Lectures on Non Perturbative Field Theory and Quantum Impurity Problems: Part II.

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Abstract

These are notes of lectures given at The NATO Advanced Study Institute/EC Summer School on “New Theoretical Approaches to Strongly Correlated Systems”, (Newton Institute, April 2000). They are a sequel to the notes I wrote two years ago for the Summer School, “Topological Aspects of Low Dimensional Systems”, (Les Houches, July 1998). In this second part, I review the form-factors technique and its extension to massless quantum field theories. I then discuss the calculation of correlators in integrable quantum impurity problems, with special emphasis on point contact tunneling in the fractional quantum Hall effect, and the two-state problem of dissipative quantum mechanics.

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Introduction

I pick up the discussion of quantum impurity problems where it is left in my Lecture Notes form the 1998 Les Houches Summer School \cite{1} - consult these notes for the necessary physical and technical background. I have so far explained how integrability provides a convenient basis of quasiparticles which scatter nicely among one another and at the impurity. I have showed how to use this basis to compute DC transport properties with a combination of thermodynamic Bethe ansatz and Landauer type arguments.

To proceed, I now would like to discuss correlation functions.

It is a natural idea to compute correlators using the same basis of quasiparticles, and inserting completness relations in the Green functions of interest. In this approach, the key ingredient is the matrix elements of the physical operators. Despite the “almost free” structure of the DC transport properties, a rather unpleasant surprise is met here however: these matrix elements are all non zero in general, that is, physical operators (eg the current) are able to create/destroy arbitrarily large numbers of particles when they act on an arbitrary state.

These matrix elements - usually called form-factors - turn out to be a major topic \cite{2, 3, 4} in recent developments (for even more recent works, see \cite{5}) about integrable quantum field theories, and are known for a large variety of integrable masssive quantum field theories.

The idea to compute correlators in integrable quantum impurity problems is first to extend the form-factors approach to the case of theories which are massless in the bulk. The next step is to take the impurity (boundary) interaction into account properly, and finally, to sum over all the intermediate states to obtain physical correlations. The last step is the most difficult, and cannot be done analytically for the moment. Fortunately, and somewhat surprisingly, it turns out that the sums over intermediate states converge very quickly, everywhere along the trajectory between the UV and IR fixed points. As a result, the form-factors approach gives extremely accurate results (and with a controlled accuracy) for most correlators of interest.

In some cases, the form-factors expansion is plagued with an “infrared catastrophe” due to the proligeration of soft modes in the massless scattering description. This problem can usually be controlled by simple means, and meaningful correlators still obtained.

The main caveat of the form-factors approach is that it seems badly behaved when one approaches isotropic limits: as a result, although correlators for the anisotropic Kondo model up to $g = 3/4$ or so are now determined, the Kondo limit is still unaccessible. This is not much of a problem in applications to the fractional quantum Hall effect, where the interesting filling fractions are far from the isotropic limit of the underlying field theories.

1 Some generalities on form-factors

I follow here the notations and conventions of \cite{1}, in particular section 5 of the latter notes.

The space of states is generated by the vectors

$$|\alpha_1, \ldots, \alpha_n\rangle_{a_1, \ldots, a_n} = Z_{a_1}^\dagger(\alpha_1) \ldots Z_{a_n}^\dagger(\alpha_n)|0\rangle$$

and the dual space by

$$\langle a_n, \ldots, a_1|\alpha_n, \ldots, \alpha_1\rangle = \langle 0| Z_{a_n}(\alpha_n) \ldots Z_{a_1}(\alpha_1)$$

Here, the labels $a$ stand for instance for solitons/antisolitons ($\pm$) and breathers in the sine-Gordon model, $\alpha$ denotes the rapidity, and the $Z, Z^\dagger$ are the annihilation creation operators of the Faddeev Zamolodchikov algebra \cite{1}. For a given set of rapidities and quantum numbers, the states with different orderings are related through S matrix elements. One usually call “in” states the states for which $\alpha_1 > \alpha_2 \ldots > \alpha_n$, and “out” states those for which $\alpha_1 < \alpha_2 \ldots < \alpha_n$. In or out states form a complete basis of the set of states, but it is convenient to keep some redundancy and consider all possible rapidity orderings, after division by the appropriate degeneracy factors.
Form-factors (more correctly, “generalized form-factors” since no order of the rapidities is prescribed) of an operator $O$ in a bulk theory are defined as:

$$f(\alpha_1, ..., \alpha_n)_{a_1, ..., a_n} = <0|O(0,0)|Z_{a_1}^1(\alpha_1) \cdots Z_{a_n}^1(\alpha_n)|0>$$ (3)

where $|0>$ is the ground state. We chose to take the operator at the origin, since relativistic invariance determines trivially the dependence of the matrix elements on the coordinates of insertion (we set $\hbar = 1$)

$$<0|O(x,t)|Z_{a_1}^1(\alpha_1) \cdots Z_{a_n}^1(\alpha_n)|0> = e^{i(Px-Et)}f(\alpha_1, ..., \alpha_n)_{a_1, ..., a_n}$$ (4)

$S$ matrices have in many cases been obtained by explicit solution of properly regularized quantum field theories. The strategy, which was implemented quite successfully in the massive Thirring model for instance [3], is to find out eigenstates, fill the ground state, determine the possible excitations over it, and finally compute their scattering, which is the result of a combination of the bare scattering and “dressing” effects coming from interaction with the ground state. Needless to say, the procedure is laborious, and it has proven much faster to obtain the $S$ matrices in a more abstract way, using general axioms of $S$ matrix theory, factorizability, and assumptions of maximal analyticity [3, 4, 6].

It is even harder to obtain the matrix elements from the solution of bare theories. A program to do this is under way [6], but very few results of interest for our problem have been obtained so far. Form-factors can however be determined (though not quite as easily as $S$ matrices) using an axiomatic approach similar in spirit to what was done for the $S$ matrices, and this is what we would like to discuss briefly here.

One of the consequences of integrability is that multiple particle processes can be reexpressed in terms of two-particle ones. The basic objects in this approach are therefore those involving only a pair of particles. To start, consider the two particle $S$ matrix: by relativistic invariance, it is expected to depend only on the Mandelstam variable

$$s = (p_a(\alpha_1) + p_b(\alpha_2))^2 = M_1^2 + M_2^2 + 2M_1M_2 \cosh(\alpha_1 - \alpha_2)$$ (5)

The function $S(s)$ is expected to be an analytic function of $s$, with a cut along the real axis running from $s = (M_1 + M_2)^2$ to $s = \infty$, and another running from $s = -\infty$ to $s = (M_1 - M_2)^2$. The exchange of in and out states corresponds to exchanging the upper lip of the right cut Im $s = 0+$ with the lower lip of the right cut Im $s = 0-$.

This plane with the cuts transforms into the “physical” strip $0 \leq \text{Im} \alpha \leq \pi$ in terms of the variable $\alpha = \alpha_1 - \alpha_2$. The line Im $\alpha = 0$ corresponds to the right cut, while Im $\alpha = \pi$ corresponds to the left cut. Upper and lower lips correspond to Re $\alpha > 0$, resp. Re $\alpha < 0$.

In terms of the $\alpha$ variable, the standard properties of the $S$ matrix are:

- $S_{a_1; a_2}^{a_1; a_2}(\alpha) = S_{a_2; a_1}^{a_2; a_1}(\alpha)$ C
- $S_{a_1; a_2}^{a_1; a_2}(\alpha) = S_{b_1; b_2}^{a_1; a_2}(\alpha)$ P
- $S_{a_1; a_2}^{a_1; a_2}(\alpha) = S_{a_2; a_1}^{a_1; a_1}(\alpha)$ T

(6)

and

- $S(\alpha)$ is real for $\alpha$ purely imaginary
- Unitarity $S_{a_1; a_2}^{b_1; b_2}(\alpha)S_{b_1; b_2}^{c_1; c_2}(-\alpha) = \delta_{a_1}^{c_1} \delta_{a_2}^{c_2}$
- Crossing $S_{a_1; a_2}^{b_1; b_2}(\alpha) = S_{a_2; a_1}^{b_1; b_2}(i\pi - \alpha)$

(7)

In addition of course, the $S$ matrix is a solution of the Yang Baxter equation, which reads in components

$$S_{a_1; a_2}^{b_1; b_2}(\alpha_1 - \alpha_2)S_{b_1; a_3}^{c_1; c_2}(\alpha_1 - \alpha_3)S_{b_2; b_3}^{c_2; c_3}(\alpha_2 - \alpha_3) = S_{a_2; a_3}^{b_2; b_3}(\alpha_2 - \alpha_3)S_{a_1; c_3}^{b_1; c_2}(\alpha_1 - \alpha_3)S_{b_1; b_2}^{c_1; c_2}(\alpha_1 - \alpha_2)$$

(8)

[1]
The space of states is conveniently built using the Faddeev Zamolodchikov algebra

\[
Z_{a_1}(\alpha_1)Z_{a_2}(\alpha_2) = S_{a_1a_2}^{a_1a_2} (\alpha_1 - \alpha_2) Z_{a_2}^{a_1} (\alpha_2) Z_{a_1}^{a_1} (\alpha_1)
\]

\[
Z_{a_1}^{\dagger}(\alpha_1)Z_{a_2}^{\dagger}(\alpha_2) = S_{a_1a_2}^{a_1a_2} (\alpha_1 - \alpha_2) Z_{a_2}^{\dagger} (\alpha_1) Z_{a_1}^{\dagger} (\alpha_1)
\]

\[
Z_{a_1}^{a_1}(\alpha_1)Z_{a_2}^{a_1}(\alpha_2) = \delta_{a_2a_1} S_{a_2a_1}^{a_1a_1} (\alpha_1 - \alpha_2) Z_{a_2}^{a_1} (\alpha_2) Z_{a_1}^{a_1} (\alpha_1) + 2 \pi \delta_{a_2a_1} \delta (\alpha_1 - \alpha_2)
\]

for which several of the S matrix properties (but not those involving crossing) are just consistency relations.

In terms of \( \alpha \), \( S \) is a meromorphic function, whose only singularities in the physical strip are poles on the imaginary axis, corresponding in general to bound states.

Consider now the two particle form-factor \( f(\alpha_1, \alpha_2) \). Up to some simple dimensionful terms (which are absent if the operator is scalar), this is also a function of the Mandelstam variable \( s \). Its analytical properties are similar although a bit different than for \( S \); in particular, the left cut does not exist for form-factors.

From the second relation of (9), it follows immediately that

\[
f(\alpha_1, \alpha_2)_{a_1a_2} = S_{a_1a_2}^{a_1a_2} (\alpha_1 - \alpha_2) f(\alpha_2, \alpha_1)_{a_2a_1}
\]

(10)

If \( \alpha_1 > \alpha_2 \), the state \(|\alpha_1, \alpha_2 >_{a_1a_2}\) is an in state, with the corresponding out state \(|\alpha_2, \alpha_1 >_{a_2a_1}\). The exchange of the two on the other hand corresponds to going from the upper to the lower lip of the cut in the s-plane, ie from \( \text{Im} \alpha = 0 \) to \( \text{Im} \alpha = 2 \pi \) in the \( \alpha \) plane. It follows that

\[
f(\alpha_1 + 2 i \pi, \alpha_2)_{a_1a_2} = f(\alpha_2, \alpha_1)_{a_2a_1}
\]

(11)

Again, the minimal solution of these equations is obtained by assuming that \( f \) does not have poles nor zeroes (except the threshold pole at \( \alpha = 0 \)) in the strip \( \text{Im} \alpha \in [0, 2 \pi] \). Consider now a closed contour \( C \) enclosing this strip. By Cauchy’s theorem one has

\[
\frac{d}{d\alpha} \ln f^{\text{min}}(\alpha) = \frac{1}{8 i \pi} \int_C \frac{dz}{\sinh^2(z - \alpha)/2} \ln f(z)
\]

\[
= \frac{1}{8 i \pi} \int_{-\infty}^{\infty} \frac{dz}{\sinh^2(z - \alpha)/2} \ln \frac{f(z)}{f(z + 2 i \pi)}
\]

\[
= \frac{1}{8 i \pi} \int_{-\infty}^{\infty} \frac{dz}{\sinh^2(z - \alpha)/2} \ln S(z)
\]

(13)

Very often, the S matrix can be cast in the form

\[
S(\alpha) = - \exp \left[ \int_0^{\infty} dx g(x) \sinh(x \alpha / i \pi) \right]
\]

(14)

from which one finds

\[
f^{\text{min}} = \text{cst} \frac{\sinh \alpha}{2} \exp \left\{ \int_0^{\infty} dx \frac{g(x)}{\sinh x} \sin^2 [x(i \pi - \alpha)/2 \pi] \right\}
\]

(15)
In general, form-factors are expressed in terms of $f^{\text{min}}$ and simple functions which encode the required pole structure, see below.

