On the tropical Lefschetz–Hopf trace formula

Johannes Rau

Received: 30 September 2021 / Accepted: 18 January 2023 / Published online: 29 March 2023
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract

In this follow-up to Rau (The tropical Poincaré–Hopf theorem, 2020. arXiv:2007.11642), our main result is a tropical Lefschetz–Hopf trace formula for matroidal automorphisms. We show that both sides of the formula are equal to the (generalized) beta invariant of the lattice of fixed flats.

Keywords

Tropical geometry · Matroid fans · Beta invariant · Matroid theory · Tropical intersection theory

1 Introduction

In [20], the author proves a tropical version of the Poincaré–Hopf theorem and conjectures a tropical version of a tropical Lefschetz–Hopf trace formula. The main result of this paper is the proof of the tropical trace formula in the case of matroidal automorphisms. In this case, the formula can be refined by giving a third description of the value in terms of the (generalized) beta invariant of the lattice of fixed flats.

Theorem 1.1 Let $M$ be a loopless matroid and let $\Psi : \Sigma_M \to \Sigma_M$ be a matroidal automorphism. Then,

$$\deg(\Gamma_\Psi \cdot \Delta) = (-1)^n \beta(\text{Fix}(L(M))) = \sum_{p=0}^{n} (-1)^p \text{Tr}(\Psi^*, F_p(\Sigma_M)).$$

The theorem is an instance of the conjecture stated in [20, Conjecture 1.2] using standard instead of Borel–Moore tropical homology (cf. [20, Remark 1.8]), since
$H_{p,0}(\Sigma_M) = F_p(\Sigma_M)$ and $H_{p,q}(\Sigma_M) = 0$ for $q > 0$. We actually hope that the theorem can be used as the local ingredient to prove more general cases of [20, Conjecture 1.2].

Let us quickly review the ingredients of the theorem (precise definitions follow in the later sections). In the following, $M$ is a loopless matroid of rank $n + 1$ on the set $E = \{0, \ldots, N\}$. We denote by $\Sigma_M \subseteq \mathbb{R}^N$ the (projective) matroid fan of $M$. A matroidal automorphism $\Psi: \Sigma_M \to \Sigma_M$ is a tropical automorphism of $\Sigma_M$ which is induced by a matroid automorphism $\psi: M \to M$. More precisely, the relationship is given by

$$\Psi(v_S) = v_{\psi(S)}$$

for any $S \subseteq E$ and associated indicator vector $v_S$.

Given $\Psi$, we denote by $\Gamma_\psi, \Delta \subseteq \Sigma_M \times \Sigma_M$ the graph of $\Psi$ and the diagonal of $\Sigma_M$, respectively. They carry the structure of tropical cycles. Using the tropical intersection theory for matroid fans constructed in [11, 22], we can define their intersection product $\Gamma_\psi \cdot \Delta$. The degree of this product is the intersection-theoretic side of the tropical trace formula.

For $p = 0, \ldots, n$, the framing group $F_p(\Sigma_M) \subseteq \bigwedge^p \mathbb{R}^N$ is the vector space generated by wedges of $p$ vectors contained in the same cone of $\Sigma_M$ [14, 18]. Since $\Psi$ is the restriction of a linear map on $\mathbb{R}^N$, $\Psi$ induces a map $\Psi_*: \bigwedge^p \mathbb{R}^N \to \bigwedge^p \mathbb{R}^N$ which can be restricted to $\Psi_*: F_p(\Sigma_M) \to F_p(\Sigma_M)$. We denote the trace of this automorphism by $\text{Tr}(\Psi_*, F_p(\Sigma_M))$. The alternating sum of traces is the trace side of the tropical trace formula.

To connect the two sides, we use an intermediate, combinatorially defined invariant which is the (generalized) beta invariant of a lattice with rank function. In our case, the lattice in question is the lattice of flats $F \in L(M)$ which are fixed by $\psi$,

$$\text{Fix}(L(M)) = \{ F \in L(M) : F = \psi(F) \}$$

The beta invariant is defined using the Möbius function $\mu^\psi$ of $\text{Fix}(L(M))$ and the rank function $\text{rk}$ of $M$ (or $L(M)$) by

$$\beta(\text{Fix}(L(M))) = (-1)^{n+1} \sum_{F \in L(M) : F = \psi(F)} \mu^\psi(\emptyset, F) \text{rk}(F).$$

To prove the trace formula, we will show that both sides agree with the beta invariant (up to sign). Note that, this is a generalization of the situation in [20], where the two sides of the Poincaré–Hopf theorem are shown to be equal to the (ordinary) beta invariant of $M$. Even though we only use elementary properties of the generalized beta invariant and the fixed lattice $\text{Fix}(L(M))$, we hope that this results encourages the further study of their combinatorial properties.

**Remark 1.2** Let us emphasize that $\Sigma_M$ denotes the projective matroid fan in $\mathbb{R}^N \cong \mathbb{R}^{N+1}/\mathbb{R}$. The reasons for preferring it over the affine matroid fan $\Sigma'_M$ are essentially
the ones mentioned in the beginning of [20, Section 2.3]. First, the analogous intersection product \( \Gamma_{\psi'} \cdot \Delta' \subset \Sigma'_M \times \Sigma'_M \) is zero for trivial reasons. Second, when trying to compute \( \Gamma_{\psi} \cdot \Delta \) by taking pullbacks to \( \Sigma'_M \times \Sigma'_M \), we are leaving the class of matroid fans which makes certain combinatorial arguments less transparent and natural. To overcome this, one would have to perform a break of symmetry analogous to the one we will encounter for example in Eq. 5, but without benefits.

**Remark 1.3** In general, the notion of (auto-)morphism commonly used in tropical geometry is based on locally \( \mathbb{Z} \)-linear affine maps. This allows for more general automorphisms of matroid fans than the matroidal automorphisms considered here. A trivial example is \( \mathbb{R}^N = \Sigma_{U_{N+1},N+1} \) with \( \text{Aut}(\mathbb{R}^N) = \text{GL}(N, \mathbb{Z}) \rtimes \mathbb{R}^N \) but \( \text{Aut}(U_{N+1},N+1) = S_{N+1} \). For this particular case, the trace formula for general automorphisms follows from [20, Section 4.5]. For arbitrary matroid fans, the situation is unclear. However, the case of matroidal automorphisms seems to be interesting in its own right, since its purely combinatorial nature yields interesting combinatorial features such as the generalized beta invariants considered here. Moreover, a future goal is to decompose the (general) automorphism group of a matroid fan into various subgroups such as the matroidal automorphisms, the ones that fix the coarse subdivision of \( \Sigma_M \) but act by \( \mathbb{Z} \)-linear automorphisms on the linearity space, possibly certain exceptional ones, etc., and reduce the general trace formula to considerations for each subgroup. Encouragement for this approach is provided for example by the results in [21, Theorems 1.2, 6.3, 9.5] which establish that in certain cases, matroidal automorphisms together with certain exceptional automorphisms generate (a big subgroup of) all automorphisms. So, we hope that the present analysis is not only of interest for its particular combinatorial features, but also can serve in the future to establish more general tropical trace formulas.

## 2 Preliminaries

### 2.1 Notational summary

In order to compute the intersection-theoretic side of the trace formula, we will use the same approach as in [20]. That is to say, we will rely on the expression of the diagonal in terms of generic chains of matroids from [11]. In order to keep the overlap to a minimum, we will just give a quick summary of the relevant notation and statements here and refer to reader to the aforementioned sources for more details.

Running through all chains \( C \) of (arbitrary) subsets of \( E \), the collection of cones \( \sigma_C \) forms a unimodular subdivision of \( \mathbb{R}^E \) (and \( \mathbb{R}^E / \mathbb{R}1 \)) which we call the permutohedral fan (since it is the normal fan of the permutohedron). Throughout the following, matroid fans (as well as all other fans to come) will always be represented as subfans of the permutohedral fan. This representation is called the fine subdivision of \( \Sigma_M \). Since the permutohedral fan is unimodular, in order to describe a piecewise \( \mathbb{Z} \)-linear function \( f \) on it, it suffices to prescribe its values on the indicator vectors \( v_S \). We will often use the shorthand \( f(S) \) instead of \( f(v_S) \).
Given two matroids \( M, Q \) on the ground set \( E \), it is obvious that \( \Sigma_Q \subset \Sigma_M \) (both as sets and fans) if and only if \( L(Q) \subset L(M) \) (or, in matroid terminology, \( Q \) is a quotient of \( M \)). In such a case, there exists a canonical sequence of matroids \( Q = M_0, M_1, \ldots, M_s = M \) (called the generic chain) such that \( \text{rk}(M_i) = \text{rk}(Q) + i \) and with rank functions

\[
\text{rk}_{M_i}(S) = \min\{\text{rk}_Q(S) + i, \text{rk}_M(S)\}. \tag{2}
\]

Moreover, there is a sequence of piecewise \( \mathbb{Z} \)-linear functions (on the permutahedral fan) \( g'_1, \ldots, g'_s : \mathbb{R}^{N+1} \to \mathbb{R} \) such that

\[
\Sigma_{M_{s-i}} = g'_1 \cdot g'_{i-1} \cdots g'_1 \cdot \Sigma_M. \tag{3}
\]

