On outer fluctuations for internal DLA

Amine Asselah ∗ Alexandre Gaudillièr †

Abstract

We had established in [2] inner and outer fluctuation for the internal DLA cluster when all walks are launched from the origin. In obtaining the outer fluctuation, we had used a deep lemma of Jerison, Levine and Sheffield of [3], which estimate roughly the possibility of fingering, and had provided in [2] a simple proof using an interesting estimate for crossing probability for a simple random walk. The application of the crossing probability to the fingering for the internal DLA cluster contains a flaw discovered recently, that we correct in this note. We take the opportunity to make a self-contained exposition.

1 Introduction

In this short note, we correct a mistake in an alternative proof we gave in [2] (proof of Lemma 1.5) of Lemma A of David Jerison, Lionel Levine and Scott Sheffield in [3]. This result bounds the probability the cluster of internal DLA, with particles starting outside a ball, eventually covers the center of this ball when the number of particles is small compared to the volume of the ball. Lemma A controls the possibility the cluster makes fingers protruding out of the spherical shape it likely adopts. This estimate is used in turn to produce an outer error bound when all internal DLA particles are launched from the origin of \( \mathbb{Z}^d \) and \( d \geq 2 \).

Recently Lionel Levine and Yuval Peres noticed a flaw in our (simple) proof. We correct this flaw by adding one step to our initial proof, and since we believe the result is of independent interest, we take the opportunity to present a self-contained argument by including an estimate of the probability one random walk crosses a shell while staying inside a given region in term of its volume. This estimate is Lemma 1.6 of [2] and concerns one random walk.

The walk is denoted \( S : \mathbb{N} \to \mathbb{Z}^d \), and we call \( P_z \) the law of the simple random walk when \( S(0) = z \). If \( \Lambda \) is a subset of \( \mathbb{Z}^d \), \( T(\Lambda) \) denotes the hitting time of \( \Lambda \). Also, we call \( \| \cdot \| \) the euclidean distance, \( B(z, \rho) \) the trace on \( \mathbb{Z}^d \) of the ball of radius \( \rho > 0 \) and center \( z \in \mathbb{R}^d \), and \( \partial B(z, \rho) \) the boundary of \( B(z, \rho) \), that is \( \partial B(z, \rho) := \{ y \in \mathbb{Z}^d \setminus B(z, \rho) : \exists x \in B(z, \rho), \| x - y \| = 1 \} \). We can now state one key Lemma of [2].

∗Université Paris-Est, LAMA (UMR 8050), UPEC, UPEMLV, CNRS, F-94010, Créteil, France; amine.asselah@u-pec.fr

†Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France; alexandre.gaudilliere@math.cnrs.fr
Lemma 1.1 Assume dimension is $d \geq 2$. There are constants $\kappa_d > 0, C_d > 1$ such that for any $r > 0, 0 < h < r/2$ and $z \in \partial B(0, r)$, and $\mathcal{V} \subset B(0, r) \setminus B(0, r - h)$ we have
\[
P_z(T(B(0, r - h)) < T(\mathcal{V}^c)) \leq C_d \exp \left( -\kappa_d \left( \frac{h^d}{|\mathcal{V}|} \right)^{\frac{1}{d-1}} \right) \tag{1.1}
\]
Note that the estimate (1.1) is useful only if $h^d/|\mathcal{V}|$ is large enough to counter $C_d$. To state our next result, we need to introduce internal DLA and more notation. A configuration of random walkers is denoted $\eta$ and is an element of $\mathbb{N}^{\mathbb{Z}^d}$. The number of walks in $\eta$ is $|\eta| = \sum_{z \in \mathbb{Z}^d} \eta(z)$. The cluster of internal DLA made of random walks initially in $\eta$ with $|\eta| < \infty$, is itself a configuration of $\{0, 1\}^{\mathbb{Z}^d}$ that we build inductively as follows. Choose an arbitrary ordering of the random walks, and run the walks one at a time following their order. When the running walk steps on an empty site, it stops, or settles and the site becomes part of the cluster. At this moment, send the next random walk until it settles and so on. This cluster has a law independent of the ordering of the walks. This is the celebrated abelian property. Note that only one random walk settles in each site of the final cluster that we call $A(\eta)$ and which can be seen as a subset of $\mathbb{Z}^d$. The random walks with the rules for settling are called explorers.

