LOCALIZATION AND FLEXIBILIZATION IN SYMPLECTIC GEOMETRY

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ABSTRACT. We introduce the critical Weinstein category – the result of stabilizing the category of Weinstein sectors and inverting subcritical morphisms – and for every finite collection $P$ of prime numbers, construct a $P$-flexibilization endofunctor. Our main result is that $P$-flexibilization is a localization of the critical Weinstein category, allowing us to characterize the essential image of the endofunctor by a universal property. This localization has the effect of replacing every Weinstein sector with one in which $P$ is invertible in the wrapped Fukaya category and hence we view it as a symplectic analogue of topological localization. We prove that this construction generalizes the flexibilization operation introduced by Cieliebak-Eliashberg and Murphy and is a variant of the ‘homologous recombination’ construction of Abouzaid-Seidel. In particular, we give an h-principle-free proof that flexibilization is idempotent and independent of presentation, up to subcriticals and stabilization. Moreover, we show that $P$-flexibilization is symmetric monoidal, and hence gives rise to a new way of constructing commutative algebra objects from symplectic geometry. Our constructions work more generally for any finite collection of regular Lagrangian disks in $T^*D^n$, where the corresponding endofunctor in particular nullifies those disks as objects in the wrapped Fukaya category.

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1. Introduction

In this work we introduce the critical ∞-category of stable Weinstein sectors

\[ \mathcal{W}_{\text{Wein}}^{\text{crit}} \]

whose morphisms are obtained by formally inverting a class of morphisms realizing subcritical handle attachments. Along the way, we also invert maps resulting from Weinstein homotopies; see Definition 2.62. This is a natural context for studying wrapped Fukaya categories, which are known to remain unchanged by stabilization, subcritical handle attachments, and Weinstein homotopies [10].

We share here the discovery that this setting allows one to geometrically create a rich algebraic operation: inverting a prime; in fact, any finite collection P of primes. Given any Weinstein sector, we construct its \textit{P-flexibilization} – which we expect to be the universal stabilized Weinstein sector in which \( P \)-torsion symplectic phenomena vanish – and we show
that $P$-flexibilization is both functorial and multiplicative. Our work generalizes and renders functorial Murphy and Cieliebak-Eliashberg’s flexibilization [4], Abouzaid-Seidel’s homological recombination [11], and Lazarev-Sylvan [16]. Upon a construction of the conjectural spectral wrapped Fukaya category, our methods are expected to yield purely symplectic constructions of various localizations of the stable homotopy category, and symmetric monoidally so. In the reverse direction, the critical $\infty$-category also allows us to give a clean categorical description of previously known geometry: Flexibilization is a localization.

Remark 1.1. Central to our constructions is a class of exact symplectic manifolds called Weinstein domains. These domains can be equipped with symplectic handle-body decompositions analogous to the cellular decomposition of CW complexes. We will often pass between “Weinstein sectors” and “Weinstein domains equipped with a Weinstein hypersurface” (these data are equivalent; see Section 2.2).

1.1. Background and motivation. Because this work produces a symplectic analog of topological localization that simultaneously generalizes symplectic flexibilization, we review both ideas briefly.

1.1.1. Localization in classical topology. Localization in algebra and topology allows one to study global phenomena one prime at a time. In algebraic topology, a concrete way to localize a simply-connected space $X$ is to begin with a CW presentation of $X$ and replace all standard CW cells of $X$ by “$P$-cells” to create a new CW complex $X[\frac{1}{P}]$. (See the classic works of Sullivan [26, 27].) This operation satisfies the following properties:

1. **Homotopy invariance:** The assignment $X \mapsto X[\frac{1}{P}]$ preserves homotopy equivalences. In particular, while our description of $X[\frac{1}{P}]$ depends on the CW presentation of $X$, a posteriori the homotopy type of $X[\frac{1}{P}]$ depends only on the homotopy type of $X$.

2. **Localization on homology:** There is a natural function $\eta_X : X \to X[\frac{1}{P}]$ exhibiting the homology groups of $X[\frac{1}{P}]$ as the localization away from $P$ of the homology groups of $X$, so $H_i(X[\frac{1}{P}]; \mathbb{Z}) \cong H_i(X; \mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{P}]$ for $i > 0$.

3. **Idempotency:** The map $\eta_{X[\frac{1}{P}]}$ is a natural homotopy equivalence $X[\frac{1}{P}] \cong (X[\frac{1}{P}])[\frac{1}{P}]$.

We have mentioned Sullivan’s construction as it parallels $X_P$ below closely. A different construction allows one to include all (not necessarily simply-connected) CW complexes $X$ and is manifestly homotopy invariant by construction; see for example Bousfield’s work [3].

Remark 1.2. Let us point out two related but distinct notions of localization appear above: (a) A construction that inverts certain multiplicative operations on algebraic invariants (e.g., multiplication by $p$), and (b) an endofunctor – which, in many nice cases, is idempotent – of the category of the objects giving rise to the invariants. (For example, Bousfield localization is an endofunctor from the category of spaces to itself.) When one can exhibit a localization of type (a) as arising from one of type (b), one may sleep at night knowing they have constructed something natural. And (b) is the sense in which we mean flexibilization is a localization.
1.1.2. Flexibilization in symplectic geometry. Symplectic flexibility refers to phenomena where the underlying smooth topology (and a bit of tangential, homotopy-theoretic data) determines the symplectic geometry. One of the first instances of symplectic flexibility is Gromov’s h-principle [11] for subcritical isotropic submanifolds – i.e., isotropic submanifolds whose dimensions are less than half the ambient dimension. If two subcritical isotropics are isotopic through smoothly embedded submanifolds (plus a bit of tangential, homotopy-theoretic data), then they are isotopic through isotropics. Consequently, subcritical Weinstein domains, which are by definition built out of handles attached along subcritical isotropics, are also determined by their smooth topology.

Gromov’s result [11] was generalized by Cieliebak-Eliashberg [4] and Murphy [21], who showed that there is a special class of flexible Weinstein structures satisfying the following h-principle: If two Weinstein manifolds \( X_1, X_2 \) with flexible structures are diffeomorphic (plus a bit of bundle-theoretic data), then they are actually symplectomorphic. In fact, for any Weinstein manifold \( X \), Cieliebak and Eliashberg [4] construct the flexibilization \( X_{\text{flex}} \) of \( X \), a flexible Weinstein manifold that is diffeomorphic (but not symplectomorphic) to \( X \). Just as with Sullivan’s model of localization for CW complexes, \( X_{\text{flex}} \) is constructed by replacing all standard Weinstein handles of \( X \) with ‘flexible’ Weinstein handles, where attaching spheres are loose Legendrians [21]. \( X_{\text{flex}} \) has the following properties, analogous to the properties of localization of topological spaces:

(F1) **Homotopy invariance:** \( X_{\text{flex}} \) depends only the Weinstein homotopy type of \( X \) (in fact, only the diffeomorphism type of \( X \), plus a bit of tangential data). In particular, though the construction of \( X_{\text{flex}} \) depends on the Weinstein presentation of \( X \), its Weinstein homotopy type depends a posteriori only on the Weinstein homotopy type of \( X \).

(F2) **Fukaya category localizes:** \( X_{\text{flex}} \) is a Weinstein subdomain of \( X \) [15] and the wrapped Fukaya category of \( X_{\text{flex}} \) is trivial, so \( \mathcal{W}(X_{\text{flex}}) \cong 0 \cong \mathcal{W}(X) \otimes \mathbb{Z}_{[\frac{1}{2}]} \).

(F3) **Idempotency:** One can arrange for the subdomain inclusion \((X_{\text{flex}})_{\text{flex}} \hookrightarrow X_{\text{flex}}\) to be a Weinstein homotopy equivalence.

Property \( \text{(F2)} \) indicates that flexibilization localizes invariants away from the integer zero. The proofs of the other two properties crucially rely on the h-principle for flexible Weinstein structures [4].

1.1.3. **Toward \( P \)-flexibilization.** Given the utility of localization in topology, one ought to generalize flexibilization to invert non-zero numbers as well. For a collection of integers \( P \), Abouzaid and Seidel [1] showed that for any Weinstein domain \( X \) with \( \dim X \geq 12 \), there is a Weinstein domain \( X_{P_{AS}} \) diffeomorphic to \( X \) that abstractly admits a group isomorphism between symplectic cohomologies as in property \( \text{(F2)} \). Their work further asked whether this construction can be viewed as a symplectic analog of topological localization; see Remark 3.14 of [1].

However, it was not (and is still not) clear whether there is a geometrically defined map between \( X \) and \( X_{P_{AS}} \) whose induced map on \( SH \) realizes \( SH(X_{P_{AS}}) \) as the \( P \)-inversion of \( SH(X) \). To remedy this, the first two authors [16] introduced a variant construction \( X_P \).
(which they conjectured is equivalent to $X^\text{AS}_P$) defined for any Weinstein sector $X$ with $\dim X \geq 10$.

Because we will need it momentarily, let us recall the construction of $X_P$ in the case $X = T^*D^n$ (which also gives the local construction for arbitrary $X$). For any collection $P$ of primes, and for $n \geq 5$, one first creates a regular Lagrangian disk $D_P \subset T^*D^n$ built out of a $P$-Moore space $M_P$ (really, a wedge of $p$-Moore spaces for $p \in P$). The sector

$$(T^*D^n)_P := T^*D^n \setminus D_P$$

(1.1)

is constructed by carving out this disk and endowing the result with an appropriate Weinstein structure; see Section 4.2.1. For general $X$, we perform this construction in a neighborhood $T^*D^n$ of every co-core of $X$ (which uses the data of the Weinstein structure of $X$). This remedies the above issue as follows: $X_P$ is a Weinstein subdomain of $X$, and the resulting Viterbo functor realizes the wrapped category of $X_P$ as a localization inverting $P$ [16]. The naive intuition is that by removing this disk, we “kill” objects representing the Moore space, thereby inverting $P$. Because of the parallel with the cell-by-cell construction of [26, 27], we refer to $X_P$ as the Sullivan-style construction.

However, $X^\text{AS}_P$ and $X_P$ depend very much on the Weinstein presentation of $X$ (e.g., to identify the cocores of $X$), so are a priori not homotopy invariant (see Example 4.11). As discussed in Section 1.1.2, the h-principle was crucial to showing that flexibilization is independent of the presentation. However, an h-principle cannot exist for $X^\text{AS}_P$ or $X_P$ since there are plenty of diffeomorphic Weinstein domains $X,Y$ for which $X_P,Y_P$ are not symplectomorphic, e.g. the exotic cotangent bundles from [1] or [8]. This is because $X^\text{AS}_P$ or $X_P$ still retain non-trivial Fukaya categories at primes other than $P$, unlike the flexibilization $X_{\text{flex}}$.

For similar reasons, the h-principle cannot be applied to establish idempotency.

Thus, a new notion of equivalence has been needed to articulate (F1) and (F3).

1.2. $P$-flexibilization is localization. In this paper, we propose that the natural notion of equivalence is generated by two operations of stabilization and subcritical morphisms. (These change the symplectic geometry only as much as the smooth topology is changed and, as we mentioned, do not change the wrapped Fukaya category.) The critical Weinstein $\infty$-category $\text{Wein}^\circ_{\text{crit}}$ is precisely the minimal $\infty$-category for which these operations are homotopy-invertible. (See Section 5 for details, and Section 1.5 for more motivation.)

We caution the reader that, for the rest of this introduction, “Weinstein domain” means a domain admitting (but not equipped with) a Weinstein Lyapunov function.

Fix a finite set of primes $P$ and let $(T^*D^n)_P$ be as in (1.1). Consider the functor

$$- \times (T^*D^n)_P : \text{Wein}^\circ_{\text{crit}} \to \text{Wein}^\circ_{\text{crit}}, \quad X \mapsto X \times (T^*D^n)_P$$

(1.2)

taking any Weinstein domain to its direct product with $(T^*D^n)_P$. The proper inclusion $T^*D^n \to (T^*D^n)_P$ induces a natural transformation

$$\eta : - \times T^*D^n \to - \times (T^*D^n)_P$$

from the identity functor. \footnote{By construction, in $\text{Wein}^\circ_{\text{crit}}$, any sector $X$ is naturally identified with its stabilizations $X \times T^*D^n$ for any $n \geq 0$.}
Theorem 1.3. The functor (1.2) is idempotent. More precisely, the natural transformation \( \eta \) evaluated at \((T^*D^n)_P\)

\[
\eta_{(T^*D^n)_P}: (T^*D^n)_P \times T^*D^n \to (T^*D^n)_P \times (T^*D^n)_P
\]

is an equivalence in \( \text{Wein}^\circ_{\text{crit}} \).

The above theorem is the geometric fact that gives rise to all categorical results of our paper. As an immediate consequence we see that, up to subcritical equivalence and stabilization, direct product with \((T^*D^n)_P\) is a natural notion of a \(P\)-flexibilization of \(X\). Indeed, the critical analogue of (F1) is obviously satisfied because taking direct products preserves (Weinstein homotopy) equivalences. The transformation \( \eta_X : X \times T^*D^n \to X \times (T^*D^n)_P \), by virtue of the Kunneth theorem, realizes (F2). \(\text{Idempotency of (1.2)}\) is (F3).

Now we have two potential candidates for \(P\)-flexibilization; that is, two sectors \(X_P\) and \(X \times (T^*D^n)_P\) with equivalent wrapped Fukaya categories. Our second main result is that these seemingly different constructions are in fact naturally equivalent in the critical category, giving a geometric explanation for this algebraic equivalence.

Theorem 1.4 (A special case of Theorem 5.1). Let \(X\) be a Weinstein sector with \(\text{dim } X = 2n \geq 10\). Then for any choice of Weinstein structure on \(X\), there is an equivalence \(\varphi_X : X_P \to X \times (T^*D^n)_P \) in \(\text{Wein}^\circ_{\text{crit}}\) satisfying the following: For every \(i : X \hookrightarrow Y\) a strict proper inclusion of Weinstein sectors, there is a homotopy commutative diagram in \(\text{Wein}^\circ_{\text{crit}}\):

\[
\begin{array}{ccc}
X_P & \overset{i_{DP}}{\longrightarrow} & Y_P \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
X \times (T^*D^n)_P & \overset{i \times \text{Id}_{(T^*D^n)_P}}{\longrightarrow} & Y \times (T^*D^n)_P.
\end{array}
\]

(1.4)

Accordingly, we propose the following definition:

Definition 1.5. A Weinstein sector \(X\) is \(P\)-flexible if \(X\) is equivalent in \(\text{Wein}^\circ_{\text{crit}}\) to an object in the image of (1.2). In other words, \(X\) is \(P\)-flexible if – up to stabilization and subcritical handle attachment/removal – \(X\) admits a \((T^*D^n)_P\) factor.

This property does not depend on a Weinstein presentation of \(X\), unlike the classical definition of flexibility [4].

We can now combine Theorem 1.4 and Theorem 1.3 to prove (F1) and (F3) for the \(X_P\) construction:

Corollary 1.6 (Porism). Fix two Weinstein sectors \(X\) and \(Y\). If \(X, Y\) are isomorphic up to Weinstein (Liouville) homotopy, then \(X_P, Y_P\) are isomorphic up to Weinstein (Liouville) homotopy, stabilization and subcritical handle attachment. Furthermore, \((X_P)_P\) is isomorphic to \(X_P\) up to Weinstein homotopy, stabilization and subcritical handle attachment.
(By an isomorphism up to Weinstein homotopy, we mean a diffeomorphism $f : X \to Y$ such that the pulled back Weinstein structure may be endowed with a homotopy of Weinstein structures to the Weinstein structure of $X$; in other literature [1] this is called a Weinstein equivalence, a term we do not use here to avoid confusion.)

Remark 1.7. As explained in Section 1.1.3, there was little hope of answering Abouzaid-Seidel’s question regarding localization given the absence of an $h$-principle. Yet in $\text{Wein}^\omega_{\text{crit}}$, we find an affirmative and complete answer to Abouzaid and Seidel’s question (for $X_P$ instead of $X_{P^{AS}}$). Indeed, we have seen that the Sullivan-style $P$-flexibilization construction satisfies all three properties satisfied by classical topological localization and flexibilization.

Next, we consider the case $P = \{0\}$. (See Example 1.15, and its preceding discussions for non-prime numbers.) Then $X_0$ is flexible in the sense of Cieliebak-Eliashberg [4], and is diffeomorphic to $X$ up to some smooth subcritical handles. Corollary 1.6 gives a new proof that this flexibilization is homotopy invariant and idempotent, up to subcritical handles.

Corollary 1.8. If $X, Y$ are isomorphic up to Weinstein (Liouville) homotopy, then $X_0, Y_0$ are isomorphic up to Weinstein (Liouville) homotopy, stabilization, and subcritical handles. Furthermore, $(X_0)_0$ is isomorphic to $X_0$, up to Weinstein homotopy, stabilization, and subcritical handles.

The main novel feature of our proof is that it does not use the $h$-principle for flexible domains [4] and loose Legendrians [21] and hence presents a new approach to studying flexibility. We refer to Section 1.4 for examples and further discussion. See also Section 1.6 for the role $h$-principles (do or do not) play in our paper.

We turn to more structural results. The fact that $P$-flexibilization is an idempotent functor (Theorem 1.3) immediately implies the following categorical fact:

Theorem 1.9. The functor (1.2) is a localization.

In other words, the image of (1.2)—otherwise known as the $\infty$-category of Weinstein sectors critically divisible by $(T^*D^n)_P$—is characterized by a universal property: Any functor from $\text{Wein}^\omega_{\text{crit}}$ that does not distinguish a sector from its $P$-flexibilization (more precisely, that sends $\eta$ to equivalences) automatically factors through this image. This is a homotopy theorist’s “favorite kind” of localization, in which the localization is found as a full subcategory. (Example: The category of $\mathbb{Z}[1/p]$-modules, which is a localization of the category of $\mathbb{Z}$-modules, is found as a full subcategory of $\mathbb{Z}\text{Mod}$.) We view Theorem 1.9 as evidence elevating the critical category from being natural to being algebraically robust.

Remark 1.10. A functor being a localization is an a priori distinct notion from a functor inducing a localization of invariants. It is natural to ask whether $P$-flexibilization (and its versions equipped with tangential structures allowing for the definition of Floer-theoretic linear invariants) is the universal functor localizing wrapped-Floer invariants (and their spectral versions).
1.3. **P-flexibilization is symmetric monoidal.** In Section 3, we will see that \( \text{Wein}_{\text{crit}}^c \) has a symmetric monoidal structure given by direct product of sectors. Idempotency, together with the fact that (1.2) is induced by direct product with an object of \( \text{Wein}_{\text{crit}}^c \), implies the following:

**Theorem 1.11.** The localization (1.2) may be promoted to be a symmetric monoidal functor to its image.

The fact that we can formally obtain such symmetric monoidal structures greatly simplifies applications to high algebra; we refer the reader to Section 1.8. For now, let us mention a geometric curiosity. By Theorem 1.11, the symmetric monoidal unit of \( \text{Wein}_{\text{crit}}^c \), \( T^*D^n \), has image given by the symmetric monoidal unit in the target, \( (T^*D^n)_P \). Purely formally, we obtain:

**Corollary 1.12.** \( (T^*D^n)_P \) is a commutative algebra (that is, an \( E_\infty \)-algebra) in \( \text{Wein}_{\text{crit}}^c \).

Most commutative symplectic objects arise from SYZ fibrations, or standard variations thereof (e.g., \( \mathbb{R}^n \)-fibers as opposed to torus fibers). At present, we do not know if the commutative structure on \( (T^*D^n)_P \) from Corollary 1.12 arises from such fibrations.

Our final main result is that our results above remain true if we wish to geometrically nullify arbitrary finite CW complexes, and not just \( P \)-Moore spaces. This is because any finite CW complex may be represented by a regular Lagrangian disk and our results hold in this generality.

**Theorem 1.13.** For any regular Lagrangian disk \( L \subset T^*D^n \), the functor \( - \times (T^*D^n) \setminus L \) is a symmetric monoidal localization of \( \text{Wein}_{\text{crit}}^c \).

In particular, for any finite CW complex \( K \), there is a Weinstein sector \( (T^*D^n)_K := T^*D^n \setminus D_K \) so that \( - \times (T^*D^n)_K \) is a symmetric monoidal localization of \( \text{Wein}_{\text{crit}}^c \).

In fact, the equivalence (Theorem 1.4) between the direct product model and the Sullivan-style model of \( P \)-flexibilization extends also in this generality. See Theorem 5.1.

1.4. **Examples.** As indicated by Theorem 1.13, our first results do not require \( P \) to be a collection of prime numbers; indeed, the geometric constructions never rely on primeness.

**Example 1.14.** If \( P = \{1\} \) or \( P = \emptyset \), then \( (T^*D^n)_P \) is \( T^*D^n_{\text{std}} \), the usual cotangent bundle of \( D^n \) with its standard Weinstein structure.

**Example 1.15.** If \( P \) contains 0, then \( (T^*D^n)_P \) is flexible in the sense of Cieliebak-Eliashberg [4], as observed in [16]. See also Corollary 1.8.

In particular, the \( P \)-flexible Weinstein sectors should be the viewed as interpolating between symplectic rigidity as represented by an arbitrary Weinstein sector \( X = X_\emptyset \) and symplectic flexibility as represented by the flexible sector \( X_0 \); indeed [16] proved that any Weinstein subdomain of \( X = T^*S^n \) has the same Fukaya category as a \( P \)-flexibilization of \( T^*S^n \).

**Example 1.16.** Our proof of Theorem 1.3 shows that if \( P \) and \( Q \) are two finite sets of integers, then \( (T^*D^n)_P \times (T^*D^n)_Q \) is equivalent to \( (T^*D^n)_{P \cup Q} \) in \( \text{Wein}_{\text{crit}}^c \); see Remark 6.10.
Furthermore, if \( m = \prod_{p_i \in P} p_i \) is a product of distinct primes, then \((T^*D^n)_m\) is equivalent to \((T^*D^n)_P\) and therefore equivalent to \( \prod_{p_i \in P} (T^*D^n)_{p_i} \); see Remark 4.17. We expect a careful analysis of the coherences to show that \( P\)-flexibilization is a functor from the symmetric monoidal category of Zariski-closed subsets of \( \text{spec } \mathbb{Z} \) (where morphisms are inclusions, and the symmetric monoidal structure is union) to \( \text{Wein}^\diamond_{\text{crit}} \).

**Example 1.17.** Another immediate consequence of Theorem 1.4 is that \( X_P \times Y \) is equivalent to \((X \times Y)_P\) in \( \text{Wein}^\diamond_{\text{crit}} \), which generalizes to \( P\)-flexibility the fact that classical flexibility is preserved by taking products \([22]\). Additionally, \( X_P \times Y_Q \) is equivalent to \( X_Q \times Y_P \) in \( \text{Wein}^\diamond_{\text{crit}} \), which is non-obvious from the usual definition of \( X_P \) in \([16]\) or \([2]\).

**Remark 1.18.** Previous constructions \( X_P \) and \( X_{AS}^P \) explicitly used the Weinstein structure of \( X \) and did not apply to Liouville \( X \). However, \( X \times (T^*D^n)_P \) makes sense even if \( X \) is a Liouville sector, and thus is a candidate definition for \( P\)-flexibilization in the Liouville setting. However, because of the absence of Kunneth formulas for general Liouville sectors, we do not know if \([F2]\) holds in the Liouville setting. See also Remark 2.64.

1.5. The benefits of \( \text{Wein}^\diamond_{\text{crit}} \). One purpose of this paper is to argue that the critical Weinstein \( \infty \)-category is a better category for studying wrapped Floer theory than the (ordinary) category of Weinstein sectors. Let us list, as a sequence of remarks, some reasons for this perspective.

**Remark 1.19.** We show here that a Weinstein subdomain inclusion \( X_0 \xrightarrow{i} X_1 \) induces a morphism \( X_1 \to X_0 \) in \( \text{Wein}^\diamond_{\text{crit}} \); (Proposition 2.73). In fact, though we do not do so here, one can show there is a contravariant functor from a suitable \( \infty \)-category of Weinstein domains, with subdomain inclusions as morphisms, to the critical Weinstein \( \infty \)-category. This is a significant utility of \( \text{Wein}^\diamond_{\text{crit}} \) – it can simultaneously house the covariant pushforward functoriality of sectorial inclusions and the contravariant Viterbo functoriality of subdomain inclusions. Indeed, the geometric ingredients of Proposition 2.73 are those in the third author’s Viterbo sector construction \([29]\); the literature has already recorded that such constructions should covariantly realize Viterbo functoriality \([10]\).

In particular, any functorial invariant unchanged by stabilization and subcritical handle attachments automatically satisfies Viterbo functoriality. We were not aware of this philosophy prior to our work. As one application, we see that any functorial invariant – Floer-theoretic or not – preserved under stabilization and subcritical attachments admits Viterbo restriction.

**Example 1.20.** The \( P\)-inversion maps in \([F2]\) and in the Sullivan-style model for \( X_P \), were introduced as Viterbo restriction functors \([16]\). We can alternatively realize these maps as pushforward functors along sectorial inclusions (by pushing forward along \( \eta \)) – and indeed, this perspective makes \( P\)-flexibilization manifestly functorial, as already discussed.
Remark 1.21. Theorem 1.4 geometrically operationalizes the algebraic fact that tensor products commute with colimits. The tool showing $X \times (T^*D^n)_P$ localizes the wrapped category of $X$ is the Kunneth formula for wrapped Floer theory. On the other hand, the main tool used in [16] to prove that $X_P$ localizes the wrapped category of $X$ is a localization formula premised on a local-to-global property of the wrapped category (expressing the wrapped category of $X_P$ as a colimit of other categories – this is the Kontsevich cosheaf conjecture, proven as a descent formula in [10]). The geometry of carving out disks likewise commutes with products in the critical setting; see Section 5.1.

Remark 1.22 (It is necessary to invert subcriticals, and to stabilize). Let us note that inverting stabilization and subcritical morphisms is not only natural from a wrapped-Fukaya-theoretical perspective, they are a “minimal” geometric class of morphisms for achieving the results of our work.

First, Theorem 1.4 is false unless we invert subcritical morphisms. For example, if $X$ is a Weinstein domain (without sectorial boundary), then $X_P \times T^*D^n$ has sectorial divisor $X_P \times T^*S^{n-1}$ while $X \times (T^*D^n)_P$ has divisor $X \times T^*S^{n-1}$. These are not symplectomorphic for general $X$ (even if we pick a model for $X_P$ that is diffeomorphic to $X$). However, adding subcritical handles can change the divisor by a loose hypersurface and hence resolves this issue; see the discussion in Example 2.23. We also note that with our definition of $(T^*D^n)_P$, $(T^*D^n)_P \times T^*D^n$ and $(T^*D^n)_P \times (T^*D^n)_P$ have different cohomology in degree $2n-1$ and hence subcritical handles are required to make them equivalent.

As for stabilization, we note that (i) $n$ must be large to make sense of $D^n_P$ for arbitrary $P$, and (ii) when all morphisms are demanded to be codimension zero embeddings, most sectors have no hope of being a (commutative) algebra object, as $X \times X$ has no map to $X$ unless $X$ is a point. Thus, stabilization is necessary to not only have plenty of commutative algebra objects, but to also localize the unit ring $\ast = T^*D^0 \simeq T^*D^n$.

