Abstract

In the context of the Anderson theory of high $T_c$ cuprates, we develop a BCS theory for Luttinger liquids. If the Luttinger interaction is much stronger than the BCS potential we find that the BCS equation is quite modified compared to usual BCS equation for Fermi liquids. In particular $T_c$ predicted by the BCS equation for Luttinger liquids is quite higher than the usual $T_c$ for Fermi liquids.

1 The anomalous BCS equation

The equivalent of BCS theory for a Luttinger liquid has not formally worked out, despite the relevance of such theory for the problem of superconductivity in the high-$T_c$ cuprates, see [1] or the discussion in the following section. Such theory should describe superconductors which in their normal state are Luttinger liquids. In this paper we develop such theory using constructive quantum field theory techniques, applied in many other Luttinger liquid problems, see [2], [3]. In particular, we compute in a rigorous way the BCS self-consistence equation for a spinning Luttinger liquid (the Mattis model, see [4]) coupled with a BCS anomalous potential. We stress that it should be difficult to find similar results using different techniques: in fact the Mattis model plus a BCS interaction is not exactly solvable, and bosonization or conformal quantum field theory cannot be used to compute the correlation functions of theories with gap like the present one (see for instance [5]).

We will assume\footnote{See [1] pag.209} that an anomalous self energy has been introduced by some external influence in a Luttinger liquid so that the model is described by the following hamiltonian

$$ H = H_a + H_{BCS} $$

where $H_a$ is the Mattis model hamiltonian

$$ 2\pi \sum \frac{k}{L} (\omega k - p_F) \psi_{+}^{k,\omega,\sigma} \psi_{-}^{k,\omega,\sigma} - \lambda \sum \frac{2\pi}{L} k \sum \langle \omega k - p \rangle : \psi_{+}^{k_1,\omega,\sigma} \psi_{-}^{k_1,\omega,\sigma} : $$

$$ + \Delta \sum \langle \omega k - p \rangle : \psi_{+}^{k_2,\omega,\sigma} \psi_{-}^{k_2,\omega,\sigma} : $$

$$ \sum \langle \omega k - p \rangle : \psi_{+}^{k_1,\omega,\sigma} \psi_{-}^{k_1,\omega,\sigma} $$

$$ \sum \langle \omega k - p \rangle : \psi_{+}^{k_2,\omega,\sigma} \psi_{-}^{k_2,\omega,\sigma} $$

We assume that $\lambda > 0$. The BCS interaction is described as usual, if $g > 0$

$$ H_{BCS} = -\Delta \sum \frac{2\pi}{L} \sum_{k,\omega} \psi_{+}^{k,\omega,\sigma} \psi_{-}^{k,\omega,\sigma} - \Delta \sum \langle \omega k - p \rangle : \psi_{+}^{k_1,\omega,\sigma} \psi_{-}^{k_1,\omega,\sigma} : $$

$$ \sum \langle \omega k - p \rangle : \psi_{+}^{k_2,\omega,\sigma} \psi_{-}^{k_2,\omega,\sigma} : $$

$$ \sum \langle \omega k - p \rangle : \psi_{+}^{k_1,\omega,\sigma} \psi_{-}^{k_1,\omega,\sigma} $$

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$$ \sum \langle \omega k - p \rangle : \psi_{+}^{k_2,\omega,\sigma} \psi_{-}^{k_2,\omega,\sigma} $$

where $\psi_{+}^{k,\omega,\sigma}$ are fermionic creation or annihilation operators with momentum $k = 2\pi n/L$, $-L/2 \leq n \leq L/2$, quasi-particle index $\omega = \pm 1$, spin $\sigma = \pm \frac{1}{2}$, $|v(p)| \leq e^{-\kappa |p|}$.

We assume that $\Delta \in R$ and $\Delta \geq 0$. Note that if $\Delta = 0$ the model is exactly solvable [4]; the system is a Luttinger liquid (it is perhaps the simplest spinning model showing Luttinger liquid behaviour) and shows spin-charge separation.

The ground state energy $E_0(\Delta)$ depends on $\Delta$: the BCS equation is the extremizing equation $\frac{\partial E_0(\Delta)}{\partial \Delta} = 0$ which has the form (see [6] for the proof of a similar statement):

$$ \Delta = g \sum \frac{2\pi}{L} \sum_{k,\omega} \langle \psi_{+}^{k,\omega,\frac{1}{2}} \psi_{+}^{k,\omega,-\frac{1}{2}} \psi_{-}^{k,\omega,-\frac{1}{2}} \psi_{-}^{k,\omega,\frac{1}{2}} \rangle $$

We are considering the Mattis model on a lattice, while the original model introduced and solved in [4] is defined on the continuum.
If $\lambda = 0$ the r.h.s of eq. (4) can be calculated very easily, as the Hamiltonian can be put in diagonal form by performing a Bogolubov transformation. In the interacting case an exact solution is not available and we compute the r.h.s of eq. (4) writing it as a convergent series for $\Delta, \lambda$ suitably small i.e. $\Delta, \lambda < \epsilon$, if $\epsilon$ is a suitable number $<< 1$ (but no restriction is imposed on their ratio).

