Solving ptychography with a convex relaxation

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Ptychography is a powerful computational imaging technique that transforms a collection of low-resolution images into a high-resolution sample reconstruction. Unfortunately, algorithms that are currently used to solve this reconstruction problem lack stability, robustness, and theoretical guarantees. Recently, convex optimization algorithms have improved the accuracy and reliability of several related reconstruction efforts. This paper proposes a convex formulation of the ptychography problem. This formulation has no local minima, it can be solved using a wide range of algorithms, it can incorporate appropriate noise models, and it can include multiple a priori constraints. The paper considers a specific algorithm, based on low-rank factorization, whose runtime and memory usage are near-linear in the size of the output image. Experiments demonstrate that this approach offers a 25% lower background variance on average than alternating projections, the current standard algorithm for ptychographic reconstruction.

Introduction

Over the past two decades, ptychography [1,2] has surpassed all other imaging techniques in its ability to produce high-resolution, wide-field-of-view measurements of microscopic and nanoscopic phenomena. Whether in the X-ray regime at third-generation synchrotron sources [3–6], in the electron microscope for atomic scale phenomena [7,8], or in the optical regime for biological specimens [9], ptychography has shown an unparalleled ability to acquire hundreds of megapixels of sample information near the diffraction limit. The standard ptychography principle is simple: a series of diffraction patterns are recorded from a sample as it is scanned through a focused beam. These intensity-only measurements are then computationally converted into a reconstruction of the complex sample (i.e., its amplitude and phase), which contains more pixels than a single recorded diffraction pattern.

A recently introduced imaging procedure, termed Fourier ptychography (FP), uses a similar principle to create gigapixel images with a conventional microscope [9]. The only required hardware modification is an LED array, which illuminates a stationary sample from different directions as the microscope captures a sequence of images. As in standard ptychography, FP must also recover the sample’s phase as it merges together the captured image sequence into a high-resolution output. Standard and Fourier ptychographic data are connected via a linear transformation [10], which allows both setups to use nearly identical image reconstruction algorithms.

Standard and Fourier ptychography both avoid the need for a large, well-corrected lens to image at the diffraction-limit. Instead, they shift the majority of resolution-limiting factors into the computational realm. Unfortunately, an accurate and reliable solver does not yet exist. For ptychography to succeed, we must reconstruct the phase of the scattered field, which is a non-convex problem. To date, nearly all ptychography algorithms solve the phase retrieval problem with an iterative technique called alternating projections (AP) [11,12]. However, AP and simple variants [13,14] often converge to incorrect local minima or they stagnate [15]. Few guarantees exist regarding convergence, let alone convergence to a reasonable solution. Despite these shortcomings, many authors have pushed beyond the basic algorithms [16] to account for unknown system parameters [17,18], to improve outcomes by careful initialization [19], and to enable multiplexed acquisition [20].

In this article, we formulate a convex program for the ptychography problem, which allows us to use efficient computational methods to obtain a reliable image reconstruction. Convex optimization has recently matured into a powerful computational tool that now solves a variety of challenging problems [21]. However, many sub-disciplines of imaging, especially those involving phase retrieval, have been slow to reap its transformative benefits. Several prior works [22–26] have connected convex optimization with abstract phase retrieval problems, but this is the first work that applies convex optimization to the quickly growing field of high-resolution ptychography.

The convex optimization approach to ptychographic reconstruction has many advantages over AP. Our formulation has no local minima, so we can always obtain a solution with minimum cost. The methodology is significantly more noise-tolerant than AP, and the results are more reproducible. There are also opportunities to establish theoretical guarantees using machinery from convex analysis.

![Fourier Ptychography Setup](image)

**Fig. 1.** Diagram of the Fourier ptychography setup (top), where we use an LED array to illuminate a sample from different directions and acquire an image set. This paper introduces a convex phase retrieval algorithm to transform this image set into a high-resolution complex sample estimate ψ. Included image set and reconstructed resolution target are experimental results.
Furthermore, there are many efficient algorithms for our convex formulation of the ptychography problem. To obtain solutions at scale, we apply a factorization method due to Burer and Monteiro [27,28]. This method avoids the limitations of earlier convex algorithms for abstract phase retrieval, whose storage and complexity scale cubically in the number of reconstructed pixels [29]. Moreover, recent results establish that this factorization technique converges globally under certain conditions [29], offering a promising theoretical guarantee. The end result is a new, noise-tolerant algorithm for ptychographic reconstruction that is efficient enough to process the multi-gigapixel images that future applications will demand.

