New metamaterials with macroscopic behavior outside that of continuum elastodynamics

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Abstract. Metamaterials are constructed such that, for a narrow range of frequencies, the momentum density depends on the local displacement gradient and the stress depends on the local acceleration. In these models the momentum density generally depends not only on the strain, but also on the local rotation, and the stress is generally not symmetric. A variant is constructed for which, at a fixed frequency, the momentum density is independent of the local rotation (but still depends on the strain) and the stress is symmetric (but still depends on the acceleration). Generalizations of these metamaterials may be useful in the design of elastic cloaking devices.

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1. Introduction

The possibility of cloaking objects to make them invisible has attracted substantial recent attention. Alú and Engheta (2005, 2007) following earlier work of Kerker (1975), found that the scattering from an object could be substantially reduced by surrounding it by a plasmonic or metamaterial coating. Milton and Nicorovici (2006), expanding on the work of Nicorovici et al (1994) and Milton et al (2005), proved that the dipole moments of a polarizable point dipole or clusters of polarizable line dipoles would vanish when placed within a cloaking region surrounding a superlens, and this has been confirmed numerically by Nicorovici et al (2007). Bruno and Lintner (2007) show that only partial cloaking occurs when the polarizable object is not discrete. Ramm (1996) and Miller (2006) found that cloaking could be achieved by sensing and manipulating the fields near the boundary of the object.

Perhaps the greatest interest has been generated by transformation based cloaking. This type of cloaking was first discovered by Greenleaf et al (2003a, 2003b) in the context of electrical conductivity. Their idea was to apply the well-known fact (see, for example, Kohn and Vogelius (1984)) that the equations of electrical conductivity retain their form under coordinate transformations, using a singular transformation which mapped a point to a sphere (the cloaking region), and which was the identity outside a larger sphere. Perturbing the conductivity in the cloaking region corresponds to changing the conductivity at a point in the equivalent problem, and this has no effect on the fields outside the larger sphere. This type of cloaking was generalized to the full-time harmonic Maxwell equations by Pendry et al (2006), and at the same time Leonhardt (2006) discovered a different transformation based, two-dimensional (2D), geometric optics or high frequency acoustic cloaking scheme which only utilized isotropic materials. The former cloaking has been verified numerically (Cummer et al 2006, Schurig et al 2006, Zolla et al 2007), placed on a firm theoretical foundation (Greenleaf et al 2007), and experimentally substantiated in the microwave regime in an approximate way in 2D by Schurig et al (2006) using Schelkunoff and Friis’s idea (Schelkunoff and Friis 1952) of utilizing metamaterials with split ring resonators to achieve artificial magnetism. (Such artificial magnetic materials have also lead to the construction of materials having a negative refractive index as studied by Veselago (1967): see the review of Shalaev (2007) for more recent references.) Cai et al (2007b) proposed another design which may experimentally achieve approximate 2D cloaking in the visible regime. Additionally, an improved approximate cloak has been suggested which minimizes reflection at the boundary (Cai et al 2007a).

Milton et al (2006) found that applying transformation based cloaking to continuum elastodynamics would require new materials with very unusual properties. This is because the continuum elastodynamic equations do not retain their form under coordinate transformations, but rather take the form of the equations that Willis (1981a, 1981b, 1997) found described the ensemble averaged behavior of composite materials. One interesting feature is that the density at a given frequency is required to be matrix valued. Willis (1985) had proved that his density operator had this property. It also follows from the work of Ávila et al (2005), which is an extension of work of Bouchitté and Felbacq (2004), that there exist high contrast materials with a local anisotropic effective density. Simple models with anisotropic and possibly complex density were obtained by Milton and Willis (2007) (see also the caption of figure 1). These microstructures generalized a construction of Sheng et al (2003) and Liu et al (2005), who, following experimental work of Liu et al (2000), established that the density can be negative over a range of frequencies. The same conclusion was reached and rigorously proved in the
2D antiplane case by Bouchitté and Felbacq (2004) (see also Movchan and Guenneau (2004)) and in the 3D case by Avila et al (2005). We remark in passing that over a fixed frequency range it is possible to obtain not only negative mass, but also negative elastic moduli, as shown experimentally by Lakes (2001) and Fang et al (2006).