Note that the restriction on the smallness of $\Delta, \lambda$ is done only for technical reasons in order to ensure convergence of the perturbative series. However it is very likely that our bound for the convergence radius are far from to be optimal, and instead our results hold also for $\lambda = O(1)$; if $\Delta = 0$ we know that this is true directly from the exact solution [4].

Our main result is

$$\Delta = 0$$

The role of the lattice should play no role.

Note also the crucial role of the sign of the Luttinger interaction.

2 A model for the high-$T_c$ cuprates

In the previous section we have written a BCS equation for a Luttinger liquid and we have seen that, if the Luttinger interaction is much stronger than the BCS interaction, there is a large deviation from the Fermi liquid BCS equation and in particular the gap is much larger. Can this result be applied to the physics of high-$T_c$ cuprates?

The Anderson theory of high-$T_c$ cuprates is based on the following points:

1) the conduction electrons are confined on layers;
2) the interaction between electrons in the same layer is much stronger than the interlayer interaction;
3) the normal state of the electrons on the layers is a Luttinger liquid;
4) the interlayer pairing allows Cooper pairs to tunneling into an adjacent layer by the Josephson mechanism.

The interlayer interaction is assumed to be given by the Josephson pair tunneling Hamiltonian and some physical arguments for motivating this choice and for not considering single particle tunneling are given. The model is then studied by the usual mean field BCS approximation. However by point 3) one has to take into account in the resulting BCS equation the Luttinger liquid nature of the fermions in their normal state. This is a big problem as no theory showing the Luttinger liquid behaviour of bidimensional strongly interacting fermions exists, so that the form of a Luttinger liquid BCS equation is essentially guessed by replacing in the usual BCS the Fermi propagator with the one dimensional Luttinger model propagator.

As the above theory is a mean field theory, its results should be, as usual, independent from the dimensions; this

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4See for instance pag.46-55 of [1]
5See also pag. 308 of [1]
6See for instance eq. (7) pag 321 of [1], or eq.(9) (in $d = 2$) below
7See for instance the considerations based on the holon-spinon coherence at pag.50 of [1] or in the Clarke reprint [o]
8See eq. (16) pag.218 of [1] or the considerations after eq.(10) below
9In [1] this is essentially derived from experimental results
10See pag.213 eq.(12) of [1]
means that we can expect that coupling \( d = 1 \) or \( d = 2 \) Luttinger liquids and performing a BCS approximation on the pair tunnelling interaction, the results are qualitatively the same. In other words we expect that the predictions of a BCS theory for a superconductor whose normal state is a Luttinger liquid are qualitatively the same whatever its dimensions are. This is just what happens for the usual superconductors whose normal state is a Fermi liquid; the BCS approximation leads for instance to a gap or a critical temperature exponentially small in \( g \) in any dimension. The crucial and very non trivial assumption is that there exist \( d = 2 \) fermions with a Luttinger liquid behaviour in their normal state; but, if they exist, the results of a BCS theory of coupled Luttinger liquids should be essentially independent on their dimensions.

Considering one dimensional coupled Luttinger liquids, the BCS equation can computed in a rigorous way. The Luttinger liquids can be described by the Mattis model hamiltonian eq.(2), giving an extra chain index to the Fermi operators \( i = a, b \). The two Mattis hamiltonians will be called \( H_a \) and \( H_b \). The pair-hopping hamiltonian \( H_{\text{int}} \) is, following [1],

\[
H_{\text{int}} = -g\left(\frac{2\pi}{L} \sum_{k,\omega} \psi_{k,\omega}^+ a \psi_{-k,\omega,-\frac{1}{2},a}^+ \right) \tag{9}
\]

The total hamiltonian is \( H_a + H_b + H_{\text{int}} \). Let we write in eq.(3)

\[
\frac{2\pi}{L} \sum_{k,\omega} \psi_{k,\omega}^+ \psi_{-k,\omega,-\frac{1}{2},i}^+ \psi_{-k,\omega,-\frac{1}{2},i}^< + \psi_{k,\omega}^+ \psi_{-k,\omega,-\frac{1}{2},i}^< \psi_{-k,\omega,-\frac{1}{2},i}^+ \tag{10}
\]

The BCS approximation consists in replacing \( H_{\text{int}} \) with \( H_{\text{int}}^{\text{app}} \) obtained neglecting the terms bilinear in the "fluctuations" i.e. in the second addend in eq.(10). We obtain

\[
-H_{\text{int}}^{\text{app}} = -\frac{\Delta}{g} \sum_{k,\omega} \psi_{k,\omega}^+ \psi_{-k,\omega,-\frac{1}{2},a}^+ + \text{c.c.}
\]

and self-consistency requires eq.(10). Replacing \( H_{\text{int}} \) with \( H_{\text{int}}^{\text{app}} \) has the effect that the model is described by two independent hamiltonians, \( H = H_a + H_b \), each one given by eq.(1).

