Continuity of Minima: Local Results

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Abstract Berge’s maximum theorem is an important statement in set-valued analysis that has significant applications in mathematical economics, operations research, and control. For a minimization problem for a continuous function of two variables and continuous compact-valued set-valued mapping defining feasible sets, this theorem states continuity of the value function and upper semi-continuity of the solution multifunction. One of the main limitations of this theorem is that the set-valued mapping is compact-valued. Recently the authors of this paper and their coauthors generalized Berge’s maximum theorem to set-valued mappings that may not be compact-valued. Here we formulate and prove the local Berge’s maximum theorem for possibly noncompact feasible sets and show that it is more general than the recently established Berge’s maximum theorem for possibly noncompact feasible sets and than the known formulations of the local Berge’s maximum theorem.

Keywords Berge’s maximum theorem · Set-valued mapping · Continuity

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1 Introduction

Berge’s maximum theorem is an important statement in set-valued analysis that has significant applications in mathematical economics, operations research, and control. For a
minimization problem depending on a parameter, this theorem states sufficient conditions for continuity of the value function and upper semi-continuity of the solution multifunction in the parameter. The conditions in Berge’s maximum theorem are: (i) the set-valued mapping, that defines feasible solutions, is compact-valued and upper semi-continuous, and (ii) the objection function, which is a function of two variables representing a parameter and a solution, is continuous.

One of the limitations of applications of Berge’s maximum theorem is that the sets of feasible actions are assumed to be compact. This limitation was overcome in Feinberg et al. [5] by introducing the notion of $\mathcal{K}\mathcal{N}$-inf-compact functions and showing that the conclusions of Berge’s maximum theorem hold, if the assumptions that the set-valued mapping is compact-valued and upper-semicontinuous is replaced with a single assumption that the objective function is $\mathcal{K}\mathcal{N}$-inf-compact. This result is a generalization of Berge’s theorem because under the conditions of Berge’s theorem the objective function is $\mathcal{K}\mathcal{N}$-inf-compact.

Continuity properties in the assumptions and conclusions of Berge’s maximum theorem are local properties. Therefore, it is natural to formulate Berge’s theorem in a local form. This was done in Bonnans and Shapiro [3, Proposition 4.4].

The original motivation of this paper was to investigate whether the approach based on $\mathcal{K}\mathcal{N}$-inf-compactness leads to a more general result than the local minimum theorem in Bonnans and Shapiro [3, Proposition 4.4]. The main result of this paper, Theorem 7, strengthens the local minimum theorem in Bonnans and Shapiro [3, Proposition 4.4] and provides a new insight to it by showing that the assumption on the solution multifunction in [3, Proposition 4.4] is the necessary and sufficient condition for upper semi-continuity of the solution multifunction, if other assumptions of [3, Proposition 4.4] hold. This is also true under the weaker assumptions stated in Theorem 7.

The remaining parts of this section contain major definitions, formulations of Berge’s maximum theorem for noncompact action sets from Feinberg et al. [5] and of the local maximum theorem from Bonnans and Shapiro [3, Proposition 4.4], and comparisons of these statements based on splitting each of them into three statements implying the following properties: (i) lower semi-continuity of minuma, (ii) upper semi-continuity of minuma, and (iii) upper semi-continuity of the solution multifunctions. Section 2 contains formulations of the main results. Section 3 describes local properties of $\mathcal{K}\mathcal{N}$-inf-compact functions. Section 4 provides the proofs of the theorems formulated in Sections 2, and 5 contains examples and counterexamples illustrating the theorems and comparing them.

Let $X$ and $Y$ be Hausdorff topological spaces, and $\Phi : X \to \mathcal{S}(Y)$ be a set-valued map, where $\mathcal{S}(Y) := 2^Y \setminus \{\emptyset\}$ is the family of all nonempty subsets of the set $Y$. Consider the graph of $\Phi$, defined as $\text{Gr}_X(\Phi) = \{(x, y) \in X \times Y : y \in \Phi(x)\}$, and let $u : \text{Gr}_X(\Phi) \subseteq X \times Y \to \mathbb{R}$, where $\mathbb{R} := \mathbb{R} \cup \{\pm \infty\}$ is the extended real line. Define the value function

$$v(x) := \inf_{y \in \Phi(x)} u(x, y), \quad x \in X,$$

and the solution multifunction

$$\Phi^*(x) := \{y \in \Phi(x) : v(x) = u(x, y)\}, \quad x \in X.$$

To clarify the above definitions, consider a Hausdorff topological space $A$. For a nonempty set $A \subseteq A$, the notation $f : A \subseteq A \to \mathbb{R}$ means that for each $a \in A$ the value $f(a) \in \mathbb{R}$ is defined. In general, the function $f$ may be also defined outside of $A$. The notation $f : A \to \mathbb{R}$ means that the function $f$ is defined on the entire space $A$. This notation is equivalent to the notation $f : A \subseteq A \to \mathbb{R}$, which we do not write explicitly. For a function $f : A \subseteq A \to \mathbb{R}$ we sometimes consider its restriction $f : B \subseteq A \to \mathbb{R}$ to the set $B \subseteq A$. Sometimes we consider functions with values in $\mathbb{R}$ rather than in $\mathbb{R}$. 

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