Phase reduction of stochastic limit cycle oscillators

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(Dated: February 9, 2022)

PACS numbers: 05.45.-a, 05.45.Xt

Many physical systems can be mathematically modeled by limit cycle oscillators. It is well known that the oscillator systems could exhibit a variety of behaviors. A fundamental theoretical technique for studying the oscillator dynamics is the phase reduction method (see e.g. 1). This method has been widely and successfully applied to coupled oscillators or an oscillator subjected to a regular external signal such as a periodic one. Considerable theoretical progress has been made in understanding their dynamics by using this method.

Recently, the dynamics of oscillators subjected to external stochastic signals has also attracted much interest in connection with entrainment of independent oscillators subjected to a common external noise. This common-noise-induced entrainment has been experimentally found in several systems as diverse as neuronal networks 2, ecological systems 3, and lasers 4. Limit cycle oscillators driven by white Gaussian noise have been used as simple models for theoretically studying this entrainment 2, 6, 7, 8. In these theoretical studies, the phase reduction method is applied to the noise-driven oscillators to derive a one dimensional equation for the phase variable only. Based on this phase equation, several reasonable theoretical results have been obtained. However, as we will show, the phase equation used in the above references is incorrect in the sense that in general it does not correctly describe the dynamics of the original full oscillator system even in the weak noise limit.

The phase reduction is a powerful method for describing the essential dynamics of oscillators. Application field of this method is expected to grow also in the case of stochastic oscillators. Therefore, it is essential to develop a phase reduction method for stochastic oscillators. In the present paper, we consider a general class of limit cycle oscillators, which are subjected to white Gaussian noises, and develop the phase reduction method for these systems. Based on some numerical results, it is demonstrated that the present phase equation properly approximate the dynamics of the original full oscillator system while the incorrect version of phase equation fails. Finally, we make remarks on the results concerning the common-noise-induced entrainment, which have been obtained in Refs. 2, 6, 7, 8.

Let \(X = (x_1, \ldots, x_N) \in \mathbb{R}^N\) be a state variable vector and consider the equation

\[
\dot{X} = F(X) + G(X)\xi(t),
\]

where \(F\) is an unperturbed vector field, \(G \in \mathbb{R}^N\) is a vector function, and \(\xi(t)\) is the white Gaussian noise such that \(\langle \xi(t) \rangle = 0\) and \(\langle \xi(t)\xi(s) \rangle = 2D \delta(t-s)\), where \(\langle \cdots \rangle\) denotes averaging over the realizations of \(\xi\) and \(\delta\) is Dirac’s delta function. We call the constant \(D > 0\) the noise intensity. The noise-free unperturbed system \(\dot{X} = F(X)\) is assumed to have a limit cycle with a frequency \(\omega\). We employ the Stratonovich interpretation for the stochastic differential equation (1). This interpretation allows us to use the conventional variable transformations in differential equations. A more general form of the noise term has been assumed in 2, 6, 7, 8. However, we assume the form of Eq. (1) for simplicity. An extension of the present derivation to a more general case is straightforward.

Consider the unperturbed system \(\dot{X} = F(X)\) and let \(X_0\) be its limit cycle solution. A phase coordinate \(\phi\) can be defined in a neighbourhood \(U\) of the limit cycle \(X_0\) in phase space. According to a conventional definition, we define the phase variable \(\phi\) so that \((\text{grad}_x \phi) \cdot F(X) = \omega\) may hold for any points in \(U\). We can define the other \(N-1\) coordinates \(r = (r_1, \ldots, r_{N-1})\) such that \(\partial \phi(r)/\partial X \neq 0\) in \(U\). We assume that \(r = a\) on the limit cycle, where \(a = (a_1, \ldots, a_{N-1})\) is a constant vector. If we perform the transformation \((x_1, \ldots, x_N) \mapsto (\phi, r_1, \ldots, r_{N-1})\) in Eq. (1), we have the equation of the form

\[
\dot{\phi} = \omega + h(\phi, r)\xi(t),
\]

\[
\dot{r}_i = f_i(\phi, r) + g_i(\phi, r)\xi(t),
\]

where \(i = 1, \ldots, N-1\).

