Article

New Criteria for Sharp Oscillation of Second-Order Neutral Delay Differential Equations

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Abstract: In this paper, new oscillation criteria for second-order half-linear neutral delay differential equations are established, using a recently developed method of iteratively improved monotonicity properties of a nonoscillatory solution. Our approach allows removing several disadvantages which were commonly associated with the method based on a priori bound for the nonoscillatory solution, and deriving new results which are optimal in a nonneutral case. It is shown that the newly obtained results significantly improve a large number of existing ones.

Keywords: half-linear neutral differential equation; delay; second-order; oscillation

1. Introduction

The aim of this work is to study the asymptotic and oscillatory properties of solutions of the second-order half-linear neutral delay differential equation

\[
\left( r(t)\left( z'(t)\right)^{\alpha} \right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0 > 0,
\]

where

\[
z(t) = x(t) + p(t)x(\tau(t)).
\]

The following assumptions will be made without further mention:

Hypothesis 1 (H1). \( \alpha > 0 \) is a quotient of odd positive integers.

Hypothesis 2 (H2). \( r(t) \in C([t_0, \infty), (0, \infty)) \) satisfies

\[
R(t, t_0) := \int_{t_0}^{t} r^{-1/\alpha}(s)ds \to \infty \quad \text{as } t \to \infty;
\]

Hypothesis 3 (H3). \( \sigma(t), \tau(t) \in C([t_0, \infty), \mathbb{R}) \), \( \sigma(t) \leq t \), either \( \tau(t) \leq t \) or \( \tau(t) \geq t \), and \( \lim_{t \to \infty} \sigma(t) = \lim_{t \to \infty} \tau(t) = \infty \).

Hypothesis 4 (H4). \( q(t) \in C([t_0, \infty), (0, \infty)) \).

Hypothesis 5 (H5). \( p(t) \in C([t_0, \infty), [0, 1]) \) and there exists a constant \( p_0 \in [0, 1) \) such that

\[
p_0 \geq p(t) \quad \text{for } \tau(t) \leq t, \quad p_0 \geq p(t) \frac{R(\tau(t), t_0)}{R(t, t_0)} \quad \text{for } \tau(t) \geq t.
\]

Under a solution of (1), we mean a function \( x \in C([t_k, \infty), \mathbb{R}) \) with

\[
t_k = \min \{ \tau(t), \sigma(t) \},
\]
for some \( t_{1} \geq t_{0} \), which has the property \( z(t) \in C^{1}([t_{k}, \infty), R) \), \( r(t)(z'(t))^\alpha \in C^{1}([t_{k}, \infty), R) \) and satisfies (1) on \( [t_{j}, \infty) \). Only those solutions of (1) which exist on some half-line \( [t_{j}, \infty) \) and satisfy the condition

\[
\sup \{|x(t)| : t_{0} \leq t < \infty \} > 0 \quad \text{for any} \quad t_{0} \geq t_{j}
\]

will be considered. As usual, a nontrivial solution \( x(t) \) of (1) is termed oscillatory or nonoscillatory according to whether it does or does not have infinitely many zeros. Equation (1) is called oscillatory if all its solutions are oscillatory.

In a neutral delay differential equation, the highest order derivative of the unknown function appears both with and without delay. Such equations arise in a variety of phenomena including mixing liquids, vibrating masses attached to an elastic bar, automatic control problems, and population dynamics, see [1]. In particular, second-order neutral delay differential equations find application in explaining human self-balancing [2]. With regard to their practical importance, oscillation of second-order neutral differential equations has been studied extensively during recent decades, see [3–39] for the recent contributions on the subject.

The purpose of this paper is to continue the previous author’s work [17], where Equation (1) was investigated under the additional restriction

\[
\tau(t) \leq t, \quad \sigma(t) < t, \quad \text{and} \quad \sigma'(t) > 0.
\]

(4)

Among others, we showed in [17] how the inequality

\[
\frac{r^{1/\alpha}(\sigma(t))z'(\sigma(t))}{r^{1/\alpha}(t)z'(t)} \geq 1,
\]

(5)

playing an important role in several related contributions [4,9,10,18,23,36,39–42], can be strengthened. For the reader’s convenience, we recall the related result.

**Lemma 1** (See ([17], Lemma 4)). Let

\[
Q(t) := (1 - p(\sigma(t)))^\alpha q(t),
\]

\[
\bar{R}(t_{1}, t_{0}) := R(t_{1}, t_{0}) + \frac{1}{\alpha} \int_{t_{0}}^{t_{1}} R(s, t_{0}) R(s) Q(s) ds,
\]

\( \tau(t) \leq t, \sigma(t) < t \) be strictly increasing, and assume that for any \( t_{1} \geq t_{0} \) and \( t \) sufficiently large, there exists \( \rho > 0 \) such that

\[
\int_{\sigma(t)}^{t} Q(s) \bar{R}(\sigma(s), t_{1}) ds \geq \rho, \quad t \geq t_{1}.
\]

If \( x(t) \) is an eventually positive solution of (1), then the corresponding function \( z(t) \) eventually satisfies

\[
\frac{r^{1/\alpha}(\sigma(t))z'(\sigma(t))}{r^{1/\alpha}(t)z'(t)} \geq f_{n}(\rho),
\]

(6)

where \( f_{n}(\rho) \) is defined by

\[
f_{0}(\rho) = 1, \quad f_{n+1}(\rho) = e^{f_{n}(\rho)}, \quad n \in \mathbb{N}_{0}.
\]

It is worth noting that if \( \rho \in (0, 1/e) \), the sequence \( f_{n}(\rho) \) is increasing and bounded from above, and so there is a limit

\[
\lim_{t \to \infty} f_{n}(\rho) = f(\rho) \in [1, e],
\]
where \( f(\rho) \) is a real root of the equation

\[
f(\rho) = e^{\rho f(\rho)}.
\]

The estimate (6) was subsequently used in the Riccati Technique to obtain sharper oscillation criteria for (1), see also the recent paper [30] for similar application of (6) in extending the modified Riccati Technique from half-linear equations to (1). However, an obvious disadvantage of Lemma 1 is that it needs \( \sigma'(t) > 0 \) and \( \sigma\sigma'(t) > 0 \), which are not required in this work. As one of the main results of this paper, we provide a new variant of (6) (see Lemma 6), which is unimprovable in certain sense.

To take a broader look at the subject, we refer the reader to [15,16] for a nice survey of existing methods for investigating neutral equations of the form (1). As in our previous work [17], we will use the method of a lower bound of the ratio \( x(t)/z(t) \) (see Lemma 3 (iii)). The most important advantage of this method is that it does not require any assumptions on the mutual relationships between \( \tau(t) \) and \( \sigma(t) \), such as \( \tau(\sigma(t)) = \sigma(\tau(t)) \) or \( \tau'(t) > 0 \) and \( (\sigma^{-1}(t))' > 0 \), required by the other methods based on the initial shift of (1) from \( \sigma(t) \) to \( \sigma^{-1}(t) \), which were used e.g., in the works [13–16]. On the other hand, we recall the two main disadvantages associated with the lower bound estimation method:

1. the method gives usually sharp results only if \( p(t) \to 0 \);
2. the method is not capable of detecting the potential dependence of the oscillation criteria on \( \tau(t) \).

Our technique allows removing both the above mentioned disadvantages and to derive new results involving an unimprovable oscillation constant in a nonneutral case, by extending the method of iteratively improved monotonicity properties presented for second-order half-linear delay differential equations of the form (1) with \( p(t) = 0 \) in the recent author’s works [43,44]. First, such results for neutral equations of the form (1) were given in [8], under the assumption that the integral (2) is convergent in a neighbourhood of infinity, see Remark 2 for more details.

The paper is organized as follows. In Section 2, we introduce the basic notations and the core of the method developed in the sequel. In Section 3, we present the main results—oscillation criteria for (1)—as a result of a series of lemmas, iteratively improving monotonicity properties of nonoscillatory solutions. As usual, the improvement made over the existing results from the literature is illustrated via Euler type differential equations. Finally, further remarks and future research directions are proposed in Section 4.

