Horizons and the cosmological constant

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Abstract

A new solution of the Einstein equations for the point mass immersed in the de Sitter Universe is presented. The properties of the metric are very different from both the Schwarzschild black hole and the de Sitter Universe: it is everywhere smooth, light can propagate outward through the horizon, there is an antitrapped surface enclosing the point mass and there is necessarily an initial singularity. The solution for any positive cosmological constant is qualitatively different from the Schwarzschild solution and is not its continuous deformation.

1. Introduction

There is an extensive literature on the solutions of the Einstein equations describing a point mass immersed in the expanding Universe (see [1] and the recent review [2] and references therein). The first exact solution for a point mass in the de Sitter Universe was constructed as early as 1918 and is known as the Kottler metric [3]:

\[ ds^2 = -\left(1 - H^2 r^2 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - H^2 r^2 - \frac{2GM}{r}} + r^2 d\Omega^2 \]  

(1)

where \( H \) is related to the cosmological constant \( \Lambda \) by \( \Lambda = \frac{3H^2}{8\pi G} \). For \( H = 0 \) the metric is equal to the standard Schwarzschild metric [4]

\[ ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2 \]  

(2)

However, for \( M = 0 \) the metric requires a diffeomorphism to transform it to the usual de Sitter metric [5]

\[ ds^2 = -dt^2 + e^{2Ht} \left(dr^2 + r^2 d\Omega^2\right) \]  

(3)

It is the purpose of this paper to present a new solution that in the above mentioned limits goes over to the standard Schwarzschild (at least outside of the black hole...
horizon) or de Sitter metrics. We will show that in the limit of small cosmological constant the metric indeed goes to the Schwarzschild metric but only outside the Schwarzschild horizon – inside it is everywhere singular.

2. The metric

The metric that solves

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \Lambda g_{\mu\nu} \]  

with required properties reads

\[ ds^2 = -f(t, r) dt^2 + \frac{e^{2Ht}dr^2}{f(t, r)} + e^{2Ht}r^2 d\Omega^2 \]  

where \( \Lambda = \frac{3H^2}{8\pi G} \),

\[ f(t, r) = h(t, r) + \sqrt{h(t, r)^2 + H^2 r^2 e^{2Ht}} \]  

and

\[ h(t, r) = \frac{1}{2} \left( 1 - H^2 r^2 e^{2Ht} - \frac{r_S}{re^{Ht}} \right) \]  

where \( r_S = 2GM \). We see that if \( H \neq 0 \) (however small) \( f(t, r) \) never vanishes. It is important to note that in the limit \( H \to 0 \) the solution tends to the Schwarzschild metric only outside of the horizon and is undefined inside. The solution is connected to \( \text{[1]} \) by a singular diffeomorphism transformation so they cover different patches of spacetime.

For some purposes it may be useful to write the metric in coordinates \((t, \rho) = (t, re^{Ht})\) (the metric is then independent of time):

\[ ds^2 = -2h(0, \rho) dt^2 - \frac{2H \rho dt d\rho}{f(0, \rho)} + \frac{d\rho^2}{f(0, \rho)} + \rho^2 d\Omega^2 \]  

with the inverse metric in these coordinates

\[ g^{tt} = -\frac{1}{f(0, \rho)}, \quad g^{t\rho} = -\frac{H \rho}{f(0, \rho)}, \quad g^{\rho\rho} = 2h(0, \rho) \]  

This form shows that there are 4 Killing vectors (1 connected with time translations and 3 with rotations) as in the Schwarzschild case. In general there are two radii \( \rho_1, \rho_2 \) solving the equation

\[ h(0, \rho_i) = 0 \]  

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and defining two null surfaces. The second null vector is in both cases directed outside the surfaces therefore the inner horizon is of different type than the usual black hole horizon.

3. Singularities

We calculate the Riemann tensor for the metric (5)

\[ R_{01}^{01} = R_{23}^{23} = H^2 + \frac{r_S}{r^3 e^{3Ht}} \]

\[ R_{02}^{02} = R_{03}^{03} = R_{12}^{12} = R_{13}^{13} = H^2 - \frac{r_S}{2r^3 e^{3Ht}} \]  

(11)

with all other components (except with permuted indices) vanishing. Therefore the Ricci tensor and the curvature scalar read

\[ R_{\mu\nu} = 3H^2 \delta_{\mu\nu}, \quad R = 12H^2 \]  

(12)

The only singularity of the scalars constructed from the Riemann tensor like \( R_{\nu\rho\sigma\mu} R^{\rho\sigma}_{\mu\nu} \) occurs for \( r \to 0 \) (as is the case also for the Schwarzschild solution) or for \( t \to -\infty \).

