An Avant-Garde Construction for Subclasses of Analytic Bi-Univalent Functions

Feras Yousef 1,*, Ala Amourah 2, Basem Aref Frasin 3 and Teodor Bulboacă 4

1 Department of Mathematics, The University of Jordan, Amman 11942, Jordan
2 Department of Mathematics, Irbid National University, Irbid 21110, Jordan; ala.amourah@siswa.ukm.edu.my
3 Department of Mathematics, Al al-Bayt University, Mafraq 25113, Jordan; bafrasin@aabu.edu.jo
4 Faculty of Mathematics and Computer Science, Babeș-Bolyai University, 400084 Cluj-Napoca, Romania; bulboaca@math.ubbcluj.ro
* Correspondence: fyousef@ju.edu.jo

Abstract: The zero-truncated Poisson distribution is an important and appropriate model for many real-world applications. Here, we exploit the zero-truncated Poisson distribution probabilities to construct a new subclass of analytic bi-univalent functions involving Gegenbauer polynomials. For functions in the constructed class, we explore estimates of Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$, and next, we solve the Fekete–Szegö functional problem. A number of new interesting results are presented to follow upon specializing the parameters involved in our main results.

Keywords: analytic bi-univalent functions; zero-truncated Poisson distribution; Gegenbauer polynomials; Fekete–Szegö functional problem

MSC: 30C45; 33C45; 60E05

1. Introduction

In discrete probability distributions, the Poisson distribution has found an extensive and varied application in formulating probability models for a wide variety of real-life phenomena dealing with counts of rare events, such as reliability theory, queueing systems, epidemiology, medicine, industry, and many others. In some practical situations, only positive counts would be available and the zero count is ignored or is impossible to be observed at all. For instance: the length of stay in a hospital is recorded as a minimum of at least one day, the number of journal articles published in different disciplines, the number of occupants in passenger cars, etc. An appropriate Poisson distribution that applies to such a case is called a zero-truncated Poisson distribution.

The probability density function of a discrete random variable $X$ that follows a zero-truncated Poisson distribution can be written as

$$P_m(X = s) = \frac{m^s}{(e^m - 1)s!}, \quad s = 1, 2, 3, \ldots,$$

where the parameter mean $m > 0$.

Now, we introduce a novel power series whose coefficients are probabilities of the zero-truncated Poisson distribution

$$P(m, z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m - 1)(n-1)!} z^n, \quad z \in U,$$

where $m > 0$ and $U := \{z \in \mathbb{C}: |z| < 1\}$ is the open unit disk. By ratio test, it is clear that the radius of convergence of the above series is infinity.
Orthogonal polynomials have been extensively studied in recent years from various perspectives due to their importance in mathematical statistics, probability theory, mathematical physics, approximation theory, and engineering. From a mathematical point of view, orthogonal polynomials often arise from solutions of ordinary differential equations under certain conditions imposed by certain model. Orthogonal polynomials that appear most commonly in applications are the classical orthogonal polynomials (Hermite polynomials, Laguerre polynomials, and Jacobi polynomials). The general subclass of Jacobi polynomials is the set of Gegenbauer polynomials, this class includes Legendre polynomials and Chebyshev polynomials as subclasses. To study the basic definitions and the most important properties of the classical orthogonal polynomials, we refer the reader to [1–4]. For a recent connection between the classical orthogonal polynomials and geometric function theory, we mention [5–10].

Gegenbauer polynomials $C_\alpha^n(x)$ for $n = 2, 3, \ldots$, and $\alpha > -\frac{1}{2}$ are defined by the following three-term recurrence formula

$$
\begin{align*}
C_\alpha^0(x) &= 1; \\
C_\alpha^1(x) &= 2\alpha x; \\
C_\alpha^n(x) &= \frac{1}{n} [2x(n + \alpha - 1)C_\alpha^{n-1}(x) - (n + 2\alpha - 2)C_\alpha^{n-2}(x)].
\end{align*}
$$

It is worth mentioning that by setting $\alpha = \frac{1}{2}$ and $\alpha = 1$ in Equation (1), we immediately obtain Legendre polynomials $P_n(x) = C_{\frac{1}{2}}^n(x)$ and Chebyshev polynomials of the second kind $U_n(x) = C_1^n(x)$, respectively.

