W-GAUGE STRUCTURES AND THEIR ANOMALIES:
AN ALGEBRAIC APPROACH

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Abstract

Starting from flat two-dimensional gauge potentials we propose the notion of W-gauge structure in terms of a nilpotent BRS differential algebra. The decomposition of the underlying Lie algebra with respect to an SL(2) subalgebra is crucial for the discussion conformal covariance, in particular the appearance of a projective connection. Different SL(2) embeddings lead to various W-gauge structures.

We present a general soldering procedure which allows to express zero curvature conditions for the W-currents in terms of conformally covariant differential operators acting on the W gauge fields and to obtain, at the same time, the complete nilpotent BRS differential algebra generated by W-currents, gauge fields and the ghost fields corresponding to W-diffeomorphisms. As illustrations we treat the cases of SL(2) itself and to the two different SL(2) embeddings in SL(3), viz. the $W_3^{(1)}$- and $W_3^{(2)}$-gauge structures, in some detail. In these cases we determine algebraically W-anomalies as solutions of the consistency conditions and discuss their Chern-Simons origin.

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1 INTRODUCTION

\(\mathcal{W}\)-symmetry \cite{1} intertwines internal symmetries with space-time symmetries in two dimensions. A dynamical realization of \(\mathcal{W}\)-symmetry arises in the reduction of WZNW theories to Toda field theory \cite{2},\cite{3}. In this approach, the original set of Lie algebra valued Kac-Moody currents of the WZNW theory is reduced to a set of \(\mathcal{W}\)-currents which are primary conformal fields of well-defined (integral or half-integral) conformal weights. An important ingredient in this construction is the identification of a \(SL(2)\) subalgebra of the Lie algebra which underlies the WZNW theory: the remaining generators are then arranged in irreducible representations with respect to the \(SL(2)\) subalgebra.

In general \cite{4}, \cite{5}, for a given Lie algebra \(\mathcal{G}\) there are several possibilities to identify such a \(SL(2)\) subalgebra \cite{6},\cite{7},\cite{8},\cite{9}. Different embeddings lead to different \(\mathcal{W}\)-structures. The first examples were found for the case of \(SL(3)\) by Zamolodchikov \cite{10} and by Polyakov \cite{11} and Bershadsky \cite{12}, the \(\mathcal{W}_3^{(1)}\)- and \(\mathcal{W}_3^{(2)}\)-algebras. In any case, the \(\mathcal{W}\)-currents correspond to the highest weight generators in the \(SL(2)\) decomposition. They are all conformally covariant tensors except for the one in the \(SL(2)\) subalgebra itself, which behaves as a projective connection (a property shared by the energy-momentum tensor in
two-dimensional conformal field theory \[13\]). All the currents in these reduced WZNW theories are holomorphic quantities, the holomorphicity conditions can be understood in terms of zero-curvature conditions, reflecting the integrability properties of the Toda theories \[14\] in terms of a Lax-pair formulation. The theory which results from the reduction procedure exhibits \(\mathcal{W}\)-symmetry, the \(\mathcal{W}\)-transformations are identified as the residual gauge transformation which survive the reduction of the original theory.

Another dynamical context in which \(\mathcal{W}\)-symmetry arises is \(\mathcal{W}\)-gravity (reviews and references may be found in \[15\] or \[16\]). These theories are conceived as generalizations of usual induced gravity where the energy-momentum tensor couples to the metric and its conservation is spoiled by the conformal anomaly. The most convenient parametrization for this system is that where the metric (resp. moving frame) is described by the Beltrami differentials and the conformal factor (resp. an additional chiral Lorentz factor for the frame). In this factorized formulation the conformal anomaly is given in terms of a covariantly chiral third order differential operators acting on the Beltrami differential (see \[13\] for a review and references). The same differential operator arises in the second hamiltonian structure of the KdV hierarchy. Integration of the conformal anomaly gives rise to induced gravity.

In the case of \(\mathcal{W}\)-gravity one imagines that the \(\mathcal{W}\)-currents, which are considered as covariant higher conformal spin generalizations of the energy-momentum tensor, couple to certain \(\mathcal{W}\)-gauge fields which in turn are considered to be generalizations of the Beltrami differentials (hence the notions \(\mathcal{W}\)-frame and \(\mathcal{W}\)-diffeomorphisms). In a conformal quantum field theory which realizes \(\mathcal{W}\)-symmetry one expects then that the \(\mathcal{W}\)-currents are no longer holomorphic quantities, the Ward identities arising from \(\mathcal{W}\)-transformations are anomalous. These questions have been addressed in \[17\],\[18\],\[19\],\[20\]), in particular detail for the case of the \(\mathcal{W}\)-structures pertaining to \(SL(3)\). In these papers special emphasis was on possible interpretations of \(\mathcal{W}\) Ward identities in terms of zero curvature conditions and in relation with integrable hierarchies and their Lax pair formulation.

On the other hand, as it is well known, anomalies must satisfy the Wess-Zumino consistency conditions, and one can define anomalies algebraically as nontrivial solutions to these consistency conditions. The most powerful and elegant formulation of this approach is in terms of the BRS differential algebra of gauge and ghost fields (and matter fields as well).

It is this purely algebraic attitude that will be pursued in this paper: we propose the notion of \(\mathcal{W}\)-gauge structure in terms of differential BRS algebra generated by \(\mathcal{W}\)-currents, \(\mathcal{W}\)-gauge fields and \(\mathcal{W}\)-ghost fields, with nilpotent
operation of the exterior space-time derivative and the BRS operator. This \(W\)-gauge structure is obtained from a generalized zero curvature formulation. In turn, the zero curvature condition can be viewed as a compatibility relation for covariantly constant fields, which will be included in our BRS analysis as well. With this algebraic formalism at hand, in particular the nilpotent BRS differential algebra, one can search for nontrivial solutions of the consistency equations.

The paper is organized as follows: in chapter 2, based on [21], we present the general framework and define our notations, in chapter 3 we treat in detail the case of \(SL(2)\) itself, and the Chern-Simons origin of the anomaly. Chapter 4 is devoted to the presentation of the two different \(W\)-gauge structures deriving from \(SL(3)\), including, in each case, the structure of matter fields and the algebraic construction of the anomalies as solutions of consistency conditions and Chapter 5 contains some concluding remarks of more conceptual nature concerning the issue of \(W\)-geometry.

## 2 GENERAL STRUCTURE

### 2.1 Gauge potentials, ghosts and their BRS structure

For a given simple Lie algebra \(\mathcal{G}\) we consider a decomposition of the set of generators with respect to some \(SL(2)\) subalgebra. In general there are several possibilities to identify such a subalgebra and therefore different decompositions, as for instance for \(SL(3)\) where the two different decompositions correspond to the two different \(W_3\)-algebras of Zamolodchikov and of Polyakov and Bershadsky. For a given decomposition of the Lie algebra \(\mathcal{G}\) we denote the generators of the \(SL(2)\) subalgebra by \(L_k\), with \(k = -1, 0, +1\) and commutators

\[
[L_k, L_l] = (k - l)L_{k+l}.
\]  

The remaining generators are arranged in irreducible representations with respect to this subalgebra, they will be denoted \(T_{\rho \, L}^a\). The index \(a \geq -1\), integer or half-integer, characterizes the representation (spin \(a + 1\)) and \(k\) runs from \(a - 1\) to \(a + 1\) in integer steps. Finally the index \(\rho\) serves to distinguish different copies of the same spin which may occur in the decomposition (for instance, in the second \(SL(3)\) decomposition two spin 1/2 occur, carrying different hypercharge).
For the commutators of the $SL(2)$ generators with the remaining ones $L_0$ measures as usual the third component of the spin,

$$[L_0, T^a_{\rho k}] = -k T^a_{\rho k},$$

(2.2)

while $L_-$ and $L_+$ act as step operators, we define

$$[L_-, T^a_{\rho k}] = \sigma^-_{\rho k} T^a_{\rho k-1},$$

(2.3)

$$[L_+, T^a_{\rho k}] = \sigma^+_{\rho k} T^a_{\rho k+1},$$

(2.4)

with structure constants $\sigma^\pm_{\rho k} a$ in some convenient normalization. The remaining commutation relations are parametrized as

$$[T^a_{\rho k}, T^b_{\sigma l}] = \sigma_{\rho k \sigma l}^{\tau m} T^c_{\tau m} + \sigma_{\rho k \sigma l}^{b m} L_m.$$  

(2.5)

We define Lie algebra valued gauge potentials with respect to this decomposition,

$$A = A^k L_k + A^\rho_{ a} T^a_{\rho k},$$

(2.6)

which are differential forms in two dimensions

$$A = dz A_z(z, \bar{z}) + d\bar{z} A_{\bar{z}}(z, \bar{z}).$$

(2.7)

The gauge transformations are given as

$$gA = gAg^{-1} + gdg^{-1},$$

(2.8)

where the group elements $g$ depend on the parameters $\alpha^-, \alpha^0, \alpha^+$ for the $SL(2)$ subalgebra and $\alpha^\rho_{ a}$ for the remaining generators. The parameters are functions of $z$ and $\bar{z}$.

We define, as usual, covariant derivative and field strength. For $\Sigma(z, \bar{z})$ transforming as

$$g\Sigma = g\Sigma,$$

(2.9)

in some representation if the Lie group, the covariant derivative is

$$D\Sigma = d\Sigma + A\Sigma,$$

(2.10)

\footnote{In the general case, the $SL(2)$ decomposition makes it necessary to use this triple index notation, which we hope not to be too confusing, in particular for the structure constants $\sigma^-_{\rho k} T^a_{\rho k-1}$}
with $A$ in the appropriate representation. We will, in the sequel, frequently use the term *matter fields* for $\Sigma$. Applying the covariant exterior derivative once more one obtains

$$DD\Sigma = F\Sigma, \quad (2.11)$$

with

$$F = dA - AA, \quad (2.12)$$

the covariant field strength which satisfies Bianchi identities

$$dF = FA - AF, \quad (2.13)$$

as a consequence of the nilpotency of the exterior derivative.

The BRS symmetry, arising originally in the quantization of gauge theories, has an interpretation \cite{22,23} as a differential algebra with an additional grading, related to the appearance of the Lie algebra valued Faddeev-Popov ghost fields

$$\omega = \omega^k L_k + \omega^a \rho \, T^a \rho, \quad (2.14)$$

Correspondingly a nilpotent BRS operation is defined as

$$sA = -d\omega + A\omega + \omega A, \quad (2.15)$$

$$s\omega = \omega\omega, \quad (2.16)$$

and

$$s\Sigma = -\omega\Sigma. \quad (2.17)$$

The BRS grading is often referred to as ghost number. While the exterior derivative raises the form degree by one unit and leaves the ghost number unchanged, the BRS operator raises the ghost number by one unit and does not change the form degree. The nilpotency properties are $d^2 = 0$, $s^2 = 0$ and $ds + sd = 0$.

Hence this system of fields and derivations describes a gauge structure in a special basis of its Lie algebra in terms of a bigraded differential algebra.

It is important to observe that the gauge potentials have well defined conformal properties, $A_z$ and $A_{\bar{z}}$ are conformally covariant of weights $(1,0)$ and $(0,1)$, respectively. Under a conformal change of coordinates

$$z \mapsto w(z), \quad \bar{z} \mapsto \bar{w}(\bar{z}), \quad (2.18)$$
they change as

$$A_w = \frac{1}{w'} A_z, \quad A_{\bar{w}} = \frac{1}{\bar{w}'} A_{\bar{z}}.$$  \hfill (2.19)

In this local description the internal gauge symmetry and conformal transformations do not interfere with each other.

So far we have presented the standard gauge BRS structure mostly in order to fix our notations. We shall now describe a procedure which allows to identify for each $SL(2)$ spin occurring in the Lie algebra decomposition:

- a primary field $W^\rho_{a+2}$ of conformal weight $(a+2, 0)$ corresponding to the $\mathcal{W}$-currents,
- a conformally covariant field $v^\rho_{-a-1}$ of conformal weight $(-a-1, 1)$, corresponding to the $\mathcal{W}$-gauge fields,
- a ghost field $c_{-a-1}$ of weight $(-a-1, 0)$, conformally covariant as well, corresponding to $\mathcal{W}$-gauge transformations.

Together with the complete nilpotent BRS algebra realized on this set of fields and on those arising in the $SL(2)$ substructure itself, which are

- a projective connection $\Lambda_{zz}$ which ensures conformal covariance due to its inhomogeneous transformation law,
- a covariant $(-1, 1)$ differential $v^\rho z$, which was proposed to play the role of a Beltrami differential in refs. \cite{17}, \cite{18}, \cite{19},
- a ghost field $c^z$ of weight $(-1, 0)$ which has a certain resemblance with the diffeomorphism ghost arising in other contexts \cite{24}, \cite{25}.

The basic ingredients in this prescription are field dependent redefinitions of the gauge potentials which have the form of gauge transformations, highest weight parametrization, and zero curvature conditions. The $SL(2)$ substructure plays an important special role in the discussion of conformal covariance.

### 2.2 Highest weight and conformal parametrization

In a first step we consider a gauge transformation which depends on the parameter $\alpha^0$ only,

$$g_0 = e^{\alpha^0 L_0}.$$  \hfill (2.20)
Since $L_0$ measures the $SL(2)$ spin component, all the gauge fields transform covariantly according to

$$g_0 A^\rho {}_a = A^\rho {}_a e^{-k\alpha_0}, \quad (2.21)$$

except for $A^0$, the gauge potential pertaining to the generator $L_0$, which picks up an inhomogeneous term:

$$g_0 A^0 = A^0 - d\alpha^0. \quad (2.22)$$

In particular, for

$$A^- = dz A^-_z + d\bar{z} A^-_{\bar{z}}, \quad (2.23)$$

one has

$$g_0 A^- = A^- e^{\alpha^0}. \quad (2.24)$$

We define now what we will call the *conformal parametrization*. It consists in a redefinition on the set of gauge potentials which has the form of a gauge transformation, denoted $\hat{g}_0$ and chosen such that (cf. also [11]).

$$\hat{g}_0 A^-_z = 1, \quad (2.25)$$

i.e. $\hat{\alpha}^0 = -\log A^-_z$. In general, in the conformal parametrization, we define

$$\Gamma = \hat{g}_0 A. \quad (2.26)$$

This redefinition assigns now definite conformal weights to any of the gauge potentials due to the transformation properties of $A^-_z$ and the definition

$$\Gamma^\rho {}_a = A^\rho {}_a \left(A^-_z\right)^k, \quad (2.27)$$

i.e. $\Gamma^\rho {}_a$ has conformal weight $(k,0)$.