Generalizing the two equations \([12]\) we have, for \(n\) particles,

\[
\begin{align*}
  f(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n)_{a_1 \ldots a_i a_{i+1} \ldots a_n} &= f(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n)_{a_1 \ldots a_i a_{i+1} \ldots a_n} \\
  &\quad \times S_{a_i a_{i+1}}^{b_i b_{i+1}} (a_i - a_{i+1}) \\
  f_{a_1 \ldots a_n} (a_1 + 2i\pi, \ldots, a_n) &= f_{a_2 \ldots a_n a_1} (a_2, \ldots, a_n, a_1)
\end{align*}
\]

(16)

The form-factors are meromorphic functions of the rapidity differences \(\alpha_{ij}\) in the strip \(0 \leq \text{Im} \alpha_{ij} \leq 2\pi\), with two kinds of simple poles.

One type of poles is called annihilation, or kinematical, pole. Such poles are always expected, even if the theory has no bound states, when some of the rapidities in the form-factor differ by \(i\pi\), corresponding physically to the presence of a pair particle-antiparticle in the process. The residue is a form-factor with two fewer particles

\[
i \text{Res}_{\alpha' = \alpha + i\pi} f(\alpha', \alpha, \alpha_1, \alpha_2, \ldots, \alpha_n)_{a_1 a_2 \ldots a_n} = [\delta^{b_1}_{a_1} \ldots \delta^{b_n}_{a_n} - S_{a_1 \ldots a_n}^{b_1 \ldots b_n} (\alpha_1 \ldots \alpha_n | \alpha)] f(\alpha_1, \ldots, \alpha_n)_{b_1 \ldots b_n},
\]

(17)

where, we have defined the \(S\) matrix element

\[
S_{a_1 \ldots a_n}^{b_1 \ldots b_n} (\alpha_1 \ldots \alpha_n | \alpha) = S_{a_1 c_1}^{b_1 c_1} (\alpha_1 - \alpha) S_{a_2 c_2}^{b_2 c_2} (\alpha_2 - \alpha) \ldots S_{a_n c_n}^{b_n c_n} (\alpha_n - \alpha)
\]

(18)

This provides a recursive relation between form-factors with \(n + 2\) and \(n\) particles, which proves crucial in determining the multiple particle form-factors.

The other type of poles has a more physical origin, and corresponds to the appearance of bound states. In words, form-factors obey relations that mimic the bootstrap structure of the theory: if a particle appears as a bound state of two others, its form-factors are residues of form-factors involving these particles. In formulas, things are a bit more complicated. If \(c\) is a bound state of particles \(a\) and \(b\), such that the \(S\) matrix has a simple pole

\[
S_{ab}^{de} (\alpha) \approx \frac{ig_{ab}g_{de}}{\alpha - iu_{ab}^c} \tag{19}
\]

then the form factor \(f(a_1, \alpha_2, \ldots, a_{n+2})_{a_1 \ldots a_n ab}\) has a pole when \(\alpha_{n+1} - \alpha_{n+2} = iu_{ab}^c\), with residue

\[
i \text{Res}_{\alpha = \alpha + i\pi} f(a_1, \alpha_2, \ldots, a_{n+2})_{a_1 \ldots a_n ab} = g_{ab}^c f(a_1, \alpha_2, \ldots, a_n, a_{n+1} - iu_{ab}^c/2)_{a_1 \ldots a_n c}
\]

(20)

where the value of the \(n + 1\)th rapidity follows from conservation of energy and momentum at the three particle vertex \(ab \to c\). Formulas \((17, 20)\) are usually called LSZ reduction formulas, from the pioneering paper of Lehmann, Symanzik and Zimmermann \([11]\).

In the simple case where there is only one type of particle (like in the sinh-Gordon model, see below), the general solution of these equations can always be written in the form

\[
f(\alpha_1, \ldots, \alpha_n) = K(\alpha_1, \ldots, \alpha_n) \prod_{i<j} f^{\text{min}}(\alpha_i - \alpha_j)
\]

(21)

where \(K\) is a completely symmetric, doubly periodic function, which contains all the physical poles.

Finally, we point out that other form-factors can be obtained by crossing

\[
a_1', \ldots, a_m' \langle \alpha_1', \ldots, \alpha_m' | \mathcal{O} | \alpha_1, \ldots, \alpha_n \rangle_{a_1 \ldots a_n} = f(\alpha_1' + i\pi, \ldots, \alpha_m' + i\pi, \alpha_1, \ldots, \alpha_n)_{a_1' \ldots a_m' a_1 \ldots a_n}
\]

(22)

Let us now discuss indeed the sinh-Gordon model in more details - here, I follow closely the paper \([13]\). The action is

\[
S = \frac{1}{16\pi g} \int_{-\infty}^{\infty} dx dy \left[ (\partial_x \Phi)^2 + (\partial_y \Phi)^2 + \Lambda \cosh \Phi \right].
\]

(23)
Note that here the free boson is normalized differently than in [1]: this will prove more convenient below.

Eq. (23) defines an integrable massive theory; the conformal weights of the perturbing operator are \( h = \hat{h} = -g \). The spectrum is very simple and consists of a single particle of mass \( M \), and \( S \) matrix:

\[
S(\alpha) = \frac{\tanh \frac{1}{2}(\alpha - i\frac{\pi}{2}B)}{\tanh \frac{1}{2}(\alpha + i\frac{\pi}{2}B)},
\]

where:

\[
B = \frac{2g}{1 + g}
\]

Observe the remarkable duality of the \( S \) matrix in \( B \to 2 - B \), i.e. in \( g \to 1/g \). This duality is certainly not obvious at the level of the action, and is deeply non perturbative in nature.

To determine the minimal form-factor, it is convenient to replace trigonometric functions with \( \Gamma \) functions using the basic identity

\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}
\]

together with the integral representation

\[
\ln \Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-t} - e^{-(z-1)}}{1-e^{-t}}\right] \frac{dt}{t}
\]

One finds then

\[
S(\alpha) = -\exp \left[2 \int_0^\infty \frac{dx \cosh x(1-B)/2}{\cosh x/2} \sinh(\alpha x/i\pi)\right]
\]

Using a similar representation for \( \ln \sinh \alpha/2 \), we find finally

\[
f_{\min}(\alpha) = \mathcal{N} \exp \left\{ 8 \int_0^\infty \frac{dx \sinh(xB) \sinh(\frac{x(2-B)}{4}) \sinh(\frac{x}{2})}{\sinh^2(x)} \sin^2[x(i\pi - \alpha)/2\pi] \right\}.
\]

Here, we have put the normalization which is standard in the literature [12]:

\[
\mathcal{N} = \exp \left[-4 \int_0^\infty \frac{dx \sinh(xB) \sinh(\frac{x(2-B)}{4}) \sinh(\frac{x}{2})}{\sinh^2(x)} \right]
\]

Consider now the form-factors of the field \( \Phi \) itself. Since \( \Phi \) as well as the creation operators of the sinh-Gordon particle, are odd under the \( \mathbb{Z}_2 \) symmetry \( \Phi \to -\Phi \), only form-factors with an even number of particles are non vanishing. One finds the general formula [12]

\[
f(\alpha_1, \ldots, \alpha_{2n+1}) = \mu \left(\frac{4 \sin \frac{\pi B}{2}}{F_{\min}(i\pi, B)}\right)^n \sigma_{2n+1}(\alpha_1, \ldots, \alpha_{2n+1}) \prod_{i<j} \frac{f_{\min}(\alpha_i - \alpha_j)}{x_i + x_j},
\]

where we introduced \( x = e^\alpha \) and the \( \sigma \)'s are the basic symmetric polynomials:

\[
\sigma_p = \sum_{i_1 < i_2 < \cdots < i_p} x_{i_1} x_{i_2} \cdots x_{i_p},
\]

with the convention \( \sigma_0 = 1 \) and \( \sigma_p = 0 \) if \( p \) is greater than the number of variables. The \( P_{2n+1} \)'s are symmetric polynomials, which can be obtained by solving LSZ [11] recursion relations. The first ones read:

\[
\begin{align*}
P_3(x_1, \ldots, x_3) &= 1 \\
P_5(x_1, \ldots, x_5) &= \sigma_2 \sigma_3 - c_1^2 \sigma_5 \\
P_7(x_1, \ldots, x_7) &= \sigma_2 \sigma_3 \sigma_4 \sigma_5 - c_1^2 (\sigma_4 \sigma_5^2 + \sigma_1 \sigma_2 \sigma_5 \sigma_6 + \sigma_3^2 \sigma_3 - c_1^2 \sigma_2 \sigma_5) \\
&\quad - c_2 (\sigma_1 \sigma_6 \sigma_7 + \sigma_1 \sigma_2 \sigma_4 \sigma_7) + \sigma_3 \sigma_5 \sigma_6) + c_1 c_2^2 \sigma_7.
\end{align*}
\]
with \( c_1 = 2 \cos \pi B/2, c_2 = 1 - c_1^2 \). Observe that except for the overall normalization \( \mu(g) \), these expressions are invariant in the duality transformation \( g \rightarrow \frac{1}{g} \). This is expected from the duality of the S matrix itself - as for the role of the overall normalization, it will be discussed later.

The conventional normalization \([12]\) is \( \mu = 1 \), which corresponds to choosing \( <0|\Phi(0)\alpha> = \frac{1}{\sqrt{2}} \). We shall make a different choice later on, when we consider the massless limit.

The method can be generalized to models with several types of particles, like the sine-Gordon model: this will be discussed in the following sections. Of course, all the foregoing equations for form-factors have been established within the context of ordinary massive integrable field theories. The case of massless theories (first considered in a slightly different context in \([13]\)) will be handled simply by taking the appropriate massless (ultra violet) limit in all the equations. This is somewhat safer than trying directly to formulate axioms for a massless theory per se, massless scattering presenting some physical ambiguities (see \([14]\) for a detailed discussion of this point).

We shall mostly consider physical properties related with the \( U(1) \) currents \( \partial \Phi \); hence, our discussion will be centered on the form-factors of this operator. In the last section however, we give the example of a correlator involving vertex operators.

We now discuss in more details the questions at stake in the case of the sinh-Gordon model. It has little physical interest in the present context, but is pedagogically quite useful.

### 2 Example: The sinh-Gordon model.

#### 2.1 Massless form-factors and the bulk current-current correlators.

In most of the following calculations, we shall work in Euclidian space with \( x,y \) coordinates. Imaginary time is at first considered as running along \( x \). The action and \( S \) matrix for the massive sinh-Gordon model were given in the introduction (23).

Let us now try to describe the free boson theory as a massless limit of this model. First, recall the current correlators (we could use here the notation \( \phi, \bar{\phi} \) for the chiral components, but since these get mixed in the presence of the boundary, we will not do so)

\[
< \partial_\tau \Phi(z, \bar{z}) \partial_\tau \Phi(z', \bar{z}') > = -\frac{2g}{(z - z')^2} \\
< \partial_\tau \Phi(z, \bar{z}) \partial_{z'} \Phi(z', \bar{z}') > = -\frac{2g}{(z - z')^2}.
\] (31)

To start, we wish to recover these correlators using form-factors. We thus take the massless limit \( \Lambda \rightarrow 0 \) of the factorized scattering description of (23). As discussed in \([1]\), we start by writing \( \alpha = \pm(A + \theta) \) and take simultaneously \( A \rightarrow \infty \) and \( M \rightarrow 0 \) with \( Me^4/2 \rightarrow m \), finite. In that case, the spectrum separates into Right and Left movers with respectively \( E = P = me^\theta \) and \( E = -P = me^\theta \). The scattering of R and L movers is still given by (24) where \( \alpha \rightarrow \pm \theta \). The RL and LR scattering becomes a simple phase, \( e^{\pm i\pi B/2} \). This phase will turn out to cancel out at the end of all computations, but is confusing to keep along. We just set it equal to unity in the following, that is we consider all L and R quantities as commuting.

In this new description of the massless theory, we will need form factors in order to compute (31). By taking the massless limit of the formulas (24) given in the introduction, it is easy to check that \( \Phi \) can alter only the right or left content of states; in other words, matrix elements of \( \Phi \) between states which have different content both in the left and right sectors vanish (that is, \( \Phi = \phi + \bar{\phi}! \)).