The generating function of Gegenbauer polynomials is given as

$$
H_\alpha(x,z) = \frac{1}{(1 - 2xz + z^2)^{\alpha}},
$$

where $x \in [-1, 1]$ and $z \in \mathbb{U}$. For fixed $x$, the function $H_\alpha$ is analytic in $\mathbb{U}$, so it can be expanded in a Taylor–Maclaurin series, as follows:

$$
H_\alpha(x,z) = \sum_{n=0}^{\infty} C_\alpha^n(x)z^n, \quad z \in \mathbb{U}. 
$$

2. Preliminaries and Definitions

Let $A$ denote the class of all normalized analytic functions $f$ written as

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}.
$$

Differential subordination of analytic functions provides excellent tools for study in geometric function theory. The earliest problem in differential subordination was introduced by Miller and Mocanu [11], see also [12]. The book of Miller and Mocanu [13] sums up most of the advancement in the field and the references to the date of its publication.

**Definition 1.** Let $f$ and $g$ be two analytic functions in $\mathbb{U}$. The function $f$ is said to be subordinate to $g$, written as $f(z) \prec g(z)$, if there is an analytic function $\omega$ in $\mathbb{U}$ with the properties

$$
\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1, \quad z \in \mathbb{U},
$$

such that

$$
f(z) = g(\omega(z)), \quad z \in \mathbb{U}.
$$

**Definition 2.** A single-valued one-to-one function $f$ defined in a simply connected domain is said to be a univalent function.
Let $S$ denote the class of all functions $f \in A$, given by (3), that are univalent in $U$. Hence, every function $f \in S$ has an inverse given by
\[
    f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \ldots
\]  
(4)

**Definition 3.** A univalent function $f$ is said to be bi-univalent in $U$ if its inverse function $f^{-1}(w)$ has an analytic univalent extension in $U$.

Let $\Sigma$ denote the class of all functions $f \in A$ that are bi-univalent in $U$ given by (3). For interesting subclasses of functions in the class $\Sigma$, see [14–24].

The coefficient functional
\[
    \Delta_\eta(f) = a_3 - \eta a_2^2 = \frac{1}{6} \left(f'''(0) - \frac{3\eta}{2} (f''(0))^2\right)
\]  
(5)

of the analytic function $f$ given by (3) is very important in the theory of analytic and univalent functions. Thus, it is quite natural to ask about inequalities for $\Delta_\eta(f)$ corresponding to subclasses of bi-univalent functions in the open unit disk $U$. The problem of maximizing the absolute value of the functional $\Delta_\eta(f)$ is called the Fekete–Szegö problem [25]. There are now several results of this type in the literature, each of them dealing with $|a_3 - \eta a_2^2|$ for various classes of functions defined in terms of subordination (see, e.g., [26–31]).

Now, let us define the linear operator 
\[
    \chi : A \rightarrow A
\]
by
\[
    \chi_m f(z) := f(m, z) \ast f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m - 1)(n - 1)!} a_n z^n, \quad z \in U,
\]
where the symbol “$\ast$” denotes the Hadamard product of the two series.

To obtain our results we need the following lemma:

**Lemma 1** ([32], p. 172). Assume that $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n, z \in U$, is an analytic function in $U$ such that $|\omega(z)| < 1$ for all $z \in U$. Then, $|\omega_1| \leq 1$, $|\omega_n| \leq 1 - |\omega_1|^2, \ n = 2, 3, \ldots$.

Motivated essentially by the earlier work of Amourah et al. [33], we construct, in the next section, a new subclass of bi-univalent functions governed by the zero-truncated Poisson distribution series and Gegenbauer polynomials. Then, we investigate the optimal bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and solve the Fekete–Szegö functional problem for functions in our new subclass.

3. The Class $\zeta_\Sigma(x, \alpha, \delta, \mu)$

Consider the function $f \in \Sigma$ given by (3), the function $g = f^{-1}$ given by (4), and $H_a$ is the generating function of Gegenbauer polynomials given by (2). Now, we are ready to define our new subclass of bi-univalent functions $\zeta_\Sigma(x, \alpha, \delta, \mu)$ as follows.

