Passivity of Electrical Transmission Networks modelled using Rectangular and Polar D-Q variables

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Abstract—The increasing penetration of converter-interfaced distributed energy resources has brought out the need to develop decentralized criteria that would ensure the small-signal stability of the inter-connected system. Passivity of the D-Q admittance or impedance is a promising candidate for such an approach. It is facilitated by the inherent passivity of the D-Q impedance of an electrical network. However, the passivity conditions are generally restrictive and cannot be complied with in the low frequency range by the D-Q admittance of devices that follow typical power control strategies. However, this does not imply that the system is unstable. Therefore, alternative formulations that use polar variables (magnitude/phase angle of voltages and real/reactive power injection instead of the D-Q components of voltages and currents) are investigated. Passivity properties of the electrical network using these different formulations are brought out in this paper through analytical results and illustrative examples.

Index Terms—Passive systems, small-signal stability, T&D network passivity, Network Jacobian, Grid resonance.

I. INTRODUCTION

A power system consists of a large number of devices like conventional and renewable energy generators, storage systems, FACTS and HVDC converters, which are connected to the Transmission & Distribution (T&D) network. Sometimes, adverse dynamic interactions between these devices and the network occur, leading to oscillatory instabilities [1]. These instabilities can be analyzed with the help of stability assessment tools like time-domain simulation and eigenvalue analysis. These studies generally need to consider a large set of operating conditions of the network and the connected devices. Therefore, the possibility of specifying decentralized criteria which, if complied with by individual devices, would ensure the stability of the interconnected system, has great appeal.

Passivity [2] is a concept well-suited for this purpose because it is a sufficient stability criterion and is easy to evaluate in the frequency domain. Moreover, a T&D network consisting of transmission lines, transformers, capacitors and inductors is inherently passive when it is formulated with currents and voltages as the interface (input or output) variables. Hence the task is reduced to assessing passivity of the devices connected to the network. The synchronously rotating (D-Q) coordinate system is convenient to do the passivity analysis as most devices are time-invariant in this frame of reference. Therefore, passivity of the D-Q based admittance of the shunt-connected devices has been used in the past to prevent adverse device-grid interactions [3]–[6].

The passivity of the admittance of converters and synchronous machines with their controllers (small-signal models) have been analyzed in [7]. It has been found that the frequency domain passivity conditions are invariably violated in the low frequency range, when droop based frequency and voltage control strategies are used. However, this inherent non-passivity does not imply that the system will be unstable. Therefore, the passivity constraints on the D-Q based admittance are too restrictive for wide-band dynamic models of these devices. Therefore, it may be necessary to consider the high and low frequency models in a decoupled fashion (with the assumption that transients are time-scale separated), and formulate the low frequency device models with other input-output variables. These include the active and reactive power injections, and the polar components of the bus voltage and their derivatives. However, whether the T&D network retains its passivity with these formulations needs to be investigated.

With this motivation, this paper derives analytical results pertaining to the passivity behaviour of the T&D network model when represented using the alternative input-output variables. It is found that the wide-band dynamical model of the T&D network is not passive when represented using these alternative interface variables. Further, it is shown that the low frequency model of the T&D network can be passivated if shunt-connected devices to the network can provide voltage regulation capabilities. The results are verified with numerical simulations on the T&D network of the IEEE 9-bus system [8].

II. PASSIVE SYSTEMS: DEFINITION

A dynamical system between the inputs $u(t)$ and an equal number of outputs $y(t)$ is called passive if it satisfies [2]

$$u(t)^T y(t) \geq S(x)$$

for all inputs and initial conditions. $S(x)$ is a continuously differentiable positive semi-definite function of the state variables $x(t)$, and is called the “storage function” of the system. The passivity definition for linear time-invariant (LTI) systems is now presented.
A. Passivity of LTI Systems (Time Domain Conditions)

A LTI system represented by the state space model \( (A, B, C, D) \), is passive if there exist matrices \( P, Q \) and \( W \) of appropriate dimensions such that

\[
PA + AT P = -Q^T Q, \quad PB = CT - QT W, \quad D + DT = WT W
\]

where the matrix \( P \) is symmetric positive definite.

