NON-EXISTENCE FOR FRACTIONALLY DAMPED FRACTIONAL DIFFERENTIAL PROBLEMS

Mohammed D. KASSIM
King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics,
Dhahran, 31261, Saudi Arabia
E-mail: dahan@kfupm.edu.sa

Khaled M. FURATI
King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics,
Dhahran, 31261, Saudi Arabia
E-mail: kmfurati@kfupm.edu.sa

Nasser-eddine TATAR
King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics,
Dhahran, 31261, Saudi Arabia
E-mail: tatarn@kfupm.edu.sa

Abstract  In this paper, we are concerned with a fractional differential inequality containing a lower order fractional derivative and a polynomial source term in the right hand side. A non-existence of non-trivial global solutions result is proved in an appropriate space by means of the test-function method. The range of blow up is found to depend only on the lower order derivative. This is in line with the well-known fact for an internally weakly damped wave equation that solutions will converge to solutions of the parabolic part.

Key words  Nonexistence, global solution, fractional differential equation, Riemann-Liouville fractional integral and fractional derivative

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1 Introduction

In this paper we consider the problem

\[ \begin{cases} D_0^\sigma y(t) + D_0^\beta y(t) = f(t, y(t)), & t > 0, \\ I_0^{1-\alpha} y(t) |_{t=0} = b, \end{cases} \tag{1.1} \]

where \( D_0^\sigma \) is the Riemann-Liouville fractional derivative of order \( \sigma > 0 \), \( 0 < \beta \leq \alpha \leq 1 \).

A nonexistence result of non-trivial global solutions for the problem (1.1) will be proved when
\[ f(t, y(t)) \geq t^\gamma |y(t)|^m \text{ for some } m > 1 \text{ and } \gamma \in \mathbb{R}. \] That is we consider the problem:

\[
\begin{cases}
D_0^\alpha y(t) + D_0^\beta y(t) \geq t^\gamma |y(t)|^m, & t > 0, \\
I_0^{1-\alpha} y(t)|_{t=0} = b,
\end{cases}
\tag{1.2}
\]

where \(0 < \beta \leq \alpha \leq 1\) and show that no solutions can exist for all time for certain values of \(\gamma\) and \(m\). In particular, we find the range of values of \(m\) for which solutions do not exist globally.

Clearly, sufficient conditions for nonexistence provide necessary conditions for existence of solutions.

The interest to fractional calculus has been accelerated the past three decades after the publication of the three papers of Bagley and Torvik [3–5] and the paper by Podlubny [28]. Many phenomena in diverse fields of science and engineering can be described by differential equations of non-integer order. Namely, they arise naturally in viscoelasticity, porous media, electrochemistry, control and electromagnetic, etc [25–27].

In fact it has been shown by experiments that derivatives of non-integer order can describe many phenomena better than derivatives of integer order specially hereditary phenomena and processes.

Some recent applications arose in viscoelasticity, rheology, control systems, synthesis, robots and nanotechnology, etc (see [11, 14, 19, 20, 22, 23, 29]).

Regarding the existence of solutions for various classes of fractional differential equations, there are many results (e.g. see [1, 2, 4, 13, 24, 31]). For the issue of nonexistence of solutions for fractional differential equations, we refer to [10, 12, 21, 30] and to [15–18] for partial differential equations involving fractional derivatives (see also references therein).

The existence and uniqueness of solutions for problem (1.1) has been discussed in [14]. In case \(\alpha = \beta = 1\) and \(f(t, y(t)) = 2y^m(t)\) in (1.1) we obtain

\[
\begin{cases}
y'(t) = y^m(t), \\
y(t)|_{t=0} = b,
\end{cases}
\]

This problem has, for \(m > 1\), the solution

\[ y(t) = [(1-m)(t+c)]^{1/(1-m)}, \]

where

\[ c = \frac{b^{1-m}}{1-m}. \]

Observe that, for \(m > 1\), the solution blows-up in finite time.

