Rate-Distortion-Perception Tradeoff of Variable-Length Source Coding for General Information Sources

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Abstract: Blau and Michaeli recently introduced a novel concept for inverse problems of signal processing, that is, the perception-distortion tradeoff. We introduce their tradeoff into the rate distortion theory of variable-length lossy source coding in information theory, and clarify the tradeoff among information rate, distortion and perception for general information sources. We also discuss the fixed-length coding with average distortion criterion that was missing in the previous letter.

Keywords: perception-distortion tradeoff, rate-distortion theory, data compression, variable-length coding

Classification: Fundamental theories for communications

References

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1 Introduction

An inverse problem of signal processing is to reconstruct the original information from its degraded version. It is not limited to image processing, but it often arises in the image processing. When a natural image is reconstructed, the reconstructed image sometimes does not look natural while it is close to the original image by a reasonable metric, for example mean squared error. When the reconstructed information is close to the original, it is often believed that it should also look natural.

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Blau and Michaeli [1] questioned this unproven belief. In their research [1], they mathematically formulated the naturalness of the reconstructed information by a distance of the probability distributions of the reconstructed information and the original information. The reasoning behind this is that the perceptual quality of a reconstruction method is often evaluated by how often a human observer can distinguish an output of the reconstruction method from natural ones. Such a subjective evaluation can mathematically be modeled as a hypothesis testing [1]. A reconstructed image is more easily distinguished as the variational distance $\sigma(P_R, P_N)$ increases [1], where $P_R$ is the probability distribution of the reconstructed information and $P_N$ is that of the natural one. They regarded the perceptual quality of reconstruction as a distance between $P_R$ and $P_N$. The distance between the reconstructed information and the original information is conventionally called as distortion. They discovered that there exists a tradeoff between perceptual quality and distortion, and named it as the perception-distortion tradeoff.

Claude Shannon [2, Chapter 5] initiated the rate-distortion theory in 1950’s. It clarifies the tradeoff between information rate and distortion in the lossy source coding (lossy data compression). The rate-distortion theory has served as a theoretical foundation of image coding for past several decades, as drawing a rate-distortion curve is a common practice in research articles of image coding. Since distortion and perceptual quality are now considered two different things, it is natural to consider a tradeoff among information rate, distortion and perceptual quality. Blau and Michaeli [1] briefly mentioned the rate-distortion theory, but they did not clarify the tradeoff among the three. Then the author [3] clarified the tradeoff among the three for fixed-length coding, but did not clarified variable-length coding, where fixed and variable refer the length of a codeword is fixed or variable, respectively [2, Chapter 5]. The variable-length lossy source coding is practically more important than the fixed-length counterpart because most of image and audio coding methods are variable-length.

The purpose of this letter is to mathematically define the tradeoff of variable-length lossy source coding for general information sources, and to express the tradeoff in terms of information spectral quantities introduced by Han and Verdú [2]. We also discuss the fixed-length coding with average distortion criterion that was missing in the previous letter [3].

Since the length limitation is strict in this journal, citations to the original papers are replaced by those to the textbook [2], and the mathematical proof is a bit compressed. The author begs readers’ kind understanding. The base of log is an arbitrarily fixed real number $> 1$ unless otherwise stated.

