Toward good families of codes from towers of surfaces

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Abstract. We introduce in this article a new method to estimate the minimum distance of codes from algebraic surfaces. This lower bound is generic, i.e., can be applied to any surface, and turns out to be “liftable” under finite morphisms, paving the way toward the construction of good codes from towers of surfaces. In the same direction, we establish a criterion for a surface with a fixed finite set of closed points $P$ to have an infinite tower of $\ell$-étale covers in which $P$ splits totally. We conclude by stating several open problems. In particular, we relate the existence of asymptotically good codes from general type surfaces with a very ample canonical class to the behaviour of their number of rational points with respect to their $K^2$ and coherent Euler characteristic.

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Introduction

In the early 80’s, V. D. Goppa proposed a construction of codes from algebraic curves [21]. A pleasant feature of these codes from curves is that they benefit from an elementary but rather sharp lower bound for the minimum distance, the so-called Goppa designed distance. In addition, an easy lower bound for the dimension can be derived from Riemann-Roch theorem. The very simple nature of these lower bounds for the parameters permits to formulate a concise criterion for a sequence of curves to provide asymptotically good codes: roughly speaking, the curves should have a large number of rational points compared to their genus. Ihara [32] and independently Tsfasman, Vlăduţ and Zink [50] proved that this criterion was satisfied by some modular and Shimura curves. This led to an impressive breaktrough in coding

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theory: the unexpected existence of sequences of codes beating the asymptotic Gilbert-Varshamov bound.

As suggested by Manin and Vlăduţ, Goppa’s original construction extends to higher dimensional varieties. However, in comparison to the rich literature involving codes from curves, codes from higher dimensional varieties have been subject to very few developments. One of the reasons for this gap is that for varieties of dimension two and higher, divisors and points are objects of different nature, and this difference makes the estimate of the minimum distance much more difficult. For this reason, most of the works on codes from surfaces in the literature concern specific subfamilies of surfaces, and estimate the parameters by ad hoc techniques relying on specific arithmetic and geometric properties of the involved surfaces. For instance, codes from quadric and Hermitian surfaces are considered in [16, 17, 13], toric surface codes are studied in [27], and codes from Hirzebruch surfaces in [42].

For this reason, codes from surfaces in the literature concern specific subfamilies of surfaces, and estimate the parameters by ad hoc techniques relying on specific arithmetic and geometric properties of the involved surfaces. For instance, codes from quadric and Hermitian surfaces are considered in [16, 17, 13], toric surface codes are studied in [27], and codes from Hirzebruch surfaces in [42]. Toward an analog of having a large number of points compared to their genus, Voloch and Zarzar suggested to seek surfaces with a low Picard number [54, 52], leading to further discoveries of good codes [38, 9]. A significant part of the previously described code constructions appear in the excellent survey [37].

On the other hand, a few references propose “generic” lower bounds for the minimum distance, i.e. bounds that could be applied to codes from any surface. The first one is proposed by Aubry in [3], resting on a counting argument on the maximum number of points of a curve in a very ample linear system. Next, S. H. Hansen proposed two other approaches to bound the minimum distance from below [28]. The first one involves an auxiliary set of curves, while the second one involves the Seshardi constant at the evaluation points. Note that the latter approach is very close to that of Bouganis [10], while Bouganis’ code construction is not stricto sensu an algebraic geometry code from a surface. Unfortunately, it should be emphasized that, even with the knowledge of these bounds, no result is known on the possible asymptotic performance of codes from algebraic surfaces, and the principle of getting good sequences of codes from sequences of curves whose number of rational points grows quickly compared to their genus has no known counterpart in the surfaces’ setting.

It is worth noting that besides the highly difficult objective of getting new families of codes with excellent asymptotic parameters, the geometry of algebraic surfaces could provide codes with many other interesting features. Indeed, in the last decade, new questions brought by distributed storage problems arose and motivated a focus on codes with good local properties. Concerning the use of algebraic geometry codes for constructing locally recoverable codes, one can cite for instance (and the list is very far from being exhaustive) [49, 8, 7, 30, 36, 41, 44]. It should be pointed out that algebraic geometry codes from surfaces are of deep interest from this point of view. Indeed, given a code from a surface $X$, to access to some digit of the code, which corresponds to some point of the surface, one can consider a curve contained in the surface and containing this point and restrict to the code on the curve. This point of view, which is in particular investigated in [44], provides another motivation to study codes from surfaces or higher dimensional varieties. Note finally that these interesting local properties of codes on surfaces have been remarked more than 15 years ago by Bouganis [10].

Our contribution. This article expects to be a first step toward the construction of asymptotically good families codes from sequences of algebraic surfaces. For this sake, we provide several tools and results that would be helpful to progress in this direction.
In §1 we propose a new “generic” lower bound for the minimum distance of an algebraic geometry code from a surface. This bound involves a linear system of divisors with a property called $\mathcal{P}$-interpolating (see Definition 1.4). The existence and the construction of $\mathcal{P}$-interpolating divisors could appear as a limitation to handle this result. However, we also prove that for a surface $X$ over the finite field $\mathbb{F}_q$, the linear system associated to $\mathcal{O}_X(q + 1)$ is always $\mathcal{P}$-covering. Further we compare this new bound with previous ones in the literature. Finally, we discuss the behaviour of these generic bounds, including ours, under finite morphisms of projective surfaces and show that our bound such as those involving Seshadri constants are easy to “lift” under such a finite map, and hence could be used to study codes from towers of surfaces.

In §2.1 we propose a class field-like criterion for a surface with a fixed set of rational points $\mathcal{P}$ to have an infinite tower of $\ell$-étale covers in which $\mathcal{P}$ splits totally. Next, in §2.2 we sieve the Kodaira classification of algebraic surfaces to determine which types of surfaces may have infinite totally split towers of étale covers. Despite we acknowledge that we have not been able to apply the criterion established in §2.1 in order to provide new families of codes, we however show in §2.3 that this criterion can be applied on a product of two hyperelliptic curves.

We conclude the article by presenting some open problems in §3. In particular, we show that the existence of asymptotically good families of codes can be deduced from that of families of general type surfaces with very ample canonical class $K$ whose number $N$ of rational points goes to infinity, whose $K^2$ and coherent Euler characteristic $\chi(\mathcal{O}_X)$ are proportional to $N$ and whose asymptotic ratio $\chi(\mathcal{O}_X)/K_X$ is the largest possible.

**Note.** The published version of the present article includes an appendix by Alexander Schmidt [46].

1. Codes from surfaces

1.1. Context and notation.

1.1.1. Context. In what follows, $X$ denotes a smooth projective geometrically connected surface over $\mathbb{F}_q$ and $\mathcal{P}$ a non empty set of rational points on $X$ of cardinality $n$. The surface $\overline{X}$ is defined as $\overline{X} := X \times_{\text{Spec}(\mathbb{F}_q)} \text{Spec}(\mathbb{F}_q)$. In addition, $G$ is a divisor on $X$ whose support avoids the elements of $\mathcal{P}$. Our point is to study the parameters of the linear code $C(X, \mathcal{P}, G)$ introduced in [21, 51] defined as the image of the map:

$$e_{\mathcal{P}}: \left\{ \begin{array}{rcl} H^0(X, G) & \rightarrow & \mathbb{F}_q^n \\ f & \mapsto & (f(P))_{P \in \mathcal{P}} \end{array} \right.,$$

where elements of $H^0(X, G)$ are regarded as rational fractions on $X$.

**Remark 1.1.** Actually, the condition “the support of $G$ avoids the elements of $\mathcal{P}$” can be removed. In such a case, one needs to choose at each point $P$ a generator $s_P$ of the stalk of the sheaf $\mathcal{O}(G)$ at $P$. Then, any global section $f$ can be locally written $f = fs_P$ for some $f_P \in \mathcal{O}_{X, P}$, and one can evaluate $f_P$ at $P$.

1.1.2. Codes. Recall that a code is a vector subspace $C$ of $\mathbb{F}_q^n$ for some finite field $\mathbb{F}_q$ and some positive integer $n$, called the length of $C$. The dimension of the code is the dimension of $C$ as a $\mathbb{F}_q$-vector space. The Hamming weight $w_H(x)$ of a vector $x \in \mathbb{F}_q^n$ is the number of nonzero entries of $x$, and that the minimum distance of a code $C \subseteq \mathbb{F}_q^n$ is the minimum weight of a nonzero vector of $C$. The objective of this first section is to investigate various manners to estimate the parameters of codes from algebraic surfaces, namely the dimension and the minimum distance.
1.1.3. **Equivalence of divisors and cycles.** In the sequel, we frequently deal with linear and numerical equivalence of divisors. One is denoted by $\sim$ and the second one by $\equiv$. We refer to [29, Chapters II.6 & V.1] for the definitions. Recall that linear equivalence entails numerical equivalence. Further we also deal with rational equivalence of cycles which is a generalisation of linear equivalence of divisors and hence will also be denoted by $\sim$. We refer to [19] for the definition of rational equivalence.

1.2. The dimension of codes from surfaces. To bound from below the dimension of a code $C(X, \mathcal{P}, G)$, the natural tool is Riemann-Roch theorem which asserts that

$$h^0(X, G) - h^1(X, G) + h^2(X, G) = \frac{1}{2}G \cdot (G - K_X) + \chi(O_X),$$

and hence

$$h^0(X, G) + h^2(X, G) \geq \frac{1}{2}G \cdot (G - K_X) + \chi(O_X).$$

The following lemma is useful in what follows. This criterion can be found in [29] and has been previously suggested for applications to codes on surfaces by Bouganis in [10].

**Lemma 1.2.** Let $H$ be an ample divisor on $X$. If $G \cdot H > K_X \cdot H$ then

$$h^0(X, G) \geq \frac{1}{2}G \cdot (G - K_X) + \chi(O_X).$$

**Proof.** From [29, Lemma V.1.7], if $G \cdot H > K_X \cdot H$ then $h^2(X, G) = 0$ which yields the proof. \qed

1.3. A new lower bound for the minimum distance. In this subsection, we present a general manner to bound from below the minimum distance of a code from a surface. This bound can be regarded as the counterpart for codes from surfaces of Goppa’s designed distance for codes from curves. Indeed, in the case of curves, an algebraic geometry code of length $n$ associated to a divisor $G$ has minimum distance bounded from below by $n - \deg G$. In the sequel, we prove that on a surface, a code of length $n$ associated to a divisor $G$ has minimum distance bounded from below by $n - \Gamma \cdot G$ for some divisor $\Gamma$ with a special property. Here, such as in many situations in algebraic geometry, the degree for divisors on a curve is replaced by the intersection product for divisors on surfaces. For this sake, the introduction of an auxiliary divisor $\Gamma$ is necessary. Our work to follow consists in providing de relevant definition for $\Gamma$. Therefore, the statement of this new bound requires first to recall some basic notions on linear systems of divisors.

1.3.1. **Linear systems on a surface.** Recall that a linear system of divisors $\Gamma$ on $X$ is a family of positive divisors which are all linearly equivalent, and which is parametrised by a projective space. Given a divisor $D$ on $X$, the complete linear system denoted as $|D|$ is the set of all positive divisors on $X$ which are linearly equivalent to $D$. This set is parametrised by $P(H^0(X, D))$ and a general linear system is a subset of some complete linear system $|D|$ parametrised by some proper linear subspace of $P(H^0(X, D))$.

1.3.2. **$\mathcal{P}$-interpolating linear systems.**

**Notation 1.3.** Let $\Gamma$ be a linear system of curves on $X$ and let $\mathcal{Y}$ be a proper closed sub-scheme of $X$. Then, we denote by $\Gamma - \mathcal{Y}$ the maximal linear subsystem of $\Gamma$ of elements whose base locus contains $\mathcal{Y}$.

**Definition 1.4.** Given a surface $X$ and a set of rational points $\mathcal{P}$, a linear system $\Gamma$ of divisors on $X$ is said to be $\mathcal{P}$-interpolating if

(i) $\Gamma - \mathcal{P}$ is non empty;
(ii) the base locus of $\Gamma - \mathcal{P}$ has dimension 0.
Remark 1.5. About condition (ii), obviously, the base locus of \( \Gamma - \mathcal{P} \) contains \( \mathcal{P} \). In addition it may contain a finite number of other points. In particular, this base locus cannot be empty.

Remark 1.6. Note that the notion of \( \mathcal{P} \)-interpolating system such as lower bound for the minimum distance we get in Theorem 1.10 to follow are of geometric and not arithmetic nature: the \( \mathcal{P} \)-interpolating linear system \( \Gamma \) is not assumed to be defined over \( \mathbb{F}_q \).

The following statement provides an equivalent definition of \( \mathcal{P} \)-interpolating systems that will be useful in the sequel.

Proposition 1.7. Conditions (i) and (ii) of Definition 1.4 hold if and only if the following condition holds

(\( i' \)) There exist two sections \( A, B \) of \( \Gamma - \mathcal{P} \) such that the supports of \( A \) and \( B \) have no common component.

Proof. Suppose (\( i' \)) holds. Then, the base locus of \( \Gamma - \mathcal{P} \) is contained in the intersection of the supports of \( A \) and \( B \). From Remark 1.5 this intersection contains \( \mathcal{P} \), then it is non empty and by (\( i' \)) has dimension 0.

