How Will the Presence of Autonomous Vehicles Affect the Equilibrium State of Traffic Networks?

Negar Mehr and Roberto Horowitz*

January 17, 2019

Abstract

It is known that connected and autonomous vehicles are capable of maintaining shorter headways and distances when they form platoons of vehicles. Thus, such technologies can result in increases in the capacities of traffic networks. Consequently, it is envisioned that their deployment will boost the network mobility. In this paper, we verify the validity of this impact under selfish routing behavior of drivers in traffic networks with mixed autonomy, i.e. traffic networks with both regular and autonomous vehicles. We consider a nonatomic routing game on a network with inelastic (fixed) demands for the set of network O/D pairs, and study how replacing a fraction of regular vehicles by autonomous vehicles will affect the mobility of the network. Using the well known US bureau of public roads (BPR) traffic delay models, we show that the resulting Wardrop equilibrium is not necessarily unique even in its weak sense for networks with mixed autonomy. We state the conditions under which the total network delay is guaranteed not to increase as a result of autonomy increase. However, we show that when these conditions do not hold, counter intuitive behaviors may occur: the total delay can grow by increasing the network autonomy. In particular, we prove that for networks with a single O/D pair, if the road degrees of asymmetry are homogeneous, the total delay is 1) unique, and 2) a nonincreasing continuous function of network autonomy fraction. We show that for heterogeneous degrees of asymmetry, the total delay is not unique, and it can further grow with autonomy increase. We demonstrate that similar behaviors may be observed in networks with multiple O/D pairs. We further bound such performance degradations due to the introduction of autonomy in homogeneous networks.

Keywords: autonomous vehicles, Wardrop equilibrium, game theory, Braess’s paradox, routing games, traffic networks.

1 Introduction

Connected and autonomous vehicles technology have attracted significant attention as a result of their potentials for increasing vehicular safety and drivers’ comfort. Connected technologies can be used to inform drivers about the existing hazards through vehicle to vehicle (V2V) or vehicle to infrastructure (V2I) communication. Aligned with these safety considerations, automobile companies have started to equip vehicles with autonomous capabilities. In fact, some of these capabilities, such as driver assistive technologies and adaptive cruise control (ACC) have already been deployed in vehicles.

The impact of these technologies is not limited to vehicles safety. Connected and autonomous vehicles technology can facilitate vehicle platooning. Vehicle platoons are groups of more than one

*N. Mehr and R. Horowitz are with the Department of Mechanical Engineering, University of California, Berkeley, Berkeley, CA, 94720 USA e-mails: negar.mehr@berkeley.edu, horowitz@berkeley.edu
vehicle, capable of maintaining shorter headways; thus, platooning can lead to increases in the capacities of network links [LPTV17]. Such increases can be up to three-fold [LPTV17] if all the vehicles are autonomous and connected. In addition to mobility benefits, platooning can have sustainability benefits, it can also reduce energy consumption for heavy duty vehicles [AAGJ10, LMJ13, ABT+15].

The mobility benefits of platooning and autonomous capabilities of vehicles are not limited to increasing network capacities. There has been a focus on how to utilize vehicle autonomy and connectedness to remove signal lights from intersections and coordinate conflicting movements such that the network throughput is improved [ZMC16, TC15, MK14, FV18]. However, in order for such approaches to be implemented, all vehicles in the network need to have autonomous capabilities. To reach the point where all vehicles are autonomous, transportation networks need to face a transient era, when both regular and autonomous vehicles coexist in the networks. Therefore, it is crucial to study networks with mixed autonomy.

In [AFKV16], the performance of traffic networks with mixed autonomy was studied via simulations. Moreover, it was shown in multiple works that in networks with mixed autonomy, autonomous vehicles can be utilized to stabilize the low-level dynamics of traffic networks and damp congestion shockwaves [WKVB17, DR99, YH06, PvA10, SCDM18]. In [MLH18b, MLH18a] altruistic lane choice of autonomous vehicles was studied. In [LCP17a], the capacity of network links was modeled in a traffic setting with mixed autonomy. This modeling framework was further used in [LCP17b] to calculate the price of anarchy of traffic networks with mixed autonomy, where the price of anarchy is an indicator of how far the equilibrium of networks with mixed autonomy is from their social optimum that could have been achieved if a social planner had routed all the vehicles. In [LCPS18], it was shown that local actions of the autonomous vehicles on the road can lead to optimal vehicle orderings for the global network properties such as link capacities.

It is well known that due to the selfish route choice behavior of drivers, traffic networks normally operate in an equilibrium state, where no vehicle can decrease its trip time by unilaterally changing its route [Smi79]. In this paper, we wish to study how the introduction of autonomous vehicles in the network will affect the equilibrium state of traffic networks compared to the case when all vehicles are nonautonomous. We extend our initial results presented in [MH18]. In particular, given a fixed demand of vehicles, we study how replacing a fraction of regular vehicles by autonomous vehicles will affect the equilibrium state of traffic networks. We study the system behavior when both regular and autonomous vehicles select their routes selfishly to investigate the necessity of centrally enforcing autonomous vehicles routing by a network manager. We state the conditions under which increasing network autonomy fraction is guaranteed to reduce the overall network delay. Moreover, we show that when these conditions do not hold, counter intuitive and undesirable behaviors might occur, such as the case when increasing the portion of autonomous vehicles in the network can increase the overall network delay. Such behaviors are similar to Braess’ paradox, where the construction of a new road or expanding link capacities may increase total network delay.

We model the network in a macroscopic framework where vehicle route choices are taken into account. We model the selfish route choice behavior of the drivers as a nonatomic routing game [Rou02] where drivers choose their routes selfishly until a Wardrop Equilibrium is achieved [War52]. We represent a traffic network by a directed graph with a certain set of origin destination (O/D) pairs. For each O/D pair, we consider two classes of vehicles, regular and autonomous. For a given fixed demand profile along O/D pairs, we study how increasing the autonomy fraction of O/D pairs will affect the total delay of the network at equilibrium.

We first show that the equilibrium may not be unique even in the weak sense of total link utilization. Then, we study networks with a single O/D pair and prove that if the degrees of road capacity asymmetry are homogeneous in the network, the social or total delay of the network is unique, and further it is a monotone nonincreasing function of the network autonomy ratio. However, in networks
with heterogeneous degrees of road asymmetry, we first show that the social delay is not unique. Then, we demonstrate that, surprisingly, increasing the autonomy ratio of the network may lead to an increase in the overall network delay. This is a counter intuitive behavior as we might expect that having more autonomous vehicles in the network will always be beneficial in terms of total network delay. For the networks with multiple O/D pairs, we show that similar complicated behaviors may occur, namely increasing autonomy fraction of an O/D pair might worsen the social delay of the network. Our work in fact shows that traffic paradoxes similar to the well known Braess’s Paradox [Bra68] can occur due to capacity increases provided by autonomous vehicles. We further bound such performance degradations that can arise from the presence of autonomous vehicles.

The organization of this paper is as follows. In Section 2, we describe our notation and model. We review the prior relevant results in Section 3. Then, in Section 4, we study the uniqueness of equilibrium in our routing setting. Next, in Section 5, we analyze mixed autonomy networks with a single O/D pair in Section 5. Subsequently, we study mixed--autonomy networks with multiple O/D pairs in Section 6. Finally, we conclude the paper and provide relevant future directions in Section 7.

