Diameter diminishing to zero IFSs

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Received: 6 September 2020 / Accepted: 16 September 2021 / Published online: 28 September 2021
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Abstract
In this paper we introduce the notion of diameter diminishing to zero iterated function system, study its properties and provide alternative characterizations of it.

Keywords Iterated function system (IFS) · (Hyperbolic)$\varphi$-contractive IFS · (Hyperbolic)locally uniformly point fibred IFS · (Hyperbolic)uniformly point fibred IFS · IFS having (hyperbolic)attractor · (Hyperbolic)diameter diminishing to zero IFS

Mathematics Subject Classification 28A80 · 37C70 · 54H20

1 Introduction

The notion of iterated function system, which is due to Hutchinson (see [9]), was popularized by M. Barnsley (see [3]). It represents one of the main ways to generate fractal sets and (since it has numerous applications) various generalizations of this concept were introduced. Among them we mention the one exhibited by Kameyama (see [11]) under the label of self-similar system which is a topological generalization of the attractor of an iterated function system.
In connection with Kamayema’s work, Atkins et al. (see [1]) presented a theorem that characterizes hyperbolic affine iterated function systems defined on \( \mathbb{R}^m \). Results connected with the aforementioned theorem are included in [2,4,13–16,18,22] and [23].

Along these lines of research, in this paper, we introduce the concept of (hyperbolic) diameter diminishing to zero iterated function system (see Definition 2.17 and Remarks 2.19) and study its properties (see Propositions 3.1–3.7).

In addition, via the concepts of hyperbolic \( \varphi \)-contractive iterated function system (see Definition 2.11 and Remarks 2.19), hyperbolic (locally) uniformly point fibred iterated function system (see Definitions 2.13 and 2.14 and Remarks 2.19) and iterated function system having hyperbolic attractor (see Definition 2.15 and Remarks 2.19), we provide alternative characterizations of hyperbolic diameter diminishing to zero iterated function systems (see Theorem 4.1). Consequently we also come up with different ways to prove that an iterated function system has hyperbolic attractor.

2 Preliminaries

Given two sets \( A \) and \( B \), by \( B^A \) we mean the set of functions from \( A \) to \( B \).

By \( \mathbb{N} \) we mean the set \( \{0, 1, 2, \ldots, n, \ldots\} \) and by \( \mathbb{N}^* \) we mean the set \( \{1, 2, \ldots, n, \ldots\} \).

For a set \( X \), a function \( f : X \to X \) and \( n \in \mathbb{N}^* \), by \( f^n \) we mean the composition of \( f \) by itself \( n \) times. By \( f^0 \) we mean the identity function \( Id_X : X \to X \) given by \( Id_X(x) = x \) for every \( x \in X \).

Given a metric space \((X, d)\), by:

- \( P_b(X) \) we mean the set of non-empty bounded subsets of \( X \)
- \( P_{b,cl}(X) \) we mean the set of non-empty bounded and closed subsets of \( X \)
- \( P_{cp}(X) \) we mean the set of non-empty compact subsets of \( X \)
- by \( B(x, r) \) we mean the set \( \{y \in X \mid d(x, y) < r\} \), where \( x \in X \) and \( r > 0 \).

The Hausdorff–Pompeiu metric

**Definition 2.1** Given a metric space \((X, d)\), \( H_d : P_{b,cl}(X) \times P_{b,cl}(X) \to [0, +\infty) \) given by

\[
H_d(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}
\]

\[
= \inf\{\varepsilon \in [0, \infty) \mid A \subseteq E_\varepsilon(B) \text{ and } B \subseteq E_\varepsilon(A)\},
\]

for all \( A, B \in P_{b,cl}(X) \), where

\[
d(x, A) = \inf_{y \in A} d(x, y)
\]

and

\[
E_\varepsilon(A) \overset{def}{=} \{y \in X \mid \text{there exists } x \in A \text{ such that } d(x, y) < \varepsilon\} = \bigcup_{x \in A} B(x, \varepsilon),
\]
turns out to be a metric, which is called the Hausdorff–Pompeiu metric.

**Proposition 2.2** For a metric space \((X, d)\), we have

\[
H_d \left( \bigcup_{i \in I} H_i, \bigcup_{i \in I} K_i \right) \leq \sup_{i \in I} H_d(H_i, K_i),
\]

for every \((H_i)_{i \in I}\) and \((K_i)_{i \in I}\) families of elements from \(P_{b,cl}(X)\) such that \(\bigcup_{i \in I} H_i, \bigcup_{i \in I} K_i \in P_{b,cl}(X)\).

**Remark 2.3** The metric spaces \((P_{b,cl}(X), H_d)\) and \((P_{cp}(X), H_d)\) are complete provided that \((X, d)\) is complete.

**The shift space**

Let \(I\) be a non-empty set.

We denote the set \(I^{\mathbb{N}^+}\) by \(\Lambda(I)\). Thus \(\Lambda(I)\) is the set of infinite words with letters from the alphabet \(I\) and a standard element \(\omega\) of \(\Lambda(I)\) can be presented as \(\omega = \omega_1\omega_2...\omega_n\omega_{n+1}...\).

