Transpose symmetry of the Jones Matrix and topological phases

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The transmission Jones matrix of an arbitrary stack of reciprocal plane parallel plates which has been turned through 180° about an axis in the plane of the stack is, in an appropriate basis, the transpose of the transmission matrix of the unturned slab with a change in the sign of the off-diagonal elements. We prove this convention-free result for the case where reflection at the interfaces can be ignored and use it to devise an experimental scheme to separate isotropic and topological phase changes in a reciprocal optical medium.

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1. Introduction

Our interest in this problem originated in an attempt to answer the following question: Can one find anything in nature or put together anything that could act as a “one-way window” for unpolarized light without the use of magnetic fields, i.e. anything that would transmit a different fraction of unpolarized light in one direction as compared to the reverse direction. It is generally believed that such a possibility is ruled out by fundamental laws of nature. This implies that there must exist a simple relation between the 2 x 2 transmission Jones matrices of an arbitrary stack of reciprocal plane parallel plates (films) for forward and backward propagation. Several different formulations have been used to deal with this question. A convenient and useful way of asking the above question is: If \( M \) is the transmission Jones matrix of a stack of reciprocal, anisotropic and absorbing plane parallel plates with surfaces parallel to the \((\hat{x}, \hat{y})\) plane, expressed in the basis of \(\hat{x}\) and \(\hat{y}\) linearly polarized states, with the light beam propagating in the \(\hat{z}\) direction and the plate is turned through 180° about \(\hat{x}\) or \(\hat{y}\), how is the new Jones matrix \(M^B\) related to \(M\)? The first answer that comes to mind, following a quick reading of the pioneering work of Jones [1, 2] is, \(M^B = M^T\), where \(M^T\) is the transpose of \(M\). We give below a simple argument to show why this is not true.
Consider the case where the sample is a pure reciprocal optical rotator (for example sugar solution) represented by a Jones matrix $R$. Under a $180^\circ$ rotation about an axis perpendicular to the beam, the rotator is invariant, hence must be represented by the same matrix $R$. But in the basis of linearly polarized states, $R$ is given by,

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \quad (1)$$

where $\phi$ is the angle of rotation produced by the rotator. It is obvious from Eqn.(1) that the transpose $R^T(\phi)$ of $R(\phi)$ is not equal to $R(\phi)$. The transmission matrix of the turned sample cannot therefore be equal to $R^T(\phi)$.

2. **Statement of the theorem and proof**

The correct answer to the above question is,

For $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$, $M^B = \begin{pmatrix} m_{11} & -m_{21} \\ -m_{12} & m_{22} \end{pmatrix} \quad (2)$

This can be proved through the following steps:

(A) Define an operation “n-transpose” on any NxN complex matrix $G$ as being one under which $G$ goes to $\bar{G}$ such that

$$\bar{G}_{ij} = (-1)^{i+j}G_{ji} \quad (3)$$

From Eqn.(3) it can easily be shown that

$$(G_1G_2...G_n)^* = \bar{G}_n\bar{G}_{n-1}...\bar{G}_1 \quad (4)$$

where $G_1, G_2, ..., G_n$ are NxN complex matrices.

(B) Any arbitrary Jones matrix $M$ can be written as a product

$$M = p \ a \ S, \quad (5)$$

where $p = \exp(i\alpha), a = \exp(-\beta); \ \alpha, \ \beta$ being real numbers and $S$ is a 2x2 complex matrix with determinant +1, i.e. an element of the group SL(2,C), also called SL(2).
(C) It can be shown that $S$ can always be written as a product

$$S = K_1 K_2 ... K_6,$$

where $K_1, K_2 ... K_6$ are matrices of the form $R$, $L_0$ or $D_0$ where $R$ is given by Eqn.11 and $L_0, D_0$ are given by,

$$L_0(\delta) = \text{diag}[\exp(-\frac{\delta}{2}), \exp(\frac{\delta}{2})],$$

$$D_0(\gamma) = \text{diag}[\exp(-\frac{\gamma}{2}), \exp(\frac{\gamma}{2})].$$

The matrix $L_0(\delta)$ represents a linear retarder with retardation $\delta$ and $D_0(\gamma)$ represents an element of dichroism with relative attenuation coefficient $\gamma$, the eigenstates of both matrices being linear polarizations along $\hat{x}$ and $\hat{y}$. Note that the matrices $R(\phi), L_0(\delta)$ and $D_0(\gamma)$ have the property

$$\bar{R}(\phi) = R(\phi), \quad \bar{L}_0(\delta) = L_0(\delta) \quad \text{and} \quad \bar{D}_0(\gamma) = D_0(\gamma).$$