## 2 Nonatomic Selfish Routing

We model a traffic network by a directed graph $G = (N, L, W)$, where $N$ and $L$ are respectively the set of nodes and links in the network. Each link $l \in L$ in the network is a pair of distinct nodes $(v, w)$ and represents a directed edge from $v$ towards $w$. We assume that each link joins two distinct nodes; thus, no self loops are allowed. Define $W = \{(a_1, d_1), (a_2, d_2), \ldots, (a_k, d_k)\}$ to be the set of origin destination (O/D) vertex pairs of the network. A node $n \in N$ can appear in multiple O/D pairs. In a nonatomic selfish routing game, if each O/D pair has a fixed given nonzero demand, then it is called a nonatomic selfish routing game with inelastic demands. Each O/D pair consists of infinitesimally small agents where every agent decides on each path such that their own delay is minimized. The delay of each path depends on how network paths are shared among different O/D pairs. For each O/D pair $w = (a_i, d_i), 1 \leq i \leq k$, we let $P_w$ denote the set of all possible network paths from $a_i$ to $d_i$. We assume that the network topology is such that for each O/D pair $w \in W$, there exists at least one path from its origin to its destination, i.e. $P_w \neq \emptyset$. We further let $P = \cup_{w \in W} P_w$ denote the set of all network paths.

For an O/D pair $w \in W$, let $r_w$ be the given fixed demand of vehicles associated with $w$. Furthermore, for a path $p \in P_w$, let $f_p$ be the flow of the O/D pair $w$ along path $p$. Note that each path connects exactly one origin to one and only one destination; thereby, once a path is fixed, its origin and destination are uniquely determined. Consequently, there is no need to explicitly include path O/D pairs in the notation used for $f_p$. It is important to note that in our setting, each O/D pair $w$ has two classes of vehicles: autonomous and regular. Consequently, for each $w \in W$, we define $\alpha_w$ to be the fraction of vehicles in $r_w$ that are autonomous. We let $r = (r_w : w \in W)$ and $\alpha = (\alpha_w : w \in W)$ be the vectors of network demand and autonomy fraction respectively. Also, for each path $p \in P_w$, we use $f^r_p$ and $f^a_p$ to respectively denote the flow of regular and autonomous vehicles along path $p$. Note that for each path $p \in P$, we have $f_p = f^r_p + f^a_p$. Moreover, for each O/D pair $w \in W$, due to flow conservation, we must have $\sum_{p \in P_w} f^r_p = r_w(1 - \alpha_w)$, and $\sum_{p \in P_w} f^a_p = r_w\alpha_w$. The network flow vector $f$ is a nonnegative vector of regular and autonomous flows along network paths, i.e. $f = (f^r_p, f^a_p : p \in P)$. A flow vector $f$ is called feasible for a given network $G$, if for each $w \in W$,

$$\sum_{p \in P_w} f^r_p = (1 - \alpha_w)r_w, \quad \text{and} \quad \sum_{p \in P_w} f^a_p = \alpha_w r_w, \quad (1a)$$

$$f^r_p \geq 0, \text{ and } f^a_p \geq 0, \forall p \in P_w. \quad (1b)$$
For each link $l \in L$, $f_l$ is the total flow of vehicles in link $l$, i.e. $f_l = \sum_{p \in \mathcal{P}: l \in p} f_p$. Since we need to decompose the total link flow into regular and autonomous vehicles, we let $f_l^r$ and $f_l^a$ be the total flow of regular and autonomous vehicles along link $l$ respectively. In fact, $f_l^r$ and $f_l^a$ are the summation of the flow of regular and autonomous vehicles on all routes containing link $l$,

$$f_l^r = \sum_{p \in \mathcal{P}: l \in p} f_p^r, \quad \text{and} \quad f_l^a = \sum_{p \in \mathcal{P}: l \in p} f_p^a.$$ 

Note that if all vehicles are regular for an O/D pair $w \in W$, i.e. $\alpha_w = 0$, then, we only have a single class of regular vehicles along that O/D pair, and for each path $p \in \mathcal{P}_w$, $f_p = f_p^r$. If for all network O/D pairs $w \in W$, the autonomy fraction $\alpha_w = 0$; then, the same argument holds for link flows, $f_l = f_l^r$ for all links $l \in L$. In fact, if all vehicles are regular, our routing game reduces to a single class game

$$(\forall w \in W, \alpha_w = 0) \iff (\forall p \in \mathcal{P}, f_p = f_p^r). \quad (2)$$

In order to be able to model the incurred delays when vehicles are routed throughout the network, it is assumed that each link $l \in L$ has a delay per unit of flow function $e_l : \mathbb{R}^2 \to \mathbb{R}$. We assume that the delay per unit of flow for each path $p \in \mathcal{P}$ is obtained by the summation of the link delays over the links that form $p$,

$$e_p(f) = \sum_{l \in L : l \in p} e_l(f_l^r, f_l^a). \quad (3)$$

Equation (3) implies that the delay of each path $p \in \mathcal{P}$ depends not only on the flows of regular and autonomous vehicles along path $p$, but also on the flows along other paths. The overall network delay or social delay is given by

$$J(f) = \sum_{p \in \mathcal{P}} f_p e_p(f). \quad (4)$$

### 2.1 Wardrop Equilibrium

It is well known in the transportation literature that if there are many noncooperative agents, namely, flows that behave selfishly [Rou06], a network is at an equilibrium if the well known Wardrop conditions hold [War52]. The Wardrop conditions state that at equilibrium, no user has any incentive for unilaterally changing its path. This implies that for an equilibrium flow vector $f$, if there exists a path $p \in \mathcal{P}_w$ such that either $f_p^r \neq 0$ or $f_p^a \neq 0$, we must have that $e_p(f) \leq e_p'(f)$, for all paths $p' \in \mathcal{P}_w$.

**Definition 1.** Given a network $G = (N, L, W)$, a flow vector $f$ is a Wardrop equilibrium if and only if for every O/D pair $w \in W$ and every $p, p' \in \mathcal{P}_w$

$$f_p^r (e_p(f) - e_{p'}(f)) \leq 0, \quad (5a)$$

$$f_p^a (e_p(f) - e_{p'}(f)) \leq 0. \quad (5b)$$

Note that an implication of the above definition is that for each O/D pair $w \in W$, and any two paths $p, p' \in \mathcal{P}_w$ such that $f_p \neq 0$ and $f_{p'} \neq 0$, we must have that $e_p(f) = e_{p'}(f)$.

**Definition 2.** Given an equilibrium flow vector $f$ for the network $G = (N, L, W)$, we define the delay of travel for each O/D pair $w \in W$ to be

$$e_w(f) := \min_{p \in \mathcal{P}_w} e_p(f). \quad (6)$$
Motivated by the above discussion, \( e_w(f) \) is precisely the delay across all paths \( p \in \mathcal{P}_w \) which have a nonzero flow. Moreover, the equilibrium condition implies that for a path \( p \in \mathcal{P}_w \) with zero flow, we have \( e_p(f) \geq e_w(f) \).

It is worth mentioning that when there are no autonomous vehicles, i.e. for all \( w \in W, \alpha_w = 0 \), since \( f_p' = f_p \) for all \( p \in \mathcal{P} \), Conditions (5) reduce to

\[
f_p(e_p(f) - e_p(f)) \leq 0, \quad \forall w \in W, \forall p, p' \in \mathcal{P}_w.
\]

2.2 Delay Characterization

We first specify the structure of our delay functions. If there is only a single class of regular vehicles in the network, the US Bureau of Public Roads (BPR) [Man64] suggests the following form of delay functions.

Assumption 1. When network links are shared by only regular vehicles, the link delay functions \( e_l : \mathbb{R} \rightarrow \mathbb{R} \) are of the following form

\[
e_l(f_l) = a_l \left( 1 + \gamma_l \left( \frac{f_l}{C_l} \right)^{\beta_l} \right),
\]

where \( C_l \) is the capacity of link \( l \), and \( a_l, \gamma_l, \) and \( \beta_l \) are nonnegative link parameters.

