2} + 2^{(m-3)/2}. \] (21)
Combining (21)(15)(17), we have
\[
\begin{cases}
  c_5 + c_8 = 2^{m-3} + 2^{(m-3)/2} \text{Tr}_1^m(\frac{\mu}{\lambda}) \\
  c_5 + c_6 = 2^{m-3} + 2^{(m-3)/2} \text{Tr}_1^m(\frac{\Delta}{\mu}).
\end{cases}
\] (22)
By (19) and (22) we have
\[ c_6 = 2^{m-4} + 2^{(m-5)/2} (\text{Tr}_1^m(\frac{\lambda}{\mu}) - \text{Tr}_1^m(\frac{\lambda}{\mu})) \] (23)
and
\[ c_8 = 2^{m-4} + 2^{(m-5)/2} (\text{Tr}_1^m(\frac{\lambda}{\mu}) - \text{Tr}_1^m(\frac{\lambda}{\mu})). \] (24)
Further, we have
\[ c_5 = 2^{m-4} + 2^{(m-5)/2} \left( \text{Tr}_1^m\left(\frac{\lambda}{\mu}\right) + \text{Tr}_1^m\left(\frac{\mu}{\lambda}\right) \right) \] (25)
according to (21)(15)(24), and
\[ c_9 = 2^{m-4} - 2^{(m-5)/2} \left( \text{Tr}_1^m\left(\frac{\lambda}{\mu}\right) + \text{Tr}_1^m\left(\frac{\mu}{\lambda}\right) \right) \] (26)
by (26) and (20). Thus, we have obtained the values of \( c_i \) for all \( 1 \leq i \leq 9 \) and then we get the weight distribution of the codewords in \( S_\lambda \) according to the definitions of \( c_i \) and (13). This completes the proof.

For any integer \( m > 0 \), the Kloosterman sums over \( \mathbb{F}_{2^m} \) are defined as \( K(a) = \sum_{x \in \mathbb{F}_{2^m}} (-1)^{\text{Tr}_1^m(a^x + ax)} \), where \( a \in \mathbb{F}_{2^m} \). In fact, the Kloosterman sums are generally defined on the multiplicative group \( \mathbb{F}_{2^m}^* \). We extend them to 0 by assuming \( (-1)^0 = 1 \).

**Lemma 9 ([40]).** For any integer \( m > 0 \), \( K(1) = 1 - \sum_{t=0}^{\lfloor m/2 \rfloor} (-1)^{m-t} \frac{m-t}{m-t} 2^t \).

By the definition of Kloosterman sums, the following lemma can be easily obtained.
Lemma 10. For any integer $m > 0$, we have $\sum_{x \in \mathbb{F}_2} (-1)^{Tr^m_1(1/x)} = 1/2 \mathcal{K}(1)$ and $\sum_{x \in \{z: Tr^m_1(z) = 0\}} (-1)^{Tr^m_1(1/x)} = -1/2 \mathcal{K}(1)$.

Lemma 11. For any odd integer $m > 0$ and arbitrary $\mu \in \mathbb{F}_2^*$, we denote by $H'_\mu$ the set $\{x \in \mathbb{F}_2^*: Tr^m_1(\mu^{-1}x) = 0\}$ and by $A_{(i,j)}$ the number of $(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ appeared in the multi-set $\{(Tr^m_1(x/(x + \mu)), Tr^m_1((x + \mu)/x)) \in \mathbb{Z}_2 \times \mathbb{Z}_2 : x \in H'_\mu\}$. Then the values of $A_{(i,j)}$ is given in Table 5.

Proof. Note that $H'_\mu = \{\mu(y + y^2): y \in \mathbb{F}_2^m\}$. We have

$$
\sum_{x \in H'_\mu} (-1)^{Tr^m_1(x/x+\mu)} = \frac{1}{2} \sum_{y \in \mathbb{F}_2^m} (-1)^{Tr^m_1(\frac{\mu(y + y^2)}{y+y^2+\mu})}
= \frac{1}{2} \sum_{y \in \mathbb{F}_2^m} (-1)^{Tr^m_1(\frac{y+y^2}{y+y^2+1})}
= \sum_{x \in H'_\mu} (-1)^{Tr^m_1(\frac{x}{x+1})} (\text{by changing } x \text{ into } x + 1)
= -\sum_{x \in \{z: Tr^m_1(z) = 1\}} (-1)^{Tr^m_1(\frac{x}{x})}
= \frac{1}{2} \mathcal{K}(1).
$$