Here is an outline for the paper. First, we develop a linear algebraic framework to illustrate the Fourier ptychographic image formation process. Second, we manipulate this framework to pose its sample recovery problem as a convex program. This initial algorithm, termed “convex lifted ptychography” (CLP), supports a-priori knowledge of noise statistics to significantly increase the accuracy of image reconstruction in the presence of noise.

Convex Lifted Ptychography

In this section, we outline the image capture process for Fourier ptychography (see Fig. 1). At the end of this section, we discuss how a simple exchange of variables yields a nearly equivalent mathematical description of “standard” (i.e., diffraction imaging-based) ptychography data, which our proposed algorithm may also process. In addition, while the following analysis considers a two-dimensional optical geometry for simplicity, extension to three dimensions is direct.

We assume that a distant plane \( L(x') \) contains \( q \) different quasi-monochromatic LEDs (central wavelength \( \lambda \)) evenly distributed along \( x' \) with a spacing \( r \). We assume each LED acts as an effective point source and illuminates a sample \( \psi(x) \) at a plane \( S(x) \) a distance \( l \) away from \( L(x') \). Under this assumption, the \( j \)th LED illuminates the sample with a spatially coherent plane wave at angle \( \theta_j = \tan^{-1}(q/r/j) \), where \(-q/2 \leq j \leq q/2\). Additionally assuming the sample \( \psi(x) \) is thin, we may express the optical field exiting the thin sample as the product,

\[
s(x,j) = \psi(x)e^{ikxj/p_j},
\]

where the wavenumber \( k = 2\pi/\lambda \) and \( p_j = \sin \theta_j \) describes the off-axis angle of the \( j \)th LED. The \( j \)th illuminated sample field \( s(x,j) \) then enters an imaging system with a low numerical aperture (NA). Neglecting scaling factors and a quadratic phase factor for simplicity, Fourier optics gives the field at the imaging system pupil plane \( A(x') \) as \( F[s(x,j)] = \psi(x' - p_j) \), where \( F \) represents the Fourier transform between conjugate variables \( x \) and \( x' \). \( \psi \) is the Fourier transform of \( \psi \), and we have applied the Fourier shift property. The shifted spectrum field \( \psi(x' - p_j) \) is then modulated by the imaging system’s aperture function \( a(x') \), which acts as a low-pass filter. It is now useful to consider the spectrum \( \psi \) discretized into \( n \) pixels with a maximum spatial frequency \( k \). We denote the bandpass cutoff of the aperture function \( a \) as \( k \cdot m/n \), where \( m \) is an integer less than \( n \). The modulation of \( \psi \) by \( a \) results in a field characterized by \( m \) discrete samples, which propagates to the camera imaging plane and is critically sampled by an \( m \)-pixel digital detector. This forms a reduced-resolution image, \( g \):

\[
g(x,j) = |F[a(x')\psi(x' - p_j)]|^2.
\]

The Fourier ptychography setup in Fig. 1 may be converted into a standard ptychography experiment by interchanging the sample plane \( S \) and the aperture plane \( A \). This results in a standard ptychographic data matrix taking the form of Eq. 2 but now with a sample spectrum \( \psi \) as an \((n \times 1)\) vector. Again, \( n \) is the known sample resolution before it is reduced by the finite bandpass of the lens aperture. Second, the \( j \)th detected image becomes an \((m \times 1)\) vector \( g_j \), where again \( m \) is the number of pixels in each low-resolution image. The ratio \( n/m \) defines the ptychographic resolution improvement factor, set by the

**Convex Lifted Ptychography**

We begin the process of solving Eq. 2 as a convex program by expressing it in matrix form. First, we represent the unknown sample spectrum \( \psi \) as an \((n \times 1)\) vector. Again, \( n \) is the known sample resolution before it is reduced by the finite bandpass of the lens aperture. Second, the \( j \)th detected image becomes an \((m \times 1)\) vector \( g_j \), where again \( m \) is the number of pixels in each low-resolution image. The ratio \( n/m \) defines the ptychographic resolution improvement factor, set by the

- **Image set** \( b \)
- **Block-diagonal** \( F \)
- **Sparse** \( A \)
- **\( \hat{\psi}(x) \)**

**Fig. 2.** A set of images captured by Fourier ptychography stack together into a long data vector, \( b \). Each associated matrix transform is similarly stacked and combined to form our final measurement matrix, \( D = FA \). Here, we show stacking of just two images for simplicity. Typically, over 200 images are stacked.
ratio between the maximum LED angle and the imaging lens acceptance angle.

