CRT and Fixed Patterns in Combinatorial Sequences

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Abstract. In this paper, new context of Chinese Remainder Theorem (CRT) based analysis of combinatorial sequence generators has been presented. CRT is exploited to establish fixed patterns in LFSR sequences and underlying cyclic structures of finite fields. New methodology of direct computations of DFT spectral points in higher finite fields from known DFT spectra points of smaller constituent fields is also introduced. Novel approach of CRT based structural analysis of LFSR based combinatorial sequence is given both in time and frequency domain. The proposed approach is demonstrated on some examples of combiner generators and is scalable to general configuration of combiner generators.

Keywords: CRT, LFSR, DFT, combinatorial generators.

1 Introduction

Chinese Remainder Theorem (CRT) is known for centuries as a solution of congruences in number theory and was appeared in a mathematical classics of Sun Tzu, a mathematician in ancient China. It is termed as one of the jewels of mathematics and has diverse applications in number theory, abstract algebra, theory of automata, digital signal processing and cryptology. Its magical applications have been classified in three 'C’s' which are Computing with various aspects of algorithmics and modular computations, Theory of Codes and Cryptography [3]. From an analytical perspective, CRT is basically a manifestation of addressing complex problems through divide and conquer approach. In other words big structures represented mathematically through their smaller parts mapping the harder problems to their smaller equivalents and making the analysis easy. In the filed of cryptology, CRT has been known for secret sharing schemes, RSA-CRT and rebalanced RSA-CRT. Continual to new contexts of CRT, new results on applications of CRT in analysis of LFSR based sequences have been presented in this paper.

This paper shows that there exist hidden structures in underlying finite fields related to LFSR based combinatorial sequences which can be exploited through CRT. Number of constituent LFSRs in a combiner generator posses certain fixed patterns in their base finite fields which can be directly mapped through CRT to resultant fields even being combined through non linear functions. These results
are consistent both in time and frequency domain. Direct computation of spectral components in higher fields from smaller field spectral components through CRT is yet a new idea introduced in this paper. CRT based direct relevance of components of smaller fields to higher fields is novel in associated finite fields theory of combinatorial sequence generators and has obvious usefulness in coding theory and cryptology.

The paper is organized as follows: Section 2 describes the mathematical preliminaries on the subject. In section 3, CRT based fixed patterns existing in the product sequences both in time and frequency domain have been deliberated upon. Section 4 covers the generalized case of combinatorial sequence generators and new methodology to compute spectral components in higher fields from spectral components of constituent fields is given. Comparison of computational complexity of proposed methodology of DFT computations viz-a-viz classical DFT methods is also included in this section. In Section 5, applications of our results on CRT based fixed structures in cryptanalysis are discussed with small example of a combiner generator. The paper is finally concluded in Section 6.

2 Mathematical Preliminaries

Classical theory on LFSR sequences and their applications in cryptology can be found in [5], [4] and [12]. In this section, basic fundamentals related to algebraic theory of LFSR sequences and their frequency domain representations have been presented. By analyzing the sequences in both time and frequency domain simultaneously, fixed structures related to LFSR sequences and underlying finite fields are highlighted which are considered useful in coding theory and cryptanalysis.

Discrete Fourier Transform (DFT) is considered one of the most important discovery in the area of signal processing. DFT presents us with an alternate mathematical tool that allows us to examine the frequency domain behaviour of signals, often revealing important information not apparent in time domain. DFT $S_k$ of an $n$-point sequence $s_i$ is expressed in terms inner product between the sequence and set of complex discrete frequency exponentials:

$$S_k = \sum_{i=0}^{n-1} s_i e^{-j\frac{2\pi ik}{n}}, \quad k = 0, 1, 2, \ldots, n-1$$  \hspace{1cm} (1)

The term $e^{-j\frac{2\pi ik}{n}}$ represents discrete set of exponentials. Alternatively, $e^{-j\frac{2\pi k}{n}}$ can be viewed as $n^{th}$ root of unity.

Analogous to the classical DFT, a DFT for a periodic signal $s_t$ with period $n$ defined over a finite field $GF(2^m)$ is represented as

$$S_k = \sum_{t=0}^{n-1} s_t \alpha^{tk}, \quad k = 0, 1, 2, \ldots, n-1$$  \hspace{1cm} (2)
where \( S_k \) is \( k \)-th frequency component of DFT and \( \alpha \) is the primitive element; generator of \( GF(2^m) \) with period \( n \). For Inverse DFT, we will have a relation

\[
s_t = \sum_{k=0}^{n-1} S_k \alpha^{-tk}, \quad k = 0, 1, 2, \ldots, n - 1
\]  (3)

Similarly for polynomials, we have a relation for DFT and IDFT. Having a correspondence between a minimum polynomial and its associated sequence \( s_t \) with \( s(x) = \sum_{t=0}^{n-1} s_t x^t \) and \( S(x) = \sum_{k=0}^{n-1} S_k x^k \), following relation holds for DFT \[4\]:

\[
S_k = s(\alpha^{-k}), \quad k = 0, 1, 2, \ldots, n - 1
\]  (4)

and similarly for IDFT:

\[
s_t = S(\alpha^t), \quad t = 0, 1, 2, \ldots, n - 1
\]  (5)

