Sparse estimation for generalized exponential marked Hawkes process

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Received: 15 October 2021 / Accepted: 11 March 2022 / Published online: 13 April 2022
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Abstract
We established a sparse estimation method for the generalized exponential marked Hawkes process by the penalized method to ordinary method (P–O) estimator. Furthermore, we evaluated the probability of the correct variable selection. In the course of this, we established a framework for a likelihood analysis and the P–O estimation when there might be nuisance parameters, and the true value of the parameter might be at the boundary of the parameter space. Finally, numerical simulations are given for several important examples.

Keywords Hawkes process · Sparse estimation · P–O estimator · Quasi-likelihood analysis · Statistical inference · Generalized exponential kernel

1 Introduction

The Hawkes process is a self-exciting point process introduced by Hawkes (1971) and has a wide range of applications including seismic (see Ogata 1981), finance (see Abergel et al. 2016), and web data analysis (see Goda et al. 2021). For the properties as a stochastic process, a class with exponential kernels has attracted much attention since its intensity process has Markov, geometric ergodic, and mixing properties, for example, see Clinet and Yoshida (2017) and Goda (2021). As an extension of the Hawkes process, the marked Hawkes process is known. By the marked Hawkes process, it is possible to consider a model that adds the scale of the event and other characteristics to the occurrence times of the events. The properties of the marked Hawkes process were investigated by Clinet (2021), and in which an important class called the generalized exponential marked Hawkes process (GEMHP) was introduced. The GEMHP is a marked Hawkes process whose kernel has the flexible form, which is generated by terms multiplying $e^{-r\cdot w}$, where $r > 0$, by a polynomial or trigonometric function. The GEMHP can be represented in terms of a Markov process, and we can establish the geometric ergodicity of the GEMHP. Moreover, the convergence of moments for the...
quasi maximum likelihood estimator (QMLE) and the quasi Bayesian estimator (QBE) was established, in Clinet (2021), for a class of the GEMHP with a linear intensity. In particular, a polynomial type large deviation inequality was established, in the results of the quasi-likelihood analysis in Yoshida (2011). The GEMHP with a linear intensity has been used for the models of earthquakes marked with the magnitude (see Ogata 1981), the limit order book in finance (see Rambaldi et al. 2017), etc. In this paper, we introduce the Hawkes process marked by “topic” as an example of the GEMHP with a linear intensity.

Model selection is one of the most important topics in statistical inference. In particular, sparse estimation methods like the least absolute shrinkage and selection operator (LASSO) in Tibshirani (1996), the elastic net in Zou and Hastie (2005), and so on, have been widely studied starting from linear regression problems. By the sparse estimation method, we can execute parameter estimation and variable selection simultaneously. Let \( \theta^* = (\theta^*_j)_{j=1,\ldots,p} \) be the true parameter of a parameter \( \theta = (\theta_j)_{j=1,\ldots,p} \) for \( p \in \mathbb{N} \). Moreover, let \( J^0 = \{ j | \theta^*_j = 0 \} \) and \( J^1 = \{ j | \theta^*_j \neq 0 \} \). We write \( \theta_J = (\theta_j)_{j \in J} \) for a vector \( \theta \) and an index set \( J \). The following two properties are called the oracle properties (see Fan and Li 2001) that the sparse estimator \( \hat{\theta}_T \) should satisfy:

- Selection consistency: \( P[\hat{\theta}_{J^0} = 0] \to 1 \),
- Asymptotic normality: \( \sqrt{T}(\hat{\theta}_{J^1} - \theta^*_{J^1}) \to^d N(0, \Gamma^{-1}) \),

as \( T \to \infty \) for some positive definite matrix \( \Gamma \), where \( T \) represents an observation time, and we omit \( T \) in the expression \( \hat{\theta}_{J^0} \) and \( \hat{\theta}_{J^1} \). We focus on the penalized method to ordinary method (P–O) estimator proposed in Suzuki and Yoshida (2020). The P–O estimation is a sparse estimation method using the least-squares approximation method given a prior estimator with the consistency, and retuning using an ordinary estimation method such as the maximum likelihood estimator. It satisfies the oracle properties under the suitable conditions. Moreover, we can evaluate the probability of the correct variable selection.

A sparse estimation for the multivariate Hawkes process is useful to identify disconnections in networks. It is also possible to identify which marks do not affect the trend by a sparse estimation for the marked Hawkes process. Furthermore, it is also important that a sparse estimation prevents the overfitting of the model. In a previous study applying a sparse estimation to the Hawkes process, Hansen et al. (2015) proposed an adaptive \( L^1 \)-penalized methodology for the nonparametric case and evaluated the oracle inequality. In the parametric case, there are several studies with respect to the Hawkes process using the exponential kernel. In Zhou et al. (2013), they designed the log-likelihood function penalized by the nuclear and \( L^1 \) norm and an algorithm ADM4 for their estimation method, while they investigated the performance of their method through numerical experiments. In Bacry et al. (2020), they proposed the least-squares method with the entry-wise weighted nuclear and \( L^1 \) norm penalization and proved a sharp oracle inequality for their procedure. In Goda et al. (2021), they introduced a hybrid method combined with the QMLE and the \( L^1 \)-penalized QMLE and investigated the accuracy of model selection and the asymptotic normality through numerical experiments. However, the oracle properties have not been established even for the exponential Hawkes process with no marks.

We apply the P–O estimation to the GEMHP, and we prove the oracle properties and evaluate the probability of the correct variable selection. In this application, when the GEMHP contains zero parameters, it is often necessary to assume the existence of nuisance parameters. For example, let \( N_t = (N^1_t, \ldots, N^d_t) \) be a \( d \)-dimensional exponential Hawkes process with...
the intensity process

\[ \lambda_i^t = \mu_i + \sum_{j=1}^{d} \int_{0}^{t} \alpha_{ij} e^{-\beta_{ij}(t-s)} dN_i^j, \quad i = 1, \ldots, d, \] (1.1)

where \( \mu_i, \alpha_{ij}, \) and \( \beta_{ij} \) are parameters for \( i, j = 1, \ldots, d \). Then, \( \beta_{ij} \) is undefined, that is, \( \beta_{ij} \) is a nuisance parameter, when \( \alpha_{ij} = 0 \). We confirm that the polynomial type large deviation inequality for the quasi-likelihood of the GEMHP with nuisance parameters holds and thus that the consistency for the QMLE and the QBE holds. Furthermore, in most cases of statistical inference, the true value of a parameter in the GEMHP is zero, it might be at the boundary of the parameter space. For an example of an exponential Hawkes process with the intensity (1.1), \( \alpha_{ij} \)'s are often assumed to take value in a compact subset in \([0, \infty)\). Then, \( \alpha_{ij} = 0 \) is realized on the boundary of the parameter space. We prove that the P–O estimator works well even in such a situation where the true value is on the boundary.

We explain the GEMHP and its properties in Sect. 2. In Sect. 3, we discuss the P–O estimator under the condition that some parameters are nuisance parameters and the true parameter is possibly on the boundary of the parameter space. The main results about the application of the P–O estimation to the sparse GEMHP are in Sect.4. Finally, Sect. 5 presents the results of some numerical experiments. We introduce the Hawkes process marked by “topic” in this section. The proofs of each statement are given in Appendix A. Moreover, we give Additional numerical experiments in Appendix B.

2 Generalized exponential marked Hawkes process

In this section, we review the theory in Clinet (2021). In particular, we define the generalized exponential marked Hawkes process (GEMHP). The GEMHP is a class of marked Hawkes processes that satisfies the Markov, ergodic, and mixing properties under the stability conditions. In this article, we only focus on the GEMHP with the linear form intensity for our application in Sect.4. On the other hand, we note that the Markov, ergodic, and mixing properties are proved for more general non-linear (sub-linear) form intensities.

2.1 Marked point process

First, we define the general marked point process. Let \( \mathbb{B} = (\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, P) \) be a stochastic basis, and \( (\mathbb{X}, \mathcal{X}) \) be a measurable space. For \( d \in \mathbb{N} \), we consider a sequence of couples \( (T_n^i, X_n^i)_{n \in \mathbb{N}, i=1,\ldots,d} \). Suppose that \( T_n^i \)'s are \( \mathbb{F} \)-stopping times such that almost surely \( T_0^i = 0 < T_1^i < \cdots < T_n^i < \cdots < \infty \) and \( T_n^i \to \infty \) as \( n \to \infty \) hold for each \( i \), and \( X_n^i \)'s are \( \mathbb{X} \)-valued \( \mathcal{F}_{T_n^i} \)-measurable random variables. We define the \( d \)-dimensional marked point process \( \tilde{N} = (\tilde{N}^1, \ldots, \tilde{N}^d) \) as a family of random measures on \( \mathbb{R}_+ \times \mathbb{X} \) such that \( \tilde{N}^i(ds, dx) = \sum_{n \in \mathbb{N}} \delta_{(T_n^i, X_n^i)}(ds, dx) \). Moreover, we call a random measure \( \nu^i(ds, dx) \) the compensator of \( \tilde{N}^i(ds, dx) \) when \( \tilde{N}^i([0, t] \times F) - \nu^i([0, t] \times F) \) is a local martingale for any \( F \in \mathcal{X} \), see Theorem 1.8 in Jacod and Shiryaev (2003).
2.2 Generalized exponential marked Hawkes process

We use the same notations in Sect. 2.1. We write the counting process associated with $\tilde{N}^i_t$ as $N^i_t = \tilde{N}^i(0, t] \times X$ and jump times of the global counting process $\sum_{i=1}^d N^i_t$ as $(T^i_n)_{n \in \mathbb{N}}$. Furthermore, let $(X^i_n)_{n \in \mathbb{N}}$ be a permutation of $(\tilde{X}^i_n)_{n \in \mathbb{N}, i=1, \ldots, d}$ similar to the relationship between jump times $(T^i_n)_{n \in \mathbb{N}}$ and $(\tilde{T}^i_n)_{n \in \mathbb{N}, i=1, \ldots, d}$. Then, we define the mark process $X_t$ as a piecewise constant and right continuous stochastic process such that $X_t = X_n$ for $t \in [T_n, T_{n+1})$ where $T_0 = 0$ and $X_0 = X^1_0$. The marked Hawkes process is defined as below.

Definition 2.1 A $d$-dimensional marked point process $\tilde{N}$ is called a $d$-dimensional marked Hawkes process if the intensity process of the associated counting process $N^i_t$ has the form

$$\lambda^i_t = \phi_i \left( \left( \int_{(0,t] \times X} h_{ij}(t - s, x) \tilde{N}^j(ds, dx) \right)_{j=1, \ldots, d}, X_{t^-} \right), \quad i = 1, \ldots, d,$$

where $\phi_i : \mathbb{R}^d_+ \times X \rightarrow \mathbb{R}_+$ is a continuous function and $h_{ij} : \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+$ is a measurable function for each $i, j = 1, \ldots, d$.

Then, the linear GEMHP is defined by restricting the form of the function $\phi_i$ and the kernel function $h_{ij}$. For $p \in \mathbb{N}$, we denote the Frobenius inner product on a real matrix space $\mathbb{R}^{p \times p}$ as $\langle \cdot | \cdot \rangle$.

Definition 2.2 A $d$-dimensional marked Hawkes process $\tilde{N}$ is called a $d$-dimensional linear generalized exponential marked Hawkes process if the function $\phi_i$ and the kernel function $h_{ij}$ have the representations, for $i, j = 1, \ldots, d$,

$$\phi_i(u, x) = \mu_i(x) + \sum_{j=1}^d u_j$$

and

$$h_{ij}(s, x) = \langle A_{ij} | e^{-sB_{ij}} \rangle g_{ij}(x),$$

where $\mu_i : X \rightarrow \mathbb{R}_+$ and $g_{ij} : X \rightarrow \mathbb{R}_+$ are measurable functions, $A_{ij}, B_{ij} \in \mathbb{R}^{p \times p}$ for some $p \in \mathbb{N}$, and $e^{-sB_{ij}}$ is the matrix exponential for each $i, j = 1, \ldots, d$. That is, its intensity process has the form

$$\lambda^i_t = \mu_i(X_{t^-}) + \sum_{j=1}^d \int_{(0,t] \times X} \langle A_{ij} | e^{-sB_{ij}} \rangle g_{ij}(x) \tilde{N}^j(ds, dx)$$

for $i = 1, \ldots, d$.

The temporal part $\langle A | e^{-sB} \rangle$ of the kernel is represented as a linear combination of terms $P(s)(1 + C_1 \cos(\xi s) + C_2 \sin(\xi s))e^{-r}$, where $\xi, C_1, C_2 \in \mathbb{R}$ and $r > 0$ are constants, and $P(s) = \sum_{p=0}^P a_p s^p$ is a polynomial for $a_0, \ldots, a_P \in \mathbb{R}$ and some $P \in \mathbb{N}$, see Proposition 3.1 of Clinet (2021).

The remainder of this section is devoted to the explanation of the sufficient conditions of Theorem 2.3. We restrict the distribution of the mark process $X_t$ to maintain the Markov structure of the intensity process. Let $(\kappa_n)_{n \in \mathbb{N}}$ be labels of the jumps of the global counting process $\sum_{i=1}^d N^i_t$, i.e., $\kappa_n$ is a $\{1, \ldots, d\}$-valued random variable such that $\Delta N^\kappa_{T_n} = 1$ for
\( n \in \mathbb{N}. \) We write \( \Delta T_n = T_n - T_{n-1}. \) Then, we assume that there exists a family of Feller transition kernels \( \{ Q_i \}_{i=1,\ldots,d} \) on \( \mathcal{X} \times \mathcal{X} \) such that

\[
P\left[ X_n \in F \mid \kappa_n, \Delta T_n, \mathcal{F}_{T_{n-1}} \right] = Q_{\kappa_n}(X_{n-1}, F) \tag{2.1}\]

for any \( F \in \mathcal{X} \) and \( n \in \mathbb{N}. \) For the stability of the process, we assume that \( B_{ij} \) has eigenvalues with positive real parts. We define the intensity process of the GEMHP. We write the transition kernel of the global mark process \( X_t \) and there exists a family of Feller transition kernels \( \{ \Phi_{ij} \}_{i,j=1,\ldots,d} \) that refers to the conditional expectation of a jump size in \( \lambda^i \) when \( N^j \) jumps.

