INTRODUCTION

The purpose of this work is to extend Howard’s results on the variation of Heegner points in Hida families of modular forms \( (24) \) to a general quaternionic setting. In fact, while analogues
of the constructions by Howard of systems of big Heegner points on towers of classical modular curves have been proposed by Fouquet [11, 12] for Shimura curves attached to indefinite quaternion algebras over totally real number fields, to the best of our knowledge the case where modular curves need to be replaced by Shimura curves coming from definite quaternion algebras has never been investigated. However, the philosophy behind the so-called “parity conjectures” suggests that the definite and indefinite cases are equally significant from an arithmetic point of view, so in our opinion it would be desirable to have both sides of the quaternionic setting well understood and developed. With this goal in mind, in this article we offer a systematic construction of big Heegner points and classes attached to Hida families in both the definite case and the indefinite case, and study the arithmetic of the relevant extended Selmer groups as defined by Nekovář. We remark that we exclusively work with quaternion algebras over $\mathbb{Q}$, but we expect that our constructions could be extended to the more general situation in which $\mathbb{Q}$ is replaced by a totally real number field (see [11] and [12] for results in this direction in the indefinite case). Now let us describe the subject of the paper more in detail.

Fix an integer $N$, a prime $p \nmid 6N$ and an ordinary $p$-stabilized newform

$$f(q) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(Np), \omega^j)$$

where $\omega$ is the Teichmüller character and $j \equiv k \mod 2$. Let $F$ be a finite extension of $\mathbb{Q}_p$ containing all the eigenvalues of the Hecke operators acting on $f$ and let $\mathcal{O}_F$ denote its ring of integers. Assume also that the residual representation attached to $f$ is absolutely irreducible.

Fix an imaginary quadratic field $K$ of discriminant prime to $Np$ and consider the factorization $N = N^+N^-$ induced by $K$: a prime number $\ell$ divides $N^+$ (respectively, $N^-$) if and only if $\ell$ splits (respectively, is inert) in $K$. Assume throughout that $N^-$ is square-free and say that we are in the definite (respectively, indefinite) case if the number of primes dividing $N^-$ is odd (respectively, even). In this introduction, for simplicity suppose that $p$ does not divide the class number of $K$.

Hida’s theory ([17], [18]) incorporates the modular form $f$ and the $p$-adic Galois representation $\rho_f : G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(F)$ attached to $f$ by Deligne into an analytic family of modular forms and Galois representations. More precisely, Hida defines the universal ordinary Hecke algebra $\mathfrak{h}_\infty$ by taking the inverse limit over $m$ of the (classical) Hecke algebras $\mathfrak{h}_m$ over $\mathcal{O}_F$ acting on weight 2 cusp forms with coefficients in $\mathcal{O}_F$ of level $\Gamma_1(Np^m)$ and then projecting to the ordinary part. Out of $\mathfrak{h}_\infty$ one then constructs a local domain $\mathcal{R}$, finite and flat over the Iwasawa algebra $\Lambda := \mathcal{O}_F[1 + p\mathbb{Z}_p]$, such that certain prime ideals $p$ of $\mathcal{R}$ (called arithmetic) correspond to modular forms $f_p$ of suitable weight $k_p$, level $\Gamma_1(Np^{mr})$ and character $\psi_p$ with coefficients in the residue field $F_p$ of the localization of $\mathcal{R}$ at $p$; moreover, $f_p = f$ for a certain arithmetic prime $\mathfrak{p}$ of weight $k$. Finally, taking inverse limits over $m$ of the $p$-adic Tate modules of the Jacobian varieties of the modular curves $X_1(Np^m)$ one can introduce a $G_{\mathbb{Q}}$-representation $\mathbf{T}$ which is free of rank two over $\mathcal{R}$ and has the property that $V_\mathfrak{p} := \mathbf{T} \otimes_\mathcal{R} F_\mathfrak{p}$ is a twist of the representation $V(f_p)$ associated with $f_p$.

**Big Selmer groups.** In recent years, the systematic study of certain Selmer groups attached to the $G_{\mathbb{Q}}$-representation $\mathbf{T}$ has been pursued, among others, by Nekovář and Plater ([35], Nekovář ([36]), Ochiai ([39]), Howard ([24]) and Delbourgo ([8]). More precisely, the $G_{\mathbb{Q}}$-representation $\mathbf{T}$ admits a twist $\mathbf{T}^\dagger$ which has a perfect alternating pairing $\mathbf{T}^\dagger \times \mathbf{T}^\dagger \to \mathcal{R}(1)$, and for every arithmetic prime $p$ of $\mathcal{R}$ the representation $V_\mathfrak{p}^\dagger := \mathbf{T}^\dagger \otimes_\mathcal{R} F_\mathfrak{p}$ is a self-dual twist of $V(f_p)$. Then, using Nekovář’s theory of Selmer complexes ([36]), for any number field $L$ one can define extended
Selmer groups $\tilde{H}^1_f(L, T^1)$ and $\tilde{H}^1_f(L, V_p^1)$, whose arithmetic is the main theme of the present paper.

Now we briefly sketch the work of Howard which was the original inspiration for our article. In order to study the arithmetic of Nekovář’s Selmer groups, when all primes $\ell | N$ split in $K$ (i.e., when $N^- = 1$) Howard introduced in [24] canonical cohomology classes

$$x_c \in \tilde{H}^1_f(H_c, T^1),$$

which he calls “big Heegner points”, where $c \geq 1$ is an integer prime to $N$ and $H_c$ is the ring class field of $K$ of conductor $N$. These classes are constructed by taking an inverse limit of cohomology classes arising from Heegner points in the Jacobians of classical modular curves via Kummer maps, and satisfy suitable Euler system relations in the sense of Kolyvagin: see [24, §§2.2–2.4]. In particular, denoting by $K_n$ the $n$-th layer of the anticyclotomic $\mathbb{Z}_p$-extension $K_\infty$ of $K$, the Euler system compatibilities enjoyed by the classes $x_c$ make it possible to introduce cohomology classes

$$3_0 := \text{Cor}_{H_1/K}(x_1) \in \tilde{H}^1_f(K, T^1),$$

$$3_n := \text{Cor}_{H_p^{n+1}/K_n}(U_p^{-n}x_{n+1}) \in \tilde{H}^1_f(K_n, T^1) \quad \text{for } n \geq 1,$$

$$3_\infty := \lim_n 3_n \in \tilde{H}^1_{f,lw}(K_\infty, T^1).$$

Here $\tilde{H}^1_{f,lw}(K_\infty, T^1) := \lim_n \tilde{H}^1_f(K_n, T^1)$ is a module over the Iwasawa algebra $\mathcal{R}_\infty$ of the profinite Galois group $G_\infty := \text{Gal}(K_\infty/K)$ with coefficients in $\mathcal{R}$, as described in [24, §3.3].

These classes are used to obtain various results on the arithmetic of the above-mentioned Selmer groups; in particular, a vertical nonvanishing theorem (generalizing results of Cornut and Vatsal in [6]) is proved in [24, §3.1 and §3.2], while an horizontal nonvanishing theorem is obtained in [24, §3.4]. Moreover, in [24, Conjecture 3.3.1] Howard proposes a two-variable Iwasawa main conjecture for $\tilde{H}^1_{f,lw}(K_\infty, T^1)$ which extends the Heegner point main conjecture formulated by Perrin-Riou in [41].

In this paper we are interested in results and conjectures of the type described above in the more general case where one allows for the existence of primes dividing $N$ which are inert in $K$. In other words, the integer $N^-$ is not necessarily equal to 1. In the indefinite case (i.e., when the number of primes dividing $N^-$ is even) our constructions and results should be compared with those obtained by Fouquet in [11] and [12] for Shimura curves over totally real fields; on the contrary, as far as we know the definite case (corresponding to an odd number of primes dividing $N^-$) is considered here for the first time and represents the most significant novelty in our approach. In the rest of the introduction we give a brief description of the paper, referring to the main body of the text for all details.

**Families of optimal embeddings on Shimura curves.** Let $B$ denote the quaternion algebra over $\mathbb{Q}$ of discriminant $N^-$ (thus $B$ is split at the archimedean place $\infty$ of $\mathbb{Q}$ in the indefinite case and is ramified at $\infty$ in the definite case) and for every integer $m \geq 0$ choose an Eichler order $\mathcal{O}_m$ of $B$ of level $N^+p^m$ such that $\mathcal{O}_m \subset \mathcal{O}_{m-1}$ for all $m \geq 1$. If the hat denotes adelicizations, one then defines open compact subgroups $U_m \subset \hat{\mathcal{O}}^\times$ by imposing an extra $\Gamma_1(p^m)$-level structure on $\hat{\mathcal{O}}_m$ and considers the Shimura curves $\tilde{X}_m$ associated with $U_m$ (precise definitions in terms of double cosets are given in [11] and [12]). In the definite case these are disjoint unions of genus zero curves defined over $\mathbb{Q}$, while in the indefinite case they are compact Riemann surfaces admitting
canonical models over \( \mathbb{Q} \). For any integer \( c \geq 1 \) prime to \( N \) and the discriminant of \( K \) we define the Heegner points of conductor \( c \) on \( \tilde{X}_m \) as those pairs \([g, f]\) in the subset
\[
\tilde{X}^{(K)}_m := U_m \left( \tilde{B}^\times \times \text{Hom}(K, B) \right) / B^\times \subset \tilde{X}_m(\mathbb{C})
\]
such that \( f \) is an optimal embedding of the order \( \mathcal{O}_c \) of \( K \) of conductor \( c \) into the Eichler order \( B \cap (g^{-1}\tilde{R}_m,g) \) of \( B \) (of course, a priori this set of special points might be empty). In \( \S 2.3 \) we prove that in the indefinite case these Heegner points are rational over \( H_c(\mu_{p^m}) \), where \( H_c \) is the ring class field of \( K \) of conductor \( c \) and \( \mu_{p^m} \) is the group of \( p^m \)-th roots of unity. If \( a \in \tilde{K}^\times \) and \( \tilde{f} : \tilde{K}^\times \to \tilde{B}^\times \) is the adelicization of \( f \), in both the definite and the indefinite cases the map
\[
[(g, f)] \mapsto [(g\tilde{f}(a), f)]
\]
induces a (free) action of \( \text{Gal}(H_c(\mu_{p^m})/K) \) on the set of Heegner points of conductor \( c \). Furthermore, the group \( \text{Div}(\tilde{X}_m) \) of divisors on \( \tilde{X}_m \) is endowed with an action of the usual Hecke operators \( T_\ell \) for primes \( \ell \nmid Np^m \) and \( U_\ell \) for primes \( \ell | Np^m \) and of diamond operators \( (\ell) \) for \( \ell \in (\mathbb{Z}/p^m\mathbb{Z})^\times \). In Section \( 3 \) we provide an explicit construction of suitably compatible families of Heegner points on our tower of Shimura curves. More precisely, the main result of Part \( 1 \) which is Theorem \( 3.18 \) in the text can be stated as follows.

**Theorem A.** For every integer \( m \geq 0 \) and every integer \( c \geq 1 \) prime to \( N \) and the discriminant of \( K \) there is a Heegner point \( \tilde{P}_{c,m} \in \tilde{X}^{(K)}_m \) of conductor \( cp^m \), rational over \( H_{cp^m}(\mu_{p^m}) \) in the indefinite case, such that the following conditions are satisfied.

1. **Vertical Compatibility.** The image of \( \tilde{P}_{c,m} \in \tilde{X}_m \) under the covering \( \tilde{X}_m \to \tilde{X}_n \) is a Heegner point of conductor \( cp^m \) on \( \tilde{X}_n \) for all \( n \in \{0, \ldots, m\} \). Furthermore, the divisor \( U_p(\tilde{P}_{c,m-1}) \) equals the image under the map \( \tilde{X}_m \to \tilde{X}_{m-1} \) of the trace from \( H_{cp^m}(\mu_{p^m}) \) to \( H_{cp^{m-1}}(\mu_{p^m}) \) of \( \tilde{P}_{c,m} \).

2. **Horizontal Compatibility.** If \( m \geq 1 \) and \( n \geq 1 \) are fixed integers then the divisor \( U_p(\tilde{P}_{cp^{n-1},m}) \) is equal to the trace from \( H_{cp^{n+1}}(\mu_{p^{n+1}}) \) to \( H_{cp^{n-1}}(\mu_{p^{n+1}}) \) of \( \tilde{P}_{cp^n,m} \). Similar relations are valid for the Hecke operator \( T_\ell \) with \( \ell \nmid cNp^m \).

3. **Galois Compatibility.** Set \( p^* := (-1)^{(p-1)/2}p \), let \( \epsilon_{\text{cyc}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^* \) be the \( p \)-adic cyclotomic character and let \( \vartheta : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{p^*})) \to \mathbb{Z}_p^*/\{\pm 1\} \) be the unique continuous homomorphism such that \( \vartheta^2 \) coincides with the restriction of \( \epsilon_{\text{cyc}} \). Then \( \tilde{P}^*_{c,m} = (\vartheta(\sigma))\tilde{P}_{c,m} \) for all \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{p^*})) \) in \( \text{Div}(\tilde{X}_m) \).

The existence of compatible sequences of CM points as in Theorem A was shown by Howard in \[24\] when the \( \tilde{X}_m \) are classical modular curves and by Fouquet in \[11\] for Shimura curves attached to indefinite quaternion algebras over totally real fields \( F \) having exactly one split archimedean place (thus, finite unions of compact Riemann surfaces if \( F \neq \mathbb{Q} \)). In \[24\] these points are built using the interpretation of modular curves as moduli spaces for elliptic curves with suitable level structures, while in \[11\] one fixes an embedding \( \psi : K \hookrightarrow B \); takes the fixed point \( z \) for the action of \( \psi(K^\times) \) on the complex upper half plane via fractional linear transformations and finally considers sequences of points of the form \((z, b_m)\) on Shimura curves which are analogues of our indefinite curves \( \tilde{X}_m \).

As remarked before, in this paper our families of CM points are introduced via a systematic use of the theory of optimal embeddings as described in \[13\] and \[1\], and this approach (although technically more intricate than those of Howard and Fouquet) has at least two advantages:
it offers a uniform setting for dealing with both the definite case and the indefinite case, as in [1];

• in the indefinite case it gives us a fine control on the fields of rationality of our Heegner points, as provided in the classical context of [24] by the modular point of view.

The first point is worth emphasizing, since treating the definite and the indefinite cases on an equal footing was in fact the central motivation behind our work. The importance of dealing with the definite case as well when studying the representation associated with a Hida family stems from the fact that, conjecturally, the indefinite case should take care of situations in which the rank of the Selmer group is odd (typically, one), while the definite case should describe even rank (typically, rank zero) settings. Observe, moreover, that the definite case cannot be treated by means of the tools developed in [24], [11] and [12].

In Part 2 of the paper we use these families to construct big Heegner points and classes that are the counterparts of those defined in [24], and finally in Part 3 we prove results and formulate conjectures which generalize those obtained in loc. cit. by Howard. In the rest of the introduction we focus our attention on the main results obtained in our work. For clarity of exposition, it will be convenient to treat the definite case and the indefinite case separately.

The definite case. This represents the newest contribution of the paper and corresponds to Sections 8 and 9. In this case, we consider the Hecke modules

$$J_m := \mathcal{O}_F[U_m \backslash \hat{B}^\times / B^\times] \simeq \text{Pic}(\hat{X}_m) \otimes \mathbb{Z} \mathcal{O}_F$$

and define the inverse limit $J_\infty := \varprojlim J_m$ with respect to the canonical projection maps. By the Jacquet–Langlands correspondence, this $\mathcal{O}_F$-module is endowed with an action of the $N^-$-new quotient $T_\infty$ of the universal ordinary Hecke algebra $\mathfrak{h}_\text{ord}^\infty$. One can then consider the ordinary part $J_\text{ord}_\infty$ of $J_\infty$ and introduce the finitely generated $\mathcal{R}$-module $J := J_\text{ord}_\infty \otimes_{T_\infty} \mathcal{R}$. Under a suitable hypothesis (Assumption 8.2), we prove that $J$ is free of rank one over $\mathcal{R}$ and fix an isomorphism

$$J \simeq \mathcal{R}.$$  

The compatible sequence of Heegner points on the tower of definite Shimura curves whose existence is guaranteed by Theorem A can be combined with the isomorphisms (3) to produce canonical elements in $J$. By the isomorphism (4) one then obtains an element $J_\mathcal{R} \in \mathcal{R}$ defined in terms of the $\mathcal{R}$-component of a suitable big Heegner point. Theorem 8.6 can be stated as follows.

**Theorem B.** Assume that $J_\mathcal{R} \neq 0$ and that Conjecture 8.5 is true. Then $\widetilde{H}^1_f(K, T^\dagger)$ is a torsion $\mathcal{R}$-module.

We expect that Conjecture 8.5, for the statement of which we refer to the text, can be proved (at least for arithmetic primes of $\mathcal{R}$ of weight two) by suitably extending the arguments of [3] and [28] to the case of forms with non-trivial character.

Now set $G_n := \text{Gal}(K_n/K)$ for all integers $n \geq 1$. Essentially by corestricting to the finite layers of $K_\infty$, one also gets elements

$$J_{n, \mathcal{R}}^\sigma \in \mathcal{R}$$

for all $\sigma \in G_n$ and all $n \geq 1$. The compatibility properties of these points allow us to define

$$\theta_n := \alpha_p^{-n} \sum_{\sigma \in G_n} J_{n, \mathcal{R}}^\sigma \otimes \sigma^{-1} \in \mathcal{R}[G_n], \quad \theta_\infty := \varprojlim_n \theta_n \in \mathcal{R}_\infty$$

where $\alpha_p \in \mathbb{R}^\times$ is the image of the Hecke operator $U_p$ under the natural map $\mathfrak{h}_\text{ord}^\infty \to \mathcal{R}$. As explained in [8.3] one introduces an $\mathcal{R}_\infty$-module $\widetilde{H}^1_{f,1w}(K_\infty, \mathbf{A}^\dagger)$ where $\mathbf{A}^\dagger := \text{Hom}(T^\dagger, \mu_p^\infty)$. 

Finally, write $x \mapsto x^*$ for the involution of $\mathcal{R}_\infty$ given by $\sigma \mapsto \sigma^{-1}$ on group-like elements. The following statement (which is Conjecture 8.10) must be seen as a main conjecture of Iwasawa theory in the definite setting.

**Conjecture C.** Assume that the local ring $\mathcal{R}$ is regular. The group $\widetilde{H}^1_{f,Iw}(K_\infty, A^\dagger)$ is a finitely generated torsion module over $\mathcal{R}_\infty$ and there is an equality

$$(\theta_\infty \cdot \theta^*_\infty) = \text{Char}_{\mathcal{R}_\infty}(\widetilde{H}^1_{f,Iw}(K_\infty, A^\dagger)^\vee)$$

of ideals of $\mathcal{R}_\infty$.

Here the symbol $^\vee$ denotes the Pontryagin dual and the product $\theta_\infty \cdot \theta^*_\infty$ is interpreted as a $p$-adic $L$-function. Note that these definitions are reminiscent of the constructions performed by Bertolini and Darmon in, e.g., [1] and [3].

Along a different line of investigation, in the definite case we also propose conjectures on the vanishing of the special values of twists of the (classical) $L$-functions over $K$ of modular forms living on the same Hida branch of $f$. Namely, fix an integer $c \geq 1$ as before and a character $\chi : \tilde{G}_c \to \mathcal{O}_F^\times$ where $\tilde{G}_c := \text{Gal}(H_c/K)$, then extend $\chi$ to an $\mathcal{R}$-linear homomorphism $\chi : \mathcal{R}[\tilde{G}_c] \to \mathcal{R}$. In §8.2 we define a theta element $D_{c,\mathcal{R}} \in \mathcal{R}[\tilde{G}_c]$ coming from the $\mathcal{R}$-component of our big Heegner point at level $c$. We restate here Conjecture 9.1.

**Conjecture D.** Let $p$ be an arithmetic prime of $\mathcal{R}$ of weight $k_p \geq 2$ and let $\chi$ be as above. The special value $L_K(f_p, \chi, k_p/2)$ is non-zero if and only if $\chi(D_{c,\mathcal{R}}) \in \mathcal{R}$ does not belong to $p$.

Analogously, Conjecture 9.2 deals with the vanishing of the special values of twists by finite order characters of the $p$-profinite Galois group $G_\infty$.

### The indefinite case

This is the direct generalization of the classical modular curves setting originally studied by Howard in [24], and has also been considered in [11] and [12] by Fouquet (who works in the broader context of Shimura curves attached to indefinite quaternion algebras over totally real fields). In particular, the reader is suggested to compare our approach and results to Fouquet’s. We deal with the indefinite case in Section 10.

By taking the inverse limit of the $p$-adic Tate modules of the Jacobian varieties of $\tilde{X}_m$ as in [18], we construct a $G_\mathbb{Q}$-representation $T_{\text{Sh}}$ which, as a consequence of results in [11], is free of rank two over $\mathcal{R}$ and locally behaves like $T$ at unramified primes. In particular, we prove that

$$T \simeq T_{\text{Sh}}, \quad T^\dagger \simeq T_{\text{Sh}}^\dagger$$

as $G_\mathbb{Q}$-modules. Following [24], the compatible sequence of Heegner points of Theorem A can then be used to define cohomology classes $\kappa_{c,\mathcal{R}} \in H^1(H_c, T^\dagger)$. In this general quaternionic setting, the problem of showing that these classes belong to Šikavč’s Selmer group presents extra complications. More precisely, due to the presence of primes dividing $N$ which are inert in $K$ and so split completely in $H_c/K$ for all $c$ (prime to $N$), we are only able to show that $\lambda \cdot \kappa_{c,\mathcal{R}} \in \overline{H}^1_c(H_c, T^\dagger)$ for any choice of $\lambda \in \mathcal{R}$ in the annihilator of the $\mathcal{R}$-module $\prod_{\ell \mid N} H^1(K_\ell, T^\dagger)$. We
fix once and for all a non-zero $\lambda$ in this annihilator and define, in analogy with [24], the classes $X_{c,\mathcal{R}} := \lambda \cdot \kappa_{c,\mathcal{R}} \in \tilde{H}^1_f(H_c, T^\dagger)$,

$3_{0,\mathcal{R}} := \text{Cor}_{H_1/K}(X_{1,\mathcal{R}}) \in \tilde{H}^1_f(K, T^\dagger)$,

$3_{n,\mathcal{R}} := \text{Cor}_{H_{p^{n+1}}/K_\lambda}(U_p^{-n}x_{n+1,\mathcal{R}}) \in \tilde{H}^1_f(K_n, T^\dagger)$ for $n \geq 1$,

$3_{\infty,\mathcal{R}} := \lim_{\leftarrow n} 3_{n,\mathcal{R}} \in \tilde{H}^1_{f,1w}(K_n, T^\dagger)$.

The following two results generalize theorems of Howard.

**Theorem E.** Let $p$ be a non-exceptional arithmetic prime of $\mathcal{R}$ with trivial character and even weight. If $3_{0,\mathcal{R}}$ has non-trivial image in $\tilde{H}^1_f(K, V_{p^\dagger})$ then $\text{dim}_{F_p} \tilde{H}^1_f(K, V_{p^\dagger}) = 1$.

This is Theorem 10.8 in the text.

**Theorem F.** If $3_{0,\mathcal{R}}$ is not $\mathcal{R}$-torsion then $\tilde{H}^1_f(K, T^\dagger)$ is an $\mathcal{R}$-module of rank one.

We prove this statement in Theorem 10.10 and we expect the condition on the class $3_{0,\mathcal{R}}$ to be always true.

We conclude with a main conjecture of Iwasawa theory (Conjecture 10.14) that can be viewed as the counterpart of Conjecture C in the indefinite setting.

**Conjecture G.** Suppose that $\mathcal{R}$ is a regular local ring and that $\kappa_{p^m,\mathcal{R}} \in \tilde{H}^1_f(H_{p^m}, T^\dagger)$ for all $m \geq 0$. The square of the characteristic ideal of the $\mathcal{R}_\infty$-module $\tilde{H}^1_{f,1w}(K_\infty, T^\dagger)/(3_{\infty,\mathcal{R}})$ is equal to the characteristic ideal of the $\mathcal{R}_\infty$-torsion submodule of $\tilde{H}^1_{f,1w}(K_\infty, A^\dagger)^\vee$.

Conjecture G extends both the conjecture proposed by Howard in [24, Conjecture 3.3.1] and the classical Heegner point main conjecture for elliptic curves formulated by Perrin-Riou in [41]. We remark that when $N^- = 1$ one divisibility in the statement of the above conjecture was proved by Fouquet in [12, Theorem A].

**Part 1. Construction of compatible families of Heegner points**

This part of our work is devoted to the construction of suitably compatible families of Heegner points on a certain tower of Shimura curves over $\mathbb{Q}$. The fact that we can deal with both definite quaternion algebras and indefinite quaternion algebras in a uniform way is the peculiarity of our approach, which is based on the description of CM points in terms of optimal embeddings of quadratic orders of imaginary quadratic fields into Eichler orders of the relevant quaternion algebras. In particular, the interpretation of (indefinite) Shimura curves as moduli spaces of abelian surfaces with quaternionic multiplication plays no role in our arguments. The families of Heegner points built here will allow us to define in Part 2 big Heegner points and classes à la Howard in our definite/indefinite setting.

1. **Towers of Shimura curves**

As a piece of notation, for any ring $A$ denote by $\hat{A} := A \otimes \mathbb{Z} \prod_\ell \mathbb{Z}_\ell$ its profinite completion, where the product is over all prime numbers $\ell$, by $A_\ell := A \otimes \mathbb{Z}_\ell$ its $\ell$-adic completion at a prime number $\ell$ and by $A_\infty := A \otimes \mathbb{R}$ its archimedean completion. An element $x \in \hat{A}$ is denoted by $(x_\ell)_\ell$. 

Let $N^-$ be a positive square free integer and $N^+$ a positive integer prime to $N^-$. Define
\[ N := N^+ N^- \]
and let $p \nmid N$ be an odd prime number. Denote by $B$ the (unique, up to isomorphism) quaternion algebra over $\mathbb{Q}$ of discriminant $N^-$. If the number of primes dividing $N^-$ is odd (respectively, even) then $B$ is definite (respectively, indefinite), that is, $B \otimes \mathbb{Q} \mathbb{R}$ is isomorphic to the Hamilton skew field (respectively, to the matrix algebra $M_2(\mathbb{R})$). For every integer $m \geq 0$, let $R_m \subset B$ be an Eichler order of level $N^+ p^m$ such that $R_{j+1} \subset R_j$ for all $j \geq 0$. Fix an isomorphism $\phi_p : B_p \xrightarrow{\sim} M_2(\mathbb{Q}_p)$ such that
\[ \phi_p(R_m \otimes \mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \mid c \equiv 0 \pmod{p^m} \right\} \]
for all integers $m \geq 0$. Finally, for all $m \geq 0$ let $U_m \subset \hat{R}_m^\times$ be the subgroup of elements $(x_\ell)_\ell$ satisfying $\phi_p(x_\ell) \equiv \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \pmod{p^m}$ for some $b \in \mathbb{Z}_p$ and some $d \in \mathbb{Z}_p^\times$.

**Convention.** In order not to burden the notation, in the rest of the paper we will sometimes identify $B_p$ with $M_2(\mathbb{Q}_p)$ via the isomorphism $\phi_p$ – we will do so according to convenience, without explicit warning. Thus the reader should always bear in mind that when we write “the adele $b \in \hat{B}$ has $p$-component $b_p$ equal to $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Q}_p)$” we really mean that $b_p$ is equal to $\phi_p^{-1}(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix})$.

### 1.1. Definite Shimura curves.

Let $B$ be definite. Denote by $\mathbb{P} = \mathbb{P}_{N^-}$ the curve of genus 0 defined over $\mathbb{Q}$ by setting
\[ \mathbb{P}(A) := \{ x \in B \otimes \mathbb{Q} A \mid \text{Norm}(x) = \text{Trace}(x) = 0 \} \]
for any $\mathbb{Q}$-algebra $A$, where Norm and Trace are the reduced norm and trace of $B \otimes \mathbb{Q} A$. The group $B^\times$ acts on $\mathbb{P}$ by conjugation and this action is algebraic and defined over $\mathbb{Q}$. Note that $\mathbb{P}(\mathbb{C})$ is canonically identified with $\text{Hom}_\mathbb{Q}(\mathbb{C}, B^\times)$, where $\text{Hom}_\mathbb{Q}$ denotes homomorphisms of $\mathbb{Q}$-algebras. More generally, $\mathbb{P}(K)$ is identified with $\text{Hom}_\mathbb{Q}(K, B)$ for any imaginary quadratic field $K$. Define the definite Shimura curve of level $R_m$ (respectively, $U_m$) and discriminant $N^-$ to be the double coset spaces
\[ X_m := \hat{R}_m^\times \backslash \hat{B}^\times \times \mathbb{P} / B^\times, \quad \hat{X}_m := U_m \backslash \hat{B}^\times \times \mathbb{P} / B^\times \]
where $\hat{R}_m$ and $U_m$ act by left multiplication on $\hat{B}^\times$ and trivially on $\mathbb{P}$, while $B^\times$ acts by conjugation on $\mathbb{P}$ and by right multiplication on $\hat{B}^\times$.

If $K$ is an imaginary quadratic field write
\[ X_m^{(K)} := \hat{R}_m^\times \backslash \hat{B}^\times \times K / B^\times, \quad \hat{X}_m^{(K)} := U_m \backslash \hat{B}^\times \times K / B^\times. \]
As remarked in [14, p. 131], $X_m^{(K)} = X_m(K)$ and $\hat{X}_m^{(K)} = \hat{X}_m(K)$. However, in the following we use the above symbols in order to keep our notation uniform with the one adopted in the indefinite case (see below), where the points in $X_m^{(K)}$ or $\hat{X}_m^{(K)}$ are in general rational only over (abelian) extensions of $K$.

By strong approximation, choose representatives $g_1, \ldots, g_{h(m)}$ and $\tilde{g}_1, \ldots, \tilde{g}_{h(m)}$ of the double cosets $\hat{R}_m^\times \backslash \hat{B}^\times / B^\times$ and $U_m \backslash \hat{B}^\times / B^\times$, respectively. Define the arithmetic groups
\[ \Gamma^i_m := g_i^{-1} \hat{R}_m^\times g_i \cap B^\times, \quad \tilde{\Gamma}^j_m := \tilde{g}_j^{-1} U_m \tilde{g}_j \cap B^\times \]
with \( i \in \{1, \ldots, h(m)\} \) and \( j \in \{1, \ldots, \tilde{h}(m)\} \). Then \( X_m \) and \( \tilde{X}_m \) can be expressed as disjoint unions

\[
X_m = \prod_{i=1}^{h(m)} \mathbb{P}/\Gamma^i_m, \quad \tilde{X}_m = \prod_{i=1}^{\tilde{h}(m)} \mathbb{P}/\tilde{\Gamma}^i_m
\]

of curves of genus 0.