Conversely, suppose that (i) and (ii) hold. From (i), there exists at least one section \( A \) of \( \Gamma - \mathcal{P} \). Let \( A_1, \ldots, A_s \) be the geometrically irreducible components of its support. Then, by (ii), for any \( i \in \{1, \ldots, s\} \), the linear system \( \Gamma - \mathcal{P} - A_i \) is strictly contained in \( \Gamma - \mathcal{P} \). Consequently, consider the projective space \( \mathbb{P}^r \) parametrizing \( \Gamma - \mathcal{P} \), then the sublinear systems \( \Gamma - \mathcal{P} - A_i \) correspond to finitely many of proper linear subvarieties of \( \mathbb{P}^r(\mathbb{F}_q) \) and since \( \mathbb{F}_q \) is infinite, there exists a point in \( \mathbb{P}^r(\mathbb{F}_q) \) avoiding them all. Therefore, there exists a section \( B \) of \( \Gamma - \mathcal{P} \) whose support has no common component with that of \( A \). \( \square \)

Remark 1.8. If \( \Gamma \) is a sub–linear system of a linear system \( \Delta \), then, \( \Delta \) is \( \mathcal{P} \)-interpolating too. On the other hand, for any \( \mathcal{P}' \subseteq \mathcal{P} \), any \( \mathcal{P} \)-interpolating linear system is \( \mathcal{P}' \)-interpolating.

The following statement is useful in what follows.

Proposition 1.9. Let \( \Gamma \) be a \( \mathcal{P} \)-interpolating system, then

\[ \Gamma^2 \geq |\mathcal{P}|. \]

Proof. From Proposition 1.7, there exist \( A, B \in \Gamma - \mathcal{P} \) with no common irreducible component. The points of \( \mathcal{P} \) lying at the intersection of \( A \) and \( B \), we have thus \( \Gamma^2 = A \cdot B \geq |\mathcal{P}|. \) \( \square \)

1.3.3. A lower bound for the minimum distance.

Theorem 1.10. Let \( X \) be a smooth geometrically connected surface over \( \mathbb{F}_q \), with a set of rational points \( \mathcal{P} \) and \( G \) be a divisor on \( X \) whose support avoids \( \mathcal{P} \). Let \( \Gamma \) be a \( \mathcal{P} \)-interpolating linear system on \( X \). Then the minimum distance \( d \) of \( C_L(X, \mathcal{P}, G) \) satisfies

\[ d \geq n - \Gamma \cdot G, \]

where \( n = |\mathcal{P}| \).

Proof. Let \( f \) be an element of \( H^0(X, G) \) providing a nonzero codeword \( c \). Let \( D \) be the positive divisor \( D = (f) + G \). Then,

\[ n - w_H(\text{ev}(f)) \leq |\text{Supp}(D) \cap \mathcal{P}|, \]

where \( \text{Supp}(D) \) denotes the support of the divisor \( D \).

We claim that there exists \( E \in \Gamma - \mathcal{P} \) whose support has no common irreducible component with that of \( D \). Indeed, by Definition 1.4, \( \Gamma - \mathcal{P} \) is nonempty.
Therefore it is parametrised by some projective space $\mathbb{P}^\ell$ for $\ell \geq 0$. Next, by Definition 1.11, for any $\mathbb{F}_q$-irreducible component $Y$ of the support of $D$, the linear system $\Gamma - (P \cup Y)$ is distinct from $\Gamma - P$. Thus, there is a proper linear subvariety $V_Y$ of $\mathbb{P}^\ell(\mathbb{F}_q)$ parametrising $\Gamma - (P \cup Y)$. Let $W \subseteq \mathbb{P}^\ell(\mathbb{F}_q)$ be defined as

$$W := \bigcup_Y V_Y,$$

where $Y$ runs over all the $\mathbb{F}_q$-irreducible components of the support of $D$. Then, any element of $\mathbb{P}^\ell(\mathbb{F}_q) \setminus W$ provides such a divisor $E$. Consequently,

$$|\text{Supp}(D) \cap P| \leq |\text{Supp}(D) \cap \text{Supp}(E)| \leq D \cdot E = G \cdot \Gamma.$$

Several examples of applications of this lower bound are given further in §1.5 and 1.6. Before, let us state a brief summary of our estimates.

1.4. Summary on the parameters of codes for a surface. Our previous results lead to the following statement.

**Theorem 1.11.** Let $\Gamma$ be a $P$-interpolating linear system such that $n > \Gamma \cdot G$, and assume moreover that there exists an ample divisor $H$ on $X$ such that $G \cdot H > K_X \cdot H$. Then the code $C(X,P,G)$ has parameters

$$k \geq \frac{1}{2} \Gamma \cdot (G - K_X) + \chi(O_X) \quad \text{and} \quad d \geq n - \Gamma \cdot G.$$

**Proof.** The condition $n > \Gamma \cdot G$ entails that the evaluation map is injective and hence asserts that the minimum distance of the code is that of the Riemann-Roch space. \hfill \Box

1.5. A “universal” example of application of our bound. A natural question is: how to find a $P$-interpolating system? The following lemma provides an example that can be applied to any surface with a very ample divisor.

**Theorem 1.12.** Let $\mathcal{L}$ be a very ample sheaf on $X$, then for any $P$ contained in $X(\mathbb{F}_q)$, the sheaf $\mathcal{L}^{(q-1)}$ is $P$-interpolating and, in the conditions of Theorem 1.10, we get

$$d \geq n - (q + 1)G \cdot L.$$

If in addition there exists $L \in |\mathcal{L}|$ such that $P$ is contained in the affine chart $X \setminus \text{Supp}(L)$, then $\mathcal{L}^{(q)}$ is $P$-interpolating and

$$d \geq n - qG \cdot L.$$

**Remark 1.13.** In particular, if $X$ is embedded in $\mathbb{P}^\ell$, then $O_X(q + 1)$ is $P$-interpolating.

The proof of Theorem 1.12 rests on the two following well-known statements.

**Lemma 1.14.** Let $x_1, \ldots, x_\ell$ be a system of coordinates in the affine space $\mathbb{A}^\ell$, then the intersection of the hypersurfaces of equation $x_i^q - x_i$ for $i = 1, \ldots, \ell$ equals $\mathbb{A}^\ell(\mathbb{F}_q)$.

**Lemma 1.15.** Let $\ell$ be a positive integer and $x_0, \ldots, x_\ell$ be a system of homogeneous coordinates in $\mathbb{P}^\ell_{\mathbb{F}_q}$. For any pair $(i, j)$ with $0 \leq i < j \leq \ell$, let $H_{ij}$ be the hypersurface of equation $x_i^q x_j - x_i x_j^q$. Then, the variety $\cap_{0 \leq i < j \leq \ell} H_{ij}$ has dimension 0 and is the union of the rational points of $\mathbb{P}^\ell$. 
Proof of Theorem 1.12. Let \( \phi : X \to \mathbb{P}^g \) with \( \ell = h^0(X,\mathcal{L}) - 1 \) be the embedding associated to \( \mathcal{L} \). Hence \( \mathcal{L} \) is isomorphic to \( \phi^*\mathcal{O}_{\mathbb{P}^g}(1) \) and we denote by \( x_0, \ldots, x_\ell \) a system of homogeneous coordinates of \( \mathbb{P}^g \).

The reduced base locus of \( |\mathcal{L}^{\otimes(q+1)}| - \mathcal{P} \) is contained in the intersection of the zero loci of the sections \( (x_i^q - x_j s_j^{q-1})x_j \) of \( \mathcal{L}^{\otimes(q+1)} \) for \( 0 \leq i < j \leq \ell \). From Lemma 1.15, this zero locus equals \( X(\mathbb{F}_q) \). Therefore, the sheaf \( \mathcal{L}^{\otimes(q+1)} \) is \( \mathcal{P} \)-interpolating.

For the second situation, we can apply Lemma 1.14 to get the result. \( \square \)

1.6. Further examples. In § 1.5 we proposed a generic linear system that is \( \mathcal{P} \)-interpolating for any set \( \mathcal{P} \) of rational points. When we have further information on the geometry of the surface it is possible to construct more specific \( \mathcal{P} \)-interpolating linear systems as suggested in the examples to follow.

1.6.1. Product of curves. Let \( X = C \times D \) be a product of curves and denote by \( \pi_C, \pi_D \) the corresponding projections. Assume moreover that \( \mathcal{P} \) is a “grid” of rational points, i.e. there is a set \( \mathcal{P}_C \) of rational points of \( C \) and a set \( \mathcal{P}_D \) of rational points of \( D \) such that \( \mathcal{P} = \mathcal{P}_C \times \mathcal{P}_D \). Note that \( \mathcal{P}_C, \mathcal{P}_D \) can be regarded as reduced positive divisors respectively of \( C \) and \( D \). Assume moreover that these divisors are base point free. For instance, assume that \( \mathcal{P}_C, \mathcal{P}_D \) have respective degrees \( t_C, t_D \) larger than or equal to \( 2g_C \) and \( 2g_D \) where \( g_C, g_D \) denote the respective genera of \( C \) and \( D \). Then, the linear system

\[
\Gamma := |\pi_C(\mathcal{P}_C) + \pi_D(\mathcal{P}_D)|
\]

is \( \mathcal{P} \)-interpolating. Indeed, by hypothesis, \( \mathcal{P}_C \) is base point free and hence, there exists a positive divisor \( E_C \) on \( C \) (resp. \( E_D \) on \( D \)) which is linearly equivalent to \( \pi_C(\mathcal{P}_C) \) (resp. \( \mathcal{P}_D \)) and with disjoint support. Hence, the linear system \( \Gamma \) contains \( \pi_C^*E_C + \pi_D^*\mathcal{P}_D \) and \( \pi_C^*\mathcal{P}_C + \pi_D^*E_D \). The supports of these divisors have no common component. We deduce from Proposition 1.7 that \( \Gamma \) is \( \mathcal{P} \)-interpolating.

As a consequence, \( \Gamma \) is numerically equivalent to

\[
\Gamma \equiv t_C F_C + t_D F_D,
\]

where \( F_C, F_D \) denote the respective numerical equivalences of a fibre by \( \pi_C \) and \( \pi_D \) and \( t_C, t_D \) denote the respective degrees of the divisors \( \mathcal{P}_C, \mathcal{P}_D \) respectively on \( C \) and \( D \). Hence, given a divisor \( G \) on \( X \), the code \( C(X, \mathcal{P}, G) \) has minimum distance

\[
d \geq n - t_C F_C \cdot G - t_D F_D \cdot G
\]

by Theorem 1.10.

Example 1.16. On the product of two projective lines, the divisor class group is generated by two classes \( H, V \). Let \( G = aH + bV \). For the choice of \( \Gamma \), one can easily see that \( H + V \) is very ample and hence, from Theorem 1.12 the system \( |(q+1)(H + V)| \) is \( \mathcal{P} \)-interpolating. Therefore, we get

\[
d \geq n - (q+1)(a + b).
\]

Actually the exact minimum distance is known to be \( n - (q+1)(a + b) + ab \) (see Theorem 2.1 & Remark 2.2).

1.6.2. Fibred surfaces. Let \( \pi : X \to C \) be a fibred surface over a curve \( C \) of genus \( g \). Let \( F_1, \ldots, F_r \) be the fibres under \( \pi \) of \( r \) distinct rational points of \( C \), let \( C_1, \ldots, C_s \) be \( s \) distinct sections of \( \pi \) and consider the set of rational points \( \mathcal{P} \) given by

\[\mathcal{P} = (F_1 \cup \ldots \cup F_r) \cap (C_1 \cup \ldots \cup C_s)\]

Suppose in addition that \( r > 2g_C \) where \( g_C \) denotes the genus of \( C \) and that there exists \( C \in |C_1 + \cdots + C_s| \) whose support avoids any element of \( \mathcal{P} \). Then, the complete linear system \( |F_1 + \cdots + F_r + C_1 + \cdots + C_s| \) is \( \mathcal{P} \)-interpolating.
Indeed, similarly to the previous case, \( F_1 + \cdots + F_r \) is the pullback by \( \pi \) of a base point free divisor on \( C \) and there exists a positive divisor \( F \sim F_1 + \cdots + F_r \) whose support does not contain any of the \( F_i \)'s. Next, the divisors

\[
F_1 + \cdots + F_r + C \quad \text{and} \quad F + C_1 + \cdots + C_s
\]

have no common component and are both in \( |F_1 + \cdots + F_r + C_1 + \cdots + C_s| \). Thus, according to Proposition 1.7, the linear system \( \Gamma \) is \( \mathcal{P} \)-interpolating.

1.6.3. Hirzebruch surfaces. To get a more explicit example, consider the case of a rational ruled surface, i.e. a Hirzebruch surface \( \Sigma_e \). Such a surface is ruled, i.e. there is a morphism \( \pi_e : \Sigma_e \to \mathbb{P}^1 \) with a section whose image in \( \Sigma_e \) is denoted by \( S_e \). This surface has a discrete Picard group generated by \( S_e \) and a fibre \( F_e \) and \( F_2 = 0 \), \( F_e \cdot S_e = 1 \) and \( S_2 = -e \).

The surface can be obtained from \( \mathbb{P}^2 \) by a sequence of blow up and blow down:

- \( \Sigma_1 \) is the blow up of \( \mathbb{P}^2 \) at a point \( P \). Fix a line \( L \subseteq \mathbb{P}^2 \) containing \( P \). We denote by \( S_1 \) the exceptional divisor and by \( F_1 \) the strict transform of \( L \). We have \( S_1^2 = -1 \) and \( F_1^2 = 0 \).
- \( \Sigma_{e+1} \) is obtained from \( \Sigma_e \) as follows. Let \( P \) be the point at the intersection of \( F_e \) and \( S_e \). Blow up \( \Sigma_e \) at \( P \) and denote by \( \tilde{E}, \tilde{F}, \tilde{S_e} \) the respective strict transforms of \( S, F \) by the blowup map. Blow down \( \tilde{F} \) and set \( F_{e+1} \) and \( S_{e+1} \) the respective images of \( E \) and \( S_e \) by the blow down map. See Figure 1 for an illustration.

\[
\begin{array}{c}
\Sigma_e \\
\text{(0)} \\
\Sigma_{e+1}
\end{array}
\quad
\begin{array}{c}
\Sigma_e \\
\text{(0)} \\
\Sigma_{e+1}
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\quad
\begin{array}{c}
\Sigma_e \\
\text{(0)}
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\begin{array}{c}
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\Sigma_{e+1}
\end{array}

\text{Figure 1. From } \Sigma_e \text{ to } \Sigma_{e+1}

Such surfaces are rational, in particular there exists a birational map

\[
\psi : \mathbb{P}^2 \xrightarrow{\sim} \Sigma_e
\]

which induces an isomorphism between the affine chart \( \mathbb{P}^2 \setminus L \) and the affine chart \( \Sigma_e \setminus (S_e \cup F_e) \).