B. Passivity of LTI Systems (Frequency Domain Conditions)

A LTI system represented by a \( n \times n \) rational, proper transfer function matrix \( G(s) \) is passive if

1) there are no poles in the right half \( s \)-plane (complex plane).
2) The matrix \( G^R(j \Omega) = G(j \Omega) + G^T(-j \Omega) \) is positive semi-definite for all \( \Omega \in (-\infty, \infty) \) which is not a pole of \( G(s) \).
3) For all \( j \Omega_p \) that are poles of \( G(s) \), the poles must be simple and the residue of that pole \( \lim_{s \to j \Omega_p} (s - j \Omega_p) G(s) \) should be positive semi-definite Hermitian.

For LTI systems, the time domain and frequency domain conditions are equivalent. The usefulness of the passivity criterion for stability assessment stems from the following properties.

a) Passivity is a sufficient condition for stability.

b) The inverse of a passive system is also passive, assuming that the state-space representation of the inverse system is well-defined.

c) A system formed by a negative feedback connection of passive sub-systems is also passive.

The passivity of the small-signal model of the T&D network, when represented using different input-output variables, is now presented.

III. MODEL I VARIABLES: \((\Delta v_D, \Delta v_Q)\) AND \((\Delta i_D, \Delta i_Q)\)

In general, a T&D network can be modelled as a multi-port admittance/impedance transfer function; the currents/voltages being the interface (input-output) variables. The storage function for this system can be chosen to be the electro-magnetic energy stored in the inductive and capacitive components, which is dissipated in the resistive parts of these components. Topological changes in the network (due to the addition or removal of lines), or changes in the operating conditions do not affect the passivity of the network.

Since the admittance/impedance of a T&D network is passive regardless of the topology or the operating condition, this can be exploited to develop a local small-signal stability assessment scheme, as shown in Figure 2. In this scheme, the D-Q domain admittance (represented by \( Y_{sj} \)) for device \( j \) in Figure 2 of the shunt-connected devices need to be individually passive in order to ensure the stability of the system. However, the violation of the frequency domain passivity conditions by controlled power injection devices in the low frequency range restricts the applicability of this scheme [7]. To overcome this difficulty, the passivity of these components, when modelled using other input-output variables, is also examined. The choice of these alternative variables are motivated by the variables used in typical steady-state real and reactive power control strategies.

IV. MODEL II VARIABLES: \((\Delta \phi, \Delta V_n)\) AND \((\Delta P, \Delta Q)\)

The interface variables in this case are the active and reactive power injections and the polar components of the bus voltage. Let \( P_j \) and \( Q_j \) denote the active and reactive power injection at bus \( j \) respectively, which are represented as follows.

\[
P_j = v_{DJ} i_{DJ} + v_{Qj} i_{Qj}, \quad Q_j = v_{DJ} i_{Qj} - v_{Qj} i_{DJ}
\]

where \( v_{DJ}, v_{Qj}, i_{DJ}, i_{Qj} \) represent the instantaneous D-Q components of the bus voltage and current injection at bus

Figure 1: Schematic of R-L-C series circuit

Figure 2: Scheme of passivity based stability criterion

1The zero sequence variables are generally stable, decoupled from the D-Q variables, and localized to a small part of the network. Therefore, they are not considered in this analysis.
The schematic of a network with con-

The lossless approximation does not alleviate

corresponding variable. The subscript \(j\) is dropped from

P and Q represent the vector of active and reactive power

j = \tan^{-1}\left(\frac{v_{Dj}}{v_{Qj}}\right), \quad V_{nj} = \sqrt{\frac{v_{Dj}^2 + v_{Qj}^2}{v_{Dj0}^2 + v_{Qj0}^2}}

where the subscript \(o\) denotes the quiescent value of the

P, and \(Q\) represent the vector of active and reactive power

\(v_D = \text{diag}(v_{D10}, \ldots, v_{Dn0}), v_Q = \text{diag}(v_{Q10}, \ldots, v_{Qn0})\)

\(i_D = \text{diag}(i_{D10}, \ldots, i_{Dn0}), i_Q = \text{diag}(i_{Q10}, \ldots, i_{Qn0})\)

Note that \(\text{diag}(a, b)\) represents a diagonal matrix with \(a\) and \(b\) as the diagonal entries. The equations of (2) are linearized, and their small-signal variations are represented as follows.

\[
\begin{bmatrix}
\Delta P(s) \\
\Delta Q(s)
\end{bmatrix} =
\begin{bmatrix}
J_{11}(s) & J_{12}(s) \\
J_{21}(s) & J_{22}(s)
\end{bmatrix}
\begin{bmatrix}
\Delta \phi(s) \\
\Delta V_n(s)
\end{bmatrix} = J(s) \begin{bmatrix}
\Delta \phi(s) \\
\Delta V_n(s)
\end{bmatrix}
\]

