Lax Pair Formulation and Multi-soliton Solution of the Integrable Vector Nonlinear Schrödinger Equation

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Abstract
The integrable vector nonlinear Schrödinger (vector NLS) equation is investigated by using Zakharov-Shabat (ZS) scheme. We get a Lax pair formulation of the vector NLS model. Multi-soliton solution of the equation is also derived by using inverse scattering method of ZS scheme. We also find that there is an elastic and inelastic collision of the bright and dark multi-solitons of the system.

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1. Introduction
The integrable coupled nonlinear Schrödinger equation of Manakov\textsuperscript{1} type is widely used in recent developments in the field of optical solitons in fibers. The use and applications of the equation is to explain how the solitons waves transmit in optical fiber, what happens when the interaction among optical solitons influences directly the capacity and quality of communication and so on.\textsuperscript{2−5} In optical communications system, the information is coded in binary pulses which modulate the light carrier wave. We can use bright solitons or dark solitons in the communications system. The performance of a high-speed fiber communications system is limited by dispersion, losses, and parasitic effects induced by electronic repeaters. The use of a single soliton as a bit of information is interesting and solves the first problem, since the dispersion broadening effects are compensated for by the nonlinear focusing effects. The attenuation of solitons due to fiber loss can be compensated for by amplifying the solitons periodically, in order that they recover their original width and power. In an all optical communications system, which does not use electronic repeaters, this amplification can be achieved by using stimulated Raman scattering.\textsuperscript{6}

The generalization of Manakov type equation to be the integrable vector nonlinear Schrödinger equation (vector NLS model) can be written as follows\textsuperscript{7−10}

\[
\left( i \frac{\partial}{\partial x} + \chi \frac{\partial^2}{\partial t^2} + 2\mu \sum_{b=1}^{m} |q_b|^2 \right) q_c = 0, \quad c = 1, 2, 3, \ldots, m, \quad (1.1)
\]

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where $q_c$ are slowly varying envelopes of the $m$ interacting optical modes, describing a charged field with $m$ colours, the variables $t$ and $x$ are the normalized retarded time and distance along the fiber, $\chi$ and $\mu$ are arbitrary real parameters.

The Hamiltonian of the model in eq.(1.1), for $\chi = 1$, is

$$ H = \int \left[ \sum_{c=1}^{m} \left| \frac{\partial q_c}{\partial t} \right|^2 + \mu \left( \sum_{c=1}^{m} |q_c|^2 \right)^2 \right] dt, $$

(1.2)

where integration takes account of the boundary conditions which extend those for the ordinary NLS model.

The set of equations (eq.(1.1), for $\chi = 1/2$) can also be described as a propagation of $m$ self-trapped mutually incoherent wave packets in media with Kerr-like nonlinearity. Related to this case, we can also interpret that $q_c$ denotes the $c$th component of the beam, $2\mu$ is the coefficient representing the strength of nonlinearity, $t$ is the transverse coordinate, $x$ is the coordinate along the direction of propagation, and $\sum_{b=1}^{m} |q_b|^2$ is the change in refractive index profile created by all incoherent components of the light beam. The response time of the nonlinearity is assumed to be long compared to temporal variations of the mutual phases of all components, so the medium response to the average light intensity, and this just a simple sum of modal intensities expressed by the relation $\sum_{b=1}^{m} |q_b|^2$.

5 We investigate and derive a Lax pair formulation of the vector NLS model using inverse scattering method of ZS scheme. This Lax pair has involved the Lax pair of the integrable single and coupled NLS equation. After getting the Lax pair, we solve the equation. We then find multi-soliton solution of the equation. The solution has involved the solution of the integrable coupled NLS equation of Manakov type appeared in our previous papers in Ref.5 and 10. In this solution, we also get an elastic and inelastic collision of the bright and dark multi-solitons.

This paper is organized as follows. In section 2, we will perform the ZS scheme for the Lax pair of the integrable vector nonlinear Schrödinger equation. In section 3, we will solve the bright and dark multi-soliton solution of the integrable vector NLS equation. We also compare our reduced results with the results in Ref.8, 9, 17 and 19 provided by Seppard and Kivshar, Akhmediev et al., Radhakrishnan et al., and Shchesnovich, respectively. Section 4 is devoted for discussions and conclusions. In this section, we also state that there must be a multi-solitons solution in multidimensions $(p+1)$ of the integrable vector NLS model.

