Probabilistic representation of parabolic stochastic variational inequality with Dirichlet-Neumann boundary and variational generalized backward doubly stochastic differential equations

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Abstract

We derive the existence and uniqueness of the generalized backward doubly stochastic differential equation with sub-differential of a lower semi-continuous convex function under a non Lipschitz condition. This study allows us give a probabilistic representation (in stochastic viscosity sense) to the parabolic variational stochastic partial differential equations with Dirichlet-Neumann conditions.

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1 Introduction

Since the pioneering work of Pardoux and Peng in [17], many others work on connection between backward doubly stochastic differential equations (BDSDEs, for short) and semilinear parabolic and elliptic systems of second order stochastic partial differential equations (SPDEs, for short) have been established. Among these, the notion of stochastic viscosity solution for semi-linear SPDEs has been introduced firstly by Lions and Souganidis in [11, 12]. Their idea is to remove the stochastic integral from a SPDEs using a so-called "stochastic characteristic". A years later, two others definitions of stochastic viscosity solution of SPDEs have been considered respectively in [6, 7, 5] using the "Doss-Sussman" transformation to remove the stochastic integrals from a SPDEs.

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Using this similar approach, Boufoussi et al. [2] derive the stochastic viscosity solution of SPDEs with nonlinear Dirichlet-Neumann boundary condition via a so-called generalized BDSDEs.

More precisely, for $\phi \in C^3_b(\mathbb{R}^d)$, let $\Theta$ be an open connected and smooth bounded subset of $\mathbb{R}^d$ defined by

$$\Theta = \{x \in \mathbb{R}^d : \phi(x) > 0\}$$

and its boundary

$$Bd(\Theta) = \{x \in \mathbb{R}^d : \phi(x) = 0\}.$$ 

For any $x \in Bd(\Theta), \nabla \phi(x)$, the gradient of $\phi$ at $x$ is the unit normal vector pointing towards the interior of $\Theta$. The outward normal derivative at $u(t, .)$ at point $x \in Bd(\Theta)$ is

$$\Gamma u(t, x) = \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i}(x) \frac{\partial u}{\partial x_i}(t, x).$$  \hspace{1cm} (1.1)$$

Given continuous mappings: $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}, b : \mathbb{R}^d \to \mathbb{R}^d, f, g : \Omega \times [0, T] \times \Theta \to \mathbb{R}, \chi : \mathbb{R}^d \to \mathbb{R}$, we define the collection of second order PDE operators

$$Lu(t, x) = \sum_{i,j=1}^{d} (\sigma \sigma^*)(i,j)(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}(t, x)$$  \hspace{1cm} (1.2)$$

and consider the following second order semilinear parabolic SPDE

$$\begin{cases}
\frac{\partial u}{\partial t}(t, x) + Lu(t, x) + f(t, x, u(t, x)) + h(t, x, u(t, x)) = 0, (t, x) \in [0, T] \times \Theta, \\
\Gamma u(t, x) + g(t, x, u(t, x)) = 0, (t, x) \in [0, T] \times \partial \Theta, \\
u(T, x) = \chi(x), x \in \partial \Theta.
\end{cases}$$  \hspace{1cm} (1.3)$$

where $\frac{\partial u}{\partial t}$ denotes the "white noise" defined formally as the derivative of the Brownian motion.

In the case $h \equiv 0$ and functions $f, g$ are deterministic, SPDE (1.3) becomes a well-know PDE studied by Pardoux and Zhang [20].

For all $(t, x) \in [0, T] \times \Theta$, setting

$$u(t, x) = Y^{t,x}_t,$$

where $\{Y^{t,x}_s, s \in [t, T]\}$ satisfied the following decoupled forward generalized doubly backward stochastic differential equation (FBDSDE, for short)

$$\begin{cases}
X^{t,x}_s = x + \int_t^s b(X^{r,x}_r)dr + \int_t^s \nabla \phi(X^{r,x}_r)dA^{r,x}_r + \int_t^s \sigma(r, X^{r,x}_r)dW_r \\
Y^{t,x}_s = \chi(X^{T,x}_T) + \int_t^T f(r, X^{r,x}_r, Y^{r,x}_r, Z^{r,x}_r)dr \\
+ \int_t^T g(r, X^{r,x}_r, Y^{r,x}_r)dA^{r,x}_r + \int_t^T h(r, X^{r,x}_r, Y^{r,x}_r, Z^{r,x}_r)dB_r - \int_t^T Z^{r,x}_r dW_r
\end{cases}$$  \hspace{1cm} (1.4)$$

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Boufoussi et al. in [2] prove that \( u \) is the stochastic viscosity solution of (1.3).

In certain applications, the solution \( u(t,x) \) of SPDE (1.3) need to be maintained in two non-empty closed convex real sets \( D_1 \) and \( D_2 \) respectively for all \( x \in \Theta \) and \( x \in Bd(\Theta) \). Practically, this requirement can be realized by adding the supplementary sources \( \partial I_{D_i}(u(t,x)) \) and \( \partial I_{D_2}(u(t,x)) \) to the system, where \( \partial I_{D_1} \) and \( \partial I_{D_2} \) design respectively the sub-differential of the convex indicators functions \( I_{D_1} \) and \( I_{D_2} \) defined by: for \( i = 1,2 \),

\[
\mathbb{I}_{D_i}(y) = \begin{cases} 
0 & \text{if } y \in D_i \\
+\infty & \text{if } y \notin D_i.
\end{cases}
\]

These sources inward pushes process \( u(t,x) \) in \( D_1 \), for all \( x \in \Theta \) and in \( D_2 \), for all \( x \in \partial \Theta \) in a minimal way (i.e. only when \( u(t,x) \) arrives respectively on the boundary of \( D_1 \) and \( D_2 \)) such that \( \partial I_{D_1}(u(t,x)) \) and \( \partial I_{D_2}(u(t,x)) \) represent perfect feedback flux controls.

In this paper, we study an existence result for this parabolic variational inequality with mixed nonlinear Neumann-Dirichlet boundary condition:

\[
\begin{cases}
\frac{d\mathbf{u}}{dt}(t,x) + Lu(t,x) + f(t,x,u(t,x)) + h(t,x,u(t,x)) \partial \mathbf{u} \cdot \partial \mathbf{u} \in \partial \varphi(u(t,x)), (t,x) \in [0,T] \times \Theta, \\
\Gamma u(t,x) + g(t,x,u(t,x)) \in \partial \psi(u(t,x)), (t,x) \in [0,T] \times \partial \Theta), \\
u(T,x) = \chi(x), \ x \in \Theta,
\end{cases}
\]

(1.5)

where \( \partial \varphi \) and \( \partial \psi \) are sub-differential operators of the convex function \( \varphi \) and \( \psi \).

The special case (when \( h \equiv 0 \)) of SPDE (1.5) corresponds to now well-know variational PDE with Dirichlet-Neumann boundary condition studied in [15].

Our method is purely probabilistic and use a Markovian version of this following variational generalized backward doubly stochastic differential equations (VGBDSDEs, in short)

\[
\begin{cases}
dY_t + f(t,Y_t,Z_t)dt + g(t,Y_t)dA_t + h(t,Y_t,Z_t)d\mathbf{B}t \in \partial \varphi(Y_t)dt + \partial \psi(Y_t)dA_t + \mathbf{Z_t}dW_t, \\
Y_T = \xi
\end{cases}
\]

(1.6)

under "Main assumption" introduced by X. Mao in [13] and defined as follows:

There exist two constants \( C > 0 \) and \( 0 < \alpha < 1 \) such that

\[
\begin{align*}
|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 & \leq \rho(|y_1-y_2|^2) + C \|z_1-z_2\|^2, \\
|h(t,y_1,z_1) - h(t,y_2,z_2)|^2 & \leq \rho(|y_1-y_2|^2) + \alpha \|z_1-z_2\|^2,
\end{align*}
\]

where \( \rho(.) \) is continuous, concave and non-decreasing function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \) such that \( \rho(0) = 0, \) for all \( u > 0, \rho(u) > 0 \) and

\[
\int_{0^+} \frac{du}{\rho(u)} = +\infty.
\]
Several kinds of BSDEs have been established under this assumption and it relaxed version (see Owo and N’zi [16], Wang and Wang [24], Wang and Huang [23]). It is now well-know the continuity of the map \((t,x) \mapsto Y^{t,x}\) plays an important role for viscosity solution of SPDE (1.5). In this paper, we prove this result in Proposition 3.11. It can be view as a stochastic version of continuity result established by Pardoux and Răşcanu in [19]. Since the solution \(u\) of (1.5) is a stochastic fields, we use weak convergence to derive this continuity result.

The rest of this paper is organized as follows. In Section 2, we study VGBDSDEs (1.6) under "main assumptions". The last section derives and establishes stochastic viscosity solution for a one dimensional variation stochastic Dirichlet-Neumann problems in the special case (Lipschitz coefficients).

2 Variational Generalized Backward Doubly SDEs

This section aims to study VGBDSDEs (1.6) under above assumptions. In subsection 2.1, we give the statement of our study. We derive existence and uniqueness result in subsection 2.2 and subsection 2.3 is devoted to comparison theorem.

2.1 Preliminaries and notations

For a final time \(T > 0\), we consider \(\{W_t; 0 \leq t \leq T\}\) and \(\{B_t; 0 \leq t \leq T\}\) two standard Brownian motion defined respectively on complete probability spaces \((\Omega_1, \mathcal{F}_1, \mathbb{P}_1)\) and \((\Omega_2, \mathcal{F}_2, \mathbb{P}_2)\) with respectively \(\mathbb{R}^d\) and \(\mathbb{R}^I\) values. For any process \(\{K_t, t \in [0,T]\}\), we define the following family of \(\sigma\)-algebra \(\mathcal{F}^K_t = \sigma\{K_r - K_s, s \leq r \leq t\}\). In particular, \(\mathcal{F}^K_t = \mathcal{F}^K_{0,t}\). Next, we consider the product space \((\Omega, \mathcal{F}, \mathbb{P})\), where

\[
\Omega = \Omega_1 \times \Omega_2, \quad \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2, \quad \mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2
\]

and \(\mathcal{F}_t = \mathcal{F}^W_t \otimes \mathcal{F}^B_{T,t}\). We should note that since \(\mathcal{F}^W_t = (\mathcal{F}^W_t)_{t \in [0,T]}\) and \(\mathcal{F}^B_t = (\mathcal{F}^B_t)_{t \in [0,T]}\) are respectively increasing and decreasing filtration, the collection \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}\) is neither increasing nor decreasing such that it is not a filtration. Further, for random variables \(\zeta\) and \(\pi\) defined respectively in \(\Omega_1\) and \(\Omega_2\) are viewed as random variables on \(\Omega\) via the following identification:

\[
\zeta(\omega) = \zeta(\omega_1); \quad \pi(\omega) = \pi(\omega_2), \quad \omega = (\omega_1, \omega_2).
\]

Let \(\mathcal{M}^2(\mathcal{F}, [0,T]; \mathbb{R}^{d \times k})\) denote the set of \(d \mathbb{P} \otimes dt\) a.e. equal and \((d \times k)\)-dimensional jointly measurable random processes \(\{\varphi_t; 0 \leq t \leq T\}\) such that

\[
(i) \quad \|\varphi\|_{\mathcal{M}^2}^2 = \mathbb{E} \left( \int_0^T \|\varphi_t\|^2 dt \right) < +\infty
\]

\[
(ii) \quad \varphi_t \text{ is } \mathcal{F}_t\text{-measurable, for a.e. } t \in [0,T].
\]

We denote by \(\mathcal{S}^2(\mathcal{F}, [0,T]; \mathbb{R}^k)\) the set of continuous \(k\)-dimensional random processes such that

\[
(i) \quad \|\varphi\|_{\mathcal{S}^2}^2 = \mathbb{E}( \sup_{0 \leq t \leq T} |\varphi_t|^2 ) < +\infty
\]

\[
(ii) \quad \varphi_t \text{ is } \mathcal{F}_t\text{-measurable, for any } t \in [0,T].
\]
We denote also by $A^2(F, [0, T]; \mathbb{R}^k)$ the set of $d\mathbb{P} \otimes dA_t$ a.e. equal and $k$-dimensional jointly measurable random processes $\{\varphi_t; 0 \leq t \leq T\}$ such that

(i) $\|\varphi\|_{A^2}^2 = E \left( \int_0^T |\varphi_t|^2 dA_t \right) < +\infty$

(ii) $\varphi_t$ is $F_t$-measurable, for a.e. $t \in [0, T]$.

In the sequel, for simplicity, we shall set $S^2_\mathcal{F}(\mathbb{R}^k)$, $M^2_\mathcal{F}(\mathbb{R}^{d \times k})$ and $A^2_\mathcal{F}(\mathbb{R}^k)$ instead of $S^2(F, [0, T]; \mathbb{R}^k)$, $M^2(F, [0, T], \mathbb{R}^{d \times k})$ and $A^2(F, [0, T], \mathbb{R}^k)$ respectively.

The coefficients $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^k$, $h : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^{1 \times k}$, $g : \Omega \times [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}^k$, the terminal value $\xi : \Omega \rightarrow \mathbb{R}^k$ and $\varphi, \psi : \mathbb{R}^k \rightarrow \mathbb{R}$ satisfy the following conditions:

**H1** $(A_t)_{t \geq 0}$ denotes a continuous one-dimensional increasing $F$-measurable process such that $A_0 = 0$.

**H2** $\xi$ is a $F_T$-measurable random variable such that $E(|\xi|^2 + \varphi(\xi) + \psi(\xi)) < +\infty$.

**H3** There exist three constants $K > 0$, $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ such that

$$
\begin{cases}
(i) & f(.0,0) \in M^2(\mathbb{R}^k), \; h(.0,0) \in M^2(\mathbb{R}^{1 \times k}), \; g(s,0) \in A^2(\mathbb{R}^k), \\
(ii) & |f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \leq \rho(|y_1 - y_2|^2) + K ||z_1 - z_2||^2 \\
(iii) & |h(t,y_1,z_1) - h(t,y_2,z_2)|^2 \leq \rho(|y_1 - y_2|^2) + \alpha ||z_1 - z_2||^2 \\
(iv) & |y_1 - y_2, g(t,y_1) - g(t,y_2)| \leq \beta |y_1 - y_2|^2
\end{cases}
$$

for all $(t,y_i,z_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$ $i = 1,2$, and $\rho$ continuous, concave and non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ satisfies $\rho(0) = 0$, $\rho(u) > 0$ for all $u > 0$ and

$$
\int_0^+ \frac{du}{\rho(u)} = +\infty. \quad (2.1)
$$

**H4** There exist a constant $K' > 0$ and $\gamma, \eta : [0, +\infty[ \times \Omega \rightarrow [0, +\infty[$

$$
\begin{cases}
(i) & |f(t,y_1,z_1)| \leq \gamma + K'(|y_1| + ||z_1||) \\
(ii) & |g(t,y_1)| \leq \eta + K'|y_1|
\end{cases}
$$

for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$.

**H5** The functions $\varphi$ and $\psi$ are proper convex and lower semi-continuous such that $\varphi(y) \geq \varphi(0) = 0$, $\psi(y) \geq \psi(0) = 0$.

