Implementation of quantum and classical discrete fractional Fourier transforms

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Fourier transforms, integer and fractional, are ubiquitous mathematical tools in basic and applied science. Certainly, since the ordinary Fourier transform is merely a particular case of a continuous set of fractional Fourier domains, every property and application of the ordinary Fourier transform becomes a special case of the fractional Fourier transform. Despite the great practical importance of the discrete Fourier transform, implementation of fractional orders of the corresponding discrete operation has been elusive. Here we report classical and quantum optical realizations of the discrete fractional Fourier transform. In the context of classical optics, we implement discrete fractional Fourier transforms of exemplary wave functions and experimentally demonstrate the shift theorem. Moreover, we apply this approach in the quantum realm to Fourier transform separable and path-entangled biphoton wave functions. The proposed approach is versatile and could find applications in various fields where Fourier transforms are essential tools.
Two hundred years ago, Joseph Fourier introduced a major concept in mathematics, the so-called Fourier transform (FT). It was not until 1965, when Cooley and Tukey developed the ‘fast Fourier transform’ algorithm, that Fourier analysis became a standard tool in contemporary sciences1. Two crucial requirements in this algorithm are the discretization and truncation of the domain, where the signals to be transformed are defined. These requirements are always satisfied, since observable quantities in physics must be well behaved and finite in extension and magnitude.

In 1980, Namias made another significant leap with the introduction of the fractional Fourier transform (FrFT), which contains the FT as a special case2. Several investigations quickly followed, leading to a more general theory of joint time-frequency signal representations3 and fractional Fourier optics4. The vast scope of the FrFT has been demonstrated in areas such as wave propagation, signal processing and differential equations3,5–7.

So far, the FT of fractional order was realized only by single-lens systems8,9, although other theoretical suggestions, including multi-lens systems10 or graded index fibre exist11. The aim to discretize this generalized FT led to the introduction of the discrete fractional Fourier transform (DFrFT) operating on a finite grid in a way similar to that of a discrete FT12. Along those lines, several versions of the DFrFT have been introduced9, however, without any experimental realization, so far. In this work we focus on the optical implementation of the so-called Fourier–Kravchuk transform12 that can be equally applied to the classical and quantum states. Throughout our paper, we simply refer to this transform as DFrFT whose application reaches from the demonstration of the Fourier suppression law13, N00N-state generation14,15 and qubit storage16 to the realization of perfect propagation, signal processing and differential equations 3,5–7.

Meanwhile, j represents an arbitrary positive or half-integer that determines the total number of waveguides via N = 2j + 1 (Fig. 1b).

Coupled mode theory states that the evolution of light in the Jk-waveguide array is governed by the following set of equations20

\[ \frac{d}{dZ}E_m(Z) = \frac{1}{\kappa_0} \sum_{n=-j}^{j} (J_k)_{m,n} E_n(Z). \]  

Here, \( E_n(Z) \) is the complex electric field amplitude at site n. In the quantum optics regime, single photons traversing such devices are governed by a set of Heisenberg equations that are isomorphic to equation (1). The only difference is that in the quantum case \( E_n(Z) \) must be replaced by the photon creation operator \( a_n^\dagger(Z) \).

In a spinor context, the evolution parameter Z is associated with time, whereas in the framework of integrated quantum optics, Z represents the propagation distance, see Fig. 1b–e. A spectral decomposition of the \( J_k \)-matrix yields the eigenvectors

\[ u_m^{(j)} = 2^n \sqrt{\frac{(j+n)!}{(j-n)!}} \frac{1}{m-n+j} \]  

(0)14,19, which in combination with the eigenvalues, \( \beta_m = -j, \ldots, j \), render the closed-form point-spread function

\[ G_{p,q}(Z) = \exp \left\{ -q \frac{1}{2} \frac{\sin \left[ \frac{Z}{2} \right]}{\sin \left[ \frac{Z}{2} \right]} \right\} \]

