Poisson Structures for Dispersionless Integrable Systems and Associated W-Algebras

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Abstract

In analogy to the KP theory, the second Poisson structure for the dispersionless KP hierarchy can be defined on the space of commutative pseudodifferential operators \( L = p^n + \sum_{j=-\infty}^{n-1} u_j p^j \). The reduction of the Poisson structure to the symplectic submanifold \( u_{n-1} = 0 \) gives rise to the \( w \)-algebras. In this paper, we discuss properties of this Poisson structure, its Miura transformation and reductions. We are particularly interested in the following two cases: a) \( L \) is pure polynomial in \( p \) with multiple roots and b) \( L \) has multiple poles at finite distance. The \( w \)-algebra corresponding to the case a) is defined as \( w_{[m_1,m_2,\cdots,m_r]} \), where \( m_i \) means the multiplicity of roots and to the case b) is defined by \( w(n,[m_1,m_2,\cdots,m_r]) \) where \( m_i \) is the multiplicity of poles. We prove that \( w(n,[m_1,m_2,\cdots,m_r]) \)-algebra is isomorphic via a transformation to \( w_{[m_1,m_2,\cdots,m_r]} \bigoplus w_{n+m} \bigoplus U(1) \) with \( m = \sum m_i \). We also give the explicit free fields representations for these \( w \)-algebras.
It is well-known that the second Poisson structures of integrable hierarchies give rise to the classical realizations of conformal W-algebras. The Miura transformation that transforms the second Poisson structures of systems to vastly simpler ones provides the free fields representations of correspondent W-algebras. (see[1-3] for reviews)

The purpose of this paper is to discuss some features dealing with the (second) Poisson structure for the dispersionless Kadomtsev-Petviashvili (dKP) hierarchy, its reductions and associated w-algebras. We are particularly interested in the reduced case that the Lax pair of the hierarchy have either multiple roots or multiple poles at finite distance.

The dKP hierarchy is known as the dispersionless limit of the (dispersionful) KP hierarchy. It is written in the following Lax form [4,5,6]

\[
\frac{\partial L}{\partial t_n} = \left[\left(\frac{L^n}{\partial t_n}\right)_+, L\right],
\]

where \( L \) is the commutative pseudodifferential operators (cΨDOs)

\[
L = p + \sum_{j=1}^{\infty} u_j p^{-j}
\]

and the subscript ”+” is defined as usual to take the polynomial part of a cΨDO. The double bracket in (1) is defined by

\[
\left[\left[A, B\right]\right] = \frac{\partial A}{\partial p} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial p},
\]

for any two cΨDOs \( A \) and \( B \) with coefficients being smooth functions of \( x \). Similar to the KP hierarchy, the dKP hierarchy admits some reductions. If \( L^n \) is restricted to be polynomial in \( p \) for a fixed integer \( n \), then dKP is reduced to the dispersionless Gelfand-Dickey hierarchy [6]. The Zakharov reduction is represented by the restriction that \( L^n \) has multiple poles at finite distance [7]

\[
L^n = p^n + \sum_{j=0}^{n-2} u_j p^j + \sum_{i=1}^{r} \sum_{j=1}^{m_i} a_{i,j} \frac{a_{i,j}}{(p + p_i)^j}
\]

and the associated system is called the extended Benney hierarchy.

The Poisson structures for dKP are built first on the space \( M_n \) of

\[
L = p^n + \sum_{j=-\infty}^{n-1} u_j p^j
\]

and then on the submanifold \( u_{n-1} = 0 \) by constraint. They can be formulated either by taking dispersionless limit of the Poisson structures for the KP hierarchy [8] or by a
dispersionless analogue of the Adler-Gelfand-Dickey construction \[4,5\]. In particular the second Poisson structure on the space \(M_n\) is given by

\[
\{\tilde{f}, \tilde{g}\}^{(n)} = \int \text{res}(\left[\left(\frac{\delta f}{\delta L}\right)_+, L\right][\left(\frac{\delta g}{\delta L}\right)_+, L\left(\frac{\delta g}{\delta L}\right)_-])dx
\]

