Some Additions to a Family of Integrals related to Hurwitz’ Zeta Function

Michael Milgram
Consulting Physicist, Geometrics Unlimited, Ltd.
Box 1484, Deep River, Ont. Canada. K0J 1P0

May 8, 2020

MSC classes: 11M06, 11M35, 11M99, 26A09, 30B40, 30E20, 33C20, 33C47, 33B99, 33E20

Keywords: evaluation of improper integrals, Hurwitz zeta function, sech kernel, arctan, log, Parseval identity, Riemann zeta function

Abstract
Integrals involving the kernel function $\text{sech}(\pi x)$ over a semi-infinite range are of general interest in the study of Riemann’s function $\zeta(s)$ and Hurwitz’ function $\zeta(s,a)$. Such integrals that include the $\text{arctan}$ and $\log$ functions in the integrand are evaluated here in terms of $\zeta(s,a)$, thereby adding some new members to a known family of related integrals. A claimed connection between the function $\zeta(2m+1)$ and such integrals is verified.

1 Introduction

Integral representations of Riemann’s Zeta function $\zeta(s)$ and Hurwitz’ Zeta function $\zeta(s,a)$ have a long and storied history. For example, in 1895 Jensen [1] presented (without proof) a result equivalent to

$$\zeta(s) = 2^s \int_0^\infty \frac{\sinh(\pi t) \sin(s \arctan(2t))}{(4t^2 + 1)^{s/2} \cosh(\pi t)} \, dt, \quad (1.1)$$

that being a specialized ($a=1$) case of Hermite’s more general result

$$\zeta(s,a) = \frac{1}{(2a^s)} + \frac{a^{-s+1}}{s-1} + 2 \int_0^\infty \frac{\sin(s \arctan(t/a))}{(a^2 + t^2)^{s/2} (a^2 \pi t - 1)} \, dt. \quad (1.2)$$

More recently, Boros, Espinosa and Moll [2] and Adamchik [3, Section 3], have studied a family of integrals

$$\int_0^\infty f(t) K(a,t) \, dt \quad (1.3)$$

where the kernels

$$K(a,t) = \frac{1}{(\exp(2\pi at) - 1)},$$

$$= \frac{1}{(\exp(2\pi at) + 1)},$$

and

$$= \frac{1}{\sinh(2\pi at)}$$

mike@geometrics-unlimited.com

1
generalize the older results. Following that, Patkowski [4], attempted to extend Boros, Espinosa
and Moll’s results to include the kernel
\[ K(a, t) = \frac{1}{\cosh(\pi t)}. \] (1.4)
Unfortunately, that paper contains a large number of misprints (see the Appendix for corrections)
as well as a fundamental error in the analysis of the purported result that incorporates kernel
\[ (1.4). \] However, Patkowski’s paper makes a significant contribution by showing the importance of
the associated Laguerre polynomials [5]
\[ L(n, a, x) = \sum_{j=0}^{n} \frac{(-1)^j \Gamma(n + a + 1) x^j}{\Gamma(n - j + 1) \Gamma(a + j + 1) \Gamma(j + 1)} \] (1.5)
that arise from the identity [4, Eq. (2.6)], [6, Eq. 3.769(4)]
\[ \int_{0}^{\infty} t^{2n} \sin(\omega t) \sin(s \arctan \left( \frac{t}{a} \right)) \frac{\sinh(\pi t)}{(a^2 + t^2)^{s/2}} dt = \frac{(-1)^n \pi \Gamma(2n + 1) L(2n, s - 2n - 1, aw) e^{-aw}}{2 \Gamma(s) w^{2n+s}}. \] (1.6)
Approaching from a different direction, Milgram (2020 - in preparation) has noted an integral
representation related to \( \zeta(2m+1) \) that involves the kernel (1.4), specifically
\[ (2^{2m+1} - 1) \zeta(2m+1) = A(m) + (-1)^m \pi^2 m \sum_{j=0}^{m} \frac{\mathcal{E}(2j) \psi(1 - 2j + 2m)}{\Gamma(1 - 2j + 2m) \Gamma(2j + 1)}, \] (1.7)
where
\[ A(m) = \frac{2^{3+2m} \pi^2 m}{\Gamma(3+2m)} \int_{0}^{\infty} t^{2m+2} \mathcal{F}_2 \frac{3F_2(1, 1, 3/2; 2 + m, 3/2 + m; -4t^2)}{\cosh(\pi t)} dt \] (1.8)
and \( \mathcal{E}(2j) \) are Euler numbers (see below for a compendium of notation).

It will be shown in Section 6 that the hypergeometric function in (1.8) reduces to combinations of
functions that include \( \arctan(t/a) \) and \( \log(1 + 4t^2) \); hence the interest in integrals with kernel
\[ (1.4). \] Finally, Patkowski (also Glasser - private communication) demonstrates the power of
Parseval’s identity (q.v.) that eventually leads to the main result of this work, that being a
related form of [4, Theorem 4] and corrected form of [4, Eq. (3.20)] (see also [6, Section 3.975]):

**Theorem 1**

\[ \int_{0}^{\infty} t^{2n+2} \sin(s \arctan \left( \frac{t}{a} \right)) \frac{\sinh(\pi t)}{(a^2 + v^2)^{s/2}} dv = 2^{2n-2s-1}(-1)^{1+n} a \int_{0}^{\infty} t^{2n+2} \mathcal{F}_2 \frac{3F_2(1, 1, 3/2; 2 + m, 3/2 + m; -4t^2)}{\cosh(\pi t)} dt \] (1.9)
where [7, Eq. (2.2(5))] \( S(s, q) \equiv \zeta(s, q) - \zeta(s, q + 1/2) = 2^s \zeta_a(s, 2q). \) (1.10)

In the above, and throughout this work
\[ \zeta_a(s, a) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j}{(j + a)^s}, \] (1.11)
is the alternating Hurwitz function, \( \Re(a) > 0 \) and
\[
q \equiv \frac{a}{4} + \frac{1}{4}.
\] (1.12)
Throughout, the symbols \( j, k, m, n \) are non-negative integers, \( E(n, x) \) is an Euler polynomial, \( E(n) \) is an Euler number, \( B(n, x) \) is a Bernoulli polynomial and \( B(n) \) is a Bernoulli number. \( \gamma \) is the Euler-Mascheroni constant and \( \psi(x) \) is the digamma function. \( A \) and \( G \) are Glaisher’s and Catalan’s constants respectively. Unless otherwise specified, the prime symbol ‘ applied to \( \zeta(s, a) \) or \( S(s, a) \) always refers to differentiation with respect to the first argument. Thus
\[
S'(-n, a) \equiv \frac{\partial}{\partial s} S(s, a)_{s=-n}.
\] (1.13)
Also, throughout, I employ, without further comment, the identities
\[
\sin \left( \arctan \left( \frac{x}{a} \right) \right) = \frac{x}{\sqrt{x^2 + 1}}
\] (1.14) and
\[
\cos \left( \arctan \left( \frac{x}{a} \right) \right) = \frac{1}{\sqrt{x^2 + 1}}.\] (1.15)

