Euclidean integers, Euclidean ultrafilters, and Euclidean numerosities

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Abstract

We introduce axiomatically the ring $\mathbb{Z}_\kappa$ of the Euclidean integers, that can be viewed as the “integral part” of the field $\mathbb{E}$ of Euclidean numbers of [4], where the transfinite sum of ordinal indexed $\kappa$-sequences of integers is well defined. In particular any ordinal might be identified with the transfinite sum of its characteristic function, preserving the so called natural operations.

The ordered ring $\mathbb{Z}_\kappa$ may be obtained as an ultrapower of $\mathbb{Z}$ modulo suitable ultrafilters, thus constituting a ring of nonstandard integers. Most relevant is the algebraic characterization of the ordering: a Euclidean integer is positive if and only if it is the transfinite sum of natural numbers. This property requires the use of special ultrafilters called Euclidean, here introduced to this end.

The ring $\mathbb{Z}_\kappa$ allows to assign a “Euclidean” size (numerosity) to “ordinal Punktmengen”, i.e. sets of tuples of ordinals, as the transfinite sum of their characteristic functions: so every set becomes equinumerous to a set of ordinals, the Cantorian definitions of order, addition and multiplication are maintained, while the Euclidean principle “the whole is greater than the part” is fulfilled.

Introduction

The paper [4] introduces the ordered field $\mathbb{E}$ of the Euclidean numbers, constituted by all the transfinite sums of real numbers, of length less than a fixed (strong) limit cardinal $\kappa$. Similarly, we introduce here axiomatically, for any cardinal $\kappa$, the ordered domain $\mathbb{Z}_\kappa$ of the Euclidean integers, characterized as the collection of all the transfinite sums of ordinal indexed $\kappa$-sequences of integers. Only ordinal indexed sums are considered, so as to avoid the antinomies and paradoxes that might affect the summation up of infinitely many numbers. In particular, the choice of ordinal numbers as indices seems particularly appropriate, given their natural wellordered structure, combined with the lattice structure inherited from their finite subsets (see Subsection I.1).
We call \( \mathbb{Z}_\kappa \) the ring of the \textit{Euclidean integers} because it arises in a "numerosity theory" of ordinal-labelled sets, whose main aim is to save all the Euclidean common notions, including the fifth "the whole is greater than the part", but still maintaining the Cantorian definitions of ordering, addition and multiplication of sets. Having at disposal transfinite sums of integers, a natural way of assigning numbers (numerosities) to sets is to take the transfinite sums of their respective characteristic functions, which are Euclidean integers after an appropriate labelling of sets by ordinals. These numerosities being nonnegative Euclidean integers, their arising arithmetic, in contrast to the awkward Cantorian cardinal arithmetic, shows the best algebraic properties, since these numerosities constitute the non-negative part of the ring \( \mathbb{Z}_\kappa \), a semiring of \textit{hypermixed numbers} of Nonstandard Analysis (see [3]).

The paper is organized as follows.

In Section 1 we introduce axiomatically the ring \( \mathbb{Z}_\kappa \) of the Euclidean integers, an ordered non-Archimedean ring with a supplementary structure given by the operation of \textit{transfinite sum} \( \sum \alpha a_\alpha \), where \( \langle a_\alpha \mid \alpha < \kappa \rangle \) is a \( \kappa \)-sequence integers, subject to four axioms stated in Subsection 1.2. Every Euclidean integer is obtained as a transfinite sum of ordinary integers, and more generally, any transfinite sum of integers is well defined. Moreover the ring of the Euclidean integers is characterized, in Subsection 1.3, as an \textit{ultrapower of ordinal-indexed finite partial sums}, modulo suitable ultrafilters, hence \( \mathbb{Z}_\kappa \) is a ring of \textit{hypermixed numbers}.

In Subsection 1.4 any ordinal \( \beta < \kappa \) is associated to the transfinite sum \( \sum \alpha \chi_\beta(\alpha) \) of its characteristic function \( \chi_\beta(\alpha) = \begin{cases} 1 & \text{if } \alpha < \beta, \\ 0 & \text{otherwise}. \end{cases} \), consistently with the so called \textit{natural ordinal operations} \( \oplus \) and \( \otimes \), so the ring \( \mathbb{Z}_\kappa \) might be considered as a sort of \textit{natural extension} both of the ring of the ordinary integers \( \mathbb{Z} \) and of the \textit{semiring} of ordinals \((\kappa; \oplus, \otimes)\).

Subsection 1.5 is dedicated to the main result of the paper, namely the existence of Euclidean ultrafilters, or equivalently of rings \( \mathbb{Z}_\kappa \) of Euclidean integers satisfying the strong axiom (SRA), which allows for the most wanted identification of the non-negative part \( \mathbb{Z}_\kappa^{\geq 0} \) with the set of all transfinite sums of \textit{natural numbers}.

In Section 2 we specialize the general Euclidean principles for dealing with the size of sets, and we explicit the axioms of our Euclidean theory, grounded on these principles, together with appropriate properties of multiplication and (total) ordering of \textit{numerosities}. In particular we deal with the proper superset property (SupP), and with the equivalent difference property (diff), which follow from the strong representation axiom (SRA).

In Subsection 2.4 Euclidean integers are assigned as numerosities to all point sets of finitely dimensional linear spaces over an "ordinal line" in such a way that all properties corresponding to the Euclidean common notions are satisfied. Moreover one obtains the supplementary benefits that every point set is equinumerous to a set of ordinals, and conversely that every nonnegative Euclidean integer is the numerosity of a set \( X \) of ordinals, namely the transfinite sum of
the characteristic function $\chi_X$.

A few final remarks and open questions can be found in Section 3.

In general, we refer to [12] for the set-theoretical notions and facts used in this paper, and to [6] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models.

1 The Euclidean integers

We present here an axiomatization of the ordered domain $\mathbb{Z}_\kappa$ of the Euclidean integers, where all and only the transfinite sums of $\kappa$-sequences of ordinary integers represent any element of $\mathbb{Z}_\kappa$. This axiomatization is independent of, but essentially the same as the Euclidean numbers of [4]. The main new result is the possibility of postulating the stronger axiom ($\text{SRA}$), which allow to make the semiring $\mathbb{Z}_\kappa^{\geq 0}$ of the nonnegative Euclidean integers coincide exactly with the set of all transfinite sums of natural numbers.