We can now state a weaker result than Lemma 1.5 of [2]. It is a direct consequence of the (corrected) proof of Lemma 1.5.

Lemma 1.2 Assume that $d = 2$. There are $a, \kappa_2 > 0$, such that for $r$ large enough, and for any configuration $\eta$ of explorers outside of $B(0, 2r)$, with $|\eta| \leq ar^2/\log^2 r$, we have
\[
P(0 \in A(\eta)) \leq \exp \left( -\kappa_2 \frac{r^2}{\log(r)} \right). \tag{1.2}
\]
Assume $d \geq 3$. There are $a, \kappa_d > 0$, such that for $r$ large enough, and for any configuration $\eta$ of explorers outside of $B(0, 2r)$, with $|\eta| \leq ar^2/\log r$, we have
\[
P(0 \in A(\eta)) \leq \exp \left( -\kappa_d r^2 \right). \tag{1.3}
\]
Finally, we state Lemma A of Jerison, Levine and Sheffield, (and Lemma 1.5 of [2]) as a Corollary of Lemma 1.2.

Corollary 1.3 The inequalities (1.2) and (1.3) hold as soon as $r$ is large enough and any configuration $\eta$ of explorers outside $B(0, 2r)$ with $|\eta| \leq \epsilon r^d$, for some positive constant $\epsilon$ depending only on dimension.

We prove Lemma 1.1 in Section 2. We prove Lemma 1.2 in Section 3 and we correct our proof in Section 4.
2 One explorer crossing a shell

In this Section, we reproduce the short proof of Lemma 1.1 of [2].

Take a positive integer \( n < h \), and consider a partition of the shell \( S := B(0, r) \setminus B(0, r - h) \) into \( n \) shells \( \{S_k, k < r\} \) of width \( 2\delta := h/n \). For \( k < n \), set \( \Sigma_k := \partial B(0, r - (2k + 1)\delta) \). Let \( \{S(n), n \in \mathbb{N}\} \) be the underlying random walk with which we build an explorer.

With each explorer of internal DLA is associated a so-called flashing explorer which can settle only on some random sites, when they are empty. Thus, we define the random sites \( \{Z_k, 0 \leq k < n\} \) as follows. For each \( k < n \), we draw a continuous random variable \( R_k \) on \([0, \delta]\) with density in \( x \in [0, \delta] \mapsto dx^{d-1}/\delta^d \), and \( Z_k \) is the exit site of \( S \) from \( B(S(T(\Sigma_k)), R_k) \) after time \( T(\Sigma_k) \). Then, the flashing explorer settles on the first \( Z_k \) not belonging to \( V \). The purpose of the flashing construction is that (i) the flashing sites are each distributed almost uniformly inside the ball \( B(S(T(\Sigma_k)), \delta) \) (and this is Proposition 3.1 of [4]), and (ii) \( P_z(T(B(0, r)) < T(V^c)) \) is bounded above by the probability that the flashing explorer crosses \( S \).

Now, for a small \( \beta \) to be chosen later, we say that \( y \in \Sigma_k \) has a dense neighborhood if \(|B(y, \delta) \cap V| \geq \beta \delta^d \), and we call \( D_k \subset \Sigma_k \) the set of such \( y \). There is \( \kappa > 0 \) such that knowing that the explorer has crossed \( D_1, \ldots, D_{k-1} \), we have the following.

