

A CONJECTURE ON $H_3(1)$ FOR CERTAIN STARLIKE FUNCTIONS

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Abstract. We prove a conjecture concerning the third Hankel determinant, proposed in “Anal. Math. Phys., https://doi.org/10.1007/s13324-021-00483-7”, which states that $|H_3(1)| \leq 1/9$ is sharp for the class $S_3^* = \{z f'(z)/f(z) \prec \varphi(z) := 1 + z\}$. In addition, we also establish bounds for sixth and seventh coefficient, and $|H_3(1)|$ for functions in $S_3^*$. The general bounds for two and three-fold symmetric functions related to the Ma-Minda classes $S^*(\varphi)$ of starlike functions are also obtained.

1. Introduction

For the given positive integers $n$ and $q$, the Hankel determinant $H_q(n)$ related to the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$, the class of normalized analytic functions, given by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

where $a_1 = 1$, was defined by Noonan and Thomas [17]. For various choices of $q$ and $n$, the growth of $H_q(n)$ was explored for many subfamilies of univalent functions. Jan
teng et al. [5] discovered the sharp estimates on the second Hankel determinant, for the classes of starlike and convex functions. Krishna et al. [9] calculated the best estimates of $H_2(2)$ for the class of Bazilević functions.

Moreover, $H_3(1) := 2a_2 a_3 a_4 + a_3(a_5 - a_3^2) - a_4^2 - a_3^2 a_5$ (1.1)
is the third order Hankel determinant. Zaprawa [21] evaluated the non sharp bounds on the third Hankel determinant as $|H_3(1)| \leq 1$ and $|H_3(1)| \leq 49/540$ for the classes of starlike and convex functions, respectively. The presence of higher order coefficients in the expression of $H_3(1)$ makes it difficult to solve, and the sharpness of the solution is crucial. Overcoming the challenges, Kowacz et al. [7] established the sharp bound $|H_3(1)| \leq 4/135$ for the class of convex functions in 2018. Kwon et al. [12] provided the best known estimate for starlike functions given by $|H_3(1)| \leq 4/135$. Later on, in 2018, Lecko et al. [14] proved that $|H_3(1)| \leq 1/9$ is the sharp bound for starlike functions of order 1/2. By choosing specific values of $\varphi(z)$ in the class $S^*(\varphi)$ of Ma-Minda [16] defined by

$$S^*(\varphi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

authors in [14] have obtained the sharp estimates for $|H_3(1)| \leq 1/36$ for the choice $\varphi(z) = \sqrt{1 + z}$. Since, a proper, careful and infact precise re-arrangement of terms is highly required to obtain the best possible bound and it results in few research articles, which are available in this area for the sharp bound of $H_3(1)$, see [18,20]. The most important step in obtaining the sharp bound of $H_3(1)$ was to rewrite equation (1.1) in terms of Carathéodory coefficients, especially $p_1, p_2, p_3, p_4$ and $p_5$, where $p_i's$ are coefficients of the class $P := \{p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n : \text{Re } p(z) > 0\}$. Libera

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and Zlotkiewicz presented the formula for $p_2, p_3$ in $[14,15]$ and Kwon et al. gave the expression for $p_4$ in $[11]$, which is stated below in the form of a lemma:

**Lemma 1.1.** Let $p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{P}$. Then,
\[ 2p_2 = p_1^2 + \gamma(4 - p_1^2), \]
\[ 4p_3 = p_1^3 + 2p_1(4 - p_1^2)\gamma - p_1(4 - p_1^2)\gamma^2 + 2(4 - p_1^2)(1 - |\gamma|^2)\eta \]
and
\[ 8p_4 = p_1^4 + (4 - p_1^2)\gamma(p_1^2(\gamma^2 - 3\gamma + 3) + 4\gamma) - 4(4 - p_1^2)(1 - |\gamma|^2)(p_1(\gamma - 1)\eta + \gamma\eta^2 - (1 - |\eta|^2)p), \]
for some $\gamma$, $\eta$ and $p$ such that $|\gamma| \leq 1$, $|\eta| \leq 1$ and $|p| \leq 1$.