**Definition 4.** A function $f$ is said to be in the class $\zeta_\Sigma(x, \alpha, \delta, \mu)$, if the following subordinations are fulfilled:
\[
    (1 - \mu) \frac{\chi_m f(z)}{z} + \mu (\chi_m f(z))' + \delta z (\chi_m f(z))'' < H_a(x, z),
\]
and
\[
    (1 - \mu) \frac{\chi_m g(w)}{w} + \mu (\chi_m g(w))' + \delta w (\chi_m g(w))'' < H_a(x, w),
\]
where $\alpha > 0$, $\mu, \delta \geq 0$, and $x \in \left(\frac{1}{2}, 1\right]$. 
Upon allocating the parameters $\mu$ and $\delta$, one can obtain several new subclasses of $\Sigma$, as illustrated in the following two examples.

**Example 1.** A function $f$ is said to be in the class $\zeta_\Sigma(x, \alpha, \mu) := \zeta_\Sigma(x, \alpha, 0, \mu)$, if the following subordinations are fulfilled:

\[
(1 - \mu)\frac{X_m(f(z))}{z} + \mu(\chi_m(f(z)))' < H_\alpha(x, z),
\]

and

\[
(1 - \mu)\frac{X_m(g(w))}{w} + \mu(\chi_m(g(w)))' < H_\alpha(x, w),
\]

where $\alpha > 0$, $\mu \geq 0$, and $x \in \left(\frac{1}{2}, 1\right]$.

**Example 2.** A function $f$ is said to be in the class $\zeta_\Sigma(x, \alpha) := \zeta_\Sigma(x, \alpha, 0, 1)$, if the following subordinations are fulfilled:

\[
(\chi_m(f(z)))' < H_\alpha(x, z),
\]

and

\[
(\chi_m(g(w)))' < H_\alpha(x, w),
\]

where $\alpha > 0$ and $x \in \left(\frac{1}{2}, 1\right]$.

**4. Main Results**

**Theorem 1.** If the function $f$ belongs to the class $\zeta_\Sigma(x, \alpha, \delta, \mu)$, then

\[
|a_2| \leq \frac{2\alpha x(e^m - 1)^{\frac{m}{2}}}{m \sqrt{\left[2\alpha(1 + 2\mu + 6\delta)(e^m - 1) - 2(1 + \alpha)(1 + \mu + 2\delta)^2\right] x^2 + (1 + \mu + 2\delta)^2}},
\]

and

\[
|a_3| \leq \frac{4\alpha^2 x^2(e^m - 1)^2}{m^2(1 + \mu + 2\delta)^2} + \frac{4\alpha x(e^m - 1)^{\frac{m}{2}}}{m\sqrt{(1 + 2\mu + 6\delta)^2}}.
\]

**Proof.** If $f \in \zeta_\Sigma(x, \alpha, \delta, \mu)$, from the Definition 4 there exist two analytic functions in $U$ that are $w$ and $v$, such that $w(0) = v(0) = 0$ and $|w(z)| < 1$, $|v(w)| < 1$ for all $z, w \in U$, and

\[
(1 - \mu)\frac{X_m(f(z))}{z} + \mu(\chi_m(f(z)))' + \delta z(\chi_m(f(z)))'' = H_\alpha(x, \omega(z)), z \in \mathbb{U},
\]

and

\[
(1 - \mu)\frac{X_m(g(w))}{w} + \mu(\chi_m(g(w)))' + \delta w(\chi_m(g(w)))'' = H_\alpha(x, \upsilon(w)), w \in \mathbb{U},
\]

From the equalities (7) and (8), we obtain

\[
(1 - \mu)\frac{X_m(f(z))}{z} + \mu(\chi_m(f(z)))' + \delta z(\chi_m(f(z)))'' = 1 + C_1(x)c_1z + \left[C_1^n(x)c_2 + C_2^n(x)c_1^2\right]z^2 + \ldots, z \in \mathbb{U},
\]

and

\[
(1 - \mu)\frac{X_m(g(w))}{w} + \mu(\chi_m(g(w)))' + \delta w(\chi_m(g(w)))'' = 1 + C_1^n(x)d_1w + \left[C_1^n(x)d_2 + C_2^n(x)d_1^2\right]w^2 + \ldots, w \in \mathbb{U},
\]
where
\[ \omega(z) = \sum_{j=1}^{\infty} c_j z^j, \quad z \in \mathbb{U}, \] and \[ v(w) = \sum_{j=1}^{\infty} d_j w^j, \quad w \in \mathbb{U}. \] (11)

According to Lemma 1, if the above function \( \omega \) and \( v \) has the form (11), then
\[ |c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \] (12)