When \(\alpha = 1, \beta = 0\) and \(\gamma = 0\), the problem (1.2) with an equality instead of inequality is equivalent to the Bernoulli differential problem

\[
\begin{cases}
y'(t) + y(t) = y^m(t), & t > 0, \\
y(t)|_{t=0} = b.
\end{cases}
\tag{1.3}
\]

The solution of (1.3) is given by

\[ y(t) = [1 + (b^{1-m} - 1) \exp (m-1)t]^{1/(1-m)}. \]
Clearly $y(t)$ blows up in the finite time

$$c = \frac{1}{1-m} \ln \left(1 - b^{1-m}\right), \ m, \ b > 1.$$ 

In case $\alpha = \beta$ in (1.2) we obtain the problem with only one fractional derivative

$$\begin{cases}
2D^\alpha_0 y(t) \geq t^\gamma |y(t)|^m, \ t > 0, \\
I^{1-\alpha}_0 y(t) |_{t=0} = b.
\end{cases} \tag{1.4}$$

Problem (1.4) has been considered by Laskri and Tatar [21]. It was shown that if $\gamma > -\alpha$ and $1 < m \leq \frac{-\alpha+1}{\alpha}$, then, Problem (1.4) does not admit global nontrivial solutions when $b \geq 0$.

Here, we would like to investigate the case where a lower order fractional derivative is present in the equation (or inequality). It is known that for hyperbolic equations, say the wave equation with an internal fractional damping represented by the first derivative (i.e. $\alpha = 2, \beta = 1$ also known as the Telegraph equation), this damping has a dissipation effect. It will compete with the polynomial source and may take it over this blowing-up term under certain circumstances. Moreover, it has been shown for the telegraph problem that solutions approach the solution of the same problem without the highest derivative when $t$ goes to infinity (that is the parabolic equation). This result has been generalized to the fractional derivative case in [6] and in [30].

For our problem here (1.2), we would like to see how much influential $D^\beta_0 y$ will be on the blow-up phenomenon. In particular, how the range of values $m$ ensuring blow-up in finite time would be affected. We reached the conclusion that here also it is the lower order derivative (i.e. $\beta$) which determines the range of blow-up just like the parabolic part in the hyperbolic problem.

The rest of the paper is divided into two sections. In Section 2, we present some definitions, notations, and lemmas which will be needed later in our proof. Section 3 is devoted to the nonexistence result.

## 2 Preliminaries

In this section we present some definitions, lemmas, properties and notation which will be used in our result later.

**Definition 2.1** The Riemann-Liouville left-sided fractional integral $I^\alpha_a f$ of order $\alpha > 0$ is defined by

$$I^\alpha_a f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ t > a, \ \alpha > 0, \tag{2.1}$$

provided that the integral exists. Here $\Gamma(\alpha)$ is the Gamma function. When $\alpha = 0$, we define $I^0_a f = f$.

**Definition 2.2** The Riemann-Liouville right-sided fractional integral $I^\alpha_{b-} f$ of order $\alpha > 0$ is defined by

$$I^\alpha_{b-} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \ t < b, \ \alpha > 0, \tag{2.2}$$

provided that the integral exists. When $\alpha = 0$, we define $I^0_{b-} f = f$.

**Definition 2.3** The Riemann-Liouville left-sided fractional derivative $D^\alpha_a f$ of order $\alpha$, $0 < \alpha < 1$, is defined by
\[ D_a^\alpha f(t) = \frac{d}{dt} I^{1-\alpha}_a f(t), \]

that is,
\[ D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha} ds, \quad t > a, \quad 0 < \alpha < 1, \quad (2.3) \]

when \( \alpha = 1 \) we have \( D_a^0 f = Df \). In particular, when \( \alpha = 0 \), \( D_a^0 f = f \).

**Definition 2.4** The Riemann-Liouville right-sided fractional derivative \( D_{b+}^\alpha f \) of order \( \alpha \), \( 0 < \alpha < 1 \), is defined by
\[ D_{b+}^\alpha f(t) = -\frac{d}{dt} I^{1-\alpha}_{b+} f(t), \]
that is,
\[ D_{b+}^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^\alpha} ds, \quad t < b, \quad 0 < \alpha < 1. \quad (2.4) \]

In particular, when \( \alpha = 0 \), \( D_{b+}^0 f = f \).

