Periodic orbits and Birkhoff sections of Stable Hamiltonian structures

Robert Cardona

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joint work with A. Rechtman (Université de Strasbourg)
General setting

Throughout the talk:

\( M = \) orientable closed three-manifold
\( X = \) non-vanishing smooth vector field preserving some volume form \( \mu \)
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- Does $X$ have periodic orbits?
- How many / How often?
- Does $X$ have a *Birkhoff section*?
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- Does \( X \) have a Birkhoff section?

In \( S^3 \) it is not known if \( X \) always admits a periodic orbit. It does not if \( X \) is only \( C^1 \) as shown by Kuperberg '96.
Motivation

- $(W, \omega)$ a four-dimensional symplectic manifold. Given $H \in C^\infty(M)$, let $X_H$ be the Hamiltonian vector field. If $M = H^{-1}(c)$ where $c$ is regular then $X = X_H|_M$ is non-vashing and volume-preserving in $M$. 

Remark: A vector field $X$ is Eulerisable if there exists a metric for which $X$ is a stationary solution to the Euler equations. Reeb fields defined by contact forms and by stable Hamiltonian structures are Eulerisable (Sullivan, Etnyre-Ghrist, Rechtman, Cieliebak-Volkov).
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A global section (or cross section) of $X$ is an embedded closed surface $\Sigma$ transverse to $X$ and that intersects all its orbits.
Global sections

**Definition**

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The existence of such a surface allows us to study the dynamics of $X$ via an area-preserving diffeomorphism $f: \Sigma \to \Sigma$ (the first-return map). The vector field is orbit equivalent to the suspension of $f$.

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If $M$ is a compact with boundary, a global section is an embedded surface with boundary $\Sigma$ satisfying $\partial \Sigma \subset \partial M$. 
Birkhoff sections

Definition

A Birkhoff section of $X$ is an immersed compact surface with boundary $\Sigma$ satisfying:

1. its interior is embedded and transverse to $X$,
2. its boundary is mapped to periodic orbits of $X$,
3. there exists some $T > 0$ such that for each $p \in M$, the flow segment $\varphi_{[0,T]}(p)$ intersects $\Sigma$.

Some classes of flows with Birkhoff sections: geodesic flows on positively curved spheres and negatively curved surfaces (Birkhoff '17), transitive Anosov flows (Fried '83), transitive pseudo-Anosov flows (Brunella '95, see also Tsang 2022).

Techniques of Birkhoff for more general geodesic flows (Contreras – Knieper – Mazzucchelli – Schulz 2022).
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Every Reeb field admits a periodic orbit.
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- Known in dimension three (Hofer ’93, Taubes ’07)
- Two or infinitely many for strictly convex hypersurfaces in $\mathbb{R}^4$ (Hofer-Wysocki-Zehnder ’98)
- Two or infinitely many for nondegenerate Reeb flows defined by torsion contact forms (Cristofaro-Gardiner–Hutchings–Pomerleano ’19)
- Two or infinitely many for nondegenerate Reeb flows (Colin-Dehornoy-Rechtman ’20)
- Complete understanding of Reeb flows with two periodic orbits (Cristofaro-Gardiner–Hryniewicz-Hutchings-Liu ’21, Hutchings-Taubes ’09)
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- Strongly nondegenerate contact forms admit a Birkhoff section (Contreras-Mazzucchelli ’21)
- Open and dense set of contact forms admits a Birkhoff section on any three-manifold (Colin-Dehornoy-Hryniewicz-Rechtman ’22)
Stable Hamiltonian structures

First defined by Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder (2003), foundations in 3D by Cieliebak-Volkov (2015).

**Definition**

A stable Hamiltonian structure is a pair \((\lambda, \omega) \in \Omega^1(M) \times \Omega^2(M)\) such that:

- \(\lambda \wedge \omega > 0\),
- \(d\omega = 0\),
- \(\ker \omega \subseteq \ker d\lambda\).

It a Reeb field by \(\lambda(X) = 1\), \(\iota_X \omega = 0\).
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Given a volume-preserving vector field \(X\), the following are equivalent:

1. \(X\) is the Reeb field of a SHS
2. \(X\) preserves some transverse plane field
3. there is a metric on \(M\) making \(X\) of unit length and the flowlines geodesics
Introduced by Hofer-Zehnder (1994). Identified with the previous definition by Eliashberg-Kim-Polterovich and Cieliebak-Mohnke around 2005.