\[
= \int \text{res}(\left[\left(\frac{\delta f}{\delta L}\right)_+, L\right][\left(\frac{\delta g}{\delta L}\right)_+, L\left(\frac{\delta g}{\delta L}\right)_-] + \left[\left(\frac{\delta g}{\delta L}\right)_-, L\right][\left(\frac{\delta f}{\delta L}\right)_-])dx, \tag{6}
\]

where \(L\) is in the form of (5), \(\tilde{f} = \int f(u)dx\) with \(f(u) = f(u_{n-1}, u_{n-2}, \cdots)\) being polynomial in \((u_{n-1}, u_{n-2}, \cdots)\). The variation \(\delta f/\delta L\) is defined by

\[
\frac{\delta f}{\delta L} = \sum_{j=-\infty}^{n-1} \frac{\partial f}{\partial u_j} p^{-j-1} \tag{7}
\]

and similarly for \(\tilde{g}\) and \(\delta g/\delta L\). The second equality in (6) was obtained by using

\[
\int \text{res}(\left[\left[A, B\right]\right]C)dx = \int \text{res}(\left[\left[B, C\right]\right]A)dx \tag{8}
\]

for arbitrary \(A, B\) and \(C\). If (6) is restricted to \(u_{n-1} = 0\), then the following condition \(4,5,8\)

\[
\text{res}[\left[\frac{\delta f}{\delta L}, L\right]] = 0 \tag{9}
\]

must to be taken into account. From (6), the Poisson brackets among fields \((u_{n-1}, u_{n-2}, \cdots)\) are expressed by

\[
\{u_i(x), u_j(y)\}^{(n)} = J^{(n)}_{ij}(u)\delta(x-y), i, j \leq n-1 \tag{10}
\]

which provides the \(\tilde{w}^{(n)}_{dKP}\)-algebras. The constraint \(u_{n-1} = 0\) is the second class one and the reduced brackets

\[
\{u_i(x), u_j(y)\}_D^{(n)} = \tilde{J}^{(n)}_{ij}(u)\delta(x-y), \ i, j \leq n-2 \tag{11}
\]

represent the \(\tilde{w}^{(n)}_{\infty}\)-algebra, where

\[
\tilde{J}^{(n)}_{ij} = J^{(n)}_{ij} - J^{(n)}_{i,n-1}(J^{(n)}_{n,n-1})^{-1}J^{(n)}_{n-1,j}. \tag{12}
\]

We will show both \(w^{(n)}_{dKP}\) and \(\tilde{w}^{(n)}_{\infty}\) are independent on \(n\).

In this paper, analogous to our previous construction for the KP hierarchy \[9,10\], We give a general Miura transformation by expressing \(L \in M_n\) into a factorization from \(L = L_1 L_2\) where \(L_j \in M_{n_j}\) and \(n_1 + n_2 = n\). As a result the second Poisson brackets (6) is decomposed into a summation of two brackets associated with \(L_1\) and \(L_2\) respectively. It
is then no difficulty to extend the above factorization to a rational from $L = L_1L_2^{-1}$, since in this case, we may think that $L_1$ is factorized as $L_1 = LL_2$. The rational factorization will give rise to the Miura transformation for the extended Benney hierarchy, which also leads to a decomposition of the associated second Poisson structure. In other words, we may decompose the correspondent $w$-algebra to a direct sum of others.

We emphasize that although in the generic case, the dKP and associated $w$-algebra can be obtained from the counterpart for the KP by taking dispersionless limit, as far as we know, there is no similar result for the systems, such as the extended Benney hierarchy, that the associated $c\Psi DO_L$ has multiple roots or multiple poles.

Let $L$ in (5) is factorized by

$$L = L_1L_2,$$

where $L_j = p^{n_j} + v_{j,n_j-1}p^{n_j-1} + \cdots, j = 1, 2$ are two $c\Psi DO$s and $n_1 + n_2 = n$. Compare the same powers of $p$ in both sides of (13), we obtain the Miura transformation

$$u_{n-1} = v_{1,n_1-1} + v_{2,n_2-1},$$

$$u_j = \sum_{j-n_1+1 \leq k \leq n_1-1} v_{1,k}v_{2,j-k}.$$  \hfill (14)