In practice, \( a_l \) is the free flow travel time on \( l \), \( \gamma_l \) is normally 0.15, and \( \beta_l \) is a positive integer ranging from 1 to 4. In order to characterize the delay functions in networks with mixed autonomy, where we have two classes of vehicles, we first need to model the impact of autonomous vehicles on link capacities. In each network link \( l \in L \), the link capacity \( C_l \) restricts the maximum possible flow of vehicles. It was shown in [LCP17a] that in networks with mixed autonomy, \( C_l \) depends on the autonomy ratio of link \( l \) defined as \( \alpha_l := \frac{f_{a_l}^l}{f_{a_l}^l + f_{r_l}^l} \). We use \( C_l(\alpha_l) \) to emphasize this dependence. Let \( m_l \) and \( M_l \) be the capacity of link \( l \) when all vehicles are regular and autonomous respectively. Since autonomous vehicles are capable of maintaining shorter headways, it is normally the case that \( \frac{m_l}{M_l} \leq 1 \). When the two classes of regular and autonomous vehicles are present in the network, using the results in [LCP17a], we have

\[
C_l(\alpha_l) = \frac{m_l M_l}{\alpha_l m_l + (1 - \alpha_l) M_l}.
\]

We adopt this model throughout this paper to investigate the mobility impact of autonomous vehicles on the network. Since for each link \( l \in L \), \( \alpha_l = \frac{f_{a_l}^l}{f_{a_l}^l + f_{r_l}^l} \) and \( f_l = f_{a_l}^l + f_{r_l}^l \), using (9), for networks with mixed autonomy, the delay function (8) can be modified as:

\[
e_l(f_{a_l}^l, f_{r_l}^l) = a_l \left( 1 + \gamma_l \left( \frac{f_{a_l}^l + f_{r_l}^l}{m_l M_l (f_{a_l}^l + f_{r_l}^l)} \right)^{\beta_l} \right).
\]

\[
e_l(f_{a_l}^l, f_{r_l}^l) = a_l \left( 1 + \gamma_l \left( \frac{f_{a_l}^l + f_{r_l}^l}{M_l m_l} \right)^{\beta_l} \right).
\]

Note that when only regular vehicles are present in the network, for each link \( l \in L \) since \( f_l = f_{a_l}^l \), the link delay function reverts to
3 Prior Work

3.1 Existence of Equilibrium

We state the following proposition from [BK79] which studies the conditions under which a Wardrop Equilibrium exists for a multiclass traffic network.

**Proposition 1.** Given a network \( G = (N, L, W) \), if the link delay functions are continuous and monotone in the link flow of each class; then, there exists at least one Wardrop equilibrium.

**Remark 1.** Using (11), since our assumed delay functions are nonnegative, continuous, and monotone in the flow of each class, Proposition 1 implies that there always exists at least one Wardrop equilibrium for a routing game with mixed autonomy.

3.2 Uniqueness of Equilibrium

In this part, we review the known results regarding the uniqueness of the Wardrop Equilibrium. When multiple classes of vehicles are present in the network, the uniqueness of the equilibrium flow vector does not hold. However, uniqueness in a weak sense is known to hold from [ABEA+06].

**Proposition 2.** For a general topology network \( G \) with multiple classes of vehicles on each O/D pair, if the delay functions are of the form (8), and the link capacities \( C_l \) are fixed and the same for all vehicle classes, for a given demand vector \( r \), we have

1. The equilibrium is unique in a weak sense, i.e. for each link \( l \), the total flow \( f_l \) for all Wardrop equilibrium flow vectors \( f \) is unique.

2. For each O/D pair \( w \in W \), the delay of travel \( e_w(f) \) is unique for all Wardrop equilibrium flow vectors \( f \). Thus, the delay of travel for each O/D pair in equilibrium, i.e. \( e_w(f) \), only depends on the network demand vector \( r \). Hence, we may unambiguously define \( e_w(r) \) to denote this unique value.

**Remark 2.** Note that a routing game that has only a single class of vehicles can be viewed as an instance of the games described in Proposition 2. Therefore, uniqueness in the weak sense applies to games with a single class of vehicles too.
3.3 Monotonicity of Social Delay

As we discussed above, in general, the equilibrium is not unique. However, if the conditions of Proposition 2 hold for a network, the social delay and the delay of travel for each O/D pair are unique. For a single class routing game on \( G = (N, L, W) \), recall the following from [Hal78].

**Proposition 3.** Consider a network \( G = (N, L, W) \), where only one class of vehicles exists for each O/D pair \( w \in W \). Assume that for each link \( l \in L \), \( e_l(\cdot) \) is continuous, positive valued, and monotonically increasing. Then, for each \( w \in W \), the delay of travel \( e_w(r) \) is a continuous function of the demand vector \( r \). Furthermore, \( e_w(\cdot) \) is nonincreasing in \( r_w \) when all other demands \( r_{w'}, w' \neq w \), are fixed.

4 Uniqueness in the Mixed-Autonomy Setting

Now we study equilibrium uniqueness in our setting. Using Remark 1, we know that there exists at least one equilibrium. However, since in our setting, for each link \( l \), \( C_l \) depends on the autonomy ratio \( \alpha_l \), Proposition 2 does not apply. Indeed, we demonstrate through an example that the equilibrium is not unique even in the weak sense introduced in Proposition 2.

**Example 1.** Consider the network of Figure 1. Let \( p_1 \) and \( p_2 \) be the ABD and ACD paths respectively. For each link \( l = 1, \cdots, 4 \), let the link parameters be \( \beta_l = 1, a_l = 1, m_l = 1 \), and \( M_l = 2 \). Thus, for each link \( l \in L \), the link delay function is \( e_l = a_l + f_{r_l} + f_{r_l}^2 \). Assume that the demand from node A to D is \( r = 2 \), and \( \alpha = 0.5 \). The example is simple enough so that we can compute the equilibrium flows manually. Let \( f_{r_1} \) and \( f_{r_2} \) be the regular and autonomous vehicles flows along \( p_1 \), and \( f_{a_1} \) and \( f_{a_2} \) be the regular and autonomous flows along \( p_2 \). At equilibrium, using the symmetry of the network, we must have

\[
2 + 2f_{r_1} + f_{r_2}^2 = 2 + 2f_{a_2} + f_{a_2}^2 \\
f_{r_1} + f_{a_1} = 1 \\
f_{r_2} + f_{a_2} = 1 \\
f_{r_1}, f_{r_2}, f_{a_1}, f_{a_2} \geq 0.
\]

Clearly, there is no unique solution to the above set of equations. Moreover, among the set of all possible equilibrium flow vectors, for each link, the maximum link flow at equilibrium is 1.25, whereas the minimum link flow is 0.75 at equilibrium. This implies that equilibrium uniqueness does not hold even in the weak sense for traffic networks with mixed autonomy.

5 Networks with a Single O/D Pair

In this section, we study two terminal networks which have a single O/D pair in the presence of autonomy. For such networks, since there is only one O/D pair, all paths originate from a common source \( o \) and end in a common destination \( d \). Since \( W \) is singleton, we omit the subscript \( w \) from \( r_w, e_w \) and \( \alpha_w \) throughout this section. Note that when the network has a single O/D pair, \( r \) and \( \alpha \) are scalars.

Having observed that in the mixed-autonomy setting, the equilibrium is not unique even in the weak sense, it is important to study if the social delay is unique for all network equilibrium flow vectors. To this end, in the following, we study the properties of the social delay including its uniqueness. To this end, we need to define the notion of road degree of capacity asymmetry introduced in [LCP17b].
Given a network $G = (N, L, W)$, for each link $l \in L$, we define $\mu_l := m_l/M_l$ to be the degree of capacity asymmetry of link $l$. Note that since we assumed that autonomous vehicles headway is less than or equal to that of regular vehicles, for each link $l \in L$, $\mu_l \leq 1$. In the sequel, we consider two scenarios for investigating the properties of social delay:

1. Homogeneous degrees of road capacity asymmetry, where $\mu_l$ is the same for all links, i.e. $\mu_l = \mu$, for all links $l \in L$, where $\mu$ is the common value of capacity asymmetry.