Consider in \( \mathbb{P}^2 \) a line \( L' \) distinct from \( L \) and not containing \( P \). Let \( Q \) be the point at the intersection of \( L \) and \( L' \). The Zariski closure of the image of \( L' \setminus \{Q\} \) by the map \( \psi \) of (1.1) is a genus 0 curve \( C \) on \( \Sigma_e \) which is another section of \( \pi_e \) which does not meet \( S_e \). Therefore, it satisfies

\[
C \cdot S_e = 0 \quad \text{and} \quad C \cdot F_e = 1.
\]

Thus,

\[
C \sim eF_e + S_e.
\]
Let \((X : Y : Z)\) be a system of homogeneous coordinates in \( \mathbb{P}^2 \) such that \( L = \{X = 0\} \), \( P = \{0 : 1 : 0\} \) and \( Q = \{0 : 0 : 1\} \). Let \( P_0 \) be the set of rational points \( \{(1 : x : y) \mid x \in A, y \in B\} \) for some sets \( A \subseteq \mathbb{F}_q \) and \( B \subseteq \mathbb{F}_q \) of respective cardinalities \( a \), \( b \). Finally, let \( P \) be the image of \( P_0 \) by \( \psi \). The points in \( P \) and \( P_0 \) are represented by filled black dots in Figure 2.

Note first that in \( \mathbb{P}^2 \setminus L \) the set of points \( P_0 \) is contained in the following two unions of parallel lines:

\[
\bigcup_{t \in A} \{Y = tX\} \quad \text{and} \quad \bigcup_{u \in B} \{X = uZ\}.
\]

On the left–hand part of Figure 2 the first family of lines is represented by \( \ldots \) and the second one by \( \ldots \). The Zariski closures of the respective images of these lines by \( \psi \) are denoted as \( V \) and \( H \) and, from [1,2], we have

\[
V \sim aC \sim a(eF_e + S_e) \quad \text{and} \quad H \sim bF_e.
\]

Figure 2. Behaviour of the family of lines under \( \psi \)

We claim that the linear system \( \Gamma := [(a + be)F_e + bS_e] \) is \( P \)–interpolating. Indeed, since \( |F_e| \) is base point free, then so is \( |bF_e| \) and there exists \( H' \) in \( |bF_e| \) such that \( P \cap \text{Supp}(V') = \emptyset \). Similarly, the divisor \( eF_e + S_e \) is very ample [29 Corollary 2.18] and hence the corresponding linear system is base point free. Thus there exists \( V' \sim V \) whose support avoids any point of \( P \). Finally, the divisors

\[
V + F' \quad \text{and} \quad V' + F
\]

are both in \( \Gamma \) and the intersection of their supports is \( P \). Therefore, \( \Gamma \) is \( P \)–interpolating.

Now, consider a divisor \( G \sim uF_e + vS_e \) on \( \Sigma_e \). Using the previous context, we get a code \( C(\Sigma_e, P, G) \) with length \( n = ab \) and minimum distance

\[
d \geq n - (a + be)v - ub + ebv.
\]

On the other hand if one wants to take \( P = \Sigma_e(\mathbb{F}_q) \), that is to say evaluating global sections at all the rational points, one can use §1.5. For this sake, one needs a very ample divisor. According to [29 Corollary V.2.18(a)], the divisor \( H = (e + 1)F_e + S_e \) is very ample. Therefore, one can take \( \Gamma = [(q + 1)(e + 1)F_e + (q + 1)S_e] \). For a divisor \( G \sim uF_e + vS_e \) leads to the following lower bound for the minimum distance.

\[
\begin{align*}
d &\geq \frac{a}{n} (q + 1)^2 - (q + 1) ((e + 1)v + u - ev) \\
&\geq (q + 1)^2 - (q + 1)(u + v).
\end{align*}
\]
Remark 1.17. The exact minimum distance and dimension of codes from Hirzebruch surfaces when the evaluation set is the full set of rational points have been computed by Nardi in [42]. According to this reference, when \( v \geq 1 \), the genuine minimum distance is \( d = q(q - u + 1) \). In particular, it does not depend on \( v \). Here we can estimate the defect of our bound compare to the actual minimum distance:

\[
q(q - u + 1) - (q + 1)^2 + (q + 1)(u + v) = u + v - 1 + q(v - 1).
\]

In particular, the larger the \( v \), the worst our estimate.

1.7. Previous estimates for the minimum distance in the literature.

Let us first list some previous estimates given in the literature, which can roughly speaking be divided into three categories.

1.7.1. Using the maximum number of points of curves in a linear system: Aubry’s bound.

The first category of bounds consists in observing that the minimum distance is related to the maximum number of rational points of an element of the linear system \( [G] \), that is to say,

\[
d \geq n - \max_{C \in [G]} |C(F_q)|.
\]

The problem is then translated into that of getting an upper bound on such a maximum. This is the approach used for instance by Aubry in [2] who obtained the following result.

Proposition 1.18 ([2 Proposition 3.1(ii)]). Let \( D \) be a very ample divisor on a nonsingular projective surface \( X \). Then the minimum distance of \( C(X,P,D) \) satisfies

\[
d \geq n - D^2(q + 1).
\]

When \( G \sim O_X(1) \), and taking as \( P \)-interpolating system \( \Gamma \sim O_X(q + 1) \) as suggested in §1.5, our bound yields the very same result as Aubry’s bound. The examples to follow show that our bound can actually be much better than Aubry’s one in other situations. Another advantage of Theorem 1.10 is that it does not require the very ampleness of the divisor \( G \).

Example 1.19. Consider the case of a surface \( X \) together with a very ample sheaf \( O_X(1) \) and \( G \) be a divisor such that \( G \sim O_X(d) \) for some positive integer \( d \). Thus, Aubry’s result yields

\[
(1.3) \quad d \geq n - (q + 1)G^2 = n - (q + 1)d^2 \deg(X),
\]

while our bound yields

\[
(1.4) \quad d \geq n - G \cdot O(q + 1) = n - d(q + 1) \deg(X).
\]

Thus a \( d^2 \) is replaced by a \( d \).

Example 1.20. Back to Example 1.16, for any positive integers \( a,b \), the divisor \( G = aH + bV \) is very ample and Aubry’s bound yields

\[
d \geq n - 2ab(q + 1).
\]

Here our “\( a + b \)” term replaces an “\( ab \)” term in the above lower bound.

Caution. In the sequel, we discuss two lower bounds for the minimum distance, that are due to S. H. Hansen. One is presented in §1.7.2 and rests on the use of a set of auxiliary curves. The second one is introduced in §1.7.3 and involves the Seshardi constant. To avoid confusion, the first kind of bound will be referred to Hansen bound (A), where (A) stands for Auxiliary curves and the second one to Hansen bound (S), where the (S) stands for Seshadri constant.
1.7.2. **Using an auxiliary set of curves : Hansen’s bound (A).** The second category of estimates consists in using an auxiliary family of irreducible curves \( C_1, \ldots, C_a \) such that the curve \( C := C_1 \cup \cdots \cup C_a \) contains \( \mathcal{P} \). Bounds of this kind appear in [10] Theorem 4 and [28] §3.2. For instance, we have the following statement.

**Proposition 1.21** (Hansen bound (A), [28] Proposition 3.2). Let \( X \) be a normal projective variety defined over \( \mathbb{F}_q \) of dimension at least two. Let \( C_1, \ldots, C_a \) be (irreducible) curves on \( X \) and \( \mathcal{P} = \{ P_1, \ldots, P_n \} \) be a set of \( \mathbb{F}_q \)-rational points of \( X \). Assume that for all \( 1 \leq i \leq a \), the number of \( \mathbb{F}_q \)-rational points on \( C_i \) is less than an integer \( N \). Let \( \mathcal{L} \) be a line bundle on \( X \) over \( \mathbb{F}_q \) such that \( \mathcal{L} \cdot C_i > 0 \) for all \( i \). Let

\[
\ell := \sup_{s \in W^0(X, \mathcal{L})} \{ i \mid Z(s) \text{ contains } C_i \},
\]

where \( Z(s) \) denotes the vanishing locus of \( s \). Then the code \( C(X, \mathcal{P}, \mathcal{L}) \) has length \( n \) and minimum distance

\[
d \geq n - \ell N - \sum_{i=1}^a \mathcal{L} \cdot C_i.
\]

Moreover, if \( \mathcal{L} \cdot C_i = \eta \leq N \) for all \( i \), then

\[
d \geq n - \ell N - (a - \ell)\eta.
\]

1.7.3. **Using Seshadri constants, Hansen bound (S).** The third kind of estimate is based on Seshadri constants whose definition is recalled below.

**Definition 1.22** (Seshadri constant). Let \( \mathcal{L} \) be a line bundle on \( X \) and \( \mathcal{P} \) be a union of closed points of \( X \). Let \( \pi : \text{Bl}_\mathcal{P} X \rightarrow X \) be the blowup of \( X \) at \( \mathcal{P} \). Then the local Seshadri constant is defined as

\[
\varepsilon(X, \mathcal{P}, \mathcal{L}) := \sup \{ \varepsilon \in \mathbb{Q} \mid \pi^* \mathcal{L} - \varepsilon E \text{ is nef} \},
\]

where \( E \) denotes the exceptional divisor of \( \text{Bl}_\mathcal{P}(X) \).

**Proposition 1.23** (Hansen bounds (S), [28] Proposition 3.1). Let \( X \) be a smooth projective surface defined over \( \mathbb{F}_q \). Let \( \mathcal{L} \) be a line bundle on \( X \).

1. **(S1) Suppose \( \mathcal{L} \) is ample with Seshadri constant \( \varepsilon(X, \mathcal{P}, \mathcal{L}) \geq \varepsilon \). Then the corresponding code has minimum distance

\[
d \geq n - \frac{\mathcal{L}^2}{\varepsilon}.
\]

2. **(S2) Let \( \mathcal{I} \) be the ideal sheaf of the \( \mathbb{F}_q \)-rational points \( \mathcal{P} = \{ P_1, \ldots, P_n \} \). Suppose \( \mathcal{L} \otimes \mathcal{I} \) is generated by global sections (such \( \xi \in \mathbb{N} \) exists for instance if \( \mathcal{L} \) is ample). Then,

\[
d \geq n - \xi \mathcal{L}^2.
\]

**Remark 1.24.** Actually, in [28] the author states the result for codes on an arbitrary variety.

Note that using Proposition 1.21 Hansen [28] obtains a better lower bound, namely

\[
d \geq n - (q + 1)(a + b) + ab
\]

which turns out to be the actual minimum distance as proved in [13] Theorem 2.1 & Remark 2.2.

**Remark 1.25.** Actually, Hansen bound (S) is very close to Aubry’s one. In particular, when \( \mathcal{L} \) is a very ample line bundle on \( X \), then \( \mathcal{L} \otimes (q+1) \otimes \mathcal{I} \) is always generated by its global sections. Indeed, consider the embedding \( X \rightarrow \mathbb{P}^l \) associated to \( \mathcal{L} \), using the notation of Lemma 1.15 one can check that the global sections of the form \( x_1^i x_2^j \cdots x_n^j \) generate locally the sheaf \( \mathcal{L} \otimes (q+1) \otimes \mathcal{I} \) at any point. Consequently, for the choice \( \xi = q + 1 \), Hansen’s bound of Proposition 1.23 is nothing but Aubry’s bound.
1.8. Further discussion about Seshadri constants. Our Theorem 1.10 which provides a lower for the minimum distance can be related with the Seshadri constant as follows.

**Theorem 1.26.** Let $d^* = n - \Gamma \cdot G$ be the lower bound for the minimum distance given by Theorem 1.10 of $C(X, \mathcal{P}, G)$. We have

$$d^* \leq (1 - \varepsilon(X, \mathcal{P}, G))n.$$

**Proof.** Let $a \in \mathbb{Q}$ such that $\pi^*G - aE$ is nef. Then, consider the strict transform $\tilde{\Gamma}$ of $\Gamma$ by the blowup map $\pi : Bl_\mathcal{P}(X) \to X$. We have

$$(\pi^*G - aE) \cdot \tilde{\Gamma} \geq 0,$$

that is to say,

$$(\pi^*G - aE) \cdot (\pi^*\Gamma - \sum_{P \in \mathcal{P}} \text{mult}_P(\Gamma)E_P) \geq 0,$$

where $\text{mult}_P(\Gamma) = \min\{\text{mult}_P(C) | C \in \Gamma\}$. This leads to

$$\Gamma \cdot G - a \sum_{P \in \mathcal{P}} \text{mult}_P(\Gamma) \geq 0.$$

By definition of a $\mathcal{P}$-interpolating system, $\text{mult}_P(\Gamma) \geq 1$ for all $P \in \mathcal{P}$. This entails $\Gamma \cdot G \geq an$. This holds for any $a \in \mathbb{Q}$ such that $\pi^*G - aE$ is nef, and hence, we get

$$(1.5) \quad \Gamma \cdot G \geq \varepsilon(X, \mathcal{P}, G)n. \quad \square$$

**Remark 1.27.** Inequality (1.5) suggest another interesting application of $\mathcal{P}$-interpolating linear systems. They permit to get upper bounds for Seshadri constants.

1.9. Behaviour in towers. Another interest of our approach is that our criterion can be lifted by a finite morphism. Hence it can be used to estimate the asymptotic parameters of codes from towers of surfaces.

