ON SUPER-EXponential DIVERGENCE OF PERIODIC POINTS FOR PARTIALLY HYPERBOLIC SYSTEMS

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(Communicated by Shaobo Gan)

Abstract. We say that a diffeomorphism f is super-exponentially divergent if for every \( b > 1 \) the lower limit of \( \#\text{Per}_n(f)/b^n \) diverges to infinity, where \( \text{Per}_n(f) \) is the set of all periodic points of \( f \) with period \( n \). This property is stronger than the usual super-exponential growth of the number of periodic points. We show that for any \( n \)-dimensional smooth closed manifold \( M \) where \( n \geq 3 \), there exists a non-empty open subset \( \mathcal{O} \) of \( \text{Diff}^1(M) \) such that diffeomorphisms with super-exponentially divergent property form a dense subset of \( \mathcal{O} \) in the \( C^1 \)-topology. A relevant result about the growth rate of the lower limit of the number of periodic points for diffeomorphisms in a \( C^r \)-residual subset of \( \text{Diff}^r(M) \) (1 \( \leq r \leq \infty \)) is also shown.

1. Introduction.

1.1. Backgrounds. The investigation of the growth of the number of periodic points for dynamical systems is a fundamental problem. For uniformly hyperbolic systems, we know that the growth of this number cannot be faster than some exponential functions. Then a natural question is what happens for systems which fail to be uniformly hyperbolic in a robust fashion.

A fundamental result is given by Artin and Mazur, which asserts that for a dense subset of \( C^r \)-maps of a compact manifold into itself with the uniform \( C^r \)-topology (1 \( \leq r < \infty \)), the number of isolated periodic points grows at most exponentially [2]. Meanwhile, there are some opposite results for locally generic maps. For instance, Bonatti, Díaz and Fisher show that generically in \( \text{Diff}^1(M) \), if a homoclinic class contains periodic points of different indices, then it exhibits super-exponential growth of number of periodic points [8]; for certain semi-group actions on the interval, Asaoka, Shinohara and Turaev construct a \( C^r \)-open set \( (r \geq 1) \) in which \( C^r \)-generic maps exhibit super-exponential growth of this number.

2020 Mathematics Subject Classification. Primary: 37C20, 37C25, 37C29, 37D30.

Key words and phrases. Non-uniform hyperbolicity, partial hyperbolicity, heterodimensional cycles, super-exponential growth, number of periodic points.

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for \( C^r \)-diffeomorphisms of compact smooth manifolds, they also construct locally \( C^r \)-generic subset with super-exponential growth of this number under certain conditions [4]; Berger shows that for \( 2 \leq r \leq \infty \) and some manifolds of dimension greater than 2, there exists a non-empty open set \( O \subset \text{Diff}^r(M) \) in which \( C^r \)-generic \( f \) displays a super-exponential growth of this number [5]. Thus one may consider the difference of the growth rate as a probe of the degree of non-hyperbolicity which the system exhibits.

Let us be more precise. Given a set \( X \) and a map \( f : X \to X \), we say that \( x \in X \) is a periodic point of period \( n \) (where \( n \geq 1 \)) if \( f^n(x) = x \) and \( n \) is the least positive integer for which this equality holds. By \( \pi(x) \) we denote the period of \( x \). In particular, \( x \) is called a fixed point if \( f(x) = x \). A periodic point is called \( n \)-periodic if its period is \( n \). We write the set of \( n \)-periodic points of \( f \) by \( \text{Per}_n(f) \), whose cardinal is denoted by \( \#\text{Per}_n(f) \).

For the investigation of the number of periodic points, we mainly focus on the ratio of \( \#\text{Per}_n(f) \) to \( b^n \) (\( b > 1 \)). We customarily consider the upper limit (lim sup) of the ratio when \( n \) goes to infinity. \( f \) is called super-exponential if for every \( b > 1 \) the sequence \( b^{-n}\#\text{Per}_n(f) \) has upper limit equals to \( +\infty \). One motivation of this definition comes from the fact that the finiteness of the upper limit implies the positivity of the convergence radius of the dynamical zeta function (see [2] for related information).

Apart from the original motivation, as a measure of non-uniform hyperbolicity it would be a natural question to ask what happens for the lower limit (lim inf) of the ratio. The condition of lim inf to be infinity is a property stronger than lim sup, because the former implies the ratio tends to large for every sufficiently large \( n \), while the latter implies the largeness for some subsequence. Thus it is not easy to construct a diffeomorphism around which, maps whose lower limit of the ratio diverging to \( +\infty \) exist in a robust way, such as in a dense or residual subset of a neighborhood of the initial diffeomorphism. The aim of this paper is to provide such an example.

In this paper, we say that a diffeomorphism \( f \) is super-exponentially divergent if for every \( b > 1 \) the sequence \( b^{-n}\#\text{Per}_n(f) \) has lower limit equals to \( +\infty \). This is equivalent to say that the limit is equal to \( +\infty \). Let \( \text{Diff}^1(M) \) denote the space of \( C^1 \)-diffeomorphisms of a manifold \( M \) endowed with the \( C^1 \)-topology. The following is one of the main result of this paper.

**Theorem 1.1.** For any \( n \)-dimensional smooth closed manifold \( M \) where \( n \geq 3 \), there exists a non-empty open set \( O \subset \text{Diff}^1(M) \) and a dense subset \( D \) of \( O \) such that every diffeomorphism in \( D \) is super-exponentially divergent.

Let us make some comments. First, we discuss the detail of \( O \) and \( D \) in the next subsection, see Theorem 1.2, 1.3 and 1.4. Second, our construction is based on the bifurcation of heterodimensional cycles. Since heterodimensional cycles exist only for manifolds whose dimension is greater than two, we do not know whether a similar result holds for surface diffeomorphisms. Third, we prove this result under \( C^1 \)-regularity. As we will see, our technique heavily depends on the nature of the \( C^1 \)-topology. Thus the \( C^r \)-case for \( r > 1 \) is open.

**1.2. Results.** Let us introduce some notations. Let \( M \) be an \( n \)-dimensional smooth closed Riemannian manifold. We fix a Riemannian metric \( \| \cdot \| \) on \( TM \) and a metric \( d \) on \( M \). For \( 1 \leq r \leq \infty \), denote by \( \text{Diff}^r(M) \) the space of \( C^r \)-diffeomorphisms of
of unstable index where $u-\text{ind}(P)$ if there exists $x$ such that $E \oplus F$ be non-trivial subbundles of $TM|_{\Lambda}$ which are invariant under $Df$ respectively such that $E_{x} \cap F_{x} = \{0\}$ for every $x \in \Lambda$. We say that $E \oplus F$ is a dominated splitting if there exists $\alpha \in (0, 1)$ such that for every $x \in \Lambda$ we have $\|Df|_{E_{x}}\| \cdot \|Df^{-1}|_{F_{x}(x)}\| < \alpha$, where $\|Df|_{E_{x}}\|$ denotes the operator norm of $Df|_{E_{x}}$ with respect to the Riemannian metric.

We say that $\Lambda$ is strongly partially hyperbolic if there is a splitting

$$TM|_{\Lambda} = E^{s} \oplus E^{c} \oplus E^{u}$$

such that $E^{s} \oplus (E^{c} \oplus E^{u})$ and $(E^{s} \oplus E^{c}) \oplus E^{u}$ are dominated splittings with $\dim E^{c} = 1$, $\dim E^{s} \geq 1$ and $\dim E^{u} \geq 1$, where $E^{s}$ is uniformly contracting and $E^{u}$ is uniformly expanding. Notice that in this paper we will consider only the case where center bundle is one-dimensional. We say that a strongly partially hyperbolic set is orientation preserving if $E^{c}$ can be oriented in such a way that $Df$ preserves the orientation.

Let $\Lambda$ be a strongly partially hyperbolic set. Suppose we have a pair of hyperbolic periodic points $P_{1}, P_{2} \in \Lambda$. We say that they are adapted if

$$u-\text{ind}(P_{1}) = \dim E^{u} + 1 \quad \text{and} \quad u-\text{ind}(P_{2}) = \dim E^{u},$$

where $u-\text{ind}(P)$ denotes the dimension of the unstable subspace of $P$, called the unstable index of $P$.

We say that $f$ is transitive on $\Lambda$ if there is an orbit which is dense in $\Lambda$, that is, if there exists $x \in \Lambda$ such that its orbit $\text{orb}(x) := \{f^{n}(x) : n \in \mathbb{Z}\}$ is dense in $\Lambda$. If $\Lambda = M$, we just say $f$ is transitive.

A $C^{r}$-diffeomorphism $f$ is said to satisfy some property $C^{r}$-robustly if there is a $C^{r}$-open neighborhood $U$ of $f$ in $\text{Diff}^{r}(M)$ such that every $g \in U$ satisfies it.

The following result says that we have $C^{1}$-dense super-exponential divergence for a certain class of robustly transitive diffeomorphisms.

**Theorem 1.2.** Let $M$ be an $n$-dimensional smooth closed manifold where $n \geq 3$ and $f \in \text{Diff}^{1}(M)$. Suppose that $f$ is $C^{1}$-robustly transitive, the entire manifold $M$ is an orientation preserving strongly partially hyperbolic set $C^{1}$-robustly and $f$ has two adapted hyperbolic fixed points $P_{1}$ and $P_{2}$. Then there exist a $C^{1}$-neighborhood $U$ of $f$ in $\text{Diff}^{1}(M)$ and a dense subset $D$ of $U$ such that every $g \in D$ is super-exponentially divergent.

Robustly transitive partially hyperbolic diffeomorphisms are one of the important classes of partially hyperbolic systems. Thus Theorem 1.2 shows the largeness of the class of diffeomorphisms for which Theorem 1.1 holds. On the other hand, there are some restrictions on the manifolds which support robustly transitive partially hyperbolic diffeomorphisms. Later, we will give Theorem 1.4 which has no restrictions on the topology of the manifold.

Before the discussion of Theorem 1.4, let us see the idea of the proof of Theorem 1.2. We introduce some notations. For a hyperbolic periodic point $P$ of $f \in \text{Diff}^{1}(M)$, $W^{s}(P)$ (resp. $W^{u}(P)$) denotes the stable (resp. unstable) manifold of $P$. For a periodic point, by eigenvalues of $P$ we mean the eigenvalues of the linear
map $Df^\pi(P)$ at $P$. Suppose $P$ has a unique weakest contracting (resp. weakest expanding) eigenvalue, that is, it has a unique eigenvalue whose absolute value is the largest (resp. smallest) among contracting (resp. expanding) eigenvalues. Then, we have the strong stable (resp. strong unstable) manifold of $P$ corresponding to the other eigenvalues. We denote it by $W^{ss}(P)$ (resp. $W^{uu}(P)$).

Let us discuss the basic idea of our construction of super-exponentially divergent diffeomorphisms under $C^1$-regularity. For a large integer $n$, our first goal is to construct a large number of periodic points with period $n$. For that, a convenient setting is the dynamical configuration called heterodimensional cycles. It is one of the typical mechanisms leading to non-hyperbolicity. Let us recall its definition. Given two hyperbolic periodic points $P_1$ and $P_2$ of $f$ with different unstable indices, if there exist $Q_1 \in W^u(P_1) \cap W^s(P_2)$ and $Q_2 \in W^u(P_1) \cap W^s(P_2)$, we say that $f$ exhibits a heterodimensional cycle associated to $P_1, P_2$ and heteroclinic points $Q_1, Q_2$.

