Heavy-to-light form factors: sum rules on the light cone and beyond

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We report the first systematic analysis of the off-light-cone effects in sum rules for heavy-to-light form factors. These effects are investigated in a model based on scalar constituents, which allows a technically rather simple analysis but has the essential features of the analogous QCD calculation. The correlator relevant for the extraction of the heavy-to-light form factor is calculated in two different ways: first, by adopting the full Bethe–Salpeter amplitude of the light meson and, second, by performing the expansion of this amplitude near the light cone $x^2 = 0$. We demonstrate that the contributions to the correlator from the light-cone term $x^2 = 0$ and the off-light-cone terms $x^2 \neq 0$ have the same order in the $1/m_Q$ expansion. The light-cone correlator, corresponding to $x^2 = 0$, is shown to systematically overestimate the full correlator, the difference being $\sim \Lambda_{\text{QCD}}/\delta$, with $\delta$ the continuum subtraction parameter of order 1 GeV. Numerically, this difference is found to be $10 \div 20\%$.

1. INTRODUCTION

QCD sum rules on the light cone \cite{1} have been extensively applied to various exclusive form factors, in particular, to weak heavy-to-light transition form factors (see, e.g., \cite{2}). Within this method, the relevant correlators are obtained as power expansions in $x^2$ near the light cone (LC) $x^2 = 0$ in terms of the distribution amplitudes of increasing twist. In practice, however, only few lowest-twist distribution amplitudes are known with a reasonable accuracy. Thus one has to rely on calculations which take into account only lowest powers of $x^2$, at most up to terms linear in $x^2$, without systematic study of the effects related to higher powers of $x^2$ (\cite{2,3,4} and references therein). However, the off-LC ($x^2 \neq 0$) contributions to the correlator are not parametrically suppressed compared to the contribution evaluated at $x^2 = 0$: for instance, Braun and Halperin \cite{5} studied the pion elastic form factor with light-cone sum rules and found that contributions to the correlator from terms corresponding to higher powers of $x^2$ in the pion Bethe–Salpeter (BS) amplitude $\langle 0 | T \bar{u}(x) \gamma_\mu \gamma_5 d(0) | \pi(p) \rangle$ have in general the same $\sim 1/q^2$ behaviour. As we show here, a similar situation occurs for the heavy-to-light form factor: the contributions to the correlator coming from the LC and the off-LC terms in the BS amplitude of the light meson have the same dependence in $1/m_Q$, $m_Q$ being the heavy-quark mass. Therefore, relying on calculations performed at $x^2 = 0$ does not seem to be safe and the effects of all powers of $x^2$ should be summed and taken into account in the sum rule.

In this paper, we study the correlator relevant for the extraction of the heavy-to-light form factors with sum rules and propose a method which allows one to obtain and take into account the full $x^2$-dependence of the BS amplitude of the light meson. For the sake of argument, we present the analysis for the model with scalar constituents, the heavy scalar $Q$ of the mass $m_Q$ and the light scalar $\varphi$ of the mass $m$ (which we name “quarks” throughout the paper). We study the weak transition of a heavy spinless bound state $M_Q(Q\varphi)$ to the light spinless bound state $M(\varphi\varphi)$ (which we refer to as mesons) induced by the weak heavy-to-light $Q \to \varphi$ quark transition. The analysis of this model is technically simpler but at the same time allows one to study some essential features of the corresponding QCD case.

The paper is organized as follows:

In Sec. 2 we consider the BS amplitude of the light meson $\Psi_{\text{BS}}(x, p) = \langle 0 | T \varphi(x) \varphi(0) | M(p) \rangle$ and its expansion in powers of $x^2$. In Sec. 3 we study the correlator $i \int d^4 x \exp(ipx) \langle 0 | T \varphi(x) Q(x) Q(0) \varphi(0) | M(p') \rangle$, relevant for the extraction of the $M_Q \to M$ form factor within the method of sum rules. First, we show that the use of the Nakanishi representation for the BS amplitude of the light meson amounts to a summation of all the off-LC effects. We refer to the correlator obtained by this procedure as the full correlator. Second, we make use of the expansion of the BS amplitude near the LC and obtain a different form of the correlator corresponding to this LC expansion. We show that for the quark–quark interaction dominated by one-boson exchange at small distances (similar to QCD, where the interaction is dominated by one-gluon exchange) any term corresponding to $(x^2)^n$ in the BS amplitude gives a contribution to the correlator which behaves as $1/n!m_Q^2$, independent of $n$. Keeping the term $x^2 = 0$ only, gives the LC correlator. We then discuss the procedure of making the Borelization of the correlator and obtaining the sum rule for the full and the LC correlators. Section 4 presents the numerical analysis of the full and the LC correlators for charm and beauty decays. It is shown that at $q^2 \simeq 0$ the LC correlator systematically overestimates the full correlator, the difference being numerically $10 \div 20\%$, independent of the mass of the heavy quark. Section 5 summarizes our results.

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2. THE BETHE–SALPETER AMPLITUDE AND ITS EXPANSION NEAR THE LIGHT CONE

The BS amplitude is defined according to

$$\Psi_{BS}(x, p') = \langle 0| T\varphi(x)\varphi(0)| M(p') \rangle = \Psi(x^2, xp', p^2 = M^2). \quad (2.1)$$

As a function of the variable $xp'$, the amplitude may be represented by the Fourier integral

$$\Psi_{BS}(x, p') = \int_0^1 d\xi \exp(-i\xi p'x)K(x^2, \xi), \quad (2.2)$$

where the $\xi$-integration runs from 0 to 1 as follows from the analytic properties of Feynman diagrams. Nakanishi proposed to parametrize the kernel $K(x^2, \xi)$ as [6]

$$K(x^2, \xi) = \frac{1}{(2\pi)^4i} \int_0^\infty dz G(z, \xi) \int d^4k' \frac{\exp(-ik'x)}{[z + m^2 - \xi(1 - \xi)M^2 - k'^2 - i0]^3}, \quad (2.3)$$

where the function $G(z, \xi)$ has no singularities in the integration regions in $z$ and $\xi$. The BS equation for the function $\Psi_{BS}$ leads to an equation for the function $G(z, \xi)$.

The $k'$-integral in (2.3) is the second derivative w.r.t. $\mu^2$ of the Feynman propagator of a scalar particle of mass $\mu$ [with $\mu^2 = z + m^2 - \xi(1 - \xi)M^2$] in coordinate space. Making use of the explicit expression for this propagator [7], we obtain the following expansion near the light cone $x^2 = 0$:

$$\frac{1}{(2\pi)^4i} \int d^4k' \exp(-ik'x) \frac{1}{(\mu^2 - k'^2 - i0)^3} = \frac{1}{2} \frac{\partial^2}{(\partial\mu^2)^2} \left\{ \frac{1}{(2\pi)^4i} \int d^4k' \exp(-ik'x) \frac{1}{\mu^2 - k'^2 - i0} \right\}
= \frac{1}{32\pi^2 \mu^2} + \frac{x^2}{128\pi^2} \left[ 1 - 2\gamma_E - \log\left(-\mu^2 x^2/4\right) \right] + O(x^4). \quad (2.4)$$

Interestingly, this is not a pure power series but it contains also terms involving $\log(-x^2)$. Inserting (2.4) into (2.3) leads to the light-cone expansion of $K(x^2, \xi)$:

$$K(x^2, \xi) = g_0(\xi) + x^2 \left[ g_1(\xi) + \log(-x^2m^2)h_1(\xi) \right] + x^4 \left[ g_2(\xi) + \log(-x^2m^2)h_2(\xi) \right] + \cdots. \quad (2.5)$$