1.6. Relation to h-principles. Our proofs of Theorem 1.3 through Corollary 1.12 for $P$ a collection of integers do not use any h-principles. So our techniques in fact give a new proof of properties [F1], [F2], and [F3] for classical flexibility, in our critical setting, independent of any h-principles. We also do not use the theory of wrinkled Legendrian embeddings (on which the h-principle for flexible Weinsteins was originally based).

However, h-principles do play a role for various extensions and modifications of our results. We briefly explain this in the following remarks.

Remark 1.23. The generalization of $P$ to arbitrary regular disks (the first part of Theorem 1.13) does use the h-principle for subcritical isotropics; see Remark 6.9.

Remark 1.24. We never use the h-principle for loose Legendrians [21] in this paper. However, to compare our constructions to others, one can employ this h-principle. For example, $X_0$ (which is manifestly flexible) can be shown via a (local) h-principle to be Weinstein homotopic to $X_{flex}$ plus some subcritical Weinstein handles. See Section 4.2.2 for details.

Remark 1.25. We note that flexible handles also do not affect the wrapped Fukaya category [3, 6] and satisfy an h-principle [21]. Hence one could contemplate forming a category of Weinstein sectors where ‘flexible morphisms’ are all inverted. Therefore it is surprising
that Theorems 1.3 and 1.4 do not require inverting flexibles. Indeed, one of the motivations for considering arbitrary Lagrangian disks \( L \subset T^*D^n \), as in Theorem 1.13, instead of just \( D_P \) disks, is that this allows us to avoid inverting flexibles while still having access to certain flexible cobordisms; see Section 4.2.2.

**Remark 1.26.** There are several models for the smooth topology of \((T^*D^n)_P\) \([16]\), which all differ by subcritical or flexible Weinstein handles. In this paper, we take a model only diffeomorphic to \( T^*D^n \) up to smooth subcritical handles (but not Weinstein subcritical handles since they have different Fukaya categories). It is possible to make \((T^*D^n)_P\) and \( T^*D^n \) diffeomorphic but this naturally employs a construction relying on the h-principle for loose Legendrians, which we avoid in this paper.

### 1.7. Geometric ingredients of the proofs

We mention key geometric ingredients in the proofs of our main results, all of which require stabilization in essential ways:

- For Lagrangians \( L, K \subset T^*D^n \), the Lagrangian \( L \times K \) is Lagrangian isotopic to \( K \times L \) for \( n \) even, and hence for any \( n \) after stabilization (Proposition 5.9).
- Lagrangian links are unlinked, after stabilization (Proposition 6.11).
- For any regular Lagrangian disk \( L \subset T^*D^n \), after stabilization, there is a particularly simple Weinstein presentation of \( T^*D^n \) with exactly two co-cores consisting of \( L \) and a disjoint cotangent fiber (Proposition 6.6).

The first fact is used in the comparison theorem (Theorem 1.4). All three facts are used in the idempotency theorem (Theorem 1.3); in fact, the Sullivan-style model renders the idempotency statement easier to verify in the critical category, so we employ the comparison to prove idempotency.

### 1.8. Future applications: Higher algebra

Prior to the present work, all works regarding \( P \)-inversion in symplectic geometry were linear, but not fully multiplicative. For example, while there were symplectic constructions of the module \( \mathbb{Z}[1/P] \), there were no symplectic constructions of its natural commutative ring structure. Moreover, writing down a commutative ring structure “by hand” is rarely feasible for spectra. Given this difficulty, and given the emergence of spectral sectorial invariants \([23, 13]\), we faced the important task of producing higher-algebraic structures formally from geometric facts.

Our main results accomplish a great deal of this task, establishing all the \( \infty \)-categorical coherences one could hope for (Theorem 1.3) without engaging with holomorphic disks. Let us explain. It is widely expected that any spectral wrapped Fukaya category of Weinstein sectors is:

- Preserved under stabilization, subcritical handle attachments, and trivial inclusions.
- Symmetric monoidal with respect to direct product of sectors and strict proper inclusions. (One need only verify this for sets of inclusions, not spaces of them.)

It is then formal that the spectral wrapped Fukaya category will send an \( E_\infty \)-algebra in the critical Weinstein category to a symmetric monoidal stable \( \infty \)-category. In particular, the unit of this stable \( \infty \)-category is an example of a commutative ring spectrum.
Theorem 1.13 shows that the critical Weinstein category $\mathcal{W}_{\text{crit}}^\diamond$ has a plethora of commutative algebra objects. In fact, because the spectral wrapped Fukaya category of a point (equipped with standard tangential structures) must be the $\infty$-category of finite spectra:

- The process of replacing $T^*D^n$ with $T^*D^n \setminus D_K$ — as in Theorem 1.13 — nullifies (the suspension spectrum of) $K$.

So we expect $(T^*D^n)_K$ to be a purely symplectic way to encode “the universal commutative ring spectrum” whose modules nullify $\Sigma^\infty_+ K$. For example, when $K = M_p$ is the $p$-Moore space, we expect the cotangent fiber of $(T^*D^n)_K$ to have endomorphism spectrum $S[1/p]$, the sphere with $p$ inverted.

**Remark 1.27.** One must specify certain tangential structures of Weinstein sectors to define a spectrally enriched wrapped category. As in [17], incorporating such tangential structures preserve all arguments involving localizations and symmetric monoidal structures.

**Outline.** In Section 2 we give some background on Liouville geometry, introduce the critical Weinstein category, and prove some helpful properties. Section 3 reviews the necessary material on localization functors. In Section 4 we describe the two $P$-flexibilization functors $X_P$ and $X \times (T^*D^n)_P$, reviewing the construction from [16]. We prove Theorem 1.4 comparing these two functors in Section 5 and we prove Theorem 1.3 (that these functors are localizing) in Section 6.

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2. Liouville geometry background

2.1. Liouville and Weinstein sectors.

**Definition 2.1** (Liouville sector). Fix an exact symplectic manifold $(X, \omega = d\lambda)$ possibly with corners, together with the data, for each $x \in \partial^i X$ in a codimension $i$ corner $(i \geq 1)$, of a neighborhood $\text{Nbhd}(x)$ inside $X$ and a codimension-preserving symplectic submersion

$$\pi_x : \text{Nbhd}(x) \to T^*[0,1]^i. \quad (2.1)$$

We say this collection of data is a **Liouville sector** if it satisfies the following:

1. $\lambda$ has finite type. This means that $X$ admits a proper, smooth function $X \to \mathbb{R}_{\geq 0}$ which, outside some compact subset of $X$, is linear with respect to the Liouville flow of $X$.
2. Each $\pi_x$ is flat, and $\lambda$ is split with respect to $\{\pi_x\}_{x \in \partial X}$.
3. If $y \in \partial^j X \cap \text{Nbhd}(x)$ with $j \leq i$, then on the overlap we have $\pi_y = \pi_{yx} \circ \pi_x$, where $\pi_{yx} : T^*[0,1]^i \to T^*[0,1]^j$ is a projection to $j$ components (not necessarily respecting the order of coordinates).
The splitting in (2) means that $\text{Nbhd}(x)$ is a product of a fiber $F$ of $\pi_x$ and some neighborhood of $\pi_x(x)$ so that $\lambda|\text{Nbhd}(x) = \lambda^F + \pi^*_x p dq$. One can check that $\lambda^F$ renders $F$ as an open subset of a Liouville sector.

As in the boundaryless setting (also called a completed Liouville domain or a Liouville manifold), the skeleton $\text{skel} X$ is the smallest attracting set for the negative Liouville flow.

We will usually denote a Liouville sector by $(X, \lambda)$ or just $X$, leaving the family of projections $\{\pi_x\}$ implicit.

**Remark 2.2.** Note that the splitting condition implies that any trajectory of the Liouville vector field $v_\lambda$ which begins away from $\partial X$ must remain away from $\partial X$. This definition of Liouville sector agrees with the notion from [10] of a straightened Liouville sector with corners.

**Notation 2.3** ($X_{\text{cpt}}$). We will sometimes identify $X$ with a compact, codimension 0 submanifold $X_{\text{cpt}} \subset X$ for which $X \setminus X_{\text{cpt}}$ is the positive half of a symplectization of a contact manifold with (convex) boundary.

**Definition 2.4** (Sectorial boundary). Let $X$ be a Liouville sector. Then the **sectorial boundary** of $X$ is the boundary of $X$ when considered as a smooth manifold with corners – in other words, the union of all boundary and corner strata of $X$.

As a consequence of Definition 2.1 after smoothing the corners, we have a splitting of a neighborhood of $\partial X$:

$$ (H \times \mathbb{R}_+^+ \times \mathbb{R}_y, \lambda_H + y dx). \tag{2.2} $$

**Definition 2.5.** We will call the identification (2.2) of a neighborhood of $\partial X$ a **bordering** and call the Liouville sector $H$ from (2.2) the **sectorial divisor** of $X$. It is a Liouville sector without boundary—i.e., the completion of a Liouville domain. We will use the notation $\left[X, H\right]$ to denote a sector with its sectorial divisor.

**Remark 2.6.** The bordering condition (2.2) implies that the Liouville vector field is tangent to $\partial X$ and in a neighborhood of $\partial X$, the zero locus of $v_\lambda$ is the product of $[0, 1]$ with the zero locus of the Liouville vector field on $H$.

**Remark 2.7.** There is another approach to dealing with a sector with corners—for example, immersing sectorial hypersurfaces to cover $\partial X$ [10]. This is similar to a common smooth-topology convention that treats the boundary of $[0, 1]^2$ not as a topological circle, but as a disjoint union of four intervals.

We instead choose a smoothing of $\partial X$ to associate a single $H$ (well-defined up to deformation of Liouville structure). Note that any sector with corners is equivalent (in the sense of Section 2.7, though not isomorphic) to its boundary-smoothing. This is the same way in which $[0, 1]^2$ is isotopy equivalent to $D^2$.

In this paper, we will consider Liouville sectors with Morse-Bott-with-corners Weinstein structures.
**Definition 2.8.** A smooth function $f$ is *Morse-Bott-with-corners* if, in a neighborhood $U$ of each critical point $p$, we can find coordinates $x_i$ centered at $p$ so that

$$f(x) = \sum_i f_i(x_i)$$

for all $x \in U$, where each $f_i(x_i)$ is one of

1. $\pm x_i^2$
2. a cutoff function which is zero for $x_i \leq 0$ and has strictly positive derivative for $x_i > 0$
3. 0.

If we additionally allow the possibility $f_i = x_i^3$ (i.e. a birth-death singularity), then we’ll say $f$ is *generalized Morse-Bott-with-corners*.

**Notation 2.9.** We denote the critical locus of $f$ by $\text{Crit}(f)$.

**Remark 2.10.** If $f$ is Morse-Bott-with-corners, $\text{Crit}(f)$ is a smooth manifold with corners.

**Definition 2.11 (Morse index).** If $f$ is Morse-Bott-with-corners, we define the *Morse index* of a connected component $C \subset \text{Crit}(f)$ to be the sum of the dimensions of the non-positive eigenspaces of the Hessian $Hf(p)$ for $p \in C$, or equivalently the number of coordinates $x_i$ above for which $f_i(x_i) \neq +x_i^2$. The index of $C$ does not depend on the choice of $p$.

**Definition 2.12.** A Weinstein sector is a Liouville sector $(X, \lambda)$ so that

- $v_\lambda$ is gradient-like for a Morse-Bott-with-corners function $\varphi : X \to \mathbb{R}$
- $\varphi$ can be arranged to be split with respect to the boundary projections $\pi_x$. In other words, a neighborhood of $\partial X$ has the form

$$(H \times \mathbb{R}_x^+ \times \mathbb{R}_y, \lambda_H + ydx, \varphi_H + e^f p^2)$$

for a Weinstein sector $H$.

**Remark 2.13.** In the case without corners, our definition is similar to the notion of a *Morse-Bott* Weinstein structure appearing in Starkston’s work [25]. We will not attempt to address whether it is in fact equivalent.

Another similar definition is given in [7] and called an *adjusted* Weinstein structure.

**Definition 2.14.**

- A connected component $C \subset \text{Crit}(\varphi)$ is called *subcritical* if it either has Morse index less than $n$ or has free boundary in the sense that its boundary not entirely contained in $\partial X$.
- $C$ is called *critical* if it is not subcritical.
- A Weinstein sector is called subcritical if all components of the zero locus are subcritical.

**Remark 2.15.** In this paper, we further require that all critical components consist of isolated points, i.e. are already Morsified. A general Morse-Bott Weinstein sector can be put into this form by a $C^0$-small Liouville homotopy. Thanks to this assumption, any component intersecting the sectorial boundary is subcritical.
Figure 1. Two Morse-Bott Weinstein structures on $T^*D^n$, depicted via their corresponding vector fields on $D^n$; the zero locus of the vector fields is in red. The left figure has sectorial boundary $T^*S^{n-1}$, equipped with a Morse-Bott Weinstein structure having critical locus $S^{n-1}$. The right figure has the same sectorial boundary $T^*S^{n-1}$, but for which the boundary is equipped with a Morse Weinstein structure consisting of two points.

Example 2.16 (Cotangent bundles of disks). There are two convenient Weinstein structures on the symplectic manifold $T^*D^n$, both with sectorial divisor $T^*S^{n-1}$.

The first arises from the Morse-Bott structure on the divisor $T^*S^{n-1}$, which has $S^{n-1}$ as the zero locus. We call this the standard Weinstein structure.

The second arises naturally when the divisor is given the Weinstein structure induced by a Morse function on $S^{n-1}$ with two isolated critical points of index 0 and $n-1$.

Both structures further contain an index $n$ critical point on the interior. See Figure 1, where we depict the Liouville vector fields restricted to the zero-section $D^n$. Note that any vector field on $M$ has a canonical extension to a Liouville field on $T^*M$.

2.2. Stopped domains.

Definition 2.17. Let $X_0$ and $\Lambda$ be (compact) Liouville domains, together with a strict embedding of $\Lambda$ into the contact boundary of $X_0$. We will call the pair

$$(X_0, \Lambda)$$

a stopped domain. When both $X_0$ and $\Lambda$ are Weinstein, we call the pair a Weinstein pair. (We demand no compatibility between Weinstein Morse functions.)

Remark 2.18. In [28], such a hypersurface $\Lambda$ in a contact manifold is called a stop while in the Weinstein setting of [7], it is called a Weinstein hypersurface.

A common maneuver in the world of Liouville geometry passes between a compact exact symplectic manifold-with-contact-boundary (where the Liouville vector field points outward along the boundary) and a complete, non-compact exact symplectic manifold-without-contact-boundary (obtained by attaching a cylinder along the Liouville vector field). There is a corresponding maneuver in the world of sectors allowing us to pass between (compact) stopped domains $(X_0, \Lambda)$ (Definition 2.17) and (non-compact) Liouville sectors $X$, as established in [9].

We review the constructions briefly.
Construction 2.19 (From sectors to stopped domains). Fix a Liouville sector $X$ and let $H$ be the sectorial divisor (Definition 2.5). To construct $X_0$, we consider $DT^*[-1, 1] \cong T^*[-1, 1]_{cpt}$ (the unit disk cotangent bundle) with the Weinstein structure induced by a vector field on $[-1, 1]$ that is pointing towards $-1$ along $[-1, 0]$ and vanishing along $[0, 1]$. Note that the bordering (2.2) allows us to identify $H \times T^*[0, 1]$ with a neighborhood inside $X$. So we glue $H_{cpt} \times T^*[-1, 1]_{cpt}$ to $X_{cpt}$ (Notation 2.3) along $H_{cpt} \times T^*[0, 1]_{cpt}$ and call the resulting domain $X_0$. Note that $X$ has a proper inclusion into the completion of $X_0$.

Next we observe that the Liouville form on $X_0$ restricts to the Liouville form on $H_{cpt} \sim H_{cpt} \times \{-1\} \subset H_{cpt} \times T^*[0, 1]_{cpt}$. Then $(X_0, H_{cpt})$ is a stopped domain. Note also that if $X$ is Weinstein, then so is $X_0$.

Construction 2.20 (From stopped domains to sectors). As explained in [7], any stopped domain $(X_0, \Lambda)$ can be converted into a Liouville sector that we denote $(\overline{X_0}, \Lambda)$. Namely, we consider $T^*[0, 1]_{cpt}$ with the Weinstein structure induced by a vector field on $[0, 1]$ that vanishes on $[0, 1/4]$, pointing towards 1 along $(1/4, 1/2)$, has an index 1 critical point at $1/2$, and pointing towards 0 along $(1/2, 1)$. Then we attach $\Lambda \times T^*[0, 1]_{cpt}$ to the stopped domain $(X_0, \Lambda)$ along $\Lambda \times T^*_1[0, 1]_{cpt}$. Here $T^*_1[0, 1]_{cpt}$ is the unit cotangent fiber over $1 \in [0, 1]$; the gluing identifies these fibers with small integral curves of the Reeb vector field on $X$. See Figure 2. We define $(\overline{X_0}, \Lambda)$ by completing. When $(X_0, \Lambda)$ is Weinstein, so is $(\overline{X_0}, \Lambda)$.

Remark 2.21. In particular, $X_0$ is a Weinstein subdomain of $(\overline{X_0}, \Lambda)$; see Section 2.12 for a definition. The Liouville vector field on $(\overline{X_0}, \Lambda)$ has zero locus that corresponds to the zero locus of $X_0$ and and $([0, 1/4] \cup \{1/2\}) \times C$, where $C$ is the zero locus on $\Lambda$. If $\Lambda$ has isolated critical locus, then so does the resulting Weinstein structure on $(\overline{X_0}, \Lambda)$.

Remark 2.22. In general, if $(X_0, \Lambda)$ arises from a sector $X$ as in Construction 2.19, then $(\overline{X_0}, \Lambda)$ is Weinstein homotopic to (a slightly larger version of) $X$; see Section 2.5 below.

Example 2.23. Let $X$ be a subcritical Weinstein sector (Definition 2.14) with associated domain $X_0$ and stop $\Lambda$. Since $X$ is a subcritical sector, $X_0$ is a subcritical domain. However, the stop $\Lambda$ itself need not be subcritical. For example, the sector $\Lambda \times \mathbb{C}_{Re \geq 0} = [\Lambda \times D^2, \Lambda \times \{1\}]$ is subcritical for any Weinstein domain $\Lambda$ since the Weinstein sector $[D^2, \{1\}]$ has only critical point of index 0, lying in its boundary. A subcritical sector has no Lagrangian co-cores since it has no isolated index $n$ critical points but it does have a collection of Lagrangian 'linking'
disks of its sectorial divisor. As explained in Remark 2.28, these disks can be realized as co-cores of a homotopic Weinstein structure; see [10] for a definition of linking disks.

If $X$ is a subcritical sector, then these linking disks are isotopic to unknots and $\Lambda \subset \partial X_0$ is a loose Weinstein hypersurface, as defined in [7]. We will prove this only in the case when $X$ is obtained by attaching subcritical handles to $\Lambda \times (D^2, \{1\})$. We claim that $\Lambda \times \{1\} \subset \Lambda \times D^2$ is a loose Weinstein hypersurface. To see this, we proceed by induction. Let $H^n$ be a Weinstein handle of $\Lambda^{2n}$ and $C^n$ be the core of this handle. Then $C^n$ is a Legendrian disk in the boundary of $H^n \times D^2$ (a handle of $\Lambda \times D^2$) that intersects the belt sphere of $H^n \times D^2$ exactly once; so $C^n$ is loose relative to its boundary by the criteria in [4]. The linking disks of the stop $\Lambda$ are unknots in $\Lambda \times D^2$; they remain unknots when we add the stop $\Lambda \times \{1\}$ since we can still isotope them to infinity near $\Lambda \times \{0\}$ which is not a stop. Adding subcritical handles preserves loose-ness and the property that the linking disks are isotopic to the Lagrangian unknots.

2.3. Products of sectors. Given two Liouville sectors $(X, \lambda_X), (Y, \lambda_Y)$, we define the product sector $X \times Y$ to be $(X \times Y, \lambda_X + \lambda_Y)$. If $X$ and $Y$ further admit Weinstein functions $\varphi_X$ and $\varphi_Y$, then $\varphi_X + \varphi_Y$ is a Weinstein function on $X \times Y$.

Remark 2.24. Note that critical points of $\varphi_X + \varphi_Y$ correspond to pairs of critical points of $\varphi_X$ and $\varphi_Y$. Furthermore, the unstable manifold of the critical point $p$ of $\varphi_X + \varphi_Y$ corresponding to a pair of critical points of $\varphi_X, \varphi_Y$ is the product of the associated unstable manifolds.

Example 2.25. If $X, Y$ are sectors associated to stopped domain domains $(X_0, H_X), (Y_0, H_Y)$, then the associated stopped Weinstein domain to $X \times Y$ is $(X_0 \times Y_0, X_0 \times H_Y \coprod_{H_X \times H_Y} H_X \times Y_0)$.

Definition 2.26. For any integer $k \geq 1$, the sector $X \times T^*D^k$ is called a stabilization of the sector $X$.

2.4. Strict proper inclusions. From here through Section 2.7, we discuss maps between Liouville and Weinstein sectors. Here is the most basic class:

Definition 2.27. For Liouville sectors $(X, \lambda_X), (Y, \lambda_Y)$, a strict proper inclusion otherwise known as a strict proper embedding, is a smooth embedding $i : (X, \lambda_X) \hookrightarrow (Y, \lambda_Y)$ that is proper, and that strictly preserves the Liouville forms: $i^* \lambda_Y = \lambda_X$.

Remark 2.28. Note we allow the sectorial boundary of $X$ to intersect the sectorial boundary of $Y$. (Compare with Convention 3.1 of [9].) If $X$ is contained in the interior of $Y$, then the complement $Y \setminus i(X)$ is also a Liouville sector, with a strict proper inclusion into $Y$; if $Y$ is further a Weinstein sector, then so is $Y \setminus i(X)$.

Remark 2.29. If $i : X \hookrightarrow Y$ is a strict proper inclusion of Weinstein sectors, then the index $n$ Lagrangian co-cores of $X^{2n}$ are identified with a subset of the Lagrangian co-cores of $Y$. Furthermore, since the Liouville vector field on $Y$ is tangent to $\partial X$, points in $Y \setminus i(X)$ cannot flow into $i(X)$. So the Lagrangian co-cores of $Y$ are one of the co-cores of $i(X)$ (and hence identified with a cocore of $X$) or are entirely contained in $Y \setminus i(X)$.

Remark 2.30. We note that the formation of products is compatible with strict proper inclusions. That is, if $i$ and $i'$ are strict proper embeddings, so is the product $i \times i'$.
2.5. **Liouville deformations.** In this paper, we will also consider certain classes of non-strict Liouville embeddings, which we will call just Liouville embeddings. These embeddings allow certain deformations of the Liouville form, which we now discuss.

**Definition 2.31** (Homotopies/deformations of Liouville structures). Let \( \{ \lambda_t \}_{t \in [0,1]} \) be a smooth, 1-parameter family of Liouville structures on \( Y \). As usual, we will say that the family is *exact* if \( \lambda_t = \lambda_0 + dh_t \) for some smooth family of smooth functions \( h_t \). Throughout the paper, unless explicitly stated otherwise, it is assumed that every family is exact.

Finally, we demand a tameness condition on our families at infinity: We demand there exists a *proper* smooth function \( R : Y \times [0,1] \to \mathbb{R}_{\geq 0} \) and a single compact subset \( K \subset Y \) such that, for all \( t \in [0,1] \), \( R_t \) is \( \lambda_t \)-linear outside of \( K \)—that is, we demand that \( d(R_t)(v_{\lambda_t}) \) equals \( R_t \) outside of \( K \).

We will call such a family—exact, and satisfying the tameness condition at infinity—a *deformation*, or equivalently a *homotopy*, of \( \lambda = \lambda_0 \).

Exact deformations will further be called (in order of increasing restrictiveness):

- **Bordered** if for each \( t \), \( \lambda_t \) respects the splitting (2.2).
- **Interior** if \( \lambda_t \) is constant (i.e., \( t \)-independent) near \( \partial Y \).
- **Compactly supported** if there exists a compact set \( K \subset Y \setminus \partial Y \) for which \( \text{supp}(\lambda_t - \lambda_0) \subset K \).

**Remark 2.32.** For a deformation \( \lambda_t \) to be bordered means that \( \lambda_t \), in a neighborhood of \( \partial Y \), is a direct product of deformations—a deformation of Liouville structure of the divisor \( H \), and a constant (non-)deformation of the structure on \( T^*[0,1]^k \)—with respect to the splitting in (2.2).

**Remark 2.33.** We warn the reader that we take “compact support” to be a condition checked on \( Y \setminus \partial Y \) (not on \( Y \) itself). In particular, if \( \lambda_t \) is a compactly supported deformation, the functions \( h_t \) can be chosen to vanish near \( \partial Y \) and \( \lambda_t \) is constant (i.e., \( t \)-independent) near this boundary.

Put another way, we abusively use compactly supported to mean “interior and compactly supported.”

**Remark 2.34.** (On Weinstein homotopies) For clarity, we say that a *Weinstein homotopy* is a family \( \lambda_t \) of Liouville forms admitting generalized Morse-Bott-with-corners Lyapunov functions. More generally, the reader can consider a class of Liouville structures with Lyapunov functions whose singularities are invariant under products with other sectors, i.e. if \( (X,\lambda_t) \) is a Weinstein homotopy and \( Z \) is a Weinstein sector, then \( (X,\lambda_t) \times Z \) is also a Weinstein homotopy. For any such choice, all of our results, e.g. Proposition 2.42 and Theorem 5.1, involve only Weinstein homotopies (assuming the regular Lagrangians \( L \subset T^*D^n \) which are the input for Theorem 5.1 are defined using the same class of generalized Weinstein structures).

We have already seen that we can move between sectors and stopped domains (Constructions 2.19 and 2.20). The following proposition makes precise the idea that these operations are invertible up to a natural notion of equivalence; it further shows that these operations respect families of Liouville deformations/homotopies. This was proven in a form in [10].
Figure 3. Homotoping an arbitrary sector to a (sector induced by a) stopped domain, with stop in blue and additional critical point in red.

**Proposition 2.35.** Let $X$ be a Liouville/Weinstein sector, and $(X_0, F)$ its associated stopped domain (Construction 2.19). Then $X$ is interior Liouville/Weinstein homotopic to $X' := (X_0, F)$ (Construction 2.20), and this interior homotopy can be chosen to be supported in a standard neighborhood of $\partial X$. Similarly, any bordered Weinstein homotopy $X_t$ between Weinstein sectors associated to stopped domains is homotopic through interior homotopies to a homotopy through Weinstein sectors associated to stopped domains.

**Proof.** Consider a Weinstein sector $X$, so that in a neighborhood of the sectorial boundary the Weinstein structure agrees with $F \times T^*[0,1]$, where we take the canonical Morse-Bott Weinstein structure on $T^*[0,1]$ induced by the zero vector field on $[0,1]$. Then there is an interior Weinstein homotopy to $X'$ so that in a neighborhood of the sectorial boundary the Weinstein structure agrees with $F \times T^*[0,1]$, where the Weinstein structure on $T^*[0,1]$ is induced by a vector field on $[0,1]$ that is zero on $[0,1/4]$, pointing towards 1 on $(1/4,1/2)$, has an index 1 critical point at $1/2$, is pointing towards 0 on $(1/2,1)$. See Figure 3. We note that Weinstein structure $X'$ is induced by a stopped domain $(X_0, F)$, where $X_0 \subset X'$ is a Weinstein subdomain. The second claim follows from a parametrized version of the proof of the first claim. □

**Remark 2.36.** For every index $n - 1$ critical point of $F^{2n-2}$, there is an index $n$ critical point in $(X')^{2n}$, lying over the index 1 critical point on $[0,1]$ and the co-cores of these critical points are called the linking disks of the sectorial divisor $F$.

