Pseudo-stopping times and the hypothesis \((\mathcal{H})\)

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Abstract

The main goals of this study are to relate the class of pseudo-stopping times to the hypothesis \((\mathcal{H})\), to provide alternative characterizations of the hypothesis \((\mathcal{H})\) and to study the relationships between pseudo-stopping times, honest times and barrier hitting times. Our main result states that given two filtrations \(F \subset G\), then every \(F\)-martingale is a \(G\)-martingale, or equivalently, the hypothesis \((\mathcal{H})\) is satisfied for \(F\) and \(G\), if and only if every \(G\)-stopping time is an \(F\)-pseudo-stopping time.

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1 Introduction

Based on the example given in Williams [14], the concept of a pseudo-stopping time was formally introduced by Nikeghbali and Yor in [13]. As its name suggest, the class of pseudo-stopping time is larger than the class of stopping times and enjoys stopping times like properties. In this paper, we study the properties of pseudo-stopping times (see Definition 2.2) in the context of the theory of enlargement of filtrations.

Let us describe first our setting and results. We work on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ denotes a filtration satisfying the usual conditions and we set $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. A process on this space is said to be $\mathbb{F}$-martingale if it is adapted to the filtration $\mathbb{F}$. As convention, for any martingale, we work always with its càdlàg modification, while for any random process $(X_t)_{t \geq 0}$, we set $X_0 = 0$ and $X_\infty = \lim_{t \to \infty} X$ a.s. if it exists.

In the theory of enlargement of filtration, we consider another filtration $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ such that $\mathbb{F} \subset \mathbb{G}$, i.e., for each $t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t$. Then, one has the following hypothesis linked to enlargement of filtration problem:

**Definition 1.1.** [2] The hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$ if every (bounded) $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. For ease of language, when the hypothesis $(\mathcal{H})$ is satisfied for $\mathbb{F} \subset \mathbb{G}$, we shall often say $\mathbb{F}$ is immersed in $\mathbb{G}$ and write $\mathbb{F} \hookrightarrow \mathbb{G}$.

One special way of enlarging a filtration is the progressive enlargement with a random time. Let $\tau$ be a random time, i.e., a measurable mapping $\tau : (\Omega, \mathcal{A}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Then, the progressive enlargement of $\mathbb{F}$ with a random time $\tau$, which we denote by $\mathbb{F}_\tau := (\mathcal{F}_\tau_t)_{t \geq 0}$, is the smallest right-continuous filtration containing $\mathbb{F}$ such that $\tau$ is an $\mathbb{F}_\tau$-stopping time. More precisely, for each $t \geq 0$, $\mathcal{F}_\tau_t := \bigwedge_{s \geq t} \mathcal{F}_s \vee \sigma(\tau \wedge s)$.

Our results are as follows. In Nikeghbali and Yor [13], the authors have remarked that given two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F}$ is immersed in $\mathbb{G}$, then every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time. The main result of the present work is fostered in Theorem 3.3 where we complete this observation by showing that the converse is true and provide alternative characterizations of the hypothesis $(\mathcal{H})$ as done in Brémaud and Yor [2], but based on pseudo-stopping times and (dual) optional projections of processes with finite variation.

Assuming that the filtration $\mathbb{F}$ is immersed in the filtration $\mathbb{G}$, in Proposition 3.5, we re-examine and generalize a known result in the literature (see Gapeev [5] and Jeanblanc [9]), which states that every finite $\mathbb{G}$-stopping time that avoids $\mathbb{F}$-stopping times (see Assumption 2.1 (A)) is a $\mathbb{F}$-barrier hitting time (see Definition 2.9) with a uniformly distributed barrier, which is independent of $\mathcal{F}_\infty$. As an application of Proposition 3.5 we show in Corollary 3.7 that every $\mathbb{G}$-stopping time can be written as the minimum of two $\mathbb{G}$-stopping times, which are $\mathbb{F}$-barrier hitting time and $\mathbb{F}$-thin time respectively.

To conclude, in Lemma 3.9 we extend Proposition 6 in [13] by removing the assumption that all $\mathbb{F}$-martingales are continuous (see Assumption 2.1 (C)) to show that in general, a pseudo-stopping time is an honest time (see Definition 2.5) if and only if it is also a stopping time. This result, when combined with Theorem 3.3 gives an alternative proof to the classical result that the hypothesis $(\mathcal{H})$ is not satisfied between $\mathbb{F}$ and the progressive enlargement of $\mathbb{F}$ with a honest time.

2 Tools and background

The main tools used in this study are the optional projection and the dual optional projection onto the reference filtration $\mathbb{F}$. We record here some known results from the general theory of stochastic processes, for more details of the theory the reader is referred to He et al. [5] or Jacod and Shiryaev [7]. For any pre-locally integrable variation process $V$ ([6] section 5.18, 5.19), we denote the $\mathbb{F}$-optional projection of $V$ by $\mathcal{O}_V$ and the dual $\mathbb{F}$-optional projection of $V$ by $V^\circ$. It is known that the process $N^V := \mathcal{O}_V - V^\circ$ is an uniformly integrable $\mathbb{F}$-martingale with $N^V_0 = 0$ and $\mathcal{O}(\Delta V) = \Delta V^\circ$. 

Specializing to the study of random times, for an arbitrary random time \( \tau \), we set \( A^\tau := \mathbb{1}_{[\tau,\infty[} \) and define

- the supermartingale \( Z^\tau \) associated with \( \tau \), \( Z^\tau := \omega (\mathbb{1}_{[0,\tau[}) = 1 - \omega (A^\tau) \),
- the supermartingale \( \tilde{Z}^\tau \) associated with \( \tau \), \( \tilde{Z}^\tau := \omega (\mathbb{1}_{[0,\tau[}) = 1 - \omega (A^-) \),
- the martingale \( m^\tau := 1 - (\omega (A^\tau) - (A^\tau)^\omega) \).

Those processes are linked through the following relationships:

\[
Z^\tau = m^\tau - (A^\tau)^\omega \quad \text{and} \quad \tilde{Z}^\tau = Z^\tau + \Delta (A^\tau)^\omega. \tag{1}
\]

Typically, one finds in the literature the two following assumptions related to the reference filtration \( F \) and random time \( \tau \):

**Assumption 2.1.**

Assumption \((C)\) is satisfied if all \( F \)-martingales are continuous,

Assumption \((A)\) is satisfied if \( \tau \) avoids all \( F \)-stopping times or equivalently \( \Delta (A^\tau)^\omega = 0 \).

