The Aharonov-Bohm-Effect, Non-commutative Geometry, Dislocation Theory, and Magnetism

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Abstract
The four items mentioned in the title are put into context in an informal way.

1 Introduction
This is an informal paper, not intended for publication: the items mentioned in the title

1. the Aharonov-Bohm effect
2. non-commutative geometry
3. dislocation theory, in particular concerning the role of the Burgers vector
4. magnetism (mainly spin-orbit interaction)

are considered and put into context. In this way we hope to remove certain high-browed features from the issue, at the same time clarifying the general meaning of the first and second terms, and also putting some emphasis on the work of the community studying the third or fourth one, often without relation to those working on one of the two first-mentioned subjects.

2 The Aharonov-Bohm Effect
The Aharonov-Bohm effect, see [1], is an important quantum-mechanical phenomenon showing explicitly that quantum-mechanics is not a classical theory as usual, for example, as Newtonian mechanics or conventional electromagnetism. A magnetic induction \( \vec{B} \) is considered, which gives rise to an interference effect of electrons, which are definitely outside the range where the Lorentz forces act and some effect could naturally be expected. Nevertheless, a well-defined interference is observed, since in quantum mechanics it is not \( \vec{B} \), but the magnetic vector potential \( \vec{A} \) that counts; of course, the closed-loop property of the integration path also counts, see below.

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In fact, in quantum mechanics the behaviour of a charged particle is described by the Hamiltonian

$$\mathcal{H} = \frac{\hat{p}^2 - q\vec{A}}{2m} + q\Phi(\vec{r}, t). \quad (1)$$

Here we work in the SI-system; \( q \) is the charge and \( m \) the mass of the particle, \( \hat{p} \) is the momentum operator, \( \vec{A}(\vec{r}, t) \) the vector potential and \( \Phi(\vec{r}, t) \) the scalar potential of the electromagnetic field, \( i.e.: \)

$$\vec{B} = \text{rot} \vec{A}. \quad (2)$$

and

$$\vec{E} = -\text{grad} \Phi - \frac{\partial \vec{A}}{\partial t}. \quad (3)$$

In the following, for simplicity, let us consider only situations where neither the electrical field \( \vec{E} \) nor the scalar potential \( \Phi \) plays a role.

As a consequence, classically, only the magnetic induction \( \vec{B} \) is important, namely through the Lorentz force, and the Newtonian equation of motion is simply

$$m\ddot{\vec{v}} = q[\vec{v} \times \vec{B}], \quad (4)$$

where \( \vec{v} \) is the velocity and \( \ddot{\vec{v}} \) the acceleration of the particle. Usually one considers a thin magnetic wire magnetized longitudinally, such that the magnetic induction outside the wire vanishes. But the vector potential \( \vec{A} \) does not vanish outside!

To keep gauge invariance, \( i.e. \) invariance of the electromagnetic fields against changes involving an arbitrary gauge function \( f \) in the form

$$\vec{B} = \text{rot} \vec{A} + \text{grad} f, \quad (5)$$

$$\vec{E} = -\text{grad} \phi - \frac{\partial f}{\partial t}. \quad (6)$$

it is necessary to concentrate on closed loop integration paths \( \Gamma \), \( i.e. \) by representing the closed loop property by the integration symbol \( \oint \) one has:

$$\oint_{\Gamma} d\vec{r} \cdot \vec{A} = \int_{\Gamma} B \cdot \vec{n} d^2A =: \Phi_F. \quad (7)$$

Here \( F \) is any surface clamped into \( \Gamma \) (\( i.e. \) \( \Gamma \) is the boundary to \( F \), \( \Gamma = \partial F \)). Note that there are many different surfaces, of which the same \( \Gamma \) is the boundary; this is the genuine reason for the gauge degree of freedom.

In \( (7) \) the vector \( \vec{n} \) denotes the normal to the surface \( F \) and \( d^2A \) is the area element. The result of \( (7) \) is the magnetic induction \( \Phi_F \) flux through \( F \), \( i.e. \) through the wire cross section, although \( F \) can be much larger.

So, the Aharonov-Bohm effect, which was experimentally realized, \( e.g., \) by Börsch et al., \( 2 \), shows in a specific way that quantum-mechanics is more than a classical theory: It is not \( \vec{B} \) and the corresponding Lorentz force, which counts, but the magnetic flux \( \Phi_F \) through an arbitrary closed loop \( \Gamma \), or equivalently the circumferential integral \( \oint_{\Gamma} d\vec{r} \cdot \vec{A} \), which is identical with \( \Phi_F \). Thereby the essential point is to note that the flux may correspond to a very small part of the interior of \( \Gamma \), and not to the totality of it.
3 Non-commutative Geometry