Third, we express each lens aperture function \( a(x + p_j) \) as an \((n \times 1)\) discrete aperture vector \( a_j \), which modulates the unknown sample spectrum \( \psi \). To rewrite Eq. 2 as a matrix product, we define \( \{A_j\}_{j=1}^q \) to be the sequence of \((m \times n)\) rectangular matrices that contain a deterministic aperture function \( a_j \) along a diagonal. For an aberration-free rectangular aperture, each matrix \( A_j \) has ones originating at \((0, p_j)\) and terminating at \((m, p_j + n - 1)\), where \( p_j \) is now a discretized version of our shift variable \( p_j \). Finally, we introduce an \((m \times m)\) discrete Fourier transform matrix \( F^{(m)} \) to express the transformation of the low-pass filtered sample spectrum through our fixed imaging system for each low-resolution image \( g_j \):

\[
g_j = |F^{(m)} A_j \psi|^2, \quad 1 \leq j \leq q.
\]

Fourier ptychography acquires a series of \( q \) images, \( \{g_j\}_{j=1}^q \). We combine this image set into a single vector by “stacking” all images in Eq. 3

\[
b = |FA\psi|^2 = |D\psi|^2.
\]

Here, \( b \) is \( \{g\} \) expressed as a \((q \cdot m \times 1)\) stacked image vector (see Fig. 3). In addition, we define \( D = FA \), where \( F \) is a \((q \cdot m \times q \cdot m)\) block diagonal matrix containing \( q \) copies of the low-resolution DFT matrices \( F^{(m)} \) in its diagonal blocks, and \( A \) has size \((q \cdot m \times n)\) and is formed by vertically stacking each aperture matrix \( A_j \):

\[
F = \begin{pmatrix} F^{(m)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F^{(m)} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ \vdots \\ A_q \end{pmatrix}.
\]

We denote the transpose of the \( i \)th row of \( D \) as \( d_i \), which is a column vector. The set \( \{d_i\} \) forms our measurement vectors. The measured intensity in the \( i \)th pixel is the square of the inner product between \( d_i \) and the spectrum \( \psi: b_i = \left| \langle d_i, \psi \rangle \right|^2 \). Next, we “lift” the solution \( \psi \) out of the quadratic relationship in Eq. 4. As suggested in [23], we may instead express it in the space of \((n \times n)\) positive-semidefinite matrices:

\[
b_i = \operatorname{Tr} \left( \psi^* d_i d_i^T \psi \right) = \operatorname{Tr} \left( d_i d_i^T \psi \psi^* \right) = \operatorname{Tr} (DX),
\]

where \( D = d_i d_i^T \) is a rank-1 measurement matrix constructed from the \( i \)th measurement vector \( d_i \). \( X = \psi \psi^* \) is an \((n \times n)\) rank-1 outer product, and \( 1 \leq i \leq q \cdot m \). Equation 6 states that our quadratic image measurements \( \{b_i\}_{i=1}^{q \cdot m} \) are linear forms of \( \psi \) in a higher dimensional space. We may combine these \( q \cdot m \) linear transforms into a single linear operator \( \mathcal{A} \) to summarize the relationship between the stacked image vector \( b \) and the matrix \( X \) as, \( \mathcal{A}(X) = b \).

