Analysis of normality of quotient subgroups on matrices integers modulo prime

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Abstract. A set of matrices integers modulo prime forms a finite group. This group has trivial and nontrivial subgroups. The trivial subgroup is normal and for a non-trivial subgroup the normal properties will be investigated. Furthermore, if a nontrivial subgroup is a normal subgroup, then a quotient subgroup can be constructed. This paper discusses the characteristics of normality of quotient subgroup on matrices integers modulo prime. By study of literatures, the order of group play an important role in this quotient subgroup. We derive some characteristics of normality of quotient subgroup. The result is that the order of subgroup determine on the normality of a quotient subgroup.

1. Introduction

In matrices integers modulo prime has interesting characteristic when we derive some quotient subgroups. Because this matrices has nontrivial matrices which is normal, so we can construct the quotient subgroups [1]. discusses on subgroups of finite groups by quotient concept groups by presented the algorithm using basic operation. Group of matrices become finite group if the set of this matrices is finite. In special set of 2x2 matrices integers modulo prime, when we choose prime number equal 2, we have the number of the elements of set is 6 and for prime number equal 3 we get 48 element. By multiplication matrix, we can construct finite group. In general, this set is finite depends on the value of prime number [2]. If we extend the set become 3x3 of matrices integers modulo prime, for the same prime number, the number of the elements become 168 elements [3]. Furthermore, we have the number of element in general order in this set [4]. The subgroups of this matrices is discussed in Hadi where the normality of this group cannot determined directly [5]. One of the problems is because the numbers of the elements probably determine which subgroup will normal. Moreover, the order of subgroups will give influence to the normality of subgroup.

In this paper, we evaluate some normal subgroups from group of matrices integers modulo prime and construct of quotient subgroup of this normal subgroup. The discussion focus in this set, which are the special cases different from [6]. We investigated the normality of subgroup by looking the cases start from simple order of matrices. Some theory of finite group and subgroup will be used to derived and comparing with another result such as in Kuzmanovich [7].
2. Materials and method

2.1. Materials

Let \( GL_n(Z_p) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{L} & a_{nn} \end{pmatrix} \bigg| a_{ij}, a_{mn} \in Z_p, \det \begin{pmatrix} a_{11} & a_{12} & \mathbf{L} & a_{1n} \\ a_{21} & a_{22} & \mathbf{L} & a_{2n} \\ \mathbf{M} & \mathbf{M} & \mathbf{O} & \mathbf{M} \\ a_{n1} & a_{n2} & \mathbf{L} & a_{nn} \end{pmatrix} \neq 0 \right\} \) be a set of \( n \times n \) non singular matrices integers modulo prime. Under matrix multiplication this set form a finite group depends on the number of \( n \) and \( p \). For example, if we choose \( n = 2 \), and \( p = 2 \) (\( p \) is a prime number), we have

\[
G_{22} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| a, b, c, d \in \mathbb{Z}_2, ad - bc \neq 0 \right\}
\]

Is a noncommutative finite group where the order is 6. Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, F = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \) then we have the table below represent finite group of matrix under multiplication where \( A \) is identity element.

**Table 1.** Matrix operation under \( GL_2(Z_2) \).

| *  | A   | B   | C   | D   | E   | F   |
|----|-----|-----|-----|-----|-----|-----|
| A  | A   | B   | C   | D   | E   | F   |
| B  | B   | A   | D   | C   | F   | E   |
| C  | C   | F   | A   | E   | D   | B   |
| D  | D   | E   | B   | F   | C   | A   |
| E  | E   | D   | F   | B   | A   | C   |
| F  | F   | C   | E   | A   | B   | D   |

From the table, we can define invers element of each matrix, \( A^{-1} = A, B^{-1} = B, C^{-1} = C, D^{-1} = F, E^{-1} = E, F^{-1} = D \). If we choose another number \( n \) and \( p \), we have large number of the set and it is cumbersome to define the table. It is interesting to define nontrivial subgroup which will construct to quotieny subroups.