In the $SL(2)$ substructure itself the conformal parametrization is particularly relevant, there we define

$$\Gamma^- = dz + d\bar{z} \frac{A^-_z}{A^-_{\bar{z}}} \overset{\text{def}}{=} dz + d\bar{z} v_{\bar{z}}^z = v^z, \quad (2.28)$$

inducing a constant term which will be crucial in the subsequent investigations. At $k = 0$ inhomogeneous derivative terms appear,

$$\Gamma^0 = dz \left(A^0_\bar{z} + \partial_z \log A^-_{\bar{z}}\right) + d\bar{z} \left(A^0_z + \partial_{\bar{z}} \log A^-_z\right) \overset{\text{def}}{=} dz \chi_z + d\bar{z} \chi_{\bar{z}} = \chi, \quad (2.29)$$
and for $k = +1$ one obtains

$$\Gamma^+ = dz A_z^+ A_z^- + d\bar{z} A_{\bar{z}}^+ A_{\bar{z}}^- \overset{\text{def}}{=} dz \lambda_{zz} + d\bar{z} \lambda_{\bar{z}z} = \lambda_z. \quad (2.30)$$

In these equations we have used suggestive index notations to account for the *soldering* of internal $SL(2)$ and conformal properties. By construction, the new quantities appearing here are inert under $g^0$ gauge transformations. In exchange, they acquire well-defined conformal properties. For instance, $v_z^w$ is now a conformally covariant tensor of weight $(-1, 1),$

$$v_w^w = v_z^w \frac{w'}{w}, \quad (2.31)$$

$\lambda_{zz}$ is a quadratic differential of weight $(2, 0),$

$$\lambda_{ww} = \frac{1}{(w')^2} \lambda_{zz}, \quad (2.32)$$

whereas $\chi_z$ transforms inhomogeneously according to

$$\chi_w = \frac{1}{w'} \left( \chi_z - \frac{w''}{w'} \right). \quad (2.33)$$

Due to this particular transformation law, $\chi_z$ will play the role of a gauge potential - it will serve to define covariant derivatives with respect to conformal transformations. All the remaining gauge potential components are conformally covariant, their weights are determined from the respective index structure.

In analogy with the gauge potentials we define

$$c = \hat{g}_0 \hat{\omega} \hat{g}_0^{-1} + \hat{g}_0 \hat{s} \hat{g}_0^{-1}, \quad (2.34)$$

the conformal parametrization for the ghost fields (remember the gauge-like redefinition induced by $\hat{g}_0$ is field-dependent). With this definition the BRS transformations in the conformal parametrization take the form

$$s \Gamma = -dc + \Gamma c + c \Gamma, \quad (2.35)$$

$$s c = c c. \quad (2.36)$$

The ghost fields are decomposed as

$$c = c^k L_k + c^\rho {}^k_a T^a_{\rho k}, \quad (2.37)$$
with conformally covariant coefficients $c^k$ and $c^\rho a^k$ of weights $(k,0)$.

Similarly, for the fields $\Sigma$ we define the conformal parametrization

$$\Psi = \hat{g}_0 \Sigma, \quad (2.38)$$

with covariant derivative

$$\mathcal{D}(\Gamma)\Psi = d\Psi + \Gamma \Psi = \hat{g}_0 D(A)\Sigma, \quad (2.39)$$

and BRS transformation

$$s \Psi = -c\Psi. \quad (2.40)$$

So far we have established the conformal parametrization for the gauge potentials, the ghost sector and the matter fields. We come now back to the discussion of the gauge potentials where we impose, in a next step, the highest weight gauge condition

$$A^\rho z^a_k = 0, \quad \text{for} \quad -a - 1 \leq k \leq a, \quad (2.41)$$

i.e. all the $z$-components of the gauge potentials are constraint to be zero, except for the highest weight components at $k = a + 1$, where we define

$$W^\rho a+2 = \Gamma^\rho z^a a+1, \quad (2.42)$$

indicating that this component is a conformal tensor of weight $(a + 2,0)$. As to the existence of this highest weight gauge and its implications for residual gauge transformations we refer to [26] and [7] and references quoted there.

For the $\bar{z}$-components of the gauge potentials, on the other hand, we define at lowest weight, $k = -a - 1$, for each $SL(2)$ spin,

$$v^\rho z^a-a-1 = \Gamma^\rho z^a-a-1, \quad (2.43)$$

whereas for the $\mathcal{W}$-ghost fields at lowest weight we introduce the notation

$$c^\rho -a-1 = c^\rho a-a-1. \quad (2.44)$$

Recall that $v^\rho z^a-a-1$ are conformal $(-a-1,1)$ differentials and the ghosts $c^\rho -a-1$ have conformal weight$(-a-1,0)$. We also shall use the convention $c^- = c^z$ and $c^+ = c_z$ in the $SL(2)$ subsector to emphasize the conformal tensorial properties.
In the discussion at the end of this chapter we will see that zero field strength conditions in the conformal parametrization reduce the number of variables considerably. At the end one is left with

\[ v^\rho - a - 1 \bar{z}, \quad W^\rho a + 2, \quad c^\rho - a - 1 \bar{z}, \]

for each irreducible representation and with \( v^z, \chi_z, \lambda_{zz} \) and \( c^z, c_z \) for the \( SL(2) \) substructure. All the other components of the gauge and ghost fields will be recursively expressed in terms of these few basic variables and their conformally covariant derivatives. Moreover, the BRS algebra for these basic variables emerges and one derives expressions for \( \partial \bar{z} W^\rho_{a+2} \) in terms of conformally covariant operators and differential polynomials, with similar remarks applying for the sector of the covariantly constant fields \( \Psi \).

2.3 Projective parametrization

Before turning to the detailed discussion of zero curvature conditions we will now introduce the definition of what we call projective parametrization. This will again be a redefinition which has the form of a gauge transformation. It will have the effect to eliminate \( \chi_z \) and to introduce, at the same time, a projective connection \( \Lambda_{zz} \), replacing the quadratic differential \( \lambda_{zz} \). Likewise, the ghost \( c_z \) disappears from the set of independent variables. This goes as follows. We consider a gauge transformation which depends on the parameter \( \alpha^+ \) only,

\[ g^+ = e^{\alpha^+ L_+}. \] (2.45)

As a general property, this gauge transformation acts inside a given representation of \( SL(2) \) spin. Since \( L_+ \) is the positive step operator, \( g_+ \) acts always as a finite polynomial in \( \alpha^+ \) such that in the transformation law of a given component \( A^k \) only contributions of gauge potentials \( A^k \) with \( k' \leq k \) occur. It follows that the lowest weight fields \( v^z, c^z \) and \( v^\rho - a - 1, c^\rho - a - 1 \) are inert under those transformations. Moreover, in the highest weight gauge the fields \( W^\rho_{a+2} \) are invariant as well.

Let us first have a closer look to what happens in the \( SL(2) \) substructure itself. Since gauge transformations are defined on the original fields \( A_z, A_{\bar{z}} \), the transformation laws of the fields in the conformal parametrization are easily obtained from direct substitution in the relevant definitions. It is not hard to see that for \( \chi_z \) and \( \lambda_{zz} \) one obtains

\[ g^+ \chi_z = \chi_z + 2\eta_z, \] (2.46)
\[ g^+ \lambda_{zz} = \lambda_{zz} - \partial \bar{z} \eta_z + \chi_z \eta_z + \eta_z \eta_z, \] (2.47)
observing that the gauge parameter $\alpha + 1$ appears always in the combination
\[ \eta_z = \alpha z + 1, \] (2.48)
assigning conformal dimension $(1, 0)$ to the gauge parameter $\eta_z$ in the conformal parametrization. We define now the projective parametrization,
\[ \Pi = \hat{g}^+ \Gamma. \] (2.49)
It is obtained from the conformal parametrization through a redefinition which has the form of a gauge transformation
\[ \hat{g}^+ = g(0,0,\hat{\alpha} + 1), \] (2.50)
such that
\[ \hat{\eta}_z = \hat{\alpha} z + 1 = -\frac{1}{2} \chi_z. \] (2.51)
It is, of course, understood that the gauge transformations are evaluated on the original variables and then substituted in the conformal parametrization. Following this prescription one obtains
\[ \hat{g}^+ \chi_z = 0, \] (2.52)
\[ \hat{g}^+ \lambda_{zz} = \frac{1}{2} \Lambda_{zz} = \lambda_{zz} + \frac{1}{2} \partial_z \chi_z - \frac{1}{4} \chi_z \chi_z. \] (2.53)
As a consequence of the inhomogeneous transformations of $\chi_z$ under conformal transformations one finds that the combination
\[ \pi_{zz} = \partial_z \chi_z - \frac{1}{2} \chi_z \chi_z, \] (2.54)
transforms as a projective connection,
\[ \pi_{ww} = \frac{1}{(w')^2} \left( \pi_{zz} - \{ w, z \} \right), \] (2.55)
with a Schwarzian derivative
\[ \{ w, z \} = \left( \frac{w''}{w'} \right)' - \frac{1}{2} \left( \frac{w''}{w'} \right)^2, \] (2.56)
as inhomogeneous term. Since $\lambda_{zz}$ is a covariant quadratic differential, the combination
\[ \Lambda_{zz} = 2 \lambda_{zz} + \pi_{zz}, \] (2.57)
transforms as a projective connection as well.

This discussion shows that, in the projective parametrization, \( \chi_z \) disappears and \( \lambda_{zz} \) is replaced by the projective connection \( \Lambda_{zz} \). Moreover, as already remarked above, for each \( SL(2) \) spin, the lowest weight fields \( v_z^\rho \), \( v_z^{\rho - a - 1} \) and their ghosts \( c^\rho \) and \( c^{\rho - a - 1} \) as well as the highest weight fields \( W_{\rho a+2}^\rho \) do not change in the transition from the conformal to the projective parametrization, i.e. they remain conformally covariant. On the other hand, the remaining fields \( \chi_{\bar{z}}, \lambda_{\bar{z}z} \) and \( \Gamma_{\rho k}^k_{\bar{z} a} \) for \( k \geq -a \), as well as the corresponding ghosts are no longer conformally covariant when expressed in the conformal parametrization due to the appearance of \( \chi_z \) in the redefinitions. But these fields will be eliminated recursively by means of the zero curvature conditions as will be discussed below.

### 2.4 Zero curvature conditions

We turn now to the detailed discussion of the zero curvature conditions. On the one hand we shall argue in terms of the conformal parametrization, where conformal covariance is manifest due to the presence of \( \chi_z \) which behaves as a gauge potential under conformal transformation and appears in the recursive procedure such that at each step successively covariant derivatives emerge. This can be seen quite clearly in the explicit expressions for the field strength two-forms: in the conformal parametrization they read

\[
F^- = (d - \chi)v^z - \frac{1}{2}\Gamma^\rho_a k_{b} \Gamma^\sigma_l c^a b_{\rho k \sigma l}, \quad (2.58)
\]
\[
F^0 = d\chi + 2v^z \lambda_z - \frac{1}{2}\Gamma^\rho_a k_{b} \Gamma^\sigma_l c^a b_{\rho k \sigma l} - \lambda_z \Gamma^\tau m c_{- \rho k \sigma l}, \quad (2.59)
\]
\[
F^+ = (d + \chi)\lambda_z - \frac{1}{2}\Gamma^\rho_a k_{b} \Gamma^\sigma_l c^{a b}_{\rho k \sigma l} + \lambda_z \Gamma^\tau m c_{- \rho k \sigma l}, \quad (2.60)
\]

for the \( SL(2) \) substructure whereas for the remaining generators one obtains

\[
F^\tau m c_{\tau c} = (d + m\chi)\Gamma^\tau m c_{\tau c} - v^z \Gamma^\tau m\tau_{m+1} c_{\tau c} - \lambda_z \Gamma^\tau m\tau_{m-1} c_{\tau c} - \frac{1}{2}\Gamma^\rho a k_{b} \Gamma^\tau m\tau_{m+1} c_{\tau c} b_{\rho k \sigma l}, \quad (2.61)
\]

We note first of all the appearance of conformally covariant derivatives \( d + m\chi \). Moreover, we observe that the quadratic terms involving \( v^z \) actually have a linear piece due to the constant term in the definition of \( v^z = dz + d\bar{z} v_{\bar{z}} \).

For a given \( a \), these linear terms occur in all components \(-a - 1 \leq k \leq a\) of the field strengths except in the highest weight ones \( k = a+1 \). For vanishing field strength this means then that the coefficients \( \chi_{\bar{z}}, \lambda_{\bar{z}z} \) and \( \Gamma^k_{\bar{z} a} \) for \( k \geq -a \) are recursively expressed in terms of the basic covariant variables \( v_{\bar{z}}^z, \lambda_{zz} \) of the \( SL(2) \) subsector and \( v_{\bar{z}}^{\rho - a - 1}, W_{a+2}^\rho \) of the various spins occurring in
the decomposition. Conformal covariance is ensured at each step through the covariant derivatives involving $\chi_z$. The zero field strength conditions at the highest weights give then rise to equations which express $\partial_z \lambda_{zz}$ and $\partial_z W_{\rho a+2}$ in terms of those basic variables and their conformally covariant derivatives. In particular, for each $a$ a differential operator of order $2a + 3$ occurs which maps conformal weight $-a - 1$ into conformal weight $a + 2$, and which is of the form

\[ (\partial_z + (a + 1)\chi_z) \cdots (\partial_z - (a + 1)\chi_z) v_\xi^{-a-1}. \quad (2.62) \]

But there are also other contributions which have the form of differential polynomials in the basic variables. The explicit form of these "anholomorphicity equations" depends of course on the structure of the Lie algebra and the particular decomposition under consideration.

