Characterizing Attributed Tree Translations in Terms of Macro Tree Transducers

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Abstract

It is well known that attributed tree transducers can be equipped with “regular look-around” in order to obtain a more robust class of translations. We present two characterizations of this class in terms of macro tree transducers (MTTs): the first one is a static restriction on the rules of the MTTs, where the MTTs need to be equipped with regular look-around. The second characterization is a dynamic one, where the MTTs only need regular look-ahead.

Keywords: Macro tree transducers, Attributed tree transformations

1. Introduction

Attributed tree transducers (ATTs)\cite{1} are instances of attribute grammars\cite{2,3} where the only semantic domain is trees, and the only available operation is tree top-concatenation. Most definitions of ATTs define them as total deterministic devices, thus realizing total functions from input trees to output trees (cf., e.g.,\cite{4}). ATTs strictly generalize top-down tree transducers: an ATT without inherited attributes is exactly a top-down tree transducer. On the other hand, there are bottom-up tree transformations that
cannot be realized by ATTs [4]. The latter “deficiency” can be remedied if the ATTs are equipped with regular look-around; this yields a robust class of translations which coincides with the class of tree-to-dag (to tree) translations definable in MSO logic [5]. Intuitively, regular look-around amounts to preprocessing the input tree via an “attributed relabeling”; such a relabeling is equivalent to a top-down relabeling with regular look-ahead (or, to preprocessing the input tree first via a bottom-up relabeling, followed by a top-down relabeling) [6].

While ATTs can be seen as an operational model, macro tree transducers (MTTs) [7] (introduced independently in [8] and in [9,10]) are a denotational model, more akin to functional programming, see e.g. [11]. MTTs are strictly more powerful than ATTs. For instance, they can translate a monadic input tree of height \( n \) into a monadic output tree of height \( 2^n \). An ATT cannot do this, because the number of distinct subtrees of an output tree is linearly bounded in the size of the corresponding input tree (property “\text{LIN}”; see, e.g., [4]). In the context of XML translation, ATTs with nested pebbles have been considered [12] which can be simulated by compositions of MTTs [13].

Fülöp and Vogler [14] were the first to present a characterization of ATTs in terms of MTTs. Their characterization is based on a particular construction from ATTs to MTTs: each synthesized attribute of the ATT becomes a state of the MTT, and each inherited attribute becomes a parameter of a state of the MTT. This causes that for a given input symbol the parameter trees of all state calls in the rules for that symbol are equal. This equality is the first part of their restriction (called consistency) on MTTs. This may be a somewhat inflexible syntactic restriction. In fact the non-deleting normal
form of some consistent MTT is not consistent. In their construction from MTT to ATT, it may happen that the resulting ATT is circular (and hence not well-defined). Thus, the second part of their restriction requires that the ATT resulting from their construction is non-circular. Let us refer to their (full) restriction as $FV$-orig.

In this paper we formulate restrictions on MTTs to characterize ATTs. Our first restriction, called $FV$ property is similar to consistency (part one of FV-orig) and differs in the following ways: (1) it is defined for MTTs with regular look-around, (2) the MTTs are nondeleting (with respect to the parameters), and (3) for such MTTs we can construct ATTs with regular look-around that are guaranteed to be non-circular. In the FV property we relax the requirement of the consistency on equality between the parameter trees of all state calls in the rules for the same symbols by using a “parameter renaming mapping”.

Our second characterization of ATTs with regular look-around in terms of MTTs (with regular look-ahead) is called the dynamic $FV$ property. It generalizes the FV property in several ways. First, we do not compare argument trees of different states. Second, argument trees (of the same state) are not required to be syntactically equal, but are only required to be semantically equal (i.e., must evaluate to the same output tree). And third, we only consider states and argument trees that are reachable. In this characterization, it suffices to consider MTTs with regular look-ahead. The reason for this is that MTTs with the dynamic FV property are closed under left-composition with top-down relabelings; a property that we conjecture not to hold for MTTs with the FV property and regular-look ahead only. It should be noted
that the generality of the dynamic FV property comes at a price: while the FV property is easily decidable, we do not know how to decide the dynamic FV property. In fact, we are able to show that deciding the dynamic FV property is at least as difficult as deciding equivalence of ATTs. The equivalence problem of ATTs is a long standing open problem; for some subclasses of ATTs with regular look-around, e.g., MSO-definable tree-to-tree translations, equivalence is decidable [8, 15]. Note that MTTs with monadic output trees (i.e., a class incomparable to ATTs) has decidable equivalence [15] (see also, [16, 17]). All our results also hold for partial transducers; this can be achieved by allowing the look-ahead and look-around to be partial. The characterization by Fülöp and Vogler [14] is defined for total transducers only.

2. Preliminaries

For a non-negative integer \( k \), we denote by \([k]\) the set \( \{1, \ldots, k\} \). For functions \( f, g \) we denote by \( f \circ g \) their sequential composition that maps \( a \in \text{dom}(f) \) to \( g(f(a)) \). For classes of functions \( F \) and \( G \), \( F \circ G \) denotes \( \{f \circ g \mid f \in F, g \in G\} \).

Trees and Tree Substitution. A ranked alphabet \( \Sigma \) consists of an alphabet together with a mapping \( \text{rk}_\Sigma \) that associates to each symbol of the alphabet a non-negative integer (its rank). By \( \Sigma^{(k)} \) we denote the set of symbols \( \sigma \) in \( \Sigma \) for which \( \text{rk}_\Sigma(\sigma) = k \). We also write \( \sigma^{(k)} \) to indicate that the symbol \( \sigma \) has rank \( k \). The set \( T_\Sigma \) of trees (over \( \Sigma \)) is the smallest set \( T \) of strings such that if \( t_1, \ldots, t_k \in T, k \geq 0 \), and \( \sigma \in \Sigma^{(k)} \), then \( \sigma(t_1, \ldots, t_k) \) is in \( T \). We write \( \sigma \) for a tree of the form \( \sigma() \). For a set \( S \) disjoint with \( \Sigma \)
we define $T_\Sigma(S)$ as $T_\Sigma'$ where $\Sigma' = \Sigma \cup \{s^{(0)} \mid s \in S\}$. The set $V(t)$ of nodes of a tree $t = \sigma(t_1, \ldots, t_k)$ is defined as $\{\varepsilon\} \cup \{iu \mid i \in [k], u \in V(t_i)\}$. For $u, u' \in V(t)$, we say that $u$ is a descendant of $u'$ if $u'$ is a prefix of $u$. If $u$ is a descendant of $u'$ and $u \neq u'$, we say that $u$ is a proper descendant of $u'$. We let $u_0 = u$ for every node $u$. For $u, u' \in V(t)$, we say that $u$ is a descendant of $u'$ if $u'$ is a prefix of $u$. If $u$ is a descendant of $u'$ and $u \neq u'$, we say that $u$ is a proper descendant of $u'$. We let $u_0 = u$ for every node $u$. For $u \in V(t)$, $t[u]$ denotes the label of $u$, $t/u$ denotes the subtree rooted at $u$, and for a tree $t'$, $t[u \leftarrow t']$ denotes the tree obtained from $t$ by replacing the subtree rooted at $u$ by the tree $t'$.

Let $t, t_1, \ldots, t_k$ be trees and $\sigma_1, \ldots, \sigma_k$ be symbols of rank zero. Then $t[\sigma_i \leftarrow t_i \mid i \in [k]]$ denotes the result of replacing in $t$ each occurrence of $\sigma_i$ by the tree $t_i$. We also define a more powerful type of tree substitution, where inner nodes of a tree may be replaced. We fix the set $Y = \{y_1, y_2, \ldots\}$ of parameters, and denote $\{y_1, \ldots, y_m\}$ by $Y_m$. Now let $\sigma_1, \ldots, \sigma_k$ be symbols of arbitrary rank $m_1, \ldots, m_k$, respectively, and $t_i \in T_\Sigma(Y_{m_i})$ for $i \in [k]$. Then $t[\sigma_i \leftarrow t_i \mid i \in [k]]$ denotes the result of replacing in $t$ each subtree $\sigma_i(s_1, \ldots, s_{m_i})$ by the tree $t_i[y_j \leftarrow s_j' \mid j \in [m_i]]$ where $s_j' = s_j[\sigma_i \leftarrow t_i \mid i \in [k]]$.

**Macro Tree Transducers.** We fix the set $X = \{x_1, x_2, \ldots\}$ of input variables and assume it to be disjoint from $Y$ and all other alphabets. We denote $\{x_1, \ldots, x_k\}$ by $X_k$. A (total deterministic) macro tree transducer ($MTT$ for short) is a tuple $M = (Q, \Sigma, \Delta, q_0, R)$ where $Q$ is a ranked alphabet of states, $\Sigma$ and $\Delta$ are ranked alphabets of input and output symbols, respectively, $q_0 \in Q^{(0)}$ is the initial state, and $R$ is a set of rules. For every $q \in Q^{(m)}$ and $\sigma \in \Sigma^{(k)}$ there is exactly one rule in $R$ of the form $\langle q, \sigma(x_1, \ldots, x_k)\rangle(y_1, \ldots, y_m) \rightarrow \zeta$ where $\zeta \in T_{\Delta \cup (Q, X_k)}(Y_m)$. Note that we follow the style of the definition of MTTs in [6]. For a set $S$, $\langle Q, S \rangle$
denotes the ranked set \( \{(q, s)^{(n)} | q \in Q^{(n)}, n \geq 0, s \in S\} \). Note that we use this notation even for an infinite set \( S \), e.g., \( \langle Q, X \rangle \) for the variable set \( X \). The tree \( \zeta \) in the right-hand side is also denoted \( \text{rhs}_M(q, \sigma) \).

The semantics of an MTT is defined as follows. Let \( \sigma \in \Sigma^{(k)}, k \geq 0, s_1, \ldots, s_k \in T_{\Sigma} \), and \( q \in Q^{(m)} \) with \( m \geq 0 \). Then \( M_q(\sigma(s_1, \ldots, s_k)) \) denotes the tree \( \text{rhs}_M(q, \sigma)[\langle q', x_i \rangle \leftarrow M_q(s_i) | q' \in Q, i \in [k]] \). The translation realized by \( M \), denoted \( \tau_M \), is defined as \( \{(s, t) | s \in T_{\Sigma}, t = M_{q_0}(s)\} \). The class of all translations realized by MTTs is denoted \( \text{MTT} \).

**Attributed Tree Transducers.** An *attributed tree transducer* is a tuple \( A = (S, I, \Sigma, \Delta, \alpha_0, R) \) where \( S \) and \( I \) are disjoint finite sets of synthesized and inherited attributes, respectively, \( \Sigma \) and \( \Delta \) are ranked alphabets of input and output symbols, respectively, \( \alpha_0 \in S \) is the output attribute, and \( R \) is a collection of sets \( R_\sigma, \sigma \in \Sigma \) of rules. Let \( \sigma \in \Sigma^{(k)} \) and \( k \geq 0 \). Let \( \pi \) be a variable for paths, and we define \( \pi_0 = \pi \). For every \( \alpha \in S \) the set \( R_\sigma \) contains a rule of the form \( \alpha(\pi) \rightarrow t \) and for every \( \beta \in I \) and \( i \in [k] \) the set \( R_\sigma \) contains a rule of the form \( \beta(\pi_i) \rightarrow t' \), and the trees \( t, t' \) are in \( T_\Delta(\{\alpha'(\pi_i) | \alpha' \in S, i \in [k]\} \cup \{\beta'(\pi) | \beta' \in I\}) \). The right-hand sides \( t, t' \) of these rules are denoted \( \text{rhs}_A(\sigma, \alpha(\pi)) \) and \( \text{rhs}_A(\sigma, \beta(\pi_i)) \), respectively.

To define the semantics of the ATT \( A \) on an input tree \( s \in T_{\Sigma} \), we first define the dependency graph of \( A \) on \( s \) as \( D_A(s) = (V, E) \), where \( V = \{(\alpha_0, \varepsilon) \cup (S \cup I) \times (V(s) - \{\varepsilon\}) \) and \( E = \{(\alpha', uj), (\gamma, wi) | u \in V(s), \alpha'(\pi_j) \text{ occurs in } \text{rhs}_A(s[u], \gamma(\pi_i)), 0 \leq i, j \leq \text{rk}_\Sigma(s[u]), \gamma, \gamma' \in S \cup I\} \). If \( D_A(s) \) contains a cycle for some \( s \in T_{\Sigma} \) then \( A \) is called circular. Let \( N = \{\alpha_0(\varepsilon) \cup \{a(u) | a \in S \cup I, u \in V(s) - \{\varepsilon\}\} \). For trees \( t, t' \in T_\Delta(N), t \Rightarrow_{A,s} t' \) holds if \( t' \) is obtained from \( t \) by replacing a node labeled \( \gamma(\varepsilon) \)
by \(\text{rhs}_A(s[u], \gamma(\pi i))|\gamma'(\pi i) \leftarrow \gamma'((u)i) \mid \gamma' \in S \cup I, 0 \leq i \leq \text{rk}_\Sigma(s[u])\). If \(A\) is non-circular, then every \(t \in T_\Delta(N)\) has a unique normal form with respect to \(\Rightarrow_{A,s}\) which we denote by \(nf(\Rightarrow_{A,s}, t)\). The translation realized by \(A\), denoted \(\tau_A\), is defined as \(\{(s, nf(\Rightarrow_{A,s}, \alpha_0(\bar{s}))) \mid s \in T_S\}\). The class of all translations realized by non-circular ATTs is denoted \(ATT\).

**Regular Look-Ahead and Regular Look-Around.** For the classes \(MTT\) and \(ATT\) we define regular look-ahead and regular look-around by means of pre-composition with the classes \(B-REL\) and \(T^R-REL\), respectively. Here \(B-REL\) is the class of deterministic bottom-up finite state relabelings \((P, \Sigma, \Sigma', F, R)\) where \(P\) is a finite set of states, \(\Sigma, \Sigma'\) are ranked alphabets, the set \(F \subseteq P\) of final states, and a set \(R\) of relabeling rules. The set \(R\) contains for every \(p_1, \ldots, p_k \in P\) and \(\sigma \in \Sigma^{(k)}, k \geq 0\), exactly one rule of the form \(\sigma(p_1(x_1), \ldots, p_k(x_k)) \rightarrow p(\sigma'(x_1, \ldots, x_k))\) where \(p \in P\) and \(\sigma' \in \Sigma'(k)\). They are evaluated in the obvious way, like a bottom-up tree automaton with the set \(F\) of final states. Note that we assume \(F = P\) in this paper since we treat only total transducers for simplicity of discussion, but our results can be extended to partial transducers by allowing relabeling transducers to be partial. The class \(T^R-REL\) is defined as \(B-REL \cap T-REL\), where \(T-REL\) is a deterministic top-down relabeling \((Q, \Sigma, \Sigma', q_0, R)\) which contains for every \(q \in Q\) and \(\sigma \in \Sigma^{(k)}\) exactly one rule of the form \(q(\sigma(x_1, \ldots, x_k)) \rightarrow \sigma'(q_1(x_1), \ldots, q_k(x_k))\), where \(\sigma' \in \Sigma'(k)\) and \(q_1, \ldots, q_k \in Q\). We use the superscripts \(R (U)\) to indicate the presence of regular look-ahead (look-around). For instance \(ATT^U = T^R-REL; ATT\) is the class of translations realized by attributed tree transducers with regular look-around (cf. Section 4 of [6]); this is the class for which we give new
characterizations in terms of MTTs in this paper.

3. FV Property for Nondeleting MTTs with Regular Look-Around

Let us recall the definition of the consistency of Fülöp and Vogler [14]. In the following, let $M = (Q, \Sigma, \Delta, q_0, R)$ be an MTT.

**Definition 1.** Let $\sigma \in \Sigma^{(k)}$, $k \geq 0$, and $q \in Q$. Let $\zeta = \text{rhs}_M(q, \sigma)$ and $v \in V(\zeta)$. The node $v$ is important in $\zeta$ for $s_1, \ldots, s_k \in T_\Sigma$ if the symbol $\ast$ occurs in $\zeta[v \leftarrow \ast][\langle q', x_i \rangle \leftarrow M_{q'}(s_i) | q' \in Q, i \in [k]]$. If $v$ is important in $\zeta$ for some $s_1, \ldots, s_k \in T_\Sigma$, then we say that $v$ is important in $\zeta$.

