New bounds for the b-chromatic number of vertex deleted graphs

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Abstract. A b-coloring of a graph is a proper coloring of its vertices such that each color class contains a vertex adjacent to at least one vertex of every other color class. The b-chromatic number of a graph is the largest integer $k$ such that the graph has a b-coloring with $k$ colors. In this work we present lower bounds for the b-chromatic number of a vertex-deleted subgraph of a graph, particularly regarding two important classes, quasi-line and chordal graphs. We also get bounds for the b-chromatic number of $G - \{x\}$, when $G$ is a graph with large girth.

Key words: b-coloring, quasi-line graph, chordal graph, girth

Mathematics Subject Classification: 05C15.

1 Introduction

All graphs considered in this work are simple and undirected. Let $G = (V(G), E(G))$ be an undirected graph where $V(G)$ and $E(G)$ are the sets of its vertices and edges, respectively. If $A \subseteq V(G)$, we denote by $\langle A \rangle$ the induced subgraph generated by $A$. For any vertex $x$ of a graph $G$, the neighborhood of $x$ is the set $N(x) = \{y \in V(G) | xy \in E(G)\}$. The degree of a vertex $x$ is the cardinality of $N(x)$ and it is denoted by $d(x)$. We denote by $\Delta(G)$ the maximum degree of $G$. If $x, y \in V(G)$, the distance between $x$ and $y$ (that is, the length of the shortest $x$–$y$-path) is represented by $d(x, y)$ and $N^2(x) = \{y \in V(G) | d(x, y) = 2\}$. The girth of $G$ is the length of its shortest cycle, and is denoted by $g(G)$.

Let $G$ be a graph with a proper vertex coloring. Let us denote by $C_i$ the set of vertices of color $i$, herein called the class of color $i$. Let $x_i$ denote a vertex $x$ of color $i$; $x_i$ is said a color-dominating vertex (or, b-dominating vertex) if $x_i$ is adjacent to at least one vertex in each of the other classes. A
color $i$ is a \textit{dominating color} if there is at least one vertex $x_i$ that is color-dominating. If $y_i$ is a vertex which is not color-dominating, at least one color $j \neq i$ does not appear in $N(y_i)$. The color $j$ is said a \textit{missing color} in $N(y_i)$ or simply a missing color of $y_i$.

A \textit{b-coloring} is a proper coloring of its vertices such that each color class contains a color-dominating vertex.

The b-chromatic number $b(G)$ is the largest integer $k$ such that $G$ admits a b-coloring with $k$ colors. Since this parameter has been introduced by R. W. Irving and D. F. Manlove \cite{6}, it aroused the interest of many researchers as we can see in \cite{5}, \cite{2} and \cite{3} and, more recently, in \cite{7}.

For a vertex $x$ of $G$, let $G - \{x\}$ be the vertex-deleted subgraph of $G$ obtained by deleting $x$ and all edges incident to $x$. It is known that the chromatic number of $G - \{x\}$ can have a maximum variation of one unit compared to the chromatic number of $G$. However, this is not true for the b-chromatic number - the difference between $b(G)$ and $b(G - \{x\})$ can be arbitrarily large. This fact motivates the search for bounds to the b-chromatic number (see \cite{1} and \cite{8}). For general graphs S.F.Raj and R.Balakrishnan proved that:

\textbf{Theorem 1} \cite{1} \textit{For any connected graph of order $n \geq 5$, and for any vertex $x \in V(G)$,}
\[ b(G) - \left(\lceil n/2 \rceil - 2\right) \leq b(G - \{x\}) \leq b(G) + \left(\lfloor n/2 \rfloor - 2\right) \]

The bounds are sharp.

In \cite{9}, some upper bounds of $b(G - \{x\})$ have been established in some classes of graphs as quasi-line graphs, graphs of large girth and chordal graphs. A \textit{chordal} graph is a graph such that every cycle of length at least 4 has a chord. A graph is a \textit{quasi-line} graph if the neighborhood of each vertex is covered by at most two cliques. In particular, claw-free graphs (i.e. graphs without induced $K_{1,3}$) are quasi-line graphs. The following results have been shown.

