Counting paths in Bratteli diagrams for SU(2)k

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Abstract – It is known that the Hilbert space dimensionality for quasiparticles in an SU(2)k
Chern-Simons-Witten theory is given by a number of directed paths in certain Bratteli diagrams.
We present an explicit formula for these numbers for arbitrary k. This is on the basis of a relation
with Dyck paths and Chebyshev polynomials.

Introduction. – The so-called Bratteli diagrams have been introduced by Bratteli in 1972 [1] for the classification of some classes of C∗-algebras. For our scope, it is sufficient to give here an “algebra-free” description of this notion (see, e.g., [2] or [3]). More specifically, lattice points will be labeled by numbers instead of Young diagrams. A Bratteli diagram for SU(2)k is defined as a finite digraph Dk = (V, A) with set of vertices V and set of arcs A, where

- The vertices of Dk are associated to lattice points in the positive quadrant of the Cartesian plane:
  \[(j, i) \in (\mathbb{Z}^{>0})^2 \quad \text{with} \quad j \geq i.\]

The vertex set V is \{(j, i) : 0 \leq i \leq j; i and j have the same parity\}. Here 0 is assumed to be even (note that when \(j = 0\), we have only one vertex in V, namely \((0, 0)\)).

- There is an arc \((j, i), (j', i')\) in A if \(j' = j + 1\) and \(i' = i \pm 1\).

- A directed path from a vertex \((j, i)\) to a vertex \((j', i')\) of length n is a sequence of n arcs of the form \((j, i), (j + 1, l_1), \ldots, (j + n - 1, l_{n-1}), (j', i')\). Let \(D_k(x, y)\) be the number of directed paths from the vertex \((0, 0)\) to the vertex \((j, i)\) in the Bratteli diagram \(D_k\). In a Bratteli diagram \(D_k\), the vertex \((j, i)\) is labeled by the number \(D_k(i, j)\). In our notation, the function \(f\) introduced in the general

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\[\text{Fig. 1: Bratteli diagram } D_2.\]

\[\text{definition of a Bratteli diagram is exactly } D_k(i, j).\]

Notice that

\[D_k(i, j) = D_k(i - 1, j - 1) + D_k(i + 1, j - 1)\]

with the initial conditions \(D_k(0, j) = D_k(1, j - 1)\) and \(D_k(k, j) = D_k(k - 1, j - 1)\).

Let

\[d^+(i) = |\{j : (i, j) \in A\}|\]

and

\[d^-(i) = |\{j : (j, i) \in A\}|\]

be the indegree and the outdegree of a vertex \(i\), respectively. Notice that \(d^+(0, 0) = 1, d^-(0, 0) = 0,\) and \(d^+(0, j) = d^+(k, i) = d^-(l, l) = 1\). All other vertices have indegree and outdegree 2. Figure 1 illustrates the Bratteli diagram \(D_2\).

The number \(D_k(i, j)\) corresponds to the Hilbert space dimensionality for \(i\) q-spin 1/2 quasiparticles having total q-spin \(j\) in an SU(2)k Chern-Simons-Witten theory (see, e.g., [4] and the references therein). An asymptotic expression and a recurrence relation for \(D_k(i, j)\) have been
pointed out by Slingerland and Bais (see [5], sect. 2.4). More precisely, they showed that

\[ D_k(i, j) = \left(2 \cos \left(\frac{k}{k+2} \right) \right)^j, \]

where \( i + j = 0 (\text{mod} 2) \). (From the definitions we have that \( D_k(i, j) = 0 \) for all \( i + j = 1 (\text{mod} 2) \).) We present here an explicit formula for \( D_k(i, j) \), which implies the asymptotic expression in (1). In particular, we find an explicit formula for the total number of directed paths from the vertex \((0, 0)\) to the vertex \((i, j)\) in the Bratteli diagram \( D_k \), for any \( k \), as shown in the next section. Our tools will be Dyck paths and Chebyshev polynomials.