2 Proof of Theorem 1

Following Patkowski, the Fourier sine transform of the function \( 1/\cosh(\pi t/2) \) is required in order to apply Parseval’s identity to evaluate the integral [4, Theorem 4], which transforms to (1.9) with even moments. Although such an identity exists [6, Eq. 3.981.9], it is rather complicated, and related discussion will be deferred to Section 5. Alternatively, consider the well-known [6, Eq. 3.524.13] Fourier sine transform of the associated function \( t/\cosh(\pi t/2) \)
\[
\int_0^\infty \frac{t \sin(wt)}{\cosh(\pi t/2)} \, dt = \frac{\sinh(w)}{\cosh^2(w)},
\] (2.1)
from which, employing Parseval’s identity along with (1.6) and (2.1) and choosing \( s > 2n - 1 \), we have
\[
\int_0^\infty \frac{t^{2n+1} \sin(s \arctan \left( \frac{t}{a} \right))}{(a^2 + t^2)^{s/2} \cosh(\pi t/2)} \, dt = (-1)^n \frac{\Gamma(2n + 1)}{\Gamma(s)} \int_0^\infty \frac{L(2n, s - 2n - 1, aw) \sinh(w) e^{-aw}}{w^{2n+1-s} \cosh^2(w)} \, dw
\]
\[
= (-1)^n \frac{\Gamma(2n + 1)}{\Gamma(s)} \sum_{j=0}^{2n} \frac{(-a)^j}{\Gamma(2n - j + 1) \Gamma(s - 2n + j) \Gamma(j + 1)} \int_0^\infty \frac{w^{s-2n-1+j} \sinh(w) e^{-aw}}{\cosh^2(w)} \, dw
\] (2.2)
after applying (1.6) and inverting the order of operations because the integral converges under the condition specified and the sum is finite. To evaluate the integral in the second equality of (2.2), consider the well-known [8, Eq. 25.11.31] representation (and corrected form of [4, Eq. (3.3)]), which generalizes [6, Eq. 3.552.3]
\[
\int_0^\infty \frac{v^{s-1} e^{-av}}{\cosh(v)} \, dv = 2^{1-s} \Gamma(s) S(s, q).
\] (2.3)
After integration by parts, (2.3) becomes
\[
a \int_0^\infty \frac{e^{-av} v^s}{\cosh(v)} \, dv + \int_0^\infty \frac{v^s e^{-av} \sinh(v)}{\cosh^2(v)} \, dv = \Gamma(s + 1) 2^{1-s} S(s, q),
\] (2.4)
from which, using (2.3) with \( s := s + 1 \) it follows that
$$\int_0^\infty \frac{e^{-aw} w^s \sinh(w)}{\cosh^2(w)} \, dw = 2^{-2s} \Gamma(s+1) \left( 2S(s, q) - \frac{a}{2} S(s+1, q) \right). \quad (2.5)$$

Set $s \to s - 2n - 1 + j$ and substitute (2.5) into (2.2), to arrive at the following

$$\int_0^\infty \frac{t^{2n+1} \sin \left( s \arctan \left( \frac{t}{a} \right) \right)}{(a^2 + t^2)^{s/2} \cosh(\pi t/2)} \, dt = (-1)^n 2^{s+2n-2} \sum_{j=0}^{2n} \binom{2n}{j} (-a/4)^j \left( S(s - 2n + j - 1, q) - \frac{a}{4} S(s - 2n + j, q) \right). \quad (2.6)$$

Since (2.6) involves two separate finite sums, shift the index of one sum by unity so that the arguments of $S(s - 2n + j, q)$ and $S(s - 2n + j - 1, q)$ coincide, and after some minor simplification, including reversal of the series, and a minor change of variables in the integration term, the result (1.9) will be found. QED

3 Corollaries

3.1 Corollary 3.1

$$\sum_{j=0}^{2n} \frac{(-a/2)^{-j} \mathcal{E}(j+1, 2q)}{\Gamma(j+1) \Gamma(2n-j+1)} = \frac{\mathcal{E}(2n)}{2 \Gamma(2n+1) a^{2n-1}}. \quad (3.1)$$

Proof: Set $s=0$ in (2.6) to find

$$\sum_{j=0}^{2n} \frac{(-a/4)^j (S(j - 2n - 1, q) - \frac{a}{4} S(j - 2n, q))}{\Gamma(2n-j+1) \Gamma(j+1)} = 0, \quad (3.2)$$

and invoke [2, Lemma 3.1], that is

$$\zeta(1-m, q) = -B(m, q)/m \quad m > 0. \quad (3.3)$$

Then reverse the sum, so that, with (1.10), (3.2) can be rewritten in terms of Bernoulli polynomials as

$$\sum_{j=0}^{2n} \frac{(-a/4)^j \left( B(j+1, q) - \frac{a}{4} B(j+1, q+1) \right)}{\Gamma(j+2) \Gamma(2n-j+1)} = 0. \quad (3.4)$$

The third and fourth sums in (3.4) can each be summed using [3, Eq. 24.4.12], which, in terms of the variables used here reads

$$\sum_{j=0}^{2n} \frac{(-a/4)^j B(j+1, q+1/2)}{\Gamma(j+2) \Gamma(2n-j+1)} = -\frac{B(2n+1, q-a/4) + (-a/4)^{2n+1}}{\Gamma(2n+1) (-a/4)^{2n}}. \quad (3.5)$$

The first two sums in (3.4) can be reduced by reference to [8, Eq. 24.4.23], that being

$$B(j+2, q+1/2) - B(j+2, q) = \frac{(j+2) \mathcal{E}(j+1, 2q)}{2j+2}. \quad (3.6)$$

In the case that $q = 1/4$

$$B(2n+1, 1/4) = -B(2n+1, 3/4) = -\frac{(2n+1) \mathcal{E}(2n)}{4^{2n+1}}. \quad (3.7)$$
which, applied to (3.5) along with the identity

$$\mathcal{E}(2n, 1/2) = 2^{-2n}\mathcal{E}(2n)$$

(3.8)

eventually yields (3.1). Q.E.D.

Note that (3.1) differs from the well-known result [8, Eq. (24.4.13)] which, in the present notation, reads

$$\sum_{j=0}^{2n} \frac{(-a/2)^{-j}}{\Gamma(j+1)\Gamma(2n-j+1)} = \frac{\mathcal{E}(2n)}{\Gamma(1+2n)a^{2n}}$$

(3.9)

Remark: (3.9) can be obtained from (1.9) by setting \( s = 0 \) and employing the above identities. See [9, Section 5] for similar results.

### 3.2 Corollary 3.2

$$\int_{0}^{\infty} \frac{v^{2+2n}}{(a^2 + 4v^2) \cosh(\pi v)} \, dv = (-1)^n 2^{2n-3} (a/4)^2n$$

$$\times \left( \Gamma(2+2n) \sum_{j=0}^{2n} \frac{(-2/a)^{j} \mathcal{E}(j, 2q)}{\Gamma(j+2)\Gamma(2n-j+1)} - \frac{a}{2} (\psi(q+1/2) - \psi(q)) \right), \quad n \geq 0.$$ (3.10)

Proof: In (1.9), set \( s = 1 \) noting the limit [8, Eq. 25.11.31]

$$\lim_{s \to 1} S(s, q) = \psi(q + 1/2) - \psi(q).$$

(3.11)

Apply the identity (3.3) and after some simplification, find

$$\int_{0}^{\infty} \frac{v^{2+2n}}{(a^2 + 4v^2) \cosh(\pi v)} \, dv = 2^{2n-4} (-1)^{n+1} (a/4)^2n$$

$$\times \left( 4\Gamma(2+2n) \sum_{j=0}^{2n} \frac{(B(j+1,q) - B(j+1,q+1/2)) (-4/a)^{j}}{(j+1)\Gamma(j+2)\Gamma(2n-j+1)} + \frac{a}{2} (\psi(q+1/2) - \psi(q)) \right).$$

(3.12)

