PERMANENCE PROPERTIES OF PROPERTY A AND COARSE EMBEDDABILITY FOR LOCALLY COMPACT GROUPS

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Abstract. If $\Gamma \subset G$ is a lattice in a locally compact second countable group $G$, then we show that $G$ has property A (respectively is coarsely embeddable into Hilbert space) if and only if $\Gamma$ has property A (respectively is coarsely embeddable into Hilbert space). Moreover, we show three interesting generalizations of this result. If $H \subset G$ is a closed subgroup of $G$ that is co-amenable in $G$, and if $H$ has property A (respectively, is coarsely embeddable into Hilbert space), then we show that $G$ has property A (respectively, is coarsely embeddable into Hilbert space). We also show that an extension of property A groups still has property A. On the coarse embeddability side, we show that if $e \to H \to G \to Q \to e$ is a short exact sequence, and if either $H$ is coarsely embeddable into Hilbert space and $Q$ has property A, or $H$ is compact and $Q$ is coarsely embeddable into Hilbert space, then $G$ is coarsely embeddable into Hilbert space. We extend the theory of measure equivalence to locally compact non-unimodular groups. In a natural way, we can also define measure equivalence subgroups. We show that property A and uniform embeddability into Hilbert space pass to measure equivalence subgroups. Using the same techniques, we show that also the Haagerup property, weak amenability and the weak Haagerup property pass to measure equivalence subgroups.

Introduction and statement of the main results

In [Gro93], Gromov introduced the notion of uniform embeddability of metric spaces. Nowadays, this is often called coarse embeddability, and we stick to the more modern terminology in this paper. Gromov suggested that a discrete finitely generated group $\Gamma$ that coarsely embeds into Hilbert space, would satisfy the Novikov conjecture. It was later shown that this is indeed the case: in [Yu00], Yu showed that it is true for all discrete groups that are uniformly embeddable into Hilbert space, and whose classifying space $B\Gamma$ is a finite CW-complex. In the same paper, he introduced a condition on $\Gamma$, which he called property A, that ensures coarse embeddability of $\Gamma$ into a Hilbert space. Higson and Roe showed that $\Gamma$ has property A if and only if $\Gamma$ has a topologically amenable action on a compact Hausdorff space [HR00]. Ozawa showed that this is equivalent to exactness [Oza00]. In [Hig00], Higson showed that all discrete groups with property A satisfy the Novikov conjecture, even when the classifying space is not a finite CW-complex. Skandalis, Tu and Yu [STY02] could then show that all discrete groups that coarsely embed into Hilbert space, do indeed satisfy the Novikov conjecture. In fact, the results by Higson and Skandalis, Tu and Yu are slightly stronger: they showed...
that for all discrete groups with property A (respectively that are coarsely embeddable into Hilbert space), the Baum-Connes assembly map with coefficients is split-injective. Baum, Connes and Higson showed in \[ \text{BCH94} \] that this implies the Novikov conjecture.

Similar results also hold for locally compact second countable (from now on l.c.s.c) groups. In \[ \text{Roe05} \], Roe extended the definition of property A to proper metric spaces with bounded geometry. Every l.c.s.c. group has a proper compatible left-invariant metric, and has bounded geometry with respect to this metric (see \[ \text{Str74, HP06} \]). So Roe’s definition applies in particular to l.c.s.c. groups. In a previous paper \[ \text{DL13} \], we showed that a l.c.s.c. group \( G \) has property A in this sense if and only if \( G \) admits a topologically amenable action on a compact Hausdorff space. Chabert, Echterhoff and Oyono-Oyono showed that the Baum-Connes assembly map with coefficients is split-injective for every l.c.s.c. group that admits a topologically amenable action on a compact Hausdorff space, see \[ \text{CEO04} \]. In \[ \text{DL13} \], we extended this result and showed that coarse embeddability into Hilbert space implies split-injectivity of the Baum-Connes assembly map with coefficients.

In the present paper, we study permanence properties of property A and coarse embeddability into Hilbert space for l.c.s.c. groups. For brevity, we will drop the “into Hilbert space”, so the phrase “\( G \) is coarsely embeddable” will mean that \( G \) is coarsely embeddable into Hilbert space. For the convenience of the reader, we review the relevant definitions in section 1. It is clear that property A and coarse embeddability pass to closed subgroups of l.c.s.c. groups. We will be interested in the other direction: if \( G \) is a l.c.s.c. group and a closed subgroup \( H \subset G \) has property A (resp. is coarsely embeddable), under which conditions on the inclusion \( H \subset G \) can we conclude that \( G \) has property A (resp. is coarsely embeddable)? The first of our results of this type is that this holds when \( H \) is a lattice in \( G \). This result can be extended in a number of ways, and we obtain the following result.

**Theorem 0.1.** Let \( G, H \) be l.c.s.c. groups. In each of the following situations, if \( H \) has property A (resp. is coarsely embeddable), then \( G \) has property A (resp. is coarsely embeddable).

1. \( H \subset G \) is a lattice
2. \( H \subset G \) is a closed subgroup with finite covolume
3. \( H \subset G \) is a closed co-amenable subgroup, in the sense of Eymard \[ \text{Eym72} \]
4. \( H \) is a closed normal subgroup of \( G \) and the quotient group \( G/H \) has property A
5. \( H = G/Q \) where \( Q \subset G \) is a compact normal subgroup
6. \( G \) is a measure equivalence subgroup of \( H \). We give a careful definition of this notion in definition 3.5, inspired by \[ \text{Gro93} \]

All of the above statements are special cases of a more general result. The crucial ingredients of this result are proper cocycles (inspired by Jolissaint \[ Jol00 \]) and property A for pairs (inspired by amenable pairs \[ Eym72, Gre69, Zim78, Jol96 \]). We explain both notions below.

In \[ Jol00 \], Jolissaint introduced proper cocycles in order to prove permanence properties of the Haagerup property for l.c.s.c. groups. Recently, he used proper cocycles to derive similar
permanence properties of weak amenability and of the weak Haagerup property for l.c.s.c. groups [Jol14]. We introduce a slightly weaker notion of proper cocycle.

**Definition 0.2** (inspired by [Jol00]). Let $G, H$ be l.c.s.c. groups and let $G \curvearrowright (X, \mu)$ be a non-singular Borel action on a standard probability space. A Borel cocycle $\omega : G \times X \to H$ is said to be

- proper with respect to a family $\mathcal{A}$ of Borel sets in $X$ if
  
  1. for every compact subset $K \subset G$ and every $A, B \in \mathcal{A}$, we find a precompact set $L(K, A, B) \subset H$ such that, for every $g \in K$ we get that
     
     $\omega(g, x) \in L(K, A, B)$ for almost all $x \in A \cap g^{-1}B$
  
  2. for every compact subset $L \subset H$ and every $A, B \in \mathcal{A}$, we get that the set $K(L, A, B)$ of all $g \in G$ such that
     
     $\mu\{x \in X \mid x \in A, gx \in B, \omega(g, x) \in L\} > 0$
     
     is precompact in $G$.

- proper if $\omega$ is proper with respect to some family $\mathcal{A}$ of Borel sets in $X$ such that for every $\varepsilon > 0$ there is a set $A \in \mathcal{A}$ with $\mu(X \setminus A) < \varepsilon$.

Our notion of proper cocycle has a few advantages over Jolissaint’s notion. First of all, it is more natural because it is invariant under cohomology, while Jolissaint’s notion is not. In section 2, we give an example of a cocycle $\omega$ that is cohomologous to a cocycle that is proper in Jolissaint’s sense, but $\omega$ itself is not proper in Jolissaint’s sense (see example 2.5). More importantly, we have more examples. In theorem 4.1 we show that the cocycles coming from measure equivalence subgroups are proper in our sense. Every cocycle that is proper in Jolissaint’s sense is also proper in our sense, see proposition 2.7. So example 2.5 gives an example of a cocycle that is proper in our sense but not in Jolissaint’s sense. Jolissaint’s main result about proper cocycles is [Jol14, theorem 1.4]. With theorem 0.6 below, we show that this result also holds for our weaker notion of proper cocycle.

From now on, when we say that $G \curvearrowright (X, \mu)$ is a non-singular action, we mean that $G \curvearrowright (X, \mu)$ is a non-singular Borel action of a l.c.s.c. group on a standard probability space. Moreover, we assume all cocycles to be Borel cocycles.

**Examples 0.3** ([Jol00]). Jolissaint gives the following elementary examples of proper cocycles.

- Let $H \subset G$ be a closed subgroup of a l.c.s.c. group. Set $X = G/H$ and consider any quasi-invariant probability measure $\mu$ on $X$. Let $s : X \to G$ be a regular Borel section (see [Mac52] lemma 1.1), i.e. $s(x)H = x$ for all $x \in X$ and $s(K)$ is precompact in $G$, for all compact subsets $K \subset X$. Then the cocycle $\omega : G \times X \to H$ that is defined by $\omega(g, x) = s(gx)^{-1}gs(x)$ is a proper cocycle with respect to the family of all compact subsets of $X$.

- Let $Q \subset G$ be a closed normal subgroup and set $H = G/Q$. Consider $X$ to be the one-point space and consider the quotient morphism $\pi : G \to H$ to be a cocycle $\pi : G \times X \to H$. Such a cocycle is proper if and only if $Q$ is compact.
We add a non-obvious example to the list, coming from measure equivalence of locally compact groups. Measure equivalence was introduced for discrete groups by Gromov in section 0.5.E, as a measure-theoretic counterpart to coarse equivalence. Even though he only mentions discrete groups, his definition works well for unimodular l.c.s.c. groups, see for example [BFS13]. In the non-unimodular case, we need to take a little more care. We first introduce the much weaker notion of a measure correspondence: we say that a non-singular action \( G \times H \) is a \textit{measure correspondence} between \( G \) and \( H \) if there are standard probability spaces \((X, \mu)\) and \((Y, \nu)\) and measure class preserving Borel isomorphisms \( \varphi : X \times H \to \Omega \) and \( \psi : Y \times G \to \Omega \) such that \( \varphi \) commutes with the \( H \) action and \( \psi \) commutes with the \( G \) action. We consider the Haar measure on the groups \( G \) and \( H \).

Between every two l.c.s.c. groups \( G, H \), there is a measure correspondence, namely \( \Omega = G \times H \) with the left translation action of \( G \times H \). There is also a "composition" operation on measure correspondences: if \( \Omega_1 \) is a correspondence between \( G, H \) and \( \Omega_2 \) is a correspondence between \( H, K \), then \( \Omega_1 \circ_H \Omega_2 \) is a measure correspondence between \( G \) and \( K \).

Given a measure correspondence \( \Omega \) between \( G \) and \( H \), we can transfer the \( G \)-action from \( \Omega \) to \( X \times H \), using \( \varphi \). This action is of the form

\[
g \varphi(x, h) = \varphi(gx, h \omega(g, x)^{-1}) \text{ for all } g \in G \text{ and almost all } (x, h) \in X \times H,
\]

for some non-singular action \( G \rhd X \) and some Borel cocycle \( \omega : G \times X \to H \). In a similar way, we find an action of \( H \) on \( Y \) and a cocycle \( \beta : H \times Y \to G \). We say that \( G \) is a measure equivalence subgroup of \( H \) if there is a \( G \)-invariant probability measure on \( X \) in the measure class of \( \mu \). We say that \( G \) is measure equivalent to \( H \) if there are invariant probability measures on \( X,Y \), in the measure class of \( \mu,\mu \). If \( G,H \) are unimodular, then this is equivalent to the usual notion of measure equivalence, see theorem 3.6.