Our conventions are conveniently summarized by giving the one particle form factor of the sinh-Gordon field:

\[
<0|\Phi(x, y)|\theta_R> = \mu \exp \left[ me^\theta(x + iy) \right] \\
<0|\Phi(x, y)|\theta_L> = \mu \exp \left[ me^\theta(x - iy) \right],
\] (32)
and we will use the obvious notation:

\[ f(\theta_1, \ldots, \theta_{2n+1}) = \langle 0 | \Phi | \theta_1, \ldots, \theta_{2n+1} >_{R, \ldots, R}, \quad (33) \]

with the normalization of asymptotic states \( R < \theta | \theta' >_R = 2\pi \delta(\theta - \theta') \). Here, \( f \) depends on the interaction strength \( g \), but we do not indicate it explicitly for simplicity. We have the following properties:

\[ \begin{align*}
\langle 0 | \Phi | \theta_1, \ldots, \theta_{2n+1} >_{R, \ldots, R} &= (\langle 0 | \Phi | \theta_1, \ldots, \theta_{2n+1} >_{L, \ldots, L})^* \\
\langle 0 | \Phi | \theta_1, \ldots, \theta_{2n+1} >_{R, \ldots, R} &= \langle 0 | \Phi | \theta_{2n+1}, \ldots, \theta_1 >_{L, \ldots, L}.
\end{align*} \quad (34) \]

These form factors are expressed just as in the massive case

\[ f(\theta_1, \ldots, \theta_{2n+1}) = \mu \left( \frac{4 \sin \frac{\pi B}{2}}{F_{\min}(i\pi, B)} \right)^n \sigma_{(2n+1)} P_{2n+1}(x_1, \ldots, x_{2n+1}) \prod_{i<j} \frac{f_{\min}(\theta_i - \theta_j)}{x_i + x_j}, \quad (35) \]

where we introduced \( x \equiv e^\theta \) and the \( \sigma \)'s are the same symmetric polynomials as before.

We will now choose the normalization \( \mu \) by demanding that the result (31) be recovered. Using form factors, this two point function expands, assuming \( \text{Re} z < \text{Re} z' \), as:

\[ \begin{align*}
\langle 0 | \partial_z \Phi(z, \bar{z}) \partial_{\bar{z}} \Phi(z', \bar{z}') | 0 > &= - \sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_{2n+1}}{(2\pi)^{2n+1}(2n+1)!} m^2 (e^{\theta_1} + \ldots + e^{\theta_{2n+1}})^2 \\
& \quad \times \exp \left[ m(z - z') (e^{\theta_1} + \ldots + e^{\theta_{2n+1}}) \right] | f(\theta_1, \ldots, \theta_{2n+1})|^2. \quad (36) \end{align*} \]

Now, by relativistic invariance, all the form factors depend only on differences of rapidities. Setting \( m(z - z') \equiv e^{\theta_0} \), (where \( \theta_0 \) will in general be complex), one can shift all the \( \beta \)'s by \( \theta_0 \) to factor out, for any \( 2n+1 \) particle contributions, a factor \( e^{-\frac{z - z'}{2\eta}} \). Hence, the form factor expansion gives the result (31) provided \( \mu \) is chosen such that

\[ \sum_{n=0}^{\infty} I_{2n+1} = 2g, \quad (37) \]

where

\[ I_{2n+1} = \int \frac{d\theta_1 \ldots d\theta_{2n+1}}{(2\pi)^{2n+1}(2n+1)!} (e^{\theta_1} + \ldots + e^{\theta_{2n+1}})^2 e^{-(e^{\theta_1} + \ldots + e^{\theta_{2n+1}})} | f(\theta_1, \ldots, \theta_{2n+1})|^2. \quad (38) \]

In practice, this sum cannot be computed analytically, but it can be easily evaluated numerically. The convergence is extremely fast with \( n \), and for most practical purposes, the consideration of up to five particles is enough to get correct results up to \( 10^{-4} \). Similar convergence properties were observed in [14] in the massive case; note however that in this massless case, contributions with a higher number of particles are not damped-off by exponential mass terms.

It must be emphasized that this result is very peculiar to the current operator. For most other chiral operators, the correct \( z - z' \) dependence involves a non trivial anomalous dimension, instead of the naive engineering dimension. Hence, this dependence is not obtained term by term, as observed here, but rather once the whole series is summed up. Truncating the series to any finite \( n \) does not, in such cases, give reliable results all the way from short to large distances, unless some additional tricks are performed (see below).

### 2.2 Current current correlators with a boundary

Having fixed the form-factors normalization, let us now consider the theory with a boundary. The geometry of the problem is such that the boundary stands at \( x = 0 \), and runs parallel to the \( y = it \) axis. The action is now:

\[ S = \frac{1}{16\pi g} \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[ (\partial_x \Phi)^2 + (\partial_y \Phi)^2 + \Lambda \cosh \Phi \right] + \lambda \int_{-\infty}^{\infty} dy \cosh \frac{1}{2} \Phi(x = 0, y). \quad (39) \]
This model is also integrable for any choice of $\Lambda, \lambda$. The boundary dimension of the perturbing operator is $x = -g$. We can in particular take the limit where $\Lambda \to 0$ while $\lambda$ remains finite. It then describes a theory which is conformal invariant in the bulk but has a boundary interaction that breaks this invariance and induces a flow from Neumann boundary conditions at small $\lambda$ to Dirichlet boundary conditions at large $\lambda$. As discussed in [1], the boundary interaction is characterized by an energy scale, which one can represent as $T_B = me^{\theta_B}$. $T_B$ is related with the bare coupling in the action (39) by $\lambda \propto T_B^{1+g}$. In the following, since obviously changes of $m$ (which is not a physical scale) can be absorbed in rapidity shifts, we set $m = 1$. The effect of the boundary is then expressed by the reflection matrix:

$$R(\theta) = \tanh \left[ \frac{\theta}{2} - i \frac{\pi}{4} \right]. \quad (40)$$

In the picture where imaginary time is along $x$, the effect of the boundary is represented by a boundary state. Following [16] we can represent it in terms of the boundary scattering matrix:

$$|B> = \exp \left[ \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} K(\theta_B - \theta) Z_L^\dagger(\theta) Z_R^\dagger(\theta) \right] |0> \quad (41)$$

In this formula, $Z^\dagger$ are again the Zamolodchikov Fateev creation operators, $K$ is related with the reflection matrix by:

$$K(\theta) = R \left( i\frac{\pi}{2} - \theta \right) = -\tanh \frac{\theta}{2}. \quad (42)$$

We do not prove the expression (41) directly here. It follows however from the compatibility between the present computation, and the one we will do next, in a modular transformed point of view where the direction of imaginary time will have been switched.

One can expand the boundary state into the convenient form:

$$|B> = \sum_{n=0}^{\infty} \prod_{0 < \theta_1 \cdots < \theta_n} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} K(\theta_B - \theta_1) \cdots K(\theta_B - \theta_n) \times Z_L^\dagger(\theta_1) \cdots Z_L^\dagger(\theta_n) Z_R^\dagger(\theta_1) \cdots Z_R^\dagger(\theta_n) |0> \quad (43)$$

Here, we have ordered the rapidities, but the result is the same as the unordered integral with the additional symmetry factor $1/n!$, the contributions from L and R scattering cancelling out after reordering. Observe now that by analyticity, the matrix elements of $\partial_z \Phi$ between the ground state and an arbitrary state with at least one L moving particle are identical with the ones without a boundary. More generally, the only non vanishing matrix elements of $\partial_z \Phi$ are those where bra and ket have the same L moving part. The same results apply by exchanging $\partial_{\bar{z}}$ with $\partial_z$ and L with R moving particles. As a result one gets immediately two of the four current correlators:

$$< \partial_{\bar{z}} \Phi(z, \bar{z}) \partial_{\bar{z}} \Phi(z', \bar{z}') > = -\frac{2g}{(z - z')^2},$$

$$< \partial_z \Phi(z, \bar{z}) \partial_z \Phi(z', \bar{z}') > = -\frac{2g}{(\bar{z} - \bar{z}')^2}, \quad (44)$$

which are identical with the ones without a boundary.

The two other correlators are more difficult to get. Let us consider for instance:

$$< 0 | \partial_z \Phi(z, \bar{z}) \partial_{\bar{z}} \Phi(z', \bar{z}') | B > \quad (45)$$

The first non-trivial contribution comes from the two particle term in the expansion of the boundary state:

$$\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} K(\theta_B - \theta) < 0 | \partial_{\bar{z}} \Phi(z, \bar{z}) \partial_{\bar{z}} \Phi(z', \bar{z}') \times Z_L^\dagger(\theta) Z_R^\dagger(\theta) | 0 > =$$

$$\times \mu^2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} K(\theta_B - \theta) e^{2\theta} \exp \left[ e^\theta (\bar{z} + z') \right]. \quad (46)$$
More generally, because $|B>$ is a superposition of states with equal numbers of left and right moving particles, and $\partial_z \phi$, respectively $\partial_{\bar{z}} \phi$ act only on R, respectively L, particles, the expansion of (45) takes a very simple form:

$$\sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_{2n+1}}{(2\pi)^{2n+1}(2n+1)!} K(\theta_B - \theta_1) \ldots K(\theta_B - \theta_{2n+1}) \left(e^{\theta_1} + \ldots + e^{\theta_{2n+1}}\right)^2 \exp \left[(\bar{z} + z') \left(e^{\theta_1} + \ldots + e^{\theta_{2n+1}}\right)\right] |f(\theta_1, \ldots, \theta_{2n+1})|^2.$$

(47)

This correlation function depends on the product $e^{\theta_B}(\bar{z} + z')$. It is scale invariant at the UV and IR fixed point. These correspond respectively to sending $\theta_B$ to $\mp \infty$, that is the coupling $\lambda$ in the action to 0 or $\infty$, in other words Neumann or Dirichlet boundary conditions. In the first case, $K = 1$, in the second, $K = -1$. Comparing with (36) and (37) we find, as expected, that:

$$<0|\partial_{\bar{z}} \Phi(z, \bar{z}) \partial_z \Phi(z', \bar{z}')|B> = \pm \frac{2g}{(\bar{z} + z')^2},$$

(48)

for Neumann, respectively Dirichlet boundary conditions. Although trivial, this result shows that the form factor expansion is well behaved, and allows us to study the correlator all the way from the UV to the IR fixed point when there is a boundary perturbation. In figures 1 and 2 we show the one particle (which is independent of $B$) and three particles contributions. We observe that indeed the convergence, by looking at the respective contributions, is very rapid.

![Figure 1: One particle contribution.](image1)

![Figure 2: Three particles contribution for $B = 1, 0.1$.](image2)
The only drawback of this expansion is that it is not suited for studying the correlation of two operators right at the boundary. Indeed in that case, $Re(z + z') = 0$, and the integrals in (47) do not converge. To solve this problem, we can introduce a modular transformed picture. We now consider the imaginary axis. Now the boundary is not represented as a state; rather, the whole space of states is different, since now we have only a half space to deal with. The asymptotic states are not pure L or R moving, but are mixtures. For instance, one particle states are:

$$||\theta| = |\theta > R + R(\theta)|\theta > L.$$ (49)

More generally, asymptotic states are obtained by adding to $|\theta_1, \ldots, \theta_n > R, \ldots, R$ all combinations with different choices of $R$ particles transformed into $L$ particles, via action of the boundary. Only the following two terms contribute:

$$|\theta_1, \ldots, \theta_n >= |\theta_1, \ldots, \theta_n > R, \ldots, R + \ldots + R(\theta_1) \ldots R(\theta_n)|\theta_1, \ldots, \theta_1 > L, \ldots, L + \ldots.$$ (50)

Although we used the same notation as previously, different things are meant by L,R. To make it clear, we now use the conventions:

$$< 0|\Phi(x, y)|\theta > R = \mu \exp[me^\theta(-y + ix)]$$

$$< 0|\Phi(x, y)|\theta > L = \mu \exp[me^\theta(-y - ix)].$$ (51)

To keep the notations as uniform as possible, we introduce the new coordinates:

$$w(z) \equiv iz = -y + ix,$$ (52)

so here R movers depend on $w$, L movers on $\bar{w}$. The normalization $\mu$ is of course the same as before, and as before the LL and RR correlators do not depend on the boundary interaction. One finds:

$$< 0|\partial_\omega \Phi(w, \bar{w})|\partial_\omega \Phi(w', \bar{w}')|0 > = \frac{2g}{(w - w')}^2,$$

$$< 0|\partial_\omega \Phi(w, \bar{w})|\partial_\omega \Phi(w', \bar{w}')|0 > = \frac{2g}{(w - w')}^2,$$ (53)

where we used the fact that $|R(\theta)|^2 = 1$. When compared with (44), these correlators have an overall minus sign due to the dimension $h = 1, \bar{h} = 0$ (resp. $h = 0, \bar{h} = 1$) of the operators.