(2)
Note that $q$ and $p$ represent the excited and observed sites, respectively, and $P_n^{(4,0)}(x)$ are the Jacobi polynomials of order $n$ (see Methods section for discussion). Using equation (2), we can compute the response of the system to any input signal, which in turn gives the DFrFT$^{12}$. Accordingly, DFrFT of any particular order arises at one specific propagation distance $Z$ lying between $0$ and $\pi/2$. In the limit $N \to \infty$, the eigenvectors of $I_x \cdot u_{x/N}^{(m)}$ become the continuous Hermite–Gauss polynomials $H_m(x)$, which are known to be the eigenfunctions of the fractional Fourier operator$^{12}$. As a result, in the continuous limit, the DFrFT described by equation (2) converges to the continuous FrFT$^{5,12}$; and the standard FT is recovered at $Z = 0$. The experimental data (blue crosses) is compared with the numeric FrFT (red curves). (Fig. 2a).

Note that in general, the DFrFT obtained in our devices and the usual DFT become equal only in the continuous limit $N \to \infty$ and at $Z = \pi/2$.

**Experiments with classical light.** To experimentally demonstrate the functionality of the suggested waveguide system, we use $N=21$ waveguides to perform FTs of simple wave packets. We first consider a Gaussian wave packet with a full-width at half-maximum (FWHM) covering the five central sites (Fig. 2a). The input signal is prepared by focusing a Gaussian beam from a HeNe laser onto the front facet of the sample. By exploiting the fluorescence from colour centres within the waveguides$^{21}$, we monitor the full intensity evolution from the input to the output plane. The fluorescence image, Fig. 2a, shows a gradual transition from an initially narrow Gaussian distribution at the input to a broader one at the Fourier plane (left and right panels Fig. 2a), demonstrating that narrow signals in space correspond to broad signals in Fourier space. For intermediate propagation distances ($Z \in [0, \pi/2]$) we extract other orders of the DFrFT, simultaneously. For comparison, we plot the continuous FrFT produced by the corresponding continuous Gaussian profile (red curves Fig. 2a). The agreement between the computed FrFT and the experimental DFrFT proves that for the considered Gaussian input signal, $N=21$ is sufficient to achieve the continuous limit. We now shift the input Gaussian beam by six channels towards the edge. Since the separations between adjacent waveguides at the edges are bigger than the separations between adjacent waveguides in the centre, the discretization grid is not perfectly homogeneous. Strictly speaking, the discretized shifted Gaussian just at the input plane covers slightly less than five waveguides FWHM. We observe that the well-approximated off-centre Gaussian travels to the centre at $Z = \pi/2$ (Fig. 2b), hereby showing the famous shift theorem. In additional experiments, extended signals, for example, a shifted top-hat function, are found to be well transformed according to equation (2) as well. However, we find that for this type of excitation $N > 21$ would be required to discuss the continuous limit (see Supplementary Note 1 with Supplementary Fig. 1).

An unequivocal criterion, for the functionality of devices that perform the DFrFT, equation (2), can be formulated by evaluating $G_p^{(q)}(Z)$. At this particular distance, point-like excitations will give rise to signal magnitudes that perfectly resemble the magnitudes of one of the eigenstates of the transform. More specifically, for transforms such as equation (2), one finds that an excitation of the $q$th site excites the $q$th system eigenstate up to local phases ($G_p^{(q)}(Z) = \beta^{-q} u_p^{(q)}$) (see the Methods section for explanations). The experimental demonstration of this intriguing

**Figure 2 | DFrFT of classical light.** (a) Transformation of a Gaussian input into a Gaussian profile of larger width along the evolution in the $J_x$-array. The FT is obtained at $Z = \pi/2$. The experimental data (blue crosses) is compared with the numeric FrFT (red curves). (b) A shifted input Gaussian profile evolves towards the centre of the array and acquires the same width as in (a).
effect is shown in the subpanels of Fig. 3a–d along with the theoretical predictions. It can be argued that for any point-like excitation, the continuous limit cannot be met experimentally (see the Methods for discussion). Instead, equation (2) creates a non-uniform amplitude distribution with a phase difference of $\pi/2$ between adjacent sites. Nevertheless, in the continuous limit, $G_{p,q}(Z)$ tends to the usual FT kernel$^{12}$. At this point, it is worth emphasizing the formal equivalence to the quantum Heisenberg XY model in condensed matter physics$^{22,23}$. In this respect, our observations demonstrate the capability of the here-presented systems to store quantum information in XY Hamiltonians by converting specific inputs into eigenstates of the system$^{16}$. To our knowledge, this rather rare property has never been thoroughly investigated before.

