Solitons of the Resonant Nonlinear Schrödinger Equation with Nontrivial Boundary Conditions and Hirota Bilinear Method

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Abstract
Physically relevant soliton solutions of the resonant nonlinear Schrödinger (RNLS) equation with nontrivial boundary conditions, recently proposed for description of uniaxial waves in a cold collisionless plasma, are considered in the Hirota bilinear approach. By the Madelung representation, the model transformed to the reaction-diffusion analog of the NLS equation for which the bilinear representation, soliton solutions and their mutual interactions are studied.

1 Introduction
Recently for description of low dimensional gravity (the Jackiw-Teitelboim model), and a medium response to the action of a quasimonochromatic wave with complex amplitude $\psi(x,t)$, which is slowly varying function of the coordinate and the time, a novel integrable version of NLS equation namely

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\Lambda}{4}|\psi|^2 \psi = s\frac{1}{|\psi|} \frac{\partial^2 |\psi|}{\partial x^2} \psi$$

(1)

was introduced [1]. This has been termed the resonant nonlinear Schrödinger (RNLS) equation. It can be considered as a third version of the NLS equation, intermediate between the defocusing and focusing cases. Even though the RNLS model is integrable for arbitrary values of the coefficient $s$, the critical value $s = 1$ separates two distinct regions of behavior. Thus, for $s < 1$ the model is reducible to the conventional NLS equation. However, for $s > 1$ it is not reducible to the usual NLS equation, but rather to a reaction-diffusion (RD) system. In this case the model exhibits resonance solitonic phenomena [1].
The RNLS equation can be interpreted as a particular realization of the NLS soliton propagating in the so called "quantum potential" $U_Q(x) = \psi_{xx}/|\psi|$. This potential, responsible for producing the quantum behavior, was introduced by de Broglie [2] and was subsequently used by Bohm [3] to develop a hidden-variable theory in quantum mechanics. It also appears in stochastic mechanics [4]. Connections between such non-classical motions with the internal spin motion and the zitterbewegung have been considered in a series of papers (see [5]). Quantum potentials also appear in proposed nonlinear extensions of quantum mechanics with regard both to stochastic quantization [6], [7] and to corrections from quantum gravity [8]. It is noted that the RNLS equation, like the conventional NLS equation, may also be derived in the context of capillarity models [9], [10].

Very recently [11] it was shown that the RNLS equation appears in plasma physics, where it describes the propagation of one-dimensional long magnetoacoustic waves in a cold collisionless plasma subject to a transverse magnetic field. The complex wave function satisfying RNLS equation is a combination of plasma density and the velocity fields as in the Madelung representation. The Backlund-Darboux transformations along with novel associated nonlinear superposition principle were presented and used to generate solutions descriptive of the interaction of solitonic magnetoacoustic waves. This application requires to consider a solution of the RNLS equation with nontrivial boundary conditions at infinity. The goal of the present paper is to derive such solutions in the Hirota bilinear approach and study their mutual interactions.

\section{Magnetoacoustic waves in cold plasma}

The dynamics of two-component cold collisionless plasma in the presence of an external magnetic field $B$ [12], [13] for uni-axial plasma propagation
\[ u = u(x,t)e_x, \quad B = B(x,t)e_z \]
reduces to the form [14]
\begin{equation}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{3}
\end{equation}
\begin{equation}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{B}{\rho}\frac{\partial B}{\partial x} = 0, \tag{4}
\end{equation}
\begin{equation}
\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial B}{\partial x}\right) = B - \rho. \tag{5}
\end{equation}
(where we set $B = 1$ and plasma density $\rho = 1$ at infinity) This system is equivalent to that of Whitham and has also been derived by Gurevich and Meshcherkin [15]. It describes the propagation of nonlinear magnetoacoustic waves in a cold plasma with a transverse magnetic field. It has been shown recently by El, Khodorovskii and Tyurina [19] that a system of the type (3)-(5) also occurs in the context of hypersonic flow past slender bodies.
3 A shallow water approximation

Here, we consider a shallow water approximation to the magnetoacoustic system (3)-(5). Thus, rescaling the space and time variables via $x' = \beta x$ and $t' = \beta t$, we have

\[
\begin{align*}
\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u) &= 0 \quad (6) \\
\frac{\partial u}{\partial t'} + u \frac{\partial u}{\partial x'} + \frac{B}{\rho} \frac{\partial B}{\partial x'} &= 0 \quad (7) \\
\beta^2 \frac{\partial}{\partial x'} \left( \frac{1}{\rho} \frac{\partial B}{\partial x'} \right) &= B - \rho \quad (8)
\end{align*}
\]

On expansion of $B$ as a power series in the parameter $\beta^2$ according to

\[
B = \rho + \beta^2 b_2(\rho, \rho x', \rho x' x', ...) + O(\beta^4),
\]

insertion into (8) yields

\[
b_2 = \frac{\partial}{\partial x'} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x'} \right). \quad (10)
\]

Substitution of (9) into (7) yields

\[
\begin{align*}
\frac{\partial u}{\partial t'} + u \frac{\partial u}{\partial x'} + \frac{\partial \rho}{\partial x'} + \beta^2 \left[ \frac{1}{\rho} \frac{\partial^3 \rho}{\partial x'^3} - \frac{2}{\rho^2} \frac{\partial^2 \rho}{\partial x' \partial x'^2} + \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x'} \right)^3 \right] &= 0 \quad (11)
\end{align*}
\]

to $O(\beta^2)$. Accordingly, the following system results:

\[
\begin{align*}
\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u) &= 0, \quad (12) \\
\frac{\partial u}{\partial t'} + u \frac{\partial u}{\partial x'} + \frac{\partial \rho}{\partial x'} + \beta^2 \frac{\partial}{\partial x'} \left[ \frac{1}{\rho} \frac{\partial^2 \rho}{\partial x'^2} - \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x'} \right)^2 \right] &= 0. \quad (13)
\end{align*}
\]

This describes the propagation of long magnetoacoustic waves in a cold plasma of density $\rho$ moving with velocity $u$ across the magnetic field as given by (2), (9) (see [16], [17]). In this system the dispersion is negative, i.e., the wave velocity decreases with increasing wave vector $k$.

