PATTERN RECOGNITION ON ORIENTED MATROIDS: THREE–TOPE COMMITTEES

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Abstract. A three-tope committee $K^*$ for a simple oriented matroid $M$ is a 3-subset of its maximal covectors such that every positive halfspace of $M$ contains at least two topes from $K^*$. We consider three-tope committees as the vertex sets of triangles in graphs associated with the topes and enumerate them making use of the properties of the poset of convex subsets of the ground set of $M$.

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1. Introduction

Throughout this note, which is a sequel to [9, 10], $M := (E, T)$ denotes a simple oriented matroid that is not acyclic, on ground set $E$, with set of topes $T$; we say that $M$ is simple if it has no loops, parallel or antiparallel elements. For an element $e \in E$, we denote by $T^+_e := \{ T \in T : T(e) = +\}$ the positive halfspace of $M$, corresponding to $e$.

A tope committee $K^*$ for $M$ is a subset of its maximal covectors such that $|\{ K \in K^* : K(e) = +\}| > |\{ K \in K^* : K(e) = -\}|$ or, in other words, $|T^+_e \cap K^*| > \frac{1}{2}|K^*|$, for each element $e \in E$. This construction can serve as an abstract analogue of building blocks of decision rules in contradictory problems of pattern recognition [9].

A comprehensive survey of (in)feasibility studies is given in [3].

When the oriented matroid $M$ interprets in the pattern recognition problem as (a reorientation of) an abstract training set, the three-tope committee becomes the most preferred approximation to the notion of solution of

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an infeasible system of constraints, and we restrict our attention here to three-tope committees.

General three-tope committees are considered and enumerated in Section 2 while Section 3 is devoted to the committees (that are the most important constructions for recognition goals) whose tope s have inclusion-maximal positive parts.

We use the following notation: $|\cdot|$ and $\#$ denote the cardinality of a set and the number of sets in a family, respectively. If $\mathcal{F}$ is a set of sign vectors, then $-\mathcal{F} := \{-F : F \in \mathcal{F}\}$. We let $F^+$ and $F^-$ denote the positive and negative parts of a sign vector $F$, respectively.

For a bounded poset $P$ we denote by $P_c$ the set of its coatoms. The singleton has no atoms and coatoms. For a chain $x \preceq y$ of unit length we set $\{x, y\}^c := \{x\}$. See, e.g., [1, §IV.4.B] on a link between the number of coatoms in $P$ and incidence functions on $P$. $[x, z] := \{y \in P : x \preceq y \preceq z\}$ is a closed interval in $P$, and $\mu_P(\cdot, \cdot)$ denotes the Möbius function on $P$.

We adopt from [6] the following terminology and statements: A subset $A \subset E$ is acyclic if the restriction of $\mathcal{M}$ to $A$ is an acyclic oriented matroid. The convex hull $\text{conv}(A)$ of an acyclic set $A$ is the set $B \supseteq A$ such that for every covector $F$ and for every element $b \in B$, the implication $F(a) = +, \forall a \in A \implies F(b) = +$ holds. A convex set $A$ is convex if $\text{conv}(A) = A$; its subset of extreme points $\text{ex}(A)$ is defined by $\text{ex}(A) := \{a \in A : a \not\in \text{conv}(A - \{a\})\}$. For any acyclic set $A$ it holds $\text{conv}(\text{ex}(A)) = \text{conv}(A)$. A convex set $A \subset E$ is free if $\text{ex}(A) = \text{conv}(A) = A$.

The meet-semilattice $L_{\text{conv}}(\mathcal{M})$ is defined as the family of convex subsets of $E$ ordered by inclusion.

We regard the least element $\hat{0}$ of $L_{\text{conv}}(\mathcal{M})$ as the empty subset of $E$. If a subset $H \subseteq E$ is not acyclic then we set $\text{conv}(H) := E$ and $\text{ex}(E) := E$.

For a simple graph $G$ we let $\mathcal{V}(G)$ and $\mathcal{E}(G)$ denote its vertex set and the family of edges, respectively.

If $\mathcal{F}$ is a set family then the Kneser graph $KG(\mathcal{F})$ of $\mathcal{F}$ is the graph with $\mathcal{V}(KG(\mathcal{F})) := \mathcal{F}$; if $F', F'' \in \mathcal{F}$ then $\{F', F''\} \in \mathcal{E}(KG(\mathcal{F}))$ iff $|F' \cap F''| = 0$, see [8, §3.3].