In the following, we introduce the classes of random times studied in this work. The main object is the class of pseudo-stopping times introduced by Nikeghbali and Yor in \([13]\) as an extension of William’s example in \([14]\). Pseudo-stopping times provide examples of random times where the supermartingale \( Z \) is decreasing, but the hypothesis \((H)\) is not satisfied for the progressive enlargement.

We first recall their definition of pseudo-stopping times, with a slight modification, that is the random time is allowed to take the value infinity.

**Definition 2.2.** \(([14], [13])\) A random time \( \tau \) is an \( F \)-pseudo-stopping time if for every bounded \( F \)-martingale \( M \), we have \( \mathbb{E}(M_\tau) = \mathbb{E}(M_0) \).

**Theorem 2.3** \(([13])\). The following conditions are equivalent:

(i) \( \tau \) is a finite \( F \)-pseudo-stopping time;

(ii) \( (A^\tau)^\infty = 1 \);

(iii) \( m^\tau = 1 \) or equivalently \( \omega (A^\tau) = (A^\tau)^\omega \);

(iv) for every \( F \)-local martingale \( M \), the process \( (M_t)_{t \leq \tau} \) is an \( F^\tau \)-local martingale.

Moreover, if either \((C)\) or \((A)\) holds then the process \( Z^\tau \) is decreasing \( F \)-predictable process.

**Remark 2.4.** We would like to point out that one motivation of this work is to better understand the property that \( \omega (A^\tau) = (A^\tau)^\omega \), which is somewhat hidden in \([13]\). In essence, this property says that the optional projection is equal to the dual optional projection, which is not true in general.

The present paper is devoted mostly to the study of pseudo-stopping times. However, we explore also the relationship between pseudo-stopping times and other classes of random times, namely honest times, thin times and barrier hitting times.

**Definition 2.5.** \(([8] \text{ ch. 5, p. 73})\) A random time \( \tau \) is an \( F \)-honest time if for every \( t > 0 \) there exists an \( F_t \)-measurable random variable \( \tau_t \) such that \( \tau = \tau_t \) on \( \{ \tau < t \} \).

**Proposition 2.6** \(([8] \text{ Proposition (5.1) p.73})\). Let \( \tau \) be a random time. Then, the following conditions are equivalent:

(i) \( \tau \) is an honest time;

(ii) there exists an optional set \( \Gamma \) such that \( \tau(\omega) = \sup \{ t : (\omega, t) \in \Gamma \} \) on \( \{ \tau < \infty \} \);

(iii) \( \tau = \sup \{ t : \tilde{Z}^\tau_t = 1 \} \) a.s. on \( \{ \tau < \infty \} \);

\(^1\text{We shall not make use of stopped processes, hence no confusion of notation can take place.}\)
Definition 2.7. [I] A random time $\tau$ is an $\mathcal{F}$-thin time if its graph $[\tau]$ is contained in a thin set, i.e., if there exists a sequence of $\mathcal{F}$-stopping times $(T_n)_{n=1}^\infty$ with disjoint graphs such that $[\tau] \subset \bigcup_n [T_n]$. We say that such a sequence $(T_n)_n$ exhausts the thin random time $\tau$ or that $(T_n)_n$ is an exhausting sequence of the thin random time $\tau$.

Lemma 2.8 ([I]). Any random time $\tau$ can be written as $\tau^c \wedge \tau^d$, where $\tau^c$ is a random time that avoids finite $\mathcal{F}$-stopping times and $\tau^d$ is an $\mathcal{F}$-thin time. More precisely $\tau^c = \tau 1_{\{\Delta(A^\tau) = 0\}} + \infty 1_{\{\Delta(A^\tau) > 0\}}$ and $\tau^d = \tau 1_{\{\Delta(A^\tau) = 0\}} + \infty 1_{\{\Delta(A^\tau) > 0\}}$.

Definition 2.9. A random time $\tau$ is an $\mathcal{F}$-barrier hitting time if it is of the form

$$\tau = \inf \{ s : X_s \geq U \}$$

where $X$ is an $\mathcal{F}$-adapted non-decreasing process and $U$ is a random barrier that is $\mathcal{A}$ measurable.

If the random barrier $U$ is uniformly distributed on $[0, 1]$ and independent of $\mathcal{F}_\infty$, then we are in the case of the Cox construction which is often used in credit risk modeling (see Bielecki et al. [?]).

3 Pseudo-stopping times and hypothesis ($\mathcal{H}$)

Before proceeding to the main results of the paper, we give an auxiliary lemma which characterises the main property of our interest, that is, given a process of finite variation, when is its optional projection equal to its dual optional projection.

Lemma 3.1. Given an raw increasing process $V$, the following properties (i) $o(V_-)$ is a càglàd increasing process or $o(V_-) = V^0$, (ii) $o(V_-) = oV_-$ and (iii) $oV = V^o$ are equivalent.

Proof. For any raw increasing process $V$, from classic theory we know that the process $N^V := oV - V^o$ is an uniformly integrable martingale with $N^V_0 = 0$ and $o(\Delta V) = \Delta V^o$. The combination of these two properties gives

$$N^V = o(V_-) - V^o \quad \text{and} \quad N^V_0 = oV_- - V^o. \quad (2)$$

If $o(V_-)$ is a càglàd increasing process, then from (2), we see that $N^V$ is a predictable martingale of finite variation, therefore is constant and equal to zero, since predictable martingales are continuous. This shows (i) $\implies$ (iii), while for the converse, it is enough to use the definition of $N^V$. Since $N^V$ is càdlàg, we know that $N^V = 0$ if and only if $N^V_0 = 0$. This fact combined with (2) gives the equivalence between (i) and (ii).

In the following, we extend Theorem 2.3 due to Nikeghbali and Yor to non-finite pseudo-stopping times and remove (C) and (A) assumptions from the last statement in Theorem 2.3.

Theorem 3.2. The following conditions are equivalent:

(i) $\tau$ is an $\mathcal{F}$-pseudo-stopping time;
(ii) $(A^\tau)^o = \mathbb{P}(\tau < \infty | \mathcal{F}_\infty);$
(iii) $m^\tau = 1$ or equivalently $o(A^\tau) = (A^\tau)^o;$
(iv) for every $\mathcal{F}$-local martingale $M$, the process $(M_{\tau^{\wedge} t})_t$ is an $\mathcal{F}^\tau$-local martingale;
(v) the process $\tilde{Z}^\tau$ is a càglàd decreasing $\mathcal{F}$-adapted process.