The functions $g_n$ and $h_n$ here may be expressed in terms of $G(z, \xi)$. For instance,

$$g_0(\xi) = \frac{1}{32\pi^2} \int_0^\infty dz G(z, \xi) \frac{1}{z + m^2 - \xi(1 - \xi)M^2},$$

$$g_1(\xi) = \frac{1}{128\pi^2} \int_0^\infty dz G(z, \xi) \left[ 1 - 2\gamma_E - \log\left(\frac{z + m^2 - \xi(1 - \xi)M^2}{4m^2}\right) \right],$$

$$h_1(\xi) = -\frac{1}{128\pi^2} \int_0^\infty dz G(z, \xi). \quad (2.6)$$

The presence of the logarithmic terms leads to complications: Let us go back to Eq. (2.3) and expand the exponential $\exp(-ik'x)$. Only the first term corresponding to $x^2 = 0$ (the light-cone) is finite, whereas all higher terms lead to divergent $k'$-integrals; this is a consequence of the presence of the terms $\log(-x^2)$. After performing the Wick rotation in the $k'$-space, we may cut the $k'_E$-integration at $k'^2_E = \Lambda^2$. In this way, we obtain a regularized pure power series in $x^2$ for $K(x^2, \xi)$ and thus for $\Psi_{BS}(x, p')$. The introduction of the cutoff $\Lambda$ in the momentum-space integrals is equivalent to the introduction of a regulator $\lambda$ in coordinate space:

$$\log(-x^2m^2) \rightarrow \log(\lambda - x^2m^2).$$

The initial expansion (2.5) is reproduced by setting $\lambda \rightarrow 0$. For a nonzero $\lambda$, we can represent $\log(\lambda - x^2m^2)$ as a power series in $x^2$ and insert this expansion in (2.3), obtaining

$$K(x^2, \xi) = \phi_0(\xi) + x^2\phi_1(\xi, \lambda) + x^4\phi_2(\xi, \lambda) + \cdots, \quad (2.8)$$
with the first three distribution amplitudes $\phi_n$ given by

$$
\begin{align*}
\phi_0(\xi) &= g_0(\xi), \\
\phi_1(\xi, \lambda) &= g_1(\xi) + \log(\lambda) h_1(\xi), \\
\phi_2(\xi, \lambda) &= g_2(\xi) + \log(\lambda) h_2(\xi) - \frac{1}{\lambda} m^2 h_1(\xi).
\end{align*}
$$

As a result, instead of the series expansion for the BS wave function in terms of powers and logarithms of $x^2$ with finite coefficients, we obtained a pure power series, in which only the first term, corresponding to $x^2 = 0$, is finite, whereas all higher distribution amplitudes depend on the regulator parameter $\lambda$ and become infinite in the limit $\lambda \to 0$ ($\Lambda \to \infty$). The regularised LC expansion of the BS amplitude reads

$$
\Psi_{BS}(x, p') = \sum_{n=0}^{\infty} (x^2)^n \int_0^1 d\xi \exp(-i p' x \xi) \phi_n(\xi, \lambda).
$$

Notice that in our case the full BS amplitude does not depend on the regulator parameter $\lambda$. However, any truncation of the series at some $n$ leads to an explicit dependence of $\Psi_{BS}(x, p)$ on $\lambda$ in such a way, that it diverges as $\lambda \to 0$.

The BS amplitude in momentum space reads

$$
\Psi_{BS}(k, p') = \int d^4 x \exp(ikx) \Psi_{BS}(x, p') = \frac{1}{\pi} \int_0^1 d\xi \int_0^\infty dz \frac{G(z, \xi)}{[z + m^2 - \xi(1 - \xi) M^2 - (k - \xi p')^2 - i0]^3}.
$$

The kernel $G(z, \xi)$ determines now the bound-state properties. For instance, the decay constant of a bound state $M$ is given by the relation

$$
f_M = \langle 0|\varphi(0)\varphi(0)|M(p')\rangle = \Psi_{BS}(x = 0, p').
$$

Performing the $k'$-integration in Eq. (2.13) for $x = 0$ leads to

$$
f_M = \frac{1}{32\pi^2} \int_0^1 d\xi \int_0^\infty dz \frac{G(z, \xi)}{z + m^2 - \xi(1 - \xi) M^2}.
$$

The light-cone wave function is related to the kernel $G(z, \xi)$ as follows [8]:

$$
\Psi_{LC}(\xi, k_\perp) = \xi(1 - \xi) \int_0^\infty dz \frac{G(z, \xi)}{z + k_\perp^2 + m^2 - \xi(1 - \xi) M^2}.
$$

Respectively, the distribution amplitudes $\phi_n$ may be calculated as $k_\perp^2$-moments of $\Psi_{LC}(\xi, k_\perp)$. For the function (2.14), only the zero $k_\perp^2$-moment leading to $\phi_0$ is finite, all higher moments require a cut-off in $k_\perp^2$ precisely as in (2.9).

For the case of an interaction between the constituents via exchange of a massless boson, the solution to the BS equation in the ladder approximation takes a simple form [8]:

$$
G(z, \xi) = \delta(z) G(\xi), \quad G(\xi) = \xi(1 - \xi) f(\xi).
$$

The function $f(\xi)$ here is a smooth function which takes finite nonzero values at the end-points. Only the end-point behavior is crucial for the heavy-to-light correlator, therefore the precise form of the function $f(\xi)$ and its normalization are not essential for our analysis; of importance for us is only the fact that $f(\xi = 0) \neq 0$. Hereafter we just set $G(\xi) = m^2 \xi(1 - \xi)$ and do not care about the overall factor $\propto f(0)$. This factor turns out to be the same in the full and the light-cone correlators which we calculate and compare in the next section.

Closing this section, let us emphasize that the $\delta$-functional $z$-dependence of the kernel $G(z, \xi)$ leads to a nontrivial $k_\perp^2$-dependence of the light-cone wave function

$$
\Psi_{LC}(\xi, k_\perp) = \frac{\xi(1 - \xi) G(\xi)}{[k_\perp^2 + m^2 - \xi(1 - \xi) M^2]^2}.
$$

The $1/k_\perp^4$ tail of the bound-state LC wave function is a consequence of the massless-boson exchange (and is similar in this respect to the power-like behavior of the pion light-cone wave function in QCD).
3. HEAVY-TO-LIGHT CORRELATOR AND THE DECAY FORM FACTOR

In this section, we consider the correlator

\[ \Gamma(p^2, q^2) = i \int d^4x \exp(ipx) \langle 0 | \varphi(x)Q(x)Q(0)\varphi(0) | M(p') \rangle \]  

(3.1)

and its Borel transform in the variable \( p^2 \) relevant for the extraction of the \( M_Q \to M \) form factor. Since \( p^2 = M^2 \) is fixed, this correlator depends on the two variables \( p^2 \) and \( q^2 \).

A. Borel sum rule for \( \Gamma(p^2, q^2) \)

Making use of the hadronic degrees of freedom, by inserting the complete system of hadron states into the correlator, we obtain the so-called phenomenological representation for \( \Gamma(p^2, q^2) \), which we denote by \( \Gamma_{\text{phen}}(p^2, q^2) \):

\[
\Gamma_{\text{phen}}(p^2, q^2) = \frac{1}{(2\pi)^4} \int d^4x \exp(ipx) \langle 0 | \varphi(x)Q(x) | M_Q(p) \rangle \frac{d^4p}{M_Q^2 - p^2 - i0} \langle M_Q(p)|Q(0)\varphi(0) | M(p') \rangle + \text{hadr. cont.}
\]

(3.2)

\[ = F_{M_Q \to M}(q^2) \frac{1}{M_Q^2 - p^2 - i0} f_{M_Q} + \int_{s_{\text{cont}}} \frac{ds}{s-p^2-i0} \Delta_{\text{phen}}(s, q^2), \]

where \( s_{\text{cont}} \) is the threshold of the hadronic continuum, containing the \( M_Q \)-meson plus light mesons, i.e., \( M_Q M, M_Q M M \), etc. states. Respectively, \( s_{\text{cont}} = (M_Q + M)^2 \). The decay constant \( f_{M_Q} \) of the \( M_Q \)-meson is defined as

\[ f_{M_Q} = \langle 0 | T\varphi(0)Q(0) | M_Q(p) \rangle, \]