**Remark 2.1.** \( \text{(i) Lipschitz condition on generators } f, h \text{ with respect to the variable } y \text{ is the special case of (H3). In addition to the case of Lipschitz, there exist these two following examples } \rho_1 \text{ and } \rho_2 \text{ defined by: for } \delta \in (0,1) \text{ be sufficiently small,}

$$
\rho_1(u) = \begin{cases}
  u \log(u^{-1}), & 0 \leq u \leq \delta, \\
  \delta \log(\delta^{-1}) + \kappa_1(\delta)(u-\delta), & u > \delta
\end{cases}
$$

\( \text{and}} \)
where \( \partial \theta \)

Moreover, the subdifferential operator

For example, if \( \theta \)

\( Y \) osida approximation of their sub-differential operator. Let \( \phi \)

to

The quadruplet of processes

Definition 2.2. The quadruplet of processes \( (Y, U, V, Z) \) is called a solution of VGBDSDE \( (1.6) \) if

(i) \( (Y, Z, U, V) \in S^2(\mathbb{R}^k) \times M^2(\mathbb{R}^{d \times k}) \times M^2(\mathbb{R}^k) \times A^2(\mathbb{R}^k) \)

(ii) \( (Y_t, U_t) \in \partial \phi \) and \( (Y_t, V_t) \in \partial \psi \)

(iii)

\[
Y_t + \int_t^T U_s ds + \int_t^T V_s dA_s = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dA_s + \int_t^T h(s, Y_s, Z_s) d\bar{B}_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,
\]

Now, let us define some object relative to function \( \phi \) and \( \psi \) which permit us to understand the notion of Yosida approximation of their sub-differential operator. Let \( \theta : \mathbb{R}^k \rightarrow ]-\infty, +\infty[ \) be equal to \( \phi \) or \( \psi \), we define

\[
Dom(\theta) = \{ u \in \mathbb{R}^k : \theta(u) < +\infty \},
\]

\[
\partial \theta(u) = \{ u^* \in \mathbb{R}^k : \langle u^*, v - u \rangle + \theta(u) \leq \theta(v), \text{ for all } v \in \mathbb{R}^k \},
\]

\[
Dom(\partial \theta) = \{ u \in \mathbb{R}^k : \partial \theta(u) \neq \emptyset \},
\]

\[
(u, u^*) \in \partial \theta \iff u \in Dom(\theta), u^* \in \partial \theta(u).
\]

Moreover, the subdifferential operator \( \partial \theta \) is a maximal monotone operator, which means that

\[
\langle u - v, u^* - v^* \rangle \geq 0, \ \forall (u, u^*), (v, v^*) \in \partial \theta.
\]

For example, if \( \theta : \mathbb{R} \rightarrow ]-\infty, +\infty[ \), then every \( y \in Dom(\theta) \) we have

\[
\partial \theta(y) = \mathbb{R} \cap [\theta'_l(y), \theta'_r(y)],
\]

where \( \theta'_l(y) \) and \( \theta'_r(y) \) are respectively left and right derivative at \( y \).
Let now consider the Yosida approximation of the sub-differential operator $\partial \theta$ (see [4] and the references therein) as follows: for $\varepsilon > 0$,

$$\theta_\varepsilon(x) = \min_y \left( \frac{1}{2\varepsilon} |x - y|^2 + \theta(y) \right) = \frac{1}{2\varepsilon} |x - J_\varepsilon(x)|^2 + \theta(J_\varepsilon(x)),$$

with $J_\varepsilon(x) = (I + \varepsilon \partial \theta)^{-1}(x)$ called the resolvent of the monotone operator $\partial \theta$. Then, one can show that

$$\nabla \theta_\varepsilon(x) = \frac{x - J_\varepsilon(x)}{\varepsilon}.$$

The following results come from [4] or [18].

**Proposition 2.3.**

(i) The function $\theta_\varepsilon$ is convex with a Lipschitz continuous derivatives

(ii) for all $x \in \mathbb{R}^k$, $\nabla \theta_\varepsilon(x) = \frac{1}{\varepsilon} (x - J_\varepsilon(x)) \in \partial \theta(J_\varepsilon(x));$

(iii) for all $x, y \in \mathbb{R}^k$, $|\nabla \theta_\varepsilon(x) - \nabla \theta_\varepsilon(y)| \leq \frac{1}{\varepsilon} |x - y|;$

(iv) for all $x, y \in \mathbb{R}^k$, $(\nabla \theta_\varepsilon(x) - \nabla \theta_\varepsilon(y), x - y) \geq 0;$

(v) for all $x, y \in \mathbb{R}^k$ and $\varepsilon, \delta > 0$, $(\nabla \theta_\varepsilon(x) - \nabla \theta_\delta(y), x - y) \geq -(\varepsilon + \delta) \langle \nabla \theta_\varepsilon(x), \nabla \theta_\delta(y) \rangle$.

To end this subsection, let us set Bihari’s inequality which is useful in the proof of existence and uniqueness.

**Proposition 2.4.** Let $u$ and $f$ be non-negative continuous functions defined on $[0, +\infty)$, and let $w$ be a continuous non-decreasing function defined on $[0, +\infty)$ and $w(u) > 0$ on $(0, +\infty)$. If $u$ satisfies the following integral inequality,

$$u(t) \leq \alpha + \int_0^t f(s)w(u(s))ds$$

where $\alpha$ is a non-negative constant, then

$$u(t) \leq G^{-1} \left( G(\alpha) + \int_0^t f(s)ds \right),$$

with the function $G$ is defined by

$$G(x) = \int_{x_0}^x \frac{1}{w(y)}dy, \quad x > 0, \quad x_0 > 0.$$

### 2.2 Existence and uniqueness result

The real difficulty to derive the existence and uniqueness result for VGBDSDEs (1.6) resides in the fact that when we use the Yosida approximations, we obtain this following doubly BSDEs

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s)dA_s + \int_t^T h(s,Y_s,Z_s)dB_s - \int_t^T Z_sdW_s,$$

introduced in [2] by Boufoussi et al. Under at least Lipschitz assumption on the coefficients, they establish existence and uniqueness result. Unfortunately, this assumption doesn’t work in our context. Therefore before starting our main existence and uniqueness result, we need to establish a general existence and uniqueness result for GBDSDEs (2.3) under "Main assumptions".
**Theorem 2.5.** Assume assumptions (H1)-(H3) hold. Then, GBDSDEs (2.3) has a unique solution.

**Remark 2.6.** We emphasize that when \( g \equiv 0 \), equation (2.3) becomes the classical backward doubly stochastic equations (BDSDEs, in short) introduced by Pardoux and Peng [17] under Lipschitz generator and recently relax by many authors. We can see in [16], Owo and N’zi’s work and references therein.

**Proof.**

(i) **Existence**

To prove existence result, we follow the well-know idea by constructing a Picard scheme and show its convergence. In this fact, let set \( Y^n_0 = 0 \) and define for \( n \in \mathbb{N} \) recursively

\[
Y^n_t = \xi + \int_t^T f(s,Y^{n-1}_s,Z^n_s)ds + \int_t^T g(s,Y^n_s)dA_s + \int_t^T h(s,Y^{n-1}_s,Z^n_s)dB_s - \int_t^T Z^ndW_s, \quad 0 \leq t \leq T. \tag{2.4}
\]

Let us remark that for \( Y^{n-1} \in S^2(\mathbb{R}^k) \), generators \( f \) and \( h \) depend only of \( z \) and are Lipschitz. Therefore thanks to Theorem 2.1 in [2], BDSDEs (2.4) admits a unique solution \( (Y^n,Z^n) \in S^2(\mathbb{R}^k) \times M^2(\mathbb{R}^{d \times k}) \). Our purpose is to prove that the sequence of processes \( (Y^n,Z^n) \) converges in \( S^2(\mathbb{R}^k) \times M^2(\mathbb{R}^{d \times k}) \) and its limit solves GBDSDEs (2.3). The proof is subdivided into four steps

**Step 1:** There exist \( T_1 \in [0,T) \) and \( M_1 \geq 0 \) such that for each \( n \geq 1 \),

\[
\mathbb{E} \left( |Y^n_t|^2 + \int_t^T |Y^n_s|^2 dA_s + \int_t^T \|Z^n_s\|^2 ds \right) \leq M_1, \quad \forall t \in [T_1,T]. \tag{2.5}
\]

Applying again the generalized Itô’ formula, we obtain

\[
\mathbb{E}(|Y^n_t|^2) + \int_t^T \|Z^n_s\|^2 ds = \mathbb{E}(|\xi|^2) + 2\mathbb{E} \left[ \int_t^T \langle Y^n_s, f(s,Y_s^{n-1},Z^n_s) \rangle ds \right]
\]

\[
+ 2\mathbb{E} \left[ \int_t^T \langle Y^n_s, g(s,Y^n_s) \rangle dA_s \right]
\]

\[
+ 2\mathbb{E} \left[ \int_t^T \|h(s,Y_s^{n-1},Z^n_s)\|^2 ds \right]
\]

There exist by Young inequality a constants \( \lambda > 0 \) such that

\[
2\langle Y^n_s, f(s,Y_s^{n-1},Z^n_s) \rangle = 2\langle Y^n_s, f(s,Y_s^{n-1},Z^n_s) - f(s,0,0) \rangle + 2\langle Y^n_s, f(s,0,0) \rangle
\]

\[
\leq \left( \frac{1}{\lambda} + 1 \right) |Y^n_s|^2 + \lambda \rho(|Y_s^{n-1}|^2) + \lambda K \|Z^n_s\|^2 + |f(s,0,0)|^2,
\]

and

\[
\|h(s,Y_s^{n-1},Z^n_s)\|^2 \leq (1+\lambda)\|h(s,Y_s^{n-1},Z^n_s) - h(s,0,0)\|^2 + (1 + \frac{1}{\lambda}) \|h(0,0)\|^2
\]

\[
\leq (1+\lambda)\rho(|Y_s^{n-1}|^2) + \alpha(1+\lambda) \|Z^n_s\|^2 + (1 + \frac{1}{\lambda}) \|h(s,0,0)\|^2.
\]
For the second term of the right side since $\beta$ appear in $(\text{H}3)$ is negative, we have

$$\begin{align*}
2(Y^n, g(s, Y^n)) &= 2(Y^n, g(s, Y^n) - g(s, 0)) + 2(Y^n, g(s, 0)) \\
&\leq 2|\beta||Y^n|^2 + |\beta||Y^n|^2 + \frac{1}{|\beta|}|g(s, 0)|^2 \\
&= -|\beta||Y^n|^2 + \frac{1}{|\beta|}|g(s, 0)|^2.
\end{align*}$$

Then, it follows by choosing $0 < \lambda < \frac{1}{\alpha + \kappa}$ there exist $0 < \alpha_1 = 1 - (\alpha(1 + \lambda) + \lambda\kappa)$ and $\mu = 1 + \frac{1}{\kappa}$ such that

$$\begin{align*}
\mathbb{E} \left[ |Y_t^n|^2 + |\beta| \int_t^T |Y^n_s|^2 dA_s + \alpha \int_t^T ||Z^n_s||^2 ds \right] \\
&\leq \mathbb{E} \left[ \mu \int_t^T |Y^n_s|^2 ds + (1 + 2\lambda) \int_t^T \rho(|Y^n-1|^2) ds \right] \\
&\quad + CE \left[ ||\xi||^2 + \int_t^T (|f(s, 0, 0)|^2 + ||h(s, 0, 0)||^2) ds + \int_t^T |g(s, 0)|^2 dA_s \right] + \rho(\mathbb{E}(|Y^n_t|^2)) ds,
\end{align*}$$

(2.7)

Therefore Gronwall’s lemma yields

$$\mathbb{E}(|Y^n_t|^2) \leq \Gamma_t + M \int_t^T \rho(\mathbb{E}(|Y^n_s|^2)) ds,$$

where

$$\Gamma_t = e^{tT} C \mathbb{E} \left[ ||\xi||^2 + \int_t^T (|f(s, 0, 0)|^2 + ||h(s, 0, 0)||^2) ds + \int_t^T |g(s, 0)|^2 dA_s \right] < +\infty$$

and $M = (1 + 2\lambda)e^{tT}$. In view of (2.7), the proof of (2.5) require an induction method.

For this, recalling again (2.7), since $Y^0 \equiv 0$, $\rho(0) = 0$ and (H3)-(i), we have

$$\begin{align*}
\mathbb{E}(|Y^1_t|^2) &\leq \Gamma_t + M \int_t^T \rho(\mathbb{E}(|Y^0_s|^2)) ds \\
&\leq 2\Gamma_0 \\
&< +\infty.
\end{align*}$$

Since $\rho$ is non-decreasing, we have

$$\begin{align*}
\mathbb{E}(|Y^2_t|^2) &\leq \Gamma_t + M \int_t^T \rho(\mathbb{E}(|Y^1_s|^2)) ds \\
&\leq \Gamma_0 + M(T - t)\rho(2\Gamma_0).
\end{align*}$$

By virtue of (H3), $T\rho(2\Gamma_0)$ is finite so that we can find $T_1 \in [0, T)$ depends only on $\rho$ such that $M(T - T_1)\rho(2\Gamma_0) \leq \Gamma_0$. Then for $t \in [T_1, T]$, we have

$$\begin{align*}
\mathbb{E}(|Y^2_t|^2) &\leq \Gamma_0 + M(T - T_1)\rho(2\Gamma_0) \\
&\leq 2\Gamma_0.
\end{align*}$$
For $t \in [T_1, T]$, we suppose that there exists $n \in \mathbb{N}^*$ such that

$$
\mathbb{E}(\|Y_t^n\|^2) \leq M_1,
$$

where $M_1 = 2\Gamma_0$. Next, it follows from inequalities (2.7) and (2.8), and the non increase of $\rho$ that for $t \in [T_1, T]$

$$
\mathbb{E}(\|Y_t^{n+1}\|^2) \leq \Gamma_0 + M \int_t^T \rho(\mathbb{E}(\|Y_s^{n+1}\|^2))ds
\leq \Gamma_0 + M(T-T_1)\rho(M_1)
\leq M_1,
$$

which according to (2.6) prove Step 1.

**Step 2:** For $n, m \in \mathbb{N}$, we claim that there exist $\bar{M} > 0$ such that

$$
\mathbb{E}\left[\|Y_t^{n+m} - Y_t^n\|^2 + \int_t^T \|Z_t^{n+m} - Z_t^n\|^2 ds\right] \leq \bar{M} \int_t^T \rho(\mathbb{E}(\|Y_s^{n+m-1} - Y_s^{n-1}\|^2))ds.
$$

Indeed, for any $\mu$, and in view of Itô’s formula, we get

$$
\mathbb{E}(e^{\mu A_t} |Y_t^{n+m} - Y_t^n|^2) + \mu \int_t^T e^{\mu A_s} |Y_s^{n+m} - Y_s^n|^2 ds + \int_t^T e^{\mu A_s} \|Z_s^{n+m} - Z_s^n\|^2 ds
= 2\mathbb{E}\left[\int_t^T e^{\mu A_s} (Y_s^{n+m} - Y_s^n, f(s, Y_s^{n+m-1}, Z_s^{n+m}) - f(s, Y_s^{n-1}, Z_s^n))ds\right]
+ 2\mathbb{E}\left[\int_t^T e^{\mu A_s} (Y_s^{n+m} - Y_s^n, g(s, Y_s^{n+m}) - g(s, Y_s^n)ds)\right]
+ 2\mathbb{E}\left[\int_t^T e^{\mu A_s} \|h(s, Y_s^{n+m-1}, Z_s^{n+m}) - h(s, Y_s^{n-1}, Z_s^n)\|^2 ds\right]
$$

Choosing $\mu = |\beta|$, it follows from (H3) and same estimates as in step 1, that

$$
\mathbb{E}(e^{\mu A_t} |Y_t^{n+m} - Y_t^n|^2) + \alpha \int_t^T e^{\mu A_s} \|Z_s^{n+m} - Z_s^n\|^2 ds
\leq \mathbb{E}\left[\frac{1}{\lambda} \int_t^T e^{\mu A_s} |Y_s^{n+m} - Y_s^n|^2 ds\right]
+ (1 + \lambda) \int_t^T \rho(\mathbb{E}(e^{\mu A_s} |Y_s^{n+m-1} - Y_s^{n-1}|^2))ds,
$$

where $\alpha = 1 - \lambda \kappa - \alpha$. Choosing $\lambda$ such that $\lambda > 0$, it follows from Gronwall’s and Jensen’s inequalities that

$$
\mathbb{E}(e^{\mu A_t} |Y_t^{n+m} - Y_t^n|^2) \leq \bar{M} \int_t^T \rho(\mathbb{E}(e^{\mu A_t} |Y_s^{n+m-1} - Y_s^{n-1}|^2))ds,
$$

where $\bar{M} = e^{\xi(1+\lambda)}$.