\(J(s)\) is related to the D-Q admittance \(Y_{DQ}(s)\) as follows.

\[
J(s) = (EY_{DQ}(s) + C) \mathcal{F}
\]

where

\[
\mathcal{E} = \begin{bmatrix}
v_D & -v_Q \\
v_Q & v_D
\end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix}
i_D & i_Q \\
i_Q & -i_D
\end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix}
v_Q & v_D \\
v_D & -v_Q
\end{bmatrix}
\]

Illustrative Example: The schematic of a network with controllable generators and loads is shown in Figure 4. The transmission line parameters and the equilibrium power flows are taken from the three-machine system of [5].

The non-passivity of \(J_{LF}\) can be alleviated by requiring that some devices connected to the network “contribute” to the diagonal terms of \(J_{LF22}\). This is in order to compensate the effect of the shunt capacitances. In practical terms, this means that at least some devices should contribute to voltage regulation through \(\Delta Q - \Delta V_n\), droop control (“grid-forming” devices), as shown in Figure 3. The contribution, denoted by \(k_{qv}\), has to be specified for each device by the T&D operator based on an evaluation of the eigenvalues of \(J_{LF\cdot}\). The aim is to ensure that no eigenvalue of \(J_{LF}\) is positive.
This is also numerically verified using the three-machine system shown in Figure 2. The line resistances are neglected, and then \( J_{LF} \) is calculated. The eigenvalues of \( J_{LF}^{R} \), with and without shunt voltage regulation contributions, are presented in Table II. Therefore, the low-frequency lossless T&D network model can also be passivated by borrowing voltage regulation capabilities from the shunt-connected devices.

### Table II: Eigenvalues of \( J_{LF}^{R} \) of the three-machine system

| Base Case | Modified Case |
|-----------|---------------|
| \(-0.84, 0.9, 8.83, 10.32, 13.11, 32.72, 35.46, 35.74, 36.79, 41.72, 43.16, 94.37, 95.34, 109, 109.33, 116.92, 117.26\) | \(0.027, 8.15, 9.15, 10.59, 13.31, 33.44, 35.86, 36.11, 37.07, 42.45, 43.25, 94.77, 95.91, 109.33, 110.1, 117.2, 118.09\) |

**Decoupled Model** \((J_{LF12} = J_{LF21} = 0):\) For transmission network parameters, the off-diagonal blocks of \( J_{LF} \) are usually much smaller than the diagonal blocks. Therefore, a further simplified model is also considered where \( J_{LF12} = J_{LF21} = 0 \). This will be referred to as the “decoupled” model in this paper. The passivity behaviour is as follows.

(a) Decoupled lossy network: \( J_{LF} \) is not passive.

(b) Decoupled lossless network: \( J_{LF} \) is not passive if shunt capacitances are considered. However, for both (a) and (b), voltage regulation by the shunt-connected devices can alleviate the non-passivity.

(c) \( J_{LF} \) is passive in the case of decoupled lossless model, with shunt capacitances also neglected.

### V. Model III Variables: \((\Delta P, \Delta Q) - (\Delta \bar{\omega}, \Delta \bar{V}_n)\)

In contrast to the input-output variables used in the previous case, the bus frequency deviation \(\bar{\omega}\) is considered here instead of the bus phase angle. The derivative is approximated by using a small time-constant \(\tau\) as given in (6).

\[
\Delta \bar{\omega}(s) = \frac{s}{1 + s\tau}\Delta \phi(s) \tag{6}
\]

Let the transfer function of the T&D network in these variables be denoted by \(J_{dp}(s)\). This is related to \(J(s)\) as follows.

\[
J_{dp}(s) = \begin{bmatrix}
J_{11}(s) & J_{12}(s) \\
J_{21}(s) & J_{22}(s)
\end{bmatrix} \begin{bmatrix}\frac{(1+s\tau)}{s} & 0 \\
0 & 1\end{bmatrix} \tag{7}
\]

where \(I\) is the identity matrix. The following property reflects the passivity behaviour of the dynamical model of the T&D network, when these input-output variables are considered.

#### A. Wide-band dynamical model

**Property 2.** The dynamical model of a R-L-C network with \((\Delta P, \Delta Q) - (\Delta \bar{\omega}, \Delta \bar{V}_n)\) as interface variables is not passive.

