LUSIN SPACES AS IMAGES OF LOCALLY COMPACT POLISH SPACES

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Abstract. A Lusin space is a Hausdorff space being the image of a Polish space under a continuous bijection. Such spaces have multiple applications, in particular, as state spaces of various stochastic systems. In this work, we consider the spaces obtained as the images of a noncompact and locally compact Polish space \((X, T)\), which we call \(c\)-Lusin. The main result is the statement that a \(c\)-Lusin space \(Y = f(X)\), can be written as \(Z \cup Y_1\), where \(Z\) is a locally compact Polish space whereas \(Y_1\) is \(c\)-Lusin. At the same time, \(Y_1\) is the set of the discontinuity points of \(f^{-1}\) which is a closed subset of \(Y\). Moreover, \(Y_1\) is nowhere dense if (and only if) \(Y\) is a Baire space. By the same arguments, \(Y_1\) can also be decomposed as \(Z_1 \cup Y_2\) with the properties as above. In the case where \(f\) can be extended to a continuous map \(f : X \cup \{\infty\} \to Y\), and thus \(Y_1\) is a singleton, we construct a metric on \(X\) such that the corresponding metric space is compact and homeomorphic to the \(c\)-Lusin space \((f(X), T')\).

1. Introduction

A Polish space is a separable space the topology of which is consistent with a complete metric. Polish spaces, as well as their images, cf. [5] pages 239–277 and [10] [12], have multiple applications. A Lusin space is a Hausdorff space that is the image of a Polish space under a continuous bijection, see [1] or [5, page 273].

To have the freedom of dealing with different topologies defined on the same underlying set \(X\), we use the notation \((X, T)\) for the corresponding topological space. Let \((X, T)\) and \((Y, T')\) be a Polish space and \(Y = f(X)\) for a continuous bijection \(f\). An important feature of this pair is that the spaces are Borel isomorphic, see [5, Proposition 8.6.13, page 275]. In view of this fact, along with Polish spaces Lusin spaces are frequently used as state spaces of a broad range of stochastic systems, see, e.g., [2] [3] [6] [7] [14].

Lusin spaces have a plenty of topological properties, see e.g., [1] and the literature quoted therein. Some of them may be quite different from those of Polish spaces. In particular, a Lusin space need not be metrizable. An example can be an infinite dimensional linear space, \(X\), equipped with the weak topology relative to the topology that makes \(X\) a separable Banach space. The situation with the mentioned properties can get more controllable if one assumes additional properties of the Polish space \((X, T)\). If it is compact, then \((f(X), T')\) is also compact and \(f\) is a homeomorphism. In this work, we study the spaces obtained as the images of a noncompact and locally compact Polish space \((X, T)\). For convenience, we call them \(c\)-Lusin spaces. Our main result is the following statement, see Theorem 2.4 below. Let \(Y_1\) be the set of all discontinuity points of \(f^{-1}\). Then \(Y_1\) is a closed subset of \(Y = f(X)\), which is a \(c\)-Lusin space in the subspace topology – since its preimage \(X_1\) is a locally compact Polish space (by the continuity of \(f\)). At the same time, \(Z := Y \setminus Y_1\) is a locally compact Polish space for it is homeomorphic to its preimage \(W\). Moreover, \(Y_1\) is nowhere dense (hence
Proposition 2.2. \cite[Theorem 1.15]{9}

Proof. Let \( X \) be a Baire space. If \( f \) is \textit{feeably} open, which implies that \( Y \) is Baire, then also \( X_1 \) is a nowhere dense subset of \( X \). By repeating the same arguments one obtains the decomposition \( Y_1 = Z_1 \cup Y_2 \), where \( Z_1 \) is a locally compact Polish space and \( Y_2 \) is \( c \)-Lusin. This procedure can be continued ad infinitum, or to the step at which \( Z_k = Y_k \) or \( Z_k = \emptyset \), hence \( Y_{k+1} = Y_k \). In Sect. 3, we study the case where \( f \) can be continuously extended to the Alexandroff compactification \( X \cup \{ \infty \} \), and thus \( Y_1 \) is at most singleton. For \( Y_1 = \{ y_0 \} \), hence \( X_1 = \{ x_0 \} \), \( x_0 = f^{-1}(y_0) \), in Theorem 3.2 we show that \((Y, T')\) is a compact Polish space homeomorphic to \((X, T_{x_0})\), where \( T_{x_0} \) is the metric topology corresponding to an explicitly constructed complete \( x_0 \)-dependent metric.