### 2. Preliminaries and the Method Description

Similarly as in the earlier author’s works [43,44], all the results presented in the paper rely on the existence of a positive limit inferior \( \beta_* \) defined by

\[
\beta_* := \frac{1}{\alpha} \liminf_{t \to \infty} t^{1/\alpha} \frac{\sum_{j=1}^{n} R(t, \sigma_j(t)) R(\sigma_j(t), t_0)}{R(t, t_0)} R(\sigma(t), t_0).
\]

In addition, we put

\[
\lambda_* := \liminf_{t \to \infty} \frac{R(t, t_0)}{R(\sigma(t), t_0)},
\]

\[
\delta_* := \liminf_{t \to \infty} \frac{R(t, t_0)}{R(\tau(t), t_0)} \quad \text{for} \quad \tau(t) \leq t,
\]

\[
\omega_* := \liminf_{t \to \infty} \frac{R(\tau(t), t_0)}{R(t, t_0)} \quad \text{for} \quad \tau(t) \geq t.
\]
It is useful to note that in view of (H2) and (H3), $\lambda_s \geq 1$, $\omega_s \geq 1$, and $\delta_s \geq 1$. In the proofs, we will often use the fact that there exists a $t_1 \geq t_0$ sufficiently large such that, for arbitrary but fixed $\beta \in (0, \beta_s)$, $\lambda \in [1, \lambda_s)$, $\omega \in [1, \omega_s)$, and $\delta \in [1, \delta_s)$, we have

\[
r^{1/\alpha}(t)q(t)R(t, t_0)R^\alpha(\sigma(t), t_0) \geq \beta,
\]

\[
\frac{R(t, t_0)}{R(\sigma(t), t_0)} \geq \lambda,
\]

\[
\frac{R(t, t_0)}{R(\tau(t), t_0)} \geq \delta \quad \text{for } \tau(t) \leq t,
\]

\[
\frac{R(\tau(t), t_0)}{R(t, t_0)} \geq \omega \quad \text{for } \tau(t) \geq t,
\]

on $[t_1, \infty)$.

2.1. Definitions of the Sequences $\{\beta_n\}_{n \in \mathbb{N}_0}$ and $\{\gamma_n\}_{n \in \mathbb{N}_0}$

The method used in this paper will often refer to the sequences $\{\beta_n\}_{n \in \mathbb{N}_0}$ and $\{\gamma_n\}_{n \in \mathbb{N}_0}$, which we define (as long as they exist) as follows. For positive and finite $\beta_s$ and $\lambda_s$, we set

\[
\beta_0 := (1 - p_0)^{\alpha} \beta_s,
\]

\[
\gamma_0 := \sqrt[\alpha]{\beta_0} = (1 - p_0)^{\frac{\alpha}{2}} \sqrt[\alpha]{\beta_s},
\]

and for $n \in \mathbb{N}_0$, we put

1. for $(\tau(t) \leq t$ and $\delta_s = \infty$) or $(\tau(t) \geq t$ and $\omega_s = \infty)$:

\[
\begin{align*}
\beta_{n+1} & := \frac{\beta_s \lambda_s^{\beta_n}}{(1 - \beta_n)^{\alpha}}, \\
\gamma_{n+1} & := \frac{\sqrt[\alpha]{\beta_s \lambda_s^{\beta_n}}}{\sqrt[\alpha]{1 - \gamma_n}},
\end{align*}
\]

2. for $(\tau(t) \leq t$ and $\delta_s < \infty)$:

\[
\begin{align*}
\beta_{n+1} & := \frac{\beta_0 \lambda_s^{\beta_n} (1 - p_0 \delta_s^{-\gamma_n})^{\alpha}}{(1 - \beta_n)^{\alpha}} = \frac{\beta_s \lambda_s^{\beta_n} (1 - p_0 \delta_s^{-\gamma_n})^{\alpha}}{(1 - \beta_n)^{\alpha}}, \\
\gamma_{n+1} & := \frac{\gamma_0 \lambda_s^{\beta_n} (1 - p_0 \delta_s^{-\gamma_n})}{\sqrt[\alpha]{1 - \gamma_n}},
\end{align*}
\]

3. for $(\tau(t) \geq t$ and $\omega_s < \infty)$:

\[
\begin{align*}
\beta_{n+1} & := \frac{\beta_0 \lambda_s^{\beta_n} (1 - p_0 \omega_s^{-\beta_n})^{\alpha}}{(1 - \beta_n)^{\alpha}} = \frac{\beta_s \lambda_s^{\beta_n} (1 - p_0 \omega_s^{-\beta_n})^{\alpha}}{(1 - \beta_n)^{\alpha}}, \\
\gamma_{n+1} & := \frac{\gamma_0 \lambda_s^{\beta_n} (1 - p_0 \omega_s^{-\beta_n})}{\sqrt[\alpha]{1 - \gamma_n}},
\end{align*}
\]

By induction, one can easily verify that if for some $n \in \mathbb{N}_0$, $\beta_1 < 1$ and $\gamma_i < 1$, $i = 0, 1, \ldots, n$, then $\beta_{n+1}$ and $\gamma_{n+1}$ exist and

\[
\begin{align*}
\beta_{n+1} &= \ell_n \beta_n > \beta_n, \\
\gamma_{n+1} &= h_n \gamma_n > \gamma_n,
\end{align*}
\]

where $\ell_n$ and $h_n$ are defined as follows:
1. for $(\tau(t) \leq t$ and $\delta_s = \infty$) or $(\tau(t) \geq t$ and $\omega_s = \infty$):

$$\ell_0 := \left[ \frac{\lambda_s^0}{1 - \beta_0} \left( \frac{1 - p_0 \delta - \gamma_0}{1 - p_0} \right) \right]^a,$$

$$\ell_{n+1} := \left[ \lambda_s^{\beta_n(\ell_n-1)} \left( \frac{1 - \beta_n}{1 - \ell_n \delta_a} \right) \left( \frac{1 - p_0 \delta - \gamma_n}{1 - p_0 \delta - \gamma_n} \right) \right]^a,$$

and

$$h_0 := \frac{\lambda_s^0}{1 - \gamma_0} \left( \frac{1 - p_0 \delta - \gamma_0}{1 - p_0} \right),$$

$$h_{n+1} := \lambda_s^{\beta_n(\ell_n-1)} \left( \frac{1 - \gamma_n}{1 - \ell_n \gamma} \right) \left( \frac{1 - p_0 \delta - \gamma_n}{1 - p_0 \delta - \gamma_n} \right).$$

2. for $(\tau(t) \leq t$ and $\delta_s < \infty$):

$$\ell_0 := \left[ \frac{\lambda_s^0}{1 - \beta_0} \left( \frac{1 - p_0 \delta - \gamma_0}{1 - p_0} \right) \right]^a,$$

$$\ell_{n+1} := \left[ \lambda_s^{\beta_n(\ell_n-1)} \left( \frac{1 - \beta_n}{1 - \ell_n \delta_a} \right) \left( \frac{1 - p_0 \delta - \gamma_n}{1 - p_0 \delta - \gamma_n} \right) \right]^a,$$

and

$$h_0 := \frac{\lambda_s^0}{\sqrt{1 - \gamma_0}} \left( \frac{1 - p_0 \delta - \gamma_0}{1 - p_0} \right),$$

$$h_{n+1} := \lambda_s^{\beta_n(\ell_n-1)} \left( \frac{1 - \gamma_n}{1 - \ell_n \gamma} \right) \left( \frac{1 - p_0 \delta - \gamma_n}{1 - p_0 \delta - \gamma_n} \right).$$

3. for $(\tau(t) \geq t$ and $\omega_s < \infty$):

$$\ell_0 := \left[ \frac{\lambda_s^0}{1 - \beta_0} \left( \frac{1 - p_0 \omega - \beta_0}{1 - p_0} \right) \right]^a,$$

$$\ell_{n+1} := \left[ \lambda_s^{\beta_n(\ell_n-1)} \left( \frac{1 - \beta_n}{1 - \ell_n \delta_a} \right) \left( \frac{1 - p_0 \omega - \beta_n}{1 - p_0 \omega - \beta_n} \right) \right]^a,$$

and

$$h_0 := \frac{\lambda_s^0}{\sqrt{1 - \gamma_0}} \left( \frac{1 - p_0 \omega - \beta_0}{1 - p_0} \right),$$

$$h_{n+1} := \lambda_s^{\beta_n(\ell_n-1)} \left( \frac{1 - \gamma_n}{1 - \ell_n \gamma} \right) \left( \frac{1 - p_0 \omega - \beta_n}{1 - p_0 \omega - \beta_n} \right).$$

**Lemma 2.** Let $\beta_s > 0, \lambda_s < \infty$, and the sequence $\{\beta_n\}_{n \in \mathbb{N}_0}$ be well-defined and bounded from above. Then

1. for $(\tau(t) \leq t$ and $\delta_s = \infty$) or $(\tau(t) \geq t$ and $\omega_s = \infty$), the equation

$$\beta_s = \alpha(1 - m)^{\lambda_s - \alpha m}$$

has a solution $m \in (0, 1)$;
2. for \((\tau(t) \leq t \text{ and } \delta < \infty)\), the system

\[
\beta^* = \frac{m(1-m)\lambda^{-am}}{(1-p_0\delta^k)^a}
\]

\[
\beta^* = \frac{k(1-k)\lambda^{-am}}{(1-p_0\omega^k)^a}
\]

has a solution \(\{m \in (0,1), k \in (0,1)\}\);

3. for \((\tau(t) \geq t \text{ and } \omega < \infty)\), the equation

\[
\beta^* = \frac{m(1-m)\lambda^{-am}}{(1-p_0\omega^m)^a}
\]

has a solution \(m \in (0,1)\).

