In this paper we will consider the family of elliptic curves:

\[ E_{D,\alpha} : y^2 = x^3 + 16D^2\alpha^3, \quad D \in \mathbb{Z}, \quad \alpha \in \mathbb{Z}[\omega], \]

where \( \omega = \frac{-1 + \sqrt{-3}}{2} \). The elliptic curves \( E_{D,\alpha} \) are quadratic twists over \( K = \mathbb{Q}[\sqrt{-3}] \) of the elliptic curve \( y^2 = x^3 - 432D^2 \), which is a Weierstrass equation for the famous family of elliptic curves

\[ E^{(D)} : x^3 + y^3 = D. \]

In particular, \( E_{D,\alpha} \) are sextic twists over \( K \) of the familiar elliptic curve \( E_1 : y^2 = x^3 + 1 \). The equation \( E^{(D)} \) was extensively studied in the literature over \( \mathbb{Q} \), with the goal to answer the question for which integers we can write \( D \) as the sum of two rational cubes.

However, very little is known about the twists of \( E^{(D)} \) over \( K \). The goal of this paper is to study the family of twists \( E_{D,\alpha} \) and, in particular, the central values \( L(E_{D,\alpha}/K, 1) \) of their \( L \)-functions. From the Birch and Swinnerton-Dyer (BSD) conjecture, the vanishing of \( L(E_{D,\alpha}/K, 1) \) is equivalent to having rational solutions for \( E_{D,\alpha}(K) \). Without assuming BSD, when \( L(E_{D,\alpha}/K, 1) \neq 0 \) we have no rational solutions \( E_{D,\alpha}(K) \) from the work of Coates and Wiles [CW].

We define the invariant

\[ S_{D,\alpha} = \frac{1}{c_{E_{D,\alpha}}\Omega_{D,\alpha}}L(E_{D,\alpha}/K, 1), \]

where \( c_{E_{D,\alpha}} \) is the product of Tamagawa numbers \( c_v \) and \( \Omega_{D,\alpha} \in \mathbb{C}^\times \) is a period of \( E_{D,\alpha} \), more precisely \( \Omega_{D,\alpha} = \prod_{v|\Omega_{D,\alpha}} \frac{1}{\mu_{D,\alpha}(\Gamma(1/3)^3)} \). We have defined \( S_{D,\alpha} \) such that when \( \text{Nm}(\alpha) > 1 \), if \( L(E_{D,\alpha}/K, 1) \neq 0 \), from the BSD conjecture we have:

\[ S_{D,\alpha} = \#\text{Tate-Shafarevich}(E_{D,\alpha}/K), \]

where \( \text{Tate-Shafarevich}(E_{D,\alpha}/K) \) is the order of the Tate-Shafarevich group of \( E_{D,\alpha} \) over \( K \). We note that we have used here the fact that the torsion subgroup \( E_{D,\alpha}(K)_{\text{tor}} \) is trivial for \( \text{Nm} \alpha > 1 \).

When \( L(E_{D,\alpha}/K, 1) \neq 0 \), the order of \( \text{Tate-Shafarevich}(E_{D,\alpha}/K) \) is known to be finite from the work of Rubin [R], and its order is a square as proved by Cassels [C] via the Cassels-Tate pairing. We will show in

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**Abstract**

We prove a new formula for the central value of the \( L \)-function \( L(E_{D,\alpha}, 1) \) corresponding to the family of sextic twists over \( \mathbb{Q}[\sqrt{-3}] \) of elliptic curves \( E_{D,\alpha} : y^2 = x^3 + 16D^2\alpha^3 \) for \( D \) an integer and \( \alpha \in \mathbb{Q}[\sqrt{-3}] \). The formula generalizes the result of cubic twists over \( \mathbb{Q} \) of Rodriguez-Villegas and Zagier for a prime \( D = 1(9) \) and of Rosu for general \( D \). For \( \alpha \) prime and all integers \( D \), we also show that the expected value from the Birch and Swinnerton-Dyer conjecture of the order of the Tate-Shafarevich group is an integer square in certain cases, and an integer square up to a factor \( 2^{20}3^{20} \) in general.
the current paper that, for certain cases, $S_{D\alpha}$ is indeed an integer square, and in general $16S_{D\alpha}$ is an integer square up to an even power of 3.

By computing $S_{D\alpha}$, we can check if we have solutions in (1) and, if $S_{D\alpha} \neq 0$, we get the value of the analytic rank $S_{D\alpha}$ of the Tate-Shafarevich group. To summarize:

(i) $S_{D\alpha} \neq 0$ $\implies$ no solutions in (1)

(ii) $S_{D\alpha} \neq 0$ $\implies$ $S_{D\alpha} = \#\Sha$ integer square

(iii) $S_{D\alpha} = 0$ $\implies$ have solutions in (1).

From the work of Rubin [Ru], it is known that $v_p(S_{D\alpha}) = v_p(\Sha_{E,D,\alpha/K})$ for all primes $p \neq 2, 3$, where $v_p$ is the valuation at $p$. We show that further $v_p(S_{D\alpha}) \equiv v_p(\Sha_{E,D,\alpha/K}(2))$ for $p = 2, 3$.

We also note that the $L$-function $L(E_{D,\alpha}/K, s)$ has a functional equation from $s$ to $2 - s$ that is expected to have constant global root number $w = 1$ (see [BK]), thus we do not have a priori expectations for any of the central values $L(E_{D,\alpha}/K, 1)$ to vanish.

In the current paper, we will compute several formulas for $S_{D\alpha}$. In previous work, Rodriguez-Villegas and Zagier computed in [RV-Z] the special values $L_{\text{c}}$ is expected to have constant global root number $w = 1$ (see [BK]), thus we do not have a priori expectations for any of the central values $L(E_{D,\alpha}/K, 1)$ to vanish.

We will state now the results of the paper. Without loss of generality, we choose $\alpha$ square-free with $Nm\alpha > 1$ and $D$ cube-free. Moreover, we note that the elliptic curve $E_{D,\alpha}$ is invariant when multiplying $\alpha$ by a cubic root of unity, thus we will fix throughout the paper the representatives

$$\alpha \equiv \pm 1, \pm \sqrt{-3}(4).$$

We denote $m = Nm\alpha$.

**Theorem 1.1.** For $\alpha$ prime and any integer $D$ such that $(D,6\alpha) = (\alpha,6) = 1$, $S_{D\alpha}$ is an integer square up to a factor $2^{2a} \cdot 3^{2b}$, $0 \leq a \leq 2$. More precisely:

- $v_p(S_{D\alpha})$ is even for all $p$,
- $v_p(S_{D\alpha}) \geq 0$, for all $p \neq 2, 3$ and
- $v_3(c_{E,D,\alpha}S_{D\alpha}) \geq -1$, $v_2(c_{E,D,\alpha}S_{D\alpha}) \geq 2e$, where $e = 1$ for $m \equiv 1(4)$ and $e = 0$ for $m \equiv 3(4)$.

Moreover, as a particular case, $S_{D\alpha}$ is an integer square for $c_{E,D,\alpha} = 1.4$ when $m \equiv 1(4)$, or $c_{E,D,\alpha} = 1$ when $m \equiv 3(4)$. Taking $\left(\frac{\alpha}{7}\right)$ to be the quadratic character over $K$ and $\chi_{2D^2}$ the cubic character (see Section 2.1 for the definitions), then we have:

**Theorem 1.2.** $S_{D\alpha}$ is an integer square for $\alpha$ prime with $m \equiv 1(4)$ such that

- $D \equiv \pm 1(9)$, $\alpha \equiv -1(3)$ and $\left(\frac{\alpha}{p}\right) = -1$ for all prime ideals $p|D$
- $D \equiv \pm 4(9)$, $\alpha \equiv 1(3)$ and $\left(\frac{\alpha}{p}\right) = -1$ for all prime ideals $p|D$

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\bullet D \equiv \pm 2(9) \text{ and } \chi_{2D^*}(\alpha) \neq 1

For \( m \equiv 3(4) \), \( S_{D\alpha} \) is an integer square for \( D \equiv \pm 1, \pm 4 \) such that \( \chi_{2D^*}(\alpha) \neq 1 \) and satisfying the conditions above.

Moreover, when \( D = 1 \), for the elliptic curve \( E_{\alpha} : y^2 = x^3 + 16\alpha^3 \) defined over \( K \), we have:

**Corollary 1.3.** For \( \alpha \equiv -1(3) \) prime, \( S_{\alpha} \) is an integer square.

Theorem 1.1 follows from the explicit formula of \( S_{D\alpha} \) that we compute in Theorem 1.4.

**Theorem 1.4.** Let \( \alpha \) prime and \( D \) any integer such that \( (\alpha, 3) = 1 \), \( (D, 6\alpha) = 1 \). Then for \( c_{E_{D, \alpha}} = 4^a : 3^b \), \( 0 \leq b \leq 2 \), the Tamagawa number of \( E_{D, \alpha} \), we have

\[
S_{D\alpha} = \frac{1}{4^a(-3)^{b+1}} Z^2,
\]

where \( Z \) is an integer if \( b \) is odd, and \( Z/\sqrt{-3} \) is an integer if \( b \) is even.

Here

\[
Z = u \operatorname{Tr}_{H_{3D^*}/K} \frac{\Theta_M(D^*\tau_0/m^*)}{\Theta_K(\tau_0/m)} D^{1/3} \alpha^{1/2},
\]

\bullet \( \tau_0 = \frac{b_0 + \sqrt{-3}}{2} \) CM point with \( b_0 \equiv \sqrt{-3}(\alpha) \), \( b_0 \equiv 1(2) \)

\bullet \( M = \mathbb{Q}[\sqrt{-3m}] \) of discriminant \( -3m^* \) and class number \( h_M \), \( H_{3m^*D} \) is the ring class field over \( K \) corresponding to the order \( \mathcal{O}_{3Dm^*} = \mathbb{Z} + 3Dm^* \mathcal{O}_K \)

\bullet \( \Theta_M(z) = h_M + 2 \sum_{N \geq 1} \sum_{d|N} \left( \frac{a}{3m^*} \right) e^{2\pi i N z} \) and \( \Theta_K(z) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)z} \) theta functions of weight 1

\bullet \( D^* = \begin{cases} D & \alpha \equiv 1(4) \\ 4D & \alpha \equiv -1, \pm \sqrt{3}(4) \end{cases} \), \( u = 2^e \omega^l \), for \( \omega \) a cubic root of unity, and \( e = \begin{cases} 1 & m \equiv 1(4) \\ 0 & m \equiv 3(4) \end{cases} \)

We get immediately Theorem 1.1 as well as Theorem 1.2 and Corollary 1.3. We also note the surprisingly simple formulas for \( \alpha \) prime:

**Corollary 1.5.** For \( \alpha \) prime such that \( \alpha \equiv 1(4) \), we have

\[
S_{\alpha} = \left( \frac{2}{\sqrt{3}} \frac{\Theta_M(\tau_0/m)}{\Theta_K(\tau_0/m)} \alpha^{1/2} \right)^2.
\]

For \( \alpha \) prime such that \( \alpha \equiv -1, \pm \sqrt{-3}(4) \), we have \( S_{\alpha} = \left( \frac{2}{\sqrt{3}} \operatorname{Tr}_{H_1/K} \frac{\Theta_M(\tau_0/m)}{\Theta_K(\tau_0/m)} \alpha^{1/2} \right)^2 \).

As mentioned in Corollary 1.3, for \( \alpha \equiv -1(3) \), \( S_{\alpha} \) is an integer square. Similarly, in the case \( \alpha \equiv 1(3) \), \( 9S_{\alpha} \) is an integer square.

Theorem 1.4 is based on the more general result Theorem 1.6 proved for all \( \alpha \) not necessarily prime. From CM theory we can find a Hecke character \( \chi \) defined over \( \mathbb{A}_K \), the ideles of \( K \), such that:

\[
L(E_{D, \alpha}/K, s) = L(s, \chi)L(s, \overline{\chi}).
\]

By computing the value of each \( L(1, \chi) \), we get:

\[ \text{...} \]
Theorem 1.6. For $D \in \mathbb{Z}$ and $\alpha \in \mathcal{O}_K$ such that $(D, 6\alpha) = 1$, $(\alpha, 3) = 1$, we have:

$$S_{Da} = \frac{1}{c_{ED,a}} \left| c \text{Tr}_{H_{3D', m}*}^*(K) \frac{\Theta_M(D')}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3} \right|^2,$$

and $S_{Da}/|c|^2 \in \mathbb{Z}$. Here $c = \frac{2L(1, \chi)}{\sqrt{3m}}$, where $L(1, \chi) = \prod_{p | (\alpha)} (1 - \chi(p) \text{Nm} p^{-1})^{-1}$, and $D' = \begin{cases} D & \alpha \equiv 1, \pm \sqrt{-3}(4) \\ 4D & \alpha \equiv -1(4) \end{cases}$.

The paper is structured as follows. In Section 2.1 we present the background on the Hecke characters, in particular the properties of the quadratic character $\left[ \frac{4}{\alpha} \right]$. In the rest of Section 2 we cover the properties of the weight 1 classical Eisenstein series and prove several properties of the theta function $\Theta_M(z)$, including a Siegel-Weil type result and an inverse transformation.

The goal of Section 3 and Section 4 is to prove Theorem 1.6. The proof is similar to that of Theorem 1.6. We note that one can computationally check the values of $S_{Da}$, which consist of computing a formula for $L(s, \chi)$ by using Tate’s thesis, which we cover in section 2.3. We obtain a finite linear combination of Eisenstein series and characters and by further applying the Siegel-Weil type result we get a linear combination of ratios of theta functions evaluated at CM points. These CM points correspond to the inverse transformation.

In the rest of Section 2 we cover the technical results proved using Shimura reciprocity law that are used in Section 5. Finally, in the Appendix we present the explicit computations involving the Galois conjugates of modular functions used in the proofs of Theorem 1.6. These are proved using Shimura’s reciprocity law and properties of the theta function $\Theta_M$ proved in Section 2.4. We also compute in this section the explicit values of the cubic and quadratic characters for various ideals.

We note that one can computationally check the values of $S_{Da}$. These computations, while delicate to program, are much faster than the direct computations of the $L$-functions in Magma.

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2 Background

Let $K = \mathbb{Q}[\sqrt{-3}]$, $\mathcal{O}_K = \mathbb{Z}[\omega]$ its ring of integers for $\omega = \frac{-1 + \sqrt{-3}}{2}$ a fixed cubic root of unity, and $K_v$ the completion of $K$ at the place $v$. We denote $K_p = K \otimes \mathbb{Q}_p$ and $\mathcal{O}_K = \mathcal{O}_K \otimes \mathbb{Z}_p$ the semilocal ring of integers.

We also remark that we can write the primitive ideals $\mathcal{A}$ in $\mathcal{O}_K$ as $\mathcal{A}$-modules $\mathcal{A} = \left[ a, \frac{b + \sqrt{-3}}{2a} \right]$, where $a = \text{Nm} A$ and $b$ is chosen (non-uniquely) such that $b^2 \equiv -3(4a)$. We will denote $\tau = \frac{b + \sqrt{-3}}{2a}$. 

4
2.1 Hecke characters

We recall that from CM theory we can find a Hecke character \( \chi : \mathbb{A}_K^\times \to \mathbb{C}^\times \) corresponding to the elliptic curve \( E_{D,\alpha} \) such that:

\[
L(E_{D,\alpha}/K, s) = L(s, \chi)L(s, \overline{\chi}).
\]

We can explicitly write the Hecke character \( \chi \) as a product of Hecke characters \( \varphi, \chi_D, \varepsilon \) defined over \( K \):

\[
\chi = \varphi \chi_D \varepsilon.
\]

We define the characters \( \varphi, \chi_D, \varepsilon \) in classical language. The character \( \varphi \) has conductor 3 and, for an ideal \( \mathcal{A} \) prime to 3, it is defined by

\[
\varphi(\mathcal{A}) = k_\mathcal{A},
\]

for \( k_\mathcal{A} \) the unique generator of \( \mathcal{A} \) such that \( k_\mathcal{A} \equiv 1(3) \). As \( K \) is a PID and \( \mathcal{O}_K^* \) is generated by \( -\omega \), the character is well-defined.

2.1.1 The cubic character \( \chi_D \)

The cubic character \( \chi_D : \{1, \omega, \omega^2\} \to \{1, \omega, \omega^2\} \) is defined for the space of ideals prime to 3D to be \( \chi_D = (\overline{\omega})_3 \), the complex conjugate of the cubic character defined in Ireland and Rosen [IR]. More precisely, \( (\overline{\omega})_3 \) is the unique cubic root of unity for which we have \( \left( \frac{\alpha}{\mathfrak{p}} \right)_3 \equiv D \overline{\omega}^{\frac{D-1}{3}} \mod \mathfrak{p} \). Moreover, \( \chi_D \) is well-defined on \( \text{Cl}(\mathcal{O}_{3D}) \), the class group for the order \( \mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K \). We note the property:

\[
(D^{1/3})^{\sigma_\mathcal{A}^{-1}} = D^{1/3} \chi_D(\mathcal{A}),
\]

where \( \sigma_\mathcal{A} \) is the Galois action corresponding to the ideal \( \mathcal{A} \) via the Artin map.

