GRAVITY AS A HIGGS FIELD.
I. GEOMETRIC EQUIVALENCE PRINCIPLE

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Abstract

If gravity is a metric field by Einstein, it is a Higgs field. Gravitation theory meets spontaneous symmetry breaking in accordance with the equivalence principle reformulated in the spirit of Klein-Chern geometries of invariants. In gravitation theory, the structure group of the principal linear frame bundle $LX$ over a world manifold $X^4$ is reducible to the connected Lorentz group $SO(3,1)$. The physical underlying reason of this reduction is Dirac fermion matter possessing only exact Lorentz symmetries. The associated Higgs field is a tetrad gravitational field $h$ represented by a global section of the quotient $\Sigma$ of $LX$ by $SO(3,1)$. The feature of gravity as a Higgs field issues from the fact that, in the presence of different tetrad fields, Dirac fermion fields are described by spinor bundles associated with different reduced Lorentz subbundles of $LX$, and we have nonequivalent representations of cotangent vectors to $X^4$ by Dirac’s matrices. It follows that a fermion field must be regarded only in a pair with a certain tetrad field. These pairs fail to be represented by sections of any product bundle $S \times \Sigma$, but sections of the composite spinor bundle $S \rightarrow \Sigma \rightarrow X^4$. They constitute the so-called fermion-gravitation complex where values of tetrad gravitational fields play the role of coordinate parameters, besides the familiar world coordinates. In Part 1 of the article, geometry of the fermion-gravitation complex is investigated. The goal is the total Dirac operator into which components of a connection on $S \rightarrow \Sigma$ along tetrad coordinate directions make contribution. The Part II will be devoted to dynamics of fermion-gravitation complex. It is a constraint system to describe which we use the covariant multisymplectic generalization of the Hamiltonian formalism when canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the time.

1 Introduction

Gravitation theory is theory with spontaneous symmetry breaking. Spontaneous symmetry breaking is quantum phenomenon modelled by a classical Higgs field. In the algebraic quantum field theory, Higgs fields characterize nonequivalent Gaussian states of algebras of quantum fields [17]. They are sui generis fictitious fields describing collective phenomena. In gravitation theory, spontaneous symmetry breaking displays on the classical level.
It is established by the equivalence principle reformulated in the terms of Klein-Chern geometries of invariants [9, 16, 19].

In Einstein’s General Relativity, the equivalence principle is called to provide transition to Special Relativity with respect to some reference frames. In the spirit of F.Klein’s Erlanger program, the Minkowski space geometry can be characterized as geometry of Lorentz invariants. The geometric equivalence principle then postulates that there exist reference frames with respect to which Lorentz invariants can be defined everywhere on a world manifold \( X^4 \). This principle has the adequate mathematical formulation in terms of fibre bundles.

We follow the generally accepted geometric description of classical fields as sections of a fibred manifold

\[ \pi : Y \to X \]

over a world manifold \( X^4 \). In gauge theory, \( Y \to X \) is a bundle with a structure group.

Let \( LX \) be the principal bundle of linear frames in tangent spaces to \( X^4 \). Its structure group is

\[ GL_4 = GL^+(4, \mathbb{R}). \]

The geometric equivalence principle requires that this structure group is reduced to the connected Lorentz group

\[ L = SO(3, 1). \]

It means that there is given a reduced subbundle \( L^h X \) of \( LX \) whose structure group is \( L \). They are atlases of \( L^h X \) with respect to which Lorentz invariants can be defined.

In accordance with the well-known theorem, there is the 1:1 correspondence between the reduced \( L \) subbundles \( L^h X \) of \( LX \) and the tetrad gravitational fields \( h \) represented by global sections of the quotient bundle

\[ \Sigma = LX/L \to X^4. \]

Its standard fibre is the quotient space \( GL_4/L \). The bundle (1) is isomorphic to the 2-fold covering of the bundle \( \Sigma_g \) of pseudo-Riemannian forms in cotangent spaces to \( X^4 \). A global section of \( \Sigma_g \) is a pseudo-Riemannian metric on \( X^4 \).

Thereby, the geometric equivalence principle provides a world manifold with the so-called \( L \)-structure [24]. From the physical point of view, it singles out the Lorentz group as the exact symmetry subgroup of world symmetries broken spontaneously [9]. The associated classical Higgs field is a tetrad (or metric) gravitational field.

For the first time, the conception of a graviton as a Goldstone particle corresponding to violation of Lorentz symmetries in a curved space-time had been advanced in mid 60s by Heisenberg and Ivanenko in discussion on cosmological and vacuum asymmetries. This idea was revived in connection with constructing the induced representations of the group \( GL_4 \) [7, 12, 13] and then in the framework of the approach to gravitation theory as a nonlinear \( \sigma \)-model [14]. In geometric terms, the fact that a pseudo-Riemannian metric is similar a Higgs field has been pointed out by Trautman [25] and by us [15]. To prove it, the new geometric formulation of the equivalence principle has been suggested [8, 9].
The underlying physical reason of the geometric equivalence principle is Dirac fermion matter possessing only exact Lorentz symmetries.

Let us consider a bundle of complex Clifford algebras $C_{3,1}$ over $X^4$. Its subbundles are both a spinor bundle $S_M \to X^4$ and the bundle $Y_M \to X^4$ of Minkowski spaces of generating elements of $C_{3,1}$. There is the bundle morphism

$$\gamma : Y_M \otimes S_M \to S_M$$

which defines representation of elements of $Y_M$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_M$. To describe Dirac fermion fields on a world manifold, one must require that the bundle $Y_M$ is isomorphic to the cotangent bundle $T^*X$ of $X^4$. It takes place if $Y_M$ is associated with some reduced $L$ subbundle $L^hX$ of the linear frame bundle $LX$. Then, there exists the representation

$$\gamma_h : T^*X \otimes S_h \to S_h$$

of cotangent vectors to a world manifold $X^4$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$ associated with the lift of $L^hX$ to a $SL(2, \mathbb{C})$ principal bundle. Sections of $S_h$ describe Dirac fermion fields in the presence of a tetrad gravitational field $h$.

It follows that, in the presence of Dirac fermion matter, we must handle two types of reference frames. They are holonomic atlases of $LX$ and atlases of a reduced Lorentz subbundle $L^hX$ of $LX$. The corresponding tetrad gravitational field $h$ determines transformations between these reference frames. The pecularity of gravity thus is clarified. In contrast to other fields, a tetrad gravitational field itself defines reference frames. The Higgs field character of gravity issues from the fact that the reference frames and other characteristics corresponding to different gravitational fields are not equivalent in a sense.

