Monoidal functional dependencies

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Abstract
We propose and prove completeness of logic for reasoning with functional dependencies (FDs) with semantics defined by general non-idempotent aggregation functions. Our approach is based on the idea of preserving similarity of attribute values and allows us to express and reason with stronger relationships between attribute values than the ordinary FDs. In our setting, the FDs not only express that certain values are determined by others but also express that similar values of attributes imply similar values of other attributes, formalizing a type of continuity of FDs. We show that in order to handle such rules, it is sufficient to interpret FDs over partially ordered monoidal structures instead of Boolean algebras which are implicitly used for the ordinary FDs. We present syntax and interpretation of the rules over classes of commutative integral partially ordered monoids and complete residuated lattices. The main result shows complete axiomatization of the semantic entailment by Armstrong-like axioms. We also comment on the related computational issues, the relational vs. propositional semantics of the monoidal FDs, and the relationship to the ordinary FDs.

1 Introduction

Rank-aware approaches in database systems [25] represent a popular alternative to traditional database systems which consider answers to queries as sets of

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objects (e.g., sets of tuples of values in relational systems). In contrast, rank-aware databases represent query results as sets of objects together with scores. The role of scores is to express degrees to which objects match queries. The primary interpretation of scores is comparative—higher scores represent better matches. Most of the existing rank-aware approaches focus on issues related to efficient query evaluation in order to show only $k$ best matches (results with $k$ best scores) to a query. The various approaches differ in how they achieve this goal, see [25] for a survey.

In this paper, we study a logic for a new type of dependencies that appear in particular rank-aware database systems. Namely, we are interested in approaches which (i) evaluate atomic queries as sets of objects with scores and (ii) express scores in results of composed conjunctive queries by applying monotone aggregation functions to the scores obtained from evaluating the subqueries. In fact, these are particular types of queries which appear in the influential paper of R. Fagin [16] (cf. also [17]) dealing with monotone query evaluation. In sense of [16], answers to a query like

\[
\text{LOCATION} = "\text{Byron St}" \land \text{AREA} = 2,400 \land \text{PRICE} = 800,000 \quad (1)
\]

which represents a request for houses in (or near) Byron St, with floor size of (or about) 2,400 square feet, and sold at $800,000 (or similar price) are determined by evaluating all three subqueries for each object (a house for sale) in the database, obtaining three scores. Then, the three scores for each object are aggregated by a monotone function to get the score for the object in the result of the conjunctive query (1). In the same way as users may be interested only in the best few answers to queries like (1), we may argue that maintainers of the database may be interested in imposing constraints which take the scores (i.e., degrees of matches) into account. For instance,

\[ (\text{LOCATION} \land \text{AREA}) \Rightarrow \text{PRICE} \quad (2) \]

is syntactically an ordinary FD but we can give it a new semantics from the point of view of the scores and the aggregation function: A relation $r$ satisfies (2) if for
any two tuples in \( r \), similar values of locations and similar values of areas imply similar prices. For any two tuples \( r_1 \) and \( r_2 \), we may formalize the condition as

\[
(r_1(LOCATION) \approx r_2(LOCATION)) \otimes (r_1(AREA) \approx r_2(AREA)) \leq r_1(PRICE) \approx r_2(PRICE),
\]

where \( \otimes \) is the monotone aggregation function which interprets the conjunction denoted above as \& and \( \approx \) assigns to any two values \( d_1 \) and \( d_2 \) of the same type a score which is the result of atomic query \( d_1 = d_2 \). Let us note that \( \leq \) in (3) is used to interpret the material implication. This reflects the fact that in the classical propositional logic, a formula \( \varphi \Rightarrow \psi \) is true under evaluation \( e \) iff the truth value of \( \varphi \) under \( e \) is less than or equal to the truth value of \( \psi \) under \( e \).

In (3), we have just applied this principle to scores instead of the logical 0 and 1 (which may be seen as two borderline scores). In general, (3) represents a stronger relationship than that represented by the ordinary FD semantics: the condition can be violated if two tuples have close values of locations and area but considerably larger difference between prices. In this sense, the illustrative formula (2) can be seen as a constraint in a rank-aware database, ensuring that houses of similar properties (locations and area) should be offered for similar prices, thus avoiding unwanted situations of underpriced or overpriced offers.

The approach in [16] of efficient query execution relies on aggregation functions defined on the real unit interval which are monotone and strict. Typically, triangular norms are used for the job but [16] is even more general (it has been exploited in various approached which are not truth functional, cf. [9]). We consider more general structures than those defined on the real unit interval. In order to interpret (2) as in (3), it suffices to have a set \( L \) of scores which can be compared by a partial order relation \( \leq \) on \( L \) and with \( 1 \in L \) being the highest score (representing a full match). Moreover, we need an aggregation function \( \otimes \) which should be associative and commutative (because the bracketing and the order of the conjunctive subqueries should not matter) with \( 1 \) being its neutral element. In addition, \( \otimes \) should be monotone w.r.t. \( \leq \) which ensures that better matches of subqueries yield higher scores in the result. These conditions imply
the condition of strictness from [16]. Altogether, we base our considerations on structures of scores which are in fact partially ordered Abelian monoids from which comes the term “monoidal FDs” (shortly, an MFD).

In this paper, we primarily focus on logic for reasoning with formulas like (2) which is different from the logic for reasoning with FDs, because we interpret the formulas over general monoidal structures and not Boolean algebras. For instance, $\otimes$ is not idempotent in general (on $L = [0,1]$ with its natural ordering, the only idempotent $\otimes$ is the minimum). In practice this means that the number of occurrences of propositional variables (i.e., the names of attributes in database terminology) in formulas matters and it enables us to express weaker or stronger relationships between attributes. For illustration,

$$(\text{LOCATION} \& \text{AREA} \& \text{AREA}) \Rightarrow \text{PRICE}$$

is a formula which prescribes a weaker constraint than (2) because the truth value of its antecedent (under a given evaluation) is in general lower than (or equal to) the truth value of the antecedent of (2) (under the same evaluation). Thus, if (2) is satisfied then so is (4) but not vice versa in general. Analogously,

$$(\text{LOCATION} \& \text{AREA}) \Rightarrow (\text{PRICE} \& \text{PRICE})$$

prescribes a stronger constraint than (2). Indeed, the truth value of its consequent is in general lower than or equal to the truth value of the consequent of (2), i.e., if (5) is satisfied then so is (2) but not vice versa in general. So, the very presence of non-idempotent conjunctions allows us to put more/less emphasis on similarity-based constraints. Let us also note that [16] considers general non-idempotent functions interpreting $\&$ as well. As a result, writing $\text{AREA} = 2,400$ twice in a query like (1) changes the meaning of the query by putting more emphasis on the area being close to the specified value and the query may produce a different result. So, accepting non-idempotent interpretations of $\&$ in rank-aware approaches to query evaluation as in [16] or data dependencies as we present here should not be surprising, cf. also [28] for an informal discussion on topics related to non-idempotent conjunctions.
Using a different technique than is usual in the ordinary case, we establish a complete axiomatization of our logic which resembles the well known Armstrong rules \[1\]. This makes our approach different from other approaches which tackle similar issues but focus almost exclusively on idempotent conjunctions; we present more details on the relationship to other approaches in Section \[4\]. A survey and a comparison of relevant approaches in this direction can be found in \[4\].

Our approach is not limited only to the database (relational) semantics. In fact, we start with a propositional semantics over monoidal structures and later prove that there is a relational semantics which yields the same notion of semantic entailment (and thus have the same axiomatization). This is analogous to \[15\] (cf. also \[11\]) which shows that the logic of the classic FDs is in fact a particular propositional fragment. In this sense, the logic of MFDs we describe in the paper is a particular propositional fragment of Höhle’s monoidal logic \[24\].

In much the same way as the classic functional dependencies, MFDs serve two basic purposes. First, they can be used as formulas prescribing constraints. Second, they can be used as formulas derived from database instances, describing dependencies that hold in data. While the first role may be expected and is traditionally studied in databases, the second one seems to be of equal importance and is more related to data analysis and data mining. Our paper offers a sound and complete logic system which can be used as a formal basis for both types of problems.

The paper is organized as follows. In Section \[2\] we present preliminaries from partially ordered structures we utilize in the paper. In Section \[3\] we present the syntax and semantics of our logic and in Section \[4\] we prove its completeness. In Section \[5\] we deal with related computational issues. In Section \[6\] we discuss the relationship between two possible interpretations of formulas used in this paper. Finally, in Section \[7\] we present conclusions and open problems.
2 Preliminaries

We assume that readers are familiar with the basic notions of partially ordered sets (posets) and lattices. A partially ordered monoid (shortly, a pomonoid) is a structure \( L = \langle L, \leq, \otimes, 1 \rangle \) where \( \langle L, \otimes, 1 \rangle \) is a monoid (i.e., a semigroup with neutral element 1), and \( \leq \) is a partial order on \( L \) so that \( \otimes \) is monotone w.r.t. \( \leq \): If \( a \leq b \), then \( a \otimes c \leq b \otimes c \) and \( c \otimes a \leq c \otimes b \). Furthermore, if 1 is the greatest element of \( L \) w.r.t. \( \leq \), then \( L \) is called integral pomonoid. In the paper, we work mostly with integral commutative pomonoids (i.e., \( \otimes \) is in addition commutative). Given \( L, a \in L \) and non-negative integer \( n \), we define the \( n \)th power \( a^n \) of \( a \) by putting \( a^0 = 1 \) and \( a^{n+1} = a \otimes a^n \) for each natural \( n \).