Proof. To see that (i) are equivalent (ii), suppose $\tau$ is an $\mathcal{F}$-pseudo-stopping time then, by properties of optional and dual optional projection, we have

$$\mathbb{E}(M_\tau 1_{\{\tau < \infty\}}) = \mathbb{E}\left(\int_{[0, \infty]} M_s d(A^\tau)^o_s\right) = \mathbb{E}(M_\infty (A^\tau)^o_\infty).$$
Therefore, the equality, $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty)$ holds true for every bounded $\mathbb{F}$-martingale $M$ if and only if $(A^\tau)_t^\infty = P(\tau < \infty | \mathcal{F}_\infty)$, since $\mathbb{E}(M_\tau) = \mathbb{E}(M_\infty((A^\tau)_t^\infty + \mathbb{1}_{\{\tau=\infty\}}))$.

On the other hand, $\sigma(A^\tau)_t^\infty = \lim_{s\to\infty} P(\tau \leq s | \mathcal{F}_s) = P(\tau < \infty | \mathcal{F}_\infty)$ a.s., and from the definition of $m^\tau$, we note that (ii) holds if and only if (iii) holds, that is $m^\tau = 1$ or equivalently $\sigma(A^\tau) = (A^\tau)^\circ$.

The equivalence of (iii) and (v) follows directly from Lemma 3.1.

To see the equivalence between (i) and (iv) also holds in the case of non-finite pseudo stopping times. For any $\mathbb{G}$-stopping time $\nu$, from page 186 of Dellacherie et al. [7], we know there exists an $\mathbb{F}$-stopping time $\sigma$ such that $\rho \land \nu = \rho \land \sigma$. Therefore, together with the definition of pseudo-stopping time, we have

$$\mathbb{E}(M_{\rho \land \nu}) = \mathbb{E}(M_{\rho \land \sigma}) = \mathbb{E}(M_0),$$

which is an uniformly integrable $\mathbb{G}$-martingale by Theorem 1.42 [7].

Theorem 3.3. Given two filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$, the following conditions are equivalent

(i) every bounded $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale (i.e. the hypothesis (H) is satisfied for $\mathbb{F} \subset \mathbb{G}$),

(ii) every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time,

(iii) the $\mathbb{F}$-dual optional projection of any $\mathbb{G}$-optional process of integrable variation is equal to its $\mathbb{F}$-optional projection.

Proof. To show (i) $\implies$ (ii), let $M$ be any bounded $\mathbb{F}$-martingale and $\nu$ an $\mathbb{G}$-stopping time. Then, from hypothesis (H), $M$ is a $\mathbb{G}$-martingale and $\mathbb{E}(M_\nu) = \mathbb{E}(M_0)$, which implies $\nu$ is an $\mathbb{F}$-pseudo-stopping time.

To show (ii) $\implies$ (i), suppose that $M$ is a bounded $\mathbb{F}$-martingale and $\nu$ is any $\mathbb{G}$-stopping time. Since every $\mathbb{G}$-stopping time is an $\mathbb{F}$-pseudo-stopping time, we have $\mathbb{E}(M_\nu) = \mathbb{E}(M_0)$ for every $\mathbb{G}$-stopping time $\nu$, which by Theorem 1.42 in [7], implies that $M$ is a uniformly integrable $\mathbb{G}$-martingale.

The implication (iii) $\implies$ (ii) follows directly from Theorem 2.3 (iii), therefore we show only the implication (i) $\implies$ (iii). Under the hypothesis (H), the $\mathbb{F}$-optional projection of any bounded process is equal to its optional projection on to the constant filtration $\mathcal{F}_\infty$ (see Bremaud and Yor [2]). More explicitly, for any given increasing $\mathbb{G}$-adapted process $V$, we have $\sigma(V_{\sigma^{-}}) = \mathbb{E}(V_{\sigma^{-}} | \mathcal{F}_\infty)$ for all $\mathbb{F}$-stopping time $\sigma$. From this we see that the process $\sigma(V_{\nu})$ is increasing càglàd and (iii) follows from Lemma 3.1.

Example 3.4. For any $\mathbb{F}$-stopping time $\sigma$, one can shrink $\mathbb{F}$ to $\mathbb{F}_\sigma := (\mathcal{F}_\sigma \land t)_{t \geq 0}$. It can be shown that $\mathbb{F}_\sigma$ is immersed in $\mathbb{F}$ and therefore every $\mathbb{F}$-stopping time is an $\mathbb{F}_\sigma$-pseudo-stopping time.

We suppose that the hypothesis (H) is satisfied for $\mathbb{F} \subset \mathbb{G}$. We characterize all $\mathbb{G}$-stopping times in terms of $\mathbb{F}$-barrier hitting times and $\mathbb{F}$-thin pseudo-stopping times in a similar way to Lemma 2.8. We first present an auxiliary result in Proposition 3.5, which is to some extent known in the current literature (see Remark 3.2 in Gapeev [5] and Jeanblanc [9]) for finite random times, however the exact assumptions on the invertibility of the supermartingale $Z$ is unclear to us. Therefore, we will re-examine the result in the general framework and give a short and concise proof.

Proposition 3.5. If the hypothesis (H) is satisfied for $\mathbb{F} \subset \mathbb{G}$ and $\nu$ is a $\mathbb{G}$-stopping time that avoids all finite $\mathbb{F}$-stopping times then

(i) the $\mathcal{F}_\infty$-conditional distribution of $(A^\nu)^\circ_t$ is uniform on the interval $[0, (A^\nu)^\circ_\infty)$, with an atom of size $1 - (A^\nu)^\circ_\infty$ at $(A^\nu)^\circ_\infty$.

(ii) the $\mathbb{G}$-stopping time $\nu$ is an $\mathbb{F}$-barrier hitting time, that is $\nu = \inf \{ t > 0 : (A^\nu)^\circ_t \geq (A^\nu)^\circ_\infty \}$.