The key point of our consideration consists in the fact that, for different tetrad fields $h$ and $h'$, the representations $\gamma_h$ and $\gamma_{h'}$ fail to be equivalent [16, 19]. It follows that every Dirac fermion field must be regarded only in a pair with a certain tetrad gravitational field $h$. These pairs constitute the so-called fermion-gravitation complex [12]. They can not be represented by sections of any product $S \times \Sigma$ where $S \to X^4$ is some standard spinor bundle. At the same time, there is the 1:1 correspondence between these pairs and the sections of the composite bundle

$$S \to \Sigma \to X^4$$

where $S \to \Sigma$ is a spinor bundle associated with the $L$ principal bundle $LX \to \Sigma$ [18, 19]. In particular, every spinor bundle $S_h \to X^4$ is isomorphic to restriction of $S$ to $h(X^4) \subset \Sigma$.

To show the physical relevance of the composite bundle (2), we aim constructing the total Dirac operator $\mathcal{D}$ on its section.

We however observe that the composite bundle (3) fails to possess a structure group. Therefore, connections on $S \to X^4$ which we consider are not principal connections. They are so-called composite connections constructed by means of principal connections.
Recall that the $k$-order jet manifold $J^k Y$ of a fibred manifold $Y \to X$ comprises the equivalence classes $j^k_x s$, $x \in X$, of sections $s$ of $Y$ identified by the $(k + 1)$ terms of their Taylor series at $x$. It is a finite-dimensional manifold. Jet manifolds have been widely used in the theory of differential operators. Their application to differential geometry is based on the 1:1 correspondence between the connections on a fibred manifold $Y \to X$ and the global sections of the jet bundle $J^1 Y \to Y$. 

Dynamics of fields represented by sections of a fibred manifold $Y \to X$ is phrased in terms of jet manifolds $J^1 Y \to Y$. In the first order Lagrangian formalism, the jet manifold $J^1 Y$ plays the role of a finite-dimensional configuration space of fields. Given fibred coordinates $(x^\lambda, y^i)$ of $Y \to X$, it is endowed with the adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$ where coordinates $y^i_\lambda$ make the sense of values of partial derivatives $\partial_\lambda y^i(x)$ of field functions $y^i(x)$. A Lagrangian density on $J^1 Y$ is defined by a form

$$L L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda) dx^1 \wedge \ldots \wedge dx^n, \quad n = \dim X.$$ 

Dynamics of the fermion-gravitation complex will be investigated in Part II of the work. Its Lagrangian density on the configuration space $J^1 S$ is chosen in such a way that the associated Euler-Lagrange operator $\mathcal{E}_L$ reproduces the total Dirac operator $D$ mentioned above. This Lagrangian density however is degenerate.

If a Lagrangian density is degenerate, the corresponding Euler-Lagrange equations are underdetermined and need supplementary gauge-type conditions. In gauge theory, they are the familiar gauge conditions. In general case, the above-mentioned supplementary conditions remain elusive. To describe constraint field systems, one can use the covariant multimomentum Hamiltonian formalism where canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the time. Given a fibred manifold $Y \to X$, the corresponding multimomentum phase space is the Legendre manifold

$$\Pi = \mathcal{R} T^* X \otimes TX \otimes VY$$

endowed with the canonical coordinates $(x^\lambda, y^i, p^i_\lambda)$.

The feature of a tetrad gravitational field as a Higgs field consists in the fact that, on the constraint space, its canonical momenta are equal to zero, otherwise in the presence of fermion fields. Fermion fields deform the constraint space in the gravitation sector, and this deformation makes contribution into the energy-momentum conservation law. In the framework of the multimomentum Hamiltonian formalism, we have the fundamental identity whose restriction to a constraint space can be treated as the energy-momentum conservation law. In Part II, peculiarity of this conservation law in gravitation theory will be considered.
2 Geometric Preliminary

This Section aims to summarize the necessary prerequisites of jet manifolds and connections [11, 21, 23].

All morphisms throughout are differentiable maps of class $C^\infty$. Manifolds are real, Hausdorff, finite-dimensional, second-countable and connected.

We use the standard symbols $\otimes, \wedge$ and $\i$ for the tensor, exterior and interior products respectively.

Given a manifold $Z$ with an atlas of local coordinates $(z^\lambda)$, the tangent bundle $T^Z$ and the cotangent bundle $T^* Z$ of $Z$ are provided with the induced coordinates $(z^\lambda, \dot{z}^\lambda)$ and $(z^\lambda, \dot{z}^\lambda)$ with respect to the holonomic bases $\partial_\lambda$ and $dz^\lambda$. If $f : Z \to Z'$ is a manifold map, by $Tf : TZ \to TZ'$ is meant the morphism tangent to $f$.

A fibred manifold $Y$ is defined to be a surjective submersion

$$\pi : Y \to X$$

where, unless otherwise stated, $X$ is an $n$-dimensional manifold. A fibred manifold $Y$ is provided with an atlas of fibred coordinates

$$(x^\lambda, y^i), \quad x^\lambda \to x'^\lambda(x^\mu), \quad y^i \to y'^i(x^\mu, y^j).$$

A locally trivial fibred manifold $Y$ is called a bundle, and its fibred coordinates $y^i$ are coordinates of its standard fibre $V$.

By a fibred morphism of a fibred manifold $Y \to X$ to a fibred manifold $Y' \to X'$ is meant a fibre-to-fibre manifold map $\Phi : Y \to Y'$ over a map $f : X \to X'$. If $f = Id_X$, one says briefly that $\Phi$ is a fibred morphism $\to_X$ over $X$.

Given a fibred manifold $Y \to X$, every map $f : X' \to X$ induces the pullback $f^* Y \to X'$ comprising the pairs

$$\{(y, x') \in Y \times X' \mid \pi(y) = f(x')\}.$$  

In particular, the product

$$Y = Y \times_X Y'$$

of fibred manifolds over $X$ is defined. For instance, given the tangent bundle $TX$ and the cotangent bundle $T^* X$ of $X$, we have the products

$$Y \times_X TX, \quad Y \times_X T^* X$$

over $X$. For the sake of simplicity, we denote them by the symbols $TX$ and $T^* X$ respectively.

The tangent bundle of a fibred manifold $Y$ contains the vertical tangent subbundle

$$V Y = \text{Ker}(T\pi)$$
with the induced coordinates \((x^\lambda, y^i, \dot{y}^i)\). Given a fibred morphism \(\Phi : Y \to Y'\), the tangent morphism \(T\Phi\) yields the corresponding vertical tangent morphism \(V\Phi : VY \to VY'\). With \(VY\), we have the exact sequence

\[
0 \to VY \to TY \to \frac{Y \times TX}{Y} \to 0. \tag{3}
\]

Its different splittings

\[
Y \times TX \to TY
\]

correspond to different connections on the fibred manifold \(Y \to X\).