1.1 Recalls on ordinals

The so called "natural" product and sum of ordinals will be denoted by $\alpha \otimes \beta$ and $\alpha \oplus \beta$, respectively, whereas $\alpha \beta$, $\alpha + \beta$, and $\alpha^\beta$ denote the ordinary ordinal operations.

Recall that, given ordinals $\alpha, \beta$, there exist uniquely determined ordinals $\gamma \leq \alpha$ and $\delta < 2^\beta$, such that

$$\alpha = (2^\beta \gamma) + \delta.$$ 

Hence each ordinal has a unique base-2 normal form

$$\alpha = \sum_{n=1}^{N} 2^{\alpha_n}, \text{ where } i < j \Rightarrow \alpha_i > \alpha_j,$$

and one has $\alpha = \bigoplus_{n=1}^{N} 2^{\alpha_n}$, independently of the ordering of the exponents.

In particular $2^\omega = \omega$, and the power $2^\alpha = \omega^\alpha$ whenever $\alpha = \omega \alpha$. It follows that the fixed points of the function $\alpha \mapsto 2^\alpha$ are $\omega$ and the so called $\varepsilon$-numbers $\varepsilon$ such that $\omega^\varepsilon = \varepsilon$.

Recalling the antilexicographic wellordering of finite sets of ordinals

$$L < L' \text{ if and only if } \max (L \triangle L') \in L',$$

the $\alpha$th finite set of ordinals, is

$$L_\alpha = \{\alpha_1, ..., \alpha_n\} \text{ where } \alpha = \bigoplus_{i=1}^{n} 2^{\alpha_i}.$$

In particular

$$L_0 = \emptyset, \quad L_\alpha = \{\alpha\}, \quad \text{and} \quad L_{2^{\alpha} + \beta} = \{\alpha\} \cup L_\beta \text{ for all } \beta < 2^\alpha.$$
So \( \mathcal{P}_{\text{fin}}(\alpha) \) can be naturally indexed by \( 2^\alpha \). The correspondence \( \alpha \mapsto L_\alpha \) induces two restrictions \( \sqsubset, < \) of the ordinal ordering, called respectively formal inclusion\(^1\) corresponding to ordinary set-inclusion between finite sets \( \alpha \sqsubset \beta \iff L_\alpha \subset L_\beta \), and formal membership corresponding to ordinary membership \( \alpha < \beta \iff \alpha \in L_\beta \). Then we have:

1. Given \( \alpha = \bigoplus_{i \in I} 2^{\alpha_i}, \beta = \bigoplus_{h \in H} 2^{\beta_h} \), one has \( \alpha \sqsubset \beta \iff I \subset H \) and \( \alpha < \beta \iff 2^\alpha \subseteq \beta \iff \exists h \in H \alpha = \alpha_h \). 

2. For \( A \subseteq \kappa \) let \( \hat{A} = \{\alpha \mid L_\alpha \subseteq A\} \): then the map \( \ell : \alpha \mapsto L_\alpha \) is a lattice isomorphism of \( (\hat{A}, \sqsubset) \) onto \( ([|A|]^{<\omega}, \sqsubset) \). In particular, for \( |X| = \kappa \), 
   \[
   ([|X|]^{<\omega}, \sqsubset) \cong ([|\kappa|]^{<\omega}, \sqsubset) \cong (\kappa, \sqsubset) \cong (\varepsilon, \sqsubset) \text{ for } \kappa \leq \varepsilon = 2^\varepsilon < \kappa^+.
   \]

3. The following properties hold:
   - \( 0 \sqsubset \alpha \) for all \( \alpha \), and \( \alpha \sqsubset \beta \Rightarrow \alpha < \beta ; \)
   - \(|\{\beta \mid \beta \sqsubset \alpha\}| = 2^{L_\alpha} \) and \(|\{\beta \mid \beta < \alpha\}| = |L_\alpha| \) are finite for all \( \alpha \);
   - for all \( \alpha, \beta, \gamma \) one has \( \alpha, \beta < \gamma \iff 2^\alpha \sqcup 2^\beta \subseteq \gamma ; \)
   - for all \( \alpha \) and all \( \beta, \gamma, \xi < 2^\theta \) one has the following useful criteria:
     \[
     (C) \quad 2^\theta \alpha + \beta \sqsubset \delta \iff 2^\theta \alpha, \beta \sqsubset \delta \\
     (D) \quad \delta = 2^{\theta^2} + 2^\theta \xi + \xi \implies \left( 2^\theta \gamma + \beta \sqsubset \delta \iff \gamma, \beta \sqsubset \delta \right)
     \]

4. The cones \( C(\theta) = \{\alpha \mid \theta \sqsubset \alpha\} \), for \( \theta < \kappa \), generate a filter \( \mathcal{C}_\kappa \) on \( \kappa \): call \( \text{fine} \) a filter \( \mathcal{F} \) on \( \kappa \) that contains \( \mathcal{C}_\kappa \).

5. For \( \eta < \kappa \), let \( D(\eta) = \{\delta = 2^{\eta^2} + 2^\eta \xi + \xi \mid \xi < 2^\eta, \alpha < \kappa\} \): the family of sets \( D(\eta, \theta) = D(\eta) \cap C(\theta) \) has the FIP, and generates a \( \text{fine filter} \) \( \mathcal{D}_\kappa \) on \( \kappa \): call \( \text{superfine} \) a filter \( \mathcal{F} \) on \( \kappa \) that contains \( \mathcal{D}_\kappa \).

