PERIOD-TWO SOLUTION FOR A CLASS OF DISTRIBUTED DELAY DIFFERENTIAL EQUATIONS

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Abstract. We study the existence of periodic solutions for differential equations with distributed delay. It is shown that, for a class of distributed delay differential equation, a symmetric period-2 solution is obtained as a periodic solution of a Hamiltonian system of ordinary differential equations, where the period is twice the maximum delay. This work extends the result of Kaplan and Yorke (J. Math. Anal. Appl., 1974) for a discrete delay differential equation with an odd nonlinear function. To illustrate the result, we present distributed delay differential equations that have periodic solutions expressed in terms of Jacobi elliptic functions.

1. Introduction

In the study of delay differential equations, it has been shown that delay is often responsible for causing oscillation of the solution. Periodic solutions of delay differential equations (DDEs) have been extensively studied in the literature, see e.g. [19, 27, 28, 30, 38] and references therein. The theory of delay differential equations has been extensively developed and widely applied to problems in several disciplines, see [7, 14, 33].

In the paper [16], Kaplan and Yorke investigate the existence of a symmetric period-4 solution to the following DDE:

\[ x'(t) = -f(x(t-1)), \]

where \( f \) is an odd continuous function mapping \( \mathbb{R} \) to \( \mathbb{R} \). It is assumed that the function \( f \) satisfies \( xf(x) > 0 \) for \( x \neq 0 \) with \( f(0) = 0 \). The authors specifically focused on a periodic solution that satisfies the following symmetry constraint:

\[ x(t) = -x(t+2), \]

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implying the period of the solution is 4. They showed that the following planar system of ordinary differential equations (ODEs)

\[ x' = -f(y), \ y' = f(x) \]

yields such a symmetric periodic solution, satisfying (1.2), to DDE (1.1). The proof of Kaplan and Yorke [16] was elaborated by Nussbaum in [30]. See also [38, 19]. Stability of the so-called Kaplan-Yorke type periodic solution has been analyzed in [6, 13].

The study [16] has been extended to a scalar DDE and a system of DDEs, including (1.1) as a special case. In [8, 9, 15], a scalar DDE incorporating \( x(t) \) inside the feedback function is studied. Fei employs a variational method to analyze the existence of a symmetric periodic solution for a higher-order Hamiltonian systems of ODEs to study a scalar DDE with multiple discrete delays [10, 11]. Similar results are available for a system of DDEs with multiple discrete delays [12, 40]. Recently, the author in [22] characterizes the set of periodic solutions for a scalar DDE, assuming that feedback function satisfies a certain symmetry condition and (positive and negative) monotonicity.

In [25] the author of this paper studies the existence of a period-2 solution for a logistic type differential equation with a distributed delay:

\[ x'(t) = rx(t) \left( 1 - \int_0^1 x(t-s)ds \right). \]

It is shown that a similar ansatz to that in Kaplan and Yorke [16] leads to a system of ODEs that yields a periodic solution to (1.4). It turns out that equation (1.4) has a periodic solution expressed in terms of Jacobi elliptic functions. In [26] the same author also shows that a pendulum equation yields a periodic solution of a differential equation with a distributed delay. The study [26] is recently generalized in [39], where the authors used a variational method.

Distributed delay appears in modeling of biological processes and population dynamics, see [33, Chapter 7] and references therein, transmission of information [32] and so on. Linear stability of distributed delay differential equations (distributed DDEs) has been studied in the literature, see [2, 5, 20, 24, 29, 37] and references therein. In [36] a non-ejective fixed point theorems is applied to prove the existence of a periodic solution for distributed DDEs. We also refer [11] for the study of a coupled system of distributed DDEs. Distributed delay can be seen as a limit case of multiple discrete delays. Kennedy studies the existence of periodic solutions for a differential equation with multiple discrete delays [17]. In [21], employing a computational fixed-point method, the authors study the existence of periodic solutions for DDEs
with multiple discrete delays and show coexistence of multiple periodic solutions. The authors in [3] study the existence of a symmetric periodic solution for a nonlinear renewal equation, applying the connection between DDE (1.1) and the planar system (1.3).

In this paper, applying the idea of Kaplan and Yorke [16], we study the existence of a symmetric periodic solution to the following distributed DDE:

\begin{equation}
    x'(t) = -g \left( \int_0^1 f(x(t-s))ds \right),
\end{equation}

where \( f \) and \( g \) are continuous functions, satisfying certain additional properties. If \( f \) and \( g \) are suitable symmetric odd functions, it is shown that a period-2 solution, satisfying the symmetric condition \( x(t) = -x(t-1) \) (\( \forall t \in \mathbb{R} \)), is given as a solution of the following system of ODEs:

\begin{align}
    x'(t) &= -g(y(t)), \quad (1.6a) \\
    y'(t) &= 2f(x(t)). \quad (1.6b)
\end{align}

The system of ODEs (1.6) represents a planar Hamiltonian system with the following Hamiltonian function:

\begin{equation}
    \int_0^y g(\xi)d\xi + 2 \int_0^x f(\xi)d\xi.
\end{equation}

Since (1.6) is a Hamiltonian system, for some particular nonlinear function \( f \) and \( g \), it is possible to obtain an explicit form of the periodic solution. The form would help our understanding to the nature of the periodic solutions to (1.5). In this paper, we find a symmetric periodic solution of (1.5), exploiting the planar Hamiltonian system (1.6), and obtain a sufficient condition for the existence of such a symmetric periodic solution of (1.5) in Theorem 10. We then recover the results obtained in [25, 26] from the viewpoint of equation (1.5). Kennedy [18] studied the existence of a periodic solution for a general distributed DDE including (1.5) as a special case, employing the fixed point theorem. We here obtain a slightly different condition from the one obtained in [18], due to our different approach.

