On Some Algebraic Properties of Semi-Discrete Hyperbolic Type Equations

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Abstract

Nonlinear semi-discrete equations of the form \( t_x(n+1) = f(t(n), t(n+1), t_x(n)) \) are studied. An adequate algebraic formulation of the Darboux integrability is discussed and the attempt to adopt this notion to the classification of Darboux integrable chains has been undertaken.

1 Introduction

Investigation of the class of hyperbolic type differential equations of the form

\[
  u_{xy} = f(x, y, u, u_x, u_y)
\]

has a very long history. Various approaches have been developed to look for particular and general solutions of these kind equations. In the literature one can find several definitions of integrability of the equation. According to one given by G. Darboux (see \([1, 2]\)), equation (1) is called integrable if it is reduced to a pair of ordinary (generally nonlinear) differential equations, or more exactly, if its any solution satisfies the equations of the form

\[
  F(x, y, u, u_x, u_{xx}, ..., D_x^m u) = a(x), \quad G(x, y, u, u_y, u_{yy}, ..., D_y^n u) = b(y),
\]

for appropriately chosen functional parameters \(a(x)\) and \(b(y)\), where \(D_x\) and \(D_y\) are operators of differentiation with respect to \(x\) and \(y\), \(u_x = D_x u, u_{xx} = D_x u_x\) and so on. Functions \(F\) and \(G\) in (2) are called \(y\)- and \(x\)-integrals of the equation (1) respectively.

An effective criterion of Darboux integrability has been proposed by G. Darboux himself. Equation (1) is integrable if and only if the Laplace sequence of the linearized equation terminates at both ends. A rigorous proof of this statement has been found only recently in \([3, 4]\). A complete list of the Darboux integrable equations of the form (1) is given in \([5]\).

An alternative method of investigation and classification of the Darboux integrable equations has been developed by A. B. Shabat in \([6]\), based on the notion of characteristic Lie algebra. Let us give a brief explanation of this notion. Begin with the basic property of the integrals. Evidently each \(y\)-integral satisfies the condition: \(D_y F(x, y, u, u_x, u_{xx}, ..., D_x^m u) = 0\). Taking the derivative by applying the chain rule one defines a vector field \(T_1\) such that

\[
  T_1 F = \left( \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + f \frac{\partial}{\partial u_x} + D_x(f) \frac{\partial}{\partial u_{xx}} + ... \right) F = 0.
\]

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So the vector field $T_1$ solves the equation $T_1 F = 0$. But in general, the coefficients of the vector field depend on the variable $u_y$ while the solution $F$ does not. This puts a severe restriction on $F$, actually $F$ should satisfy one more equation: $T_2 F = 0$, where $T_2 = \frac{\partial}{\partial u_y}$.

Now the commutator of these two operators $T_1$ and $T_2$ will also annihilate $F$. Moreover, for any $T$ from the Lie algebra generated by $T_1$ and $T_2$ one gets $T F = 0$. This Lie algebra is called the characteristic Lie algebra of the equation (1) in the direction of $y$. Characteristic Lie algebra in the $x$-direction is defined in a similar way. Now by virtue of the famous Jacobi theorem, equation (1) is Darboux integrable if and only if both of its characteristic algebras are of finite dimension. In [6] and [7], the characteristic Lie algebras for the systems of nonlinear hyperbolic equations and their applications are studied.

The characteristic Lie algebra has proved to be an effective tool for classifying nonlinear hyperbolic partial differential equations. This concept can be extended to discrete versions of partial differential equations (see [8]). Discrete models have become rather popular in the last decade because of their applications in physics and biology (see survey [9]). The problem of classification of the discrete Darboux integrable equations is very important and, to our knowledge, still open. Emphasize that the notion of integrability has a various of meanings. Different approaches and methods are applied for classifying different types of integrable equations (see [10] - [18]).

In this paper we study semi-discrete chains of the following form

$$t_{1x} = f(t, t_1, t_x)$$

(3)

from the Darboux integrability point of view. Here the unknown $t = t(n, x)$ is a function of two independent variables: one discrete $n$ and one continuous $x$. It is assumed that

$$\frac{\partial f}{\partial t_x} \neq 0.$$  

(4)

Subindex means shift or derivative, for instance, $t_1 = t(n+1, x)$ and $t_x = \frac{\partial}{\partial x} t(n, x)$. Below we use $D$ to denote the shift operator and $D_x$ to denote the $x$-derivative: $D h(n, x) = h(n+1, x)$ and $D_x h(n, x) = \frac{\partial}{\partial x} h(n, x)$. For the iterated shifts we use the subindex: $D^j h = h_j$.

Introduce now notions of the $n$-integral for the semi-discrete chain (3). It is natural, in accordance with the continuous case, to call a function $I = I(x, n, t, t_x, t_{xx}, ...)$ a $n$-integral of the chain (3) if it is in the kernel of the difference operator: $(D - 1) I = 0$. In other words, $n$-integral should still be unchanged under the action of the shift operator $D I = I$, (see also [19]). One can write it in an enlarged form

$$I(x, n+1, t_1, f, f_x, f_{xx}, ...) = I(x, n, t, t_x, t_{xx}, ...).$$

(5)

Notice that it is a functional equation, the unknown is taken at two different "points". This circumstance causes the main difficulty in studying discrete chains. Such kind problems appear when one tries to apply the symmetry approach to discrete equations (see [20], [21]). However, the concept of the Lie algebra of characteristic vector fields can serve as a basis for chains' investigation.

Introduce vector fields in the following way. Concentrate on the main equation (5). Evidently the left hand side of it contains the variable $t_1$ while the right hand side does not.
Hence the total derivative of the function $DI$ with respect to $t_1$ should vanish. In other words the $n$-integral is in the kernel of the operator $Y_1 := D^{-1} \frac{\partial}{\partial t_1} D$. Similarly one can check that $I$ is in the kernel of the operator $Y_2 := D^{-2} \frac{\partial}{\partial t_1} D^2$. Really, the right hand side of the equation $D^2 I = I$ which immediately follows from (5) does not depend on $t_1$, therefore the derivative of the function $D^2 I$ with respect to $t_1$ vanishes. Proceeding this way one can easily prove that for any $j \geq 1$ the operator

$$Y_j = D^{-j} \frac{\partial}{\partial t_1} D^j$$

solves the equation $Y_j I = 0$.

So far we shifted the argument $n$ forward, shift it now backward and use the main equation (5) written as $D^{-1} I = I$. Rewrite the original equation (3) in the form

$$t_{-1x} = g(t, t_{-1}, t_x).$$

(7)

This can be done because of the condition (4) assumed above. In the enlarged form the equation $D^{-1} I = I$ looks like

$$I(x, n - 1, t_{-1}, g, g_x, g_{xx}, ...) = I(x, n, t, t_x, t_{xx}, ...).$$

(8)

The right side of equation (8) does not depend on $t_{-1}$ so the total derivative of $D^{-1} I$ with respect to $t_{-1}$ is zero, i.e. the operator $Y_{-1} := D^{-1} \frac{\partial}{\partial t_{-1}} D^{-1}$ solves the equation $Y_{-1} I = 0$. Moreover, the operators

$$Y_{-j} = D^j \frac{\partial}{\partial t_{-1}} D^{-j},$$

(9)

$j \geq 1$, also satisfy similar conditions $Y_{-j} I = 0$.

