Entropy production is a key concept of thermodynamics and allows one to analyze the operation of engines. For the Scovil-Schulz-DuBois heat engine, the archetypal three-level thermal maser coupled to thermal baths, it was argued that the common definition of heat flow may provide negative entropy production for certain parameters [E. Boukobza and D. J. Tannor, Phys. Rev. Lett. 98, 240601 (2007)]. Here, we show that this can be cured, if corrections for detuning are properly applied to the energies used for the bath transitions. This method can be used more generally for the thermodynamical analysis of optical transitions where the modes of the light field are detuned from the transition energy.

I. INTRODUCTION

With the realization of masers and lasers quantum optics has proved fertile ground for thermodynamic research in open quantum systems. An archetypal example is the Scovil-Schulz-DuBois heat engine based on a three-level maser driven by two heat baths of different temperatures [1]. This system served as a model to develop a variety of approaches for the microscopic description of quantum systems in contact with thermal baths in interaction with classical [2–6] or quantized light fields [7–9].

Of key relevance is the formulation of work and heat in the quantum realm. Refs. [10, 11] defined work flow (power) and heat flow by partitioning the time derivative of the expectation value of the full Hamiltonian, i.e. including the time-dependent interaction with classical degrees of freedom, such as a microwave field. Later, Boukobza and Tannor [5] proposed an alternative definition of power and heat flow by restricting to the bare Hamiltonian, which describes the system itself and lacks explicit time dependence. This is sometimes conceptually simpler and was, e.g., also used in Refs. [12, 13]. The authors argued [14], that the bare heat flows always provide a positive entropy production [15] for the three level maser, while this was not the case for full heat flows in their treatment, which thus may violate the second law of thermodynamics. (We use full and bare in the sense that they relate to the Hamiltonian from which the flows are derived.) The correct definition of heat and work is actually still an open issue, see, e.g., the discussion on page 339 of [16], where further references are given.

In this work we study the definitions for work and heat for the three-level maser coupled to a classical microwave field, where the bath couplings are treated by a Lindblad dissipator as outlined in Sec. II. Sec. III focuses on the different definitions of heat and work, where we essentially follow Ref. [14] showing a violation of entropy production for the full approach. In Sec. IV we present a reformulation of their expressions, which allows for the correct identification of energies supplied by the baths.

FIG. 1. Energy diagram of the three-level maser subjected to a modulating field (dotted arrow), where the transitions $g \leftrightarrow u$ and $g \leftrightarrow l$ (full arrows) are coupled to different reservoirs. A finite value of $\Delta = \omega_d - (\omega_u - \omega_l)$ reflects the detuning between the modulating field and the energy level difference.

Using these we recover a strictly positive entropy production for the full heat flows.

II. THE SYSTEM

We consider the three-level system of Scovil and Schulz-DuBois [1] consisting of an upper ($u$) and lower ($l$) maser level and the ground level ($g$), see Fig 1. Throughout this article we set $\hbar = k_B = 1$ in order to simplify the notation. The full system Hamiltonian, $H = H_0 + V(t)$, consists of the bare Hamiltonian $H_0 = \omega_u \sigma_{uu} + \omega_l \sigma_{ll}$ and a modulating external field $V(t) = \epsilon (e^{i \omega_d t} \sigma_{lu} + e^{-i \omega_d t} \sigma_{ul})$, where $\epsilon$ is the strength of the driving field, $\omega_d$ its modulating frequency, and we use the operators $\sigma_{ij} = \ket{i} \bra{j}$. Without loss of generality the energy of the ground state $\ket{g}$ is set to zero. The three-level system is connected to two bosonic reservoirs (baths), which are labeled by $\alpha$, where $\alpha \in \{u,l\}$. The bath $\alpha$ couples to the transition $\ket{g} \leftrightarrow |\alpha\rangle$ with strength $\gamma_\alpha$, where an average number of excitations $n_\alpha$ is available in the bath. The model and the analysis of its steady state behavior summarized below follows recent work by [5, 13, 14].