The difference of terms involving Bernoulli polynomials can be rewritten with reference to (3.6) after which (3.12) is easily reduced to (3.10). Q.E.D

Special cases:

- \( a = 1 \)

$$\int_{0}^{\infty} \frac{v^{2+2n}}{(4v^2 + 1) \cosh(\pi v)} \, dv = (-1)^n 2^{-2} 2^{2n} \left( n + (1 - \ln(2))/2 + \Gamma(2+2n) \sum_{j=0}^{n-1} \frac{2^{2j} \mathcal{E}(2j+1,0)}{\Gamma(3+2j)\Gamma(2n-2j)} \right)$$

(3.13)

- \( a = 2 \)
\[
\int_0^\infty \frac{v^{2n+2}}{(v^2 + 1) \cosh(\pi v)} \, dv = (-1)^{n+1} \left( \Gamma \left( 2n + 2 \right) \sum_{j=1}^{n} \frac{\mathcal{E}(2j)}{\Gamma(2j+2)\Gamma(2n-2j+1)} - \frac{\pi}{2} + 2^{-2n} + n + 1/2 \right) ;
\]
\[
= (-1)^{n+1} \left( \Gamma \left( 2n + 2 \right) \sum_{j=1}^{n} \frac{\mathcal{E}(2j)}{\Gamma(2j+2)\Gamma(2n-2j+1)} - \frac{\pi}{2} + 2^{-2n} + n + 1/2 \right) ;
\]
\[
(3.14)
\]

- \(a = 4\)

\[
\int_0^\infty \frac{v^{2n+2}}{(v^2 + 4) \cosh(\pi v)} \, dv = (-1)^n \left( 2^{2n-1} \Gamma \left( 2n + 2 \right) \sum_{j=1}^{n} \frac{2^{-4j} \mathcal{E}(2j)}{\Gamma(2j+2)\Gamma(2n-2j+1)} - (\pi - n - 1/2) 2^{2n} + 2^{-2n} (3^{2n+1} - 1/3) \right) .
\]
\[
(3.15)
\]

Note that (3.12) extends results found in \[6, Section 3.522\]. Simplification of the above special cases employ the following recursions as well as \[8, Eq. 24.5.2\] and \[8, Eq. 24.4.13\] (equivalent to (3.9)):

\[
\mathcal{E}(j, 5/2) = 2 \left( \frac{3}{2} \right)^j - \mathcal{E}(j, 3/2) \quad (3.16)
\]
\[
\mathcal{E}(j, 3/2) = 2 \left( \frac{1}{2} \right)^j - \mathcal{E}(j) \quad (3.17)
\]

3.3 Corollary 3.3

\[
\int_0^\infty \frac{v^{2n+1} \arctan \left( \frac{2v}{a} \right)}{\cosh(\pi v)} \, dv = 2 \left( -1 \right)^n \left( \frac{a}{4} \right)^{2n} \left( \frac{\Gamma(q)}{\Gamma(q+1/2)} \right)
\]
\[
\times \left( \Gamma(2 + 2n) \sum_{j=0}^{2n} \frac{(-a/4)^{-j} S'(-j - 1, q)}{\Gamma(2n - j + 1)\Gamma(j+2)} - \left( \frac{\Gamma(q)}{\Gamma(q+1/2)} \right) \right) \quad (3.18)
\]

**Proof:**

In (1.9), operate with \(\frac{\partial}{\partial s}\) and set \(s = 0\), to obtain

\[
\int_0^\infty \frac{v^{2n+1} \arctan \left( \frac{2v}{a} \right)}{\cosh(\pi v)} \, dv = T_1(a, s = 0) + T_2(a, s = 0)
\]
\[
(3.19)
\]

where

\[
T_1(a, s) \equiv (-1)^{n+1} \left( \frac{a}{4} \right)^{2n} \ln(2) 2^{2n-2s} \left( 4\Gamma(2 + 2n) \sum_{j=0}^{2n} \frac{S(s - 1 - j) (-4/a)^j}{\Gamma(j+2)\Gamma(2n - j + 1)} + a S(s) \right)
\]
\[
(3.20)
\]

and

\[
T_2(a, 0) \equiv 2^{2n+1} \left( -1 \right)^n \left( \frac{a}{4} \right)^{2n} \frac{\partial}{\partial s} \left( \Gamma(2 + 2n) \sum_{j=0}^{2n} \frac{(-4/a)^j S(s - 1 - j)}{\Gamma(j+2)\Gamma(2n - j + 1)} - 2 S(s) \right) \bigg|_{s=0} .
\]
\[
(3.21)
\]

From \[8, Eq. 25.11.18\]

\[
\zeta'(0, q) = \ln(\Gamma(q)) - \frac{1}{2} \ln(2\pi)
\]
\[
(3.22)
\]
and, from (3.1), it is easily shown that $T_1(a, 0) = 0$. Thus, from (3.19), (3.21) and (3.22) after some simplification the result is (3.18). QED.

The following special cases can profitably be compared with examples given in Section 5 of [2].

**Special case** $n = 0$.

**Note:** None of the following cases appear to be known to Mathematica [10].

\[
\int_0^\infty \frac{v \arctan \left( \frac{2v}{\cosh(\pi v)} \right)}{\cosh(\pi v)} \, dv = \frac{a}{2} \left( - \ln(\Gamma(q)) + \ln(\Gamma(q + 1/2)) \right) + 2(\zeta'(-1, q) - \zeta'(-1, q + 1/2))
\]

(3.23)

- If $a = 1$ then (3.23) reduces to

\[
\int_0^\infty \frac{v \arctan \left( \frac{2v}{\cosh(\pi v)} \right)}{\cosh(\pi v)} \, dv = -3\zeta'(-1) + \ln(2) / 6 - \ln(2\pi) / 4
\]

(3.24)

and by writing $\zeta'(-1)$ in the form of its underlying sum, we identify [11]

\[
\zeta'(-1) = -\gamma / 12 - \ln(2\pi) / 12 + 1 / 12 + \zeta'(2) / 2\pi^2.
\]

(3.25)

Furthermore, from [10]

\[
\zeta'(2) = \pi^2(\gamma + \ln(2\pi) - 12 \ln(A)) / 6,
\]

all of which identifies

\[
\int_0^\infty \frac{v \arctan \left( \frac{2v}{\cosh(\pi v)} \right)}{\cosh(\pi v)} \, dv = - \ln(\pi) / 4 - 1/4 + 3 \ln(A) - \ln(2) / 12.
\]

(3.27)

- If $a = 2$ then (3.23) reduces to

\[
\int_0^\infty \frac{v \arctan \left( \frac{v}{\cosh(\pi v)} \right)}{\cosh(\pi v)} \, dv = -2 \ln(\Gamma(3/4)) - \ln(2) / 2 + \ln(\pi) + 2\zeta'(-1, 3/4) - 2\zeta'(-1, 1/4).
\]

(3.28)

From Miller and Adamchik [12] Eq. (5) (also [8] Eq. 25.11.21)]

\[
\zeta'(-1, 3/4) - \zeta'(-1, 1/4) = -\frac{G}{4\pi} + \frac{\psi'(3/4)}{32\pi} - \pi / 32
\]

(3.29)

and again from those same sources

\[
\psi'(3/4) = \zeta(2, 3/4) = \pi^2 - 8 G
\]

(3.30)

all of which leads to

\[
\int_0^\infty \frac{v \arctan \left( \frac{v}{\cosh(\pi v)} \right)}{\cosh(\pi v)} \, dv = - \frac{1}{2} \ln(2) - 2 \ln(\Gamma(3/4)) + \ln(\pi) - G / \pi.
\]