2. Heterogeneous degrees of capacity asymmetry, where $\mu_l$ varies on different links.

### 5.1 Homogeneous Degrees of Capacity Asymmetry

In this case, we can establish the uniqueness of the social delay, and characterize the relationship between social delay and network autonomy ratio.

**Theorem 1.** Given a network $G = (N, L, W)$ with a single O/D pair and a homogeneous degree of capacity asymmetry $\mu$, for any demand vector $r > 0$, we have:

1. For a fixed autonomy ratio $0 \leq \alpha \leq 1$, the social delay $J(f)$ is unique for all Wardrop equilibrium flow vectors $f$.

2. If for each $0 \leq \alpha \leq 1$, we denote the common value of social delay in the above by $J(\alpha)$, then $J(\cdot)$ is continuous and nonincreasing.
Proof. Fix \( r > 0 \) and \( 0 \leq \alpha \leq 1 \). Recalling Remark 1, we know that a Wardrop equilibrium exists. Let \( f = (f^a_p, f^w_p : p \in \mathcal{P}) \) be such an equilibrium flow vector where \( f^a_p = f^a_p + f^w_p \) for each path \( p \) in \( \mathcal{P} \). Define \( e_{\min}(f) := \min_{p \in \mathcal{P}} e_p(f) \). Since the network has only one O/D pair, and the delay associated with all paths with nonzero flows are the same, denoting this uniform path delay by \( e_{\min}(f) \), we realize that the social delay is given by \( J(f) = r e_{\min}(f) \). For each path \( p \in \mathcal{P} \), define the fictitious single-class regular flow \( \tilde{f}_p := f^a_p + \mu f^w_p \). We claim that the flow vector \( \tilde{f} = (\tilde{f}_p : p \in \mathcal{P}) \) is a Wardrop equilibrium for a routing game on \( G \) with a single class of regular vehicles and a total demand of \( \tilde{r} = r(1-\alpha) + r\alpha \mu \) with the delay function \((\tilde{e}_l : l \in L)\) defined as

\[
\tilde{e}_l(\tilde{f}_l) = a_l \left( 1 + \gamma_l \left( \frac{\tilde{f}_l}{\mu_l} \right)^{\beta_l} \right).
\]

To see this, for each \( p \in \mathcal{P} \), we show that relations (7) hold. Fix \( p, p' \in \mathcal{P} \) and note that since \( f \) was a Wardrop equilibrium in the original setting, we have \( f^a_p (e_p(f) - e_{p'}(f)) \leq 0 \), and \( f^w_p (e_p(f) - e_{p'}(f)) \leq 0 \). Multiplying the latter by the positive constant \( \mu \) and adding the two inequalities, we have

\[
\tilde{f}_p (e_p(f) - e_{p'}(f)) \leq 0, \quad \forall p, p' \in \mathcal{P}. \tag{13}
\]

Now, we claim that for all \( p \in \mathcal{P} \), we have \( e_p(f) = \tilde{e}_p(\tilde{f}) \). Note that for each link \( l \in L \), we have \( \tilde{f}_l = f^a_l + \mu f^w_l \). Using the fact that \( \mu = m_l / M_l \) for all \( l \in L \), we get

\[
\tilde{e}_p(\tilde{f}) = \sum_{l \in p} a_l \left( 1 + \gamma_l \left( \frac{f^a_l + \mu f^w_l}{m_l} \right)^{\beta_l} \right) = \sum_{l \in p} a_l \left( 1 + \gamma_l \left( \frac{f^a_l}{m_l} + \frac{f^w_l}{M_l} \right)^{\beta_l} \right) = e_p(f). \tag{14}
\]

Substituting into (13), we realize that

\[
\tilde{f}_p (\tilde{e}_p(\tilde{f}) - \tilde{e}_{p'}(\tilde{f})) \leq 0, \quad \forall p, p' \in \mathcal{P}, \tag{15}
\]

which means that \( \tilde{f} \) is an equilibrium flow vector. Clearly, the total demand of this new routing game is \( \tilde{r} = \sum_{p \in \mathcal{P}} \tilde{f}_p = \sum_{p \in \mathcal{P}} f^a_p + \mu f^w_p = (1-\alpha)\tilde{r} + \mu \alpha \tilde{r} \). Moreover, define \( e_{\min}(\tilde{f}) \) to be the minimum of \( \tilde{e}_p(\tilde{f}) \) among \( p \in \mathcal{P} \). Since \( w \) is the single O/D pair of the network, \( e_{\min}(\tilde{f}) \) is indeed equal to \( \tilde{e}_w(\tilde{f}) \), the travel delay of the single O/D pair of the network associated with \( \tilde{f} \). Note that Proposition 2 implies that \( \tilde{e}_{\min}(\tilde{f}) \) is a function of \( \tilde{r} \) only. On the other hand, (14) implies that \( \tilde{e}_{\min}(\tilde{f}) = e_{\min}(f) \). Putting these together, we realize that

\[
J(f) = re_{\min}(f) = r\tilde{e}_{\min}(\tilde{f}) = r\tilde{e}_w(\tilde{r}).
\]

Note that the right hand side of the above identity does not depend on \( f \), which establishes the proof of the first part. In fact, this shows that

\[
J(\alpha) = r\tilde{e}_w(r(1-\alpha) + \alpha \mu\tilde{r}).
\]

From Proposition 3, we know that \( \tilde{e}_w(.) \) is continuous and nonincreasing. Also, since \( \mu \leq 1 \), the map \( r \mapsto r(1-\alpha) + \alpha \mu\tilde{r} \) is continuous and nonincreasing. This completes the proof of the second part. \( \square \)
5.2 Heterogeneous Degrees of Capacity Asymmetry

Now, we allow $\mu_l$ to vary among the network links. We show that this makes the behavior of the system more complex. First, we show via the following example that the social delay is not necessarily unique in this case.

**Example 2.** Consider the network shown in Figure 2. Assume that $\gamma_l = 1, \beta_l = 1$, for $l = 1, 2, \cdots, 5$. Let the other link parameters be the following: $\{a_1 = 1, m_1 = 1, M_1 = 1\}$, $\{a_2 = 2, m_2 = 1, M_2 = 3\}$, $\{a_3 = 1, m_3 = 1, M_3 = 2\}$, $\{a_4 = 1, m_4 = 1, M_4 = 4\}$, and $\{a_5 = 3, m_5 = 1, M_5 = 3\}$. Moreover, let the total flow from origin A to destination D be 2. Now, if we compute the social delay for this network for any $\alpha > 0$ at the different equilibria of the system, we observe that the social delay is not unique. In particular, Figure 3 shows the plots of the maximum and minimum social delay of the system at equilibrium for every value of $\alpha$. As Figure 3 shows, as soon as $\alpha$ starts to increase from 0, uniqueness of social delay is lost. Once, $\alpha = 1$, the uniqueness of social delay is again preserved. This behavior implies that the change in the social delay due to increasing the autonomy ratio of the network is dependent on which equilibrium the system will be at.

Now, we study the effect of increasing network autonomy on the social delay. In the previous example, both the maximum and minimum social delays decreased as a function of $\alpha$. But, is this necessarily the case? We use the following examples to demonstrate that it may not be true in general, as increasing network autonomy may increase social delay in some networks.