Summarizing the reasonings above one can conclude that the $n$-integral is annihilated by any operator from the Lie algebra $\tilde{L}_n$ generated by the operators

$$..., Y_{-2}, Y_{-1}, Y_0, Y_1, Y_2, ...,$$

(10)

where

$$Y_0 = \frac{\partial}{\partial t_1} \quad \text{and} \quad Y_{-0} = \frac{\partial}{\partial t_{-1}}$$

(11)

The algebra $\tilde{L}_n$ consists of the operators from the sequence (10), all possible commutators, and linear combinations with coefficients depending on the variables $n$ and $x$. Evidently equation (3) admits a nontrivial $n$-integral only if the dimension of the algebra $\tilde{L}_n$ is finite. But it is not clear that the finite dimension of $\tilde{L}_n$ is enough for existence of $n$-integrals. By this reason we introduce another Lie algebra called the characteristic Lie algebra $L_n$ of the equation (3) in the direction of $n$. First we define in addition to the operators $Y_1, Y_2, ...$ differential operators

$$X_j = \frac{\partial}{\partial t_{-j}}$$

(12)

for $j \geq 1$.

The following theorem (see [22]) allows us to define this characteristic Lie algebra.
Theorem 1.1. Equation (3) admits a nontrivial n-integral if and only if the following two conditions hold:
1) Linear space spanned by the operators \( \{Y_j\}_1^{\infty} \) is of finite dimension, denote this dimension by \( N \);
2) Lie algebra \( L_n \) generated by the operators \( Y_1, Y_2, ..., Y_N, X_1, X_2, ..., X_N \) is of finite dimension. We call \( L_n \) the characteristic Lie algebra of (3) in the direction of \( n \).

Remark 1.2. If dimension of the linear space \( L_Y \) generated by \( \{Y_j\}_1^{\infty} \) is \( N \) then the set \( \{Y_j\}_1^{N} \) constitutes a basis in \( L_Y \).

The \( x \)-integral and the characteristic algebra in the \( x \)-direction of equation (3) are defined similar to the continuous case. We call a function \( F = F(x, n, t, t_1, t_2, ...) \) depending on a finite number of shifts an \( x \)-integral of the chain (3), if the following condition is valid
\[
D_x F = 0, \quad i.e. \quad K_0 F = 0,
\]
where
\[
K_0 = \frac{\partial}{\partial x} + t_x \frac{\partial}{\partial t} + f \frac{\partial}{\partial t_1} + g \frac{\partial}{\partial t_{-1}} + f_1 \frac{\partial}{\partial t_2} + g_{-1} \frac{\partial}{\partial t_{-2}} + \ldots \tag{13}
\]
Vector fields \( K_0 \) and
\[
X = \frac{\partial}{\partial t_x} \tag{14}
\]
as well as any vector field from the Lie algebra generated by \( K_0 \) and \( X \) annulate \( F \). This algebra is called the characteristic Lie algebra \( L_x \) of the chain (3) in the \( x \)-direction. The following result is essential, its proof can be found in [6].

Theorem 1.3. Equation (3) admits a nontrivial \( x \)-integral if and only if its Lie algebra \( L_x \) is of finite dimension.

The article is organized as follows. In Section 2 we study the algebra \( L_n \) introduced in Theorem 1.1. Section 3 is devoted to properties of the Lie algebra \( L_x \). These algebras \( L_n \) and \( L_x \) can be used as a new classifying tool for equations on a lattice. From this viewpoint the system of equations (36) is of special importance. Actually, the consistency condition of this overdetermined system of ”ordinary” difference equations provides necessary conditions of the Darboux integrability of the original equation (3). As an illustration of efficiency of our approach in the last Section 4 we study in details equation (3) admitting characteristic Lie algebras \( L_n \) and \( L_x \) of minimal possible dimensions equal 2 and 3 respectively. It is proved that in this case the equation (3) can be reduced to \( t_{1x} = t_x + t_1 - t \).

2 Characteristic Lie Algebra \( L_n \)

The proof of the first two lemmas can be found in [22]. Here, we still present their short proofs for the reader’s convenience.

Lemma 2.1. If for some integer \( N \) the operator \( Y_{N+1} \) is a linear combination of the operators with less indices:
\[
Y_{N+1} = \alpha_1 Y_1 + \alpha_2 Y_2 + \ldots + \alpha_N Y_N \tag{15}
\]
then for any integer \( j > N \), we have a similar expression
\[
Y_j = \beta_1 Y_1 + \beta_2 Y_2 + \ldots + \beta_N Y_N. \tag{16}
\]
Proof. Due to the property \( Y_{k+1} = D^{-1}Y_k D \), we have from (15)

\[
Y_{N+2} = D^{-1}(\alpha_1)Y_2 + D^{-1}(\alpha_2)Y_3 + \ldots + D^{-1}(\alpha_N)Y_N.
\]

Now by using induction one can easily complete the proof of the Lemma. □

Lemma 2.2. The following commutativity relations take place:

\[
[Y_0, Y_{-0}] = 0, \quad [Y_0, Y_{1}] = 0, \quad [Y_{-0}, Y_{-1}] = 0.
\]

Proof. The first of the relations is evident (see the definition (11) of \( Y_0 \) and \( Y_{-0} \)). In order to prove two others find the coordinate representation of the operators \( Y_1 \) and \( Y_{-1} \), defined by (6) and (9), acting on the class of locally smooth functions of the variables \( x, n, t, t_x, t_{xx}, \ldots \).

By direct computations

\[
Y_1 H = D^{-1} \frac{d}{dt} D H = D^{-1} \frac{d}{dt} H(t; f, f_x, \ldots)
\]

\[
= \left\{ \frac{\partial}{\partial t} + D^{-1} \left( \frac{\partial f}{\partial t_1} \right) \frac{\partial}{\partial t_x} + D^{-1} \left( \frac{\partial f_x}{\partial t_1} \right) \frac{\partial}{\partial t_{xx}} + \ldots \right\} H(t, t_x, t_{xx}, \ldots)
\]

one gets

\[
Y_1 = \frac{\partial}{\partial t} + D^{-1} \left( \frac{\partial f}{\partial t_1} \right) \frac{\partial}{\partial t_x} + D^{-1} \left( \frac{\partial f_x}{\partial t_1} \right) \frac{\partial}{\partial t_{xx}} + \ldots.
\] (17)

Now notice that all of the functions \( f, f_x, f_{xx}, \ldots \) depend on the variables \( t_1, t, t_x, t_{xx}, \ldots \) and do not depend on \( t_2 \) hence the coefficients of the vector field \( Y_1 \) do not depend on \( t_1 \) and therefore the operators \( Y_1 \) and \( Y_0 \) commute. In a similar way by using the explicit coordinate representation

\[
Y_{-1} = \frac{\partial}{\partial t} + D \left( \frac{\partial g}{\partial t_{-1}} \right) \frac{\partial}{\partial t_x} + D \left( \frac{\partial g_x}{\partial t_{-1}} \right) \frac{\partial}{\partial t_{xx}} + \ldots,
\]

where \( g \) is defined by (7), one can prove that \([Y_{-0}, Y_{-1}] = 0\). □

The following statement turned out to be very useful for studying the characteristic Lie algebra \( L_n \).