If \( Q \) and \( M \) correspond to the hyperplane arrangements associated to the projective subspaces \( K \subset L \subset \mathbb{CP}^n \), then the \( M_i \) corresponds to a generic flag of subspaces \( K \subset S_1 \subset \cdots \subset S_s = L \) (see also Lemma 2.2 for a tropical version). The functions are given by

\[
g'_i(S) = \begin{cases} -1 & \text{rk}_M(S) \geq \text{rk}_Q(S) + s + 1 - i, \\ 0 & \text{otherwise.} \end{cases} \tag{4}
\]

Let us note that we use the max-convention here. Moreover, given a weighted unimodular fan \( \Sigma \) and a function \( g : |\Sigma| \to \mathbb{R} \) which is linear on the cones of \( \Sigma \), we denote by \( g \cdot \Sigma \) the weighted fan defined in [3, Definition 3.4]. It is given by the codimension 1 skeleton of \( \Sigma \) equipped with the weights

\( \Diamond \) Springer
\[
\omega_{g, \Sigma}(\tau) = \sum_{\tau \subset \sigma \in \Sigma} \omega_{\Sigma}(\sigma) g(v_{\sigma/\tau}) - g \left( \sum_{\tau \subset \sigma \in \Sigma} \omega_{\Sigma}(\sigma) v_{\sigma/\tau} \right).
\]

Here, \(v_{\sigma/\tau}\) denotes the primitive generator of the unique ray in \(\sigma\) not contained in \(\tau\) (since \(\Sigma\) is unimodular). See Remark 4.4 for a more detailed analysis of our particular case.

We denote by \(M \oplus_0 M\) the parallel connection of \(M\) with itself along the element 0. Its base set \(E \sqcup 0 E\) is the disjoint union of \(E\) with itself, but the two zeros identified. By convention, we write a subset of \(E \sqcup 0 E\) as a pair \((F, G), F, G \subset E\) such that either \(0 \in F \cap G\) or \(0 \notin F \cup G\). We will always identify \(R^{E/R1} = R^N\) by setting the coordinate corresponding to \(0 \in E\) to zero — \(x_0 = 0\). This induces a natural identification of

\[
R^{E\sqcup_0 E}/R1 = R^{2N} = R^{E/R1} \times R^{E/R1}.
\]

Under this identification, we have \(\Sigma_{M \oplus_0 M} = \Sigma_M \times \Sigma_M\). Using this setup, the diagonal \(\Delta \subset \Sigma_M \times \Sigma_M\) can also be represented as a matroid fan. The rank function of the associated matroid is given by

\[
\text{rk}_\Delta(F, G) = \text{rk}(F \cup G).
\]

Applying the construction from Eq. 3 to \(\Delta \subset \Sigma_M \times \Sigma_M\), we obtain (dehomogenised) functions \(g_1, \ldots, g_n : R^{2N} \to R\) given by

\[
g_i(F, G) = \begin{cases} 
-1 & 0 \notin F, \text{rk}(F) + \text{rk}(G) \geq \text{rk}(F \cup G) + n + 1 - i, \\
+1 & 0 \in F, \text{rk}(F) + \text{rk}(G) \leq \text{rk}(F \cup G) + n + 1 - i, \\
0 & \text{otherwise},
\end{cases}
\]

and such that

\[
\Delta = g_n \cdots g_1 \cdot (\Sigma_M \times \Sigma_M)
\]

(see [20, Equation (12) and Proposition 2.6]). Our computation of \(\text{deg}(\Gamma_\psi \cdot \Delta)\) is based on this description of \(\Delta\).

### 2.2 Hyperplane sections

A special case of Eq. 3 that we will use briefly is when \(Q = U_{1,N+1}\) is the unique loopless matroid of rank 1 with ground set \(E\). In this case, \(\Sigma_Q = \{0\}\). Given any loopless matroid \(M\), let \(M^{\leq i}\) denote the matroid whose flats are the flats of \(M\) of rank at most \(i\) as well as \(E\) (called the \(i\)-th truncation of \(M\)). Indeed, this defines a matroid by an obvious check of axioms or by the following observation.

**Lemma 2.1** Let \(M\) be a loopless matroid and set \(Q = U_{1,N+1}\). Then, the intermediate matroids \(M_i\) from Eq. 3 are equal to \(M^{\leq i}\) for all \(i = 0, \ldots, s\).
Proof This is straightforward using Eq. 2 and $\text{rk}_Q(S) = 1$ for all $S \neq \emptyset$.

As mentioned before, the intermediate matroids $M_i$ should be thought of as generic hyperplane sections. This can be made precise tropically by the following statement, which is originally proven in [13, Lemma 5.1]. As a warm-up, we include a short proof based on the construction from Eqs. 3 and 4.

Lemma 2.2 Let $H$ denote the standard hyperplane in $\mathbb{R}^N$. For $i = 0, \ldots, n$, we have

$$H^{n-i} \cdot \Sigma_M = \Sigma_{M_{\leq i}}.$$  

Proof By induction, it is sufficient to prove $H \cdot \Sigma_M = \Sigma_{M_{\leq n-1}}$. Setting

$$h' = \max\{x_0, \ldots, x_N\},$$

we have $H \cdot \Sigma_M = (h' \cdot \Sigma_M')/R1$, hence, it remains to prove $h' \cdot \Sigma_M' = \Sigma_{M_{\leq n-1}}$. By Lemma 2.1 and Eq. 3, we can write $\Sigma_{M_{\leq n-1}}$ as $g_1' \cdot \Sigma_M'$. By Eq. 4, $g_1'(S) = -1$ if $\text{rk}(S) = n + 1$ and $g_1'(S) = 0$ otherwise. For flats $S = F$ of $M$, $\text{rk}(F) = n + 1$ is equivalent to $F = E$. On the other hand, we have $h'(S) = -1$ if and only if $S = E$. It follows that the functions $h'$ and $g_1'$ agree on flats of $M$ and hence on $\Sigma_M'$, which proves the claim.

2.3 Cutting out matroid fans

As a side remark, let us have a quick look at the opposite situation to the previous subsection, the inclusion $\Sigma_M \subset \mathbb{R}^N = \Sigma_{U_{N+1,N+1}}$. In this case, Eq. 3 allows us to write $\Sigma_M$ as a complete intersection

$$\Sigma_M = g_{N-n} \cdots g_1 \cdot \mathbb{R}^N.$$  

Of course, this is a complete intersection in a rather weak sense. For example, the functions need not be tropically linear nor convex (tropical polynomials) in general. Nevertheless, this description of arbitrary matroid fans might be useful in some contexts. For example, the CSM classes of matroid fans studied in [8] can be written in terms of these functions as follows. Here, given a fan cycle $X \subset \mathbb{R}^N$ and a piecewise linear function $g$ on $X$, we consider $1 + g$ as the operator on the group $Z_*(X)$ of fan cycles in $X$ given by

$$(1 + g): Z_*(X) \to Z_*(X), \quad Z \mapsto Z + g \cdot Z.$$  

The inverse of this operator is given by

$$\frac{1}{1 + g} = 1 - g + g^2 - g^3 + \cdots: Z_*(X) \to Z_*(X).$$

(More formally, $1 + g$ and $1/(1 + g)$ are piecewise polynomial functions or tropical Chow cohomology elements, but this is not needed here).
Proposition 2.3 Let \( \Sigma_M \) be a matroid fan cut out by the functions \( g_i \) as in Eq. 7. Then, the CSM classes of \( \Sigma_M \) are equal to

\[
CSM_*(\Sigma_M) = \prod_{i=1}^{N-n} \frac{g_i}{1 + g_i} \cdot R^N = \prod_{i=1}^{N-n} \frac{1}{1 + g_i} \cdot \Sigma_M.
\]

Proof In the classical case, the analogous formula is essentially a consequence of the definitions and the adjunction formula. Here, the quick and dirty proof goes as follows: We proceed by induction on the codimension. The induction start \( n = N \) is trivial, so let us assume \( n < N \). Consider the generic chain \( M = M_0, M_1, \ldots, M_{N-n} = U_{N+1,N+1} \) associated to \( \Sigma_M \subset R^N \). We set \( \Sigma_0 = \Sigma_M = \Sigma_{M_0} \) and \( \Sigma_1 = \Sigma_{M_1} \). By induction assumption, we have

\[
CSM_*(\Sigma_1) = \prod_{i=1}^{N-n-1} \frac{g_i}{1 + g_i} \cdot R^N.
\]

We set \( g = g_{N-n} \), hence, \( g \cdot \Sigma_1 = \Sigma_0 \). It remains to show that

\[
CSM_*(\Sigma_0) = \frac{g}{1 + g} \cdot CSM_*(\Sigma_1).
\]

Since \( M_0 \) is an elementary quotient of \( M_1 \), there exists a unique matroid \( L \) on the base set \( E \sqcup \{ e \} \) such that \( L \setminus e = M_1 \) and \( L \setminus e = M_0 \). Here, \( e \) is some additional element which is neither loop nor coloop of \( L \). We set \( \Sigma = \Sigma_L \subset R^{N+1} \). Let \( \pi : R^{N+1} \to R^N \) denote the projection forgetting the coordinate \( x_e \). Then, \( \pi : \Sigma \to \Sigma_1 \) is an elementary tropical modification along the function \( g : \Sigma_1 \to R \). By [8, Proposition 5.4], we have

\[
CSM_*(\Sigma) = \pi^*CSM_*(\Sigma_1) - \pi^*CSM_*(\Sigma_0).
\] (8)

