Comparison of realizations of Lie algebras

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Abstract. The notion of equivalence of Lie algebra realizations is revisited and the quantities stable under the equivalence transformations are proposed. As a result we formulate a practical algorithm that allows to establish the existence of equivalence between any two realizations of a Lie algebra. Several illustrative examples are considered.

1. Introduction
Notion of “realization” is widely explored in mathematical and physical literature and denotes a number of different objects, in particular, there are several definitions and types of Lie algebra realizations, e.g. in [3], realizations are considered in form of derivations of formal power series over a field of zero characteristic, and in [1] and [2] two different “physical” types of realizations are considered. The “Lie algebra realization” term also appears in different contexts, for example, it can be connected to a realization of the corresponding group action, see e.g. [5]. In this paper we consider realizations of Lie algebras by vector fields, i.e., representations by first-order differential operators.

This type of realizations was first considered by S. Lie and has a major place in modern group analysis of differential equations (general theory, group classification and integration of differential equations). Representation of Lie algebras by vector fields is also of great interest and widely applicable in classification of gravity fields of a general form with respect to motion groups, in geometric control theory and in the theory of systems with superposition principles and in difference schemes for numerical solutions of differential equations.

One of the key problems that usually emerge with realizations is to detect if two given realizations are the same (equivalent) or not and this forms the main objective of the paper. To start with we give all necessary notions and definitions in the first section, then we introduce a number of quantities and objects that are invariant and helpful in the establishment of equivalence, and, finally, we propose the practical algorithm that allows to detect the equivalence existence.

2. Definitions and conventions
Let $V$ be an $n$-dimensional vector space over the field of real numbers. Consider a Lie algebra $\mathfrak{g}$ on $V$ spanned by a basis $\{e_1, e_2, \ldots, e_n\}$ with the structure constants $C_{jk}^i \in \mathbb{R}$, here and below $i, j, k = 1, 2, \ldots, n$. We denote an open domain of $\mathbb{R}^m$ as $M$ and $\text{Vect}(M)$ is the Lie algebra of smooth vector fields on $M$ with the Lie product defined as commutator (i.e. the Lie algebra of first-order linear differential operators with analytical function coefficients).
Definition 1. A realization of a Lie algebra $\mathfrak{g}$ in vector fields on $M$ is a homomorphism $R(\mathfrak{g}) = R: \mathfrak{g} \rightarrow \text{Vect}(M)$. The realization is called faithful if $\ker R = \{0\}$ and unfaithful otherwise.

In Lie theory realizations are considered locally at some neighborhood $U_x \subset M \subset \mathbb{R}^m$ of a point $x \in M$ and in most of the cases without loss of generality the realization can be considered in a neighborhood of a zero point $x = 0$.

Definition 2. A local realization $R|_x$ of a Lie algebra $\mathfrak{g}$ at $x \in M$ is the realization $R$ of $\mathfrak{g}$ in vector fields actions of which are restricted to $U_x$.

Denote local coordinates of a point $x \in M$ as $(x_1, \ldots, x_m)$, then in coordinate form a realization $R(\mathfrak{g})$ is performed by the images $\Xi_i(x)$ of the basis elements $e_i$ of a general form

$$\Xi_i(x) = R(e_i) = \sum_{l=1}^{m} \xi_{il}(x_1, x_2, \ldots, x_m) \partial_l,$$

hereafter $\partial_l = \frac{\partial}{\partial x_l}$ and the coefficients $\xi_{il}(x_1, x_2, \ldots, x_m)$ are smooth (analytic) functions.

Let us fix a point $x \in M$ and let $R_x$ be a realization of $\mathfrak{g}$ at this point. Consider the linear map $R_x : \mathfrak{g} \rightarrow \text{Vect}(M)(x)$ that transforms a vector $v \in \mathfrak{g}$ to its image $R(v(x))$ at $x$. The matrix that corresponds to this linear map is the $n$ by $m$ matrix $\xi$ formed by the coefficients of the realization (1)

$$\xi(x) = \begin{pmatrix} 
\xi_{11}(x) & \xi_{12}(x) & \cdots & \xi_{1m}(x) \\
\xi_{21}(x) & \xi_{22}(x) & \cdots & \xi_{2m}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{m1}(x) & \xi_{m2}(x) & \cdots & \xi_{mm}(x) 
\end{pmatrix}. \tag{2}$$

Definition 3. The rank of the linear map $R_x$, or, equivalently, the rank of the matrix $\xi(x)$ at a point $x$ is called a rank of realization $R$ at point $x$ and is denoted rank $R_x$.