On fibred manifolds, we consider the following types of differential forms:

(i) exterior horizontal forms \(Y \to \wedge T^*X\);

(ii) tangent-valued horizontal forms \(Y \to \hat{T}^*X \otimes TY\) and, in particular, soldering forms \(Y \to T^*X \otimes VY\);

(iii) pullback-valued forms

\[
Y \to \hat{T}^*Y \otimes TX, \\
Y \to \hat{T}^*Y \otimes T^*X.
\]

Horizontal \(n\)-forms are called horizontal densities.

Given a fibred manifold \(Y \to X\), the first order jet manifold (or simply the jet manifold) \(J^1Y\) of \(Y\) comprises the equivalence classes \(j^1_x s\), \(x \in X\), of sections \(s\) of \(Y\) so that different sections \(s\) and \(s'\) belong to the same jet class \(j^1_x s\) if and only if the tangent morphisms \(T_s\) and \(T_{s'}\) consist with each other on the tangent space \(T_x\) to \(X\). In other words, sections \(s, s' \in j^1_x s\) are identified by their values \(s^i(x) = s'^i(x)\) and values of their partial derivatives \(\partial^\mu s^i(x) = \partial^\mu s'^i(x)\) at \(x\). The jet manifold \(J^1Y\) is provided with the adapted coordinates

\[
(x^\lambda, y^i, y^i_\lambda), \quad y^i_\lambda(j^1_x s) = \partial^\lambda s^i(x),
\]

\[
y^i_\lambda = \left(\frac{\partial y^i}{\partial y^j} y^j_\mu + \frac{\partial y^i}{\partial x^\mu} \partial x^\mu \right) \frac{\partial x^\lambda}{\partial x^\lambda}.
\]

It is both a fibred manifold \(j^1_x s \to x\) over \(X\) and an affine bundle \(j^1_x s \to s(x)\) over \(Y\). There exists the canonical monomorphism

\[
\lambda : J^1Y \to T^*X \otimes TY,
\]

\[
\lambda = dx^\lambda \otimes (\partial^\lambda + y^i_\lambda \partial^i),
\]

over \(Y\) whose image is an affine bundle modelled on the vector bundle

\[
T^*X \otimes VY. \tag{4}
\]
The monomorphism $\lambda$ is called the contact map. It enables us to handle jets as tangent-valued forms.

Every fibred morphism $\Phi : Y \to Y'$ over a diffeomorphism of $X$ admits the jet prolongation

$$J^1\Phi : J^1Y \to J^1Y',$$

$$y'^i_\lambda \circ J^1\Phi = \left( \frac{\partial \Phi^i}{\partial y^j} y'^j_\mu + \frac{\partial \Phi^i}{\partial x^\mu} \right) \frac{\partial x^\mu}{\partial x'^\lambda}.$$ 

In particular, every section $s$ of a fibred manifold $Y$ has the jet prolongation

$$(x^\lambda, y^i, y'^i_\lambda) \circ J^1s = (x^\lambda, s^i(x), \partial_\lambda s^i(x))$$

to the section $J^1s$ of the fibred jet manifold $J^1Y \to X$.

Note that the jet manifold machinery is naturally generalized to complex bundles over a real base manifold.

Application of jet manifolds to differential geometry is based on the horizontal splitting

$$J^1Y \times TX \to J^1Y \times TY,$$

$$\dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i = \dot{x}^\lambda (\partial_\lambda + y'^i_\lambda \partial_i) + (\dot{y}^i - \dot{x}^\lambda y'^i_\lambda) \partial_i,$$ (5)

of the tangent bundle $TY$ over $J^1Y$. 

A connection on a fibred manifold $Y \to X$ is defined to be a global section

$$\Gamma : Y \to J^1Y,$$

$$(x^\lambda, y^i, y'^i_\lambda) \circ \Gamma = (x^\lambda, y^i, \Gamma^i_\lambda(y)),$$

$$\Gamma^i_\lambda = \left( \frac{\partial y'^i_\lambda}{\partial y^j} \Gamma^j_\mu + \frac{\partial y'^i_\lambda}{\partial x^\mu} \right) \frac{\partial x^\mu}{\partial x'^\lambda},$$

of the affine bundle $J^1Y \to Y$. For instance, a linear connection on a vector bundle $Y$ reads

$$\Gamma^i_\lambda(y) = \Gamma^i_{j\lambda}(x)y^j.$$ 

By means of the contact map $\lambda$, any connection $\Gamma$ on $Y \to X$ can be represented by the tangent-valued 1-form $\lambda \circ \Gamma$ on $Y$ which we denote by the same symbol

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda(y) \partial_i).$$ (6)

Substituting (6) into the canonical horizontal splittings (3), we obtain the familiar horizontal splitting of the tangent bundle $TY$ with respect to a connection on $Y$.

Since $J^1Y \to Y$ is an affine bundle modelled on the vector bundle (4), connections on a fibred manifold $Y$ constitute an affine space modelled on the linear space of soldering forms on $Y$. Namely, if $\Gamma$ is a connection and $\sigma$ is a soldering form on a fibred manifold
$Y$, then $\Gamma + \sigma$ is a connection on $Y$. Conversely, if $\Gamma$ and $\Gamma'$ are connections on a fibred manifold $Y$, then $\Gamma - \Gamma'$ is a soldering form on $Y$.

Every connection $\Gamma$ on a fibred manifold $Y$ yields the affine bundle morphism

\[
D_\Gamma : J^1Y \to \frac{T^*X \otimes VY}{Y},
\]

\[D_\Gamma = (y^i_\lambda - \Gamma^i_\lambda)dx^\lambda \otimes \partial_i,
\]

which is called the covariant differential. The corresponding covariant derivative of a section $s$ of $Y$ reads

\[
\nabla_\Gamma s = D_\Gamma \circ J^1s = (\partial_\lambda s^i - (\Gamma \circ s)^i_\lambda)dx^\lambda \otimes \partial_i.
\]

A section $s$ of a fibred manifold $Y$ is called an integral section for a connection $\Gamma$ on $Y$ if $\Gamma \circ s = J^1s$, that is, $\nabla_\Gamma s = 0$.

The general approach to connections as jet fields is suitable to formulate the classical concept of principal connections.

Let $P \to X$ be a principal bundle with a structure group $G$ which is assumed to be a real finite-dimensional Lie group. By $r_g$, $g \in G$, we denote the canonical action of $G$ on $P$ on the right.

In case of a principle bundle with a structure group $G$, the exact sequence (3) reduces to the exact sequence

\[0 \to V^GP \to T^GP \to TX \to 0\]

where

\[T^GP = TP/G, \quad V^GP = VP/G\]

are respectively the quotients of the tangent bundle $TP$ and the vertical tangent bundle $VP$ of $P$ by the tangent and vertical tangent morphisms to the canonical mappings $r_g$.