Proof. To show (i), we compute the $\mathcal{F}_\infty$-conditional distribution of $(A^\nu)^\circ_t$, that is

$$\mathbb{E}_\mathbb{P} \left( \mathbb{1}_{\{(A^\nu)^\circ_t \leq u\}} \mid \mathcal{F}_\infty \right) = \mathbb{E}_\mathbb{P} \left( \mathbb{1}_{\{(A^\nu)^\circ_t \leq u\}} \mid \mathcal{F}_\infty \right) \mathbb{1}_{\{\nu < (A^\nu)^\circ_\infty\}} + \mathbb{1}_{\{\nu \geq (A^\nu)^\circ_\infty\}}$$
Let us set $C$ to be the right inverse of $(A^\nu)^\circ$, then the first term in the right hand side above is
\[
\mathbb{E}_\nu \left( \mathbb{I}_{\{ (A^\nu)\circ \leq u \}} \mathbb{I}_{\{ C_n < \infty \}} \, | \, \mathcal{F}_\infty \right) = \mathbb{E}_\nu \left( \mathbb{I}_{\{ \nu \leq C_n \}} \mathbb{I}_{\{ C_n < \infty \}} \, | \, \mathcal{F}_C \right)
\]
\[
= (A^\nu)\circ_{C_n} \mathbb{I}_{\{ C_n < \infty \}}
\]
\[
= (A^\nu)^\circ_{C_n} \mathbb{I}_{\{ C_n < \infty \}}
\]
\[
= u \mathbb{I}_{\{ u < (A^\nu)^\circ_{C_n} \}}
\]
where we apply [Theorem 3.3](#) in the third equality, while last equality follows from the fact that $(A^\nu)^\circ_{C_n} = u$, since $(A^\nu)^\circ$ is continuous except perhaps at infinity. This implies that the $\mathcal{F}_\infty$-conditional distribution of $A^\nu_{\circ}$ is uniform on $[0, (A^\nu)^\circ_{\infty})$.

To show $(ii)$, we first define another random time $\nu^*$ by setting
\[
\nu^* := \inf \{ t > 0 : (A^\nu)^\circ_t \geq (A^\nu)^\circ_u \}.
\]
To see that $\nu^* = \nu$ (it is obvious that $\nu^* \leq \nu$), we use Lemma 4.2 of [8] which states that the left-support of the measure $dA^\nu$, i.e.,
\[
\{ (\omega, t) : \forall \varepsilon > 0 \quad A^\nu_t(\omega) > A^\nu_{t-\varepsilon}(\omega) \} = [\nu]
\]
belongs to the left-support of $(A^\nu)^\circ$, i.e., to the set $\{ (\omega, t) : \forall \varepsilon > 0 \quad (A^\nu)^\circ_t(\omega) > (A^\nu)^\circ_{t-\varepsilon}(\omega) \}$.

**Remark 3.6.** In [9] the process $Z$ was assumed to be continuous and strictly increasing (i.e invertible), therefore $\nu^* = \nu$ holds without the result of Jeulin [8]. While in Remark 3.2 in [10], the equality $\nu^* = \nu$ appears to be obvious from continuity and the author do not refer to other results.

As a special case, if the hypothesis $(\mathcal{H})$ is satisfied for filtration $\mathbb{F} \subset \mathbb{G}$ and that $\nu$ is a finite $\mathbb{G}$-stopping time, then $(A^\nu)^\circ_{\infty} = 1$ and $(A^\nu)^\circ_{\infty}$ is independent of $\mathbb{F}_\infty$ and uniformly distributed on the interval $[0, 1]$. In this case, the stopping time $\nu$ is a random time constructed from the Cox construction.

**Corollary 3.7.** If the hypothesis $(\mathcal{H})$ is satisfied for filtration $\mathbb{F} \subset \mathbb{G}$ and $\nu$ is a $\mathbb{G}$-stopping time, then it can be written as $\nu^* \wedge \nu^{\prime \prime}$, where $\nu^*$ is a $\mathbb{G}$-stopping time which is an $\mathbb{F}$-barrier hitting time avoiding $\mathbb{F}$-stopping times and $\nu^{\prime \prime}$ is a $\mathbb{G}$-stopping time whose graph is contained in the graphs of a sequence of $\mathbb{F}$-stopping times.

**Proof.** By [Lemma 2.8](#) it has the representation $\nu = \nu^{\prime \prime} \wedge \nu^*$. We only show that $\nu^*$ is a $\mathbb{G}$-stopping time as for $\nu^{\prime \prime}$ the proof is analogical. Note that $\{ \nu^* \leq t \} = \{ \nu \leq t \} \cap \{ \Delta(\nu)^\circ_{\nu} = 0 \}$, thus $\nu^*$ is $\mathbb{G}$-stopping time if and only if $\{ \Delta(\nu)^\circ_{\nu} = 0 \} \in \mathbb{G}_\nu$. By Corollary 3.23 in [6], $\Delta(\nu)^\circ_{\nu} \mathbb{I}_{\{ \nu < \infty \}}$ is $\mathbb{G}_\nu$-measurable as $\Delta(\nu)^\circ$ is $\mathbb{G}$-optional process and the assertion follows. The $\mathbb{G}$-stopping time $\nu^*$ avoids all $\mathbb{F}$-stopping times and we conclude by applying [Proposition 3.3](#).

Unlike stopping times, the minimum and maximum of two $\mathbb{F}$-pseudo-stopping times is in general not an $\mathbb{F}$-pseudo-stopping time. In the following, we explore extensions to Proposition 4. in [13], which states that the minimum of a pseudo-stopping time $\rho$ with an $\mathbb{F}^{\rho}$-stopping time is again a pseudo-stopping time.

**Lemma 3.8.** (i) Let $\rho$ be a $\mathcal{F}_\infty$-measurable $\mathbb{F}$-pseudo-stopping time and $\tau$ be a random time such that $\mathbb{F}$ is immersed in $\mathbb{F}^\tau$, then $\tau \wedge \rho$ is again an $\mathbb{F}$-pseudo-stopping time.

(ii) Let $\rho$ be an $\mathbb{F}$-pseudo-stopping time and if $\tau$ is an $\mathbb{F}^{\rho}$-pseudo-stopping time, then $\tau \wedge \rho$ is again an $\mathbb{F}$-pseudo-stopping time.

**Proof.** (i) To compute the $\tilde{\mathcal{T}}^{\tau \wedge \rho}$, we note that from the fact that $\rho$ is $\mathcal{F}_\infty$ measurable and hypothesis $(\mathcal{H})$ holds between $\mathbb{F}$ and $\mathbb{F}^\tau$, we have
\[
\mathbb{P}(\tau \geq t, \rho \geq t \mid \mathcal{F}_\infty) = \mathbb{I}_{\{ \rho \geq t \}} \mathbb{P}(\tau \geq t \mid \mathcal{F}_\infty) = \mathbb{I}_{\{ \rho \geq t \}} \mathbb{P}(\tau \geq t \mid \mathcal{F}_t).
\]
This shows that $\tilde{Z}^{\tau \wedge \rho} = \tilde{Z}^\tau \tilde{Z}^\rho$ and is decreasing càdlàg. We conclude by using Theorem 3.2 (v).