- If \( S(T(\Sigma_k)) \notin D_k \), then the probability that the explorer does not settle in \( S_k \) is smaller than \( \kappa\beta \) (Proposition 3.1 of [4]).
- The probability that \( S(T(\Sigma_k)) \in D_k \) is smaller than \( \kappa|D_k|/h^{d-1} \) (see Lemma 5 of [4]) uniformly over \( Z_{k-1} \).

If the explorer has crossed \( S \), the flashing has also done so, which means that \( Z_k \in V \) for all \( k < n \). By successive conditioning, we obtain

\[
P_z(T(B(0, r - h)) < T(V^c)) \leq \prod_{k<n} \left( \kappa\beta + \frac{\kappa|D_k|}{\delta^{d-1}} \right). \tag{2.1}
\]

By the arithmetic-geometric inequality, we obtain

\[
P_z(T(B(0, r - h)) < T(V^c)) \leq \left( \kappa\beta + \frac{\kappa}{n} \sum_{k<n} \frac{|D_k|}{\delta^{d-1}} \right)^n. \tag{2.2}
\]

Note that while each \( y \in D_k \) satisfies \(|B(y, \delta) \cap V| \geq \beta \delta^d \), each site in \( B(y, \delta) \cap V \) is at a distance less than \( \delta \) from a number of sites of \( D_k \) of order at most \( \delta^{-1} \). Thus, for some \( c > 0 \)

\[
\sum_{k<n} \frac{\beta|D_k|\delta^d}{\delta^{d-1}} \leq c|V|, \quad \text{i.e.,} \quad \frac{1}{n} \sum_{k<n} \frac{|D_k|}{\delta^{d-1}} \leq \frac{2c|V|}{\beta h^{d-1}} \quad \text{(we recall} \quad \frac{1}{n\delta} = \frac{2}{h}).
\]

We choose now \( \beta \) such that \( 4\kappa\beta < 1 \), and we choose the smallest \( \delta \) such that

\[
\delta^{d-1} \geq \frac{2c|V|}{\beta^2 h}. \tag{2.3}
\]
Thus, (2.2) reads
\[ P_z(T(B(0, r - h)) < T(V^c)) \leq \left( \frac{1}{2} \right)^{h/(2\delta)}. \] (2.4)

Requiring that \( 2\delta < h \) adds a constraint on \( |\mathcal{V}| \):
\[ |\mathcal{V}| \leq \frac{\beta^2}{2^d c} h^d. \] (2.5)

Instead of including (2.5) as a condition of our Lemma, we find it more convenient to note that the probability we estimate is less than 1, so that we obtain (1.1), with constant \( C_d \).

### 3 Cloud of Explorers Crossing a Shell

We prove in this Section Lemma 1.2. Our initial problem consists in estimating the crossing probability of one explorer out of \( \eta \) when the positions of \( \eta \) lay in the boundary of a ball of radius \( 2r \). The key idea of the proof is to divide the shell \( B(0, 2r) \setminus B(0, r) \) into a sequence of smaller shells with random widths. The choice is however different in the proofs of Corollary 1.3 and of Lemma 1.2. In proving Corollary 1.3 the width of a shell depends on the number of explorers that reach the shell boundary. This is natural in view of Lemma 1.1 to estimate the crossing probability of a shell, the quantity one needs to control is the number of explorers having settled in this very shell, which is bounded by the number of explorers arriving on its external boundary. However, in proving Lemma 1.2 the width depends on the number of explorers having settled in the previous shell.

Also, instead of considering one configuration \( \eta \), we find it convenient to take a Poisson cloud of explorers. This simplifies some large deviation estimates.