The time evolution of the system density matrix $\rho$ is assumed to be Markovian and governed by the Lindblad

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master equation [17]
\[ \dot{\rho} = -i[H(t), \rho] + \mathcal{L}_u[\rho] + \mathcal{L}_l[\rho] \]  
(1)
where the coupling to the baths are described by \( \mathcal{L}_a[\rho] = \gamma_a n_a D_{\sigma a}[\rho] + \gamma_a (n_a + 1) D_{\sigma a}[\rho] \) with the dissipator \( D_a[\rho] = \sigma \rho a^\dagger - \frac{1}{2} \{ \sigma^\dagger \sigma \rho + \rho \sigma^\dagger \sigma \} \).

To simplify the master equation we remove the time dependence of the Hamiltonian by transforming the system to a rotating frame [13, 14]. For \( x = \omega_t \sigma_{ul} + (\omega_d) \sigma_{uu} \), we define \( A^{rot} = U(t) A U(t)^\dagger \) according to the unitary operator \( U(t) = e^{i X t} \). While the dissipative terms are unaffected by the choice of the rotating frame, the rotated Hamiltonian for unitary time evolution can be rewritten \( H^{rot} = -\Delta \sigma_{uu} + \epsilon (\sigma_{ul} + \sigma_{lu}) \) with the detuning parameter \( \Delta = \omega_d - (\omega_u - \omega_l) \).

Thus Eq. (1) for the steady state in the rotating frame (details are given in App. A) yields the net transition rate \( R_{u \rightarrow l} \) from the upper to the lower level
\[ R_{u \rightarrow l} = \frac{A(\gamma_{ul}, \gamma_u, n_u, n_l, \epsilon)}{F(\gamma_{ul}, \gamma_u, n_u, n_l, \epsilon, \Delta)} (n_u - n_l) \]  
(2)
where the \( A \) and \( F \) are both positive [14], see Eq. (A7). Thus \( R_{u \rightarrow l} \) has the same sign as the difference \( n_u - n_l \) between bath occupations which is driving the transitions.

### III. WORK, HEAT, AND ENTROPY

Let the average energy in the system be \( \langle E \rangle = \text{Tr}(\rho H) \). The typical definitions of full power and full heat flows in the density matrix formalism are [10, 11]
\[ P = \dot{W} = \text{Tr}(\rho H), \quad \dot{Q} = \text{Tr}(\rho H) \]  
(3)
Alternatively, some authors apply an alternative definition of the work and heat for systems coupled to a time-dependent external field which is based on the bare Hamiltonian, \( \langle E_0 \rangle = \text{Tr}(\rho_0 H) \) [5, 12]. Based on the first law of thermodynamics the bare flows are identified from
\[ \dot{E}_0 = -i \text{Tr}(\rho [H_0, V(t)]) + \sum \text{Tr}(\mathcal{L}_a[\rho] H_0) \]  
(4)
where the first (unitary) term is interpreted as the bare power \( P_0 \) and the second (dissipative) term as the sum of bare heat flows \( \dot{Q}_{0\alpha} \) from the respective baths to the system. These terms can be either evaluated in the original or the rotating frame due to the invariance of the trace under cyclic permutations of operators. (This is simpler compared to the first definition with full power and heat flow Eq. (3), where the transformations \( A^{rot} \neq dA^{rot}/dt \) for \( A = H, \rho \) are more involved.)

From these definitions the steady state bare power and heat flow become (see App. B)
\[ P_0 = -R_{u \rightarrow l} (\omega_u - \omega_l) \]
\[ \dot{Q}_{0u} = + R_{u \rightarrow l} \omega_u \]
\[ \dot{Q}_{0l} = - R_{u \rightarrow l} \omega_l \]  
(5)
We note that the bare power and heat flow correspond to the net transition rate \( R_{u \rightarrow l} \) multiplied with the respective bare transition energies from \( H_0 \).