**Definition**

A hypersurface $M$ on a symplectic four-manifold $(W, \omega)$ is **stable** if there exists a tubular neighborhood $U \cong M \times (-\varepsilon, \varepsilon)$ such that the characteristic foliations of $M \times \{t\}$ are all conjugate via a family of diffeomorphisms depending smoothly on $t$. 

It is a natural boundary condition for compactness results in SFT and appears in other works in symplectic topology. Natural examples arise in regular energy level sets of magnetic flows on surfaces. A steady solution to the Euler equations of Beltrami type is the (reparametrized) Reeb field of a SHS.
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Concrete examples

**Contact forms.** Given $\alpha$ a contact form, the pair $(\alpha, d\alpha)$ defines a stable Hamiltonian structure.
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**Suspension flows.** Given an area-preserving diffeomorphism of a surface $f : \Sigma \to \Sigma$, it induces a stable Hamiltonian structure $(\lambda, \omega)$ with $d\lambda = 0$ on the suspended manifold $M = \Sigma \times [0, 1]/\sim$. The flow admits a global cross section.

Reeb flows with a first integral. Let $\alpha$ be a contact form defining a Reeb field $X$ with a first integral $g \in C^\infty(M)$, that we assume positive. Then $(\alpha, gd\alpha)$ defines a SHS whose Reeb field is the Reeb field of $\alpha$.
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Let \( N_{c_i} = f^{-1}[c_i - \delta, c_i + \delta] \) for each singular value \( c_i \) of \( f \).
In general $d\lambda = f\omega$, and $f$ is a first integral. The one-form $\lambda$ is of (positive or negative) contact type where $f \neq 0$. Let $N_{c_i} = f^{-1}[c_i - \delta, c_i + \delta]$ for each singular value $c_i$ of $f$.

In each integrable region $U_i \cong T^2 \times I$, the flow is fiberwise linear, the “slope” of $X$ is constant (rational or irrational) or non-constant.
The Weinstein conjecture

Theorem (Hutchings-Taubes ’09, Rechtman ’10 (some cases))

Let $M$ be a closed three-manifold that is not a torus bundle over $S^1$. Then any Reeb field of any SHS on $M$ admits a closed orbit.
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Let $M$ be a closed three-manifold that is not a torus bundle over $S^1$. Then any Reeb field of any SHS on $M$ admits a closed orbit.

**Theorem**

Let $X$ be an aperiodic Reeb field defined by a SHS $(\lambda, \omega)$ in $M$. Then one of the following holds:

- $M$ is a three-torus or a positive parabolic bundle and $X$ is orbit equivalent to the suspension of an aperiodic symplectomorphism of the two-torus,
- $M$ is a hyperbolic bundle and $X$ does not admit a global section, but after cutting open along an invariant tori the flow is orbit equivalent to the suspension of a pseudorotation of the closed annulus.
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Robert Cardona (ICMAT and UPC)
Dynamics of SHS
joint work with Ana Rechtman
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Theorem (Cieliebak-Volkov ’15)
The flow in $N_0$ admits a global section (a surface with boundary).

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Recall that the vector field $X$ is:

- **non-degenerate** if for each periodic orbit, no root of the unity is an eigenvalue of the linearized Poincaré map,

- **strongly non-degenerate** if for every pair of closed hyperbolic orbits $\gamma_1, \gamma_2$ we have $W_s(\gamma_1) \prec W_u(\gamma_2)$.

**Definition**

A SHS $(\lambda, \omega)$ is called contact non-degenerate if the Reeb field is non-degenerate in $N_{\text{cont}}$.

Analogously, one can define as well a contact strongly non-degenerate SHS.

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Contact non-degenerate SHS are $C^1$-dense in the set of stable Hamiltonian structures of $M$. 

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*Contact non-degenerate SHS are $C^1$-dense in the set of stable Hamiltonian structures of $M$.***
Theorem

Let $(\lambda, \omega)$ be a contact non-degenerate SHS with at least one periodic orbit. It has infinitely many periodic orbits unless:

- the flow orbit equivalent to the suspension of a symplectomorphism of a surface $\Sigma_g$ with finitely many periodic points.

- $M$ is the 3-sphere or a lens space, there are exactly two closed orbits and they are core circles of a genus one Heegaard splitting of $M$.

Hence except on some surface-bundles, there are two or infinitely many periodic orbits.

Remark

It follows from the proof that the degenerate case would follow from a proof for (contact) Reeb fields.
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- if \((\lambda, \omega)\) is contact non-degenerate then it is carried by a broken book decomposition.

