A relativistic toy model for back-reaction

Günter Plunien1, Marcus Ruser1,2 and Ralf Schützhold1

1 Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany
2 Département de Physique Théorique, Université de Genève, 24 quai E Ansermet, CH-1211 Genève 4, Switzerland

E-mail: plunien@physik.tu-dresden.de, marcus.ruser@physics.unige.ch and schuetz@theory.phy.tu-dresden.de

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Abstract

We consider a quantized massless and minimally coupled scalar field within a (1+1)-dimensional spacetime described by a circle with a time-dependent radius \( R(t) \). Within the semi-classical treatment, the back-reaction of the quantum field onto the \( R(t) \)-dynamics is taken into account in terms of the renormalized expectation value of the energy–momentum tensor including the trace anomaly. In case the classical energy of the circle is positive, the results indicate that the back-reaction (induced by the trace anomaly) could prevent the collapse of the spacetime \( R \downarrow 0 \); however, the semi-classical picture fails to describe the \( R(t) \)-dynamics at the turning point (i.e., possible bounce) at finite values of \( R \) and \( \dot{R} \). The fate of the interacting system after that point (e.g., oscillation or eternal acceleration) cannot be determined within the semi-classical picture and thus probably requires the full quantum treatment. Allowing also for negative classical energies (similar to the Newtonian gravitational energy, for example), the Casimir effect yields the dominant back-reaction contribution and might also prevent the collapse of the spacetime—leading to eternal oscillations of \( R(t) \neq 0 \).

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

With incomplete knowledge of the underlying theory including quantum gravity, the back-reaction of quantum fields onto the geometry is usually described by the semi-classical Einstein equations:

\[
\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = - \kappa \langle \mathcal{T}_{\mu\nu} \rangle_{\text{ren}}.
\]  

(1)

Although intrinsically incomplete, this semi-classical approach might—after proper renormalization of \( \langle \mathcal{T}_{\mu\nu} \rangle_{\text{ren}} \) is employed—shed light onto the following questions:
Could the back-reaction of quantum fields prevent the collapse (i.e., singularity) of the spacetime?

Under which circumstances does the semi-classical picture apply and when does it fail?

During the last decades, numerous investigations related to the first question were devoted to effects of quantum fields on the singularities and particle horizons in the early universe by solving the semi-classical back-reaction equations for homogeneous and isotropic spacetimes in \((3 + 1)\) dimensions (see, e.g., [1, 2] and references therein). Despite the enlightenings obtained from studies of the behaviour of solutions of the higher-order, nonlinear differential equations for the scale factor (together with that for the quantum field under consideration) assuming plausible initial conditions and ranges of the parameters involved, one is still left with the perhaps more difficult second question that has to be addressed simultaneously. In view of the inherent complexity of the \((3+1)\)-dimensional case, it is yet desirable to investigate completely a more simple but nevertheless generic model for back-reactions, which allows us to also shed light on the issue concerning the range of applicability of the semi-classical approach.

As a model for gauge field theories (vector \(A_\mu\) and spinor \(\Psi\) fields) in a \((3+1)\)-dimensional gravitational background, we shall consider the conformally invariant theory of a massless and minimally coupled scalar field \(\Phi\) in \((1+1)\) dimensions. Since the Einstein tensor \(R_{\mu\nu} - g_{\mu\nu} R/2\) vanishes identically in \((1+1)\) dimensions, one has to start with an alternative geometric action leading to a preferably simple dynamics which still preserves major aspects such as relativistic invariance as well as generic features of higher-dimensional spacetimes.

In order to construct a simple (and yet not completely trivial) model for the \((1+1)\)-dimensional geometry, let us outline the generic features we want to reproduce. First, a finite spatial volume \(V\) facilitates a simple and unambiguous definition of the collapse via \(V \downarrow 0\). Second, in analogy to cosmology, we assume spatial translation-invariance for simplicity. Owing to these two conditions, the \((1+1)\)-dimensional geometry is just a circle with a time-dependent radius \(R(t)\), i.e., the time-dependent radius \(R\) is the only degree of freedom of the geometry. The dynamics of the geometry is supposed to be governed by an action which becomes singular at \(R = 0\) (collapse) and \(R = \infty\) (blow-up) as well as at \(\dot{R} = \pm 1\), where \(\dot{R}\) denotes the derivative with respect to the physical time coordinate (divided by the speed of light), such that superluminal regions \(\dot{R}^2 > 1\) cannot be reached. Hence, the most simple ansatz for the Lagrangian depending on \(R\) and \(\dot{R}\) reads

\[
L(R, \dot{R}) \propto R^\alpha (1 - \dot{R}^2)^\beta, \tag{2}
\]

where \(\alpha\) and \(\beta\) are appropriately chosen constants. Deriving the Hamiltonian/energy

\[
E \propto R^\alpha (1 - \dot{R}^2)^{\beta + \frac{1}{2}(2\beta - 1)} \frac{1 + (2\beta - 1) \dot{R}^2}{1 - \dot{R}^2}, \tag{3}
\]

and demanding that the only critical points are at \(\dot{R} = \pm 1\) (as well as \(R = 0\) and \(R = \infty\)), we obtain \(\beta = 1/2\). As one would expect, this is the same exponent as for a relativistic particle. In this sense, the above action is relativistic (in contrast to any \textit{ad hoc} kinetic terms like \(\dot{R}^2 = (\partial_t R)^2\)). In addition, if we assume that the energy is an extensive quantity and thus scales with the volume (i.e., the radius \(R\)), we have \(\alpha = 1\).

For \(\beta = 1/2\) and \(\alpha = 1\), equation (2) leads to the action \((\hbar = c = 1)\)

\[
\mathcal{A} = -\frac{\sigma}{2} \int d^2x \sqrt{-g}, \tag{4}
\]

with \(g\) being the determinant of the metric in the usual time-gauge (see, e.g., [3, 4])

\[
ds^2 = [1 - (\partial_t R)^2] dr^2 - R^2 d\varphi^2, \tag{5}
\]
where $t$ is the physical time and $\phi$ is the angular coordinate. The factor $\sigma$ required by dimensional reasons can be interpreted as an effective cosmological constant for the (1+1)-dimensional geometry and could be positive as well as negative. In the latter case, however, the energy would be negative as well. Since the gravitational energy in Newtonian gravity is also negative, this case will not be excluded a priori.