One can now pose the phase retrieval problem as a rank minimization procedure:

\[
\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad \mathcal{A}(X) = b, \\
& \quad X \succeq 0,
\end{align*}
\]

where \( X \succeq 0 \) denotes \( X \) is positive-semidefinite. This rank minimization problem is not convex and is a computational challenge. Instead, adapting ideas from [23], we form a convex relaxation of Eq. 7 by replacing the rank of matrix \( X \) with its trace. This creates a convex semidefinite program:

\[
\begin{align*}
\text{minimize} & \quad \operatorname{Tr}(X) \\
\text{subject to} & \quad \mathcal{A}(X) = b, \\
& \quad X \succeq 0.
\end{align*}
\]

Several recent results establish that the relaxation in Eq. 8 is equivalent to Eq. 7 under certain conditions on the operator \( \mathcal{A} \) [32,33]. Although not necessarily equivalent in general, this relaxation consistently offers us highly accurate experimental performance. To account for the presence of noise, we may reform Eq. 8 such that the measured intensities in \( b \) are no longer strictly enforced constraints, but instead appear in the objective function:

\[
\begin{align*}
\text{minimize} & \quad \operatorname{Tr}(X) + \frac{1}{2} \|\mathcal{A}(X) - b\|_2^2 \\
\text{subject to} & \quad X \succeq 0.
\end{align*}
\]

Here, \( \alpha \) is a scalar regularization variable that directly trades off goodness for complexity of fit. Its optimal value depends upon the assumed noise level. Equation 9 forms our final convex problem to recover a solution-improved complex sample \( \psi \) from a set of obliquely illuminated images in \( b \). Many standard tools are available to solve this semidefinite program (see Methods section). Its solution defines our Convex Lifted Ptychography (CLP) approach.

In practice, CLP returns a low-rank matrix \( X \), with a rapidly decaying spectrum, as the optimal solution of Eq. 9. We obtain our final complex image estimate \( \psi \) by first performing a singular value decomposition of \( X \). Given low-noise imaging conditions and spatially coherent illumination, we set \( \psi \) to the Fourier transform of the largest resulting singular vector. Viewed as an autocorrelation matrix, we may also find useful statistical measurements within the remaining smaller singular vectors of \( X \). We note that one may also identify \( X \) as the discrete mutual intensity matrix of a partially coherent optical field [34], and view Eq. 9 as an alternative solver for the stationary measured states of a ptychography setup [35].

Without any further modification, three points distinguish Eq. 9 from existing AP-based ptychography solvers. First, the convex solver has a larger search space. If AP is used to iteratively update an \( n \times n \) pixel estimate, Eq. 9 must solve for an \( n \times n \) positive-semidefinite matrix. Second, this boost in the solution space dimension guarantees the convex program may find its global optimum with tractable computation. This allows CLP to avoid AP’s frequent convergence to local minima (i.e., failure to approach the true image). Finally, Eq. 9 implicitly

![Fig. 3. Simulation of the CLP algorithm.](image)
considers the presence of noise by offering a parameter (α) to tune with an assumed noise level. AP-based solvers lack this parameter and can be easily led into incorrect local minima with an assumed noise level. AP-based solvers lack this parameter.

CLP simulations and noise performance. We simulate Fourier ptychography following the setup in Fig. 1. We capture multiple two-dimensional images in (x, y) from a three-dimensional optical geometry. The simulated FP setup contains a detector with m = 12² pixels that are each 4µm wide, a 0.1 NA lens (6° collection angle) with unity magnification, and an LED array designed to offer an illumination NA of 0.2 (θmax = 11.5° maximum illumination angle). These parameters define the reconstructed resolution of our complex sample as n = 36² pixels, increasing the pixel count of one raw image by a factor n/m = 9.

Fig. 3(b) shows example simulated raw images from a sample of absorptive beads modulated by a quadratic phase envelope. Within each raw image, the set of absorptive beads is not clearly resolved. Here, we simulate the capture of q = 8² low-resolution images, each uniquely illuminated from one of q = 8² LEDs in the square array. We then input this image set into both the standard AP algorithm [16] as well as CLP in Eq. 8 (setting α = .001) to recover a high-resolution (36 × 36 pixel) complex sample. Even in the noiseless case, 5 iterations of nonlinear AP introduces unpredictable artifacts to both the recovered amplitude and phase (Fig. 3(d)), while CLP offers near perfect recovery (Fig. 3(c)). A constant phase offset is recovered amplitude and phase (Fig. 3(d)), while CLP offers near perfect recovery (Fig. 3(c)). A constant phase offset is recovered amplitude and phase (Fig. 3(d)), while CLP offers near perfect recovery (Fig. 3(c)).