**Definition 2.** The MTT $M$ is consistent if the following condition holds for every $\sigma \in \Sigma^{(k)}$, $k \geq 0$, and $q_1, q_2 \in Q$: for $i = 1, 2$ let $\zeta_i = \text{rhs}_M(q_i, \sigma)$ and $w_i \in V(\zeta_i)$ such that $\zeta_i[w_i] \in \langle Q, \{ x_l \} \rangle$ for some $l \in [k]$. Let $\langle p_i^{(m_i)}, x_i \rangle = \zeta_i[w_i]$ for $i = 1, 2$. Then, for every $j \in [\min\{ m_1, m_2 \}]$, if $w_1j$ and $w_2j$ are important in $\zeta_1$ and $\zeta_2$, respectively, then $\text{top}(\zeta_1/w_1j) = \text{top}(\zeta_2/w_2j)$ where $\text{top}(\zeta)$ is defined as follows:

$$\text{top}(\zeta) = \begin{cases} q'(\pi i) & \text{if } \zeta[\epsilon] = \langle q', x_i \rangle \in \langle Q, X \rangle \\ \delta(\text{top}(\zeta_1), \ldots, \text{top}(\zeta_k)) & \text{if } \zeta = \delta(\zeta_1, \ldots, \zeta_k) \text{ where } \delta \in \Delta \\ y_j(\pi) & \text{if } \zeta = y_j \in Y. \end{cases}$$

The class of all translations realized by consistent MTTs is denoted $\text{MTT}_C$.

In Figure 1 on the left we see the dependency graph of an ATT which translates monadic trees of the form $\#(a^n(e))$ to monadic output trees of the form $a^n(b^n(c^n(d^n(e))))$; here shown for $n = 2$. The top right part shows the rules of a consistent MTT. Observe the two bottom-most rules, where the
Figure 1: An ATT and MTTs that translate \rceil(a^n(e)) \rightarrow a^n(b^n(c^n(d^n(e))))

state $q_1$ deletes its second parameter ($y_2$) and the state $q_2$ deletes its first parameter, and the two rules above which have $\bot$ at the non-important argument positions in their right-hand sides. Note that non-important parameters just for padding are necessary for an MTT to satisfy the consistency. The bottom right part in Figure 1 shows an equivalent MTT without the redundant parameters which does not satisfy the consistency.

Fülöp and Vogler construct from a consistent MTT an ATT which may possibly be circular. Let us review such an example from [14]. Consider the MTT with the rules shown in the left of Figure 2 where all rules with left-hand sides that are not shown, have as right-hand side the output leaf \# . As the reader may verify, this MTT indeed is consistent: the $(q_0, \sigma)$-rule is consistent, because the parameter of $q$ is important (for inputs of the form
\[ \langle q_0, \sigma(x_1, x_2) \rangle \rightarrow \langle q, x_1 \rangle(\langle q_2, x_2 \rangle(\langle q_1, x_1 \rangle(\varepsilon))) \]
\[ \langle q, e \rangle(y_1) \rightarrow y_1 \]
\[ \langle q_2, e \rangle(y_1) \rightarrow y_1 \]
\[ \langle q_1, e' \rangle(y_1) \rightarrow y_1 \]

Figure 2: The consistent MTT \( M_c \) and the circular ATT obtained by the construction of \([14]\), but the one of \( q_1 \) is not important (because it is deleted by \( q_1 \) for inputs \( \sigma(e, s) \), or by \( q \) for inputs \( \sigma(e', s) \)). A part of the dependency graph for the input tree \( \sigma(e', e) \) is shown in Figure 2. The idea is as follows. Since \( \langle q, x_1 \rangle \) appears in a non-nested position in \( \text{rhs}(q_0, \sigma) \), there is an edge from \( (q, 1) \) to \( (q_0, \varepsilon) \). The inherited attribute “\( y_1 \)” at the \( e' \)-node has an incoming edge from \( (q_2, 2) \), because the call \( \langle q_2, x_2 \rangle \) appears in the first parameter position of \( \langle q, x_1 \rangle \) in \( \text{rhs}(q_0, \sigma) \). The bottom-most edges from the nodes \( (y_1, 1) \) and \( (y_1, 2) \) stem from rules with right-hand side \( y_1 \).

We now define the FV property for nondeleting MTTs, by adapting the definition of consistency to nondeleting MTTs. An MTT is **nondeleting**, if every parameter that appears in the left-hand side of a rule, also appears in the right-hand side. In order to formalize MTTs such as \( M_a \) in the bottom-right of Figure 1 we use mappings of the form \( \rho : Q \times \mathbb{N} \rightarrow \mathbb{N} \). Intuitively, if \( \rho(q, j) = j' \), then the \( j \)th parameter of state \( q \) of an MTT is renamed by the parameter with index \( j' \) in the corresponding consistent MTT. So, for \( M_a \) it holds that \( \rho(q_1, 1) = 1 \) and \( \rho(q_2, 1) = 2 \). As we will state, in our construction of an equivalent ATT via a consistent MTT, the parameter with the index
of the state \( q \) corresponds to the \( j' \)th inherited attribute of the resulting ATT.

**Definition 3.** Let \( \rho : Q \times \mathbb{N} \to \mathbb{N} \). For \( q \in Q^{(m)} \), we denote by \( \Psi_q^\rho \) the renaming of parameters \([y_l \leftarrow y_{l'} | l \in [m], l' = \rho(q, l)]\). For \( q, q' \in Q \), we define the relation \( \sim_{\rho,q,q'} \) on trees \( \xi_1, \xi_2 \in T_{\Delta \cup \{Q,X\}} \) such that \( \xi_1[\varepsilon] = \langle q_1, x_1 \rangle \) and \( \xi_2[\varepsilon] = \langle q_2, x_2 \rangle \) for some \( q_1, q_2 \in Q \) and \( x \in X \), and for every \( j_1 \in [m_1] \) and \( j_2 \in [m_2] \), if \( \rho(q_1, j_1) = \rho(q_2, j_2) \) then \((\xi_1/j_1)\Psi_q^\rho = (\xi_2/j_2)\Psi_{q'}^\rho\).

**Definition 4.** Let \( \rho : Q \times \mathbb{N} \to \mathbb{N} \) such that for every \( q \in Q \), \( \rho(q, i) \neq \rho(q, j) \) if \( i \neq j \) and \( i, j \in [\text{rk}_Q(q)] \). Suppose that the MTT \( M \) is nondeleting. The MTT \( M \) has the FV property with \( \rho \) if the following condition holds for every \( \sigma \in \Sigma^{(k)} \), \( k \geq 0 \), and \( q_1, q_2 \in Q \): let \( \xi_1 \) and \( \xi_2 \) be any subtrees of \( \text{rhs}_M(q_i, \sigma) \) for \( i = 1, 2 \), respectively, such that \( \xi_1[\varepsilon], \xi_2[\varepsilon] \in \langle Q, \{x_j\} \rangle \) for some \( j \in [k] \). Then \( \xi_1 \sim_{\rho,q_1,q_2} \xi_2 \) must hold. We say that \( M \) has the FV property if it has the FV property with some \( \rho \). We denote by \( MTT_{\text{FV}} \) the class of all translations realized by MTTs with the FV property.

We now present our construction from MTTs with the FV property to consistent MTTs. Intuitively, according to the mapping \( \rho \), in the construction, we increase the number of parameters of every state to the maximum number \( \tilde{m} \) of the range of \( \rho \), and each \( j \)th argument tree of each call \( \langle q, x_i \rangle \) in the right-hand side of rules are moved to the \( \rho(q, j) \)th argument position of the corresponding call \( \langle q, x_i \rangle^{(\tilde{m})} \). The remaining argument parts of state calls are filled by a dummy label \( \perp \).
Definition 5. Suppose that the nondeleting MTT $M$ satisfies the FV property with $\rho : Q \times \mathbb{N} \to \mathbb{N}$. Let $\tilde{m} = \max\{\rho(q,j) \mid q \in Q, j \in [\text{rk}_Q(q)]\}$. The consistent MTT associated with $M$, denoted by $E(M)$, is the MTT $(Q_e, \Sigma, \Delta, q_0, R_e)$ with $Q_e = Q^{(0)} \cup \{q^{(\tilde{m})} \mid q \in Q - Q^{(0)}\}$ and $R_e$ is the smallest set $R'$ satisfying the following. Note that each state of $E(M)$ corresponds to the state with the same name in $M$ but its rank is $\tilde{m}$ if its original rank is not zero, and zero otherwise. For every $q \in Q_e$ and $\sigma \in \Sigma^{(k)}$ with $k \geq 0$, $R'$ contains the $(q,\sigma)$-rule with $e_q(\text{rhs}_M(q,\sigma))$ in the right-hand side where $e_q(\zeta)$ is defined as follows:

- $e_q(\zeta) = \langle q', x_i \rangle$ if $\zeta = \langle q', x_i \rangle \in \langle Q^{(0)}, X_k \rangle$.
- $e_q(\zeta) = \langle q', x_i \rangle(s_1, \ldots, s_{\tilde{m}})$ if $\zeta = \langle q', x_i \rangle(\zeta_1, \ldots, \zeta_m)$ such that $\langle q', x_i \rangle \in \langle Q^{(m)}, X_k \rangle$ and $m > 0$, where for $j \in [\tilde{m}]$, $s_j = e_q(\zeta_i)$ if there exists $i$ such that $\rho(q',i) = j$, and $s_j = \bot$ otherwise, where $\bot$ is any label in $\Delta^{(0)}$.
- $e_q(\zeta) = \delta(e_q(\zeta_1), \ldots, e_q(\zeta_m))$ if $\zeta = \delta(\zeta_1, \ldots, \zeta_m)$ with $\delta \in \Delta$.
- $e_q(\zeta) = y_{\rho(q,j)}$ if $\zeta = y_j \in Y_{\text{rk}_Q(q)}$.

We can see that the above construction transforms the bottom right MTT with the FV property in Figure 1 into the top right MTT, which is consistent.

We shall prove the correctness of the construction of Definition 5. We show that $E(M)$ is a consistent MTT equivalent with $M$ if the nondeleting MTT $M$ has the FV property.

Lemma 1. Suppose that the nondeleting MTT $M$ has the FV property with $\rho$. Then $M$ and $E(M)$ are equivalent.
Proof. Let $M' = E(M) = (Q', \Sigma, \Delta, q_0, R')$. Recall that $\Psi_q^\rho = [y_l \leftarrow y_r \mid l \in [m], l' = \rho(q, l)]$ for $q \in Q^{(m)}$ and $m \geq 0$. We prove the following two statements (i) and (ii). The lemma follows from (i). Let $s$ be an arbitrary tree in $T_\Sigma$.

(i) For every $q \in Q$, $M_q(s) \Psi_q^\rho = M_q'(s)$.

(ii) For every $q \in Q$, $u \in V(\zeta_q)$, and $k \geq 0$ such that $s[\varepsilon] \in \Sigma^{(k)}$,

$$(\zeta_q/u)\theta \Psi_q^\rho = e_q(\zeta_q/u)\theta'$$

where $\zeta_q = \text{rhs}_M(q, \sigma)$, $\theta = \{ \langle r, x_i \rangle \leftarrow M_r(s/i) \mid r \in Q, i \in [k] \}$, and $\theta' = \{ \langle r, x_i \rangle \leftarrow M_r'(s/i) \mid r \in Q', i \in [k] \}$. Assume that (ii) holds for $s$. With (ii) for $u = \varepsilon$, we get $M_q(s) \Psi_q^\rho = \zeta \theta \Psi_q^\rho = e_q(\zeta)\theta' = M_q'(s)$ because $\text{rhs}_{M'}(q, \sigma) = e_q(\zeta_q)$ by the construction of Definition 5.

Next, with the property that (ii) implies (i), we prove that (ii) holds for all $s \in T_\Sigma$ by induction on the structure of $s$. They imply that (i) also holds for all $s$.

Base case of (ii). Let $s = \sigma \in \Sigma^{(0)}$ and $k = 0$. Let $q \in Q^{(m)}$, $\zeta = \text{rhs}_M(q, \sigma)$, and let $u \in V(\zeta)$ and $\xi = \zeta/u$. Note that $\zeta \in T_\Delta(Y_m)$ and thus $\xi \in T_\Delta(Y_m)$. By the definition of $e_q$ and $\xi \in T_\Delta(Y_m)$, we get $\xi \Psi_q^\rho = e_q(\xi)$. Since $\theta$ and $\theta'$ are empty substitutions when $k = 0$, $\xi \theta \Psi_q^\rho = e_q(\xi)\theta'$.

Induction step of (ii). Let $s = \sigma(s_1, \ldots, s_k) \in T_\Sigma$ where $\sigma \in \Sigma^{(k)}$, $k > 0$, and $s_1, \ldots, s_k \in T_\Sigma$. Let $q \in Q^{(m)}$. Let $\zeta_q = \text{rhs}_M(q, \sigma)$, $\theta = \{ \langle r, x_i \rangle \leftarrow M_r(s/i) \mid r \in Q, i \in [k] \}$. Note that $\zeta_q \in T_\Delta(Y_m)$ and thus $\xi \in T_\Delta(Y_m)$. By the definition of $e_q$ and $\xi \in T_\Delta(Y_m)$, we get $\xi \Psi_q^\rho = e_q(\xi)$. Since $\theta$ and $\theta'$ are empty substitutions when $k > 0$, $\xi \theta \Psi_q^\rho = e_q(\xi)\theta'$.
Now we prove that \( \xi \theta \Psi_q^0 = e_q(\xi) \theta' \) holds for every subtree \( \xi \) of \( \zeta_q \) by induction on the structure of \( \xi \), which implies that (ii) holds for \( s \). Let us denote by (IHs) the induction hypothesis of the outer induction on \( s \), and by (IH\( \xi \)) that of the inner induction on \( \xi \).

**Case 1.** \( \xi = y_j \). It is trivial from the fact that \( \Psi_q^0(y_j) = y_{\rho(q,j)} = e_q(y_j) \).

**Case 2.** \( \xi = \delta(\xi_1, \ldots, \xi_l) \). It holds that \( \xi \theta \Psi_q^0 = \delta(\zeta_1 \theta \Psi_q^0, \ldots, \zeta_l \theta \Psi_q^0) \) and \( e_q(\xi) \theta' = \delta(e_q(\xi_1) \theta', \ldots, e_q(\xi_l) \theta') \). It follows by (IH\( \xi \)) that \( \zeta_i \theta \Psi_q^0 = e_q(\xi_i) \theta' \) for every \( i \in [l] \).

**Case 3.** \( \xi = \langle q', x_i \rangle \) with \( q' \in Q^{(0)} \). By applying (IHs) to \( s/i \) and using the fact that (ii) implies (i), \( M_{q'}(s/i) \Psi_q^0 = M_{q'}(s/i) \) holds. Thus, we get \( \xi \theta \Psi_q^0 = M_{q'}(s/i) \Psi_q^0 = M_{q'}(s/i) = \langle q', x_i \rangle \theta' = e_q(\xi) \theta' \).

**Case 4.** \( \xi = \langle q', x_i \rangle(\xi_1, \ldots, \xi_{m'}) \) with \( q' \in Q^{(m')} \) and \( m' > 0 \). We get (a) \( M_{q'}(s/i) \Psi_q^0 = M_{q'}(s/i) \) by (IHs) and (ii) \( \Rightarrow \) (i). Recall that (b) \( e_q(\xi) = \langle q', x_i \rangle(\zeta_1', \ldots, \zeta_{\tilde{m}}') \) where for \( j \in [\tilde{m}] \), \( \zeta_j' = e_q(\xi_j) \) if \( \rho(q,l) = j \), and \( \zeta_j' = \bot \) otherwise. In addition, it follows from (a) that (c) the parameters
appearing in $M'_q(s/i)$ are in $\{y_j \mid j = \rho(q', l), l \in [m']\}$. Hence,

\[
\begin{align*}
\xi \theta \Psi^\rho_q &= M'_q(s/i)[y_l \leftarrow \xi_l \theta \mid l \in [m']]\Psi^\rho_q \\
&= M'_q(s/i)[y_l \leftarrow \xi_l \theta \Psi^\rho_q \mid l \in [m']] \\
&= M'_q(s/i)[y_l \leftarrow e_p(\xi_i)\theta' \mid l = \rho(q', l), l \in [m']] \\
&= M'_q(s/i)[y_j \leftarrow e_p(\xi_i)\theta' \mid j = \rho(q', l), l \in [m']] \quad \text{(by (IH$\xi$))} \\
&= M'_q(s/i)[y_j \leftarrow e_p(\xi_i)\theta' \mid j = \rho(q', l), l \in [m']] \quad \text{(by (a))} \\
&= M'_q(s/i)[y_j \leftarrow \xi'_j \theta' \mid j \in [\bar{m}]] \\
&= \langle q', x_i \rangle(\xi'_1, \ldots, \xi'_{\bar{m}})\theta' = e_q(\xi)\theta'.
\end{align*}
\]

\[\square\]

The resulting consistent MTT $\mathcal{E}(M)$ has additional properties. We say that the $j$th parameter of state $q$ in $M$ is \textit{permanent} if for every $q' \in Q$, $\sigma \in \Sigma$, $v \in V(\zeta)$ where $\zeta = \text{rhs}_M(q', \sigma)$, if $\zeta[v] = \langle q, x_i \rangle$ for some $x_i \in X$ then $v_j$ is important in $\zeta$.