(3.31)

- If $a = 4$ then (3.23) reduces to

\[
\int_0^\infty \frac{v \arctan \left( \frac{v/2}{\cosh(\pi v)} \right)}{\cosh(\pi v)} \, dv = 2 \ln \left( \frac{\Gamma(7/4)}{\Gamma(5/4)} \right) + 2\zeta'(-1, 1/4) - \ln(2) - 2\zeta'(-1, 3/4) - 3/2 \ln(3/4)
\]

(3.32)
where the recursion
\[ \zeta'(-1, q + 1/2) = \zeta'(-1, q - 1/2) + (q - 1/2) \ln (q - 1/2) \]  
(3.33)
has been used to convert the original reduction of (3.23) into a range where [12, Eq. (5)] is valid when \( q > 1 \). Following the use of (3.29) and (3.30) we eventually find
\[ \int_0^\infty \frac{v \arctan (v/2)}{\cosh (\pi v)} \, dv = \ln (2) + 4 \ln (\Gamma (3/4)) + \frac{1}{2} \ln (3) - 2 \ln (\pi) + G/\pi . \]  
(3.34)

• The above results can be associated with well-known arctangent identities to obtain other interesting identities. From [13] we find two identities valid for all values of their arguments, those being
\[ \arctan (v/2) = \arctan (v) - \arctan \left( \frac{v}{v^2 + 2} \right) \]  
(3.35)
\[ \arctan (v) = 2 \arctan (v/2) - \arctan \left( \frac{v^3}{3 v^2 + 4} \right) \]  
(3.36)

Substitution into (3.31) and (3.34) yields
\[ \int_0^\infty \frac{v \arctan \left( \frac{v}{v^2 + 2} \right)}{\cosh (\pi v)} \, dv = 3 \ln (\pi) - 6 \ln (\Gamma (3/4)) - 3/2 \ln (2) - \ln (3) /2 - \frac{2 G}{\pi} \]  
(3.37)
and
\[ \int_0^\infty \frac{v^3 \arctan \left( \frac{v^3}{3 v^2 + 4} \right)}{\cosh (\pi v)} \, dv = 10 \ln (\Gamma (3/4)) + 5 \ln (2) /2 - 5 \ln (\pi) + \frac{3 G}{\pi} + \ln (3) . \]  
(3.38)

**Special case** \( n = 1 \)
\[ \int_0^\infty \frac{v^3 \arctan \left( \frac{2 v}{\pi} \right)}{\cosh (\pi v)} \, dv = \frac{a^3}{8}  \ln \left( \frac{\Gamma (q)}{\Gamma (q + 1/2)} \right) - \frac{3a^2}{2} S' (-1, q) + 6 a S' (-2, q) - 8 S' (-3, q) . \]  
(3.39)

• If \( a = 1 \)..., we find
\[ \int_0^\infty \frac{v^3 \arctan \left( \frac{2 v}{\pi} \right)}{\cosh (\pi v)} \, dv = 3/2 (\zeta' (-1) - \zeta' (-1, 1/2)) + 6 \zeta' (-2, 1/2) - 6 \zeta' (-2) - 8 \zeta' (-3, 1/2) + 8 \zeta' (-3) + \ln (\pi) /16 . \]  
(3.40)

From Mathematica [10]
\[ \zeta'(-1, 1/2) = - \ln (2) /24 - 1/24 + \ln (A) /2 \]  
(3.41)
\[ \zeta'(-2, 1/2) = \frac{3 \zeta (3)}{16 \pi^2} \]  
(3.42)
\[ \zeta'(-3, 1/2) = \frac{\ln (2)}{960} - \frac{7 \zeta'(-3)}{8} \]  
(3.43)

and from [14],
\[ \zeta'(-2) = -\zeta(3)/(4\pi^2) . \]  
(3.44)
Together with other identifications noted previously, after simple calculation, we obtain

$$\int_{0}^{\infty} \frac{v^3 \arctan(2v)}{\cosh(\pi v)} \, dv = \frac{13 \ln(2)}{240} - 9 \ln(A)/4 + \ln(\pi)/16 + 15 \zeta'(-3) + 3/16 + \frac{21 \zeta(3)}{8 \pi^2} \quad (3.45)$$

From [15, Section 5] we have

$$\zeta'(-3) = -\frac{11}{720} - \ln(A_3), \quad (3.46)$$

where $A_3$ is a generalized Glaisher constant.

- If $a = 2$, we find

$$\int_{0}^{\infty} \frac{v^3 \arctan(v)}{\cosh(\pi v)} \, dv = 6 \{\zeta'(-1, 1/4) - \zeta'(-1, 3/4)\} + 8 \{\zeta'(-1, 1/4) - \zeta'(-3, 3/4)\}$$

$$- 12 \{\zeta'(-2, 1/4) - \zeta'(-2, 3/4)\} - 7/4 \ln(2) + \ln \left(\frac{2\sqrt{2} (\Gamma(3/4))^2}{\pi}\right). \quad (3.47)$$

In addition to the identities utilized previously, we also have, from [12, Eq. (5)], [12, Eq. (6)] and [12, Eq. (12)] respectively,

$$\zeta'(-3, 1/4) - \zeta'(-3, 3/4) = \frac{\pi}{128} - \frac{\psi(3)(1/4)}{1024 \pi^3}, \quad (3.48)$$

$$\zeta'(-2, 1/4) - \zeta'(-2, 3/4) = - \ln(\pi)/32 - \frac{3 \ln(2)}{32} + \frac{3}{64} - \gamma/32 + \frac{\zeta(3, 1/4) - \zeta(3, 3/4)}{64 \pi^3}, \quad (3.49)$$

and

$$\zeta'(3, 3/4) = 120 \ln(2) \zeta(3) + 56 \zeta'(3) - \zeta'(3, 1/4). \quad (3.50)$$

In the above, $\psi(3)(1/4)$, the polygamma function, can also be written

$$\psi(3)(1/4) = 8 \pi^4 + 768 \beta(4) \quad (3.51)$$

where $\beta(4)$ is Dirichlet’s Beta function (see [16]). These identities eventually yield

$$\int_{0}^{\infty} \frac{v^3 \arctan(v)}{\cosh(\pi v)} \, dv = 2 \ln(\Gamma(3/4)) + 7 \ln(2)/8 - 5 \ln(\pi)/8 + 3\gamma/8 - 9/16 + 3G/\pi$$

$$+ \frac{1}{\pi^2} (45 \ln(2) \zeta(3)/2 + 21 \zeta'(3)/2 - 3 \zeta'(3, 1/4)/2 - 6 \beta(4)), \quad (3.52)$$

which is likely the simplest form that exists in terms of independent fundamental constants (see [17, 18] and [12, Eq. (21)]). Results equivalent to (3.37) and (3.38) can now be obtained using (8.35) and (8.36).

3.4 Corollary 3.4

$$a \int_{0}^{\infty} \frac{v^{2n+1} \arctan(2v)}{(a^2 + 4v^2) \cosh(\pi v)} \, dv - \frac{a \zeta'_a(1, 2q)}{4} + \frac{\Gamma(2n + 2)}{2} \left(\sum_{j=0}^{2n} \frac{S'(-j, q)}{\Gamma(j + 1) \Gamma(2n - j + 1)} - \ln(2) \sum_{j=0}^{2n} \frac{\zeta(j, 2q)(-2/a)^j}{\Gamma(j + 2) \Gamma(2n - j + 1)}\right) \quad (3.53)$$
where
\[ ζ_{a}'(s, 2q) \equiv -\sum_{j=0}^{∞} \frac{(-1)^j \ln (j + 2q)}{(j + 2q)^s} \] (3.54)
is the first derivative of the alternating Hurwitz zeta function with respect to s (see (1.11)).