In 2021, Kumar and Gangania $[10]$, introduced a new class $S_\psi^*$ by choosing $\varphi(z) = 1 + ze^z$ in $[12]$ defined as
\[ S_\psi^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < 1 + ze^z \right\}. \]
They used the similar strategy and obtained the bound as, $|H_3(1)| \leq 0.150627$ for $S_\psi^*$ and also proposed a conjecture which is stated as follows:

**Conjecture 1.2.** $[10]$ Page no. 33 If $f \in S_\psi^*$, then the sharp bound for the third Hankel determinant is given by
\[ |H_3(1)| \leq \frac{1}{9} \approx 0.1111\ldots, \]
with the extremal function $f(z) = z \exp\left(\frac{1}{3}(e^z - 1)\right) = z + \frac{1}{3}z^4 + \frac{1}{9}z^7 + \ldots$.

In this article, together with the proof of conjecture, the estimates for $a_6$ and $a_7$ in association with the fourth order Hankel determinant for the functions in the class $S_\psi^*$ are obtained. For $S^*(\varphi)$, the general third Hankel determinant for second and third fold symmetric functions are also estimated.

2. **Proof of the Conjecture** $[12]$

The initial coefficients for the functions in class $S_\psi^*$ given in $[10]$, are as follows:
\[ a_2 = \frac{p_1}{2}, \quad a_3 = \frac{1}{4}\left(p_2 + \frac{p_1^2}{2}\right), \quad a_4 = \frac{1}{6}\left(p_3 + \frac{3}{4}p_1p_2\right) \]
\[ a_5 = \frac{1}{8}\left(p_1^4 + \frac{p_3^2}{4} + 2p_1p_5 - \frac{p_1^2p_2}{3} - p_1p_4\right) \]
\[ a_6 = \frac{1}{4}\left(-\frac{p_1^5}{240} + \frac{19p_1^3p_2}{480} - \frac{7p_1^2p_3}{80} + \frac{p_1^2p_4}{15} + \frac{p_2p_3}{6} + \frac{p_1p_4}{4} + \frac{2p_5}{5}\right) \]
and
\[ a_7 = \frac{1}{4}\left(\frac{17p_1^6}{11520} - \frac{37p_1^4p_2}{1920} + \frac{29p_1^2p_3^2}{480} + \frac{p_3^3}{32} - \frac{13p_1^3p_3}{360} - \frac{p_1p_2p_3}{6} + \frac{p_2^3}{18} + \frac{p_1p_4}{16} + \frac{p_2p_4}{8} + \frac{p_1p_5}{5} + \frac{p_6}{3}\right). \]

Now, we proceed by providing a positive response to the conjecture $[12]$ in the form of a new theorem which states:

**Theorem 2.1.** Let $f \in S_\psi^*$. Then,
\[ |H_3(1)| \leq \frac{1}{9}. \]

The result is sharp.
Proof. As the Carathéodory class is rotationally invariant, \( p_1 \) lies inside the interval \([0, 2]\). On substituting the expressions of \( a_i \)'s from equations (2.1) and (2.2) in equation (1.1) assuming \( p_1 := p \), we get

\[
H_3(1) = \frac{1}{9216} \left( 3p^6 - 12p^4p_2 + 96p^3p_3 - 192pp_2p_3 - 144p^2p_2^2 + 144p^2p_4 + 72p_2^3 - 256p_3^2 + 288p^2p_4 \right).
\]

On simplification using (1.3)-(1.5), we obtain

\[
H_3(1) = \frac{1}{9216} \left( \nu_1(p, \gamma) + \nu_2(p, \gamma)\eta + \nu_3(p, \gamma)\eta^2 + \phi(p, \gamma, \eta)\rho \right)
\]

where \( \gamma, \eta, \rho \in \mathbb{D} \),

\[
\nu_1(p, \gamma) := -4p^6 - 25p^2\gamma^2(4 - p^2)^2 - 5p^2\gamma^3(4 - p^2)^2 + 2p^2\gamma^4(4 - p^2)^2 + 5p^4\gamma(4 - p^2) + 36\gamma^3(4 - p^2)^2 - 16p^4\gamma^2(4 - p^2),
\]

\[
\nu_2(p, \gamma) := 8(1 - |\gamma|^2)(4 - p^2)(4p^3 - (4 - 2p)(p\gamma + 10p\gamma^2)),
\]

\[
\nu_3(p, \gamma) := 8(1 - |\gamma|^2)(4 - p^2)^2(-8 - |\gamma|^2)
\]

\[
\phi(p, \gamma, \eta) := 72(1 - |\gamma|^2)(4 - p^2)^2(1 - |\eta|^2)\gamma.
\]