Next, we show that any bordered homotopy can be converted into an interior homotopy on a slightly larger sector which agrees with the original homotopy away from the sectorial boundary.

**Proposition 2.37.** Let $[Y, F]$ be a sector with sectorial divisor $F$ and $[Y', F]$ be an enlargement by gluing $(F, \lambda_{F,0}) \times (T^*[-1,1], pdq)$ to $Y$ along $F \times T^*[0,1]$.

1. For every (abstract) Liouville homotopy $(F, \lambda_{F,t})$ of $F$, one may choose a bordered Liouville homotopy $[(Y', \lambda_{Y',t}), (F, \lambda_{F,t})]$ of the sector $Y'$ which is constant on $Y$.

2. For every bordered Liouville homotopy $(Y, \lambda_{Y,t})$, one may choose an interior Liouville homotopy $(Y', \lambda_{Y',t, \text{int}})$ of $Y'$ which agrees with $(Y, \lambda_{Y,t})$ on $Y$.

**Remark 2.38.** Instead of attaching $F \times T^*[-1,0]$ to $Y$ to form $Y'$ and modifying $Y'$ in $F \times T^*[-1,0]$, we can identify a neighborhood of $\partial Y$ in $Y$ with $F \times T^*[0,1]$ and apply
Proposition 2.37 to this neighborhood (without affecting the part of $Y$ away from $\partial Y$). In particular, we can assume that $Y'$ is $Y$. In this way, we have the following diagram, which we include for readability.

$$
\begin{align*}
\text{Def}(F) & \\
\xrightarrow{\text{Prop 2.37} (1)} & \text{Interior}(Y) \longrightarrow \text{Bordered}(Y) \xrightarrow{\text{Prop 2.37} (2)} \text{Interior}(Y)
\end{align*}
$$

The vertical downward arrow is a section of the natural forgetful map taking a bordered deformation of $Y$ to a deformation of the sectorial divisor $F$. Though we do not prove this here, there is a natural topology we may give to all sets in the diagram (see Section 2.7 of [17]) for which the functions of Proposition 2.37 are continuous, and for which the horizontal arrows of (2.3) are homotopy equivalences.

The proof of this result will use the following construction from [7, Section 3.3], which we will use repeatedly in this paper.

**Construction 2.39 (Movie construction).** Let $\{\lambda_t\}_{t \in [0,1]}$ be a Liouville homotopy on $Y$. Then we define a Liouville sector structure $\lambda_{\text{movie}}$ on the manifold $Y \times T^*[0,1]$ following [7, Section 3.3]. Namely, if $\lambda_t = \lambda_0 + dh_t$ for a function $h_t : Y \to \mathbb{R}$ (constant in $t$ near 0, 1), then $\lambda_{\text{movie}} = \pi_Y^*\lambda_0 + \pi_T^*[0,1] \lambda_{T^*[0,1]} + dh$, where $h : Y \times [0,1] \to \mathbb{R}$ is defined by $h(t, y) := h_t(y)$ and $\pi_Y : Y \times [0,1] \to Y$ and $\pi_T : Y \times T^*[0,1] \to T^*[0,1]$ are projections. See Section 2.2 of [17] for a proof that this is a sector.

**Proof of Proposition 2.37**. We can first form the movie construction of the homotopy $(F, \lambda_{\Lambda,s})$ [7, Section 3.3], for $0 \leq s \leq t$, i.e. a Liouville sector structure $(F \times T^*[-1,0], \lambda_{\text{movie},t})$ which looks like $(F, \lambda_{F,t}) \times T^*[-1,-1+\varepsilon]$ and $(F, \lambda_{F,0}) \times T^*[-\varepsilon,0]$ near its sectorial boundary. Then we can append $(F \times T^*[-1,0], \lambda_{\text{movie},t})$ to $(F, (F, \lambda_{\Lambda,0}))$ along $F \times 0$ to get a bordered deformation $(Y', \lambda_{Y,t})$ of $Y'$.

Conversely, a bordered Liouville homotopy $(Y, \lambda_{Y,t})$ gives a Liouville homotopy $(F, \lambda_{F,t})$ of the sectorial divisor $F$ of $Y$. Then we can form the ‘flipped’ movie construction of the homotopy $(F, \lambda_{F,t})$ to get $(F \times T^*[-1,0], \lambda_{\text{movie},t})$, which looks like $(F, \lambda_{F,0}) \times T^*[-1,-1+\varepsilon]$ and $(F, \lambda_{F,0}) \times T^*[-\varepsilon,0]$ near its sectorial boundary. Then we can append $(F \times T^*[-1,0], \lambda_{\text{movie},t})$ to $(Y', (F, \lambda_{\Lambda,0}))$ along $F \times 0$ to get an interior deformation $(Y', \lambda_{Y,t,\text{int}})$ of $Y'$ which agrees with $(Y, \lambda_{Y,t})$ on $Y$.

**Remark 2.40.** Since these two constructions are appending the movie construction and the ‘flipped’ movie construction, the concatenation of these homotopies is an interior homotopy of $Y$ which is homotopic through interior homotopies to the constant homotopy.

**Remark 2.41.** In this paper, we will mostly use bordered or interior homotopies but not compactly supported homotopies. By the movie construction (Proposition 2.37), any bordered homotopy can be converted into an interior homotopy. Then by Moser’s theorem, any interior homotopy can be converted into a compactly supported homotopy; however the resulting compactly supported homotopy is not very explicit and so we prefer to work with interior homotopies.
Next to prove a Weinstein version of Proposition 2.37, we first construct a modified Weinstein movie.

**Proposition 2.42. (Weinstein movie construction)** Suppose that \((X, \lambda_t = \lambda_0 + dh_t)\) is a Liouville homotopy between Weinstein structures \((X, \lambda_0, \varphi_0)\), \((X, \lambda_1, \varphi_1)\) which have Lagrangian co-cores \(C_0, C_1\) respectively. Then there is a function \(F : T^*[0, 1] \rightarrow \mathbb{R}\) so that \((X \times T^*[0, 1], \lambda_{\text{movie}}^W := \lambda_{\text{movie}} + dF)\) admits a Weinstein structure whose only Lagrangians co-cores are \(C_1 \times T^*_3[0, 1]\). If \((X, \lambda_t), t \in [0, 1]\) is a Weinstein homotopy, then \((X \times T^*[0, 1], \lambda_{\text{movie}}^W)\), constructed using the restricted homotopy \((X, \lambda_s), s \in [0, t]\), is also a Weinstein homotopy on \(X \times T^*[0, 1]\).

**Proof.** Suppose that \(\varphi_0, \varphi_1\) are Lyapunov functions on \((X, \lambda_0)\) and \((X, \lambda_1)\) respectively that have linear growth rate \(d\varphi_i(Z_i) = \varphi_i\) (outside a compact subset of \(M\)) and \(\lambda_t - \lambda_0 = dh_t\). We will explain how to construct the Weinstein structure \((\lambda_{\text{movie}}^W, \Phi)\) over the region where \(\frac{\partial}{\partial \tau} \lambda_t\) is nonzero, which after reparametrizing we take to be \(\left[\frac{1}{3}, \frac{2}{3}\right]\). There, the only requirement for the Lyapunov condition is that \(d\Phi(\lambda_{\text{movie}}^W) > \varepsilon > 0\). To begin, pick a family \(r_t\) of symplectization coordinates for \(\lambda_t\) with \(r_0 = \varphi_0\) and \(r_1 = \varphi_1\), again assuming this family is constant on \([0, 1/3] \cup [2/3, 1]\). Let \(f : [0, 1] \rightarrow \mathbb{R}_{\geq 0}\) be a Morse-Bott-with-corners function which is zero near \(\{0, 1\}\), has a unique index 1 critical point at \(\frac{3}{4}\), and has non-zero, constant gradient \(\xi = \nabla f\) in \([\frac{1}{3}, \frac{2}{3}]\) (that points toward the critical point at \(\frac{3}{4}\)). We will consider the induced functions \(f\) (by abuse of notation) and \(p(\xi)\) on \(T^*[0, 1]\) (the former by pullback, the latter by pairing the vector field \(\xi\) on \([0, 1]\) with a covector \(p\)). Consider the modified movie form \(\lambda_{\text{movie}}^W\) and function \(\Phi\)

\[
\lambda_{\text{movie}}^W = \lambda_t + \tau dt + dh + ad(p(\xi)) = \lambda_{\text{movie}} + ad(p(\xi))
\]

\[
\Phi = r_t + \tau^2 + Af
\]

Then \((\lambda_{\text{movie}}^W, \Phi)\) is a Weinstein pair for \(a > 0\) sufficiently small and \(A > 0\) sufficiently large depending on \(a\); here \(h(x,t) = h_t(x)\). So in the notation of the statement of this proposition, \(F = p(\xi) : T^*[0, 1] \rightarrow \mathbb{R}\).

To see this note that the corresponding Liouville vector field is

\[
Z_{\text{movie}} = Z_t + (\tau - \dot{h} - a\dot{\xi}\tau)\partial_\tau + a\xi,
\]

so

\[
d\Phi(Z_{\text{movie}}) = d_Mr_t(Z_t) + 2\tau(\tau - \dot{h} - a\dot{\xi}\tau) + ad_t r_t(\xi) + Aadf(\xi) \tag{2.4}
\]

First, we observe that on \([0, 1/3]\), the Liouville homotopy is constant and so \(dh = 0\). So on \(M \times T^*[0, 1/3]\), the structure \((\lambda_{\text{movie}}^W, \Phi)\) is the product of the Weinstein structure \((\lambda_M, \varphi_0)\) on \(M\) and the Morse-Bott Weinstein structure on \(T^*[0, 1/3]\) associated to the Morse-Bott vector field \(\xi\) on \([0, 1/3]\) as discussed in Example 11.12 of [4]. Concretely, here

\[
d\Phi(Z_{\text{movie}}) = d_Mr_t(Z_t) + 2\tau^2(1 - a\dot{\xi}) + Aadf(\xi) \tag{2.5}
\]

since \(\dot{h}\) and \(d_t r_t\) vanish. The key is that for sufficiently small \(a\), the middle term is a positive multiple of \(\tau^2\) and so is bounded from below by a positive multiple of the norm squared of \((\tau - a\dot{\xi}\tau)\partial_\tau\). The first and last term satisfy the Lyapunov inequality using the facts that
(M, \lambda_0, \varphi_0) is Weinstein and the fact that \xi is the gradient of f. The analogous result holds on M \times T^*[2/3, 1], where the homotopy and r_t is also constant in t.

Next, we consider the pair (\lambda_{\text{movie}}^{\text{Weinstein}}, \Phi) on M \times T^*[1/3, 2/3], where \xi is bounded away from zero. We first observe that the compact region of M \times T^*[0, 1] bounded by r_t = C and \tau^2 = D with D large compared to C is attracting for the negative Liouville flow and hence it suffices to prove the Lyapunov inequality in this region. This region is attracting because all points in M \times T^*[0, 1] flow into the non-compact region \{r_t \leq C\} \subset M \times T^*[0, 1] for some large C by considering just the Z_t component of Z_{\text{movie}}. The projection of the region \{r_t \leq C\} to M is compact and so \dot{h} is bounded in \{r_t \leq C\}. Therefore, for all points with sufficiently large \tau-coordinate, say D, we have that \tau - \dot{h} - a \xi \tau is positive (assuming that a is sufficiently small), and similarly for all points with sufficiently negative \tau-coordinate. In this compact region, all the terms of (2.4) are bounded, and the last term is positive and bounded away from zero, since we further assume that we are in M \times T^*[1/3, 2/3]. So we can make the whole equation positive and bounded away from zero by making A sufficiently large. This proves the Lyapunov inequality since |Z_{\text{movie}}|, |df| are both positive in this region.

Since \xi is non-constant in [1/3, 2/3], the Weinstein Lyapunov function \Phi has no critical points in M \times T^*[1/3, 2/3]. In M \times T^*[0, 1/3] and M \times T^*[2/3, 1], we have the product Weinstein structure. So the only critical points of maximal index correspond to (x, 3/4) \subset M \times \{3/4\} \subset M \times T^*[0, 1], where x is a maximal index critical of (M, \lambda_1, \varphi_1). The Lagrangian co-core of this critical point is the product of co-core of p and the co-core of \{3/4\} \subset T^*[0, 1], which is T^*_{3/4}[0, 1]. This finishes the proof of the first claim in the proposition.

For the second claim, we consider the Weinstein movie structure \(X \times T^*[0, 1], \lambda_{\text{movie}}^{\text{Weinstein}}\) that is constructed using the restricted homotopy \((X, \lambda_s), s \in [0, t]\) (and reparametrizing [0, t] to [0, 1]); we call this the generalized Weinstein movie construction since \((X, \lambda_t)\) may be generalized Weinstein. Note that for all t, the Liouville vector field on \((X \times T^*[1/3, 2/3], \lambda_{\text{movie}}^{\text{Weinstein}})\) has no zeroes while the Weinstein structure \((X \times T^*[0, 1/3], \lambda_{\text{movie}}^{\text{Weinstein}})\) is just \((X, \lambda_0) \times T^*[0, 1/3]\) (with the standard Weinstein structure on structure on T^*[0, 1/3]) and the Weinstein structure \((X \times T^*[2/3, 1], \lambda_{\text{movie}}^{\text{Weinstein}})\) is just \((X, \lambda_t) \times T^*[2/3, 1]\) (with the Weinstein structure on structure on T^*[2/3, 1] induced from the vector field \(\xi)\). Since \((X, \lambda_t)\) is a Weinstein homotopy, so is \((X, \lambda_t) \times T^*[2/3, 1]\) since by assumption Weinstein homotopies are preserved by stabilization.

Now we use this Weinstein movie construction to prove the Weinstein analogs of Proposition 2.37.

**Proposition 2.43.**

1. If \([Y, F]\) is a Weinstein sector and \((F, \lambda_{F,t})\) is a Weinstein homotopy with co-cores \(C_{F_t}\) (at times t when \(\lambda_t\) is a Weinstein structure), there is a bordered Weinstein homotopy \([Y', \lambda_{Y',t}), (F, \lambda_{F,t})]\) constant on Y and with the Lagrangian co-cores in \(F \times T^*[-1, 0]\) equal to \(C_{F_t} \times T^*_{-3/4}[-1, 0]\).

2. If \([\{Y, \lambda_{Y,t}\}, (F, \lambda_{F,t})]\) is a bordered Weinstein homotopy, then there is an interior Weinstein homotopy \([\{Y, \lambda_{Y,t,\text{int}}\}, (F, \lambda_{F,0})]\) whose only Lagrangian co-cores in \(F \times T^*[-1, 0]\) equal to \(C_{F_0} \times T^*_{-3/4}\).
Proof. As in Proposition 2.37, For part 1), we first form the (generalized) Weinstein movie $(F \times T^*[-1,0], \lambda_{F,movie, t})$ by applying Proposition 2.42 to the (restricted) Weinstein homotopy $(F, \lambda_{F,s})$ between 0 and $t$ (we use the diffeomorphism $\times -1 : [0,1] \rightarrow [-1,0]$) to make the movie construction be on $F \times T^*[-1,0]$ instead of $F \times T^*[0,1]$). Note that the movie $(F \times T^*[-1,0], \lambda_{F,movie, t})$ can be generalized Weinstein at particular $t$, with possibly birth-death singularities, if $(F, \lambda_{F,t})$ is generalized Weinstein at those $t$. Then we append $(F \times T^*[-1,0], \lambda_{F,movie, t})$ to $Y$ along $F$ to construct $[(Y', \lambda_{Y,t'}, (F, \lambda_{F,t}))$. The co-cores are $C_{F,t} \times T^*[-3/4,-1,0]$ by the construction in Proposition 2.42.

The second part of the proposition is exactly the same except that now we use the flipped movie construction $\lambda_{t, int}$ and so the co-cores are given by the product of the co-cores of $F_0$ and the fiber $T^*[-3/4] \subset T^*[-1,0]$.

Furthermore, if the Weinstein homotopy $(Y, \lambda_t)$ was associated to a stopped domain $((Y_0, \lambda_t), F_t)$ then the original sectorial structure has the form $T^*[0,1]$ for a vector field
Figure 5. Bordered homotopy at time $t$ from Part 3) of Proposition 2.43, before cancellation of critical points and after cancellation. The stopped domain $Y_0$ is modified to the stopped domain $Y_0'$, which has the same interior critical points.

that is zero on $[0, 1/4]$, has an index 1 critical point at $1/2$, and is inward pointing near $1$. Once we attach $T^*[-1, 0] \times F$ with the movie construction, this $T^*[0, 1]$ is in the interior and hence we can cancel the subcritical zero locus on $[0, 1/4]$, with the index 1 critical locus at $1/2$. The result will be a vector field on $T^*[-1, 1]$ which has an index 1 critical point at $-3/4$ and is outward pointing everywhere on $(-3/4, 1)$. So this is precisely a sector associated to a stopped domain $((Y_0, \lambda_t'), F_0)$; here $(Y_0, \lambda_t', F_0)$ is a slight enlargement of the domain $(Y_0, \lambda_t)$ obtained by attaching $F \times T^*[-1/2, 0]$ with a Liouville vector field that has no zeroes in this region. Since the Lagrangians co-cores of $(Y_0, \lambda_t)$ are disjoint from the stop $F$, the co-cores of $(Y_0, \lambda_t')$ and $(Y_0, \lambda_t')'$ agree. See Figure 5.

2.6. Non-strict Liouville embeddings. We will want to consider smooth proper embeddings that only respect Liouville forms after some deformation.

Definition 2.44. Fix a (not necessarily strict) smooth, proper embedding $f : (X, \lambda_X) \to (Y, \lambda_Y)$. We say that the pair $(f, \lambda_Y)$ is a bordered, interior, or compactly supported deformation embedding if $\lambda_Y$ is a bordered, interior, or compactly supported Liouville deformation from $(Y, \lambda_Y')$ to $(Y, \lambda_Y')$ so that $f^*\lambda_Y' = \lambda_X$.

We may also have occasion to require the deformation of $Y$ to be a Weinstein homotopy.
Remark 2.45. We note that interior and compactly supported embeddings can be composed and form the morphisms of a category; see [17] for details. However these properties of these deformations are not invariant under products, nor under stabilization. For example, if \( Y, Y' \) are interior or compactly supported Liouville homotopic, then there is a canonical bordered homotopy of sectors between \( Y \times T^* D^1, Y' \times T^* D^1 \) but it is not interior or compactly supported.

The special case when \( \varphi \) is a diffeomorphism will appear often in this paper:

**Definition 2.46.** If \( \varphi : (X, \lambda_X) \to (Y, \lambda_Y) \) is a bordered, interior, or compactly supported deformation embedding which is also a diffeomorphism, then we call \( \varphi \) an *isomorphism, up to bordered, interior, or compactly supported deformation* respectively.

This is called a Weinstein homotopy equivalence in [4]; we do not use this term in this paper to avoid confusion with equivalences in the categories discussed in Section 2.8.

2.7. **Notions of equivalence between sectors.** There are various notions of equivalence one may define for Liouville sectors—trivial inclusions, sectorial equivalences, bordered deformation equivalences, and movie inclusions. We refer to Section 12 of [17] for details. Here, we recall only one notion:

**Definition 2.47.** Let \( M \times T^*[0, 1] \) be the movie construction for some bordered deformation of Liouville structures on \( M \), where the deformation is constant near \( t = 0, 1 \) as usual. (In particular, there are well-defined Liouville forms \( \lambda_0 \) and \( \lambda_1 \) on \( M \).) Then for any \( \varepsilon \) small enough, we call the inclusions

\[
(M, \lambda_0) \otimes T^*[0, \varepsilon] \hookrightarrow M \times T^*[0, 1], \quad (M, \lambda_1) \otimes T^*[1 - \varepsilon, 1] \hookrightarrow M \times T^*[0, 1]
\]

movie inclusions. Note that both are strict proper embeddings.

**Remark 2.48.** Movie inclusions induce equivalences of wrapped Fukaya categories; see Section 12 of [17].

2.8. **The \( \infty \)-category of stabilized Liouville sectors.** Detailed descriptions of the following appear in [17].

**Notation 2.49.** We let \( \text{Liou}_{\text{str}} \) denote the category whose objects are Liouville sectors, and whose morphisms are strict proper embeddings (Definition 2.27).

The category \( \text{Liou}_{\text{str}} \) admits a symmetric monoidal structure under direct product. (See Remark 2.30.) Accordingly, we have an endofunctor

\[
- \times T^*[0, 1] : \text{Liou}_{\text{str}} \to \text{Liou}_{\text{str}}.
\]

**Notation 2.50.** We let

\[
\text{Liou}_{\text{str}}^\diamond := \text{colim} \left( \text{Liou}_{\text{str}} \xrightarrow{- \times T^*[0, 1]} \text{Liou}_{\text{str}} \xrightarrow{- \times T^*[0, 1]} \ldots \right)
\]

denote the colimit, which one can model as an increasing union of categories.

We call \( \text{Liou}_{\text{str}}^\diamond \) the category of stabilized Liouville sectors.
Remark 2.51. Concretely, an object of $\text{Liou}^\circ_{\text{str}}$ is an equivalence class of a pair $(X, k)$ where $X$ is a Liouville sector and $k \geq 0$ is an integer. By construction, we identify $(X, k) \sim (X \times T^* [0, 1]^n, k + n)$.

Moreover, given two objects (represented by) $(X, k)$ and $(X', k')$, there is a morphism between them if and only if $\dim X - 2k = \dim X' - 2k'$. By definition, the set of morphisms is given by

$$\text{hom}_{\text{Liou}^\circ_{\text{str}}}( (X, k), (X', k') ) = \bigcup_{l \geq 0} \{ i : X \times T^* [0, 1]^l \to X' \times T^* [0, 1]^l + k - k' \}$$

where each $i$ is required to be a strict proper embedding, and the union is taken by identifying any $i$ with $i \times \text{id}_{T^* [0, 1]}$.

Thus $\text{Liou}^\circ_{\text{str}}$ may be thought of as a disjoint union of categories indexed by the invariant $\dim X - 2k$.

Notation 2.52. Let $X$ be a Liouville sector. By abuse of notation, we will denote by $X$ the object of $\text{Liou}^\circ_{\text{str}}$ represented by the pair $(X, \frac{1}{2} \dim X)$.

Notation 2.53. We let $\text{eq}_{\text{movie}} \subset \text{Liou}_{\text{str}}$ denote the collection of strict Liouville embeddings that happen to be movie inclusions (Definition 2.47). We let $\text{eq}_{\text{movie}}^\circ$ denote the collection of strict movie inclusions in $\text{Liou}^\circ_{\text{str}}$.

Now we enter the realm of $\infty$-categories:

Notation 2.54. We let

$$\mathcal{L}\text{iou}^\circ := \text{Liou}^\circ_{\text{str}}[ (\text{eq}_{\text{movie}}^\circ)^{-1} ]$$

denote the ($\infty$-categorical) localization.

Remark 2.55. In fact, we show in Section 12 of [17] that one may localize $\text{Liou}^\circ_{\text{str}}$ with respect to other natural classes of symplectic equivalences such as (i) strict trivial inclusions, (ii) strict sectorial equivalences, (iii) strict bordered deformation equivalences, and (iv) movie inclusions, and the resulting $\infty$-categories are all equivalent.

Remark 2.56. A priori, it is unclear what geometric information this $\infty$-categorically formal process creates. We show in [17] that in fact, the $\infty$-category $\mathcal{L}\text{iou}^\circ$ recovers, up to homotopy equivalence, the stabilized mapping spaces (not just sets) of compactly supported deformation embedding. We do not need this powerful result in the present work, but we illustrate some of the geometric utility in Proposition 2.61 below.

Example 2.57 (Cotangent bundles of disks and cubes). For every $k \geq 2$, there are two a priori different objects $T^* [0, 1]^k$ (which is identified with $T^* D^0$ in $\text{Liou}^\circ_{\text{str}}$) and $T^* D^k$. As smooth manifolds, the former has corners, while the latter does not.

However, there are smooth embeddings $[0, 1]^k \to D^k$ and $D^k \to [0, 1]^k$ whose compositions are smoothly isotopic to the identity. Results of [17] show that, therefore, $T^* [0, 1]^k$ and $T^* D^k$ are equivalent objects in $\mathcal{L}\text{iou}^\circ$, even though they are not isomorphic objects in $\text{Liou}_{\text{str}}$. Notice also that the (strict) proper embeddings $T^* [0, 1]^k \to T^* D^k$ are sectorial equivalences, but not trivial inclusions in the sense of [9] Section 2.4.
Remark 2.58. One of the main results of [17] is that $\text{Liou}^\circ$ admits a symmetric monoidal structure whose action on objects is given by direct product of (stabilized) sectors. More precisely, given two objects $X$ and $X'$ – i.e., pairs $(X, k)$ and $(X', 2k')$ with $\dim X - 2k = \dim X' - 2k' = 0$ (Notation 2.52) – their monoidal product is given by $X \times X'$. Note in particular that the symmetric monoidal unit is the point $T^*D^0$ (and hence $T^*[0, 1]^k$ for $k \geq 0$).

This is far from formal, and is a consequence of the fact that, after localizing with respect to sectorial equivalences, the orientation-preserving permutations of $T^*[0, 1]^k$ are homotopic to the identity. The existence of such homotopies is one of the non-trivial ways in which the categorically formal process of localization detects geometrically meaningful phenomena.

Notation 2.59 (The subcategories of Weinsteins). Finally, we let

$$\text{Wein}_{\text{str}} \subset \text{Liou}_{\text{str}}, \quad \text{Wein}^\circ \subset \text{Liou}^\circ$$

denote the full subcategory of those sectors that admit a Weinstein structure.

Remark 2.60. Note that $\text{Wein}_{\text{str}}$ and $\text{Wein}^\circ$ are defined to be full subcategories – in particular, objects are not equipped with a Weinstein structure, though they are abstractly known to admit one. We emphasize that the morphisms in these $\infty$-categories need not respect Weinstein structures in any way.

The reader may well wonder why we do not stabilize $\text{Wein}_{\text{str}}$ and localize. This is immaterial: We will prove in later work that $\text{Wein}_{\text{str}}^\circ[\{\text{eq}^\circ_{\text{movie}} \cap \text{Wein}_{\text{str}}^\circ\}^{-1}]$ is equivalent as an $\infty$-category to $\text{Wein}^\circ$. The main reason for considering $\text{Wein}^\circ$ instead of $\text{Wein}_{\text{str}}^\circ[\{\text{eq}^\circ_{\text{movie}} \cap \text{Wein}_{\text{str}}^\circ\}^{-1}]$ in this paper is because there is a concrete geometric model for $\text{Wein}^\circ$, as proven in [17]; see Remark 2.56.