2.2. The Method Description

For the function \(z(t)\) corresponding to the nonoscillatory, say positive solution \(x(t)\) of (1), the purpose of the method of iteratively improved monotonicity properties developed herein is to find optimal values of positive constants \(a\) and \(b\) such that

\[
az(t) > r^{1/a}(t)z'(t)R(t, \cdot)
\]

and

\[
bz(t) < r^{1/a}(t)z'(t)R(t, \cdot),
\]

which correspond to the monotonicities

\[
\left(\frac{z(t)}{R^a(t, \cdot)}\right)' < 0
\]

and

\[
\left(\frac{z(t)}{R^b(t, \cdot)}\right)' > 0,
\]

respectively.

It turns out that the iterative procedure that converges to these optimal values essentially uses the above-defined sequences \(\{\beta_n\}_{n \in \mathbb{N}_0}\) and \(\{\gamma_n\}_{n \in \mathbb{N}_0}\). As a side-product of this finding, it follows that if (1) has a nonoscillatory solution \(x(t)\), then the sequence \(\{\beta_n\}_{n \in \mathbb{N}_0}\) (as well as \(\{\gamma_n\}_{n \in \mathbb{N}_0}\)) is well-defined and bounded from above, see Corollary 1. Hence, the existence of a nonoscillatory solution of (1) implies, in view of Lemma 2, the existence of a solution to one of Equations (9)–(11). By contradiction, if these particular equations have no root on \((0,1)\), we can conclude that (1) is oscillatory. This is stated in the main result of this paper—Theorem 2.

For the sake of completeness, we conclude this section by stating that all functional inequalities occurring in the sequel are assumed to hold eventually, i.e., they are satisfied for all sufficiently large \(t\). Without loss of generality, we only need to be concerned with positive solutions of (1) since the proofs for eventually negative solutions are similar.

3. Main Results

We start with a simple result which can be seen as an extension of ([43], Lemma 1) (given for (1) with \(p(t) = 0\)) or an analogue of ([8], Lemma 1) (given for (1) when \(K(\infty, t_0) < \infty\), see Remark 2 for more details).
Lemma 3. Assume that $\beta_* > 0$. If $x(t)$ is an eventually positive solution of (1), then the corresponding function $z(t)$ eventually satisfies

(i) $z(t) \geq x(t) > 0$ and $(r(z')^{\alpha})'(t) < 0$;  
(ii) $z'(t) > 0$;  
(iii) $x(t) \geq z(t)(1 - p_0)$;  
(iv) $\lim_{t \to \infty} z(t)/R(t, t_0) = 0$;  
(v) $(z(t)/R(t, t_0))' < 0$;  
(vi) the function

$$h(t) := z(t) - r^{1/\alpha}(t)z'(t)R(t, t_0)$$

is positive and for any $\beta \in (0, \beta_*)$

$$h'(t) \geq p\left(\frac{r^{1/\alpha}(t)z'(t)}{R^{1/\alpha}(\sigma(t), t_0)}\right)^{1-\alpha}x^{\alpha}(\sigma(t))$$

and

$$r(t)(z'(t))^{\alpha} \geq \beta^{\alpha}\int_1^\infty \frac{x^{\alpha}(\sigma(s))}{r^{1/\alpha}(s)R(s, t_0)R^{1/\alpha}(\sigma(s), t_0)}ds,$$

eventually.

Proof. Pick $t_1 \geq t_0$ large enough such that $x(t) > 0, x(\sigma(t)) > 0,$ and $x(\tau(t)) > 0$ for $t \geq t_1$. 

(i) and (ii) This is a simple consequence of (H2) and (H3). For the proof, see, e.g., ([4], Lemma 3).

(iii) If $\tau(t) \leq t$, then

$$x(t) \geq z(t) - p(t)x(\tau(t))$$

$$\geq z(t) - p(t)z(\tau(t))$$

$$\geq (1 - p(t))z(t)$$

$$\geq (1 - p_0)z(t).$$

If $\tau(t) \geq t$, then using the monotonicity of $r^{1/\alpha}(t)z'(t)$, we have

$$z(t) = z(t_1) + \int_{t_1}^{t} r^{-1/\alpha}(s)r^{1/\alpha}(s)z'(s)ds$$

$$\geq r^{1/\alpha}(t)z'(t)R(t, t_1)$$

and so

$$\left(\frac{z(t)}{R(t, t_1)}\right)' \leq 0.$$  

(14)

Hence, for any $\varepsilon \in (0, 1)$ and $t$ large enough, we obtain

$$x(t) \geq z(t) - p(t)x(\tau(t))$$

$$\geq z(t) - p(t)z(\tau(t))$$

$$\geq \left(1 - p(t)\frac{R(\tau(t), t_1)}{R(t, t_1)}\right)z(t)$$

$$\geq \varepsilon(1 - p_0)z(t),$$

(15)

where we used that

$$\lim_{t \to \infty} \frac{R(t, t_1)}{R(t, t_0)} = 1.$$
in view of (H2). As a consequence of point (v) below, we will show that we can put \( \varepsilon = 1 \) in (15).

(iv) By l’Hospital’s rule, it suffices to show that

\[
\lim_{t \to \infty} r^{1/\sigma}(t)z'(t) = \ell = 0.
\]

If not and \( \ell > 0 \), then \( z'(t) \geq \ell r^{-1/\alpha}(t) \) and by integrating from \( t_2 \geq t_1 \) to \( t \), we obtain \( z(t) \geq \ell R(t, t_2) \). Now, there exists \( t_3 \geq t_2 \) such that

\[
x(t) \geq \varepsilon(1 - p_0)z(t) \geq \frac{\ell}{2} R(t, t_0), \quad t \geq t_3.
\]

Using (16) in (1), we find

\[
\left( r(t)(z'(t))^\alpha \right)' + \frac{(1 - p_0)^{\alpha \beta \ell^\alpha}}{2\alpha r^{1/\alpha}(t)R(t, t_0)} \leq 0.
\]

By integrating the above inequality from \( t_3 \) to \( t \), we arrive at

\[
r(t_3)(z'(t_3))^\alpha \geq r(t)(z'(t))^\alpha + \frac{\alpha(1 - p_0)^{\alpha \beta \ell^\alpha}}{2\alpha} \ln \frac{R(t, t_0)}{R(t_3, t_0)},
\]

which is a contradiction, since the right-hand side is unbounded. Hence \( \ell = 0 \) and (iv) is proved.