2. **ZS Scheme for the Lax Pair Formulation of the Integrable Vector NLS Equation**

We start by choosing the following two operators related to Zakharov-Shabat (ZS) scheme

$$ \Delta_0^{(1)} = I \left( i\alpha_1 \frac{\partial}{\partial x} - \frac{\partial^2}{\partial t^2} \right), $$

(2.1a)

and

$$ \Delta_0^{(2)} = \begin{pmatrix} \alpha_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_{m+1} \end{pmatrix} \frac{\partial}{\partial t}, $$

(2.1b)

where $\alpha_1, \alpha_2, \ldots, \alpha_{m+1}$ are arbitrary real values, and $I$ is the $(m+1) \times (m+1)$ unit matrix. We can then define the following operators by using this scheme related to inverse scattering techniques:

$$ \Delta^{(1)} = \Delta_0^{(1)} + U(t,x), $$

(2.2a)

and

$$ \Delta^{(2)} = \Delta_0^{(2)} + V(t,x). $$

(2.2b)

Here operators $\Delta^{(i)} (i = 1, 2)$ satisfy the following equation...
\[
\Delta^{(i)} (I + \Phi_+) = (I + \Phi_+) \Delta^{(i)}_0, \tag{2.3}
\]

where the Volterra integral operators \( \Phi_\pm (\psi) \) are defined according to equation
\[
\Phi_\pm (\psi) = \int_{-\infty}^{\infty} k_\pm (t, z) \psi(z) dz. \tag{2.4}
\]

We now suppose that operators \( \Phi_F (\psi) \) and \( \Phi_\pm (\psi) \) are related to the following operator identity
\[
(I + \Phi_+) (I + \Phi_F) = (I + \Phi_-), \tag{2.5}
\]

where the integral operator \( \Phi_F (\psi) \) is
\[
\Phi_F (\psi) = \int_{-\infty}^{\infty} F(t, z) \psi(z) dz. \tag{2.6}
\]

Both \( k_+ \) and \( F \) in eq.(2.4) and (2.6) are the \((m + 1) \times (m + 1)\) matrices chosen as follows
\[
k_+ = \begin{pmatrix}
a(t, z; x) & q_1(t, z; x) & q_2(t, z; x) & \ldots & q_m(t, z; x) \\
\pm q_1^*(t, z; x) & d_{11}(t, z; x) & d_{12}(t, z; x) & \ldots & d_{1m}(t, z; x) \\
\pm q_2^*(t, z; x) & d_{21}(t, z; x) & d_{22}(t, z; x) & \ldots & d_{2m}(t, z; x) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\pm q_m^*(t, z; x) & d_{m1}(t, z; x) & d_{m2}(t, z; x) & \ldots & d_{mm}(t, z; x)
\end{pmatrix}, \tag{2.7a}
\]

and
\[
F = \begin{pmatrix}
0 & (A_1)_n & (A_2)_n & \ldots & (A_m)_n \\
\pm (A_1^*)_n & 0 & 0 & \ldots & 0 \\
\pm (A_2^*)_n & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\pm (A_m^*)_n & 0 & 0 & \ldots & 0
\end{pmatrix}. \tag{2.7b}
\]

Here \( a, q_1, q_2, \ldots, q_m, \pm q_1^*, \pm q_2^*, \ldots, \pm q_m^*, d_{11}, d_{12}, \ldots, d_{mm}, (A_1)_n, (A_2)_n, \ldots, (A_m)_n, \) and \( (A_1^*)_n, (A_2^*)_n, \ldots, (A_m^*)_n \) (where \( n = 1, 2, 3, \ldots, N \)) are parameters which will be calculated in section 3.

In eq.(2.5), we have assumed that \((I + \Phi_+)\) is invertible, then
\[
(I + \Phi_F) = (I + \Phi_+)^{-1} (I + \Phi_-), \tag{2.8}
\]

so that operator \((I + \Phi_F)\) is factorisable. From eq.(2.5), we derive Marchenko matrix equations,
\[
k_+ (t, z; x) + F(t, z; x) + \int_t^\infty k_+ (t, t'; x) F(t', z; x) dt' = 0, \quad \text{for } z > t, \tag{2.9a}
\]

and
\[
k_- (t, z; x) = F(t, z; x) + \int_t^\infty k_+ (t, t'; x) F(t', z; x) dt', \quad \text{for } z < t. \tag{2.9b}
\]