2.9. Converting sectorial equivalences into equivalences in the stable Liouville category. The following is a formal consequence of [17], but we give an explicit proof for the sake of being self-contained in our geometry. It demonstrates how geometric structures that cannot be categorically captured using only strict proper inclusions in $\text{Liou}_{\text{str}}$ become visible in $\text{Liou}^\circ$ by passing to movies.

**Proposition 2.61.**

1. If $(X, \lambda_{X,t})$ is a Liouville homotopy, then $(X, \lambda_{X,0})$ and $(X, \lambda_{X,1})$ are equivalent in $\text{Liou}^\circ$.

2. Furthermore, suppose there is a commutative diagram of smooth maps

$$
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
$$

where $f_0, f_1$ are strict proper inclusions and $\varphi_X, \varphi_Y$ are isomorphisms up to bordered deformation, and that the bordered deformation on $Y_1$ may be chosen to extend $f_1$ of
the bordered deformation of $X_1$. Then there is a diagram in $\mathcal{L}_{\text{Liou}}^\circ$:

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
\downarrow{\overline{\varphi}_X} & & \downarrow{\overline{\varphi}_Y} \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]

where $\overline{\varphi}_X$ and $\overline{\varphi}_Y$ are equivalences (i.e., homotopy invertible morphisms) in $\mathcal{L}_{\text{Liou}}^\circ$.

Proof. (1) The movie construction of $\lambda_{X,t}$ defines a Liouville sector $(X \times T^*[0,1], \lambda_{X,movie})$, which admits strict proper inclusions

\[
i_{X,0} : (X, \lambda_{X,0}) \times T^*[0,\varepsilon] \to (X \times T^*[0,1], \lambda_{X,movie})
\]

and

\[
i_{X,1} : (X, \lambda_{X,1}) \times T^*[1-\varepsilon,1] \to (X \times T^*[0,1], \lambda_{X,movie})
\]

These two strict proper inclusions are movie inclusions (Definition 2.47), hence are equivalences in $\mathcal{L}_{\text{Liou}}^\circ$. In $\mathcal{L}_{\text{Liou}}^\circ$, $(X, \lambda_{X,0})$ and $(X, \lambda_{X,1})$ are identified with their stabilizations, so we see they are equivalent in $\mathcal{L}_{\text{Liou}}^\circ$.

(2) Because $\varphi_X : (X_0, \lambda_{X_0}) \to (X_1, \lambda_{X_1})$ is an isomorphism up to deformation, there is a homotopy of forms $\lambda_{X_{1,t}}$ on $X_1$ from $\lambda_{X_1} := \lambda_{X_{1,0}}$ to $\lambda_{X_{1,1}}$ and $\varphi : (X_0, \lambda_{X_0}) \to (X_1, \lambda_{X_{1,1}})$ is a strict isomorphism. As in the previous paragraph, we have strict inclusions

\[
(X_0, \lambda_{X_0}) \times T^*D^1 \to (X_1, \lambda_{X_{1,1}}) \times T^*D^1 \to (X_1 \times T^*D^1, \lambda_{X_{1,movie}}) \leftarrow (X_1, \lambda_{X_1}) \times T^*D^1
\]

where the first map is an isomorphism in $\mathcal{L}_{\text{Liou}}^\text{str}$ (it is in fact the diffeomorphism $\varphi_X$) and the last two morphisms are (strict) movie inclusions (given by $i_{X,0}$ and $i_{X,1}$). Because $\mathcal{L}_{\text{Liou}}^\circ$ is a localization of $\mathcal{L}_{\text{Liou}}^\text{str}$ along movie inclusions, this zig-zag defines an equivalence $\overline{\varphi}_X : (X_0, \lambda_{X_0}) \to (X_1, \lambda_{X_1})$, well-defined up to contractible space of choices, in $\mathcal{L}_{\text{Liou}}^\circ$. This also defines $\overline{\varphi}_Y$.

To see that the square (2.6) can be made to commute, use the assumption that $\lambda_{Y_{1,t}}$ extends $\lambda_{X_{1,t}}$ to construct the strict proper inclusion $f_{\text{movie}} : (X_1 \times T^*[0,1], \lambda_{X_{1,movie}}) \to (Y_1 \times T^*[0,1], \lambda_{Y_{1,movie}})$ fitting into the following commutative diagram in $\mathcal{L}_{\text{Liou}}^\text{str}$:

\[
\begin{array}{ccc}
(X_0, \lambda_{X_0}) \times T^*D^1 & \xrightarrow{f_0} & (X_1, \lambda_{X_{1,1}}) \times T^*D^1 & \xrightarrow{f_1} & (X_1 \times T^*D^1, \lambda_{X_{1,movie}}) & \rightarrow & (X_1, \lambda_{X_{1,0}}) \times T^*D^1 \\
\downarrow{f_{\text{movie}}} & & \downarrow{f_0} & & \downarrow{f_{\text{movie}}} & & \downarrow{f_1} \\
(Y_0, \lambda_{Y_0}) \times T^*D^1 & \rightarrow & (Y_1, \lambda_{Y_{1,1}}) \times T^*D^1 & \rightarrow & (Y_1 \times T^*D^1, \lambda_{Y_{1,movie}}) & \rightarrow & (Y_1, \lambda_{Y_{1,0}}) \times T^*D^1
\end{array}
\]

Thus the induced diagram (2.6) in $\mathcal{L}_{\text{Liou}}^\circ$ also commutes (up to canonical homotopy in $\mathcal{L}_{\text{Liou}}^\circ$).

\[
\square
\]

2.10. Subcritical morphisms.

**Definition 2.62.** A strict proper inclusion $i : X \hookrightarrow Y$ is said to realize a subcritical handle removal if, after attaching some subcritical Weinstein handles to $Y \setminus i(X)$ to produce a new sector $Y'$, the induced inclusion $i' : X \to Y'$ is a movie inclusion. Likewise, a strict proper inclusion $i : X \hookrightarrow Y$ is said to realize a subcritical handle attachment if, after removing some subcritical Weinstein handles to $Y \setminus i(X)$ to produce a new sector $Y'$, the inclusion $i : X \to Y'$ is a movie inclusion.
More generally, we say that $i$ is subcritical if it can be written as a composition of strict proper inclusions realizing subcritical handle attachments or subcritical handle removals.

In particular, a movie inclusion is a subcritical proper inclusion (obtained by removing or attaching no subcritical handles to $Y \setminus i(X)$).

The next proposition shows that subcritical morphisms are preserved under taking products.

**Proposition 2.63.** Let $Y_i$ be Weinstein sectors and let $f_i : X_i \to Y_i$ be subcritical morphisms for $i = 1, 2$. Then the map $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is subcritical.

**Proof.** Without loss of generality, we may assume $f_1$ realizes a subcritical handle attachment/removal, and likewise for $f_2$. So there are sectors $Y'_i$ so that $f'_i : X_i \to Y'_i$ is a movie inclusion and $Y'_1$ is obtained from $Y_1 \setminus i(X_1)$; similarly, $Y'_2$ is obtained from $Y_2$ by removing/attaching some subcritical cobordism $C_2$ in $Y_2 \setminus i(X_2)$. Then $f'_1 \times id : X_1 \times X_2 \to Y'_1 \times X_2$ and $id \times f'_2 : X_1 \times X_2 \to X_1 \times Y'_2$ are movie inclusions. Moreover, $Y'_1 \times X_2$ is obtained from $Y_1 \times X_2$ by removing/attaching some cobordism of the form $C_1 \times X_2$. Since $C_1$ is subcritical and $X_2$ is Weinstein, $C_1 \times X_2$ is subcritical for dimension reasons. This shows $f_1 \times id$ is subcritical. Likewise, we see $id \times f_2$ is subcritical. Then $f_1 \times f_2 = (id \times f_2) \circ (f_1 \times id)$ is also subcritical. \qed

**Remark 2.64.** One may have wondered why our main results concern $\text{Wein}^\circ_{\text{crit}}$ rather than $\text{Liou}^\circ_{\text{crit}}$. Proposition 2.63’s proof shows why. In general, the product of two subcritical morphisms of Liouville sectors need not be subcritical. One may of course define a natural class of morphisms among Liouville sectors that are generated by subcriticaals under direct product, but we prove no notable properties of such a class of morphisms here.

2.11. The critical $\infty$-category.

**Notation 2.65.** We let $s \subset \text{Wein}_{\text{str}}$ denote the collection of strict proper embeddings which happen to be subcritical morphisms (Definition 2.62). We let $s^\circ$ denote the image of the collection of such morphisms in $\text{Wein}^\circ$. We let

$$\text{Wein}^\circ_{\text{crit}} := \text{Wein}^\circ[(s^\circ)^{-1}]$$

denote the localization.

**Remark 2.66.** Note that it also makes sense to ask whether a morphism between Liouville sectors (not Weinstein sectors) is subcritical. Thus, one could also profitably consider

$$\text{Liou}^\circ_{\text{crit}} := \text{Liou}^\circ[(s^\circ)^{-1}]$$

but we do not do so in this paper. See Remark 2.64.

2.12. Subdomain embeddings and sectorial cobordisms. It will also be convenient to consider a class of symplectic embeddings between Liouville or Weinstein sectors that are not necessarily proper inclusions.

**Definition 2.67.** Let $X$ and $Y$ be Liouville sectors. We say that a (not necessarily proper) smooth, codimension zero embedding $i : X \to Y$ with $i^*\lambda_Y = \lambda_X$ is a strict subdomain embedding, or strict subdomain inclusion, if $i(\partial X) \subset \partial Y$. 
**Notation 2.68.** We will use the notation \( i : X \rightarrowtail Y \) for subdomain inclusions and the notation \( i : X \hookrightarrow Y \) for strict proper inclusions.

**Remark 2.69.** As we will see later (Proposition 2.73), subdomain inclusions can be converted (contravariantly and up to homotopy) into proper inclusions in the stable critical Weinstein category.

**Definition 2.70.** If \( i : X \rightarrowtail Y \) is a strict subdomain inclusion, then we say that \( Y \setminus i(\text{skel} X) \) is a Liouville sectorial cobordism. We will frequently abuse notation and write \( Y \setminus i(X) \) to mean \( Y \setminus i(\text{skel} X) \).

We can also define an abstract Liouville sectorial cobordism to be an exact symplectic manifold satisfying all conditions of a Liouville sector except that there are two ‘boundaries at infinity’, namely \( \partial\pm\infty C \), and the Liouville vector field is outward, inward pointing at \( \partial_{+\infty} C, \partial_{-\infty} C \) respectively.

**Definition 2.71.** We say that \( i : X \rightarrowtail Y \) is a (subcritical) Weinstein subdomain inclusion if the sectorial cobordism \( Y \setminus i(X) \) admits a (subcritical) Weinstein structure.

**Remark 2.72.** Note that a subcritical proper inclusion \( i : X \rightarrow Y \) as in Definition 2.62 implies that there is a subcritical subdomain inclusion \( Y \rightarrowtail Y' \) or \( Y' \rightarrowtail Y \).

### 2.13. Converting subdomain inclusions into morphisms in the critical category.

In this section, we explain how to convert sectorial subdomain inclusions into morphisms in \( \text{Wein}^\circ_{\text{crit}} \), i.e. strict inclusions up to stabilization and subcritical inclusions.

**Proposition 2.73.** Suppose there is a commutative diagram of symplectic embeddings

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
\downarrow{\varphi_X} & & \downarrow{\varphi_Y} \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]  

(2.7)

where \( f_0, f_1 \) are strict proper inclusions and \( \varphi_X, \varphi_Y \) are strict Weinstein subdomain inclusions; furthermore, assume that this is a pullback diagram of sets, i.e. \( X_1 \cap Y_0 = X_0 \). Then there is a homotopy commutative diagram in \( \text{Wein}^\circ_{\text{crit}} \):

\[
\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
\varphi_X^{*} & & \varphi_Y^{*} \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]  

(2.8)

Furthermore, if \( \varphi_X, \varphi_Y \) respectively are subcritical subdomain inclusions, then \( \varphi_X^{*}, \varphi_Y^{*} \) are isomorphisms in \( \text{Wein}^\circ_{\text{crit}} \) respectively.

**Proof.** We first discuss the case of a single subdomain inclusion \( \varphi_X : X_0 \rightarrowtail X_1 \). We construct an intermediate sector, which we call the *Viterbo sector*, that admits strict proper inclusions from the stabilization of both \( X_0 \) and \( X_1 \); this sector was introduced by the second author [29]. The stabilization of \( X_1 \) is \( X_1 \times T^*D^1 = [X_1 \times T^*D^1, X_1 \times 0 \coprod X_1 \times 1] \).
Then we define the Viterbo sector $V(X_0, X_1)$ by removing $(X_1 \setminus X_0) \times \{1\}$ from the sectorial divisor $X_1 \times \{0, 1\}$ of $X_1 \times T^*D^1$ by making the Liouville vector field point outward along $(X_1 \setminus X_0) \times \{1\}$. Since $V(X_0, X_1)$ is obtained by stop removal from $X_1 \times T^*D^1$, there is a proper inclusion

$$i_{X_1,V} : X_1 \times T^*D^1 \hookrightarrow V(X_0, X_1).$$

The sectorial divisor of $V(X_0, X_1)$ looks like $X_0 \times \{0, 1\}$ as a set and using the condition that $\varphi_0(\partial X_0) \subset \partial X_1$ (since $\varphi_X : X_0 \xhookrightarrow{} X_1$ is a subdomain inclusion), we get a proper inclusion

$$i_{X_0,V} : X_0 \times T^*D^1 \hookrightarrow V(X_0, X_1)$$

We note that if $S = \varphi_X(\partial X_0) \setminus \partial X_1$ is non-empty, then we would only get a proper inclusion $(X_0 \setminus S) \times T^*D^1 \hookrightarrow V(X_0, X_1)$, where $(X_0 \setminus S)$ is the result of stop removal of $S$ from $X_0$. After homotopy, the proper inclusion $i_{X_0,V}$ is always a subcritical morphism. Namely, the sector $V(X_0, X_1)$ can be homotoped to $V'(X_0, X_1)$ which is obtained from $X_0 \times T^*D^1$ by attaching handles, one for each handle of $X_1 \setminus X_0$, and these handles have the same index in $V'(X_0, X_1)$ as in $X_1 \setminus X_0$ (which has lower dimension than $V'(X_0, X_1)$) since the Liouville vector field is outward pointing along $(X_1 \setminus X_0) \times \{0, 1\}$. Hence we have a zig-zag of strict proper inclusions

$$i_{X_1,V} : X_1 \times T^*D^1 \hookrightarrow V(X_0, X_1) \hookrightarrow X_0 \times T^*D^1$$

with the second map a subcritical morphism, which defines the morphism $\varphi_X^V : X_1 \to X_0$ in the critical category $\text{Wein}_{\text{crit}}^\circ$.

Next, suppose have a commutative diagram of symplectic embeddings as in the second part of the proposition. Then we have a commutative diagram in $\text{Wein}_{\text{str}}^\circ$ of the form

$$\xymatrix{ X_1 \times T^*D^1 \ar[r]^{f_1 \times \text{Id}_{T^*D^1}} \ar[d]^{i_{X_1,V}} & Y_1 \times T^*D^1 \ar[d]^{i_{Y_1,V}} \\
V(X_0, X_1) \ar[r]^{f_0 \times \text{Id}_{T^*D^1}} & V(Y_0, Y_1) \ar[r]^{i_{Y_0,V}} & X_0 \times T^*D^1 }$$

The middle horizontal map from $V(X_0, X_1)$ to $V(Y_0, Y_1)$ is constructed as follows. By the commutative diagram 2.7 and the assumption that $X_1 \cap Y_0 = X_0$, we have $f_1 (X_1 \setminus X_0)$ is a subset of $(Y_1 \setminus Y_0)$. Since $V(X_0, X_1), V(Y_0, Y_1)$ are obtained from $X_1 \times T^*D^1, Y_1 \times T^*D^1$ by making the Liouville vector fields outward pointing on $(X_1 \setminus X_0) \times \{0, 1\}, (Y_1 \setminus Y_0) \times \{0, 1\}$ respectively, we get an induced map $f_0 \times \text{Id}_{T^*D^1}$ from $V(X_0, X_1)$ to $V(Y_0, Y_1)$ making the diagram commute. The bottom vertical maps in Equation 2.9 are subcritical morphisms and hence this induces the desired commutative diagram in $\text{Wein}_{\text{crit}}^\circ$.

Finally, we note that if $X_0 \subset X_1$ is a subcritical subdomain inclusion, then the first map $i_{X_1,V}$ is also an equivalence in $\text{Wein}_{\text{crit}}^\circ$. In general, we can construct a sector $V(X_0, X_1)'$ by attaching handles to $V(X_0, X_1)$ in the complement of the image of $i_{X_1,V}$ so that $i_{X_1,V} : X_1 \times T^*D^1 \to V(X_0, X_1)'$ is a strict movie inclusion, after an interior homotopy of $V(X_0, X_1)'$. More precisely, for each handle of $X_1 \setminus X_0$ of index $i$, we attach a Weinstein cobordism with a pair of cancelling handles of index $i, i+1$ (with the index $i$ handle in the sectorial boundary of
Therefore, if all handles of $X_1 \setminus X_0$ are subcritical, then $V(X_0, X_1)'$ is obtained by attaching subcritical handles to $V(X_0, X_1)$ (the index of one of the handles is one larger, but the half-dimension of $V(X_0, X_1)$ is one larger than that of $X_0$). See Figure 6. □

2.14. Lagrangians. In this paper, we consider properly embedded Lagrangians $L^n$ in $(X^{2n}, \lambda)$ that are exact, i.e. $\lambda|_L$ is an exact 1-form. From the perspective of stopped domains, the properness condition means exactly that the Lagrangian boundary of $L$ avoids the stop. We will also equip $L$ with the data of a bordered Liouville homotopy $\lambda_{L,t}$ (Definition 2.31 and Remark 2.32) from $\lambda$ to $\lambda_L$ for which $L$ is strictly exact, i.e. $\lambda_L|_L = 0$. Then by Proposition 2.37, we can also assume that this bordered homotopy is an interior homotopy. We will keep track of $(L, \lambda_{L,t})$ as a tuple. If $L$ is already strictly exact for $\lambda$, then we say that a homotopy $\lambda_t$ of $\lambda$ is relative to $L$ if $L$ is strictly exact for all $\lambda_t$.

Example 2.74. For example, if $L$ has connected Legendrian boundary for $\lambda$, then there is a compactly supported function $h : X \to \mathbb{R}$ so that $\lambda + dh|_L = 0$ and $\lambda + dh$ is (canonically) compactly supported homotopic to $\lambda$.

Example 2.75. The issue is that the condition of having Legendrian boundary is not invariant under taking products. If $L \subset X, K \subset Y$ are exact Lagrangians, then $L \times K \subset X \times Y$ is another exact Lagrangian in the product. If $\lambda_X, \lambda_Y$ do not vanish everywhere on $L, K$, then $L \times K$ does not have Legendrian boundary with respect to the product Liouville form $\lambda_X + \lambda_Y$. However, $L \times K$ is strictly exact for $\lambda_{X,L} + \lambda_{Y,K}$, which is bordered Liouville homotopic to $\lambda_X + \lambda_Y$. In particular, there is an interior Liouville homotopy of $\lambda_X + \lambda_Y$ to a new form $\lambda_{X \times Y, L \times K}$ for which $L \times K$ is strictly exact. Note that there cannot be a compactly supported deformation of $\lambda_X + \lambda_Y$ to a form for which $L \times K$ is strictly exact since this would imply that $L \times K$ has Legendrian boundary for $\lambda_X + \lambda_Y$.

Definition 2.76. If $(X, \lambda)$ is Weinstein, a Lagrangian $L \subset X$ is called regular if the Liouville deformation $\lambda_{L,t}$ from $\lambda$ to $\lambda_L$ is a bordered Weinstein homotopy.

Lemma 2.2 of [8] proves that one can apply a further Weinstein homotopy (with only birth-death singularities) supported near $L$ so that a neighborhood $N$ of $L$ for which $(N, \lambda_L)$ can...
be identified with \((T^*L, \lambda_{T^*L, std})\) equipped with its canonical Weinstein structure (induced by any proper Morse function on \(L\) or the zero function). By further applying Proposition 2.37, we can also assume that there is an interior Weinstein homotopy from \(\lambda\) to such \(\lambda_L\).

3. \(\infty\)-CATegorical Background

Let \(A\) be a Weinstein sector of dimension \(N\), and suppose that there is a sectorial embedding \(u : T^*\mathbb{R}^n \to A\) so that

\[
\text{id}_A \times u : A \times T^*\mathbb{R}^N \to A \times A
\]

is, after localizing \(\text{Wein}^\circ\) along some collection of morphisms \(S\), an equivalence. (For example, \(\text{id}_A \times u\) could itself be in \(S\).) The goal of this section is to prove that – so long as \(S\) is closed under direct product of sectors – the existence of \(u\) implies that

1. the functor \(\text{Wein}^\circ[S^{-1}] \to \text{Wein}^\circ[S^{-1}]\) given by \(\_ \times A\) is a localization onto its image, and

2. \(A\) inherits a natural \(E_\infty\)-algebra structure in \(\text{Wein}^\circ[S^{-1}]\) with unit \(u\).

See Corollary 3.14. We will apply this result when \(A = (T^*D)_p\) and \(S\) is the class of subcritical morphisms to obtain Theorem 1.3.

Remark 3.1. There are geometric consequences of the \(\infty\)-categorical results here. For example, by applying Theorem 1.3 we can conclude that the diagram of sectorial embeddings

\[
\begin{array}{ccc}
(T^*D)_p \times (T^*D)_p & \text{swap} & (T^*D)_p \times (T^*D)^n \\
\downarrow \text{swap} & & \downarrow \\
(T^*D)_p \times (T^*D)_p
\end{array}
\]

commutes up to homotopy in \(\text{Wein}^\circ_{\text{crit}}\). (Further, all maps are equivalences in \(\text{Wein}^\circ_{\text{crit}}\).)

The statements in the rest of this section are purely \(\infty\)-categorical. All of them are already contained in, or are immediate consequences of, the extensive machinery constructed in [19] [18].

Finally, for further preliminaries on \(\infty\)-categories and their localizations streamlined for our uses, we refer the reader to [17]. Further references include [18] [19] [12] [24].

3.1. Idempotent functors and localizations. We review the passage between idempotent endofunctors and localizations. More specifically, let us declare an idempotent structure on an endofunctor \(L : \mathcal{C} \to \mathcal{C}\) to be a natural transformation \(\eta : \text{id}_\mathcal{C} \to L\) for which the induced maps \(\eta_{L(X)}, L(\eta(X)) : L(X) \to L \circ L(X)\) are both equivalences.

Proposition 3.2. Let \(L : \mathcal{C} \to \mathcal{C}\) be a functor. The following are equivalent:

1. \(L\) admits an idempotent structure.
2. Think of \(L\) as a functor from \(\mathcal{C}\) to the full subcategory \(\mathcal{C}_L \subset \mathcal{C}\) spanned by the image of \(L\). Then \(L\) is a left adjoint to the (fully faithful) inclusion \(\mathcal{C}_L \to \mathcal{C}\).

Proof. This follows from Proposition 5.2.7.4 of [18]. \(\square\)
Remark 3.3. Recall that the localization along some class of morphisms $S$ is the initial $\infty$-category inverting all morphisms in $S$. Proposition 3.6 below (which is an immediate consequence of Proposition 5.2.7.12 of [18]) shows that any functor satisfying (2) of Proposition 3.2 is a localization.

Let us mention for the benefit of the reader that not all localizations satisfy (2). (For example, the localizations used in [17] do not.) This fact conflicts with the convention of [18], which only declares functors satisfying (2) to qualify as localizations. In this work, this distinction will not matter; all localizations will be left adjoints to fully faithful inclusions.

Example 3.4 (Smashing localizations). Let $C^\otimes$ be a symmetric monoidal $\infty$-category and fix $A \in \text{ob} C$. Let $1_C$ be the monoidal unit, and fix a map $u : 1_C \to A$. This defines a natural transformation $\eta : \text{id}_C \to A \otimes -$ , and for any object $X \in C$, we have induced maps

$$\eta_{A \otimes X} : A \otimes X \simeq 1_C \otimes (A \otimes X) \xrightarrow{u \otimes \text{id}_A \otimes X} A \otimes A \otimes X,$$

$$A \otimes \eta_X : A \otimes X \simeq A \otimes 1_C \otimes X \xrightarrow{\text{id}_A \otimes u \otimes \text{id}_X} A \otimes A \otimes X.$$

Of course, by permuting the two $A$ factors (using the symmetric monoidal structure) either both of these maps are equivalences, or neither is. So a sufficient condition for $u$ to induce an idempotent structure on the endofunctor $X \mapsto A \otimes X$ is for

$$A \simeq 1 \otimes A \xrightarrow{u \otimes \text{id}_A} A \otimes A$$

to be an equivalence.

Example 3.5. Here are three examples; they are all the “same” example in spirit.

1. Let $C^\otimes$ be the category (in the classical sense) of $(\mathbb{Z}$-linear) rings, with the usual symmetric monoidal structure of tensor product (over $\mathbb{Z}$). Letting $A = \mathbb{Z}[1/p]$, we see that the unit map $u : \mathbb{Z} \to \mathbb{Z}[1/p]$ – otherwise known as the inclusion of the integers – satisfies the property that $u \otimes \text{id}_A$ is an isomorphism from $A$ to $A \otimes A$.

2. More generally, let $C^\otimes$ be the opposite category (in the classical sense) of schemes over some base scheme $S$, equipped with the symmetric monoidal structure of fiber product over $S$. Let $A$ be any open subscheme of $S$, and let $u$ be (the opposite of) the inclusion of $A$ into $S$ – e.g., a degree one open immersion. Then $u \times_S \text{id}_A$ is an isomorphism from $A$ to $A \times_S A$.

3. Likewise, let $C^\otimes$ be the category (in the classical sense) of topological spaces equipped with a continuous map to a fixed space $X$, equipped with symmetric monoidal structure given by fiber product over $X$. Then any open subset $U \subset X$, equipped with the inclusion $u : U \to X$, has the property that $u \times_X \text{id}_U : U \to U \times_X U$ is a homeomorphism.

Proposition 3.6. Let $L : C \to C$ be a functor satisfying either of the conditions of Proposition 3.2. Let $S \subset C$ be the class of morphisms that are sent to equivalences under $L$. Then the natural map $C[S^{-1}] \to LC$ from the localization along $S$ is an equivalence.

Proof. This follows from Proposition 5.2.7.12 of [18], which shows that $LC$ satisfies the mapping property that characterizes $C[S^{-1}]$.  $\square$
Example 3.7. Following the enumeration of Example 3.5:

1. $L_C$ is the full subcategory consisting of those rings in which $p$ is multiplicatively invertible.
2. $L_C$ is the full subcategory consisting of those objects $A \to S$ for which the map from $A$ factors through $U$.
3. $L_C$ is the full subcategory consisting of those objects $A \to X$ for which the map from $A$ factors through $U$.

3.2. Symmetric monoidal localization. Let $C^\otimes$ be a symmetric monoidal $\infty$-category with underlying $\infty$-category $C$. Fix also a collection of morphisms $S \subset C$.

Definition 3.8. We say that $S$ is compatible with $\otimes$ if for all $f, g \in S \times S$ we have $f \otimes g \in S$.