### 1.2. Indefinite Shimura curves

Let \( B \) be indefinite. In this case, for all \( m \geq 0 \), both \( \tilde{R}^x_m \backslash \tilde{B}^x \backslash B^x \) and \( U_m \backslash \tilde{B}^x \backslash B^x \) consist of a single class. Fix an isomorphism \( \phi_\infty : B_\infty \xrightarrow{\cong} M_2(\mathbb{R}) \); then \( \phi_\infty(R^x_m) \) is a discrete subgroup of \( GL_2(\mathbb{R}) \). Denote by \( \Gamma_m \) the subgroup of \( \phi_\infty(R^x_m) \) consisting of matrices with determinant 1 and let \( \tilde{\Gamma}_m \) be the analogous subgroup of \( \phi_\infty(U_m \cap B) \). Define the Riemann surfaces

\[
Y_m(\mathbb{C}) := \mathcal{H}/\Gamma_m, \quad \tilde{Y}_m := \mathcal{H}/\tilde{\Gamma}_m
\]

where \( \mathcal{H} \) is the complex upper half plane and the groups \( \Gamma_m \) and \( \tilde{\Gamma}_m \) act on \( \mathcal{H} \) by Möbius (i.e., fractional linear) transformations. Let \( X_m(\mathbb{C}) \) (respectively, \( \tilde{X}_m(\mathbb{C}) \)) denote the Baily-Borel compactification of \( Y_m(\mathbb{C}) \) (respectively, \( \tilde{Y}_m(\mathbb{C}) \)). If \( B \neq M_2(\mathbb{Q}) \) then \( X_m(\mathbb{C}) = Y_m(\mathbb{C}) \) and \( \tilde{X}_m(\mathbb{C}) = \tilde{Y}_m(\mathbb{C}) \). The Riemann surface \( X_m(\mathbb{C}) \) (respectively, \( \tilde{X}_m(\mathbb{C}) \)) has a model over \( \mathbb{Q} \) which will be denoted by \( X_m \) (respectively, \( \tilde{X}_m \)) and referred to as the indefinite Shimura curve of level \( R_m \) (respectively, \( U_m \)) and discriminant \( N^- \). Setting \( \mathbb{P} := \mathbb{C} - \mathbb{R} \) for the union of the complex upper and lower half planes yields

\[
Y_m(\mathbb{C}) = \tilde{R}^x_m \backslash (\tilde{B}^x \times \mathbb{P}) / B^x, \quad \tilde{Y}_m(\mathbb{C}) = U_m \backslash (\tilde{B}^x \times \mathbb{P}) / B^x
\]

where, as above, \( \tilde{R}^x_m \) and \( U_m \) act by left multiplication on \( \tilde{B}^x \) and trivially on \( \mathbb{P} \), while \( B^x \) acts by Möbius transformations via \( \phi_\infty \) on \( \mathbb{P} \) and by right multiplication on \( \tilde{B}^x \). Observe that there is a \( B^x \)-equivariant identification \( \mathbb{P} = \text{Hom}_\mathbb{Q}(\mathbb{C}, B_\infty) \) (here \( B^x \) acts on the homomorphisms by conjugation), so \( Y_m(\mathbb{C}) \) and \( \tilde{Y}_m(\mathbb{C}) \) can also be described as

\[
Y_m(\mathbb{C}) = \tilde{R}^x_m \backslash (\tilde{B}^x \times \text{Hom}_\mathbb{Q}(\mathbb{C}, B_\infty)) / B^x
\]

and

\[
\tilde{Y}_m(\mathbb{C}) = U_m \backslash (\tilde{B}^x \times \text{Hom}_\mathbb{Q}(\mathbb{C}, B_\infty)) / B^x.
\]

Finally, for any imaginary quadratic field \( K \) there are injections

\[
X_m^{(K)} := \tilde{R}^x_m \backslash (\tilde{B}^x \times \text{Hom}_\mathbb{Q}(K, B)) / B^x \hookrightarrow X_m(\mathbb{C}),
\]

\[
\tilde{X}_m^{(K)} := U_m \backslash (\tilde{B}^x \times \text{Hom}_\mathbb{Q}(K, B)) / B^x \hookrightarrow \tilde{X}_m(\mathbb{C})
\]

induced by the map \( \text{Hom}_\mathbb{Q}(K, B) \to \text{Hom}_\mathbb{Q}(\mathbb{C}, B_\infty) \) which is obtained by extending scalars from \( \mathbb{Q} \) to \( \mathbb{R} \). Actually, the subsets \( X_m^{(K)} \) and \( \tilde{X}_m^{(K)} \) are contained in \( X_m(\mathbb{Q}) \) and \( \tilde{X}_m(\mathbb{Q}) \), respectively, where \( \mathbb{Q} \) is the algebraic closure of \( \mathbb{Q} \) in \( \mathbb{C} \).

As a piece of notation, both in the definite case and in the indefinite case write \( \text{Div}(X_m) \) and \( \text{Div}(\tilde{X}_m) \) for the groups of divisors on \( X_m \) and on \( \tilde{X}_m \), respectively.
1.3. **The tower of curves.** The inclusions $R_{m+1} \subset R_m$, $U_{m+1} \subset U_m$ and $U_m \subset R_m$ for $m \geq 0$ yield a commutative diagram of curves

$$
\begin{array}{c}
\cdots \xrightarrow{\alpha_{m+1}} X_m \xrightarrow{\alpha_m} X_{m-1} \xrightarrow{\alpha_{m-1}} \cdots \\
\downarrow \beta_m & \quad & \downarrow \beta_{m-1} \\
\cdots \xrightarrow{\alpha_{m+1}} X_m \xrightarrow{\alpha_m} X_{m-1} \xrightarrow{\alpha_{m-1}} \cdots 
\end{array}
$$

in which all maps are finite coverings that are defined over $\mathbb{Q}$.

1.4. **The Hecke operator $U_p$.** The action of the Hecke operator $U_p$ on $\text{Div}(X_m)$ and $\text{Div}(\tilde{X}_m)$ can be described as follows. For all $a \in \{0, \ldots, p-1\}$ denote $\pi_a$ the idele whose $p$-component is equal to $(\frac{1}{a} \, \frac{a}{p})$ and whose components at all other places are equal to 1. Then

$$
\tilde{R}_m \times \pi_0 \tilde{R}_m = \bigcup_{a=0}^{p-1} \tilde{R}_m \pi_a, \quad U_m \pi_0 U_m = \bigcup_{a=0}^{p-1} U_m \pi_a.
$$

Analogously to the classical, non-adelic situation (see, e.g., [46, Ch. 3]), the operator $U_p$ on $\text{Div}(X_m)$ and $\text{Div}(\tilde{X}_m)$ can be defined as

$$
U_p([(g, f)]) := \sum_{a=1}^{p-1} (\pi_a g, f).
$$

Let $\mathcal{T}_p$ denote the Bruhat–Tits tree of $\text{GL}_2(\mathbb{Q}_p)$. The vertices of this tree correspond to the maximal orders of $\text{M}_2(\mathbb{Q}_p)$ and two vertices are connected by an edge if and only if the intersection of the corresponding orders is an Eichler order of level $p$ in $\text{M}_2(\mathbb{Q}_p)$. In general, the distance of two vertices is $m$ if and only if the intersection of the corresponding orders is an Eichler order of level $p^m$. Since every Eichler order of level $p^m$ is determined in a unique way by the intersection of two maximal orders corresponding to vertices $v_0$ and $v_m$ at distance $m$, an Eichler order of level $p^m$ can be represented by exactly two paths $p = (v_0, v_1, \ldots, v_m)$ and $\tilde{p} := (v_m, v_{m-1}, \ldots, v_0)$. Write $\bar{\mathcal{T}}_p(m)$ for the set of paths of $\mathcal{T}_p$ of length $m$ with no backtracking. For every $p = (v_0, v_1, \ldots, v_m)$ in $\bar{\mathcal{T}}_p(m)$ and all $i \in \{0, \ldots, m\}$ set $v_i(p) := v_i$ and define $s(p) := v_0(p)$ to be the source of $p$ and $t(p) := v_m(p)$ to be the target of $p$. Since we are moving along a tree, a path $p$ is uniquely determined by its source and its target. Observe that $s(\pi) = t(p)$ and $t(\bar{p}) = s(p)$.

For any $g \in \text{GL}_2(\mathbb{Q}_p)$ and any vertex $v$ of $\mathcal{T}_p$ corresponding to a maximal order $R$ of $\text{M}_2(\mathbb{Q}_p)$ let $g^{-1}vg$ be the vertex corresponding to the maximal order $g^{-1}Rg$. Likewise, for a path $p = (v_0, v_1, \ldots, v_m)$ let $g^{-1}pg$ be the path $(g^{-1}v_0g, g^{-1}v_1g, \ldots, g^{-1}v_mg)$ corresponding to the Eichler order obtained by taking the intersection of the maximal orders corresponding to the vertices $g^{-1}v_0g$ and $g^{-1}v_mg$. For all integers $m$, let $v^{(m)}$ be the vertex corresponding to the maximal order $(p^m z_p, p^{-m} z_p)$. Fix the base point

$$
p_0 := (v^{(0)}, v^{(1)}, \ldots, v^{(m)})
$$

and define a map $\text{GL}_2(\mathbb{Q}_p) \rightarrow \bar{\mathcal{T}}_p(m)$ by $g \mapsto g^{-1}p_0g$. Since the action of the matrices of the form $(\frac{a}{0} \frac{0}{d})$, with $a \in \mathbb{Q}_p^*$, on an Eichler order is trivial, the above map induces a well-defined map $\text{PGL}_2(\mathbb{Q}_p) \rightarrow \bar{\mathcal{T}}_p(m)$. Now define

$$
\Gamma_0(p^m) := \{(a \, b \, c \, d) \in \text{GL}_2(\mathbb{Z}_p) \mid c \equiv 0 \pmod{p^m}\}.
$$

**Lemma 1.1.** The map $[g] \mapsto g^{-1}p_0g$ from $\Gamma_0(p^m) \backslash \text{PGL}_2(\mathbb{Q}_p)$ to $\bar{\mathcal{T}}_p(m)$ is a bijection.
Proof. First of all, the map is well defined because \( \Gamma_0(p^m) \) is the stabilizer in \( \GL_2(\mathbb{Q}_p) \) of the pairs of vertices of \( T_p \) at distance \( m \). This also shows injectivity. Finally, surjectivity follows because \( \PGL_2(\mathbb{Q}_p) \) acts transitively on \( \mathcal{P}_p(m) \). \( \square \)

Remark 1.2. The bijection exhibited in Lemma 1.1 depends on the path \( p_0 \), hence it is not canonical. However, since our choice of the distinguished path \( p_0 \) is in some sense the natural one, in the following we will interpret Lemma 1.1 as providing an identification of \( \Gamma_0(p^m) \backslash \PGL_2(\mathbb{Q}_p) \) with \( \mathcal{P}_p(m) \).

Define \( \mathcal{R}_m(p) := \{ x \in \mathcal{R}_m \mid x_p = 1 \} \) and \( \Delta_m := B^\times \cap \mathcal{R}_m(p)^\times \GL_2(\mathbb{Q}_p) \). By strong approximation,

\[
\mathcal{R}_m^\times \backslash (\mathcal{B}^\times \times \mathcal{P}) / B^\times = \Gamma_0(p^m) \backslash (\GL_2(\mathbb{Q}_p) \times \mathcal{P}) / \Delta_m.
\]

Since \( \Gamma_0(p^m) \supset \mathbb{Z}_p^\times \) and \( \Delta_m \supset \mathbb{Q}_p^\times \cap \mathbb{Z}[1/p] \), strong approximation and the fact that \( \Gamma_0(p^m) \) acts trivially on \( \mathcal{P} \) give further identifications

\[
\mathcal{R}_m^\times \backslash (\mathcal{B}^\times \times \mathcal{P}) / B^\times = \Gamma_0(p^m) \backslash (\PGL_2(\mathbb{Q}_p) \times \mathcal{P}) / \Delta_m
\]

\[
= \left( \left( \Gamma_0(p^m) \backslash \PGL_2(\mathbb{Q}_p) \right) \times \mathcal{P} \right) / \Delta_m
\]

\[
(6) = (\mathcal{P}_p(m) \times \mathcal{P}) / \Delta_m,
\]

with the last equality coming from Lemma 1.1. The next result describes the action of \( U_p \) on \( \Div(X_m) \).

**Proposition 1.3.** The action of \( U_p \) on \( \Div(X_m) \) is represented by the map

\[
(p, z) \mapsto \sum (p', z)
\]

on \( \mathcal{P}_p(m) \times \mathcal{P} \), where if \( p = (v_0, \ldots, v_m) \) the the sum is over all paths \( p' \) which can be written as \( p' = (v', v_0, \ldots, v_{m-1}) \) for some vertex \( v' \) at distance one from \( v_0 \), with the additional requirement that \( p' \neq \bar{p} \) if \( m = 1 \).

Proof. Keep equation (6) in mind and suppose that \( v_i(p) = g^{-1}v(i)g \) for some \( g \in \GL_2(\mathbb{Q}_p) \).

Then \( \pi_a g \) represents the path \( p' \) with \( v_i(p') = g^{-1}v(i-1)g \) for \( i = 0, \ldots, m \). Upon noticing that \( \pi_a^{-1}v(i)\pi_a = v(i-1) \) for \( i = 1, \ldots, m \) and that \( \pi_a v(0) \) is the vertex of \( T_p \) corresponding to the maximal order \( v(-1) = \left( \frac{z_p}{p-tz_p}, \frac{p^2z_p}{z_p} \right) \), which is at distance one from \( v(0) \) and different from \( v(1) \), the result follows. \( \square \)

2. Heegner Points

Let \( K \) be an imaginary quadratic field of discriminant \( D = D_K \) prime to \( pN \) and denote by \( \mathcal{O}_K \) its ring of algebraic integers. Assume that the following *Heegner hypothesis* is satisfied:

- a prime number \( \ell \) divides \( N^+ \) (respectively, \( N^- \)) if and only if \( \ell \) splits (respectively, is inert) in \( K \).

No conditions are imposed on \( p \).
2.1. Heegner points on $X_m$ and $\tilde{X}_m$. For any order $\mathcal{O} \subset K$ and any Eichler order $R \subset B$, we say that a morphism $f \in \text{Hom}_\mathbb{Q}(K, B)$ is an optimal embedding of $\mathcal{O}$ in $R$ if

$$f(\mathcal{O}) = R \cap f(K) \quad (\text{i.e., } f^{-1}(R) = \mathcal{O}).$$

We say that a point $P = [(g, f)] \in X_m^{(K)}$ for some integer $m \geq 0$ is a Heegner point of conductor $\text{cond}(\mathcal{O})$ on $X_m$ if $f$ is an optimal embedding of $\mathcal{O}$ into the Eichler order $g^{-1}R_m g \cap B$, where $\text{cond}(\mathcal{O})$ is the conductor of $\mathcal{O}$. Note that both in the definite and in the indefinite case Heegner points are contained in $X_m(\mathbb{Q})$. More precisely, suppose that $P$ is a Heegner point of conductor $\text{cond}(\mathcal{O})$ on $X_m$; if $P$ is definite then $P \in X_m(K)$, while if $P$ is indefinite then $P \in X_m(H_{\text{cond}(\mathcal{O})})$ where $H_{\text{cond}(\mathcal{O})}$ is the ring class field of $K$ of conductor $\text{cond}(\mathcal{O})$. It can be shown that the set $\text{Heeg}_m(\text{cond}(\mathcal{O}))$ of Heegner points on $X_m$ of conductor $\text{cond}(\mathcal{O})$ is always finite (possibly empty) and to compute its cardinality as $\text{cond}(\mathcal{O})$ and $X_m$ vary: see [40, Theorem 1] and [50, Ch. III, Théorème 5.11].

Finally, we say that a point $\tilde{P} \in \tilde{X}_m^{(K)}$ is a Heegner point of conductor $\text{cond}(\mathcal{O})$ on $\tilde{X}_m$ if the same is true of $\beta_m(\tilde{P}) \in X_m$, where $\beta_m$ is as in diagram (5). In other words, the Heegner points of a certain conductor $r$ on $\tilde{X}_m$ are simply the lifts via $\beta_m$ of the Heegner points of conductor $r$ on $X_m$ as defined above.

2.2. Hecke relations on $X_m$. Define an action of $\text{Gal}(K^{ab}/K) \simeq \hat{K}^\times/K^\times$ on $X_m^{(K)}$ by the formula

$$(7) \quad \quad P^a := [(g\bar{f}(a), f)]$$

for all $a \in \hat{K}^\times/K^\times$ and all $P = [(g, f)] \in X_m^{(K)}$, where $\bar{f} : \hat{K}^\times \to \hat{B}^\times$ is the map obtained from $f$ by passing to the adelizations of $K$ and $B$. Let $\text{Pic}(\mathcal{O}) := \hat{K}^\times/K^\times\hat{\mathcal{O}}^\times$ be the Picard group of $\mathcal{O}$. By class field theory, $\text{Pic}(\mathcal{O}) \simeq \text{Gal}(H_{\text{cond}(\mathcal{O})}/K)$ and (7) defines an action of $\text{Pic}(\mathcal{O})$ on the set $\text{Heeg}_m(\text{cond}(\mathcal{O}))$. By Shimura’s reciprocity law ([46, Theorem 9.6]), if $B$ is indefinite the action of $\text{Pic}(\mathcal{O})$ defined in (7) corresponds under the isomorphism $\text{Gal}(H_{\text{cond}(\mathcal{O})}/K) \simeq \text{Pic}(\mathcal{O})$ to the usual Galois action of $\text{Gal}(H_{\text{cond}(\mathcal{O})}/K)$ on $X_m(H_{\text{cond}(\mathcal{O})})$. If $B$ is definite, formula (7) may be regarded instead as the definition of the Galois action on the set $\text{Heeg}_m(\text{cond}(\mathcal{O}))$; note that this action is not induced by the natural action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the geometric points of $X_m$, since in this case $X_m^{(K)} = X_m(K)$ as remarked before.

For every integer $c \geq 1$ prime to $N$ and every integer $n \geq 0$ denote by $\mathcal{O}_{cp^n} := \mathbb{Z} + cp^n\mathcal{O}_K$ the order of $K$ of conductor $cp^n$.

Proposition 2.1. Let $P = [(g, f)]$ be a point in $X_m^{(K)}$ for some $m \geq 1$. Suppose that $P$ is a Heegner point of conductor $cp^n$ for some integer $n$ and that $\pi P = [(\pi g, f)]$ is a Heegner point of conductor $cp^{n+1}$. Then

$$U_p(P) = \text{Tr}_{H_{cp^{n+1}}/H_{cp^n}}(\tilde{\pi}P)$$

in $\text{Div}(X_m)$.

Proof. Let $P$ be represented by $(\gamma, f')$ under (3), so that $g = r\gamma b$ and $f' = bfb^{-1}$ with $\gamma \in B^\times$, $r \in \hat{R}_m^{(p)}$ and $b \in B^\times$, and let $\gamma$ correspond to the path $p = (v_0, \ldots, v_m)$ (cf. (1,4)). By Proposition 1.3 the elements in $U_p(P)$ are represented by the pairs $(p(a), f')$ where $p(a) = (v(a), v_0, \ldots, v_{m-1})$ and the $v(a)$ for $a = 0, \ldots, p - 1$ are the vertices adjacent to $v_0$ and different from $v_1$. In particular, the path $p(0)$ is $(\nu(-1), \nu(0), \ldots, \nu(m-1))$. 
Note that \( \hat{\pi}P = [(\hat{\pi}g, f)] \). We have \( \hat{\pi}g = \hat{\pi}r \gamma b = r((\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \gamma)b \), so \( \hat{\pi}P \) is represented under (6) by the pair \( ((\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \gamma, f') \) and thus corresponds to the path \( p(0) \). Let now \( \sigma \in \text{Gal}(H_{cp^n}/H_{cp^{n+1}}) \) be represented by the element \( a \in \mathcal{O}_{cp^n}^\times \) under the isomorphism

\[
\text{Gal}(H_{cp^n}/H_{cp^{n+1}}) \simeq \mathcal{O}_{cp^n}^\times / \mathcal{O}_{cp^{n+1}}^\times .
\]

Since \( f \) is an optimal embedding of \( \mathcal{O}_{cp^n} \) into \( g^{-1} \mathcal{R}_m g \cap B \), we see that \( \hat{f}(a) = g^{-1}xg \) for some \( x \in \mathcal{R}_m^\times \) and so we can write

\[
\hat{\pi}P^a = [(\hat{\pi}g \hat{f}(a), f)] = [(\hat{\pi}xg, f)]
\]

for some \( x \in \mathcal{R}_m^\times \). As above, we have \( \hat{\pi}xg = \hat{\pi}x^{(p)}x_pr\gamma b = (x^{(p)}r)((\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix})x_p\gamma) b \) where \( x^{(p)} \) and \( x_p \) denote the product of the components of \( x \) outside \( p \) and the component of \( x \) in \( p \), respectively. Thus the path associated with \( \hat{\pi}P^a \) is the one associated with the matrix \( (\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix})x_p\gamma \). Now the image of \( p_0 \) under the action of \( (\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \) is \( p(0) \) and, since \( x_p \in (R_m \otimes \mathbb{Z}_p)^\times \), the action of this matrix on \( p(0) \) stabilizes the vertices \( v(0), v(1), \ldots, v(n-1) \), so it is of the form \( p(a) \) for some \( a \in \{0, \ldots, p-1\} \). Since \( \text{Gal}(H_{cp^n}/H_{cp^{n+1}}) \) acts without fixed points and the cardinality of this group is \( p-1 \), the result follows.

2.3. Fields of rationality. Define an action of \( \text{Gal}(K^{ab}/K) \simeq \tilde{K}^\times /K^\times \) on \( \tilde{X}_m(K) \) by the formula

\[
P^a := [(g \hat{f}(a), f)]
\]

for all \( a \in \tilde{K}^\times /K^\times \) and all \( P = [(g, f)] \in \tilde{X}_m(K) \). As in [22] if \( B \) is indefinite then Shimura’s reciprocity law ensures that this action corresponds to the usual Galois action of \( \text{Gal}(K^{ab}/K) \) on \( \tilde{X}_m(K^{ab}) \), and if \( f \) is an embedding of \( K \) into \( B \) then \( P \in \tilde{X}_m(K^{ab}) \). If \( B \) is definite then the action of \( \text{Gal}(K^{ab}/K) \) on \( \tilde{X}_m(\mathbb{Q}) \) is instead defined as in [8].

To study the fields of rationality of Heegner points on \( \tilde{X}_m \) some class field theory is needed. To begin with, define

\[
Z_m := \hat{f}^{-1}(U_m \cap \hat{f}(\mathcal{O}_{cp^n}^{\times})).
\]

Lemma 2.2. \( Z_m = \{ a = (a_q) \in \mathcal{O}_{cp^n}^{\times} \mid a_p \equiv 1 \pmod{p^m} \} \).

Proof. As usual, let \( f_p \) denote the map obtained from \( f \) by extending scalars to \( \mathbb{Z}_p \). Note that \( a \in Z_m \) if and only if \( a \in \mathcal{O}_{cp^n}^{\times} \) and

\[
f_p(a_p) = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} + p^m A
\]

for some \( b \in \mathbb{Z}_p \), \( d \in \mathbb{Z}_p^\times \) and \( A \in M_2(\mathbb{Z}_p) \). Since \( \mathcal{O}_{cp^n} \otimes \mathbb{Z}_p = \mathbb{Z}_p + p^m \mathcal{O}_K \otimes \mathbb{Z}_p \), write \( a_p = \alpha + p^m \beta \) with \( \alpha \in \mathbb{Z}_p^\times \) and \( \beta \in \mathcal{O}_K \otimes \mathbb{Z}_p \). It follows that

\[
f_p(a_p) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + p^m B
\]

with \( B = f_p(\beta) \in M_2(\mathbb{Z}_p) \). Thus \( a \in Z_m \) precisely when \( a \in \mathcal{O}_{cp^n}^{\times} \) and \( a_p = \alpha + p^m \beta \) with \( \alpha \equiv 1 \pmod{p^m} \), whence the claim.

For any number field \( F \) denote by \( I_F \) its idèle group (so \( \hat{F}^\times \) is the finite part of \( I_F \)). Write \( \tilde{H}_{cp^n} \) for the class field of \( Z_m^\infty := Z_m \times \mathbb{C}^\times \), so that

\[
\text{Gal}(\tilde{H}_{cp^n}/K) \simeq \tilde{K}^\times /K^\times Z_m.
\]
Proposition 2.3. Let \( P \in \tilde{X}_m \) be a Heegner point. Then
\begin{enumerate}
  \item \( P \in H^0(\Gal(K^{ab}/\tilde{H}_{cp^m}), \tilde{X}_m(K)) \) in the definite case;
  \item \( P \in \tilde{X}_m(\tilde{H}_{cp^m}) \) in the indefinite case.
\end{enumerate}

Proof. Use the fact that \( P \) is fixed by the action of \( \Gal(K^{ab}/\tilde{H}_{cp^m}) \) and that in the indefinite case \( P \) is rational over \( K^{ab} \) by, for example, [6, Lemma 3.11]. \( \Box \)

We give a more explicit description of \( \tilde{H}_{cp^m} \). As a general notation, for every integer \( n \geq 1 \) let \( \mu_n \) be the \( n \)-th roots of unity. Set \( p^* := (-1)^{(p-1)/2}p \).

Lemma 2.4. \( \mathbb{Q}(\sqrt{p^*}) \subset H_{cp^m} \).

Proof. It is enough to show that \( \mathbb{Q}(\sqrt{p^*}) \subset H_p \), as \( H_p \subset H_{cp^m} \) for all integers \( m \geq 1 \). The field \( \mathbb{Q}(\sqrt{p^*}) \) is known to be the unique quadratic extension of \( \mathbb{Q} \) contained in \( \mathbb{Q}(\mu_{p^m}) \). Its norm group contains the group \( W_{m,\infty} \) introduced in the proof of Proposition 2.5 below and is contained with index 2 in the trivial norm group \( \mathbb{Z}_p^\times \). Define
\[
Q := \prod_{\ell \neq p} \mathbb{Z}_\ell^\times \times \{ y \in \mathbb{Z}_p^\times \mid y \text{ is a square modulo } p \};
\]
the characterization of \( \mathbb{Q}(\sqrt{p^*}) \) implies that \( \Gal(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}) \cong I_\mathbb{Q}/\mathbb{Q}^\times \). Moreover, it is easy to show that \( \Norm_{K/\mathbb{Q}}(\tilde{O}_p^\times) \subset Q \). Indeed,
\[
\Norm_{K/\mathbb{Q}}(\tilde{O}_p^\times) \subset \mathbb{Z}_\ell^\times
\]
for all \( \ell \neq p \). On the other hand, an element \( x \in \mathcal{O}_p \otimes \mathbb{Z}_p \) in the \( p \)-component can be written as \( x = \alpha + p\beta \) with \( \alpha \in \mathbb{Z}_p^\times \) and \( \beta \in \mathcal{O}_K \otimes \mathbb{Z}_p \). It follows that
\[
\Norm_{K_p/\mathbb{Q}_p}(x) \equiv \alpha^2 \pmod{p}.
\]
Since \( \mathbb{Q}^\times \Norm_{K/\mathbb{Q}}(\tilde{O}_p^\times) \subset \mathbb{Q}^\times Q \) and
\[
\mathbb{Q}^\times \Norm_{H_p/\mathbb{Q}}(\tilde{H}_p^\times) = \mathbb{Q}^\times \Norm_{K/\mathbb{Q}}(\Norm_{H_p/K}(\tilde{H}_p^\times)) = \mathbb{Q}^\times \Norm_{K/\mathbb{Q}}(\tilde{O}_p^\times),
\]
the maximal abelian extension of \( \mathbb{Q} \) in \( H_p \) contains the field corresponding to \( \mathbb{Q}^\times Q \), namely \( \mathbb{Q}(\sqrt{p^*}) \). This proves the lemma. \( \Box \)

Now we can prove

Proposition 2.5. \( \tilde{H}_{cp^m} = H_{cp^m}(\mu_{p^m}) \).

Proof. Write \( \Gal(K/\mathbb{Q}) \cong I_\mathbb{Q}/\mathbb{Q}^\times C \) where \( C := \Norm_{K/\mathbb{Q}} I_K \) is the norm group of \( K \). Define
\[
W_m := \prod_{\ell \neq p} \mathbb{Z}_\ell^\times \times \{ \alpha \in \mathbb{Z}_p^\times \mid \alpha \equiv 1 \pmod{p^m} \}
\]
and set \( W_{m,\infty} := W_m \times \mathbb{R}_+ \) where \( \mathbb{R}_+ \) is the group of positive real numbers. The extension \( K(\mu_{p^m})/\mathbb{Q} \) is abelian, and since \( \Gal(\mathbb{Q}(\mu_{p^m})/\mathbb{Q}) \cong I_\mathbb{Q}/\mathbb{Q}^\times W_{m,\infty} \) it follows by global class field theory (cf. [37, Ch. IV, Theorem 7.1]) that
\[
\Gal(K(\mu_{p^m})/\mathbb{Q}) \cong I_\mathbb{Q}/\mathbb{Q}^\times (C \cap W_{m,\infty}).
\]
Now \( \Gal(K(\mu_{p^m})/K) \cong \tilde{K}^\times /K^\times V_m \) where \( V_m \) denotes the finite part of the norm group \( \Norm_{K(\mu_{p^m})/K}(I_K(\mu_{p^m})) \). Hence
\[
\Gal(H_{cp^m}(\mu_{p^m})/K) \cong \tilde{K}^\times /K^\times (V_m \cap \tilde{O}_{cp^m}^\times).
\]
Since the finite part of \( \text{Norm}_{K(\mu_{p^n})/Q}(I_K(\mu_{p^n})) \) equals \( \text{Norm}_{K/Q}(V_m) \) and
\[
\mathbb{Q}^\times \text{Norm}_{K(\mu_{p^n})/Q}(I_K(\mu_{p^n})) = \mathbb{Q}^\times(C \cap W_{m,\infty}),
\]
it follows that
\[
V_m \subset \{ x \in \tilde{\mathbb{K}}^\times | \text{Norm}_{K/Q}(x) \in \mathbb{Q}^\times W_m \}.
\]
Let \( x \in V_m \cap \tilde{\mathcal{O}}_{cp^m} \) and write \( x = \alpha + cp^m \beta \) with \( \alpha \in \hat{\mathbb{Z}} \) and \( \beta \in \tilde{\mathcal{O}}_K \). Then \( \text{Norm}_{K/Q}(x) \in \mathbb{Q}^\times W_m \cap \hat{\mathbb{Z}}^\times = W_m \). On the other hand, locally at \( p \):
\[
\text{Norm}_{K_{p}/Q}(x_p) \equiv \alpha_p^2 \pmod{p^m}.
\]
It follows that \( \alpha_p \equiv \pm 1 \pmod{p^m} \), and the description of \( Z_m \) provided by Lemma 2.2 gives the inclusion
\[
K^\times(V_m \cap \tilde{\mathcal{O}}_{cp^m}) \subset K^\times Z_m.
\]
The isomorphism (9) finally yields:
\[
\tilde{H}_{cp^m} \subset H_{cp^m}(\mu_{p^n}).
\]
It is easily seen that the Galois group \( \text{Gal}(H_{cp^m}(\mu_{p^n})/H_{cp^m}) \) is isomorphic to \( \tilde{\mathcal{O}}_{cp^m}^\times / \mathcal{O}_{cp^m}^\times Z_m \). Since \( \tilde{\mathcal{O}}_{cp^m}^\times / Z_m \) is isomorphic to \( (\mathbb{Z}/p^m\mathbb{Z})^\times \) via the map which sends \( a = (a_q)_q \in \tilde{\mathcal{O}}_{cp^m}^\times \) to \( a_p \pmod{p^m} \in (\mathbb{Z}/p^m\mathbb{Z})^\times \), and \( \mathcal{O}_{cp^m}^\times = \{ \pm 1 \} \) for \( m \geq 1 \), we get that
\[
[\tilde{H}_{cp^m} : H_{cp^m}] = \varphi(p^m)/2.
\]
The result follows from (10) and (11) upon noticing that \([H_{cp^m}(\mu_{p^n}) : H_{cp^m}] \leq \varphi(p^m)/2 \) because \( H_{cp^m} \supset Q(\sqrt{p^m}) \) by Lemma 2.4.

\begin{flushright}
\( \square \)
\end{flushright}

**Convention.** In light of Proposition 2.5, from now on we adopt the explicit notation \( H_{cp^m}(\mu_{p^n}) \) in place of the shorthand \( \tilde{H}_{cp^m} \). The reason why we do so is that whenever \( c = p^n \) with \( n \geq 1 \) we have \( \tilde{H}_{cp^m} \neq \tilde{H}_{(c/p)p^{m+1}} \), so the previous notation would be ambiguous.

2.4. **Hecke relations on \( \tilde{X}_m \).** Let \( r, s \geq 1 \) be integers. Then
\[
\text{Gal}(H_{cp^s}(\mu_{p^r})/K) \simeq \tilde{\mathbb{K}}^\times / K^\times(V_r \cap \tilde{\mathcal{O}}_{cp^s}^\times)
\]
where \( V_r \) is the finite part of the norm group \( \text{Norm}_{K(\mu_{p^r})/K}(I_K(\mu_{p^r})) \). Hence for every pair of integers \( t, u \) with \( t \geq s \) and \( u \geq r \) there is an isomorphism
\[
K^\times(V_r \cap \tilde{\mathcal{O}}_{cp^s}^\times)/K^\times(V_u \cap \tilde{\mathcal{O}}_{cp^r}^\times) \xrightarrow{\sim} \text{Gal}(H_{cp^r}(\mu_{p^u})/H_{cp^s}(\mu_{p^t})).
\]
As pointed out in the proof of Proposition 2.5, every element \( x = \alpha + cp^s \beta \in V_r \cap \tilde{\mathcal{O}}_{cp^s}^\times \) (with \( \alpha \in \hat{\mathbb{Z}} \) and \( \beta \in \tilde{\mathcal{O}}_K \)) satisfies the local conditions
\[
\text{Norm}_{K_{p}/Q}(x_p) \equiv \alpha_p^2 \pmod{p^r}, \quad \text{Norm}_{K_{p}/Q}(x_p) \equiv 1 \pmod{p^r}.
\]
It follows that every \( \sigma \in \text{Gal}(H_{cp^r}(\mu_{p^u})/H_{cp^s}(\mu_{p^t})) \) can be represented by an element \( x = \alpha + cp^s \beta \) as above such that \( \alpha_p \equiv 1 \pmod{p^r} \).