Let us now consider:

$$< 0|\partial_\omega \Phi(w, \bar{w})|\partial_\omega \Phi(w', \bar{w}')|0 > .$$ (54)

To compute it, we insert a complete set of states which are of the form (49). In the massless case however, since $\partial_\omega \Phi$ is a R operator, $\partial_\omega \Phi$ a L operator, the only terms that contribute are in fact the ones with either all L or all R moving particles, as written in (50). Thus, (54) expands simply as:

$$\sum_{n=0}^\infty \int \frac{d\theta_1 \ldots d\theta_{2n+1}}{(2\pi)^{2n+1}(2n + 1)} R(\theta_1 - \theta_B) \ldots R(\theta_{2n+1} - \theta_B) (e^{\theta_1} + \ldots + e^{\theta_{2n+1}})^2$$

$$\exp \left[(\bar{w} - w')(e^{\theta_1} + \ldots + e^{\theta_{2n+1}}) \right] |f(\theta_1, \ldots, \theta_{2n+1})|^2.$$ (55)

Observe the crucial minus sign when compared to (47). It occurs because in one geometry the correlator depends on $z + z'$, while in the other on $\bar{w} - w'$. This now converges provided $y > y'$, even if $x = x' = 0$ ie the operators are sitting right on the boundary. Now, using the fact that from factors depend only on differences of rapidities, this expression can be mapped with (47) if we formally set $\theta = \theta' + i\pi/2$, provided one has:

$$K(\theta) = R \left(\frac{i\pi}{2} - \theta\right),$$ (56)

as claimed above.
To summarize, we can write the left right current current correlator in two possible ways. By using the boundary state one finds:

\[
\langle \partial_x \Phi(x, y) \partial_x \Phi(x', y') \rangle = \int_0^\infty dE \mathcal{G}(E) \exp \left[ E(x + x') - iE(y - y') \right],
\]

(recall that \(x, x' < 0\)). One obtains \(\mathcal{G}(E)\) simply by fixing the energy to a particular value in (47). When this is done, the remaining integrations occur on a finite domain for each of the individual particle energies since \(\sum_{i=1}^{2n+1} e^{\theta_i} = E\), and there is no problem of convergence anymore. One then gets:

\[
\mathcal{G}(E) = \sum_{n=0}^{\infty} \int_{-\infty}^{\ln E} d\theta_1 \cdots d\theta_{2n} \frac{E^2}{(2\pi)^{2n+1}(2n+1)!} \frac{1}{1 - e^{\theta_1} - \cdots - e^{\theta_{2n}}} \\
\times K(\theta_B - \theta_1) \cdots K(\theta_B - \theta_{2n}) K[\theta_B - \ln(E - e^{\theta_1} - \cdots - e^{\theta_{2n}})] \\
\times \left[ f[\theta_1 \cdots \theta_{2n}, \ln(E - e^{\theta_1} - \cdots - e^{\theta_{2n}})] \right]^2,
\]

(58)

with the constraint \(\sum_{i=1}^{2n} e^{\theta_i} \leq E\). The denominator might suggest some possible divergences; it is important however to realize that it vanishes if and only if the particle with rapidity \(\theta_{2n+1}\) has vanishing energy, in which case the form factor vanishes too. We can now shift the integrands to write equivalently:

\[
\mathcal{G}(E) = E \sum_{n=0}^{\infty} \int_{-\infty}^{0} d\theta_1 \cdots d\theta_{2n} \frac{1}{(2\pi)^{2n+1}(2n+1)!} \frac{1}{1 - e^{\theta_1} - \cdots - e^{\theta_{2n}}} \\
\times K(\theta_B - \theta_1) \cdots K(\theta_B - \theta_{2n}) K[\ln(T_B/E) - \ln(1 - e^{\theta_1} - \cdots - e^{\theta_{2n}})] \\
\times \left[ f[\theta_1 \cdots \theta_{2n}, \ln(1 - e^{\theta_1} - \cdots - e^{\theta_{2n}})] \right]^2,
\]

(59)

where the constraint \(\sum_{i=1}^{2n} e^{\theta_i} \leq 1\) is implied, we used the fact that form-factors depend only on rapidity differences, and \(T_B = e^{\theta_{2n}}\).

By using the dual picture, one finds, instead of (57):

\[
\langle \partial_x \Phi(x, y) \partial_x \Phi(x', y') \rangle = \int_0^\infty dEF(E) \exp \left[ -iE(x + x') - E(y - y') \right],
\]

(60)

where:

\[
F(E) = -E \sum_{n=0}^{\infty} \int_{-\infty}^{0} d\theta_1 \cdots d\theta_{2n} \frac{1}{(2\pi)^{2n+1}(2n+1)!} \frac{1}{1 - e^{\theta_1} - \cdots - e^{\theta_{2n}}} \\
\times R(\theta_1 - \ln(T_B/E)) \cdots R(\theta_{2n} - \ln(T_B/E)) R[\ln(1 - e^{\theta_1} - \cdots - e^{\theta_{2n}}) - \ln(T_B/E)] \\
\times \left[ f[\theta_1 \cdots \theta_{2n}, \ln(1 - e^{\theta_1} - \cdots - e^{\theta_{2n}})] \right]^2,
\]

(61)

where in (59) and (61) the constraint \(\sum_{i=1}^{2n} e^{\theta_i} \leq 1\) is implied. The two expressions are in correspondence by the simple analytic continuation:

\[
\mathcal{G}(E) = iF(iE).
\]

### 3 The sine-Gordon Model.

In this section we follow the same line of thought for the sine-Gordon model. This is the massive deformation of the free boson which preserves integrability with either boundary interactions used in the fractional quantum Hall problem and the anisotropic Kondo model \([1]\). Thus the form factors of the sine-Gordon model in the massless limit will be the quantities we need. The solitons/anti-solitons and breathers quasi-excitation make the problem more complicated but the results presented before hold with the addition of a few indices (and rather more complicated form factors).
We start as before with the massive sine-Gordon model, whose action reads:

\[
S = \frac{1}{16\pi g} \int_{-\infty}^{\infty} dxdy \left[ (\partial_x \Phi)^2 + (\partial_y \Phi)^2 + \Lambda \cos \Phi \right].
\] (63)

Notice again that we have used a different normalization than in [1]: this will avoid carrying factors of \(2\pi\) all along the following paragraphs. Recall that \(g = \frac{\beta^2}{2\pi}\), \(\beta\) the usual sine-Gordon parameter.

The form factors approach is formally the same, albeit more complicated because the particle content is much richer, and depends on \(g\). For \(1/2 < g < 1\), only solitons/anti-solitons appear in the spectrum of the theory. This is the so called repulsive case, with \(g = 1/2\) the free fermion point (giving rise, in quantum impurity problems, to so called “Toulouse limits”. When \(0 < g < 1/2\), the particle content is enriched by \([1/g - 2]\) bound states, called breathers. In the following we will denote by the indices \(a = \pm\) the solitons and anti-solitons, and \(a = 1, 2, \ldots, [1/g - 2]\) the breathers. The solitons form factors in the massive case were written by Smirnov [3] and we obtain the massless form factors by taking the appropriate limit of the massive ones. Only right and left moving form factors survive in this limit, as in the sinh-Gordon case. Moreover, the symmetry of the action dictates that only form factors with total topological charge zero are non-zero for the current operator. As an example, the soliton/anti-soliton form factor is given by:

\[
< 0| \frac{1}{2\pi} \partial_x \Phi(z, \bar{z})| \theta_1, \theta_2 >_{RR}^{R_R} = a' \mu md e^{(\theta_1 + \theta_2)/2} \times \frac{\zeta(\theta_1 - \theta_2)}{\cosh \frac{1-\beta}{2\beta}(\theta_1 - \theta_2 + i\pi)} \exp \left[ m(e^{\theta_1} + e^{\theta_2})z \right],
\] (64)

with \(a + a' = 0\) and \(a = \pm\) stands for soliton (resp. antisoliton). From [3] one has:

\[
\zeta(\theta) = c \sinh \frac{\theta}{2} \exp \left( \int_0^{\infty} \sin^2 \frac{x(i\pi - \theta)}{2\pi} \sinh \frac{(1-2g)x}{2(1-g)} dx \right),
\] (65)

(this is essentially the minimum form-factor discussed in the introduction) with the constant \(c\) given by:

\[
c = \left( \frac{4(1-g)}{g} \right)^{1/4} \exp \left( \frac{1}{4} \int_0^{\infty} \sin \frac{x}{2} \sinh \frac{(1-2g)x}{2(1-g)} dx \right),
\] (66)

and \(d\) by:

\[
d = \frac{1}{2\pi c} \frac{(1-g)}{g},
\] (67)

The normalization constant \(\mu\) can be determined from first principles. Indeed, the operator \(\partial_x \Phi\) being related with the \(U(1)\) charge, we need that

\[
^+ \quad \delta^+ \theta_1 \int_{-\infty}^{\infty} \frac{1}{2\pi} \partial_x \Phi| \theta_2 > R^+_+ = 2\pi \delta(\theta_1 - \theta_2),
\] (68)

using the fact that a soliton for the bulk theory (63) obeys \(\Phi(\infty) - \Phi(-\infty) = 2\pi\). On the other hand, using the dependence of the form-factor on spatial coordinates, the left hand side is

\[
^+ \quad \delta^+ \theta_1 \frac{1}{2\pi} \partial_x \Phi| \theta_2 > R^+_+ \int dxe^{i(m\theta_1 - \theta_2)x} = \frac{2\pi}{m\pi} \delta^+ \theta_1 \frac{1}{2\pi} \partial_x \Phi| \theta_1 > R \delta(\theta_1 - \theta_2),
\] (69)

Comparing the two and using crossing leads to the identity

\[
i\mu d \zeta(i\pi) = 1
\] (70)
or
\[ \mu = 2\pi \frac{g}{1-g} = \frac{1}{c^d}. \]  

The other soliton-antisoliton form factors follow from the sophisticated analysis of [3]. Their expression simplifies in the case \( g = 4, t \) an integer. This is the physically relevant case for the \( \nu = \frac{1}{2} \) fractional quantum Hall effect. One then finds:

\[
< 0 \left| \frac{1}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \Phi(z, \bar{z}) \phi_1, \ldots, \phi_{2n} \right|_{\mathcal{R}^{-\infty} \mathcal{R}^{\infty}} = \mu n (2d)^n e^{(\phi_1 + \cdots + \phi_{2n})/2} \prod_{i<j} \zeta(\theta_i - \theta_j)
\]

\[
\sinh \left[ \frac{t-1}{2} \sum_{p=1}^{n} (\theta_{p+n} - \theta_p - i\pi) \right] \prod_{p=1}^{n} \prod_{q=n+1}^{2n+1} \sinh^{-1}(t-1)(\theta_q - \theta_p) \det H.
\]

The matrix \( H \) is obtained as follows. First introduce the function:

\[
\psi(\alpha) = 2^{t-2} \prod_{j=1}^{t-2} \sinh \left( \frac{1}{2} \left( \alpha - \frac{\pi j}{t-1} + \frac{\pi}{4} \right) \right).
\]

One then defines the matrix elements as:

\[
H_{ij} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} \prod_{k=1}^{2n} \psi(\alpha - \beta_k) \exp \left[ (n-2j-1)\alpha + (n-2i)(t-1)\alpha \right],
\]

where \( i, j \) run over \( 1, \ldots, n-1 \). It is not difficult to convince oneself that this produce a symmetric polynomial of the right degree. Although cumbersome, it is an easy task to extract these determinants, as examples we find for \( g = 1/3 \):

\[
det H = \exp \left( -\frac{1}{2} \sum_{j=1}^{2n} \theta_j \right) \sigma_1(e^{\theta_j}), \quad n = 2,
\]

\[
det H = \exp \left( -\sum_{j=1}^{2n} \theta_j \right) \sigma_1(e^{\theta_j})\sigma_2(e^{\theta_j}), \quad n = 3,
\]

up to irrelevant phases and with the \( \sigma_q \)'s defined previously. Having these expression we can get all form factors using the axiomatics sketched in the introduction. For example, the slitons form factors with different positions of the indices \( a_i \), we use the symmetry property (16):

\[
f(\theta_1, \ldots, \theta_i, \theta_i+1, \ldots, \theta_n)_{a_1, a_2, a_3, \ldots, a_n} = f(\theta_1, \ldots, \theta_{i+1}, \theta_i+1, \ldots, \theta_n)_{a_1, a_2, a_3, \ldots, a_n} \delta_{a_{i+1} a_i} \delta_{a_i a_{i+1}} \theta_i - \theta_{i+1}.
\]

Here again, we omit the distinction between left and right moving form factor, they are simply related by complex conjugation. At the points \( g = 1/t \) the soliton \( S \) matrix used in the last expression is reflectionless and basically just permutes the rapidities up to a phase. When there are breathers, the soliton form factors from these poles:

\[
(i \pi \sigma_{n-1-a_n} = i \pi g^{\frac{1}{2}} \mu \phi_{n-1} f(\theta_1, \ldots, \theta_{n-1}, \theta_n)_{a_1, a_2, \ldots, a_{n-1}, a_n} = r_m (-1)^{\frac{n-2a_n+1}{2}} \delta_{a_{n-1} a_n} \times f(\theta_1, \ldots, \theta_{n-1} + \frac{i\pi}{2} - \frac{i\pi g m}{2(1-g)})_{a_1, \ldots, a_{n-2}, m}.
\]

and \( r_m \) is given by the residue at \( i \pi g^{\frac{1}{2}} \mu \phi_{n} = i \pi - \frac{i\pi g m}{(1-g)} m \):

\[
r_m = \left[ S^{++}_{1+} \left( i \pi g^{\frac{1}{2}} \mu \frac{1}{2} \right) \left( \frac{1}{1-g} \right) \sin \pi \left( \frac{1-g}{g} \right) \right]^{1/2}.
\]
Having these relations, we possess all ingredients to compute all form factors for \( g = 1/t \). Using them for the computation of the current correlations is then merely an extension of the previous results for sinh-Gordon with indices. The normalisation of the form factors, \( \mu \), should also ensure that \( [4] \) is reproduced. This is fixed by introducing a complete basis of states:

\[
1 = \sum_{n=0}^{\infty} \sum_{a_i} \int \frac{d\theta_1...d\theta_n}{(2\pi)^n n!} |\theta_1, ..., \theta_n >_{a_1,...,a_n} a_n,...,a_1 < \theta_n, ..., \theta_1| \tag{79}
\]

and computing the correlations exactly like in the sinh-Gordon case.