The BCS equation for coupled Luttinger liquids is then given by eq.(5). In the range of parameters physically reasonable (see point 2) above) i.e. if \( \lambda >> g \) and noting that, as usual, \( T_c \) is proportional to \( \Delta \) (see below) we find that \( T_c \approx \beta e^{-\frac{\beta}{\pi}}\log\left[\frac{\Delta}{\pi}\right] \) i.e. much higher than \( T_c \) for normal superconductor \( T_c \approx \beta e^{-\frac{1}{4}} \).

In order to compare our result with [1], note that the r.h.s. of eq.(4) computed[13] in [1] is similar to our eq.(22) below (see especially the presence of the wave function renormalization) but \( \sigma_h \approx \Delta e^{-\beta(\beta+\cdots)} \) is replaced by \( \Delta \) i.e. the anomalous flow of the BCS gap is neglected. Such effect is on the other hand crucial for our analysis: all the conclusion drawn from eq.(5) in the preceding section are based on the fact that \( \beta_1 \neq 0 \). In fact it is this anomalous enlarging of the gap due to the Luttinger interaction which produces a much larger solution of the BCS equation. On the other hand the gap renormalization in Luttinger liquids is a well known fact, see for instance the gap of the XYZ model, [9].

Finally we stress that one can try to study directly a model of coupled chains, see [7], without any approximation, and it could happen that the model is not really described by our BCS approximation. This has not real relevance for our analysis, as our aim was just to find a BCS equation for coupled system of fermions on planes with a Luttinger liquid behaviour, and we make a BCS computation in \( d = 1 \) using the fact that a mean field theory like BCS should be insensitive to the dimensions. So it could be that our BCS equation eq.(5) could be applied to coupled planes and not chains, and on the other hand a detailed analysis of the model given of eq.(10) could give no insight on the problem of coupled planes as the behavior of the system depend on the dimensions. This is in fact just what happens in the usual BCS theory: the

\[\text{See eq.(7) pag.320}\]
\[\text{See eq.(16) pag.218 of [1]}\]
\[\text{The above computation follows the Anderson gap derivation, see chap. 7 in [1], but for simplicity we have neglected the electron phonon interaction, see eq.(21) pag. 219 of [1]}\]
\[\text{13 see eq.(12) pag.213}\]
BCS equation is qualitatively the same in any dimension, but in $d = 1$ the mean field approximation is not correct. The real question is if really bidimensional Luttinger exists, but this question is not addressed here and we refer to [1].

### 3 Renormalization group analysis

We discuss now how to compute the r.h.s. of eq. (4). We introduce as usual a set of Grassman variables $\psi_{k,\omega,\sigma}^\pm$, $\vec{k} = (k_0, k)$ and $k = \frac{2\pi n_0}{L}$, $k_0 = \frac{2\pi(n_0+2^{-1})}{\beta}$, if $n_0, n_1$ are integers and $-\frac{L}{2} \leq n_1 \leq (L-1)/2$, $-\frac{\beta}{2} \leq n_2 \leq \frac{\beta-1}{2}$, if $1/\beta$ is the temperature. The Grassmanian integration $P(d\psi)$ is defined by the anticommutative Wick rule with propagator

$$g_\omega(\vec{k}) = \frac{1}{-ik_0 + \omega k - p_F}$$

In general we denote by $\int \{ D\psi e^{-\int d\psi^+ h(\vec{k})^{-1} \psi^+} \}$ the Grassmanian integration with propagator $h(\vec{k})$, where $\int d\vec{k} = \frac{(2\pi)^3}{\beta} \sum \vec{k}$; in particular

$$P(d\psi) = \left\{ D\psi \prod_{\vec{k},\omega,\sigma} e^{-\int d\psi^+_{\vec{k},\omega,\sigma} (-ik_0 + \omega k - p_F) \psi^+_{\vec{k},\omega,\sigma}} \right\}$$

Then we write the r.h.s. side of eq. (4) as a functional integral

$$\int P(d\psi) e^{-V(\psi)} = \frac{1}{\int P(d\psi) e^{-V(\psi)}} \int P(d\psi) e^{-V(\psi)} \left( \psi_{-1,1,\omega}^+ \psi_{1,-1,\omega}^+ \right)$$

where $V(\psi) = \lambda \vec{V} + \Delta P$ and

$$P = \sum_{\omega} \int d\vec{k} \left( \psi_{\vec{k},\omega,\sigma}^+ \psi_{\vec{k},\omega,\sigma}^- + \psi_{\vec{k},\omega,\sigma'}^+ \psi_{\vec{k},\omega,\sigma'}^- \right)$$

$$V = \int d\vec{k} \beta \left( \sum_{\omega,\sigma,\sigma'} \epsilon_{i}(\vec{k}_i) \psi_{\vec{k}_i,\omega,\sigma}^+ \psi_{\vec{k}_i,\omega,\sigma'}^- \right)$$

We evaluate the above Grassman integral using Wilsonian renormalization group techniques. It is convenient to write $k = k' + \omega p_F$, where $k'$ is the momentum measured from the Fermi surface.