#### 3.1 Poisson Cloud

Let \( \prec \) be the usual partial order on the space of configurations, and consider an increasing sequence of configurations \( (\zeta_k, k \in \mathbb{N}) \) on the boundary of \( B(2r) \). We require also that this sequence satisfies \( |\zeta_k| = k \), and
\[ \forall k \geq |\eta|, \quad \eta \prec \zeta_k. \]
We call \( u(\eta) \) the probability that one explorer settles in \( B(r) \) when starting with an initial configuration \( \eta \). Note that the event we consider is increasing: with more explorers, it is easier to make one of them cross. In other words,
\[ \forall k \geq |\eta|, \quad u(\zeta_k) \geq u(\eta). \]

Let \( K \) be a Poisson variable of parameter \( \lambda_0 := \epsilon r^d \). The sequence \( (\zeta_k, k \in \mathbb{N}) \) being given, we have
\[ E[u(\zeta_K)] = \sum_{k \in \mathbb{N}} u(\zeta_k) \cdot \frac{e^{-\lambda_0} \lambda_0^k}{k!}. \]
Since \( P(K > \frac{1}{2} \lambda_0) \geq 1/8 \) when \( \lambda_0 \geq 1 \), and \( k \mapsto u(\zeta_k) \) is increasing, we have that if \( k^* \) is the integer part of \( \frac{1}{2} \lambda_0 \), and \( r \) is large enough so that \( k^* > 1 \), then

\[
\quad u(\zeta_{k^*}) \cdot P(K > k^*) \leq E[u(\zeta_K)] \iff u(\zeta_{k^*}) \leq 8E[u(\zeta_K)].
\]

Thus, we need to estimate the expectation \( E[u(\zeta_K)] \), where the number of explorers is Poisson, whereas the initial configuration is arbitrary. To this end we will make a repeated use of the following deviation bound for a generic Poisson random variable \( K \) of parameter \( \lambda \): for any positive \( \theta \)

\[
P(K > \theta) \geq \exp \left\{ -\theta \log \left( \frac{\theta}{e \lambda} \right) \right\}
\]

Let \( \eta_0 := \zeta_K \), and subdivide the shell \( B(2r) \setminus B(r) \) into successive shells of random width \( H_0, H_1, \ldots \) defined by induction as follows. For some constant \( \gamma \),

\[
H_0^d = \gamma |\eta_0| = \gamma K, \quad \text{and} \quad S_0 := B(2r) \setminus B(2r - H_0).
\]

Imagine we have labelled the explorers, and have send \( k - 1 \) of them, that we stop either if they settle or when they enter \( B(2r - H_0) \). Let \( \mathcal{V} \subset S_0 \) be the domain where some have settled, whereas at most \( k - 1 - |\mathcal{V}| \) are stopped on entering \( B(2r - H_0) \). The probability that the \( k \)-th explorer, with \( k \leq K \), crosses \( S_0 \), knowing \( K \) and \( \mathcal{V} \), is bounded as we use (1.1) and \( |\mathcal{V}| \leq K \),

\[
P\left( \text{the } k \text{-th explorer crosses } S_0 \mid K, \mathcal{V} \right) \leq \sup_{z \in \partial B(2r)} \sup_{\mathcal{V} \subset S_0: |\mathcal{V}| \leq K} P_z(T(B(r - H_0)) < T(\mathcal{V}^c))
\]

\[
\quad \leq \sup_{\mathcal{V} \subset S_0: |\mathcal{V}| \leq K} C_d \exp \left( -\kappa_d \frac{H_0^d}{|\mathcal{V}|} \right)^{1/(d-1)}
\]

\[
\leq C_d \exp \left( -\kappa_d \gamma^{1/(d-1)} \right).
\]

Now, we define \( \gamma \) large enough so that the right hand side of (3.2) is less than \( 1/e \). Thus, we have an estimate valid for any explorer:

\[
P\left( \text{an explorer crosses } S_0 \right) \leq \frac{1}{e}.
\]