Keeping a finite number of form-factors and demanding that the two point function is properly normalized gives rise to values of \( \mu \) which are slightly different from \( [7] \). How different is a good measure of the convergence of the expansion, and the validity of the truncation. For \( g = 1/3 \), the one breather and 2 solitons form factors normalise to \( \mu = 3.14 \) which is very close to the exact \( \pi \). Similarly for \( g = 1/4 \) we found from the contributions up to two solitons that \( \mu = 2.05 \) to compare with 2.094 = \( 2\pi/3 \).

Calculations in the presence of an integrable boundary interaction are also done like in the sinh-Gordon case. The boundary state is now given by:

\[
|B> = \sum_{n=0}^{\infty} \int_{0<\theta_1<...<\theta_n} \frac{d\theta_1}{2\pi} ... \frac{d\theta_n}{2\pi} K^{ab_1}(...K^{a_n b_n} (\theta_B - \theta_1) ... (\theta_B - \theta_n) \\
\times Z^{a_1 b_1}(\theta_1) ... Z^{a_n b_n}(\theta_n) Z^{b_1 b_n}(\theta_1) ... Z^{b_n b_n}(\theta_n), \tag{80}
\]

with an implicit sum on the indices. The matrix \( K^{ab} \) is related to the boundary \( R \) matrix in the following way:

\[
K^{ab}(\theta) = R^b_\theta \left( i\frac{\pi}{2} - \theta \right). \tag{81}
\]

The \( \tilde{b} \) means that we take the conjugate of the indices i.e. \( \pm \rightarrow \mp \) and \( m \rightarrow m \).

From the previous expressions, we can compute de current-current correlation function in the presence of a boundary for \( g = 1/t \). The results we will get depends on the boundary interaction, in the next subsection we present some results for the boundary sine-Gordon model, which is of relevance to tunneling experiments in fractional quantum Hall devices \([1]\) \([7]\) \([8]\) \([9]\).

4 Conductance in the fractional quantum Hall effect

4.1 General remarks.

The boundary sine-Gordon action is

\[
S = \frac{1}{16\pi g} \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[ (\partial_x \Phi)^2 + (\partial_y \Phi)^2 \right] + \lambda \int_{-\infty}^{\infty} dy \cos \frac{1}{2} \Phi (x = 0, y). \tag{82}
\]

The reflection matrices have been worked out in \([6]\) \([20]\). For generic values of the coupling \( g \), the amplitude for the processes \(+ \rightarrow + \) and \(- \rightarrow - \) is \( R^\pm_\pm (\theta - \theta_B) \), and for the processes \(+ \rightarrow - \) and \(- \rightarrow + \) it is \( R^\pm_\mp (\theta - \theta_B) \) (with the notation is slightly changed with respect to \([4]\), where outgoing indices were not raised):

\[
R^\pm_\pm (\theta) = e^{(l_0 l_0)\pm i\theta} R(\theta) \\
R^\pm_\mp (\theta) = i e^{(l_0 l_0)\pm i\theta} R(\theta) \tag{83}
\]

where the function \( R \) reads:

\[
R(\theta) = \frac{1}{2 \cosh \frac{\lambda g}{2\pi} i \theta} \prod_{l=0}^{\infty} \frac{Y_l(\theta)}{Y_l(-\theta)} \\
Y_l(\theta) = \frac{\Gamma\left(\frac{1}{4} + l(1/2) \frac{l(1/2)\theta}{\lambda g} \right) \Gamma\left(\frac{1}{4} + l(1/2) \frac{-l(1/2)\theta}{\lambda g} \right)}{\Gamma\left(\frac{1}{4} + (l+1/2) \frac{l(1/2)\theta}{\lambda g} \right) \Gamma\left(\frac{1}{4} + (l+1/2) \frac{-l(1/2)\theta}{\lambda g} \right)}. \tag{84}
\]
In [83], our conventions are such that in the UV limit \((\theta_B \to -\infty)\) the scattering is totally off-diagonal so a soliton bounces back as an anti-soliton, in agreement with classical limit results for Neumann boundary conditions. A useful integral representation of \(R\) is given by:

\[
R(\theta) = \frac{e^{\gamma_1}}{2 \cosh \left( \frac{(1-g)\theta}{2g} - i\frac{\pi}{4} \right)} \exp \left( i \int_{-\infty}^{\infty} dy \frac{2(1-g)\theta y}{g\pi} \frac{\sinh \left( \frac{1-2g}{g} \right) y}{\sinh 2y \cosh \left( \frac{(1-g)y}{g} \right)} \right). \tag{85}
\]

Recall that the spectrum is made of one breather and the pair soliton antisoliton in the whole domain \(1/3 \leq g < 1/2\). More breathers appear for \(g < 1/3\) (and the reflection matrix of the 1-breather is always the same as in the sinh-Gordon case.) There are no breathers for \(g > 1/2\).

The physical quantity of interest in this case corresponds to the AC conductance at vanishing temperature in the edge states tunneling problem: this problem has been described in details in [1], to which we refer the reader.

A standard way of representing the conductance is through the Kubo formula [21]:

\[
G(\omega_M) = -\frac{1}{8\pi\omega_M L^2} \int_{-L}^{L} dx \int_{-\infty}^{\infty} dy \ e^{i\omega_M y} < j(x,y)j(x',0) >, \tag{86}
\]

where \(\omega_M\) is a Matsubara frequency, \(y\) is imaginary time, \(y = it\). One gets back to real physical frequencies by letting \(\omega_M = -i\omega\). In [83], \(j\) is the physical current in the unfolded system [1]. Without impurity, the AC conductance of the Luttinger liquid is frequency independent, \(G = g\). When adding the impurity, it becomes \(G = \frac{g}{2} + \Delta G\). After some simple manipulations using the folding [1], one finds:

\[
\Delta G(\omega_M) = \frac{1}{8\pi\omega_M L^2} \int_{-L}^{L} dx dx' \int_{-\infty}^{\infty} dy e^{i\omega_M y} \times \left[ < \partial_x \Phi(x,y)\partial_x \phi(x',0) > + < \partial_y \Phi(x,y)\partial_y \Phi(x',0) > \right], \tag{87}
\]

where \(z = x + iy\). The strategy is simply to evaluate the current-current correlator using form-factors, and extract the conductance from [83].

We will do so for special values of \(g\), but first, we can extract some general features of the UV and IR expansions easily. To do so, consider the soliton antisolitons reflection matrix. Evaluating the integral in [83] by the residues method leads to a double expansion of the elements \(R_{\pm}^{\pm} \) in powers of \(\exp(\theta)\) and \(\exp(\frac{1}{g} - 1)\theta\). This leads for the conductance to a double power series in \((\omega/T_B)^{-2+2/g}\) and \((\omega/T_B)^2\) in the IR, \((T_B/\omega)^2-2g\) and \((T_B/\omega)^2\) in the UV. Breathers do not change this result. For instance for the 1-breather, since the reflection matrix is the same as in the sinh-Gordon case, and therefore \(g\) independent, the contributions expand as a series in \((\omega/T_B)^2\) in the IR, \((T_B/\omega)^2\) in the UV. This holds for any coupling \(g\). Therefore, as first argued by Guinea et al. [22], at low frequency, the conductance goes as \(\omega^2\) for \(g < 1/2\), \(\omega^{-2+2/g}\) for \(g > 1/2\). The \(\omega^2\) power would seem to indicate that there should be a \(T^2\) term in the DC conductance, but this is not correct because only the modulus square of \(R_{\pm}^{\pm}\) contribute to the DC conductance, and this expands only as powers of \(\exp(\frac{1}{g} - 1)\)\(\beta\).

The presence of analytic terms in the IR is a straightforward consequence of the fact that IR perturbation theory involves an infinity of counter-terms, in particular polynomials in derivatives of \(\Phi\) [22]. More surprising maybe is the fact that we find analytical terms in the UV. This requires some discussion. The UV terms follow from the short distance behaviour of the correlation function of the current. For any operator \(O\) we could write formally,

\[
< O(x',y')O(x,y) > = \sum_{n=0}^{\infty} \Lambda^{2n} \int d1...dn < O(x',y')O(x,y) \cos \frac{1}{2} \Phi(1) ... \cos \frac{1}{2} \Phi(n) >. \tag{88}
\]

From (88), one would naively expect that the two point function of the current expands as a power series in \(\lambda^2\), which would lead to a power series in \((\omega/T_B)^{2g-2}\). This is incorrect however because, even if integrals are convergent at short distance for \(g < 1/2\), they are always divergent at large distances. It is
known that these IR divergences give precisely rise to non analyticity in the coupling constant $\lambda$. One usually writes:

$$<O(x', y')O(x, y)> = \sum_i C_{iO}^i(x' - x, y' - y)O_i(x, y),$$\hspace{1cm} (89)

where $O_i$ are a complete set of local operators in the theory and the $C_i$’s are structure functions. These, being local quantities, have analytic behaviour in $\lambda$. However, $<O_i(x, y)>$ being non local is in general non analytic - actually, on dimensional grounds,

$$<O_i(x, y)> \propto \lambda^{\Delta/(1-\eta)} \propto T_B^{\Delta},$$\hspace{1cm} (90)

where $\Delta = h + \bar{h}$ is the (bulk) dimension of the field $O_i$. If we computed the conductance perturbatively using Matsubara formula, we would use (88) with $O$ the electrical current operator. The case $O_i$ the identity operator gives rise to an analytical expression in $\lambda$, but eg the case $O_i = \partial_x \partial_y \Phi$ gives $\lambda^{2/(1-\eta)}$ times an analytical expression in $\lambda$ (its mean value can be non zero because there is a boundary). More generally, since the only operators $O_i$ appearing in the case of the current are polynomials in derivatives of $\Phi$, all with integer dimensions, we expect that the two point function of the current will expand as a double series of the form $\lambda^{2n}\lambda^{2m/(1-\eta)}$, ie going back to $T_B$ variable, that the conductance will expand as a double series of the form $(T_B/\omega)^{2n(1-\eta)}(T_B/\omega)^{2m}$, in agreement with the form factors result.

### 4.2 The free case.

In the case $g = 1/2$ one has simply:

$$R_{\mp}^\pm(\theta) = P(\theta) = \frac{e^{\theta}}{e^{\theta} + 1},$$

$$R_{\pm}^\pm(\theta) = Q(\theta) = \frac{i}{e^{\theta} + 1}.\hspace{1cm} (91)$$

In that case, only the soliton-antisoliton form factor is non zero, $f(\theta_1, \theta_2) = i\mu e^{\theta_1/2}e^{\theta_2/2}$, with the normalization $\mu = 2\pi$ and we have set $m = 1$. Hence, $F(\omega)$ is readily evaluated

$$F(\omega) = \int_{-\infty}^{\infty} d\theta_1 d\theta_2 e(\theta_1 + e^{\theta_2} - \omega)[Q(\theta_1)Q(\theta_2) - P(\theta_1)P(\theta_2)]e^{\theta_1}e^{\theta_2},$$\hspace{1cm} (92)

so

$$F(\omega) = \omega \int_0^1 dx \frac{x(1 - x) + 1}{(x + (2\mu \omega)/(\omega - x + (2\mu \omega))},$$\hspace{1cm} (93)

from which it follows that

$$\Delta G(\omega) = \frac{1}{4} - \frac{T_B}{2\omega} \tan^{-1}(\omega/T_B).$$\hspace{1cm} (94)

Thus we find

$$G(\omega) = \frac{1}{2} \left(1 - \frac{T_B}{\omega} \tan^{-1}(\omega/T_B)\right).$$\hspace{1cm} (95)

This is in agreement with the solution of [17].

### 4.3 $G(\omega)$ at $g = 1/3$. 

The conductance for $g = 1/3$ has a direct application to the quantum Hall effect. Comparing with the free case, previously treated, we now have a breather in the spectrum and non zero form factors for all number of rapidities. Still the convergence is such that evaluating the first few form factors give results to a very good accuracy, independently of the regime, UV or IR, in which we make the computation.