**Quantum experiments.** To demonstrate the applicability of our approach in the quantum domain, we now analyse intensity correlations of separable and path-entangled photon pairs propagating through these Fourier transformers. To do so, we fabricated $J_x$-arrays involving $N=8$ channels. The importance of exploring FTs of such states has been highlighted in several investigations, demonstrating interesting effects such as suppression of states and portraying biphoton spatial correlations$^{24–26}$.

In this discrete quantum optical context, pure separable two-photon states are readily produced by coupling pairs of indistinguishable photons into two distinct lattice sites $(m, n)$, this state is mathematically described by $|\Psi(0)\rangle=a_m^\dagger a_n^\dagger |0\rangle$. Conversely, path-entangled two-photon states are created by simultaneously launching both photons at either site $m$ or $n$ with exactly the same probability, that is, $|\Psi(0)\rangle=(|a_m^\dagger|^2 + |a_n^\dagger|^2)|0\rangle/2$. Furthermore, the probability of observing one of the photons at site $k$ and its twin at site $l$ is given by the intensity correlation matrix $\Gamma_{kl}(Z)=\langle a_k^\dagger a_l^\dagger a_k a_l \rangle$ (ref. 27). An intriguing and unique property of the $J_x$-systems is that at $Z=\pi/2$ the correlation matrices are...
given in terms of the eigenstates, as noticed above. Hence, for the separable case, $|\Psi(0)\rangle=a^\dagger a^\dagger|0\rangle$, the correlation matrices are given by $\Gamma_{kk}=|u^{(m)}_k u^{(n)}_k + u^{(n)}_k u^{(m)}_k|^2$; whereas for the path-entangled state, $|\Psi(0)\rangle=\frac{1}{2}|(a^\dagger)^2 + (a^\dagger)^2|0\rangle/2$, we have $\Gamma_{kk}=|(-i)^2 u^{(m)}_k u^{(m)}_k + (-i)^2 u^{(n)}_k u^{(n)}_k|^2$. Of particular interest is the separable case, where the photons are symmetrically coupled into the outermost waveguides, $|\Psi(0)\rangle=a^\dagger a^\dagger|0\rangle$. In this scenario, only the correlation matrix elements for which $(k+l)$ are odd are nonzero, and are given by $\Gamma_{kk}=4(k+l+1)(j+k)! (j-k)! (j-l)! (j-l)! \frac{P_{j+k}^{j+k}(0)P_{j-k}^{j-k}(0)P_{j+l}^{j+l}P_{j-l}^{j-l}(0)/[(N-1)!]^2$. These effects are demonstrated for the initial state $|\Psi(0)\rangle=a^\dagger a^\dagger|0\rangle$ in Fig. 4a, where concentration and absence of probability in the correlation matrix clearly show that some states are completely suppressed—a hallmark of any Fourier unitary process. An estimation of the statistical significance of the data set, along with a short discussion on incoherence effects, can be found in the Supplementary Note 2 involving Supplementary Figs 2 and 3.