**Lemma 2.5** (Fractional Integration by Parts) Let \( \alpha > 0 \), \( p \geq 1 \), \( q \geq 1 \) and \( \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha \) \((p \neq 1 \text{ and } q \neq 1 \text{ in the case when } \frac{1}{p} + \frac{1}{q} = 1 + \alpha)\). If \( \varphi \in L_p(a, b) \) and \( \psi \in L_q(a, b) \), then
\[ \int_a^b \varphi(t) (I^\alpha_a \psi(t)) dt = \int_a^b \psi(t) (I^\alpha_a \varphi(t)) dt. \quad (2.5) \]

**Definition 2.6** We consider the weighted spaces of continuous functions
\[ C_\gamma [a, b] = \{ f : (a, b) \to \mathbb{R} : (t-a)^\gamma f(t) \in C [a, b] \}, \quad 0 < \gamma < 1, \]
\[ C_0 [a, b] = C [a, b], \]
and
\[ C_{1-\alpha}^\alpha [a, b] = \{ f \in C_{1-\alpha} [a, b] : D_a^\alpha f \in C_{1-\alpha} [a, b] \}, \quad 0 < \alpha < 1. \quad (2.6) \]

**Lemma 2.7** Let \( 0 \leq \gamma < 1 \) and \( f \in C_\gamma [a, b] \). Then
\[ I^\alpha_a f(a) = \lim_{t \to a^+} I^\alpha_a f(t) = 0, \quad 0 \leq \gamma < \alpha. \]

**Proof** Since \( f \in C_\gamma [a, b] \) then \((t-a)^\gamma f(t)\) is continuous on \([a, b]\) and on \([a, b]\) we have
\[ |(t-a)^\gamma f(t)| < M, \]
for some positive constant \( M \). Therefore
\[ |I^\alpha_a f(t)| < M \left[ I^{\alpha} (s-a)^{-\gamma} \right] (t-a) = M \frac{\Gamma(1-\gamma)}{\Gamma(\alpha + 1 - \gamma)} (t-a)^{\alpha-\gamma}. \]
As \( \alpha > \gamma \) we see that
\[ I^\alpha_a f(a) = \lim_{t \to a^+} I^\alpha_a f(t) = 0, \quad 0 \leq \gamma < \alpha \]
which completes the proof of Lemma 2.7.

**Lemma 2.8** Let \( \varphi \in C^1 [0, \infty) \) be a test function, that is: \( \varphi(t) \geq 0 \), \( \varphi(t) \) is non-increasing and such that
\[ \varphi(t) := \begin{cases} 1, & t \in [0, T/2] \\ 0, & t \in [T, \infty), \end{cases} \]
for $T > 0$. Then

$$I(T) = \int_{T/2}^{T} \left( \frac{1 - \alpha}{T - t} \right)^m \left( \frac{\varphi'}{\varphi} (t) \right) \, dt \leq K_{\alpha,m} T^{1-\alpha m}, \quad 0 < \alpha < 1, \ T, \ p, \ m > 0 \quad (2.7)$$

where

$$K_{\alpha,m} = \frac{K_1^m}{2m(1-\alpha)+1} \frac{\Gamma^m (2-\alpha) \mid m (1-\alpha) + 1 \mid}.$$ \quad (2.8)

and $K_1$ is a bound for $\frac{\varphi'(r)}{\varphi(r)}$.

**Proof** Using (2.2), we see that

$$I(T) = \int_{T/2}^{T} \left( \frac{1}{\Gamma (1-\alpha)} \int_t^T (s-t)^{-\alpha} \left| \frac{\varphi'(s)}{\varphi(s)} \right|^p ds \right)^m \, dt. \quad (2.9)$$

The change of variable $\sigma T = t$ in (2.9) yields

$$I(T) = \int_{1/2}^{1} \left( \frac{1}{\Gamma (1-\alpha)} \int_{\sigma T}^{T} (s - \sigma T)^{-\alpha} \left| \frac{\varphi'(s)}{\varphi(s)} \right|^p ds \right)^m \, d\sigma. \quad (2.10)$$

Another change of variable $s = r T$ in (2.10) gives

$$I(T) = \frac{T^{1-\alpha m}}{\Gamma^m (1-\alpha)} \int_{1/2}^{1} \left( \int_{\sigma}^{1} (r - \sigma)^{-\alpha} \left| \frac{\varphi'(r)}{\varphi(r)} \right|^p dr \right)^m \, d\sigma. \quad (2.11)$$

Since $\varphi \in C^1([0, \infty))$, we may assume without loss of generality that

$$\frac{\varphi'(r)}{\varphi(r)} \leq K_1,$$

for some positive constant $K_1$, for otherwise we consider $\varphi^\lambda(r)$ with some sufficiently large $\lambda$.