We denote the set \(I^{\{1,2,...,n\}}\) by \(\Lambda_n(I)\). Thus \(\Lambda_n(I)\) is the set of words with letters from the alphabet \(I\) of length \(n\) and a standard element \(\omega\) of \(\Lambda_n(I)\) can be presented as \(\omega = \omega_1\omega_2...\omega_n\). The length of \(\omega\) is denoted by \([\omega]\). By \(\Lambda_0(I)\) we mean the set having only one element, namely the empty word denoted by \(\lambda\).

We denote the set \(\bigcup_{n \in \mathbb{N}} \Lambda_n(I)\) by \(\Lambda^*(I)\). Thus \(\Lambda^*(I)\) is the set of words with letters from the alphabet having finite length.

Given \(m, n \in \mathbb{N}\) and two words \(\omega = \omega_1\omega_2...\omega_n \in \Lambda_n(I)\) and \(\theta = \theta_1\theta_2...\theta_m \in \Lambda_m(I)\) or \(\theta = \theta_1\theta_2...\theta_m\theta_{m+1}... \in \Lambda(I)\), by \(\omega\theta\) we mean the concatenation of the words \(\omega\) and \(\theta\), i.e. \(\omega\theta = \omega_1\omega_2...\omega_n\theta_1\theta_2...\theta_m\) and respectively \(\omega\theta = \omega_1\omega_2...\omega_n\theta_1\theta_2...\theta_m\theta_{m+1}...\).

For \(m \in \mathbb{N}^+\) and \(\omega = \omega_1\omega_2...\omega_n\omega_{n+1}... \in \Lambda(I)\), by \([\omega]_m\) we mean \(\omega_1\omega_2...\omega_m\).

For \(\omega = \omega_1\omega_2...\omega_n \in \Lambda_n(I)\), the word \(\theta = \theta_1\theta_2...\theta_m\theta_{m+1}... \in \Lambda(I)\), where \(\theta_{nk+i} = \omega_i\) for every \(k \in \mathbb{N}\) and every \(i \in \{1, ..., n - 1, n\}\), will be denoted by \(\dot{\omega}\).

For \(i \in I\) one can consider the function \(\tau_i : \Lambda(I) \to \Lambda(I)\) given by

\[
\tau_i(\omega) = i\omega,
\]

for every \(\omega \in \Lambda(I)\).

\(\Lambda(I)\) becomes a metric space if we endow it with the distance described by

\[
d_\Lambda(\omega, \theta) = \begin{cases} 
0, & \text{if } \omega = \theta \\
\frac{1}{2^\min\{i \in \mathbb{N}^+: \omega_i \neq \theta_i\}}, & \text{if } \omega \neq \theta,
\end{cases}
\]

where \(\omega = \omega_1\omega_2\omega_3...\omega_n\omega_{n+1}...\) and \(\theta = \theta_1\theta_2\theta_3...\theta_m\theta_{m+1}...\).

**Remark 2.4**

(a) The convergence in the metric space \((\Lambda(I), d_\Lambda)\) is the convergence on components.

(b) \((\Lambda(I), d_\Lambda)\) is a complete metric space.

(c) If \(I\) is finite, then \((\Lambda(I), d_\Lambda)\) is compact.

Note that points a) and c) follow from the fact that the metric \(d_\Lambda\) induces the Tychonoff product topology.
Proposition 2.5 Let us consider \((\omega_n)_{n\in \mathbb{N}^*} \subseteq \Lambda(I)\) and \(\omega \in \Lambda(I)\) such that \(\lim_{n \to \infty} \omega_n = \omega\). Then for every \(m \in \mathbb{N}^*\) there exists \(n_m \in \mathbb{N}^*\) such that \([\omega_n]_m = [\omega]_m\) for every \(n \in \mathbb{N}^*, n \geq n_m\).

Proof Ignoring those \(\omega_n\) which are equal with \(\omega\) and supposing that \(\omega_n = \omega_1^1 \omega_2^2 \cdots \omega_k^k \cdots\) and \(\omega = \omega_1 \omega_2 \cdots \omega_k \cdots\), for every \(m \in \mathbb{N}^*\) there exists \(n_m \in \mathbb{N}^*\) such that

\[
\frac{1}{2 \min\{l \in \mathbb{N}^* \mid \omega_n^l \neq \omega_l\}} < \frac{1}{2m}.
\]

for every \(n \in \mathbb{N}^*, n \geq n_m\). As the previous inequality means that

\[m < \min\{l \in \mathbb{N}^* \mid \omega_n^l \neq \omega_l\},\]

we get \(\omega_1^1 = \omega_1, \omega_2^2 = \omega_2, \ldots, \omega_m^m = \omega_m\), i.e

\([\omega_n]_m = [\omega]_m,\]

for every \(n \in \mathbb{N}^*, n \geq n_m\). \(\square\)

Comparison functions and \(\varphi\)-contractions

Definition 2.6 (comparison function). A function \(\varphi : [0, \infty) \to [0, \infty)\) is called a comparison function if:

(i) \(\varphi\) is increasing;
(ii) \(\varphi(t) < t\) for every \(t > 0\);
(iii) \(\varphi\) is right-continuous.

Remark 2.7 If \(\varphi : [0, \infty) \to [0, \infty)\) is a comparison function, then \(\lim_{n \to \infty} \varphi^{[n]}(t) = 0\) for every \(t \in [0, \infty)\).

Definition 2.8 (\(\varphi\)-contraction). Given a metric space \((X, d)\) and a comparison function \(\varphi\), a function \(f : X \to X\) is called \(\varphi\)-contraction if \(d(f(x), f(y)) \leq \varphi(d(x, y))\) for all \(x, y \in X\).