Related structures which appear in various substructural logics are residuated lattices \cite{12, 36}. An integral commutative residuated lattice (shortly, a residuated lattice) is a structure \( L = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle \) such that \( \langle L, \wedge, \vee, 0, 1 \rangle \) is a bounded lattice, \( \langle L, \leq, \otimes, 1 \rangle \) is an integral commutative pomonoid (\( \leq \) is the lattice order from \( L \), i.e., \( a \leq b \) iff \( a = a \wedge b \)), and \( \rightarrow \) satisfies, for all \( a, b, c \in L \), \( a \otimes b \leq c \) iff \( a \leq b \rightarrow c \) (so-called adjointness property). The operations \( \otimes \) (called a multiplication) and \( \rightarrow \) (called a residuum) serve as general interpretations of logical connectives “conjunction” and “implication”. In addition, \( L \) is called complete if \( \langle L, \wedge, \vee, 0, 1 \rangle \) is a complete lattice. The adjointness property ensures that \( \otimes \) and \( \rightarrow \) are general enough and still have desirable properties—the important role of the adjointness condition in logics has been discovered by J. A. Goguen \cite{20}. We mention here one property that is relevant to this paper: As a consequence of the adjointness, \( a \leq b \) iff \( a \rightarrow b = 1 \) (easy to see). The class of residuated lattices is definable by identities and therefore it forms a variety. The variety has interesting subvarieties, including a subvariety which is term-equivalent to the variety of Boolean algebras. In particular, \( L = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle \), where \( L = \{0, 1\} \), \( \otimes = \wedge \), and \( \wedge, \vee, \rightarrow \) are truth functions of the classic conjunction, disjunction, and implication, respectively, is the structure of truth degrees of the classic propositional logic \cite{29}. Most widely known multiple-valued (fuzzy) logics based on subclasses of resid-
uated lattices are BL [22] and MTL [13] which are the logics of all continuous and left-continuous triangular norms [26], respectively. More details on residuated structures and their role in logic and relational systems may be found in [3, 6, 18, 37], cf. also recent edited book [8].

3 Monoidal Functional Dependencies:

Syntax and Semantics

In this section, we formalize the rules, present their interpretation, and introduce an inference system for deriving rules from sets of other rules. From the logical point of view, our rules are implications between two formulas containing conjunctions of propositional variables which can occur in the formulas multiple times. This is in contrast to the classic FDs where the number of occurrences does not matter—what matters is whether a propositional variable (in database systems called an attribute) is present in the formula or not. Consequently, classic FDs are often presented as implications between sets of propositional variables which simplifies many considerations on FDs including their axiomatization and computation of closures.

In our setting, we cannot make such simplification because conjunctions are interpreted by aggregation functions which are not idempotent in general. On the other hand, the functions are still commutative and associative. Therefore, we can disregard the order in which propositional variables appear in formulas and the bracketing. We may therefore introduce the following notation: If \( \text{Var} \) is a denumerable set of propositional variables, we consider maps of the form

\[
A: \text{Var} \rightarrow \mathbb{Z}
\]

satisfying both of the following conditions:

1. \( A(p) \geq 0 \) for all \( p \in \text{Var} \),
2. \( \{ p \in \text{Var}; A(p) > 0 \} \) is finite.

The maps can be seen as finite multi-subsets of \( \text{Var} \) and we use them to formalize antecedents and consequents of if-then formulas. In particular, we denote by \( \top \)
a map of the form (6) such that $\top(p) = 0$ for all $p \in \text{Var}$, i.e., $\top$ can be seen as an empty multi-subset of $\text{Var}$.

Now, by a monoidal functional dependency (shortly, an MFD), we mean any expression of the form $A \Rightarrow B$, where both $A, B$ are of the form (6). Clearly, such rules can be seen as shorthands for formulas like (2); in this case, $A(\text{LOCATION}) = 1, A(\text{AREA}) = 1, B(\text{PRICE}) = 1$, and $= 0$ otherwise.

MFDs are interpreted with respect to evaluations of propositional variables which assign to each propositional variable an element from the support of an integral commutative pomonoid. The situation is fully analogous to evaluations in the classic case which assign to propositional variables two logical values 0 and 1.

Formally, let $L = \langle L, \leq, \otimes, 1 \rangle$ be an integral commutative pomonoid. An L-evaluation (shortly, an evaluation if $L$ is clear from context) is any map $e: \text{Var} \rightarrow L$. Each evaluation can be uniquely extended to all maps (6): For $A$ of the form (6), we define $e(A) \in L$ as follows

$$e(A) = e(p_1)^{A(p_1)} \otimes \cdots \otimes e(p_n)^{A(p_n)},$$

(7)

where $\{p \in \text{Var}; A(p) > 0\} \subseteq \{p_1, \ldots, p_n\}$. Recall from the preliminaries that the powers which appear in (7) are considered with respect to the monoidal operation $\otimes$ in $L$. Thus, $e(p_1)^{A(p_1)}$ means $e(p_1)$-multiplied by itself $A(p_1)$-times. Also note that by definition, we get $a^0 = 1$. Thus, the value of (7) depends only on variables $p \in \text{Var}$ such that $A(p) > 0$. As a special case, we have $e(\top) = 1$ because 1 is neutral with respect to $\otimes$.

For $A \Rightarrow B$ and L-evaluation $e$, we say that $A \Rightarrow B$ is satisfied under $e$, written $e \models A \Rightarrow B$ whenever $e(A) \leq e(B)$, where $\leq$ is the partial order in $L$. Furthermore, $A \Rightarrow B$ is called an L-tautology if it is satisfied in any L-evaluation (with $L$ fixed); $A \Rightarrow B$ is called trivial if it is L-tautology for any $L$.

We now introduce semantic entailment of MFDs in terms of models. Suppose that $\text{Var}$ is fixed. A set $\Gamma$ of MFDs is called a theory (over $\text{Var}$). Each L-evaluation $e$ such that $e \models A \Rightarrow B$ for all $A \Rightarrow B \in \Gamma$ is called an L-model of $\Gamma$. An MFD $A \Rightarrow B$ is semantically entailed by $\Gamma$, written $\Gamma \models A \Rightarrow B$, if
e \models A \Rightarrow B$ for any $L$-model $e$ of $\Gamma$ with $L$ being any integral commutative pomonoid.

**Remark 1.** Our notion of the semantic entailment is not dependent on a particular choice of $L$ because $\Gamma \models A \Rightarrow B$ iff for any $L$ and any $L$-evaluation $e$, we get $e(A) \leq e(B)$. Also note that our logic is consistent in that each theory has an $L$-model $e$ for any $L$ (take $e(p) = 1$ for all $p \in \text{Var}$).

In the paper we show that $\models$ can be characterized syntactically. In case of MFDs, the need for a syntactic characterization of $\models$ seems to be more important than in the case of classic FDs because the semantic entailment, by its definition, involves checking $e \models A \Rightarrow B$ over all $L$-models where $L$ ranges over all integral commutative pomonoids which is a proper class of algebras. In contrast, the entailment of FDs can be checked by efficient linear-time algorithms \cite{2}.

In the inference rules introduced below, we use the following notation. For maps $A, B$ of the form (6), we define a map $AB$ : $\text{Var} \rightarrow \mathbb{Z}$ by

$$(AB)(p) = A(p) + B(p)$$

for any $p \in \text{Var}$. In addition, we put $A^0 = \top$ and $A^{n+1} = AA^n$ for any natural $n$ and call $A^n$ the $n$th power of $A$. Obviously, \{$p \in \text{Var}; (AB)(p) > 0$\} is a finite set and therefore $AB$ as well as $A^n$ are maps of the form (6). Our use of maps like (8) is analogous to the set-theoretic union which is used in inference rules for the classic FDs.

In our logic, we consider the following two inference rules:

(\text{Ax}) infer $AB \Rightarrow B$,

(\text{Cut}) from $A \Rightarrow B$ and $BC \Rightarrow D$ infer $AC \Rightarrow D$,

where $A, B, C, D$ are arbitrary maps (6). As usual, a sequence $\varphi_1, \ldots, \varphi_n$ of MFDs is called a proof of $\varphi_n$ by a theory $\Gamma$ if each $\varphi_i$ is in $\Gamma$ or is derived from $\varphi_1, \ldots, \varphi_{i-1}$ by (Ax) or (Cut). Notice that (Ax) is in fact a nullary rule (an axiom scheme) which derives $AB \Rightarrow B$ from no input formulas. In this sense,
(Cut) is the only (non-trivial) inference rule in our system which infers new formulas from existing ones. In database literature [28], the classic counterpart of (Cut) is often called pseudotransitivity. An MVD $A \Rightarrow B$ is called provable by $\Gamma$, written $\Gamma \vdash A \Rightarrow B$, if there is a proof of $A \Rightarrow B$ by $\Gamma$.

Remark 2. (a) For convenience, we may write (Ax) and (Cut) in a “fraction notation” like

\[
\frac{AB \Rightarrow B}{\text{(Ax)}}, \quad \frac{A \Rightarrow B, \ BC \Rightarrow D}{AC \Rightarrow D} \quad \text{(Cut)},
\]

and write proofs by $\Gamma$ as trees with leaves corresponding to formulas in $\Gamma$ and internal nodes given by instances of (Ax) and (Cut).

(b) Let us note that complete systems of inference rules for the classic FDs (Armstrong systems [1]) are usually presented in less compact way using (Ax) (sometimes called the axiom of reflexivity) and the following rules

\[
\frac{A \Rightarrow B, \ B \Rightarrow C}{A \Rightarrow C} \quad \text{(Tra)}, \quad \frac{A \Rightarrow B}{AC \Rightarrow BC} \quad \text{(Aug)},
\]

instead of (Cut). This can also be done in our case. Indeed, (Tra) is a particular case of (Cut) for $C = \top$ and (Aug) results by (Cut) from $A \Rightarrow B$ and $BC \Rightarrow BC$ which is an instance of (Ax). Conversely, in order to show that (Cut) is derivable from (Tra) and (Aug), observe that

\[
\frac{A \Rightarrow B, \ BC \Rightarrow D}{AC \Rightarrow BC} \quad \text{(Aug)}, \quad \frac{A \Rightarrow B}{AC \Rightarrow BC} \quad \text{(Tra)}.
\]

Let us note that even if (Ax) and (Cut) as well as the other rules look syntactically similar to their classic counterparts, the rules do not operate on implications between sets of attributes and, therefore, represent different rules. In general, (Ax) and (Cut) in our logic are weaker rules than their set-theoretic counterparts. For instance, our system admits the following weaker form of additivity:

\[
\frac{A \Rightarrow B, \ BC \Rightarrow BC}{AC \Rightarrow BC} \quad \text{(Ax)}, \quad \frac{A \Rightarrow C, \ AC \Rightarrow BC}{AA \Rightarrow BC} \quad \text{(Cut)}.
\]
i.e., a rule which from $A \Rightarrow B$ and $A \Rightarrow C$ infers $AA \Rightarrow BC$ but in general, the ordinary-style additivity \cite{28} which infers $A \Rightarrow BC$ from $A \Rightarrow B$ and $A \Rightarrow C$ is not sound and thus not derivable in our logic as we shall see in the next section.