A comparison of Maxwell’s equations, which can be written in the form

\[
\frac{\partial}{\partial x_i} \left( C_{ijkl} \frac{\partial E_j}{\partial x_k} \right) = \{\omega^2 \varepsilon \mathbf{E}\}_j, \tag{1.1}
\]

where

\[
C_{ijkl} = \varepsilon_{ijm} \varepsilon_{kln} (\mu^{-1})_{mn}, \tag{1.2}
\]

(in which \( \mathbf{E} \) is the electric field, \( \varepsilon \) the electric permittivity tensor, \( \mu \) the magnetic permeability tensor, and \( \varepsilon_{ijm} = 1 (-1) \) if \( ijm \) is an even (odd) permutation of 123 and is zero otherwise) with the equations of continuum elastodynamics

\[
\frac{\partial}{\partial x_i} \left( C_{ijkl} \frac{\partial u_j}{\partial x_k} \right) = -\{\omega^2 \rho \mathbf{u}\}_j, \tag{1.3}
\]

(in which \( \mathbf{u} \) is the displacement field, \( \rho \) is the density tensor, and \( \mathbf{C} \) is now the elasticity tensor) suggests that \( \rho \) as a function of the frequency \( \omega \) could share many of the same properties as \( \varepsilon(\omega) \). This has been established, and in fact for every function \( \rho(\omega) \) satisfying these properties one can construct a model which has approximately that function as its effective density as a function of frequency (Milton and Willis 2007). Cummer and Schurig (2007) investigated transformation based acoustic cloaking in 2D and, due to a mathematical equivalence with the electromagnetic problem, found that it could be achieved provided one could construct the necessary materials with anisotropic density.

Anisotropic density is not the only new property needed to achieve transformation based elasticity cloaking. One needs materials where the constitutive law, like that in the Willis equations (1981a, 1981b, 1997), couples the stress not only with the strain but also with the velocity and couples the momentum density not only with the velocity but also with the displacement gradient through the strain. This coupling is non-local in time, so that the stress depends on the velocity at previous times. Of course if the velocity is constant then this coupling must vanish as demanded by Galilean invariance. For time harmonic motions the (complex) velocity is proportional to the (complex) acceleration, so the stress depends on the acceleration. The constitutive law of Willis appears strange, but as remarked by Milton and Willis (2007), it is the precise analog for elastodynamics of the bianisotropic constitutive law for electromagnetism, where \( \mathbf{D} \) and \( \mathbf{H} \) are cross-coupled with \( \mathbf{E} \) and \( \mathbf{B} \) (Serdikukov et al 2001).

Unlike in the Willis equations, one wants this dependence to be local in space and to apply to a single microgeometry rather than to an ensemble average of microgeometries. Here, we show that such unusual behavior can be realized in a model such that, for a narrow range of frequencies, the associated waves have wavelength much larger than the microstructure, and the momentum density depends on the displacement gradient, and the stress depends on the acceleration. Curiously the momentum density depends not only on the strain, but also on the local rotation, and the stress is not symmetric. We also construct a variant of the model for which (at a fixed frequency) the momentum density is independent of the local rotation (but still depends on the strain) and the stress is symmetric (but still depends on the acceleration). Both models rely heavily on the discovery of Sheng et al (2003) that one can design structures which,
for a certain range of frequencies, respond as if they have negative mass. Figure 1 illustrates their idea (see also the simplified constructions of Milton and Willis (2007)) of a structure with apparent negative mass at a given frequency above resonance. In the discrete models considered in this paper both positive and negative masses are idealized as point masses, although in the summary section we suggest how the models may be approximated by a realistic continuum model where the mass is positive everywhere, using the construction of figure 1 (see figure 4). We emphasize that the masses only have a negative effective mass over a range of frequencies (which implies that the inertial force depends on the acceleration at previous times) and thus an extended spring attached to such a mass should not generate a divergent motion.