2. The main result

We begin by making precise the notions used throughout. A set the closure of which has empty interior is called nowhere dense.

Definition 2.1. A topological space is called a Baire space if the union of any countable collection of its closed subsets each with empty interior also has empty interior.

A detailed presentation of the properties of Baire spaces can be found in \cite{9}. Among them there are the following ones: (a) every nonempty open subset of a Baire space is of second category \cite{9} page 11, and hence is a Baire space in the subspace topology; (b) closed subsets need not be Baire; (c) both locally compact Hausdorff and completely metrizable spaces are Baire, see \cite{9} Theorems 2.3 and 2.4. Thus, our Polish space \((X, \mathcal{T})\) is a Baire space. However, its continuous image \( Y = f(X) \) need not be such, which can be seen from the following example. Consider \((\mathbb{Q}, \mathcal{T})\) and \((\mathbb{Q}, \mathcal{T}_{[1]}),\) where \(\mathbb{Q}\) is the set of rational numbers, \(\mathcal{T} = 2^\mathbb{Q}\), i.e., contains all subsets of \(\mathbb{Q}\), and \(\mathcal{T}_{[1]}\) is the norm topology related to absolute value \(|\cdot|\). The metric \(d(x, y) = 1\) for \(x \neq y\), and \(d(x, y) = 0\) for \(x = y\), is consistent with \(\mathcal{T}\); hence, the former is a Polish space, which is obviously locally compact. Since the embedding map \(f : (\mathbb{Q}, \mathcal{T}) \to (\mathbb{Q}, \mathcal{T}_{[1]})\) is continuous, the latter is a \(c\)-Lusin space, which is apparently not Baire. Noteworthy, these spaces are Borel isomorphic as their Borel \(\sigma\)-fields coincide with \(2^\mathbb{Q}\) in this case.

The following is known.

Proposition 2.2. \cite[Theorem 1.15]{9} Every space which contains a dense Baire subspace is a Baire space.

A map \(f : (X, \mathcal{T}) \to (Y, \mathcal{T}')\) is called feeably open if for each nonempty \(A \in \mathcal{T}\), there exists a nonempty \(B \subset \mathcal{T}'\) such that \(B \subset f(A)\).

Proposition 2.3. \cite[Theorem 4.1]{9} Let \((X, \mathcal{T})\) be a Baire space and \(f : (X, \mathcal{T}) \to (Y, \mathcal{T}')\) a continuous and feeably open surjection. Then \((Y, \mathcal{T}')\) is also a Baire space.

Our main result is the following statement.

Theorem 2.4. Let \((Y, \mathcal{T}')\) be a \(c\)-Lusin space obtained by means of a continuous bijection \(f : (X, \mathcal{T}) \to (Y, \mathcal{T}')\). Then there exists an open subset \(Z \subset Y\) such that:

(a) \((Z, \mathcal{T}_Z')\) is a locally compact Polish space;
(b) \((Y_1, \mathcal{T}_1')\) is a \(c\)-Lusin space. Here \(\mathcal{T}_Z' = \{ A \cap Z : A \in \mathcal{T}' \}\), \(Y_1 = Y \setminus Z\) and \(\mathcal{T}_1' = \{ A \cap Y_1 : A \in \mathcal{T}' \}\). \(Y_1\) is a nowhere dense subset of \(Y\) if and only if \((Y, \mathcal{T}')\) is a Baire space. If \(f\) is feeably open, hence \(Y\) is Baire, then \(X_1 := f^{-1}(Y_1)\) is nowhere dense in \(X\).