As a second case, we consider a fully symmetric path-entangled two-photon state of the form $|\Psi(0)\rangle=|a^\dagger|^2 + |a^\dagger|^2|0\rangle/2$. Physically, both photons are entering together into the array at either site $j$ or $-j$ with equal probability. The correlations are determined by $\Gamma_{kk}=|u^{(m)}_k u^{(n)}_k + u^{(n)}_k u^{(m)}_k|^2$, from which we infer that the probability of measuring photon coincidences at coordinates $(k, l)$ vanishes at sites where the sum $(k+l)$ is odd. In contrast, at coordinates where $(k+l)$ is even, the correlation function collapses to the expression $\Gamma_{kk}=4(k+l+1)(j+k)! (j-k)! (j+l)! (j-l)! \frac{P_{j+k}^{j+k}(0)P_{j-k}^{j-k}(0)P_{j+l}^{j+l}P_{j-l}^{j-l}(0)/[(N-1)!]^2$. This indicates that in this path-entangled case the correlation map appears rotated by 90° with respect to the matrix obtained with separable two-photon states. We performed an experiment to demonstrate these predictions using states of the type $|\Psi(0)\rangle=|a^\dagger|^2 + |a^\dagger|^2|0\rangle/2$, which were prepared using a 50:50 directional coupler acting as a beam splitter. The whole experiment is achieved using a single chip containing both the state preparation stage followed by a $J_{10}$-system, yielding high interferometric control over the field dynamics (Fig. 5). The experimental measurements are presented in Fig. 4b. Similarly, suppression of states occurs as a result of destructive quantum interference. As predicted, a closer look into the correlation pattern reveals that indeed the correlation map appears rotated by 90° with respect to the matrix obtained with separable two-photon states.

**Discussion**

We emphasize that our quantum measurements feature interference fringes akin to the ones observed in quantum Young's
two-slit experiments of biphoton wave functions in free space as demonstrated in ref. 24. In such free-space experiments, however, far-field observations were carried out using lenses and the two slits were emulated by optical fibres3,4. Along those lines, we have created a fully integrated quantum interferometer to observe fundamental quantum mechanical features5,25. This additionally suggests an effective way to generate quantum states containing only even (odd) non-vanishing inter-particle distance probabilities for the separable input state (symmetric path-entangled state). In addition, the eigenfunctions associated with the Hamiltonian system explored in our work are specific Jacobi polynomials, which are well known as the optimal basis for quantum optics34,35. Also, FrFTs appear naturally in optics as the FrFT6,33. The Radon–Wigner function is a basic tool for the Radon–Wigner transform given by the squared modulus of the wavefunction and its FT, a phase-retrieval algorithm for signals that are square integrable in the continuous limit. Thus, the operator representation of the quantum harmonic oscillator2. Let us consider the eigenvalue equation for this matrix

\[ J_x \psi_{m} = \pm \sqrt{m+1} \psi_{m+1} + \sqrt{m} \psi_{m-1} \]

(10)

The indices \( m \) and \( n \) range from \(-j\) to \( j \) in unit steps and \( j \) is an arbitrary positive integer or half-integer. The dimension of the \( J_z \)-matrix is \( N = 2j + 1 \). We now introduce the variable \( \gamma = (j+1) \) \((N^2 - 1)/4\), which implies that \( (J_m)_{mn} \) can be written as

\[ (J_m)_{mn} = \frac{\sqrt{\gamma}}{2} \left( \sqrt{1 - \frac{m}{\gamma}} \psi_{m+1} + \sqrt{1 - \frac{m}{\gamma}} \psi_{m-1} \right) \]

(11)

We let consider the eigenvalue equation for this matrix

\[ \sqrt{\gamma} \left( \sqrt{1 - \frac{m}{\gamma}} \psi_{m+1} + \sqrt{1 - \frac{m}{\gamma}} \psi_{m-1} \right) = \beta_n \psi_m, \]

(12)

Considering the region \( m \ll j \), since \( \gamma \approx N^2 \), in the limit \( N \rightarrow \infty \), the terms \( m (m+1) / \gamma \ll 1 \). Hence, in the domain far from the edge of the array a Taylor expansion yields

\[ \sqrt{\gamma} \left( \sqrt{1 - \frac{m}{\gamma}} \psi_{m+1} + \sqrt{1 - \frac{m}{\gamma}} \psi_{m-1} \right) = \beta_n \psi_m \]

(13)

By defining \( m = y/\sqrt{1/4} \) (or \( x = m/\sqrt{1/4} \)), we obtain

\[ \sqrt{\gamma} \left( \sqrt{1 - \frac{m}{\gamma}} \psi_{m+1} + \sqrt{1 - \frac{m}{\gamma}} \psi_{m-1} \right) \]