This work on metamaterials with macroscopic behavior outside that of continuum elastodynamics, even though they are governed by continuum elastodynamics at the microscale, is preceded by work on metamaterials with macroscopic non-Ohmic, possibly non-local, conducting behavior, even though they conform to Ohm’s law at the microscale (Briane 1998, 2002, Briane and Mazliak 1998, Camar-Eddine and Seppecher 2002, Cherednichenko et al 2006, Khruslov 1978) by work on metamaterials with non-Maxwellian macroscopic electromagnetic behavior, even though they conform to Maxwell’s equations at the microscale (Shin et al 2007), and by work on metamaterials with a macroscopic higher order gradient or non-local elastic response even though they are governed by usual linear elasticity equations at the microscale (Alibert et al 2003, Bouchitté and Bellieud 2002, Camar-Eddine and Seppecher 2003).

For simplicity, we consider a 2D model, although it can be generalized to 3D. The model is illustrated in figure 2 (and its continuum approximation in figure 4).
Figure 2. The model with strange elastic behavior. The large solid black circles have positive mass, while the neighboring large white circles have negative mass at the given frequency. They are connected to the spring network by rods of fixed length, which alternatively can be regarded as springs with infinite spring constants.

2. The momentum density in the model

Figure 3 shows the unit cell in the model. The masses are approximated as point masses, and the springs may be taken to have all the same spring constant $hK$, which scales in proportion to the size of the unit cell. This ensures that the spring network, by itself, responds as a elastic material in the limit $h \to 0$. Without loss of generality let us place the origin at the center $x_0$ of the unit cell under consideration. Then, when the material is at rest the points A, B, C, D, E and F in figure 3 are at positions

$$
x_A = (-h, 0), \quad x_B = (0, -h), \quad x_C = (h, 0), \quad x_D = (0, h), \quad x_E = (-ch, 0), \quad x_F = (ch, 0),
$$

(2.1)

where $c$ is a parameter between 0 and 1 which controls the inclination of the rods joining E and F to B and D: these rods have length $h\sqrt{1+c^2}$. We assume that the motion of the structure is time harmonic so that the physical displacement at any given point is obtained by modulating a time-independent complex displacement by a factor of $e^{-i\omega t}$ and then taking the real part. In the limit $h \to 0$, we assume the (infinitesimal) physical displacements at the points $x_A, x_B, x_C$ and $x_D$ derive from some smooth complex valued displacement field $u(x)$, and in particular

$$
\begin{align*}
    u_A & = \Re[ue^{-i\omega t}] \approx \Re[(u_0 - hq)e^{-i\omega t}], \\
    u_B & = \Re[ue^{-i\omega t}] \approx \Re[(u_0 -hw)e^{-i\omega t}], \\
    u_C & = \Re[ue^{-i\omega t}] \approx \Re[(u_0 +hq)e^{-i\omega t}], \\
    u_D & = \Re[ue^{-i\omega t}] \approx \Re[(u_0 +hw)e^{-i\omega t}],
\end{align*}
$$

(2.2)
Figure 3. The unit cell of the model.

where

\[ u_0 = u(x_0), \quad q = \frac{\partial u}{\partial x_1} \bigg|_{x=x_0}, \quad w = \frac{\partial u}{\partial x_1} \bigg|_{x=x_0}, \] (2.3)

and \( x_0 = 0 \) because the origin is at the center of the unit cell.

At the macroscopic level, we choose not to keep track of the displacements \( u_E \) and \( u_F \). These are internal hidden variables which however can be recovered from \( u_B \) and \( u_D \). They have the form

\[ u_E \approx \Re e[(u_0 + hs)e^{-i\omega t}], \quad u_F \approx \Re e[(u_0 - hs)e^{-i\omega t}], \] (2.4)

where \( s \) is defined as the complex vector such that the approximation for \( u_E \) holds to first-order in \( h \). To obtain an explicit expression for \( s = (s_1, s_2) \), we observe that since there is a rigid rod connecting the points D and E we have, in the limit \( h \rightarrow 0 \), the constraint that \( w - s \) must be perpendicular to the line joining D and E, i.e. perpendicular to \( x_D - x_E = h(c_1, 1) \), implying

\[ 0 = (w - s) \cdot (c, 1) = c(w_1 - s_1) + (w_2 - s_2). \] (2.5)

Similarly, since there is a rigid rod connecting the points B and E we have, in the limit \( h \rightarrow 0 \), the constraint

\[ 0 = (w + s) \cdot (-c, 1) = -c(w_1 + s_1) + (w_2 + s_2). \] (2.6)

These equations have the solution

\[ s_1 = w_2/c = \frac{1}{c} \frac{\partial u_2}{\partial x_2}, \quad s_2 = cw_1 = c \frac{\partial u_1}{\partial x_2}. \] (2.7)
One can easily see that if \( c \) is replaced by \(-c\) then \( s \) gets replaced by \(-s\), which justifies the formula (2.4) for \( u_F \).