**Theorem 1** The factorization (13) leads to the following decomposition of the second Poisson bracket (6) associated with $L$ in (5)

$$\{\tilde{f}, \tilde{g}\}^{(n)} = \{\tilde{f}, \tilde{g}\}^{(n_1)} + \{\tilde{f}, \tilde{g}\}^{(n_2)},$$

and the constraint condition $u_{n-1} = v_{1,n_1-1} + v_{2,n_2-1} = 0$ is equivalent to

$$\text{res}[\frac{\delta f}{\delta L}, L] = \text{res}[\frac{\delta f}{\delta L_1}, L_1] + \text{res}[\frac{\delta f}{\delta L_2}, L_2] = 0.$$  \hfill (16)

Proof: First we can express $\delta f/\delta L_j$ in terms of $\delta f/\delta L$

$$\frac{\delta f}{\delta L_1} = \frac{\delta L}{\delta L_2} \frac{\delta f}{\delta L_2}, \quad \frac{\delta f}{\delta L_2} = \frac{\delta f}{\delta L} L_1.$$  \hfill (17)

They can be derived by using the Miura transformation (14)

$$\frac{\partial f}{\partial v_{1k}} = \sum_j \frac{\partial f}{\partial u_j} \frac{\partial u_j}{\partial v_{1k}} = \sum_j \frac{\partial f}{\partial u_j} v_{2,j-k},$$

so

$$\frac{\delta f}{\delta L_1} = \sum_k \frac{\partial f}{\partial v_{1k}} p^{-k-1} = \sum_k \sum_j \frac{\partial f}{\partial u_j} v_{2,j-k}p^{-k-1} = \frac{\delta f}{\delta L} L_2.$$  \hfill (18)
and similarly for the second expression in (17). Then by using the Leibniz rule
\[
[[A, BC]] = [[A, B]]C + [[A, C]]B
\]
(19)
of the double square bracket (3), we have
\[
\begin{align*}
&\left(\left[\frac{\delta f}{\delta L_1}, L_1\right] + \left[\frac{\delta f}{\delta L_2}, L_2\right]\right) \frac{\delta g}{\delta L_1} \\
&= \left(\left[\frac{\delta f}{\delta L_1}, L_1\right]L_2 + \left[\frac{\delta f}{\delta L_2}, L_2\right]L_1\right) \frac{\delta g}{\delta L} \\
&= \left[\frac{\delta f}{\delta L}L, L\right] \frac{\delta g}{\delta L}
\end{align*}
\]
which immediately implies (15). The equation in (16) can also be derived by using (17), i.e.
\[
\left[\frac{\delta f}{\delta L_1}, L_1\right] + \left[\frac{\delta f}{\delta L_2}, L_2\right] = \left[\frac{\delta f}{\delta L}, L_1\right] + \left[\frac{\delta f}{\delta L}, L_2\right] = \left[\frac{\delta f}{\delta L}, L\right].
\]
(20)
Thus we complete the proof.

**Corollary 1** If \( L \) is factorized in the following form
\[
L = L_1 L_2 \cdots L_r,
\]
(21)
with \( L_j \) being cΨDOs of order \( n_j \), then
\[
\{\tilde{f}, \tilde{g}\}_{(n)} = \sum_{j=1}^{r} \{\tilde{f}, \tilde{g}\}_{(n_j)}
\]
(22)
and the constraint condition \( u_{n-1} = 0 \) is given by
\[
\text{res}\left[\frac{\delta f}{\delta L}, L\right] = \text{res} \sum_{j=1}^{r} \left[\frac{\delta f}{\delta L_j}, L_j\right] = 0.
\]
(23)

**Corollary 2** If \( L \) is factorized in the rational form
\[
L = L_1 L_2^{-1},
\]
(24)
where \( L_1 \) and \( L_2 \) are \((n + m)^{th}\)-order and \((m)^{th}\)-order polynomials respectively
\[
\begin{align*}
L_1 &= p^{n+m} + v_{n+m-1}p^{n+m-1} + \cdots + v_0; \\
L_2 &= p^m + w_{m-1}p^{m-1} + \cdots + w_0,
\end{align*}
\]
(25)
than we have
\[
\{\tilde{f}, \tilde{g}\}_{(n)} = \{\tilde{f}, \tilde{g}\}_{(n+m)} - \{\tilde{f}, \tilde{g}\}_{m}.
\]
(26)
Theorem 2 If $L$ is the $k^{th}$ power of the ΨDO $L_1$ of order $l$

$$L = L_1^k,$$  \hfill (27)

then we have

$$\{\tilde{f}, \tilde{g}\}^{(kl)} = \frac{1}{k} \{\tilde{f}, \tilde{g}\}^{(l)}. \hfill (28)$$

Proof: Similar to the proof of Theorem 1, we first have

$$\frac{\delta f}{\delta L_1} = k \frac{\delta f}{\delta L} L_1^{k-1} \hfill (29)$$

By substituting this expression into the right hand side of (26), we can derive (28).