The same sequence \( s_t \) can also be expressed in terms of its trace representation \[7\]: a linear operator from \( GF(2^m) \) to its subfield \( GF(2) \). Let \( Tr^m_1(x) = \sum_{k=0}^{2^m-2} x^{2^k} \) be the trace mapping from \( GF(2^m) \) to \( GF(2) \), then \( m \)-sequence \( s_t \) can be represented as:

\[
s_t = \text{Tr}^m_1(\beta \alpha^t)
\]  (6)

where \( \alpha \) is a generator of a cyclic group \( GF(2^m)^* \) and is called as primitive element of \( GF(2^m) \). Note that \( \beta \in GF(2^m) \) and each of its nonzero value corresponds to cyclic shift of the \( m \)-sequence generated by an LFSR with primitive polynomial \( f(x) \). Importance of this interpretation of \( m \)-sequence is that different sequences constructed from root \( \alpha \) of primitive polynomial \( f(x) \) are cyclic shifts of the same \( m \)-sequence. The associated linear space \( G(f) \) of dimension \( m \) contains \( 2^m \) different binary sequences including all 0s sequence as:

\[
G(f) = \{ \tau^i s \mid 0 \leq i \leq 2^m - 2 \} \cup \{0\}
\]  (7)

where \( \tau \) is a left shift operator and represents a linear transformation of sequence \( s_t \). According to Blahut’s famous theorem, the linear complexity of a periodic sequence over \( GF(2^m) \) of period \( n \) is equal to the hamming weight of its fourier transform, provided a fourier transform of block length \( n \) exists \[1\]. All DFT components of an LFSR sequence \( \in GF(2^m) \).

The zero components in the Fourier spectrum of a sequence over \( GF(2^m) \) are related to the roots of a polynomial of that sequence. For example, DFT of an LFSR sequence with feedback polynomial \( f(x) = x^3 + x + 1 \) initialized with state 001 is 0, 0, 0, \( \alpha^4 \), 0, \( \alpha^2 \), \( \alpha \). As roots of \( f(x) \) are \( \alpha \) along with its conjugates i.e. \( \alpha^2 \) and \( \alpha^4 \), so first, second and fourth spectral components are zero. Indices of non zero DFT points for LFSR with minimum polynomial and no multiple roots also follow a fixed pattern. If \( k \)-th component of spectral sequence is non zero then all \((2^j k) \mod n \) components will be harmonics of the \( k \)-th component where \( 1 \leq j \leq m-1 \). As DFT of a time domain signal comprises of a fundamental frequency and its harmonics, DFT of an LFSR sequence based on a minimal polynomial with
Theorem 1. Let \( s_t \in GF(2^m) \) be a reference product sequence with period \( n \mid 2^m - 1 \) having two constituent LFSRs defined over primitive polynomials with individual periods \( n_1 \) and \( n_2 \). With different shifts \( k_1 \) and \( k_2 \) in initial states of LFSRs, resulting output sequence \( u_t \) is correlated to \( s_t \) by \( u_t = s_{t + \tau} \) where shift \( \tau \) is determined through CRT as

\[
\tau \equiv k_1 \pmod{n_1} \\
\tau \equiv k_2 \pmod{n_2}
\]
Proof. Within a cyclic group $GF(2^m)$, associated linear space $G(f)$ of dimension $m$ contains $2^m - 1$ non-zero binary sequences by (7). As $s_t$ and $u_t$ both $\in GF(2^m)$, they are shift equivalents by (??) with unknown shift value of $\tau$. The product sequence $s_t$ of $a_t$ and $b_t$ can be expressed as

$$s_i = a_j b_v$$

where $0 \leq i \leq n - 1$, $0 \leq j \leq n_1 - 1$ and $0 \leq v \leq n_2 - 1$.

**Axiom 1.** While contributing towards a product sequence of length $n$ with two LFSRs, stream of LFSR-1 defined over $GF(2^p)$ with primitive polynomial and its maximum period $2^p - 1$ is repeated $\delta_1$ times while LFSR-2 defined over $GF(2^q)$ with primitive polynomial as well and corresponding period $2^q - 1$ is repeated $\delta_2$ where

$$\delta_1 = \frac{\text{lcm}(n_1, n_2)}{n_1}, \quad \text{and}$$

$$\delta_2 = \frac{\text{lcm}(n_1, n_2)}{n_2}$$

**Axiom 2.** Within a sequence of period $n$ for a product sequence, each value of index $j$ corresponds to all values of index $v$ if and only if $\gcd(n_1, n_2) = 1$.

From Axioms 1 and 2 any shift in LFSRs initial states will produce output corresponding to some fixed indices of $j$ and $v$ which already existed in the reference sequence at some fixed place with initial states of LFSRs without shift.

With known values of $j$ and $v$ i.e. $k'_s$, CRT will give us the value of $\tau \mod n$ as

$$\tau \equiv k_1 \pmod{n_1}$$

$$\tau \equiv k_2 \pmod{n_2}$$

Let we explain the facts with an example.

**Example 1.** Let we have a sequence $s_t$ generated from product of two LFSRs having primitive polynomials of $g_1(x) = x^2 + x + 1$ and $g_2(x) = x^3 + x + 1$. The period $n_1$ of stream $a_t$ corresponding to LFSR-1 is 3 and $n_2$ of $b_t$ corresponding to LFSR-2 is 7. The period $n$ of $s_t$ is 21.

Table 1 demonstrates product of two sequences generated from these two LFSRs.
We analyze the impact of shift on LFSR sequences and their behaviour in cyclic structures of finite fields involved. We will shift the LFSR sequences one by one and observe the fixed patterns which can be exploited in cryptanalysis of the combiner generators in particular. We can represent shifts in LFSR sequences with \( k \) and \( l \) as

\[
s_t = a_{i+k} b_{i+l}, \quad \text{with } 0 \leq i \leq n - 1
\]

where \( k \in [0, n_1 - 1] \) and \( l \in [0, n_2 - 1] \). Table 2 demonstrates the scenario where \( a_t \) is left shifted by one bit while keeping the \( b_t \) fixed with initial state of '1'.