By the assumption on \( B_{ij} \), we have the representation \( \Phi_{ij}(x) = (A_{ij} | B_{ij}^{-1})G_{ij}(x) \) for any \( i, j = 1, \ldots, d. \)

Let

\[
\xi_t^{ij} = \int_{[0,t] \times \mathcal{X}} e^{-(t-s)B_{ij}} g_{ij}(x) \mathcal{N}_j(ds, dx) \tag{2.2}
\]

for \( i, j = 1, \ldots, d \) and \( Z_t = (\xi_t, X_t) = ((\xi_t^{ij})_{i,j=1,\ldots,d}, X_t) \). \( Z_t \) obviously drives the intensity process of the GEMHP. We write the transition kernel of the global mark process \( X \) by \( Q \). It satisfies

\[
Q(Z_{T_{n-1}}, \cdot) = \frac{1}{\xi(\Delta T_i, Z_{T_{n-1}})} \sum_{i=1}^d \xi_t^{i}(\Delta T_i, Z_{T_{n-1}}) Q_i(X_{i-1}, \cdot),
\]

where \( \xi(t, z) = \sum_{i=1}^d \xi^{i}(t, z) \) and \( \xi^{i}(t, z) = \mu_i(x) + \sum_{j=1}^d \langle A_{ij} | e^{-tB_{ij}} \rangle \) for \( t \geq 0 \) and \( z = (\xi_{ij})_{i,j,x} \), see Proposition 3.2 of Clinet (2021). We call a non-negative function \( f \) a norm-like function if \( f(x) \to \infty \) as \( |x| \to \infty \) holds. The following statements are the sufficient conditions for the geometric ergodicity and the geometric mixing property of the GEMHP.

[L1] There exist norm-like functions \( f \) and \( u \) such that

\[
\sum_{i=1}^d \mu_i(x) = O(u_X(x)) \text{ as } |x| \to \infty, \tag{2.3}\]

\[
\sum_{i=1}^d \mu_i(x) \int_{\mathcal{X}} \left\{ f_X(y) - f_X(x) \right\} Q_i(x, dy) \leq -u_X(x) \text{ for any } x \in \mathcal{X}, \tag{2.4}\]

\[
\sum_{i=1}^d \mu_i(x) \sum_{j=1}^d G_{ji}(x) = o(u_X(x)) \text{ as } |x| \to \infty, \tag{2.5}\]

and there exists \( \tilde{c} > 0 \) such that

\[
\sup_{x \in \mathcal{X}, j=1,\ldots,d} \int_{\mathcal{X}} e^{\tilde{c} \left[ \sum_{i=1}^d g_{ij}(y) + f_X(y) - f_X(x) \right]} Q_j(x, dy) < \infty. \tag{2.6}\]

[L2] There exist \( \kappa \in \mathbb{R}^d \) with positive coefficients and \( \rho \in [0, 1) \) such that, component-wise, \( \sup_{x \in \mathcal{X}} (\Phi(x)^T \kappa) \leq \rho \kappa. \)

[ND1] There exist \( \phi, g > 0 \) such that \( \phi_i > \phi \) and \( g_{ij} > g \) for any \( i, j = 1, \ldots, d. \)

[ND2] The transition kernel \( Q \) admits a reachable point \( x_0 \in \mathcal{X}. \) Moreover, for any \( j = 1, \ldots, d, \) the transition kernel \( Q_j \) admits a sub-component \( T_j, \) such that there exist
a lower semi-continuous function $r_j: \mathbb{X}^2 \to \mathbb{R}_+$ and a non-trivial measure $\sigma_j$ on $\mathcal{X}$, such that

- $\sigma_j(O) > 0$ for any non-empty open set $O \in \mathcal{X}$,
- $T_j(x, F) = \int_F r_j(x, y)\sigma_j(\text{d}y)$ and $T_j(x, \mathcal{X}) > 0$ for any $x \in \mathcal{X}$ and $F \in \mathcal{X}$.

We remark that $\nu_\theta$ that might take the value $\nu_1$ and $N$.

We write the transition kernel of $Z$ as $P^t$ for $t \geq 0$. Finally, the $V$-norm of a measure $\mu$ on a measurable space $(S, \mathcal{S})$ is defined as

$$\|\mu\|_V = \sup_{\psi \leq V} \left| \int_S \psi(s)\mu(\text{d}s) \right|$$

for a positive function $V$, where the supremum is taken over all the measurable functions $\psi$ such that $\psi(s) \leq V(s)$ for all $s \in S$. The following theorem is the main theorem in this section, which is shown in Theorem 3.7 of Clinet (2021).

**Theorem 2.3** (Theorem 3.7 in Clinet 2021) Under $[L1]$–$[L2]$ and $[ND1]$–$[ND2]$, $Z$ is $V$-geometrically ergodic, i.e., there exist a unique invariant measure $\pi$ and constants $(a_{ij})_{i,j=1,\ldots,d} \in \mathbb{R}^{p^2d^2}$, $\eta > 0$, $C \geq 0$, $r \in [0,1)$ such that for any $t > 0$ and $z = (\epsilon, x) \in \mathbb{R}^{p^2d^2} \times \mathbb{X}$

$$\|P^t(z, \cdot) - \pi\|_V \leq C(1 + V(z))r^t$$

where $V(z) = \exp\{ \sum_{i,j=1}^d (a_{ij} - \eta f_x(x)) \}$. Moreover, $Z$ is $V$-geometrically mixing, i.e., there exist positive constants $C' > 0$, $r' \in [0,1)$ such that for any $t, u \geq 0$ and measurable functions $\phi, \psi$ with $\phi \leq V$, $\psi \leq V$,

$$\left| E[\phi(Z_{t+u})\psi(Z_t) | Z_0 = z] - E[\phi(Z_{t+u}) | Z_0 = z]E[\psi(Z_t) | Z_0 = z] \right| \leq C'V(z)r'^u.$$

### 3 P–O estimator with nuisance parameter

The penalized method to ordinary method (P–O) estimator was introduced by Suzuki and Yoshida (2020). The P–O estimator allows us to execute parameter estimation and variable selection simultaneously. In this section, we extend the applicable condition of the P–O estimation to the case where nuisance parameters might exist, and the true parameter might be at the boundary of the parameter space. In general, a parameter that is not subject to estimation is called a nuisance parameter. In particular, we admit a nuisance parameter whose true value might be undefined.

Let $\vartheta = (\theta^0, \theta^1, \nu^0, \nu^1) \in \Sigma = \Theta^0 \times \Theta^1 \times N^0 \times N^1$ be a parameter, where $\Theta^0$, $\Theta^1$, $N^0$, and $N^1$ are open convex bounded subsets of $\mathbb{R}^{p_0}$, $\mathbb{R}^{p_1}$, $\mathbb{R}^{n_0}$, and $\mathbb{R}^{n_1}$ for $p_0, p_1, n_0, n_1 \in \mathbb{N}$, respectively. $\theta^0 = (\theta^0_1, \ldots, \theta^0_{p_0})$ is a parameter that might take the value $\theta^0_i = 0$ for some $i$. $\theta^1 = (\theta^1_1, \ldots, \theta^1_{p_1})$ is a non-zero parameter. $\nu^0 = (v^0_1, \ldots, v^0_{n_0})$ is a nuisance parameter that might take the value $\nu^0_i = 0$ for some $i$. $\nu^1 = (v^1_1, \ldots, v^1_{n_1})$ is a nuisance parameter that takes a non-zero value. Note that $\nu^0$ and $\nu^1$ might not have the true value. Suppose that $(\theta^{0*}, \theta^{1*}) = (\theta^{0*}_1, \ldots, \theta^{0*}_{p_0}, \theta^{1*}_1, \ldots, \theta^{1*}_{p_1}) \in \Theta^0 \times \Theta^1$ is the true parameter, and let $\mathcal{J}^0 = \{ j = 1, \ldots, p_0 | \theta^{0*}_j = 0 \}$ and $\mathcal{J}^1 = \{ j = 1, \ldots, p_0 | \theta^{1*}_j \neq 0 \}$. We write $d_1 = \#\mathcal{J}^1$. We remark that $\theta^{0*}$ might be on the boundary of $\Theta^0$. 

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Example 3.1 Let $\tilde{N}$ be a GEMHP with the intensity $\lambda_i^j(\vartheta^*)$: Here, for $i = 1, \ldots, d$,

$$\lambda_i^j(\vartheta) = \mu_i + \sum_{k=1}^d \int_{0,t} p_k \alpha_{ijk} s^k \left( 1 + \sum_{l=1}^{d'} m_{ijl} \varphi_l^2 \right) e^{-(u-s)\beta_{ij}} N_j(ds, dx),$$

where $\vartheta = (\mu_i, (\alpha_{ijk})_{i,j,k}, (\beta_{ij})_{i,j}, (m_{ijl})_{i,j,l})$ are non-negative parameters and $\vartheta^* = (\mu_i^*, (\alpha_{ijk})^*_{i,j,k}, (\beta_{ij})^*_{i,j}, (m_{ijl})^*_{i,j,l})$ is the true parameters with nuisance parameters. We consider the situation where some $\alpha_{ijk}$ and $m_{ijl}$ might be 0 besides $\mu_i$ and $\beta_{ij}$ are always positive. In this case, $\beta_{ij}^*$ and $m_{ijl}^*$ are undefined parameters, and thus nuisance parameters, when $\alpha_{ijk}^*$ $= 0$ for all $k$. In other words, we can write $\vartheta^0 = ((\alpha_{ijk})_{i,j,k}, (m_{ijl})_{i,j,l})$.

Let $T > 0$ be an observation time index. We often consider the case where $T \in \mathbb{N}$ with discrete time observation or $T \in \mathbb{R}_+$ with continuous time observation. For convenience of explanation, we consider two objective functions $L^1_T: \mathbb{E} \to \mathbb{R}$ and $L^2_T: \mathbb{E} \to \mathbb{R}$. For example, $L^1_T$ and $L^2_T$ are log-likelihood functions multiplied by $-1$. The objective function $Q^T(q): \tilde{N}^0 \times \tilde{N}^0 \to \mathbb{R}$ will be defined later. We denote the $j$-th component of each estimator $\theta_T$ as $\theta_j$ by omitting $T$. If argmin is realized at multiple points, we take one of them arbitrarily. The P–O estimation is done in the following three steps:

Step 1 We obtain the first estimator of $\vartheta$ by

$$\left( \hat{\theta}_T^0, \hat{\vartheta}_T^0, \check{\vartheta}_T^0, \check{\vartheta}_T^1 \right) = \arg\min_{\vartheta \in \mathbb{E}} \mathbb{L}^1_T(\vartheta).$$

Step 2 We obtain the second estimator of $(\theta^0, \nu^0)$ by

$$\left( \hat{\theta}_T^0(q), \hat{\nu}_T^0(q) \right) = \arg\min_{(\theta^0, \nu^0) \in \tilde{N}^0 \times \tilde{N}^0} Q^T(q)(\theta^0, \nu^0),$$

where the objective function $Q^T(q)$ depends on the first estimator $(\hat{\theta}_T^0, \check{\vartheta}_T^0)$. 

Step 3 We obtain the third estimator of $\vartheta$ by

$$\left( \hat{\vartheta}_T^0, \hat{\vartheta}_T^1, \check{\vartheta}_T^0, \check{\vartheta}_T^1 \right) = \arg\min_{\vartheta \in \hat{\vartheta}_T^0 \times \hat{\vartheta}_T^1 \times \tilde{N}^0 \times \tilde{N}^1} \mathbb{L}^2_T(\vartheta),$$

where $\hat{\vartheta}_T^0 = \{\theta^0 \in \tilde{N}^0 | \theta_j^0 = 0, j = 1, \ldots, p^0 \}$, $\hat{\vartheta}_T^1 = \{j = 1, \ldots, p^0 | \theta_j^0 = 0 \}$, $\check{\vartheta}_T^0 = \{\nu^0 \in \tilde{N}^0 | \nu_j^0 = 0, j = 1, \ldots, n_0 \}$, and $\check{\vartheta}_T^1 = \{j = 1, \ldots, n_0 | \nu_j^0 = 0 \}$. We call $\tilde{\vartheta}_T = (\hat{\vartheta}_T^0, \hat{\vartheta}_T^1, \check{\vartheta}_T^0, \check{\vartheta}_T^1)$ the P–O estimator. The sparse model selection is derived from Step 2. The objective function $Q^T(q)(\theta^0, \nu^0)$ is constructed by using the first estimators $\hat{\vartheta}_T^0, \check{\vartheta}_T^0$ as below:

$$Q^T(q)(\theta^0, \nu^0) = \sum_{j=1}^{p^0} \left( (\theta_j^0 - \check{\theta}_j^0)^2 + \kappa_j^0 |\theta_j^0| \right) + \sum_{j=1}^{n_0} \left( (\nu_j^0 - \check{\nu}_j^0)^2 + \kappa_j^0 |\nu_j^0| \right), \quad (3.1)$$

where $q \in (0, 1]$, $\kappa_j^0 = \alpha_T |\nu_j^0 + \nu_j^0|$, and $\alpha_T$ are deterministic sequences. We remark that $\kappa_j^0$ and $\kappa_j^0$ depend on $T$. We often choose $\epsilon_T$ and $\alpha_T$ to converge to 0 as $T \to \infty$, see Remark 3.4. Let $a_T = \max_{j \in J^1} \kappa_j^0$ and $b_T = \min_{j \in J^0} \kappa_j^0$. 