Remark 1. The realization rank value possess the obvious inequality $0 \leq \text{rank } R_x \leq n$, where $n$ is the dimension of a Lie algebra $\mathfrak{g}$. The second inequality is dictated by the number of rows in matrix $\xi$, which is equal to the number of basis vector fields of $\mathfrak{g}$.

Definition 4. A realization $R$ of a Lie algebra $\mathfrak{g}$ is called transitive if the action of the local Lie group corresponding to $R$ is transitive. Or, equivalently (see [4]), a realization $R$ of a Lie algebra $\mathfrak{g}$ is called transitive if rank $R_p = m$ for all $p \in M$.

Definition 5. A point $x \in M$ is called a regular point of a realization $R$ if there exists a neighborhood $U_x$ of $x$, such that rank $R_y$ is constant for all $y \in U_x$. A point that is not regular is called a singular point.

3. Equivalence of realizations

For many practical applications it is necessary to define if two given sets of first order differential operators (with the isomorphic commutation relations) can be transformed to each other or not. In particular this problem arises as subproblem in methodization and matching of differential equations and in classification of realizations of a given Lie algebra. This task is rather complicated even in the case of small number of operators and variables.

Example 1. Consider two realizations of the smallest Poincaré algebra $p(1,1)$:

$R_1 : \{\partial_1, \ \cosh x_1 \partial_2, \ -\sinh x_1 \partial_2\}$ and
$R_2 : \{\partial_1, \ x_2 \partial_1, \ x_1 x_2 \partial_1 - (1-x_2^2) \partial_2\}.$

This realizations are equivalent and connected by the change of variables $y_2 = -\tanh x_2.$
So the main target of this work is the constructive procedure matching the two given realizations. For further study let us introduce an equivalence relation on the set of all possible realizations of a fixed Lie algebra.

Denote Aut(\(g\)) \(\subset GL_n\) the whole automorphism group of \(g\) and let Int(\(g\)) \(\subset Aut(\(g\))\) denote the inner automorphism group of \(g\).

Consider notions of “weak” and “strong” equivalences introduced in [11].

Let \(A\) be a subgroup of Aut(\(g\)). The realizations \(R_1: A \rightarrow \text{Vect}(M_1)\) and \(R_2: A \rightarrow \text{Vect}(M_2)\) are called A-equivalent if there exist \(\alpha \in A\) and a diffeomorphism \(f\) from \(M_1\) to \(M_2\) such that \(R_2(v) = f_\ast R_1(\alpha(v))\) for all \(v \in A\). Here \(f_\ast\) is the isomorphism from \(\text{Vect}(M_1)\) to \(\text{Vect}(M_2)\) induced by \(f\). If \(A\) contains only the identical transformation, the realizations are called strongly equivalent. The realizations are weakly equivalent if \(A = \text{Aut}(g)\). All the diffeomorphisms are local.

The above definition of equivalence does not work well in singular points, to demonstrate this let us consider the one-dimensional Lie algebra span \(\{e_1\}\) and its realization \(R(e_1(x)) = x_1 \partial_1\) on some open domain \(M \subset \mathbb{R}^m\), such that \(0 \in M\). Since the application of Lie algebras to differential equations [6,10] is based on the local correspondence between Lie groups and Lie algebras, in Lie theory it is supposed that any one-dimensional vector field is equivalent to the vector field corresponding to shifts (locally the rectification theorem is supposed to hold). In our case this means that \(x_1 \partial_1\) has to be equivalent to the operator \(\partial_1\). But there is no necessary diffeomorphism at zero point and these two realizations are inequivalent in frames of the above definition. To avoid the equivalence collisions at the singular points we introduce a relation of local equivalence.