The second law of thermodynamics requires a positive definite energy production. Spohn’s entropy production function for the engine reads [15]
\[ \sigma = \frac{\partial S}{\partial t} - \frac{\dot{Q}_u}{T_u} - \frac{\dot{Q}_l}{T_l} \]  
(6)
where \( S = \text{Tr}(\rho \ln \rho) = \text{Tr}(\rho^{rot}_{\text{steady state}} \ln \rho^{rot}_{\text{steady state}}) \) is the von Neumann entropy [18] of the three-level system, which is constant in steady state. The temperatures of the baths are commonly related to the mean occupations as
\[ T_\alpha = \frac{\omega_{\alpha}}{\log (1 + \frac{1}{n_\alpha})} \]  
(7)
by using the appropriate Bose distribution function.

Using the bare heat flows from Eq. (5) Boukobza and Tannor found [14]
\[ \sigma_0 = R_{u \rightarrow l} \left[ \log (1 + \frac{1}{n_l}) - \log (1 + \frac{1}{n_u}) \right] > 0 \]  
(8)
which is positive definite as both factors have the same sign of \( (n_u - n_l) \), see Eq. (2). In contrast, using the full heat flows from Eq. (3) with temperatures by Eq. (7), Boukobza and Tannor [14] detected negative entropy production for some operation points. This suggested that the definition of work and heat based on the bare Hamiltonian (4) should be preferred.

### IV. RESOLUTION BY EFFECTIVE ENERGIES

Here, we rewrite the results for the full power and heat flows evaluated from Eq. (3) in the form:
\[ P = -R_{u \rightarrow l} \omega_d, \quad \dot{Q}_h = +R_{u \rightarrow l} \omega_u, \quad \dot{Q}_c = -R_{u \rightarrow l} \omega_l \]  
(9)
with effective energies (see App. C)
\[ \check{\omega}_u = \omega_u + \frac{\Delta \gamma_u (n_u + 1)}{\gamma_u (n_u + 1) + \gamma_l (n_l + 1)} \]
\[ \check{\omega}_l = \omega_l - \frac{\Delta \gamma_l (n_l + 1)}{\gamma_u (n_u + 1) + \gamma_l (n_l + 1)} \]  
(10)
Comparing Eq. (5) with Eq. (9) we note that all flows are proportional to the transition rate \( R_{u \rightarrow l} \), describing the round trip rate of the engine. However, there are different energy factors in each term.

For vanishing detuning, \( \Delta = 0 \), the respective energy factors in Eq. (5) and Eq. (9) agree. Here, the heat fluxes from the baths are determined by the level energies \( \omega_{\alpha} \)
and the power transferred from the light field is given by the photon energy $\omega_d$, as expected.

However, for finite detuning, i.e., $\Delta = \omega_d - \omega_u + \omega_l \neq 0$, energy conservation does not allow for this structure, where the full and the bare approach provide different remedies: In the bare approach based on Eq. (4), the power supplied from the ac field changes its energy factor $\omega_d \rightarrow \omega_u - \omega_l$, see Eq. (5). This appears not physical, as a quantized ac field, should have energies in portions of $\hbar \omega_d$ and thus may result in an error of the order $\Delta R_{u\rightarrow l}$ in the power. In contrast, for the full approach, the bare level energies are replaced by effective ones $\omega_u \rightarrow \tilde{\omega}_u$, see Eq. (9), which satisfy $\omega_d = \tilde{\omega}_u - \tilde{\omega}_l$, so that energy conservation holds with the ac-frequency of the field.

Here we argue that the effective energies Eq. (10) should be taken seriously in the full approach and thus be used in the definitions of the bath temperatures

$$\tilde{T}_\alpha = \frac{\tilde{\omega}_\alpha}{\log (1 + \frac{1}{n_\alpha})} \quad (11)$$

Then Eq. (6) provides

$$\sigma = \frac{\dot{Q}_u}{T_u} - \frac{\dot{Q}_l}{T_l} = R_{u\rightarrow l} \left[ \frac{\tilde{\omega}_l}{T_l} - \frac{\tilde{\omega}_u}{T_u} \right] = R_{u\rightarrow l} \left[ \log \left( 1 + \frac{1}{n_l} \right) - \log \left( 1 + \frac{1}{n_u} \right) \right] \quad (12)$$

which is identical with the entropy production function Eq. (8) from the bare approach and, most importantly, positive definite.