We decompose the integration $P(d\psi)$ into a product of independent integrations. This can be done writing

$$g_\omega(\vec{k} + \omega p_F) = \sum_{h=0}^{\infty} g_h^0(\vec{k} + \omega p_F) + g_h^1(\vec{k} + \omega p_F)$$

with

$$g_h^0(\vec{k} + \omega p_F) = \frac{f_h(\vec{k})}{-ik_0 + \omega k'}$$

and $f_1(\vec{k}) = 1 - \chi(\vec{k})$, $\chi(\vec{k}) \equiv \chi(|\vec{k}|)$ is a smooth compact support function such that $\chi(|\vec{k}|) = 1$ for $|\vec{k}| \leq \gamma^{-1}$ and $\chi(|\vec{k}|) = 0$ for $|\vec{k}| \geq 1$, if $\gamma > 1$; moreover for $h \leq 0$

$$f_h(\vec{k}) = \chi(\gamma^{-h} |\vec{k}|) - \chi(\gamma^{-h+1} |\vec{k}|)$$

is a smooth compact support function non vanishing only for $\gamma^{-h-2} \leq |\vec{k}| \leq \gamma^{-h}$. Then $g_h^0(\vec{k} + \omega p_F)$ is the ultraviolet part of the propagator, while $\sum_{h=0}^{\infty} g_h^1(\vec{k})$ is the infrared part. Note that, from the compact support properties of $g_h$, the sum in eq.(12) is from 0 to $h_\beta$, where $\gamma^{h_\beta} = \frac{\beta}{\gamma}$, as $|k_0| > \frac{\beta}{\gamma}$. Let be $\sum_{h=0}^{h_\beta} f_k$. The ultraviolet integration is somehow special (and essentially trivial for the presence of the lattice) and we will not discuss it here, see [2]. If $\lambda = 0$ the infrared integration can be done by performing the well known Bogolubov transformation to diagonalize the BCS hamiltonian. If $\lambda \neq 0$ the BCS gap and the wave function renormalization have a non trivial RG flow so that we have to perform a different Bogolubov transformation at each iteration of the RG. We set $Z_0 = 1$: once the fields $\psi(0), \ldots, \psi(1)$ have been integrated we have:

$$\int \{ D\psi^{(\leq h)} \prod_{\omega=\pm 1} e^{-\int d\vec{k} C_{h_\beta} \psi_{h_\beta}^{(\leq h)} + \psi_{h}^{(\leq h)}(\vec{k})^{-1} \psi_{h}^{(\leq h)}(\vec{k})} \}$$

$$e^{-V_h(\sqrt{Z_{h_\beta}} \psi^{(\leq h)})}$$

if

$$\psi_{k,\omega}^{(\leq h)} = (\psi_{k,\omega+\omega p_F,\sigma}^{(\leq h)})$$

and $G^{(h)}(\vec{k})^{-1}$ is defined by

$$\begin{pmatrix}
-ik_0 + \omega k' & \sigma_h(\vec{k}') \\
\sigma_h(\vec{k}) & -ik_0 - \omega k'
\end{pmatrix}$$

$V^h$ is called the effective potential at scale $h$ and is given by

$$V^h(\psi^{(\leq h)}) = \sum_{n=2}^{\infty} \int \prod_{i=1}^{n} d\vec{k}_i$$

(14)
\[ W_n^h(\vec{k}_1, \ldots, \vec{k}_n) \prod_{i=1}^n \psi^{(\leq h)}_{k_i,\omega,\sigma_i} \delta(\sum_{i=1}^n \epsilon_i(\vec{k}_i' + \omega_i p_F)) \]

We define a localization operator \( \mathcal{L} \) extracting the relevant or marginal part of the effective potential \( V^h \):

i) If \( n > 4 \) then \( \mathcal{L} W_n^h = 0 \);

ii) Let be \( n = 4 \). In this case \( \mathcal{L} W_4^h = 0 \) unless \( \sum_{i=1}^4 \epsilon_i \omega_i p_F = 0 \), \( \sum_{i=1}^4 \epsilon_i = 0 \) in which case the action is non trivial and it is given by

\[
\mathcal{L} W_4^h(\vec{k}_1 + \omega_1 p_F, \ldots, \omega_4 p_F) = 0
\]

iii) if \( n = 2 \) then if \( \sum_{i=1}^2 \epsilon_i = 0 \)

\[
\mathcal{L} W_2^h(\vec{k}_1 + \omega_1 p_F, \vec{k}_2 + \omega_2 p_F) = | W_2^h(\omega_1 p_F, \omega_2 p_F) + \omega_1 E(k' + \omega_1 p_F) \partial_{\vec{k}_1} W_2^h(\omega_1 p_F, \omega_2 p_F) + k^0 \partial_{\vec{k}_2} W_2^h(\omega_1 p_F, \omega_2 p_F) |
\]

while if \( \sum_{i=1}^2 \epsilon_i \neq 0 \) then

\[
\mathcal{L} W_2^h(\vec{k}_1 + \omega_1 p_F, \vec{k}_2 + \omega_2 p_F) = W_2^h(\omega_1 p_F, \omega_2 p_F)
\]