(3.3)

and the form factor is given by

\[ F_{M_Q \to M}(q^2) = \langle M_Q(p)|Q(0)\varphi(0) | M(p') \rangle, \quad q \equiv p - p'. \]

(3.4)

Eq. (3.2) gives the single dispersion representation for \( \Gamma(p^2, q^2) \) in terms of the hadron degrees of freedom. One can calculate the correlator \( \Gamma(p^2, q^2) \) also by making use of the underlying field-theoretic degrees of freedom and obtain in this way a different — theoretical — expression for \( \Gamma(p^2, q^2) \), which we call \( \Gamma_{\text{th}}(p^2, q^2) \). This expression may also be written as a single dispersion representation in \( p^2 \):

\[
\Gamma_{\text{th}}(p^2, q^2) = \int \frac{ds}{s - p^2 - i0} \Delta_{\text{th}}(s, q^2).
\]

(3.5)

Quark–hadron duality \[9, 10] postulates that \( \Gamma_{\text{th}} \) and \( \Gamma_{\text{phen}} \) are equal to each other after a proper smearing is applied to both of them. The smearing may be implemented, e.g., by applying the Borel transform \[9\]. Calculating the Borel image of \( \Gamma_{\text{th}} \) and \( \Gamma_{\text{phen}} \) with the parameter \( 2\mu_B^2 \), such that \( 1/(s - p^2) \rightarrow \exp(-s/2\mu_B^2) \), we obtain the sum rule for \( \Gamma(p^2, q^2) \):

\[
f_{M_Q} F_{M_Q \to M}(q^2) \exp(-M_Q^2/2\mu_B^2) + \int ds \theta(s - s_{\text{cont}}) \exp(-s/2\mu_B^2) \Delta_{\text{phen}}(s, q^2) = \int ds \exp(-s/2\mu_B^2) \Delta_{\text{th}}(s, q^2).
\]

(3.6)

Let us ignore for a moment the dependence of \( \Delta \) on \( q^2 \) and introduce the continuum threshold, different from the physical continuum threshold \( s_{\text{cont}} \) according to

\[
\int ds \theta(s - s_0) \exp(-s/2\mu_B^2) \Delta_{\text{th}}(s) = \int ds \theta(s - s_{\text{cont}}) \exp(-s/2\mu_B^2) \Delta_{\text{phen}}(s).
\]

(3.7)

For sufficiently large \( s \), one expects \( \Delta_{\text{th}}(s) = \Delta_{\text{phen}}(s) \) (up to oscillating terms \[10\]). The region near the continuum threshold \( s \simeq s_{\text{cont}} \) is clearly below the duality region: recall that \( \Delta_{\text{phen}}(s_{\text{cont}}) = 0 \), while \( \Delta_{\text{th}}(s_{\text{cont}}) > 0 \). Thus, in the vicinity of \( s_{\text{cont}} \), \( \Delta_{\text{th}} > \Delta_{\text{phen}} \). As a result, the continuum subtraction \( s_0 \) as defined by (3.7) depends on the Borel parameter (and on \( q^2 \)) and lies above the physical threshold: \( s_0(\mu_B^2) > s_{\text{cont}} \). Then the sum rule (3.6) takes the form

\[
f_{M_Q} F_{M_Q \to M}(q^2) = \exp(M_Q^2/2\mu_B^2) \tilde{\Gamma}_{\text{th}}(\mu_B^2, q^2, s_0(\mu_B^2, q^2)),
\]

(3.8)
Fig. 1: Feynman diagrams for the correlator \( \Gamma(p, q) \). The normal and bold lines denote the heavy \( Q \) and the light \( \varphi \) particles, respectively, the wavy line corresponds to the scalar particle which mediates the \( Q - \varphi \) interaction (a scalar analog of the gluon in QCD); \( |M(p')\rangle \) is the state-vector of the \( \varphi\varphi \) bound-state with the momentum \( p' \). The contribution of the first diagram is expressable through the BS wave function of the light meson and the full \( Q \)-propagator. In the region of not very large \( q, q^2 \ll m_Q^2 \), the second and the third terms are suppressed by higher powers of the heavy-quark mass \( (1/m_Q \text{ and } 1/m_Q^2, \text{ respectively}) \) compared to the first term and will not be considered.

with the correlator to which the \( s \)-cut is applied (hereafter referred to as the cut correlator)

\[
\hat{\Gamma}_{th}(\mu_B^2, q^2, s_0) = \int ds \theta(s < s_0) \exp (-s/2\mu_B^2) \Delta_{th}(s, q^2).
\] (3.9)

If one had known the \( \mu_B \)- and \( q^2 \)-dependent solution \( s_0(\mu_B^2, q^2) \) to (3.7), which encodes the full nonperturbative dynamics, the expression (3.8) would have been exact. Taking \( s_0 \) constant makes it approximate. One may then impose the stability criterium which is expected to guarantee the extraction of the true form factor in several different ways (see, e.g., [9, 11, 12]).

We shall now concentrate on different possibilities to calculate \( \Gamma_{th}(p^2, q^2) \) and compare the corresponding results.

**B. Theoretical calculation of \( \Gamma(p^2, q^2) \)**

We now calculate this correlator using the underlying field-theoretic degrees of freedom, and denote the result as \( \Gamma_{th} \). The corresponding Feynman diagrams are shown in Fig. 1. In the region of not too large timelike momentum transfer \( q, q^2 \ll m_Q^2 \), the first diagram gives the main contribution whereas the contributions of the other diagrams are suppressed by higher powers of the heavy-particle mass \( (1/m_Q \text{ and } 1/m_Q^2, \text{ respectively}) \) compared to the first term and will not be considered.

\[
\Gamma_{th}(p^2, q^2) = i \int dx \exp(ipx) D_Q(x) |0\rangle T \varphi(x) \varphi(0) |M(p')\rangle,
\] (3.10)

where \( D_Q \) is the full propagator of the heavy quark. Approximating the full propagator with the free propagator gives

\[
\Gamma_{th}(p^2, q^2) = \frac{1}{(2\pi)^4} \int d^4k dx \exp(ipx - ikx) \frac{1}{m_Q^2 - k^2 - i0} (0)|T \varphi(x) \varphi(0)|M(p')\rangle.
\] (3.11)

We can proceed further in two different ways:

A. First, the correlator (3.11) may be expressed in terms of the BS amplitude in momentum space

\[
\Gamma_{th}(p^2, q^2) = \frac{1}{(2\pi)^4} \int d^4k \frac{\Psi_{BS}(k, p')}{m_Q^2 - (p - k)^2 - i0}.
\] (3.12)

Making use of (2.11) gives

\[
\Gamma_{th}(p^2, q^2) = \frac{1}{(2\pi)^4} \int d\xi \frac{1}{\int_0^\infty dz G(z, \xi) \int \frac{d^4k}{|z + m^2 - \xi(1 - \xi)M^2 - (k - \xi p')^2 - i0|^3}}.
\] (3.13)
Introducing the Feynman parameter $\alpha$ and performing the $k$-integration, we find

$$
\Gamma_{th}(p^2, q^2) = \frac{1}{32\pi^2} \int_0^1 d\xi \int_0^\infty dz G(z, \xi) \int_0^1 d\alpha (1-\alpha)^2 \frac{1}{[\mu^2(1-\alpha) + m_Q^2 + p^2 - \alpha(1-\alpha)]^2},
$$

$$
\mu^2 = z + m^2 - \xi (1-\xi) M^2, \quad P^2 = p^2 - (1-\xi) M^2 + q^2 \xi.
$$

From this expression, it is straightforward to obtain the single dispersion representation for $\Gamma_{th}$ in the form

$$
\Gamma_{th}(p^2, q^2) = \int_0^\infty \frac{ds}{(s - p^2 - i0)^2} \sigma_{th}(s, q^2),
$$

with

$$
\sigma_{th}(s) = \frac{1}{32\pi^2} \int_0^1 d\xi \int_0^\infty dz G(z, \xi) \int_0^1 d\alpha \frac{\delta(s - s_\alpha)}{\alpha^2}
$$

$$
\frac{z + m^2 - M^2 \xi (1-\xi) + m_Q^2}{\alpha} \cdot \frac{1}{1-\xi} + M^2 \xi - q^2 \xi \frac{1}{1-\xi}.
$$

The spectral density $\sigma_{th}(s, q^2)$ vanishes at the lower limit of the $s$-integration at $s = (m_Q + m)^2$. Notice that the quantity $m$ in the lower limit of the $s$-integration is the mass of the light spectator. For more details about the analytic properties of the three-point functions, we refer to [13].