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We retune the estimator in Step 3. We rewrite the notation in Step 3 to describe the properties of components with the non-zero true values in the third estimator. Without loss of generality, we can assume that \( \theta^0 = (\phi, \psi) \in \bar{\Theta}_\phi^0 \times \bar{\Theta}_\psi^0 = \bar{\Theta}^0 \) and its true value \( \theta^{0*} = (\phi^*, \psi^*) = (\phi^*, 0) \). Let \( \bar{l}^2_T(\theta, \theta^0, v^0, v^1) = \bar{l}_T^2(\phi, 0, \theta^1, v^0, v^1) \) and
\[
(\bar{\theta}_T^0, \bar{\theta}_T^1, \bar{v}_T^0, \bar{v}_T^1) = \arg\min_{(\phi, \theta^1, v^0, v^1) \in \bar{\Theta}_\phi^0 \times \Theta_1 \times \bar{\Theta}_\psi^0 \times \mathbb{R}^N} \bar{l}_T^2(\phi, \theta^1, v^0, v^1).
\]

By contrary, we write \( \bar{\theta}_T^0 = (\bar{\theta}_T^0)_{j \in J_1} \). We call a stochastic process \( X_T \) is \( L^\infty \)-bounded if
\[
sup_T E [|X_T|^p] < \infty \text{ holds for all } p \geq 1.
\]
We introduce sufficient conditions for the oracle properties of the P–O estimator. Note that the following assumptions demand the properties of the first and third-step estimators. In other words, it is not necessary to assume the objective functions \( \bar{L}_T^1 \) and \( \bar{L}_T^2 \) to establish the oracle properties of the P–O estimator.

**Assumption 3.2**

(i) \( \sqrt{T} \bar{\theta}_T^0 = \sqrt{T}(\bar{\theta}_T^0 - \theta^{0*}) = O_p(1) \) as \( T \to \infty \).

(i)’ \( \sqrt{T}(\bar{\theta}_T^0 - \theta^{0*}) \) is \( L^\infty \)-bounded.

(ii) \( T^{1/2} a_T = O_p(1) \) and \( T^{(2-q)/2} b_T \to^p \infty \) as \( T \to \infty \).

(iii) \( \sqrt{T} \{|(\bar{\theta}_T^0, \bar{\theta}_T^1) - (\phi^*, \theta^{1*})|\} \) → \( d \Lambda^{-1/2}(\psi^*) \xi \) as \( T \to \infty \), where \( \Lambda(\psi^*) \in \mathbb{R}^{(d_1 + p_1) \times (d_1 + p_1)} \) is a positive definite matrix for any \( \psi^* \in \bar{\Theta}_\psi^0 \times \mathbb{R}^N \), and \( \xi \) is a \( (d_1 + p_1) \)-dimensional standard Gaussian random vector.

(iv) There exists \( \epsilon \in (-1 + q, \gamma) \) such that \( T^{-(1 + \gamma - \epsilon)}/2 \alpha_T^{-1} = O(1) \) as \( T \to \infty \).

(v) \( \sqrt{T} \{|(\bar{\theta}_T^0, \bar{\theta}_T^1) - (\phi^*, \theta^{1*})|\} \) is \( L^\infty \)-bounded.

**Remark 3.3** Assumption 3.2 (i)’ is obviously a stronger condition than (i). We need (i)’ to evaluate the probability of a correct variable selection for the P–O estimator. In the case where \( \bar{L}_T^1 \) and \( \bar{L}_T^2 \) are the quasi log-likelihood functions, (i)’ and (v) are satisfied if the polynomial type large deviation inequality for the quasi likelihood ratio random field holds, see Proposition 1 in Yoshida (2011).

**Remark 3.4** Assumptions 3.2 (ii) and (iv) are conditions on the weight of the penalty term. These conditions are satisfied, for example, in the following setup under Assumption 3.2 (i). We take \( a \in (0, 1 - q + \gamma) \) for given \( q \in (0, 1] \) and \( \gamma > -1 + q \). Moreover, we take positive deterministic sequences \( a_T, \epsilon_T \) that satisfy \( a_T = O(T^{-(1+q)/2}) \) and \( \epsilon_T = O(T^{-1/2}) \). Then, as \( T \to \infty \),
\[
T^{1/2} a_T = O(T^{-\frac{q}{2}}) \left| O(T^{-\frac{1}{2}}) + \min_{j \in J_1} \left| \bar{\theta}_j^0 - \theta_j^{0*} \right| + \max_{j \in J_1} \theta_j^{0*} \right|^{-\gamma} = o_p(1)
\]
and
\[
T^{2-q} b_T = O(T^{\frac{q - 2 - \gamma - q}{2}}) \left| O(1) + \max_{j \in J_0} \sqrt{T} \left| \bar{\theta}_j - \theta_j^{0*} \right| \right|^{-\gamma} \to^p \infty.
\]
Moreover, \( T^{-(1+\gamma-\epsilon)/2} \alpha_T^{-1} = O(1) \) as \( T \to \infty \) holds for \( \epsilon = \gamma - a \in (-1 + q, \gamma) \).

The following theorem is the main theorem in this section. The statement (i) means the selection consistency in Step 2. Moreover, the statements (ii) and (iv) say that the probability of a correct variable selection converges to 1 in a polynomial decay as \( T \to \infty \). The statement (iii) is about the asymptotic normality of the P–O estimator. In particular, the statements (iii) and (iv) mean the oracle properties of the P–O estimator. Compared to the result of Suzuki and Yoshida (2020), each statement is extended to the situation where nuisance parameters might exist and the true parameter might be at the boundary of the parameter space.
Theorem 3.5  
(i) Under Assumption 3.2 (i) and (ii),

\[ P \left[ \bar{\mathcal{J}}^0_T = \mathcal{J}^0 \right] \to 1 \text{ as } T \to \infty. \]

(ii) Under Assumption 3.2 (i)', (ii) and (iv), for all \( L > 0 \), there exists \( C_L > 0 \) such that

\[ P \left[ \bar{\mathcal{J}}^0_T = \mathcal{J}^0 \right] \geq 1 - C_L T^{-L} \]

for all \( T > 0 \).

(iii) Under Assumption 3.2 (i), (ii) and (iii)

\[ \sqrt{T} \{ (\bar{\mathcal{J}}^0, \bar{\mathcal{J}}^0) - (\phi^*, \theta^1) \} \to^d A^{-1/2} (v^*) \xi \text{ as } T \to \infty. \]

(iv) Under Assumption 3.2 (i)', (ii), (iv) and (v), \( \sqrt{T} \{ (\mathcal{J}^0, \mathcal{J}^0) - (\phi^0, \theta^1) \} \) is \( L^\infty_- \)-bounded. Moreover, for all \( L > 0 \), there exists \( C_L > 0 \) such that

\[ P \left[ \bar{\mathcal{J}}^0_T = \mathcal{J}^0 \right] \geq 1 - C_L T^{-L} \]

for all \( T > 0 \), where \( \bar{\mathcal{J}}^0_T = \{ j = 1, \ldots, p_0 \mid \bar{\mathcal{J}}^0_j = 0 \} \).

4 Application to GEMHP

In this section, we apply the penalized method to ordinary method (P–O) estimator to the generalized exponential marked Hawkes process (GEMHP).

For tractability, we restrict the intensity of the GEMHP to a linear form, that is, \( \phi_i(u, x) = \mu_i(x) + \sum_{j=1}^{d} u_j \) for any \( i = 1, \ldots, d \). When \( \phi_i \) is sub-linear, more involved formulations for the conditions [AH1]–[AH2] below are needed, and we let it aside for future works as in Section 4 of Clinet (2021). We set a parameter \( \vartheta = (\theta^0, \theta^1, v^0, v^1) \in \mathcal{Z} = \mathcal{G}^0 \times \mathcal{G}^1 \times \mathcal{N}^0 \times \mathcal{N}^1 \) as in Sect. 3. For some \( \vartheta^* \in \mathcal{Z} \), we assume that \( \tilde{\mathcal{N}} \) is a \( d \)-dimensional GEMHP with the following intensity process:

\[
\lambda_i^\vartheta(x, \vartheta^*) = \mu_i(X_{t-}, \vartheta^*) + \sum_{j=1}^{d} \int_{(0,t) \times \mathcal{X}} \langle A_{ij}(\vartheta^*) e^{-\vartheta B_{ij}(\vartheta^*)} \rangle_{g_{ij}}(x, \vartheta^*) \tilde{\mathcal{N}}^j(ds, dx),
\]

for \( i = 1, \ldots, d \), where \( \mu_i : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}_+ \), \( g_{ij} : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}_+ \), \( A_{ij} : \mathcal{Z} \to \mathbb{R}_+^{p \times p} \), \( B_{ij} : \mathcal{Z} \to \mathbb{R}_+^{p \times p} \) are measurable functions, and the real parts of the eigenvalues of \( B_{ij}(\vartheta) \) are dominated by some \( r > 0 \) independently of \( \vartheta \in \mathcal{Z} \), for some \( p \in \mathbb{N} \) and each \( i, j = 1, \ldots, d \). We assume that there exist the Feller transition kernels of the mark process similar to (2.1). Moreover, we restrict the structure of the mark transition kernel \( \bar{Q}_j(x, y, \vartheta) \). We assume that there exists a dominating measure \( \rho \) on \( \mathcal{X} \) which induces the density \( p_j \), i.e.,

\[ Q_j(x, dy, \vartheta) = p_j(x, y, \vartheta) \rho(dy) \]

for any \( \vartheta \in \mathcal{Z} \) and \( j = 1, \ldots, d \). Then, we have the following representation of the compensator of \( \tilde{\mathcal{N}}^i(ds, dx) \):

\[ v^i(ds, dx) = \lambda^\vartheta_i(X_{t-}, x, \vartheta^*) ds \rho(dx), \]
where $X$ is the mark process defined as in Sect. 2.1. We write $q^i_t(x, \cdot) = p_t(X_{t-}, x, \cdot)$. Let $T > 0$ be an observation time index. We consider the quasi log-likelihood function with respect to the above GEMHP:

$$
I_T(\theta) = \sum_{i=1}^d \left( \int_{[0,T] \times \mathbb{X}} \log (\lambda_{i,T}(\theta)q^i_t(x, \cdot)) \tilde{N}^i(dt, dx) - \int_{[0,T] \times \mathbb{X}} \lambda_{i,T}(\theta)q^i_t(x, \cdot) dt \rho(dx) \right)
$$

$$
= \sum_{i=1}^d \left( \int_0^T \log \lambda_{i,T}(\theta) dN^i_t - \int_0^T \lambda_{i,T}(\theta) dt \right) + \sum_{i=1}^d \int_{[0,T] \times \mathbb{X}} \log q^i_t(x, \cdot) \tilde{N}^i(dt, dx)
$$

$$
=: I_T^{(1)}(\theta) + I_T^{(2)}(\theta).
$$

(4.2)

Here, $I_T^{(1)}(\theta)$ is the part related to the counting process $N^i_t = \tilde{N}([0, t] \times \mathbb{X})$ and $I_T^{(2)}(\theta)$ to the mark process $X_t$. We remark that $I_T(\theta)$ is actually equivalent to the log-likelihood function related to $\tilde{N}$ if the filtration $\mathcal{F}$ satisfies the appropriate condition, see the condition (2.12) and Theorem 5.43 in Jacod and Shiryaev (2003). However, we call $I_T(\theta)$ the “quasi” log-likelihood function since we allow here to take a more general filtration $\mathcal{F}$. We consider the P–O estimator whose first and third objective functions are set to this quasi log-likelihood function.

The quasi maximum likelihood estimator (QMLE) $\tilde{\theta}_T$ is defined as a quantity satisfying

$$
\tilde{\theta}_T = (\tilde{\theta}_T^0, \tilde{\theta}_T^1, \tilde{v}_T^0, \tilde{v}_T^1) \in \text{argmax}_{\theta \in \Xi} I_T(\theta),
$$

and we set this QMLE as the first-step estimator. Similar to as in Sect. 3, we set the second-step estimator of $(\theta^0, \nu^0)$ by

$$
(\tilde{\theta}_T^{0,(q)}, \tilde{v}_T^{0,(q)}) = \text{argmin}_{(\theta^0, \nu^0) \in \Theta^0 \times \tilde{N}^0} Q_T^{(q)}(\theta^0, \nu^0),
$$

and the third-step estimator of $\theta$ by

$$
\tilde{\theta}_T = (\tilde{\theta}_T^0, \tilde{\theta}_T^1, \tilde{v}_T^0, \tilde{v}_T^1) = \text{argmax}_{\theta \in \tilde{\Theta}_T^0 \times \Theta^1 \times \tilde{N}^0 \times \tilde{N}^1} I_T(\theta),
$$

where $Q_T^{(q)}(\theta^0, \nu^0)$ is the same as (3.1), and $\tilde{\Theta}_T^0, \tilde{N}_T^0$ are defined in Step 3 of the P–O estimator in Sect. 3.

**Remark 4.1** The results in this section also hold if we consider the quasi Bayesian estimator (QBE)

$$
\tilde{\theta}_T = \frac{\int_{\Xi} \theta \exp(l_T(\theta)) p(\theta) d\theta}{\int_{\Xi} \exp(l_T(\theta)) p(\theta) d\theta}
$$

instead of the QMLE by changing notations in Step 1 and Step 3 in Sect. 3, where $\rho : \Xi \to \mathbb{R}_+$ is a continuous prior density with $0 < \inf_{\theta \in \Xi} p(\theta) < \sup_{\theta \in \Xi} p(\theta) < \infty$.

We define the quasi likelihood ratio random field as

$$
\mathbb{Z}_T(u, \nu^*) = \exp \{ I_T(\theta^* + u/\sqrt{T}, \nu^*) - I_T(\theta^*) \}.
$$

(4.3)

As mentioned in Remark 3.3, if $\mathbb{Z}_T$ satisfies the polynomial type large deviation inequality (PLD), we can show that Assumption 3.2 (i’) and (v) holds for the above P–O estimator. Therefore, to obtain the PLD, we prepare additional conditions [AH1]–[AH3] beside the conditions [L1]–[L2], [ND1]–[ND2]. We sometimes rewrite $\theta = (\theta^0, \theta^1, \nu^0, \nu^1)$ as $(\theta, \nu)$,
where \( \theta = (\theta^0, \theta^1) \in \Theta = \Theta^0 \times \Theta^1 \) and \( v = (v^0, v^1) \in N = \bar{N}^0 \times \bar{N}^1 \). We also write \( \theta^* = (\theta^0, v^*) = (\theta^{0*}, \theta^{1*}, v^{0*}, v^{1*}) \), where \( \theta^{0*}, \theta^{1*} \) is the true value and \( v^{0*}, v^{1*} \) is an arbitrary point in \( N \). We call that \( f(\theta) \) is of class \( C^4(\Xi) \) for some \( i \in \mathbb{N} \) if \( f(\theta) \) is of class \( C^4(\Xi) \) and its derivatives admit continuous extensions on \( \partial \Xi \).