We can write then the relevant part of the effective potential as:

\[
\mathcal{L} V^h = \gamma^h n_h F_{\nu'}^h + s_h F^h + s_h F_{\rho}^h + a_h F_{\alpha}^h + g_{2,h} F^4_2 + g_{4,h} F^4_4
\]

where

\[
F_\nu^h = \sum_{\omega} \int \frac{d\vec{k}'}{4} f_{\nu'}(\vec{k}' + \omega p_F, \omega, \sigma)^{-} \psi^{(h)}_{\nu',\omega,\sigma}^{-} \psi^{(h)}_{\nu,\omega,\sigma}^{-}
\]

\[
F_\rho^h = \sum_{\omega} \int \frac{d\vec{k}'}{4} f_{\rho'}(\vec{k}' + \omega p_F, \omega, \sigma)^{-} \psi^{(h)}_{\rho',\omega,\sigma}^{-} \psi^{(h)}_{\rho,\omega,\sigma}^{-}
\]

\[
F_2^h = \int \frac{4}{4} \frac{d\vec{k}_1}{4} \delta(\sum_{i=1}^4 \vec{k}_i) \sum_{\omega,\sigma,\sigma'} \left[ \psi^{(h)}_{\nu',\omega,\sigma'} \psi^{(h)}_{\rho,\omega,\sigma} \psi^{(h)}_{\rho',\omega,\sigma'} \psi^{(h)}_{\nu,\omega,\sigma} \right]
\]

\[
F_4^h = \int \frac{4}{4} \frac{d\vec{k}_1}{4} \delta(\sum_{i=1}^4 \vec{k}_i) \sum_{\omega,\sigma,\sigma'} \left[ \psi^{(h)}_{\nu',\omega,\sigma'} \psi^{(h)}_{\rho,\omega,\sigma} \psi^{(h)}_{\rho',\omega,\sigma'} \psi^{(h)}_{\nu,\omega,\sigma} \right]
\]

where \( i = \nu, \alpha, \xi, f_\nu = 1, f_\omega = \omega k', f_\xi = -i k_0 \). Moreover \( g_{2,0} = \tilde{v}(0) \Lambda + O(\Lambda^2), |g_{4,0}| \leq C \Lambda^2, s_0 = \Delta + O(\Delta \Lambda), a_0, z_0 = O(\Lambda), n_0 = O(\Lambda) \). We write eq.(13) as:

\[
\int \mathcal{D}\psi^{(h)}_{\nu'} e^{\int \frac{d^2 k'}{4} \psi^{(h)}_{\nu'}(\vec{k}') - \frac{1}{2} \overline{\psi^{(h)}_{\nu'}}(\vec{k}') \mathcal{L} \psi^{(h)}_{\nu'}} \int \mathcal{D}\psi^{(h)}_{\nu} e^{\int \frac{d^2 k}{4} \psi^{(h)}_{\nu}(\vec{k}) - \frac{1}{2} \overline{\psi^{(h)}_{\nu}}(\vec{k}) \mathcal{L} \psi^{(h)}_{\nu}} e^{-V^h(\sqrt{Z_{h-1}}\psi^{(h)})}
\]

where \( G^{(h-1)}(\vec{k}')^{-1} \) is defined as in eq.(13), with \( h - 1 \) instead of \( h \), \( \tilde{V}^h = \mathcal{L} \tilde{V}^h + (1 - \mathcal{L}) \tilde{V}^h \)

\[
\mathcal{L} \tilde{V}^h = \gamma^h n_h F_{\nu'}^h + (a_h - z_h) F_{\alpha}^h + g_{2,h} F_{\omega}^4 + g_{4,h} F_{\omega}^4
\]

and

\[
Z_{h-1}(k') = \frac{Z_{h-1}(k') + Z_{h} C_{h}^{-1} Z_{h} z_{h}}{Z_{h} C_{h}^{-1} z_{h}}
\]

This means that we extract from the effective potential the terms leading to a mass and wave function renormalization. Now one can perform the integration respect to \( \psi^{(h)} \), rescaling the effective potential \( \tilde{V}^h(\psi) = V^h(\sqrt{Z_{h-1}} \psi) \) and

\[
\mathcal{L} \tilde{V}^h = \gamma^h (\varepsilon_{\nu'}^h + \delta_h F^h + \lambda_2 F_{\omega}^h + \lambda_4 F_{\omega}^4)
\]

with \( \varepsilon_{\nu'} = \frac{Z_{h-1}}{Z_{h-1}} \varepsilon_{\nu} \), \( \delta_h = \frac{Z_{h-1}}{Z_{h-1}} (a_h - z_h) \), \( \lambda_2 = (\frac{Z_{h-1}}{Z_{h-1}})^2 g_{2,h} \) and \( \lambda_4 = \frac{Z_{h-1}}{Z_{h-1}} \). We can rewrite eq.(18) as:

\[
\int \mathcal{D}\psi^{(h)}(\vec{k}') \int \frac{d^2 k'}{4} \psi^{(h)}(\vec{k}') \mathcal{L} \psi^{(h)}(\vec{k}') \int \mathcal{D}\psi^{(h)}(\vec{k}) \int \frac{d^2 k}{4} \psi^{(h)}(\vec{k}) \mathcal{L} \psi^{(h)}(\vec{k}) e^{-\tilde{V}^h(\sqrt{Z_{h-1}} \psi^{(h)})}
\]

and the integration of \( \psi^{(h)} \) has propagator

\[
\tilde{g}_{\omega, \omega'}(x - y) = \frac{1}{Z_{h-1}} \int \frac{d^2 k'}{4} \psi^{(h)}(x') \tilde{f}_{\omega}^{(h)}(\vec{k}') \psi^{(h)}(\vec{k}') \psi^{(h)}(\omega')
\]

with \( G^{(h-1)}(\vec{k}') \) given by

\[
\tilde{f}_{\omega}^{(h)} = \frac{1}{A_{\omega}^h} \left( \begin{array}{cc} -i k_0 + \omega k' & -\sigma_{h-1}(k') \\ -\sigma_{h-1}(k') & (-i k_0 - \omega k') \end{array} \right)
\]

where \( A_{\omega} = -k_0^2 - k'^2 - \sigma_{h-1}(k')^2 \) and \( Z_{h-1} = \sigma_{h-1}(0) \).

We then replace \( h \) by \( \tilde{h} \). The result of this integration is in the same form as eq.(13) with \( h \) replaced by \( \tilde{h} \) and we can iterate.

Let us explain the main motivations of the integration procedure discussed above. In a renormalization group approach one has to identify the relevant, marginal and irrelevant interactions. By a power counting argument
one sees that the terms bilinear in the fields are relevant and the quartic terms (or the bilinear ones with a derivative respect to some coordinate acting on the fields) are marginal. However there are too many kinds of marginal terms, depending on the labels $\omega_i$ and $\varepsilon_i$ on each fields, so that their renormalization group flow seems impossible to study. However (see [2], [3] for a similar procedure) the power counting can be improved and many marginal terms are indeed irrelevant; in particular all the marginal terms with four or two fields with $\sum_i \varepsilon_i \neq 0$ are indeed irrelevant. The reason is that such terms are generated contracting at least a non diagonal propagator and such propagators are smaller than the diagonal ones by a factor $\sigma_h \gamma^{-h}$; see eq. (17) below; this will be sufficient for improving the power counting. (see the last paper in [3] for the proof of a similar statement in the XYZ chain). Moreover also the marginal terms with $\sum_i \varepsilon_i \omega_i p_F \neq 0$ are irrelevant, by momentum conservation considerations. In fact if $\sum_i \varepsilon_i \omega_i p_F \neq 0$ then the momenta of the fermions cannot be all close to the Fermi surface; mathematically this means that, for the compact support properties of the propagators, there is an $\hbar$ such that all scattering process involving fermions such that $\sum_i \varepsilon_i \omega_i p_F \neq 0$ with scale $h \leq \hbar$ are vanishing.

The relevant terms are of two kinds: the $\nu$ terms, reflecting the renormalization of the Fermi momentum, and the $\sigma$ terms, related to the presence of a gap in the spectrum. The presence of the $\nu$ terms is due to the renormalization of the chemical potential, and in general one introduces a counterterm in the hamiltonian to fix the Fermi momentum, see [2]. In this case however there is no necessity of adding this counterterm; roughly speaking, $\mu$ can vary in the gap whithout changing the Fermi momentum i.e. the position of the singularity of the propagator. This is a crucial point: if one had to put a $\Delta$-dependent counterterm in the hamiltonian, then considering $\frac{\partial E_0(\Delta)}{\partial \Delta}$ one would be forced to derive also such counterterm, and a much more complex BCS equation would appear. Regarding the other relevant term, they are related to the BCS gap generation. However due to the interaction the BCS gap has a non trivial flow, so that one has to perform different Bogoliubov transformations at each integration.

Regarding the marginal terms, there is an anomalous wave function renormalization which one has to take into account, what is expected as if $\Delta = 0$ the theory is a Luttinger liquid. In general the flow of the marginal terms can be controlled using some cancellations due to the fact that the Beta function is "close" (for small $u$) to the Mattis model Beta function. In eq. (20) we write the propagator as a Mattis model propagator plus a remainder, so that the Beta function is equal to the Mattis model Beta function plus a "remainder" which is small if $\sigma_h \gamma^{-h}$ is small.