Thus, upon comparing and equating the corresponding coefficients in (9) and (10), we have
\[ \frac{(1 + \mu + 2\delta)m}{e^m - 1} a_2 = C_1^a(x) c_1, \] (13)
\[ \frac{(1 + 2\mu + 6\delta)m^2}{2(e^m - 1)} a_3 = C_1^a(x) c_2 + C_2^a(x) c_1^2, \] (14)
\[ - \frac{(1 + \mu + 2\delta)m}{e^m - 1} a_2 = C_1^a(x) d_1, \] (15)
and
\[ \frac{(1 + 2\mu + 6\delta)m^2}{2(e^m - 1)} \left[ 2a_2^2 - a_3 \right] = C_1^a(x) d_2 + C_2^a(x) d_1^2. \] (16)

It follows from (13) and (15) that
\[ c_1 = -d_1, \] (17)
and
\[ \frac{2(1 + \mu + 2\delta)^2 m^2}{(e^m - 1)^2} a_2 = \left[ C_1^a(x) \right]^2 \left( c_1^2 + d_1^2 \right). \] (18)

If we add (14) and (16), we get
\[ \frac{(1 + 2\mu + 6\delta)m^2}{(e^m - 1)} a_2^2 = C_1^a(x) (c_2 + d_2) + C_2^a(x) \left( c_1^2 + d_1^2 \right). \] (19)

Substituting the value of \( (c_1^2 + d_1^2) \) from (18) in the right hand side of (19), we deduce that
\[ \left[ (1 + 2\mu + 6\delta) - \frac{2(1 + \mu + 2\delta)^2 C_1^a(x)}{e^m - 1} \right] \frac{m^2}{(e^m - 1)} a_2^2 = C_1^a(x) (c_2 + d_2). \] (20)

Now, using (1), (12) and (20), we find that (6) holds.
Moreover, if we subtract (16) from (14), we obtain
\[ \frac{(1 + 2\mu + 6\delta)m^2}{(e^m - 1)} \left( a_3 - a_2^2 \right) = C_1^a(x) (c_2 - d_2) + C_2^a(x) \left( c_1^2 - d_1^2 \right). \] (21)

Then, in view of (17) and (18), Equation (21) becomes
\[ a_3 = \frac{(e^m - 1)^2 \left[ C_1^a(x) \right]^2}{2m^2(1 + \mu + 2\delta)^2} \left( c_1^2 + d_1^2 \right) + \frac{(e^m - 1)C_1^a(x)}{m^2(1 + 2\mu + 6\delta)} (c_2 - d_2). \]

Thus, applying (1), we conclude that
\[ |a_3| \leq \frac{4\alpha^2 x^2 (e^m - 1)^2}{m^2(1 + \mu + 2\delta)^2} + \frac{4\alpha x (e^m - 1)}{m^2(1 + 2\mu + 6\delta)}, \]
and the proof of the theorem is complete. \( \square \)
The following result addresses the Fekete–Szegő functional problem for functions in the class \( \zeta(x, \alpha, \delta, \mu) \).

**Theorem 2.** If the function \( f \) belongs to the class \( \zeta(x, \alpha, \delta, \mu) \), then

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{4ax(e^m - 1)}{m^2(1 + 2\mu + 6\delta)} & \text{if } |\eta - 1| \leq M, \\
\frac{8a^2 x^2 |e^m - 1| |1 - \eta|}{m^2 \left[2a(1 + 2\mu + 6\delta)(e^m - 1) - 2(1 + \alpha)(1 + \mu + 2\delta)^2x^2 + (1 + \mu + 2\delta)^2\right]} & \text{if } |\eta - 1| \geq M,
\end{cases}
\]

where

\[
M := \left|1 - \frac{(1 + \mu + 2\delta)^2 [2(1 + x) x^2 - 1]}{2ax^2 (e^m - 1)(1 + 2\mu + 6\delta)}\right|.
\]