Proof. Let \((X, \mathcal{T})\) and \(Y = f(X)\) be as assumed. For \(y \in Y\), by \(\mathcal{T}'(y)\) we denote the set of all \(C \in \mathcal{T}'\) such that \(y \in C\). For \(x \in X\), \(\mathcal{T}(x)\) is defined analogously. Now we define

\[
Z = \{ y \in Y : \exists C \in \mathcal{T}'(y) \ f^{-1}(C) \text{ has compact closure} \}.
\] (2.1)
Clearly, \( Z \in \mathcal{T}' \) and \( f^{-1} \) is discontinuous at each \( y \in Y_1 := Y \setminus Z \) since each \( x = f^{-1}(y) \), also for \( y \in Y_1 \), has a compact neighborhood. Let us show now that \( f^{-1} \) is continuous at each \( z \in Z \). Consider a net \( \{y_n\} \subset Y \) convergent to a given \( z \in Z \), and set \( x_n = f^{-1}(y_n) \). Let also \( C \in \mathcal{T}'(z) \) be as in (2.1), and then \( K \subset X \) be the compact closure of \( f^{-1}(C) \). Then the preimage of the part \( \{y_n\} \cap C \) lies in \( K \), and hence possesses accumulation points. By the continuity of \( f \) it follows that all of them should coincide with the preimage of \( z \), which yields the continuity in question.

Denote \( X_1 = f^{-1}(Y_1) \) and \( W = f^{-1}(Z) \), and also \( \mathcal{T}_W = \{A \cap W : A \in \mathcal{T}\} \), \( \mathcal{T}_1 = \{A \cap X_1 : A \in \mathcal{T}\} \). Since \( W \) is open and \( X_1 \) is closed, both \((W, \mathcal{T}_W)\) and \((X_1, \mathcal{T}_1)\) are Polish spaces, see [5, Proposition 8.1.2, page 240]. These spaces are also locally compact, see [8, Theorem 3.3.8]. Hence, \((Y_1, \mathcal{T}_1)\) is a c-Lusin space, which yields the validity of claim (b). At the same time, by the continuity of \( f^{-1}|_Z \) the restriction \( f_W := f|_W \) is a homeomorphism between \((W, \mathcal{T}_W)\) and \((Z, \mathcal{T}_2)\), which means that the latter is a locally compact Polish space, which proves claim (a).

Let \( Y \) be a Baire space. Every locally compact and second countable space is also \( \sigma \)-compact. That is, there exists a sequence of compact subsets, \( \{K_n\}_{n \in \mathbb{N}} \), such that: (a) each \( K_n \) is contained in the interior of \( K_{n+1} \); (b) \( X = \bigcup_n K_n \). For such a sequence, we set \( Q_n = f(K_n) \) and let \( C_n \) be the interior of \( Q_n \). Then \( Y = \bigcup_n Q_n \). Since \( Y \) is a Baire space, \( C_n \neq \emptyset \) for at least some of \( n \in \mathbb{N} \). On the other hand, \( C_n \subset Z \) for each \( n \), see (2.1). Set \( Q_{n,1} = Q_n \cap Y_1 \), which means that \( Y_1 = \bigcup_n Q_{n,1} \). Let \( C_{n,1} \) be the interior of \( Q_{n,1} \). Then \( C_{n,1} \subset C_n \subset Z \); hence, \( C_{n,1} = \emptyset \) for all \( n \); thus, \( Y_1 \) is nowhere dense as the closed set of first category, see Definition 2.1. On the other hand, if \( Z \) is dense in \( Y \), then \( Y \) is a Baire space – by Proposition 2.2 and the openness of \( Z \) shown above.

Finally, let \( f \) be feebly open. If the interior of \( X_1 \) is nonempty, then the interior of \( Y_1 \) should also be nonempty which is impossible in this case by Proposition 2.3. Thus, \( X_1 \) is a nowhere dense subset of \( X \). This completes the whole proof. 