Therefore from (2.11) we get

$$I(T) \leq \frac{K_1^m T^{1-\alpha m}}{\Gamma^m (1-\alpha)} \int_{1/2}^{1} \left( \int_{\sigma}^{1} (r - \sigma)^{-\alpha} dr \right)^m \, d\sigma = \frac{K_1^m T^{1-\alpha m}}{\Gamma^m (2-\alpha)} \int_{1/2}^{1} (1 - \sigma)^{m(1-\alpha)} \, d\sigma$$

$$= \frac{K_1^m}{2m(1-\alpha)+1} \frac{\Gamma^m (2-\alpha) \mid m (1-\alpha) + 1 \mid} T^{1-\alpha m}.$$

Therefore

$$I(T) \leq K_{\alpha,m} T^{1-\alpha m}.$$

**Remark 2.9** Lemma 2.8 is true also for the case $\alpha = 1$. We prove this fact in the following lemma.

**Lemma 2.10** Let $\varphi$ be as in Lemma 2.8 Then

$$I(T) = \int_{T/2}^{T} \left( \frac{\varphi'(t)}{\varphi^p (t)} \right) \, dt \leq \frac{1}{2} K_1^m T^{1-m}, \quad T, \ p, \ m > 0, \quad (2.12)$$

with

$$\frac{\varphi'(r)}{\varphi(r)} \leq K_1.$$
3 Nonexistence result

In this section, we consider the problem

$$\begin{cases}
D_0^\alpha y(t) + D_0^\beta y(t) \geq t^\gamma |y(t)|^m, & t > 0, \ m > 1, \ 0 < \beta < \alpha < 1, \\
I_0^{1-\alpha} y(t) \big|_{t=0} = b,
\end{cases}$$

(3.1)

where $D_0^\alpha$ is defined in (2.3). Nonexistence of non-trivial solutions is investigated in the space $C^{\alpha}_{1-\alpha}$ defined in (2.6).

**Theorem 3.1** Assume that $\gamma > -\beta$ and $1 < m \leq \frac{\gamma + 1}{\beta - 1}$. Then, Problem (3.1) does not admit global nontrivial solutions in $C^{\alpha}_{1-\alpha}$, when $b \geq 0$.

**Proof** Assume, on the contrary, that a nontrivial solution $y$ exists for all time $t > 0$. Let $\varphi$ be as in Lemma 2.8. Multiplying the inequality in (3.1) by $\varphi(t)$ and integrating over $(0,T)$ we get

$$I_1 := \int_0^T t^\gamma |y(t)|^m \varphi(t) \, dt \leq \int_0^T D_0^\alpha y(t) \varphi(t) \, dt + \int_0^T D_0^\beta y(t) \varphi(t) \, dt. \tag{3.2}$$

Let

$$I_2 := \int_0^T \varphi(t) D_0^\alpha y(t) \, dt,$$

and

$$I_3 := \int_0^T \varphi(t) D_0^\beta y(t) \, dt.$$

From the definition of $D_0^\alpha y$ in (2.3) we can write

$$I_2 = \int_0^T \varphi(t) \frac{d}{dt} I_0^{1-\alpha} y(t) \, dt.$$

An integration by parts yields

$$I_2 = [\varphi(t) I_0^{1-\alpha} y(t)]_0^T - \int_0^T \varphi'(t) I_0^{1-\alpha} y(t) \, dt.$$

Since $\varphi(T) = 0$, $\varphi(0) = 1$ and $I_0^{1-\alpha} y(0) = b$, then

$$I_2 = -b - \int_0^T \varphi'(t) I_0^{1-\alpha} y(t) \, dt.$$

As $b \geq 0$, we have

$$I_2 \leq - \int_0^T \varphi'(t) I_0^{1-\alpha} y(t) \, dt \leq \int_0^T |\varphi'(t)| (I_0^{1-\alpha} |y|)(t) \, dt$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \, ds \, dt. \tag{3.3}$$

Because $\varphi(t)$ is nonincreasing $\varphi(s) \geq \varphi(t)$ for all $t \geq s$, and therefore