Remark 1 Theorem 1 is the dispersionless analogue of our previous result for the KP hierarchy [9,10] but the proof is simplified. The rational factorization and their resulted decomposition formula of the second Poisson structure for the KP hierarchy was particularly derived in [11], which can also be, however, considered as the consequence of our previous result in [9,10].

Remark 2 If we choose $L_1 = p + v_0 + v_1 p^{-1} + \cdots$, then (26) implies that the second Poisson structure associated with $L_1^k$ for any integer $k$ is proportional to that associated with $L_1$. In other words, $w^{(k)}_{dKP}$-algebra is essentially independent on the value of $k$ [12]. There is no such an analogue for the KP hierarchy and $W^{(k)}_{KP}$-algebra, since we known that in the KP theory, the second Poisson structures associated with $L_1^k$ (here $L_1$ is the non-commutative ΨDOs) are not compatible with different values of $k$ [13].

In the following, we give some applications of the above general theory. For the simplicity, we use notation $\{\ , \ \}$ instead of the Poisson bracket $\{\ , \ \}^{(n)}$ associated with generic cΨDO $L$.

We first consider the Poisson structure and associated $w$-algebra on the space of

$$L = p^m + \sum_{j=1}^{m-1} u_j p^j = \prod_{j=1}^{r} (p + p_j)^{m_j}, \hfill (30)$$

where

$$m = \sum_{j=1}^{r} m_j, \hfill (31)$$

namely we assume that $L$ is a polynomial with multiple roots. Notice that the Poisson structure (6) for $L = p + p_j$ is simply $\partial_x$, thus by applying Corollary 2 and Theorem 2, the Poisson brackets among $p_j$ are

$$\{p_i(x), p_j(y)\} = \frac{1}{m_i} \delta_{ij} \delta'(x - y). \hfill (32)$$
The constraint \( u_{m-1} = \sum_{j=1}^{r} m_j p_j = 0 \) is the second type and the reduced Poisson brackets are

\[
\{ p_i(x), p_j(y) \} = \left( \frac{1}{m_i} \delta_{ij} - \frac{1}{m} \right) \delta'(x - y). \tag{33}
\]

Since (30) provides expressions of \( u_j \) in terms of \( p_j \), therefore from (32) one may derive the Poisson brackets among \( u_{m-2}, \ldots, u_0 \), which is defined as \( w_{[m_1, \ldots, m_r]} \)-algebra. If \( r = m \) and all \( m_j = 1 \), then we recover \( w_m \)-algebra \([12]\), i.e. \( w_{[1, \ldots, 1]} = w_m \).

To present the free fields realization of \( w_{[m_1, \ldots, m_r]} \)-algebra, we introduce \( r - 1 \) free fields

\[
\phi = (\varphi_1, \ldots, \varphi_{r-1}), \tag{34}
\]

with Poisson bracket

\[
\{ \varphi'_i(x), \varphi'_j(y) \}_D = \delta_{ij} \delta'(x - y), \tag{35}
\]

and an overcomplete set of vectors \( \mathbf{h}_j, \ j = 1, 2, \ldots, r \) in \((r - 1)\)-dimensional Euclidean space with

\[
\sum_{j=1}^{r} m_j \mathbf{h}_j = 0, \quad \mathbf{h}_i \cdot \mathbf{h}_j = \left( \frac{1}{m_i} \delta_{ij} - \frac{1}{m} \right). \tag{36}
\]