Proposition 3.9. Suppose $S$ is compatible with $\otimes$. Then there is an induced symmetric monoidal structure on $C[S^{-1}]$ so that the functor $C \to C[S^{-1}]$ may be promoted to a symmetric monoidal functor.

In fact, more is true: for any symmetric monoidal $\infty$-category $D^\otimes$, the $\infty$-category of symmetric monoidal functors Fun$^\otimes(C[S^{-1}]^\otimes, D^\otimes)$ is identified with the full subcategory of Fun$^\otimes(C^\otimes, D^\otimes)$ sending morphisms in $S$ to equivalences in $D$. This identification is given by composing with the symmetric monoidal functor $C^\otimes \to C[S^{-1}]^\otimes$.

Proof. This is a consequence of Proposition 4.1.3.4 of [19].

In [17] we show that the $\infty$-category $\mathcal{L}$iou$^\otimes$ of stabilized Liouville sectors admits a symmetric monoidal structure, which on objects acts by the direct product of stabilized sectors (Remark 2.58). This restricts to a symmetric monoidal structure on $\mathcal{W}$ein$^\otimes$. Because $\mathfrak{s}$ is compatible with direct product (Proposition 2.63), we see that $\mathfrak{s}^\otimes$ is also compatible with the symmetric monoidal structure of $\mathcal{W}$ein$^\otimes$. Thus, we have

Corollary 3.10. $\mathcal{W}$ein$^\otimes_{\text{crit}}$ admits a symmetric monoidal structure, which on objects is given by direct product of (stabilized) Liouville sectors.

3.3. Idempotent algebras.

Remark 3.11. In the context of symmetric monoidal $\infty$-categories, the term “commutative algebra” is synonymous with the term “$E_\infty$-algebra.” See Definition 2.1.3.1 of [19].

Proposition 3.12. Let $C^\otimes$ be a symmetric monoidal $\infty$-category and fix $A \in \text{ob}C$. Endow the endofunctor $L : C \to C$ given by $X \mapsto A \otimes X$ with an idempotent structure. (For example, via Example 3.4.)

Then the essential image of $L$ — i.e., the full subcategory $LC \subset C$ — can be given a symmetric monoidal structure for which the functor $X \mapsto A \otimes X$ is symmetric monoidal, and for which $A$ is the symmetric monoidal unit.

Proof. Let $S$ be the class of morphisms $f : X \to Y$ in $C$ such that the induced map $\text{id}_A \otimes f : A \otimes X \to A \otimes Y$ is an equivalence. It follows that $S$ is compatible with $\otimes$ by using the idempotent structure. So $C[S^{-1}]$ is symmetric monoidal and the map $C \to C[S^{-1}]$ may be promoted to be symmetric monoidal by Proposition 3.9. Further, the natural map from
The endofunctor $L : X \mapsto X \otimes A$ is idempotent. By Proposition 3.12, the induced localization $C \to LC$ may be promoted to be symmetric monoidal. This proves the first claim.

Moreover, any right adjoint to a symmetric monoidal functor is automatically lax symmetric monoidal. (This follows from Corollary 7.3.2.7 of [19]. In the notation of loc. cit., one sets $O^\otimes$ to be the nerve of the category of finite pointed sets, and notices that if $F$ is a symmetric monoidal functor, then the fact that the underlying functor $C \to LC$ is a left adjoint implies that the product functors $C^n \to (LC)^n$ are also left adjoints. Finally, note that a map of $\infty$-operads between symmetric monoidal $\infty$-categories is precisely a lax symmetric monoidal functor -- see the comments before Definition 2.1.3.7 of [19].) Noting that $L$ factors as a composition

$$C \to LC \hookrightarrow C$$

where the first arrow may be promoted to be symmetric monoidal, and the latter arrow to a lax symmetric monoidal functor, we see that $L : C \to C$ is lax symmetric monoidal. This proves the second claim.

Because lax symmetric monoidal functors send commutative algebras to commutative algebras, we conclude that $A$ (the unit of $LC$, hence a commutative algebra in $LC$) is sent to a commutative algebra object in $C$. This proves the last claim. □

**Remark 3.15.** If one likes, one need not use the “lax” terminology in the above proof. Corollary 7.3.2.7 of [19] guarantees that the right adjoint becomes a map of $\infty$-operads $j : (LC)^\otimes \to C^\otimes$, hence one can compose any map of $\infty$-operads from the category of finite
pointed sets to $LC$ with $j$. On the other hand, an $\infty$-operad map from the category of finite pointed sets is the definition of a commutative algebra object.

**Remark 3.16.** Though we will not need this, we note that Corollary 3.14 is functorial in the $(A,u)$ and $C^\otimes$ variables. Call an $E_0$-algebra $u : 1_c \to A$ idempotent if the condition in Corollary 3.14 is satisfied. It is straightforward to modify the proof to exhibit a functor from the $\infty$-category of idempotent $E_0$-algebras in $C^\otimes$ to the $\infty$-category of commutative algebra objects in $C^\otimes$. This can further be promoted to be a map of coCartesian fibrations over the $\infty$-category of symmetric monoidal categories, mapping the coCartesian fibration classifying idempotent $E_0$-algebras to the coCartesian fibration classifying commutative algebras.

**Remark 3.17.** In the setting of Corollary 3.14, let $m : A \otimes A \to A$ denote the product of $A$ associated to the guaranteed commutative algebra structure of $A$. Since $A$ is a unital (commutative) algebra, we know that the composition

$$A \simeq 1 \otimes A \xrightarrow{u \otimes id_A} A \otimes A \xrightarrow{m} A$$

is homotopic to the identity. On the other hand, $u \otimes id_A$ is an equivalence by assumption. Thus, we see that $m$ is in fact an equivalence, and is naturally (up to structure maps guaranteed by the symmetric monoidal structure of $C$) identified as a homotopy inverse to $u \otimes id_A$.

**Remark 3.18.** In the setting of Corollary 3.14 (2), the lax monoidal structure maps

$$L(X) \otimes L(Y) \to L(X \otimes Y)$$

are all equivalences. To see why, one can unwind the definitions and observe that the structure maps are the equivalences

$$A \otimes X \otimes A \otimes Y \to A \otimes X \otimes Y$$

guaranteed by the idempotent structure.

However, the structure map from the unit $1_c \to L(1_c) \in C$ may not be an equivalence—see Example 3.5. This is the only reason that the right adjoint is not symmetric monoidal, and is only lax.

### 4. Two models for P-flexibilization

In this section, we review a construction from the first two authors’ previous work [16] that localizes the wrapped Fukaya category. This construction has the advantage of taking Weinstein sectors to equi-dimensional Weinstein domains (and takes Weinstein domains to domains, instead of sectors). A priori it is not Weinstein homotopy invariant and not defined for general Liouville sectors (as opposed to Weinstein sectors). In Section 1.3 we introduce an alternative P-flexibilization functor, which manifestly preserves homotopy equivalences.
4.1. Carving out Lagrangian disks. The construction from [16] relies on a procedure for removing Lagrangian disks, which we now explain. Let $T^*D_{1/2}$ denote the cotangent bundle of the disk of radius $1/2$, with Liouville structure inherited from $T^*D^n$ (see Example 2.16). Note that the Liouville vector field points inward near $\partial T^*D_{1/2}$ so that $T^*D_{1/2}$ is not a sector. Let $X^{2n}$ be a Liouville sector with a properly embedded Lagrangian disk $C$ so that $\lambda_X|_C = 0$.

**Definition 4.1.** A strict proper inclusion

$$\varphi_C : T^*D_{1/2} \hookrightarrow X$$

that sends the cotangent fiber $T^*_0D^n$ to the Lagrangian $C$ is called a parametrization of a neighborhood of $C$.

**Remark 4.2.** We note that for any Lagrangian disk $C$ with Legendrian boundary, there is a Liouville homotopy supported in a neighborhood of $C$ (hence an interior Liouville homotopy) so that a neighborhood of $C$ has a parametrization for the new Liouville form. If $C$ is strictly exact, then there is a Liouville homotopy $\lambda_t$ relative to $C$, i.e. vanishes on $C$, so that $C \subset (X, \lambda_1)$ has a parametrization.

Since the Liouville vector field points into $\varphi_C(T^*D_{1/2})$ near $\varphi_C(\partial T^*D_{1/2})$, $X \setminus \varphi_C(T^*D_{1/2}/2)$ is a new Liouville sector. Note that since $\varphi_C(T^*D_{1/2})$ is contained in the interior or $X$, $X \setminus \varphi_C(T^*D_{1/2})$ has the same sectorial boundary as $X$ but has different contact boundary.

**Notation 4.3 ($X \setminus C$).** Often, we will drop the choice of parametrization $\varphi_C$ and use the notation $X \setminus C$ to denote this sector, which we say is obtained from $X$ by carving out $C$.

**Remark 4.4.** The following fact will be used repeatedly in the proofs of the main results Theorems 1.3, 1.4: If $(X, \lambda_t)$ is a homotopy relative to $C$, then there is an induced homotopy $(X \setminus C, \lambda_t)$.

Next we discuss the Weinstein case.

**Example 4.5.** Any Weinstein sector $X^{2n}$ has a distinguished collection of Lagrangian disks, namely the co-cores $C_i$ of the index $n$ critical points, which can be parametrized if they are properly embedded. Note that if the critical values of the index $n$ critical points is larger than the critical values of the lower index critical points, then the Lagrangians co-cores are properly embedded. Any Weinstein structure can be homotoped through Weinstein structures to one of this form and hence to one whose index $n$ co-cores are properly embedded. Then for each $i$, $X \setminus \varphi_{C_i}(T^*D_{1/2})$ is a Weinstein sector and $X \setminus \bigcup_i \varphi_{C_i}(T^*D_{1/2})$ is a subcritical Weinstein sector.

More generally, by Proposition 2.3 of [8], any regular Lagrangian disk (where the Liouville vector field is taken to point outward near $\partial T^*D_{1/2}$) has a parametrized neighborhood (where the Liouville vector field points inward near $\partial T^*D_{1/2}$) after a further Weinstein homotopy.

**Definition 4.6.** Let $\text{Wein}_{\text{param}}$ be the category whose objects are $(X, \{\varphi_{C_X}\})$, where $X$ is a Weinstein sector with parametrized properly embedded Lagrangian co-cores $\{C_X\}$ and $\varphi_{C_X} : T^*D_{1/2} \hookrightarrow X$ is a strict proper inclusion parametrizing $C_X$. Morphisms are strict
proper inclusions of Liouville sectors that respect this parametrization. Namely, if \( i : X \hookrightarrow Y \) is a strict proper inclusion, then each co-core \( C_X \) of \( X \) is also a co-core of \( Y \) and we require \( i \circ \varphi_{C_X} = \varphi_{C_Y} \), where \( \varphi_{C_X}, \varphi_{C_Y} \) are the parametrizations of the same co-core in \( X, Y \) respectively. Note that there is a forgetful functor from \( \text{Wein}_{\text{param}} \) to \( \text{Wein}_{\text{str}} \).

### 4.2. \( P \)-flexibilization for strict, parametrized Weinstein sectors.
Recall that if \( L \subset (T^*D^n, \lambda_{T^*D^n, \text{std}}) \) is a regular disk (Definition 2.76), then there is an interior Weinstein homotopy from the standard structure to a Weinstein structure \( (T^*D^n, \lambda_L) \) for which \( L \) is strictly exact and has a parametrized neighborhood.

**Notation 4.7.** We set
\[
(T^*D^n)_L := (T^*D^n, \lambda_L) \setminus \varphi_L(T^*D^n_{1/2}).
\]
(See Notation 4.3.)

This is a Weinstein sector whose sectorial boundary has a canonical identification with the sectorial boundary \( \partial T^*D^n \) of \( T^*D^n \). In fact, we will assume that \( \varphi_L(T^*D^n_{1/2}) \) is contained in \( T^*D^n_{1/2} \) and that the Weinstein homotopy from \( (T^*D^n, \lambda_{T^*D^n, \text{std}}) \) to \( (T^*D^n, \lambda_L) \) is supported inside this region; hence, we can also consider the subset \( \varphi_L(T^*D^n_{1/2}) \) of \( (T^*D^n)_L := (T^*D^n_{1/2}, \lambda_L) \setminus \varphi_L(T^*D^n_{1/2}) \). Later we will take a more explicit model for \( (T^*D^n)_L \) suitable to our purposes. Finally, we also fix parametrizations of the Lagrangian co-cores of \( (T^*D^n)_L \), which we can assume are properly embedded.

More generally, consider a Weinstein sector \( (X, \varphi) \) with parametrized middle-dimensional co-cores \( \{C_X\} \). So we have, for each middle-dimensional co-core \( C_X \), a strict proper inclusion \( \varphi_{C_X} : T^*D^n_{1/2} \hookrightarrow X \) that takes \( T^*_0D^n \) to \( C_X \).

**Notation 4.8.** We define
\[
X_L := (X \setminus \bigsqcup_{C_X} \varphi_{C_X}(T^*D^n_{1/2})) \cup \bigsqcup_{C_X} (T^*D^n_{1/2})_L
\]
where we glue \( \varphi_{C_X}(\partial T^*D^n_{1/2}) \) and the copy of \( \partial(T^*D^n_{1/2})_L \) corresponding to \( C_X \).

This is a Weinstein sector and has parametrized critical Weinstein handles since this is true for \( (X \setminus \bigsqcup_{C_X} \varphi_{C_X}(T^*D^n_{1/2})) \), which is subcritical, and also true for \( \bigsqcup_{C_X} (T^*D^n_{1/2})_L \). Note that \( X_L \) makes sense even if \( L \subset T^*D^n \) is a finite set of several disjoint Lagrangian disks.

**Remark 4.9.** The whole construction of \( X_L \) can be summarized by taking a certain Weinstein homotopy \( (X, \lambda_X) \) to \( (X, \lambda_{X,L}) \) (supported near the index \( n \) co-cores) which now has many copies of \( L \) as co-cores, and then carving out those copies of \( L \).

More precisely, if \( C_X \) is a co-core of \( X \), then we let \( C^L_X := \varphi_{C}(L) \subset X \) denote the copy of \( L \) in the tubular neighborhood \( \varphi_{C_X}(T^*D^n) \) of \( C_X \); then we also have
\[
X_L \cong (X, \lambda_{X,L}) \setminus \bigsqcup_{C_X} C^L_X
\]
or just
\[
X \setminus \bigsqcup_{C_X} C^L_X
\]
when \( \lambda_{X,L} \) is clear.
By construction, there is a sectorial subdomain inclusion $X_L \subset X$; by Proposition 2.73, this gives rise to a morphism $X \to X_L$ in the critical category $\mathcal{W}_{\text{crit}}^\circ$. By [10], this implies that the Fukaya category of $X_L$ is a localization of the Fukaya category of $X$ obtained by nullifying $L$ viewed as an object the Fukaya category; that is, $Tw\mathcal{W}(X_L) \simeq Tw\mathcal{W}(X)/\coprod_{(C_X)} C_{X_L}^L$.

Next we discuss functoriality.

**Notation 4.10 ($i_L$).** Given a strict proper inclusion of Weinstein sectors $i : X \hookrightarrow Y$ preserving parametrizations of properly embedded Lagrangian co-cores, there is an induced strict proper inclusion

$$i_L : X_L \hookrightarrow Y_L$$

of Weinstein sectors (again preserving parametrizations of co-cores). To see this, recall that the Lagrangian co-cores of $X$ are a subset of the Lagrangian co-cores of $Y$ (Remark 2.29) so when we carve out copies of $L$ near the Lagrangian disks of $Y$, we do the same for $X$. Furthermore, $(j \circ i)_L = j_L \circ i_L$ for the same reason. Then $(\_)_L$ defines an endofunctor of $\text{Wein}_{\text{param}}$.

The main issue is that the functor $(\_)_L$ does not preserve Weinstein (or Liouville) homotopy equivalences. More precisely, if $X, X'$ are Weinstein homotopic Weinstein sectors, then $X_L$ and $X'_L$ need not be Weinstein homotopic, i.e. $X_L$ depends on the Weinstein presentation of $X$. For example, we can Weinstein homotope $X$ to a Weinstein presentation $X'$ with many more index $n$ handles, in which case $X'_L$ is not even homotopy equivalent to $X_L$ as topological spaces. Next, we give a more drastic example.

**Example 4.11.** Consider the standard Weinstein presentation $T^*S^n$ with a single co-core $T_x^*S^n$ or homotopic structure $T^*S^n'$ with two co-cores $T^*S^n_1, T^*S^n_2$; these structures are obtained by taking Morse functions on $S^n$ with either one or two index $n$ critical points. Then $(T^*S^n)_L = T^*S^n \setminus (T^*_xS^n)^L$, while $(T^*S^n')_L = T^*S^n \setminus (T^*_xS^n)^L \coprod (T^*_xS^n)^L$. So in the former case we carve out one copy of $L$ while in the latter case we carve out two copies of $L$. Even if we add flexible handles to make these two domains diffeomorphic, it is not at all clear that they would become symplectomorphic (and we do not know whether this is the case). The issue is that $(T^*_xS^n)^L \coprod (T^*_xS^n)^L$ is not necessarily a parallel Lagrangian link, as we discuss further in Section 6.4. However, the realization that these spaces become Weinstein homotopic after stabilizing and inverting subcriticals was the original impetus for this project. To show that $P$-flexibilization is completely Weinstein homotopy invariant, one further needs to consider arbitrary Weinstein homotopic structures $T^*S^n'$ on $T^*S^n$, not just the simple structure considered above.

Furthermore, $X_L$ is only defined for Weinstein sectors, but not general Liouville sectors. We will remedy these issues in the next section.

4.2.1. Abouzaid-Seidel and Lazarev-Sylvan constructions. Next, we explain how to construct $X_P$ for a set of integers $P$ and any Weinstein sector $X$ with $\dim X \geq 10$. We follow previous work of the first two authors [10], which is a variant of the construction of Abouzaid and Seidel [1]. First for $p \in P$, we consider a certain Lagrangian disk $D_p \subset T^*D^n$ from [1], Section 3b. We briefly recall its definition: $D_p$ is the graph of the differential $d(f_p) \subset T^*D^n$
for a function \( f_p : D^n \rightarrow \mathbb{R} \), which near \( \partial D^n \) looks like \( r^2 g_p(\theta) \) for a function \( g_p : S^{n-1} \rightarrow \mathbb{R} \) that is positive on the tubular neighborhood of a \( p \)-Moore space \( U_p \subset S^{n-1} \) and negative on the complement of this \( p \)-Moore space; here \( r \) is radial coordinate and \( \theta \) is a coordinate on \( S^{n-1} = \partial D^n \). This construction requires embedding a \( p \)-Moore space in \( S^{n-1} \) and hence works only for \( n \geq 5 \). Then consider the disjoint embedding \( \bigsqcup_{p \in P} D_p \subset T^* D^n \) (obtained by embedding \( |P| \) copies of \( D^n \) disjointly into \( D^n \) and considering the induced proper inclusion \( \bigsqcup_{p \in P} T^* D^n \hookrightarrow T^* D^n \)). The definition of \( X_P \) by the first two authors \[\text{16}\] is essentially

**Notation 4.12.** \( X_P := X_{\bigsqcup_{p \in P} D_p} \)

More precisely, they also added flexible handles to \( X_{\bigsqcup_{p \in P} D_p} \) to ensure that the result is diffeomorphic to \( X \). We will not do this in this paper; see the discussion in the next section.

**Remark 4.13.** Note the construction of the Lagrangian disk \( D_p \) can be generalized by using any codimension zero subdomain \( U \subset S^{n-1} \) instead of just the \( p \)-Moore space \( U_p \). Indeed, Abouzaid and Seidel \[\text{11}\] showed that the Lagrangian disk \( D_U \cong \mathcal{C}^{n-1}(U) \otimes T^*_0 D^n \) in the wrapped Fukaya category \( Tw \mathcal{W}(T^* D^n) \); see \[\text{16}\] for details. So if \( U \) is a \( p \)-Moore space, then \( D_U = D_p \cong (\mathbb{Z}[1] \xrightarrow{\delta} \mathbb{Z}) \otimes T^*_0 D^n \) in \( Tw \mathcal{W}(T^* D^n) \). Using this result along with the localization formula of Ganatra-Pardon-Shende \[\text{10}\], the first two authors \[\text{16}\] proved that

\[
Tw \mathcal{W}(X_P) \cong Tw \mathcal{W}(X) \otimes \mathbb{Z} \left[ \frac{1}{P} \right].
\]

We observe that since they are all graphical Lagrangians, the disks \( D_U \subset T^* D^n \) are all Lagrangian isotopic to the zero-section \( D^n \subset T^* D^n \) if we allow the boundary of \( D_U \) to intersect the sectorial divisor, i.e. \( D_U \) is isotopic to \( D^n \) in the *unstopped* domain \( B^{2n} \).

**4.2.2. Smooth topology of \( X_L \) and flexible handles.** From the point of view of classifying symplectic structures on a fixed smooth manifold, it can be desirable to have symplectic constructions preserve diffeomorphism type. For example, the flexibilization \( X_{flex} \) of \( X \) defined by Cieliebak and Eliashberg \[\text{4}\] is a flexible domain diffeomorphic to \( X \). As we will see in Proposition \[\text{4.14}\] in this paper we can ensure that all constructions preserve the diffeomorphism type up to smoothly subcritical handles; since we work in \( \text{Wein}^2_{\text{crit}} \), Weinstein sectors up to subcritical handles and stabilization, this is the only sensible notion.

Since \( X_L \) is obtained by removing a disk from \( X \), it is never diffeomorphic to \( X \). Furthermore, if the (Poincaré dual) class \( [C^L_X] \in H^n(X; \mathbb{Z}) \) is non-zero, then \( X \) and \( X_L \) have different middle-dimensional cohomology and hence fail to be diffeomorphic up to subcriticals. However, if \( L \subset T^* D^n \) is *smoothly* isotopic to an unknot, then \( X_L \) is diffeomorphic to \( X \) (but not necessarily symplectomorphic to \( X \)) up to adding subcritical handles. We also observe that for any Lagrangian disk \( L \subset T^* D^n, n \geq 5 \), the double \( L^2 T \) is *smoothly* isotopic to the unknot; here \( L \) is a disk with the opposite orientation and \( L \) is the isotropic boundary connected sum. Furthermore, if \( L \) is regular, then so is \( L^2 T \).

**Proposition 4.14.** \[\text{14}\] For any regular Lagrangian disk \( L \subset T^* D^n, n \geq 3 \), \( X_{L^2 T} \) is Weinstein homotopic to \( X_L \) plus some flexible handles and is diffeomorphic to \( X \) up to subcritical handles.
So whatever we can prove about $X_L$ (for arbitrary regular Lagrangians $L$), we can also prove about $X_{L^\natural\mathcal{L}}$, which is diffeomorphic to $X$ up to smooth subcritical handles. We also note that $Tw\,\mathcal{W}(X_{L^\natural\mathcal{L}}) \cong Tw\,\mathcal{W}(X_L)$ since they are related by flexible handles.

In Proposition 4.14, $X_{L^\natural\mathcal{L}}$ is Weinstein homotopic to $X_L \cup H^{n-1} \cup H^n$, where one first attaches a subcritical handle $H^{n-1}$ and a flexible handle $H_{flex}$ (attached along a loose Legendrian) for each handle of $X$. In particular, there is no need to discuss flexible cobordisms separately in this paper since they appear naturally by removing the Lagrangian $L^\natural\mathcal{L}$ (instead of $L$). Using the h-principle for loose Legendrians in a certain local setting, the first author [14] also showed that the flexible handle $H^n_{flex}$ can be attached before the subcritical handle $H^{n-1}$ and hence $X_{L^\natural\mathcal{L}} = (X_L \cup H^{n-1}) \cup H^n_{flex}$ is Weinstein homotopic to $(X_L \cup H^n_{flex}) \cup H^{n-1} =: X'_L \cup H^{n-1}$, where $X'_L$ is actually diffeomorphic to $X$; as before, $Tw\,\mathcal{W}(X'_L) \cong Tw\,\mathcal{W}(X_L)$ since attaching subcritical handles doesn’t affect the Fukaya category. So in fact, $X'_L$ and $X_{L^\natural\mathcal{L}}$ are equivalent in $\text{Wein}^\circ_{crit}$, assuming this local form of h-principle.

**Example 4.15.** Let $L = T^*_0 D^n$. Then $X_{T^*_0 D^n}$ is the subcritical part $X_{sub}$ of $X$. Also, $X_{T^*_0 D^n \cup T^*_0 D^n} = X_{sub} \cup H^{n-1} \cup H^n_{flex}$ is flexible and diffeomorphic to $X$ up to some smooth subcritical handles. Using the h-principle for loose Legendrians in the local setting, [15, 14] verified that $X_{T^*_0 D^n \cup T^*_0 D^n}$ is actually Weinstein homotopic to $X_{flex} \cup H^{n-1}$, where $X_{flex}$ is the flexibilization defined in [14] (a flexible domain diffeomorphic to $X$). In particular, $X_{T^*_0 D^n \cup T^*_0 D^n}$ is equivalent to $X_{flex}$ in $\text{Wein}^\circ_{crit}$.

However, if one is not interested in comparing $X_{T^*_0 D^n \cup T^*_0 D^n}$ to $X_{flex} \cup H^{n-1}$, there is no need to use the h-principle for loose Legendrians. We take this approach, viewing $X_{T^*_0 D^n \cup T^*_0 D^n}$ as the flexibilization (since it is flexible and diffeomorphic to $X$ up to subcriticals) and proving idempotency and independence of presentation for this domain. Since the h-principle for loose Legendrians is used in a single local setting, we also expect that it is possible to compare these domains explicitly without using the h-principle at all.

Finally, we note that $T^*_0 D^n \cup T^*_0 D^n$ is Lagrangian isotopic to $D_0$, the Abouzaid-Seidel disk constructed using a 0-Moore space $S^1 \vee S^2$; see [16] for a proof. So by definition $X_0$ is $X_{T^*_0 D^n \cup T^*_0 D^n}$. Therefore $X_0$ is flexible and diffeomorphic to $X$ up to subcritical handles (and equivalent to $X_{flex}$ in $\text{Wein}^\circ_{crit}$ assuming the local h-principle).