**Proposition 1.28.** Let $\varphi : Y \to X$ be a finite morphism of smooth projective surfaces. Let $\mathcal{P}$ be a set of rational points of $X$, and $\mathcal{P}_0$ be a set of rational points of $Y$ such that $\varphi(\mathcal{P}_0) \subseteq \mathcal{P}$. If $\Gamma$ is a $\mathcal{P}$-interpolating linear system on $X$, then $\varphi^*\Gamma$ is $\mathcal{P}_0$-interpolating.

**Proof.** Let $H \in \Gamma - \mathcal{P}$, then $\varphi^*H$ is in $\varphi^*\Gamma - \mathcal{P}_0$ since $\varphi(\mathcal{P}_0) \subseteq \mathcal{P}$. Assume now that $\varphi^*\Gamma - \mathcal{P}_0$ had a base curve and let $Y_0$ be an irreducible component of this base curve. This entails that $\varphi(Y_0)$ is in the base locus of $\Gamma - \mathcal{P}$. Since by assumption this linear system has no base curve, $\varphi$ maps $Y$ into a single point. This contradicts the finiteness of $\varphi$. \square

Note that no previous work in the literature investigated the behaviour of a lower bound for the minimum distance in such a relative case. For this reason, and in order to push further the comparison with other known bounds, we conclude this section by investigating the behaviour of these bounds under morphisms.

1.9.1. Aubry’s bound. This bound is simple to apply and hence could be used for estimates in towers. However, it requires to have a very ample divisor at each level of the tower. Note that even for a finite morphism, $\pi : Y \to X$, given a very ample divisor $G$ on $X$, then, from [23] Prop 5.1.12], $\pi^*G$ is ample on $Y$ but not necessarily very ample.
1.9.2. \textit{Hansen bound (A).} Contrarily to our bound, Hansen’s bound (A) does not seem to be usable for asymptotics. Even in the case of a single morphism \( \pi : Y \to X \). Let \( \mathcal{P}_X \) be a set of rational points of \( X \), \( G \) be a divisor and set of curves \( C_1, \ldots, C_a \) as in Proposition [1,21]. In addition, suppose there exists a set of rational points \( \mathcal{P}_Y \) of \( Y \) such that \( \pi(\mathcal{P}_Y) = \mathcal{P}_X \). To study the code \( C(\mathcal{P}, \mathcal{P}_Y, \pi^* G) \) one could consider the set of curves \( \pi^* C_1, \ldots, \pi^* C_a \). In the lower bound from Proposition [1,21]

\[
d \geq |\mathcal{P}_Y| - \ell_Y N_Y - \sum_{i=1}^{a} \pi^* G \cdot \pi^* C_i
\]

Thus, a part of the lower bound can be deduced from the lower bound for the minimum distance of \( C(X, \mathcal{P}, G) \) and \( \deg \pi \). Even \( N_Y \) can roughly be bounded above by \( (\deg \pi) N \). However, there does not seem to be a manner to deduce \( \ell_Y \) from \( \ell \).

1.9.3. \textit{Hansen bounds (S).} Contrarily to the two previous ones, the bounds given in Proposition [1,23] behave well under finite morphisms, and can be applied to towers of finite morphisms. To prove it, first notice that the criterion of Proposition [1,23] rests on an assumption of ampleness. Fortunately, as we already noticed, the pullback of an ample divisor by a finite map between smooth surfaces is ample [23 Prop 5.1.12]. Next, to prove the good behaviour of Hansen criteria in the relative case, we use the following statements.

\textbf{Proposition 1.29.} Let \( Y \) be a smooth projective geometrically connected surface over \( \mathbb{F}_q \) and \( \pi : Y \to X \) be a finite morphism. Then,

\[ \varepsilon(X, \mathcal{P}, G) = \varepsilon(Y, \mathcal{Q}, \pi^* G), \]

where \( \mathcal{Q} \) is the set of points of \( Y \) defined as \( \mathcal{Q} := \pi^{-1}(\mathcal{P}) \).

\textbf{Proof.} Consider the following diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi_Y} & \text{Bl}_\mathcal{Q}(Y) \\
\downarrow \pi & & \downarrow \bar{\pi} \\
X & \xleftarrow{\pi_X} & \text{Bl}_\mathcal{P}(X)
\end{array}
\]

where \( \text{Bl}_\mathcal{P}(X) \) (resp. \( \text{Bl}_\mathcal{Q}(Y) \)) denotes the blowup of \( X \) at \( \mathcal{P} \) (resp. the blowup of \( Y \) at \( \mathcal{Q} \)). In addition, we denote respectively by \( E_X \) and \( E_Y \) the exceptional divisor of \( \text{Bl}_\mathcal{P}(X) \) and \( \text{Bl}_\mathcal{Q}(Y) \). Let \( s \in \mathcal{Q} \) be such that \( p_X^* G - sE_X \) is nef. That is \( s \leq \varepsilon(X, \mathcal{P}, G) \). Let \( C \) be a curve on \( \text{Bl}_\mathcal{Q}(Y) \) and suppose that

\[ C \cdot (p_Y^* \pi^* G - sE_Y) < 0. \]

By the commutativity of the diagram, we get

\[ C \cdot (\bar{\pi}^* p_X^* G - s\bar{\pi}^* E_X) < 0. \]

By the projection formula,

\[ \bar{\pi}^* C \cdot (p_X^* G - sE_X) < 0, \]

which contradicts the original assumption on \( s \). As a consequence, we get

\[ \varepsilon(X, \mathcal{P}, G) \leq \varepsilon(Y, \mathcal{Q}, \pi^* G). \]

Conversely, consider a rational number \( s \leq \varepsilon(Y, \mathcal{Q}, \pi^* G) \) and a curve \( D \) on \( \text{Bl}_\mathcal{P}(X) \) such that \( D \cdot (p_X^* G - sE_X) < 0 \). Then,

\[ \bar{\pi}^* D \cdot (\bar{\pi}^* p_X^* G - s\bar{\pi}^* E_X) < 0, \]

that is \( \bar{\pi}^* D \cdot (p_Y^* \pi^* G - sE_Y) < 0, \)
which contradicts the original assumption on $s$ and concludes the proof. \hfill \Box

**Proposition 1.30.** Let $Y$ be a smooth projective geometrically connected surface over $\mathbb{F}_q$ and $\pi : Y \to X$ be a finite morphism. Let $Q$ be the set of points of $Y$ defined as $Q := \pi^{-1}(P)$. Let $I_P$ (resp. $I_Q$) be the ideal sheaf of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) associated to the set of rational points $P$ (resp. $Q$). Suppose that the rational points in $P$ do not lie in the ramification locus of $\pi$. Let $L$ be a line bundle on $X$ and $\xi$ a positive integer such that $L^{\otimes \xi} \otimes I_P$ is generated by its global sections, then $\pi^* L^{\otimes \xi} \otimes I_Q$ is generated by its global sections.

**Proof.** The sheaf $L^{\otimes \xi} \otimes I_P$ is generated by its global sections, which is equivalent to the existence of a surjective morphism $\mathcal{O}_X^{\oplus m} \to L^{\otimes \xi} \otimes I_P$ for some positive integer $m$. Since $\pi$ is finite, it is flat ([29, Exercise 9.3(a)]) and hence the functor $\pi^*$ is exact. Thus, by pulling back by $\pi$ and using the exactness of $\pi^*$ we get a surjective morphism $\mathcal{O}_Y^{\oplus m} \to \pi^* L^{\otimes \xi} \otimes \pi^* I_P$ and the non ramification hypothesis entails that $\pi^* I_P = I_Q$. \hfill \Box

Thanks to the previous results, we can assert that Hansen’s bounds (S) stated in Proposition 1.23 applies in the relative case. However, one needs to be careful about one fact: to get a code over the same ground field, the set of points $Q := \pi^{-1}(P)$ should contain only rational points. In addition, for Proposition 1.23 (S2) to hold, $\pi$ should not be ramified at the points of $P$ and hence, $P$ should split completely.

1.9.4. **Summary of the behaviours of various bounds under finite morphisms.** Given a surface $X$ with a finite set of rational points $P$ and a divisor $G$, an interesting question would be the following. Consider a tower of finite covers

\[ \cdots \to X_{n+1} \to X_n \to \cdots \to X_2 \to X_1 \to X \]

in which $P$ split totally. Is it possible to deduce lower bounds for the parameters of the code $C(X_n, \pi_{n}^{-1}(P), \pi_{n}^* G)$, where $\pi_n$ denotes the composed morphism $\pi_n : X_n \to X$, from some information defined only on the bottom surface $X$?

- **Aubry’s bound (Proposition 1.18).** Partially possible but a very ampleness condition on $\pi_n^* G$ to assert for every $n$;
- **Hansen bound (A) (Proposition 1.21).** Seems difficult;
- **Hansen bounds (S) (Proposition 1.23).** Yes : one only needs to know some lower bound for a Seshadri constant on the bottom surface $X$ or the integer $\xi$ such that $L^{\otimes \xi} \otimes I_P$ is globally generated.
- **Our bound (Theorem 1.10).** Yes, one only needs a $P$–interpolating linear system on the bottom surface $X$. 
1.10. How to get good towers of surfaces? This is clearly a natural question since codes from towers of curves are well-known to provide the best known sequences of codes. In the case of curves, one can see three approaches in the literature:

- modular curves, see for instance [50];
- recursive towers of function fields, see for instance [20];
- class field towers, see for instance [35].

In § 2.1 we investigate a class field like approach towards by providing a criterion for a surface to have an infinite tower of étale $\ell$–covers that splits totally above a fixed finite set of points.

2. Infinite étale towers of surfaces

In § 2.1 to follow, we investigate a class–field like approach for the existence of an infinite tower of étale covers of a given surface $X$. In particular, we prove Theorem 2.7 a sufficient condition for the existence of such a tower in which some finite set $T$ of points splits totally.

This criterion raises the following question: which explicit surfaces may have an infinite tower of étale covers in which some non-empty fixed set of rational points splits totally? For this sake, we prove in § 2.2 that one can find such an example on a given surface $X$ if, and only if, one can find an example on some of its relatively minimal models. Then, we consider the Kodaira classification of minimal surfaces, explaining why some cases can quickly be excluded. In particular, note that for a surface to have an infinite étale tower that splits completely at some finite set of rational points, a necessary condition is that its geometric étale fundamental group is infinite.

We conclude this Section with an example in § 2.3 of surface for which Theorem 2.7 can be entirely worked out.

2.1. Étale covers of marked surfaces.

2.1.1. The étale site of a marked scheme. Throughout this section, let $X$ be a scheme and $T$ a finite set of closed points. For a morphism $\phi : Y \to X$, we denote by $T_Y$ the set $\phi^{-1}(T)$.

Caution. Compared to the previous section where the finite set of points on the surfaces was referred to as $P$, in this section we denote it by $T$ to be coherent with the usual notation and to emphasize that what follows holds for a finite set of closed points and not only a set of rational points.

Basic definitions and properties. In this paragraph, we briefly recall the definition of the marked étale site of a marked scheme which has been introduced by A. Schmidt [45] in the case of curves and later generalized to general schemes (see [46]). A marked scheme is a pair $(Y, S)$ where $Y$ is a scheme and $S$ is a set of points of $Y$. A morphism of marked schemes $f : (Z, R) \to (Y, S)$ is a morphism of schemes $f : Z \to Y$ such that $f(R) \subset S$.

We consider the marked scheme $(X, T)$ and we denote by $(X, T)_{\text{ét}}$ its marked étale site. It consists in the category of morphisms $\phi : (U, S) \to (X, T)$ such that $\phi : U \to X$ is étale and $S = T_U$ (called $T$-marked étale morphisms) together with the following coverings: surjective families $(p_i : (U_i, S_i) \to (U, S))_i$ such that, for any $s \in S$, there is a $i$ and a $u_i \in S_i$ such that $p_i(u_i) = s$ and the induced field homomorphism $\kappa(s) \to \kappa(u_i)$ is an isomorphism.

We can easily check that these data define a site (i.e. a Grothendieck topology in [11 Def. 1.1.1]). Thus, the general properties of Grothendieck topologies apply here.
Let $P(X,T)$ and $S(X,T)$ denote respectively the category of presheaves and sheaves of abelian groups on $(X,T)_{\text{et}}$. An example of such a sheaf is given by $O_{(X,T)}(U) = \Gamma(U, O_U)$. We denote by $i : S(X,T) \to P(X,T)$ the inclusion functor and by $\# : P(X,T) \to S(X,T)$ its left adjoint (see [1] §2.1 for details of the construction). It is a general fact for Grothendieck topologies that $i$ is left exact and $\#$ is exact (see [1] Th. 2.1.4).

As usual, if $\pi : Y \to X$ is a morphism and $\mathcal{G}$ is a presheaf on $(Y,T_Y)_{\text{et}}$, we define a presheaf $\pi^*\mathcal{G}$ by putting $\pi^*\mathcal{G}(U) = \mathcal{G}(U \times_X Y)$ for any $T$-marked étale morphism $U \to X$. It is clear that it satisfies the axiom of being a sheaf as soon as $\mathcal{G}$ is a sheaf, and that the induced functor $\pi_* : S(Y,T_Y) \to S(X,T)$ is left exact.

We can define (see [1] Th. 1.3.1) a left adjoint $\pi^*$ to $\pi_*$. Indeed, for a presheaf $\mathcal{G}$ on $(X,T)_{\text{et}}$, we define first a presheaf $\pi^*(\mathcal{G})$ on $(Y,T_Y)$ by putting, for any $T_Y$-marked étale map $V \to Y$, $\pi^*(\mathcal{G})(V) = \lim \mathcal{G}(U)$, where the direct limit is taken on the $T$-marked étale maps $U \to X$ making the diagram

$$
\begin{array}{ccc}
V & \longrightarrow & U \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
$$

commute. If $\mathcal{F}$ is a sheaf, we define $\pi^*(\mathcal{F}) = \#\pi^!(i(\mathcal{F}))$.