On the other hand, for positive $\sigma$, equation (4) is just the Nambu–Goto action [3] which describes the propagation of relativistic membranes (in this case strings) embedded in a higher-dimensional spacetime. In this case, $\sigma$ would be the tension (which is usually assumed to be positive). However, we shall not consider the effects of this embedding but just treat equations (4) and (5) as a (1+1)-dimensional toy model for the geometry.

Left alone, i.e., without additional forces, the free dynamics of the circle governed by $\delta A/\delta R = 0$, i.e.,

$$R\ddot{R} + (1 - R^2) = 0, \quad (6)$$

with the initial conditions $R(0) = R_0$, $\dot{R}(0) = 0$, for example, leads to a collapse $R \downarrow 0$ and $\dot{R} \uparrow 1$ (i.e., a singularity) after a finite time. The question we want to address is: how does the presence of a field change this picture?

2. Toy model

Let us consider this circle (time-dependent background metric) being endowed with a real massless scalar field $\phi$ such that the total action reads (see, e.g., [5])

$$A = \frac{1}{2} \int d^2x \sqrt{-g} (\partial_{\mu} \Phi \partial^{\mu} \Phi - \sigma). \quad (7)$$

The (classical) equations of motion for $R(t)$ can be derived by variation $\delta A/\delta R = 0$

$$\sqrt{1 - R^2} \left[ \frac{\sigma}{2} + T^1 \right] + \frac{d}{dt} \left[ \frac{R \ddot{R}}{\sqrt{1 - R^2}} \left( \frac{\sigma}{2} + T^0 \right) \right] = 0, \quad (8)$$

with the energy–momentum tensor of the scalar field

$$T_{\mu\nu} = \partial_{\mu} \Phi \partial_{\nu} \Phi - \frac{1}{2} g_{\mu\nu} \partial_{\sigma} \Phi \partial^{\sigma} \Phi. \quad (9)$$

The equation of motion for the scalar field $\delta A/\delta \Phi = 0$ follows as

$$\Box \Phi = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi \right) = 0, \quad (10)$$

where $\Box$ denotes the d’Alembertian with respect to the metric in equation (5).

The first integral of the equations of motion (8) and $\Box \Phi = 0$ corresponds to the (conserved) total energy

$$E = \frac{2\pi R}{\sqrt{1 - R^2}} \left[ \frac{\sigma}{2} + T^0 \right]. \quad (11)$$

3. Classical case

For calculating the dynamics of the field $\Phi$, it is convenient to introduce another time coordinate via the transformation

$$\tau = \int dt \frac{\sqrt{1 - (\dot{R}/R)^2}}{R(t)}. \quad (12)$$
leading to conformally flat metric

\[ ds^2 = R^2(\tau)[d\tau^2 - d\phi^2]. \] (13)

Note that the insertion of this metric into the action in equation (4) does not yield the same equations of motion for \( R(t) \) since equation (12) represents a non-algebraic transformation.

Let us first consider the classical back-reaction of a spatially homogeneous field \( \partial \Phi / \partial \phi = 0 \) as a solution of the field equation \( \Box \Phi = \partial^2 \Phi / \partial \tau^2 = 0 \) which simply reads \( \Phi = \mathcal{E}_b \tau \) with \( \mathcal{E}_b = \) constant, leading to the kinetic term

\[ \dot{\Phi}^2 = \mathcal{E}_b^2 \frac{1 - R^2}{R^2}. \] (14)

with respect to the time \( t \). Insertion into the total energy in equation (11) yields

\[ E = \frac{2\pi R}{\sqrt{1 - \dot{R}^2}} \left[ \frac{\sigma}{2} + \frac{1}{2} \mathcal{E}_b^2 R^2 \right]. \] (15)

Since this energy becomes arbitrarily large for both, \( R \downarrow 0 \) and \( R \uparrow \infty \) (as well as for \( \dot{R} \uparrow 1 \)), the presence of this (classical) scalar field solution prevents the collapse \( R \downarrow 0 \) for \( \mathcal{E}_b \neq 0 \). The dynamics is governed by

\[ R \ddot{R} [R^2 + \mathcal{E}_b^2] + (1 - \dot{R}^2) [R^2 - \mathcal{E}_b^2] = 0, \] (16)

with \( \mathcal{E}_b^2 = \mathcal{E}_b^2 / \sigma > 0 \) (assuming \( \sigma > 0 \)) denoting the square of the radius which minimizes the energy in equation (15) for \( \dot{R} = 0 \). Initial conditions \( R(t = 0) \neq R_E \) or \( R(t = 0) \neq 0 \) lead to an eternal oscillation between \( R \downarrow 0 \) and \( R \uparrow \infty \) whereas for \( R(t = 0) = R_E \) and \( R(t = 0) = 0 \), the ring remains static forever.

However, it should be emphasized here that a spatially homogeneous solution such as \( \partial \Phi / \partial \phi = 0 \), which is also called a zero-mode, does not usually exist in higher-dimensional situations with vector fields, for example (see also the following section). On the other hand, one would obtain the same result for a thermal bath of the \( \Phi \)-field with \( \langle E \rangle_\beta \) representing the (classical) ensemble average of the energy for the inverse temperature \( \beta \). The calculation of \( \langle T_0^0 \rangle_\beta \) may proceed in basically the same way as the derivation starting from equation (23), provided that one neglects all quantum effects such as the trace anomaly. As we shall see in the following section, the Casimir effect would contribute to the total energy in the same way if the Casimir energy was positive. Therefore, the above (classical) contribution and the induced stabilization of the radius of the circle can be negated by the (quantum) Casimir effect, in particular since the amplitude of the field or the temperature can decrease, whereas the Casimir effect remains. This observation motivates the consideration of the quantum field effects.