\textbf{Lemma 2.} Suppose that the nondeleting MTT $M$ has the FV property with $\rho$. Let $M' = \mathcal{E}(M)$. Then $M'$ has the following properties.

(i) In the right-hand side $\zeta$ of any rule of $M'$, for every $v \in V(\zeta)$, if $v = v'j$ with some $v' \in V(\zeta)$ and $j \in [\bar{m}]$ such that $\zeta[v] = \langle q', x_i \rangle \in \langle Q_e, X \rangle$ and $\rho(q', l) \neq j$ for all $l \in \text{rk}_Q(q')$, then $v$ is not important in $\zeta$ and $\zeta[v] = \bot$; otherwise, $v$ is important in $\zeta$.

(ii) The $j$th parameter of $q$ is permanent in $M'$ if $y_j \in Y$ appears in $\text{rhs}_{M'}(q, \sigma)$ for some $\sigma \in \Sigma$. 

\[15\]
PROOF. Let $\zeta = \text{rhs}_{M'}(q, \sigma)$. We can prove Property (i) by induction on the distance from the root in $\zeta$. The root is trivially important in $\zeta$. Suppose $v = v'j \in V(\zeta) - \{\varepsilon\}$. Since the rank of $\zeta[v']$ is greater than zero and thus $\zeta[v'] \neq \bot$, $v'$ is important in $\zeta$ by the induction hypothesis. If $\zeta[v'] \in \Delta \cup Y$, $v$ is also important in $\zeta$ trivially from the importance of $v'$. Suppose $\zeta[v'] = \langle q', x_i \rangle$. Let $m'$ be the rank of $q'$ in $M$. It follows from Statement (i) in the proof of Lemma 1 that $\rho(q', l) = j$ for some $l \in [m']$ if and only if $y_j$ appears in $M'_q(s)$ for all $s \in T_\Sigma$. Thus, $v$ is important in $\zeta$ if and only if $\rho(q', l) = j$ for some $l \in [m']$. Recall that $\text{rhs}_{M'}(q, \sigma) = e_q(\text{rhs}_M(q, \sigma))$ from Definition 5. By the definition of $e_q$, if $\rho(q', l) \neq j$ for all $l \in [m']$, we get $\zeta[v] = \bot$.

Assume that $y_j$ occurs in $\text{rhs}_{M'}(q, \sigma)$ for some $\sigma \in \Sigma$. Since $\text{rhs}_{M'}(q, \sigma) = e_q(\text{rhs}_M(q, \sigma))$, by the definition of $e_q$, $y_l$ occurs in $\text{rhs}_M(q, \sigma)$ and $\rho(q, l) = j$ for some $l \in [m]$ where $m$ is the rank of $q$ in $M$. Let $q' \in Q$, $\sigma' \in \Sigma$, $v \in V(\zeta)$ where $\zeta = \text{rhs}_{M'}(q', \sigma)$. Suppose that $\zeta[v] = \langle q, x_i \rangle$ for some $x_i \in X$. Since $\rho(q, l) = j$, it follows from Property (i) that $v_j$ is important in $\zeta$. Hence, the $j$th parameter of $q$ is permanent in $M'$.

$\square$

**Lemma 3.** Suppose that the nondeleting MTT $M$ has the FV property with $\rho$. Then $E(M)$ is consistent.

**Proof.** Let $M' = E(M) = (Q_e, \Sigma, \Delta, q_0, R_e)$. Let $\sigma \in \Sigma^{(k)}$ with $k \geq 0$. Let $q_1, q_2 \in Q_e$. Let $p_1^{(m_1)}, p_2^{(m_2)} \in Q_e$ and $l \in [k]$. For $i = 1, 2$, let $\zeta_i = \text{rhs}_{M'}(q_i, \sigma)$, $w_i \in V(\zeta_i)$ such that $\zeta_i[w_i] = \langle p_i, x_i \rangle$. Let $j \in [\min\{m_1, m_2\}]$ such that $w_1j$ and $w_2j$ are important in $\zeta_1$ and $\zeta_2$, respectively. Note that $k > 0$ and $m_1 = m_2 = \bar{m}$ because of the existence of $w_i$ and $j$. From Lemma 2(i), we get $\rho(p_1, l_1) = \rho(p_2, l_2) = j$ for some $l_1, l_2 \in [\bar{m}]$. Since $\zeta_i = e_q(\zeta_i)$ where
ζ_q_i = rhs_M(q_i, σ), from the definition of e_q, there exists node w'_i ∈ V(ζ_q_i) such that ζ_q_i[w'_i] = (p_i, x_i) and e_q_i(ζ_q_i/w'_il_i) = ζ_i/w_ij. By the FV property of M, (ζ_q_i/w'_il_i)^Ψ_q_i = (ζ_q_2/w'_2l_2)^Ψ_q_2. Thus, we get e_q_i(ζ_q_i/w'_il_1) = e_q_2(ζ_q_2/w'_2l_2), and thus ζ_1/w_1j = ζ_2/w_2j. This implies top(ζ_1/w_1j) = top(ζ_2/w_2j). □

We shall show that E(M) satisfies FV-orig if the nondeleting M has the FV property. That is, the resulting ATT obtained from E(M) by the construction Ω given in [13] is circular. Note that this does not hold for consistent MTTs in general. For the proof, we give a simplified version of the same construction here because the style of the definition of MTTs is different with that of [13] and we focus on the consistent MTTs with the properties in Lemma 2 as inputs. The difference with the original Ω is just that important/non-important nodes in the right-hand side of rules are explicit in E(M), and thus we define Ω without the recursive functions SUB and DECOMPOSE in the original Ω. It is straightforward that the following definition of Ω is equivalent with the original one for E(M).

**Definition 6.** Suppose that the nondeleting MTT M satisfies the FV property with ρ. Let M′ = E(M) = (Q, Σ, Δ, q_0, R). Let ̂m be the maximum rank of states in Q. The attributed tree transducer associated with M′, denoted by Ω(M′), is the ATT (S, I, Σ, Δ, q_0, R') defined as follows: S = Q, I = [y_i | i ∈ [̂m]], and R' = ∪_{σ ∈ Σ} R_σ. For every σ ∈ Σ^{(k)} with k ≥ 0, R_σ is constructed as follows:

1. For q ∈ Q, let ζ = rhs_M′(q, σ),
   
   • let the rule q(π) → top(ζ) be in R_σ, and
• for every $v \in V(\zeta)$ such that $\zeta[v] = \langle q', x_i \rangle$ where $q' \in Q(\tilde{m})$ and $i \in [k]$, let the rule $y_j(\pi i) \rightarrow \text{top}(\zeta/v_j)$ be in $R_\sigma$ for each $j \in [\bar{m}]$ such that $v_j$ is important in $\zeta_q$.

2. If there is no rule with $\beta(\pi i)$ in the left-hand side for $\beta \in I$ and $i \in [k]$, let the dummy rule $\beta(\pi i) \rightarrow \bot$ be in $R_\sigma$ where $\bot$ is any label in $\Delta^{(0)}$.

Now we show that $\Omega(\mathcal{E}(M))$ is non-circular. For this, we give the following technical lemma (Lemma 4) which deduces from the existence of certain paths in the dependency graph of $\Omega(\mathcal{E}(M))$ certain properties about occurrences of state calls in the right-hand sides of $\mathcal{E}(M)$, which the proof of the non-circularity of $\Omega(\mathcal{E}(M))$ (Lemma 5) is based on.

**Lemma 4.** Suppose that $M$ is the consistent MTT obtained by $\mathcal{E}$ from a nondeleting MTT with the FV property. Let $\Omega(M) = (S, I, \Sigma, \Delta, q_0, R')$. Let $s$ be an arbitrary tree in $T_\Sigma$.

(i) For every $q \in S$, and $y_\ell \in I$, if there exists a path from $(y_\ell, \varepsilon)$ to $(q, \varepsilon)$ in $D_{\Omega(M)}(s)$, then the $\ell$th parameter of $q$ is permanent.

(ii) For every $q, q' \in S$ and $i, i' \in [\text{rk}_\Sigma(s[\varepsilon])])$, if there exists a nonempty path from $(q, i)$ to $(q', i')$ in $D_{\Omega(M)}(s)$ such that $\langle q', x_{i'} \rangle = \zeta[v']$ for some $\tilde{q} \in Q$ and $v' \in V(\zeta)$ where $\zeta = \text{rhs}_M(\tilde{q}, s[\varepsilon])$, then there exists a proper descendant $v$ of $v'$ in $\zeta$ such that $\zeta[v] = \langle q, x_i \rangle$.

**PROOF.** We prove that (i) and (ii) hold for every $s \in T_\Sigma$ by induction on the structure of $s$ in the following way.

• Base case. Let $s \in \Sigma^{(0)}$. Since (ii) follows from $\text{rk}_\Sigma(s[\varepsilon]) = 0$, we prove only that (i) holds for $s$ below.
• Induction step. Let \( s = \sigma(s_1, \ldots, s_k) \) be an arbitrary tree in \( T_\Sigma \) such that \( k > 0 \), \( \sigma \in \Sigma^{(k)} \), and \( s_1, \ldots, s_k \in T_\Sigma \). Under the induction hypothesis of (i), denoted by (IHi) henceforth, for the subtrees of \( s \), we first prove that (ii) holds for \( s \), and then by using (ii) for \( s \) as well, we prove that (i) holds for \( s \).

Base case of (i). Let \( s = \sigma \in \Sigma^{(0)} \). Let \( q \in S \) and \( y_\ell \in I \). Assume that there exists a path from \((y_\ell, \varepsilon)\) to \((q, \varepsilon)\) in \( D_{\Omega(M)}(s) \). Since \( \sigma \in \Sigma^{(0)} \), the path consists of only a direct edge from \((y_\ell, \varepsilon)\) to \((q, \varepsilon)\). The edge originates from a rule in \( R'_\sigma \) such that \( q(\pi) \) and \( y_\ell(\pi) \) occur in the lhs and rhs, respectively. From the construction \( \Omega(M) \) in Definition 6, the rhs is \( \text{top}(\text{rhs}_M(q, \sigma)) \). Since \( y_\ell(\pi) \) occurs in the rhs, by the definition of \( \text{top} \), \( y_\ell \) occurs in \( \text{rhs}_M(q, \sigma) \). From Lemma 2(ii), the \( \ell \)th parameter of \( q \) is permanent.

Induction step of (ii). Let \( s = \sigma(s_1, \ldots, s_k) \in T_\Sigma \) with \( k > 0 \) and \( \sigma \in \Sigma^{(k)} \). Let \( q, q' \in S \) and \( i, i' \in [k] \). Hereafter, we abbreviate \( \text{rhs}_M(p, \sigma) \) as \( \zeta_p \) for \( p \in Q \). Assume that there exists a nonempty path \( w \) from \((q, i)\) to \((q', i')\) in \( D_{\Omega(M)}(s) \), and that there exists \( \tilde{q} \in Q \) and \( v' \in V(\zeta_{\tilde{q}}) \) such that \( \langle q', x_{i'} \rangle = \zeta_{\tilde{q}}[v'] \).

We can regard the path \( w \) as a sequence of nodes of \( D_{\Omega(M)}(\sigma) \) such that every two consecutive nodes are connected by an edge of \( D_{\Omega(M)}(\sigma) \), or a path of a subgraph \( D_{\Omega(M)}(s_i) \) for some \( i \in [k] \). Let \( \leftarrow_\varepsilon \) and \( \leftarrow_i \) denote connections by an edge of \( D_{\Omega(M)}(\sigma) \) and a path of a subgraph \( D_{\Omega(M)}(s)/i \) for \( i \in [k] \), respectively. Then, let \( q_1 = q' \), \( i_1 = i' \), \( q_n = q \), and \( i_n = i \), and we can write
the path $w$ as follows:

$$(q', i') = (q_1, i_1) \leftarrow_{i_1} (y_{\ell_1}, i_1) \leftarrow_\varepsilon (q_2, i_2) \leftarrow_{i_2} \cdots \leftarrow_\varepsilon (q_{n-1}, i_{n-1}) \leftarrow_{i_{n-1}} (y_{\ell_{n-1}}, i_{n-1}) \leftarrow_\varepsilon (q_n, i_n) = (q, i)$$

for some $n > 1$ where $i_j \in [k]$ and $q_j \in S$ for all $j \in [n]$, and $y_{\ell_j} \in I$ for all $j \in [n-1]$. For such $w$, now we prove that there exists a proper descendant $v$ of $v'$ in $\zeta_{\hat{q}}$ such that $\zeta_{\hat{q}}[v] = \langle q, x_i \rangle$. To achieve this, we show that for every $c$ where $2 \leq c \leq n$, there exists a proper descendant $v$ of $v'$ in $\zeta_{\hat{q}}$ such that $\zeta_{\hat{q}}[v] = \langle q_c, x_{i_c} \rangle$. It can be shown by induction on $c$. Let us denote the induction hypothesis of the inner induction on $c$ by (IHc).

(Base case) Let $c = 2$. We have $(q_1, i_1) \leftarrow_{i_1} (y_{\ell_1}, i_1) \leftarrow_\varepsilon (q_2, i_2)$. Since the edge $(q_1, i_1) \leftarrow_{i_1} (y_{\ell_1}, i_1)$ is in $D_{\Omega(M)}(s)/i_1$, an edge from $(y_{\ell_1}, \varepsilon)$ to $(q_1, \varepsilon)$ is in $D_{\Omega(M)}(s/i_1)$. By (IH$i$) to the edge, the $\ell_1$th parameter of $q_1$ is permanent. Recall that $\zeta_{\hat{q}}[v'] = \langle q', x_{i'} \rangle = (q_1, x_{i_1})$. Thus, $v'\ell_1$ is important in $\zeta_{\hat{q}}$. Moreover, the edge from $(q_2, i_2)$ to $(y_{\ell_1}, i_1)$ originates from a rule in $R'_\sigma$ such that $y_{\ell_1}(\pi i_1)$ and $q_2(\pi i_2)$ occur in the lhs and rhs, respectively. From the construction of Definition 6, for some $\hat{q}, \hat{q} \in Q$, $u \in V(\zeta_{\hat{q}})$, the rhs is $\text{top}(\zeta_{\hat{q}}/u\ell_1)$, $\zeta_{\hat{q}}[u] = \langle \hat{q}, x_{i_1} \rangle$, and $u\ell_1$ is important in $\zeta_{\hat{q}}$. Since $\zeta_{\hat{q}}[v'] = (q_1, x_{i_1})$, $\zeta_{\hat{q}}[u] = \langle \hat{q}, x_{i_1} \rangle$, and $v'\ell_1$ and $u\ell_1$ are important in $\zeta_{\hat{q}}$ and $\zeta_{\hat{q}}$, respectively, we get $\text{top}(\zeta_{\hat{q}}/v'\ell_1) = \text{top}(\zeta_{\hat{q}}/u\ell_1)$ by the consistency of $M$. Since $\langle q_2, x_{i_2} \rangle$ occurs in $\text{top}(\zeta_{\hat{q}}/u\ell_1)$, it also occurs in $\text{top}(\zeta_{\hat{q}}/v'\ell_1)$. By the definition of $\text{top}$, there exists a proper descendant $v$ of $v'$ in $\zeta_{\hat{q}}$ such that $\zeta_{\hat{q}}[v] = \langle q_2, x_{i_2} \rangle$.

(Induction step) By applying (IHc) to the sub-path $(q_1, i_1) \leftarrow_{i_1} \cdots \leftarrow_\varepsilon (q_{c-1}, i_{c-1})$, it holds that there exists a descendant $v_{c-1}$ of $v'$ in $\zeta_{\hat{q}}$ such that
\[ \zeta_q[v_{c-1}] = \langle q_{c-1}, x_{i_{c-1}} \rangle. \]

From the part \((q_{c-1}, i_{c-1}) \leftarrow_{i_{c-1}} (y_{c-1}, i_{c-1}) \leftarrow_{\varepsilon} (q_c, i_c)\), with a similar discussion as (Base case), we can show that there exists a proper descendant \(v_c\) of \(v_{c-1}\) in \(\zeta_q\) such that \(\zeta_q[v_c] = \langle q_c, x_{i_c} \rangle\). Therefore, the node \(v_c\) is a proper descendant of \(v'\) such that \(\zeta_q[v_c] = \langle q_c, x_{i_c} \rangle\).