(c) In Section 5 we utilize an alternative system of inference rules which resemble the classic B-axioms \cite{28, page 52}. Namely, we consider the following rules of reflexivity, rewriting, and projectivity:

\[
\frac{\quad}{A \Rightarrow A}^{(\text{Ref})}, \quad \frac{A \Rightarrow BC, C \Rightarrow D}{A \Rightarrow BD}^{(\text{Rwt})}, \quad \frac{A \Rightarrow BC}{A \Rightarrow B}^{(\text{Pro})}.
\]

Note that the original B-axioms use (Ref) and (Pro) together with the rule of accumulation which infers $A \Rightarrow BCD$ from $A \Rightarrow BC$ and $C \Rightarrow DE$. As in the case of additivity, we can show that accumulation in this form is not derivable in our system. Our rule (Rwt) may be seen as a weaker form of the accumulation and its name reflects the fact that $C$ appearing in $A \Rightarrow BC$ is replaced by $D$ and is not kept in the derived formula $A \Rightarrow BD$. The inference rules (Ref), (Rwt), and (Pro) are equivalent to (Ax) and (Cut). Indeed, (Ref) is an instance of (Ax), (Rwt) is obtained by (Aug) and (Tra), and (Pro) is obtained by (Ax) and (Tra). Conversely, (Ax) is obtained by (Ref) and (Pro), and (Cut) is obtained by (Ref) and (Rwt) applied twice:

\[
\frac{AC \Rightarrow AC}{AC \Rightarrow BC}^{(\text{Rwt})}, \quad \frac{BC \Rightarrow D}{AC \Rightarrow D}^{(\text{Rwt})},
\]

showing that (Ax) and (Cut) are equivalent to (Ref), (Rwt), and (Pro).

4 Completeeness

We start investigating soundness and completeness of the inference system with respect to the semantic entailment introduced in the previous section. First, note that directly from (7),

\[
e(AB) = e(A) \otimes e(B)
\]
for any maps \(A, B\) like (6). As a consequence, \(e(A^n) = e(A)^n\). Our first observation identifies trivial MFDs and instances of (Ax). In its proof, we use a special notation for writing particular maps of the form (6). Namely, for \(p \in \text{Var}\), we consider \(\alpha_p\) such that

\[
\alpha_p(q) = \begin{cases} 
1, & \text{if } p = q, \\
0, & \text{otherwise}, 
\end{cases}
\]  

(10)

for all \(q \in \text{Var}\). Note that for any \(L\)-evaluation \(e\) and any \(p \in \text{Var}\), we have \(e(p) = e(\alpha_p)\). Therefore, if there is no danger of confusing propositional variables and maps of the form (6), we write just \(p, q, \ldots\) to denote \(\alpha_p, \alpha_q, \ldots\), and the like. This allows us to write, e.g., \(ppq\) as an abbreviation for \(\alpha_p \alpha_p \alpha_q\) and we have \(e(ppq) = e(p) \otimes e(p) \otimes e(q) = e(\alpha_p \alpha_p \alpha_q)\) according to (7) and (8).

**Theorem 1.** \(A \Rightarrow B\) is trivial iff \(A \Rightarrow B\) is an instance of (Ax).

**Proof.** Consider an \(L\)-evaluation \(e\). We get \(e(AB) = e(A) \otimes e(B) \leq 1 \otimes e(B) = e(B)\). Indeed, the first equality comes from (9); the next inequality is a consequence of the monotony of \(\leq\) and the fact that 1 is the greatest element of \(L\); and the last equality follows from the fact that 1 is the neutral element of \(\otimes\). Hence, \(e(AB) \leq e(B)\) yields \(e \models AB \Rightarrow B\), i.e., instances of (Ax) are trivial.

Conversely, we find an \(L\)-model which satisfies only the trivial MFDs. Let \(L = \langle L, \leq, \cdot, \top \rangle\) be a structure where \(L\) is the set of all maps (6) for fixed \(\text{Var}\), \(\cdot\) is a binary operation defined by \(A \cdot B = AB\) as in (8), and \(A \leq B\) iff \(B(p) \leq A(p)\) for all \(p \in \text{Var}\). Clearly, \(L\) is an integral commutative pomonoid. In addition, consider \(L\)-evaluation \(e\) such that \(e(p) = \alpha_p\) with \(\alpha_p\) defined as in (10). It is easily seen that \(e\) extends to all maps like (6) so that \(e(A) = A\) for any \(A \in L\). Now, if \(A \Rightarrow B\) is not an instance of (Ax), then there is \(p\) such that \(A(p) < B(p)\) and thus \(e(A) = A \not\leq B = e(B)\), showing \(e \not\models A \Rightarrow B\).

**Theorem 2** (soundness). If \(\Gamma \vdash A \Rightarrow B\) then \(\Gamma \models A \Rightarrow B\).

**Proof.** Assume that \(e \models A \Rightarrow B\) and \(e \models BC \Rightarrow D\) for \(L\)-evaluation \(e\). It means \(e(A) \leq e(B)\) and \(e(BC) = e(B) \otimes e(C) \leq e(D)\). Thus, utilizing the monotony
of $\otimes$ and the transitivity of $\leq$, $e(A) \otimes e(C) \leq e(D)$, meaning that $e(AC) \leq e(D)$ which proves $e \models AC \Rightarrow D$. The rest follows by induction on the length of a proof, utilizing Theorem 1.

Remark 3. Take $L = \langle [0,1], \leq, \otimes, 1 \rangle$ which is the commutative monoid of reals restricted to the interval $[0,1]$ with $\leq$ and $\otimes$ being the genuine ordering and multiplication of reals, respectively. Take $L$-evaluation $e$ such that $e(p) = 0.5$ and $e(q) = e(r) = 0.6$. Thus, $e(p) = 0.5 \leq 0.6 = e(q)$ and analogously for $p$ and $r$. On the other hand, $e(p) \not\leq 0.36 = 0.6 \otimes 0.6 = e(qr)$. Therefore, $e \models p \Rightarrow q$, $e \models p \Rightarrow r$, and $e \not\models p \Rightarrow qr$, showing that the classic rule of additivity is not derivable in our system, cf. Remark 2(b). In a similar way, one can show that $p \Rightarrow qrs$ is not provable by $\{p \Rightarrow qr, r \Rightarrow st\}$ and thus the classic rule of accumulation is not derivable in our system (consider $e$ such that $e(q) = e(t) = 1$ and $e(p) = e(r) = e(s) = 0.6$), cf. Remark 2(c).

The classic proof of completeness of inference rules for the classic FDs involves closures of sets of attributes and exploits the property that for each $A \subseteq R$ (where $R$ is a finite set of attributes) the set $\{B \subseteq R; \Gamma \vdash A \Rightarrow B\}$ has a greatest element with respect to $\subseteq$. This property no longer holds in our case (hint: see the previous Remark). Nevertheless, we are able to prove strong completeness (for general infinite $\Gamma$) by a technique which involves construction of a model from equivalence classes based on provability by $\Gamma$. The procedure in the proof of the following theorem can be seen as construction of the Lindenbaum algebra [35] for a logic with a restricted set of formulas which only take form of implications between conjunctions of propositional variables.

Theorem 3 (completeness). $\Gamma \vdash A \Rightarrow B$ iff $\Gamma \models A \Rightarrow B$.

Proof. The only-if part follows by Theorem 2. We prove the if-part indirectly. Assuming that $\Gamma \not\vdash A \Rightarrow B$, we find an $L$-model $e$ of $\Gamma$ such that $e(A) \not\leq e(B)$.

Let $L$ denote the set of all maps of the form (11) for a fixed denumerable $\text{Var}$ such that all propositional variables which occur in all formulas in $\Gamma$ are
contained in Var. Furthermore, consider the commutative monoid \( \langle L, \cdot, \top \rangle \) as in the proof of Theorem 1 (the partial order is not considered at this point). The monoid is further used to express the desired model of \( \Gamma \) in which \( A \Rightarrow B \) is not satisfied.

Define binary relation \( \equiv_{\Gamma} \) on \( L \) as follows: \( E \equiv_{\Gamma} F \) iff \( \Gamma \vdash E \Rightarrow F \) and \( \Gamma \vdash F \Rightarrow E \). We claim that \( \equiv_{\Gamma} \) is a congruence relation on \( \langle L, \cdot, \top \rangle \). In order to see that, we must check that \( \equiv_{\Gamma} \) is equivalence and is compatible with \( \cdot \) from \( \langle L, \cdot, \top \rangle \). Obviously, \( \equiv_{\Gamma} \) is reflexive because of (Ax) and is symmetric by its definition. Since (Tra) is a special case of (Cut), we can also conclude that \( \equiv_{\Gamma} \) is transitive, i.e., it is an equivalence relation. Now, assume that \( E \equiv_{\Gamma} F \) and \( G \equiv_{\Gamma} H \). We have

\[
\begin{align*}
G \Rightarrow H, \\
F \Rightarrow FH \quad \text{(Ax)}
\end{align*}
\]

\[
\begin{align*}
E \Rightarrow F, \\
FH \Rightarrow FH \quad \text{(Cut)}
\end{align*}
\]

\[
\begin{align*}
FG \Rightarrow FH \quad \text{(Cut)}
\end{align*}
\]

\[
\begin{align*}
EG \Rightarrow FH \quad \text{(Cut), (11)}
\end{align*}
\]

i.e., from \( \Gamma \vdash E \Rightarrow F \) and \( \Gamma \vdash G \Rightarrow H \), it follows that \( \Gamma \vdash EG \Rightarrow FH \). Dually, \( \Gamma \vdash F \Rightarrow E \) and \( \Gamma \vdash H \Rightarrow G \) yield \( \Gamma \vdash FH \Rightarrow EG \), showing \( EG \equiv_{\Gamma} FH \).

