The heavy baryons in the nonperturbative string approach

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Abstract

We present some piloting calculations of the short-range correlation coefficients for the light and heavy baryons and masses of the doubly heavy baryons $\Xi_{QQ'}$ and $\Omega_{QQ'}$ ($Q, Q' = c, b$) in the framework of the simple approximation within the nonperturbative QCD approach.
1. Introduction

The observation of $B_{c}^{+}$ meson by the CDF collaboration \cite{1} opens a new direction in the physics of hadrons containing two heavy quarks. Presently at the LHC, $B$-factories, and the Tevatron with high luminosity, several experiments have been proposed, in which there is a possibility to identify and study hadrons containing two heavy quarks, like doubly-charm baryons ($ccq$) or baryons ($bcq$) with charm and beauty\textsuperscript{1}. In the more distant future the next generation experiments with high bottom quark production rate will provide excellent possibilities for the study bottom baryons and their decays. In view of this project, it is important to have safe theoretical predictions for heavy baryon masses as a guide to the experimental search of these hadrons.

A number of authors \cite{2}-\cite{12} have already considered baryons containing two heavy quarks in anticipation of future experiments which may discover these particles. In most of these works, however, theoretical predictions are somewhat biased by the introduction of the additional dynamical assumptions and supplementary dynamical parameters like constituent quark masses in addition to the only one parameter really pertinent to QCD – the overall scale of the theory $\Lambda_{QCD}$.

The purpose of this paper is to calculate the masses of the heavy baryons in a simple approximation within the nonperturbative QCD, developed in \cite{13}-\cite{16}. This method has been already applied to study baryon Regge trajectories \cite{15} and, very recently, for computation of magnetic moments of light baryons \cite{17}. The essential point of this paper is that it is very reasonable that the same method should also hold for hadrons containing heavy quarks. In this work we will concentrate on the masses of doubly heavy baryons. As in \cite{17} we take as the universal parameter the QCD string tension $\sigma$, fixed in experiment by the meson and baryon Regge slopes. We

\textsuperscript{1} Here, and throughout this paper, $q$ denotes a light quark $u$ or $d$. 
also include the perturbative Coulomb interaction with the frozen coupling 
\( \alpha_s(1 \text{ GeV}) = 0.39 \). The basic feature of the considered approach is the 
dynamical calculation of the quark constituent masses \( m_i \) in terms of the 
quark current masses \( m_i^{(0)} \). This is done using the einbein (auxiliary fields) 
formalism, which is proven to be rather accurate in various calculations for 
relativistic systems. The einbeins are treated as the variational parameters 
which are to be found form the condition of the minimum of baryon eigen 
energies [18].

2. **Formalism**

The starting point of the approach is the Feynman–Schwinger representa-
tion of the 3q Green’s function, where the role of ”time” parameter along 
a quark path is played by the Fock–Schwinger proper time. The final step 
is the derivation of the c.m. Effective Hamiltonian (EH) containing the 
dynamical quark masses as parameters. For many details see the original 
papers [13]–[16].

Consider a baryon consisting of three quarks with arbitrary masses \( m_i, 
\( i = 1, 2, 3 \). In what follows we confine ourselves to consideration of the 
ground state baryons without radial and orbital excitations in which case 
tensor and spin-orbit forces do not contribute perturbatively. Then only 
the spin-spin interaction survives in the perturbative approximation. The 
EH has the following form:

\[
H = \sum_{i=1}^{3} \left( \frac{m_i^{(0)2}}{2m_i} + \frac{m_i}{2} \right) + H_0 + V, \tag{1}
\]

where \( m_i^{(0)} \) are the current quark masses and \( m_i \) are the dynamical quark 
masses to be found from the minimum condition (see Eq. (2) below). Since 
\( m_i \gg m_i^{(0)} \) for light quarks, but \( m_i \sim m_i^{(0)} \) for heavy quarks, each light 
quark contributes to the baryon mass an additional mass \( \sim m_i/2 \) (not \( m_i \) 
as in the ordinary nonrelativistic quark model), whereas each heavy quark
contributes an additional mass \( \sim m_i \). The dynamical quark masses are evaluated from the equations defining the stationary points of the baryon mass \( M_B \) as function of \( m_i \)

\[
\frac{\partial M_B(m_i)}{\partial m_i} = 0. \tag{2}
\]

