Solving Nonlinear Second Order Delay Eigenvalue Problems
By Collocation Method

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Abstract. The goal of this research is to study the nonlinear second order delay eigenvalue problems which consists of delay ordinary differential equations, in fact one of the expansion methods that is called the Collocation method will be developed to solve this kind of problems.

1. Introduction
The nonlinear second order delay eigenvalue problems consist of delay nonlinear ordinary differential equations with the boundary conditions defined on some intervals; this kind of equations has many applications in different scientific fields, such as physical, biological and engineering science. Also it is one of the most important applications referred to as a delay nonlinear eigenvalue problem. This delay eigenvalue problem belongs to a wide class of problems whose eigenvalues and eigenfunctions have particularly nice properties, [1].

In the 1st half of the 18th century, Johan and Bernoulli, d’Alembert and Euler faced the eigenvalue problems when studying the activity of a cord , which they deemed to be a weightless chain bearing with a number of masses, Laplace and Lagrange continued their work in the second half of the century. They recognize that the eigenvalues are associated to the constancy of the activity. They also used eigenvalues methods in their study of the solar system [2].

In this paper we study and solve this kind of problems by Collocation method analytically, to get exact solution.

2. Basic Definitions and Remarks
This section recalls some basic definitions and remarks that needed in this work. We start with the following definition.

Definition 2.1
The delay differential equation is the equation that the unidentified function and some of its derivatives, evaluated at cases which are different by any of fixed number or function of values.

Consider the n-th order delay differential equation:
\[ F(x,y(x), y(x - \tau_1), ..., y(x - \tau_m), y'(x), y'(x - \tau_1), ..., y'(x - \tau_m), y''(x), y''(x - \tau_1), ..., y''(x - \tau_m)) = y(x) \] ...2.1

Where F is a given function and \( \tau_1, \tau_2, ..., \tau_m \) are given fixed positive numbers called the time delays, [3].
We say that equation 2.1 is homogenous delay differential equation in case \( y(x) \geq 0 \), which we handle in this paper, otherwise it is called non-homogenous delay differential equation, [4].

Definition 2.2
The delay differential equation is said to be nonlinear when it is nonlinear with respect to the unknown function that enter with different arguments and their derivatives that appeared in it, [5].

Hence, the new concept of this work is given by the following definition.

Definition 2.3
The delay eigenvalue problem consist of delay ordinary differential equation is said to be nonlinear when it is nonlinear with respect to the unknown eigenfunction enter with different arguments and their derivatives that appeared in it,[6].

Next, consider the following nonlinear second order delay eigenvalue problem:

\[ -(p(x)y' (x))' + q(x)y (x - \tau) - f (x, \lambda, y (x - \tau)) = 0 \]  ...2.2

with the associated conditions:

\[ a_1 y(a) + a_2 y' (a) = 0, \quad x \in [a - \tau, a] \]  ...2.3

\[ b_1 y(b) + b_2 y' (b) = 0, \quad x \in [b - \tau, b] \]

\[ y(x - \tau) = \varphi(x - \tau), \text{if } x - \tau < a \]

where , \( a_1, a_2, b_1, b_2, p, p' \) and \( q \) are given real continuous functions defined on the interval \([a,b]\), \( p \) is positive, not both coefficients in one condition are zero, \( \tau > 0 \) is the time delay, \( f \) is a well-defined nonlinear function with respect to \( y \). \( \varphi \) is the initial function defined on \( x \in [x_0 - \tau, x_0] \). The problem here is to determine the eigenvalue \( \lambda \) in which a nontrivial solution \( y \) for the problem given by equations 2.2-2.3 occurs. In this case \( \lambda \) is said to be a delay eigenvalue and \( y \) is the associated delay eigenfunction.

In other words \( y \) is an eigenfunction for the variable \( x \) and the nonlinear function \( f(x, \lambda, y(x-\tau)) \) with respect to the eigenvalue \( \lambda \).

Like the linear second order eigenvalue problems, the problem given by equations 2.2-2.3 satisfies the following remarks, [7].

Remarks 2.4
1. The linear delay operator: \( L = -\frac{d^2}{dx^2} p(x) \frac{d}{dx} p'(x) + A(x) q(x) \), where \( A(x) \) is an operator defined by \( A(x)y(x) = y(x - \tau) \), is self-adjoint, [8].
2. The delay eigenfunctions are orthogonal.
3. There are infinite numbers of delay eigenvalues forming a monotone increasing sequence with \( \lambda_j \rightarrow \infty \) as \( j \rightarrow \infty \). Moreover, the delay eigenfunctions corresponding to the delay eigenvalues has exactly \( j \) roots on the interval \([a,b]\).
4. The delay eigenfunctions are complete and normal in \( L^2[a,b] \).
5. Each delay eigenvalue corresponds only one delay eigenfunction in \( L^2[a,b] \).

To check remarks (2-4), see[9].