Let be $h^* = \inf_h \{ \gamma^h \geq |\sigma_h| \}$. Note that, if $h^*$ is finite uniformly in $L, \beta$ so that $|\sigma_{h^* - 1}| \gamma^{-h^* + 1} \geq 1$ one has

$$|g^{<h^*}(\vec{x})| \leq \frac{1}{Z_h} \frac{C_M \gamma^{h^*}}{1 + (\gamma^h |\vec{x}|)^M}$$

Moreover if $h \geq h^*$ we have

$$|g^{h}_{\omega,\omega}(\vec{x})| \leq \frac{1}{Z_h} \frac{C_M \gamma^{h}}{1 + (\gamma^h |\vec{x}|)^M}$$

and

$$|g^{h}_{\omega,-\omega}(\vec{x})| \leq \frac{1}{Z_h} \frac{|\sigma_h| \gamma^{h}}{1 + (\gamma^h |\vec{x}|)^M}$$

Moreover for $h \geq h^*$ the bound for the non diagonal propagator has a factor more $\frac{|\sigma_h|}{\gamma^h}$ with respect to the diagonal propagator. This is the reason for which the quartic terms with $\sum_i \varepsilon_i \neq 0$ are irrelevant, despite dimensionally marginal. Finally

$$g^{h}_{\omega,\omega}(x - y) = g^{h}_{\omega,L}(x - y) + C^h_{2,\omega}(x - y) \quad \text{(20)}$$

with

$$g^{h}_{\omega,L}(x - y) = \int d\vec{k} e^{i \vec{k} \cdot \vec{x}} \frac{e^{i \vec{k} \cdot \vec{y}}}{Z_h} \frac{f_a(v)}{-\epsilon_{0h} + \omega k^2}$$

which is just the propagator "at scale $h^*$" of the Mattis model, and the other term verify the bound of $g^{h}_{\omega,\omega}(\vec{x}, \vec{y})$ with an extra factor $\frac{|\sigma_h|}{\gamma^h}$.

We see from the above bounds that the propagator of the integration of all the scale between $h^*$ and $h$ has the same bound as the propagator of the integration of a single scale greater than $h^*$; this will be used to perform the integration of all the scales $< h^*$ in a single step, i.e. integrating directly $\psi^{(<h^*)}$. In fact $\gamma^{h^*}$ is a momentum scale and, roughly speaking, for momenta bigger than $\gamma^{h^*}$ the theory is "essentially" a massless theory (up to $O(\sigma_h \gamma^{-h})$ terms) while for momenta smaller than $\gamma^{h^*}$ is a "massive" theory with mass $O(\gamma^{h^*})$.

15 Of course if $h^* \leq \hbar$ there is no such integration.
One can prove that the effective potentials $V^h$ are well defined, if the running coupling constants are small enough. More precisely, let we write eq.(14) in coordinate space and let be $W_n^h$ the corresponding kernel; it holds that

**Lemma:** Assume that $h^*$ is finite uniformly in $L, \beta$ and that for any $h > k > h^*$ there exists an $\varepsilon$ such that $|\tilde{U}^h| \leq \varepsilon$ and $|\tilde{U}^{h,\alpha_k}| \leq \varepsilon^c \gamma^{h',h}$ with $c, c_b$ positive constants. Then there exist a constant $C$ such that

$$||\tilde{W}_n^h|| \leq N|\varepsilon|C^{-k(h^* - 2)}$$

The proof of the above lemma is an immediate modification of ones existing in literature, see in particular [3].

In order to prove that the effective potentials are well defined we have to show that the above conditions of smallness in the above lemma on the running coupling constants are verified.

The beta function for $\nu_h$ is

$$\nu_{h-1} = \gamma \nu_h + \beta_h + \tilde{\beta}_h$$

where $\beta_h$ is the contribution to $\nu_h$ obtained setting $\sigma_h = 0$, $0 \leq k' \leq h$, so it is exactly equal to 0 by the parity properties of the Mattis model, and

$$|\tilde{\beta}_h| \leq C \frac{\sigma_h}{\gamma^h} \lambda^2$$

is the remaining part. Iterating the above relation we find

$$\nu_{h-1} = \gamma^{h'} \sum_{k=0}^h \gamma^k \tilde{\beta}_k \leq \gamma^{h'} \lambda^2 \sum_k |\sigma_k| \leq C|\lambda|$$

as $\sigma_k \simeq \Delta \gamma^{n_k}$, $\eta_1 = -\beta_1 \lambda + O(\lambda^2)$, see below.

The Beta function can be written, for $0 \leq h \leq h^*$:

$$\begin{align*}
\lambda_{2,h-1} &= \lambda_h + G_{1,h}^2 + G_{2,h}^2 \\
\lambda_{4,h-1} &= \lambda_h + G_{1,h}^4 + G_{2,h}^4 \\
\sigma_{h-1} &= \sigma_h + G_{1,h}^4 \\
\delta_{h-1} &= \delta_h + G_{1,h}^4 + G_{2,h}^4 \\
Z_{h-1} &= 1 + G_{1,h}^2 + G_{2,h}^2
\end{align*}$$

(21)

where ($i = 2, 4$)

a) $G_{1,h}^2, G_{1,h}^4$ and $G_{2,h}^4$ depend only on $\lambda_i, 0; \delta_0; \lambda, h, \delta_0$ and are given by series of terms involving only the Mattis model part of the propagator $g_{L,h}^k(x - y)$, so they coincide with the Mattis model Beta function

b) $G_{1,h}^2, G_{1,h}^4, G_{2,h}^4, G_{2,h}^2$ are given by a series of terms involving at least a propagator $G_{2,h}^2(x - y)$ or $g_{L,h}^k(x - y)$ with $k \geq h$.