Lemma 2.3. (1) Suppose that the vector field

\[
Y = \alpha(0) \frac{\partial}{\partial t} + \alpha(1) \frac{\partial}{\partial t_x} + \alpha(2) \frac{\partial}{\partial t_{xx}} + \ldots,
\]

where \( \alpha_x(0) = 0 \), solves the equation \([D_x, Y] = 0\), then \( Y = \alpha(0) \frac{\partial}{\partial t} \).

(2) Suppose that the vector field

\[
Y = \alpha(1) \frac{\partial}{\partial t_x} + \alpha(2) \frac{\partial}{\partial t_{xx}} + \alpha(3) \frac{\partial}{\partial t_{xxx}} + \ldots
\]

solves the equation \([D_x, Y] = hY\), where \( h \) is a function of variables \( t, t_x, t_{xx}, \ldots, t_{\pm1}, t_{\pm2}, \ldots \), then \( Y = 0 \).
The proof of Lemma 2.3 can be easily derived from the following formula

\[
[D_x, Y] = -(\alpha(0)f_t + \alpha(1)f_{tx})\frac{\partial}{\partial t_1} + (\alpha_x(0) - \alpha(1))\frac{\partial}{\partial t} \\
+ (\alpha_x(1) - \alpha(2))\frac{\partial}{\partial t_{xx}} + (\alpha_x(2) - \alpha(3))\frac{\partial}{\partial t_{xxx}} + \ldots 
\]  

(18)

In the formula (17) we have already given an enlarged coordinate form of the operator \(Y_1\). One can check that the operator \(Y_2\) is a vector field of the form

\[
Y_2 = D^{-1}(Y_1(f))\frac{\partial}{\partial t_x} + D^{-1}(Y_1(f_x))\frac{\partial}{\partial t_{xx}} + D^{-1}(Y_1(f_{xx}))\frac{\partial}{\partial t_{xxx}} + \ldots 
\]

(19)

It immediately follows from the equation \(Y_2 = D^{-1}Y_1D\) and the coordinate representation (17). By induction one can prove similar formulas for arbitrary \(Y_{j+1}, j \geq 1:\n
\[
Y_{j+1} = D^{-1}(Y_j(f))\frac{\partial}{\partial t_x} + D^{-1}(Y_j(f_x))\frac{\partial}{\partial t_{xx}} + D^{-1}(Y_j(f_{xx}))\frac{\partial}{\partial t_{xxx}} + \ldots 
\]

(20)

Lemma 2.4. For any \(n \geq 0\), we have

\[
[D_x, Y_n] = -\sum_{j=0}^{n} D^{-j}(Y_{n-j}(f))Y_j.
\]

(21)

In particular,

\[
[D_x, Y_0] = -Y_0(f)Y_0, \quad \text{(22)}
\]

\[
[D_x, Y_1] = -Y_1(f)Y_0 - D^{-1}(Y_0(f))Y_1. \quad \text{(23)}
\]

Proof. We have,

\[
[D_x, Y_0]H(t, t_1, t_x, t_{xx}, \ldots) = D_xH_{t_1} - Y_0D_xH
\]

\[
= (H_{tt_x} + H_{tx_1}) - \frac{\partial}{\partial t_1}(H_t + H_{tx_1})
\]

\[
= -H_{tx_1} + -Y_0(f)Y_0H,
\]

i.e. equation (22) holds.

We prove equations (21), \(n \geq 1\), by induction. By (17), (18) and (22),

\[
[D_x, Y_1] = -Y_1(f)\frac{\partial}{\partial t_1} - D^{-1}(Y_0(f))\frac{\partial}{\partial t} + D^{-1}[D_x, Y_0]f\frac{\partial}{\partial t_x} + D^{-1}[D_x, Y_0]f_x\frac{\partial}{\partial t_{xx}} + \ldots
\]

\[
= -Y_1(f)Y_0 - D^{-1}(Y_0(f))Y_0(f)\frac{\partial}{\partial t} - D^{-1}(Y_0(f)Y_0(f))\frac{\partial}{\partial t_x} - D^{-1}(Y_0(f)Y_0(f))\frac{\partial}{\partial t_{xx}} + \ldots
\]

\[
= -Y_1(f)Y_0 - D^{-1}(Y_0(f))Y_1,
\]
that shows that the base of Mathematical Induction holds for \( n = 1 \). By (20), (18) and the inductive assumption, for \( n \geq 1 \) we have,

\[
[D_x, Y_{n+1}] = -D^{-1}(Y_n(f))f_{t_x} \frac{\partial}{\partial t_1} - D^{-1}(Y_n(f)) \frac{\partial}{\partial t} + D^{-1}[D_x, Y_n]f \frac{\partial}{\partial t_x} + \ldots
\]

By Lemma 2.2, the coefficients of the vector field \( Y \) do not depend on the variable \( t \). If the operator \( f \) is defined by Lemma 2.6.

**Proof.** By Lemma 2.2 \([Y_1, Y_0] = 0\). The reason for this equality is that the coefficients of the vector field \( Y_1 \) do not depend on the variable \( t_1 \). They might depend only on \( t_{-1}, t, t_x, t_{xx}, t_{xxx}, \ldots \). The coefficients of the vector field \( Y_2 \) being of the form \( D^{-1}(Y_1(D_x f)) \) (see (19)) also do not depend on the variable \( t_1 \). They might depend only on \( t_{-2}, t_{-1}, t, t_x, t_{xx}, t_{xxx}, \ldots \). Therefore, we have \([Y_2, Y_0] = 0\). Continuing this reasoning we see that for any \( n \geq 1 \) the commutativity relation

\([Y_n, Y_0] = 0\)

takes place. Consider now the commutator \([Y_n, Y_{n+m}], n \geq 1, m \geq 1\). We have,

\([Y_n, Y_{n+m}] = [D^{-n}Y_0D^n, D^{-(n+m)}Y_0D^{n+m}] = D^{-n}[Y_0, Y_m]D^n = 0\),

that finishes the proof of the Lemma. □

**Lemma 2.5.** Lie algebra generated by the operators \( Y_1, Y_2, Y_3, \ldots \) is commutative.

**Proof.** By Lemma 2.2 \([Y_1, Y_0] = 0\). The reason for this equality is that the coefficients of the vector field \( Y_1 \) do not depend on the variable \( t_1 \). They might depend only on \( t_{-1}, t, t_x, t_{xx}, t_{xxx}, \ldots \). The coefficients of the vector field \( Y_2 \) being of the form \( D^{-1}(Y_1(D_x f)) \) (see (19)) also do not depend on the variable \( t_1 \). They might depend only on \( t_{-2}, t_{-1}, t, t_x, t_{xx}, t_{xxx}, \ldots \). Therefore, we have \([Y_2, Y_0] = 0\). Continuing this reasoning we see that for any \( n \geq 1 \) the commutativity relation

\([Y_n, Y_0] = 0\)

takes place. Consider now the commutator \([Y_n, Y_{n+m}], n \geq 1, m \geq 1\). We have,