Given a fan cycle \( X \in R^{N+1} \), we denote by \( D_e \cdot X \) the star fan \( \text{Star}_{-v_e}(X) \subset R^N \) along the direction \(-v_e\) (we think of this geometrically as the intersection of \( X \) with the divisor \( D_e = \{ x_e = -\infty \} \) at infinity). It follows from [11, Lemma 8.8] and [17, Proposition 5.3.13] that \( D_e \cdot \pi^*X = g \cdot X \) for any fan cycle \( X \subset \Sigma_1 \). Moreover, by the local definition of CSM classes, it is clear that \( D_e \cdot CSM_*(\Sigma) = CSM_*(D_e \cdot \Sigma) = CSM_*(\Sigma_0) \). Hence, intersecting Eq. 8 with \( D_e \), we obtain

\[
CSM_*(\Sigma_0) = g \cdot (CSM_*(\Sigma_1) - CSM_*(\Sigma_2)).
\]

Solving for \( CSM_*(\Sigma_2) \), the result follows. \(\square\)

The \( n_\ast \)-classes appearing in [8, Section 9.3] in connection with Speyer’s \( g \)-polynomial [23] can be written as

\[
n_\ast(\Sigma_M) = \prod_{i=1}^{N-n} \frac{1 + g_i}{1 + h} \cdot \Sigma_M.
\]

Here, \( h = \max\{ x_0, \ldots, x_N \} - x_0 \). Finally, note that, \( g_1 = h \) if and only if \( M \) has no coloops. Hence, in this case \( n_\ast \) can be further simplified by clearing the denominator. In view of the \( g \)-polynomial conjecture, it is interesting to study the positivity properties of the functions \( g_i \) and the expressions above.
3 Matroidal automorphisms and their beta invariant

3.1 Generalized beta invariants

Beta invariants are usually defined in the context of geometric lattices. We want to extend the definition to the case of arbitrary lattices \( L \) equipped with a rank function \( \text{rk} \). The application we have in mind is the sublattice \( L \subseteq \text{Fix}(L(M)) \) of fixed flats of an matroidal automorphism (with the restricted rank function), see Sect. 3.2. We define the generalized beta invariant as (up to signs) the convolution of \( \text{rk} \) with the Möbius function \( \mu \) of \( L \). The main property that we will need later (Lemma 3.3) is based purely on this definition, without further requirements on \((L, \text{rk})\).

Let \( L \) be a finite lattice with partial order \( \subseteq \), minimal element \( \emptyset \) and maximal element \( E \). Let \( \text{rk} : L \to \mathbb{Z} \) be an arbitrary function on \( L \), called the rank function. We set \( n := \text{rk}(E) - \text{rk}(\emptyset) - 1 \).

**Convention.** We emphasize that in this text, the symbol \( \subseteq \) is used synonymously to \( \subseteq \) or \( \leq \), that is, equality is included.

**Definition 3.1** The beta invariant of \( L \) (or rather, \((L, \text{rk})\)) is

\[
\beta(L) := (-1)^{n+1} \sum_{F \in L \cup E} \mu(\emptyset, F) \text{rk}(F).
\]

For any \( F \in L \), we equip the interval \([F, E]\) with the restricted rank function \( \text{rk} |_{[F, E]} \) and set

\[
\beta(F) := \beta([F, E]) = (-1)^{\text{rk}(E) - \text{rk}(F)} \sum_{F \subseteq G \subseteq L} \mu(F, G) \text{rk}(G). \tag{9}
\]

**Lemma 3.2** For any \( G \in L \), we have

\[
\text{rk}(G) = \sum_{G \subseteq F \subseteq L} (-1)^{\text{rk}(E) - \text{rk}(F)} \beta(F).
\]

**Proof** This is just Möbius inversion for Eq. 9. \( \square \)

For our purposes, it will be useful to rewrite this formula in a particular, asymmetric way.

**Lemma 3.3** Fix an element \( G \in L, G \neq \emptyset \). Then,

\[
(-1)^{n} \beta(L) = \text{rk}(G) - \text{rk}(\emptyset) - \sum_{\emptyset \neq F \notin [G, E]} (-1)^{\text{rk}(E) - \text{rk}(F) - 1} \beta(F).
\]

**Proof** Since \( \beta(\emptyset) = \beta(L) \), the formula is a rearrangement (including some sign yoga) of

\[
\text{rk}(\emptyset) - \text{rk}(G) = \sum_{F \notin [G, E]} (-1)^{\text{rk}(E) - \text{rk}(F)} \beta(F).
\]
This follows from using Lemma 3.2 twice. □

**Remark 3.4** Let $K \subset L$ be a sublattice with $\emptyset, E \in K$. We equip $K$ with the restricted rank function $\text{rk} |_{K}$. For any $S \subset L$, we set

$$\text{cl}(F) = \bigwedge_{G \in K, F \subset G} G.$$ 

By general properties of Möbius functions, the Möbius function $\mu^K$ of $K$ can be computed in terms of the Möbius function $\mu^L$ of $L$ by

$$\mu^K(\emptyset, G) = \sum_{F \in L, \text{cl}(F) = G} \mu^L(\emptyset, F).$$

It follows that the beta invariant of $K$ can be expressed as

$$\beta(K) = (-1)^{n+1} \sum_{G \in K} \mu^K(\emptyset, G) \text{rk}(G) = (-1)^{n+1} \sum_{F \in L} \mu^L(\emptyset, F) \text{rk}(\text{cl}(F)).$$ (10)

### 3.2 Matroidal automorphisms

A matroid automorphism $\psi : M \to M$ is a bijection $\psi : E \to E$ such that $F$ is a flat if and only if $\psi(F)$ is a flat. This induces an automorphism of geometric lattices $\psi : L(M) \to L(M)$ which we denote by the same letter. Vice versa, a lattice automorphism $\psi : L(M) \to L(M)$ defines a matroid automorphism up to the ambiguity of permuting parallel elements. For what follows, it is actually enough to fix the lattice automorphism $\psi : L(M) \to L(M)$ (since the linear span of $\Sigma_M$ is generated by the indicator vectors of flats).

**Definition 3.5** Let $M$ be a loopless matroid. Given an matroid automorphism $\psi : M \to M$, the associated *matroidal automorphism* $\Psi : \Sigma_M \to \Sigma_M$ is the restriction of the linear map on $\mathbb{R}^N$ given by

$$\Psi : v_S \mapsto v_{\psi(S)}$$

for all $S \subset E$.

It follows from the definition that $\Psi$ is a tropical automorphism of $\Sigma_M$ which respects its fine subdivision.

In the context of the trace formula, we are interested in the subset

$$\text{Fix}(L(M)) = \{ F \in L(M) : F = \psi(F) \}.$$

**Lemma 3.6** The set $\text{Fix}(L(M))$ is a sublattice of $L(M)$. 

 Springer
**Proof** We need to show that $\text{Fix}(L(M))$ is closed under $\wedge$ (intersection) and $\vee$ (union and closure). Since $\psi$ is a bijection, taking images commutes with intersection/unions, and since it takes flats to flats, it also commutes with taking closures. The statement follows. \hfill $\square$

**Remark 3.7** As a sublattice of $L(M)$, $\text{Fix}(L(M))$ is distributive. Note, however, that in general, it is neither graded, atomic, coatomic nor complemented.

In the context of Sect. 3.1, we will always equip $\text{Fix}(L(M))$ with the rank function $\text{rk}$ of $L(M)$ restricted to $\text{Fix}(L(M))$. In other words, we set

$$
\beta(\text{Fix}(L(M))) = (-1)^{n+1} \sum_{F \in L(M)} \mu^\psi(\emptyset, F) \text{rk}(F)
$$

and for any $F \in \text{Fix}(L(M))$

$$
\beta(\text{Fix}([F, E])) = (-1)^{\text{rk}(E)-\text{rk}(F)} \sum_{G \in L(M)} \mu^\psi(F, G) \text{rk}(G). \quad (11)
$$

Here, $\mu^\psi$ denotes the Möbius function on $\text{Fix}(L(M))$ (in contrast to the Möbius function of $L(M)$). So, to emphasize, the definition mixes the Möbius function of $\text{Fix}(L(M))$ with the rank function of $L(M)$. Note that, given $F \in \text{Fix}(L(M))$, we could alternatively consider the contracted matroid $M/F$ with rank function $\text{rk}' = \text{rk} - \text{rk}(F)$ and induced automorphism $\psi'$. The beta invariant

$$
\beta(\text{Fix}(M/F)) = (-1)^{\text{rk}(E)-\text{rk}(F)} \sum_{G \in L(M)} \mu^\psi(F, G)(\text{rk}(G) - \text{rk}(F)). \quad (12)
$$

is equal to $\beta(\text{Fix}([F, E]))$ from Eq. 11, except for $F = E$ (since summing the Möbius function over a non-trivial interval gives zero). In the (non-relevant) case $F = E$, we have $\beta([E]) = \text{rk}(E) = n+1$. Finally, note that, Eq. 10 applied to $K = \text{Fix}(L(M)) \subset L(M) = L$ allows us to rewrite the beta invariant as

$$
\beta(\text{Fix}(L(M))) = (-1)^{n+1} \sum_{F \in L(M)} \mu(\emptyset, F) \text{rk}(F^\psi)
$$

where $F^\psi$ denotes smallest flat containing $F$ and fixed under $\psi$,
We recall from the introduction that our main theorem consists in the following equations.

\[ \deg(\Gamma \psi \cdot \Delta) = (-1)^n \beta(\text{Fix}(L(M))) = \sum (-1)^p \text{Tr}(\Psi, F_p(\Sigma_M)) \]

The next two sections are devoted to the proofs of the first and second equality, respectively.