4 Resonant NLS

Introducing the velocity potential $S(x, t) = -\frac{1}{2} \int x u(x, t) dx$ so that $u = -2 \partial S/\partial x$ and integrating once equation (13) we get the system

\[
\begin{align*}
\frac{\partial \rho}{\partial t} - 2 \frac{\partial}{\partial x} \left( \frac{\partial S}{\partial x} \right) &= 0 \quad (14) \\
- \frac{\partial S}{\partial t} + \left( \frac{\partial S}{\partial x} \right)^2 + \frac{1}{2} \rho + \frac{\beta^2}{2} \left[ \frac{1}{\rho} \frac{\partial^2 \rho}{\partial x'^2} - \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right)^2 \right] &= 0 \quad (15)
\end{align*}
\]
(here we skip prime superscript). Combining \( \rho \) and \( S \) as one complex function

\[
\psi = \sqrt{\rho} e^{-iS}
\]  

the system (14),(15) can be represented as the nonlinear Schrödinger equation with quantum potential, the RNLS (1),

\[
i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2} \left| \psi \right|^2 \psi = (1 + \beta^2) \frac{1}{|\psi|} \frac{\partial^2 |\psi|}{\partial x^2} \psi
\]  

where \( \Lambda = -2 \), \( s = 1 + \beta^2 \), and we have used expression for the quantum potential

\[
\frac{1}{\rho} \frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right)^2 = 2 \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2}
\]  

Since parameter \( s > 1 \) equation (17) can not be transformed to the NLS (1). But by combining \( \rho \) and \( S \) as a couple of real functions

\[
e^{(+)} = \sqrt{\rho} e^{+S}, \quad e^{(-)} = -\sqrt{\rho} e^{-S}
\]  

so that \( e^{(+)} > 0, e^{(-)} < 0 \), equations (14),(15) can be written as the reaction-diffusion (RD) system

\[
\pm \frac{\partial e^{(\pm)}}{\partial \tau} + \frac{\partial^2 e^{(\pm)}}{\partial x'^2} - \frac{1}{2\beta^2} e^{(+)e^{(-)}e^{(\pm)} = 0}
\]  

where \( \tau = \beta t' \).

Linearization of (17) near the "condensate" solution \( \psi = \sqrt{\rho_0} e^{-i\rho_0 t/2} \) gives dispersion \( \omega = \sqrt{\rho_0} k \sqrt{1 - \frac{k^2}{\rho_0}} k^2 \). This dispersion is negative, i.e. the wave velocity decreases with increasing wave vector \( k \) and is unstable for short waves with \( k > k_{cr} \), where \( k_{cr} = \sqrt{\rho_0}/\beta \). However this instability results from the truncation of the dispersion relation

\[
\omega^2 = \rho_0 \frac{k^2}{1 + \frac{\beta^2}{\rho_0} k^2}
\]  

for the system (8), (9), (10) and does not correspond to any actual physical effect for shallow water waves [18]. Though this system is linearly stable for all wave numbers \( k \), it is not known to be integrable. Therefore, it is not as suitable for studying wave interactions as the RNLS, which is completely integrable and admits a rich variety of the exact solutions.

5 Steady State Flow and Solitons

Now we consider the system (12),(13) for the steady state flow. It describes the motion with fixed velocity \( u(t,x) = u_0 = const \) for which the continuity equation (12) implies \( \frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} = 0 \) or that fluid density has the travelling
wave form $\rho = \rho(x - u_0 t)$, where $x' = \beta x$, $t' = \beta t$. Equation (13) in this case gives

$$\rho + \left[ \frac{1}{\rho} \frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right)^2 \right] = a = \text{const.}$$

(22)

This has a simple physical interpretation. The motion with fixed velocity implies that sum of all forces acting on the system is zero. In our case Eq. (22) describes compensation of the nonlinearity by the quantum potential. Using (18) this equation gives

$$\rho + 2 \frac{1}{\sqrt{\rho}} \frac{\partial \sqrt{\rho}}{\partial x^2} = a$$

(23)

In terms of $y(z) = \sqrt{\rho}$, $z = x - u_0 t$, then we have nonlinear equation

$$\frac{d^2 y}{dz^2} - \frac{a}{2} y + \frac{1}{2} y^3 = 0$$

(24)

or multiplying with $y'$ and integrating once

$$\left( \frac{dy}{dz} \right)^2 - \frac{a}{2} y^2 + \frac{1}{4} y^4 = 2b = \text{const.}$$

(25)