[AH1]

(i) For any \( x, y \in \mathbb{X} \) and \( i, j = 1, \ldots, d, \mu_i(x, \cdot), g_{ij}(x, \cdot), A_{ij}(\cdot), B_{ij}(\cdot), p_i(x, y, \cdot) \) are in \( C^4(\Xi) \).

(ii) For any \( \theta \in \Xi \) and \( i, j = 1, \ldots, d, \lambda_i^j(\theta)q_{ij}^*(x, \theta) = 0 \) if and only if \( \lambda_i^j(\theta)q_{ij}^*(x, \theta^*) = 0, dt \rho(dx)P(d\omega) \text{-} a.e. \)

[AH2] For any \( p > 1, x \in \mathbb{X} \), and \( i, j = 1, \ldots, d \), there exists \( C_p > 0 \) which may depend on \( p \) such that

(i) \( \sup_{\theta \in \Xi} \sum_{n=0}^{\infty} \left[ |\partial^p_\theta \mu_i(x, \theta)|_p + |\mu_i(x, \theta)|_p \right] < \infty \)

\( \leq C_p e^{\eta f(x)} \),

(ii) \( \sup_{\theta \in \Xi, v \in \mathbb{N}} \sum_{n=0}^{\infty} \left[ |\partial^p_\theta g_{ij}(x, \theta)|_p + |g_{ij}(x, \theta)|_p \right] < \infty \)

\( \leq C_p e^{\eta f(x)} \),

where \( f(x) \) is a norm-like function in \( [L1] \), and \( \eta \) is a positive constant in Theorem 2.3.

The conditions [AH1] and [AH2] are sufficient conditions to leads the conditions [A1]–[A3] in Appendix A.2. [AH1] is the regularity condition for the intensity process, which leads to [A1]. [AH2] is closely related to [A2] and is used for the evaluation of moments of the quasi log-likelihood process. Let the quasi log-likelihood random field be

\( \mathbb{V}_T(\theta, v^*) = \frac{1}{T} \left[ l_T(\theta) - l_T(\theta^*) \right] \).

The \( V \)-geometric ergodicity of the GEMHP and [AH1]–[AH2] guarantee the existence of the limit field

\( \mathbb{V}(\theta, v^*) = \sum_{i=1}^{d} E \left[ \int_{\mathbb{X}} \log \left( \frac{f_{0i}^*(x, \theta)}{f_{0i}^*(x, \theta^*)} \right) \left( f_{0i}^*(x, \theta^*) - \left( f_{0i}^*(x, \theta) - f_{0i}^*(x, \theta^*) \right) \rho(dx) \right) \right] \)

where \( f_{0i}^*(x, \theta) = \lambda_{0i}^*(\theta)q_{0i}^*(x, \theta) \) is the density of the predictable compensator of the stationary version \( \bar{N}^0 \) at time 0, see Lemma 4.1 in Clinet (2021). For this \( \mathbb{V}(\theta, v^*) \), we consider the following identifiability condition.

[AH3] \( \inf_{\theta \in \Theta \setminus \{\theta^*\}, v, v^* \in N} \frac{\mathbb{V}(\theta, v^*)}{|\theta - \theta^*|^2} > 0 \).

Moreover, we write \( \Gamma(v^*) = -\partial^2_{\theta^*} \mathbb{V}(\theta^*, v^*) \).

Remark 4.2 \( \mathbb{V}(\theta, v^*) \) is of class \( C^2(\Xi) \) by the condition [A3] in Appendix. Then, the condition [AH3] is equivalent to the two following conditions:

(i) \( \mathbb{V}(\theta, v^*) < 0 \) for any \( \theta \in \Theta \setminus \{\theta^*\} \) and \( v, v^* \in N \).

(ii) \( \Gamma(v^*) \) is positive definite uniformly in \( v^* \in N \).

Let \( U_T = \{ u \in \mathbb{R}^{p_0^0 + p_1} | \theta^* + u/\sqrt{T} \in \Theta \} \) and \( V_T(r) = \{ u \in U_T | |u| \geq r \} \). The following theorem says that the quasi likelihood ratio random field \( Z_T \) satisfies the PLD.
Theorem 4.3 (Polynomial type large deviation inequality) Under \([L1]–[L2], [ND1]–[ND2],\) and \([AH1]–[AH3],\) for any \(L > 0,\) there exists \(C_L > 0\) such that

\[
\sup_{r > 0, T > 0} \sup_{u \in \mathcal{V}_T(r), \nu^* \in \mathcal{N}} P\left[ Z_T(u, \nu^*) \geq e^{-r} \right] \leq \frac{C_L r L}{rL},
\]

where \(Z_T(u, \nu^*)\) is defined in (4.3). In particular, \(\sqrt{T}(\tilde{\theta}_{0T}^0 - \theta_{0*})\) is \(L^{\infty^-}\)-bounded.

On the other hand, the asymptotic normality of the QMLE, that is, Assumption 3.2 (iii), is shown in Corollary 4.2 of Clinet (2021). With the help of the PLD and the asymptotic normality, we obtain the following theorem, which is the main theorem in this article.

Theorem 4.4 Under \([L1]–[L2], [ND1]–[ND2],\) and \([AH1]–[AH3],\) Assumption 3.2 are satisfied by \(L^1_T(\vartheta) = L^2_T(\vartheta) = l_T(\vartheta)\) and \(\alpha_T, \epsilon_T\) in Remark 3.4. In particular, following statements hold:

(i) \(\sqrt{T}\left\{\left(\tilde{\theta}_{0T}^0, \tilde{\theta}_{1T}^0\right) - \left(\vartheta_{0*}, \vartheta_{1*}\right)\right\} \rightarrow^d \Lambda^{-1/2}(\nu^*)\xi\) as \(T \to \infty,\) where \(\Lambda(\nu^*) = -\delta_{i,j}^2(\vartheta_{0*}, \nu^*)\) and \(\xi\) is a standard Gaussian random vector.

(ii) \(\sqrt{T}\left\{\left(\tilde{\theta}_{0T}^0, \tilde{\theta}_{1T}^0\right) - \left(\vartheta_{0*}, \vartheta_{1*}\right)\right\}\) is \(L^{\infty^-}\)-bounded. Moreover, for all \(L > 0,\) there exists \(C_L > 0\) such that

\[
P\left[ \tilde{\mathcal{J}}_T^0 = \mathcal{J}_T^0 \right] \geq 1 - C_L T^{-L}
\]

for all \(T > 0,\) where \(\mathcal{J}_T^0 = \{j = 1, \ldots, p_0 \mid \theta_{jT}^0 = 0\}\) and \(\tilde{\mathcal{J}}_T^0 = \{j = 1, \ldots, p_0 \mid \tilde{\theta}_{jT}^0 = 0\}.

5 Examples and simulation results

In this section, we show the numerical simulation result for Theorem 4.4. Experiments in the scenario with no zero parameters and a comparison with previous studies are presented in Appendix B. All experiments are done by using Python3.1 We consider the same notations and restrictions in Sect. 4. Moreover, let \(\Phi(\varphi)\) be as in Sect. 2. We write \(Z_t = (\mathcal{E}_t, X_t),\) where \(X_t\) is the mark process and \(\mathcal{E}_t\) is the generalized elementary excitation process defined by (2.2). The value of the GEMHP’s intensity process \(\lambda_t\) is updated based on the value of \(Z_u\) for \(u < t\) as follows:

\[
\lambda_{it} = \mu_i(X_{t-}) + \sum_{j=1}^{d} \left\{ A_{ij} e^{-((t-u)B_{ij} \mathcal{E}_{ui})} + \int_{(u,t) \times \mathbb{X}} A_{ij} e^{-sB_{ij}} g_{ij}(x) N^j(ds, dx) \right\},
\]

for \(i = 1, \ldots, d.\) Then, a path of the GEMHP can be simulated by using the above computation and Ogata’s method, see Ogata (1988). In the following subsections, we see the oracle properties of the P–O estimator for the GEMHP via numerical experiments for two basic models. For each model, we calculate the QMLE and the P–O estimator 300 times, respectively, while changing the observation time to \(T = 100, 500,\) and 3000. Here, all optimizations are done by the limited memory Broyden-Fletcher-Goldfarb-Shanno method for bound-constrained (L-BFGS-B), see Zhu et al. (1997).

1 The code is available on the GitHub page https://github.com/goda235/Sparse-estimation-for-GEMHP.
5.1 Multivariate exponential Hawkes process

First, we focus on the non-marked, however, quite important, Hawkes process.

5.1.1 Definition

We consider the multivariate exponential Hawkes process \( N_t = (N_t^1, \ldots, N_t^d) \) with the intensity
\[
\lambda_i^j(\theta) |_{\theta = \theta^*} = \mu_i + \sum_{j=1}^d \int_{[0,t]} \alpha_{ij} e^{-(t-s)\beta_{ij}} N_j^j(ds) |_{\theta = \theta^*},
\]
for \( i = 1, \ldots, d \), where \( \theta = (\alpha_{ij})_{ij}, (\mu_i)_i, (\beta_{ij})_{ij} \in \mathcal{X} \) is a parameter, \( \theta^* = ((\alpha_{ij}^*)_{ij}, (\mu_i^*)_i, (\beta_{ij}^*)_{ij}) \in \mathcal{X} \) is the true value, and \( \mathcal{X} = \Theta_\alpha \times \Theta_\mu \times \Theta_\beta \subset \mathbb{R}^d_+ \times \mathbb{R}^d_{>0} \times \mathbb{R}^d_{>0} \) is an open convex bounded parameter space. We assume the following conditions.

Assumption 5.1 (i) Some \( \alpha_{ij}^* \) might be 0 besides all \( \mu_i^* \) and \( \beta_{ij}^* \) are positive. Moreover, \( \Theta_\beta \subset \mathbb{R}^d_+ \).

(ii) The spectral radius of \( \Phi = \left( \frac{\alpha_{ij}^*}{\beta_{ij}^*} \right)_{ij} \) is less than 1.

In (i), we assume that \( \beta_{ij} \) are away from 0 to control the oscillation of the nuisance parameter. (ii) is a stability condition related to [L2]. We write \( \theta^0 = (\alpha_{ij})_{ij}, \theta^1 = ((\mu_i)_i, (\beta_{ij})_{ij}) \in \mathcal{X} \) is a parameter, \( \theta^* = ((\alpha_{ij}^*)_{ij}, (\mu_i^*)_i, (\beta_{ij}^*)_{ij}) \in \mathcal{X} \) is the true value, and \( \mathcal{X} = \Theta_\alpha \times \Theta_\mu \times \Theta_\beta \subset \mathbb{R}^d_+ \times \mathbb{R}^d_{>0} \times \mathbb{R}^d_{>0} \) is an open convex bounded parameter space. We assume the following conditions.

Assumption 5.1 (i) Some \( \alpha_{ij}^* \) might be 0 besides all \( \mu_i^* \) and \( \beta_{ij}^* \) are positive. Moreover, \( \Theta_\beta \subset \mathbb{R}^d_+ \).

(ii) The spectral radius of \( \Phi = \left( \frac{\alpha_{ij}^*}{\beta_{ij}^*} \right)_{ij} \) is less than 1.

5.1.2 Simulation results

Let \( N_t = (N_t^1, N_t^2, N_t^3) \) be a 3-dimensional exponential Hawkes process with the following parameters:
\[
\mu^* = (0.2, 0.1, 0.1), \quad \alpha^* = \begin{pmatrix} 0.0 & 0.2 & 0.0 \\ 0.2 & 0.1 & 0.4 \\ 0.0 & 0.0 & 0.2 \end{pmatrix}, \quad \beta^* = \begin{pmatrix} * & 0.9 & * \\ 0.5 & 1.2 & 0.6 \\ * & * & 0.7 \end{pmatrix},
\]
where * means a non-definite value. We set the hyperparameters of the P–O estimator in Remark 3.4 to be \( q = 1.0, r = 1.0, a = 0.5 \), the observation times \( T = 100, 500, 3000 \), and the number of the Monte Carlo simulation \( MC = 300 \). Table 1 shows the fraction of
Fig. 1 Examples of Hawkes graphs by the 3-dimensional multivariate exponential Hawkes process. The left figure shows the case where all $\alpha_{ij}^*$’s are positive. The right figure shows the case where $\alpha_{11}^* = \alpha_{13}^* = \alpha_{31}^* = \alpha_{32}^* = 0$.

trials in which the parameter $\alpha_{ij}$’s are estimated to be completely zero. We can see that the variable selection is performed more accurately by the P–O estimator than by the QMLE as $T$ becomes larger.

**Remark 5.3** In Table 1, the QMLE asymptotically correctly estimates about 50% of the zero parameters. This phenomenon is derived from the local asymptotic normality on the restricted parameter space to a positive region, see Goda et al. (2021) for an intuitive but more detailed explanation. There are few studies of the maximum likelihood method on the constrained parameter space using as a variable selection method. In an i.i.d. case, the asymptotic behavior of the maximum likelihood estimator is discussed under general assumptions where we can consider such a constrained parameter space, see Le Cam (1970). The same phenomenon is observed in Table 3 below.

Figure 2 shows histograms of the error distribution of the P–O estimator, that is, histograms of the values of $\sqrt{T}(\hat{\theta}_T - \vartheta^*)$. It seems that the distribution is close to the normal distribution as $T$ becomes larger.

Table 2 shows the averages of squared errors of the QMLE $(\hat{\theta}_T - \vartheta^*)^2$ and the P–O estimator $(\hat{\vartheta}_T - \vartheta^*)^2$. For non-zero parameters, owing to the asymptotic normality, both the QMLE and the P–O estimator have asymptotically the same level of variance. For zero parameters, we can guess that the P–O estimator asymptotically has a smaller error than the QMLE, due to the accurate model selection. However, when the observation time is small, the performance of the QMLE is better due to the miss model selection of the P–O estimator.