The paper is organized as follows. In Section 2, we characterize a symmetric period-2 solution of (1.5). It is shown that a period-2 solution yields a family of periodic solutions for a distributed DDE with a multiplicative parameter. In Section 3, we introduce the notion of Special Symmetric Periodic Solution (SSPS for short), following [8, 9]. We show that an SSPS of (1.5) is related to a periodic solution of
Hamiltonian system of ODEs (1.6), see Theorem 7. In Section 4, following [31], we obtain an explicit condition for (1.6) to have a period-2 solution, thus a condition for (1.5) to have a period-2 solution, that is an SSPS. In Section 5, we reconsider particular examples of (1.5), from the previous studies [25, 26]. In the special cases, period-2 solutions are expressed in terms of the Jacobi elliptic functions. Detailed computations and materials are summarized in Appendices A, B and C.

2. Period-2 solution

Definition 1. If a solution to (1.5) satisfies $x(t) = x(t + 2)$ for any $t \in \mathbb{R}$, then the solution is called a period-2 solution.

We impose the following hypotheses on $f$:

(f1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

and the following hypotheses on $g$:

(g1) $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.
(g2) $xg(x) > 0$ for $x \in \mathbb{R} \setminus \{0\}$ and $g(0) = 0$.
(g3) $g$ is an odd function, i.e., $g(x) = -g(-x)$ for $x \in \mathbb{R}$.

The following Proposition motivates the study of the period-2 solution to (1.5).

Proposition 2. Suppose that (f1), (g1-3) hold. Let $x : \mathbb{R} \rightarrow \mathbb{R}$ be a solution of equation (1.5). Then the following statements are equivalent.

1. There exists $t_0 \in \mathbb{R}$ for any $t \in \mathbb{R}$ such that $x(t_0 + t) = x(t_0 - t)$ holds, i.e., $x$ is an even function about $t = t_0$.

2. There exists $c \in \mathbb{R}$ such that

\[(2.1) \quad \forall t \in \mathbb{R}, \; x(t) + x(t - 1) = c\]

thus the solution $x$ is a period-2 solution.

Proof. (1) $\Rightarrow$ (2) From the time translation invariance of equation (1.5), one can assume that $t_0 = 0$ without loss of generality. Suppose that
\( x(t) = x(-t), \forall t \in \mathbb{R}. \) Then, one obtains

\[
    x'(t) = -x'(-t) \\
    = g \left( \int_0^1 f(x(-t-s))ds \right). \\
    = g \left( \int_0^1 f(x(t+s))ds \right) \quad (\because x(t) = x(-t)) \\
    = g \left( \int_0^1 f(x(t+1-s))ds \right) \\
    = -x'(t+1),
\]

which implies that \( x(t) + x(t-1) \) is a constant. Therefore, we obtain the statement (2).

(2) \( \Rightarrow \) (1) Let us assume that there exists a constant \( c \) such that for \( \forall t \in \mathbb{R}, x(t) + x(t-1) = c. \) Since \( x(t-1) = c - x(t), \) the solution of (1.5) satisfies the following system of ODEs

\[
(2.2a) \quad x' = -g(y), \\
(2.2b) \quad y' = f(x) - f(c - x),
\]

where \( y = \int_0^1 f(x(t-s))ds. \) Consider the solution of (2.2) with the initial condition \((x(t_0), y(t_0)) = (x_0, 0)\) where \( x_0 \) is not a zero of \( f(x) - f(c - x) \) and \( t_0 \in \mathbb{R} \) is the initial time. One can easily see that \((\tilde{x}(t+t_0), \tilde{y}(t+t_0)) := (x(t_0 - t), -y(t_0 - t))\) solves (2.2) with the initial condition \((\tilde{x}(t_0), \tilde{y}(t_0)) = (x(t_0), y(t_0)) = (x_0, 0).\) From the uniqueness of the solution (see e.g. [35]), we obtain \( x(t_0 + t) = x(t_0 - t). \) We thus obtain the statement (1). \( \square \)

**Remark 3.** Similar results are obtained in [1, Lemmas 3.1, 3.2] for a nonlinear renewal equation.

We have the following result.

**Lemma 4.** Let \( g(x) = x. \) Suppose that (f1) holds. Let \( x : \mathbb{R} \to \mathbb{R} \) be a solution of equation (1.5). Assume that there exists \( c \in \mathbb{R} \) such that (2.1) holds. Then, it holds that

\[
(2.3) \quad \forall t \in \mathbb{R}, \quad \int_0^2 f(x(t-s))ds = 0.
\]

**Proof.** From equation (1.5), one has

\[
x'(t) + x'(t-1) = - \int_0^2 f(x(t-s))ds.
\]
Since \( x(t) + x(t-1) \) is a constant, \( x'(t) + x'(t-1) = 0 \) for \( \forall t \in \mathbb{R} \) follows. Thus (2.3) holds.

Then, we show that the period-2 solution of equation (1.5) gives rise to a family of periodic solutions to the equation of the following form

\[
(2.4) \quad x'(t) = -r \int_0^1 f(x(t-s))ds,
\]

where \( r > 0 \) is a multiplicative parameter.

**Proposition 5.** Let \( g(x) = x \). Suppose that (f1) holds. Let \( x : \mathbb{R} \to \mathbb{R} \) be a solution of equation (1.5). Assume that there exists \( c \in \mathbb{R} \) such that (2.1) holds. Then, for every \( n \in \mathbb{N} \), equation (2.4) with \( r = (2n - 1)^2 \) has a periodic solution \( x((2n - 1)t) \) of period \( 2/(2n - 1) \).