This equation has solution

$$y = 2p \, dn[p(x - u_0 t), \kappa]$$

(26)

where $dn$ is the Jacobian elliptic function with the modulus $\kappa$, and $p$ is an arbitrary constant. It is connected with integration constants $a > 0$ and $b < 0$ by equations

$$\frac{a}{2p^2} = 1 + \kappa'^2, \quad \frac{b}{2p^4} = -\kappa'^2$$

(27)

where $\kappa' = \sqrt{1 - \kappa^2}$ is the complimentary modulus of the Jacobian elliptic function. These equations fix relation between the constants

$$a = 2p^2 \left( 1 - \frac{b}{2p^4} \right)$$

(28)

and the modulus $\kappa = \sqrt{1 + b/2p^4}$. Then for density $\rho$ we have the travelling wave solution

$$\rho(x, t) = 4p^2 \, dn^2[p(x - u_0 t), \kappa].$$

(29)

With fixed potential (23) from equation (15) and (29) for $u_0 = -(2/\beta)\partial S/\partial x$, we have the Hamilton-Jacobi equation

$$- \frac{1}{\beta} \frac{\partial S}{\partial t} + \frac{1}{\beta^2} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{a}{2} = 0$$

(30)

which has solution in the form

$$S = \beta \left[ S_0 - \frac{u_0}{2} x + \left( \frac{u_0^2}{4} + \frac{a}{2} \right) t \right]$$

(31)
or using (28)

\[
S = \beta \left[ S_0 - \frac{u_0}{2} x + \left( \frac{u_0^2}{4} + p^2(2 - \kappa^2) \right) t \right]
\]  

(32)

For the degenerate case \( \kappa = 1 \) which corresponds to \( b = 0 \) and as follow \( a = 2p^2 \), the elliptic solution (29), (32) becomes the soliton

\[
\rho(x, t) = 4p^2 \text{sech}^2[p(x - u_0t)]
\]  

(33)

\[
S = \beta \left[ S_0 - \frac{u_0}{2} x + \left( \frac{u_0^2}{4} + p^2 \right) t \right]
\]  

(34)

More general form of the travelling wave appears if we consider solution of the system (12), (13) in the form \( u(x, t) = u(x - u_0t), \rho(x, t) = \rho(x - u_0t) \). Then the first equation (12) implies

\[
\frac{d}{dz} \left[ \left( u - u_0 \right) \rho \right] = 0
\]

where \( z = x - u_0t \) and

\[
u = u_0 + \frac{C}{\rho}
\]  

(35)

where \( C = \text{const.} \). Substituting to the second equation (13) and integrating once we have

\[
\frac{1}{2} C^2 + \rho + \left[ \frac{1}{\rho} \frac{d^2 \rho}{dz^2} - \frac{1}{2} \left( \frac{d \rho}{dz} \right)^2 \right] = A = \text{const.}
\]  

(36)

Multiplying with \( \rho^2 \) and differentiating once we have (travelling wave form of the KdV equation)

\[
\rho \left( \frac{d^3 \rho}{dz^3} + 3\rho \frac{d^2 \rho}{dz^2} - 2A \frac{d \rho}{dz} \right) = 0
\]  

(37)

which implies after one integration

\[
\frac{d^2 \rho}{dz^2} + \frac{3}{2} \rho^2 - 2A \rho = B = \text{const.}
\]  

(38)

This gives equation

\[
\left( \frac{d \rho}{dz} \right)^2 = -\rho^3 + 2A \rho^2 + 2B \rho + C^2
\]  

(39)

and solution

\[
\rho(x - u_0t) = \alpha_1 + (\alpha_3 - \alpha_1) \text{dn}^2 \left[ \frac{1}{\sqrt{2}} (\alpha_3 - \alpha_1)^{1/2} (x - u_0t), \kappa \right]
\]  

(40)

with modulus of elliptic function \( \kappa^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1) \), constants \( \alpha_1, \alpha_2, \alpha_3 \) and

\[
u(x - u_0t) = u_0 - \frac{(\alpha_1 \alpha_2 \alpha_3)^{1/2}}{\rho}
\]  

(41)

This solution was first reported in [17]. We have particular reductions:
1) \( \alpha_1 = \alpha_2 = 0 \) and corresponding \( \kappa^2 = 1 \) so that (40) reduces to (29), with following identification of parameters \( \alpha_3 = 4k^2, b = -\alpha_2 k^2/2 \) and velocity \( u = u_0 \).

2) \( \alpha_1 = \alpha_2 \neq 0 \), then again \( \kappa^2 = 1 \) but solution

\[
\rho(x - u_0 t) = \alpha_1 + (\alpha_3 - \alpha_1) \text{sech}^2 \left[ \frac{1}{2}(\alpha_3 - \alpha_1)^{1/2}(x - u_0 t) \right]
\]

(42)

has nontrivial asymptotics. The physical value relevant for plasma physics is \( \alpha_1 = 1 \) leading to \( \lim_{|x| \to \infty} \rho = 1 \). Denoting \( \alpha_3 \equiv \sigma^2 \) we can write it in the form

\[
\rho(x - u_0 t) = 1 + (\sigma^2 - 1) \cosh^2 \left[ \sqrt{\sigma^2 - 1} \left( x - u_0 t \right) \right]
\]

(43)

Using the above results we can now construct solutions of RNLS (17). Substituting Eqs. (29) (32) to Eq. (16) and changing parameter \( k \equiv p/\beta \), we have quasi-periodic solution

\[
\psi(x', t') = 2\beta k \frac{e^{-i(\phi_0 - \frac{v}{\beta} x' + \sqrt{k^2 + 2(2 - \kappa^2)} t')}}{\cosh k(x' - u_0 t')}
\]