Let \( \tilde{P} = [(g, f)] \) be a point on \( \tilde{X}_m \) for some \( m \geq 1 \) and let \( \tilde{\pi} \) be the idele introduced in 2.2. Suppose that \( \tilde{P} \) is a Heegner point of conductor \( cp^n \) for some \( n \geq 0 \) and that \( \tilde{\pi}\tilde{P} = [(\tilde{\pi}g, f)] \) is a Heegner point of \( \tilde{X}_m \) of conductor \( cp^{n+1} \). Moreover, suppose that the following condition holds.
Assumption 2.6. The local embedding $g_p^{-1} f_p g_p : K_p \hookrightarrow B_p$ sends the elements of $O_{cp^n} \otimes \mathbb{Z}_p$ which are congruent to 1 modulo $p^n$ to matrices in $M_2(\mathbb{Z}_p)$ which are congruent to the identity modulo $p^n$.

When applying the results below to our compatible family of Heegner points, these conditions will always be satisfied (see Proposition 3.11 and Corollary 3.13).

Take $\tilde{\sigma} \in \text{Gal}(H_{cp^n+1}(\mu_{p^n+1})/H_{cp^n}(\mu_{p^n+1}))$ and let it be represented by $a \in \hat{O}_{cp^n}^\times$. By the above discussion, we can (and do) assume that

$$a = \alpha + cp^n \beta, \quad \alpha_p \equiv 1 \pmod{p^n}.$$

Since

$$\text{Gal}(H_{cp^n+1}(\mu_{p^n+1})/H_{cp^n}(\mu_{p^n+1})) \subset \text{Gal}(H_{cp^n+1}(\mu_{p^n+1})/H_{cp^n}),$$

it makes sense to consider the projection $\sigma$ of $\tilde{\sigma}$ to $\text{Gal}(H_{cp^n+1}/H_{cp^n})$, which is represented by the same $a$. Let $P$ be the image of $\tilde{P}$ on $X_m$ via $\beta_m$. The proof of Proposition 2.1 shows that if $P$ is represented by a pair $(p, f')$ via (12) then both $U_p(P)$ and $\text{Tr}_{H_{cp^n+1}/H_{cp^n}}(\hat{\pi} P)$ are represented by $\sum_{i=0}^{p-1} (p(a), f')$. Hence there exist $i \in \{0, \ldots, p-1\}$ and $\Xi \in \hat{R}_m^\times$ such that

$$\hat{\pi}_p g_p \hat{f}(a)_p = (\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix})(\begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix}) g_p$$

with $\Xi_p = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ (recall the convention about the adeles in $\hat{B}$ introduced at the beginning of Section 1). By (12) and Assumption 2.6, there also exists $\Upsilon \in \hat{R}_m^\times$ such that

$$\left( \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \cdot \left( \begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix} \right) = \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \cdot \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$$

with $\Upsilon_p = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ and $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \equiv (\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}) \pmod{p^n}$. The equality (13) of matrices in $M_2(\mathbb{Z}_p)$ immediately implies that $\Xi \in \hat{R}_m^\times \cap U_n$. In particular, if $n \geq m$ then $\Xi \in U_m$. In this case, the point

$$\hat{\pi} \tilde{P} = [(\pi, g, f)]$$

appears in $U_p(\tilde{P})$.

Proposition 2.7. Let $\tilde{P} = [(g, f)]$ be a Heegner point of conductor $cp^n$ on $X_m$ for some $n \geq m$. Suppose also that $\hat{\pi} \tilde{P} = [(\hat{\pi} g, f)]$ is a Heegner point of conductor $cp^{n+1}$ on $X_m$ and that Assumption 2.6 holds. Then

$$U_p(\tilde{P}) = \text{Tr}_{H_{cp^n+1}/H_{cp^n}}(\mu_{p^n+1}) \cdot (\hat{\pi} \tilde{P})$$

in $\text{Div}(X_m)$.

Proof. Since the fields $H_{cp^n+1}$ and $H_{cp^n}(\mu_{p^n+1})$ are linearly disjoint over $H_{cp^n}$, the projection

$$\text{Gal}(H_{cp^n+1}(\mu_{p^n+1})/H_{cp^n}(\mu_{p^n+1})) \xrightarrow{\sim} \text{Gal}(H_{cp^n+1}/H_{cp^n})$$

is an isomorphism. Thus $\text{Gal}(H_{cp^n+1}(\mu_{p^n+1})/H_{cp^n}(\mu_{p^n+1}))$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, and the claim of the proposition follows from (14).
3. Explicit construction of families of Heegner points

The purpose of this section is to construct a family of Heegner points on the tower of Shimura curves which satisfies suitable compatibility properties with respect to the natural covering maps in the tower. These points will be the building blocks in our definition of big Heegner points and classes that will be performed in Section 6. Unlike what done in [24], to achieve our goal we systematically adopt the language and formalism of optimal embeddings, and this approach allows us to treat in a uniform way both the definite and the indefinite case. As will be apparent below, the construction (though as explicit as possible) is technically rather intricate: the reader who is exclusively interested in the formal properties enjoyed by our system of points is suggested to skip directly to §3.5.

3.1. Heegner points of conductor $cp^m$ on $X_m$ and $\tilde{X}_m$. In order to introduce the systems of Heegner points that we shall work with, we need to recall some auxiliary results and definitions. As a preliminary remark, note that the Heegner hypothesis and [40, Theorems 1 and 2] ensure that the set $\text{Heeg}_{m}(cp^m)$ of Heegner points of conductor $cp^m$ on $X_m$ is not empty.

Let $O$ be an order of $K$ and $R$ an order of $B$. Let $\ell$ be a prime number. Define $K_\ell := K \otimes_\mathbb{Z} \mathbb{Z}_\ell$ and $B_\ell := B \otimes_\mathbb{Z} \mathbb{Z}_\ell$. An injective homomorphism $\varphi : K_\ell \hookrightarrow B_\ell$ of $\mathbb{Q}_\ell$-algebras is said to be an optimal embedding of $O \otimes \mathbb{Z}_\ell$ into $R \otimes \mathbb{Z}_\ell$ if

$$\varphi(O \otimes \mathbb{Z}_\ell) = \varphi(K_\ell) \cap (R \otimes \mathbb{Z}_\ell) \quad \text{(i.e., } \varphi^{-1}(R \otimes \mathbb{Z}_\ell) = O \otimes \mathbb{Z}_\ell).$$

Two optimal embeddings $\varphi$ and $\psi$ of $O \otimes \mathbb{Z}_\ell$ into $R \otimes \mathbb{Z}_\ell$ are said to be equivalent if there exists an element $u \in (R \otimes \mathbb{Z}_\ell)^\times$ such that $\varphi = u^{-1}\psi u$.

If $f : K \hookrightarrow B$ is an injective homomorphism of $\mathbb{Q}$-algebras and $\ell$ is a prime number, denote by $f_\ell = f \otimes \text{id}_{\mathbb{Z}_\ell} : K_\ell \hookrightarrow B_\ell$ the homomorphism which is obtained from $f$ by extension of scalars.

The next lemma says that a global embedding is optimal if and only if it induces at every prime an optimal embedding of the local orders.

**Lemma 3.1.** An injective homomorphism of $\mathbb{Q}$-algebras $f : K \hookrightarrow B$ is an optimal embedding of $O$ into $R$ if and only if $f_\ell$ is an optimal embedding of $O \otimes \mathbb{Z}_\ell$ into $R \otimes \mathbb{Z}_\ell$ for all primes $\ell$.

**Proof.** Let us prove the “if” part. First of all, observe that $K \cap (O \otimes \mathbb{Z}_\ell) = O \otimes \mathbb{Z}(\ell)$ and $B \cap (R \otimes \mathbb{Z}_\ell) = R \otimes \mathbb{Z}(\ell)$ where $\mathbb{Z}(\ell)$ is the localization of $\mathbb{Z}$ at $\ell$ and the intersections are taken in $K_\ell$ and $B_\ell$, respectively. On the one hand,

$$f^{-1}_\ell(R \otimes \mathbb{Z}_\ell) \cap K = (O \otimes \mathbb{Z}_\ell) \cap K = O \otimes \mathbb{Z}(\ell).$$

On the other hand,

$$f^{-1}_\ell(R \otimes \mathbb{Z}_\ell) \cap K = f^{-1}(B \cap (R \otimes \mathbb{Z}_\ell)) = f^{-1}(R \otimes \mathbb{Z}(\ell)).$$

It follows that

$$f^{-1}(R) = f^{-1}\left(\bigcap_\ell (R \otimes \mathbb{Z}(\ell))\right) = \bigcap_\ell f^{-1}(R \otimes \mathbb{Z}(\ell)) = \bigcap_\ell (O \otimes \mathbb{Z}(\ell)) = O,$$

which proves the claim.

To show the “only if” implication proceed as follows. By assumption,

$$f(O) = f(K) \cap R.$$
Then one has
\[ f_\ell(O \otimes \mathbb{Z}_\ell) = f(O) \otimes \mathbb{Z}_\ell = (f(K) \cap R) \otimes \mathbb{Z}_\ell = (f(K) \otimes \mathbb{Z}_\ell) \cap (R \otimes \mathbb{Z}_\ell) = f_\ell(K\ell) \cap (R \otimes \mathbb{Z}_\ell), \]
where the third equality follows from \[24\] Theorem 7.4 (i)] since \( \mathbb{Z}_\ell \) is flat as a \( \mathbb{Z} \)-module. See \[13\] Lemma 4.9] for a quick proof of this lemma using the elementary divisor theorem. □

Let \( R \) be an Eichler order of \( B \), let \( I_1, \ldots, I_h \) be representatives of all the distinct classes of left \( R \)-ideals and denote by \( R_i \) the right order of \( I_i \) for \( i = 1, \ldots, h \). The number \( h \) depends only on the level of \( R \) and the discriminant of the quaternion algebra, and the set \( \{R_1, \ldots, R_h\} \) consists of representatives for all the conjugacy classes of Eichler orders in \( B \) with the same level. For every \( i \in \{1, \ldots, h\} \) fix an element \( \gamma_i \in \hat{B}^\times \) such that \( \hat{R}_i = \gamma_i^{-1} \hat{R} \gamma_i \) and write \( \gamma_{i,\ell} \) for the \( \ell \)-component of \( \gamma_i \) at a prime \( \ell \).

**Proposition 3.2.** Let \( O \) be an order of \( K \) and \( R \) an Eichler order of \( B \), and let \( \{\varphi_\ell\}_\ell \) be a collection of optimal embeddings of \( O \otimes \mathbb{Z}_\ell \) into \( R \otimes \mathbb{Z}_\ell \) for all primes \( \ell \). Then there exists an optimal embedding \( f : K \hookrightarrow B \) of \( O \) into \( R_i \) for some \( i \in \{1, \ldots, h\} \) such that \( \gamma_{i,\ell} f \varphi_\ell \gamma_{i,\ell}^{-1} \) is equivalent to \( \varphi_\ell \) for all \( \ell \).

**Proof.** This is essentially a consequence of Eichler's trace formula ([50] Ch. III, Théorème 5.11); for the convenience of the reader, we give here a direct proof (see [50] Ch. III, §5 or [42] §3 for more details). By the assumption on \( K \), there exists an injective homomorphism \( g : K \hookrightarrow B \) of \( \mathbb{Q} \)-algebras. By the Skolem–Noether theorem, for each prime \( \ell \) there exists \( a_\ell \in B_\ell^\times \) such that \( g_\ell = a_\ell^{-1} \varphi_\ell a_\ell \). For almost all primes \( \ell \) which do not divide the discriminant of \( B \), the level of \( R \) and the conductor of \( O \) the map \( g_\ell \) is an optimal embedding of \( O \otimes \mathbb{Z}_\ell \) into \( R \otimes \mathbb{Z}_\ell \); this is so because \( g(O) \) is contained in a maximal order whose \( \ell \)-adic completion is equal to \( R \otimes \mathbb{Z}_\ell \) for almost all \( \ell \). Hence we can assume that \( a_\ell \in (R \otimes \mathbb{Z}_\ell)^\times \) for almost all \( \ell \); in fact, by [50] Ch. II, §3], if \( \ell \) does not divide the discriminant of \( B \) and the level of \( R \) there is only one equivalence class of optimal embeddings of \( O \otimes \mathbb{Z}_\ell \) into \( R \otimes \mathbb{Z}_\ell \). Write \( a \) for the idele \( (a_\ell)_\ell \). By the strong approximation theorem, there exist a unique index \( i \in \{1, \ldots, h\} \), a global element \( b \in B^\times \) and a unit \( u \in \hat{R}^\times \) such that \( a = u \gamma_i b \). Then \( f := bgb^{-1} \) is a global embedding of \( K \) into \( B \) such that \( f_\ell \) is conjugate to \( \varphi_\ell \) for all primes \( \ell \). In fact, for every prime \( \ell \) one has
\[ \gamma_{i,\ell} f \varphi_\ell \gamma_{i,\ell}^{-1} = \gamma_{i,\ell} b \gamma_{i,\ell} b^{-1} \gamma_{i,\ell}^{-1} = (\gamma_{i,\ell} b a \gamma_{i,\ell}^{-1}) \varphi_\ell (\gamma_{i,\ell} b a \gamma_{i,\ell}^{-1})^{-1} = u_\ell^{-1} \varphi_\ell u_\ell, \]
which shows that \( \gamma_{i,\ell} f \varphi_\ell \gamma_{i,\ell}^{-1} \) is equivalent to \( \varphi_\ell \). In particular, \( f_\ell \) is an optimal embedding of \( O \otimes \mathbb{Z}_\ell \) into \( \gamma_{i,\ell}^{-1} (R \otimes \mathbb{Z}_\ell) \gamma_{i,\ell} = R_\ell \otimes \mathbb{Z}_\ell \) for every prime \( \ell \), hence \( f \) is an optimal embedding of \( O \) into \( R_i \) by Lemma 4.1. □

For every integer \( m \geq 0 \), let \( R_0^{(m)} \) denote the order of level \( N^+ \) in \( B \) such that \( R_m = R_0 \cap R_0^{(m)} \). The order \( R_0^{(m)} \) is determined by the following local conditions:
\begin{align*}
(15) & \quad R_0^{(m)} \otimes \mathbb{Z}_\ell = R_0 \otimes \mathbb{Z}_\ell \quad \text{for all } \ell \neq p; \\
(16) & \quad \phi_p(R_0^{(m)} \otimes \mathbb{Z}_p) = \left( \begin{array}{cc} 1 & 0 \\ 0 & p^m \end{array} \right) \mathbb{M}_2(\mathbb{Z}_p) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} p^m & 1 \\ 0 & 1 \end{array} \right); \end{align*}

In particular, \( R_0^{(0)} = R_0 \). Furthermore, for every \( r \in \{0, \ldots, m\} \) define Eichler orders \( R_{m-r}^{(m)} := R_0^{(m)} \cap R_0^{(r)} \). Note that \( R_0^{(m)} \) has level \( N^+p^k \) for all \( k \in \{0, \ldots, m\} \) and that, from equations (15)
We distinguish when
\[ R_k^{(m)} \otimes \mathbb{Z}_\ell = R_0 \otimes \mathbb{Z}_\ell \quad \text{for all } \ell \neq p; \]
\[ \phi_p(R_k^{(m)} \otimes \mathbb{Z}_p) = \left( \frac{z_p}{p^{m \cdot p^h} z_p} \right). \]

The following result will be used to prove optimality properties of the local embeddings \( \varphi_p^{(m)} \)
introduced below.

**Lemma 3.3.** Fix an integer \( m \geq 1 \) and suppose that \( \vartheta_p : K_p \hookrightarrow B_p \) is an optimal embedding of \( \mathcal{O}_c \otimes \mathbb{Z}_p \) into \( R_0^{(m)} \otimes \mathbb{Z}_p \). Then \( \vartheta_p \) is an optimal embedding of \( \mathcal{O}_c \otimes \mathbb{Z}_p \) into \( R_0^{(m)} \otimes \mathbb{Z}_p \) for all \( n \in \{0, \ldots, m\} \) if and only if \( \vartheta_p^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) \subset \vartheta_p^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) \) for all \( n \in \{0, \ldots, m-1\} \).

**Proof.** The condition is clearly necessary; to prove that it is also sufficient we proceed by induction on \( n \). The case \( n = 0 \) is guaranteed by our assumption, so let us suppose that \( \vartheta_p \) is an optimal embedding of \( \mathcal{O}_c \otimes \mathbb{Z}_p \) into \( R_0^{(m)} \otimes \mathbb{Z}_p \) for a certain \( k \in \{0, \ldots, m-1\} \).

We want to show that \( \mathcal{O}_c \otimes \mathbb{Z}_p \) is equal to \( \vartheta_p^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) \). Take \( x \) in \( \mathcal{O}_c \otimes \mathbb{Z}_p \) and write \( x = t + cp^k y \) with \( t \) in \( \mathbb{Z}_p \) and \( y \) in \( \mathcal{O}_K \). Then \( x = t + p(cp^ky) \) with \( cp^ky \) in \( \mathcal{O}_c \), hence if \( z \) in \( \mathbb{Z}_p \) we conclude that \( \vartheta_p(x \otimes z) = \vartheta_p(t \otimes z) + p\vartheta_p(cp^ky \otimes z) \in R_k^{(m)} \otimes \mathbb{Z}_p \)
because \( \vartheta_p(cp^ky \otimes z) \in R_k^{(m)} \otimes \mathbb{Z}_p \) by the inductive step and \( pR_k^{(m)} \subset R_{k+1}^{(m)} \) by (17) and (18). Therefore:
\[ \mathcal{O}_c \otimes \mathbb{Z}_p \subset \vartheta_p^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) \subset \vartheta_p^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) = \mathcal{O}_c \otimes \mathbb{Z}_p, \]
with the equality on the right due to the fact that \( \vartheta_p \) is an optimal embedding of \( \mathcal{O}_c \otimes \mathbb{Z}_p \) into \( R_k^{(m)} \otimes \mathbb{Z}_p \). But \( \vartheta_p^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) \subset \vartheta_p^{-1}(R_k^{(m)} \otimes \mathbb{Z}_p) \) by assumption and \( \mathcal{O}_c \otimes \mathbb{Z}_p \) has index \( p \) in \( \mathcal{O}_c \otimes \mathbb{Z}_p \), so \( \vartheta_p^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) = \mathcal{O}_c \otimes \mathbb{Z}_p \) and the lemma is proved. \( \square \)

Proposition 3.2 reduces the construction of global optimal embeddings to that of local ones. In the following the local component at \( p \) is studied. Let \( p^h \) be the power of \( p \) dividing \( c \) exactly (i.e., \( p^h | c \) but \( p^{h+1} \not| c \)). Write \( K = \mathbb{Q}(\sqrt{-D}) \) with \( D > 0 \) and recall the chosen isomorphism \( \phi_p : B_p \cong M_2(\mathbb{Q}_p) \).

We distinguish when \( p \) is inert and when \( p \) splits in \( K \):
\[ K_p = \begin{cases} \mathbb{Q}_p \oplus \sqrt{-D} \mathbb{Q}_p & \text{if } p \text{ is inert in } K, \\ \mathbb{Q}_p \oplus \mathbb{Q}_p & \text{if } p \text{ splits in } K. \end{cases} \]

Then we consider the following embeddings \( \psi_p : K_p \hookrightarrow B_p \):

1) \( p \) inert
\[ K_p \rightarrow B_p \]
\[ (\alpha, \sqrt{-D}\beta) \rightarrow \phi_p^{-1}\left( \left( \begin{smallmatrix} \alpha & -D\beta^h \\ \beta^h & \alpha \end{smallmatrix} \right) \right); \]

2) \( p \) split
\[ K_p \rightarrow B_p \]
\[ (\alpha, \beta) \rightarrow \phi_p^{-1}\left( \left( \begin{smallmatrix} \alpha & 0 \\ \beta & \alpha \end{smallmatrix} \right) \right). \]
In both cases, since \( \phi_p(R_0 \otimes \mathbb{Z}_p) = M_2(\mathbb{Z}_p) \), we have

\[
\psi_p^{-1}(R_0 \otimes \mathbb{Z}_p) = \mathcal{O}_c \otimes \mathbb{Z}_p.
\]

Define

\[
\varphi_p^{(m)} := \phi_p^{-1}((0 \begin{smallmatrix} 0 & m \\ -m & 0 \end{smallmatrix})) \psi_p \phi_p^{-1}((0 \begin{smallmatrix} 0 & -m \\ 1 & 0 \end{smallmatrix})).
\]

For all \( m \geq 0 \), it follows from equations (15) and (19) that

\[
(\varphi_p^{(m)})^{-1}(R_0^{(m)} \otimes \mathbb{Z}_p) = \mathcal{O}_c \otimes \mathbb{Z}_p,
\]

that is, \( \varphi_p^{(m)} \) is an optimal embedding of \( \mathcal{O}_c \otimes \mathbb{Z}_p \) into \( R_0^{(m)} \otimes \mathbb{Z}_p \). Moreover, a direct computation shows that

\[
(\varphi_p^{(m)})^{-1}(R_{n+1}^{(m)} \otimes \mathbb{Z}_p) \subseteq (\varphi_p^{(m)})^{-1}(R_n^{(m)} \otimes \mathbb{Z}_p).
\]

for all \( n \in \{0, \ldots, m\} \). Now we study \( \varphi_p^{(m)} \) more closely.

**Lemma 3.4.** The map \( \varphi_p^{(m)} \) is an optimal embedding of \( \mathcal{O}_{cp^n} \otimes \mathbb{Z}_p \) into \( R_n^{(m)} \otimes \mathbb{Z}_p \) for all \( n \in \{0, \ldots, m\} \).

**Proof.** Since \( \varphi_p^{(m)} \) is an optimal embedding of \( \mathcal{O}_c \otimes \mathbb{Z}_p \) into \( R_0^{(m)} \otimes \mathbb{Z}_p \) by (20), the result follows from (21) combined with Lemma 3.3. \( \square \)

**Lemma 3.5.** The map \( \varphi_p^{(m)} \) is an optimal embedding of \( \mathcal{O}_{cp^n} \otimes \mathbb{Z}_p \) into \( R_n \otimes \mathbb{Z}_p \) for all \( n \in \{0, \ldots, m\} \).

**Proof.** The case \( n = m \) follows because \( \varphi_p^{(m)} \) is an optimal embedding of \( \mathcal{O}_{cp^n} \otimes \mathbb{Z}_p \) into \( R_0^{(m)} \otimes \mathbb{Z}_p = R_m \otimes \mathbb{Z}_p \) by Lemma 3.3. So suppose that \( \varphi_p^{(m)} \) is not an optimal embedding of \( \mathcal{O}_{cp^n} \otimes \mathbb{Z}_p \) into \( R_n \otimes \mathbb{Z}_p \) for some \( n \in \{0, \ldots, m-1\} \). Hence

\[
\varphi_p^{(m)}(\mathcal{O}_{cp^{n-1}} \otimes \mathbb{Z}_p) \subset R_n \otimes \mathbb{Z}_p.
\]

Since \( R_{m-1}^{(m)} \otimes \mathbb{Z}_p \subset R_n^{(m)} \otimes \mathbb{Z}_p \) and \( \varphi_p^{(m)} \) is an optimal embedding of \( \mathcal{O}_{cp^{m-1}} \otimes \mathbb{Z}_p \) into \( R_{m-1}^{(m)} \otimes \mathbb{Z}_p \), it follows that

\[
\varphi_p^{(m)}(\mathcal{O}_{cp^{m-1}} \otimes \mathbb{Z}_p) \subset R_n^{(m)} \otimes \mathbb{Z}_p.
\]

But we know that

\[
R_n = R_0 \cap R_0^{(n)}, \quad R_n^{(m)} = R_0^{(m)} \cap R_0^{(m-n)},
\]

thus we get

\[
\varphi_p^{(m)}(\mathcal{O}_{cp^{m-1}} \otimes \mathbb{Z}_p) \subset (R_0 \cap R_n^{(m)}) \otimes \mathbb{Z}_p \subset (R_0 \cap R_0^{(m)}) \otimes \mathbb{Z}_p = R_m \otimes \mathbb{Z}_p.
\]

Hence \( \varphi_p^{(m)} \) cannot be an optimal embedding of \( \mathcal{O}_{cp^n} \otimes \mathbb{Z}_p \) into \( R_m \otimes \mathbb{Z}_p \), which is a contradiction. \( \square \)

For all primes \( \ell \neq p \) choose an optimal embedding \( \varphi_\ell : K_\ell \rightarrow B_\ell \) of \( \mathcal{O}_K \otimes \mathbb{Z}_\ell \) into \( R_0 \otimes \mathbb{Z}_\ell \); this can be done by [40, Theorem 2]. For primes \( \ell \nmid Np \) it is possible to choose \( \varphi_\ell \) in such a way that for every integer \( n \geq 0 \) it induces an optimal embedding of \( \mathcal{O}_{\ell^n} \otimes \mathbb{Z}_\ell \) into a maximal order \( R(\ell, n) \) of \( M_2(\mathbb{Q}_\ell) \) with the property that the collection \( \{v_i\}_{i=0, \ldots, n} \), where \( v_i \) is the vertex of the Bruhat–Tits tree \( T_\ell \) of \( GL_2(\mathbb{Q}_\ell) \) representing \( R(\ell, i) \), determines a path of length \( n \) with no backtracking. See [1, §2.4] for details.
Corollary 3.8. There exists an element \( z \) such that the end of the point of conductor \( O' \). Proof. is fixed, in the following we will simply set \( f \) the map \( \gamma \) prime \( p \). Finally, \( \varphi_p \) is an optimal embedding of \( O_{cp^m} \otimes \mathbb{Z}_\ell \) into \( R_{m}(c) \otimes \mathbb{Z}_\ell \). For primes \( \ell | N \) the map \( \varphi_{p} \) is an optimal embedding of \( O_{cp^m} \otimes \mathbb{Z}_\ell \) into \( R_{m}(c) \otimes \mathbb{Z}_\ell \). Hence, since \( R_{m}(c) \) and \( R_{m} \) have the same level, by Proposition 3.2 we can choose an optimal embedding \( f^{(m,c)} \) of \( O_{cp^m} \) into \( R_{m}(c) \otimes \mathbb{Z}_{\ell} = R_{m} \otimes \mathbb{Z}_{\ell} \). For the prime \( p \) observe that \( f^{p}_{\ell} \) is \( (\gamma_{m,k_{m,c}})^{-1}(R_{m} \otimes \mathbb{Z}_{p})(\gamma_{m,k})_{p} \) equivalent to the map \( (\gamma_{m,k})^{-1} \varphi_{p}^{(m)}(\gamma_{m,k})_{p} \). Hence, there exists an element \( u \in (\gamma_{m,k})_{p}^{-1}(R_{n} \otimes \mathbb{Z}_{p})(\gamma_{m,k})_{p} \) such that
\[
\varphi_{p}^{(m)} = u^{-1}(\gamma_{m,k})_{p}^{-1} \varphi_{p}^{(m)}(\gamma_{m,k})_{p} u.
\]
Since \( R_{n} \supset R_{m} \), the map \( f^{p}_{\ell} \) is an optimal embedding of \( O_{cp^m} \otimes \mathbb{Z}_{p} \) into \( (\gamma_{m,k})_{p}^{-1}(R_{n} \otimes \mathbb{Z}_{p})(\gamma_{m,k})_{p} \). We are done.

Corollary 3.8. The image of \( [(\gamma_{m,k_{m}}, f^{(m)})] \in X_{m}^{(K)} \) via the maps in diagram 5 is a Heegner point of conductor \( cp^m \) on \( X_{n} \) for all \( n \in \{0, \ldots, m\} \).

Proof. Immediate from Proposition 3.7.
Corollary 3.9. The image of $[(\gamma_{m,k}, f^{(m)})] \in \widetilde{X}_m^{(K)}$ via the maps in diagram (6) is a Heegner point of conductor $cp^m$ on $\widetilde{X}_m$ for all $n \in \{0, \ldots, m\}$.

Proof. Immediate from Corollary 3.8 by the commutativity of (6). □

3.2. Compatible families of Heegner points. We slightly modify the sequence of points $m \mapsto [(\gamma_{m,k}, f^{(m)})]$ in order to make them compatible with respect to the Hecke action (3.3) and the Galois action (3.4). As in the proof of Proposition 3.2, fix an injection $g : K \hookrightarrow B$ of $\mathbb{Q}$-algebras and for every integer $m \geq 0$ write $f^{(m)} = b_m g b_m^{-1}$ with $g_\ell = a_{m,\ell}^{-1} \varphi_{m,\ell}$ for $\ell \neq p$, $g_p = a_{m,p}^{-1} \varphi_{m,p}$ and $a_m = (a_{m,t})_t = u_m \gamma_{m,k,m} b_m$ (here $u_m \in \widehat{R}_m$ and $b_m \in B^\times$). Recall that $\hat{\pi} = \hat{\pi}_0$ is the idele in $\widehat{B}^\times$ with all components equal to 1 save the $p$-component $\hat{\pi}_p$ which is equal to $\pi_p = (1_p \ 0)$. Let $n \in \{0, \ldots, m\}$ and note that $\varphi^{(m)}_p = \pi_p^{m-n} \varphi^{(n)}_p \pi_p^{-(m-n)}$. A direct calculation shows that the product $\pi_p^{-(m-n)} a_{m,p} a_{n,p}^{-1}$ commutes with every element in $\varphi^{(n)}_p(K_p)$, thus $a_{m,p} = \pi_p^{m-n} \varphi^{(n)}_p(c_p) a_{n,p}$ for some $c_p \in K_p^{\times}$. It follows that there exists $c_{m,n} \in \widehat{K}^\times$ such that its $p$-component $(c_{m,n})_p$ is equal to $c_p$ and

$$u_m \gamma_{m,k,m} = \hat{\pi}^{m-n} u_n \gamma_{n,k,n} f^{(n)}(c_{m,n}) b_n b_n^{-1}.$$  

(22)

On the other hand, since $f^{(m)} = b_m g b_m^{-1}$ and $f^{(n)} = b_n g b_n^{-1}$, it follows that

$$f^{(m)} = b_m b_n^{-1} f^{(n)} b_n b_n^{-1}.$$  

(23)

Lemma 3.10. Fix an integer $m \geq 0$ and let $n \in \{0, \ldots, m\}$. Then

$$[(u_m \gamma_{m,k,m}, f^{(m)})] = [(\hat{\pi}^{m-n} u_n \gamma_{n,k,n} f^{(n)}(c_{m,n}), f^{(n)})]$$

both on $X_m$ and on $\widetilde{X}_m$, and if $x \in \widehat{K}^\times$ then

$$[(u_m \gamma_{m,k,m} f^{(m)}(x), f^{(m)})] = [(\hat{\pi}^{m-n} u_n \gamma_{n,k,n} f^{(n)}(c_{m,n} x), f^{(n)})]$$

both on $X_m$ and on $\widetilde{X}_m$.

Proof. Straightforward from (22) and (23). □

Define

$$P_{0,0} = \tilde{P}_{0,0} := [(u_{0,0} \gamma_{0,0}, f^{(0)})] \in X_0^{(K)} = \widetilde{X}_0^{(K)}.$$  

For all integers $m \geq 1$ set $x_m := \prod_{n=0}^{m-1} c_{n+1,n}^{-1} \in \widehat{K}^\times$. Then define

$$P_{m,0} := [(\gamma_{m,k,m} f^{(m)}(x_m), f^{(m)})] \in X_m^{(K)}$$

and

$$\tilde{P}_{m,0} := [(u_m \gamma_{m,k,m} f^{(m)}(x_m), f^{(m)})] \in \widetilde{X}_m^{(K)}.$$  

Each $P_{c,m}$ (respectively, $\tilde{P}_{c,m}$) is a Heegner point on $X_m$ (respectively, $\widetilde{X}_m$) of conductor $cp^m$. In fact, if we set

$$Q_{c,m} := [(\gamma_{m,k,m}, f^{(m)})] \in X_m^{(K)}$$

and let $\sigma_m$ be the Galois element represented by $x_m$ under the isomorphism

$$\text{Gal}(H_{cp^m}/K) \simeq \widehat{K}^\times/K^\times \hat{O}_{cp^m}^{\times}$$

then Corollary 3.8 ensures that $Q_{c,m}$ is a Heegner point on $X_m$ of conductor $cp^m$, while formula (7) shows that

$$P_{c,m} = (Q_{c,m})^{\sigma_m}. $$
The point $\tilde{P}_{c,m}$ is a suitable lift of $P_{c,m}$ to $\tilde{X}_m^{(K)}$. More generally, we have the following

**Proposition 3.11.** The image of $P_{c,m} \in X_m$ (respectively, $\tilde{P}_{c,m} \in \tilde{X}_m$) via the maps in diagram (5) is a Heegner point of conductor $cp^n$ on $X_n$ (respectively, $\tilde{X}_n$) for all $n \in \{0, \ldots, m\}$.

**Proof.** A direct consequence of Corollary 3.8 (respectively, Corollary 3.9).

For all $m \geq 0$, every point $P = [(g, f)] \in X_m$ and every $x \in \tilde{B}^\times$ define

$$xP := [(xg, f)] \in X_m.$$

Moreover, for $a \in \tilde{K}^\times$ and $P = [(g, f)] \in X_m^{(K)}$ set

$$P^a := [(g\hat{f}(a), f)] \in X_m^{(K)}.$$

This is nothing other than the action of the class of $a$ in $\tilde{K}^\times/K^\times$ as defined in (7). Analogous conventions will be adopted for points on $\tilde{X}_m$.