(v) Taking the monotonicity of \( r^{1/\alpha}(t)z'(t) \) and (iv) into account, we obtain

\[
z(t) \geq z(t_1) + r^{1/\alpha}(t)z'(t)R(t, t_1) = z(t_1) - r^{1/\alpha}(t)z'(t)R(t_1, t_0) + r^{1/\alpha}(t)z'(t)R(t, t_0)
\]

for \( t \geq t_4 \), where \( t_4 \geq t_1 \) is large enough so that

\[
z(t_1) - r^{1/\alpha}(t)z'(t)R(t_1, t_0) > 0, \quad t \geq t_4.
\]

Hence,

\[
\left( \frac{z(t)}{R(t, t_0)} \right)' < 0.
\]

Turning back to (iii) case \( \tau(t) \geq t \), we see that

\[
x(t) \geq \left( 1 - p(t) \frac{R(\tau(t), t_0)}{R(t, t_0)} \right)z(t) \geq (1 - p_0)z(t).
\]

(vi) As in [43], differentiating \( h(t) \), using the chain rule and (1), we obtain

\[
h'(t) = -(r^{1/\alpha}(t)z'(t))^\alpha R(t, t_0)
\]

\[
= -\frac{1}{\alpha} (r^{1/\alpha}(t)z'(t))^{1-\alpha} \left( r(t)(z'(t))^\alpha \right)' R(t, t_0)
\]

\[
\geq \frac{1}{\alpha} (r^{1/\alpha}(t)z'(t))^{1-\alpha} R(t, t_0) q(t) x^\alpha(\sigma(t)),
\]

which in view of (7) implies (12). Finally, (13) follows from integrating (1) from \( t \) to \( \infty \) and (7).

The proof is complete. \( \square \)
Next, we present a result that initiates the procedure of iterative improvement of monotonicity properties of the function \( z(t) \) (see below points (vii) and (ix)), which are subsequently used to obtain a more accurate relation between \( x \) and \( z \) than (iii), see below point (xi).

**Lemma 4.** Assume that \( \beta_* > 0 \). If \( x(t) \) is an eventually positive solution of (1), then the corresponding function \( z(t) \) eventually satisfies for any \( \beta \in (0, \beta_*): 

(vii) \( \beta(1 - p_0)^a < 1 \);  

(ix) Using (iii) in (12), we have

\[
\beta(1 - p_0)^a \left( R(t, t_0) \right)^{1/\alpha} z(t) \geq \frac{z(s)}{R^{1/\alpha}(s, t_0)} ds.
\]

Now, taking (v) and (ii) into account in a given order, we find

\[
r(t) \left( z'(t) \right)^a \geq \beta(1 - p_0)^a \left( \frac{z^a(s)}{R^{1/\alpha}(s, t_0)} \right) ds
\]

\[
\geq \beta(1 - p_0)^a z^a(t) \int_{l}^{\infty} \frac{1}{R^{1/\alpha}(s, t_0)} ds
\]

\[
= \beta(1 - p_0)^a \frac{z^a(t)}{R^a(t, t_0)}.
\]

that is,

\[
R(t, t_0) \frac{z^a(t)}{R^{1/\alpha}(t, t_0)} \geq \beta(1 - p_0) z(t)
\]

and so (vii) holds.

(ix) Using (iii) in (12), we have

\[
h'(t) \geq \beta(1 - p_0) \left( \frac{r^{1/\alpha}(t) z'(t) z(t)}{r^{1/\alpha}(t)} \right)^{1-\alpha} \frac{z(a(s))}{R^a(s, t_0)}.
\]

Applying part (v) twice, we obtain

\[
h'(t) \geq \beta(1 - p_0) \left( \frac{r^{1/\alpha}(t) z'(t) z(t)}{r^{1/\alpha}(t)} \right)^{1-\alpha} \frac{z(t)}{R^a(t, t_0)}
\]

\[
> \beta(1 - p_0)^a \frac{r^{1/\alpha}(t) z'(t) R(t, t_0)}{R^a(t, t_0)}
\]

\[
= \beta(1 - p_0)^a z'(t) > 0.
\]

Set

\[
k(t) := \frac{z(t)}{R^{1-\beta(1-p_0)^a}(t, t_0)}.
\]
Then
\[ \left( r^{1/a}(t) R^{2 - \beta (1-p_0)\alpha} (t, t_0) k'(t) \right)' \]
\[ = \left( r^{1/a}(t) z'(t) R(t, t_0) - (1 - \beta (1-p_0)\alpha) z(t) \right)' \]
\[ = (r^{1/a}(t) z'(t))' R(t, t_0) + z'(t) - (1 - \beta (1-p_0)\alpha) z'(t) \]
\[ = -h'(t)\beta (1-p_0)\alpha z'(t) < 0. \]

Since
\[ \int_{t_1}^{\infty} \frac{1}{r^{1/a}(s) R^{2 - \beta (1-p_0)\alpha}(s, t_0)} \, ds = \frac{1}{(1 - \beta (1-p_0)\alpha) R^{1 - \beta (1-p_0)\alpha}(t_1, t_0)} < \infty, \]
either \( k'(t) > 0 \) or \( k'(t) < 0 \), eventually. Suppose that \( k'(t) > 0 \). Then using (v) and \( k'(t) > 0 \) in (17) implies
\[ r(t)(z'(t))^{\alpha} \geq \beta\alpha(1-p_0)\alpha \int_{t}^{\infty} \frac{z^{\alpha}(s)}{r^{1/a}(s) R^{1 + \alpha}(s, t_0)} \, ds \]
\[ \geq \beta\alpha(1-p_0)\alpha \int_{t}^{\infty} \frac{1}{r^{1/a}(s) R^{1 + \alpha}(s, t_0)} \frac{z^{\alpha}(s)}{R^{\alpha(1 - \beta (1-p_0)\alpha)}(s, t_0)} \, ds \]
\[ \geq \beta\alpha(1-p_0)\alpha \int_{t}^{\infty} \frac{z^{\alpha}(s)}{R^{\alpha(1 - \beta (1-p_0)\alpha)}(s, t_0)} \, ds \]
\[ = \frac{z^{\alpha}(t)}{R^{\alpha}(t, t_0)}, \]
that is,
\[ R(t, t_0) r^{1/a}(t) z'(t) \geq z(t), \]
which contradicts (v). Hence \( k'(t) < 0 \) and this part is proved.

(x) If \( \lambda_\ast = \infty \), then for any fixed \( \beta \) we can choose \( \lambda \) satisfying (7) such that
\[ \lambda > \frac{1}{\beta (1-p_0)\alpha}. \] (18)

Using (ix), (ii), (7), and (18) in a given order in (17), we find
\[ r(t)(z'(t))^{\alpha} \]
\[ \geq \beta\alpha(1-p_0)\alpha \int_{t}^{\infty} \frac{z^{\alpha}(\sigma(s))}{r^{1/a}(s) R(s, t_0) R^{\alpha}(s, t_0)} \, ds \]
\[ = \beta\alpha(1-p_0)\alpha \int_{t}^{\infty} \frac{1}{r^{1/a}(s) R(s, t_0) R^{\alpha(1 - \beta (1-p_0)\alpha)}(s, t_0)} \frac{z^{\alpha}(\sigma(s))}{R^{\alpha(1 - \beta (1-p_0)\alpha)}(s, t_0)} \, ds \]
\[ \geq \beta\alpha(1-p_0)\alpha \int_{t}^{\infty} \frac{1}{r^{1/a}(s) R(s, t_0) R^{\alpha(1 - \beta (1-p_0)\alpha)}(s, t_0)} \frac{z^{\alpha}(s)}{R^{\alpha(1 - \beta (1-p_0)\alpha)}(s, t_0)} \, ds \]
\[ \geq \beta\alpha(1-p_0)\alpha \lambda^{\alpha(1 - \beta (1-p_0)\alpha)} z^{\alpha}(t) \int_{t}^{\infty} \frac{1}{r^{1/a}(s) R^{1 + \alpha}(s, t_0)} \, ds \]
\[ \geq \beta(1-p_0)\alpha \lambda^{\alpha(1 - \beta (1-p_0)\alpha)} z^{\alpha}(t) \int_{t}^{\infty} \frac{1}{r^{1/a}(s) R^{1 + \alpha}(s, t_0)} \, ds \]
\[ > \frac{z^{\alpha}(t)}{R^{\alpha}(t, t_0)}, \]
which is a contradiction by means of the same argument as in (ix).
(xi) Let \( \tau(t) \leq t \). From (vii), we have
\[
\begin{align*}
\tau(t) & \leq (R(\tau(t), t_0))^{\sqrt{\beta(1-\rho)}} \\
& \leq R(t_0)z(t)\delta^{-\sqrt{\beta(1-\rho)}}.
\end{align*}
\]

Hence,
\[
\begin{align*}
x(t) & \geq z(t) - p(t)z(\tau(t)) \\
& \geq (1 - p(t)\tau^{-\sqrt{\beta(1-\rho)})} z(t) \\
& \geq (1 - p_0\tau^{-\sqrt{\beta(1-\rho)})} z(t).
\end{align*}
\]