2 Preliminaries

The following definitions are borrowed from Han’s textbook [2]. Let

$$X = \left\{ X^n = (X^{(n)}_1, \ldots, X^{(n)}_n) \right\}_{n=1}^\infty$$

be a general information source, where the alphabet of the random variable $X^n$ is the $n$-th Cartesian product $\mathcal{X}^n$ of some finite alphabet $\mathcal{X}$. For a sequence of real-valued
random variables $Z_1$, $Z_2$, … we define
\[
\text{p-} \lim_{n \to \infty} \sup Z_n = \inf \left\{ \alpha \mid \lim_{n \to \infty} \Pr[Z_n > \alpha] = 0 \right\}.
\]
For two general information sources $X$ and $Y$ we define
\[
T(X, Y) = \text{p-} \lim_{n \to \infty} \frac{1}{n} \log \frac{P_{X^n Y^n}(X^n, Y^n)}{P_{X^n}(X^n) P_{Y^n}(Y^n)},
\]
\[
H_K(X) = \lim_{n \to \infty} \frac{1}{n} H_K(X^n),
\]
\[
F_K(R) = \lim_{n \to \infty} \Pr \left[ \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} \geq R \right],
\]
where $H_K(X^n)$ is the Shannon entropy of $X^n$ in $\log K$.

For two distributions $P$ and $Q$ on an alphabet $\mathcal{X}$, we define the variational distance $\sigma(P, Q)$ as $\sum_{x \in \mathcal{X}} |P(x) - Q(x)|/2$. In the rate-distortion theory, we usually assume a reconstruction alphabet different from a source alphabet. In order to consider the distribution similarity of reconstruction, in this letter we assume $\mathcal{X}^n$ as both source and reconstruction alphabets.

3 Variable-length source coding

An encoder of length $n$ is a stochastic mapping $f_n : \mathcal{X}^n \to \mathcal{U}^n$, where $\mathcal{U} = \{1, \ldots, K\}$ and $\mathcal{U}^n$ is the set of finite-length sequences over $\mathcal{U}$. By stochastic we mean that the encoder output $f_n(x^n)$ is probabilistic with a fixed input $x^n \in \mathcal{X}^n$. The corresponding decoder of length $n$ is a deterministic mapping $g_n : \mathcal{U}^n \to \mathcal{X}^n$. We denote by $|f_n(x^n)|$ the (random variable of) length of sequence $f_n(x^n) \in \mathcal{U}^n$ for $x^n \in \mathcal{X}^n$. We denote by $\delta_n : \mathcal{X}^n \times \mathcal{X}^n \to [0, \infty)$ a general distortion function.

3.1 Average distortion criterion

Definition 1 A triple $(R, D, S)$ is said to be va-achievable if there exists a sequence of encoder and decoder $(f_n, g_n)$ such that
\[
\limsup_{n \to \infty} \frac{\mathbb{E}[\log |f_n(X^n)|]}{n} \leq R, \quad (1)
\]
\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[\delta_n(X^n, g_n(f_n(X^n)))] \leq D, \quad (2)
\]
\[
\limsup_{n \to \infty} \sigma(P_{g_n(f_n(X^n))}, P_{X^n}) \leq S. \quad (3)
\]

Define the function $R_{\text{va}}(D, S)$ by
\[
R_{\text{va}}(D, S) = \inf \{ R \mid (R, D, S) \text{ is va-achievable } \}.
\]

Theorem 2
\[
R_{\text{va}}(D, S) = \inf_{Y} H_K(Y)
\]
where the infimum is taken with respect to all general information sources $Y$ satisfying
\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[\delta_n(X^n, Y^n)] \leq D, \quad (4)
\]
\[
\limsup_{n \to \infty} \sigma(P_{Y^n}, P_{X^n}) \leq S. \quad (5)
\]
Proof: Let a pair of encoder $f_n$ and decoder $g_n$ satisfies Eqs. (1)–(3). Let $Y^n = g_n(f_n(X^n))$, and define the general information source $Y$ from $Y^n$. We immediately see that $Y$ satisfies Eqs. (4) and (5). By the same argument as [2, p. 349] we immediately see

$$\limsup_{n \to \infty} \frac{E[\log |f_n(X^n)|]}{n} \geq H_K(Y).$$

On the other hand, suppose that a general information source $Y$ satisfies Eqs. (4) and (5). Let $f'_n$ and $g'_n$ be a lossless variable-length encoder and its decoder [2, Section 1.7] for $Y$ such that $Y^n = g'_n(f'_n(Y^n))$