$$\frac{1}{\varphi(s)^{1/m}} \leq \frac{1}{\varphi(t)^{1/m}}, \ m > 1.$$
Also we have
\[ \varphi'(t) = 0, \quad t \in [0, T/2]. \]
Thus
\[
I_2 \leq \frac{1}{\Gamma(1 - \alpha)} \int_0^{T/2} \left| \varphi'(t) \right| \left( \int_0^t \frac{|y(s)|}{(t-s)\alpha} \varphi(s)^{1/m} ds \right) dt \\
\leq \frac{1}{\Gamma(1 - \alpha)} \int_0^{T/2} \left| \varphi'(t) \right| \left( \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \varphi(s)^{1/m} ds \right) dt \\
\leq \frac{1}{\Gamma(1 - \alpha)} \int_0^{T/2} \varphi(t)^{1/m} \left( \int_0^t \frac{|y(s)|}{(t-s)^\alpha} \varphi(s)^{1/m} ds \right) dt \\
\leq \int_0^{T/2} \left| \varphi'(t) \right| \left( I_{T/2}^{1-\alpha} \varphi^{1/m} |y| \right) (t) dt.
\]
A fractional integration by parts [26], in the last expression yields
\[
I_2 \leq \int_0^{T/2} \left( I_{T/2}^{1-\alpha} |\varphi'| \varphi^{1/m} \right) (t) \varphi(t)^{1/m} |y(t)| dt.
\]
Next, we multiply by \( t^\gamma/m t^{-\gamma/m} \) inside the integral in the right hand side
\[
I_2 \leq \int_0^{T/2} \left( I_{T/2}^{1-\alpha} |\varphi'| \varphi^{1/m} \right) (t) \varphi(t)^{1/m} t^\gamma/m t^{-\gamma/m} |y(t)| dt.
\]
For \( \gamma < 0 \) we have \( t^{-\gamma/m} < T^{-\gamma/m} \) (because \( t < T \)) and for \( \gamma > 0 \) we get \( t^{-\gamma/m} < 2 t^\gamma/m T^{-\gamma/m} \) (because \( T/2 < t \)): that is
\[
t^{-\gamma/m} < \max \left\{ 1, 2^\gamma/m \right\} T^{-\gamma/m}.
\]
Then
\[
I_2 \leq \max \left\{ 1, 2^\gamma/m \right\} T^{-\gamma/m} \int_0^{T/2} \left( I_{T/2}^{1-\alpha} |\varphi'| \varphi^{1/m} \right) (t) t^\gamma/m \varphi(t)^{1/m} |y(t)| dt. \tag{3.4}
\]
By Hölder’s inequality, it is clear that
\[
I_2 \leq \max \left\{ 1, 2^\gamma/m \right\} T^{-\gamma/m} \left( \int_0^{T/2} t^\gamma \varphi(t) |y(t)|^m dt \right)^{1/m} \left( \int_0^{T/2} \left( I_{T/2}^{1-\alpha} |\varphi'| \varphi^{1/m} \right) (t) dt \right)^{1/m}.
\]
Lemma [2.8] implies that
\[
I_2 \leq \max \left\{ 1, 2^\gamma/m \right\} T^{-\gamma/m} \left( \int_0^{T/2} t^\gamma \varphi(t) |y(t)|^m dt \right)^{1/m} \left( K_{\alpha,m'} T^{1-\alpha m'} \right)^{1/m}. \tag{3.5}
\]
where \( K_{\alpha,m'} \) is the constant appearing in Lemma [2.8] corresponding to the present exponents. Therefore from [3.5] we have the estimate
\[
I_2 \leq \max \left\{ 1, 2^\gamma/m \right\} \frac{K_{\alpha,m}}{T^{1/m' - \alpha - \gamma/m}} I_1^{1/m'}. \tag{3.6}
\]
Now, we turn to \( I_3 \). First, since \( y \in C_{1-\alpha} [0, T] \) and \( 1 - \alpha < 1 - \beta \), then by Lemma [2.7] we have
\[
I_0^{1-\beta} y(0) = \lim_{t \to 0} I_0^{1-\beta} y(t) = 0.
\]
An integration by parts in
\[
I_3 = \int_0^T \varphi(t) D_0^\beta y(t) dt = \int_0^T \varphi(t) \frac{d}{dt} I_0^{1-\beta} y(t) dt.
\]
gives
\[ I_3 = \left[ \varphi(t) I_0^{1-\beta} y(t) \right]_{t=0}^T - \int_0^T \varphi'(t) I_0^{1-\beta} y(t) dt. \]

Since \( \varphi(T) = 0 \) and \( I_0^{1-\beta} y(0) = 0 \), it follows that
\[ I_3 = - \int_0^T \varphi'(t) I_0^{1-\beta} y(t) dt \leq \int_0^T |\varphi'(t)| \left( I_0^{1-\beta} |y| \right)(t) dt \]
\[ \leq \frac{1}{\Gamma(1-\beta)} \int_0^T |\varphi'(t)| \int_0^t \frac{|y(s)|}{(t-s)^\beta} ds dt. \]