Therefore, \( \equiv_{\Gamma} \) is a congruence relation and we may consider the quotient algebra \( L / \Gamma \) of \( L \) modulo \( \equiv_{\Gamma} \). In a more detail, \( L / \Gamma = \langle L / \Gamma, \circ, [\top]_\Gamma \rangle \), where \( L / \Gamma \) consists of all the equivalence classes \([\cdot \cdot \cdot]_\Gamma\) of \( \equiv_{\Gamma} \), \( [E]_\Gamma \circ [F]_\Gamma = [EF]_\Gamma \), and \([\top]_\Gamma \) is the equivalence class containing \( \top \). Since commutative monoids form a variety \( [37] \), \( L / \Gamma = \langle L / \Gamma, \circ, [\top]_\Gamma \rangle \) is also a commutative monoid. In addition, it can be equipped with a relation \( \leq_{\Gamma} \) as follows: We put \([E]_\Gamma \leq_{\Gamma} [F]_\Gamma \) whenever \( \Gamma \vdash E \Rightarrow F \). Again, using (Ax) and (Cut), it follows that \( \leq_{\Gamma} \) is a partial order on \( L / \Gamma \) and its definition does not depend on the choice of elements from the equivalence classes—this is easy to see, we omit details. Moreover, \( \circ \) is monotone with respect to \( \leq_{\Gamma} \). Indeed, if \([E]_\Gamma \leq_{\Gamma} [F]_\Gamma \) and \([G]_\Gamma \leq_{\Gamma} [H]_\Gamma \), then from \( \Gamma \vdash E \Rightarrow F \) and \( \Gamma \vdash G \Rightarrow H \), we get \( \Gamma \vdash EG \Rightarrow FH \) as in (11), showing \([EG]_\Gamma \leq_{\Gamma} [FH]_\Gamma \). In addition, we can see that \([E]_\Gamma \leq_{\Gamma} [\top]_\Gamma \) on account of \( \Gamma \vdash E \Rightarrow \top \) which is a trivial consequence of (Ax), i.e., \([\top]_\Gamma \) is the greatest element with respect to \( \leq_{\Gamma} \). Altogether, \( L / \Gamma = \langle L / \Gamma, \leq_{\Gamma}, \circ, [\top]_\Gamma \rangle \) is a commutative
integral pomonoid.

Take \( L/\Gamma \)-evaluation \( e \) such that \( e(p) = [\alpha_p]_\Gamma \), where \( \alpha_p : \text{Var} \rightarrow \mathbb{Z} \) is defined as in (10). Observe how \( e \) extends to all maps \( E \) of the form (6). According to (7),

\[
e(E) = e(p_1)^{E(p_1)} \circ \cdots \circ e(p_n)^{E(p_n)} = [\alpha_{p_1}]_\Gamma \circ \cdots \circ [\alpha_{p_n}]_\Gamma
\]

(12)

where \( \{ p \in \text{Var}; E(p) > 0 \} \subseteq \{ p_1, \ldots, p_n \} \). We now show that such \( e \) is an \( L/\Gamma \)-model of \( \Gamma \). Take any \( E \Rightarrow F \in \Gamma \). Trivially, \( \Gamma \vdash E \Rightarrow F \) and thus (12) yields

\[
e(E) = [E]_\Gamma \leq [F]_\Gamma = e(F), \text{ showing } e \models E \Rightarrow F.
\]

Since we have assumed \( \Gamma \not\vdash A \Rightarrow B \), we get \( e(A) = [A]_\Gamma \not\leq [B]_\Gamma = e(B) \) which shows that \( e \not\models A \Rightarrow B \) and therefore \( \Gamma \not\models A \Rightarrow B \).

As a further demonstration of properties of \( \vdash \) which is weaker than the provability of classic FDs, we show the following variant of a deduction-like theorem [29]:

**Theorem 4** (local deduction theorem). Let \( \Gamma \) be a theory. Then, the following are equivalent:

(i) there is natural \( n \) such that \( \Gamma \vdash A^n \Rightarrow B \),

(ii) \( \Gamma \cup \{ \top \Rightarrow A \} \vdash \top \Rightarrow B \).

**Proof.** Assume that \( \Gamma \vdash A^n \Rightarrow B \) for some natural \( n \). Since \( \vdash \) is monotone, we get \( \Gamma \cup \{ \top \Rightarrow A \} \vdash A^n \Rightarrow B \). Applying (Cut), we get \( \Gamma \cup \{ \top \Rightarrow A \} \vdash \top A^{n-1} \Rightarrow B \). Since \( \top A^{n-1} \) equals \( A^{n-1} \), we may repeat the argument \( n \)-times to get \( \Gamma \cup \{ \top \Rightarrow A \} \vdash \top \Rightarrow B \).

Conversely, let \( \Gamma \cup \{ \top \Rightarrow A \} \vdash \top \Rightarrow B \), i.e., there is a proof \( A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n \) of \( \top \Rightarrow B \) by \( \Gamma \cup \{ \top \Rightarrow A \} \). By induction on the length of the proof, we show there is natural \( i \) such that \( \Gamma \vdash A^n A_i \Rightarrow B_i \). Hence, (i) will result as a special case for \( A_n \Rightarrow B_n \) being \( A \Rightarrow B \). If \( A_i \Rightarrow B_i \in \Gamma \), then

\[
\frac{A_i \Rightarrow B_i}{\frac{B_i A \Rightarrow B_i^{(Ax)}}{AA_i \Rightarrow B_i^{(Cut)}}}
\]
proves that $\Gamma \vdash A^n_i A_i \Rightarrow B_i$ for $n_i = 1$. If $A_i \Rightarrow B_i$ is an instance of $(Ax)$ then so is $AA_i \Rightarrow B_i$, i.e., $\Gamma \vdash A^n_i A_i \Rightarrow B_i$ for $n_i = 1$. Finally, if $A_i \Rightarrow B_i$ results from $A_j \Rightarrow B_j$ and $A_k \Rightarrow B_k$ ($j, k < i$) by (Cut), then using the induction hypothesis $\Gamma \vdash A^n_j A_j \Rightarrow B_j$ and $\Gamma \vdash A^n_k A_k \Rightarrow B_k$ for some natural $n_j$ and $n_k$. In addition to that, the fact that $A_i \Rightarrow B_i$ results from $A_j \Rightarrow B_j$ and $A_k \Rightarrow B_k$ by (Cut) yields that $B_i = B_k, A_i = A_j C$ for some $C$, and $A_k = B_j C$. Then,

$$
\begin{align*}
A^n_j A_j &\Rightarrow B_j, \\
B_j A^n_k &\Rightarrow B_j A^n_k (Ax) \\
A^n_j A^n_k A_j &\Rightarrow A^n_k B_j \quad \text{(Cut),} \\
A^n_k B_j &\Rightarrow B_i \\
A^n_j A^n_k A_j C &\Rightarrow B_i \quad \text{(Cut)}
\end{align*}
$$

shows that $\Gamma \vdash A^n_j A^n_k A_j C \Rightarrow B_i$, meaning that $\Gamma \vdash A^n_i A_i \Rightarrow B_i$ for $n_i = n_j + n_k$.

Remark 4. Analogously as in the case of the rule of additivity, our logic does not admit a classic form of the deduction theorem. In other words, the exponent in Theorem 4(i) cannot be omitted.

The semantic entailment can be formulated in terms of other classes of algebras than the integral commutative pomonoids. For instance, we may use models and thus semantic entailment based on complete residuated lattices and we still be able to establish the completeness using the same axiomatization. The completeness over complete residuated lattices shown in the following assertion is an important observation because most of the modern fuzzy logics use residuated lattices as structures of degrees [8].

**Theorem 5** (completeness over complete residuated lattices). $\Gamma \vdash A \Rightarrow B$ iff $A \Rightarrow B$ is satisfied by each $L$-model of $\Gamma$, where $L$ is arbitrary complete residuated lattice.

**Proof.** The only-if part follows directly from the fact that from each complete residuated lattice $L = \langle L, \land, \lor, \otimes, \rightarrow, 0, 1 \rangle$ we can take its reduct $\langle L, \otimes, 1 \rangle$ and equip it with $\leq$ defined by $a \leq b$ iff $a \rightarrow b = 1$. Clearly, $\langle L, \leq, \otimes, 1 \rangle$ is a commutative integral pomonoid. Now, apply Theorem 2.
In order to prove the if-part, it suffices to show that each commutative integral pomonoid can be embedded into a complete residuated lattice. The rest then follows by using Theorem 3. Take any commutative integral pomonoid \( \langle L, \leq, \otimes, 1 \rangle \). Consider the system \( \mathcal{L} \) of all downward closed subsets of \( L \) with respect to \( \subseteq \). It is well known that \( \mathcal{L} \) with \( \subseteq \) is a complete lattice. Put

\[
X \ast Y = \{ z \in L; z \leq x \otimes y \text{ for some } x \in X \text{ and } y \in Y \},
\]

\[
X \rightarrow Y = \{ z \in L; X \ast \{ z \} \subseteq Y \}.
\]

for any \( X, Y \in \mathcal{L} \). Using the result of Galatos [18, Lemma 3.39], \( \mathcal{L} = \langle \mathcal{L}, \cap, \cup, \ast, \rightarrow, \emptyset, L \rangle \) is a complete residuated lattice and \( h: L \rightarrow \mathcal{L} \) defined by \( h(y) = \{ x \in L; x \leq y \} \) is an embedding.

We now turn our attention to the relationship of our rules and the classic FDs. From the syntactic point of view, the classic FDs can be seen as MFDs in which we allow to arbitrarily duplicate all occurrences of propositional variables. From the semantic point of view, it turns out that FDs are just MFDs with the semantics defined over the class of Boolean algebras. We show details in the next theorem, where we use the following notation. For any \( \Gamma \), put

\[
\Gamma_2 = \Gamma \cup \{ \alpha_p \Rightarrow \alpha_p \alpha_p; p \in \text{Var} \},
\]

where \( \alpha_p \) is defined as in (10). Now, we have:

**Theorem 6** (Boolean case extension). \( \Gamma_2 \vdash A \Rightarrow B \) iff \( A \Rightarrow B \) is satisfied by each \( L \)-model of \( \Gamma \), where \( L \) is the two-element Boolean algebra.