Such vectors can be written exactly, they are

\[
\begin{align*}
\mathbf{h}_1 & = \left( \frac{m_2}{m_1(m_1 + m_2)}, \frac{m_3}{(m_1 + m_2)(m_1 + m_2 + m_3)} \right), \ldots, \left( \frac{m_r}{(\sum_{i=1}^{r-1} m_i)(\sum_{i=1}^{r} m_i)} \right); \\
\mathbf{h}_2 & = \left( \frac{-m_1}{m_2(m_1 + m_2)}, \frac{m_3}{(m_1 + m_2)(m_1 + m_2 + m_3)} \right), \ldots, \left( \frac{m_r}{(\sum_{i=1}^{r-1} m_i)(\sum_{i=1}^{r} m_i)} \right); \\
\mathbf{h}_3 & = \left( 0, \frac{-m_1 + m_2}{m_3(m_1 + m_2)} \right), \ldots, \left( \frac{m_r}{(\sum_{i=1}^{r-1} m_i)(\sum_{i=1}^{r} m_i)} \right); \\
\vdots & \quad \vdots \\
\mathbf{h}_j & = \left( 0, \ldots, 0, \frac{-\sum_{i=1}^{j-1} m_i}{m_j \sum_{i=1}^{j} m_i} \right), \ldots, \left( \frac{m_r}{(\sum_{i=1}^{r-1} m_i)(\sum_{i=1}^{r} m_i)} \right); \\
\vdots & \quad \vdots \\
\mathbf{h}_r & = \left( 0, \ldots, 0, \frac{-\sum_{i=1}^{r-1} m_i}{(m_r \sum_{i=1}^{r} m_i)} \right).
\end{align*} \tag{37}
\]

Then using the identification

\[
p^m + \sum_{j=1}^{m-2} u_j p^j = \prod_{j=1}^{r} (p + \mathbf{h}_j \cdot \phi')^{m_j}. \tag{38}
\]
we obtain the free fields realization of the $w_{[m_1,\ldots,m_r]}$-algebra. The particular expression $u_{m-2}$ is straightforward to derive from (38),

$$u_{m-2} = -\frac{1}{2} \sum_{j=1}^{r} m_j (h_j \cdot \phi')^2. \quad (39)$$

It satisfies

$$\{u_{m-2}(x), u_{m-2}(y)\}_D = -(u_{m-2}(x) \partial + \partial u_{m-2}(x)) \delta(x - y). \quad (40)$$

Next we consider the Poisson structure on the space of

$$L = p^n + \sum_{j=0}^{n-1} u_j p^j + \sum_{i=1}^{r} \sum_{j=1}^{m_i} \frac{a_{ij}}{(p + p_i)^j}, \quad (41)$$

and then take the reduction $u_{n-1} = 0$, where

$$\sum_{j=1}^{r} m_j = m, \quad (42)$$

and $p_j$ are distinct poles. Such type of $L$ can be written as

$$L(p) = L_1(p)L_2^{-1}(p), \quad (43)$$

with

$$L_1(p) = \prod_{j=1}^{n+m} (p + q_j); \quad L_2(p) = \prod_{j=1}^{r} (p + p_j)^{m_j}, \quad (44)$$

here for the simplicity we assume all $q_j$ are distinct. By the application of Corollary 2 and results associated with (29), we find

$$\{p_i, p_j\} = -\frac{1}{m_i} \delta_{ij} \delta'; \quad (45)$$

$$\{q_k, q_l\} = \delta_{kl} \delta'; \quad \{p_i, q_k\} = 0, \quad 1 \leq i, j \leq r, 1 \leq k, l \leq n + m.$$

The constraint

$$u_{m-1} = \sum_{k=1}^{n+m} q_k - \sum_{j=1}^{r} m_j p_j = 0 \quad (46)$$

is the second type and the reduced brackets are given by

$$\{p_i, p_j\}_D = -(\frac{1}{m_i} + \frac{1}{n}) \delta'; \quad (47)$$

$$\{q_k, q_l\}_D = (\delta_{kl} - \frac{1}{n}) \delta'; \quad \{p_i, q_k\}_D = -\frac{1}{n} \delta'.$$
for $1 \leq i, j \leq r, \ 1 \leq k, l \leq n+m$. We denote the associated w-algebra by $w(n, [m_1, \cdots, m_r])$.

**Theorem 3** The $w(n, [m_1, \cdots, m_r])$-algebra is isomorphic via a transformation to the direct sum of a $w_{[m_1, \cdots, m_r]}$-algebra, a $w_{n+m}$-algebra and a $U(1)$ current algebra.