As in any Grothendieck topology, $S(X,T)$ has enough injectives (see [1] Misc. 2.1.1 or [22] Th. 1.10.1). For a sheaf $\mathcal{F}$ of abelian groups on $(X,T)_{\text{et}}$, the functor $\mathcal{F} \mapsto \mathcal{F}(X,T) = \Gamma(X,T,\mathcal{F})$ is left exact and we denote by $H^p(X,T,\mathcal{F})$ its associated cohomology groups. One can also define the cohomology groups with support in a closed subscheme $Z$, as the right derivatives $H^p_Z(X,T,\mathcal{F})$ of the left exact functor $\mathcal{F} \mapsto \ker(\mathcal{F}(X,T) \to \mathcal{F}(U,T \cap U))$, where $U$ is $X \setminus Z$.

As well as for the étale site, we can prove excision (see [46] §2–3) and a limit result, which is the main tool to compute marked étale cohomology groups. For $x \in T$, put $X^h_x = \text{Spec}(O_{X,x}^h)$, where $O_{X,x}^h$ denotes the henselianization of the local ring $O_{X,x}$. Let us quote here the results of [46] we will use.

**Proposition 2.1.** Let $X$ be a scheme and $T$ a finite set of closed points. Then, for every sheaf of abelian groups $\mathcal{F}$ on $(X,T)$, there is a long exact sequence

$$
\cdots \to H^p_T(X,T,\mathcal{F}) \to H^p_{\text{et}}(X,T,\mathcal{F}) \to H^p_{\text{et}}(X \setminus T,\mathcal{F}) \to \cdots
$$

and, for all $i \geq 0$,

$$
H^p_T(X,T,\mathcal{F}) \cong \bigoplus_{x \in T} H^p_{\text{et}}(X^h_x,x,\mathcal{F}).
$$

Following Schmidt (see [46] §5), one can define, for a noetherian, normal and connected scheme $X$ a fundamental group $\pi_1(X,T)$ which is profinite and classifies étale covers of $X$ where the points of $T$ split completely. Be cautious that fundamental groups are defined in [46] for more general schemes and that what we denote here by $\pi_1(X,T)$ for convenience is denoted by $\hat{\pi}^\text{et}_1(X,T)$ in [46] §5 (in our setting, there is no possible confusion, thanks to [46] Proposition 5.1).

We denote by $(X,T)(\ell)$ the universal pro-\(\ell\)-cover of $(X,T)$. The projection $(X,T)(\ell) \to X$ is Galois with Galois group the maximal pro-\(\ell\)-quotient $\pi_1(X,T)(\ell)$ of $\pi_1(X,T)$. Finally, the Hochschild-Serre spectral sequence relates Galois cohomology and étale cohomology groups (see [46] §4–5). Let us also recall it here.

**Proposition 2.2.** Let $X$ be a connected noetherian scheme, $T$ a finite set of closed points of $X$. Let $M$ be a discrete $\ell$-torsion $\pi_1(X,T)(\ell)$-module. Then we have the following spectral sequence:

$$
E_2^{pq} = H^p(\pi_1(X,T)(\ell), H^q((X,T)(\ell), T, M)) \Rightarrow H^{p+q}(X,T,M).
$$
It induces a five term exact sequence:
\[ 0 \to H^1(\pi_1(X,T)(\ell), M) \to H^1_{\text{et}}(X, T, M) \to H^1((\overline{X,T})(\ell), M)_{\pi_1}^{(X,T)(\ell)} \to \]
\[ \to H^2(\pi_1(X,T)(\ell), M) \to H^2_{\text{et}}(X, T, M). \]

As \( H^1((\overline{X,T})(\ell), M) = 0 \), because \((\overline{X,T})(\ell)\) admits no non-trivial \( \ell \)-cover, we see that we have the following isomorphisms.

**Corollary 2.3.** In the setting of the proposition, the Hochschild-Serre spectral sequence induces isomorphisms
\[ H^i(\pi_1(X,T)(\ell), M) \simeq H^i_{\text{et}}(X, T, M) \]
for \( i = 0, 1 \)

and an injection
\[ H^2(\pi_1(X,T)(\ell), M) \to H^2_{\text{et}}(X, T, M). \]

2.1.2. The cohomology of marked surfaces. Let \( X \) be a 2-dimensional noetherian regular scheme defined over \( \mathbb{F}_q \) and let \( T \) be a finite set of closed points. Let \( \Lambda = \mathbb{F}_q \), where \( \ell + q \) is prime number. The aim of this paragraph is to compute the étale cohomology groups \( H^i(X,T, \Lambda) \) of the marked surface \((X,T)\). Notice that this can be immediately generalized to higher dimensions.

**Local computations.** Let \( X^h_x \) denote the spectrum \( \text{Spec}(\mathcal{O}^h_{X,x}) \) of the henselisation of the local ring \( \mathcal{O}_{X,x} \) of \( X \) at a point \( x \in T \) with finite residue field \( k \). The scheme \( X^h_x \) is henselian with closed point \( x \), and therefore
\[ H^i_{\text{et}}(X^h_x, \Lambda) = H^i_{\text{et}}(x, \Lambda) = H^i(k, \Lambda) \begin{cases} \Lambda & \text{for } i = 0, 1 \\ 0 & \text{for } i \geq 2. \end{cases} \]

Let \( U_x = X^h_x - x \). We wish to understand the groups
\[ H^i_x(X^h_x, \Lambda) \text{ and } H^i_x(X^h_x, x, \Lambda). \]

The first groups arise in the sequence
\[ \cdots \to H^i_x(X^h_x, \Lambda) \to H^i_x(X^h_x, x, \Lambda) \to H^i(U_x, \Lambda) \to \cdots. \]

They are computed thanks to Gabber purity theorem [18]:
\[ H^i_x(X^h_x, \Lambda) \approx H^{i-1}(x, \Lambda(-2)) \]
as \( x \) is regular of codimension 2 in the regular affine scheme \( X^h_x \). Then, they are 0 for \( i \neq 4, 5 \) and \( H^4_x(X^h_x, \Lambda) = \text{Hom}(\mu_4(k) \otimes \mu_4(k), \Lambda) \), \( H^5_x(X^h_x, \Lambda) = \text{Hom}(\mu_5(k) \otimes \mu_5(k), \Lambda) \) (thus \( \Lambda \)-vector spaces of dimension 1 for \( i = 4, 5 \) if \( \ell \) divides \( q - 1 \) and trivial otherwise).

We deduce that \( H^1(U_x, \Lambda) \approx H^1(X^h_x, \Lambda) \), that
\[ H^i(U_x, \Lambda) \approx H^{i+1}_x(X^h_x, \Lambda) \text{ for } i = 3, 4 \]
and that \( H^i(U_x, \Lambda) = 0 \) for \( i = 2 \) or \( i \geq 5 \).

The second groups appear in
\[ \cdots \to H^i_x(X^h_x, x, \Lambda) \to H^i_x(X^h_x, x, \Lambda) \to H^i(U_x, \Lambda) \to \cdots. \]

Since the identity of \((X^h_x,x)\) is cofinal among the covering families of \((X^h_x,x)\), we have \( H^i(X^h_x, x, \Lambda) = 0 \) for \( i \geq 1 \). Indeed, if \( f : Y \to X_x^h \) is a finite étale morphism marked at \( x \), then it admits a section (see [39], Th. I.4.2.) inducing an isomorphism between \( X^h_x \) and a connected component of \( Y \) (as \( f \) is separated). If \( Y \) is connected then the map \( f \) is an isomorphism.

It implies that \( H^i_x(X^h_x, x, \Lambda) = \ker(H^i(X^h_x, x, \Lambda) \to H^i(U_x, \Lambda)) = 0 \) and that \( H^i_x(X^h_x, x, \Lambda) \approx H^{i-1}(U_x, \Lambda) \) for any \( i \geq 2 \). Therefore, we have shown the following:
Proposition 2.4. Let $T$ be $\{x\}$ or $\emptyset$. The groups $H^i_x(X^h_x, T, \Lambda)$ are trivial for $i \leq 1$, $i = 3$ or $i \geq 6$. Moreover, we have:

$$\dim H^i_x(X^h_x, T, \Lambda) = \begin{cases} 
\|T \| & \text{if } i = 2 \\
1 & \text{if } i = 4, 5 \text{ and } \ell \mid q - 1 \\
0 & \text{otherwise}
\end{cases}$$

Global computations. Let us now use excision and the local computations in order to get information on the cohomology groups. Put $h^i(X, \ldots)$ for the dimension $\dim \Lambda H^i(X, \ldots)$. As soon as it is well defined, put $\chi(X, \ldots) := \sum_{i \geq 0} (-1)^i h^i(X, \ldots)$.

Proposition 2.5. Let $X$ be a 2-dimensional noetherian regular scheme defined over $F_q$ and let $T$ be a finite set of closed points. Let $\Lambda = F_\ell$.

$$h^2(X, T, \Lambda) - h^1(X, T, \Lambda) = \|T \| + h^2(X, \Lambda) - h^1(X, \Lambda) = \dim H^2(\overline{X}, \Lambda)^{G_{F_q}} + \|T \| - 1,$$

and

$$h^5(X, T, \Lambda) - h^4(X, T, \Lambda) + h^3(X, T, \Lambda) = h^5(X, \Lambda) - h^4(X, \Lambda) + h^3(X, \Lambda),$$

where $G_{F_q} = \text{Gal}(F_q, F_q)$ and $X = X \times_{F_q} F_q$.

Proof. We use excision to compute the $H^i(X, T, \Lambda)$. It provides exact sequences

$$\cdots \to \bigoplus_{x \in T} H^i_x(X^h_x, \Lambda) \to H^i(X, \Lambda) \to H^i(X - T, \Lambda) \to \cdots$$

and

$$\cdots \to \bigoplus_{x \in T} H^i_x(X^h_x, T_x, \Lambda) \to H^i(X, T, \Lambda) \to H^i(X - T, \Lambda) \to \cdots.$$

The first sequence leads to isomorphisms $H^i(X, T, \Lambda) \cong H^i(X - T, \Lambda)$ for $i = 1, 2$, and from the second one can deduce the first equality of the proposition by the local computations.

Moreover, the Hochschild-Serre spectral sequence leads to the following, for any $i \geq 1$:

$$0 \to H^1(G_{F_q}, H^{i-1}(\overline{X}, \Lambda)) \to H^i(X, \Lambda) \to H^i(\overline{X}, \Lambda)^{G_{F_q}} \to 0.$$

Together with the fact that for any finite module $M$, $H^0(G_{F_q}, M)$ and $H^1(G_{F_q}, M)$ have same cardinality, we obtain

$$h^2(X, \Lambda) - h^1(X, \Lambda) = \dim H^2(\overline{X}, \Lambda)^{G_{F_q}} - 1,$$

which leads to

$$h^2(X, T, \Lambda) - h^1(X, T, \Lambda) = \dim H^2(\overline{X}, \Lambda)^{G_{F_q}} + \|T \| - 1.$$

The second part of the theorem comes from the isomorphisms $H^i_x(X^h_x, x, \Lambda) \cong H^i_x(X^h_x, \Lambda)$ for $i \geq 3$, and their triviality for $i = 3$ and 6. Note that if $\ell \nmid q - 1$, then the exact sequences imply that $H^i(X, T, \Lambda) \cong H^i_{\text{cl}}(X, \Lambda)$ for $i \geq 3$.

Corollary 2.6. Let $X$ be a smooth projective surface defined over $F_q$. Let $T$ be a finite set of closed points of $X$. Then, the cohomology groups $H^i(X, T, F_\ell)$ vanish for $i \geq 6$, and we have $\chi(X, T, F_\ell) = \|T \|.$

Proof. Indeed, we have for $i \geq 6$, $H^i(X, T, F_\ell) \cong H^i(X - T, F_\ell) \cong H^i(X, F_\ell) = 0$, as the local cohomology groups vanish, as $X$ is smooth. Moreover, since $X$ is projective, $\chi(X, F_\ell)$ vanishes by the Poincaré duality, which gives the desired result by the proposition.
Choice of the closed points.

Let $X$ be a smooth projective (absolutely irreducible) surface defined over $F_q$ and $T$ be a finite set of closed points containing at least a point $x_0$. Class field theory (see [47 VI.§4–5] or [48]) for a concise exposition) defines a reciprocity map $\rho$ from the Chow group $\text{CH}_0(X)$ of zero-cycles of $X$ to $\pi_1^{ab}(X)$, by sending closed points $x$ to their Frobenius (the image in $\pi_1(X)^{ab}$ of the Frobenius of $\text{Gal}(\kappa(x)/\kappa(x))^{ab}$). Its image is exactly the set $\pi_1^{ab}(X)$ of elements of $\pi_1^T(X)$ whose image in $\text{Gal}(F_q/F_q)$ by the natural map is an element of $\mathbb{Z}$ (in $\mathbb{Z}$). By [47 VI.n° 16 Théorème 1]), we have the following isomorphism of short exact sequences:

$$
\begin{array}{c}
0 \longrightarrow \text{CH}_0^0(X) \longrightarrow \text{CH}_0(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0 \\
0 \longrightarrow \pi_1^{ab,0}(X) \longrightarrow \pi_1^{ab}(X) \longrightarrow \mathbb{Z} \longrightarrow 0
\end{array}
$$

The map $\rho$ induces an isomorphism from $\text{CH}_0(X)/(x_0)$ to $\pi_1(X)^{ab}/(\rho(x_0))$. Then the group $\pi_1(X)^{ab}/(\rho(x_0))$ is the finite quotient of $\pi_1(X)^{ab}$ corresponding to the maximal étale abelian cover of $X$, totally split at $x_0$ (see [47 VI.§5] : a finite étale cover is totally split at $x_0$ if and only if its associated Frobenius is trivial). Therefore, we have an isomorphism $\text{CH}_0(X)/(\ell, x_0) \simeq \pi_1(X, \{x_0\})^{ab}/\ell$.

Similarly, adding one more point $x$ to $T$ lowers $h^1$ by one or enlarge $h^2$ by one, depending on the fact that $x$ belongs to the group generated by the previous split points in $\text{CH}_0(X)/\ell$ or not.