Similarly, if \( \tau(t) \geq t \), then by (ix), we have
\[
\begin{align*}
z(\tau(t)) & \leq z(t) \left( \frac{R(\tau(t), t_0)}{R(t, t_0)} \right)^{1-\beta(1-\rho)\alpha} \\
& = z(t) \frac{R(\tau(t), t_0)}{R(t, t_0)} \omega^{-\beta(1-\rho)\alpha}.
\end{align*}
\]

Therefore,
\[
\begin{align*}
x(t) & \geq z(t) - p(t)z(\tau(t)) \\
& \geq \left( 1 - p(t) \frac{R(\tau(t), t_0)}{R(t, t_0)} \omega^{-\beta(1-\rho)\alpha} \right) z(t) \\
& \geq \left( 1 - p_0 \omega^{-\beta(1-\rho)\alpha} \right) z(t).
\end{align*}
\]

Moreover, if \( \tau(t) \leq t \) and \( \delta_\alpha = \infty \) (\( \tau(t) \geq t \) and \( \omega_\alpha = \infty \)), then \( \delta (\omega) \) can be chosen such that
\[
p_0\omega^{-\sqrt{\beta(1-\rho)}} < \epsilon \quad (p_0\omega^{-\beta(1-\rho)\alpha} < \epsilon)
\]
for any \( \epsilon \in (0, 1) \). The proof is complete. \( \square \)

The following result iteratively improves the previous one.

**Lemma 5.** Assume that \( \beta_\alpha > 0 \). If \( x(t) \) is an eventually positive solution of (1), then the corresponding function \( z(t) \) eventually satisfies
\[
\begin{align*}
(i)_n & \quad \frac{z(t)}{R(\tau_n(t), t_0)} \geq 0; \\
(ii)_n & \quad \frac{z(t)}{R^1 - \beta_\alpha(t, t_0)} < 0; \\
(iii)_n & \quad x(t) \geq \epsilon z(t) \text{ for } (\tau(t) \leq t, \delta_\alpha = \infty) \text{ or } (\tau(t) \geq t, \omega_\alpha = \infty) \text{ and any } \epsilon \in (0, 1); \\
& \quad x(t) \geq z(t)(1 - p_0\omega^{-\beta_\alpha\alpha}) \text{ for } (\tau(t) \leq t, \delta_\alpha < \infty) \text{ and any } \delta \in [1, \delta_\alpha); \\
& \quad x(t) \geq z(t)(1 - p_0\omega^{-\beta_\alpha\alpha}) \text{ for } (\tau(t) \geq t, \omega_\alpha < \infty) \text{ and any } \omega \in [1, \omega_\alpha). 
\end{align*}
\]

**Proof.** Pick \( t_1 \geq t_0 \) large enough such that
\[
x(t) > 0, \quad x(\sigma(t)) > 0, \quad \text{and} \quad x(\tau(t)) > 0 \quad \text{for} \quad t \geq t_1.
\]

The proof will proceed in two steps.

1. We show by induction on \( n \) that for arbitrary \( \gamma_\epsilon n \in (0, 1) \) and \( \beta_\epsilon n \in (0, 1) \):
\[
(i)_n \quad \frac{z(t)}{R^{\gamma_\epsilon n(t)}(t_0)} \geq 0,
\]

2. Next, we show by induction that
\[
(ii)_n \quad \frac{z(t)}{R^1 - \beta_\epsilon n(t, t_0)} < 0.
\]
\[
\text{\text{(II)}_n} \quad \left( \frac{z(t)}{R^{(1-\beta \alpha \beta_n)}}(t, t_0) \right)' < 0,
\]

and
\[
\text{\text{(III)}_n} \quad x(t) \geq a_n z(t),
\]

where
\[
a_n = \left\{ \begin{array}{ll}
\epsilon & \text{for } (\tau(t) \leq t, \delta_s = \infty) \text{ or } (\tau(t) \geq t, \omega_s = \infty); \\
1 - p_0 \delta^{-\gamma \epsilon \gamma_n} & \text{for } (\tau(t) \leq t, \delta_s < \infty); \\
1 - p_0 \omega^{-\beta \alpha \beta_n} & \text{for } (\tau(t) \geq t, \omega_s < \infty).
\end{array} \right.
\]

For \( n = 0 \), the conclusion follows from (vii), (ix) and (xi) with
\[
\gamma \epsilon_0 = \beta \epsilon_0 = \frac{\beta}{\beta_s}.
\]

Clearly,
\[
\lim_{\beta \to \beta_s} \epsilon_0 = 1.
\]

Now, assume that (I)\(_n\)–(III)\(_n\) hold for some \( n \geq 1 \) and we will show that they hold for \( n + 1 \), with \( \beta \epsilon_{n+1} \) and \( \gamma \epsilon_{n+1} \) defined by:

(a) for either \( (\tau(t) \leq t \text{ and } \delta_s = \infty) \text{ or } (\tau(t) \geq t \text{ and } \omega_s = \infty) \):
\[
\beta \epsilon_{n+1} = \beta \epsilon_0 \lambda_1^{\alpha \epsilon_{n-1} \beta \beta_{n-1}} \left( \frac{1 - \beta \beta_{n-1}}{1 - \beta \epsilon_{n-1} \beta \beta_{n-1}} \right)^{\alpha},
\]
\[
\gamma \epsilon_{n+1} = \sqrt[\alpha]{\gamma \epsilon_0} \lambda_1^{\alpha \epsilon_{n-1} \beta \beta_{n-1}} \left( \frac{1 - \gamma \gamma_{n-1}}{1 - \gamma \epsilon_{n-1} \gamma \gamma_{n-1}} \right)^{\alpha},
\]

(b) for \( (\tau(t) \leq t \text{ and } \delta_s < \infty) \):
\[
\beta \epsilon_{n+1} = \beta \epsilon_0 \lambda_1^{\alpha \epsilon_{n-1} \beta \beta_{n-1}} \left( \frac{1 - \beta \beta_{n-1}}{1 - \beta \epsilon_{n-1} \beta \beta_{n-1}} \right)^{\alpha} \left( \frac{1 - p_0 \delta^{-\gamma \epsilon \gamma_n}}{1 - p_0 \delta^{-\gamma \epsilon \gamma_n}} \right)^{\alpha},
\]
\[
\gamma \epsilon_{n+1} = \sqrt[\alpha]{\gamma \epsilon_0} \lambda_1^{\alpha \epsilon_{n-1} \beta \beta_{n-1}} \left( \frac{1 - \gamma \gamma_{n-1}}{1 - \gamma \epsilon_{n-1} \gamma \gamma_{n-1}} \right)^{\alpha} \left( \frac{1 - p_0 \omega^{-\beta \alpha \beta_1}}{1 - p_0 \omega^{-\beta \alpha \beta_1}} \right)^{\alpha},
\]

(c) for \( (\tau(t) \geq t \text{ and } \omega_s < \infty) \):
\[
\beta \epsilon_{n+1} = \beta \epsilon_0 \lambda_1^{\alpha \epsilon_{n-1} \beta \beta_{n-1}} \left( \frac{1 - \beta \beta_{n-1}}{1 - \beta \epsilon_{n-1} \beta \beta_{n-1}} \right)^{\alpha} \left( \frac{1 - p_0 \omega^{-\beta \alpha \beta_1}}{1 - p_0 \omega^{-\beta \alpha \beta_1}} \right)^{\alpha},
\]
\[
\gamma \epsilon_{n+1} = \sqrt[\alpha]{\gamma \epsilon_0} \lambda_1^{\alpha \epsilon_{n-1} \beta \beta_{n-1}} \left( \frac{1 - \gamma \gamma_{n-1}}{1 - \gamma \epsilon_{n-1} \gamma \gamma_{n-1}} \right)^{\alpha} \left( \frac{1 - p_0 \omega^{-\beta \alpha \beta_1}}{1 - p_0 \omega^{-\beta \alpha \beta_1}} \right)^{\alpha},
\]

for \( n \in \mathbb{N} \). Clearly, in all three cases, we have
\[
\lim_{(\beta, \lambda) \to (\beta_s, \lambda_s, 1)} \beta \epsilon_n = \lim_{(\beta, \lambda) \to (\beta_s, \lambda_s, 1)} \gamma \epsilon_n = 1,
\]
\[
\lim_{(\beta, \lambda) \to (\beta_s, \lambda_s, 1)} \beta \epsilon_n = \lim_{(\beta, \lambda) \to (\beta_s, \lambda_s, 1)} \gamma \epsilon_n = 1,
\]
and
\[
\lim_{(\beta, \lambda \omega) \to (\beta, \lambda, \omega_1^*)} \beta \varepsilon_n = \lim_{(\beta, \lambda \omega) \to (\beta, \lambda, \omega_1^*)} \gamma \varepsilon_n = 1,
\]
respectively.