\textbf{Theorem 2} \cite{9} \textit{Let $G = (V, E)$ a graph.}
1) If $G$ is a quasi-line-graph, then for each vertex $x$,
\[ b(G - \{x\}) \leq b(G) + 2 \]
2) If $G$ is any graph of girth at least 5, then for each vertex $x$,
\[ b(G - \{x\}) \leq b(G) + 1 \]
Theorem 3 Let $G = (V, E)$ be a chordal graph of clique-number $\omega$ and $b$-chromatic number $b(G)$. Then, for each vertex $x$,

$$b(G - \{x\}) \leq b(G) + 1 + \sqrt{d(x)} - 1$$

$$b(G - \{x\}) \leq b(G) + 1 + \sqrt{\omega - 1}$$

In this work we present lower bounds for $b(G - \{x\})$ in terms of $b(G)$, particularly regarding two important classes of graphs, quasi-line and chordal graphs. We also obtain a lower bound for $b(G - \{x\})$ for graphs of large girth.

Besides this introduction we have three more sections. In the second one we present a lower bound for $b(G - \{x\})$, when $G$ is a general graph and another bound for quasi-line graphs. The third section is devoted to the study of chordal graphs, obtaining also a lower bound for $b(G - \{x\})$ in this class. Finally, in the last section, we analyse graphs with girth at least 5, presenting also here a lower bound for $b(G - \{x\})$.

2 General bound and quasi-line graphs

We begin this section with a lower bound for the $b$-chromatic number of a vertex deleted subgraph of any graph.

Proposition 1 For every vertex $x \in V(G)$, $b(G - \{x\}) \geq b(G) - d(x)$.

Proof. Let $x \in V(G)$ be a fixed vertex. For a $b$-coloring of $G$, let $i$ be the color of $x$. We consider two cases. First suppose that each color has at least one color-dominating vertex in $G - \{x\}$; then the $b$-coloring of $G$ is also a $b$-coloring of $G - \{x\}$, so $b(G - \{x\}) \geq b(G)$. Now, let us consider that there is a color with no color-dominating vertices in $G - \{x\}$. We have then two possibilities:

- There is no color-dominating vertex of color $i$ in $G - \{x\}$, that is, $C_i$ has no color-dominating vertex then; for each vertex $z$ in $C_i$, there is at least one color missing in $N(z)$. We can change the color of each vertex $z$ in $C_i$ by a missing color in $N(z)$, eliminating the color $i$. As $C_i$ is a stable set, the new coloring is proper. For this case we have $b(G - \{x\}) \geq b(G) - 1 \geq b(G) - d$. 

• There is a vertex $y \in N(x)$ such that $y$ was the color-dominating vertex of color $s$, $s \neq i$ in $G$ and there is no more color-dominating vertex of color $s$ in $G - \{x\}$. As $C_s$ has no color-dominating vertices, we then change the color $s$ of $y$ by $i$ and, for each other vertex $z$ in $C_s$, we change the color $s$ for a missing color of $z$, eliminating the color $s$. As $C_s$ is a stable set, the new coloring is proper. We repeat this process for all vertices in $N(x)$ in the same conditions as $y$. We do this for at most $d(x)$ vertices. In this case we obtain $b(G - \{x\}) \geq b(G) - d(x)$.

If $\Delta(G) < \lceil \frac{n}{2} \rceil - 2$, this bound is better than the lower bound in [1].

Note that there exist chordal (resp. quasi-line) graphs $G$ such that $b(G - \{x\})$ is strictly less than $b(G)$. For example, let $G_0$ be a chordal graph obtained from a chordal graph $H$ and a new vertex $x$ joined to every vertex of $H$. Then $b(G_0 - x) = b(G_0) - 1$.

