ON A TIME-DEPENDENT EXTRA SPATIAL DIMENSION

PETER K.F. KUHFITTIG

Abstract. In the usual brane-world scenario matter fields are confined to the four-dimensional spacetime, called a 3-brane, embedded in a higher-dimensional space, usually referred to as the bulk spacetime. In this paper we assume that the 3-brane is a de Sitter space; there is only one extra spatial dimension, assumed to be time dependent. By using the form of the brane-world energy-momentum tensor suggested by Shiromizu et al. in the five-dimensional Einstein equations, it is proposed that the cosmological expansion of the 3-brane may provide a possible explanation for the collapse of the extra dimension, as well as for the energy stored in the resulting curled-up dimension. More precisely, whenever the bulk cosmological constant $\Lambda$ is negative, the extra spatial dimension rapidly shrinks during the inflation of the brane. When $\Lambda$ is positive, on the other hand, the extra spatial dimension either completely follows the cosmological expansion of the brane or completely ignores it, thereby shrinking relative to the expanding space. This behavior resembles the all-or-nothing behavior of ordinary systems in an expanding universe, as recently demonstrated by R.H. Price.

PACS number(s): 04.20.Cv, 04.50.+h

1. Introduction

The notion that our world may contain more than three spatial dimensions can be traced to the pioneer work of Kaluza and Klein starting in 1919. The development of string/M-theory has resulted in a revival of this idea. At this stage in the evolution of the Universe the extra spatial dimensions are hidden from us four-dimensional observers. It has been conjectured that the extra dimensions had suddenly compactified to become unobservable, but the mechanisms for this dimension breaking has remained somewhat of a mystery [1]. It is proposed in this paper that cosmic inflation may provide a possible explanation, provided that certain conditions are met. It is also proposed that the same

Date: March 28, 2017.
mechanism would be the source of the energy stored in the resulting curled-up dimensions.

We are going to confine ourselves to a single extra spatial dimension with a scale factor that is necessarily time dependent to allow the size to vary. Accordingly, our starting point is the spacetime topology $M \times S^1$, where $M$ refers to a de Sitter space and $S^1$ to an extra-dimensional 1-sphere. So our metric is given by

\begin{equation}
    ds^2 = -dt^2 + [R(t)]^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right] + [\rho(t)]^2 d\chi^2,
\end{equation}

where $\chi$ is the coordinate in the fifth dimension. (Note the time-dependent scale factor $\rho(t)$.) In the usual brane-world picture matter fields are confined to the four-dimensional spacetime (or 3-brane), while gravity acts in five dimensions (the bulk). Our basic assumption is that when dealing with the Early Universe we may apply the five-dimensional Einstein field equations $G_{\mu\nu} = k^2 T_{\mu\nu}$ using a particular form of the energy-momentum tensor: following Ref. [2], $(M, q_{\mu\nu})$ denotes our 3-brane in a five-dimensional spacetime $(V, g_{\mu\nu})$ and

\begin{equation}
    T_{\mu\nu} = -\Lambda g_{\mu\nu} + \delta(\chi)(-\lambda q_{\mu\nu} + \tau_{\mu\nu}).
\end{equation}

Here $\Lambda$ is the cosmological constant of the bulk spacetime, while the hypersurface $\chi = 0$ corresponds to our 3-brane. The $\delta$-function expresses the confinement of matter in the brane; its appearance here is due to the boundary surface $\chi = 0$, where $T_{\mu\nu}$ is proportional to the $\delta$-function. Also, $\lambda$ and $\tau_{\mu\nu}$ are the vacuum energy and the energy-momentum tensor, respectively, of the 3-brane.

Concerning our basic assumption, it must be kept in mind that the coefficients in Eq. (1) are independent of $\chi$, an assumption more in line with the Kaluza-Klein model than the brane-world model. It is not unreasonable to assume, however, that during inflation the enormous rate of expansion is so dominant that outside influences, including the existence of an extra spatial dimension, are negligible. Moreover, the resulting simplification leads directly to several interesting conclusions that may not be apparent otherwise.