**Step 3:** The sequence of process $(Y^n, Z^n)$ converge in $S^2(\mathbb{R}^k) \times M^2(\mathbb{R}^{d \times k})$
For $M$ and $M_1$ obtained in Step 1, let consider $(\phi_n)_{n \geq 1}$ the sequence of processes defined recursively by

$$\phi_1(t) = M \rho (M_1) (T - t), \quad \phi_{n+1}(t) = M \int_t^T \rho (\phi_n(s)) ds, \quad t \in [0, T]$$

and $(\phi_{n,m})_{n,m \geq 1}$ defined by

$$\phi_{n,m}(t) = E(e^{\mu t} | Y_t^{n+m} - Y_t^n|^2), \quad t \in [T_1, T].$$

For any $m \geq 1$ and all $n \geq 1$, let us prove by induction method on $n$ that

$$\phi_{n,m}(t) \leq \phi_n(t) \leq \phi_{n-1}(t) \leq \cdots \leq \phi_1(t), \quad t \in [T_1, T]. \tag{2.11}$$

First, in view of Step 2 and Step 1,

$$\phi_{1,m}(t) = E(e^{\mu t} | Y_t^{1+m} - Y_t^1|^2) \leq \bar{M} \int_t^T \rho (E(e^{\mu s} | Y_s^m(s)^2)) ds \leq M(T-t) \rho (M_1) = \phi_1(t). \tag{2.12}$$

Now recall Step 2, it follows from (3.18) that

$$\phi_{2,m}(t) = E(e^{\mu t} | Y_t^{2+m} - Y_t^2|^2) \leq \bar{M} \int_t^T \rho (E(e^{\mu s} | Y_s^{1+m}(s) - Y_s^1(t)^2)) ds$$

$$= \bar{M} \int_t^T \rho (\phi_{1,m}(s)) ds \leq M \int_t^T \rho (\phi_1(s)) ds = \phi_2(t).$$

On other hand, since $\phi_1(t) = M(T-t) \rho (M_1) \leq M_1$,

$$\phi_2(t) = M \int_t^T \rho (\phi_1(s)) ds \leq M(T-t) \rho (M_1) = \phi_1(t).$$

In the other word, we get

$$\phi_{2,m}(t) \leq \phi_2(t) \leq \phi_1(t), \quad t \in [T_1, T].$$

Next, let us assume that (2.11) holds for some $n \geq 2$. Using again Step 2, we get for all $t \in [T_1, T]$,

$$\phi_{n+1,m}(t) \leq M \int_t^T \rho (\phi_{n,m}(s)) ds \leq M \int_t^T \rho (\phi_n(s)) ds = \phi_{n+1}(t) \leq \int_t^T \rho (\phi_{n-1}(s)) ds = \phi_n(t),$$

which proved (2.11) for $n + 1$. Therefore using induction principle, (2.11) holds. Moreover, for $t, t' \in [T_1, T]$, we have

$$\sup_{n \geq 0} |\phi_n(t) - \phi_n(t')| \leq M \rho (M_1) |t - t'|. \tag{2.13}$$
that proves that the decreasing sequence \((\phi_n)_{n \geq 0}\) is uniformly equicontinuous. Therefore, in view of Ascoli-Arzela theorem, it converge to a limit \(\phi\) as \(n\) goes to \(+\infty\) such that

\[
\phi(t) = M \int_t^T \rho(\phi(s))ds.
\]

Hence with the help of Bihari’s inequality we have \(\phi \equiv 0\) on \([T_1, T]\). Finally according to \((3.17)\) and \((2.10)\) \((Y^n, Z^n)\) is a Cauchy sequence in \(\mathcal{M}^2([T_1, T], \mathbb{R}^k) \times \mathcal{M}^2([T_1, T], \mathbb{R}^{d \times k})\). Furthermore by adapted calculus,

\[
\mathbb{E}\left( \sup_{T_1 \leq t \leq T} e^{\mu_A t} |Y^n_{t} - Y^n_{t_0}|^2 \right) \leq C \mathbb{E} \left[ \int_t^T e^{\mu_A s} |Y^n_{s} - Y^n_{s_0}|^2 ds + \phi_n(T_1) \right. \\
+ \left. \int_t^T e^{\mu_A s} \|Z^n_{s} - Z^n_{s_0}\|^2 ds \right],
\]

which implies that \((Y^n)\) is also a Cauchy sequence in a classical Banach space \(S^2([T_1, T], \mathbb{R}^k)\). Therefore there exist a process \((Y, Z) \in S^2([T_1, T], \mathbb{R}^k) \times \mathcal{M}^2([T_1, T], \mathbb{R}^{d \times k})\) such that for \(t \in [T_1, T] \),

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} e^{\mu_A t} |Y^n_{t} - Y_{t_0}|^2 + \int_t^T e^{\mu_A s} \|Z^n_{s} - Z_{s_0}\|^2 ds \right) \to 0 \ \text{as} \ \ n \to +\infty.
\]  

\textit{Step 4:} \((Y, Z)\) solves GBDSDE \((2.3)\).

Letting \(n \to +\infty\) in \((2.4)\) provides that \((Y, Z)\) solves GBDSDE \((2.3)\) on \([T_1, T]\). If \(T_1 = 0\), then we have proved existence result. But if \(T_1 \neq 0\), we must continuous by study now the existence result for this equation

\[
Y_t = Y_{T_1} + \int_t^{T_1} f(s, Y_s, Z_s)ds + \int_t^{T_1} g(s, Y_s)dA_s + \int_t^{T_1} h(s, Y_s, Z_s)d\widehat{B}_s - \int_t^{T_1} Z_s dB_s, \ \ 0 \leq t \leq T_1.
\]

With the same analysis, there exist \(T_2 \in [0, T_1]\) and a process that also denoted \((Y, Z)\) defined on \([T_2, T_1]\) solution of \((2.16)\). If \(T_2 = 0\) the proof of existence is complete. Otherwise, we repeat the previous analysis. Therefore we built a decreasing sequence \((T_p)_{p\geq1}\) such that on each \([T_p, T_{p+1}]\), GBDSDE associated to the data \((Y_{p+1}, f, g, h)\) admit a solution. We also prove as like in \((16)\) that there exists a finite \(p\) such that \(T_p = 0\). Finally by the path continuity of the solution on each interval, we obtain the solution on \([0, T]\) and end the proof of existence.

\textbf{(ii) Uniqueness}

Let \((Y, Z)\) and \((\bar{Y}, \bar{Z})\) be two solutions of GBDSDEs \((2.3)\). By use the now classical computation as above (Step 1 and Step 3) we have

\[
\mathbb{E}\left[ e^{\mu_A t} |Y_t - \bar{Y}_t|^2 + \int_t^T e^{\mu_A s} \|Z_s - \bar{Z}_s\|^2 ds \right] \leq C \int_t^T \rho(\mathbb{E}(e^{\mu_A s} |Y_s - \bar{Y}_s|^2))ds,
\]

which yields by Bihari’s inequality \(Y = \bar{Y}\) and also \(Z = \bar{Z}\).

\[\square\]

\textbf{Remark 2.7.} Let consider the following assumption
(H3')

\[
(i) \quad f(\cdot, 0, 0) \in M^2(\mathbb{R}), \quad h(\cdot, 0, 0) \in M^2(\mathbb{R}^d), \quad g(\cdot, 0) \in A^2(\mathbb{R}),
\]

\[
(ii) \quad |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq \rho(t, |y_1 - y_2|^2) + C |z_1 - z_2|^2,
\]

\[
(iii) \quad |h(t, y_1, z_1) - h(t, y_2, z_2)|^2 \leq \rho(t, |y_1 - y_2|^2) + \alpha |z_1 - z_2|^2,
\]

\[
(iv) \quad (y_1 - y_2, (g(t, y_1) - g(t, y_2)) \leq \beta |y_1 - y_2|^2
\]

for all \((t, y_i, z_i) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, i = 1, 2\), where \(C > 0, 0 < \alpha < 1\) and \(\beta \in \mathbb{R}\) are three constants and \(\rho : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+\) satisfies:

(a) for fixed \(t \in [0, T]\), \(\rho(t, \cdot)\) is continuous, concave and non-decreasing such that \(\rho(0) = 0\)

(b) for all \(u \geq 0\), \(\int_0^T \rho(t, u)dt < +\infty\),

(c) For any \(M > 0\), the following ODE

\[
\begin{cases}
  u'(t) = -Mp(t, u) \\
  u(T) = 0
\end{cases}
\]

has a unique solution \(u(t) = 0, t \in [0, T]\).

If we replace the hypothesis (H3) by (H3'), then Theorem 2.5 still works. In the evidence, it would have adjustments in the proof to take account the fact that \(\rho\) depends on \(t\). In this case the Bihari inequality could no longer be applied and will have to be replaced by (H3')-(c). We refer the reader to the works of Owo and N’zi [16] for more details.

Now we are ready to give the main result of this section.

**Theorem 2.8.** Assume the assumptions (H1)-(H5) hold. Then, MGBDSDEs (1.6) has a unique solution.

**Proof.** (i) Existence

Let follow the well-know idea by Yosida approximation to built a following GBDSDE

\[
Y^\varepsilon_t + \int_t^T \nabla \psi(Y^\varepsilon_s) ds + \int_t^T \nabla \phi(Y^\varepsilon_s) dA_s = \xi + \int_t^T f(s, Y^\varepsilon_s, Z^\varepsilon_s) ds + \int_t^T g(s, Y^\varepsilon_s) dA_s
\]

\[
+ \int_t^T h(s, Y^\varepsilon_s, Z^\varepsilon_s) dB_s - \int_t^T Z^\varepsilon_s dW_s, \quad (2.17)
\]

where \(\xi, f, h, g, \phi, \psi\) satisfy (H1)-(H4). Therefore in virtue of Theorem 2.5 GBDSDE (2.17) admits a unique solution \((Y^\varepsilon, Z^\varepsilon)\).

**Remark 2.9.** Since \(\nabla \psi\) and \(\nabla \phi\) are monotone, it follows from comparison theorem

Our goal is to provide that the family of processes \((Y^\varepsilon, Z^\varepsilon)_{\varepsilon > 0}\) converges and its limit is solution of BDSGVI (1.6). In the sequel, \(C > 0\) is a constant which can change its value from line to line.

**Step 1: A first priori estimate**

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^\varepsilon_t|^2 + \int_0^T (|Y^\varepsilon_t|^2 dA_t + ||Z^\varepsilon_t||^2 dt \right] \leq C.
\]
By using a generalized Itô’s formula (cf. [17] Lemma 1.3) and taking expectation, we get,

\[
\mathbb{E} \left[ |Y_t^e|^2 + \int_t^T \|Z_s^e\|^2 ds + \int_t^T \langle Y_s^e, \nabla \varphi_e(Y_s^e) \rangle ds + \int_t^T \langle Y_s^e, \nabla \psi_e(Y_s^e) \rangle dA_s \right] = \mathbb{E} \left[ |\xi_t^e|^2 + 2 \int_t^T \langle Y_s^e, f(s, Y_s^e, Z_s^e) \rangle ds + 2 \int_t^T \langle Y_s^e, g(s, Y_s^e) \rangle dA_s + 2 \int_t^T \|h(s, Y_s^e, Z_s^e)\|^2 ds \right].
\]

From (iv) of Proposition 2.3, we obtain

\[
\mathbb{E} \left[ |Y_t^e|^2 + \int_t^T \|Z_s^e\|^2 ds \right] \leq \mathbb{E} \left[ |\xi_t^e|^2 + 2 \int_t^T \langle Y_s^e, f(s, Y_s^e, Z_s^e) \rangle ds + 2 \int_t^T \langle Y_s^e, g(s, Y_s^e) \rangle dA_s + 2 \int_t^T \|h(s, Y_s^e, Z_s^e)\|^2 ds \right]
\]

On the other hand, Schwartz’s inequality and assumptions (H3) imply that for all \( r, r' > 0 \)

\[
2 \langle y, f(s, y, z) \rangle = \langle y, f(s, y, z) - f(s, 0, 0) \rangle + \langle y, f(s, 0, 0) \rangle \\
\leq r p(|y|^2) + \left( \frac{K}{r} + 1 \right) |y|^2 + r |z|^2 + |f(s, 0, 0)|^2,
\]

\[
2 \langle y, g(s, y) \rangle = 2 \langle y, g(s, y) - g(s, 0) \rangle + 2 \langle y, g(s, 0) \rangle \\
\leq (2\beta + |\beta|)|y|^2 + \frac{1}{|\beta|}|g(s, 0)|^2
\]

and

\[
\|h(s, y, z)\|^2 \leq (1 + \frac{1}{r'}) \|h(s, y, z) - h(s, 0, 0)\|^2 + (1 + r') \|h(s, 0, 0)\|^2 \\
\leq (1 + \frac{1}{r'}) p(|y|^2) + \alpha (1 + \frac{1}{r'}) |z|^2 + (1 + r') \|h(s, 0, 0)\|^2.
\]

Choosing \( r = \frac{1}{2K} \) and \( r' = \frac{3\alpha}{1-\alpha} \) and since \( \beta < 0 \), we get

\[
\mathbb{E} \left[ |Y_t^e|^2 + |\beta| \int_t^T |Y_s^e|^2 dA_s + \frac{1-\alpha}{6} \int_t^T \|Z_s^e\|^2 ds \right] \\
\leq C \mathbb{E} \left[ |\xi_t^e|^2 + \int_t^T (|Y_s^e|^2 + \rho(|Y_s^e|^2)) ds \\
+ \int_t^T (|f(s, 0, 0)|^2 + \|h(s, 0, 0)\|^2) ds + \int_t^T |g(s, 0)|^2 dA_s \right].
\]

Finally, Bihari-LaSalle inequality yields

\[
\sup_{0 \leq t \leq T} \mathbb{E}(|Y_t^e|^2) + C \mathbb{E} \left( \int_t^T |Y_s^e|^2 dA_s + \int_t^T \|Z_s^e\|^2 dt \right) \\
\leq C \mathbb{E} \left[ |\xi_t^e|^2 + \int_t^T (|f(s, 0, 0)|^2 + \|h(s, 0, 0)\|^2) ds + \int_t^T |g(s, 0)|^2 dA_s \right].
\]
Furthermore, applying again generalized Itô’s formula together with Burkholder-Davis-Gundy’s inequality, we obtain

\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^e|^2 \right) \leq C \mathbb{E}\left[|\xi|^2 + \int_0^T (|f(s,0,0)|^2 + \|h(s,0,0)\|^2)ds + \int_0^T |g(s,0)|^2dA_s \right]
\]

\[
+ C \mathbb{E}\left(\int_0^T |Y_t^e|^2 ||Z_t^e||^2 dt \right)^{1/2} + C \mathbb{E}\left(\int_0^T |Y_t^e||h(s,Y_s^e,Z_s^e)||^2 dt \right)^{1/2}
\]

(2.18)

We estimate the last term as follows

\[
C \mathbb{E}\left(\int_0^T |Y_t^e|^2 ||h(s,Y_s^e,Z_s^e)||^2 ds \right)^{1/2}
\]

\[
\leq \frac{1}{4} \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^e|^2 \right) + C \mathbb{E}\left(\int_0^T ||h(s,Y_s^e,Z_s^e)||^2 ds \right)
\]

\[
\leq \frac{1}{4} \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^e|^2 \right) + C \mathbb{E}\left[|\xi|^2 + \int_0^T (|f(s,0,0)|^2 + \|h(s,0,0)\|^2)ds + \int_0^T |g(s,0)|^2dA_s \right]
\]

\[
+ C \int_0^T \rho(\mathbb{E}(\sup_{0 \leq u \leq s} |Y_u^e|^2))ds.
\]

The second term on the right side of (2.18) is treated analogously to obtain

\[
C \mathbb{E}\left(\int_0^T |Y_t^e|^2 ||Z_t^e||^2 dt \right)^{1/2}
\]

\[
\leq \frac{1}{4} \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^e|^2 \right)
\]

\[
+ C \mathbb{E}\left[|\xi|^2 + \int_0^T (|f(s,0,0)|^2 + \|h(s,0,0)\|^2)ds + \int_0^T |g(s,0)|^2dA_s \right].
\]

Therefore we deduce

\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^e|^2 \right) \leq C \mathbb{E}\left[|\xi|^2 + \int_0^T (|f(s,0,0)|^2 + \|h(s,0,0)\|^2)ds + \int_0^T |g(s,0)|^2dA_s \right]
\]

\[
+ \int_0^T \rho(\mathbb{E}(\sup_{0 \leq u \leq s} |Y_u^e|^2))ds,
\]

which by using again Bihari-LaSalle inequality provides

\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^e|^2 \right) \leq C \mathbb{E}\left[|\xi|^2 + \int_0^T (|f(s,0,0)|^2 + \|h(s,0,0)\|^2)ds + \int_0^T |g(s,0)|^2ds \right].
\]

Finally according to (H2) and (H3) we get (2.22).