### 1.2 Axiomatic introduction of the Euclidean integers

From the algebraic point of view, the Euclidean integers are a non-Archimedean discretely ordered superring \( \mathbb{Z}_\kappa \) of \( \mathbb{Z} \), with a supplementary structure, the Euclidean structure, introduced axiomatically via the operation of transfinite sum \( \Sigma(a) = \sum_\alpha a_\alpha \), where \( a = \langle a_\alpha \mid \alpha < \kappa \rangle \in \mathbb{Z}_\kappa \) is any \( \kappa \)-sequence of integers

We don’t need the full field \( \mathbb{E} \) of the Euclidean numbers, so we do not ground on their general theory of \([4]\). We present here instead a simpler independent axiomatization of the ring of the Euclidean integers \( \mathbb{Z}_\kappa \), which is best suited for assigning Euclidean sizes to sets, in particular if integrated with the stronger axiom (SRA), whose consistency was not known before.

Let \( \mathbb{Z}_\kappa \) denote a discretely ordered commutative domain, endowed with the supplementary “Euclidean” structure given by the transfinite sum

\[
\sum_\alpha a_\alpha = \Sigma(a), \quad \text{where } \langle a_\alpha \mid \alpha < \kappa \rangle = a \in \mathbb{Z}_\kappa,
\]

\(^1\) The name formal inclusion should also recall that the respective base-2 normal forms are indeed contained one inside of the other one.
Remark that we intend that any transfinite sum comprehends all summands \( a_\alpha, \alpha < \kappa \). When needed, we restrict the sum to a subset \( K \subseteq \kappa \) by putting

\[
\sum_{\alpha \in K} a_\alpha = \sum_\alpha b_\alpha, \quad \text{with} \quad b_\alpha = a_\alpha \cdot \chi_K(\alpha), \quad \text{and} \quad \chi_K(\alpha) = \begin{cases} 1 & \text{if } \alpha \in K, \\ 0 & \text{otherwise.} \end{cases}
\]

We make the natural assumption that a transfinite sum coincides with the ordinary sum of the ring \( \mathbb{Z}_\kappa \) when the number of non-zero summands is finite, and, similarly to \( 4 \), we postulate for the ring \( \mathbb{Z}_\kappa \) the following axioms.

First, an axiom representing each Euclidean integer as a transfinite sum of (ordinary) integers.

\textbf{(RA) (Representation Axiom)}

For any \( \mathfrak{r} \in \mathbb{Z}_\kappa \) there exists \( x = \langle x_\alpha \mid \alpha < \kappa \rangle \in \mathbb{Z}^{\kappa} \) such that \( \mathfrak{r} = \sum_\alpha x_\alpha \).

Next, a linearity axiom:

\textbf{(LA) (Linearity Axiom)} The transfinite sum is \( \mathbb{Z} \)-linear, i.e.

\[
u \sum_\alpha x_\alpha + \nu \sum_\alpha y_\alpha = \sum_\alpha (\nu x_\alpha + \nu y_\alpha) \quad \text{for all } \nu, v, x_\alpha, y_\alpha \in \mathbb{Z}.
\]

Then an axiom for comparing transfinite sums:

\textbf{(CA) (Comparison Axiom)} For all \( x, y \in \mathbb{Z}^{\kappa} \)

\[
\exists \theta < \kappa \forall \delta \exists \theta \left( \sum_{\alpha \subseteq \delta} x_\alpha \leq \sum_{\alpha \subseteq \delta} y_\alpha \right) \implies \sum_\alpha x_\alpha \leq \sum_\alpha y_\alpha
\]

(Remark that any sum \( \sum_{\alpha \subseteq \delta} x_\alpha \) is an ordinary finite sum of integers.)

Finally an axiom for multiplying transfinite sums:

\textbf{(PA) (Product axiom)}:

\[
(\sum_\alpha x_\alpha)(\sum_\beta y_\beta) = \sum_{\alpha, \beta} x_\alpha y_\beta,
\]

where the transfinite double sum \( \sum_{\alpha, \beta} \) is defined by:

\[
\sum_{\alpha, \beta} = \sum_\gamma \left( \sum_{\alpha \vee \beta = \gamma} x_\alpha y_\beta \right) \quad \text{where} \quad \bigoplus 2^\delta \vee \bigoplus 2^\delta = \bigoplus 2^\delta
\]

(i.e. the ordinal \( \alpha \vee \beta \) is the supremum w.r.t. the lattice ordering \( \sqsubseteq \).)

\textsuperscript{2} For sake of clarity, we tend to denote general Euclidean numbers by fractures \( a, b, c, s, t, \theta, g, \beta \), and integers by latin letters \( a, b, c, m, n, p, q, u, v, w, x, y, z \). The ordinal indices are denoted by greek letters \( \alpha, \beta, \gamma, \delta, \eta \), with \( \kappa, \nu, \mu \) reserved for cardinals; \( \kappa \)-sequences are denoted by boldface letters \( a, b, x, \ldots \).
So \( \sum_{\gamma = \alpha \lor \beta} x_{\alpha \beta} \) is an ordinary finite sums of integers, and the corresponding comparison criterion holds:

\[
\exists \theta < \kappa \; \forall \delta \geq \theta \left( \sum_{\alpha, \beta \subseteq \delta} x_{\alpha \beta} \leq \sum_{\alpha, \beta \subseteq \delta} y_{\alpha \beta} \right) \implies \sum_{\alpha, \beta} x_{\alpha \beta} \leq \sum_{\alpha, \beta} y_{\alpha \beta}
\]

Then the following useful property follows:

**Translation invariance:** If \( \eta, \gamma < \kappa \) and \( x_\alpha = 0 \) for \( \alpha \geq 2^n \eta \), then

\[
(TI) \sum_\alpha x_\alpha = \sum_\delta y_\delta, \quad \text{where } y_\delta = \begin{cases} 
  x_\alpha & \text{if } \delta = 2^n \gamma + \alpha \\
  0 & \text{otherwise}
\end{cases}.
\]

In fact, \( \sum_{\alpha \subseteq \beta} x_\alpha = \sum_{\delta = 2^n \gamma + \alpha \subseteq \beta} y_\delta \), for \( \alpha < 2^n \) and \( \beta \supseteq 2^n \gamma \).