**Proposition 3.12.** For all $m \geq 1$ and every $a \in \tilde{K}^\times$:

$$\alpha_m(P_{c,m}^a) = (\hat{P}_{c,m-1})^a \quad \text{in } X_{m-1}$$

and

$$\tilde{\alpha}_m(\tilde{P}_{c,m}^a) = (\hat{\tilde{P}}_{c,m-1})^a \quad \text{in } \tilde{X}_{m-1}.$$

**Proof.** Note that $c_{m,m-1}x_m = x_{m-1}$ for all $m \geq 1$. It follows from Lemma 3.10 that

$$[(um\gamma_{m,m}^{-1}f(m)(x_m, f(m))] = [(\tilde{u}m\gamma_{m-1,m}^{-1}f(m-1)(x_{m-1}, f(m-1))],$$

and we are done. □

The following immediate consequence will be crucially used when studying Hecke relations.

**Corollary 3.13.** For all $m \geq 1$, the point $\hat{\pi}P_{c,m-1}$ (respectively, $\hat{\pi}\tilde{P}_{c,m-1}$) is a Heegner point of conductor $cp^n$ on $X_m$ (respectively, $\tilde{X}_m$).

**Proof.** Take $a = 1$ in Proposition 3.12 and then apply Proposition 3.11 with $n = m - 1$. □

### 3.3. Hecke relations in compatible families.

The results we prove in this § justify our choice of the points $P_{c,m}$ and $\tilde{P}_{c,m}$. Write

$$\alpha_{m,*} : \text{Div}(X_m) \rightarrow \text{Div}(X_{m-1}), \quad \tilde{\alpha}_{m,*} : \text{Div}(\tilde{X}_m) \rightarrow \text{Div}(\tilde{X}_{m-1})$$

for the maps between divisor groups induced by $\alpha_m$ and $\tilde{\alpha}_m$ by covariant functoriality. In other words, $\alpha_{m,*}(P_1 + \cdots + P_s) = \alpha_m(P_1) + \cdots + \alpha_m(P_s)$ for all points $P_1, \ldots, P_s$ on $X_m$, and similarly for $\tilde{\alpha}_{m,*}$.

**Proposition 3.14.** Let $m \geq 2$. Then

$$U_P(P_{c,m-1}) = \alpha_{m,*}(\text{Tr}_{H_{cp^n}/H_{cp^{m-1}}}(P_{c,m}))$$

in $\text{Div}(X_{m-1})$.

**Proof.** The point $P_{c,m-1}$ is (by construction) a Heegner point of conductor $cp^{m-1}$ on $X_{m-1}$, and $\hat{\pi}P_{c,m-1}$ is a Heegner point of conductor $cp^n$ on $X_{m-1}$ by Corollary 3.13. Hence the assumptions of Proposition 2.1 are satisfied, and we get the equality

$$U_P(P_{c,m-1}) = \text{Tr}_{H_{cp^n}/H_{cp^{m-1}}}(\hat{\pi}P_{c,m-1}).$$

Now the claim follows from Proposition 3.12.
The next result is the counterpart for the curves $\tilde{X}_m$ of Proposition 3.14. Before stating it, we remark that a straightforward (but somewhat tedious) computation, starting from the explicit expression of the embeddings $\psi_p$ given in (3.1) shows that the Heegner points $\tilde{P}_{c,m}$ satisfy Assumption 2.6.

**Proposition 3.15.** Let $m \geq 2$. Then
\[ U_p(\tilde{P}_{c,m-1}) = \tilde{\alpha}_{m,*}(\text{Tr}_{H_{p^m}}(\mu_{p^m})/H_{p^m-1}(\mu_{p^m})(\tilde{P}_{c,m})) \]
in $\text{Div}(\tilde{X}_{m-1})$.

**Proof.** Proceed exactly as in the proof of Proposition 3.14, this time using Proposition 2.7 in place of Proposition 2.4. \hfill \Box

The two propositions below study Hecke relations between Heegner points of the form $P_{c,m}$ and $\tilde{P}_{c,m}$ when we multiply the integer $c$ by powers of $p$.

**Proposition 3.16.** Let $m \geq 1$ and $r \geq 1$ fixed integers. Then
\[ U_p(P_{c p^r-1,m}) = \text{Tr}_{H_{p^m+r}}(\mu_{p^m+r})/H_{p^m+r-1}(\mu_{p^m+r})(P_{c p^r,m}) \]
in $\text{Div}(X_m)$.

**Proof.** Combine Propositions 2.7 and 3.7. \hfill \Box

**Proposition 3.17.** Let $m \geq 1$ and $r \geq 1$ fixed integers. Then
\[ U_p(\tilde{P}_{c p^r-1,m}) = \text{Tr}_{H_{p^m+r}}(\mu_{p^m+r})/H_{p^m+r-1}(\mu_{p^m+r})(\tilde{P}_{c p^r,m}) \]
in $\text{Div}(\tilde{X}_m)$.

**Proof.** Combine Propositions 2.7 and 3.7. \hfill \Box

3.4. *Galois relations in compatible families.* Set $G_\mathbb{Q} := \text{Gal}(\mathbb{Q}/\mathbb{Q})$ and let
\[ \epsilon_{\text{cyc}} : G_\mathbb{Q} \longrightarrow \mathbb{Z}_p^\times \]
be the cyclotomic character describing the action of the absolute Galois group of $\mathbb{Q}$ on the group $\mu_{p^\infty}$ of roots of unity of $p$-power order. Since the restriction of $\epsilon_{\text{cyc}}$ to $\text{Gal}(\mathbb{Q}/\mathbb{Q}(\sqrt{p^\ell}))$ takes values in $(\mathbb{Z}_p^\times)^2$, there is a unique continuous homomorphism
\[ \vartheta : \text{Gal}(\mathbb{Q}/\mathbb{Q}(\sqrt{p^\ell})) \longrightarrow \mathbb{Z}_p^\times /\{\pm 1\} \]
such that $\vartheta^2 = \epsilon_{\text{cyc}}(\bar{\sigma})$. Fix $\sigma \in \text{Gal}(H_{p^m}(\mu_{p^m})/H_{p^m})$. Since $\text{Gal}(H_{p^m}(\mu_{p^m})/H_{p^m})$ is isomorphic to $\mathbb{O}_{p^m}/\mathbb{O}_{p^m}^\times \mathbb{Z}_m$, it follows that $\sigma$ can be represented by an element $x \in \mathbb{O}_{p^m}^\times$ such that $x_{\ell} = 1$ for $\ell \neq p$. Write $x_p = \alpha + p^m \beta$ with $\alpha \in \mathbb{Z}_p^\times$ and $\beta \in \mathbb{O}_K \otimes \mathbb{Z}_p$. The image $\bar{\sigma}$ of $\sigma$ via the map
\[ \text{Gal}(H_{p^m}(\mu_{p^m})/H_{p^m}) \subset \text{Gal}(H_{p^m}(\mu_{p^m})/\mathbb{Q}(\sqrt{p^\ell})) \longrightarrow \text{Gal}(\mathbb{Q}(\mu_{p^m})/\mathbb{Q}(\sqrt{p^\ell})), \]
where the arrow on the right is the canonical projection, is represented by $\text{Norm}_{K_p/\mathbb{Q}}(x_p)$ and, by class field theory, $\text{Norm}^{-1}_{K_p/\mathbb{Q}}(x_{p}) = \epsilon_{\text{cyc}}(\bar{\sigma})$. Hence $\epsilon_{\text{cyc}}(\bar{\sigma}) \equiv \alpha^2 \pmod{p^m}$, so the map $\sigma \mapsto x$ is equal up to $\pm 1$ to the restriction of $\vartheta$ to $\text{Gal}(H_{p^m}(\mu_{p^m})/H_{p^m})$. Thus
\[ \tilde{P}_{c,m}^{\sigma} = [(u_m \gamma_{m,k,m}, f(\alpha))] = [(u_m \gamma_{m,k,m} \vartheta(\sigma), f(\alpha))] \pmod{\pm 1}. \]
For any $\ell \in \mathbb{Z}/p^m\mathbb{Z}$, let $\langle \ell \rangle$ denote the diamond operator acting on $\tilde{X}_m$, whose action may be defined by the map $[(g, z)] \mapsto [(g\ell, z)]$. Since the action of $(-1)$ on $\text{Div}(\tilde{X}_m)$ is trivial, it follows that for all $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{H}_{p^m})$ there is an equality
\begin{equation}
\tilde{P}^\sigma_{c,m} = \langle \vartheta(\sigma) \rangle \tilde{P}_{c,m}
\end{equation}
in $\text{Div}(\tilde{X}_m)$.

3.5. Summary of the properties of the Heegner points system. Before turning to the construction of big Heegner points, for the convenience of the reader we isolate in a single statement the main features of our system of Heegner points which were discussed in the previous sections. Note that some of these properties will be proved in \textbf{7.2} where we describe the action of the Hecke operator $T_\ell$ for primes $\ell \nmid Np^m$.

\textbf{Theorem 3.18.} For every integer $m \geq 0$ and every integer $c \geq 1$ prime to $N$ and the discriminant of $K$ there is a Heegner point $\tilde{P}_{c,m} \in \tilde{X}^{(K)}_m$ of conductor $cp^m$, rational over $H_{cp^m}(\mu_{p^m})$ in the indefinite case, such that the following conditions are satisfied.

1. Vertical compatibility. The image of $\tilde{P}_{c,m}$ under the covering $\tilde{X}_m \rightarrow \tilde{X}_n$ is a Heegner point of conductor $cp^m$ on $\tilde{X}_n$ for all $n \in \{0, \ldots, m\}$. Furthermore, if $m \geq 2$ then the equality

\[ U_{\varpi}(\tilde{P}_{c,m-1}) = \tilde{\alpha}_{m,\varpi}(\text{Tr}_{H_{p^m}(\mu_{p^m})/H_{cp^m-1}(\mu_{p^m})}(\tilde{P}_{c,m})) \]

holds in $\text{Div}(\tilde{X}_{m-1})$.

2. Horizontal compatibility. Let $m \geq 1$ and $n \geq 1$ be integers. Then the equality

\[ U_{\varpi}(\tilde{P}_{cp^{n-1},m}) = \text{Tr}_{H_{cp^m+n}(\mu_{p^{m+n}})/H_{cp^m+n-1}(\mu_{p^m})}(\tilde{P}_{cp^m,m}) \]

holds in $\text{Div}(\tilde{X}_m)$. Furthermore, for primes $\ell \nmid cp^m$ one has

\[ \text{Tr}_{H_{c\ell^{n+1}p^m}(\mu_{p^m})/H_{c\ell^{n}p^m}(\mu_{p^m})}(\tilde{P}_{c\ell^{n+1},m}) = T_\ell(\tilde{P}_{c\ell^n,m}) - \tilde{P}_{c\ell^n-1,m} \]

and

\[ u\text{Tr}_{H_{c\ell p^m}/H_{c\ell m}}(\tilde{P}_{c\ell,m}) = \begin{cases} T_\ell(\tilde{P}_{c,m}) & \text{if } \ell \text{ is inert in } K \\ (T_\ell - \sigma_{\varpi} - \sigma_2)(\tilde{P}_{c,m}) & \text{if } \ell \text{ is split in } K \end{cases} \]

in $\text{Div}(\tilde{X}_m)$, where $u$ is the cardinality of $\mathcal{O}_{cp^m}^*/\{\pm 1\}$ and, in the split case, $(\ell) = l_1 l_2$ in $\mathcal{O}_K$ and $\sigma_{\varpi}$ is a Frobenius element at $\varpi$. (3) Galois compatibility. Set $p^* := (-1)^{(p-1)/2}p$, let $\epsilon_{\text{cyc}} : G_\mathbb{Q} \rightarrow \mathbb{Z}_p^\times$ be the $p$-adic cyclotomic character and let $\vartheta : \text{Gal}(\mathbb{Q}/\sqrt{p^*}) \rightarrow \mathbb{Z}_p^\times/\{\pm 1\}$ be the unique continuous homomorphism such that $\vartheta^2$ coincides with the restriction of $\epsilon_{\text{cyc}}$. Then for all $\sigma \in \text{Gal}(\mathbb{Q}/H_{cp^m})$ the equality

\[ \tilde{P}^\sigma_{c,m} = \langle \vartheta(\sigma) \rangle \tilde{P}_{c,m} \]

holds in $\text{Div}(\tilde{X}_m)$.

\textit{Proof.} Part (1) is a combination of Proposition 3.11 and Proposition 3.15 while part (2) is Proposition 3.17 plus (62) and (63). Finally, part (3) is equality (24) above. \qed
Part 2. Construction of big Heegner points

In the following sections we use the compatible family of Heegner points built in Part 1 to define big Heegner points and classes associated with towers of Shimura curves over $\mathbb{Q}$ both in the definite case and in the indefinite case. The system of cohomology classes so obtained satisfies formal properties which are analogous to those enjoyed by the big Heegner points introduced by Howard in [24] in the case of classical modular curves (see also the work [11] and [12] by Fouquet for similar constructions for Shimura curves attached to indefinite quaternion algebras over totally real fields).

4. Hida theory on $GL_2$

Throughout this paper we choose an (algebraic) isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$, and view any subring of $\mathbb{C}_p$ as a subring of $\mathbb{C}$ via this fixed isomorphism.

In the next few lines we use Shimura’s notations $T(n)$ and $T(n,n)$ (with $n$ an integer) for the Hecke operators defined as in [46, §3.1-§3.3] by double cosets. For every integer $m \geq 0$ denote $\mathcal{H}_m$ the abstract Hecke algebra generated over $\mathbb{Z}$ by $T_\ell := T(\ell)$ for primes $\ell \nmid Np^m$, $U_\ell := T(\ell)$ for primes $\ell \mid Np^m$.

Fix an isomorphism $\mathbb{Z} \times \mathbb{Q}_p^\times \cong \Gamma \times \Delta$ with $\Delta \cong \mu_{p-1}$ the torsion subgroup of $\mathbb{Z} \times \mathbb{Q}_p^\times$ and $\Gamma := 1 + p\mathbb{Z}_p$.

Define the two Iwasawa algebras
\[ \Lambda := \mathcal{O}_F[\Gamma], \quad \tilde{\Lambda} := \mathcal{O}_F[\mathbb{Z}_p^\times] \]
where $F$ is a finite extension of $\mathbb{Q}_p$ (which will eventually contain the Fourier coefficients of our modular form $f$) and $\mathcal{O}_F$ is its ring of integers, so that we have a natural inclusion $\Lambda \subset \tilde{\Lambda}$. Finally, denote by $z \mapsto [z]$ the inclusions of group-like elements $\Gamma \hookrightarrow \Lambda$ and $\mathbb{Z}_p^\times \hookrightarrow \tilde{\Lambda}$.

4.1. Ordinary Hecke algebras. For any ring $A$, any subgroup $\Delta \subset SL_2(\mathbb{Z})$ and any character $\psi : \Delta \to \mathbb{Q}_p^\times$, let $S_k(\Delta, \psi, A)$ denote the $A$-module of weight $k$ modular forms of level $\Delta$ with coefficients in $A$.

We follow [24] for the presentation of Hida’s Hecke algebras. Define
\[ \Gamma_{0,1}(N, p^m) := \Gamma_0(N) \cap \Gamma_1(p^m) \]
and write $\mathfrak{h}_{k,m}$ for the image in $\text{End}(S_k(\Gamma_{0,1}(N, p^m), \mathbb{C}))$ of the Hecke algebra $\mathcal{H}_m \otimes \mathbb{Z} \mathcal{O}_F$.

**Proposition 4.1.** For all $m \geq 0$ the Hecke algebra $\mathfrak{h}_{k,m}$ is a finite product of complete local rings.

**Proof.** The algebra $\mathfrak{h}_{k,m}$ is a finite $\mathcal{O}_F$-module, hence by [9] Corollary 7.6] it is the direct product of its localizations at its finitely many maximal ideals, which are complete local rings finite over $\mathcal{O}_F$. \qed

Keeping Proposition 4.1 in mind, let $\mathfrak{h}_{k,m}^{\text{ord}}$ be the ordinary part of $\mathfrak{h}_{k,m}$, i.e. the product of those local factors on which the image of $U_p$ is a unit. Alternatively, if
\[ e_m^{\text{ord}} := \lim_{n \to \infty} U_p^n! \]
is Hida’s idempotent in $\mathfrak{h}_{k,m}$ (see [46] (4.3)) for details) then $\mathfrak{h}_{k,m}^{\text{ord}} := e_m^{\text{ord}} \cdot \mathfrak{h}_{k,m}$.

For all integers $m \geq n \geq 1$ there are canonical injections
\[ i_{n,m} : S_k(\Gamma_{0,1}(N, p^n), \mathbb{C}) \hookrightarrow S_k(\Gamma_{0,1}(N, p^m), \mathbb{C}) \]
which are compatible with the actions of $\mathcal{H}_n \otimes \mathbb{Z} \mathcal{O}_F$ and $\mathcal{H}_m \otimes \mathbb{Z} \mathcal{O}_F$ on the source and the target of $i_{n,m}$, respectively; so if $m \geq n \geq 1$ there are canonical projections

$$ h_{k,m} \rightarrow h_{k,n}, \quad h_{k,m}^{\text{ord}} \rightarrow h_{k,n}^{\text{ord}} $$

which allow us to define the Hecke algebras of weight $k$ as

$$ h_{k,\infty} := \lim_{m} h_{k,m}, \quad h_{k,\infty}^{\text{ord}} := \lim_{m} h_{k,m}^{\text{ord}}, $$

the projective limits being taken with respect to these maps. Alternatively, if

$$ e_{\text{ord}} := \lim_{m} e_{m} $$

is Hida’s idempotent in $h_{k,\infty}$ (see loc. cit.) then $h_{k,\infty}^{\text{ord}} := e_{\text{ord}} \cdot h_{k,\infty}$.

The $\mathcal{O}_F$-algebras $h_{k,\infty}$ and $h_{k,\infty}^{\text{ord}}$ can be endowed with structures of $\hat{\Lambda}$ and $\Lambda$-algebras in such a way that if $a$ is an integer prime to $Np$ and $T(a,a)_k$ denotes the image of $T(a,a)$ in $\text{End}(S_k(\Gamma_0(1), \mathbb{Z}^m, \mathbb{C}))$ then the image of $a$ in $h_{k,m}$ is the diamond operator $(a)$ defined by the formula $T(a,a)_k = a^{k-2}(a)_k$ (here we adopt the conventions of [24] rather than those of [17]). Since this $\Lambda$-algebra structure commutes with the $U_p$-action, the algebra $h_{k,\infty}^{\text{ord}}$ inherits a structure of $\Lambda$-algebra. Finally, the inclusion $\Lambda \subset \hat{\Lambda}$ induces a $\Lambda$-structure on $h_{k,\infty}^{\text{ord}}$.

**Theorem 4.2** (Hida). The $\Lambda$-algebra $h_{k,\infty}^{\text{ord}}$ is finite and flat over $\Lambda$. For every pair of weights $(k,k')$ there is a unique isomorphism of algebras

$$ \rho_{k,k'} : h_{k,\infty}^{\text{ord}} \overset{\cong}{\rightarrow} h_{k',\infty}^{\text{ord}} $$

taking the images of $T(\ell)$ and $T(\ell,\ell)$ in $h_{k,\infty}^{\text{ord}}$ to the images of the same operators in $h_{k',\infty}^{\text{ord}}$.

**Proof.** The basic result in this direction can be found in [18, Theorem 1.1]. For the particular arithmetic groups we are considering here, see also the more general results in [20, Theorems 2.3 and 2.4].

**Remark 4.3.** In the notation of Theorem 4.2 one has $\rho_{k',k} = \rho_{k,k'}^{-1}$ for all weights $k,k'$.

**Convention.** It will usually be convenient to identify the Hecke algebras $h_{k,\infty}^{\text{ord}}$ for all weights $k$ by means of the isomorphisms of Theorem 4.2, so we simply set

$$ h_{k,\infty}^{\text{ord}} := h_{2,\infty}^{\text{ord}}. $$

This notation will be in force throughout the rest of the paper.

### 4.2. New quotients.

We are especially interested in the $\mathbb{C}$-vector space $S_{k}^\text{new}(\Gamma_0(1), \mathbb{Z}^m, \mathbb{C})$ consisting of those forms which are new at all the primes dividing $N$. Write $T_{k,m}$ for the image of $h_{k,m}$ in the endomorphisms ring $\text{End}(S_{k}^\text{new}(\Gamma_0(1), \mathbb{Z}^m, \mathbb{C}))$ and set

$$ T_{k,\infty} := \lim_{m} T_{k,m}, \quad T_{k,m}^{\text{ord}} := e_{m} \cdot T_{k,m}, \quad T_{k,\infty}^{\text{ord}} := e_{\infty} \cdot T_{k,\infty} = \lim_{m} T_{k,m}^{\text{ord}}. $$

The isomorphisms of Theorem 4.2 yield isomorphisms of $\Lambda$-modules $T_{k,\infty}^{\text{ord}} \simeq T_{k',\infty}^{\text{ord}}$ for all pairs of weights $(k,k')$, so in analogy to what done before we use these isomorphisms to justify the following

**Convention.** We usually identify the algebras $T_{k,\infty}^{\text{ord}}$ for all weights $k$, and set $T_{\infty}^{\text{ord}} := T_{2,\infty}^{\text{ord}}$. 

4.3. Maximal ideals of Hecke algebras. Following [35 §1.4.4], we briefly describe the decompositions of our Hecke algebras into products of local components. Since $h_\infty^{\text{ord}}$ and $T_\infty^{\text{ord}}$ are finite $\Lambda$-modules (Theorem 4.2), they split by [9 Corollary 7.6] as finite products
\[(25) \quad h_\infty^{\text{ord}} = \prod_{\tilde{m}} h_{\infty,\tilde{m}}^{\text{ord}}, \quad T_\infty^{\text{ord}} = \prod_{m} T_{\infty,m}^{\text{ord}}\]
of their localizations at their (finitely many) maximal ideals $\tilde{m}$ and $m$. Every summand appearing in these decompositions is a complete local ring, finite over $\Lambda$. If $L$ is the fraction field of $\Lambda$ then $h_{\infty,m}^{\text{ord}} \otimes_\Lambda L$ and $T_{\infty,m}^{\text{ord}} \otimes_\Lambda L$ are finite dimensional artinian algebras over $L$, so they are sums of local artinian algebras. We want to explicitly point out an immediate consequence of the decompositions (25):
- if $\tilde{m}$ (respectively, $m$) is a maximal ideal of $h_{\infty}^{\text{ord}}$ (respectively, of $T_\infty^{\text{ord}}$) then $h_{\infty,\tilde{m}}^{\text{ord}} \otimes_\Lambda L$ (respectively, $T_{\infty,m}^{\text{ord}} \otimes_\Lambda L$) is a direct factor of $h_{\infty}^{\text{ord}} \otimes_\Lambda L$ (respectively, of $T_{\infty}^{\text{ord}} \otimes_\Lambda L$).

There are splittings of $L$-algebras
\[(26) \quad h_{\infty}^{\text{ord}} \otimes_\Lambda L = \left( \prod_{i \in I} F_i \right) \bigoplus M, \quad T_{\infty}^{\text{ord}} \otimes_\Lambda L = \left( \prod_{j \in J} K_j \right) \bigoplus N\]
where $F_i$ and $K_j$ are finite field extensions of $L$ while $M$ and $N$ are nilpotent. According to Hida’s terminology, $F_i$ and $K_j$ are called the \textit{primitive components} of $h_{\infty}^{\text{ord}} \otimes_\Lambda L$ and $T_{\infty}^{\text{ord}} \otimes_\Lambda L$, respectively. As explained in [17 §3], one has $I = J$ and there are canonical isomorphisms
\[(27) \quad F_i \xrightarrow{\sim} K_i\]
for all $i \in I$. We say that $F_i$ (respectively, $K_i$) \textit{belongs to} $\tilde{m}$ (respectively, $m$) if it is a direct summand of $h_{\infty,\tilde{m}}^{\text{ord}} \otimes_\Lambda L$ (respectively, of $T_{\infty,m}^{\text{ord}} \otimes_\Lambda L$).

Now fix a modular form
\[(28) \quad f = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma_0(Np), \omega^j, \mathcal{O}_F)\]
with $j \equiv k \mod 2$, where $\mu_{p-1}$ is the group of $(p-1)$-st roots of unity and $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mu_{p-1}$ is the Teichmüller character. Assume that $f$ is a normalized eigenform for the Hecke operators $T_\ell$ (with $\ell \nmid Np$) and $U_\ell$ (with $\ell | Np$). Here, as before, $F$ is a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_F$ is its ring of integers. Let $\rho_f : G_\mathbb{Q} \to \text{GL}_2(F)$ be the $p$-adic Galois representation attached to $f$ by Deligne. We make the following

\textbf{Assumption 4.4.} Throughout this article we assume that
i) $f$ is an \textit{ordinary} $p$-stabilized newform in the sense that $a_p \in \mathcal{O}_F^\times$ and the conductor of $f$ is divisible by $N$ (cf. [13 Definition 2.5]), i.e., $f$ arises from a newform of level $N$ or $Np$;
ii) the residual representation $\bar{\rho}_f$ is absolutely irreducible.

\textbf{Remark 4.5.} Condition i) in Assumption 4.4 ensures that $\bar{\rho}_f$ is ramified at all prime numbers dividing $N$. Duality between modular forms and Hecke algebras yields morphisms
\[(29) \quad \theta_f : T_\infty^{\text{ord}} \longrightarrow \mathcal{O}_F, \quad \bar{\theta}_f : h_\infty^{\text{ord}} \longrightarrow \mathcal{O}_F\]
such that $\theta_f$ factors through $T_{k_\ell}^{\text{ord}}$ and is characterized by $\theta_f(T(\ell)) = a_\ell$ for all primes $\ell$, $\theta_f([\delta]) = \omega^{k+j-2}(\delta)$ for $\delta \in \Delta$, $\theta_f([\gamma]) = \gamma^{k-2}$ for $\gamma \in \Gamma$, while $\bar{\theta}_f$ is the composition of the canonical projection $h_\infty^{\text{ord}} \to T_\infty^{\text{ord}}$ with $\theta_f$. Let $\tilde{m}_f$ and $m_f$ be the maximal ideals corresponding to the
unique local factors of $\mathfrak{h}_\infty^{\text{ord}}$ and $T_\infty^{\text{ord}}$ through which $\tilde{\theta}_f$ and $\theta_f$ factor. Since $f$ satisfies i) in Assumption 4.4, we can consider the unique primitive component $F$ of $\mathfrak{h}_\infty^{\text{ord},\tilde{m}_f} \otimes_\Lambda \mathcal{L}$ appearing in (26) to which $f$ belongs in the sense of [17] Corollary 3.7 or [19] pp. 316–317 (see also [24] p. 95 for the more general type of arithmetic groups we are working with here). Thanks to the isomorphisms (27), there is a unique primitive component $\mathcal{K}$ of $T_\infty^{\text{ord},m_f} \otimes_\Lambda \mathcal{L}$ (appearing in (26)) which is isomorphic to $F$. In the above terminology, $F$ belongs to $\mathfrak{m}_f$ and $\mathcal{K}$ belongs to $m_f$. In the sequel we let $A \subset F$ and $B \subset \mathcal{K}$ denote the integral closures of $\Lambda$ in $F$ and in $\mathcal{K}$, respectively; notice that $A$ is nothing other than the ring $\mathcal{I}(\mathcal{K})$ introduced by Hida in [18] p. 554. The isomorphism (27) between $F$ and $\mathcal{K}$ takes $A$ isomorphically onto $B$.

**Proposition 4.6.** The ring $A$ is a complete noetherian local domain which is finitely generated as a $\Lambda$-module.

**Proof.** See, e.g., [48, Theorem 4.3.4].

**Remark 4.7.** The reader should always keep in mind that the fields $F$ and $\mathcal{K}$ (and so the rings $A$ and $B$) depend on the form $f$; however, since in our arguments we shall view $f$ as fixed once and for all, such a dependence will not explicitly appear in the notation.

Observe that $A$ (respectively, $B$) is an $\mathfrak{h}_\infty^{\text{ord},\tilde{m}_f}$-algebra (respectively, a $T_\infty^{\text{ord},m_f}$-algebra). Indeed, the field $F$ is an $\mathfrak{h}_\infty^{\text{ord},\tilde{m}_f}$-algebra; moreover, $\mathfrak{h}_\infty^{\text{ord},\tilde{m}_f}$ identifies by (25) with a $\Lambda$-subalgebra of $\mathfrak{h}_\infty^{\text{ord}}$, hence it is integral over $\Lambda$ by Theorem 4.2, and this implies that $\mathfrak{h}_\infty^{\text{ord},\tilde{m}_f}$ preserves the subring $A$ of $\mathcal{K}$. Analogous arguments work for $B$ and $T_\infty^{\text{ord},m_f}$.

Now consider the composition

$$f_\infty : \mathfrak{h}_\infty^{\text{ord}} \longrightarrow \mathfrak{h}_\infty^{\text{ord},\tilde{m}_f} \longrightarrow A$$

in which the first arrow is the natural projection and the second arrow is the structure map of $A$ as an $\mathfrak{h}_\infty^{\text{ord},\tilde{m}_f}$-algebra. The map $f_\infty$ is univocally determined by the primitive component $F$ to which $f$ belongs.

**Definition 4.8.** The local $\Lambda$-algebra $\mathfrak{h}_\infty^{\text{ord},\tilde{m}_f}$ is the *Hida family* of $f$ and $A$ is the *branch* of the Hida family on which $f$ lives. Finally, we call the map $f_\infty$ the *primitive morphism* associated with $f$.

A slightly different definition of “branch” is given in [24] p. 95. Clearly, the morphism $f_\infty$ determines the Hida family of $f$.

4.4. **Critical characters.** Factor $e_{\text{cyc}} : G_\mathbb{Q} \rightarrow \mathbb{Z}_p^\times$ as a product $e_{\text{cyc}} = e_{\text{tame}} e_{\text{wild}}$ with

$$e_{\text{tame}} : G_\mathbb{Q} \longrightarrow \mu_{p-1}, \quad e_{\text{wild}} : G_\mathbb{Q} \longrightarrow \Gamma$$

and define the *critical character* $\Theta : G_\mathbb{Q} \rightarrow \Lambda^\times$ by

$$\Theta := e_{\text{tame}}^{(k+j-2)/2} [e_{\text{wild}}]^{1/2},$$

where $e_{\text{wild}}$ is the unique square root of $e_{\text{wild}}$ taking values in $\Gamma$. Using the Teichmüller character to identify $\Delta$ and $\mu_{p-1}$, the idempotent

$$e_i := \frac{1}{p-1} \sum_{\delta \in \Delta} \omega^{-1}(\delta)[\delta] \in \mathcal{O}_F[\mathbb{Z}_p^\times],$$
where $i \in \mathbb{Z}/(p-1)\mathbb{Z}$, satisfies the relation
\begin{equation}
(30) \quad e_i[\zeta] = \zeta^i e_i
\end{equation}
for all $\zeta \in \mu_{p-1}$.

Since $f(e_i) = 0$ if $i \neq k + j - 2$, we have $e_{k+j-2}(h_{k,\infty}^{\text{ord}}) = (h_{k,\infty}^{\text{ord}})$. So we also have
\begin{equation}
(31) \quad e_{k+j-2}(h_{k,\infty}^{\text{ord}}) = (h_{k,\infty}^{\text{ord}})
\end{equation}
and it follows that in $(h_{k,\infty}^{\text{ord}})$ we have
\[ [\epsilon_{\text{tame}}(\sigma)] = [\epsilon_{\text{tame}}^{-1}(\sigma)] \]
for all $\sigma \in \mathbb{Q}$. Furthermore, by definition of $\Theta$, in $(h_{k,\infty}^{\text{ord}})$ there are also equalities
\[ \Theta^2(\sigma) = [\epsilon_{\text{tame}}^{-2}(\sigma)][\epsilon_{\text{wild}}(\sigma)] = [\epsilon_{\text{cyc}}(\sigma)] \]
for all $\sigma \in \mathbb{Q}$.