**Theorem 4** If $G$ is a quasi-line graph then, for every vertex $x \in V(G)$, $b(G - \{x\}) \geq b(G) - 2$.

**Proof.** Let $x \in V(G)$ be a fixed vertex. $N(x)$ is covered by at most two cliques $K_1$ and $K_2$. Let $i$ be the color of $x$.

Considering a b-coloring of $G$, there is at most two vertices $u_k, u'_k \in N(x)$ with the same color $k$. Again, by the fact that $G$ is quasi-line, $N^2(x) \cap N(u_k)$ is a clique as it is independent from the neighbour $x$ of $u_k$. Analogously $N^2(x) \cap N(u'_k)$ is a clique.

We delete the vertex $x$. If in $G - \{x\}$ each color is dominating, then $b(G - \{x\}) \geq b(G)$. Let $i$ be the color of $x$ in $G$. We may suppose that $G - \{x\}$ has a color-dominating vertex of color $i$ otherwise we color each vertex of color $i$ by a missing color and we get a b-coloring of $G - \{x\}$ by $b(G) - 1$ colors.

We choose a color $s$ that is no more dominating, which means that there is no more color-dominating vertices of color $s$. Each vertex of color $s$ has a missing color.

If the color $s$ had more than one color-dominating vertex in $G$, then it had exactly two color-dominating vertices $w_s \in K_1$ and $w'_s \in K_2$. We recolor both of them by $i$. We then recolor each other vertex of color $s$ by a missing color. In this way we obtain a b-coloration of $G - \{x\}$, eliminating one color.
If in $G$, there was only one color-dominating vertex $w_s$ of color $s$ in $N(x)$, say in $K_1$, we recolor $w_s$ by $i$. We eliminate the color $s$ by coloring each vertex of $C_s$ by a missing color. If there is a color $t$ no more dominating we choose one, then the color-dominating vertex $w_t$ was necessarily in $K_2$. We color $w_t$ by $i$. We color any other vertex of $C_t$ by a missing color. Necessarily all the remaining colors are dominating. We conclude that $b(G - \{x\}) \geq b(G) - 2$.

Note that there exists a quasi-line graph $G$ such that $b(G - x) = b(G) - 1$ for at least a vertex $x$. We give an example. Let $\omega \geq 3$ be an integer, and let $p = 2\omega - 1$. Let $P = \{x_0, x_1, \ldots, x_p, x_{p+1}\}$ be a path. We consider the graph $G_1$ obtained by replacing each edge $[x_i, x_{i+1}]$ by a clique $K_i$ of order $\omega$. The graph $G_1$ is a claw-free graph; we have $b(G_1) = p$ and $b(G_1 - \{x\}) = b(G_1) - 1$.

3 Chordal graphs

We want to show the following result.

**Theorem 5** Let $G$ be a chordal graph and $x$ be a fixed vertex of $G$. Then $b(G - \{x\}) \geq b(G) - \omega_G$ where $\omega_G$ is the clique number of $G$.

We will need first the next lemma, about the adjacencies in chordal graphs.

**Lemma 1** Let $G$ be a chordal graph, and $a, x, b$, be three consecutive vertices of a cycle $\Gamma$ of $G$. Suppose that the vertex $x$ of $G$ has no neighbours in $\Gamma - \{a, b\}$. Then $a$ and $b$ are adjacent in $G$.

**Proof.** The proof is by contradiction. We suppose $a$ and $b$ independent. Suppose $\Gamma$ is a shortest cycle containing the path $axb$. If the length of $\Gamma$ is at least 4, then as $G$ is chordal, and by minimality of $\Gamma$, it contains a chord incident with $x$ whose second endvertex is distinct from $a$ and $b$, a contradiction.
In what follows, consider a $b$-coloring of $G$, with $b = b(G)$ colors. Let $x \in V(G)$ be a fixed vertex and let $i$ be the color of $x$. Let $I_a$ be the set of colors without color-dominating vertices in $G - \{x\}$ and let $J_a$ be their set of color-dominating vertices in $G$. We remark that $J_a \subset N(x)$ and no vertex in $J_a$ is neighbour of a vertex of color $i$ in $G - \{x\}$.