**Step 2: A second priori estimate**
We have for all \( t \in [0, T] \),

\[
(a) \quad \mathbb{E} \left( \int_0^T |\nabla \varphi_e(Y^e_s)|^2 \, ds + \int_0^T |\nabla \psi_e(Y^e_s)|^2 \, dA_s \right) \leq C \Lambda \\
(b) \quad \mathbb{E} \left( \varphi(J_e(Y^e)) + \psi(J_e(Y^e)) + \int_0^T \varphi(J_e(Y^e)) \, dt + \int_0^T \psi(J_e(Y^e)) \, dA_s \right) \leq C \Lambda, \tag{2.19}
\]

\[
(c) \quad \mathbb{E} \left( |Y^e_t - J_e(Y^e)|^2 + |Y^e_t - \bar{J}_e(Y^e)|^2 \right) \leq \varepsilon C \Lambda,
\]

Like in Pardoux and Răşcanu in [18] (see Proposition 2.2), For \( t = t_0 < t_1 < \cdots < t_n = T \), where \( t_i - t_{i-1} = \frac{1}{n} \), let write the subdifferential inequality

\[
\varphi_e(Y^e_{t_{i+1}}) \geq \varphi_e(Y^e_{t_i}) + \langle \nabla \varphi(Y^e), Y^e_{t_{i+1}} - Y^e_{t_i} \rangle
\]

and

\[
\psi_e(Y^e_{t_{i+1}}) \geq \psi_e(Y^e_{t_i}) + \langle \nabla \psi(Y^e), Y^e_{t_{i+1}} - Y^e_{t_i} \rangle.
\]

Using the approximation equation \( \text{(2.17)} \), summing up over \( i \), and passing to the limit as \( n \) goes to \( +\infty \), we deduce

\[
\varphi_e(Y^e_t) + \psi_e(Y^e_t) + \int_t^T |\nabla \varphi_e(Y^e_s)|^2 \, ds + \int_t^T |\nabla \psi_e(Y^e_s)|^2 \, dA_s
\]

\[
+ \int_t^T \langle \nabla \varphi_e(Y^e_s), \nabla \psi_e(Y^e_s) \rangle (ds + dA_s)
\]

\[
\leq \varphi_e(\xi) + \psi_e(\xi) + \int_t^T \langle \nabla \varphi_e(Y^e_s) + \nabla \psi_e(Y^e_s), f(s, Y^e_s, Z^e_s) \rangle \, ds
\]

\[
+ \int_t^T \langle \nabla \varphi_e(Y^e_s) + \nabla \psi_e(Y^e_s), g(s, Y^e_s) \rangle \, dA_s + \int_t^T \langle \nabla \varphi_e(Y^e_s) + \nabla \psi_e(Y^e_s), h(s, Y^e_s, Z^e_s) \rangle dB_s
\]

\[
- \int_t^T \langle \nabla \varphi_e(Y^e_s) + \nabla \psi_e(Y^e_s), Z^e_s \rangle \, dW_s. \tag{2.20}
\]

From (H3), for \( \theta = \varphi \) or \( \psi \) we get

\[
\langle \nabla \theta_e(y), f(s, y, z) \rangle \leq \frac{1}{4} |\nabla \theta_e(y)|^2 + K (\gamma_s^2 + |y|^2 + \|z\|^2)
\]

and

\[
\langle \nabla \theta_e(y), g(s, y) \rangle \leq \frac{1}{4} |\nabla \theta_e(y)|^2 + K (\eta_s^2 + |y|^2).
\]

Hence taking expectation in (2.20), we obtain

\[
\mathbb{E} \left[ \int_t^T |\nabla \varphi_e(Y^e_s)|^2 \, ds + |\nabla \psi_e(Y^e_s)|^2 \, dA_s \right]
\]

\[
\leq C \mathbb{E} \left[ \varphi(\xi) + \psi(\xi) + \sup_{0 \leq s \leq T} |Y^e_t|^2 + \int_0^T \|Z^e_t\|^2 \, dt
\]

\[
+ \int_0^T |Y^e_t|^2 \, dA_t + \int_0^T \gamma^2_t \, ds + \int_0^T \eta^2_t \, dA_t \right], \tag{2.21}
\]

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where we have used the fact that $\varphi_e(Y^e_t) + \psi_e(Y^e_t) > 0$ and $\varphi_e(\xi) + \psi_e(\xi) \leq e(\varphi(\xi) + \psi(\xi))$ successively. Therefore (a) holds by combining (2.21) with the result of Step 1.

(b) From (2.20), we have

$$
\mathbb{E}[\varphi_e(Y^e_T) + \psi_e(Y^e_T)] \leq \mathbb{E}\left[\epsilon(\varphi(\xi) + \psi(\xi)) + \left(\frac{2}{\delta_1} + \frac{2}{\delta_2}\right) \int_T^T (|\nabla \varphi_e(Y^e_s)|^2 ds + |\nabla \psi_e(Y^e_s) dA_s|^2) + \delta_1 \int_T^T |f(s, Y^e_s, Z^e_s)|^2 ds + \delta_2 \int_T^T |g(s, Y^e_s)|^2 dA_s\right].
$$

Since $\varphi(J^e(Y^e_t)) \leq \varphi_e(Y^e_t)$ and $\psi(J^e(Y^e_t)) \leq \psi_e(Y^e_t)$, by the same arguments as before we obtain

$$
\mathbb{E}[\varphi(J^e(Y^e_T)) + \psi(J^e(Y^e_T))] \leq C\mathbb{E}\left[\varphi(\xi) + \psi(\xi) + |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds + \int_0^T |g(s, 0)|^2 dA_s + \int_0^T |h(s, 0, 0)|^2 ds + \int_0^T \eta_s^2 dA_s\right].
$$

The assumption (c) is proved similarly since $\frac{1}{\epsilon}|J^e(y) - y|^2 \leq \varphi_e(y)$ and $\frac{1}{\delta_2}|J^e(y) - y|^2 \leq \psi_e(y)$.

Step 3: Convergence of the family of processes $(Y^e, Z^e)$.

For arbitrary $\epsilon, \delta > 0$, we have

$$
\mathbb{E}\left[\sup_{0 \leq s \leq T} |Y^e_s - Y^\delta_s|^2 + \int_0^T \|Z^e_s - Z^\delta_s\|^2 dt\right] \leq C(\epsilon + \delta)\Lambda. \tag{2.22}
$$

Indeed, by generalized Itô’s formula,

$$
|Y^e_t - Y^\delta_t|^2 + 2 \int_t^T \langle Y^e_s - Y^\delta_s, \nabla \varphi_e(Y^e_s) - \nabla \varphi_0(Y^\delta_s) \rangle ds
$$

$$
+ 2 \int_t^T \langle Y^e_s - Y^\delta_s, \nabla \psi_e(Y^e_s) - \nabla \psi_0(Y^\delta_s) \rangle dA_s
$$

$$
= 2 \int_t^T \langle Y^e_s - Y^\delta_s, f(s, Y^e_s, Z^e_s) - f(s, Y^\delta_s, Z^\delta_s) \rangle ds + 2 \int_t^T \langle Y^e_s - Y^\delta_s, g(s, Y^e_s) - g(s, Y^\delta_s) \rangle dA_s
$$

$$
- \int_t^T \|Z^e_s - Z^\delta_s\|^2 ds + \int_t^T \|h(s, Y^e_s, Z^e_s) - h(s, Y^\delta_s, Z^\delta_s)\|^2 ds - 2 \int_t^T \langle Y^e_s - Y^\delta_s, (Z^e_s - Z^\delta_s) dW_s \rangle
$$

$$
+ 2 \int_t^T \langle Y^e_s - Y^\delta_s, (h(s, Y^e_s, Z^e_s) - h(s, Y^\delta_s, Z^\delta_s)) \rangle dB_s).
$$

On the other hand by assumption (H3), we have

$$
2\langle Y^e_s - Y^\delta_s, f(s, Y_s^e, Z_s^e) - f(s, Y_s^\delta, Z_s^\delta) \rangle \leq \frac{1}{2} |Y^e_s - Y^\delta_s|^2 + r \rho(\|Y^e_s - Y^\delta_s\|^2) + rC\|Z^e_s - Z^\delta_s\|^2,
$$

$$
2\langle Y^e_s - Y^\delta_s, g(s, Y_s^e) - g(s, Y_s^\delta) \rangle \leq \beta |Y^e_s - Y^\delta_s|^2
$$

and

$$
\|h(s, Y_s^e, Z_s^e) - h(s, Y_s^\delta, Z_s^\delta)\| \leq \rho(\|Y^e_s - Y^\delta_s\|^2) + \alpha \|Z^e_s - Z^\delta_s\|^2.
$$
Since $\beta < 0$, if we take $r = \frac{1-g}{2g}$, we obtain

$$|Y_t^\varepsilon - Y_0^\delta|^2 + \int_0^T \|Z_s^\varepsilon - Z_s^\delta\|^2 ds$$

$$\leq \int_0^T p(|Y_s^\varepsilon - Y_s^\delta|^2) ds - 2 \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, \nabla \varphi_e(Y_s^\varepsilon) - \nabla \varphi_0(Y_s^\delta) \rangle ds$$

$$- 2 \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, \nabla \psi_e(Y_s^\varepsilon) - \nabla \psi_0(Y_s^\delta) \rangle dA_s - 2 \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, (Z_s^\varepsilon - Z_s^\delta) \rangle dW_s$$

$$+ 2 \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, (h(s, Y_s^\varepsilon, Z_s^\varepsilon) - h(s, Y_s^\varepsilon, Z_s^\delta)) \rangle dB_s.$$ 

By (v) of Proposition 2.2.3, it follows that

$$|Y_t^\varepsilon - Y_t^\delta|^2 + \int_0^T \|Z_s^\varepsilon - Z_s^\delta\|^2 ds$$

$$\leq \int_0^T p(|Y_s^\varepsilon - Y_s^\delta|^2) ds + 2(\varepsilon + \delta) \int_0^T |\nabla \varphi_e(Y_s^\varepsilon)||\nabla \varphi_0(Y_s^\delta)| ds$$

$$+ 2(\varepsilon + \delta) \int_0^T |\nabla \psi_e(Y_s^\varepsilon)||\nabla \psi_0(Y_s^\delta)| dA_s - 2 \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, (Z_s^\varepsilon - Z_s^\delta) \rangle dW_s$$

$$+ 2 \int_0^T \langle Y_s^\varepsilon - Y_s^\delta, (h(s, Y_s^\varepsilon, Z_s^\varepsilon) - h(s, Y_s^\varepsilon, Z_s^\delta)) \rangle dB_s.$$ 

Now, from Step 2 (a), we have

$$2(\varepsilon + \delta) \left[ \int_0^T |\nabla \varphi_e(Y_s^\varepsilon)||\nabla \varphi_0(Y_s^\delta)| ds + \int_0^T |\nabla \psi_e(Y_s^\varepsilon)||\nabla \psi_0(Y_s^\delta)| dA_s \right] \leq (\varepsilon + \delta) \Lambda,$$ 

(2.23)

which implies that

$$\mathbb{E}(|Y_t^\varepsilon - Y_t^\delta|^2) + \mathbb{E} \left( \int_0^T \|Z_s^\varepsilon - Z_s^\delta\|^2 ds \right) \leq \int_0^T p(\mathbb{E}(|Y_s^\varepsilon - Y_s^\delta|^2)) ds + (\varepsilon + \delta) \Lambda.$$ 

Bihari-Lasalle inequality yields

$$\sup_{0 \leq t \leq T} \mathbb{E}(|Y_t^\varepsilon - Y_t^\delta|^2) + \mathbb{E} \left( \int_0^T \|Z_s^\varepsilon - Z_s^\delta\|^2 ds \right) \leq C(\varepsilon + \delta) \Lambda.$$ 

(2.24)

The result follows from Burkholder-Davis-Gundy’s inequality. Therefore the family of process $(Y^\varepsilon, Z^\varepsilon)$ is Cauchy family in a Banach space $S^2(\mathbb{R}^k) \times M^2(\mathbb{R}^{d \times k})$. Then there exist $(Y, Z) \in S^2(\mathbb{R}^k) \times M^2(\mathbb{R}^{d \times k})$ such that $\lim_{\varepsilon \to 0} (Y_t^\varepsilon, Z_t^\varepsilon) = (Y_t, Z_t)$. Furthermore, from Step 2 we have

$$\lim_{\varepsilon \to 0} J_t^\varepsilon(Y^\varepsilon) = Y_t, \text{ in } M^2(\mathbb{R}^k) \text{ and } \lim_{\varepsilon \to 0} \mathbb{E}(J_t^\varepsilon(Y^\varepsilon) - Y_t^2) = 0, \; t \in [0, T].$$ 

(2.25)

Moreover, for all $\varepsilon > 0$, we set

$$U_t^\varepsilon = \nabla \varphi_e(Y^\varepsilon) \quad \text{and} \quad \bar{U}_t^\varepsilon = \int_0^t \nabla \varphi_e(Y^\varepsilon) ds$$

$$V_t^\varepsilon = \nabla \psi_e(Y^\varepsilon) \quad \text{and} \quad \bar{V}_t^\varepsilon = \int_0^t \nabla \psi_e(Y^\varepsilon) dA_s.$$
Then with the approximating equation (2.17) and the help of Step 3, we get

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( |\bar{U}^\varepsilon_t - U^\delta_t|^2 + |\bar{V}^\varepsilon_t - V^\delta_t|^2 \right) \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| Y^\varepsilon_t - Y^\delta_t \right|^2 + \int_0^T \|Z^\varepsilon_t - Z^\delta_t\|^2 dt \right].
\]

Since the right side converges to 0 as \( \varepsilon, \delta \to 0 \), the family of process \((\bar{U}^\varepsilon, \bar{V}^\varepsilon)\) is a Cauchy family and there exists a measurable process \(\bar{U}_t, \bar{V}_t\) such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( |\bar{U}^\varepsilon_t - \bar{U}_t|^2 + \int_0^T |\bar{V}^\varepsilon_t - \bar{V}_t|^2 dt \right) \right] \to 0 \text{ as } \varepsilon \to 0.
\]

**Step 4: A quadruplet of process \(Y,Z,U,V\) solves the MGBDSDE (1.6).**

From Step 2 there exists a constant independent of \(\varepsilon\) such that

\[
\sup_{\varepsilon} \mathbb{E} \left[ \int_0^T |U^\varepsilon_t|^2 dt + \int_0^T |V^\varepsilon_t|^2 dA_t \right] \leq \sup_{\varepsilon} \mathbb{E} \left[ \int_0^T |\nabla \psi(Y^\varepsilon_t)|^2 dt + |\nabla \psi(Y_t)|^2 dA_t \right] \leq C,
\]

which means that \(U^\varepsilon\) and \(V^\varepsilon\) are bounded respectively in \(L(\Omega, H^1([0,T], dt))\) and \(L(\Omega, H^1([0,T], dA_t))\), where \(H^1([0,T], dt)\) and \(H^1([0,T], dA_t)\) are the classical Sobolev space with respect respectively \(dt\) and \(dA_t\). This implies that family of process \((\bar{U}^\varepsilon)_{\varepsilon>0}\) and \((\bar{V}^\varepsilon)_{\varepsilon>0}\) converge weakly to \(\bar{U}\) and \(\bar{V}\) respectively. In particular, \(\bar{U}\) and \(\bar{V}\) are absolutely continuous, i.e. there exist two progressively measurable processes \(U\) and \(V\) such that

\[
\bar{U}_t = \int_0^t U_s ds \quad \text{and} \quad \bar{V}_t = \int_0^t V_s dA_s.
\]

Moreover by Fatou’s Lemma, \(U\) and \(V\) belong respectively in \(\mathcal{M}^2(\mathbb{R}^k)\) and \(\mathcal{A}^2(\mathbb{R}^k)\). Assertion (i) in Definition 2.2 is then proved. Let now prove \((ii)\) i.e. \((Y_t, U_t) \in \partial \phi, dP \otimes dt\)-a.e and \((Y_t, V_t) \in \partial \psi, dP \otimes dA_t\)-a.e on \([0,T]\). For this instance, for \(a < b \leq T\), \(u \in \mathcal{M}^2(\mathbb{R}^k)\) and \(v \in \mathcal{A}^2(\mathbb{R}^k)\), since \(U^\varepsilon, \bar{V}^\varepsilon\) and \(Y^\varepsilon\) converge uniformly to \(\bar{U}, \bar{V}\) and \(Y\), we deduce from Lemma 5.8 in Gegout-Petit and Pardoux [9] that

\[
\int_a^b \langle U^\varepsilon_t, u_t - Y^\varepsilon_t \rangle dt \to \int_a^b \langle U_t, u_t - Y_t \rangle dt
\]

and

\[
\int_a^b \langle V^\varepsilon_t, v_t - Y^\varepsilon_t \rangle dA_t \to \int_a^b \langle V_t, v_t - Y_t \rangle dA_t
\]

as \(\varepsilon\) goes to 0 in probability, which together with (2.25) imply

\[
\int_a^b \langle U^\varepsilon, f^\varepsilon(Y^\varepsilon) - Y^\varepsilon \rangle dt \to 0
\]

and

\[
\int_a^b \langle V^\varepsilon, f^\varepsilon(Y^\varepsilon) - Y^\varepsilon \rangle dA_t \to 0
\]