4. Renormalization of \( \langle \hat{T}_{\mu\nu} \rangle \) and trace anomaly

In order to investigate the back-reaction of the quantum field \( \hat{\Phi} \) in analogy to the semi-classical Einstein equation (1), we have to calculate the renormalized expectation value of the energy–momentum tensor and insert it into the equation of motion (8) for \( R(t) \). In both cases, the renormalization of \( \langle \hat{T}_{\mu\nu} \rangle \), i.e., the absorption of the divergences, requires a redefinition of the involved coupling constants—the cosmological constant \( \Lambda \) and Newton’s constant \( \kappa \) in gravity, and the constant \( \sigma \) in our model.

Adopting the point-splitting renormalization procedure, we need the two-point function \( W \). Fortunately, a massless and minimally coupled scalar field in (1+1) dimensions is conformally invariant and hence the two-point function of the conformal vacuum (see, e.g., [6])
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in terms of the conformal coordinates in equation (13) has the same form as in flat spacetime \((R = \text{constant})\)

\[ \langle \hat{\Phi}(\tau, \varphi) \hat{\Phi}(\tau', \varphi') \rangle = W(\tau - \tau', \varphi - \varphi'). \]  

(17)

In the coincidence limit, i.e., for \(\tau \rightarrow \tau'\) and \(\varphi \rightarrow \varphi'\), the Wightman function \(W\) behaves as \(\ln[(\tau - \tau')^2 - (\varphi - \varphi')^2].\)

Note that \(W\) is only determined up to an additive constant—in the usual \((1+1)\)-dimensional spacetime \(R \times R\), this fact reflects the infrared problem (in \((1+1)\) dimensions), and, in our case \(R \times S_1\), this undetermined additive constant corresponds to the zero mode \(\partial \Phi/\partial \varphi = 0\).

However, zero modes do not have a zero-point energy (the energy spectrum reaches zero) and decouple from the rest of the modes (in our situation). Therefore, we omit those modes and the related problems with their quantization in the following.

In any covariant regularization method in \((1+1)\) dimensions (where only one principal divergence exists), for example, point-splitting after omitting the direction-dependent terms, the regularized (but not yet renormalized) expectation value reads (see also appendix A)

\[ \langle \hat{T}_{\mu\nu} \rangle_{\text{bare}} = \frac{1}{\epsilon} g_{\mu\nu} + \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}, \]  

(18)

where \(\epsilon \rightarrow 0\) is the regularization parameter (e.g., the geodesic distance in point-splitting) and the remaining \(\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}\) is finite.

In the expressions for the energy (11) as well as the equations of motion (8), the above divergence can be completely absorbed by a renormalization of \(\sigma\) via

\[ \sigma_{\text{bare}} = -\frac{2}{\epsilon} + \sigma_{\text{ren}}. \]  

(19)

The question of whether \(\sigma_{\text{ren}}\) can acquire any dependence on \(R\) or \(\dot{R}\) via the renormalization procedure (roughly similar to the running coupling in quantum electrodynamics, for example) will be answered below.

Owing to the conformal invariance of the scalar field, the two-point function of the conformal vacuum can be calculated easily in the conformal metric. For example, for a single scalar field with periodic boundary conditions, we obtain

\[ \langle \partial_\varphi \hat{\Phi}(\tau, \varphi) \partial_\varphi \hat{\Phi}(\tau, \varphi') \rangle = \langle \partial_\varphi \hat{\Phi}(t, \varphi) \partial_\varphi \hat{\Phi}(t, \varphi') \rangle = \frac{(4\pi)^{-1}}{\cos(\varphi - \varphi')} - 1. \]  

(20)

By means of this example, one can read off the symmetries \(\varphi \rightarrow \varphi + \varphi_0\) and \(\varphi \rightarrow -\varphi\) which reflect the homogeneity and \(Z_2\)-isotropy of the underlying spacetime. These symmetries of the vacuum state and the geometry are inherited by the (renormalized) expectation value of the energy–momentum tensor (see, e.g., [6, 7])

\[ \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} = \langle \hat{T}^0_0 \rangle_{\text{ren}} = 0, \]  

(21)

i.e., no flux, as well as

\[ \partial_\mu \langle \hat{T}^\mu_0 \rangle_{\text{ren}} = 0. \]  

(22)

Furthermore, one has to demand that the renormalization procedure of \(\langle \hat{T}^\mu_0 \rangle_{\text{ren}}\) respects the property of a vanishing covariant divergence (see also appendix A)

\[ \nabla_\mu \langle \hat{T}^\mu_0 \rangle_{\text{ren}} = 0 \quad \Rightarrow \quad \partial_\mu (\sqrt{-g} \langle \hat{T}^\mu_0 \rangle_{\text{ren}}) = \frac{\sqrt{-g}}{2} \langle \hat{T}^\mu_0 \rangle_{\text{ren}} \partial_\mu g^{\alpha\beta}. \]  

(23)

As it is well known [7], one has to give up the classical feature \(T_\rho^\rho = 0\) in this process, i.e., \(\langle \hat{T}^\mu_0 \rangle_{\text{ren}}\) acquires an anomalous trace during the renormalization

\[ \langle \hat{T}^\mu_\mu \rangle_{\text{ren}} = C_\alpha R, \]  

(24)
with \( C_{tr} \) denoting a constant related to the central charge. In (1+1) dimensions, the trace anomaly is completely determined by the Ricci scalar (see, e.g., [6, 7]):

\[
\mathcal{R} = \frac{2}{R(1 - R^2)^2} = \frac{2RR'' - (R')^2}{R^4},
\]

(25)

with \( \dot{R} = dR/d\tau \) and \( R' = dR/d\tau \), respectively.

Similar to the calculation of the Hawking radiation in (1+1) dimensions [7] via the trace anomaly, we can integrate equation (23) with the proper boundary/initial conditions and symmetries. The case \( \nu = 1 \) in equation (23) is trivial and in the remaining equation with \( \nu = 0 \)—when expressed in terms of the conformal metric—the trace anomaly acts somewhat similar to a source

\[
\partial_\tau \left( R^2 \langle \hat{T}^0_{0,\text{ren}} \rangle \right) = \frac{2RR'' - (R')^3}{R^3} C_{tr}.
\]

(26)

The conformal invariance of the scalar field in (1+1) dimensions leading to the absence of particle creation (conformal vacuum) manifests itself in the fact that the rhs of the above equation is a total differential. Thus, integrating equation (26) yields

\[
\langle \hat{T}^0_{0,\text{ren}} \rangle = C_{tr} \frac{(R')^2}{R^4} + C_{Cas} \frac{1}{R^2},
\]

(27)

where \( C_{Cas} \) is a priori an integration constant. However, if we consider a static circle \( R = \text{constant} \), the only contribution to the vacuum energy is the Casimir effect induced by the compactness of the space \((S_1)\). In this case, \( C_{Cas}/R^2 \) is just the Casimir energy density which allows us to determine the constant \( C_{Cas} \) and we keep this nomenclature also for the general time-dependent situation \( R' \neq 0 \).