**Remark 4.16** (Carving Lagrangians may be assumed connected). Returning briefly to the construction of $X_P$ in the previous section, we note that $D_p$ is trivial in $H^n(X)$ (this is true for $D_U$ for any $U$ with Euler characteristic $\chi(U) = 1$). So by the smooth h-cobordism theorem, $X_P$ is diffeomorphic to $X$, up to subcritical handles. We also mention a slight variant which uses connected Lagrangian disks. Namely, we form the Lagrangian disk $\sharp_{p\in P} D_p$, the isotropic boundary connected sum of $\coprod_{p\in P} D_p$. Then, as proven in [14], $X_{\sharp_{p\in P} D_p}$ is obtained by adding a flexible cobordism to $X_P := X_{\coprod_{p\in P} D_p}$. However, since $D_p$ (and hence both $\coprod_{p\in P} D_p$ and $\sharp_{p\in P} D_p$) is trivial in $H^n(X)$, this flexible cobordism is in fact a subcritical Weinstein cobordism; so $X_{\sharp_{p\in P} D_p}$ and $X_P$ are equivalent in $\text{Wein}^\circ_{crit}$. In particular, we can assume that $X_P$ is constructed by carving out the connected Lagrangian disk $\sharp_{p\in P} D_p$ near the co-cores of $X$. 
Remark 4.17. If \( n = p_1 \cdots p_k \) is a product of distinct primes (or relatively coprime integers), then the \( n \)-Moore space \( M_n \) is homotopy equivalent to the wedge sum \( M_{p_1} \vee \cdots \vee M_{p_k} \) (assuming the Moore spaces are simply-connected). To see this, note that there are always maps \( M_n \to \vee_{p_i \in P} M_{p_i} \) and the coprime condition implies that this is a homology isomorphism (and hence a homotopy equivalence by Whitehead’s theorem). The construction of Lagrangian disks \( D_U \) in \( T^*D^n \) from spaces \( U \) in \( S^{n-1} \) takes wedge sum to isotropic connected sum and therefore the Lagrangian disk \( D_n \) is Lagrangian isotopic to the Lagrangian disk \( D_{p_1} \natural \cdots \natural D_{p_k} = \natural_{p_i} D_{p_i} \). So by the previous remark, \( X_n \) is equivalent to \( X_P \), where \( P \) is the set of primes \( \{p_1, \ldots, p_k\} \).

4.3. A homotopy invariant P-flexibilization functor. Next, we discuss the present work’s P-flexibilization functor which is manifestly homotopy invariant, and which we prove in Section 5 to be equivalent to the non-homotopy-invariant construction \( (\ )_L \) defined in the previous section.

To do so, we fix a regular Lagrangian disk \( L \subset T^*D^n \) and a Weinstein structure \( (T^*D^n)_L \). Then taking the product with \( (T^*D^n)_L \) defines a functor
\[
\times(T^*D^n)_L : \text{Lioustr} \to \text{Lioustr}
\]
that takes a strict proper inclusion \( f \) to \( f \times \text{Id}_{(T^*D^n)_L} \), as well as a restricted functor
\[
\times(T^*D^n)_L : \text{Weinstr} \to \text{Weinstr}
\]
since the product of Weinstein sectors is a Weinstein sector. Furthermore, it is clear that if \( f \) is a movie inclusion or subcritical morphism, then so is \( f \times \text{Id}_{(T^*D^n)_L} \), so that \( \times(T^*D^n)_L \) descends to \( \text{Wein}^\text{crit}_{\text{str}} \). Furthermore, \( X \times (T^*D^n)_L \) has the advantage that it can be defined without parametrizing the co-cores of \( X \); in fact, the definition of \( X \times (T^*D^n)_L \) makes sense when \( X \) is an arbitrary Liouville sector.

If \( X \) is Weinstein, or the Kuneth formula holds for \( X \), and \( L = D_P \), then
\[
Tw \ W(X \times (T^*D^n)_P) \cong Tw \ W(X) \otimes Tw \ W((T^*D^n)_P)
\]
\[
\cong Tw \ W(X) \otimes Tw \ Z \left[ \frac{1}{P} \right] \cong Tw \ W(X) \left[ \frac{1}{P} \right]
\]
so that \( Tw \ W(X \times (T^*D^n)_P) \) is equivalent to \( Tw \ W(X_P) \), the Fukaya category of the non-homotopy-invariant P-flexibilization \( X_P \) defined in the previous section. In the next section, we show that in fact \( X_L \) and \( X \times (T^*D^n)_L \) are equivalent in the critical category \( \text{Wein}^\text{crit}_{\text{str}} \).

Finally, we observe that there exist Weinstein homotopic structures \( (T^*D^n)_L \) with different number of critical points; however, the resulting functors \( \times(T^*D^n)_L \) are all equivalent in \( \text{Wein}^\text{crit}_{\text{str}} \). Hence we are free to take slightly different models for \( (T^*D^n)_L \) at different stages of the proof.

Remark 4.18. More generally, it is a priori possible that the Liouville homotopy type \( (T^*D^n)_L \) depends on more than just \( L \). If \( L \) is strictly exact with respect to two forms \( \lambda_L, \lambda'_L \) that have the same behavior at infinity (and hence are linearly homotopic), then \( (T^*D^n, \lambda_L) \setminus L, (T^*D^n, \lambda'_L) \setminus L \) are Liouville homotopic by the linear homotopy, which necessarily vanishes on \( L \). However, if \( L \) does not have Legendrian boundary with respect to \( \lambda_{T^*D^n, \text{str}} \), then we need to pick a homotopic form \( \lambda_L \) (or \( \lambda'_L \)) for which \( L \) is strictly exact. Then \( \lambda_L, \lambda'_L \) are
Liouville homotopic but not necessarily through a family of forms vanishing on $L$; hence
the homotopy does not descend to a homotopy on $T^*D^n \setminus L$. Said differently, we need to
Lagrangian isotope $L$ to make it have Legendrian boundary with respect to $\lambda_{T^*D^n, std}$ and
there are different ways of doing this. It is better to think of $\lambda_L$ as a whole path from $\lambda_{T^*D^n, std}$
to $\lambda_L$ and we need a path of paths to the path from $\lambda_{T^*D^n, std}$ to $\lambda_L'$; at the endpoints of this
path, we get a Liouville homotopy that vanishes on $L$ as desired.

5. Comparison of P-flexibilization functors

The goal of this section is to prove the following result comparing $X_L := X \setminus \bigsqcup C^L_X$ and
$X \times (T^*D^n)_L$.

**Theorem 5.1.** Consider a regular Lagrangian disk $L^n \subset T^*D^n$ (equipped with a Weinstein
homotopy $\lambda_{T^*D^n,L,t}$ as in Definition 2.76). Then for any parametrized Weinstein sector $X^{2n}$,
there is an equivalence $\varphi_X : X_L \to X \times (T^*D^n)_L$ in $\text{Wein}_\text{crit}$. Furthermore, for any strict proper
inclusion $f : X^{2n} \hookrightarrow Y^{2n}$ in $\text{Wein}_\text{param}$, the following is a homotopy commuting diagram in
the critical Weinstein category $\text{Wein}_\text{crit}$:

$$
\begin{array}{ccc}
X_L & \xrightarrow{f_L} & Y_L \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
X \times (T^*D^n)_L & \xrightarrow{f \times Id_{(T^*D^n)_L}} & Y \times (T^*D^n)_L
\end{array}
$$

(5.1)

**Remark 5.2.** Since $\text{Wein}_\text{crit}$ is an $\infty$-category, it does not have a notion of strictly commuting
diagrams; instead, it only has a notion of homotopy commuting diagrams, as in Theorem 5.1. Furthermore, since there are no strict composition of morphisms in an infinity category,
it does not have a notion of isomorphism, only equivalence (which gives an isomorphism in
the homotopy category).

5.1. Compatibility of taking products and carving out disks. In this section we start
the proof of Theorem 5.1 comparing the functors $\times (T^*D^n)_L$ and $(\phantom{X}_L)$. To do so, we prove
some slightly more general results concerning the compatibility between taking products and
carving out Lagrangian disks.

Recall that given a strict proper inclusion $f : X \hookrightarrow Y$ of Weinstein sectors $X,Y$ with
parametrized Weinstein handles, there is an induced strict proper inclusion $i_L : X_L \hookrightarrow Y_L$
(Notation 4.10). Also, for any Weinstein sector $Z$ with parametrized handles, there are induced inclusions $f \times Id : X \times Z \hookrightarrow Y \times Z$ and $(f \times Id)_L : (X \times Z)_L \hookrightarrow (X \times Z)_L$.

The following proposition compares these maps.

**Proposition 5.3.** Let $X^{2n}, Y^{2n}, Z^{2m}$ be Weinstein sectors with parametrized Weinstein han-
dles, $f : X \hookrightarrow Y$ a strict proper inclusion, and $L \subset T^*D^n, K \subset T^*D^m$ parametrized regular
Lagrangian disks (equipped with Weinstein homotopies $\lambda_{T^*D^n,L,t}, \lambda_{T^*D^m,K,t}$ as in Definition
Then there are homotopy commuting diagrams in $\text{Wein}^\circ_{\text{crit}}$

$$
\begin{array}{ccc}
X_L \times Z & \xrightarrow{f_L \times \text{Id}} & Y_L \times Z \\
\downarrow \phi_{(X,L),Z} & & \downarrow \phi_{(Y,L),Z} \\
(X \times Z)_{L \times T^n_D m} & \xrightarrow{(f \times \text{Id})_{T^n_D m \times K}} & (Y \times Z)_{L \times T^n_D m}
\end{array}
$$

(5.2)

$$
\begin{array}{ccc}
X \times Z_K & \xrightarrow{f \times \text{Id}_K} & Y \times Z_K \\
\downarrow \phi_{X,(Z,K)} & & \downarrow \phi_{Y,(Z,K)} \\
(X \times Z)_{T^n_D m \times K} & \xrightarrow{(f \times \text{Id})_{T^n_D m \times K}} & (Y \times Z)_{T^n_D m \times K}
\end{array}
$$

(5.3)

where all vertical maps are equivalences in $\text{Wein}^\circ_{\text{crit}}$.

Remark 5.4. If Weinstein sectors $X, Y, Z$ have Weinstein stops $H_X, H_Y, H_Z$, then $X_L \times Z$ has stop $H_X \times Z \coprod X_L \times H_Z$ while $(X \times Z)_{L \times T^n_D m}$ has stop $H_X \times Z \coprod X \times H_Z$. So the second components $X_L \times H_Z, X \times H_Z$ of these two respective stops are different. Part of the content of this proposition is that these two stops differ by a loose Legendrian hypersurface coming from a subcritical Weinstein cobordism, that gives rise to the equivalences in $\text{Wein}^\circ_{\text{crit}}$.

Proof of Proposition 5.3. The proof of Proposition 5.3 follows from the following Proposition 5.5 combined with Proposition 2.73 and Proposition 2.61 which convert subcritical subdomain inclusions and isomorphisms up to deformation into equivalences in the critical Weinstein category.

Recall that $C_X$ is a co-core of $X$ with its original Weinstein structure $\lambda_X$, $C^L_X$ is a copy of $L$ embedded in a neighborhood of $C_X$, $\lambda_{X,L}$ is the homotopic Weinstein structure for which $C^L_X$ is a co-core, and $X_L := (X, \lambda_{X,L}) \setminus \bigsqcup_{C_X} C^L_X$.

Proposition 5.5. Let $X^{2n}, Y^{2n}, Z^{2m}$ be Weinstein sectors with parametrized Weinstein handles, $f : X \hookrightarrow Y$ a strict proper inclusion, and $L \subset T^* D^n, K \subset T^* D^m$ parametrized regular Lagrangian disks (equipped with Weinstein homotopies as in Definition 2.76). Then the following hold:

1. There is a strict subcritical subdomain inclusions

$$
i_{(X,L),Z} : X^{2n}_L \times Z^{2m} := ((X, \lambda_{X,L}) \setminus \bigsqcup_{C_X} C^L_X) \times Z \subseteq (X, \lambda_{X,L}) \times Z \setminus \bigsqcup_{C_X,C_Z} C^L_X \times C_Z$$

where $C_X, C_Z$ are the co-cores of $X, Z$ respectively.

2. The identity map $1d_{(X,L),Z}$

$$(X, \lambda_{X,L}) \times Z \setminus \bigsqcup_{C_X,C_Z} C^L_X \times C_Z \rightarrow (X \times Z)_{L \times T^n_D m} := (X \times Z, \lambda_{X \times Z, L \times T^n_D m}) \setminus \bigsqcup_{C_X \times Z} C^L_X \times C_Z$$

is an isomorphism, up to bordered Weinstein homotopy, i.e. these two sectors are bordered Weinstein homotopic.
(3) There is a strictly commuting diagram of symplectic embeddings:

\[
\begin{array}{ccc}
X_L \times Z & \xrightarrow{f_L \times Id} & Y_L \times Z \\
\downarrow i_{(X,L),Z} & \downarrow i_{(Y,L),Z} & \\
(X, \lambda_{X,L}) \times Z \setminus \bigsqcup_{C_X,C_Z} C^L_X \times C_Z & \xrightarrow{f \times Id} & (Y, \lambda_{Y,L}) \times Z \setminus \bigsqcup_{C_X,C_Z} C^L_X \times C_Z \\
\downarrow Id_{(X,L),Z} & \downarrow Id_{(Y,L),Z} & \\
(X \times Z)_L \times T_0^* D^m & \xrightarrow{(f \times Id)_{T_0^* D^n \times K}} & (Y \times Z)_{L \times T_0^* D^m}
\end{array}
\]  

(5.4)

Furthermore, the top square is a pullback diagram of sets and the bordered Weinstein homotopies for \( Id_{(Y,L),Z} \) extend those for \( Id_{(X,L),Z} \). There are similar maps for \( Z \) and a strictly commuting diagram of exact symplectic embeddings:

\[
\begin{array}{ccc}
X \times Z_K & \xrightarrow{f \times Id_{Z_K}} & Y \times Z_K \\
\downarrow i_{X(Z,K)} & \downarrow i_{Y(Z,K)} & \\
X \times (Z, \lambda_{Z,K}) \setminus \bigsqcup_{C_X,C_Z} C_X \times C^K_Z & \xrightarrow{f \times Id_{Z_K}} & Y \times (Z, \lambda_{Z,K}) \setminus \bigsqcup_{C_X,C_Z} C_X \times C^K_Z \\
\downarrow Id_{X(Z,K)} & \downarrow Id_{Y(Z,K)} & \\
(X \times Z)_{T_0^* D^n \times K} & \xrightarrow{(f \times Id)_{T_0^* D^n \times K}} & (Y \times Z)_{T_0^* D^n \times K}
\end{array}
\]  

(5.5)

Proof of Proposition 5.5. There is a commutative diagram of symplectic embeddings of the form

\[
\begin{array}{ccc}
X_L & \xrightarrow{f_L} & Y_L \\
\downarrow i_{(X,L)} & \downarrow i_{(Y,L)} & \\
(X, \lambda_{X,L}) & \xrightarrow{f} & (Y, \lambda_{Y,L})
\end{array}
\]  

(5.6)

where \((X, \lambda_{X,L}), (Y, \lambda_{Y,L})\) are Weinstein homotopic to \(X, Y\) respectively as in the construction of \(X_L, Y_L\); in particular, \(i_{(X,L)}, i_{(Y,L)}\) are strict Weinstein subdomain inclusions. This diagram commutes since the map \(f\) takes parametrized co-cores of \(X\) to those of \(Y\) and hence the construction of \(Y_L\) extends that of \(X_L\). In particular, this is a pullback diagram of sets.

Next, we can take the product of this diagram with \(Z\) and the identity map \(Id_Z\) to obtain another commutative diagram of symplectic embeddings:

\[
\begin{array}{ccc}
X_L \times Z & \xrightarrow{f_L \times Id_Z} & Y_L \times Z \\
\downarrow i_{(X,L) \times Id_Z} & \downarrow i_{(Y,L) \times Id_Z} & \\
(X, \lambda_{X,L}) \times Z & \xrightarrow{f \times Id_Z} & (Y, \lambda_{Y,L}) \times Z
\end{array}
\]  

(5.7)

This is also a pullback diagram of sets. Since \(X_L \preceq (X, \lambda_{X,L})\) is a strict Weinstein subdomain inclusion, then \(X_L \times Z \preceq (X, \lambda_{X,L}) \times Z\) is a strict sectorial subdomain inclusion, and similarly for \(Y\). In particular, \((X, \lambda_{X,L}) \times Z \setminus X_L \times Z\) is a Weinstein cobordism. Let \(p_X^L\) be the index \(n\) critical points of the Weinstein cobordism \((X, \lambda_{X,L}) \setminus i_{(X,L)}(X_L)\), whose co-core is \(C_X^L\). Then the index \(n + m\) critical points of the cobordism \((X, \lambda_{X,L}) \times Z \setminus X_L \times Z\) correspond to pairs consisting of an index \(n\) critical point \(p_X^L\) and an index \(m\) critical point \(p_Z\) of the Weinstein
structure on $Z^{2m}$. The co-cores of these critical points are the products of the co-cores of the respective critical points, i.e. $C^L_X \times C_Z$. Hence, if we carve out the Lagrangian co-core $C^L_X \times C_Z$ from this Weinstein cobordism, we get a subcritical Weinstein cobordism. This is the desired strict subcritical subdomain inclusion $i_{(X,L),Z}$. Since the co-cores $C^L_X \times C_Z$ are a subset of the co-cores $C^L_Y \times C_Z$, the commutative diagram in Equation 5.7 induces a commutative diagram which is the top square in Equation 5.7, this remains a pullback diagram of sets since the co-cores $C^L_Y \times C_Z$ are a subset of the co-cores $C^L_Y \times C_Z$.

Next we need to relate $(X, \lambda_{X,L}) \times Z \setminus \coprod_{C_X} C^L_X \times C_Z$ to

$$(X \times Z)_{L \times T^*_0 D^m} := (X \times Z, \lambda_{X \times Z, L \times T^*_0 D^m}) \setminus \coprod_{C_X \times Z} C^L_{X \times Z} \times C^L_{X \times Z}$$

where $\lambda_{X \times Z, L \times T^*_0 D^m}$ is a Weinstein structure on $X \times Z$ that has $C^L_{X \times Z} \times C^L_{X \times Z}$ as co-cores and is Weinstein homotopic to $\lambda_X + \lambda_Z$ via a homotopy supported near the co-cores $C_{X \times Z}$ of $X \times Z$. We observe that $(X, \lambda_{X,L}) \times Z = (X \times Z, \lambda_{X,L} + \lambda_Z)$ and $(X \times Z, \lambda_{X \times Z, L \times T^*_0 D^m})$ are Weinstein homotopic structures on $X \times Z$ (as both are homotopic to $\lambda_X + \lambda_Z$). Using the identification between the product $C_X \times C_Z$ of co-cores and the co-core $C_{X \times Z}$ of the product $X \times Z$ and the fact that $C^T_{Z} \times C^T_{m} = C_Z$, we have that $C^L_X \times C_Z$ equal to $C^L_{X \times Z} \times C^L_{X \times Z}$. Hence the identity map is a symplectomorphism between these two sectors.

We need the stronger statement that the identity map is an isomorphism up to deformation. To do this, we need to show that the homotopic structures $\lambda_{X,L} + \lambda_Z$ and $\lambda_{X \times Z, L \times T^*_0 D^m}$ are homotopic relative to $C^L_X \times C_Z = C^L_{X \times Z}$, i.e. through a family of forms that vanish on this Lagrangian. Note that the the Weinstein structure $\lambda_{X,L} + \lambda_Z$ near neighborhoods $T^* D^n \times T^* D^m$ of the co-cores $C_{X \times Z}$ is $\lambda_{T^* D^n} + \lambda_{T^* D^m}$. Let $\lambda_{T^* D^n \times L,t}$ be the interior Weinstein homotopy from $\lambda_{T^* D^n \times L}$ to $\lambda_{T^* D^m \times L}$ and let $\lambda_{X,L,t}$ be the homotopy from $\lambda_{X,L}$ to $\lambda_X$ obtained by extending by the identity. This induces a homotopy $\lambda_{X,L,t} + \lambda_Z$ on $X \times Z$, is not relative to $L$ since for example the form $\lambda_{T^* D^m \times L} \lambda_{T^* D^m \times L}$ does not vanish on $L \times T^*_0 D^m$. We explain how to modify it and produce a homotopy that does vanish on $L \times T^*_0 D^m$.

Consider the homotopy $\lambda_{T^* D^n \times L,t} + \lambda_{T^* D^m}$ on $T^* D^n \times T^* D^m$ and the induced homotopy $s_t$ on the sectorial boundary $\partial(T^* D^n \times T^* D^m) = \partial T^* D^n \times T^* D^m \cup T^* D^n \times \partial T^* D^m$. We view $s_t$ as a homotopy of the stop for the fixed Weinstein structure $\lambda_{T^* D^n \times L} + \lambda_{T^* D^m}$; so we can proceed as in the first part of Proposition 2.43 and insert the movie construction of $s_t$ near the sectorial boundary of $T^* D^n \times T^* D^m$. The result is a bordered Weinstein homotopy $\lambda_{T^* D^n \times T^* D^m, L \times T^*_0 D^m}$ on a slightly larger copy $(T^* D^n \times T^* D^m)'$ that agrees with the original structure $\lambda_{T^* D^n \times L} + \lambda_{T^* D^m}$ on $T^* D^n \times T^* D^m \subset (T^* D^n \times T^* D^m)'$ and agrees with $s_t$ near the sectorial boundary of $(T^* D^n \times T^* D^m)'$. Since $\lambda_{T^* D^n \times T^* D^m, L \times T^*_0 D^m}$, this homotopy vanishes on $L \times T^*_0 D^m$. See the top-right square of Figure 7, which is $(T^* D^n \times T^* D^m)'$ (the smaller subrectangle is $T^* D^n \times T^* D^m \subset (T^* D^n \times T^* D^m)'$).

By construction, $\lambda_{T^* D^n \times T^* D^m, L \times T^*_0 D^m}$ agrees with the homotopy $\lambda_{X,L,t} + \lambda_Z$ near the sectorial boundary of $T^* D^n \times T^* D^m$ (since $s_t$ is the restriction of $\lambda_{X,L,t} + \lambda_Z$ to a neighborhood of $\partial(T^* D^n \times T^* D^m)$). Hence we can take a homotopy on $X \times Z$ that is $\lambda_{T^* D^n \times T^* D^m, L \times T^*_0 D^m}$ in $T^* D^n \times T^* D^m$ and is $\lambda_{X,L,t} + \lambda_Z$ elsewhere on $X \times Z$. This is a bordered homotopy and
is relative to $L \times T_0^* D^n$. Note that the resulting form at time $1 \lambda_{X \times Z,L \times T_0^* D^n}$ agrees with $\lambda_X + \lambda_Z$ except near the co-cores $C_{X \times Z}$; so after carving out $C_{X \times Z}^{L \times T_0^* D^n}$, we call the resulting sector $(X \times Z)_{L \times T_0^* D^n}$. See Figure 7. Finally, we observe that since all our constructions happen locally near the co-cores of the index $n$ handles, this Weinstein homotopy on $X \times Z$ extends to a Weinstein homotopy on $Y \times Z$ and hence the bottom square in Equation 5.2 also commutes.

The second commuting diagram Equation 5.3 for $Z$ and $K$ is constructed similarly, except that the homotopies occur in the $Z$-coordinates instead of the $X$-coordinates as in the previous paragraphs. In particular, the resulting local model for $(T^* D^n \times T^* D^n, \lambda_{T^* D^n \times T^* D^n, T_0^* D^n \times K})$ is the pullback of $(T^* D^n \times T^* D^n, \lambda_{T^* D^n \times T^* D^n, K \times T_0^* D^n})$ via the the swap map $\varphi : T^* D^n \times T^* D^n \to T^* D^n \times T^* D^n$ given by $\varphi(x, y) = (y, x)$; this observation will be important for the proof of Theorem 5.1 below. □

Remark 5.6. Note that in general, the homotopy from $(X, \lambda_{X,L}) \times Z \setminus \bigsqcup_{C_X,C_Z} C_{X}^{L} \times C_{Z}$ to $(X \times Z)_{L \times T_0^* D^n} := (X \times Z, \lambda_{X \times Z,L \times T_0^* D^n}) \setminus \bigsqcup_{C_X,C_Z} C_{X \times Z}^{L \times T_0^* D^n}$ is a homotopy of the movie constructions. These movie constructions are induced by the Weinstein homotopy that is the product of the homotopy $(T^* D^n, \lambda_{T^* D^n, std})$ to $(T^* D^n, \lambda_{T^* D^n, L})$ with the constant Weinstein structure on $Z \setminus C_Z$. So if the homotopy from $(T^* D^n, \lambda_{T^* D^n, std})$ to $(T^* D^n, \lambda_{T^* D^n, L})$ is a Weinstein homotopy, then so the homotopy from $(X, \lambda_{X,L}) \times Z \setminus \bigsqcup_{C_X,C_Z} C_{X}^{L} \times C_{Z}$. This uses our assumption that the class of Weinstein homotopies are preserved under taking products.

Remark 5.7. For the proof of this key result, it is crucial that we carve out the same Lagrangian $L$ from all $n$-handles of $X$ when constructing $X_L$; in principle it is possible to construct a subdomain $X_{L,K}$ by carving out arbitrary Lagrangian disks $L$ and $K$ from different $n$-handles in an unrelated way. However, this $X_{L,K}$ will not be equivalent to $X \times (T^* D^n)_L$.
or $X \times (T^*D^n)_K$. In fact, such a ‘mixed’ construction will not be homotopy invariant, even in $\text{Wein}^\circ_{\text{crit}}$. In the proof of Proposition 5.5, this appears in the statement that we carve out $\coprod_{C_X,C_Z} C^L_X \times C_Z$ and this is the same as $\coprod_{C_X \times Z} C^L_{X \times Z}$.

Example 5.8. Setting $Z = T^*D^k$ in part 1) of Proposition 5.5 shows that there is a strict subcritical cobordism from $X_L \times T^*D^k$ to $(X, \lambda_{X,L}) \times T^*D^k \setminus \coprod_{C_X} C^L_X \times T^*_0D^k$ and an isomorphism up to deformation of the latter to $(X \times T^*D^k)_{L \times T^*_0D^k}$; however, this subcritical cobordism is non-trivial since the stop of $X_L \times T^*D^k$ is $X_L \times T^*S^{k-1}$ while the stop of $(X \times T^*D^k)_{L \times T^*_0D^k}$ is $X \times T^*S^{k-1}$, i.e. the stop of $X \times T^*D^k$ itself. In particular, to prove results about $X_L$ and maps between such sectors in $\text{Wein}^\circ_{\text{crit}}$, it suffices to prove results about $(X \times T^*D^k)_{L \times T^*_0D^k}$ for any $k$ and maps between such sectors.

5.2. Swapping Lagrangian disks. Note that if we take $L = K$ and $Z = T^*D^n$ in Proposition 5.5, the map in top row of Equation 5.2 is the top row in Theorem 5.1 and the top row of Equation 5.3 is the bottom row in Theorem 5.1. By Proposition 5.5, the top row of Equation 5.2, 5.3 is related to the bottom row of these equations by equivalences in $\text{Wein}^\circ_{\text{crit}}$. Hence it suffices to prove that the bottom rows of Equation 5.2, 5.3

\[ (X \times T^*D^n)_{L \times T^*_0D^n} \xrightarrow{(f \times \text{Id})_{L \times T^*_0D^n}} (Y \times T^*D^n)_{L \times T^*_0D^n} \]

\[ (X \times T^*D^n)_{T^*_0D^n \times L} \xrightarrow{(f \times \text{Id})_{T^*_0D^n \times L}} (Y \times T^*D^n)_{T^*_0D^n \times L} \]

are equivalent in $\text{Wein}^\circ_{\text{crit}}$. The data used to define these first, second maps is just the Lagrangian embeddings $L \times T^*_0D^n \subset T^*D^n \times T^*D^n$, $T^*_0D^n \times L \subset T^*D^n \times T^*D^n$ respectively. Hence it suffices to show that two Lagrangian disks are isotopic in $T^*D^n \times T^*D^n$. In the following, we prove this if $n$ is even; a different version of this result via Lagrangian cobordisms appeared in the third author’s previous work [30].