**Table 2.** Product sequence with \( a_t \) shifted left

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_5 \) | \( a_6 \) | \( a_7 \) | \( a_8 \) | \( a_9 \) | \( a_{10} \) | \( a_{11} \) | \( a_{12} \) | \( a_{13} \) | \( a_{14} \) | \( a_{15} \) | \( a_{16} \) | \( a_{17} \) | \( a_{18} \) | \( a_{19} \) | \( a_{20} \) |
| \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( b_6 \) | \( b_7 \) | \( b_8 \) | \( b_9 \) | \( b_{10} \) | \( b_{11} \) | \( b_{12} \) | \( b_{13} \) | \( b_{14} \) | \( b_{15} \) | \( b_{16} \) | \( b_{17} \) | \( b_{18} \) | \( b_{19} \) | \( b_{20} \) |
| \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) | \( s_5 \) | \( s_6 \) | \( s_7 \) | \( s_8 \) | \( s_9 \) | \( s_{10} \) | \( s_{11} \) | \( s_{12} \) | \( s_{13} \) | \( s_{14} \) | \( s_{15} \) | \( s_{16} \) | \( s_{17} \) | \( s_{18} \) | \( s_{19} \) | \( s_{20} \) |

Comparison of Table 1 with Table 2 reveals that shifting one bit left of \( a_t \) and fixing the \( b_t \) to reference initial state of '1' shifts \( s_t \) by seven units left. Similarly, shifting another bit of \( a_t \) to left, brings \( a_3 \) corresponding to \( b_1 \) which can be located in Table 1 at shift position 14. So two left shifts of \( a_t \) shifts \( s_t \) by 14 units left with reference to bit positions in Table 1. Now we analyze the impact of left shift of \( b_t \) on \( s_t \). Table 3 demonstrates the scenario where \( b_t \) is left shifted by one bit while keeping the \( a_t \) fixed with initial state of '1'.

**Table 3.** Product sequence with \( b_t \) shifted left

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_5 \) | \( a_6 \) | \( a_7 \) | \( a_8 \) | \( a_9 \) | \( a_{10} \) | \( a_{11} \) | \( a_{12} \) | \( a_{13} \) | \( a_{14} \) |
| \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( b_5 \) | \( b_6 \) | \( b_7 \) | \( b_8 \) | \( b_9 \) | \( b_{10} \) | \( b_{11} \) | \( b_{12} \) | \( b_{13} \) | \( b_{14} \) |
| \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) | \( s_5 \) | \( s_6 \) | \( s_7 \) | \( s_8 \) | \( s_9 \) | \( s_{10} \) | \( s_{11} \) | \( s_{12} \) | \( s_{13} \) | \( s_{14} \) |
It can be easily seen that one left shift in $b_t$ shifts $s_t$ by 15 units where $b_2$ is corresponding to $a_1$. Similarly, another left shift in $b_t$ shifts $s_t$ by another 15 units bringing the $b_3$ corresponding to $a_1$. Subsequently, three left shifts in $b_t$ with reference to initial state of ‘1’ brings $b_4$ corresponding to $a_1$ which is at shift index-3 in Table 1. Similar fixed patterns can be observed for simultaneous shifts of LFSRs and it will be discussed with more detail in following paragraphs.

Let us model this fixed patterns in LFSRs cyclic structures and shifts in initial states of LFSRs through CRT as

$$x \equiv k \pmod{n_1}$$
$$x \equiv l \pmod{n_2}$$

where $k$ and $l$ denote the amount of shifts in initial state of individual LFSRs with reference to initial state of ‘1’. The solution of CRT i.e. $x \pmod{r}$ gives the amount of shift in $s_t$ with reference to $u_t$ as depicted in (??). Consider a scenario again where $a_t$ is shifted left by one bit and $b_t$ is fixed with initial state of ‘1’ and can be expressed as

$$x \equiv 1 \pmod{3}$$
$$x \equiv 0 \pmod{7}$$

The CRT gives the solution of $7 \pmod{21}$ which is index position of $a_2$ corresponding to $b_1$ in Table 1 shifting the product sequence $s_t$ by seven units left. Consider another scenario of simultaneous shifts in both LFSRs sequences where $a_t$ is shifted left by one bit and $b_t$ is shifted left by 3 bits with reference to their initial states of ‘1’ and can be expressed as

$$x \equiv 1 \pmod{3}$$
$$x \equiv 3 \pmod{7}$$

The CRT gives value of $-11$ which is $10 \pmod{21}$, representing the product sequence $u_t$ as 10 units left shifted version of $s_t$. This value matches to index position of $b_4$ corresponding to $a_2$ in Table 1.

Our Observations related to direct correspondence of shift index with initial states of LFSRs and CRT calculations done modulo periods of individual LFSRs are valid for any number of LFSRs in different configurations of nonlinear sequence generators. These observations on classical theory of LFSR cyclic structures with their CRT based interpretation are considered significant for cryptanalysis.

In addition to the results of Blahut’s theorem on time and frequency domain relationship of sequences, an important corollary establishes new facts related fourier transform in binary fields.

**Corollary 1.** Let $s_t \in GF(2^m)$ be a product sequence with period $n \mid 2^m - 1$ having two constituent sequences $a_t \in GF(2^p)$ and $b_t \in GF(2^q)$ of LFSRs each
defined over primitive polynomials with individual periods \( n_1 = 2^p - 1 \) and \( n_2 = 2^q - 1 \). If \( A \) be a DFT spectra of \( a_t \), \( B \) be a DFT spectra of \( b_t \) and \( S \) be a DFT spectra of \( s_t \), non zero spectral components of \( S \) will only exist at those indices where spectral components of \( A \) and \( B \) are non zero.

we have another associated corollary here:-

**Corollary 2.** With known non zero spectral components of \( A \) and \( B \), non zero spectral components of \( S \) can be directly determined through Chinese Remainder Theorem (CRT) as:

\[
\begin{align*}
x &\equiv k_1 \pmod{n_1} \\
x &\equiv k_2 \pmod{n_2}
\end{align*}
\]

where \( k_1 \) and \( k_2 \) are non zero index positions of \( A_k \) and \( B_k \) respectively and \( x \) is the position of non zero component of DFT spectra of \( s_t \) within its period \( n \).