In this case, the first few non zero form factors are $f_1, f_{\pm, \mp}, f_{\pm, 1, 0}, f_{1, 1, 1}$, etc... Here the subscript “1” denotes the breather. The first step is to compute the normalisation in order to satisfy $\langle \mathbf{1} \rangle$. When computing this normalisation, we find that the first two form factors account for the whole result to more
than one percent accuracy. Then including the 1 breather-2 solitons form factor is sufficient to get the result to a very good accuracy (\( f_{1,1,1} \) is negligible). Actually one observes that the speed of convergence of the form factor expansion varies geometrically with the number of solitons (counting the breathers as 2 solitons).

In order to get the conductance we need the reflection matrices, they were given in previous expressions and reduce to a simpler form for this value of \( g \):

\[
R(\theta) = \frac{1}{2 \cosh(\theta - \frac{i\pi}{4})} \frac{\Gamma(3/8 - \frac{i\theta}{2\pi})\Gamma(5/8 + \frac{i\theta}{2\pi})}{\Gamma(5/8 - \frac{i\theta}{2\pi})\Gamma(3/8 + \frac{i\theta}{2\pi})}
\]

(96)

and the breather reflection matrix is:

\[
R^1(\theta) = \tanh \left( \frac{\theta - \frac{i\pi}{4}}{2} \right).
\]

(97)

From the pole of the 2 solitons form factor, the one breather form factor is found using (77) and its contribution to the conductance is:

\[
\Delta G(\omega)^{(1)} = -\mu^2 \frac{d^2}{8} \Re e \left[ \log \left( \frac{\omega}{\sqrt{T_B}} \right) \right] \tanh \left[ \log \left( \frac{\omega}{\sqrt{T_B}} \right) - \frac{i\pi}{4} \right],
\]

(98)

here \( \mu = \pi \) is fixed by (71) and \( d = 0.1414... \). The contribution from the two solitons form factors is computed similarly, we find:

\[
\Delta G(\omega)^{(2)} = -\mu^2 d^2 \Re e \int_{-\infty}^{0} d\theta \frac{R(\theta) + \log \left( \frac{\omega}{\sqrt{T_B}} \right) R(\log \left( 1 - e^\theta \right))}{\cosh^2(\theta - \log(1 - e^\theta))} \left| \zeta(\theta - \log(1 - e^\theta)) \right|^2 e^\theta \left[ e^{\theta} (1 - e^\theta) \left( \frac{\omega}{\sqrt{T_B}} \right)^2 + e^{\theta} (1 - e^\theta) \left( \frac{1}{\sqrt{T_B}} \right)^2 \right],
\]

(99)

where \( \zeta(\theta) \) is the function defined in (64). We can similarly write the following contribution, and we find that these last two expressions are sufficient for any reasonable purpose, they give the frequency dependent conductance to more than one percent accuracy. We give the full function \( G(\omega) \) in figure 3.

![Figure 3: Frequency dependent conductance at T=0.](image)

Observe that in the UV and in the IR we obtain the \( \omega \) dependence discussed previously, even with the truncation to a few form-factors. The form-factors expansion is indeed very different from the perturbative expansion in powers of the coupling constant in the UV, or in powers of the inverse coupling constant in the IR. Each form-factor contribution has by itself the same analytical structure as the whole sum; contributions with higher number of particles simply determine coefficients to a greater accuracy.
5 Anisotropic Kondo model and dissipative quantum mechanics.

It turns out that the anisotropic Kondo model is related with the famous two state problem of dissipative quantum mechanics [23]. The bosonised form of the hamiltonian is:

\[
H = \frac{1}{2} \int_{-\infty}^{0} dx \left[ 8\pi g \Pi^2 + \frac{1}{8\pi g} (\partial_x \Phi)^2 \right] + \lambda \left( S^+ e^{i\Phi(0)/2} + S^- e^{-i\Phi(0)/2} \right),
\]

(100)

where \( S \) are Pauli matrices. As for the quantum Hall problem, we keep using as a basis the massless excitations of the sine-Gordon model; however the boundary interaction is different: this will result in different reflection matrices.

5.1 A computation of \( C(t) \).

We first work in imaginary time. We consider therefore the anisotropic Kondo problem at temperature \( T \). Let us consider the quantity \( X(y) \equiv \langle [S^z(y) - S^z(0)]^2 \rangle \). On the one hand, using that \( S^z = \pm 1 \), it reads \( 2[1 - C(y)] \), where \( C(y) \) is the usual spin correlation

\[
C(y) = \frac{1}{2} \left[ \langle S^z(y)S^z(0) \rangle + \langle S^z(0)S^z(y) \rangle \right].
\]

(101)

On the other hand, we can write a perturbative expansion for \( X(y) \) by expanding evolution operators in powers of the coupling constant \( \lambda \). At every order, we get ordered monomials which are a product of a monomial in \( S^z \) and vertex operators of charge \( \pm 1/2 \). We must then evaluate \( S^z(y) - S^z(0) \) for each such term, trace over the two possible spin states, and average over the quantum field. Since we deal with spin \( 1/2 \), terms \( S^+ \) and \( S^- \) must alternate, and there must be an overall equal number of \( S^+ \) and \( S^- \), and an equal number of \( 1/2 \) and \(-1/2 \) electric charges.

Now, since each \( S^z(y) \) comes with a \( e^{-i\Phi(y)/2} \) and each \( S^z(0) \) comes with a \( e^{i\Phi(y)/2} \), \( S^z(y) = S^z(0) \) if there is a vanishing electric charge inserted between 0 and \( y \), and \( S^z(y) = -S^z(0) \) if the charge inserted between 0 and \( y \) is non zero (and then it has to be \( \pm 1/2 \)). Therefore, we can write the perturbation expansion of \( X(y) \) in such a way that the spin contributions all disappear:

\[
X(y) = \frac{1}{Z} \sum_{n=0}^{\infty} \lambda^{2n} \left[ \sum_{\text{alternating } \epsilon_i = \pm 1} \sum_{p=0}^{2n} \int_{y_1}^{y} dy_1 \int_{y_2}^{y} dy_2 \cdots \int_{y_{p-1}}^{y} dy_{p} \int_{y_{p+1}}^{1/T} dy_{p+1} \cdots \int_{y_{2n-1}}^{1/T} dy_{2n} \right] \\
\times 4(\epsilon_1 + \cdots + \epsilon_{p})^2 \left\langle e^{-i\epsilon_1 \phi(y_1)/2} \cdots e^{-i\epsilon_{2n} \phi(y_{2n})/2} \right\rangle_N,
\]

(102)

where \( Z \) is the partition function, the factor 4 occurs because of the normalization \( S^z = \pm 1 \), for every configuration of \( \epsilon \)'s, only one value of \( S^z(0) \) gives a non vanishing contribution. Here, the label \( N \) indicates correlation functions for the free boson evaluated with Neumann boundary conditions (the conditions as \( \lambda \to 0 \)).

On the other hand, let us consider the correlator

\[
\langle \partial_x \Phi(x, y) \Phi(0, y') \rangle_N = -8g \frac{x}{x^2 + (y - y')^2},
\]

(103)

which goes to \( -8g\pi \delta(y - y') \) as \( x \to 0 \). We have then, by Wick’s theorem,

\[
\langle e^{-i\epsilon_1 \Phi(y_1)/2} \cdots e^{-i\epsilon_{2n} \Phi(y_{2n})/2} \partial_x \Phi(x, y) \rangle_N = \frac{8ig}{x} \left( 2n \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y - y_i)^2} \right) \left\langle e^{i\epsilon_1 \Phi(y_1)/2} \cdots e^{i\epsilon_{2n} \Phi(y_{2n})/2} \right\rangle_N,
\]

(104)

where
and therefore
\[
-(8g)^2 \left( \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y - y_i)^2} \right) \left( \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y' - y_i)^2} \right) \left\langle e^{-i \epsilon_1 \Phi(y_1)/2} \cdots e^{-i \epsilon_{2n} \Phi(y_{2n})/2} : \partial_x \Phi(x, y) \partial_x \Phi(x, y') : \right\rangle_N = \]
\[
\left( \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y - y_i)^2} \right) \left( \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y' - y_i)^2} \right) \left\langle e^{-i \epsilon_1 \Phi(y_1)/2} \cdots e^{-i \epsilon_{2n} \Phi(y_{2n})/2} : \partial_x \Phi(x, y) \partial_x \Phi(x, y') : \right\rangle_N, \tag{105}
\]
where contractions between the dots are discarded. In (103), contractions between the dots would lead to a term factored out as the product of the two point function of \( \partial_x \Phi \) and the 2n point function of vertex operators, both evaluated with N boundary conditions. Now, we are going to be interested in the \( x \to 0 \) limit where, with N boundary conditions, \( \partial_x \Phi \) vanishes. As a result we can actually forget the subtraction in (102) and write simply obtain
\[
X(y, \lambda) = -\frac{1}{(4g\pi)^2} \lim_{x \to 0} \int_0^y \int_0^y dy' dy'' < \partial_x \Phi(x, y') \partial_x \Phi(x, y'') > _\lambda, \tag{106}
\]
where the label \( \lambda \) designates the correlator evaluated at coupling \( \lambda, N \) corresponding to \( \lambda = 0 \). Hence, we can get \( C(y) \) from the current current correlator. The latter can then be obtained using form factors along the above lines. The only difference is the boundary matrix. If we restrict to the repulsive regime where the bulk spectrum contains only a soliton and an antisoliton, one has:
\[
R_\pm = \tanh \left( \frac{\theta}{2} - \frac{i\pi}{4} \right), \quad R_\pm = 0. \tag{107}
\]
Here again our conventions are such that a soliton bounces back as an antisoliton, in agreement with the UV and the IR limit that have Neumann boundary conditions. In the attractive regime we need to add the breathers with:
\[
R_m^\pm = \frac{\tanh \left( \frac{\theta}{2} - \frac{i\pi m}{4(1/g - 1)} \right)}{\tanh \left( \frac{\theta}{2} + \frac{i\pi m}{4(1/g - 1)} \right)}, \tag{108}
\]
Writing, as in (13):
\[
< \partial_x \Phi(x, y') \partial_x \Phi(x, y'') > _\lambda = \int_0^\infty \mathcal{G}(E, \beta_B) \exp \left[ 2Ex - iE(y' - y'') \right], \tag{109}
\]
we have that:
\[
\lim_{x \to 0} < \partial_x \Phi(x, y') \partial_x \Phi(x, y'') > _\lambda = \int_0^\infty dE \left[ \mathcal{G}(E, \beta_B) - \mathcal{G}(E, -\infty) \right] \exp [-iE(y' - y'')] + c.c., \tag{110}
\]
where the \( < \partial_x \Phi \partial_x \Phi > \) part and its complex conjugate (which are \( \lambda \) independent) have been evaluated by requiring that the correlator vanishes as \( \lambda \to 0 \) due to \( N \) boundary conditions. Hence, using the fact that \( \mathcal{G} \) is real,
\[
X(y) = \frac{1}{2(2g\pi)^2} \int_0^\infty \frac{dE}{E^2} \left[ \mathcal{G}(E, \theta_B) - \mathcal{G}(E, -\infty) \right] \sin^2 (Ey/2). \tag{111}
\]
Therefore, if we write:
\[
C(y) - 1 = \int_0^\infty A(\omega_M) \cos(\omega_M y) \ d\omega_M, \tag{112}
\]
where \( \omega_M \) is a Matsubara frequency, we have:
\[
A(\omega_M) = \frac{1}{(2g\pi)^2} \frac{1}{\omega_M^2} \left[ \mathcal{G}(\omega_M, \theta_B) - \mathcal{G}(\omega_M, -\infty) \right]. \tag{113}
\]
An observation is now in order. From the foregoing results we see that:
\[
< S^z(0) S^z(y) > -1 = \frac{1}{(2g\pi)^2} \int_0^\infty \frac{dE}{E^2} \left[ \mathcal{G}(E, \theta_B) - \mathcal{G}(E, -\infty) \right] \cos Ey. \tag{114}
\]
On the other hand, consider the expression
\[
< \int_{-\infty}^{0} dx' \int_{-\infty}^{0} dx'' [\partial_{x'} \Phi(x', y) \partial_{x''} \Phi(x'', 0)] > \chi - \partial_{x} \Phi(x', y) \partial_{x} \Phi(x'', 0) > N.
\]  
(115)

By using the same representation (109), this is
\[
\int_{-\infty}^{0} dx' \int_{-\infty}^{0} dx'' \int_{0}^{\infty} dE [\mathcal{G}(E, \theta_B) - \mathcal{G}(E, -\infty)] \exp [E(x' + x'') - iEy] + cc,
\]
which coincides with (114) after performing the integrations. We conclude that
\[
< S^z(0)S^z(y) > -1 = < J_x(0)J_x(y) > - < J_x(0)J_x(y) > N,
\]
(116)
where we defined
\[
J_x = \frac{1}{2g\pi} \int_{-\infty}^{0} \partial_x \Phi(x, y) dx.
\]
(117)