**Remark 1.1.** An axiom much stronger than (RA), connected with the Subtraction principle of numerosities (diff), dealt with in Subsection 2.2, could characterize the non-negative Euclidean integers as transfinite sums of natural numbers:

\[
(\text{SRA}) \text{(Strong Representation Axiom)}
\]

For all \( \tau \geq 0 \) in \( \mathbb{Z}_\kappa \) there exists \( n = \langle n_\alpha \mid \alpha < \kappa \rangle \in \mathbb{N}_\kappa \) such that \( \tau = \sum_\alpha n_\alpha \).

The axiom (SRA) directly yields the representation axiom (RA) above, since every Euclidean integer is the difference of two disjoint transfinite sums of natural numbers \( z = \sum_{\alpha \in A^+} z_\alpha - \sum_{\alpha \in A^-} z_\alpha \), where \( A^\pm = \{ \alpha \mid z_\alpha > 0 \} \).

More important, the full axiom (SRA) has the interesting consequence that all Euclidean integers are actually of the simpler form \( z = \pm \sum_{\alpha \subseteq \beta} n_\alpha \) for some \( n \in \mathbb{N}_\kappa \): this fact is of great importance for the theory of numerosities.

### 1.3 The counting functions

To each \( \kappa \)-sequence of integers \( \mathbf{x} \in \mathbb{Z}_\kappa \) we associate its counting function \( f_\mathbf{x} : \kappa \to \mathbb{Z} \) defined by \( f_\mathbf{x}(\alpha) = \sum_{\beta \subseteq \alpha} x_\beta \).

**Lemma 1.2.** Every function \( \psi : \kappa \to \mathbb{Z} \) is the counting function \( f_\mathbf{x} \) of some \( \kappa \)-sequence \( \mathbf{x} \in \mathbb{Z}_\kappa \).

**Proof.** Given \( \psi : \kappa \to \mathbb{Z} \) define \( x_\alpha \) inductively by putting

\[
x_0 = \psi(0), \quad x_\alpha = \psi(\alpha) - \sum_{\beta \subseteq \alpha} x_\beta.
\]

Thus \( \psi(\alpha) = \sum_{\beta \subseteq \alpha} x_\beta \). \qed

Then we obtain the following characterization
Theorem 1.3. There exist a fine ultrafilter $U$ over $\kappa$, corresponding to a prime ideal $p$ of the ring $\mathbb{Z}^\kappa$, and an isomorphism $\sigma$ of the ring $\mathbb{Z}_\kappa$ of the Euclidean integers onto the ultrapower $\mathbb{Z}_U^\kappa \cong \mathbb{Z}^\kappa / p$ that makes the diagram $(\ast)$ commute:

\[
\begin{array}{cccc}
\mathbb{Z}^\kappa & \xrightarrow{f} & \mathbb{Z}^\kappa \\
\Sigma & \downarrow & \downarrow \pi_U \\
\mathbb{Z}_\kappa & \overset{\sigma}{\longrightarrow} & \mathbb{Z}^\kappa / p \cong \mathbb{Z}_U^\kappa
\end{array}
\]

($f$ maps $x \in \mathbb{Z}^\kappa$ to its counting function $f_x \in \mathbb{Z}^\kappa$, $\Sigma$ maps $x \in \mathbb{Z}^\kappa$ to its transfinite sum $\Sigma(x) \in \mathbb{Z}_\kappa$, while $\pi_p$ and $\pi_U$ are the canonical projections of $\mathbb{Z}^\kappa$ onto the quotient ring mod $p$ and onto the ultrapower mod $U$, respectively)

Moreover $\Sigma$ maps the semiring $\mathbb{N}^\kappa$ of all $\kappa$-sequences of natural numbers onto the nonnegative part $\mathbb{Z}^\kappa \geq 0$ of $\mathbb{Z}^\kappa$ if and only if $\mathbb{Z}_\kappa$ satisfies the strong representation axiom (SRA).

Proof. Let $U \ni C_\kappa$ be the ultrafilter generated by the zero-sets of $\Sigma$, i.e. the sets $Z(\mathbb{x}) = \{ \alpha < \kappa \mid f_\mathbb{x}(\alpha) = 0 \}$ of those $\mathbb{x} \in \mathbb{Z}^\kappa$ that give $\Sigma(\mathbb{x}) = 0$. Then $\sigma$ is consistently and uniquely defined by putting $\sigma(\Sigma(\mathbb{x})) = \pi_U(f_\mathbb{x})$.

It turns out that any fine ultrafilter $U$ on $\kappa$ validates the axioms (RA),(LA), (CA), and (PA). (Remark that $U$ must intersect all cones $C(\theta)$, by axiom (CA), hence it should be fine.)

Thus the existence of rings of Euclidean integers is granted, and problems might arise only in the search for suitable ultrafilters validating the strong representation axiom (SRA), which, differently from the other axioms, is independent even of the strong axioms of the Euclidean field $\mathbb{E}$ of [4].

Remark 1.4. By virtue of the fine ultrafilter $U$, the comparison axiom may receive the following formulation

(CA) (Comparison Axiom) For all $x, y \in \mathbb{Z}^\kappa$

\[
\sum_\alpha x_\alpha \leq \sum_\alpha y_\alpha \iff \exists U \in U \forall \delta \in U \left( \sum_{\alpha \leq \delta} x_\alpha \leq \sum_{\alpha \leq \delta} y_\alpha \right)
\]

In particular, if the ultrafilter $U$ is superfine, i.e. contains, besides the cones $C(\theta)$, also the sets $D(\eta) = \{ \delta = 2^{\eta^2} \alpha + 2^\eta \xi + \xi \mid \xi < 2^\eta, \alpha < \kappa \}$, for all $\eta < \kappa$, then criterion (D) of Subsection [1] gives the following property:

Double sum linearization: If $\eta < \kappa$ and $x_{\beta, \gamma} = 0$ for $\gamma, \beta \geq 2^\eta$, then

\[
(DSL) \quad \sum_{\beta, \gamma} x_{\beta, \gamma} = \sum_\alpha y_\alpha, \quad \text{where} \quad y_\alpha = \begin{cases} x_{\beta, \gamma} & \text{if } \alpha = 2^\eta \beta + \gamma \\ 0 & \text{otherwise} \end{cases}
\]

In fact, by taking $\delta \in D(\eta)$, $\sum_{\beta, \gamma \subseteq \delta} x_{\beta, \gamma} = \sum_{2^{\eta \beta} + \gamma \subseteq \delta} x_{\beta, \gamma} = \sum_{\alpha \subseteq \delta} y_\alpha$. 7
1.4 Embedding the ordinals

The fact that the ordinals less than $\kappa$, with the so called “natural sum and product” can be naturally embedded into $\mathbb{Z}_\kappa$ as an ordered subsemiring, could be proved exactly as in [4], but it should be remarked that, in dealing with products, [4] uses the property (DSL), that follows there from the axiom (DSS), that is not assumed here. However, as proved above, the property (DSL) holds when the ultrafilter $U$ of Theorem 1.3 is superfine.