It is important to observe here that indices of non zero spectral components present in a complete spectrum of resultant stream are determined while working in base fields of component LFSRs and without computing DFT of \( s_t \) in a larger field. Let we explain these corollaries through a small example here.

**Example 2.** Following the Example [1] consider a product sequence \( s_t \) generated from two LFSRs with minimum polynomials \( g_1(x) = x^3 + x + 1 \) and \( g_2(x) = x^2 + x + 1 \).

1. In time domain representation, we have following sequences.

   Sequence \( a_t \): 011 \( (\text{of period 3}) \)
   Sequence \( b_t \): 0010111 \( (\text{of period 7}) \)
   Sequence \( s_t \): 00101100000101001001110111011101 \( (\text{of period 21}) \)

2. From (2), frequency domain representations of these sequences are:

   (a) \( A = 0, 1, 1 \)
   (b) \( B = 0, 0, 0, \alpha^4, 0, \alpha^2, \alpha \)
   (c) To compute \( S \), associated minimum polynomial is determined through Berlekamp-Massey algorithm which is \( g(x) = x^6 + x^4 + x^2 + x + 1 \).

\[
S = 0, 0, 0, 0, 0, 0, 0, 0, 0, \alpha^{18}, 0, 0, \alpha^{15}, 0, 0, 0, 0, \alpha^{18}, 0, \alpha^9, \alpha^{15}
\]

Non-zero DFT points in \( S \) clearly follow a linear behaviour as of time domain representation where any \( k \)-th component is non-zero if and only if \( A_k \) and \( B_k \) are both non-zero. Through non zero indices of \( A \) and \( B \), CRT can be directly used to determine non-zero spectral points of \( S \). For instance,

\[
\begin{align*}
x &\equiv 1 \pmod{3} \\
x &\equiv 3 \pmod{7}
\end{align*}
\]

results into index 10 where \( \alpha^{18} \) is a non zero spectral component of \( S \). These results on determining non zero spectral indices for product of two sequences are valid for product sequences containing more number of LFSRs as well.
Harmonic pattern of DFT spectra are visible for A, B and S. Non-zero indices of DFT sequences also follow a fixed pattern. In case of S, non zero DFT element at index 5 has its harmonics at indices 10, 20, 19 (40 mod 21), 17 (80 mod 21) and at 13 (160 mod 21). The zero components in the fourier transform of a product sequence $s_t$ defined over $GF(2^m)$ are related to roots of $g(x) = x^6 + x^4 + x^2 + x + 1$. As roots of $g(x)$ are $\alpha, \alpha^4, \alpha^8$ and $\alpha^{16}$ so first, second, fourth, eigth and sixteenth spectral components are zero.

4 Computing the Spectral Components in $GF(2^m)$ through CRT

Computing DFT of a sequence $s \in GF(2^m)$ by equation (2) over binary fields requires determining the associated minimum polynomial $m(x)$ of $s$. The most efficient method which computes the linear complexity $l$ of a periodic sequence $s$ and gives its minimum polynomial is berlekamp massey algorithm [4]. The algorithm further requires $2l$ bits of the sequence to determine the linear complexity and minimum polynomial $m(x)$. Based on the root of minimum polynomial $m(x)$, equation (2) requires complete period of the sequence to compute each spectral component of $S$. Faster method to compute DFT in binary fields proposed in [8] requires lesser number of bits equal to linear complexity $l$ or in few cases lesser than that. However, in all these cases computations have to be in $GF(2^m)$ to which sequence $s$ belongs. In this section, new method has been introduced which allows mapping of spectral components of smaller constituent fields to larger finite fields with few limitations of choice of particular indices. We will develop our idea progressively from product of sequences in time domain to a general case of boolean functions where addition of bits in $GF(2)$ is encompassed as well.

4.1 Product of Arbitrary Number of $m$-Sequences

In this subsection, case of product sequence is considered where any arbitrary number of LFSR sequences are multiplied together. Starting with simple case of two LFSRs, we will establish facts for more number of LFSRs where direct computation of spectral points for product sequence is done from DFT points of individual LFSR sequences. We have an important theorem here.

**Theorem 2.** Let $s_i \in GF(2^m)$ be a product sequence with period $n_i | 2^m - 1$ having $r$ constituent sequences $a_i \in GF(2^{p_i})$ of LFSRs each defined over primitive polynomials with individual periods $n_i = (2^{p_i} - 1)$, where all $n_i$ are coprime to each other and $0 \leq i \leq r - 1$. Let $A_i$ be a DFT spectra of $a_i$, a $k$-th spectral component of $S$ corresponding to each non-zero spectral components of $A_{(k \ mod \ n_i)}$ can be determined directly through CRT as
From equation (13), there exists a unique mapping for $\gamma$ can be computed using CRT as

$$d \equiv d_1 \pmod{n_1}$$

$$d \equiv d_2 \pmod{n_2}$$

$$\ldots \ldots$$

$$\ldots \ldots$$

$$d \equiv d_r \pmod{n_r}$$

where $d$, $d_1$, $d_2$, $\ldots$, $d_r$ are degrees of non-zero spectral components i.e. $S_k$, $A_{1(\mod{n_1})}, \ldots, A_{r(\mod{n_r})}$ represented in terms of associated roots $\gamma \in GF(2^m)$, $\alpha_1 \in GF(2^{n_1})$, $\alpha_2 \in GF(2^{n_2})$, $\ldots$ and $\alpha_r \in GF(2^{n_r})$ of minimal polynomials of $s$, $a_1$, $a_2$, $\ldots$ and $a_r$ respectively.