As the size of X scales with n², processable image sizes are limited to an order of 10⁴ pixels on current desktop machines. This scaling limit does not prevent large-scale CLP processing of ptychography data. It is common practice to segment each collected image into as few as 10² pixels, process each segment separately, and then “tile” the resulting reconstructions back together into a final full-resolution algorithm output [9], CLP may also parallelize its computation with this strategy.

While such tiling parallelization offers significant speedup, a simple observation helps avoid the poor scaling of CLP altogether: the desired solution of the ptychography problem in Eq. 7 is low-rank. Instead of solving for an n × n matrix X, it is thus natural to adopt a low-rank ansatz and factorize the matrix X as X = RRᵀ, where our decision variable R is an n × r rectangular matrix with r < n [27, 28]. Inserting this factorization into our optimization problem in Eq. 8 and writing the constraints in terms of the measurement matrix D, dDdᵀ creates the non-convex program,

\[
\text{minimize } \text{Tr}(RRᵀ)
\]

subject to \(\text{Tr}(DDᵀ) = b_i \) for all i.

Besides removing the positive semidefinite constraint in Eq. 6, the factored form of Eq. 10 presents two more key adjustments to our original convex formulation. First, using the relationship \(\text{Tr}(RRᵀ) = \|R\|_F^2\), it is direct to rewrite the objective function and each constraint in Eq. 10 with just one n × r matrix R. Now instead of storing an n × n matrix like CLP, LRP must only store an n × r matrix. Since most practical applications of ptychography require coherent optics, the desired solution rank r will typically be close to 1, thus significantly relaxing storage requirements (i.e., LRP memory usage now scales linearly instead of quadratically with n).

Second, the feasible set of Eq. 10 is no longer convex. We thus must shift our solution strategy away from a simple semidefinite program. Prior work in [27, 28] suggests that an efficient and practically successful method of solving Eq. 10 is to minimize the following augmented Lagrangian function:

\[
L(R, y, σ) = \text{Tr}(RRᵀ) - \sum_i y_i \cdot (\text{Tr}(DDᵀ) - b_i) + \frac{σ}{2} \sum_i \left(\text{Tr}(DDᵀ) - b_i\right)^2
\]

where \(R ∈ \mathbb{R}^{n×r}\) is the unknown decision variable and the two variables \(y ∈ \mathbb{R}^m\) and \(σ ∈ \mathbb{R}^+\) are new parameters to help
guide our algorithm to its final reconstruction. The first term in Eq. (11) is the objective function from Eq. (10) indirectly encouraging a low-rank factorization product. This tracks our original assumption of a rank-1 solution within a “lifted” solution space. The second term contains the known equality constraints in Eq. (10) (i.e., the measured intensities), each assigned a weight \( y \). The third term is a penalized fitting error that we abbreviate with label \( \sigma \). It is weighted by one penalty parameter \( \sigma \), mimicking the role of a Lagrangian multiplier. With an appropriate fixed selection of \( y \)'s and \( \sigma \), the minimization of \( L(\mathbf{R}, \mathbf{y}, \sigma) \) with respect to \( \mathbf{R} \) will produce our desired solution. As an unconstrained function, the minimum of \( L \) is quickly found via standard tools (e.g., a quasi-Newton approach such as the LBFGS algorithm [33]).

The goal of our low-rank ptychography (LRP) algorithm may now be stated as follows: determine a suitable set of \( (y, \sigma) \) such that Eq. (11) may be minimized with respect to \( \mathbf{R} \) to find the optimal solution. We use the iterative algorithm suggested in [27] to sequentially minimize \( L \) with respect to \( \mathbf{R} \) at iteration \( k \), and then update a new parameter set \((g^{k+1}, \gamma^{k+1})\) at iteration \( k + 1 \). We update parameters \((g^{k+1}, \gamma^{k+1})\) to ensure their associated term’s contribution to the summation forming \( \mathbf{R} \) is relatively small. This suggests \( R^{k+1} \) is proceeding to a more feasible solution. The relative permissible size of the second and third terms in \( L \) are controlled by two important parameters, \( \eta < 1 \) and \( \gamma > 1 \): if the third term \( \gamma \) sufficiently decreases such that \( v^{k+1} \leq \gamma v^{k} \), then we hold its multiplier \( \gamma \) fixed and update the equality constraint multipliers, \( y \). Otherwise, we increase \( \gamma \) by a factor of \( \gamma \) such that \( \sigma^{k+1} = \sigma^{k} \). A detailed discussion of these algorithmic steps is in [27][28].