**Proof.** The only-if part is easy to see since \( \otimes \) in the two-element Boolean algebra is the truth function of the classic conjunction which is idempotent. In order to see the if-part, inspect the proof of Theorem 3 and observe that \( E \equiv_{\Gamma_2} E^n \) for any \( E \) and any natural \( n \). Indeed, \( \Gamma_2 \vdash E^n \Rightarrow E \) follows from (Ax) while \( \Gamma_2 \vdash E \Rightarrow E^n \) results by a repeated application of

\[
E \Rightarrow EE, \quad EEE \Rightarrow EEE \quad \text{(Ax)}
\]

\[
EE \Rightarrow EEE \quad \text{(Cut)}
\]

\[
E \Rightarrow EEE \quad \text{(Cut)}.
\]
Therefore, the operation ◦ in \( L/\Gamma_2 \) is idempotent and thus \( \langle L/\Gamma_2, \circ, [\top]_{\Gamma_2} \rangle \) is a semilattice. In addition, we can show that \( \leq_{\Gamma_2} \) coincides with the meet-semilattice order induced by \( \circ \). In order to see that, it suffices to show \([E]_{\Gamma_2} \leq_{\Gamma_2} [F]_{\Gamma_2} \circ [E]_{\Gamma_2} = [E]_{\Gamma_2}\) iff \([E]_{\Gamma_2} \leq_{\Gamma_2} [E]_{\Gamma_2}\). The latter condition can be rewritten as \([EF]_{\Gamma_2} = [E]_{\Gamma_2}\) which is true iff \(EF \equiv_{\Gamma_2} E\), i.e., if \(\Gamma_2 \vdash EF \Rightarrow E\) and \(\Gamma_2 \vdash E \Rightarrow EF\). Since \(EF \Rightarrow E\) is an instance of \((Ax)\), it suffices to check that \(\Gamma_2 \vdash E \Rightarrow F\) iff \(\Gamma_2 \vdash E \Rightarrow EF\) which is indeed the case: The if-part follows by

\[
E \Rightarrow EF, \quad EF \Rightarrow F \quad (Ax) \\
\hline
E \Rightarrow F \quad \text{(Cut)}
\]

and the only-if part follows by

\[
E \Rightarrow F, \quad FE \Rightarrow EF \quad (Ax) \\
\hline
EE \Rightarrow EF \quad \text{(Cut)} \\
\hline
E \Rightarrow EF \quad \text{(Cut)}
\]

As a consequence, there is an \( L/\Gamma_2 \)-model \( e \) of \( \Gamma_2 \) such that \( e(A) \circ e(B) \neq e(A) \), where \( L/\Gamma_2 = \langle L/\Gamma_2, \circ, [\top]_{\Gamma_2} \rangle \) is a meet-semilattice. Using standard arguments, \( L/\Gamma_2 \) can be embedded into a (complete) Boolean algebra \( L' \) of sets which is a subdirect product of two-element Boolean algebras \([6]\). Hence, for the two-element Boolean algebra \( L \) on \( \{0, 1\} \) with \( 0 < 1 \) there must be an \( L \)-evaluation \( e \) which is a model of \( \Gamma_2 \), \( e(A) = 1 \), and \( e(B) = 0 \), proving the claim.

Remark 5. Let us comment on the relationship to other approaches which study formulas expressing if-then dependencies whose semantics involves degrees coming from general structures of truth values. First, let us note that there exists a vast amount of papers on “fuzzy functional dependencies”, often with questionable technical quality, which combine (in various ways) the concepts of fuzzy sets and functional dependencies in order to formalize vague dependencies between attributes. While this idea is tempting and close what we present here, our objection is that most of these papers are purely definitional or just experimental and are not interested in the underlying logic in the narrow sense of it (i.e., in logic as a study of consequence). From one viewpoint this is not surprising since a number of papers in this category predate the beginning of systematic
formalization of various types of fuzzy logics which appeared in the late 90’s, see [22] as a standard reference and a historical overview. One of the most influential early approaches is [34]. Since our paper is not a survey, we do not write further details on such approaches and refer interested readers to [41] where they can find further comments. Our approach is also related to approaches to graded if-then rules which are motivated by formal concept analysis [19] of data with graded attributes. In [33], Polland proposed graded if-then rules with semantics defined using complete residuated lattices as structures of degrees. The approach has been later extended and more developed in [5] by considering formalizations of linguistic hedges [14, 38] as additional parameters of semantics of the if-then rules. Compared to the present paper, there are significant technical and epistemic differences. First, the approaches in [5, 33] use arbitrary, but fixed, structures of degrees. That is, instead of focusing on formulas which may be true in $L$-models where $L$ ranges over a class of structures of degrees (like the class of all integral commutative pomonoids), the papers fix $L$ and define semantics with respect to the fixed $L$. Second, the formulas in [5, 33] are syntactically different. Namely, they involve idempotent conjunctions instead of general non-idempotent ones. On the other hand, the formulas use degrees in $L$ to express lower bound of degrees to which attributes in antecedents and consequents of formulas are present—this is possible because $L$ is fixed. As a consequence, the formulas in [5, 33] allow to express dependencies like “if $x$ is true at least to degree $a$ and $y$ is true at least to degree $b$, then $z$ is true at least to degree $c$” with $a, b, c$ being degrees in the fixed $L$. Third, unlike our logic, the logic for such rules is Pavelka-style complete [30, 31, 32] which means that degrees of semantic consequence agree with (suitably defined) degrees of provability. In our case, Pavelka-style completeness cannot be considered because $L$ is not fixed. On the other hand, [5] shows that in order to obtain Pavelka-style completeness for a general (infinite) $L$, one has to resort to admitting infinitary inference rules which is not our case.
5 Computational Issues

In this section, we discuss computational issues of the logic of monoidal functional dependencies. We start by observing that the logic is decidable and show that the provability in our logic may be expressed as reducibility in an abstract rewriting system [37]. Based on that, we show that for theories consisting of formulas in a special form, there is a polynomial closure-like algorithm for deciding whether \( A \Rightarrow B \) is provable by a finite \( \Gamma \).

**Theorem 7.** If \( \Gamma \) is finite, then \( \Gamma^+ = \{ A \Rightarrow B; \Gamma \vdash A \Rightarrow B \} \) is decidable.

**Proof.** Given a finite \( \Gamma \), its deductive closure \( \Gamma^+ = \{ A \Rightarrow B; \Gamma \vdash A \Rightarrow B \} \) is obviously recursively enumerable. In addition, using Theorem 5 and the fact that the variety of residuated lattices has the finite embeddability property (every finite partial residuated sublattice can be embedded into a finite residuated lattice) and therefore the strong finite model property (every quasi-identity that fails in a residuated lattice fails in some finite one) [7], we conclude that

\[
\{ A \Rightarrow B; \Gamma \not\vdash A \Rightarrow B \} = \{ A \Rightarrow B; A \Rightarrow B \not\in \Gamma^+ \}
\]

is recursively enumerable. As a consequence, it is decidable whether \( A \Rightarrow B \) is provable by a finite \( \Gamma \). \( \square \)

As a consequence of Theorem 7, we obtain a naive approach to decide whether \( A \Rightarrow B \) is provable by a finite \( \Gamma \) which consists in enumerating all proofs by \( \Gamma \) and, simultaneously, generating finite residuated lattices to find counterexamples. The enumeration of proofs can be simplified since finding a proof of \( A \Rightarrow B \) may be seen as a process in which we sequentially reduce \( A \) in finitely many steps using formulas in \( \Gamma \). In order to formalize the rewriting process, to each \( \Gamma \) we associate a rewriting system \( \langle \mathcal{A}, \twoheadrightarrow_\Gamma \rangle \) where \( \mathcal{A} \) is the set of all maps of the form (6) and \( \twoheadrightarrow_\Gamma \) is a binary relation on \( \mathcal{A} \) such that

\[
A \twoheadrightarrow_\Gamma B \tag{14}
\]

for \( A, B \in \mathcal{A} \) whenever the following conditions are satisfied:
1. $A = EG$ for some $E, G \in A$,
2. $E \Rightarrow F \in \Gamma$, and
3. $B = FG$.

The transitive and reflexive closure $\rightarrow_\Gamma^*$ of $\rightarrow_\Gamma$ is called the reducibility by $\Gamma$.

The basic relationship between the provability by $\Gamma$ and $\rightarrow_\Gamma^*$ is described by the following assertion.

**Theorem 8.** $\Gamma \vdash A \Rightarrow B$ iff there is $C \in A$ such that $A \rightarrow_\Gamma^* BC$.

**Proof.** Assume that there is $C \in A$ such that $A \rightarrow_\Gamma^* BC$. By definition of $\rightarrow_\Gamma^*$, there are $A = D_0, \ldots, D_k = BC$ such that $D_0 \rightarrow_\Gamma D_1 \rightarrow_\Gamma \cdots \rightarrow_\Gamma D_k$. By induction, assume that $\Gamma \vdash A \Rightarrow D_i$ and observe that $D_i \rightarrow_\Gamma D_{i+1}$ means that $D_i = EG_i$ and $D_{i+1} = FG_i$ for some $E \Rightarrow F \in \Gamma$ and $G_i \in A$. Therefore, from $A \Rightarrow D_i$ we can infer $A \Rightarrow D_{i+1}$ by (Rwt) and so $\Gamma \vdash A \Rightarrow D_{i+1}$ because (Rwt) is a derived inference rule, cf. Remark 2 (c). Therefore, $\Gamma \vdash A \Rightarrow D_k$ means $\Gamma \vdash A \Rightarrow BC$ and so $\Gamma \vdash A \Rightarrow B$ by (Pro).

Conversely, we first argue that if $\Gamma \vdash A \Rightarrow B$ then there is a proof $\phi_1, \ldots, \phi_n$ of $A \Rightarrow B$ by $\Gamma$ which uses only the inference rules (Ref), (Rwt), and (Pro). In addition, we claim that the proof can be found so that the following additional properties are all satisfied:

1. $\phi_1$ is $A \Rightarrow A$ and it is the only instance of (Ref) in the proof;
2. each $\phi_i$ such that $1 < i < n$ is a formula in one of the following forms:
   (a) $\phi_i \in \Gamma$, or
   (b) $\phi_i$ results by (Rwt) applied to some $\phi_j$ ($j < i$) of the form $A \Rightarrow X$ for some $X \in A$ and a formula in $\Gamma$;
3. $\phi_n$ results from $\phi_{n-1}$ by (Pro) and it is the only application of (Pro) used in the proof.