The functions \(h, f_i,\) and \(g_i\) are defined as follows:

\[
h(\phi, r) = (\text{grad}_x \phi) \cdot G(X(\phi, r)),
\]

\[
f_i(\phi, r) = (\text{grad}_x r_i) \cdot F(X(\phi, r)),
\]

\[
g_i(\phi, r) = (\text{grad}_x r_i) \cdot G(X(\phi, r)),
\]

where the gradients are evaluated at the point \(X(\phi, r)\). These functions are periodic with respect to \(\phi\); i.e., \(h(\phi + 2\pi, r) = h(\phi, r), f_i(\phi + 2\pi, r) = f_i(\phi, r),\) and \(g_i(\phi + 2\pi, r) = g_i(\phi, r)\).
Equations (2) and (3) are Stratonovich stochastic differential equations. They can be converted into equivalent Ito type reduced phase equations. The phase component of this Ito type equation is obtained as follows:

\[
\dot{\phi} = \omega + D \left[ \frac{\partial h(\phi, r)}{\partial \phi} h(\phi, r) + \sum_{i=1}^{N-1} \frac{\partial h(\phi, r)}{\partial r_i} g_i(\phi, r) \right] + h(\phi, r) \xi(t).
\]  

(7)

In the case of weak noise \(0 < D/\omega \ll 1\), the deviation of \(r\) from \(a\) is expected to be small. Thus, we can use the approximation \(r = a\) in Eq. (7). Using this approximation, we arrive at

\[
\dot{\phi} = \omega + D \left[ Z'(\phi) Z(\phi) + W(\phi) \right] + Z(\phi) \xi(t),
\]  

(8)

where \(Z(\phi)\) and \(W(\phi)\) are given by

\[
Z(\phi) = h(\phi, a),
\]  

(9)

\[
W(\phi) = \sum_{i=1}^{N-1} \frac{\partial h(\phi, a)}{\partial r_i} g_i(\phi, a),
\]  

(10)

respectively. Since \(h\) and \(g_i\) are periodic functions, \(Z(\phi)\) and \(W(\phi)\) are also periodic: \(\phi \rightarrow \phi + 2\pi \equiv Z(\phi)\) and \(\phi \rightarrow \phi + 2\pi = W(\phi)\). We may conclude that the reduced phase equation for the noise-driven oscillator (1) is given by Eq. (8). The oscillator dynamics is often studied by assuming a phase model instead of multidimensional differential equations. We emphasize that an equation of the form (8) has to be assumed in studying the dynamics of oscillators with white Gaussian noises.

In the previous studies \([5, 6, 7, 8]\), the authors assumed the Ito type reduced phase equation of the form

\[
\dot{\phi} = \omega + D Z(\phi) Z'(\phi) + Z(\phi) \xi(t).
\]  

(11)

Comparison of the present phase equation (8) and Eq. (11) clearly shows that the term \(D W(\phi)\) is dropped in the previously used equation (11). This term is \(O(D)\) and is of the same order as \(D Z(\phi) Z'(\phi)\). Thus, in general, equation (11) does not correctly describe the essential dynamics of the oscillator even in the lowest order approximation as will be demonstrated. In the exceptional case \(W \approx 0\), it gives reasonable results.

The reduced phase equation (8) is useful to calculate statistical quantities, which characterize the dynamics of oscillators subjected to white Gaussian noises. Using Eq. (8), we derive analytical expressions for two fundamental statistical quantities, which are the steady probability distribution of the phase variable and the mean frequency \(\Omega\). We will compare these quantities obtained by using a two dimensional oscillator model with those obtained by using its reduced phase model.

Let \(P(\phi, t)\) be the time-dependent probability distribution function for the phase \(\phi\). The stochastic differential equation (8) is equivalent to the Fokker-Planck equation

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi} \left[ (\omega + D Z(\phi) Z'(\phi) + D W(\phi)) P \right]
\]  

\[+ D \frac{\partial^2}{\partial \phi^2} \left[ Z(\phi)^2 P \right].
\]  

(12)

We consider Eq. (12) over the interval \(\phi \in [0, 2\pi]\) and assume the periodic boundary condition \(P(0, t) = P(2\pi, t)\). The steady solution \(P(\phi)\) is obtained by assuming \(\partial P/\partial t = 0\) in Eq. (12). If we construct an asymptotic solution for \(P(\phi)\) in the power of \(\varepsilon \equiv D/\omega\), then up to the first order we can obtain

\[
P(\phi) = \frac{1}{2\pi} + \frac{\varepsilon}{2\pi} \left[ Z(\phi) Z'(\phi) - W(\phi) + \bar{W} \right] + O(\varepsilon^2),
\]  

(13)

where \(\bar{W}\) is defined by \(\bar{W} = (2\pi)^{-1} \int_0^{2\pi} W(\phi) d\phi\).