**5.2 Hawkes process marked with “Topic”**

Second, we introduce the marked Hawkes process useful in the field of natural language processing.
Table 1  Percentage of estimated to be zero

|               | QMLE: T = 100 |       |       |       |       |       |
|---------------|--------------|-------|-------|-------|-------|-------|
|               | α_{11}       | α_{12} | α_{13} |       |       |       |
| QMLE: T = 100 | 71.7%        | 11.7%  | 52.7%  |       |       |       |
|               | α_{21}       | α_{22} | α_{23} |       |       |       |
| QMLE: T = 500 | 62.7%        | 0.00%  | 54.0%  |       |       |       |
|               | α_{31}       | α_{32} | α_{33} |       |       |       |
| QMLE: T = 3000| 58.0%        | 0.00%  | 50.0%  |       |       |       |
|               | α_{11}       | α_{12} | α_{13} |       |       |       |
| P–OE: T = 100 | 84.3%        | 30.7%  | 66.3%  |       |       |       |
|               | α_{21}       | α_{22} | α_{23} |       |       |       |
| P–OE: T = 500 | 83.3%        | 3.67%  | 72.7%  |       |       |       |
|               | α_{31}       | α_{32} | α_{33} |       |       |       |
| P–OE: T = 3000| 87.7%        | 0.00%  | 79.0%  |       |       |       |
|               | α_{11}       | α_{12} | α_{13} |       |       |       |
| P–OE: T = 100 | 87.3%        | 88.7%  | 0.00%  |       |       |       |
| Bold values are parameters whose true values are zero

5.2.1 Definition

We consider a web service where \( d \) types of users post texts while reading each other’s posts, like social network services such as Twitter and Facebook, product reviews on Amazon, and so on. In this subsection, we model a sequence of posting times \((T_n^i)_{i=1,...,d,n\in\mathbb{N}}\) by using a GEMHP. Since we can consider the distribution of future posting times naturally depends on the content of the previous posts, we will regard the content of texts as marks.

Techniques to quantify the amount of “topic” in a sentence have been studied in the field of natural language processing. For example, Latent Dirichlet Allocation (LDA) is a hierarchical Bayesian model in which a sentence \( w = (w_1,\ldots,w_N) \) consisting of \( N \) words is generated by a conditional multinomial distribution given a “topic” \( z \), see Blei et al. (2003). The topic \( z \) in the LDA model is a \( \{1,\ldots,d'\} \)-valued random variable (where \( d' \) is the number of topics), and its distribution is a conditional multinomial distribution whose parameters are generated by the Dirichlet distribution. Conversely, we can consider the conditional probabilities \( p(z = l|w) \) \( l=1,...,d' \), and it can be assumed as a proportion of each topic in a sentence \( w \).

For \( i = 1,\ldots,d \) and \( n \in \mathbb{N} \), let \( w_n^i = (w_{n,1}^i,\ldots,w_{n,M_n^i}^i) \) be the \( n \)-th post by the \( i \)-th user, where \( M_n^i \) is the number of words in a post \( w_n^i \). We assume that the conditional probabilities
\[ \sqrt{T} \left( \hat{\vartheta}_T - \vartheta^* \right) \] are given for a \( \{1, \ldots, d'\} \)-valued random topic \( z \). Then, we regard \( p(z = l | w = w_n^i), \ldots, p(z = d' | w = w_n^i) \) as a mark \( X_n^i \).

Now, we consider the model for the above sequence of a couple \((T_n^i, X_n^i)_{i,n}\). Let \( \tilde{N} \) be the GEMHP with the intensity

\[ \lambda_i^*(\vartheta^*) = \mu_i + \sum_{j=1}^{d'} \int_{[0, t] \times \mathbb{X}} e^{-\beta_{ij} s} \left( \sum_{l=1}^{d''} m_{ijl} x_l \right) \tilde{N}_j(ds, dx) \bigg|_{\vartheta^*} \],

for \( i = 1, \ldots, d \), and suppose that its mark process takes values on the \((d' - 1)\)-simplex\(^2\) and has the transition kernel

\[ Q_j(x, dy, \theta_M) = p_j(x, y, \theta_M) dy, \]

where \( \vartheta = (m_{ijl}, \mu_i, \beta_{ij}) \in \Xi \) is a parameter, \( \vartheta^* = (m_{ijl}^*, \mu_i^*, \beta_{ij}^*) \in \Xi \) is the true value, and \( \Xi = \Theta_m \times \Theta_\mu \times \Theta_\beta \times \Theta_M \subset \mathbb{R}^{d' \times d''} \times \mathbb{R}^d_{>0} \times \mathbb{R}^2_{>0} \times \mathbb{R}^{d''} \) is an open convex bounded parameter space. For a vector

\(^2\) The \((d' - 1)\)-simplex is the set \( \{ x = (x^1, \ldots, x^{d''}) \in \mathbb{R}^{d''} \mid \sum_{i=1}^{d'} x^i = 1 \} \).
Table 2  Average of squared errors

|   | Method  | \(\mu_1\) | \(\mu_2\) | \(\mu_3\) | \(\alpha_{11}\) | \(\alpha_{12}\) | \(\alpha_{13}\) |
|---|---------|------------|------------|------------|----------------|----------------|----------------|
| 100 | QMLE    | 6.80e-03  | 5.63e-03  | 3.09e-03  | 5.90e-01     | 1.15e-00     | 4.89e-00     |
|     | P–OE    | 5.79e-03  | 6.73e-03  | 1.80e-03  | 6.34e-01     | 1.19e-00     | 5.01e-00     |
| 500 | QMLE    | 1.58e-03  | 1.81e-03  | 8.06e-04  | 1.61e-02     | 1.41e-02     | 1.03e-01     |
|     | P–OE    | 1.24e-03  | 1.80e-03  | 3.62e-04  | 2.16e-02     | 1.39e-02     | 9.51e-02     |
| 3000| QMLE    | 2.59e-04  | 2.44e-04  | 1.84e-04  | 1.81e-03     | 1.28e-03     | 4.19e-03     |
|     | P–OE    | 1.68e-04  | 2.43e-04  | 6.17e-05  | 1.72e-03     | 1.26e-03     | 4.76e-03     |

|   | Method  | \(\alpha_{21}\) | \(\alpha_{22}\) | \(\alpha_{23}\) | \(\alpha_{31}\) | \(\alpha_{32}\) | \(\alpha_{33}\) |
|---|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| 100 | QMLE    | 3.78e-01    | 1.38e-00    | 1.47e-00    | 1.66e-00     | 5.75e-01     | 4.73e-01     |
|     | P–OE    | 4.89e-01    | 1.51e-00    | 1.77e-00    | 1.77e-00     | 1.06e-00     | 6.64e-00     |
| 500 | QMLE    | 9.07e-03    | 1.01e-01    | 2.26e-02    | 8.13e-03     | 5.51e-02     | 1.87e-02     |
|     | P–OE    | 9.26e-03    | 1.02e-01    | 2.20e-02    | 8.17e-03     | 5.50e-02     | 1.86e-02     |
| 3000| QMLE    | 1.09e-03    | 3.81e-03    | 3.15e-03    | 1.07e-03     | 1.82e-03     | 1.95e-03     |
|     | P–OE    | 1.09e-03    | 3.96e-03    | 3.13e-03    | 1.03e-03     | 1.77e-03     | 1.91e-03     |

|   | Method  | \(\beta_{11}\) | \(\beta_{12}\) | \(\beta_{13}\) | \(\beta_{21}\) | \(\beta_{22}\) | \(\beta_{23}\) |
|---|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| 100 | QMLE    | *              | 4.07e+02      | *              | 6.88e+01      | 2.77e+02      | 5.97e+01      |
|     | P–OE    | *              | 4.17e+02      | *              | 7.79e+01      | 2.81e+02      | 8.79e+01      |
| 500 | QMLE    | *              | 1.70e-00      | *              | 1.84e-01      | 1.37e+01      | 1.30e-01      |
|     | P–OE    | *              | 1.51e-00      | *              | 1.67e-01      | 1.36e+01      | 1.38e-01      |
| 3000| QMLE    | *              | 5.67e-02      | *              | 1.04e-02      | 6.99e-00      | 8.98e-03      |
|     | P–OE    | *              | 5.13e-02      | *              | 1.01e-02      | 6.94e-00      | 8.88e-03      |

|   | Method  | \(\beta_{31}\) | \(\beta_{32}\) | \(\beta_{33}\) |
|---|---------|----------------|----------------|----------------|
| 100 | QMLE    | *              | *              | 4.66e+01       |
|     | P–OE    | *              | *              | 6.24e+01       |
| 500 | QMLE    | *              | *              | 1.76e-00       |
|     | P–OE    | *              | *              | 1.65e-00       |
| 3000| QMLE    | *              | *              | 3.58e-02       |
|     | P–OE    | *              | *              | 3.14e-02       |
(ii) The spectral radius of \( \Phi(x) = \left( \frac{G_{ij}(x)}{p_{ij}} \right) \) is less than 1 uniformly in \( x \in \mathbb{X} \), where

\[
G_{ij}(x) = \int_{\mathbb{X}} \left( \sum_{l=1}^{d'} m_{ijl}y_l \right) p_j(x, y, \theta^n_M)dy.
\]

(iii) The transition kernel \( Q \) admits a reachable point \( x_0 \in \mathbb{X} \). Moreover, there exists a lower semi-continuous function \( r_j : \mathbb{X}^2 \rightarrow \mathbb{R}_+ \) such that \( Q_j \) admits a sub-component \( T_j \) with \( T_j(x, \mathbb{X}) > 0 \) and \( T_j(x, F) = \int_{F} r_j(x, y)dy \) for any \( x \in \mathbb{X} \) and \( F \in B(\mathbb{X}) \).

(iv) For any \( l = 1, \ldots, d' \), the mark process \( X^l \) is not almost surely constant. Moreover, for any \( l_1, l_2 = 1, \ldots, d' \), \( X^{l_1} \) and \( X^{l_2} \) are distinguishable\(^3\).

(v) For any \( p > 1 \), \( x \in \mathbb{X} \), and \( i = 1, \ldots, d \),

\[
\sup_{\theta_M \in \Theta_M} \sum_{n=0}^{3} \int_{\mathbb{X}} |\partial^2_{\theta_M} \log p_i(x, y, \theta_M)|^p p_i(x, y, \theta^n_M)\rho(dy) \leq C_p \eta^p f_X(x)
\]

holds, where \( f_X \) is a norm-like function in \( [L1] \), and \( \eta \) is a positive constant in Theorem \( 2.3 \). Moreover, for any \( t \geq 0 \) and \( i = 1, \ldots, d \), the transition densities \( q_{ij}^n(x, \theta_M) \)'s satisfy the following conditions.

(a) If there exists \( \theta_M \in \Theta_M \) such that \( q_{ij}^n(x, \theta_M) = q_{ij}^n(x, \theta^n_M) \), \( dx P(d\omega) - a.e. \), then \( \theta_M = \theta^n_M \).

(b) If there exists \( x \in \mathbb{R}^{d''} \) such that \( x^T \partial_{\theta_M} \log q_{ij}^n(x, \theta^n_M) = 0 \), \( dx P(d\omega) - a.e. \), then \( x = 0 \).

(c) Almost surely,

\[
\int_{\mathbb{X}} \left( \partial_{\theta_M} \log q_{ij}^n(x, \theta^n_M) \right)^2 q_{ij}^n(x, \theta^n_M)dx = -\int_{\mathbb{X}} \partial^2_{\theta_M} \log q_{ij}^n(x, \theta^n_M)q_{ij}^n(x, \theta^n_M)dx.
\]

Remark 5.5 We can prove that the above GEMHP satisfies the conditions \([L1]–[L2], [ND1]–[ND2] \) without Assumption \( 5.4 \) (v). In other words, Theorem \( 2.3 \) holds without Assumption \( 5.4 \) (v). Then, the statement of Assumption \( 5.4 \) (v) has a meaning under the other assumptions. See the proof of Proposition \( 5.6 \).

(i) is a constraint on the parameters, in particular, requiring \( p_{ij} \) to be away from \( 0 \) to control the oscillation of the nuisance parameter. (ii) and (iii) are assumptions for the sake of \([L2] \) and \([ND2] \), respectively. (iv) ensures the identifiability of the parameters. (v) is an assumption related to the probability density of marks to guarantee \([AH2] \) and \([AH3] \). For this model, the oracle properties of the P–O estimation holds by Theorem \( 4.4 \).

Proposition 5.6 Under Assumption \( 5.4 \), the GEMHP with the intensity \( 5.2 \) satisfies the conditions \([L1]–[L2], [ND1]–[ND2], \) and \([AH1]–[AH3] \).

Remark 5.7 Referring to the expression of the quasi log-likelihood function in \( (4.2) \), for the above model, the intensity’s parameters \( (m_{ijl})_{ijl}, (\mu_i)_i, (\beta_{ij})_{ij} \) and the mark’s parameter \( \theta_M \) are only included in \( l_T^{(1)}(\vartheta) \) and \( l_T^{(2)}(\vartheta) \), respectively. Therefore, we can obtain the estimator by optimizing \( l_T^{(1)}(\vartheta) \) and \( l_T^{(2)}(\vartheta) \) independently. Then, Assumption \( 5.4 \) (v) is irrelevant to the estimation for \( (m_{ijl})_{ijl}, (\mu_i)_i, (\beta_{ij})_{ij} \).

\(^3\) That is, \( P[X_i^{l_1} = X_i^{l_2} \text{ for all } t > 0] < 1 \).
Table 3  Percentage of estimated to be zero

| Method     | $T = 100$ | $T = 500$ | $T = 3000$ |
|------------|-----------|-----------|------------|
| QMLE       |           |           |            |
| $m_1$      | 30.0%     | 6.00%     | 0.00%      |
| $m_2$      | 58.7%     | 54.0%     | 51.7%      |
| $m_3$      | 18.0%     | 0.00%     | 0.00%      |
| P–OE       |           |           |            |
| $m_1$      | 45.3%     | 19.7%     | 0.00%      |
| $m_2$      | 73.0%     | 77.3%     | 85.7%      |
| $m_3$      | 0.00%     | 7.33%     | 0.00%      |

Bold values are parameters whose true values are zero

Fig. 3  Histograms of the error of the P–O estimator $\sqrt{T}(\hat{\vartheta}_T - \vartheta^*)$

5.2.2 Simulation results

As a simple case, suppose that only one user posts texts and set the number of topics in texts to 3. Moreover, we assume that the proportion of each topic in a text follows simply the Dirichlet distribution, although it has a more complicated distribution in the LDA model.