Using (III), in (13), we see that
\[
r(t)(z'(t))^a \geq \beta \alpha a_n^a \int_1^\infty \frac{z^n(\sigma(s))}{r^{1/n}(s)R(s, t_0)R^n(\sigma(s), t_0)} ds.
\]
As in the case \( n = 0 \), using (II) and (7), we obtain
\[
r(t)(z'(t))^a \geq \beta \alpha a_n^a \int_1^\infty \frac{z^n(\sigma(s))}{r^{1/n}(s)R(s, t_0)R^{n(1-\rho_n\beta_n)}(\sigma(s), t_0)R^{n(1-\rho_n\beta_n)}(\sigma(s), t_0)} ds \geq \beta \alpha a_n^a \lambda^{\rho_n \beta_n} \int_1^\infty \frac{z^n(\sigma(s))}{r^{1/n}(s)R^{1+n(1-\gamma_n \alpha)}(s, t_0)} ds.
\]
Employing (I), in the above inequality, we obtain
\[
r(t)(z'(t))^a = \beta \alpha a_n^a \lambda^{\rho_n \beta_n} \int_1^\infty \frac{z^n(\sigma(s))}{r^{1/n}(s)R^{1+n(1-\gamma_n \alpha)}(s, t_0)} ds \geq \beta \alpha a_n^a \lambda^{\rho_n \beta_n} \frac{z^n(t)}{R^{1+n(1-\gamma_n \alpha)}(t, t_0)} = \frac{z^n(t)}{1-\gamma_n \alpha} \frac{z^n(t)}{R^{1+n(1-\gamma_n \alpha)}(t, t_0)} = \frac{z^n(t)}{\gamma_n+1} \frac{z^n(t)}{R^{1+n(1-\gamma_n \alpha)}(t, t_0)},
\]
that is,
\[
R(t, t_0)r^{1/n}(t)z'(t) \geq \gamma_{n+1} z(t).
\]
Hence, (I) holds.

Now, we apply (III) in (12) and use (II) twice: once as a monotone property and then as a corresponding inequality:
\[
h'(t) \geq \beta a_n^a \frac{(r^{1/n}(t)z'(t))^{1-a}}{r^{1/n}(t)} z^n(\sigma(t)) \frac{z^n(\sigma(t))}{R^{n}(\sigma(t), t_0)} \geq \beta a_n^a \frac{(r^{1/n}(t)z'(t))^{1-a}}{r^{1/n}(t)} R^{n(1-\rho_n\beta_n)}(\sigma(t), t_0)R^{n(1-\rho_n\beta_n)}(\sigma(t), t_0) \geq \beta a_n^a \lambda^{\rho_n \beta_n} \frac{z^n(t)}{R^{1+n(1-\gamma_n \alpha)}(t, t_0)} \frac{z^n(t)}{R^{1+n(1-\gamma_n \alpha)}(t, t_0)} \geq \beta a_n^a \lambda^{\rho_n \beta_n} \frac{(r^{1/n}(t)z'(t))^{1-a}}{r^{1/n}(t)} \frac{z^n(t)}{R^{n}(t, t_0)} = \beta \varepsilon_n+1 \beta_n+1 z'(t).
\]
Set
\[
k(t) := \frac{z(t)}{r^{1-n(\beta_n+1+\varepsilon_n)}(t, t_0)}.
\]
By straightforward computation, we see that

\[
\left( r^{1/\kappa} (t) R^{2 - \beta \epsilon_{n+1} \beta_{n+1}} (t, t_0) k'(t) \right)'
\]

\[
= \left( r^{1/\kappa} (t) z'(t) R(t, t_0) - (1 - \beta \epsilon_{n+1} \beta_{n+1}) z(t) \right)'
\]

\[
= (r^{1/\kappa} (t) z'(t) )' R(t, t_0) + z'(t) - (1 - \beta \epsilon_{n+1} \beta_{n+1}) z'(t)
\]

\[
= -k'(t) + \beta \epsilon_{n+1} \beta_{n+1} z'(t) < 0,
\]

where we use (20). Hence, either \( k'(t) > 0 \) or \( k'(t) < 0 \) eventually. If we suppose that \( k'(t) > 0 \) holds, then, in view of (19), we find

\[
r(t)(z'(t))^\alpha \geq \beta a_n^\alpha \gamma_\beta \epsilon_n \beta_{n+1} \int_t^\infty \frac{z^\alpha(s)}{r^{1/\kappa}(s) R^{1+\kappa}(s, t_0)} ds
\]

\[
\geq \beta a_n^\alpha \gamma_\beta \epsilon_n \beta_{n+1} \int_t^\infty \frac{z^\alpha(t)}{R^{1+\alpha}(t, t_0)} ds
\]

\[
= \beta a_n^\alpha \gamma_\beta \epsilon_n \beta_{n+1} \frac{z^\alpha(t)}{R^{1+\alpha}(t, t_0)}
\]

\[
= (1 - \beta \epsilon_n \beta_{n+1})^\alpha \frac{z^\alpha(t)}{R^\alpha(t, t_0)}
\]

that is,

\[
R(t, t_0) r^{1/\kappa}(t) z'(t) \geq (1 - \beta \epsilon_n \beta_{n+1}) z(t),
\]

which is in contradiction with

\[
\left( \frac{z(t)}{R^{1 - \beta \epsilon_n \beta_{n+1}} (t, t_0)} \right)'< 0.
\]

Hence, \( k'(t) < 0 \) and so \((\Pi)_{n+1}\) holds. The proof of \((\Pii)_{n+1}\) proceeds in the same way as for \( n = 0 \) and hence we omit it.

2. To prove the statement, we claim that \((\I)_{n}\) and \((\II)_{n}\) implies \((\I)_{n-1}\) and \((\II)_{n-1}\) for \( n \in \mathbb{N} \). Note that \((\III)_{n-1}\) is only a simple consequence of the first two parts. Clearly, \((\I)_{n}\) and \((\II)_{n}\) correspond to

\[
\gamma \epsilon_n \gamma_{n-1} z(t) \leq r^{1/\kappa}(t) z'(t) R(t, t_0)
\]

(21)

and

\[
(1 - \beta \epsilon_n \beta_{n+1}) z(t) > r^{1/\kappa}(t) z'(t) R(t, t_0),
\]

(22)

respectively. Then, by virtue of (ii) and (v), it is easy to see that

\[
\beta \epsilon_n \beta_{n+1} < 1 \quad \text{and} \quad \gamma \epsilon_n \gamma_{n-1} < 1.
\]

Using this and (8), we have

\[
1 > \beta \epsilon_n \beta_{n+1} = \beta \epsilon_n \ell_{n-1} \beta_{n-1} > \beta_{n-1}
\]

(23)

and

\[
1 > \gamma \epsilon_n \gamma_{n-1} = \gamma \epsilon_n \ell_{n-1} \gamma_{n-1} > \gamma_{n-1},
\]

(24)

where we used that \( \beta \epsilon_n \in (0, 1) \) and \( \gamma \epsilon_n \in (0, 1) \) are arbitrary. Therefore, (21) and (22) become

\[
\gamma_{n-1} z(t) \leq r^{1/\kappa}(t) z'(t) R(t, t_0)
\]
and

\[(1 - \beta_{n-1})z(t) > r^{1/a}(t)z'(t)R(t,t_0),\]

for \(n \in \mathbb{N}\), which proves our claim.

The proof is complete. \(\square\)

Next, we state an improved version of Lemma 3.