Now suppose the masses at the points E and F, which we call a mass pair, are respectively chosen to have the form

\[
m_E = hm, \quad m_F = -hm + \delta h^2,
\]

where \( m \) is a positive real constant, and \( \delta \) is a possibly complex constant with a non-negative imaginary part. Then at any time \( t \) the physical momentum in the unit cell is, to second-order in \( h \),

\[
\Re\{ -i\omega [m_E(u_0 + hs) + m_F(u_0 - hs)]e^{-i\omega t} \} \approx h^2 \Re\{ -i\omega [2ms + \delta u_0]e^{-i\omega t} \}. \tag{2.9}
\]

Since the area of the unit cell is \( 2h^2 \), we see that the complex momentum density is

\[
p = -i\omega ms + (\delta/2)(-i\omega u_0), \tag{2.10}
\]

in which \(-i\omega u_0\) is the complex velocity of the unit cell, and \( s \) depends on the deformation gradient through (2.7).

3. The stress in the model

The acceleration of the material will also generate stress. To see this note that the masses at E and F must be accelerated by physical forces

\[
f_E = -\omega^2 m_E u_E, \quad f_F = -\omega^2 m_F u_F, \tag{3.1}
\]

and from (2.4) and (2.8) we see that to first-order in \( h \)

\[
f_E \approx \Re\{ F e^{-i\omega t} \}, \quad f_f \approx \Re\{ -F e^{-i\omega t} \}, \quad \text{where} \quad F = -\omega^2 m h u(x_0). \tag{3.2}
\]

Thus to leading order in \( h \), we can think of complex forces \( F \) and \(-F\) acting on the respective masses. Let \( F_{ED} \) and \( F_{EB} \) be the forces which the rods ED and EB exert on the vertex E and let \( F_{FD} \) and \( F_{FB} \) be the forces which the rods FD and FB exert on the vertex F. Balance of forces at E and F requires that

\[
F_{ED} + F_{EB} = F, \quad F_{FD} + F_{FB} = -F. \tag{3.3}
\]

Since these forces are aligned with their respective rods, we have

\[
F_{ED} = -F_{FB} = \alpha(c, 1), \quad F_{EB} = -F_{FD} = \beta(c, -1), \tag{3.4}
\]

where the notation \( \alpha(c, 1) \) means that the force \( F_{ED} \) has Cartesian components \( \alpha c \) and \( \alpha \), and is thus aligned with the vector \((hc, h) = x_D - x_E\) in the direction of the line DE, and the constants \( \alpha \) and \( \beta \) are such that (3.3) is satisfied:

\[
(\alpha + \beta)c = F_1, \quad \alpha - \beta = F_2. \tag{3.5}
\]

Forgetting for a moment the springs and surrounding material, a force

\[
h t = F_{ED} + F_{FD} = \alpha(c, 1) - \beta(c, -1) = (c F_2, F_1/c) = -\omega^2 m h (cu_2, u_1/c), \tag{3.6}
\]