**Example 0.4** (see theorem 4.1). Let \( G,H \) be l.c.s.c. groups and let \( (\Omega, \eta) \) be a measure correspondence between \( G \) and \( H \). Let \( \omega : G \times X \to H \) be a cocycle as defined above. Then \( \omega \) is a proper cocycle.

Property A for pairs is based on amenability for pairs. Let \( G \rhd (X,\mu) \) be a non-singular action and denote its Radon-Nikodym cocycle by \( \chi : G \times X \to \mathbb{R}_+^\times \). Consider the Koopman representation \( \pi_X : G \to U(L^2(X, \mu)) \), i.e. \( (\pi_X(g)\xi)(x) = \xi(g^{-1}x)\sqrt{\chi(g^{-1}, x)} \) for all \( g \in G \), \( \xi \in L^2(X, \mu) \) and almost all \( x \in X \). Remember that the pair \((G,X)\) is said to be amenable if the Koopman representation \( \pi_X \) has almost invariant vectors (see [Eym72, Gre69, Zim78, Jo96] for more background and equivalent definitions). In this spirit, we make the following definition.

**Definition 0.5.** Let \( G \rhd (X,\mu) \) be a non-singular action and let \( \mathcal{A} \) be a family of Borel subsets of \( X \). We say that the pair \((G,X)\) has property A with respect to the family \( \mathcal{A} \) if for every compact set \( K \subset G \) and every \( \varepsilon > 0 \), there exists a continuous family \( (\xi_g)_{g \in G} \) of unit vectors in \( L^2(X, \mu) \) such that

- \( ||\xi_g - \xi_h||_2 < \varepsilon \) whenever \( g^{-1}h \in K \)
- there is a set \( A \in \mathcal{A} \) such that each \( \xi_g \) is supported in \( gA \).
It is clear that every amenable pair has property A: take a $(K, \varepsilon)$-invariant vector $\xi \in L^2(X, \mu)$. We can assume that $\xi$ is supported in $A$ for some set $A \in A$. Set now $\xi_g = \pi_X(g)\xi$. In particular, if the measure class of $\mu$ contains a $G$-invariant probability measure, then we know that the pair $(G, X)$ is amenable and hence has property A. When $X$ is of the form $G/H$ where $H$ is a closed normal subgroup, then the pair $(G, G/H)$ has property A with respect to the family of compact sets in $G/H$ if and only if the group $G/H$ has property A as a locally compact group. Slightly more generally, when $H \subset G$ is a closed subgroup and $X = G/H$, then the pair $(G, X)$ has property A with respect to the family of all compact sets in $G/H$ if and only if $X = G/H$ has property A in the sense of Roe [Roe05], as a metric space.

In [Jol14], Jolissaint showed the following theorem. He proved the case with the Haagerup property before in [Jol00]. The statement about the weak Haagerup property was shown before by Knudby [Knu14], in the case where the action preserves an infinite measure and satisfies the Følner condition.

**Theorem 0.6** ([Jol14] theorem 1.4, see theorem 6.1 for our proper cocycle). Let $G, H$ be l.c.s.c. groups, let $G \rtimes (X, \mu)$ be a non-singular action such that the pair $(G, X)$ is amenable. Let $\omega : G \times X \to H$ be a proper cocycle. If $H$ has the Haagerup property, is weakly amenable, respectively has the weak Haagerup property, then so does $G$. Moreover, the weak amenability and weak Haagerup constant of $G$ is less than that of $H$:

$$\Lambda_{WA}(G) \leq \Lambda_{WA}(H) \text{ and } \Lambda_{WH}(G) \leq \Lambda_{WH}(H).$$

Together with example 0.4, this shows that the Haagerup property, weak amenability and the weak Haagerup property for locally compact groups pass to measure equivalence subgroups.

The main result of this paper is the following theorem.

**Theorem 0.7** (see theorem 6.2). Let $G, H$ be l.c.s.c. groups and let $G \rtimes (X, \mu)$ be a non-singular action. Suppose that $\omega : G \times X \to H$ is a proper cocycle with respect to some family $A$ and that the pair $(G, X)$ has property A with respect to the same family $A$. If $H$ has property A (respectively is coarsely embeddable), then $G$ has property A (respectively is coarsely embeddable).

It is clear from the above that theorem 0.1 is a direct consequence of theorem 0.7.

## 1. Preliminaries on approximation properties

Let $G$ be a l.c.s.c. group. A continuous function $\varphi : G \to \mathbb{C}$ is called a Herz-Schur multiplier if there exist bounded continuous functions $\xi, \eta : G \to \mathcal{H}$ from $G$ into Hilbert space $\mathcal{H}$, such that

$$\varphi(g^{-1}h) = \langle \xi_g, \eta_h \rangle \text{ for all } g, h \in G.$$ 

For a bounded function $\xi : G \to \mathcal{H}$, we set $\|\xi\|_\infty = \sup_{g \in G} \|\xi_g\|$. The Herz-Schur norm $\|\varphi\|_{HS}$ of a Herz-Schur multiplier $\varphi : G \to \mathbb{C}$ is the infimum of $\|\xi\|_\infty \|\eta\|_\infty$ where $\xi, \eta$ runs over all
pairs of continuous bounded functions \( \xi, \eta : G \to H \) from \( G \) into Hilbert space \( H \), satisfying 
\[ \varphi(g^{-1}h) = \langle \xi_g, \eta_h \rangle \]
for all \( g, h \in G \).

Suppose that \( \xi, \eta : G \to H \) are bounded Borel maps from \( G \) into a separable Hilbert space and that there is a function \( \varphi : G \to \mathbb{C} \) such that \( \varphi(g^{-1}h) = \langle \xi_g, \eta_h \rangle \). Then it follows automatically that \( \varphi \) is continuous, so it is a Herz-Schur multiplier. Moreover, its Herz-Schur norm is bounded above by \( \| \varphi \|_{HS} \leq \| \xi \|_{\infty} \| \eta \|_{\infty} \). So in the following statements, it does not matter if we take continuous maps or Borel maps. This was proven in the unpublished manuscript [Haa86]. Knudby included this proof also in his paper [Knu14, lemma C.1].

One of the possible definitions of amenability is the following.

**Definition 1.1.** A l.c.s.c. group \( G \) is called amenable if there exists a sequence \(( \varphi_n )_n \) of continuous compactly supported positive type functions \( \varphi_n : G \to \mathbb{C} \) such that \( \varphi_n(g) \) converges to \( 1 \) uniformly on compact subsets of \( G \).

If in the above definition, you replace the condition that \( \varphi_n \) is compactly supported by the condition that \( \varphi_n \) tends to 0 at infinity, you get the Haagerup property. If instead you fix a constant \( C \geq 1 \) and replace the condition that \( \varphi_n \) is of positive type by the condition that \( \varphi_n \) is a Herz-Schur multiplier with Herz-Schur norm bounded by \( \| \varphi_n \|_{HS} \leq C \), then you get the definition of weak amenability with weak amenability constant (Cowling-Haagerup constant) less than \( C \). Recently, Knudby [Knu14] studied the weak Haagerup property, where you apply both replacements above.

**Definition 1.2.** Let \( G \) be a l.c.s.c. group.

- We say that \( G \) has the Haagerup property if there exists a sequence \(( \varphi_n )_n \) of positive type functions \( \varphi_n \in C_0(G) \) such that \( \varphi_n(g) \) converges to \( 1 \) uniformly on compact subsets of \( G \).

- We say that \( G \) is weakly amenable if there exists a constant \( C \geq 1 \) and a sequence \(( \varphi_n )_n \) of continuous compactly supported Herz-Schur multipliers such that \( \varphi_n(g) \) converges to \( 1 \) uniformly on compact subsets of \( G \), and such that \( \| \varphi_n \|_{HS} \leq C \) for all \( n \in \mathbb{N} \). The infimum of all such \( C \) is called the weak amenability constant \( \Lambda_{WA}(G) \) of \( G \). This constant is also called the Cowling-Haagerup constant of \( G \) and is often denoted by \( \Lambda(G) \).

- We say that \( G \) is weakly Haagerup if there exists a constant \( C \geq 1 \) and a sequence \(( \varphi_n )_n \) of continuous Herz-Schur multipliers \( \varphi_n \in C_0(G) \) such that \( \varphi_n(g) \) converges to \( 1 \) uniformly on compact subsets of \( G \), and such that \( \| \varphi_n \|_{HS} \leq C \) for all \( n \in \mathbb{N} \). The infimum of all such \( C \) is called the weak Haagerup constant \( \Lambda_{WH}(G) \) of \( G \).

In [DL13], we introduced several equivalent definitions of property A and coarse embeddability of l.c.s.c. groups. We review the definitions that are relevant for the current paper.

**Definition 1.3** (see [DL13, theorem 2.3]). Let \( G \) be a l.c.s.c. group with left Haar measure \( \mu \). We say that \( G \) has property A if one of the following equivalent conditions holds

1. For every compact subset \( K \subset G \) and any \( \varepsilon > 0 \), there is a continuous family \(( \xi_g )_{g \in G} \) of unit vectors in \( L^2(G, \mu) \) such that
• \( \| \xi_g - \xi_h \| < \varepsilon \) whenever \( g^{-1}h \in K \)
• there is a compact set \( L \subseteq G \) such that each \( \xi_g \) is supported in \( gL \).

(2) For every compact subset \( K \subseteq G \) and any \( \varepsilon > 0 \) there is a continuous positive definite kernel \( k : G \times G \to \mathbb{C} \) such that
• \( |k(g, h) - 1| < \varepsilon \) whenever \( g^{-1}h \in K \)
• the set \( \{g^{-1}h \in G \mid k(g, h) \neq 0 \} \) is precompact.

**Definition 1.4** (see [DL13] theorem 3.4). Let \( G \) be a l.c.s.c. group. We say that \( G \) is coarsely embeddable (into a Hilbert space) if one of the following equivalent conditions holds

(1) there exists a Hilbert space \( H \) and a continuous map \( u : G \to H \) such that
• For every compact set \( K \subseteq G \), there is a real number \( R > 0 \) such that \( \| u(g) - u(h) \| < R \) for all \( g, h \in G \) with \( g^{-1}h \in K \).
• For every real number \( R > 0 \), the set \( \{g^{-1}h \in G \mid \| u(g) - u(h) \| < R \} \) is precompact.

(2) there exists a continuous conditionally negative definite kernel \( k : G \times G \to \mathbb{C} \) such that
• \( k \) is bounded on tubes, i.e. for every compact subset \( K \subseteq G \) there is a real number \( R > 0 \) such that \( k(g, h) < R \) for all \( g, h \in G \) with \( g^{-1}h \in K \)
• \( k \) is proper, i.e. for every real number \( R > 0 \), the set \( \{g^{-1}h \in G \mid k(g, h) < R \} \) is precompact.