\([Y_n, Y_{n+m}] = [D^{-n}Y_0D^n, D^{-(n+m)}Y_0D^{n+m}] = D^{-n}[Y_0, Y_m]D^n = 0\),

that finishes the proof of the Lemma. □

**Lemma 2.6.** If the operator \( Y_2 = 0 \) then \([X_1, Y_1] = 0\).

**Proof.** By (19), \( Y_2 = 0 \) implies that \( Y_1(f) = 0 \). Due to (17), \( Y_1(f) = 0 \) means that \( f_{t_x} + D^{-1}(f_{t_x})f_{t_x} = 0 \) and, therefore, \( D^{-1}(f_{t_x}) \) does not depend on \( t_{-1} \). Together with Lemma 2.4 and the fact that \([D_x, X_1] = 0\) (see the definition (12)) it allows us to conclude that

\[
[D_x, [X_1, Y_1]] = -[X_1, [Y_1, D_x]] - [Y_1, [D_x, X_1]]
\]

\[
= -[X_1, D^{-1}(f_{t_x})Y_1]
\]

\[
= -D^{-1}(f_{t_x})[X_1, Y_1],
\]

i.e. \([D_x, [X_1, Y_1]] = -D^{-1}(f_{t_x})[X_1, Y_1]\). By Lemma 2.4 part (2), it follows that \([X_1, Y_1] = 0\). □

**Lemma 2.7.** The operator \( Y_2 = 0 \) if and only if we have

\[
f_t + D^{-1}(f_{t_x})f_{t_x} = 0.
\]
Lemma 3.1. We have, 
$$f_1 = \frac{\partial}{\partial t} + X(f) \frac{\partial}{\partial t_1} + X(g) \frac{\partial}{\partial t_2} + \ldots,$$
that implies, by Lemma 2.3, part (2), that
$$[D_x, Y_2] = -D^{-2}(Y_0(f))Y_2$$

Proof. Assume $Y_2 = 0$. By (19), $Y_1(f) = 0$. Due to (17) equality $Y_1(f) = 0$ is another way of writing (24).

Conversely, assume (24) holds, i.e. $Y_1(f) = 0$. It follows from (19) that $Y_2(f) = 0$. Due to Lemma 2.4, we have
$$[D_x, Y_2] = -D^{-2}(Y_0(f))Y_2$$
that implies, by Lemma 2.3 part (2), that $Y_2 = 0$. □

Corollary 2.8. The dimension of the Lie algebra $L_n$ associated with $n$-integral is equal to 2 if and only if (24) holds, or the same $Y_2 = 0$.

Proof. By Theorem 1.1 the dimension of $L_n$ is 2 if and only if $Y_2 = \lambda_1 X_1 + \mu_1 Y_1$ and 
$[X_1, Y_1] = \lambda_2 X_1 + \mu_2 Y_1$ for some $\lambda_i, \mu_i, i = 1, 2$.

Assume the dimension of $L_n$ is 2. Then $Y_2 = \lambda_1 X_1 + \mu_1 Y_1$. Since among $X_1, Y_1, Y_2$ differentiation by $t_{-1}$ is used only in $X_1$, differentiation by $t$ is used only in $Y_1$, then
$$\lambda_1 = \mu_1 = 0.$$ Therefore, $Y_2 = 0$, or the same, by Lemma 2.7 (24) holds.

Conversely, assume (24) holds, that is $Y_2 = 0$. By Lemma 2.6 $[X_1, Y_1] = 0$. Since $Y_2$ and $[X_1, Y_1]$ are trivial linear combinations of $X_1$ and $Y_1$ then the dimension of $L_n$ is 2. □

3 Characteristic Lie Algebra $L_x$

Denote by 
$$K_1 = [X, K_0], \quad K_2 = [X, K_1], \quad \ldots, \quad K_{n+1} = [X, K_n], \quad n \geq 1,$$ 
where $X$ and $K_0$ are defined by (14) and (13).

It is easy to see that
$$K_1 = \frac{\partial}{\partial t} + X(f) \frac{\partial}{\partial t_1} + X(g) \frac{\partial}{\partial t_2} + \ldots,$$
$$K_n = \sum_{j=1}^{\infty} \left\{ X^n(f_{j-1}) \frac{\partial}{\partial t_j} + X^n(g_{j-1}) \frac{\partial}{\partial t_{j-1}} \right\}, \quad n \geq 2,$$
where $f_0 := f$ and $g_0 := g$.

Lemma 3.1. We have,
$$DXD^{-1} = \frac{1}{f_t} X, \quad DK_0 D^{-1} = K_0 - \frac{t_x f_t + f(t_x)}{f_t} X,$$
$$DK_1 D^{-1} = \frac{1}{f_{t_x}} K_1 - \frac{t_x f_t + f(t_x)}{f_{t_x}^2} X, \quad DK_2 D^{-1} = \frac{1}{f_{t_x}^2} K_2 - \frac{f_t t_x f_t}{f_{t_x}^3} K_1 + \frac{f_{t_x} t_x f_t}{f_{t_x}^4} X,$$
$$DK_3 D^{-1} = \frac{1}{f_{t_x}^3} K_3 - \frac{3 f_{t_x} t_x f_t}{f_{t_x}^4} K_2 + \left(3 \frac{f_{t_x}^2}{f_{t_x}^3} - \frac{f_{t_x} t_x f_t}{f_{t_x}^4} \right) K_1 - \frac{f_t f_{t_x}}{f_{t_x}^5} \left(3 \frac{f_{t_x}^2}{f_{t_x}^3} f_t - \frac{f_{t_x} t_x f_t}{f_{t_x}^4} \right) X.$$
Proof. In the proof of this lemma whenever we write $H$ and $H^*$ we mean functions $H(x, t, t_x, t_{t_x}, t_{t_{t_x}}, \ldots)$ and $H^* = D^{-1}H = H(x, t_{t_x}, t_{t_{t_x}}, \ldots)$.

Since

$$DX D^{-1}H = DH_{t_x} = D \left\{ g_{t_x} \frac{\partial H^*}{\partial g} \right\} = D(g_{t_x}) \frac{\partial H}{\partial t_x} = D(g_{t_x})XH$$

and $D(g_{t_x}) = \frac{1}{f_{t_x}}$ then $DX D^{-1}H = \frac{1}{f_{t_x}}XH$.

Since

$$DK_0 D^{-1}H = D \left( \frac{\partial H^*}{\partial x} + t_x \frac{\partial H^*}{\partial g} + \frac{\partial H^*}{\partial t_x} + f_1 \frac{\partial H^*}{\partial t_1} + g_{t_x} \frac{\partial H^*}{\partial t_{t_x}} \right) H^* = D \left( \frac{\partial H}{\partial x} + t_x g_{t_x} + t_x \frac{\partial H}{\partial g} + f_1 \frac{\partial H}{\partial t_1} + g_{t_x} \frac{\partial H}{\partial t_{t_x}} + \ldots \right)$$

$$= \left( \frac{\partial H}{\partial x} + t_x \frac{\partial H}{\partial g} + f \frac{\partial H}{\partial t_1} + f_1 \frac{\partial H}{\partial t_2} + \ldots \right) + (t_x D(g_{t_x}) + f D(g_t)) \frac{\partial H}{\partial t_x}$$

and $D(g_{t_{t_x}}) = -\frac{f_t}{f_{t_x}}$, $D(g_t) = -\frac{f_{t_t}}{f_{t_x}}$ then $DK_0 D^{-1}H = K_0 H - \frac{t_x f_t + f f_{t_t}}{f_{t_x}}XH$.