In eq.(2.9b), \( k_- \) is obviously defined in terms of \( k_+ \) and \( F \). Both eq.(2.9a) and (2.9b) require \( F \), and \( F \) is supplied by the solution of equations:
\[
\left[ \Delta^{(1)}_0, \Phi_F \right] = 0, \tag{2.10a}
\]
and
\[
\left[ \Delta_0^{(2)}, \Phi_F \right] = 0. \tag{2.10b}
\]

After a little algebraic manipulation, we get
\[
\alpha_1 F_x + F_{zz} - F_{tt} = 0, \tag{2.11a}
\]
and
\[
\begin{pmatrix}
\alpha_1 & 0 & \ldots & 0 \\
0 & \alpha_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \alpha_{m+1}
\end{pmatrix}
F_t + F_x
\begin{pmatrix}
\alpha_1 & 0 & \ldots & 0 \\
0 & \alpha_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \alpha_{m+1}
\end{pmatrix}
= 0, \tag{2.11b}
\]
where \( F_t \equiv \frac{\partial F}{\partial t}, \ F_x \equiv \frac{\partial F}{\partial x}, \) etc.

\( U(t, x) \) and \( V(t, x) \) can be found by solving eq.\((2.3)\) in which we have substituted eq.\((2.7a)\) and \((2.4)\) (for \( k_+ \)) to that equation:
\[
V(t, x) = \begin{pmatrix}
0 & (1 - \alpha_2) q_1 & \ldots & (1 - \alpha_m) q_{m-1} & (1 - \alpha_{m+1}) q_m \\
\pm (\alpha_2 - 1) q_{1}^* & 0 & \ldots & (\alpha_2 - \alpha_m) d_{1(m-1)} & (\alpha_2 - \alpha_{m+1}) d_{1m} \\
\pm (\alpha_3 - \alpha_2) q_{2}^* & (\alpha_3 - \alpha_2) d_{21} & \ldots & (\alpha_3 - \alpha_m) d_{2(m-1)} & (\alpha_3 - \alpha_{m+1}) d_{2m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\pm (\alpha_m - \alpha_1) q_{m-1}^* & (\alpha_m - \alpha_2) d_{(m-2)1} & \ldots & 0 & (\alpha_m - \alpha_{m+1}) d_{(m-1)m}
\end{pmatrix}, \tag{2.12a}
\]
and
\[
U(t, x) = -2k_+ - 2 \begin{pmatrix}
a_t & q_{11} & q_{21} & \ldots & q_{m1} \\
\pm q_{11}^* & d_{111} & d_{112} & \ldots & d_{11m} \\
\pm q_{21}^* & d_{211} & d_{212} & \ldots & d_{21m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\pm q_{m1}^* & d_{m11} & d_{m12} & \ldots & d_{m1m}
\end{pmatrix}. \tag{2.12b}
\]

Based on the solution of equation \( \Delta^{(2)}(I + \Phi_+) = (I + \Phi_+) \Delta_0^{(2)}, \) we get that \( k_+(t, z; x) \) must obey the following equation:
\[
\begin{pmatrix}
\alpha_1 & 0 & \ldots & 0 \\
0 & \alpha_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \alpha_{m+1}
\end{pmatrix}
k_{1t} + k_{1z} \begin{pmatrix}
\alpha_1 & 0 & \ldots & 0 \\
0 & \alpha_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \alpha_{m+1}
\end{pmatrix}
+ V(t, x) \ k_+ = 0, \tag{2.13}
\]
and if we evaluate this eq.\((2.13)\) on \( z = t, \) we find
\[
a_t = \mp \frac{1}{\alpha_1} \sum_{b=1}^{m} (\alpha_1 - \alpha_{b+1}) \ | \ q_b |^2, \tag{2.14a}
\]
\[
d_{11t} = -\frac{1}{\alpha_2} \left[ \pm (\alpha_2 - 1) |q_1|^2 + \sum_{b=1}^{m} (\alpha_2 - \alpha_{b+1}) d_{1b} d_{b1} \right], \tag{2.14b}
\]
\[
d_{12t} = -\frac{1}{\alpha_2} \left[ \pm (\alpha_2 - 1) q_1^* q_2 + \sum_{b=1}^{m} (\alpha_2 - \alpha_{b+1}) d_{1b} d_{b2} \right], \tag{2.14c}
\]
\[
\cdots \cdots 
\]
\[
d_{1mt} = -\frac{1}{\alpha_2} \left[ \pm (\alpha_2 - 1) q_1^* q_m + \sum_{b=1}^{m} (\alpha_2 - \alpha_{b+1}) d_{1b} d_{bm} \right], \tag{2.14d}
\]
The above matrix components can be simplified by choosing $\alpha$ an arbitrary real parameter.\footnote{The term which arises for general $\alpha_2, \alpha_3, \ldots, \alpha_{m+1}$ is an addition term of the vector NLS equation.}\footnote{\textsuperscript{10}} We then only find the following matrix components