(2.27)
as $\varepsilon$ goes to $0$. On the other hand, in virtue of (ii) of Proposition 2.3 we have $U^E_t \in \partial \varphi(J^E(Y^E_t))$ and $V^E_t \in \partial \psi(\bar{J}^E(Z^E_t))$ which imply

\[
(U^E_t, u_t - J^E(Y^E_t)) + \varphi(J^E(Y^E_t)) \leq \varphi(u_t), \quad d\mathbb{P} \otimes dt \text{-a.e}
\]

and

\[
(V^E_t, v_t - J^E(Y^E_t)) + \psi(\bar{J}^E(Z^E_t)) \leq \psi(v_t), \quad d\mathbb{P} \otimes dA_t \text{-a.e.}
\]

Therefore we obtain

\[
\int_a^b \langle U^E_t, u_t - J^E(Y^E_t) \rangle dt + \int_a^b \varphi(J^E(Y^E_t)) dt \leq \int_a^b \varphi(u_t) dt
\]

and

\[
\int_a^b \langle V^E_t, v_t - J^E(Y^E_t) \rangle dA_t + \int_a^b \psi(\bar{J}^E(Z^E_t)) dA_t \leq \int_a^b \psi(v_t) dA_t.
\]

Taking the lim inf in (2.28), we have

\[
\int_a^b U_t(u_t - Y_t) dt + \int_a^b \varphi(Y_t) dt \leq \int_a^b \varphi(u_t) dt
\]

and

\[
\int_a^b V_t(v_t - Z_t) dA_t + \int_a^b \psi(Z_t) dA_t \leq \int_a^b \psi(v_t) dA_t
\]

in probability. We have used (2.27), (2.28) and the fact that $\varphi$ and $\psi$ are lower semi continuous. As $a, b, u$ and $v$ be taken arbitrary we get

\[
\langle U_t, u_t - Y_t \rangle + \varphi(Y_t) \leq \varphi(u_t), \quad d\mathbb{P} \otimes dt \text{-a.e}
\]

and

\[
\langle V_t, v_t - Z_t \rangle + \psi(Z_t) \leq \psi(v_t), \quad d\mathbb{P} \otimes dt \text{-a.e},
\]

which implies (ii).

Finally taking limit in (2.17) yields (iii) and ended the step 4 and also the proof of existence.

(ii) Uniqueness

Let $(Y, Z, U, V)$ and $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{V})$ be two solutions of MGBDSDE (2.1). In light of the computation used in Step 3, we get

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t - \tilde{Y}_t|^2 + \int_0^T \|Z_t - \tilde{Z}_t\| dt \right] \leq C \int_0^T \rho(\mathbb{E}(\sup_{0 \leq s \leq t} |Y_s - \tilde{Y}_s|^2)) dt.
\]

Next, Bihari-Lasalle inequalities yields

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t - \tilde{Y}_t|^2 \right] = 0.
\]

In view of (2.29), and since $\rho(0) = 0$, we obtain

\[
\mathbb{E} \left[ \int_0^T \|Z_t - \tilde{Z}_t\| dt \right] = 0.
\]

Finally we have $Y = \tilde{Y}$ and $Z = \tilde{Z}$. \hfill \Box
2.3 Comparison principle for variational GBDSDEs under non-Lipschitz condition

In this subsection, we only consider one-dimensional variational generalized BDSDEs, i.e., \( k = 1 \).

We consider the following variational GBDSDEs: \( 0 \leq t \leq T \) for \( i = 1, 2 \)

\[
Y^i_t + \int_t^T U^i_s ds + \int_t^T V^i_s dA_s = \xi^i_t + \int_t^T f^i(s, Y^i_s, Z^i_s) ds + \int_t^T g^i(s, Y^i_s) dA_s + \int_t^T h(s, Y^i_s, Z^i_s) dB_s - \int_t^T Z^i_s dW_s,
\]

(2.30)

where \( (Y^i_t, U^i_t) \in \partial \varphi \) and \( (Y^i_t, V^i_t) \in \partial \psi \).

**Remark 2.10.** Since for \( i = 1, 2 \), \( (Y^i_t, U^i_t) \in \partial \varphi \) and \( (Y^i_t, V^i_t) \in \partial \psi \), we have

\[
(Y^2 - Y^1)(U^2 - U^1) \geq 0 \quad \text{and} \quad (Y^2 - Y^1)(V^2 - V^1) \geq 0.
\]

(2.31)

If for \( f^i, g^i \), \( i = 1, 2 \) and \( h \) satisfy the conditions of Theorem 2.8, then there exist a unique quadruplet of measurable processes \( (Y^i, U^i, V^i, Z^i)_{i=1,2} \) solution of (2.30). Assume

(H6) \( \begin{array}{ll}
(i) & \xi^1 \leq \xi^2, \text{ a.s.,}
(ii) & f^1(t, Y^1_t, Z^1_t) \leq f^2(t, Y^2_t, Z^2_t) \text{ and } g^1(t, Y^1_t) \leq g^2(t, Y^1_t) \text{ a.s., a.e. } t \in [0, T], \text{ or }
& f^1(t, Y^1_t, Z^1_t) \leq f^2(t, Y^2_t, Z^2_t) \text{ and } g^1(t, Y^2_t) \leq g^2(t, Y^2_t), \text{ a.s., a.e. } t \in [0, T].
\end{array} \)

We have the following comparison result.

**Theorem 2.11.** Assume the conditions of Theorem 2.8. Let for \( i = 1, 2 \), \( (Y^i_t, U^i_t, V^i_t, Z^i_t) \) be solutions of GBDSDE (2.30). If (H6) holds, then \( Y^1_t \leq Y^2_t \) a.s. \( \forall t \in [0, T] \).

The proof of this result follows the proof of Theorem 3.1 appear in [22].

**Proof.** Let us assume that \( f^1(t, Y^2_t, Z^2_t) \leq f^2(t, Y^2_t, Z^2_t) \) and \( g^1(t, Y^2_t) \leq g^2(t, Y^2_t) \), a.s., a.e. \( t \in [0, T] \).

Denoting \( \overline{Y}_t = Y^2_t - Y^1_t, \overline{Z}_t = Z^2 - Z^1, \overline{U}_t = U^2_t - U^1_t, \overline{V}_t = V^2_t - V^1_t \), it follows from Itô formula to \( e^{\mu t} |\overline{Y}_t| \)

\[
|e^{\mu t} |\overline{Y}_t|^2 + \mu \int_t^T e^{\mu s} |\overline{Y}_s|^2 ds - \int_t^T e^{\mu s} \overline{U}_s ds - \int_t^T e^{\mu s} \overline{V}_s dA_s
\]

\[
= |e^{\mu t} (\xi^2 - \xi^1)|^2 - \int_t^T e^{\mu s} (f^1(s, Y^2_s, Z^2_s) - f^1(s, Y^1_s, Z^1_s)) ds - \int_t^T e^{\mu s} (g^1(s, Y^2_s) - g^1(s, Y^1_s)) dA_s
\]

\[
- \int_t^T e^{\mu s} (h(s, Y^2_s, Z^2_s) - h(s, Y^1_s, Z^1_s)) dB_s + \int_t^T e^{\mu s} \overline{Z}_s dW_s + \int_t^T 1_{\{\overline{Y}_s < 0\}} e^{\mu s + \lambda A_t} |h(s, Y^2_s, Z^2_s) - h(s, Y^1_s, Z^1_s)|^2 ds
\]

\[
\leq |e^{\mu t} (\xi^2 - \xi^1)|^2 - \int_t^T e^{\mu s} (f^1(s, Y^2_s, Z^2_s) - f^1(s, Y^1_s, Z^1_s)) ds - \int_t^T e^{\mu s} (g^1(s, Y^2_s) - g^1(s, Y^1_s)) dA_s
\]

\[
+ \int_t^T e^{\mu s} (h(s, Y^2_s, Z^2_s) - h(s, Y^1_s, Z^1_s)) dB_s + \int_t^T e^{\mu s} \overline{Z}_s dW_s - \int_t^T e^{\mu s} \overline{Y}_s dA_s
\]

\[
+ \int_t^T 1_{\{\overline{Y}_s < 0\}} e^{\mu s} |h(s, Y^2_s, Z^2_s) - h(s, Y^1_s, Z^1_s)|^2 ds - \int_t^T 1_{\{\overline{Y}_s < 0\}} e^{\mu s} |\overline{Z}_s|^2 ds.
\]

(2.32)
The last inequality is obtained thanks to
\[ f^2(s, Y_s^2, Z_s^2) - f^1(s, Y_s^2, Z_s^2) \geq 0 \]
\[ g^2(s, Y_s^2) - f^1(s, Y_s^2) \geq 0, \]
which follows from \((H6)\). Recall again \((H6)\), we have \(e^{\mu s} (\xi^2 - \xi^1) \geq 0\) so that
\[ E((e^{\mu s} (\xi^2 - \xi^1)^2) = 0. \tag{2.33} \]
Since \((Y^i, Z^i), i = 1, 2\) are in \(S^2(\mathbb{R}) \times \mathbb{M}^2(\mathbb{R}^d)\), we have
\[ E \left( \int_t^T e^{\mu s} Y_s Z_s dW_s \right) = 0. \tag{2.34} \]
and
\[ E(\left( \int_t^T e^{\mu s} Y_s \left( h(s, Y_s^2, Z_s^2) - h(s, Y_s^1, Z_s^1) \right) dB_s \right) = 0. \tag{2.35} \]
By Assumptions \((H3)\) and the basic inequality \(2ab \leq \delta a^2 + \frac{1}{\delta} b^2\) we get
\[ \int_t^T e^{\mu s} Y_s (f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2)) ds \leq \delta \int_t^T e^{\mu s} |Y_s|^2 ds + \frac{1}{\delta} \int_t^T e^{\mu s} p(|Y_s|^2) 1_{[Y_s < 0]} ds \]
\[ + \frac{c}{\delta} \int_t^T e^{\beta s} |Z_s|^2 1_{[Y_s < 0]} ds, \tag{2.36} \]
\[ - \int_t^T e^{\mu s} Y_s (g^1(s, Y_s^2) - g^1(s, Y_s^1)) dA_s \leq \beta \int_t^T e^{\mu s} |Y_s|^2 1_{[Y_s < 0]} dA_s \tag{2.37} \]
and
\[ \int_t^T 1_{[Y_s < 0]} e^{\mu s} |h(s, Y_s^2, Z_s^2) - h(s, Y_s^1, Z_s^1)|^2 ds \]
\[ \leq \int_t^T e^{\mu s} p(|Y_s|^2) 1_{[Y_s < 0]} ds + \alpha \int_t^T e^{\mu s} |Z_s|^2 1_{[Y_s < 0]} ds. \tag{2.38} \]
Putting \((2.33)-(2.38)\) in \((2.32)\) and since \(\beta < 0, 0 < \alpha < 1\), it follows from \((2.31)\) that
\[ E(e^{\mu t} |Y_t|^2) + (\mu - \beta - \delta) E \left( \int_t^T e^{\mu s} |Y_s|^2 ds \right) + \left( 1 - \frac{\alpha - \frac{\alpha}{\delta}}{2} \right) \]
\[ \leq \left( \frac{1}{\delta} + 1 \right) E \left( \int_t^T e^{\mu s} p(|Y_s|^2) ds \right). \]
Finally choosing \(\mu > 0\) and \(\delta > 0\) such that \(\mu - \beta - \delta > 0\) and \(1 - \alpha - \frac{\alpha}{\delta} > 0\), we have
\[ E(e^{\mu t} |Y_t|^2) \leq C \int_t^T e^{\mu s} p(|Y_s|^2) ds, \]
which by using Fubini’s theorem and Jensen’s inequality leads to
\[ E(|Y_T|^2) \leq C \int_t^T p(E(|Y_s|^2)) ds. \]

Then we can use Bihari’s inequality to obtain \(E(|Y_T|^2) = 0, \quad \forall t \in [0, T]\), and so \(Y^1_t \leq Y^2_t\), a. s., for all \(t \in [0, T]\).
3 Stochastic viscosity solutions of variational SPDEs with a nonlinear Neumann-Dirichlet boundary condition

In this section, we defined the notion of stochastic viscosity solution for variational SPDE (1.5). Next, via variational GBDSDEs studied in the previous section, we give a probabilistic representation of this variational SPDE in a such stochastic viscosity sense.

3.1 Preliminaries and definitions

Let \( F^B = \{ F^B_t \}_{0 \leq t \leq T} \) be the backward filtration defined in Section 2. We also denote by \( M^B_{\tau t} \) all \( F^B \)-stopping times \( \tau \) such 0 \( \leq \tau \leq T \), a.s. For generic Euclidean spaces \( E \) and \( E_1 \), we introduce the following spaces:

1. \( C^{k,n}([0,T] \times E;E_1) \) denotes the space of all \( E_1 \)-valued functions defined on \([0,T] \times E\) which are \( k \)-times continuously differentiable in \( t \) and \( n \)-times continuously differentiable in \( x \); and \( C_b^{k,n}([0,T] \times E;E_1) \) denotes the subspace of \( C^{k,n}([0,T] \times E;E_1) \) in which all function have uniformly bounded partial derivatives.

2. \( C^{k,n}(F^B_t,[0,T] \times E;E_1) \) (resp. \( C_b^{k,n}(F^B_t,[0,T] \times E;E_1) \)) is the space of all random fields \( \gamma \in C^{k,n}([0,T] \times E;E_1) \) (resp. \( C_b^{k,n}([0,T] \times E;E_1) \)), such that for fixed \( x \in E \),

the mapping \((t,\omega) \mapsto \gamma(t,\omega,x)\) is \( F^B \)-progressively measurable.

3. For a real number \( p \geq 1 \), let \( L^p(F^B_t;E) \) be a set of all \( E \)-valued, \( F^B_t \)-measurable random variable \( \xi \) such that \( \mathbb{E}|\xi|^p < +\infty \).

For \((t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^k\), we denote \( D_x = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}) \), \( D_{xx} = (\frac{\partial^2}{\partial x_i \partial x_j})_{i,j=1}^d \), \( D_y = \frac{\partial}{\partial y} \), \( D_t = \frac{\partial}{\partial t} \).

Let recall \( \Theta \) and \( \Gamma \) defined respectively by

\[
\Theta = \{ x \in \mathbb{R}^d : \phi(x) > 0 \}, \quad Bd(\Theta) = \{ x \in \mathbb{R}^d : \phi(x) = 0 \}, \quad \text{for any } \phi \in C_b^2(\mathbb{R}^d)
\]

and

\[
\Gamma u(t,x) = \sum_{j=1}^d \frac{\partial \phi}{\partial x_j}(x) \frac{\partial u}{\partial x_j}(t,x) = \langle \nabla \phi(x), \nabla u(t,x) \rangle, \quad \text{for all } x \in Bd(\Theta).
\]

For exemple let \( \phi : \mathbb{R}^d \to \mathbb{R} \) define as follows: for \( x = (x_1, \cdots, x_d) \), \( \phi(x) = x_1 \). Hence

\[
\Theta = \{ x \in \mathbb{R}^d : x_1 > 0 \}, \quad Bd(\Theta) = \{ x \in \mathbb{R}^d : x_1 = 0 \} \cong \mathbb{R}^{d-1}.
\]

Moreover,

\[
\Gamma u(t,x) = \frac{\partial u}{\partial x_1}(t,x) = (e_1, \nabla u(t,x)),
\]

where \( e_1 = (1,0,\cdots,0) \).