**Theorem 1.5.** (Thm. 2.2 of [4]) Define $\Psi : \kappa \rightarrow \mathbb{Z}_\kappa$ by $\Psi(\beta) = \sum_\alpha \chi(\alpha)$, where $\chi$ is the characteristic function of $\beta$, i.e. $\chi(\alpha) = \begin{cases} 1 & \text{if } \alpha < \beta, \\ 0 & \text{otherwise} \end{cases}$. Then

$\alpha < \beta \iff \Psi(\alpha) < \Psi(\beta)$, and $\Psi(\alpha \oplus \beta) = \Psi(\alpha) + \Psi(\beta)$,

hence $\Psi$ is an isomorphic embedding of $(\kappa; <, \oplus)$, as ordered semigroup, into the nonnegative part of $\mathbb{Z}_\kappa$.

Moreover, if the property (DSL) holds, then also $\Psi(\alpha \otimes \beta) = \Psi(\alpha) \cdot \Psi(\beta)$, and $\Psi$ is an isomorphic embedding of $(\kappa; <, \oplus, \otimes)$, as ordered semiring, into the nonnegative part of the ring $\mathbb{Z}_\kappa$.

(Recall that $\oplus, \otimes$ denote the natural sum and product of ordinals.)

**Proof.** The values of $\Psi$ being transfinite sums of ones “without holes”, the assertion on $<$ is immediate. Moreover, commutativity, associativity and distributivity holding both in the ring $\mathbb{Z}_\kappa$ and in the semiring $(\kappa, \oplus, \otimes)$, it suffices to refer to the base-2 normal form and prove that, for $\theta \geq \eta$,

$$\Psi(2^\theta) + \Psi(2^\eta) = \Psi(2^\theta \oplus 2^\eta) \quad \text{and} \quad \Psi(2^\theta \otimes 2^\eta) = \Psi(2^\theta \cdot 2^\eta).$$

The first equality follows directly by Translation Invariance (TI), whereas the latter follows when the property (DSL) of Double Sum Linearization holds:

$$\Psi(2^\theta) \cdot \Psi(2^\eta) = \left( \sum_\beta \chi(\beta) \right) \left( \sum_\gamma \chi(\gamma) \right) = \sum_{\beta, \gamma} w_{\beta\gamma}, \quad w_{\gamma\beta} = \begin{cases} 1 & \text{if } \beta < 2^\theta, \gamma < 2^\eta \\ 0 & \text{otherwise} \end{cases}$$

$$\Psi(2^\theta \otimes 2^\eta) = \Psi(2^{\theta+\eta}) = \sum_\alpha \chi(\alpha) = \sum_\alpha z_\alpha, \quad z_\alpha = \begin{cases} w_{\gamma\beta} & \text{if } \alpha = 2^\theta \gamma + \beta \\ 0 & \text{otherwise} \end{cases}$$

Actually, by taking any $\delta \in D(\theta)$, criterion (D) gives

$$\sum_{\alpha \in \delta} z_\alpha = \sum_{2^\gamma \gamma+\beta \in \delta} w_{\gamma\delta} = \sum_{\gamma, \beta \in \delta} w_{\gamma\beta},$$

and so $\sum_{\alpha} z_\alpha = \sum_{\gamma, \beta} w_{\gamma\beta}$ follows by (DSL). \qed

It is worth noticing that the meaning of the natural product between ordinal numbers, defined through order types, might seem quite involved and not easily intuitive. On the contrary, thinking of an ordinal number as a particular Euclidean integer, namely as a transfinite sum of ones “without holes”, makes appear quite natural the meaning of the product, as given by the product formula. And by the same reason the natural ordering of ordinals obviously agrees with that induced by the ordering of $\mathbb{Z}_\kappa$. 

8
1.5 The Euclidean ultrafilters.

Given a function $\psi \in \mathbb{Z}^\kappa$ that is positive modulo $\mathcal{U}$, in order to validate the strong representation axiom (SRA) one needs a nonnegative sequence $x \in \mathbb{N}^\kappa$ such that $\{\alpha < \kappa \mid f_\alpha(x) = \sum_{\beta \leq \alpha} x_\beta = \psi(\alpha)\} \in \mathcal{U}$. Now, nonnegative $\kappa$-sequences $x$ give rise to nondecreasing counting functions $f_\alpha$, and conversely, so one needs a fine ultrafilter $\mathcal{U}$ including, for each $\psi \in \mathbb{N}^\kappa$, a set $U_\psi$ such that

$$\forall \alpha, \beta \in U_\psi (\alpha \vartriangleleft \beta \implies \psi(\alpha) \leq \psi(\beta)).$$

Let $[\kappa]^2 = \{(\alpha, \beta) \mid \alpha \vartriangleleft \beta\}$ be the set of all $\subseteq$-ordered pairs, and let $G : [\kappa]^2 \rightarrow \{0, 1\}$ be a 2-partition of $[\kappa]^2$; a set $H \subseteq \kappa$ is $i$-homogeneous for $G$ if $\forall \alpha, \beta \in H_i (\alpha \vartriangleleft \beta \implies G(\alpha, \beta) = i)$.