The mean frequency \(\Omega\) of the oscillator is defined by

\[
\Omega = \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{\phi}(t) dt.
\]  

(14)

This can be calculated by replacing the time average (14) with the ensemble average: \(\langle \cdot \rangle\). In the Ito equation, unlike in Stratonovich formulation, the correlation between \(\phi\) and \(\xi\) vanishes. If we take the ensemble average of Eq. (8), then we have

\[
\langle \phi \rangle = \omega + D \langle Z(\phi) Z'(\phi) + W(\phi) \rangle,
\]  

(15)

where we used the fact \(\langle Z(\phi) \xi(t) \rangle = \langle Z(\phi) \rangle \langle \xi(t) \rangle = 0\). For an arbitrary function \(A(\phi)\), the ensemble average can be calculated by using the steady probability distribution \(P(\phi):\ i.e., \langle A \rangle = \int_0^{2\pi} A(\phi) P(\phi) d\phi\). If we use Eq. (13), we can obtain \(\Omega\) up to the second order in \(\varepsilon\) as follows:

\[
\frac{\Omega}{\omega} = 1 + \varepsilon \bar{W} + \varepsilon^2 \left[ (ZZ')^2 - \bar{W}^2 + \bar{W}^2 \right] + O(\varepsilon^3),
\]  

(16)

where \(\langle ZZ' \rangle^2 = (2\pi)^{-1} \int_0^{2\pi} (Z(\phi) Z'(\phi))^2 d\phi\) and \(\bar{W}^2 = (2\pi)^{-1} \int_0^{2\pi} (W(\phi))^2 d\phi\). Since the white Gaussian noise has no characteristic frequency, intuitively, one might expect that the noise does not cause any change in the oscillator frequency. However, this is not the case. Equation (16) clearly shows that an external white Gaussian noise changes the mean frequency \(\Omega\) in a general class of oscillators. It depends on the sign of \(\bar{W}\) whether \(\Omega\) increases or decreases as the noise intensity increases.

Equations (13) and (16) show that the term \(W(\phi)\) in Eq. (8) significantly affects both the steady probability distribution \(P(\phi)\) and the mean frequency \(\Omega\) in the first order of \(\varepsilon\). In particular, as shown by Eq. (16), the first order frequency shift is determined only from \(W(\phi)\). Therefore, it is crucially important to include the term \(W(\phi)\) into the reduced phase equation as in Eq. (8). It is clear that the previously used phase equation (11) cannot give proper approximations for \(P(\phi)\) and \(\Omega\).

In order to validate the above phase reduction method, we carried out numerical calculations for an example of noise-driven oscillator. We compare \(P(\phi)\) and \(\Omega\) between the theoretical and numerical results. We consider the Stuart-Landau (SL) oscillator

\[
\dot{x} = x - c_0 y - (x^2 + y^2)(x - c_2 y) + G_\nu \xi(t),
\]  

(17)

\[
\dot{y} = c_0 x + y - (x^2 + y^2)(c_2 x + y) + G_\nu \xi(t),
\]  

(18)
where \( c_0 \) and \( c_2 \) are constants, \( \mathbf{G} = (G_x, G_y) \) is a vector function of \((x, y)\), and \( \xi \) is the white Gaussian noise with the properties \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t) \xi(s) \rangle = 2D \delta(t-s) \). The SL oscillator has the limit cycle solution \( \mathbf{X}_0 = (\cos \omega t, \sin \omega t) \), where the natural frequency \( \omega \) is given by \( \omega = c_0 - c_2 \). If we define the coordinates \((\phi, r)\) by the transformation

\[
x = r \cos(\phi + c_2 \ln r), \quad y = r \sin(\phi + c_2 \ln r),
\]
then \( \phi \) gives the phase variable and the limit cycle is represented by \( r = 1 \).

The functions \( Z(\phi) \) and \( W(\phi) \) can be obtained from Eq. (19). As examples, we consider the following two types of \( \mathbf{G} \): \( \mathbf{G}_1 = (1, 0) \) and \( \mathbf{G}_2 = (x, 0) \). For the first example, \( Z(\phi) \) and \( W(\phi) \) are given by \( Z(\phi) = - (\sin \phi + c_2 \cos \phi) \) and \( W(\phi) = \{(1 + c_2^2) / 2\} \sin 2\phi \). For the second example, they are given by \( Z(\phi) = - \cos \phi (\sin \phi + c_2 \cos \phi) \) and \( W(\phi) = c_2 \cos^2 \phi - \cos 2\phi + c_2 \sin 2\phi \). Approximations for \( P(\phi) \) and \( \Omega \) can be obtained by substituting these expressions for \( Z(\phi) \) and \( W(\phi) \) into Eqs. (13) and (18). As for \( \Omega \), we can obtain \( \Omega/\omega = 1 + O(\varepsilon^3) \) for the first example \( \mathbf{G}_1 \) and \( \Omega/\omega = 1 - (c_2^2 / 32) \varepsilon^2 + O(\varepsilon^4) \) for the second example \( \mathbf{G}_2 \). The former indicates that \( \Omega \) is independent of \( c_2 \) and constant up to the second order in the first example. In contrast, the latter indicates that \( \Omega \) can either increase or decrease in the first order depending on \( c_2 \) in the second example.