A very good reference on comparison functions and \(\varphi\)-contractions is the survey [10]. See also [8].

Iterated function systems

Definition 2.9 (iterated function system). A pair \(((X, d), (f_i)_{i \in I})\) is called an iterated function system (IFS for short) if:

i) \((X, d)\) is a complete metric space;
ii) \(I\) is a finite set;
iii) \(f_i : X \to X\) is continuous for each \(i \in I\);
iv) \(f_i(B) \in P_b(X)\) for every \(B \in P_b(X)\) and every \(i \in I\).
Notations

1. We shall denote the IFS \(((X, d), (f_i)_{i \in I})\) by \(S\).

2. In the framework of the above definition, for \(\omega = \omega_1 \omega_2 ... \omega_n \in \Lambda_n(I)\) and \(B\) subset of \(X\), by \(f_\omega(B)\) we mean \((f_{\omega_1} \circ ... \circ f_{\omega_n})(B)\).

Definition 2.10 (fractal operator). Given an IFS \(S = ((X, d), (f_i)_{i \in I})\), the function \(F_S : P_{b,cl}(X) \to P_{b,cl}(X)\), given by

\[
F_S(B) = \bigcup_{i \in I} f_i(B),
\]

for every \(B \in P_{b,cl}(X)\), is called the fractal operator associated to \(S\).

Definition 2.11 (\(\varphi\)-contractive IFS). Given a comparison function \(\varphi\), an iterated function system \(S = ((X, d), (f_i)_{i \in I})\) is called \(\varphi\)-contractive if \(f_i\) is a \(\varphi\)-contraction for each \(i \in I\).

Definition 2.12 (point fibred IFS). An iterated function system \(S = ((X, d), (f_i)_{i \in I})\) is called point fibred if for every \(\omega \in \Lambda(I)\) there exists \(a_\omega \in X\) such that

\[
\lim_{n \to \infty} f_{[\omega]_n} (x) = a_\omega,
\]

for all \(x \in X\).

Definition 2.13 (locally uniformly point fibred IFS). An iterated function system \(S = ((X, d), (f_i)_{i \in I})\) is called locally uniformly point fibred if it is point fibred and for each \(x \in X\) there exists an open set \(D_x\) containing \(x\) such that

\[
\lim_{n \to \infty} \sup_{\omega \in \Lambda(I)} \sup_{y \in D_x} d(f_{[\omega]_n}(y), a_\omega) = 0.
\]

Note that the concept of locally uniformly point fibred IFS is the same as the “condition C” that was introduced in Definition 3.1 from [15].

Definition 2.14 (uniformly point fibred IFS). An iterated function system \(S = ((X, d), (f_i)_{i \in I})\) is called uniformly point fibred if it is point fibred and

\[
\lim_{n \to \infty} \sup_{\omega \in \Lambda(I)} \sup_{x \in B} d(f_{[\omega]_n}(x), a_\omega) = 0,
\]

for every \(B \in P_{b,cl}(X)\).

Definition 2.15 (IFS having attractor). We say that an iterated function system \(S = ((X, d), (f_i)_{i \in I})\) has an attractor if there exists \(A_S \in P_{b,cl}(X)\) such that:

(i)

\[
F_S(A_S) = A_S;
\]
Remark 2.16 $F_S$ has a unique fixed point, namely $A_S \in P_{cp}(X)$, which is called the attractor of $S$.

Indeed, if for some $B \in P_{b,cl}(X)$ we have $F_S(B) = B$, then

$$0 = \lim_{n \to \infty} H_d(F_S^n(B), A_S) = \lim_{n \to \infty} H_d(B, A_S) = H_d(B, A_S),$$

so $B = A_S$. In addition, $\lim_{n \to \infty} H_d(F_S^n(K), A_S) = 0$ for every $K \in P_{cp}(X) \subseteq P_{b,cl}(X)$ and in view of Proposition 2.7 from [17] we conclude that $A_S \in P_{cp}(X)$.

Definition 2.17 (diameter diminishing to zero IFS). An iterated function system $S = ((X, d), (f_i)_{i \in I})$ is called diameter diminishing to zero iterated function systems if for every $B \in P_{b,cl}(X)$ there exists $M_B \in P_{b,cl}(X)$ such that:

(i) $B \subseteq M_B$;

(ii) $F_S(M_B) \subseteq M_B$;

(iii) $\lim_{n \to \infty} \max_{\omega \in \Lambda_n(I)} \text{diam}(f_\omega(M_B)) = 0$.

Definition 2.18 (hyperbolic $\varphi$-contractive IFS). Given a comparison function $\varphi$, an iterated function system $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ is called hyperbolic $\varphi$-contractive if there exists a distance $d_1$ on $X$ such that:

(i) $d$ and $d_1$ are topologically equivalent;

(ii) $(X, d_1)$ is complete;

(iii) $S_1 = ((X, d_1), (f_i)_{i \in I})$ is $\varphi$-contractive.

Remark 2.19 (a) The concepts of hyperbolic locally uniformly point fibred IFS, hyperbolic uniformly point fibred IFS, IFS having hyperbolic attractor and hyperbolic diameter diminishing to zero IFS could be defined having as model Definition 2.18.

(b) An iterated function system $S$ which is uniformly point fibred is locally uniformly point fibred.