Let \( r_i \) be the quark coordinates. The kinetic momentum operator \( H_0 \) in Eq. (2) acquires the familiar form

\[
H_0 = -\frac{1}{2m_1} \frac{\partial^2}{\partial r_1^2} - \frac{1}{2m_2} \frac{\partial^2}{\partial r_2^2} - \frac{1}{2m_3} \frac{\partial^2}{\partial r_3^2}. \tag{3}
\]

\( V \) is the sum of the perturbative Coulomb-like one gluon exchange potential and the string potential. The Coulomb-like potential is

\[
V_c = -\frac{2\alpha_s}{3} \sum_{i<j} \frac{1}{|r_{ij}|}, \tag{4}
\]

where the factor \( 2/3 \) is the value of the quadratic Casimir operator for the group \( SU_c(3) \). The string potential has been calculated in \cite{15} as the static energy of the three heavy quarks

\[
V_{\text{string}}(r_1, r_2, r_3) = \sigma R_{\text{min}}, \tag{5}
\]

where \( R_{\text{min}} \) is the sum of the three distances \( |r_i| \) from the string junction point, which for simplicity is chosen as coinciding with the centre-of-mass coordinate \( R_{\text{cm}} \).

### 3. Hyper Radial Approximation

We use the hyperspherical formalism approach (for detail see original papers \cite{19}). In the hyperradial approximation (HRA) corresponding to the truncation of the wave function \( \psi(\{r_i\}) \) by the component with grand orbital momentum \( K = 0 \) the three-quark wave function depends only on the hyperradius \( R^2 = \rho^2 + \lambda^2 \), where \( \rho \) and \( \lambda \) are the three-body Jacobi
variables, and does not depend on angular variables. The confining potential (3) has a specific three-body character. However, this potential as well as the Coulomb potential in Eq. (4) is smooth in the sense that the HRA (where only the part of the potential which is invariant under rotation in the six-dimensional space spanned by the Jacobi coordinates is taken into account) is already an excellent approximation. The HRA neglects the mixed symmetry components of the three-quark wave function, which appear in the higher approximations of the hyperspherical formalism [19]. Introducing the reduced function $\chi(R) = R^{5/2}\psi(R)$ and averaging $V = V_c + V_{\text{string}}$ over the six-dimensional sphere one obtains the Schrödinger equation

$$\frac{d^2\chi(R)}{dR^2} + 2\mu \left[ E - W(R) - \frac{15}{8\mu R^2} \right] \chi(R) = 0,$$

where $\mu$ is an arbitrary parameter with the dimension of mass which drops off in the final expressions. The last term in (6) represents the three-body centrifugal barrier and $W(R)$ is the average of the three-quark potentials over the six-dimensional sphere:

$$W(R) = \langle V \rangle = -\frac{a}{R} + bR,$$

with

$$a = \frac{2\alpha_s}{3} \cdot \frac{16}{3\pi} \sum_{i<j} \alpha_{ij}, \quad b = \sigma \cdot \frac{32}{15\pi} \sum_{i,j} \gamma_{ij}. \quad (8)$$

The mass depending constants $\alpha_{ij}$ and $\gamma_{ij}$ are defined by Eqs. (A.2) and (A.13) in the Appendix.

It is convenient to introduce a new variable $x = R\sqrt{\mu}$, to eliminate an artificial dependence of Eq. (6) on $\mu$, then the equation (6) becomes

$$\chi''(x) + 2 \left( E - U(x) - \frac{15}{8x^2} \right) \chi(x) = 0,$$

for their definition see Appendix

$^3$In what follows we omit the value of $K = 0$ to avoid subscripts. Note that the radially symmetric component with $K = 0$ is the dominant one in the three-quark wave function.
where

\[ U(x) = -\frac{a\sqrt{\mu}}{x} + \frac{b}{\sqrt{\mu}}x. \]  

(10)

Since \( a \sim 1/\sqrt{\mu}, \ b \sim \sqrt{\mu} \) (see Eqs. (A.2), (A.13)), the eigenvalue \( E \) in (9) does not depend on \( \mu \).