From the proof made above, it readily follows that \( X_1 \) can be characterized by the following property. Let \( X \cup \{\infty\} \) be the Alexandroff compactification of \((X, \mathcal{T})\). Let also \( \mathfrak{X} \) stand for the set of all sequences \( \{x_n\} \subset X \) that converge to \( \infty \). That is, each \( \{x_n\} \in \mathfrak{X} \) is eventually in \( X \setminus K \) for each compact \( K \). Then \( X_1 = f^{-1}(Y_1) \) can be written in the form

\[
X_1 = \{x \in X : \exists \{x_n\} \in \mathfrak{X} \text{ s.t. } \lim_{n \to +\infty} f(x_n) = f(x)\}. \tag{2.2}
\]

By (2.2) one can see that the structure of \( X_1 \) (and thus of \( Y_1 \)) is predetermined by the properties of \( f \) in the vicinity of \( \infty \). In particular, \( X_1 \) is at most singleton if \( f \) has a continuous extension to \( X \cup \{\infty\} \). In the aforementioned example with \( X = Y = \mathbb{Q} \), \( \{x_n\} \in \mathfrak{X} \) means that, for some \( k \), \( x_n \neq x_m \) for \( n, m \geq k \). Take any \( x \in \mathbb{Q} \) and set \( x_n = x + 1/n \). Then this sequence belongs to \( \mathfrak{X} \). At the same time, \( |x_n - x| = 1/n \), which means that \( x \in X_1 \), and hence \( X_1 = X \).

By repeating the arguments used in the proof of Theorem 2.3 we obtain the following statement that establishes the structure of the c-Lusin space \((Y, \mathcal{T}')\).

**Corollary 2.5.** There exists a descending sequence \( \{Y_k\}_{k \in \mathbb{N}} \) of closed subsets of \( Y \) such that, for each \( k \in \mathbb{N} \), \((Y_k, \mathcal{T}_k')\) and \((Z_k, \mathcal{T}'_{Z_k})\) are a c-Lusin space and a locally compact Polish space, respectively. Here \( \mathcal{T}_k' = \{A \cap Y_k : A \in \mathcal{T}'\} \) and \( \mathcal{T}'_{Z_k} = \{A \cap Z_k : A \in \mathcal{T}'\} \).

Note that the aforementioned sequence may end up with \( Y_{k+1} = \emptyset \) or \( Y_{k+1} = Y_k \) for some \( k \in \mathbb{N}_0 \). Here by \( Y_0 \) and \( Z_0 \) we mean \( Y \) and \( Z \), respectively. For \( A \in \mathcal{T}' \), set
\[ A_{k+1} = A \cap Z_k, \ k \in \mathbb{N}_0. \] Then \( A_k \cap A_{k'} = \emptyset \) for all distinct \( k \) and \( k' \). Therefore,

\[ A = \bigcup_{k \in \mathbb{N}} A_k \]

is a disjoint decomposition of \( A \) in which \( A_k \in T'_{Z_k-1}, \ k \in \mathbb{N} \).

Now we introduce a complete metric \( \delta_k \), consistent with \( T'_{Z_k}, \ k \geq 0 \). In this context, we also set \( W_k = f^{-1}(Z_k) \) and \( X_k = f^{-1}(Y_k), \ k \geq 0 \). Let \( d \) be a complete metric consistent with \( T \). For \( x \in X_k \) and \( y \in Z_k, \ k \in \mathbb{N}_0 \), we define

\[ d(x, X_{k+1}) = \inf_{v \in X_{k+1}} d(x, v), \quad \varphi_k(y) = 1/d(f^{-1}(y), X_{k+1}). \quad (2.3) \]

That is, \( d(f^{-1}(y), X_{k+1}) \) is the distance from the pre-image of \( y \) to \( X_{k+1} = X_k \setminus W_k \).