Proof. To prove the theorem for a generalized case of $r$ LFSRs multiplied together, let we consider first a simple case of product of two LFSRs only.

Let $s_t \in GF(2^m)$ be a product sequence with period $n \mid 2^m - 1$ having two constituent sequences $a_t \in GF(2^m)$ and $b_t \in GF(2^m)$ of LFSRs each defined over primitive polynomials with individual periods $n_1 \equiv (2^p - 1)$ and $n_2 \equiv (2^q - 1)$, where $n_1$ and $n_2$ are coprime to each other. Let $A$ be a DFT spectra of $a_t$, $B$ be a DFT spectra of $b_t$ and $S$ be a DFT spectra of $s_t$.

Let $d$, $d_1$ and $d_2$ are degrees of non-zero spectral components i.e. $S_k$, $A_{1(\mod{n_1})}$ and $B_{1(\mod{n_2})}$ represented in terms of associated roots $\gamma \in GF(2^m)$, $\alpha \in GF(2^p)$ and $\beta \in GF(2^q)$ of minimal polynomials of $s_t$, $a_t$ and $b_t$ respectively. All roots of minimum polynomials of $a$, $b$ and $s$ lie within their respective fields i.e. $\alpha \in GF(2^p)$, $\beta \in GF(2^q)$ and $\gamma \in GF(2^m)$ respectively. $n_1$ and $n_2$ being coprime, $n = lcm(n_1, n_2)$. By corollary 2 spectral components of $S$ are non zero at all indices where corresponding spectral components of $A$ and $B$ are non zero. As all DFT spectral components of $S$ lie within $GF(2^m)$ and correspond to $\gamma^h$, where $0 \leq h \leq m - 1$. Let we consider any $k$-th component of spectra of $S$ corresponding to non zero DFT components of $A$ and $B$, where we only need to prove that both non zero spectral components of $A$ and $B$ has one to one mapping to $S$ through CRT.

Transforming the relationship of $s_t = a_t b_t$ into roots of associated polynomials of each sequence in their respective binary fields by using definitions of $GF(2^m)$ by $\gamma^h \ (0 \leq h \leq n)$, $GF(2^p)$ by $\alpha^i \ (0 \leq i \leq n_1)$ and $GF(2^q)$ by $\beta^j \ (0 \leq j \leq n_2)$, we have

$$\gamma^d = \alpha^d \beta^d, \quad d = 0, 1, 2, \ldots, n - 1$$

(12)

As we can write $s_t = a_{(t \mod n_1)} b_{(t \mod n_2)}$, equation (12) can be expressed as

$$\gamma^d = \alpha^{d \mod{n_1}} \beta^{d \mod{n_2}}, \quad t = 0, 1, 2, \ldots, n - 1$$

(13)

From equation (13), there exists a unique mapping for $\gamma^d$, $\alpha^{d_1}$, and $\beta^{d_2}$ which can be computed using CRT as

$$d \equiv d_1 \pmod{n_1}$$

$$d \equiv d_2 \pmod{n_2}$$
Mapping these facts on a product sequence having $r$ constituent sequences, it becomes trivial to see

\[
d \equiv d_1 \pmod{n_1} \\
d \equiv d_2 \pmod{n_2} \\
\ldots \ldots \ldots \\
d \equiv d_r \pmod{n_r}
\]

As $GF(2^m)$ considered here is implicitly constituted by product of elements of $GF(2^p)$ and $GF(2^q)$, convolution of $\alpha_i \in GF(2^p)$, $\beta_j \in GF(2^q)$ should result into spectral component $\gamma_h \in GF(2^m)$ ideally at each index $k$. For convolutions in finite fields, readers may refer to [11]. However, when elements belong to different binary fields, not much is known to us. Nevertheless, CRT based method of computing DFT components in higher binary fields from constituent DFT components in lower order fields is considered novel in this regard. Let us illustrate our results through an example.

**Example 3.** Consider a product sequence $s$ having three LFSRs with primitive polynomials as $g_1(x) = x^2 + x + 1$, $g_2(x) = x^3 + x + 1$ and $g_3(x) = x^5 + x^2 + 1$. The outputs of LFSRs in this case are m-sequences, denoted as $a^1$, $a^2$ and $a^3$ respectively. Product stream $s$ is obtained as

\[
s_t = a^1_t.a^2_t.a^3_t \quad \text{where} \quad 0 \leq t \leq n - 1
\]  

where period $n$ of $s$, in this case becomes 651 as $\text{lcm}(3, 7, 31) = 651$. DFT components of $a$, $b$, and $c$ with primitive elements $\alpha \in GF(2^2)$, $\beta \in GF(2^3)$ and $\delta \in GF(2^5)$ respectively are 

- $A^1 = \{0, 1, 1\}$
- $A^2 = \{0, 0, 0, \beta^4, 0, \beta^2, \beta\}$
- $A^3 = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \delta^{29}, 0, 0, 0, 0, 0, 0, \delta^{30}, 0, 0, 0, \delta^{15}, 0, \delta^{23}, \delta^{27}\}$

To compute DFT of $s$, we need to compute its associated minimum polynomial through berlekamp massey algorithm which in this case is \(m(x) = x^{30} + x^{25} + x^{24} + x^{20} + x^{19} + x^{17} + x^{16} + x^{13} + x^{10} + x^9 + x^8 + x^7 + x^4 + x^2 + 1\) with generator $\gamma \in GF(2^{30})$.