Corollary

On any closed three-manifold, there exists a \(C^1\)-dense, \(C^2\)-open set of SHS whose Reeb field admits a Birkhoff section. Concretely, given any SHS, it is exact stable homotopic to a \(C^1\)-close SHS with a Birkhoff section. Cieliebak-Volkov (2014) showed that any SHS is stable homotopic to one supported by an open book decomposition.
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- if \((\lambda, \omega)\) is contact strongly non-degenerate, then it admits a Birkhoff section.

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Idea of the proof: finitely many periodic orbits

Assume that a contact non-degenerate $X$ defined by $(\lambda, \omega)$ has finitely many periodic orbits. Recall that $f = \frac{d\lambda}{\omega}$.

1. If $f = c > 0$, we have a non-degenerate Reeb field so the theorem follows from Colin-Dehornoy-Rechtman.

2. If $f = 0$, the flow is a suspension of a symplectomorphism of a closed surface with finitely many periodic points (see Le Calvez 2022).

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Trivial observation: the invariant tori given by regular level sets of $f$ are all irrational.
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**Proposition**

Let $X$ be a non-degenerate Reeb vector field in a three-manifold with boundary. Assume that near the boundary it is foliated by irrational invariant tori, and that it has finitely many periodic orbits. Then

- $M \cong D^2 \times S^1$ and $X$ is the suspension of an irrational pseudorotation of the disk,
- $M \cong T^2 \times I$ and $X$ is the suspension of an irrational rotation of the annulus.

Each connected component of the ”contact region” is as above.
For the $N_0$ region:

**Theorem**

Let $\varphi : \Sigma \to \Sigma$ be a symplectomorphism of a surface with boundary. Assume that it has no periodic points in the boundary. Then it has periodic points of arbitrarily large period unless:

- $M \cong D^2 \times S^1$ and $X$ is the suspension of an irrational pseudorotation of the disk,
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The proof involves the Nielsen-Thurston decomposition, working in the universal cover of the surface and Franks theorem. We have decomposed our manifold as a union of $T^2 \times I$ and $D^2 \times S^1$. There is at least one $D^2 \times S^1$ component, from which we get that there are exactly two and that $M$ is a lens space.
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3. General case: \(f\) is non-constant.

\[ T^2 \times I \]

\[ M \]

\[ N_{c_2} \]

\[ N_0 \]

\[ N_{c_3} \]

\[ N_{c_4} \]

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By Cieliebak-Volkov, there is a global section in $N_0$, hence there is $\Sigma_0 \hookrightarrow N_0$.

We end up with several $T^2 \times I$ regions, with non-constant slope, with sections to the flow near the boundary.

**Theorem**

Let $X$ be a $T^2$-invariant flow on $T^2 \times I$ with a non-constant slope. Then given two families of curves $\Gamma_0, \Gamma_1$ such that $\Gamma_0 \subset T^2 \times \{0\}$ and $\Gamma_1 \subset T^2 \times \{1\}$ with $X \pitchfork \Gamma_i$, there exists a Birkhoff section $S$ such that $S \cap T^2 \times \{0\} = \Gamma_0$ and $S \cap T^2 \times \{1\} = \Gamma_1$. 

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Dynamics of SHS  
joint work with Ana Rechtman  
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Main tool: Helix boxes

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Boundary segments
Surfaces in cube
Smooth versions

Dynamics of SHS

joint work with Ana Rechtman
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Choose generators $\eta_1$ and $\eta_2$ on two rational tori with close slope. 

$[\Gamma_\varepsilon] = [\Gamma_0] + k_1[\eta_1] + k_2[\eta_2]$. 

Key point: make sure that the intermediate section remains transverse before reaching $\gamma_1$. 

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$C^2$-openness

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- In the contact region, the Birkhoff section is $\delta$-strong and has non-degenerate binding: by Colin-Dehornoy-Hryniewicz-Rechtman, any $C^1$-close vector field has a Birkhoff section.
C²-openness

To see that in a C²-neighborhood around any such SHS that has a Birkhoff section:

- A C²-close SHS admits a decomposition as in the structure theorem that is C¹-close to the previous one, the flow is C¹-close, and the integral regions still have non-constant slope (Cieliebak-Volkov),
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- In the suspension region, having a global section is C¹-stable.

Conclusion: we can apply again our theorem.

Question: Does every Reeb field defined by a SHS (perhaps non-aperiodic) admit a Birkhoff section?
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Thanks for your attention