**Example 3.** Consider the network of Figure 2. Let $\gamma_l = 1$ and $\beta_l = 1$ for all links. Select the other network parameters to be the following, $\{a_1 = 0, m_1 = 0.1, M_1 = 0.1\}$, $\{a_2 = 50, m_2 = 1, M_2 = 1\}$, $\{a_3 = 50, m_3 = 1, M_3 = 1\}$, $\{a_4 = 0, m_4 = 0.1, M_4 = 0.1\}$, $\{a_5 = 10, m_5 = 0.5, M_5 = 1\}$. Let the total O/D demand be $r = 6$. In the absence of autonomy ($\alpha = 0$), the social delay is $J = 504.3$. However, if we increase the autonomy ratio to $\alpha = \frac{1}{10}$, $J = 518.6$. Clearly, in this case, the social delay increases when the network autonomy ratio $\alpha$ is increased. Note that since $\mu_l = 1$ for $l = 1, 2, 3, 4$ and $\mu_5 = 0.5 < 1$, this can be viewed as an instance of the classical Braess’s Paradox [Bra68], where an increase in the capacity of the middle link of a Wheatstone network can paradoxically lead to an increase in the social delay.

One might argue that if we allow $\mu_l$ to be strictly less than 1 for all network links $l \in L$, the network
social delay will decrease. We use the following example to show that even in this case, increasing autonomy can worsen social delay.

**Example 4.** Consider the previous example with the total flow $r = 6$, but change $M_l$'s to be, $M_1 = \frac{1}{5}$, $M_2 = 1.1$, $M_3 = 1.1$, $M_4 = \frac{1}{5}$, and $M_5 = 1$. In this case, clearly, $\mu_l < 1$, for all $l \in L$. We computed the maximum and minimum social delay at equilibrium for every autonomy fraction $\alpha$. Figure 4 shows the maximum and minimum social delay in this example for different values of $\alpha$. Figure 4 demonstrates that the maximum social delay increases as we increase $\alpha$ from 0, until we reach a local maximum. The minimum social delay decreases as we increase $\alpha$ from 0, until we reach a local minimum, and then, it increases sharply to values that are higher than the social delay at $\alpha = 0$. Surprisingly, when all vehicles are autonomous ($\alpha = 1$) the social delay is greater than the social delay when $\alpha = 0$, i.e. $J(\alpha = 1) > J(\alpha = 0)$. This might be counter intuitive as we expect the network with full autonomy to have smaller social delay. However, this example shows that when capacity increases are heterogeneous across the network, the selfish behavior of the vehicles when making their route choices might actually lead to worsening the social delay of the network. Therefore, the mobility benefits obtained from the introduction of autonomous vehicles in the network, in terms of decreasing network social delay, are not obvious.

As mentioned previously, the increase in social delay due to an increase in the fraction of autonomous vehicles is in fact a particular instance of Braess’s paradox. Braess’s Paradox is the counterintuitive but well known fact that removing edges from a network or increasing the delay functions on certain links can improve social delay [Rou06]. In our problem setting, replacing a fraction of regular vehicles by autonomous vehicles can be interpreted as replacing the link delay function $a_l \left( 1 + \gamma_l \left( \frac{f_a}{M_l} + \frac{f_r}{M_l} \right)^{\beta_l} \right)$ by $a_l \left( 1 + \gamma_l \left( \frac{f_a}{M_l} + \phi \frac{f_r}{M_l} \right)^{\beta_l} \right)$ for every link $l \in L$. It was shown in previous studies that Braess paradox is prevalent and can be arbitrarily severe [SZ83, Rou06]. Despite the price of anarchy, the occurrence of Braess’s paradox heavily depends on network topology and the parameters of link delay functions [Rou01, HA01, Mil03].

6 Networks with Multiple O/D Pairs

So far, we have seen that even in a network with only one O/D pair, the introduction of autonomous vehicles can result in complex behaviors. Thus, it should be expected that a general network with multiple O/D pairs will exhibit similar counter intuitive behaviors. In the previous section, we saw that the existence of a homogeneous degree of capacity asymmetry throughout the network is sufficient for guaranteeing improvements in the social delay by increasing the fraction of autonomous vehicles. We now show, via the following example, that this is not the case for networks with multiple O/D pairs.
Example 5. Consider the network shown in Figure 5 which was first introduced in [Fis79]. There are three O/D pairs, $W = \{(A,B), (B,C), (A,C)\}$. The total demand of the network O/D pairs are $r_{AB} = 17, r_{AC} = 20$, and $r_{BC} = 90$. Assume that $\beta_l = 1, \forall l \in L$. Let the link parameters be $\{a_1 = 0, m_1 = 1, M_1 = 4\}, \{a_2 = 0, m_2 = 1\}$, and $\{a_3 = 90, m_3 = 1\}$. Let the vehicles that travel from A to C, and from B to C be all regular vehicles, i.e. $\alpha_{AC} = \alpha_{BC} = 0$. Figure 6 shows a plot of the network social delay versus the fraction of autonomous vehicles traveling along O/D pair AB, $\alpha_{AB}$. As the figure shows, as vehicle autonomy increases, so does the social delay. Note that the social delay is unique in this case. This example shows that existence of vehicle autonomy along certain network O/D pairs can result in worsening the overall or social delay of the network even if the road degrees of capacity asymmetry are homogeneous. This is of paramount importance in practice. For instance, if O/D pair AB belongs to a high-income neighborhood, autonomous vehicles may first be deployed along this path, while other neighborhood or O/D pairs may still travel via regular vehicles. Then, although the early adoption of autonomous vehicles along O/D pair AB will lead to a decrease in travel delay of O/D pair AB, it worsens the social delay in the network and increases the delays experienced by users along other O/D pairs. This example shows that even with homogeneous degrees of capacity asymmetry, when there exist multiple O/D pairs, different autonomy fractions along network O/D pairs can be another source of heterogeneity in the network; hence, counterintuitive behaviors might occur for networks with mixed autonomy.

It was shown in [Fis79, DN84] that a decrease in the total demand of a single O/D pair, might lead to an increase in delay of travel along other network O/D pairs and the social delay. In the previous example, we showed that a similar behavior can also be observed due to the presence of autonomous vehicles. In fact, what we have shown so far is that the long known paradoxical traffic behavior resulting from constructing more roads or reducing demands can actually happen in networks with mixed autonomy due to the presence of autonomous vehicles. Thus, the mobility benefits of increasing autonomy in a network are not immediate, and in order to take advantage of the full mobility potential of autonomous vehicles, control and routing strategies that guarantee mobility benefits must be developed for the next generation of traffic networks.

Now that we have shown, the social delay can increase as a consequence of the presence of autonomous vehicles in networks with multiple O/D pairs, we wish to study whether we can bound this degradation in the network performance, to see how much worse the social delay can get with increasing the fraction of autonomous vehicles. To answer this, we derive a bound on the performance degradation that can result from all possible demand and autonomy fraction vectors in general networks that have a homogeneous degree of capacity asymmetry. To this end, for a given network $G$ and a demand vector $r$, define the vector of fictitious reduced demand $\tilde{r} = (\tilde{r}_w : w \in W)$ to be $\tilde{r}_w = (1 - \alpha_w)r_w + \mu \alpha_w r_w$ for each O/D pair $w \in W$. Consider an auxiliary fictitious routing game with a total demand $\tilde{r}$ of only regular vehicles on $G$. For this auxiliary game, similar to Theorem 1, define $(\tilde{e}_l : l \in L)$ to be

$$\tilde{e}_l = a_l \left(1 + \gamma_l \left(\tilde{r}_l / \hat{m}_l\right)^{\beta_l}\right),$$

and let $\tilde{e}_w(\tilde{r})$ be the delay of travel for each $w \in W$ in this auxiliary game. Then, using the auxiliary fictitious game, we can state the following proposition.

Proposition 4. Consider a general network $G = (N, L, W)$ with a homogeneous degree of capacity asymmetry $\mu \leq 1$ in all of its links. For any demand vector $r$, fix the vector of autonomy fraction $\alpha = (\alpha_w : w \in W)$ such that $0 \leq \alpha_w \leq 1$ for all $w \in W$. Then, we have

1. The social delay $J(f)$ is unique for all Wardrop equilibrium flow vectors $f$. 

12
2. The social delay of the original game is $J(f) = \sum_{w \in W} r_w \hat{e}_w(\hat{r}_w)$ for all Wardrop equilibrium flow vectors $f$.