(c) An iterated function system $S$ which is hyperbolic uniformly point fibred is hyperbolic locally uniformly point fibred.

(d) As one of the referees of this paper noted, Definition 2.15 raises the question whether there exists an IFS with hyperbolic attractor and not having attractor.
3 The properties of diameter diminishing to zero iterated function systems

Proposition 3.1 Given a diameter diminishing to zero IFS $S = ((X, d), (f_i)_{i \in I})$, for every $\omega \in \Lambda(I)$ there exists $a_{\omega, B} \in X$ such that

$$\bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]}_n(B)} = \{a_{\omega}\},$$

for every $B \in P_{b,cl}(X)$ having the property that $F_S(B) \subseteq B$.

Proof Let $\omega \in \Lambda(I)$ be fixed.

Claim 1 For every $B \in P_{b,cl}(X)$ such that $F_S(B) \subseteq B$ there exists $a_{\omega, B} \in X$ such that

$$\bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]}_n(B)} = \{a_{\omega, B}\}.$$ 

Justification of Claim 1. As $f_i(B) \subseteq B$ for each $i \in I$, we have

$$f_{[\omega]}_{n+1}(B) \subseteq f_{[\omega]}_n(B),$$

for every $n \in \mathbb{N}^*$. Since

$$diam(\overline{f_{[\omega]}_n(B)}) = diam(f_{[\omega]}_n(B))$$

$$\leq \max_{\theta \in \Lambda_n(I)} diam(f_\theta(B)) \leq \max_{\theta \in \Lambda_n(I)} diam(f_\theta(M_B)),$$ (1)

for every $n \in \mathbb{N}^*$, where $M_B$ is the element of $P_{b,cl}(X)$ whose existence is stated in Definition 2.17. Using iii) of the same Definition, via (1), we conclude that

$$\lim_{n \to \infty} diam(\overline{f_{[\omega]}_n(B)}) = 0.$$ 

Therefore, according to Cantor’s theorem, there exists $a_{\omega, B} \in X$ such that

$$\bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]}_n(B)} = \{a_{\omega, B}\}.$$ 

The justification of the claim is done.

Claim 2 For every $B, C \in P_{b,cl}(X)$ such that $F_S(B) \subseteq B$, $F_S(C) \subseteq C$ and $B \subseteq C$, we have

$$a_{\omega, B} = a_{\omega, C}.$$ 

Justification of Claim 2. Indeed

$$\{a_{\omega, B}\} \overset{\text{Claim 1}}{=} \bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]}_n(B)} \subseteq \bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]}_n(C)} \overset{\text{Claim 1}}{=} \{a_{\omega, C}\}.$$
and the justification of the claim is done.

**Claim 3** For every $B, C \in P_{b,cl}(X)$ such that $F_S(B) \subseteq B$ and $F_S(C) \subseteq C$, we have

$$a_{\omega, B} = a_{\omega, C}.$$

**Justification of Claim 3.** Since $B \cup C \in P_{cl,b}(X)$ and

$$F_S(B \cup C) \subseteq F_S(B) \cup F_S(C) \subseteq B \cup C,$$

we infer that

$$a_{\omega, B} \overset{\text{Claim 2}}{=} a_{\omega, B \cup C} \overset{\text{Claim 2}}{=} a_{\omega, C},$$

and the justification of the claim is done.

Hence $\{a_{\omega, B} \mid B \in P_{b,cl}(X) \text{ such that } F_S(B) \subseteq B\}$ has only one element, which is denoted by $a_\omega$. $\square$

In view of the previous Proposition, given a diameter diminishing to zero IFS $S = ((X, d), \{f_i\}_{i \in I})$ one can consider the function $\pi : \Lambda(I) \to X$ given by

$$\pi(\omega) = a_\omega,$$

for each $\omega \in \Lambda(I)$.

**Proposition 3.2** Each diameter diminishing to zero IFS is uniformly point fibred.

**Proof** If $S = ((X, d), \{f_i\}_{i \in I})$, is a diameter diminishing to zero IFS $\{f_i\}_{i \in I}$, then, according to Definition 2.17, for each $B \in P_{b,cl}(X)$ exists $M_B \in P_{b,cl}(X)$ such that $B \subseteq M_B$ and $F_S(M_B) \subseteq M_B$. Consequently, for every $\omega \in \Lambda(I)$, we have

$$\sup_{x \in B} d(f_{[\omega]_n}(x), a_\omega) \leq \sup_{x \in M_B} d(f_{[\omega]_n}(x), a_\omega) \overset{a_{\omega, f_{[\omega]_n}(M_B)}}{\leq} \leq diam(f_{[\omega]_n}(M_B)) \leq \max_{\theta \in \Lambda_n(I)} diam(f_{\theta}(M_B)),$$

for every $n \in \mathbb{N}^\ast$. Hence

$$\sup_{\omega \in \Lambda(I)} \sup_{x \in B} d(f_{[\omega]_n}(x), a_\omega) \leq \max_{\omega \in \Lambda_n(I)} diam(f_{\omega}(M_B)),$$

for every $n \in \mathbb{N}^\ast$, so, in view of Definition 2.17, we obtain

$$\lim_{n \to \infty} \sup_{\omega \in \Lambda(I)} \sup_{x \in B} d(f_{[\omega]_n}(x), a_\omega) = 0,$$

$\square$ Springer
Proposition 3.3  For each diameter diminishing to zero IFS $\mathcal{S} = ((X, d), (f_i)_{i \in I})$, we have

$$f_i \circ \pi = \pi \circ \tau_i,$$

for every $i \in I$.