Before proving our main result, we introduce a necessary definition.

**Definition 1** Let $x'_i$ be a fixed color-dominating vertex of color $i$, different from $x$. Let $W = \{w \mid w \notin C_i, w$ color-dominating vertex in $G\}$. Let $k \leq b$ be a fixed integer. We denote by $W_k$ the set of color-dominating vertices of color $k$. A path $P$ of $G - \{x\}$ is said a pseudo-alternating path of $G - \{x\}$, and denoted by $P_k[x'_i, z]$, if it is a path of endvertices $x'_i$ and $z$, such that:

- $V(P_k) \subset C_i \cup C_k \cup W \cup \{z\}$
- each $w \in W \cap V(P_k)$, $w \neq x'_i$ and $w \neq z$, is preceded by a vertex of color $C_k$ (resp. of $C_i$) and succeeded by a vertex of $C_i$ (resp. of $C_k$).
- $V(P_k - \{z\}) \cap N(x) = \emptyset$.

A pseudo-alternating path $P[x'_i, z]$ is an alternating path if $z$ is neighbour of $x$ in $G$. We remark that if $z \notin C_i \cup C_k$, $z$ is necessarily preceded by a vertex of $C_i \cup C_k$ and, if $P$ is maximal, $z$ has neighbours in $C_i$ and $C_k$, belonging to $P$ or $z \in I_a$.

**Proof of the Theorem 5**

We consider a $b$-coloring of $G$, with $b(G)$ colors. Suppose $x \in C_i$. We may assume that:

There is no $b$-coloring of $G - \{x\}$ by $b(G) - 1$ colors (a) otherwise we have the inequality of the theorem. If $C_i - \{x\}$ contains no color-dominating vertex, we recolor each vertex of that set by a missing color in its neighborhood. We get a $b$-coloring of $G - \{x\}$ by $b(G) - 1$ colors, a contradiction with assumption (a).

We may suppose from now that $C_i - \{x\}$ has color-dominating vertices $x'_i, x''_i, \ldots, x'^r_i$. Let us take $k$ in $I_a$. We recolor each vertex of $C_k$ by a missing color. We eliminate color $k$. In this new coloring, if there is a color $k'$ with no color-dominating vertex, then $k' \in I_a$. We recolor $C_{k'}$ and we eliminate color $k'$. Repeating this process, we get finally a $b$-coloring by at least $b(G) - |I_a|$.
In view to establish the bound of the theorem, we want to bound $|I_a|$. The bound will be established by three claims.

We denote by $Q[y, v]$ any path of $G - \{x\}$ such that $V(Q) \cap N(x) = \{v\}$. Let $v^-$ be the neighbour of $v$ in that path. Let $x_i$ be a color-dominating vertex.

Let $F(x_i)$ be the set of neighbours $z$ of $x$ such that $z$ is extremity of an alternating path $P_k[x_i, z]$ and $k$ is the color of $z$. i.e., $F(x_i) = F_1(x_i) \cup F_2(x_i)$, where $F_1(x_i)$ and $F_2(x_i)$ are defined below:

$F_1(x_i) = \{z \in N(x)|z = z_k \text{ and in some } P_k[x_i, z_k], z_k^- \text{ in } C_i \}$

and

$F_2(x_i) = \{z \in N(x)|z = z_k \text{ and in some } P_k[x'_i, z_k], z_k^- \text{ in } W \} \setminus F_1(x_i)$.

Let $G$ be a component of $G - (\{x\} \cup N(x))$ Consider $X_i$ the set of color-dominating vertices of color $i$ contained in $G$.

Let $F'(x_i) = \{v \in N(x)|\text{there exists } Q[x_i, v]\}$. Let $F' = \{v \in N(x)|\text{there exists } x_i \text{ in } X_i \text{ and } Q[x_i, v]\}$. Note that for any $x_i$, we have $F'(x_i) = F' = N(x) \cap N(G)$. Let $F_1 = \cup \{F_1(x'_i), x'_i \in G\}$. It is a subset of $F'$.