**Proof.** For \( n \in \mathbb{N} \), we let \( x_n(t) := x(\alpha_n t) \), where \( \alpha_n := 2n - 1 \). From the direct computation,

\[
(2.5) \quad x_n'(t) = \alpha_n x'(\alpha_n t) = -\alpha_n \int_0^1 f(x(\alpha_n t-s))ds.
\]

Applying (2.3) in Lemma 4, we see that

\[
\int_1^{\alpha_n} f(x(\alpha_n t-s))ds = 0
\]

holds. Therefore,

\[
(2.6) \quad \int_0^1 f(x(\alpha_n t-s))ds = \int_0^{\alpha_n} f(x(\alpha_n t-s))ds = \alpha_n \int_0^1 f(x_n(t-s))ds.
\]

From (2.6) and (2.5), we see that \( x_n(t) \) solves equation (2.4) with \( r = \alpha_n^2 = (2n - 1)^2 \). \qed

### 3. Distributed DDE (1.5) and Hamiltonian System (1.6)

In this section, we show that distributed DDE (1.5) and Hamiltonian system of ODEs (1.6) are related via a period-2 solution.

We now impose the following hypotheses on \( f \):

\begin{itemize}
  \item [(f2)] \( xf(x) > 0 \) for \( x \in \mathbb{R} \setminus \{0\} \) and \( f(0) = 0 \).
  \item [(f3)] \( f \) is an odd function, i.e., \( f(x) = -f(-x) \) for \( x \in \mathbb{R} \).
\end{itemize}

Then, analogously to [8, 9], we define a special symmetric periodic solution (SSPS for short) for equation (1.5).

**Definition 6.** Let \( x : \mathbb{R} \to \mathbb{R} \) be a solution of equation (1.5). \( x \) is called a special symmetric periodic solution (SSPS) of (1.5) if \( x \) is a periodic solution of (1.5) with minimal period 2, satisfying

\[
(3.1) \quad x(t) + x(t-1) = 0, \ \forall t \in \mathbb{R}.
\]
Similar to the well-known connection established in Kaplan and Yorke [16] for the symmetric periodic solution between the discrete DDE (1.1) and the Hamiltonian system (1.3), we establish the following connection between the distributed DDE (1.5) and the Hamiltonian system (1.6).

**Theorem 7.** Suppose that (f1-3) and (g1-3) hold. Then the following statements are true.

1. Let \( x : \mathbb{R} \to \mathbb{R} \) be an SSPS of equation (1.5). Then
   \[
   (x(t), y(t)) = \left( x(t), \int_0^1 f(x(t-s)) \, ds \right)
   \]
   is a periodic solution in \( \mathbb{R}^2 \) with minimal period 2 for Hamiltonian system (1.6).

2. Let \( \{(x(t), y(t)) : t \in \mathbb{R} \} \subset \mathbb{R}^2 \) be a closed orbit with minimal period 2 around the origin of Hamiltonian system (1.6). Then \( x : \mathbb{R} \to \mathbb{R} \) is an SSPS of equation (1.5).

Thus equation (1.5) has an SSPS if and only if Hamiltonian system (1.6) has a closed orbit in \( \mathbb{R}^2 \) with minimal period 2.

**Proof.**

1. Let \( x \) be an SSPS of equation (1.5). Define
   \[
   y(t) := \int_0^1 f(x(t-s)) \, ds.
   \]
   Then
   \[
   y'(t) = f(x(t)) - f(x(t-1)).
   \]
   From (f3) and the property of SSPS that is (3.1), one has
   \[
   -f(x(t-1)) = f(-x(t-1)) = f(x(t)).
   \]
   Therefore, we see that \( (x(t), y(t)) \) satisfies the Hamiltonian system (1.6). It is obvious that \( (x(t), y(t)) \) is a closed symmetric orbit around the origin of minimal period 2.

2. Let \( \{(x(t), y(t)) : t \in \mathbb{R} \} \) be a closed symmetric trajectory around the origin of minimal period 2 for the Hamiltonian system (1.6). Since \( \{(-x(t), -y(t)) : t \in \mathbb{R} \} \) represents the same trajectory, one can find that
   \[
   (x(t+1), y(t+1)) = (-x(t), -y(t)), \quad t \in \mathbb{R}.
   \]
   Since \( x(t) \) is of period 2, the property \( x(t-1) = -x(t) \) holds. Therefore, it follows that, using the condition (f3), \( f(x(t)) = -f(-x(t)) = -f(x(t-1)) \), thus
   \[
   y'(t) = 2f(x(t)) = f(x(t)) - f(x(t-1)),
   \]
from which we obtain $y(t) = \int_{t-1}^{t} f(x(s))ds + \text{Const}$. We show that the constant is zero. Consider a closed orbit around the origin of system (1.6) satisfying $x(\frac{1}{2}) = 0$ and $y(\frac{1}{2}) > 0$. From Lemma 14 in Appendix A, $(x, y) = (-x(-t), y(-t))$ also solve (1.6). Therefore, from the uniqueness of the solution of (1.6), we have $(x(\frac{1}{2}+t), y(\frac{1}{2}+t)) = (-x(\frac{1}{2}-t), y(\frac{1}{2}-t))$. Thus $x$ is an odd function about $t = \frac{1}{2}$, i.e., $x(t + \frac{1}{2}) = -x(t - \frac{1}{2})$. It then follows that

$$y(t) = \int_{t-1}^{t} f(x(s))ds.$$  

Consequently, the first component of the solution of equation (1.6) yields an SSPS $x(t)$ for distributed DDE (1.5). \hfill \Box