**Proof:**

Differentiate (1.9) with respect to s, reverse the resulting sums and evaluate the limit \( s \to 1 \). The two summation terms in (3.53) follow immediately with the help of Corollary 1 and the arguments used in its derivation. It remains to show that

Lemma

\[ \lim_{s \to 1} (2 \ln (2) S_s (q) - S'_s (q)) = 2 \sum_{j=0}^{∞} \frac{(-1)^j \ln (j + 2q)}{(j + 2q)^s} + \ln (2) (\psi (q + 1/2) - \psi (q)) \] (3.55)

By definition, the Stieltjes constants \( γ_n (q) \) are the coefficients of the Laurent expansion of \( ζ(s, q) \) about \( s = 1 \), that is

\[ ζ(s, q) = (s - 1)^{-1} + \sum_{n=0}^{∞} \frac{(-1)^n γ_n (a) (s - 1)^n}{\Gamma (n + 1)}. \] (3.56)

After differentiating once with respect to s, we have

\[ ζ'(s, q) = -(s - 1)^{-2} + \sum_{n=0}^{∞} \frac{(-1)^{n+1} γ_{n+1} (q) (s - 1)^n}{\Gamma (n + 1)} \] (3.57)

and we are interested in

\[ \lim_{s \to 1} (ζ'(s, q) - ζ'(s, q + 1/2)) = γ_1(q + 1/2) - γ_1(q). \] (3.58)

Differentiate (1.10) with respect to s, giving

\[ ζ'(s, q) - ζ'(s, q + 1/2) = 2^s \ln (2) \sum_{j=0}^{∞} \frac{(-1)^j \ln (j + 2q)}{(j + 2q)^s} - 2^s \sum_{j=0}^{∞} \frac{(-1)^j \ln (j + 2q)}{(j + 2q)^s}. \] (3.59)

Since the first sum in (3.59) is simply identified as

\[ \sum_{j=0}^{∞} \frac{(-1)^j}{(j + 2q)^s} = 2^{-s} S_s (q) \] (3.60)

the requisite Lemma is proven by taking the limit \( s \to 1 \) and taking note of (3.11). QED

- **Special case n=0.**

\[
a \int_{0}^{∞} \frac{v \arctan \left( \frac{2v}{a^2 + 4 v^2} \right)}{(a^2 + 4 v^2) \cosh (π v)} \, dv - \int_{0}^{∞} \frac{v^2 \ln (a^2 + 4 v^2)}{(a^2 + 4 v^2) \cosh (π v)} \, dv
\]
\[
= a \ln (2) (\psi (q + 1/2) - \psi (q)) /8 + a/2 \sum_{j=0}^{∞} \frac{(-1)^j \ln (j + 2q)}{j + 2q}
\]
\[
+ (\ln (\Gamma (q)) - \ln (\Gamma (q + 1/2)) - \ln (2)) /2
\] (3.61)
• Special case \( n=0, a=1 \)

The identity \[ \sum_{j=0}^{\infty} \frac{(-1)^j \ln(j+1)}{j+1} = \ln(2) (\ln(2) - 2\gamma) / 2 \] \hspace{1cm} (3.62)

leads to

\[ \int_0^\infty \frac{v \arctan(2v)}{(4v^2+1) \cosh(\pi v)} \, dv - \int_0^\infty \frac{v^2 \ln(4v^2+1)}{(4v^2+1) \cosh(\pi v)} \, dv = \frac{1}{4} \left( \frac{3}{2} \ln^2(2) - \gamma \ln(2) + \ln(\pi/4) \right); \hspace{1cm} (3.63) \]

• Special case \( n=0, a=2 \)

With recourse to (3.14) with \( n = 0 \), we find

\[ 2 \int_0^\infty \frac{v \arctan(v)}{(v^2+1) \cosh(\pi v)} \, dv - \int_0^\infty \frac{v^2 \ln(v^2+1)}{(v^2+1) \cosh(\pi v)} \, dv = \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-1)^j \ln(2j+3)}{2j+3} + \pi \ln(2) + 4 \ln(\Gamma(3/4)) - 2 \ln(2\pi); \hspace{1cm} (3.64) \]

• Special case \( n=0, a=4 \)

With recourse to (3.15) with \( n = 0 \), we find

\[ 4 \int_0^\infty \frac{v \arctan(v/2)}{(v^2+4) \cosh(\pi v)} \, dv - \int_0^\infty \frac{v^2 \ln(v^2+4)}{(v^2+4) \cosh(\pi v)} \, dv = \frac{8}{2m+5} \sum_{m=0}^{\infty} \frac{(-1)^m \ln(2m+5)}{2m+5} - 2\pi \ln(2) - 4 \ln(\Gamma(3/4)) + 16 \ln(2)/3 + 2 \ln(\pi/3). \hspace{1cm} (3.65) \]

(\textbf{Remark}: Further evaluations are accessible by differentiating any of the above with respect to the parameter \( a \).)

4 \hspace{1cm} \textbf{Theorem 2 and Proof}

\textbf{Theorem 2}

\[ \int_0^\infty \frac{v^{2n} \cos(s \arctan(\frac{2v}{\pi}))}{(a^2 + 4v^2)^{3/2} \cosh(\pi v)} \, dv = 2^{2(n-s)} (-1)^n \Gamma(2n+1) \sum_{j=0}^{2n} \frac{(-a/4)^j S(j+s-2n,q)}{\Gamma(2n-j+1) \Gamma(j+1)} \hspace{1cm} (4.1) \]

\textbf{Proof:}

From [6, Eq. 4.111(7)], after invocation of minor arctangent identities, we have the Fourier sine transform

\[ \int_0^\infty \frac{t^{-1} \sin(wt)}{\cosh(\pi t/2)} \, dt = 2 \arctan(\tanh(w/2)). \hspace{1cm} (4.2) \]

In complete analogy to (2.2) this leads to

\[ \int_0^\infty \frac{t^{2n-1} \sin(s \arctan(\frac{v}{\pi}))}{(a^2 + t^2)^{3/2} \cosh(\pi t/2)} \, dt = 2 (-1)^n \Gamma(2n+1) \sum_{j=0}^{2n} \frac{(-a)^j J(j+s-2n-1)}{\Gamma(2n-j+1) \Gamma(j+s-2n) \Gamma(j+1)}, \hspace{1cm} (4.3) \]
where
\[ J(s) \equiv \int_{0}^{\infty} w^{s} e^{-aw} \arctan (\tanh (w/2)) \, dw \tag{4.4} \]
obes the recursion
\[ J(s) = \frac{sJ(s-1)}{a} + \frac{2^{-1-s} \Gamma(1+s) S(1+s,q)}{2a} \tag{4.5} \]
derived by integration by parts and the use of \(2.3\). Substitution of \(2.3\) into \(4.3\) with
\[ s \to j + s - 2n - 1 \]
leads to
\[ \int_{0}^{\infty} t^{2n-1} \sin \left( s \arctan \left( \frac{t}{a} \right) \right) \frac{dt}{(a^2 + t^2)^{1/2+s/2} \cosh (\pi t/2)} = \frac{2}{a} \frac{(-1)^n \Gamma(2n+1)}{\Gamma(2n-j+1) \Gamma(j+1)} \sum_{j=0}^{2n} \frac{(-a/4)^j S(j+s-2n,q)}{\Gamma(j+1)} \tag{4.6} \]
Now let \( s := s - 1 \) in \(4.3\) to find
\[ \int_{0}^{\infty} t^{2n-1} \sin \left( (s-1) \arctan \left( \frac{t}{a} \right) \right) \frac{dt}{(a^2 + t^2)^{1/2+s/2} \cosh (\pi t/2)} = 2 \frac{(-1)^n \Gamma(2n+1)}{\Gamma(2n-j+1) \Gamma(j+1)} \sum_{j=0}^{2n} \frac{(-a/4)^j S(j+s-2n,q)}{\Gamma(j+1)} \tag{4.7} \]
Notice that the sum on the right-hand side of \(4.7\) coincides with the first sum on the right-hand side of \(4.6\), so substitute, expand the term \( \sin ((s-1) \arctan (t/a)) \) using a simple trigonometric identity and simplify. The result is \(4.1\). QED