(44)

where \( x' = \beta x, t' = \beta t \). In the limit \( \kappa = 1 \) it gives the envelope soliton solution

\[
\psi(x', t') = 2\beta k \frac{e^{-i(\phi_0 - \frac{v}{\beta} x' + \sqrt{k^2 + 2(2 - \kappa^2)} t')}}{\cosh k(x' - u_0 t')}
\]

(45)

For the reaction-diffusion system (20) correspondingly, we have the "dissipative" periodic solution

\[
e^{(\pm)}(x', \tau) = \pm 2\beta k \frac{e^{\pm i(\phi_0 - \frac{v}{\beta} x' + \sqrt{k^2 + 2(2 - \kappa^2)} \tau)}}{\cosh k(x' - v \tau)}
\]

(46)

where velocity \( v \equiv u_0/\beta, k \equiv p/\beta \) and the dissipative analog of the envelope soliton

\[
e^{(\pm)}(x', \tau) = \pm 2\beta k \frac{e^{\pm i(\phi_0 - \frac{v}{\beta} x' + \sqrt{k^2 + 2(2 - \kappa^2)} \tau)}}{\cosh k(x' - v \tau)}
\]

(47)

the so called "dissipaton" solution [1]

6 Bilinear form and solitons. Trivial Boundary Conditions

The reaction-diffusion representation (20) of the RNLS (17) is the key point for construction multisoliton solutions. Due to algebraic similarity of RD system with NLS it is easy to write the bilinear representation for it. Representing two real functions \( e^{(+)}, e^{(-)} \) in terms of three real functions

\[
e^{(\pm)} = 2\beta \frac{G^\pm}{F}
\]

(48)
we have the next bilinear system of equations

\[(\pm D_x - D_{xx}^2)(G^{(\pm)} \cdot F) = 0, \quad D_{xx}^2(F \cdot F) = -2G^{(\pm)}G^{(-)}\]  

(49)

The corresponding solution of the RNLS (17) has

\[|\psi(x, t)|^2 = \rho = -e^{(+)e^{(-)} = 2\beta^{2}D_{x}^{2}(F \cdot F) = 4\beta^{2}\frac{\partial^{2} \ln F}{\partial x^{2}}\]  

(50)

The one-dissipaton is given by the following solution of system (49): \(G^{\pm} = \pm e^{\eta^{\pm}}, \quad F = 1 + e^{\eta^{\pm}} + \eta^{\pm}, \quad e^{\phi_{1,1}} = (k^{\pm}_{1} + k^{\mp}_{1})^{-2}, \) where \(\eta^{\pm}_{1} \equiv k_{1}^{\pm}x \pm (k_{1}^{\pm})^{2}\tau + \eta^{\pm}(0)\) and \(k_{1}^{\pm}, \eta^{\pm}(0)\) are constants. In terms of redefined parameters, \(k \equiv (k_{1}^{+} + k_{1}^{-})/2, \ v = -(k_{1}^{+} - k_{1}^{-})\) it acquires the form (17). In the space of parameters \((v, k)\) there exist the critical value \(v_{\text{crit}} = 2k\) for solution (??) so that when \(v < v_{\text{crit}},\) one has \(e^{\pm} \to 0\) at infinities, so the vanishing b.c. for the dissipaton. At the critical value the solution is a kink steady state in the moving frame \(e^{\pm} = \pm ke^{\pm i\xi} (1 \mp \text{tanh} \ k \xi), \) with constant asymptotics \(e^{\pm} \to \pm 2ke^{\pm i\xi}\) for \(x \to \mp \infty\) and \(e^{\pm} \to 0\) for \(x \to \pm \infty.\) In the over-critical case \(v > v_{\text{crit}},\) \(e^{\pm} \to \pm \infty\) for \(x \to \mp \infty\) and \(e^{\pm} \to 0\) for \(x \to \pm \infty.\)

For the two-dissipaton solution we have

\[G^{\pm} = \pm [e^{\eta^{\pm}_{1}} + e^{\eta^{\pm}_{2}} + (\frac{\bar{F}^{\pm}_{12}}{\bar{F}^{\pm}_{21}} + \frac{\bar{F}^{\pm}_{12}}{\bar{F}^{\pm}_{21}})\eta^{\pm}_{1} + \eta^{\pm}_{2} + (\frac{\bar{F}^{\pm}_{12}}{\bar{F}^{\pm}_{21}})\eta^{\pm}_{2} + \eta^{\pm}_{1}]; \quad (51)\]

\[F = 1 + \frac{e^{\eta^{\pm}_{1}} + \eta^{\pm}_{1}}{(k^{\pm}_{11})^{2}} + \frac{e^{\eta^{\pm}_{2}} + \eta^{\pm}_{2}}{(k^{\pm}_{12})^{2}} + \frac{e^{\eta^{\pm}_{2}} + \eta^{\pm}_{2}}{(k^{\pm}_{22})^{2}} + \frac{e^{\eta^{\pm}_{2}} + \eta^{\pm}_{2}}{(k^{\pm}_{22})^{2}}
+ \frac{e^{\eta^{\pm}_{2}} + \eta^{\pm}_{2}}{(k^{\pm}_{22})^{2}} + \frac{e^{\eta^{\pm}_{2}} + \eta^{\pm}_{2}}{(k^{\pm}_{22})^{2}} + \frac{e^{\eta^{\pm}_{2}} + \eta^{\pm}_{2}}{(k^{\pm}_{22})^{2}} + \frac{e^{\eta^{\pm}_{2}} + \eta^{\pm}_{2}}{(k^{\pm}_{22})^{2}}, \quad (52)\]

where \(k^{ab}_{ij} = k_{ai}^{b} + k_{aj}^{b}, \quad \bar{k}_{ij}^{ab} = k_{ai}^{b} - k_{aj}^{b}, \quad \eta^{\pm}_{1} \equiv k_{1}^{\pm}x \pm (k_{1}^{\pm})^{2}\tau + \eta^{\pm}(0).\) This solution shows the resonance character of dissipatons interaction [1]