Having a complete period (651 bits) of $s$, we compute DFT through equation[2] Corresponding to degree of minimum polynomial, we get thirty non-zero DFT components at indices shown in Table[4] below.
Table 4. Non Zero Spectral Points of $S$

| Index   | Spectral Component |
|---------|--------------------|
| 61      | $\gamma^{492}$    |
| 89      | $\gamma^{387}$    |
| 122     | $\gamma^{333}$    |
| 139     | $\gamma^{246}$    |
| 178     | $\gamma^{123}$    |
| 209     | $\gamma^{585}$    |
| 215     | $\gamma^{309}$    |
| 244     | $\gamma^{240}$    |
| 271     | $\gamma^{15}$     |
| 278     | $\gamma^{30}$     |

| Index   | Spectral Component |
|---------|--------------------|
| 325     | $\gamma^{492}$    |
| 370     | $\gamma^{387}$    |
| 395     | $\gamma^{333}$    |
| 418     | $\gamma^{246}$    |
| 430     | $\gamma^{123}$    |
| 461     | $\gamma^{585}$    |
| 488     | $\gamma^{309}$    |
| 492     | $\gamma^{240}$    |
| 519     | $\gamma^{15}$     |
| 585     | $\gamma^{30}$     |

| Index   | Spectral Component |
|---------|--------------------|
| 523     | $\gamma^{492}$    |
| 542     | $\gamma^{387}$    |
| 556     | $\gamma^{333}$    |
| 587     | $\gamma^{246}$    |
| 619     | $\gamma^{123}$    |
| 635     | $\gamma^{585}$    |
| 647     | $\gamma^{309}$    |
| 649     | $\gamma^{240}$    |
| 650     | $\gamma^{15}$     |

From corollary 2, non zero indices of $S$ can be determined directly from knowing the individual DFTs of three LFSRs separately. For instance,

\[
\begin{align*}
    x &\equiv 1 \pmod{3} \\
    x &\equiv 3 \pmod{7} \\
    x &\equiv 15 \pmod{31}
\end{align*}
\]

gives result of 325 which exists amongst thirty non-zero DFT computations as well. Similarly with known spectral points of $A_1 = \alpha^6$, $A_2 = \beta^4$ and $A_3 = \delta^{29}$, spectral component $S_{325}$ can be determined directly by theorem 2 as

\[
\begin{align*}
    d &\equiv 0 \pmod{3} \\
    d &\equiv 4 \pmod{7} \\
    d &\equiv 29 \pmod{31}
\end{align*}
\]

CRT gives the result of 60. So the spectral component $S_{325} \in GF(2^{30})$ becomes $\gamma^{60}$. Similarly all non zero points of $S \in GF(2^{30})$ can be computed directly by theorem 2 without the requirement of minimum polynomial $m(x)$, $n$ number of bits of $s$ and classical computations of DFT by equation 2. Conversely, from known DFT spectra of $S$ only, individual DFT spectral points of $A_1$, $A_2$ and $A_3$ can also be computed. For instance, having known $\gamma^{492}$ at $S_{61}$, $A_1$ is directly computed as $\alpha^6$, $A_2$ is computed as $\beta^4$ and $A_3$ is computed as $\delta^{27}$. These results are considered very useful in cryptanalysis of LFSR based sequences.

4.2 Generic Combinatorial Sequences

Having considered the product sequences of multiple LFSRs, generic case of combinatorial sequences is discussed now where outputs of multiple LFSRs are combined through a non linear function involving multiplication and addition of bits in $GF(2)$. From the established fact of theorem 2 for product sequences, we now generalize the case for combinatorial generators here.

Consider a combinatorial generator consisting of $r$ constituent LFSRs. Let $z_t \in GF(2^m)$ be the output sequence of generator with period $n \mid 2^m - 1$ and
Let $m(x)$ be the associated minimum polynomial. Let $\gamma \in GF(2^m)$ be the root of $m(x)$, $\alpha_1 \in GF(2^{p_1})$, $\alpha_2 \in GF(2^{p_2})$, ..... and $\alpha_r \in GF(2^{p_r})$ of minimal polynomials of $z, a_1, a_2, ....$ and $a_r$ respectively. The nonlinear function $f(x_1, x_2, ..., x_{r-1})$ combines outputs of $r$ LFSRs and produces the resultant stream $z$ as

$$z_t = f(a_1^t, a_2^t, ..., a_r^t) \quad \text{where} \quad 0 \leq t \leq n - 1$$

(15)

As $f(x)$ is not only a product function, we have

$$m(x) \neq m_1(x).m_2(x)....m_r(x)$$

(16)

$$\Rightarrow \gamma \neq \alpha_1.\alpha_2....\alpha_r$$

(17)

To take DFT of $z$ by Equation (2), we require $n$ bits of $z$ and DFT will be computed with respect to $\gamma \in GF(2^m)$ having order $n$. Non zero DFT terms termed as linear span of $z$ will be equivalent to degree of associated minimum polynomial $m(x)$. These results are consistent to known theory of DFT in binary fields. However, few additional results are noted which are correlated to CRT based fixed patterns of sequences.