We find also by the same manipulations that
\[
< S^z(0)S^z(y) > -1 = < J_y(0)J_y(y) > - < J_y(0)J_y(y) > N,
\]
(118)
where
\[
J_y = \frac{1}{2g\pi} \int_{-\infty}^{0} \partial_y \Phi(x, y) dx.
\]
(119)

We now continue to real frequencies to find the response function :
\[
\chi''(\omega) = \frac{1}{2} \int dt e^{i\omega t} \langle [S^z(t), S^z(0)] \rangle,
\]
(120)
to find :
\[
\chi''(\omega) = \frac{1}{(2g\pi)^2} \frac{1}{\omega^2} \text{Im} \left[ \mathcal{G}(-i\omega, \theta_B) - \mathcal{G}(-i\omega, -\infty) \right].
\]
(121)

As a first example, let us consider the so called Toulouse limit or free fermion case. Then the only contribution comes from the soliton antisoliton form factors, which as discussed above is \( f(\theta_1, \theta_2) = i\mu e^{\theta_1/2} e^{\theta_2/2} \). Hence,
\[
\chi''(\omega) = \frac{1}{\pi^2} \text{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\theta_1 d\theta_2 \frac{e^{\theta_1} e^{\theta_2}}{(e^{\theta_1} + i\theta_B)(e^{\theta_2} + i\theta_B)} \frac{1}{e^{\theta_1} + e^{\theta_2} - \omega},
\]
(122)
that is
\[
\chi''(\omega) = \frac{2}{\pi^2} \frac{T_B}{\omega} \text{Im} \left( \int_{0}^{\omega} dx \frac{1}{(x + iT_B)(\omega - x + iT_B)} \right),
\]
(123)
and
\[
\chi''(\omega) = \frac{4}{\pi^2} \frac{T_B}{\omega} \text{Im} \left( \frac{1}{\omega + 2iT_B} \ln \left( \frac{\omega + iT_B}{iT_B} \right) \right) \]
\[
= \frac{1}{\pi^2} \frac{4T_B^2}{\omega^2 + 4T_B^2} \left[ \frac{1}{\omega} \ln \left( \frac{T_B^2 + \omega^2}{T_B^2} \right) + \frac{1}{T_B} \tan^{-1} \left( \frac{\omega}{T_B} \right) \right].
\]
(124)

In general, observe that, since the reflection matrix for solitons and antisolitons expands as a series in \( e^{\theta_1} \), \( \chi''(\omega) \), will, for any coupling, expand as a series of the form \( (\omega/T_B)^{2n} \) in the IR. In particular, this leads to a behaviour \( C(t) \propto \frac{1}{t^{1+\theta}}, t >> 1 \) for any \( g \). In the UV, one has to split integrals in two pieces. Since the soliton-soliton form factors expansion involves powers of \( \exp(\frac{4}{g} - 1) \), \( \chi''(\omega) \) expands as a double series
in \((T_B/\omega)^{2-2g}\) and \((T_B/\omega)^2\) in the UV. Hence at short times, \(C(t) \sim t^{2-2g}\). This is in agreement with the qualitative analysis of [24].

Results for \(g \neq 1/2\) are more involved because there are non zero form factors at all levels. Still, when working out the first few form factors we observe a very rapid convergence with the number of rapidities and again we can give precise results for different values of \(g\). As an example, let us show the results for \(g = 1/3\).

The computation for \(g = 1/3\) is very similar to the previous conductance computations. The boundary matrices are much more simpler though. In this case we have:

\[
R_\pm^+ = \tanh \left( \frac{\theta}{2} - \frac{i\pi}{4} \right), \quad R_\pm^- = 0, \tag{125}
\]

and:

\[
R_1^1 = \frac{\tanh(\frac{\theta}{2} - \frac{i\pi}{8})}{\tanh(\frac{\theta}{2} + \frac{i\pi}{8})}. \tag{126}
\]

Then, as was found for the conductance, we find that the first two contributions are sufficient for most purposes, they are given by:

\[
\Delta \chi''(\omega)^{(1)} = -\frac{9\mu^2d^2}{8\pi\omega} \text{Re} \left[ \frac{\tanh \left( \frac{\log(\frac{\omega}{\sqrt{2}T_B})}{2} - \frac{i\pi}{8} \right)}{\tanh \left( \frac{\log(\frac{\omega}{\sqrt{2}T_B})}{2} + \frac{i\pi}{8} \right)} - 1 \right] \tag{127}
\]

and:

\[
\Delta \chi''(\omega)^{(2)} = - \left( \frac{3\mu d}{2\pi} \right)^2 \frac{1}{\omega} \text{Re} \int_{-\infty}^{0} d\theta \frac{|\zeta(\theta - \log(1-e^\theta))|^2}{\cosh^2(\theta - \log(1-e^\theta))} e^\theta \times \left[ R_\pm^+(\theta + \log(\omega/T_B)) R_\pm^-((1-e^\theta)\omega/T_B) - 1 \right]. \tag{128}
\]

Again these two expressions are sufficient to get a very precise result. Similar computations give rise to the results in figure 4 where we plotted \(S(\omega) \equiv \chi''(\omega)/\omega\) for the values, \(g = 3/5, 1/2, 1/3, 1/4\).

![Figure 4: Spectral function for \(T_B = 0.1\).](image)

When making these calculations we have to be careful about which terms are needed for a good convergence. Our observation is that keeping the form factors up to two rapidities give very good results.
precise to 1%. It is possible to go further and get a better precision if needed. The last statements are true for \( g \in [0.6, 0.2] \) and we believe even further (we can get rough bounds on the higher contributions and have an idea of the precision). Still, for the moment the isotropic Kondo point is difficult to treat.

To a good approximation, \( S(\omega) \) can be represented for \( g < \frac{1}{2} \) by a Lorentzian shape built around the four poles of the one-breather term:

\[
S(\omega) \approx \frac{\text{cst}}{[\frac{\omega}{\Omega}]^2 - 1 + [\frac{\omega}{\Gamma}]^2 + 4 [\frac{\omega}{\Gamma}]^2}
\]

where the poles are located at \( \omega = \pm \Omega + \pm \Gamma \). Here, it follows from the one breather reflection matrix that

\[
\Omega = \frac{\pi g}{2(1-g)},
\]

and \( \Omega = T_B \sin \frac{\pi g}{(1-g)} \). The approximation improves as \( g \) gets smaller.

According to (129), the pic in \( S(\omega) \) disappears at \( g = \frac{1}{3} \), where \( \Omega = \Gamma \). The poles meanwhile reach the imaginary axis at \( g = \frac{1}{4} \) where \( \frac{\Omega}{\Gamma} = 0 \). These features are shared by the exact solution, as far as can be established from numerical study of the form-factors series: the pic in \( S(\omega) \) disappears at \( g = \frac{1}{3} \), while the tail oscillations of \( C(t) \) disappear at \( g = \frac{1}{4} \). Depending on one’s definition of the “transition” from coherent to incoherent regime, the latter therefore occurs at \( g = \frac{1}{3} \) or \( g = \frac{1}{4} \). This feature was somewhat unexpected from previous approximate studies, but is fully confirmed by recent numerical work [25, 26]. In any case, there is no real transition per se, and it is possible that the study of different quantities (for instance multi point correlations) would exhibit a qualitative change of behaviour at still another value of \( g \).

5.2 Shiba’s Relation.

Up untill now, we showed results for certain values of \( g \) more or less limited by our ability (or tenacity) to write the form factors corresponding to that value of the anisotropy, and make them converge. It is not impossible to find general relations though; for example the behaviours in the UV and the IR in different models were inferred in all generality.

Here we present a generalisation of Shiba’s relation [27], which was proven for the Anderson model and generalised to Luttinger liquids by Sassetti and Weiss [28]. The relation states that:

\[
\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega} = 2\pi g \chi_0^2
\]

with \( \chi_0 \) the static susceptibility. If we look at the quantity:

\[
\mathcal{G}(E) = E \sum_{n=0}^{\infty} \int_{-\infty}^{0} \frac{d\theta_1 \ldots d\theta_{2n-1}}{(2\pi)^{2n}(2n)!} \frac{1}{1 - e^{\theta_1} - \ldots - e^{\theta_{2n-1}}} \times K^{a_1b_1} \left(\ln(T_B/E) - \theta_1\right) \ldots K^{a_{n-1}b_{2n-1}} \left(\ln(T_B/E) - \theta_{n-1}\right) \times K^{a_{2n}b_{2n}} \left[\ln(T_B/E) - \ln \left(1 - e^{\theta_1} - \ldots - e^{\theta_{2n-1}}\right)\right] \times \int_{a_1 \ldots a_2n} f_{b_1 \ldots b_{2n}} \left[\theta_1 \ldots \theta_{2n-1}, \ln \left(1 - e^{\theta_1} - \ldots - e^{\theta_{2n-1}}\right)\right],
\]

insert it in the expression for \( \chi''(\omega) \) and expand it around \( E \approx 0 \) we find that the contributions from the \( K \) matrices all cancel and only a constant is left (we have to take into account the fact that the soliton/anti-solitons \( K \) matrices always appear in pair). Then comparing this with the UV normalisation we find that:

\[
\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega} = \frac{1}{\pi^2 g T_B^2}.
\]
The total susceptibility is $\chi = \chi' + i\chi''$ and the static susceptibility $\chi_0$ which is the zero frequency limit of $\chi'$ can also be inferred from the previous expressions for the spin-spin correlation. We just need to take the real part when continuing (113) to real frequencies, which leads to:

$$\chi_0 = \frac{1}{\pi^2 gT_B}.$$ (134)

Finally, in order to make contact with the usual form of Shiba’s relation, we need to renormalise the spins to $1/2$ and use the usual normalization for Fourier transforms, which leads indeed to (131).

6 Friedel oscillations: correlations involving Vertex operators

As exemplified in the previous sections, the method works naively indeed for currents, i.e. for operators with no anomalous dimension (calculations for the stress energy tensor for instance, would be very similar). Many physical properties are however described by more complicated operators, that is operators which have a non trivial anomalous dimension. As an example, I would like to discuss here the equivalent of Friedel oscillations in Luttinger liquids: more precisely, the $2k_F$ part of the charge density profile in a one dimensional Luttinger liquid away from an impurity. This is a problem which has attracted a fair amount of interest recently [29], [30].

We start with the bosonised form of the model. The Hamiltonian is:

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ 8\pi g\Pi^2 + \frac{1}{8\pi g} (\partial_x \phi)^2 \right] + \lambda \cos \phi(0),$$ (135)

where we have set $v_F = g$. Then for the Friedel oscillations, the charge density operator is just:

$$\rho(x) = \rho_0 + 2\partial_x \phi + \frac{k_F}{\pi} \cos [2k_F x + \phi(x)].$$ (136)

with $\rho_0 = \frac{k_F \pi}{2}$ the background charge.

We decompose this system into even and odd basis (this is was explained already in [1]) by writing $\phi = \phi_L + \phi_R$ and setting:

$$\varphi_e(x + t) = \frac{1}{\sqrt{2}} [\phi_L(x,t) + \phi_R(-x,t)]$$

$$\varphi_o(x + t) = \frac{1}{\sqrt{2}} [\phi_L(x,t) - \phi_R(-x,t)]$$ (137)

Observe that these two field are left movers. We now fold the system by setting (prefactors here are slightly different from [1] because of the normalizations in the Hamiltonian):

$$\phi_L^e = \sqrt{2} \varphi_e(x + t), \quad x < 0 \quad \phi_R^e = \sqrt{2} \varphi_e(-x + t), \quad x < 0$$

$$\phi_L^o = \sqrt{2} \varphi_o(x + t), \quad x < 0 \quad \phi_R^o = -\sqrt{2} \varphi_o(-x + t), \quad x < 0$$ (138)

and introduce new fields $\phi^{e,o} = \phi_L^{e,o} + \phi_R^{e,o}$. The density oscillations now read:

$$\frac{\langle \rho(x) - \rho_0 \rangle}{\rho_0} = \cos (2k_F x + \eta_F) \cos \frac{\phi_o^e(x)}{2} \cos \frac{\phi_o^o(x)}{2},$$ (139)

with $\eta_F$ the additional phase shift coming from the unitary transformation to eliminate the forward scattering term [29]. $\phi^o$ is the odd field with Dirichlet boundary conditions, at the origin $\phi^o(0) = 0$ leading to [3]:

$$\langle \cos \frac{\phi_o^o}{2} \rangle \propto \left( \frac{1}{x} \right)^{g/2},$$ (140)
and the $\phi^e$ part is computed with the hamiltonian:

$$H^e = \frac{1}{2} \int_{-\infty}^{0} dx \left[ 8\pi g \Pi x^2 + \frac{1}{8\pi g} (\partial_x \phi^e)^2 \right] + \lambda \cos \frac{\phi^e(0)}{2}.$$  \hfill (141)

On general grounds, we expect the scaling form:

$$\langle \cos \frac{\phi^e}{2} \rangle \propto \left( \frac{1}{x} \right)^{g/2} F(\lambda x^{1-g}),$$  \hfill (142)

where $F$ is a scaling function to be determined. Note that even the small $x$ behaviour of this function was not known in general.