**Proof.** If \( f \in \zeta(x, \alpha, \delta, \mu) \), from (20) and (21) we get

\[
a_3 - \eta a_2^2 = (1 - \eta) \frac{(e^m - 1)^2 [C^m_1(x)]^3 (c_2 + d_2)}{m^2 \left(e^m - 1)(1 + 2\mu + 6\delta) [C^m_1(x)]^2 - 2(1 + \mu + 2\delta)^2 C^m_2(x)\right]}
+ \frac{(e^m - 1) C^m_1(x)}{m^2 (1 + 2\mu + 6\delta)} (c_2 - d_2)
\]

\[
= C^m_1(x) \left[h(\eta) + \frac{(e^m - 1)}{m^2 (1 + 2\mu + 6\delta)} \right] c_2 + \left[h(\eta) - \frac{(e^m - 1)}{m^2 (1 + 2\mu + 6\delta)} \right] d_2,
\]

where

\[
h(\eta) = \frac{(e^m - 1)^2 [C^m_1(x)]^2 (1 - \eta)}{m^2 \left(e^m - 1)(1 + 2\mu + 6\delta) [C^m_1(x)]^2 - 2(1 + \mu + 2\delta)^2 C^m_2(x)\right]}
\]

Then, in view of (1), we conclude that

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{4ax(e^m - 1)}{m^2(1 + 2\mu + 6\delta)} & \text{if } 0 \leq |h(\eta)| \leq \frac{(e^m - 1)}{m^2(1 + 2\mu + 6\delta)}, \\
4ax|h(\eta)|, & \text{if } |h(\eta)| \geq \frac{(e^m - 1)}{m^2(1 + 2\mu + 6\delta)},
\end{cases}
\]

which completes the proof of Theorem 2. \( \square \)

5. Corollaries and Consequences

Corresponding essentially to the Example 1 (setting \( \delta = 0 \)) and Example 2 (setting \( \delta = 0 \) and \( \mu = 1 \)), from Theorems 1 and 2 we get the following consequences, respectively.

**Corollary 1.** If the function \( f \) belongs to the class \( \zeta(x, \alpha, \mu) \), then

\[
|a_2| \leq \frac{2ax(e^m - 1)\sqrt{2x}}{m^2 \left[2a(1 + 2\mu)(e^m - 1) - 2(1 + \alpha)(1 + \mu)^2x^2 + (1 + \mu)^2\right]},
\]

\[
|a_3| \leq \frac{4a^2 x^2 (e^m - 1)^2}{m^2 (1 + \mu)^2} + \frac{4ax(e^m - 1)}{m^2(1 + 2\mu)}.
\]
and

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{4ax(e^m-1)}{m^2(1+2\mu)}, & \text{if } |\eta - 1| \leq N, \\
\frac{8a^2x^2(e^m-1)^2(1-\eta)}{m^2 \left\{ 2(a(1+2\mu)(e^m-1)-2(1+\alpha)(1+\mu)^2)x^2+(1+\mu)^2 \right\}}, & \text{if } |\eta - 1| \geq N,
\end{cases}
\]

where

\[N := \left| 1 - \frac{(1+\mu)^2}{2ax^2(e^m-1)(1+2\mu)} \right| \]

**Corollary 2.** If the function \(f\) belongs to the class \(\zeta_{\Sigma}(x, \alpha)\), then

\[
|a_2| \leq \frac{2ax(e^m-1)\sqrt{2x}}{m\sqrt{\left| 6a(e^m-1)-8(1+\alpha)x^2+4 \right|}},
\]

\[
|a_3| \leq \frac{a^2x^2(e^m-1)^2}{m^2} + \frac{4ax(e^m-1)}{3m^2},
\]

and

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{4ax(e^m-1)}{3m^2}, & \text{if } |\eta - 1| \leq L, \\
\frac{8a^2x^2(e^m-1)^2(1-\eta)}{m^2 \left\{ 6a(e^m-1)-8(1+\alpha)x^2+4 \right\}}, & \text{if } |\eta - 1| \geq L,
\end{cases}
\]

where

\[L := \left| 1 - \frac{2(1+\alpha)x^2-1}{3ax^2(e^m-1)} \right| \]

**6. Concluding Remarks**

In the present work we have constructed a new subclass \(\zeta_{\Sigma}(x, \alpha, \delta, \mu)\) of normalized analytic and bi-univalent functions governed with the zero-truncated Poisson distribution series and Gegenbauer polynomials. For functions belonging to this class, we have made estimates of Taylor–Maclaurin coefficients, \(|a_2|\) and \(|a_3|\), and solved the Fekete–Szegö functional problem. Furthermore, by suitably specializing the parameters \(\delta\) and \(\mu\), one can deduce the results for the subclasses \(\zeta_{\Sigma}(x, \alpha, \mu)\) and \(\zeta_{\Sigma}(x, \alpha)\) which are defined, respectively, in Examples 1 and 2.