Performing an integration by parts in (3.15), we obtain the spectral representation in the “standard” form (3.5) with $\Delta_{th}(s, q^2) = \frac{\partial}{\partial s} \sigma_{th}(s, q^2)$. The Borel images of the two representations (3.15) and (3.5) read, respectively,

$$
\hat{\Gamma}_{th}(\mu_B^2, q^2) = \frac{1}{2\mu_B^2} \int_{(m_Q + m)^2}^\infty ds \exp(-s/2\mu_B^2) \sigma_{th}(s, q^2)
$$

(3.17)

and

$$
\hat{\Gamma}_{th}(\mu_B^2, q^2) = \int_{(m_Q + m)^2}^\infty ds \exp(-s/2\mu_B^2) \Delta_{th}(s, q^2).
$$

(3.18)

When the $s$-integration extends to infinity, the equality of both formulas is demonstrated by integration by parts.

We now want to obtain the cut Borel image of Eq. (3.9). Recall, that the cut at $s_0$ is applied to the Borel transform of the spectral representation for the correlator in the form (3.11). Then, the relation

$$
\int_{(m_Q + m)^2}^{s_0} ds \exp(-s/2\mu_B^2) \Delta_{th}(s, q^2) = \frac{1}{2\mu_B^2} \int_{(m_Q + m)^2}^{s_0} ds \exp(-s/2\mu_B^2) \sigma_{th}(s, q^2) + \exp(-s_0/2\mu_B^2) \sigma_{th}(s_0, q^2)
$$

(3.19)

leads to the appearance of an additional surface term in the cut spectral representation of the form Eq. (3.17):

$$
\hat{\Gamma}_{th}(\mu_B^2, q^2, s_0) = \frac{1}{2\mu_B^2} \int_{(m_Q + m)^2}^{s_0} ds \exp(-s/2\mu_B^2) \sigma_{th}(s, q^2) + \exp(-s_0/2\mu_B^2) \sigma_{th}(s_0, q^2).
$$

(3.20)

Moreover, if one works with the Borel image in the form (3.20), the surface term gives the main contribution to the correlator for large values of $\mu_B$.

The sum rule for the form factor can now be written in two equivalent ways:

$$
f_{M_Q \to M}(q^2) = \frac{1}{2\mu_B^2} \int_{(m_Q + m)^2}^{s_0} ds \exp\left(-s/2\mu_B^2\right) \sigma_{th}(s, q^2) + \exp\left(-s_0/2\mu_B^2\right) \sigma_{th}(s_0, q^2)
$$

(3.21)

$$
\int_{(m_Q + m)^2}^{s_0} ds \exp\left(-s/2\mu_B^2\right) \Delta_{th}(s, q^2).
$$

(3.22)
Another possibility to calculate $\Gamma_{th}(p^2, q^2)$ is to substitute the light-cone expansion of $\Psi_{BS}(x, p')$, Eq. (2.10), into Eq. (3.11):

$$\Gamma_{th}(p^2, q^2) = \frac{1}{(2\pi)^3} \int d^4k \, d^4x \exp(ipx - ikx) \frac{1}{m_Q^2 - k^2 - i0} \sum_{n=0}^{\infty} (x^2)^n \int_0^1 d\xi \exp(-ip'x\xi)\phi_n(\xi, \lambda).$$  (3.23)

The functions $\phi_i(\xi)$ are related to $G(z, \xi)$ via Eqs. (2.6) and (2.9). Performing the $x$ and $k$ integrations gives

$$\Gamma_{th}(p^2, q^2) = \int_0^1 \frac{d\xi}{m_Q^2 - (p - \xi p')^2} - \frac{1}{m_Q^2} \int_0^1 \frac{d\xi}{(m_Q^2 - (p - \xi p')^2)^3} \cdots.$$  (3.24)

Here $(p - \xi p')^2 = p^2(1 - \xi) - M^2\xi(1 - \xi) + q^2\xi$. This series (5.24) may be written as the series of spectral representations

$$\Gamma_{th}(p^2, q^2) = \int_0^\infty \frac{ds}{s - p^2}\Delta^{(0)}(s, q^2) + \int_0^\infty \frac{ds}{(s - p^2)^3}\Delta^{(1)}(s, q^2) + \cdots,$$  (3.25)

with $\Delta^{(n)}$ calculable via $\phi_n$. For instance, for $q^2 = 0$ one finds

$$\Delta^{(0)}(s, q^2 = 0) = \frac{1}{m_Q^2} (1 - \xi^*)\phi_0(\xi^*, \lambda), \quad \Delta^{(1)}(s, q^2 = 0) = -\frac{8\phi_1(\xi^*, \lambda)}{1 - \xi^*}, \quad \xi^* = 1 - m_Q^2/s.$$  (3.26)

The corresponding uncut Borel image reads

$$\hat{\Gamma}_{th}(\mu_B^2, q^2) = \int_0^\infty \frac{ds}{m_Q^2} \exp(-s/2\mu_B^2)\Delta^{(0)}(s, q^2) + \frac{1}{2(2\mu_B^2)^2} \int_0^\infty \frac{ds}{m_Q^2} \exp(-s/2\mu_B^2)\Delta^{(1)}(s, q^2) + \cdots,$$  (3.27)

or, equivalently,

$$\hat{\Gamma}_{th}(\mu_B^2, q^2) = \int_0^1 \frac{d\xi}{1 - \xi} \left[ \frac{m_Q^2 \phi_0(\xi)}{\mu_B^2 (1 - \xi)^2} + \cdots \right] \exp(-s\xi/2\mu_B^2).$$  (3.28)

with

$$s\xi = \frac{m_Q^2}{1 - \xi} + M^2\xi - q^2 \frac{\xi}{1 - \xi}.$$  (3.29)

In (3.27) and (3.28) the dots stand for terms corresponding to higher $n$. Clearly, their contributions are suppressed with powers of the Borel parameter $\mu_B$ and vanish in the limit $\mu_B \to \infty$.

Usually, the Borel parameter has the following dependence on the heavy quark mass: $\mu_B^2 = m_Q^2\beta$ [2], where $\beta$ stays finite in the limit $m_Q \to \infty$. Then, the terms corresponding to higher $n$ are not suppressed at $m_Q \to \infty$.