needs to act on the vertex D to balance the forces which the rods ED and FD exert on this vertex. Similarly a force \(-ht\) needs to act on the vertex B to balance the forces which the rods EB and FB exert on this vertex. On the other hand, no additional forces need to be exerted on the vertices A and C.
Now consider a rectangular sample (with dimensions, say of order $\sqrt{h}$, small compared to the wavelength) as in figure 2 with (complex) forces acting on the boundary vertices to maintain the motions. These forces will have two components: an elastic (or possibly viscoelastic) component to compensate for the (complex) stress $\mathbf{\sigma}_E$ caused by the (complex) strain in the spring network, and an inertial component to compensate for the inertial stress $\mathbf{\sigma}_I$ caused by the acceleration $-\omega^2 \mathbf{u}$ of the material. The compensating inertial component will be zero on the left and right sides of the network. At the top five vertices in figure 2 the compensating inertial component will be approximately $2h \mathbf{t}$ at each vertex. The extra factor of 2 arises because of the extra load that the springs at the top boundary carry to support the inertial component of the forces at the four vertices immediately below the top. (At the other interior nodes these inertial forces are balanced.) Thus, since the distance between these boundary vertices is $2h$ the compensating inertial component per unit length, is $\mathbf{t}$ on the top, $-\mathbf{t}$ on the bottom, and zero on the sides. By the definition of stress this should be equated with $\mathbf{\sigma}_I \mathbf{n}$, where $\mathbf{n}$ is the unit normal to the boundary. Thus, we deduce that

$$\mathbf{\sigma}_I = \begin{pmatrix} 0 & t_1 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega^2 m c u_2 \\ 0 & -\omega^2 m u_1/c \end{pmatrix}. \quad (3.7)$$

Of course, in the limit $h \rightarrow 0$, the stress in the spring network (excluding the masses and connecting rods) will be governed by a normal relation

$$\mathbf{\sigma}_E = \mathbf{C} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right], \quad (3.8)$$

where $\mathbf{C}$ is the elasticity tensor of the network.

Our choice to define the macroscopic stress through the forces at the vertices of the unit cell (rather than as a volume average of the microscopic stress field, which would result in a symmetric stress) makes good physical sense and is consistent with our choice to define the macroscopic displacement field through the values of the displacement at the vertices of the unit cell. These choices bear some resemblance to the way Pendry et al (1999) define macroscopic electromagnetic fields through field values at the boundary of the unit cell. In defining the stress in this way it is important that the unit cell be chosen so that each mass pair (of positive and negative masses) is not split by the boundary of the unit cell: each mass pair and their four connecting rods is regarded as an indivisible unit in the same way that in defining the electric polarization field through induced surface charges on an infinitesimal element one would not choose to place the boundary in a dielectric material so as to split the positive and negative charges of the constituent atoms.

4. Summary

In the limit $h \rightarrow 0$, the constitutive law takes the form

$$\left( \begin{array}{c} \mathbf{\sigma} \\ \mathbf{p} \end{array} \right) = \begin{pmatrix} \mathbf{C} & \mathbf{S} \\ \mathbf{D} & \rho \end{pmatrix} \left( \begin{array}{c} \nabla \mathbf{u} \\ \mathbf{v} \end{array} \right), \quad (4.1)$$

where $\mathbf{\sigma} = \mathbf{\sigma}_E + \mathbf{\sigma}_I$ is the total stress, $\mathbf{v} = -i \omega \mathbf{u}$ is the velocity, and $\rho = (\delta/2) \mathbf{I}$ is the density.

In the basis where $\mathbf{\sigma}$, $\mathbf{p}$, $\nabla \mathbf{u}$ and $\mathbf{v}$ are represented by the vectors

$$\mathbf{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \nabla \mathbf{u} = \begin{pmatrix} \partial u_1/\partial x_1 \\ \partial u_2/\partial x_1 \\ \partial u_1/\partial x_2 \\ \partial u_2/\partial x_2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (4.2)$$
the third-order tensors \( S \) and \( D \), from (2.7), (2.10) and (3.7) are represented by the matrices

\[
S = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & -i\omega mc \\
-i\omega mc & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -i\omega mc \\
0 & i\omega mc & 0
\end{pmatrix} = S^T.
\]

Thus, the matrix entering the constitutive law (4.1) is symmetric. Since the constitutive law and the equation of motion are asymptotically independent of \( h \), the waves associated with their solutions will have wavelength much larger that the microstructure. In this circumstance, the usual continuum elastodynamic equations normally apply. However this model shows that this is not always the case.

