Relaxation to steady states and dynamical exponents in deposition models

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(November 20, 2018)

Considering some deposition models with limited mobility, we show that the typical decay of the interface width to its saturation value is exponential, which defines the crossover or saturation time \( \tau \). We present a method to calculate a characteristic time \( \tau_0 \) proportional to \( \tau \) and estimate the dynamical exponent \( z \). In one dimensional substrates of lengths \( L \leq 2048 \), the method is applied to the Family model, the restricted solid-on-solid (RSOS) model and the ballistic deposition. Effective exponents \( z_L \) converge to asymptotic values consistent with the corresponding continuum theories. For the two-dimensional Family model, the expected dynamic scaling hypothesis suggests a particular definition of \( \tau_0 \) that leads to \( z = 2 \), improving previous calculations based on data collapse methods. For the two-dimensional RSOS model, we obtain \( z \approx 1.6 \) and \( \alpha < 0.4 \), in agreement with recent large scale simulations.

PACS numbers: 05.40.+j, 05.50.+q
Keywords: Growth models; thin films; Universality classes; Dynamical exponent; Relaxation

I. INTRODUCTION

Statistical deposition models have attracted much attention in the last years because they may describe real systems’ features by representing the basic growth mechanisms as simple stochastic processes, thus neglecting the details of the microscopic interactions. Some examples are the Family model, the restricted solid-on-solid (RSOS) model of Kim and Kosterlitz and the ballistic deposition (BD) model. The former one is representative of the Edwards-Wilkinson (EW) universality class of linear growth, while the other ones are representative of the Kardar-Parisi-Zhang (KPZ) class of non-linear growth. More complex models involve the competition among deposition, diffusion and aggregation and may represent quantitatively real systems’ features.

Often one is interested in surface properties, then the main quantity to be measured is the interface width of the deposit. If deposition occurs in a \( d \)-dimensional substrate of length \( L \), then the interface width at time \( t \) is defined as

\[
W(L, t) = \left[ \frac{1}{L^d} \sum_i (h_i - \bar{h})^2 \right]^{1/2}
\]

(1)

or as

\[
\xi(L, t) = \left[ \frac{1}{L^d} \sum_i (h_i - \bar{h})^2 \right]^{1/2}.
\]

(2)

In Eqs. (1) and (2), \( h_i \) is the height of column \( i \), the bar in \( \bar{h} \) denotes a spatial average and the angular brackets denote a configurational average, i.e., an average over many realizations of the noise. \( W \) and \( \xi \) have different values but the same scaling properties, then in the following any definition or discussion concerning \( W \) is also valid for \( \xi \).

At early times, in the so-called growth regime, the interface width increases as

\[
W \sim t^\beta,
\]

(3)

while at long times finite-size effects lead to its saturation at

\[
W_{\text{sat}}(L) \equiv W(L, t \to \infty) \sim L^\alpha.
\]

(4)

Relations (3) and (4) are included in the dynamic scaling relation proposed by Family and Vicsek,

\[
W = L^\alpha f(tL^{-z}),
\]

(5)

in which \( f \) is a scaling function and the dynamical exponent \( z \) is given by
In Eq. (5), the argument of the scaling function is a ratio of the deposition time and the crossover (or saturation) time \( \tau \sim L^z \).

Most numerical works on deposition models focus on the calculation of exponents \( \alpha \) (Eq. 4) and \( \beta \) (Eq. 3) - see e.g. the summary of numerical results in Ch. 8 of Ref. [1]. The calculation of \( \alpha \) requires the identification of the steady-state regime, in which \( W \) saturates. Eventually, the extrapolation of effective exponents is essential, such as in the BD model [1]. The calculation of \( \beta \) depends on the definition of the growth regime, which must be a region of high linear correlation in a \( \log W \times \log t \) plot. Finding such region may be very difficult for \( d \leq 2 \) due to small lattice lengths [1]. The usual methods to estimate exponent \( z \) are based on data collapse of the scaled quantities appearing in Eq. (5) or on the scaling of the structure factor \( S(k, t) \equiv \langle h(k, t) h(-k, t) \rangle \) \( (h(k, t) \) is the Fourier transformation of the surface height \( h(x, t) \) [12–14]. In a recent work, Kim et al. presented a method to estimate \( z \) from the exponential decay of a relaxation function, starting the growth process from a sinusoidal surface [15]. In \( d = 1 \), they obtained estimates for the Family, the RSOS and the Krug models which were consistent with the expected universality classes [14]. However, for the most common problem of deposition in a flat surface, a precise definition of the crossover time \( \tau \) and a method to estimate \( z \) accurately are still lacking.