The proof is given in Appendix A.2. The passivity behaviour of the low frequency model is now presented.

#### B. Low frequency model

The low frequency transfer function \(N_p(s)\) is given in (8).

\[
N_p(s) = J(0) \begin{bmatrix}\frac{(1+s\tau)}{s} & 0 \\
0 & 1\end{bmatrix} = \begin{bmatrix} J_{LF11}(1+s\tau) & J_{LF12} \\
J_{LF21}(1+s\tau) & J_{LF22}\end{bmatrix} \tag{8}
\]

Note that \(N_p(s)\) has a simple pole at \(s = 0\). For the system to be passive, the residue at \(s = 0\) must be positive semi-definite Hermitian. The expression of the residue of \(N_p(s)\) evaluated at \(s = 0\), denoted by \(S_{dp}\), is as follows.

\[
S_{dp} = \lim_{s \to 0} sN_p(s) = \begin{bmatrix} J_{LF11} & 0 \\
J_{LF21} & 0\end{bmatrix} \tag{9}
\]

Note that \(S_{dp}\) cannot be Hermitian if \(J_{LF12} \neq 0\). Therefore, the low frequency model of the network in these variables cannot be passive, if coupled power flow models are considered.

**Decoupled model:** \(S_{dp}\) in (9) is positive semi-definite if \(J_{LF11}\) is Hermitian positive semi-definite. Note that \(J_{LF11}\) is Hermitian only for lossless networks. Therefore, \(N_{dp}(s)\) of decoupled lossy T&D network models is also not passive.

**Lossless network:** For decoupled lossless networks at \(\Omega \neq 0\),

\[
N_p^R(j\Omega) = N_p(j\Omega) + N_p^T(-j\Omega) = \begin{bmatrix} 0 & 0 \\
0 & J_{LF22} + J_{LF22}^{T}\end{bmatrix} \tag{*}
\]

This may not be passive due to the presence of shunt capacitors. However, the non-passivity can be alleviated by contribution of voltage regulation by the shunt-connected devices. If the shunt capacitors are also neglected, then the network model is passive in these variables. This network model is used for passivity based stability analysis in VI.

### VI. Model IV variables: \((\Delta P, \Delta Q) - (\Delta \bar{\omega}, \Delta \bar{V}_n^d)\)

The input variables that are considered here are \(\Delta \bar{\omega}\) and the derivative of \(\Delta \bar{V}_n\), denoted by \(\Delta \bar{V}_n^d\). Similar to the evaluation of \(\bar{\omega}\) as given in (6), the derivative here is also approximated by the same time-constant \(\tau\), as given in (10).

\[
\Delta \bar{V}_n^d(s) = \frac{s}{(1 + s\tau)}\Delta \bar{V}_n(s) \tag{10}
\]

The transfer function in these variables \(J_{df}(s)\) is related to \(J(s)\) as follows.

\[
J_{df}(s) = J(s) \times \frac{(1 + s\tau)}{s} \tag{11}
\]

The following property reflects the passivity behaviour of the dynamical model of the T&D network, when these input-output variables are considered.

#### A. Wide-band dynamical model

**Property 3.** The dynamical model of a R-L-C network with \((\Delta P, \Delta Q) - (\Delta \bar{\omega}, \Delta \bar{V}_n^d)\) as interface variables is not passive.

The proof is given in Appendix A.3. The passivity behaviour of the low frequency model is now presented.

#### B. Low frequency model

The low frequency model of the T&D network in these variables is represented by \(N_{df}(s)\), which is given as follows.

\[
N_{df}(s) = J_{LF} \times \frac{(1 + s\tau)}{s} \tag{12}
\]

\(N_{df}(s)\) has a simple pole at \(s = 0\). For \(N_{df}(s)\) to be passive, the residue, evaluated at \(s = 0\) has to be positive semi-definite Hermitian. The residue is \(J_{LF}\), which is not Hermitian for
lossy networks. Therefore, the low frequency model of lossy networks will not be passive in these input-output variables.

**Lossless network:** For lossless networks, \( J_{LF} \) is passive if and only if \( J_{LF} \) is positive semi-definite. Although \( J_{LF} \) may not be positive semi-definite with shunt capacitances considered, it can be achieved with voltage regulation contribution from the shunt-connected devices. If the shunt capacitances are neglected and decoupling is considered, then \( J_{LF} \) is positive semi-definite Hermitian.