4.5. Arithmetic primes and Galois representations. For every integer $m \geq 1$ denote by $X_{0,1}(N, p^m)$ the compact modular curve of level structure $\Gamma_{0,1}(N, p^m)$, by $\text{Jac}(X_{0,1}(N, p^m))$ its Jacobian variety and by $\text{Tate}(\text{Jac}(X_{0,1}(N, p^m)))$ the $p$-adic Tate module of the Jacobian. As in [24, §2.1], for every integer $m \geq 1$ we define the $h_{\infty}^{\text{ord}}$-modules
\[ \text{T}_m := e_m(\text{Tate}(\text{Jac}(X_{0,1}(N, p^m)))) \otimes \mathbb{Z}_p \mathcal{O}_F, \quad \text{T}^\text{ord} := \varprojlim \text{T}_m, \]
\[ \text{T}_m := \text{T}^\text{ord} \otimes h_{\infty}^{\text{ord}} h_{\infty, \overline{m}_f} \mathcal{O}_F, \quad \text{T} := \text{T}_m \otimes h_{\infty, \overline{m}_f} \mathcal{O}_F. \]

All these modules are endowed with $h_{\infty}^{\text{ord}}$-linear actions of the Galois group $G_{\mathbb{Q}}$.

Let $A^\dagger$ denote $A$ viewed as a module on itself with $G_{\mathbb{Q}}$ acting through $\Theta^{-1}$ and define the critical twist of $T$ to be the $G_{\mathbb{Q}}$-module
\[ T^\dagger := T \otimes_A A^\dagger = T_m \otimes h_{\infty, \overline{m}_f} \mathcal{O}_F. \]

It turns out that $T$, $A$ and $T^\dagger$ enjoy the following properties:

1. The $A$-module $T$ is free of rank two;
2. The $G_{\mathbb{Q}}$-module $T$ is unramified outside $Np$ and the arithmetic Frobenius at a prime $\ell \nmid Np$ acts with characteristic polynomial $X^2 - T_{\ell} X + [\ell] \ell$;
3. There is a perfect $A$-bilinear pairing
\[ T^\dagger \times T^\dagger \rightarrow A(1), \]
where $A(1)$ is the Tate twist of the trivial $G_{\mathbb{Q}}$-module $A$.

For a proof of these facts see, e.g., [24, Proposition 2.1.2] and [32, Théorème 7].

Write $m_A$ for the maximal ideal of the local ring $A$ and set
\[ \mathbb{F}_A := A/m_A, \quad \mathbb{F}_{\text{ord}} := h_{\infty, \overline{m}_f}/\overline{m}_f h_{\infty, \overline{m}_f} \]
for the residue fields of $A$ and $h_{\infty, \overline{m}_f}$. Since $A$ is finite over $\Lambda$ by Proposition 4.6, the map $h_{\infty, \overline{m}_f} \rightarrow A$ is also finite, hence integral. Thus $\mathbb{F}_A$ is naturally a finite extension of $\mathbb{F}_{\text{ord}}$ and hence of $\mathbb{F}_p$. The next result will be exploited in Section 10.

**Proposition 4.9.** The residual $G_{\mathbb{Q}}$-representation $T/m_A T$ is equivalent up to finite base change to the residual representation $\tilde{\rho}_f$ of $f$. In particular, it is absolutely irreducible.
Proof. First of all, if the claimed equivalence of representations is true then the absolute irreducibility follows from condition ii) in Assumption 1.4. With notation as above, there are canonical isomorphisms of $G_{\mathbb{Q}}$-modules
\[
T / \mathfrak{m}_A T \simeq (T_{m_f}^{\mathrm{ord}} \otimes_{b_{\infty,m_f}^{\mathrm{ord}}} \mathcal{A}) \otimes_{\mathcal{A}} \mathbb{F}_A
\]
\[
\simeq T_{m_f}^{\mathrm{ord}} \otimes_{b_{\infty,m_f}^{\mathrm{ord}}} \mathbb{F}_A
\]
\[
\simeq (T_{m_f}^{\mathrm{ord}} \otimes_{b_{\infty,m_f}^{\mathrm{ord}}} \mathbb{F}_{b_{\infty,m_f}^{\mathrm{ord}}}) \otimes_{b_{\infty,m_f}^{\mathrm{ord}}} \mathbb{F}_A
\]
\[
\simeq (T_{m_f}^{\mathrm{ord}} / \mathfrak{m}_f T_{m_f}^{\mathrm{ord}}) \otimes_{b_{\infty,m_f}^{\mathrm{ord}}} \mathbb{F}_A.
\]

As explained in [17] p. 251 (see also [10] §1), all modular forms in the Hida family $b_{\infty,m_f}^{\mathrm{ord}}$ have residual representation equivalent to $\tilde{\rho}_f$. On the other hand, by [24] Proposition 2.1.2, the local ring $b_{\infty,m_f}^{\mathrm{ord}}$ is a Gorenstein $\Lambda$-algebra, and then [18] §9 (see also [32] §3) shows that the residual $G_{\mathbb{Q}}$-representation $T_{m_f}^{\mathrm{ord}} / \mathfrak{m}_f T_{m_f}^{\mathrm{ord}}$ is equivalent (up to finite base change) to $\tilde{\rho}_f$. The proposition is proved. $\square$

Now recall that $\Gamma := 1 + p\mathbb{Z}_p$. If $A$ is a finitely generated $\Lambda$-algebra then a homomorphism of $\mathcal{O}_F$-algebras $\kappa : A \to \mathbb{Q}_p$ is said to be arithmetic if its restriction to (the image of) $\Gamma$ is of the form
\[
\gamma \mapsto \psi(\gamma)\gamma^{r-2}
\]
for some integer $r \geq 2$ (called the weight of $\kappa$) and some finite order character $\psi$ of $\Gamma$ (called the wild character of $\kappa$). The kernel of an arithmetic homomorphism, which is a prime ideal of $\mathcal{A}$, is said to be an arithmetic prime of $\mathcal{A}$ if its restriction to (the image of) $\Gamma$ is of the form
\[
\gamma \mapsto \psi(\gamma)(\gamma^{r-2})^a
\]
for some integer $a \geq 0$. The kernel of a non-arithmetic homomorphism, which is a non-prime ideal of $\mathcal{A}$, is said to have arithmetic weight $r$.

Remark 4.10. The homomorphisms of $\mathcal{O}_F$-algebras
\[
\tilde{\theta}_f : b_{\infty}^{\mathrm{ord}} \longrightarrow \mathcal{O}_F, \quad \theta_f : T_{\infty}^{\mathrm{ord}} \longrightarrow \mathcal{O}_F
\]
that were attached in [4](4.3) to the modular form $f \in S_k(\Gamma_0(Np), \omega^j, \mathcal{O}_F)$ are arithmetic of weight $k$ and trivial character. Let $p$ be an arithmetic prime of $\mathcal{A}$ of weight $r_p$ and character $\psi_p$, and set
\[
m_p := \max\{1, \text{ord}_p(\text{cond}(\psi_p))\}.
\]
By [18] Corollary 1.3, the morphism obtained by composing the maps
\[
b_{\infty}^{\mathrm{ord}} \longrightarrow b_{\infty,m_f}^{\mathrm{ord}} \longrightarrow \mathcal{A} \longrightarrow F_p
\]
factors through $b_{\infty,m_f}^{\mathrm{ord}}$ and determines, by duality, an ordinary $p$-stabilized newform
\[
f_p = \sum_{n \geq 1} a_n(f_p)q^n \in S_{r_p}(\Gamma_{0,1}(N,p^{r_p}), \chi_p, F_p)
\]
where, for simplicity, we put $\chi_p := \psi_p\omega^{k+j-r_p}$.
Denote by $V(f_p)$ the $G_{\mathbb{Q}}$-representation over $F_p$ attached to $f_p$ by Deligne. Thanks to a result of Ribet, it is well known that $V(f_p)$ is (absolutely) irreducible. If $T_p := T \otimes_A A_p$ then the $G_{\mathbb{Q}}$-module

$$V_p := T_p/pT_p$$

is isomorphic to the dual representation $V^*(f_p) := \text{Hom}(V(f_p), F_p)$ of $V(f_p)$. Moreover, $V^*(f_p)$ is isomorphic to $V(f_p)(r_p - 1) \otimes [\chi_p^{-1}]$, where $[\chi_p]$ is the one-dimensional representation of $G_{\mathbb{Q}}$ over $F_p$ defined by $[\chi_p](\text{Fr}(\ell)_{\text{geom}}) = \chi(\ell)$ and $M(s)$ is the $s$-th Tate twist of the $G_{\mathbb{Q}}$-module $M$; see [35, §1.5.5] for details. Note, in particular, that if $k$ is even and $\chi_p$ is trivial then $V(f_p)$ is self-dual.

Set $T_p^\dagger := T^\dagger \otimes_A A_p$ and define the $G_{\mathbb{Q}}$-representation $V_p^\dagger$ over $F_p$ as

$$V_p^\dagger := T \otimes_A F_p \simeq T_p^\dagger/pT_p^\dagger.$$

The above discussion shows that $V_p^\dagger$ is a twist of the classical representation attached to $f_p$. There is an alternating, non-degenerate pairing

$$V_p^\dagger \times V_p^\dagger \longrightarrow F_p(1)$$

where $F_p(1)$ is the Tate twist of the trivial $G_{\mathbb{Q}}$-modules $F_p$. See [24, §2.1] and [35, §1.5 and §1.4] for details. Let now $v$ be a place of $\mathbb{Q}$ above $p$, let $D_v \subset G_{\mathbb{Q}}$ be a decomposition group at $v$ and let $I_v \subset D_v$ be the inertia subgroup. Denote by

$$\eta_v : D_v/I_v \longrightarrow A^\times$$

the character defined by sending the arithmetic Frobenius to $U_p$. Then [24, Proposition 2.4.1] ensures that there is a short exact sequence of $A[D_v]$-modules

$$0 \longrightarrow F_v^+(T) \longrightarrow T \longrightarrow F_v^-(T) \longrightarrow 0$$

where $F_v^+(T)$ and $F_v^-(T)$ are free $A$-modules of rank one and $D_v$ acts on $F_v^+(T)$ (respectively, on $F_v^-(T)$) through $\eta_v^{-1}\epsilon_{\text{cyc}}[\epsilon_{\text{cyc}}]$ (respectively, through $\eta_v$). Furthermore, if $M$ denotes one of $T_p$, $V_p$, $T_p^\dagger$ and $V_p^\dagger$ then twisting by $\Theta^{-1}$ and tensoring by $A_p$ or $F_p$ the above exact sequence yields another exact sequence of $D_v$-modules

$$0 \longrightarrow F_v^+(M) \longrightarrow M \longrightarrow F_v^-(M) \longrightarrow 0,$$

where $F_v^+(M)$ and $F_v^-(M)$ are free modules of rank one over either $A_p$ or $F_p$, depending on whether $M \in \{T_p, T_p^\dagger\}$ or $M \in \{V_p, V_p^\dagger\}$, respectively. Moreover, the Galois group $G_{\mathbb{Q}}$ acts on $F_v^+(M)$ and $F_v^-(M)$ either by $\eta_v^{-1}\epsilon_{\text{cyc}}[\epsilon_{\text{cyc}}]$ and $\eta_v$ or by $\Theta^{-1}\eta_v^{-1}\epsilon_{\text{cyc}}[\epsilon_{\text{cyc}}]$ and $\Theta^{-1}\eta_v$, depending on whether $M \in \{T_p, V_p\}$ or $M \in \{T_p^\dagger, V_p^\dagger\}$, respectively.

### 4.6. Selmer groups.

We recall the definitions of the various Selmer groups that are relevant for our purposes. The reader may also wish to consult [24, §2.4], [35, §2.1] and [15, Ch. 1].

Let $L$ be a finite extension of $\mathbb{Q}$ and for any prime $v$ of $L$ let $L_v$ be the completion of $L$ at $v$ and $L_v^{\text{unr}}$ the maximal unramified extension of $L_v$. Let $M$ be one of the left $R[G_{\mathbb{Q}}]$-modules $T$, $T^\dagger$, $T_p$, $T_p^\dagger$, $V_p$, $V_p^\dagger$ where $R$ denotes the ring $A$ in the first two cases, the ring $A_p$ in the middle two cases and the field $F_p$ in the last two cases. Fix a prime $v$ of $L$ and define the Greenberg local subgroup at $v$ by

$$H_{\text{Gr}}^1(L_v, M) := \begin{cases} \ker(H^1(L_v, M) \longrightarrow H^1(L_v^{\text{unr}}, M)) & \text{if } v \nmid p, \\ \ker(H^1(L_v, M) \longrightarrow H^1(L_v, F_v^-(M))) & \text{if } v \mid p. \end{cases}$$
Then the Greenberg Selmer group is by definition the group
\[ \text{Sel}_{\text{Gr}}(L, M) := \ker \left( H^1(L, M) \to \prod_v H^1(L_v, M)/H^1_{\text{Gr}}(L_v, M) \right). \]

Let \( A^\dagger := \text{Hom}_{\mathbb{Z}_p}(T^\dagger, \mu_{p^\infty}) \). For \( M = T^\dagger, A^\dagger \) or \( V_p^\dagger \) one can also consider the Nekovář Selmer group \( \tilde{H}^1_f(L, M) \), which for \( M = T^\dagger \) or \( V_p^\dagger \) sits in the short exact sequence
\[ 0 \to \bigoplus_{v \mid p} H^0(L_v, F_v^-(M)) \to \tilde{H}^1_f(L, M) \to \text{Sel}_{\text{Gr}}(L, M) \to 0, \]
the direct sum being extended over the primes of \( L \) above \( p \). See [36, Ch. 6] for definitions and [36, Lemma 9.6.3] for a proof of (34).

Remark 4.11. Unlike Greenberg’s Selmer group \( \text{Sel}_{\text{Gr}}(L, M) \), the group \( \tilde{H}^1_f(L, M) \) is defined by imposing local conditions on the level of cochain complexes rather than on cohomology. More precisely, this group was introduced by Nekovář in terms of his theory of Selmer complexes in derived categories, and the reader is referred to [36] (and, especially, to [36, Ch. 6]) for the original treatment of the subject. However, for the aims of the present paper it will be enough for us to view Nekovář’s theory as a “black box”, simply referring to [36] for the formal properties of Selmer groups that we need.

Finally, if \( M = V_p \) or \( V_p^\dagger \) one has the Bloch–Kato Selmer group \( H^1_f(L, M) \) as well, whose definition (in terms of Fontaine’s ring \( B_{cris} \)) can be found in [5, §3 and §5]. In particular, if \( M = V_p^\dagger \) and \( p \) has even weight then, by [36, Proposition 12.5.9.2], this group fits into the short exact sequence
\[ 0 \to \bigoplus_{v \mid p} H^0(L_v, F_v^-(V_p^\dagger)) \to \tilde{H}^1_f(L, V_p^\dagger) \to H^1_f(L, V_p^\dagger) \to 0. \]
An arithmetic prime \( p \) of \( A \) is said to be exceptional if \( r_p = 2 \), the character \( \psi_p \) is trivial and the image of \( U_p \) under the map \( A \to F_p \) is equal to \( \pm 1 \). The relations between the Selmer groups that we introduced above are then summarized by the exact sequences (34) and (35) and the following result.

Proposition 4.12. (1) If \( p \) is a non-exceptional arithmetic prime of \( A \) then
\[ \tilde{H}^1_f(L, V_p^\dagger) \simeq \text{Sel}_{\text{Gr}}(L, V_p^\dagger). \]

(2) If \( p \) is an arithmetic prime of even weight then
\[ H^1_f(L, V_p^\dagger) = \text{Sel}_{\text{Gr}}(L, V_p^\dagger). \]

Proof. The first assertion is [24 (22)], which follows from [24, Lemma 2.4.4]. The last claim is [24 (23)], which is immediate from (34) and (35). \( \square \)

5. Hida theory on quaternion algebras

Recall the quaternion algebra \( B/\mathbb{Q} \) of discriminant \( N^- \) and the Eichler orders \( R_m \), with \( m \geq 0 \), fixed at the beginning of the paper. In the following we use Hida’s notations \( T(n) \) and \( T(n,n) \) (with \( n \) an integer) for the Hecke operators defined as in [19, p. 309] by double cosets. (Observe that there is a notational conflict, since we wrote \( T(n) \) and \( T(n,n) \) also for the Hecke operators introduced in Section 4. However, we are confident that this will cause no confusion, and apologize with the reader for the inconvenience.) See also [33, p. 217], which covers the case where \( B \) is
5.1. Weight two modular forms on quaternion algebras. We first assume that $B$ is definite and set $U := \hat{\mathbb{R}}_m^\times$ or $U := U_m$ for an integer $m \geq 0$. For any ring $A$ let $\mathcal{M}(U, A)$ denote the $A$-module of functions

$$f : U \backslash \hat{B}^\times / B^\times \rightarrow A.$$ 

There is a natural isomorphisms of $A$-modules

$$\mathcal{M}(U, A) \simeq A[U \backslash \hat{B}^\times / B^\times]$$

where $A[U \backslash \hat{B}^\times / B^\times]$ is the free $A$-module over $U \backslash \hat{B}^\times / B^\times$; this isomorphism sends $f \in \mathcal{M}(U, A)$ to the $A$-linear combination $\sum_b f([b])[b]$, the sum being made over representatives of $U \backslash \hat{B}^\times / B^\times$.

Let $\mathcal{I}(U, A)$ denote the $A$-submodule of $\mathcal{M}(U, A)$ consisting of constant functions and define the $A$-module of weight 2 modular forms on $\hat{B}$ to be the quotient

$$\mathcal{S}(U, A) := \mathcal{M}(U, A)/\mathcal{I}(U, A).$$

Now we assume that $B$ is indefinite and set, as before, $U := \hat{\mathbb{R}}_m^\times$ or $U := U_m$ for an integer $m \geq 0$. Let $\Gamma(U)$ denote the subgroup of elements of norm 1 in $\phi_{\infty}(B^\times \cap U)$ and consider the compact Riemann surface $X_U(\mathbb{C}) := \mathcal{H}/\Gamma(U)$ and its Jacobian variety $\text{Jac}(X_U(\mathbb{C}))$. Write $\mathcal{S}(U, \mathbb{C})$ for the $\mathbb{C}$-vector space of holomorphic functions $f : \mathcal{H} \rightarrow \mathbb{C}$ such that

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^2 f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(U)$. This $\mathbb{C}$-vector space, referred to as the space of weight 2 modular forms on $\hat{B}$, is (non-canonically) isomorphic to the $\mathbb{C}$-vector space of invariant differentials of degree one on $\text{Jac}(X_U(\mathbb{C}))$.

In any case, definite and indefinite, there is a canonical action of $\mathcal{H}_m^{(B)} \otimes_{\mathbb{Z}} \mathcal{O}_F$ on $\mathcal{S}(U_m, \mathbb{C})$, for which we refer to [23 §2.3.5]. We write $\mathbb{B}_m$ for the image of $\mathcal{H}_m^{(B)} \otimes \mathcal{O}_F$ in $\text{End}_\mathbb{C}(\mathcal{S}(U_m, \mathbb{C}))$. 

indeterminate. For every integer $m \geq 0$ denote $\mathcal{H}_m^{(B)}$ the abstract Hecke algebra generated over $\mathbb{Z}$ by $T_\ell := T(\ell)$ for primes $\ell \nmid Np^m$, $U_\ell := T(\ell)$ for primes $\ell | Np^m$ and $T(\ell, \ell)$ for primes $\ell \nmid Np^m$.

In this section we always assume that $B$ is a division algebra, the theory for the split case $B \simeq M_2(\mathbb{Q})$ having already been considered in Section 4.

Fix an integer $m \geq 0$. Recall that $\text{Div}(\tilde{X}_m)$ and $\text{Div}^0(\tilde{X}_m)$ denote the group of divisors and degree zero divisors, respectively, on $\tilde{X}_m$. For any extension $L/\mathbb{Q}$ write $\text{Div}_L(\tilde{X}_m)$ (respectively, $\text{Div}^0_L(\tilde{X}_m)$) for the group of divisors (respectively, degree zero divisors) on $\tilde{X}_m$ which are invariant under the action of $\text{Gal}(\overline{\mathbb{Q}}/L)$. The Hecke operators $T(\ell)$ for all primes $\ell$ and $T(\ell, \ell)$ for primes $\ell$ not dividing $Np$ act by correspondences on $\tilde{X}_m \times \tilde{X}_m$. Let $\text{Pr}(\tilde{X}_m)$ denote the group of principal divisors on $\tilde{X}_m$ and define, as usual, the Picard groups

$$\text{Pic}(\tilde{X}_m) := \text{Div}(\tilde{X}_m)/\text{Pr}(\tilde{X}_m), \quad \text{Pic}^0(\tilde{X}_m) := \text{Div}^0(\tilde{X}_m)/\text{Pr}(\tilde{X}_m).$$

The groups $\text{Pic}(\tilde{X}_m)$ and $\text{Pic}^0(\tilde{X}_m)$ are connected by the exact sequence

$$0 \rightarrow \text{Pic}^0(\tilde{X}_m) \rightarrow \text{Pic}(\tilde{X}_m) \xrightarrow{\text{deg}} \mathbb{Z} \rightarrow 0,$$

where $\text{deg}$ is the degree map.

Let $\mathcal{M}(U, A)$ denote the $A$-module of functions

$$f : U \backslash \hat{B}^\times / B^\times \rightarrow A.$$ 

There is a natural isomorphisms of $A$-modules

$$\mathcal{M}(U, A) \simeq A[U \backslash \hat{B}^\times / B^\times]$$

where $A[U \backslash \hat{B}^\times / B^\times]$ is the free $A$-module over $U \backslash \hat{B}^\times / B^\times$; this isomorphism sends $f \in \mathcal{M}(U, A)$ to the $A$-linear combination $\sum_b f([b])[b]$, the sum being made over representatives of $U \backslash \hat{B}^\times / B^\times$.

Let $\mathcal{I}(U, A)$ denote the $A$-submodule of $\mathcal{M}(U, A)$ consisting of constant functions and define the $A$-module of weight 2 modular forms on $\hat{B}$ to be the quotient

$$\mathcal{S}(U, A) := \mathcal{M}(U, A)/\mathcal{I}(U, A).$$

Now we assume that $B$ is indefinite and set, as before, $U := \hat{\mathbb{R}}_m^\times$ or $U := U_m$ for an integer $m \geq 0$. Let $\Gamma(U)$ denote the subgroup of elements of norm 1 in $\phi_{\infty}(B^\times \cap U)$ and consider the compact Riemann surface $X_U(\mathbb{C}) := \mathcal{H}/\Gamma(U)$ and its Jacobian variety $\text{Jac}(X_U(\mathbb{C}))$. Write $\mathcal{S}(U, \mathbb{C})$ for the $\mathbb{C}$-vector space of holomorphic functions $f : \mathcal{H} \rightarrow \mathbb{C}$ such that

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^2 f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(U)$. This $\mathbb{C}$-vector space, referred to as the space of weight 2 modular forms on $\hat{B}$, is (non-canonically) isomorphic to the $\mathbb{C}$-vector space of invariant differentials of degree one on $\text{Jac}(X_U(\mathbb{C}))$.

In any case, definite and indefinite, there is a canonical action of $\mathcal{H}_m^{(B)} \otimes_{\mathbb{Z}} \mathcal{O}_F$ on $\mathcal{S}(U_m, \mathbb{C})$, for which we refer to [23 §2.3.5]. We write $\mathbb{B}_m$ for the image of $\mathcal{H}_m^{(B)} \otimes \mathcal{O}_F$ in $\text{End}_\mathbb{C}(\mathcal{S}(U_m, \mathbb{C}))$. 

5.2. **Jacquet–Langlands correspondence.** In order to simplify notations, set $T_* := T_{2,*}$ and $T_{2,*}^\text{ord} := T_{2,*}^{\text{ord}}$ for $* \geq 0$ or the symbol $\infty$: this is an extension of the convention introduced at the end of [12].

The Jacquet–Langlands correspondence (see [23 §4]) gives an $H_m^{(B)}$-equivariant isomorphism between $S_2(\Gamma_{0,1}(N, p^m), \mathbb{C})$ and $S(U_m, \mathbb{C})$, from which we deduce a canonical isomorphism of $O_F$-algebras

$$JL_m : T_m \xrightarrow{\sim} B_m$$

taking $T(\ell)_2$ and $T(\ell, \ell)_2$ to the images of $T_{\ell}$ and $T(\ell, \ell)$ in $B_m^{\text{ord}}$, respectively. Denote by $B_m^{\text{ord}} := JL_m(T_m^{\text{ord}})$ the *ordinary part* of $B_m$, which is the product of those local rings of $B_m$ where $U_p$ is invertible. Alternatively, one can define the ordinary projector $e_m^{\text{ord}} := \lim_{n \to \infty} U_{p^n}$ exactly as in the case of elliptic modular forms (see [23 §3.1.4]), and then $B_m^{\text{ord}} := e_m^{\text{ord}} \cdot B_m$.

In the present context too, for $m \geq n \geq 1$ there are injective homomorphisms

$$i_{n,m} : S(U_n, \mathbb{C}) \hookrightarrow S(U_m, \mathbb{C})$$

which are equivariant for the action of $H_n^{(B)} \otimes O_F$ and $H_m^{(B)} \otimes O_F$, so we get projection maps $B_m \to B_n$ and $B_m^{\text{ord}} \to B_n^{\text{ord}}$, and we can form the $O_F$-modules

$$B_\infty := \lim_m B_m, \quad B_m^{\text{ord}} := \lim_m B_m^{\text{ord}}.$$ 

Alternatively, $B_m^{\text{ord}} := e_m^{\text{ord}} \cdot B_\infty$ where $e_m^{\text{ord}} := \lim_m e_m^{\text{ord}}$ is the ordinary projector in $B_\infty$.

We define a continuous structure of $\hat{\Lambda}$-algebra on $B_\infty$ and $B_m^{\text{ord}}$ as in [23 §3.2.8], and denote $[z] \mapsto \langle z \rangle$ the image of group-like elements of $\hat{\Lambda}$. We normalize this action so that if $n$ is an integer coprime with $Np$ then $T(n, n) = \langle n \rangle$ as operators in $B_m$ (as in the case of elliptic modular forms, we adopt the normalization in [24] instead of the one usually found in Hida’s papers).

In the present context, the Jacquet–Langlands correspondence can be formulated as follows.

**Proposition 5.1 (Jacquet–Langlands).** There is a canonical isomorphism of $\hat{\Lambda}$-algebras

$$JL_\infty^{\text{ord}} : T_\infty^{\text{ord}} \xrightarrow{\sim} B_\infty^{\text{ord}}.$$

**Proof.** For $m \geq 1$ there are commutative diagrams

$$\begin{array}{ccc}
T_m^{\text{ord}} & \xrightarrow{\text{JL}_m} & B_m^{\text{ord}} \\
\downarrow & & \downarrow \\
T_{m-1}^{\text{ord}} & \xrightarrow{\text{JL}_{m-1}} & B_{m-1}^{\text{ord}}
\end{array}$$

where the vertical arrows are the canonical projections. The claim of the proposition then follows by taking inverse limits and noticing that the two $\hat{\Lambda}$-algebra structures agree on the set of integers prime to $Np$, so (by a continuity argument) they must be equal. \qed

In particular, it follows from Proposition 5.1 that there is a $\Lambda$-algebra decomposition

$$B_\infty^{\text{ord}} = \prod_n B_{\infty,n}^{\text{ord}}$$

of $B_\infty^{\text{ord}}$ into a product of complete local rings, finite over $\Lambda$. Here, as before, the sum (38) is indexed by the maximal ideals of $B_\infty^{\text{ord}}$ and $B_{\infty,n}^{\text{ord}}$ denotes the localization of $B_\infty^{\text{ord}}$ at $n$. Furthermore, the map $JL_\infty^{\text{ord}}$ respects the decompositions of $T_\infty^{\text{ord}}$ and $B_\infty^{\text{ord}}$ into local summands.
Recall the modular form \( f \) fixed at the beginning of Section 1.4 the maximal ideal \( \mathfrak{m}_f \) of \( T_\infty^{\text{ord}} \) (i.e., the maximal ideal corresponding to the unique local summand of \( T_\infty^{\text{ord}} \) through which the morphism \( \theta_f \) in (29) factors) and the integral domain \( B \subset T_\infty^{\text{ord},m_f} \otimes_\Lambda \mathcal{L} \) introduced in [41]. We use the Jacquet–Langlands isomorphism \( \text{JL}_\infty^{\text{ord}} \) of Proposition 5.1 to define the maximal ideal
\[
\mathfrak{n}_f := \text{JL}_\infty^{\text{ord}}(\mathfrak{m}_f) \subset \mathbb{B}_\infty^{\text{ord}}.
\]
Of course, there is an induced isomorphism
\[
\text{JL}_\infty^{\text{ord}} : T_\infty^{\text{ord},m_f} \otimes_\Lambda \mathcal{L} \xrightarrow{\simeq} \mathbb{B}_\infty^{\text{ord},\mathfrak{n}_f} \otimes_\Lambda \mathcal{L},
\]
and we can consider the integral domain
\[
(39) \quad \mathcal{C} := \text{JL}_\infty^{\text{ord}}(B) \subset \mathbb{B}_\infty^{\text{ord},\mathfrak{n}_f} \otimes_\Lambda \mathcal{L}.
\]
Note that there are ring isomorphisms
\[
A \simeq B \simeq C.
\]

**Remark 5.2.** Since, by construction, the rings \( A \) and \( B \) depend on the modular form \( f \) (cf. Remark 1.1), the notation \( \mathcal{C} \) for the ring defined in (39) should be interpreted as a shorthand for the more correct \( \mathcal{C}_f \).

Finally, we write \( \mathbb{B}_\infty^{\text{ord,}+} \) and \( \mathbb{B}_m^{\text{ord,}+} \) for the twisted \( G_\mathbb{Q} \)-modules \( \mathbb{B}_\infty^{\text{ord}} \) and \( \mathbb{B}_m^{\text{ord}} \), respectively, where the action of \( G_\mathbb{Q} \) is via \( \Theta^{-1} \).

### 5.3. Hecke modules in the definite case

Assume that \( B \) is definite and fix an integer \( m \geq 0 \).

As pointed out in [11] §1.4 and [14] §4, in this case \( \text{Pic}(\tilde{X}_m) \) can be identified with the free abelian group \( \mathbb{Z}[U_m \setminus \hat{B}_\infty^x/B_\infty^x] \) on the finite set of double cosets \( U_m \setminus \hat{B}_\infty^x/B_\infty^x \) and \( \text{Pic}^0(\tilde{X}_m) \) corresponds to the degree zero elements in this group. With notation as in §1.1 the sequence \( m \mapsto h(m) \) is unbounded because the same is true, by [12] Theorem 16, of the sequence \( m \mapsto h(m) \), and \( h(m) \leq \hat{h}(m) \). Hence the ranks of the free abelian groups \( \text{Pic}(\tilde{X}_m) \) and \( \text{Pic}^0(\tilde{X}_m) \) are unbounded as \( m \) varies. Now define
\[
J_m := \text{Pic}(\tilde{X}_m) \otimes_\mathbb{Z} \mathcal{O}_F, \quad J_m^0 := \text{Pic}^0(\tilde{X}_m) \otimes_\mathbb{Z} \mathcal{O}_F.
\]
Tensoring (36) by \( \mathcal{O}_F \otimes \mathbb{Z} \) yields a short exact sequence of \( \mathcal{O}_F \)-modules
\[
0 \longrightarrow J_m^0 \longrightarrow J_m \xrightarrow{\deg} \mathcal{O}_F \longrightarrow 0.
\]
By what has been said a few lines before, there is an identification of \( \mathcal{O}_F \)-modules
\[
(41) \quad J_m = \mathcal{O}_F[U_m \setminus \hat{B}_\infty^x/B_\infty^x],
\]
which will usually be viewed as an equality. The \( \mathcal{O}_F \)-module \( J_m \) is finitely generated, so, by [9] Proposition 2.10, it follows that \( \text{End}_\mathbb{Z}(\text{Pic}(\tilde{X}_m)) \otimes_\mathbb{Z} \mathcal{O}_F \) is canonically isomorphic to \( \text{End}_{\mathcal{O}_F}(J_m) \). A similar remark also applies to \( J_m^0 \).

In [11] §1.4 (see also [14] §4 and [49] §2.2) it is explained how equality (41) can be used to define an \( \mathcal{O}_F \)-linear action of \( \mathcal{H}_m^{(B)} \otimes \mathcal{O}_F \) on \( J_m \) and \( J_m^0 \): if \( UXU = \bigsqcup_{i \in I} Ux_i \) (finite disjoint union), then \( b|UXU| := \sum_i x_ib \). Enumerate once and for all the elements of \( U_m \setminus \hat{B}_\infty^x/B_\infty^x \) and write
\[
U_m \setminus \hat{B}_\infty^x/B_\infty^x = \{ [b_1], \ldots, [b_{\hat{h}(m)}] \}.
\]
The ordered \( \hat{h}(m) \)-tuple at the right member of (42) will be our “canonical” basis of \( J_m \) over \( \mathcal{O}_F \) under the identification (41). The matrices \( B_\ell, B'_\ell \) and \( B_{\ell,\ell} \) representing the action of \( T_\ell, U_\ell \) and
commutative and $\mathcal{H}_m^{(B)} \otimes \mathcal{O}_F$-equivariant. If $\mathbb{B}_m$ denotes the image of $\mathcal{H}_m^{(B)} \otimes \mathbb{Z} \mathcal{O}_F$ in the endomorphism ring $\text{End}(\mathcal{M}(U_m, \mathbb{C}))$, then $\mathcal{M}(U_m, \mathcal{O}_F)$ is $\mathbb{B}_m$-stable, and hence $J_m$ is canonically a $\mathbb{B}_m$-module. Furthermore, since $J_m^0$ is stable under the action of Hecke operators, we see that $J_m^0$ is a $\mathbb{B}_m$-module too.