A principal connection $A$ on a principal bundle $P \to X$ is defined to be a $G$-equivariant connection on $P$ such that

\[A \circ r_g = J^1r_g \circ A, \quad g \in G,\]

\[A = dx^\lambda \otimes (\partial_\lambda + A^m_\lambda(q)\tau_m), \quad q \in P,\]

\[A^m_\lambda(qg) = A^m_\lambda(q)\text{adj}^{-1}(\tau_m),\]

where $\tau_m$ are the fundamental vector fields on $P$. Given an atlas $\Psi^P = \{U_\xi, z_\xi\}$ of $P$ where $\{z_\xi\}$ is a family of local sections of $P$, we reproduce the familiar local connection 1-forms

\[A_\xi = A^m_\lambda(x)dx^\lambda \otimes I_m \quad A^m_\lambda(x) = A^m_\lambda(z_\xi(x)),\]

where $I_m$ is a basis for the left Lie algebra of the group $G$.

There is the 1:1 correspondence between the principal connections on a principal bundle $P \to X$ and the global sections of the quotient

\[C = J^1P/G \to X\]
of $J^1P$ by the jet prolongations of the canonical morphisms $r_q$. We call $C$ the principal connection bundle. It is an affine bundle modelled on the vector bundle

$$\overline{C} = T^*X \otimes V^G P.$$  

Given a bundle atlas $\Psi^P$ of $P$, the principal connection bundle $C$ is provided with the fibred coordinates $(x^\mu, k^m_\mu)$ so that

$$(k^m_\mu \circ A)(x) = A^m_\mu(x)$$

are coefficients of the local connection 1-form \([8]\). The first order jet manifold $J^1C$ of $C$ is endowed with the adapted coordinates $(x^\mu, k^m_\mu, k^m_{\mu \lambda})$.

Let $Y \rightarrow X$ be a bundle associated with a principal bundle $P \rightarrow X$. The structure group $G$ of $P$ acts freely on the standard fibre $V$ of $Y$ on the left. The total space of $Y$, by definition, is the quotient

$$Y = (P \times V)/G$$

of the product $P \times V$ by identification of elements $(qg \times gv)$ for all $g \in G$. The $P$ associated bundle $Y$ is provided with atlases $\Psi = \{U_\xi, \psi_\xi\}$ associated with atlases $\Psi^P = \{U_\xi, z_\xi\}$ of $P$ as follows:

$$\psi^{-1}_\xi(x \times V) = [z_\xi(x)]_V(V), \quad x \in U_\xi,$$

where by $[q]_V$ is meant the restriction of the canonical map $P \times V \rightarrow Y$ to $q \times V$.

Every principal connection $A$ on a principal bundle $P$ yields the associated principal connection $\Gamma$ on a $P$ associated bundle $Y$. With respect to associated atlases $\Psi$ of $Y$ and $\Psi^P$ of $P$, this connection reads

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + A^m_\mu(x)I_m^{ij}y^i\partial_j)$$

where $A^m_\mu(x)$ are coefficients of the local connection 1-form \([8]\) and $I_m$ are generators of the structure group $G$ on the standard fibre $V$ of the bundle $Y$.

3 Composite Manifolds

A composite manifold is defined to be composition of surjective submersions

$$\pi_{\Sigma X} \circ \pi_{Y \Sigma} : Y \rightarrow \Sigma \rightarrow X.$$  \hspace{1cm} (10)

It is a fibred manifold $Y \rightarrow X$ provided with the particular class of coordinate atlases $(x^\lambda, \sigma^m, y^i)$:

$$x^\lambda = f^\lambda(x^\mu), \quad \sigma^m = f^m(x^\mu, \sigma^n), \quad y^i = f^i(x^\mu, \sigma^n, y^j),$$

where $(x^\mu, \sigma^m)$ are fibred coordinates of $\Sigma \rightarrow X$. We further suppose that $Y \rightarrow \Sigma$ is a bundle denoted by $Y_\Sigma$.  

9
Application of composite manifolds to field theory is based on the following assertions [18, 21].

(i) Let $Y$ be the composite manifold (10). Given a section $h$ of $\Sigma \to X$ and a section $s_\Sigma$ of $Y \to \Sigma$, their composition $s_\Sigma \circ h$ is a section of the composite manifold $Y$. Conversely, every global section $s$ of the fibred manifold $Y \to X$ is represented by some composition $s_\Sigma \circ h$ where $h = \pi_{Y\Sigma} \circ s$ and $s_\Sigma$ is an extension of the local section $h(X) \to s(X)$ of the bundle $Y_\Sigma$ over the closed imbedded submanifold $h(X) \subset \Sigma$.

(ii) Given a global section $h$ of $\Sigma \to X$, the restriction $Y_h = h^*Y_\Sigma$ of the bundle $Y \to \Sigma$ to $h(X)$ is a fibred imbedded submanifold of $Y \to X$. Moreover, there is the 1:1 correspondence between the sections of $Y_h$ and the sections $s$ of the composite manifold $Y$ such that $\pi_{Y\Sigma} \circ s = h$.

Therefore, one can say that sections $s_h$ of $Y_h \to X$ describe fields in the presence of a background field $h$, whereas sections of the composite manifold $Y \to X$ describe all pairs $(s_h, h)$. In accordance with the assertion (ii), there is the 1:1 correspondence between them. It is important when physical systems in the presence of different background fields $h$ and $h'$ are nonequivalent and can not be represented by sections of the product $\Sigma \times Y$ where $Y \to X$ is some standard bundle.

Let $Y_\Sigma$ be the composite manifold (10) and $J^1\Sigma$, $J^1Y_\Sigma$ and $J^1Y$ the jet manifolds of $\Sigma \to X$, $Y \to \Sigma$ and $Y \to X$ respectively. Given fibred coordinates $(x^\lambda, \sigma^m, y^i)$ of $Y$, the corresponding adapted coordinates of $J^1\Sigma$, $J^1Y_\Sigma$ and $J^1Y$ are

$$(x^\lambda, \sigma^m, \sigma^m_\lambda), \quad (x^\lambda, \sigma^m, y^i, \bar{y}^i_\lambda, y^i_m), \quad (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda).$$

There exists the canonical surjection

$$\rho : J^1\Sigma \times J^1Y_\Sigma \to J^1Y,$$

$$\rho(j^1_xh, j^1_{h(x)}s_\Sigma) = j^1_x(s_\Sigma \circ h),$$

$$y^i_\lambda \circ \rho = y^i_m\sigma^m_\lambda + \bar{y}^i_\lambda,$$

where $s_\Sigma$ and $h$ are sections of $Y \to \Sigma$ and $\Sigma \to X$ respectively [23].