Transforming back to the laboratory time, we obtain (note that \( \tau = \tau(t) \rightarrow t \) and hence \( T^0_0 \rightarrow T^0_0 \))

\[
\langle \hat{T}^0_{0,\text{ren}} \rangle = \frac{1}{R^2} \left[ \frac{\dot{R}^2}{1 - \dot{R}^2} C_{tr} + C_{Cas} \right].
\]

(28)

The remaining non-trivial component \( \langle \hat{T}^1_{1,\text{ren}} \rangle \) can be determined via the trace anomaly

\[
\langle \hat{T}^1_{1,\text{ren}} \rangle = 2C_{tr} \frac{R}{R(1 - R^2)^2} - \langle \hat{T}^0_{0,\text{ren}} \rangle.
\]

(29)

5. Back-reaction

At this stage we are in the position to calculate the back-reaction in analogy to the semi-classical Einstein equations by inserting the renormalized expectation values of the energy–momentum tensor \( T^\mu_\nu \rightarrow \langle \hat{T}^\mu_\nu \rangle_{\text{ren}} \) in equations (28) and (29) into the equation of motion (8)

\[
R \ddot{R} \left( \frac{\sigma_{\text{ren}}}{2} R^2 + C_{Cas} + \frac{2 + R^2}{1 - R^2} C_{tr} \right) = (1 - \dot{R}^2) \left( C_{Cas} - \frac{\sigma_{\text{ren}}}{2} R^2 \right) + \ddot{R} C_{tr}.
\]

(30)

The consistency of the above equation, assuming a constant \( \sigma_{\text{ren}} \), with the conservation of the total (renormalized) energy of the system,

\[
E = \frac{2\pi R}{\sqrt{1 - R^2}} \left( \frac{\sigma_{\text{ren}}}{2} + \langle \hat{T}^0_{0,\text{ren}} \rangle \right)
\]

\[
= \frac{2\pi R}{\sqrt{1 - R^2}} \left( \frac{\sigma_{\text{ren}}}{2} + \frac{1}{R^2} \left[ \frac{\dot{R}^2}{1 - \dot{R}^2} C_{tr} + C_{Cas} \right] \right),
\]

(31)

3 If particles were created, the energy density \( \langle \hat{T}^0_{0,\text{ren}} \rangle \) would depend on the history of the spacetime and not just on the values \( R(t) \) and \( \dot{R}(t) \), etc, at present.
A relativistic toy model for back-reaction can be considered as a cross-check that $\sigma_{\text{ren}}$ does not acquire any non-trivial dependence on $R(t)$ or $\dot{R}(t)$ during the renormalization procedure (see also appendix A).

It turns out that the equation of motion (30) can be derived from the following effective Lagrangian:

$$L_{\text{eff}} = 2\pi \sqrt{1 - \frac{\dot{R}^2}{R}} \left( -\frac{\sigma_{\text{ren}} R^2}{2} - \xi_{\text{Cas}} + \xi_{\text{tr}} \dot{R}^2 \right),$$

and that the total (renormalized) energy in equation (31) is just the associated Hamiltonian $E = H = P_{\text{eff}} \dot{R} - L_{\text{eff}}$.

6. Discussion

As for any conservative system, possible trajectories $R(t)$ can be inferred from the energy landscape according to equation (31), see figures 1 and 2. To this end, we have to specify the constants $\xi_{\text{tr}}$ and $\xi_{\text{Cas}}$. For the example of a single scalar field with periodic boundary conditions, we have

$$\xi_{\text{tr}} = -\frac{1}{24\pi}, \quad \xi_{\text{Cas}} = -\frac{\pi}{6}. \quad (33)$$

For multiple scalar fields and/or possibly different boundary conditions, one would obtain other values. For instance, in the case of ‘twisted fields’, the Casimir energy for a single scalar field is positive

$$\xi_{\text{Cas}} = \frac{\pi}{12}. \quad (34)$$

and thus contributes to the total energy analogous to a thermal bath of the $\Phi_1$-field.

In what follows, we discuss the possible trajectories $(R, \dot{R})$ for positive as well as negative $\sigma_{\text{ren}}$ in dependence on the possible values of $\xi_{\text{Cas}}$ and $\xi_{\text{tr}}$. Being interested in the qualitative behaviour of the trajectories, we consider the values $\xi_{\text{Cas}} = -1, 0, +1$, $\xi_{\text{tr}} = -1, 0$ and $\sigma_{\text{ren}} = -0, 20$. The corresponding energy landscapes are depicted in figures 1 and 2. Note that we do not consider the case $\xi_{\text{tr}} = +1$ since the corresponding energy landscapes can be obtained via inverting the pictures for $\xi_{\text{tr}} = -1$, i.e., $\xi_{\text{Cas}} \rightarrow -\xi_{\text{Cas}}$ and $\sigma_{\text{ren}} \rightarrow -\sigma_{\text{ren}}$.