**Lemma 6.** Assume that \(\beta_* > 0\). If \(x(t)\) is an eventually positive solution of (1), then the corresponding function \(z(t)\) eventually satisfies for any \(n \in \mathbb{N}_0\),

\[
\liminf_{t \to \infty} \frac{r^{1/a}(\sigma(t))z'(\sigma(t))}{r^{1/a}(t)z'(t)} \geq \lambda_{n}^p,
\]

where

\[p_n := \frac{\beta_* \lambda_{n}^{a(1 - \beta_n)} r_n}{(1 - \beta_n)^a}\]

with

\[r_n = \begin{cases} 
1 - p_0 \delta_*^{-\gamma_n} & \text{for } \tau(t) \leq t; \\
1 - p_0 \omega_* \beta_n & \text{for } \tau(t) \geq t.
\end{cases}\]

**Proof.** Pick \(t_1 \geq t_0\) large enough such that

\[x(t) > 0, \quad x(\sigma(t)) > 0, \quad \text{and } x(\tau(t)) > 0 \quad \text{for } t \geq t_1.\]

Using (7) and (iii)_n in (1), we have

\[-\left( r(t)(z'(t))^a \right)' = q(t)x^a(\sigma(t)) \geq \beta_0 e_n^a r_n^{a(1 - \beta_n)} \frac{z^a(\sigma(t))}{r^{1/a}(t)R(t,t_0)R^a(\sigma(t),t_0)},\]  \hspace{1cm} (25)

where

\[\varepsilon_n = \begin{cases} 
\varepsilon & \text{for } (\tau(t) \leq t, \delta_* = \infty) \text{ or } (\tau(t) \geq t, \omega_* = \infty); \\
1 - p_0 \delta_*^{-\gamma_n} & \text{for } (\tau(t) \leq t, \delta_* < \infty); \\
1 - p_0 \omega_* \beta_n & \text{for } (\tau(t) \geq t, \omega_* < \infty).
\end{cases}\]

Clearly \(\varepsilon_n \in (0, 1)\) is arbitrary. Using (ii)_n twice in (25), similarly as before, we obtain

\[-\left( r(t)(z'(t))^a \right)' \geq \beta_0 e_n^a r_n^{a(1 - \beta_n)} \frac{r(t)(z'(t))^a}{(1 - \beta_n)^a} \frac{r^{1/a}(t)(z'(t))^a}{R(t,t_0)},\]

where

\[\xi_n = \frac{\beta}{\beta_*} \left( \frac{\lambda}{\lambda_*} \right)^{a(1 - \beta_n)} \in (0, 1)\]

is arbitrary. Hence,

\[\left( \frac{r(t)(z'(t))^a}{R^{a_n}(t,t_0)} \right)' \leq 0\]

and so

\[
\frac{r^{1/a}(\sigma(t))z'(\sigma(t))}{r^{1/a}(t)z'(t)} \geq \lambda_{n}^p.
\]

The proof is complete. \(\square\)
In view of the monotonicities (i) and (ii), the following result is immediate.

**Theorem 1.** Let \( \gamma_i < 1 \) and \( \beta_i < 1 \) for \( i = 0, 1, \ldots, n \) for some \( n \in \mathbb{N}_0 \). If

\[
\gamma_{n+1} + \beta_{n+1} > 1,
\]

then (1) is oscillatory.

To provide our final criteria, we rely on another simple consequence of Lemma 5 (see (23) and (24)).

**Corollary 1.** Let \( \beta_* > 0 \). If \( x(t) \) is an eventually positive solution of (1), then both sequences \( \{\beta_n\}_{n \in \mathbb{N}_0} \) and \( \{\gamma_n\}_{n \in \mathbb{N}_0} \) are well-defined and bounded from above.

Now we are prepared to state the main result of this paper.

**Theorem 2.** If one of the conditions

(C1) \( \beta_* > 0 \) and \( \lambda_* = \infty \);

(C2) either \( (\tau(t) \leq t \text{ and } \delta_* = \infty) \) or \( (\tau(t) \geq t \text{ and } \omega_* = \infty) \), and Equation (9) has no solution \( m \in (0, 1) \), i.e.,

\[
\beta_* > \max \left\{ m(1-m)^a \lambda_*^{-am} : 0 < m < 1 \right\};
\]

(C3) \( (\tau(t) \leq t \text{ and } \delta_* < \infty) \), and the system (10) has no solution \( \{m \in (0, 1), k \in (0, 1)\} \);

(C4) \( (\tau(t) \geq t \text{ and } \omega_* < \infty) \), and Equation (11) has no solution \( m \in (0, 1) \), i.e.,

\[
\beta_* > \max \left\{ m(1-m)^a \lambda_*^{-am} \left(1 - \frac{p_0}{\omega_*^2}m^a\right) : 0 < m < 1 \right\};
\]

is satisfied, then (1) is oscillatory.

**Proof.** Assume on the contrary that (1) has a nonoscillatory, say positive solution \( x \). By (x), we have \( \lambda_* < \infty \), which contradicts (C1). On the other hand, by combining Corollary 1 and Lemma 2, we see that cases (C2)–(C4) are impossible as well. Hence (1) is oscillatory. □

In a nonneutral case, we obtain a partly result given for a single delay equation \( (m = 1) \) in [44, Theorem 1, Theorem 2].

**Corollary 2.** Let \( p(t) = 0 \). If

\[
\beta_* > \begin{cases} 
0 & \text{for } \lambda_* = \infty, \\
\max \left\{ m(1-m)^a \lambda_*^{-am} : 0 < m < 1 \right\} & \text{for } \lambda_* < \infty,
\end{cases}
\]

then

\[
\left( r(t) (x'(t))^a \right)' + q(t) x^a (r(t)) = 0
\]

is oscillatory.

To end this section, we wish to illustrate the novelty of our results via Euler differential equations.

**Example 1.** As in [17], let us consider the second-order neutral differential equation of Euler type

\[
\left( \left((x(t) + p(t) x(\tau(t)))^a\right)' + \frac{q_0}{\lambda_*} x^a (\lambda_* t) = 0, \quad (27)
\]
where \( \alpha > 0 \) is a quotient of odd positive integers, either \( \tau(t) \leq t \) or \( \tau(t) \geq t \), \( q_0 > 0 \), and \( p(t) \geq 0 \) is such that

\[
1 > p_0 = \begin{cases} 
p(t) & \text{for } \tau(t) \leq t, \\
p(t) \frac{\tau(t)}{t} & \text{for } \tau(t) \geq t.
\end{cases}
\]

We recall that a usual criterion for (27) with \( \tau(t) \leq t \) resulting from the Riccati substitution method (see e.g., [30, 36–38, 45]) requires

\[
q_0 > \left( \frac{\alpha}{\alpha + 1} \right)^{a+1} \frac{1}{(1 - p_0)^a \lambda_2^a}.
\]

(28)

If we put \( p_0 = 0 \) and \( \lambda_2 = 1 \), then (28) reduces to

\[
q_0 > \left( \frac{\alpha}{\alpha + 1} \right)^{a+1},
\]

which is sharp for the oscillation of the Euler type half-linear ordinary differential equation

\[
\left( (x'(t))^a \right)' + \frac{q_0}{\lambda_2^a} x^a(t) = 0.
\]

Obviously, the oscillation constant in (28) increases when

1. \( \lambda_2 \to 1 \);
2. \( p_0 \to 0 \).

In [17], we showed that (27) with \( \tau(t) \leq t \) is oscillatory if

\[
\rho := (1 - p_0)^a q_0^a \lambda_2^a \frac{a + (1 - p_0)^a q_0^a \lambda_2^a)^a}{a^a} \ln \frac{1}{\lambda_2} > e.
\]

If \( \rho \leq 1/e \), we were able to improve (28) to

\[
q_0 > \left( \frac{\alpha}{\alpha + 1} \right)^{a+1} \frac{1}{(1 - p_0)^a \lambda_2^a f(\rho)},
\]

(29)

where

\[
f(\rho) = -\frac{W(-\rho)}{\rho},
\]

and \( W \) stands for the principal branch of the Lambert \( W \) function. Clearly, the oscillation constant in (29) is better than that in (28). Now, let us illustrate the progress made in this work.