Let $\tilde{N}$ be the 1-dimensional GEMHP whose intensity is

$$
\lambda_t(\vartheta^*) = \mu + \int_{[0,t] \times \mathbb{X}} e^{-\beta(t-s)}(m_1x_1 + m_2x_2 + m_3x_3)\tilde{N}(ds, dx) \bigg|_{\vartheta = \vartheta^*}
$$

$$
= 1.5 + \int_{[0,t] \times \mathbb{X}} e^{-0.5(t-s)}(0.4x_1 + 0.0x_2 + 0.4x_3)\tilde{N}(ds, dx),
$$

where its marks independently and identically follow the 3-dimensional Dirichlet distribution with a parameter $\alpha = (2, 2, 5)$. Here, we only estimate the parameters $m_1, m_2, m_3, \mu, \beta$ since $\alpha$ is estimated as the conventional MLE, see Remark 5.7. Furthermore, we assume that only
parameters $m$’s can take the zero value, i.e., we set $\theta^0 = (m_1, m_2, m_3)$ and $\theta^1 = (\mu, \beta)$. Then, we can immediately confirm the conditions in Assumption 5.4. We set the hyperparameters of the P–O estimator in Remark 3.4 to be $q = 1.0$, $\gamma = 2.0$, $a = 0.5$, the observation times $T = 100, 500, 3000$, and the number of the Monte Carlo simulation $MC = 300$.

Table 3 shows the fraction of trials in which the parameters $m_1, m_2, m_3$ are estimated to be completely zero. We can see that the variable selection is performed more accurately by the P–O estimator than by the QMLE as $T$ becomes larger.

Figure 3 show histograms of the error distribution of the P–O estimator, that is, histograms of the values of $\sqrt{T}(\hat{\theta}_T - \theta^*)$. It seems that the distribution is close to the normal distribution as $T$ becomes larger.

Table 4 shows the averages of squared errors of the QMLE $(\bar{\theta}_T - \theta^*)^2$ and the P–O estimator $(\hat{\theta}_T - \hat{\theta}^*)^2$. For non-zero parameters, owing to the asymptotic normality, both the QMLE and the P–O estimator have asymptotically the same level of variance. For zero parameters, we can guess that the P–O estimator asymptotically has a smaller error than the QMLE, due to the accurate model selection. However, when the observation time is small, the performance of the QMLE is better due to the miss model selection of the P–O estimator.

**Acknowledgements** I am deeply grateful to Professor Yoshida. Without his guidance and help, I could not have completed this article. This research was supported by the FMSP program of The University of Tokyo and Japan Science and Technology Agency CREST JPMJCR14D7.

**Appendix A: Proofs**

**A.1: Proofs of Sect. 3**

**Proof of Theorem 3.5 (i)** By considering $0 \geq Q^{(q)}_T(\hat{\theta}^{0,(q)}, \nu^{0,(q)}_T) - Q^{(q)}_T(\theta^0, \nu^{0,(q)}_T)$ similarly to the proof of Theorem 1 in Suzuki and Yoshida (2020), we obtain the $\sqrt{T}$-consistency of $\hat{\theta}^{0,(q)}_T$ from Assumption 3.2 (i) and (ii). Let

$$\Omega_{T,1} = \left\{ \omega \in \Omega \mid \exists j \in J \text{ s.t. } \hat{\theta}^{0,(q)}_j = 0 \right\},$$

and

$$\Omega_{T,2} = \left\{ \omega \in \Omega \mid \exists j \in J \text{ s.t. } \hat{\theta}^{0,(q)}_j \neq 0 \right\}.$$
Then we obtain $\{ \tilde{J}_T^0 \neq J^0 \} \subset \Omega_{T,1} \cup \Omega_{T,2}$. From the consistency of $\hat{\theta}_T^{0,(q)}$,

$$P[\Omega_{T,1}] \leq P\left[ \sqrt{T}\left| \hat{\theta}_T^{0,(q)} - \theta^{0*} \right| \geq T^{\frac{1}{2}}c_0 \right] \to 0 \quad (A1)$$

holds as $T \to \infty$, where $c_0 = \min_{j \in J} \theta_j^{0*} > 0$. On the other hand, to handle the case where the true value is at the boundary of the parameter space, we consider the following sets:

$$\Omega_{T,2,1} = \{ \omega \in \Omega \mid \exists j \in J^0 \text{ s.t. } \left| \hat{\theta}_T^{0,(q)} - \theta_j^{0*} \right| \geq c_j \}$$

and

$$\Omega_{T,2,2} = \{ \omega \in \Omega \mid \exists j \in J^0 \text{ s.t. } \hat{\theta}_T^{0,(q)} \neq 0 \text{ and } \hat{\theta}_T^{0,(q)} \neq \partial \Theta_j^0 \},$$

where $\Theta_j^0 = \{ \theta_j^0(\theta_j^0, \ldots, \theta_j^0) \in \Theta_j^0 \}$ and $c_j = \inf_{\theta_j^0 \in \partial \Theta_j^0 \setminus \{0\}} \theta_j^0 > 0$. Then, $\Omega_{T,2} \subset \Omega_{T,2,1} \cup \Omega_{T,2,2}$ holds. The consistency of $\hat{\theta}_T^{0,(q)}$ immediately yields

$$P[\Omega_{T,2,1}] \to 0 \quad (A2)$$

as $T \to \infty$. On $\Omega_{T,2,2}$, $Q_T^{(q)}(\theta_0^0, \nu^0)$ is differentiable at $(\theta_0^0, \nu^0) = (\hat{\theta}_T^{0,(q)}, \nu_T^{0,(q)})$ with respect to some $j$-th component. Thus, the same way as the proof of Theorem 2 in Suzuki and Yoshida (2020) leads

$$P[\Omega_{T,2,2}] \leq P\left[ \exists j \in J^0 \text{ s.t. } 2\sqrt{T}\left| \hat{\theta}_T^{0,(q)} - \theta_j^0 \right|\sqrt{T}\left| \hat{\theta}_T^{0,(q)} \right|^{1-q} \geq qT^{\frac{2-q}{2}}b_T \right] \to 0. \quad (A3)$$

From the equations (A1), (A2), and (A3), we get the conclusion. \hfill \square

We write $X_T \lesssim T^{-L}$ for a sequence $X_T$ and a positive constant $L$ if there exists a positive constant $C_L$ such that $X_T \leq C_LT^{-L}$ for all $T > 0$.

**Proof of Theorem 3.5 (ii)** Same as the proof of Theorem 4 in Suzuki and Yoshida (2020), we have the $L^\infty$-boundedness of $\sqrt{T}\left( \hat{\theta}_T^{0,(q)} - \theta^{0*} \right)$, and $P[\Omega_{T,1}] \lesssim T^{-L}$, $P[\Omega_{T,2,2}] \lesssim T^{-L}$.

Moreover, we immediately obtain an inequality

$$P[\Omega_{T,2,1}] \leq \frac{1}{(T^{\frac{1}{2}} \min_j c_j)^{2L}} E\left[ \left| \sqrt{T}\left( \hat{\theta}_T^{0,(q)} - \theta^{0*} \right) \right|^{2L} \right] \lesssim T^{-L}$$

for $c_j = \inf_{\theta_j^0 \in \partial \Theta_j^0 \setminus \{0\}} \theta_j^0 > 0$. Thus, $P[\tilde{J}_T^0 \neq J^0] \lesssim T^{-L}$ holds. \hfill \square

Theorem 3.5 (iii) is obvious. The same way as the proof of Theorem 5 (b) in Suzuki and Yoshida (2020) leads to Theorem 3.5 (iv).

**A.2: Proofs of Sect. 4**

Let $E = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{P_1}$, and $D_t(E, \mathbb{R})$ be a set of functions $\phi: E \to \mathbb{R}$ such that:

1. $\phi$ is of class $C^1$ on $(\mathbb{R}_+ - \{0\}) \times (\mathbb{R}_+ - \{0\}) \times \mathbb{R}^{P_1}$,
2. $\phi$ and $|\Delta \phi|$ are polynomial growth in $(u, v, w, u^{-1}_{\{u \neq 0\}}, v^{-1}_{\{v \neq 0\}})$ for $(u, v, w) \in E$,
3. $\phi(0, v, w) = \phi(u, 0, w) = 0$.
The sufficient conditions for the PLD are proposed by Clinet (2021). Here, modifying the conditions [A1]–[A3] in Clinet (2021) to allow the model with nuisance parameter, we consider the following conditions. Here, we again call that \( f(\theta) \) is of class \( C^i(\Xi) \) for some \( i \in \mathbb{N} \) if \( f(\theta) \) is of class \( C^i(\Xi) \) and its derivatives admit continuous extensions on \( \partial \Xi \).

[A1] For any \( i = 1, \ldots, d \),

(i) \( \lambda^i_1(\theta)q_i(t, \theta) \) is a predictable on \( \Omega \times \mathbb{R}_+ \times \mathbb{X} \) for any \( \theta \in \Xi \).

(ii) \( \theta \mapsto \lambda^i_1(\theta)q_i(t, \theta) \) is almost surely in \( C^4(\Xi) \) for any \( (t, x) \in \mathbb{R}_+ \times \mathbb{X} \),

(iii) \( \lambda^i_1(\theta)q_i(t, \theta) = 0 \) if and only if \( \lambda^i_1(\theta^*)q_i(t, \theta^*) = 0 \), \( dt \rho(dx)P(\omega) \)-a.e. for any \( \theta \in \Xi \) and \( \nu^* \in \mathcal{N} \).

[A2] For any \( p > 1 \) and \( i = 1, \ldots, d \),

(i) \( \sup_{t \in \mathbb{R}_+} \sum_{n=0}^3 \sup_{\theta \in \Xi} \| \sup_{\theta \in \Xi} |\partial^p_\theta \lambda^i_1(\theta)| \|_p < \infty \),

(ii) \( \sup_{t \in \mathbb{R}_+} \sup_{\theta \in \Xi} |\lambda^i_1(\theta)|^{-1} \| \lambda^i_1(\theta)\|_p < \infty \),

(iii) \( \sup_{t \in \mathbb{R}_+} \sum_{n=0}^3 \int_\mathbb{X} E \left[ \sup_{\theta \in \Xi, \nu^* \in \mathcal{N}} |\partial^i_\theta \log q_i^t(x, \theta)|^{n+i} q_i^t(x, \theta) \right] \rho(dx) < \infty \),

(iv) \( \sup_{t \in \mathbb{R}_+} \sum_{n=0}^3 \int_\mathbb{X} E \left[ \sup_{\theta \in \Xi, \nu^* \in \mathcal{N}} |\partial^i_\theta \log q_i^t(x, \theta)|^{n+i} q_i^t(x, \theta) \right] \rho(dx) < \infty \).

[A3] There exist \( \gamma \in (0, 1/2) \), \( \pi_i : D^1_+(E, \mathbb{R}) \times \Xi \times \mathcal{N} \to \mathbb{R} \), and \( \chi_i : \{0, 1, 2\} \times \Xi \times \mathcal{N} \to \mathbb{R} \) such that

\[
\sup_{\theta \in \Xi, \nu^* \in \mathcal{N}} T^\gamma \left\| \frac{1}{T} \int_0^T \phi \left( \lambda^i_1(\theta^*), \lambda^i_2(\theta), \partial_\theta \lambda^i_2(\theta) \right) ds - \pi_i(\phi, \theta, \nu^*) \right\|_p \to 0
\]

as \( T \to \infty \) for any \( \phi \in D^1_+(E, \mathbb{R}) \) and \( p \geq 1 \), and

\[
\sup_{\theta \in \Xi, \nu^* \in \mathcal{N}} T^\gamma \left\| \frac{1}{T} \int_{[0, T] \times \mathbb{X}} \partial_{\theta}^i \log q_i^t(x, \theta)q_i^t(x, \theta^*) \rho(dx)\lambda^i_1(\theta^*)ds - \chi_i(k, \theta, \nu^*) \right\|_p \to 0
\]

as \( T \to \infty \) for any \( k \in \{0, 1, 2\} \).

In [A1], which is a regularity condition to ensure the existence of the quasi log-likelihood process, we extended the differentiability of each function to the boundary of the parameter space. [A2] gives moment and smoothness conditions, and here, we consider the finiteness uniformly in \( \nu^* \). [A3] is the condition for the ergodicity of \( \lambda^i \) and \( q^i \) uniformly in \( (\theta, \nu^*) \in \Xi \times \mathcal{N} \). These conditions are derived from the ergodicity of the GEMHP and the conditions [AH1]–[AH2].

**Lemma A.1** Under \([L1]–[L2], [ND1]–[ND2], and [AH1]–[AH2]\), the conditions \([A1]–[A3]\) hold.

**Proof** The condition [A1] immediately follows from [AH1]. Same as Lemma 6.6 in Clinet (2021), (i)–(ii) of [A2] hold. From [AH2] and Theorem 2.3, we readily obtain (iii)-(iv) of [A2]. With the help of uniformity in \( \nu^* \) in [A2], the condition [A3] holds similar to the proof of Lemma 6.7 in Clinet (2021). \(\square\)

We write \( \Delta_T(\nu^*) = T^{-1/2}\partial_\theta l_T(\theta^*) \) and \( \Gamma_T(\theta) = -T^{-1}\partial^2_\theta l_T(\theta) \). Furthermore, we decompose \( Y_T(\theta, \nu^*) \), \( \Delta_T(\nu^*) \), and \( \Gamma_T(\theta) \) as

\[
Y_T(\theta, \nu^*) = Y^{(1)}(\theta, \nu^*) + Y^{(2)}(\theta, \nu^*) \]

\[
:= \frac{1}{T} \left[ l_1^{(1)}(\theta) - l_1^{(1)}(\theta^*) \right] + \frac{1}{T} \left[ l_2^{(2)}(\theta) - l_2^{(2)}(\theta^*) \right],
\]
\[ \Delta_T(v^*) = \Delta_T^{(1)}(v^*) + \Delta_T^{(2)}(v^*) := T^{-1/2} \partial_\theta l_T^{(1)}(\theta^*) + T^{-1/2} \partial_\theta l_T^{(2)}(\theta^*), \]

and

\[ \Gamma_T(\theta) = \Gamma_T^{(1)}(\theta) + \Gamma_T^{(2)}(\theta) := -T^{-1} \partial_\theta^2 l_T^{(1)}(\theta) - T^{-1} \partial_\theta^2 l_T^{(2)}(\theta). \]

For the proof of Theorem 4.3, we prepare the following lemmas.