By assumption (a) there exists a component $G$ for which there is no recoloring of $G$ by $\{1, ..., b_G\} - \{i\}$ such that all the colors of $W \cap (G \cup N(x))$ have a color-dominating vertex in $G - x$. From now we use such a component.

We get the following assertion as corollary of Lemma 1.

**Claim 1:** For vertex $x_i$ of $X_i$, $F'(x'_i)$ is a clique containing $F_1$.

Proof of claim 1: It is sufficient to note that $F'(x_i)$ is not empty, otherwise there is no alternating path, we choose $k \neq i$ and we exchange colors $k$ and $i$ in the pseudo-alternating paths. No color-dominating vertex loses a color. $X_i = \{x'_i, \ldots, x'_r\}$ is recolored by $k$. A contradiction with the definition of $G$. $F'(x_i)$ is a clique by Lemma 1.

At this moment we need to introduce another definition.

Let $P_s[x'_i, z_1]$ be an alternating path for $s \neq i, s \neq 1$, with $z_1 \in F_1(x'_i)$. An extension of $P_s$, denoted by $R_s[x'_i, z']$, is a path of the form $P_s[x'_i, z_1] \cup [z_1, y_i] \cup L(y_i, z')$, where

- $V(L) \subset C_i \cup C_s \cup W$; $(V(L) - \{z'\}) \cap N(x) \subset W$. 

• For each \( z \in V(L) - \{z'\} \),

• if \( z = z_k \in N(x) \), then \( z \in W \) a color-dominating vertex of color \( k \) and \( W_k \subset N(x) \); \( z_k \) is preceded in \( R_s \) by a vertex of \( C_s \), followed by a vertex \( y_i(k) \). If \( z \in F_i(x^i) \), then \( y_i(k) \in P_k(x^i, z_k) \)

• if \( z \in W - N(x) \) then \( z \) is preceded by a vertex of \( C_r \), followed by a vertex of \( C_r' \), where \( \{r, r'\} = \{i, s\} \)

• If \( z' \in W \), \( z' \) is preceded by a vertex of \( C_i \cup C_s \).

And now, we have two claims:

Let \( \mathcal{K}(A) \) be the set of colors which appear in the subset \( A \) of \( V \).

**Claim 2:** Let \( P_s[x_i', z'] \) be an alternating extended path. Then \( V(L) \cap N(x) \) is a subset of \( F_1 \). So if \( z' \in W \) then \( z' \notin J_a \). If \( s \notin \mathcal{K}(F') \), then \( z' \in W - J_a \).

Proof of claim 2:

Let \( z_1, z_2, ..., z_t, ..., z_p \) be the successive vertices of \( V(R_s) \cap N(x) \) and \( z' = z_p \). We show by induction on \( t \), that \( z_t \) is in \( F_1 \). Suppose that \( z_{t-1} \in F_1 \) for some \( t \). So there is a path \( P_{t-1}[x_i^t, y_i^{(t-1)}] \). Composing it with \( R_s(y_i^{(t-1)}, z_t) \), we get a path \( Q[x^{(t)}, z_t] \). If \( z_t \notin F_1 \), we do \( \tau(i, t) \) from \( x^t \) for any \( t ' \), where \( \tau(i, t) \) means exchanging the colors \( i \) and \( t \) in \( G \). No color-dominating vertex loses color \( t \) even if it is an extremity of an alternating path \( P_i \) in this later case it is neighbour of \( z_t \). Some color-dominating neighbours of \( N_t(x) \) may lose color \( i \). We recolor each \( v \in X_i \) by a missing color. No color-dominating vertex of color \( i \) is created by uniqueness of the color-dominating of color \( t \). We get a coloring by \( b(G) - 1 \) colors. A contradiction. So \( z_t \in F_1 \) and \( z_t \notin J_1 \).