The results offered in this paper would lead to other different new results for the classes \(\zeta_{\Sigma}(x, 1/2, \delta, \mu)\) for Legendre polynomials and \(\zeta_{\Sigma}(x, 1, \delta, \mu)\) for Chebyshev polynomials.

It remains an open problem to derive estimates on the bounds of \(|a_n|\) for \(n \geq 4\), \(n \in \mathbb{N}\), for the subclasses that have been introduced here.

**Author Contributions:** Conceptualization, F.Y. and A.A.; methodology, A.A.; validation, F.Y., A.A., B.A.F. and T.B.; formal analysis, A.A.; investigation, F.Y., B.A.F. and T.B.; writing—original draft preparation, F.Y. and A.A.; writing—review and editing, F.Y., T.B.; supervision, B.A.F. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** No data were used to support this study.

**Conflicts of Interest:** The authors declare no conflict of interest.
References

1. Agarwal, P.; Agarwal, R.P.; Ruzhansky, M. Special Functions and Analysis of Differential Equations; CRC Press: Boca Raton, FL, USA, 2020.
2. Doman, B. The Classical Orthogonal Polynomials; World Scientific: Singapore, 2015.
3. Chihara, T.S. An Introduction to Orthogonal Polynomials; Courier Corporation: Mineola, NY, USA, 2011.
4. Ismail, M.; Ismail, M.E.; van Assche, W. Classical and Quantum Orthogonal Polynomials in One Variable; Cambridge University Press: Cambridge, UK, 2005.
5. Wanas, A.K. New Families of Bi-univalent Functions Governed by Gegenbauer Polynomials. *Ear. J. Math. Sci.* 2021, 7, 403–427. [CrossRef]
6. Frasin, B.A.; Yousef, F.; Al-Hawary, T.; Aldawish, I. Application of Generalized Bessel Functions to Classes of Analytic Functions. *Afr. Mat.* 2021, 32, 431–439. [CrossRef]
7. Frasin, B.A.; Al-Hawary, T.; Yousef, F.; Aldawish, I. On Subclasses of Analytic Functions Associated with Struve Functions. *Nonlinear Func. Anal. Appl.* 2022, 27, 99–110. [CrossRef]
8. Altınkaya, Ş.; Yalçın, S. Estimates on Coefficients of a General Subclass of Bi-univalent Functions Associated with Symmetric q-Derivative Operator by Means of the Chebyshev Polynomials. *Asia Pac. J. Math.* 2017, 8, 90–99.
9. Bulut, S.; Magesh, N.; Balaji, V.K. Initial Bounds for Analytic and Bi-Univalent Functions by Means of Chebyshev Polynomials. *J. Class. Anal.* 2017, 11, 83–89. [CrossRef]
10. Amourah, A.; Frasin, B.A.; Abdeljawad, T. Fekete-Szegő Inequality for Analytic and Bi-univalent Functions Subordinate to Gegenbauer Polynomials. *Axioms* 2022, 11, 92. [CrossRef]
11. Miller, S.S.; Mocanu, P.T. Second Order Differential Inequalities in the Complex Plane. *J. Math. Anal. Appl.* 1978, 65, 289–305. [CrossRef]
12. Miller, S.S.; Mocanu, P.T. Differential Subordinations and Univalent Functions. *Mich. J. Math.* 1981, 28, 157–172. [CrossRef]
13. Miller, S.S.; Mocanu, P.T. Differential Subordinations. *Theory and Applications*; Marcel Dekker, Inc.: New York, NY, USA, 2000.
14. Ahmad, I.; Ali Shah, S.G.; Hussain, S.; Darus, M.; Ahmad, B. Fekete-Szegő Functional for Bi-univalent Functions Related with Gegenbauer Polynomials. *J. Math.* 2022, 2022, 2705203. [CrossRef]
15. Murugusundaramoorthy, G.; Balboacă, T. Subclasses of Yamakawa-Type Bi-Starlike Functions Associated with Gegenbauer Polynomials. *Axioms* 2022, 11, 92. [CrossRef]