For any symplectic manifold $M$, there is a swap symplectomorphism $S : M \times M \to M \times M$ given by $S(x, y) = (y, x)$. We will need to use the following proposition for the swap symplectomorphism when $M = T^*D^n$.

Proposition 5.9. If $n$ is even, then $S : (T^*D^n \times T^*D^n, \lambda_{T^*D^n, \text{std}} + \lambda_{T^*D^n, \text{std}}) \to (T^*D^n \times T^*D^n, \lambda_{T^*D^n, \text{std}} + \lambda_{T^*D^n, \text{std}})$ is isotopic to the identity through strict sectorial isomorphism. Furthermore, there is a strict sectorial isomorphism $\varphi : T^*D^n \times T^*D^n \to T^*D^n \times T^*D^n$ which is the identity map near the sectorial boundary and agrees with $S$ in a smaller copy of $T^*D^n \times T^*D^n$ in the interior of $T^*D^n \times T^*D^n$.

Proof. There is a symplectomorphism $\varphi : T^*D^n \times T^*D^n \to T^*(D^n \times D^n)$ given by the pullbacks of the projection maps $D^n \times D^n \to D^n$. We first observe that there is swap map $s : D^n \times D^n \to D^n \times D^n$ similarly given by $s(x_1, x_2) = (x_2, x_1)$ on the zero-section. This is a diffeomorphism and hence there is an induced map $T^*s : T^*(D^n \times D^n) \to T^*(D^n \times D^n)$ given by $T^*s(x_1, x_2, p_1dx_1 + p_2dx_2) = (s^{-1}(x_1, x_2), s^*(p_1dx_1 + p_2dx_2) = (x_2, x_1, p_1dx_2 + p_2dx_1) = (x_2, x_1, p_2dx_1 + p_1dx_2)$. In particular, $T^*s$ is the swap map $S : T^*D^n \times T^*D^n \to T^*D^n \times T^*D^n$.

Now we note that since $n$ is even, $s$ is an orientation-preserving linear map and hence is isotopic to the identity through diffeomorphisms of $D^n \times D^n$ (viewed as a manifold with
boundary after smoothing the corners). Let $s_t$ be this diffeotopy between $Id$ and $s$. Then $T^*s_t$ is an isotopy of sectorial symplectomorphisms of $T^*D^n \times T^*D^n$ between the identity and the swap symplectomorphism as desired. Furthermore, $T^*s_t$ preserves the standard Liouville form on $T^*D^n \times T^*D^n$ since it is induced by a diffeotopy $s_t$ of the zero-section. Finally, we take $\varphi$ to be $T^*s_{\text{movie}}$, where $s_{\text{movie}}$ is a diffeomorphism of $D^n \times D^n$ which is the identity near the boundary, $s_t$ in a smaller copy of $D^n \times D^n$ in the interior, and interpolates via $s_t$ between these two regions.

We will call $\varphi$ the cut-off swap map since it is the identity map near the sectorial boundary of $T^*D^n \times T^*D^n$.

**Corollary 5.10.** If $i_L : L \to T^*D^n, i_K : K \to T^*D^n$ are Lagrangian embeddings and $n$ is even, then $(i_L, i_K) : L \times K \to T^*D^n \times T^*D^n$ is isotopic to $S \circ (i_L, i_K) : L \times K \to T^*D^n \times T^*D^n$.

**Proof.** By Proposition 5.9, there is an isotopy $T^*s_t$ of sectorial symplectomorphisms of $T^*D^n \times T^*D^n$ from $Id$ to $S$. Hence $T^*s_t \circ (i_L, i_K) : L \times K \to T^*D^n \times T^*D^n$ is an isotopy of Lagrangian embeddings between $(i_L, i_K)$ and $S \circ (i_L, i_K)$. The key point is that a sectorial symplectomorphism (or more generally, proper inclusion of sectors) takes Lagrangians to Lagrangians. □

### 5.3. Proof of Theorem 5.1

In this section, we complete the proof of the comparison result Theorem 5.1. We first show that the cut-off swap map defines strict isomorphisms between the relative sectors.

**Proposition 5.11.** If $n = \frac{1}{2} \dim X = \frac{1}{2} \dim Z$ is even, then there is a strictly commuting diagram where the vertical maps are strict isomorphisms:

$$
\begin{align*}
(X \times Z)_{L \times T_0^*D^n} & \xrightarrow{\varphi_{X,Z,L,T_0^*D^n}} (Y \times Z)_{L \times T_0^*D^n} \\
& \downarrow \varphi_{X,Z,L,T_0^*D^n} \\
(X \times Z)_{T_0^*D^n \times L} & \xrightarrow{\varphi_{Y,Z,L,T_0^*D^n}} (X \times Z)_{T_0^*D^n \times L}
\end{align*}
$$

(5.9)

**Proof.** Recall that the cut-off swap map $\varphi : (T^*D^n \times T^*D^n, \lambda_{T^*D^n \times T^*D^n, std}) \to (T^*D^n \times T^*D^n, \lambda_{T^*D^n \times T^*D^n, std})$ from Proposition 5.9 is a strict Liouville isomorphism that agrees with the swap map $S$ in the interior and is the identity near the sectorial boundary. In particular, $\varphi(L \times T_0^*D^n) = T_0^*D^n \times L$. Hence we can take $\varphi_{X,Z,L,T_0^*D^n} : X \times Z \to X \times Z$ to be $\varphi$ near all the co-cores $C_{X,Z}$ of $X \times Z$ and the identity elsewhere. Then $\varphi_{X,Z,L,T_0^*D^n}$ induces a symplectomorphism $\varphi_{X,Z,L,T_0^*D^n} : (X \times Z)_{L \times T_0^*D^n} \to (X \times Z)_{T_0^*D^n \times L}$.

Next, we observe that this map is actually a strict Liouville isomorphism. This is because by construction, the swap map is a strictly isomorphism between $(T^*D^n \times T^*D^n, \lambda_{T^*D^n \times T^*D^n, L \times T_0^*D^n})$ and $(T^*D^n \times T^*D^n, \lambda_{T^*D^n \times T^*D^n, T_0^*D^n \times L})$; see the last paragraph of the proof of Proposition 5.5. Furthermore, these forms are standard near their sectorial boundaries and so on a trivial enlargement of these sectors, the cut-off swap map $\varphi$ is also a strict isomorphism. It also induces a strict isomorphism once we carve out the disks $L \times T_0^*D^n$ and $\varphi(L \times T_0^*D^n) = T_0^*D^n \times L$. So the map $\varphi_{X,Z,L,T_0^*D^n}$, which extends $\varphi$ by the identity, is also a strict isomorphism. Since $\varphi_{X,Z,L,T_0^*D^n}$ is defined near the co-cores of the handles of $X \times Z$, it extends to a similar map on $Y \times Z$ and get the desired commutative diagram. □
Proof of Theorem 5.1. If dim $L = \frac{1}{2}$ dim $X$ is even, then Theorem 5.1 follows from Proposition 5.5 and Proposition 5.11 applied to $Z = (T^*D^n, \lambda_{T^*D^n,\text{std}})$, which has a single isolated critical point of index $n$ so that $(T^*D^n)_L$ is obtained from $T^*D^n$ by removing a single copy of $L$; we also need to use Propositions 2.61, 2.73 converting isomorphisms up to deformation and subcritical subdomain inclusions into equivalences in $\mathcal{W}_\text{crit}$. Suppose $\dim L = \frac{1}{2} \dim X$ is odd, we first note that

$$f_L : X_L \rightarrow Y_L$$

is equivalent in $\mathcal{W}_\text{crit}$ to

$$(f \times \text{Id}_{T^*D^1})_{L\times T^*_0 D^1} : (X \times T^*D^1)_{L\times T^*_0 D^1} \rightarrow (Y \times T^*D^1)_{L\times T^*_0 D^1}$$

by part 1) of Proposition 5.5 and Proposition 2.73 transforming subcritical subdomain inclusions to equivalences in $\mathcal{W}_\text{crit}$. Then $L \times T^*_0 D^1$ has even dimension and we proceed as before to show that $(f \times \text{Id})_{L\times T^*_0 D^1}$ is equivalent to

$$(f \times \text{Id}_{T^*D^1}) \times \text{Id}_{(T^*D^{n+1})_{L\times T^*_0 D^1}} : (X \times T^*D^1) \times (T^*D^{n+1})_{L\times T^*_0 D^1} \rightarrow (Y \times T^*D^1) \times (T^*D^{n+1})_{L\times T^*_0 D^1}$$

in $\mathcal{W}_\text{crit}$. Since there is a subcritical cobordism from $(T^*D^n)_L \times T^*D^1$ to $(T^*D^{n+1})_{L\times T^*_0 D^{n+1}}$ again by Part 1) of Proposition 5.5, this proper inclusion is equivalent to

$$f \times \text{Id}_{T^*D^1} \times \text{Id}_{(T^*D^n)_L} \times \text{Id}_{T^*D^1} : (X \times T^*D^1) \times T^*D^n_L \times T^*D^1 \rightarrow (Y \times T^*D^1) \times T^*D^n_L \times T^*D^1$$

Finally, we observe that this morphism is conjugate to the morphism

$$f \times \text{Id}_{(T^*D^n)_L} \times \text{Id}_{T^*D^1} \times \text{Id}_{T^*D^1} : X \times (T^*D^n)_L \times T^*D^1 \times T^*D^1 \rightarrow Y \times (T^*D^n)_L \times T^*D^1 \times T^*D^1$$

via the swap maps $X \times T^*D^1 \times (T^*D^n)_L \times T^*D^1 \rightarrow X \times (T^*D^n)_L \times T^*D^1 \times T^*D^1$. The latter is the stabilization of $f \times (T^*D^n)_L$ as desired.

Finally, we note that the above proof of Theorem 5.1 proves a slightly stronger result than in the statement of that theorem. Namely, the only non-strict maps that appear in the proof are isomorphisms up to Weinstein homotopy (i.e. the non-strict map in Proposition 5.5 is the identity map, a diffeomorphism).

Corollary 5.12. $X_L$ and $X \times (T^*D^n)_L$ are isomorphic up to Weinstein homotopy, stabilization, and subcritical cobordism.

Using the fact that $\times (T^*D^n)_L$ preserves isomorphisms up to Weinstein (Liouville) homotopy, we have the following corollary, which implies Corollary 1.6 from the Introduction.

Corollary 5.13. If $X, X'$ are isomorphic up to Weinstein (Liouville) homotopy, then $X_L$ and $X'_L$ are isomorphic up to Weinstein (Liouville) homotopy, stabilization, and subcritical cobordism. In particular, this holds for flexibilization.

6. Idempotency of $P$-flexibilization

In this section, we prove that $\times (T^*D^n)_L$ (and also $(\ )_L$) is an idempotent functor of $\mathcal{W}_\text{crit}$, i.e. there is a natural transformation $\eta : \text{Id} \rightarrow \times (T^*D^n)_L$ so that $\eta_X \times \text{Id}_{(T^*D^n)_L}, \eta_X \times (T^*D^n)_L$ are equivalences in $\mathcal{W}_\text{crit}$ for all Weinstein sectors $X$. 
6.1. A natural transformation for $\times (T^*D^n)_L$. First, we define a natural transformation $\eta : Id \to \times (T^*D^n)_L$, using a slightly modified model for $(T^*D^n)_L$. Let $D^n' = [0,2] \times D^{n-1}$ be a larger $D^n = [0,1]^n$. Then we take a Morse-Bott function $f'$ on $D^n'$, homotopic relative to the boundary to the standard one $f$, so that there is an embedding $\varphi_1 \coprod \varphi_2 : (D^n, f) \coprod (D^n, f') \hookrightarrow (D^n', f')$ with $\varphi_1(D^n) = [0,1/2] \times D^{n-1}$, $\varphi_2(D^n) = [1/2,1] \times D^{n-1}$ so that $f'$ pulls back to $f$. See Figure 8. Then the function $f'$ induces a Morse-Bott Weinstein structure $(T^*D^n, \lambda_{T^*D^n})$ on $T^*D^n'$. It has two co-cores which are cotangent fibers over two different points in $D^n$, say at $x_1, x_2$.

Then there are induced strict proper inclusion

$$T^*\varphi_1, T^*\varphi_2 : (T^*D^n, \lambda_{T^*D^n, std}) \hookrightarrow (T^*D^n', \lambda_{T^*D^n'})$$

so that $T^*\varphi_i(T^n_0, D^n) = T^n_0, D^n$ for $i = 1, 2$. Then as in the previous sections, there is an interior Weinstein homotopy of $(T^*D^n, \lambda_{std})$ to $(T^*D^n, \lambda_{T^*D^n,L})$ that has $L$ as a co-core. We can apply this homotopy to the subset $T^*\varphi_2(T^*D^n) \subset T^*D^n'$ and denote this sector $(T^*D^n', \lambda_{T^*D^n', T^*\varphi_2(L)})$; it has $T^*\varphi_1(T^n_0, D^n) = T^n_{x_1}, D^n$ and $T^*\varphi_2(L) = (T^n_{x_2}, D^n)^L$ as co-cores (and possibly some other co-cores). Since this homotopy occurs in the complement of $T^*\varphi_1(T^*D^n)$, there is still a strict proper inclusion

$$T^*\varphi_1 : (T^*D^n, \lambda_{std}) \hookrightarrow (T^*D^n', \lambda_{T^*D^n', T^*\varphi_2(L)})$$

Since $T^*\varphi_2(L)$ is in the complement of the image of $T^*\varphi_1$, there is an induced proper inclusion

$$\eta_{T^*D^n} := T^*\varphi_1 : T^*D^n \hookrightarrow (T^*D^n')_L$$

where $(T^*D^n)_L := (T^*D^n', \lambda_{T^*D^n', T^*\varphi_2(L)}) \setminus T^*\varphi_2(L)$. It will be helpful to keep $(T^*D^n)_L$ and the larger version $(T^*D^n')_L$ separate, even though $T^*\varphi_2 : (T^*D^n)_L \to (T^*D^n')_L$ is a strict sectorial equivalence. For the rest of this section, we consider the functor $\times (T^*D^n)_L$ instead of the functor $\times (T^*D^n)_L$ (which are equivalent functors by the previous sentence). Then the morphism $\eta_{T^*D^n}$ induces a natural transformation

$$\eta : Id \to \times (T^*D^n')_L$$

in $\text{Wein}^\s$, where $\eta_X : X \to X \times (T^*D^n')_L$ is defined by the proper inclusion on the stabilized version of $X$:

$$\text{id}_X \times \eta_{T^*D^n} : X \times T^*D^n \to X \times (T^*D^n')_L$$
Since $\times (T^*D^n)_L$ is a functor of $\text{Wein}_{\text{str}}$, we can apply it to the morphism $\eta_X$ to produce the morphism

$$\text{Id}_X \times \eta_{T^*D^n} \times \text{Id}_{(T^*D^n)_L} : X \times T^*D^n \times (T^*D^n)_L \to X \times (T^*D^n)_L \times (T^*D^n)_L$$

Similarly, there is the morphism $\eta_{X \times (T^*D^n)_L}$ defined by the proper inclusion

$$\text{Id}_X \times \text{Id}_{(T^*D^n)_L} \times \eta_{T^*D^n} : X \times (T^*D^n)_L \times T^*D^n \to X \times (T^*D^n)_L \times (T^*D^n)_L$$

Using a swapping identifications of $X \times (T^*D^n)_L \times T^*D^n$ with $X \times T^*D^n \times (T^*D^n)_L$, these two morphisms are conjugate in $\text{Wein}_{\text{str}}$. So one morphism is an equivalence if and only if the other is an equivalence.

The following theorem is the main result of this section.

**Theorem 6.1.** For any Liouville sector $X$, the proper inclusions $\text{Id}_{(T^*D^n)_L} \times \eta_X, \eta_{X \times (T^*D^n)_L}$ are equivalences in $\text{Wein}^\circ_{\text{crit}}$. In particular, $(T^*D^n)_L$ is a localization functor of $\text{Wein}^\circ_{\text{crit}}$ via the natural transformation $\eta : \text{Id} \to \times (T^*D^n)_L$.

To prove this result, note that it suffices to prove this for $X = T^*D^0$, i.e. that the (unstabilized) map

$$\text{Id}_{(T^*D^n)_L} \times \eta_{T^*D^n} : (T^*D^n)_L \times T^*D^n \to (T^*D^n)_L \times (T^*D^n)_L$$

is an equivalence in $\text{Wein}^\circ_{\text{crit}}$. Note that attaching subcritical handles to these sectors is crucial, since $T^*D^n \times (T^*D^n)_L$ and $(T^*D^n)_L \times (T^*D^n)_L$ have different singular cohomology in degree $2n-1$ (since $(T^*D^n)_L$ is diffeomorphic to $T^*D^n$ with a subcritical handle attached, as discussed in Section 4.2.2) and hence are not even diffeomorphic after stabilizing.

### 6.2. A natural transformation for $(\ )_L$.

To prove Theorem 6.1, we introduce a natural transformation $\theta : \text{Id} \to (\ )_L$ for the non-homotopy invariant $\text{P}$-flexibilization functor $(\ )_L$. As we explain, it will be easier to prove idempotency for this natural transformation. Recall that $X_L$ is a non-strict Weinstein subdomain of $X$. By Proposition 2.73, this yields a morphism $X \to X_L$ in $\text{Wein}^\circ_{\text{crit}}$. To make this morphism explicit, we use a similar construction as in Section 6.1. As we discuss in Remark 6.2 below, the definition of $(\ )_L$ here will be slightly different than in Sections 4 and 5 but are all equivalent in $\text{Wein}^\circ_{\text{crit}}$ by the comparison result Theorem 5.1.

As in Section 6.1, we consider the Morse-Bott function $f'$ on $D^{n'}$ which induces the Weinstein structure $(T^*D^{n'}, \lambda_{T^*D^{n'}})$ and have two strict proper inclusions

$$\text{Id}_X \times T^*\varphi_1, \text{Id}_X \times T^*\varphi_2 : X \times (T^*D^n, \lambda_{\text{std}}) \to X \times (T^*D^{n'}, \lambda_{T^*D^{n'}})$$

The co-cores of $X \times (T^*D^{n'}, \lambda_{T^*D^{n'}})$ are $C_X \times T^*_x D^n$ and $C_X \times T^*_x D^{n'}$. Next, recall that there is an interior Weinstein homotopy of $X \times (T^*D^n, \lambda_{\text{std}}) = (X \times T^*D^n, \lambda_{X \times T^*D^n})$ to $(X \times T^*D^n, \lambda_{X \times T^*D^n, \lambda_{\text{std}}})$ that has $C^L_X \times T^*_0 D^n$ as co-cores; we recall that near the product of co-cores $T^*D^n \times T^*D^{n'}$, the Weinstein structure $\lambda_{T^*D^n \times T^*D^{n'}, \lambda_{\text{std}}}$ is obtained by swapping the Weinstein structure $\lambda_{T^*D^n \times T^*D^{n'}, \lambda_{\text{std}}}$, $L \times T^*D^n \times L$. We can apply this homotopy to $\text{Id}_X \times T^*\varphi_2(X \times T^*D^n) \subset X \times T^*D^{n'}$, while keeping the Weinstein structure on $\text{Id}_X \times T^*\varphi_1(X \times T^*D^n) \subset X \times T^*D^n$ fixed. We will call the resulting structure $(X \times T^*D^{n'}, \lambda_{X \times T^*D^{n'}, \lambda_{T^*D^n, \lambda_{\text{std}}}})$ which has $C^L_X \times T^*\varphi_2(T^*_0 D^n) = C^L_X \times T^*_x D^n$ as a co-core.
There is still a strict proper inclusion

\[ \text{Id}_X \times T^*\varphi_1 : X \times T^*D^n \to (X \times T^*D^n, \lambda_{X \times T^*D^n, L \times T^*\varphi_2(T_0^n D^n)}) \]

Since \( C^L_X \times T^*\varphi_2(T_0^n D^n) \) are disjoint from the image of \( \text{Id}_X \times T^*\varphi_1 \), there is an induced proper inclusion

\[ \theta_{X,L} := \text{Id}_X \times T^*\varphi_1 : X \times T^*D^n \to (X \times T^*D^n)_{L \times T^*_L D^n} \]

where we define

\[ (X \times T^*D^n)_{L \times T^*_L D^n} := (X \times T^*D^n, \lambda_{X \times T^*D^n, L \times T^*\varphi_2(T_0^n D^n)}) \backslash \coprod_{C_X} C^L_X \times T^*_L D^n \tag{6.1} \]

Using \( \theta_{X,L} \), we get a natural transformation

\[ \theta : \text{Id} \to (_{-} \times T^*D^n)_{L \times T^*_L D^n} \]

in \( \text{Wein}_{\text{param}} \); that is, for any proper inclusion \( i : X \hookrightarrow Y \) of parametrized Weinstein sectors, the following diagram commutes

\[
\begin{array}{ccc}
X \times T^*D^n & \xrightarrow{f \times \text{Id}_{T^*D^n}} & Y \times T^*D^n \\
\downarrow^{\theta_{X,L}} & & \downarrow^{\theta_{Y,L}} \\
(X \times T^*D^n)_{L \times T^*_L D^n} & \xrightarrow{(f \times \text{Id})_{L \times T^*_L D^n}} & (Y \times T^*D^n)_{L \times T^*_L D^n} 
\end{array}
\tag{6.2}
\]

Remark 6.2. Note that the definition of \((X \times T^*D^n)_{L \times T^*_L D^n}\) in Equation 6.1 differs slightly from the definition of \((X \times T^*D^n)_{L \times T^*_0 D^n}\) given in Section 5 since we use \((T^*D^n, \lambda_{T^*D^n})\) instead of \((T^*D^n, \lambda_{\text{std}})\). However, there is a strict sectorial equivalence from \((X \times T^*D^n)_{L \times T^*_0 D^n}\) to \((X \times T^*D^n)_{L \times T^*_L D^n}\) via the map \(T^*\varphi_2\). Also, \((X \times T^*D^n)_{L \times T^*_0 D^n}\) is equivalent to \(X_L\) by Example 5.8. Hence, we can consider \(\theta_{X,L}\) as a natural transformation between \(\text{Id}\) and the usual functor \(\_\times T^*D^n\).

6.3. Comparison of natural transformations. In this section, we prove a comparison result between the two natural transformations \(\eta\) and \(\theta\) discussed in the previous sections.

Theorem 6.3. For any Weinstein sector \(X\), there is an equivalence \(\varphi'_X : (X \times T^*D^n)_{L \times T^*_0 D^n} \to X \times (T^*D^n)_L\) and a homotopy commutative diagram in \(\text{Wein}_{\text{crit}}^2\):

\[
\begin{array}{ccc}
X \times T^*D^n & \xrightarrow{\theta_X} & (X \times T^*D^n)_{L \times T^*_L D^n} \\
\downarrow^{\eta_X} & & \downarrow^{\varphi'_X} \\
X \times (T^*D^n)_L & & 
\end{array}
\tag{6.3}
\]

Remark 6.4. We note that \(\varphi'_X\) is conjugate to the equivalence \(\varphi_X : X_L \times T^*D^n \to X \times (T^*D^n)_L\) from the previous section.

The proof of Theorem 6.3 follows from the following result, which is its analog before stabilizing and inverting subcritical handles.
Proposition 6.5. If \( n = \dim L = \frac{1}{2} \dim X \) is even, there is a strictly commuting diagram of symplectic embeddings:

\[
\begin{array}{ccc}
X \times T^*D^n & \xrightarrow{\eta_X = Id_X \times \eta_{T^*D^n}} & X \times (T^*D^n')_L \\
\downarrow Id & & \downarrow i_{X,(T^*D^n,L)} \\
X \times T^*D^n & \xrightarrow{\lambda_{T^*D^n,T^*\varphi_2(L)}} & X \times (T^*D^n', \lambda_{T^*D^n',T^*\varphi_2(L)}) \\
\downarrow Id & & \downarrow Id \\
(X \times T^*D^n)_{T_0^*D^n \times L} & \xrightarrow{\varphi_{X,T^*D^n,L,T_0^*D^n}} & (X \times T^*D^n')_{T_0^*D^n \times L} \\
\end{array}
\]

where the horizontal maps are strict proper inclusions, \( i_{X,(T^*D^n,L)} \) is a strict subcritical subdomain inclusion, and the map \( Id \) on the middle-right is an isomorphism up to Weinstein homotopy, and \( \varphi_{X,T^*D^n,L,T_0^*D^n} \) is a strict Liouville isomorphism.

Proof. The proof is essentially a relative version of the proof of Proposition 5.5. There is a strictly commuting diagram

\[
\begin{array}{ccc}
T^*D^n & \xrightarrow{\eta_{T^*D^n}} & (T^*D^n')_L \\
\downarrow Id & & \downarrow i_{(T^*D^n,L)} \\
T^*D^n & \xrightarrow{T^*\varphi_1} & (T^*D^n', \lambda_{T^*D^n',T^*\varphi_2(L)}) \\
\end{array}
\]

and taking the product with \( X \), we get the strictly commuting diagram

\[
\begin{array}{ccc}
X \times T^*D^n & \xrightarrow{\eta_X} & X \times (T^*D^n')_L \\
\downarrow Id & & \downarrow Id \times i_{(T^*D^n,L)} \\
X \times T^*D^n & \xrightarrow{Id_X \times T^*\varphi_1} & X \times (T^*D^n', \lambda_{T^*D^n',T^*\varphi_2(L)}) \\
\end{array}
\]

Then as in the proof of Proposition 5.5, \( X \times (T^*D^n')_L \xrightarrow{\subset} X \times (T^*D^n', \lambda_{T^*D^n',T^*\varphi_2(L)}) \) is a strict Weinstein subdomain inclusion, and the complementary cobordism has critical points with co-cores \( C_X \times T^*\varphi_2(L) \). Since these co-cores are in the complement of the inclusion \( Id_X \times T^*\varphi_1 \), this induces the top square of Equation 6.4.

Next, we observe that the identity is a bordered Weinstein equivalence from

\[
X \times (T^*D^n', \lambda_{T^*D^n',T^*\varphi_2(L)}) \setminus \bigcup C_X \times T^*\varphi_2(L)
\]

to

\[
(X \times T^*D^n')_{T_0^*D^n \times L} := (X \times T^*D^n', \lambda_{X \times T^*D^n',T_0^*D^n \times T^*\varphi_2(L)}) \setminus \bigcup C_X \times T^*\varphi_2(L)
\]

To see this, we proceed as in the proof of Proposition 5.5 and note that the form \( \lambda_X + \lambda_{T^*D^n',T^*\varphi_2(L)} \) looks like \( (T^*D^n \times T^*D^n, \lambda_{T^*D^n,\text{std}} + \lambda_{T^*D^n,L}) \) near \( C_X \times T^*_2D^n \subset X \times T^*\varphi_2(T^*D^n) \). Then we use part 1) of Proposition 2.43 to obtain a homotopy of forms on
\[ T^*D^n \times T^*D^n \text{ from } \lambda_{T^*D^n,\text{std}} + \lambda_{T^*D^n,L} \text{ to } \lambda_{T^*D^n \times T^*D^n,T_0^*D^n \times L}, \] which agrees with \( \lambda_{T^*D^n,\text{std}} + \lambda_{T^*D^n,L} \) near \( \partial T^*D^n \times T^*D^n \), is constant near \( T^*D^n \times \partial T^*D^n \), and is \( \lambda_{T^*D^n,\text{std}} + \lambda_{T^*D^n,L} \) on a slightly smaller copy of \( T^*D^n \times T^*D^n \). We then extend this homotopy to \( X \times T^*D^n \) by taking the homotopy \( \lambda_X + (T^*\varphi_2)_*(\lambda_{T^*D^n,L}) \) on the rest of \( X \times T^*\varphi_2(T^*D^n) \) and the constant homotopy on \( X \times T^*\varphi_1(T^*D^n) \). Since this homotopy happens in the complement of \( X \times T^*\varphi_1(T^*D^n) \), the second square of Equation 6.4 also commutes. See Figure 7 to make that figure appropriate for this proof, the Z in that figure there should be \( X \) and the \( X \) should be \( T^*D^n \).