In particular, let

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m),$$

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \bar{A}^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i)$$

be connections on fibred manifolds $\Sigma \to X$ and $Y \to \Sigma$ respectively. Building on the canonical morphism (11), one can construct the composite connection

$$A = dx^\lambda \otimes [\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^{i_m}_m \Gamma^m_\lambda + \bar{A}^i_\lambda) \partial_i]$$

on $Y \to X$ [18, 21]. Composite connections (13) are by no means the unique type of connections on a composite manifold. We consider them since, if a connection $\Gamma$ on $\Sigma \to X$ has an integral section $h$, the composite connection (13) on $Y$ is reducible to the connection

$$A_h = dx^\lambda \otimes [\partial_\lambda + (A^{i_m}_m \partial_m h^m + \bar{A}^i_\lambda) \partial_i]$$

on the fibred submanifold $Y_h$ of $Y \to X$. 

10
4 Spontaneous Symmetry Breaking

In classical field theory, spontaneous symmetry breaking is modelled by classical Higgs fields. In geometric terms, the necessary condition of spontaneous symmetry breaking consists in reduction of a structure group $G$ of a principal bundle $P$ to its closed subgroup $K$ of exact symmetries. Classical Higgs fields are represented by global sections of the bundle $P/K \to X$ or isomorphic homogeneous bundles.

Let $\pi_P : P \to X$ be a principal bundle with a structure Lie group $G$ and $K$ its closed subgroup. Then $\Sigma = P/K \to X$ is the associated bundle

$$P/K = (P \times G/K)/G$$

with the standard fibre $G/K$ on which the structure group $G$ acts on the left. We have the composite manifold

$$\pi_{\Sigma X} \circ \pi_P : P \to P/K \to X$$

(15)

where $P_{\Sigma} = P \to P/K$ is a principal bundle with the structure group $K$. Note that (15) fails to be a principal bundle.

Let the structure group $G$ be reducible to its closed subgroup $K$. By the well-known theorem, there is the 1:1 correspondence

$$\pi_{P\Sigma}(P_h) = (h \circ \pi_P)(P_h)$$

between global sections $h$ of the bundle $P/K \to X$ and the $K$ reduced subbundles $P_h$ of $P$ which are restrictions of the principal bundle $P_{\Sigma}$ over $h(X)$. Every principal connection $A_h$ on a reduced subbundle $P_h$ is lifted to a principal connection on $P$. Conversely, a principal connection $A$ on $P$ is reducible to a principal connection on $P_h$ if and only if the global section $h$ of the bundle $P/K \to X$ is an integral section of the connection $A$.

One says that sections $s_h$ of a vector bundle $Y_h \to X$ with a standard fibre $V$ describe matter fields in the presence of a Higgs field $h$ if $Y_h$ is associated with the reduced subbundle $P_h$ of the principal bundle $P$, that is,

$$Y_h = (P_h \times V)/K.$$  (16)

Matter fields $s_h$ in the presence of different Higgs fields $h$ and $h'$ are described by sections of the matter bundles $Y_h$ and $Y_{h'}$ associated with different reduced subbundles $P_h$ and $P_{h'}$ of $P$. If the standard fibre $V$ admits only representation of the exact symmetry subgroup $K \subset G$, there is no canonical isomorphism between $Y_h$ and $Y_{h'}$. In this case, a connection on $Y_h$ is assumed to be associated with a principal connection on the reduced subbundle $P_h$. A principal connection $A_h$ on $P_h$ is extended to a principal connection on $P$ which however fails to be reducible to a connection on another reduced subbundle $P_{h'} \neq h$. It follows that matter fields and gauge potentials possessing only exact symmetries must be regarded only in pairs with a certain Higgs field.

To describe this spontaneous symmetry breaking, composite manifolds have been suggested [18].
Given the composite manifold (13), the canonical morphism (11) results in the surjection
\[ J^1 P_\Sigma / K \times J^1 \Sigma \to J^1 P / K \] (17)
over \( J^1 \Sigma \). In particular, let \( A_\Sigma \) be a principal connection on \( P_\Sigma \) and \( \Gamma \) be a connection on \( \Sigma \). The corresponding composite connection (13) on the composite manifold (13) is equivariant under the canonical action of \( K \) on \( P \). If a connection \( \Gamma \) has an integral global section \( h \) of \( P / K \to X \), the composite connection (13) is reducible to the connection (14) on \( P_h \) which is a principal connection on \( P_h \).

Let us consider the composite manifold
\[ Y = (P \times V) / K \to P / K \to X \] (18)
where the bundle
\[ Y_\Sigma = (P \times V) / K \to P / K \]
is associated with the \( K \) principal bundle \( P \to P / K \). Given a reduced subbundle \( P_h \) of \( P \), the associated bundle
\[ Y_h = (P_h \times V) / K \]
is isomorphic to the restriction of \( Y_\Sigma \) over \( h(X) \). The composite manifold (18) can be provided with the composite connection (13) where the connection \( A_\Sigma \) on the bundle \( Y \to P / K \) is a principal connection on the \( K \) principal bundle \( P_\Sigma \) and the connection \( \Gamma \) on \( \Sigma \) is a principal connection on some reduced subbundle \( P_h \) of \( P \) [18, 21]. In this case, the composite connection \( A \) is reducible to the connection (14) on the bundle \( Y_h \) (11) as a subbundle of the composite manifold \( Y \). This connection appears to be some principal connection \( A_h \) on \( P \).

Thus, they are sections of the composite manifold (18) which describe the above-mentioned pairs \((s_h, h)\) of matter fields possessing only exact symmetries and Higgs fields.

Accordingly, the pairs \((A_h, h)\) of exact symmetry gauge potentials \( A_h \) and Higgs fields \( h \) can be represented by sections of the composite manifold
\[ C_K = J^1 P / K \to P / K \to X \] (19)
which are projected onto jet prolongations \( J^1 h \) of global sections \( h \) of the bundle \( \Sigma \).

Note that the manifold \((P \times V) / K\) possesses the structure of the bundle associated with the principal bundle \( P \). Its standard fibre is \((G \times V) / K\) on which the structure group \( G \) of \( P \) (and its subgroup \( K \)) acts by the law
\[ G \ni g : (G \times V) / K \to (gG \times V) / K. \]
However it differs from the action of the structure group \( K \) of \( P_\Sigma \) on it by the law
\[ K \ni g : (G \times V) / K \to (Gg^{-1} \times V) / K. \]
5 Dirac Fermion Fields

Spontaneous symmetry breaking in gravitation theory where Dirac fermion fields possess only Lorentz symmetries belongs to the type mentioned in previous Section.

By $X^4$ is further meant an oriented world manifold which satisfies the well-known global topological conditions in order that gravitational fields, space-time structure and spinor structure can exist. To summarize these conditions, we assume that $X^4$ is not compact and the linear frame bundle $LX$ is trivial \[19\].