Now similarly as in [5] page 240 we set

\[ \delta_k(y, y') = d(f^{-1}(y), f^{-1}(y')) + |\varphi_k(y) - \varphi_k(y')|, \quad y, y' \in Z_k, \ k \geq 0. \]

By this construction, it follows that

\[ \delta_k(y, y') = d_k(x, x') := d(x, x') + \left| \frac{1}{d(x, X_{k+1})} - \frac{1}{d(x', X_{k+1})} \right|, \]

where \( x = f^{-1}(y), \ x' = f^{-1}(y') \). One can show, see the proof of Proposition 8.1.2 in [5], that \( d_k \) is a complete metric consistent with \( T'_W \). Then \( \delta_k \) is the metric in question.

### 3. A SPECIAL CASE

Here we consider the case where \( f \) has a continuous extension to the Alexandroff compactification of \( X \). As mentioned above, this property corresponds to \( X_1 \) consisting of at most one element. If \( X_1 = \emptyset \), then \( f \) is a homeomorphism. Recall that \((X, T)\) is noncompact.

**Proposition 3.1.** \((Y, T')\) is compact if \( X_1 \) is a singleton.

**Proof.** Set \( X_1 = \{x_0\} \). Let \( \{y_k\} \subset Y \) be a net, for which we have the corresponding net of \( x_k = f^{-1}(y_k) \). Then either \( \{x_k\} \subset K \) for some compact \( K \subset X \), or it contains a sub-net, \( \{x_k\} \), convergent to \( \infty \). In the former case, the net \( \{y_k\} \) is contained in the compact \( f(K) \), and hence has accumulation points. Otherwise, the sub-net \( \{y_k\} \) converges to \( f(x_0) \), which yields the compactness of \((Y, T')\). \( \square \)

A priori, even being compact \((Y, T')\) need not be metrizable. In the next statement, we nevertheless show that it is.

**Theorem 3.2.** Assume that \( X_1 = \{x_0\} \), and hence \( f \) has a continuous extension to the \( X \cup \{\infty\} \). Then there exists a metric, \( \delta \), on \( X \) such that \((X, \delta)\) is a compact metric space homeomorphic to \((Y, T')\).

The proof will be done by an explicit construction of \( \delta \). Its main idea stems from the proof of Proposition 3.1 by which \((Y, T')\) is a one-point compactification of \((Z, T'_Z)\). To figure it out, we take two (disjoint) sequences \( \{x_k\}, \{x'_k\} \subset X \cup \{\infty\} \) such that \( x_k \to \infty \) and \( x'_k \to x_0 \). Then the closures in \( Y \) of their \( f \)-images contain \( f(x_0) \). This yields that the map \( f^{-1} \) from \( Z \) to \( X \cup \{\infty\} \) is not uniformly \( T'/T \)-continuous. Hence, by Taimanov’s theorem, see [13], it does not have a continuous extension to \( Y \) in this case. At the same time, it may get such an extension if one identifies \( \infty \) and \( x_0 \), and thus their neighborhoods. Then \( \delta \) is obtained by applying the corresponding construction of [11], modified to take into account the mentioned identification. We thus begin by making this step.
For a nonempty $D \subset X$ and $r > 0$, we set $D' = \{x \in X : \exists v \in D \text{ } d(x, v) < r\}$, where $d$ is as in (2.3). Let $\{K_n\} \subset X$ be an ascending sequence of compact subsets that exhausts $X$ and is such that $K_n$ is contained in the interior of $K_{n+1}$, $n \in \mathbb{N}$. Then one finds $\{r_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ such that: (a) $K^{r_n} \subset K_{n+1}$; (b) $r_n > r_{n+1}$; (c) $r_n \to 0$ as $n \to +\infty$. Of course, we can also assume that $x_0 \in K_1$. Let us now define the following functions
\[
g(x) = \max_{n \in \mathbb{N}} [r_n - d(x, K_n)], \quad h(x) = \min\{d(x, x_0); g(x)\}, \quad x \in X. \tag{3.1}
\]
Note that $h(x_0) = 0$ and $g(x_0) = r_1 > 0$. Moreover, $g(x) > 0$ for all $x$. Let us prove that
\[
|h(x) - h(y)| \leq d(x, y), \quad x, y \in X, \tag{3.2}
\]
i.e., $x \mapsto h(x)$ is Lipschitz-continuous. Since the right-hand side of (3.2) is symmetric with respect to the interchange $x \leftrightarrow y$, it is enough to show that
\[
h(x) - h(y) \leq d(x, y), \quad x, y \in X. \tag{3.3}
\]
For $h(y) = d(y, x_0)$, (3.3) holds true by the triangle inequality for $d$. Indeed, by (3.1) we have
\[
h(x) - d(y, x_0) \leq d(x, x_0) - d(y, x_0) \leq d(x, y).
\]
Assume now that $h(y) = g(y)$. By (3.1) we then have $h(x) - h(y) \leq g(x) - g(y)$, which means that (3.3) will follow by
\[
g(x) - g(y) \leq d(x, y).
\]
First we consider the case of $g(x) = r_n$ for some $n \in \mathbb{N}$, which corresponds to $x \in K_n$. For this $n$, by (3.1) we have $g(y) \geq r_n - d(y, K_n)$, which implies
\[
g(x) - g(y) \leq d(y, K_n) \leq d(x, y).
\]
For $g(x) = r_n - d(x, K_n)$, similarly we have
\[
g(x) - g(y) \leq d(y, K_n) - d(x, K_n) = d(y, K_n) - d(x, z)
\]
\[
\leq d(y, z) - d(x, z) \leq d(x, y),
\]
where $z \in K_n$ is such that $d(x, K_n) = d(x, z)$. This completes the proof of (3.2).