We work in this section with the following assumptions:

\textbf{(H6)} \( f,g \) are functions defined from \( \Omega \times [0,T] \times \overline{\Theta} \times \mathbb{R} \) to \( \mathbb{R} \) such that for all \( t \in [0,T] \), \( (x,y) \mapsto f(t,x,y,.) \) is uniformly Lipshitz and \( g \) is Lipshitz continuous in \( t,x,y \).
(H7) The functions $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are bounded and uniformly Lipschitz continuous, with common Lipschitz constant $K > 0$.

(H8) The function $\chi : \mathbb{R}^d \to \mathbb{R}$ is continuous, such that for some constants $K, p > 0$,

$$|\chi(x)| + |\varphi(\chi(x))| + |\psi(\chi(x))| \leq K(1 + |x|^p).$$

(H9) $h \in C_b^{0.2,3}([0, T] \times \overline{\Theta} \times \mathbb{R}; \mathbb{R}^l)$.

Remark 3.1. The Lipschitz condition on $g$ with respect time variable is necessary in order to obtain the continuity of the function $(t, x) \mapsto Y^{t,x}$ where $(t, x)$ is initial data of the forward SDE (3.18) (i.e. $X^{t,x}_s = x$ for all $s \in [0, t]$).

3.2 Notion of stochastic viscosity solution of variational SPDEs with a nonlinear Neumann-Dirichlet boundary condition

The aim of this subsection is to set up the notion of stochastic viscosity solution of variational-stochastic partial differential equations (VSPDEs, in short) with nonlinear Neumann-Dirichlet boundary condition:

$$\left\{ \begin{array}{ll}
(i) & \left( \frac{\partial u}{\partial t}(t,x) + Lu(t,x) + f(t,x,u(t,x)) + h(t,x,u(t,x)) \frac{\partial h}{\partial u} \right) \in \partial \varphi(u(t,x)), (t,x) \in [0, T] \times \Theta, \\
(ii) & \left( \frac{\partial u}{\partial t}(t,x) + g(t,x,u(t,x)) \right) \in \partial \psi(u(t,x)), (t,x) \in [0, T] \times Bd(\Theta), \\
(iii) & u(T,x) = \chi(x), x \in \Theta.
\end{array} \right. \tag{3.1}$$

Our method is purely probabilistic and follows the idea in [5]. To this fact, let us recall their statement.

Definition 3.2. Let $\tau \in M^B_{0,T}$, and $\xi \in \mathcal{F}_{\tau,T}$. We say that a sequence of random variables $(\tau_k, \xi_k)$ is a $(\tau, \xi)$-approximating sequence if for all $k$, $(\tau_k, \xi_k) \in M^B_{\infty} \times L^2(\mathcal{F}_{\tau,T}, \Theta)$ such that

(i) $\xi_k \to \xi$ in probability;

(ii) either $\tau_k \uparrow \tau$ a.s., and $\tau_k < \tau$ on the set $\{\tau > 0\}$; or $\tau_k \downarrow \tau$ a.s., and $\tau_k > \tau$ on the set $\{\tau < T\}$.

Definition 3.3. Let $(\tau, \xi) \in M^B_{0,T} \times L^2(\mathcal{F}_{\tau,T}; \Theta)$ and $u \in C_0^B([0, T] \times \overline{\Theta})$. A triplet of $(a,p,X)$ is called a stochastic $h$-superjet of $u$ at $(\tau, \xi)$ if $(a,p,X)$ is an $\mathbb{R} \times \mathbb{R}^d \times S(d)$-valued, $\mathcal{F}_{\tau,T}$-measurable random vector, such that setting $b = h(\tau, \xi, u(\tau, \xi))$, $c = (h_d h)(\tau, \xi, u(\tau, \xi))$ and $q = \partial_x h(\tau, \xi, u(\tau, \xi)) + \partial_{\tau} h(\tau, \xi, u(\tau, \xi))$ and for all $(\tau, \xi)$-approximating sequence $(\tau_k, \xi_k)$, we have

$$u(\tau_k, \xi_k) \leq u(\tau, \xi) + a(\tau_k - \tau) + b(B_{\tau_k} - B_\tau) + \frac{c}{2}(B_{\tau_k} - B_\tau)^2 + \langle p, \xi_k - \xi \rangle$$

$$+ \langle q, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \frac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle$$

$$+ o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2). \tag{3.2}$$

We denote by $J^1_{h} u(\tau, \xi)$ the set of all stochastic $h$-superjet of $u$ at $(\tau, \xi)$. Similarly, the triplet of $(a,p,X)$ is a stochastic $h$-subject of $u$ at $(\tau, \xi)$ if the inequality in (3.2) is reversed and $J_{h}^{1.2,-} u(\tau, \xi)$ denotes the set of all stochastic $h$-subject of $u$ at $(\tau, \xi)$.
We define the stochastic viscosity solution of MSPDE (3.1). To simplify let us set

$$V_f(\tau, \xi, a, p, X) = -a - \frac{1}{2} \text{Trace}(\sigma^2(\xi)X) - \langle p, b(\xi) \rangle - f(\tau, \xi, u(\tau, \xi)) \cdot$$

**Definition 3.4.** Let \( u \in C(\mathbb{F}^B, [0, T] \times \Theta) \) satisfying \( u(T, x) = \chi(x) \), for all \( x \in \Theta \). Moreover, \( \forall (\tau, \xi) \in M_{0,T} \times L^2(\mathcal{F}_{\tau,T}^B; \Theta) \)

$$u(\tau, \xi) \in \text{Dom}(\phi), \text{ on } \{ \xi \in \Theta \}$$

and

$$u(\tau, \xi) \in \text{Dom}(\psi), \text{ on } \{ \xi \in Bd(\Theta) \} \cdot$$

The function \( u \) is called a stochastic viscosity subsolution of MSPDE (3.1) if at any \( (\tau, \xi) \in M_{0,T} \times L^2(\mathcal{F}_{\tau,T}^B; \Theta) \), for any \( (a, p, X) \in J_{\tau}^{1,2} u(\tau, \xi) \), we have \( \mathbb{P} \)-a.s.

(a) on \( \{ 0 < \tau < T \} \cap \{ \xi \in \Theta \} \)

$$V_f(\tau, \xi, a, p, X) + \phi_f(u(\tau, \xi)) - \frac{1}{2} (h \partial_a h)(\tau, \xi, u(\tau, \xi)) \leq 0; \quad (3.5)$$

(b) on \( \{ 0 < \tau < T \} \cap \{ \xi \in Bd(\Theta) \} \)

$$\min \left( V_f(\tau, \xi, a, p, X) + \phi_f(u(\tau, \xi)) - \frac{1}{2} (h \partial_a h)(\tau, \xi, u(\tau, \xi)), \right.$$  

$$\langle \nabla \phi(\xi), p \rangle + \psi_f(u(\tau, \xi)) - g(\tau, \xi, u(\tau, \xi)) \right) \leq 0. \quad (3.6)$$

The function \( u \) is called a stochastic viscosity supersolution of MSPDE (3.1) if at any \( (\tau, \xi) \in M_{0,T} \times L^2(\mathcal{F}_{\tau,T}^B; \Theta) \), for any \( (a, p, X) \in J_{\tau}^{1,2} u(\tau, \xi) \), it hold \( \mathbb{P} \)-a.s.

(a) on \( \{ 0 < \tau < T \} \cap \{ \xi \in \Theta \} \)

$$V_f(\tau, \xi, a, p, X) + \phi_f(u(\tau, \xi)) - \frac{1}{2} (h \partial_a h)(\tau, \xi, u(\tau, \xi)) \geq 0; \quad (3.7)$$

(b) on \( \{ 0 < \tau < T \} \cap \{ \xi \in Bd(\Theta) \} \)

$$\max \left( V_f(\tau, \xi, a, p, X) + \phi_f(u(\tau, \xi)) - \frac{1}{2} (h \partial_a h)(\tau, \xi, u(\tau, \xi)), \right.$$  

$$\langle \nabla \phi(\xi), p \rangle + \psi_f(u(\tau, \xi)) - g(\tau, \xi, u(\tau, \xi)) \right) \geq 0 \quad (3.8)$$

Finally, a random field \( u \in C_{\text{F}^B, [0, T] \times \Theta} \) is called a stochastic viscosity solution of variational SPDE (3.1) if it is both a stochastic viscosity subsolution and a stochastic viscosity supersolution.

**Remark 3.5.** We observe that if \( f \) and \( g \) are deterministic and \( h \equiv 0 \), Definition 3.4 coincides with the definition of (deterministic) viscosity solution of MPDE given by Maticiuc and Raşcanu in [15].

We now state the notion of random viscosity solution which will be a bridge link to the stochastic viscosity solution and its deterministic counterpart.

**Definition 3.6.** A random field \( u \in C_{\text{F}^B, [0, T] \times \Theta} \) is called an \( \varpi \)-wise viscosity solution if for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), \( u(\omega, \cdot, \cdot) \) is a (deterministic) viscosity solution of MSPDE associated to data \( (f, g, 0, \chi, \varphi, \psi) \).
3.3 Doss-Sussmann transformation

In this subsection, using the Doss-Sussman transformation, our goal is to establish the link between the notion of stochastic viscosity solution for variational SPDE (3.1) and the notion of viscosity solution for PDE with random coefficient. For this fact, let introduce the process \( \eta \in C(\mathbb{F}_T^B, [0, T] \times \mathbb{R}^n \times \mathbb{R}) \), as the unique solution of the stochastic differential equation written in Stratonovich form

\[
\eta(t,x,y) = y + \int_t^T \langle h(s,x,\eta(s,x,y)), \circ dB_s \rangle.
\] (3.9)

Note that due to the direction of Itô integral, (3.9) should be viewed as going from \( T \) to \( t \) (i.e. \( y \) should be understood as the initial value). Under the assumption (H9), the mapping \( y \mapsto \eta(t,x,y) \) is a diffeomorphism for all \( (t,x), \mathbb{P}\text{-a.s.} \) (see Protter [21]). Hence if we denote its \( y \)-inverse by \( \varepsilon \), we get

\[
\varepsilon(t,x,y) = y - \int_t^T D_\varepsilon e(s,x,y) \langle h(s,x,\eta(s,x,y)), \circ dB_s \rangle.
\]

Let us recall the following important proposition in [3] (see Lemma 4.8).

**Proposition 3.7.** Assume (H6)-(H9) hold. Let \( (\tau, \xi) \in \mathcal{M}_0^B \times L^2(\mathcal{F}_{T}^B; \Theta), u \in C(\mathbb{F}_T^B, [0, T] \times \Theta) \) and \( (a_u, X_u, p_u) \in J_{T}^{1.2,+} u(\tau, \xi) \). Define \( v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot)) \). Then, for any \( (\tau, \xi) \)-approximating sequence \( (\tau_k, \xi_k) \), and for \( \mathbb{P}\text{-a.e.} \), it holds that

\[
v(\tau_k, \xi_k) \leq v(\tau, \xi) + a_v(\tau_k - \tau) + b_v(B_{\tau_k} - B_\tau) + \langle p_v, \xi_k - \xi \rangle + \langle q_v, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \frac{1}{2} \langle X_v(\xi_k - \xi), \xi_k - \xi \rangle + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2).
\]

where

\[
\left\{
\begin{aligned}
a_v &= D_v e(\tau, \xi, u(\tau, \xi)) a_u \\
p_v &= D_v e(\tau, \xi, u(\tau, \xi)) p_u + D_\xi e(\tau, \xi, u(\tau, \xi)) \\
X_v &= D_v e(\tau, \xi, u(\tau, \xi)) X_u + 2 D_{vv} e(\tau, \xi, u(\tau, \xi)) p_u^* + D_{v\xi} e(\tau, \xi, u(\tau, \xi)) p_u + D_{\xi\xi} e(\tau, \xi, u(\tau, \xi)) p_u p_v^*
\end{aligned}
\right.
\]

Namely, \( (a_v, X_v, p_v) \in J_{T}^{1.2,+} v(\tau, \xi) \)

Conversely, let \( (\tau, \xi) \in \mathcal{M}_0^B \times L^2(\mathcal{F}_{T}^B; \Theta), v \in C(\mathbb{F}_T^B, [0, T] \times \Theta) \) and \( (a_v, X_v, p_v) \in J_{T}^{1.2,+} v(\tau, \xi) \). Define \( u(\cdot, \cdot) = \eta(\cdot, \cdot, v(\cdot, \cdot)) \). Then, the triplet \( (a_u, X_u, p_u) \) given by

\[
\left\{
\begin{aligned}
a_u &= D_\tau \eta(\tau, \xi, v(\tau, \xi)) a_v \\
p_u &= D_\tau \eta(\tau, \xi, v(\tau, \xi)) p_v + D_{\xi} \eta(\tau, \xi, v(\tau, \xi)) \\
X_u &= D_\tau \eta(\tau, \xi, v(\tau, \xi)) X_v + 2 D_{\tau v} \eta(\tau, \xi, v(\tau, \xi)) p_u^* + D_{\tau\xi} \eta(\tau, \xi, v(\tau, \xi)) p_u + D_{\xi\xi} \eta(\tau, \xi, v(\tau, \xi)) p_u p_v^*
\end{aligned}
\right.
\]

satisfies \( (a_u, X_u, p_u) \in J_{T}^{1.2,+} u(\tau, \xi) \).

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One of the key ideas of Buckdahn and Ma is to use the Doss-Sussman transformation to convert a SPDE to a PDE with random coefficients, so that the stochastic viscosity solution can be studied otherwise. However, if we apply Doss-Sussman transformation to MSPDE (3.1) the resulting equation is not necessarily the multivalued PDE studied by Matciciuc and Răşcanu in [15], because of the presence of the subdifferential term. For this reason we will need the following version of Doss-Sussman transformation.

**Corollary 3.8.** Assume that the assumptions (H6)-(H9) hold. Let \((\tau, \xi) \in \mathcal{M}^B_{0,T} \times L^2(\mathcal{F}^B_{\tau,T}; \Theta)\), \(u \in C(\mathcal{F}^B_0, [0,T] \times \Theta)\) and define \(v(\cdot, \cdot) = \varepsilon(\cdot, \cdot, u(\cdot, \cdot))\).