We describe Dirac fermion fields as follows. Given a Minkowski space $M$ with the Minkowski metric $\eta$, let

$$A_M = \bigoplus_n M^n, \quad M^0 = \mathbb{R}, \quad M^{n>0} = \otimes^n M,$$

be the tensor algebra modelled on $M$. The complexified quotient of this algebra by the two-sided ideal generated by elements

$$e \otimes e' + e' \otimes e - 2\eta(e, e') \in A_M, \quad e \in M,$$

constitutes the complex Clifford algebra $C_{1,3}$. A spinor space $V$ is defined to be a linear space of some minimal left ideal of $C_{1,3}$ on which this algebra acts on the left. We then have the representation

$$\gamma : M \otimes V \to V,$$

of elements of the Minkowski space $M \subset C_{1,3}$ by $\gamma$-matrices on $V$:

$$\gamma(e^a \otimes v^A V) = \gamma^a_B v^B v^A,$$

where $\{e^0...e^3\}$ is a fixed basis for $M$, $v^A$ is a basis for $V$, and $\gamma^a$ are Dirac's matrices of a fixed form \[4\].

Let us consider the transformations preserving the representation (20). These are pairs $(l, l_s)$ of Lorentz transformations $l$ of the Minkowski space $M$ and invertible elements $l_s$ of $C_{1,3}$ such that

$$lM = l_s M l_s^{-1},$$

$$\gamma(lM \otimes l_s V) = l_s \gamma(M \otimes V).$$

Elements $l_s$ form the Clifford group whose action on $M$ however is not effective. We here restrict ourselves to its spinor subgroup

$$L_s = SL(2, \mathbb{C}), \quad L = L_s/\mathbb{Z}_2,$$

whose generators act on $V$ by the representation

$$I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b].$$
Let us consider a bundle of complex Clifford algebras $C_{3,1}$ over $X^4$. Its subbundles are both a spinor bundle $S_M \rightarrow X^4$ and the bundle $Y_M \rightarrow X^4$ of Minkowski spaces of generating elements of $C_{3,1}$. To describe Dirac fermion fields on a world manifold, one must require $Y_M$ be isomorphic to the cotangent bundle $T^*X$ of a world manifold $X^4$. It takes place only if the structure group of the principal linear frame bundle $LX$ is reducible to the Lorentz group $L$ and $LX$ contains a reduced $L$ subbundle $L^hX$ such that

$$Y_M = (L^hX \times M)/L = T^*X.$$ 

In this case, the spinor bundle $S_M$ is associated with the $L_s$ lift $P_h$ of $L^hX$:

$$r : P_h \rightarrow L^hX = P_h/\mathbb{Z}_2,$$

$$S_M = (P_h \times V)/L_s = S_h.$$ 

The geometric equivalence principle thereby is the necessary condition in order that Dirac fermion fields can be defined on a world manifold.

There is the above-mentioned 1:1 correspondence between the reduced subbubdles $L^hX$ of $LX$ and the tetrad gravitational fields $h$ identified with global sections of the bundle $\Sigma (1)$.

Given a tetrad field $h$, let $\Psi^h = \{z^h_\xi\}$ be an atlas of $LX$ which is extention of an atlas of $L^hX$, that is, the local sections $z^h_\xi$ take their values into $L^hX$. With respect to $\Psi^h$, the pseudo-Riemannian metric $g$ associated with a gravitational field $h$ comes to the Minkowski metric which exemplifies a Lorentz invariant defined in accordance with the geometric equivalence principle.

With respect to an atlas $\Psi^h$ and a holonomic atlas $\Psi^T = \{\psi^T_\xi\}$ of $LX$, a tetrad field $h$ can be represented by a family of $GL_4$-valued tetrad functions

$$h_\xi = \psi^T_\xi \circ z^h_\xi. \quad h = \pi_{PS} \circ z^h_\xi.$$ 

They are $GL_4$-valued functions of atlas transformations

$$dx^a = h^a_\xi(x)h^a$$ \hspace{1cm} (21)$$

between the bases $dx^a$ and $h^a$ for the cotangent spaces to $X^4$ which are associated with atlases $\Psi^T$ and $\Psi^h$ respectively.

Given a tetrad field $h$, one can define the representation

$$\gamma_h : T^*X \otimes S_h = (P_h \times (M \otimes V))/L_s \rightarrow (P_h \times \gamma(M \otimes V))/L_s = S_h$$ \hspace{1cm} (22)$$

of cotangent vectors to a world manifold $X^4$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$. With respect to an atlas $\{z_\xi\}$ of $P_h$ and the associated atlas $\{z^h_\xi = r \circ z_\xi\}$ of $LX$, the morphism (22) reads

$$\gamma_h(h^a \otimes y^A v_A(x)) = \gamma^A B y^B v_A(x).$$
where \( \{h^a\} \) and \( \{v_A(x)\} \) are the associated bases for fibres \( T^*_xX \) of \( T^*X \) and \( V_x \) of \( S_h \) respectively. As a shorthand, we can write
\[
\hat{h}^a = \gamma_h(h^a) = h^a,
\]
\[
\hat{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda(x)\gamma^a.
\]

We shall say that, given the representation \( (22) \), sections of the spinor bundle \( S_h \) describe fermion fields in the presence of the tetrad gravitational field \( h \).

Let \( A_h \) be a principal connection on \( S_h \) and
\[
D : J^1S_h \to T^*_xX \otimes \tilde{V}S_h,
\]
\[
D = (y^A - A^{ab}_\lambda(x)I_{ab}^A y^B)dx^\lambda \otimes \partial_A,
\]
the corresponding covariant differential (7). Given the representation \( (22) \), one can construct the Dirac operator
\[
D_h = \gamma_h \circ D : J^1S_h \to T^*_xX \otimes \tilde{V}S_h \to \tilde{V}S_h, \tag{23}
\]
\[
\tilde{y}^A \circ D_h = h^\lambda(x)\gamma^A_B(y^B - A^{ab}_\lambda I_{ab}^A y^B).
\]
We use the fact that the vertical tangent bundle \( \tilde{V}S_h \) admits the canonical splitting \( \tilde{V}S_h = S_h \times S_h \), and \( \gamma_h \) in the expression \( (23) \) is the pullback
\[
\gamma_h : T^*_xX \otimes \tilde{V}S_h \to \tilde{V}S_h,
\]
\[
\gamma_h(h^a \otimes \tilde{y}^A \partial_A) = \gamma^A_B \tilde{y}^B \partial_B,
\]
over \( S_h \) of the bundle morphism \( (22) \). On sections \( \phi_h \) of \( S_h \), we have the familiar expression
\[
D_h \circ \phi_h = \hat{dx}^\lambda \nabla_\lambda \phi_h = h^\lambda_a(x)\hat{h}^a \nabla_\lambda \phi_h = h^\lambda_a(x)\gamma^a \nabla_\lambda \phi_h
\]
for the Dirac operator in the presence of a tetrad gravitational field \( h \).