**Induction step of (i).** Let \(s = \sigma(s_1, \ldots, s_k) \in T_\Sigma\) with \(k > 0\) and \(\sigma \in \Sigma^{(k)}\). Let \(q \in S\) and \(\ell \in I\). Assume that there exists a path \(w\) from \((y_\ell, \varepsilon)\) to \((q, \varepsilon)\) in \(D_{O(M)}(s)\). Hereafter, we abbreviate \(\text{rhs}_M(p, \sigma)\) as \(\zeta_p\) for \(p \in Q\).

For the case that \(w\) consists of only a direct edge from \((y_\ell, \varepsilon)\) to \((q, \varepsilon)\), we can complete this case with the same discussion as the base case of the proof of Lemma 4.

For the other case, we can assume that the path \(w\) goes from \((y_\ell, \varepsilon)\) to \((q, \varepsilon)\) via at least one synthesized attribute node. As in the proof of Induction step of (ii), we can regard the path \(w\) as \((q, \varepsilon) \leftarrow_{\varepsilon} (q_1, i_1) \leftarrow^* (q_n, i_n) \leftarrow_{i_n} (y_{\ell_n}, i_n) \leftarrow_{\varepsilon} (y_\ell, \varepsilon)\) where \(i_1, i_n \in [k], q, q_1, q_n \in S, y_\ell, y_{\ell_n} \in I\), and \(\leftarrow^*\) is the reflexive transitive closure of \(\bigcup_{i \in [k]} \leftarrow_i\). First, we show from the sub-path \((q, \varepsilon) \leftarrow_{\varepsilon} (q_1, i_1) \leftarrow^* (q_n, i_n)\) that there exists a node \(v_n\) in \(\zeta_q\) such that \(\zeta_q[v_n] = \langle q_n, x_{i_n} \rangle\). The edge \((q, \varepsilon) \leftarrow_{\varepsilon} (q_1, i_1)\) originates from a rule such that \(q(\pi)\) and \(q_1(\pi i_1)\) occur in the lhs and rhs. From the construction of Definition 6, the rhs is \(\text{top}(\zeta_q)\). By the definition of \(\text{top}\) and the fact that \(q_1(\pi i_1)\) occurs in \(\text{top}(\zeta_q)\), \(\zeta_q[v_1] = \langle q_1, x_{i_1} \rangle\) for some \(v_1 \in V(\zeta_q)\). From the sub-path \((q_1, i_1) \leftarrow^* (q_n, i_n)\) in \(w\), we can obtain the fact that there exists a descendant \(v_n\) of \(v_1\) in \(\zeta_q\) such that \(\zeta_q[v_n] = \langle q_n, x_{i_n} \rangle\) as follows. If the sub-path \((q_1, i_1) \leftarrow^* (q_n, i_n)\) in \(w\) is empty, we have \(q_n = q_1\) and \(i_n = i_1\), and thus \(\zeta_q[v_1] = \langle q_n, x_{i_n} \rangle\). Otherwise, since (ii) holds for \(s\) under (IHi) for \(s_1, \ldots, s_k\) (by the induction step of (ii)), by the existence of the nonempty
path from \((q_n, i_n)\) to \((q_1, i_1)\) with \(\zeta_q[v_1] = \langle q_1, x_{i_1} \rangle\), there exists a proper descendant \(v_n\) of \(v_1\) in \(\zeta_q\) such that \(\zeta_q[v_n] = \langle q_n, x_{i_n} \rangle\). Next, by (IIIi) to the sub-path \((q_n, i_n) \leftarrow_{i_n} (y_{\ell_n}, i_n)\), the \(\ell_n\)th parameter of \(q_n\) is permanent. Thus, \(v_n\ell_n\) is important in \(\zeta_q\). The edge \((\ell_n, i_n) \leftarrow_{\varepsilon} (\ell, \varepsilon)\) originates from a rule such that \(\ell_n(\pi i_n)\) and \(\ell(\pi)\) occur in the lhs and rhs, respectively. From the construction of Definition \([6]\) for some \(q', q'' \in Q\) and \(u \in V(\zeta'_{q'})\), the rhs is \(top(\zeta'_q/u\ell_n)\) and \(\zeta'_q[u] = \langle q'', x_{i_n} \rangle\) and \(u\ell_n\) is important in \(\zeta_q\). Since \(\zeta_q[v_n] = \langle q_n, x_{i_n} \rangle\), \(\zeta'_q[u] = \langle q'', x_{i_n} \rangle\), and \(v_n\ell_n\) and \(u\ell_n\) are important in \(\zeta_q\) and \(\zeta'_q\), we get \(top(\zeta_q/v_n\ell_n) = top(\zeta'_q/u\ell_n)\) by the consistency of \(M\). since \(y_{\ell'}\) appears in \(top(\zeta'_q/u\ell_n)\), \(y_{\ell'}\) appears in \(top(\zeta_q/v_n\ell_n)\). Thus, \(y_{\ell'}\) appear in \(\zeta_q/v_n\ell_n\). From Lemma \([2]\)(ii), the \(\ell\)th parameter of \(q\) is permanent. \(\square\)

**Lemma 5.** If the nondeleting MTT \(M\) has the FV property then \(\Omega(\mathcal{E}(M))\) is non-circular.

**Proof.** Let \(M' = \mathcal{E}(M)\). Let \(A = \Omega(M') = (S, I, \Sigma, \Delta, q_0, R')\). The proof is by contradiction. Assume that \(A\) is circular. Then there exists a tree \(s \in T_\Sigma\) such that \(D_A(s)\) has a cycle. Let \(u \in V(s)\) be a node such that \(s/u\) is a minimal subtree that includes the cycle. Then \(s/u \in T_\Sigma - \Sigma^{(0)}\) because no cycle can be made in \(D_A(\sigma)\) for any \(\sigma \in \Sigma^{(0)}\). Let \(\sigma = s[u] \in \Sigma^{(k)}\) with \(k > 0\). There is a cycle from some synthesized attribute node \((q, i)\) to itself via at least one inherited attribute node in \(D_A(s/u)\) for some \(q \in S\) and \(i \in [k]\). Since there exists an outgoing edge of \((q, i)\), by the construction, there exist \(\bar{q} \in Q\) and \(v' \in V(\zeta)\) such that \(\zeta[v'] = \langle q, x_i \rangle\) where \(\zeta = rhs_{M'}(\bar{q}, \sigma)\). From Lemma \([4]\) there exists a proper descendant \(v\) of \(v'\) in \(\zeta\) such that \(\zeta[v] = \langle q, x_i \rangle\). Then, let us consider the loop end of state calls in \(\zeta\) involved by the existence of such nodes \(v\) and \(v'\), that is, we can choose distinct two nodes \(u\) and \(u'\).
in $\zeta$ such that $\zeta[u] = \zeta[u'] = \langle q', x_j \rangle \in \langle Q, X \rangle$, $u'$ is a descendant of $u_j$ for some $j \in [\bar{m}]$, any call $\langle q'', x_i \rangle \in \langle Q, X \rangle$ over the path from $u$ to $u'$ does not occur in $\zeta/u'j$. This implies $\text{top}(\zeta/u_j) \neq \text{top}(\zeta/u'j)$. On the other hand, $\zeta[u_j] \neq \bot$. From Lemma 2(i), $u_j$ is important in $\zeta$ and thus $y_j$ occurs in $\text{rhs}_{M'}(q', \sigma')$ for some $\sigma' \in \Sigma$. Since the $j$th parameter of $q'$ is permanent from Lemma 2(ii), $u'j$ is also important in $\zeta$. Since $M'$ is consistent from Lemma 3, $\text{top}(\zeta/u_j) = \text{top}(\zeta/u'j)$, which is a contradiction. □

For the nondeleting MTT $M$ with the FV property, from Lemmas 1, 3, and 5, $E(M)$ satisfies the FV-orig property (i.e., attributed-like in [14]). Lemma 3.18 in [14] says that for an MTT with the FV-orig property, there is an equivalent ATT. From that, we obtain the following lemma and corollary.

**Lemma 6.** $\text{MTT}_{FV} \subseteq \text{ATT}$.

**Corollary 1.** $\text{MTT}_{FV}^R \subseteq \text{ATT}^R$ and $\text{MTT}_{FV}^U \subseteq \text{ATT}^U$.

Next, we show that the converse of the inclusion of Corollary 1. For this, we will show $\text{MTT}_C \subseteq \text{MTT}_{FV}^R$. Before that, let us consider an example. For the consistent MTT $M_c$ of Figure 2, let us construct its nondeleting normal form $B ; M'_c$. We will give the details of the construction in Lemma 7, and the normal form is in $\text{MTT}_{FV}^R$. We first define the bottom-up relabeling $B$ that realizes the look-ahead:

- $e \to p_1(e)$ for $p_1 = \{ q \mapsto \{1\}, q_1 \mapsto \emptyset, q_2 \mapsto \{1\}\}$
- $e' \to p_2(e')$ for $p_2 = \{ q \mapsto \emptyset, q_1 \mapsto \{1\}, q_2 \mapsto \{1\}\}$
- $\sigma(p(x_1), p'(x_2)) \to p_3([\sigma, p, p'](x_1, x_2))$ for $p_3 = \{ q \mapsto \emptyset, q_1 \mapsto \emptyset, q_2 \mapsto \emptyset\}$

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where \( p, p' \in \{p_1, p_2, p_3\} \). The rules of the MTT \( M'_c \) are as follows:

\[
\begin{align*}
\text{r}_1 : \quad & \langle (q_0, \emptyset), [\sigma, p_1, p_1/p_2](x_1, x_2) \rangle \\
& \quad \quad \quad \rightarrow \langle (q, \{1\}), x_1 \rangle (\langle (q_2, \{1\}), x_2 \rangle (\langle (q_1, \emptyset), x_1 \rangle))
\end{align*}
\]

\[
\begin{align*}
\text{r}_2 : \quad & \langle (q_0, \emptyset), [\sigma, p_1, p_3](x_1, x_2) \rangle \\
& \quad \quad \rightarrow \langle (q, \{1\}), x_1 \rangle (\langle (q_2, \emptyset), x_2 \rangle)
\end{align*}
\]

\[
\begin{align*}
\text{r}_3 : \quad & \langle (q_0, \emptyset), [\sigma, p_2/p_3, \_](x_1, x_2) \rangle \\
& \quad \quad \rightarrow \langle (q, \emptyset), x_1 \rangle
\end{align*}
\]

Here the symbol “\( \_ \)” denotes any state, or any input symbol (and look-ahead combination) and the notation \( p_1/p_2 \) means that for both input symbols rules with the same right-hand side exist. As the reader may verify, this transducer indeed has the FV property (with \( \rho((\_, \{1\})) = 1 \)). Note how the bottom-most symbol of \( \text{rhs}_{M_c}(q_0, \sigma) \), i.e., the \( $ \)-labeled leaf does not occur in any rule of \( M'_c \). This is because only three possibilities exist how the first rule \( r \) of \( M_c \) is evaluated on an input tree \( s \) (as expressed by the rules \( r_1, r_2, r_3 \) of \( M'_c \)):

1. if \( s = [\sigma, p_1, p_1/p_2](e, e/e') \), then \( r \) evaluates to \( \langle q_1, e \rangle \) (viz. \( r_1 \))
2. if \( s = [\sigma, p_1, p_3](e, \sigma(\ldots)) \), then \( r \) evaluates to \( \langle q_2, \sigma(\ldots) \rangle \) (viz. \( r_2 \))
3. if \( s = [\sigma, p_2/p_3, \_](e'/\sigma(\ldots), \ldots) \), then \( r \) evaluates to \( \langle q, e'/\sigma(\ldots) \rangle \) (viz. \( r_3 \)).

In each case, the MTT \( M_c \) finally applies a rule with \# in the right-hand side, i.e., a deleting rule.

\textbf{Lemma 7.} \( \text{ATT} \subseteq MTT_C \subseteq MTT_{RFV}^{R} \).

\textbf{Proof.} It is well known that for any ATT there exists an equivalent consistent MTT \([14]\). It should be noted that in Theorem 5.11 of \([6]\) it is proved
that $ATT \subseteq MTT^R$. The MTT constructed in the proof of that theorem in fact has the FV property, so we could simply use that construction. However, to make this paper more self-contained we prefer to give another proof, purely in terms of MTTs.

Let $M = (Q, \Sigma, \Delta, q_0, R)$ be a consistent MTT. We show that the non-deleting normal form $M'$ of $M$ obtained by the construction in the proof of Lemma 6.6 of [6] has the FV property. We give the construction again, in our setting.

We denote by $P$ the set of all functions which associate with every $q \in Q^{(m)}$ a subset of $[m]$. For a subset $I$ of $\mathbb{N}$ we denote by $|I|$ the cardinality of $I$ and by $I(j)$ the $j$th element of $I$ with respect to <. We define the B-REL $B = (P, \Sigma, \Sigma', P, R_B)$, where $\Sigma' = \{[\sigma, p_1, \ldots, p_k] \mid \sigma \in \Sigma^{(k)}, p_1, \ldots, p_k \in P\}$. For $\sigma \in \Sigma^{(k)}$ and $p_1, \ldots, p_k \in P$, let the rule

$$\sigma(p_1(x_1), \ldots, p_k(x_k)) \rightarrow p_0([\sigma, p_1, \ldots, p_k](x_1, \ldots, x_k))$$

be in $R_B$, where for every $q \in Q^{(m)}$, $p_0(q) = \text{oc}(\text{rhs}_M(q, \sigma))$ and for $t \in T_{[Q, X_k] \cup \Delta}(Y_m)$,

$$\text{oc}(t) = \begin{cases} \{j\} & \text{if } t = y_j \in Y_m \\ \bigcup_{i=1}^l \text{oc}(t_i) & \text{if } t = \delta(t_1, \ldots, t_l) \text{ where } \delta \in \Delta^{(l)} \\ \bigcup_{j \in P_i(r)} \text{oc}(t_j) & \text{if } t = \langle r, x_i \rangle(t_1, \ldots, t_l) \text{ where } \langle r, x_i \rangle \in \langle Q, X_k \rangle^{(l)}.
\end{cases}$$

We define the transition function $h$ such that $h_{\sigma}(p_1, \ldots, p_k) = p_0$ if rule $\sigma(p_1(x_1), \ldots, p_k(x_k)) \rightarrow p_0([\sigma, p_1, \ldots, p_k](x_1, \ldots, x_k))$ is in $R_B$.

Let $M' = (Q', \Sigma', \Delta \cup \{d^{(2)}\}, (q_0, \emptyset), R')$ be an MTT where $Q' = \{(q, I)^{(l)} \mid q \in Q^{(m)}, I \subseteq [m]\}$. For every $(q, I) \in Q'$, $\sigma \in \Sigma^{(k)}$, $k \geq 0$, and $p_1, \ldots, p_k \in
\( P \), let the rule
\[
\langle (q, I), [\sigma, p_1, \ldots, p_k](x_1, \ldots, x_k) \rangle(y_1, \ldots, y_{|I|}) \rightarrow \zeta
\]
be in \( R' \), where \( p_0 = h_\sigma(p_1, \ldots, p_k) \) and for \( I \neq p_0(q) \) let \( \zeta = y_1 \) if \(|I| = 1\) and otherwise let \( \zeta = d(y_1, d(y_2, \ldots, d(y_{|I|-1}, y_{|I|}) \cdot \cdot \cdot) \); if \( I = p_0(q) \) let \( \zeta = \text{rhs}_M(q, \sigma) \Theta_p \theta_I \), where \( \Theta_p \) with \( \bar{p} = (p_1, \ldots, p_k) \) denotes the substitution
\[
[[r, x_i] \leftarrow ((r, I_r), x_i) \langle y_{t(1)}, \ldots, y_{t(n)} \rangle ] \langle r, x_i \rangle \in \langle Q, X_k \rangle, I_r = p_i(r), n = |I_r|\]
and \( \theta_I = [y_{I(j)}] \leftarrow y_j \mid j \in [|I||] \). We let \( \rho((q, I), j) = I(j) \) for every \((q, I) \in Q'\) and \( j \in [|I|] \).