What happens in the ghost sector? To discuss this issue consider the explicit form of the BRS transformations, which, in the $SL(2)$ sector are given as

\[
\begin{align*}
sv^z &= -(d - \chi) c^z + v^z c^0 + \Gamma^\rho_{ k a} c^\sigma_{ b l} \sigma_{\rho k \sigma l} b^b - , \\

s\chi &= - dc + 2\lambda_z c^z - 2v^zc_z + \Gamma^\rho_{ k a} c^\sigma_{ b l} \sigma_{\rho k \sigma l} b^b , \\

s\lambda_z &= -(d + \chi) c_z + \lambda_z c^0 + \Gamma^\rho_{ k a} c^\sigma_{ b l} \sigma_{\rho k \sigma l} b^b + ,
\end{align*}
\]

and for the remaining generators take the form

\[
\begin{align*}
sg_{m c} &= -(d + m\chi) c_{m c} + m c^0 \Gamma_{m c} + \left( v^z c_{m+1 c} + c_z \Gamma_{m+1 c} \right) \sigma_{m+1 c} \\
&+ \left( \lambda_z c_{m-1 c} + c_z \Gamma_{m-1 c} \right) \sigma_{m-1 c} + \Gamma^\rho_{ k a} c^\sigma_{ b l} \sigma_{\rho k \sigma l} b^b ,
\end{align*}
\]

Here, a similar mechanism as before takes place. Note first of all, that these equations are differential one forms of BRS grade one. As a consequence each of these equations has a component in the direction $dz$ and another one in the direction $d\bar{z}$ both of ghost number one.

Let us first discuss the $SL(2)$ subsector. In its $dz$ component, the first of the three equations determines $c^0$ as a dependent variable, the second equation shows that the BRS transformation of $\chi_z$ has a term linear in $c_z$,

\[ s\chi_z = - 2c_z + \cdots , \quad (2.67) \]

and the third one determines $s\lambda_{zz}$. In the $d\bar{z}$ sector the first equation yields $s v_{\bar{z}}$, while the two other ones contain no new information. For the remaining generators, in the $dz$ components, the constant term allows to express the ghost fields $c^\rho_{ k a}$, for $k \geq -a$, in terms of the basic fields. The highest weight equations
in $dz$ determine then the BRS transformation of $W_{a+2}$. Here again, among other things, the differential operator of order $2a + 3$ shows up, acting on $c^\rho {}_a^k$. On the other hand, the $d\bar{z}$ components of these equations determine, at lowest weight, the BRS transformations of $v_{\bar{z}}^\rho {}_{-a-1}$, whereas all the other $d\bar{z}$ equations, i.e. for $k \geq -a$ should be identically satisfied with the information extracted so far.

It remains to discuss the equations at BRS grade two. In the $SL(2)$ subsector they read

$$
sc^z = c^0 c^z + \frac{1}{2} c^\rho {}_a^k c^\sigma {}_b^l \sigma^a {}_{\rho k} \sigma^b {}_{\sigma l} - ,
$$

(2.68)

$$
sc^0 = 2c_z c^z + \frac{1}{2} c^\rho {}_a^k c^\sigma {}_b^l \sigma^a {}_{\rho k} \sigma^b {}_{\sigma l}^0 ,
$$

(2.69)

$$
sc_z = c_z c^0 + \frac{1}{2} c^\rho {}_a^k c^\sigma {}_b^l \sigma^a {}_{\rho k} \sigma^b {}_{\sigma l}^+ ,
$$

(2.70)

where the first and the third equation determine the BRS operation on $c^z$ and $c_z$, respectively, while the second equation contains no new information. Likewise, for the remaining ghosts,

$$
sc^m {}^m c = c^z c^z {}^m c + \frac{1}{2} c^\rho {}_a^k c^\sigma {}_b^l \sigma^a {}_{\rho k} \sigma^b {}_{\sigma l}^m ,
$$

(2.71)

at each $a$, the lowest weight equations determine the BRS transformations of the independent ghost fields $c^\rho {}_{-a-1}$, and all the remaining equations are identically satisfied.

This completes the discussion of zero curvature conditions and the construction of a nilpotent BRS differential algebra in terms of the conformal parametrization and the highest weight gauge. The presentation was deliberately rather qualitative with the intention to explain rather the general structure than detailed quantitative features. Those can be most conveniently studied in explicit examples, which will be given in detail later on in this paper for the cases of $SL(2)$ and $SL(3)$.

One of the most important features on which we would like to insist here, however, is the conformal covariance of the whole construction, as a consequence of the presence of conformally covariant derivatives in terms of $\chi_z$.

In the transition from the conformal to the projective parametrization by means of the $\hat{g}^+$ redefinition, on the other hand, $\chi_z$ (as well as $c_z$, the corresponding ghost) disappear and the basic field and ghost variables left over after the recursive procedure of the zero curvature conditions are

$$
v_{\bar{z}}^z, \quad c^z, \quad \Lambda_{zz},
$$

(2.72)

in the $SL(2)$ subsector and

$$
v_{\bar{z}}^\rho {}_{-a-1}, \quad c^\rho {}_{-a-1}, \quad W_{a+2}^\rho ,
$$

(2.73)
for each $SL(2)$ spin occurring in the decomposition. All the fields are conformally covariant except $\Lambda_{zz}$, the projective connection.

What happens to conformal covariance in the projective parametrization? The answer is that conformal covariance is maintained in terms of the projective connection. This is due to the fact that the transition from the conformal to the projective parametrization consists in redefinitions which have the form of gauge transformations and leave therefore invariant the zeros in the curvature conditions.

As a consequence, the conformal covariance of the differential operators and differential polynomials occurring in the anholomorphicity relation for projective connection $\Lambda_{zz}$ and for the $W$-currents $W^\rho_{a+2}$ and in the BRS differential algebra is achieved solely in terms of $\Lambda_{zz}$.

In particular, one recovers, for each value of $a$, a covariant differential operator $\Delta^{2a+3}$ which maps $(-a-1,k)$-differentials into $(a+2,k)$-differentials \[27\], i.e.

\[ \Delta^{2a+3} : \mathcal{V}^{-a-1} \mapsto \mathcal{V}^{a+2}. \] (2.74)

So much for the general discussion, we shall turn now to more detailed descriptions of specific examples.

3 \textit{SL(2) Gauge Structure: The Corner-Stone}

As we have pointed out in the preceding general discussion, the Lie algebra $SL(2)$ plays a crucial role in the construction of $W$-gauge structures: a given Lie algebra can give rise to various different $W$-gauge structures according to different $SL(2)$ embeddings. The particular structure of the $SL(2)$ embedding is responsible for the special properties of a soldering of internal and conformal symmetries, especially the appearance of higher conformal spins.

It is therefore worthwhile to study first the case of $SL(2)$ itself in some detail. We shall consider here $SL(2)$-valued gauge potentials together with a doublet field transforming covariantly under gauge transformations. On this basic set of fields we will then rediscuss in full detail the properties of the conformal and the projective parametrizations, in particular the appearance of the projective connection, and the explicit form of the residual gauge transformations \[28\].

The zero curvature conditions express the anholomorphicity of the projective connection in terms of a conformally covariant differential operator, known also from the second Hamiltonian structure of the KdV hierarchy.

We consider then the zero curvature condition as \textit{integrability condition} for a covariantly constant doublet field. The condition of vanishing covariant
derivative determines one of the doublet fields in terms of the other one, which
is then subject to the (conformally covariant) Sturm-Liouville equation.

Finally we present the nilpotent differential BRS algebra, which strongly
resembles with the factorized diffeomorphism BRS algebra encountered in the
context of two dimensional conformal field theory and construct algebraically
the corresponding consistent anomaly.

3.1 Conformal and projective parametrizations

We start from a gauge potential one-form

\[ A = A^k L_k, \tag{3.1} \]

which takes its values in the Lie algebra of SL(2) with generators \( L_k, k = -1, 0, +1 \) and commutation relations

\[ [L_k, L_l] = (k - l) L_{k+l}. \tag{3.2} \]

Correspondingly the gauge transformations depend on three parameters \( \alpha^k(z, \bar{z}) \) and we denote an element of the gauge group \( g(\alpha^k) = g(\alpha^-, \alpha^0, \alpha^+). \)

In addition, we consider a doublet with respect to these SL(2) gauge trans-
formations, which we denote

\[ \Sigma = \begin{pmatrix} \Sigma_{+1/2} \\ \Sigma_{-1/2} \end{pmatrix}. \tag{3.3} \]

In this representation the generators are taken to be

\[ \lambda_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \lambda_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \lambda_+ = \begin{pmatrix} 0 & +1 \\ 0 & 0 \end{pmatrix}. \tag{3.4} \]

The covariant exterior derivative is defined as

\[ D(A) \Sigma = (d + A^k \lambda_k) \Sigma, \tag{3.5} \]

with the usual definition of field strength

\[ F(A) = dA - AA. \tag{3.6} \]

This is our basic set of classical fields.
In a first step, we consider now a special gauge transformation \( g^0 = g(0, \alpha^0, 0) = \exp(\alpha^0 L_0) \). On the gauge potentials themselves this gives rise to

\[
\begin{align*}
g^0 A^- &= A^- e^{+\alpha^0}, \\
g^0 A^0 &= A^0 - d\alpha^0, \\
g^0 A^+ &= A^+ e^{-\alpha^0},
\end{align*}
\]

whereas the doublet transforms as

\[
\begin{align*}
g^0 \Sigma^{+1/2} &= e^{-\alpha^0/2} \Sigma^{+1/2}, \\
g^0 \Sigma^{-1/2} &= e^{+\alpha^0/2} \Sigma^{-1/2}.
\end{align*}
\]

Taking into account the one-form nature of the gauge potentials, we consider in particular

\[
g^0 A^- = dz A^- e^{+\alpha^0} + d\bar{z} A^- e^{+\alpha^0},
\]

In view of these equations we will now perform a particular redefinition of the gauge potential components and of the doublet field, which has the form of such a gauge transformation, \textit{i.e.} for non-vanishing \( A^- \) we define the \textit{conformal parametrization}:

\[
\Gamma = g^0 A, \quad \Psi = g^0 \Sigma,
\]

of field dependent parameter such that

\[
\hat{g}^0 = g(0, \hat{\alpha}^0, 0), \quad \hat{\alpha}^0 = -\log A^-.
\]

Let us look at these redefinitions in some more detail. For the gauge potential components at \( k = -1 \),

\[
\Gamma^- = dz + d\bar{z} \frac{A^-}{A^-} \overset{\text{def}}{=} dz + d\bar{z} v_z \overset{\text{def}}{=} v^z,
\]

a constant term is induced. At \( k = 0 \), inhomogeneous derivative terms appear,

\[
\Gamma^0 = dz \left( A^0 + \partial_z \log A^- \right) + d\bar{z} \left( A^0 + \partial_{\bar{z}} \log A^- \right) \overset{\text{def}}{=} dz \chi_z + d\bar{z} \chi_{\bar{z}} = \chi,
\]

and for \( k = +1 \) one obtains

\[
\Gamma^+ = dz A^+_z A^-_z + d\bar{z} A^+_z A^-_z \overset{\text{def}}{=} dz \lambda_{zz} + d\bar{z} \lambda_{\bar{z}z} = \lambda_z.
\]
In these equations we have used suggestive notations to account for the soldering of internal $SL(2)$ and conformal properties. By construction, the new quantities appearing here are inert under $g^0$ gauge transformations. In exchange, they acquire well-defined conformal properties. For instance, $v^z_w$ is now a conformally covariant tensor of weight $(-1,1)$,

$$v^w_w = v^z_w \frac{w'}{w'},$$

(3.15)

$\lambda_{zz}$ is a quadratic differential of weight $(2,0)$,

$$\lambda_{ww} = \frac{1}{(w')^2} \lambda_{zz},$$

(3.16)

whereas $\chi_z$ transforms inhomogeneously,

$$\chi_w = \frac{1}{w'} \left( \chi_z - \frac{w''}{w'} \right).$$

(3.17)

Due to this particular transformation law, $\chi_z$ will play the role of a gauge potential, it will serve to define covariant derivatives with respect to conformal transformations. All the remaining fields are conformally covariant, their weights are determined from the respective index structure.

A similar soldering occurs for the matter fields, where we define

$$\Psi_{+1/2} = \sqrt{A^-} \Sigma_{+1/2} \overset{\text{def}}{=} \psi_\zeta, \quad \Psi_{-1/2} = \frac{1}{\sqrt{A^-}} \Sigma_{-1/2} \overset{\text{def}}{=} \psi^\zeta.$$  

(3.18)

Here greek indices are used to indicate the occurrence of half-integer conformal dimensions,

$$\psi_\omega = \frac{1}{\sqrt{w'}} \psi_\zeta, \quad \psi^{\omega} = \sqrt{w'} \psi^\zeta,$$

(3.19)

in other words, $\psi_\zeta$ is a $(\frac{1}{2},0)$ and $\psi^\zeta$ a $(-\frac{1}{2},0)$-differential.

Since the redefinitions used to define the conformal parametrization have the form of gauge transformations we define field strength and covariant derivative in the conformal parametrization as

$$\mathcal{F}(\Gamma) = \hat{g}^0 F(A), \quad \mathcal{D}(\Gamma)\Psi = \hat{g}^0 D(A) \Sigma.$$  

(3.20)

In some more detail, this yields

$$\mathcal{F}^{-} = dv^z + v^z \chi, \quad \mathcal{F}^{0} = d\chi + 2v^z \lambda_z, \quad \mathcal{F}^{+} = d\lambda_z - \lambda_z \chi,$$

(3.21)

(3.22)

(3.23)
for the field strength and
\begin{align*}
D\psi_\zeta &= \left(d - \frac{1}{2} \chi \right) \psi_\zeta + \lambda_z \psi_\zeta, \\
D\psi_\zeta' &= \left(d + \frac{1}{2} \chi \right) \psi_\zeta' - v^z \psi_\zeta,
\end{align*}
(3.24)
(3.25)
for the covariant derivatives.

In a second step we consider transformations which depend on the parameter \( \alpha^+ \) only, \( g^+ = g(0, 0, \alpha^+) = \exp(\alpha^+ L_+) \), defined on the original fields \( A_z, A_{\bar{z}} \) and \( \Sigma \). The transformation laws of the fields in the conformal parametrization are obtained from substitution in the relevant definitions. It is not hard to see that for \( v^z, \chi \) and \( \lambda_z \) one obtains
\begin{align*}
g^+ v^z &= v^z, \\
g^+ \chi &= \chi + 2 \eta_z v^z, \\
g^+ \lambda_z &= \lambda_z - d \eta_z + \eta_z \chi + \eta_z \eta_z v^z.
\end{align*}
(3.26)
The gauge parameter \( \alpha^+ \) appears always in the combination
\[ \eta_z = \alpha^+ A_z^-, \]
(3.27)
it acquires conformal dimension \((1, 0)\). Observe that \( v^z \) is invariant under these gauge transformation, and the same holds for \( \psi_\zeta' \).