### 4 The intersection-theoretic side

In this section, our goal is to prove

\[ \deg(\Gamma \psi \cdot \Delta) = (-1)^n \beta(\text{Fix}(L(M))). \] (13)

#### 4.1 General approach

We denote by \( \Gamma : \Sigma_M \to \Sigma_M \times \Sigma_M, x \mapsto (x, \psi(x)) \) the graph map. We denote by \( f_i := \Gamma^*(g_i) \) the pullbacks of the functions \( g_i \) constructed in Eq. 5.

**Lemma 4.1** With the notations from above, we have

\[ \deg(\Gamma \psi \cdot \Delta) = \deg(f_n \cdots f_1 \cdot \Sigma_M). \]

**Proof** This follows from Eq. 6, [11, Theorem 4.5 (6)] and the projection formula [3, Proposition 7.7]. \( \square \)

We set \( X_k := f_k \cdots f_1 \cdot X \). Our goal is to compute information about these intermediate intersection products inductively. The first new technical difficulty that we encounter in comparison with [20] is the fact that \( \Gamma : \Sigma_M \to \Sigma_M \otimes_0 M \) is in general not a map of fans, i.e. the cones of \( \Sigma_M \) are not necessarily mapped to cones of \( \Sigma_M \otimes_0 M \). Consequently, the functions \( f_k \) are in general not linear when restricted to cones \( \sigma \mathcal{F} \subset \Sigma_M \). In principal, this could lead to fan structures which cannot be described as subfans of the permutahedral fan. However, somewhat mysteriously, it turns out that no such refinements are necessary. In fact, we will show that \( f_k \) is linear when restricted to a (nonzero) cone of \( X_{k-1} \) (but not of \( \Sigma_M \)). By induction, this is enough to ensure that \( X_k \) can (still) be written as a subfan of the permutahedral fan. The following lemma contains the technical key piece of this argument.

**Lemma 4.2** Pick \( k \in \{1, \ldots, n\} \) and let \( \mathcal{F} = (F_i) \) be a chain of flats such that one of the following conditions holds:

(a) For any \( i \), either \( 0 \in F_i \cap \psi(F_i) \) or \( 0 \not\in F_i \cup \psi(F_i) \).
(b) The gap sequence of \( \mathcal{F} \) is \( \text{gap}(\mathcal{F}) = (k-1, 0, \ldots, 0) \).

Then, the function \( f_k \) is linear on \( \sigma \mathcal{F} \).
Proof If property (a) holds, then $\Gamma(\sigma F)$ is a cone in the fine subdivision of $\Sigma_{M \otimes_0 M}$, given by the chain of flats $((F_i, \psi(F_i)))$. Since the $g_k$ are linear on such a cone by definition, the $f_k$ are linear on $\sigma F$ as well.

Let us now assume property (b) holds. We set $G_i = \psi(F_i)$ and choose $p$ and $q$ maximal with the property that $0 \in F_p$ and $0 \in G_q$, respectively. If $p = q$, we are back in the case of property (a). By symmetry of the function $g_k$ in the two factors, we can assume $p > q$ without loss of generality. Given a point $x \in \sigma F$, we write it as a positive linear combination of the $\psi(F_i)$ with coefficients $a_i$. We claim that $f_k|_{\sigma F}(x)$ is equal to the linear function $A(x) := \sum_{i=1}^p a_i$.

Set $y := \Gamma(x) = (x, \psi(x))$. Using indicator vectors, we can write $y$ as

$$y = a_1 v(F_1, G_1) + \cdots + a_q v(F_q, G_q)$$
$$+ a_{q+1}(v(F_{q+1}, E) + v(\emptyset, G_{q+1})) + \cdots + a_p (v(F_p, E) + v(\emptyset, G_p))$$
$$+ a_{p+1} v(F_{p+1}, G_{p+1}) + \cdots + a_l v(F_l, G_l).$$

Here, we use the formula $(v_F, v_G) = v(F, E) + v(\emptyset, G)$ if $0 \in F \setminus G$, which follows from our conventions. Note that, when considered as an equation in $\mathbb{R}^{E'} = \mathbb{R}^{2n+1}$, $y$ is normalised in the sense that its minimal coordinate is 0. Moreover, $y_0 = A(x)$. More generally, the coefficient $y_e$ of $y$, depending on whether $e$ is chosen from the first or second copy of $E$, is equal to $b_i := a_1 + \cdots + a_i$ or $c_i := (a_{q+1} + \cdots + a_p) + b_i$ where $i \in \{0, \ldots, l\}$ denotes the maximal index such that $e \in F_i$ or $e \in G_i$, respectively. It follows that the cone $\sigma \subset \Sigma_{M \otimes_0 M}$ that contains $y$ in its relative interior corresponds to a chain of flats of the form $(F_i, G_j)$. Indeed, given $t \in \mathbb{R}$ denote by $i(t)$ and $j(t)$ the maximal indices such that $b_{i(t)} \leq t$ and $c_{j(t)} \leq t$. As we increase $t$, the tuples $(F_i(t), G_j(t))$ yield the required chain. Note that, $b_p = c_q$, so indeed each of the tuples $(F_i, G_j)$ in this chain satisfies $0 \in F_i \cap G_j$ or $0 \notin F_i \cup G_j$, which can also be rewritten as either $i \leq p$, $j \leq q$ or $i > p$, $j > q$. Note that, $p \geq 1$, so in the latter case $i \geq 2$.

By our assumption, we have $rk(F_i) = rk(G_i) \leq n + 1 - k$ for $i \geq 1$ and $rk(F_i) = rk(G_i) < n - k + 1$ for $i \geq 2$. It follows that $rk(F_i) + rk(G_j) \leq rk(F_i \cup G_j) + n + 1 - k$ if $(i, j) \neq (0, 0)$ and the inequality is strict if $i > p$, $j > q$. Using the description of $g_k$ in Eq. 5, we find

$$g_k(F_i, G_j) = \begin{cases} 1 & i \leq p, j \leq q, (i, j) \neq (0, 0), \\ 0 & i > p, j > q \text{ or } (i, j) = (0, 0). \end{cases}$$

We now write $y$ as a positive sum of the cone generators of $\sigma$. Setting the coefficient $v_{(E, E)}$ to zero, this sum is normalised as above (minimal coordinate is 0) and hence equal to Eq. 14 in $\mathbb{R}^{2n+1}$. Moreover, by the previous computation, $g_k(y)$ (and hence $f_k(x)$) is equal to the sum of coefficients of the vectors $v_{(F_i, G_j)}$ with $0 \in F_i \cap G_j$, which clearly is equal to $y_0$. Since $y_0 = A(x)$, this proves the claim. \qed
4.2 Combinatorial properties of $X_k$

We will now proceed by formulating certain properties of the intermediate intersection products $X_k$. We split the properties into two statements: Here, we consider combinatorial properties, in the next subsection, we discuss weights.

Lemma 4.3 For all $k \in \{0, \ldots, n\}$, the following statements hold.

(a) The tropical cycle $X_k$ can be represented as a weighted subfan of the permutahedral fan. (By a facet of $X_k$, we mean a cone of nonzero weight in $X_k$.)

(b) For a facet $\sigma F$ of $X_k$, the gap sequence of $F$ has one of the following two forms.

$$\text{gap}(F) = (r, s, 0, \ldots, 0) =: (A), \quad r + s = k,$$

$$\text{gap}(F) = (r, s, 0, \ldots, 0, 1, 0, \ldots, 0) =: (B), \quad r + s = k - 1. \quad (15)$$

(c) If $\text{gap}(F) = (A), 0 \notin F_1 \cup \psi(F_1)$ and $s \geq 1$, then $F_1 = \psi(F_1)$.

(d) If $\text{gap}(F) = (A), 0 \in F_1 \cup \psi(F_1)$ and $s \geq 1$, then $0$ and $\psi^{-1}(0)$ are linearly independent in $F_1/F_2$. If moreover $s \geq 2$, then $F_1 = \psi(F_1)$.

(e) If $\text{gap}(F) = (B)$ and $G \supset H$ denotes the part of $F$ corresponding to the final $1$ in $\text{gap}(F)$, then $0$ and $\psi^{-1}(0)$ are linearly independent in $G/H$ (and hence, $G = \text{cl}(H \cup \{0, \psi^{-1}(0)\})$). If moreover $s \geq 1$, then $F_1 = \psi(F_1)$.

(f) For $k < n$, $f_k+1$ is linear when restricted to a facet of $X_k$.

The proof of this lemma mostly consists of simple, but tedious calculations of weights in the intersection product $f_{k+1} \cdot X_k$. To make it as transparent as possible, we will first collect a couple of recurrent arguments and case distinctions that occur in this calculation.

Remark 4.4 Let $\tau$ be a codimension one cone in $X_k$ and let $F$ be the corresponding chain of flats. Our general goal is to compute the weight $\omega(\tau)$ of $\tau$ in $f_{k+1} \cdot X_k$.