The last identity follows from Lemma 10. Similarly, we have

$$
\sum_{x \in H'_\mu} (-1)^{Tr^m_1(x+\mu)} = -\frac{1}{2} \mathcal{K}(1) + 2 \text{ and } \sum_{x \in H'_\mu} (-1)^{Tr^m_1(\frac{x}{x+\mu} + \frac{x}{x})} = \frac{1}{2} \mathcal{K}(1).
$$

Then we can easily obtain Table 5 according to the values of $\sum_{x \in H'_\mu} (-1)^{Tr^m_1(\frac{x}{x+\mu})}$, $\sum_{x \in H'_\mu} (-1)^{Tr^m_1(\frac{x+\mu}{x})}$ and $\sum_{x \in H'_\mu} (-1)^{Tr^m_1(\frac{x}{x+\mu} + \frac{x}{x})}$.

| $(i,j)$ | $A_{(i,j)}$ |
|---------|-------------|
| (0,0)   | $2m^{-3} + \frac{1}{8} \mathcal{K}(1) + \frac{1}{2}$ |
| (0,1)   | $2m^{-3} + \frac{3}{8} \mathcal{K}(1) - \frac{1}{2}$ |
| (1,0)   | $2m^{-3} - \frac{3}{8} \mathcal{K}(1) + \frac{1}{2}$ |
| (1,1)   | $2m^{-3} - \frac{1}{8} \mathcal{K}(1) - \frac{1}{2}$ |
Theorem 6. Let \( F(x) = x^e \) be the Gold functions in odd dimension \( m \geq 5 \), where \( e = 2^i + 1 \) and \( i \) is such that \( \gcd(i, n) = 1 \) and \( 1 \leq i \leq (n - 1)/2 \). Let \( f_\nu = \text{Tr}^n_1(\nu F) \) and \( H_\nu = \{ x \in \mathbb{F}_{2^m} : \text{Tr}^m_1(\nu^{-1} x) = 0 \} \), where \( \nu \in \mathbb{F}_{2^m}^* \). Then \( C_{D_f,H_\nu} \) given by (10) is a \([2^{m-1}, 2m - 1, 2^{m-2} - 2^{(m-1)/2}] \) five-weight binary code with the weight distribution in Table 6.

Table 6: The weight distribution of the code of Theorem 6

| Weight \( w \) | Multiplicity \( A_w \) |
|-----------------|----------------------|
| \( 2^{m-2} - 2^{(m-1)/2} \) | \( 2^{2m-5} + 2^{m-2} - 2^{m-3}/2 \) |
| \( 2^{m-2} - 2^{(m-3)/2} \) | \( 2^{2m-3} + 2^{m-2} \) |
| \( 2^{m-2} \) | \( 3 \cdot 2^{m-4} + 2^{m-3} - 1 \) |
| \( 2^{m-2} + 2^{(m-3)/2} \) | \( 2^{2m-3} - 2^{m-2} \) |
| \( 2^{m-2} + 2^{(m-1)/2} \) | \( 2^{2m-5} - 2^{m-2} + 2^{m-3}/2 \) |

*where the value of \( \mathcal{K}(1) \) is given in Lemma 9.

Proof. Recall that \( F \) is a permutation over \( \mathbb{F}_{2^m} \) and hence \( f_\nu \) is balanced. This implies that the length of code \( C_{D_f,H_\nu} \) is equal to \( 2^{m-1} \). Recall also that \( F \) has nonlinearity \( 2^{m-1} - 2^{m/2-1} \). By Theorem 4 we have \( C_{D_f,H_\nu} \) is a \([2^{m-1}, 2m - 1, 2^{m-2} - 2^{(m-1)/2}] \) five-weight binary code.

We now determine the weight distribution of \( C_{D_f,H_\nu} \). We can see that the code \( C_{D_f,H_\nu} \) does not include the all-one codeword. Thus, according to the proof of Theorem 2, we can assume that the nonzero codewords in \( C_{D_f,H_\nu} \) have weights:

\[
\begin{align*}
    w_1 &= 2^{m-2} - 2^{m-3}, \\
    w_2 &= 2^{m-2} - 2^{m-3}, \\
    w_3 &= 2^{m-2}, \\
    w_4 &= 2^{m-2} + 2^{m-3}, \\
    w_5 &= 2^{m-2} + 2^{m-3}.
\end{align*}
\]

We now see the number \( A_{w_i} \) of codewords with weight \( w_i \) in \( C_{D_f,H_\nu} \), where \( i = 1, 2, 3, 4, 5 \). Note that \( C_{D_f,H_\nu} = \bigcup_{\lambda \in H_\nu \setminus \{0\}} S_\lambda \cup C' \), where \( S_\lambda, \lambda \in H_\nu \setminus \{0\} \), is defined in Lemma 8 by replacing \( \lambda + \mu \) by \( \nu \) and \( C' \) is defined as \( C' = \{ c_{x,0} \in C_{D_f,H_\nu} : x \in \mathbb{F}_{2^m} \} \).