Finally, the sector \( (X \times T^*D^n)_{L \times T_0^*D^n} \) is constructed via the local model \( (T^*D^n \times T^*D^n, \lambda_{T^*D^n \times T*^D^n,L \times T_0^*D^n}) \), which is the swap of \( (T^*D^n \times T^*D^n, \lambda_{T^*D^n \times T*^D^n,T_0^*D^n \times L}) \). So the cut-off swap map \( \varphi_{X,T^*D^n,L,T_0^*D^n} \) is a strict Liouville isomorphism. Since this map is supported away from \( Id_X \times T^*\varphi_1(X \times T^*D^n) \), the last square also strictly commutes. 

**Proof of Theorem 6.3.** If \( n \) is even, this follows from Propositions 6.3 and Propositions 2.61, 2.73. We note that the top commuting square in Proposition 6.5 is a pullback diagram since the left vertical map is the identity map; hence Proposition 2.73 can be applied in this setting.

If \( n \) is odd, we first have to stabilize and show that the resulting maps \( \eta_{X \times T^*D^1}, \theta_{X \times T_0^*D^1} \) for Lagrangian \( L \times T_0^*D^1 \) are related by a homotopy commutative diagram in \( \text{Wein}^\circ_{\text{crit}} \) to the unstabilized maps \( \eta_X, \theta_X \). This follows from the first half of Proposition 6.5 applied to \( X = T^*D^1 \) that discusses the maps \( i_{T^*D^1,(T^*D^n,L)} \) and \( Id \) (the proposition is stated for \( X \) with \( \frac{1}{2} \text{dim } X = \text{dim } L = n \) but the first half holds for any \( X \)). 

**6.4. Proof of idempotency.** Recall that to prove Theorem 6.1 it suffices to prove that

\[ Id_{(T^*D^n)_L \times \eta_{T^*D^n}} : (T^*D^n)_L \times T^*D^n \to (T^*D^n)_L \times (T^*D^n)_L \]

is an equivalence in \( \text{Wein}^\circ_{\text{crit}} \). Since \( (T^*D^n)_L \) is equivalent to \( (T^*D^n)_L \) in \( \text{Wein}^\circ_{\text{crit}} \), it suffices to prove that

\[ Id_{(T^*D^n)_L \times \eta_{T^*D^n}} : (T^*D^n)_L \times T^*D^n \to (T^*D^n)_L \times (T^*D^n)_L \]

where we use \( (T^*D^n)_L \) instead of \( (T^*D^n)_L \) for the first term in the product; this will turn out to be slightly more convenient. Then by the comparison result Theorem 6.3 applied to \( X = (T^*D^n)_L \), it suffices to prove that

\[ \theta_{(T^*D^n)_L} : (T^*D^n)_L \times T^*D^n \to ((T^*D^n)_L \times (T^*D^n)_L)_{L \times T_0^*D^n} \]

is an equivalence in \( \text{Wein}^\circ_{\text{crit}} \). We will do this for a particular special choice of model for \( (T^*D^n)_L \), after first stabilizing \( L \).

Let \( C \) be a Lagrangian co-core of \( (T^*D^n)_L \) and \( C^L \) be a copy of \( L \) embedded in a Weinstein neighborhood of \( C \). Then \( \theta_{(T^*D^n)_L} \) is the proper inclusion

\[ (T^*D^n)_L \times T^*D^n \to (T^*D^n)_L \times T^*D^n \setminus \bigcup C \times T^*x_2D^n \]

induced by \( T^*D^n \to T^*D^n \). Next, suppose that each \( C^L \subset (T^*D^n)_L \) is the co-core of an index \( n \) handles that is in cancelling position with an index \( n - 1 \) handles. That is, there is a Weinstein homotopy that cancels both of these handles (which is the case if the attaching
sphere of the index \( n \) handle intersects the belt sphere of the index \( n-1 \) handle exactly once). In this case, the union of the index \( n \) and index \( n-1 \) handles is a trivial cobordism (up to Weinstein homotopy) and \( C^L \subset (T^*D^n)_L \) is a Lagrangian unknot. Then \( \theta_{(T^*D^n)_L} \) is an equivalence since carving out \( C^L \) from \((T^*D^n)_L\) is the same as adding a subcritical handle to \((T^*D^n)_L\) (and similarly, carving out \( C^L \times T^*_0 D^n \) from \((T^*D^n)_L \times T^*D^n\) is the same as adding subcritical handles to the complement of the image of \((T^*D^n)_L \times T^*D^n \rightarrow (T^*D^n)_L \times T^*D^n\).

However a priori, \((T^*D^n)_L\) might have many co-cores \( C \) and so we cannot control \( C^L \) and assume that it is a Lagrangian unknot. The following key proposition shows that we can control the new co-cores after stabilization.

**Proposition 6.6.** For any regular Lagrangian disk \( L^n \subset (T^*D^n, \lambda_{T^*D^n, L,L}) \), there is an interior Weinstein homotopy of \((T^*D^n \times T^*D^1, \lambda_{T^*D^n, L} + \lambda_{T^*D^1, std})\), relative to \( L \times T^*_0 D^n \), to a Weinstein structure \( \lambda_{T^*D^n \times T^*D^1, L \times T^*_0 D^n} \) with exactly two index \( n + 1 \) critical points whose Lagrangian co-cores are \( T^*_0 D^n \times T^*_{-1/2} D^1 \) and \( L \times T^*_0 D^1 \).

We momentarily postpone the proof of Proposition 6.6 until the end of this section and now explain how to prove idempotency for the functor \( \times (T^*D^{n+1})_{L \times T^*_0 D^1} \) assuming this result. By Proposition 6.6, \( T^*D^{n+1} \) has a Weinstein presentation with two co-cores \( C_1 = T^*_0 D^n \times T^*_{-1/2} D^1 \) and \( C_2 = L \times T^*_0 D^1 \). So \( T^*D^{n+1}_{L \times T^*_0 D^1} = T^*D^{n+1} \setminus L \times T^*_0 D^1 \) has a single co-core \( C_1 = T^*_0 D^n \times T^*_{-1/2} D^1 \). Recall that we need to show that \( C^L_{L \times T^*_0 D^1} \) is the co-core of an index \( n + 1 \) handle that cancels with a subcritical handle. To do so, it is helpful to consider

\[
C^L_{L \times T^*_0 D^1} \coprod C_2
\]

as a Lagrangian link, i.e. union of disjointly embedded Lagrangians. We say that a link \( C \coprod C' \subset X \) of Lagrangian disks is parallel if there is a strict proper inclusion \( i: T^*D^n \hookrightarrow X \) so that \( i(T^*_0 D^n \coprod T^*_x D^n) = C \coprod C' \) for some \( x \neq 0 \) in \( D^n \). This perspective is helpful in light of the following observation.

**Proposition 6.7.** Let \( C \coprod C' \subset X \) be a parallel Lagrangian link of Lagrangian disks. Then \( C' \subset X \setminus C \) is Lagrangian co-core of an index \( n \) handle that cancels an index \( n-1 \) handle.

**Proof.** Take a Morse function \( f \) on \( D^n \) whose interior critical points are two index \( n \) critical points (at \( 0 \) and \( x \)) and one critical point of index \( n-1 \); this Morse function can be obtained by starting with the standard structure that has a single critical point of index \( n \) and creating a cancelling pair of critical points of index \( n \) and \( n-1 \). Then the induced Weinstein structure on \( T^*D^n \) has two index \( n \) critical points with Lagrangian co-cores \( T^*_0 D^n \) and \( T^*_x D^n \). Then \( T^*D^n \setminus T^*_0 D^n \) has a single index \( n \) handle which is cancelling with the other index \( n-1 \) handle and has co-core \( T^*_x D^n \); this proves the result for \( T^*D^n \). Since \( C \coprod C' \subset X \) is modelled on \( T^*_0 \coprod T^*_x D^n \subset T^*D^n \), the same result holds for \( X \).

Finally, we note that

\[
C^L_{L \times T^*_0 D^1} \coprod C_2
\]

is the link

\[
L \times T^*_{-1/2} D^1 \coprod L \times T^*_0 D^1
\]
which is parallel. Indeed, if \( \varphi_L : T^* D^n \to T^* D^n \) is a parametrization of \( L \), then \( \varphi_L \times Id_{T^* D^1} : T^* D^n \times T^* D^1 \to T^* D^n \times T^* D^1 \) has the property that \( \varphi_L \times Id_{T^* D^1}(T^*_0 D^n \times T^*_{-1/2} D^1) = L \times T^*_{-1/2} D^1 \) and \( \varphi_L \times Id_{T^* D^1}(T^*_0 D^n \times T^*_0 D^1) = L \times T^*_0 D^1 \). Hence

\[
L \times T^*_{-1/2} D^1 \subset (T^* D^{n+1})_L \times T^*_0 D^1 = T^* D^{n+1} \setminus L \times T^*_0 \]

is the co-core of a cancelling \( n + 1 \) handle. This completes the proof of Theorem 6.1 for the functor \( \times (T^* D^{n+1})_L \times T^*_0 D^1 \) using the stabilized Lagrangian \( L \times T^* D^1 \).

6.4.1. Simple Weinstein presentation, after stabilization. Next, we complete the proof of Proposition 6.6 showing that \( T^* D^{n+1} \) admits a simple Weinstein presentation with two co-cores \( T^*_0 D^n \times T^*_{-1/2} D^1, L \times T^*_0 D^1 \). Then we will complete the proof of Theorem 6.1 idempotency for the functor \( \times T^* D^1_L \).

Proof of Proposition 6.6. In this proof, it will be useful to consider Weinstein sectors as associated to stopped Weinstein domains since we will consider separate homotopies of the stop and of the domain and then combine them into sectorial homotopies. To that end, we recall the following notation from Definitions 2.5, 2.17 \( [X, F] \) denotes a sector \( X \) and its sectorial divisor \( F \) and \( (X_0, \Lambda) \) denotes the sectorial completion of a stopped domain \( (X_0, \Lambda) \). Since we will need to carve out the Lagrangian \( L \) or \( L \times T^*_0 D^1 \), all our homotopies are relative to these Lagrangians, i.e. through forms vanishing on the Lagrangians.

Since \( L^n \subset [T^* D^n, T^* \partial D^n] \) is a regular disk, there is a bordered Weinstein homotopy of \( [T^* D^n, T^* \partial D^n] \) to a sector \( (T^* D^n, \Lambda_{T^* D^n, L}) = (T^* L^+ \cup C^{2n}, \partial D^n) \) so that \( L \subset T^* D^n \) corresponds to the zero-section of \( T^* L^+ \). Here \( (T^* L^+ \cup C^{2n}, \partial D^n) \) is a stopped domain; \( T^* L^+ \) is the canonical Weinstein domain structure on cotangent bundles with outward pointing Liouville vector field everywhere and an index 0 critical point on \( L \), with \( \partial L \) mapping to the contact boundary; \( C^{2n} \) is a Weinstein cobordism and the attaching spheres of \( C^{2n} \) are disjoint from \( \partial L \). Similarly, \( F = T^*_0 D^1 \subset T^* D^1 \) is a regular Lagrangian and so we can Weinstein homotope \( [T^* D^1, T^* \partial D^1] \) to \( (T^* F^+, \partial D^1) \). Here \( (T^* F^+, \partial D^1) \) is just \( (B^2, \pm 1) \), a ball with a single index 0 critical point in the interior and two points for stops. So \( (T^* F^+, \partial D^1) \) has two index 1 critical points in the interior (corresponding the linking disks of the two stops), one index 0 critical point in the interior (corresponding to the index 0 critical point of \( B^2 \)), and two index 0 critical points on the boundary (corresponding to the stops).

Next, we consider the product of these two sectors \( (T^* D^n, \Lambda_{T^* D^n, L}) \times T^* D^1 \). The sector associated to the product of the stopped domains coincides with the product of sectors associated to two stopped domains. So this sector is \( ((T^* L^+ \cup C) \times T^* F^+, \Lambda) \), where \( \Lambda = (T^* L^+ \cup C) \times T^* \partial D^1 \cup T^* \partial D^n \times T^* F^+ \). Since this sector is bordered homotopy to \( (T^* D^n, \Lambda_{T^* D^n, std}) \times (T^* D^1, \Lambda_{T^* D^1, std}) \), the stop \( \Lambda \) is a Weinstein homotopic to the stop of \( T^* D^n \times T^* D^1 \) which is \( T^* \partial D^{n+1} \). Next, we proceed as in Part 3) of Proposition 2.43 and append the movie construction of this homotopy to \( ((T^* L^+ \cup C) \times T^* F^+, \Lambda) \) and get a bordered homotopy rel \( L \times F \) from \( (T^* D^n, \Lambda_{T^* D^n, L}) \times T^* D^1 \) to \( \left((T^* L^+ \cup C) \times T^* F^+, (T^* \partial D^{n+1}) \right) \). Here \( ((T^* L^+ \cup C) \times T^* F^+) \) is a slight enlargement of \( (T^* L^+ \cup C) \times T^* F^+ \) with the same Lagrangian co-cores; hence we will identify these two domains.

Next, we will Weinstein homotope the domain part \( (T^* L^+ \cup C) \times T^* F^+ \) of the stopped domain \( ((T^* L \cup C) \times T^* F^+, T^* \partial D^{n+1}) \). First, we observe that \( (T^* L \cup C) \times T^* F^+ \) is a
subcritical Weinstein domain since $T^*F^+$ is subcritical. In fact, the Weinstein cobordism $C \times T^*F^+$ in the complement of $T^*(L \times F)^+$ is subcritical. Since $C$ is a disk, this cobordism is a smooth h-cobordism and hence is smoothly trivial if $n \geq 2$; if $n = 1$, there is only two Lagrangian disks in $T^*D^1$, the fiber and the unknot in which case the result can be checked explicitly. So $C^{2n} \times T^*F^+$ is smoothly trivial Weinstein cobordism, which is also subcritical, and hence by the h-principle for subcritical cobordisms \[\Pi\], it is Weinstein homotopic to the trivial cobordism. Furthermore, this Weinstein homotopy is relative to $L \times F$ since the smoothly isotopies of the attaching spheres can done away from $\partial(L \times F)$ and the isotropic isotopies are $C^\infty$-close to the smooth isotopies. In particular, $(T^*L^+ \cup C) \times T^*F^+$ is homotopic, relative $L \times F$, to $T^*(L \times F)^+$; note that $T^*(L \times F)^+$ is abstractly just a Weinstein ball $B^{2n+2}$.

Now we return to the sector $(T^*(L \times F)^+, T^*\partial D^{n+1})$ has the following critical points. The domain $T^*(L \times F)^+$ has one index 0 critical point (lying in $L \times F$) while the stop $T^*\partial D^{n+1}$ has two critical points of index 0, $n$, which give rise to index 1, $n+1$ critical points in a neighborhood $T^*\partial D^{n+1} \times T^*[0, 1]$ of the stop $T^*\partial D^{n+1}$. The co-core of the index $n+1$ critical point is precisely a cotangent fiber of $T^*\partial D^{n+1} \times T^*[0, 1]$, which viewed in $T^*D^{n+1}$ is $T^*_x D^{n+1}$ for some $x$ near $\partial D^{n+1}$. After further Weinstein homotopy, we can assume that $x = 0 \times \{-1/2\} \in D^n \times D^1$.

Next, we again use the fact that $L$ (and hence $L \times F$) is a disk. In this case, there is a Weinstein homotopy of $(T^*(L \times F)^+, T^*\partial D^{n+1})$ supported in a small neighborhood of $L \times F$ to a new structure with an additional index $n$ and $n+1$ critical point so that the co-core of the latter is $L \times F$; see [8]. Since this Weinstein homotopy is supported in a small neighborhood of $L \times F$, it does not create any new critical points outside and does not change the co-core $T^*_0 D^{n+1}$ of the other index $n+1$ critical point lying in $T^*\partial D^{n+1} \times T^*[0, 1]$. Finally, we can homotope this structure so that this index 0 and index 1 critical point lying in $T^*\partial D^{n+1} \times T^*[0, 1]$ are cancelled, producing a structure with a total of three critical points, one of index $n$ and two of index $n+1$ whose co-cores are $T^*_0 D \times T^*_{-1/2} D^1$ and $L \times F = L \times T^*_0 D^1$. □

**Remark 6.8.** Proposition [6.6] is false unless we first stabilize the Lagrangian disk $L \subset T^*D^n$. Otherwise $T^*D^n \setminus L \cup T^*_0 D^n = B^{2n} \cup H^{n-1} \setminus L$ would be subcritical, which is not always the case for Lagrangian disks in $L$. Indeed, by [22], for every $n \geq 4$, there are Lagrangian disks $L \subset B^{2n}$ so that $B^{2n} \setminus L$ is not subcritical. However, when we stabilize, $L \times T^*_0 D^1$ becomes isotopic to the Lagrangian unknot and so $B^{2n+2} \setminus L \times T^*_0 D^1$ is automatically subcritical.
For general $L \subset T^*D^n$, there cannot be a Weinstein structure on $T^*D^{n+1}$ with $L \times T_0^*D^1$ as the only co-core since this implies that $L \times T_0^*D^1$ (and also $L$) is the generator of the Fukaya category of $T^*D^{n+1}$. Therefore, Proposition 6.6, where there are precisely two co-cores, is the best possible scenario.

**Remark 6.9.** Proposition 6.6 is the only place in the paper where we actually use any h-principle (Gromov’s h-principle for subcritical isotropics [III]). We also observe that if $L \subset T^*D^n$ is already isotopic to the zero-section $D^n$ as a Lagrangian in the unstopped domain $B^{2n}$, then there is no need to stabilize or use this h-principle since we are already in the situation that $(T^*D^n, \lambda_{T^*D^n,L})$ is homotopic, relative $L$, to $(T^*L^+, T^*\partial D^n)$. For example, the $D_P$ disks from [I] used to construct $X_P$ are graphical and hence isotopic to $D^n \subset B^{2n}$. Hence our proofs that $X_P$ (including the flexible case $X_0$) is idempotent and independent of presentation does not depend on any h-principle.

Finally, we complete the proof that the functor $\times (T^*D^n)_L$ is idempotent, using the already proven fact that $\times (T^*D^{n+1})_{L \times T_0^*D^1}$ is idempotent.

**Proof of Theorem 6.1.** Suppose that $L \subset T^*D^n$ is a regular Lagrangian and we fix a Weinstein homotopic form $\lambda_{T^*D^n,L}$ on $T^*D^n$ so that $\lambda_{T^*D^n,L}|_L = 0$. Then by Proposition 6.6, $\lambda_{T^*D^n,L} + \lambda_{T^*D^n,\text{std}}$ is homotopic, relative $L \times T_0^*D^1$, to the structure $\lambda_{T^*D^{n+1},L \times T_0^*D^1}$ which has only two co-cores. Setting $X = T^*D^1$ in Proposition 6.5, we have that the morphism

$$\eta_{T^*D^n} : T^*D^n \to (T^*D^n)_L$$

which is defined via $\lambda_{T^*D^n,L}$, is conjugate in $\text{Wein}^\odot_{\text{crit}}$ to the stabilized morphism

$$\eta_{T^*D^{n+1}} : T^*D^{n+1} \to (T^*D^{n+1})_{L \times T_0^*D^1}$$

which is defined via $\lambda_{T^*D^{n+1},L \times T_0^*D^1}$. More precisely, this is because the homotopy used to construct $\lambda_{T^*D^{n+1},L \times T_0^*D^1}$ consists of two steps, a movie of a boundary homotopy (precisely as in the proof of Proposition 6.5) and an interior homotopy, relative to $L \times T_0^*D^1$, (which does not affect the proof of Proposition 6.5). We also have already proven idempotency for $(T^*D^{n+1})_{L \times T_0^*D^1}$, i.e. that the morphism

$$\text{Id}_{(T^*D^{n+1})_{L \times T_0^*D^1}} \times \eta_{T^*D^{n+1}} : (T^*D^n)_{L \times T_0^*D^1} \times T^*D^{n+1} \to (T^*D^n)_{L \times T_0^*D^1} \times (T^*D^n)_{L \times T_0^*D^1}$$

is an equivalence in $\text{Wein}^\odot_{\text{crit}}$. Since $(T^*D^{n+1})_{L \times T_0^*D^1}$ is equivalent to $(T^*D^n)_L$ in $\text{Wein}^\odot_{\text{crit}}$ by Example 5.8,

$$\text{Id}_{(T^*D^n)_L} \times \eta_{T^*D^n} : (T^*D^n)_L \times T^*D^n \to (T^*D^n)_L \times (T^*D^n)_L$$

is also an equivalence in $\text{Wein}^\odot_{\text{crit}}$ as desired. \qed

**Remark 6.10.** The construction of the simple Weinstein presentations in Proposition 6.6 can be used to prove the claim in Example 1.16 - that $(T^*D^n)_P \times (T^*D^n)_Q$ is equivalent to $(T^*D^n)_{P \cup Q}$ in $\text{Wein}^\odot_{\text{crit}}$. More generally, we prove that $(T^*D^n)_L \times (T^*D^n)_K$ is equivalent to $(T^*D^n)_{L \sqcup K}$. First note that $(T^*D^n)_L \times (T^*D^n)_K$ is equivalent to $((T^*D^n)_L)_K$ in $\text{Wein}^\odot_{\text{crit}}$. 


by Theorem 5.1. Now, by Proposition 6.6, $T^*D^{n+1}$ has a Weinstein presentation with two co-cores $L \times T_0^*D^1$ and $T_0^*D^n \times T_{-1/2}^*D^1$. Therefore, we have the following equalities:

$$(T^*D^{n+1})_{L \times T_0^*D^1} = (T^*D^{n+1})_{L \times T_0^*D^1} \setminus (L \times T_0^*D^1) = (T^*D^{n+1})_{L \times T_0^*D^1} \setminus (L \times T_0^*D^1)$$

The latter sector is precisely $(T^*D^{n+1})_{L \times T_0^*D^1} \setminus (L \times T_0^*D^1)$. Assuming that $L \coprod K$ is disjointly embedded into $T^*D^n$ via a map $T^*\varphi_1$ $\coprod T^*\varphi_2$ : $T^*D^n \coprod T^*D^n \hookrightarrow T^*D^n$, then $(T^*D^n)_{L \coprod K}$ is equivalent to $(T^*D^{n+1})_{L \times T_0^*D^1} \setminus (L \times T_0^*D^1)$ by Example 5.8, which is isomorphic to $(T^*D^{n+1})_{L \times T_0^*D^1} \setminus (L \times T_0^*D^1)$. This proves that $(T^*D^n)_{P \coprod Q}$ is equivalent to $(T^*D^n)_{P \times (T^*D^n)_{Q, P \coprod Q}}$, where we allow possibly repeated integers in the set $P \coprod Q$. However by Theorem 6.1, $(T^*D^n)_{P \coprod Q}$ is equivalent to $(T^*D^n)_{P \cup Q}$, where no integers are repeated in the set $P \cup Q$.

6.4.2. Unlinking, after stabilization. In this section, we briefly discuss certain unlinking phenomena which appeared implicitly in the proof of Theorem 6.1. A crucial step in the proof was the fact that

$$C_1^{L \times T_0^*D^1} \coprod C_2$$

is a parallel link. This fact was easy to check since this link was the stabilization via two different cotangent fibers $T_{-1/2}^*D^1, T_0^*D^1$ (stabilization was already used to prove that Proposition 6.6).

Furthermore, in Example 4.11, the main issue was that it is not clear that the Lagrangian link $(T^*_{x_1}S^n)_{L \coprod (T^*_{x_2}S^n)}$ is parallel, even though the two components of this link are isotopic to each other; see Figure 9 for a schematic depiction of the situation. Indeed, if this link were parallel, then $T^*S^n \setminus (T^*_{x_1}S^n) \coprod (T^*_{x_2}S^n)$ would coincide with $T^*S^n \setminus (T^*_{x_1}S^n)$ up to subcritical handles by Proposition 6.7 (and so $T^*S_{L}^n$ and $T^*S_{L'}^n$ would be equivalent up to subcritical handles). However, note that $(T^*_{x_1}S^n) \coprod (T^*_{x_2}S^n)$ is component-wise isotopic to a parallel link $(T^*_{x_1}S^n) \coprod (T^*_{x_2}S^n)$ (obtained by taking a small push-off of $(T^*_{x_1}S^n)$ in its Weinstein neighborhood) since $(T^*_{x_2}S^n)$ is isotopic to $(T^*_{x_1}S^n)$.

More generally, consider two Lagrangian links $L \coprod K$ and $L' \coprod K' \subset X$ that are component-wise isotopic; that is, $L$ is Lagrangian isotopic to $L'$ and $K$ is Lagrangian isotopic to $K'$ in $X$. This does not imply that $L \coprod K$ is Lagrangian isotopic as a link to $L' \coprod K'$ since the isotopy from $L$ to $L'$ may intersect the isotopy from $K$ to $K'$, i.e. $L \coprod K$ are linked differently from $L' \coprod K'$; see [20] for examples of Lagrangian linking. In the following proposition, we show
that all Lagrangian links become unlinked after a single stabilization, which as explained above is a key phenomena underlying Theorems 5.1 and 6.1.

**Proposition 6.11.** Let $L \amalg K, L' \amalg K' \subset X$ be Lagrangian links that are component-wise isotopic. Then the stabilized Lagrangian links $L \times T^*_0 D^1 \amalg K \times T^*_0 D^1, L' \times T^*_0 D^1 \amalg K' \times T^*_0 D^1$ are isotopic as links in $X \times T^* D^1$.

**Proof.** First, the link $L \times T^*_0 D^1 \amalg K \times T^*_0 D^1$ is isotopic to $L \times T^*_0 D^1 \amalg K \times T^*_{1/2} D^1$ obtained by isotoping $T^*_0 D^1$ to $T^*_{1/2} D^1$ in $T^* D^1$ on the second component of the link, keeping $X$ coordinates fixed. Then we isotope $L \times T^*_0 D^1$ to $L' \times T^*_0 D^1$ and simultaneously isotope $K \times T^*_{1/2} D^1$ to $K' \times T^*_{1/2} D^1$, keeping the $T^* D^1$ coordinates fixed. This is an isotopy of Lagrangian links since the two component-wise isotopies occur at different cotangent fibers $T^*_0 D^1$ and $T^*_{1/2} D^1$. Finally, we isotope the link $L' \times T^*_0 D^1 \amalg K' \times T^*_{1/2} D^1$ back to $T^*_0 D^1$ in $T^* D^1$ on the second component, again keeping the $X$ coordinates fixed. □

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