2. THREE-TOPE (ANTI-)COMMITTEES

A tope anti-committee $A^*$ for $\mathcal{M}$ is a subset of its maximal covectors such that the set $-A^*$ is a tope committee for $\mathcal{M}$. We denote by $K^*_k(\mathcal{M})$ and $A^*_k(\mathcal{M})$ the families of all tope committees and anti-committees of cardinality $k$ for $\mathcal{M}$, respectively. Recall that due to axiomatic symmetry (L1) [2, §4.1.1] we have

$$T = -T$$

and thus it follows from the definition that

$$\#K^*_k(\mathcal{M}) = \#A^*_k(\mathcal{M}) = \#K^*_{|T|-k}(\mathcal{M}) = \#A^*_{|T|-k}(\mathcal{M})$$
because for any tope committee $K^*$ for $M$ its complement $T - K^*$ is an anti-committee.

Since a three-tope anti-committee meets each positive halfspace in at most one tope, we have

$$\#A_3^* (M) = \sum_{S_1 \in E, S_2 \in E, S_3 \in E: |S_1|, |S_2|, |S_3| > 0, |S_1 \cap S_2| = |S_1 \cap S_3| = |S_2 \cap S_3| = 0} \prod_{k \in \{1, 2, 3\}} \left| \bigcap_{s \in S_k} T_s^+ - \bigcup_{s \in E - S_k} T_s^+ \right| ; \tag{2.2}$$

note that $\bigcap_{s \in S_k} T_s^+ - \bigcup_{s \in E - S_k} T_s^+ = \bigcap_{s \in S_k} T_s^+ \cap \left( - \bigcap_{s \in E - S_k} T_s^+ \right)$.

On the right-hand side of (2.2) the (extreme points of) convex sets $S_1$, $S_2$ and $S_3$ only contribute to the total sum, therefore we turn to the poset $L_{\text{conv}} (M)$ of convex subsets of the ground set of $M$.

Let $\hat{L}_{\text{conv}} (M)$ be the poset $L_{\text{conv}} (M)$ augmented by a greatest element $\hat{1}$; it is convenient to set $\hat{1} := E$. For brevity, we will write $\hat{L}$ instead of $\hat{L}_{\text{conv}} (M)$.

If we let $T_B^+ := \bigcap_{b \in B} T_b^+$ denote the intersection of the positive halfspaces corresponding to the elements of a subset $B \subseteq E$, then expression (2.2) reformulates in the following way:

$$\#K_3^* (M) = \#A_3^* (M) = \sum_{\{A_1, A_2, A_3\} \in \hat{L} - \{\hat{0}, \hat{1}\}: A_1 \wedge A_2 = A_1 \wedge A_3 = A_2 \wedge A_3 = 0} \prod_{k \in \{1, 2, 3\}} \left| T_{\text{ex}(A_k)}^+ \cap \left( - T_{\text{ex}(\text{conv}(E - A_k))}^+ \right) \right| . \tag{2.3}$$

The related quantity

$$8 \left( \frac{|T|}{3} \right)^2 \sum_{A \in \hat{L} - \{\hat{0}, \hat{1}\}} \mu_{\hat{L}} (\hat{0}, A) \left( \frac{|T_{\text{ex}(A)}^+|}{3} \right) = 8 \left( \frac{|T|}{3} \right)^2 + \sum_{A \in \hat{L} - \{\hat{0}, \hat{1}\}: A \text{ free}} (-1)^{|A|} \left( \frac{|T_A^+|}{3} \right)$$

is the number of three-tope sets containing no opposites and meeting every positive halfspace of $M$. Recall that if $A \in \hat{L} - \{\hat{0}, \hat{1}\}$ then $\mu_{\hat{L}} (\hat{0}, A) = (-1)^{|A|}$ whenever the convex set $A$ is free and, as a consequence, $[\hat{0}, A]$ is order-isomorphic to the Boolean lattice of rank $|A|$; otherwise, $\mu_{\hat{L}} (\hat{0}, A) = 0$, see [6].

Let $G := G (M)$ be a graph with the vertex set $\mathcal{V} (G) := T$; a pair $\{T', T''\} \subset T$ by definition belongs to the edge family $\mathcal{E} (G)$ of $G$ iff no positive halfspace of $M$ contains this pair, that is, $|(T')^+ \cap (T'')^+| = 0$ or, in other words, $\{T', T''\}$ is a 1-dimensional missing face of the abstract simplicial complex $\Delta_{\text{acyclic}} (M)$ of acyclic subsets of $E$. Thus, $G$ is isomorphic to the Kneser graph $K_{\text{G}} (\{T^+ : T \in T\})$ of the family of the positive parts of topes of $M$.