1. First, let

\[
\tau(t) = t^k, \quad k \in (0,1). \quad \left( \tau(t) = t^k, \quad k > 1. \right)
\]

Since

\[
\delta_* = \lim_{t \to \infty} t^{1-k} = \infty, \quad \left( \omega_* = \lim_{t \to \infty} t^{k-1} = \infty. \right)
\]

condition (C2) from Theorem 2 reduces to

\[
q_0 > \max \{a m (1-m)^a \lambda_2^{a(m-1)} : 0 < m < 1 \}.
\]

(30)

Note that (30) is sharp for the oscillation of the Euler type half-linear delay differential equation

\[
\left( (x'(t))^a \right)' + \frac{q_0}{\lambda_2^a} x^a(t) = 0,
\]

(31)
since (31) has a nonoscillatory solution $x(t) = t^{1-m}$, if
\[ q_0 \leq \max\{am(1-m)^a\lambda_2^{a(m-1)} : 0 < m < 1\}. \]

Obviously, the function $\tau(t)$ completely suppresses the influence of the function $p(t)$ in the final oscillation criterion, while the existing conditions (28) and (29) (as well as all the other ones from the references used in the paper) do not depend on $\tau(t)$ at all.

2. Now, let
\[ \tau(t) = \lambda_1 t, \quad \lambda \in (0, 1]. \]

Since
\[ \delta_+ = \frac{1}{\lambda_1}, \]
condition (C3) from Theorem 2 requires that the system
\[
\begin{align*}
q_0 &= \frac{am(1-m)^a\lambda_2^{a(m-1)}}{(1-p_0\lambda_1^a)^a} \\
q_0 &= \frac{ak^a(1-k)\lambda_2^{a(m-1)}}{(1-p_0\lambda_1^a)^a}
\end{align*}
\]
does not have a solution $\{m \in (0, 1), k \in (0, 1)\}$. Clearly, if $p_0 = 0$, then the first equation in (32) has no solution on $(0, 1)$ if (30) holds. If $p_0 \neq 0$, then a computer algebra can be used to find whether there is a solution of (32).

3. Finally, let
\[ \tau(t) = \lambda_1 t, \quad \lambda_1 \geq 1. \]

Since
\[ \omega_+ = \lambda_1, \]
condition (C4) from Theorem 2 reduces to
\[ q_0 > \max\left\{ \frac{am(1-m)^a\lambda_2^{a(m-1)}}{(1-p_0\lambda_1^a)^a} : 0 < m < 1 \right\}. \]

As well as in cases 1 and 2, this condition becomes sharp for $p_0 = 0$. Moreover, it is obvious that the delay function $\tau(t)$ affects the criterion via the term $(1 - p_0\lambda_1^{-m})^a$.

**Example 2.** Now, consider
\[
\left( \left( (x(t) + p(t)x(\tau(t)))^{\alpha} \right)' \right)' + \frac{q_0}{\mu_1^{\alpha+1}} x^{\alpha}(t^{\lambda_2}) = 0
\]
with the same assumptions as for (27). Here,
\[ \lambda_+ = \lim_{t \to \infty} t^{1-\lambda_2} = \infty \]
and
\[ \beta_+ = \frac{q_0}{\alpha} > 0, \]
hence by Theorem 2 condition (C1), (33) is oscillatory.

4. Further Remarks, Open Problems and Research Directions

**Remark 1.** The methods mostly used in investigating the oscillatory behavior of (1) have been based on a reduction of order and comparison with oscillation of first-order delay differential equations, or on reducing (1) to a first-order Riccati inequality, based on a suitable Riccati type substitution, see e.g., [17] for more details. We note that none of the related results [3–7,10,12–18,20–22,26,28–36,39,42,46]
involving (1) with \( \alpha = 1, r(t) = 1, p(t) = 0 \), gives a sharp result when applied to the Euler linear delay differential equation
\[
x''(t) + \frac{q_0}{t^{\alpha}} x(\lambda_2 t) = 0, \quad q_0 > 0, \quad \lambda_2 \in (0, 1),
\]
which is oscillatory if and only if
\[
q_0 > \max\{k(1 - k)\lambda_2^{-k} : 0 < k < 1\}.
\]
On the other hand, in [43,44], we have presented an oscillation criterion for (1) with \( p(t) = 0 \) which is sharp for the Euler half-linear delay differential Equation (31). In the present paper, the method developed in [43,44] has been extended for neutral differential equations of the form (1), under the assumptions (H1)–(H5). The main results of this paper also include those from [43,44] given in a nonneutral case.

In contrast with all previous works on neutral equations based on the method of a lower bound of the ratio \( x(t)/z(t) \), our method significantly depends on the function \( \tau(t) \) and produces effective oscillation criteria even if \( p_0 \) is not close to zero. Furthermore, we stress that our method does not require that \( \sigma'(t) > 0 \) or \( \tau'(t) > 0 \).

The results based on the method of iteratively improved monotonicity properties reveal many fruitful problems for further research.

**Remark 2.** The first contribution for neutral equation (1) involving an unimprovable oscillation constant in the nonneutral case was made in the author’s work [8], replacing (2) in (H2) with
\[
\pi(t_0) := \int_{t_0}^{\infty} r^{-1/\alpha}(s)ds < \infty,
\]
and (3) in (H5) by
\[
p_0 \geq p(t) \frac{\pi(\tau(t))}{\pi(t)} \quad \text{for} \quad \tau(t) \leq t,
\]
\[
p_0 \geq p(t) \quad \text{for} \quad \tau(t) \geq t.
\]
For
\[
\beta_* := \frac{1}{\alpha} \liminf_{t \to \infty} t^{1/\alpha}(t) \pi^{\alpha+1}(t) q(t),
\]
\[
\lambda_* := \liminf_{t \to \infty} \frac{\pi(\sigma(t))}{\pi(t)},
\]
we showed that if
\[
\beta_* > \begin{cases} 
0 & \text{for } \lambda_* = \infty, \\
\frac{\max\{m^\alpha (1 - m) \lambda_*^{-\alpha m} : 0 < m < 1\}}{(1 - p_0)^\alpha} & \text{for } \lambda_* < \infty,
\end{cases}
\]
then (1) is oscillatory, see (Theorem 1, Theorem 2, [8]), which improved earlier related results from [3,7,18,25,36,42,47–51].

Following the approach presented in this paper, it is, therefore, possible to extend and improve the results from [8] by refining the relation between \( x \) and \( z \) in each iteration of the procedure, depending on the limits
\[
\delta_* := \liminf_{t \to \infty} \frac{\pi(\tau(t))}{\pi(t)} \quad \text{for} \quad \tau(t) \leq t,
\]
\[
\omega_* := \liminf_{t \to \infty} \frac{\pi(t)}{\pi(\tau(t))} \quad \text{for} \quad \tau(t) \geq t,
\]
which would lead to the analogue of Theorem 2.

**Remark 3.** In addition to the above mentioned problems, it is also interesting to extend the method and establish corresponding results for

1. **neutral differential equations of the form** (1) **with advanced argument** (i.e., if \( \sigma(t) \geq t \) under the assumption (2) or (34) (for sharp results in a nonneutral case, see also [52,53]));
2. **neutral differential equations of the form** (1) **with different ranges of** \( p(t) \) **than those in** (HS), mainly \( p(t) > 1 \) or \(-1 < p(t) \leq 0\);
3. **neutral differential equations of the form** (1) **with more general function** \( z(t) \), involving, e.g.,
   a. mixed (delayed and advanced) neutral terms:
   \[
   z(t) = x(t) + p_1(t)x(\tau_1(t)) + p_2(t)x(\tau_2(t)),
   \]
   b. mixed (sublinear and superlinear) neutral terms
   \[
   z(t) = x(t) + p_1(t)x^{\alpha}(\tau_1(t)) + p_2(t)x^{\beta}(\tau_2(t)),
   \]
   c. mixed (positive and negative) neutral terms
   \[
   z(t) = x(t) + p_1(t)x(\tau_1(t)) - p_2(t)x(\tau_2(t)).
   \]

It is also open how to extend the approach presented in this paper for

1. **neutral differential equations of higher-order** \( (n \geq 3) \) (for sharp results obtained for third-order linear delay differential equations, see also [54,55]);
2. **half-linear delay differential equations of higher-order** \( (n \geq 3) \) (for sharp results for second-order half-linear delay differential equations, see also [43,44,52,53,56]);
3. corresponding classes of functional difference equations (for first such extension of the approach, see the very recent contribution [57]).

Finally, there is a wish to provide a unified approach for investigation of oscillatory and asymptotic properties of solutions to second-order half-linear neutral delay dynamic equations on time scales via the method of iteratively improved monotonicities. It is worth noting that the application of at least a first iteration of Lemma 5 in the methods developed in related works for second-order delay dynamic equations [42,45,58–68] would immediately improve the oscillation results stated therein.

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