Also, each explorer having crossed \( S_0 \) is stopped upon entering \( B(2r - H_0) \). We call \( \eta_1 \) their configuration. The key observation is that \( |\eta_1| \) is bounded by a Poisson variable \( \mathcal{N}_1 \) with parameter \( \lambda_0/e \). Now, we define the second shell so that its width is

\[
H_1^d = \gamma \mathcal{N}_1, \quad \text{and} \quad S_1 := B(2r - H_0) \setminus B(2r - H_1),
\]

and so forth. Thus, after considering \( i \geq 1 \) crossings, we have a Poisson variable \( \mathcal{N}_i \) with parameter \( e^{-i} \lambda_0 \) (with \( \lambda_0 = er^d \)) bounding the number of explorers stopped upon entering of \( B(2r - H_0 - \cdots - H_{i-1}) \), and a width \( H_i^d = \gamma \mathcal{N}_i \). This ensures that any explorer has probability less than \( 1/e \) to cross the \( i \)-th shell.

Now, note that the event that one of the explorer of \( \eta_0 \) crosses \( B(2r) \setminus B(r) \) is contained in the event that \( \{ H_0 + \cdots + H_L > r \} \) where \( L = \inf \{ k : H_k = 0 \} \).
Define \( i^* \) to be the smallest integer so that \( e^{-i^*} r^d \) is less than 1. Our starting point is
\[
\{ \sum_{i<L} H_i > r \} \subset \{ \sum_{i<i^*} H_i > \frac{r}{2} \} \cup \{ \sum_{i<i^*} H_i \leq \frac{r}{2}, \sum_{i^* \leq i<L} H_i \geq \frac{r}{2} \}. \tag{3.3}
\]
Now, \( N_{i^*} \) is bounded by a Poisson variable of parameter 1, and we further divide the second event of (3.3) according to dimension.

### 3.2 Dimension Two

We write, for some small \( \delta \)
\[
\{ \sum_{i<L} H_i > r \} \subset \{ \sum_{i<i^*} H_i > \frac{r}{2} \} \cup \{ \sum_{i<i^*} H_i \leq \frac{r}{2} \}
\subset \{ \sum_{i<i^*} H_i^{1/2} > 4\rho \} \cup \{ N_{i^*} > \frac{\delta r^2}{\log^2 r} \} \quad \text{with} \quad 4\rho := \frac{r}{2\gamma^{1/2}} \tag{3.4}
\]
\[\cup \{ \text{less than} \frac{\delta r^2}{\log^2 r} \text{ explorers cross a shell of width } r/2 \}.\]

We now proceed in estimating the three events separately. Note that the event that less than \( \delta r^2/\log^2 r \) explorers have to cross a shell of width \( r/2 \) is dealt with Lemma 1.2. Also, the fact that \( N_{i^*} \) is bounded by \( \mathcal{P}(1) \) and (3.1) yield
\[
P(N_{i^*} \geq \delta \frac{r^2}{\log^2 r}) \leq \exp \left( -\delta \frac{r^2}{\log^2 r} \log\left( \frac{r^2}{e \log^2 r} \right) \right)
\leq \exp \left( -2\delta \frac{r^2}{\log r} (1 + o(1)) \right).
\]

We now deal with the deviation \( \{ \sum_{i<i^*} N_i^{1/2} > 4\rho \} \). A union bound allows us to treat this term after we distinguish three regimes: (i) when \( i \) is small, the deviation asks \( N_i \) to be larger than \( e^{-i} r^2 \), and small means for \( i \leq j^* \) with \( \exp(j^*) \cdot \log(r) = 1 \), (ii) when \( j^* \leq i \leq 2j^* \), we ask \( N_i \) to be larger than \( r^2/(i^2 \cdot \log^2 2) \), and finally (iii) when \( i \) is large the deviation asks \( N_i \) to be larger than \( r^2/(i \cdot i^*) \), and we shall see that this gives the correct bound for \( i \geq 2j^* \).

The first regime will fix the value for \( \epsilon \).