7 Bilinear form and solitons. Non-trivial Boundary Conditions

As was first noticed by Hirota, for NLS equation of defocusing type with nonvanishing boundary conditions, the bilinear form of equations should be modified [20]. After substituting representation (48) to the system (20) the decoupling system is choosen in the form

\[(\pm D_x - D_{xx}^2 + \lambda)(G^{\pm} \cdot F) = 0, \quad (D_{xx}^2 - \lambda)(F \cdot F) = -2G^{(\pm)}G^{(-)}\]  

(53)

where we have introduced constant \(\lambda\) to be determined. Equation (48) and the second one of (53) imply \(-e^{(+)e^{(-)} = -4\beta^{2}[\frac{\lambda}{2} - (\ln F)_{xx}].\) Expanding \(G^{\pm}\) and \(F\) in Hirota’s power series

\[G^{\pm} = \pm g_{0}^{\pm} (1 + \epsilon_{1}g_{1}^{\pm} + \epsilon^{2}g_{2}^{\pm} + \ldots), \quad F = 1 + \epsilon f_{1} + \epsilon^{2}f_{2} + \ldots, \]

(54)
and requiring \( \lim_{|x| \to \infty} (\ln F)_{xx} = 0 \) we have the boundary condition

\[
\alpha_1 = \lim_{|x| \to \infty} \left[ -e^{(+)} e^{(-)} \right] = \lim_{|x| \to \infty} -4\beta^2 \left[ \frac{\lambda}{2} - (\ln F)_{xx} \right] = -2\beta^2 \lambda \quad (55)
\]

which fixes constant \( \lambda = -\alpha_1/2\beta^2 \). In the zero order approximation we have the system

\[
(\pm D_r - D_x^2 + \lambda)(g_0^\pm \cdot 1) = 0, \quad (D_x^2 - \lambda)(1 \cdot 1) = 2g_0^+ g_0^- \quad (56)
\]

It has a solution in the form \( g_0^\pm = \beta^\pm e^{\theta^\pm} \), where \( \beta_0^\pm = \beta^+ \beta^- = -\lambda/2 = \alpha_1/4\beta^2 \), \( \theta^\pm = \pm kx \pm (k^2 - \lambda)t \), \( \beta^\pm = \beta_0 e^{\pm \gamma_0} \). Using following properties of Hirota’s derivatives

\[
D_x(fg \cdot h) = \frac{\partial f}{\partial x} gh + fD_x(g \cdot h) \quad (57)
\]

\[
D_x^2(fg \cdot h) = \frac{\partial^2 f}{\partial x^2} gh + 2\frac{\partial f}{\partial x} D_x(g \cdot h) + fD_x^2(g \cdot h) \quad (58)
\]

the bilinear system becomes

\[
(\mp D_r \pm 2kD_x + D_x^2)((1 + \epsilon g_1^+ + \epsilon^2 g_2^+ + \ldots) \cdot (1 + \epsilon f_1 + \epsilon^2 f_2 + \ldots)) = 0, \quad (59)
\]

\[
(D_x^2 + 2\beta_0^2)((1 + \epsilon f_1 + \ldots) \cdot (1 + \epsilon f_1 + \ldots)) = 2\beta_0^2(1 + \epsilon g_1^+ + \ldots)(1 + \epsilon g_1^- + \ldots). \quad (60)
\]

### 7.1 One soliton solution

In the first order we have the system

\[
(\mp \partial_r \pm 2k\partial_x + \partial_x^2)g_1^\pm + (\pm \partial_r \mp 2k\partial_x + \partial_x^2)f_1 = 0, \quad (61)
\]

\[
(\partial_x^2 + 2\beta_0^2)f_1 = \beta_0^2(g_1^+ + g_1^-). \quad (62)
\]

Considering solution in the form \( g_1^\pm = a_1^\pm e^{\eta_1} \), \( f_1 = b_1 e^{\eta_1} \), where \( \eta_1 = k_1 x + \omega_1 \tau + \eta_1^0 \) we get \( a_1^+ = \gamma_1 b_1, \gamma_1 = (\omega_1 - 2kk_1 + k_1^2)/(\omega_1 - 2kk_1 - k_1^2) \), where dispersion formula is

\[
\omega_1^\pm = k_1(2k \pm \sqrt{k_1^2 + 4\beta_0^2}) \quad (63)
\]

It is worth to notice that in contrast to the dark soliton of the defocusing NLS equation, in our case no restrictions on values of \( k_1 \) appear.