On taking \( x = |\gamma|, y = |\eta| \) and using the fact that \( |\rho| \leq 1 \), we have

\[
|H_3(1)| \leq \frac{1}{9216} \left( |\nu_1(p, \gamma)| + |\nu_2(p, \gamma)|y + |\nu_3(p, \gamma)|y^2 + |\phi(p, \gamma, \eta)| \right) \leq B(p, x, y),
\]

where

\[
B(p, x, y) = \frac{1}{9216} \left( b_1(p, x) + b_2(p, x)y + b_3(p, x)y^2 + b_4(p, x)(1 - y^2) \right)
\]

with

\[
b_1(p, x) := 4p^6 + 25p^2x^2(4 - p^2)^2 + 5p^2x^3(4 - p^2)^2 + 2p^2x^4(4 - p^2)^2 + 5p^4x(4 - p^2) + 36x^3(4 - p^2)^2 + 16p^4x^2(4 - p^2),
\]

\[
b_2(p, x) := 8(1 - x^2)(4 - p^2)(4p^3 + (4 - 2p)(px + 10px^2)),
\]

\[
b_3(p, x) := 8(1 - x^2)(4 - p^2)^2(8 + x^2),
\]

\[
b_4(p, x) := 72(1 - x^2)(4 - p^2)^2x.
\]

We must maximise \( B(p, x, y) \) in the cuboid \( V = [0, 2] \times [0, 1] \times [0, 1] \). So, we use the maximum values in the interiors of the six faces, the twelve edges, and the interior of \( V \).

(1) To begin with, all the interior points of \( V \) are taken into consideration. On partially differentiating equation (2.6) with respect to \( y \) to determine its points of maxima in the interior of \( V \) by considering \( (p, x, y) \in (0, 2) \times (0, 1) \times (0, 1) \). We get

\[
\frac{\partial B}{\partial y} = \frac{1}{1152}(4 - p^2)(1 - x^2) \left( 4p^3 + (4 - 2p)(px + 10x^2) \right).
\]

Now \( \frac{\partial B}{\partial y} = 0 \) gives

\[
y = y_0 := \frac{4px(1 + 10x) - p^3(10x^2 + x - 4)}{2(x - 1)(x - 8)(p^2 - 4)}.
\]

We note that \( y_0 \in (0, 1) \) assuring the existence of critical points and it is possible when

\[
4px(1 + 10x) - p^3(10x^2 + x - 4) + 64 - 72x + 2x^2(4 - p^2) < 2p^2(8 - 9x).
\]

(2.7)
Now we will apply the hit and trial method to find solutions that satisfy the inequalities (2.7) for the existence of a critical point. If we assume that \( p \) tends to 0 and 2, there is no \( x \in (0, 1) \) that can satisfy the equation (2.7). Similarly, if \( x \) tends to 0 and 1, there is no \( p \in (0, 2) \) satisfying equation (2.7). As a result, the function \( B \) does not have a critical point in \((0, 2) \times (0, 1) \times (0, 1)\).

(2) Now, the interiors of all the faces of \( V \) are taken into consideration.

On the face \( p = 0, B(p, x, y) \) becomes

\[
d_1(x, y) := B(0, x, y) = \frac{9x^3 + 2(1 - x^2)((8 + x^2)y^2 + 9x(1 - y^2))}{144}
\]

with \( x, y \in (0, 1) \). We notice that \( d_1 \) has no critical points in \((0, 1) \times (0, 1)\) as

\[
\frac{\partial d_1}{\partial y} = \frac{-(1 - x)^2(x + 1)(x - 8)y}{36} \neq 0, \quad x, y \in (0, 1).
\]  

On the face \( p = 2, B(p, x, y) \) becomes

\[
B(2, x, y) := \frac{1}{36}, \quad x, y \in (0, 1).
\]

On the face \( x = 0, B(p, x, y) \) becomes

\[
d_2(p, y) := B(p, 0, y) = \frac{(p^3 + 16y - 4p^2y)^2}{2304}
\]

with \( y \in (0, 1) \) and \( p \in (0, 2) \). To identify the points of maxima, we solve \( \partial d_2/\partial p \) and \( \partial d_2/\partial y \). On solving \( \partial d_2/\partial y = 0 \), we obtain

\[
y = -\frac{p^3}{4(4 - p^2)} (=: y_1).
\]

Based on simple calculations, we can conclude that such \( y_1 \) does not belong to \((0, 1)\). As a result, there is no critical point in \((0, 2) \times (0, 1)\) for \( d_2 \).