Returning to Eq. (1), if $(M, q_{\mu\nu})$ is to be a de Sitter space, we need to let $R(t) = e^{Ht}$, where $H = \sqrt{\Lambda_4/3}$ and $\Lambda_4$ is the cosmological constant of the 3-brane. Universes with exponential expansion are usually called inflationary.

Since the derivations in Ref. [2] do not depend on the sign of $\Lambda$, we are justified in considering the cases $\Lambda > 0$ and $\Lambda < 0$ separately. In the latter case, discussed in Sec. 3, the extra spatial dimension rapidly shrinks during the inflation of the brane. The former case, discussed next, is the more interesting of the two: the extra spatial dimension
either completely follows the cosmological expansion of the brane or completely ignores it, thereby shrinking relative to the expanding space.

2. **The Case \( \Lambda > 0 \)**

Our first step is to calculate the nonzero components of the Einstein tensor in the orthonormal frame. These are given next:

\[
G_{\hat{t}\hat{t}} = 3 \left[ \frac{R'(t)}{R(t)} \right]^2 + 3 \frac{R'(t)\rho'(t)}{R(t)\rho(t)},
\]

\[
G_{\hat{r}\hat{r}} = -2 \frac{R''(t)}{R(t)} - \frac{[R'(t)]^2}{[R(t)]^2} - 2 \frac{R'(t)\rho'(t)}{R(t)\rho(t)} - \frac{\rho''(t)}{\rho(t)},
\]

\[
G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -2 \frac{R''(t)}{R(t)} - \frac{[R'(t)]^2}{[R(t)]^2} - \frac{\rho''(t)}{\rho(t)} - 2 \frac{R'(t)\rho'(t)}{R(t)\rho(t)},
\]

\[
G_{\hat{\chi}\hat{\chi}} = -3 \frac{R''(t)}{R(t)} - 3 \frac{[R'(t)]^2}{[R(t)]^2}.
\]

(The components of the Riemann curvature and Ricci tensors are given in the Appendix.)

2.1. **Solutions.** From Eq. (3), we have

\[
3 \frac{[He^{Ht}]^2}{[e^{Ht}]^2} + 3 \frac{He^{Ht}\rho'(t)}{e^{Ht}\rho(t)} = k_5^2 T_{\hat{t}\hat{t}},
\]

which reduces to

\[
\frac{\rho'(t)}{\rho(t)} = -H + \frac{k_5^2 T_{\hat{t}\hat{t}}}{3H}.
\]

Let \( A_{in} \) be the initial value of \( \rho(t) \) (at the onset of inflation), i.e., \( \rho(0) = A_{in} \). Then the solution is

\[
\rho(t) = A_{in}e^{-Ht}e^{k_5^2 \Lambda t/3H}.
\]

By Eq. (2), since \( \delta(\chi) = 0 \) in the bulk,

\[
\rho(t) = A_{in}e^{-Ht}e^{k_5^2 \Lambda t/3H}.
\]

Similarly, from both Eqs. (1) and (5), we get

\[
\rho''(t) + 2H\rho'(t) + (3H^2 - k_5^2 \Lambda)\rho(t) = 0
\]

and

\[
\rho(t) = c_1 e^{-Ht}e^{\sqrt{-2H^2 + k_5^2 \Lambda} t} + c_2 e^{-Ht}e^{-\sqrt{-2H^2 + k_5^2 \Lambda} t},
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants. Solutions (8) and (10) must agree, particularly at \( t = T \), the end of inflation. It follows that \( c_1 = \)
\(A_{in}\) and \(c_2 = 0\). The second term can also be eliminated for physical reasons: since \(HT \approx 100\) \(3, 4\), \(e^{-HT}e^{-\sqrt{-2H^2+k_5^2}\Lambda t}\) is many orders of magnitude below Planck length and would therefore not make physical sense. So

\[
\rho(t) = A_{in}e^{-Ht}e^{\sqrt{-2H^2+k_5^2}\Lambda t}.
\]

Eqs. (8) and (11) now yield

\[
\frac{k_5^2\Lambda}{3H} = \sqrt{-2H^2+k_5^2}\Lambda
\]

with \(H = \sqrt{\Lambda_4/3}\). The only solutions are \(\Lambda = \Lambda_4/k_5^2\) and \(\Lambda = \Lambda_4/(\frac{1}{2}k_5^2)\).