Consider a regular point \(p \in M\) and two systems \((x_1, \ldots, x_m)\) and \((y_1, \ldots, y_m)\) of local coordinates of the mentioned point \(p\) on \(M\). Then there exists a local diffeomorphism \(\varphi: U(p_x) \rightarrow U(p_y)\), such that \(y_i = \varphi_i(x_1, \ldots, x_m)\). Here \(U(p_x) \subset \mathbb{R}^m\) and \(U(p_y) \subset \mathbb{R}^m\) are the neighborhoods of the images (under the coordinate transformation) \(p_x\) and \(p_y\) of the point \(p\) in “\(x\)” and “\(y\)” coordinate spaces \(\mathbb{R}^m\) respectively. Since the manifold \(M\) is an open subspace in \(\mathbb{R}^m\) we do not need to consider additional coordinate spaces and can introduce the diffeomorphism \(\varphi\) providing the change of coordinates directly on the manifold \(M\) between the neighborhoods of points \(x \in M\) and \(y \in M\), that is \(\varphi: U_x \rightarrow U_y\).

**Definition 6.** The realization \(R_1: \mathfrak{g} \rightarrow \text{Vect}(M)\) at a point \(x \in M\) and the realization \(R_2: \mathfrak{g} \rightarrow \text{Vect}(M)\) at a point \(y \in M\) are locally equivalent if there exist two neighborhoods \(U_x\) and \(U_y\) and a diffeomorphism \(\varphi: U_x \rightarrow U_y\), such that \(y = \varphi(x)\) and \(R_2(v(\varphi(x))) = R_1(v(x))\) for all \(v \in \mathfrak{g}\).

The introduced notion of local equivalence is indeed an equivalence relation, as far as it is symmetric, reflexive and transitive.

**Definition 7.** The realization \(R_1: \mathfrak{g} \rightarrow \text{Vect}(M)\) at a point \(x \in M\) and the realization \(R_2: \mathfrak{g} \rightarrow \text{Vect}(M)\) at a point \(y \in M\) are called Aut-equivalent if there exist \(\alpha \in \text{Aut}(\mathfrak{g})\), two neighborhoods \(U_x\) and \(U_y\) and a diffeomorphism \(\varphi: U_x \rightarrow U_y\), such that \(y = \varphi(x)\) and \(R_2(v(\varphi(x))) = R_1(\alpha(v(x)))\) for all \(v \in \mathfrak{g}\).

**Definition 8.** The realization \(R_1: \mathfrak{g} \rightarrow \text{Vect}(M)\) at a point \(x \in M\) and the realization \(R_2: \mathfrak{g} \rightarrow \text{Vect}(M)\) at a point \(y \in M\) are called Int-equivalent if there exist \(\alpha \in \text{Int}(\mathfrak{g})\), two neighborhoods \(U_x\) and \(U_y\) and a diffeomorphism \(\varphi: U_x \rightarrow U_y\), such that \(y = \varphi(x)\) and \(R_2(v(\varphi(x))) = R_1(\alpha(v(x)))\) for all \(v \in \mathfrak{g}\).

If it is necessary to consider realizations not only on a neighborhood of a point, but on \(M\), then the equivalence has to be established for all regular points (set of regular points of a vector field is an open domain in \(\mathbb{R}^m\)). Another possibility is to take an analytic continuation of the local realization. Hereafter the term “equivalence” means the Aut-equivalence if the type of equivalence is not indicated explicitly.
3.1. Induced faithful realizations and essential variables

Roughly speaking we can conclude that two realizations are equivalent, if they can be transformed to the identical form by means of non-singular automorphic basis changes \( (e_i \mapsto \tilde{e}_i) \) and 1 to 1 changes of variables \( (x_i \mapsto y_i = \varphi_i(x)) \) with non-zero Jacobi determinant.

Let us have a diffeomorphism of \( M \) such that for the corresponding \( x, y \in M \) we have \( y_1 = \varphi_1(x_1, \ldots, x_m), \ y_2 = \varphi_2(x_1, \ldots, x_m), \ldots, y_m = \varphi_m(x_1, \ldots, x_m) \). Then the realization of the form (1) transforms to the following:

\[
\tilde{R}(e_i) = \sum_{l=1}^{m} \tilde{\xi}_{il}(y) \partial_{y_l} = \sum_{l=1}^{m} \left( \sum_{l'=1}^{m} \tilde{\xi}_{ll'}(x) \frac{\partial \varphi_l(x)}{\partial x_{l'}} \right) \partial_{y_l}.
\]

Note, that the coefficients \( \tilde{\xi}_{il}(y) \) are written in terms of \( y \) using the inverse transformation \( \varphi^{-1} \).