For $\psi \in \mathbb{N}^\kappa$, define the partition

$$G_\psi : [\kappa]^2 \rightarrow \{0, 1\} \text{ by } G_\psi(\alpha, \beta) = \begin{cases} 0 & \text{if } \psi(\alpha) > \psi(\beta), \\ 1 & \text{otherwise}. \end{cases}$$

Then the partition $G_\psi$ does not admit any sequence $\langle \alpha_n \mid n < \omega \rangle$ such that, for $n < \omega, \alpha_n \vartriangleleft \alpha_{n+1}$ and $G(\alpha_n, \alpha_{n+1}) = 0$ (call such a sequence a 0-chain).

Call Euclidean a fine ultrafilter $\mathcal{U}$ on $\kappa$ if for all $\psi \in \mathbb{N}^\kappa$ the 2-partition $G_\psi$ of $[\kappa]^2$ has a homogeneous set $U_\psi \in \mathcal{U}$. Clearly $U_\psi$ is $\subseteq$-cofinal, hence it cannot be 0-homogeneous, so it satisfies the condition (#). Hence Theorem 1.3 yields the following characterization:

**Corollary 1.6.** The ring of the Euclidean integers $\mathbb{Z}_\kappa \cong \mathbb{Z}_{\kappa}^{\omega}$ satisfies the strong representation axiom (SRA) if and only if the ultrafilter $\mathcal{U}$ is Euclidean. \qed

We are left with the question of the existence of Euclidean ultrafilters.

**Remark 1.7.** The partition property $[A]^\omega \rightarrow (\text{cofin})^r_\kappa$. (see [13] [14])

The partition property $[A]^\omega \rightarrow (\text{cofin})^r_\kappa$ means that any finite partition of all $\subseteq$-ordered $r$-tuples of finite subsets of $A$ admits a $\subseteq$-cofinal homogeneous subset $H \subseteq A$, i.e. every $u \in [A]^\omega$ is included in some $v \in [H]^\omega$, and all $\subseteq$-ordered $r$-tuples from $[H]^\omega$ belong to the same piece of the partition.

Clearly, the partition property $[A]^\omega \rightarrow (\text{cofin})^r_\kappa$ depends only on $|A|$. For countable $A$ it follows immediately from Ramsey’s Theorem, and for $|A| = \aleph_1$ it has been proved by Jech and Shelah in [14], while the general problem, already posed by Thomas Jech in 1973, remains still open (see [13]).

Considering the corresponding notion for the formal inclusion $\subseteq$, the validity of $\kappa \rightarrow (\subseteq-\text{cofin})^r_\kappa$ would yield directly the existence of Euclidean ultrafilters on $\kappa$; however, the full property $\kappa \rightarrow (\subseteq-\text{cofin})^r_\kappa$ might be stronger than needed for the existence of Euclidean ultrafilters: to be sure, a (possibly weaker) property providing an appropriate version of the Erdős-Dushnik-Miller partition property $\kappa \rightarrow (\omega, \kappa)^2$ suffices.
1.5.1 The partition property $\kappa \to (\omega, \kappa)^2_\Box$.

**Definition 1.8.** The partition property $\kappa \to (\omega, \kappa)^2_\Box$ affirms that any 2-partition $G : [\kappa]^2_\Box \to \{0, 1\}$ either admits a 0-chain (i.e., a $\Box$-increasing sequence $\alpha_n$ with $G(\alpha_n, \alpha_{n+1}) = 0$), or it has a $\Box$-cofinal homogeneous set $H$ (hence necessarily 1-homogeneous).

This partition property is all that is needed in order to have Euclidean ultrafilters, namely

**Lemma 1.9.** If $\kappa \to (\omega, \kappa)^2_\Box$ holds, then there are Euclidean ultrafilters on $\kappa$.

**Proof.** Given $\psi_1, \ldots, \psi_n \in \mathbb{N}^\kappa$, define $G_\psi$, as above: then the product partition $G_\psi = \prod_1^n G_{\psi_i}$ cannot admit 0-chains, so there is a $\Box$-cofinal 1-homogeneous set $H_\psi$, which is simultaneously 1-homogeneous for all $G_{\psi_i}, 1 \leq i \leq n$. Hence the family $\mathcal{H} = \{H_\psi \mid \psi \in \mathbb{N}^\kappa\}$ has the FIP, and any fine ultrafilter $\mathcal{U}$ on $\kappa$ including $\mathcal{H}$ is Euclidean. $\square$

It turns out that the above property $\kappa \to (\omega, \kappa)^2_\Box$, has been recently stated for all cardinals $\kappa$ in [9], where the authors follow the track traced in section 2 of [14], with the ordinal $\alpha$ identified with the finite set $L_\alpha$, and formal inclusion $\Box$ replacing ordinary set-inclusion $\subset$. So the existence of rings of Euclidean integers satisfying the strong representation axiom (SRA) is granted.

2 Euclidean measures of size for sets

In set theory the usual measure of the size of sets is is given by the classical Cantorian notion of “cardinality”, whose ground is the so called Hume’s Principle

**Hume’s Principle**

Two sets have the same size if and only if there exists a biunique correspondence between them.

This assumption might seem natural, and even implicit in the notion of counting; but it strongly violates the equally natural Euclid’s principle applied to sets

\[ A \text{ set is greater than its proper subsets,} \]

which in turn seems implicit in the notion of magnitudo, even for sets.

So one could distinguish two basic kinds of size theories for sets:

- A size theory is Cantorian if, for all $A, B$:
  \[ (\text{HP}) \quad A \simeq B \iff \exists f : A \to B \text{ biunique} \]

- A size theory is Euclidean if, for all $A, B$:
  \[ (\text{EP}) \quad A < B \iff \exists C \text{ s.t. } A \subset C \text{ & } B \simeq C. \]
  (Remark the use of proper inclusion in defining strict comparison of sets.)
The consistency of the principle (EP) for uncountable sets appeared problematic from the beginning, and this question has been posed in several papers (see [1, 2, 7]), where only the literal set-theoretic translation of the fifth Euclidean notion, i.e. the sole left pointing arrow of (EP),

\[(E5) \quad A \subset B \implies A \prec B,\]

has been obtained. (On the other hand, it is worth recalling that also the totality of the Cantorian weak cardinal ordering had to wait more than two decades till Zermelo’s new axiom of choice to be established!)