We shall prove statement (C) in the context where the 6-parameter group SL(2,C) represents polarization transformations. Choose the $\hat{x}$ and $\hat{y}$ linearly polarized states as the basis states for all the unitary and nonunitary polarization transformation matrices. We first note from the theory of the group SL(2,C), which is homomorphic to the Lorentz Group SO(3,1) [3], that any element $S$ of the group can be written as a product

$$S = NU$$

where $U$ is a unitary matrix, i.e. an element of pure birefringence with a pair of orthogonal eigenstates $|u>, |\tilde{u}>$ and $N$ is a nonunitary matrix, i.e. an element of pure dichroism with a pair of orthogonal eigenstates $|v>, |\tilde{v}>$. Further, if $U_0$ and $N_0$ represent elements of birefringence and dichroism respectively which are diagonal in the chosen basis and with the same eigenvalues as $U$ and $N$ respectively, there always exist unitary transformations $F$ and $G$ with linearly polarized eigenstates such that $F |x>=|u>, F |y>=|\tilde{u}>$, $G |x>=|v>$ and $G |y>=|\tilde{v}>$ so that

$$U = FU_0F^\dagger \quad \text{and} \quad N = GN_0G^\dagger.$$ 

Note $U_0$ and $N_0$ are matrices of the form $L_0$ and $D_0$ respectively. Further, since $F$ and $G$ have linearly polarized eigenstates,
\[ F = L_\psi(\delta_1) = R(\psi)L_0(\delta_1)R(-\psi) \quad \text{and} \quad G = L_\xi(\delta_2) = R(\xi)L_0(\delta_2)R(-\xi) \quad (12) \]

where \( L_\psi(\delta_1) \) and \( L_\xi(\delta_2) \) are linear retarders with retardations \( \delta_1 \) and \( \delta_2 \) and fast axes making angles \( \psi \) and \( \xi \) respectively with \( \hat{y} \). Eqs. (10), (11) and (12) together constitute a proof of the statement (C).

(D) Consider now an infinitely thin sample whose Jones matrix in the chosen basis is given by a matrix \( M \) which can be expressed as in Eqn.(5). The matrix \( S \) can then be expressed as in Eqn.(6) where \( K_1, K_2...K_6 \) represent infinitesimal transformations given by matrices of the type \( R, L_0 \) or \( D_0 \). Now rotate the sample through \( 180^\circ \) about the \( \hat{x} \) or \( \hat{y} \) axis. The Jones matrix \( S^B \) of the reversed sequence of elements is given by

\[ S^B = K_6^B.K_5^B...K_1^B \quad (13) \]

where \( K_n^B \) is the matrix of the reversed version of \( K_n \). Since each of the elements \( R, L_0 \) and \( D_0 \) is physically invariant under such a rotation, \( K_n^B = K_n \) for all \( n \). Hence

\[ S^B = K_6.K_5...K_1 \quad (14) \]

Then, from Eqns.(9) and (14), we have,

\[ S^B = \bar{K}_6.\bar{K}_5...\bar{K}_1 = \bar{S} \quad (15) \]

Since the isotropic factors of \( M \) commute with all operations, we therefore have,

\[ M^B = \bar{M} \quad (16) \]

(E) We now make the only assumption in this proof which is necessary in view of possible absorption in the sample. We assume that if two infinitesimally thin samples have the same transmission Jones matrix \( M \) for forward propagation they must have, in the absence of non-reciprocal effects, the same transmission Jones matrix \( M^B \) for reverse propagation. Eqn.(16) therefore holds for the original infinitesimal sample. Now since a sample of finite thickness can be looked upon as an infinite sequence of infinitesimally thin samples and since the above sequence of arguments can be repeated for such a sequence, it follows that Eqn.(16) holds for a finite sample, which is our main result.

The reason for the difference between our result and that of Jones is that in [1] the sample is kept fixed and the beam is reversed. This requires, in addition to an effective \( 180^\circ \) rotation
of the sample about $\hat{y}$, a convention for the relative phase between the basis states of the matrices for the two directions of propagation. In the convention used in [1], if the rotation is about the $\hat{y}$ axis, there is a $\pi$ phase difference between the two $\hat{x}$ polarized basis states and none between the two $\hat{y}$ polarized ones. This implies a relative phase shift of $\pi$ between the $\hat{x}$ and $\hat{y}$ polarizations in switching the direction of propagation, corresponding to a unitary transformation by means of $\sigma_3=\text{diag}[1,-1]$. Indeed it can be verified that $\bar{M}$ and $M^T$ are related by

$$M^T = \sigma_3 \bar{M} \sigma_3^\dagger.$$  \hspace{1cm} (17)

Jones’ result is therefore consistent with ours. In our formulation however, the direction of propagation remains fixed. We therefore use the same set of basis states with the same phases for the forward and the backward matrices which is the simple and natural thing to do and one does not require a phase convention. Eqn. (16) can be verified in a sequence of simple null interference experiments which will be described elsewhere. From Eqns.(16) and (3) it follows that for an incident unit intensity unpolarized beam, the intensity transmitted by the reversed sample $I^B_T = \frac{1}{2} Tr(M^B M^B)$ is the same as that transmitted by the original sample i.e. $I_T = \frac{1}{2} Tr(M^M M)$.

3. Separation of isotropic and topological phases

It is now well known [4, 5] that if a beam of polarized light passes through an arbitrary reciprocal medium, then through a 45° Faraday rotator and is then reflected normally off an isotropic plane mirror so that it retraces its path through the medium, the effect of any reciprocal birefringence or dichroism in the medium is cancelled. Eqn. (16) together with the method of analysis described in [7] provides a very compact proof of this result.