By a simple explicit computation

$$\begin{align*}
G_{2}^{1,h} &= \lambda h_1 \left[ \beta_2 + \tilde{G}_h^2 \right] \\
G_{1}^{1,h} &= \lambda h \sigma_h \left[ \beta_1 + \tilde{G}_h^2 \right]
\end{align*}$$

with $\beta_1, \beta_2 > 0$ and $\tilde{G}_h^2, G_{1,h}^h = O(\lambda_h)$. Moreover $G_{1,h}^{1,h}, G_{2,h}^{1,h}$ coincide by definition with the Mattis model Beta function, and it was proved in [2],[3] that it is vanishing at any order, i.e.

$$G_{1,h}^{1,h} = G_{2,h}^{1,h} = 0$$

Finally as $|G_{1,h}^{2,h}|, |G_{2,h}^{2,h}|, |G_{2,h}^{2,h}| \leq K \varepsilon^2 |\lambda|^{\gamma - h}$, one finds, for $h > h^*$,

$$|\lambda_{i,h-1} - \lambda_i, 0| < c_1 \lambda^2 |\delta_{h-1} - \delta_0| \leq c_1 \lambda^2$$

and $\sigma_h \simeq (\Delta) \gamma^{n,h}$, $Z_h \simeq \gamma^{-n_h}$ for $h \geq h^*$, with $\eta_1 = -\beta_1 \lambda + O(\lambda^2)$, $\eta_2 = \beta_2 \lambda^2 + O(\lambda^3)$. As usual in models to which the RG is successfully applied the flow is essentially described by the second order truncation of the beta function. This shows that it is possible to choose $\lambda$ so small that the conditions of the above lemma are fulfilled. From the definition of $h^*$ and the fact that $\sigma_h \simeq (\Delta) \gamma^{n,h}$ it follows $\sigma_h = \Delta \lambda^2 + O(\lambda^3)$.

As we said the integrations of the $\psi^{(\lambda^2)}$ (if $h^* \geq h_\beta$) is essentially equivalent to the integration of a single scale $h \geq h^*$, so it is well defined by the preceding arguments. If $h^* < h_\beta$ there is no such integration, and the last scale to be integrated is $h_\beta$; from this consideration one obtains easily that the critical temperature is proportional as usual to the gap amplitude.

An expansion for the two points Schwinger function can be derived in a standard way [2] and from the proof of the convergence of the expansion for the effective potential one obtains easily the convergence of the series for the Schwinger function. We can write the r.h.s. of eq.(4) as

$$\sum_{h = \max[h^*, h_0]}^{0} \frac{(2\pi)^2}{L \beta} \sum_{k} \frac{\sigma_h}{Z_h \sum_{k'} f_h(k')} \left[ 1 + \lambda \tilde{S}_h(k') \right]$$

(22)
where $S_h(\vec{k}')$ is a convergent series bounded by a constant; from the above expression one can easily derive the BCS equation eq.(5) as well as the critical temperature.

Note the crucial role of the renormalization of the BCS gap $\sigma_h$; it is sensitive to the sign of the interaction and it eliminates or enlarges the singularity of the r.h.s. of eq.(5); neglecting such renormalization one obtains completely different results. In fact our model belongs to the class of universality of the massive Luttinger model, for which it is well known that the bare mass $\Delta$ is renormalized by the interaction to be given by $\Delta^{1-\eta}$, $\eta = O(\lambda)$; other models belonging to this class are the XYZ chain or the Yukawa$_2$ model, see [3],[9]. Note also that no role is played in the above analysis by the spin-charge separation; in fact $|v_c - v_s| = O(\lambda)$ and such effect is incorporated in the term $\tilde{S}^h$ in eq.(22).

4 References

[1] P.W. Anderson The theory of high $T_c$ superconductivity, Princeton Press, 1997; and reprints by various authors therein

[2] G. Benfatto, G. Gallavotti, A. Procacci, B. Scoppola Comm. Math. Phys 160, 93-171;

[3] F. Bonetto, V. Mastropietro, Comm. Math. Phys. 172, 57-93 (1995); Phys. Rev. B 56 1296-1308 (1997); Nucl. Phys. B 497 541-554 (1997); V.Mastropietro, in print on Comm. Math. Phys.; V.Mastropietro, submitted to Math. Phys. Lett.

[4] D. Mattis, Physics, 1, 183-193 (1964)

[5] A.M. Tsvelik, Quantum field theory in condensed matter physics, Cambridge Un. Press, 1997

[6] G.Benfatto, G. Gentile, V.Mastropietro, Jour, Stat. Phys. 92,314, 1998

[7] K.Penc, J.Solyom: Phys. Rev. B, 41, 1, 704-716 (1990); M.Fabrizio, A. Parola, E. Tosatti, 46, 5, 3159-3162 (1992); A.A. Nerseyan, A. Luther, F.V.Kusmartsev, Physics Letters A, 176, 363-370 (1993)

[8] Solyom, Advances in Physics, 28, 201-303 (1978)