Once we have such a cycle under the assumption of the strong partial hyperbolicity, following the argument in [6], one can linearize the dynamics around $P_1$ and $Q_i$ $(i = 1, 2)$ by an arbitrarily small $C^1$-perturbation. This locally affine model of a heterodimensional cycle is called a simple cycle. A direct calculation shows that, nearby such a cycle there exists a sequence of periodic points whose periods tend to infinity and whose center Lyapunov exponents tend to zero (see Section 2.3 for the definition of center Lyapunov exponents). We call them weak periodic points.

Then, by exploiting the flexibility of the $C^1$-topology, we can increase the number of periodic points as large as we want by perturbing in the center direction. This is the construction carried out in [8].

In our problem, we furthermore need to investigate the frequency of the periods of the weak periodic points. More precisely, we need to confirm the occurrence of periods of weak periodic points for all sufficiently large integers. By investigating the above calculation carefully and using the condition that $P_1$ and $P_2$ are fixed points, together with the information of the orientation, we observe that under certain quantitative assumption on the characteristics of the simple cycle, we can construct a sequence of weak periodic orbits whose periods exhaust all sufficiently large integers (see Proposition 3).

In order to complete our construction, we need to establish the additional quantitative condition on the simple cycle. We introduce a special kind of simple cycles which we call SH-simple cycles. The prefix SH stands for “Strongly Heteroclinic”, meaning that the unstable manifold $W^u(P_1)$ intersects the strong stable manifold $W^{ss}(P_2)$, see Definition 2.3. We will see that such simple cycles satisfy our quantitative requirement (Lemma 5.1) and obtain the abundance of SH-simple cycles among certain class of diffeomorphisms, see Lemma 3.3, 3.4 and Section 3.1. This completes our construction.

The following Theorem 1.3 and Proposition 1 recapitulate our construction.

**Theorem 1.3.** Let $M$ be an $n$-dimensional smooth closed manifold where $n \geq 3$. Suppose $f \in \text{Diff}^1(M)$ satisfies the following:

1. **(T1) (Codimension-1 property)** There are hyperbolic fixed points $P_1$ and $P_2$ of $f$ with $u \text{-ind}(P_1) = u \text{-ind}(P_2) + 1$.
2. **(T2) (A strong heteroclinic intersection)** There exists a point $Q_1 \in W^u(P_1) \cap W^{ss}(P_2)$ at which $W^s(P_1)$ intersects $W^{ss}(P_2)$ transversely.
3. **(T3) (A quasi-transverse intersection)** There exists a point $Q_2 \in W^u(P_2) \cap W^s(P_1)$. 

This completes our construction.
(T4) (Partial hyperbolicity property) The compact invariant set $P_1 \cup P_2 \cup \text{orb}(Q_1) \cup \text{orb}(Q_2)$ is an orientation preserving strongly partially hyperbolic set and $P_1$ and $P_2$ are adapted.

Then, there exists an arbitrarily small $C^1$-perturbation $g$ of $f$ such that $g$ is super-exponentially divergent.

Remark 1. By condition (T4), we know that the weakest expanding eigenvalue of $P_1$ and the weakest contracting eigenvalue of $P_2$ are both real, positive and have multiplicity one.

One important condition in the assumptions of Theorem 1.3 is that we assume $P_1, P_2$ are fixed points of $f$. In many situations, the difference whether the periodic orbit we are interested in has non-trivial period or not can be overcome by taking power of the dynamics. However, as we will see later, this strategy does not work in a simple way for the investigation of the lower limit of $b^{-n} \# \text{Per}_n(f)$. The investigation of to what extent we can relax this assumption would be an interesting topic, but we will not pursue this problem in this paper.

The following perturbation result tells us that for systems which satisfy the hypothesis of Theorem 1.2, we can obtain the assumptions of Theorem 1.3 up to an arbitrarily small $C^1$-perturbation and taking inverse if necessary. Thus Theorem 1.2 is a straightforward consequence of Theorem 1.3 and the following proposition.

Proposition 1. Let $f$ be a $C^1$-robustly transitive diffeomorphism on a smooth $n$-dimensional closed manifold ($n \geq 3$) such that the entire manifold is an orientation preserving strongly partially hyperbolic set $C^1$-robustly. Assume that there are two adapted hyperbolic fixed points $P_1$ and $P_2$. Then, there exists an arbitrarily small $C^1$-perturbation $g$ of $f$ such that $g$ or $g^{-1}$ satisfies assumptions (T1-4) of Theorem 1.3.

The proof of Proposition 1 will be given in Section 3.

Let us give a “local version” of Theorem 1.2. In Theorem 1.2, we stated the result under the condition that the diffeomorphism is robustly transitive. While this condition is easy to understand, it is not the essential one which we need to reach the conclusion. Below we give Theorem 1.4, in which the assumption for the super-exponential divergence is stated in terms of homoclinic classes and this statement makes it easier to grasp the mechanism of the result. See [14] for related investigations.

Let us recall the notion of homoclinic classes. The homoclinic class of a hyperbolic periodic saddle $P$ of a diffeomorphism $f$, denoted by $H(P)$, is defined to be the closure of the set of the points of transversal intersections between the stable and the unstable manifolds of $P$. We can equivalently define $H(P)$ as the closure of all hyperbolic periodic saddles $Q$ homoclinically related to $P$. Recall that two hyperbolic periodic points $P$ and $Q$ are said to be homoclinically related if the stable manifold of $P$ transversally intersects the unstable manifold of $Q$ and vice versa. Homoclinic classes are $f$-invariant and transitive, but not necessarily hyperbolic in general, see for instance [1]. We say that a homoclinic class $H(P)$ of $f$ satisfies some property $C^r$-robustly if there is a $C^r$-neighborhood $\mathcal{U} \subset \text{Diff}^r(M)$ of $f$ such that for every $g \in \mathcal{U}$, we have the continuation $P^g$ of $P$ and $H(P^g)$ satisfies the same property.

Theorem 1.4. Let $M$ be an $n$-dimensional smooth closed manifold ($n \geq 3$). Suppose $f \in \text{Diff}^1(M)$ satisfies the following:
There are hyperbolic fixed points $P_1$ and $P_2$ of $f$ satisfying
\[ u\text{-ind}(P_1) = u\text{-ind}(P_2) + 1; \]
- $W^u(P_1)$ intersects $W^{ss}(P_2)$ transversally;
- $H(P_1)$ admits an orientation preserving strongly partially hyperbolic splitting $C^1$-robustly;
- the homoclinic class $H(P_1)$ contains $P_2$;

Then, there exists an arbitrarily small $C^1$-perturbation $g$ of $f$ such that $g$ is super-exponentially divergent.

The proof of Theorem 1.4 will be given in Section 3.

Let us briefly see how Theorem 1.4 implies Theorem 1.1. We denote the $n$-dimensional closed unit disc by $D^n$. By standard techniques, one can construct a diffeomorphism $f \in \text{Diff}^1(D^n)$ ($n \geq 3$) satisfying the following (in the following, “a property holds $C^r$-generically in an open set” means there is a $C^r$-residual subset of the open set in which every diffeomorphism satisfies the property):
- $f$ coincides with the identity map near the boundary of $D^n$;
- $f$ satisfies the first three conditions of Theorem 1.4 $C^1$-robustly;
- in some $C^1$-small open neighborhood of $f$, the last condition of Theorem 1.4 holds $C^1$-generically for the continuations of $P_1$ and $P_2$.

By realizing this diffeomorphism on any manifold whose dimension is greater than two, we obtain the conclusion of Theorem 1.1. Note that the construction of such local map can be done by standard techniques: For instance see [1] for the $C^1$-locally generic coincidence of homoclinic classes of periodic points of different indices and [20] for the concrete method of local constructions of homoclinic classes.

A natural question regarding Theorem 1.1 is whether one could replace “dense” to some stronger condition such as “residual” or “open and dense”. For instance, one might wonder the following:

**Problem.** Does there exist an open subset $\mathcal{U}$ of $\text{Diff}^s(M)$ such that $C^1$-generically in $\mathcal{U}$, diffeomorphisms are super-exponentially divergent?

While we do not have an answer to this problem, in Section 6 we will prove one result about the lower limit which is valid for diffeomorphisms in a residual subset of $\text{Diff}^r(M)$ where $1 \leq r \leq \infty$, based on the argument of Kaloshin [19]. We say that a sequence $\{a_n\}$ of positive integers is super-exponential if for every $b > 1$, we have $\lim_{n \to \infty} b^n/a_n = 0$.

**Theorem 1.5.** Let $M$ be a smooth closed manifold. Given $1 \leq s \leq \infty$ and a super-exponential sequence $\{a_n\}$, there exists a $C^s$-residual subset $\mathcal{R}$ of $\text{Diff}^s(M)$ such that the following holds: For every $f \in \mathcal{R}$, we have
\[ \liminf_{n \to \infty} \frac{\#\text{Per}_n(f)}{a_n} = 0. \]

This result suggests that we cannot extend Theorem 1.1 in a straightforward way and shows the significance of Theorem 1.1. Notice that Theorem 1.5 is not an answer to the above problem, because there is no “slowest” super-exponential sequence.

We would like to discuss one possible direction of future research in an informal way. For maps which exhibit super-exponential divergence, we know that the sequence $\{\#\text{Per}_n(f)\}$ turns to be uniformly super-exponentially large as $n$ tends
to infinity. Meanwhile, if some of the conditions which we posed on the maps are disturbed, this may fail to be true. In other words, there may be some periods for which we cannot increase the number of periodic points by small perturbations. The question is: Can we obtain some information of such “missing periods” for systems which we cannot increase the number of periodic points by small perturbations. The disturbed, this may fail to be true. In other words, there may be some periods for infinity. Meanwhile, if some of the conditions which we posed on the maps are not preserved are investigated. The pursuit of the relation between \{#\text{Per}_n(f)\} and the configuration of heterodimensional cycles would be an intriguing future research topic.

Finally, let us see the organization of this paper. In Section 2, after giving some basic definitions and notations on SH-simple cycles, we provide the proof of Theorem 1.3 by assuming several results which will be proved in the following sections. In Section 3, we will discuss the proof of Theorem 1.4 and how to prove Theorem 1.2 from Theorem 1.3. Section 4 is devoted to the proof of Proposition 2, a perturbation result for obtaining SH-simple cycles. In Section 5, we prove Proposition 3 and an analytic result for SH-simple cycles. In Section 6, by using a theorem of [19], we prove Theorem 1.5, a generic result about the growth rate of the lower limit of the number of periodic points with respect to a given speed.

2. Preliminaries and strategies. In this section, we give some definitions and cite known results which are used throughout this paper. Then, we state some propositions which will be used for the proof of Theorem 1.3. Finally, assuming these propositions we give the proof of Theorem 1.3.

2.1. SH-simple cycles. In this paper, it is more convenient if we define a specific class of simple cycles, which we call \textit{SH-simple cycles}. Let us give the definition of it.