4. Period map of Hamiltonian system (1.6)

In this section, following [31, Chapters I, II], we derive an explicit condition for Hamiltonian system (1.6) to have the periodic orbit of period-2 around the origin. We suppose that (f1-3) and (g1-3) hold. Consider the solution of the Hamiltonian system (1.6) with the following initial condition

$$\tag{4.1} (x(0), y(0)) = (x_0, 0), \quad x_0 \in \mathbb{R}.$$  

The trajectory of the solution $(x(t), y(t))$ with the initial condition (4.1) is a closed orbit in $\mathbb{R}^2$, satisfying

$$\tag{4.2} 2F(x) + G(y) = 2F(x_0),$$  

where

$$F(x) = \int_{0}^{x} f(\xi)d\xi, \quad G(y) = \int_{0}^{y} g(\xi)d\xi.$$  

It is easy to see that the closed orbit given by (4.2) with $x_0 \neq 0$ is symmetric about the $x$-axis and the $y$-axis.

For $x, y \in \mathbb{R}$, we implicitly define two variables $u$ and $v$ via

$$\tag{4.3a} 2F(x) = \frac{1}{2}u^2, \quad \text{sign}x = \text{sign}u,$$

$$\tag{4.3b} G(y) = \frac{1}{2}v^2, \quad \text{sign}y = \text{sign}v.$$  

Note that $u$ and $v$ are uniquely determined and that $(u, v) = (0, 0)$ for $(x, y) = (0, 0)$.

We suppose that
(H1) There exist
\[ a = \lim_{x \to 0} \frac{f(x)}{x}, \quad A = \lim_{x \to \infty} \frac{f(x)}{x}, \]
\[ b = \lim_{y \to 0} \frac{g(y)}{y}, \quad B = \lim_{y \to \infty} \frac{g(y)}{y} \]
allowing that those quantities are 0 and \( \infty \).

(H2) It holds that
\[ \lim_{x \to \infty} F(x) = \infty, \quad \lim_{x \to \infty} G(x) = \infty. \]

Note that \( 0 \leq a, b, A, B \leq \infty \) and that \( u \to \infty \) (as \( x \to \infty \)) and \( v \to \infty \) (as \( y \to \infty \)), from (f1-2) and (g1-2).

From (4.3), it is easy to see that, for \( u \neq 0 \), \( x \) is differentiable with respect to \( u \). Similarly, for \( v \neq 0 \), \( y \) is differentiable with respect to \( v \). Denote by \( x_u \) the derivative of \( x \) with respect to \( u \) and \( y_v \) the derivative of \( y \) with respect to \( v \). Then one has
\[ x_u = \frac{u}{2f(x)}, \quad y_v = \frac{v}{g(y)} \]
for \( u \neq 0 \) and \( v \neq 0 \).

**Lemma 8.** Suppose that (H1-2) hold. If \( a, b \in (0, \infty) \), then \( x \) and \( y \) are differentiable at \( u = 0 \) and \( v = 0 \), respectively, and
\[ (4.5a) \quad x_u(0) = \lim_{u \to 0} x_u(u) = \frac{1}{\sqrt{2a}}, \]
\[ (4.5b) \quad y_v(0) = \lim_{v \to 0} y_v(v) = \frac{1}{\sqrt{b}}. \]

If \( A, B \in (0, \infty) \), then it also holds that
\[ (4.6a) \quad \lim_{u \to \infty} x_u(u) = \frac{1}{\sqrt{2A}}, \]
\[ (4.6b) \quad \lim_{v \to \infty} y_v(v) = \frac{1}{\sqrt{B}}. \]

**Proof.** First, we consider \( x_u(0) = \lim_{u \to 0} x/u \). From (4.3a), one has
\[ \lim_{u \to 0} \frac{x}{u} = \lim_{x \to 0} \frac{x}{2\sqrt{F(x)}.} \]

By l’Hôpital’s rule and (4.3a),
\[ \lim_{x \to 0} \frac{x^2}{4F(x)} = \lim_{x \to 0} \frac{2x}{4f(x)} = \frac{1}{2a}. \]
Thus we obtain \( x_u(0) = 1/\sqrt{2a} \). One also sees that
\[
\lim_{u \to 0} x_u(u) = \lim_{u \to 0} \frac{u}{2f(x)}
\]
Then by l'Hôpital’s rule and (4.3a),
\[
\lim_{u \to 0} u^2 = \lim_{x \to 0} \frac{F(x)}{x^2} = \lim_{x \to 0} \frac{f(x)}{x} \cdot \frac{1}{2x} \cdot \frac{1}{f(x)/x^2} = \frac{1}{2a}.
\]
Hence we obtain (4.5a). Similarly, we obtain other equalities in (4.5) and (4.6).