4. The quark dynamical masses

Equation (9) applied to the nucleon \( (m_q^{(0)} \sim 0) \) yields the dynamical mass \( m_q \) of the light quark, and applied to the strange hyperons gives the strange quark mass \( m_s \). In the same manner application of this equation to the charm and beauty baryons yields the constituent masses of \( c \)- and \( b \)-quarks. In our calculations we use the same parameters as in [22], namely \( \sigma = 0.17 \) GeV, \( \alpha_s = 0.4, m_q^{(0)} = 0.009 \) GeV, \( m_s^{(0)} = 0.17 \) GeV, \( m_c^{(0)} = 1.4 \) GeV, and \( m_b^{(0)} = 4.8 \) GeV.

We solve Eq. (9) using both the quasiclassical and variational solutions. The first approach is based on the well-known fact that interplay between the centrifugal term and the confining potential produces a minimum of the effective potential specific for the three-body problem. The numerical solution of (9) for the ground state eigen energy may be reproduced on a per cent level of accuracy by using the parabolic approximation for the effective potential [20], [21]. This approximation provides an analytical expression for the eigen energy. The potential \( \tilde{U}(x) = U(x) + \frac{15}{8x^2} \) has the minimum at a point \( x = x_0 \), which is defined by the condition \( \tilde{U}'(x_0) = 0 \), i.e.:

\[ \frac{b}{\sqrt{\mu}}x_0^3 + (a\sqrt{\mu})x_0 - 15/4 = 0. \]  

(11)

Expanding \( \tilde{U}(x) \) in the vicinity of \( x = x_0 \) one obtains:

\[ \tilde{U}(x) \approx \tilde{U}(x_0) + \frac{1}{2}\tilde{U}''(x_0)(x - x_0)^2, \]
i.e. the potential of the harmonic oscillator with the frequency
\( \omega = \sqrt{\tilde{U}''(x_0)} \). Therefore the energy eigenvalue is

\[
E_0 \approx \tilde{U}(x_0) + \frac{1}{2} \omega.
\]  

(12)

In Table 1 we show the dynamical masses \( m_i \) and the ground state eigenvalues \( E_0 \) for various baryons calculated using the procedure described above. Our values of light quark mass \( m_q \) qualitatively agree with the results of [22] obtained from the analysis of the heavy-light ground meson states, but \( \sim 60 \text{ MeV} \) higher than those of [15], [17]. This difference is due to the different treatment of the Coulomb and spin-spin interactions. In [15] both interactions have not been included and the light quark mass has been calculated from the fit of the mass of \( \Delta(1232) \) where the Coulomb-like potential and the spin-spin interaction seem to balance each other. In [17] the smeared spin-spin interaction for the light quarks has been included into Eq. (2) defining the dynamical mass of the light quark. In our calculation as in [22] we include the Coulomb-like term, but neglect the spin-spin interaction.

There is no good theoretical reason why dynamical quark masses need to be the same in different mesons and baryons. From the results of Table 1 we conclude that the dynamical masses of the light quarks (\( u, d, \) or \( s \)) are increased by \( \sim 100 - 150 \text{ MeV} \) when going from the light to heavy baryons. For the heavy quarks (\( c \) and \( b \)) the variation in the values of their dynamical masses is marginal. In Table 2 we compare the quark masses in \( \Lambda_Q \) and \( \Xi_Q \) baryons with those calculated in [22] in \( D \) and \( B \) mesons. One observes that the masses of the light quarks in baryons are slightly smaller than those in the mesons. The small variations in the values of \( m_c \) and \( m_b \) are within the accuracy of our calculations.
5. Correlation functions for the baryons

For many applications the quantities $\langle \psi | \delta^{(3)}(r_j - r_i) | \psi \rangle$ are needed. To estimate effects related to the baryon wave function we solve Eq. (9) by the variational method. We introduce a simple variational ansatz for $\chi(x)$

$$\chi(x) = 2\sqrt{2}p^3 x^{5/2} e^{-p^2 x^2},$$

(13)

where $p$ is the variational parameter, and the numerical factor is chosen so that $\int \chi^2(x) dx = 1$. The trial three-quark Hamiltonian admits explicit solutions for the energy, the wave function, and the density matrix:

$$E_0 \approx \min_p E(p),$$

(14)

where

$$E(p) = \langle \chi | H | \chi \rangle = 3p^2 - (a\sqrt{\mu}) \cdot \frac{3}{4} \sqrt{\frac{\pi}{2}} \cdot p + (b/\sqrt{\mu}) \cdot \frac{15}{16} \sqrt{\frac{\pi}{2}} \cdot p^{-1}. \quad (15)$$