$$a_k = \mp \frac{(\alpha_1 - \delta)}{\alpha_1} \sum_{b=1}^{m} |q_b|^2,$$  

$$d_{1i} = \mp \frac{(\delta - \alpha_1)}{\delta} [q_1^* q_c],$$  

$$d_{2i} = \mp \frac{(\delta - \alpha_1)}{\delta} [q_2^* q_c],$$  

$$d_{3i} = \mp \frac{(\delta - \alpha_1)}{\delta} [q_3^* q_c],$$  

$$d_{m+1} = \mp \frac{(\delta - \alpha_1)}{\delta} [q_m^* q_c],$$  

$$d_{21t} = -\frac{1}{\alpha_3} \left[ \pm (\alpha_3 - \alpha_1) q_2^* q_1 + \sum_{b=1}^{m} (\alpha_3 - \alpha_{b+1}) d_{2bd_1} \right],$$  

$$d_{22t} = -\frac{1}{\alpha_3} \left[ \pm (\alpha_3 - \alpha_1) |q_2|^2 + \sum_{b=1}^{m} (\alpha_3 - \alpha_{b+1}) d_{2bd_2} \right],$$  

$$\cdots$$  

$$d_{2m+1} = -\frac{1}{\alpha_3} \left[ \pm (\alpha_3 - \alpha_1) q_2^* q_m + \sum_{b=1}^{m} (\alpha_3 - \alpha_{b+1}) d_{2bd_m} \right],$$  

$$d_{31i} = -\frac{1}{\alpha_4} \left[ \pm (\alpha_4 - \alpha_1) q_3^* q_1 + \sum_{b=1}^{m} (\alpha_4 - \alpha_{b+1}) d_{3bd_1} \right],$$  

$$d_{32i} = -\frac{1}{\alpha_4} \left[ \pm (\alpha_4 - \alpha_1) q_3^* q_2 + \sum_{b=1}^{m} (\alpha_4 - \alpha_{b+1}) d_{3bd_2} \right],$$  

$$\cdots$$  

$$d_{3m+1} = -\frac{1}{\alpha_4} \left[ \pm (\alpha_4 - \alpha_1) q_3^* q_m + \sum_{b=1}^{m} (\alpha_4 - \alpha_{b+1}) d_{3bd_m} \right],$$  

$$d_{m1i} = -\frac{1}{\alpha_{m+1}} \left[ \pm (\alpha_{m+1} - \alpha_1) q_m^* q_1 + \sum_{b=1}^{m} (\alpha_{m+1} - \alpha_{b+1}) d_{mbd_1} \right],$$  

$$d_{m2i} = -\frac{1}{\alpha_{m+1}} \left[ \pm (\alpha_{m+1} - \alpha_1) q_m^* q_2 + \sum_{b=1}^{m} (\alpha_{m+1} - \alpha_{b+1}) d_{mbd_2} \right],$$  

$$\cdots$$  

$$d_{mm+1} = -\frac{1}{\alpha_{m+1}} \left[ \pm (\alpha_{m+1} - \alpha_1) |q_m|^2 + \sum_{b=1}^{m} (\alpha_{m+1} - \alpha_{b+1}) d_{mbd_m} \right].$$  

The above matrix components can be simplified by choosing $\alpha_2 = \alpha_3 = \ldots = \alpha_{m+1} = \delta$ and $\alpha_1 \neq \delta$, where $\delta$ is an arbitrary real parameter.\footnote{\textsuperscript{10}}
Plugging the above equations into eq. (2.12b) yields

\[
U(t, x) = -2 \left( \sum_{b=1}^{m} |q_b|^2 \right) \begin{pmatrix}
\pm q_1^1 \\
\pm q_2^2 \\
\vdots \\
\pm q_m^m 
\end{pmatrix} + \frac{(\delta - \alpha_1)}{\delta} |q_1|^2 + \frac{(\delta - \alpha_2)}{\delta} |q_2|^2 + \ldots + \frac{(\delta - \alpha_m)}{\delta} |q_m|^2,
\]