The above generalizes the classic result (see \([19\), Eq. (26)] and references therein) corresponding to \( n = 0, a = 1 \) and Glasser’s more recent generalization \([20\), corresponding to the case \( n = 0 \). Additionally, various specific cases extend results found in \([6\), Section 3.522\].

**Special Cases**

- **Case:** \( s = 0 \). When reduced by use of \(3.3\) and \(3.5\), this case simply reproduces a known result \([6\), Eq. 3.523(4)]\.

- **Case:** \( s = 1 \).

  After an obvious change of variables using \( n = 0 \) and \( a = 1/2 \) we find
  \[ \int_{0}^{\infty} \frac{1}{(t^2 + 1) \cosh (\pi t/4)} \, dt = (\psi(7/8) - \psi(3/8))/2 \tag{4.8} \]
Comparison with \([6\), Eq. 3.522(10)] yields the identity
\[ \psi(7/8) - \psi(3/8) = \sqrt{2} \left( \pi - 2 \ln \left( \sqrt{2} + 1 \right) \right), \tag{4.9} \]
which could alternatively have been obtained from Gauss’ classic expression (see \([21\), Eq. (1.3)]\).

**4.1 Corollary 4.1**

\[ \int_{0}^{\infty} t^{2n} \ln \left( a^2 + 4v^2 \right) \frac{dv}{\cosh (\pi v)} = (-1)^n \left( 2^{1-2n} \zeta(2n) \ln (2) - 2^{2n+1} S'(-2n,q) \right. \]
\[ - 2 (a/2)^{2n} \Gamma(2n+1) \sum_{j=0}^{2n-1} \frac{(-4/a)^j S'(-j,q)}{\Gamma(2n-j+1) \Gamma(j+1)} \tag{4.10} \]
\[ - 2 (a/2)^{2n} \Gamma(2n+1) \sum_{j=0}^{2n-1} \frac{(-4/a)^j S'(-j,q)}{\Gamma(2n-j+1) \Gamma(j+1)} \tag{4.11} \]

**Proof:** Operate on \(4.1\) with \( \frac{\partial}{\partial s} \) and let \( s = 0 \). The proof follows exactly the same steps as that of Section 3.4. QED
Case \( n = 0 \): (also see [6, Eqs. 4.373(1) and (2)])

\[
\int_0^\infty \ln \left( \frac{a^2 + 4v^2}{\cosh(\pi v)} \right) dv = 2\ln(2) - 2\ln(\Gamma(q)) + 2\ln(\Gamma(q + 1/2))
\]

(4.12)

Case: \( n = 1 \)

\[
\int_0^\infty \frac{v^2 \ln \left( \frac{a^2 + 4v^2}{\cosh(\pi v)} \right)}{\cosh(\pi v)} dv = \frac{a^2}{2} \left( \ln(\Gamma(q)) - \ln(\Gamma(q + 1/2)) \right) - 4aS'(-1,q) + 8S'(-2,q) + \frac{\ln(2)}{2}
\]

(4.13)

Case \( n = 1 , a = 1 \)

From (3.25), (3.26), (3.41), (3.42) and (3.44)

\[
\int_0^\infty \frac{v^2 \ln \left( \frac{a^2 + 4v^2}{\cosh(\pi v)} \right)}{\cosh(\pi v)} dv = \ln(\pi)/4 + \frac{2\ln(2)}{3} + \frac{1}{2} - 6\ln(A) + \frac{7\zeta(3)}{2\pi^2}
\]

(4.14)

Case \( n = 2 , a = 1 \)

\[
\int_0^\infty \frac{v^4 \ln \left( \frac{a^2 + 4v^2}{\cosh(\pi v)} \right)}{\cosh(\pi v)} dv = \frac{23}{40} \ln(2) + \frac{93\zeta(5)}{2\pi^4} - 60\zeta'(-3) - \frac{21\zeta(3)}{4\pi^2} - \frac{1}{4} + 3\ln(A) - \ln(\pi)/16
\]

(4.15)

4.2 Corollary 4.2

\[
\int_0^\infty \frac{v^{2n+1} \arctan \left( \frac{2v}{a^2 + 4v^2} \right)}{\cosh(\pi v)} dv + \frac{a}{4} \int_0^\infty \frac{v^{2n} \ln \left( \frac{a^2 + 4v^2}{\cosh(\pi v)} \right)}{\cosh(\pi v)} dv = 2^{2n-3} (-1)^n (a/4)^{2n}
\]

\[
\left( -\frac{4\Gamma(2n+1)}{a} \right) \left( \ln(2) - \sum_{j=0}^{2n-1} \frac{\mathcal{E}(j,2q)(-2/a)^j}{\Gamma(j+2)\Gamma(2n-j)} - \sum_{j=0}^{2n-1} \frac{(-4/a)^j S'(-j,q)}{\Gamma(j+2)\Gamma(2n-j)} \right)
\]

\[
+ 2\sum_{j=0}^{\infty} \frac{(-1)^j \ln(j+2q)}{j+2q} + \ln(2) \left( \psi(q+1/2) - \psi(q) \right)
\]

(4.16)

Proof: Operate on (1.1) with \( \frac{d}{ds} \) and let \( s \to 1 \). The proof follows exactly the same steps as that of Section 3.4.

QED

5 A Comment on the Even moments of Theorem 1

As noted previously, in his paper, Patkowski purports to resolve the case where the identity (1.9) contains even rather than odd moments. As discussed in the Appendix, the derivation of that result is flawed; additionally the putative result does not satisfy numerical testing. To utilize the method developed by Patkowski, the Fourier sine transform of \( t^{2n}/\cosh(at) \) for any \( n \geq 0 \), must be employed, and in fact, such an identity is known for \( n = 0 \). From [6, Eq. 3.981(2)] we find

\[
\int_0^\infty \frac{\sin(wx)}{\cosh(\pi x/2)} dx = \frac{i}{\pi} \psi \left( \frac{1}{4} + \frac{iw}{2\pi} \right) - \frac{i}{\pi} \psi \left( \frac{1}{4} - \frac{iw}{2\pi} \right) - \tanh(w),
\]

which may also be written as

\[
\int_0^\infty \frac{\sin(wx)}{\cosh(\pi x/2)} dx = -\tanh(w) + \frac{2}{\pi} 3 \left( \psi \left( \frac{1}{4} + \frac{iw}{2\pi} \right) \right).
\]