On the face \( x = 1, B(p, x, y) \) becomes

\[
d_3(p, y) := B(p, 1, y) = \frac{576 + 224p^2 - 136p^4 + 15p^6}{9216}, \quad p \in (0, 2).
\]

During the calculations, \( p_0 := p \approx 0.991758 \) comes out to be the critical point when \( \partial d_3/\partial p = 0 \). Moreover, using basic mathematics, \( d_3 \) reaches its maximum value of approximately 0.0736789 at \( p_0 \). On the face \( y = 0, B(p, x, y) \) becomes

\[
B(p, x, 0) = \frac{1}{9216} \left( 4p^4x(23 - 34x - 19x^2 - 4x^3) + p^6(4 - 5x + 9x^2 + 5x^3 + 2x^4) \\
+ 576x(2 - x^2) + 16p^2x(-36 + 25x + 23x^2 + 2x^3) \right)
\]

\( =: d_4(p, x) \).

On computing,

\[
\frac{\partial d_4}{\partial x} = \frac{1}{9216} \left( -4p^4(-23 + 34x + 19x^2 + 4x^3) + p^6(-5 + 18x + 15x^2 + 8x^3) \\
- 4p^4x(34 + 38x + 12x^2) + 16p^2(-36 + 25x + 23x^2 + 2x^3) \\
- 1152x^2 - 576(x^2 - 2) + 16p^2x(25 + 46x + 6x^2) \right)
\]
and
\[ \frac{\partial d_4}{\partial p} = \frac{p}{4608} \left( 16x(-36 + 25x + 23x^2 + 2x^3) - 8p^2x(-23 + 34x + 19x^2 + 4x^3) \\
+ 3p^4(4 - 5x + 9x^2 + 5x^3 + 2x^4) \right). \]

According to a numerical computation, the critical points obtained by solving the system of equations \( \partial d_4/\partial x = 0 \) and \( \partial d_4/\partial p = 0 \) in \((0,2) \times (0,1)\) implies
\[ B(0,0.00115734, 0.816497, 0) \leq 0.0680414. \]

On the face \( y = 1, B(p, x, y) \) reduces to
\[ B(p, x, 1) = \frac{1}{9216} \left( 4p^3(16 + 5x - 48x^2 - x^3 - 6x^4) - 8p^5(4 - x - 14x^2 + x^3 + 10x^4) \right. \\
+ p^6(4 - 5x + 9x^2 + 5x^3 + 2x^4) + 16p^2(-32 + 53x^2 - 13x^3 + 6x^4) \\\n+ 128px(1 + 10x - x^2 - 10x^3) + 64(16 - 14x^2 + 9x^3 - 2x^4) \\\n\left. + 64p^3(2 - x - 12x^2 + x^3 + 10x^4) \right) =: d_5(p, x). \]

We observe that the system of equations \( \partial d_5/\partial x = 0 \) and \( \partial d_5/\partial p = 0 \) has no solution in \((0,2) \times (0,1)\).

(3) Now, we determine the maximum values attained on the edges of the cuboid \( V \) by \( B(p, x, y) \). We get \( B(p, 0, 0) = c_1(p) := 4p^6/9216 \) from equation (2.11). It is apparent that \( c_1(p) = 0 \) for \( p = \beta_0 := 0 \) and \( p = \beta_1 := 2 \) as points of minima and maxima in the interval \([0,2]\), respectively. Maximum value of \( c_1(p) \) is \(\approx 0.0277778\). Hence,
\[ B(p, 0, 0) \leq 0.0277778. \]

With \( y = 1 \), we get \( B(p, 0, 1) = c_2(p) := (16 - 4p^2 + p^3)2/2304 \) from the equation (2.11). Since \( c_2(p) < 0 \) in the range \([0,2]\), \( p = 0 \) acts as its point of maxima. Thus
\[ B(p, 0, 1) \leq \frac{1}{9}, \quad p \in [0,2]. \]

Based on computations, \( B(0, 0, y) \) in equation (2.11) reaches its maximum value at \( y = 1 \). This leads to
\[ B(0, 0, y) \leq \frac{1}{9}, \quad y \in [0,1]. \]

Since, the equation (2.13) is free from \( x \), we have \( B(p, 1, 1) = B(p, 1, 0) = c_3(p) := (15p^6 - 136p^4 + 224p^3 + 576)/9216 \). Now, \( c_3'(p) = 448p - 544p^3 + 90p^5 = 0 \) when \( p = \beta_2 := 0 \) and \( p = \beta_3 := 0.991758 \) in the interval \([0,2]\) where \( \beta_2 \) and \( \beta_3 \) are the minimum and maximum points, respectively. Hence
\[ B(p, 1, 1) = B(p, 1, 0) \leq 0.0736789, \quad p \in [0,2]. \]