From the proof in Lemma 8, we also have the following equalities:
\[
\begin{align*}
\lim_{u \to 0} x_u(u) &= \begin{cases} 
\infty & \text{if } a = 0, \\
0 & \text{if } a = \infty,
\end{cases} \\
\lim_{v \to 0} y_v(v) &= \begin{cases} 
\infty & \text{if } b = 0, \\
0 & \text{if } b = \infty,
\end{cases}
\end{align*}
\]
Since \( x' = x_uu' \) and \( y' = y_vv' \), where \( u' \) and \( v' \) are derivative of \( u \) and \( v \) with respect to \( t \), from (1.6) and (4.4), one obtains
\[
(4.7) \quad x_u y_v u' = -v, \quad x_u y_v v' = u.
\]
Consider the time transformation defined by
\[
\frac{dt}{d\theta} = x_u y_v, \quad t(0) = 0,
\]
then, from (4.7), we obtain the following equations:
\[
\frac{du}{d\theta} = -v, \quad \frac{dv}{d\theta} = u.
\]
Hence we have
\[
(u, v) = (p \cos \theta, p \sin \theta),
\]
where \( p = 2\sqrt{F(x_0)} \) from (4.3a). From (f3) and (g3), the period of the solution \( T = T(p) \) is given as
\[
T(p) = 4 \int_0^{\pi/2} x_u(p \cos \theta) y_v(p \sin \theta) d\theta.
\]
Finally, applying the Lebesgue’s Dominated Convergence Theorem, as in \[31\] Proposition 1.6.1 and \[31\] Section 5, we obtain the asymptotic behaviour of the period map as follows.
Lemma 9. Suppose that (H1-2) hold. One has
\[ \lim_{p \rightarrow +0} T(p) = \sqrt{\frac{2}{ab}} \pi, \quad \lim_{p \rightarrow \infty} T(p) = \sqrt{\frac{2}{AB}} \pi, \]
with the convention \( \frac{1}{0} = \infty \) and \( \frac{1}{\infty} = 0 \).

Therefore, we obtain the existence of the period-2 solution to (1.5), according to Theorem 7.

Theorem 10. Suppose that (f1-3), (g1-3) and (H1-2) hold. If \( a, b, A \) and \( B \) satisfy either
\[ ab < \frac{\pi^2}{2} < AB, \quad \text{or} \quad AB < \frac{\pi^2}{2} < ab, \]
then system (1.6) has a closed orbit with minimal period 2, thus equation (1.5) has an SSPS.

5. Examples

In this section we study distributed DDE (1.5) with particular non-linear functions. We express the periodic solutions in terms of the Jacobi elliptic functions.

Let us introduce the complete elliptic integrals of the first kind and of the second kind respectively given as
\[ K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta, \]
\[ E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta \]
for \( 0 \leq k < 1 \). We refer [4, 23] for the fundamental properties of the Jacobi elliptic functions.

5.1. \( f(x) = r \sin(x) \) and \( g(x) = x \). In [26], we consider the existence of an SSPS to the following equation:
\[ x'(t) = -g \left( \int_0^1 x(t - s) ds \right), \]
with \( g(x) = r \sin x \), where \( r > 0 \). In Appendix B, we show a transformation between equations (5.1) and (1.5) with \( g(x) = x \). Here, we consider the existence of an SSPS of (1.5) with \( f(x) = r \sin x \) and \( g(x) = x \). One can see that the system of ODEs (1.6) with \( f(x) = r \sin x \) and \( g(x) = x \) is a well-known pendulum equation. We denote by \( cn, sn, dn \) Jacobi elliptic functions. See [4, 23] for the fundamental properties of the Jacobi elliptic functions. We then obtain the following result.
**Theorem 11.** Let \( f(x) = r \sin x, \) \( r > 0 \) and \( g(x) = x. \) Equation (1.5) has an SSPS if and only if \( r > \frac{\pi^2}{2}. \) The SSPS is expressed as

\[
x(t) = 2 \arcsin \left( k \operatorname{sn} \left( \sqrt{2r} t, k \right) \right),
\]

where \( k \in (0, 1) \) is uniquely determined as the solution of the following equation:

\[
\frac{2K(k)}{\sqrt{2r}} = 1.
\]

**Proof.** We seek an SSPS \( x(t) \) satisfying \( -\pi < x(t) < \pi \) for \( t \in \mathbb{R} \) for (1.5), applying Theorem 7. In Appendix C, we compute a periodic orbit for (1.6), surrounding the origin. (1.6) has a periodic solution given as in (C.5) in Lemma 16, where the period is \( T = T(k) \) given as in (C.6). Since \( K(k) \) is an increasing function of \( k \) with \( K(0) = \frac{\pi}{2} \), there exists a unique \( k \in (0, 1) \) such that \( T(k) = 2 \) if and only if \( r > \frac{\pi^2}{2}. \) Thus, for \( r > \frac{\pi^2}{2} \), one can see that there exists a unique \( k \in (0, 1) \) such that (5.3) holds and that (5.2) is the SSPS for (1.5). \( \square \)

5.2. \( f(x) = r (e^x - 1) \) and \( g(x) = x. \) In [25] we study logistic equation with distributed delay (1.4). Changing the variable \( x \mapsto \log x, \) we obtain equation (1.5) with \( f(x) = r (e^x - 1) \) and \( g(x) = x. \) Note that \( f(x) \) is not an odd function, thus we may not directly apply Theorem 7 to obtain the period-2 solution of (1.5).

Consider the existence of a period-2 solution, satisfying

\[
x(t) + x(t - 1) = c,
\]

where \( c \) denotes a constant. The period-2 solution satisfies the following system of ODEs:

\[
\begin{align*}
x' &= -y, \\
y' &= f(x) - f(c - x),
\end{align*}
\]

(see also the proof of Proposition 2). Note that \( c \) is a parameter to be determined.