6.1. Positive classical energy $\sigma_{\text{ren}} > 0$

As one may infer from the qualitative behaviour of the energy landscapes for $\xi_{\text{tr}} = -1$ in figures 1(a)–(c), a typical trajectory $R(t)$ corresponding to a positive energy $E > 0$ does never reach $R = 0$ and also cannot approach $\dot{R} = 1$ for finite $R$, i.e., we avoid the collapse to a singularity $R = 0$, provided that the semi-classical treatment does not breakdown completely (cf the following section). For the example depicted in figure 1(a), let us discuss possible trajectories $R(t)$ for the case $\xi_{\text{tr}} = -1$ in more detail, assuming that we start with a circle with a large initial radius $R(t = 0) = R_0$ and a vanishing initial velocity $\dot{R}(t = 0) = 0$ where the quantum effects are negligible. In the energy landscape in figure 1(a), this corresponds to the region in the middle (around the axis $\dot{R} = 0$) high up the mountain. Starting in normal region with $R(t = 0) = R_0$, the radius begins to shrink $R < 0$ and the negative velocity gradually increases in magnitude. After a while, i.e., for a sufficiently large velocity (and, correspondingly, small radius), quantum effects become important. Finally, the trajectory reaches the ridge, where the radius $R$ cannot decrease anymore. Therefore, the velocity $\dot{R}$ should change its sign, i.e., the trajectory $R(t)$ should turn around. But these two ridges correspond to finite velocities $\dot{R}$—one positive and the other negative—such that a change in
the sign is only possible by jumping from one ridge to the other (corresponding to the same values of \( R, |R| \) and \( E \)). Obviously, this requires a diverging \( \dot{R} \) and seems to be a very strange property of the semi-classical system.

Nevertheless, it turns out that such a strange property is a necessary consequence of the weird features of the energy landscape in figure 1(a). The energy \( E(R, \dot{R}) \) equals the Hamiltonian \( H(R, P) \) derived from the effective Lagrangian \( L(R, \dot{R}) \) in equation (32). For a given and fixed radius \( R \), there are three values of \( \dot{R} \) for which

\[
\left( \frac{\partial E}{\partial \dot{R}} \right)_R = 0,
\]

the usual \( \dot{R} = 0 \) (in the middle, i.e., the normal region) and two anomalous points (on the two ridges) where \( R \neq 0 \). In view of the Hamilton equation

\[
\dot{R} = \left( \frac{\partial H}{\partial \dot{P}} \right)_R = \left( \frac{\partial H}{\partial \dot{R}} \right)_R \left( \frac{\partial P}{\partial \dot{R}} \right)_R,
\]

for these two anomalous points where \( \dot{R} \neq 0 \), we must have

\[
\left( \frac{\partial P}{\partial \dot{R}} \right)_R = \left( \frac{\partial^2 L}{\partial \dot{R}^2} \right)_R = 0,
\]

i.e., the ridges are critical points where the Euler–Lagrange equation,

\[
\left( \frac{\partial L}{\partial \dot{R}} \right)_R = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{R}} \right)_R = \ddot{R} \left( \frac{\partial^2 L}{\partial \dot{R}^2} \right)_R + \dot{R} \frac{\partial^3 L}{\partial \ddot{R} \partial \dot{R}} = 0,
\]

cannot determine \( \ddot{R} \) anymore, i.e., \( \ddot{R} \) diverges. Indeed, the pre-factor in front of the \( \ddot{R} \)-term in the equation of motion (8) goes to zero at these critical points.

By inspection of equation (30), one observes that close to the critical point (i.e., the ridge) the acceleration \( \ddot{R}(t) \) diverges as \( 1/\sqrt{t_0 - t} \) while the velocity behaves as \( \dot{R}(t) = V_0 + \text{const.}\sqrt{t_0 - t} \). Hence, in that case, the trajectory \( R(t) \) exhibits a non-analytic behaviour at that point \( t = t_0 \) (where the semi-classical treatment breaks down) which cannot be analytically continued to later times \( t > t_0 \).

Since \( \ddot{R} \) approaches infinity there, whereas \( \dot{R} \) and \( R \) remain finite, the Ricci scalar diverges at the ridges, i.e., these critical points still represent a curvature singularity. Without any back-reaction, \( \xi_{\text{tr}} = \xi_{\text{Cas}} = 0 \) (cf figure 1(d)), the circle collapses to \( R = 0 \) and \( \dot{R} = 1 \) after a finite laboratory time, i.e., the metric, the action and the Ricci scalar become singular. If we take into account the Casimir effect only with \( \xi_{\text{Cas}} = -1 \), which is rather artificial, the circle reaches \( \ddot{R} = 1 \) and hence develops a singularity of the metric, the action and the Ricci scalar, respectively, after a finite laboratory time and at a finite radius \( R > 0 \) (cf figure 1(c)). Calculating the full expectation value \( \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} \) in the conformal vacuum, including the trace anomaly, we find in the case of a negative Casimir energy \( \xi_{\text{Cas}} = -1 \) that the circle reaches the critical point (ridge) after a finite laboratory time, cf figure 1(a). However, such a critical point is characterized by a singular acceleration \( \ddot{R} \) and hence the Ricci scalar only—while the radius \( R > 0 \), the velocity \( \dot{R} < 1 \), and hence both, the action as well as the metric, remain regular. Note that the trace anomaly alone, i.e., \( \xi_{\text{tr}} = -1, \xi_{\text{Cas}} = 0 \), already prevents the collapse of the circle to \( R = 0 \) (cf figure 1(b)). For a positive Casimir energy \( \xi_{\text{Cas}} = 1 \), all of these singularities are avoided and the circle oscillates smoothly forever in both cases, with and without the trace anomaly, cf figures 1(e) and (f), respectively. This is not too surprising as the latter case is equivalent to a purely classical back-reaction (e.g., a thermal bath). Note how the shape of the energy landscape changes when going from negative Casimir energies \( \xi_{\text{Cas}} = -1 \) (cf figure 1(a)) across \( \xi_{\text{Cas}} = 0 \) (cf figure 1(b)) to positive Casimir energies \( \xi_{\text{Cas}} = 1 \) (cf figure 1(e)), whereby the ridges remain.
A relativistic toy model for back-reaction

Figure 1. Energy landscape (in arbitrary units). We plot $\mathcal{E} = E/(2\pi|\sigma_{\text{ren}}|)$ with $\sigma_{\text{ren}} = 20$ where negative energies have been cut off for convenience. Panel (a): $C_{\text{tr}} = C_{\text{Cas}} = -1$, negative Casimir energy and trace anomaly, i.e., full back-reaction. Panel (b): $C_{\text{tr}} = -1, C_{\text{Cas}} = 0$, i.e., trace anomaly only. Panel (c): $C_{\text{tr}} = 0, C_{\text{Cas}} = -1$, i.e., negative Casimir energy, no trace anomaly. Panel (d): $C_{\text{tr}} = 0, C_{\text{Cas}} = 0$, i.e., no back-reaction. Panel (e): $C_{\text{tr}} = -1, C_{\text{Cas}} = 1$, i.e., full back-reaction with positive Casimir energy. Panel (f): $C_{\text{tr}} = 0, C_{\text{Cas}} = 1$, i.e., positive Casimir contribution and no trace anomaly, equivalent to a thermal bath of particles.