More precisely, our first sum runs up to \( j^* = \log(\log r) \), such that \( \exp(j^*) = \log r \). We now use that
\[
\{ \sum_{i<j^*} N_i^{1/2} \geq \rho \} \subset \bigcup_{i<j^*} \{ N_i > \frac{e^{-i}}{(1-e^{-1/2})^2} \rho^2 \}.
\]
For any \( i \leq j^* \), we have for \( \epsilon \) small enough, and some constant \( \kappa \)
\[
P(N_i > \frac{e^{-i}}{(1-e^{-1/2})^2} \rho^2) \leq \exp \left( -\frac{\rho^2}{(1-e^{-1/2})^2 e\epsilon} \log\left( \frac{\rho^2}{e\epsilon(1-e^{-1/2})^2 \rho^2} \right) \right)
\leq \exp \left( -\kappa e^{-j^*} r^2 \right) = \exp \left( -\kappa \frac{r^2}{\log r} \right).
Now, we consider case (ii). Note that
\[
\{ \sum_{j^* \leq i \leq 2j^*} N_i^{1/2} \geq \rho \} \subset \bigcup_{j^* \leq i \leq 2j^*} \{ N_i > \frac{\rho^2}{i^2 \log^2 2} \}.
\]
Since \( j^* = \log \log r, e^{-i^*} \) is smaller than \( 4(\log \log r)^2 / \log r \) for \( j^* \leq i < 2j^* \). Hence, by (3.1) there is a positive constant \( c_0 \) such that, for any such \( i \),
\[
P \left( N_i > \frac{\rho^2}{i^2 \log^2 2} \right) \leq \exp \left\{ -\frac{\rho^2}{i^2 \log^2 2} \log \left( \frac{c_0 \log r}{(\log \log r)^2} \right) \right\}
\]
and there is positive constant \( \kappa \) such that
\[
P \left( N_i > \frac{\rho^2}{i^2 \log^2 2} \right) \leq \exp \left\{ -\kappa \frac{r^2}{(\log \log r)^2} \right\}.
\]
Finally, consider \( 2j^* \leq i \leq i^* \). First, note that
\[
\left\{ \sum_{2j^* \leq i \leq i^*} N_i^{1/2} > 2 \rho \right\} \subset \bigcup_{2j^* \leq i \leq i^*} \left\{ N_i^{1/2} > \frac{\rho}{\sqrt{i^* \cdot i}} \right\}.
\]
Secondly, there is a constant \( C > 1 \), such that for any \( \epsilon \), and \( r \) large enough
\[
i \cdot i^* \cdot e^{-i} \leq \left( i \cdot e^{-i/4} \right) \times \left( i^* \cdot e^{-j^*} \right) \times e^{-i/4} \leq \frac{C \cdot e^{-i/4}}{\epsilon}.
\]
Indeed, we have used that \( i^* \) is of order \( 2 \log(r) \) whereas \( \exp(j^*) = \log(r) \). Thus, for \( 2j^* \leq i \leq i^* \), there is a constant \( \kappa > 0 \) such that for \( r \) large enough
\[
P \left( N_i > \frac{\rho^2}{i^* \cdot i} \right) \leq \exp \left( -\frac{\rho^2}{i^* \cdot i} \log \left( \frac{\rho^2}{i^* \cdot e^{-i^*} \cdot e^{-i/4}} \right) \right)
\leq \exp \left( -\frac{\rho^2}{i^*} \times \frac{i/4 + \log(C)}{i} \right) \leq \exp \left( -\frac{\kappa r^2}{\log(r)} \right).
\]
This concludes the proof of the Corollary in the case \( d = 2 \).