(2.16)

Now we substitute equations (2.12a) and (2.16) into eq. (2.2) (for \(\alpha_2 = \alpha_3 = \ldots = \alpha_{m+1} = \delta\) and \(\alpha_1 \neq \delta\), we get

\[
\Delta^{(1)} = I \left( \alpha_1 \frac{\partial}{\partial x} - \frac{\partial^2}{\partial t^2} \right) - 2 \left( \sum_{b=1}^{m} |q_b|^2 \right) \begin{pmatrix}
\pm q_1^1 \\
\pm q_2^2 \\
\vdots \\
\pm q_m^m 
\end{pmatrix} + \frac{(\delta - \alpha_1)}{\delta} |q_1|^2 + \frac{(\delta - \alpha_2)}{\delta} |q_2|^2 + \ldots + \frac{(\delta - \alpha_m)}{\delta} |q_m|^2,
\]

(2.17a)

and

\[
\Delta^{(2)} = \begin{pmatrix}
\alpha_1 & 0 & \ldots & 0 \\
0 & \alpha_2 & \ldots & 0 \\
0 & 0 & \ldots & \alpha_{m+1}
\end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix}
0 & (\alpha_1 - \alpha_2) q_1 & \ldots & (\alpha_1 - \alpha_m) q_m & (\alpha_1 - \alpha_{m+1}) q_m \\
\pm (\alpha_2 - \alpha_1) q_1^* & 0 & \ldots & 0 & 0 \\
\pm (\alpha_3 - \alpha_1) q_2^* & 0 & \ldots & 0 & 0 \\
\pm (\alpha_4 - \alpha_1) q_3^* & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ldots & \ldots & \ldots \\
\pm (\alpha_m - \alpha_1) q_m^* & 0 & \ldots & 0 & 0 \\
\pm (\alpha_{m+1} - \alpha_1) q_{m+1}^* & 0 & \ldots & 0 & 0
\end{pmatrix},
\]

(2.17b)

Finally, we get a Lax pair of the integrable vector NLS equation as follows

\[
i \alpha_1 L_x + [L, M] = 0,
\]

(2.18)

here,

\[
L = \Delta^{(2)} = L_0 + V(t, x),
\]

(2.19a)

where

\[
L_0 = \Delta_0^{(2)},
\]

(2.19b)

and

\[
M = I \left( i \alpha_1 \frac{\partial}{\partial x} \right) - \Delta^{(1)} = M_0 - U(t, x),
\]

(2.19c)

where

\[
M_0 = I \frac{\partial^2}{\partial t^2}.
\]

(2.19d)

Since \(\Delta^{(1)}\) commutes with \(\Delta^{(2)}\), we find the equation which is satisfied by \(q_c\)

\[
i q_{ct} + \chi q_{cxt} + 2 \mu \sum_{b=1}^{m} |q_b|^2 q_c = 0, \quad c = 1, 2, \ldots, m,
\]

(2.20a)
and its complex conjugate
\[-i q_{cz}^* + \chi q_{ctt}^* + 2\mu \sum_{b=1}^{m} |q_b|^2 q_c^* = 0, \quad c = 1, 2, ..., m.\] (2.20b)

Here parameters \(\mu\) and \(\chi\) are arbitrary real parameters which are defined as follows
\[\mu = \pm \frac{\alpha_1^2 - \delta^2}{\alpha_1^2 \delta},\] (2.21a)
and
\[\chi = \frac{\alpha_1 + \delta}{\alpha_1 (\alpha_1 - \delta)}.\] (2.21b)

It is obvious that eq.(2.20a) and (2.20b) are a general form of the integrable vector nonlinear Schrödinger equation (vector NLS model) and its complex conjugate.