The density matrix (the correlation function) $f_{ijk}(r_{ij})$ in a baryon $\{ijk\}$ is defined as:

$$f_{ijk}(r_{ij}) = \alpha_{ij}^3 \int |\psi(\alpha_{ij}r_{ij}, \lambda_{ij})|^2 \lambda_{ij}^3 \lambda_{ij} d^3 \lambda_{ij} \quad (16)$$

so that

$$\int f_{ijk}(r_{ij}) d^3 r_{ij} = \int \int |\psi(\rho_{ij}, \lambda_{ij})|^2 \lambda_{ij}^3 \lambda_{ij} d^3 \lambda_{ij} d^3 \rho_{ij} = 1. \quad (17)$$

For the trial function (13) $f_{ijk}(r_{ij})$ are evaluated explicitly:

$$f_{ijk}(r_{ij}) = \left( \frac{\xi_{ij}}{\pi} \right)^{3/2} e^{-\xi_{ij}|r_{ij}|^2},$$

(18)

with

$$\xi_{ij} = 2p_0^2 \cdot \mu_{ij},$$

(19)

where $\mu_{ij}$ is the reduced mass of the quarks $i$ and $j$, and $p_0$ is to be found from the condition

$$\frac{dE}{dp} \bigg|_{p=p_0} = 0.$$
The expectation values \( f_{ijk}(r_{ij}) \) depend on the third or ‘spectator’ quark through the three-quark wave function. 

Let us define the quantities 

\[
R_{ijk} = f_{ijk}(0) = \left( \frac{\xi_{ij}}{\pi} \right)^{3/2}
\]  

(20)

The corresponding quantity for a meson is denoted as \( R_{ij} \). The results of the variational calculations are given in Table 3 where for each baryon we show the variational parameters \( p_0 \), the quantities \( R_{ijk} \) (in units of GeV\(^3\)), and the average distances \( \bar{r}_{ij} = \sqrt{\langle r_{ij}^2 \rangle} \) (in units of fm). The variational estimations of \( E_0 \) and quark dynamical masses do not differ from those shown in Table 1.

Comparing the results of Table 3 with those of [22] we obtain (see Table 4)

\[
R_{ijk} < \frac{1}{2} R_{ij},
\]

(21)

and

\[
R_{ijk} \gtrsim R_{ijl}, \text{ if } m_k \leq m_l
\]

(22)

Note, however, that if \( i, j \) are the light quarks, and the quarks \( k \) and \( l \) are the heavy, then \( R_{ijk} \approx R_{ijl} \) (e.g., \( R_{qqc} \approx R_{qqb} \)) in agreement with the limit of the heavy quark effective theory.

Our estimations for the ratios \( R_{ijk}/R_{ij} \) agree with the results obtained using the nonrelativistic quark model or the bag model [23] - [26] or QCD sum rules [27] which are typically in the range \( 0.1 \sim 0.5 \). On the other hand, our result for \( \Lambda_b \) disagrees with the one by Rosner [28] who estimated the heavy-light diquark density at zero separation in \( \Lambda_b \) from the ratio of hyperfine splittings between \( \Sigma_b \) and \( \Sigma_b^* \) baryons and \( B \) and \( B^* \).\(^4\)

\(^4\) Inequalities (21) and (22) were first suggested in [23] from the observed mass splitting in mesons and baryons.
$B^*$ mesons and found $R_{qbu}/R_{\bar{b}d} \sim 0.9 \pm 0.1$, if the baryon splitting is taken to be $m_{\Sigma^*_b}^2 - m_{\Sigma_b}^2 \sim m_{\Sigma^*_c}^2 - m_{\Sigma_c}^2 = (0.384 \pm 0.035)$ GeV$^2$, or even to $R_{ubd}/R_{\bar{b}d} \sim 1.8 \pm 0.5$, if the surprisingly small and not confirmed yet DELPHI result $m_{\Sigma^*_b} - m_{\Sigma_b} = (56 \pm 16)$ MeV is used.