The model has some unusual features. Although the net amount of mass in a region of unit area is independent of \( h \), the total amount of positive (or negative) mass in a region of unit area scales like \( 1/h \). However in any physical model, as usual, one never in fact takes the limit \( h \rightarrow 0 \), but instead one sets the microstructure as small as required to get a reasonable approximation to the limiting behavior for a desired set of applied fields. Also the amount of positive (or negative) mass in the unit cell only represents the amount of apparent mass, not gravitational mass, which could be quite different if the apparent mass arises from substructures which are close to resonance (as in the continuum model of figure 4). Of course the analysis given here needs some rigorous justification to check, for example, that the displacements can be approximated by a smooth field \( u(x) \) in the limit \( h \rightarrow 0 \). Also from a practical point of view, it needs to be checked that the behavior is stable to sufficiently small variations in the masses or spring constants from cell to cell.

To reproduce the unusual constitutive behavior in an experiment one would need to design a continuum model which approximates the response of the discrete structure. Figure 4 presents a structure that we expect should respond in the desired way within the framework of linear elasticity. Of course the vibrations must be sufficiently small that the narrow elastic beams, which approximate the springs, do not flex and buckle.

The model can easily be generalized. One simple generalization has additional masses and connecting rods as illustrated in figure 5, with masses

\[
m_G = -hm' + \delta h^2, \quad m_H = hm',
\]

at the points G and H which, when the material is at rest, are located at

\[
x_G = (-c'h, 0), \quad x_H = (c'h, 0),
\]

where \( c' \) and \( m' \) are positive constants. By superposition the constitutive law will take the form (4.1) with a density \( \rho = \delta I \) and with the third-order tensors \( S \) and \( D \) represented by the matrices

\[
S = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & i\omega (m' c' - mc) \\
i\omega (m' c' - mc) & 0 & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0 & 0 & 0 \\
0 & i\omega (m' c' - mc) & 0 \\
i\omega (m' c' - mc) & 0 & 0
\end{pmatrix}.
\]
Figure 4. The unit cell of a continuum model which we conjecture should approximate the behavior of the discrete model. The black disks are heavy masses, and the remaining black areas are rigid but light material. The shaded areas are compressible material. The material surrounding the two black disks on the left side of the unit cell is sufficiently stiff that their motion is in phase with that of the surrounding rigid material, giving a positive effective mass. The material surrounding the two black disks on the right side of the unit cell is sufficiently compliant that their motion is out of phase with that of the surrounding rigid material, giving a negative effective mass. The triangular regions of material are highly compressible and are introduced so the junctions behave like hinges. The remaining area is void, or to make it a proper continuum, filled with a light highly compressible material.

In particular if, at a given frequency, \( m'c' = mc \), we obtain a model in which the momentum does not depend on the local rotation and the stress is symmetric. With this condition satisfied one has the relations

\[
\mathbf{\sigma} = C \nabla \mathbf{u} + S' \mathbf{u}, \quad \nabla \cdot \mathbf{\sigma} = -i\omega \mathbf{p} = D' \nabla \mathbf{u} - \omega^2 p \mathbf{u}, \tag{4.7}
\]

where

\[
S' = -i\omega S, \quad D' = -i\omega D, \tag{4.8}
\]

and the only non-zero Cartesian elements of these two tensors are from (4.1), (4.2) and (4.6)

\[
S'_{221} = D'_{122} = \omega^2 (m'/c' - m/c). \tag{4.9}
\]

The general form of these tensors \( S' \) and \( D' \) corresponds with that required for elasticity cloaking: they are real and satisfy the identities \( S'_{ijk} = S'_{jik} = D'_{kij} \). They may serve as a good starting point in the quest to find materials with a suitable combination of properties, \( C, S', D' \) and \( \rho \) needed for elasticity cloaking. However, quite apart from this, the novel properties of this new class of metamaterials may have other important applications.
In the design of these models perhaps the key feature is that in the narrow frequency range under consideration the amount of negative mass in the unit cell (due to hidden internal masses moving out of phase with the motion) almost balances the amount of positive mass in the unit cell. This ensures that the momentum density approaches a constant as the cell size approaches zero, while keeping non-trivial the stresses caused by the accelerating masses in the unit cell. It will be interesting to see if other models with this key feature also have a local constitutive law of the form (4.1), with non-zero coupling terms $S$ and $D$.