**Claim 1.** The following two properties hold.

1. For every \((q, I) \in Q', \sigma \in \Sigma^k, k \geq 0, \) and \( p_1, \ldots, p_k \in P \), if 
\( \text{rhs}_{M'}((q, I), [\sigma, \bar{p}]) \), where \( \bar{p} \) is the sequence \( p_1, \ldots, p_k \), is not the dummy right-hand side, then there is an injective mapping \( \psi : V(\zeta) \rightarrow V(\xi) \) where \( \zeta = \text{rhs}_{M'}((q, I), [\sigma, \bar{p}]) \) and \( \xi = \text{rhs}_M(q, \sigma) \) such that for every \( w \in V(\zeta) \),
   (a) \( \zeta/w = (\xi/\psi(w)) \Theta_p \theta_I \) and \( \psi(w) \) is important in \( \xi \) for some trees \( s_1, \ldots, s_k \in T_\Sigma \) such that \( B(\sigma(s_1, \ldots, s_k))[\varepsilon] = [\sigma, \bar{p}] \), and
   (b) if \( \zeta[w] = \langle (q', I'), x_i \rangle \in \langle Q', X \rangle \) then \( \psi(wj) = \psi(w)\rho((q', I'), j) \) for every \( j \in [|I'|] \).

2. Let \( \xi_1 = \text{rhs}_M(q_1, \sigma) \) and \( \xi_2 = \text{rhs}_M(q_2, \sigma) \) for states \( q_1, q_2 \in Q \). Let 
\( p_1, \ldots, p_k \in P \) and \( \bar{p} \) be the sequence \( p_1, \ldots, p_k \). Assume that neither \( \text{rhs}_{M'}(q_1, [\sigma, \bar{p}]) \) nor \( \text{rhs}_{M'}(q_2, [\sigma, \bar{p}]) \) is the dummy right-hand side.
Let \( v_1 \in V(\xi_1) \) and \( v_2 \in V(\xi_2) \). Assume that \( v_1 \) and \( v_2 \) are important in \( \xi_1 \) and \( \xi_2 \), respectively, for some \( s_1, \ldots, s_k \in T_\Sigma \) such that
Let \(B(\sigma(s_1, \ldots, s_k))[\varepsilon] = [\sigma, \bar{p}]\). Then, \(\text{top}(\xi_1/v_1) = \text{top}(\xi_2/v_2)\) implies 
\((\xi_1/v_1)\Theta_{\bar{p}} = (\xi_2/v_2)\Theta_{\bar{p}}\). 

Now we show that \(M'\) has the FV property with \(\rho\) by using the above claim. Let \((q_1, I_1), (q_2, I_2) \in Q', \sigma \in \Sigma^{(k)}, k \geq 0, \) and \(p_1, \ldots, p_k \in P\). Let \(\zeta_i = \text{rhs}_{M'}((q_i, I_i), \sigma')\) for \(i \in [2]\) where \(\sigma' = [\sigma, p_1, \ldots, p_k]\). Let \(w_1 \in V(\zeta_1)\) and \(w_2 \in V(\zeta_2)\) such that \(\zeta_1[w_1] = (q, I, x_i)\) and \(\zeta_2[w_2] = (q', I', x_i)\). Let \(j \in [\text{rk}_{M'}((q, I))]\) and \(j' \in [\text{rk}_{M'}((q', I'))]\) such that \(\rho((q, I), j) = \rho((q', I'), j') = j''\). Let \(\xi_i = \text{rhs}_{M'}(q_i, \sigma)\) for \(i \in [2]\). Let \(\psi_i\) be the mapping defined by the property [1] of the claim. By property [1], \(\psi_1(w_1, j)\) and \(\psi_2(w_2, j')\) are important in \(\xi_1\) and \(\xi_2\), respectively, for some trees \(s_1, \ldots, s_k\) such that 
\(B(\sigma(s_1, \ldots, s_k))[\varepsilon] = [\sigma, \bar{p}]\). In addition, we get 
\(\zeta_1/w_1, j = (\xi_1/\psi_1(w_1, j))\Theta_{\bar{p}}\Theta_{I_1}\) and 
\(\zeta_2/w_2, j' = (\xi_2/\psi_2(w_2, j'))\Theta_{\bar{p}}\Theta_{I_2}\). Since \(\Psi_{(q_i, I_i)} = \theta_i^{-1}\) for each \(i \in [2]\), 
\((\zeta_1/w_1, j)\Psi_{(q_1, I_1)} = (\xi_1/\psi_1(w_1, j))\Theta_{\bar{p}}\) and 
\((\zeta_2/w_2, j')\Psi_{(q_2, I_2)} = (\xi_2/\psi_2(w_2, j'))\Theta_{\bar{p}}\). By the fact that \(M\) is consistent, 
\(\text{top}(\xi_1/\psi(w_1, j)) = \text{top}(\xi_2/\psi(w_2, j'))\). By the second property of the claim, we get 
\((\xi_1/\psi_1(w_1, j))\Theta_{\bar{p}} = (\xi_2/\psi_2(w_2, j'))\Theta_{\bar{p}}\). Thus, 
\((\zeta_1/w_1, j)\Psi_{(q_1, I_1)} = (\zeta_2/w_2, j')\Psi_{(q_2, I_2)}\). Hence, \(M'\) has the FV property with \(\rho\).

Before we state our main theorem of this section, let us compare the circular ATT obtained by the construction of [14], and the ATT obtained from the nondeleting \(M'_c\) with the FV property given before Lemma [7]. We apply the construction of Definitions [5] and [6] to the MTT \(M'_c\) and generate the ATT \(A_c = \Omega(\mathcal{E}(M'_c))\). We have \(S = \{q_0, 0, q_0, 1, q_1, 0, q_2, 0, q_2, 0\}\); these are the states of \(M'_c\) (in condensed notation). The inherited attributes of \(A_c\) are
$I = \{1\}$ ($y_1$ is abbreviated as 1). The set $R_{[\sigma,p_1,p_1/p_2]}$ contains the rule

$q_00(\pi) \rightarrow \text{top}(e_{q_00}(\text{rhs}_{M_c}(q_00, [\sigma,p_1,p_1/p_2]))) = q_1(\pi_1)$. 

Since the right-hand side of $r_1$ has two parameter trees, we obtain two more rules:

$\rho(q_1, 1)(\pi_1) \rightarrow \text{top}(e_{q_00}(\langle q_21, x_2 \rangle(\langle q_10, x_1 \rangle))) = q_21(\pi_2)$

and $1(\pi_2) \rightarrow \text{top}(e_{q_00}(\langle q_10, x_1 \rangle)) = q_10(\pi_1)$. The dependency graph for the input tree $\sigma(e, e)$ depicted in Figure 3 shows the full set of rules $R_{[\sigma,p_1,p_1/p_2]}$. Note that for the input tree $\sigma(e', e)$ which causes the construction of [14] for $M_c$ to generate a circular ATT, the corresponding dependency graph for $A_c$ looks rather innocent, see Figure 4.

By Corollary 1 and Lemma 7, and the facts that $B-REL \subseteq T^{R-REL}$ and
that $B\text{-REL}$ and $T^R\text{-REL}$ are closed under composition we obtain the main result of this section.

**Theorem 1.** $ATT^R = MTT^R_C = MTT^R_{FV}$ and $ATT^U = MTT^U_C = MTT^U_{FV}$.

### 4. Dynamic FV Property

We present a new property that characterizes ATT with regular look-around in terms of MTTs with regular look-ahead. The dynamic FV property is a strict generalization of the FV property. The idea is to require that during any derivation the argument trees of any state should be semantically equal.

In the following, let $M = (Q, \Sigma, \Delta, q_0, R)$ be an MTT.

**Definition 7.** Let $q \in Q$, $s \in T_\Sigma$, and $u \in V(s)$. The set of call trees of $q$ at $u$ on $s$ is defined as $ct_M(s,u,q) = \{\xi/v \mid v \in V(\xi), \xi[v] = \langle q, x \rangle\}$ where $\xi = M_{q_0}(s[u \leftarrow x])$.

**Definition 8.** The MTT $M$ has the dynamic FV property on a set $L$ of trees if for every $s \in L$, $u \in V(s)$, $q \in Q^{|m|}$, $j \in [m]$, and $\xi_1, \xi_2 \in ct_M(s,u,q)$, $\xi_1 / j[s/u] = \xi_2 / j[s/u]$ where $[s/u] = [\langle q', x \rangle \leftarrow M_{q'}(s/u) \mid q' \in Q]$. We say that an MTT$_{DFV}^R$ (resp. MTT$_{DFV}^U$) $E; M$, where $E$ is a relabeling and $M$ is an MTT, has the dynamic FV property if $M$ has the dynamic FV property on the image $E(T_\Sigma)$.

We denote by MTT$_{DFV}^R$ and MTT$_{DFV}^U$ the class of all translations realized by MTT$_R$s and MTT$_U$s with the dynamic FV property, respectively. As example we consider the MTT $M_{\text{dyn}}$ which does not have the FV property.
with any parameter renaming mapping $\rho$ but has the dynamic FV property. These are the rules of $M_{dyn}$:

\[
\begin{align*}
\langle q_0, a(x_1) \rangle & \rightarrow f(\langle q_1, x_1 \rangle(\langle q_2, x_1 \rangle), \langle q_1, x_1 \rangle(\langle q_3, x_1 \rangle(e))) \\
\langle q_1, a(x_1)(y_1) \rangle & \rightarrow a(\langle q_1, x_1 \rangle(b(y_1))) \\
\langle q_2, a(x_1) \rangle & \rightarrow a(\langle q_2, x_1 \rangle) \\
\langle q_3, a(x_1)(y_1) \rangle & \rightarrow a(\langle q_3, x_1 \rangle(a(y_1))) \\
\langle q_0, e \rangle & \rightarrow e \\
\langle q_2, e \rangle & \rightarrow e \\
\langle q_1, e \rangle(y_1) & \rightarrow y_1 \\
\langle q_3, e \rangle(y_1) & \rightarrow y_1
\end{align*}
\]

The MTT $M_{dyn}$ translates trees of the form $a^n(e)$ to trees of the form $f(t, t)$ where $t = a^{n-1}b^{n-1}a^{2n-2}(e)$. Clearly, $M$ does not satisfy the FV property. On the other hand, the two argument trees of $q_1$ always evaluate to the same trees and hence $M$ satisfies the dynamic FV property. We show that the dynamic FV property is a generalization of the FV property.

**Lemma 8.** $MTT_{FV} \subseteq MTT_{DFV}$.

**Proof.** Let $M = (Q, \Sigma, \Delta, q_0, R)$ be an MTT that has the FV property with $\rho$. We give the following claim to prove the lemma.

**Claim 2.** For every $s \in T_{\Sigma}$, $u \in V(s)$, $q^{(m)} \in Sts_M(\{q_0\}, s, u)$, $m \geq 0$, $j \in [m]$, and $t_1, t_2 \in ct_M(s, u, q)$, it holds that $t_1/j = t_2/j$.

To show that the claim holds, we prove the following property by induction on the length of $u$: for every $u \in \mathbb{N}^*$, $s \in T_{\Sigma}$ such that $u \in V(s)$, $q_1^{(m_1)}, q_2^{(m_2)} \in Sts_M(\{q_0\}, s, u)$ with $m_1, m_2 \geq 0$, $t_1 \in ct_M(s, u, q_1)$, $t_2 \in ct_M(s, u, q_2)$, $j_1 \in$
If \( \rho(q_1, j_1) = \rho(q_2, j_2) \) then \( t_1/j_1 = t_2/j_2 \). Since \( \rho(q, j_1) = \rho(q, j_2) \) implies \( j_1 = j_2 \), we obtain the claim.

It is clear from Claim 2 that \( M \) has the dynamic FV property.

We say that \( M \) is nonerasing if for every \( q \in Q \) and \( \sigma \in \Sigma \), \( \text{rhs}_M(q, \sigma) \notin Y \). The nondeleting and nonerasing normal form can be obtained by the construction given in Lemmas 6.6 and 7.11 in [6]. The construction preserves the dynamic FV property.

**Lemma 9.** Let \( L \subseteq T_\Sigma \) and let \( M = (Q, \Sigma, \Delta, q_0, R) \) be an MTT that has the dynamic FV property on \( L \). Let \( B : M' \) be the nondeleting MTT obtained from \( M \) by the construction given in the proof of Lemma 7. Then \( M' \) has the dynamic FV property on \( B(L) \).

**Proof.** Let \( s \in L \) and \( u \in V(s) \). Let \( s' = s[u \leftarrow x] \) and \( s'' = B(s)[u \leftarrow x] \). Let \( p_0 = h_\sigma(p_1, \ldots, p_k) \) and \( (\sigma, p_1, \ldots, p_k) = B(s)[u] \). Since the construction of the nondeleting normal form is the same as one given in Lemma 6.6 in [6], we get the following claim from Claims 2 and 3 in the proof of Lemma 6.6 in [6].

**Claim 3.** The following two properties holds.

1. \( M'(s'') = M(s')\left[ (q, x) \leftarrow ((q, I), x)(y_{I(1)}, \ldots, y_{I(n)}) \right| I = p_0(q), n = |I| \right] \).
2. \( M'_{(q, I)}(B(s)/u)\theta_I = M_q(s/u) \) for every \( q \in Q \) where \( I = p_0(q) \).

From the above claim, we get the injective mapping \( \psi : V(M'(s'')) \rightarrow V(M(s')) \) such that
Claim 4. Then, the following property trivially holds.

Let $\langle \text{dynamic FV property on } M \rangle$ for every $j \in [m]$, and

if $M'(s'')[w] = \langle (q, I), x \rangle \in \langle Q', \{x\} \rangle$ then $M'(s'')[w] = \langle q, x \rangle$ and $\psi(w_j) = \psi(w)I(j)$ for every $j \in ||I||$.

Then, the following property trivially holds.

Claim 4. $M'(s'')[w] = M(s')/\psi(w)\Theta_{p_0}$ for every $w \in V(M'(s''))$.

Let $w_1, w_2 \in V_{Q',\{x\}}(M'(s''))$ such that $M'(s'')[w_1] = M'(s'')[w_2]$, and let $\langle (q, I), x \rangle = M'(s'')[w_1]$. It follows from the third property of $\psi$ that $M'(s'')[w_1] = M(s')/\psi(w_2) = \langle q, x \rangle \in V_{Q',\{x\}}(M(s'))$. Since $M$ has the dynamic FV property on $L$, $M(s')/\psi(w_1)j[s/u] = M(s')/\psi(w_1)j[s/u]$ for every $j \in [\text{rk}_M(q)]$. By the first claim we get that $M'(s'')[w_1]j'[B(s)/u] = M'(s'')[w_2]j'[B(s)/u]$ for every $j' \in ||I||$. Hence, $M'$ has the dynamic FV property on $B(L)$.

Next, we give a construction of the nonerasing normal form of an MTT in the style of this paper according to the proof of Lemma 7.11 of [6]. Let $M = (Q, \Sigma, \Delta, q_0, R)$ be a nondeleting MTT. For $p_1, \ldots, p_k \subseteq Q^{(1)}$, let $\Theta_{p_1 \ldots p_k} = \langle q', x_i \rangle \mapsto y_i \mid \langle q', x_i \rangle \in \langle Q, X_k \rangle, q' \in p_i \rangle$. We first define a B-REL $B = (2^{Q^{(1)}}, \Sigma, \Sigma_M, 2^{Q^{(1)}}, R_B)$ where $\Sigma_M = \{[\sigma, p_1 \ldots p_k]^{(k)} \mid \sigma \in \Sigma^{(k)}, k \geq 0, p_1, \ldots, p_k \subseteq Q^{(1)} \}$. For $\sigma \in \Sigma^{(k)}$ and $p_1, \ldots, p_k \subseteq Q^{(1)}$, let the rule

$$\sigma(p_1(x_1), \ldots, p_k(x_k)) \rightarrow p([\sigma, p_1 \ldots p_k](x_1, \ldots, x_k))$$
be in \( R_B \) where \( p = \{ q \in Q \mid \text{rhs}_M(q, \sigma) \Theta_{p_1 \cdots p_k} = y_1 \} \). We construct \( M' = (Q, \Sigma, \Delta, q_0, R') \). For \( q \in Q^{(m)} \), \( [\sigma, p_1 \cdots p_k] \in \Sigma^{(k)}_M \), and \( k, m \geq 0 \), let the rule
\[
(q, [\sigma, p_1 \cdots p_k]([x_1, \ldots, x_k]) (y_1, \ldots, y_m) \rightarrow \zeta
\]
be in \( R' \) where \( \zeta = \text{rhs}_M(q, \sigma) \Theta_{p_1 \cdots p_k} \) if \( \text{rhs}_M(q, \sigma) \Theta_{p_1 \cdots p_k} \neq y_1 \), and otherwise \( \zeta = \bot(y_1) \) with \( \bot \in \Delta^{(1)} \).