**Lemma A.2** Under \([A1]–[A3]\), we have, for any \(p \geq 1\),

\[ \sup_{T \in \mathbb{R}_+} \left\| \sup_{v^* \in \mathcal{N}} |\Delta_T(v^*)| \right\|_p < \infty. \]

**Proof** By Sobolev’s inequality of Theorem 4.12 in Adams and Fournier (2003), there exists a constant \(A(\mathcal{N}, p)\) such that

\[
\left\| \sup_{v^* \in \mathcal{N}} |\Delta_T(v^*)| \right\|_p \leq A(\mathcal{N}, p) \left\{ \int_{\mathcal{N}} E \left[ |\Delta_T(v^*)|^p \right] dv^* + \int_{\mathcal{N}} E \left[ |\partial_{v^*} \Delta_T(v^*)|^p \right] dv^* \right\} \leq A(\mathcal{N}, p) \text{diam}(\mathcal{N}) \left\{ \sup_{v^* \in \mathcal{N}} E \left[ |\Delta_T(v^*)|^p \right] + \sup_{v^* \in \mathcal{N}} E \left[ |\partial_{v^*} \Delta_T(v^*)|^p \right] \right\}. \tag{A4}
\]

Since we immediately get that

\[
\Delta_T(v^*) = \frac{1}{\sqrt{T}} \sum_{i=1}^d \int_{[0,T] \times \mathbb{R}} \frac{\partial_\theta \left( \lambda_i^t(\theta^*) q_i^t(x, \theta^*) \right)}{\lambda_i^t(\theta^*) q_i^t(x, \theta^*)} \mathbf{1}_{\{\lambda_i^t(\theta^*) q_i^t(x, \theta^*) \neq 0\}} \tilde{M}^i(dt, dx),
\]

where \(\tilde{M}^i(dt, dx) = \tilde{N}^i(dt, dx) - \lambda_i^t(\theta^*) q_i^t(x, \theta^*) dt \rho(dx)\), the last term in (A4) is bounded uniformly in \(T\) by Burkholder-Davis-Gundy inequality along with Hölder’s inequality and [A2]. Thus, we have the conclusion. \(\square\)

**Lemma A.3** Under \([A1]–[A3]\), we have, for any \(p \geq 1\),

\[ \sup_{T \in \mathbb{R}_+} T^\gamma \left\| \sup_{v^* \in \mathcal{N}} |\Gamma_T(\theta^*) - \Gamma(v^*)| \right\|_p < \infty. \]

**Proof** By applying Sobolev’s inequality of Theorem 4.12 in Adams and Fournier (2003), we can choose \(B(\mathcal{N}, p)\) such that

\[
\left( T^\gamma \left\| \sup_{v^* \in \mathcal{N}} |\Gamma_T^{(1)}(\theta^*) - \Gamma^{(1)}(v^*)| \right\|_p \right)^p \leq B(\mathcal{N}, p) T^{\gamma p} \left\{ \int_{\mathcal{N}} E \left[ |\Gamma_T^{(1)}(\theta^*) - \Gamma^{(1)}(v^*)|^p \right] dv^* + \int_{\mathcal{N}} E \left[ |\partial_{v^*} \Gamma_T^{(1)}(\theta^*) - \partial_{v^*} \Gamma^{(1)}(v^*)|^p \right] dv^* \right\} \leq B(\mathcal{N}, p) \text{diam}(\mathcal{N}) \left\{ \sup_{v^* \in \mathcal{N}} T^{\gamma p} E \left[ |\Gamma_T^{(1)}(\theta^*) - \Gamma^{(1)}(v^*)|^p \right] + \sup_{v^* \in \mathcal{N}} T^{\gamma p} E \left[ |\partial_{v^*} \Gamma_T^{(1)}(\theta^*) - \partial_{v^*} \Gamma^{(1)}(v^*)|^p \right] \right\}
\]
where the last convergence is a consequence of the condition \([A3]\). Similarly, we get

\[
T \rightarrow \infty \quad \implies \quad 0,
\]

and thus we have the conclusion. \(\square\)

**Proof of Theorem 4.3** We only have to check that the conditions \((A1'')\), \((A4')\), \((A6)\), \((B1)\), and \((B2)\) in Theorem 3 (c) in Yoshida (2011) are satisfied. Note that the nuisance parameter \(\tau\) in Yoshida (2011) is replaced by \(\nu^*\) in our literature. Set

\[
\beta_1 = \gamma, \quad \beta_2 = \frac{1}{2} - \gamma, \quad \rho_1 \in (0, 2), \quad \rho_2 \in (0, 2\gamma), \quad \alpha \in (0, 2), \quad \alpha \in (0, \min\left(1, \frac{\alpha}{1-\alpha}, \frac{2\gamma}{1-\alpha}\right))
\]

for satisfying \((A4')\). By Lemma A.1, we have

\[
T \rightarrow \infty, \quad \text{as} \quad p > 1 \quad \text{in the same way as Theorem 2.2 in Clinet (2021).}
\]

Thus, \((A5)\) and Lemma A.2 lead to \((A6)\). Moreover, we have

\[
\sup_{T \in \mathbb{R}^+} \left\| \frac{1}{T} \sup_{\theta \in \Xi, \nu^* \in \mathbb{N}} |\partial_\theta^3 l_T(\theta)| \right\|_p < \infty
\]

for any \(p > 1\) by Sobolev’s inequality, Hölder’s inequality, and \([A2]\). Now \((A1'')\) is satisfied by \((A6)\) and Lemma A.3. Finally, from \([AH3]\), we see that the conditions \((B1)\) and \((B2)\) follow immediately; see Remark 4.2. \(\square\)

**Proof of Theorem 4.4** We immediately get the conclusion from Theorem 4.3, Remark 3.3, Remark 3.4 in this article, and Corollary 4.2 in Clinet (2021). \(\square\)

### A.3: Proofs of Sect. 5

**Proof of Proposition 5.2** Since there are no marks, we can assume that \(\Xi = \mathbb{R}, \quad Q_j(x, dy) = \delta_0(dx), \quad g_{ij}(x) = 1\) for all \(i, j = 1, \ldots, d\). Then, the conditions \([L1], [ND1], [ND2]\) hold, for example, for \(f_X(x) = |x|\) and \(u_X(x) = \left(\sum_{l=1}^d \mu^*_l \right)|x|\). The condition \([L2]\) holds by the Perron-Frobenius’ theorem and the assumption that the spectral radius of \(\Phi\) is less than 1. Thus, the multivariate exponential Hawkes process with sparse structure has geometric ergodicity.

Since each \(\mu^*_j\) takes a positive value, the conditions \([AH1]\) and \([AH2]\) hold. Finally, the condition \([AH3]\) is satisfied in the same way as Lemma A.7 in Clinet and Yoshida (2017). To obtain \([AH3]\), we prove that \(Y = 0\) if there exists a vector \(y\) such that \(\inf_{\nu^* \in \mathbb{N}} y^T \Gamma(\nu^*) y = 0\). In this proof, we use Assumption 5.1 (i) to take positive \(\nu^*\) realizing the infimum. \(\square\)

**Proof of Proposition 5.6** Let

\[
g_{ij}(x) = \begin{cases} 1 & \text{if } m^*_{ij} = 0 \text{ for all } l = 1, \ldots, d', \\ \sum_{l=1}^{d'} m^*_{ijl} x_l & \text{otherwise,} \end{cases}
\]

and

\[
\alpha^*_{ij} = \begin{cases} 0 & \text{if } m^*_{ijl} = 0 \text{ for all } l = 1, \ldots, d', \\ 1 & \text{otherwise,} \end{cases}
\]
and then we can write

$$\lambda_i^i(\theta^*) = \mu_i^* + \sum_{j=1}^{d} \int_{[0,t] \times \mathcal{X}} a_{ij}^* e^{-(t-s)\beta_{ij}^*} g_{ij}(x) \tilde{N}_j(ds, dx), \quad i = 1, \ldots, d.$$  

We check the conditions [L1]–[L2] and [ND1]–[ND2] for this GEMHP. First, the condition [ND1] obviously holds. Let $f_X(x) = |x|$ and $u_X(x) = -\sum_{i=1}^{d} \mu_i^* f_X(|y| - |x|) Q_i(x, dy)$ for $x \in \mathbb{R}^d$, where we extend the first domain of $Q_i(x, dy)$ as $Q_i(x, dy) = \delta_{(1,0,\ldots,0)}(dy)$ for $x \notin \mathcal{X}$. Then, we can easily check the condition [L1]. The conditions [L2] and [ND2] are assumed in Assumption 5.4 (ii) and (iii). Thus, this model has geometric ergodicity.

Now, we consider the conditions [AH1]–[AH3]. We define $g_{ij}(x, \vartheta) = \sum_{l=1}^{d'} m_{ijl} x_l$, and then we can write

$$\lambda_i^i(\vartheta) = \mu_i + \sum_{j=1}^{d} \int_{[0,t] \times \mathcal{X}} e^{-(t-s)\beta_{ij}} g_{ij}(x, \vartheta) \tilde{N}_j(ds, dx), \quad i = 1, \ldots, d.$$  

[AH1] obviously holds, and [AH2] follows from Assumption 5.4 (v). We write $\vartheta = (\theta, \nu)$ as in Sect. 4. Finally, for the sake of the condition [AH3], we only have to show that (i) $\Upsilon(\vartheta, \nu^*) < 0$ for any $\vartheta \in \Theta - \{\theta^*\}$ and $\nu, \nu^* \in \mathcal{N}$, and (ii) $\Gamma(\nu^*)$ is positive definite uniformly in $\nu^* \in \mathcal{N}$, see Remark 4.2. Let there be $\theta \in \Theta - \{\theta^*\}$ and $\nu, \nu^* \in \mathcal{N}$ such that $\Upsilon(\vartheta, \nu^*) = 0$. We have

$$0 = -\Upsilon(\vartheta, \nu^*) = \sum_{i=1}^{d} \left\{ E \left[ \lambda_i^i(\vartheta) - \lambda_i^i(\vartheta^*) - \log \left( \frac{\lambda_i^i(\vartheta)}{\lambda_i^i(\vartheta^*)} \right) \lambda_i^i(\vartheta^*) \right] + E \left[ \int_{\mathcal{X}} \log \left( \frac{q_{i}^{M}(x, \theta_M^*)}{q_{i}^{M}(x, \theta_M)} \right) q_{i}^{M}(x, \theta_M^*) \lambda_i^i(\vartheta^*) dx \right] \right\}. $$

Since each term on the right-hand side is non-negative, we have

$$\lambda_i^M(\vartheta) - \lambda_i^M(\vartheta^*) - \log \left( \frac{\lambda_i^M(\vartheta)}{\lambda_i^M(\vartheta^*)} \right) \lambda_i^M(\vartheta^*) = 0, \quad a.s.$$  

and

$$\log \left( \frac{q_{i}^{M}(x, \theta_M^*)}{q_{i}^{M}(x, \theta_M)} \right) q_{i}^{M}(x, \theta_M^*) \lambda_i^M(\vartheta^*) = 0, \quad dx P(d\omega)-a.e.$$  

Thus, we obtain $\lambda_i^M(\vartheta) = \lambda_i^M(\vartheta^*)$ a.s. and $q_{i}^{M}(x, \theta_M^*) = q_{i}^{M}(x, \theta_M)$ a.e. Then, we get $\theta_M = \theta_M^*$ by Assumption 5.4 (v-a). On the other hand, we have

$$\mu_i^* - \mu_i = \sum_{j=1}^{d} \int_{(-\infty, t] \times \mathcal{X}} e^{-\beta_{ij}(t-s)} \left( \sum_{l=1}^{d'} m_{ijl} x_l \right)$$

$$-e^{-\beta_{ij}(t-s)} \left( \sum_{l=1}^{d'} m_{ijl} x_l \right) \tilde{N}_j(ds, dx), \quad a.s., \quad (A7)$$

for any $i = 1, \ldots, d$. Since the left-hand side is constant, the right-hand side must only have jumps of size zero. Then, by Assumption 5.4 (iv), we get $m_{ijl}^* = m_{ijl}$ for all $i$, $j$, and $l$. By
taking the derivative with respect to \( t \), we have

\[
0 = \sum_{j=1}^{d} \int_{(\infty, t) \times \mathbb{X}} \beta_{ij} e^{-\beta_{ij}(t-s)} \left( \sum_{l=1}^{d'} m_{ijl} x_l \right) \\
- \beta_{ij} e^{-\beta_{ij}(t-s)} \left( \sum_{l=1}^{d'} m_{ijl} x_l \right) \tilde{N}^{ij}(ds, dx), \quad a.s.,
\]

and thus, we get \( \beta_{ij}^* = \beta_{ij} \) for all \( i \) and \( j \) such that \( m_{ij} > 0 \) for some \( l \) by Assumption 5.4 (iv). Finally, the right-hand side of (A7) becomes zero, and \( \mu_{ij}^* = \mu_i \) holds. Therefore, \( \theta = \theta^* \) holds and contradicts \( \theta \in \Theta - \{ \theta^* \} \).

Next, we assume that there exists \( y \in \mathbb{R}^n \) such that \( \inf_{v \in \mathcal{N}} y^T \Gamma(v^*) y = 0 \). We write \( y = (y_\lambda, y_M) \), where \( y_\lambda \) and \( y_M \) are related to \( \theta_\beta \) and \( \theta_M \), respectively, and \( \theta_\beta \) denotes non-nuisance parameters \( \mu, \beta, m \).

From Assumption 5.4 (v-c), we have

\[
0 = \inf_{v \in \mathcal{N}} y^T \Gamma(v^*) y = \inf_{v \in \mathcal{N}} \sum_{l=1}^{d} \left\{ E \left[ \frac{\left( y^T \partial_{\theta_M} \lambda_i^q(\theta^*) \right)^2}{\lambda_i^q(\theta^*)} \right] \right\} \\
+ E \left[ \int_{\mathbb{X}} (y_M \partial_{\theta_M} \log q_i^q(x, \theta^*_{M}) \right] q_i^q(x, \theta^*_M) \lambda_i^q(\theta^*) dx \right\},
\]

and then it is necessary that almost surely

\[
\inf_{v \in \mathcal{N}} \left( y^T \partial_{\theta_M} \lambda_i^q(\theta^*) \right)^2 = 0 \quad (A8)
\]

and

\[
y_M \partial_{\theta_M} \log q_i^q(x, \theta^*_M) = 0 \quad (A9)
\]

hold for any \( t \geq 0 \) and \( i = 1, \ldots, d \). Then, (A9) and Assumption 5.4 (v-b) lead to \( y_M = 0 \).