We can see that \( C' \) is a subcode of \( C_{D_f,H_\nu} \) with length \( 2^{m-1} \) and dimension \( m \). By (7) and (8) we have the nonzero codewords in \( C' \) only have weights \( w_2, w_3, w_4 \). We now determine the number \( A'_{w_i} \) of codewords with weights \( w_2, w_3, w_4 \). It can be easily seen from [18, Theorem 10] that the minimum weight of the dual code of \( C' \) is no less than 3. According to the first three Pless Power Moments [37, p. 260], we have

\[
\begin{align*}
    A'_{w_2} + A'_{w_3} + A'_{w_4} &= 2^m - 1 \\
    w_2 A'_{w_2} + w_3 A'_{w_3} + w_4 A'_{w_4} &= 2^{m-2} \\
    w_2^2 A'_{w_2} + w_3^2 A'_{w_3} + w_4^2 A'_{w_4} &= (2^{m-1} + 1)2^{2m-3}.
\end{align*}
\]
By solving this system of equations, we have $A'_{w_3} = 2^{m-2} - 2^{(m-3)/2}$, $A'_{w_4} = 2^{m-1} - 1$ and $A'_{w_3} = 2^{m-2} + 2^{(m-3)/2}$. By (7) and (8), we can see that the codewords with weight $w_1, w_5$ only appear in the set $\bigcup_{\lambda \in H_\nu \setminus \{0\}} S_\lambda$. By Lemma 8, we have
\[
A_{w_1} = \sum_{z \in H_\nu \setminus \{0\}} \left( 2^{m-4} + 2^{m-5} \left( T_{r_1}^m \left( \frac{z}{z + \nu} \right) - T_{r_1}^m \left( \frac{z + \nu}{z} \right) \right) \right)
\]
\[
= \left( 2^{m-4} (A_{(0,0)} - 1) \right) + \left( 2^{m-4} - 2^{m-5} \right) A_{(0,1)} + \left( 2^{m-4} + 2 \frac{m-5}{2} \right) A_{(1,0)} + \left( 2^{m-4} A_{(1,1)} \right)
\]
\[
= 2^{m-4} |H_\nu - 1| + 2^{m-5} \left( 1 - \frac{1}{2} \mathcal{K}(1) \right)
\]
\[
= 2^{m-5} + 2^{m-7} (1 - \frac{1}{2} \mathcal{K}(1)) - 2^{m-4}
\]
in which the values of $A_{(i,j)}$ come from Table 5. Similarly, we could deduce that
\[
A_{w_5} = 2^{2m-5} + 2^{m-7} \mathcal{K}(1) - 2^{m-5} - 2^{m-4}.
\]

We now calculate the values of $A_{w_2}, A_{w_3}, A_{w_4}$. Note that the codewords with weight $w_2, w_3, w_4$ appear in both $\bigcup_{\lambda \in H_\nu \setminus \{0\}} S_\lambda$ and $C_{D_\nu}$. By Lemma 8 and recall that $A'_{w_2} = 2^{m-2} - 2^{(m-3)/2}$, we have
\[
A_{w_2} = \sum_{z \in H_\nu \setminus \{0\}} \left( 2^{m-2} + 2 \frac{m-3}{2} \left( T_{r_1}^m \left( \frac{z + \nu}{z} \right) - T_{r_1}^m \left( \frac{z}{z + \nu} \right) \right) \right) + \left( 2^{m-2} - 2^{m-3} \right)
\]
\[
= \left( 2^{m-2} (A_{(0,0)} - 1) \right) + \left( 2^{m-2} + 2 \frac{m-3}{2} \right) A_{(0,1)} + \left( 2^{m-2} - 2^{m-3} \right) A_{(1,0)} + 2^{m-2} A_{(1,1)}
\]
\[
= 2^{m-2} |H_\nu - 1| + 2^{m-3} \left( \frac{1}{2} \mathcal{K}(1) - 1 \right) + \left( 2^{m-2} - 2^{m-3} \right)
\]
in which the values of $A_{(i,j)}$ are given in Table 5. Similarly, we have
\[
A_{w_4} = 2^{2m-3} - 2^{m-5} \mathcal{K}(1) + 2^{m-1}
\]
and
\[
A_{w_3} = \left( 3 \cdot 2^{m-3} |H_\nu - 1| \right) + \left( 2^{m-1} - 1 \right)
\]
\[
= 3 \cdot 2^{m-4} + 2^{m-3} - 1.
\]