Now, let us consider the cut Borel image of Eq. (3.23). As explained above, the cut is applied to the dispersion representation in the form (3.11). Therefore, if we want to work with the spectral densities $\Delta^{(i)}$, we should take into account the surface terms, similar to Eq. (3.20). The cut Borel image takes the form

$$\hat{\Gamma}_{th}(\mu_B^2, q^2, s_0) = \int_{m_Q^2}^{s_0} ds \exp(-s/2\mu_B^2)\Delta^{(0)}(s, q^2)$$

$$+ \int_{m_Q^2}^{s_0} \frac{ds}{2(2\mu_B^2)^2} \left[ \frac{\partial \Delta^{(1)}(s, q^2)}{\partial s_0} + \frac{\Delta^{(1)}(s, q^2)}{(2\mu_B^2)^2} \right] \exp(-s/2\mu_B^2) + \text{contributions of terms corresponding to higher } n.$$
Making use of the distribution amplitudes $\phi_n$, we rewrite the correlator (3.30) as

$$\hat{\Gamma}_{1b}(\mu_B^2, q^2, s_0) = \int_0^1 \frac{d\xi}{1 - \xi} \left[ \phi_0(\xi) - \frac{m_Q^2}{\mu_B^2} \phi_1(\xi, \lambda) \right] \exp(-s_\xi / 2\mu_B^2) \theta(s_\xi < s_0)$$

$$- 4 \exp(-s_0 / 2\mu_B^2) \left[ \frac{\phi_1(0)}{m_Q^2} + \frac{\phi_1'(0)}{4\mu_B^2(1 - 0)} + \frac{\phi_1''(0)}{s_0} \right] + \cdots,$$

(3.31)

where $\xi_0 = 1 - m_Q^2 / s_0$ and $\cdots$ stand for contributions of the terms corresponding to $n \geq 2$.

The light-cone correlator corresponds to the first term in the expansions (3.30) and (3.31), calculated with the cut parameter $s_0^{LC}$ specific for the LC approximation (see the next section for details):

$$\hat{\Gamma}_{LC}(\mu_B^2, q^2, s_0^{LC}) = \int_{m_Q^2}^{s_0^{LC}} ds \exp(-s / 2\mu_B^2) \Delta(0)(s, q^2) = \int_0^1 \frac{d\xi}{1 - \xi} \phi_0(\xi) \exp(-s_\xi / 2\mu_B^2) \theta(s_\xi < s_0^{LC}).$$

(3.32)

Keeping only this term leads to the following sum rule:

$$I_{MQ} F_{MQ \rightarrow M}(q^2) = \int_0^1 \frac{d\xi}{1 - \xi} \phi_0(\xi) \exp\left(-\frac{s_\xi - M_Q^2}{2\mu_B^2}\right) \theta(s_\xi < s_0^{LC}).$$

(3.33)

Let us study the conditions under which the contributions of higher $n$ are parametrically suppressed compared to the contribution of the light-cone $n = 0$.

1. The heavy-meson mass is related to the heavy quark mass by $M_Q = m_Q + \varepsilon_Q, \varepsilon_Q \simeq \Lambda_{QCD}$. One can then check that the dominant contribution to the integral comes from the end-point region $\xi \sim \Lambda_{QCD} / m_Q$. Therefore, in order to have contributions of higher $n$ suppressed by powers of $1/m_Q$ compared to the $n = 0$ contribution, one needs, e.g., the following end-point behavior:

$$\phi_n(\xi) \sim \xi^{n+1}.$$  

(3.34)

Such a behavior can be obtained, e.g., for the light-cone wave function

$$\Psi_{LC}(\xi, k_\perp^2) = \exp\left(-\frac{k_\perp^2}{2\beta_M^2} \xi(1 - \xi)\right),$$

(3.35)

$\beta_M$ being the size parameter of the light meson. In this specific case the functions $\phi_n(\xi)$ have the necessary behavior (3.34) in the end-point region. However, for realistic wave functions obtained as solutions to the BS equation, the distribution amplitudes $\phi_n(\xi)$ for different $n$ have a similar behavior in the end-point region and therefore the LC expansion for the form factor has no small parameter. In this case, the term $n = 0$ is not parametrically enhanced compared to terms of higher $n$, and, in order to obtain the form factor, terms for all $n$ should be summed. In other words, the transverse motion of the light quark is essential, the longitudinal light-cone distribution amplitude $g_0(\xi)$ is not sufficient, and the knowledge of the full wave function $\Psi(\xi, k_\perp)$ is necessary to obtain the $B \rightarrow M$ form factor.

2. One might expect to have a power suppression of the off-LC terms corresponding to $n \geq 1$ in the limit $\mu_B \rightarrow \infty$, within the “local-duality” sum rule [14], taking into account that in standard SVZ sum rules the choice $\mu_B \rightarrow \infty$ suppresses condensate contributions. Within the context of light-cone sum rules, the limit $\mu_B \rightarrow \infty$ is, however, tricky: In the uncut light-cone correlator the limit $\mu_B \rightarrow \infty$ indeed suppresses the off-LC effects ($x^2 \neq 0$ terms). However, after applying the cut, this property is lost: obviously, the surface terms in the cut correlator (3.30) corresponding to $n \geq 1$ remain finite in the limit $\mu_B \rightarrow \infty$ and can give a sizeable contribution. There is also another subtlety: A specific feature of the local-duality sum rule is that the observables are determined to great extent by the value of the continuum subtraction point $s_0$. Therefore one cannot apply the standard sum-rule stability criteria to extract the physical value of an observable. Nevertheless, assuming the duality interval to be process-independent and fixing $s_0$ from one observable allows one to predict the other observables. However, as we shall see in the next section, when

---

1 We have shown recently that the local-duality sum rule should provide a good description of the pion elastic form factor for all spacelike momentum transfers [14].
the standard criterion to fix the continuum subtraction \( s_0 \) is applied, its value for the full and for the LC correlators turn out to be different from each other.

Finally, we conclude that for the realistic case of the interaction dominated by one-boson exchange at short distances, the LC contribution does not dominate the cut correlator parametrically, i.e., the off-LC terms are not suppressed by any large parameter compared to the LC term. To understand how well the LC contribution numerically compares with the full result, we calculate the correlator \( \hat{\Gamma}_{th} \), Eq. (3.20), and the LC correlator, Eq. (3.32), making use of the BS wave function obtained as solution to the BS equation with a realistic potential dominated by a one-boson exchange at small separations.

4. NUMERICAL RESULTS

In this section we address the case \( q^2 = 0 \), and therefore do not write explicitly the argument \( q^2 \). For the BS kernel \( G(z, \xi) = m^2 \delta(z) \xi (1 - \xi) \) it is straightforward to calculate the spectral densities \( \Delta_{th} \) and \( \Delta_{LC} \). We shall analyse the correlators for the light meson in the final state. We may therefore set \( M = 0 \) and find

\[
\Delta_{th}(s) = \frac{1}{s^3} \left[ (m_Q^2 + m^2) \lambda^{1/2}(s, m_Q^2, m^2) + 4m^2 m^2 \log \left( \frac{s - m_Q^2 - m^2 + \lambda^{1/2}(s, m_Q^2, m^2)}{2m_Q m} \right) \right] \theta(s - (m_Q + m)^2),
\]

\[
\Delta_{LC}(s) = \frac{m_Q^2(s - m_Q^2)}{s^3} \theta(s - m_Q^2).
\]

Here \( \lambda(s, m_Q^2, m^2) = (s - m_Q^2 - m^2)^2 - 4m_Q^2 m^2 \) and \( \Delta_{LC}(s) \) is just \( \Delta^{(0)}(s, q^2 = 0) \) of Eq. (3.25). Clearly, in the limit \( m \to 0 \), both quantities coincide. Notice the relation

\[
\int_{(m_Q + m)^2}^{\infty} ds \Delta_{th}(s) = \int_{m_Q^2}^{\infty} ds \Delta_{LC}(s),
\]

which can be checked explicitly.

Let us now discuss the values of the parameters to be used in numerical estimates.

For the heavy-quark mass the situation is obvious: to study the beauty-meson decay, we choose the value \( m_Q = 4.8 \) GeV used in light-cone sum-rules [3, 4]. The same value is employed in quark-model calculations [16]. To describe decay of the charm meson, we set \( m_Q = 1.4 \) GeV according to sum rules [17].