It remains to show that these solutions are consistent with \(G_{\dot{\chi}\dot{\chi}}\). Observe that this component is completely independent of \(\rho(t)\), suggesting that \(\chi = 0\) in Eq. (2). Retaining the notation \(\delta(\chi)\), Eq. (2) becomes

\[
T_{\dot{\chi}\dot{\chi}} = -\Lambda g_{\dot{\chi}\dot{\chi}} + \delta(\chi)(-\lambda g_{\dot{\chi}\dot{\chi}} + \tau_{\dot{\chi}\dot{\chi}}),
\]

and, since \(R(t) = e^{Ht}\), with \(H = \sqrt{\Lambda_4/3}\),

\[
G_{\dot{\chi}\dot{\chi}} = -3\frac{R''(t)}{R(t)} - 3\frac{|R'(t)|^2}{|R(t)|^2} = -\Lambda_4 - \Lambda_4.
\]

In Eq. (13), \(q_{\dot{\chi}\dot{\chi}} = 0\) and \(g_{\dot{\chi}\dot{\chi}} = 1\), so that

\[
-\Lambda_4 - \Lambda_4 = k_5^2[-\Lambda + \delta(\chi)\tau_{\dot{\chi}\dot{\chi}}].
\]

It now follows that \(\delta(\chi)\tau_{\dot{\chi}\dot{\chi}} = 0\) if, and only if, \(\Lambda = \Lambda_4/(\frac{1}{2}k_5^2)\). So if \(\delta(\chi)\tau_{\dot{\chi}\dot{\chi}} \neq 0\), then \(\Lambda = \Lambda_4/k_5^2\), the other solution. For this case, Eq. (15) implies that

\[
-2\Lambda_4 = k_5^2[-\Lambda_4/k_5^2 + \delta(\chi)\tau_{\dot{\chi}\dot{\chi}}].
\]

One of the properties of the \(\delta\)-function is that \(\delta(y)f(y) = \delta(y)f(0)\) for any continuous function \(f\). Applying this property to \(\tau_{\dot{\chi}\dot{\chi}}\), we have

\[
\delta(\chi)\tau_{\dot{\chi}\dot{\chi}} = \delta(\chi)\left(\tau_{\dot{\chi}\dot{\chi}}|_{\chi = 0}\right),
\]

emphasizing the confinement to the brane. So Eq. (16) becomes

\[
\delta(\chi)\left(\tau_{\dot{\chi}\dot{\chi}}|_{\chi = 0}\right) = -\frac{\Lambda_4}{k_5^2}.
\]

This form is similar to the classical de Sitter forms

\[
T_{\dot{\theta}\dot{\theta}} = T_{\dot{\phi}\dot{\phi}} = T_{\dot{\chi}\dot{\chi}} = -\frac{\Lambda_4}{8\pi}
\]

obtainable from Eqs. (4) and (5) by letting \(k_5^2 = 8\pi\) and eliminating \(\rho(t)\).
2.2. **Analysis.** We now examine each solution in turn:

(1) Suppose \( \Lambda = \frac{\Lambda_4}{k^2} \). Substituting in Eqs. (8) and (11) and recalling that \( H = \sqrt{\Lambda_4/3} \), we get in both cases,

\[
\rho(t) = A_{\text{in}}e^{-Ht}e^{Ht} = A_{\text{in}}.
\]

(2) For the other solution, \( \Lambda = \frac{\Lambda_4}{(\frac{1}{2}k^2)} \), we obtain from Eqs. (8) and (11),

\[
\rho(t) = A_{\text{in}}e^{-Ht}e^{2Ht} = A_{\text{in}}e^{Ht}.
\]

So the “radius” of the extra dimension expands by the factor \( e^{Ht} \), the same as for any other distance in the brane. To see this, consider the proper circumference \( C \) of the circle in the “equatorial slice” \( \theta = \pi/2 \) of the sphere \( r = a \):

\[
C = \int_0^{2\pi} e^{Ht}a \, d\phi = e^{Ht}(2\pi a), \quad t = \text{constant}.
\]

So \( C \) is simply \( e^{Ht} \) times the initial proper circumference. Similarly, the radial proper distance separating two points \( A \) and \( B \) for any fixed \( t \) is given by

\[
\ell(t) = \int_{r_A}^{r_B} e^{Ht}dr = e^{Ht}(r_B - r_A).
\]

So \( \ell(t) \) is just \( e^{Ht} \) times the initial proper radial separation.