This completes the proof. \qed
**Example 2.** For \( m = 9 \), we define \( F(x) = x^3 \), where \( x \in \mathbb{F}_{2^9} \). Let \( \lambda = 1 \) in (10). Thus we have \( f_\lambda = \text{Tr}_1^9(x^3) \) and \( H_\lambda = \{ x \in \mathbb{F}_{2^9} : \text{Tr}_1^9(x) = 0 \} \). By our Magma program, we can get the linear code \( C_{D_{f_\lambda}, H_\lambda} \) defined by (10) is a \([256, 17, 112]\)-code with weight enumerator \( 1 + 8172z^{112} + 32736z^{120} + 49215z^{128} + 32800z^{136} + 8148z^{144} \), which confirms the result of Theorem 6.

## 5 Conclusion

Inspired by a generic recent construction developed by Ding et al., we constructed several classes of binary linear codes from vectorial Boolean functions and determined their parameters. Firstly, by employing PN functions and AB functions we obtained several classes of six-weight linear codes which contain the all-one codeword. Secondly, we defined a subcode in any linear code we constructed and considered its parameter. When the vectorial Boolean function is a PN function or a Gold AB function (in odd dimension), we completely determined the weight distribution of this subcode. Besides, our linear codes have more larger dimensions than the ones by Ding et al.’s generic construction.

**References**

[1] Irving S. Reed. A class of multiple-error-correcting codes and the decoding scheme. *Trans. of the IRE Professional Group on Information Theory (TIT)*, 4:38–49, 1954.

[2] D. E. Muller. Application of boolean algebra to switching circuit design and to error detection. *Transactions of the I.R.E. Professional Group on Electronic Computers*, EC-3(3):6–12, Sept 1954.

[3] Anthony M. Kerdock. A class of low-rate nonlinear binary codes. *Information and Control*, 20(2):182–187, 1972.

[4] Claude Carlet. A simple description of kerdock codes. In *Coding Theory and Applications, 3rd International Colloquium, Toulon, France, November 2-4, 1988, Proceedings*, pages 202–208, 1988.

[5] Claude Carlet. The automorphism groups of the kerdock codes. *Journal of Information and Optimization Sciences*, 12(3):387–400, 1991.

[6] Ross J. Anderson, Cunsheng Ding, Tor Helleseth, and Torleiv Kløve. How to build robust shared control systems. *Des. Codes Cryptography*, 15(2):111–124, 1998.
[7] Claude Carlet, Cunsheng Ding, and Jin Yuan. Linear codes from perfect non-linear mappings and their secret sharing schemes. *IEEE Trans. Information Theory*, 51(6):2089–2102, 2005.

[8] Jin Yuan and Cunsheng Ding. Secret sharing schemes from three classes of linear codes. *IEEE Trans. Information Theory*, 52(1):206–212, 2006.

[9] Kelan Ding and Cunsheng Ding. A class of two-weight and three-weight codes and their applications in secret sharing. *IEEE Trans. Information Theory*, 61(11):5835–5842, 2015.

[10] Cunsheng Ding and Xuesong Wang. A coding theory construction of new systematic authentication codes. *Theor. Comput. Sci.*, 330(1):81–99, 2005.

[11] Cunsheng Ding, Tor Helleseth, Torleiv Kløve, and X. Wang. A generic construction of cartesian authentication codes. *IEEE Trans. Information Theory*, 53(6):2229–2235, 2007.

[12] Philippe Delsarte and Vladimir I. Levenshtein. Association schemes and coding theory. *IEEE Trans. Information Theory*, 44(6):2477–2504, 1998.

[13] Cunsheng Ding. Linear codes from some 2-designs. *IEEE Transactions on Information Theory*, 61(6):3265–3275, 2015.

[14] Cunsheng Ding. A construction of binary linear codes from boolean functions. *Discrete Mathematics*, 339(9):2288–2303, 2016.