On the other hand, from Assumption 5.4 (i), the infimum in (A8) for each path is realized by some positive \( \beta_{ij}^* \), and we again write this point as \( \beta_{ij}^* \). Then, for \( i = 1, \ldots, d \),

\[
0 = y^T \partial_{\theta_M} \lambda_i^q(\theta^*) \\
= y_{\mu_i} + \sum_{j=1}^{d} \int_{(\infty, t) \times \mathbb{X}} \left\{ - (t-s) \left( \sum_{l=1}^{d'} m_{ijl} x_l \right) y_{\beta_{ij}} + \sum_{l=1}^{d'} x_l y_{m_{ijl}} \right\} \\
e^{-\beta_{ij}(t-s)} \tilde{N}^{ij}(ds, dx), \quad a.s.,
\]

where \( y_{\mu_i}, y_{\beta_{ij}}, \) and \( y_{m_{ijl}} \) are components of \( y \) corresponding to non-nuisance parameters \( \mu_i, \beta_{ij}, \) and \( m_{ijl} \), respectively. Differentiating both sides \( n \)-times with respect to \( t \), we almost surely get

\[
0 = \sum_{j=1}^{d} \int_{(\infty, t) \times \mathbb{X}} \left\{ (t-s) \beta_{ij}^* - n \right\} \left( \sum_{l=1}^{d'} m_{ijl} x_l \right) y_{\beta_{ij}} - \beta_{ij}^* \sum_{l=1}^{d'} x_l y_{m_{ijl}} \\
( - \beta_{ij}^* )^{n-1} e^{-\beta_{ij}(t-s)} \tilde{N}^{ij}(ds, dx). \quad (A10)
\]

Let \( \Omega_{j'} = \{ \omega \in \Omega | T_1^{j'} < 1 \text{ and } T_1^{k} > 1 \text{ for } k \neq j' \} \), where \( T_1^{j} \) represents the first jump time of the counting process \( N_t^{j} = \tilde{N}^{ij}([0, t) \times \mathbb{X}) \). We easily see \( P[\Omega_{j'}] > 0 \). By taking
Table 5 Percentage of estimated to be zero

|                | QMLE: $T = 100$ | QMLE: $T = 500$ | QMLE: $T = 3000$ | P–OE: $T = 100$ | P–OE: $T = 500$ | P–OE: $T = 3000$ |
|----------------|-----------------|-----------------|------------------|-----------------|-----------------|------------------|
| $m_1$          | 22.0%           | 3.67%           | 0.00%            | 35.3%           | 11.0%           | 0.00%            |
| $m_2$          | 35.3%           | 12.0%           | 1.67%            | 57.3%           | 36.3%           | 36.3%            |
| $m_3$          | 16.7%           | 0.33%           | 0.00%            | 44.0%           | 6.00%           | 6.00%            |

The limit $t \downarrow T_1^j$ in (A10) on the set $\Omega_{j'}$, we obtain

$$0 = \left( \sum_{l=1}^{d'} m_{ijl}^* X_{T_1^l} \right) y_{\beta_{ij}} + \frac{p_{ijl}}{n} \sum_{l=1}^{d'} X_{T_1^l} y_{m_{ijl}} \to \left( \sum_{l=1}^{d'} m_{ijl}^* X_{T_1^l} \right) y_{\beta_{ij}} \text{ as } n \to \infty,$$

and thus we get $y_{\beta_{ij}} = 0$ since $m_{ijl}^* > 0$ holds for some $l$ by the definition of $y_{\beta_{ij}}$. Then, we immediately obtain $y_{m_{ijl}} = 0$ by Assumption 5.4 (iv), and thus $y_{\mu_i} = 0$ also holds. Now $y = 0$, and we get the conclusion. \hfill \Box

Appendix B: Additional numerical experiments

B.1: Scenario with no zero coefficients

We considered scenarios with no zero coefficients to see if the P–O estimator and QMLE perform similarly. Here, we deal with the Hawkes process marked with “Topic” introduced in Sect. 5.2. Let $\bar{N}$ be the 1-dimensional GEMHP whose intensity is

$$\lambda_t(\bar{\vartheta}^*) = \mu + \int_{[0,t) \times \mathbb{R}} e^{-\beta(t-s)}(m_1 x_1 + m_2 x_2 + m_3 x_3) \bar{N}(ds, dx) \bigg|_{\vartheta = \bar{\vartheta}^*}$$

$$= 1.2 + \int_{[0,t) \times \mathbb{R}} e^{-0.5(t-s)}(0.5 x_1 + 0.3 x_2 + 0.4 x_3) \bar{N}(ds, dx),$$

where its marks independently and identically follow the 3-dimensional Dirichlet distribution with a parameter $\alpha = (2, 2, 5)$. Same as Sect. 5.2.2, we estimate the parameters $m_1, m_2, m_3, \mu, \beta$ and assume that only parameters $m$’s can take the zero value, i.e., we set $\theta^0 = (m_1, m_2, m_3)$ and $\theta^1 = (\mu, \beta)$. We also set the hyperparameters of the P–O estimator to be $q = 1.0, \gamma = 2.0, a = 0.5$, the observation times $T = 100, 500, 3000$, and the number of the Monte Carlo simulation $MC = 300$.

Table 5 shows the fraction of trials in which the parameters $m_1, m_2,$ and $m_3$ are estimated to be completely zero. We see that both methods asymptotically make correct model selections, but that the probability of incorrectly estimating zero is higher for the P–O estimator.

Table 6 shows the averages of squared errors of the QMLE $(\hat{\bar{\vartheta}_T} - \bar{\vartheta}^*)^2$ and the P–O estimator $(\hat{\vartheta}_T - \vartheta^*)^2$. When the observation time is small, we see that the QMLE perform...
Table 6  Average of squared errors

| $T$ | Method | $\mu$   | $\beta$   | $m_1$    | $m_1$    | $m_3$    |
|-----|--------|---------|-----------|---------|---------|---------|
| 100 | QMLE   | 3.69e–01| 4.20e–02  | 2.81e–01| 1.97e–01| 7.09e–02|
|     | P–OE   | 7.58e–01| 8.85e–01  | 5.45e–01| 2.40e–01| 9.63e–02|
| 500 | QMLE   | 7.41e–02| 5.30e–03  | 8.55e–02| 5.39e–02| 2.01e–02|
|     | P–OE   | 7.54e–02| 5.43e–03  | 1.00e–01| 6.80e–02| 2.58e–02|
| 3000| QMLE   | 1.13e–02| 7.02e–04  | 1.32e–02| 1.28e–02| 3.36e–03|
|     | P–OE   | 1.12e–02| 7.00e–04  | 1.39e–02| 1.53e–02| 3.51e–03|

better than the P–O estimator due to the mismodel selection of the P–O estimator. However, the difference becomes smaller as the observation time is longer. We note that both the QMLE and the P–O estimator have asymptotic normality with the same variance.

B.2: Comparison with previous studies

In this subsection, we compare the performance of the P–O estimator with the mixed method of Lasso and nuclear regularization, introduced in Zhou et al. (2013), and the elastic net. These classical methods are implemented in tick library\(^4\) in Python3, see Bacry et al. (2018), and work only for an exponential Hawkes model whose decay parameter is given.

Here, we consider the 4-dimensional exponential Hawkes process $N_t = (N^1_t, \ldots, N^4_t)$, see Eq. (5.1), whose intensity with the following parameters:

$$\mu^* = (0.05, 0.05, 0.05), \quad \alpha^* = \begin{pmatrix} 0.15 & 0 & 0 & 0 \\ 0.15 & 0 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 & 0.1 \end{pmatrix},$$

and the decay parameter is given by $\beta^*_{ij} = 1.0$ for all $i, j = 1, \ldots, 4$. We only estimate the parameters $\vartheta = (\mu, \alpha)$ and assume that only parameters $\alpha$ can take the zero value, i.e., we set $\theta^0 = \alpha$ and $\theta^1 = \mu$.

The mixed method of Zhou et al. (2013) is defined by

$$\tilde{\vartheta}_T \in \text{argmin}_{\vartheta \in \mathbb{R}} \left[ -l_T(\vartheta) + C_m \left\{ (1 - \rho_m)\|\alpha\|_* + \rho_m\|\alpha\|_1 \right\} \right], \quad (B1)$$

where $C_m$ and $\rho_m$ are hyperparameters, $\| \cdot \|_*$ is the nuclear norm of a matrix, which is defined to be the sum of its singular value, $\| \cdot \|_1$ is the $L^1$ norm, and $l_T$ is the log-likelihood process of the Hawkes process. On the other hand, the elastic net is given by

$$\tilde{\vartheta}_T \in \text{argmin}_{\vartheta \in \mathbb{R}} \left[ R_T(\vartheta) + C_e \left\{ (1 - \rho_e)\|\alpha\|_1 + \rho_e\|\alpha\|_2 \right\} \right], \quad (B2)$$

where $C_e$ and $\rho_e$ are hyperparameters, $\| \cdot \|_2$ is the $L^2$ norm, and $R_T$ is the least-squares function for the Hawkes process, that is,

$$R_T(\vartheta) = \frac{1}{T} \sum_{i=1}^d \left\{ \int_0^T (\lambda_i^i(\vartheta))^2 \, dt - 2 \int_0^T \lambda_i^i(\vartheta)N^i(dt) \right\}.$$  

\(^4\) The documentation is available here https://x-datainitiative.github.io/tick/.
Table 7  Percentage of estimated to be zero

|               | Mixed Method in Zhou et al. (2013) | Elastic Net | P–O Estimator |
|---------------|------------------------------------|-------------|---------------|
| $\alpha_{11}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{12}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{13}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{14}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{21}$| 40.3%                              | 56.7%       | 93.3%         |
| $\alpha_{22}$| 40.3%                              | 60.7%       | 92.7%         |
| $\alpha_{23}$| 39.7%                              | 60.3%       | 92.7%         |
| $\alpha_{24}$| 40.3%                              | 61.0%       | 95.7%         |
| $\alpha_{31}$| 32.0%                              | 59.7%       | 90.3%         |
| $\alpha_{32}$| 43.7%                              | 55.0%       | 92.7%         |
| $\alpha_{33}$| 40.0%                              | 0.67%       | 1.00%         |
| $\alpha_{34}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{35}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{41}$| 41.7%                              | 57.7%       | 90.3%         |
| $\alpha_{42}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{43}$| 0.00%                              | 0.00%       | 0.00%         |
| $\alpha_{44}$| 40.0%                              | 0.00%       | 0.00%         |

Bold values are parameters whose true values are zero

We set the hyperparameters in Eqs. (B1) and (B2) to be $C_m = C_e = 1000$, $\rho_m = 0.5$, and $\rho_e = 0.95$, and the hyperparameters of the P–O estimator to be $q = 1.0$, $\gamma = 1.0$, $a = 0.5$. Let the observation times $T = 3000$ and the number of the Monte Carlo simulation $MC = 300$.

Table 7 shows the fraction of trials in which the parameter $\alpha_{ij}$’s are estimated to be completely zero. Here, we regarded estimated values less than $1.0\text{e}–8$ as zero by taking into account the numerical error in the tick library. We can see that the variable selection is performed more accurately by the P–O estimator than by the other methods.

Table 8 shows the averages of squared errors of each method. For non-zero parameters, each method has almost the same level of variance. For zero parameters, we get a smaller error by the P–O estimator than by the other methods due to the accurate model selection.
### Table 8  Average of squared errors

| Method       | $\mu_1$    | $\mu_2$    | $\mu_3$    | $\mu_4$    | $\alpha_{11}$ | $\alpha_{12}$ |
|--------------|------------|------------|------------|------------|----------------|----------------|
| Mixed Method | 2.71e–05   | 2.17e–05   | 2.69e–05   | 2.52e–05   | 1.38e–03       | 2.79e–04       |
| Elastic Net  | 2.71e–05   | 2.14e–05   | 3.06e–05   | 2.76e–05   | 1.73e–03       | 2.71e–04       |
| P–OE         | 2.43e–05   | 2.06e–05   | 2.72e–05   | 2.55e–05   | 1.38e–03       | 1.70e–04       |

| Method       | $\alpha_{13}$ | $\alpha_{14}$ | $\alpha_{21}$ | $\alpha_{22}$ | $\alpha_{23}$ | $\alpha_{24}$ |
|--------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Mixed Method | 2.67e–04       | 2.77e–04       | 1.47e–03       | 3.39e–04       | 2.15e–04       | 1.78e–04       |
| Elastic Net  | 2.32e–04       | 2.56e–04       | 1.81e–03       | 2.38e–04       | 1.88e–04       | 1.45e–04       |
| P–OE         | 1.71e–04       | 1.81e–04       | 1.51e–03       | 2.26e–04       | 1.21e–04       | 8.14e–05       |

| Method       | $\alpha_{31}$ | $\alpha_{32}$ | $\alpha_{33}$ | $\alpha_{34}$ |
|--------------|----------------|----------------|----------------|----------------|
| Mixed Method | 3.98e–04       | 1.36e–03       | 1.09e–03       | 1.16e–03       |
| Elastic Net  | 4.14e–04       | 1.71e–03       | 1.24e–03       | 1.27e–03       |
| P–OE         | 2.76e–04       | 1.52e–03       | 1.14e–03       | 1.21e–03       |

| Method       | $\alpha_{41}$ | $\alpha_{42}$ | $\alpha_{43}$ | $\alpha_{44}$ |
|--------------|----------------|----------------|----------------|----------------|
| Mixed Method | 4.02e–04       | 1.31e–03       | 1.21e–03       | 1.09e–03       |
| Elastic Net  | 4.15e–04       | 1.46e–03       | 1.37e–03       | 1.29e–03       |
| P–OE         | 3.00e–04       | 1.45e–03       | 1.31e–03       | 1.17e–03       |

Bold values are parameters whose true values are zero

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