(a) The facets in $X_k$ containing $\tau$ correspond to filling a (non-trivial) gap $G \supset H$ of $F$ with an additional flat $G \supset F \supset H$. The balancing condition around $\tau$ naturally splits into separate equations, one for each gap of $F$ (and the facets corresponding to it). In particular, the calculation of $\omega(\tau)$ can be split into a calculation for each gap.

(b) Let $F^1, \ldots, F^m$ denote the flats corresponding to the facets of $X_k$ for a given gap $G \supset H$ of $F$. The typical situation will be that the sets $F^i \setminus H$ form a partition of $G$. In this situation, the involved indicator vectors satisfy a unique linear relation (up to multiples), namely

$$\sum_{i=1}^{m} v_{F^i} = v_G + (m - 1)v_H.$$ 

By uniqueness, it follows that the weights of the corresponding facets in $X_k$ are all equal, say, to $\omega \in \mathbb{Z} \setminus \{0\}$. 

$\square$ Springer
(c) By induction, we may assume that \( f_{k+1} \) is linear on the facets of \( X_k \) (which are cones of the permutahedral fan). It follows that in order to compute \( \omega(\tau) \), it is sufficient to know the values \( f_{k+1}(F) \). Using Eq. 5, these values are given by

\[
 f_{k+1}(F) = \begin{cases} 
 -1 & 0 \notin F \cup \psi(F) \text{ and } 2 \text{rk}(F) \geq \text{rk}(F \cup \psi(F)) + n - k, \\
 0 & 0 \notin F \cap \psi(F) \text{ and } 2 \text{rk}(F) = \text{rk}(F \cup \psi(F)) + n - k, \\
 0 & 0 \in (F \cup \psi(F)) \setminus (F \cap \psi(F)) \text{ and } \text{rk}(F) \leq n - k, \\
 +1 & \text{otherwise.} 
\end{cases} 
\]  

(17)

In the third case, we use the fact that \( \Gamma(\nu_F) = \nu_{(F, E)} + \nu_{(\emptyset, \psi(F))} \) or \( \Gamma(\nu_F) = \nu_{(F, \emptyset)} + \nu_{(E, \psi(F))} \), depending on whether \( 0 \in F \) or \( 0 \in \psi(F) \). It is convenient to list a few particular cases, focusing on the critical rank \( n - k \).

\[
 f_{k+1}(F) = \begin{cases} 
 -1 & 0 \notin F = \psi(F) \text{ and } \text{rk}(F) \geq n - k, \\
 0 & 0 \notin F \cup \psi(F), F \neq \psi(F) \text{ and } \text{rk}(F) = n - k, \\
 0 & 0 \notin F \cup \psi(F) \text{ and } \text{rk}(F) < n - k, \\
 0 & 0 \in F = \psi(F) \text{ and } \text{rk}(F) \geq n - k + 1, \\
 +1 & 0 \in F \cap \psi(F), \text{ and } \text{rk}(F) = n - k + 1, \\
 +1 & 0 \in F \cap \psi(F) \text{ and } \text{rk}(F) \leq n - k. 
\end{cases} 
\]  

(18)

(d) Going back to the partition case of item (b), let \( q \) be the number of flats \( F^i \) such that \( 0 \in F^i \cup \psi(F^i) \). The possible values are \( q = 0, 1, 2, m \). The first and latter case correspond to \( 0 \notin G \cup \psi(G) \) and \( 0 \notin H \cup \psi(H) \), respectively. The case \( q = 2 \) occurs if \( 0 \text{ and } \psi^{-1}(0) \) are linearly independent in \( G/H \). The remaining cases correspond to \( q = 1 \). Based on all the previous comments, we list in Table 1 the computation of \( \omega(\tau) \) (or rather, the contribution of a fixed gap \( G \supset H \) to it) for the various values of \( q \) and with various extra conditions.

**Proof of Lemma 4.3** The initial case \( k = 0 \) is trivial (except for item (f), which follows by the same argument as below).

Let us consider the induction step \( k \to k + 1 \). Item (a), note that, by induction assumption, \( X_k \) can be represented on the permutahedral fan and \( f_{k+1} \) is linear on the facets of this representation. Hence, \( X_{k+1} = f_{k+1} \cdot X_k \) can also be represented on the permutahedral fan.

For item (b), let \( \tau \) be a cone of dimension \( n - k - 1 \) whose weight \( \omega(\tau) \) in \( X_{k+1} \) is nonzero. Let \( \mathcal{F} \) be the corresponding chain of flats with gap sequence \( \text{gap}(\mathcal{F}) = (r_0, \ldots, r_l) \). We need to show that \( S := \sum_{i=2}^l r_i \leq 1 \). If \( S > 2 \), then by induction assumption, \( \tau \) is not contained in any facet of \( X_k \), a contradiction.

So let us assume \( S = 2 \). Note that, hence \( r_1 + r_2 = k - 1 \) and therefore \( \text{rk}(F_2) = n - k \).

If \( (r_2, \ldots, r_l) = (\ldots, 1, \ldots, 1, \ldots) \) (the dots represent zeros), one of the two gaps must be as in item (e) and the facets of \( X_k \) correspond to filling the other gap. Depending on the ordering of the gaps, \( \omega(\tau) \) is computed according to Table 1, row 1, \( q = 0 \) or \( q = m \). In both cases, \( \omega(\tau) = 0 \). If \( (r_2, \ldots, r_l) = (\ldots, 2, \ldots) \), with gap \( G \supset H \), then \( 0 \text{ and } \psi^{-1}(0) \) must be linearly independent in \( G/H \) and the possible fillings are given by \( F = \text{cl}(H \cup \{0, \psi^{-1}(0)\}) \) and \( F \notin \text{cl}(H \cup \{0, \psi^{-1}(0)\}) \), \( \text{rk}(F) = \text{rk}(H) + 1 \).
Table 1 The computation of the weight ω(τ) for various types of facets τ in $f_{k+1} \cdot X_k$. The notation is borrowed from Remark 4.4.

| Condition | $q = 0$ | $q = 1$ | $q = 2$ | $q = m$ |
|-----------|---------|---------|---------|---------|
| $f_{k+1}(G)$ | $rk G < n - k$ | $rk G \leq n - k$ | $rk G \leq n - k$ | $rk G \leq n - k$ |
| $f_{k+1}(F^i)$ | $0, \ldots, 0$ | $1, 0, \ldots, 0$ | $1, 1, 0, \ldots, 0$ | $1, \ldots, 1$ |
| $f_{k+1}(H)$ | $0$ | $0$ | $0$ | $0$ |
| $\omega(τ)$ | $0$ | $0$ | $0$ | $0$ |
| $G \neq \neq ψ(G)$ | $G \neq \neq ψ(G)$ | $G \neq \neq ψ(G)$ | $G \neq \neq ψ(G)$ | $G \neq \neq ψ(G)$ |
| $f_{k+1}(G)$ | $−1/0$ | $0/1$ | $0/1$ | $0/1$ |
| $f_{k+1}(F^i)$ | $0, \ldots, 0$ | $1, 0, \ldots, 0$ | $1, \ldots, 1$ | $1, \ldots, 1$ |
| $f_{k+1}(H)$ | $0$ | $0$ | $0$ | $0$ |
| $ω(τ)$ | $ω_0/0$ | $ω_0/0$ | $ω_0/0$ | $ω_0/0$ |
| $G = ψ(G)$ | $G = ψ(G)$ | $G = ψ(G)$ | $G = ψ(G)$ | $G = ψ(G)$ |
| $rk F^i = n - k - 1$ | $0$ | $0$ | $0$ | $0$ |
| $f_{k+1}(G)$ | $−1$ | $0$ | $0$ | $0$ |
| $f_{k+1}(F^i)$ | $0, \ldots, 0$ | $1, 0, \ldots, 0$ | $1, \ldots, 1$ | $1, \ldots, 1$ |
| $f_{k+1}(H)$ | $0$ | $0$ | $0$ | $0$ |
| $ω(τ)$ | $ω$ | $ω$ | $ω$ | $ω$ |

Note that, this still induces a partition of $G \setminus H$ and $q = 1$, so by Table 1, row 1, $q = 1$, we get $ω(τ) = 0$ again. This finishes item (b).

We proceed with item (c), so $\text{gap}(F) = (r, s, \ldots)$ with $r + s = k + 1$, $s \geq 1$ and $0 \notin F_1 \cup ψ(F_1)$. We want to show $F_1 = ψ(F_1)$. Note that, the only possible (nonzero) fillings must have gap sequence $(r, s - 1, \ldots)$ (since type (B) is excluded by $0 \notin F_1 \cup ψ(F_1)$ and item (e)). For $s > 1$, the statement follows by induction assumption. For $s = 1$, note that, $r = k$, and hence, $rk F_1 = n - k$. So the statement follows from Table 1, row 2, $q = 0$.

For item (d), we assume $\text{gap}(F) = (r, s, \ldots)$ with $r + s = k + 1$, $s \geq 1$ and $0 \in F_1 \cup ψ(F_1)$. Assume that $0$ and $ψ^{-1}(0)$ are not linearly independent in $F_1/F_2$. Then, we must have $s = 1$ and the only possible gap sequence for fillings is $(r, 0, \ldots)$ (all other possibilities have no facets adjacent). In this case, $rk F_1 = n - k$ and Table 1, row 1, $q = 1$ or $q = m$ gives $ω(τ) = 0$, a contradiction.