Let us now assume $m \geq 1$. Since $\mathbb{B}_m$ is a finitely generated $\mathcal{O}_F$-module, we can define an idempotent $\tilde{e}_m^{\text{ord}} \in \mathbb{B}_m$ attached to the Hecke operator $U_p$ and introduce the ordinary parts

$$\mathcal{X}^{\text{ord}} := \tilde{e}_m^{\text{ord}} \cdot \mathcal{X}$$

for $\mathcal{X} \in \{ \mathbb{B}_m, J_m, J_m^0, \mathcal{M}(U_m, \mathcal{O}_F) \}$. It is immediate to see that $\tilde{e}_m^{\text{ord}} \mapsto e_m^{\text{ord}} \in \mathbb{B}_m$ under the canonical projection map $\mathbb{B}_m \to \mathbb{B}_m$. Since $U_p$ has degree $p$, exact sequence (40) implies that

$$J_m^{\text{ord}} = J_m^{0, \text{ord}}.$$  

(43)

The following trivial result in linear algebra will be used in the proof of the subsequent proposition.

**Lemma 5.3.** Let $A$ be a ring, let $M$ be an $A$-module and let $I$ be an $A$-submodule of $M$. If $a \in A$ is such that $aI = 0$ then there is an isomorphism $a(M/I) \simeq aM$ of $A$-modules.

**Proof.** Let $[\ast]$ denote an equivalence class in $M/I$. The map $a[m] \mapsto am$ from $a(M/I)$ to $aM$ gives the searched-for isomorphism. \hfill $\square$

Now observe that $\mathbb{B}_m^{\text{ord}}$ naturally identifies with the image of $\mathbb{B}_m$ (or of $\mathcal{H}_m^{(B)} \otimes \mathcal{O}_F$) in

$$\text{End}_{\mathcal{O}_F}(\mathcal{M}^{\text{ord}}(U_m, \mathbb{C}));$$

an analogous statement for $\mathbb{B}_m^{0, \text{ord}}$ is true if one replaces $\mathcal{M}^{\text{ord}}(U_m, \mathbb{C})$ with $\mathcal{S}^{\text{ord}}(U_m, \mathbb{C})$.

**Proposition 5.4.** There are canonical isomorphisms

1. $\mathcal{M}^{\text{ord}}(U_m, \mathcal{O}_F) \simeq \mathcal{S}^{\text{ord}}(U_m, \mathcal{O}_F)$ as $\mathcal{O}_F$-modules;
2. the images of $\mathbb{B}_m^{\text{ord}}$ and $\mathbb{B}_m^{0, \text{ord}}$ in $\text{End}(\mathcal{M}^{\text{ord}}(U_m, \mathcal{O}_F))$ and $\text{End}(\mathcal{S}^{\text{ord}}(U_m, \mathcal{O}_F))$, respectively, are isomorphic as $\mathcal{O}_F$-algebras.

**Proof.** Since the degree of $U_p$ is $p$, we see that $\tilde{e}_m^{\text{ord}} : I(U_m, \mathcal{O}_F) = 0$, so (since $\tilde{e}_m^{\text{ord}}$ maps to $e_m^{\text{ord}}$ under the canonical projection $\mathbb{B}_m \to \mathbb{B}_m$) part (1) follows from Lemma 5.3. Claim (2) is then a consequence of (1) and the above discussion. \hfill $\square$
with respect to these maps. In particular, $J^\text{ord}_\infty$ is a $\mathbb{B}_\infty^\text{ord}$-module, while $J^0_\infty$ and $J^0_\infty$ are $\widehat{\mathbb{B}}_\infty$-modules.

5.4. Hecke modules in the indefinite case. Now suppose that $B$ is indefinite and fix an integer $m \geq 0$. Then Pic$^0(\tilde{X}_m)$ can be identified with the Jacobian variety Jac$(\tilde{X}_m)$ of $\tilde{X}_m$, which is an abelian variety defined over $\mathbb{Q}$ whose dimension equals the genus of $\tilde{X}_m$, while (36) shows that Pic$(\tilde{X}_m)$ is an extension of $\mathbb{Z}$ by Pic$^0(\tilde{X}_m)$. More precisely, Pic$(\tilde{X}_m)$ identifies with the $\mathbb{Q}$-points of the Picard scheme of $\tilde{X}_m$ and Pic$^0(\tilde{X}_m)$ with the identity component of this scheme. If $L$ is an extension of $\mathbb{Q}$ then we denote by Pic$(\tilde{X}_m)(L)$ and Pic$^0(\tilde{X}_m)(L)$ the $L$-rational points of Pic$(\tilde{X}_m)$ and Pic$^0(\tilde{X}_m)$, respectively. Unlike what done in the definite case, in the indefinite case by $J^0_m$ and $J^0_m$ we mean the $\mathcal{O}_F$-module schemes defined over $\mathbb{Q}$ which associate with any field extension $L/\mathbb{Q}$ the $\mathcal{O}_F$-modules

$$J_m(L) := \text{Pic}(\tilde{X}_m)(L) \otimes_{\mathbb{Z}} \mathcal{O}_F, \quad J^0_m(L) := \text{Jac}(\tilde{X}_m)(L) \otimes_{\mathbb{Z}} \mathcal{O}_F,$$

respectively, which are endowed with a canonical action of $\mathbb{B}_m$ via Albanese functoriality.

Let $L$ be an algebraic extension of $\mathbb{Q}$ and set $G_L := \text{Gal}(\bar{\mathbb{Q}}/L)$. Since Jac$(\tilde{X}_m)$ is defined over $\mathbb{Q}$, the $\mathcal{O}_F$-module $J^0_m(L)$ has a natural left $G_L$-action and so is canonically a left $\mathbb{B}_m[G_L]$-module. Furthermore, the ordinary part

$$J^0_m,\text{ord}(L) := e^\text{ord}_m \cdot J^0_m(L)$$

inherits a canonical structure of left $\mathbb{B}_m[\mathcal{O}_F[\mathcal{G}_L]]$-module.

Suppose that $L/\mathbb{Q}$ is a finite extension. Tensoring (36) (with values in $L$) by $\mathcal{O}_F$ over $\mathbb{Z}$ yields, as above, a short exact sequence of left $\mathcal{O}_F[\mathcal{G}_L]$-modules

$$0 \longrightarrow J^0_m(L) \longrightarrow J_m(L) \xrightarrow{\text{deg}} \mathcal{O}_F \longrightarrow 0. \quad \text{(44)}$$

Denote by $\mathbb{B}_m(L)$ the image of $\mathcal{H}^m(\mathbb{B}) \otimes_{\mathbb{Z}} \mathcal{O}_F$ in $\text{End}_{\mathcal{O}_F}(J_m(L))$. Since $J_m(L)$ is a finitely generated $\mathcal{O}_F$-module, it makes sense to introduce the idempotent $e^\text{ord}_m(L) := \lim_{n \to \infty} U^{n!}p$ in $\mathbb{B}_m(L)$ attached to the Hecke operator $U_p$ and define the ordinary part of $J_m(L)$ to be

$$J_m^0(L) := e^\text{ord}_m(L) \cdot J_m(L).$$

Now observe that, since $U_p$ has degree $p$, sequence (44) shows that

$$J_m^\text{ord}(L) = J_m^0,\text{ord}(L) \quad \text{for every } m \geq 0 \text{ and every finite extension } L/\mathbb{Q}. \text{ Let now } L/\mathbb{Q} \text{ be an arbitrary field extension, which can be written as a direct limit } L = \lim_{i} L_i \text{ of finite extensions. Since the } J_m(L_i) \text{ have ordinary parts which are compatible with direct limits, we can define the ordinary part of } J_m(L) \text{ as}

$$J_m^\text{ord}(L) := \lim_{i} J_m^\text{ord}(L_i).$$

Thanks to (45), and the fact that $J^0_m(L) = \lim_{i} J^0_m(L_i)$ because direct limits commute with tensor products, we see that

$$J_m^\text{ord}(L) = J^0_m,\text{ord}(L) \quad \text{for every } m \geq 0 \text{ and every extension } L/\mathbb{Q}. \text{ Although we did not define a } \mathbb{B}_m\text{-module structure on } J_m(L), \text{ equality (46) shows that } J_m^\text{ord}(L) \text{ has a structure of } \mathbb{B}_\infty^\text{ord}\text{-module for every } m \text{ and every field extension } L/\mathbb{Q}, \text{ this structure being deduced from that of } J^0_m,\text{ord}(L).$
As above, for every extension $L/\mathbb{Q}$ and every $m \geq 1$ we can define by covariant functoriality maps $\tilde{\alpha}_{m,*} : J_m(L) \to J_{m-1}(L)$ and $\tilde{\alpha}_{m,*} : J_m^0(L) \to J_{m-1}^0(L)$ which preserve the ordinary parts, so we can form the projective limits

$$J_\infty(L) := \lim_{m} J_m(L), \quad J_\infty^0(L) := \lim_{m} J_m^0(L), \quad J_\infty^{\text{ord}}(L) := \lim_{m} J_m^{\text{ord}}(L)$$

with respect to these maps. In particular, $J_\infty^0(L)$ is a left $B_\infty[G_L]$-module and $J_\infty^{\text{ord}}(L)$ is a left $B_\infty^{\text{ord}}[G_L]$-module.

Finally, write $T_a p(Jac(\tilde{X}_m))$ for the $p$-adic Tate module of $\text{Jac}(\tilde{X}_m)$ and define

$$T_m := T_a p(Jac(\tilde{X}_m)) \otimes_{\mathbb{Z}_p} \mathcal{O}_F, \quad T_\infty := \lim_{m} T_m$$

where the inverse limit is with respect to the canonical projection maps. Then $T_\infty$ and $T_m$ are $B_\infty$ and $B_m$-modules, respectively, and one can define the ordinary parts

$$T_m^{\text{ord}} := e_\text{ord} \cdot T_m, \quad T_\infty^{\text{ord}} := e_\text{ord} \cdot T_\infty,$$

which are left $B_m^{\text{ord}}[G_\mathbb{Q}]$ and $B_\infty^{\text{ord}}[G_\mathbb{Q}]$-modules, respectively.

5.5. Critical twists. In the following, let $W_\text{ord}^\star$ denote one of the left $B_\infty^{\text{ord}}[G_\mathbb{Q}]$-modules $J_\text{ord}^\star$ (definite and indefinite case) and $T_\text{ord}^\star$ (indefinite case), for $\star$ an integer $m \geq 1$ or the symbol $\infty$. Since $B_m^{\text{ord}}$ is a quotient of $B_\infty^{\text{ord}}$, it follows that $W_m^{\text{ord}}$ is naturally a $B_m^{\text{ord}}$-module via the projection map. The critical twist $W_\text{ord}^\star$ of $W_\text{ord}^\star$ is the left $B_\infty^{\text{ord}}[G_\mathbb{Q}]$-module

$$W_\star^{\text{ord},\dagger} := W_\star^{\text{ord}} \otimes_{B_\infty^{\text{ord}}} B_\infty^{\dagger}. $$

With notation as in \S 5.2 define also the left $B_\infty^{\text{ord}}[G_\mathbb{Q}]$-modules

$$W_\star, n_f := W_\star^{\text{ord}} \otimes_{B_\infty^{\text{ord}}} B_\infty^{\text{ord}}, \quad W_\star, n_f^{\text{ord},\dagger} := W_\star^{\text{ord},\dagger} \otimes_{B_\infty^{\text{ord}}} B_\infty^{\text{ord}}, \quad W_\star, C := W_\star, n_f^{\text{ord},\dagger} \otimes_{B_\infty^{\text{ord}}} C.$$

Finally, let $C^{\dagger}$ stand for the Galois module $C$ on which the action of $G_\mathbb{Q}$ is via $\Theta^{-1}$. The critical twist $W_\text{ord}^\star, C^{\dagger}$ of $W_\text{ord}^\star, C$ is the left $B_\infty^{\text{ord}}[G_\mathbb{Q}]$-module

$$W_\star, C^{\dagger} := W_\star^{\text{ord}} \otimes_{B_\infty^{\text{ord}}} C^{\dagger} = W_\star, n_f^{\text{ord},\dagger} \otimes_{B_\infty^{\text{ord}}} C.$$

In the next \S 6 we perform analogous twists on groups of divisors.

5.6. Critical twists on divisors. Recall the subsets $\tilde{X}_m^{(K)}$ defined in \S 1.1 (definite case) and \S 1.2 (indefinite case), where $K$ is an imaginary quadratic field admitting injections $K \hookrightarrow B$. In both cases, let us denote by $\text{Div}(\tilde{X}_m^{(K)})$ and $\text{Div}^0(\tilde{X}_m^{(K)})$ the submodules of $\text{Div}(\tilde{X}_m)$ and $\text{Div}^0(\tilde{X}_m)$, respectively, supported on points in $\tilde{X}_m^{(K)}$. The group $G_K^{ab} := \text{Gal}(K^{ab}/K)$ acts naturally on $\tilde{X}_m^{(K)}$, so $\text{Div}(\tilde{X}_m^{(K)})$ and $\text{Div}^0(\tilde{X}_m^{(K)})$ are $\mathbb{Z}[G_K^{ab}]$-modules. Furthermore, they are also $\mathcal{H}_m^{(B)}$-modules, so one can define the $B_m[G_K^{ab}]$-modules

$$D_m := \text{Div}(\tilde{X}_m^{(K)}) \otimes_{\mathcal{H}_m^{(B)}} B_m, \quad D_0 := \text{Div}^0(\tilde{X}_m^{(K)}) \otimes_{\mathcal{H}_m^{(B)}} B_m$$

and the $B_m^{\text{ord}}[G_K^{ab}]$-modules

$$D_m^{\text{ord}} := D_m \otimes_{B_m} B_m^{\text{ord}}, \quad D_0^{\text{ord}} := D_0 \otimes_{B_m} B_m^{\text{ord}}.$$
For \( m \geq 2 \), the maps \( \tilde{\alpha}_m : \tilde{X}_m^{(K)} \to \tilde{X}_{m-1}^{(K)} \) induce maps \( \tilde{\alpha}_m : D_m \to D_{m-1} \) by covariant functoriality which respect the ordinary parts, so we can define the \( B_\infty[G_K^{ab}] \)-modules
\[
D_\infty := \lim_{m} D_m, \quad D_\infty^0 := \lim_{m} D_\infty^0
\]
and the \( B_\infty[G_K^{ab}] \)-modules
\[
D_\infty^{ord} := \lim_{m} D^{ord}_m, \quad D_\infty^{0,ord} := \lim_{m} D^{0,ord}_m.
\]
In the indefinite case, when we want to emphasize the field of definition \( H \subset K^{ab} \) of a divisor or a limit of divisors we write \( D_*(H) \) and \( D^{ord}(H) \), for \( * = m \) or \( \infty \), respectively. (In the definite case, all the points in \( D_m \) are rational over \( K \), so there is no need to specify fields of definition.)

Now we proceed as in \( \S 5.5 \). Let \( W_*^{ord} \) denote one of the \( \mathbb{B}_\infty[G_Q] \)-modules \( D_*^{ord} \) and \( D_*^{0,ord} \), for \( * \) an integer \( m \geq 2 \) or the symbol \( \infty \). Since \( \mathbb{B}_\infty[G_K^{ab}] \) is a quotient of \( \mathbb{B}_\infty \), it follows that \( W_*^{ord} \) is naturally a \( \mathbb{B}_\infty \)-module via the projection map. The \textit{critical twist} \( W_*^{ord,\dagger} \) of \( W_*^{ord} \) is the \( \mathbb{B}_\infty[G_K^{ab}] \)-module
\[
W_*^{ord,\dagger} := W_*^{ord} \otimes_{\mathbb{B}_\infty} \mathbb{B}_\infty^{\dagger}.
\]
Again, we define the \( \mathbb{B}_\infty[G_K^{ab}] \)-modules
\[
W_*,n_f := W_*^{ord} \otimes_{\mathbb{B}_\infty^{\dagger}} \mathbb{B}_\infty^{\dagger, n_f}, \quad W_*,n_f^{ord} := W_*^{ord,\dagger} \otimes_{\mathbb{B}_\infty^{\dagger}} \mathbb{B}_\infty^{\dagger, n_f}, \quad W_*,C := W_*^{ord} \otimes_{\mathbb{B}_\infty^{\dagger}} \mathbb{B}_\infty^{\dagger, n_f} C.
\]
Finally, let \( C^{\dagger} \) denote the Galois module \( C \) on which the action of \( G_Q \) is via \( \Theta^{-1} \). The \textit{critical twist} \( W_*^{ord,\dagger,\dagger} \) of \( W_*^{ord} \) is defined to be
\[
W_*^{ord,\dagger,\dagger} := W_*^{ord} \otimes_{C} C^{\dagger} = W_*^{ord,\dagger} \otimes_{\mathbb{B}_\infty^{\dagger, n_f}} C^{\dagger},
\]
with its natural structure of \( \mathbb{B}_\infty[G_K^{ab}] \)-module.

6. Big Heegner Points

In this section we introduce big Heegner points and big Heegner classes, and prove their main compatibility properties. Note that the first three \( \S \) apply both to the definite and to the indefinite case. These results generalize the construction of Galois cohomology classes out of Heegner points attached to indefinite quaternion algebras over totally real fields.

6.1. Galois relations. Let \( \sigma \in \text{Gal} (\overline{Q}/H_{cp,m}) \). Since, by Lemma 2.4, \( \sigma \) is the identity on \( Q(\sqrt{p^\alpha}) \), it follows that there exists \( \xi_\sigma \in \mu_{p-1} \) such that \( \xi_\sigma^{2} = \epsilon_{\text{tame}}(\sigma) \). Hence
\[
(47) \quad \xi_\sigma^{1/2} \epsilon_{\text{wild}}(\sigma) = \pm \theta(\sigma),
\]
with \( \theta \) as in \( \S 3.1 \). By definition, \( \Theta(\sigma) = \xi_\sigma^{k+j-2} [\epsilon_{\text{wild}}^{1/2}(\sigma)] \). From (30) it follows that
\[
(48) \quad \Theta(\sigma) \epsilon_{k+j-2} = \xi_\sigma^{k+j-2} [\epsilon_{\text{wild}}^{1/2}(\sigma)] \epsilon_{k+j-2} = \epsilon_{k+j-2} [\xi_\sigma \epsilon_{\text{wild}}^{1/2}(\sigma)]
\]
in \( \Lambda \). By (31), we know that \( \epsilon_{k+j-2}(\tau_\sigma^{ord},n_f) = \epsilon_{k+j-2}(\tau_\sigma^{ord},n_f) \), so \( \epsilon_{k+j-2}(\tau_\sigma^{ord},n_f) = \epsilon_{k+j-2}(\tau_\sigma^{ord},n_f) \). Hence (47) and (48) imply that
\[
(49) \quad \Theta(\sigma) P = [\pm \theta(\sigma)] P = (\theta(\sigma)) P
\]
for all $P \in D^{\text{ord}}_{m,n_f}$, where $(\ell)$ is the diamond operator at $\ell$ as in §3.1. Now recall the point $\tilde{P}_{c,m} \in \tilde{X}_m^{(K)}$ defined in §3.2, and write $\tilde{P}^{\text{ord}}_{c,m}$ for its image in $D^{\text{ord}}_{m,n_f}$. For all $\sigma \in \Gal(\bar{Q}/H_{cp^n})$, equations (24) and (49) give the equality

\[(\tilde{P}^{\text{ord}}_{c,m})^\sigma = \Theta(\sigma)\tilde{P}_{c,m}\]

in $D^{\text{ord}}_{m,n_f}$, from which it follows that

\[\tilde{P}^{\text{ord}}_{c,m} \in H^0(\Gal(K_{ab}/H_{cp^n}), D^{\text{ord},\dagger}_{m,n_f}),\]

by definition of the $G^{ab}_K$-module $D^{\text{ord},\dagger}_{m,n_f}$.

6.2. Hecke relations. For any pair of positive integers $(s, t)$ let $\Cor_{H_{st}/H_s}$ denote the corestriction map from $H_{st}$ to $H_s$. Explicitly, for all $\eta \in \Gal(H_{st}/H_s)$ choose an extension $\tilde{\eta} \in \Gal(K_{ab}/H_{s})$ of $\eta$; if $Q \in H^0(\Gal(K_{ab}/H_s), D^{\text{ord},\dagger}_{m,n_f})$ then

\[\Cor_{H_{st}/H_s}(Q) = \sum_{\eta \in \Gal(H_{st}/H_s)} \Theta(\tilde{\eta}^{-1})Q^{\tilde{\eta}}\]

in $H^0(\Gal(K_{ab}/H_s), D^{\text{ord},\dagger}_{m,n_f})$. As usual, the maps $\tilde{\alpha}_m : \tilde{X}_m \to \tilde{X}_{m-1}$ induce maps

\[\tilde{\alpha}_m = \tilde{\alpha}_{m,s} : H^0(\Gal(K_{ab}/H_{cp^n}), D^{\text{ord},\dagger}_{m,n_f}) \to H^0(\Gal(K_{ab}/H_{cp^n}), D^{\text{ord},\dagger}_{m-1,n_f})\]

by covariant functoriality.

Proposition 6.1. The equality

\[\tilde{\alpha}_m \left( \Cor_{H_{cp^n}/H_{cp^{m-1}}} \left( \tilde{P}^{\text{ord}}_{c,m} \right) \right) = U_p(\tilde{P}^{\text{ord}}_{c,m-1})\]

holds in $H^0(\Gal(K_{ab}/H_{cp^{m-1}}), D^{\text{ord},\dagger}_{m-1,n_f})$ for all $m \geq 1$.

Proof. Since $H_{cp^n}$ and $H_{cp^{m-1}}(\mu_{p^n})$ are linearly disjoint over $H_{cp^{m-1}}$, we can fix a finite set $S_m \subset \Gal(K_{ab}/H_{cp^{m-1}})$ of extensions of the elements in $\Gal(H_{cp^n}/H_{cp^{m-1}})$ such that every $\sigma \in S_m$ acts trivially on $\mu_{p^n}$. Applying $e_{k+j-2}\text{ord}$ to the equation of Proposition 3.15 yields the analogous relation

\[\tilde{\alpha}_m \left( \sum_{\sigma \in S_m} (\tilde{P}^{\text{ord}}_{c,m})^\sigma \right) = U_p(\tilde{P}^{\text{ord}}_{c,m-1})\]

in $D^{\text{ord}}_{m-1}$. Now we calculate the corestriction $\Cor_{H_{cp^n}/H_{cp^{m-1}}}$ by choosing the elements $\tilde{\eta}$ of (50) in $S_m$, and this yields precisely

\[\tilde{\alpha}_m \left( \Cor_{H_{cp^n}/H_{cp^{m-1}}} \left( \tilde{P}^{\text{ord}}_{c,m} \right) \right) = \tilde{\alpha}_m \left( \sum_{\sigma \in S_m} (\tilde{P}^{\text{ord}}_{c,m})^\sigma \right).\]

The result follows.

Define

\[\mathcal{P}^{\text{ord}}_{c,m} := \Cor_{H_{cp^n}/H_c} \left( \tilde{P}^{\text{ord}}_{c,m} \right) \in H^0(\Gal(K_{ab}/H_c), D^{\text{ord},\dagger}_{m,n_f}).\]

Corollary 6.2 (Hecke relations). The equality

\[\tilde{\alpha}_m(\mathcal{P}^{\text{ord}}_{c,m}) = U_p(\mathcal{P}^{\text{ord}}_{c,m-1})\]

holds in $H^0(\Gal(K_{ab}/H_c), D^{\text{ord},\dagger}_{m-1,n_f})$ for all $m \geq 1$. 

Proof. Straightforward from Proposition 6.1 on applying Corollary 2.3.5. □

6.3. Definition of big Heegner points. Thanks to Corollary 6.2 and the isomorphism
\[
\lim_{m} H^0(\text{Gal}(K_{\text{ab}}^\chi/H_c), D_{m,\eta_f}^{\text{ord},\dagger}) \simeq H^0(\text{Gal}(K_{\text{ab}}^\chi/H_c), D_{\infty,\eta_f}^{\text{ord},\dagger}),
\]
the following definition makes sense.

**Definition 6.3.** The big Heegner point of conductor \(c\) is the element
\[
P_c := \lim_{m} U_{p}^{-m}(\mathcal{P}_{c,m}^{\text{ord}}) \in H^0(\text{Gal}(K_{\text{ab}}^\chi/H_c), D_{\infty,\eta_f}^{\text{ord},\dagger}).
\]

The canonical map \(\pi_c : D_{\infty,\eta_f}^{\text{ord},\dagger} \to D_{\infty,\eta_f}^{\text{ord},\dagger}, x \mapsto x \otimes 1\) induces a map in cohomology
\[
\pi_c : H^0(\text{Gal}(K_{\text{ab}}^\chi/H_c), D_{\infty,\eta_f}^{\text{ord},\dagger}) \to H^0(\text{Gal}(K_{\text{ab}}^\chi/H_c), D_{\infty,\eta_f}^{\text{ord},\dagger})
\]
which is denoted by the same symbol.

**Definition 6.4.** The cohomology class
\[
P_{c,C} := \pi_c(P_c) \in H^0(\text{Gal}(K_{\text{ab}}^\chi/H_c), D_{\infty,\eta_f}^{\text{ord},\dagger})
\]
is the \(C\)-component of \(P_c\).

6.4. Big Heegner classes in the indefinite case. Suppose we are in the *indefinite* case. Let \(H_c^{(Np)}\) be the maximal extension of \(H_c\) unramified outside \(Np\) and set
\[
G_c^{(Np)} := \text{Gal}(H_c^{(Np)}/H_c).
\]
Define a twisted Kummer map
\[
\delta_m : H^0(H_c, J_{m,\eta_f}^{\text{ord},\dagger}(H_{\text{cp}^m}(\mu_{p^m}))) \to H^1(G_c^{(Np)}, T_{m,\eta_f}^{\text{ord},\dagger})
\]
as in [24, p. 101]. Write
\[
\tilde{P}_{c,m}^{\text{ord}} \in H^0(H_c, J_{m,\eta_f}^{\text{ord},\dagger}(H_{\text{cp}^m}(\mu_{p^m})))
\]
for the image of \(\mathcal{P}_{c,m}^{\text{ord}} \in H^0(\text{Gal}(K_{\text{ab}}^\chi/H_c), D_{m,\eta_f}^{\text{ord},\dagger})\) under the natural map, and set
\[
\kappa_{c,m}^{\text{ord}} := \delta_m(\tilde{P}_{c,m}^{\text{ord}}).
\]
Because of the equivariance of the Kummer map with respect to the action of the Hecke operator \(U_p\), the Hecke relations of Corollary 6.2 imply the corresponding relations
\[
(51)
\]
between the classes \(\kappa_{c,m}^{\text{ord}}\). Here the map
\[
\tilde{\alpha}_m : H^1(G_c^{(Np)}, T_{m,\eta_f}^{\text{ord},\dagger}) \to H^1(G_c^{(Np)}, T_{m-1,\eta_f}^{\text{ord},\dagger})
\]
is induced by covariant functoriality by the map \(\tilde{\alpha}_m\) of diagram (51).

**Lemma 6.5.** There is an isomorphism of \(E_{\infty,\eta_f}^{\text{ord},\dagger}\)-modules
\[
\lim_{m} H^1(G_c^{(Np)}, T_{m,\eta_f}^{\text{ord},\dagger}) \simeq H^1(G_c^{(Np)}, T_{\infty,\eta_f}^{\text{ord},\dagger}).
\]

Proof. By [38, Corollary 2.3.5], it suffices to show that the groups \(H^0(G_c^{(Np)}, T_{m,\eta_f}^{\text{ord},\dagger})\) are finite for all \(m\). But the \(G_c^{(Np)}\)-module \(T_{m,\eta_f}^{\text{ord},\dagger}\) is isomorphic to the Galois representation associated with any modular form \(g\) that factors through \(\eta_f\), and the claim follows. □
Thanks to (51) and Lemma 6.5, we can give the following

**Definition 6.6.** The big Heegner class of conductor $c$ is the element

$$\kappa_c := \lim_{\leftarrow} U_p^{-m}(\kappa_{c,m}^{\text{ord}}) \in H^1(G_c^{(Np)}, T_{\text{ord},c}^{\text{ord},\dagger})$$

Put $H_{c,p}^{\infty}(\mu_p^{\infty}) := \cup_{m \geq 1} H_{c,p}^m(\mu_p^{m})$. By Lemma 6.5, taking the inverse limit with respect to the maps $\delta_m$ yields a twisted Kummer map

$$\delta_{\infty} : H^0(G_c^{(Np)}, J_{\text{ord},c}^{\text{ord},\dagger}(H_{c,p}^{\infty}(\mu_p^{\infty}))) \longrightarrow H^1(G_c^{(Np)}, T_{\text{ord},c}^{\text{ord},\dagger})$$

Write

$$\tilde{P}_c \in H^0(Gal(K^{ab}/H_c), J_{\text{ord},c}^{\text{ord},\dagger}(H_{c,p}^{\infty}(\mu_p^{\infty})))$$

for the image of $\mathcal{P}_c \in H^0(Gal(K^{ab}/H_c), D_{\text{ord},c}^{\text{ord},\dagger})$ under the natural map. The next lemma will be used in the proof of Corollary 7.2.

**Lemma 6.7.** $\delta_{\infty}(\tilde{P}_c) = \kappa_c$.

*Proof.* Recall that, by definition, $\delta_m(\tilde{P}_{c,m}^{\text{ord}}) = \kappa_{c,m}^{\text{ord}}$ and pass to the inverse limit over $m$. \qed

As above, the canonical map $\pi_c : T_{\text{ord},c}^{\text{ord},\dagger} \rightarrow T_{\text{ord},c}^{\text{ord},\dagger}$, $x \mapsto x \otimes 1$ induces a map in cohomology

$$\pi_c : H^1(G_c^{(Np)}, T_{\text{ord},c}^{\text{ord},\dagger}) \longrightarrow H^1(G_c^{(Np)}, T_{\text{ord},c}^{\text{ord},\dagger})$$

which is denoted by the same symbol.

**Definition 6.8.** The cohomology class

$$\kappa_{c,C} := \pi_c(\kappa_c) \in H^1(G_c^{(Np)}, T_{\text{ord},c}^{\text{ord},\dagger})$$

is the $C$-component of $\kappa_c$.

**Remark 6.9.** As will become clear in the next sections, our big Heegner points and classes satisfy the same Euler system relations as those defined by Howard in [21, §2.2]. Note, however, the (slight) difference in terminology: the big Heegner points of Howard correspond to our indefinite big Heegner classes of Definition 6.6 while our big Heegner points of Definition 6.3 are the counterparts of the points denoted by $x_{c,s}$ in [24, p. 101].

**Remark 6.10.** In the special case where $N^- = 1$ (i.e., when $B \simeq M_2(\mathbb{Q})$) we expect that our system of big Heegner classes essentially coincides with the system of big Heegner points considered by Howard in [24]. On the contrary, we have not investigated the existence of an explicit relation between our indefinite cohomology classes and the specialization to the base field $F = \mathbb{Q}$ of the ones introduced by Fouquet in [11].

### 7. Euler system relations

This section is devoted to the proof of the “Euler system” relations satisfied by the classes $\tilde{P}_c^{\text{ord}}$ and $P_c^{\text{ord}}$ introduced above. The formulas obtained, which are the counterparts in our definite/indefinite quaternionic setting of the results in [24, §2.3], will be used in §10.2 to control the size of certain Selmer groups.
7.1. The operator $U_p$. We begin with an analysis of the action of the Hecke operator $U_p$.

**Proposition 7.1.** For all $m \geq 1$ the equality

$$\text{Cor}_{H_{cp,m+1}/H_{cp,m}}(\tilde{P}_{c,m}^{\text{ord}}) = U_p(\tilde{P}_{c,m}^{\text{ord}}).$$

holds in $H^0(\text{Gal}(K^{ab}/H_{cp,m}), D_{m,\eta}^{\text{ord,}^\dagger})$.