From the results of Table 3 it follows that the correlation between two quarks depends on the third one. Note also that the wave function calculated in HRA shows the marginal diquark clustering in the doubly heavy baryons. This is principally kinematic effect related to the fact that in the HRA the difference between the various $r_{ij}$ in a baryon is due to the factor $\sqrt{1/\mu_{ij}}$ which varies between $\sqrt{2/m_i}$ for $m_i = m_j$ and $\sqrt{1/m_i}$ for $m_i \ll m_j$.

In Table 5 we compare the short-range correlation coefficients in the doubly heavy baryons with those calculated in [7] using the pair-wise quark interaction with the power-law potential, and in [9] using the non-relativistic model with the Buchmüller–Tye potential.

6. Masses of doubly heavy baryons

To calculate hadron masses we, as in [15], first renormalise the string potential

$$V_{\text{string}} \rightarrow V_{\text{string}} + \sum_i C_i,$$

where the constants $C_i$ take into account the residual self-energy (RSE) of quarks. In principle, these constants can be expressed in terms of the two scalar functions entering covariant expansion of the bilocal cumulants of gluonic fields in the QCD vacuum [14, 15]. In the present work we treat them phenomenologically. To find $C_i$ in (23) we assume, first, that the spin splittings of hadrons with a given quark content arise from the colour-magnetic interaction in QCD. Indeed, for the ground state hadrons the hadron wave functions have no orbital angular momentum, so tensor...
and spin-orbit forces do not contribute. The second assumption is that the colour-magnetic interaction can be treated perturbatively \[29, 30\]:

$$\Delta E_{\text{spin}} = \frac{16\pi\alpha_s}{9} \sum_{i<j} \frac{s_is_j}{m_im_j} R_{ijk}.$$  

(24)

Because the colour-magnetic interaction between two quarks goes inversely as the product of their masses, the perturbative approximation improves as the quark mass increases. However, this approximation may not be good for the baryons containing light quarks\[5\]. In what follows we adjust the RSE constants $C_i$ to reproduce the centre-of-gravity for baryons with a given flavour. To this end we consider the spin-averaged masses, such as:

$$\frac{M_N + M_\Delta}{2} = 1.085 \text{ GeV}, \quad \text{and} \quad \frac{M_\Lambda + M_\Sigma + 2M_{\Sigma^*}}{4} = 1.267 \text{ GeV},$$  

(25)

and analogous combinations for $qqc$ and $qqb$ states. Then we obtain

$$C_q = 0.34 \text{ GeV}, \quad C_s = 0.19 \text{ GeV}, \quad C_c \sim C_b \sim 0.$$  

(26)

We keep these parameters fixed to calculate the masses given in Table 6, namely the spin-averaged masses (computed without the spin-spin term) of the lowest doubly heavy baryons. Our results are very similar to those obtained in \[7\] using the pair-wise power-law potential.

7. Conclusions

In this paper we employ the general formalism for the baryons, which is based on nonperturbative QCD and where the only inputs are the string tension $\sigma$, the strong coupling constant $\alpha_s$, and two additive constants, $C_q$ and $C_s$, the residual self-energies of the light quarks. We present some piloting calculations of the dynamical quark masses for various baryons (see Table 1). The latters are computed solely in terms of $\sigma$ and $\alpha_s$ and depend on a baryon.

\[5\] Note that $1/m_im_j$ dependence in Eq. (24), if treated literally in the EH method, results in a collapse both in the pseudoscalar $q\bar{q}$ channel and the proton. That may be a signal of the Nambu–Goldstone phenomenon.
The second important point of our investigation is the calculation of the correlation functions for baryons. They are given, among the other things, in Table 3. We have also performed the calculations of the spin-averaged masses of baryons with two heavy quarks. One can see from Table 6 that our predictions are especially close to those obtained in [7] using a variant of the power-law potential adjusted to fit ground state baryons.

Evaluation of the spin-spin interactions requires inclusion of the $K = 2$ hyperspherical components and/or more sophisticated treatment of the colour-magnetic interaction. We shall consider these calculations in the next publication.