By Schoenberg’s theorem, we can translate definition 1.4.2 to a statement that resembles definition 1.3.2. The proof is a standard application of Schoenberg’s theorem, see for example [CCJ+01] theorem 2.1.1. We give a complete proof, for the convenience of the readers that are not familiar with that argument.

**Theorem 1.5.** A l.c.s.c. group \( G \) is coarsely embeddable iff. for every compact subset \( K \subseteq G \) and any \( \varepsilon > 0 \) there is a continuous positive definite kernel \( k : G \times G \to \mathbb{C} \) such that

• \( |k(g, h) - 1| < \varepsilon \) whenever \( g^{-1}h \in K \)
• for every \( \delta > 0 \), the set \( \{g^{-1}h \in G \mid k(g, h) > \delta \} \) is precompact.

**Proof.** Let \( k : G \times G \to \mathbb{R} \) be a conditionally negative definite kernel as in definition 1.4.2. Observe that \( k \) takes positive values. Let \( K \subseteq G \) be a compact set and let \( \varepsilon > 0 \). By Since \( k \) is bounded on tubes, there is a real number \( R > 0 \) such that \( k(g, h) < R \) for all \( g, h \in G \) with \( g^{-1}h \in K \). Let \( t > 0 \) be such that \( 1 - \exp(-tR) < \varepsilon \). Schoenberg’s theorem asserts that the kernel \( k_0(g, h) = \exp(-tk(g, h)) \) is positive definite. It is easy to see that

\[
|k_0(g, h) - 1| = 1 - \exp(-tk(g, h)) \leq 1 - \exp(-tR) < \varepsilon
\]

whenever \( g^{-1}h \in K \). Moreover, let \( \delta > 0 \). Since \( k \) is a proper kernel, we know that the set

\[
\{g^{-1}h \mid |k_0(g, h)| > \delta\} = \{g^{-1}h \mid \exp(-tk(g, h)) > \delta\} = \left\{g^{-1}h \mid k(g, h) < \frac{-\log(\delta)}{t}\right\}
\]
is precompact.

On the other hand, suppose that $G$ satisfies the condition in theorem \[\text{[L5]}\]. We show that $G$ is coarsely embeddable. Fix an increasing sequence of compact sets $(K_n)_n$ in $G$ such that $G = \bigcup_n \overline{K_n}$, where $\overline{K_n}$ denotes the interior of $K_n$. We can assume that $e \in K_1$. Take a sequence $(\varepsilon_n)_n$ of positive real numbers that tends to infinity and let $(\varepsilon_n)_n$ be a sequence of strictly positive real numbers such that \(\sum_n c_n \varepsilon_n < \infty\). We can assume that all the $\varepsilon_n < \frac{1}{2}$. Then we find continuous positive definite kernels $(k_n)_n$ such that

- $|k_n(g, h) - 1| < \varepsilon_n$ whenever $g^{-1}h \in K_n$
- for every $\delta > 0$, the set \(\{g^{-1}h \in G \mid |k_n(g, h)| > \delta\}\) is precompact.

Define a new kernel $k : G \times G \to \mathbb{R}_+$ by the formula

\[
k(g, h) = \sum_n c_n \left(1 - \frac{|k_n(g, h)|^2}{k_n(g, g)k_n(h, h)}\right).
\]

An elementary computation shows that $k$ is conditionally negative definite. The series that defines $k$ is uniformly convergent on sets of the form \(\{(g, h) \in G \times G \mid g^{-1}h \in L\}\) for compact sets $L \subset G$: let $L$ be compact, then $L \subset K_N$ for some $N \in \mathbb{N}$. It follows that \(1 - \frac{|k_n(g, h)|^2}{k_n(g, g)k_n(h, h)} < \varepsilon_n\) for all $g, h \in G$ with $g^{-1}h \in L$ and $n \geq N$. Therefore we also get that

\[
\sum_{n > N} c_n \left(1 - \frac{|k_n(g, h)|^2}{k_n(g, g)k_n(h, h)}\right) < 8 \sum_{n > N} c_n \varepsilon_n \to 0 \text{ as } N \to \infty
\]

for all $g, h \in G$ with $g^{-1}h \in L$. As a consequence, we see that $k$ is well-defined, continuous and bounded on tubes.

It remains to show that $k$ is proper. Let $R > 0$ and take $n \in \mathbb{N}$ such that $c_n > 4R$. Then we see that $L = \{g^{-1}h \mid |k_n(g, h)| > \frac{1}{4}\}$ is precompact. Whenever $g, h \in G$ are such that $g^{-1}h \notin L$, we compute that

\[
k(g, h) \geq c_n \left(1 - \frac{|k_n(g, h)|^2}{k_n(g, g)k_n(h, h)}\right) \geq c_n \frac{1}{4} > R.
\]

\[\square\]

In fact, the continuity of the kernels $k : G \times G \to \mathbb{C}$, the families $(\xi_g)_{g \in G}$ and the coarse embedding $u : G \to H$ is not important, neither for property A nor for uniform embeddability. In the case of the family $(\xi_g)_g$ and the coarse embedding $u$, an argument was given in \[\text{[DL13]}\]. We give a short and elementary argument for the positive definite kernels $k$. The main idea of the argument is that we restrict the non-continuous kernel to a metric lattice in $G$ and then extend it back to a continuous kernel on $G \times G$, using a partition of unity in $C_c(G)$. A very similar argument was given in \[\text{[Roed05]}\] to show that property A passes from a metric lattice to the complete space.
Proposition 1.6. A l.c.s.c. group $G$ has property A (is coarsely embeddable) if and only if for every compact subset $K \subset G$ and every $\varepsilon > 0$, there is a (not necessarily continuous) positive definite kernel $k : G \times G \to \mathbb{C}$ such that

- $|k(g,h) - 1| < \varepsilon$ whenever $g, h \in G$ satisfy $g^{-1}h \in K$
- (in the property A case) the set $\{ g^{-1}h \mid k(g,h) \neq 0 \}$ is precompact
- (in the coarse embeddability case) for every $\delta > 0$, the set $\{ g^{-1}h \mid k(g,h) > \delta \}$ is precompact

Proof. It is clear that property A (respectively coarse embeddability) implies our condition. Suppose now that $G$ is a l.c.s.c. group that satisfies our condition. Fix a compact neighborhood $U$ of identity in $G$, and take continuous functions $f_n : G \to [0, 1]$, $(n \in \mathbb{N})$ with the following properties

- $\sum_n f_n(g) = 1$ for all $g \in G$ and the convergence is uniform on compact subsets of $G$.
- for every $g \in G$, there are only finitely many $n \in \mathbb{N}$ such that $f_n(g) \neq 0$.
- for every $n \in \mathbb{N}$, there is a group element $g_n \in G$ such that $\text{supp} f_n \subset g_nU$.

Let $K \subset G$ be a compact subset, and let $\varepsilon > 0$. By our condition, we find a positive definite kernel $k_0 : G \times G \to \mathbb{C}$ with the properties

- $|k_0(g,h) - 1| < \varepsilon$ whenever $g, h \in G$ satisfy $g^{-1}h \in UKU^{-1}$
- (in the property A case) the set $\{ g^{-1}h \mid k_0(g,h) \neq 0 \}$ is precompact
- (in the coarse embeddability case) for every $\delta > 0$, the set $\{ g^{-1}h \mid k_0(g,h) > \delta \}$ is precompact

Define a new kernel $k : G \times G \to \mathbb{C}$ by the formula

$$k(g,h) = \sum_{n,m} f_n(g)f_m(h)k_0(g_n,g_m) \text{ for all } g,h \in G.$$ 

This sum converges uniformly on compact subsets of $G \times G$, so it follows that $k$ is continuous. It remains to show that it satisfies the following conditions

1. $k$ is positive definite
2. $|k(g,h) - 1| < \varepsilon$ whenever $g, h \in G$ satisfy $g^{-1}h \in K$
3. (in the property A case) the set $\{ g^{-1}h \mid k(g,h) \neq 0 \}$ is precompact
4. (in the coarse embeddability case) for every $\delta > 0$, the set $\{ g^{-1}h \mid k(g,h) > \delta \}$ is precompact.
To prove (1), take $h_1, \ldots, h_s \in G$ and $c_1, \ldots, c_s \in \mathbb{C}$ arbitrarily. Then we compute that

$$
\sum_{i,j=1}^{s} c_i c_j k(h_i, h_j) = \sum_{n,m \in \mathbb{N}} \left( \sum_{i} c_i f_n(h_i) \right) \left( \sum_{j} c_j f_m(h_j) \right) k_0(g_n, g_m)
$$

$$
= \sum_{n,m \in \mathbb{N}} \left( \sum_{i} c_i f_n(h_i) \right) \left( \sum_{j} c_j f_m(h_j) \right) k_0(g_n, g_m).
$$

Observe that only finitely many terms in this last sum are non-zero, so the result is positive because $k_0$ is positive definite.

We prove condition (2) as follows. Let $g, h \in G$ be such that $g^{-1}h \in K$. If $n \in \mathbb{N}$ is such that $f_n(g) \neq 0$, then $g_n^{-1}g \in U$. Thus for every $n, m \in \mathbb{N}$ with $f_n(g)f_m(h) \neq 0$, we see that $g_n^{-1}g_m \in UKU^{-1}$, and hence $|k_0(g_n, g_m) - 1| < \varepsilon$. It follows that

$$
|k(g, h) - 1| = \left| \sum_{n,m} f_n(g)f_m(h)k_0(g_n, g_m) - \sum_{n,m} f_n(g)f_m(h) \right|
$$

$$
\leq \sum_{n,m} f_n(g)f_m(h) |k_0(g_n, g_m) - 1|
$$

$$
< \sum_{n,m} f_n(g)f_m(h) \varepsilon
$$

$$
= \varepsilon.
$$

To prove (3) and (4) at once, we show that for every $\delta \geq 0$,

$$
\{ g^{-1}h \mid |k(g, h)| > \delta \} \subset U^{-1} \{ g^{-1}h \mid |k_0(g, h)| > \delta \} U.
$$

The property A case corresponds to the case where $\delta = 0$. Let $g, h \in G$ be such that $|k(g, h)| > \delta$. Then there is at least one pair $n, m \in \mathbb{N}$ such that $f_n(g)f_m(h) \neq 0$ while $|k_0(g_n, g_m)| > \delta$. But then we get that

$$
g^{-1}h = g^{-1}g_n^{-1}g_m^{-1}h \in U^{-1} \{ g_n^{-1}h_0 \mid |k_0(g_n, h_0)| > \delta \} U.
$$


2. Proper cocycles

In this section, we introduce proper cocycles. Proper cocycles were first introduced by Jolissaint in [Jol00]. Our definition is inspired by his notion of proper cocycles, but we use a slightly weaker version. We did this because we could not show that measure correspondences give rise to proper cocycles in Jolissaint’s sense, but they do give rise to proper cocycles in our sense. Jolissaint’s results based on proper cocycles remain valid for our proper cocycles, as we show in section 6.