**Lemma 10.** Let \( L \subseteq T_\Sigma \) and let \( M = (Q, \Sigma, \Delta, q_0, R) \) be a nondeleting MTT that has the dynamic FV property on \( L \). Let \( B ; M' \) be the nondeleting and nonerasing MTT\(^R\) obtained from \( M \) by the construction given above. Then \( M' \) has the dynamic FV property on \( B(L) \).

**Proof.** From the construction, we have the following properties.

1. \( M'(s'') = M(s')[[q, x] \leftarrow y_1 \mid M_q(s/u) = y_1] \).
2. \( M'_q(B(s)/u) = M_q(s/u) \) for every \( q \in Q \) such that \( M_q(s/u) \neq y_1 \).

Similar to the discussion in the last part of the proof of Lemma 9 using a correspondence between \( V(M'(s'')) \) and \( V(M(s')) \), we obtain the lemma by the above properties. \( \square \)

It follows from Lemmas 9 and 10 that for every MTT\(^R\)\(_{DFV}\) (resp. MTT\(^U\)\(_{DFV}\)) there exists a nondeleting and nonerasing MTT\(^R\)\(_{DFV}\) (resp. MTT\(^U\)\(_{DFV}\)) equivalent with it.

Next we show that the tree translation realized by any MTT\(^U\)\(_{DFV}\) can be expressed by an ATT\(^U\).

**Lemma 11.** MTT\(^U\)\(_{DFV}\) \( \subseteq \) ATT\(^U\).
Henceforth, we assume w.l.o.g. that an MTT\textsuperscript{U}\textsubscript{DFV} is nondeleting and non-erasing. We define the notion of reachable state.

**Definition 9.** Let $Q' \subseteq Q$ and $s \in T_{\Sigma}(X)$. For $u \in V(s)$, the set of reachable states of $s$ at $u$ from $Q'$, denoted by $\text{Sts}_M(Q', s, u)$, is the set $\bigcup_{q' \in Q'} \{ q \in Q \mid \exists \xi' \in V(\xi_{q'}) \text{ s.t. } \xi'[u] = \langle q, x \rangle \}$. The idea for proving that for every MTT\textsubscript{DFV} there exists an equivalent ATT\textsuperscript{U} is as follows. The dynamic FV property demands (semantic) equivalence of parameter trees only for those states that are processing a given input node simultaneously. Therefore, we use the top-down relabeling to add to each input node $u$ the information which states are processing $u$, i.e., the set $\text{Sts}_M(\{q_0\}, s, u)$, where $s$ is the input tree. Then, we construct for regular look-around a composition of the bottom-up relabeling of the given MTT\textsuperscript{R}\textsubscript{DFV} and the top-down relabeling. Further, the constructed ATT has a synthesized attribute for each state $q$ of the MTT, and an inherited attribute for each pair $\langle q, j \rangle$, where $j \in [\text{rk}_Q(q)]$. Last but not least, of those parameter trees which are known to be equivalent, we pick one and construct from its “top-part” a rule for the corresponding inherited attribute. The top-part is defined similarly to the definition of $\text{top}$ in Definition 2, except that if we are computing the top-part of a parameter tree of a state appearing in $\text{rhs}_M(q, \sigma)$, then an occurrence of the parameter $y_j$ is now replaced by the inherited attribute $\langle q, j \rangle$. The construction from MTT\textsuperscript{U}’s with the dynamic FV to ATT\textsuperscript{U} following the idea is given in Definition 10.

To illustrate our idea, consider the above example MTT $M_{\text{dyn}}$. We construct an ATT that has a rule set $R[\sigma, Q']$ for each label $\sigma$ and subset $Q'$ of states reachable from $\{q_0\}$ (see Figure 5). Here we focus on $Q' = \{q_0\}$ and
Figure 5: The ATT constructed from the MTT $M_{\text{dyn}}$ with the dynamic FV property

$\sigma = a$. For the synthesized attribute $q_0$ in $Q'$, we construct its rule as the top part of $\text{rhs}_M(q_0, a): q_0(\pi) \rightarrow f(q_1(\pi 1), q_1(\pi 1))$, where state calls $\langle q, x_1 \rangle$ were replaced by $q(\pi 1)$. For the inherited attribute $\langle q_1, 1 \rangle$ which corresponds to the first parameter of $q_1$, we construct the rhs of $\langle q_1, 1 \rangle(\pi 1)$ by choosing a candidate from the first arguments of $\langle q_1, x_1 \rangle$ appearing in $\text{rhs}_M(q', \sigma)$ for $q' \in Q'$. Here we choose $\langle q_2, x_1 \rangle$ and construct the rule $\langle q_1, 1 \rangle(\pi 1) \rightarrow q_2(\pi 1)$, where $\langle q_2, x_1 \rangle$ was replaced by the synthesized attribute $q_2(\pi 1)$. The non-dummy rules of the ATT are shown in Figure 5.

**Definition 10.** Let $H = (P, \Sigma, \Gamma, F, R_H)$ and $M = (Q, \Gamma, \Delta, q_0, R)$ be a $T^R$-REL and an MTT such that $H ; M$ has the dynamic FV property. Let
1. The top-down relabeling $E(M) = (2^Q, \Gamma, \Gamma_M, \{q_0\}, R_E)$ is defined as follows. Let $\Gamma_M = \Gamma \times 2^Q$. For every $Q' \subseteq Q$ and $\sigma \in \Gamma^{(k)}$, let $\sigma' = [\sigma, Q']$ be in $\Gamma_M^{(k)}$ and let $(Q', \sigma(x_1, \ldots, x_k)) \rightarrow \sigma'(\langle Q_1, x_1 \rangle, \ldots, \langle Q_k, x_k \rangle)$ be a rule in $R_E$, where $Q_i = Sts_M(Q', \sigma(x_1, \ldots, x_k), i)$ for $i \in [k]$. It should be clear that for every input tree $s \in T_\Gamma$ and node $u \in V(s)$, $D(s)[u] = [s/u, Sts_M(\{q_0\}, s, u)]$.

2. The ATT $A(M) = (S, I, \Gamma_M, \Delta, q_0, R')$ is defined as follows. $S = Q$, $I = \{(q, j) \mid q \in Q^{(m)}, m \geq 0, j \in [m]\}$, and $R' = \bigcup_{\sigma' \in \Gamma_M} R'_{\sigma'}$ where $R'_{\sigma'}$ is defined below. Let $\sigma' = [\sigma, Q'] \in \Gamma_M$ and $k = \text{rk}_\Sigma(\sigma)$. For every $i \in [k], q \in Q$, and $j \in [\text{rk}_Q(q)]$, we fix trees $\zeta_{q, j}^{\sigma'}$ that will be used to define the $\text{rhs}_A(\sigma', \langle q, j \rangle(\pi_i))$. — Let $F = \emptyset$. We fix an order $q_1, q_2, \ldots, q_n$ on the states in $Q'$. We now traverse, starting with $\nu = 1$ the trees $\zeta = \text{rhs}_M(q_\nu, \sigma)$ in post-order. Whenever a state call $\langle q, x_i \rangle$ (at node $v$ of $\zeta$) is encountered and $\langle q, x_i \rangle \notin F$, then we change $F$ to $F \cup \{\langle q, x_i \rangle\}$ and we define $\zeta_{q, j}^{\sigma'} = \text{top}_{q_\nu}(\zeta/v_j)$ for all $j \in [\text{rk}_Q(q)]$. The function $\text{top}_{q_\nu} : T_{\Delta \cup \{\langle q, x_i \rangle\}}(Y_m) \rightarrow T_\Delta(\{\alpha(\pi_i) \mid \alpha \in S, i \geq 0\} \cup \{\beta(\pi) \mid \beta \in I\})$ for $q \in Q^{(m)}$ is defined as: $\text{top}_{q_\nu}(\zeta)$ is obtained from $\text{top}(\zeta)$ by replacing $y_j \in Y_m$ with $\langle q, j \rangle$. For every $\sigma' = [\sigma, Q'] \in \Gamma_M^{(k)}$ with $k \geq 0$, we construct the set $R'_{\sigma'}$ in the following way:

- For $q \in Q'$, let the rule $q(\pi) \rightarrow \text{top}_{q}(\text{rhs}_M(q, \sigma))$ be in $R'_{\sigma'}$. For $q \in Q - Q'$, let the rule $q(\pi) \rightarrow \bot$ be in $R'_{\sigma'}$.
- For every $i \in [k], q \in Q^{(m)}$, and $j \in [\text{rk}_Q(q)]$, if $\zeta_{q, j}^{\sigma'}$ is defined then we add $\langle q, j \rangle(\pi_i) \rightarrow \zeta_{q, j}^{\sigma'}$ to $R'_{\sigma'}$; otherwise add $\langle q, j \rangle(\pi_i) \rightarrow \bot$ to $R'_{\sigma'}$.

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Henceforth, we assume that MTT_U H ; M has the dynamic FV property where and M is non-deleting and non-erasing. Let \( E = E(M) \), \( E_A = H ; E \), and \( A = A(M) \). To prove Lemma 11, we show the non-circularity of \( A \) on \( E_A(T_\Sigma) \) (Lemma 13) and then the equivalence of \( H ; M \) and \( E_A ; A \) (Lemma 14). Note that the ATT A may not be non-circular on \( T_\Gamma - E_A(T_\Sigma) \). However, by Theorem 15 of [5], we can construct a TR-REL \( E' \) and an ATT \( A' \) such that \( A' \) is non-circular on all input trees and \( \tau_{E'} ; \tau_{A'} = \tau_{E_A} ; \tau_A \). Thus, \( MTT^{U}_{DFV} \subseteq ATT^{U} \) follows the two lemmas.

Let \( H = (P, \Sigma, \Gamma, F, R_H) \), \( M = (Q, \Gamma, \Delta, q_0, R) \), and \( [s] = [(r, x) \leftarrow M_r(s) | r \in Q] \) for \( s \in T_\Gamma \). Let \( A = (S, I, \Gamma_M, \Delta, q_0, R') \). First, we give a proof of the non-circularity of \( A \) on the image of \( E_A \). To prove it, we use the following lemma.

**Lemma 12.** The following properties hold.

1. For every \( s \in T_\Gamma \), \( s_0 \in H(T_\Sigma) \), \( u \in V(s_0) \) such that \( s_0/u = s \), \( \langle q_1, j \rangle \in I \), and \( q_2 \in S \), if there exists a path from \( \langle \langle q_1, j \rangle, \varepsilon \rangle \) to \( \langle q_2, \varepsilon \rangle \) in \( D_A(E_Q(s)) \) where \( Q' = Sts_M(\{q_0\}, s_0, u) \), then \( ct_M(s_0, u, q_1) \not= \emptyset \) and \( size(t_1/j[s]) < size(t_2[s]) \) for every \( t_1 \in ct_M(s_0, u, q_1) \) and \( t_2 \in ct_M(s_0, u, q_2) \).

2. For every \( s \in T_\Gamma \), \( s_0 \in H(T_\Sigma) \), \( u \in V(s_0) \) such that \( s_0/u = s \) and \( s_0[u] = \sigma \in \Gamma^{(k)} \) with \( k > 0 \), \( \langle q_1, j \rangle \in I \), \( a \in S \cup I \), and \( i, i' \in [k] \), if there exists a path from \( \langle \langle q_1, j \rangle, i \rangle \) to \( \langle a, i' \rangle \) in \( D_A(E_Q(s)) \) where \( Q' = Sts_M(\{q_0\}, s_0, u) \), then \( ct_M(s_0, u, q_1) \not= \emptyset \) and

   - if \( a = q_2 \in S \) then \( size(t_1/j[s/i]) < size(t_2[s/i']) \) for every \( t_1 \in ct_M(s_0, u, q_1) \) and \( t_2 \in ct_M(s_0, u, q_2) \).
• if $a = \langle q_2, j' \rangle \in I$ then $\text{size}(t_1/j[s/i]) < \text{size}(t_2/j'[s/i'])$ for every $t_1 \in ct_M(s_0, u, q_1)$ and $t_2 \in ct_M(s_0, u', q_2)$.

PROOF. We prove Statements 1 and 2 by two (nested) inductions on the structure of $s$ as follows:

• Base case: prove that Statement 1 holds for $s \in \Gamma(0)$. Statement 2 holds for $s$ because $k = 0$.

• Induction step I: by using the induction hypothesis of Statement 1 on the subtrees of $s$, prove that Statement 2 holds for $s \in T_\Sigma$ by induction on the length of the path in $D_A(E_Q'(s))$.

• Induction step II: by using the fact that Statement 2 holds for $s \in T_\Gamma$, prove that Statement 1 holds for $s$.

Base case. Let $s = \sigma \in \Gamma(0)$. Let $s_0 \in H(T_\Sigma)$ and $u \in V(s_0)$ such that $s_0/u = s$. Let $Q' = Sts_M(\{q_0\}, s_0, u)$, $\langle q_1, j \rangle \in I$, and $q_2 \in S$. Assume that there exists a path from $\langle (q_1, j), \varepsilon \rangle$ to $\langle q_2, \varepsilon \rangle$ in $D_A(E_Q'(\sigma))$. Since $E_Q'(\sigma) = [\sigma, Q'] \in \Gamma_M(0)$, the path is only a direct edge from $\langle (q_1, j), \varepsilon \rangle$ to $\langle q_2, \varepsilon \rangle$ in $D_A([\sigma, Q'])$. From the construction of rules for labels with rank 0, the edge originates from the rule $q_2(\pi) \rightarrow top_{q_2}(\text{rhs}_M(q_2, \sigma))$ of $A$, and $q_2 \in Q'$. The edge from $\langle (q_1, j), \varepsilon \rangle$ to $\langle q_2, \varepsilon \rangle$ exists only if $\langle q_1, j \rangle(\pi)$ occurs in the rhs of the rule of $A$. By the definition of $top_{q_2}$, inherited attributes with states other than $q_2$ do not occur in the rhs of the rule. Hence, $q_1$ must be equal to $q_2$. Since $q_2 \in Q'$, we get $ct_M(s_0, u, q_1) = ct_M(s_0, u, q_2) \neq \emptyset$. Let $t \in ct_M(s_0, u, q_2)$. Since $t[\varepsilon] = \langle q_2, x \rangle$, $t[s] = \text{rhs}_M(q_2, \sigma)[y_l \leftarrow t[l[s]] \mid l \in [\text{rk}_Q(q_2)]]$. Since $M$ is nonerasing and nondeleting, $\text{size}(t/j[s]) < \text{size}(t[s])$. 

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By the dynamic FV property of $M$, we get $\text{size}(t_1/j[s]) < \text{size}(t_2[s])$ for every $t_1, t_2 \in \text{ct}_M(s_0, u, q_1)$.

**Induction step I.** Let $s = \sigma(s_1, \ldots, s_k)$ and $k > 0$. Let $s_0 \in H(T_\Sigma)$ and $u \in V(s_0)$ such that $s_0/u = s$. Let $Q' = \text{Sts}_M\{s_0, u\}$, $(q_1, j) \in I$, and $a \in S \cup I$. Let $i, i' \in [k]$. Assume that there exists a path from $(q_1, j, i)$ to $(a, i')$ in $D_A(E_Q(s))$. By the definition of $E$, $E_Q(s)[\varepsilon] = [\sigma, Q']$ and $E_Q(s)/i = E_Q(s_i)$ for $i \in [k]$ where $Q_i = \text{Sts}_M(Q', s, i)$.

We can regard the path $w$ from $(q_1, j, i)$ to a node $v \in (S \cup I) \times [k]$ in $D_A(E_Q(s))$ as a sequence of nodes of $D_A([\sigma, Q'])$ such that every two consecutive nodes are connected by an edge of $D_A([\sigma, Q'])$, or a path of a subgraph $D_A(E_{Q_i}(s_i))$ for some $i \in [k]$. We denote by $\rightarrow_\varepsilon$ and $\rightarrow_i$ connections by an edge of $D_A([\sigma, Q'])$ and a path of a subgraph $D_A(E_{Q_i}(s_i))$, respectively. Then, we prove by induction on the length of $w$, by using the induction hypothesis of Statement 1 on the subtrees of $s$, that for every node $v$ in $(S \cup I) \times [k]$ on $w$,

- if $v = (q, j', i')$ where $(q, j') \in I$ and $i' \in [k]$ then $\text{size}(t_1/j[s_i]) < \text{size}(t_2/j'[s_{i'}])$ for every $t_1 \in \text{ct}_M(s_0, ui, q_1)$ and $t_2 \in \text{ct}_M(s_0, ui', q)$.