If \( \beta \in \mathcal{O}_K \) such that \( \beta \equiv \omega(D), \beta \equiv 1(3) \) we will write \( \chi_D(\beta) = \chi_\omega(D) \) and this equals \( \overline{\omega}^{D^2-1} \) for \( D \equiv 2(3) \), and \( \overline{\omega}^{2D-1} \) for \( D \equiv 1(3) \), respectively. Thus explicitly we have:

\[
\chi_\omega(D) = \begin{cases} 
1 & \text{if } D \equiv \pm 1(9) \\
\omega & \text{if } D \equiv \pm 4(9) \\
\omega^2 & \text{if } D \equiv \pm 2(9).
\end{cases}
\]

2.1.2 The quadratic character \( \varepsilon \)

The quadratic character \( \varepsilon : \{1, \pm 1\} \to \{1\} \) is defined on the space of ideals prime to 2\( \alpha \) and it has conductor \( \alpha \) for \( \alpha \equiv 1(4) \) and conductor 4\( \alpha \) for \( \alpha \equiv -1, \pm \sqrt{-3}(4) \). We have \( \varepsilon = [\overline{\alpha}] \), where \( [\overline{\alpha}] \) is defined as in [IR] by taking

\[
\left[ \frac{\alpha}{\mathfrak{p}} \right] \equiv \alpha^{N_{\mathfrak{p}-1}} \mod \mathfrak{p}
\]

for a prime ideal \( \mathfrak{p} \) prime to 2\( \alpha \), and extending by multiplication. We also write \( [\overline{\alpha}] = \left[ \frac{\alpha}{\mathfrak{p}} \right] \) and we note the property:

\[
(\alpha^{1/2})^{\sigma_\mathcal{A}^{-1}} = \alpha^{1/2} \varepsilon(\mathcal{A}),
\]

where \( \sigma_\mathcal{A} \) is the Galois action as above. We note \( [\overline{\alpha}] = \left[ \frac{\alpha}{\mathfrak{p}} \right] \), thus \( [\overline{\alpha}] = \left[ \frac{\alpha}{\mathfrak{p}} \right] \).

We will follow Lemmermeyer [L2] in presenting several properties of the quadratic character \( [\overline{\alpha}] \) for \( \alpha \in \mathbb{Q}(\sqrt{-3}) \), in particular the Eisenstein quadratic reciprocity law. We note that \( \omega \) in the notation of [L2] is \( \omega^2 \) in our notation.
Clearly \( \left[ \frac{\alpha}{p} \right] = \left[ \frac{\alpha'}{p} \right] \) if \( \alpha \equiv \alpha' \mod p \), and, moreover, for \( a \in \mathbb{Z} \) and \( \beta \in \mathcal{O}_K \), we have:

\[
(6) \quad \left[ \frac{\beta}{a} \right] = \left( \text{Nm} \frac{\beta}{a} \right), \quad \left[ \frac{a}{\beta} \right] = \left( \frac{a}{\text{Nm} \beta} \right),
\]

where \( \left( \frac{\alpha}{p} \right) \), \( \left( \frac{\alpha}{\beta} \right) \) are the usual quadratic characters over \( \mathbb{Q} \).

\textit{Eisenstein’s quadratic reciprocity law} for \( \alpha, \beta \) of odd norm is given by:

\[
(7) \quad \left[ \frac{\alpha}{\beta} \right] = \left[ \frac{\beta}{\alpha} \right] [\alpha] [\beta],
\]

for a symbol \( [\gamma] \) that is uniquely determined by the value of \( \gamma \) modulo \( 4 \). For general \( \gamma = a - b\omega^2 \) of odd norm, we take as definition \( [\gamma] = (-1)^{\frac{\text{Nm} \gamma}{2} - 1} \left( \frac{2}{\text{Nm} \gamma} \right)^r \), where \( b = 2^r b' \) for \( b' \) odd (see Lemma 12.11 of \textbf{[L2]}). However, for \( \gamma = a - b\omega^2 \) a primitive element of odd norm, the definition simplifies to

\[
[\gamma] = \left( \frac{b}{\gamma} \right).
\]

We have the supplementary reciprocity laws:

(i) \( \left[ \frac{1}{\gamma} \right] = (-1)^{|\text{Tr} \frac{1}{\gamma}|} \),

(ii) \( \left[ \frac{2}{\gamma} \right] = (-1)^{|\text{Tr} \frac{2}{\gamma}|} \).

We note that the usual quadratic reciprocity law holds for \( \alpha \equiv 1(4) \). For the remaining cases \( \alpha \equiv -1, \pm \sqrt{-3} \) the reciprocity law does not hold in general, but depends on \( \beta \mod 4 \). Noting that we can modify \( \beta \) by a cubic root of unity to get \( \beta \equiv \pm 1, \pm \sqrt{-3}(4) \), we compute explicitly in the following lemma:

\textbf{Lemma 2.1.} For \( \beta \) prime to \( 2\alpha \), we have:

(i) For \( \alpha \equiv 1(4) \), we have the reciprocity law \( \left[ \frac{\alpha}{\beta} \right] = \left[ \frac{\beta}{\alpha} \right] \).

(ii) For \( \alpha \equiv -1(4) \), we have:

\[
\left[ \frac{\beta}{\alpha} \right] \left[ \frac{\alpha}{\beta} \right] = \begin{cases} 
1 & \text{for } \beta \equiv \pm 1 (4) \\
\left( \frac{\beta/\sqrt{-3}}{\alpha} \right) & \text{for } \beta \equiv \pm \sqrt{-3} (4)
\end{cases}
\]

(iii) For \( \alpha \equiv \pm \sqrt{-3}(4) \), we have:

\[
\left[ \frac{\beta}{\alpha} \right] \left[ \frac{\alpha}{\beta} \right] = \begin{cases} 
\left( \frac{\beta}{\alpha} \right) & \text{for } \beta \equiv \pm 1 (4) \\
\left( \frac{\alpha/\sqrt{-3}}{\beta} \right) \left( \frac{\beta/\sqrt{-3}}{\alpha} \right) & \text{for } \beta \equiv \pm \sqrt{-3} (4)
\end{cases}
\]

\textbf{Proof:} We compute \( [1] = [-1] = [-\sqrt{-3}] = 1 \) and \( [\sqrt{-3}] = -1 \) and apply (7).

We get immediately from Lemma 2.1 and (6):

\textbf{Corollary 2.2.} For an ideal \( \mathcal{A} \) with generator \( k_\mathcal{A} \) such that \( \begin{cases} 
k_\mathcal{A} \equiv n(m) & \text{for } \alpha \equiv 1(4) \\
k_\mathcal{A} \equiv n(4m) & \text{for } \alpha \equiv -1, \pm \sqrt{-3}(4)
\end{cases} \),

with \( n \in \mathbb{Z} \), we have

\[
\varepsilon(\mathcal{A}) = \left( \frac{n}{m^*} \right).
\]

Corollary 2.2 will be used in Section 3 when choosing Schwartz-Bruhat functions \( \Phi_p \) such that \( \chi_p \Phi_p \) to be invariant on \( (\mathbb{Z} + 3D'm^* \mathcal{O}_K, \Phi_p) \) for \( p|3D'm^* \).

We further present the following lemma that will be used in Section 5.1.
Lemma 2.3. For $\alpha \equiv 1$ primitive, we have $\varepsilon(\alpha) = \varepsilon(\sqrt{-3})$ and for $\alpha \equiv -1, \pm \sqrt{-3}(4)$ primitive, we have $\varepsilon(\alpha) = -\varepsilon(\sqrt{-3})$.

Proof: By definition, $[\alpha] = \left[\frac{(\alpha - \overline{\alpha})(\sqrt{-3})}{\alpha}\right]$, which equals $[\alpha] = \left(\frac{-1}{\alpha}\right)\left[\frac{\sqrt{-3}}{\alpha}\right]$. By the reciprocity law (7), we have $\left[\frac{-3}{\alpha}\right] = \left[\frac{\alpha}{\overline{\alpha}\overline{\alpha}}\right][\alpha][-\sqrt{-3}][3\alpha]$, thus $[\alpha] = \left(\frac{-1}{\alpha}\right)\left[\frac{\alpha}{\overline{\alpha}\overline{\alpha}}\right][\alpha][-\sqrt{-3}][3\alpha]$, which implies $\left[\frac{-3}{\alpha}\right] = \left(\frac{-1}{\alpha}\right)\left[\frac{\alpha}{\overline{\alpha}\overline{\alpha}}\right]$. But $[-\sqrt{-3}] = 1$ for $\alpha \equiv 1, \pm \sqrt{-3}(4)$ and $[-\sqrt{-3}] = -1$ for $\alpha \equiv -1(4)$. Thus $\varepsilon(\sqrt{-3}) = \varepsilon(\alpha)$ for $\alpha \equiv 1(4)$ and $\varepsilon(\sqrt{-3}) = -\varepsilon(\alpha)$ for $\alpha \equiv \pm \sqrt{-3}(4), -1$.

2.2 Adelic Hecke characters

Each Hecke character $\chi$ can be written adelically as $\chi = \prod \chi_v$, where $\chi_v : K_v^\times \to \mathbb{C}$, such that at the unramified places we have $\chi_v(O_K^\times) = 1$ and $\chi_v(\overline{\omega}_v) = \chi(p_v)$, where $\overline{\omega}_v$ is the uniformizer corresponding to the prime ideal $p_v$.

In our case, we have:

- $\varphi_{e}(z) = z^{-1}$, $\varphi_{p}(p) = -p$ for $p \equiv 2(3)$, $\varphi_{v}(\overline{\omega}_v) = \overline{\omega}_v$, where $\overline{\omega}_v \equiv 1(3)$ uniformizer for $p_v$ prime such that $p_v\overline{p}_v = p$ prime in $\mathbb{Q}$
- $\chi_{D,v}(z) = 1$, $\chi_{D,p}(p) = 1$ for $p \equiv 2(3)$ and $\chi_{D,v}(p)\chi_{D,p}(p) = 1$ for $p_v\overline{p}_v = p$ prime in $\mathbb{Q}$
- $\varepsilon_{v}(z) = 1$, $\varepsilon_{v}(\overline{\omega}_v) = \left(\frac{\alpha}{p_v}\right)$

2.3 Eisenstein series

In the current section we will discuss the properties of a family of weight 1 Eisenstein series $E_N(s, z)$ and relate their central values $E_N(0, z)$ to theta functions $\Theta_N(z)$. The Eisenstein series $E_N(0, z)$ will appear in Section 3 in the computation of the $L$-function $L(1, \chi)$ for $N = 3m^*$ and we will discuss further properties of the theta function $\Theta_M$ corresponding to $E_{3m^*}$ in the following section.

For a positive integer $N$, we define the following classical Eisenstein series:

$$E_s(z; a_1, a_2, N) = \sum_{d = a_2(\text{mod } N)} \frac{1}{(cz + d)^s},$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$. They were extensively studied by Hecke in [H]. While the series does not converge absolutely for $s = 0$, we can still compute their Fourier expansion using Hecke’s trick. More precisely, Hecke computed:

$$(8) \quad E_0(z; a_1, a_2, N) = C(a_1, a_2, N) \frac{2\pi i}{N} \sum_{c = 0, c \equiv a_1(\text{mod } N)} (\text{sgn}(n))e^{2\pi ina_2/N}e^{2\pi incz/N},$$

where the constant part is $C(a_1, a_2, N) = \delta(a_1/N) \sum_{d = a_2(\text{mod } N)} \frac{\text{sgn}(d)}{|d|^s} |_{s=0} - \frac{\pi i}{N} \sum_{c = a_1(\text{mod } N)} \frac{\text{sgn}(c)}{|c|^s} |_{s=0}$, for

$$\delta(a_1/N) = \begin{cases} 1, & \text{if } N|a_1; \\ 0, & \text{if } N \nmid a_1. \end{cases}$$

We are interested in the Eisenstein series

$$(9) \quad E_N(s, z) = \sum_{a_2 = 0}^{N-1} \left(\frac{a_2}{N}\right) E_s(z; 0, a_2, N).$$
We will relate its central value $E_N(0, z)$ to the theta function:

\begin{equation}
\Theta_N(z) = h_N + 2 \sum_{n \geq 1} \left( \sum_{d \mid n} \left( \frac{d}{N} \right) \right) e^{2\pi i n z},
\end{equation}

where $-N$ is the discriminant of the number field $L = \mathbb{Q}[\sqrt{-N}]$ and $h_N$ is the class number of $\mathcal{O}_L$.

More precisely, comparing the Fourier expansions, we compute below a variation of the Siegel-Weil theorem:

**Lemma 2.4.** For $N \neq 3$, we have $E_N(0, z) = \frac{2\pi}{\sqrt{N}} \Theta_N(z)$.

**Proof:** As $E_N(0, z) = \sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) E_0(z; a_2, N)$, we use the Fourier expansion (8) at $s = 0$ for each $E(z; 0, a_2, N)$. We compute first the constant term:

\[ \frac{1}{\sqrt{N}} \sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) C(0, a_2, N) = 2 \sum_{d=1}^{\infty} \frac{(d/N)}{d} \left( \frac{a_2}{N} \right) \sum_{n \geq 1} \frac{1}{|c|^n} \mid_{s=0} = 2\pi \left( \sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) \right) \sum_{c \equiv 2 \mod{N}} \frac{1}{|c|^s} \mid_{s=0} = 2L(1, \left( \frac{-1}{N} \right)), \]

which further equals $\frac{2\pi h_N}{\sqrt{N}}$. In the first equality we have used $\left( \frac{-1}{N} \right) = -1$, as $N$ is the discriminant of $\mathbb{Q}[\sqrt{-N}]$. We also compute the non-constant terms:

\[ \frac{2\pi}{\sqrt{N}} \sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) \sum_{c \equiv 2 \mod{N}} \frac{1}{|c|^s} \mid_{s=0} = 2\pi \sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) e^{\frac{2\pi i a_2}{N}} \]

Computing the Gauss sum $\sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) e^{\frac{2\pi i a_2}{N}} = \left( \frac{N}{N} \right) i\sqrt{N}$, and changing the notation to $T = \frac{2\pi}{\sqrt{N}}$ we get $\frac{4\pi}{\sqrt{N}} \sum_{T=1}^{\infty} \sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) e^{\frac{2\pi i a_2}{N} e^{2\pi i T z / N}}$. Finally this gives us the Fourier expansion

\[ \frac{2\pi}{\sqrt{N}} \left( h_N + 2 \sum_{T=1}^{\infty} \left( \sum_{n \mid T, n \equiv 2 \mod{N}} \left( \frac{n}{N} \right) \right) q^T \right) \]

for $q = e^{2\pi i z}$, and we recognize that this is $\frac{2\pi}{\sqrt{N}} \Theta_N(z)$.

Similarly, we also prove:

**Lemma 2.5.** For $N \neq 3$, we have $\sum_{c,d \equiv 2 \mod{N}} \frac{1}{cz + d} = -2\pi i \Theta_N(z)$.

**Proof:** We write similarly to Lemma 2.4

\[ \sum_{c,d \equiv 2 \mod{N}} \frac{1}{cz + d} = \sum_{a_2=0}^{N-1} \left( \frac{a_2}{N} \right) E_1(z; a_1, a_2, N). \]

We compute the constant term

\[ \sum_{a_1,a_2=0}^{N-1} \left( \frac{a_1}{N} \right) C(a_1, a_2, N) = -2\pi \sum_{c=1}^{\infty} \left( \frac{c}{N} \right) \mid_{s=0} = -2\pi L(0, \left( \frac{-1}{N} \right)) = -2\pi i h_N. \]

For the non-constant part, we get:

\[ \sum_{a_2=0}^{N-1} \left( \frac{a_1}{N} \right) \sum_{c \equiv 2 \mod{N}} \frac{1}{|c|^s} \mid_{s=0} = -2\pi \sum_{c \equiv 2 \mod{N}} \frac{1}{|c|^s} \mid_{s=0} \]

As $\sum_{a_2=0}^{N-1} e^{2\pi i a_2/N} \neq 0$ only for $N|c$, we denote $T = nc/N$ and we get $-4\pi i \sum_{T=1}^{\infty} \left( \sum_{c \mid T} \left( \frac{c}{N} \right) \right) e^{2\pi i T z}$, which gives us the result.
2.4 Properties of $\Theta_M$

We will specialize now the previous section to the case $N = 3m^*$. Let $M = \mathbb{Q}\sqrt{-3m^*}$ of discriminant $-3m^*$ and define the Eisenstein series

$$E_{3m^*}(s, z) = \sum_{c,d \in \mathbb{Z}} \frac{d_{3m^*}}{(cz + d)^3},$$

that will appear in the computation of $L(1, \chi)$ in Section 3. For $m \neq 1$, we define the theta function

$$\Theta_M(z) = h_M + 2 \sum_{n \geq 1} \left( \sum_{d | n} \frac{d_{3m^*}}{d} \right) e^{2\pi inz},$$

This is a theta function of weight 1 for $\Gamma_1(3m^*)$ (see Lemma 2.7 below). In general we can define $\Theta_M(z) = h_M + u_M \sum_{n \geq 1} \left( \sum_{d | n} \frac{d_{3m^*}}{d} \right) e^{2\pi inz}$, where $u_M$ is the number of units in $\mathcal{O}_M$. We note that for $m = 1$ this is the theta function

$$\Theta_K(z) = 1 + 6 \sum_{m,n \in \mathbb{Z}} e^{2\pi i(m^2+n^2-3m)z}$$

of weight 1 and level 3.

The first property we mention is immediate from Lemma 2.4.

**Corollary 2.6.** $E_{3m^*}(0, z) = \frac{2\pi}{\sqrt{3m^*}} \Theta_M(z)$.

This is a generalization of Theorem 11 in [Ro] that states $E_3(0, z) = \frac{2\pi}{3} \Theta_K(z)$.

We present now several properties of $\Theta_M$ that are used in Section 4 and Section 5, in particular the modularity transformation of Lemma 2.7 and the inverse transformation of Lemma 15.