3. The Bright and Dark Multi-Soliton Solution of the Vector NLS Equation

We consider a general matrix function \(F\) in eq.(2.7b) and substitute it into eq.(2.11a), we find the following differential equations
\[i\alpha_1 (A_c)_{nx} + (A_c)_{nxx} - (A_c)_{ntt} = 0,\] (3.1a)
and
\[i\alpha_1 (A_c^*)_{nx} + (A_c^*)_{nxx} - (A_c^*)_{ntt} = 0.\] (3.1b)

The solution of the above equations can be derived by using separable variable method. We then find
\[(A_c)_n (t, z; x) = \sum_{n=1}^{N} (A_c)_{n0} e^{-\alpha_1 \rho_n z} \left[ e^{\rho_n (\delta t + i \rho_n (\alpha_1^2 - \delta^2) x)} \right],\] (3.2a)
and
\[(A_c^*)_n (t, z; x) = \sum_{n=1}^{N} (A_c^*)_{n0} e^{-\delta \sigma_n z} \left[ e^{\sigma_n (\alpha_1 t + i \sigma_n (\delta^2 - \alpha_1^2) x)} \right],\] (3.2b)
where \(\sigma_n, \rho_n, (A_c)_{n0}\) and \((A_c^*)_{n0}\) are arbitrary complex parameters.

To get the final solution of the integrable vector NLS equation, we have to substitute eq.(2.7b), eq.(3.2a) and eq.(3.2b) into Marchenko matrix equation (eq.(2.9a)). We get (for \(a, q_1, q_2, ..., q_m\))
\[a (t, z; x) = \sum_{n=1}^{N} \sum_{c=1}^{m} \int_{t}^{\infty} q_c (A_c^*)_{n0} \exp \left( i \frac{\sigma_n^2}{\alpha_1} (\delta^2 - \alpha_1^2) x + \sigma_n (\alpha_1 t' - \delta z) \right) dt',\] (3.3a)
and
\[q_c = - \sum_{n=1}^{N} \left( e^{-\alpha_1 \rho_n z} (A_c)_{n0} \right) e^{\delta \rho_n t} e^{i \frac{\sigma_n^2}{\alpha_1} (\alpha_1^2 - \delta^2) x} \] (3.3b)
\[= \sum_{n=1}^{N} \left( (A_c)_{n0} \right) \int_{t}^{\infty} a (t, z; x) e^{i \frac{\sigma_n^2}{\alpha_1} (\alpha_1^2 - \delta^2) x} e^{-\alpha_1 \rho_n z} e^{\delta \rho_n t'} dt'.\]
The final solution is work on \( z = t \). Hence, by substituting eq.(3.3a) to eq.(3.3b), we find the solution:

\[
q_c = \sum_{n=1}^{N} \frac{-(A_c)_{n0} e^{\rho_n (\delta - \alpha_1) t} e^{-i \frac{\alpha_2^2}{\alpha_1} (\delta^2 - \alpha_2^2) x}}{1 + \left( \frac{\mu}{\sum_{b=1}^{m} |(A_b)_{n0}|^2} \right) e^{\rho_n (\delta - \alpha_1) t} e^{-i \frac{\alpha_2^2}{\alpha_1} (\delta^2 - \alpha_2^2) x} e^{-\sigma_n (\delta - \alpha_1) t} e^{i \frac{\alpha_2^2}{\alpha_1} (\delta^2 - \alpha_2^2) x}}. \tag{3.4}
\]

We define

\[
\eta_n = k_n (t + ik_n \chi x), \tag{3.5a}
\]

and

\[
\eta_n^* = k_n^* (t - ik_n^* \chi x), \tag{3.5b}
\]

where

\[
k_n = (\delta - \alpha_1) \rho_n, \tag{3.5c}
\]

\[
k_n^* = -(\delta - \alpha_1) \sigma_n, \tag{3.5d}
\]

and

\[
\tau = \left( \frac{\alpha_2^2 - \delta^2}{\alpha_1} \right)^{1/2} \left( \frac{1}{\delta - \alpha_1} \right). \tag{3.5e}
\]

Here arbitrary complex parameter \( \rho_n^* = -\sigma_n \).

Now \( q_c \) can be rewritten as

\[
q_c = \sum_{n=1}^{N} \frac{-(A_c)_{n0} e^{\eta_n}}{1 + e^{R_n} + \eta_n + \eta_n^*}, \quad c = 1, 2, ..., m, \tag{3.6}
\]

where

\[
e^{R_n} = \frac{\mu}{\sum_{b=1}^{m} |(A_b)_{n0}|^2} \left( \frac{\tau k_n + \tau k_n^*}{\tau k_n + \tau k_n^*} \right). \tag{3.7}
\]