Let \( w = x - \frac{1}{2}c. \) Then system (5.5) becomes

\[
\begin{align*}
w' &= -y, \\
y' &= f \left( w + \frac{1}{2}c \right) - f \left( \frac{1}{2}c - w \right).
\end{align*}
\]
For \( f(x) = r(e^x - 1) \), we see
\[
f \left( w + \frac{1}{2}c \right) - f \left( \frac{1}{2}c - w \right) = r \left( e^{w + \frac{1}{2}c} - 1 \right) - r \left( e^{-w + \frac{1}{2}c} - 1 \right)
= 2re^{\frac{1}{2}c} \sinh w.
\]
Thus we obtain the following system of ODEs:
\[
(5.6a) \quad w' = -y,
(5.6b) \quad y' = 2re^{\frac{1}{2}c} \sinh w.
\]
Let us first find a period-2 solution to system (5.6). Consider the periodic solution of (5.6) with the following initial condition
\[
(5.7) \quad (w(0), y(0)) = (w_0, 0), \quad w_0 > 0.
\]
Let
\[
\gamma = re^{\frac{1}{2}c}, \quad \alpha = \sinh \left( \frac{w_0}{2} \right) > 0.
\]
Define
\[
(5.8) \quad \beta = \sqrt{2\gamma(1 + \alpha^2)}, \quad k = \frac{\alpha}{\sqrt{1 + \alpha^2}} \in (0, 1).
\]
Note that
\[
(5.9) \quad \beta k = \alpha \sqrt{2\gamma}, \quad \alpha = \frac{k}{\sqrt{1 - k^2}}
\]
hold. In terms of \( \beta \) and \( k \), the solution of (5.6) is expressed as follows, see Appendix C for the proof.

**Lemma 12.** The solution of system (5.6) with the initial condition (5.7) is expressed as
\[
(5.10a) \quad w(t) = \log \left( \frac{\text{dn}(\beta t, k) + k \text{cn}(\beta t, k)}{\text{dn}(\beta t, k) - k \text{cn}(\beta t, k)} \right),
(5.10b) \quad y(t) = 2\beta k \text{sn}(\beta t, k).
\]
The period of the solution (5.10) for (5.6) is 2 if and only if
\[
(5.11) \quad \beta = 2K(k),
\]
where \( K \) denotes the complete elliptic integrals of the first kind.
Furthermore, \( x = w + \frac{1}{2}c \) satisfies (3.2) with \( y(0) = 0 \) if and only if
\[
0 = -\int_0^1 f(x(-s))ds.
\]
Finally, we obtain the following result.
Theorem 13. Let \( f(x) = r(e^x - 1), \) \( r > 0 \) and \( g(x) = x. \) Equation (1.5) has a period-2 solution, satisfying (5.4) for some \( c \in \mathbb{R}, \) if and only if \( r > \frac{\pi^2}{2}. \) The period-2 solution is expressed as

\[
(5.12) \quad x(t) = \log \left( \frac{K(k)(dn(2K(k)t, k) + kcn(2K(k)t, k))^2}{2E(k) - K(k)(1 - k^2)} \right),
\]

where \( k \in (0, 1) \) is a unique solution of

\[
(5.13) \quad r = 2K(k)(2E(k) - K(k)(1 - k^2))
\]

for \( r > \frac{\pi^2}{2}. \)

Proof. From Lemma 12, we can express \( x(t) \) as

\[
(5.14) \quad x(t) = \log \left( \frac{dn(\beta t, k) + kcn(\beta t, k)}{dn(\beta t, k) - kcn(\beta t, k)} \right) + \frac{1}{2}c.
\]

We determine two parameters \( k \) and \( c \) so that (5.14) is a period-2 solution of equation (1.5).

Equation (5.11) holds if and only if the period of (5.10a) and (5.10b) is 2. We thus assume that (5.11) holds. Then

\[
(5.15) \quad re^{\frac{1}{2}c} = 2K^2(k)(1 - k^2).
\]

Furthermore we require that

\[
y(t) = \int_0^1 f(x(t - s)) ds
\]

holds for \( t \in \mathbb{R}, \) where \( y(t) \) is (5.10b).

Since, from (5.11), \( y(1) = 2 \beta k sn(\beta, k) = 0 \) follows, we must have

\[
(5.16) \quad 0 = \int_0^1 (e^{x(t)} - 1) dt.
\]

Substituting (5.14) into (5.16), we obtain

\[
0 = e^{\frac{1}{2}c} \int_0^1 \frac{dn(\beta t, k) + kcn(\beta t, k)}{dn(\beta t, k) - kcn(\beta t, k)} dt - 1.
\]

We now compute

\[
\frac{dn(\beta t, k) + kcn(\beta t, k)}{dn(\beta t, k) - kcn(\beta t, k)} = \frac{(dn(\beta t, k) + kcn(\beta t, k))^2}{1 - k^2}
\]

\[
= \frac{2dn^2(\beta t, k) + 2kdn(\beta t, k)cn(\beta t, k)}{1 - k^2} - 1.
\]

Then, we obtain

\[
\int_0^1 \frac{dn(\beta t, k) + kcn(\beta t, k)}{dn(\beta t, k) - kcn(\beta t, k)} dt = \frac{2}{1 - k^2} E(k) - 1,
\]
thus

\begin{equation}
(5.17) \quad e^{\frac{1}{2}c} \left( \frac{2}{1-k^2} \frac{E(k)}{K(k)} - 1 \right) - 1 = 0.
\end{equation}

Here, we use

\[
\int_0^1 \text{dn}^2(\beta t, k) \, dt = \frac{2}{\beta} E(k) = \frac{E(k)}{K(k)},
\]

(See also 314.02 in [4] for the former identity) and

\[
\int_0^1 \text{dn}(\beta t, k) \, \text{cn}(\beta t, k) \, dt = \frac{1}{\beta} \left[ \text{sn}(\beta t, k) \right]_{t=0}^{t=1} = 0,
\]

from the condition (5.11).