The relevant choice of the light-quark mass parameter \( m \) requires clarification. Our interest is to understand with the help of the simplified model under consideration the corresponding QCD calculation. We therefore choose the numerical parameters relevant for QCD. Since the light-quark mass parameter \( m \) appears in the framework of the BS equation, it should be understood as the effective quark mass which takes into account nonperturbative effects related to spontaneous chiral symmetry breaking in the soft region, i.e., the constituent quark mass. In [18, 19], the constituent quark mass was calculated through the quark condensate in QCD with the result \( m \simeq 220 \) MeV at the chiral-symmetry breaking scale 1 GeV. The scale-dependence of the constituent mass of the light quark above 1 GeV was also reported in [18]. In sum-rule analyses [3, 4] it was found, that the relevant infrared factorization scale for a heavy-meson decay is \( \sqrt{M_Q^2 - m_Q^2} \). We therefore make use of the constituent quark mass evaluated at this scale. Employing the results from [18], we find the relevant value of the constituent mass of the light quark \( m = 150 \) MeV for beauty-meson decay, and \( m = 200 \) MeV for charm-meson decay. These values of the quark masses will be used in numerical estimates below.

Fig. 2 shows the quantities \( m_Q^2 \Delta_{th} \) and \( m_Q^2 \Delta_{LC} \) for two cases: (a) \( m_Q = 4.8 \) GeV and \( m = 150 \) MeV relevant for \( B \)-decay and (b) \( m_Q = 1.4 \) GeV and \( m = 200 \) MeV relevant for \( D \)-decay. Important for the following is that the thresholds in \( \Delta_{th} \) and \( \Delta_{LC} \) do not coincide: in the light-cone spectral density the threshold is \( m_Q^2 \) whereas in the full spectral density it is \( (m_Q + m)^2 \). The region near the threshold provides the main contribution to the cut Borel-transformed correlators, therefore the mismatch of the thresholds is responsible for the nonvanishing of the off-light-cone effects in sum rules.

Let us briefly address the uncut Borel-transformed full and LC correlators

\[
\hat{\Gamma}_{th}(\mu_B^2) = \int_{(m_Q + m)^2}^{\infty} ds \exp(-s/2\mu_B^2) \Delta_{th}(s), \quad \hat{\Gamma}_{LC}(\mu_B^2) = \int_{m_Q^2}^{\infty} ds \exp(-s/2\mu_B^2) \Delta_{LC}(s).
\]
Eq. (4.2) shows that for large values of the Borel mass $\mu_B$ these two quantities coincide. The same conclusion may be obtained directly from Eq. (3.27): the terms proportional to the off-LC distribution amplitudes are suppressed by powers of the Borel parameter $\mu_B^2$. Thus, in the case of the uncut Borel transform there is a clear limit — large values of the Borel parameter — in which the off-LC effects vanish, independently of the specific value of the light-quark mass. However, the uncut Borel-transformed correlators contain contributions of all possible hadronic states containing the heavy quark, and therefore cannot provide information on the properties of the single heavy meson.

Now, let us study the cut Borel-transformed correlators which are relevant for the extraction of the form factors within QCD sum rules. We shall see that for the cut Borel-transformed correlator the situation is different: namely, there is no physical limit in which the off-LC effects are negligible.\footnote{As we have seen, the full and the LC spectral densities coincide in the limit $m \to 0$. However, the parameter $m$ should be identified with the effective quark mass, which stays finite of order $\Lambda_{\text{QCD}}$ in the chiral limit. Therefore, the limit $m \to 0$ does not correspond to a realistic situation.}

We introduce the parameters $\delta$ related to the continuum subtraction $s_0$ by

$$s_0 = (m_Q + \delta)^2. \quad (4.4)$$

To fix $\delta$, we follow the standard procedure\cite{4}: namely, we require that the quantity

$$\langle s(\beta, q^2, \delta) \rangle \equiv \frac{\int_{s_{\text{low}}}^{(m_Q + \delta)^2} ds \exp \left( -s - \frac{M_Q^2}{2\mu_B^2} \right) s \Delta(s, q^2)}{\int_{s_{\text{low}}}^{(m_Q + \delta)^2} ds \exp \left( -s - \frac{M_Q^2}{2\mu_B^2} \right) \Delta(s, q^2)} \mu_B^2 = m_Q \beta, \quad (4.5)$$

reproduces the heavy-meson mass, i.e.,

$$\langle s(\beta, q^2, \delta) \rangle = M_Q^2, \quad (4.6)$$

both for the LC and the full spectral densities, where $s_{\text{low}} = m_Q^2$ for the LC correlator and $s_{\text{low}} = (m_Q + m)^2$ for the full correlator. The quantity $\delta$ as defined by (4.6) depends on $q^2$ and $\mu_B$: Eq. (4.6) is just the definition of the implicit function $\delta(q^2, \mu_B^2)$. Such a procedure of fixing $\delta$ for the light-cone correlator was employed e.g. in\cite{4}. Recall, however, that there is no unique way to fix $\delta$: one may require instead that $\langle s^n \rangle = (M_Q^2)^n$ for $n > 1$. Moreover, since the spectral densities and the thresholds in Eq. (4.5) are different for $\delta_{\text{th}}$ and $\delta_{\text{LC}}$, also the numerical values of $\delta_{\text{th}}$ and $\delta_{\text{LC}}$ obtained from Eq. (4.6) are different. We discuss here only the case $q^2 = 0$. Taking into account the lack of a unique way to introduce $\delta$, we shall not consider the $\mu_B$-dependent $\delta_{\text{th}}$ and $\delta_{\text{LC}}$. We rather determine the constant values $\delta_{\text{th}}$ and $\delta_{\text{LC}}$ such that the relation (4.6) is satisfied only for one specific value of $\beta$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Spectral densities $m_Q^2 \Delta_{\text{th}}(s)$ (solid red line) and $m_Q^2 \Delta_{\text{LC}}(s)$ (dashed blue line) for parameter values corresponding to beauty-meson decay, $m_Q = 4.8$ GeV, $m = 150$ MeV (left), and to charm-meson decay, $m_Q = 1.4$ GeV, $m = 200$ MeV (right).}
\end{figure}
Fig. 3 presents the numerical results for beauty-meson decay: \( M_Q = 5.27 \) GeV, \( m_Q = 4.8 \) GeV and \( \delta \) fixed by tuning \( \sqrt{\langle s \rangle} \) to \( M_Q = 5.27 \) GeV at two different values of \( \beta \): The plots in the left column correspond to \( \delta_{\text{LC}} = 0.96 \) GeV and \( \delta_{\text{th}} = 0.79 \) GeV, which are obtained from the relation \( \sqrt{\langle s \rangle_{\text{LC}}} = \sqrt{\langle s \rangle_{\text{th}}} = 5.27 \) GeV for \( q^2 = 0 \) and \( \beta = 0.5 \) GeV; the plots in the right column correspond to \( \delta_{\text{LC}} = 0.755 \) GeV and \( \delta_{\text{th}} = 0.69 \) GeV, which are obtained from the relation \( \sqrt{\langle s \rangle_{\text{LC}}} = \sqrt{\langle s \rangle_{\text{th}}} = 5.27 \) GeV for \( q^2 = 0 \) and \( \beta = 4 \) GeV. **First row:** \( \sqrt{\langle s \rangle_{\text{th}}} \) (solid red line) and \( \sqrt{\langle s \rangle_{\text{LC}}} \) vs \( \beta \) (dashed blue line). The horizontal (green) line is \( M_Q = 5.27 \) GeV. **Second row:** \( \Gamma(\beta, q^2, \delta) \) vs \( \beta \) (dashed blue line). The plots in the left column show \( \Gamma(\beta, 27 \) GeV, \( m_Q = 4.8 \) GeV and \( m = 150 \) MeV. **Third row:** The ratio \( \Gamma_{\text{th}}(\beta, q^2, \delta_{\text{th}})/\Gamma_{\text{LC}}(\beta, q^2, \delta_{\text{LC}}) \) vs \( \beta \) for \( q^2 = 0 \).
Fig. 4: Plots corresponding to charm-meson decay $m_Q = 1.4$ GeV, for $\delta$ fixed by tuning $\sqrt{\langle s \rangle}$ to $M_Q = 1.87$ GeV at $\beta = 2$ GeV, and for two different values of the light-quark mass: $m = 200$ MeV (left column) and $m = 100$ MeV (right column).