Let \( \mathbb{D}_n := \{ x \in \mathbb{R}^n : \|x\| < r \} \), where \( \| \cdot \| \) denotes the Euclidian norm and \( r \) is some positive real number. Let \( d_s, d_c, d_u \) be positive integers, put \( d = d_s + d_c + d_u \) and \( \mathbf{d} = (d_s, d_c, d_u) \). A subset \( \mathbb{D}^d = \mathbb{D}_{d_s}^d \times \mathbb{D}_{d_c}^d \times \mathbb{D}_{d_u}^d \) of \( \mathbb{R}^d \) is called a polydisc of \( \mathbb{R}^d \) of index \( \mathbf{d} \). We call the triple \( \mathbf{d} \) and \( (r_s, r_c, r_u) \) the \textit{index} and the \textit{size} of the polydisc \( \mathbb{D}^d \) respectively. In the following, we only consider the case \( d_c = 1 \). Thus we have \( \mathbf{d} = (d_s, 1, d_u) \). In the same way, we say that a strongly partially hyperbolic splitting \( E^s \oplus E^c \oplus E^u \) has index \( \mathbf{d} \) if \( (\dim E^s, \dim E^c, \dim E^u) = \mathbf{d} \).

We first give two definitions, which describe the local dynamics around a fixed point and the transition dynamics near the heteroclinic point of an SH-simple cycle respectively.

\textbf{Definition 2.1.} Let \( f \in \text{Diff}^1(M) \) and \( P \) be a fixed point contained in a strongly partially hyperbolic set of index \( \mathbf{d} \). We say that a coordinate neighborhood \((U, \phi)\) around \( P \) is a \textit{linearized neighborhood of index \( \mathbf{d} \)} if the following hold:

- \( \phi(U) \subset \mathbb{D}^d \) is a polydisc of index \( \mathbf{d} \), that is, \( \phi(U) = \mathbb{D}_{r_s}^{d_s} \times \mathbb{D}_{r_c}^{d_c} \times \mathbb{D}_{r_u}^{d_u} \) for some size \( (r_s, r_c, r_u) \);
- there exist a contracting linear map \( \Lambda : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s} \), a linear map \( t : \mathbb{R} \to \mathbb{R} \) and an expanding linear map \( M : \mathbb{R}^{d_u} \to \mathbb{R}^{d_u} \) such that the following holds: For every \( x \in U \), if \( \phi(x) = (x_s, x_c, x_u) \) satisfies \( t(x_c) \in \mathbb{D}_{r_c}^{1} \) and \( M(x_u) \in \mathbb{D}_{r_u}^{d_u} \).
then we have
\[
\phi(f(x)) = (\Lambda(x_s), t(x_c), M(x_u)).
\]
We call the linear map \((x_s, x_c, x_u) \mapsto (\Lambda(x_s), t(x_c), M(x_u))\) the linearization of \(f\) near \(P\). We also call the set of the points \(x\) satisfying \(t(x_c) \in \mathbb{D}_{r_c}\) and \(M(x_u) \in \mathbb{D}_{r_u}\) the linearized region.

**Definition 2.2.** Let \(f \in \text{Diff}^1(M)\), \(P_1\) and \(P_2\) be two hyperbolic fixed points of \(f\) with \(u\)-ind\((P_1) = u\)-ind\((P_2) + 1\). For \(i = 1, 2\), suppose that \(P_i\) have linearized neighborhoods \((\phi_i, U_i)\) of index \(d\) with linearization
\[
(x_s, x_c, x_u) \mapsto (\Lambda_i(x_s), t_i(x_c), M_i(x_u))
\]
and assume that we have a heteroclinic point \(Q \in W^u(P_1) \cap W^s(P_2) \cap U_1\). Let \(\sigma\) be the least positive integer such that \(f^\sigma(Q) \in U_2\) holds. We say that \(Q\) is an adapted heteroclinic point with respect to \((U_1, \phi_1)\) and \((U_2, \phi_2)\) if there exist positive real numbers \(\kappa_s, \kappa_c\) and \(\kappa_u\) such that the following hold:

- There exists a neighborhood \(K \subset U_1\) of \(Q\) such that \(\phi_1(K)\) has the form \(\phi_1(Q) + \mathbb{D}^1_{\kappa_c} \times \mathbb{D}^1_{\kappa_c} \times \mathbb{D}^d_{\kappa_u}\), that is, \(\phi_1(K)\) is a polydisc centered at \(\phi(Q)\) of size \((\kappa_s, \kappa_c, \kappa_u)\);
- furthermore, \(f^i(K) \cap (U_1 \cup U_2) = \emptyset\) for every \(i = 1, \ldots, \sigma - 1\) and \(f^\sigma(K) \subset U_2\);
- there exist three linear maps \(\Lambda : \mathbb{R}^d \to \mathbb{R}^d\), \(\tilde{t} : \mathbb{R} \to \mathbb{R}\) and \(\tilde{M} : \mathbb{R}^d \to \mathbb{R}^d\) such that the following holds: For every \((X, Y, Z) \in \mathbb{R}^d\) such that \(\phi_1(Q) + (X, Y, Z) \in \phi_1(K)\) holds, we have
\[
\phi_2 \circ f^\sigma \circ \phi_1^{-1}(\phi_1(Q) + (X, Y, Z)) = \phi_2(f^\sigma(Q)) + (\Lambda(X), \tilde{t}(Y), \tilde{M}(Z)).
\]

We call \(\tilde{t}\) the center multiplier of the transition map \(f^\sigma\) and \(K\) the transition region.

Now we are ready to state the definition of SH-simple cycles.

**Definition 2.3.** Let \(f \in \text{Diff}^1(M)\), \(P_1\) and \(P_2\) be two hyperbolic fixed points of \(f\) with \(u\)-ind\((P_1) = u\)-ind\((P_2) + 1\). Suppose there is a heterodimensional cycle associated to \(P_1\), \(P_2\) and heteroclinic points \(Q_1 \in W^u(P_1) \cap W^s(P_2)\) and \(Q_2 \in W^s(P_1) \cap W^u(P_2)\). Furthermore, we assume that \(P_1 \cup P_2 \cup \text{orb}(Q_1) \cup \text{orb}(Q_2)\) is a strongly partially hyperbolic set and \(P_1, P_2\) are adapted. We say that this heterodimensional cycle is SH-simple if the following hold (see Figure. 1):

- There are linearized neighborhoods \((U_i, \phi_i)\) around \(P_i\) for \(i = 1, 2\);
- \(Q_i \in U_i\) and they are adapted heteroclinic points with respect to \((U_i, \phi_i)\) and \((U_{i+1}, \phi_{i+1})\) with transition maps \(f^{\sigma_i}\), for \(i = 1, 2\), where we set \((U_3, \phi_3) = (U_1, \phi_1)\);
- the \(\mathbb{R}^{d_u}\)-coordinate of \(\phi_1(Q_1)\) is \(0^{d_u}\);
- the \(\mathbb{R}^{d_c}\)-coordinate of \(\phi_2(f^{\sigma_1}(Q_1))\) is \(0^{d_c}\);
- the center multipliers of the transition maps \(f^{\sigma_i}\) \((i = 1, 2)\) are one.

**Remark 2.** According to the coordinates of \(Q_1\) and \(Q_2\), we may assume that they have the following forms:
\[
\begin{align*}
\phi_1(Q_1) &= (0^{d_c}, q_1, 0^{d_u}), & \phi_2(f^{\sigma_1}(Q_1)) &= (q_1', 0, 0^{d_u}), \\
\phi_2(Q_2) &= (0^{d_c}, 0, q_2), & \phi_1(f^{\sigma_2}(Q_2)) &= (q_2, 0, 0^{d_c}),
\end{align*}
\]
where \(q_1 \in \mathbb{R}, q_1', q_2' \in \mathbb{R}^{d_c}\) and \(q_2 \in \mathbb{R}^{d_u}\).
2.2. Main perturbation result. In [6], it was proven that given a diffeomorphism having a certain kind of heterodimensional cycle, by an arbitrarily small $C^1$-perturbation one can make the heterodimensional cycle to be a simple cycle, that is, the local dynamics around it are given by locally affine maps. One of the main steps of our proof is to show that by an arbitrarily small $C^1$-perturbation, one can obtain similar affine dynamics (i.e. SH-simple cycles) from an orientation preserving strongly partially hyperbolic heterodimensional cycle having a strong heteroclinic connection.

**Proposition 2.** Let $f \in \text{Diff}^1(M)$ which has two hyperbolic fixed points $P_1$, $P_2$ and satisfies assumptions (T1-4) in Theorem 1.3. Then, there exists an arbitrarily small $C^1$-perturbation $g$ of $f$ such that the continuations of $P_1$ and $P_2$ form an SH-simple cycle.

The proof of Proposition 2 is given in Section 4.

2.3. Main analytic result. Let us state the main analytic result about the existence of periodic points. We give a definition. For a hyperbolic periodic point $P$ contained in a strongly partially hyperbolic set with index $d = (d_s, 1, d_u)$, its center Lyapunov exponent, denoted by $\lambda_c(P)$, is the real number given as follows:

$$
\lambda_c(P) := \frac{1}{\pi(p)} \log \|Df^{\pi(p)}|_{E^c(P)}\|
$$

where $E^c(P)$ denotes the one-dimensional center direction at $P$. It is easy to see that periodic points in the same orbit share the same center Lyapunov exponent. So we also say center Lyapunov exponent of some orbit. Notice that this definition coincides with the usual one if we consider the uniformly distributed Dirac measure along the orbit of $P$.

The following proposition shows that for an SH-simple cycle, we can find a sequence of periodic orbits whose periods are successive integers and whose center Lyapunov exponents tend to zero. This is a key feature of our system. Its proof will be given in Section 5.

**Proposition 3.** Let $f \in \text{Diff}^1(M)$ with a heterodimensional cycle associated to hyperbolic fixed points $P_1$ and $P_2$. Suppose that they form an SH-simple cycle with respect to the coordinates $(U_i, \phi_i)$ ($i = 1, 2$). Then, there is an integer $\tilde{l}$ such that for every $j \geq \tilde{l}$ there exists a periodic point $R_j$ of period $j$ whose orbit admits
an orientation preserving strongly partially hyperbolic splitting with the following properties:

- For every $R \in \text{orb}(R_j)$, the angles between $E^u(R), E^c(R)$ and $E^s(R)$ are bounded from below by some positive constant independent of $j$;
- let $\lambda_c(R_j)$ be the center Lyapunov exponent of $R_j$. Then $\lambda_c(R_j) \to 0$ as $j \to \infty$;
- there exists a neighborhood $V_j$ of $\text{orb}(R_j)$ such that $\{V_j\}_{j \geq i}$ are pairwise disjoint. In particular, the sequence $\{\text{orb}(R_k)\}_{k > j}$ does not accumulate to $\text{orb}(R_j)$.

**Remark 3.** It follows from the orientation property that the center eigenvalue of $R_j$ is positive.

**2.4. Proof of Theorem 1.3.** Using Proposition 2 and Proposition 3, let us complete the proof of Theorem 1.3. For the proof, we state a lemma. This is used to perturb a periodic orbit having small Lyapunov exponent into one with zero Lyapunov exponent.