Using Lemma 2.4 we first prove the modular transformation:

**Lemma 2.7.** For $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(3m^*)$, we have the transformation:

$$\Theta_M(\gamma z) = \left( \frac{d}{3m^*} \right) (cz + d) \Theta_M(z).$$

**Proof:** Note first that for $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(3m^*)$, we have $E_{3m^*}(s, \gamma z) = \sum_{p,q \in \mathbb{Z}} \frac{d_{3m^*}}{3m^*} \sum_{p,q \in \mathbb{Z}} \frac{d_{3m^*}}{3m^*} \frac{p}{(pa+qc)z + pb + qd}$, or equivalently

$$(\frac{d}{3m^*})(cz + d)^3E_{3m^*}(s, z).$$

Taking the limit $s \to 0$ as in [Ro], we get:

$$E_{3m^*}(0, \gamma z) = \left( \frac{d}{3m^*} \right) (cz + d) E_{3m^*}(0, z)$$

and together with Lemma 2.4 we get the result of the lemma.

We note as a particular case the modular transformation of $\Theta_K$ from [Ro]:

$$\Theta_K(\gamma z) = \left( \frac{d}{3} \right) (cz + d) \Theta_K(z), \quad \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(3),$$

which was proved similarly.

Using Lemma 2.4 and Lemma 2.5 we show the inverse transformation:
Lemma 2.8. We have the transformation

\[ \Theta_M(z) = \frac{-1}{3m^*} \sqrt{3m^*} i \Theta_M \left( -\frac{1}{3m^*} z \right) \]

Proof: We denote \( N = 3m^* \) and we have

\[ \sum_{c,d \in \mathbb{Z}} \left( \frac{d}{Nc+d} \right) |Nc+d|^s = -\frac{1}{Nz|Nz|^s} \sum_{c,d \in \mathbb{Z}} \left( \frac{d}{Nz} \right) |c+d\frac{1}{Nz}|^s \]

Taking the limit to \( s = 0 \), we get from Lemma 2.4 and Lemma 2.5:

\[ \frac{2\pi}{\sqrt{3m^*}} \Theta_M(z) = -2\pi i \frac{-1}{3m^*} \Theta \left( -\frac{1}{3m^*} z \right), \]

which gives us the result.

This is a generalization of the transformation for \( m = 1 \) mentioned in \([\text{Ro}]\):

\[ \Theta_K(z) = -\frac{1}{3z} i \Theta_K \left( -\frac{1}{3z} \right). \]

Finally, we present a result relating sums of theta functions that will be used in Section 5:

Lemma 2.9. \( \sum_{0 \leq j \leq d-1} \Theta_M \left( z + \frac{d}{d} \right) = d\Theta_M(dz) \) for \( d|3m^* \).

Proof: Using the Fourier expansion \([12]\) \( \Theta_M(z) = \sum_{N \geq 0} c_N e^{2\pi i N z} \), with \( c_0 = 1 \), \( c_N = 2 \sum_{d|N} \left( \frac{d}{3m^*} \right) \)

for \( N \geq 1 \), we rewrite:

\[ \sum_{0 \leq j \leq d-1} \Theta_M \left( z + \frac{j}{d} \right) = \sum_{N \geq 0} c_N \sum_{0 \leq j \leq d-1} e^{2\pi i N (z+j/d)} = \sum_{N \geq 0} c_N e^{2\pi i N z} \sum_{j=0}^{d-1} e^{2\pi i N j/d}, \]

which equals \( \sum_{N \geq 0} c_N e^{2\pi i N z} \). As \( c_N = c_{N/d} \), we get on the LHS the Fourier expansion \( d \sum_{N \geq 0} c_N e^{2\pi i N' dz} = d\Theta_M(dz) \).

3 Computing \( L(1, \chi) \)

In this section we use the method of Rosu \([\text{Ro}]\) to compute the value of \( L(1, \chi) \) using Tate’s thesis as a linear combination of values of theta functions and Hecke characters.

3.1 Schwartz-Bruhat functions

We choose the Schwartz-Bruhat function \( \Phi_f \in S(A_K,f) \) such that Tate’s zeta function \( \zeta(s, \Phi, \chi) \) defined below to be nonzero. More precisely, \( \Phi_f = \prod_{v \mid \mathcal{N}} \Phi_v \), where:

- \( \Phi_v = \text{char}_{\mathcal{O}_{K_v}} \) for \( v \nmid 3D \),
- \( \Phi_p = \sum_{(a,p) = 1} \left( \frac{a}{p} \right) \text{char}_{(a+p\mathcal{O}_K)} \) for \( p|m \),
\[ \Phi_p = \text{char}(\mathbb{Z} + 3\mathcal{O}_K) \] for \( p | D \)

\[ \Phi_v = \text{char}(1 + 3\mathcal{O}_K_v) \] for \( v = \sqrt{-3} \).

\[ \Phi_2 = \begin{cases} 
\text{char}_{\mathbb{Z}_p[\omega]} & \text{for } \alpha \equiv 1(4) \\
\text{char}_{\mathbb{Z}_p[\omega]} & \text{for } \alpha \equiv -1(4) \\
\text{char}_{1 + 3\mathbb{Z}_p[\omega]} - \text{char}_{3 + 4\mathbb{Z}_p[\omega]} & \text{for } m \equiv 3(4).
\end{cases} \]

We note that we have chosen the local Schwartz-Bruhat functions such that

\[ \Phi_v \chi_v = \text{char}_{\mathbb{Z} + 3m^*D'} \mathcal{O}_{K_v} \quad v \mid 3m^*D' \]

\[ \Phi_v \chi_v = \text{char}_{1 + 3\mathcal{O}_{K_v}} \quad v = \sqrt{-3}, \]

where \( \chi_v \) the local component of the Hecke character \( \chi : \mathbb{A}_K^* / K^* \to \mathbb{C} \) defined in (5), corresponding to the elliptic curve \( E_{D,\alpha} \) from CM theory.

We also recall \( D' = \begin{cases} 
D & \text{for } \alpha \equiv 1, \pm \sqrt{-3}(4) \\
4D & \text{for } \alpha \equiv -1(4)
\end{cases} \).

### 3.2 Tate’s zeta function

For a Hecke character \( \chi : \mathbb{A}_K^* / K^* \to \mathbb{C}^* \) and a Schwartz-Bruhat function \( \Phi \in \mathcal{S}(\mathbb{A}_K) \), Tate’s zeta function is defined locally as

\[ Z_v(s, \chi, \Phi_v) = \int_{\mathcal{K}_v^*} \chi_v(\alpha_v)\alpha_v^{-s} \Phi_v(\alpha_v) d\alpha_v, \]

where we choose the self-dual Haar measure as in Tate’s thesis (see [Bu]). Globally \( Z(s, \chi, \Phi) = \prod_v Z_v(s, \chi, \Phi_v) \) has meromorphic continuation to all \( s \in \mathbb{C} \) and in our case it is entire.

We will compute \( Z_f(s, \chi_f, \Phi_f) = \prod_{v \nmid \infty} Z_v(s, \chi, \Phi_v) \) for \( \chi \) the Hecke character (5) corresponding to \( E_{D,\alpha} \) via CM theory, and the Schwartz-Bruhat function \( \Phi_f \) chosen above in Section 3.1. From Tate’s thesis (see [Bu], Proposition 3.1.4), we have the equality of local factors \( L_v(s, \chi) = Z_v(s, \chi) \) at all the unramified places \( v \nmid 3m^*D' \). We define for \( v | D \) the local \( L \)-function:

\[ L_v(s, \chi) = \prod_{\mathfrak{p} | D} \frac{\mathcal{O}_v}{\mathcal{O}_{K_v}}, \]

where \( L_v(s, \chi) = (1 - \varepsilon(\mathfrak{p}_v) \chi_D(\mathfrak{p}_v) \omega_v q_v^{-s})^{-1} \) for \( \omega_v \) the unique generator of \( \mathfrak{p}_v \) such that \( \omega_v \equiv 1(3) \), and \( q_v = \text{Nm} \alpha_v \).

We fix the Schwartz-Bruhat function \( \Phi_f \) from section 3.1 and it will be an immediate consequence of Tate’s thesis:

**Lemma 3.1.** For all \( s \in \mathbb{C} \), we have:

\[ \frac{L(s, \chi)}{L(1, \chi)} = Z_f(s, \chi_f, \Phi_f) \frac{1}{2} \prod_{\mathfrak{p} | 3D^*m^*} \text{vol}(\mathbb{Z} + 3D^*m^* \mathcal{O}_{K_v})^{1/2}. \]

**Proof:** As \( L_v(s, \chi) = Z_v(s, \chi) \) at all the unramified places \( v \nmid 3m^*D' \), we have the equality \( L(s, \chi) = Z_f(s, \chi, \Phi) \prod_{v \nmid 3D^*m^*} \frac{L_v(s, \chi)}{Z_v(s, \chi, \Phi_v)}. \) At the ramified places of the \( L \)-function, we have \( \Phi_v(x) \chi(x) = 1 \), when \( \Phi_v \) is nonzero. Thus, from (17), we have \( \prod_{v \nmid 3m^*D'} Z_v(s, \chi, \Phi_v) = \frac{1}{2} \prod_{\mathfrak{p} | 3D^*m^*} \text{vol}(\mathbb{Z} + 3D^*m^* \mathcal{O}_{K_v})^{1/2}. \)

The 1/2 factor occurs from considering \( \text{vol}(1 + 3\mathcal{O}_{K_v}) \) at \( v = \sqrt{-3} \). Finally, by definition, the
terms \( L_v(s, \chi) = 1 \) for \( v \mid 3D\alpha \) for \( \alpha \equiv 1(4) \), respectively for \( v \mid 6D\alpha \) for \( \alpha \equiv -1, \pm \sqrt{-3}(4) \). Thus \( \prod_{v \mid 3D^*D'} L_v(s, \chi) = L_{\mathbb{Q}}(s, \chi) \).

Next we use \([13]\) to get the value of \( L(s, \chi) \) by computing the value of \( Z_f(s, \chi_f, \Phi_f) \) as a linear combination of Hecke characters:

**Lemma 3.3.** For all \( s \in \mathbb{C} \), we have:

\[
\frac{L(s, \chi)}{L_{\mathbb{Q}}(s, \chi)} = \sum_{\beta_f \in U(3D^*m^*) \backslash \mathbb{A}_K, /K^\times} I(s, \beta_f, \Phi_f) \chi_f(\beta_f),
\]

where \( I(s, \beta_f, \Phi_f) = \sum_{k \in \mathbb{K}^\times} \frac{k}{|k|^2_{\mathbb{C}}^s} \Phi_f(k\beta_f) \) and \( U(3D^*m^*) = (1 + 3\mathcal{O}_K, \mathfrak{m}) \prod_{p \mid D^*m^*} (\mathbb{Z} + D^*m^*\mathcal{O}_K)^\times \prod_{v \mid 3D} \mathcal{O}_K^\times . \)

**Proof:** We take the quotient by \( K^\times \) in the integral defining \( Z_f(s, \chi_f, \Phi_f) \). Noting that \( \chi_f(k\beta_f) = \chi_{\mathbb{C}}^{-1}(k) \chi_f(\beta_f) = k \chi_f(\beta_f) \) and \( |k\beta_f|^p = |k|^{-2s}|\beta_f|^p \), where \( |\cdot|_C \) is the usual absolute value over \( \mathbb{C} \), we get:

\[
Z_f(s, \chi_f, \Phi_f) = \int_{\mathbb{A}^\times /K^\times} \left( \sum_{k \in \mathbb{K}^\times} \frac{k}{|k|^2_{\mathbb{C}}^s} \Phi_f(k\beta_f) \right) \chi_f(\beta_f)|\beta_f|^p \ d^x \beta_f.
\]

As \( \varphi \) and \( |\cdot| \) are unramified on \( U(3D^*m^*) \), and \( \Phi_f(k\beta_f) \) is 1 on \( U(3D^*m^*) \), we get a finite sum \( Z_f(s, \chi_D, \Phi_f) = \text{vol}(U(3D^*m^*)) \sum_{U(3D^*m^*) \backslash \mathbb{A}_K^\times} I(s, \beta_f, \Phi_f) \chi_f(\beta_f)|\beta_f|^p \). As \( \text{vol}(U(3D^*m^*)) = \frac{1}{2} \prod_{p \mid 3D^*m^*} \text{vol}(\mathbb{Z} + 3D^*m^*\mathcal{O}_K)^\times \) and using \([13]\), we get the result of the lemma.

We recall the Eisenstein series from Section 2.4:

\[
E_{3m^*}(s, z) = \sum_{c \in \mathbb{Z}, d \in \mathbb{Z}, \frac{d}{3m^*} | c + z} \frac{1}{(cz + d)[cz + d]^s}.
\]

Then we have:

**Lemma 3.3.** \( \frac{L(s, \chi)}{L_{\mathbb{Q}}(s, \chi)} = \sum_{[\mathcal{A}] \in \text{Cl}(O_{3m^*D'})} E_{3m^*}(2s - 2, D^*\mathcal{A}) \chi(\mathcal{A}) \alpha^{1-2s} . \)

**Proof:** For representatives \( \beta_f \in \prod_{\mathfrak{p} \mid m^*} \mathcal{O}_{K_\mathfrak{p}} \), \( \beta_0 = 1(3) \), we take \( \beta_0 \in \mathcal{O}_K \) such that \( \beta_0 \equiv \beta \mod 3D^*m^* \).

Thus, for \( k \in K \), we have \( \Phi_f(k\beta_0) = \Phi_f(k\beta_0) \) and this is nonzero only for \( k \in \mathcal{O}_K \). Let \( \mathcal{A} = (\beta_0) \) be the corresponding ideal. Moreover, \( \varphi_f(\beta_f) = \varphi_f((\beta_0)p|3D) = 1 \), \( \chi_{D, f}(\beta_f) = \chi_{D, f}((\beta_0)p|3D) = \chi_{D, v}((\beta_0)v|\beta_0) = \chi_{D}(\mathcal{A}) \) and similarly \( \varepsilon_f(\beta_f) = \varepsilon^{-1}(\mathcal{A}) = \varepsilon(\mathcal{A}) \). We note \( U(3m^*D') \text{\backslash} \mathbb{A}_K^\times /K^\times \cong \text{Cl}(\mathcal{O}_{3m^*D'}) \), the ring class field corresponding to the order \( O_{3m^*D'} = \mathbb{Z} + 3m^*D'^* \mathcal{O}_K \).

Thus we have:

\[
\frac{L(s, \chi)}{L_{\mathbb{Q}}(s, \alpha)} = \sum_{[\mathcal{A}] \in \text{Cl}(O_{3m^*D'})} \left( \sum_{k \in \mathbb{K}^\times} \frac{k}{|k|^2_{\mathbb{C}}^s} \Phi_f(k\beta_0) \right) \chi_D(\mathcal{A}) \varepsilon(\mathcal{A}),
\]

where the ideals \( \mathcal{A} \) are representatives of the classes of \( Cl(O_{3m^*D'}) \).

As \( k \in \mathcal{O}_K \), we have \( \Phi_{3D^*m^*}(k\beta_0) \neq 0 \) for \( k|\beta_0 \in \mathcal{A} \) and \( k|\beta_0 \in (\mathbb{Z} + 3D^*m^*\mathcal{O}_K)^\times \) with \( k|\beta_0 = 1(3) \). Write the ideal \( \mathcal{A} = [a, \frac{b + \sqrt{-3}}{2}] \) with \( a = \text{Nm} \mathcal{A}, b^2 = -3(4a) \). Thus:

\[
\frac{|\beta_0|_{\mathbb{C}}^s}{\beta_0} \sum_{k \in \mathcal{O}_K} \frac{\beta_0 k}{|\beta_0 k|^2_{\mathbb{C}}} \Phi_{3m^*D'}(\beta_0 k) = a^s \sum_{c, d \geq \mathbb{Z}, \frac{d}{3m^*} | c + \beta_0} \frac{d}{|d + 3m^*D'\mathcal{C}|^s} \left( \frac{d}{3m^*} \right) \chi_D(\mathcal{A}) \varepsilon(\mathcal{A}).
\]

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We further rewrite this as \( \frac{1}{2} \frac{L(s, \chi)}{L_\pi(s, \chi)} (\frac{m}{3m*}) E_{3m*}(2s - 2, D'^{2} \tau_{\mathcal{A}}) \). Note further that \( (\frac{m}{3m*}) = (\frac{m}{m*}) = [\frac{m}{\mathcal{A}}] = [\frac{\alpha}{\mathcal{A}}] = \varepsilon(\mathcal{A}) \varepsilon(\mathcal{A}) \), and we get the sum:

\[
\frac{L(s, \chi)}{L_\pi(s, \chi)} \Bigg| \frac{1}{2} \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_{3m*D'})} E_{3m*}(2s - 2, D'^{2} \tau_{\mathcal{A}}) \chi(\mathcal{A}) a^{1-2s}.
\]

Changing \( \mathcal{A} \to \mathcal{A} \mathcal{A} \mathcal{A} \mathcal{A} \) in \( \frac{1}{2} \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_{3m*D'})} E_{3m*}(2s - 2, D'^{2} \tau_{\mathcal{A}}) \chi(\mathcal{A}) a^{1-2s} \) we get the result of the lemma.