We get \( B(0, 1, y) = 1/16 \) when \( p = 0 \) is substituted in equation (2.13). The equation (2.10) do not involve any variables namely \( p, x \) and \( y \). Thus, on the edges \( p = 2, x = 1; p = 2, x = 0; p = 2, y = 0; \) and \( p = 2, y = 1 \), the maximum value of \( B(p, x, y) \) is determined by
\[ B(2, 1, y) = B(2, 0, y) = B(2, x, 0) = \frac{1}{36}, \quad x, y \in [0,1]. \]

From equation (2.11), we obtain \( B(0, 0, y) = y^2/36 \). A simple calculation shows that
\[ B(0, 0, y) \leq \frac{1}{36}, \quad y \in [0,1]. \]
We get $B(0, x, 1) = c_4 := (8 - 7x^2 - x^4)/72$ using equation (2.8). We observe that $c_4$ is a decreasing function in the range $[0, 1]$, and so reaches its maximum value at $x = 0$, according to a simple calculation. Hence

$$B(0, x, 1) \leq \frac{1}{9}, \quad x \in [0, 1].$$

$B(0, x, 0) = c_5(x) := x(1 - x^2)/8$ is obtained using equation (2.8) once again. We get $c_5'(x) = 0$ after further calculations for $x = x_0 := 1/\sqrt{3}$. Also, $c_5(x)$ is an increasing function in $[0, x_0]$ and decreasing in $(x_0, 1]$. As a result, maximum value is attained at $x_0$. Thus

$$B(0, x, 0) \leq 0.0481125, \quad x \in [0, 1].$$

Thus, the inequality (2.5) holds for all the cases. The sharpness of the result is governed by the function $f : \mathbb{D} \to \mathbb{C}$ defined as

$$f(z) = z \exp \left( \frac{1}{3} (e^{z^3} - 1) \right) = z + \frac{z^4}{3} + \frac{2z^7}{9} + \cdots, \quad (2.14)$$

with $f(0) = 0$ and $f'(0) = 1$. For the values of $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/3$ the function specified in equation (2.14) serves as an extremal function for the bounds of $H_3(1)$.

To prove our next result, we must first recall the following lemma:

**Lemma 2.2.** Let $p = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$. Then

$$|p_n| \leq 2, \quad n \geq 1, \quad (2.15)$$

$$|p_{n+k} - \mu p_n p_k| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & elsewhere, \end{cases} \quad (2.16)$$

and

$$|p_1^3 - \mu p_3| \leq \begin{cases} 2|\mu - 4|, & \mu \leq 4/3; \\ 2\mu \sqrt{\frac{\mu}{\mu - 1}}, & 4/3 < \mu. \end{cases} \quad (2.17)$$

**Lemma 2.3.** Let $f \in S^*$. Then $|a_6| \leq 47/60 \approx 0.7833$ and $|a_7| \leq 503/480 \approx 1.0479$.

**Proof.** From equation (2.3), we have

$$|a_6| = \left| \frac{1}{4} \left( -\frac{p_1^5}{240} + \frac{19p_1^3p_2}{480} - \frac{7p_1p_2^2}{80} - \frac{p_1^2p_3}{15} + \frac{p_2p_3}{6} + \frac{p_1p_4}{4} + \frac{2p_5}{5} \right) \right|$$

$$= \left| \frac{1}{10} \left( p_5 + \frac{5}{8}p_1p_4 \right) - \frac{p_1^2}{60} \left( p_3 - \frac{19}{32}p_1p_2 \right) + \frac{p_2}{24} \left( p_3 - \frac{21}{40}p_1p_2 \right) - \frac{2}{1920}p_1^5 \right|.$$

By using equation (2.15), (2.16) and triangle inequality, we get

$$|a_6| \leq \frac{47}{60}.$$

Similarly, from equation (2.4), we have

$$46080a_7 = 17p_1^5 - 222p_1^4p_2 + 696p_1^3p_2^2 - 360p_1^2p_2^3 + 416p_1^3p_3 - 1920p_1p_2p_3 + 640p_3^2 - 720p_1^2p_4$$

$$+ 1440p_2p_4 + 2304p_1p_5 + 3840p_6$$
or

$$46080|a_7| \leq 3840 \left| p_6 + \frac{3p_2p_4}{8} + 416|p_1|^3|p_3 - \frac{111p_1p_2}{208} + 360|p_2|^2 \right| - p_2 + \frac{29p_1^2}{15} + 2304|p_1| \left| p_5 - \frac{5p_2p_3}{6} \right| + |17p_1^6 - 720p_1^2p_4| + 640|p_3|^2.$$  