The projective parametrization, defined as
\[ \Pi = \hat{g}^+ \Gamma, \quad \Psi_\pi = \hat{g}^+ \Psi, \]
(3.28)
is obtained from the conformal parametrization through a redefinition which has the form of a gauge transformation
\[ \hat{g}^+ = g(0, 0, \hat{\alpha}^+), \]
(3.29)
such that
\[ \hat{\eta}_z = \hat{\alpha}^+ A_z^- = -\frac{1}{2} \chi_z. \]
(3.30)
Using the explicit form of the \( g^+ \) gauge transformations given above one finds easily from the transformation laws (3.26)
\begin{align*}
\hat{g}^+ v_z z &= v_z z, \\
\hat{g}^+ \chi_z &= 0, \\
\hat{g}^+ \lambda_{zz} &= \lambda_{zz} + \frac{1}{2} \partial_z \chi_z - \frac{1}{4} \chi_z \chi_z \overset{\text{def}}{=} \frac{1}{2} \Lambda_{zz}.
\end{align*}
(3.31)
(3.32)
(3.33)
Clearly, the new variable $\Lambda_{zz}$ will transform inhomogeneously under conformal transformations. This is due to the appearance of the combination

$$\pi_{zz} = \partial_z \chi_z - \frac{1}{2} \chi_z \chi_z,$$  \hspace{1cm} (3.34)

Taking into account eq.\((3.17)\) for the transformation of $\chi_z$ one recognizes easily that $\pi_{zz}$ behaves in the same way as a projective connection, \textit{i.e.}

$$\pi_{ww} = \frac{1}{(w')^2} \left( \pi_{zz} - \{ w, z \} \right), \hspace{1cm} (3.35)$$

with a Schwarzian derivative

$$\{ w, z \} = \left( \frac{w''}{w'} \right)' - \frac{1}{2} \left( \frac{w''}{w'} \right)^2, \hspace{1cm} (3.36)$$

as inhomogeneous term. Since $\lambda_{zz}$ is a covariant quadratic differential, the combination

$$\Lambda_{zz} = 2 \lambda_{zz} + \pi_{zz}, \hspace{1cm} (3.37)$$

transforms as a projective connection as well.

So far we made use of the $g^0$ and $g^+$ gauge transformations to reduce the $SL(2)$ gauge structure to the conformal and the projective parametrizations. Recall that the fields in the conformal parametrization are inert under $g^0$ transformations while the fields in the projective parametrization are invariant under both, $g^0$ and $g^+$ transformations. What about the remaining residual gauge transformations of parameter $\alpha^-$?

Performing a gauge transformation $g^- = g(\alpha^-, 0, 0) = \exp(\alpha^- L_-)$ on the original variables and keeping track of the various redefinitions one learns that $\alpha^-$ always occurs in the combination

$$\eta^z = \frac{\alpha^-}{A_z}, \hspace{1cm} (3.38)$$

\textit{i.e.} it acquires conformal dimension $(-1, 0)$ and thus looks like a vector field. In terms of this parameter the corresponding residual transformations read then

$$g^- v^z_{\bar{z}} = \frac{1}{\Omega} \left( v^z_{\bar{z}} - \left( \partial_{\bar{z}} + \chi_{\bar{z}} \right) \eta^z + \lambda_{\bar{z}z} \eta^z \eta^z \right), \hspace{1cm} (3.39)$$

$$g^- \lambda_{zz} = \lambda_{zz} \Omega \hspace{1cm} (3.40)$$

$$g^- \chi_z = \chi_z - 2 \lambda_{zz} \eta^z + \frac{1}{\Omega} \partial_z \Omega. \hspace{1cm} (3.41)$$
with
\[ \Omega = 1 - (\partial_z + \chi_z) \eta^z + \lambda_{zz} \eta^z \eta^z, \] (3.42)

It is instructive to consider the infinitesimal version of these transformations. For \( v_z^z \) one obtains
\[ \delta_{g^-} : \ v_z^z \mapsto v_z^z - \partial_z \eta^z + v_z^z \partial_z \eta^z - \eta^z (\chi_z - v_z^z \chi_z), \] (3.43)

whereas the projective connection \( \Lambda_{zz} \) transforms as
\[ \delta_{g^-} : \ \Lambda_{zz} \mapsto \Lambda_{zz} - \eta^z \partial_z \Lambda_{zz} - 2\Lambda_{zz} \partial_z \eta^z - \partial_z \partial_z \partial_z \eta^z. \] (3.44)

For the doublet fields we obtain for a finite transformation
\[ g^- \psi^\zeta = \frac{1}{\sqrt{\Omega}} (\psi^\zeta - \eta^z \psi^\zeta), \] (3.45)
\[ g^- \psi^\zeta = \sqrt{\Omega} \psi^\zeta, \] (3.46)

while for the infinitesimal version
\[ \delta_{g^-} : \ \psi^\zeta \mapsto \psi^\zeta + \frac{1}{2} \psi^\zeta (\partial_z + \chi_z) \eta^z - \eta^z \psi^\zeta, \] (3.47)
\[ \delta_{g^-} : \ \psi^\zeta \mapsto \psi^\zeta - \frac{1}{2} \psi^\zeta (\partial_z + \chi_z) \eta^z. \] (3.48)

Although these are special \( SL(2) \) transformations we observe certain similarities with diffeomorphism transformations of parameter \( \eta^z \). This will become even more striking after having taken into account the zero curvature conditions in the next section (they will have the effect to replace \( \chi_z - v_z^z \chi_z \) by \( \partial_z v_z^z \)).

### 3.2 Zero curvature conditions

We consider now the doublet field \( \Sigma \) to be covariantly constant, \( i.e. \ D(A) \Sigma = 0 \). Applying covariant exterior derivative to this condition implies vanishing field strength, \( F(A) = 0 \). Since the transition to the conformal parametrization has the form of a gauge transformation, these conditions are invariant, and we shall investigate them here in the conformal parametrization:
\[ D(\Gamma) \Psi = 0, \quad F(\Gamma) = 0. \] (3.49)

In some more detail, for the doublet \( \Psi \), the conditions of zero covariant derivative read
\[ (d - \frac{1}{2} \chi) \psi^\zeta + \lambda_z \psi^\zeta = 0, \] (3.50)
\[ (d + \frac{1}{2} \chi) \psi^\zeta - v^z \psi^\zeta = 0, \] (3.51)
whereas the zero curvature conditions for the gauge potentials are

\[ dv^z + v^z \chi = 0, \quad (3.52) \]
\[ d\chi + 2v^z \lambda_z = 0, \quad (3.53) \]
\[ d\lambda_z - \lambda_z \chi = 0. \quad (3.54) \]

Since \( v^z \) contains a constant term, three of the above equations allow to express certain fields in terms of others and their derivatives. For the doublet this yields

\[ \psi_\zeta = \left( \partial_z + \frac{1}{2} \chi_z \right) \psi^\zeta, \quad (3.55) \]

while for the gauge potentials, the coefficients \( \chi_{\bar{z}} \) and \( \lambda_{\bar{z}z} \) can be expressed as

\[ \chi_{\bar{z}} = (\partial_z + \chi_z) v_{\bar{z}z}, \quad (3.56) \]
\[ \lambda_{\bar{z}z} = v_{\bar{z}z}\lambda_{zz} + \frac{1}{2} \partial_z (\partial_z + \chi_z) v_{\bar{z}z} - \frac{1}{2} \partial_{\bar{z}} \chi_z. \quad (3.57) \]

Observe that all the derivatives occurring here are conformally covariant thanks to the inhomogeneous transformation law of \( \chi_z \).

Substitution of these expressions in the other equations gives then rise to differential expressions involving the remaining basic fields \( v_{\bar{z}z}, \chi_z, \lambda_{zz} \) and \( \psi^\zeta \). As to the latter, the zero covariant derivative conditions yield

\[ \partial_{\bar{z}} \psi^\zeta = v_{\bar{z}z} \partial_z \psi^\zeta - \frac{1}{2} \psi^\zeta \partial_z v_{\bar{z}z}, \quad (3.58) \]

and

\[ \left( \partial_z - \frac{1}{2} \chi_z \right) \left( \partial_z + \frac{1}{2} \chi_z \right) \psi^\zeta + \lambda_{zz} \psi^\zeta = 0. \quad (3.59) \]

Again, the differential operator appearing here is conformally covariant. Straightforward manipulation shows that it can be rewritten in terms of \( \pi_{zz} \):

\[ \left( \partial_z - \frac{1}{2} \chi_z \right) \left( \partial_z + \frac{1}{2} \chi_z \right) = \partial_z \partial_z + \frac{1}{2} \pi_{zz}, \]

and one obtains the conformally covariant Sturm-Liouville equation

\[ \left( \partial_z \partial_z + \frac{1}{2} \Lambda_{zz} \right) \psi^\zeta = 0, \quad (3.60) \]

in terms of the second-order differential operator

\[ \Delta^{(2)} = \partial_z \partial_z + \frac{1}{2} \Lambda_{zz}. \quad (3.61) \]

Absorbing \( \chi_z \) in a redefinition of \( \Lambda_{zz} \) amounts to pass from the conformal to the projective parametrization, note also that this particular redefinition has the
form of a Miura transformation \([29]\). The covariance of the resulting equation is ensured by construction. The differential operator \(\Delta^{(2)}\) provides a map from the space \(\mathcal{V}^{-1/2}\) of covariant \((-\frac{1}{2}, k)\)-differentials into \(\mathcal{V}^{+3/2}\), the space of covariant \((\frac{3}{2}, k)\)-differentials:

\[
\Delta^{(2)} : \mathcal{V}^{-1/2} \mapsto \mathcal{V}^{+3/2}.
\] (3.62)

On the other hand, in the zero curvature conditions, substituting for \(\chi_{\bar{z}}\) and \(\lambda_{\bar{z}z}\) in the third equation yields, with very little algebraic effort, the conformally covariant equation

\[
\partial_{\bar{z}} \left( \lambda_{zz} + \frac{1}{2} \partial_{\bar{z}} \chi_{z} - \frac{1}{4} \chi_{z} \chi_{z} \right) = 2 \lambda_{zz} v_{\bar{z} z} + v_{\bar{z} z} \partial_{\bar{z}} \lambda_{zz} + \frac{1}{2} (\partial_{\bar{z}} - \chi_{z}) \partial_{\bar{z}} (\partial_{\bar{z}} + \chi_{z}) v_{\bar{z} z}.
\] (3.63)

Again, a straightforward reshuffling in the third order differential operator gives rise to

\[
(\partial_{\bar{z}} - \chi_{z}) \partial_{\bar{z}} (\partial_{\bar{z}} + \chi_{z}) v_{\bar{z} z} = (\partial_{\bar{z}} \partial_{\bar{z}} \partial_{\bar{z}} + \partial_{\bar{z}} \cdot \pi_{zz} + \pi_{zz} \partial_{\bar{z}}) v_{\bar{z} z},
\] (3.64)

absorbing again \(\chi_{z}\) in the same redefinition as already encountered above, reflecting the transition from the conformal to the projective parametrization by means of a Miura transformation of \(\lambda_{zz}\). As a result we are simply left with

\[
\partial_{\bar{z}} \Lambda_{zz} = (\partial_{\bar{z}} \partial_{\bar{z}} \partial_{\bar{z}} + \partial_{\bar{z}} \cdot \Lambda_{zz} + \Lambda_{zz} \partial_{\bar{z}}) v_{\bar{z} z},
\] (3.65)

where the third order differential operator

\[
\Delta^{(3)} = \partial_{\bar{z}} \partial_{\bar{z}} \partial_{\bar{z}} + \partial_{\bar{z}} \cdot \Lambda_{zz} + \Lambda_{zz} \partial_{\bar{z}},
\] (3.66)

maps covariant \((-1, k)\)-differentials into covariant \((2, k)\)-differentials:

\[
\Delta^{(3)} : \mathcal{V}^{-1} \mapsto \mathcal{V}^{+2}.
\] (3.67)

Observe that the same differential operator appears also in the second hamiltonian structure of the KdV equation. These intriguing structures gave rise to investigations concerning possible relations between the anomalous conservation equation of the energy momentum tensor in two dimensional conformal theory and the KdV hierarchy \([17], [18]\).

As already anticipated above, after taking into account the zero curvature conditions, the infinitesimal \(\eta^z\) transformations of \(v_{\bar{z} z}\), \(\Lambda_{zz}\), and \(\psi^z\) acquire a form which is very similar to diffeomorphisms. This is in particular the case for
which transforms under $\eta^z$ transformations in the same way as the Beltrami differential transforms under changes of coordinates, \textit{i.e.}

$$\delta_{g^-} : \ v^z \mapsto v^z + \partial_z \eta^z + v^z \partial_z \eta^z - \eta^z \partial_z v^z, \quad (3.68)$$

the transformation of the projective connection has already been given above,

$$\delta_{g^-} : \ \Lambda_{zz} \mapsto \Lambda_{zz} - \eta^z \partial_z \Lambda_{zz} - 2\Lambda_{zz} \eta^z - \partial_z \partial_z \eta^z. \quad (3.69)$$

Finally, using (3.55), we obtain for the doublet field

$$\delta_{g^-} : \ \psi^\zeta \mapsto \psi^\zeta - \eta^z \partial_z \psi^\zeta + \frac{1}{2} \psi^\zeta \partial_z \eta^z. \quad (3.70)$$

The analogues of these gauge structures for other Lie algebras are sometimes called $\mathcal{W}$-diffeomorphisms, in the $\text{SL}(2)$ case one might call them $\mathcal{W}_2$-diffeomorphisms. A very convenient way to treat these structures is in terms of BRS differential algebra, the subject of the next section.