Using the arguments in Remark 2 (c), there indeed is a proof of $A \Rightarrow B$ by $\Gamma$ which uses only (Ref), (Pro), and (Rwt). It remains to show that the proof may be transformed into a proof satisfying 1.–3. This can be shown using analogous
arguments as in [28, Theorem 4.2] which shows this in the classic setting with the rule of accumulation instead of (Rwt) and proves the existence of the so-called RAP-derivation sequences, cf. also [27]. A moment’s reflection shows that the procedure in the proof of [28, Theorem 4.2] may be carried over with the weaker rule (Rwt) by performing the following steps during which we

- add $A \Rightarrow A$ at the beginning of the proof (if it is not there);
- add an application of (Pro) at the end of the proof (if it is not there);
- eliminate all applications of (Pro) except for the last one using the argument that (Pro) commutes with (Rwt) and therefore a formula derived by first using (Pro) and then using (Rwt) may be derived by first using (Rwt) and then using (Pro);
- eliminate applications of (Rwt) which do not conform to either of (a) and (b) specified above by substituting each such an application by a series of applications of (Rwt) which yield formulas with $A$ as the antecedent and use only formulas in $\Gamma$. This can be done by going backwards through the proof and using the observation that

$$
\frac{A \Rightarrow DE, \ E \Rightarrow FG, \ G \Rightarrow H}{A \Rightarrow DFH}
$$

(Rwt)

$$
\frac{E \Rightarrow FH}{A \Rightarrow DFH}
$$

(Rwt)

can equivalently be expressed as

$$
\frac{A \Rightarrow DE, \ E \Rightarrow FG}{A \Rightarrow DFG}
$$

(Rwt), $G \Rightarrow H$

$$
\frac{A \Rightarrow DFG}{A \Rightarrow DFH}
$$

(Rwt)

cf. [28, Theorem 4.2].

At this point we have shown that if $\Gamma \vdash A \Rightarrow B$ then there is a proof $\varphi_1, \ldots, \varphi_n$ of $A \Rightarrow B$ by $\Gamma$ satisfying 1.–3. Let $A \Rightarrow X_1, \ldots, A \Rightarrow X_k$ be the subsequence of $\varphi_1, \ldots, \varphi_n$ which consists of all formulas with the antecedent $A$. By induction, we prove that $A \rightarrow^*_X X_i$ for all $i = 1, \ldots, k$. We distinguish three cases. First, if $X_i = A$, then trivially $A \rightarrow^*_X X_i$. Second, if $A \Rightarrow X_i \in \Gamma$, then directly by the
definition of $\rightarrow_\Gamma$, we get $A \rightarrow_\Gamma X_i$ and so $A \rightarrow^{\ast}_{\Gamma} X_i$. Third, if $A \Rightarrow X_i$ results from $A \Rightarrow X_j$ (for some $j < i$) and some $E \Rightarrow F \in \Gamma$ by (Rwt), then $X_j = EG$ and $X_i = FG$ for some $G \in A$ and so $X_j \rightarrow_\Gamma X_i$, meaning $A \rightarrow^{\ast}_{\Gamma} X_j \rightarrow_\Gamma X_i$, i.e., $A \rightarrow^{\ast}_{\Gamma} X_i$. Altogether, $A \rightarrow^{\ast}_{\Gamma} X_i$ for all $i = 1, \ldots, k$ and as a special case for $i = k$, we get $A \rightarrow^{\ast}_{\Gamma} X_k = BC$ for some $C \in A$ because $A \Rightarrow B$, being the last formula in $\varphi_1, \ldots, \varphi_n$, results from $A \Rightarrow X_k$ by (Pro).

Theorem \[\text{8}\] may be used to find proofs of $A \Rightarrow B$ by a finite $\Gamma$ in a more convenient way than the naive approach because instead of storing proofs, one can just store representations of maps of the form \[6\] and in order to find a proof one may perform a breadth-first search through a (possibly infinite) tree of derivations starting with $A$. Needless to say, the procedure is still very expensive because the memory consumed by the process can grow exponentially. More importantly, in general it is still necessary to simultaneously generate counterexamples in order to decide whether $A \Rightarrow B$ follows by $\Gamma$ because the search space is infinite.

In the rest of this section, we show that considerably more efficient decision procedures may be find in case of theories consisting only of particular formulas. We describe a procedure which exploits the rewriting process and the result of Theorem \[\text{8}\] and which resembles the well-known CLOSURE algorithm \[28\] Algorithm 4.2. We confine ourselves only to so-called non-contracting theories.

A formula $A \Rightarrow B$ is called non-contracting whenever $B$ can be written as $AC$ for some $C$. A theory $\Gamma$ is non-contracting whenever all its formulas are non-contracting.

Clearly, if $\Gamma$ is non-contracting and $A \rightarrow^{\ast}_{\Gamma} B$, then $A(y) \leq B(y)$ for all $y \in \text{Var}$. From the point of view of the inference rules, (Rwt) applied to non-contracting formulas acts like the classic accumulation rule. In contrast to the classic properties of closures of sets of attributes, there still is no guarantee that for $A$ there is a greatest $B$ such that $A \rightarrow^{\ast}_{\Gamma} B$. Nevertheless, for non-contracting theories, we may propose an algorithm as in Figure \[\text{4}\] which generalize the well-known algorithm MEMBER \[28\] Algorithm 4.3. 23
Data: a finite non-contracting theory $\Gamma$ and a formula $A \Rightarrow B$

Result: boolean value

1. $\Delta := \Gamma \cup \{B \Rightarrow By\}$; /* $y \in \text{Var}$ is unused in $A,B,\Gamma$ */
2. $W := A$; /* $W$ is auxiliary map */
3. $N := \sum_{E \Rightarrow F \in \Delta} \sum_{p \in \text{Var}} E(p)$; /* counter */

4. repeat
5. 
6. 
7. if $W = EX$ for some $X$ of the form $[\text{0}]$ then
8. 
9. end
10. end
11. $N := N - 1$; /* decrease the counter */
12. until $L = W$ or $N \leq 0$ or $W(y) > 0$;

13. if $W(y) > 0$ then
14. 
15. else
16. 
17. end

Figure 1: Algorithm for deciding $\Gamma \vdash A \Rightarrow B$ for non-contracting $\Gamma$. 
The algorithm in Figure 1 accepts a finite non-contracting theory Γ and arbitrary formula $A \Rightarrow B$ as its input. It is obvious that the algorithm terminates after finitely many steps (check the condition at line 12) and returns a value true or false. The following assertion shows that the algorithm decides $\vdash$.

**Theorem 9.** The algorithm in Figure 1 is correct: For a non-contracting finite $\Gamma$, it terminates after finitely many steps and returns “true” iff $\Gamma \vdash A \Rightarrow B$.

**Proof.** The algorithm uses $W$ as an auxiliary variable which represents a working multi-set in Var whose initial value is $A$ (see line 2). In addition, $\Delta$ is set to $\Gamma$ which is extended by a formula $B \Rightarrow By$, see line 1, where $y$ is a fresh new propositional variable which does not appear in either formula in $\Gamma$ or in $A \Rightarrow B$. Recall that using the abbreviated notation for (10), for the antecedent $By$ of $B \Rightarrow By$ we have $By(y) = 1$ and $By(z) = B(z)$ for all $z \neq y$. The algorithm utilizes an additional counter $N$ which is initially set to the total number of occurrences of propositional variables in all antecedents in $\Delta$, see line 3.

The repeat-unit loop updates $W$ as long as it can be updated (the auxiliary variable $L$ is used to detect no update) based on the formulas in $\Delta$ and the property which is maintained after each update is that $A \xleftarrow{\Delta} W$. This is the same as in the ordinary Closure. Whenever an antecedent of a formula in $\Delta$ is contained in $W$, its consequent is added to $W$, see line 8.

We now inspect the halting condition of the repeat-until loop. If $W(y) > 0$, it means that $B \Rightarrow By$ has been used in line 8. Therefore, $A \xleftarrow{\Delta} W$ such that $W = BX$ for some $X$ and thus $\Gamma \vdash A \Rightarrow B$ in which case the algorithm returns true. If the repeat-until loop terminates and we have $W(y) = 0$, false is returned. It suffices to show that in this case $\Gamma \not\vdash A \Rightarrow B$. To see this, observe that if $E \Rightarrow F \in \Delta$ passes the condition in line 7, then it passes the condition in all consecutive iterations of the loop and $W$ is repeatedly updated by this formula (this is because all formulas in $\Delta$ are non-contracting, so the antecedent of $E \Rightarrow F$ cannot “vanish” from $W$). As a consequence, if $L \neq W$ holds when the algorithm reaches line 12 for the first time, then $L \neq W$ for all consecutive iterations. Therefore, the repeat-until loop can be terminated.
because of \( L = W \) only at the end of the first iteration in which case there is no formula in \( \Delta \) which may update the value of \( W \) and so \( \Gamma \not
Rightarrow A \Rightarrow B \).

Let us assume that \( L \neq W \), \( W(y) = 0 \), and \( N = 0 \). We use the argument that \( B \Rightarrow By \) is either used to update \( W \) (line 8) in \( N \) steps with the initial value of \( N \) given as in line 3, or it cannot be used to update \( W \) at all. To see that, assume the worst case in which for \( \Delta = \{ E_1 \Rightarrow F_1, \ldots, E_n \Rightarrow F_n, B \Rightarrow By \} \), only \( E_1 \Rightarrow F_1 \) is used to update \( W \) during the first \( m_1 \) iterations, then \( E_1 \Rightarrow F_1 \) and \( E_2 \Rightarrow F_2 \) are used simultaneously to update \( W \) during the next \( m_2 \) iterations, etc., so that finally \( B \Rightarrow By \) is used to update \( W \). The key observation here is that \( m_1 \) cannot be strictly greater than the number of attributes in the antecedent of \( E_2 \) because in the worst case, the attributes (including their multiple occurrences) are added to \( W \) one by one. That is, \( m_1 \leq \sum_{p \in \text{Var}E_2(p)} \) and analogously, \( m_2 \leq \sum_{p \in \text{Var}E_3(p)} \), etc. So, in the worst case, the use of \( B \Rightarrow By \) to update \( W \) is bounded from above by

\[
\sum_{E \Rightarrow F \in \Delta} \sum_{p \in \text{Var}E(p)} E(p)
\]

iterations. As a conclusion, if \( N \) initially set to the value in line 3 reaches 0 and \( W(y) = 0 \), there is no \( X \) such that \( A \prec^*_\Gamma BX \), i.e., \( \Gamma \not
Rightarrow A \Rightarrow B \). \( \square \)

Remark 6. It is clear that the algorithm in Figure 1 is polynomial since it only represents an extension of Closure and Member which results in more iterations of the main loop than in the case of Closure but the number of iterations is bounded by the size of the input. In fact, our algorithm has quadratic worst-case time complexity, the same as Closure.