Based on our solution in eq.(3.6), we can see that our results are the bright and dark multi-soliton solution since \( \mu = \pm \frac{\alpha_2^2 - \delta^2}{\alpha_1^2 \delta} \). If we choose parameter \( (\chi \mu) > 0 \) then we get the bright \( N \)-solitons solution. In the case of bright solitons described by the vector NLS equation, the coherent interaction of solitons depends on the relative phase between them, so that identical solitons with opposite phases repel each other, whereas in-phase solitons attract each other.\(^{14}\) On the other hand, the dark \( N \)-solitons solution is found when \( (\chi \mu) < 0 \). Interaction of the dark solitons is unconditionally repulsive, in all types of models described by the generalized (vector) NLS equation.\(^{15}\)

The results of \( q_c \) show that there are also an elastic and inelastic collision of the bright and dark multi-soliton. If a nonlinear system supports propagation of two, or more, waves of different frequencies or polarization, vector solitons consisting of more than one components can be formed. Indeed, when two vector solitons are closely separated, they may form a bound state if the sum of all forces acting between different soliton components is zero. As an example, let us consider interaction of two vector solitons consisting of dark and bright components in a defocusing optical medium. Two dark solitons always repel each other, and as a result they can’t form a bound state.\(^{15}\) However, if we introduce out-of-phase bright components guided by each of the dark solitons, their attractive interaction creates a proper balance of forces, which results in a stationary two-soliton bound state.
In eq. (3.6), there are several arbitrary complex parameters \((A_c)_{\omega_0}, (A^c)_{\omega_0}\), \(\rho_n\) and \(\sigma_n\) which can directly influence the phase of the solitons interaction. The results of \(q_1, q_2, \ldots, q_m\) can also be rewritten in the more conventional form by introducing \(\rho_n = l_n + i\lambda_n\) (where \(l_n\) and \(\lambda_n\) are arbitrary real parameters),

\[
q_c = \sum_{n=1}^{N} \frac{(-\frac{\alpha^2-\delta^2}{\alpha_1})^{1/2} l_n (A_c)_{\omega_0}}{\sqrt{\mu \sum_{b=1}^{m} |(A_b)_{\omega_0}|^2}} \exp \left( i (\delta - \alpha_1) \left[ l_n (t + \chi (\delta - \alpha_1) (l_n^2 - \lambda_n^2) x) \right] \right) \cosh \left( (\delta - \alpha_1) l_n (t - 2\chi (\delta - \alpha_1) \lambda_n x) + \varphi_n \right), \quad c = 1, 2, \ldots, m ,
\]

where \(\varphi_n\) is a real multi-soliton phase,

\[
\varphi_n = \frac{1}{2} R_n.
\]

and the amplitudes of the multi-soliton are

\[
Ampl^{(c)} = -\frac{(-\frac{\alpha^2-\delta^2}{\alpha_1})^{1/2} l_n (A_c)_{\omega_0}}{\sqrt{\mu \sum_{b=1}^{m} |(A_b)_{\omega_0}|^2}} c = 1, 2, \ldots, m .
\]

Parameter \(\lambda_n\) contributes to the velocities of the multi-soliton.\(^{16}\)

The result \(q_c\) in eq.(3.8) can be reduced to our general multi-soliton solution of the integrable coupled NLS equation of Manakov type appeared in Ref.10 if we put \(\chi = \xi, (A_c)_{\omega_0}\), for \(c = 1, 2\) and \(\sum_{b=1}^{m} |(A_b)_{\omega_0}|^2\) in the equation. On the other hand, we also get that the bright \(N\)-solitons solution (for \((\chi\mu) > 0\)) can be reduced to the bright one and two soliton solutions related to that in the works that have been done before by Radhakrishnan, et.al. using Hirota method in Ref.17 and 18 if we put \((A_c)_{\omega_0}\), for \(c = 1, 2\) and \(\sum_{b=1}^{m} |(A_b)_{\omega_0}|^2\) in eq.(3.8). According to the comparison of the methods, their results of the bright one soliton solution is equal to our results when \(\alpha e^{\eta(0)} = - (A_1)_{\omega_0}, \beta e^{\eta(0)} = - (A_2)_{\omega_0}\), \(\chi = 1, \tau \simeq 1, k_1 = (\delta - \alpha_1) (l_1 + i\lambda_1), \mu = +\frac{\alpha^2-\delta^2}{\alpha_1^2}\) and \(|\alpha|^2 + |\beta|^2 = \left( |(A_1)_{\omega_0}|^2 + |(A_2)_{\omega_0}|^2 \right)^2\). However, their results of the inelastic collision of the bright two soliton can be reduced to our elastic collision of the solution if we put \(\alpha_1 : \alpha_2 = \beta_1 : \beta_2\) in their result.