We recall the theta function \( \Theta \mathcal{K}(z) = 1 + 6 \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)z} \) of weight 1 and level 3. From Lemma 3.8 of [Ro], for \( \mathcal{A} = \big[ a, \frac{b+\sqrt{-3}}{2} \big] \), we have:

\[
\Theta \mathcal{K}(\tau_{\mathcal{A}}) = \varphi(\mathcal{A}) \Theta \mathcal{K}(\omega)
\]

Plugging in \( s = 1 \) in Lemma 3.3, together with (20), we get:

**Proposition 3.4.** \( L(1, \chi) = \frac{L(1, \chi)}{2D'^{1/2}} \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_{3m*D'})} \frac{E_{3m*}(0, D'^{2} \tau_{\mathcal{A}})}{\Theta \mathcal{K}(\tau_{\mathcal{A}})} \chi(\mathcal{A}) \varepsilon(\mathcal{A}) D^{1/3} \alpha^{1/2}. \)

We recall from Corollary 2.5 that we have:

\[
E_{3m*}(0, z) = \frac{2\pi}{\sqrt{3m*}} \Theta \mathcal{M}(z).
\]

Thus we get in Proposition 3.4:

\[
L(1, \chi) = L_{\mathcal{K}}(1, \chi) \frac{\pi}{\sqrt{3m*}} \frac{\Theta \mathcal{K}(\omega)}{D^{1/2}} \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_{3m*D'})} \frac{\Theta \mathcal{M}(D'^{2} \tau_{\mathcal{A}})}{\Theta \mathcal{K}(\tau_{\mathcal{A}})} \chi(\mathcal{A}) \bigg| \frac{\alpha}{\mathcal{A}} \bigg| D^{1/3} \alpha^{1/2}.
\]

Recall from [Ro] that \( \Theta \mathcal{K}(\omega) = \Gamma \left( \frac{1}{3} \right)^3 / (2\pi^2) \). Denoting:

\[
X_{\mathcal{D}_a} = \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_{3m*D'})} \frac{\Theta \mathcal{M}(D'^{2} \tau_{\mathcal{A}})}{\Theta \mathcal{K}(\tau_{\mathcal{A}})} \chi(\mathcal{A}) \bigg| \frac{\alpha}{\mathcal{A}} \bigg| D^{1/3} \alpha^{1/2},
\]

we get:

**Corollary 3.5.** \( L(\mathcal{E}_{\mathcal{D}_a/K}, 1) = \Omega_{\mathcal{D}_a} \frac{4|L_{\mathcal{K}}(1, \chi)|^2}{3m*} |X_{\mathcal{D}_a}|^2. \)

### 4 Galois conjugates

Our final step in showing Theorem 1.4 is to prove that the individual terms in the sum \( X_{\mathcal{D}_a} \) are Galois conjugates to each other, using Shimura’s reciprocity law. For \( f \) a modular function of level \( N \) and \( \tau \in \mathcal{H} \) a CM point, \( f(\tau) \) is an algebraic integer and Shimura’s reciprocity law gives an explicit way to compute its Galois conjugates. More precisely, for \( \tau \in \mathcal{H} \cap K \) and \( \sigma \in \text{Gal}(K^{ab}/K) \) Shimura’s reciprocity law states:

\[
f(\tau)^s = f^{g_s}(\tau),
\]

where \( g_s \in \text{GL}_2(\mathbb{A}) \) acts on the modular function \( f \). For the explicit action see [La], [Ro] or [GI].

We will recall an explicit version of Shimura’s reciprocity law. For a general treatment see [St], [GS], [G], as well as [Ro] for more details for the current approach. Let \( f \) be a modular function of level dividing \( 3N \) and \( \mathcal{A} \) an primitive ideal prime to \( 3N \). We write \( \mathcal{A} = \big[ a, \frac{b+\sqrt{-3}}{2} \big] \) and let
\( k_A = ta + s \frac{-b + \sqrt{3}}{2} \) be a generator of \( A \) and \( c = \frac{b^2 + 3}{4a} \). Then for \( \sigma_A \) the Galois action corresponding to the ideal \( A \) via the Artin map, we have for \( \tau = \frac{-b + \sqrt{3}}{2} \):

\[
(f(\tau))^{f_A^{-1}} = f((\frac{ta - sb - sc}{s t})_{p|3N}(\tau)),
\]

Moreover, if \( f \) has rational coefficients at \( \infty \), we have:

\[
(23) \quad f(\tau)^{f_A^{-1}} = f(\tau_A),
\]

where \( \tau_A = \frac{-b + \sqrt{3}}{2} \).

We will use \( (23) \) and \( (24) \) at various points in the paper. At the moment we are interested in the Galois conjugates of the modular function \( F(z) = \Theta_{\mu}(D_z) \) at \( z = \tau \). Our goal is to show that \( F(\tau)D^{1/3}\alpha^{1/2} \in H_{3\text{D}^6\mu} \) and its Galois conjugates are the terms appearing in the sum defining \( X_{D_\alpha} \).

Using Shimura’s reciprocity law \( (23) \) we compute the Galois conjugates of \( F(\tau) \) under \( \text{Gal}(K^{ab}/H_{3\text{D}^6\mu}) \) below:

**Lemma 4.1.** For \( A = [a, -\frac{b+\sqrt{3}}{2}] \) a primitive ideal with generator \( k_A = ta + s\frac{-b+\sqrt{3}}{2} \), \( 3m^*D'|s \), \( t \equiv 1(3) \), we have:

\[
(F(\tau))^{f_A^{-1}} = \left( \frac{t}{m^*} \right) F(\tau).
\]

**Proof:** Using the remarks above, for \( c = \frac{b^2 + 3}{4a} \), and \( \tau = \frac{-b + \sqrt{3}}{2} \) we need to compute:

\[
F((\frac{ta - sb - sc}{s t})\tau) = \frac{\Theta_M((\frac{ta - sb - sc}{s t})D')}{\Theta_K((\frac{ta - sb - sc}{s t})\tau)}
\]

Using the modular transformations proved in Lemma \( (24) \) for \( 3m^*|(s/D') \) and \( (14) \) from Section \( 2.4 \) we get

\[
F((\frac{ta - sb - sc}{s t})\tau) = \left( \frac{t}{m^*} \right) \frac{\Theta_M(D')}{\Theta_K(\tau)}
\]

which proves the lemma.

It follows that

**Lemma 4.2.** \( F(\omega)\alpha^{1/2}D^{1/3} \in H_{3m^*D'} \).

**Proof:** We want to show that \( F(\omega)\alpha^{1/2}D^{1/3} \) is invariant under the action of \( \text{Gal}(H_{3m^*D'}/K) \). Let \( A = [a, -\frac{b+\sqrt{3}}{2}] \) a primitive ideal with generator \( k_A = ta + s\frac{-b+\sqrt{3}}{2} \), \( 3m^*D'|s \), \( t \equiv 1(3) \). Then \( (\alpha^{1/2})^{f_A^{-1}} = \left( \frac{\alpha}{k_A} \right) \alpha^{1/2} \) and by quadratic reciprocity (see Section \( 2.1.2 \)), we have \( \left[ \frac{\alpha}{k_A} \right] = \left[ \frac{k_A}{\alpha} \right] \left( \frac{ta}{4} \right) \) for \( \alpha \equiv \pm 1(4) \) and \( \left[ \frac{\alpha}{k_A} \right] = \left[ \frac{k_A}{\alpha} \right] \left( \frac{ta}{4m} \right) \) for \( \alpha \equiv \pm \sqrt{3}(4) \). Thus we get \( \left[ \frac{\alpha}{k_A} \right] = \left( \frac{ta}{m} \right) \) for \( m \equiv 1(4) \) and \( \left[ \frac{\alpha}{k_A} \right] = \left( \frac{ta}{4m} \right) \) for \( m \equiv 3(4) \). Combining this with the lemma above, we get:

\[
(F(\omega)\alpha^{1/2})^{f_A^{-1}} = \left( \frac{a}{m^*} \right) F(\omega)\alpha^{1/2}.
\]
Note that as $3m^8|s$, we have $a = Nm k_A \equiv t^2 a^2 \mod 3m^8$, thus $1/t^2 \equiv a \mod 3m^8$, implying $(\frac{a}{a}) = 1$. As $D^{1/3} \in H_{3D'}m^9$, this finishes our proof.

Moreover, using (24) for the modular function $F(z) = \frac{\Theta_M(D_z)}{\Theta_M(z)}$, we have for $\tau_A = \frac{-k + \sqrt{-3}}{2a}$:

$$F(\tau)^{\sigma_A} = F(\tau_A).$$

Furthermore, by definition we have $(D^{1/3})^{\sigma_A} = \chi_D(A)D^{1/3}$ and $(\alpha^{1/2})^{\sigma_A} = \left[\frac{a}{A}\right] \alpha^{1/2}$. Then we can rewrite (24):

$$X_{Da} = \sum_{[A] \in \Omega(3m^8 D')} (F(\tau)D^{1/3} \alpha^{1/2})^{\sigma_A}.$$

Then combining with Lemma 4.2 we get:

**Proposition 4.3.** $X_{Da} = \text{Tr}_{H_{3D'}m^9/K}(F(\omega)D^{1/3} \alpha^{1/2})$ and $X_{Da} \in \Omega_K$.

To see that $X_{Da} \in \Omega_K$, it is enough to show that $F(\omega)$ is an algebraic integer. This follows from the fact that $\omega$ is a CM point and $F(\gamma z)$ is a modular function that has integer coefficients in its Fourier expansion at $\infty$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

Together with Corollary 3.3, this implies:

**Theorem 4.4.** $L(E_{Da}/K, 1) = \Omega_{Da} \Omega_{Da}\left|\frac{\text{Tr}_{H_{3D'}m^9/K}(\frac{\Theta_M(D')(\omega)}{\Theta_M(\omega)}D^{1/3} \alpha^{1/2})}{c}\right|^2$, where $c = \frac{4 | \Theta(1, \chi)|^2}{3m^8}$.

## 5 Value as a square

To simplify the calculations, we only consider the case of $\alpha$ prime. The goal of this section is to show Theorem 1.4 from the Introduction, that we restate below as Theorem 5.1. Starting with Theorem 4.4 that states

$$L(E_{Da}/K, 1) = \frac{4^e}{3} \Omega_{Da} \Omega_{Da}\left|\frac{L(1, \chi)}{\alpha}X_{Da}\right|^2,$$

we will show that $L(1, \chi)X_{Da}/\alpha$ equals, up to a sextic root of unity, the following trace:

$$Z = \omega^l \text{Tr}_{H_{3D^8}/K}\left(\frac{\Theta_M(D^* \tau_0/m^*)}{\Theta_M(\tau_0/m^*)}D^{1/3} \alpha^{1/2}\right),$$

where $\tau_0 = \frac{B + \sqrt{-3}}{2}$ with $B \equiv \sqrt{-3(\alpha)}$, $B$ odd and $D^* = \begin{cases} D & \alpha = 1(4) \\ 4D & \alpha = -1, \pm \sqrt{3}(4) \end{cases}$.

Using Theorem 4.4 we will show:

**Theorem 5.1.** Let $\alpha \in \Omega_K$ prime and $D$ any integer such that $(D, 6\alpha) = 1$, $(\alpha, 3) = 1$. Then:

$$L(E_{Da}/K, 1) = \frac{4^e}{3} \Omega_{Da} \Omega_{Da} Z_{Da}/\sqrt{-3},$$

where $c_0 =(-1)^{c_0(c_E, \omega)}$ and $e = 1$ for $m \equiv 1(4)$, $e = 0$ for $m \equiv 3(4)$ as above.

Moreover, $Z \in \Omega_K$ and $Z/\sqrt{-3} = c_0(Z/\sqrt{-3})$. In particular, $Z/\sqrt{-3} \in \mathbb{Z}$ for $c_0 = 1$ and $Z \in \mathbb{Z}$ for $c_0 = -1$, respectively.
Denote $S_{D,\alpha} = \frac{L(E_{D,\alpha}(K))}{\Omega_{D,\alpha} \Omega_{D,\alpha}} (\#E_{D,\alpha}(K)_{\text{tor}})^2$. As $\#E_{D,\alpha}(K)_{\text{tor}} = 1$ for $Nm\alpha > 1$, we will immediately get that $S_{D,\alpha}$ is a square when

- $c_{E_{D,\alpha}} = 1,4$ for $m = 1(4)$
- $c_{E_{D,\alpha}} = 1$ for $m = 3(4)$

These are the situations described below:

**Corollary 5.2.** For $m = 1(4)$, $\text{III}_{an,E_{D,\alpha}}$ is an integer square in the cases:

- $D \equiv \pm 1(9), \alpha \equiv -1(3)$ and $\frac{\alpha}{p} = -1$ for all $p|D$
- $D \equiv \pm 4(9), \alpha \equiv 1(3)$ and $\frac{\alpha}{p} = -1$ for all $p|D$
- $D \equiv \pm 2(9), \chi_{2D^2}(\alpha) \neq 1$

For $m = 3(4)$, $\text{III}_{an,E_{D,\alpha}}$ is an integer square in the cases:

- $D \equiv \pm 1(9)$ and $\frac{\alpha}{p} = -1$ for all $p|D, \chi_{2D^2}(\alpha) \neq 1$
- $D \equiv \pm 4(9), \alpha \equiv 1(3)$ and $\frac{\alpha}{p} = -1$ for all $p|D, \chi_{2D^2}(\alpha) \neq 1$

For $D = 1$ and $Nm\alpha > 1$, we have $(E_{\alpha}(K))_{\text{tor}} = \{\infty\}$ and $c_{E_{\alpha}} \in \{1,3\}$, thus we get:

**Corollary 5.3.** For $\alpha = -1(3)$ prime, $S_{\alpha}$ is an integer square.

In order to show Theorem 5.1 we want to change $L_{\alpha}(1,\chi)X/\alpha$ by a cubic root of unity and show that this is an integer or $\sqrt{-3}$ times an integer. The proof proceeds as follows. We introduce several traces $T, U, W, Y, \text{and } Z$ (see the beginning of Section 5.1 and Section 5.2).

In Section 5.1 we compute two formulas that relate $X$ to $T$ and $U$, showing in Corollary 5.17 that:

$$L_{\alpha}(1,\chi)X = U,$$

with $U/\alpha$ by definition being equal to $Z$ up to a cubic root of unity. The main goal of section 5.2 is to relate $U/\alpha$ to its complex conjugate and we start by noting that $U$ equals $Y/D'$ up to a sextic root of unity. Thus we are interested in the complex conjugate of $Y/\alpha$. We first compute several relations between $Y_f$ and $Y$, and their complex conjugates, which leads to showing in Proposition 5.13 and Proposition 5.19 that $Y/\alpha$, and respectively $U/\alpha$, are real or purely imaginary up to a third root of unity.

This culminates with the proof of Corollary 5.17 in which we write up $L(E_{D,\alpha}(K))$ as $4^v|Z|^2/3$, where $Z$ is real or purely imaginary, with $Z$ equal to $U/\alpha$ up to a cubic root of unity. We finally show in Corollary 5.18 that $Z$ is an integer or $\sqrt{-3}$ times an integer, which finishes the proof of Theorem 5.1

We note that we have to treat the cases $m = 1(4)$ and $m = 3(4)$ separately, due to the formula for the transformation (15) of $\Theta_M$ from $z$ to $-1/3m^2z$. Thus we will define different traces for $m = 1(4)$ and $m = 3(4)$ with the change:

$$\text{Tr}_{H_{D'}^{2,0,m}} \frac{\Theta_M(D'f, \frac{\tau/2^0}{m^2 \cdot 3^0})}{\Theta_K(\frac{\tau}{3^0})}, \text{ where } e_0 = \begin{cases} 0 & m = 1(4) \\ 1 & m = 3(4) \end{cases}.$$

Here $f = \pm 1, e', e'', e'' \in \{0, 1\}$.

Throughout the proofs we will use several results proved in Section 6 that are applications of the Shimura reciprocity law. In particular, we show that the traces we take are well defined and take values in $O_K$. Moreover, the computations of the characters $\chi_D$ and $\varepsilon$ at various ideals, as well as the explicit computation of various Galois conjugates of modular functions are treated in the Appendix.
5.1 Computing $X$

For $m \equiv 3(4)$, we define $X^{(i)} = \text{Tr}_{H_D/m/K} \Theta_M(D^{(i)/2}) \alpha^{1/2} D^{1/3}$ with $i \in \{1, 3\}$, $b_1 \equiv 1(8)$, $b_3 \equiv -1(8)$. Then from Lemma 6.6 in Section 6, we have $X = X^{(i)}$ for $m \equiv 3(4)$ under the condition

\[(27) \quad [\alpha] \left( -\frac{b_i}{4} \right) = 1,
\]

where the symbol $[\alpha]$ was defined in Section 2.1.2.

We will fix $b_i(8)$ to satisfy (27) and we will use the notation $X^*$ for $X^{(i)}$ when $m \equiv 3(4)$ and for $X$ when $m \equiv 1(4)$. Clearly $X = X^*$ and our goal will be to compute $L(1, \chi) X^*/\alpha$, for which we get:

\[
L(E_D, \alpha, 1) / \Omega_{D\alpha} \Omega_{D\alpha} = \frac{4}{3m^*} |L(1, \chi) X^*| \cdot 1,
\]

In this section, we define the traces $U$ and $T$(see table below) and compute two formulas that relate $X^*$ to $T$ and to $U$ in Lemmas 5.3 and 5.6. They give us Proposition 5.4, the main result of the section, in which we show that $L(1, \chi) X = U$.

Below are the important traces we take:

| $\alpha$ | $X^*$ | $U$ | $T$ |
|---------|-------|-----|-----|
| $\pm 1(4)$ | $\text{Tr}_{H_D/m/K} \Theta_M(D^{(i)/2}) \alpha^{1/2} D^{1/3}$ | $\text{Tr}_{H_D/m/K} \Theta_M(D^{(i)/2}) \Theta_K(\omega) \alpha^{1/2} D^{1/3}$ | $\text{Tr}_{H_D/m/K} \Theta_M(D^{(i)/2}) \Theta_K(\omega) \alpha^{1/2} D^{1/3}$ |
| $\pm 3(4)$ | $\text{Tr}_{H_D/m/K} \Theta_M(D^{(i)/2}) \Theta_K(\omega) \alpha^{1/2} D^{1/3}$ | $\text{Tr}_{H_D/m/K} \Theta_M(D^{(i)/2}) \Theta_K(\omega) \alpha^{1/2} D^{1/3}$ | $\text{Tr}_{H_D/m/K} \Theta_M(D^{(i)/2}) \Theta_K(\omega) \alpha^{1/2} D^{1/3}$ |

Here $\tau_0^* = -b_0^* + \sqrt{-3}/2$, $\tau_i = -b_i + \sqrt{-3}/2$, where we choose $b_i, b_0, b_0^*$ such that $b_i \equiv b_0 \equiv b_0^* \equiv 1(6D)$, $b_i \equiv b_0 \equiv b_0^*(8)$, differing mod $\alpha$ as follows:

- $b_0 \equiv -\sqrt{-3} \alpha$
- $b_0^* \equiv -\sqrt{-3} \alpha$
- $b_i - b_0 \equiv i(m)$

We will also fix in all cases only $b_i$ such that $b_i \equiv \pm 1(8)$. This does not change the result, but simplifies some of the calculations.