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References

Alibert J-J, dell’Isola F and Seppecher P 2003 Truss modular beams with deformation energy depending on higher displacement gradients *Math. Mech. Solids* **8** 51–74

Alú A and Engheta N 2005 Achieving transparency with plasmonic and metamaterial coatings *Phys. Rev. E* **72** 0166623

Alú A and Engheta N 2007 Plasmonic materials in transparency and cloaking problems: Mechanism, robustness, and physical insights *Opt. Express* **15** 3318–32

Ávila A, Griso G and Miara B 2005 Bandes phononiques interdites en élasticité linéarisée *Phys. Rev. E* **340** 933–8

*New Journal of Physics* **9** (2007) 359 (http://www.njp.org/)
Bouchitté G and Bellieud M 2002 Homogenization of a soft elastic material reinforced by fibers *Asymptotic Anal.* **32** 153–83

Bouchitté G and Felbacq D 2004 Homogenization near resonances and artificial magnetism from dielectrics *C. R. Acad. Sci. I. Math.* **339** 377–82

Briane M 1998 Homogenization in some weakly connected domains *Ricerche di Matematica (Napoli)* **47** 51–94

Briane M 2002 Homogenization of non-uniformly bounded operators: Critical barrier for nonlocal effects *Arch. Ration. Mech. Anal.* **164** 73–101

Briane M and Mazliak L 1998 Homogenization of two randomly weakly connected materials *Port. Math.* **55** 187–207

Bruno O P and Lintner S 2007 Superlens-cloaking of small dielectric bodies in the quasistatic regime, submitted

Cai W, Chettiar U K, Kildishev A V, Milton G W and Shalaev V M 2007a Non-magnetic cloak with minimized scattering *Appl. Phys. Lett.* **91** 111105 (Preprint 0707.3641)

Cai W, Chettiar U K, Kildishev A V and Shalaev V M 2007b Optical cloaking with metamaterials *Nat. Photonics* **1** 224–6

Camar-Eddine M and Seppecher P 2002 Closure of the set of diffusion functionals with respect to the Mosco-convergence *Math. Models Methods Appl. Sci.* **12** 1153–76

Camar-Eddine M and Seppecher P 2003 Determination of the closure of the set of elasticity functionals *Arch. Ration. Mech. Anal.* **170** 211–45

Cherednichenko K D, Smyshlyaev V P and Zhikov V V 2006 Non-local homogenized limits for composite media with highly anisotropic periodic fibres *Proc. R. Soc. Edinb.* **136A** 87–114

Cummer S A, Popa B-I, Schurig D, Smith D R and Pendry J 2006 Full-wave simulation of electromagnetic cloaking structures *Phys. Rev. E* **74** 036621

Cummer S A and Schurig D 2007 One path to acoustic cloaking *New J. Phys.* **9** 45

Fang N, Xi D, Xu J, Ambati M, Sirituravanich W, Sun C and Zhang X 2006 Ultrasonic metamaterials with negative modulus *Nat. Mater.* **5** 452–6

Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Full-wave invisibility of active devices at all frequencies *Commun. Math. Phys.* **275** 749–89 (Preprint math/0611185v3)

Greenleaf A, Lassas M and Uhlmann G 2003a Anisotropic conductivities that cannot be detected by EIT *Physiol. Meas.* **24** 413–9

Greenleaf A, Lassas M and Uhlmann G 2003b On non-uniqueness for Calderón’s inverse problem *Math. Res. Lett.* **10** 685–93

Kerker M 1975 Invisible bodies *J. Opt. Soc. Am.* **65** 376–9

Khruslov E Y 1978 Asymptotic behavior of the solutions of the second boundary value problem in the case of the refinement of the boundary of the domain *Matematicheskii Sbornik* **106** 604–21

Khruslov E Y 1979 *Math. USSR Sbornik* **35** 266–82 (Engl. Transl.)