First row: $\sqrt{\langle s \rangle}_{\text{LC}}$ and $\sqrt{\langle s \rangle}_{\text{th}}$ [Eq. (4.5)] vs $\beta$. The horizontal (green) line is $M_Q = 1.87$ GeV. The parameters $\delta_{\text{LC}} = 0.93$ GeV and $\delta_{\text{th}} = 0.72$ GeV for $m = 200$ MeV (left column) and $\delta_{\text{th}} = 0.85$ GeV for $m = 100$ MeV (right column) are obtained by requiring that at $q^2 = 0$ and $\beta = 2$ GeV $\sqrt{\langle s \rangle}_{\text{LC}} = \sqrt{\langle s \rangle}_{\text{th}} = 1.87$ GeV. Second row: $\tilde{\Gamma}(\beta, q^2, \delta)$ [Eq. (4.7)] at $q^2 = 0$: $\tilde{\Gamma}_{\text{th}}(\beta, q^2, \delta_{\text{th}})$ (solid red line) and $\tilde{\Gamma}_{\text{LC}}(\beta, q^2, \delta_{\text{LC}})$ (dashed blue line). Third row: The ratio $\tilde{\Gamma}_{\text{th}}(\beta, q^2, \delta_{\text{th}})/\tilde{\Gamma}_{\text{LC}}(\beta, q^2, \delta_{\text{LC}})$ at $q^2 = 0$.

the quantity

$$\tilde{\Gamma}(\beta, q^2, \delta) = m_Q^2 \exp \left( \frac{M_Q^2}{2\mu_B^2} \right)^2 \tilde{\Gamma}(\mu_B^2, q^2, s_0), \quad \mu_B^2 = m_Q\beta, \quad s_0 = (m_Q + \delta)^2,$$

(4.7)

for the LC and the full correlators. Finally, the third row gives the ratio of the full to the LC correlators.

Fig. 4 gives the results for charm-meson decay: $M_Q = 1.87$ GeV, $m_Q = 1.4$ GeV and $m = 200$ MeV (left column). To illustrate the influence of the light-quark mass on the off-LC effects, we present also the results for $m = 100$ MeV (right column). The continuum subtraction parameter $\delta$ is fixed from the relation $\sqrt{\langle s(\beta, \delta) \rangle} = M_Q$ at $\beta = 2$ GeV.
To show the origin of the difference between the cut full and light-cone correlators, we consider the limit $m_Q \to \infty$ and $\mu_B \to \infty$, with $\mu_B \gg m_Q$. In this case explicit expressions for the correlators may be obtained:

\[
m^2 \hat{\Gamma}_{LC}(\mu^2_B \to \infty, q^2 = 0, \delta_{LC}) = 2\delta^2_{LC} + O(\delta^3_{LC}/m_Q),
\]

\[
m^2 \hat{\Gamma}_{th}(\mu^2_B \to \infty, q^2 = 0, \delta_{th}) = 2\delta^2_{th} - m^2 \left[ \log \left( \frac{4\delta^2_{th}}{m^2} \right) + 1 \right] + O(m^4/\delta^3_{th}) + O(\delta^3_{th}/m_Q).
\]

(4.8)

For $\mu_B \gg m_Q$ ($\beta \sim m_Q$), the uncut and the cut correlators (both full and LC) behave quite differently for large $m_Q$:

\[
\Gamma(\mu^2_B \to \infty, m^2_Q) = O(1), \quad \Gamma(\mu^2_B \to \infty, m^2_Q, \delta) = O(\delta^2/m^2_Q).
\]

(4.9)

Thus, the cut correlator picks up only a small fraction of the full correlator from the region not far from the threshold.

(For $\beta \ll m_Q$, both the cut and the uncut correlators have a similar behavior $\sim 1/m^2_Q$.)

Now, fixing $\delta_{th}$ and $\delta_{LC}$ according to the standard procedure [1,6], we express these quantities via the binding energy of the heavy meson $\varepsilon_Q$ defined according to $M_Q = m_Q + \varepsilon_Q$:

\[
\delta_{LC} = \frac{3}{2} \varepsilon_Q,
\]

\[
\delta_{th} = \frac{3}{2} \varepsilon_Q - \frac{2m^2}{3\varepsilon_Q} \left[ \log \left( \frac{3\varepsilon_Q}{m} \right) - 1 \right] + \cdots,
\]

(4.10)

leading to

\[
m^2 \hat{\Gamma}_{LC}(\mu^2_B \to \infty, q^2 = 0, \delta_{LC}) = \frac{9}{2} \varepsilon_Q^2,
\]

\[
m^2 \hat{\Gamma}_{th}(\mu^2_B \to \infty, q^2 = 0, \delta_{th}) = \frac{9}{2} \varepsilon_Q^2 - 6m^2 \log \left( \frac{3\varepsilon_Q}{\sqrt{em}} \right) + \cdots,
\]

(4.11)

and thus

\[
\frac{\hat{\Gamma}_{th}(\mu^2_B \to \infty, q^2 = 0, \delta_{th})}{\hat{\Gamma}_{LC}(\mu^2_B \to \infty, q^2 = 0, \delta_{LC})} = \left[ 1 - \frac{4m^2}{3\varepsilon_Q} \right] \log \left( \frac{3\varepsilon_Q}{\sqrt{em}} \right) + \cdots.
\]

(4.12)

In the expressions above the dots denote terms containing higher powers of $m/\varepsilon_Q$. This example illustrates that the off-LC effects may play an essential role in the cut correlators, as their contribution is not suppressed by any large parameter: the quantities $m$ and $\varepsilon_Q$ have the same order of magnitude.

Let us emphasize that we compare the full and the light-cone correlators evaluated at different values of the cut parameters $\delta_{LC}$ and $\delta_{th}$. From our point of view this very comparison is relevant if one wants to understand the error due to taking into account only the light-cone ($x^2 = 0$) contribution to the correlator and neglecting terms containing higher powers of $x^2$.

One could also compare the correlators for the same value of $\delta$. The difference between the full and the LC correlators is only slightly reduced in this case, the ratio still remaining $O(m^2/\delta^2)$. This can be seen by setting $\delta_{LC} = \delta_{th}$ in (4.8).

The following lessons may be drawn from the results presented in this section:

a. The off-LC effects play an essential role in the cut correlator, as they are not suppressed by any large parameter.

Numerically, the difference between the full and the LC correlators, evaluated at the same value of the Borel parameter, is $10 \pm 20\%$. This difference is due to the off-LC effects.

b. The functions $\hat{\Gamma}_{LC}(\beta)$ and $\hat{\Gamma}_{th}(\beta)$ have the same shape, but $\hat{\Gamma}_{LC}(\beta)$ lies well above $\hat{\Gamma}_{th}(\beta)$, if the standard procedure of fixing $\delta_{LC}$ and $\delta_{th}$ [1,6] is used. The values of both (th and LC) correlators obtained with $\delta_{th}$ and $\delta_{LC}$ tuned at $\beta = 0.5$ GeV (left column in Fig. 3) are greater than the values of the respective correlators obtained with $\delta_{th}$ and $\delta_{LC}$ tuned at $\beta = 4$ GeV (right column in Fig. 3). The local stability is better when one fixes $\delta$ from (4.10) at a larger value of $\beta$ (right column in Fig. 3). In this case, both Borel curves for $\hat{\Gamma}_{LC}$ and $\hat{\Gamma}_{th}$ show a good stability in $\beta$. Nevertheless, still $\hat{\Gamma}_{LC}$ is much larger than $\hat{\Gamma}_{th}$! This illustrates that the Borel stability per se does not guarantee the extraction of the correct physical value.

c. The difference between $\hat{\Gamma}_{LC}$ and $\hat{\Gamma}_{th}$ increases with increasing mass of the light quark. Therefore, this difference is expected to be greater for the heavy mesons $B_s$ and $D_s$, containing the strange $s$-quark, than for $B$ and $D$.