**Definition 2.1.** Let $G \curvearrowright (X, \mu)$ be a non-singular Borel action of a l.c.s.c. group $G$ on a standard probability space, and let $H$ be another l.c.s.c. group.
• A Borel map \( \omega : G \times X \to H \) is called a cocycle if for every \( g,h \in G \), the relation 
\[ \omega(gh,x) = \omega(g,hx)\omega(h,x) \]
holds for almost every \( x \in X \).

• Two cocycles \( \omega_1, \omega_2 : G \times X \to H \) are cohomologous if there is a Borel map \( \varphi : X \to H \) such that for every \( g \in G \) separately, we have that 
\[ \omega_2(g,x) = \varphi(gx)^{-1}\omega_1(g,x)\varphi(x) \]
for almost every \( x \in X \).

Whenever we say that \( \omega \) is a cocycle, we mean that \( \omega \) is a Borel cocycle. Similarly, when we say that \( G \curvearrowright (X,\mu) \) is a non-singular action, we mean that the action is Borel and that \( (X,\mu) \) is a standard probability space.

**Definition 2.2** (see also [Jol00]). Let \( G,H \) be l.c.s.c. groups and let \( G \curvearrowright (X,\mu) \) be a non-singular action.

• A cocycle \( \omega : G \times X \to H \) is said to be proper with respect to a family \( \mathcal{A} \) of Borel sets in \( X \) if
  1. for every compact subset \( K \subset G \) and every \( A,B \in \mathcal{A} \), we find a precompact set \( L(K,A,B) \subset H \) such that, for every \( g \in K \) we get that 
     \[ \omega(g,x) \in L(K,A,B) \]
     for almost all \( x \in A \cap g^{-1}B \)
  2. for every compact subset \( L \subset H \) and every \( A,B \in \mathcal{A} \), we get that the set \( K(L,A,B) \) of all \( g \in G \) such that 
     \[ \mu\{x \in X \mid x \in A, gx \in B, \omega(g,x) \in L \} > 0 \]
     is precompact in \( G \).

• A family \( \mathcal{A} \) of Borel sets in \( X \) is said to be large if it is closed under finite unions and under taking Borel subsets, and if for every \( \varepsilon > 0 \) there is a set \( A \in \mathcal{A} \) such that 
\[ \mu(X \setminus A) < \varepsilon \].

• A cocycle \( \omega : G \times X \to H \) is said to be proper if \( \omega \) is proper with respect to some large family \( \mathcal{A} \).

Throughout this section, we will use the following examples of large families.

**Observation 2.3.** Let \( (X,\mu) \) be a standard probability space.

• If \( \varphi : X \to H \) is a Borel map into a \( \sigma \)-compact space, then the family
  \[ \mathcal{A} = \{ A \subset X \mid A \text{ is Borel and } \varphi(A) \text{ is precompact} \} \]
is a large family.

• If \( (\mathcal{A}_n)_{n \in \mathbb{N}} \) is a countable sequence of large families in \( X \), then the intersection 
  \[ \mathcal{A} = \bigcap_{n \in \mathbb{N}} \mathcal{A}_n \]
is still large.

**Proof.** For the first point, observe that \( \mathcal{A} \) is clearly closed under finite unions and under taking Borel subsets. We can write \( H \) as a countable union \( H = \bigcup_n L_n \) of compact sets. So \( X \) is the countable union \( X = \bigcup_n A_n \), where \( A_n = \varphi^{-1}(L_n) \in \mathcal{A} \) for all \( n \). Since \( \mu \) is a probability measure, we get that the measure \( \mu(X \setminus A_n) \) tends to 0.
For the second point, it is clear that $A$ is closed under finite unions and under taking Borel subsets. Moreover, because the $A_n$ are closed under taking Borel subsets, we see that

$$A = \bigcap_n A_n = \left\{ \bigcap_n A_n : A_n \in A \right\}.$$  

We only have to show that, for every $\varepsilon > 0$, there is a set $A \in A$ such that $\mu(X \setminus A) < \varepsilon$. Let $\varepsilon > 0$. For every $n \in \mathbb{N}$, take a set $A_n \in A_n$ with measure $\mu(X \setminus A_n) < \frac{1}{2^n} \varepsilon$. Observe that $A = \bigcap_{n=1}^\infty A_n \in A$ and that

$$\mu(X \setminus A) \leq \sum_{n=1}^\infty \mu(X \setminus A_n) < \sum_{n=1}^\infty \frac{1}{2^n} \varepsilon = \varepsilon.$$  

We want to compare our notion of proper cocycle with Jolissaint’s notion. In order to do that, we call his type of proper cocycles “Jolissaint-proper”.

**Definition 2.4** ([Jol00]). Let $G, H$ be l.c.s.c. groups and let $G \acts (X, \mu)$ be a non-singular action. We say that a cocycle $\omega : G \times X \to H$ is Jolissaint-proper if there is a family $A$ such that

1. for every $A \in A$ and for every compact set $K \subset G$, the set $\omega(K \times A) \subset H$ is precompact.
2. for every compact set $L \subset H$ and every $A \in A$, we get that the set $K_J(L, A)$ of all $g \in G$ such that

$$\mu\{x \in X \mid x \in A, gx \in A, \omega(g, x) \in L\} > 0,$$

is precompact in $G$.
3. for every $\varepsilon > 0$, there is a set $A \in A$ such that $\mu(X \setminus A) < \varepsilon$.

Our first observation is that Jolissaint’s notion of proper cocycle is not invariant under cohomology, while ours is.

**Example 2.5.** Consider the action $\mathbb{R} \acts \mathbb{S}^1 = \mathbb{R} / \mathbb{Z}$, and the cocycle $\omega : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ that is defined by $\omega(g, x) = g$. This cocycle is clearly Jolissaint-proper, where we can take $A$ to be the family of all Borel subsets of $\mathbb{S}^1$. Take now any unbounded Borel function $\varphi : \mathbb{S}^1 \to \mathbb{R}$. Then the formula $\omega_1(g, x) = \varphi(gx)^{-1} \omega(g, x) \varphi(x)$ defines a new cocycle that is cohomologous to $\omega$. But $\omega_1$ is not Jolissaint-proper, because for every fixed $x \in X$, we get that $gx$ runs over all of $\mathbb{S}^1$ when $g \in K = [0, 1]$. So $\omega_1(K \times A)$ is not precompact for any non-empty set $A$. This shows that $\omega_1$ can never satisfy condition (1) from definition 2.4.

**Observation 2.6.** Let $G, H$ be l.c.s.c. groups and let $G \acts (X, \mu)$ be a non-singular action. Suppose that $\omega_1, \omega_2 : G \times X \to H$ are two cocycles that are cohomologous. If $\omega_1$ is proper, then $\omega_2$ is also proper.

**Proof.** Let $\varphi : X \to H$ be a Borel function such that for all $g \in G$, we have that $\omega_2(g, x) = \varphi(gx) \omega_1(g, x) \varphi(x)^{-1}$ for almost all $x \in X$. Suppose that $\omega_1$ is proper with respect to the
large family $\mathcal{A}$. Consider the large family $\mathcal{A}_1$ of all Borel sets $A \subset X$ such that $\varphi(A)$ is a precompact set in $H$. Then it is clear that $\omega_2$ is proper with respect to the large family $\mathcal{A} \cap \mathcal{A}_1$. □

If a cocycle $\omega$ is proper in our sense, with respect to a family $\mathcal{A}$, it is clear that $\omega$ is also proper with respect to the larger family $\mathcal{A}$ that consists of all Borel subsets of finite unions of elements in $\mathcal{A}$. So we can always assume that $\omega$ is a proper cocycle with respect to a family that is closed under finite unions and under taking Borel subsets. This last fact is also true for Jolissaint-properness, though it is a little bit less obvious.

**Proposition 2.7.** Let $G, H$ be l.c.s.c. groups and let $G \ltimes (X, \mu)$ be a non-singular action. If $\omega$ is Jolissaint-proper with respect to a family $\mathcal{A}$, then $\omega$ is also Jolissaint-proper with respect to a large family $\mathcal{A}$. In particular, it is a proper cocycle in our sense.

**Proof.** The only thing that is not clear is that we can assume that $\mathcal{A}$ is closed under taking finite unions. Since $X$ is the countable union of sets in $\mathcal{A}$ (up to measure 0), we find a sequence of sets $A_n \in \mathcal{A}$ such that $X$ is the countable disjoint union of their saturations, i.e. $X = \bigsqcup_n G A_n$, up to measure 0. The family $\mathcal{A}$ is now the set of all Borel subsets of sets of the form $K_1 A_1 \sqcup \ldots \sqcup K_n A_n$ where $n \in \mathbb{N}$ and $K_1, \ldots, K_n, A \subset G$ are compact. It is clear that $\mathcal{A}$ is a large family. We show that $\omega$ is Jolissaint-proper with respect to $\mathcal{A}$. For the first condition, let $K \subset G$ be compact and let $A \subset K_1 A_1 \sqcup \ldots \sqcup K_n A_n \in \mathcal{A}$. Then we see that

$$\omega(K \times A) \subset \bigcup_{k=1}^n \omega(K \times (K_k A_k)) \subset \bigcup_{k=1}^n \omega(K K_k \times A_k) \omega(K_k \times A_k)^{-1},$$

and this last set is precompact.

For the second condition, let $L \subset H$ be a compact set, and take a set $A \subset \bigsqcup_{k=1}^n K_k A_n$ in $\mathcal{A}$. Consider the precompact sets $L_k = \omega(K_k \times A_k)^{-1} L \omega(K_k \times A_k)$ in $H$ and define $K = \bigsqcup_{k=1}^n K_k K_f(L_k, A_k) K_k^{-1}$. Observe that $K$ is precompact. Let $g \in G$ be such that

$$\mu\{x \in X \mid x \in A, gx \in A, \omega(g, x) \in L\} > 0.$$

We show that $g \in K$. For every $x \in X$ with $x \in A$, $gx \in A$ and $\omega(g, x) \in L$, we find $k, l \leq n$ and $h_1 \in K_k, h_2 \in K_l$ such that $h_1^{-1} x \in A_k$ and $h_2^{-1} gx \in A_l$. Because the saturations of $A_k, A_l$ are disjoint for different $k, l$, we see that $k = l$. So $h_2^{-1} gh_1$ is an element in $G$ for which the measure

$$\mu\{x \in X \mid x \in A_k, h_2^{-1} gh_1 x \in A_k, \omega(h_2^{-1} gh_1, x) \in L_k\} > 0.$$

So $g \in K_k K_f(L_k, A_k) K_k^{-1} \subset K$. □

### 3. Measure correspondences

In this section, we introduce measure correspondences between arbitrary l.c.s.c. groups. This is used to define measure equivalence between l.c.s.c. groups. We show that this more general notion coincides with the classical one for unimodular groups. A good exposition of the unimodular case is given in [BFS13, Appendix A]. We follow a similar strategy in the general case.
**Definition 3.1.** Let $G, H$ be two l.c.s.c. groups and let $G \times H \acts (\Omega, \eta)$ be a non-singular action. In the following we consider the Haar measure on $G, H$. We say that $\Omega$ is a measure $G$-$H$-correspondence if there exist standard probability spaces $(X, \mu)$ and $(Y, \nu)$ and (almost everywhere defined) measure class preserving Borel isomorphisms $\varphi : X \times H \to \Omega$ and $\psi : Y \times G \to \Omega$ such that for all $h \in H$ we have that $\varphi(x, h) = h\varphi(x, k)$ for almost all $(x, k) \in X \times H$, and such that for every $g \in G$, we have that $\psi(y, gk) = g\psi(y, k)$ for almost all $(y, k) \in Y \times G$.