Now assume $s \geq 2$. We need to show $F_1 = ψ(F_1)$. The only possible gap sequences of fillings are $(r, s - 1, 0, \ldots)$ and $(r, s - 2, 1, \ldots)$. For $s > 2$, the statement follows by the induction assumption. For $s = 2$, we have $rk(F_1) = n - k + 1$ and the possible fillings agree with the ones described in the last case of item (a). This is covered by Table 1, row 2, $q = 1$. 
Let us now consider item (e), so \( \text{gap}(\mathcal{F}) = (r, s, \ldots, 1, \ldots) \). Assume first that 0 and \( \psi^{-1}(0) \) are not linearly independent in \( G \supseteq H \) (the final gap in \( \mathcal{F} \)). Then, the only possible fillings are fillings of \( G \supseteq H \). But the corresponding weight can be computed according to Table 1, row 1, and we get a nonzero weight only if \( q = 2 \), a contradiction.

Now let \( s \geq 1 \) and assume that \( F_1 \neq \psi(F_1) \). Since by the previous argument 0 \( \notin F_2 \cap \psi(F_2) \), the only possible fillings have gap sequence \( (r, s-1, \ldots, 1, \ldots) \). Again, if \( s > 1 \), the statement follows from the induction assumption. If \( s = 1 \), we have \( \text{rk} F_1 = n - k + 1 \) and hence the claim follows from Table 1, row 2, \( q = m \). This finishes the proof of item (e).

Finally, let us prove item (f). Note that, up to now, we established that \( X_{k+1} \) can be represented as a weighted subfan of the permutahedral fan and that its facets satisfy the properties of item (c)–item (e). Then, the linearity of \( f_{k+2} \) follows from Lemma 4.2. Indeed, note that, all facets of \( X_{k+1} \) satisfy condition (a) of that lemma, except for the facets from item (c) or item (d) with \( s = 0 \). These ones satisfy condition (b) instead. \( \square \)

### 4.3 Weights on \( X_k \)

Based on our understanding of the combinatorics of \( X_k \), we can now describe the weights of some of its facets.

**Lemma 4.5** Fix \( k \in \{0, \ldots, n\} \) and let \( \sigma = \sigma_{\mathcal{F}} \) be a facet of \( X_k \). Then, the following statements hold.

(a) If \( \text{gap}(\mathcal{F}) = (k, 0, \ldots, 0) \) and \( 0 \in F_1 \cup \psi(F_1) \), then \( \omega(\sigma) = 1 \).

(b) If \( \text{gap}(\mathcal{F}) = (k-1, 0, \ldots, 0, 1, 0, \ldots, 0) \) and \( F_1 \neq \psi(F_1) \), then \( \omega(\sigma) = \text{rk} F_1 - \text{rk} F_1 \).

(c) If \( 0 \notin F_1 \) and \( F_1 = \psi(F_1) \), then \( \omega(\sigma) = (-1)^{n - \text{rk} F_1} \beta(\text{Fix}(M/F_1)) \).

Before proving the lemma, let us check that it implies Eq. 13 as promised.

**Proof** (Equation 13) In the case \( k = n \), the only chain of correct dimension is the trivial flag \( \mathcal{F} = (E \supset \emptyset) \). We have \( \text{gap}(\mathcal{F}) = (n) \) and \( 0 \notin F_1 \cup \psi(F_1) = \emptyset \). Hence, by item (c) of Lemma 4.5 and Lemma 4.1, we conclude

\[
\deg(\Gamma \psi \cdot \Delta) = \deg(X_n) = \omega(\sigma_{(E \supset \emptyset)}) = (-1)^n \beta(\text{Fix}^\psi(M)).
\]

\( \square \)

We now want to prove Lemma 4.5.

**Proof** (Lemma 4.5) We (again) proceed by induction on \( k \). For \( k = 0 \), the statements are trivial (only item (a) and item (c) occur, and both give weight 1 in this case).

We now prove the induction step \( k \to k+1 \). Let \( \sigma = \sigma_{\mathcal{F}} \) be a facet of \( X_{k+1} \). We start with item (a). In this case, the facets of \( X_k \) containing \( \sigma \) correspond to fillings \( E \supset F \supset F_1 \) with gap sequences of the form \( (r, s, \ldots), r + s = k \). Note that, since
We proceed with item (b). In this case, we have two gaps that can potentially be filled, namely $E \supseteq F_1$ and $G \supseteq H$, the gap corresponding to the final 1. By Lemma 4.3, 0 and $\psi^{-1}(0)$ are independent in $G/H$. Hence, the contribution $\omega_1$ of the fillings of $G \supseteq H$ to $\omega(\sigma)$ can be computed according to Table 1, row 1, $q = 2$. Note that, these fillings correspond to facets of $X_k$ of the type discussed in item (a), hence by induction assumption, all have weight 1. It follows that $\omega_1 = 1$.

Let us now consider the gap $E \supseteq F_1$. This is the first time that we encounter a facet structure that is not just given by a partition of $E \setminus F_1$. In fact, by Lemma 4.3, the possible fillings are given by flats $F \supseteq F_1$ which satisfy at least one of the following conditions: Either $rk F = rk F_1 + 1 = n - k + 1$, or $F = \psi(F)$. Denoting the weights of the corresponding facets in $X_k$ by $\omega(F)$, the balancing condition for $X_k$ states that there are coefficients $\omega(E), \omega(F_1) \in \mathbb{Z}$ such that

$$
\sum_F \omega(F)v_F = \omega(E)v_E + \omega(F_1)v_{F_1}.
$$

(19)

Pick an element $i \in E \setminus F_1$. Then, we can express the coefficients $\omega(E), \omega(F_1)$ as

$$
\omega(E) = \sum_{F \ni i} \omega(F),
$$

$$
\omega(F_1) = \sum_{F} \omega(F) - \omega(E) = \sum_{F \ni i} \omega(F).
$$

(20)

By assumption, $F_1 \neq F_1^\psi$ and we can choose $i \in F_1^\psi \setminus F_1$. This implies $i \in F$ for all $F = \psi(F)$, so for this choice of $i$, the sum for $\omega(F_1)$ can be restricted to $F$ with $F \neq \psi(F)$ (and hence, $rk F = rk F_1 + 1$). Finally, by comparing with Eq. 18, we find that the values of the indicator vectors in Eq. 19 under $f_{k+1}$ are zero except for the cases $F_1$ and $F \neq \psi(F)$, in which case the value is 1. We conclude that the contribution $\omega_2$ of $E \supseteq F_1$ is

$$
\omega_2 = \sum_{F \ni i} \omega(F) - \omega(F_1)
$$

$$
= \sum_{F \neq \psi(F)} \omega(F) - \sum_{F \ni i} \omega(F) = \begin{cases} 
\omega(F') & \text{if } F' \neq F_1^\psi \\
0 & \text{if } F' = F_1^\psi
\end{cases}
$$

using the shorthand $F' = \text{cl}(F_1 \cup \{i\})$. Note that, we have $(F')^\psi = F_1^\psi$, so either by induction assumption (if $F' \neq F_1^\psi$) or trivially (if $F' = F_1^\psi$), we have

$$
\omega_2 = rk F_1^\psi - rk \text{cl}(F_1 \cup \{i\}) = rk F_1^\psi - rk F_1 - 1.
$$
So finally, we get \( \omega(\sigma) = \omega_1 + \omega_2 = \operatorname{rk} F_1^\psi - \operatorname{rk} F_1 \), which proves item (b).

It remains to prove item (c). In this case, \( 0 \notin F_1 \cup \psi(F_1) \), and hence, \( \operatorname{gap}(\mathcal{F}) = (r, s, \ldots) \), \( r + s = k + 1 \). Assume first that \( s \geq 1 \). Then, by Lemma 4.3, the only possible fillings have gap sequence \( (r, s - 1, \ldots) \). Note that, \( \operatorname{rk} F_2 = n - k - 2 \), while the ranks of the fillings are \( \operatorname{rk} F = \operatorname{rk} F_2 + 1 = n - k - 1 \). We can thus use Table 1, row 3, \( q = 0 \), which proves the claim in this case.

Assume now \( s = 0 \), so \( \operatorname{gap}(\mathcal{F}) = (k + 1, \ldots) \). This is another case where the balancing condition cannot be expressed in terms of a partition of \( E \setminus F_1 \). Note that, \( \operatorname{rk} F_1 = n - k - 1 \), and hence, \( f_{k+1}(F_1) = f_{k+1}(E) = 0 \). So, in order to compute \( \omega(\sigma) \), we are only interested in fillings \( E \supseteq F \supseteq F_1 \) which correspond to facets of \( X_k \), on the one hand, and for which \( f_{k+1}(F) \neq 0 \), on the other hand. By comparing Eq. 18 and Lemma 4.3, we see that such flats \( F \) belong to one of the following three subcases:

(i) \( 0 \notin F, F = \psi(F) \)
(ii) \( 0 \in F \cup \psi(F), \operatorname{rk} F = n - k \)
(iii) \( 0 \in F \cup \psi(F), \operatorname{rk} F = n - k + 1, F = \cl(F_1 \cup \{0, \psi^{-1}(0)\}), F \neq \psi(F) \)

We set \( G = \cl(F_1 \cup \{0\}) \). Note that, \( f_{k+1}(F) = -1 \) for item (i) and \( f_{k+1}(F) = 1 \) for items (ii) and (iii). By the induction assumption, the flats of type (i) account for the last term in Lemma 3.3 applied to \( L = \Fix^\psi(M/F_1) \) and \( G^\psi \). Hence, it remains to show that the contribution \( \omega' \) of items (ii) and (iii) to \( \omega(\sigma) \) is \( \operatorname{rk}(G^\psi) - \operatorname{rk}(F_1) \) (the first two terms in Lemma 3.3). This can easily be checked by going through the following case consideration.