(ii) Since $\tau$ is an $\mathbb{F}^p$-pseudo-stopping time and $(M_{\rho \wedge s})_{t \geq 0}$ is an $\mathbb{F}^p$ martingale, we have $\mathbb{E}(M_{\rho \wedge s}) = \mathbb{E}(M_0)$ from the definition of pseudo-stopping time. Therefore $\tau \wedge \rho$ is an $\mathbb{F}$-pseudo-stopping time. \qed

Finally, we relate pseudo-stopping times with honest times. Under (C), a result of similar spirit was presented in Proposition 6 in [13], where distributional argument were given. Here, we use sample path properties to show that the same kind of result holds in full generality.

**Lemma 3.9.** Let $\tau$ be a random time. Then, the following conditions are equivalent
(i) $\tau$ is equal to an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$;
(ii) $\tau$ is an $\mathbb{F}$-pseudo-stopping time and an $\mathbb{F}$-honest time.

**Proof.** The implication (i) $\implies$ (ii) holds by [Theorem 3.2 (iii)] and [Proposition 2.6 (iii)]. To show (ii) $\implies$ (i), let us note that the honest time property of $\tau$ implies $\tau = \sup\{t : Z^\tau_t = 1\}$ on $\{\tau < \infty\}$ by [Proposition 2.6 (iii)], and by [Theorem 3.2 (v)], the pseudo-stopping time property of $\tau$ implies that $\tilde{Z}^\tau = 1 - (A^\tau)_0^-$. Therefore, on $\{\tau < \infty\}$,
\[
\tau = \sup\{t : Z^\tau_t = 1\} = \sup\{t : (A^\tau)_0^- = 0\} = \inf\{t : (A^\tau)_0^- > 0\},
\]
so, $\tau$ is equal to an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$. \qed

As a simple consequence of [Lemma 3.9] and [Theorem 3.3], we recover the following classical result found in Jeulin [8], where the result follows from $G$-semimartingale decompositions of $\mathbb{F}$-martingales.

**Corollary 3.10.** If $\tau$ is an $\mathbb{F}$-honest time which is not equal to an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$, then the hypothesis (H) is not satisfied for $\mathbb{F} \subset \mathbb{F}^\tau$.

### 4 Time change construction of pseudo-stopping times

In this subsection, we revisit and extend the construction of pseudo-stopping times suggested in [13]. In that paper, an honest time $\tau$ is given and under the assumptions (C) and (A), a pseudo-stopping time $\rho$ is constructed by setting
\[
\rho := \sup\{t \leq \tau : Z^\tau_s = \inf Z^\tau_s\}. \tag{3}
\]

The goal of this section is to demonstrate that the honest time $\tau$ which appears in [13] can be replaced by an arbitrary finite random time and that assumptions (C) and (A) can be relaxed in certain cases, which allows us to construct an $\mathbb{F}$-pseudo-stopping time which is also an $\mathbb{F}$-thin time. In the following, we assume that an arbitrary random time $\tau$ is given and we define the processes $D$ and $G$:
\[
D_t := \inf\{s > t : Z^\tau_s = \inf Z^\tau_u\}, \quad G_t := \sup\{s \leq t : Z^\tau_s = \inf Z^\tau_u\}.
\]

**Lemma 4.1.** The processes $D$ and $G$ are increasing and càdlàg. The process $D$ is the right-inverse of $G$.

**Proof.** From [3] Paragraph 1 Chapter XX, we know that $D$ is a right-continuous process, and, since the set $\Gamma := \{(\omega, t) : Z^\tau_t(\omega) = \inf_{s \leq t} Z^\tau_s(\omega)\}$ is right closed (as $Z^\tau$ is càdlàg process), $G$ is a right-continuous process as well. To show that the process $D$ is the right-inverse of $G$, i.e.,
\[
D_t = \inf\{u : G_u > t\}
\]
we fix $t$ and $\omega$. Then we have:
1) If $u \in [G_t, D_t]$ then, since $[G_t, D_t] \notin \Gamma$, we get $G_u = G_t \leq t$.
2) If $D_t = t$ and $u > D_t$ then, since there exists $s \in [D_t, u]$ such that $s \in \Gamma$, we get $G_u \geq G_s \geq s > t$.
3) If $D_t < t$ and $u > D_t$ then, since there exists $s \in [D_t, u]$ such that $s \in \Gamma$, we get $G_u \geq D_t > t$.
And the proof is completed. \qed
We denote by \( d \) the left limit of \( D \), i.e., \( d_t := D_{t-} \). Note that \( d \) is the left-inverse of \( G \), i.e., \( d_t = \inf \{ s : G_s \geq t \} \). For each \( \mathbb{F} \)-stopping time \( T \), \( D_T \) and \( d_T \) are \( \mathbb{F} \)-stopping times. Let \( \rho \) be defined as in (3), then we have

\[
\{ \rho \geq T \} = \{ \tau \geq d_T \}. \tag{4}
\]

**Lemma 4.2.** Let \( \rho \) be defined by (3). Then

(i) the \( \mathbb{F} \)-dual optional projection of \( A^\rho \) is given by \( A_D^\rho = (A_D^\tau)^o \),

(ii) the \( \mathbb{F} \)-supermartingale \( \tilde{Z}^\rho \) is given by \( \tilde{Z}^\rho = \sigma(\tilde{Z}^\tau_d) \).

**Proof.** (i) For any \( \mathbb{F} \)-optional process \( X \) we have

\[
E(X_{\rho}) = E(X_{G_\rho}) = E\left( \int_{[0,\infty]} X_{G_s} dA^{\tau,o}_s \right) = E\left( \int_{[0,\infty]} X_{G_C} I_{\{C_s < \infty\}} ds \right),
\]

where the second equality follows from the fact that \( X_{G_\rho} \) is \( \mathbb{F} \)-optional (see [12]) and the third from a time change in the integrals with \( C \) being the right-inverse of \( A^{\tau,o} \) (see [10] Lemma 1.38). Using once again time change in the integrals, as \( G_C \) is right-inverse of \( A_D^{\tau,o} \), we obtain:

\[
E(X_{\rho}) = E(\int_{[0,\infty]} X_s dA^{\tau,o}_s).
\]