Therefore, we have $h^1(X, T, \Lambda) = h^1(X) - r_T$, where $r_T$ is the $F_q$-dimension of the space generated by the points of $T$ in $\text{CH}_0(X)/\ell$. If $T \neq \emptyset$, $1 \leq r_T \leq \# T$.

2.1.3. Golod-Shafarevich criterion. Our strategy to produce infinite covers of $X$ with splitting properties is the Golod-Shafarevich criterion. For this purpose, we need the following inequality:

$$h^2(\pi_1(X, T)(\ell), \mathbb{Z}/\ell\mathbb{Z}) \leq \frac{h^1(\pi_1(X, T)(\ell), \mathbb{Z}/\ell\mathbb{Z})^2}{4}.$$

Because of the previous comparison maps (see Corollary 2.3), it is sufficient to check that it is satisfied for the étale cohomology groups of the marked surface $(X, T)$.

**Theorem 2.7.** If

$$\dim H^1(X, \Lambda)^{Gr_{\ell}} \geq r_T + 1 + 2\sqrt{\dim H^2(X, \Lambda)^{Gr_{\ell}} + \# T},$$

then $\pi_1(X, T)(\ell)$ is infinite, i.e. $X$ admits a $T - \ell$ infinite classfield tower.

**Proof.** The group $\pi_1(X, T)(\ell)$ is infinite if

$$h^2(\pi_1(X, T)(\ell), \mathbb{Z}/\ell\mathbb{Z}) \leq \frac{h^1(\pi_1(X, T)(\ell), \mathbb{Z}/\ell\mathbb{Z})^2}{4}.$$

As $h^1(\pi_1(X, T)(\ell), \mathbb{Z}/\ell\mathbb{Z}) = h^1_T(X, T; \Lambda) := h^1_T$ and

$$h^2(\pi_1(X, T)(\ell), \mathbb{Z}/\ell\mathbb{Z}) \leq h^2(X, T, \mathbb{Z}/\ell\mathbb{Z}) =: h^2_T,$$

because of Corollary 2.3 Therefore it suffices to have $4h^2_T \leq \left(\frac{h^1_T}{4}\right)^2$.

Let $\tilde{h}^i := \dim H^i(X, \Lambda)$ and $\tilde{h}^{1, G} := \dim H^1(X, \Lambda)^{Gr_{\ell}}$. The previous inequality is equivalent to

$$(h^1_T - 2)^2 \geq 4(h^2_T - h^1_T + 1)$$

and to

$$(\tilde{h}^{1, G} - r_T - 1)^2 \geq 4(\tilde{h}^{2, G} + \# T)$$

and the proposition follows. \qed
The next proposition seems to be more useful in a computing point of view, as the Euler-Poincaré characteristic is computable. Let \( \alpha = \bar{h}^1 - h^{1,G} \) and \( \bar{\chi} = \chi(X, F_t) = \chi(X, Q_t) \) (cf. \cite{paper} p166).

**Proposition 2.8.** The conclusion also holds if
\[
(\bar{h}^1 - \alpha + r_T - 5)^2 \geq 4(\bar{\chi} + 2\alpha + 2r_T + 4 + \# T).
\]

**Proof.** We introduce \( \bar{\chi} = \bar{h}^2 - 2\bar{h}^1 + 2 \) in what precedes and note that \( \bar{h}^{2,G} \leq \bar{h}^2 \). \( \square \)

Remark that the first \( \ell \)-adic Betti number \( b_1 \) of \( X \) satisfies \( b_1 \leq \bar{h}^1 \). We may hope to be able to compute \( b_1 \) and \( \bar{\chi} \) by computing the Zeta function of \( X \), and thus to verify whether the previous inequality holds for \( b_1 \), in particular after extending the constants so that \( \alpha = 0 \). When \( \bar{\chi} \) is very small the condition should be satisfied.

### 2.2. Using the classification of surfaces.

Theorem \cite{classification} raises the question: which surfaces may have an infinite tower of étale covers in which some finite set \( T \) splits completely? In the sequel, we sieve the Kodaira classification of surfaces in order to study which classes should be excluded and which ones may provide good candidates. The following statement will be useful to exclude some cases.

**Proposition 2.9.** Let \( \pi : Y \to X \) be a finite morphism of smooth connected projective surfaces over an algebraically closed field. Suppose that \( Y \) has a finite fundamental group, then \( X \) cannot have an infinite tower of étale covers.

**Proof.** Since \( Y \) has a finite fundamental group, then its universal cover \( \tilde{Y} \to Y \) is a finite map and hence, replacing \( Y \) by its universal cover, one can suppose that \( Y \) is simply connected.

Now, denote by \( s \) the degree of the morphism \( Y \to X \) and consider an étale cover \( X_1 \to X \) of degree \( m \). Its pullback on \( Y \) provides an étale cover of \( Y \) and since \( Y \) is supposed to be simply connected, this cover is the disjoint union of \( m \) copies of \( Y \)

\[
\begin{array}{ccc}
Y & \leftarrow & \bigsqcup_{i=1}^m Y \\
\downarrow & & \downarrow g \\
X & \leftarrow & X_1
\end{array}
\]

and the restriction of \( f \) to any component is an isomorphism. Consider the restriction of \( g \) to a connected component of \( \bigsqcup_{i=1}^m Y \), then the map \( Y \to X \) is factorised by a dominant map \( Y \to X_1 \). As a consequence the degree \( s \) of \( Y \to X \) divides \( m \) and hence any étale cover of \( X \) has a bounded degree. \( \square \)

#### 2.2.1. Relatively minimal models.

Recall that a smooth surface \( X \) over a field \( k \) is said to be relatively minimal if there is no \((-1)\)-curve of genus 0 in \( X \). Equivalently, for such a surface, there does not exist a regular surface \( Y \) over \( k \) and a \( k \)-morphism \( X \to Y \) which consists in blowing down a finite set of curves in \( X \).

The following result asserts that any étale cover of a surface comes from an étale cover of a relatively minimal model, so that it is sufficient to investigate relatively minimal surfaces. It is a refinement for relatively minimal models of surfaces of the well known birational invariance property of the fundamental group. This result is probably well-known by the experts, we give a proof because of a lack of a reference.

**Theorem 2.10.** Let \( X_0 \) be a surface over an algebraically closed field \( \bar{k} \) and \( \pi : \bar{X} \to X_0 \) be a surface obtained from \( X_0 \) after blowing up a point. Then for any étale cover \( f : Y \to X \), there exists an étale cover \( f_0 : Y_0 \to X_0 \) from a smooth surface \( Y_0 \), such that \( Y = Y_0 \times_{X_0} X \).
Proof. From Stein Factorization Theorem [29], Corollary III.11.5], the map \( f \circ \pi \) factorizes as

\[
\begin{array}{c}
\pi' \\
\downarrow \\
\overline{Y}_0 \rightarrow \overline{X}_0 \\
\pi' \\
\downarrow \\
X \\
\pi \\
\downarrow \\
Y \\
\downarrow \\
f_0 \\
\overline{Y}_0 \rightarrow \overline{X}_0
\end{array}
\]

where \( \pi' \) has connected fibres and \( f_0 \) is finite.

Let us prove that \( f_0 : \overline{Y}_0 \rightarrow \overline{X}_0 \) is étale. For any variety \( V \) in this diagram, we denote by \( U_V \) the open subvariety of \( V \) above the open subsheme \( U_{\overline{X}_0} \) of \( \overline{X}_0 \) obtained by puncturing the blown-up point \( x_0 \). Let \( \iota : \overline{U} \hookrightarrow \overline{X}_0 \) be a geometric curve on \( \overline{X}_0 \), and \( U_{\overline{X}} = \iota^* U_{\overline{X}_0} \). Since \( f \) is étale, \( \pi'^* f_0^* U_{\overline{X}} = f^* \circ \pi'^* U_{\overline{X}} \) is reduced, so that \( f_0^* U_{\overline{X}} \) itself is reduced and \( f_0 \) is étale outside \( x_0 \). By Zarisky purity Theorem [53], it follows that \( f_0 \) is étale. In particular, \( \overline{Y}_0 \) is smooth since \( \overline{X}_0 \) is.

By the universal property of fiber products, there exists a unique morphism \( \varphi \) such that the following diagram commutes.

\[
\begin{array}{c}
\overline{Y} \\
\varphi \\
\downarrow \\
\overline{Y}_0 \times_{\overline{X}_0} \overline{X} \\
\varphi \\
\downarrow \\
\overline{Y}_0 \\
\downarrow \\
f_0 \\
\overline{X}_0
\end{array}
\]

Let us prove that \( \varphi \) is an isomorphism, concluding the proof. Remind that \( f = p_2 \circ \varphi \) is étale and \( p_2 \) is étale as a base change of the étale map \( f_0 \) [39], Proposition I.3.3.(b)]. Thus, from [39], Corollary I.3.6], the morphism \( \varphi \) is étale too. Therefore, to prove that \( \varphi \) is an isomorphism it is sufficient to prove that any geometric closed point \( P = (y_0, x) \in \overline{Y}_0 \times_{\overline{X}_0} \overline{X} \) has a unique pre-image. By the commutativity of the diagram, the pullback of \( P \) by \( \varphi \) is contained in \( \pi'^* (y_0) \), which is connected by Stein Theorem. In the same way, it is contained in \( f^* (x) \) which is finite since \( f \) is finite. Consequently, \( P \) has a single inverse image by \( \varphi \) and hence \( \varphi \) is an isomorphism.

\[ \square \]

2.2.2. Classification of surfaces. It follows from Theorem [210] that a surface \( X \) admits an infinite tower in which a non-empty set \( T \) of rational points splits if and only if any relatively minimal model \( \pi : X \rightarrow X_0 \) of \( X \) also admits such an infinite tower, in which \( \pi(T) \) splits. This leads us to investigate the Kodaira classification of minimal surfaces given in Table 1 (see for instance [34], §II.8.1] for the characteristic 0 case and or [11] for the general case), in which the numerical equivalence of divisors is denoted by \( \equiv \). For each class of surface in the table, we ask whether such a tower can exists. We indicate in a last column the conclusion of the discussion that follows.

We recall that the Kodaira dimension is invariant under étale covers [33], Theorem 10.9]. Thus, for an étale tower of surfaces, the Kodaira dimension is constant and hence is that of the bottom surfaces. So let us investigate surfaces of Kodaira dimension up to one.

Beginning with surfaces of Kodaira dimension \(-1\), it is well-known that rational surfaces are simply connected (see [24], Corollaire XI.1.2] for instance). Regarding
**Table 1. Classification of algebraic surfaces**

| Structure                    | $\kappa$ | canonical class $K$ | $K^2$ | existence of infinite splitting tower |
|------------------------------|----------|---------------------|-------|---------------------------------------|
| Rational                     | -1       | -                   | -     | no                                    |
| Ruled above a curve of genus $g$ | -        | 8(1 - $g$)          |       | comes from the base curve             |
| Abelian                      |          | $K \equiv 0$        | $K^2 = 0$ |                                        |
| K3                           |          |                      |       |                                        |
| Enriques                     |          |                      |       |                                        |
| Hyperelliptic                | 0        | $K \equiv 0$        | $K^2 = 0$ |                                        |
| Quasi-elliptic (only in char. 2 and 3) | 0        |                      |       | ?                                     |
| Elliptic                     | 1        | $\forall n > 0, nK \not\equiv 0$ | $K^2 = 0$ | ?                                     |
| General type                 | 2        | $K \not\equiv 0$    | $K^2 > 0$ | ?                                     |

ruled surfaces $X$ over a base $C$, since for complete varieties, the fundamental group is a birational invariant [39, Example 5.2(h)] and $X$ is birational to $C \times P^1$, then, from [24, Corollary X.1.7], $\pi_1(X) \cong \pi_1(C) \times \pi_1(P^1)$ and since $P^1$ is simply connected, we conclude that $\pi_1(X) = \pi_1(C)$. More precisely, it can be checked, with a proof similar to that of Theorem 2.10 above using Stein factorisation, that any étale cover of a ruled surface is the pullback of some étale cover of the base curve. It follows that the splitting property of some infinite tower of a ruled surface would come from some of the base, so that we are reduced to the classical problem on curves.

Next, we continue with surfaces of Kodaira dimension 0. First, K3 surfaces are simply connected [31, Remark I.2.3]. Second, in odd characteristic, Enriques surfaces are known (see [15]) to have an étale cover of degree 2 by a K3 surface. In characteristic 2 (see [15] again) an Enriques surface is either covered by a surface which is birational to a K3 surface or by a non normal rational surface. Therefore, in any characteristic, an Enriques surface is covered by a simply connected surface and hence Proposition 2.9 asserts that Enriques surfaces are bad candidates. Third, whereas abelian surfaces can have infinite towers of étale covers, such towers cannot satisfy a non trivial splitting property because each stage being an abelian surface (see [40, Chapter 4 §18]), it cannot have more than $(\sqrt{q} + 1)^2$ rational points from Weil bound. This means that the marked fundamental group of an abelian surface is finite. Fourth, hyperelliptic surfaces are a finite quotient of a product of two elliptic curves, hence are dominated under a finite map by an abelian surface whose marked fundamental group is finite. Hence, from Proposition 2.9 their marked fundamental group is finite too. Last, we are not able to conclude for quasi-elliptic surfaces.

Then, we end with surfaces of Kodaira dimension 1, that is for elliptic surfaces. Here, we are only able to prove, again with a proof very close to that of Theorem 2.10 above using Stein factorisation, that any étale cover of an elliptic surface is an elliptic surface whose base is an étale cover of the original base curve.