For different tetrad fields \( h \) and \( h' \), Dirac fermion fields are described by sections of spinor bundles \( S_h \) and \( S_{h'} \) associated with \( L \) lifts \( P_h \) and \( P_{h'} \) of different reduced \( L \) subbundles of \( LX \). Therefore, the representations \( \gamma_h \) and \( \gamma_{h'} \) \( (22) \) are not equivalent \[16, 19\].

For two arbitrary elements \( q \in P_h \) and \( q' \in P_{h'} \) over the same point \( x \in X \), there is an element \( g \in GL_4 \) so that
\[
 rq' = (r_g \circ r)q.
\]
Let \( T^*_xX \) be the cotangent space to \( X^4 \) at \( x \in X^4 \). Since
\[
T^*_xX = [rq]_M M = ([rq']_M \circ g^{-1})M,
\]
we can write
\( \gamma_h : T^*_x \otimes V_x = [rq]_M M \otimes [q]_V V \rightarrow ([q]_V \circ \gamma) (M \otimes V) \),
\( \gamma_{h'} : T^*_x \otimes V'_x = ([rq]_M \circ g) M \otimes [q']_V V \rightarrow ([q']_V \circ \gamma) (gM \otimes V) \).

If \( g \in GL_4 \setminus L \), there is no isomorphism \( l_V \) of the spinor space \( V \) such that
\[ \gamma(gM \otimes l_V V) = l_V \gamma(M \otimes V). \]

It follows that a Dirac fermion field must be regarded only in a pair with a certain tetrad gravitational field. There is the 1:1 correspondence between these pairs and sections of the composite bundle \( \Sigma \).

6 Composite Spinor Bundles

In gravitation theory, we have the composite manifold
\[ \pi_{\Sigma X} \circ \pi_{PS} : LX \rightarrow \Sigma \rightarrow X^4 \]  
(24)
where \( \Sigma \) is the bundle \( LX/L \rightarrow X^4 \) \([1]\) and \( LX_\Sigma = LX \rightarrow \Sigma \) is the \( L \) principal bundle.

Since the fibration \( GL_4 \rightarrow GL_4/L \) is trivial, the bundle \( LX_\Sigma \) is isomorphic to the pullback of some \( L \) principal bundle over \( X^4 \) by \( \pi_{\Sigma X} \). There exists the principal \( L \) lift \( P_\Sigma \) of \( LX_\Sigma \) such that
\[ P_\Sigma/L_s = \Sigma, \quad LX_\Sigma = rP_\Sigma = P_\Sigma/LZ_2. \]

In particular, there is imbedding of \( P_h \) onto the restriction of \( P_\Sigma \) onto \( h(X^4) \).

Let us consider the composite spinor bundle \( \Sigma \)
\[ S = \pi_{\Sigma X} \circ \pi_{PS} : (P_\Sigma \times V)/L_s \rightarrow \Sigma \rightarrow X^4 \]  
(25)
where \( S_\Sigma = S \rightarrow \Sigma \) is associated with the \( L_s \) principal bundle \( P_\Sigma \). It is readily observed that, given a global section \( h \) of \( \Sigma \rightarrow X^4 \), the restriction \( S \rightarrow \Sigma \) to \( h(X^4) \) is the spinor bundle \( S_h \) whose sections describe Dirac fermion fields in the presence of the tetrad field \( h \).

Let us provide the principal bundle \( LX \) with a holonomic atlas \( \{ \psi_T^T, U_\xi \} \) and the principal bundles \( P_\Sigma \) and \( LX_\Sigma \) with associated atlases \( \{ z^*_x, U_\epsilon \} \) and \( \{ z_x = r \circ z^*_x \} \). With respect to these atlases, the composite spinor bundle is endowed with the fibred coordinates \( (x^\lambda, \sigma^\mu_\alpha, y^A) \) where \( (x^\lambda, \sigma^\mu_\alpha) \) are fibred coordinates of the bundle \( \Sigma \rightarrow X \) such that \( \sigma^\mu_\alpha \) are the matrix components of the group element
\[ GL_4 \ni (\psi_T^T \circ z_\epsilon)(\sigma) : R^4 \rightarrow R^4, \quad \sigma \in U_\epsilon, \quad \pi_{\Sigma X}(\sigma) \in U_\xi. \]
Given a section $h$ of $\Sigma \to X^4$, we have
\[
\begin{align*}
  z_i^h(x) &= (z_i \circ h)(x), & h(x) &\in U_\epsilon, & x &\in U_\xi,
  
  (\sigma_\lambda^a \circ h)(x) &= h_\lambda^a(x),
\end{align*}
\]
where $h_\lambda^a(x)$ are tetrad functions (21).

The jet manifolds $J^1 \Sigma$, $J^1 S_\Sigma$ and $J^1 S$ are provided with the adapted coordinates
\[
\begin{align*}
  (x^\lambda, \sigma_\mu^a, \sigma_\mu^a_\lambda),
  
  (x^\lambda, \sigma_\mu^a, y^A, \tilde{y}_A^a, y^A_{\mu}),
  
  (x^\lambda, \sigma_\mu^a, y^A, \sigma_\mu^a_\lambda, y^A_\lambda).
\end{align*}
\]

Note that, for each section $h$ of $\Sigma$, the fibred jet manifold $J^1 S_h \to X^4$ is a fibred submanifold of $J^1 S$ given by the coordinate relations
\[
\begin{align*}
  \sigma_\mu^a &= h_\mu^a(x), & \sigma_\mu^a_\lambda &= \partial_\lambda h_\mu^a(x).
\end{align*}
\]

Let us consider the bundle of Minkowski spaces
\[
(LX \times M)/L \to \Sigma
\]
associated with the $L$ principal bundle $LX_\Sigma$. It is isomorphic to the pullback $\Sigma \times T^*X$ which we denote by the same symbol $T^*X$. Building on the morphism (20), one can define the bundle morphism
\[
\gamma_\Sigma : T^*X \otimes S_\Sigma = (P_\Sigma \times (M \otimes V))/L_s \to (P_\Sigma \times \gamma(M \otimes V))/L_s = S_\Sigma
\]
over $\Sigma$. In the coordinate form, we have
\[
\hat{dx}^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma_\lambda^a \gamma^a
\]
where $dx^\lambda$ is the basis for the fibre of $T^*X$ over $\sigma \in \Sigma$. Being restricted to the submanifold $h(X^4) \subset \Sigma$ for a section $h$ of $\Sigma$, the morphism (26) comes to the morphism $\gamma_h$ (22). Because of the canonical vertical splitting
\[
VS_\Sigma = S_\Sigma \times S_\Sigma,
\]
the morphism (26) yields the corresponding morphism
\[
\gamma_\Sigma : T^*X \otimes VS_\Sigma \to VS_\Sigma.
\]