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Appendix

Consider three quarks with arbitrary masses $m_i$, $i = 1, 2, 3$, and coordinates $r_i$. The problem is conveniently treated using Jacobi coordinates $\rho_{ij}$ and $\lambda_{ij}$:

$$\rho_{ij} = \alpha_{ij}(r_i - r_j), \quad \lambda_{ij} = \beta_{ij}\left(\frac{m_ir_i + m_jr_j}{m_i + m_j} - r_k\right),$$

(A.1)

where

$$\alpha_{ij} = \sqrt{\frac{\mu_{ij}}{\mu}}, \quad \beta_{ij} = \sqrt{\frac{\mu_{ij,k}}{\mu}}.$$  

(A.2)

Here $\mu_{ij}$ and $\mu_{ij,k}$ are the reduced masses

$$\mu_{ij} = \frac{m_im_j}{m_i + m_j}, \quad \mu_{ij,k} = \frac{(m_i + m_j)m_k}{m_i + m_j + m_k}.$$  

(A.3)
Altogether with the centre-of-mass coordinate $R_{cm}$ Jacobi coordinates determine completely the position of the system. The Jacobian of the transformation for the differential volume elements is 1, \( i.e., \)

\[
d^3\rho_{12}d^3\lambda_{12} = d^3\rho_{32}d^3\lambda_{32} = d^3\rho_{13}d^3\lambda_{13} \quad (A.4)
\]

The inverse transformations for the relative coordinates $r_{ij} = r_i - r_j$ and $r_k - R_{cm}$ are

\[
r_{ij} = \frac{1}{\alpha_{ij}}\rho_{ij}, \quad r_k - R_{cm} = -\sqrt{\frac{\mu(m_i + m_j)}{m_k(m_1 + m_2 + m_3)}} \lambda_{ij} \quad (A.5)
\]

The hyperradius $R^2$ is defined as $R^2 = \rho^2_{ij} + \lambda^2_{ij}$ and does not depend on the order of the quark numbering:

\[
R^2 = \rho^2_{12} + \lambda^2_{12} = \rho^2_{32} + \lambda^2_{32} = \rho^2_{13} + \lambda^2_{13} \quad (A.6)
\]

Written in terms of $r_{ij}$ Eq. (A.6) reads:

\[
R^2 = \sum_{i<j} \frac{m_i m_j}{\mu(m_i + m_j)} r_{ij}^2 \quad (A.7)
\]

In the centre-of-mass frame $R_{cm} = 0$ the invariant kinetic energy operator (3) is written in terms the Jacobi coordinates (A.1) as

\[
H_0 = -\frac{1}{2\mu} \left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \lambda^2} \right) = -\frac{1}{2\mu} \left( \frac{\partial^2}{\partial R^2} + \frac{5}{R} \frac{\partial}{\partial R} + \frac{K^2(\Omega)}{R^2} \right), \quad (A.8)
\]

where $K^2(\Omega)$ is angular momentum operator whose eigen functions (the hyperspherical harmonics) are

\[
K^2(\Omega) Y_{[K]} = -K(K+4) Y_{[K]}, \quad (A.9)
\]

with $K$ being the grand orbital momentum. In terms of $Y_{[K]}$ the wave function $\psi(\rho, \lambda)$ can be written in a symbolical shorthand as

\[
\psi(\rho, \lambda) = \sum_K \psi_K(R) Y_{[K]}(\Omega).
\]
In the HRA $K = 0$ and $\psi = \psi(R)$. Note that the centrifugal potential in the Schrödinger equation for the radial function $\psi_K(R)$ with a given $K$

$$\frac{(K + 2)^2 - 1/4}{R^2}$$

is not zero even for $K = 0$. For the reduced function $\chi(R) = R^{5/2}\psi(R)$ one obtains after averaging the interaction over the six-dimensional sphere Eq. (5) with

$$W(R) = \langle V(\rho, \lambda) \rangle = \int (V_c + V_{\text{string}}) \frac{d\Omega}{\pi^3} \quad (A.10)$$

One can easily see that the definition of $\langle V(\rho, \lambda) \rangle$ does not depend on the order of the quark numeration.