**2.3. Example: product of hyperelliptic curves.** The aim of this section is to provide an illustrative example of application of Theorem 2.7 in which any computation can be made explicit. It turns out that, for most surfaces $X$ for which the answer in the last column in Table 1 is not negative, it is a hard task to compute the Galois invariants of $H^1(X, F_\ell)$ and $H^2(X, F_\ell)$, and even to bound from below the first one and to bound from above the second one in an efficient way.
The authors acknowledge that they do succeed only in the case of the product of two hyperelliptic curves of genus larger than 2. Unfortunately, for a product of two curves \( C \times D \) with canonical projections \( p_C, p_D \) and a prime integer \( \ell \), one can prove that the existence of an infinite tower of \( \ell \)-\é etale covers splitting totally at a set of closed points \( \mathcal{P} \) entails either the existence of an infinite tower of \( \ell \)-\é etale covers of \( C \) splitting totally at \( p_C(\mathcal{P}) \) or the existence of an infinite tower of \( D \) splitting totally at \( p_D(\mathcal{P}) \).

**Proposition 2.11.** Let \( C, D \) be two smooth curves. Consider the product \( X = C \times D \) with canonical projections \( p_C, p_D \). Let \( P \) be a rational point of \( C \) and \( Q \) be a rational point of \( D \) and \( C_0 := p_C^{-1}(P) \) and \( D_0 := p_D^{-1}(Q) \). Let \( \ell \) be a prime integer and \( Y \) be a connected smooth surface and \( \pi : Y \rightarrow X \) be an \( \ell \)-\é etale Galois cover of \( X \), then at least one of the two curves \( \pi^*C_0 \) or \( \pi^*D_0 \) is connected. In addition, if \( \mathcal{P} \) is a set of rational points of \( X \) that is totally split in \( Y \), then \( p_C(\mathcal{P}) \) (resp. \( p_D(\mathcal{P}) \)) is totally split in \( \pi^*C_0 \rightarrow C_0 \) (resp. \( \pi^*D_0 \rightarrow D_0 \)).

**Proof.** **Step 1.** Let us prove that the divisor \( C_0 + D_0 \) is ample. Indeed, let \( g \) be a positive integer larger than the genus of \( C \) and that of \( D \). Then we will prove that \( (2g+1)(C_0 + D_0) \) is very ample.

Clearly, \( (2g+1)P \) is very ample on \( C \) and hence provides a closed immersion \( C \rightarrow \P^{n_C} \) for some positive integer \( n_C \). Similarly we \( (2g+1)Q \) provides a closed immersion \( D \rightarrow \P^{n_D} \). This yields a closed immersion \( C \times D \rightarrow \P^{n_C} \times \P^{n_D} \), then by Segre embedding, we get a closed immersion \( \varphi : C \times D \rightarrow \P^{N} \) for some positive integer \( N \) and \( (2g+1)(C_0 + D_0) \sim \varphi^*\mathcal{O}_{\P^{N}}(1) \).

**Step 2.** Since the pullback of an ample divisor is ample ([23 Prop 5.1.12]), the divisor \( \pi^*C_0 + \pi^*D_0 \) is ample and hence has a connected support ([29 Corollary III.7.9]).

**Step 3.** Suppose that both \( \pi^*C_0 \) and \( \pi^*D_0 \) are not connected. Then, using base change principle and since \( \pi \) is a Galois cover, one deduces that both are \( \ell \) copies of their base. In addition, by the projection formula we have
\[
\pi^*C_0 \cdot \pi^*D_0 = \ell
\]
and hence each copy of \( C_0 \) in \( \pi^*C_0 \) meets a single copy of \( D_0 \) in \( \pi^*D_0 \) with multiplicity 1 and avoids the other ones. Thus, one deduces that \( \pi^*C_0 + \pi^*D_0 \) has a support which is isomorphic to the disjoint union of \( \ell \) copies of the support of \( C_0 + D_0 \) on \( X \). Therefore, it is non connected which contradicts its ampleness.

**Step 4.** Suppose that the pullback of \( C_0 \) is connected and that some rational point of \( p_C(\mathcal{P}) \) has a non rational inverse image \( R \) under \( \pi^*C_0 \rightarrow C_0 \). Then, some elements of \( \pi^*\mathcal{P} \) lie in the fibre \( \{ R \} \times \pi^*D_0 \) which does not contain any rational point. This contradicts the assumption that \( \mathcal{P} \) splits completely under \( \pi \).

Therefore, the application of our infiniteness-criterion in the sequel yields an existence result that could have been proved with usual class field theory in dimension one. Moreover, it is even true that, under the usual assumptions, the marked (at a product of sets of rational points) \é etale fundamental group of a product is the fibre product of the corresponding marked \é etale fundamental groups, as shown in [46 Proposition 5.3]. It follows that if the marked \é etale fundamental group of a product \( \pi_1(C \times D, S \times T) \) is infinite, then at least one of the two marked \é etale fundamental groups \( \pi_1(C, S) \) and \( \pi_1(D, T) \) has to be infinite and this also holds for maximal pro-\( p \)-quotients.

Thus, we emphasize that the example to follow is only illustrative. The objective in only to show that the objects involved in Theorem 2.7 can be explicitly computed.
Proposition 2.12. Let \( q \) be a power of an odd prime number. Let \( f(t) \in \mathbb{F}_q[t] \) be the product of \( 2g_1 + 2 \) distinct linear factors and \( g(t) \in \mathbb{F}_q[t] \) be the product of \( g_2 + 1 \) distinct irreducible quadratic factors. Let \( C: y^2 = f(t) \) and \( D: y^2 = g(t) \) be the associated hyperelliptic curves. Let \( \rho \) be a positive integer such that:

- \( 2g_1 + 2 + 2\rho \leq \# C(\mathbb{F}_q) \);
- \( 2\rho \leq \# D(\mathbb{F}_q) \);
- \( 2g_1 + g_2 \geq 3\rho + 2 + 2\sqrt{2g_1g_2} + 2 + 4\rho \).

Then, there exists an infinite étale tower of \( C \times D \) in which some set of \( 4\rho \) rational points splits totally.

Before proving this Proposition below, let us fix some notations and state two preliminary lemmas. We begin by choosing the prime number \( \ell = 2 \), prime to \( q \). We denote by \( \sigma \) and \( \tau \) the hyperelliptic involutions on \( C \) and \( D \) respectively. Those extend on \( X = C \times D \) by \( \sigma(p,q) = (\sigma(p),q) \) and \( \tau(p,q) = (p,\tau(q)) \), generating the abelian group \( A = \langle \sigma, \tau \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2 \). We denote by \( A.P \) the orbit of a point \( P = (p,q) \in C \times D \) under \( A \). Then neither \( p \) nor \( q \) is a Weierstrass point if, and only if, this orbit have order \( 4 \).

Lemma 2.13. Let \( P_1, \ldots, P_\rho \) be \( \rho \) rational points on \( X = C \times D \) whose orbits under \( A \) have 4 elements and are disjoints. Let \( T \) be the union of these orbits. Then, \( \# T = 4\rho \) and

\[
\# T \leq 3\rho + 1 = \frac{3}{4} \# T + 1.
\]

Proof. We denote \( P_i = (p_i, q_i) \in C \times D \). Since all divisors of the form \( p + \sigma(p) \) on \( C \) are equivalent in \( \text{CH}_0(C) = \text{Jac}_C \), and considering both horizontal curves \( C \times \{q_i\} \) and \( C \times \{\tau(q_i)\} \) on \( X \), we have

\[
(p_i, q_i) + (\sigma(p_i), q_i) + (\tau(p_i), q_i) + (\sigma(p_i), \tau(q_i))
\]

and

\[
(p_i, \tau(q_i)) + (\sigma(p_i), \tau(q_i)) + (\tau(q_i), \sigma(p_i)) + (\sigma(p_i), \tau(q_i))
\]

for any \( 2 \leq i \leq \rho \).

Summing these relations, we get, for any \( 2 \leq i \leq \rho \),

(2.1) \[
(p_i, q_i) + (\sigma(p_i), q_i) + (\tau(p_i), q_i) + (\sigma(p_i), \tau(q_i))
\]

\[
= (p_i, q_i) + (\sigma(p_i), q_i) + (\tau(p_i), q_i) + (\sigma(p_i), \tau(q_i)).
\]

In the same way, considering both vertical curves \( \{p_i\} \times D \) and \( \{\sigma(p_i)\} \times D \), we get

(2.2) \[
(p_i, q_i) + (\sigma(p_i), q_i) + (\tau(p_i), q_i) + (\sigma(p_i), \tau(q_i))
\]

\[
= (p_i, q_i) + (\sigma(p_i), q_i) + (\tau(p_i), q_i) + (\sigma(p_i), \tau(q_i))
\]

for any \( 2 \leq i \leq \rho \). Both equations (2.1) and (2.2) give, for any \( 2 \leq i \leq \rho \),

\[
(p_i, q_i) + (\sigma(p_i), q_i) + (\tau(p_i), q_i) + (\sigma(p_i), \tau(q_i))
\]

\[
= (p_i, q_i) + (\sigma(p_i), q_i) + (\tau(p_i), q_i) + (\sigma(p_i), \tau(q_i))
\]

from which we deduce that, for any \( 2 \leq i \leq \rho \), we can express one point in the orbit \( A.P_i \) in term of the three other ones and those of \( A.P_1 \). It follows that

\[
\# T \leq 4\rho - (\rho - 1),
\]

and the lemma is proved.

Now, let us consider the Galois invariants of the groups \( H^1(C \times D, F_2) \) and \( H^2(C \times D, F_2) \), with coefficients in \( \Lambda = F_2 \), for the action of \( G_{F_q} = \text{Gal}(\overline{F}_q/F_q) \). The first one is easily tackled. We have \( H^1(C, F_2) = \text{Jac}_C[2] \), and so as for \( D \). From Künneth formula

\[
H^1(C \times D, F_2) \simeq H^1(C, F_2) \oplus H^1(D, F_2)
\]
as a sum of $G_{F_q}$-modules, we deduce
\begin{equation}
H^1(\overline{C} \times D, F_2)^{G_{F_q}} \simeq H^1(\overline{C}, F_2)^{G_{F_q}} \oplus H^1(D, F_2)^{G_{F_q}} \simeq \text{Jac}_C[2](F_q) \oplus \text{Jac}_D[2](F_q).
\end{equation}

The Galois invariants of $H^2(\overline{C} \times D, F_2)$ also come from Künneth formula
\begin{equation}
H^2(\overline{C} \times D, F_2) = H^1(\overline{C}, F_2) \otimes H^1(D, F_2) \oplus K
\end{equation}
as a sum of $G_{F_q}$-modules, where $K$ is a 2-dimensional vector space over $F_2$ on which $G_{F_q}$ acts trivially. It follows that
\begin{equation}
\dim H^2(\overline{C} \times D, F_2)^{G_{F_q}} \leq \dim(H^1(\overline{C}, F_2) \otimes H^1(D, F_2))^{G_{F_q}} + 2.
\end{equation}

In the following Lemma (holding in fact for any prime $\ell$, prime to $q$), we denote by $\text{Sp}_2(C)$ (resp. $\text{Sp}_2(D)$) the spectrum in $F_2$ of the Frobenius on $\text{Jac}_C[2] \simeq F_2^{2g_1}$ (resp. on $\text{Jac}_D[2] \simeq F_2^{2g_2}$), by $m_{\lambda, C}$ the dimension of the eigenspace of $\text{Jac}_C[2]$ for the eigenvalue $\lambda \in \text{Sp}_2(C)$, and by $\text{Sp}_2(C, D) = \{ \lambda \in F_2; \lambda \in \text{Sp}_2(C) \text{ and } \lambda^{-1} \in \text{Sp}_2(D) \}$.

**Lemma 2.14.** We have
\begin{equation}
\dim \left( H^1(\overline{C}, F_2) \otimes H^1(D, F_2) \right)^{G_{F_q}} = \sum_{\lambda \in \text{Sp}_2(C, D)} m_{\lambda, C} m_{\lambda^{-1}, D}.
\end{equation}

**Proof.** We consider the decomposition of $H^1(\overline{C}, F_2) = \text{Jac}_C[2]$ and that of $H^1(D, F_2) = \text{Jac}_D[2]$ as a sum of their characteristic subspaces for the action of the Frobenius
\begin{equation}
H^1(\overline{C}, F_2) = \bigoplus_{\lambda \in \text{Sp}_2(C)} E_\lambda(C)
\end{equation}
and
\begin{equation}
H^1(D, F_2) = \bigoplus_{\mu \in \text{Sp}_2(D)} E_\mu(D).
\end{equation}
The decomposition of $H^1(\overline{C}, F_2) \otimes H^1(D, F_2)$ for the action of the Frobenius is then
\begin{equation}
H^1(\overline{C}, F_2) \otimes H^1(D, F_2) = \bigoplus_{(\lambda, \mu) \in \text{Sp}_2(C) \times \text{Sp}_2(D)} E_\lambda(C) \otimes E_\mu(D).
\end{equation}
The eigenvalue on $E_\lambda(C) \otimes E_\mu(D)$ is the product $\lambda \mu$, and the uniqueness of the decomposition of a vector in the decomposition above shows that the invariants are those in
\[ \bigoplus_{\lambda, \mu = 1} E_{0, \lambda}(C) \otimes E_{0, \mu}(D), \]
where the $E_{0, \lambda}(C)$ and $E_{0, \mu}(D)$ are the eigenspaces, hence the Lemma. \(\square\)

We can now prove Proposition 2.12.

**Proof.** By hypothesis on $g(C(F_q))$, there exists at least $2p$ non-Weierstrass rational points on $C$. Since the hyperelliptic involution $\sigma$ on $C$ is defined over $F_q$, one can choose $p$ distinct pairs of non-$\sigma$-conjugated points $p_1, \sigma(p_1), \ldots, p_p, \sigma(p_p)$ in $C(F_q)$. In the same way, since no Weierstrass point of $D$ is rational by assumption on $g$ and by hypothesis on $\| D(F_q) \|$, one can choose $p$ distinct pairs of non-$\tau$-conjugated points $q_1, \tau(q_1), \ldots, q_p, \tau(q_p)$ in $D(F_q)$. It follows that Lemma 2.13 applies to $P_t = (p_1, q_1), \ldots, P_p = (p_p, q_p) \in (C \times D)(F_q)$, and we choose $T$ to be the union of the $A$-orbits of $P_1, \ldots, P_p$.