We show in Lemma 6.1 in Section 6 that these traces are well-defined over the precise ring class fields we take, thus $X^*, U, T \in K$. Moreover, as each ratio is an algebraic integer from CM theory, we get further $X^*, U, T \in O_K$.

We note that, for $\alpha \equiv \pm \sqrt{-3} \alpha$, the traces $U$ and $T$ also depend on $b_i(8)$, but we omit this from the notation.

We define $u_{\alpha, b} = \begin{cases} 1 & \text{if } \alpha \equiv 1(4) \\ -1 & \text{if } \alpha \equiv -1(4) \\ [\alpha] \left( -\frac{b_i}{4} \right) & \text{if } \alpha \equiv \pm \sqrt{-3} \alpha \end{cases}$ and note that, for $m \equiv 3(4)$, the condition (27) implies

\[
(28) \quad u_{\alpha, b}^* = -1.
\]

The goal of the section is to show:

**Proposition 5.4.**  
(i) $X^* = \frac{U}{L(1, \chi)}$,
(ii) $X^* \frac{\mathcal{L}(1;\chi)}{\alpha} = -u_{\alpha,-b_0^*} \varepsilon(\sqrt{-3}) D_{p(1-\chi_D(a))}(\alpha)\varphi(\alpha)$

We also define the ideals prime to $m$:

- $A_j = (\tau_j) = [a_j, \tau_j]$, $j \neq 0, 2b_0(m)$, of norm $a_j = \frac{b_j^2 + 3}{4}
- C_0' = (\tau_0)/\langle a \rangle = [c_0', \tau_0]$ of norm $c_0' = \frac{b_0^2 + 3}{4m}$
- $C_0'' = (\tau_0^*)/\langle \alpha \rangle = [c_0'', \tau_0^*]$ of norm $c_0'' = \frac{b_0^2 + 3}{4m}$

The idea of the proof is to write several sums of ratios of theta functions using Lemma 2.9 from the Section 2.4 that states:

\[
\sum_{0 \leq j \leq d-1} \Theta_M \left(z + \frac{j}{d}\right) = d\Theta_M(dz), \quad d|3m^*.
\]

We show that the ratios of theta functions in the sums are Galois conjugate under the Galois action given by the ideals $A_j, C_0', C_0''$ defined above (via the Artin map). This is shown using Lemma 7.3 from the Appendix. Taking the traces in the sums, and using the characters computed in the Appendix Lemma 7.1 and 7.2 we get relations between $X^*, T$ and $U$.

We will proceed now to computing these relations in Lemma 5.5 and Lemma 5.6. First we show:

**Lemma 5.5.** $X^* = -u_{\alpha,-b_0^*} \varepsilon(\sqrt{-3}) \frac{T}{D} + \frac{\chi_D(\tau_0)\varepsilon(\tau_0)}{\varphi(\alpha)}(m - 1)U$.

**Proof:** For $\alpha \equiv \pm 1(4)$, Lemma 2.9 for $d = m$ and $z = \frac{b_0 + \sqrt{m}}{6mD}$, we have $\sum_{0 \leq j \leq m - 1} \Theta_M \left(\frac{-\tau_j}{3mD'}\right) = m\Theta_M \left(\frac{-\tau_j}{3mD'}\right)$. We can rewrite this as:

\[
\sum_j D' \sqrt{m} \left(\frac{\Theta_M(D'\tau_j)}{\Theta_K(\tau_j)}\right)^{\sigma_j} + D' \sqrt{m} \left(\frac{\Theta_M(D'\tau_0/m)}{\Theta_K(\omega)}\right)^{\sigma_0} = m\Theta_M \left(\frac{-\tau_j/3D'}{\Theta_K(\tau_j)}\right),
\]

where the sum is taken over all $j$ such that $b_j^2 \neq -3(m)$. Using the Appendix Lemma 7.3 (iii), (iv), (v) we get on the LHS:

\[
\sum_j D' \sqrt{m} \left(\frac{\Theta_M(D'\tau_j)}{\Theta_K(\tau_j)}\right)^{\sigma_j} + D' \sqrt{m} \left(\frac{\Theta_M(D'\tau_0/m)}{\Theta_K(\omega)}\right)^{\sigma_0} = m\Theta_M \left(\frac{-\tau_j/3D'}{\Theta_K(\tau_j)}\right).
\]

and multiplying by $\tau^{1/2}D^{1/3}$ we get:

\[
\sum_j D' \sqrt{m} \left(\frac{\Theta_M(D'\tau_j)}{\Theta_K(\tau_j)}\right)^{\sigma_j} \varepsilon(A_j)\chi_D(\tau_j) + D' \sqrt{m} \left(\frac{\Theta_M(D'\tau_0/m)}{\Theta_K(\omega)}\right)^{\sigma_0} \chi_D(\tau_0)\varepsilon(\tau_0) = m\Theta_M \left(\frac{-\tau_j/3D'}{\Theta_K(\tau_j)}\right)D^{1/3}\alpha^{1/2}.
\]

Taking the traces to $H_{3D' \alpha}$, we get:

\[
D' \sqrt{m} \sum_j \varepsilon(A_j)\chi_D(\tau_j) + D' \sqrt{m} \chi_D(\tau_0)\varepsilon(\tau_0) = m\Theta_M \left(\frac{-\tau_j/3D'}{\Theta_K(\tau_j)}\right)D^{1/3}\alpha^{1/2}.
\]

Here we have used $\text{Tr}_{H_{3D' \alpha}/K} \left(\frac{\Theta_M(D'\tau_0/m)}{\Theta_K(\omega)}\right)^{\alpha^{1/2}}D^{1/3} = 0$ from Lemma 6.1. We also note that the trace of $U$ is defined on $H_{3D' \alpha}/K$, hence the extra $(m - 1)$-factor.
We similarly obtain for $\alpha \equiv \pm \sqrt{-3}(4)$:

$$X^{(i)}D\pi \sum \varepsilon(A_j)\chi_D(A_j) + \frac{D\pi}{\varphi(\alpha)}\chi_D(C_0^*)\varepsilon(C_0^*)(m-1)U = mT.$$  

We computed the $\varepsilon$ and $\chi_D$ characters in the Appendix. From Lemma 7.2, $\chi_D(A_j) = 1$, $\chi_D(C_0^*) = \chi_D(\overline{\alpha})$, and, from Lemma 7.1, $\varepsilon(C_0^*) = u_{\alpha,-b^0} \varepsilon(\overline{\alpha}) \varepsilon(\sqrt{-3})$, thus we have

$$\sum \varepsilon(A_j)D\pi X^* + \varepsilon(\sqrt{-3})D\pi \chi_D(\overline{\alpha})\varepsilon(\overline{\alpha})\frac{D\pi}{\varphi(\alpha)}(m-1)U = mT.$$  

As, from Lemma 7.1, $\varepsilon(A_j) = \left(\frac{-b_i + b_j}{m}\right)u'$, where $u' = 1$ for $m \equiv 1(4)$ and $u' = \left[\frac{b_i}{4}\right]$ for $m \equiv 3(4)$, we have $u'\sum \left(\frac{-b_i + b_j}{m}\right) = -u' \left[\frac{b_i}{m}\right] = -u' \left[\frac{b_i}{\alpha}\right] = u' \varepsilon(\sqrt{-3})[\alpha]\varepsilon(\sqrt{-3}[\alpha])$. This equals $-u_{\alpha,-b^0} \varepsilon(\sqrt{-3})$.

Thus the final sum equals $-\overline{\alpha} u_{\alpha,-b^0} \varepsilon(\sqrt{-3})D\pi X + u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\varphi(\alpha)} \chi_D(\overline{\alpha})\varepsilon(\overline{\alpha})\frac{D\pi}{\varphi(\alpha)}(m-1)U = mT$, or equivalently $X = -u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\varphi(\alpha)} \chi_D(\overline{\alpha})\varepsilon(\overline{\alpha})\frac{D\pi}{\varphi(\alpha)}(m-1)U$ for $\alpha \equiv 1(4)$.

For $\alpha \equiv \pm \sqrt{-3}(4)$, we get $-\overline{\alpha} u_{\alpha,-b^0} \varepsilon(\sqrt{-3})D\pi (X^{(i)}) + u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\varphi(\alpha)} \chi_D(\overline{\alpha})\varepsilon(\overline{\alpha})\frac{D\pi}{\varphi(\alpha)}(m-1)U = mT$.

We show a second linear relation between $X^*$, $U$ and $T$ below:

**Lemma 6.5.** $-u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\varphi(\alpha)} T + (m-1)U = mX^*$

**Proof:** For $\alpha \equiv \pm 1(4)$, using Lemma 2.9 for $d = m$ and $z = D\pi \frac{b_i + \sqrt{-3}}{2m}$, we have $\sum \varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m}) = m\varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})$. We rewrite this as:

$$\sum \varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m}) = m\varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m}) \alpha$$

where the sum is taken over all $j$ such that $b_j \neq -3(m)$. As $\frac{\varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})} = \frac{\varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}$, from Lemma 7.3 (vi), for the ideal $A_j = (-\overline{\tau_j})$ of norm $a_j$, then the first sum equals

$$\sum \frac{\varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})} = \chi_D(A_j) \left[ \frac{\alpha}{A_j} \right].$$

From Lemma 7.1, we have $\chi_D(A_j) = 1$, thus the taking the traces, we get

$$\frac{\alpha}{D\pi} T \sum \varepsilon(A_j) + (m-1)U = mX,$$

where we have used Lemma 6.1 to show that $\text{Tr}_{H_{d'} \frac{m}{2}} H_{d'} \frac{m}{2} \chi_D(\overline{\alpha}) = \alpha$.

We show similarly that $\frac{\alpha}{D\pi} T \sum \varepsilon(A_j) + (m-1)U = mX^*(i)$ for $\alpha \equiv \pm \sqrt{-3}(4)$.

We already computed in the previous lemma $\sum \varepsilon(A_j) = -u_{\alpha,-b^0} \varepsilon(\sqrt{-3})$. Thus we get $-u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\varphi(\alpha)} T + (m-1)U = mX^*$.  

**Proof Proposition 5.4** From Lemma 5.3 and Lemma 5.6, we get immediately Proposition 5.4 by solving the system

$$\begin{cases}
X^* = -u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\varphi(\alpha)} T + \frac{\varepsilon, \chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}(m-1)U
\end{cases}$$

Thus $X^* = U = 1 - \frac{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}(m-1)U$.

or equivalently $L_{\alpha}(1, \chi)X^* = U$. We also obtain $X^*(1-m \frac{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}) = -(1-m \frac{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}{\chi_D(D\pi \frac{b_i + \sqrt{-3}}{2m})}) u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\varphi(\alpha)} T$, which is equivalent to $X^* \frac{L_{\alpha}(1,\chi)}{\alpha} = -u_{\alpha,-b^0} \varepsilon(\sqrt{-3}) \frac{D\pi}{\chi_D(D\pi b_i + \sqrt{-3}) \varphi(\alpha)} T$.  

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5.2 Complex conjugates

In this section we will relate \( U/\alpha \) to its complex conjugate.

We choose \( b_{0,i} \equiv -i(3) \) for \( i = 0, 1, 2 \) such that

- \( b_{0,0} \equiv -i(3) \)
- \( b_{0,1} \equiv \sqrt{-3}(\alpha) \)

We choose \( b_{0,0} \equiv b_{0,1} \equiv b_{0,2}(8) \equiv -b_0^* \), which is congruent to \( \pm 1(8) \) for simplification of calculations.

We note that then we have

\[
\begin{align*}
    b_0^* &= -b_{0,1}(24\alpha),
    \end{align*}
\]

where \( b_0^* \) was defined in Section 5.1. Moreover, as \( b_0^* = -b_{0,1}(8) \), we also have \( u_{\alpha,-b_{0,i}} = u_{\alpha,b_0^*} \). For \( \tau_{0,i} = \frac{-b_{0,i} + \sqrt{-3}}{2} \), thus \( \tau_{0,i} = \frac{b_{0,i} + \sqrt{-3}}{2} \), we define:

\[
\begin{array}{|c|c|c|}
\hline
\alpha & \mp 1(4) & \mp \sqrt{3}/4(4) \\
\hline
Y_i & \text{Tr}_{H_D/K} & \Theta_M\left(\frac{-i b_{0,i}}{8}\right) \alpha^{1/2} D^{1/3} \\
\hline
Y & \text{Tr}_{H_D/K} & \Theta_M\left(\frac{-i b_{0,2}}{8}\right) \alpha^{1/2} D^{1/3} \\
\hline
W_i & \text{Tr}_{H_D/K} & \Theta_M\left(\frac{-i b_{0,2}}{8}\right) \alpha^{1/2} D^{1/3} \\
\hline
\end{array}
\]

Here we define \( W_i \) for \( i = 1 \) and \( Y_i \) for \( i = 0, 1, 2 \). We note that the traces are well defined from Lemma 6.1 and that \( W_i, Y_i, Y \in \mathcal{O}_K \), as they are traces of algebraic integers (again from CM-theory applied to modular functions evaluated at CM points).

We define the ideals:

- \( C_0 = (\tau_{0,0})/\alpha\sqrt{-3} \) of norm \( c_0 \)
- \( C_i = (\tau_{0,i})/\alpha \) of norm \( c_i \)
- \( A_0 = t_A a + 3m^* s' \tau_{0,0} \), with \( D'|t_A \), of norm \( a_0 \)

As \( U \) will equal \( Y/D' \) up to a sixth root of unity (see Corollary 5.9), the goal of the section is to show Proposition 5.16 in which we relate \( U/\alpha \) to its complex conjugate via Proposition 5.13 which related \( Y/\alpha \) to its complex conjugate.

From Lemma 2.9 for \( d = 3 \), we have

\[
(30) 3Y = Y_0 + Y_1 + Y_2.
\]

The goal is to obtain a relation between \( Y \) and \( \overline{Y} \). The general idea is to relate the traces to their complex conjugates by looking at various Galois conjugates, using the Galois action corresponding to the ideals \( A_0, C_i \) defined above. These Galois conjugates are computed explicitly in Lemma 7.3 in the Appendix. The characters \( \varepsilon \) and \( \chi_0 \) are computed in the Lemmas 7.1 and 7.2 in the Appendix.

In Lemma 5.8 we relate the values of \( W_i \) to \( \overline{Y} \), which also implies Corollary 5.11 relating the values of \( Y_i \) to \( \overline{Y} \) for \( i = 1, 2 \). The essential tool is Lemma 6.4 proved in the Section 6 which allows us to freely move the \( D' \) between the numerator and denominator in the traces. More precisely, for \( \tau = \frac{b_0^* + \sqrt{-3}}{2 \sqrt{3} m^*} \) with \( e, e' \in \{0, 1\} \) and \( b_0 \equiv \sqrt{3}(\alpha) \), we have:

\[
(31) \frac{\Theta_M(D')}{\Theta_K(\tau)} = \frac{1}{D'} \left( \frac{D}{3m^*} \right) \left( \frac{t_A}{m^*} \right) \left( \frac{\Theta_M(\tau/D')}{\Theta_K(\tau)} \right)^{\alpha_{A_0}^{-1}}.
\]
Via a similar approach, we relate $Y_0$ to its complex conjugate in Lemma 5.12 and show that in certain cases $Y_0 = 0$.

Plugging back in (30), the values for $Y_i$, we get Proposition 5.13 in which we show that $Y/\alpha$ is real or purely imaginary up to a third root of unity. Finally this gives us Proposition 5.16 that $U/\alpha$ is real or purely imaginary up to a third root of unity.

We also note that we can define $U^0 = U/\alpha$, $Y^0 = Y/\alpha$, $Y_i^0 = Y_i/\alpha$ for $i = 0, 1, 2$ and $W_1^0 = W_1/\alpha$. In particular, using (20) we have $\Theta_K(-\tau_{0,i/m}) = \alpha \Theta_K(\omega)$ and $\Theta_K(-\tau_{0,i/(3m)}) = \alpha \Theta_K(-\tau_{0,i}/3)$. Then, using Lemma 5.8 from Section 5, $U^0, Y^0, Y_i^0, W_1^0$ are in $\mathcal{O}_K$, and our results state:

- $U^0 \not\in \mathcal{O}_K$, $U^0 \in \mathcal{O}_K$
- $Y^0 \not\in \mathcal{O}_K$, $Y^0 \in \mathcal{O}_K$
- $W_1^0 \not\in \mathcal{O}_K$, $W_1^0 \in \mathcal{O}_K$.

where by $\not\in$ we mean equality up to a $6^{th}$ root of unity. To ease the calculations, we will keep the notation $U, Y, W_1, Y$ in the proofs and statements below. We will denote the constant

$$c_\alpha = u_{\alpha - b_0} \left( \frac{D}{3m^*} \right) \epsilon(\sqrt{-3})\xi(\alpha)\chi_D(\alpha)\alpha,$$

which equals a sixth root of unity. We also note that, under (27), from Lemma 5.7 we have $u_{\alpha - b_0} \epsilon(\sqrt{-3})\xi(\alpha) = 1$. Thus actually:

$$c_\alpha = \left( \frac{D}{3m^*} \right) \chi_D(\alpha)\alpha.$$

We also recall that we computed the value of the character $\chi_\omega(D)$ in Section 2.1.1.