Replacing \( \alpha \) by \( \beta \) in the argument above allows us to write
\[ I_3 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_{T/2}^T \left( \frac{1-\beta}{\varphi^{1/m}} \right)(t) t^{\gamma/m} \varphi(t) (t)^{1/m} |y(t)| dt, \quad (3.7) \]
or simply
\[ I_3 \leq K_{\beta,m} \max \left\{ 1, 2^{\gamma/m} \right\} T^{1/m'-\beta-\gamma/m} I_1^{1/\beta}. \quad (3.8) \]

From (3.2), (3.6) and (3.8), we have
\[ I_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} K_{\alpha,m} \frac{1}{\varphi^{1/m}} T^{1/m'-\alpha-\gamma/m} I_1^{1/\beta} + \max \left\{ 1, 2^{\gamma/m} \right\} T^{1/m'-\beta-\gamma/m} I_1^{1/\beta} \]
\[ \leq \max \left\{ K_{\alpha,m}, K_{\beta,m}^{1/\beta} \right\} \max \left\{ 1, 2^{\gamma/m} \right\} \left( T^{1/m'-\alpha-\gamma/m} + T^{1/m'-\beta-\gamma/m} \right) I_1^{1/\beta}. \]

Therefore
\[ I_1^{1/m'} \leq K_2 \left( T^{1/m'-\alpha-\gamma/m} + T^{1/m'-\beta-\gamma/m} \right), \quad (3.9) \]
with
\[ K_2 := \max \left\{ K_{\alpha,m}, K_{\beta,m}^{1/\beta} \right\} \max \left\{ 1, 2^{\gamma/m} \right\}. \]

Raising both sides of (3.9) to the power \( m' \) we obtain
\[ I_1 \leq K_3 \left( T^{1-\alpha m'-\gamma m'/m} + T^{1-\beta m'-\gamma m'/m} \right), \quad (3.10) \]
with
\[ K_3 = 2^{1-m'} K_2^{m'}. \]

If \( m < \frac{\gamma+1}{1-\beta} \), we see that \( 1 - \beta m' - \gamma m'/m < 0, 1 - \alpha m' - \gamma m'/m < 0 \), and consequently \( T^{1-\beta m'-\gamma m'/m} \rightarrow 0 \) and \( T^{1-\alpha m'-\gamma m'/m} \rightarrow 0 \) as \( T \rightarrow \infty \). Then, from (3.10), we obtain
\[ \lim_{T \rightarrow \infty} I_1 = \lim_{T \rightarrow \infty} \int_0^T t^{\gamma} |y(t)|^m \varphi(t) dt = 0. \]

We reach a contradiction since the solution is not supposed to be trivial.

If \( m = \frac{\gamma+1}{1-\beta} \) we have \( 1 - \beta m' - \gamma m'/m = 0, 1 - \alpha m' - \gamma m'/m \leq 0 \), and the relation (3.10) ensures that
\[ \lim_{T \rightarrow \infty} \int_0^T t^{\gamma} |y(t)|^m \varphi(t) dt \leq K_4. \quad (3.11) \]

Further, in view of (3.2), (3.4) and (3.7), we see that
\[ I_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_{T/2}^T t^{\gamma/m} \varphi(t) (t)^{1/m} \left( \left( I_{\alpha}^{1-\alpha} \frac{|\varphi|}{\varphi^{1/m}} \right)(t) + \left( I_{-\beta}^{1-\beta} \frac{|\varphi|}{\varphi^{1/m}} \right)(t) \right) dt. \]
Thanks to Hölder’s inequality, it is clear that

\[ I_1 \leq \max \left\{ 1, 2^\gamma/m \right\} T^{-\gamma/m} \left[ \int_{T/2}^{T} t^\gamma \varphi (t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \]

\[ \times \left\{ \int_{T/2}^{T} \left[ \left( I^{1-\alpha}_{T-} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) + \left( I^{1-\beta}_{T-} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) \right]^{m'} \, dt \right\}^{\frac{1}{m'}} \]

\[ \leq \max \left\{ 1, 2^\gamma/m \right\} 2^{1/m} T^{-\gamma/m} \left[ \int_{T/2}^{T} t^\gamma \varphi (t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \]

\[ \times \left\{ \int_{T/2}^{T} \left[ \left( I^{1-\alpha}_{T-} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) + \left( I^{1-\beta}_{T-} \frac{|\varphi'|}{\varphi^{1/m}} \right) (t) \right]^{m'} \, dt \right\}^{\frac{1}{m'}}. \]