By hypothesis on $f$, the $2$–torsion of $C$ is rational and hence
\[ \text{Jac}_C[2](F_q) \simeq F_2^{2g_1} \]
on which the Frobenius acts by identity. By hypothesis on $g$, we have
\[ \text{Jac}_2[2](F_q) \simeq F_2^{2g_2} \]
on which the Frobenius acts by a block-diagonal matrix with \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] on the diagonal.

Since \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] is conjugated to \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\] over \(F_2\), we deduce from (2.3) that

\[
H^1(C \times D, F_2)^{G_{F_2}} \cong F_2^{2g_1} \oplus F_2^{g_2},
\]

and from (2.4) together with Lemma 2.14 that

\[
\dim H^2(X, F_\ell)^{G_{F_\ell}} = 2g_1g_2 + 2
\]

since the only eigenvalue on \(C\) is 1, with \(m_{1,C} = 2g_1\) and \(m_{1,D} = g_1\).

We can then deduce from Theorem 2.7, Lemma 2.13, (2.5) and (2.6) that their exists an infinite 2-tower in which \(T\) splits as soon as

\[
2g_1 + g_2 \geq 3\rho + 2 + 2\sqrt{2g_1g_2 + 2 + 4\rho},
\]

completing the proof of Proposition 2.12.

\[\square\]

3. Open problems

In the theory of codes from curves, the usual problem to get asymptotically good families of codes reduces to construct sequences of curves whose sequence of numbers of rational points grows “quickly”, and whose sequence of genera grows “slowly”. To get asymptotically good families of codes from surfaces, we first need sequences of surfaces whose number of rational points goes to infinity. Next, the natural question is: the number of rational points should be large, but compared to what? What will play the part of the genus in the case of surfaces? In [43], Papikian studies families of surfaces whose number of rational points is large compared to the sum of the Betti numbers. Despite the interest of this problem for itself, it is up to now unclear that such a feature would permit to construct good codes.

In the present section, we show that for families of surfaces of general type, both invariants \(K^2\) and the topological (i.e. étale) Euler characteristic could be an analog of the genus for this coding theoretic problem. In particular, since the sum of Betti number is larger than the topological Euler characteristic, Papikian’s investigations are relevant for the coding theoretic setting.

Caution. The open problem introduced in what follows is independent from the question of existence of infinite étale towers and can be applied to any sequence of surfaces of general type with very ample canonical class.

3.1. From surfaces of general type to the domain of codes.

3.1.1. The asymptotic domain of surfaces of general type. We define here a domain \(\mathcal{S}_q\) in \((\mathbb{R} \cup \{+\infty\})^2\) as follow.

**Definition 3.1.** The asymptotic domain of surfaces of general type \(\mathcal{S}_q\) is the set of points \((\kappa, \chi) \in (\mathbb{R} \cup \{+\infty\})^2\) for which there exists a sequence \(X_n\) of smooth projective absolutely irreducible surfaces defined over \(F_q\), such that:

- for any \(n\), the canonical divisor \(K_{X_n}\) is very ample;
- \(\# X_n(F_q)\) goes to +\(\infty\) as \(n\) goes to +\(\infty\);
- the ratio \(\frac{K_{X_n}^2}{\# X_n(F_q)}\) goes to a limit \(\kappa \in \mathbb{R}_+ \cup \{+\infty\}\);  
- the ratio \(\frac{\chi(\mathcal{O}_{X_n})}{\# X_n(F_q)}\) goes to a limit \(\chi \in \mathbb{R} \cup \{+\infty\}\).

**Remark 3.2.** Note that \(K_{X_n}\) being ample, we have \(\kappa \geq 0\), and that \(\chi \geq 0\) in case \(q\) is odd, since from [25] we have \(\chi(\mathcal{O}_X) > 0\) for any surface \(X\) of general type in odd characteristic.
Remark 3.3. Note also that by Noether formula:
\[(3.1)\]
\[12\chi(O_X) = K^2 + \chi_{\text{\text{\text{\text{\text{\text{et}}}}}}},\]
one can choose as parameters for this domain any pair among \(\lim \chi(O_X), \lim K\chi^2, \lim \chi_n(F_q^\ast)\) and \(\lim \chi_n(F_q^\ast)\).

To bound this domain, we use the following well–known inequality \((3.2)\) listed for instance in [25]:
\[(3.2)\]
\[5K^2 + 36 > \chi_{\text{\text{\text{\text{\text{\text{et}}}}}}}\]
which, together with Noether formula \((3.1)\), leads asymptotically to
\[\chi \leq \frac{\kappa}{2}\]
It follows that the domain \(S_q\) lies below the line \(\chi = \frac{\kappa}{2}\). Moreover, Theorem 1.12 together with Proposition 1.9 entail the following new asymptotic bound.

Theorem 3.4. For any point \((\kappa, \chi) \in S_q\), we have \(\kappa \geq \frac{1}{(q+1)^2}\).

Proof. Let \(X\) be a member of a family of surfaces whose parameters go to \((\kappa, \chi)\). From Theorem 1.12 the linear system \(\Gamma = (q + 1)K_X\) is \((X(F_q))^\ast\)-interpolating. Hence by Proposition 1.9 we have \(\Gamma^2 \geq \|X(F_q)\\), that is
\[(q + 1)^2K^2 \geq \|X(F_q)\|,\]
from which the Theorem follows. \(\square\)

3.1.2. Some maps between the domain \(S_q\) and the domain of codes. We recall that the domain of codes \(D_q\) is the set of points \((\delta, R) \in [0, 1]^2\), such that there exists a family of \([N_n, k_n, d_n]_q\)-codes for which:
- \(N_n\) goes to infinity;
- \(\frac{k_n}{N_n}\) goes to \(R\);
- \(\frac{d_n}{N_n}\) goes to \(\delta\).
A well-known Plotkin bound asserts that this domain lies under the line from \((0, 1)\) to \((1 - \frac{1}{2}, 0)\).

Proposition 3.5. For any integer \(2 \leq g \leq q\), the affine map
\[\varphi_g: (\kappa, \chi) \mapsto \left(\delta = 1 - g(q + 1)\kappa, R = \frac{g(g - 1)}{2}\kappa + \chi\right)\]
sends the part of \(S_q\) for which \(\kappa < \frac{1}{g(q+1)}\) into the domain of codes \(D_q\).

The map \(\varphi_g\) is illustrated in Figure 3.

Remark 3.6. We emphasize the following.

1. If the maps \(\varphi_g\) are defined for any \(g \geq 2\), they are irrelevant for \(g \geq q + 1\) since then, the part of \(S_q\) for which \(\kappa < \frac{1}{g(q+1)}\) is empty by Theorem 3.4.
2. It is easily checked that the whole part \(\{(\kappa, \chi): \chi \leq \frac{\kappa}{2}\}\) does map under \(\varphi_g\) in the area below Singleton bound \(R + \delta \leq 1\), and even under Plotkin bound adding the restriction \(\kappa \geq \frac{1}{(q+1)^2}\) given by Theorem 3.4.

Proof. Let \(g \geq 2\) and \(X\) be a member of a family whose parameters go to \((\kappa, \chi)\) in \(S_q\). We consider the code \(C(X, P, G)\) for \(P = X(F_q)\) and \(G = gK\), whose length is \(N = \|X(F_q)\|\). Since \(K\) is very ample, \(\Gamma = ((q + 1)K)\) is \(P\)-interpolating by Theorem 1.12. Moreover, since \(g \geq 2\), we have \(G \cdot K = gK^2 \geq K^2\) for the ample divisor \(H = K\). Consequently, from Theorem 1.11 this code has minimum distance at least
\[d \geq N - \Gamma \cdot G = N - g(q + 1)K^2,\]
Figure 3. The affine map \( \varphi_g(\kappa, \chi) = (1 - g(q + 1)\kappa, \frac{g(g-1)}{2} \kappa + \chi) \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). The poorly known domain \( S_q \) is arbitrarily drawn as the hashed area in the left hand side, and its image under \( \varphi_g \) as the hashed area in the right hand side. The part of \( S_q \), lying on the left of the line \( \kappa = \frac{1}{g(q+1)} \) is contained inside the grey polygon \( P_1 = A_1B_1C_1D_1 \). The point \( A_1 = \left( \frac{1}{g(q+1)}, 0 \right) \) maps to \( A_2 = (1 - \frac{g}{q+1}, \frac{g^2-g}{2(q+1)}), B_1 = (\frac{1}{g(q+1)}, 0) \) maps to \( B_2 = (0, \frac{g^2-g}{2g(q+1)}), C_1 = (\frac{1}{g(q+1)}, \frac{1}{2g(q+1)}) \) maps to \( C_2 = (0, \frac{g^2-g+1}{2g(q+1)}) \) and \( D_1 = (\frac{1}{g(q+1)}, \frac{1}{2g(q+1)}) \) maps to \( D_2 = (1 - \frac{g}{q+1}, \frac{g^2-g+1}{2g(q+1)}) \). Hence the grey polygon \( P_1 \) maps onto the grey polygon \( P_2 = A_2B_2C_2D_2 \) inside \([0,1]^2\), below the Plotkin bound. The dotted line \((C_1D_1)\) of equation \( \chi = \frac{g}{2} \) on the left maps to the dotted line \((C_2D_2)\) of slope \(-\frac{g^2-g+1}{2g(q+1)}\) on the right; since \( g < q \), this slope is less than \(-\frac{1}{2}\), close to \(-\frac{1}{2}\) if \( g \) is close to \( q \). The oriented dotted path \([B_1D_1]\) on the left is mapped on the oriented dotted path \([B_2D_2]\) on the right.

and hence its asymptotic relative distance satisfies

\[ \delta \geq 1 - g(q + 1)\kappa. \]

In addition, its dimension is at least

\[ k \geq \frac{1}{2} G \cdot (G - K) + \chi(O_X) = \frac{g(g-1)}{2} K^2 + \chi(O_X), \]

hence, its asymptotic transmission rate satisfies

\[ R \geq \frac{g(g-1)}{2} \kappa + \chi, \]

and the Proposition is proved. \( \square \)

3.1.3. A first new open problem. Considering these maps, it follows that the study of \( S_q \) is of interest for coding theoretic purposes. More precisely, proving the existence of a family \((X_n)_{n \in \mathbb{N}}\) of surfaces of general type with very ample canonical divisors with

\[
\left( \kappa = \lim \frac{K_{X_n}^2}{\# X_n(F_q)}, \chi = \lim \frac{\chi(O_{X_n})}{\# X_n(F_q)} \right)
\]
as close as possible to the line \( \chi = \frac{2}{g} \) and with \( \frac{1}{(q+1)^2} \leq \kappa < \frac{1}{2(q+1)^2} \) would yield to a family of codes near the line \((C_2D_2)\) of Figure 3. Moreover, the closer the point \((\kappa, \chi)\) to \(D_1 = \left( \frac{1}{(q+1)^2}, \frac{1}{2(q+1)^2} \right)\), the closer the point to the Plotkin bound.

**Remark 3.7.** For any non-hyperelliptic curves \(C_1\) and \(C_2\) of genus \(g_1 \geq 3\) and \(g_2 \geq 3\), the surface \(C_1 \times C_2\) is of general type and the canonical divisor \(K_{C_1 \times C_2} = (2g_1 - 2)V + (2g_2 - 2)H\) is very ample using the Segre embedding. Moreover, \(\chi(\mathcal{O}_{C_1 \times C_2}) = (g_1 - 1)(g_2 - 1)\) and \(K^2_{C_1 \times C_2} = 8(g_1 - 1)(g_2 - 1)\), so that this kind of surfaces with large genus and high number of rational points seems to provide good candidates since, asymptotically \(\chi = \frac{2}{8}\), whose slope is a quarter of the one of the line \(\chi = \frac{2}{2}\). Unfortunately, the asymptotic Drinfeld-Vladut bounds \(\frac{L(C, F_q)}{q} \leq \sqrt{q} - 1\) for \(i = 1, 2\) yield to \(\kappa \geq \frac{8}{(\sqrt{q} - 1)^2}\), larger than \(\frac{1}{2(q+1)^2}\) for any value of \(q\). These product surfaces thus lie on the right of the line \(\kappa = \frac{1}{2(q+1)^2}\), mapping under any \(\varphi_q\) in the area \(\delta < 0\).

### 3.2. Asymptotic theory for surfaces: algebraic and analytic side.

Kunyavskii–Tsfasman, Hindry–Pacheco, Zykin and Lebacque studied the asymptotic behavior of \(L\)-functions of elliptic curves over rational function field or of Zeta functions of varieties defined over a finite field. Zykin put it in a general context but his study is well adapted to the analytic context and his point of view does not suit the algebraic properties of families.

In the case of curves, we compare the number of their rational points with their genus. The beauty of the theory comes from the fact that everything is closely connected: good families of global fields for the algebraic point of view (unramified, with many points of small norms) are good for the analytic point of view (giving rise to good infinite Zeta functions) and they give rise to asymptotic good families of codes and sphere packings.

For surfaces, we have (at least) two reasonable choices for what plays the role of the genus: the sum of the \(\ell\)-adic Betti-numbers or their alternate sum—their \(\ell\)-adic Euler-Poincaré characteristic. The first appears when we consider the analytic side, and the second appears naturally in the algebraic one, as it is multiplied by the degree in \(\mathbb{Z}\) étale covers. As far as we understand now, the three points of view—algebraic, analytic and applications—seem to involve different quantities and the relations between them is not so clear as in the case of curves. An asymptotically good family should be a family that is good for the algebraic and analytic point of view, and one should be able to construct from them good codes.

**Questions:** What should be the definition of an asymptotically good family? What should be the normalization? What should be their algebraic and analytic properties?

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