Now, we want to highlight the particular meaning of the energies $\tilde{\omega}_k$, $\tilde{\omega}_l$ from Eq. (10). Due to life-time broadening the energies of the levels $u$ and $l$ are smeared out by Lorentzian spectral functions (here normalized to one)

$$A_\alpha(\omega) = \frac{1}{2\pi} \frac{\gamma_\alpha (1 + n_\alpha)}{(\omega - \omega_\alpha)^2 + \gamma_\alpha^2 (1 + n_\alpha)^2 / 4}$$

with a full width at half maximum (FWHM) $\gamma_\alpha (1 + n_\alpha)$ resulting from the decay of the sates by relaxation to the ground level. This allows for energy-conserving transitions between the levels $u$ and $l$ at the energy $\omega_d$ imposed by the ac field, even if $\omega_d \neq \omega_u - \omega_l$, see Fig. 2. Fermi’s golden rule provides the transition rate from the initial level $u$ with the energy $\omega$ (similar but not necessarily equal to $\omega_u$).

$$W_{u\rightarrow l} = 2\pi e^2 A_l(\omega - \omega_d) \quad (13)$$

Weighting with the density $A_\alpha(\omega)$ of the initial state and multiplying with the difference in occupation $f_u(\omega) - f_l(\omega - \omega_d)$ of the levels (technically, $f_\alpha$ is the ratio between the imaginary part of the lesser Green’s function and the spectral function$)$, we obtain the net transition rate

$$R_{u\rightarrow l} = 2\pi e^2 \int d\omega A_u(\omega) A_l(\omega - \omega_d) [f_u(\omega) - f_l(\omega - \omega_d)] \quad (13)$$

Neglecting the energy dependence of $f_\alpha$ over the width of the spectral functions (which would be relevant to study dispersive/Bloch gain$)$, we set $f_u(\omega) - f_l(\omega - \omega_d) \approx \rho_{uu} - \rho_{ll}$. Then, some algebra, see Eq. (D1), results in the expression (A6). This shows the equivalence of this Green’s function based treatment with the density matrix calculations used above.

Eq. (13) shows that there is not a single definite energy involved for the upper and lower level, if broadening is taken into account. However, as the transitions occur with the weight $A_u(\omega) A_l(\omega - \omega_d)$, we can identify the average energy for the upper level involved in transitions

$$\langle \omega \rangle_u = \frac{\int d\omega \omega A_u(\omega) A_l(\omega - \omega_d)}{\int d\omega A_u(\omega) A_l(\omega - \omega_d)} \quad (14)$$
V. CONCLUSION

Both definitions of heat and work, applying either the full or the bare system Hamiltonian, provide identical (and positive definite) expressions for the entropy production for the common three-level maser driven by thermal baths. For the case of the full heat flow, it is crucial to carefully analyze the energies exchanged with the baths. These differ from the bare level energies if the ac field does not match the transition frequency. Disregarding this can provide violations of the second law as reported earlier [14]. While both the full and bare approach are thermodynamically consistent and provide identical expressions for entropy production, the full approach requires an elaborate description of the energies transferred to the bath. On the other hand, the bare approach provides the transition frequency rather than the ac frequency in the work output, which provides a (typically small) error.

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Appendix A: Detailed derivations for the steady state solution

After transforming to the rotating frame, Eq. (1) provides the equations of motion for $\rho_{ij} = \langle i|\rho_{\text{rot}}|j\rangle$

\[
\begin{align*}
\frac{d}{dt} \rho_{gg} &= \gamma_u(n_u + 1)\rho_{uu} + \gamma_l(n_l + 1)\rho_{ll} - (n_u\gamma_u + n_l\gamma_l)\rho_{gg} \\
\frac{d}{dt} \rho_{uu} &= \gamma_u n_u \rho_{gg} - \gamma_u (n_u + 1) \rho_{uu} + i\epsilon(\rho_{ul} - \rho_{ul}^*) \\
\frac{d}{dt} \rho_{ll} &= \gamma_l n_l \rho_{gg} - \gamma_l (n_l + 1) \rho_{ll} - i\epsilon(\rho_{ul} - \rho_{ul}^*) \\
\frac{d}{dt} \rho_{ul} &= i\Delta \rho_{ul} + i\epsilon(\rho_{uu} - \rho_{ll}) - [(\gamma_u (n_u + 1) + \gamma_l (n_l + 1))]\rho_{ul}/2
\end{align*}
\]