**Lemma 2.4** (Franks’ Lemma, [15]). Let $f \in \text{Diff}^1(M)$, $\varepsilon > 0$ and $P$ be a hyperbolic periodic point of period $\pi$. Let $\{G_i : TM|_{f^i(P)} \to TM|_{f^{i+1}(P)}\}_{i=0,\ldots,\pi-1}$ be a sequence of linear maps such that $\|Df^i(P) - G_i\| < \varepsilon$ holds for every $i$. Then, given a neighborhood $V$ of $\text{orb}(P)$, there exists $g \in \text{Diff}^1(M)$ such that the following hold:

- $\text{dist}_1(f, g) < \varepsilon$;
- $g(x) = f(x)$ for every $x \in M \setminus V$;
- $g$ preserves the orbit of $P$, that is, for every $i \in \mathbb{N}$ we have $g^i(P) = f^i(P)$;
- $Dg(g^i(P)) = G_i$.

**Proof of Theorem 1.3.** Let $f \in \text{Diff}^1(M)$ have a heterodimensional cycle associated to fixed points $P_1$ and $P_2$ satisfying assumptions (T1-4) of Theorem 1.3. We fix an arbitrarily small $\varepsilon_* > 0$.

Let us apply Proposition 2 to this heterodimensional cycle. Then we obtain a diffeomorphism $f_1$ with an SH-simple cycle associated to the hyperbolic continuations of $P_1$ and $P_2$ for $f_1$. Notice that $f_1$ can be chosen arbitrarily $C^1$-close to $f$, in particular, $\varepsilon_*/3$-close to $f$ in the $C^1$-distance.

Now we can apply Proposition 3: For $f_1$ we know that there exist $\hat{l} \in \mathbb{N}$ and a sequence of periodic points $\{R_j\}_{j \geq \hat{l}}$ satisfying the conclusion of Proposition 3. For each $j \geq \hat{l}$, we have a neighborhood $V_j$ of $\text{orb}(R_j)$ in such a way that $V_j \cap V_{j'} = \emptyset$ holds for $j \neq j'$.

Since $\text{orb}(R_j)$ admits an orientation preserving partially hyperbolic splitting and $\lambda_c(R_j) \to 0$ as $j \to \infty$, there exists $L > \hat{l}$ such that for every $j \geq L$, we can find an $\varepsilon_* / 3 C^1$-small perturbation whose support is contained in $V_j$ such that it preserves the orbit of $R_j$ and the resulted center Lyapunov exponent of $R_j$ is equal to zero. Furthermore, we can assume that the perturbations are arbitrarily small as $j \to \infty$.

Let us state this more precisely. Take a positive integer $L > \hat{l}$ large enough such that $\lambda_c(R_L)$ is sufficiently close to zero. For every $j \geq L$, using Franks’ Lemma we take a $C^3$ diffeomorphism $\rho_j \in \text{Diff}^1(M)$ such that the following hold:

- $\text{supp}(\rho_j) \subset V_j$ where we put $\text{supp}(\rho_j) := \{x \in M \mid \rho_j(x) \neq x\}$;
- for every $k \in \mathbb{N}$, we have $(\rho_j \circ f_1)^k(R_j) = f_1^k(R_j)$. In particular, $R_j$ is still a periodic point of period $j$ for $\rho_j \circ f_1$;
Thus, the diffeomorphism $h$ductively as follows:

$$\lambda \quad \text{The existence of such a sequence of diffeomorphisms can be confirmed by the fact that } \lambda(R_j, f_1) \to 0 \text{ and the boundedness of the angles of partially hyperbolic splittings over } \{\text{orb}(R_j)\}. \text{ We define the sequence of diffeomorphisms } \{g_j\}_{j \geq L} \text{ inductively as follows:}

1. $g_L = \rho_L \circ f_1$;
2. $g_{j+1} = \rho_{j+1} \circ g_j$ for $j > L$.

Using the disjointness of $\{\text{supp}(\rho_j)\}_{j \geq L}$, we can see that for every $k \in \mathbb{N}$, the sequence $\text{dist}(g_{j+k}, g_j)$ converges to zero as $j \to \infty$, uniformly with respect to $k$.

Consequently, $\{g_j\}_{j \geq L}$ is a Cauchy sequence in $\text{Diff}^1(M)$. By the completeness of $\text{Diff}^1(M)$ (see [18] for instance), the sequence $\{g_j\}$ converges to a $C^1$-diffeomorphism in the $C^1$-distance. Let $g_{\infty}$ be the limit diffeomorphism. Notice that, again by the disjointness of $\{V_j\}_{j \geq L}$, we see that for every $j$, $\text{orb}(R_j)$ is the same for $f_1$ and $g_{\infty}$, having center eigenvalue which is equal to one (recall that by the conclusion of Proposition 3, the strongly partially hyperbolic splitting over $TM|_{\text{orb}(R_j)}$ is orientation preserving). Furthermore, by the continuity of the distance function we have $\text{dist}(f_1, g_{\infty}) \leq \varepsilon_*/4 < \varepsilon_*/3$.

To give the final perturbation, let us fix some super-exponential sequence $\{a_n\}$, that is, $\lim_{n \to \infty} b^n/a_n = 0$ for every $b > 1$. We can take $a_n = n!$ for instance. Since $\{\text{orb}(R_j)\}_{j \geq L}$ are the same for $g_{\infty}$ and $f_1$, still $\{V_j\}$ are pairwise disjoint neighborhoods of $\{\text{orb}(R_j)\}$. For each $j \geq L$, we take a diffeomorphism $\eta_j \in \text{Diff}^1(M)$ such that

1. $\text{supp}(\eta_j) \subset V_j$;
2. $\eta_j \circ g_{\infty}$ has $j \cdot a_j$ distinct periodic points of period $j$ in $V_j$;
3. $\text{dist}(\eta_j \circ g_{\infty}, g_{\infty}) < \varepsilon_*/4$ for every $j \geq L$ and converges to zero as $j \to \infty$.

The existence of such $\{\eta_j\}$ can be deduced by using the fact that the center eigenvalue of $R_j$ equals one. Indeed, we only need to choose $\eta_j$ which “oscillates” $a_j$ times in the center direction of each point of $\text{orb}(R_j)$, keeping the derivative close to the identity map. See for instance [3, Remark 5.2] for the concrete construction of such a perturbation.

Then, put

$$h_n := \eta_n \circ \cdots \circ \eta_L \circ g_{\infty}.$$  

By the same reason as above, one can check that the limit $h_{\infty} := \lim_{n \to \infty} h_n$ exists and it is a $C^1$-diffeomorphism. Furthermore, one can see that $h_{\infty}$ has at least $j \cdot a_j$ periodic points of period $j$ for every $j \geq L$ and $\text{dist}(g_{\infty}, h_{\infty}) < \varepsilon_*/3$. Finally, we have $\text{dist}(f, h_{\infty}) \leq \text{dist}(f, f_1) + \text{dist}(f_1, g_{\infty}) + \text{dist}(g_{\infty}, h_{\infty}) < \varepsilon_*$

and for every $b > 1$,

$$\liminf_{n \to \infty} \frac{\#\text{Per}_n(h_{\infty})}{b^n} \geq \liminf_{n \to \infty} \frac{na_n}{b^n} = +\infty.$$  

Thus, the diffeomorphism $h_{\infty}$ satisfies the conclusion of Theorem 1.3.  

$\square$
3. Creation of strong heterodimensional cycles. In this section, we prove Theorem 1.4 (assuming Theorem 1.3) and Proposition 1. In the proof, we use the following perturbation lemma by Hayashi [16] which allows us to create a heterodimensional cycle by connecting invariant manifolds of saddles by $C^1$-small perturbations.

**Lemma 3.1** (Connecting lemma). Let $a_f$ and $b_f$ be a pair of saddles of $f \in \text{Diff}^1(M)$ such that there are sequences of points $\{y_n\}$ and of natural numbers $\{k_n\}$ satisfying:

- $y_n \to y \in W^u(a_f)$ ($n \to \infty$), $y \neq a_f$;
- $f^{k_n}(y_n) \to z \in W^s(b_f)$ ($n \to \infty$), $z \neq b_f$.

Then, there is a diffeomorphism $g$ arbitrarily $C^1$-close to $f$ such that $W^u(a_g)$ and $W^s(b_g)$ have a non-empty intersection arbitrarily close to $y$, where $a_g$ (resp. $b_g$) is the hyperbolic continuation of $a_f$ (resp. $b_f$) for $g$.

### 3.1. Proof of Theorem 1.4

We begin with the proof of Theorem 1.4. Let us recall a general result on the transitivity of the systems. The readers could find a proof in [12, Proposition 2.2.2] for instance.

**Lemma 3.2.** Let $X$ be a compact metric space without isolated point and $f : X \to X$ be a transitive homeomorphism. Put $\text{orb}^+(x) := \{f^n(x) : n \in \mathbb{N}\}$ and call it the forward orbit of $x$. Then there is a residual subset $R \subset X$ such that for every $x \in R$, $\text{orb}^+(x)$ is dense in $X$.

Let us give the proof of Theorem 1.4. Note that the first part of the proof is similar to the argument appearing in [1, Lemma 2.8].

**Proof of Theorem 1.4.** Let $f$ be a diffeomorphism satisfying the assumption of Theorem 1.4. We will prove that $f$ can be approximated by diffeomorphisms which satisfy the assumption of Theorem 1.3. Then the conclusion follows from Theorem 1.3 immediately.

Let us take an arbitrarily small $\varepsilon > 0$. We fix fundamental domains of $W^s(P_1)$ and $W^u(P_2)$ and denote their closures by $K_1$ and $K_2$ respectively. Notice that they are compact sets. Then, by the transitivity of $f$ on $H(P_1)$ and the hyperbolicity near $P_1$ and $P_2$, we can choose the sequences of orbits $\{y_n\}$ and of integers $\{k_n\}$ satisfying the assumption of Lemma 3.1, letting $a_f = P_2$ and $b_f = P_1$. That is, first we choose a point $x \in H(P_1)$ whose forward orbit is dense in $H(P_1)$ (see Lemma 3.2). Notice that $H(P_1)$ has no isolated points since it is non-trivial). Then, by using the hyperbolicity of $P_1$ and $P_2$, we can see that $\text{orb}^+(x)$ has accumulating points in $K_1$ and $K_2$. Let $y$ be one of the accumulating point in $K_2$ and $z$ be one in $K_1$. Then the constructions of $\{y_n\}$ and $\{k_n\}$ are straightforward.

Now, by applying Lemma 3.1, we obtain an $\varepsilon/2$-small $C^1$-perturbation $g$ of $f$ such that $W^s(P^g_1) \cap W^u(P^g_2) \neq \emptyset$, where $P^g_i$ (i = 1, 2) denote the hyperbolic continuation of $P_i$ for $g$. Notice that the transversal intersection of $W^u(P_1)$ and $W^{ss}(P_2)$ is $C^1$-robust, which implies that (by shrinking $\varepsilon$ if necessary) $W^u(P^g_1)$ and $W^{ss}(P^g_2)$ still have a non-empty intersection. Thus $P^g_1$ and $P^g_2$ form a heterodimensional cycle.