Using formulas (28) for $DX D^{-1}$, $DK_0 D^{-1}$ and the definition (25) of $K_1$ we have

$$DK_1 D^{-1} = [DX D^{-1}, DK_0 D^{-1}] = \left[ \frac{1}{f_{t_x}}X, K_0 - \frac{f_t f_{t_t} + f f_{t_t}}{f_{t_x}}X \right]$$

$$= \frac{1}{f_{t_x}} K_1 - K_0 \left( \frac{1}{f_{t_x}}X - \frac{1}{f_{t_x}} X \left( \frac{t_x f_t + f f_{t_t}}{f_{t_x}} \right) X + \frac{t_x f_t + f f_{t_t}}{f_{t_x}} \left( -\frac{f_{t_x} t_{t_x}}{f_{t_x}^2} \right) X \right)$$

$$= \frac{1}{f_{t_x}} K_1 - \frac{f_t + f_t f_{t_t}}{f_{t_x}^2} X.$$

Using formulas (28) and (29) for $DX D^{-1}$, $DK_1 D^{-1}$ and the definition (25) of $K_2$ we have

$$DK_2 D^{-1} = [DX D^{-1}, DK_1 D^{-1}] = \left[ \frac{1}{f_{t_x}} X, \frac{1}{f_{t_x}} K_1 - \frac{f_t + f_t f_{t_t}}{f_{t_x}^2} X \right]$$

$$= -\frac{f_{t_t}}{f_{t_x}^2} f_{t_{t_x}} K_1 - \frac{1}{f_{t_x}} K_1 \left( \frac{1}{f_{t_x}} X + \frac{1}{f_{t_x}^2} K_2 - \frac{1}{f_{t_x}} X \left( \frac{f_t + f_t f_{t_t}}{f_{t_x}} \right) \right) X$$

$$= -\frac{f_t + f_t f_{t_t}}{f_{t_x}} f_{t_{t_x}} X$$

$$= \frac{1}{f_{t_x}} K_2 - \frac{f_{t_t}}{f_{t_x} f_{t_{t_x}}} X.$$
Using formulas (28) and (29) for $DXD^{-1}$, $DK_2D^{-1}$ and the definition (25) of $K_3$ we have

$$DK_3D^{-1} = [DXD^{-1}, DK_2D^{-1}] = \left[ \frac{1}{f_{tx}}X, \frac{1}{f_{tx}}K_2 - \frac{f_{tx_t}f_t}{f_{tx}}K_1 + \frac{f_{tx_t}f_t}{f_{tx}}X \right]$$

$$= \frac{1}{f_{tx}^3}K_3 - \frac{f_{tx_t}f_t}{f_{tx}^4}K_2 - \frac{2f_{tx_t}f_t}{f_{tx}^4}K_2 - \frac{1}{f_{tx}}X\left( \frac{f_{tx_t}f_t}{f_{tx}^4}K_1 + X\left\{ \frac{1}{f_{tx}}X\left( \frac{f_{tx_t}f_t}{f_{tx}^4} \right) \right\} \right)$$

$$- \frac{1}{f_{tx}^2}K_2\left( \frac{1}{f_{tx}} \right) + \frac{f_{tx_t}f_t}{f_{tx}^4}K_1\left( \frac{1}{f_{tx}} \right) - \frac{f_{tx_t}f_t}{f_{tx}}X\left( \frac{1}{f_{tx}} \right)$$

$$= \frac{1}{f_{tx}^3}K_3 - \frac{3f_{tx_t}f_t}{2f_{tx}^4}K_2 - \frac{f_{tx_t}t_xf_t}{f_{tx}^5} - \frac{3f_{tx_t}^2f_t}{f_{tx}^5}K_1 + \frac{f_{tx} f_{tx_t}f_t f_{tx}^2}{f_{tx}^5} - \frac{3f_{tx_t}^2}{f_{tx}^5}X,$$

that finishes the proof of the Lemma. □

**Lemma 3.2.** For any $n \geq 1$ we have,

$$DK_nD^{-1} = a^{(n)}_nK_n + a^{(n)}_{n-1}K_{n-1} + a^{(n)}_{n-2}K_{n-2} + \ldots + a^{(n)}_1K_1 + b^{(n)}X,$$  \hspace{1cm} (31)

where coefficients $b^{(n)}$ and $a^{(n)}_k$ are functions that depend only on variables $t$, $t_1$ and $t_x$ for all $k$, $1 \leq k \leq n$. Moreover,

$$\begin{align*}
    a^{(n)}_n &= \frac{1}{f_{tx}^n}, \quad n \geq 1, \\
    a^{(n)}_{n-1} &= -\frac{n(n-1)}{2f_{tx}^{n+1}}, \quad n \geq 2, \\
    b^{(n)} &= -\frac{f_{tx}}{f_{tx}}a^{(n)}_1, \quad n \geq 2, \\
    a^{(n)}_{n-2} &= \frac{(n-2)(n^2-1)}{4} \frac{f_{tx_t}^2}{2f_{tx}^{n+2}} - \frac{(n-2)(n-1)}{3} \frac{f_{tx_t}t_x}{2f_{tx}^{n+1}}, \quad n \geq 3.
\end{align*}$$ \hspace{1cm} (32) \hspace{1cm} (33)

**Proof.** We use induction to prove the Lemma. As Lemma 3.1 shows the base of Mathematical Induction holds.

Assuming the representation (31) for $DK_nD^{-1}$ is true and all coefficients $a^{(n)}_k$ are functions of $t$, $t_1$, $t_x$ only, consider $DK_{n+1}D^{-1}$. We have,

$$DK_{n+1}D^{-1} = [DXD^{-1}, DK_nD^{-1}]$$

$$= \left[ \frac{1}{f_{tx}}X, a^{(n)}_nK_n + a^{(n)}_{n-1}K_{n-1} + a^{(n)}_{n-2}K_{n-2} + \ldots + a^{(n)}_1K_1 + b^{(n)}X \right]$$

$$= a^{(n+1)}_{n+1}K_{n+1} + a^{(n+1)}_nK_n + a^{(n+1)}_{n-1}K_{n-1} + \ldots + a^{(n+1)}_1K_1 + b^{(n+1)}X,$$

where

$$\begin{align*}
    a^{(n+1)}_{n+1} &= \frac{1}{f_{tx}}a^{(n)}_n, \\
    a^{(n+1)}_{n-k} &= \frac{1}{f_{tx}}X(a^{(n)}_{n-k}) + \frac{1}{f_{tx}}a^{(n)}_{n-k-1}, \quad 0 \leq k \leq n - 2, \\
    a^{(n+1)}_1 &= \frac{1}{f_{tx}}X(a^{(n)}_1).
\end{align*}$$
Lemma 3.3. Suppose that the vector field

\[ a_{n-k}^{(n+1)}, \quad 0 \leq k \leq n-1, \] are functions of \( t, t_1, t_x \) only as well.
Assuming formulas \((32)\) and \((33)\) for \( a_{n}^{(n)}, a_{n-1}^{(n)} \) and \( a_{n-2}^{(n)} \) are true, the following equality