We initialize the LRP algorithm with an estimate of the unknown high-resolution complex sample function \( \psi_0 \), contained within a low-rank matrix \( \mathbf{R}^0 \). We terminate the algorithm either if it reaches a sufficient number of iterations, or if the minimizer fulfills some convergence criterion. We form \( \mathbf{R}^0 \) using a spectral method, which can help increase solver accuracy and decrease computation time [30]. Specifically, we select the \( r \) columns of \( \mathbf{R}^0 \) as the leading \( r \) eigenvectors of \( \mathbf{D} \text{diag}(\mathbf{b}) \mathbf{D} \), where \( \mathbf{D} \) is the measurement matrix in Eq. (2). While this spectral approach works quite well in practice, a random initialization of \( \mathbf{R}^0 \) also often produces an accurate reconstruction.

**LRP simulations and noise performance.** Following the same procedure used to simulate the CLP algorithm, we test the MSE performance of the LRP algorithm as a function of SNR in Fig. 5. We again add different amounts of uncorrelated Gaussian noise to each simulated raw image set and compare the LRP reconstruction with ground truth. This simulated sample is the experimentally obtained amplitude and phase of a human blood smear. It is qualitatively similar to the sample used in Fig. 3. Unlike with the simulations in Fig. 3, the AP algorithm no longer malfunctions at lower spectrum overlap percentages (i.e., lower values of \( \sigma t \)). Despite this apparent success, the AP minimizer MSE is still an average 5 dB worse than the LRP minimizer MSE across all levels of SNR, achieved without additional parameter optimization or explicit noise modeling.

In these simulations, we somewhat arbitrarily fix \( \eta \) and \( \gamma \) at 0.5 and 1.5, respectively, and set the desired rank of the solution, \( r \), to 1. In practice, these free variables offer significant freedom to tune the response of LRP to noise. For example, similar to the noise parameter \( \alpha \) in Eq. (4), the multiplier \( \gamma \) (controlled via \( \gamma \)) in Eq. (11) helps trade off complexity for goodness of fit by re-weighting the quadratic fitting error term.

**Experimental results.** We experimentally verify the accuracy and improved noise stability of LRP using a Fourier ptychographic microscope, which captures a sequence of images under unique LED illumination following the protocol in [9] (see Methods section). We first quantitatively verify that LRP accurately measures high resolution and sample phase. This verification highlights how LRP avoids certain undesirable artifacts commonly found in AP reconstructions. Second, we qualitatively compare the high-resolution AP and LRP reconstructions of a biological sample. This set of experiments demonstrates the improved noise stability of our LRP solver.

**Quantitative performance.** To quantitatively verify resolution improvement, we turn on each of 15 × 15 LEDs in an array placed 80 mm beneath a calibration target (U.S. Air Force [USAF] resolution chart), capturing a total of 225 low-resolution images. Each LED has an approximate 20 nm spectral bandwidth centered at \( \lambda = 632 \) nm. Using a 0.08 NA microscope objective (5° collection angle) and a 0.35 illumination
Experimental reconstruction of a USAF target showing 5X resolution improvement along both x and y using two ptychography algorithms: (a) AP and (b) LRP. Here we only show reconstructed intensity. LRP avoids artifacts (e.g., boxed in green) commonly encountered in the AP approach. Variances measured in blue boxes.

(c) Same cropped region of one low-resolution raw image, for comparison.

NA (θ_{max} = 20° illumination angle), this setup is designed to offer a total complex field resolution gain of n/m = 25. Each image spectrum overlaps by θ ≈ 70° in area with each neighboring image spectrum.