In terms of the Jacobi coordinates the Coulomb and string potentials read:

$$V_c = -\frac{2}{3} \alpha_s \sum_{i<j} \frac{\alpha_{ij}}{\rho_{ij}}, \quad (A.11)$$

$$V_{\text{string}} = \sigma \sum_{i<j} \gamma_{ij} |\lambda_{ij}|, \quad (A.12)$$

where

$$\gamma_{ij} = \sqrt{\frac{\mu(m_i + m_j)}{m_k(m_1 + m_2 + m_3)}}. \quad (A.13)$$

Using the relations [20]

$$\langle \frac{1}{|\rho_{ij}|} \rangle = \frac{16}{3\pi} \cdot \frac{1}{R}, \quad \langle |\lambda_{ij}| \rangle = \frac{32}{15\pi} \cdot R,$$

valid for any pair (ij), one obtains Eqs. (3).
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Table 1. The constituent quark masses $m_i$ and the ground state eigen energies $E_0$ (in units of GeV) for the various baryon states. (The results obtained from the quasiclassical solution and from the variational one practically coincide.)

| baryon   | $m_1$ | $m_2$ | $m_3$ | $E_0$  |
|----------|-------|-------|-------|--------|
| $(qqq)$  | 0.446 | 0.446 | 0.446 | 1.438  |
| $(qqs)$  | 0.451 | 0.451 | 0.485 | 1.414  |
| $(qss)$  | 0.457 | 0.490 | 0.490 | 1.392  |
| $(sss)$  | 0.495 | 0.495 | 0.495 | 1.370  |
| $(qqc)$  | 0.519 | 0.519 | 1.502 | 1.176  |
| $(qsc)$  | 0.522 | 0.555 | 1.505 | 1.157  |
| $(ssc)$  | 0.589 | 0.589 | 1.507 | 1.138  |
| $(qcb)$  | 0.564 | 0.564 | 4.836 | 1.057  |
| $(qsb)$  | 0.567 | 0.601 | 4.837 | 1.038  |
| $(ssb)$  | 0.604 | 0.604 | 4.838 | 1.019  |
| $(qcc)$  | 0.569 | 1.555 | 1.555 | 0.926  |
| $(scc)$  | 0.604 | 1.557 | 1.557 | 0.908  |
| $(qcb)$  | 0.606 | 1.616 | 4.866 | 0.783  |
| $(scb)$  | 0.642 | 1.618 | 4.867 | 0.765  |
| $(qbb)$  | 0.636 | 4.931 | 4.931 | 0.582  |
| $(sbb)$  | 0.673 | 4.931 | 4.931 | 0.565  |
Table 2. The dynamical quark masses for the ground state \((qc)\), \((sc)\), \((qb)\), \((sb)\) mesons [22] and for the corresponding ground state baryons.

| State | \(m_q\) | \(m_s\) | \(m_c\) | \(m_b\) |
|-------|---------|---------|---------|---------|
| \((qc)\) | 0.529   | 1.497   |         |         |
| \((sc)\) | 0.569   | 1.501   |         |         |
| \((qqc)\) | 0.519   | 1.502   |         |         |
| \((qsc)\) | 0.555   | 1.505   |         |         |
| \((qb)\) | 0.619   |         | 4.84    |         |
| \((sb)\) | 0.658   | 4.842   |         |         |
| \((qqb)\) | 0.564   |         | 4.836   |         |
| \((qsb)\) | 0.601   | 4.838   |         |         |
Table 3. $R_{ijk}$ in units of GeV$^3$ and $\tilde{r}_{ij} = \sqrt{\langle r_{ij}^2 \rangle}$ in units of fm. (The results are obtained from the trial functions (13) with the variational parameters $p_0$ given in units of GeV$^{(1/2)}$ in the first column. The results for light baryons are presented for completeness.)