\[
DK_{n+1}D^{-1} = a_{n+1}^{(n+1)}K_{n+1} + a_{n}^{(n+1)}K_{n} + a_{n-1}^{(n+1)}K_{n-1} + \ldots + a_{1}^{(n+1)}K_{1} + b^{(n+1)}X
\]

implies that

\[
a_{n+1}^{(n+1)} = \frac{1}{f_{t_x}^{n+1}},
\]

\[
a_{n}^{(n+1)} = \frac{1}{f_{t_x}^{n}}X\left(\frac{1}{f_{t_x}^{n}}\right) + \frac{1}{f_{t_x}^{n}}a_{n-1}^{(n)}
= -\frac{n f_{t_x}^{n}}{f_{t_x}^{n+2}} - \frac{n(n-1)f_{t_x}^{n+1}}{2f_{t_x}^{n+2}} = -\frac{n(n+1)f_{t_x}^{n+1}}{2f_{t_x}^{n+2}},
\]

\[
a_{n-1}^{(n+1)} = \frac{1}{f_{t_x}^{n}}X(a_{n-1}^{(n)}) + \frac{1}{f_{t_x}^{n}}a_{n-2}^{(n)}
= \frac{(n-1)n(n+1)(n+2)}{4} \frac{f_{t_x}^{n+3}}{2f_{t_x}^{n+3}} - \frac{(n-1)n(n+1)}{3} \frac{f_{t_x}^{n+1}}{2f_{t_x}^{n+2}}.
\]

Using the same notation for \( H \) and \( H^* \) as in Lemma \((3.1)\) by \((27)\), we have (for \( n \geq 2 \)),

\[
DK_{n}D^{-1}H = D\left\{X^{n}(f)\frac{\partial}{\partial t_1} + X^{n}(g)\frac{\partial}{\partial t_{-1}} + X^{n}(f_1)\frac{\partial}{\partial t_2} + X^{n}(g_{-1})\frac{\partial}{\partial t_{-2}} + \ldots\right\}H^*
= D\left\{X^{n}(f)\frac{\partial H^*}{\partial t_1} + X^{n}(g)\frac{\partial H^*}{\partial t_{-1}} + X^{n}(g_t_{-1})\frac{\partial H^*}{\partial t} + X^{n}(f_1)\frac{\partial H^*}{\partial t_2} + \ldots\right\}
= D(X^{n}(f))\frac{\partial H}{\partial t} + D(X^{n}(g))\frac{\partial H}{\partial t} + D(X^{n}(g))D(g_{t_{-1}})\frac{\partial H}{\partial t} + D(X^{n}(f_1))\frac{\partial H}{\partial t} + \ldots
= D(X^{n}(g))\frac{\partial H}{\partial t} - f_{t_x}D(X^{n}(g))XH + \sum_{k=1}^{\infty}\left(a_{k}^{(n)}\frac{\partial}{\partial t_k} + \beta_{k}^{(n)}\frac{\partial}{\partial t_{-k}}\right)H
= (b^{(n)}X + a_{1}^{(n)}K_{1} + a_{2}^{(n)}K_{2} + \ldots + a_{n}^{(n)}K_{n})H.
\]

Since among \( X, K_{k}, 1 \leq k \leq n \) differentiation by \( t \) is used only in \( K_{1} \) and differentiation by \( t_x \) is used only in \( X \) then \( a_{1}^{(n)} = D(X^{n}(g)) \) and \( b^{(n)} = -\frac{f_{t}}{f_{t_x}}D(X^{n}(g)) \), that implies that

\[ b^{(n)} = -\frac{f_{t}}{f_{t_x}}a_{1}^{(n)}. \] The fact that \( b^{(n)} \) is a function of \( t, t_1 \) and \( t_x \) only follows then from the analogous result for \( a_{1}^{(n)} \). \( \Box \)

**Lemma 3.3.** Suppose that the vector field

\[ K = \sum_{j=1}^{\infty}\left\{ \alpha(k)\frac{\partial}{\partial t_k} + \alpha(-k)\frac{\partial}{\partial t_{-k}}\right\}
\]

solves the equation \( DKD^{-1} = hK \), where \( h \) is a function of variables \( t, t_{\pm 1}, t_{\pm 2}, \ldots, t_{x}, t_{xx}, \ldots \), then \( K = 0 \).
The proof of Lemma 3.3 can be easily derived from the following formula.

\[
DKD^{-1} = -\frac{f_t}{f_{tx}} D(\alpha(-1))X + D(\alpha(-1)) \frac{\partial}{\partial t} + D(\alpha(-2)) \frac{\partial}{\partial t} - 1
+ \sum_{j=2}^{\infty} \left\{ D(\alpha(1-j)) \frac{\partial}{\partial t^j} + D(\alpha(-j-1)) \frac{\partial}{\partial t} \right\}.
\] (34)

Consider the linear space \( L^* \) generated by \( X \) and \( K_n \), \( n \geq 0 \). It is a subset in the finite dimensional Lie algebra \( L_x \). Therefore, there exists a natural number \( N \) such that

\[
K_{N+1} = \mu X + \lambda_0 K_0 + \lambda_1 K_1 + \ldots + \lambda_N K_N
\] (35)

and \( X, K_n, 0 \leq n \leq N \) are linearly independent. It can be proved that the coefficients \( \mu, \lambda_i, 0 \leq i \leq N \), are functions of finite number of dynamical variables. Since \( \mu = \lambda_0 = \lambda_1 = 0 \), then the equality above should be studied only if \( N \geq 2 \), or the same, if the dimension of \( L_x \) is 4 or more. Case, when the dimension of \( L_x \) is equal to 3 must be considered separately.

Assume \( N \geq 2 \). Then

\[
DK_{N+1}D^{-1} = D(\lambda_2)DK_2D^{-1} + D(\lambda_3)DK_3D^{-1} + \ldots + D(\lambda_{N-1})DK_{N-1}D^{-1}
+ D(\lambda_N)DK_N D^{-1}.
\]

Rewriting \( DK_k D^{-1} \) in the last equation for each \( k, 2 \leq k \leq N+1 \), using formulas (31), and \( K_{N+1} \) as a linear combination (35) allows us to compare coefficients before \( K_k, 2 \leq k \leq N \) and obtain the following system of equations.

\[
a^{(N+1)}_{N+1} \lambda_N + a^{(N+1)}_N = D(\lambda_N)a^{(N)}_N
\]

\[
a^{(N+1)}_{N+1} \lambda_{N-1} + a^{(N+1)}_{N-1} = D(\lambda_{N-1})a^{(N-1)}_{N-1} + D(\lambda_N)a^{(N)}_{N-1}
\]

\[
a^{(N+1)}_{N+1} \lambda_{N-2} + a^{(N+1)}_{N-2} = D(\lambda_{N-2})a^{(N-2)}_{N-2} + D(\lambda_{N-1})a^{(N-1)}_{N-2} + D(\lambda_N)a^{(N)}_{N-2}
\]

\[
\vdots
\]

\[
a^{(N+1)}_{N+1} \lambda_k + a^{(N+1)}_k = D(\lambda_k)a^{(k)}_k + D(\lambda_{k+1})a^{(k+1)}_k + \ldots + D(\lambda_N)a^{(N)}_k,
\]

for \( 2 \leq k \leq N \). Using the fact that coefficients \( \lambda_k, 2 \leq k \leq N \), depend on a finite number of arguments, it is easy to see that all of them are functions of only variables \( t \) and \( t_x \).