Therefore, by Lemma 2.8 we obtain

\[ I_1 \leq K_5 T^{-\gamma/m} \left[ \int_{T/2}^{T} t^\gamma \varphi (t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \left[ K_{\alpha,m'} T^{1-\alpha m'} + K_{\beta,m'} T^{1-\beta m'} \right]^{\frac{1}{m}} \]

\[ = K_5 \left[ \int_{T/2}^{T} t^\gamma \varphi (t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \left[ K_{\alpha,m'} T^{1-\alpha m' - \gamma m'/m} + K_{\beta,m'} T^{1-\beta m' - \gamma m'/m} \right]^{\frac{1}{m}}, \]

with

\[ K_5 = \max \left\{ 1, 2^\gamma/m \right\} 2^{1/m}. \]

Since \( m = \frac{\gamma + 1}{1-\beta} \), then \( 1 - \beta m' - \gamma m'/m = 0 \) and \( 1 - \alpha m' - \gamma m'/m \leq 0 \). Therefore

\[ I_1 \leq K_6 \left[ \int_{T/2}^{T} t^\gamma \varphi (t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \]

for some positive constant \( K_6 \), with

\[ \lim_{T \to \infty} \int_{T/2}^{T} t^\gamma \varphi (t) |y(t)|^m \, dt = 0 \]

due to the convergence of the integral in (3.11). This is again a contradiction. The proof is complete.

Next, we take \( \alpha = 1 \) and \( 0 < \beta < 1 \), that is

\[ \begin{cases} 
  y'(t) + D^-_b \beta y(t) \geq t^\gamma |y(t)|^m, & t > 0, \ m > 1, \quad 0 < \beta < 1, \\
  y(t) |_{t=0} = b \in \mathbb{R}. 
\end{cases} \quad (3.12) \]

**Theorem 3.2** Assume that \( \gamma > -\beta \) and \( 1 < m \leq \frac{\gamma + 1}{1-\beta} \). Then, Problem (3.12) does not admit global nontrivial solutions when \( b \geq 0 \).

**Proof** Assume, on the contrary, that a nontrivial solution \( y \) exists for all time \( t > 0 \). Let \( \varphi \) be as in Lemma 2.8. Multiplying the inequality in (3.12) by \( \varphi(t) \) and integrating we get

\[ J_1 = \int_{0}^{T} t^\gamma |y(t)|^m \varphi(t) \, dt \leq \int_{0}^{T} y'(t) \varphi(t) \, dt + \int_{0}^{T} D^\beta_0 y(t) \varphi(t) \, dt. \quad (3.13) \]
Let
\[ J_2 = \int_0^T \varphi(t) y'(t) \, dt, \]  
(3.14)
and
\[ J_3 = \int_0^T \varphi(t) D_\beta y(t) \, dt. \]  
(3.15)
Following procedure as in the proof of Theorem 3.1, we obtain the following estimates for \( J_2 \) and \( J_3 \)
\[ J_2 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_0^{T/2} \varphi(t) \, dt \int_0^T |y(t)| \varphi(t)^{1/m} t^{\gamma/m} \, dt, \]  
(3.16)
(or By using Hölder’s inequality and Lemma 2.10)
\[ J_2 \leq \max \left\{ 1, 2^{\gamma/m} \right\} K_1 T^{1/m' - 1 - \gamma/m} J_1 \]  
(3.17)
and
\[ J_3 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_0^{T/2} \varphi(t) \, dt \int_0^T |y(t)| \varphi(t)^{1/m} t^{\gamma/m} \, dt, \]  
(3.18)
(or By using Hölder’s inequality and Lemma 2.8)
\[ J_3 \leq \max \left\{ 1, 2^{\gamma/m} \right\} K_2^{-1/\beta \gamma/m} T^{1/m' - \beta - \gamma/m} J_1 \]  
(3.19)
From (3.13), (3.17) and (3.19), we have
\[ J_1^{1/m'} \leq K_2 T^{1/m' - 1 - \gamma/m} + T^{1/m' - \beta - \gamma/m}, \]  
(3.20)
with
\[ K_2 := \max \left\{ 1, 2^{\gamma/m} \right\} \max \left\{ K_{\beta, m'}, K_3 \right\}. \]
Raising both sides of (3.20) to the power \( m' \) we obtain
\[ J_1 \leq K_3 \left( T^{1/m' - \gamma/m'} + T^{1 - \beta m' - \gamma/m'} \right), \]  
(3.21)
with
\[ K_3 = 2^{1 - m'} K_2^{m'}. \]
If \( m < \frac{\gamma + 1}{1 - \beta} \), we see that \( 1 - m' - \gamma/m < 0, 1 - \beta m' - \gamma/m < 0 \). Then from (3.21) we obtain
\[ \lim_{T \to \infty} J_1 = \lim_{T \to \infty} \int_0^T t^\gamma |y(t)|^m \varphi(t) \, dt = 0. \]
We are contradict since the solution is not supposed to be trivial.
In the case \( m = \frac{\gamma + 1}{1 - \beta} \) we have \( 1 - m' - \gamma/m \leq 0, 1 - \beta m' - \gamma/m = 0 \), and the relation (3.21) ensures that
\[ \lim_{T \to \infty} \int_0^T t^\gamma |y(t)|^m \varphi(t) \, dt \leq K_4. \]  
(3.22)
Also from (3.13), (3.16) and (3.18), we have
\[ J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \int_0^T \varphi(t) \, dt \int_0^{T/2} t^{\gamma/m} \varphi(t)^{1/m} |y(t)| \left[ \frac{\varphi'(t)}{\varphi(t)^{1/m}} + \left( t^{1-\beta} \frac{\varphi'}{\varphi^{1/m}} \right)(t) \right] \, dt. \]
By using Hölder’s inequality, it is clear that
\[ J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \left[ \int_{T/2}^T t^{\gamma} \varphi(t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \times \left[ \int_{T/2}^T \left( \frac{|\varphi'(t)|}{\varphi(t)^{1/m}} + \left( \frac{t^{1-\beta}}{t^{1-\beta/m}} \right)^{m'} \right) \, dt \right]^{\frac{1}{m'}}. \]