Our results can also be reduced to the results in Ref.19 provided by Shchesnovich if we put \(\chi = \frac{1}{2} \mu, \tau = 1, x = z, t = \tau, \theta = (A_c)_{\omega_0}, l = c = 1, 2, n = 1\), and \(|\theta_1|^2 + |\theta_2|^2 = |(A_1)_{\omega_0}|^2 + |(A_2)_{\omega_0}|^2 = 1\). In his results, Shchesnovich had also derived and discussed polarization scattering by soliton-soliton collisions.

If we compare our multi-solitons solution of the vector NLS equation of Manakov type with the results calculated by Sheppard and Yu. S. Kivshar in Ref.8, we find that their Manakov type results are identical with our results when \(\chi = \frac{1}{2}, \mu = -1, x = z, t = x, \) and \(q_e = e_{\pm}, \) for \(c = 1, 2\). However, their Hirota method solutions have been modified and extended in complicated solutions related to single dark-bright soliton solution and dark-bright multi-soliton solution.

On the other hand, our multi-solitons solution of the vector NLS equation are identical with the results in Ref.9 which is provided by Akhmediev et. al. when \(\chi = \frac{1}{2}, \mu = \alpha, x = z, t = \tau, \delta_n (\psi_i) = \sum_{b=1}^{N} |q_b|^2\), and \(q_e = \psi_i\), for \(i, c = 1, 2, \ldots, m\). However, Akhmediev et. al. results were derived by using stationary solutions. Related to the stationary solutions, we can choose the solution as follows

\[
q_c (t, x) = \frac{1}{\sqrt{2\mu}} G_c (t) \exp \left( i \frac{q_c^2}{2} x \right),
\]

then eq.(1.1) can be reduced to
\[ \frac{\partial^2 G_c}{\partial t^2} + 2 \left[ \sum_{b=1}^{m} G_b^2 \right] G_c = \theta_c G_c, \quad c = 1, 2, \ldots, m. \] (3.12)

Eq. (3.12) is the same as the related equation in Ref. 9 when \( G_c(t) = u_i(\tau) \) and \( \theta_c = \lambda_i \). So, we can generalize that our result is the solution of the bright and dark multi-solitons collisions of the vector NLS model.

4. Discussions and Conclusions

We have presented a Lax pair and its bright and dark multi-soliton solution of the integrable vector NLS equation using the inverse scattering Zakharov-Shabat scheme. We can conclude that the solution of the equation can be solved by using an expanded inverse scattering Zakharov-Shabat scheme in which we have chosen a certain operator in eq. (2.1), and a certain \( k_+ \) (eq. (2.7a)) and \( F \) (eq. (2.7b)) in our solution.

Finally, we find that our results correspond to an elastic collision of the bright and dark vector multi-solitons, as long as

\[ [(A_1)_{1_0} : (A_1)_{2_0} : \ldots : (A_1)_{N_0}] = [(A_2)_{1_0} : (A_2)_{2_0} : \ldots : (A_2)_{N_0}] = \ldots = [(A_m)_{1_0} : (A_m)_{2_0} : \ldots : (A_m)_{N_0}]. \] (4.1)

If in our results,

\[ [(A_1)_{1_0} : (A_1)_{2_0} : \ldots : (A_1)_{N_0}] \neq [(A_2)_{1_0} : (A_2)_{2_0} : \ldots : (A_2)_{N_0}] \neq \ldots \neq [(A_m)_{1_0} : (A_m)_{2_0} : \ldots : (A_m)_{N_0}], \] (4.2)

then we get an inelastic collision of the bright and dark vector multi-solitons. From eq. (3.8), we can conclude that although the collision between the bright and dark vector multi-solitons, their velocities and amplitudes or intensities do not change, their phases do change. In the eq. (3.8), we also conclude that the bright and dark multi-solitons solution of the integrable vector NLS equation is exactly solved by using an extended ZS scheme. The applications of our results can widely contribute to some experiments related to optical fiber in communications system.

We then propose that there must be an analytical solution for vector solitons governed by the system of the vector NLS model exhibit a novel type (in Ref. 20) of solitons collision-the solitons polarization states switching. Here, the solitons polarization states switching can be used for the construction of logic elements on bright vector solitons. We also propose that there is a vector multi-solitons solution for the other nonlinear equations. Investigations concerning this problem are now in progress.

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