A general discussion of different ways for comparing and measuring the size of sets can be found in [8]. Here we present a Euclidean numerosity theory for suitable collections \(W\) of point sets of finite dimensional spaces over lines \(L\) of arbitrary cardinality, satisfying the full principle (EP). This numerosity might be extended to the whole universe under simple set theoretic assumptions, e.g. Von Neumann’s axiom, that gives a (class-)bijection between the universe \(V\) and the class \(Ord\) of all ordinals.

2.1 Natural congruences

First of all, once the general Hume’s principle cannot be assumed, the fourth Euclid’s common notion

*Things exactly applying onto one another are equal to one another*

is left in need of an adequate choice of natural “exact applications” that preserve size (\(=\)congruences): so we isolate the group \(\mathfrak{G}(W)\) of the “natural transformations” of tuples, i.e. those preserving the support (the set of components) of a tuple \((\text{supp}(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}\)\(^3\) that seem appropriate for a Euclidean theory involving sets of tuples, and postulate

\[(CP) \quad (\text{Congruence Principle}) \quad \sigma \in \mathfrak{G}(W) \implies \forall A \in W \ (\sigma[A] \in W \& \sigma[A] \simeq A).\]

2.2 Addition of numerosities

One wants not only compare, but also add and subtract magnitudines, according to the second and third Euclidean common notions

*... if equals be added to equals, the wholes are equal.*

*... if equals be subtracted from equals, the remainders are equal.*

When dealing with sets, it is natural to take addition to be (disjoint) union, and subtraction to be (relative) complement, and the following

\[(AP) \quad (\text{Aristotle’s Principle}) \quad A \simeq B \iff A \setminus B \simeq B \setminus A.\]

\(^3\) In particular \(A \times B \simeq B \times A\) and \((A \times B) \times C \simeq A \times (B \times C)\).

\(^4\) This principle has been named Aristotle’s Principle in [3, 11], because it resembles Aristotle’s preferred example of a “general axiom”. It is especially relevant in this context, because (AP) implies both the second and the third Euclidean common notions, and also the fifth whenever no nonempty set is equivalent to \(\emptyset\), as stated in the proposition below.
is convenient, because it yields both the second and third Euclidean common notions, see \[5\].

Define the addition of numerosities, i.e. equivalence classes modulo \(\simeq\), by

\[
[A] + [B] = [A \cup B] \quad \text{for all } A, B \text{ such that } A \cap B = \emptyset;
\]

then the quotient set \(\mathcal{R} = \mathbb{W}/\simeq\) becomes a positive semigroup, i.e. the non-negative part of an ordered abelian group. In particular one gets the “most wanted Subtraction Principle” of \([1]\):

\[
\text{(diff) } A \prec B \iff \exists C \neq \emptyset \ (C \cap (A \cup B) = \emptyset, \ (C \cup A) \simeq B).
\]

The consistency problem of the Subtraction Principle, studied in several papers dealing with Euclidean (also called Aristotelian) notions of size for sets, received a positive answer only for countable sets in \([7, 5, 11]\). A positive answer for sets of arbitrary cardinality is obtained in \([8]\), and follows from the existence of Euclidean ultrafilters, see Subsubsection 2.4.

2.3 Multiplication of numerosities

In classical mathematics, geometric figures having different dimensions are never compared, so a multiplicative version of Euclid’s second common notion

\[
... \text{if equals be multiplied by equals, the products are equal}
\]

was not considered, for “dishomogeneous” magnitudes.

On the other hand, in modern mathematics a single set of “numbers”, the real numbers \(\mathbb{R}\), is used as a common scale for the size of figures of any dimension. In a general set theoretic context it seems natural to consider abstract sets as homogeneous mathematical objects, without distinctions based on dimension, and a satisfying arithmetic of numerosities needs a product (with a corresponding unit): we adhere to the natural Cantorian choice of introducing multiplication through Cartesian products\(^5\) and taking singletons as unitary.\(^6\)

So any \(A \times \{b\}, b \in B\) may be viewed as a disjoint equinumerous copy of \(A\), thus making their (disjoint) union \(A \times B\) the sum of “\(B\)-many copies of \(A\)”, in accord with the intuitive idea of product.

2.4 Euclidean numerosities as Euclidean integers

The properties of the ring \(\mathbb{Z}_\kappa\) of the Euclidean integers allow for a simple engrafting of a Euclidean numerosity for “Punktmengen”, i.e. sets of tuples, over any line \(L\) of arbitrary size \(\kappa\). Since in a general set-theoretic context there

\(^5\) Although the Cartesian product is neither commutative nor associative \textit{stricto sensu}, nevertheless the corresponding natural transformations should be taken among the congruences in the group \(\Phi(W)\).

\(^6\) \textit{CAVEAT:} the Cartesian product is optimal when any two sets \(A, B\) are multipliable in the sense that their Cartesian product is disjoint from their union, but when transitive universes like \(V_\kappa, H(\kappa)\), or \(L\) are considered becomes untenable \(\text{e.g.} \) already \(V_\kappa \times \{x\} \subset V_\kappa\) for any \(x \in V_\kappa\) \(\text{e.g.} \) the discussion in \([1, 8]\)), hence not all singletons may be “suitable” for a Euclidean theory.
are no "geometric" or "analytic" properties to be considered, the sole relevant characteristic of the line \( L \) remains cardinality, so a convenient choice seems to be simply identifying \( L \) with its cardinal \( \kappa \), thus obtaining the fringe benefit that no pair of ordinals is an ordinal, and Cartesian products may be freely used. Grounding on the preceding discussion, we pose the following definition

**Definition 2.1.** A (Euclidean) numerosity for "Punktmengen" over \( L \) is a pair \((\mathcal{W}, \preceq)\), where \( \preceq \) is a total preordering on a set \( \mathcal{W} \subseteq \mathcal{P}(\bigcup_{n \in \mathbb{N}} L^n) \) such that

\[
A \cup B, \ A \times B \in \mathcal{W} \iff A, B \in \mathcal{W}, \ C \subseteq A \in \mathcal{W} \implies C \in \mathcal{W}
\]

and the following conditions are satisfied for all \( A, B, C \in \mathcal{W} \):

1. **(EP)** \( A \prec B \iff \exists B'(A \subset B' \simeq B) \), where inclusion and preordering are strict;
2. **(CP)** \( \tau[A] \simeq A \) for all \( \tau \in G(L) \), the group of all support-preserving bijections\(^7\);
3. **(AP)** \( A \simeq B \iff A \setminus B \simeq B \setminus A \);
4. **(PP)** \( A \simeq B \iff A \times C \simeq B \times C \) (for all \( C \neq \emptyset \));
5. **(UP)** \( A \simeq A \times \{w\} \) for all \( w \in W = \bigcup \mathcal{W} \).