Proof. Fix $r$ and $\alpha$, such that for each $w \in W$, $0 < r_w$ and $0 \leq \alpha_w \leq 1$. Recalling Lemma 1, we know that there exists at least one equilibrium. Let $f = (f^r_p, f^a_p : p \in \mathcal{P})$ be such an equilibrium flow vector for $G$. For each path $p \in \mathcal{P}$, define $\hat{f}_p := f^r_p + \mu f^a_p$. By generalizing the proof of Theorem 1, it is easy to see that $\hat{f} = (\hat{f}_p : p \in \mathcal{P})$ is an equilibrium for the defined auxiliary routing game on $G$ with reduced demand $\hat{r}$ of only regular vehicles. Moreover, for each path $p \in \mathcal{P}$, $e_p(f) = e_p(\hat{f})$. Therefore, for each O/D pair $w \in W$, $\hat{e}_w(\hat{f}) = \min_{p \in \mathcal{P}_w} e_p(f) = \hat{e}_w(\hat{f})$. Hence,

$$J(f) = \sum_{w \in W} r_w \hat{e}_w(f) = \sum_{w \in W} r_w \hat{e}_w(\hat{f}).$$

(17)

Since $\hat{f}$ contains only regular vehicles, recalling Remark 2 and Proposition 2, for each $w \in W$, the delay of travel $\hat{e}_w(f)$ is unique for a given $\hat{r}$; thus,

$$J(f) = \sum_{w \in W} r_w \hat{e}_w(\hat{r}).$$

(18)

As $\hat{r}$ is uniquely determined for a given demand vector $r$ and a vector of autonomy fraction $\alpha$, the social delay $J(f)$ is unique for all Wardrop equilibrium flow vectors $f$ and can be obtained via (18).

The uniqueness of social delay established by Proposition 4 implies that for a fixed demand vector $r$, the social delay is a well defined function of autonomy fraction $\alpha$. With a slight abuse of notation, we use $J(\alpha)$ to emphasize the dependence of the social delay on the vector of autonomy fraction $\alpha$. Note that Proposition 4 establishes a connection between our original routing game, which has two classes of vehicles, with a fictitious auxiliary routing game, which has only regular vehicles and a reduced demand vector $\hat{r}$. We exploit this connection in the remainder of the paper. Since the auxiliary game has only one class of vehicles, the results in [CSSM08] hold for this game. Before proceeding, we need to adopt and review some of the definitions in [CSSM08] for our proposed auxiliary game.
In the auxiliary game, for a given O/D demand vector \( \tilde{r} \), a flow vector \( \tilde{f} \) is feasible if \( \tilde{f}_p \geq 0 \) for all paths \( p \in \mathcal{P} \), and \( \sum_{p \in \mathcal{P}} \tilde{f}_p = \tilde{r}_w \) for all \( w \in \mathcal{W} \). Let \( \phi \in \mathbb{R}^{|L|} \) be a vector of link flows that result from a feasible flow vector \( \tilde{f} \), where \( |L| \) is the number of links in the network. Also, let \( \Phi \) represent the set of all feasible link flow vectors \( \phi \) for a given reduced demand vector \( \tilde{r} \). Then, for a vector of link delay functions \( (\tilde{e}_l : l \in L) \) of the form (16) and any vector \( v \in \Phi \), define

\[
\lambda((\tilde{e}_l : l \in L), v) := \max_{x \in \mathbb{R}_{\geq 0}^{|L|}} \frac{\sum_{l \in L} (\tilde{e}_l(v_l) - \tilde{e}_l(x_l))x_l}{\sum_{l \in L} \tilde{e}_l(v_l)v_l},
\]

(19)

where \( 0/0 \) is considered to be \( 0 \). Additionally, let \( \mathcal{E} \) be the class of delay functions represented by (16). Define

\[
\lambda(\mathcal{E}) := \sup_{(\tilde{e}_l : l \in L) \in \mathcal{E}, v \in \Phi} \lambda((\tilde{e}_l : l \in L), v).
\]

(20)

It is important to mention that since the class of delay functions \( \mathcal{E} \) is monotone, \( \lambda(\mathcal{E}) \leq 1 \) in our setting (See Section 4 in [CSSM08]). Note that \( \lambda(\mathcal{E}) \) can be easily computed for certain classes of delay functions such as polynomials. For instance, \( \lambda(\mathcal{E}) = \frac{1}{4} \) for the class of linear delay functions.

Now, we can bound the network performance degradation due to the introduction of autonomy in homogeneous networks via the following theorem.

**Theorem 2.** Consider a general network \( G = (N, L, W) \) with a homogeneous degree of capacity asymmetry \( \mu \). Fix the demand vector \( r \). Let \( J^\alpha \) be the social delay when all vehicles are nonautonomous, i.e., \( \alpha_w = 0 \) for all \( w \in \mathcal{W} \). Then, for any other vector of autonomy fraction \( \alpha \) such that \( 0 \leq \alpha_w \leq 1 \) for all \( w \in \mathcal{W} \), we have

\[
J(\alpha) \leq (1 - \lambda(\mathcal{E}))^{-1}J^\alpha,
\]

(21)

where \( J(\alpha) \) is the social delay for the vector of autonomy fraction \( \alpha \). Note that using Proposition 4, \( J(\alpha) \) and \( J^\alpha \) are unique, and, thus, well defined.

**Proof.** Fix the demand vector \( r \). Let \( f^\alpha = (f^\alpha_p : p \in \mathcal{P}) \) be an equilibrium flow vector when all vehicles are regular. We further use \( f^\alpha_l \) to denote the flow along link \( l \in L \) in this case. Note that using Proposition 2, we know that \( f^\alpha_l \) is unique for every link \( l \in L \). Moreover, for each path \( p \in \mathcal{P} \), we use \( e^\alpha_p \) to represent the delay along path \( p \) when all vehicles are regular. Using Remark 2 and Proposition 2, in the absence of autonomy, the delay of travel for each O/D pair \( w \in \mathcal{W} \) is unique. Thus, in this case, the unique social delay \( J^\alpha = \sum_{w \in \mathcal{W}} r_w e^\alpha_w(r) \), where \( e^\alpha_w(r) \) is the delay of travel along \( w \in \mathcal{W} \) when all vehicles are regular.

On the other hand, when there are autonomous vehicles with a given autonomy fraction \( \alpha \) in the network, as defined in Proposition 4, construct the auxiliary game on \( G \) with fictitious reduced demand \( \tilde{r} = (\tilde{r}_w : w \in \mathcal{W}) \) of only regular vehicles, where \( \tilde{r}_w = (1 - \alpha_w)r_w + \mu r_w \alpha_w \) for every \( w \in \mathcal{W} \). Let \( \tilde{f} = (\tilde{f}_p : p \in \mathcal{P}) \) be an equilibrium flow vector for this auxiliary game. Using Proposition 4, the social delay of the network with autonomous vehicles is given by \( J(\alpha) = \sum_{w \in \mathcal{W}} r_w \tilde{e}_w(\tilde{r}) \). First, we claim that

\[
J(\alpha) = \sum_{w \in \mathcal{W}} r_w \tilde{e}_w(\tilde{r}) \leq \sum_{l \in L} f^\alpha_l \tilde{e}_l(\tilde{r}).
\]

(22)