16. Sakar, F.M.; Doğan, E. Problem on Coefficients of Bi-Univalent Function Class Using Chebyshev Polynomials. In *Mathematical, Computational Intelligence and Engineering Approaches for Tourism, Agriculture and Healthcare*; Lecture Notes in Networks and Systems; Srivastava, P., Thakur, S.S., Oros, G.I., AlJarrah, A.A., Laohakosol, V., Eds.; Springer: Singapore, 2022; Volume 214. [CrossRef]
17. Wanas, A.K.; Cotlera, L-I. Applications of (M,N)-Lucas Polynomials on a Certain Family of Bi-Univalent Functions. *Mathematics* 2022, 10, 595. [CrossRef]
18. Frasin, B.A.; Swamy, S.R.; Nirmala, J. Some Special Families of Holomorphic and Al-Oboudi Type Bi-Univalent Functions Related to k-Fibonacci Numbers Involving Modified Sigmoid Activation Function. *Afr. Mat.* 2021, 32, 631–643. [CrossRef]
19. Srivastava, H.M.; Wanas, A.K.; Srivastava, R. Applications of the q-Srivastava-Atiya Operator Involving a Certain Family of Bi-univalent Functions Associated with the Horadam Polynomials. *Symmetry* 2021, 13, 1230. [CrossRef]
20. Kanas, S.; Siwaşnarković, P.V.; Karthikeyan, R.; Sivasubramanian, S. Second Hankel Determinant for a Certain Subclass of Bi-Close to Convex Functions Defined by Kaplan. *Symmetry* 2021, 13, 567. [CrossRef]
21. Atshan, W.G.; Rahman, I.A.R.; Lupas, A.A. Some Results of New Subclasses for Bi-Univalent Functions Using Quasi-Subordination. *Symmetry* 2021, 13, 1653. [CrossRef]
22. Yousif, F.; Alroud, S.; Ilafe, M. New Subclasses of Analytic and Bi-univalent Functions Endowed with Coefficient Estimate Problems. *Anal. Math. Phys.* 2021, 11, 58. [CrossRef]
23. Bulut, S. Coefficient Estimates for a Class of Analytic and Bi-univalent Functions. *Novi. Sad. J. Math.* 2013, 43, 59–65.
24. Murugusundaramoorthy, G.; Magesh, N.; Prameela, V. Coefficient Bounds for Certain Subclasses of Bi-univalent Function. *Abstr. Appl. Anal.* 2013, 2013, 573017. [CrossRef]
25. Fekete, M.; Szegő, G. Eine Bemerkung über ungerade schlichte funktionen. *J. Lond. Math. Soc.* 1933, 1, 85–89. [CrossRef]
26. Ilafe, M.; Amourah, A.; Haji Mohd, M. Coefficient Estimates and Fekete-Szegő Functional Inequalities for a Certain Subclass of Analytic and Bi-Univalent Functions. *Axioms* 2022, 11, 147. [CrossRef]
27. Wanas, A.K.; Yalçın, S. Coefficient Estimates and Fekete-Szegő Inequality for Family of Bi-Univalent Functions Defined by the Second Kind Chebyshev Polynomial. *Int. J. Problems Compt. Math.* 2020, 13, 25–33.
28. Karthikeyan, K.R.; Murugusundaramoorthy, G. Unified Solution of Initial Coefficients and Fekete-Szegő Problem for Subclasses of Analytic Functions Related to a Conic Region. *Afr. Mat.* 2022, 33, 44. [CrossRef]
29. Swamy, S.R.; Altınkaya, Ş. Fekete-Szegő Functional for Regular Functions Based on Quasi-Subordination. *Int. J. Nonlinear Anal. Appl.* 2022, 14, 1–11.
30. Swamy, S.R.; Sailaja, Y. On the Fekete-Szegő Coefficient Functional for Quasi-Subordination Class. *Palas. J. Math.* 2021, 10, 666–672.
31. Bulut, S. Fekete-Szegő Problem for Starlike Functions Connected with k-Fibonacci Numbers. *Math. Slov.* 2021, 71, 823–830. [CrossRef]
32. Nehari, Z. *Conformal Mapping*; McGraw-Hill: New York, NY, USA, 1952.
33. Amourah, A.; Frasin, B.A.; Ahmad, M.; Yousef, F. Exploiting the Pascal Distribution Series and Gegenbauer Polynomials to Construct and Study a New Subclass of Analytic Bi-Univalent Functions. *Symmetry* 2022, 14, 147. [CrossRef]