If \( s \notin \mathcal{K}(F') \), then no vertex of \( C_s \) is in \( F_1 \). So \( z' \in W \). As \( F_1 \cap J_a = \emptyset \), then \( z' \in W - J_a \).

**Claim 3:** \( \mathcal{K}(F') \) contains \( I_a \)

Proof of claim 3:

It is by contradiction. We suppose that there is a color \( s \) such that \( s \in I_a \ \setminus \mathcal{K}(F') \). Thus no vertex of color \( s \) belongs to \( F_1 \).

**Case 1:** There is no path \( P_s[x_i^t, z] \) with \( x_i^t \in G \) and \( z \in N(x) \).

We do \( \tau(i, s) \) along the pseudo-alternating paths \( P_s[x_i^t, u] \), \( u \notin N(x) \) simul-
taneously. No color-dominating vertex loses color. We recolor the remaining vertices of $C_i$ in $G$ and this leads to a contradiction with the assumption.

Case 2: There exists $x_i^r$ in $G$ and a path $P_s[x_i^r, z_t]$ with $z_t \in N(x), x_i^r \in G$.

This case will be divided in two sub-cases:

Case 2.1: There exists $P_s[x_i^l, z_t]$ with $z_t / \in F^1$. By claim 1, $F'$ does not contain $z_t'$, with $z_t' \neq z_t$. We do $\tau(i, t)$ simultaneously in all alternating $P_t[x_i^l, z]$ with $x_i^l$ in $G$. As $z_t \in F'$ by Claim 1, the color-dominating vertices contained in $F'$ do not lose color $t$, they may lose color $i$. The color-dominating vertices which may lose a color are the color-dominating vertices preterminal in $P_t[x_i^l, z_t]$; these color-dominating vertices may lose color $i$. We recolor $C_i$ by missing colors in $G$.

Case 2.2: For any $P_s$, the extremity contained in $N(x)$ is in $F_1$. So this extremity does not belong to $J_a$.

If for any $x_i^l \in X_i$, there is no alternating path $P_s[x_i^l, z]$ with color $s$ preceding $z$, then we do $\tau(i, s)$ in all the pseudo-alternating paths and the paths $P_s$. We have the same conclusion as in case 2.1.

If for some $l, z$ is preceded by a vertex of color $s$ in $P_s[x_i^l, z]$ we consider the extended paths $R_s$. We do $\tau(i, s)$ simultaneously in all alternating and pseudo-alternating paths $P_s$ and the extended paths $R_s$. The color-dominating vertices which may lose a color are either in $N(x)$ or in $N^2(x)$, they are among terminal vertices and preterminal vertices of the alternating paths $P_s$ and $R_s$; and they may lose color $i$. We then recolor each remaining vertex of $C_i \cap G$ by a missing color.

In each case we have a contradiction with the definition of the component $G$. So $I_a \subset K(F')$.

By claim 1 and claim 3 there is a clique containing $x$ and $F'$. So we have $\omega(G) \geq |F'| + 1 \geq |I_a| + 1$, and this finishes the proof of the theorem.

\[\blacksquare\]

4 Graphs with large girth

The $m$-degree of a graph $G$, denoted by $m(G)$, is the largest integer $m$ such that $G$ has $m$ vertices of degree at least $m - 1$. It is known that, for any graph $G$, $b(G) \leq m(G)$ (see [6]).
Note that A. Campos et al. [4] have shown that graphs of girth at least 7 have high b-chromatic number; for each such a graph $G$ this number is at least $m(G) - 1$.

We can verify that $m(G - \{x\}) \geq m(G) - 1$. Indeed, we have three possibilities to consider: $x$ is one of the $m$ vertices of degree $m - 1$; $x$ is a neighbor of one of the $m$ vertices of degree $m - 1$, or $x$ is not in any of the previous situations. In the first case, there remain $m - 1$ vertices of degree at least $m - 2$ and, thus $m(G - \{x\}) \geq m(G) - 1$. In the second case, there remain $m$ vertices of degree at least $m - 2$, and again, $m(G - \{x\}) \geq m(G) - 1$. In the latter case the m-degree does not change, that is, $m(G - \{x\}) = m(G)$.