In the following, we denote one of the heteroclinic point in $W^u(P_1) \cap W^{ss}(P_2)$ by $Q_1$ and in $W^u(P^g_1) \cap W^{ss}(P^g_2)$ by $Q_2$.

To see that this heterodimensional cycle satisfies assumptions (T1-4) of Theorem 1.3, it remains to show (T4). Note that, by assumption, $H(P^g_1)$ is an orientation preserving strongly partially hyperbolic set if $g$ is sufficiently $C^1$-close to $f$. Thus to
can be found arbitrarily close to $Q$. By the robust transitivity, we can approximate the diffeomorphism by one that is super-exponentially divergent. Then we have

$$\text{dist}(h, f) \leq \text{dist}(h, g) + \text{dist}(g, f) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

Since $\varepsilon$ can be chosen arbitrarily small, the proof of Theorem 1.4 is completed.

3.2. Proof of Proposition 1. Let us give the proof of Proposition 1. The proof is divided into two steps.

**Lemma 3.3.** Let $M$ be an $n$-dimensional closed manifold for $n \geq 3$. Let $U$ be an open set of $\text{Diff}^1(M)$ such that every $f \in U$ satisfies the assumption of Theorem 1.2. Then, there is a set $V \subset U$ which is open and dense in $U$ such that for every $g \in V$ either $W^u(P^g_1) \cap W^{ss}(P^g_2) \neq \emptyset$ or $W^{uu}(P^g_1) \cap W^s(P^g_2) \neq \emptyset$ holds, where $P^g_1$ and $P^g_2$ are the continuations of $P_1$ and $P_2$. Furthermore, these intersections are transversal.

**Proof.** By the robust transitivity, we can approximate the diffeomorphism by one such that either the strong stable foliation or the strong unstable foliation is minimal in $M$ (i.e., every leaf is dense in $M$) see [17, Theorem 3] (see also [10, Theorem 1.3] for a three-dimensional version of this result). For such a diffeomorphism, we have either $W^u(P_1) \cap W^{ss}(P_2) \neq \emptyset$ or $W^{uu}(P_1) \cap W^s(P_2) \neq \emptyset$. Due to the existence of strong partial hyperbolicity, we know that these intersections are automatically transversal. Since this is an $C^1$-open condition, we obtain the conclusion.

**Lemma 3.4.** Let $f \in V$ in Lemma 3.3. Then, $f$ can be $C^1$-approximated by $g$ such that $g$ or $g^{-1}$ satisfies conditions (T1-4) in Theorem 1.3.

**Proof.** Let us take $f \in V$. We assume $W^u(P_1) \cap W^{ss}(P_2) \neq \emptyset$. The other case can be done by similar argument. Let $U$ be the open set in Lemma 3.3. Since $f \in U$, by Lemma 3.1 we can perturb $f$ so that $W^s(P_1) \cap W^u(P_2) \neq \emptyset$ (see the argument in Section 3.1 for the detail). Thus we can obtain (T3) by an arbitrarily small $C^1$-perturbation $g$ of $f$. In addition, this perturbation can be made small enough such that for the resulted diffeomorphism, the whole manifold still admits an orientation preserving strongly partially hyperbolic splitting. Thus, condition (T4) is automatically satisfied. Finally, since the other conditions (T1-2) are $C^1$-robust, we have that $g$ satisfies all the conditions (T1-4).

4. Perturbation to SH-simple cycles. In this section, we prove Proposition 2. The strategy of the proof is close to the proof of [6, Proposition 3.5], which is based on [9, Lemma 3.2].

**Proof of Proposition 2.** Let $f$ be a $C^1$-diffeomorphism with two hyperbolic fixed points $P_1$ and $P_2$ that satisfy assumptions (T1-4) in Theorem 3. We will construct an arbitrarily small $C^1$-perturbation $g$ of $f$ such that $g$ exhibits an SH-simple cycle.
associated to the continuations of $P_1$ and $P_2$. In fact, such a perturbation will be obtained by finitely many steps and the $C^1$-size of the perturbation can be controlled arbitrarily small in each step. Let us fix an arbitrarily small $\varepsilon > 0$.

**STEP 1.** First, we perturb $f$ in such a way that the resulted diffeomorphism acts as an affine map in small neighborhoods of the fixed points and the heteroclinic points. Namely, by performing $C^1$-small perturbations near $P_1, P_2$ and the heteroclinic points, we take a diffeomorphism $f_1$ with $\dist_1(f_1, f) < \varepsilon/4$ and local charts $(U_i, \phi_i)$ of $P_i$ ($i = 1, 2$) such that the following hold:

- $\phi_i(U_i) = \mathbb{R}^d$ and
- $\phi_i(P_i) = (0^d, 0, 0^d)$;
- for every $(x_s, x_c, x_u) \in \phi_i(U_i \cap f_1^{-1}(U_i))$, we have

$$
\phi_i \circ f_1 \circ \phi_i^{-1}(x_s, x_c, x_u) = \left( DF_1|_{E_{f_1}^c}(x_s), DF_1|_{E_{f_1}^s}(x_c), DF_1|_{E_{f_1}^u}(x_u) \right),
$$

where $DF_1$ denotes the linear map $D(\phi_i \circ f \circ \phi_i^{-1})(P_i)$;

- there exist $Q_1 \in W^u(P_1) \cap W^s(P_2)$, $Q_2 \in W^u(P_2) \cap W^s(P_1)$ and $\sigma_1, \sigma_2 \in \mathbb{N}$ such that $Q_i \in U_i$, $f_1^k(Q_i) \notin U_1 \cup U_2$ for $k = 1, 2, \ldots, \sigma_i - 1$ and $f_1^\sigma_i(Q_i) \in U_{i+1}$, where we set $P_3 = P_1$ and $(U_3, \phi_3) = (U_1, \phi_1)$. In the following we refer the integer $\sigma_i$ as the *transition time* of $Q_i$ into $U_{i+1}$;

- $\phi_2 \circ f_1^i \circ \phi_1^{-1}$ and $\phi_1 \circ f_1^i \circ \phi_2^{-1}$ are affine in small neighborhoods of $Q_1$ and $Q_2$;

- we also require the following: Let $t_1$ (resp. $t_2$) denote the center unstable (resp. stable) eigenvalue of $DF_1(P_1)$ (resp. $DF_1(P_2)$). Then we have that $t_1$ and $t_2$ are positive and $\log t_1$ and $\log t_2$ are rationally independent.

Such a perturbation can be done in the same way as [9, Lemma 3.2].

For $i = 1, 2$, there are locally invariant foliations in $U_i$ which are parallel to the corresponding coordinate planes. We define as follows:

- $\mathcal{F}_i^s$ is the $d_s$-dimensional foliations in $U_i$ ($i = 1, 2$) with leaves parallel to $\mathbb{R}^{d_s}$ in the local coordinate, called the strong stable foliation;

- similarly, we define foliations $\mathcal{F}_i^u, \mathcal{F}_i^c, \mathcal{F}_i^{cs}$ and $\mathcal{F}_{i}^{cu}$ and call them the strong unstable, center, center stable, center unstable foliations respectively. Their leaves are of dimensions $d_u, 1, (d_s + 1), (d_u + 1)$ respectively.

In the following, for $* \in \{s, u, c, cs, cu\}$, by $(\mathcal{F}_i^*)_{x}$ we mean the leaf of the foliation $\mathcal{F}_i^*$ containing $x$ and by $W^*_i(P_i)$ we mean the connected component of $W^*(P_i) \cap U_i$ containing $P_i$. For instance, $W^{uc}_{i\text{loc}}(P_i)$ denotes the connected component of $W^u(P_i) \cap U_i$ containing $P_i$. Then, we have the following:

- Inside $U_1$, we have $(\mathcal{F}_{1}^{cu})_{P_1} = W^{uc}_{\text{loc}}(P_1)$ and $(\mathcal{F}_{1}^{u})_{P_1} = W^{s}_{\text{loc}}(P_1)$;

- inside $U_2$, we have $(\mathcal{F}_{2}^{cs})_{P_2} = W^{uc}_{\text{loc}}(P_2)$ and $(\mathcal{F}_{2}^{u})_{P_2} = W^{u}_{\text{loc}}(P_2)$.

**STEP 2.** In this step, we construct a perturbation of $f_1$ such that the heteroclinic point $Q_1$ is located in $(\mathcal{F}_1)_{P_1}$. First, by an arbitrarily small $C^1$-perturbation, we can always assume that $Q_1 \notin (\mathcal{F}_1)_{P_1}$. By the domination of $E_{f_1}^c \oplus E_{f_1}^u$, we have

$$
\frac{d_u(\phi_1(f_1^{-k}(Q_1)), \phi_1(f_1^{-(k+1)}(Q_1)))}{d_c(\phi_1(f_1^{-k}(Q_1)), \phi_1(f_1^{-(k+1)}(Q_1)))} \to 0 \quad (k \to +\infty),
$$
where \( d_w(A, B) \) (resp. \( d_c(A, B) \)) denotes the distance between points \( A \) and \( B \) along the \( F^u \) (resp. \( F^s \)) direction. We take \( k \in \mathbb{N} \) sufficiently large such that

\[
\frac{d_w(\phi_1(f_1^{-k}(Q_1)), \phi_1(f_1^{-(k+1)}(Q_1)))}{d_c(f_1^{-k}(Q_1)), \phi_1(f_1^{-(k+1)}(Q_1)))} < \frac{\varepsilon}{10M_0},
\]

where

\[
M_0 = \sup\{\|D(\phi_1 g \phi_1^{-1})(x)\| + \|D(\phi_1 g^{-1} \phi_1^{-1})(x)\| : x \in \phi_1(U_1), \ \text{dist}_1(g, f_1) < C\}
\]

and \( C \) is some positive constant such that \( M_0 \) is finite.

Then, we take a diffeomorphism, denoted by \( \alpha \), such that

- \( \text{dist}_1(\alpha, \text{id}) < \frac{\varepsilon}{8M_0} \), where \( \text{id} \) is the identity map;
- \( \alpha \) coincides with the identity map outside a small neighborhood \( U \) of \( f_1^{-k}(Q_1) \).

Here, \( U \) can be taken so small that \( U \cap \text{orb}(Q_1) = f_1^{-k}(Q_1) \);
- \( \alpha \circ f_1^{-k}(Q_1) \in (F_1^s)_{P_1} \).

Thus, \( f_2 = f_1 \circ \alpha^{-1} \) is a perturbation of \( f_1 \) with \( \text{dist}_1(f_2, f_1) < \varepsilon/4 \) satisfying \( f_2^{-k}(Q_1) \in (F_1^s)_{P_1} \). Notice that the forward iterations of \( Q_1 \) are not affected by this perturbation and the heterodimensional cycle associated to \( P_1 \) and \( P_2 \) also survives. By shrinking \( U_1 \) and replacing \( Q_1 \) by some backward iteration of it (still denoted by \( Q_1 \), for notational simplicity), we also get a new transition time of \( Q_1 \) (still denoted by \( \sigma_1 \)) such that \( Q_1 \in U_1 \cap (F_1^s)_{P_1} \), \( f_2^{\sigma_1}(Q_1) \in (F_2^s)_{P_2} \) and \( f_2^{k}(Q_1) \notin U_1 \cup U_2 \) for \( j = 1, 2, \ldots, \sigma_1 - 1 \). Finally, for \( i = 1, 2 \), we fix small neighborhoods \( K_i \subset U_i \) of \( Q_i \) such that \( \phi_i(K_i) \) is a polydisk.