In Figs. 1(a)–(d), numerical and theoretical results for \( P(\phi) \) are compared: filled circle and solid line represent \( P(\phi) \) obtained by numerically solving Eqs. (17) and (18) and that given by Eq. (19), respectively. Theoretical predictions made by Eq. (11), which are obtained just by setting \( W = 0 \) in Eq. (19), are also shown by dashed line. Figures 1(a) and (b) are for the case of \( \mathbf{G}_1 \) while figures 1(c) and (d) are for the case of \( \mathbf{G}_2 \). It is clear that the present phase model (9) gives precise approximations in all the cases. The agreements are excellent. In contrast, the incorrect version of phase equation (11) does not give proper approximations at all in spite of the weak noise intensity.

Figures 2(a) and (b) show the mean frequency \( \Omega \) plotted as a function of \( \varepsilon = D / \omega \) for the cases of \( \mathbf{G}_1 \) and \( \mathbf{G}_2 \), respectively. In all the numerical calculations, the natural frequency is set as \( \omega = 1 \). The numerical results obtained by solving Eqs. (17) and (18) are shown by filled or open circle. The theoretical estimations given by Eq. (16) are also shown by solid or dashed line. The theoretical estimation is \( \Omega/\omega = 1 + O(\varepsilon^3) \) for \( \mathbf{G}_1 \), which is independent of \( \varepsilon \) and constant up to the second order in \( \varepsilon \). In Fig. 2(a), the numerically obtained \( \Omega \) is almost constant for \((c_0, c_2) = (1, 0)\). This coincides with the above theoretical estimation. In the case of \((c_0, c_2) = (2, 1)\), there is a deviation between the numerical and theoretical results: the numerical result shows an increase with increasing \( \varepsilon \). However, this increase in not linear with respect to \( \varepsilon \) but a higher order one as shown in the inset. In this sense, an agreement between the numerical and theoretical results is confirmed up to the first order. In the case of \( \mathbf{G}_2 \), the theoretical estimation is given by \( \Omega/\omega = 1 - (c_2^2 / 4) \varepsilon + \{(1 + c_2^2) / 32\} \varepsilon^2 + O(\varepsilon^4) \), which has a non-vanishing term of \( O(\varepsilon) \) except for \( c_2 = 0 \). In Fig. 2(b), a good agreement between the numerical result and this estimation is obtained in each of the cases \((c_0, c_2) = (2, 1) \) and \((0, -1) \). If we use Eq. (11) instead of Eq. (8), then we obtain the estimation \( \Omega/\omega = 1 + O(\varepsilon^2) \), in which the \( O(\varepsilon) \) term vanishes. This estimation apparently disagrees with the numerical results.

Figures 1 and 2 clearly demonstrate that the reduced phase equation (8) precisely approximate the dynamics of stochastic oscillators with weak white Gaussian noises. In addition, it is apparent that the previously used equation (11) is erroneous.

It is known that an entrainment could occur between two independent oscillators subjected to a common ex-
ternal white Gaussian noise. We discuss this entrainment phenomenon, paying particular attention to an effect of the term $W(\phi)$. Consider the two equations

$$X_i = F(X_i) + \delta F_1(X_i) + G_i(X_i)\xi(t), \quad i = 1, 2, \quad (20)$$

where $X_i \in \mathbb{R}^N$, $F$ is an unperturbed vector field, $\delta F_1$ and $\delta F_2$ are small deviations from it, $G_1$ and $G_2$ are slightly different vector functions, and $\xi(t)$ is the common white Gaussian noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(s) \rangle = 2D\delta(t-s)$. The phase is defined by the unperturbed system $\dot{X} = F(X)$. Equation (20) can be reduced into the phase equation

$$\dot{\phi}_i = \omega_i + D Z_i(\phi_i) + W_i(\phi_i), \quad (21)$$

where $\omega_i$ represents the natural frequency. For simplicity, we assume $\omega_i$ is a constant. We introduce the average $Z(\phi) = (Z_1(\phi) + Z_2(\phi))/2$ and the difference $\delta Z(\phi) = (Z_1(\phi) - Z_2(\phi))/2$. The difference $\delta Z(\phi)$ is small since $G_1 \simeq G_2$ is assumed.