It is also important to recall that there are controversies on the universality classes of various statistical growth models even in \( d = 1 \) [16]. At this point, the development of reliable techniques to estimate critical exponents is also helpful.

The aim of this work is to analyze the convergence of the scaling function in Eq. (5) to an asymptotic constant value (consistently with Eq. 4) and to propose and test methods to estimate crossover times and the exponent \( z \).

First we will consider the Family, the RSOS and the BD models in \( d = 1 \). We will show that the typical decay of \( W \) to its saturation value is exponential, which uniquely defines the crossover or saturation time \( \tau \). Then we will present a method to calculate a characteristic time \( \tau_0 \) that is proportional to \( \tau \), and we will estimate the exponent \( z \) from \( \tau_0 \). The reliability of the method will be proved with comparisons with exact results for the corresponding hydrodynamic equations.

We will also consider the Family and the RSOS models in \( d = 2 \). Results for the RSOS model confirm the exponential decay of \( W \) to the saturation value and give \( z \approx 1.6 \). Estimates of the exponent \( \alpha \), obtained from \( W_{sat} \) data, are in good agreement with the prediction of Marinari et al. [17] that \( \alpha < 0.4 \) for KPZ in \( d = 2 \). For the Family model, we will consider the expected dynamic scaling hypothesis, which involves a \( W \sim \log L \) growth, and we will show results consistent with \( z = 2 \). A recent numerical work [18] showed large discrepancies from that value, using data collapse techniques, which proves the relevance of the methods presented here.

The rest of this paper is organized as follows. In Sec II we present the results of simulations in \( d = 1 \) for the Family, the RSOS and the BD models, show the exponential decay of \( W \) to its saturation value and the methods to calculate \( z \). In Sec. III we present the simulations’ results of the Family and the RSOS models in \( d = 2 \), including estimates of exponent \( \alpha \) for the RSOS model. In Sec. IV we summarize our results and present our conclusions.

**II. DEFINITION OF THE MODELS AND RESULTS IN ONE-DIMENSIONAL SUBSTRATES**

First we recall the definition of the models studied here.

In the Family model [2], a column of the deposit is randomly chosen and the incident particle searches for the smallest height in its neighborhood. If no neighboring column has a smaller height than the column of incidence, the particle sticks at the top of this one. Otherwise, it sticks at the top of the column with the smallest height among the neighbors. If two or more neighbors have the same minimum height, the sticking position is randomly chosen among them.

In the RSOS model [3], the incident particle may stick at the top of the column of incidence if the differences of heights of all pairs of neighboring columns do not exceed \( \Delta H_{MAX} = 1 \). Otherwise, the aggregation attempt is rejected.

In BD, the incident particle follows a straight trajectory perpendicular to the surface and sticks upon first contact with a nearest neighbor occupied site. It leads to the formation of a porous deposit, contrary to the previous solid-on-solid models.

In all models, a time unit corresponds to the deposition of \( L \) particles. We simulated them in one-dimensional substrates with lengths \( L = 2^n \), with integer \( 7 \leq n \leq 11 \) (128 \( \leq L \leq 2048 \)). For each \( L \), typically \( 10^4 \) different deposits were generated, and very long steady-state regions were observed.

In order to obtain reliable estimates of \( W_{sat} \), we computed it over different time ranges to ensure that it was fluctuating around an average value, and not increasing systematically in time (the latter possibility would suggest
that the steady-state region was not attained yet. $W_{sat}$ was obtained with accuracy smaller than 0.2% in all cases, which enables the calculation of the deviation from $W_{sat}$,

$$\Delta W (t) \equiv W_{sat} - W (t),$$

also with good accuracy.

In Fig. 1a we show log $\Delta W \times t$ for the Family model in $L = 1024$, and in Fig. 1b we show log $\Delta W \times t$ for the RSOS and BD models, also in $L = 1024$. In Figs. 2a and 2b we show log $\Delta \xi \times t$ in the same systems. It clearly shows an exponential decay of $W$ and $\xi$ to $W_{sat}$ and $\xi_{sat}$, in the form

$$\Delta W (t) = A \exp (-t/\tau),$$

with constants $A$ and $\tau$. The estimates of relaxation times shown in Figs. 1 and 2 were obtained from fits of the data in regions of high linear correlation. For the other lengths $L$, similar linear behaviors in log $\Delta W \times t$ plots were obtained.