(1) \(u\) is a subsolution of VSPDE (3.1) if and only if \(v(\cdot, \cdot)\) satisfies

(a) on the event \(\{0 < \tau < T\} \cap \{\xi \in \Theta\}\)

\[
V_f(\tau, \xi, a_v, p_v, X_v) + \frac{\phi'_I(\eta(\tau, \xi, \nu(\tau, \xi)))}{D_I \eta(\tau, \xi, \nu(\tau, \xi))} \leq 0; \quad (3.10)
\]

(b) on \(\{0 < \tau < T\} \cap \{\xi \in Bd(\Theta)\}\)

\[
\min \left(V_f(\tau, \xi, a_v, p_v, X_v) + \frac{\phi'_I(\eta(\tau, \xi, \nu(\tau, \xi)))}{D_I \eta(\tau, \xi, \nu(\tau, \xi))}, \langle \nabla \phi(\xi), p_v \rangle - \bar{g}(\tau, \xi, u(\tau, \xi)) + \frac{\psi'_I(\eta(\tau, \xi, \nu(\tau, \xi)))}{D_I \eta(\tau, \xi, \nu(\tau, \xi))}\right) \leq 0. \quad (3.11)
\]

(2) \(u\) is a supersolution of VSPDE (3.1) if and only if \(v(\cdot, \cdot)\) satisfies

(a) on \(\{0 < \tau < T\} \cap \{\xi \in \Theta\}\)

\[
V_f(\tau, \xi, a_v, p_v, X_v) + \frac{\phi'_I(\eta(\tau, \xi, \nu(\tau, \xi)))}{D_I \eta(\tau, \xi, \nu(\tau, \xi))} \geq 0; \quad (3.12)
\]

(b) on \(\{0 < \tau < T\} \cap \{\xi \in Bd(\Theta)\}\)

\[
\max \left(V_f(\tau, \xi, a_v, p_v, X_v) + \frac{\phi'_I(\eta(\tau, \xi, \nu(\tau, \xi)))}{D_I \eta(\tau, \xi, \nu(\tau, \xi))}, \langle \nabla \phi(\xi), p_v \rangle - \bar{g}(\tau, \xi, u(\tau, \xi)) + \frac{\psi'_I(\eta(\tau, \xi, \nu(\tau, \xi)))}{D_I \eta(\tau, \xi, \nu(\tau, \xi))}\right) \geq 0. \quad (3.13)
\]

**Proof.** Suppose \(u\) a stochastic subsolution of VSPDE (3.1). Hence for \((\tau, \xi) \in \mathcal{M}^B_{0,T} \times L^2(\mathcal{F}^B_{\tau,T}; \Theta)\) and \((a_v, p_v, X_v) \in \mathcal{F}^{1,2}_h u(\tau, \xi), u(\tau, \xi) \in \text{Dom}(\phi)\) on \(\{\xi \in \Theta\}\) and \(u(\tau, \xi) \in \text{Dom}(\psi)\) on \(\{\xi \in Bd(\Theta)\}\). Moreover,

on \(\{0 < \tau < T\} \cap \{\xi \in \Theta\}\)

\[
V_f(\tau, \xi, a_v, p_v, X_v) + \phi_I(u(\tau, \xi)) - \frac{1}{2} (h \partial_h \partial h)(\tau, \xi, u(\tau, \xi)) \leq 0; \quad (3.14)
\]

and
on \( \{0 < \tau < T\} \cap \{\xi \in \partial(\Theta)\} \)

\[
\min\left( V_f(\tau, \xi, a_v, p_v, X_v) + \frac{q'_f(u(\tau, \xi))}{D_y \eta(\tau, \xi, v(\tau, \xi))}, \right.
\]

\[
\langle \nabla \phi(\xi), p_v \rangle + \psi_f(u(\tau, \xi)) - g(\tau, \xi, u(\tau, \xi)) \leq 0.
\]

(3.15)

Using Proposition 3.7, there exist \((a_v, p_v, X_v) \in f_0^{1.2} \cdot v(\tau, \xi)\) such that

\[
V_f(\tau, \xi, a_v, p_v, X_v)
= D_y \eta(\tau, \xi, v(\tau, \xi))(\langle a_v - \frac{1}{2} \text{Trace}(\sigma \sigma^*) (\xi) X_v \rangle - \langle p_v, b(\xi) \rangle)
- \frac{1}{2} \text{Trace}(\sigma \sigma^*) (\xi) D_y \eta(\tau, \xi, v(\tau, \xi))
- \frac{1}{2} D_y y \eta(\tau, \xi, v(\tau, \xi)) \sigma^* (\xi) p_v^2 - \langle \sigma^* (\xi) D_y \eta(\tau, \xi, v(\tau, \xi)), \sigma^* (\xi) p_v \rangle
- \langle D_y \eta(\tau, \xi, v(\tau, \xi)), b(\xi) \rangle - f(\tau, \xi, \eta(\tau, \xi, v(\tau, \xi)))
\]

and

\[
\langle \nabla \phi(\xi), p_v \rangle - g(\tau, \xi, \eta(\tau, \xi, v(\tau, \xi)))
= D_y \eta(\tau, \xi, v(\tau, \xi)) \langle \nabla \phi(\xi), p_v \rangle + \langle \nabla \phi(\xi), D_y \eta(\tau, \xi, v(\tau, \xi)) \rangle - g(\tau, \xi, \eta(\tau, \xi, v(\tau, \xi))).
\]

Setting

\[
f(\tau, x, y) = \frac{1}{D_y \eta(\tau, x, y)} \left[ f(\tau, x, \eta(\tau, x, y) - \frac{1}{2} (h \partial_h h)(\tau, x, \eta(\tau, x, y)) + L \eta(\tau, x, y) \right],
\]

and

\[
g(\tau, x, y) = \frac{1}{D_y \eta(\tau, x, y)} (g(\tau, x, y) - \langle \nabla \phi(\xi), D_y \eta(\tau, x, y) \rangle),
\]

it follows from (3.14) and (3.15) that

on \( \{0 < \tau < T\} \cap \{\xi \in \Theta\} \),

\[
V_f(\tau, \xi, a_v, p_v, X_v) + \frac{q'_f(u(\tau, \xi))}{D_y \eta(\tau, \xi, v(\tau, \xi))} \leq 0,
\]

on \( \{0 < \tau < T\} \cap \{\xi \in \partial(\Theta)\} \)

\[
\min\left( V_f(\tau, \xi, a_v, p_v, X_v) + \frac{q'_f(\eta(\tau, \xi, v(\tau, \xi))))}{D_y \eta(\tau, \xi, v(\tau, \xi))}, \right.
\]

\[
\langle \nabla \phi(\xi), p_v \rangle - g(\tau, \xi, v(\tau, \xi)) + \frac{\psi_f(\eta(\tau, \xi, v(\tau, \xi)))}{D_y \eta(\tau, \xi, v(\tau, \xi))} \leq 0.
\]

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The converse part of (1) can be proved similarly. In the same manner one can show the second assertion (2).

\[\hfill\]

**Example 3.9.** Let us consider a special case of \(h\) by \(h(s, x, y) = \beta y\), one get \(\eta(t, x, y) = y \exp(\beta(B_T - B_t))\) and \(\varepsilon(t, x, y) = y \exp(-\beta(B_T - B_t))\). In this context,

\[\nu(t, x) = \varepsilon(t, x, u(t, x)) = \exp(-\beta(B_T - B_t))u(t, x), \quad \forall (t, x) \in [0, T] \times \Theta.\]

On the other hand, for \((\tau, \xi) \in M_{\Theta}^{\beta} \times L^2(\mathcal{F}_{\tau}, \mathcal{F}_{T})\), if \((a, p, A)\) belongs to \(\mathcal{J}^{1.2+}_{\Theta} v(\tau, \xi)\), we derive that \((\tilde{a}, \tilde{p}, \tilde{A})\) defined by

\[\tilde{a} = \exp(-\beta(B_T - B_t))a,\]
\[\tilde{p} = \exp(-\beta(B_T - B_t))p + \exp(-\beta(B_T - B_t))\nabla u(\tau, \xi),\]
\[\tilde{A} = \exp(\beta(B_T - B_t))A + \exp(-\beta(B_T - B_t))D^2u(t, x),\]

belongs to \(\mathcal{J}^{1.2+}_{\Theta} v(\tau, \xi)\). Moreover,

\[\tilde{f}(t, x, y) = \exp(-\beta(B_T - B_t)) \left[f(t, x, y \exp(\beta(B_T - B_t))) - \frac{1}{2} y \beta^2 \exp(\beta(B_T - B_t))\right],\]
\[\tilde{g}(t, x, y) = \exp(-\beta(B_T - B_t))g(t, x, y \exp(\beta(B_T - B_t))).\]

and inequalities \((3.10)\) and \((3.11)\) become respectively

(a) on the event \(\{0 < \tau < T\} \cap \{\xi \in \Theta\}\)

\[V_f(\tau, \xi, \tilde{a}, \tilde{p}, \tilde{A}) + \exp(-\beta(B_T - B_t))\phi'_f(\nu(\tau, \xi) \exp(\beta(B_T - B_t))) \leq 0; \quad (3.16)\]

(b) on \(\{0 < \tau < T\} \cap \{\xi \in \text{bd}(\Theta)\}\)

\[\min \left(V_f(\tau, \xi, \tilde{a}, \tilde{p}, \tilde{A}) + \exp(-\beta(B_T - B_t))\phi'_f(\nu(\tau, \xi) \exp(\beta(B_T - B_t)))\right), \quad (3.17)\]
\[\langle \nabla \phi(\xi), \tilde{p} \rangle - \bar{g}(\tau, \xi, \nu(\tau, \xi)) + \exp(-\beta(B_T - B_t))\psi'_f(\nu(\tau, \xi) \exp(\beta(B_T - B_t))) \leq 0,\]

In addition if \(\varphi(x) = I_{[a, b]}\) and \(\psi(x) = I_{[c, d]}\) where

\[I_{[a, b]}(x) = \begin{cases} 0, & \text{if } x \in [a_1, a_2], \\ +\infty, & \text{if } x \notin [a_1, a_2] \end{cases}\]

we derive that \(\theta'_f(y) = \theta'_f(y) = 0 \text{ si } y \in [a_1, a_2], \theta'_f(a_1) = -\infty, \theta'_f(a_1) = 0, \theta'_f(a_2) = 0, \theta'_f(a_2) = +\infty.\)

Therefore, inequality \((3.18)\) and \((3.17)\) is equivalent to

\[\hfill\]
The main objective of this subsection is to use the solution of VGBDSDE introduced in Section 2 in the Markovian framework to give a probabilistic representation (in stochastic viscosity sense) for the variational SPDEs.

3.4 Probabilistic representation result for stochastic viscosity solution to variational SPDE

The main objective of this subsection is to use the solution of VGBDSDE introduced in Section 2 in the Markovian framework to give a probabilistic representation (in stochastic viscosity sense) for the variational stochastic partial differential equations (3.1).

Roughly speaking, for \( (t,x) \in [0,T] \times \mathbb{Θ} \), let consider the forward backward doubly SDE

\[
\begin{align*}
X_t^{\tau,x} &= x + \int_s^T b(X_t^{\tau,x}) \, dr + \int_s^T \nabla \phi(X_t^{\tau,x}) \, dA_t^{\tau,x} + \int_s^T \sigma(r,X_t^{\tau,x}) \, dW_r \\
Y_t^{\tau,x} + \int_s^T U_t^{\tau,x} \, dr + \int_s^T V_t^{\tau,x} \, dA_t^{\tau,x} &= \chi(X_t^{\tau,x}) + \int_s^T f(r,X_t^{\tau,x},Y_t^{\tau,x},Z_t^{\tau,x}) \, dr \\
+ \int_s^T g(r,X_t^{\tau,x},Y_t^{\tau,x}) \, dA_t^{\tau,x} + \int_s^T h(r,X_t^{\tau,x},Y_t^{\tau,x},Z_t^{\tau,x}) \, dB_r - \int_s^T Z_s^{\tau,x} \, dW_r, \quad s \in [t,T],
\end{align*}
\]

which respectively from (10) (for, forward SDE) and Theorem 2.8 (for, VGBDSDE) admit a unique solution \( \{(X_t^{\tau,x}, A_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x}, U_t^{\tau,x}, V_t^{\tau,x}); s \in [t,T]\} \) satisfying the following:

\[
s \mapsto A_s^{\tau,x} \text{ is non-decreasing such that } A_s^{\tau,x} = \int_t^{\tilde{t}_s} \mathbf{1}_{\{X_r^{\tau,x} \in Bd(\Theta)\}} \, dA_r^{\tau,x}, \quad (3.19)
\]

\[
(Y_s^{\tau,x}, U_s^{\tau,x}) \in \partial \phi, \ \partial \psi \otimes ds, \quad (Y_s^{\tau,x}, V_s^{\tau,x}) \in \partial \psi, \ \partial \Phi \otimes dA_s. \quad (3.20)
\]

As announced in the introduction, the continuity of the different processes with respect to the initial data \( (t,x) \) is very essential in the determination of our result. The first concerns the forward SDE as follows.

**Proposition 3.10.** Assume (H7) holds. Then for any \( \kappa, p \geq 2 \), there exists a constant \( C > 0 \) such that for all \( 0 \leq t < t' \leq T \) and \( x, x' \in \mathbb{Θ} \) we have,

(a)

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| X_s^{\tau,x} - X_s^{\tau,x'} \right|^p + \sup_{0 \leq s \leq T} \left| A_s^{\tau,x} - A_s^{\tau,x'} \right|^p \right] \leq C \left[ |t' - t|^{p/2} + |x - x'|^p \right]
\]
(b) \[ \mathbb{E} \left( |A_{s}^{I_{s},X}|^{p} \right) \leq C(1 + |s \vee t|^{p}). \]

(c) \[ \mathbb{E} \left( e^{\kappa sX_{s}^{I_{s},X}} \right) \leq C. \]

(d) \([0,T] \times \Theta \ni (t,x) \mapsto \mathbb{E} \left( \int_{t}^{T} h_{1}(s,X_{s}^{I_{s},X})ds \right) + \mathbb{E} \left( \int_{t}^{T} h_{2}(s,X_{s}^{I_{s},X})dA_{s}^{I_{s},X} \right) \text{ is continuous, for every} \]

**Proof.** This proof follows the similar argument used in [20]. We apply Itô’s formula to the semi-martingale

\[ \exp \left[ \delta \left( \phi(X_{s}^{I_{s},X}) + \phi(X_{s}^{I_{s},X}') \right) \right] \left| X_{s}^{I_{s},X} - X_{s}^{I_{s},X}' \right|^{p}, \]

where \( \delta \) is a strictly positive constant (which exists due to (13), Theorem 4.47) such that

\[ -\left( X_{s}^{I_{s},X} - X_{s}^{I_{s},X}', \nabla \phi(X_{s}^{I_{s},X})dA_{s}^{I_{s},X} - \nabla \phi(X_{s}^{I_{s},X}')dA_{s}^{I_{s},X} \right) \leq \delta |X_{s}^{I_{s},X} - X_{s}^{I_{s},X}'|^{2} \left( dA_{s}^{I_{s},X} + dA_{s}^{I_{s},X}' \right), \]

a.s. Hence by the standard SDE estimates we obtain

\[ \mathbb{E} \left( \sup_{0 \leq s \leq T} |X_{s}^{I_{s},X} - X_{s}^{I_{s},X}'|^{p} \right) \leq C \left( |t - t'|^{p/2} + |x - x'|^{p} + \mathbb{E} \int_{0}^{T} |X_{s}^{I_{s},X} - X_{s}^{I_{s},X}'|^{p}ds \right). \]

Next, by Itô formula we have

\[ A_{s}^{I_{s},X} = \phi(X_{s}^{I_{s},X}) - \phi(x) - \int_{t}^{s} L\phi(X_{r}^{I_{s},X})dr - \int_{t}^{s} \nabla \phi(X_{r}^{I_{s},X})\sigma(X_{r}^{I_{s},X})dW_{r}, \]

where \( L \) is a second order differential operator associated to SDE of (3.18) and defined by (1.2).

The second result in this way concerns the continuity of the stochastic process \((Y_{t}^{I_{s},X})_{0 \leq s \leq T}\) with respect to the initial data \((t,x)\). Since this result has already been established in [19] when \( h \equiv 0 \) and \( f \) and \( g \) are deterministic, our proof follows their approach. However, because of the presence of stochastic integral in our case, we use respectively the Meyer-Zheng and S-topology to drive our convergence result.

**Proposition 3.11.** Let \((Y_{t}^{I_{s},X}, U_{t}^{I_{s},X}, V_{t}^{I_{s},X}, Z_{t}^{I_{s},X})_{t \in [0,T]}\) be the unique solution of the VGBDSDE (3.18). Then, the random field \((t,x) \mapsto Y_{t}^{I_{s},X}\) is a.s. continuous.