**STEP 3.** We construct a small perturbation of \( f_2 \) which keeps the foliations invariant under the transition maps. First, we consider the perturbation along \( \text{orb}(Q_1) \). Without loss of generality, we can assume that \( f_2^{\sigma_1}((F_1^s)_{Q_1}) \) is in the general position with respect to \( (F_2^s)_{P_1} \).

Due to the existence of the domination of \( E^s \oplus (E^c \oplus E^u) \), the forward image of \( f_2^{\sigma_1}((F_1^s)_{Q_1}) \) under \( f_2 \) tends to \( (F_2^u)_{P_2} \). Thus, by replacing \( \sigma_1 \) by \( \sigma_1 + k \) for some large \( k \), we can make a small perturbation \( f_3 \) of \( f_2 \) which keeps the foliation \( F^u_k \) invariant under \( f_3^{\sigma_1} \) on a smaller \( K_1 \).

Now we consider the invariance of \( F^s_k \). For the strong stable foliation, we can assume that \( f_3^{\sigma_1}(F_2^s(f_1^{\sigma_1}(Q_1))) \) is in a general position with respect to \( (F_1^s)_{Q_1} \). Consider the backward iterations of \( f_3^{\sigma_1}((F_2^s)_{f_1^{\sigma_1}(Q_1)}) \) under \( f_3 \), which tends to \( (F_1^s)_{P_1} \). Replacing \( Q_1 \) by \( f_3^{-k}(Q_1) \), \( K_1 \) by some neighborhood of \( f_3^{-k}(Q_1) \) and \( \sigma_1 \) by \( \sigma_1 + k \) for some large \( k \), we can take a small perturbation \( f_4 \) of \( f_3 \), such that the foliation \( F^s_k \) is invariant under \( f_4^{\sigma_1} \) on a smaller \( K_1 \), preserving the invariance of center unstable foliations \( F^u_k \) in \( K_1 \).

Repeating the above argument to \( F^s_k \) and \( F^u_k \), we obtain a small perturbation \( f_5 \) of \( f_4 \) such that \( f_5^{\sigma_1}|_{K_1} \) preserves the foliations \( F^s_k \) and \( F^u_k \), in addition to \( F^s_\alpha \) and \( F^u_\alpha \). Then, the preservation of \( F^s_\alpha \) and \( F^u_\alpha \) implies the preservation of \( F^c_\alpha \). Thus, we have seen the preservation of all the five foliations under \( f_5^{\sigma_1}|_{K_1} \).

Completely in a similar way, by an arbitrarily small \( C^1 \)-perturbation (still denoted by \( f_5 \)), we have \( f_5^{\sigma_2}|_{K_2} \) also preserves these foliations. Each perturbation in this **STEP 3** can be made arbitrarily small in the \( C^1 \)-distance, thus we can have \( \text{dist}_1(f_5, f_2) < \varepsilon/4 \).
STEP 4. We give the final perturbation $f_6$ of $f_5$ such that the transition map $f_6^{\sigma_1}|_{\mathcal{K}_1}$ and $f_6^{\sigma_2}|_{\mathcal{K}_2}$, restricted to the center direction, are isometries (i.e., their center multipliers are equal to one).

Since $F^c_5$ is invariant under $f_5$, we only need to consider the restriction of $f_5^{\sigma_1}$ to $F^c_5$, which has the following form (recall that $\text{orb}(Q_1)$ admits an orientation preserving strongly partially hyperbolic splitting whose center bundle has dimension one):

$$q_1 + Y \mapsto bY \quad (b > 0).$$

For $m, n \in \mathbb{N}$, let us consider $f_5^{-n}(Q_1)$ and $f_5^{\sigma_1+m}(Q_1)$ instead of $Q_1$ and $f_5^{\sigma_1}(Q_1)$ respectively. Since $f_5$ acts as a linear map $Df_5(P_i)$ in $U_i$ ($i = 1, 2$), the derivative of $f_5^{\sigma_1+m}$ at $(f_5^{-n}(Q_1))$ is of the following form:

$$Y \mapsto t_1^n t_2^m bY,$$

where $t_1 > 1$ and $t_2 \in (0, 1)$ are the center multipliers of $Df_5(P_1)$ and $Df_5(P_2)$ respectively. We remind the reader that $\log t_1$ and $\log t_2$ are rationally independent by STEP 1 of our perturbation. Thus we are allowed to choose $n$ and $m$ sufficiently large such that

$$|n \log t_1 + m \log t_2 + \log b| < \varepsilon/4,$$

which is equivalent to

$$|t_1^n t_2^m b - 1| < \varepsilon/4.$$

Consider a linear perturbation $A$ of $Df_5(f_5^{\sigma_1+m-1}(Q_1))$ satisfying

- $A = Df_5(f_5^{\sigma_1+m-1}(Q_1))$ in $E^s \oplus E^n$ direction;
- $A = (t_1^n t_2^m b)^{-1} Df_5(f_5^{\sigma_1+m-1}(Q_1))$ in $E^c$ direction;
- $||A - Df_5(f_5^{\sigma_1+m-1}(Q_1))|| < \varepsilon/4$.

Applying Franks’ Lemma to $f_5$, we get a $C^1$-perturbation $f_6$ of $f_5$ with

$$\text{dist}_1(f_6, f_5) < \varepsilon/4$$

such that

$$Df_6(f_5^{\sigma_1+m-1}(Q_1)) = A.$$

By our construction, the center multiplier of $f_6^{\sigma_1+n+m}(Q_1)$ equals one. Let us rewrite $(\sigma_1 + n + m)$ by $\sigma_1$, $f_6$ by $g$, $f_6^{-n}(Q_1)$ and $f_6^{\sigma_1+m}(Q_1)$ by $Q_1$ and $g^{\sigma_1}(Q_1)$ respectively. We also shrink $U_1$ and $U_2$, take small neighborhood $K_1 \subset U_1$ of $Q_1$ such that $\sigma_1$ is the transition time of $Q_1$ into $U_2$.

In a similar way, we give another arbitrarily small $C^1$-perturbation (still denoted by $g$) to make the center multiplier of $g^{\sigma_2}(Q_2)$ equals one. Now, $g$ has an SH-simple cycle associated to $P_1$ and $P_2$ with adapted heteroclinic points $Q_1$ and $Q_2$. Moreover, we have $\text{dist}_1(g, f) < \varepsilon$, since $\varepsilon$ is taken arbitrarily small in advance, the $C^1$-size of the perturbation can be made arbitrarily small. This completes the proof of Proposition 2.

5. Proof of analytic result. In this section, we prove Proposition 3. The proof is technical and will be done by purely analytic argument.
5.1. Setting. In this subsection, we introduce the maps in the local coordinates and give formal calculations of the coordinates of the periodic points around the heterodimensional cycle.

Let us consider a diffeomorphism $f$ having an SH-simple cycle associated to fixed points $P_1, P_2$ and heteroclinic points $Q_1, Q_2$ with linearized neighborhoods $(U_i, \phi_i)$ ($i = 1, 2$), see Figure 1. By definition, we know that for $i = 1, 2$,

$$\phi_i(U_i) = \mathbb{D}_{r^1_i}^{d^1_x} \times \mathbb{D}^1_{r^1_c} \times \mathbb{D}_{r^1_u}^{d^1_u}.$$ 

We put $F_i := \phi_i \circ f \circ \phi_i^{-1}$. There are linear maps $F_i : \mathbb{R}^{d^1_x} \to \mathbb{R}^{d^1_x}$ and $M_i : \mathbb{R}^{d^1_u} \to \mathbb{R}^{d^1_u}$ and $t_i : \mathbb{R} \to \mathbb{R}$ ($i = 1, 2$), which describe the local dynamics around $P_i$. In the following, we put $t_1 := \mu, \ t_2 := \lambda$ and identify $\mu$ and $\lambda$ with real numbers which give the center multipliers of these linear maps.

Then we have the following local dynamics: For $i = 1$, for every $(x_s, x_c, x_u) \in \phi_1(U_1)$, if $\mu(x_c) \in \mathbb{D}^{d^1_x} \times \mathbb{D}^1_{\kappa^1_c} \times \mathbb{D}^{d^1_u}$ then we have

$$F_1(x_s, x_c, x_u) = (A_1(x_s), \mu(x_c), M_1(x_u)).$$

Also, for $i = 2$, for every $(x_s, x_c, x_u) \in \phi_2(U_2)$, if $M_2(x_u) \in \mathbb{D}^{d^1_u}$ then we have

$$F_2(x_s, x_c, x_u) = (A_2(x_s), \lambda(x_c), M_2(x_u)).$$

Next, let us recall the local dynamics around the transition region. Let $K_i \subset U_i$ be the transition region from $U_i$ to $U_{i+1}$ (we put $U_3 = U_1$). Then (see Remark 2)

$$\phi_1(K_1) = (0^{d^1_x}, q_1, 0^{d^1_u}) + \mathbb{D}_\kappa^1 \times \mathbb{D}^1_{\kappa^1_c} \times \mathbb{D}^{d^1_u} \quad \text{and} \quad \phi_2(K_2) = (0^{d^1_x}, 0^{d^1_u}) + \mathbb{D}_\kappa^2 \times \mathbb{D}^1_{\kappa^2_c} \times \mathbb{D}^{d^1_u}.$$

Let $\sigma_i$ be the transition time of $Q_i$ into $U_{i+1}$. We put $\hat{F}_i = \phi_{i+1} \circ f^{\sigma_i} \circ \phi_i^{-1}$. Then we have

$$\hat{F}_1(0^{d^1_x}, q_1, 0^{d^1_u}) = (q'_1, 0^{d^1_u}) \quad \text{and} \quad \hat{F}_2(0^{d^1_x}, 0^{d^1_u}) = (q'_2, 0^{d^1_u}).$$

Again, there are linear maps $\hat{A}_i : \mathbb{R}^{d^1_x} \to \mathbb{R}^{d^1_x}$ and $\hat{M}_i : \mathbb{R}^{d^1_u} \to \mathbb{R}^{d^1_u}$ ($i = 1, 2$), which describe the local dynamics of the transition maps. That is, for $i = 1$, for every $(X, q_1 + Y, Z) \in K_1$ we have (recall the definition of SH-simple cycles, both of the center multipliers of the two transition maps are one)

$$\hat{F}_1(X, q_1 + Y, Z) = (q'_1 + \hat{A}_1(X), Y, \hat{M}_1(Z)).$$

Similarly, for $i = 2$, for every $(X, Y, q_2 + Z) \in K_2$ we have

$$\hat{F}_2(X, Y, q_2 + Z) = (q'_2 + \hat{A}_2(X), Y, \hat{M}_2(Z)).$$

5.2. Formal calculation. Given an SH-simple cycle, we are interested in finding periodic points whose orbits turn around it. Let us assume that there exists a periodic point $R \in K_1$ with the following itinerary (denoted by $*$):