Standard examples of measure correspondences are the following:

- For every l.c.s.c. group $G$ with Haar measure, we have the identity $G$-$G$-correspondence $\Omega = G$ with the left-right action of $G \times G$.
- When $H_1, H_2 \subseteq G$ are closed subgroups, then we find an $H_1$-$H_2$-correspondence $\Omega = G$ with the left action of $H_1$ and the right action of $H_2$.
- For every two l.c.s.c. groups $G, H$ with Haar measure, we define the coarse $G$-$H$-correspondence $\Omega = G \times H$ with the left translation action of $G \times H$.
- A non-singular action $G \acts (X, \mu)$, induces a $G$-$G$-correspondence $\Omega = X \times G$ with an action of $G \times G$ that is defined by $(g, h) \cdot (x, k) = (gx, gkh^{-1})$.

In the rest of this section, all Borel maps are almost everywhere defined, and on all the l.c.s.c. groups we consider a probability measure that is equivalent to the Haar measure. We only consider equality almost everywhere.

**Observation 3.2.** Let $H$ be a l.c.s.c. group and let $(X, \mu)$ be a standard probability space. Let $\alpha_1 : X \to X$ be a measure class preserving Borel isomorphism, and let $\alpha_2 : X \to H$ be a Borel map. Then the formula

$$\alpha(x, h) = (\alpha_1(x), h\alpha_2(x)^{-1}) \quad \text{for almost all } (x, h) \in X \times H$$

defines a measure class preserving Borel isomorphism of $X \times H$ that commutes with the action of $H$. Moreover, all measure class preserving Borel isomorphisms of $X \times H$ that commute with the action of $H$ are of this form.

**Proof.** It is clear that every $\alpha$ of this form is a measure class preserving Borel isomorphism that commutes with the action of $H$.

When $\alpha$ is a measure class preserving Borel isomorphism, then we find Borel maps $\alpha_1 : X \times H \to X$ and $\alpha_2 : X \times H \to H$ such that $\alpha(x, h) = (\alpha_1(x, h), h\alpha_2(x, h)^{-1})$ for all $(x, h) \in X \times H$. When $\alpha$ commutes with the action of $H$, these maps $\alpha_1, \alpha_2$ are invariant under the action of $H$, so they essentially depend only on $x$. The map $\alpha_1$ is a measure class preserving Borel isomorphism because $\alpha$ is.

Suppose that $\Omega$ is a measure correspondence between two l.c.s.c. groups $G, H$. Then there exists a measure class preserving Borel isomorphism $\varphi : X \times H \to \Omega$ that commutes with the action of $H$. By observation 3.2 we find a non-singular action $G \acts X$ and a Borel cocycle $\omega : G \times X \to H$ such that

$$g\varphi(x, h) = \varphi(gx, h\omega(g, x)^{-1}) \quad \text{for almost all } (x, h) \in X \times H.$$
This action and cocycle are completely determined by the map \( \varphi \). We say that two non-singular actions \( G \curvearrowright (X, \mu) \) and \( G \curvearrowright (X', \mu') \) are isomorphic if there is a measure class preserving Borel isomorphism \( \Delta : X \to X' \) that commutes with the action of \( G \). Choosing a different Borel isomorphism \( \varphi \) yields an isomorphic action of \( G \) and a cohomologous cocycle \( \omega \). So the action and cocycle are well-defined by \( \Omega \) up to isomorphism and cohomology. Similarly, the Borel isomorphism \( \psi : Y \times G \to \Omega \) yields a non-singular action \( H \curvearrowright Y \) and a cocycle \( \beta : H \times Y \to G \). We say that the actions \( G \curvearrowright X, H \curvearrowright Y \) and the cocycles \( \omega : G \times X \to H \) and \( \beta : H \times Y \to G \) are associated to \( \Omega \).

There are two important operations on measure correspondences: composition and the opposite.

**Definition 3.3.** Let \( G, H, K \) be l.c.s.c. groups.

- If \( \Omega \) is a measure correspondence between \( G, H \), then the opposite measure correspondence \( \overline{\Omega} \) between \( H \) and \( G \) is \( \overline{\Omega} = \Omega \) with the obvious action of \( H \times G \).
- If \( \Omega_1 \) and \( \Omega_2 \) are measure correspondences between \( G, H \) and \( H, K \) respectively. Then we define \( \Omega_1 \otimes_H \Omega_2 \) to be the quotient of \( \Omega_1 \times \Omega_2 \) by the action of \( H \) that is given by \( h \cdot (x, y) = (hx, hy) \). On \( \Omega_1 \otimes_H \Omega_2 \) we consider the probability measure that is induced by the quotient map from \( \Omega_1 \times \Omega_2 \), and together with the action of \( G \times K \) that is given by \( (g, k) \cdot (x, y) = (gx, ky) \). Proposition 3.4 below shows that \( \Omega_1 \otimes_H \Omega_2 \) is a measure correspondence between \( G, K \).

**Proposition 3.4.** Let \( G, H, K \) be l.c.s.c. groups and let \((\Omega_1, \eta_1)\) and \((\Omega_2, \eta_2)\) be measure correspondences between \( G, H \) and \( H, K \) respectively. Let \( \varphi_1 : X_1 \times H \to \Omega_1, \psi_1 : Y_1 \times G \to \Omega_2, \varphi_2 : X_2 \times K \to \Omega_2 \) and \( \psi_2 : Y_2 \times H \to \Omega_2 \) be measure class preserving Borel isomorphisms as in definition 3.3. Consider the quotient map \( \pi : \Omega_1 \times \Omega_2 \to \Omega_1 \otimes_H \Omega_2 = \Omega \). Then the following maps are measure class preserving Borel isomorphisms.

\[
\varphi : X_1 \times X_2 \times K \to \Omega \quad \varphi(x_1, x_2, k) = \pi(\varphi_1(x_1, e), \varphi_2(x_2, k))
\]

\[
\psi : Y_1 \times Y_2 \times G \to \Omega \quad \psi(y_1, y_2, g) = \pi(\psi_1(y_1, g), \psi_2(y_2, e))
\]

The map \( \varphi \) clearly commutes with action of \( K \) and \( \psi \) commutes with the action of \( G \). This shows that \( \Omega_1 \otimes_H \Omega_2 \) is a measure correspondence.

**Proof.** By definition, it is clear that \( \varphi, \psi \) are Borel maps such that \( \varphi^{-1}(E) \) has measure 0 for every set \( E \) of measure 0, and similar for \( \psi \). So it suffices to give an inverse Borel map \( \varphi' \) for \( \varphi \) and \( \psi' \) for \( \psi \). If we write \( \varphi_1^{-1} : \Omega_1 \to X_1 \times H \) as \( \varphi_1^{-1}(x) = (u(x), s(x)) \) for almost all \( x \in \Omega_2 \), then we can define \( \varphi_0' : \Omega_1 \otimes_H \Omega_2 \to X_1 \times X_2 \times K \) by

\[
\varphi_0'(x, y) = (u(x), \varphi_2^{-1}(s(x)^{-1}y)).
\]

This function \( \varphi_0' \) is invariant under the action of \( H \) and hence defines a map \( \varphi' : \Omega_1 \otimes_H \Omega_2 \to X_1 \times X_2 \times K \). An elementary computation shows that \( \varphi' \) is the inverse of \( \varphi \). The same idea works for \( \psi \).

**Definition 3.5.** Let \((\Omega, \eta)\) be a measure correspondence between l.c.s.c. groups \( G, H \). Consider the non-singular actions \( G \curvearrowright (X, \mu) \) and \( H \curvearrowright (Y, \nu) \) associated to \( \Omega \). We say that \( G \curvearrowright (X, \mu) \) has an invariant probability measure if there exists a \( G \)-invariant probability measure on \( X \) that is equivalent with \( \mu \).
• We say that $\Omega$ is a measure equivalence coupling if both $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ have invariant probability measures. In that case we say that $G$ and $H$ are measure equivalent.

• We say that $\Omega$ is a measure equivalence subgroup coupling if $G \curvearrowright (X, \mu)$ has an invariant probability measure. In that case we say that $G$ is a measure equivalence subgroup of $H$.

It is now easy to see that the composition of two measure equivalence (subgroup) couplings is still a measure equivalence (subgroup) coupling.

We show that definition 3.5 is equivalent to the classical one for unimodular groups.

**Proposition 3.6.** Let $G, H$ be unimodular l.c.s.c. groups and let $(\Omega, \eta)$ be a measure equivalence coupling. Consider standard probability spaces $(X, \mu)$ and $(Y, \nu)$ together with measure class preserving Borel isomorphisms $\varphi : X \times H \rightarrow \Omega$ and $\psi : Y \times G \rightarrow \Omega$ as in definition 3.1. Then the following measures exist:

- An infinite $G \times H$-invariant measure $\eta'$ on $\Omega$, that is equivalent to $\eta$.
- Finite measures $\mu', \nu'$ on $X, Y$ that are equivalent to $\mu, \nu$ and such that the Borel isomorphisms

\[
\varphi : (X, \mu') \times (H, \mu_H) \rightarrow (\Omega, \eta')
\]

\[
\psi : (Y, \nu') \times (G, \mu_G) \rightarrow (\Omega, \eta')
\]

are measure preserving. The measures $\mu_G, \mu_H$ are Haar measures on $G, H$.

In particular, $(\Omega, \eta')$ is a measure equivalence coupling in the sense of [BFS13, definition 1.1].

**Proof.** Consider the actions $G \curvearrowright (X, \mu)$, $H \curvearrowright (Y, \nu)$ and cocycles $\omega, \beta$ associated to $\Omega$. Since $\Omega$ is a measure equivalence coupling, we can assume that $\mu$ and $\nu$ are invariant probability measures on $X$ and $Y$. Because $H$ is unimodular, we see that the action $G \times H \curvearrowright (X \times H)$ defined by $(g, h) \cdot (x, k) = (gx, hkw(g, x)^{-1})$ preserves the measure $\mu \times \mu_H$ where $\mu_H$ denotes the Haar measure on $H$. The push-forward measure $\eta_1 = \varphi_\#(\mu \times \mu_H)$ is preserved by the $G \times H$ action on $\Omega$. Similarly, $\eta_2 = \psi_\#(\nu \times \mu_G)$ is a $G \times H$-invariant measure on $\Omega$ that is equivalent to $\eta_1$. So we find a measurable $G \times H$-invariant function $f : \Omega \rightarrow (0, \infty)$ such that $d\eta_2(x) = f(x)d\eta_1(x)\cdot$. Consider the $G \times H$-invariant measure $\eta'$ on $\Omega$ that is defined by $d\eta'(x) = \min(1, f(x))d\eta_1(x)$. Then $\eta'$ is still $G \times H$-invariant and equivalent to $\eta$, and $\eta'$ is smaller than both $\eta_1$ and $\eta_2$.