We use this morphism in order to construct the total Dirac operator on sections of the composite spinor bundle (25). We are based on the following fact.
Let $Y \to \Sigma \to X$ be a composite manifold (11). Every connection (12) on the bundle $Y \to \Sigma$ defines the splitting

$$ VY = VY_\Sigma \oplus (Y \times V_\Sigma), $$

$$ \dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A^i_m \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A^i_m \partial_i), $$

of the vertical tangent bundle $VY$ to $Y \to X$. As a consequence, we have the following morphism over $Y$:

$$ \tilde{D} : J^1 Y \to T^* X \otimes VY_\Sigma, $$

$$ \tilde{D} = dx^\lambda \otimes (y^i_\lambda - \dot{A}^i_\lambda \sigma^m_\lambda) \partial_i. \tag{28} $$

Let

$$ \tilde{A} = dx^\lambda \otimes (\partial_\lambda + \tilde{A}^B_\lambda \partial_B) + d\sigma^a_\mu \otimes (\partial_\mu + A^B_a \partial_B) \tag{29} $$

be a connection on the bundle $S_\Sigma$. Then, the composition of morphisms (27) and (28) results in the following morphism over $S$:

$$ D = \gamma_\Sigma \circ \tilde{D} : J^1 S \to T^* X \otimes VS_\Sigma \to VS_\Sigma, \tag{30} $$

$$ \dot{y}^A \circ D = \sigma^a_\mu \gamma^{A}_B (y^B_\lambda - \dot{A}^B_\lambda - A^B_a \sigma^a_\mu). $$

One can treat this morphism as the total Dirac operator since, for each tetrad field $h$, the restriction of $D$ to $J^1 S_h \subset J^1 S$ comes to the Dirac operator $D_h$ (23) in the presence of a principal connection

$$ A_h = dx^\lambda \otimes [\partial_\lambda + (\dot{A}^B_\lambda + A^B_a \partial_\lambda h^a_\mu) \partial_B]. $$

The Dirac operator (30) thus characterizes the fermion-gravitation complex only in the presence of background gauge gravitational potentials $A_h$.

In the gauge gravitation theory, classical gravity is described by pairs $(h, A_h)$ of tetrad gravitational fields $h$ and gauge gravitational potentials $A_h$ identified with principal connections on the reduced $L$ subbundles $L^h X$ of $LX$. Every connection on $L^h X$ is extended to a Lorentz connection on $LX$ which however fails to be reducible to a principal connection on another reduced subbundle $L^{h'} X$ if $h \neq h'$. It follows that gauge gravitational potentials also must be regarded in pairs with a certain tetrad gravitational field $h$. Following Section 4, one can describe these pairs $(h, A_h)$ by sections of the bundle (13) where $P = LX$ and $K = L$. The corresponding configuration space is the jet manifold $J^1 C_L$.

The bundle $C_L = J^1 (LX)$ is endowed with the local fibre coordinates

$$(x^\mu, \sigma^a_\mu, k^{ab}_\lambda = -k^{ba}_\lambda, \sigma^a_\lambda)$$

where $(x^\mu, \sigma^a_\mu, \sigma^a_\lambda)$ are coordinates of the jet bundle $J^1 \Sigma$. Given a section $s$ of $C_L$, we have familiar tetrad functions and Lorentz gauge potentials

$$(\sigma^a_\mu \circ s)(x) = h^a_\mu(x), \quad (k^{ab}_\lambda \circ s)(x) = A^{ab}_\lambda(x)$$
respectively. The jet manifold $J^1C_L$ of $C_L$ is provided with the adapted coordinates

$$(x^\mu, \sigma^\mu_a, k^{ab}_\lambda = -k^{ba}_\lambda, \sigma^{\mu\lambda}_a = \sigma^{\mu\lambda}_a, k_{\mu\lambda}, \sigma^{\mu}_{a\lambda\nu}).$$

The total configuration space of the fermion-gravitation complex is the product

$$J^1C_L \times J^1S.$$ 

Part II of the work will be devoted to dynamics of the fermion-gravitation complex on this configuration space and the associated phase space.

References

1. K. Bugajska, *Journal of Mathematical Physics*, 27, 143 (1986).
2. J. Cariñena, M. Crampin and L. Ibort, *Differential Geometry and its Application*, 1, 345 (1991).
3. J. Crowford, *Journal of Mathematical Physics*, 32, 576 (1991).
4. G. Giachetta and L. Mangiarotti, *International Journal of Theoretical Physics*, 29, 789 (1990).
5. M. Gotay: in *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, ed. M.Francaviglia (Elseiver Science Publishers B.V., 1991) p. 203.
6. C. Günther, *Journal of Differential Geometry*, 25, 23 (1987).
7. C. Isham, A. Salam and J. Strathdee, *Annals of Physics*, 62, 98 (1971)
8. D. Ivanenko and G. Sardanashvily, *Letters al Nuovo Cimento*, 31, 220 (1981)
9. D. Ivanenko and G. Sardanashvily, *Physics Reports*, 94, 1 (1983).
10. B. Kupershmidt, *Geometry of Jet Bundles and the Structure of Lagrangian and Hamiltonian Formalisms*, Lect. Notes in Math., 775, 162 (1980).
11. L. Mangiarotti and M. Modugno, in *Geometry and Physics*, ed. M.Modugno (Pitagora Editrice, Bologna, 1982) p. 135.
12. J. Ne’eman and Dj. Šijački, *Annals of Physics*, 120, 292 (1979).
13. V. Ogievetsky and I. Polubarinov, *Soviet Physics JETP*, 21, 1093 (1965).
14. R. Percacci, *Geometry of Nonlinear Field Theories* (World Scientific, Singapore, 1986).

15. G. Sardanashvily, *Physics Letters*, **75A**, 257 (1980).

16. G. Sardanashvily, *International Journal of Theoretical Physics*, **30**, 721 (1991).

17. G. Sardanashvily, *Il Nuovo Cimento*, **104A**, 105 (1991).

18. G. Sardanashvily, *Journal of Mathematical Physics* **33**, 1546 (1992).

19. G. Sardanashvily and O. Zakharov, *Gauge Gravitation Theory* (World Scientific, Singapore, 1992).

20. G. Sardanashvily and O. Zakharov, *Differential Geometry and its Applications*, **3**, 245 (1993).

21. G. Sardanashvily, *Gauge Theory in Jet Manifolds* (Hadronic Press, Palm Harbor, 1993).

22. G. Sardanashvily, Multimomentum Hamiltonian Formalism in Field Theory, LaTeX preprint: [hep-th/9403172](http://arxiv.org/abs/hep-th/9403172).

23. D. Saunders, *The Geometry of Jet Bundles*, L.M.S. Lect. Notes Ser., **142** (Cambridge Univ. Press, Cambridge, 1989).

24. R. Sulanke and P. Wintgen, *Differentialgeometrie und Faserbündel* (Veb Deutsher Verlag der Wissenschaften, Berlin, 1972).

25. A. Trautman, *Czechoslovac Journal of Physics*, **B29**, 107 (1979).