If we take DFT of $z$ with respect to generator $\sigma$ of its minimum polynomial $m(x)$, experimental results reveal that irrespective of combining function $f(x)$, a fixed relationship between frequency components of $Z$ and individual spectral components of $A^1, A^2, ..., A^r$ exists at all those indices of $Z$ where corresponding spectral components of $A^1, A^2, ..., A^r$ are all non zero. Let we represent $d, d_1, d_2, ..., d_r$ as degrees of non-zero spectral components of $Z_k, A^{1}_{(k \mod n_1)}, ..., A^{r}_{(k \mod n_r)}$ represented in terms of associated roots of respective minimum polynomials. At any index $k$, where all corresponding spectral components of LFSR sequences are non zero, $Z_k \in GF(2^m)$ can be directly determined using through CRT as described in theorem (2). Similarly, from corollary (2) and (2), non zero indices of $S$ corresponding to non zero spectral components of $A^1, A^2, ..., A^r$ are directly determined. Let we validate our observations through an example of a simple combiner generator.

**Example 4.** With the same assumptions of Example (3) with three LFSRs and notations used therein, output stream $z_t$ of a combiner is obtained as

$$z_t = a_1^t.a_2^t + a_2^t.a_3^t + a_3^t.a_1^t \quad \text{where} \quad 0 \leq t \leq n - 1$$

(18)

Taking the DFT of $z$ by Equation (2) with respect to $\gamma \in GF(2^{30})$ as a generator of $(x^{30} + x^{25} + x^{24} + x^{20} + x^{19} + x^{17} + x^{16} + x^{13} + x^{10} + x^9 + x^8 + x^7 + x^4 + x^2 + 1)$ with generator $\gamma \in GF(2^{30})$, 31 non zero DFT points are mentioned in Table (5) below.
Linear complexity of $z_t$ is determined to be 31 through berlekamp-massey algorithm and the corresponding minimum polynomial $m(x)$ in this case is: 

$$(x^{31} + x^{29} + x^{28} + x^{27} + x^{24} + x^{23} + x^{22} + x^{20} + x^{18} + x^{17} + x^{16} + x^{15} + x^{13} + x^{11} + x^{10} + x^{9} + x^{8} + x^{7} + x^{5} + x^{4} + x^{2} + x + 1).$$

Now DFT is taken with respect to generator $\sigma \in GF(2^{31})$ of $m(x)$ with order 651. We will only mention spectral points at those indices where constituent LFSR sequences have all non zero spectral points.

Now we apply our observations on CRT based fixed patterns in sequences and compute spectral components of $Z$ directly by using Theorem 2. From individual DFTs of LFSR sequences as computed in Example 2, corresponding to non zero indices of $A_1$, $A_2$ and $A_3$, we first determine non zero index of $Z$ through CRT using Equation 2 as:

$$x \equiv 2 \pmod{3}$$

$$x \equiv 6 \pmod{7}$$

$$x \equiv 30 \pmod{31}$$
we get index of 650. Now we compute spectral value of $Z_{650}$ represented in terms of $\sigma \in GF(2^{31})$ through CRT using Theorem 2 as

\[
d \equiv 0 \pmod{3} \\
d \equiv 1 \pmod{7} \\
d \equiv 27 \pmod{31}
\]

we get $Z_{650} = \sigma_1^{120}$. Spectral components of other non zero indices of $Z$ along with all values of $\sigma^i$ with order 651 are mentioned at appendix A. These results reveal that irrespective of non linear function $f(x)$, degree of spectral components corresponding to spectra of constituent LFSR sequences is consistent even being in different fields. For instance $Z_{650} = \sigma_1^{120}$ for a generalized combiner case and $S_{650} = \gamma_1^{120}$ for a product case (Example [2] have same degree with different values of spectral components being $\sigma \in GF(2^{31})$ and $\gamma \in GF(2^{30})$).

4.3 Complexity of CRT Based DFT Computations

In this subsection, discussion on computational complexity of CRT based DFT calculations in comparison to classical DFT is presented. DFT in binary fields from Equation 5 dictates that the complexity for computing each $S_k$ is equivalent to cost for evaluating polynomial $s(x)$ at $\alpha^{-k}$ [6] where $\psi = \alpha^{-k}$. In terms of exclusive-or operations, we have:-

1. The complexity of computing minimum polynomial of a sequence $\in GF(2)$ through berlekamp massey algorithm is $O(m \log m)$.
2. The complexity of multiplying two polynomials of degree $m$ is $O(m \log m \log \log m)$.
3. The complexity of solving system of $r$ linear equations over $GF(2^m)$ is $O(r^2 \cdot 37 \cdot m \log m \log \log m)$.
4. The complexity in terms of Xor operations for computing each $S_k$ using the Equation 5 is $O((m \log m \log \log m)(\log (k) + \deg(s(x))))$.

For CRT based computations of spectral components from constituent spectral components, we will consider a case of product of two LFSR sequences which can be generalized for a combiner generator. Let we have two sequences $a \in GF(2^p)$ and $b \in GF(2^q)$. It is trivial to mention that

Complexity of DFT ($s$) $\gg$ Complexity of [DFT (a) + DFT (b)]

For each $S_k$, additional computational complexity for CRT is $O(len(n)^2)$. As non zero terms of $S_k$ are equivalent to the linear span of the sequence, thus total cost of CRT based computational step of spectral components is $O(LS(s) \cdot len(n)^2)$, where $LS$ is linear span of the sequence $s$. Thus CRT based computations of spectral components of $s$ for combiner generators are far efficient than classical methods of DFT computations in binary fields.
5 CRT and Cryptanalysis of Combiner Generators

In this subsection, discussion on application of our novel results on CRT based fixed patterns in cryptanalysis of combiner sequences is made. From discussion made in Section 3 on established linkage between period of LFSR sequence, effect of left shifts of LFSR initial states and mathematical rationale through CRT, let we demonstrate application of our observations on analysis of combiner generators.

Example 5. With same structure of combiner generator mentioned in Example 4, suppose we know 10 bits of keystream $u_t = [101110001]$. During off-line computations, we will generate 651 bits of reference stream i.e. $s_t$ with initial fills of all three LFSRs as ‘1’ which comes out to be:

$00101101011101101101101101101010101101101100101110001101101101100101100101110110110010100101100101100101110110010111011011001010010110.................... 101011010010111011000111010111012$.