The measure $\varphi_\#^{-1}\eta'$ is an $H$-invariant measure on $X \times H$, so it is of the form $\mu' \times \mu_H$ with $\mu'$ equivalent to $\mu$. Moreover, $\mu'$ is smaller that $\mu$ so it is still a finite measure, and $\varphi : (X, \mu') \times (H, \mu_H) \rightarrow (\Omega, \eta')$ is measure preserving. Similarly we find a finite measure $\nu'$ such that $\psi : (Y, \nu') \times (G, \mu_G) \rightarrow (\Omega, \eta')$ is measure preserving. \qed
4. Measure correspondences give rise to proper cocycles

In this section, we show that measure correspondences give rise to proper cocycles. For discrete groups, this is relatively easy, see [Jo114]. We give a complete proof for the general case.

**Theorem 4.1.** Let $G, H$ be l.c.s.c. groups and let $(\Omega, \eta)$ be a measure correspondence between $G$ and $H$. Consider the non-singular actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ associated with $\Omega$, together with the cocycles $\omega : G \times X \to H$ and $\beta : H \times Y \to G$. Then both $\omega$ and $\beta$ are proper cocycles.

**Proof.** By symmetry, we only have to show that $\omega$ is a proper cocycle. By results of Mackey, we can assume that the actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are everywhere defined Borel actions [Mac62], and by the Mackey cocycle theorem, we can assume that $\omega$ and $\beta$ are strict Borel cocycles. This means that the cocycle relation $\omega(gh, x) = \omega(g, hx)\omega(h, x)$ holds for all $g, h \in G$ and all $x \in X$ instead of almost all $x \in X$, and similarly for $\beta$.

We proceed in three steps.

**step 1:** For every compact set $K \subset G$, there is an increasing sequence $(A_n)_n$ of Borel sets in $X$ with $X = \bigcup_n A_n$ and such that for all $n \in \mathbb{N}$, the set

$$\{\omega(g, x) \mid x \in A_n, gx \in A_n, g \in K\}$$

is precompact.

Take an increasing sequence of compact sets $L_n \subset H$ such that $H = \bigcup_n L_n$. Denote $U_n = \omega^{-1}(L_n) \subset G \times X$. For every $x \in X$ we denote $U_{n,x} = \{g \in G \mid (g, x) \in U_n\}$.

Consider the left Haar measure on $G$ and fix a strictly positive Borel function $f : G \to \mathbb{R}$, with $\|f\|_2 = 1$. For every $n \in \mathbb{N}$ and $x \in X$, we write $f_{n,x} = \chi_{U_{n,x}} f$. It follows that, for every $x \in X$, we get that

$$\|f_{n,x} - f\|_2 \to 0 \text{ when } n \to \infty.$$

For two functions $f_1, f_2 \in L^2(G)$, we define a function $f_1 \star f_2 : G \to \mathbb{R}$ by the formula

$$(f_1 \star f_2)(g) = \int_G f_1(hg^{-1})f_2(h)dh.$$ 

It is easy to see that $\star$ is bilinear and that $|\langle f_1 \star f_2 \rangle(g)| \leq \Delta_G(g) \|f_1\|_2 \|f_2\|_2$ for all $g \in G$, where $\Delta_G$ is the modular function of $G$. Using a similar argument as in [Fol99] theorem 8.8, it is easy to see that $f_1 \star f_2$ is a continuous function.

Consider the functions $\tilde{f} = f \star f$ and $\tilde{f}_{n,x,y} = f_{n,y} \star f_{n,x}$. By the above, these functions are continuous and positive. Moreover, $\tilde{f}$ is strictly positive. Hence $\varepsilon = \min\{\tilde{f}(g)\Delta_G(g)^{-1} \mid g \in K\}$ is strictly positive. Moreover, we see that

$$\left|\langle \tilde{f} - \tilde{f}_{n,x,y} \rangle(g)\right| \leq \Delta_G(g)(\|f - f_{n,y}\|_2 + \|f - f_{n,x}\|_2),$$
for all \( n \in \mathbb{N} \) and \( x, y \in X \).

For every \( n \in \mathbb{N} \), we define an increasing sequence of Borel sets \( A_n \) in \( X \) by the relation

\[
A_n = \left\{ x \in X \middle| \| f_{n,x} - f \|_2 < \frac{1}{3^n} \right\}.
\]

It is clear that \( X = \bigcup_n A_n \). Fix \( n \in \mathbb{N} \) and suppose that \( x \in A_n \), \( gx \in A_n \) and \( g \in K \). Then we see that \( \bar{f}_{n,x,gx}(g) \geq \frac{1}{2} f(g) > 0 \). In particular, there is an \( h \in U_{n,x} \) with \( k = hg^{-1} \in U_{n,gx} \), so

\[
\omega(g, x) = \omega(k^{-1} h, x) = \omega(k, gx)^{-1} \omega(h, x) \in L_n^{-1} L_n.
\]

This last set is precompact.

**step 2:** There exist Borel maps \( s : X \to G \) and \( u : X \to Y \) such that for all \( g \in G \) we have that

\[
u(gx) = \omega(g, x)u(x) \text{ for almost all } x \in X
\]

\[
g = s(gx) \beta(\omega(g, x), u(x)) s(x)^{-1} \text{ for almost all } x \in X.
\]

Fix measure class preserving Borel isomorphisms \( \varphi : X \times H \to \Omega \) and \( \psi : Y \times G \to \Omega \) as in definition 3.1. We find Borel maps \( u_0 : X \times H \to Y \) and \( s_0 : X \times H \to G \) such that

\[
\varphi(x, h) = \psi(h u_0(x, h), s_0(x, h) \beta(h, u_0(x, h))^{-1})
\]

for almost all \((x, h) \in X \times H\). For every \( k \in H \), we see that

\[
\varphi(x, kh) = k \varphi(x, h)
\]

\[
\varphi(x, kh) = \psi(kh u_0(x, h), s_0(x, h) \beta(kh, u_0(x, h))^{-1})
\]

and \( \varphi(x, kh) = \psi(kh u_0(x, kh), s_0(x, kh) \beta(kh, u_0(x, kh))^{-1}) \)

for almost all \((x, h) \in X \times H\). It follows that \( u_0 \) and \( s_0 \) are invariant under the action of \( H \) and hence there are Borel functions \( u : X \to Y \) and \( s : X \to G \) such that \( u_0(x, h) = u(x) \) and \( s_0(x, h) = s(x) \) almost everywhere. These maps \( u, s \) satisfy

\[
\varphi(x, h) = \psi(h u(x), s(x) \beta(h, u(x))^{-1})
\]

for almost all \((x, h) \in X \times H\).

For every \( g \in G \), we compute that for almost all \((x, h) \in X \times H\),

\[
g \varphi(x, h) = \varphi(gx, h \omega(g, x)^{-1})
\]

\[
= \psi(h \omega(g, x)^{-1} u(gx), s(gx) \beta(h \omega(g, x)^{-1}, u(gx))^{-1})
\]

and \( g \varphi(x, h) = \psi(h u(x), g s(x) \beta(h, u(x))^{-1}) \)

It follows that for all \( g \in G \) we have that

\[
u(gx) = \omega(g, x)u(x) \text{ for almost all } x \in X
\]

\[
gs(x) = s(gx) \beta(\omega(g, x)^{-1}, u(gx))^{-1}
\]
This finishes the proof of step 2.

**Step 3:** The cocycle $\omega$ is a proper cocycle.

Fix an increasing sequence $(K_n)_n$ of precompact open subsets of $G$ such that $G = \bigcup_n K_n$. For every $n \in \mathbb{N}$, step 1 gives us an increasing sequence $A_{n,k}$ of Borel sets in $X$ with $X = \bigcup_k A_{n,k}$ and such that for all $n, k \in \mathbb{N}$, we get that the set

$$\{ \omega(g, x) \mid x, gx \in A_{n,k}, g \in K_n \}$$

is precompact.

Similarly, for an increasing sequence $(L_n)_n$ of precompact open subsets of $H$ with $H = \bigcup_n L_n$, we find similar increasing sequences of Borel sets $B_{n,k}$ in $Y$ such that $\bigcup_k B_{n,k} = Y$ and for all $n, k \in \mathbb{N}$, we get that

$$\{ \beta(h, y) \mid y, hy \in B_{n,k}, h \in L_n \}$$

is precompact.

By step 2, we find Borel maps $s : X \to G$ and $u : X \to Y$ such that for all $g \in G$ we have that

$$u(gx) = \omega(g, x)u(x) \quad \text{for almost all } x \in X$$

$$g = s(gx) \beta(\omega(g, x), u(x))s(x)^{-1} \quad \text{for almost all } x \in X.$$

Consider the family $\mathcal{A}$ of all Borel sets $A \subseteq X$ such that $s(A)$ is precompact and such that for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $A \subseteq A_{n,k}$ and $u(A) \subseteq B_{n,k}$. By observation 2.3 this is a large family. We show that $\omega$ is a proper cocycle with respect to $\mathcal{A}$.

To show condition (1) from definition 2.2, take a compact set $K \subseteq G$ and two sets $A, B \in \mathcal{A}$. Then there is an $n \in \mathbb{N}$ such that $K \subseteq K_n$. Moreover, there is a $k \in \mathbb{N}$ such that $A, B \subseteq A_{n,k}$. Now we see that

$$\{ \omega(g, x) \mid x \in A, gx \in B, g \in K \} \subseteq \{ \omega(g, x) \mid x, gx \in A_{n,k}, g \in K_n \},$$

and the second set was assumed to be precompact.

To show condition (2), let $L \subseteq H$ be a compact set and take $A, B \in \mathcal{A}$. Then there is an $n \in \mathbb{N}$ such that $L \subseteq L_n$, and there is a $k \in \mathbb{N}$ with $u(A), u(B) \subseteq B_{n,k}$. Moreover, $s(A)$ and $s(B)$ are precompact. Let $g \in G$. For almost all $x \in X$ with $x \in A, gx \in B$ and $\omega(g, x) \in L$, we can make the following computation

$$g = s(gx) \beta(\omega(g, x), u(x))s(x)^{-1} \in s(B) \{ \beta(h, y) \mid y, hy \in B_{n,k}, h \in L \} s(A)^{-1}.$$

Observe that this last set is precompact and denote it by $K(L, A, B) \subseteq G$. Whenever the set

$$\{ x \in A \cap g^{-1}B \mid \omega(g, x) \in L \}$$

is non-null, we see that $g \in K(L, A, B)$. □
Jolissaint showed in [Jol00] that, when \( \omega : G \times X \to H \) is a proper cocycle, \((G,X)\) is an amenable pair and \( H \) has the Haagerup property, then \( G \) has the Haagerup property. We will show that the same is true for property A and coarse embeddability instead of the Haagerup property. For property A and coarse embeddability, we will be able to weaken the condition that \((G,X)\) is an amenable pair to the case where the pair \((G,X)\) has property A. For the convenience of the reader, we also review the definition of an amenable pair, see [Eym72, Gre69, Zim78, Jol96].

**Definition 5.1.** Let \( G \curvearrowright (X, \mu) \) be a non-singular action of a l.c.s.c. group on a standard probability space. Consider the Koopman representation \( \pi : G \to U(L^2(X, \mu)) \).