The calculation of $\tau$ from those plots has some disadvantages. For instance, $\tau$ has to be estimated in different time ranges, in order to avoid spurious effects of an arbitrary choice of the fitting region. Thus we searched for an alternative method, as described below, which is not based on similar data fits.

The scaling relation (5) and Eq. (8) imply that the amplitude $A$ of Eq. (8) is proportional to $L^\alpha$. Consequently, when approaching the steady-state region, the scaling function of Eq. (5) may be written as

$$f(x) = a - b \exp (-x),$$

with constant $a$ and $b$. If a characteristic time $\tau_0$ is defined from

$$W (\tau_0) = kW_{sat},$$

with constant $k$, then

$$\tau_0 = \tau \ln \left[ \frac{b}{a (1 - k)} \right],$$

i. e., $\tau_0$ is proportional to the crossover time $\tau$. The estimates of $\tau_0$ from Eq. (10) always had higher accuracy than the relaxation times obtained from log $\Delta W \times t$ plots (typically for a factor 3).

We considered two possible values of the constant $k$ in Eq. (10), $k_1 = 1 - 1/e = 0.6321\ldots$ and $k_2 = 0.8$. The first choice would give $\tau_0 = \tau$ if $a = b$ (Eqs. 9 and 11), which seems to be approximately valid for BD. The effective exponents $z_L$ were obtained from

$$z_L = \frac{\ln [\tau_0 (L)/\tau_0 (L/2)]}{\ln 2},$$

and must be independent of the particular choice of the constant $k$.

In Figs. 3a and 3b we show $z_L$ versus $1/L$ for the three models using $k = k_1$ and $k = k_2$, respectively, in the calculation of $\tau_0$ (Eq. 10). $W$ data were used, but the effective exponents obtained from $\xi$ are nearly the same. The uncertainties in $W_{sat}$ are responsible for the error bars, thus results with $k = k_1$ (larger $\Delta W$) are more accurate.

In all cases shown in Figs. 3a and 3b, $z_L$ converges to the exactly known values as $L \to \infty$. In the Family and in the RSOS models, $z = 2$ and $z = 3/2$ are recovered with small finite-size corrections. In BD, $1/L$ corrections to $z_L$ lead to expected $z = 3/2$ asymptotic value. It is interesting to notice that these corrections in BD are weaker than the corrections found in effective exponents $\beta_L$ and $\alpha_L$ in Ref. [11], which proves the relevance of developing methods to estimate $z$ independently. Moreover, this result complements that systematic study of BD in $d = 1$, which discussed the discrepancies in numerical estimates of critical exponents and the controversies on the equivalence of BD and the KPZ theory [11].

### III. RESULTS IN TWO-DIMENSIONAL SUBSTRATES

First we consider the RSOS model in $d = 2$. Simulations were performed for lengths $32 \leq L \leq 512$, with a small number of realizations (nearly $10^3$) in the largest lattices. The exponential decay of $\Delta W (t)$ is confirmed in Fig. 4a for lattices of length $L = 512$. In Fig. 4b we show $z_L$ versus $1/L$, which gives the asymptotic estimate $z = 1.60 \pm 0.05$. This result is consistent with several previous estimates of critical exponents for that model [3].
Using $W_{\text{sat}}$ data, we were also able to obtain finite-size estimates of the exponent $\alpha$:

$$\alpha_L = \frac{\ln|W_{\infty}(L)/W_{\infty}(L/2)|}{\ln 2}.$$  \hfill (13)

In Figs. 5a and 5b we show $\alpha_L$ versus $1/L$ obtained from $W$ and $\xi$ data, respectively. Both sets of data strongly suggest an asymptotic exponent $\alpha < 0.4$, which agrees with simulations of Marinari et al [17], who obtained $\alpha = 0.393 \pm 0.003$. However, it disagrees with the proposal of $\alpha = 0.4$ based on an operator product expansion [19], which suggests further investigations on these lines.

For the Family model, it is expected the dynamic scaling relation [21]

$$W^2 = A \ln [L f (t/\tau)]$$  \hfill (14)

where $f$ is a scaling function and $A$ is constant. As $x \to \infty$, it is expected that $f(x) \to a$, with constant $a$. The characteristic time $\tau_0$ defined from Eq. (10) is not proportional to $\tau$ anymore. Then, in order to define a characteristic time $\tau_1$ which is proportional to $\tau$ in this system, we consider

$$\delta W^2 \equiv W_{\text{sat}}^2 - W(\tau_1)^2 = C,$$  \hfill (15)

with constant $C$.