To compute the correlation functions, we use the formalism introduced in the previous section (with $\phi^e \equiv \Phi$). The massless scattering description and the boundary state are the same; only the operator which is studied changes.

The one point function of interest is $\langle 0 | \cos \frac{\Phi}{2} | B \rangle$. We thus need the matrix elements of the operator $\cos \frac{\Phi}{2}$ in the quasiparticle basis: these still follow easily from the massive sine-Gordon form-factors \cite{3}, like the form-factors of $\partial \Phi$ in the previous sections. Difficulties however arise when one considers the contribution of each form-factor to the one-point function: the rapidity integrals turn out to be all IR divergent! This was not the case for the current operator, whose form factor has the naive engineering dimension of an energy, leading to convergent integrals: some sort of additional regularization is thus needed here.

To explain the strategy, consider first the case $g = 1/2$. Here, the friedel oscillations are simply \cite{32} related to the spin one point function in an Ising model with boundary magnetic field. By using the same approach as the one outlined before, one finds the following form-factors expansion:

$$\langle \sigma(x) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left\{ \frac{d\theta_i}{2\pi} \tanh \frac{\theta_B - \theta_i}{2} e^{-2m x e^{\theta_i}} \right\}$$

$$\times \prod_{i<j} \left( \tanh \frac{\theta_i - \theta_j}{2} \right)^2. \hfill (143)$$

The integrals are all divergent at low energies, when $\theta_i \rightarrow -\infty$ and the integrand goes to a constant. Let us then introduce an IR cut-off (we chose $\theta \geq \theta_{\min}$ and set $\Lambda \equiv e^{\theta_{\min}}$) and take the log of the previous expressions (a similar method has been used in \cite{33,34} to study the UV limit of massive correlators. See also \cite{35,36}). Ordering this log by increasing number of integrations, one can show that each term diverges as $\ln \Lambda$. Moreover, since the divergence occurs at very low energy, where the tanh goes to unity, the amplitudes of these $\ln \Lambda$ do not depend on $\theta_B$ (for $\theta_B \neq -\infty$), ie on the boundary coupling. It is then easy to get rid of the cut-off: we simply substract the log of the IR spin function, ie we substract the same formal expression with $\theta_B = \infty$. The first two terms of the resulting expression read:

$$\ln \langle \sigma(x) \rangle_{T_B} = \int_{\Lambda}^{\infty} \frac{du}{2\pi u} e^{-2u x} \left( \frac{T_B - u}{T_B + u} - 1 \right)$$

$$+ \frac{1}{2} \int_{\Lambda}^{\infty} \prod_{i=1}^{2} \frac{du_i}{2\pi u_i} e^{-2u_i x} \left( \prod_{i=1}^{2} \frac{T_B - u_i}{T_B + u_i} - 1 \right)$$

$$\times \left[ \left( \frac{u_1 - u_2}{u_1 + u_2} \right)^2 - 1 \right] + \cdots \hfill (144)$$

where we have set $\mu = 1$, $u_i = e^{\theta_i}$, $T_B = e^{\theta_B} \propto \lambda^{1/(1-g)}$.

Clearly, the integrals are now convergent at low energies, and we can send $\Lambda$ to zero. Since the IR value of the one point function is easily determined by other means, $\langle \sigma(x) \rangle_{IR} \propto x^{-1/8}$ \cite{37}, we can now obtain $\langle \sigma(x) \rangle_{T_B}$ from (144). Hence the procedure involves a double regularization. Of course, there
remains an infinity of terms to sum over. However, as in the case of current operators, the convergence of
the form-factors expansion is very quick, and the first few terms are sufficient to get excellent accuracy
all the way from UV to IR. To illustrate this more precisely, we recall that for $g = 1/2$ (144) can be
resummed in closed form [33, 32], giving rise to:

$$
R_{\text{exact}} = \frac{\langle \sigma(x) \rangle_{T_B}}{\langle \sigma(x) \rangle_{IR}} = \frac{\sqrt{2\pi}}{\sqrt{x_{TB}}} e^{x_{TB}} K_0(x_{TB}).
$$

(145)

By reexponentiating the two first terms in (144), one gets a ratio differing from (145) by at most 1/100
for $x_{TB} \in [0, \infty)$ (see figure 1).

By reexponentiating the first three terms, accuracy is improved to more than 1/1000. Clearly, the form-
factors approach thus provides analytical expressions that can be considered as exact for most reasonable
purposes.

It is fair to mention however that, at any given order in (144), the exponent controlling the $x \to 0$
behaviour is not exactly reproduced, as could be seen on a log-log plot. For instance, the first term is
immediately found to produce a behaviour $R(x) \propto x^{1/\pi}$, to be compared with the result $R_{\text{exact}}(x) \propto
x^{1/2} \ln x$. The comparison of the exact result (145) and of (144) show that the form factors expression
has, term by term, the correct asymptotic expansion ie the IR expansion in powers of $\frac{1}{x_{TB}}$. Adding
terms with more form-factors simply gives a more accurate determination of the coefficients. This is to
be compared with the results of [35] for eg the frequency dependent conductance, where the form factors
expression had the correct functional dependence both in the UV and in the IR. This is not to say that
the method is inefficient in the UV, because we know, at least formally, all the terms. In fact, we will
show in what follows how the expansion (144) can always be resummed in the UV, and that the exponent
can be exactly obtained from the form-factors approach too.

The regularization is the same for other values of $g$. Let us discuss here the cases $g = 1/t$ with $t$
integer again. For these values, the scattering is diagonal and the form factors are rather simple. To
obtain them, we again take the massless limit of the results in [3] (but this time for vertex operator
$\cos \Phi/2$ (in normalizations where the bulk sine-Gordon perturbation is $\cos \Phi$) instead of the current)
and impose that half of the quasiparticles become right movers and half become left movers, since the
boundary state always involve pairs of right and left moving particles. It is in fact easier to take that limit
if we change basis from the solitons and anti-solitons to $\frac{1}{\sqrt{2}}(|S > \pm|A >)$. In that case, the boundary
scattering matrix becomes diagonal and the isotopic indices always come in pairs. The reflection matrices

Figure 5: Accuracy of the finite $T_B$ over the IR part of the envelope of $\rho(x)$ for $g = 1/2$. 

\[\text{Diagram:}\]
in this new basis are given by:

\[
R_-(\theta) = -e^{i\frac{\pi}{4}(2-t)} \tanh \left[ \frac{(t-1)\theta}{2} + i\frac{\pi(t-2)}{4} \right] R(i\frac{\pi}{4} - \theta)
\]
\[
R_+(\theta) = e^{i\frac{\pi}{4}(2-t)} R(i\frac{\pi}{4} - \theta)
\]

(146)

with (note the slight change of notation compared with (83)):

\[
R(\theta) = \exp \left( i \int_{-\infty}^{\infty} dy \frac{2(y-1)y\theta}{\pi} \frac{\sinh(2y)\cosh(t-1)y}{\sinh(2y)\cosh(t-1)y} \right).
\]

(147)

The breathers reflection matrices follow from [36] (see also (97).

The case \( g = 1/2 \) having already been worked out, let us concentrate on \( g = 1/3 \). There, in addition to the solitons and anti-solitons, there is also one breather. The first contribution to the one point function comes from the two breathers form-factor, with one right moving and one left moving breather. It is given by a constant:

\[
f(\theta, \theta')_{11}^L = c_1,
\]

(148)

and this obviously leads to IR divergences. Other contributions come from \( 2n \) breathers form-factors, and \( 4n \) solitons form-factors The whole expression can be controlled as for \( g = 1/2 \), by taking the log, and factoring out the IR part. Setting \( c(x) = \cos \frac{\theta(2)}{2} \), we organize the sum as follows:

\[
\ln \left( \frac{\langle c(x) \rangle_{IR}}{\langle c(x) \rangle_{LR}} \right) = \ln R^{(2)} + \ln R^{(4)} + \cdots
\]

(149)

with the subscript denoting the number of intermediate excitations.

Then, using the explicit expressions for \( g = 1/3 \) we find:

\[
\ln R^{(2)} = 2c_1 e^{2\sqrt{2}T_B x} Ei(-2\sqrt{2}T_B x),
\]

(150)

where \( E_i \) is the standard exponential integral. The next term \( \ln R^{(4)} \) is a bit bulky to be written here, but it is very easy to obtain - similar expressions have been explicitly given in the previous sections. This is all what is needed for an accuracy better than 1 percent. In figure 6 we present the results of the ratio at \( g = 1/2, 1/3, 1/4 \) for the Friedel oscillations. It should be noted that this ratio is just the pinning function of reference [24] and our results agree well qualitatively with the results found there.

As mentioned before, the deep UV behaviour is a little more difficult to obtain: the accuracy is good because the ratio goes to zero anyway, but the numerical evaluation of the power law would not be too accurate with the number of terms we consider. Fortunately, the full form-factors expansion allows the analytic determination of this exponent. First, observe for instance that in (144) the integrals converge for all \( T_B \neq 0 \), but strictly at \( T_B = 0 \), they do not. To get the dependence of \( \langle c(x) \rangle \) as \( T_B \to 0 \), we will consider, the logarithm of another ratio, \( \ln \frac{\langle c(x) \rangle_{LR}}{\langle c(x) \rangle_{TB}} \), where \( x \) and \( x' \) are two arbitrary coordinates.

For this ratio, even at \( T_B = 0 \), the integrals are convergent. But \( T_B = 0 \) is the UV fixed point, with Neumann boundary conditions. While the one point function \( \langle c(x) \rangle_{UV} \) vanishes, the ratio of two such one point functions is well defined, and can be computed by putting an IR cut-off (a finite system). One finds that it goes as \( (x/x')^{g/2} \). By regularity as \( T_B \to 0 \), the same is true for the ratio close to \( T_B = 0 \), and thus one has

\[
\langle c(x) \rangle \propto (xT_B)^{g/2}, x(T_B) \to 0
\]

(151)

This shows that the universal scaling function in (142) behaves as \( F(y) \propto y^{\frac{g}{1-2g}} \) for \( g < \frac{1}{2} \). This exponent can actually be obtained by perturbation theory. Indeed, the first term in the perturbative expansion of \( \langle c(x) \rangle \) is

\[
\lambda x^{g/2} \int_{-\infty}^{\infty} \frac{dy}{(x^2 + y^2)^{g}}
\]

(152)

For \( g < 1/2 \), this integral diverges in the IR. To regulate it, we need to put a new cut-off: since there is no other length scale in the problem, this can be nothing but \( 1/T_B \). Changing variables, the leading
behaviour is \(x^{g/2}T_B^g \propto x^{g/2} \lambda^{g/1-g}\), in agreement with the previous discussion. The exponent coincides with the result of the self-consistent harmonic approximation [29]; but it is important to stress that the latter is valid only for \(g << 1\).

The function \(F(y)\) behaves as \(y \ln y\) for \(g = 1/2\). For \(g > 1/2\), its behaviour is simply \(F(y) \propto y\), as can be easily shown since the perturbative approach is now convergent. As we approach \(g = 1/2\) this exponent seems to become asymptotic and is more difficult to get numerically [29].

7 Conclusion

The description of the Hilbert space in terms of the integrable quasiparticles basis allows, in a wide variety of cases, extremely accurate computations of time and space dependent correlators. The related form-factors method is not especially elegant, but gives rise to surprisingly good results: in fact, it is not clear whether exact analytical expressions of the correlation functions - if ever obtained - will be more useful. Although we limited ourselves to \(g = 1/t\) with \(t\) integer in these notes, all values of \(g < 1/2\) are accessible, but general computations are more complicated since the bulk scattering is non diagonal. The methods explained here should be generalizable to other problems, in particular the determination of the screening cloud in the anisotropic Kondo model [39].

The region \(g > 1/2\) - in particular the \(SU(2)\) symmetric point \(g = 1\) presents additional difficulties, which are not yet solved. The situation does not look desperate however.

As a last but important comment, I would like to stress that the form-factors technique has also been applied successfully to the computation of quantities of experimental interest in the case of (bulk) massive theories. See the lectures by F. Essler in this school, or the papers [40, 41, 42]. One point functions in massive theories and in the presence of a boundary have also been recently studied in [43].

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\(^1\)Of course the problem of Friedel oscillations for \(g = 1\) can be solved by fermionization [29], since it corresponds to non interacting fermions. In our approach, this point is non trivial because of the folding. This folding however is necessary for any value \(g \neq 1\): except at \(g = 1\), the problem on the whole line would not be integrable.
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