$$\limsup_{n \to \infty} \frac{E[\log |f'_n(Y^n)|]}{n} = H_K(Y).$$

For a given information sequence $x^n \in \mathcal{X}^n$, the encoder randomly chooses $y^n \in \mathcal{Y}^n$ according to the conditional distribution $P_{Y|X}(\cdot|x^n)$, and define the codeword as $f'_n(y^n)$. The decoding result is $y^n = g'_n(f'_n(y^n))$. Since the probability distribution of decoding result $g'_n(f'_n(y^n))$ is $P_{Y^n}$, we see that the constructed encoder and decoder satisfy Eqs. (2) and (3).

3.2 Maximum distortion criterion

Definition 3 A triple $(R, D, S)$ is said to be $vm$-achievable if there exists a sequence of encoder and decoder $(f_n, g_n)$ such that

$$\limsup_{n \to \infty} \frac{E[\log |f_n(X^n)|]}{n} \leq R,$$

$$p- \limsup_{n \to \infty} \frac{1}{n} \delta_n(X^n, g_n(f_n(X^n))) \leq D,$$

$$\limsup_{n \to \infty} \sigma(P_{g_n(f_n(X^n))), P_{X^n}) \leq S.$$

Define the function $R_{vm}(D, S)$ by

$$R_{vm}(D, S) = \inf \{ R \mid (R, D, S) is \text{ vm-achievable} \}.$$

Theorem 4

$$R_{vm}(D, S) = \inf_Y H_K(Y)$$

where the infimum is taken with respect to all general information sources $Y$ satisfying Eq. (3) and

$$p- \limsup_{n \to \infty} \frac{1}{n} \delta_n(X^n, Y^n) \leq D.$$

Proof: The proof is almost the verbatim copy of that of Theorem 2 and is omitted.

Remark 5 The tradeoff for variable-length coding with the average distortion criterion and without the perception criterion was also determined by using stochastic encoders [2, Section 5.7], but with the maximum distortion criterion without the perception criterion, only the deterministic encoders were sufficient to clarify the tradeoff [2, Section 5.6]. It is not clear at present whether or not we can remove the randomness from encoders in Theorem 4.
4 Fixed-length coding with the average distortion criterion

In this section we state the tradeoff for fixed-length coding with the average distortion criterion, because it has never been stated elsewhere. The proof is almost the same as [3]. Note that the definition of encoder will be different from Section 3 and that an assumption on the distortion $\delta_n$ will be added.

An encoder of length $n$ is a deterministic mapping $f_n : \mathcal{X}^n \to \{1, \ldots, M_n\}$, and the corresponding decoder of length $n$ is a deterministic mapping $g_n : \{1, \ldots, M_n\} \to \mathcal{X}^n$. We require an additional assumption that $\delta_n(x^n, x^n) = 0$ for all $n$ and $x^n \in \mathcal{X}^n$.

Definition 6 A triple $(R, D, S)$ is said to be $f_a$-achievable if there exists a sequence of encoder and decoder $(f_n, g_n)$ such that

$$\limsup_{n \to \infty} \frac{\log M_n}{n} \leq R,$$
$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[\delta_n(X^n, g_n(f_n(X^n)))] \leq D,$$
$$\limsup_{n \to \infty} \sigma(P_{g_n(f_n(X^n))}, P_{X^n}) \leq S.$$

Define the function $R_{fa}(D, S)$ by

$$R_{fa}(D, S) = \inf \{ R \mid (R, D, S) \text{ is } f_a\text{-achievable} \}.$$

Theorem 7

$$R_{fa}(D, S) = \max \left\{ \inf_Y I(X; Y), \inf \{ R \mid F_X(R) \leq S \} \right\}$$

where the infimum is taken with respect to all general information sources $Y$ satisfying

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[\delta_n(X^n, Y^n)] \leq D.$$

Proof: Proof is almost the verbatim copy of that of [3].