For reconstruction, we select n = 25 · m and use the same aperture parameters with AP and LRP to create the high-resolution images in Fig. 6 (details in Methods section). Both ~1 megapixel reconstructions achieve their maximum expected resolving power (Group 9, Element 3: 1.56 μm line pair spacing). This is approximately 5 times sharper than the smallest resolved feature in one raw image (Group 7, Element 2 in Fig. 6(c)). Our new LRP algorithm avoids certain artifacts that are commonly observed during the nonlinear descent of AP (boxed in green). Both reconstructions slowly fluctuate across areas in the background that we expect to be uniformly bright or dark. These fluctuations are caused in part by imperfect estimates of the aperture function a and LED shift values p_i. In a representative background area marked in Fig. 6 by a 40° pixel blue box, AP and LRP exhibit normalized background amplitude variances of σ_4^2 = 5.4 × 10^{-2} and σ_2^2 = 5.9 × 10^{-4}, respectively. Accounting for experimental uncertainty in the values of a and p_i (e.g., following 51-52) may reduce this error.

To verify that our LRP solver reconstructs quantitatively accurate phase, we next image a monolayer of polystyrene microspheres (index of refraction n_m = 1.587) immersed in oil (n_o = 1.515, both indexes for λ = 632 nm light). To demonstrate the LRP algorithm easily generalizes to any ptychographic arrangement, we perform this experiment on a new FP microscope using a larger 0.5 NA objective lens (collection angle = 30°). We now turn on each of 29 LEDs in an array placed 40 mm beneath the sample (0.7 illumination NA, θ_{max} = 45° illumination angle), capturing 29 unique images.

The synthesized numerical aperture of this new FP microscope, computed as the sum of the illumination NA and objective lens NA, is NA_s = 1.2. With a greater-than-unity synthetic NA, our reconstructions should offer oil-immersion quality resolution (~385 nm smallest resolvable feature spacing), without requiring any immersion medium between the sample and objective lens.

Using the same data and parameters for AP and LRP input, we obtain the high-resolution phase reconstructions of two adjacent microspheres (diameters 3 μm and 6 μm) in Fig. 7. We have subtracted a constant phase offset from the LRP solution in (b) to allow for direct comparison to the AP solution in (a). The two reconstructions appear qualitatively similar except at the center of the 6 μm sphere, where the AP phase profile unexpectedly flattens. We highlight this flattening by selecting phase values along each marked dashed line to plot the resulting sample thickness in Fig. 7(c). Phase φ and sample thickness t are related via t = kΔφ(n_m − n_o)^{-1}, where k is the average wavenumber and Δφ = φ − φ_0 is the reconstructed phase minus a constant offset. LRP closely matches the optical thickness of a ground-truth sphere (GT, black curve): the length of the vertical chord connecting the top and bottom halves of a 6 μm diameter circle. The normalized amplitude variances from a 40°-pixel background area are σ_4^2 = 9.2 × 10^{-4} and σ_2^2 = 5.8 × 10^{-4}, respectively, again supporting our observation that LRP reconstructions are more accurate than AP reconstructions.

Biological sample reconstruction. Our third imaging example uses the same high-NA FP configuration (collection angle = 30°, θ_{max} = 45°) to resolve a biological phenomenon: the infectious spread of malaria in human blood. The early stages of a *Plasmodium falciparum* infection in erythrocytes (i.e., red blood cells) includes the formation of small parasitic “rings”. It is challenging to resolve these parasitic rings under a microscope without using a high-NA lens and an immersion medium, even after appropriate staining (here, a modified Giemsa stain). Oil-immersion is required for an accurate diagnosis of infection.

We use FP to resolve *Plasmodium falciparum*-infected cells with a 0.5 NA objective lens and no oil. In Fig. 5, we capture 29 uniquely illuminated images of a sample region containing several infected cells, with the same FP microscope and LED array used for Fig. 7 (sample preparation details in Methods section). Figure 6(c) contains an example normally illuminated raw image from this setup, which does not clearly resolve the parasite infection. For comparison, we capture the same sample region with a conventional high-NA oil-immersion microscope (NA = 1.25) using Kohler illumination, which clearly resolves parasites within two cells (Fig. 6(a)).

Fig. 7(c) presents our phase retrieval reconstructions using the standard AP algorithm (sharpest solutions after 6 iterations). Reconstructions from three data sets are included: images captured with a 1 second exposure (top), a 0.25 second exposure (middle), and 0.1 second exposure (bottom). A shorter exposure implies increased noise within each raw image. While the 1 second exposure-based reconstruction resolves each parasite, blurred cell boundaries and non-uniform fluctuations in amplitude suggest an inaccurate AP convergence.
Discussion and conclusion

Through the relaxation in Eq. 8, we first transformed the traditionally nonlinear phase retrieval process for ptychography into a convex problem. We then suggest a practically efficient algorithm to solve the resulting semidefinite program with an appropriate factorization. The result is a new ptychographic imaging reconstruction algorithm for phase retrieval. We demonstrated successful performance in three unique experiments, concluding with a practical biological imaging scenario: the identification of malaria infection without an oil immersion medium and under short-exposure imaging conditions.