To emphasize the rescaling of the \( r \) coordinate on each \( t = \text{constant} \) slice, one could write

\[
\frac{\ell(t)}{e^{Ht}} = r_B - r_A.
\]

In the first case, Eq. (17), the corresponding equation for a fixed \( \ell_1(t) = r_B - r_A \) can be written

\[
\frac{\ell_1(t)}{e^{Ht}} = e^{-Ht}(r_B - r_A),
\]

that is, if \( \rho(t) \) remains fixed, then the original size has the appearance of having shrunk by a factor of \( e^{-Ht} \) relative to the expanded space at the end of inflation.

The shrinking case does have one obvious consequence: for the extra dimension to be very small, say Planck size, at the end of inflation, we must have

\[
A_{\text{in}}e^{-Ht} \approx A_{\text{in}}e^{-100} \approx 10^{-35} \text{ m},
\]
leading to the rather large value $A_{\text{in}} \approx 2.7 \times 10^8$ m. Not being part of the 3-brane, such a large extra dimension cannot be ruled out. Consider also the following model proposed by Chodos and Detweiler (1980 [5]):

\begin{equation}
    ds^2 = -dt^2 + \left( \frac{t}{t_0} \right) (dx^2 + dy^2 + dz^2) + \frac{t_0}{t} d\chi^2.
\end{equation}

Here the size of the extra spatial dimension exceeds that of the other three for $t < t_0$. In fact, at the singularity $t = 0$, there is only one spatial dimension whose size approaches infinity.

The last possibility is actually an exception to the shrinking case: if $A_{\text{in}}$ is infinitely large to start with, it remains infinitely large. This outcome may be more than just a curiosity: an infinite extra dimension is required in certain models [6].

3. The case $\Lambda < 0$

If $\Lambda < 0$, the solution to Eq. (7) retains the form

\begin{equation}
    \rho(t) = A_{\text{in}} e^{-Ht} e^{k^2 \Lambda t/3H}.
\end{equation}

As noted earlier, at the end of inflation ($t = T$), we have $HT \approx 100$, so that $\rho(t)$ is now many orders of magnitude below Planck length and is therefore not an acceptable solution. But from Eq. (9) we get

\begin{equation}
    \rho(t) = e^{-Ht} \left( A_{\text{in}} \cos \sqrt{2H^2 - k_5^2 \Lambda t} + c_2 \sin \sqrt{2H^2 - k_5^2 \Lambda t} \right).
\end{equation}

This solution is plausible, based on the discussion at the end of Sect. 2, but it also shows that $\rho(t)$ has shrunk significantly at the end of inflation.

4. The stored energy

It is generally believed that the extra spatial dimensions are tightly curled up, thereby storing huge amounts of potential energy. This stored energy may actually be the source of dark energy that causes the expansion of the Universe to accelerate. The existence of such energy is confirmed in the present model. Consider, for example, the radial tidal constraint $|R_{\hat{r}\hat{r}}|$ [7]. From the Appendix,

\[ |R_{\hat{r}\hat{r}}| = \left| -\frac{R''(t)}{R(t)} \right| = H^2 = \frac{\Lambda_4}{3}. \]

Since $HT \approx 100$, we have $H(10^{-34} + 10^{-32}) \approx 100$, so that $H \approx 10^{34}$ [3]. So $|R_{\hat{r}\hat{r}}| \approx 10^{34}/3$. As the Universe keeps expanding, $\Lambda_4/3$ becomes very small. Suppose, on the other hand, that a very tiny observer moves
radially in the extra-dimensional 1-sphere. Since \( \rho(t) = A e^{-Ht} \), the tidal constraint (from the Appendix) is given by
\[
|\hat{R}^{i}_{\hat{t} \hat{t}}| = \frac{\Lambda}{3} = \frac{10^{34}}{3}.
\]
Unlike the expanding case discussed above, \( \rho(t) \) cannot continue to shrink beyond Planck length, thereby terminating with a huge value for the tidal constraint. This points to a large amount of stored potential energy, the source of which is inflation.