We start by relating $W_1$ to $\overline{U}$:

**Lemma 5.7.** $W_1^\alpha = D' \left( \frac{D}{3m^*} \right) \overline{\xi} \overline{c_\alpha}$

**Proof:** From Lemma 1.9(ii) from the Appendix, we get $W_1 = D' \frac{\alpha}{\varphi(\alpha)}\chi_D(\overline{U})\xi(\overline{U})$. Then, using the characters computed in Lemma 7.1 and Lemma 7.2 in the Appendix, we further compute $W_1 = D' \frac{\alpha}{\varphi(\alpha)}\chi_D(\overline{U})u_{\alpha - b_0} \epsilon(\sqrt{-3})\xi(\alpha)\overline{U}$.

We will now relate $W_1$ to $\overline{Y}$ below:

**Lemma 5.8.** $W_2^\alpha = \chi_\omega(D)c_\alpha \overline{Y}$ and $W_2^\alpha = c_\alpha \overline{Y}$

**Proof:** For $\alpha = \pm 1(4)$, from Lemma 5.8 (ii) in the Appendix, we have $\Theta_M(-\tau_{0,i}/3mD') = \Theta_K(-\tau_{0,i}/3)\left( \frac{\Theta_M(D')}{\Theta_K(\omega)} \right)^{\sigma_{\alpha_0}}$. We use Lemma 6.3 to rewrite $\Theta_M(D')^{\sigma_{\alpha_0}} \chi_D(\omega) = \left( \frac{\Theta_M(D')}{\Theta_K(\omega)} \right)^{\sigma_{\alpha_0}} \chi_D(\omega)$ for the ideal $A_0 = (t, \alpha + 3s'\sqrt{3})$, $D'|t_A$. Thus we get:

$$\Theta_M(-\tau_{0,i}/3mD') \Theta_K(-\tau_{0,i}/3) \alpha^{1/2}D^{1/3} = \left( \frac{t_A}{m^*} \right) \left( \frac{D}{3m^*} \right) \alpha \chi_D(\omega) \left( \frac{\Theta_M(D')}{\Theta_K(\omega)} \right)^{\sigma_{\alpha_0}} \epsilon(\overline{A_0C_i}) \chi_D(\overline{A_0C_i}).$$

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Taking the traces, this is \( W_i = u^\top \), where
\[
u = \left( \frac{t_A}{m^*} \right) \left( \frac{D}{3m^*} \right) \frac{\alpha}{\varphi(\overline{\alpha})} \varepsilon(\mathcal{A}_0C_i) \chi_D(\mathcal{A}_0C_i).
\]
Similarly we obtain \( W_i = u^\top \) for \( \alpha \equiv \pm \sqrt{-3} \).

Then from Lemma 7.1 and 7.2, where we computed the characters, we get:
\[
\varepsilon(\mathcal{A}_0C_i) \chi_D(\mathcal{A}_0C_i) = u_{\alpha, b, i} \left( \frac{t}{m^*} \right) \varepsilon(\overline{\alpha}) \chi_\omega(D)^{2i-1} \chi_D(\overline{\alpha})
\]
Thus \( W_i = u_{\alpha, b, i} \chi_\omega(D)^{2i-1} \varepsilon(\sqrt{-3}) \frac{D}{3m^*} \) \( Y \), which gives us the result of the lemma.

From the above two lemmas, we get immediately:

**Corollary 5.9.** \( D^\prime U = \chi_\omega^2(D) \left( \frac{D}{3m^*} \right) Y \)

Our goal is now to relate \( Y/\alpha \) to its complex conjugate. We first note:

**Lemma 5.10.** \( 3Y = Y_0 + Y_1 + Y_2 \).

**Proof:** For \( \alpha \equiv 1(4) \), we apply Lemma 2.9 for \( d = 3 \) and \(-\frac{2}{3}m^*/3D^\prime m\) and get:
\[
\Theta_M \left( -\frac{2}{3}m^*/3D^\prime m \right) + \Theta_M \left( -\frac{2}{3}m^*/3D^\prime m \right) + \Theta_M \left( -\frac{2}{3}m^*/3D^\prime m \right) = 3\Theta_M \left( -\frac{2}{3}m^*/D^\prime m \right).
\]

Multiplying by \( \frac{D^{1/3}3^{1/2}}{\Theta_M(\omega)} \) and taking traces we get \( 3Y = Y_0 + Y_1 + Y_2 \). Similarly we show the same relation for \( \alpha \equiv \pm \sqrt{-3}(4) \).

The goal is now to relate each of the terms \( Y \) either to \( Y \), or to their own complex conjugate. Noting that \( Y_1 = (\omega^2 \sqrt{-3})W_1 \) and \( Y_2 = (\omega^3 \sqrt{-3})W_2 \), we get immediately from Lemma 5.8

**Corollary 5.11.** (i) \( \frac{Y_1}{\alpha} = (\omega^2 \sqrt{-3})\chi_\omega(D) c_{\alpha} \frac{Y}{\alpha} \)

(ii) \( \frac{Y_2}{\alpha} = (\omega^3 \sqrt{-3})c_{\alpha} \frac{Y}{\alpha} \)

Using similar methods as in Lemma 5.8 we relate below \( Y_0 \) to its complex conjugate:

**Lemma 5.12.** (i) \( Y_0 = 0 \), if \( D \equiv \pm 1, \pm 4(9), \)

(ii) If \( D \equiv \pm 2 \), let \( \zeta = \left( \frac{b + \sqrt{-3}}{2} \right) / (\sqrt{-3}), b \equiv 0(9), \) where \( b \equiv -1(2D), b \equiv \sqrt{-3}(a) \), we have
\[
\frac{Y_0}{\alpha} = -\omega^3 \chi_\omega(\zeta) \varepsilon(\overline{\sqrt{-3}}) c_\alpha \frac{Y_0}{\alpha},
\]

**Proof:** For \( \alpha \equiv 1(4) \), from Lemma 7.3(i) in the Appendix, we have
\[
\Theta_M \left( -\frac{\tau_{0,0}/3D^\prime m}{\tau_0(\omega)} \right) = \frac{\varphi(\alpha)D^\prime}{\tau_{0,0}/3} \left( \frac{\Theta_M(D^\prime \tau_{0,0}/3m)}{\Theta_K(\tau_{0,0}/9)} \right) \sigma_{\alpha_0}^{-1}.
\]
Then for the ideal \( \mathcal{A}_0 = t_A + 3m^* s^t \tau_{0,0}/3 \), with \( D^\prime t_A \), from Lemma 6.4 we get
\[
\Theta_M \left( -\frac{\tau_{0,0}/3D^\prime m}{\tau_0(\omega)} \right) !D^{1/3} = \frac{3 \varphi(\alpha)D^\prime}{\tau_{0,0}/3} \left( \frac{t_A}{m^*} \right) \Theta_K(\tau_{0,0}/9) \left( \frac{\Theta_M(\tau_{0,0}/3mD^\prime)}{\Theta_K(\tau_{0,0}/9)} \right) \sigma_{\alpha_0}^{-1} \sigma_{\alpha_0}^{-1} \alpha^{-1/2} D^{1/3}.
\]
Taking the trace from $H_{3D^*}$ to $K$ we get $Y_0 = u \overline{Y}_0$, where
\[
u = \frac{3\varphi(\alpha)}{\alpha} \left( \frac{D}{3m^*} \right) \left( \frac{t_d}{m^*} \right) \frac{\Theta_K(\omega)}{\Theta_K(\tau_0/9)} \chi_D(\overline{A_0C_0}) \epsilon(\overline{A_0C_0}).
\]

Similarly, we compute $Y_0 = u \overline{Y}_0$ for $\alpha \equiv \pm \sqrt{3}(4)$ using again Lemma 7.3 (i) and Lemma 5.4.

We compute now $u$. In Lemma 7.1 and 7.2 we computed the characters $\epsilon$ for $A_0$ and $C_1$. The character $\chi_D(C_0)$ is the only one that depends on $b_0$ mod 9. Then, if $Y_0 \neq 0$, we must get a constant for $\Theta_K(\tau_0/9)\chi_D((\omega^{b_0/3}\sqrt{3+1})$ when we vary $b_0,0 \equiv 0,3,-3$ mod 9. Let $b_0 = 3, b'_0 \equiv -3, b''_0 \equiv 0(9), b_0 = b'_0 \equiv b''_0 \equiv -1(2D)$.

We note that in all cases we have $-b_0/3\sqrt{-3+1} = -3\sqrt{-3}\omega(D)$, and $-b_0/3\sqrt{-3+1} \equiv -\omega(3), -b_0/3\sqrt{-3+1} \equiv -\omega^2(3)$ and $-b_0/3\sqrt{-3+1} \equiv -1(3)$. Denote $\zeta = -b_0/3\sqrt{-3+1}$. Then
\[
\chi_D(-b_0/3\sqrt{-3+1}) = \chi_\omega(D) \chi_D(\zeta),
\]
\[
\chi_D(-b_0/3\sqrt{-3+1}) = \chi_\omega(D) \chi_D(\zeta),
\]
\[
\chi_D(-b_0/3\sqrt{-3+1}) = \chi_D(\zeta).
\]

As $\Theta_K((-3+\sqrt{-3})/18) = -3\omega \Theta(\omega)$ and $\Theta_K((9+\sqrt{-3})/18) = -3\Theta(\omega)$, we must have $\omega^2 \chi_D((\omega^{b_0/3}\sqrt{-3+1})) = \omega \chi_D((\omega^{b_0/3}\sqrt{-3+1})) = \chi_D((\omega^{b_0/3}\sqrt{-3+1}))$. Thus $Y_0 \neq 0$ only if $\chi_D(\omega) = \omega^2$. Using the values of Lemma 5.11 and 7.2
\[
\chi_D(\overline{A_0}) \epsilon(\overline{A_0C_0}) = \chi_\omega(\overline{Y}_0) \left( \frac{1}{m} \right) \epsilon(\overline{A_0C_0})
\]

Thus $Y_0 = -u_{\alpha,b_0,0} \left( \frac{D}{3m^*} \right) \varphi(\alpha) \chi_D(\alpha) \epsilon(\overline{A_0C_0}) \chi_D(\zeta) \overline{Y}_0$ for $\chi_D(\omega) = \omega^2$ and $Y_0 = 0$ in the remaining cases.

Using Lemma 5.10, Lemma 5.12 and Corollary 5.11 we can finally relate $Y/\alpha$ to its complex conjugate:

**Proposition 5.13.** $\overline{Y} = \frac{c_{\alpha}d_{\alpha}}{\alpha} Y$, where $d_{\alpha} = \begin{cases} 1 & \text{if } D \equiv \pm 1(9) \\ -\omega^2 & \text{if } D \equiv \pm 4(9) \\ -\omega^2 \epsilon(\sqrt{-3}) \chi_D(\zeta) & \text{if } D \equiv \pm 2(9), \end{cases}$

**Proof:** From Corollary 5.11 we compute $Y_1 + Y_2 = (\omega^2 \chi_\omega(D) - \omega) \sqrt{-3}c_{\alpha}\overline{Y}$. Thus we have in Lemma 5.10
\[
3Y = Y_0 + c_{\alpha} \frac{\alpha}{\alpha} \overline{Y},
\]
where $c = (\omega^2 \chi_\omega(D) - \omega) \sqrt{-3}$.

For $\chi_\omega(D) = 1, \omega$, we have from Lemma 5.12 that $Y_0 = 0$ in these cases, hence:

(i) $\chi_\omega(D) = 1: c = 3$, thus $\frac{Y}{\alpha} = c_{\alpha} \overline{Y}$

(ii) $\chi_\omega(D) = \omega: c = -3\omega^2$, thus $\frac{Y}{\alpha} = -\omega^2 c_{\alpha} \overline{Y}$.

Finally for $\chi_\omega(D) = \omega^2$ we have $c = 0$ and thus $3Y = Y_0$. From Lemma 5.12 $Y_0 = -\omega^2 \chi_D(\zeta) \epsilon(\sqrt{-3}) c_{\alpha} \frac{\alpha}{\alpha} \overline{Y}$, thus we get $\frac{Y}{\alpha} = -\omega^2 \chi_D(\zeta) \epsilon(\sqrt{-3}) \overline{Y}$.

We could directly compute now how $\overline{U}/\alpha$ relates to $U/\alpha$ using Lemma 5.9 However, for the sake of completeness, we relate $W_1$ to $U$ directly and compute the complex conjugate of $W_1/\alpha$. We do this below:
\textbf{Proposition 5.16.} We have \( \frac{W_1}{\alpha} = c_\alpha t_\alpha \frac{W_1}{\alpha} \) for \( t_\alpha = \begin{cases} 
 1 & \text{if } D \equiv \pm 1(9) \\ 
 -1 & \text{if } D \equiv \pm 4(9) \\ 
 -\varepsilon(\sqrt{-3})\chi_D(\xi) & \text{if } D \equiv \pm 2(9), 
 \end{cases} \)

\textbf{Proof:} From Lemma 5.8 and Proposition 5.13, we compute

(i) \( \chi_\omega(D) = 1: W_1 = Y, \) thus \( W_1 = u_{\alpha,b_0} \epsilon \frac{e(\overline{\chi_\omega(D)} \alpha, \varepsilon(\sqrt{-3}))}{\varphi(\alpha)} W_1. \)

(ii) \( \chi_\omega(D) = \omega: W_1 = -\omega Y, \) which implies \( \overline{W_1} = -\omega^2 Y \) and thus \( W_1 = -u_{\alpha,b} \epsilon \frac{e(\overline{\chi_\omega(D)} \alpha, \varepsilon(\sqrt{-3}))}{\varphi(\alpha)} W_1. \)

(iii) \( \chi_\omega(D) = \omega^2: W_1 = -\omega^2 \chi_D(\xi) \varepsilon(\sqrt{-3}) Y \) and thus \( W_1 = -u_{\alpha,b} \left( \frac{D}{3m^*} \right) \chi_D(\xi) \varepsilon(\sqrt{-3}) \overline{W_1}. \)

Combining this with Lemma 5.7, we get the relation between \( U \) and \( W_1: \)

\textbf{Corollary 5.15.} \( W_1 = t_\alpha \left( \frac{D}{3m^*} \right) D'U. \)

\textbf{Proof:} We recall from Lemma 5.7 that \( W_1 = D' \frac{\alpha}{\varphi(\alpha)} \chi_D(\overline{\alpha}) u_{\alpha,-b_0} \varepsilon(\sqrt{-3}) \varepsilon(\overline{\alpha}) U. \) Combining with Proposition 5.13, we get \( W_1 = t_\alpha \left( \frac{D}{3m^*} \right) D'U. \)

Finally we relate the complex conjugate of \( U/\alpha \) to itself below:

\textbf{Proposition 5.16.} We have \( \frac{U}{\overline{\alpha}} = c_\alpha t'_\alpha \frac{U}{\alpha} \) for \( t'_\alpha = t_\alpha = \begin{cases} 
 1 & \text{for } D \equiv \pm 1(9) \\ 
 -1 & \text{for } D \equiv \pm 4(9) \\ 
 -\varepsilon(\sqrt{-3}) & \text{for } D \equiv \pm 2(9), 
 \end{cases} \)

\textbf{Proof:} From Corollary 5.15, we have \( W_1 = t_\alpha \left( \frac{D}{3m^*} \right) D'U. \) Thus applying Proposition 5.14, we get the result \( \frac{U}{\alpha} = t'_\alpha \left( \frac{D}{3m^*} \right) \varepsilon(\overline{\alpha}) \chi_D(\overline{\alpha}) \chi_D(\overline{\alpha}) \frac{U}{\alpha} \)

\subsection*{5.3 Proof of Theorem 5.1}

Finally, we are ready to prove Theorem 5.1.

We define:

\begin{equation}
Z = \chi_D(\overline{\alpha}) \chi_3(\overline{\alpha}) \text{Tr}_{H_3/D^*/K} \left( \frac{\Theta_M(D^*\tau_0/m^*)}{\Theta_K(\omega)} D^{1/3} \alpha^{-1/2} \right),
\end{equation}

where \( \chi_3(\alpha) := \frac{\alpha(\sqrt{-3})}{\varphi(\alpha)} \) is the cubic root of unity \( \omega^r \) such that \( \alpha/\omega^r = \pm \varphi(\alpha). \)

From the property \( \Theta_K(\tau_0/m) = \varphi(\alpha) \Theta_K(\omega), \) as \( (\alpha) = [m, \tau_0]. \) Thus we can rewrite

\begin{equation}
Z = \chi_D(\overline{\alpha}) \varepsilon(\sqrt{-3}) \chi_3(\alpha) \text{Tr}_{H_3/D^*/K} \left( \frac{\Theta_M(D^*\tau_0/m^*)}{\Theta_K(\tau_0/m)} D^{1/3} \alpha^{-1/2} \right),
\end{equation}

and we show in Proposition 6.2 in Section 6 that \( Z \in \mathcal{O}_K. \) We have defined \( Z \) such that

\begin{equation}
Z = \frac{U}{\alpha} \chi_D(\overline{\alpha}) \chi_3(\overline{\alpha}),
\end{equation}

in order to have from Proposition 5.10

\begin{equation}
\overline{Z} = t'_\alpha u_{\alpha,-b_0} \left( \frac{D}{3m^*} \right) \varepsilon(\overline{\alpha}) Z,
\end{equation}

giving us \( Z = \pm Z. \)
It is immediate to see (35) for \( m \equiv 1(4) \) from (33). For \( m \equiv 3(4) \), we first note that \( \frac{L}{\alpha} = Z'^{(i)} \), where \( Z'^{(i)} = \text{Tr}_{H_{D/K}} \frac{\Theta_M(D_{\tau}/(2m))}{\Theta_K(\tau/m)} \alpha^{1/2} D^{1/3} \). In Section 6.3, we will show in Lemma 6.14 that \( Z' = Z'^{(i)} \) under the condition (27), where \( Z' = \text{Tr}_{H_{D/K}} \frac{\Theta_M(D_{\tau}/(2m))}{\Theta_K(\tau/m)} \alpha^{1/2} D^{1/3} \). Thus \( Z' = Z'^{(i)} = U/\alpha \) and we again have \( Z = \frac{L}{\alpha} \chi_D(\alpha) \chi_D(3\alpha) \).