### 3.3 The BRS differential algebra and anomaly structure

As is well known, the BRS differential algebra of gauge and ghost fields has a compact formulation in terms of the generalized objects

$$\tilde{A} = A + \omega, \quad (3.71)$$

which unifies gauge and ghost fields, and the generalized nilpotent derivation

$$\tilde{d} = d + s, \quad (3.72)$$

unifying exterior derivative and the BRS operation. These generalized quantities are then used to define a generalized field strength

$$\tilde{F}(\tilde{A}) = \tilde{d} \tilde{A} - \tilde{A} \tilde{A}, \quad (3.73)$$

and a generalized covariant derivative

$$\tilde{D}(\tilde{A}) \Sigma = (\tilde{d} + \tilde{A}) \Sigma. \quad (3.74)$$

The BRS transformations of gauge and ghost fields are then recovered from the horizontality conditions \[23\]

$$\tilde{F}(\tilde{A}) = F(A), \quad (3.75)$$

24
which, for the matter fields take the form

\[ \tilde{D}(\tilde{A}) \Sigma = D(A) \Sigma. \]  

(3.76)

To be definite, we shall work in the conformal parametrization defined as

\[ \tilde{\Gamma} = g^0 \tilde{A}, \]  

(3.77)

and, in accordance with previous notations, we define

\[ \tilde{v}^z = v^z + c^z, \quad \tilde{\chi} = \chi + c, \quad \tilde{\lambda}_z = \lambda_z + c_z. \]  

(3.78)

Also, we shall work right away with the zero curvature conditions, such that the horizontality equations become

\[ \tilde{d} \tilde{v}^z + \tilde{v}^z \tilde{\chi} = 0, \]  

(3.79)

\[ \tilde{d} \tilde{\chi} + 2 \tilde{v}^z \tilde{\lambda}_z = 0, \]  

(3.80)

\[ \tilde{d} \tilde{\lambda}_z - \tilde{\lambda}_z \tilde{\chi} = 0. \]  

(3.81)

Going through this set of equations at ghost number one allows, first of all, to express the dependent variables

\[ c = (\partial_z + \chi_z)c^z, \]  

(3.82)

\[ c_z + \frac{1}{2} s \chi_z = \frac{1}{2} \partial_z c + c^z \lambda_{z2}. \]  

(3.83)

Given this information, the remaining equations reduce then to a BRS differential algebra, with \( s^2 = 0 \), which closes on the basic variables \( v^z_z, c^z \) and \( \Lambda_{z2} \) in the following simple way (see also [30]):

\[ sv^z_z = \partial_z c^z - v^z_z \partial_z c^z + c^z \partial_z v^z_z, \]  

(3.84)

\[ sc^z = -c^z \partial_z c^z, \]  

(3.85)

\[ s \Lambda_{z2} = (\partial_z \partial_z \partial_z + \partial_z \Lambda_{z2} + \Lambda_{z2} \partial_z) c^z. \]  

(3.86)

This differential algebra reproduces exactly the infinitesimal action, see eqs.(3.68) and (3.69), of the residual transformations.

Similarly, for the covariantly constant doublet field in the conformal parametrization one obtains

\[ (\tilde{d} - \frac{1}{2} \tilde{\chi}) \psi_\xi + \tilde{\lambda}_z \psi_\xi = 0, \]  

(3.87)

\[ (\tilde{d} + \frac{1}{2} \tilde{\chi}) \psi_\xi - \tilde{v}^z \psi_\xi = 0. \]  

(3.88)
Using the relations derived so far, in particular (3.55) and (3.83), this set of equations reduces to the BRS transformation of $\psi^\zeta$,

$$s\psi^\zeta = c^z \partial_z \psi^\zeta - \frac{1}{2} \psi^\zeta \partial_z c^z$$

(3.89)

clearly exhibiting the conformal nature of the field $\psi^\zeta$ as a $(-\frac{1}{2}, 0)$-differential (cf. eq.(3.70)).

This concludes our discussion of the differential BRS algebra of gauge and ghost fields in presence of the zero curvature conditions and of covariantly constant matter fields. We will use the notion of $W_2$-gauge structure for the set of fields

$$v^z, \quad \Lambda_{zz}, \quad c^z, \quad \psi^\zeta,$$

(3.90)
subject to the BRS transformations just derived and to the equations

$$\partial_z \Lambda_{zz} = (\partial_z \partial_z \Lambda_{zz} + \partial_z \cdot \Lambda_{zz} + \Lambda_{zz} \partial_z) v^z,$$

(3.91)

and

$$\partial_z \psi^\zeta = v^z \partial_z \psi^\zeta - \frac{1}{2} \psi^\zeta \partial_z v^z,$$

(3.92)

together with

$$\left( \partial_z \partial_z + \frac{1}{2} \Lambda_{zz} \right) \psi^\zeta = 0,$$

(3.93)
arising from the conditions of vanishing curvature and covariant derivative.

We come now to the discussion of possible anomalies as solutions of the consistency conditions. That is one asks for a local functional $A^{(1)}_{zz}$, which should be a $(1, 1)$ differential of ghost number one, constructed in terms of the set of basic fields, in our present case $v^z$, $\Lambda_{zz}$ and $c^z$, and which is closed under BRS transformations up to total derivatives, i.e.

$$s A^{(1)}_{zz} = \partial_z A^{(2)}_{zz} + \partial_z A^{(2)}_{zz}.$$  

(3.94)

It is rather easy to see that the expression

$$A_{zz} = c^z \partial_z \Lambda_{zz} - v^z s \Lambda_{zz} = c^z \Delta^{(3)} v^z - v^z \Delta^{(3)} c^z,$$

(3.95)

provides indeed a solution to the consistency conditions and an explicit computation allows to identify

$$A^{(2)}_{zz} = c^z \Delta^{(3)} c^z,$$

(3.96)

$$A^{(2)}_{zz} = (c^z \partial_z v^z - v^z \partial_z c^z) \partial_z \partial_z c^z - c^z \partial_z c^z \partial_z \partial_z v^z.$$  

(3.97)

⁴In fact, formally this equation might be read as $\Lambda = \partial \Sigma - \frac{1}{2} \Sigma \Sigma$, with $\Sigma = -2 \frac{1}{2} \partial \psi$. 

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We wish to emphasize that \( A_{zz}^{(1)} \) as well as \( A_{z}^{(2)} \) and \( A_{\bar{z}}^{(2)} \) are conformally covariant tensors. Also, the formal similarity with the factorized conformal anomaly (in terms of a Beltrami differential and a background holomorphic projective connection) appears quite clearly [13, 31].

Solving the consistency condition is not enough to characterize an anomaly, it must also be nontrivial: one has to convince oneself that it is not possible to express the solution given here as the BRS variation of a local functional in the basic fields \( v_{z\bar{z}}, \Lambda_{zz} \) and \( c^z \) up to derivative terms. More explicitly one has to show that
\[
A_{zz}^{(1)} \neq sB_{zz}^{(0)} + \partial_{z}B_{z}^{(1)} + \partial_{\bar{z}}B_{\bar{z}}^{(1)}. \tag{3.98}
\]
This can indeed be confirmed by explicit inspection of possible counterterms, taking into account the restrictions arising from the index structures (viz. conformal weights) and the polynomial form and degrees of derivatives in the expression for the anomaly.

On the other hand, as to the BRS transformations of the ghost number two partners of the anomaly, an explicit computation shows that
\[
sA_{z}^{(2)} = \partial_{z}A^{(3)}, \tag{3.99}
\]
\[
sA_{\bar{z}}^{(2)} = \partial_{\bar{z}}A^{(3)}, \tag{3.100}
\]
with
\[
A^{(3)} = c^z \partial_{z}c^z \partial_{\bar{z}} \partial_{\bar{z}}c^z, \tag{3.101}
\]
which is a conformally invariant tensor of ghost number three. This sequence of \( s \) modulo \( d \) equations can be compactly summarized in terms of descend equations, again in striking analogy with the usual conformal anomaly (ch. III.2 in [13]). To this end we define
\[
\tilde{a} = a_{1} + a_{2} + a_{3}, \tag{3.102}
\]
with obvious reference of the indices to form degree and BRS grading (ghost number) and identify
\[
a_{1} = -\frac{1}{2} d_{z} \wedge d_{z} A_{zz}^{(1)}, \tag{3.103}
a_{2} = \frac{1}{2} d_{z} A_{z}^{(2)} - \frac{1}{2} d_{\bar{z}} A_{\bar{z}}^{(2)}, \tag{3.104}
a_{3} = \frac{1}{2} A^{(3)}. \tag{3.105}
\]
In this notation (the factors of one half are for later convenience) we obtain the descend equations in the form
\[
s a_{2}^{1} + d a_{1}^{2} = 0, \quad s a_{1}^{2} + d a_{3}^{3} = 0, \quad s a_{3}^{3} = 0, \tag{3.106}
\]
or, even more compactly,
\[ \tilde{d} \tilde{A} = 0. \quad (3.107) \]

In our approach, \( \mathcal{W} \)-gauge structures are obtained from the usual BRS gauge structure of \( SL(2) \) valued Yang-Mills gauge potentials and their ghosts via zero curvature conditions in combination with the conformal, and, finally, projective parametrization. In the following we would like to point out that the anomaly obtained above in terms of the basic variables of the projective parametrization fits into this picture as well. It can indeed be related to the usual construction of the Yang-Mills anomaly via descend equations \[23, 32\]. Recall that there a (nontrivial) solution to the consistency conditions is identified as the ghost number one component of the generalized Chern-Simons form, constructed from \( \tilde{A} = \tilde{A}^k \lambda_k \) and \( \tilde{d} \), i.e. the generalized three form
\[ \tilde{\mathcal{Q}}(\tilde{\Gamma}) = \text{tr} \left( \tilde{A} \tilde{d} \tilde{A} - \frac{2}{3} \tilde{A} \tilde{A} \tilde{A} \right), \quad (3.108) \]
with contributions
\[ \tilde{\mathcal{Q}} = \mathcal{Q}^1_1 + \mathcal{Q}^2_1 + \mathcal{Q}^3_0, \quad (3.109) \]
at various levels of ghost number (upper index) and form degree (lower index). The consistent anomaly is then identified in \( \mathcal{Q}^1_2 \).

Keeping in mind that the transition from the original \( SL(2) \) gauge structure to the conformal parametrizations has the form of a gauge transformation, \( \tilde{\Gamma} = g^0 \tilde{A} \), one has
\[ \tilde{\mathcal{Q}}(\tilde{\Gamma}) = \tilde{\mathcal{Q}}(\tilde{\Gamma}) + \tilde{d} \tilde{\Theta}, \quad (3.110) \]
due to the well-known fact (the so-called triangular equation \[32\]) that the Chern-Simons form changes by the exterior derivative of a two form under gauge transformations. In our particular case it is straightforward to obtain explicitly
\[ \tilde{\Theta} = \frac{1}{2} \tilde{\chi} \frac{1}{A_z^+} \tilde{d} A_z^- \quad (3.111) \]
Moreover, since we are dealing here with flat gauge potentials, the Chern-Simons three form reduces to
\[ \tilde{\mathcal{Q}}(\tilde{\Gamma}) = \frac{1}{3} \text{tr} \left( \tilde{\Gamma} \tilde{d} \tilde{\Gamma} \right) = \frac{1}{3} \text{tr} \left( \tilde{\Gamma} \tilde{\Gamma} \tilde{\Gamma} \right), \quad (3.112) \]
and it is easy to convince oneself that, in the conformal parametrization (cf. also [33], [34]),

\[ \tilde{Q}(\tilde{\Gamma}) = \tilde{v}^z \tilde{\lambda}_z. \] (3.113)

Finally, an explicit calculation shows that

\[ \tilde{v}^z \tilde{\lambda}_z = \tilde{\Lambda} + \tilde{d} \tilde{\Xi}, \] (3.114)

where the trivial contributions

\[ \tilde{\Xi} = \Xi^0 + \Xi^1 + \Xi^2, \] (3.115)

are given in terms of the variables of the conformal parametrization as follows:

\[ \Xi^0 = -\frac{1}{2} dz \wedge d\bar{z} (\partial_z \partial_{\bar{z}} v_z v_{\bar{z}}^z + \chi_z \partial_z v_{\bar{z}}^z + 2 v_z v_{\bar{z}}^z \Lambda_{zz}), \] (3.116)

\[ \Xi^1 = -\frac{1}{2} dz (\partial_z \partial_{\bar{z}} c^z + \chi_z \partial_z c^z + 2 c^z \Lambda_{zz}) + \frac{1}{2} d\bar{z} \chi_z (c^z \partial_z v_{\bar{z}}^z - v_z \partial_z c^z), \] (3.117)

\[ \Xi^2 = -\frac{1}{2} \chi_z c^z \partial_z c^z. \] (3.118)

In fact, this decomposition amounts to the explicit transition from the conformal to the projective parametrization (which, as should be kept in mind, is provided by a field redefinition which has the form of a particular gauge transformation on the original $SL(2)$ variables).

To summarize, we have established the explicit relation

\[ \tilde{Q}(\tilde{A}) = \frac{1}{3} \text{tr} (\tilde{A} \tilde{A} \tilde{A}) = \tilde{\Lambda} + \tilde{d} (\tilde{\Theta} + \tilde{\Xi}). \] (3.119)

By definition, as a generalized differential three-form, $\tilde{Q}(\tilde{A})$ is a conformally invariant quantity. On the other hand, as a result of our explicit construction, $\tilde{\Lambda}$ is invariant under conformal transformations as well. As a consequence, the counterterms

\[ \tilde{U} = \tilde{\Xi} + \tilde{\Theta}, \] (3.120)

should change under a conformal transformation as

\[ \tilde{U}(w) = \tilde{U}(z) + \tilde{d} \tilde{u}, \] (3.121)

with $\tilde{u} = u_1^0 + u_1^1$. Indeed, an explicit calculation yields

\[ u_1^0 = -\frac{1}{2} \left( dz \log A_z^- + d\bar{z} v_{\bar{z}}^z \right) \frac{w''}{w'}, \] (3.122)

\[ u_1^1 = -\frac{1}{2} c^z \frac{w''}{w'}. \] (3.123)
4 \textit{SL}(3) AND $\mathcal{W}_3$-GAUGE STRUCTURES

We shall discuss now the case of $\textit{SL}(3)$ as one of the simplest examples which nevertheless illustrates already the most important features relevant to the construction in the general case.

First of all, and as is well known, $\textit{SL}(3)$ allows for two different $\textit{SL}(2)$ decompositions. The first one, which gives rise to what is usually called $\mathcal{W}_3^{(1)}$, is related to the principal $\textit{SL}(2)$ embedding, in this case the eight generators of $\textit{SL}(3)$ are split into three plus five (i.e. $\textit{SL}(2)$ spin two).