| baryon | $p_0$ | $R_{123}$ | $R_{231}$ | $R_{312}$ | $\tilde{r}_{12}$ | $\tilde{r}_{23}$ | $\tilde{r}_{31}$ |
|--------|--------|-----------|-----------|-----------|----------------|----------------|----------------|
| $(qqq)$ | 0.472  | 0.00564   | 0.00564   | 0.00564   | 0.777          | 0.777          | 0.777          |
| $(qqs)$ | 0.470  | 0.00567   | 0.00598   | 0.00598   | 0.775          | 0.762          | 0.762          |
| $(qss)$ | 0.469  | 0.00600   | 0.00633   | 0.00600   | 0.760          | 0.747          | 0.760          |
| $(sss)$ | 0.467  | 0.00636   | 0.00636   | 0.00636   | 0.746          | 0.746          | 0.746          |
| $(qqc)$ | 0.454  | 0.00626   | 0.0113    | 0.0113    | 0.750          | 0.615          | 0.615          |
| $(qsc)$ | 0.452  | 0.00656   | 0.0121    | 0.0113    | 0.738          | 0.601          | 0.615          |
| $(ssc)$ | 0.451  | 0.00688   | 0.0121    | 0.0121    | 0.727          | 0.602          | 0.602          |
| $(qqb)$ | 0.447  | 0.00681   | 0.0163    | 0.0163    | 0.729          | 0.545          | 0.545          |
| $(qsb)$ | 0.446  | 0.00711   | 0.0176    | 0.0163    | 0.719          | 0.531          | 0.545          |
| $(ssb)$ | 0.445  | 0.00742   | 0.0176    | 0.0176    | 0.708          | 0.531          | 0.531          |
| $(qcc)$ | 0.439  | 0.0116    | 0.0296    | 0.0116    | 0.611          | 0.447          | 0.611          |
| $(scc)$ | 0.438  | 0.0123    | 0.0294    | 0.0123    | 0.599          | 0.448          | 0.599          |
| $(qcb)$ | 0.436  | 0.0123    | 0.0562    | 0.0166    | 0.599          | 0.361          | 0.541          |
| $(scb)$ | 0.435  | 0.0130    | 0.0559    | 0.0178    | 0.587          | 0.361          | 0.529          |
| $(qbb)$ | 0.438  | 0.0181    | 0.165     | 0.0181    | 0.527          | 0.252          | 0.527          |
| $(sbb)$ | 0.437  | 0.0194    | 0.165     | 0.0194    | 0.515          | 0.252          | 0.515          |

Table 4. The ratios of the squares of the wave functions determining the probability to find a light quark at the location of the heavy quark inside the heavy baryon and the corresponding meson. (The meson wave functions are taken from [22].)

| $R_{ucl}/R_{uc}$ | $R_{scu}/R_{sc}$ | $R_{abd}/R_{bd}$ | $R_{sbu}/R_{sb}$ |
|------------------|------------------|------------------|------------------|
| 0.436            | 0.405            | 0.373            | 0.340            |
Table 5. Short-range correlation coefficients $R_{ijk}$. In the parentheses are shown the corresponding quantities calculated using the power-law potential \cite{7}. In the square brackets are shown correlation coefficients calculated using non-relativistic model with Buchmüller-Tye potential.

| State  | $R_{123}$ | $R_{231}$ | $R_{312}$ |
|--------|-----------|-----------|-----------|
| (ccq)  | 0.030 (0.039) [0.022] | 0.012 (0.009) | 0.012 (0.009) |
| (ccs)  | 0.030 (0.042) [0.022] | 0.012 (0.019) | 0.012 (0.019) |
| (bbq)  | 0.165 (0.152) [0.144] | 0.018 (0.012) | 0.018 (0.012) |
| (bbs)  | 0.165 (0.162) [0.144] | 0.019 (0.028) | 0.019 (0.028) |
| (bcq)  | 0.056 (0.065) [0.042] | 0.012 (0.010) | 0.017 (0.011) |
| (bcs)  | 0.056 (0.071) [0.042] | 0.013 (0.021) | 0.018 (0.025) |

Table 6. Masses of baryons containing two heavy quarks

| State  | present work | \cite{a} | \cite{b} | \cite{c} | \cite{d} |
|--------|--------------|----------|----------|----------|----------|
| $\Xi \{qcc\}$ | 3.69 | 3.70 | 3.71 | 3.66 | 3.61 | 3.48 |
| $\Omega \{scc\}$ | 3.86 | 3.80 | 3.76 | 3.74 | 3.71 | 3.58 |
| $\Xi \{qcb\}$ | 6.96 | 6.99 | 6.95 | 7.04 | 6.82 |
| $\Omega \{scb\}$ | 7.13 | 7.07 | 7.05 | 7.09 | 6.92 |
| $\Xi \{qbb\}$ | 10.16 | 10.24 | 10.23 | 10.24 | 10.09 |
| $\Omega \{sbb\}$ | 10.34 | 10.30 | 10.32 | 10.37 | 10.19 |

(a) The additive nonrelativistic quark model with the power-law potential.

(b) Relativistic quasipotential quark model.

(c) The Feynman-Hellmann theorem.

(d) Approximation of doubly heavy diquark.