On using equation (2.15) and (2.16), we get

$$3840 \left| p_6 + \frac{3p_2p_4}{8} \right| \leq 13440, \quad 416|p_1|^3 \left| p_3 - \frac{111}{208}p_1p_2 \right| \leq 6656, \quad (2.18)$$

$$360|p_2|^2 \left| p_2 - \frac{29p_1^2}{15} \right| \leq 7008, \quad 2304|p_1| \left| p_5 - \frac{5p_2p_3}{6} \right| \leq 9216 \quad (2.19)$$

and

$$|17p_1^6 - 720p_1^2p_4| + 640|p_3|^2 \leq 11968. \quad (2.20)$$

From equations (2.18), (2.19) and (2.20), we get the desired result.

### 3. Fourth Hankel Determinant and Third Hankel for n-fold Symmetric Functions

#### 3.1. Fourth Hankel estimation for the class $S_{p}^{*}$

For $q = 4$ and $n = 1$, the expression of the fourth Hankel determinant can be written as

$$H_4(1) = a_7H_3(1) - a_6Q_1 + a_5Q_2 - a_4Q_3, \quad (3.1)$$

where

$$Q_1 = a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2), \quad (3.2)$$

$$Q_2 = a_3(a_3a_5 - a_3^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3), \quad (3.3)$$

and

$$Q_3 = a_4(a_3a_5 - a_3^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_4 - a_2a_3). \quad (3.4)$$

To compute the fourth Hankel determinant for the function in class $S_{p}^{*}$, we substitute the values of $a_i$ from equations (2.1) - (2.4) in equation (3.2). Upon simplification, we have

$$92160Q_1 = 27p_1^6 - 408p_1^5p_2 + p_1^3p_2(660p_1^2 - 1224p_2) + p_1^2(336p_2p_3 - 1152p_5)$$

$$+ p_2(2304p_5 - 480p_2p_3) + p_4(1440p_1p_2 - 1920p_3) + 392p_4^4p_3.$$  

By applying Lemma 2.2, we arrive at

$$92160|Q_1| \leq |p_1|^5|27p_1^2 - 408p_2| + |p_1|^2|p_2||660p_1^2 - 1224p_2| + |p_1|^2|336p_2p_3 - 1152p_5|$$

$$+ |p_2||2304p_5 - 480p_2p_3| + |p_4||1440p_1p_2 - 1920p_3| + 392|p_1|^4|p_3|$$

$$\leq 84352.$$  

So, we obtain

$$|Q_1| \leq \frac{659}{720} \approx 0.915278. \quad (3.5)$$

In similar way, we have

$$737280Q_2 = 73p_1^8 + 1440p_1^5p_3 + p_1^3p_2(14400p_2 - 1920p_1^2) + p_1p_2p_3(4096p_1^2 - 8448p_2)$$

$$+ p_1^2p_3(-1692p_2 - 336p_1^2) + p_5(12288p_3 - 4608p_1^2) - 11520p_4^2 + 720p_4^4 + 4608p_4^2p_3.$$
Using Lemma 2.2 we get
\[ 737280 |Q_2| \leq |p_1|^573p_1^3 + 1440p_3| + |p_1|^2|p_4||14400p_2 - 1920p_2^2| + |p_1||p_2||p_3||4096p_1^2 - 8448p_2| \\
+ |p_1|^4|p_2| - 1692p_2 - 336p_1^2| + |p_5||12288p_3 - 4608p_3^2| + 11520|p_4|^2 + 720|p_2|^4 \\
+ 4608|p_1|^2|p_3|^2 \leq 759296 + 98304\sqrt{\frac{3}{5}}. \]

So, we get
\[ |Q_2| \leq \frac{759296 + 98304\sqrt{\frac{3}{5}}}{737280} \approx 1.11419. \] (3.6)