(14)

Plugging this expression into equation (12)

\[ \left( \sqrt{1/4} - \frac{x^2}{2} - \frac{x^4}{8} \right) \psi_{m+1} + \left( \sqrt{1/4} - \frac{x^2}{2} + \frac{x^4}{8} \right) \psi_{m-1} \]

\[ \approx 2 \beta_n \psi_m \]

(15)

We redefine the functions \( \psi_{m} = \psi(x) = \psi_m(\sqrt{\gamma}) \) and \( \psi_{m+1} = \psi(x + 1) = \psi_m(\sqrt{\gamma} + 1) \)

such that we can introduce the Taylor series

\[ \psi_{m+1} = \psi(x) + \frac{1}{\sqrt{\gamma}} \psi'(x) + \frac{1}{2! \sqrt{\gamma}^2} \psi''(x) + \frac{1}{3! \sqrt{\gamma}^3} \psi'''(x) + \cdots, \]

(16)

where we have kept only terms up to second order in \( 1/\sqrt{\gamma} \). Substituting equation (16) into equation (15), and using the limit

\[ \lim_{N \rightarrow \infty} \left( \frac{4}{N^2} \right) \left( 1 - \frac{m}{N} \right) = 0. \]

(17)

We obtain the time-independent Schrödinger equation for the harmonic oscillator

\[ \left( - \frac{1}{2 \alpha^2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \alpha^2 \right) \psi(x) = (\sqrt{\gamma} - \beta) \psi(x). \]

(18)

Therefore, in the continuous limit \( N \rightarrow \infty \), the difference equation describing \( J_z \)-photonic lattices becomes the time-independent Schrödinger equation for the quantum harmonic oscillator. Note, however, that due to the importance of the condition \( m \ll j \) in this derivation, this statement is only valid when dealing with signals that are square integrable in the continuous limit. Thus, the operator equation (11) can be used to define the discrete version of the quantum harmonic oscillator and thus the DFrFT.

The point-source function for \( J_z \)-photonic lattices. In this section, it is shown that at \( z = n/2 \), the green function of \( J_z \)-systems becomes proportional to the amplitude of one of the eigenstates. The evolution of light in \( J_z \)-arrays is governed by the set of \( N \) coupled differential equations (equation (11)).

The normalized propagation coordinate \( Z \) is given by \( Z = z \kappa_z \), where \( z \) is the actual propagation distance and \( \kappa_z \) is an arbitrary scale factor. The quantity \( E_{\alpha}(Z) \) denotes the mode field amplitude at site \( n \). A spectral decomposition of the \( J_z \)-matrix yields the eigensolutions

\[ u_n^{(m)} = (2)^{n} \sqrt{[j + n]! [j - n]!} \sqrt{[j + m]! [j - m]!} \beta_n \left( m - n - n \right) \]

(19)
Experiment on the characterization of two-photon correlations

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Acknowledgements
We gratefully acknowledge financial support by the German Ministry of Education and Research (Center for Innovation Competence programme, grant no. 03Z1HN31) and the Deutsche Forschungsgemeinschaft (grant no. NO462/6-1 and SZ276/7-1). M.L. thanks the Initial Training Network PICQUE (grant no. 608062) within the Seventh Framework Programme for Research of the European Commission for funding.

Author contributions
S.W. and A.P.-L. conceived the idea. S.W. and M.L. designed the samples and performed the measurements. A.P.-L. and S.W. developed the theory. S.W., M.L. and A.P.-L. analysed the data. A.S. supervised the project. All authors discussed the results and co-wrote the manuscript.

Additional information
Supplementary Information accompanies this paper at http://www.nature.com/naturecommunications

Competing financial interests: The authors declare no competing financial interests.

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How to cite this article: Weimann, S. et al. Implementation of quantum and classical discrete fractional Fourier transforms. Nat. Commun. 7:11027 doi: 10.1038/ncomms11027 (2016).

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