Counting paths in Bratteli diagrams is equivalent to obtaining the number of ways in which a given number of spin fields can be fused to give a field in the parafermion sector. This is because the braiding for a system of identical particles with hidden quantum group symmetry is described in terms of a basis that is labeled by the paths on the Bratteli diagram of the group representation carried by the particles. In our scenario, the number of elements of this basis corresponds to independent quasihole states. This gives, for instance, the dimensions of the braid group representations that govern the exchanges of the electrons and the quasiholes of Read-Rezayi states (see [6,7]).

**Main result.** — Chebyshev polynomials of the second kind are defined by

\[ U_r (\cos \theta) = \frac{\sin((r+1)\theta)}{\sin \theta}, \]

for \( r \geq 0 \). Evidently, \( U_r(x) \) is a polynomial of degree \( r \) in \( x \) with integer coefficients. Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, number theory, and the study of lattice paths in combinatorics (see [8]). For \( k \geq 0 \), we define \( R_k(x) \) by

\[ R_k(x) = \frac{U_{k-1} \left( \frac{1}{2} \right)}{2 U_k \left( \frac{1}{2} \right)}. \]

For example, \( R_0(x) = 0, R_1(x) = 1, \) and \( R_2(x) = 1/(1-x) \). It is easy to see that for any \( k \), \( R_k(x) \) is a rational function in \( x \).

A **Dyck path** is a lattice path in the plane integer lattice \((\mathbb{Z}^{\geq 0})^2\) consisting of up-steps \( u = (1, 1) \) and down-steps \( d = (1, -1) \). It follows that a Dyck path never passes below the \( x \)-axis. The length of a Dyck path is defined as the number of its up-steps and down-steps. From the definitions we can state the following observation.

**Observation 1** The number \( D_k(i, j) \) is exactly the number of Dyck paths below the line \( y = k + 1 \), starting at the origin and ending at the point with \( x \)-coordinate \( j \) and \( y \)-coordinate \( i \).

Each Dyck path \( P \) below the line \( y = k + 1 \), starting at the origin and ending at \((j, i)\) has the following form:

\[ P = P_1 u P_2 u P_3 \cdots u P_{\ell + 1}, \]

where \( P_s \) is a Dyck path of height at most \( k + 1 - s \), and the length of \( P \) is exactly \( j \) (each up-step and down-step is counted as a unit step).

Let us fix a variable \( x \) for the generating function for counting the number of up-steps and down-steps in a Dyck path. Using the fact that the generating function for the number of Dyck paths of length \( 2n \) with height at most \( s \) is given by \( R_{s+1}(x^2) \) (see [9]), we get that the generating function

\[ D_k(x; i) = \sum_{j \geq 0} D_k(i, j) x^j \]

is given by

\[ D_k(x; i) = R_{k+1}(x^2) \prod_{r=1}^{i} (x R_{k+1-r}(x^2)), \]

which is equivalent to

\[ D_k(x; i) = x^i \prod_{r=0}^{i-1} \frac{U_{k-r} \left( \frac{2r}{2k+2} \right)}{x U_{k+1-r} \left( \frac{2r}{2k+2} \right)} = \frac{U_{k-i} \left( \frac{2}{2k+2} \right)}{x U_{k+1} \left( \frac{2}{2k+2} \right)}. \]

Using the fact that the roots of \( U_k(x) \) are

\[ \cos \left( \frac{r \pi}{k+1} \right), \quad r = 1, 2, \ldots, k, \]

we obtain that the minimal positive pole of the function \( D_k(x; i) \) is given by \( \left(2 \cos \left( \frac{\pi}{k+2} \right) \right)^{-1} \). Therefore, the asymptotic behavior of the function \( D_k(i, j) \) is given by

\[ D_k(i, j) \approx \left(2 \cos \left( \frac{\pi}{k+2} \right) \right)^j, \]

where \( i + j = 0 (\text{mod} 2) \), as described in [5], eq. (26).

In order to find an explicit formula for \( D_k(i, j) \) we need the following lemma.