So $b(G - \{x\}) \geq b(G) - 2$ for graphs of girth at least 7. No particular bound is known for graphs of girth 5 or 6. In this work we show the following.

**Theorem 6** Let $G$ be a graph of girth at least 5. For each vertex $x$, $b(G - \{x\}) \geq b(G) - 2$

**Proof.** Let $B$ be a b-coloring of $G$ and let $i$ be the color of the deleted vertex $x$. Let $W$ be the set of color-dominating vertices of colors different from $i$ in $G$. Let $W_k$ be the subset of those of color $k$.

We may suppose that there is a set of color-dominating vertices $X_i$ of color $i$, different from $x$. For each vertex $u$ let $K_1(u)$ be the set of colors with at least a color-dominating vertex in $N(u)$, let us set $K_2(u) = \{1, 2, ..., b\} - (K_1(u) \cup \{i\})$.

We use the notations of the previous section. We may suppose $|I_1| \geq 3$. Note that for $u \neq x$, as the girth is at least 5, $K_1(u)$ does not contain $I_1$. So $K_2(u)$ is not empty as it intersects $I_1$. By definition of $I_1$, the color $i$ is a missing color for each vertex of $J_1$ in $G - \{x\}$. So for any $x' \in X_i$, $K_1(x')$ does not intersect $I_1$.

(a) Let $x' \in X_i$. Let $k \in K_2(x')$ be fixed.

$$(P_a)$$ If $G - N(N_k(x'))$ intersects $W_p$ for each color $p$ different from $i$, we color $x'$ by $k$, each vertex of $N_k(x')$ by a missing color. So $|X_i|$ decreases. The color-dominating vertices of $K_1(x')$ may lose color $i$ in their neighborhood.

(b) As long as there exists a vertex $x''$ of $X_i$ satisfying $(P_a)$ we do a re-coloring.

From now we may suppose that $X_i$ is not empty and no vertex of $X_i$ satisfies (a).
Lemma 2 Let $x'$ be a fixed vertex of $X_i$. For each $k \in K_2(x')$ there exists exactly one $j_k$ such that $W_{j_k}$ is contained in $N(N_k(x'))$ and $N(N_k(x')) \cap W_j = \emptyset$ for any other $j$.

Proof. We know that the property (a) is not satisfied. As $g(G) \geq 5$, if a vertex $w_j$ is in $N(N_k(x'))$ then $w_j$ is not neighbour of $N_r(x')$ for $r$ different from $k$. As (a) is not satisfied, it follows that for each $k \in K_2(x')$, $N_k(x')$ is neighbour of any vertex of a set $W_j$ for some $j$. As $g(G) \geq 5$, $j$ is in $K_2(x')$ and $W_j$ is not neighbour of $N_t(x')$ for $t \neq k$. It follows that for $k$ fixed, $j$ is unique. This finishes the proof of the lemma.

Let $s$ be a fixed element of $I_1$. Let $x' \in X_i$. We know that the set $I_1$, by definition, is a subset of $K_2(x')$. By the precedent Lemma there is exactly one $k$ such that $W_s \subset N(N_k(x'))$. We color each vertex of $N_k(x')$ by a missing color and $x'$ by $k$. If $N_k(x')$ meets $N_k(x'')$ for some $x'' \in X_i$, by the precedent Lemma, we have $W_s \subset N(N_k(x''))$; we color $x''$ by $k$ as well and we recolor $N_k(x'')$ by missing colors different from $i$. We recolor so each vertex of $X_i$. Then we recolor each vertex of color $i$ by a missing color. If, finally, the color $s$ has no vertex dominating the colors $\{1, ..., b\} - \{i\}$, we recolor each vertex $u_s$ by a missing color different from $i$.

After this recoloring of color $i$ and eventually color $s$, we get a $b$-coloring of $G$ by at least $b(G) - 2$ colors.

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