**Proof** For a fixed $B \in P_{b,cl}(X)$ such that $F_\mathcal{S}(B) \subseteq B$, we have

$$\{ f_i(\pi(\omega)) \} = \{ f_i(a_\omega) \} = f_i(\bigcap_{n \in \mathbb{N}^*} f_i(\omega)_n(B)) \subseteq \bigcap_{n \in \mathbb{N}^*} f_i(\pi(\omega)_n(B)) = \bigcap_{n \in \mathbb{N}^*} f_i(a_\omega)(N \in \mathbb{N}^*),$$

i.e.

$$(f_i \circ \pi)(\omega) = (\pi \circ \tau_i)(\omega),$$

for every $\omega \in \Lambda(I)$ and every $i \in I$.  \(\square\)

Proposition 3.4  For each diameter diminishing to zero IFS $\mathcal{S} = ((X, d), (f_i)_{i \in I})$, the function $\pi$ is continuous.

**Proof** First of all, let us chose a fixed $B \in P_{b,cl}(X)$ such that $F_\mathcal{S}(B) \subseteq B$. Now let us consider $\omega \in \Lambda(I)$ and a sequence $(\omega_n)_{n \in \mathbb{N}^*}$ of elements of $\Lambda(I)$ converging to $\omega$. Therefore, in view of Proposition 2.5, for each $m \in \mathbb{N}^*$ there exists $n_m \in \mathbb{N}^*$ such that

$$[\omega_n]_m = [\omega]_m, \quad (2)$$

for every $n \in \mathbb{N}^*$, $n \geq n_m$. Let us consider a fixed, but arbitrarily chosen $\varepsilon > 0$. Taking into account Definition 2.17, there exists $m_\varepsilon \in \mathbb{N}^*$ such that

$$\max_{\theta \in \Lambda_{m_\varepsilon}(I)} \text{diam}(f_\theta(B)) < \varepsilon. \quad (3)$$

Then for every $n \in \mathbb{N}^*$, $n \geq n_{m_\varepsilon}$, as $\pi(\omega_n) \in \bigcap_{n \in \mathbb{N}^*} f_i(\omega)_n(B)$ and $\pi(\omega) \in \bigcap_{n \in \mathbb{N}^*} f_i(\omega)_n(B)$, with the notation $[\omega_n]_{m_\varepsilon} \equiv [\omega]_{m_\varepsilon} \not= \beta_{m_\varepsilon} \in \Lambda_{m_\varepsilon}(I)$, we get

$$d(\pi(\omega_n), \pi(\omega)) \leq \text{diam}(f_{\beta_{m_\varepsilon}}(B)) = \text{diam}(f_{\beta_{m_\varepsilon}}(B)) < \varepsilon. \quad (3)$$

The last relation assures us that sequence $(\pi(\omega_n))_{n \in \mathbb{N}^*}$ converges to $\pi(\omega)$ and this shows that $\pi$ is continuous.  \(\square\)

Proposition 3.5  Each diameter diminishing to zero IFS $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ has attractor and $A_\mathcal{S} = \pi(\Lambda(I)) = \{a_\omega \mid \omega \in \Lambda(I)\}$.  \(\square\) Springer
Proof Let us note that:

(i) \( \pi(\Lambda(I)) \) \( \in \) \( P_{cp}(X) \subseteq P_{b,cl}(X) \);

(ii) 
\[
F_S(\pi(\Lambda(I))) \overset{\text{Definition 2.10}}{=} \bigcup_{i \in I} f_i(\pi(\Lambda(I))) \overset{\text{Proposition 3.3}}{=} \bigcup_{i \in I} \pi(\tau_i(\Lambda(I))) = \pi\left( \bigcup_{i \in I} \tau_i(\Lambda(I)) \right) = \pi(\Lambda(I)).
\]

Moreover,
\[
\lim_{n \to \infty} H_d(F^{[n]}(B), \pi(\Lambda(I))) = 0,
\]
for each \( B \in P_{b,cl}(X) \).

Indeed, let us consider a fixed, but arbitrarily chosen \( B \in P_{b,cl}(X) \). We have
\[
H_d(F^{[n]}(B), \pi(\Lambda(I))) = H_d(\bigcup_{\omega \in \Lambda(I)} f^{[n]}_{\omega}(B), \bigcup_{\omega \in \Lambda(I)} \{a_\omega\}) \overset{\text{Proposition 2.2}}{\leq} \sup_{\omega \in \Lambda(I)} H_d(f^{[n]}_{\omega}(B), \{a_\omega\}) = \sup_{\omega \in \Lambda(I)} \max\{\sup_{x \in B} d(f^{[n]}_{\omega}(x), a_\omega), \inf_{x \in B} d(f^{[n]}_{\omega}(x), a_\omega)\}
\]
for every \( n \in \mathbb{N}^* \) and taking into account Proposition 3.2, we conclude that \( \lim_{n \to \infty} H_d(F^{[n]}(B), \pi(\Lambda(I))) = 0 \). Consequently \( \pi(\Lambda(I)) \) is the attractor of \( S \).

Proposition 3.6 For each diameter diminishing to zero IFS \( S = ((X, d), (f_i)_{i \in I}) \), we have
\[
\{a_\omega\} = Fix(f_\omega),
\]
for each \( \omega \in \Lambda^*(I) \setminus \lambda \).