Again, by rearrangement of terms, we have,
\[ 4423680Q_3 = 57p_1^6 + 288p_3p_3 + p_1^3p_4(25920p_2 - 5760p_2^2) + p_1^4p_2(16128p_3 - 144p_1^4) \\
- p_1^3(20480p_3 + 6144p_1^2) - p_1^2p_2^2(9216p_2 + 540p_1^2) + p_1p_5(36864p_3 - 6912p_1^2) \\
- 26496p_1^2p_3 + 12528p_1^2 + 46080p_2p_3p_4 - 34560p_1^2 - 27648p_2^2p_5. \]

From Lemma 2.2 we get,
\[ 4423680|Q_3| \leq |p_1|^657p_3^3 + 288p_3 + |p_1|^3|p_4||25920p_2 - 5760p_2^2| + |p_1|^4|p_2||16128p_3 - 144p_1^4| \\
+ |p_3|^2|20480p_3 + 6144p_1^2| + |p_1|^3|p_2|^2|9216p_2 + 540p_1^2| + |p_1||p_5||36864p_3 - 6912p_1^2| \\
+ 26496|p_1|^2|p_2|^2p_3 + 12528|p_1|^2 + 46080|p_2||p_3||p_4| + 34560|p_1||p_4|^2 \\
+ 27648|p_2|^2|p_5| \leq 4029952 + 1376256\sqrt{\frac{21}{37}} + \frac{1179648}{\sqrt{13}}. \]

So, we get
\[ |Q_3| \leq \frac{4029952 + 1376256\sqrt{\frac{21}{37}} + \frac{1179648}{\sqrt{13}}}{4423680} \approx 1.21934. \] (3.7)

Based on the above computations, we make the following statement on fourth Hankel determinant:

**Theorem 3.1.** Let \( f \in S^*_\varphi \). Then \( |H_4(1)| \leq 2.54589. \)

### 3.2. Third Hankel determinant for \( 2 \& 3 \) fold symmetric functions for \( S^*(\varphi) \)

In the recent times, it has been observed that finding the sharp estimates of third Hankel determinant for general Ma-Minda class is not feasible till now. But for some classes, sharp estimates have been obtained, for instance, see [1, 8, 20] and now including Theorem 2.1 as well which motivated us to settle the Conjecture 1.2. Further looking at the difficulty of the general class, we restrict ourselves to answer the problem for \( n \)-fold symmetric functions.

**Definition 3.2.** A function \( f \in \mathcal{A} \) is called \( n \)-fold symmetric if \( f(e^{2\pi i/n}z) = e^{2\pi i/n}f(z) \) which holds for all \( z \in \mathbb{D} \) and \( n \) is a natural number. We denote the set of \( n \)-fold symmetric functions by \( \mathcal{A}^{(n)} \).

Let \( f \in \mathcal{A}^{(n)} \), then \( f \) has power series expansion
\[ f(z) = a_1z + a_{n+1}z^{n+1} + a_{2n+1}z^{2n+1} + \cdots. \]

Therefore, for \( f \in \mathcal{A}^{(3)} \) and \( f \in \mathcal{A}^{(2)} \) respectively, we have
\[ H_3(1) = -a_3^2 \quad \text{and} \quad H_3(1) = a_3(a_5 - a_3^2). \] (3.8)

Now we conclude this paper with the following result:
**Theorem 3.3.** Let \( f \in S^*(\varphi) \). Then

1. \( \hat{f} \in S^{*(3)}(\varphi) \) implies that \( |H_3(1)| \leq |B_1|^2 / 9 \).
2. \( \hat{f} \in S^{*(2)}(\varphi) \) implies that
   \[
   |H_3(1)| \leq \frac{1}{4} |B_1| \times \begin{cases} 
   \frac{1}{9}(B_2 - \frac{2}{3}B_1^2 + B_1^2), & \frac{9}{4}B_1^2 \leq 2(B_2 + B_1^2 - B_1); \\
   \frac{1}{9}B_1, & 2(B_2 + B_1^2 - B_1) \leq \frac{9}{4}B_1^2 \leq 2(B_2 + B_1^2 + B_1); \\
   \frac{1}{9}(-B_2 + \frac{2}{3}B_1^2 - B_1^2), & 2(B_2 + B_1^2 + B_1) \leq \frac{9}{4}B_1^2. 
   \end{cases}
   \]

The estimate in (1) is sharp.