In the steady state (superscript $ss$), Eq. (A4) provides

\[
\rho_{ul}^{ss} = \frac{-\epsilon(\rho_{uu}^{ss} - \rho_{ll}^{ss})}{\Delta + i[\gamma_u(n_u + 1) + \gamma_l(n_l + 1)]/2}
\]

Furthermore, we identify the net rate of transitions between $u$ and $l$ due to the ac field:

\[
R_{u\rightarrow l} = -i\epsilon(\rho_{uu}^{ss} - \rho_{ll}^{ss}) = C(\rho_{uu}^{ss} - \rho_{ll}^{ss}) \quad \text{with} \quad C = \frac{\epsilon^2[\gamma_u(n_u + 1) + \gamma_l(n_l + 1)]}{[\gamma_u(n_u + 1) + \gamma_l(n_l + 1)]^2/4 + \Delta^2}
\]

Using $\rho_{gg} = 1 - \rho_{uu} - \rho_{ll}$, Eqs. (A2,A3,A6) provide the system of equations

\[
\begin{align*}
\gamma_u n_u &= [\gamma_u (2n_u + 1) + C]\rho_{uu}^{ss} + (\gamma_u n_u - C)\rho_{ll}^{ss} \\
\gamma_l n_l &= [\gamma_l (2n_l + 1) + C]\rho_{ll}^{ss} + (\gamma_l n_l - C)\rho_{uu}^{ss}
\end{align*}
\]

with the solution

\[
\begin{align*}
\rho_{uu}^{ss} &= \frac{\gamma_u \gamma_l n_u(n_l + 1) + C(\gamma_u n_u + \gamma_l n_l)}{[\gamma_l(2n_l + 1) + C][\gamma_u(2n_u + 1) + C] - (\gamma_l n_l - C)(\gamma_u n_u - C)} \\
\rho_{ll}^{ss} &= \frac{\gamma_u \gamma_l n_l(n_u + 1) + C(\gamma_u n_u + \gamma_l n_l)}{[\gamma_l(2n_l + 1) + C][\gamma_u(2n_u + 1) + C] - (\gamma_l n_l - C)(\gamma_u n_u - C)}
\end{align*}
\]

so that

\[
\rho_{uu}^{ss} - \rho_{ll}^{ss} = \frac{\gamma_u \gamma_l n_u(n_l - n_l)}{\gamma_u \gamma_l (3n_u n_l + 2n_u + 2n_l + 1) + C[\gamma_u(3n_u + 1) + \gamma_l(3n_l + 1)]}
\]

is proportional to the occupation differences of the baths. Inserting into Eq. (A6), we obtain Eq. (2) from the main article, where

\[
\begin{align*}
A &= \frac{\gamma_u \gamma_l}{4} [\gamma_u(n_u + 1) + \gamma_l(n_l + 1)]\epsilon^2 \\
F &= \frac{\gamma_u(n_u + 1) + \gamma_l(n_l + 1)}{2} \frac{\gamma_u(3n_u + 1) + \gamma_l(3n_l + 1)}{2} \epsilon^2 \\
&+ \frac{\gamma_u \gamma_l}{4}(3n_u n_l + 2n_u + 2n_l + 1) \left\{ \frac{[\gamma_u(n_u + 1) + \gamma_l(n_l + 1)]^2}{4} + \Delta^2 \right\}
\end{align*}
\]
are quadratic polynomials in \( \epsilon \). Thus the rate \( R_{u\rightarrow l} \propto \epsilon^2 \) for small coupling \( \epsilon \) to the ac field, while it saturates for \( \epsilon^2 \gtrsim \mathcal{O}\{\gamma_\alpha^2(n_\alpha+1)^2 + \Delta_\alpha^2\} \). \( A \) and \( F \) are identical with the expressions in Eq. (13) of [14], where \( \gamma_\alpha = \gamma_\alpha/2 \) is used.