Proof Since
\[
f_\omega(a_\omega) = f_\omega(\pi(\omega)) \overset{\text{Proposition 3.3}}{=} \pi(\omega \hat{\omega}) = \pi(\hat{\omega}) = a_\omega,
\]
we conclude that
\[
\{a_\omega\} \subseteq Fix(f_\omega),
\]
for every \( \omega \in \Lambda^*(I) \setminus \lambda \).
Now we prove that
\[ Fix(f_\omega) \subseteq \{a_\omega\} , \]
for every \(\omega \in \Lambda^*(I) \setminus \{\lambda\} \).

Let us consider \(z \in Fix(f_\omega)\) (so \(z \in \bigcap_{n \in \mathbb{N}^*} \{f_{[\omega]|n|_0}(z)\}\)). Then, in view of Definition 2.17, there exists \(M(z) \in P_{b,c}(X)\) such that \(z \in M(z)\) and \(F_S(M(z)) \subseteq M(z)\). Hence
\[ f_i(M(z)) \subseteq F_S(M(z)) \subseteq M(z) , \]
for every \(i \in I\), so \(\overline{f_{[\omega]|n|_0}(M(z))}\) is a decreasing sequence and therefore
\[ \bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]|n|_0}(M(z))} = \bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]|n|_0}(M(z))} \quad \text{Proposition 3.1} \quad \{a_\omega\}. \]

Consequently
\[ z \in \bigcap_{n \in \mathbb{N}^*} \{f_{[\omega]|n|_0}(z)\} \subseteq \bigcap_{n \in \mathbb{N}^*} f_{[\omega]|n|_0}(M(z)) \subseteq \bigcap_{n \in \mathbb{N}^*} \overline{f_{[\omega]|n|_0}(M(z))} = \{a_\omega\} \]
and (4) is justified.

The Relations (4) and (5) assure us that \(Fix(f_\omega) = \{a_\omega\}\). \(\square\)

**Proposition 3.7** For each diameter diminishing to zero IFS \(S = ((X, d), (f_i)_{i \in I})\), we have
\[ A_S = \{a_\omega | \omega \in \Lambda^*(I) \setminus \{\lambda\}\}. \]

**Proof** As \(\{a_\omega | \omega \in \Lambda^*(I) \setminus \{\lambda\}\} \subseteq A_S\) and \(A_S\) is compact, it suffices to prove that
\[ A_S \subseteq \{a_\omega | \omega \in \Lambda^*(I) \setminus \{\lambda\}\}. \]

To this aim, let us consider \(x \in A_S\). For an arbitrary neighborhood \(V\) of \(x\), there exists an open subset \(D\) of \(X\) such that \(x \in D \subseteq V\). As \(A_S = \{a_\omega | \omega \in \Lambda(I)\}\), there exists \(\omega_x \in \Lambda(I)\) such that \(x = a_{\omega_x}\). Hence
\[ \bigcap_{n \in \mathbb{N}^*} f_{[\omega_x]|n|_0}(A_S) \quad \text{Proposition 3.1} \quad \{a_{\omega_x}\} \subseteq D \]
and since the sequence \((f_{[\omega_x]|n|_0}(A_S))_{n \in \mathbb{N}^*} \subseteq P_{cb}(X)\) is decreasing and \(D\) is open, there exists \(m \in \mathbb{N}\) such that
\[ f_{[\omega_x]|m}(A_S) \subseteq D \quad \text{(6)} \]
(see, for example, Corollary 3.1.5 from [7]). Therefore
\[ a_{[\omega_x]|m} \quad \text{Proposition 3.6} \quad f_{[\omega_x]|m}(a_{[\omega_x]|m}) \subseteq f_{[\omega_x]|m}(A_S) \subseteq D \subseteq V, \]
so

\[ V \cap \{a_\omega \mid \omega \in \Lambda^n(I) \setminus \{\lambda\}\} \neq \emptyset. \]

Hence \( x \in \{a_\omega \mid \omega \in \Lambda^n(I) \setminus \{\lambda\}\} \) and the proof is done. \( \square \)

### 4 The main result

**Theorem 4.1** For an iterated function system \( S = ((X, d), (f_i)_{i \in I}) \), the following statements are equivalent:

1. There exists a comparison function \( \varphi \) such that \( S \) is hyperbolic \( \varphi \)-contractive.
2. \( S \) is hyperbolic locally uniformly point fibred.
3. \( S \) is hyperbolic uniformly point fibred.
4. \( S \) is a hyperbolic diameter diminishing to zero iterated function system.
5. \( S \) has hyperbolic attractor and there exists a continuous surjection \( \pi : \Lambda(I) \to A_S \) such that

\[
  f_i \circ \pi = \pi \circ \tau_i,
\]

for all \( i \in I \).

**Proof** The argument used for the justification of Remark 3.1 from [15] ensures the validity of 1) \( \Rightarrow \) 3).

For (3) \( \Rightarrow \) (2) see Remark 2.19, c).

For (2) \( \Rightarrow \) (1) see Theorem 3.1 from [15].