Now we are ready to show:

**Corollary 5.17.** For \( Z \) defined above, we have:

\[
\frac{L(E_{D_{\alpha,1}}, 1)}{\Omega_{D_{\alpha}} \Omega_{D_{\alpha}}^*} = c_0(2^e Z/\sqrt{-3})^2,
\]

with \( \overline{Z} = -c_0 Z \) and \( c_0 = (-1)^{\gamma_3(c_{E,D_{\alpha}})} \), \( e = 1 \) for \( m \equiv 1(4) \) and \( e = 0 \) for \( m \equiv 3(4) \).

**Proof:** We recall, from Corollary 5.4, \( X^* = \frac{L}{3m(1-\chi)} \), and from Theorem 1.6, we have \( \frac{L(E_{D_{\alpha,1}}, 1)}{\Omega_{D_{\alpha}} \Omega_{D_{\alpha}}^*} = |2L\zeta(1, \chi)\frac{X^*}{\sqrt{3\alpha}}|^2 \), thus \( \frac{L(E_{D_{\alpha,1}}, 1)}{\Omega_{D_{\alpha}} \Omega_{D_{\alpha}}^*} = \frac{4}{3} \frac{L}{\alpha} \Omega_{D_{\alpha}}^* \) and further \( \frac{L(E_{D_{\alpha,1}}, 1)}{\Omega_{D_{\alpha}} \Omega_{D_{\alpha}}^*} = |2Z|^2/3 \).

Using Proposition 5.10, we have \( \overline{Z} = t'_\alpha \left( \frac{D}{3m^*} \right) \varepsilon(\overline{\alpha}) \varepsilon(Z) \), which is equivalent to:

- for \( \alpha \equiv 1(4) \): \( \overline{Z} = t'_\alpha \left( \frac{D}{3m^*} \right) \varepsilon(\overline{\alpha}) \varepsilon(\overline{\alpha}) Z \), where we have used Lemma 2.8
- for \( \alpha \equiv -1(4) \): \( \overline{Z} = -t'_\alpha \left( \frac{D}{3m^*} \right) \varepsilon(\overline{\alpha}) \varepsilon(Z) \). As \( \varepsilon(\overline{\alpha}) = -\varepsilon(\sqrt{-3}) \) from Lemma 2.3, we get \( \overline{Z} = t'_\alpha \left( \frac{D}{3m^*} \right) \varepsilon(\sqrt{-3}) Z \)
- for \( \alpha \equiv \pm \sqrt{-3}(4) \): \( \overline{Z} = -t'_\alpha \varepsilon(\sqrt{-3}) \left( \frac{D}{3m^*} \right) \varepsilon(\overline{\alpha}) \varepsilon(Z) \) for \( \alpha \equiv \pm \sqrt{-3}(4) \), as \( u_{\alpha,-b_0} = -1 \) from (27).

As \( \varepsilon(\sqrt{-3}) = -\varepsilon(\overline{\alpha}) \) from Lemma 2.3, we get \( \overline{Z} = t'_\alpha \left( \frac{D}{3m^*} \right) \varepsilon(\sqrt{-3}) Z \).

Thus \( \overline{Z} = -cZ \) for \( c = \left\{ \begin{array}{ll}
- \left( \frac{D}{3m^*} \right) & D \equiv \pm 1 \\
\left( \frac{D}{3m^*} \right) & D \equiv \pm 4(9) \\
\left( \frac{D}{3m^*} \right) & D \equiv \pm 2(9)
\end{array} \right. \), and this is exactly \( c_0 \) (see Appendix 7A).

As \( Z \in O_K \), we get immediately:

**Corollary 5.18.** \( Z \in \mathbb{Z} \) if \( c_0 = -1 \), and \( Z/\sqrt{-3} \in \mathbb{Z} \) if \( c_0 = 1 \).

### 6 Shimura reciprocity computations

#### 6.1 Fields of definition

In the current section, we will show that all the traces used in Section 6.0 are indeed well-defined. This is proved below by applying Shimura’s reciprocity law.

**Lemma 6.1.** For \( \tau_0 = \frac{-b_0 + \sqrt{-3}}{2}, b_0 \equiv \sqrt{-3}(\alpha), e = \pm 1, e', e'' \in \{0, 1\}:

(i) \( \Theta_M(D_{\tau_0} \frac{\tau_0}{\tau_{m'}}) \Theta_K(\tau_0 \psi/m') \alpha^{1/2} D^{1/3} \in H_{3D'\ast} \)

(ii) \( \Theta_M(D_{\tau_0} \frac{\tau_0}{\tau_{m'}}) \Theta_K(\tau_0 \psi/m) \alpha^{1/2} D^{1/3} \in H_{3D'm'\ast} \)

(iii) \( \text{Tr}_{H_{3D'\ast}/K} \Theta_M(D_{\tau_0} \frac{\tau_0}{\tau_{m'}}) \Theta_K(\tau_0 \psi/m') \alpha^{1/2} D^{1/3} = 0 \)

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Proof: Let $G(z) = \frac{\Theta_M\left(D^x \frac{\tau - s}{\tau \cdot m}\right)}{\Theta_K\left(\frac{m}{m}\right)}$. For (i) it is enough to show the invariance under $\text{Gal}(K^{ab}/H_{3D^*})$. Thus let $\mathcal{A}$ be an ideal of norm $a$ prime to $3D^*m$ with generator $k_{\mathcal{A}} = ta + s\tau_0$ such that $3D^*|s$. We want to show that $G(\tau_0)^{\sigma_{3^*}^{-1}} = G(\tau_0)$. Using Lemma 24 we have:

$$G(\tau_0)^{\sigma_{3^*}^{-1}} = G\left(\left(\frac{ta - sb}{s} - \frac{scm}{t}\right)\tau_0\right),$$

where $c = \frac{k^2_{\mathcal{A}}}{ta \cdot m}$. We compute $\Theta_M\left(D^x \frac{0}{3^* \cdot m}\right)\left(\frac{ta - sb}{s} - \frac{scm}{t}\right)\tau_0) = \Theta_M\left(D^x \frac{t}{3^* \cdot m}\right)\frac{\tau_0}{\tau^2}$. Then $3m^*|3^* ms/D^x$ and we get

$$\Theta_M\left(D^x \frac{0}{3^* \cdot m}\right)\left(\frac{ta - sb}{s} - \frac{scm}{t}\right)\tau_0) = \left(\frac{t}{3m^*}\right)(s\tau_0 + t)\Theta_M(\tau_0),$$

where we have used Lemma 27 for the matrix $\left(D^x \frac{0}{3^* \cdot m}\right)\left(\frac{ta - sb}{s} - \frac{scm}{t}\right)$ in $\Gamma_0(3m^*)$.

Similarly we get $\Theta_K\left(\frac{0}{3^* \cdot m}\right)\left(\frac{ta - sb}{s} - \frac{scm}{t}\right)\tau_0) = \left(\frac{1}{3}\right)(s\tau_0 + t)\Theta_K(\tau_0/3^*)$ using 34. Thus we get:

$$G(\tau_0)^{\sigma_{3^*}^{-1}} = \left(\frac{t}{m^*}\right)G(\tau_0).$$

We compute $(\tau_1^{1/2})^{\sigma_{3^*}^{-1}} = \left(\frac{ta - sb}{s} - \frac{scm}{t}\right)$. For $\alpha \equiv 1(4)$ this is immediate from the reciprocity law $\left[\frac{a}{\mathfrak{A}}\right] = \left[\frac{r_{\mathfrak{A}}}{\mathfrak{A}}\right] = \left[\frac{ta - sb}{s} - \frac{scm}{t}\right]$. For $\alpha \equiv -1, \pm \sqrt{-3}(4)$, we have $ta + s\tau_0 \equiv ta - sb \equiv \pm 1(4)$. We compute from $\left[\frac{1}{3} \left[\frac{m}{\tau}\right]\right] = \left[\frac{ta - sb}{s} \frac{ta}{m}\right] = \left[\frac{ta - sb}{s} \frac{ta}{m}\right]$. As $s(ta - sb) \equiv 1(a)$ and $t(ta - sb) \equiv 1(m^*/m)$, we get $G(\tau_0)\tau_1^{1/2}$ is invariant under $\mathcal{A}$, from which we get (i). The proof of (ii) is similar.

For (iii), we compute $(\alpha^{1/2})^{\sigma_{3^*}^{-1}} = \left[\frac{a}{\mathfrak{A}}\right]^{1/2}$ and we get $\left[\frac{a}{\mathfrak{A}}\right] = \left(\frac{ta}{m^*}\right)$. Thus $(G(\tau_0)\alpha^{1/2})^{\sigma_{3^*}^{-1}} = \left(\frac{a}{\mathfrak{A}}\right)\left(\frac{ta}{m^*}\right)G(\tau_0)\alpha^{1/2}$, which will have trace 0.

Similarly, one can show:

Lemma 6.2. Under the same conditions as above, for $m \equiv 3(4)$ we have:

(i) $\frac{\Theta_M(D^x \frac{ta - sb}{s} \frac{ta}{m^*})}{\Theta_K(\tau_0/3^*)}\tau_0^{1/2}D^1/3 \in H_{6D}$, $\frac{\Theta_M(D^x \frac{ta - sb}{s} \frac{ta}{m^*})}{\Theta_K(\tau_0/m^*)}\tau_0^{1/2}D^1/3 \in H_{12D}$.

(ii) $\frac{\Theta_M(D^x \frac{ta - sb}{s} \frac{ta}{m^*})}{\Theta_K(\tau_0)}\tau_0^{1/2}D^1/3 \in H_{6Dm}$, $\frac{\Theta_M(D^x \frac{ta - sb}{s} \frac{ta}{m^*})}{\Theta_K(\tau_0)}\tau_0^{1/2}D^1/3 \in H_{12Dm}$.

(iii) $\text{Tr}_{H_{6Dm}/K}\frac{\Theta_M(D^x \frac{ta - sb}{s} \frac{ta}{m^*})}{\Theta_K(\tau_0)}\alpha^{1/2}D^1/3 = 0$.

Thus we get $X^*$ and $T$ well-defined from (ii) of Lemma 6.1 and Lemma 6.2. From (i) of Lemma 6.1 and Lemma 6.2 for $e^\theta = 0$, $U, Y, W_i$ are well-defined, while for $e^\theta = 1$ then $Z, U^\circ, Y^\circ, W_i^\circ$ are well-defined.

We finally remark that $F(\tau_0)$ is an algebraic integer for all the modular functions $F(z) = \frac{\Theta_M(D^x \frac{\tau - s}{\tau \cdot m})}{\Theta_K(\frac{m}{m^*})}$ defined above in Lemma 6.1 and Lemma 6.2. This is a standard result from CM theory: for $\gamma \in \text{SL}_2(\mathbb{Z})$, each $F(\gamma z)$ is a modular function with algebraic coefficients in its Fourier expansion; then $F(\tau_0)$ is an algebraic integer when evaluated at a CM point $\tau_0$. Thus $F(\tau_0)D^{1/3}\alpha^{1/2}$ are algebraic integers and their traces are in $\mathcal{O}_K$. Thus we can state:

Proposition 6.3. (i) $X^*, U, T, Y, W_i \in \mathcal{O}_K$.

(ii) $Z, Y^\circ, W_i^\circ, U^\circ \in \mathcal{O}_K.$
6.2 Important Lemma

The following Lemma is essential in computing the complex conjugates of $Y_0$ and $W_1$:

**Lemma 6.4.** Let $\tau = \frac{-b_0+\sqrt{-3}}{2}$ such that $b_0 \equiv \sqrt{-3}(\alpha)$. For the ideal $A_0 = (k_0)$ ideal generated by $k_0 = \mathfrak{t}_{A_0} + s_{A_0} \tau$ of norm $a$ such that $3m^*|s_{A_0}$ and $D'|t_{A_0}$, we have:

$$
\Theta_M\left(\frac{D'}{3m^*}\right) \left(\frac{t_{A_0}}{m^*}\right) \left(\Theta_M\left(\frac{\tau}{D'|3m^*}\right)\right)^{-1}_{A_0},
$$

where $e, \epsilon, \epsilon'' \in \{0, 1\}$.

**Proof:** Let $c = \frac{b_0^2+3}{4am}$. Note that $G(z) = \Theta_M\left(\frac{\tau}{3m^*}\right)$ is a modular function of level dividing $6D'm^*$ and, from Shimura reciprocity law \(23\), we have:

$$G(\tau)^{-1} = G\left(\frac{t_{A_0} - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}}\right).$$

Explicitly, we compute:

$$
(37) \quad \Theta_M\left(\frac{D'}{3m^*}\right) \left(\frac{t_{A_0} - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}}\right) = \Theta_M\left(\frac{(t_{A_0} - s_{A_0})D' - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}D'}\right) \left(\frac{1}{3m^*}\right) \left(\Theta_M\left(\frac{\tau}{3m^*}\right)\right).
$$

As $A_0 = m^*s$ with $3|s$, we have $3m^*|s_{A_0} \mathfrak{m}^{1/3}$, and using Lemma \(27\) we get in \(37\) $\frac{t_{A_0}/D'}{3m^*} = \frac{t_{A_0}}{3}(s_{A_0} \mathfrak{m}^{1/3} + t_{A_0})\Theta_M\left(\frac{\tau}{3m^*}\right)$. Similarly we get

$$
\Theta_K\left(\frac{1}{3m^*}\right) \left(\frac{t_{A_0} - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}}\right) = \left(\frac{t_{A_0}}{3}\right) (s_{A_0} \mathfrak{m}^{1/3} + t_{A_0})\Theta_K(\tau).
$$

Thus we have $G\left(\frac{t_{A_0} - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}}\right) = \frac{1}{D'} \left(\frac{D'}{3m^*}\right) \left(\frac{t_{A_0}}{m^*}\right) \left(\Theta_M\left(\frac{\tau}{3m^*}\right)\right)$.

6.3 Condition for $m = 3(4)$

We recall $b_i = \sqrt{-3}(\alpha)$ such that $b_1 \equiv 1(8)$ and $b_2 \equiv -1(8)$, and $\tau_i = \frac{-b_i+\sqrt{-3}}{2}$. We define for $m = 3(4)$:

- $X^{(i)} = Tr_{H_{6D/m}/K} \left(\Theta_M\left(\frac{D_{\tau_i/2}}{K}\right)\right) \alpha^{1/2}D^{1/3}$, $X = Tr_{H_{12D/m}/K} \left(\Theta_M\left(\frac{D\tau_i}{K}\right)\right) \alpha^{1/2}D^{1/3}$
- $Z^{(i)} = Tr_{H_{6D/m}/K} \left(\Theta_M\left(\frac{D\tau_i}{2}\right)\right) \alpha^{1/2}D^{1/3}$, $Z = Tr_{H_{12D/m}/K} \left(\Theta_M\left(\frac{D\tau_i}{K}\right)\right) \alpha^{1/2}D^{1/3}$

We note that $Z'$ and $Z^{(i)}$ are well defined from Lemma \(6.2\) and they take values in $\mathcal{O}_K$.

**Lemma 6.5.** For $a = \pm \sqrt{-3}(4)$, we have $X^{(i)} = 0$ and $Z^{(i)} = 0$ for $[\alpha] \left(\frac{\tau_i}{4}\right) = 1$.

**Proof:** Let $\tau_i = \frac{-b_i+\sqrt{-3}}{2}$ and let $A$ be an ideal prime to $6mD$, with generator $\beta = ta + sm \tau_i$ such that $2|s$, $3D|s$. We note that $N_{\mathcal{A}} = a \equiv 3(4)$ and we fix the generator of $\mathcal{A}$ such that $\beta \equiv \sqrt{-3}(4)$. Note that then $\tau_i = -b_i(4)$.

Then we have by Shimura reciprocity law \(23\), for the modular function $F(z) = \Theta(3z/2)/\Theta(z)$ of level $6Dm$, $F(\tau_i) = F\left(\frac{t_{A_0} - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}}\right)$. Explicitly, we compute

$$
\Theta_M\left(\frac{D}{2}\right) \left(\frac{t_{A_0} - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}}\right) \tau_i = \Theta_M\left(\frac{t_{A_0} - s_{A_0} \mathfrak{m}^{1/3}}{t_{A_0}}\right) (\mathfrak{m}^{1/3}/2).$$

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We note that $12m|2sm/D$, thus the matrix above is in $\Gamma_0(3m^*)$ and we can apply Lemma 2.7. Thus we get above $(sm\tau_i + t)\left(\frac{1}{3m^*}\right)\Theta_M(D\tau_i/2)$. Similarly we compute $\Theta_K((\frac{ta-sb-smc}{t})\tau_i) = (sm\tau_i + t)\left(\frac{1}{4}\right)\Theta_K(\tau_i)$. Thus we get:

$$F(\tau_i) = \left(\frac{t}{m^*}\right)F(\tau_i).$$

On the other hand $\left(\frac{a^1/2}{p}\right) = a^{1/2}\left[\frac{a}{p}\right]$. From the reciprocity law [7] we have $\left[\frac{a}{m}\right] = \left[\frac{a}{m}\right][\beta][\beta]$. We compute $\left[\frac{m}{p}\right] = \left[\frac{m}{p}\right]$ and $[\beta][\beta] = -[\beta] = [\alpha]$, as $\beta \equiv \sqrt{-3}(4)$. Thus we get

$$F(\tau_i) = \left(\frac{t}{m}\right)[\alpha]F(\tau_i)\alpha^{1/2}D^{1/3},$$

which equals $-\left(\frac{a}{m}\right)[\alpha]F(\tau_i)\alpha^{1/2}D^{1/3}$. Thus, if $\left(\frac{a}{m}\right)[\alpha] = 1$, the trace of $X(i)$ equals 0.