(ii) From the definition of the optional projection and the identity (4), for every \( \mathbb{F} \)-stopping time \( T \),

\[
\tilde{Z}_T^\rho I_{\{T < \infty\}} = E \left( I_{\{\rho \geq T\}} I_{\{T < \infty\}} \mid \mathcal{F}_T \right)
= E \left( I_{\{\tau \geq d_T\}} I_{\{d_T < \infty\}} I_{\{T < \infty\}} \mid \mathcal{F}_T \right) + E \left( I_{\{\tau \geq \infty\}} I_{\{d_T = \infty\}} I_{\{T < \infty\}} \mid \mathcal{F}_T \right)
\]

since \( \tau \) is assumed to be finite

\[
= E \left( E \left( I_{\{\tau \geq d_T\}} I_{\{d_T < \infty\}} \mid \mathcal{F}_{d_T} \right) I_{\{T < \infty\}} \mid \mathcal{F}_T \right)
= E \left( \tilde{Z}_{d_T}^\tau I_{\{T < \infty\}} \mid \mathcal{F}_T \right),
\]

which shows the assertion. \( \square \)

**Proposition 4.3.** Suppose that for every \( \mathbb{F} \)-stopping time \( T \)

\[
\tilde{Z}_{d_T}^\tau = \inf_{s < T} Z_s^\tau. \tag{5}
\]

Then, \( \rho \) (given in (3)) is an \( \mathbb{F} \)-pseudo-stopping time and \( \tilde{Z}_T^\rho = \inf_{s \leq T} Z_s^\tau \).

**Proof.** By the assumption (5) and Lemma 4.2 we get that

\[
\tilde{Z}_T^\rho I_{\{T < \infty\}} = \sigma(\tilde{Z}_{d_T}^\tau) I_{\{T < \infty\}} = \sigma(\inf_{s < T} Z_s^\tau) I_{\{T < \infty\}} = \inf_{s < T} Z_s^\tau I_{\{T < \infty\}}.
\]

Then, by Section Theorem, we conclude that \( \tilde{Z}_T^\rho = \inf_{s \leq T} Z_s^\tau \) is a decreasing càdlàg process, thus, by Theorem 2.3 (v), \( \rho \) is an \( \mathbb{F} \)-pseudo stopping time. \( \square \)

The equality (5) is rather technical and cannot be checked in general without knowing the exact structure of the processes \( Z^\tau \) and \( \tilde{Z}^\tau \). In the following, we give examples of constructions where (4) is satisfied.

**Example 4.4.** [Poisson filtration example] We present here an example without assuming neither (C) nor (A) nor the fact that the process \( \inf_{s \leq t} Z_s^\tau \) is continuous.
Let us take a filtration $\mathcal{F}$ to be the natural filtration generated by a compound Poisson process $X$, which takes the form

$$X_t = \sum_{k=1}^{N_t} Y_k,$$

where $N$ is a Poisson process with parameter $\eta$ and the sequence of jump times are given by $(T_n)_{n \in \mathbb{N}}$. The random variables $(Y_k)_{k \in \mathbb{N}}$ are strictly positive i.i.d. and independent from $N$. We denote their distribution function by $F_Y$.

Set $\tilde{N}_t := \mu t - X_t$ with $\mu > \eta E(Y_1)$. For $a > 0$ we can define an honest time $\tau$ by

$$\tau := \sup\{t : \tilde{N}_t \leq a\}.$$

This honest time is studied in [1] and its supermartingales $\tilde{Z}^\tau$ and $\tilde{Z}^{\tau}$ are given by:

$$Z^\tau_t = \psi(\mu t - X_t - a) 1_{\{\mu t - X_t - a \geq 0\}} + 1_{\{\mu t - X_t - a < 0\}}$$

$$\tilde{Z}^{\tau}_t = \psi(\mu t - X_t - a) 1_{\{\mu t - X_t - a \geq 0\}} + 1_{\{\mu t - X_t - a \leq 0\}}$$

with $\psi(x)$ being the ruin probability associated with process $\tilde{N}_t = \mu t - X_t$ and starting point $x > 0$. That is $\psi(x) = P(R(x) < \infty)$ where $R(x) := \inf\{s : x + \mu s - X_s \leq 0\}$. From [1], we know that the function $\psi$ satisfies the following properties; (i) for $x < 0$ we have $\psi(x) = 1$, (ii) the function $\psi$ continuous and decreasing for $x > 0$ and (iii) for $x = 0$ we have $\psi(0) = \frac{\eta E(Y_1)}{\mu} < 1$.

If one looks closer, the infimum process of $Z^\tau$ has only one negative jump at the predictable stopping time

$$T_1 := \inf\{t \geq 0 : \mu t - X_t - a = 0\},$$

where the process $Z^\tau$ jumps from 1 to $\psi(0)$. That implies that $T_1$ is the only $\mathcal{F}$-stopping time which $\rho$ intersects.

**Example 4.5.** [Thin pseudo-stopping times] In the following we work under the assumption that $\inf Z^\tau$ and $A^\tau,^0$ are continuous. We show that one can systematically construct a thin pseudo-stopping times in the following manner.

Let $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$ be a positive decreasing sequence of real numbers bounded by 1. We define the following sequence of $\mathcal{F}$-predictable stopping times

$$T_n = \inf\{t \geq 0 : \inf_{s \leq t} Z^\tau_s = \varepsilon_n\}, \quad T_{N+1} = \infty$$

and the $\mathcal{F}$-adapted decreasing càglàd process $V = \varepsilon_0 1_{[0, T_0]} + \sum_{n=1}^{N} \varepsilon_n 1_{[T_{n-1}, T_n]}$. Moreover we set

$$G^\prime_t := \sup\{s \leq t : Z^\tau_s = V_t\}$$

$$d^\prime_t := \inf\{s \geq t : Z^\tau_s = V_t\}$$

and note that the process $V$ is such that $V_T = V_{d^\prime_T}$ for all stopping time $T$. The process $d^\prime$ is the left inverse of $G^\prime$ since the set $\{Z^\tau = V\}$ is a closed random set, (see [1], the reason is that for each fixed $\omega$, it contains only a finite number of points).