We now analyze the metric (5). For fixed \( t \)

\[ f(t, r) \xrightarrow{r \to \infty} 1, \quad f(t, r) \xrightarrow{r \to 0} \frac{H^2 r^3 e^{3Ht}}{r_S - re^{Ht}} \]  

(13)

while for fixed \( r \)

\[ f(t, r) \xrightarrow{t \to \infty} 1, \quad f(t, r) \xrightarrow{t \to -\infty} \frac{H^2 r^3 e^{3Ht}}{r_S - re^{Ht}} \]  

(14)

Therefore the physical radial distance to \( r \to \infty \) is infinite as well as the physical time interval to \( t \to \infty \). The physical radial distance to the origin \( r = 0 \) is also infinite (although the area is \( 4\pi r^2 e^{2Ht} \) i.e finite and small)

\[ \int_{\epsilon}^{1} e^{Ht} dr \approx 1 \frac{\sqrt{\frac{r_S}{H^2 e^{2Ht}}}}{2} \]  

(15)

so the space is geodesically complete. However the physical time interval from \( t \to -\infty \) to some \( T \) (within the applicability of (14))

\[ \int_{-\infty}^{T} dt \sqrt{f(t, r)} \approx \frac{2}{3} \sqrt{\frac{r^3 e^{3HT}}{r_S}} \]  

(16)
is finite. Therefore in the presence of a point mass and the cosmological constant there must be an initial singularity in contradistinction to the usual de Sitter Universe.

The topology of the spacetime described by (5) is $\mathbb{R}^2 \times S^2$.

4. Propagation of light

Since the $g_{\mu\nu}$ components are nowhere vanishing and nowhere singular the coordinates seem to cover the whole spacetime. To check what happens at the inner horizon (corresponding to $r = 2GM$ in the Schwarzschild case) we have to find the behaviour of light both inside and outside. To do it we solve the equation for the propagation of light i.e. stemming from $ds^2 = 0$ and directed radially outwards i.e. satisfying

$$e^{Ht} \frac{dr(t)}{dt} = f(t, r(t))$$

(17)

To simplify the discussion we assume in what follows that

$$r_S H \ll 1$$

(18)

what is extremely well satisfied for any object in the Universe. Then in these coordinates the horizon is given by

$$\frac{r_S}{r e^{Ht}} \approx 1$$

(19)

We have to distinguish two cases – the initial position at $t = 0$ $r_0 < r_S$ or $r_0 > r_S$.

In the latter case ($r_0 > r_S$) the solution reads

$$r(t) = r_0 + \frac{1}{H} (1 - e^{-Ht}) + r_S \ln \frac{r_0 H}{1 + r_0 H - e^{-Ht}} + O(r_S^2 H)$$

(20)

so that after infinite time the comoving coordinate reaches the de Sitter horizon

$$r(\infty) \approx r_0 + r_S \ln(r_0 H) + \frac{1}{H}$$

(21)

A different situation arises if $r_0 < r_S$. Then $f(t, r)$ is positive but extremely small so that $r(t)$ increases very slowly until it reaches the horizon and time grows to the value corresponding to (19) i.e.

$$T_0 = \frac{1}{H} \ln \left( \frac{r_S}{r_0} \right)$$

(22)
Then the light gets outside and the trajectory is given by (20) but with $t \rightarrow t + T_0$. Therefore the total comoving distance is approximately given by

$$r(\infty) \approx \frac{r_0}{r_S H}$$

(23)

i.e. less than the outside de Sitter horizon by a factor $\frac{r_0}{r_S}$.

It is important to check the dependence on time of the area enclosed by the outgoing or incoming light

$$\frac{dA}{dt} = 8\pi re^{2Ht} (Hr \pm f(t, r)e^{-Ht})$$

(24)

where $A = 4\pi r^2 e^{2Ht}$. Since $Hr > f(t, r)e^{-Ht}$ for sufficiently small $r$ it turns out that there is an antitrapped surface enclosing some region around $r = 0$. This is consistent with the fact that antitrapped surfaces must have a singularity in the past as we have seen is the case for the metric (5).

4. Conclusions

In the paper we have shown that the solution of the Einstein equations (5) for a point mass immersed in the universe with the positive cosmological constant has very special properties: the metric is everywhere smooth, light can propagate outward through the horizon, there is an antitrapped surface enclosing the point mass and there is necessarily an initial singularity. Although with extremely small value of $H$ such an object for all practical purposes looks like a usual black hole the conceptual difference resulting from the fact that there is no horizon for the outward propagation of light can be far-reaching – first, one should rethink a notion of a black hole entropy as proportional to the area of the horizon and second, there seems to be no information loss even classically since the communication of the inside with the outside is extremely weak but nonvanishing. It is also interesting to note that in the presence of such objects there is necessarily an initial singularity in distinction to the pure de Sitter universe and there is no continuous deformation connecting $\Lambda > 0$ solution described in this paper and the Schwarzschild metric.

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