The proof is similar for $Z(i)$, by taking $\mathcal{A}'$ be an ideal prime to $6D$, with generator $\beta = ta + st_i$ such that $2|s$, $3|s$. Then $c = \frac{b^2 + 3}{2}$ is divisible by $m$. For the modular function $F'(z) = \Theta(D_{2m})/\Theta(z)$ of level $6D$, we have $F'(\tau_i) = F'( (\frac{ta-sb-smc}{t})\tau_i)$ which gives us:

$$\Theta_M((\frac{D}{2m^*}\tau_i) = \Theta_M((\frac{ta-sb-smc}{t})\tau_i),$$

which equals $\left(\frac{a^1/2}{m^*}\right)(s\tau_i + t)\Theta_M(D\tau_i/2m)$. Similarly we get $\Theta_K((\frac{1}{4m^*}\tau_i) = \left(\frac{1}{4}\right)(s\tau_i + t)\Theta_K(\tau_i/m)$, thus $F'(\tau_i) = \left(\frac{t}{m}\right)F(\tau_i)$, and again $F'(\tau_i)\alpha^{1/2}D^{1/3} = -\left(\frac{a}{m}\right)[\alpha]F'(\tau_i)\alpha^{1/2}D^{1/3}$. This implies $Z(i) = 0$ under the condition $\left(\frac{a}{m}\right)[\alpha] = 1$.

**Corollary 6.6.** For $\alpha \equiv \sqrt{-3}(4)$, for $i$ such that $\left[\frac{a}{m}\right] = 1$, we have $X = X(i)$ and $Z' = Z(i)$.

**Proof:** For $\tau_i = \frac{-b + \sqrt{-3}}{2}$ with $b_i \equiv i(4)$, from Lemma 2.3 for $d = 2$, we have $\Theta_M(D\tau_i/2) + \Theta_M(D\tau_i/2) = 2\Theta_M(D\tau_i)$, thus by multiplying by $\alpha^{1/2}D^{1/3}/\Theta_K(\omega)$ and taking the traces from $H_{12mD}$ to $K$, we get:

$$2X(1) + 2X(3) = 2X.$$

However, since $X(3) = 0$ for $\alpha \equiv \sqrt{-3}$ and $X(1) = 0$ for $\alpha \equiv -\sqrt{-3}$ from Lemma 6.5 we get the result. The proof is similar for $Z'$.

**Remark 6.7.** We note that for $b_i$ under the condition (27), we have $Z''(i) = U$, thus $Z' = Z''(i) = U$.

7 Appendix

7.1 Tamagawa numbers

Using Tate’s algorithm (see [22]), one can compute the Tamagawa number $c_{E,D,\alpha} = c_2c_3c_Dc_{\alpha}$, where:

- $c_2 = 1$,
- $c_3 = \begin{cases} 1 & D \equiv \pm 1(9), \left[\frac{\alpha}{\sqrt{-3}}\right] = 1 \text{ or } D \equiv \pm 4(9), \left[\frac{\alpha}{\sqrt{-3}}\right] = -1 \\ 1 & D \equiv \pm 1(9), \left[\frac{\alpha}{\sqrt{-3}}\right] = -1 \text{ or } D \equiv \pm 4(9), \left[\frac{\alpha}{\sqrt{-3}}\right] = 1 \\ 1 & D \equiv \pm 2(9) \end{cases}$
- $c_D = 3^{\#(p|D)(\frac{p}{3})}) = 1$, which gives us $c_D = \begin{cases} 3^{2k} & \frac{D}{(3m^*)} = 1 \\ 3^{2k+1} & \frac{D}{(3m^*)} = -1 \end{cases}$.
\[ c_\alpha = \begin{cases} 
1 & 2D^2 \not\equiv u^3(\alpha) \\
4 & -2D^2 \equiv u^3(\alpha) 
\end{cases} = \begin{cases} 
1 & \chi_{2D^2}(\alpha) \neq 1 \\
4 & \chi_{2D^2}(\alpha) = 1 
\end{cases}. \]

Thus \( c_0 = (-1)^{v_3(\Pi_{\epsilon \in \mathcal{D}_m} c_\epsilon)} = \begin{cases} 
-\left( \frac{D}{3m} \right)^{\frac{\alpha - 3}{2}}, & D \equiv \pm 1 \\
\left( \frac{D}{3m} \right)^{\frac{\alpha - 3}{2}}, & D \equiv \pm 4(9) \\
\left( \frac{D}{3m} \right), & D \equiv \pm 2(9) 
\end{cases}. \)

7.2 Computations characters

We recall the constant \( u_{\alpha,b} = \begin{cases} 
1 & \text{if } \alpha = 1(4) \\
-1 & \text{if } \alpha = -1(4) \\
\alpha \left( \frac{-b}{\alpha} \right) & \text{if } \alpha = \pm \sqrt{3}(4) 
\end{cases} \).

We also recall the ideals

- \( C_0 = \left( \frac{-b_0,0 + \sqrt{-3}}{2} \right) / (\alpha \sqrt{-3}) \) of norm \( c_0 \), with \( b_{0,0} \equiv 0(3), b_{0,0} \equiv \sqrt{-3}(\alpha), b_{0,0} \equiv -1(2D) \)
- \( C_i = \left( \frac{-b_{0,i} + \sqrt{-3}}{2} \right) / (\alpha) \) of norm \( c_i \), with \( b_{0,i} \equiv -i(3), b_{0,i} \equiv \sqrt{-3}(\alpha), b_{0,i} \equiv -1(2D) \), for \( i = 1, 2 \)
- \( A_0 = t_A a + 3m^* s \left( \frac{b_{0,0} + \sqrt{-3}}{2} \right) \), with \( D^* \| A \), of norm \( a_0 \)
- \( A_j = \left( \frac{-b_j + \sqrt{-3}}{2} \right) \) of norm \( a_j \) with \( b_j - b_0 \equiv j(m), b_j^2 \not\equiv -3(m) \) and \( b_j \equiv 1(6D) \)
- \( C_0^* = \left( \frac{-b_0^* + \sqrt{-3}}{2} \right) / (\alpha) \) with \( b_0^* \equiv -\sqrt{-3}(\alpha), b_0^* \equiv 1(6D) \)

We also assume that all \( b_{0,i}, b_i^*, b_j \) are \( \equiv \pm 1(8) \). We compute below the character \( \varepsilon \) for each of the ideals defined:

**Lemma 7.1.**

(i) \( \varepsilon(C_0) = u_{\alpha,b_0,0} \varepsilon(\alpha) \)

(ii) \( \varepsilon(C_i) = u_{\alpha,b_0,i} \varepsilon(\alpha) \varepsilon(\sqrt{-3}) \)

(iii) \( \varepsilon(A_0) = \left( \frac{\alpha}{\mathbb{N}^*} \right) \)

(iv) \( \varepsilon(A_j) = \begin{cases} 
\left( \frac{(b_j + b_0)/2}{m} \right) \left( \frac{\alpha}{\mathbb{N}^*} \right) & \text{for } \alpha \equiv \pm \sqrt{-3}(4) \\
\left( \frac{(b_j + b_0)/2}{m} \right) & \text{for } \alpha \equiv 1(4) 
\end{cases} \)

(v) \( \varepsilon(C_0^*) = \varepsilon(\alpha) \varepsilon(\sqrt{-3})u_{\alpha,-b_0^*} \)

**Proof:** For (i), (ii) and (v), we first note that for \( b \equiv \sqrt{-3}(\alpha) \):

\[ \varepsilon\left( \frac{b + \sqrt{-3}}{2} \right) = \left[ \frac{b + \sqrt{-3}}{\alpha} \right] \frac{\alpha}{\mathbb{N}^*} \left[ \frac{b - \sqrt{-3}}{2} \right] \left[ \frac{b + \sqrt{-3}}{2} \right] \left[ \alpha(\frac{b + \sqrt{-3}}{2}) \right] \left[ \alpha(\frac{b - \sqrt{-3}}{2}) \right]. \]

We further apply the reciprocity law \( \left[ \frac{\sqrt{-3}}{\alpha} \right] = \frac{\sqrt{-3}}{\mathbb{N}^*} \left[ \frac{\sqrt{-3}}{2} \right] \left[ \frac{b + \sqrt{-3}}{2} \right] \left[ \frac{\sqrt{-3}}{2} \right] \left[ \alpha(\frac{b + \sqrt{-3}}{2}) \right] \left[ \alpha(\frac{b - \sqrt{-3}}{2}) \right] \left[ -\alpha \sqrt{-3} \right], \) which equals:

\[ \varepsilon\left( \frac{b + \sqrt{-3}}{2} \right) = \varepsilon(\sqrt{-3})u_{\alpha,b} \]

for \( b \equiv \pm 1(8) \). Then we can compute:
Lemma 7.2. We have

(i) \( \chi_D(\mathcal{C}_0) = \chi_D(D) \),
(ii) \( \chi_D(\mathcal{C}_1) = \chi_D(\mathcal{C}_2) = \chi_D(\mathcal{C}_3) \).

Proof:

(i) \( \chi_D(\mathcal{C}_0) = \chi_D(D) \).
(ii) \( \chi_D(\mathcal{C}_1) = \chi_D(\mathcal{C}_2) = \chi_D(\mathcal{C}_3) \).

7.3 Computing Galois conjugates

The following Lemma is used in Sections 5.1 and 5.2 to show that various terms are Galois conjugate to each other. The proof consists of applying Shimura reciprocity law [24] and the inverse transformations [15] and [16] for the theta functions \( \Theta_M \) and \( \Theta_K \), adjusting the formulas using [20].

Lemma 7.3. For the Galois actions corresponding to the ideals \( C_i, C^{s}_0, C_0, A_i \) defined above, we have:

(i) \( \frac{\Theta_M(-\tau_{0}/m)}{\Theta_K(\omega)} = \frac{3m(\tau_{0}/3m)}{3m(\tau_{0}/3m)} \),
(ii) \( \frac{\Theta_M(-\tau_{0}/6m)}{\Theta_K(\omega)} = \frac{3m(\tau_{0}/3m)}{3m(\tau_{0}/3m)} \).

for \( m \equiv 1(4) \)

for \( m \equiv 3(4) \)
(ii) \( \frac{\Theta_M(-\tau_m/3mD')}{\Theta_K(-\tau_m/3)} = D' \sqrt{m} \left( \frac{\Theta_M(D'\tau_m/m)}{\Theta_K(m)} \right) \sigma^{-1}_{c_0} \) for \( m \equiv 1(4) \)

\( \frac{\Theta_M(-\tau_m/3mD)}{\Theta_K(-\tau_m/3)} = D \sqrt{m} \left( \frac{\Theta_M(D\tau_m/m)}{\Theta_K(m)} \right) \sigma^{-1}_{c_0} \) for \( m \equiv 3(4) \)

(iii) \( \frac{\Theta_M(-\tau_m/3mD')}{\Theta_M(-\tau_m/3)} = D' \sqrt{m} \left( \frac{\Theta_M(D'\tau_m/m)}{\Theta_M(m)} \right) \sigma^{-1}_{A_i} \) for \( m \equiv 1(4) \)

\( \frac{\Theta_M(-\tau_m/3mD)}{\Theta_M(-\tau_m/3)} = D \sqrt{m} \left( \frac{\Theta_M(D\tau_m/m)}{\Theta_M(m)} \right) \sigma^{-1}_{A_i} \) for \( m \equiv 3(4) \)

(iv) \( \frac{\Theta_M(-\tau_m/3mD')}{\Theta_M(-\tau_m/3)} = D' \sqrt{m} \left( \frac{\Theta_M(D'\tau_m/m)}{\Theta_M(m)} \right) \sigma^{-1}_{c_0} \) for \( m \equiv 1(4) \)

\( \frac{\Theta_M(-\tau_m/3mD)}{\Theta_M(-\tau_m/3)} = D \sqrt{m} \left( \frac{\Theta_M(D\tau_m/m)}{\Theta_M(m)} \right) \sigma^{-1}_{c_0} \) for \( m \equiv 3(4) \)

(v) \( \frac{\Theta_M(D\tau/m)}{\Theta_K(\omega)} = \frac{\Theta_M(-\tau_m/3)}{\Theta_K(\omega)} \left( \frac{\Theta_M(D\tau/m)}{\Theta_K(\omega)} \right) \sigma^{-1}_{c_0} \) for \( m \equiv 1(4) \)

\( \frac{\Theta_M(D\tau/m)}{\Theta_K(\omega)} = \frac{\Theta_M(-\tau_m/3)}{\Theta_K(\omega)} \left( \frac{\Theta_M(D\tau/m)}{\Theta_K(\omega)} \right) \sigma^{-1}_{c_0} \) for \( m \equiv 3(4) \)

Proof: We sketch below the proofs for \( m \equiv 1(4) \).

(i) \( \frac{\Theta_M(\frac{a_0 + \sqrt{\Delta}}{2})}{\Theta_K(\frac{a_0 + \sqrt{\Delta}}{2})} = \varphi(\alpha) \frac{\Theta_M(\frac{b_0 + \sqrt{-1}}{2})}{\Theta_K(\frac{b_0 + \sqrt{-1}}{2})} = 3 \varphi(\alpha) D' \frac{\Theta_M(D'\frac{b_0 + \sqrt{-1}}{2})}{\Theta_K(D'\frac{b_0 + \sqrt{-1}}{2})} = D' \sqrt{m} \left( \frac{\Theta_M(D'\frac{b_0 + \sqrt{-1}}{2})}{\Theta_K(D'\frac{b_0 + \sqrt{-1}}{2})} \right) \sigma^{-1}_{c_0} \)

where we first have used (21) for the ideal \( \alpha = [m, \frac{-b_0 + \sqrt{-3}}{2}] \), followed by the transformations (15) and (16), and finally using Shimura reciprocity (24) for the ideal \( C_0 = \frac{(-b_0 + \sqrt{-3})}{(\alpha \sqrt{3})} \) of norm \( c_0 \).

(ii) \( \frac{\Theta_M(\frac{b_0 + \sqrt{-1}}{2})}{\Theta_K(\frac{b_0 + \sqrt{-1}}{2})} = D' \sqrt{m} \left( \frac{\Theta_M(D'\frac{b_0 + \sqrt{-1}}{2})}{\Theta_K(D'\frac{b_0 + \sqrt{-1}}{2})} \right) \sigma^{-1}_{c_0} \)

where we have used (15) and (16), followed by Shimura reciprocity (24) for the ideal \( C_i = \frac{(-b_0 + \sqrt{-3})}{(\alpha \sqrt{3})} \) of norm \( c_i \), and in the last step we applied (20).

(iii) \( \frac{\Theta_M(\frac{b_0 + \sqrt{-1}}{2})}{\Theta_M(\frac{b_0 + \sqrt{-1}}{2})} = D' \sqrt{m} \left( \frac{\Theta_M(D'\frac{b_0 + \sqrt{-1}}{2})}{\Theta_M(D'\frac{b_0 + \sqrt{-1}}{2})} \right) \sigma^{-1}_{A_i} \), where we have used (15) and (16), followed by (24) for the ideal \( A_i = \frac{(-b_0 + \sqrt{-3})}{(\alpha \sqrt{3})} \) with norm \( a_i = \frac{b_0^2 + 3}{2} \).

(iv) \( \frac{\Theta_M(\frac{a_0 + \sqrt{\Delta}}{2})}{\Theta_M(\frac{a_0 + \sqrt{\Delta}}{2})} = D' \sqrt{m} \left( \frac{\Theta_M(D'\frac{a_0 + \sqrt{\Delta}}{2})}{\Theta_M(D'\frac{a_0 + \sqrt{\Delta}}{2})} \right) \sigma^{-1}_{A_0} \) where we first used (15) and (16), followed by (24) for \( A_0 = \left[ a_0, \frac{-b_0 + \sqrt{-3}}{2} \right] = \frac{(-b_0 + \sqrt{-3})}{(\alpha \sqrt{3})} \), and finally (20).
\[ \Theta_M \left( \frac{\sqrt{\lambda^2 + \sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}} \right) = D \sqrt{D'} \Theta_M \left( \frac{\sqrt{\lambda^2 + \sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}} \right) \left( \Theta_M \left( \frac{\sqrt{\lambda^2 + \sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}} \right) \right)^{\frac{1}{2}} \]  

where we have used (15) and (16), followed by Shimura reciprocity (24) for the ideal \( C_0 \) of norm \( c_0 \), and in the last step we applied (20).

\[ \Theta_M \left( \frac{\sqrt{\lambda^2 + \sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}} \right) = D \sqrt{D'} \Theta_M \left( \frac{\sqrt{\lambda^2 + \sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}} \right) \left( \Theta_M \left( \frac{\sqrt{\lambda^2 + \sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}}} + \frac{1}{\sqrt{\mu^2 + \frac{1}{\sqrt{\nu^2}}}} \right) \right)^{\frac{1}{2}} \]  

where we have first used (15) and (16), followed by (24) for the ideal \( \mathcal{A}_i \) of norm \( a_i \).

The proofs for \( m \equiv 3(4) \) are completely similar.

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