Let us now introduce a candidate for another metric on $X$. Set
\[
\delta(x, y) = \min\{d(x, y); h(x) + h(y)\}. \tag{3.4}
\]

**Lemma 3.3.** It follows that $\delta$ defined in (3.4) is a metric, such that the metric space $(X, \rho)$ is compact.

**Proof.** Obviously $\delta$ is symmetric and $\delta(x, y) = 0$ implies $x = y$. Then to prove that $\delta$ is a metric it remains to show that
\[
\delta(x, y) \leq \delta(x, z) + \delta(y, z), \quad x, y, z \in X. \tag{3.5}
\]
By (3.4), it follows that each $\delta$ in (3.5) can equal either the corresponding $d$ or the sum of two corresponding $h$. If all the three $\delta$’s equal the corresponding $d$’s, then (3.5) follows by the triangle inequality for $d$. Let us consider the case where one of the $\delta$’s in (3.5) equals the sum of the $h$’s, whereas the remaining two equal the corresponding $d$’s. If this holds on the left-hand side – which by (3.4) corresponds to $h(x) + h(y) \leq d(x, y)$ – then (3.5) turns into
\[
h(x) + h(y) \leq d(x, z) + d(y, z).
\]
As just mentioned, $h(x) + h(y) \leq d(x, y)$, which yields the validity of (3.5) by the triangle inequality for $d$. Now let the mentioned equality holds on the right-hand side.
of \( (5.5) \). By the symmetry \( x \leftrightarrow y \), it is enough to consider only the case \( \delta(y, z) = h(y) + h(z) \). Then \((5.5)\) turns into
\[
d(x, y) \leq d(x, z) + h(y) + h(z).
\] (3.6)
Since \( \delta(x, y) = d(x, y) \), by \((5.5)\) and \((5.2)\) it follows that
\[
d(x, y) \leq h(x) + h(y) = h(x) - h(z) + h(y) + h(z) \leq d(x, z) + h(y) + h(z),
\]
which yields \((3.6)\). Now let two \( \delta \)'s be equal to the corresponding sums of the \( h \)'s. This case splits into the following ones
\[
d(x, y) \leq h(x) + h(y) + 2h(z),
\] (3.7)
\[
h(x) \leq d(x, z) + h(z).
\]
The validity of the first line in \((3.7)\) follows by the fact that \( d(x, y) \leq h(x) + h(y) \), see \((3.5)\). The validity of the second one follows by \((3.2)\). This completes the proof of \((3.5)\).