The $\mathcal{W}_3^{(2)}$-gauge structure, on the other hand, is based on a decomposition where the splitting is $3 + 2 + 2 + 1$, in other words two spin 1/2 doublets and a singlet in addition to the $\textit{SL}(2)$ generators, this decomposition is quite familiar in elementary particle physics since the days of the eight-fold way.

We shall separately examine these two possibilities in full detail, along the lines of the general discussion presented in the beginning of this paper.

Particular emphasis will be on the construction of the complete nilpotent BRS algebra, which may be understood to represent the infinitesimal $\mathcal{W}$-transformations and their commutators. Moreover, this differential algebra will be used to determine explicitly anomalies, both for $\mathcal{W}_3^{(1)}$ and for $\mathcal{W}_3^{(2)}$, as solutions of the consistency conditions.

Finally, the properties of covariantly constant matter fields will be presented in detail.

The presentation will proceed in two parallel tracks for the two different embeddings. In both cases we shall, after some motivating remarks, present the results right away in the projective parametrization.

4.1 $\mathcal{W}_3^{(1)}$-gauge structure

In the language of the general discussion of chapter 2, the $\textit{SL}(3)$ Lie algebra is here represented in terms of a set of three generators $L_k, k = -1, 0, +1$, for the $\textit{SL}(2)$ subalgebra and another set of five generators $T_{\rho k}$ with $a = 1$ and $m = -2, -1, 0, +1, +2$, representing spin two with respect to the $\textit{SL}(2)$ subalgebra. Since this representation of dimension five occurs just once, the label $\rho$ can be neglected and we denote these generators simply $T_m$ (omitting the index $a = 1$ as well). The commutation relations in terms of this decomposition are then given as

\begin{align*}
[L_k, L_l] &= (k - l) L_{k+l}, \quad (4.1) \\
[L_k, T_m] &= (2k - m) T_{m+k}, \quad (4.2)
\end{align*}
\[ [T_m, T_n] = -\frac{1}{3} (m - n) (2m^2 + 2n^2 - mn - 8) L_{m+n}. \quad (4.3) \]

Following the general procedure we define the gauge potential one-form and the ghost fields in this decomposition as

\[ A = A^k L_k + A^m T_m, \quad (4.4) \]
\[ \omega = \omega^k L_k + \omega^m T_m, \quad (4.5) \]

and introduce covariant matter fields as a triplet of \( SL(3) \),

\[ \Sigma = \begin{pmatrix} \Sigma_+ \\ \Sigma_0 \\ \Sigma_- \end{pmatrix}. \quad (4.6) \]

In this three-dimensional representation we use the following \( 3 \times 3 \) matrices for the generators:

\[ L_\pm = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.7) \]

The explicit form of the matrices \( L_k \) shows that \( \Sigma \) is indeed a triplet with respect to the \( SL(2) \) subalgebra (for notational simplicity we use the same symbols for the generators and their specific matrix realization).

As outlined in the general discussion, the \textit{conformal parametrization} is obtained from this set of gauge, ghost and matter fields by means of redefinitions, \((2.26), (2.37), (2.38)\), which have the form of a \( \alpha^0 \) gauge transformation with the parameter identified as \( \alpha^0 = -\log A_\pm \).

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As we have seen, these redefinitions assign well defined conformal properties. Recall that the gauge potentials in the $SL(2)$ subsector, cf. (2.28), (2.29) and (2.30), become

$$v^z = dz + d\bar{z} \, v_{\bar{z}}^z, \quad (4.8)$$
$$\chi = dz \chi_z + d\bar{z} \, \chi_{\bar{z}}, \quad (4.9)$$
$$\lambda_z = dz \lambda_{zz} + d\bar{z} \, \lambda_{\bar{z}z}. \quad (4.10)$$

They play a particularly important rôle. For the remaining five components of the $SL(3)$ gauge potential in the conformal parametrization the definitions are

$$\Gamma^m_1 = \delta^0 A^m_1 = A^m_1 (A^-_z)^m = dz \Gamma^m_{z \bar{1}} + d\bar{z} \, \Gamma^m_{\bar{z} 1}. \quad (4.11)$$

assigning conformal weights $(m,0)$ to $\Gamma^m_1$ and, as a consequence, conformal weights $(m+1,0)$ to $\Gamma^m_{z \bar{1}}$ and $(m,1)$ to $\Gamma^m_{\bar{z} 1}$.

As a next step we impose the highest weight constraints,

$$\Gamma^m_{z \bar{1}} = 0, \quad m = -2, -1, 0, +1, \quad (4.12)$$

and we denote

$$\Gamma_{z \bar{1}}^{-2} \overset{\text{def}}{=} W_3, \quad (4.13)$$

the remaining nonzero component, which has conformal weight $(3,0)$. The $\bar{z}$ components remain arbitrary, for the moment. However, the relevant quantity here (which will survive after the zero-curvature conditions) is

$$\Gamma_{z \bar{1}}^{-2} \overset{\text{def}}{=} v_{z \bar{z}}, \quad (4.14)$$

a conformal tensor of weight $(-2,1)$.

The projective parametrization is defined as a redefinition of the gauge potentials which has the form of a $\alpha^\pm$ gauge transformation (see eqs. (2.49) ff). In the $SL(2)$ subsector it has the effect to eliminate $\chi_z$ at the expense of the appearance of the projective connection $\Lambda_{zz}$. In other words, the projective parametrization establishes explicitly the highest weight gauge in the $SL(2)$ subsector. It is important to note that the fields $v_{z \bar{z}}$, $v_{z \bar{z}z}$ and $W_3$ remain unchanged in the transition from the conformal to the projective parametrization (the latter, $W_3$, due to the highest weight constraints), they remain covariant conformal tensors. The other components, $\Gamma_{z \bar{1}}^m$, for $m \geq -1$, will receive additional contributions in terms of $\chi_z$ and its derivatives and therefore acquire
non-covariant conformal transformations. However, those quantities will recursively disappear once the zero curvature conditions are imposed. The mechanism of this recursion procedure has been explained in the general discussion of chapter 2, for the case at hand we have to consider the three equations

\[
\mathcal{F}^- = dv^z + v^z \chi, \tag{4.15}
\]

\[
\mathcal{F}^0 = d\chi + 2v^z \lambda_z + 16 \Gamma_1^{+2} \Gamma_1^{-2}, \tag{4.16}
\]

\[
\mathcal{F}^+ = d\lambda_z - \lambda_z \chi + 4 \Gamma_1^{+2} \Gamma_1^{-1}, \tag{4.17}
\]

for the \( SL(2) \) part (note the appearance of the additional terms with \( \Gamma_m \) relative to the pure \( SL(2) \) case) and five equations

\[
\mathcal{F}_m = d\Gamma_m^m + m\chi \Gamma_m^m + (m + 3)v^z \Gamma_m^{m+1} + (m - 3)\lambda_z \Gamma_m^{m-1} \tag{4.18}
\]

for the spin two sector, corresponding to the values \( m = -2, -1, 0, +1, +2 \). In the zero curvature conditions two of the first three equations are recursion relations while the third one, after substitution gives rise to the equation

\[
\partial_z \Lambda_{zz} = \Delta^{(3)} v^z - 8 \left( 2v^z \partial_z W_3 + 3W_3 \partial_z v^z \right). \tag{4.19}
\]

On the other hand, in the second set the zero curvature conditions for the values \( m = -2, -1, 0, +1 \) are recursive and at \( m = +2 \) one obtains

\[
\partial_z W_3 = \frac{1}{24} \Delta^{(5)} v^z \partial_z W_3 + v^z \partial_z W_3 + 3W_3 \partial_z v^z. \tag{4.20}
\]

The third order differential operator

\[
\Delta^{(3)} = \partial^3 + \partial\Lambda + 2\Lambda\partial, \tag{4.21}
\]

appeared already in the pure \( SL(2) \) case, it provides a covariant map from the space of \((-1,k)\) into \((2,k)\)-differentials:

\[
\Delta^{(3)} : \hspace{1cm} \mathcal{V}^{-1} \mapsto \mathcal{V}^{+2}, \tag{4.22}
\]

while the fifth order differential operator

\[
\Delta^{(5)} = \partial^5 + 2\partial^3\Lambda + 10\partial\Lambda\partial^3 + 15\Lambda\partial^2\partial + 9\partial^2\Lambda\partial + 16\Lambda\partial\Lambda + 16\Lambda\Lambda\partial. \tag{4.23}
\]

defines a covariant mapping

\[
\Delta^{(5)} : \hspace{1cm} \mathcal{V}^{-2} \mapsto \mathcal{V}^{+3}. \tag{4.24}
\]
So far we have brushed over the properties of the SL(3) gauge potentials in the principal decomposition, subject to conformal and projective parametrizations, highest weight constraints and zero curvature conditions. The final result being that the remaining basic degrees of freedom

\[ v_{z^z}, \quad \Lambda_{zz}, \quad \text{and} \quad v_{z^zz}, \quad W_3, \]

are subject to the modified holomorphicity equations (4.19), (4.20).

We shall now go through the same discussion, *mutatis mutandis*, for the matter triplet \( \Sigma \). Following the general discussion we define the conformal parametrization

\[ \Psi = \hat{g}_0 \Sigma, \]  

(4.25)

as in (2.38), which in the present case of a triplet field gives rise to

\[ \Psi_+ = A_+ \Sigma_+ \overset{\text{def}}{=} \psi_z, \quad \Psi_0 = \Sigma_0 \overset{\text{def}}{=} \psi, \quad \Psi_- = \frac{1}{A_-} \Sigma_- \overset{\text{def}}{=} \psi_-, \]  

(4.26)

introducing notations which clearly exhibit the corresponding conformal weights:

As the transition to the conformal parametrization has the form of a gauge transformation, the covariant derivative is given as

\[ \mathcal{D}(\Gamma) \Psi = \hat{g}^0 D(A) \Sigma. \]  

(4.27)

In more explicit terms and taking into account the particular matrix representation (4.7) given above this reads

\[ \mathcal{D}\psi_z = (d - \chi) \psi_z + \sqrt{2} \left( \lambda_z + \Gamma_1^{+1} \right) \psi - 4 \Gamma_1^{+2} \psi_z, \]  

(4.28)

\[ \mathcal{D}\psi = \left( d + \frac{4}{3} \Gamma_1^{+1} \right) \psi - \sqrt{2} \left( v_z + \Gamma_1^{-1} \right) \psi_z + \sqrt{2} \left( \lambda_z + \Gamma_1^{+1} \right) \psi, \]  

(4.29)

\[ \mathcal{D}\psi^z = (d + \chi) \psi^z - \frac{2}{3} \Gamma_1^0 \psi^z - \sqrt{2} \left( v^z - \Gamma_1^{-1} \right) \psi - 4 v^{zz} \psi_z. \]  

(4.30)

Inspection of these equations for vanishing covariant derivatives shows easily that again, due to the presence of the terms proportional to \( v_z \), the last two equations serve to recursively eliminate the components \( \psi \) and \( \psi_z \) as functions of the other variables and their derivatives. In the transition to the projective parametrization the component \( \psi^z \) does not change. As final result of this procedure one is left with two equations, due to the \( dz \) and \( d\bar{z} \) components of the covariant derivative, and which are

\[ \left( \Delta^{(3)} - 8 W_3 \right) \psi^z = 0, \]  

(4.31)
\[
\begin{align*}
\partial_z \psi^z &= v_\frac{z}{z} \partial_z \psi^z - \psi^z \partial_z v_\frac{z}{z} + 2 v_\frac{z}{z} \partial_\frac{z}{z} \partial_z \psi^z - \partial_\frac{z}{z} v_\frac{z}{z} \partial_z \psi^z \\
&\quad+ \frac{1}{3} \psi^z \partial_z \partial_z v_\frac{z}{z} z + \frac{8}{3} v_\frac{z}{z} \partial_\frac{z}{z} \Lambda_{zz} \psi^z. \\
\end{align*}
\]  
(4.32)

Note that the last four terms in this equation are conformally covariant by virtue of the projective connection.

We come now to the discussion of the ghost sector and the corresponding differential BRS algebra. The conformal parametrization, defined in the general case in (2.34) and (2.37), reads in our case as

\[
c = c^k L_k + c^m_1 T_m,
\]  
(4.33)

where the ghost fields \( c^k \) and \( c^m_1 \) have conformal weights \((k,0)\) and \((m,0)\), respectively. Following the prescriptions of the general case, it is straightforward to convince oneself that after the transition to the projective parametrization and in presence of the highest weight constraints only the conformally covariant ghost fields

\[
c^- = c^z, \quad c^{-2}_1 = c^{zz},
\]  
(4.34)

of conformal weights \((-1,0)\) and \((-2,0)\) survive, all the other ones are recursively eliminated. Taking into account all the properties of the gauge and ghost fields, the complete nilpotent BRS differential algebra of the basic variables is given as

\[
\begin{align*}
sv_{\frac{z}{z}} &= \partial_z c^z + c^z \partial_z v_\frac{z}{z} - v_\frac{z}{z} \partial_z c^z + \frac{2}{3} \left( v_\frac{z}{z} \partial_\frac{z}{z}^3 c^{zz} - c^{zz} \partial_\frac{z}{z}^3 v_\frac{z}{z} \right) \\
&\quad+ \partial_\frac{z}{z} c^{zz} \partial_\frac{z}{z} v_\frac{z}{z} z - \partial_\frac{z}{z} v_\frac{z}{z} z \partial_\frac{z}{z} c^{zz} + \frac{16}{3} \Lambda_{zz} (v_\frac{z}{z} \partial_\frac{z}{z} c^{zz} - c^{zz} \partial_\frac{z}{z} v_\frac{z}{z}) \quad (4.35) \\
sv_{\frac{z}{z} \frac{z}{z}} &= \partial_z c^{zz} + c^z \partial_z v_\frac{z}{z} z - v_\frac{z}{z} \partial_z c^{zz} + 2 \left( c^{zz} \partial_z v_\frac{z}{z} z - v_\frac{z}{z} \partial_z c^z \right) \quad (4.36) \\
sc^z &= -c^z \partial_z c^z - \partial_z c^{zz} \partial_\frac{z}{z}^2 c^{zz} + \frac{2}{3} c^{zz} \partial_\frac{z}{z}^3 c^{zz} + \frac{16}{3} \Lambda_{zz} c^{zz} \partial_\frac{z}{z} c^{zz} \quad (4.37) \\
sc^{zz} &= -c^z \partial_z c^{zz} - 2 c^{zz} \partial_z c^z \quad (4.38) \\
s\Lambda_{zz} &= \Delta(3) c^z - 8 \left( 2 c^{zz} \partial_z W_3 + 3 W_3 \partial_z c^{zz} \right) \quad (4.39) \\
sW_3 &= \frac{1}{24} \Delta(5) c^{zz} + c^z \partial_z W_3 + 3 W_3 \partial_z c^z \quad (4.40)
\end{align*}
\]

Likewise, the BRS variation for the matter field comes out to be

\[
\begin{align*}
sv_\frac{z}{z} &= c^z \partial_z \psi^z - \psi^z \partial_z c^z + 2 c^{zz} \partial_\frac{z}{z}^2 \psi^z - \partial_z c^{zz} \partial_z \psi^z + \frac{1}{3} \psi^z \partial_\frac{z}{z}^2 c^{zz} + \frac{8}{3} \Lambda_{zz} \psi^z c^{zz}. \\
\end{align*}
\]  
(4.41)
This BRS differential algebra reflects exactly the transformations and their commutators obtained in [20] for induced $W_3$-gravity.