### Appendix B: Heat and work with bare Hamiltonian

The definition of heat flow from the bare Hamiltonian (4) provides the bare heat entering from bath \( u \) (note that the energy of the ground level is zero)

\[
\dot{Q}_{0u} = \omega_u \langle u|L_u(\rho)|L \rangle = \omega_u [\gamma_u n_u \rho_{gg} - \gamma_u (n_u+1) \rho_{uu}]
\]

Note that the diagonal elements of \( \rho(t) \) are identical in the original and rotating frame. Thus, in the steady state, Eq. (A2) provides \( \dot{Q}_{ss}^{0u} = -i \omega_u \epsilon (\rho_{ul}^0 - \rho_{lu}^0) = \omega_u R_{u\rightarrow l} \) and similarly we get \( \dot{Q}_{0l}^{ss} = -i \omega_l R_{u\rightarrow l} \). Finally, the bare work (4) done by the field on our systems is

\[
P_0 = i \text{Tr}[\rho(V[t],H_0)] = i \text{Tr}[\rho^\prime[V^\prime,H_0^\prime]] = i(\omega_u - \omega_l)(\rho_{ul} - \rho_{lu})
\]

which in the steady state provides \( P_{ss}^0 = -(\omega_u - \omega_l) R_{u\rightarrow l} \) so that \( \dot{Q}_{0u}^{ss} + \dot{Q}_{0l}^{ss} + P_{ss}^0 = 0 \), as required by energy conservation. These are the terms provide Eq. (5), where we omitted the superscript \( ss \).

### Appendix C: Heat and work with full Hamiltonian

The commonly accepted definitions for heat flow and power (addressed as full here) are

\[
\dot{Q}(t) = \text{Tr} \left\{ \frac{d\rho(t)}{dt} H(t) \right\} \quad \text{and} \quad P(t) = \text{Tr} \left\{ \rho(t) \frac{dH(t)}{dt} \right\}
\]

which naturally satisfy the first law with the system internal energy \( E(t) = \text{Tr} \{\rho H(t)\} \).

With the definition (3), we obtain the power transferred to the system

\[
P(t) = i \omega_d \text{Tr} \left\{ \rho \left( |l\rangle\langle u| e^{i \omega_d t} - |u\rangle\langle l| e^{-i \omega_d t} \right) \right\} = i \omega_d (\rho_{ul} - \rho_{lu})^{in} = -i \omega_d R_{u\rightarrow l}
\]

which corresponds to the net rate of absorbed photons (\( -R_{u\rightarrow l} \)) times the photon energy \( \omega_d \). Note, that \( \rho_{ul} \) is in the rotating frame: \( \rho_{ul} = (|u\rangle\langle u|) = (|ul\rangle\langle l|) e^{-i \omega_d t} \).

For the heat flow, the unitary evolution of \( \rho(t) \) due to the Hamiltonian does not contribute, as \( \text{Tr} \{[\rho,H]H\} = \text{Tr} \{[\rho,H]H\} = 0 \), where we used the invariance of the trace under cyclic permutations. Thus we can restrict to the non-unitarian part here. Then the part with \( H_0 \) provides the heat current \( \dot{Q}_{0u} \) from Eq. (B1). We have to add the part with \( V(t) \) and find

\[
\dot{Q}_u = \dot{Q}_{0u} + \epsilon \langle u|L_u(\rho)|l\rangle e^{i \omega_d t} + \langle l|L_u(\rho)|u\rangle e^{-i \omega_d t} = \dot{Q}_{0u} - \epsilon \frac{\gamma_u (n_u+1)}{2} (\rho_{ul} + \rho_{lu})
\]