Let us define $\rho := G^\prime_T$. Clearly it is a thin random time as $[G^\prime_T] \subset \bigcup_{n}[T_n]$. Similarly to the proof of Lemma [1.3] we compute $\tilde{Z}^{\rho}_T 1_{\{T < \infty\}}$ for an $\mathcal{F}$-stopping time $T$:

$$\tilde{Z}^{\rho}_T 1_{\{T < \infty\}} = \mathbb{E}\left(1_{\{\rho \geq T\}} 1_{\{T < \infty\}} \big| \mathcal{F}_T\right)$$

$$= \mathbb{E}\left(1_{\{\tau \geq d^\prime_T\}} 1_{\{d^\prime_T < \infty\}} \big| \mathcal{F}_T\right) 1_{\{T < \infty\}}$$

$$= \mathbb{E}\left(\tilde{Z}^{\tau}_{d^\prime_T} 1_{\{d^\prime_T < \infty\}} \big| \mathcal{F}_T\right) 1_{\{T < \infty\}}$$

$$= \mathbb{E}\left(V_{d^\prime_T} 1_{\{d^\prime_T < \infty\}} \big| \mathcal{F}_T\right) 1_{\{T < \infty\}}$$

$$= V_T 1_{\{T < \infty\}},$$
where the fourth equality comes from continuity of \(A^{\tau,o}\) (i.e., \(\tilde{Z}^T = Z^T\)) and \(Z^T_{\delta_T^+} = V_{d_T^+}\). One can conclude that \(\rho\) is an \(F\)-pseudo-stopping time (Proposition 2.3 (f)). The random time \(\rho\) is an \(F\)-thin time, since \(A^{\rho,o} = 1 - V_+\) is a pure jump process.

5 Hypothesis \((H')\) and semimartingale decomposition

In this section, we give some conditions so that hypothesis \((H')\) is valid between \(F\) and \(F^\rho\), for random times constructed in \((3)\) and, in that case, we give the semimartingale decomposition. We find it useful to change the point of view and consider the problem by examining the beginning and the end of the excursion of the process

\[
Y^T_t := Z^T_t - \inf_{s \leq t} Z^T_s
\]

straddling the random time \(\tau\). That is

\[
\rho(\tau) := \sup\{t \leq \tau : Z^T_t - \inf_{s \leq t} Z^T_s = 0\} = \sup\{s \leq \tau : Y^T_s = 0\},
\]

\[
\delta(\tau) := \inf\{t \geq \tau : Z^T_t - \inf_{s \leq t} Z^T_s = 0\} = \inf\{t \geq \tau : Y^T_s = 0\}.
\]

When \(\tau\) is an honest time, one is able to apply results from Jeulin [8] without referring to the specific structure of the excursion. However, once we replace \(\tau\) by a general random time, to retrieve the decomposition, we exploit the structure of the excursion. We notice that, in view of enlargement of filtration, the properties of the beginning and the end of the excursion are symmetric.

In the following, we shall examine properties of the three random times \(\tau\), \(\rho := \rho(\tau)\) and \(\delta := \delta(\tau)\) in view of enlargement of filtration. For simplicity, we assume that the random time \(\tau\) is finite.

**Lemma 5.1.** We have the following relations between the random times \(\tau\), \(\rho\) and \(\delta\) and associated enlarged filtrations:

(i) the random time \(\delta\) is an \(F^\rho\)-stopping time;

(ii) the random time \(\rho\) is an \(F^{\delta}\) and \(F^{\tau}\)-honest time.

**Proof.** Assertion (i) follows as the random time

\[
\delta = \inf\{t \geq \tau : Y^T_t = 0\} = \inf\{t \geq \rho : Y^T_t = 0\}
\]

is a first passage time in \(F^\rho\), therefore is an \(F^\rho\)-stopping time. To show (ii), it is enough to see that the random time

\[
\rho = \sup\{t \leq \tau : Y^T_t = 0\} = \sup\{t \leq \delta : Y^T_t = 0\}
\]

is a last passage time in \(F^{\delta}\) and \(F^{\tau}\), therefore an \(F^{\delta}\)-honest time and \(F^{\tau}\)-honest time. \(\square\)

**Proposition 5.2.** If the hypothesis \((H')\) is satisfied between filtrations \(F\) and \(F^{\tau}\), then

(i) the hypothesis \((H')\) is also satisfied between \(F^{\rho}\) and \(F^{\rho}\);

(ii) the hypothesis \((H')\) is also satisfied between \(F\) and \(F^{\delta}\).

**Proof.** (i) Note that \(\rho\) is an honest time in the filtration \(F^{\tau}\). Let us introduce the progressive enlargement of \(F^{\tau}\) with \(\rho\) which is denoted by \(F^{\tau,\rho}\). Take any \(F\)-martingale \(M\), then, since the hypothesis \((H')\) is satisfied between \(F\) and \(F^{\tau}\), the process \(M\) is an \(F^{\tau}\)-semimartingale. On the other hand, the random time \(\rho\) is an honest time in \(F^{\tau}\). Then, by classic results (see [8]), the process \(M\) is an \(F^{\tau,\rho}\)-semimartingale. Finally, from Stricker’s Theorem, the process \(M\) is an \(F^{\rho}\)-semimartingale.

(ii) It is sufficient to notice that \(F \subset F^{\delta} \subset F^{\tau}\), as \(\delta\) is an \(F^{\tau}\)-stopping time. Then, by Stricker’s Theorem, if hypothesis \((H')\) is satisfied between \(F\) and \(F^{\delta}\), it is satisfied between \(F\) and \(F^{\rho}\). \(\square\)

**Proposition 5.3.** The hypothesis \((H')\) is satisfied between \(F\) and \(F^{\rho}\) if and only if the hypothesis \((H')\) is satisfied between \(F\) and \(F^{\delta}\).
Proof. Note that $\mathbb{F} \subset \mathbb{F}^3 \subset \mathbb{F}^{\delta,\rho} = \mathbb{F}^{\rho}$, where the last equality follows from (i) of Lemma 5.1. Using (ii) of Lemma 5.1, we see that $\rho$ is an $\mathbb{F}^3$-honest time and this implies that the hypothesis $(\mathcal{H}')$ is satisfied between $\mathbb{F}^3$ and $\mathbb{F}^{\rho}$. If the hypothesis $(\mathcal{H}')$ is satisfied between $\mathbb{F}$ and $\mathbb{F}^{\rho}$ therefore the hypothesis $(\mathcal{H}')$ is satisfied between $\mathbb{F}$ and $\mathbb{F}^\rho$. The converse follows easily from Stricker’s Theorem.

In the next proposition we give a $\mathbb{F}^\rho$-semimartingale decomposition for $\mathbb{F}$-local martingales stopped at $\delta$ (which, by Lemma 5.1, is an $\mathbb{F}^\rho$-stopping time). This result is in the same spirit as [8 Corollary 5.21]. Here we consider the particular case of the beginning and the end of an excursion. We do not use properties of the middle time $\tau$. Moreover, the decomposition is given for stopped processes. It can be seen as extension of classical $\mathbb{F}^\rho$-semimartingale decomposition for $\mathbb{F}$-local martingales stopped at $\rho$ up to the end of the excursion.