- if $v = (q, i')$ where $q \in S$ and $i' \in [k]$ then $\text{size}(t_1/j[s_i]) < \text{size}(t_2[s_{i'}])$ for every $t_1 \in \text{ct}_M(s_0, ui, q_1)$ and $t_2 \in \text{ct}_M(s_0, ui', q)$.

**Base case IB.** Let $w = (q_1, j, i) \rightarrow (q, i)$. There is a path from $(q_1, j, \varepsilon)$ to $(q, \varepsilon)$ in $D_A(E_{Q_i}(s_i))$. By the induction hypothesis of Statement 1, we have $\text{ct}_M(s_0, ui, q_1) \neq \emptyset$ and $\text{size}(t_1/j[s_i]) < \text{size}(t_2[s_{i'}])$ for every $t_1 \in \text{ct}_M(s_0, ui, q_1)$ and $t_2 \in \text{ct}_M(s_0, ui, q)$. 

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Induction step I1. Let \( w = (\langle q_1, j \rangle, i) \rightarrow^+ (\langle q', j', i' \rangle, i') \rightarrow_\nu (q, i') \). Since \((\langle q_1, j \rangle, i) \rightarrow^+ (\langle q', j' \rangle, i')\), by induction hypothesis of the inner induction, we have \( ct_M(s_0, ui, q_1) \neq \emptyset \) and \( \text{size}(t_1[j][s_i]) < \text{size}(t_2[j'][s_i]) \) for every \( t_1 \in ct_M(s_0, ui, q_1) \) and \( t_2 \in ct_M(s_0, ui', q') \). Since \((\langle q', j' \rangle, i') \rightarrow_\nu (q, i')\), there is a path from \((\langle q', j' \rangle, \varepsilon \rangle\) to \((q, \varepsilon)\) in \( D_A(E_Q, (s_i)) \). By induction hypothesis of Statement 1, \( ct_M(s_0, ui', q') \neq \emptyset \) and \( \text{size}(t_2[j'][s_i]) < \text{size}(t_3[s_i]) \) for every \( t_2 \in ct_M(s_0, ui', q') \) and \( t_3 \in ct_M(s_0, ui', q) \). Thus, \( \text{size}(t_1[j][s_i]) < \text{size}(t_3[s_i]) \) for every \( t_1 \in ct_M(s_0, ui, q_1) \) and \( t_3 \in ct_M(s_0, ui', q) \).

Induction step I2. Let \( w = (\langle q_1, j \rangle, i) \rightarrow^+ (\langle q', i'' \rangle, \varepsilon \rangle \rightarrow_\varepsilon (\langle q, j' \rangle, i') \). From the construction of rules for the symbol \([\sigma, Q]\), the edge from \((q', i'')\) to \((\langle q, j' \rangle, i')\) originates from the rule \(\{q, j'\}(\pi i') \rightarrow top'_{q'}(\zeta/uj')\) such that \(q'' \in Q', \zeta = \text{rsh}_M(q'', \sigma), \zeta[v] = \{q, x_{v'}\}\) for some \(v \in V(\zeta)\), and \(q(\pi i'')\) occurs in \(top'_{q'}(\zeta/uj')\). By the definition of \(top'_{q'}\), \((q', x_{v'})\) occurs in \(\zeta/uj'\) at some node \(v'\) such that every ancestor of \(v'\) has a symbol in \(\Delta\). Since \(q'' \in Q', ct_M(s_0, u, q'') \neq \emptyset\). Let \( t \in ct_M(s_0, u, q'') \). Let \([\text{rhs}] = [\langle r, p \rangle \leftarrow \text{rsh}_M(r, \sigma) | r \in Q] \). For \(l \in \{i', i''\}\) let \([\emptyset]\) = \([\langle r, x_c \rangle \leftarrow M_r(s/c) | r \in Q, c \in \{k\} - \{l\}]\) and \([l]\) = \([\langle r, x_l \rangle \leftarrow \langle r, x \rangle | r \in Q] \), and let \([s, l] = [\text{rhs}] [\emptyset][l]\). Let \(\eta_v = \{y_l \leftarrow t[l][\text{rhs}] | l \in \text{rk}_Q(q'')\}\). Then, there exists \(\tilde{v} \in V(\eta_v)\) such that \(\eta_v/\tilde{v} = (\zeta/v)[\psi[\iota''][\iota']\] where \(\psi = [y_l \leftarrow t/l][\text{rhs}] | l \in \text{rk}_Q(q'')\]. Since \(M\) is nondeleting, \(\eta_v\) is a subtree of \(M(s_0[ui' \leftarrow x])\). Thus, \(\eta_v/\tilde{v} \in ct_M(s_0, ui', q) \neq \emptyset\). Let \(\eta_v = t[\text{rhs}][\emptyset][\iota''][\iota''\]. By the same argument, \(\eta_v/\tilde{v}' = (\zeta/v)[\psi[\iota''][\iota']\] and there exists \(\tilde{v}' \in V(\eta_v')\) such that \(\eta_v/\tilde{v}' = (\zeta/v)[\psi[\iota''][\iota']\]. Also, \(\eta_v/\tilde{v}' \in ct_M(s_0, ui'', q') \neq \emptyset\). Since \(\emptyset[\iota''][\iota''][s_{v'}] = \emptyset[\iota''][\iota''][s_{v'}]\) and \(M\) is nondeleting, \(\eta_v/\tilde{v}'[s_{v'}]\) is a subtree of \(\eta_v/\tilde{v}'[s_{v'}]\). Thus, \(\text{size}(\eta_v/\tilde{v}'[s_{v'}]) \leq \text{size}(\eta_v/\tilde{v}'[s_{v'}])\). By the dynamic FV property, \(\text{size}(t_2[s_{v'}]) \leq \text{size}(t_3[j'[s_{v'}])\).
for every \( t_2 \in ct_M(s_0, ui'', q') \) and \( t_3 \in ct_M(s_0, ui', q) \). Since \( (q_1, j), i) \rightarrow^+ (q', i'') \), by induction hypothesis of the inner induction, \( ct_M(s_0, ui, q_1) \neq \emptyset \) and \( \text{size}(t_1/j[s_i]) < \text{size}(t_2[s_{i''}]) \) for every \( t_1 \in ct_M(s_0, ui, q_1) \) and \( t_2 \in ct_M(s_0, ui'', q') \). Hence, for every \( t_1 \in ct_M(s_0, ui, q_1) \) and \( t_3 \in ct_M(s_0, ui', q) \), \( \text{size}(t_1/j[s_i]) < \text{size}(t_3/j'[s_{i''}]) \).

**Induction step II.** Let \( s = \sigma(s_1, \ldots, s_k) \) and \( k > 0 \). Let \( s_0 \in B(T_\Sigma) \) and \( u \in V(s_0) \) such that \( s_0/u = s \). Let \( Q' = Sts_M(\{q_0\}, s, u), (q_1, j) \in I \), and \( q_2 \in S \).

Assume that there exists a path \( w \) from \( (q_1, j), \varepsilon \) to \( (q_2, \varepsilon) \) in \( D_A(E_{Q'}(s)) \).

By the definition of \( E \), \( E_{Q'}(s)[\varepsilon] = [\sigma, Q'] \) and \( E_{Q'}(s)/i = E_{Q_i}(s_i) \) for \( i \in [k] \) where \( Q_i = Sts_M(Q', s, i) \).

**Case 1.** Let \( w = (\langle q_1, j \rangle, \varepsilon) \rightarrow_\varepsilon (q_2, \varepsilon) \). By the same argument of the base case, we can show that \( q_1 = q_2 \), \( ct_M(s_0, u, q_1) \neq \emptyset \), and \( \text{size}(t_1/j[s_i]) < \text{size}(t_2[s]) \) for every \( t_1, t_2 \in ct_M(s_0, u, q_1) \).

**Case 2.** Let \( w = (\langle q_1, j \rangle, \varepsilon) \rightarrow_\varepsilon (\langle q', j' \rangle, i) \rightarrow^+ (q'', i') \rightarrow_\varepsilon (q_2, \varepsilon) \). It is sufficient to show that

(i) \( ct_M(s_0, u, q_1) \neq \emptyset \) and \( \text{size}(t_1/j[s_i]) \leq \text{size}(t_2/j'[s_i]) \) for every call trees \( t_1 \in ct_M(s_0, u, q_1) \) and \( t_2 \in ct_M(s_0, ui, q') \),

(ii) \( ct_M(s_0, ui, q') \neq \emptyset \) and \( \text{size}(t_2/j'[s_i]) < \text{size}(t_3[s_{i''}]) \) for every call trees \( t_2 \in ct_M(s_0, ui, q') \) and \( t_3 \in ct_M(s_0, ui', q'') \), and

(iii) \( ct_M(s_0, ui', q'') \neq \emptyset \) and \( \text{size}(t_3[s_{i''}]) \leq \text{size}(t_4[s]) \) for every call trees \( t_3 \in ct_M(s_0, ui', q'') \) and \( t_4 \in ct_M(s_0, u, q_2) \).

Since (ii) can be shown by the induction hypothesis of Statement 2, we will show (i) and (iii) below. Let \([\text{rhs}] = [\langle r, x \rangle \leftarrow \text{rhs}_M(r, \sigma) \mid r \in Q]\), and for \( l \in \{i, i'\} \) let \( [\text{rhs}] = [\langle r, x, c \rangle \leftarrow M_r(s/c) \mid r \in Q, c \in [k] - \{l\}]\),
[\ell] = [\langle r, x_1 \rangle \leftarrow \langle r, x \rangle \mid r \in Q], and [s, l] = [\text{rhs}][\ell][\ell]. \text{ Note that the above substitutions are nondeleting because } M \text{ is nondeleting.}

(i) Since (\langle q_1, j \rangle, \varepsilon) \rightarrow_\varepsilon (\langle q', j' \rangle, i), \text{ from the construction of rules for the symbol } [\sigma, Q], \text{ the edge originates from the rule } \langle q', j' \rangle(\pi i) \rightarrow \text{top}_q'((\zeta/vj')) \text{ such that } q_1 \in Q', \zeta = \text{rhs}_M(q_1, \sigma), \zeta[v] = \langle q', x_i \rangle \text{ for some } v \in V(\zeta), \text{ and } \langle q_1, j \rangle(\pi) \text{ occurs in } \text{top}_q'((\zeta/vj')). \text{ By the definition of } \text{top}_q', y_j \text{ occurs in } \zeta/vj' \text{ at some node } v'. \text{ Since } q_1 \in Q', \text{ ct}_M(s_0, u, q_1) \neq \emptyset. \text{ Let } t_1 \in \text{ct}_M(s_0, u, q_1) \text{ and } \eta = t_1[s, i]. \text{ Since } t_1[\varepsilon] = \langle q_1, x \rangle, \eta = \zeta\psi[\iota][\iota] \text{ where } \psi = [y_i \leftarrow t_1/l[\text{rhs}] \mid l \in [\text{rk}_Q(q_1)]]]. \text{ There exists } \tilde{\nu} \in V(\eta) \text{ such that } \eta/\tilde{\nu} = (\zeta/v)\psi[\iota][\iota] \text{ and } \eta/\tilde{\nu}j'' = (\zeta/vj'')\psi[\iota][\iota] = t_1/j[\text{rhs}][\iota][\iota] \text{ because } y_j = \zeta[vj'']. \text{ Then, } \eta[\tilde{\nu}] = \langle q', x \rangle. \text{ Thus, } \eta/\tilde{\nu} \in \text{ct}_M(s_0, u_i, q') \neq \emptyset. \text{ Let } t_2 = \eta/\tilde{\nu}. \text{ Since } t_2/j'' = t_1/j[s, i], t_2/j''[s_i] = t_1/j[s, i][s_i] = t_1/j[s]. \text{ Thus, } \text{size}(t_1/j[s]) = \text{size}(t_2/j''[s_i]) \leq \text{size}(t_2/j'[s_i]). \text{ By the dynamic FV property of } M, (i) \text{ holds.}

(iii) Since (q'', i') \rightarrow_\varepsilon (q_2, \varepsilon), \text{ from the construction, the edge originates from the rule } q_2(\pi) \rightarrow \text{top}_{q_2}'(\zeta) \text{ where } \zeta = \text{rhs}_M(q_2, \sigma), \text{ and } q_2 \in Q'. \text{ By the definition of } \text{top}_{q_2}', \zeta[v] = \langle q'', x_i \rangle \text{ for some } v \in V(\zeta) \text{ and every ancestor of } v \text{ has a symbol in } \Delta. \text{ Since } q_2 \in Q', \text{ ct}_M(s_0, u, q_2) \neq \emptyset. \text{ Let } t_3 \in \text{ct}_M(s_0, u, q_2) \text{ and } \eta = t_3[s, i']. \text{ Since } t_3[\varepsilon] = \langle q_2, x \rangle, \eta = \zeta\psi[\iota'][\iota'] \text{ where } \psi = [y_i \leftarrow t_3/l[\text{rhs}] \mid l \in [\text{rk}_Q(q_2)]]]. \text{ There exists } \tilde{\nu} \in V(\eta) \text{ such that } \eta/\tilde{\nu} = (\zeta/v)\psi[\iota'][\iota']. \text{ Then, } \eta[\tilde{\nu}] = \langle q'', x \rangle. \text{ Thus, } \eta/\tilde{\nu} \in \text{ct}_M(s_0, u_i', q'') \neq \emptyset. \text{ Since } \eta[s_{i'}] = t_3[s, i'][s_{i'}] = t_3[s], \text{size}(\eta[\tilde{\nu}[s_{i'}]]) \leq \text{size}(\eta[s_{i'}]) = \text{size}(t_3[s]). \text{ By the dynamic FV property of } M, (iii) \text{ holds.} \quad \square

\textbf{Lemma 13.} \ A \text{ is non-circular on } E_A(T_\Sigma).

\textbf{Proof.} \text{ The proof is done by contradiction. Assume that } A \text{ is circular on}
$E_A(T_\Sigma) = E(H(T_\Sigma))$. Then there exists a tree $s \in H(T_\Sigma)$ such that $D_A(E(s))$ has a cycle. Let $u \in V(s)$ be a node such that $E(s)/u$ is a minimal subtree that includes the cycle. Then $s/u \in T_1 - \Gamma^{(0)}$ because no cycle can be made in $D_A(\sigma')$ for any $\sigma' \in \Gamma^{(0)}$. Let $\sigma = s[u] \in \Gamma^{(k)}$ with $k > 0$. Let $Q' = Sts_M(\{q_0\}, s, u)$. By the definition of $E$ and $A$, $Q' \neq \emptyset$ because if $Q' = \emptyset$ then there were no edges between any two nodes on $D_A(E(s))/u$. Then, there is a cycle of length greater than 0 from some $(q, j)$ to itself in $D_A(E_{Q'}(s/u))$ for some $(q, j) \in I$ and $i \in [k]$. By Lemma 12, $\text{size}(t/j[s/u\ell]) < \text{size}(t/j[s/\ell\ell])$ for any $t \in ct_M(s, u, q)$. This is a contradiction. \hfill \square

We show the equivalence of $H ; M$ and $E_A ; A$.

**Lemma 14.** $\tau_H ; \tau_M = \tau_H ; \tau_E ; \tau_A$.

**Proof.** For $s \in H(T_\Sigma)$, $u \in V(s)$, and $q \in Q'$ where $Q' = Sts_M(\{q_0\}, s, u)$, let

$$[s, u, q]_M = [y_l \leftarrow t/l[s/u] \mid t \in ct_M(s, u, q), l \in [\text{rk}_Q(q)]]$$

$$[s, u]_A = [(q, l)(\varepsilon) \leftarrow t/l[s/u] \mid q \in Q', t \in ct_M(s, u, q), l \in [\text{rk}_Q(q)]]$$

Note that by the dynamic FV property of $M$ on $H(T_\Sigma)$, $[s, u, q]_M$ and $[s, u]_A$ are well-defined. We prove the following statements for every $s \in T_\Gamma$.