**Proof.** Since \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \in S^*(\varphi) \). Let

\[
\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots
\]

and

\[
p(z) = \frac{zf'(z)}{f(z)} = 1 + b_1z + b_2z^2 + \cdots
\]

This equation shows that

\[
(n - 1)a_n = \sum_{k=1}^{n-1} b_k a_{n-k} \quad n > 1. \tag{3.9}
\]

Since \( \varphi \) is univalent and \( p < \varphi \), then the function

\[
p_1(z) = \frac{1 + \varphi^{-1}(p(z))}{1 - \varphi^{-1}(p(z))} = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots,
\]

belongs to the class \( \mathcal{P} \). Or equivalently,

\[
p(z) = \varphi \left( p_1(z) \right).
\]

Using the last equation, the coefficient \( b_i \) can be expressed in terms of \( c_i \) and \( B_i \) (\( i \in \mathbb{N} \)). We have

\[
b_1 = \frac{1}{2}B_1c_1, \quad b_2 = \frac{1}{4}((B_2 - B_1)c_1^2 + 2B_1c_2)
\]

and

\[
b_3 = \frac{1}{8}((B_1 - 2B_2 + B_3)c_1^3 + 4(B_2 - B_1)c_1c_2 + 4B_1c_3).
\]

Hence, by using the expressions for \( b_k \) in equation (3.9), we obtain

\[
a_2 = b_1 = \frac{1}{2}B_1c_1, \quad \text{and} \quad a_3 = \frac{1}{8}((B_1^2 - B_1 + B_2)c_1^2 + 2B_1c_2). \tag{3.10}
\]

(1) Since \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \in S^*(\varphi) \) if and only if

\[
\hat{f}(z) = (f(z^3))^{1/3} = z + \beta_4 z^4 + \cdots \in S^{*(3)}(\varphi).
\]

We have \( \beta_4 = b_1/3 \). Hence, for \( f \in S^{*(3)}(\varphi) \), from equation (3.8), we obtain

\[
H_3(1) = |\beta_4|^2 = \frac{1}{9} |b_1|^2 = \frac{1}{36} |B_1c_1|^2 \tag{3.11}
\]

\[
\leq \frac{1}{9} |B_1|^2. \tag{3.12}
\]

Also, the result is sharp for \( f_0(z) = z \exp \int_0^z \frac{\varphi(t) - 1}{t} dt \) and its rotations.
Since \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \in S^*(\varphi) \) if and only if
\[
\hat{f}(z) = (f(z^2))^{1/2} = z + \alpha_3z^3 + \alpha_5z^5 \cdots \in S^{*(2)}(\varphi).
\]
Upon comparing the coefficients in the following:
\[
z^2 + a_2z^4 + a_3z^6 + \cdots = (z + \alpha_3z^3 + \alpha_5z^5 \cdots)^2,
\]
we obtain
\[
\alpha_3 = \frac{1}{2}a_2 \quad \text{and} \quad \alpha_5 = \frac{1}{2}a_3 - \frac{1}{8}a_2^2.
\]
If \( f \in S^{*(2)}(\varphi) \), then from equation (3.8), we have
\[
H_3(1) = \alpha_3(a_5 - \alpha_3^2)
\]
Using equation (3.10),
\[
|H_3(1)| = \left| \frac{1}{2}a_2 \left( \frac{4a_3 - 3a_2^2}{8} \right) \right| = \left| \frac{a_2}{4} \left( a_3 - \frac{3}{4}a_2^2 \right) \right|
\leq \frac{1}{4}a_2 \left| a_3 - \frac{3}{4}a_2^2 \right|.
\]
Now, equation (3.10) and Fekete-Szego bounds [16, Theorem 3] for \( \mu = 3/4 \), we get the desired result.

**Corollary 3.4.** Let \( f \in S^*(\varphi) \) and \( H_3(1) \) is given by equation (3.12). Then

1. \( \hat{f} \in S^{*(3)}(1 + ze^z) \) implies that \( |H_3(1)| \leq 1/9 \).
2. \( \hat{f} \in S^{*(3)}((1 + z)/(1 - z)) \) implies that \( |H_3(1)| \leq 4/9 \).

The sharpness of the bounds follows from [10] and [21] respectively.

**Corollary 3.5.** Let \( f \in S^*(\varphi) \) and \( H_3(1) \) is given by equation (3.14). Then

1. \( \hat{f} \in S^{*(2)}(1 + ze^z) \) implies that \( |H_3(1)| \leq 1/24 \).
2. \( \hat{f} \in S^{*(2)}((1 + z)/(1 - z)) \) implies that \( |H_3(1)| \leq 1/6 \).

**Remark 3.6.** We observe that the bounds obtained in Corollary 3.5 are close to the sharp values and are still open for their sharpness.

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