To see this, note that for every link \( l \in L \), we have \( f^\alpha_l = \sum_{p \in \mathcal{P} : l \in p} f^\alpha_p \). Furthermore, the origin and destination of each path \( p \in \mathcal{P} \) are unique. Hence, each path \( p \) belongs to one and exactly one O/D
pair \( w \in W \). Consequently, \( f^o_l = \sum_{w \in W} \sum_{p \in P_w : l \in p} f^o_p \), and we have

\[
\sum_{l \in L} f^o_l \tilde{e}_l(\bar{r}) = \sum_{l \in L} \left( \sum_{w \in W} \left( \sum_{p \in P_w : l \in p} f^o_p \right) \right) \tilde{e}_l(\bar{r})
= \sum_{w \in W} \sum_{l \in L} \left( \sum_{p \in P_w : l \in p} f^o_p \right) \tilde{e}_l(\bar{r})
= \sum_{w \in W} \sum_{p \in P_w} f^o_p \tilde{e}_p(\bar{r})
= \sum_{w \in W} \sum_{p \in P_w} f^o_p \tilde{e}_p(\bar{r}),
\]

where \( \tilde{e}_p(\bar{r}) \) is the delay of path \( p \in P_w \) for the auxiliary game. Recalling Definition (6), for the auxiliary game, the travel delay of an O/D pair \( w \in W \) is given by \( \tilde{e}_w(\bar{r}) = \min_{p \in P_w} \tilde{e}_p(\bar{r}) \); thus, we have

\[
\sum_{w \in W} \sum_{p \in P_w} f^o_p \tilde{e}_p(\bar{r}) \geq \sum_{w \in W} \sum_{p \in P_w} f^o_p \tilde{e}_w(\bar{r})
= \sum_{w \in W} \tilde{e}_w(\bar{r}) \sum_{p \in P_w} f^o_p
= \sum_{w \in W} r_w \tilde{e}_w(\bar{r}),
\]

which proves our claim in (22). Now, since the auxiliary game has only one class of vehicles, we can use Lemma 4.1 from [CSSM08]. More precisely, since \( \tilde{f} \) is an equilibrium for the auxiliary game, then Lemma 4.1 from [CSSM08] states that for every nonnegative vector of link flows \( x \in \mathbb{R}_{\geq 0}^{\left| L \right|} \) (\( x \) is not necessarily a feasible link flow vector), we have

\[
\sum_{l \in L} x_l \tilde{e}_l(\tilde{f}_l) \leq \sum_{l \in L} x_l \tilde{e}_l(x_l) + \lambda(\mathcal{E}) \sum_{l \in L} \tilde{f}_l \tilde{e}_l(\tilde{f}_l).
\]

(23)

Since \( f^o_l \) is nonnegative for every link \( l \in L \), substituting \( x_l \) by \( f^o_l \) in (23), we get

\[
\sum_{l \in L} f^o_l \tilde{e}_l(\tilde{f}_l) \leq \sum_{l \in L} f^o_l \tilde{e}_l(f^o_l) + \lambda(\mathcal{E}) \sum_{l \in L} \tilde{f}_l \tilde{e}_l(\tilde{f}_l).
\]

(24)

Now, note that since both the auxiliary game and the game with no autonomy have only regular vehicles, utilizing (16), we realize that

\[
\tilde{e}_l(f^o_l) = a_l \left( 1 + \gamma_l \left( \frac{f^o_l}{m_l} \right)^{\beta_l} \right)
= e^o_l(f^o_l).
\]

Thus,

\[
\sum_{l \in L} f^o_l \tilde{e}_l(f^o_l) = \sum_{l \in L} f^o_l e^o_l(f^o_l) = J^o.
\]

(25)
Now, since $J(\alpha) = \sum_{w \in W} r_w \tilde{e}_w(\bar{r})$ and for all links $l \in L$, $\tilde{e}_l(\bar{r}) = \hat{e}_l(\hat{f})$ by definition, using (22), (24), and (25), we realize that

$$J(\alpha) \leq J^o + \lambda(\mathcal{E}) \sum_{l \in L} \hat{f}_l(\bar{r}).$$

(26)

As $\hat{f}$ is an equilibrium for the auxiliary routing game, $\sum_{l \in L} \hat{f}_l(\bar{r}) = \sum_{w \in W} \tilde{r}_w \tilde{e}_w(\bar{r})$. Since for each O/D pair $w \in W$, $\alpha_w \leq 1$, we have $\tilde{r}_w \leq r_w$. Therefore, using Proposition 4,

$$\sum_{w \in W} \tilde{r}_w \tilde{e}_w(\bar{r}) \leq \sum_{w \in W} r_w \tilde{e}_w(\bar{r}) = J(\alpha).$$

(27)

Using (27) and (26), we get

$$J(\alpha) \leq J^o + \lambda(\mathcal{E}) J(\alpha).$$

(28)

Hence, for the our monotone class of delay functions $\mathcal{E}$ with $\lambda(\mathcal{E}) < 1$, we can conclude that

$$J(\alpha) \leq (1 - \lambda(\mathcal{E}))^{-1} J^o,$$

which completes the proof.

Theorem 2 provides an upper bound on the severity of increases in traffic delays when a fraction of regular vehicles is replaced by autonomous vehicles.

We now postulate, as an analogous concept to the price of anarchy [RT02], the price of vehicle autonomy in homogeneous networks under every demand vector $r$ as follows:

$$\eta := \max_{\alpha: 0 \leq \alpha \leq 1, \forall w} \frac{J(\alpha)}{J^o},$$

(29)

Theorem 2 states that $\eta \leq (1 - \lambda(\mathcal{E}))^{-1}$. For polynomial delay functions of degree less than or equal to 4, $(1 - \lambda(\mathcal{E}))^{-1} = 2.151$ [CSSM08]. Interestingly, the bound that we have derived for the price of vehicle autonomy is similar to the bounds derived for the price of anarchy of routing games with a single class of users in [RT02, CSSM08]. Note that this bound for $\eta$ is different from the price of anarchy of routing games with mixed autonomy [LCP17b], it is similar to that of routing games with only a single class of vehicles. However, unlike the bounds for price of anarchy, the tightness of our bound for $\eta$ must be further investigated.

7 Conclusion and Future Work

In this paper, we studied how the coexistence of autonomous and regular vehicles in traffic networks will affect network mobility when all vehicles select their routes selfishly. We compared the total social network delay at a Wardrop equilibrium in networks with mixed autonomy with that of the networks with only regular vehicles. Having shown that the equilibrium is not unique in the mixed-autonomy setting, we proved that the total social delay is unique when the road degree of capacity asymmetry, which is the ratio between the roadway capacity with only regular vehicles and the roadway capacity with only autonomous vehicles, is homogeneous among its roadway. We further proved that the total social delay is a nonincreasing and continuous function of the fraction of autonomous vehicles on the roadways (aka the autonomy ratio $\alpha$) when the network has only one O/D pair. However, we showed that allowing for heterogeneous degrees of capacity asymmetry or multiple O/D pairs in the network
results in counter intuitive behaviors such as the fact that increasing network autonomy ratio can worsen the network total social delay. Finally, we derived an upper bound for the “price of vehicles autonomy” in networks with a homogeneous degree of capacity asymmetry, which estimates the worst possible increase in network social delay, due to the introduction of autonomous vehicles.

We believe that the results presented in this paper indicate that the expected mobility benefits resulting from wide spread utilization of autonomous vehicles in traffic networks are not immediate. Thus, in order to take advantage of the potential mobility benefits of autonomy, it will be necessary to study the stability of traffic equilibria in networks with mixed autonomy. Once the stable system equilibria are characterized, traffic management and control strategies must be developed for the traffic network that are guaranteed to steer the system to the equilibria that have lower total delay. Therefore, revisiting routing and tolling strategies for networks with mixed vehicle autonomy is essential.

Acknowledgment

This work is supported by the National Science Foundation under Grant CPS 1545116.

References

[AAGJ10] Assad Al Alam, Ather Gattami, and Karl Henrik Johansson. An experimental study on the fuel reduction potential of heavy duty vehicle platooning. In Intelligent Transportation Systems (ITSC), 2010 13th International IEEE Conference on, pages 306–311. IEEE, 2010.