[15] Markus Grassl. Bounds on the minimum distance of linear codes and quantum codes. Online available at [http://www.codetables.de](http://www.codetables.de), 2007. Accessed on 2016-08-25.

[16] Claude Carlet. Boolean functions for cryptography and error correcting codes. *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*, 2:257, 2010.

[17] Gérard Cohen, Iiro Honkala, Simon Litsyn, and Antoine Lobstein. *Covering codes*, volume 54. Elsevier, 1997.

[18] Florence Jessie MacWilliams and Neil James Alexander Sloane. *The theory of error-correcting codes*, volume 16. Elsevier, 1977.

[19] Oscar S Rothaus. On “bent” functions. *Journal of Combinatorial Theory, Series A*, 20(3):300–305, 1976.

28
[20] Kaisa Nyberg. Perfect nonlinear s-boxes. In *Advances in Cryptology-EUROCRYPT91*, pages 378–386. Springer, 1991.

[21] Sihem Mesnager. Linear codes with few weights from weakly regular bent functions based on a generic construction. *Cryptography and Communications*, pages 1–14, 2015.

[22] Chunming Tang, Yanfeng Qi, and Dongmei Huang. Two-weight and three-weight linear codes from square functions. *IEEE Communications Letters*, 20(1):29–32, 2016.

[23] Guangkui Xu, Xiwang Cao, and Shanding Xu. Two classes of p-ary bent functions and linear codes with three or four weights. *Cryptography and Communications*, pages 1–15, 2016.

[24] Chengju Li, Qin Yue, and Fang-Wei Fu. A construction of several classes of two-weight and three-weight linear codes. *Applicable Algebra in Engineering, Communication and Computing*, pages 1–20, 2016.

[25] Zhengchun Zhou, Nian Li, Cuiling Fan, and Tor Helleseth. Linear codes with two or three weights from quadratic bent functions. *Des. Codes Cryptography*, 81(2):283–295, 2016.

[26] Ziling Heng, Qin Yue, and Chengju Li. Three classes of linear codes with two or three weights. *Discrete Mathematics*, 339(11):2832–2847, 2016.

[27] Takashi Satoh, Tetsu Iwata, and Kaoru Kurosawa. On cryptographically secure vectorial Boolean functions. In *Advances in Cryptology-ASIACRYPT99*, pages 20–28. Springer, 1999.

[28] Claude Carlet. Vectorial Boolean functions for cryptography. *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*, 134:398–469, 2010.

[29] John F Dillon. *Elementary Hadamard difference sets*. PhD thesis, Univ. of Maryland, 1974.

[30] Sihem Mesnager. Bent vectorial functions and linear codes from o-polynomials. *Designs, Codes and Cryptography*, 77(1):99–116, 2015.

[31] Yuwei Xu and Chuankun Wu. On the primary constructions of vectorial Boolean bent functions.
[32] Amela Muratovic-Ribic, Enes Pasalic, and Samed Bajric. Vectorial bent functions from multiple terms trace functions. *IEEE Transactions on Information Theory*, 60(2):1337–1347, 2014.

[33] Robert Gold. Maximal recursive sequences with 3-valued recursive cross-correlation functions (corresp.). *IEEE Transactions on Information Theory*, 14(1):154–156, 1968.

[34] Tadao Kasami. The weight enumerators for several classes of subcodes of the 2nd order binary Reed-Muller codes. *Information and Control*, 18(4):369–394, 1971.

[35] Anne Canteaut, Pascale Charpin, and Hans Dobbertin. Binary m-sequences with three-valued crosscorrelation: a proof of Welch’s conjecture. *IEEE Transactions on Information Theory*, 46(1):4–8, 2000.

[36] Henk DL Hollmann and Qing Xiang. A proof of the Welch and Niho conjectures on cross-correlations of binary m-sequences. *Finite Fields and Their Applications*, 7(2):253–286, 2001.

[37] W Cary Huffman and Vera Pless. *Fundamentals of error-correcting codes*. Cambridge university press, 2003.

[38] Anne Canteaut and Pascale Charpin. Decomposing bent functions. *IEEE Transactions on Information Theory*, 49(8):2004–2019, 2003.

[39] Claude Carlet. Partially-bent functions. *Designs, Codes and Cryptography*, 3(2):135–145, 1993.

[40] Leonard Carlitz. Kloosterman sums and finite field extensions. *Acta Arithmetica*, 2(16):179–194, 1969.