Let $\Gamma := \Gamma (M)$ be the graph defined by

$$\mathcal{V} (\Gamma) := T ,$$

$$\{T', T''\} \in \mathcal{E} (\Gamma) \iff (T')^+ \cup (T'')^+ = E .$$
The vertex set of any odd cycle in $\Gamma$ is a tope committee for $\mathcal{M}$ \cite§5. Since $\{T', T''\} \in \mathcal{E}(G)$ iff $(T')^c - \cup (T'')^c = E$, for any pair of topes $\{T', T''\}$, the four graphs $\Gamma$, $G$, $K_G(\{T^+ : T \in T\})$ and $K_G(\{T^- : T \in T\})$ are all isomorphic, due to symmetry \cite(2.1). For example, the mapping $V(G) \rightarrow V(\Gamma)$, $T \mapsto -T$, is an isomorphism between the graphs $G$ and $\Gamma$. Since a subset $\{T', T'', T''\} \subset T$ is a three-tope anti-committee for $\mathcal{M}$ iff it is the vertex set of a triangle in $G$ (or, in other terms, this subset is a 2-dimensional face of the independence complex of the graph whose edges are the 1-dimensional faces of $\Delta_{\text{acyclic}}(\mathcal{M})$), the three-tope committees for $\mathcal{M}$ are precisely the vertex sets of the triangles in $\Gamma$. From the poset-theoretic point of view, the family $K^*_3(\mathcal{M})$ is regarded in \cite Proposition 4.1, Theorem 5.1] as antichains in posets associated with the topes.

We have

$$\#\mathcal{E}(\Gamma) = \binom{|T|}{2} + \sum_{A \in \hat{L} - \{0, 1\} : A \text{ free}} (-1)^{|A|} \binom{|T_A^+|}{2}$$

(2.4)

and

$$|\mathcal{Y}(\Gamma)| := |T| = \sum_{A \in \hat{L} - \{0, 1\} : A \text{ free}} (-1)^{|A| - 1} |T_A^+|$$

(2.5)

recall that in fact the most efficient tool of tope enumeration is the Las Vergnas–Zaslavsky formula, see \cite §4.6.

Now let $R$ be a graph, with the vertex set $\mathcal{Y}(R) := \{1, 2, \ldots, |T|\}$, isomorphic to either of the graphs $\Gamma$, $G$, $K_G(\{T^+ : T \in T\})$ and $K_G(\{T^- : T \in T\})$. If we let $A$ and $N(i)$ denote the adjacency matrix of $R$ and the neighborhood of the vertex $i$ in $R$, respectively, then well-known observations of graph theory imply, for example, that $\#K^*_3(\mathcal{M}) = \frac{1}{6} \text{trace}(A^3)$ and

$$\#K^*_3(\mathcal{M}) = \frac{1}{3} \sum_{\{i,j\} \in \mathcal{E}(R)} |N(i) \cap N(j)|$$

(2.6)

because the quantity $\#K^*_3(\mathcal{M}) = \#A^*_3(\mathcal{M})$ is the number of triangles in $R$.

3. Committees of Cardinality Three whose Topes Have Maximal Positive Parts

Let $\text{max}^+(T)$ denote the set of topes of the oriented matroid $\mathcal{M}$ whose positive parts are maximal with respect to inclusion; these parts are the inclusion-maximal convex subsets of the ground set $E$ and thus they are the coatoms of the lattice $\hat{L}$, that is, $\{T^+ : T \in \text{max}^+(T)\} = \hat{L}^-$. If $\mathcal{M}$ is realizable by a central arrangement of oriented hyperplanes, then the coatoms of $\hat{L}$ are the multi-indices of maximal feasible subsystems of a certain related homogeneous system of strict linear inequalities. The graph of topes with
maximal positive parts $\Gamma^+_{\text{max}} := \Gamma^+_{\text{max}}(M)$ is the subgraph of $\Gamma(M)$ with the vertex set $\text{max}^+(T)$, that is,

$$\exists(\Gamma^+_{\text{max}}) := \text{max}^+(T), \quad \{T', T''\} \in \mathcal{E}(\Gamma^+_{\text{max}}) \iff (T')^+ \cup (T'')^+ = E.$$  

The graph $\Gamma^+_{\text{max}}$ is connected [9, §5.2].