3. The Collocation Method
In this section, we use the Collocation method to solve such a type of problems. This method is based on approximating the unknown function \( y \) as a linear combination of \( n \) linearly independent functions \( \{\phi_i\}_{i=1}^n \), that is write

\[ y = \sum_{i=1}^{n} \phi_i (x) \]  ...3.1

which implies that
\[ y(x - \tau) = \sum_{i=1}^{n} \phi_i(x - \tau), \quad \text{if } x - \tau < a \]

But, this approximated solution must satisfy the boundary conditions given by equations 2.3 to get a new approximated solution. By substituting this approximated solution into equation 2.1 one can get:

\[ R(x, \lambda, \tilde{c}) = -(p(x) \sum_{i=1}^{m} \phi'(x_i))' + q(x) \sum_{i=1}^{m} \phi_i(x - \tau) - f(x, \lambda, \sum_{i=1}^{m} \phi_i(x - \tau)) \]

where \( R \) is the error in the approximation of equation 2.2 and \( \tilde{c} \) is a vector of \( n - 4 \) of \( c_i \), \( i \in \{1, 2, \ldots, n\} \).

Next, choose \( n - 3 \) points say \( \{x_h\}_{h=1}^{n-3} \), \( a \leq x_h \leq b \) where the error \( R \) will be vanished at them.

That is,

\[ R(x_h, \lambda, \tilde{c}) = -(p(x_h) \sum_{i=1}^{m} \phi'(x_h))' + q(x_h) \sum_{i=1}^{m} \phi_i(x_h - \tau) - f(x_h, \lambda, \sum_{i=1}^{m} \phi_i(x_h - \tau)) \]

Then, by evaluating equation 2.2 at \( h = 1, 2, \ldots, n - 3 \), \( m \in \mathbb{N} \), one can obtain a system of \( n - 3 \) nonlinear equations with \( n - 3 \) unknowns, [10],[11],[12].

This system can be solved to get the values of \( n - 4 \) of \( c_i \) and \( \lambda \).

To illustrate the Collocation method we shall give the following examples

### 3.1 Example

Consider the following nonlinear delay eigenvalue problem:

\[ -(xy' (x))' + 2xy(x - 1) - \lambda \left( y^2(x - 1) - \frac{3}{2} \right) = 0, \quad x \in [1.2] \]  

with the associated boundary conditions:

\[ y(1) = y'(1), \quad x \in [0.1] \]

\[ y(2) = 2y'(2), \quad x \in [1.2] \]

\[ y'(x - 1) = x - 1 \quad \text{if} \quad x - 1 < 1 \]

First, approximate the unknown function \( y \) as a polynomial of degree three, that is

\[ y(x) = c_1 + c_2(x) + c_3(x)^2 + c_4(x)^3 \]

Which implies that

\[ y(x - 1) = c_1 + c_2(x - 1) + c_3(x - 1)^2 + c_4(x - 1)^3 \]

But, this approximated solution must satisfy the boundary conditions given by equations 3.4 to get:

\[ y(x - 1) = c_1 + c_2(x - 1) - 6c_4(x - 1)^2 + c_4(x - 1)^3 \]

By substituting this approximated solution into equation 3.3 one can get:

\[ R(x, \lambda, c_1, c_4) = -x(-18c_4 + 6c_4x) - (c_1 - 18c_4x - 9c_4 + 3c_4x^2) \]

\[ + 2x(c_1 + c_4(x - 1)^2 + 6c_4(x - 1)^2 + c_4(x - 1)^3) \]

\[ - \lambda(c_1 + c_4(x - 1) - 6c_4(x - 1)^2 + c_4(x - 1)^3)^2 - \frac{1}{2} \]

Choose the points \( 1, \frac{3}{2}, 2 \) where the error \( R \) will be vanished at them.

\[ 12c_4 - (c_1 - 24c_4) + 2c_1 - \lambda(c_1 - \frac{1}{2}) = 0 \]

\[ \frac{309}{8} c_4 + 7c_1 - \lambda((\frac{3}{2}c_1 - \frac{11}{8}c_4)^2 - \frac{1}{2}) = 0 \]

\[ 25c_4 + 7c_1 - \lambda(2c_1 - 5c_4)^3 - \frac{1}{2} = 0 \]

Thus, the nontrivial solution of the above system is \( \lambda = 2, \ c_1 = 1 \) and \( c_4 = 0 \).

That is \( y(x - 1) = 1 + (x - 1) \)
3.2 Example
Consider the following nonlinear delay eigenvalue problem:
\[-y''(x) = \lambda(y(x - \frac{\pi}{2}) + \sin(x - \frac{\pi}{2})), \quad x \in [\frac{\pi}{2}, \pi]\]
\[y(\frac{\pi}{2}) + y'(\frac{\pi}{2}) = 1, \quad x \in [0, \frac{\pi}{2}]\]
\[y(\pi) - y'(\pi) = 1, \quad x \in [\frac{\pi}{2}, \pi]\]
\[y(x - \frac{\pi}{2}) = x - \frac{\pi}{2}, \quad \text{if} \quad x - \frac{\pi}{2} \leq \frac{\pi}{2}.
\]
We use the Collocation method to solve this problem. To do this, we follow the same arguments as in example 3.1.

One can get that the eigen-pair of this nonlinear delay eigenvalue problem
\[(y(x), \lambda) = (\sin(x - \frac{\pi}{2}), \frac{1}{3}), \quad \text{thus}, \quad (y(x - \frac{\pi}{2}), \lambda) = (\sin(x - \frac{\pi}{2}), \frac{1}{3})\]

4. Conclusions
From this work we can conclude the following:
1. The nonlinear second order delay eigenvalue problems consist of delay nonlinear ordinary differential equations satisfies the same properties as those consist of delay nonlinear ordinary differential equations with or without delay.
2. The Collocation method is a good choice to solve such a type of problems.

5. Future Work
Generalize the study into higher order nonlinear delay eigenvalue problems.

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