We conclude this section by a remark showing that if \( \Gamma \not
Rightarrow A \Rightarrow B \), then is may not be possible to find a linear \( L \)-model of \( \Gamma \) which serves as a counterexample. A model is linear if the order in \( L \) is total, i.e., for any \( a, b \in L \), we have \( a \leq b \) or \( b \leq a \).

Remark 7. Take \( \Gamma = \{ p \Rightarrow ux, p \Rightarrow vy, uy \Rightarrow q, vx \Rightarrow q \} \). It can be easily seen that \( \Gamma \vdash pp \Rightarrow qq \) because \( pp \rightarrow_\Gamma uxpx \rightarrow_\Gamma uxvyuq \rightarrow_\Gamma uyq \rightarrow_\Gamma qq \). On the other hand, we have \( \Gamma \not
Rightarrow p \Rightarrow q \). Indeed, we can consider \( L = \langle L, \leq, \otimes, 1 \rangle \) with
\langle L, \leq \rangle \text{ given by the Hasse diagram in Figure 2 (left) and with } \otimes \text{ given by the table in Figure 2 (right). For } e: \text{Var} \to L \text{ such that } e(p) = a, e(q) = 0, e(u) = b, e(v) = c, e(x) = b, e(y) = c, \text{we have}

\begin{align*}
e(p) &= a \leq b = b \otimes b = e(ux), \\
e(p) &= a \leq c = c \otimes c = e(vy), \\
e(uy) &= b \otimes c = 0 \leq 0 = e(q), \\
e(vx) &= c \otimes b = 0 \leq 0 = e(q),
\end{align*}

i.e., \( e \) is an \( L \)-model of \( \Gamma \). In addition, \( e(p) = a \not\leq 0 = e(q) \), showing \( \Gamma \not\models p \Rightarrow q \).

We claim there is no linear \( L \)-model of \( \Gamma \) which refutes \( p \Rightarrow q \). Indeed, suppose that \( e \) is a linear \( L \)-model of \( \Gamma \). Since \( L \) is linear, we have \( e(x) \leq e(y) \) or \( e(y) \leq e(x) \). In the first case, the monotony of \( \otimes \) gives \( e(ux) \leq e(uy) \) and so \( e(p) \leq e(ux) \leq e(uy) \leq e(q) \), meaning \( e \models p \Rightarrow q \). In the second case, \( e(p) \leq e(vy) \leq e(vx) \leq e(q) \), meaning \( e \models p \Rightarrow q \) again. Therefore, in the search for a counterexample, we cannot restrict ourselves to linear \( L \)-models, only. It also means that our logic does not admit linear completions of theories in the following sense: Given \( \Gamma \) and \( A \Rightarrow B \) such that \( \Gamma \not\models A \Rightarrow B \), in general there is no \( \Gamma' \supseteq \Gamma \) such that \( \Gamma' \not\models A \Rightarrow B \) and \( \Gamma' \vdash E \Rightarrow F \) or \( \Gamma' \vdash F \Rightarrow E \) for all \( E \) and \( F \) of the form \( \{E \Rightarrow F\} \). As a further consequence, our logic does not admit the principle of “proofs by cases”: In general the facts that \( \Gamma \cup \{E \Rightarrow F\} \vdash A \Rightarrow B \) and \( \Gamma \cup \{F \Rightarrow E\} \vdash A \Rightarrow B \) do not yield \( \Gamma \vdash A \Rightarrow B \). This also explains our choice of the name for the logic. Namely, our choice of the word “monoidal” over
the word “fuzzy” because in the modern understanding of (formal) fuzzy logics, properties like the presence of the principle of proofs by cases are considered essential, see [8] for details.

6 Propositional vs. Relational Semantics

So far, we have used a propositional semantics of the formulas. That means, MFDs have been interpreted given evaluations of propositional variables. In order to establish the desired connection to relational databases, we show that MFDs have an equivalent semantics based on evaluating MFDs in relations on relation schemes. Since relations in databases are considered on finite relations, we consider here only entailment from finite theories.

Let \( L = \langle L, \leq, \otimes, 1 \rangle \) be an integral commutative pomonoid. Let \( R \) be a relation scheme (a finite set of attributes); \( r \) be a relation on \( R \) in the usual sense; \( D^r_p \) denote the domain of attribute \( p \) in \( r \) (we consider the notion of a domain as a synonym for the notion of a type, see [10]). Furthermore, consider for any \( p \in R \) a map \( \approx^r_p : D^r_p \times D^r_p \to L \), where \( L \) is the support of \( L \). Following the discussion in Section 1, the result of \( d_1 \approx^r_p d_2 \) can be seen as a degree in \( L \) which is an answer to the atomic query: “Is \( d_1 \) similar to \( d_2 \)?” We assume that \( \approx^r_p \) are supplied along with the data and assume that \( d \approx^r_p d = 1 \) for each \( d \in D^r_p \) and \( p \in \text{Var} \) (i.e., each element is similar to itself to degree 1—the highest degree in \( L \)).

For \( r, A \) of the form (6), and any tuples \( r_1, r_2 \in r \), we put

\[
\begin{align*}
\mathrel{r_1 \approx_A^r r_2} &= \left( r_1(p_1) \approx^r_{p_1} r_2(p_1) \right)^{A(p_1)} \otimes \cdots \otimes \left( r_1(p_n) \approx^r_{p_n} r_2(p_n) \right)^{A(p_n)} \tag{15}
\end{align*}
\]

for \( R \subseteq \{ p_1, \ldots, p_n \} \). Since \( \otimes \) serves as an interpretation of a conjunction, (15) can be seen as a degree in \( L \) which is a result of conjunctive query: “Are \( r_1(p_1) \) similar to \( r_2(p_1) \) and \( \cdots \) and \( r_1(p_n) \) similar to \( r_2(p_n) \)?” Therefore, \( r_1 \approx_A^r r_2 \) is degree to which tuples \( r_1 \) and \( r_2 \) in \( r \) are similar on all attributes in \( A \). For \( r \) and \( A \Rightarrow B \) we say that \( r \) satisfies \( A \Rightarrow B \), written \( r \models A \Rightarrow B \), if for any
tuples \( r_1, r_2 \in r \), the following inequality holds:

\[
r_1 \approx^r_A r_2 \leq r_1 \approx^r_B r_2.
\]  (16)

Using the notion of satisfaction of MFDs in relations, we introduce models and semantic entailment as before. Namely, we put

\[
\text{Mod}(\Gamma) = \{ r; \ r \models E \Rightarrow F \text{ for all } E \Rightarrow F \in \Gamma \}
\]  (17)

and call each \( r \in \text{Mod}(\Gamma) \) a (relational) model of \( \Gamma \). An MFD \( A \Rightarrow B \) is semantically entailed by \( \Gamma \) (in the relational sense) if \( \text{Mod}(\Gamma) \subseteq \text{Mod}(\{A \Rightarrow B\}) \), i.e., if \( A \Rightarrow B \) is satisfied in every relational model of \( \Gamma \).

**Theorem 10.** Let \( \Gamma \) be finite. Then, \( \Gamma \models A \Rightarrow B \) iff \( A \Rightarrow B \) is semantically entailed by \( \Gamma \) in the relational sense.

**Proof.** Let \( R \) be a finite subset of \( \text{Var} \) which contains all propositional variables appearing in \( A \Rightarrow B \) and all formulas in \( \Gamma \). The if-part follows by the fact that for each \( L \)-model \( e \) of \( \Gamma \) there is \( r \in \text{Mod}(\Gamma) \) such that \( e \models E \Rightarrow F \) iff \( r \models E \Rightarrow F \) for any \( E \Rightarrow F \). Namely, we can consider \( r = \{r_1, r_2\} \) such that \( r_1(p) = 1 \) for any \( p \in R \), \( r_2(p) = e(p) \), and \( 1 \approx^p r_1(p) = e(p) \approx^p 1 = e(p) \) for any \( p \in R \). Hence, the domains of attributes in \( r \) are considered as subsets of \( L \).

Conversely, for each \( r \in \text{Mod}(\Gamma) \) with all \( \approx^p \) defined using \( L \), there is a finite set \( S \) of \( L \)-models \( e \) such that \( r \models E \Rightarrow F \) iff \( e \models E \Rightarrow F \) for all \( e \in S \). In particular, we let \( S = \{e_{r_1, r_2}; r_1, r_2 \in r\} \), where \( e_{r_1, r_2}(p) = r_1(p) \approx^p r_2(p) \) for all \( p \in R \). The rest is easy to check.

As a result of Theorem 10, the relational and propositional semantics have the same notion of semantic entailment and thus all observations on provability we have made in Section 3, Section 4, and Section 5 apply to both semantics.