- \( \operatorname{rk}(G^\psi) = \operatorname{rk}(F_1) + 1 \)—In this case, \( F = G^\psi = G \) is the only flag of type (ii) and type (iii) does not occur, so \( \omega' = 1 \).
- \( \operatorname{rk}(G^\psi) = \operatorname{rk}(F_1) + 2 \)—In this case, \( G \) and \( \cl(F_1 \cup \{\psi^{-1}(0)\}) \) are distinct and contribute to (ii), but \( \cl(F_1 \cup \{0, \psi^{-1}(0)\}) \in \Fix^\psi(L(M)) \), so again (iii) does not occur, so \( \omega' = 2 \).
- \( \operatorname{rk}(G^\psi) > \operatorname{rk}(F_1) + 2 \)—Again, \( G \) and \( \cl(F_1 \cup \{\psi^{-1}(0)\}) \) are distinct and contribute to (ii). Moreover, \( F = \cl(F_1 \cup \{0, \psi^{-1}(0)\}) \notin \Fix^\psi(L(M)) \), so it contributes to (iii). By induction assumption, the weight of the corresponding facet is

\[
\operatorname{rk} F^\psi - \operatorname{rk} F = \operatorname{rk} G^\psi - (\operatorname{rk} F_1 + 2) = \operatorname{rk}(G^\psi) - \operatorname{rk}(F_1) - 2.
\]

Hence, \( \omega' = 2 + (\operatorname{rk}(G^\psi) - \operatorname{rk}(F_1) - 2) = \operatorname{rk}(G^\psi) - \operatorname{rk}(F_1) \).

This finishes the proof of the lemma. \( \square \)

**Remark 4.6** In contrast to [20], we are unable to give a complete description of the intermediate cycles \( X_k \). Rather, the properties established in Lemmas 4.3 and 4.5 are chosen such as to be sufficient to make the induction run, on the one hand, and to prove the main statement for \( k = n \), on the other hand. Nevertheless, properties such as Lemma 4.5 item (c) indicate that the \( X_k \) exhibit an interesting recursive structure, similar to the analogous cycles in [20, Section 3]. The appearance of certain beta invariants is reminiscent of the so-called CSM classes defined in [8]. More general, a future goal is to find expressions of the cycles \( X_k \) in terms of (pullbacks/pushforwards along \( \Psi \) of) the various canonical classes on matroid fans defined in [1, 4, 5, 7–9].

\( \copyright \) Springer
5 The trace side

In this section, we prove the second half of the main theorem by showing
\[
(-1)^n \beta(\text{Fix}(L(M))) = \sum_p (-1)^p \text{Tr}(\Psi^*, F_p(\Sigma_M)).
\]  
(21)

5.1 A resolution of the framing groups

We start by constructing an explicit resolution for the framing groups \(F_p(\Sigma_M)\) of a matroid fan using chains of flats of low rank. It appears that this resolution is known to experts—a similar resolution is for example considered in [4]. We give a brief independent treatment here.

Let \(C \subseteq \bar{p} \subseteq \bar{l} \subseteq \cdots \subseteq F \subseteq \emptyset\) be a chain of flats of \(M\). Recall that \(l(F) := l\) denotes the length of \(F\). Moreover, we define the rank of \(F\) by \(\text{rk}(F) := \text{rk}(F_1)\). Hence, except for \(E\), \(F\) only involves flats of rank at most \(\text{rk}(F)\).

Let \(C \subseteq \bar{p} \subseteq \bar{l} \subseteq \cdots \subseteq F \subseteq \emptyset\) be the set of chains \(F\) of length \(l\) and rank at most \(p\) and let \(RC \subseteq \bar{l} \subseteq \bar{p} \subseteq \bar{F}\) be the real vectorspace with a basis labelled by \(C \subseteq \bar{l} \subseteq \bar{p} \subseteq \bar{F}\).

We consider the differential complex formed by the simplicial coboundary maps \(\partial: RC \subseteq \bar{l} \subseteq \bar{p} \rightarrow RC \subseteq \bar{l+1} \subseteq \bar{p}\). Explicitly, \(\partial\) maps a generator \(e_{\bar{F}}\) with \(\bar{F} \in C \subseteq \bar{l} \subseteq \bar{p}\), to a vector whose nonzero entries correspond to the chains \(G \in C \subseteq \bar{l+1} \subseteq \bar{p}\) with \(\bar{F} \subseteq \bar{G}\). Moreover, such an entry is equal to \((-1)^k\) where \(k\) denotes the index such that \(G_k \notin \bar{F}\). Given a chain \(F\) of length \(p\), we set
\[
V_F := v_{F_1} \wedge \cdots \wedge v_{F_p} \in F_p(\Sigma_M),
\]
or, in other words, \(V_F\) is the canonical volume element for the cone \(\sigma_{\bar{F}} \subseteq \Sigma_M\).

This defines, by construction of \(F_p(\Sigma_M)\), a surjective map \(RC \rightarrow F_p(\Sigma_M)\). We are interested in its restriction \(RC \subseteq \bar{l} \subseteq \bar{p} \rightarrow F_p(\Sigma_M)\) to chains only using flats of rank at most \(p\). Our goal is to prove the following statement.

**Theorem 5.1** Given a matroid fan \(\Sigma_M\) with framing groups \(F_p(\Sigma_M)\), the sequence
\[
0 \rightarrow RC_0^{\subseteq p} \rightarrow \cdots \rightarrow RC_p^{\subseteq p} \rightarrow F_p(\Sigma_M) \rightarrow 0
\]
is exact for all \(p\).

We start by proving the theorem for \(p = n = \text{dim}(\Sigma_M)\). In a second step, we show how to reduce to this case by using a tropical analogue of the Lefschetz hyperplane section theorem.

**Lemma 5.2** Given a matroid fan \(\Sigma_M\) of dimension \(n\), the sequence
\[
0 \rightarrow RC_0 \rightarrow \cdots \rightarrow RC_n \rightarrow F_n(\Sigma_M) \rightarrow 0
\]
is exact.

\[\square\] Springer
**Proof** Recall that by Poincaré duality [16], we have

\[ F_n(\Sigma_M) = H_0(\Sigma_M, F_n) \cong (H_{BM}^n(X, \mathbb{Z}))^*. \]

Moreover, note that, \( H_{BM}^k(X, \mathbb{Z}) = 0 \) for all \( k \neq 0 \) (again, by Poincaré duality, or using the well-known statement about homology groups of geometric lattices [10]). Hence, the complex of simplicial chains, completed by \( H_{BM}^n(X, \mathbb{Z}) \),

\[ 0 \to H_{BM}^n(X, \mathbb{Z}) \to R\mathcal{C}_n \to \cdots \to R\mathcal{C}_0 \to 0 \quad (23) \]

is exact and is dual to Eq. 22 under this identification. \( \square \)

**Remark 5.3** Since Poincaré duality holds over \( \mathbb{Z} \) by [15], the resolution also works over \( \mathbb{Z} \) (i.e. the sequence \( 0 \to \mathbb{Z}C_\bullet \to F_Z^n(\Sigma_1 M) \to 0 \) is exact).

The second step in our argument is to prove the (local version of) the tropical Lefschetz hyperplane section theorem for stable intersections. Even though tropical section theorems are treated in various sources (e.g. [2, 6]), it seems that this particular statement has not been covered. It is implicit in [24] however (as explained below).

**Lemma 5.4** Let \( H \) denote the standard hyperplane in \( \mathbb{R}^N \). Then, for all \( p < n \), we have

\[ F_p(H \cdot \Sigma_M) = F_p(\Sigma_M). \]

**Proof** It suffices to prove \( F_p(H^{n-p} \cdot \Sigma_M) = F_p(\Sigma_M) \). By Lemma 2.2, \( F_p(H^{n-p} \cdot \Sigma_M) = F_p(\Sigma_{M\leq p}) \). By definition, we have \( F_p(\Sigma_{M\leq p}) \subset F_p(\Sigma_M) \), and it remains to show that \( R\mathcal{C}_{\leq p} \to F_p(\Sigma_M) \) is surjective (as opposed to \( R\mathcal{C}_p \to F_p(\Sigma_M) \) which is surjective by definition).

An indirect, but short, proof works by comparing dimensions. The dimension of \( F_p(\Sigma_{M\leq p}) \) can be computed using Lemma 5.2. Conversely, the dimension of \( F_p(\Sigma_M) \) can be computed using the fact that \( F_*(\Sigma_M) \) is isomorphic to the Orlik-Solomon algebra \( OS_*(M) \) by [24]. Expressing both quantities in terms of the Möbius function, we find that they agree.