**Proposition 5.4.** Let $M$ be an $\mathbb{F}$-local martingale $\mathcal{M}$. Then, the process

$$M_t - \int_0^{\delta_{\rho}} \frac{1}{Z_{s-}^\rho} d\langle X, m^\rho \rangle_s - \int_0^{\delta_{\rho}} 1_{\{\rho < s\}} \frac{1}{Z_{s-}^\rho} d\langle X, m^\delta - m^\rho \rangle_s$$

is an $\mathbb{F}^\rho$-local martingale.

**Proof.** We use anallogical arguments to [8 section V-3]. As each local martingale is locally in $H^1_m$, it is enough to consider $M \in H^1_m(\mathbb{F})$. By Lemma 5.1 [8 Proposition (4,16) 2] and [8 Théorème (5,10)] we get that $M_{\land \delta} \in H^1_{sm}(\mathbb{F}^\rho)$ so $M_{\land \delta} \in H^1_{sm}(\mathbb{F}^\rho)$.

Let $H$ be an $\mathbb{F}^\rho$-predictable bounded process. Then $H_1[0,\delta] = J^- 1_{[0,\rho]} + J^+ 1_{[\rho,\delta]}$, where $J^-$ and $J^+$ are $\mathbb{F}$-predictable processes. The $\mathbb{F}$-predictability of $J^-$ comes from [8 Lemme (4,4) b]), by [8 Proposition (5,3) a)]. $J^+$ can be chosen to be $\mathbb{F}^\rho$-predictable. Next, using again [8 Lemme (4,4) b]), the $\mathbb{F}$-predictability of $J^+$ follows.

By the same arguments as in the proof of [8 Théorème (5,10)], based on properties of dual optional projections, we get

$$\mathbb{E}(H \cdot M_{\infty}) = \mathbb{E}((1_{[0,\delta]} J^- \cdot M_{\infty}) + \mathbb{E}((1_{[\rho,\delta]} J^+ \cdot M_{\infty})$$

$$= \mathbb{E}((J^- \cdot M_{\rho}) + \mathbb{E}((J^+ \cdot M_{\delta}) - \mathbb{E}((J^+ \cdot M_{\rho})$$

$$= \mathbb{E}\left(\int_0^{\infty} J^-_sd\langle M, m^\rho \rangle_s^\rho + \mathbb{E}\left(\int_0^{\infty} J^+_sd\langle M, m^\delta - m^\rho \rangle_s^\rho \right)\right)$$

$$= \mathbb{E}\left(\int_0^{\rho} \frac{H_s}{Z_{s-}^\rho} d\langle M, m^\rho \rangle_s^\rho + \mathbb{E}\left(\int_{\rho}^{\delta} \frac{H_s}{Z_{s-}^\rho} d\langle M, m^\delta - m^\rho \rangle_s^\rho \right)\right),$$

the proof is completed.

6 Construction from Jeanblanc-Song model

In this section, we present another technique to construct pseudo-stopping times which is based on Jeanblanc-Song model. The authors in [10] present solutions to the following problem: construct random times on an extended space with a given supermartingale $Z = Ne^{-\Gamma}$, where $N$ is a continuous local martingale and $\Gamma$ is a continuous increasing process. In our case, it would be then enough to take $N \equiv 1$. The solution is expressed through increasing family of martingales $\{(M^u_t)_{t \geq u} : u \in [0, \infty)\}$ such that $M^u_t \geq M^v_t$ for $t \geq u \geq v$. Namely, given an $\mathbb{F}$-martingale $Y$ and a Lipschitz function $f$ with $f(0) = 0$, for each $u$ the following SDE is considered:

$$\begin{cases}
    dM^u_t = M^u_t f(M^u_t - (1 - e^{-\Gamma_t}))dY_t & \text{for } t \geq u \\
    M^u_u = 1 - e^{-\Gamma_u}
\end{cases}$$

These martingales are proved to take values in $[0,1]$. 

Lemma 6.1. Let \( f(x) = x \), \( Y = B \) be a Brownian motion and \( \Gamma_\infty = \infty \). For \( M^u \) solution of SDE we have \( M^u_\infty = 0 \) or \( M^u_\infty = 1 \).

**Proof.** As \( M^u \) is a bounded martingale, we have

\[
\mathbb{E}(M^u_t - M^u_\infty)^2 = \mathbb{E} \left( \int_t^\infty M^u_s (M^u_s - (1 - e^{-\Gamma_s})) dB_s \right)^2
\]

\[
= \mathbb{E} \left( \int_t^\infty (M^u_s)^2 (M^u_s - (1 - e^{-\Gamma_s}))^2 ds \right)
\]

\[
= \int_t^\infty \mathbb{E}((M^u_s)^2 (M^u_s - (1 - e^{-\Gamma_s}))^2) ds.
\]

This implies that \( \lim_{s \to \infty} \mathbb{E}((M^u_s)^2 (M^u_s - (1 - e^{-\Gamma_s}))^2) = 0 \). From dominated convergence theorem and positivity of \( (M^u_\infty)^2 (M^u_\infty - 1)^2 \) we finally arrive at the assertion.

Lemma 6.1 and \( M^u_t \geq M^v_t \) for \( t \geq u \geq v \) allow us to define the random time \( \tau \) in the following way. Let us take the càdlàg version of the process \( (M^u_\infty)_{u \geq 0} \) and define a random time \( \tau \) as

\[
\tau := \inf \{ u : M^u_\infty = 1 \}.
\]

(8)

**Lemma 6.2.** Let \( \tau \) be defined in (8). Then
(i) \( M^u_\infty = 1_{(\tau \leq u)} \) and \( \tau \) is \( F_\infty \)-measurable;
(ii) the supermartingale \( Z^\tau \) equals \( Z^\tau = e^{-\Gamma} \) and \( \tau \) is an \( \mathbb{F} \)-pseudo-stopping time.

**Proof.** (i) comes from monotonicity of the family \( \{ M^u : u \in \mathbb{R}_+ \} \) and Lemma 6.1.
(ii) Using (i) and properties of the family \( \{ M^u : u \in \mathbb{R}_+ \} \) we write that

\[
Z^\tau = \mathbb{E} (1 - M^u_\infty | F_t) = 1 - M^t_\infty = e^{-\Gamma_t}.
\]

Thus, \( Z^\tau \) is decreasing and continuous and by Proposition 2.3 we conclude that \( \tau \) in a pseudo-stopping time.

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