(1) \( \Rightarrow \) (4) First of all, note that for every \( B \in P_{b, cl}(X) \), we have

\[
  \max_{\omega \in \Lambda_n(I)} \text{diam}(f_\omega(B)) \leq \varphi^n(\text{diam}(B))
\]

for every \( n \in \mathbb{N}^* \), so, taking into account Remark 2.7, we obtain that

\[
  \lim_{n \to \infty} \max_{\omega \in \Lambda_n(I)} \text{diam}(f_\omega(B)) = 0. \tag{7}
\]

In addition, since there exists a unique \( A_S \in P_{b, cl}(X) \) such that \( F_S(A_S) = A_S \) and \( \lim_{n \to \infty} F_S^{[n]}(B) = A_S \) for every \( B \in P_{b, cl}(X) \) (see Theorem 2.5 from [5]), one can consider the set

\[
  M_B = A_S \cup \bigcup_{n \in \mathbb{N}} F_S^{[n]}(B) \supseteq B. \tag{8}
\]

Note that as \( \lim_{n \to \infty} F_S^{[n]}(B) = A_S \), there exists \( n_1 \in \mathbb{N} \) and \( \varepsilon > 0 \) such that

\[
  F_S^{[n]}(B) \subseteq E_\varepsilon(A_S) \text{ for each } n \in \mathbb{N}, n \geq n_1. \]

Hence \( \bigcup_{n \in \mathbb{N}, n \geq n_1} F_S^{[n]}(B) \in P_b(X) \), so,
via Definition 2.9, iv), we infer that \( A_S \cup \bigcup_{n \in \mathbb{N}} F_S^n(B) \in P_b(X) \). Consequently

\[
M_B \in P_{b, cl}(X). \tag{9}
\]

Moreover, we have

\[
F_S(M_B) = F_S(A_S \cup \bigcup_{n \in \mathbb{N}} F_S^n(B)) = \bigcup_{i \in I} \bigcup_{n \in \mathbb{N}} f_i(A_S \cup \bigcup_{n \in \mathbb{N}} F_S^n(B)) \subseteq \bigcup_{i \in I} \bigcup_{n \in \mathbb{N}} F_S^n(B) \subseteq A_S \cup \bigcup_{n \in \mathbb{N}} F_S^{n+1}(B) \subseteq M_B. \tag{10}
\]

Therefore, in view of (7), (8), (9) and (10), \( S \) is a hyperbolic diameter diminishing to zero iterated function system.

4) \( \Rightarrow \) 5) See the results from Sect. 3.

5) \( \Rightarrow \) 2) Let \( d_1 \) be the distance on \( X \) whose existence is assured by Remarks 2.19, a).

First of all let us note that \( A_S \in P_{cp}(X) \) (since \( A_S = \pi(\Lambda(I)), \pi \) is continuous and \( \Lambda(I) \) compact -see Remark 2.4, c)-).

Claim 1

\[
\lim_{n \to \infty} \max_{\omega \in \Lambda_n(I)} \text{diam}(f_\omega(A_S)) = 0.
\]

*Justification of Claim 1.* See Lemma 1.6 from [11].

Claim 2 For every \( \omega \in \Lambda(I) \), the set \( \cap_{n \in \mathbb{N}^+} f_{[\omega]n}(A_S) \) consists on only one element denoted by \( b_\omega \).

*Justification of Claim 2.* Just use Cantor’s theorem and Claim 1.

Let us consider \( \varepsilon > 0 \) fixed, but arbitrarily chosen.

Then, via Claim 1, there exists \( n_0 \in \mathbb{N} \) such that

\[
\text{diam}(f_\omega(A_S)) < \frac{\varepsilon}{3}, \tag{11}
\]

for every \( \alpha \in \Lambda_{n_0}(I) \).

For every \( \alpha \in \Lambda_{n_0}(I) \) there exists \( \delta_\alpha > 0 \) such that

\[
f_\alpha(E_{\delta_\alpha}(A_S)) \subseteq E_{\varepsilon \frac{\delta_\alpha}{3}}(f_\alpha(A_S)). \tag{12}
\]

Indeed, the continuity of \( f_\alpha \) assures us that for every \( x \in A_S \) there exists \( \delta_{\alpha,x} > 0 \) such that \( f_\alpha(B(x, 2\delta_{\alpha,x})) \subseteq B(f_\alpha(x), \frac{\varepsilon \delta_{\alpha,x}}{3}) \). In view of the compactness of \( A_S \) there exist
$p \in \mathbb{N}^*$ and $x_1, \ldots, x_p \in A_S$ such that $A_S \subseteq \bigcup_{j=1}^p B(x_j, \delta_{\alpha,x_j})$. Let us consider $\delta_\alpha = \min\{\delta_{\alpha,x_1}, \ldots, \delta_{\alpha,x_p}\} > 0$. If $u \in E_{\delta_\alpha}(A_S)$, there exists $v \in A_S$ such that $d_1(u, v) < \delta_\alpha$. In addition, there exists $j_v \in \{1, \ldots, p\}$ having the property that $d_1(x_{j_v}, v) < \delta_{\alpha,x_{j_v}}$. Consequently we have $d_1(u, x_{j_v}) \leq d_1(u, v) + d_1(v, x_{j_v}) < \delta_\alpha + \delta_{\alpha,x_{j_v}} < 2\delta_{\alpha,x_{j_v}}$, so $f_\alpha(u) \in f_\alpha(B(x_{j_v}, 2\delta_{\alpha,x_{j_v}})) \subseteq B(f_\alpha(x_{j_v}), \frac{\varepsilon}{3})$, i.e. $f_\alpha(u) \in E_{\frac{\varepsilon}{3}}(f_\alpha(A_S))$. As $u$ was arbitrarily chosen in $A_S$, the justification of (12) is done.