From the definition of equivalence immediately follows that none of faithful realizations of \( g \) is equivalent to an unfaithful realization of \( g \).

Therefore we will not consider unfaithful realizations at all, they have to be replaced by the faithful ones using the following induction rule. Let \( R(g) \) be an unfaithful realization of the algebra \( g \) with the basis elements \( \{R(e_1) \neq 0, \ldots, R(e_k) \neq 0, R(e_{k+1}) \equiv 0, \ldots, R(e_n) \equiv 0\} \), then it can be replaced by the faithful realization \( R(g_0) = \text{span} \{R(e_1), \ldots, R(e_k)\} \) of the smaller Lie algebra \( g_0 = \text{span} \{e_1, \ldots, e_k\} \), which is a subalgebra of \( g \). The realization \( R(g_0) \) of \( g_0 \) is called a realization induced by \( R(g) \).

It is obvious that application of transformations from \( \text{Aut}(g) \) to the realization \( R \) does not change the rank of \( R \), and none of diffeomorphisms of \( M \) can change the realization rank either. Therefore the equivalent realizations have the same ranks.

**Lemma 1.** Let a realization \( R(x): g \rightarrow \text{Vect}(M) \) has a rank \( r = \text{rank} R < m \) at a regular point \( x \in M \), where \( m = \text{dim} M \). Then there exists a locally equivalent realization \( \tilde{R}(y): g \rightarrow \text{Vect}(M) \) at a regular point \( y \in M \) such that the coefficients of basic vector fields \( \tilde{\xi}_{il}(y) = 0 \) for all \( i = 1, \ldots, n, \ l = r + 1, \ldots, m \).

**Proof.** To prove the lemma let us construct the desired diffeomorphism. Since the realization rank is equal to \( r \) it is known from the theory of invariants \([10]\) that there are \( m - r \) functionally independent invariants \( J_1(x_1, \ldots, x_m), \ldots, J_{m-r}(x_1, \ldots, x_m) \) of the realization \( R \). The diffeomorphism of the form \( y_a = x_a, \ a = 1, \ldots, r; \ y_{r+b} = J_b, \ b = 1, \ldots, m - r \) gives the following zero coefficients of the realization \( \tilde{R}: \tilde{\xi}_{il(r+b)}(y) = R(e_i)(J_b) = 0 \) for all \( i = 1, \ldots, n, \ b = 1, \ldots, m - r \).

**Definition 9.** The above variables \( y_1, \ldots, y_r \) are called essential and the rest of non-zero variables from \( y_{r+1}, \ldots, y_m \) are called additional. The equivalence transformation described in Lemma 1 is called reduction of realization to the essential variables.

**Example 2.** Consider two-dimensional abelian Lie algebra \( 2A_1 \). It is well-known that the basis elements of this algebra can be realized by two operators of translations

\[
R_1(e_1) = \partial_1, \quad R_1(e_2) = \partial_2.
\]

It was shown in \([11]\) that there are exactly two inequivalent realizations of \( 2A_1 \), and the second one is

\[
R_2(e_1) = \partial_1, \quad R_2(e_2) = x_2 \partial_1.
\]

In these cases rank \( R_1 = 2 \) and rank \( R_2 = 1 \).

Consider the formal sum of these realizations \( R_3 = R_1 + R_2 \) \((R_1 \text{ for the variables } (x_1, x_2) \text{ and } R_2 \text{ for the variables } (x_3, x_4))\), namely

\[
R_3(e_1) = \partial_1 + \partial_3, \quad R_3(e_2) = \partial_2 + x_4 \partial_3.
\]
As far as \([\partial_1 + \partial_3, \partial_2 + x_4 \partial_3] = 0\), then \(R_3\) do realize the Lie algebra \(2A_1\) in the space of four variables \((x_1, x_2, x_3, x_4)\) and rank \(R_3 = 3\), what means that the number of essential variables is equal to \(3\) and Lemma 1 can be applied.

Indeed, the diffeomorphism \(\varphi\) given by the non-singular functions

\[
\varphi_1(x_1, \ldots, x_4) = x_1, \quad \varphi_2(x_1, \ldots, x_4) = x_2, \quad \varphi_3(x_1, \ldots, x_4) = x_1 - x_3 + x_2 x_4, \quad \varphi_4(x_1, \ldots, x_4) = x_4
\]

transforms the realization \(R_3\) to the equivalent realization \(R_1\) in essential variables.