**Remark 3.4.** The first two equations of the system (36) are

\[
\frac{1}{f_{tx}} \frac{N(N+1)}{2} f_{tx} f_{tx}, \quad f_{tx} f_{tx} = \frac{1}{f_{tx}} D(\lambda_N),
\]

\[
\frac{1}{f_{tx}} \frac{1}{4} f_{tx} f_{tx} (N^2 - 1)N(N+1) - \frac{1}{3} \frac{(N-1)N(N+1)}{2 f_{tx}} f_{tx} f_{tx} - \frac{1}{f_{tx}} D(\lambda_{N-1}) - \frac{N(N-1)}{2} \frac{f_{tx} f_{tx}}{f_{tx} f_{tx} f_{tx} f_{tx} f_{tx}} D(\lambda_N).
\] (37)
Lemma 3.5. $K_2 = 0$ if and only if $f_{t_x t_x} = 0$.

Proof. Assume $K_2 = 0$. By representation (27) we have $X^2(f) = 0$, that is $f_{t_x t_x} = 0$. Conversely, assume that $f_{t_x t_x} = 0$. By (29) we have $DK_2 D^{-1} = \frac{1}{f_{t_x}} K_2$ that implies, by Lemma 3.3, that $K_2 = 0$. □

Introduce

$$Z_2 = [K_0, K_1].$$

Lemma 3.6. We have,

$$DZ_2 D^{-1} = \frac{1}{f_{t_x}} Z_2 - \frac{t_x f_t + f f_{t t}}{f_{t_x}^2} K_2 + CK_1 - \frac{f_t}{f_{t_x}} CX,$$

where

$$C = -\frac{t_x f_{t t}}{f_{t_x}^2} - \frac{f f_{t_x t_x}}{f_{t_x}^2} + \frac{f_t}{f_{t_x}} + \frac{t_x f_t f_{t_x t_x}}{f_{t_x}^2} + \frac{f f_{t_x t_{xx}}}{f_{t_x}^2}.$$

Proof. Using the formulas (28) and (29) for $DK_0 D^{-1}, DK_1 D^{-1}$ and the definition (38) of $Z_2$ we have,

$$DZ_2 D^{-1} = [DK_0 D^{-1}, DK_1 D^{-1}] = [K_0 - AX, \frac{1}{f_{t_x}} K_1 - BX]$$

$$= K_0 \left( \frac{1}{f_{t_x}} \right) K_1 + \frac{1}{f_{t_x}} Z_2 - K_0(B)X + BK_1 - AX \left( \frac{1}{f_{t_x}} \right) K_1$$

$$- A \frac{1}{f_{t_x}} K_2 + \frac{1}{f_{t_x}} K_1(A)X + AX(B)X - BX(A)X$$

$$= \frac{1}{f_{t_x}} Z - A \frac{1}{f_{t_x}} K_2 + \left( K_0 \left( \frac{1}{f_{t_x}} \right) + B - AX \left( \frac{1}{f_{t_x}} \right) \right) K_1$$

$$+ \left( AX(B) - BX(A) - K_0(B) + \frac{1}{f_{t_x}} K_1(A) \right) X,$$

where

$$A = \frac{t_x f_t + f f_{t t}}{f_{t_x}}, \quad B = \frac{f_t + f_{t_x} f_{t_{t_x}}}{f_{t_x}^2}.$$

The coefficient before $K_1$ is

$$K_0 \left( \frac{1}{f_{t_x}} \right) + B - AX \left( \frac{1}{f_{t_x}} \right) = -t_x \frac{f_{t_x t}}{f_{t_x}^2} - \frac{f f_{t_x t}}{f_{t_x}^2} + \frac{f_t + f_{t_x} f_{t_{t_x}}}{f_{t_x}^2} + \frac{f_{t_x t} t_x f_t + f f_{t_{t_x}}}{f_{t_x}} := C.$$

It follows from (31) that the coefficient before $X$ in (39) is $-\frac{f_t}{f_{t_x}} C$. □

Lemma 3.7. The dimension of the Lie algebra $L_x$ generated by $X$ and $K_0$ is equal to 3 if and only if

$$f_{t_x t_{xx}} = 0$$

and

$$-\frac{t_x f_{t_x t}}{f_{t_x}^2} - \frac{f f_{t_x t}}{f_{t_x}^2} + \frac{f_t + f_{t_x} f_{t_{t_x}}}{f_{t_x}^2} = 0.$$
Proof. Assume the dimension of the Lie algebra $L_x$ generated by $X$ and $K_0$ is equal to 3. It means that the algebra consists of $X$, $K_0$ and $K_1$ only, and $K_2 = \lambda_1 X + \lambda_2 K_0 + \lambda_3 K_1$, $Z_2 = \mu_1 X + \mu_2 K_0 + \mu_3 K_1$ for some functions $\lambda_i$ and $\mu_i$. Since among $X$, $K_0$, $K_1$, $K_2$ and $Z_2$ we have differentiation by $t_x$ only in $X$, differentiation by $x$ only in $K_0$, then $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. Therefore, $K_2 = \lambda_3 K_1$ and $Z_2 = \mu_3 K_1$. Also, among $K_1$, $K_2$ and $Z_2$ we have differentiation by $t$ only in $K_1$ then $\lambda_3 = \mu_3 = 0$. We have proved that if the dimension of the Lie algebra $L$ is 3 then $K_2 = 0$ and $Z_2 = 0$. By Lemma 3.5 condition (10) is satisfied. It follows from (39) that

$$0 = DZ_2 D^{-1} = \frac{1}{f_{tx}} \frac{t_x f_t + f_x f_t}{f_{tx}} K_2 + CK_1 - \frac{f_t}{f_{tx}} CX = CK_1 - \frac{f_t}{f_{tx}} CX.$$ 

Since $X$ and $K_1$ are linearly independent then equality $CK_1 - \frac{f_t}{f_{tx}} CX = 0$ implies $C = 0$. Equality (41) follows from (10) and $C = 0$. Conversely, assume that properties (10) and (11) are satisfied. To prove that the dimension of the Lie algebra $L_x$ is equal to 3 it is enough to show that $K_2 = 0$ and $Z_2 = 0$. It follows from (40) and Lemma 3.5 that $K_2 = 0$. From the formula (39) for $DZ_2 D^{-1}$, property (11) and knowing that $K_2 = 0$ we have that $DZ_2 D^{-1} = \frac{1}{f_{tx}} Z_2$ that implies, by Lemma 3.3, that $Z_2 = 0$. □

4 Equations with characteristic algebras of the minimal possible dimensions.

Corollary 4.1. If Lie algebras for $n$- and $x$-integrals have dimensions 2 and 3 respectively, then equation $t_{1x} = f(t, t_1, t_x)$ can be reduced to $t_{1x} = t_x + t_1 - t$.