- $f^{\sigma_1}(R) \in U_2$;
- there exists a positive integer $m_2$ such that $f^{\sigma_1 + k}(R)$ is in the linearized region of $U_2$ for $k = 0, \ldots, m_2 - 1$;
- $f^{\sigma_1 + m_2}(R) \in K_2$ and $f^{\sigma_1 + m_2 + \sigma_2}(R) \in U_1$;
- there exists a positive integer $m_1$ such that $f^{\sigma_1 + m_2 + \sigma_2 + k}(R)$ is contained in the linearized region of $U_1$ for $k = 0, \ldots, m_1 - 1$;
- $f^{\sigma_1 + m_2 + \sigma_2 + m_1}(R) = R$. 
In this subsection, we investigate the condition that $R$ should satisfy in the local coordinates. Put
\[ \phi_1(R) = (X, q_1 + Y, Z). \] (1)
Then, $f^{\sigma_1}(R) \in U_2$ has the following form in the $(U_2, \phi_2)$ coordinates:
\[ (q_1' + \Lambda_1(X), Y, \tilde{M}_1(Z)). \]
Subsequently, the following $m_2$ times forward iterations of $f^{\sigma_1}(R)$ are in $U_2$. As a result, in the $(U_2, \phi_2)$ coordinates, the point $f^{\sigma_1+m_2}(R)$ has the following coordinates:
\[ (A_2^{m_2}[q_1' + \Lambda_1(X)], \lambda^{m_2}Y, M_2^{m_2}\tilde{M}_1(Z)) = (A_2^{m_2}[q_1' + \Lambda_1(X)], \lambda^{m_2}Y, M_2^{m_2}\tilde{M}_1(Z) - q_2 + q_2). \] (2)
Then, under $f^{\sigma_2}$, this point is mapped to $f^{\sigma_1+m_2+\sigma_2}(R)$. The local coordinates of this point with respect to $(U_1, \phi_1)$ is
\[ (q_2' + \Lambda_2A_2^{m_2}[q_1' + \Lambda_1(X)], \lambda^{m_2}Y, \tilde{M}_2[M_2^{m_2}\tilde{M}_1(Z) - q_2]). \]
Then, under $f^{m_1}$, this point is mapped back to $R$. The local coordinates are
\[ (A_1^{m_1}(q_2' + \Lambda_2A_2^{m_2}[q_1' + \Lambda_1(X)]), \mu^{m_1}\lambda^{m_2}Y, M_1^{m_1}\tilde{M}_2[M_2^{m_2}\tilde{M}_1(Z) - q_2]). \]
Since this point is equal to the point in (1), we obtain equations for $(X, Y, Z)$. Formally, the solution is
\[ X = [I - A_1^{m_1}\Lambda_2A_2^{m_2}\Lambda_1]^{-1}(A_1^{m_1}\Lambda_2A_2^{m_2}q_1' + A_1^{m_1}q_2'), \] (3)
\[ Y = \frac{q_1}{\mu^{m_1}\lambda^{m_2} - 1}, \] (4)
\[ Z = [M_1^{m_1}\tilde{M}_2M_2^{m_2}\tilde{M}_1 - I]^{-1}M_1^{m_1}\tilde{M}_2q_2 = [M_2^{m_2}\tilde{M}_1 - \tilde{M}_2^{-1}M_1^{-m_1}]^{-1}q_2, \] (5)
where $I$ denotes the identity map. This formal solution may give a true periodic point of period $(\sigma_1 + m_2 + \sigma_2 + m_1)$ depending on the choices of $m_2$ and $m_1$. In the following, we consider for what choices of $m_2$ and $m_1$ we can obtain the true orbit.

5.3. Realizability of the orbit. In order to check that the formal solution obtained in the previous subsection gives a true periodic orbit, we need to confirm that the point indeed passes the transition region at the designated moments. The following proposition states that we can judge it only by looking at the behavior in the center direction. Recall that $q_1$ is the second component of the heteroclinic point $Q_1$ in the local chart and $\kappa_c$ is the size of the transition region $K_1$ in $E_c$ direction (see Remark 2 and Definition 2.2).

**Proposition 4.** There exists a positive integer $m$ such that for every $(m_1, m_2)$ satisfying $m_1, m_2 \geq m$ the following holds: Suppose that $\mu^{m_1}\lambda^{m_2} \neq 1$ and we have the following inequality:
\[ \left| \frac{q_1}{\mu^{m_1}\lambda^{m_2} - 1} \right| \leq \kappa_c. \] (6)
Then the formal solution $\bar{R} \in U_1$ given by $\phi_1(\bar{R}) = (X, q_1 + Y, Z)$, where $X, Y, Z$ are the formal solutions $(3, 4, 5)$, gives a true periodic point whose orbit follows the itinerary $(\ast)$. 


Proof. First, we can see that the point $\bar{R} = (X,q_1 + Y,Z)$ is in the transition region $K_1$ if both $m_1$ and $m_2$ are sufficiently large. Indeed, by (3), $X \to 0^c$ as $m_1, m_2 \to +\infty$. This is because the linear map $\Lambda_1^{m_1} \Lambda_2^{m_2} \lambda_1$ goes to zero map and the point $\Lambda_1^{m_1} \Lambda_2^{m_2} q_1^1 + \Lambda_1^{m_1} q_2^1$ goes to $0^d$. Similarly, by (5) we have $Z \to 0^d$ as $m_1, m_2 \to +\infty$. Inequality (6) guarantees that the $Y$-coordinate of $\phi_1(\bar{R})$ lies in the region of $\phi_1(K_1)$. Thus the point $\bar{R}$ is indeed in $K_1$ for large $m_1$ and $m_2$.

We can see that $f^{m_1 + m_2}(\bar{R})$ is in $K_2$ for large $m_1$ and $m_2$. The condition for $X$, $Y$-coordinates are obvious for large $m_1$ and $m_2$. In fact, on the one hand, according to (2), the $X,Y$ coordinate of $f^{m_1 + m_2}(\bar{R})$ are $\Lambda_2^{m_2}[q_1^1 + \Lambda_1(X)]$ and $\lambda_2^{m_2} Y$, which are close to $0^d$ and $0$ if $m_2$ is sufficiently large. On the other hand, since

$$\phi_2(K_2) = \phi_2(Q_2) + D_{\kappa_2} \times D_{\kappa_2} = (q_2, 0, q_2),$$

we know that the $X$, $Y$-coordinates of $f^{m_1 + m_2}(\bar{R})$ are smaller than $\kappa_2$ and $\kappa_2$ respectively. Therefore, it remains to examine the condition of the $Z$-coordinate. By the definition of $Z$ in (5), it satisfies

$$Z = M_1^{m_1} M_2^{m_2} \lambda_1(Z) - q_2.$$ 

As we have observed, $Z$ is very close to $0^d$ when $m_1, m_2$ are large. It follows that $M_2^{m_2} \lambda_1(Z) - q_2$ must be close to zero since it is equal to $(M_1^{m_1} M_2^{-1}(Z)$, where $(M_1^{m_1} M_2^{-1}$ are strongly contracting linear map for large $m_1$. This means that $M_2^{m_2}(\bar{R})$, which is the $Z$-coordinate of $f^{m_1 + m_2}(\bar{R})$ in the local coordinates, converges to $q_2$ as $m_1, m_2 \to +\infty$.

To summarize, we have seen that for $m_1, m_2$ large, the itinerary of $\bar{R}$ certainly passes the transition regions with the given itinerary ($\ast$). This completes the proof of proposition 4.

5.4. **Proof of Proposition 3.** In this subsection we will complete the proof of Proposition 3 by examining the inequality (6).

By the argument of the previous subsections, we know that for $(m_1, m_2)$ sufficiently large there exists a periodic point of period $(\sigma_1 + m_2 + \sigma_2 + m_1)$ if and only if it satisfies the inequality (6). We shall show that for every $l \in \mathbb{N}$, there is a pair of integers $(m_{1,l}, m_{2,l})$ such that there is a periodic point $R_l$ of period

$$\sigma_1 + m_{2,l} + \sigma_2 + m_{1,l} := \pi(R_l)$$

satisfying

$$\pi(R_{l+1}) = \pi(R_l) + 1,$$

whose center Lyapunov exponent $\lambda_c(R_l)$ converges to zero as $l \to +\infty$.

First, by a direct calculation, we can get a sufficient condition for the inequality (6):

**Lemma 5.1.** For fixed $q_1$ and $\kappa_1^1$, under the condition $\mu^{m_1} \lambda^{m_2} > 1$, we have the inequality (6) if $\mu^{m_1} \lambda^{m_2} > \hat{\alpha}$, where

$$\hat{\alpha} := \frac{|q_1|}{\kappa_1^1} + 1 > 1.$$ 

Now we are ready to complete the proof of Proposition 3.

**End of the proof of Proposition 3.** Let

$$C := \max\{|\log \lambda|, |\log \mu|\}$$
and choose \( L, L' > \log \tilde{\alpha} \) such that \( L' - L > 2C \) holds. Then, we investigate the pair of integers \((m_1, m_2)\) satisfying

\[
L < m_2 \log \lambda + m_1 \log \mu < L'.
\]  

(7)

Now, we fix some sufficiently large \((m_{1,1}, m_{2,1})\) which satisfy the above inequality. Such pair of integers exists since \( L' - L > 2C \). Then we can inductively construct another pair of integers \((m_{1,2}, m_{2,2})\) which also satisfies (7) and

\[
m_{1,2} + m_{2,2} = m_{1,1} + m_{2,1} + 1
\]

holds. Indeed, given \((m_{1,1}, m_{2,1})\), by the condition \( L' - L > 2C \) and the fact that \( \log \lambda \) and \( \log \mu \) have opposite signatures, we can see that at least one of \((m_{1,1} + 1, m_{2,1})\) and \((m_{1,1}, m_{2,1} + 1)\) satisfies (7).

Inductively, we can choose the sequence \(\{(m_{1,t}, m_{2,t})\}_{t=1}^{\infty}\) such that

\[
\sigma_1 + m_{1,t+1} + \sigma_2 + m_{2,t+1} = \sigma_1 + m_{1,t} + \sigma_2 + m_{2,t} + 1
\]

(8)

and the inequality

\[
L < m_{2,t} \log \lambda + m_{1,t} \log \mu < L'
\]

holds for every \( t \in \mathbb{N} \). We claim that \( R_t := \phi_t^{-1}((X_t, q_1 + Y_t, Z_t)) \), where \((X_t, q_1 + Y_t, Z_t)\) is the solution of Proposition 4 (see (3, 4, 5)) corresponding to \((m_{1,t}, m_{2,t})\), gives the desired sequence of the periodic points. In fact, on the one hand, we have

\[
m_{2,t} \log \lambda + m_{1,t} \log \mu > L > \log \tilde{\alpha},
\]

thus by Lemma 5.1 and Proposition 4, there indeed exists a periodic point \( R_t \) of period \( \pi(R_t) = \sigma_1 + m_{2,t} + \sigma_2 + m_{1,t} \). In addition, it follows from (8) that \( \pi(R_{t+1}) = \pi(R_t) + 1 \).