6.2. Negative classical energy $\sigma_{\text{ren}} < 0$

A priori, the renormalized $\sigma_{\text{ren}}$ may not only take positive but also negative values leading to negative energies such as the Newtonian gravitational energy. The corresponding energy
Figure 2. Energy landscape (in arbitrary units). We plot $\mathcal{E} = \frac{E}{(2\pi |\sigma_{\text{ren}}|)}$ with $\sigma_{\text{ren}} = -20$, where negative energies have been cut off below certain values for convenience. Panel (a): $\mathcal{E}_n = \mathcal{E}_{\text{Cas}} = -1$, negative Casimir energy and trace anomaly, i.e., full back-reaction. Panel (b): $\mathcal{E}_n = -1$, $\mathcal{E}_{\text{Cas}} = 0$, i.e., trace anomaly only. Panel (c): $\mathcal{E}_n = 0$, $\mathcal{E}_{\text{Cas}} = -1$, i.e., negative Casimir energy, no trace anomaly. Panel (d): $\mathcal{E}_n = 0$, $\mathcal{E}_{\text{Cas}} = 0$, i.e., no back-reaction. Panel (e): $\mathcal{E}_n = -1$, $\mathcal{E}_{\text{Cas}} = 1$, i.e., full back-reaction with positive Casimir energy. Panel (f): $\mathcal{E}_n = 0$, $\mathcal{E}_{\text{Cas}} = 1$, i.e., positive Casimir contribution, no trace anomaly, equivalent to a thermal bath of particles.

Landscapes are depicted in figure 2. Considering the full back-reaction with a negative Casimir energy, i.e., $\mathcal{E}_{\text{Cas}} = \mathcal{E}_n = -1$, and $\sigma_{\text{ren}} = -20$, the trajectories corresponding to a given (negative) energy are regular and never reach the points $R = 0$ and $|\dot{R}| = 1$, cf
figure 2(a). The circle performs eternal oscillations with a minimum radius different from zero. Consequently, the action, the metric and the Ricci scalar remain regular and no singularities occur. Without the Casimir effect the trajectories reach the point $R = 0$ for $|\dot{R}| < 1$, cf figure 2(b), i.e., the metric and the action become singular. Hence, in contrast to the case $\sigma_{\text{ren}} > 0$, here the Casimir effect is important for the prevention of the collapse of the circle to $R = 0$. On the other hand, neglecting the trace anomaly, cf figure 2(c), does not change the actual shape of the energy landscape compared to the full back-reaction (cf figure 2(a)). It is therefore not substantial for the avoidance of singularities. Hence, for negative $\sigma_{\text{ren}}$, the Casimir effect alone prevents the collapse of the circle to $R = 0$ and the formation of singularities if the Casimir energy is negative. For positive Casimir energies, the situation changes drastically (cf figures 2(e) and (f)). Even the full back-reaction, i.e., trace anomaly $\mathcal{E}_{\text{tr}} = -1$ and positive Casimir energy $\mathcal{E}_{\text{Cas}} = 1$ may not prevent the collapse of the circle to $R = 0$ in this case, cf figure 2(e). Starting with a large value $R$ and $\dot{R} \approx 0$, the circle collapses and reaches $R = 0$ after a finite time with a finite velocity $|\dot{R}| < 1$. Neglecting in addition the contribution of the trace anomaly, the collapsing circle reaches $|\dot{R}| = 1$ before reaching $R = 0$, cf figure 2(f).

7. Conclusions

For positive $\sigma_{\text{ren}}$, we observe that the back-reaction of the quantum field $\hat{\Phi}$, in particular the trace anomaly, prevents the strong singularity of the classical scenario, but still generates a weaker singularity (unless the Casimir energy is positive, which is analogous to a thermal bath). At this critical point, i.e., the ridge in the energy landscape, cf figure 1(a), the pre-factor in front of $\ddot{R}$ obtained via the semi-classical treatment vanishes. It appears natural to expect that the quantum back-reaction beyond the semi-classical picture will become important in the vicinity of this point. In the case of a positive Casimir energy, or equivalently a thermal bath of particles, no singularities form and the circle performs eternal oscillations with a minimal radius different from zero (cf figure 1(e)).

On the other hand, for negative $\sigma_{\text{ren}}$, the back-reaction of the quantum field prevents the collapse of the circle to $R = 0$ and the formation of singularities in the case of negative Casimir energies (the trace anomaly is not substantial here, cf figures 2(a) and (c)) leading to eternal oscillations of $R$ with a minimal non-vanishing radius. For positive Casimir energies, however, the full back-reaction does not prevent the collapse of the circle to $R = 0$ and the formation of singularities, cf figures 2(e) and (f).

It is evident that the semi-classical treatment of the back-reaction problem has clear problems\(^4\). The major point is that the semi-classical Einstein equations, and similarly equation (30), imply the neglect of all quantum fluctuations by considering the expectation value $\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}$ only. This approximation is justified if the fluctuations are sufficiently small, i.e., as long as conditions, such as

$$\langle \hat{T}_{\mu\nu}^2 \rangle_{\text{ren}} \approx \langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}^2$$

hold. Assuming the validity of the renormalization procedure such as point-splitting—e.g., that all the allowed quantum states satisfy the Hadamard condition—the trace-anomaly is