1. For every $s_0 \in H(T_\Sigma)$, $u \in V(s)$ such that $s_0/u = s$, and $q \in Q'$ where $Q' = Sts_M(\{q_0\}, s_0, u)$,

$$M_q(s)_M[s_0, u, q] = nf(\Rightarrow_{A,E_{Q'}}(s, q(\varepsilon))[s_0, u]_A.$$  

2. For every $s_0 \in H(T_\Sigma)$, $u \in V(s)$ such that $s_0/u = s$ and $s_0[u] = \sigma \in \Sigma^{(k)}$ with $k \geq 0$, $q \in Q'$ where $Q' = Sts_M(\{q_0\}, s_0, u)$, and $v \in V(\zeta_q)$,

$$(\zeta_q/v)_\theta[s_0, u, q]_M = nf(\Rightarrow_{A,E_{Q'}}(s_0, \topp_{q'}(\zeta_q/v)[\pi \leftarrow \varepsilon])_M[s_0, u]_A$$  

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where \( \zeta_q = \text{rhs}_M(q, \sigma) \) and \( \theta = \left[ \langle r, x_i \rangle \leftarrow M_r(s/i) \mid r \in Q, i \in [k] \right] \).

We first prove that Statement 2 implies Statement 1 for all \( s \in T_\Gamma \).

\((\Rightarrow 1)\). Let \( s = \sigma(s_1, \ldots, s_k) \in T_\Gamma \) where \( \sigma \in \Gamma^{(k)} \), \( k \geq 0 \), and \( s_1, \ldots, s_k \in T_\Gamma \). Let \( s_0 \in H(T_\Sigma) \) and \( u \in V(s) \) such that \( s_0/u = s \), and \( s' = E_{Q'}(s) \). Let \( q \in \text{Sts}_M(\{q_0\}, s_0, u) \). Let \( \xi = \text{rhs}_M(q, \sigma) \) and \( \theta = \left[ \langle r, x_i \rangle \leftarrow M_r(s/i) \mid r \in Q, i \in [k] \right] \). We assume that Statement 2 holds for \( s \).

\[
\begin{align*}
M_q(s)[s_0, u, q]_M \\
= \xi \theta[s_0, u, q]_M \\
= nf(\Rightarrow_{A, s'}, \text{top}'_q(\xi)[\pi \leftarrow \varepsilon])[s_0, u]_A \quad \text{(by Statement 2 with } v = \varepsilon) \\
= nf(\Rightarrow_{A, s'}, q(\pi)[\pi \leftarrow \varepsilon])[s_0, u]_A \quad \text{(by the construction)}.
\end{align*}
\]

By using the above fact, we prove that Statement 2 holds for all \( s \in T_\Gamma \) by induction on the structure of \( s \).

**Base case.** Let \( s = \sigma \in \Gamma^{(0)} \). Let \( s_0 \in H(T_\Sigma) \) and \( u \in V(s_0) \) such that \( s_0/u = s \). Let \( Q' = \text{Sts}_M(\{q_0\}, s_0, u) \) and \( q \in Q' \). Let \( m = \text{rk}_Q(q) \). Let \( \zeta = \text{rhs}_M(q, \sigma) \), and let \( v \in V(\xi) \) and \( \xi = \zeta/v \). Since \( E_{Q'}(\sigma) = [\sigma, Q'] = \sigma' \in \Gamma^{(0)}_M \) and \( q \in Q' \), from the construction of rules for labels with rank 0, the edge originates from the rule \( q(\pi) \rightarrow \text{top}'_q(\xi) \) of \( A \). By the definition of \( \text{top}'_q \) and
\[ \xi \in T_{\Delta \cup Y_m}, \]

\[
\begin{align*}
\text{nf}(\Rightarrow_{A,s'}, \text{top}_q'(\xi)[\pi \leftarrow \varepsilon])[s_0, u]_A &= \text{nf}(\Rightarrow_{A,s'}, \xi[y_t \leftarrow \langle q, l \rangle(\varepsilon) \mid l \in [m]])[s_0, u]_A \\
&= \xi[y_t \leftarrow \langle q, l \rangle(\varepsilon) \mid l \in [m]][s_0, u]_A \\
&= \xi[y_t \leftarrow \langle q, l \rangle(\varepsilon) \mid l \in [m]] \\
&= \xi[y_t \leftarrow \langle q, l \rangle(\varepsilon) \mid l \in [m]](\xi \leftarrow (q', l))_0[s_0, u]_A \\
&= \xi[y_t \leftarrow \langle q, l \rangle(\varepsilon) \mid l \in [m]](\xi \leftarrow (q', l))_0[s_0, u]_A \\
&= \xi[s_0, u, q]_M.
\end{align*}
\]

Since \( \theta \) is an empty substitution when \( k = 0 \), \( \xi[s_0, u, q]_M = \xi\theta[s_0, u, q]_M \).

**Induction step.** Let \( \sigma' = [\sigma, Q'] \). We denote by \( \preceq \) the traverse order on the nodes of \( \text{rhs}_M(q, \sigma) \) for all \( q \in Q' \) when defining \( t'_{i,q,j} \) in the construction.

Let \( c(i, q) \) be the state \( q'' \) such that \( t'_{i,q,j} \) (for all \( j \in \text{rk}_q(q) \)) is picked up from \( \text{rhs}_M(q'', \sigma) \). We prove this part by induction on the total order \( \preceq \). Let \( q \in Q' \) and \( v \in V(\zeta_q) \), and let \( \xi = \zeta_q/v \). Let us denote by (IH2) the induction hypothesis of the outer induction on \( s \), by (IH\( \preceq \)) that of the inner induction on \( \preceq \), and by (IH1) the fact that Statement 1 holds for \( s \), implied by (IH2) and \( 2 \implies 1 \).

**Case 1.** \( \xi = y_j \). It is trivial from the fact that \( \text{top}_q'(y_j) = \langle q, j \rangle(\varepsilon) \).

**Case 2.** \( \xi = \delta(\xi_1, \ldots, \xi_i) \). We get \( \xi\theta[s_0, u, q]_M = \delta(\xi'_1, \ldots, \xi'_i) \) where \( \xi'_i = \xi_i\theta[s_0, u, q]_M \), and \( \text{nf}(\Rightarrow_{A,s'}, \text{top}_q(\xi)[\pi \leftarrow \varepsilon])[s_0, u]_A = \delta(\xi''_1, \ldots, \xi''_i) \) where \( \xi''_i = \text{nf}(\Rightarrow_{A,s'}, \text{top}_q'(\xi_i)[\pi \leftarrow \varepsilon])[s_0, u]_A \). Since \( \xi'_i = \xi''_i \) by (IH\( \preceq \)), this case holds.
Case 3. $\xi = (q', x_i)(\xi_1, \ldots, \xi_m)$.

\[
\xi \theta[s_0, u, q]_M = M_{q'}(s/i)[s_0, u, q']_M \\
= nf(\Rightarrow_{A,s'/i, q'(\varepsilon)})[s_0, u]_A \quad \text{(by IH1)}.
\]

On the other hand,

\[
[s_0, u]_A \\
= [(q', l)(\varepsilon) \leftarrow t/l[s/i] \mid q' \in Q_i, t \in ct_M(s_0, u, q'), l \in [rk_Q(q')]] \\
= [(q', l)(\varepsilon) \leftarrow t'_{i,q',\theta}[s_0, u, q'']_M \mid q' \in Q_i, c(i, q') = q'', l \in [rk_Q(q')]] \\
= [(q', l)(\varepsilon) \leftarrow nf(\Rightarrow_{A,s', top_{q''}(t'_{i,q',\xi})}[\pi \leftarrow \varepsilon])[s_0, u]_A \\
\quad \mid q' \in Q_i, c(i, q') = q'', l \in [rk_Q(q')]] \quad \text{(by IH2)} \\
= [(q', l)(\varepsilon) \leftarrow nf(\Rightarrow_{A,s', (q', l)(\pi i)[\pi \leftarrow \varepsilon]} \mid q' \in Q_i, l \in [rk_Q(q')])[s_0, u]_A.
\]

Hence,

\[
nf(\Rightarrow_{A,s'/i, q'(\varepsilon)})[s_0, u]_A \\
= nf(\Rightarrow_{A,s'/i, q'(\varepsilon)}) \\
\quad [(q', l)(\varepsilon) \leftarrow nf(\Rightarrow_{A,s', (q', l)(\pi i)[\pi \leftarrow \varepsilon]} \mid q' \in Q_i, l \in [rk_Q(q')])[s_0, u]_A \\
= nf(\Rightarrow_{A,s', q'(\pi i)[\pi \leftarrow \varepsilon})[s_0, u]_A \\
= nf(\Rightarrow_{A,s', top'_{q'}(\xi)[\pi \leftarrow \varepsilon})[s_0, u]_A.
\]

When $s_0 = s$ and $u = \varepsilon$, $M_{q_0}(s)[s, \varepsilon, q_0]_M = nf(\Rightarrow_{A,s', q_0(\varepsilon)}[s, \varepsilon]_A$. Since $rk_Q(q_0) = 0$ and thus $[s, \varepsilon, q_0]_M$ and $[s, \varepsilon]_A$ are empty substitutions, $M_{q_0}(s) = nf(\Rightarrow_{A,s', q_0(\varepsilon)}). \quad \square$

Next we show that $MTT_{DFV}$ is closed under pre-composition with $T-REL$. 

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**Lemma 15.** For a composition of a T-REL $E$, an MTT $M$, and a set $L$ of trees, if $M$ has the dynamic FV property on $E(L)$, then there exists an MTT $M'$ equivalent with $E;M$ such that $M'$ has the dynamic FV property on $L$.

**Proof.** Let $L \subseteq T_\Sigma$, and let $E = (Q_E, \Sigma, \Gamma, r_0, R_E)$ and $M = (Q, \Gamma, \Delta, q_0, R)$ be an T-REL and an MTT such that $M$ has the dynamic FV property on $E(L)$. We can get an MTT equivalent with $E;M$ by the product construction. Let $M' = (Q', \Sigma, \Delta, (r_0, q_0), R')$ where $Q' = Q_E \times Q$ be the MTT obtained by the product construction. Note that $\text{rk}_{M'}((r, q)) = \text{rk}_M(q)$ for $(r, q) \in Q'$. Here we show only that $M'$ has the dynamic FV property on $L$.

Let $s \in L$ and $u \in V(s)$. Let $\xi = M'(s[u \leftarrow x])$ and $\langle r, x \rangle = E(s[u \leftarrow x])$[u]. We have the following properties from the product construction.

1. $\xi = M(E(s)[u \leftarrow x])\langle q, x \rangle \leftarrow \langle (r, q), x \rangle$ if $q \in Q$.
2. $M'_{(r,q)}(s/u) = M_q(E_r(s/u))$.

Let $v_1, v_2 \in V_{(Q', (x))}(\xi)$ such that $\xi[v_1] = \xi[v_2]$. Then, it follows from the first property that $\xi[v_1] = \xi[v_2] = \langle (r, q), x \rangle$ for some $q \in Q$, and that $M(E(s)[u \leftarrow x])[v_1] = M(E(s)[u \leftarrow x])[v_2] = \langle q, x \rangle$. By the dynamic FV property of $M$ on $E(L)$, $(M(E(s)[u \leftarrow x])/v_1 j)\langle q', x \rangle \leftarrow M_q(E(s/u)| q' \in Q] = (M(E(s)[u \leftarrow x])/v_2 j)\langle q', x \rangle \leftarrow M_q(E(s/u)| q' \in Q] \quad \text{for every} \quad j \in [\text{rk}_M(q)]$. It follows from the above properties that $\langle (\xi/v_1 j), x \rangle \leftarrow M'_{(r,q')}^j(s/u)| q' \in Q] = \langle (\xi/v_2 j), x \rangle \leftarrow M_{(r,q')}^j(s/u)| q' \in Q] \quad \text{for every} \quad j \in [\text{rk}_M((r, q))]$. Therefore, $M'$ has the dynamic FV property on $L$. $\Box$

Theorem 1 and Lemmas 8, 11 and 15 yield the main result of this section.

**Theorem 2.** $\text{ATT}^U = \text{MTT}^R_{DFV} = \text{MTT}^U_{DFV}$
Let us consider the decidability of the dynamic FV property. While the FV property is easily decidable, we do not know how to decide the dynamic FV property; in fact, we are able to show that this problem is at least as difficult as deciding equivalence of ATTs. The proof constructs from two given ATTs an MTT\(^R\) which has the dynamic FV property if and only if the ATTs are equivalent. This is done by nesting the start calls for the ATTs under a new fixed state.

**Theorem 3.** Deciding the dynamic FV property for \(MTT^R\) is at least as hard as deciding equivalence of ATTs.

**Proof.** We show that the equivalence problem of ATTs can be reduced to the decision problem of the dynamic FV property for \(MTT^R\). Let \(A_1\) and \(A_2\) be ATTs. From Lemmas 7 and 8, for \(i = 1, 2\), we can get a composition of a B-REL \(E_i\) and an MTT \(M_i = (Q_i, \Sigma_i, \Delta, q_{i0}, R_i)\) such that it is equivalent with \(A_i\) and it has the dynamic FV property. We can assume with loss of generality that \(Q_1 \cap Q_2 = \emptyset\). Let \(\Sigma_M = \{((\sigma_1, \sigma_2)^{(k)} | \sigma_1 \in \Sigma_1^{(k)}, \sigma_2 \in \Sigma_2^{(k)}\}\}

We assume that \(\Sigma_M^{(1)} \neq \emptyset\), \(e \in \Delta^{(0)}\), and \(\delta \in \Delta^{(2)}\).

Let \(E\) be the relabeling such that for every input tree \(t\) \(E(t)\) is the convolution tree of \(E(t_1)\) and \(E_2(t)\), i.e., \(E(t)[u] = (E_1(t)[u], E_2(t)[u])\) for every \(u \in V(t)\). Let \(M = (Q_1 \cup Q_2 \cup \{q_0^{(0)}, q_0^{(1)}\}, \Sigma_M, \Delta, q, R_1' \cup R_2' \cup R)\) where \(q, q'\) are new distinct states not in \(Q_1 \cup Q_2\) such that for every \(a \in \Sigma_M^{(1)}\) let the rule

\[
\langle q_0, a(x_1) \rangle \rightarrow \delta((q', q_1^1), (q_{10}, q_{11})), (q', q_1)\]

be in \(R\). Let the rule \(\langle q_0, \sigma(x_1, \ldots, x_k) \rangle \rightarrow e\) be in \(R\) for every \(\sigma \in \Sigma_M^{(k)}\) with \(k \neq 1\), and let the rule \(\langle q', \sigma(x_1, \ldots, x_k) \rangle \rightarrow y_1\) be in \(R\) for every \(\sigma \in \Sigma^{(k)}\)
with \( k \geq 0 \). For \( q \in Q_i \), let the rule \( \langle q, [\sigma_1, \sigma_2](x_1, \ldots, x_k) \rangle \to \text{rhs}_{M_i}(q, \sigma_i) \) be in \( R'_i \). Since \( E_1 ; M_1 \) and \( E_2 ; M_2 \) have the dynamic FV property and \( Q_1 \cap Q_2 = \emptyset \), it follows that \( M \) has the dynamic FV property if and only if \( M_1(E_1(s)) = M_{q_{10}}(E(s)) = M_{q_{20}}(E(s)) = M_2(E_2(s)) \) for every \( s \in T_\Sigma \). \( \square \)

5. Conclusions

We have presented two new characterizations of attributed tree transformation with regular look-around in terms of macro tree transducers (with regular look-around or look-ahead): first a static restriction that is similar as the one given by Fülöp and Vogler [14], but is extended to nondeleting MTTs with regular look-ahead. We show that for every MTT with the restriction, an equivalent non-circular ATT can be constructed. Our second restriction (called dynamic FV property) requires that during any computation, all \( j \)-th parameter trees of a given state evaluate to the same output tree. This restriction captures many more MTTs than previous restrictions, however, it remains an open problem how to decide this restriction.

One may wonder if every MTT that has the \textbf{LIN} property (cf. the Introduction) also has the dynamic FV property. Alas, this is not the case:

\[
\begin{align*}
\langle q_0, \#(x_1) \rangle & \to \langle q, x_1 \rangle(e) \\
\langle q, a(x_1) \rangle(y_1) & \to f(f(\langle q, x_1 \rangle(y_1), \langle q, x_1 \rangle(y_1)), \langle q, x \rangle(f(y_1, y_1))) \\
\langle q, e \rangle(y_1) & \to y_1
\end{align*}
\]

After \( n \) applications of the second rule, state \( q \) has \( n + 1 \) distinct parameter trees. Hence the dynamic FV property is violated. Note that this MTT translates monadic trees into full binary trees (this can be done by a simple top-down or bottom-up transducer). We would like to know, if there is a
normal form that guarantees the dynamic FV property for every MTT with the \textbf{LIN} property.

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