2.2. Method

In this paper, the discussion the normality of quotient subgroups on matrices integers modulo prime is shown in Figure 1. First we define a finite group of matrices integers modulo prime. This set is an spesial form of \( GL_n(Q) \), the set of \( n \times n \) matrices. By theory of group, it is clear that this group always trivial and non trivial subgroup. Trivial subgroup is always normal, so we can define quotient group directly, but in nontrivial subgroup is not always normal so that we should find some condition to construct normal subgroup. After we define normal subgroup, from quotient group we define another subgroup from quotient group to investigate the normality of this subgroup. The order of subgroup is important to define because it depends the number of elements of subgroups, such that the quotient subgroup can be constructed. For the first step, we choose simple \( 2 \times 2 \) matrices and investigated the characteristic
associate with the normality. Furthermore, we see different forms of matrices get the special characteristic.

\[ \text{Figure 1. Scenario of normality of quotient subgroups on matrices integers modulo.} \]

3. Result and discussion

In case of \( \text{Gl}_2(Z) \), the trivial subgroup are \( A \) and \( \text{Gl}_2(Z) \) itself. By Lagrange Theorem and the definition of normal subgroup, we can construct \( \text{Gl}_2(Z)/A = \text{Gl}_2(Z) \) and \( \text{Gl}_2(Z)/\text{Gl}_2(Z) \) are quotient group where the order is similar i.e six elements. Note that \( \text{Gl}_2(Z) \) have nontrivial subgroup that is \( \{A, D, F\} \) so that we can define a quotient group \( \text{Gl}_2(Z)/\{A, D, F\} \). See Herstein [8], by this lemma below,

**Lemma.** If \( G \) is a finite group and \( N \) is a normal subgroup of \( G \), then \( o(G/N) = o(G)/o(N) \)

we get \( o(G/N) = o(G)/o(N) = 6/3 = 2 \) (let \( N = \{A, D, F\} \)). From this, we can write \( G/N = \{NA, ND\} = \{NA, NF\} \). An interesting fact of \( \text{Gl}_2(Z) \) that nontrivial subgroups \( \{A, B\}, \{A, C\}, \{A, E\} \) are not normal (where the order of subgroup is exactly two), so that \( \text{Gl}_2(Z)/\{A, B\}, \text{Gl}_2(Z)/\{A, C\}, \text{Gl}_2(Z)/\{A, E\} \) are not quotient groups. All of nontrivial subgroups of \( \text{Gl}_2(Z) \) consist of identity element and the other element which is own invers (the element equal the invers itself). By this fact, we can generalize that the subgroup which consist only two identity element and the element with own invers is not normal. Let \( \text{Gl}_2(Z, 2x2 \text{ matrices integer modulo 3}) \), we have 48 elements. It means that, for identity element \( E \in \text{Gl}_2(Z, 3) \) and for some \( X \in \text{Gl}_2(Z, 3) \) where \( X = X^{-1} \), then \( \{E, X\} \) is not normal. But if \( P, Q \in \text{Gl}_2(Z, 3) \) where \( Q = P^{-1} \) or \( Q^{-1} = P \) then \( \{E, P, Q\} \) is not normal, in general.

Let \( H = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} : a_i \in Z_3, a_{ij} \neq a_{jk} \right\} \), by [4] \( H \) normal in \( \text{Gl}_2(Z, 3) \).

Another subgroup is \( Z(\text{Gl}_3(Z, p)) = \left\{ z \in \text{Gl}_3(Z, p) \mid \exists x = xz \text{ for all } x \in \text{Gl}_3(Z, p) \right\} < \text{Gl}_3(Z, p) \). Then there
is non trivial subgroup is normal. So $Gl_2(Z_p) / H$ and $Gl_3(Z_p) / Z(Gl_3(Z_p))$ are a quotient group. Another example of normal subgroup are $D_3(Z_p) = \left\{ \begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & u \end{bmatrix} : s, t, u \in Z_p \right\}$ only for $p = 2$ and $GL_n(Z_p) = \left\{ \begin{bmatrix} a_{11} & a_{12} & L & a_{1n} \\ 0 & a_{22} & L & a_{2n} \\ 0 & 0 & O & M \\ 0 & 0 & L & a_{nn} \end{bmatrix} : a_{ij}, K, a_{nn} \in Z_p \right\}$. So far, we can find nontrivial normal subgroup of matrices integer modulo prime. The order of quotient subgroup determine the normality of subgroup. These results are differents approach about matrix groups compare with [1,6,7,9-12], which only focus in special forms of some matrices in $GL_n(Z_p)$. In this paper, we have shown only give special examples of normal subgroup. The notion is investigated only to special form of matrices. The general form of $GL_n(Z_p)$ is needed to discuss. The open problem is, in general, how we can determine the order or subgroup.

4. Conclusion
This paper shows that the order of subgroup determined the normality of quotient subgroup. The value of order should be found exactly for each subgroup so that the order of quotient subgroup can be determined. One of the open problem is procedure to find the order of group. The extended characteristic of quotient subgroups can be derived by using theory of finite group.

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