Recall that if $T \in \text{max}^+(T)$ then symmetry (2.1) and maximality of the positive part $T^+ \in \hat{L}$ imply that the set $E - T^+ = T^-$ is acyclic and convex; as a consequence, we have

$$\#\{K^* \in \mathcal{K}^*_3(M) : K^* \subseteq \text{max}^+(T)\} = \#\{D_1, D_2, D_3 \subseteq \hat{L} : \{E - D_1, E - D_2, E - D_3\} \subseteq \hat{L}, D_1 \wedge D_2 = D_1 \wedge D_3 = D_2 \wedge D_3 = \hat{0}\},$$  

cf. (2.3).

The degree of a vertex $T$ in the graph $\Gamma^+_{\text{max}}$ equals the number of coatoms in the interval $[T^-, \hat{1}]$ of the lattice $\hat{L}$. As a consequence, the number of edges in $\Gamma^+_{\text{max}}$ is

$$\#\mathcal{E}(\Gamma^+_{\text{max}}) = \frac{1}{2} \sum_{D \in \hat{L} : E - D \in \hat{L}^c} |D, \hat{1}|^c,$$

and the cyclomatic number of $\Gamma^+_{\text{max}}$ equals

$$1 + \frac{1}{2} \sum_{D \in \hat{L} : E - D \in \hat{L}^c} |D, \hat{1}|^c - |\hat{L}^c|.$$

If a tope $T' \in T$ does not belong to the set $\text{max}^+(T)$ then there exists, again thanks to symmetry (2.1), a tope $T'' \in \text{max}^+(T)$ such that the pair $\{T', T''\}$ is an edge in $\Gamma$; since the graph $\Gamma^+_{\text{max}}$ is connected, this implies that the graph $\Gamma$ is connected as well and, according to (2.4) and (2.5), the cyclomatic number of $\Gamma$ equals

$$1 + \binom{|T|}{2} + \sum_{A \in \hat{L} - \{0, 1\} : A \text{ free}} (-1)^{|A|} \left(1 + \binom{|T_A^+|}{2}\right).$$

The concluding statement is a direct consequence of expression (2.6):

**Proposition 3.1.** The number of committees of cardinality three, for the oriented matroid $M$, whose tops have inclusion-maximal positive parts is

$$\#\{K^* \in \mathcal{K}^*_3(M) : K^* \subseteq \text{max}^+(T)\} = \frac{1}{3} \sum_{\{D_1, D_2\} \subseteq \hat{L} : \{E - D_1, E - D_2\} \subseteq \hat{L}^c, D_1 \wedge D_2 = \hat{0}} |D_1 \vee D_2, \hat{1}|^c.$$

(3.1)
Example 3.2. Let $\mathcal{M} := (E_6, T)$ be the rank 3 simple oriented matroid on the ground set $E_6 := \{1, 2, \ldots, 6\}$, with the set of tope

$\{ \ldots \} \implies \{ \ldots \}$

Thus, According to (3.1) $\mathcal{L}^\ast \in K^\ast_3(\mathcal{M})$ consists of $\frac{n}{3} = 3$ committees which can be found as the vertex sets of the triangles in the graph depicted in [9, Figure 3.1]. We have

$$\{ D \in \mathcal{L} : E_6 \setminus D \in \mathcal{L}^\ast \} = \{ \{12\}, \{15\}, \{16\}, \{23\}, \{24\}, \{35\}, \{46\} \}$$

and

$$\begin{align*}
[\{12\} \lor \{35\}, \hat{1}]^c &= [\{1235\}, \hat{1}]^c = #\{1235\} = 1 \\
[\{12\} \lor \{46\}, \hat{1}]^c &= [\{1246\}, \hat{1}]^c = #\{1246\} = 1 \\
[\{15\} \lor \{23\}, \hat{1}]^c &= [\{1235\}, \hat{1}]^c = #\{1235\} = 1 \\
[\{15\} \lor \{24\}, \hat{1}]^c &= [\{1456\}, \hat{1}]^c = #\{1456\} = 1 \\
[\{16\} \lor \{23\}, \hat{1}]^c &= [\{156\}, \hat{1}]^c = #\{156\} = 1
\end{align*}$$

Thus, according to (3.1), the family $\{ \kappa^\ast \in K^\ast_3(\mathcal{M}) : \kappa^\ast \subseteq \max^+(T) \}$ consists of $\frac{n}{3} = 3$ committees which can be found as the vertex sets of the triangles in the graph depicted in [9, Figure 5.4].
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