\(^4\) As an example, let us consider the semi-classical Einstein equation (1) for a quantum state $|\Psi\rangle$ describing a macroscopic superposition $|\Psi\rangle = (|\Psi_1\rangle + |\Psi_2\rangle)/\sqrt{2}$. The state $|\Psi_1\rangle$ corresponds to a large mass $M$ at the point $x_1$ whereas $|\Psi_2\rangle$ describes the same mass $M$ at (a macroscopically) different position $x_2$. The semi-classical Einstein equation (1) predicts a gravitational field (since $\langle |\Psi_1\rangle |\hat{T}_{\mu\nu} |\Psi_2\rangle_{\text{ren}} = 0$) as if there was half the mass $M/2$ at $x_1$ and the other half $M/2$ at $x_2$. However, in reality this macroscopic superposition $|\Psi\rangle$ will exhibit an extremely fast decoherence with a negligible energy flux due to the interaction with the environment leading to a classical mixture $\hat{\varrho} = (\hat{\varrho}_1 + \hat{\varrho}_2)/2$. Hence one would naturally expect a classical mixture of gravitational fields.
independent of the quantum state and hence a c-number. Ergo, for this particular quantity one would expect the semi-classical treatment to work. However, the equations of motion do not only contain \( \langle \hat{T}^{\mu\nu}_{\text{ren}} \rangle \) but also \( \langle \hat{T}^{00}_{\text{ren}} \rangle \), etc. As the derivation of these quantities is more involved and includes other contributions (e.g., \( \langle \hat{T}^{20}_{\text{ren}} \rangle \)) as well as a time-integration, it is not clear that the entanglement between \( \hat{T}^{\mu\nu} \) and the corresponding operator \( \hat{R} \) can be omitted, for example,

\[
\langle \hat{T}^{\mu\nu} \hat{R} \rangle_{\text{ren}} \approx \langle \hat{T}^{\mu\nu}_{\text{ren}} \hat{R} \rangle_{\text{ren}}.
\]  

(40)

In summary, for negative \( \mathcal{C}_e \), the semi-classical theory suggests its own demise by predicting a trajectory \( R(t) \) which is not continuously differentiable for \( \sigma_{\text{ren}} > 0 \) (\( \dot{R} \) diverges) and hence a curvature singularity. As a result, the fate of the circle after the bounce remains unclear in the sense that it cannot be determined within the semi-classical picture without additional arguments. Assuming that the circle re-enters the region of validity of the semi-classical theory after the bounce, there are two possibilities. After jumping from one ridge to the other, the trajectory could turn around and continue towards the normal regime in the middle again (\( R \) decreases). In this case, the circle would perform an eternal oscillation. Alternatively, the trajectory could turn to the edge and enter the anomalous regime. In this situation, the velocity would increase forever and asymptotically reach the speed of light. These run-away solutions are somewhat similar to a trace-anomaly-induced inflation in the (3+1)-dimensional gravity. Interestingly, the trace anomaly of both a scalar field in (1+1) dimensions on the one hand and that of, say, a vector field in the (3+1)-dimensional gravity on the other hand are such that they prevent the collapse to a metric singularity \( R = 0 \) and admit run-away solutions. (The Casimir effect is not substantial in that respect, cf figure 1(b).)

Allowing also for negative values of \( \sigma_{\text{ren}} \), the picture changes drastically. The energy landscape shows no such weird features but has a rather simple shape. The fate of the circle—it oscillates forever with a minimal non-vanishing radius—can be determined for all times within the semi-classical treatment. Thereby the Casimir effect alone prevents the formation of singularities and the trace anomaly is not substantial if \( \sigma_{\text{ren}} < 0 \) in contrast to the case \( \sigma_{\text{ren}} > 0 \).

8. Outlook

As it became evident from the previous discussions, a more exhaustive analysis would require the investigation of the full quantum back-reaction. At a first glance, one could expect that this is provided by quantizing the effective action (32) for the geometric sector, cf equation (B.1), after integrating out the field \( \Phi \). However, it is by no means clear that such an approach is indeed equivalent to a full quantization of the geometric degrees of freedom interacting with the quantum field. In view of the non-trivial relation between the canonically conjugated momentum \( P \) and the velocity \( R \) as well as the occurrence of anomalies and the associated non-linearities, a rigorous treatment is apparently rather involved. Furthermore, an ab initio quantization has to account for non-spherical geometries, i.e., deviations from the rotational symmetry, since the monopole mode inherently couples to higher multipole modes and their quantum fluctuations—in contrast to the semi-classical case considered above.

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Appendix A. Renormalization

Starting from the obvious condition that the (renormalized) energy–momentum tensor of the total system (quantum field plus circle) is conserved with respect to the (flat) (2+1)-dimensional embedding spacetime
\[ \partial_A T_{AB}^{\text{total}} = 0, \] (A.1)
one can show the compatibility of the two conditions,
\[ \nabla_\mu \langle \hat{T}^\mu_\nu \rangle_{\text{ren}} = 0 \iff \sigma_{\text{ren}} = \text{constant}, \] (A.2)
for the renormalized energy–momentum tensor of the quantum field and \( \sigma_{\text{ren}} \), respectively.

As a consequence of equation (38), the argument works in both directions: demanding \( \nabla_\mu \langle \hat{T}^\mu_\nu \rangle_{\text{ren}} = 0 \) in order to ensure energy conservation of the quantum field separately in the presence of a Killing vector \( \xi \) of the induced metric, one deduces \( \sigma_{\text{ren}} = \text{constant} \). Conversely, one may start with the assumption \( \sigma_{\text{ren}} = \text{constant} \) and derive \( \nabla_\mu \langle \hat{T}^\mu_\nu \rangle_{\text{ren}} = 0 \) from \( \partial_A T_{AB}^{\text{total}} = 0 \).