Truncation of the Hirota expansion at this level gives one dissipative soliton

\[
e^{(\pm)} = \pm 2\beta^+ e^{\pm[kx + (k^2 + 2\beta_0^2)\tau]} \frac{1 + \gamma_1^{\pm 1} e^{\bar{\eta}_1}}{1 + e^{\eta_1}} \quad (64)
\]

we have absorbed constant \( b_1 \) to the exponential form \( \bar{\eta}_1 = k_1 x + \omega_1 \tau + \eta_1^0 + \ln b_1 \). Then we have one soliton density

\[
\rho = -e^{(+)} e^{(-)} = \alpha_1 \frac{(1 + \gamma_1 e^{\bar{\eta}_1})(1 + \gamma_1^{\prime -1} e^{\bar{\eta}_1})}{(1 + e^{\eta_1})^2} \quad (65)
\]
This solution can be represented in the form

\[ e^{(\pm)} = \pm \sqrt{\alpha_1 \mu}^\pm e^{\pm[kx+(k^2+2\beta_0^2)\tau]} \left( \frac{\gamma^{\pm1} + 1}{2} + \frac{\gamma^{\pm1} - 1}{2} \tanh \frac{\eta}{2} \right) \]  

or

\[ e^{(+)} = + \sqrt{\alpha_1 \mu}^\frac{1}{2} e^{+[kx+(k^2+2\beta_0^2)\tau]} \left( \gamma + 1 + (\gamma - 1) \tanh \frac{\eta}{2} \right) \]  

\[ e^{(-)} = - \sqrt{\alpha_1 \mu}^\frac{1}{2} e^{-[kx+(k^2+2\beta_0^2)\tau]} \left( \frac{1}{\gamma} + 1 + \left( \frac{1}{\gamma} - 1 \right) \tanh \frac{\eta}{2} \right) \]  

and the product is

\[ \rho = -e^{(+)}e^{(-)} = \alpha_1 \left[ 1 + \frac{(\gamma - 1)^2}{4\gamma \cosh^2 \frac{\eta}{2}} \right] \]  

with asymptotic \( \lim_{|x| \to \infty} \rho \to \alpha_1 \). Explicitly it is

\[ \rho = -e^{(+)}e^{(-)} = \alpha_1 \left[ 1 + \frac{k_1^2}{4\beta_0^2} \sech^2 \frac{k_1}{2} \left( x + (2k \pm \sqrt{k_1^2 + 4\beta_0^2})\tau + x_0 \right) \right] \]  

where \( \beta_0^2 = 1/(4\beta^2\alpha_1) \). For the velocity field we have

\[ u = \frac{e^{(0)} - e^{(+)} - e^{(-)}}{e^{(+)} - e^{(-)}} = -2k - \frac{(\gamma^2 - 1)k_1}{(\gamma - 1)^2 + 4\gamma \cosh^2 \frac{\eta}{2}} \]  

Let us consider particular solution for \( k = 0 \), then as follows

\[ \omega_1 = \pm k_1 \sqrt{k_1^2 + 4\beta_0^2} \]  

is the Bogolubov dispersion from the theory of superfluidity of a weakly non-ideal Bose gas. For \( k_1 >> 1 \) it is of the non-relativistic free particles form \( \omega_1 \approx k_1^2 \), while for \( k_1 << 1 \) it is of the relativistic collective form \( \omega_1 \approx 2\beta_0k_1 \). Solution for the + sign of the dispersion has the form

\[ e^{(+)} = \frac{\sqrt{\alpha_1 \mu}}{v - \sqrt{v^2 - 4\beta_0^2}} e^{+2\beta_0^2 \tau} \left( v + \sqrt{v^2 - 4\beta_0^2} \tanh \frac{v^2 - 4\beta_0^2}{2} (x + v\tau + x_0) \right) \]  

\[ e^{(-)} = -\frac{\sqrt{\alpha_1 / \mu}}{v + \sqrt{v^2 - 4\beta_0^2}} e^{-2\beta_0^2 \tau} \left( v - \sqrt{v^2 - 4\beta_0^2} \tanh \frac{v^2 - 4\beta_0^2}{2} (x + v\tau + x_0) \right) \]  

and the density is

\[ \rho = -e^{(+)}e^{(-)} = \alpha_1 \left[ 1 + \frac{v^2 - 4\beta_0^2}{4\beta_0^2} \sech^2 \frac{v^2 - 4\beta_0^2}{2} (x + v\tau + x_0) \right] \]
It shows that velocity of soliton is bounded from below by modulus $|v| > 2\beta_0$, so the soliton has the "tachionic" character. These results show first that in contrast with defocusing NLS with soliton’s velocity bounded from above (subsonic type), the RNLS soliton has velocity bounded from below (supersonic type). Another difference is that soliton of defocusing (repulsive) NLS is the hole (bubble) like excitation with $|\psi|^2 = \rho < 1$, while for the RNLS soliton we have the wall like form $\rho > 1$.