(5.1)
From Parseval’s identity, we then have

$$
\int_{0}^{\infty} \frac{t^{2n} \sin \left(s \arctan \left(\frac{t}{n}\right)\right)}{(a^{2} + t^{2})^{s/2} \cosh (\pi t/2)} \, dt = -\frac{(-1)^{n} \Gamma (2 n + 1)}{\pi \Gamma (s)} \left(\pi J_{1}(n, s, a) - 2 J_{2}(n, s, a)\right) \quad (5.3)
$$

where

$$
J_{1}(n, s, a) \equiv \int_{0}^{\infty} L (2 n, s - 2 n - 1, aw) w^{s-2n-1} \tanh (w) e^{-aw} \, dw \quad (5.4)
$$

and

$$
J_{2}(n, s, a) \equiv \int_{0}^{\infty} L (2 n, s - 2 n - 1, aw) w^{s-2n-1} \Im \left(\psi \left(\frac{1}{4} + \frac{iw}{2\pi}\right)\right) e^{-aw} \, dw . \quad (5.5)
$$

From (1.3) these become

$$
J_{1}(n, s, a) = \Gamma (s) \sum_{j=0}^{2n} \frac{(-a)^{j} \int_{0}^{\infty} w^{j+s-2n-1} \tanh (w) e^{-aw} \, dw}{\Gamma (2 n - j + 1) \Gamma (j + s - 2 n) \Gamma (j + 1)} \quad (5.6)
$$

and

$$
J_{2}(n, s, a) = \Gamma (s) \sum_{j=0}^{2n} \frac{(-a)^{j} \int_{0}^{\infty} w^{j+s-2n-1} \Im \left(\psi \left(\frac{1}{4} + \frac{iw}{2\pi}\right)\right) e^{-aw} \, dw}{\Gamma (2 n - j + 1) \Gamma (j + s - 2 n) \Gamma (j + 1)} . \quad (5.7)
$$

With $\Re(s) > -2$, the first of these leads to the related integral

$$
\int_{0}^{\infty} w^{s} \tanh (w) e^{-aw} \, dw = 2^{-2-2s} \Gamma (1 + s) \times (\zeta (1 + s, 1 + a/4) + \zeta (1 + s, a/4) - 2 \zeta (1 + s, 1/2 + a/4)) , \quad (5.8)
$$

the equality arising courtesy of Mathematica \textit{[10]}, verified numerically. The evaluation of the integral in (5.7) is more challenging. With $\Re(s) > -2$, we require an evaluation of the generic form

$$
J_{2}(s, a) \equiv \int_{0}^{\infty} w^{s} \Im \left(\psi \left(\frac{1}{4} + \frac{iw}{2\pi}\right)\right) e^{-aw} \, dw . \quad (5.9)
$$

One possibility is to identify

$$
\Im \left(\psi \left(\frac{1}{4} + \frac{iw}{2\pi}\right)\right) = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{w}{(k - 3/4)^2 + w^2/4\pi^2} . \quad (5.10)
$$

Substituted into (5.9), this leads to known identities (see \textit{[6]} Eqs. 3.356(1) and 3.356(2)), but only when $s$ is a positive integer. The case of arbitrary $s$ does not appear to be tractable. Alternatively, and following integration by parts, (5.9) leads to an interest in the real-valued integral

$$
\tilde{J}(s, a) = \int_{0}^{\infty} w^{s} e^{-aw} \left(\log \left(\Gamma \left(\frac{1}{4} + \frac{iw}{2\pi}\right)\right) + \log \left(\Gamma \left(\frac{1}{4} - \frac{iw}{2\pi}\right)\right)\right) \, dw \quad (5.11)
$$

$$
= 2 \int_{0}^{\infty} w^{s} e^{-aw} \log \left|\Gamma \left(\frac{1}{4} + \frac{iw}{2\pi}\right)\right| \, dw \quad (5.12)
$$

because

$$
J_{2}(s, a) = \pi s \tilde{J}(s - 1, a) - a \tilde{J}(s, a) . \quad (5.13)
$$

And this integral appears also to be intractable. Therefore, further analysis of the case under consideration in this Section must be left for future investigation.
6 Connection with \( \zeta(2m + 1) \)

From (1.7) and (1.8), we are interested in

\[
A(m) = \frac{2^{3+2m}m^2}{\Gamma(3+2m)} \int_0^\infty \frac{t^{2m+2}3F_2(1,1,3/2;2+m,3/2+m;-4zt)}{\cosh(\pi t)} \, dt. \tag{6.1}
\]

From [22, Eq. (3)] we obtain

\[
3F_2(1,1,3/2;2+m,m+3/2;z) = \frac{2\Gamma(2+m)\Gamma(m+3/2)}{\sqrt{\pi}} \times \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{(-1)^k(-1)^j}{\Gamma(1-k+m)\Gamma(-j+m)\Gamma(k+1)\Gamma(j+1)} \times \left( \frac{2F_1(1,3/2+j;5/2+j;z)}{(3/2+j)(-1/2+k-j)} + \frac{2F_1(1,k+1;2+k;z)}{(k+1)(1/2+j-k)} \right). \tag{6.2}
\]

Further, from [23, Eq. 7.3.1(134) and (135)] with \( t \geq 0 \) we respectively have

\[
2F_1(1,3/2+j;5/2+j;-4zt^2) = \frac{(-1)^{j+1}(2j+3)}{(4zt^2)^{j+2}} \left( 2t \arctan(2t) + \sum_{p=1}^{j+1} \frac{(-4zt^2)^p}{2p-1} \right), \tag{6.3}
\]

and

\[
2F_1(1,k+1;2+k;-4zt^2) = -\frac{k+1}{(-4zt^2)^{k+1}} \left( \ln(4zt^2+1) + \sum_{p=1}^{k} \frac{(-4zt^2)^p}{p} \right). \tag{6.4}
\]

For the case \( m = 1 \), (1.7) and the above lead to the representation

\[
\frac{\zeta(3)}{\pi^2} = 2 \int_0^\infty \frac{t^2 \ln(4t^2+1)}{\cosh(\pi t)} \, dt - \frac{1}{14} \int_0^\infty \frac{\ln(4t^2+1)}{\cosh(\pi t)} \, dt + \frac{4}{7} \int_0^\infty \frac{t \arctan(2t)}{\cosh(\pi t)} \, dt - \frac{6}{7} \int_0^\infty \frac{t^2}{\cosh(\pi t)} \, dt + \frac{3}{28}. \tag{6.5}
\]

From [1.44, 4.12, 6.27] and [6, Eq. 3.523(5)] in that order, (1.7) is verified. Similarly, for the case \( m = 2 \) we have

\[
\frac{\zeta(5)}{\pi^4} = 2 \int_0^\infty \frac{\ln(4t^2+1)t^4}{\cosh(\pi t)} \, dt - \frac{1}{31} \int_0^\infty \frac{\ln(4t^2+1)t^2}{\cosh(\pi t)} \, dt + \frac{1}{744} \int_0^\infty \frac{\ln(4t^2+1)}{\cosh(\pi t)} \, dt + \frac{8}{93} \int_0^\infty \frac{\arctan(2t)t^3}{\cosh(\pi t)} \, dt - \frac{2}{93} \int_0^\infty \frac{t \arctan(2t)}{\cosh(\pi t)} \, dt - \frac{25}{279} \int_0^\infty \frac{t^4}{\cosh(\pi t)} \, dt + \frac{7}{186} \int_0^\infty \frac{t^2}{\cosh(\pi t)} \, dt + \frac{83}{8928}. \tag{6.6}
\]

All of the above integrals are given in the various examples presented in prior Sections, save for the second last, which is given in [6, Eq. 3.523(7)]. When the various substitutions are made, (1.7) is again verified, given that

\[
\zeta'(-4) = \frac{3\zeta(5)}{4\pi^4}, \tag{6.7}
\]

and

\[
\zeta'(-4,1/2) = -\frac{45\zeta(5)}{64\pi^4}. \tag{6.8}
\]
7 Summary

A number of useful results have been obtained from a study of (1.9) and its special cases, utilizing Laguerre polynomials and Parseval’s theorem. The interesting case discussed in Section 5 remains unresolved.