- We say that the pair \((G,X)\) is amenable if there is a sequence of almost invariant unit vectors \( \xi_n \in L^2(X, \mu) \), i.e.
  \[
  \| \xi_n - \pi(g)\xi_n \| \to 0 \text{ uniformly on compact subsets of } G.
  \]

- Let \( \mathcal{A} \) be a large family of Borel subsets of \( X \). We say that the pair \((G,X)\) has property A with respect to \( \mathcal{A} \) if for every compact set \( K \subset G \) and every \( \varepsilon > 0 \), there exists a continuous family \((\xi_g)_{g \in G}\) of unit vectors in \( L^2(X, \mu) \) such that
  
  \[\begin{align*}
  &\|\xi_g - \xi_h\|_2 < \varepsilon \text{ whenever } g^{-1}h \in K \\
  &\text{there is a set } A \in \mathcal{A} \text{ such that each } \xi_g \text{ is supported in } gA.
  \end{align*}\]

The standard examples are the following. Suppose that \( H \subset G \) is a closed normal subgroup and denote \( Q = G/H \). Then the the pair \((G,Q)\) is an amenable pair if and only if the group \( Q \) is amenable. The group \( Q \) has property A if and only if the pair \((G,Q)\) has property A with respect to the family of all precompact Borel sets in \( Q \).

If \( \mu \) is a \( G \)-invariant probability measure on \( X \), then we see that the pair \((G,X)\) is amenable, because the constant function \( 1 : X \to \mathbb{C} \) is an invariant vector in \( L^2(X, \mu) \).

**Lemma 5.2.** Let \( G \curvearrowright (X, \mu) \) be a non-singular action of a l.c.s.c. group on a standard probability space. If \((G,X)\) is an amenable pair, then \((G,X)\) has property A with respect to any large family \( \mathcal{A} \).

**Proof.** Let \( K \subset G \) be compact and let \( \varepsilon > 0 \). Since \((G,X)\) is an amenable pair, we find a unit vector \( \xi \in L^2(X, \mu) \) such that \( \|\xi - \pi(g)\xi\| < \frac{\varepsilon}{3} \) for all \( g \in K \). Because \( \mathcal{A} \) is a large family, we find a set \( A \in \mathcal{A} \) such that \( \xi_0 = \chi_A\xi \) satisfies \( \|\xi_0 - \xi\| < \frac{\varepsilon}{3} \). Set \( \xi_g = \pi(g)\xi_0 \), then we see that \((\xi_g)_{g \in G}\) is a continuous family of unit vectors where every \( \xi_g \) is supported in \( gA \). Moreover, we compute that

\[\|\xi_g - \xi_h\| = \|\xi_0 - \pi(g^{-1}h)\xi_0\| \leq 2\|\xi - \xi_0\| + \|\xi - \pi(g^{-1}h)\xi\| < \varepsilon\]

for all \( g, h \in G \) with \( g^{-1}h \in K \). \( \square \)
This section is devoted to the proofs of theorems \[0.6\] and \[0.7\]. Theorem \[0.6\] is essentially the same as the main result of \[Jol14\], but we use a slightly weaker notion of proper cocycle. Although the proof of theorem \[0.6\] is very similar to the proof of \[Jol14\], we include a proof to assure the reader that \[Jol14\] remains valid for our weaker notion of proper cocycle. This shows that the Haagerup property, weak amenability and the weak Haagerup property pass to measure equivalence subgroups.

**Theorem 6.1** (theorem \[0.6\] see also \[Jol14\]). Let \( G \acts (X,\mu) \) be a non-singular action of a l.c.s.c. group. Let \( \omega : G \times X \to H \) be a proper cocycle with values in another l.c.s.c. group \( H \). Assume that the pair \((G,X)\) is amenable. If \( H \) has the Haagerup property (respectively is weakly amenable respectively has the weak Haagerup property), then so has \( G \). Moreover, the weak amenability and weak Haagerup constants satisfy \( \Lambda_{WA}(G) \leq \Lambda_{WA}(H) \) and \( \Lambda_{WH}(G) \leq \Lambda_{WH}(H) \).

**Proof.** Suppose that \( H \) has the Haagerup property (respectively is weakly amenable with constant \( \Lambda_{WA}(H) < \Lambda \), respectively has the weak Haagerup property with constant \( \Lambda_{WH}(H) < \Lambda \)). We have to show that \( G \) has the Haagerup property (respectively is weakly amenable with constant \( \Lambda_{WA}(H) < \Lambda \), respectively has the weak Haagerup property with constant \( \Lambda_{WH}(H) < \Lambda \)).

Let \( K \subset G \) be compact and let \( \varepsilon > 0 \). Since \( \omega \) is a proper cocycle, it is proper with respect to a large family \( \mathcal{A} \) of Borel subsets of \( X \). Denote the Koopman representation of \( G \acts (X,\mu) \) by \( \pi : G \to \mathcal{U}(L^2(X,\mu)) \). Amenability of the pair \((G,X)\) gives a unit vector \( \xi \in L^2(X,\mu) \) such that \( \|\xi - \pi_g\xi\|_2 < \varepsilon \) for all \( g \in K \). We can assume that \( \xi \) is supported in a set \( A \in \mathcal{A} \). Properness of the cocycle \( \omega \) gives us a compact set \( L \subset H \) such that for all \( g \in K \) we have that \( \omega(g,x) \in L \) for almost all \( x \in A \cap g^{-1}A \). We find a continuous function \( f_0 : H \to \mathbb{C} \) such that

- \( |f_0(h) - 1| < \varepsilon \) for all \( h \in L \).
- (in the case of the Haagerup property) \( f_0 \) is of positive type.
- (in the case of weak amenability and the weak Haagerup property) \( f_0 \) is a Herz-Schur multiplier with norm \( \|f_0\|_{HS} \leq \Lambda \).
- (in the case of the Haagerup property and the weak Haagerup property) \( f_0 \) is a \( C_0 \) function.
- (in the case of weak amenability) \( f_0 \) is compactly supported.

Define a Borel function \( f : G \to \mathbb{C} \) by the formula

\[
  f(g) = \int_X \overline{\xi(x)} (\pi_g\xi)(x) f_0(\omega(g,g^{-1}x)) \, d\mu(x) \quad \text{for all } g \in G.
\]

We have to show that \( f \) satisfies the following properties.

- \( |f(g) - 1| < 2\varepsilon \) for all \( g \in K \).
(in the case of the Haagerup property) \( f \) is of positive type.

1. (in the case of weak amenability and the weak Haagerup property) \( f \) is a Herz-Schur multiplier with norm \( \|f\|_{HS} \leq \Lambda \).

1. \( f \) is continuous.

1. (in the case of the Haagerup property and the weak Haagerup property) \( f \) is a \( C_0 \) function.

1. (in the case of weak amenability) \( f \) is compactly supported.

The first property follows from the following computation:

\[
|f(g) - 1| = \left| \int_X \overline{\xi(x)} (\pi_g \xi)(x) f_0(\omega(g, g^{-1} x))d\mu(x) - 1 \right|
\leq \int_X |\xi(x)| |(\pi_g \xi)(x)| |f_0(\omega(g, g^{-1} x)) - 1| d\mu(x) + |\langle \xi, \pi_g \xi \rangle - 1|
\]

When \( g \in K \), we have chosen \( \xi \) such that \( |\langle \xi, \pi_g \xi \rangle - 1| < \varepsilon \). In the same case, for almost all \( x \in X \) with \( \xi(x) \neq 0 \) and \( (\pi_g \xi)(x) \neq 0 \), we have that \( x \in A \) and \( g^{-1} x \in A \) and hence that \( \omega(g, g^{-1} x) \in L \). It follows that \( |f_0(\omega(g, g^{-1} x)) - 1| < \varepsilon \), so we also see that

\[
\int_X |\xi(x)| |(\pi_g \xi)(x)| |f_0(\omega(g, g^{-1} x)) - 1| d\mu(x) < |\langle \xi, \pi_g |\xi| \rangle \varepsilon \leq \varepsilon.
\]

This implies that \( |f(g) - 1| < 2\varepsilon \).

Observe that \( f \) is of positive type if and only if there is a separable Hilbert space \( H \) and a bounded continuous function \( \eta : G \to H \) such that \( f(g^{-1} h) = \langle \eta_g, \eta_h \rangle \). So conditions 2 and 3 follow from the property below. Moreover, continuity of \( f \) then follows from [Haa86, Appendix A] (see also [Knu14, lemma C.1]).

- If \( H_0 \) is a Hilbert space and \( \eta^0, \zeta^0 : G \to H_0 \) are bounded continuous functions such that \( f_0(g^{-1} h) = \langle \eta^0_g, \zeta^0_h \rangle \) for all \( g, h \in H \). Consider \( \mathcal{H} = L^2(X, \mu) \otimes H_0 \) and define \( \eta, \zeta : G \to \mathcal{H} \) by the formula

\[
\eta_g(x) = (\pi_g \xi)(x) \eta_{\omega(g, g^{-1} x)} \text{ and } \zeta_g(x) = (\pi_g \xi)(x) \zeta_{\omega(g, g^{-1} x)}
\]

for all \( g \in G \) and \( x \in X \). Then it follows that \( f(g^{-1} h) = \langle \eta_g, \zeta_h \rangle \) for all \( g, h \in G \) and moreover we have that \( \|\eta\|_\infty = \|\eta^0\|_\infty \) and \( \|\zeta\|_\infty = \|\zeta^0\|_\infty \).

We now prove this property. Observe that, for all \( g, h \in G \) and almost all \( x \in X \), we have that

\[
\omega(g^{-1} h, h^{-1} gx) = \omega(g, g^{-1} gx)^{-1} \omega(h, h^{-1} gx),
\]
and hence we compute that
\[
f(g^{-1}h) = \int_X \xi(x)(\pi_g^{-1}h\xi)(x)f_0(\omega(g^{-1}h, h^{-1}gx))d\mu(x)
\]
\[
= \int_X \xi(x)(\pi_g^{-1}h\xi)(x)f_0(\omega(g, g^{-1}gx)^{-1}\omega(h, h^{-1}gx))d\mu(x)
\]
\[
= \int_X (\pi_g\xi)(x)(\pi_h\xi)(x)f_0(\omega(g, g^{-1}x)^{-1}\omega(h, h^{-1}x))d\mu(x)
\]
\[
= \int_X (\pi_g\xi)(x)(\pi_h\xi)(x)\langle \eta^0_{\omega(g,g^{-1}x)}, \xi^0_{\omega(h,h^{-1}x)} \rangle d\mu(x)
\]
\[
= \langle \eta_g, \xi_h \rangle.
\]
For every \( g \in G \), the norm of \( \eta_g \) can be computed as follows:
\[
\| \eta_g \|^2 = \int_X |(\pi_g\xi)(x)|^2 \| \eta^0_{\omega(g,g^{-1}x)} \|^2 d\mu(x)
\]
\[
\leq \| \pi_g\xi \|^2_2 \| \eta^0 \|^2_\infty = \| \eta^0 \|^2_\infty.
\]

The last two properties follow from the following argument. Let \( \delta > 0 \) in the case of the 5th property and let \( \delta = 0 \) in the case of the last property. We know that \( L_0 = \{ h \in H \mid |f_0(h)| > \delta \} \) is precompact, and we have to show that \( K_0 = \{ g \in G \mid |f(g)| > \delta \} \) is precompact. Since \( \omega \) is a proper cocycle, we find a precompact set \( K_1 \) such that \( g \in K_1 \) whenever the set
\[
\{ x \in X \mid x \in A, gx \in A \text{ and } \omega(g, x) \in L_0 \}
\]
is non-null. Suppose that \( |f(g)| > \delta \), then there is a non-null set of \( x \in X \) such that \( \xi(x) \neq 0 \), \( (\pi_g\xi)(x) \neq 0 \) and \( f_0(\omega(g, g^{-1}x)) > \delta \). All these \( x \in X \) satisfy \( x \in A, g^{-1}x \in A \) and \( \omega(g, g^{-1}x) \in L_0 \). It follows that \( g \in K_1 \).