From (5.15) and (5.17), we obtain equation (5.13). From Lemma 6 in [25], one can see that (5.13) has a unique solution for \( r > \pi^2/2 \). Then, from (5.15), we obtain that

\[
\frac{1}{2} c = \log \frac{2K^2(k)(1-k^2)}{r} = \log \frac{K(k)(1-k^2)}{2E(k) - K(k)(1-k^2)}.
\]

Finally, from (5.14), it follows that

\[
x(t) = \log \left( \frac{\text{dn}(\beta t, k) + k\text{cn}(\beta t, k)}{\text{dn}(\beta t, k) - k\text{cn}(\beta t, k)} \frac{K(k)(1-k^2)}{2E(k) - K(k)(1-k^2)} \right).
\]

Using (5.11), we obtain the expression (5.12). \( \square \)

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**Appendix A. Symmetry of the solution of Hamiltonian system (1.6) with odd nonlinearity**

In this section we suppose that (f1-3) and (g1-3) hold. Consider the Hamiltonian system of ODEs (1.6).

**Lemma 14.** Suppose that (f1-3) and (g1-3) hold. Let \((x(t), y(t))\) be a solution of (1.6). Then \((-x(t), -y(t)), (x(-t), -y(-t)), (-x(-t), y(-t))\) are also solutions of (1.6).
By direct substitution, one can prove Lemma 14 using the symmetric condition on $f$ and $g$. We here omit the detail.

**Proposition 15.** Suppose that (f1-3) and (g1-3) hold. Let $(x(t), y(t))$ be a periodic solution of (1.6) of period $T$ with the initial condition (4.1). Then the following statements are true.

1. $x(t)$ is an even function of $t$ and $y(t)$ is an odd function of $t$.
2. $x\left(\frac{1}{4}T + t\right)$ is an odd function of $t$ and $y\left(\frac{1}{4}T + t\right)$ is an even function of $t$.

In particular, one has

$$(x(t), y(t)) = \left( -x\left(\frac{1}{2}T - t\right), y\left(\frac{1}{2}T - t\right) \right)$$

$$= \left( -x\left(\frac{1}{2}T + t\right), -y\left(\frac{1}{2}T + t\right) \right)$$

$$= (x(T - t), -y(T - t)).$$

**Appendix B. Equivalent formulation**

Here we show that equation (5.1) can be transformed to (1.5) with $g(x) = x$. Let $g(x) = x$. Denote by $x$ a solution of (1.5). For a solution of (1.5), we define a function $v : [−1, \infty) \to \mathbb{R}$ by

(B.1) \[ v' = -f(x) \]

with $x(0) = \int_{0}^{1} v(-s)ds$. Integrating (B.1), it follows that

$$v(t) - v(t - 1) = -\int_{t-1}^{t} f(x(s))dt.$$  

From (1.5), one sees that $v(t) - v(t - 1) = x'(t)$. Therefore, we find the following equality:

(B.2) \[ x(t) = \int_{0}^{1} v(t - s)ds. \]

Substituting (B.2) into equation (B.1), we obtain the following equation:

$$v'(t) = -f\left(\int_{0}^{1} v(t - s)ds\right).$$

**Appendix C. Explicit periodic solutions**

In this section, we derive the expressions of periodic solutions of Hamiltonian system (1.6) for $f(x) = r \sin x$, $r > 0$ and $g(x) = x$ in Section C.1 and for $f(x) = r(e^x - 1)$, $r > 0$ and $g(x) = x$ in Section C.2.
C.1. \( f(x) = r \sin x \) and \( g(x) = x \). Let \( f(x) = r \sin x, \ r > 0 \) and \( g(x) = x \). We consider the periodic solution of (1.6) with the following initial condition:

\[
(C.1) \quad (x(0), y(0)) = (x_0, 0), \ x_0 \in (0, \pi).
\]

From (1.2) we have

\[
(C.2) \quad y(t)^2 + 8r \sin^2 \frac{x(t)}{2} = 8r \sin^2 \frac{x_0}{2}.
\]

Let \( u = \sin \frac{x}{2} \). From (1.6a) and (C.2), we obtain

\[
(C.3) \quad (u')^2 = 2r \left(1 - u^2 \right) \left(k^2 - u^2 \right),
\]

where

\[
(C.4) \quad k = \sin \frac{x_0}{2} \in (0, 1).
\]

**Lemma 16.** Let \( f(x) = r \sin x, \ r > 0 \) and \( g(x) = x \). The periodic solution of (1.6) with the initial condition (C.1) is expressed as

\[
(C.5a) \quad x(t) = 2 \arcsin \left(k \text{sn} \left(\sqrt{2r} t + K(k), k \right) \right),
\]

\[
(C.5b) \quad y(t) = -2\sqrt{2r}k \text{cn} \left(\sqrt{2r} t + K(k), k \right).
\]

where \( k \) is defined in (C.4). The period of the solution is given as \( T \), where

\[
(C.6) \quad T = \frac{4K(k)}{\sqrt{2r}}.
\]

**Proof.** Equation (C.3) has the solution of the form:

\[
u(t) = k \text{ sn} \left(\sqrt{2r} t + c, k \right),
\]

where \( c \) is determined by the initial condition \( u(0) = \sin \frac{x_0}{2} = k \), so that \( \text{sn}(c, k) = 1 \). Then we see \( c = K(k) \) for \( 0 \leq c \leq 4K(k) \). From the definition, we have \( x = 2 \arcsin u \). Thus we obtain (C.5a). From (1.6a), one has \( y(t) = -x'(t) \). Thus we obtain (C.5b). \( \square \)