**Proof.** The proof is similar to that of Proposition 6.1. Applying $e_{k+j-2}c^{\text{ord}}$ to the equation of Proposition 3.17 yields the analogous relation

$$\sum_{\sigma \in S_m} (\tilde{P}_{c,m}^{\text{ord}})^{\sigma} = U_p(\tilde{P}_{c,m}^{\text{ord}})$$

in $D_{m,\eta}^{\text{ord,}^\dagger}$, and calculating corestriction $\text{Cor}_{H_{cp,m+1}/H_{cp,m}}$ by choosing the elements $\tilde{\eta}$ of (50) in $S_{m+1}$ gives the result. \hfill $\square$

**Corollary 7.2.** The following relations hold:

1. $\text{Cor}_{H_{cp}/H_c}(\mathcal{P}^{\text{ord}}_{cp,m}) = U_p(\mathcal{P}^{\text{ord}}_{c,m})$ in $H^0(\text{Gal}(K^{ab}/H_c), D_{m,\eta}^{\text{ord,}^\dagger})$;
2. $\text{Cor}_{H_{cp}/H_c}(\mathcal{P}_{cp}) = U_p(\mathcal{P}^{\text{ord}}_{c,m})$ in $H^0(\text{Gal}(K^{ab}/H_c), D_{\infty,\eta}^{\text{ord,}^\dagger})$;
3. $\text{Cor}_{H_{cp}/H_c}(\mathcal{P}^{\text{ord}}_{cp}) = U_p(\mathcal{P}^{\text{ord}}_{c,m})$ in $H^1(\text{Gal}(N_{\ell}), \mathcal{T}_{\text{ord,}^\dagger})$.

**Proof.** Relation (1) follows easily from Proposition 7.1 and the equality $\text{Cor}_{H_{cp}/H_c} \circ \text{Cor}_{H_{cp,m+1}/H_{cp,m}} = \text{Cor}_{H_{cp,m+1}/H_{cp,m}}$. Relation (2) follows from (1) by passing to the inverse limit over $m$. Finally, relation (3) is a consequence of Lemma 6.7 and the equivariance of the twisted Kummer map with respect to the action of $U_p$. \hfill $\square$

7.2. The operators $T_\ell$. We briefly review the standard description of the $T_\ell$ operator in the case of our interest. Let $m \geq 0$ be an integer and let $\ell$ be a prime number which does not divide $N_{p}^m$. In particular, the case $\ell = p$ and $m = 0$ is allowed. The action of the Hecke operator $T_\ell$ on $\text{Pic}(X_m)$ and $\text{Pic}(\bar{X}_m)$ can be described as follows. For all $a \in \{0, \ldots, \ell - 1\}$ denote $\lambda_a$ the idele whose $\ell$-component is equal to $\left(\begin{smallmatrix} 1 & a \\ 0 & \ell \end{smallmatrix} \right)$ and whose components at all other places are equal to 1. Let $\lambda_\infty$ be the adele whose $\ell$-component is equal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix} \right)$ and all other components are 1. Then

$$\mathcal{R}^{\times \ell}_m \lambda_a \mathcal{R}^{\times \ell}_m = \bigcup_{a=0}^{\ell-1} \mathcal{R}^{\times \ell}_m \lambda_a \cap \mathcal{R}^{\times \ell}_m \lambda_\infty, \quad U_m \lambda_0 U_m = \bigcup_{a=0}^{\ell-1} U_m \lambda_a \cup U_m \lambda_\infty.$$

Analogously to the classical, non-adelic situation (see, e.g., [46, Ch. 3]), the operator $T_\ell$ on $\text{Pic}(X_m)$ and $\text{Pic}(\bar{X}_m)$ can be defined as

$$T_\ell([((g,f))]) := \sum_{a=1}^{\ell} \left[ (\lambda_a g, f) \right] + \left[ (\lambda_\infty g, f) \right].$$

Define $\mathcal{R}^{(\ell)}_m := \{ x \in \mathcal{R}_m \mid x_\ell = 1 \}$ and $\Delta_{m,\ell} := B^{\times} \cap \mathcal{R}^{(\ell)}_m \times \text{GL}_2(\mathbb{Q}_\ell)$. By the strong approximation theorem,

$$\mathcal{R}^{\times \ell}_m \backslash (\hat{B}^{\times} \times \mathbb{P}) / B^{\times} = \text{GL}_2(\mathbb{Z}_p) \backslash (\text{GL}_2(\mathbb{Q}_p) \times \mathbb{P}) / \Delta_{m,\ell} = (\mathcal{V}(T_\ell) \times \mathbb{P}) / \Delta_{m,\ell}.$$
where $\mathcal{V}(T_\ell)$ is the set of vertices of the Bruhat-Tits tree $T_\ell$ of $\text{PGL}_2(\mathbb{Q}_\ell)$, which can be canonically identified with $\text{PGL}_2(\mathbb{Z}_\ell) \setminus \text{PGL}_2(\mathbb{Q}_\ell)$. The Hecke operator $T_\ell$ acting on a point $P = [(g, z)]$ can be described as follows. Let $P$ correspond to a pair $(v, z) \in \mathcal{V}(T_\ell) \times \mathbb{P}$ under the above isomorphism; then $T_\ell(P)$ can be represented by the sum $\sum_{i=0}^\ell (v_i, z)$, where the $v_i$ for $i = 0, \ldots, \ell$ are the vertices joined to $v$.

Let $c$ be an integer prime to $N\ell$. By the choice of the local embedding $\phi_\ell$ made in \textsection 3.1, the Heegner points $P_{c^m, m}$ as $n$ varies over the set of natural numbers can be represented by pairs $(v_n, f') \in \mathcal{V}(T_\ell) \times \text{Hom}(K, B)$ such that the sequence of vertices $v_n$ form a path of length $n$ with no backtracking. By \cite{[1]} \textsection 2.4, for every integer $n \geq 1$ there is an equality

$$
\text{Tr}_{H_{c^{n+1}, m}/H_{c^m}}(P_{c^{n+1}, m}) = T_\ell(P_{c^m, m}) - P_{c^{n-1}, m}.
$$

More precisely, the action of $T_\ell$ on $[(v_n, f')]$ can be represented as the sum of the points $[(v, f')]$ where $v$ varies over the vertices which are connected to $v_n$ by an edge.

Let $u$ denote the order of $\mathcal{O}_{c^m}/\{\pm 1\}$. If $\ell$ splits in $K$ let $l_1$ and $l_2$ be the prime ideals of $\mathcal{O}_K$ above $\ell$ and denote by $\sigma_{l_1}$ and $\sigma_{l_2}$ two Frobenius elements in $\text{Gal}(\overline{K}/K)$ corresponding to $l_1$ and $l_2$. By \cite{[1]} \textsection 2.4,

$$
u \text{Tr}_{H_{c^m}}(P_{c^m, m}) = \begin{cases} T_\ell(P_{c^m, m}) & \text{if } \ell \text{ is inert in } K, \\ (T_\ell - \sigma_{l_1} - \sigma_{l_2})(P_{c^m, m}) & \text{if } \ell \text{ is split in } K, \end{cases}
$$

as divisors of $X_m$. In this case too, the action of $T_\ell$ can be more explicitly represented as the sum of the points $[(v, f')]$ where $v$ varies over the vertices which are connected to $v_n$ by an edge in the inert case and the sum of the points $[(v, f')]$ where $v$ varies over the same set minus two vertices in the split case.

Define $U_m^{(\ell)} := \{ x \in U_m \mid x_\ell = 1 \}$ and $\Delta_{m, \ell} := B^\times \cap U_m^{(\ell)} \times \text{GL}_2(\mathbb{Q}_\ell)$. By the strong approximation theorem,

$$
U_m \setminus (\overline{B}^\times \times \mathbb{P}) / B^\times = \text{GL}_2(\mathbb{Z}_p) \setminus (\text{GL}_2(\mathbb{Q}_p) \times \mathbb{P}) / \Delta_{m, \ell} = (\mathcal{V}(T_\ell) \times \mathbb{P}) / \Delta_{m, \ell}.
$$

Since the fields $H_{c^{n+1}, m}$ and $H_{c^n, m}(\mu_{c^m})$ are linearly disjoint over $H_{c^n, m}$, the projection

$$
\text{Gal}(H_{c^{n+1}, m}(\mu_{c^m})/H_{c^n, m}(\mu_{c^m})) \cong \text{Gal}(H_{c^{n+1}, m}/H_{c^n, m})
$$

is an isomorphism. It follows from this and the above description of $T_\ell$ that for every integer $n \geq 1$ there are equalities

$$
\text{Tr}_{H_{c^{n+1}, m}}(\mu_{c^m})/H_{c^n, m}(\mu_{c^m})(P_{c^{n+1}, m}) = T_\ell(P_{c^m, m}) - P_{c^{n-1}, m}
$$

and

$$
u \text{Tr}_{H_{c^{m}}}(P_{c^m, m}) = \begin{cases} T_\ell(P_{c^m, m}) & \text{if } \ell \text{ is inert in } K, \\ (T_\ell - \sigma_{l_1} - \sigma_{l_2})(P_{c^m, m}) & \text{if } \ell \text{ is split in } K, \end{cases}
$$

of divisors on $X_m$.

**Proposition 7.3.** Let $\ell \nmid cNp$ be a prime number which is inert in $K$. With notation as above, for all $m \geq 1$ the equality

$$
u \text{Cor}_{H_{c^{m}}/H_{c^{m}}}(P_{c^{m}}^{\text{ord}}) = T_\ell(P_{c^m, m}^{\text{ord}})
$$

holds in $H^0(\text{Gal}(K^{ab}/H_c), D_{m, n}^{\text{ord}, \dagger})$. 

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Proof. Similar to that of Proposition 7.1. Choose a set $S \subseteq \text{Gal}(\mathbb{C}/H_{c\ell,p^m})$ of extensions of $\text{Gal}(H_{c\ell,p^m}/H_{c\ell})$ such that each $\sigma \in S$ acts trivially on $\mu_{p^\infty}$. From the above equations it follows that

$$\sum_{\sigma \in S} (\widetilde{P}_{c\ell,m})^\sigma = T_\ell(\widetilde{P}_{c,m})$$

in $D^\text{ord,\dag}_{m,n_\ell}$. Applying $c_{k+j-2}c_{p^\text{ord}}$ and calculating $\text{Cor}_{H_{c\ell,p^m}/H_{c\ell}}$ by choosing the elements $\tilde{\eta}$ of (50) in $S$ yields the desired result.

**Corollary 7.4.** Let $\ell \nmid cN_p$ be a prime number which is inert in $K$. With notation as above, the following relations hold:

1. $u\text{Cor}_{H_{c\ell}/H_c}(P_{c\ell,m}^\text{ord}) = T_\ell(P_{c,m}^\text{ord})$ in $H^0(\text{Gal}(K^{ab}/H_c), D^\text{ord,\dag}_{m,n_\ell})$;
2. $u\text{Cor}_{H_{c\ell}/H_c}(P_{c\ell}) = T_\ell(P_c)$ in $H^0(\text{Gal}(K^{ab}/H_c), D^\text{ord,\dag}_{\infty,n_\ell})$;
3. $\text{Cor}_{H_{c\ell}/H_c}(\kappa_{c\ell}^\text{ord}) = T_\ell(\kappa_c^\text{ord})$ in $H^1(\mathcal{G}(K^{(N_p)}), \tau^\text{ord,\dag}_{\infty,n_\ell})$.

**Proof.** Same proof as for Corollary 7.2 but this time to obtain relation 1 one uses the equality

$$\text{Cor}_{H_{c\ell}/H_c} \circ \text{Cor}_{H_{c\ell,p^m}/H_{c\ell}} = \text{Cor}_{H_{c\ell,p^m}/H_c} \circ \text{Cor}_{H_{c\ell,p^m}/H_{c\ell}},$$

and for relation 3 one uses the equivariance of the twisted Kummer map with respect to the action of $T_\ell$. □

### 7.3. The Eichler–Shimura congruence relation

Let $\ell \nmid cN_p$ be a prime which is inert in $K$. By class field theory, $\ell$ splits completely in the extension $H_c/K$. Fix a prime $\lambda$ of $H_c$ above $\ell$. Note that $\lambda$ is totally ramified in $H_{c\ell}$, so $\mathcal{O}_{H_{c\ell}} = \lambda^{e_\ell+1}$ for a prime ideal $\widetilde{\lambda}$ of the ring of integers $\mathcal{O}_{H_{c\ell}}$ above $\ell$. For every prime number $q$ and every integer $k \geq 1$ denote by $\mathbb{F}_{q^k}$ the field with $q^k$ elements, and for every number field $H$ and every prime ideal $q$ of the ring of integers of $H$ denote by Frobenius a Frobenius element at $q$ and by $\mathbb{F}_{H,q}$ the residue field of $H$ at $q$.

Then there are canonical isomorphisms

$$\mathbb{F}_{e_\ell} \simeq \mathbb{F}_{K,\ell} \simeq \mathbb{F}_{H_c,\lambda} \simeq \mathbb{F}_{H_{c\ell},\lambda}.$$  

Write $\widetilde{X}_{m,\ell}$ for the canonical (smooth, proper) integral model of $\widetilde{X}_m$ over $\mathbb{Z}_\ell$. By the valuative criterion of properness, any point $x \in \widetilde{X}_m$ extends uniquely to a point in $\widetilde{X}_{m,\ell}$, which will be denoted in the same fashion.

**Lemma 7.5.** Let $\ell \nmid cN_p$ be a prime number which is inert in $K$. Suppose that the group $\mathcal{O}_{c\ell,p^m}^\times/\{\pm 1\}$ is trivial. Then

$$\widetilde{P}_{c\ell,m} \equiv \text{Frob}_\lambda(\widetilde{P}_{c,m}) \pmod{\widetilde{\lambda}}$$

as points in $\widetilde{X}_{m,\ell}$.

**Proof.** Choose $S_m$ as in the proof of Proposition 7.3. For any $\sigma \in S$, there is a congruence

$$(\widetilde{P}_{c\ell,m})^\sigma \equiv \widetilde{P}_{c\ell,m} \pmod{\widetilde{\lambda}}$$

because $\lambda$ is totally ramified in $H_{c\ell}$. Hence, by (54), it follows that

$$T_\ell(\widetilde{P}_{c,m}) \equiv (\ell + 1)\widetilde{P}_{c\ell,m} \pmod{\widetilde{\lambda}}.$$  

The Eichler–Shimura congruence relation $T_\ell = \text{Frob}_p + \text{Frob}_p^\ast$ (mod $\ell$) shows that at least one of the points in the divisor $T_\ell(\widetilde{P}_{c,m})$ is congruent to $\text{Frob}_\lambda(\widetilde{P}_{c,m})$ modulo $\widetilde{\lambda}$. Thus the same holds for all the points in the divisor $T_\ell(\widetilde{P}_{c,m})$, and in particular for $\widetilde{P}_{c\ell,m}$. □
Remark 7.6. See \cite{15} Proposition 3.7 for the same argument applied in the context of Heegner points on (classical) modular curves.

**Proposition 7.7.** Let $\ell \nmid cpNp$ be a prime number which is inert in $K$. Suppose that the group $O_{\mathcal{O}^E}_{\mathcal{O}^p}/\{\pm 1\}$ is trivial. Then $\kappa_{\text{ord}}^{\ell}$ and $\text{Frob}_\lambda(\kappa_{\text{ord}}^{\ell})$ have the same image in $H^1(G_{c(Np)}, T_{\text{ord}, \mathcal{T}})$.

**Proof.** Proceed as in the proof of \cite{24} Proposition 2.3.2, using Lemma 7.5. \hfill \Box

**Part 3. Arithmetic applications and conjectures**

From here to the end of the paper, fix a modular form $f$ of weight $k$ as in (28) and let $f_{\infty} : h_{\infty}^{\text{ord}} \rightarrow A$ be the primitive morphism associated with $f$. If $p$ is an arithmetic prime of $A$ then $f_p$ is the modular form introduced in (33).

For the reader’s convenience, in the sequel we simplify our notation and write $\mathcal{R}$ for any one of the isomorphic rings $A$, $B$ of \cite{13} and $C$ of \cite{30} that we have associated with $f$: it will be clear from the context which one of these rings is meant. We explicitly highlight the properties of $\mathcal{R}$ that will be relevant for the sequel:

- $\mathcal{R}$ is a complete noetherian local domain which is finitely generated as a $\Lambda$-module (Proposition 4.6);
- if $p$ is an arithmetic prime of $\mathcal{R}$ and $P := p \cap \Lambda$ then $\Lambda_P \subset \mathcal{R}_p$ is an unramified extension of discrete valuation rings (see \cite{18} Corollary 1.4 or \cite{36} §12.7.5).

The purpose of the following sections is to apply our constructions of big Heegner points and classes to various arithmetic situations. While so far we have strived to adopt a uniform approach to the definite and indefinite cases, at this point it is inevitable to distinguish between these two settings. In fact, intuitively speaking, the philosophy behind the so-called “parity conjectures” suggests that the definite case deals with even rank (most typically, rank zero) situations while the indefinite case takes care of odd rank (most notably, rank one) contexts.

8. **Big Heegner points and Selmer groups: the definite case**

Throughout this section (and the next) we assume that we are in the definite case, i.e. that the quaternion algebra $B$ is definite.

8.1. **Algebraic results.** To simplify the above notation, set

$$D := D_{\infty, \mathcal{R}}^{\text{ord}}, \quad D^{\dag} := D_{\infty, \mathcal{R}}^{\text{ord}, \dag}, \quad J := J_{\infty, \mathcal{R}}^{\text{ord}}.$$

Let $m \geq 0$ be an integer. Since $\tilde{X}_m$ is a disjoint union of $\tilde{h}(m)$ curves of genus zero, we can (and do) fix an isomorphism of $\mathcal{O}_F$-modules

$$H_0(\tilde{X}_m(\mathbb{C}), \mathcal{O}_F) \simeq J_m,$$

where $H_*$ denotes singular homology. The above isomorphism endows $H_0(\tilde{X}_m(\mathbb{C}), \mathcal{O}_F)$ with a canonical Hecke action. Passing to the ordinary parts, one thus obtains an isomorphism of Hecke modules

$$H_0^{\text{ord}}(\tilde{X}_m(\mathbb{C}), \mathcal{O}_F) := e_m^{\text{ord}} \cdot H_0(\tilde{X}_m(\mathbb{C}), \mathcal{O}_F) \simeq J_m^{\text{ord}}. \quad (55)$$
The cohomology module $H^0(\tilde{X}_m(\mathbb{C}), F/O_F)$ with coefficients in the $p$-divisible group $F/O_F$ is also equipped with a canonical Hecke action, and its ordinary part is defined in the usual way. For any $O_F$-module $M$ let

$$M^* := \text{Hom}_{O_F}(M, F/O_F)$$

denote its Pontryagin dual, with induced Hecke action whenever $M$ is a module over the Hecke algebra. Then (see, e.g., [21 §1.9]) there is a canonical isomorphism of Hecke modules

$$H^0(\tilde{X}_m(\mathbb{C}), F/O_F)^* \simeq H_0(\tilde{X}_m(\mathbb{C}), O_F)$$

which induces an isomorphism of $O_F$-modules

$$(56)\quad H^0_{\text{ord}}(\tilde{X}_m(\mathbb{C}), F/O_F)^* \simeq H^0_{\text{ord}}(\tilde{X}_m(\mathbb{C}), O_F).$$

Following [19, Definition 8.5], set

$$V := \lim_{m} H^0_{\text{ord}}(\tilde{X}_m(\mathbb{C}), F/O_F), \quad V := V^*.$$

Then (55) and (56) yield isomorphisms of $O_F$-modules

$$(57)\quad V = \text{Hom}_{O_F}\left(\lim_{m} H^0_{\text{ord}}(\tilde{X}_m(\mathbb{C}), F/O_F), F/O_F\right) \simeq \lim_{m} H^0_{\text{ord}}(\tilde{X}_m(\mathbb{C}), F/O_F)^* \simeq \lim_{m} H^0_{\text{ord}}(\tilde{X}_m(\mathbb{C}), O_F) \simeq \lim_{m} J^0_m = J^0_{\infty}.$$

Recall that $\Gamma := 1 + p\mathbb{Z}_p$ and define $\Gamma_m := 1 + p^m\mathbb{Z}_p$; in particular,

$$\Lambda := O_F[\Gamma] = \lim_{m} O_F[\Gamma/\Gamma_m].$$

The group $\Gamma$ acts on $\hat{B}^\times$ via multiplication on the $p$-component, and this induces an $O_F$-linear action of $\Gamma/\Gamma_m$ on $J_m$. Thus $J^0_{\infty}$ is endowed with an action of $\Lambda$ which is, of course, the one induced by its $\mathbb{B}^0_{\infty}$-module structure. Furthermore, the isomorphism (57) is $\Lambda$-equivariant, the structure of $\Lambda$-module of $V$ being defined as in [19 §9]. By [19 Corollary 10.4], the $\Lambda$-modules $V$ and $J^0_{\infty}$ are free of finite rank. It follows immediately that $J^0_{\infty}$ is a finitely generated $\mathbb{B}^0_{\infty}$-module, hence $J$ is a finitely generated $\mathcal{R}$-module. If $p$ (respectively, $P$) is an arithmetic prime of $\mathcal{R}$ (respectively, $\Lambda$) and $M$ is an $\mathcal{R}$-module (respectively, a $\Lambda$-module) then we set $M_p := M \otimes_{\mathcal{R}} \mathcal{R}_p$ (respectively, $M_P := M \otimes_{\Lambda} \Lambda_P$), where $\mathcal{R}_p$ (respectively, $\Lambda_P$) is the localization of $\mathcal{R}$ at $p$ (respectively, of $\Lambda$ at $P$). To lighten the notation, put also $\mathbb{B} := \mathbb{B}^0_{\infty}$.

**Proposition 8.1.** Let $p$ be an arithmetic prime of $\mathcal{R}$. The $\mathcal{R}_p$-module $J_p$ is free of rank one.

**Proof.** By [19 Theorem 12.1], there are isomorphisms of $\mathbb{B}_P$-modules $V_P \simeq \mathbb{B}_P$ for all arithmetic primes $P$ of $\Lambda$. Since there are isomorphisms of $\mathcal{R}_P$-modules $J_P \simeq V_P \otimes_{\mathbb{B}} \mathcal{R}$ and $\mathcal{R}_P \simeq \mathbb{B}_P \otimes_{\mathbb{B}} \mathcal{R}$, we conclude that

$$(58)\quad J_P \simeq \mathcal{R}_P$$

as $\mathcal{R}_P$-modules. Fix an arithmetic prime $p$ of $\mathcal{R}$ and let $P := p \cap \Lambda$ be the arithmetic prime of $\Lambda$ which lies below $p$. There is a canonical map of rings $\mathcal{R}_P \to \mathcal{R}_p$ defined by the composition

$$\mathcal{R}_P := \mathcal{R} \otimes_{\Lambda} \Lambda_P \to \mathcal{R} \otimes_{\Lambda} \mathcal{R}_p \to \mathcal{R} \otimes_{\mathcal{R}} \mathcal{R}_p = \mathcal{R}_p.$$
There are isomorphisms of $\mathcal{R}$-modules
\[
J_P \otimes_{\mathcal{R}_p} \mathcal{R}_p = (V \otimes_{\mathcal{R}} B) \otimes_{\mathcal{R}_p} \mathcal{R}_p \\
\simeq (V \otimes_{\mathcal{R}_p} \mathcal{R}_p) \otimes_{\mathcal{R}_p} \mathcal{R}_p \\
\simeq V \otimes_{\mathcal{R}_p} \mathcal{R}_p \\
\simeq (V \otimes_{\mathcal{R}} \mathcal{R}) \otimes_{\mathcal{R}_p} \mathcal{R}_p = J \otimes_{\mathcal{R}} \mathcal{R}_p = J_p.
\] (59)

Furthermore, thanks to (58), $J_P \otimes_{\mathcal{R}_p} \mathcal{R}_p \simeq \mathcal{R}_p$ as $\mathcal{R}_p$-modules. Comparing this with (59) yields the result.

From here until the end of the section we make the following

**Assumption 8.2.** Let $m_\mathcal{R}$ be the maximal ideal of the local ring $\mathcal{R}$ and let $\mathbb{F}_\mathcal{R} := \mathcal{R}/m_\mathcal{R}$ be its residue field. The $\mathbb{F}_\mathcal{R}$-vector space $J/m_\mathcal{R}J$ has dimension one.

With this condition in force, we can prove

**Proposition 8.3.** The $\mathcal{R}$-module $J$ is free of rank one.

**Proof.** Since $J$ is finitely generated over $\mathcal{R}$ and Assumption 8.2 holds, Nakayama’s lemma ensures that there is a surjective homomorphism $\mathcal{R} \to J$ of $\mathcal{R}$-modules. If this map is not an isomorphism, there is a non-zero ideal $I \subset \mathcal{R}$ such that $\mathcal{R}/I \simeq J$ as $\mathcal{R}$-modules. By [29, Theorem 6.5], the localization $(\mathcal{R}/I)_p$ is non-zero only for a finite number of arithmetic primes $p$ of $\mathcal{R}$. Hence $J_p = 0$ for almost all arithmetic primes $p$, contradicting Proposition 8.1 (of course, the local vanishing at just one such prime $p$ would suffice). Thus $I$ is the zero ideal, and the proposition is proved. □

In light of Proposition 8.3 fix an isomorphism
\[
J \cong \mathcal{R}.
\] (60)
of $\mathcal{R}$-modules. Composing the canonical projection $D \to J$ with (60), we obtain a map
\[
D \to \mathcal{R}.
\] (61)

For any finite abelian extension $H/K$ there is a canonical map $H^0(\text{Gal}(K^{ab}/H), D^\dagger) \to D$, thus, by composing with (61), we obtain a map
\[
\eta_H : H^0(\text{Gal}(K^{ab}/H), D^\dagger) \to \mathcal{R}.
\] (62)

This map will be used in the next § to state our results on Selmer groups. In particular, [8.2 and 8.3 are motivated by [2] Theorems A and B] and [11 Corollary 4], respectively, where classical Heegner points on definite Shimura curves are used to control certain Selmer groups.

8.2. **Bounding Selmer groups.** To begin with, recall the class $\mathcal{P}_{1,\mathcal{R}} \in H^0(\text{Gal}(K^{ab}/H_1), D^\dagger)$ introduced in Definition 6.4 and set
\[
\mathcal{P}_\mathcal{R} := \text{Cor}_{H_1/K}(\mathcal{P}_{1,\mathcal{R}}) \in H^0(\text{Gal}(K^{ab}/K), D^\dagger).
\]

Then define
\[
\mathcal{J}_\mathcal{R} := \eta_{K}(\mathcal{P}_\mathcal{R}) \in \mathcal{R}.
\]

For an arithmetic prime $p$ of $\mathcal{R}$ set $F_p := \mathcal{R}_p/p\mathcal{R}_p$ and denote by $\pi_p : \mathcal{R} \to F_p$ the canonical map. We make two conjectures, the first of which predicts the non-vanishing of $\mathcal{J}_\mathcal{R}$.

**Conjecture 8.4.** $\mathcal{J}_\mathcal{R} \neq 0$.

The next one is, instead, a conjectural vanishing statement for Nekovár’s Selmer groups $\tilde{H}^1_f(K, V^\dagger_p)$.
**Conjecture 8.5.** For infinitely many non-exceptional arithmetic primes \( p \) of \( \mathcal{R} \) the following is true: if \( \pi_p(\mathcal{J}_R) \neq 0 \) then \( \tilde{H}_1^1(K, V_p) = 0 \).

We expect that Conjecture 8.5 for primes of weight two can be proved by extending the techniques and the results of [3] and [28] to the case of forms with non-trivial character. Taking the validity of the above two conjectures for granted, we can prove

**Theorem 8.6.** Assume Conjectures 8.4 and 8.5. The \( \mathcal{R} \)-module \( \tilde{H}_1^1(K, \mathbf{T}^\dagger) \) is torsion.

**Proof.** By Conjecture 8.4, the element \( \mathcal{J}_R \) is non-zero in the integral domain \( \mathcal{R} \), so it is non-torsion. Hence [24, Lemma 2.1.7] implies that \( \mathcal{J}_R \notin p\mathcal{R}_p \) for all but finitely many arithmetic primes \( p \) of \( \mathcal{R} \). Thus \( \pi_p(\mathcal{J}_R) \neq 0 \) for almost all arithmetic primes \( p \) of \( \mathcal{R} \), hence, since we are assuming Conjecture 8.5, we get that \( \tilde{H}_1^1(K, V_p) = 0 \) for infinitely many arithmetic primes \( p \). If \( p \subset \mathcal{R} \) is an arithmetic prime then there is a short exact sequence

\[
0 \rightarrow \tilde{H}_1^1(K, \mathbf{T}^\dagger)_p / p\tilde{H}_1^1(K, \mathbf{T}^\dagger)_p \rightarrow \tilde{H}_1^1(K, V_p^\dagger) \rightarrow \tilde{H}_2^1(K, \mathbf{T}^\dagger)_p[p] \rightarrow 0
\]

(see the proof of [24, Corollary 3.4.3]), which shows that

\[
(63) \quad \tilde{H}_1^1(K, \mathbf{T}^\dagger)_p / p\tilde{H}_1^1(K, \mathbf{T}^\dagger)_p = 0
\]

for infinitely many arithmetic primes \( p \) of \( \mathcal{R} \). As pointed out at the beginning of the proof of loc. cit., the \( \mathcal{R} \)-module \( \tilde{H}_1^1(K, \mathbf{T}^\dagger) \) is finitely generated, hence if some \( x \in \tilde{H}_1^1(K, \mathbf{T}^\dagger) \) were non-torsion then, by [24, Lemma 2.1.7], we would have that

\[
x \notin p\tilde{H}_1^1(K, \mathbf{T}^\dagger)_p
\]

for all but finitely many arithmetic primes \( p \). This contradicts (63), whence \( \tilde{H}_1^1(K, \mathbf{T}^\dagger) \) is \( \mathcal{R} \)-torsion. \( \square \)

For completeness, we also formulate (in Conjecture 8.7) a natural extension of Conjecture 8.5 and Theorem 8.6 that takes care of eigenspaces relative to anticyclotomic characters. Recall the class

\[
\mathcal{P}_{c, \mathcal{R}} \in H^0(\text{Gal}(K^{ab}/H_c), \mathbf{D}^\dagger)
\]

of Definition 6.4. For every \( \sigma \in \hat{G}_c := \text{Gal}(H_c/K) \) set \( \mathcal{J}_{c, \mathcal{R}}^\sigma := \eta_{H_c}(\mathcal{P}_{c, \mathcal{R}}^\sigma) \in \mathcal{R} \). Then define

\[
\mathcal{D}_{c, \mathcal{R}} := \sum_{\sigma \in \hat{G}_c} \mathcal{J}_{c, \mathcal{R}}^\sigma \otimes \sigma^{-1} \in \mathcal{R}[\hat{G}_c].
\]

Fix a character \( \chi : \hat{G}_c \rightarrow \mathcal{O}^\times \) where \( \mathcal{O} \) is a finite extension of \( \mathcal{O}_F \). After enlarging \( F \) if necessary, without loss of generality we can (and do) assume that \( \mathcal{O} = \mathcal{O}_F \). Extend \( \chi \) to an \( \mathcal{R} \)-linear homomorphism \( \chi : \mathcal{R}[\hat{G}_c] \rightarrow \mathcal{R} \), then define

\[
(64) \quad \mathcal{L}(f_\infty/K, \chi) := \chi(\mathcal{D}_{c, \mathcal{R}}) \in \mathcal{R}
\]

and

\[
(65) \quad \mathcal{L}(f_\infty/K, \chi, p) := \pi_p(\mathcal{L}(f_\infty/K, \chi)) \in F_p
\]

for any arithmetic prime \( p \) of \( \mathcal{R} \). If \( M \) is a \( \mathbb{Z}[\hat{G}_c] \)-module, set

\[
M^\chi := M \otimes_{\mathbb{Z}[\hat{G}_c]} \mathcal{O}_F
\]
where the tensor product is taken with respect to $\chi : \mathbb{Z}[\tilde{G}_c] \to \mathcal{O}_F$. Then we can formulate the following conjecture, which extends Conjecture 8.5 and Theorem 8.6 to characters of $\tilde{G}_c$ other than the trivial one.

**Conjecture 8.7.** Let $p$ be a non-exceptional arithmetic prime of $\mathcal{R}$ with trivial character and even weight. If $L(f_{\infty}/K, \chi, p) \neq 0$ then $\tilde{H}^1_f(H_c, V^\dagger_p) \chi = 0$. Furthermore, if $L(f_{\infty}/K, \chi) \neq 0$ then $\tilde{H}^1_f(H_c, T^\dagger)$ is a torsion $\mathcal{R}$-module.

Granting the first part of Conjecture 8.7, one could presumably derive the second part by using arguments close to those employed in the proof of Theorem 8.6.

**8.3. Iwasawa theory.** The goal of this § is to formulate a “main conjecture” of Iwasawa theory for Hida families (Conjecture 8.10) in our definite setting.