Therefore, by Lemma 2.8 and Lemma 2.10 and \( \varphi \in C^1[0, \infty) \), we have
\[ J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} T^{-\gamma/m} \left[ \int_{T/2}^T t^{\gamma} \varphi(t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \times \left[ K_5 T^{1-m'} + K_6 T^{1-\beta m'} \right]^{1/m'}, \]
for some positive constants \( K_5 \) and \( K_6 \), and then
\[ J_1 \leq \max \left\{ 1, 2^{\gamma/m} \right\} \left[ \int_{T/2}^T t^{\gamma} \varphi(t) |y(t)|^m \, dt \right]^{\frac{1}{m}} \times \left[ K_5 T^{1-m'-\gamma m'/m} + K_6 T^{1-\beta m'-\gamma m'/m} \right]^{1/m'}, \]
and
\[ \lim_{T \to \infty} \int_{T/2}^T t^{\gamma} \varphi(t) |y(t)|^m \, dt = 0 \]
due to the convergence of the integral in (3.22). This is again a contradiction and the proof of Theorem 3.2 is complete.

Finally, we take \( \alpha = \beta = 1 \), this mean we consider the Cauchy problem
\[
\begin{cases}
\frac{d}{dt} y(t) \geq t^{\gamma} |y(t)|^m, & t > 0, \quad m > 1, \\
y(t) \big|_{t=0} = b \in \mathbb{R}.
\end{cases}
\] (3.23)

**Theorem 3.3** Assume that \( \gamma > -1 \) and \( m > 1 \). Then, Problem (3.23) does not admit global nontrivial solutions when \( b \geq 0 \).

**Proof** Similar to the proof of Theorem 3.1.

**Conclusion 3.1** According to Theorems 3.1, 3.2 and having in mind the results in [21] it appears that the addition of the term \( D^\beta_0 y \), \( \beta < \alpha \), does not prevent the nonexistence. However, it does affect the exponent \( m \). The range of \( m \) is reduced to \( 1 < m \leq \frac{\gamma+1}{\gamma-1} \) instead of \( 1 < m \leq \frac{\gamma+1}{\gamma-\alpha} \). This shows that the range does not depend on the highest derivative. It depends on the lowest derivative. This is a well-established result for the Telegraph equation. Indeed, for this problem, it has been proved that solutions approach solutions of the corresponding parabolic part.

In case \( m \) is fixed from the beginning then we need \( \gamma > m (1-\beta)-1 \) instead of \( \gamma > m (1-\alpha)-1 \).
Therefore, it is the derivative of lower order which determines the exponent. Note that
\[ 1 < m \leq \frac{\gamma + 1}{1 - \beta} < \frac{\gamma + 1}{1 - \alpha}, \]
and
\[ \gamma > -\beta > -\alpha. \]

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