Using Eqs. (A5,A6) we get in the steady state

\[
\dot{Q}_{ss}^u = \dot{Q}_{0u}^{ss} - \frac{\gamma_u (n_u+1)}{2} \Re \{ \rho_{ul}^{ss} \} R_{u\rightarrow l} = R_{u\rightarrow l} \tilde{\omega}_u
\]

\[
\dot{Q}_{ss}^l = \dot{Q}_{0l}^{ss} - \frac{\Delta \gamma_l (n_l+1)}{2} \Im \{ \rho_{lu}^{ss} \} R_{u\rightarrow l} = -R_{u\rightarrow l} \tilde{\omega}_l
\]

with \( \tilde{\omega}_u = \omega_u + \frac{\Delta \gamma_u (n_u+1)}{\gamma_u (n_u+1) + \gamma_l (n_l+1)} \quad \text{and} \quad \tilde{\omega}_l = \omega_l - \frac{\Delta \gamma_l (n_l+1)}{\gamma_u (n_u+1) + \gamma_l (n_l+1)} \)

where \( \Re \{z\} \) and \( \Im \{z\} \) denote, respectively, the real and imaginary part of a complex value \( z \). The full power and heat flow satisfy energy conservation \( P_{ss}^0 + \dot{Q}_{ss}^u + \dot{Q}_{ss}^l = 0 \) and provide Eqs. (9,10), where we omitted the superscript \( ss \).
Appendix D: Convolution of Lorentzians

We consider the function

\[ P(\omega, \Delta) = \frac{1}{2\pi} \frac{2\gamma_u}{\omega^2 + \gamma_u^2} \frac{2\gamma_l}{(\omega - \Delta)^2 + \gamma_l^2} \]

which is the product of two spectral functions with FWHM \(2\gamma_u\). Then we find with the residue theorem

\[
\int d\omega P(\omega, \Delta) = \frac{2\gamma_u}{2\gamma_u} \frac{2\gamma_l}{(i\gamma_u - \Delta)^2 + \gamma_l^2} + \frac{2\gamma_u}{(\Delta + i\gamma_u)^2 + \gamma_l^2} \frac{2\gamma_l}{2i\gamma_u} \\
= \frac{2\gamma_l ((\Delta + i\gamma_u)^2 + \gamma_u^2) + 2\gamma_u ((\Delta - i\gamma_u)^2 + \gamma_l^2)}{(\Delta + i\gamma_u)^2 + \gamma_l^2} \\
= \frac{2(\gamma_u + \gamma_l) [\Delta^2 - 2i\Delta (\gamma_u - \gamma_l) - (\gamma_u - \gamma_l)^2]}{[\Delta^2 + (\gamma_u + \gamma_l)^2] [\Delta^2 - 2i\Delta (\gamma_u - \gamma_l) - (\gamma_u - \gamma_l)^2]} = \frac{2(\gamma_u + \gamma_l)}{\Delta^2 + (\gamma_u + \gamma_l)^2}
\]

where the third identity is verified by comparing the results of the products in numerator and denominator, respectively. The main result, is that we obtain a Lorentzian with the sum of the individual widths. Similarly we find

\[
\int d\omega \omega P(\omega, \Delta) = i \frac{2\gamma_u}{2\gamma_u} \frac{i\gamma_u 2\gamma_l}{(i\gamma_u - \Delta)^2 + \gamma_l^2} + \frac{2\gamma_u}{(\Delta + i\gamma_u)^2 + \gamma_l^2} \frac{2\gamma_l}{2i\gamma_u} \\
= 2i\gamma_u i\gamma_l ((\Delta + i\gamma_u)^2 + \gamma_u^2) + 2\gamma_u ((\Delta + i\gamma_u)^2 + \gamma_u^2) ((\Delta - i\gamma_u)^2 + \gamma_l^2) \\
= \frac{2\gamma_u \Delta [\Delta^2 - 2i\Delta (\gamma_u - \gamma_l) - (\gamma_u - \gamma_l)^2]}{[\Delta^2 + (\gamma_u + \gamma_l)^2] [\Delta^2 - 2i\Delta (\gamma_u - \gamma_l) - (\gamma_u - \gamma_l)^2]} = \frac{2\gamma_u \Delta}{\Delta^2 + (\gamma_u + \gamma_l)^2}
\]

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