### 7.2 Two soliton solution

To construct two soliton solution following Hirota [20] we consider

$$ g_1^{(+)} = a_1^+ e^{\rho_1} + a_2^+ e^{\rho_2}, \quad f_1 = e^{\rho_1} + e^{\rho_2} \quad (86) $$

Substituting to bilinear equations

$$ (\mp D_x \pm 2k D_x + D_2^2)(a_1^+ e^{\rho_1} + a_2^+ e^{\rho_2}) \cdot 1 + 1 \cdot (e^{\rho_1} + e^{\rho_2}) = 0, \quad (77) $$

$$ 2(D_x^2 + 2\beta_0^2)(1 \cdot (e^{\rho_1} + e^{\rho_2})) = 2\beta_0^2(a_1^+ e^{\rho_1} + a_2^+ e^{\rho_2} + a_1^- e^{\rho_1} + a_2^- e^{\rho_2}) \quad (78) $$

we have the system

$$ a_1^+ (\mp \partial_x \pm 2k \partial_x + \partial_x^2) e^{\rho_1} + (\pm \partial_x \mp 2k \partial_x + \partial_x^2) e^{\rho_2} + $$

$$ a_2^+ (\mp \partial_x \pm 2k \partial_x + \partial_x^2) e^{\rho_2} + (\pm \partial_x \mp 2k \partial_x + \partial_x^2) e^{\rho_1} = 0, \quad (79) $$

$$ (k_1^2 + 2\beta_0^2)(e^{\rho_1} + e^{\rho_2}) = \beta_0^2[(a_1^+ + a_1^-) e^{\rho_1} + (a_2^+ + a_2^-) e^{\rho_2}] \quad (80) $$

Using dispersion

$$ \omega_i^{\pm} = k_i \left( 2k \pm \sqrt{k_i^2 - 4\beta_0^2} \right), \quad (i = 1, 2) \quad (81) $$

we have

$$ a_1^+ = \frac{(\omega_i - 2k k_i) + k_i^2}{(\omega_i - 2k k_i) - k_i^2} e^{\phi_i}, \quad a_1^- = \frac{(\omega_i - 2k k_i) - k_i^2}{(\omega_i - 2k k_i) + k_i^2} = \frac{1}{a_1^+} e^{-\phi_i} \quad (82) $$

The last relations imply

$$ (\omega_i - 2k k_i) = k_i^2 \coth \frac{\phi_i}{2} \quad (83) $$

and

$$ k_i = 2\beta_0 \sinh \frac{\phi_i}{2} \quad (84) $$

so that

$$ (\omega_i - 2k k_i) = 2\beta_0^2 \sinh \phi_i \quad (85) $$

We note that two signs in dispersion [81] correspond in the above formulas to the simple replacement $\phi_i \rightarrow -\phi_i$. First we restrict consideration with the same sign for both frequencies. In the next order we have the system

$$ (\mp D_x \pm 2k D_x + D_2^2)(g_2^+ \cdot 1 + g_1^+ \cdot f_1 + 1 \cdot f_2) = 0, \quad (86) $$

$$ (D_x^2 + 2\beta_0^2)(2 \cdot f_2 + f_1 \cdot f_1) = 2\beta_0^2(g_2^+ + g_2^- + g_1^+ g_1^-) \quad (87) $$
or for the first equation
\[
(\mp \partial_x \pm 2k \partial_x + \partial_x^2)g_2^\pm + (\pm \partial_x \mp 2k \partial_x + \partial_x^2)f_2
+ \{a_1^\pm \text{[}\mp (\omega_1 - \omega_2) \mp 2k(k_1 - k_2) + (k_1 - k_2)^2\text{]} \\
+ a_2^\pm \text{[}\mp (\omega_2 - \omega_1) \mp 2k(k_2 - k_1) + (k_1 - k_2)^2\text{]}\}e^{n_1+n_2} = 0,
\]
(88)

It implies solution in the form
\[
g_2^\pm = a_{12}^\pm e^{n_1+n_2}, \quad f_2 = b_{12}e^{n_1+n_2}
\]
(89)

Then the second equation of the system implies relations
\[
a_{12}^\pm = a_1^\pm a_2^\pm b_{12} = e^{(\phi_1+\phi_2)}b_{12}, \quad a_{12}^- = a_1^- a_2^- b_{12} = e^{-(\phi_1+\phi_2)}b_{12}
\]
(90)

so that
\[
b_{12} = \frac{\sinh^2 \frac{\phi_1 - \phi_2}{4}}{\sinh^2 \frac{\phi_1 + \phi_2}{4}}
\]
(91)

and the first equation is satisfied automatically. As a result we have solution
\[
e^{(\pm)} = \pm 2\beta \frac{g_0^\pm (1 + g_1^\pm + g_2^\pm)}{1 + f_1 + f_2}
\]
(92)

or
\[
e^{(\pm)} = \pm 2\beta \frac{g_0^\pm (1 + e^{n_1\pm\phi_1} + e^{n_2\pm\phi_2} + b_{12}e^{n_1+n_2\pm(\phi_1+\phi_2)})}{1 + e^{n_1} + e^{n_2} + b_{12}e^{n_1+n_2}}
\]
(93)

For particular parametrization \(\beta_0 = 1/2\), implying \(\alpha_1 = 1\), we have two soliton solution for density \(\rho\),
\[
\rho = \frac{A_+ A_-}{|\sinh^2 \frac{\phi_1 + \phi_2}{4} (1 + e^{n_1} + e^{n_2}) + \sinh^2 \frac{\phi_1 - \phi_2}{4} e^{n_1+n_2}|^2}
\]
(94)

where
\[
A_\pm = \sinh^2 \frac{\phi_1 + \phi_2}{4} (1 + e^{n_1\pm\phi_1} + e^{n_2\pm\phi_2}) + \sinh^2 \frac{\phi_1 - \phi_2}{4} e^{n_1+n_2\pm(\phi_1+\phi_2)},
\]
(95)

\[
\eta_i = \sinh \frac{\phi_i}{2} x + [2k \sinh \frac{\phi_i}{2} + \frac{1}{2} \sinh \phi_i] \tau + \eta_i^{(0)}, \quad (i = 1, 2)
\]
(96)