We conclude the paper by an illustrative example of a relation and MFDs satisfied by the relation. It shows a particular relation with similarities on domains which does/does not satisfy weaker/stronger forms of constraints prescribed by MFDs with multiple occurrences of the same attribute in their antecedents and consequents, respectively.
Example 1. Consider the relation $r$ in Figure 3. The relation is defined on relation scheme consisting of attributes $\text{FOO}$, $\text{BAR}$, and $\text{BAZ}$, all having the set of integers as its domain. In addition, let us assume that all the attributes ($y \in \{\text{FOO}, \text{BAR}, \text{BAZ}\}$) have the same similarity $\approx_y$ defined by $d_1 \approx_y d_2 = 1.01^{-|d_1 - d_2|}$. Clearly, $d_1 \approx_y d_2 \in [0, 1]$. Assume that $L$ is the same as in Remark 4. Now, if we consider formula $\text{FOO} \Rightarrow \text{BAR}$, then it is satisfied by $r$ as an ordinary FD: If two tuples have equal values on $\text{FOO}$, then they have equal values on $\text{BAR}$ (the only non-trivial case applies to the first two tuples in Figure 3). If the formula is considered as an MFD, we have

$$r \not\models \text{FOO} \Rightarrow \text{BAR},$$

i.e., $r$ does not satisfy $\text{FOO} \Rightarrow \text{BAR}$. This is because

$$10 \approx_{\text{FOO}} 12 = 0.98029 \not\approx 0.97059 = 20 \approx_{\text{BAR}} 23.$$ 

Therefore, $\text{FOO} \Rightarrow \text{BAR}$ may be interpreted as a similarity-based constraint which is not satisfied by $r$. On the other hand, for the weaker formula which results by adding one more occurrence of $\text{FOO}$ to the antecedent, we get

$$r \models \text{FOO} \& \text{FOO} \Rightarrow \text{BAR}.$$ 

As a further example, consider $\text{BAZ} \Rightarrow \text{BAR}$. In the classic sense, the constraint is satisfied by $r$ trivially since there are no distinct tuples which have the same value on $\text{BAR}$. One may check that we also have

$$r \models \text{BAZ} \Rightarrow \text{BAR}.$$
Indeed, in the non-trivial cases and considering the symmetry of our similarity, we get that

\begin{align*}
190 \approx_{\text{BAZ}} 160 &= 0.74192 \leq 1.00000 = 20 \approx_{\text{BAR}} 20, \\
190 \approx_{\text{BAZ}} 130 &= 0.55045 \leq 0.97059 = 20 \approx_{\text{BAR}} 23, \\
190 \approx_{\text{BAZ}} 100 &= 0.40839 \leq 0.86135 = 20 \approx_{\text{BAR}} 35, \\
160 \approx_{\text{BAZ}} 130 &= 0.74192 \leq 0.97059 = 20 \approx_{\text{BAR}} 23, \\
160 \approx_{\text{BAZ}} 100 &= 0.55045 \leq 0.86135 = 20 \approx_{\text{BAR}} 35, \\
130 \approx_{\text{BAZ}} 100 &= 0.74192 \leq 0.88745 = 23 \approx_{\text{BAR}} 35.
\end{align*}

Observe that the previous inequalities still hold if we replace the numerical values on their right-hand sides by their squares. As a consequence,

\[ r \models \text{BAZ} \Rightarrow \text{BAR} \& \text{BAR}, \]

i.e., the stronger form of \( \text{BAZ} \Rightarrow \text{BAR} \) which results by adding one more occurrence of \( \text{BAR} \) in the consequent is also satisfied by \( r \). On the contrary,

\[ r \not\models \text{BAZ} \Rightarrow \text{BAR} \& \text{BAR} \& \text{BAR} \]

because

\[ 130 \approx_{\text{BAZ}} 100 = 0.74192 \not\leq 0.69892 = 0.88745^3 = (23 \approx_{\text{BAR}} 35)^3, \]

showing that \( \text{BAZ} \Rightarrow \text{BAR} \& \text{BAR} \& \text{BAR} \) is not satisfied by \( r \) (with this particular choice of similarity) as an MFD.

## 7 Conclusion and Open Problems

We have introduced a logic for monoidal functional dependencies (MFDs) and we proved the logic is complete with respect to the class of all integral commutative partially ordered monoids. In addition, we have shown completeness with respect to all complete residuated lattices. The logic of the classic FDs may be seen as an extension of the logic of MFDs which consists of adding formulas
expressing the idempotency of conjunction. It has two natural semantics—
propositional one and relational one. We have shown the logic is decidable and
in case of non-contracting theories there is a polynomial algorithm for deciding
whether a formula follows by a finite set of other formulas.

Further issues we consider worth studying include:

• methods for extracting non-redundant bases consisting of formulas which
entail all formulas true in given data as in [21];

• approaches to use MFDs as association rules, possible descriptions of non-
redundant rules and related algorithms, cf. [39];

• algorithms for deciding entailment of formulas which are not limited to
non-contracting theories;

• further logical and model-theoretical properties, e.g., characterization of
model classes by closure properties.

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References

[1] William Ward Armstrong, Dependency structures of data base relationships, Information Processing 74: Proceedings of IFIP Congress (Amsterdam) (J. L. Rosenfeld and H. Freeman, eds.), North Holland, 1974, pp. 580–583.

[2] Catriel Beeri and Philip A. Bernstein, Computational problems related to the design of normal form relational schemas, ACM Trans. Database Syst. 4 (1979), 30–59.

[3] Radim Belohlavek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic Publishers, Norwell, MA, USA, 2002.

[4] Radim Belohlavek and Vilem Vychodil, Codd’s relational model from the point of view of fuzzy logic, J. Log. Comput. 21 (2011), no. 5, 851–862.

[5] , Attribute dependencies for data with grades, CoRR abs/1402.2071 (2014).

[6] Garrett Birkhoff, Lattice theory, 1st ed., American Mathematical Society, Providence, 1940.
[7] Willem J. Blok and Clint J. Van Alten, *The finite embeddability property for residuated lattices, pocrims and BCK-algebras*, Algebra Universalis 48 (2002), no. 3, 253–271.

[8] Petr Cintula, Petr Hájek, and Carles Noguera (eds.), *Handbook of Mathematical Fuzzy Logic, Volume 1*, Studies in Logic, Mathematical Logic and Foundations, vol. 37, College Publications, 2011.

[9] Nilesh Dalvi, Christopher Ré, and Dan Suciu, *Probabilistic databases: diamonds in the dirt*, Commun. ACM 52 (2009), 86–94.

[10] Christopher J. Date and Hugh Darwen, *Databases, types, and the relational model: The third manifesto*, 3rd ed., Addison-Wesley, 2006.

[11] Claude Delobel and Richard G. Casey, *Decomposition of a data base and the theory of boolean switching functions*, IBM Journal of Research and Development 17 (1973), no. 5, 374–386.

[12] Robert P. Dilworth, *Abstract residuation over lattices*, Bull. Amer. Math. Soc. 44 (1938), 262–268.

[13] Francesc Esteva and Lluís Godo, *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems 124 (2001), no. 3, 271–288.

[14] Francesc Esteva, Lluís Godo, and Carles Noguera, *A logical approach to fuzzy truth hedges*, Information Sciences 232 (2013), 366–385.

[15] Ronald Fagin, *Functional dependencies in a relational database and propositional logic*, IBM Journal of Research and Development 21 (1977), no. 6, 534–544.

[16] ________, *Combining fuzzy information from multiple systems*, J. Comput. Syst. Sci. 58 (1999), no. 1, 83–99.

[17] Ronald Fagin, Amnon Lotem, and Moni Naor, *Optimal aggregation algorithms for middleware*, J. Comput. Syst. Sci. 66 (2003), no. 4, 614–656.

[18] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Volume 151*, 1st ed., Elsevier Science, San Diego, USA, 2007.

[19] Bernhard Ganter and Rudolf Wille, *Formal concept analysis: Mathematical foundations*, 1st ed., Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1997.

[20] Joseph A. Goguen, *The logic of inexact concepts*, Synthese 19 (1979), 325–373.

[21] Jean-Louis Guigues and Vincent Duquenne, *Familles minimales d’implications informatives resultant d’un tableau de données binaires*, Math. Sci. Humaines 95 (1986), 5–18.

[22] Petr Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.
[23] Petr Hájek and Jeff Paris, *A dialogue on fuzzy logic*, Soft Computing 1 (1997), no. 1, 3–5.

[24] Ulrich Hohle, *Monoidal logic*, Fuzzy-Systems in Computer Science (R. Kruse, J. Gebhardt, and R. Palm, eds.), Artificial Intelligence / Künstliche Intelligenz, Vieweg+Teubner Verlag, 1994, pp. 233–243.

[25] Ihab F. Ilyas, George Beskales, and Mohamed A. Soliman, *A survey of top-k query processing techniques in relational database systems*, ACM Comp. Surv. 40 (2008), no. 4, 11:1–11:58.

[26] Erich Peter Klement, Radko Mesiar, and Endre Pap, *Triangular Norms*, 1 ed., Springer, 2000.

[27] David Maier, *Minimum covers in relational database model*, J. ACM 27 (1980), no. 4, 664–674.

[28] ________, *Theory of Relational Databases*, Computer Science Pr, Rockville, MD, USA, 1983.

[29] Elliott Mendelson, *Introduction to Mathematical Logic*, Chapman and Hall, 1987.

[30] Jan Pavelka, *On fuzzy logic I: Many-valued rules of inference*, Mathematical Logic Quarterly 25 (1979), no. 3–6, 45–52.

[31] ________, *On fuzzy logic II: Enriched residuated lattices and semantics of propositional calculi*, Mathematical Logic Quarterly 25 (1979), no. 7–12, 119–134.

[32] ________, *On fuzzy logic III: Semantical completeness of some many-valued propositional calculi*, Mathematical Logic Quarterly 25 (1979), no. 25–29, 447–464.

[33] Silke Pollandt, *Fuzzy-Begriffe: Formale Begriffsanalyse unscharfer Daten*, Springer, 1997.

[34] K. V. S. V. N. Raju and Arun K. Majumdar, *Fuzzy functional dependencies and lossless join decomposition of fuzzy relational database systems*, ACM Transactions on Database Systems 13 (1988), no. 2, 129–166.

[35] Helena Rasiowa and Roman Sikorski, *A proof of the completeness theorem of Gödel*, Fundam. Math. 37 (1950), 193–200.

[36] Morgan Ward and Robert P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc. 45 (1939), 335–354.

[37] Wolfgang Wechler, *Universal Algebra for Computer Scientists*, EATCS Monographs on Theoretical Computer Science, vol. 25, Springer-Verlag, Berlin Heidelberg, 1992.

[38] Lotfi A. Zadeh, *A fuzzy-set-theoretic interpretation of linguistic hedges*, Journal of Cybernetics 2 (1972), no. 3, 4–34.

[39] Mohammed J. Zaki, *Mining non-redundant association rules*, Data Mining and Knowledge Discovery 9 (2004), 223–248.