*Proof:* Let

$$J = \frac{2}{m(m+n)} \sum_{i=1}^{r} m_i \bar{p}_i = \frac{2}{m(m+n)} \sum_{k=1}^{m+n} q_k;$$

$$\bar{p}_i = p_i - \frac{m+n}{2} J, \ 1 \leq i \leq r; \quad (48)$$

$$\bar{q}_k = q_k - \frac{m}{2} J, \ 1 \leq k \leq m+n.$$  

One can check that

$$\sum_{i=1}^{r} m_i \bar{p}_i = 0, \ \sum_{k=1}^{m+n} \bar{q}_k = 0. \quad (49)$$

The Poisson brackets among these new fields are given by

$$\{J, J\}_D = -\left(\frac{4}{mn(n+m)}\right) \delta';$$

$$\{\bar{p}_i, \bar{p}_j\}_D = -\left(\delta_{ij} - \frac{1}{m}\right) \delta', \ 1 \leq i, j \leq r; \quad (50)$$

$$\{\bar{q}_k, \bar{q}_l\}_D = \left(\delta_{kl} - \frac{1}{n+m}\right) \delta', \ 1 \leq k, l \leq n+m,$$

while the three groups of fields $\bar{p}_i$, $\bar{q}_k$ and $J$ mutually commute. Let

$$L_1(p) = \prod_{k=1}^{n+m} (p + \bar{q}_k) = L_1(p - \frac{m}{2} J);$$

$$\tilde{L}_2(p) = \prod_{i=1}^{r} (p + \bar{p}_i)^{m_i} = \tilde{L}_2(p - \frac{m+n}{2} J). \quad (51)$$

Thus the Poisson structures associated with $\tilde{L}_1$ and $\tilde{L}_2$ give rise to $w_{n+m}$ and $w_{[m_1, \cdots, m_r]}$ algebras respectively. Together with $U(1)$ current $J$, they indepent to each other. The transformation between original fields $(p_i, u_l, a_{ij})$ and new fields $(\bar{p}_i, \bar{q}_k, J)$ are given by the identity

$$L(p) = L_1(p + \frac{m}{2} J) L_2^{-1}(p + \frac{m+n}{2} J). \quad (52)$$

According to Theorem 3, the free fields representation of $w(n, [m_1, \cdots, m_r])$-algebra can be formulated by using the free fields representations of $w_{n+m}$ and $w_{[m_1, \cdots, m_r]}$ algebras.
Corollary 3 If \( r = m, m_i = 1 \), then \( w(n, [1, 1, \ldots, 1]) = w(n, m) \) is nothing but the classical limit of \( W(n, m) \)-algebra defined in [14], and is isomorphic to \( w_{n+m} \oplus w_m \oplus U(1) \) via a transformation. The last result is an analogue of that for \( W(n, m) \)-algebra [14].

Remark 3 The proof of Theorem 3 is to construct an explicit transformation. Similar transformation can also be constructed for the \( w(n, m) \)-algebra such that the result of [14] that \( W(n, m) \cong W_{n+m} \oplus W_m \oplus U(1) \) can be proved straightforward by this transformation.

In conclusion, we have discussed the Poisson structures for the dispersionless systems and associated \( w \)-algebra. We are particularly interested in the case that the \( n^{th} \)-order polynomial \( L \) has multiple roots or the \( c\Psi DO \) \( L \) has multiple poles at finite distance, and we gave the so-called \( w_{[m_1, \ldots, m_r]} \) or \( w(n, [m_1, \ldots, m_r]) \) algebras and their free fields representations.

In [7], a dispersionful analogue of the Benney hierarchy associated with \( c\Psi DO \) \( L \) that has simple poles at finite distance (i.e. \( r = 1 \) and \( m_i = 1 \) in (42)) was constructed. This dispersionful system can be identified with the multicomponent generalization of the constrained KP hierarchy [15]. It would be very interest to see whether there exists dispersionful analogue of the extended Benney hierarchy associated with \( L \) having multiple poles. In other words, we would ask whether there exists a \( W \)-algebra that take our \( w(n, [m_1, \cdots, m_r]) \) as its dispersionless limit.

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