If one of the parameters \(\phi_i\) is vanishing or if \(\phi_1 = \phi_2\), the solution is reduced to the one soliton form. For example if \(\phi_2 = 0\) then
\[
\rho = 1 + \frac{\sinh^2 \frac{\phi}{2}}{\cosh^2 \frac{\phi}{2}}
\]
(97)

Analyzing two soliton solution in the soliton’s moving frames we can see that it describes collision of two solitons type \([77]\), moving in the same direction with the initial position shifts
\[
\Delta x_i = (-1)^{i-1} \frac{2}{\sinh \phi_i^2} \ln \frac{\sinh \phi_i - \phi_i}{\sinh \phi_i^2} \quad (i = 1, 2)
\]
(98)
so that \( \sinh \frac{\phi_1}{2} \Delta x_1 + \sinh \frac{\phi_2}{2} \Delta x_2 = 0 \).

The different form of two soliton solution we obtain if we choose opposite
signs for frequencies (81), so that

\[
\omega^+_1 = 2\beta_0 (2k \sinh \frac{\phi_1}{2} + \beta_0 \sinh \phi_1), \\
\omega^-_2 = 2\beta_0 (2k \sinh \frac{\phi_1}{2} - \beta_0 \sinh \phi_1)
\]

(99)

(100)

Then \( a^+_1 = e^{\pm \phi_1}, a^-_1 = e^{\mp \phi_1} \),

\[
a^+_1 a^+_2 b_{12} = e^{\phi_1 - \phi_2} b_{12}, \\
a^-_1 a^-_2 b_{12} = e^{-\phi_1 + \phi_2} b_{12}
\]

(101)

(102)

and

\[
b_{12} = \frac{\cosh^2 \frac{\phi_1 + \phi_2}{4}}{\cosh^2 \frac{\phi_1 - \phi_2}{4}}
\]

(103)

For \( \beta_0 = 1/2 \) two soliton solution is

\[
\rho = \frac{B_+ B_-}{[\cosh^2 \frac{\phi_1 - \phi_2}{4} (1 + e^{\eta_1} + e^{\eta_2}) + \cosh^2 \frac{\phi_1 + \phi_2}{4} e^{\eta_1 + \eta_2}]^2}
\]

(104)

where

\[
B_\pm = \cosh^2 \frac{\phi_1 - \phi_2}{4} (1 + e^{\eta_1 \pm \phi_1} + e^{\eta_2 \mp \phi_2}) + \cosh^2 \frac{\phi_1 + \phi_2}{4} e^{\eta_1 + \eta_2 \pm \phi_1 \mp \phi_2},
\]

(105)

and

\[
\eta_1 = \sinh \frac{\phi_1}{2} (x - x_1) + [2k \sinh \frac{\phi_1}{2} + \frac{1}{2} \sinh \phi_1] \tau, \\
\eta_2 = \sinh \frac{\phi_2}{2} (x - x_2) + [2k \sinh \frac{\phi_2}{2} - \frac{1}{2} \sinh \phi_2] \tau
\]

(106)

(107)

It describes collision of two solitons in the form (97), moving in opposite
direction with initial position shifts

\[
\Delta x_i = (-1)^{i-1} \frac{2}{\sinh \frac{\phi_i}{2}} \ln \frac{\cosh \frac{\phi_i + \phi_2}{4}}{\cosh \frac{\phi_i - \phi_2}{4}}, (i = 1, 2)
\]

(108)

In Fig.1 we show 3D plot of this solution.

For the velocity field we have

\[
u = \frac{c_x^{(-)} - c_x^{(-)}}{c_x^{(+)} - c_x^{(+)}} = -2k +
\]

\[
-\frac{\cosh^2 \frac{\phi_1 - \phi_2}{4} a_+ + \cosh^2 \frac{\phi_1 - \phi_2}{4} b_-}{\cosh^2 \frac{\phi_1 - \phi_2}{4} (1 + e^{\eta_1 - \phi_1} + e^{\eta_2 + \phi_2}) + \cosh^2 \frac{\phi_1 + \phi_2}{4} e^{\eta_1 + \eta_2 - \phi_1 + \phi_2}}
\]

\[
-\frac{\cosh^2 \frac{\phi_1 - \phi_2}{4} a_- + \cosh^2 \frac{\phi_1 - \phi_2}{4} b_+}{\cosh^2 \frac{\phi_1 - \phi_2}{4} (1 + e^{\eta_1 + \phi_1} + e^{\eta_2 - \phi_2}) + \cosh^2 \frac{\phi_1 + \phi_2}{4} e^{\eta_1 + \eta_2 + \phi_1 - \phi_2}}
\]

(109)
Figure 1: Two Soliton Scattering

where

\[ a_\pm = \left( \sinh \frac{\phi_1}{2} e^{\eta_1 \mp \phi_1} + \sinh \frac{\phi_2}{2} e^{\eta_2 \pm \phi_2} \right) \] (110)

\[ b_\pm = \left( \sinh \frac{\phi_1}{2} + \sinh \frac{\phi_2}{2} \right) e^{\eta_1 \mp \phi_1 \pm \phi_2} \] (111)

It has the same phase shift and describe collision of two solitons in the form

\[ u = -2k - \frac{k_i \sinh \phi_i}{2 \cosh \frac{\eta_i + \phi_i}{2} \cosh \frac{\eta_i - \phi_i}{2}}, \quad (i = 1, 2). \] (112)

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