C.2. \( f(x) = r(e^x - 1) \) and \( g(x) = x \). Let \( f(x) = r(e^x - 1), \ r > 0 \) and \( g(x) = x \). We consider a periodic solution of (5.6) with the initial condition (5.7). From (1.2) we have

\[
(C.7) \quad y^2 + 8\gamma \sinh^2 \left(\frac{w}{2} \right) = 8\gamma \sinh^2 \left(\frac{w_0}{2} \right),
\]

using \( \cosh w = 1 + 2 \sinh^2 \frac{w}{2} \). Let

\[
(C.8) \quad u := \sinh \left(\frac{w}{2} \right).
\]
Then, from (5.6) and (C.7), we obtain

\[(C.9) \quad (u')^2 = \frac{1}{4} \cosh^2 \left(\frac{w}{2}\right) y^2 = 2\gamma (1 + u^2) \left(\alpha^2 - u^2\right) .\]

**Lemma 17.** The periodic solution of (C.9) with the initial condition
\[u(0) = \alpha\] is expressed as
\[(C.10) \quad u(t) = \alpha \cn(\beta t, k),\]
where the period is given as
\[(C.11) \quad T = \frac{4K(k)}{\beta} .\]

Here \(\beta\) and \(k\) are defined in (5.8).

**Proof.** We consider the solution of the following form:
\[u(t) = \sqrt{c_1} \cn \left(\sqrt{c_2} t, \sqrt{c_3}\right) .\]

By a direct computation, one sees
\[(u')^2 = c_1 c_2 \sn^2 \left(\sqrt{c_2} t, \sqrt{c_3}\right) \dn^2 \left(\sqrt{c_2} t, \sqrt{c_3}\right)
= c_1 c_2 \left(1 - \cn^2 \left(\sqrt{c_2} t, \sqrt{c_3}\right)\right) \left(1 - c_3 \sn^2 \left(\sqrt{c_2} t, \sqrt{c_3}\right)\right)
= c_1 c_2 \left(1 - \cn^2 \left(\sqrt{c_2} t, \sqrt{c_3}\right)\right) \left(1 - c_3 \left(1 - \cn^2 \left(\sqrt{c_2} t, \sqrt{c_3}\right)\right)\right)
= c_1 c_2 \left(1 - \frac{1}{c_1} u^2\right) \left(1 - c_3 + \frac{c_3}{c_1} u^2\right) .\]

From (C.9), we get
\[(C.12a) \quad c_1 c_2 \left(1 - c_3\right) = 2\gamma \alpha^2 ;
(C.12b) \quad c_2 \left(c_3 - \left(1 - c_3\right)\right) = 2\gamma \left(\alpha^2 - 1\right) ;
(C.12c) \quad \frac{c_2 c_3}{c_1} = 2\gamma .\]

From (C.12), eliminating \(c_1\) and \(c_2\), we have
\[\frac{\alpha^2}{(\alpha^2 - 1)^2} = \frac{c_3 (1 - c_3)}{(2c_3 - 1)^2} ,\]
from which we obtain
\[0 = c_3^2 - c_3 + \frac{\alpha^2}{(\alpha^2 + 1)^2} = \left(c_3 - \frac{\alpha^2}{1 + \alpha^2}\right) \left(c_3 - \frac{1}{1 + \alpha^2}\right) .\]
Hence
\[c_3 = \frac{\alpha^2}{1 + \alpha^2} ; \quad \frac{1}{1 + \alpha^2} \in [0, 1] .\]
Then, from (C.12b), we have
\[ c_2 = \begin{cases} 2\gamma(1 + \alpha^2) & \text{if } c_3 = \frac{\alpha^2}{1 + \alpha^2}, \\ -2\gamma(1 + \alpha^2) & \text{if } c_3 = \frac{1}{1 + \alpha^2}. \end{cases} \]

From positivity of \( c_2 \), one obtains \( c_3 = \frac{\alpha^2}{1 + \alpha^2}. \) Therefore, we now have
\[ (c_1, c_2, c_3) = \left( \alpha^2, 2\gamma(1 + \alpha^2), \frac{\alpha^2}{1 + \alpha^2} \right) = \left( \alpha^2, \beta^2, k^2 \right). \]

Hence we obtain (C.5a). The period of the solution is given as in (C.11), from the property of the Jacobi elliptic function \( cn \). \( \square \)

Proof of Lemma 12. From (C.10) in Lemma 17 we have
\[ w(t) = 2 \sinh^{-1}(\alpha \text{cn}(\beta t, k)) \]
\[ = 2 \log \left( \alpha \text{cn}(\beta t, k) + \sqrt{1 + \alpha^2 \text{cn}^2(\beta t, k)} \right), \]
where \( \beta \) and \( k \) are defined in (5.8). Then, using (5.9), we obtain
\[ w(t) = 2 \log \left( \frac{k \text{cn}(\beta t, k) + \sqrt{1 - k^2 + k^2 \text{cn}^2(\beta t, k)}}{\sqrt{1 - k^2}} \right) \]
\[ = 2 \log \left( \frac{k \text{cn}(\beta t, k) + \text{dn}(\beta t, k)}{\sqrt{1 - k^2}} \right) \]
\[ = \log \left( \frac{\text{dn}(\beta t, k) + k \text{cn}(\beta t, k)}{\text{dn}(\beta t, k) - k \text{cn}(\beta t, k)} \right). \]

Note that \( \text{dn}^2(\beta t, k) - k^2 \text{cn}^2(\beta t, k) = 1 - k^2 \) holds. Hence we obtain (5.10a). From (C.7)
\[ y(t)^2 = 8\gamma \left( \alpha^2 - \sinh^2 \left( \frac{w}{2} \right) \right) \]
\[ = 8\gamma \alpha^2 \left( 1 - \text{cn}^2(\beta t, k) \right) \]
\[ = 8\gamma \alpha^2 \text{sn}^2(\beta t, k). \]

Hence we obtain (5.10b). \( \square \)

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