Set $H_p^\infty := \bigcup_{m \geq 1} H_p^m$, denote by $K_\infty \subset H_p^\infty$ the anticyclotomic $\mathbb{Z}_p$-extension of $K$ and for every integer $n \geq 0$ let $K_n$ be the $n$-th layer of $K_\infty$, i.e. the (unique) subfield of $K_\infty$ such that $G_n := \text{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

For every integer $n \geq 1$ set

$$d(n) := \min \{ m \in \mathbb{N} \mid K_n \subset H_p^m \}.$$  

For example, if $p$ does not divide the class number of $K$ then $d(n) = n + 1$ for all $n \geq 1$. Let $G_\infty := \text{Gal}(K_\infty/K)$ (so that $G_\infty \simeq \mathbb{Z}_p$, the isomorphism depending on the choice of a topological generator of $G_\infty$) and define the completed group algebra

$$\mathcal{R}_\infty := \varprojlim_n \mathcal{R}[G_n] = \mathcal{R}[G_\infty],$$

where the inverse limit is computed with respect to the canonical maps. Throughout this § we make the following

**Assumption 8.8.** The local ring $\mathcal{R}$ is regular.

In our Iwasawa-theoretic context, this simplifying hypothesis is a natural condition to require (see, e.g., [24, §3.3] and [8, Ch. X]) and gives us some control on the behaviour of $\mathcal{R}$ and $\mathcal{R}_\infty$ under localizations.

For any finitely generated $\mathcal{R}_\infty$-module $M$ let

$$M^\vee := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$$

be its Pontryagin dual, $M_{\text{tors}}$ its torsion submodule and $\text{Char}_{\mathcal{R}_\infty}(M)$ its characteristic ideal. Recall that, by definition, $\text{Char}_{\mathcal{R}_\infty}(M)$ is the ideal of $\mathcal{R}_\infty$ given by

$$\text{Char}_{\mathcal{R}_\infty}(M) := \begin{cases} \prod_{\text{ht}(\mathfrak{p}) = 1} \mathfrak{p}^{\text{length}(M_{\mathfrak{p}})} & \text{if } M = M_{\text{tors}} \\ \{0\} & \text{otherwise} \end{cases}$$

where the product is made over all height one prime ideals of $\mathcal{R}_\infty$. Note that, thanks to the assumption that $\mathcal{R}$ is regular, the localization $\mathcal{R}_\infty, \mathfrak{p}$ is a discrete valuation ring for every prime ideal $\mathfrak{p}$ of height one in $\mathcal{R}_\infty$.

Finally, define the $\mathcal{R}_\infty$-module

$$\tilde{H}^1_{f,1w}(K_\infty, T^\dagger) := \varprojlim_n \tilde{H}^1_f(K_n, T^\dagger),$$
where the inverse limit is taken with respect to the corestriction maps, and the \( R_\infty \)-module
\[
\tilde{H}^1_{f,Iw}(K_\infty, A^\dagger) := \lim_{\rightarrow} \tilde{H}^1_f(K_n, A^\dagger),
\]
where the direct limit is taken with respect to the restriction maps.

As before, for all integers \( n \geq 1 \) take the element
\[
P_{p^m, R} \in H^0(Gal(K_{ab}^+/H_{p^m}), D^\dagger)
\]
and set
\[
Q_{n, R} := Cor_{H_{p^m}/K_n}(P_{p^m, R}) \in H^0(Gal(K_{ab}^+/K_n), D^\dagger).
\]
In other words, consider the classes \( P_c, R \) with \( c \) varying in the set of powers of the prime \( p \) and take their traces on the anticyclotomic \( \mathbb{Z}_p \)-extension \( K_\infty \) of \( K \). Then for every \( \sigma \in G_n \) define
\[
J_{\sigma} := \eta_{K_n}(Q_{n, R}) \in R \setminus [G_n].
\]
Here \( \eta_{K_n} \) is the map of (62) with \( H = K_n \). For every integer \( n \geq 1 \) we introduce the theta-element
\[
\theta_n := \alpha_p^{-n} \sum_{\sigma \in G_n} J_{\sigma} \otimes \sigma^{-1} \in R[G_n].
\]
Note that the element \( \theta_\infty \) is not entirely canonical, since it is independent of the choice of the compatible system of big Heegner points \( \{ P_{p^m, R} \}_{n \geq 1} \) only up to multiplication by an element of \( G_\infty \). To get rid of this ambiguity, we proceed as follows. Denote by \( x \mapsto x^* \) the canonical involution of \( R_\infty \) acting as \( \sigma \mapsto \sigma^{-1} \) on group-like elements. We associate a two-variable \( p \)-adic \( L \)-function with the primitive morphism \( f_\infty : h_\infty^d \to R \). Thanks to the compatibility relations enjoyed by big Heegner points (see §7.1), for all integers \( m \geq n \geq 1 \) one has
\[
\nu_{m,n}(\theta_m) = \theta_n
\]
where \( \nu_{m,n} : R[G_m] \to R[G_n] \) is the map induced by the natural surjection \( G_m \to G_n \), so one can define
\[
\theta_\infty := \lim_{\rightarrow} \theta_n \in R_\infty.
\]
Always assuming that \( R \) is regular, now we formulate our “main conjecture” relating \( L_p(f_\infty/K) \) to the characteristic ideal of the Pontryagin dual of \( \tilde{H}^1_{f,Iw}(K_\infty, A^\dagger) \).

\textbf{Conjecture 8.10.} The group \( \tilde{H}^1_{f,Iw}(K_\infty, A^\dagger) \) is a finitely generated torsion module over \( R_\infty \) and there is an equality
\[
(L_p(f_\infty/K)) = \text{Char}_{R_\infty}(\tilde{H}^1_{f,Iw}(K_\infty, A^\dagger)^\vee)
\]
of ideals of \( R_\infty \).

The reader should compare Conjecture 8.10 with the Main Conjecture of Iwasawa theory for elliptic curves in the ordinary and anticyclotomic setting that was partially proved by Bertolini and Darmon in [3] and with the main conjectures over the weight space formulated by Delbourgo in [8] §10.5].
9. Vanishing of special values in Hida families

The aim of this short section is to formulate two conjectures on the vanishing at the critical points of (twists of) the $L$-functions over $K$ of the modular forms in the Hida family of $f$ living on the same branch as $f$. As in the previous section, we work in the definite case. In fact, our conjectures will involve the elements $\mathcal{L}(f_\infty/K, \chi, p) \in F_p$ and the $p$-adic $L$-function $\mathcal{L}_p(f_\infty/K) \in \mathcal{R}_\infty$ introduced in §8.2 and §8.3, respectively. We remark that results on the vanishing of special values were obtained by Howard in [25], where it is shown that if there exists a weight two form in a Hida family whose $L$-function vanishes to exact order one at $s = 1$ then all but finitely many weight two forms in the family enjoy this same property (see [25] Theorem 8; see also [25], Theorem 7 for the analogous result for order of vanishing zero, which is a consequence of work of Kato, Kitagawa and Mazur).

To begin with, recall the fixed isomorphism $\mathbb{C} \simeq \mathbb{C}_p$, which induces an embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$ of an algebraic closure of $\mathbb{Q}_p$ into the complex field, and choose embeddings $F_p \hookrightarrow \mathbb{Q}_p$ for all arithmetic primes $p$ of $\mathcal{R}$, so that we can view the $q$-expansion coefficients of the forms $f_p$ as (algebraic) complex numbers. Then fix a character $\chi : G_\infty \rightarrow \mathcal{O}_F^\times \hookrightarrow \mathbb{C}_p^\times$, where $\mathcal{O}$ is a finite extension of $\mathcal{O}_F$.

Finally, for every arithmetic prime $p$ of $\mathcal{R}$ let $L_K(f_p, \chi, s)$ be the $L$-function of $f_p$ over $K$ twisted by $\chi$, and recall the element $\mathcal{L}(f_\infty/K, \chi, p) \in F_p$ defined in (65).

Motivated by [7 Theorem 1.11] and [51 Theorem 1.3.2] (which extend [14, Proposition 11.2] and [2, Theorem 1.1]), we propose the following

Conjecture 9.1. Let $p$ be an arithmetic prime of $\mathcal{R}$ of weight $k_p \geq 2$ and let $\chi$ be as above. The special value $L_K(f_p, \chi, k_p/2)$ is non-zero if and only if $\mathcal{L}(f_\infty/K, \chi, p)$ is non-zero.

In other words, we conjecture that $L_K(f_p, \chi, s)$ vanishes at the critical point $s = k_p/2$ precisely when the element $\mathcal{L}(f_\infty/K, \chi) \in \mathcal{R}$ introduced in (63) lies in $p$.

Now we want to formulate an analogous conjecture for twists by characters of the Galois group $G_\infty \simeq \mathbb{Z}_p$. Thus let $\chi : G_\infty \rightarrow \mathcal{O}_F^\times$ be a finite (i.e., $p$-power) order character of $G_\infty$, where $\mathcal{O}$ is a finite extension of $\mathcal{O}_F$. If $p$ is an arithmetic prime of $\mathcal{R}$ then the canonical map $\mathcal{R} \rightarrow F_p$ gives a map $\mathcal{R}_\infty \rightarrow F_p[[G_\infty]]$; composing this with the map $F_p[[G_\infty]] \rightarrow \mathbb{Q}_p$ induced by $\chi$ yields a map $\chi_p : \mathcal{R}_\infty \rightarrow \mathbb{Q}_p$.

The analogue of Conjecture 9.1 in this Iwasawa-theoretic context is the following

Conjecture 9.2. Let $p$ be an arithmetic prime of $\mathcal{R}$ of weight $k_p \geq 2$ and let $\chi$ be as above. The special value $L_K(f_p, \chi, k_p/2)$ is non-zero if and only if $\chi_p(\mathcal{L}_p(f_\infty/K))$ is non-zero.

10. Big Heegner points and Selmer groups: the indefinite case

In this section we assume that we are in the indefinite case, i.e. that the quaternion algebra $B$ is indefinite.

Results analogous to some of those which follow were also obtained by Fouquet in [11] and [12], where the more general case of Shimura curves attached to indefinite quaternion algebras over totally real number fields is considered. However, our approach is in many respects different, and (as apparent in the previous sections) the Jacquet–Langlands correspondence plays a much more prominent role in our paper than in the work of Fouquet.

Throughout this section we make the following
Assumption 10.1. The $\Lambda$-algebra $\mathbb{T}_{\infty,m_f}^\text{ord}$ is Gorenstein, that is
\[
\mathbb{T}_{\infty,m_f}^\text{ord} \simeq \text{Hom}_\Lambda(\mathbb{T}_{\infty,m_f}^\text{ord}, \Lambda)
\]
as $\mathbb{T}_{\infty,m_f}^\text{ord}$-modules.

We introduce this condition because it is taken as a working hypothesis in [11], from which we shall borrow an important result (Proposition 10.3). Note that the analogous statement for the Hida family $\mathfrak{h}_{\infty,m_f}$ of $f$ is true by [24, Proposition 2.1.2].

10.1. Galois representations. Let $p$ be an arithmetic prime of $\mathcal{R}$. To simplify our notation, set
\[
\mathbb{T}_{\text{Sh}} := \mathbb{T}_{\infty,\mathcal{R}}^\text{ord}, \quad \mathbb{T}_{\text{Sh}}^\dagger := \mathbb{T}_{\infty,\mathcal{R}}^\text{ord,\dagger}, \quad \mathbb{T}_{\text{Sh},p} := \mathbb{T}_{\text{Sh}} \otimes_{\mathcal{R}} \mathcal{R}_p,
\]
\[
V_{\text{Sh},p} := \mathbb{T}_{\text{Sh},p}/p\mathbb{T}_{\text{Sh},p}, \quad V_{\text{Sh},p}^\dagger := \mathbb{T}_{\text{Sh},p}^\dagger/p\mathbb{T}_{\text{Sh},p}^\dagger.
\]
As before, let $m_\mathcal{R}$ be the maximal ideal of the local ring $\mathcal{R}$. The next assumption plays the role of [11, Hypothèse 1.4.26].

Assumption 10.2. The residual $G_\mathcal{Q}$-representation $\mathbb{T}_{\text{Sh}}/m_\mathcal{R}\mathbb{T}_{\text{Sh}}$ is absolutely irreducible.

The result stated below, which describes the $G_\mathcal{Q}$-representation $\mathbb{T}_{\text{Sh}}$, seems to be well known to experts; we essentially reproduce its proof given in [11, Théorème 1].

Proposition 10.3. (1) The $\mathcal{R}$-module $\mathbb{T}_{\text{Sh}}$ is free of rank two.
(2) The $G_\mathcal{Q}$-representation $\mathbb{T}_{\text{Sh}}$ is unramified outside $Np$ and the arithmetic Frobenius at a prime $\ell \nmid Np$ acts with characteristic polynomial $X^2 - T_{\ell}X + [\ell]\ell$.
(3) For any arithmetic prime $p$ of $\mathcal{R}$ denote by $V(f_p)$ the $G_\mathcal{Q}$-representation over $\mathcal{F}_p$ attached to $f_p$. Then the $G_\mathcal{Q}$-representation $V_{\text{Sh},p}$ is equivalent to the dual $V^*(f_p)$ of $V(f_p)$, hence to $V(f_p)(r_p - 1) \otimes [\chi_p^{-1}].$

Proof. Keeping our assumptions on the form $f$ in mind, it can be checked that the hypotheses made in [11, §1.4.5] and used in the proof of [11, Théorème 1] are verified. We just remark that, in our context, [11, Hypothèse 1.4.28] is the analogue for Shimura curves of the main result of [31], whose generalization to Shimura curves when $N^+ = 1$ and $N^- = pq$ (with $p, q$ distinct primes) is provided by [34, Theorem 2]. Assuming that the representation associated with $f$ is ramified at all primes dividing $N^-$, we expect that this result holds in our more general situation as well (details will be given in a subsequent article). Since all assumptions are verified, the statements of the proposition follow from [11, Théorème 1].

Now recall the $\mathcal{R}$-module $\mathcal{T}$ defined in §4.5. The following consequence of Proposition 10.3 will be crucial for our arguments.

Corollary 10.4. There are isomorphisms of $G_\mathcal{Q}$-modules $\mathcal{T} \simeq \mathbb{T}_{\text{Sh}}$ and $\mathcal{T}^\dagger \simeq \mathbb{T}_{\text{Sh}}^\dagger$, which exchange the actions of $\mathbb{T}_{\infty,m_f}^\text{ord}$ on the left with the action of its isomorphic $\Lambda$-algebra $B_{\infty,m_f}^\text{ord}$ on the right.

Proof. First of all, by property (1) in §1.5 and part (1) in Proposition 10.3, both $\mathcal{T}$ and $\mathbb{T}_{\text{Sh}}$ are free $\mathcal{R}$-modules of rank two. Moreover, Proposition 1.9 and Assumption 10.2 guarantee that the residual $G_\mathcal{Q}$-representations $\mathcal{T}/m_\mathcal{R}\mathcal{T}$ and $\mathbb{T}_{\text{Sh}}/m_\mathcal{R}\mathbb{T}_{\text{Sh}}$ are absolutely irreducible. Finally, by property (2) in §1.5 and part (2) in Proposition 10.3, the arithmetic Frobenius at a prime $\ell \nmid Np$ acts on $\mathcal{T}$ and $\mathbb{T}_{\text{Sh}}$ with the same characteristic polynomial. Putting all these statements together, the isomorphisms of $G_\mathcal{Q}$-modules $\mathcal{T} \simeq \mathbb{T}_{\text{Sh}}$ and $\mathcal{T}^\dagger \simeq \mathbb{T}_{\text{Sh}}^\dagger$ follow from, e.g., [30, §5, Corollary]. The Hecke equivariance is immediate from the definitions. \qed
Corollary \[10.4\] implies that for every arithmetic prime \( p \) of \( \mathcal{R} \) there are isomorphisms of \( G_{\mathbb{Q}^c} \)-modules
\[
T_p \simeq T_{\text{Sh}, p}, \quad T_p^\dagger \simeq T_{\text{Sh}, p}^\dagger, \quad V_p \simeq V_{\text{Sh}, p}, \quad V_p^\dagger \simeq V_{\text{Sh}, p}^\dagger
\]
exchanging the actions of \( \mathbb{T}_{\text{ord}}^\infty \) on the left and of \( \mathbb{P}_{\text{ord}}^\infty \) on the right. Thanks to the isomorphism \( T_{\text{Sh}}^\dagger \simeq T^\dagger \) of \( G_{\mathbb{Q}} \)-modules, for every number field \( L \) there are isomorphisms of groups
\[
H^1(L, T_{\text{Sh}}^\dagger) \simeq H^1(L, T^\dagger), \quad \operatorname{Sel}_{\text{Gr}}(L, T_{\text{Sh}}^\dagger) \simeq \operatorname{Sel}_{\text{Gr}}(L, T^\dagger),
\]
and similarly for the other Galois modules listed above.

Let \( \ell | N^- \) be a prime number. Since \( \ell \) is inert in \( K \), the completion \( K_{\ell} \) of \( K \) at the prime \( (\ell) \) is the (unique, up to isomorphism) unramified quadratic extension of \( \mathbb{Q}_{\ell} \). By \[35\] Proposition 4.2.3, the group \( H^1(K_{\ell}, T) \) is a finitely generated \( \mathcal{R} \)-module, hence (since \( \mathcal{R} \) is noetherian) the \( \mathcal{R} \)-torsion submodule \( H^1(K_{\ell}, T_{\text{Sh}}^\dagger) \) of \( H^1(K_{\ell}, T_{\text{Sh}}^\dagger) \) is a finitely generated \( \mathcal{R} \)-module too. Define \( \mathfrak{a}_\mathcal{R} \) to be the annihilator in \( \mathcal{R} \) of the finitely generated torsion \( \mathcal{R} \)-module \( \prod_{\ell | N^-} H^1(K_{\ell}, T_{\text{Sh}}^\dagger) \). Recall the \( \mathcal{R} \)-component
\[
\kappa_{c, \mathcal{R}} \in H^1(G_{\mathbb{Q}^c}^c, T_{\text{Sh}}^\dagger)
\]
of the big Heegner class \( \kappa_c \) as introduced in Definition \[6.8\] and denote by the same symbol its image in \( H^1(H_c, T_{\text{Sh}}^\dagger) \) under inflation. The next result is a variant of \[24\] Proposition 2.4.5, to the proof of which we refer for the details we omit.

**Proposition 10.5.** For every \( \lambda \in \mathfrak{a}_\mathcal{R} \) and every integer \( c \geq 1 \) prime to \( N \) one has
\[
\lambda \cdot \kappa_{c, \mathcal{R}} \in \operatorname{Sel}_{\text{Gr}}(H_c, T_{\text{Sh}}^\dagger).
\]

**Proof.** For any place \( v \) of \( H_c \) and any \( \text{Gal}(\overline{\mathbb{Q}}/H_c) \)-module \( M \) let us denote by
\[
\text{res}_v : H^1(H_c, M) \longrightarrow H^1(H_{c,v}, M)
\]
the restriction map. Fix an integer \( c \geq 1 \) prime to \( N \) and, for simplicity, in the remainder of the proof write \( \kappa_{\mathcal{R}} \) for \( \kappa_{c, \mathcal{R}} \). If \( v \nmid Np \) then \( \kappa_{\mathcal{R}} \) satisfies the Greenberg local condition at \( v \) because its unramifiedness at \( v \) is part of Definition \[6.8\].

Now let us assume that \( v \mid Np \) and choose a place \( w \) of \( \overline{\mathbb{Q}} \) above \( v \). Let \( p \) be an arithmetic prime of \( \mathcal{R} \) of weight 2 and recall the integer \( m := m_p \) defined in \[52\]. Then the natural map \( T_{\text{Sh}}^\dagger \rightarrow V_{\text{Sh}, p} \) factors through \( T_p^\dagger \rightarrow \text{Jac}(X_m) \). Let \( \kappa_{\mathcal{R}, p} \) denote the image of \( \kappa_{\mathcal{R}} \) in \( H^1(H_c, T_p^\dagger) \). After restriction to \( H_{cp^m}(\mu_{p^m}) \), we see that \( V_{\text{Sh}, p} \simeq V_{\text{Sh}, p}^\dagger \). Furthermore, the restriction of \( \kappa_{\mathcal{R}, p} \) to \( H^1(H_{cp^m}(\mu_{p^m}), V_{\text{Sh}, p}) \) is contained in the image of the classical (untwisted) Kummer map
\[
\text{Jac}(X_m)(H_{cp^m}(\mu_{p^m}))^\text{ord} \longrightarrow H^1(H_{cp^m}, \text{Jac}(X_m)) \longrightarrow H^1(H_{cp^m}(\mu_{p^m}), V_{\text{Sh}, p}).
\]
Therefore, by \[51\] Example 3.11], the restriction of \( \kappa_{\mathcal{R}, p} \) to \( H^1(H_{cp^m}(\mu_{p^m}), V_{\text{Sh}, p}) \) lies in the Bloch–Kato Selmer group \( H^1(H_{cp^m}(\mu_{p^m}), V_{\text{Sh}, p}) \) of \( V_{\text{Sh}, p} \). By the isomorphism \( V_p \simeq V_{\text{Sh}, p} \), this group is isomorphic to \( H^1(H_{cp^m}(\mu_{p^m}), V_p) \). Thus, by Proposition \[4.12\] the isomorphic image in \( H^1(H_{cp^m}(\mu_{p^m}), V_p) \) of the restriction of \( \kappa_{\mathcal{R}, p} \) to \( H^1(H_{cp^m}(\mu_{p^m}), V_{\text{Sh}, p}) \) belongs to \( \operatorname{Sel}_{\text{Gr}}(H_{cp^m}(\mu_{p^m}), V_p) \), and thus, by \[60\], the restriction of \( \kappa_{\mathcal{R}, p} \) to \( H^1(H_{cp^m}(\mu_{p^m}), V_{\text{Sh}, p}) \) belongs to \( \operatorname{Sel}_{\text{Gr}}(H_{cp^m}(\mu_{p^m}), V_{\text{Sh}, p}^\dagger) \). Following the arguments in the proof of \[24\] Proposition 2.4.5, we thus see that
\[
\kappa_{\mathcal{R}, p} \in \operatorname{Sel}_{\text{Gr}}(H_c, V_{\text{Sh}, p}^\dagger)
\]
for all arithmetic primes \( p \) of \( \mathcal{R} \) of weight 2.
Once again by [24 Proposition 2.4.5], if \( v \mid pN^+ \) then \( \text{res}_v(\kappa_{\mathcal{R}}) \) belongs to \( H^1_{\text{Gr}}(H_{c,v}, \mathbf{T}^\dagger_{\text{Sh}}) \), while if \( v \mid N^- \) one can only show that \( \text{res}_v(\kappa_{\mathcal{R}}) \) is an \( \mathcal{R} \)-torsion element in \( H^1(H_{c,v}, \mathbf{T}^\dagger_{\text{Sh}}) \). In the latter case, let \( \ell \) be the rational prime below \( v \). As \( \ell \) is inert in \( K \), the prime \((\ell)\) of \( K \) splits completely in \( H_c \), so \( H_{c,v} = K_\ell \) and
\[
H^1(H_{c,v}, \mathbf{T}^\dagger_{\text{Sh}}) = H^1(K_\ell, \mathbf{T}^\dagger_{\text{Sh}}).
\]
Since \( \lambda \in a_{\mathcal{R}} \), the result follows. \( \square \)

In Proposition [10.5] we were only able to prove that \( \kappa_{c,\mathcal{R}} \) lies in \( \text{Sel}_{\text{Gr}}(H_c, \mathbf{T}^\dagger_{\text{Sh}}) \) after multiplication by a suitable element of \( \mathcal{R} \), which however can be chosen independently of \( c \). It would be interesting to have an answer to the following

**Question 10.6.** Does \( \kappa_{c,\mathcal{R}} \) belong to \( \text{Sel}_{\text{Gr}}(H_c, \mathbf{T}^\dagger_{\text{Sh}}) \)?

As apparent in the proof of Proposition [10.5], the obstacle towards giving a positive answer to the above question is the lack of control on the restriction of \( \kappa_{c,\mathcal{R}} \) at places dividing \( N^- \).

With notation as above, fix once and for all a non-zero \( \lambda \in a_{\mathcal{R}} \). Thanks to the isomorphisms [63] and [24 (21)], for every integer \( c \geq 1 \) prime to \( N \) the class \( \lambda \cdot \kappa_{c,\mathcal{R}} \) defines a class
\[
X_{c,\mathcal{R}} := \lambda \cdot \kappa_{c,\mathcal{R}} \in \text{Sel}_{\text{Gr}}(H_c, \mathbf{T}^\dagger) \cong \overline{H}^1_f(H_c, \mathbf{T}^\dagger).
\]
These cohomology classes are the arithmetic objects in terms of which we will formulate our results and conjectures in this indefinite setting.

### 10.2. Bounding Selmer groups.

Define the two cohomology classes
\[
\kappa_{0,\mathcal{R}} := \text{Cor}_{H_1/K}(\kappa_{1,\mathcal{R}}) \in H^1(K, \mathbf{T}^\dagger), \quad \mathfrak{Z}_{0,\mathcal{R}} := \text{Cor}_{H_1/K}(X_{1,\mathcal{R}}) = \lambda \cdot \kappa_{0,\mathcal{R}} \in \overline{H}^1_f(K, \mathbf{T}^\dagger).
\]
The following conjecture is the counterpart of [24 Conjecture 3.4.1].

**Conjecture 10.7.** The class \( \mathfrak{Z}_{0,\mathcal{R}} \) is not \( \mathcal{R} \)-torsion.

Note that Conjecture [10.7] is equivalent to the assertion that \( \kappa_{0,\mathcal{R}} \) is not \( \mathcal{R} \)-torsion. The Euler system relations satisfied by the classes \( \kappa_{c,\mathcal{R}} \) (proved in Section 7) yield a proof of the following

**Theorem 10.8.** Let \( p \) be a non-exceptional arithmetic prime of \( \mathcal{R} \) with trivial character and even weight. If \( \mathfrak{Z}_{0,\mathcal{R}} \) has non-trivial image in \( \overline{H}^1_f(K, V^\dagger_p) \) then \( \dim_{F_p} \overline{H}^1_f(K, V^\dagger_p) = 1 \).

**Proof.** We follow the proof of [24 Theorem 3.4.2], which is based on an application of the results obtained by Nekovář in [34]. Since Nekovář assumes that all primes dividing \( N \) split in \( K \) (i.e., that \( N^- = 1 \)), we briefly explain how the results in [34] can be extended to our setting. Let \( M_N \) be the modular curve over \( \mathbb{Q} \) parametrizing elliptic curves with full level \( N \)-structure, and let \( X_M \) be the Shimura curve over \( \mathbb{Q} \) which is a fine moduli scheme for abelian surfaces with quaternionic multiplication, level \( N^+ \)-structure and full level \( M \)-structure (here \( M \geq 3 \) is a fixed auxiliary integer relatively prime to \( N \)), as in [26 Definition 5.5]. There is a Galois covering \( X_M \rightarrow X_{N^+,N^-} \) where \( X_{N^+,N^-} \) is the Shimura curve of level \( N^+ \) attached to the indefinite quaternion algebra over \( \mathbb{Q} \) of discriminant \( N^- \) (see [12], loc. cit.) To obtain the desired extension of [34], one needs only replace the universal elliptic curve \( E_N \rightarrow M_N \) (denoted by \( X_N \rightarrow M_N \) in loc. cit.) with the universal abelian surface \( \mathcal{S} \rightarrow X_M \). In this setting, one can define CM cycles in terms of the Euler system of Heegner points on \( X_{N^+,N^-} \); these cycles were first introduced by Besser in [4] (see also the work [47] by Sreekantan) and satisfy the same formal properties of the CM cycles built from Heegner points on \( X_0(N) \) which are studied in [34]. We refer to [26 §8] for details...
on this circle of ideas. With these modifications, the arguments of Nekovář carry over to our more general context, and our theorem follows by considerations that are completely analogous to those in the proof of [24, Theorem 3.4.2]. □

Remark 10.9. The definition of the class $Z_{0, R}$ depends on the choice of $\lambda \in a_R$, which is not made explicit in the notation. It might be possible that for different $\lambda_1$ and $\lambda_2$ in $a_R$ the class $\lambda_1 \cdot \kappa_0$ is trivial in $\tilde{H}_{1}(K, V_1^\dagger)$ while the class $\lambda_2 \cdot \kappa_0$ is not. However, since $a_R$ is contained in only finitely many arithmetic primes $p$, this occurrence can happen only for a finite number of $p$. Furthermore, if Conjecture 10.7 is true then for any choice of $\lambda \in a_R$ the class $\lambda \cdot \kappa_0$ has non-trivial image in $\tilde{H}_{1}(K, V_p^\dagger)$ for all but finitely many primes $p$, by [24, Lemma 2.1.7]. Thus, under Conjecture 10.7 the different choices of $\lambda \in a_R$ are essentially equivalent.

The next result is essentially a consequence of Theorem 10.8.

Theorem 10.10. Assume Conjecture 10.7. The $R$-module $\tilde{H}_{1}(K, T^\dagger)$ has rank one.

Proof. Mimic the arguments in the proof of [24, Corollary 3.4.3], replacing [24, Theorem 3.4.2] with Theorem 10.8. □

Remark 10.11. By fixing a character $\chi$ of $\text{Gal}(H_c/K)$ and taking the behaviour of the $\chi$-isotypical components of $\tilde{H}_{1}(H_c, V_p^\dagger)$ and $\tilde{H}_{1}(H_c, T^\dagger)$ into account, it would be possible to formulate more general statements than those in Theorems 10.8 and 10.10.

Remark 10.12. Following [1, §1.5], it is possible to introduce an Atkin–Lehner involution on indefinite Shimura curves. Performing a careful study of the action of this involution, and proving results analogous to [24, Lemma 2.3.3 and Proposition 2.3.5], one can show that there exists $\sigma \in \text{Gal}(H_c/K)$ and $w \in \{-1, 1\}$ such that $P_c^w = w \cdot P_c^\sigma$, where $\tau \in \text{Gal}(H_c/Q)$ restricts to the non-trivial element of $\text{Gal}(K/Q)$. This allows one to complete Theorem 10.8 with the final statement of [24, Corollary 3.4.3]: the $R$-rank of $\tilde{H}_{1}(Q, T^\dagger)$ is one if $w = 1$ and is zero if $w = -1$.

10.3. Iwasawa theory. We formulate an Iwasawa-theoretic “main conjecture” (Conjecture 10.14) which is the counterpart of Conjecture 8.10 in the indefinite setting. The reader is referred to [39] for results of Ochiai on the cyclotomic Iwasawa main conjecture for Hida families.

Resume the notation of [8.3], in particular, for every integer $n \geq 1$ the field $K_n$ is the $n$-th layer of the anticyclotomic $Z_p$-extension $K_\infty$ of $K$ and $d(n)$ is the smallest natural number such that $K_n$ is a subfield of $H_{pd(n)}$. As in Assumption 8.8 we suppose that the local ring $R$ is regular. For every integer $n \geq 1$ we define the cohomology class

$$Z_{n, R} := \text{Cor}_{H_{pd(n)/K_n}}(U_{p(n)}^{1-d(n)}X_{pd(n), R}) \in \tilde{H}_{1}(K_n, T^\dagger).$$

Since the classes $Z_{n, R}$ are compatible with respect to corestriction, we can give the following

Definition 10.13. The two-variable $p$-adic $L$-function attached to the family $\{Z_{n, R}\}_{n \geq 1}$ is the element

$$Z_{\infty, R} := \lim_{n} Z_{n, R} \in \tilde{H}_{1, \text{Iw}}(K_{\infty}, T^\dagger).$$

Recall that if $M$ is a finitely generated $R_{\infty}$-module then $M^\vee$ is the Pontryagin dual of $M$. Now we propose our two-variable “main conjecture”. Since the class $Z_{\infty, R}$ depends on the element $\lambda \in a_R$ appearing in (67), in order to state our conjecture we need to assume an additional condition.
Conjecture 10.14. The group $\hat{H}_{f,1w}^{1}(K_{\infty}, T^{\dagger})/(3_{\infty}, R)$ is a finitely generated $R_{\infty}$-module. Moreover, suppose that $\kappa_{p^{m}, R}$ belongs to $\hat{H}_{f}^{1}(H_{p^{m}}, T^{\dagger})$ and set
$$X_{p^{m}, R} := \kappa_{p^{m}, R}$$
for all $m \geq 0$. There is an equality
$$\text{Char}_{R_{\infty}}\left(\hat{H}_{f,1w}^{1}(K_{\infty}, T^{\dagger})/(3_{\infty}, R)\right)^{2} = \text{Char}_{R_{\infty}}\left(\hat{H}_{f,1w}^{1}(K_{\infty}, A^{\dagger})^{\dagger\dagger}_{\text{tors}}\right)$$
of ideals of $R_{\infty}$.

Conjecture 10.14 extends both [24, Conjecture 3.3.1] and the classical Heegner point main conjecture for elliptic curves formulated by Perrin-Riou in [11]. Observe that in the special case where $N^{-} = 1$ (or, more generally, for quaternion algebras over totally real number fields satisfying suitable conditions) Fouquet shows in [12, Theorem A] that the right-hand side divides the left-hand side in (68). A refined version of Conjecture 10.14 which takes the dependence of $3_{\infty, R}$ on the element $\lambda \in a_{R}$ into account will be investigated in a future project.

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