The characteristic times $\tau_1$ were calculated using Eq. (15) with $C = 0.03$. The effective exponents $z_L$ were also obtained from Eq. (12). In Figs. 6a and 6b we show $z_L$ versus $1/L$ obtained from $W$ and $\xi$ data, respectively. In both cases, $z_L$ converges to $z = 2$ as $L \to \infty$, in agreement with the EW theory. Our result is significantly better than the value $z \approx 1.63$, obtained in Ref. [18] using data collapse methods, which again proves the reliability of our method.

IV. SUMMARY AND CONCLUSION

We proposed and tested methods to estimate crossover times of interface width scaling and the dynamic exponent $z$. We showed that the typical decays of $W$ or $\xi$ to their saturation values are exponential, which uniquely defines the crossover or saturation times $\tau$. In order to estimate $z$ with accuracy, we used a characteristic time $\tau_0$ that is proportional to $\tau$, according to the Family-Vicsek scaling. The reliability of the method was proved by testing it with the Family, the RSOS and BD models in $1 + 1$ dimensions, in which dynamic exponents of the EW and the KPZ theory were obtained. The Family and the RSOS models in $d = 2$ were also considered. For the RSOS model, the exponential decay of $W$ is confirmed, and $z \approx 1.6$ was obtained. Moreover, estimates of saturation widths suggest that $\alpha < 0.4$, in agreement with a recent prediction of large scale simulations [17]. For the two-dimensional Family model, the expected dynamic scaling hypothesis lead to an alternative definition of the characteristic time $\tau_0$, and the exact value $z = 2$ was recovered.

The relevance of our methods to estimate dynamical exponents is also proved by comparisons with previous results for related systems. For instance, using data collapse of relaxation functions and the expected scaling of $W$ at $t = L$, $z = 1.55$ was obtained for Eden clusters [12] and $z = 1.57$ for the single-step model [13] in $d = 1$. Both systems are in the $KPZ$ class (exact $z = 1.5$). These difficulties were pointed out in the paper of Kim et al [13], who proposed a method to measure $z$ from the scaling of a relaxation function and an initial sinusoidal surface. Although the efficiency of their method was clearly stated, working with a flat surface is the most usual initial situation in theoretical works and is more suitable for real applications. Moreover, our method, one is able to calculate three exponents, $\alpha$, $z$ and $\beta$, using the same set of data. The relevance of each one to determine the universality class of a particular model may depend on several factors, such as the strength of finite-size corrections. Thus we expect that this work may be useful for further studies of surface growth models.

ACKNOWLEDGMENTS

This work was partially supported by CNPq and FINEP (brazilian agencies). The author thanks the Department of Theoretical Physics at Oxford University, where part of this work was done, for the hospitality.
FIG. 1. Time evolution of the deviation from the saturation width for: (a) the 1d Family model; (b) the 1d RSOS and BD models (slower decay for RSOS). The linear fit of the data in 2000 < t < 6000 is shown for the Family model. Relaxation times calculated from such fits are shown for the three models.

FIG. 2. Time evolution of the deviation from the saturation width for the same models of Fig. 1, with ξ instead of W.

FIG. 3. Effective exponents $z_L$ versus $1/L$ in $d = 1$, obtained using (a) $k = k_1 = 1 - 1/e$ and (b) $k = k_2 = 0.8$, for the Family model (squares), the RSOS model (triangles) and the BD model (crosses). Data for the RSOS model are slightly shifted to the left.

FIG. 4. (a) Time evolution of the deviation from the saturation width for the RSOS model in $d = 2$; (b) effective exponents $z_L$ versus $1/L$ for the same model, using W data and $k = k_1$.

FIG. 5. Effective exponents $\alpha_L$ versus $1/L$ for the RSOS model in $d = 2$, obtained from (a) W and (b) ξ data.

FIG. 6. Effective exponents $z_L$ versus $1/L$ for the EW model in $d = 2$, obtained from (a) W and (b) ξ data.
\[ \ln(\xi_{\text{sat}} - \xi) \]

(a)

(b)

\[ \tau = 18894 \]

\[ \tau = 1463 \]

\[ \tau = 1007 \]