VII. CONCLUSIONS AND FUTURE WORK

The passivity behaviour of the T&D network transfer function for D-Q based rectangular and polar (and their derivatives) interface variables are summarized in Table III. The wide-band dynamic model of the impedance/admittance of the T&D network is passive. Although the wide-band models of the T&D network is not passive for the alternative interface variables considered here, the low-frequency models can be passivated by borrowing voltage regulation contributions from the shunt-connected devices. It is therefore necessary to ensure time-scale separation for decoupling the high and low frequency analysis in order to make the scheme viable. The decoupled analysis can use the rectangular variables for high frequency studies, and the polar variables for low frequency studies.

### Table III: Passivity of T&D network transfer function with different input-output variables

| Model   | Inputs          | Outputs | Transfer function | Wide-band dynamic model | Lossy (with B) | Lossless (with B) | Lossy (without B) | Lossless (without B) |
|---------|-----------------|---------|-------------------|-------------------------|---------------|------------------|-------------------|---------------------|
| I       | \( (\Delta V_D, \Delta V_Q) \) | \( (\Delta V_D, \Delta v_Q) \) | \( Z_{DQ}(s) \) | ✓                   | ✓              | ✓                | ✓                 | ✓                   |
| II      | \( (\Delta \phi, \Delta V_o) \) | \( (\Delta P, \Delta Q) \) | \( J(s) \) | ✗                   | ✗              | ✗                | ✗                 | ✗                   |
| III     | \( (\Delta \omega, \Delta V_n) \) | \( (\Delta P, \Delta Q) \) | \( J_{dp}(s) \) | ✗                   | ✗              | ✗                | ✗                 | ✗                   |
| IV      | \( (\Delta \omega, \Delta V_n^d) \) | \( (\Delta P, \Delta Q) \) | \( J_{dq}(s) \) | ✗                   | ✗              | ✗                | ✗                 | ✗                   |

*Non-passivity can be alleviated by borrowing voltage regulation capability of shunt-connected devices. *B denotes shunt capacitance.

APPENDIX A: PASSIVITY OF T&D NETWORK MODEL

Let the state space matrices of the D-Q domain admittance \( Y_{DQ}(s) \) be \( (A_y, B_y, C_y, D_y) \). Then

\[
D_y = \lim_{s \to \infty} Y_{DQ}(s) = \begin{bmatrix} D_1 & 0 \\ 0 & D_3 \end{bmatrix}
\]

(13)

Note that \( D_1 \) is symmetric.

1. **Model II variables:** \((\Delta P, \Delta Q) - (\Delta \phi, \Delta V_n)\)

If the state-space matrices of the transfer function \( J(s) \) are \((A_2, B_2, C_2, D_2)\), then \( D_2 = (E D_y + C) F \), where \( E, C \) and \( F \) are defined in \( (4) \). It can be shown that the trace of \( D_2 + D_2^T \) is zero, indicating that it cannot be positive semi-definite. This violates the time domain passivity conditions (see Section II-A). Therefore the system is not passive.

2. **Model III variables:** \((\Delta P, \Delta Q) - (\Delta \omega, \Delta V_n)\)

If the state-space matrices of the transfer function \( J_{dp}(s) \) are \((A_3, B_3, C_3, D_3)\), then

\[
D_3 = (E D_y + C) F \begin{bmatrix} \tau I & 0 \\ 0 & I \end{bmatrix}
\]

where \( I \) is the identity matrix. The diagonal entries of \( D_3 + D_3^T \) cannot be non-negative except when \( i_{DP} v_D - i_{Qo} v_{D0} = 0 \), which indicates that the quiescent reactive power injection at each node is zero, which is quite unusual. If \( i_{DP} v_D - i_{Qo} v_{D0} \neq 0 \), then \( D_3 + D_3^T \) cannot be positive semi-definite. This will violate the time-domain passivity conditions (see Section II-A). Therefore, the system is not passive.

3. **Model IV variables:** \((\Delta P, \Delta Q) - (\Delta \omega, \Delta V_n^d)\)

If the state-space matrices of the transfer function matrix \( J_{dq}(s) \) are \((A_4, B_4, C_4, D_4)\), then \( D_4 = \tau (E D_y + C) F \), where \( E, C \) and \( F \) are as given in \( (4) \). Note that the trace of \( D_4 + D_4^T \) is zero, indicating that it cannot be positive semi-definite. This violates the time domain passivity conditions (see Section II-A). Therefore, it is not passive.

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