Proof. By Lemma 3.7 and Corollary 2.8, the dimensions of $n$- and $x$-Lie algebras are 2 and 3 correspondingly means equations (24), (40), and (41) are satisfied. It follows from property (10) that $f(t, t_1, t_x) = G(t, t_1) t_x + H(t, t_1)$ for some functions $G(t, t_1)$ and $H(t, t_1)$. By (24), $G_t t_x + H_t + \{D^{-1}(G_t t_x + H_t_t)\} G = 0$, that is

$$D^{-1}(G_t t_x + H_t_t) = -\frac{G_t}{G} t_x - \frac{H_t}{G}. \quad (42)$$

Note that $t_{1x} = Gt_x + H$ implies $t_x = D^{-1}(G) t_{-1x} + D^{-1}(H)$ and, therefore, $t_{-1x} = \frac{1}{D^{-1}(G)} t_x - D^{-1}(H) / D^{-1}(G)$. We continue with (42) and obtain the following equality

$$D^{-1} \left( \frac{G_t}{G} \right) t_x - D^{-1} \left( \frac{G_t H}{G} \right) + D^{-1}(H_t) = -\frac{G_t}{G} t_x - \frac{H_t}{G}$$

that gives rise to two equations

$$D^{-1} \left( \frac{G_t}{G} \right) = -\frac{G_t}{G}, \quad D^{-1} \left( H_t - \frac{G_t H}{G} \right) = -\frac{H_t}{G}. \quad (43)$$
It is seen from the first equation of (43) that \( \frac{G_t}{G} \) is a function that depends only on variable \( t \), even though functions \( G \) and \( G_t \) depend on variables \( t \) and \( t_1 \). Denote \( \frac{G_t}{G} = a(t) \). Then \( \frac{G_t}{G} = -a(t_1) \). The last two equations imply that \( G = A_1(t_1)e^{\tilde{a}(t)} = A_2(t)e^{-\tilde{a}(t_1)} \) for some functions \( A_1(t_1) \) and \( A_2(t) \) and \( \tilde{a}(t) = \int_0^t a(\tau)d\tau \). Noticing that \( A_1(t_1)e^{\tilde{a}(t_1)} = A_2(t)e^{-\tilde{a}(t)} \) we conclude that \( A_1(t_1)e^{\tilde{a}(t_1)} \) is a constant. Denoting \( \gamma := A_1(t_1)e^{\tilde{a}(t_1)} \) and \( G_1(t) := e^{-\tilde{a}(t)} \) we have

\[
G(t, t_1) = \gamma \frac{G_1(t_1)}{G_1(t)} \quad \text{and, therefore,} \quad f(t, t_1, t_x) = \gamma \frac{G_1(t_1)}{G_1(t)} t_x + H. \tag{44}
\]

The second equation of (43) implies that

\[
\frac{H_t}{G} = -\mu(t) \quad \text{and} \quad H_{t_1} - \frac{G_{t_1}}{G} = \mu(t_1) \tag{45}
\]

for some function \( \mu(t) \). Using (44), the second equation in (45) can be rewritten as \( H_{t_1} - \frac{G(t_1)H_1(t_1)}{G_1(t_1)} = \mu(t_1) \), or the same, as \( \left\{ \frac{H(t, t_1)}{G(t_1)} \right\}_{t_1} = \frac{\mu(t_1)}{G_1(t_1)} \). It means that

\[
H(t, t_1) = G_1(t_1)H_1(t_1) + G_1(t_1)H_2(t) \tag{46}
\]

for some functions \( H_1(t_1) \) and \( H_2(t) \). By substituting \( H(t, t_1) \) from (46) and \( G(t, t_1) \) from (44) into the second equation of (45) we have,

\[
G_1(t_1)H_1(t_1) + G_1(t_1)H_1'(t_1) + G_1'(t_1)H_2(t)
- \frac{G_1'(t_1)}{G_1(t_1)}(G_1(t_1)H_1(t_1) + G_1(t_1)H_2(t)) = \mu(t_1).
\]

After all cancellations it becomes

\[
G_1(t_1)H_1'(t_1) = \mu(t_1), \quad \text{or the same,} \quad G_1(t)H_1'(t) = \mu(t). \tag{47}
\]

By substituting \( G(t, t_1) \) from (44) and \( H(t, t_1) \) from (46) into the first equation of (45) we have,

\[
H_2'(t)G_1(t) = -\gamma \mu(t). \tag{48}
\]

Combining together (47) and (48) we obtain that \( H_2'(t)G_1(t) = -\gamma G_1(t)H_1'(t) \), or the same, \( H_2'(t) = -\gamma H_1'(t) \), or \( (H_2(t) + \gamma H_1(t))' = 0 \) that implies that \( H_2(t) = -\gamma H_1(t) + \eta \) for some constant \( \eta \). Therefore,

\[
f(t, t_1, t_x) = \gamma \frac{G_1(t_1)}{G_1(t)} t_x + G_1(t_1)H_1(t_1) - \gamma G_1(t_1)H_1(t) + \eta G_1(t_1). \tag{49}
\]

Note that only properties (40) and (24) were used to obtain representation (49) for \( f(t, t_1, t_x) \).
Using (41) and (24) we have

\[ 0 = t_x \left\{ \gamma \frac{G_1(t_1)}{G_1(t)} G'_1(t) \right\} - \gamma \frac{G_1(t_1)}{G_1(t)} G'_1(t) t_x - \gamma G'_1(t_1) H'_1(t) \]

\[ -\gamma \frac{G'_1(t_1)}{G_1(t)} \left\{ \gamma \frac{G_1(t_1)}{G_1(t)} t_x + G'_1(t_1) H_1(t_1) - \gamma G_1(t_1) H_1(t) + \eta G_1(t_1) \right\} \]

\[ + \gamma \frac{G_1(t_1)}{G_1(t)} \left\{ \gamma \frac{G'_1(t_1)}{G_1(t)} t_x + G'_1(t_1) H_1(t_1) + G_1(t_1) H'_1(t_1) - \gamma G'_1(t_1) H_1(t) + \eta G'_1(t_1) \right\} \]

\[ = \gamma \frac{G_1(t_1)}{G_1(t)} \left\{ -H'_1(t) G_1(t) + G_1(t_1) H'_1(t_1) \right\}, \]

i.e. \(-H'_1(t) G_1(t) + G_1(t_1) H'_1(t_1) = 0\). This implies that \(H'_1(t) G_1(t) = c\), where \(c\) is some constant. Substituting \(G_1(t) = \frac{c}{H'_1(t)}\) into (49) we have,

\[ f(t, t_1, t_x) = \gamma \frac{H'_1(t)}{H'_1(t_1)} t_x + c \frac{H_1(t_1)}{H'_1(t_1)} - \gamma c \frac{H_1(t_1)}{H'_1(t_1)} + \eta \frac{c}{H'_1(t_1)}. \] (50)

By using substitution \(s = H_1(t)\) equation (50) is reduced to \(s_{1x} = \gamma s_x + c s_1 - c \gamma s + \eta c\). Introducing \(\ddot{x} = cx\) allows to rewrite the last equation as

\[ s_{1\ddot{x}} = \gamma s_{\ddot{x}} + s_1 - \gamma s + \eta. \] (51)

If \(\gamma = 1\) substitution \(s = \tau - n \eta\) reduces (51) to \(\tau_{1\ddot{x}} = \tau_{\ddot{x}} + \tau_1 - \tau\). If \(\gamma \neq 1\), substitution \(s = \gamma^n \tau + \eta \gamma^n \frac{1}{1 - \gamma}\) reduces (51) to \(\tau_{1\ddot{x}} = \tau_{\ddot{x}} + \tau_1 - \tau\). \(\square\)

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