Comparing the ten known bits of keystream $u_t = [101110001]$ with reference sequence $s_t$, index position of known bits is determined as $k = 632$. Thus

$u_i = s_{i+632}, \forall i \geq 0$.

After determining index position of ten known bits of $u_t$ in reference stream $s_t$, we will determine initial states of LFSR by simply applying modular computations of CRT as follows:

$k_1 \equiv 632 \pmod{3}$

$k_2 \equiv 632 \pmod{7}$

$k_3 \equiv 632 \pmod{31}$

Therefore, $k_1 \equiv 2 \pmod{3}$, $k_2 \equiv 2 \pmod{7}$ and $k_3 \equiv 12 \pmod{31}$. By using Equation 6 initial states of LFSRs is determined as given in Table below.

Remark 1. Generating the complete period of reference sequence $s_t$ followed by finding few known bits of available keystream $u_t$ in a complete period of $s_t$ may not be computationally feasible for sequence of larger periods which infact is the case of practical stream ciphers. However, the example is given to demonstrate the existing cyclic structures of LFSR based sequences designs and their CRT based interpretation.

Now, let frequency domain analysis of combiner sequences is made in the light of our results on CRT based relevance of spectral components. With the same notations as of Example 4 if spectral component $Z_k$ is computed from known ciphertext stream by any method where $k$ corresponds to all non zero unknown spectral component of constituent LFSR sequences, these individual spectral components are computed using Theorem 2 and Corollary 2.
known spectral components of constituent LFSRs, initial states of LFSRs is
determined by using Equation (8) and (6). For instance, for $Z_{650} = \sigma^{101}$, we will
do modular computations to determine the spectral component of individual
LFSRs as

\[
101 \equiv 2 \pmod{3} \\
101 \equiv 3 \pmod{7} \\
101 \equiv 10 \pmod{31}
\]

We get $\alpha^2 \in GF(2^2)$ at $A_2^{12}$, $\beta^3 \in GF(2^3)$ at $A_6^2$ and $\gamma^{10} \in GF(2^5)$ at $A_{30}^3$. Now shift value for each LFSR is computed using Equation (8) as

\[
\alpha^\tau = (Z_k.S_{k}^{-1})^{k^{-1}}
\]

where $\tau$ determines the exact amount of shift between $s_i$ and $z_i$ and $k$ is index
of any one component of DFT spectra.

Having determined the exact shift value for each LFSR, their initial states
will be computed using Equation (6) within each subfield $GF(2^j)$ as

\[
\begin{align*}
\beta_1^t &= Tr^n_1(\alpha^* \alpha^t) \\
\beta_2^t &= Tr^n_1(\beta^* \beta^t) \\
\beta_3^t &= Tr^n_1(\gamma^* \gamma^t)
\end{align*}
\]

where

\[
\begin{align*}
\alpha^*_i &= \alpha^{\tau_i} \\
\beta^*_i &= \beta^{\tau_i} \\
\gamma^*_i &= \gamma^{\tau_i}
\end{align*}
\]

The initial fills of LFSRs with referece to intial state of ‘1’ for all LFSRs
with 1 left shift in $a_2^1$, 5 left shifts in $a_2^2$ and 19 left shifts in $a_2^3$ gives:

| LFSR | Initial State |
|------|--------------|
| LFSR-1 | 10 |
| LFSR-2 | 101 |
| LFSR-3 | 01111 |

Remark 2. Application of our results on CRT based fixed patterns in combiner
sequences are valid for any configuration of non linear combining function. How-
ever, point of concern for cryptanalysis is computation of $S_k$ in a typical scenario
of ciphertext only attack where limitataion of known keystream bits is always a
driving factor for practability of the attack. For computations of DFT spectral
component, complete period of ciphertext is required which is practically not the case for cryptanalysis attacks. Fast discrete fourier spectra attacks \cite{8} provide an efficient methodology to compute particular spectral points when number of known bits are far less than the complete period of the stream. Our CRT based methodology can be utilized in conjunction with both the DFT finding algorithms proposed in \cite{8} when number of known key stream bits are equal to linear span of the sequence or even lesser than that. Detailed results on efficiency of this proposed methodology will be presented separately.

Remark 3. With regards to cryptanalysis attacks on combinatorial sequence generators, correlation attacks \cite{13} and their faster variants \cite{9} are considered to be the most efficient attacks \cite{2}. Computational cost of our proposed methodology of DFT spectral points, even by employing fast discrete fourier spectra attacks, is more than correlation attacks. However, in a scenario of correlation immune non linear boolean functions when correlation attacks are not successful, our proposed methodology of CRT based spectral computations is still valid which will be addressed at a separate forum.

6 Conclusion

In this paper, new results on CRT based analysis of combinatorial sequences have been presented. We explored inherent peculiarities of the LFSR based combiner generators through novel patterns identified with the help of a CRT based approach. These findings were then extended to the product sequences and more particularly to the combinatorial generators. An effort was made to establish the mapping of different operations from time domain to frequency domain. Novel results on fixed shift patterns of LFSRs, their relationship to cyclic structures in finite fields and CRT based interpretation of these patterns have been exploited to establish direct relevance of final keystreams of combiner generators to individual LFSR sequences. Based on these CRT based fixed structures, new methodology of direct computing the spectral components of sequences in larger finite fields from constituent spectra of smaller fields is also presented. These new results on CRT based structural analysis of LFSR based combiners are demonstrated on small scale sequence generators with brief discussion on involved computational costs and practability of these techniques in cryptanalysis attacks.

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