In Fig.(1a) let $M$ stand for any reciprocal optical medium, $FR(45)$ for a Faraday rotator that rotates the polarization about the beam axis through 45° in real space and the “mirror” stand for any isotropic reflector placed normal to the beam. Let us choose the $\hat{z}$ axis as the propagation direction and the $\hat{x}, \hat{z}$ plane to be the reflection plane [6]. The medium $M$ can be decomposed as in Eqn.(5) into (i) an isotropic refractive part $p$, (ii) an isotropic absorption part $a$ and (iii) $S$, an element of the group SL(2,C). Following [7], the reflection at the mirror is replaced by a halfwave plate $H_0$ with its fast axis along $\hat{y}$, $FR(45)$ and $M$ encountered on the reverse passage are replaced by equivalent elements which have been rotated about $\hat{y}$ through 180° and placed in the forward path of the beam. This yields an equivalent optical circuit, shown in Fig.(1b), with a round trip Jones matrix $M^{(rt)}$ given by

$$M^{(rt)} = a \ p \ S \ R(-45) \ H_0 R(45) \ S \ p \ a = a^2 p^2 S \ H_{45} \ S$$  \hspace{1cm} (18)
where we have used Eqn. (15) to represent the reversed medium by $\bar{S}$. The matrix $H_{45}$ has the following form:

$$H_{45} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The crucial part of the proof consists of the following identity which can be easily verified:

$$\bar{S}H_{45}S = H_{45}.$$  \hspace{1cm} (20)

where $S$ is any 2x2 complex matrix with determinant +1 and $\bar{S}$ is defined by Eqn. (3). Eqns. (18) and (20) then give

$$M^{(rt)} = a^2 p^2 H_{45}$$ \hspace{1cm} (21)

As shown in Figs. (1b) and (1c), the polarization evolution described by $M^{(rt)}$ is followed by a rotation of the beam about $\hat{y}$ through $180^\circ$ [7]. Eqn. (21) shows that the polarization state of the return beam is independent $S$, i.e. of any parameter representing birefringence and dichroism in the medium. Since such a cancellation is well supported by experiment and since an equation like (20) is not obtained if $S^T$ were used in place of $\bar{S}$ we consider Eqn. (20) an indirect experimental support for our Eqns. (15) and (16).

Eqn. (21) has another interesting consequence. While all phase changes that arise due to changes in the SL(2,C) part of the medium (geometric phases, Pancharatnam phases etc. [8]) are cancelled, the isotropic phase factor $p$ is not cancelled by double passage. This provides a method for separately determining isotropic and topological phase changes in a medium. An interference experiment in which the beam passes through the medium only once measures the total phase shift $\Delta \phi_1 = \phi_{iso} + \phi_{topo}$ whereas an interference experiment with double passage by means of a Faraday mirror measures $\Delta \phi_2 = 2\phi_{iso}$. From $\Delta \phi_1$ and $\Delta \phi_2$ the isotropic phase shift $\phi_{iso}$ and the topological phase shift $\phi_{topo}$ can be determined separately. In fact as shown in Fig. (2), the two interferometers can be combined in a single setup with a common arm containing the experimental medium $M$. The quantities $\Delta \phi_1$ and $\Delta \phi_2$ can then be measured simultaneously.

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Fig. 1. (a) Double passage through a medium by means of the 45° Faraday rotator reflector. (b) Equivalent optical circuit. (c) The reduced equivalent circuit.

References

1. R.C. Jones, “A new calculus for the treatment of optical systems I. Description and discussion of the calculus,” J.O.S.A. 31, 488-493 (1941).
2. R.C. Jones, “A new calculus for the treatment of optical systems VI. Experimental determination of the matrix,” J.O.S.A. 37, 110-112 (1947).
3. Wu-Ki Tung, Group theory in physics, (World Scientific 1985), Ch.10.
4. N.C. Pistoni and M. Martinelli, “Polarization noise suppression in retracing optical fiber circuits,” Opt. Lett. 16, 711-713 (1991).
5. M. O. van Deventer, “Preservation of polarization orthogonality of counterpropagating waves through dichroic birefringent media: proof and application,” Electronics Letters 27, 1538-1539 (1991).
6. It can be shown that the final state of the beam as obtained by this method is independent of the choice of the reflection plane.
7. R. Bhandari, “Geometric phase in an arbitrary evolution of a light beam,” Phys. Lett. A 135, 240-244 (1989).
8. R. Bhandari, “Polarization of light and topological phases,” Phys. Rep. 281, 1-64 (1997).
Fig. 2. An interference setup to determine simultaneously the isotropic and topological phase shifts in a reciprocal optical medium $M$. The beam splitters BS$_1'$, BS$_2'$, the mirrors M$_1'$, M$_2'$, the polarizer P' and the fringe detector D' comprise a double pass Michelson interferometer with a 45° Faraday rotator and BS$_1$, BS$_2$, M$_1$, M$_2$, D comprise a single-pass Mach-Zhender interferometer.