**Theorem 6.2** (Theorem 0.7). Let \( G \curvearrowright (X, \mu) \) be a non-singular action of a l.c.s.c. group on a standard probability space, and let \( A \) be a large family such that the pair \((G, X)\) has property \( A \) with respect to \( A \). Let \( \omega : G \times X \to H \) be a proper cocycle with respect to the same family \( A \). If \( H \) is a l.c.s.c. group with property \( A \) (respectively is coarsely embeddable), then \( G \) has property \( A \) (respectively is coarsely embeddable).

*Proof.* Denote the Koopman representation by \( \pi : G \to U(L^2(X, \mu)) \). Suppose that \( H \) has property \( A \) (resp. is coarsely embeddable). We show that \( G \) has property \( A \) (resp. is coarsely embeddable). Let \( K \subset G \) be a compact subset and let \( \varepsilon > 0 \).

Since the pair \((G, X)\) has property \( A \), we find a continuous family \((\xi_g)_{g \in G}\) of unit vectors in \( L^2(X, \mu) \) such that
\[
\| \xi_g - \xi_h \| < \frac{\varepsilon}{2} \text{ whenever } g^{-1}h \in K.
\]
\[
\text{there is a set } A \in A \text{ such that every } \xi_g \text{ is supported in } gA.
\]

Properness of the cocycle \( \omega \) gives us a compact set \( L \subset H \) such that for all \( g \in K \) and almost all \( x \in A \cap g^{-1}A \) we get that \( \omega(g, x) \in L \).
Because the group $H$ has property A (resp. is coarsely embeddable), we find a continuous positive definite kernel $k_0 : H \times H \to \mathbb{C}$ such that

- $|k_0(g, h) - 1| < \frac{\varepsilon}{2}$ whenever $g^{-1}h \in L$
- (in the property A case) the set $\{g^{-1}h \mid k_0(g, h) \neq 0\}$ is precompact in $H$
- (in the coarse embeddability case) for all $\delta > 0$ we have that the set $\{g^{-1}h \mid |k_0(g, h)| > \delta\}$ is precompact.

Define a kernel $k : G \times G \to \mathbb{C}$ by the formula

$$k(g, h) = \int_X \xi_g(x)k_0(\omega(g, g^{-1}x), \omega(h, h^{-1}x))d\mu(x).$$

We show that this kernel satisfies the conditions of definition 1.3 (resp. definition 1.4):

1. $k$ is a positive definite kernel.
2. $|k(g, h) - 1| < \frac{\varepsilon}{2}$ whenever $g^{-1}h \in K$
3. (in the property A case) the set $\{g^{-1}h \mid k(g, h) \neq 0\}$ is precompact in $G$
4. (in the coarse embeddability case) for all $\delta > 0$ we have that the set $\{g^{-1}h \mid |k(g, h)| > \delta\}$ is precompact.

To prove (1), let $g_1, \ldots, g_n \in G$ and let $c_1, \ldots, c_n \in \mathbb{C}$. Then we see that

$$\sum_{i,j=1}^n \bar{c}_i c_j k(g_i, g_j) = \int_X \sum_{i,j=1}^n \xi_{g_i}(x) \xi_{g_j}(x) c_i c_j k_0(\omega(g_i, g_i^{-1}x), \omega(g_j, g_j^{-1}x))d\mu(x).$$

The integrand in the right hand side is positive for every $x \in X$ separately, because $k_0$ is a positive definite kernel.

To prove (2), let $g, h \in G$ be such that $g^{-1}h \in K$. Then we see that, for almost all $x \in X$,

$$\omega(g, g^{-1}x)^{-1}\omega(h, h^{-1}x) = \omega(g^{-1}, hh^{-1}x)\omega(h, h^{-1}x) = \omega(g^{-1}h, h^{-1}x).$$

Moreover, when $\xi_g(x) \neq 0 \neq \xi_h(x)$, then we get that both $h^{-1}x$ and $g^{-1}hh^{-1}x = g^{-1}x$ are elements of $A$. It follows that $\omega(g^{-1}h, h^{-1}x) \in L$ for almost all $x \in X$ with $\xi_g(x) \neq 0 \neq \xi_h(x)$. In particular, we see that for almost all such $x \in X$, $|k_0(\omega(g, g^{-1}x), \omega(h, h^{-1}x)) - 1| < \frac{\varepsilon}{2}$. We compute that

$$|k(g, h) - 1| \leq \int_X |\xi_g(x)| \ |\xi_h(x)| \ |k_0(\omega(g, g^{-1}x), \omega(h, h^{-1}x)) - 1| \ d\mu(x)$$

$$\quad + \left| \int_X \xi_g(x)\xi_h(x)d\mu(x) - 1 \right|$$

$$\leq \frac{\varepsilon}{2} \ |\xi_g| |\xi_h| + |\langle \xi_g, \xi_h \rangle - 1|$$

$$\leq \frac{\varepsilon}{2} \ |\xi_g| \ |\xi_h| + \|\xi_g - \xi_h\| \ |\xi_h|$$

$$< \varepsilon.$$
In order to prove (3) and (4) we prove the following assertion

- Let $\delta \geq 0$. If $\{g^{-1}h \in H \mid |k_0(g, h)| > \delta\}$ is precompact, then also $\{g^{-1}h \in G \mid |k(g, h)| > \delta\}$ is precompact.

Statement (3) follows by setting $\delta = 0$. Assume that $\tilde{L} = \{g^{-1}h \in H \mid |k_0(g, h)| > \delta\}$ is precompact. Since $\omega$ is a proper cocycle with respect to $A$, we find that $\tilde{K} = \left\{ g \in G \mid \mu\{x \in A \cap g^{-1}A \mid \omega(g, x) \in \tilde{L}\} \geq 0 \right\}$ is precompact. Suppose that $g, h \in G$ are such that $|k(g, h)| > \delta$. It follows that there is a non-null Borel set $B \subset X$ such that $|k_0(\omega(g, g^{-1}x), \omega(h, h^{-1}x))| > \delta$ for all $x \in B$. It follows that $\omega(g^{-1}h, h^{-1}x) \in \tilde{L}$ for all $x \in B$, and hence that $g^{-1}h \in \tilde{K}$.

\[\square\]

REFERENCES

[BCH94] P. Baum, A. Connes, and N. Higson, Classifying space for proper actions and k-theory of group $C^*$-algebras, Contemp. Math., 167, Amer, Math. Soc., 1994.

[BFS13] U. Bader, A. Furman, and R. Sauer, Integrable measure equivalence and rigidity of hyperbolic lattices, Invent. Math. 194 (2013), no. 2, 313–379. MR 3117525

[CCJ+01] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette, Groups with the Haagerup property, Progress in Mathematics, vol. 197, Birkhäuser Verlag, Basel, 2001, Gromov’s a-T-menability. MR 1852148 (2002h:22007)

[CEOO04] J. Chabert, S. Echterhoff, and H. Oyono-Oyono, Going-down functors, the k"unneth formula, and the baum-connes conjecture, Geom. Funct. Anal. 14(3): 491-528, 2004.

[DL13] S. Deprez and K. Li, Property a and uniform embedding for locally compact groups, preprint (2013), arXiv:1309.7290

[Eym72] P. Eymard, Moyennes invariantes et représentations unitaires, Lecture Notes in Mathematics, Vol. 300, Springer-Verlag, Berlin, 1972. MR 0447969 (56 #6279)

[Fol99] G. B. Folland, Real analysis, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999, Modern techniques and their applications, A Wiley-Interscience Publication, MR 1681462 (2000c:00001)

[Gre69] F. P. Greenleaf, Amenable actions of locally compact groups, J. Functional Analysis 4 (1969), 295–315. MR 0246999 (40 #268)

[Gro93] M. Gromov, Asymptotic invariants of infinite groups, Geometric Group Theory, Cambridge University Press, 1-295, 1993.

[Haas86] U. Haagerup, Group $C^*$-algebras without the completely bounded approximation property, unpublished (1986).

[Hig00] N. Higson, Bivariant k-theory and the novikov conjecture, Geom. Funct. Anal. 10(3): 563-581, 2000.

[HP06] U. Haagerup and A. Przybyszewska, Proper metrics on locally compact groups, and proper affine isometric actions on banach spaces, arXiv:math/0606794v1, 2006.

[HR00] N. Higson and J. Roe, Amenable group actions and the novikov conjecture, J. Reine Angew. Math., 519, 143-153, 2000.

[Jol96] P. Jolissaint, Invariant states and a conditional fixed point property for affine actions, Math. Ann. 304 (1996), no. 3, 561–579. MR 1375626 (97d:46080)
[Jol00] ______, Borel cocycles, approximation properties and relative property $T$, Ergodic Theory Dynam. Systems 20 (2000), no. 2, 483–499. MR 1756981 (2001f:22015)

[Jol14] ______, Proper cocycles and weak forms of amenability, preprint (2014), arXiv:1403.0207

[Knu14] S. Knudby, The weak haagerup property, preprint (2014), arXiv:1401.7541

[Mac52] G. W. Mackey, Induced representations of locally compact groups. I, Ann. of Math. (2) 55 (1952), 101–139. MR 0044536 (13,434a)

[Mac62] ______, Point realizations of transformation groups, Illinois J. Math. 6 (1962), 327–335. MR 0143874 (26 #1424)

[Oza00] N. Ozawa, Amenable actions and exactness for discrete groups, C. R. Acad. Sci. Paris Ser. I Math., 330, 691–695., 2000.

[Roe05] J. Roe, Warped cones and property a, Geometry and Topology, Vol. 9, 163-178, 2005.

[Str74] R. A. Struble, Metrics in locally compact groups, Compositio Mathematica, Vol 28, no 3, 217-222., 1974.

[STY02] G. Skandalis, J. L. Tu, and G. Yu, The coarse baum-connes conjecture and groupoids, Topology 41, 807-834, 2002.

[Yu00] G. Yu, The coarse baum-connes conjecture for spaces which admit a uniform embedding into hilbert space, Invent. Math., 139, 201-240, 2000.

[Zim78] R. J. Zimmer, Amenable pairs of groups and ergodic actions and the associated von Neumann algebras, Trans. Amer. Math. Soc. 243 (1978), 271–286. MR 502907 (81e:22008)