Since the decomposition of the total energy–momentum tensor \( T_{AB}^{\text{total}} \) into the contributions of the quantum field and the geometry is not unique, one can even impose alternative renormalization conditions. When renormalizing \( \langle \hat{T}^\mu_\nu \rangle_{\text{ren}} \), its divergent part has to be absorbed into \( \sigma \), while the finite part of the counter-term remains undetermined. Here we have employed a ‘minimal subtraction scheme’, cf equations (18) and (19). As an alternative renormalization scheme, for example, one may impose the condition that the trace still vanishes \( \langle \hat{T}^\mu_\mu \rangle_{\text{ren}} = 0 \) and abandon the property \( \nabla_\mu \langle \hat{T}^\mu_\nu \rangle_{\text{ren}} = 0 \) instead. Such a renormalization procedure could be envisaged as a ‘non-minimal subtraction scheme’. Instead of employing equation (18) we could split up \( \langle \hat{T}^\mu_\nu \rangle_{\text{bare}} \) via
\[ \langle \hat{T}^\mu_\nu \rangle_{\text{bare}} = \frac{1}{\epsilon} g_{\mu\nu} + \frac{1}{2} \mathcal{C}_R g_{\mu\nu} + \langle \hat{T}^\mu_\nu \rangle_{\text{ren}}, \] (A.3)
with
\[ \langle \hat{T}^\mu_\mu \rangle_{\text{ren}} = 0 \quad \Rightarrow \quad \nabla_\mu \langle \hat{T}^\mu_\nu \rangle_{\text{ren}} \neq 0, \] (A.4)
and consequently,
\[ \sigma_{\text{bare}} = -\frac{2}{\epsilon} - \mathcal{C}_R \mathcal{R} + \tilde{\sigma}_{\text{ren}}. \] (A.5)
Within this alternative renormalization scheme, \( \tilde{\sigma}_{\text{ren}} \) depends on the geometry
\[ \tilde{\sigma}_{\text{ren}} = \tilde{\sigma}_{\text{ren}}(\mathcal{R}, \dot{\mathcal{R}}, \ddot{\mathcal{R}}), \] (A.6)
while the equation of motion for \( R(t) \) and the total energy remain the same within both schemes. Such a non-minimal subtraction scheme employed for renormalization of the quantum-field sector provides the possibility of inducing a spacetime dependence of characteristic parameters (here simply \( \sigma \)) of the classical (geometric) sector.

Appendix B. Effective action

Instead of using the equations of motion, one could investigate the back-reaction by means of the effective action, cf [8, 9]. In (1+1) dimensions, the trace anomaly can be deduced from the Liouville–Polyakov effective action [10]
\[ \mathcal{A}_{\text{eff}} = \frac{1}{96\pi^2} \int d^2x \sqrt{-g} \mathcal{R} \Box^{-1} \mathcal{R}. \] (B.1)
In view of the inverse of the d’Alembert operator $\Box^{-1}$, this expression appears to be non-local, but inserting the value for the Ricci scalar in terms of the conformal coordinates, for example, it turns out that it is in fact effectively local (no particle creation), and agrees with the term proportional to $C_{tr}$ in the effective Lagrangian in equation (32) after a coordinate transformation to the laboratory time.

However, in this naive treatment the Casimir term is missing and thus requires additional considerations. This already illustrates a potential danger of the effective action method—the fact that the total effective Lagrangian is just as a sum of the terms proportional to $C_{tr}$ and the (static) Casimir effect (in addition to the classical terms) is not a priori obvious.

In order to compare our calculation with the results of [8], a few remarks are in order:

- Switching to the Euclidean signature, as done in [8], is a well-defined procedure for the calculation of the static Casimir energy, but not for general time-dependent systems. Hence the derivation of the main result as a sum of the two contributions is somewhat ad hoc.
- Contrary to the claims in [8], the scenario under consideration does not exhibit the dynamical Casimir effect (in the sense of particle creation) due to the conformal invariance.
- Furthermore, [8] uses a non-relativistic kinetic term (introduced by hand). Therefore, the results can be compared with ours in the adiabatic regime $\dot{R} \ll 1$ only.

In addition to the problems mentioned in [11], for example, a conclusive discussion of a possible non-trivial dependence of the renormalized $\sigma_{\text{ren}}(R, \dot{R}, \ldots)$ in dependence of the renormalization scheme (see appendix A) by means of the effective action is not obvious.

Having in mind the various potential difficulties of the effective action method indicated above, we would like to emphasize that, in contrast to this approach, the calculations based on the renormalized expectation values of the energy–momentum tensor $\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}$ together with employing the equations of motion, are apparently less ambiguous and more evident.

Appendix C. Negative energies and stability

One might ask whether the breakdown of the semi-classical treatment for the toy model under consideration is just an artifact caused by the special features of this (1+1)-dimensional model or whether it represents a generic property also for the (3+1)-dimensional gravity (plus quantum fields). Even though this question cannot be addressed completely without solving the full ((3+1)-dimensional) scenario, one can compare characteristic features of both systems (at the semi-classical level).

For the classical circle (without quantum fields) as described by the Nambu–Goto action, the (conserved) energy is positive $E = \pi \sigma R / \sqrt{1 - \dot{R}^2}$. Nevertheless, the equation of motion admits singular solutions (collapse to $R = 0$) in close analogy to the classical gravity (cf the singularity theorems).

Owing to the trace anomaly and the Casimir effect, the renormalized energy of the quantum field (in the geometrical background) is negative in the vacuum state. However, that does not imply that the vacuum state of the quantum field itself (in the semi-classical treatment) is unstable. After a normal mode expansion

$$\hat{\phi}(\tau, \varphi) = \sum_{m=0}^{\infty} \hat{Q}_{m,(c)}(\tau) \cos(m \varphi) + \sum_{m=1}^{\infty} \hat{Q}_{m,(s)}(\tau) \sin(m \varphi),$$

(C.1)

the Hamiltonian decouples and is positive for each mode

$$\hat{H}_{\phi} = \sum_{m=0}^{\infty} \sum_{\xi=(c,s)} \left( \hat{p}_{m,\xi}^2 + m^2 \hat{Q}_{m,\xi}^2 \right).$$

(C.2)
Apart from the zero mode $m = 0$, all field modes possess a unique ground state—but even for the zero mode, the quantum evolution is not unstable. In summary, the dynamics of the quantum field itself (in the semi-classical treatment, i.e., for a fixed trajectory $R(\tau)$) does not display any instabilities. The stability of the full theory (circle plus field), however, lies outside the scope of the present investigations (i.e., is still unclear) in view of the breakdown of the semi-classical treatment.

The negative (renormalized) energy of the quantum field cannot be attributed to single modes. Instead, being similar to vacuum polarization effects, it is a consequence of the renormalization procedure. This is a generic feature of quantum fields in curved spacetimes and occurs in the (3+1)-dimensional scenario as well.

In view of the above similarities, one might conjecture that the results found for the (1+1)-dimensional toy model—i.e., the breakdown of the semi-classical treatment and a possible bounce caused by the trace anomaly—are not just artifacts of this toy model but shed some light on the (3+1)-dimensional situation as well.

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