The above Principles provide the set of numerosities \( \mathfrak{N} = \mathcal{W}/\simeq \) with the best arithmetic properties, namely those of the nonnegative part of an ordered domain, see e.g. \([8]\). Here, having at disposal the ring \( \mathbb{Z}_\kappa \) of the Euclidean integers, we give first a function \( n : \mathcal{P}(\kappa) \to \mathbb{Z}_\kappa \) as the transfinite sum of the characteristic functions of each subset of \( \kappa \):

\[
n(A) = \sum_{\alpha} \chi_A(\alpha), \quad \text{where} \quad \chi_A(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A \\ 0 & \text{otherwise.} \end{cases}
\]

Thus in particular \( n(\alpha) = \Psi(\alpha) \) for all ordinals.

Then we may extend the function \( n \) to \( \mathcal{P}(\kappa^n) \) by assigning to each \( n \)-tuple \( \overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \kappa^n \) the ordinal \( \psi_n(\overline{\alpha}) = \alpha_1 \lor \ldots \lor \alpha_n < \kappa \), and putting, for \( A \subseteq \kappa^n \),

\[
n(A) = \sum_\alpha \chi_A^{(n)}(\alpha), \quad \text{where} \quad \chi_A^{(n)}(\alpha) = |\{ \overline{\alpha} \in A \mid \psi_n(\overline{\alpha}) = \alpha \}|
\]

Remark that we are assigning the same ordinal \( \alpha \) to \( \alpha \in \kappa \), to \((\alpha_1, \alpha_2) \in \kappa^2\) if \( \alpha = \alpha_1 \lor \alpha_2 \), to \((\alpha_1, \ldots, \alpha_n) \in \kappa^n \) if \( \alpha = \bigvee_1^n \alpha_i \), hence the functions \( \chi^{(n)} \), for \( n > 1 \), are not properly characteristic functions, but they assume nonnegative integer values, so their sums are nonnegative Euclidean integers.

Now we can easily extend the numerosity function \( n \) to all finite dimensional point sets, \textit{i.e.} sets \( A \) such that \( \{ n \mid A \cap \kappa^n \neq \emptyset \} \) is finite, namely

\[
n(A) = \sum_n n(A \cap \kappa^n).
\]

Clearly\(^7\) see Subsubsection 2.1 in particular \( A \times B \simeq B \times A \) and \( (A \times B) \times C \simeq A \times (B \times C) \), so commutativity and associativity of multiplication follow.
3. Final remarks and open questions

3.1 The Weak Hume Principle and the Subset Property

Perhaps the best way to view a Euclidean numerosity is looking at it as a refinement of Cantorian cardinality, able to separate sets that, although equipotent, should have in fact really different sizes, in particular when they are proper subsets or supersets of one another. To this aim, the principle (EP) might be integrated by adding the clause

\[ (\text{WHP}) \quad n(A) \leq n(B) \implies \exists f: 1\text{-to-1}, f : A \to B. \]

So sums of ones of greater cardinality produce greater Euclidean integers, hence the ordering of the Euclidean numerosities refines the cardinal ordering, satisfying the “Weak Hume’s Principle”

If two sets are equinumerous, then there exists a biunique correspondence between them.

Another interesting consequence of (WHP) is the property

\[ (\text{SubP}) \quad (\text{Proper Subset Property}) \quad A \prec B \iff \exists A' \subset B \text{ s.t. } A' \simeq A. \]

In general, the set of numerosities has the same size as the universe \( W \), since one can define strictly increasing chains of sets of arbitrary length; but any set \( A \) has only \( 2^{\kappa} \) subsets, and so the Proper Subset Property (SubP) implies that the initial segment of numerosities generated by \( n(A) \) has size \( 2^{\kappa} \), contrary, e.g., to the large ultrapower models of \([1, 2]\).

This topic is dealt with in \([3]\), where it is proved that the family of sets

\[ Q_{AB}^{\kappa} = \{ \beta < \kappa \mid \sum_{\alpha \in \beta} \chi_A(\alpha) > \sum_{\alpha \in \beta} \chi_B(\alpha) \} \text{ for } |A| > |B|. \]
has the FIP together with the cones $C(\theta)$, hence may be contained in the fine ultrafilter $\mathcal{U}$. However it is not known whether that family might be included in an Euclidean ultrafilter, so the consistency of the weak Hume principle (WHP) with the difference property (diff) is still open.

3.2 The power of numerosities

The power $m^n$ of infinite numerosities is here always well-defined, since numerosities are positive euclidean numbers, hence nonstandard natural numbers. By using finite approximations given by intersections with suitable finite sets, the interesting relation

$$2^{m^n} = n([X]^{<\omega}),$$

has been obtained already in [1]. Since the comparison axiom (CA) evaluate transfinite sums by finite sums $\sum_{\alpha \subseteq \delta} \chi_A(\alpha)$, one obtains the following general set theoretic interpretation of powers:

$$m(Y)^{n(X)} = n(\{ f : X \to Y \mid |f| < \aleph_0 \}),$$

by considering sets of ordinals $X,Y$, and labelling each finite function $f$ by the $\subseteq$-supremum of the (finitely many) ordinals involved in $f$.

The interesting problem of finding appropriately defined arithmetic operations that give instead the numerosity of the full powersets and function spaces requires a quite different approach, and the history of the same problem for cardinalities suggests that it could not be properly solved.

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