Given the complete nilpotent BRS algebra we may ask for a solution of the consistency condition, i.e. the existence of a local functional of the basic variables of ghost number one which is BRS closed modulo exterior derivative. Such a quantity can indeed be constructed, and it is given as

$$\mathcal{A}_{zz}^{(1)} = (c^z \partial_{\bar{z}} - v_{\bar{z}} \bar{z} s) \Lambda_{zz} - 8 (c^{\bar{z}z} \partial_{\bar{z}} - v_{\bar{z}}^{\bar{z}z} s) W_3. \quad (4.42)$$

In more explicit terms

$$\mathcal{A}_{zz}^{(1)} = c^z \Delta^{(3)} v_{\bar{z}}^z - v_{\bar{z}}^z c^z \Delta^{(3)} c^z - \frac{1}{2} \left( c^{\bar{z}z} \Delta^{(5)} v_{\bar{z}}^{\bar{z}z} - v_{\bar{z}}^{\bar{z}z} \Delta^{(5)} c^{\bar{z}z} \right)$$

$$- 8 (c^{\bar{z}z} - c^{z} c^{\bar{z}z}) \partial_{\bar{z}} W_3$$

$$- 24 W_3 \left( c^{\bar{z}} \partial_{\bar{z}} v_{\bar{z}}^{\bar{z}z} - v_{\bar{z}}^{\bar{z}z} \partial_{\bar{z}} c^z + c^{\bar{z}z} \partial_{\bar{z}} v_{\bar{z}}^z - v_{\bar{z}}^z \partial_{\bar{z}} c^{\bar{z}z} \right), \quad (4.43)$$

where the leading terms coincide indeed with the expression obtained from the study of induced $W_3$-gravity in [20]. One should also keep in mind that the individual contributions containing $c^z$ or $c^{\bar{z}z}$ only are not separately solutions to the consistency conditions, only the particular combination given here is. This is a remnant of the Chern-Simons origin of the anomaly, i.e. the anomaly can be obtained, via conformal and projective parametrisation plus highest weight constraint from

$$\text{tr} \left( \tilde{A} d\tilde{A} \right)_1^1 \quad (4.44)$$

up to trivial terms, as we have checked by an explicit calculation for the leading terms taking into account the particular decomposition defined in the beginning of this subsection.

4.2 $W_3^{(2)}$-gauge structure

While the principal $SL(2)$ decompositions are given solely in terms of integer gradings, other $SL(2)$ embeddings allow for half-integer gradings as well. This is the case for the second $SL(3)$ decomposition which will be discussed here and which will lead us to define the $W_3^{(2)}$-gauge structure. Again, first of all, three $SL(2)$ generators $L_k$, $k = -1, 0, +1$ are identified. Note that, although we use the same symbols as in the previous case, it should be kept in mind that this $SL(2)$ is identified in a different manner among the generators $SL(3)$ . The remaining generators $T^a_{\mu \nu}$ are now arranged in two doublets of $a = -1/2$ with
\(k = \pm 1/2\), distinguished by the hypercharge index \(\rho = -1, +1\) and a singlet, which in our notation has \(a = -1, k = 0\), the hypercharge itself. The generators in the two doublets will be denoted \(T_{\rho \, k}\), neglecting the index \(a\), whereas the hypercharge will be identified as \(T_{-1}^{-1} = Y\). The commutation relations in this basis are then given as

\[
\begin{align*}
[L_k, L_l] &= (k - l) L_{k+l}, \\
[L_k, T_{\rho \, m}] &= \left(\frac{1}{2}k - m\right) T_{\rho \, m+k}, \\
[Y, L_k] &= 0, \\
[Y, T_{\rho \, k}] &= -\rho T_{\rho \, k}, \\
[T_{\rho \, k}, T_{\sigma \, l}] &= -\frac{1}{4}(\rho - \sigma) \left((k + l) + \frac{1}{2}(\rho - \sigma)(k - l)\right) L_{k+l} \\
&\quad -\frac{3}{4}(\rho - \sigma)(k - l)^2 Y.
\end{align*}
\]

(4.45)

(4.46)

(4.47)

(4.48)

(4.49)

Gauge fields and ghost fields are decomposed along this basis as follows:

\[
\begin{align*}
A &= A^k L_k + A^\rho \, m T_{\rho \, m} + A_Y Y, \\
\omega &= \omega^k L_k + \omega^\rho \, m T_{\rho \, m} + \omega_Y Y,
\end{align*}
\]

(4.50)

(4.51)

with summation over \(\rho = -1, +1\) and \(m = -1/2, +1/2\). As before we introduce a triplet of matter fields,

\[
\Sigma = \begin{pmatrix} \Sigma_{+1/2} \\ \Sigma_{-1/2} \\ \Sigma_0 \end{pmatrix},
\]

(4.52)

but now the three-dimensional representation of \(SL(3)\) is given explicitly in terms of the matrices

\[
\begin{align*}
L_- &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & L_0 &= \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, & L_+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
T_{\otimes \, +1/2} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & T_{\otimes \, -1/2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
T_{\oplus \, +1/2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & T_{\oplus \, -1/2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
Y &= \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}.
\end{align*}
\]

(4.53)
Again the explicit form of the matrices \( L_k \) justifies the notational conventions in the components of \( \Sigma \), it decomposes into a doublet and a singlet with respect to the \( SL(2) \) substructure. We denote negative and positive hypercharges by the symbols \( \ominus \) and \( \oplus \), respectively, in order to distinguish the hypercharge from the other indices.

What about the conformal parametrization in this case. We know from the general discussion, that it is always defined in the same way, i.e. as a redefinition which has the form of a \( \alpha^0 \) gauge transformation of parameter \( \hat{\alpha}^0 = -\log A_z^- \). The details depend, however, on the special basis chosen for the generators of the Lie algebra. In the present case the appearance of half-integer \( a \)- and \( k \)-values in the Lie algebra decomposition will give rise to bosonic degrees of freedom of half-integer conformal weights among the gauge, ghost and matter fields. More explicitly, the gauge potentials at \( a = -\frac{1}{2} \), \( m = \pm \frac{1}{2} \) in the conformal parametrization are defined as

\[
\Gamma^\rho \, m = \hat{\theta}^\rho \, A^\rho \, m = A^\rho \, m (A_z^-)_m, \tag{4.54}
\]

whereas for \( a = -1 \) we have

\[
\Gamma_Y = \hat{\theta}^\rho \, A_Y = A_Y. \tag{4.55}
\]

At \( a = 0 \) we employ, of course, always the same definitions in terms of \( v_z \), \( \chi \) and \( \lambda_z \). For this \( SL(2) \) subsector the curvature in the conformal parametrization reads

\[
\mathcal{F}^- = dv_z + v_z^2 \chi + \Gamma^\ominus \, -1/2 \, \Gamma^\oplus \, -1/2, \tag{4.56}
\]

\[
\mathcal{F}^0 = d\chi + 2v_z^2 \lambda_z + \Gamma^\ominus \, +1/2 \, \Gamma^\oplus \, -1/2 - \Gamma^\ominus \, -1/2 \, \Gamma^\oplus \, +1/2, \tag{4.57}
\]

\[
\mathcal{F}^+ = d\lambda_z - \lambda_z \chi - \Gamma^\ominus \, +1/2 \, \Gamma^\oplus \, +1/2. \tag{4.58}
\]

Recall that we use the symbols \( \ominus \) and \( \oplus \) to label quantities of hypercharge \( \rho = -1 \) and \( \rho = +1 \), respectively. In the singlet sector \( a = -1 \) and \( m = 0 \) the field strength components are then given as

\[
\mathcal{F}_Y = d\Gamma_Y - \frac{3}{2} \Gamma^\ominus \, -1/2 \, \Gamma^\oplus \, +1/2 - \frac{3}{2} \Gamma^\ominus \, +1/2 \, \Gamma^\oplus \, -1/2, \tag{4.59}
\]

and for the doublet at \( a = -\frac{1}{2} \) one obtains

\[
\mathcal{F}^\rho \, -1/2 = d\Gamma^\rho \, -1/2 - \frac{1}{2} \chi \, \Gamma^\rho \, -1/2 - \rho \, v_z^2 \, \Gamma^\rho \, +1/2 + \rho \, \Gamma_Y \, \Gamma^\rho \, -1/2, \tag{4.60}
\]

\[
\mathcal{F}^\rho \, +1/2 = d\Gamma^\rho \, +1/2 + \frac{1}{2} \chi \, \Gamma^\rho \, +1/2 + \rho \, \lambda_z \, \Gamma^\rho \, -1/2 + \rho \, \Gamma_Y \, \Gamma^\rho \, +1/2. \tag{4.61}
\]
The highest weight constraints in this decomposition are simply
\[ \Gamma^\rho z^{-1/2} = 0. \] (4.62)

The independent fields are then identified as follows
\[ \Gamma^Y = dz W_1 + d\bar{z} v_z, \] (4.63)
\[ \Gamma^{\rho -1/2} = d\bar{z} v^\rho \zeta, \] (4.64)
\[ \Gamma^{\rho +1/2} = dz W_{3/2}^\rho + d\bar{z} \Gamma^\rho z^{+1/2}. \] (4.65)

The zero curvature conditions yield, among other things
\[ \Gamma^{\rho +1/2} = v^\rho z W_{3/2}^\rho + v^\rho \zeta W_1 - \rho \partial_z v^\rho \zeta. \] (4.66)

After transition to conformal parametrization, highest weight constraints
and projective parametrization we are left with a set of variables which are the
pairs of \( \mathcal{W} \)-gauge potentials and currents, defined at lowest and highest weight,
respectively, for each value of \( a \). More explicitly, for \( a = 0 \) we have, as usual,
\[ v^\rho \zeta \quad \text{and} \quad \Lambda zz, \]
the \( SL(2) \) subsector with the projective connection. At \( a = -1 \) we have
\[ v_z \quad \text{and} \quad W_1, \]
of conformal weights \( (0, 1) \) and \( (1, 0) \), respectively. Finally, in the doublets at
\( a = -1/2 \) with hypercharges \( \rho = -1, +1 \), bosonic fields of half-integer conformal
weights appear, namely
\[ v^\rho \zeta \quad \text{and} \quad W_{3/2}^\rho, \]
of conformal weights \( (-1/2, 1) \) and \( (3/2, 0) \), respectively.

In terms of these quantities and after the recursive procedure the zero cur-
vature conditions then take their final form as follows
\[ \partial_z \Lambda zz = \Delta^{(3)} v^\zeta z - 2W_1 \left( v^\zeta \zeta W^\zeta 3/2 - v^\zeta \zeta W^\zeta 3/2 \right), \]
\[ -v^\zeta \zeta \partial_z W^\zeta 3/2 - 3W^\zeta 3/2 \partial_z v^\zeta \zeta - v^\zeta \zeta \partial_z W^\zeta 3/2 - 3W^\zeta 3/2 \partial_z W_{3/2} (4.67) \]
\[ \partial_z W_1 = \partial_z v^\zeta - \frac{3}{2} \left( v^\zeta \zeta W_{3/2}^\zeta - v^\zeta \zeta W_{3/2}^\zeta \right), \] (4.68)
\[ \partial_z W_{3/2}^\rho = v^\rho z \partial_z W_{3/2}^\rho + \frac{3}{2} W_{3/2}^\rho \partial_z v^\zeta z - \rho v^\rho z W_1 W_{3/2}^\rho + \rho v_z W_{3/2}^\rho \]
\[ -\rho \left( \Delta^{(2)} + W_1 W_1 \right) v^\rho \zeta + v^\rho \zeta \partial_z W_1 + 2W_1 d z v^\rho \zeta. \] (4.69)

The ghost fields which survive at the end after the reduction procedure are
\[ sv_{z}^{z} = \partial_{z}c^{z} + c^{z} \partial_{z}v_{z}^{z} - v_{z}^{z} \partial_{z}c^{z} + c^{G}c^{z} - v_{z}^{z}c^{G}c^{z}, \quad (4.70) \]

\[ sv_{z} = \partial_{z}c - 3 \left( c^{G}c^{z} - v_{z}^{z}c^{G}c^{z} \right) + \frac{3}{2} \left( c^{G}c^{z} - v_{z}^{z}c^{G}c^{z} \right) \partial_{z}c^{Gc} \]

\[ + \frac{3}{2} c^{z} \left( v_{z}^{z} W_{3/2}^{G} - W_{3/2}^{G} v_{z}^{z} \right) + \frac{3}{2} v_{z}^{z} \left( W_{3/2}^{G} c^{Gc} - c^{Gc} W_{3/2}^{G} \right), \quad (4.71) \]

\[ sv_{z}^{Gc} = \partial_{z}v_{z}^{Gc} + c^{z} \partial_{z}v_{z}^{Gc} - v_{z}^{z} \partial_{z}c^{Gc} + \frac{1}{2} \left( c^{Gc} \partial_{z}v_{z}^{Gc} - v_{z}^{Gc} \partial_{z}c^{Gc} \right) \]

\[ + \rho \left( v_{z}^{Gc} c^{Gc} - c^{Gc} v_{z}^{Gc} \right) + \rho W_{1} \left( v_{z}^{Gc} c^{Gc} - c^{Gc} v_{z}^{Gc} \right), \quad (4.72) \]

for the gauge potentials while those of the ghost fields are given as

\[ sc^{z} = -c^{z} \partial_{z}c^{z} - c^{Gc} c^{Gc}, \quad (4.73) \]

\[ sc = -\frac{3}{2} \left( c^{Gc} \partial_{z}c^{Gc} + c^{Gc} \partial_{z}c^{Gc} \right) + 3W_{1} c^{Gc} c^{Gc} \]

\[ + \frac{3}{2} c^{z} \left( W_{3/2}^{Gc} - c^{Gc} W_{3/2}^{Gc} \right), \quad (4.74) \]

\[ sc^{Gc} = -c^{z} \partial_{z}c^{Gc} - \frac{1}{2} c^{Gc} \partial_{z}c^{Gc} + \rho \left( c^{Gc} W_{1} - c \right) c^{Gc}. \quad (4.75) \]