The effect can also be interpreted as a change of geometry induced by the magnetic flux. This is called non-commutative geometry, see e.g. [3], since \( [p_j, p_k] \psi = 0 \), \( \forall \psi \), whereas \( [p_j - qA_j, p_k - qA_k] \psi = q \hbar i \cdot \{ [\partial_j A_k] \psi + [A_j \partial_k] \psi \} \neq 0 \). The term non-commutative geometry may look highbrowed to people from the solid-state community, but it turns out that this is not so. In fact, the closed line \( \Gamma \) corresponds to the Burgers loop in dislocation theory, and the magnetic induction \( \vec{B} \) to the dislocation density \( \vec{\eta} \), a tensorial quantity with two indexes, e.g., essentially the Burgers vector \( \vec{b} \) of the dislocation, times the tangent vector \( \vec{\tau} \) of the dislocation line. The fundamental equation between \( \vec{B} \) and \( \vec{A} \), namely \( B_i = \epsilon_{i,j,k} \partial_j A_k \), with the well-known antisymmetric unit tensor \( \epsilon_{i,j,k} \), corresponds to the compatibility equation between the strain \( \vec{\varepsilon} \) and \( \vec{\eta} \), namely \( \text{Ink} \ \vec{\varepsilon} = \vec{\eta} \), where the symmetric incompatibility operator, acting on a symmetric two-tensor \( \xi_{j,k} \), is given by \( (\text{Ink} \ \vec{\xi})_{i,j} := -\epsilon_{i,k,l} \cdot \epsilon_{j,m,n} \partial_k \partial_m \xi_{l,n} \). This result is symmetric in the indexes \( i \) and \( j \), and also in \( l \) and \( n \).

4 Dislocation theory

The importance of dislocation theory in the present context has already been shown in the preceding section. Additionally we mention the work of E. Krönner, see [4], who introduced a close relation between the source-tensor \( \vec{\eta} \) of the incompatibility and the eigenstresses of incompatible solids.

It is well-known that in case of compatibility, the strains \( \vec{\varepsilon} \) can be derived from a shift vector \( \vec{u}(\vec{\tau}, t) \), through the identity \( \epsilon_{i,k} = (1/2)(\partial_i u_k + \partial_k u_i) \) (for simplicity we restrict ourselves to linear elasticity).

In contrast, in case of incompatibility, the above-mentioned identity does not apply. However, strains and stresses are related as usual, and usually dislocations (Burgers vector and tangent line) are the sources of the incompatibility, see [4] and [5].

5 Magnetism

Now the Maxwellian stress tensor \( \vec{\sigma}_{\text{Maxwell}} \) comes into play (which - by the way - in a magnetic system is not symmetric, due to the torque \( dV \vec{J} \times \vec{H} \), where \( \vec{J} \) and \( \vec{H} \) have their usual meaning, i.e., \( dV \vec{J} \) is the magnetic moment of the volume element \( dV \), see for example [6]). Magnetic anisotropies, i.e., the spin-orbit forces, are particularly important at surfaces and interfaces (i.e., the spin-transfer across them should also be influenced) and one should note the effect of magnetostriction, which is often neglected, but important for the sources of incompatibility, especially the magnetostrictive stresses belong to the non-compatible eigenstresses in the sense of E. Krönner. If, e.g., by magnetostriction the magnetic domains are elongated in the direction of magnetization and compressed vertically to it, then domains with different directions usually produce incompatible strains. Here one should have a look at figure 14 in the above-mentioned book of Krönner. But in contrast to elastic and magnetostrictive energies, yielding exclusively symmetric stresses, \( \sigma_{i,k} = \sigma_{k,i} \), because the energy depends only on the symmetric part of the distortions, e.g. \( \epsilon_{i,k} = (1/2) \cdot (u_{i,k} + u_{k,i}) \), other magnetic interactions also involve the antisymmetric part, \( u_{[i,k]} = (1/2) \cdot (u_{i,k} - u_{k,i}) \). This is dual to a rotation vector \( \vec{\omega} \), e.g. \( \omega_3 := u_{[1,2]} \), with a vector.
potential $\hat{A}_3$. This must be added to Kröner’s symmetric theory, leads to torsion densities and to the appearance of an antisymmetric part of $\sigma_{i,k}$.

Moreover, the Burgers loop equation $\oint \partial F \, du^i \neq 0$, namely $\int F (g_{\text{dislocation}})^i_{\alpha, \beta, \gamma} \, dx^\alpha \cdot dx^\gamma$ corresponds exactly to the dislocation density $(g_{\text{dislocation}})^i_{\alpha, \beta, \gamma} = b^i \cdot t_{\alpha}$(areal density perpendicular to $dx^\beta \cdot dx^\gamma$), and simultaneously to the equation $R^i_{\alpha, \beta, \gamma} \, t^\alpha \, dx^\beta \cdot dx^\gamma$, with the curvature tensor $R^i_{\alpha, \beta, \gamma}$ of differential-geometric spaces Thus, dislocations, or magnetism etc., lead to curvature-like phenomena even with trivial connection (e.g. with $g_{i,k} = \delta_{i,k}$).

All this may be well-known, but usually the phenomena are looked upon only separately, if at all.

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