It is instructive to give a simple direct proof. Let \( F \in C_p \) be a chain of length \( p \). Let \( \text{gap}(F) \) be its gap sequence and let \( G \supseteq F \) denote the piece of \( F \) corresponding to the last nonzero entry of \( \text{gap}(F) \) (in particular, \( \text{rk}(G) \geq \text{rk}(F) + 2 \)), and assume \( G \neq E \). We denote by \( F_1, \ldots, F_k \) the flats of rank \( \text{rk}(F) + 1 \) such that \( G \supseteq F_i \supseteq F \). Finally, we denote by \( F_i \in C_p \) the chains obtained from \( F \) by removing \( G \) and inserting \( F_i \). The equation for indicator vectors \( v_G + (k-1)v_F = \sum v_{F_i} \) implies the equation for volume elements

\[ V_F = \sum_{i=1}^k V_{F_i}. \]

But note that the last nonzero entry of the gap sequences \( \text{gap}(F_i) \) has moved by one position to the front (compared to \( \text{gap}(F) \)). By recursion, we can express \( V_F \) in terms
of elements $V_{\mathcal{F}'}$ with gap sequences $\text{gap}(\mathcal{F}') = (n - p, 0, \ldots, 0)$, or equivalently, $\mathcal{F}' \in \mathcal{C}_{p}^{n-p}$. This proves the claim. □

**Proof** (Theorem 5.1) By Lemmas 5.4 and 2.2, the statement can be reduced to the case $p = n$ after intersecting $n - p$ times with the standard hyperplane. This case was done in Lemma 5.2. □

### 5.2 Computing the trace side

We continue by computing $\text{Tr}(\Psi_*, F_p(\Sigma_M))$ using the resolution from Theorem 5.1.

**Lemma 5.5** Let $M$ be a loopless matroid of rank $n + 1$ and let $\Psi : \Sigma_M \to \Sigma_M$ be a matroidal automorphism. Then, for any $p = 0, \ldots, n$, we have

$$(-1)^p \text{Tr}(\Psi_*, F_p(\Sigma_M)) = \sum_{F \in \text{Fix}(L(M)) \atop \text{rk } F \leq p} \mu_{\Psi}(\emptyset, F).$$

**Proof** Given a chain $\mathcal{F} = (F_i)$, we define $\psi(\mathcal{F}) = (\psi(F_i))$. This induces linear maps on $\mathcal{R}_{i}^{\leq p}$ for $i = 0, \ldots, p$ by permuting the generators. By abuse of notation, we denote these maps by $\Psi_*$. It is obvious that these maps form a morphism of the sequence from Theorem 5.1. Hence, using the Hopf trace lemma [12, §9, Theorem 2.1], we get

$$(-1)^p \text{Tr}(\Psi_*, F_p(\Sigma_M)) = \sum_{i=0}^{p} (-1)^i \text{Tr}(\Psi_*, \mathcal{R}_{i}^{\leq p}).$$

Since $\Psi_* : \mathcal{R}_{i}^{\leq p} \to \mathcal{R}_{i}^{\leq p}$ is given by a permutation of the generators, its trace is equal to the number of fixed generators, i.e. the chains $\mathcal{F}$ with $\psi(\mathcal{F}) = \mathcal{F}$. We denote the set of such chains by $\text{Fix}(\mathcal{C}_{i}^{\leq p})$. Obviously, it corresponds to the set of chains (with given length/rank) of the lattice $\text{Fix}(L(M))$. Note that, for any lattice $[\hat{0}, \hat{1}]$, the Möbius function can be expressed in terms of chains by

$$\mu(\hat{0}, \hat{1}) = \sum_{\mathcal{F}} (-1)^{l(\mathcal{F})+1}$$

(e.g. [19, Proposition 2.37]). Here, the sum runs through all chains $\mathcal{F}$ of $[\hat{0}, \hat{1}]$ and the length is measured as usual ($\hat{0}$, $\hat{1}$ are not counted). Applied to our situation, we obtain

$$(-1)^p \text{Tr}(\Psi_*, F_p(\Sigma_M)) = \sum_{i=0}^{p} (-1)^i |\text{Fix}(\mathcal{C}_{i}^{\leq p})| = \sum_{F \in \text{Fix}(L(M)) \atop \text{rk } F \leq p} \sum_{\mathcal{F} \in \text{Fix}(\mathcal{C})} (-1)^{l(\mathcal{F})}$$

(24)

Note the change of exponent from $l(\mathcal{F}) + 1$ to $l(\mathcal{F})$ due to the fact that we consider $\mathcal{F}$ as a chain of $\text{Fix}(\mathcal{C})$, hence, $F_1 = F$ is counted. □
It is now easy to finish the proof.

**Proof** (Equation 21) By Lemma 5.5, we have

\[
\sum_{p=0}^{n} (-1)^p \text{Tr}(\Psi_*, F_p(\Sigma_M)) = \sum_{p=0}^{n} \sum_{F \in \text{Fix}(L(M))} \mu_{\psi}(\emptyset, F) \mu_{\psi}(\emptyset, F) \\
= \sum_{F \in \text{Fix}(L(M))} \mu_{\psi}(\emptyset, F)(n + 1 - \text{rk}(F))
\]

\[= -\sum_{F \in \text{Fix}(L(M))} \mu_{\psi}(\emptyset, F) \text{rk}(F)
\]

\[= (-1)^n \beta(\text{Fix}(L(M))).\]  

(25)

In the second last step, we use (again) that summing the Möbius function over a non-trivial interval gives zero.  □

**Acknowledgements** I would like to thank Kristin Shaw, Karim Adiprasito and Omid Amini for useful discussions and feedback on this project. I would like to thank the two anonymous referees for careful reading of the manuscript and many helpful comments and corrections.

**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**References**

1. Adiprasito, K., Huh, J., Katz, E.: Hodge theory for combinatorial geometries. Ann. Math. 188(2), 381–452 (2018). arXiv:1511.02888
2. Adiprasito, K.A., Björner, B.: Filtered geometric lattices and Lefschetz Section Theorems over the tropical semiring. Preprint (2014). arXiv:1401.7301
3. Allermann, L., Rau, J.: First steps in tropical intersection theory. Math. Z. 264(3), 633–670 (2010). arXiv:0709.3705
4. Amini, O., Piquerez, M.: Hodge theory for tropical varieties. Preprint (2020). arXiv:2007.07826
5. Ardila, F., Denham, G., Huh, J.: Lagrangian geometry of matroids (2020)
6. Arnal, C., Renaudineau, A., Shaw, K.: Lefschetz section theorems for tropical hypersurfaces. Preprint (2019). arXiv:1907.06420
7. Berget, A., Eur, C., Spink, H., Tseng, D.: Tautological classes of matroids (2021) arXiv:2103.08021
8. de Medrano, L.L., Rincon, F., Shaw, K.: Chern–Schwartz–MacPherson cycles of matroids. Proc. Lond. Math. Soc. (3) 120(1), 1–27 (2020). arXiv:1707.07303
9. Fink, A., Speyer, D.E.: K-classes for matroids and equivariant localization. Duke Math. J. 161(14), 2699–2723 (2012). arXiv:1004.2403
10. Folkman, J.: The homology groups of a lattice. J. Math. Mech. 15(4), 631–636 (1966)
11. Francois, G., Rau, J.: The diagonal of tropical matroid varieties and cycle intersections. Collect. Math. 64(2), 185–210 (2013). arXiv:1012.3260
12. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003). https://doi.org/10.1007/978-0-387-21593-8
13. Huh, J., Katz, E.: Log-concavity of characteristic polynomials and the Bergman fan of matroids. Math. Ann. 354(3), 1103–1116 (2012). arXiv:1104.2519
14. Itenberg, I., Katzarkov, L., Mikhalkin, G., Zharkov, I.: Tropical Homology. Math. Ann. 374, 963–1006 (2019). arXiv:1604.01838
15. Jell, P., Rau, J., Shaw, K.: Lefschetz (1,1)-theorem in tropical geometry. Épijournal de Géométrie Algébrique 2(11) (2018). arXiv:1711.07900
16. Jell, P., Shaw, K., Smacka, J.: Superforms, tropical cohomology, and Poincaré duality. Adv. Geom. 19(1), 101–130 (2019). arXiv:1512.07409
17. Mikhalkin, G., Rau, J.: Tropical Geometry (in preparation) (2019). https://math.uniandes.edu.co/j.rau/downloads/main.pdf
18. Mikhalkin, G., Zharkov, I.: Tropical eigenwave and intermediate Jacobians. In: Homological Mirror Symmetry and Tropical Geometry. Springer, Cham, pp. 309–349 (2014). https://doi.org/10.1007/978-3-319-06514-4. arXiv:1302.0252
19. Orlik, P., Terao, H.: Arrangements of Hyperplanes. Springer, Berlin (1992). https://doi.org/10.1007/978-3-662-02772-1
20. Rau, J.: The tropical Poincaré–Hopf theorem. Preprint (2020). arXiv:2007.11642
21. Shaw, K., Werner, A.: On the birational geometry of matroids (2022). arXiv:2207.13639 [math.AG]
22. Shaw, K.M.: A tropical intersection product in matroidal fans. SIAM J. Discrete Math. 27(1), 459–491 (2013). arXiv: 1010.3967
23. Speyer, D.E.: A matroid invariant via the K-theory of the Grassmannian. 221 (2009), pp. 882–913. arXiv:math/0603551
24. Zharkov, I.: The Orlik–Solomon algebra and the Bergman fan of a matroid. J. Gökova Geom. Topol. GGT 7, 25–31 (2013). arXiv:1209.1651

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.