**Corollary 1.** In transitive realizations all variables are essential and, since rank \(R \leq n\), any transitive realization of a Lie algebra is realized in not more then \(n\) variables.

### 3.2. Correspondence between realization and subalgebra

Two recent papers on the construction of vector fields [7] and classification of realizations [4] establish one-to-one correspondence between inequivalent transitive realizations of a Lie algebra \(g\) and Int-inequivalent subalgebras of \(g\). Moreover, this relation was extended to the non-transitive case as well. To use this correspondence as a criteria of equivalence of two considered realizations we have to define explicitly the subalgebra that corresponds to the given realization. In practice the desired subalgebra can be found using the following lemma.

**Lemma 2.** The subalgebra that corresponds to the given realization is the kernel of its linear map at the origin of coordinates.

In other words at the point \(x = 0 \in \mathbb{R}^n\) the realization vectors that form a basis of corresponding subalgebra are identically equal to zero.

The proof of the lemma follows from the algorithm of construction of vector fields proposed in [7].

**Example 3.** Consider the realizations

\[
R_1: e_1 = \partial_1, \quad e_2 = x_2 \partial_1, \quad e_3 = x_1 \partial_1 + 2x_2 \partial_2 \quad \text{and} \\
R_2: e_1 = \partial_1, \quad e_2 = x_1 \partial_1 - x_2 \partial_2, \quad e_3 = \partial_2.
\]

At the origin of coordinates \(x = 0\) their basis vectors have the form

\[
R_1(x = 0): e_1 = \partial_1, \quad e_2 = 0, \quad e_3 = 0 \quad \text{and} \\
R_2(x = 0): e_1 = \partial_1, \quad e_2 = 0, \quad e_3 = \partial_2.
\]

Therefore the realization \(R_1\) corresponds to the subalgebra \(\langle e_2, e_3 \rangle\) and \(R_2\) corresponds to \(\langle e_2 \rangle\).

### 4. Match algorithm

Let us sum up all the above statements and formulate the procedure allowing us to compare and match two finite \(n\)-sets of linearly independent vector fields, namely

\[
S_1 = \{ \sum_{j=1}^{m} \xi_{ij}(x_1, x_2, \ldots, x_m) \partial_{x_j} \} \quad \text{and} \quad S_2 = \{ \sum_{j=1}^{m} \eta_{ij}(x_1, x_2, \ldots, x_m) \partial_{x_j} \}, \quad i = 1, 2, \ldots, n.
\]

**4.1. Preliminary testing**

The first thing that should be checked is if these two sets do realize a Lie algebra (differential operators have to be closed with respect to the Lie product).

The next question is if the considered realizations do realize the same Lie algebra. At this stage the structure constants for both sets have to be calculated and then, if they do not coincide, the isomorphism transformation from \(GL_n\) has to be found. The construction of isomorphism is rather complicated problem, but it can be simplified in some cases: for the low-dimensional \((n \leq 5)\) algebras the known classification of Lie algebras can be used as well as for semi-simple Lie algebras; for Lie algebras with the abelian ideal of codimension one the Jordan canonical form of the adjoint action of the “non-commuting” basis element can be used.
The last preparatory procedure is the elimination of unfaithful realizations from consideration or replacement of them by the faithful induced realization.

4.2. Reduction to essential variables
The second step is to calculate ranks of realizations, to compare them and, if the ranks coincide, to reduce realizations to the essential variables. Then one should test the transitivity of realizations (all variables have to be essential) and note that a transitive realization is not equivalent to a non-transitive one.

4.3. Subalgebra detection
Now one can define the corresponding subalgebras and establish their equivalence transformation using Int. Once the subalgebras coincide (note that in the non-transitive case all parameters of subalgebras have to coincide completely) then the task moves to the next step.

4.4. Construction of diffeomorphism and automorphism
At the last stage we are able to construct diffeomorphism by solving the system of differential equations arising from (3) explicitly. It is important that any partial solution of this system fits our requirement and in the most cases the construction of the solution is not necessary at all, since it is enough to know that realizations are equivalent. Sometimes the diffeomorphism transformation has to be followed by simple automorphism (e.g., sign change or sum of two basis elements).

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