**Proof.** For an arbitrary \((t,x) \in [0,T] \times \Theta\), let us consider the sequence \((t_{n},x_{n})_{n \geq 1}\) of \([0,T] \times \Theta\) such that \((t_{n},x_{n}) \to (t,x)\) as \( n \to +\infty\). To prove that \(Y_{t}^{I_{s},X} \to Y_{t}^{I_{s},X}\) a.s as \( n \to +\infty\), it suffice to show that any subsequence of \(Y_{t}^{I_{s},X}_{t_{n},x_{n}}\) converges to \(Y_{t}^{I_{s},X}\). For \((t_{n},x_{n})\) be an arbitrary subsequence of \((t_{n},x_{n})\) that we will still note by \((t_{n},x_{n})\), we set \(X^{n} = X^{I_{s},X}_{t_{n},x_{n}}, A^{n} = A^{I_{s},X}_{t_{n},x_{n}}, Y^{n} = Y^{I_{s},X}_{t_{n},x_{n}}, Z^{n} = Z^{I_{s},X}_{t_{n},x_{n}}, U^{n} = U^{I_{s},X}_{t_{n},x_{n}}, \) and \( W_{t}^{n} = W_{t}^{I_{s},X}_{t_{n},x_{n}}\), where \( W^{n} \) is a Brownian motion on \([0,T] \times \Theta\). The proof of Proposition 3.11 is similar to the proof of Proposition 3.10.
It follows from the identical computation used in Step 1 and Step 2 of Theorem 2.8 that

\[
\begin{aligned}
X^n_s &= x_n + \int_{t_n}^{s \wedge t_n} b(X^n_r) \, dr + \int_{t_n}^{s \wedge t_n} \sigma(X^n_r) \, dW_r + \int_{t_n}^{s \wedge t_n} \nabla \phi(X^n_r) \, dA^n_r, \\
A^n_s &= \int_{t_n}^{s \wedge t_n} 1_{(X^n_r \in \delta \Theta)} \, dA^n_r \quad \forall s \in [0, T]
\end{aligned}
\]

and

\[
\begin{aligned}
Y^n_s + \int_s^T U^n_r \, dr + \int_s^T V^n_r \, dA^n_r &= \chi(X^n_T) + \int_s^T f_n(r,X^n_r,Y^n_r) \, dr + \int_s^T g_n(r,X^n_r,Y^n_r) \, dA^n_r \\
&\quad + \int_s^T h_n(r,X^n_r,Y^n_r) \, dB_r - \int_s^T Z^n_r \, dW_r, \quad 0 \leq s \leq T.
\end{aligned}
\]

where

\[
f_n(r,x,y) = 1_{[\delta_n,T]} f(r,x,y), \quad g_n(r,x,y) = 1_{[\delta_n,T]} g(r,x,y), \quad h_n(r,x,y) = 1_{[\delta_n,T]} h(r,x,y).
\]

We consider the extension \(X^n_s = x_n, \quad Y^n_s = Y^n_{t_n}, \quad A^n_s = U^n_s - V^n_s = Z^n_s = 0\) if \(s < 0, t_n\]. The first part of the proof study the \(S\)-tightness of the process \((X^n,A^n,Y^n,M^n,K^{1,n},K^{2,n})\) where for all \(s \in [0, T]\),

\[
M^n_s = \int_{t_n}^{s \wedge t_n} Z^n_r \, dW_r = \int_{t_n}^{s \wedge t_n} Z^n_r \, dM^n_r, \quad K^{1,n}_s = \int_{t_n}^{s \wedge t_n} U^n_r \, dr, \quad K^{2,n}_s = \int_{t_n}^{s \wedge t_n} V^n_r \, dA^n_r,
\]

with \(dM^n_r = \sigma(X^n_r) \, dW_r\). With the notation below, equation (3.22) is written

\[
\begin{aligned}
Y^n_s + (K^{1,n}_s - K^{1,n}_T) + (K^{2,n}_s - K^{2,n}_T) &= \chi(X^n_T) + \int_s^T f_n(r,X^n_r,Y^n_r) \, dr + \int_s^T g_n(r,X^n_r,Y^n_r) \, dA^n_r \\
&\quad + \int_s^T h_n(r,X^n_r,Y^n_r) \, dB_r - (M^n_T - M^n_s), \quad 0 \leq s \leq T.
\end{aligned}
\]

It follows from the identical computation used in Step 1 and Step 2 of Theorem 2.8 that

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_t|^2 + \int_0^T |Y^n_t|^2 A^n_t + \int_0^T |Z^n_t|^2 \, ds \right. \\
\left. \quad + \int_0^T |U^n_s|^2 \, ds + \int_0^T |V^n_s|^2 \, dA^n_s \right] < +\infty.
\]

and then there exists a constant independent of \(n\) such that

\[
CV_T(Y^n) + CV_T(K^{1,n}) + CV_T(K^{2,n}) + CV_T(M^n) \leq C,
\]

where for a càdlàg stochastic process \(L\) such that \(\mathbb{E}([L_t]) < +\infty\),

\[
CV_T(L) = \sup_{\pi} \sum_{i=0}^N \mathbb{E} \left[ \left| \mathbb{E}(L_{t_{i+1}} - L_{t_i} | \mathcal{F}_t) \right| \right]
\]

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such that the supremum is taken over all partition $\pi: 0 = t_0 < t_1 < \cdots < t_N = T$. Next according to Theorem 16 in [14], it follows from (3.24) that the sequence $(Y^n,M^n,K_1^n,K_2^n)$ is tight with respect to the $S$-topology. Therefore, there exists a subsequence still denoted by $(M^n,K_1^n,K_2^n)$ and the process $(\hat{Y}^n,\hat{M}^n,K_1^n,K_2^n)$ in $D([0,T],\mathbb{R})^4$ such that $(Y^n,M^n,K_1^n,K_2^n)$ and $(\hat{Y}^n,\hat{M}^n,K_1^n,K_2^n)$ are equal in law and $(\hat{Y}^n,\hat{M}^n,K_1^n,K_2^n) \to (\hat{Y},\hat{M},\hat{K}_1,\hat{K}_2)$ in sense of $S$-topology. Finally adapted arguments used in the proof of Proposition 4.3 in [1] or Lemma 5, Lemma 6 and Lemma 7 in [14]. we obtain

$$\hat{Y} = Y^{t_1}, \hat{M} = M^{t_1}, \hat{K}_1 = K_1^{t_1}, \hat{K}_2 = K_2^{t_1}.$$ 

In particular, since $Y^n_t$ and $\hat{Y}_t^n$ are $\mathcal{F}_t^n$-measurable, they can be regarded as a random variable defined on $\Omega_2$ and $Y^n_t = \hat{Y}_t^n$. Therefore since for the same reason as below $Y^{t_1}_t$ is a $\mathcal{F}_{t,T}$-measurable random defined on $\Omega_2$, $Y^{t_1}_t \to Y^{t_1}_t$, $\mathbb{P}_2$-a.s.

Let us define

$$u(t,x) = Y^{t_1}_t.$$ 

Then $u$ is random field such that $u(t,x)$ is $\mathcal{F}_{t,T}$-measurable for each $(t,x) \in [0,T] \times \Theta$.

We are now ready to derive the main result of this section.

**Theorem 3.12.** Assume (H6)-(H9) hold. Then, the function $u$ defined above is a stochastic viscosity solution of variational SPDE (3.1).

**Proof.** First, since $u(t,x) = Y^{t_1}_t$, it follows from Proposition 3.11 that $u \in C(F^B, [0,T] \times \Theta)$ and $u(T,x) = \chi(x)$. On the other hand, it follows from (3.20) that for all $(\tau,\xi) \in M_{0,T} \times L^2(\mathcal{F}^B_{t,T};\Theta)$, $u(\tau,\xi) \in \text{Dom}(\Phi)$ on $\{\xi \in \Theta\}$ and $u(\tau,\xi) \in \text{Dom}(\psi)$ on $\{\xi \in Bd(\Theta)\}$.

Thus it remains to show (3.5)-(3.6) and (3.7) -(3.8). In other word, using Corollary 3.8 it suffices to prove that $v(t,x) = \varepsilon(t,x,u(t,x))$ satisfies (3.10)-(3.11) and (3.12). In this fact, we are going to use the Yosida approximation of (3.18), which was studied in Section 2. For each $(t,x) \in [0,T] \times \Theta$, $\delta > 0$, let $\{(Y^{t_1,x,\delta},Z^{t_1,x,\delta}), 0 \leq s \leq T\}$ denote the solution of the following GBDSDE:

$$Y^{t_1,x,\delta}_s = Y_{t_1} + \int_s^T \nabla \varphi_\delta(Y^{t_1,x,\delta}_r) \, dr + \int_s^T \nabla \varphi_\delta(Y^{t_1,x,\delta}_r) \, dA_r = \chi(X^{t_1}_r) + \int_s^T f(r,X^{t_1,x,\delta}_r,Y^{t_1,x,\delta}_r) \, dr + \int_s^T g(r,X^{t_1,x,\delta}_r,Y^{t_1,x,\delta}_r) \, dA_r + \int_s^T h(r,X^{t_1,x,\delta}_r,Y^{t_1,x,\delta}_r) \, dB_r - \int_s^T Z^{t_1,x,\delta}_r \, dW_r, \ t \leq s \leq T. \tag{3.26}$$

Define $Y^{t_1,x,\delta} = u^\delta(t,x)$, it is well known (see Theorem 4.7, [2]) that the function $v^\delta(t,x) = \varepsilon(t,x,u^\delta(t,x))$ is a $\omega_2$-wise viscosity solution to the following SPDE with nonlinear Dirichlet-Neumann boundary condition

$$\begin{align*}
(i) & \quad \left( \frac{\partial v^\delta}{\partial t}(t,x) - \left[ L v^\delta(t,x) + \tilde{f}_\delta(t,x,v^\delta(t,x)) \right] \right) = 0, \ (t,x) \in [0,T] \times \Theta, \\
(ii) & \quad \frac{\partial v^\delta}{\partial n}(t,x) + \tilde{g}_\delta(t,x,v^\delta(t,x)) = 0, \ (t,x) \in [0,T] \times \partial \Theta, \\
(iii) & \quad v(T,x) = \chi(x), \ x \in \Theta.
\end{align*} \tag{3.27}$$
where
\[ f_\delta(t, x, y, z) = f(t, x, y) - \frac{\nabla \varphi_\delta(\eta(t,x,y))}{D_\eta \eta(t,x,y)} \quad \text{and} \quad g_\delta(t, x, y) = g(t, x, y) - \frac{\nabla \psi_\delta(\eta(t,x,y))}{D_\eta \eta(t,x,y)}. \]

for all \((\tau, \xi) \in M_{\eta,T}^B \times L^2 \left( J_{t,T}^+, \Theta \right), \) it follows from section 2 that (along a subsequence) \( \check{v}_\delta(\tau, \xi) \) converge to \( v(\tau, \xi) \) almost surely as \( \delta \) goes to 0. Let \( \omega_2 \in \Omega_2 \) be fixed such
\[ |v_\delta(\tau(\omega_2), \xi(\omega_2)) - v(\tau(\omega_2), \xi(\omega_2))| \to 0 \quad \text{as} \quad \delta \to 0, \]
and consider \((a_\nu, p_\nu, X_\nu) \in J_0^{1,2,+}(v(\tau(\omega_2), \xi(\omega_2))). \) Then, it follows from Crandall- Ishii-Lions [8] that there exist sequences
\[
\begin{align*}
\{ \delta_n(\omega_2) \to 0, \\
(\tau_n(\omega_2), \xi_n(\omega_2)) \in [0, T] \times \Theta, \\
(a_\nu^n, p_\nu^n, X_\nu^n) \in J_0^{1,2,+}(v_\delta(\tau_n(\omega_2), \xi_n(\omega_2)))
\end{align*}
\]
such that
\[
(\tau_n(\omega_2), \xi_n(\omega_2), a_\nu^n, p_\nu^n, X_\nu^n, v_\delta(\tau_n(\omega_2), \xi_n(\omega_2))) \to (\tau(\omega_2), \xi(\omega_2), a_\nu, p_\nu, X_\nu, v(\tau(\omega_2), \xi(\omega_2))), \quad n \to \infty.
\]
Since \( v_\delta(\omega_2, \cdot, \cdot) \) is a (deterministic) viscosity solution to the PDE \((f_\delta(\omega_2, \cdot, \cdot, 0, g_\delta(\cdot, \cdot, \cdot), \chi), \) we obtain
\[
\begin{align*}
(a) \quad (\tau_n(\omega_2), \xi_n(\omega_2)) \in [0, T] \times \Theta \\
V_{f_\delta}(\tau_n(\omega_2), \xi_n(\omega_2), a_\nu^n, X_\nu^n, p_\nu^n) + \frac{\nabla \varphi_\delta(\eta(\tau_n(\omega_2), \xi_n(\omega_2), v_\delta(\tau_n(\omega_2), \xi_n(\omega_2))))}{D_\eta \eta(\tau_n(\omega_2), \xi_n(\omega_2), v_\delta(\tau_n(\omega_2), \xi_n(\omega_2)))} \\
\leq 0,
\end{align*}
\]
\[
(b) \quad (\tau_n(\omega_2), \xi_n(\omega_2)) \in [0, T] \times \partial \Theta \\
\min \left\{ V_{f_\delta}(\tau_n(\omega_2), \xi_n(\omega_2), a_\nu^n, X_\nu^n, p_\nu^n) + \frac{\nabla \varphi_\delta(\eta(\tau_n(\omega_2), \xi_n(\omega_2), v_\delta(\tau_n(\omega_2), \xi_n(\omega_2))))}{D_\eta \eta(\tau_n(\omega_2), \xi_n(\omega_2), v_\delta(\tau_n(\omega_2), \xi_n(\omega_2)))}, \right.
\]
\[
\left. \frac{\nabla \psi_\delta(\eta(\tau_n(\omega_2), \xi_n(\omega_2), v_\delta(\tau_n(\omega_2), \xi_n(\omega_2))))}{D_\eta \eta(\tau_n(\omega_2), \xi_n(\omega_2), v_\delta(\tau_n(\omega_2), \xi_n(\omega_2)))} \right\} \\
\leq 0.
\]

In the sequel and for simplicity, we will omit writing \( \omega_2. \) Let take \( y \in \text{Dom}(\varphi) \cap \text{Dom}(\psi) \) such that \( y \leq u(\tau, \xi) = \eta(\tau, \xi, v(\tau, \xi)), \) and similarly \( v_\delta \) converges uniformly in probability to \( v, \) there exist \( n_0 > 0 \) such that \( y < \eta(\tau_n, \xi_n, v_\delta(\tau_n, \xi_n)) \) for all \( n \geq n_0. \) Therefore, inequality (3.28) and (3.28) imply
\[
\begin{align*}
(\eta(\tau_n, \xi_n, v_\delta(\tau_n, \xi_n)) - y) V_{\check{f}}(\tau_n, \xi_n, a_\nu^n, X_\nu^n, p_\nu^n) \\
\leq \left( -\varphi_\delta(J_{\delta}(\eta(\tau_n, \xi_n, v_\delta(\tau_n, \xi_n)))) + \varphi(y) \right) \frac{1}{D_\eta \eta(\tau_n, \xi_n, v_\delta(\tau_n, \xi_n))},
\end{align*}
\]
and
\[
\min \left( \eta(\tau_n, \xi_n, v^{\delta_h}(\tau_n, \xi_n)) - y \right) V_f(\tau_n, \xi_n, a_v, X_v, p_v) + \frac{\phi_h(\eta(\tau_n, \xi_n, v^{\delta_h}(\tau_n, \xi_n))) - \phi(y)}{D_y \eta(\tau_n, \xi_n, v^{\delta_h}(\tau_n, \xi_n))},
\]

and
\[
\min \left( \eta(\tau_n, \xi_n, v^{\delta_h}(\tau_n, \xi_n)) - y \right) \left( [\nabla \phi(\xi_n, p'_v)] - \bar{g}(\tau_n, \xi_n, v^{\delta_h}(\tau_n, \xi_n)) \right)
\]

Passing in the limit in the two below inequality, we get for all \( y \leq \eta(\tau, \xi, v(\tau, \xi)) \),

\[
(V_f(\tau, \xi, a_v, X_v, p_v) \leq -\frac{\phi(\eta(\tau, \xi, v(\tau, \xi))) - \phi(y)}{\eta(\tau, \xi, v(\tau, \xi)) - y} \frac{1}{D_y \eta(\tau, \xi, v(\tau, \xi))},
\]

and
\[
\min \left( \frac{V_f(\tau, \xi, a_v, X_v, p_v) + \phi_h(\eta(\tau, \xi, v(\tau, \xi))) - \phi(y)}{\eta(\tau, \xi, v(\tau, \xi)) - y} \right) \leq 0.
\]

Taking the limit when \( y \) goes to \( \eta(\tau, \xi, v(\tau, \xi)) \) we have

\[
(V_f(\tau, \xi, a_v, X_v, p_v) + \frac{\phi_h(\eta(\tau, \xi, v(\tau, \xi)))}{D_y \eta(\tau, \xi, v(\tau, \xi))} \leq 0,
\]

and
\[
\min \left( \frac{V_f(\tau, \xi, a_v, X_v, p_v) + \phi_h(\eta(\tau, \xi, v(\tau, \xi)))}{D_y \eta(\tau, \xi, v(\tau, \xi))},
\]

which implies that \( v \) satisfies (3.10) and (3.11). Then, it follows from Corollary 3.8 that \( u \) is a stochastic viscosity subsolution of MSPDE (3.1). By similar arguments, one can prove that \( u \) is a stochastic viscosity supersolution of MSPDE (3.1) and completes the proof. \( \square \)

**Remark 3.13.** In this work, we established existence and uniqueness result for variational generalized backward doubly stochastic differential equations under non-Lipschitz condition. Then using this result, we only gave a probabilistic representation of the viscosity solution of the variational stochastic partial differential equations with nonlinear Neumann condition. Since the study of uniqueness requires additional conditions, we promise to discuss this in a complementary article.

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