### 5. Summary

It is proposed in this paper that in the case of a de Sitter 3-brane world, cosmic inflation may provide an explanation for the collapse of the extra spatial dimension, as well as the source of the energy stored in the resulting curled-up dimension. It is shown that whenever the cosmological constant \( \Lambda \) is negative, \( \rho(t) \) shrinks rapidly during the inflation of the brane. When \( \Lambda \) is positive, the extra spatial dimension either completely follows the cosmological expansion of the brane or completely ignores it. In the former case the extra dimension expands, rather than shrinks. In the latter case, \( \rho(t) \) shrinks relative to the expanding space, unless the size of the extra dimension is infinite to start with.

For \( \Lambda > 0 \), the conclusion resembles the interesting all-or-nothing behavior demonstrated in Ref. [8]: a system will either completely follow the cosmological expansion of the universe or completely ignore it.

### Appendix

The nonzero components of the Riemann curvature tensor are
\[
R_{\hat{t} \hat{r} \hat{r}} = R_{\hat{t} \hat{\theta} \hat{\theta}} = R_{\hat{t} \hat{\phi} \hat{\phi}} = -\frac{R''(t)}{R(t)},
\]
\[
R_{\hat{t} \hat{t} \hat{t}} = -\frac{\rho''(t)}{\rho(t)},
\]
\[
R_{\hat{r} \hat{\theta} \hat{\theta}} = R_{\hat{r} \hat{\phi} \hat{\phi}} = -\frac{[R'(t)]^2}{[R(t)]^2},
\]
\[
R_{\hat{r} \hat{\theta} \hat{\phi}} = R_{\hat{r} \hat{\phi} \hat{\theta}} = R_{\hat{\theta} \hat{\phi} \hat{\phi}} = -\frac{R'(t)\rho'(t)}{R(t)\rho(t)}.
\]
The components of the Ricci tensor are
\[
R_{\hat{t} \hat{t}} = -3\frac{R''(t)}{R(t)} - \frac{\rho''(t)}{\rho(t)},
\]
\[ R_{\bar{\tau}\bar{\tau}} = R_{\bar{\theta}\bar{\theta}} = R_{\bar{\phi}\bar{\phi}} = \frac{R''(t)}{R(t)} + 2\left[\frac{R'(t)}{R(t)}\right]^2 + \frac{R'(t)\rho(t)}{R(t)\rho(t)}, \]

\[ R_{\bar{\chi}\bar{\chi}} = 3\frac{R'(t)\rho'(t)}{R(t)\rho(t)} + \frac{\rho''(t)}{\rho(t)}. \]

REFERENCES

[1] M. Kaku, *Introduction to Superstrings and M-Theory* (Springer-Verlag, New York, 1999), p. 376.
[2] T. Shiromizu, K. Maeda, and M. Sasaki, The Einstein equations on the 3-brane world, Phys. Rev. D 62, 024012 (2002).
[3] T.A. Roman, “Inflating Lorentzian wormholes,” Phys. Rev. D 47, 1370 (1993).
[4] E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, New York, 1990), pp. 270-275.
[5] A. Chodos and S. Detweiler, Where has the fifth dimension gone? Phys. Rev. D 21, 2167 (1980).
[6] L. Randall and R. Sundrum, An alternative to compactification, Phys. Rev. Lett. 83, 4690 (1999).
[7] M.S. Morris and K.S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, Am. J. Phys. 56, 395 (1988).
[8] R.H. Price, In an expanding universe, what doesn’t expand? arXiv:gr-qc/0508052 (2005).

Department of Mathematics, Milwaukee School of Engineering, Milwaukee, Wisconsin 53202-3109, USA