### 3.3 Dimensions \( d \geq 3 \)

We recall (3.3), and we write for some small \( \delta \) with \( 2\rho := r/(2\gamma^{1/d}) \),
\[
\{ H_0 + \cdots + H_L > r \} \subset \left\{ \sum_{i < i^*} N_i^{1/d} > 2\rho \right\} \cup \left\{ N_i > \delta \frac{r^2}{\log r} \right\}
\cup \left\{ \text{less than } \frac{\delta r^2}{\log r} \text{ explorers cross a shell of width } r/2 \right\}.
\]

We now proceed in estimating the three events separately.
Note that $N_i^*$ is bounded by $\mathcal{P}(1)$ so

$$P(N_i^* \geq \frac{\delta r^2}{\log r}) \leq \exp \left(- \delta \frac{r^2}{\log r} \log \left(\frac{\delta r^2}{e \log r}\right)\right)$$

$$\leq \exp \left(- 2\delta r^2 (1 + o(1))\right).$$

To deal with the deviation $\{\sum_{i<i^*} N_i^{1/d} > 2\rho\}$, we use again a union bound, but here we only need to distinguish two regimes: (i) when $i$ is small, the deviation asks $N_i$ to be larger than $c_0 e^{-i r^d}$, for some fixed constant $c_0$ and thus $i$ small means that for some $\kappa > 0$,

$$P(N_i \geq c_0 e^{-i r^d}) \leq \exp \left(- c_0 e^{-i r^d} \log(c_0(ee)^{-1})\right) \ll \exp(-\kappa r^2).$$

We define $j^*$ to be the largest integer $i$ such that

$$c_0 r^{d-2} e^{-i \log(c_0(ee)^{-1})} \geq 1.$$

Note that $j^*$ is of order $(d - 2) \log r$, whereas $i^*$ is of order $d \log r$. Now, for $i > j^*$, we use that for some constant $c_1 > 0$ such that

$$\sum_{i=j^*}^{i^*} \frac{c_1}{i} \leq 1.$$

We use the estimate for $i > j^*$ and $i < i^*$, that for some constant $\kappa > 0$, and $r$ large enough

$$P(N_i^{1/d} \geq \frac{c_1}{i} \rho) \leq \exp \left(- c_1 \frac{\rho^d}{(i^*)^{d} \log(c_0(ee)^{-1})}\right) \leq \exp(-\kappa r^2).$$

### 4 Proof of Lemma 1.2

For $C_d > 1$ and $\kappa_d > 1$ appearing in Lemma 1.1 we set

$$\gamma = \max \left(1, \left(\frac{2 \log C_d}{\kappa_d}\right)^{d-1}\right).$$

We divide $B(0, r)$ into shells $S_0, S_1, \ldots$ of widths $H_0, H_1, \ldots$ We set $H_0 = h_0 = r/4$, and for $k \geq 1$ the width $H_k$ is random and depends on the number $N_{k-1}$ of explorers settling in the previous shell.

$$H_k^d = \gamma N_{k-1}.$$

We denote by $L$ the first $k$ for which $\sum_{i<k} H_i > 3r/4$ or $H_k < 1$, in which case $N_{k-1} = 0$. The shells are as follows. For $k < L$,

$$S_k = B \left(0, r - \sum_{i<k} H_i \right) \setminus B \left(0, r - \sum_{i<k} H_i \right).$$

For $r$ large enough, we have $\gamma|\eta| \leq (r/4)^d = h_0^d$, so that for all $k \geq 0$,

$$H_k^d \leq \gamma N_{k-1} \leq \gamma|\eta| \leq \left(\frac{r}{4}\right)^d = h_0^d.$$
Now, if $0 \in A(\eta)$, then $\sum_{k<L} H_k$ has to be larger than $3r/4$, which implies, since $H_1 \leq H_0 = r/4$, that $\sum_{k=2}^L H_k > r/4$. Also, for each $k < L$, $N_k$ explorers have to cross the shells $S_0$, $S_1$, $\ldots$, $S_{k-1}$. Since $L < r$ we get by Lemma 1.1 that, writing $(n_k, k < l)$ for a generic family of $l$ positive integers and writing $h_i$ for $(\gamma^{-1}n_i)$, then

$$P(0 \in A(\eta)) \leq \sum_{l<r} \sum_{(n_k, k < l)} \left( \frac{|\eta|}{|n_k|} \right) C_d^{\sum_{k<i} k n_k} \exp \left\{ - \sum_{k<l} n_k \sum_{i<k} \kappa_d \left( \frac{h_i^d}{n_i} \right)^{\frac{1}{d'}} \right\} \times 1 \left\{ \sum_{k=2}^l h_k > \frac{r}{4} \text{ and } h_k \leq h_0 \text{ for all } k < l \right\} \quad (4.1)$$