Kohn R V and Vogelius M 1984 *Inverse Problems Proc. Symp. Appl. Math. of the American Mathematical Society and the Society for Industrial and Applied Mathematics (New York, 12–13 April 1983)* ed D W McLaughlin (Providence, Rhode Island: American Mathematical Society) pp 113–23 ISBN 0-8218-1334X. LCCN QA370.S96

Lakes R S 2001 Extreme damping in compliant composites with a negative stiffness phase *Phil. Mag. Lett.* **81** 95–100

Leonhardt U 2006 Optical conformal mapping *Science* **312** 1777–80

Liu Z, Chan C T and Sheng P 2005 Analytic model of phononic crystals with local resonances *Phys. Rev. B* **71** 014103

Liu Z, Zhang X, Mao Y, Zhu Y Y, Yang Z, Chan C T and Sheng P 2000 Locally resonant sonic materials *Science* **289** 1734–6

Miller D A B 2006 On perfect cloaking *Opt. Express* **14** 12457–66

Milton G W, Briane M and Willis J R 2006 On cloaking for elasticity and physical equations with a transformation invariant form *New J. Phys.* **8** 248

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*New Journal of Physics* **9** (2007) 359 (http://www.njp.org/)
Milton G W and Nicorovici N-A P 2006 On the cloaking effects associated with anomalous localized resonance Proc. R. Soc. Lond. A 462 3027–59
Milton G W, Nicorovici N-A P, McPhedran R C and Podolskiy V A 2005 A proof of superlensing in the quasistatic regime, and limitations of superlenses in this regime due to anomalous localized resonance Proc. R. Soc. Lond. A 461 3999–4034
Milton G W and Willis J R 2007 On modifications of Newton’s second law and linear continuum elastodynamics Proc. R. Soc. Lond. A 463 855–80
Movchan A B and Guenneau S 2004 Split-ring resonators and localized modes Phys. Rev. B 70 125116
Shalaev V M 2007 Optical negative-index metamaterials Nat. Photonics 1 41–8
Nicorovici N A, McPhedran R C and Milton G W 1994 Optical and dielectric properties of partially resonant composites Phys. Rev. B 49 8479–82
Nicorovici N-A P, Milton G W, McPhedran R C and Botten L C 2007 Quasistatic cloaking of two-dimensional polarizable discrete systems by anomalous resonance Opt. Express 15 6314–23
Pendry J, Holden A J, Robbins D J and Stewart W J 1999 Magnetism from conductors and enhanced nonlinear phenomena IEEE Trans. Microw. Theory. Tech. 47 2075–84
Pendry J B, Schurig D and Smith D R 2006 Controlling electromagnetic fields Science 312 1780–2
Ramm A G 1996 Minimization of the total radiation from an obstacle by a control function on a part of its boundary J. Inverse Ill-Posed Problems 4 531–4
Schelkunoff S A and Friis H T 1952 Antennas: The Theory and Practice pp 584–5 (New York: Wiley) LCCN TK7872.A6 S3
Schurig D, Mock J J, Justice B J, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Metamaterial electromagnetic cloak at microwave frequencies Science 314 977–80
Schurig D, Pendry J B and Smith D R 2006 Calculation of material properties and ray tracing in transformation media Opt. Express 14 9794–804
Serdikukov A, Semchenko I, Tretkyakov S and Sihvola A 2001 Electromagnetics of Bi-anisotropic Materials, Theory and Applications (Amsterdam: Gordon and Breach) p 337 ISBN 9056993275
Sheng P, Zhang X X, Liu Z and Chan C T 2003 Locally resonant sonic materials Physica B 338 201–5
Shin J, Shen J-T and Fan S 2007 Three-dimensional electromagnetic metamaterials with non-maxwellian effective fields Preprint physics/0703053v1
Veselago V G 1967 The electrodynamics of substances with simultaneously negative values of $\epsilon$ and $\mu$ Usp. Fiz. Nauk 92 517–26
Veselago V G 1968 Soviet Phys. Usp. 10 509–14 (Engl. Transl.)
Willis J R 1981a Variational and related methods for the overall properties of composites Adv. Appl. Mech. 21 1–78
Willis J R 1981b Variational principles for dynamic problems for inhomogeneous elastic media Wave Motion 3 1–11
Willis J R 1985 The non-local influence of density variations in a composite Int. J. Solids Struct. 21 805–17
Willis J R 1997 Dynamics of composites (Continuum Micromechanics CISM Lecture Notes) (New York: Springer) pp 265–90
Zolla F, Guenneau S, Nicolet A and Pendry J B 2007 Electromagnetic analysis of cylindrical invisibility cloaks and the mirage effect Opt. Lett. 32 1069–71

New Journal of Physics 9 (2007) 359 (http://www.njp.org/)