[ABEA+06] Eitan Altman, Thomas Boulogne, Rachid El-Azouzi, Tania Jiménez, and Laura Wynter. A survey on networking games in telecommunications. Computers & Operations Research, 33(2):286–311, 2006.

[ABT+15] Assad Alam, Bart Besselink, Valerio Turri, Jonas Martensson, and Karl H Johansson. Heavy-duty vehicle platooning for sustainable freight transportation: A cooperative method to enhance safety and efficiency. IEEE Control Systems, 35(6):34–56, 2015.

[AFKV16] Armin Askari, Daniel Albarnaz Farias, Alex A Kurzhanskiy, and Pravin Varaiya. Measuring impact of adaptive and cooperative adaptive cruise control on throughput of signalized intersections. arXiv preprint arXiv:1611.08973, 2016.

[BK79] D Braess and G Koch. On the existence of equilibria in asymmetrical multiclass-user transportation networks. Transportation Science, 13(1):56–63, 1979.

[Bra68] Dietrich Braess. Über ein paradoxon aus der verkehrsplanung. Unternehmensforschung, 12(1):258–268, 1968.

[CSSM08] José R Correa, Andreas S Schulz, and Nicolás E Stier-Moses. A geometric approach to the price of anarchy in nonatomic congestion games. Games and Economic Behavior, 64(2):457–469, 2008.

[DN84] Stella Dafermos and Anna Nagurney. On some traffic equilibrium theory paradoxes. Transportation Research Part B: Methodological, 18(2):101–110, 1984.

[DR99] Swaroop Darbha and KR Rajagopal. Intelligent cruise control systems and traffic flow stability. Transportation Research Part C: Emerging Technologies, 7(6):329–352, 1999.
[Fis79] Caroline Fisk. More paradoxes in the equilibrium assignment problem. *Transportation Research Part B: Methodological*, 13(4):305–309, 1979.

[FV18] S Alireza Fayazi and Ardalan Vahidi. Mixed integer linear programming for optimal scheduling of autonomous vehicle intersection crossing. *IEEE Transactions on Intelligent Vehicles*, 2018.

[HA01] Jane N Hagstrom and Robert A Abrams. Characterizing braess’s paradox for traffic networks. In *Intelligent Transportation Systems, 2001. Proceedings. 2001 IEEE*, pages 836–841. IEEE, 2001.

[Hal78] Michael A Hall. Properties of the equilibrium state in transportation networks. *Transportation Science*, 12(3):208–216, 1978.

[LCP17a] Daniel A Lazar, Samuel Coogan, and Ramtin Pedarsani. Capacity modeling and routing for traffic networks with mixed autonomy. In *Decision and Control (CDC), 2017 IEEE 56th Conference on, to appear, IEEE*, 2017.

[LCP17b] Daniel A Lazar, Samuel Coogan, and Ramtin Pedarsani. The price of anarchy for transportation networks with mixed autonomy. *arXiv preprint arXiv:1710.07867*, 2017.

[LCPS18] Daniel A Lazar, Kabir Chandrasekher, Ramtin Pedarsani, and Dorsa Sadigh. Maximizing road capacity using cars that influence people. *arXiv preprint arXiv:1807.04414*, 2018.

[LMJ13] Kuo-Yun Liang, Jonas Mårtensson, and Karl Henrik Johansson. When is it fuel efficient for a heavy duty vehicle to catch up with a platoon? In *7th IFAC Symposium on Advances in Automotive Control, Tokyo, Japan, September 4-7, 2013*, 2013.

[LPTV17] Jennie Lioris, Ramtin Pedarsani, Fatma Yildiz Tascikaraoglu, and Pravin Varaiya. Platoons of connected vehicles can double throughput in urban roads. *Transportation Research Part C: Emerging Technologies*, 77:292–305, 2017.

[Man64] Traffic Assignment Manual. Bureau of public roads. *US Department of Commerce*, 1964.

[MH18] Negar Mehr and Roberto Horowitz. Can the presence of autonomous vehicles worsen the equilibrium state of traffic networks? In *Decision and Control (CDC), 2018 IEEE 57th Conference on, to appear, IEEE*, 2018.

[Mil03] Igal Milchtaich. Network topology and the efficiency of equilibrium. In *ICM Millennium Lectures on Games*, pages 233–266. Springer, 2003.

[MK14] David Miculescu and Sertac Karaman. Polling-systems-based control of high-performance provably-safe autonomous intersections. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pages 1417–1423. IEEE, 2014.

[MLH18a] Negar Mehr, Ruolin Li, and Roberto Horowitz. A game theoretic macroscopic model of bypassing at traffic diverges with applications to mixed autonomy networks. *arXiv preprint arXiv:1809.02762*, 2018.

[MLH18b] Negar Mehr, Ruolin Li, and Roberto Horowitz. A game theoretic model for aggregate bypassing behavior of vehicles at traffic diverges. In *2018 21st International Conference on Intelligent Transportation Systems (ITSC)*, pages 1968–1973. IEEE, 2018.
[PvA10] Rattaphol Pueboobpaphan and Bart van Arem. Driver and vehicle characteristics and platoon and traffic flow stability: Understanding the relationship for design and assessment of cooperative adaptive cruise control. *Transportation Research Record: Journal of the Transportation Research Board*, (2189):89–97, 2010.

[Rou01] Tim Roughgarden. Designing networks for selfish users is hard. In *Foundations of Computer Science, 2001. Proceedings. 42nd IEEE Symposium on*, pages 472–481. IEEE, 2001.

[Rou02] Tim Roughgarden. Selfish routing. Technical report, CORNELL UNIV ITHACA NY DEPT OF COMPUTER SCIENCE, 2002.

[Rou06] Tim Roughgarden. On the severity of braess’s paradox: designing networks for selfish users is hard. *Journal of Computer and System Sciences*, 72(5):922–953, 2006.

[RT02] Tim Roughgarden and Éva Tardos. How bad is selfish routing? *Journal of the ACM (JACM)*, 49(2):236–259, 2002.

[SCDM+18] Raphael E Stern, Shumo Cui, Maria Laura Delle Monache, Rahul Bhadani, Matt Bunting, Miles Churchill, Nathaniel Hamilton, Hannah Pohlmann, Fangyu Wu, Benedetto Piccoli, et al. Dissipation of stop-and-go waves via control of autonomous vehicles: Field experiments. *Transportation Research Part C: Emerging Technologies*, 89:205–221, 2018.

[Smi79] Mt J Smith. The existence, uniqueness and stability of traffic equilibria. *Transportation Research Part B: Methodological*, 13(4):295–304, 1979.

[SZ83] Richard Steinberg and Willard I Zangwill. The prevalence of braess’ paradox. *Transportation Science*, 17(3):301–318, 1983.

[TC15] Pavankumar Tallapragada and Jorge Cortés. Coordinated intersection traffic management. *IFAC-PapersOnLine*, 48(22):233–239, 2015.

[War52] John Glen Wardrop. Some theoretical aspects of road traffic research. In *Inst Civil Engineers Proc London/UK*, 1952.

[WKVB17] Cathy Wu, Aboudy Kreidieh, Eugene Vinitsky, and Alexandre M Bayen. Emergent behaviors in mixed-autonomy traffic. In *Conference on Robot Learning*, pages 398–407, 2017.

[YH06] Jingang Yi and Roberto Horowitz. Macroscopic traffic flow propagation stability for adaptive cruise controlled vehicles. *Transportation Research Part C: Emerging Technologies*, 14(2):81–95, 2006.

[ZMC16] Yue J Zhang, Andreas A Malikopoulos, and Christos G Cassandras. Optimal control and coordination of connected and automated vehicles at urban traffic intersections. In *American Control Conference (ACC)*, 2016, pages 6227–6232. IEEE, 2016.