Much future work remains to fully explore the specific benefits of our problem reformulation. Perhaps the most significant departure from prior phase retrieval solvers is the ability to tune the solution rank, $r$. As noted earlier, $r$ connects to the statistical features of the optical experiment. We expect an appropriately selected $r$ to enable our solvers to eventually determine the partial coherence of our optical sources, possible vibrations within the setup, sample auto-fluorescence, or even three-dimensional structure. As in prior work with low-rank matrix optimization, we may also artificially enlarge our desired solution rank to encourage the transfer of experimental noise into its smaller singular vectors.

Other extensions of LRP include simultaneously solving for unknown aberrations, systematic setup errors, and inserting additional sample priors such as sparsity. Our connection to the well-understood area of convex optimization will hopefully make the task of analyzing phase retrieval algorithm performance more accessible to the average experimentalist, who may tailor our problem formulation to their own unique application. What’s more, we are hopeful that tools from convex analysis will continue to provide useful theoretical guarantees regarding algorithm performance, a crucial feature missing from current nonlinear solvers.

Materials and Methods

The Fourier ptychographic microscope. The FP microscope consists of a 16×16 array of surface-mounted LEDs (model SMD 3528, center wavelength $\lambda = 632$ nm, 4 mm LED pitch, 150 $\mu$m active area diameter) placed $l = 40-80$ mm beneath the sample plane. Images are recorded with a CCD detector (Kodak KAI-2905) containing 5.5 $\mu$m pixels. The images in Fig. 6 were captured with a 2X Olympus microscope objective (apochromatic Plan APO 0.80 NA), and we set $l = 42$ mm, $n_t = 160$, $n_s = 320$, and used three rings of 8, 12 and 16 evenly spaced LEDs (ring radii =16, 32 and 40 mm, respectively) and one central (on-axis) LED to capture the 29 images. The sample for Fig. 8 was prepared by maintaining erythrocyte asexual stage cultures of the P. falciparum strain 3D7 in culture medium, then smearing, fixing with methanol and applying a Hema 3 stain (following the protocol in [41]).

Algorithm parameters. For the reconstructions in Fig. 6 we first segmented each raw image into $3 \times 3$ tiles (each with $n_t = 48$ pixels) and overlapping with neighboring tiles by 4 pixels). AP and LRP then reconstruct each tile separately before recombining the solutions into the approximate 1 megapixel images in Fig. 6. This tiling procedure matches that in [39]. We determined the optimal number of AP and LRP algorithm iterations as 6 and 15, respectively, and fixed this for each tile here (and all reconstructions in general). LRP's free/initial parameters are typically set to $\gamma = 1$, $\eta = 0.3$, $\sigma = 0$, and $\sigma = 10$. We determined each aperture function $\alpha$ a-priori using an iterative procedure [39] and used the same $\alpha$ for each algorithm and each tile. No tiling was used for the reconstructions in Fig. 6 and Fig. 8. For these two figures, we set $\alpha$ to an unmodified circ-function with radius defined by the objective lens NA.

Fig. 6 transforms each $n_t = 80$ pixel sub-image into its $n_t = 160$ pixel solutions, while Fig. 7 transforms each $n_t = 120^2$ pixel sub-image into its $n_t = 240^2$ pixel solutions. As noted in the main text, a fixed phase background was subtracted from all phase images for fair comparison.
Computational specifics. We performed all processing on a high-end desktop containing two Intel Xeon 2.0 GHz CPUs and two 3GB GeForce GTX GPUs. Code was written in Matlab with built-in GPU support. Our solved CLP semidefinite program using the TFOCS code package. Our LRP algorithm borrows concepts from the LBFGS solver in [37] for one specific minimization step. LRP’s total recovery time for the 1 megapixel example in Fig 6 was approximately 130 seconds, while AP completion in approximately 15 seconds on the same desktop.

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