Then, by the arithmetic-geometric inequality and using (4.1), it holds, for each $k > 0$ with $h_k \leq h_0$,

$$\frac{1}{k} \sum_{i<k} \left( \frac{h_i^d}{n_i} \right)^{\frac{1}{d'}} \geq \left( \prod_{i<k} \frac{h_i^d}{n_i} \right)^{\frac{1}{k(d-1)}} = \left( \frac{h_0^d}{n_k \gamma^{k-1}} \right)^{\frac{1}{k(d-1)}} = \left( \frac{h_0^d \gamma^k}{h_k^d} \right)^{\frac{1}{k(d-1)}} \geq \frac{2 \log C_d}{\kappa_d}$$

and as soon as $h_k \leq h_0$ for all $k < l$,

$$C_d^{\sum_{k<l} k n_k} \exp \left\{ - \sum_{k<l} n_k \sum_{i<k} \kappa_d \left( \frac{h_i^d}{n_i} \right)^{\frac{1}{d'}} \right\} \leq C_d^{-\sum_{k<l} k n_k} = C_d^{-\frac{1}{d} \sum_{k<l} k h_k^d}.$$ 

By Hölder’s inequality

$$\sum_{k=2}^l h_k = \sum_{k=1}^{l-1} h_{k+1} \leq \left( \sum_{k=1}^{l-1} \frac{k}{\gamma} \right)^{\frac{d}{d+1}} \left( \sum_{k=1}^{l-1} k h_k^d \right)^{\frac{1}{d}},$$

Now, there is a positive constant $c_d$ such that in $d = 2$

$$\sum_{k<l} k h_k^d \geq \frac{\left( \sum_{k=2}^l h_k \right)^2}{c_2 \log l},$$

whereas if $d \geq 3$.

$$\sum_{k<l} k h_k^d \geq \frac{\left( \sum_{k=2}^l h_k \right)^d}{c_d d^{d-2}}.$$ 

Hence, (4.1) yields in $d = 2$

$$P(0 \in A(\eta)) \leq r|\eta| r^{|\eta|} C_2^{-\frac{r^2}{2 \log^2 r}},$$

and yields in $d \geq 3$

$$P(0 \in A(\eta)) \leq r|\eta| r^{|\eta|} C_d^{-\frac{r^2}{d \log^d r}}.$$ 

These bounds establish the required asymptotics provided that $|\eta| \leq ar^2 / \log^2 r$ if $d = 2$, or $|\eta| \leq ar^2 / \log r$ if $d \geq 3$, for a small enough $a > 0$. 

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References

[1] Asselah A., Gaudilliè re A., From logarithmic to subdiffusive polynomial fluctuations for internal DLA and related growth models. Ann. Probab. 41 (2013), no. 3A, 1115–1159.

[2] Asselah A., Gaudilliè re A., Sublogarithmic fluctuations for internal DLA. Ann. Probab. 41 (2013), no. 3A, 1160–1179.

[3] Jerison, D.; Levine, L.; Sheffield, S., Logarithmic fluctuations for internal DLA. J. Amer. Math. Soc. 25 (2012), no. 1, 271–301.

[4] Lawler, G.; Bramson, M.; Griffeath, D. Internal diffusion limited aggregation. Ann. Probab. 20 (1992), no. 4, 2117–2140.

[5] Lawler, G., Intersection of Random Walks Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1991.