Let us consider

$$
\delta = \min\{\delta_\alpha \mid \alpha \in \Lambda_{n_0}(I)\}.
$$

Since $\lim_{n \to \infty} F_S^{[n]}(B(x, \eta)) = A_S$ for every $x \in X$ and $\eta > 0$, there exists $m_0 \in \mathbb{N}^*$ such that

$$
F_S^{[m]}(B(x, \eta)) \subseteq E_\delta(A_S), \quad (13)
$$

for every $m \in \mathbb{N}^*$, $m \geq m_0$. For an arbitrary $\omega \in \Lambda(I)$ let us consider $[\omega]_{n_0} \not= \alpha$. Then, for $m \in \mathbb{N}$, $m \geq m_0$, if $[\omega]_{n_0+m} = \alpha \beta$, where $\beta \in \Lambda_m(I)$, via Claim 2, we have

$$
d_1(f_{[\omega]_{n_0+m}}(y), b_\omega) = d_1(f_\alpha(f_\beta(y)), b_\omega) \leq \sup_{u \in F_S^{[m]}(B(x, \eta))} d_1(f_\alpha(u), b_\omega) \leq \sup_{u \in E_S(f_\alpha(A_S))} d_1(v, b_\omega) \leq \text{diam}(E_\varepsilon(f_\alpha(A_S))) \leq \frac{2\varepsilon}{3} + \text{diam}(f_\alpha(A_S)) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
$$

for every $y \in B(x, \eta)$, so

$$
\sup_{\omega \in \Lambda(I)} \sup_{y \in B(x, \eta)} d_1(f_{[\omega]_{n_0+m}}(y), b_\omega) \leq \varepsilon,
$$

for every $m \in \mathbb{N}$, $m \geq m_0$. Hence

$$
\lim_{n \to \infty} \sup_{\omega \in \Lambda(I)} \sup_{y \in B(x, \eta)} d_1(f_{[\omega]_n}(y), b_\omega) = 0,
$$

i.e. $S$ is hyperbolic locally uniformly point fibred. \hfill \Box

## 5 Final remarks

**A.** Let us recall the concept of topologically contractive iterated function system that was introduced by Mihail (see [19]) and by Tetenov (see [20,21]) under a different
name, namely self-similar topological structure satisfying condition (P). It is part of the last decades trend to establish purely topological conditions on an iterated function system in order to guarantee the existence of attractors.

**Definition 5.1** A pair \(((X, \tau), (f_i)_{i \in I})\) is called a topologically contractive iterated function system if:

i) \((X, \tau)\) is a topological space;

ii) \(I\) is a finite set;

iii) \(f_i : X \to X\) is continuous for every \(i \in I\);

iv) for every \(K \in P_{cp}(X)\) there exists \(C_K \in P_{cp}(X)\) such that \(K \subseteq C_K\) and \(\bigcup_{i \in I} f_i(C_K) \subseteq C_K\);

v) for every \(C \in P_{cp}(X)\) such that \(\bigcup_{i \in I} f_i(C) \subseteq C\) and every \(\omega \in \Lambda(I)\), the set \(\bigcap_{n \in \mathbb{N}^+} f_{\omega[n]}(C)\) is a singleton.

Particular cases of the above mentioned concept were considered by Edalat (see [6]) under the name of weakly hyperbolic iterated function systems and by Kieninger (see [12]) under the name of point fibred iterated function systems.

A comparison between conditions i) and ii) from Definition 2.17 and condition iv) from Definition 5.1 and between the conclusion of Proposition 3.1 and condition v) from Definition 5.1 shows that the concept of diameter diminishing to zero iterated function systems is a counterpart in terms of metric spaces of the one of topologically contractive iterated function system. Note that it is dealing with closed and bounded (not necessarily compact) sets.

**B.** Even though Propositions 3.3, 3.4 and 3.5 follow from Proposition 3.2 and Theorem 3.1 from [15] we presented their proofs as they are elementary, while Theorem 3.1 from [15] is complicated and nontrivial.

**C.** For an iterated function system \(S = (((X, d), (f_i)_{i \in I}))\) we can consider the following conditions:

(1’’) There exists a comparison function \(\varphi\) such that \(S\) is \(\varphi\)-contractive.

(2’’) \(S\) is locally uniformly point fibred.

(3’’) \(S\) is uniformly point fibred.

(4’’) \(S\) is diameter diminishing to zero iterated function system.

(5’’) \(S\) has an attractor and there exists a continuous function \(\pi : \Lambda(I) \to A_S\) such that \(\pi \circ \tau_i = f_i \circ \pi\) for all \(i \in I\).

Actually the proof of Theorem 4.1 ensures the validity of the following implications: (1’’) \(\Rightarrow\) (4’’), (4’’) \(\Rightarrow\) (3’’), (3’’) \(\Rightarrow\) (2’’), (2’’) \(\Rightarrow\) (5’’) and (5’’) \(\Rightarrow\) (2’’).

We raise the following question: is (2’’) \(\Rightarrow\) (4’’) valid?

If this is true, then we get the equivalence of (2’’), (3’’), (4’’) and (5’’).

**Acknowledgements** We want to thank the referees whose generous and valuable remarks brought improvements to the paper (specially by adding Sect. 5) and enhanced clarity.

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