**Lemma 2** The generating function

\[ \frac{U_{k-i}(x)}{U_{k+1}(x)} \]

is given by

\[ \frac{1}{k+2} \sum_{r=1}^{k+1} (-1)^{r+1} U_{k-r}(\rho_{k+1,r}) \sin^2 \frac{r \pi}{k+2}, \]

where \( \rho_{m,r} = \cos \left( \frac{r \pi}{m+1} \right) \).

**Proof.** Let us compute the partial fraction decomposition of

\[ \frac{U_{k-i}(x)}{U_{k+1}(x)}. \]
By general principles, it is

$$a_{k+1,r} = \frac{U_{k+1}(\rho_{k+1,r})}{U'_{k+1}(x)_{\rho_{k+1,r}}}$$

where $\rho_{m,r} = \cos\left(\frac{\pi r}{m+1}\right)$ are the zeros of the $m$-th Chebyshev polynomials of the second kind. Now,

$$0 < \rho_{m,r} \leq 1$$

and we obtain

$$W_k = \frac{dU'_{k+1}(x)}{dx} = \frac{d}{dx} \left(\frac{\sin(k+2)\theta}{\sin\theta}\right),$$

We work out that

$$\frac{dU_{k+1}}{d\theta} = (k+2) \cos(k+2)\theta \cdot \frac{\sin(k+2)\theta \cdot \cos\theta}{\sin^2\theta},$$

and if we plug in $x = \rho_{k+1,r}$, simplification occurs, since certain terms are just zero; we obtain that

$$\frac{dU_{k+1}}{d\theta}(\arccos \rho_{k+1,r}) = (k+2) \cos(k+2)\theta \cdot \frac{\sin(k+2)\theta \cdot \cos\theta}{\sin^2\theta}$$

$$= (k+2) \cos(\pi r) \cdot \frac{\sin \frac{\pi r}{k+2}}{\sin^2 \frac{\pi r}{k+2}}$$

$$= \frac{(k+2) \cos(\pi r) \cdot \sin \frac{\pi r}{k+2}}{\sin^2 \frac{\pi r}{k+2}}$$

Further $\frac{d}{d\theta} = -\sin\theta$, so together

$$a_{k+1,r} = \frac{U_{k+1}(\rho_{k+1,r})}{U'_{k+1}(\rho_{k+1,r})}$$

$$= \frac{1}{k+2} (-1)^{r+1} U_{k-1}(\rho_{k+1,r}) \sin^2 \frac{\pi r}{k+2},$$

which completes the proof.

Now we are ready to give an explicit formula for $D_k(i,j)$.

**Theorem 3** For all $k \geq i$ and $j \geq i$, the number $D_k(i,j)$ of directed paths from the vertex $(0,0)$ to the vertex with $x$-coordinate $j$ and $y$-coordinate $i$ in the Bratteli diagram $D_k$ is given by

$$D_k(i,j) = \frac{1}{k+2} \sum_{r=1}^{k+1} (-1)^{r+1} U_{k-1}(\rho_{k+1,r}) \sin^2 \left(\frac{\pi r}{k+2}\right)(2\rho_{k+1,r})^j,$$

where $\rho_{m,r} = \cos\left(\frac{\pi r}{m+1}\right)$.
Using the continued-fraction representation of the generating function $R_k(x)$, namely

$$R_k(x) = \frac{1}{1 - \frac{x}{1 - \frac{x}{\ddots}}}$$

with exactly $k$-levels (see [11], Lemma 3.1), we obtain that

$$R_\infty(x) = \lim_{k \to \infty} R_k(x) = C(x) := \frac{1}{1 + \sqrt{1 - 4x}}.$$ 

Hence, from (2) we obtain that the generating function

$$D_\infty(x; i) = \lim_{k \to \infty} D_k(x; i)$$

is given by

$$D_\infty(x; i) = x^i C^{i+1}(x^2).$$

The generating function $C(x)$ satisfies

$$C(x) = 1 + x C^2(x),$$

thus by the Lagrange inversion formula ([12], sect. 5.4) we obtain that

$$D_\infty(i, j) = \frac{i + 1}{j + 1} \left( \frac{j + 1}{2} \right),$$

where $i + j = 0 \pmod{2}$ (as shown in [5], eq. (21)).

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