Let us turn now to proving the compactness, which is equivalent to the completeness of \((X, \delta)\) and the total boundedness of \( \delta \). Let \( \{x_l\}_{l \in \mathbb{N}} \) and \( \{K_n\}_{n \in \mathbb{N}} \) be a \( \delta \)-Cauchy sequence and the ascending sequence as in \((3.1)\), respectively. Then either \( \{x_l\}_{l \in \mathbb{N}} \subset K_n \) for some \( n \), or there exists a subsequence \( \{x_{l_m}\}_{m \in \mathbb{N}} \subset \{x_l\}_{l \in \mathbb{N}} \) such that, for each \( n \), there exists \( m_n \) such that \( x_{l_m} \in X \setminus K_n \) for all \( m > m_n \). In the former case, there exists a subsequence \( \{x_{l_p}\}_{p \in \mathbb{N}} \subset \{x_l\}_{l \in \mathbb{N}} \) \( \delta \)-convergent to a certain \( x \in K_n \). By \((3.1)\) \( \{x_{l_p}\}_{p \in \mathbb{N}} \) is also \( \delta \)-convergent to \( x \); hence, the whole \( \{x_l\}_{l \in \mathbb{N}} \) is \( \delta \)-convergent. In the latter case, \( g(x_{l_m}) \to 0 \) as \( m \to +\infty \), see \((3.1)\). The latter implies \( h(x_{l_m}) \to 0 \), which yields
\[
0 \leq \delta(x_{l_m}, x_0) \leq h(x_{l_m}) + h(x_0) = h(x_{l_m}) \to 0.
\]
Hence, \( \{x_{l_m}\}_{m \in \mathbb{N}} \), and thus also \( \{x_l\}_{l \in \mathbb{N}} \) converge in \( \delta \) to \( x_0 \). This yields the completeness of \( \delta \). To prove the total boundedness, we have to show that, for each \( \varepsilon > 0 \), there exists a finite \( D_\varepsilon \subset X \) such that \( B_\varepsilon^\delta(x) \cap D_\varepsilon \neq \emptyset \) holding for each \( x \in X \). Here \( B_\varepsilon^\delta(x) = \{ y \in X : \delta(x, y) < \varepsilon \} \), which by \((3.1)\) contains the ball \( B_\varepsilon(x) \). Let \( \{r_n\}_{n \in \mathbb{N}} \) be the sequence that appears in \((3.1)\). For a given \( \varepsilon > 0 \), find \( n_\varepsilon \in \mathbb{N} \) such that \( r_{n_\varepsilon} < \varepsilon \) whenever \( n \geq n_\varepsilon \). By \((3.1)\) we then have that \( h(x) < \varepsilon \) for all \( x \in X \setminus K_{n_\varepsilon} \). Since \( K_{n_\varepsilon} \) is compact in \((X, T)\), one finds a finite \( C_\varepsilon \subset K_{n_\varepsilon} \) such that \( B_\varepsilon(x) \cap C_\varepsilon \neq \emptyset \), holding for each \( x \in K_{n_\varepsilon} \). Then the set in question is \( D_\varepsilon = C_\varepsilon \cup \{x_0\} \). This completes the proof.

\textbf{Proof of Theorem 3.3} In view of Lemma 3.3 it remains to show that the spaces \((X, T_{\delta y})\) and \((Y, T')\) are homeomorphic. Here by \( T_{\delta y} \subset T \) we mean the metric topology associated with \( \delta \). It is clear now that for any two disjoint \( \delta \)-convergent sequences \( \{x_k\}, \{x'_k\} \subset X \), the closures of \( \{f(x_k)\} \) and \( \{f(x'_k)\} \) in \((Y, T')\) are disjoint, which means that the map \( f^{-1} : Z \to X \) can now be continuously extended to the whole \( Y \). As such, it turns into a homeomorphism, cf. \cite[Theorem 7.7, page 19]{4}, which yields the proof.

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