Let $\theta$ and $\psi$ be defined by $\theta = \phi_1 - \phi_2$ and $\psi = \phi_1 + \phi_2 - 2\omega t$, where $\omega$ is the average natural frequency defined by $\omega = (\omega_1 + \omega_2)/2$. The variable $\theta$ measures the phase difference between the two oscillators. In the case of weak noise, $\theta$ and $\psi$ can be regarded as slow variables and thus the averaging approximation can be applied. If we perform the time-averaging and neglect the terms which are of the order of $D|\delta Z|$, then we obtain the Fokker-Planck equation for the probability distribution $Q(t, \theta, \psi)$ as follows:

$$\frac{\partial Q}{\partial t} = -\{\delta \omega + D(\overline{W}_1 - \overline{W}_2)\} \frac{\partial Q}{\partial \theta} - D \frac{\partial^2 Q}{\partial \psi^2} u(\theta)Q + D \frac{\partial^2 Q}{\partial \psi^2} v(\theta)Q,$$

where $\delta \omega = \omega_1 - \omega_2$. The functions $u$ and $v$ are defined as $u(\theta) = 2\{\Gamma'(0) - \Gamma(\theta)\}$ and $v(\theta) = 2\{\Gamma'(\theta) + \Gamma(\theta)\}$, where $\Gamma(\theta) = (2\pi)^{-1} \int_0^{2\pi} Z(\phi)Z(\phi+\theta)d\phi$. Equation (22) has a steady solution $\tilde{Q}(\theta)$, which is a function of $\theta$ only. The entrainment phenomenon is characterized by $\tilde{Q}(\theta)$. The steady solution is determined by the equation

$$D \frac{d[u(\theta)Q]}{d\theta} - \{\delta \omega + D(\overline{W}_1 - \overline{W}_2)\} Q = C, \quad (23)$$

where $C$ is an integration constant. In Eq. (23), it has been shown that the mean frequency $\Omega$ shifts by $D\overline{W}$ in the lowest order. Equation (23) indicates that this frequency shift effect appears as the effective detuning $D(\overline{W}_1 - \overline{W}_2)$. The profile of $Q(\theta)$ depends on the coefficient of $Q$ in Eq. (23). It has been shown that $Q(\theta)$ has peaks at the zero points of $u(\theta)$ and these peaks become narrower and higher, which corresponds to better synchronization quality, as the ratio $[\delta \omega + D(\overline{W}_1 - \overline{W}_2)]/D$ between the coefficient of $Q$ and $D$ becomes smaller. Therefore, the profile of $Q(\theta)$ depends on the functional form of $W_i(\phi)$ in Eq. (21). It may be concluded that the contribution due to $W(\phi)$ in Eq. (21) is not negligible in the common-noise-induced entrainment. In addition, equation (23) suggests that the synchronization quality could be improved if $\delta \omega$ and $D(\overline{W}_1 - \overline{W}_2)$ cancel with each other.

The previous works [3, 6, 7, 8] have assumed the case $G_1 = G_2$. In this particular case, $W_1 = W_2$ holds and thus the effective detuning $D(\overline{W}_1 - \overline{W}_2)$ vanishes in Eq. (23). Therefore, the same equation for $Q(\theta)$ can be obtained even if $W_i(\phi)$ in Eq. (21) is not taken into account. Because of this fact, fortunately, an analysis based on the incorrect phase equation (11) also leads to correct results.

In conclusion, we have developed the phase reduction method for a general class of limit cycle oscillators subjected to white Gaussian noises. Applying the present reduced phase equation, we derived analytical expressions for the steady probability distribution $P(\phi)$ of phase and the mean frequency $\Omega$. It has been found that an external white Gaussian noise gives rise to a frequency shift. We showed that these analytical estimations of $P(\phi)$ and $\Omega$ are in good agreement with numerical results to demonstrate that the present phase equation properly approximates the dynamics of the original full oscillator system. In addition, we pointed out that an effect due to the frequency shift emerges also in the common-noise-induced entrainment.

The authors would like to thank the members of NTT Communication Science Laboratories for their continual encouragements.

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