8 Acknowledgements

The author thanks Larry Glasser for his helpful insights and independent derivation(s) of some of the special cases obtained here.

References

[1] M. Jensen. L’Intermédiaire des Mathématiciens, 1895. page 346.

[2] George Boros, Olivier Espinosa, and Victor H. Moll. On some families of integrals solvable in terms of polygamma and negapolygamma functions. Integral Transforms and Special Functions, 14:187–203, 2003. Also available from arXiv Mathematics e-prints math/0305131.

[3] V.S. Adamchik. Contributions to the theory of the Barnes function. 2003. https://arxiv.org/abs/math/0308086.

[4] Alexander E Patkowski. On Some Families of Integrals Connected to the Hurwitz Zeta Function. 2015. available from arXiv Mathematics e-prints math arXiv:1502.02743.

[5] E.W. Weisstein. Associated Laguerre polynomial. Mathworld – a Wolfram Web Resource., 2016. Retrieved from http://mathworld.wolfram.com/AssociatedLaguerrePolynomial.html.

[6] I.S. Gradshteyn and I.M. Ryzhik. Tables of Integrals, Series and Products, corrected and enlarged Edition. Academic Press, 1980.

[7] H.M. Srivastava and Junesang Choi. Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier, 32 Jamestown Rd.,London, NW1 7BY, first edition, 2012.

[8] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. NIST Handbook of Mathematical Functions. Cambridge University Press, New York, NY, 2010. Print companion to [24].

[9] Navas, Luis, Ruiz, Fernando, and Verona, Juan. Old and new identities for Bernoulli polynomials via Fourier series. International Journal of Mathematics and Mathematical Sciences, 2012:129126–129139, 2012. doi=https://doi.org/10.1155/2012/129126.

[10] Wolfram Research, Champagne, Illinois. Mathematica, version 12, 2019.

[11] Maplesoft, a division of Waterloo Maple Inc. Maple.

[12] Jeff Miller and Victor S. Adamchik. Derivatives of the Hurwitz zeta function for rational arguments. Journal of Computational and Applied Mathematics, 100(2):201 – 206, 1998.

[13] Weisstein E. https://mathworld.wolfram.com/InverseTangent.html.

[14] https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function.

[15] Victor S. Adamchik. Polygamma functions of negative order. Journal of Computational and Applied Mathematics, 100(2):191 – 199, 1998.

[16] Kölbig K.S. The polygamma function ψ(k)(x) for x = 1/4 and x = 3/4. Journal of Computational and Applied Mathematics, 75:43–46, 1996.
[17] Murty M. Ram and N. Saradha. Transcendental values of the digamma function. *Journal of Number Theory*, 125(2):298 – 318, 2007.

[18] Waldschmidt Michel. Recent Diophantine results on zeta values: a survey. 2009. http://www.math.jussieu.fr/ miw/articles/pdf/ZetaValuesRIMS2009.pdf.

[19] M.S. Milgram. Integral and series representations of Riemann’s Zeta function, Dirichlet’s Eta function and a medley of related results. *Journal of Mathematics, Article ID 181724*, 2013. http://dx.doi.org/10.1155/2013/181724.

[20] Glasser M.L. The evaluation of lattice sums. I. analytic procedures. *J. Math. Phys.*, 14(3), 1972. also https://www.researchgate.net/publication/234847928

[21] Choi Junesang and Cvijović Djurdje. Values of the polygamma functions at rational arguments. *J. Phys. A: Math. Theor.*, 40:15019–15028, 2007.

[22] M. A. Shpot and H. M. Srivastava. The Clausenian hypergeometric function \(3F_2\) with unit argument and negative integral parameter differences. *Applied Mathematics and Computation*, 259:819–827, May 2015. also available from ArXiv e-prints,1411.2455v3, April 15, 2015, doi/10.1016/j.amc.2015.03.031.

[23] A.P. Prudnikov, Yu. A. Brychkov, and O.I. Marichev. *Integrals and Series: More Special functions*, volume 3. Gordon and Breach Science Publishers, New York, 1986.

[24] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.9 of 2014-08-29.

## Appendices

### A Patkowski’s paper [4] Errata

The following are corrections to [4]: all numbered references to [4] in this Appendix are enclosed in square brackets (viz. “[..]”).

Eq. [3.1] is missing a factor of two and should read

\[
\int_0^\infty \frac{t^{s-1} e^{-at}}{\sinh(t)} \, dt = 2 \Gamma(s) \left( \zeta(s, a) - 2^{-s} \zeta(s, a/2) \right)
\] (A.1)

Eq. [3.3] is missing a factor of two and should read as given in (2.3).

Eq. [3.8] [i.e. proof of Theorem 2] has missing and extraneous factors and should read

\[
\int_0^\infty \frac{t^{2n} \sin(s \arctan(t/\alpha))}{(a^2 + t^2)^{s/2} \sinh(\pi t)} \, dt = \frac{(-1)^n \Gamma(2n + 1)}{2 \Gamma(s)} \left( L(2n, s - 2n - 1, a, \omega) \tanh(\omega/2) e^{-\omega \omega^s - 2n - 1} \right)
\] (A.2)

Eq. [3.4] [i.e. Theorem 2] correctly reads

\[
\int_0^\infty \frac{t^{2n} \sin(s \arctan(t/\alpha))}{(a^2 + t^2)^{s/2} \sinh(\pi t)} \, dt = 1/2 \sum_{m=0}^{2n} (-1)^{m+n} \binom{2n}{m} \alpha^m P_2(a, m + s - 2n)
\] (A.3)

if \(P_2(a, \alpha)\) had been correctly defined as follows:
\[ P_2(a, s) = 2^{2-s} \zeta(s, a/2) - 2 \zeta(s, a) \quad \text{(A.4)} \]

Eq. [3.11] [i.e. Theorem 3], correctly reads

\[ \int_0^\infty \frac{t^{2n} \sin(s \arctan \left(\frac{1}{a}\right))}{(a^2 + t^2)^{s/2}(e^{2\pi t} + 1)} \, dt = \sum_{m=0}^{2n} (-1)^{m+n} \binom{2n}{m} a^m P_3(a, m + s - 2n) \quad \text{(A.5)} \]

if \( P_3(a, s) \) had been correctly defined as follows:

\[ P_3(a, s) = \frac{a^{1-s}}{s-1} + \zeta(s, a) - 2^s \zeta(s, 2a) \quad \text{(A.6)} \]

The derivation of Eq. [3.15] [i.e. Theorem 4] employs the following correct identity

\[ \int_0^\infty \frac{\cos(\omega t)}{\cosh(\beta t)} \, dt = \frac{\pi}{2 \beta \cosh \left(\frac{\pi \omega}{2 \beta}\right)} \quad \text{(A.7)} \]

However, this identity is a Fourier cosine transform, and thus cannot be used in Parseval’s identity. Thus Eq. [3.15] and Eq. [3.20], based on (A.7) are fundamentally flawed. See (1.9) for the correct result corresponding to [3.20] and Section 5 for comments on Eq. [3.15] - i.e. [Theorem 4].