By a direct calculation, the center Lyapunov exponent of \( R_t \) is given as follows:

\[
\lambda_c(R_t) = \frac{m_{2,t} \log \lambda + m_{1,t} \log \mu}{\sigma_1 + m_{2,t} + \sigma_2 + m_{1,t}}.
\]

The numerator has absolute value bounded by \( L' \) from above, and as \( t \) tends to infinity, \((m_{1,t} + m_{2,t})\) tends to infinity. This shows that the center Lyapunov exponent converges to zero as \( l \to +\infty \). Finally, by translating the subscript of \( R_t \), we have \( \pi(R_t) = l \) for every sufficiently large \( l \in \mathbb{N} \).

Let us confirm that there is no self accumulation of the sequence of \( \{\text{orb}(R_t)\} \). Indeed, by construction one can check that the set of accumulation points of \( \{\text{orb}(R_t)\} \) is contained in the set

\[
\{P_1\} \cup \{P_2\} \cup \text{orb}(Q_1) \cup \text{orb}(Q_2)
\]

and it does not contain any point of \( \text{orb}(R_t) \). This implies the conclusion.

Furthermore, by construction we know that every \( \text{orb}(R_t) \) admits an orientation preserving strongly partially hyperbolic splitting with bounded angles, deriving from the SH-simple cycle (indeed, the splitting is orthogonal in the local coordinate).

Thus the proof is completed. \( \square \)

6. **On the lower limit of generic diffeomorphisms.** In this section, we provide the proof of Theorem 1.5. Recall that a sequence \( \{a_n\} \) of positive integers is said to be super-exponential if for every \( b > 1 \) we have \( \lim_{n \to \infty} b^n/a_n = 0 \).

The following result by Kaloshin [19] is the main ingredient of the proof.

**Proposition 5.** Given \( 1 \leq s < \infty \), there exists a subset \( D^s \subset \text{Diff}^s(M) \) which is \( C^s \)-dense in \( \text{Diff}^s(M) \) such that for every \( f \in D^s \) the following hold:
(I) Every periodic point of \( f \) is hyperbolic; 
(II) there exists a positive real number \( C_f \) such that 
\[
\#\text{Per}_n(f) < \exp(C_f n)
\]
holds for every \( n \in \mathbb{N} \).

**Remark 4.** Proposition 5 is not proved for \( s = \infty \). We do not know whether the result is true for \( s = \infty \) or not. Nonetheless, we can prove Theorem 1.5 for \( s = \infty \).

Let us recall some definitions. By \( \text{dist}(\cdot, \cdot) \) we denote the distance function on \( \text{Diff}^s(M) \) which is compatible with the \( C^s \)-topology. A periodic point is called \( n \)-periodic if its period is \( n \).

For the proof, we state a lemma about the number of hyperbolic periodic points. Its proof is elementary and we omit the proof.

**Lemma 6.1.** Let \( 1 \leq s \leq \infty \), \( f \in \text{Diff}^s(M) \) and \( n \in \mathbb{N} \). Suppose that all \( n \)-periodic points of \( f \) are hyperbolic. Then, there exists a \( C^s \)-neighborhood \( \mathcal{U} \) in \( \text{Diff}^s(M) \) of \( f \) such that for every \( g \in \mathcal{U} \) the number of \( n \)-periodic points is the same as that of \( f \).

Now, let us prove Theorem 1.5.

**Proof of Theorem 1.5.** Let \( \{a_n\} \) be a super-exponential sequence as in the assumption of Theorem 1.5.

First, we consider the case \( 1 \leq s < \infty \). For every \( L \in \mathbb{N} \), put 
\[
\mathcal{O}_L^s = \{ f \in \text{Diff}^s(M) : \exists N > L \text{ such that } \#\text{Per}_N(f) < a_N/N \}.
\]
Let us show that \( \mathcal{O}_L^s \) contains a subset of \( \text{Diff}^s(M) \) which is \( C^s \)-open and \( C^s \)-dense in \( \text{Diff}^s(M) \). Indeed, for every \( f \in \mathcal{D}^s \) (see Proposition 5 for the definition of \( \mathcal{D}^s \)), by property (II) of Proposition 5, there is a constant \( C_f > 0 \) with \( \#\text{Per}_n(f) < \exp(C_f n) \) holds for every \( n \in \mathbb{N} \). Since \( \{a_n\} \) is super-exponential, i.e., 
\[
\lim_{n \to \infty} a_n^{-1} \exp(C_f n) = 0,
\]
we can take \( N = N(f) > L \) such that 
\[
a_N^{-1} \exp(C_f N) < 1/N.
\]
Then, we have 
\[
\#\text{Per}_N(f) < \exp(C_f N) < a_N/N,
\]
which implies \( \mathcal{D}^s \subset \mathcal{O}_L^s \). Then, by applying Lemma 6.1 to each \( f \in \mathcal{D}^s \), we can find a \( C^s \)-open neighborhood \( \mathcal{U}_f \subset \text{Diff}^s(M) \) of \( f \) such that for every \( g \in \mathcal{U}_f \), the number of \( N \)-periodic points of \( g \) is the same as that of \( f \). This yields \( \mathcal{U}_f \subset \mathcal{O}_L^s \).

Now, it is straightforward to see that every diffeomorphism in \( \mathcal{D}^s \) satisfies the conclusion of Theorem 1.5.

Let us consider the case \( s = \infty \). For each positive integer \( L \), we define the set of \( C^\infty \)-diffeomorphisms \( \mathcal{O}_L^\infty \) as in the previous case. Namely, we put 
\[
\mathcal{O}_L^\infty = \{ f \in \text{Diff}^\infty(M) : \exists N > L \text{ such that } \#\text{Per}_N(f) < a_N/N \}.
\]
We make the following claim.

**Claim 1.** For every \( L \in \mathbb{N} \) and \( f \in \text{Diff}^\infty(M) \), there is a \( C^\infty \)-open set \( \mathcal{W}_L^f \) of \( \text{Diff}^\infty(M) \) such that the following holds:

- The \( C^\infty \)-closure of \( \mathcal{W}_L^f \) contains \( f \);
• for every $g \in \mathcal{W}_L^f$, there exists $N > L$ such that $\#\text{Per}_N(g) < a_N/N$.

Note that the second condition implies $\mathcal{W}_L^f \subset \mathcal{O}_L^\infty$. It follows from Claim 1 that $\mathcal{O}_L^\infty$ contains a $C^\infty$-open and $C^\infty$-dense subset $\cup_{f \in \text{Diff}^\infty(M)} \mathcal{W}_L^f$ of $\text{Diff}^\infty(M)$. Then, as in the previous case, we can complete the proof of Theorem 1.5.

Now, we need to prove Claim 1. The proof is reduced to the following claim.

Claim 2. Let $L \in \mathbb{N}$ and $f \in \text{Diff}^\infty(M)$. Given a $C^\infty$-neighborhood $\mathcal{Y}$ of $f$ in $\text{Diff}^\infty(M)$, there is a non-empty $C^\infty$-open set $\mathcal{Z}_L^f$ of $\text{Diff}^\infty(M)$ contained in $\mathcal{Y}$ such that the following holds: For every $g \in \mathcal{Z}_L^f$, there exists $N > L$ with $\#\text{Per}_N(g) < a_N/N$.

Let us see how Claim 1 follows from Claim 2. Given $L$ and $f$, we first take a nested sequence of $C^\infty$-neighborhoods $\{\mathcal{Y}_k\}$ of $f$ such that $\cap_k \mathcal{Y}_k = \{f\}$ holds. For each $k$, we apply Claim 2: It provides us a non-empty $C^\infty$-open set $\mathcal{Z}_L^f(k)$. Then $\cap_k \mathcal{Z}_L^f(k)$ gives a $C^\infty$-open set $\mathcal{W}_L^f$ in the conclusion of Claim 1.

It remains to prove Claim 2. Let $L$, $f$ and $\mathcal{Y}$ be given. First, for small $\varepsilon > 0$, we choose a positive integer $t$ and $\delta > 0$ such that the following holds: For any $g_1 \in \text{Diff}^\infty(M)$ with $\text{dist}_t(f,g_1) < \delta$, we have $\text{dist}_\infty(f,g_1) < \varepsilon$. Note that the existences of such $t$ and $\delta$ are guaranteed by the definition of the $C^\infty$-topology.

Then, since $\text{Diff}^\infty(M) \subset \text{Diff}^t(M)$, Proposition 5 allows us to choose $h_f \in D^t$ such that $\text{dist}_t(f,h_f) < \delta/2$ holds, see Proposition 5 for the definition of $D^t$. By the same reason as in the case of $1 \leq s < \infty$, we know $h_f \in \mathcal{O}_L^\infty$. In particular, there is $N > L$ with $\#\text{Per}_N(h_f) < a_N/N$. Applying Lemma 6.1 to $h_f$, we obtain a $C^t$-open neighborhood $\mathcal{V}_f \subset \text{Diff}^t(M)$ of $h_f$ such that for every $g \in \mathcal{V}_f$, we have

\begin{enumerate}
  \item $\text{dist}_t(h_f,g) < \delta/2$;
  \item $\#\text{Per}_N(g) = \#\text{Per}_N(h_f) < a_N/N$.
\end{enumerate}

Then, we choose $\hat{h}_f \in \text{Diff}^\infty(M) \cap \mathcal{V}_f$. Such $\hat{h}_f$ exists because $\text{Diff}^\infty(M)$ is $C^t$-dense in $\text{Diff}^t(M)$. Then, we choose a $C^\infty$-open neighborhood $\mathcal{Z}_L^f$ of $\hat{h}_f$ in $\text{Diff}^\infty(M)$ satisfying the following:

$$\mathcal{Z}_L^f \subset \mathcal{V}_f \cap \text{Diff}^\infty(M).$$

Note that the existence of such $\mathcal{Z}_L^f$ is guaranteed by the definition of the $C^\infty$-topology.

Now, for every $g \in \mathcal{Z}_L^f$, condition (i) implies

$$\text{dist}_t(f,g) < \text{dist}_t(f,h_f) + \text{dist}_t(h_f,g) < \delta/2 + \delta/2 = \delta,$$

which immediately implies that $\text{dist}_\infty(f,g) < \varepsilon$ by the choice of $t$ and $\delta$. By condition (ii), we know that $\mathcal{Z}_L^f$ satisfies the desired property of Claim 2. Note that the choice of $\varepsilon > 0$ is arbitrary. Hence, we can choose $\mathcal{Z}_L^f$ in such a way that it is contained in $\mathcal{Y}$.

\[\square\]

Acknowledgments. The paper has been supported by the JSPS KAKENHI Grant Number 18K03357, 21K03320. XL is supported by the National Natural Science Foundation of China (11701199) and the Fundamental Research Funds for the Central Universities, HUST: 2017 KFYXJJ095. KS thank the warm hospitality of Huazhong University of Science and Technology. We thank Masayuki Asaoka and Ken-ichiro Yamamoto for useful conversations. We also acknowledge the anonymous referees for their helpful suggestions.
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Received January 2021; 1st revision January 2021; final revision September 2021; early access November 2021.

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