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3 Notice that $\delta_{LC}$ turns out to be different from $\delta_{th}$. This has the following origin: If we include $N$ terms in the LC expansion of the cut correlator $\Delta^{(N)}$, cut them at $\delta_{LC}$, and determine the latter from Eqs. (4.9) and (4.10), then $\lim_{N \to \infty} \delta^{(N)}_{LC} = \delta_{th}$. Since we have included only one ($n = 0$) term, we obtain $\delta^{(0)}_{LC} \neq \delta_{th}$.
5. CONCLUSIONS

In this paper, we studied the correlator

\[ i \int dx \exp(ipx) \langle 0 | T \varphi(x)Q(x)Q(0)\varphi(0)|M(p') \rangle, \]

which is one of the basic objects for extracting the heavy-to-light form factor within the method of sum rules. We have shown that to leading 1/mQ-accuracy this correlator may be calculated through the BS amplitude of the light meson

\[ \langle 0 | T \varphi(x)\varphi(0)|M(p') \rangle. \]

Expanding the BS amplitude near the light cone \( x^2 = 0 \) generates the light-cone expansion of the correlator.

Making use of the Nakanishi representation for the BS amplitude, we obtained dispersion representations for the full and the LC correlators in terms of the kernel \( G(z, \xi) \) of the Nakanishi representation. We studied the full and the light-cone correlators and their Borel transforms depending on the properties of the kernel \( G(z, \xi) \).

We then made use of the known solution for \( G(z, \xi) \) in a model with light scalar particles interacting by an exchange of a massless boson. This relatively simple model provides a good laboratory for studying QCD since the corresponding bound-state wave functions have properties similar to the properties of hadron wave functions in QCD. We calculated the full and the light-cone correlators and their Borel transforms in the variable \( p^2 \), and studied these correlators for various prescriptions to fix the heavy-hadron continuum subtraction and in various regions of the parameters relevant for extracting the heavy-to-light form factors. This work thus represents the first systematic study of the off-light-cone effects in QCD sum rules for heavy-to-light form factors.

Our main results may be summarized as follows:

1. We have seen that — after performing the Borel transform — the light-cone correlator provides numerically the bulk of the full correlator, although parametrically the off-LC effects are not suppressed compared to the LC contribution. This observation holds for various prescriptions of fixing the heavy-hadron continuum subtraction point (i.e., a cut applied to the correlator for isolating the contribution of the heavy hadron of interest in the initial state) and a wide range of masses of particles involved in the decay process.

2. We demonstrated that, nevertheless, the difference between the cut full and the cut light-cone correlators always remains nonvanishing. For example, fixing the continuum subtraction points the by standard criteria, we have found the following relation for the cut Borel transforms of the full and the LC correlators for \( m_Q \to \infty \), \( \mu_B \to \infty \), and \( q^2 = 0 \):

\[
\frac{\hat{\Gamma}_{\text{th}}(\mu_B^2 \to \infty, q^2 = 0, \delta_{\text{th}})}{\hat{\Gamma}_{\text{LC}}(\mu_B^2 \to \infty, q^2 = 0, \delta_{\text{LC}})} = 1 - \frac{4m^2}{3\varepsilon_Q^2} \log \left( \frac{3\varepsilon_Q}{\sqrt{em}} \right) + \cdots ,
\]

(5.1)

the correction being always negative. Here \( m \) is the effective constituent mass of the light quark, which emerges from the BS equation, \( m \approx \Lambda_{\text{QCD}} \), and \( \varepsilon_Q \) is the binding energy of the heavy meson \( M_Q = m_Q + \varepsilon_Q \). Taking into account that the constituent quark mass remains finite in the chiral limit, we come to the following important conclusion: In heavy-to-light decays, there exists no rigorous theoretical limit in which the cut LC correlator coincides with the cut full correlator.

Thus, the off-light-cone effects in sum rules for heavy-to-light correlators are not negligible and should be taken into account.

3. We note that the Borel curves for the full and the LC correlators have similar shapes. However the light-cone correlator systematically overestimates the full correlator, the difference at small \( q^2 \) being 10 \( \div \) 20\% in a wide range of the heavy-quark mass relevant for charm and beauty decays. We want to point out that the similarity of the Borel curves for the full and the LC correlators implies that the systematic difference between the correlators cannot be diminished by a relevant choice of the criterion for extracting the heavy-to-light form factor.

The observed effect might suggest a systematic uncertainty in the results for form factors obtained within light-cone sum rules. As follows from the relation 5.1, this uncertainty is expected to be larger for decays of heavy mesons containing the strange quark, \( B_s \) and \( D_s \), than for the \( B \) and \( D \) mesons. This issue deserves further investigation.

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4 The Nakanishi representation leads to technical simplifications, but conceptually any other form of the BS amplitude may be used.
Finally, we point out the following: Although the model, discussed here, in many aspects differs from QCD, this model mimics correctly those features which are essential for the effects discussed. Therefore, many of the results obtained in this paper are valid also for QCD. In particular, the expression \((5.1)\) suggests the following relationship between the light-cone and the full correlators in QCD for large values of \(m_Q\) and \(\mu_B\):

\[
\frac{\hat{\Gamma}_{lh}(\mu^2_B,q^2,\delta_{lh})}{\hat{\Gamma}_{LC}(\mu^2_B,q^2,\delta_{LC})} = 1 - \mathcal{O}\left(\frac{\Lambda_{QCD}}{\delta}\right).
\]

In numerical estimates, we used the parameters relevant for \(B\) and \(D\) decays. We therefore believe that also the numerical estimates for higher-twist effects obtained in this work provide a realistic estimate for higher-twist effects in QCD.

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[1] I. I. Balitsky, V. M. Braun, and A. V. Kolesnichenko, Nucl. Phys. B312, 509 (1989); V. M. Braun and I. Filyanov, Z. Phys. C44, 157 (1989); V. I. Chernyak and I. R. Zhitnitsky, Nucl. Phys. B345, 137 (1990).
[2] P. Colangelo and A. Khodjamirian, QCD sum rules: a modern perspective, in At the Frontier of Particle Physics, edited by M. Shifman (World Scientific, Singapore, 2001), vol. 3, p. 1495 [hep-ph/0010175].
[3] P. Ball and V. M. Braun, Phys. Rev. D58, 094016 (1998).
[4] P. Ball and R. Zwicky, Phys. Rev. D71, 014015 (2005).
[5] V. M. Braun and I. Halperin, Phys. Lett. B328, 457 (1994).
[6] N. Nakanishi, Phys. Rev. 130, 1230 (1963).
[7] N. N. Bogolyubov and D. V. Shirkov, Quantum Fields, Nauka, Moscow (1980).
[8] V. A. Karmanov and J. Carbonell, Eur. Phys. J. A27, 1 (2006).
[9] M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. B147, 1 (1979).
[10] M. Shifman, Prog. Theor. Phys. Suppl. 131, 1 (1998) [hep-ph/9802214].
[11] M. Jamin and B. Lange, Phys. Rev. D65, 056005 (2002).
[12] T. Kleinschmidt, [hep-ph/0409039] (unpublished).
[13] D. Melikhov, Phys. Rev. D53, 2460 (1996); Phys. Rev. D56, 7089 (1997); Eur. Phys. J. direct C4, 2 (2002) [hep-ph/0110087]; W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D75, 016001 (2007).
[14] V. A. Nesterenko and A. V. Radyushkin, Phys. Lett. 115B, 410 (1982).
[15] W. Lucha and D. Melikhov, Phys. Rev. D73, 054009 (2006).
[16] D. Melikhov and B. Stech, Phys. Rev. D62, 014006 (2000).
[17] P. Ball, Phys. Lett. B641, 50 (2006).
[18] D. Melikhov and S. Simula, Eur. Phys. Journal C37, 437 (2004).
[19] W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D74, 054004 (2006).