Finally, for the \( W \)-currents, one arrives at

\[ s\lambda_{z} = \Delta^{(3)} c^{z} - c^{Gc} \partial_{z}W_{3/2}^{G} - 3W_{3/2}^{G} \partial_{z}c^{Gc} - c^{Gc} \partial_{z}W_{3/2}^{G} - 3W_{3/2}^{G} \partial_{z}c^{Gc} \]

\[ + \Delta^{(3)} c^{z} - c^{Gc} \partial_{z}W_{3/2}^{G} - 3W_{3/2}^{G} \partial_{z}c^{Gc} - c^{Gc} \partial_{z}W_{3/2}^{G} - 3W_{3/2}^{G} \partial_{z}c^{Gc} \]

\[ + 2W_{1} \left( W_{3/2}^{Gc} - c^{Gc} W_{3/2}^{Gc} \right), \quad (4.76) \]

\[ sW_{1} = \partial_{z}c + \frac{3}{2} \left( W_{3/2}^{Gc} - c^{Gc} W_{3/2}^{Gc} \right), \quad (4.77) \]

\[ sW_{3/2}^{G} = -\rho \left( \Delta^{(2)} W_{1} \right) c^{Gc} + c^{z} \partial_{z}W_{3/2}^{G} + \frac{3}{2} W_{3/2}^{G} \partial_{z}c^{Gc} \]

\[ + c^{Gc} \partial_{z}W_{1} + 2W_{1} \partial_{z}c^{Gc} + \rho \left( c^{Gc} W_{1} - c \right) W_{3/2}^{G}. \quad (4.78) \]
The consistent anomaly, expressed in terms of the basic variables, is a \((1, 1)\) differential of ghost number one and hypercharge zero. As before it is a particular combination of contributions from the different ghosts:

\[
\mathcal{A}^{(1)}_{\Delta z} = (c^z \partial_z - v_z \bar{\partial} s) \Lambda_{zz} - \frac{4}{3} (c \partial_z - v_z s) W_1
\]

\[
-2 \left( c^{\bar{\zeta}} \partial_{\bar{z}} - v_{\bar{z}} \partial^\zeta s \right) W_{3/2}^\emptyset - 2 \left( c^{\bar{\zeta}} \partial_{\bar{z}} - v_{\bar{z}} \bar{\partial} \zeta s \right) W_{3/2}^\emptyset. \tag{4.79}
\]

In some more detail this anomaly takes the following explicit form

\[
\mathcal{A}^{(1)}_{\Delta z} = c^z \Delta^{(3)} v_z \bar{\partial} - v_z \Delta^{(3)} c^z + \frac{4}{3} (v_z \partial_z c - c \partial_z v_z)
\]

\[
+ 2 \left( c^{\bar{\zeta}} \Delta^{(2)} v_{\bar{z}} \zeta + v_{\bar{z}} \zeta \Delta^{(2)} c^{\bar{\zeta}} - c^{\bar{\zeta}} \Delta^{(2)} v_{\bar{z}} \zeta - v_{\bar{z}} \zeta \Delta^{(2)} c^{\bar{\zeta}} \right)
\]

\[
+ 4W_1 \left( v_{\bar{z}} \zeta \partial_z c^{\bar{\zeta}} - c^{\bar{\zeta}} \partial_z v_{\bar{z}} \zeta + v_{\bar{z}} \zeta \partial_z c^{\bar{\zeta}} - c^{\bar{\zeta}} \partial_z v_{\bar{z}} \zeta \right)
\]

\[
+ 3W_{3/2}^\emptyset \left( v_{\bar{z}} \zeta \partial_z c^\zeta - c^\zeta \partial_z v_{\bar{z}} \zeta + v_{\bar{z}} \zeta \partial_z c^\zeta - c^\zeta \partial_z v_{\bar{z}} \zeta \right)
\]

\[
+ \left( c^\zeta v_{\bar{z}} \zeta - v_{\bar{z}} c^\zeta \right) \partial_z W_{3/2}^\emptyset
\]

\[
+ 3W_{3/2}^\emptyset \left( v_{\bar{z}} \zeta \partial_z c^\zeta - c^\zeta \partial_z v_{\bar{z}} \zeta + v_{\bar{z}} \zeta \partial_z c^\zeta - c^\zeta \partial_z v_{\bar{z}} \zeta \right)
\]

\[
+ \left( c^\zeta v_{\bar{z}} \zeta - v_{\bar{z}} c^\zeta \right) \partial_z W_{3/2}^\emptyset
\]

\[
+ 4W_{3/2}^\emptyset \left( v_z c^{\bar{\zeta}} - c v_{\bar{z}} \bar{\partial} \zeta \right) + 4W_{3/2}^\emptyset \left( c v_{\bar{z}} \zeta - v_z c^{\bar{\zeta}} \right)
\]

\[
+ 4W_1 W_{3/2}^\emptyset \left( c^\zeta v_{\bar{z}} \zeta - v_{\bar{z}} c^\zeta \right) - 4W_1 W_{3/2}^\emptyset \left( c^\zeta v_{\bar{z}} \zeta - v_{\bar{z}} c^\zeta \right) \tag{4.80}
\]

where the terms are arranged such that conformal covariance becomes as transparent as possible. Again, this expression can be obtained, modulo trivial terms, from

\[
\text{tr} \left( \tilde{A} d \tilde{A} \right)_{1/2}^1, \tag{4.81}
\]

when expanded in the projective parametrization and subject to the highest weight gauge, taking into account the second \(SL(2)\) decomposition of this subsection in the evaluation of the trace.

Let us now come back to the discussion of the triplet

\[
\Sigma = \begin{pmatrix} \Sigma_{+1/2} \\ \Sigma_{-1/2} \\ \Sigma_0 \end{pmatrix}. \tag{4.82}
\]

The conformal parametrization,

\[
\Psi = \tilde{g}_0 \Sigma, \tag{4.83}
\]

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is now obtained using the explicit matrix representation (4.53) of this subsection. As a consequence, we arrive at the redefinitions

$$\psi_\zeta = \sqrt{A_z} \Sigma_{+1/2}, \quad \psi^\xi = \frac{1}{\sqrt{A_z}} \Sigma_{-1/2}, \quad \psi = \Sigma_0.$$ (4.84)

introducing conformal weights (+1/2, 0), (−1/2, 0) and (0, 0), respectively. The covariant derivatives on these components in the conformal parametrization are

$$\mathcal{D}\psi_\zeta = (d - \frac{1}{2} \chi) \psi_\zeta + \lambda_2 \psi^\xi + \Gamma^{+1/2}_Y \psi_\zeta,$$ (4.85)

$$\mathcal{D}\psi^\xi = (d + \frac{1}{2} \chi) \psi^\xi - v^z \psi_\zeta + \Gamma^{-1/2}_Y \psi^\xi,$$ (4.86)

$$\mathcal{D}\psi = d\psi + \Gamma^{+1/2}_Y + \Gamma^{-1/2}_Y - \frac{2}{3} \Gamma_Y \psi.$$ (4.87)

Imposing the constraint on Ψ to be covariantly constant shows, by virtue of the second equation, that ψ_ζ becomes a dependent variable and we are left, in the matter sector, with the two basic fields ψ^ξ and ψ. Taking into account the projective and highest weight gauges, these basic fields are subject to the conditions

$$\partial_\bar{z} \psi^\xi = v^\xi \partial_\bar{z} \psi^\xi - \frac{1}{2} \psi^\xi \partial_\bar{z} v^\xi + \frac{1}{3} (v^\xi W_1 - v^\xi) \psi^\xi - v^\xi \zeta \psi,$$ (4.88)

$$\left(\partial_\bar{z} \partial_\bar{z} + \frac{1}{2} \Lambda_{zz}\right) \psi^\xi + \frac{2}{3} \psi^\xi \partial_\bar{z} W_1 + \frac{4}{3} W_1 \partial_\bar{z} \psi^\xi + W_{3/2}^\xi \psi = 0,$$ (4.89)

and

$$\partial_\bar{z} \psi = -v^\xi \zeta \partial_\bar{z} \psi^\xi + \psi^\xi \partial_\bar{z} v^\xi + \frac{2}{3} v^\xi \psi - \frac{4}{3} v^\xi W_{3/2}^\xi \psi^\xi - v^\xi W_{3/2}^\xi \psi^\xi,$$ (4.90)

$$\partial_\bar{z} \psi - \frac{2}{3} W_1 \psi + W_{3/2}^\xi \psi^\xi = 0.$$ (4.91)

Finally, the BRS transformations of the basic variables in the matter sector are obtained as

$$s\psi^\xi = c^z \partial_\bar{z} \psi^\xi - \frac{1}{2} \psi^\xi \partial_\bar{z} c^z + \frac{1}{3} (c^z W_1 - c) \psi^\xi - c^{\xi \xi} \psi,$$ (4.92)

$$s\psi = -c^\xi \xi \partial_\bar{z} \psi^\xi + \psi^\xi \partial_\bar{z} c^{\xi \xi} + \frac{2}{3} c^\xi \psi - \frac{4}{3} c^{\xi \xi} W_1 \psi^\xi - c^z W_{3/2}^\xi \psi^\xi,$$ (4.93)

completing our presentation of the $W_{3}^{(2)}$-gauge structure.
5 SUMMARY AND CONCLUSIONS

In this paper we have presented a procedure to identify what we have called \(\mathcal{W}\)-gauge structures, starting from two dimensional flat Lie algebra valued gauge potentials (and ghosts), with special emphasis on conformal covariance properties. The crucial point of our construction was a soldering process which, for a given \(SL(2)\) decomposition of the corresponding Lie algebra and in a certain highest weight parametrization, gave rise to gauge structures in terms of a number of triplets

\[
W_{a+2}, \quad v_{\bar{z}}^{-a-1}, \quad c^{-a-1}, \quad (5.1)
\]

suggestively called \(\mathcal{W}\)-currents (of conformal weight \((a + 2, 0)\)), \(\mathcal{W}\)-gauge fields (of conformal weight \((-a - 1, 1)\)), and \(\mathcal{W}\)-ghost fields (of conformal weight \((-a - 1, 0)\)). The values of \(a\) occurring (i.e. among \(a \geq -1\), integer or half-integer) and their multiplicities depend on the properties of the particular \(SL(2)\) decomposition chosen. In all cases there is, in addition to this set of triplets, another one,

\[
\Lambda_{zz}, \quad v_{z}^{z}, \quad c^{z}, \quad (5.2)
\]

pertaining to the the \(SL(2)\) subsector, featuring the projective connection \(\Lambda_{zz}\), crucial for conformal covariance.

The notion of gauge structure is justified by the action of the nilpotent BRS antiderivation \(s\) pertaining to the residual \(\mathcal{W}\)-gauge transformations, which is concisely defined on the full set of currents, gauge and ghost fields. It is instructive to discuss this BRS differential algebra schematically. For instance, the projective connection and the conformally covariant currents transform as

\[
s \Lambda_{zz} \sim \Delta^{(3)} c^{z} + \text{diff. pol.}, \quad (5.3)
\]

\[
s W_{a+2} \sim \Delta^{(2a+3)} c^{-a-1} + \text{diff. pol.}, \quad (5.4)
\]

exhibiting conformally covariant differential operators \(\Delta^{(2a+3)}\) and \(\text{diff. pol.}\) standing for other conformally covariant differential polynomials in terms of the basic fields, detailed properties depending on the special case under consideration. Observe that only \(\partial\) derivatives (and no \(\bar{\partial}\) derivatives) occur here. As to the BRS differential algebra, the \(\bar{\partial}\) derivatives occur only in the transformations of the \(\mathcal{W}\)-gauge fields and there only linearly and in the particular combinations

\[
s v_{\bar{z}}^{z} - \bar{\partial} c^{z} \sim \text{diff. pol.}, \quad (5.5)
\]

\[
s v_{\bar{z}}^{-a-1} - \bar{\partial} c^{-a-1} \sim \text{diff. pol.} \quad (5.6)
\]
This asymmetry between $\partial$ and $\bar{\partial}$ derivatives manifests itself also in the zero curvature conditions, appearing as certain anholomorphy conditions on the projective connection and the currents, of the form

$$\bar{\partial} \Lambda_{zz} \sim \Delta^{(3)} v_{\bar{z}}^{\bar{z}} + \text{diff. pol.}, \quad (5.7)$$

$$\bar{\partial} W_{a+2} \sim \Delta^{(2a+3)} v_{z}^{-a-1} + \text{diff. pol.}, \quad (5.8)$$

very similar in structure to the above BRS transformations. This qualitative presentation of the general case has been illustrated in full detail for the examples of $SL(2)$ and $SL(3)$, with special emphasis on covariantly chiral matter fields and the anomaly structure.

As it stands, the procedure presented in this paper might be understood as a purely algebraic algorithm which allows to derive in a concise way the consistent BRS differential algebra pertaining to $\mathcal{W}$-geometry. A number of points like the explicit construction of the highest weight parametrization (which was assumed here as some constraint, consistent with the general structure, and without any further justification) or the features of the covariant constant matter fields in the general case deserve further study. Likewise, possible dynamical realizations of the geometric concepts presented here should be investigated. It should also be worthwhile to study possible relations with the quantum Drinfeld-Sokolov reduction scheme as e.g. in [35].

A more profound mathematical understanding might appeal, at least in the $SL(2)$ case, to concepts of complex and projective structures in the context of flat complex vector bundles over Riemann surfaces [36], [37], [38] for an interpretation of the soldering procedure in terms of a special representative for the corresponding connections. The projective parametrization in such a picture might be related to the change of complex structure on the Riemann surface through smooth diffeomorphisms, with $v_{z}^{\bar{z}}$ playing eventually the role of a Beltrami differential.

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