On $H_\sigma$-permutably embedded and $H_\sigma$-subnormaly embedded subgroups of finite groups

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Abstract

Let $G$ be a finite group. Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and $n$ an integer. We write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$, $\sigma(G) = \sigma(|G|)$. A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member of $\mathcal{H} \setminus \{1\}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i$ and $\mathcal{H}$ contains exact one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$. A subgroup $A$ of $G$ is called: (i) a $\sigma$-Hall subgroup of $G$ if $\sigma(|A|) \cap \sigma(|G : A|) = \emptyset$; (ii) $\sigma$-permutable in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$. We say that a subgroup $A$ of $G$ is $H_\sigma$-permutably embedded in $G$ if $A$ is a $\sigma$-Hall subgroup of some $\sigma$-permutable subgroup of $G$.

We study finite groups $G$ having an $H_\sigma$-permutably embedded subgroup of order $|A|$ for each subgroup $A$ of $G$. Some known results are generalized.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $n$ is an integer, $\mathbb{P}$ is the set of all primes, and if $\pi \subseteq \mathbb{P}$, then $\pi' = \mathbb{P} \setminus \pi$. The symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$. We use $n_\pi$ to denote the $\pi$-part of $n$, that is, the largest $\pi$-number dividing $n$; $n_p$ denotes the largest degree of $p$ dividing $n$.

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In what follows, $\sigma = \{\sigma_i|i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $\Pi$ is a subset of $\sigma$ and $\Pi' = \sigma \setminus \Pi$.

Let $\sigma(n) = \{\sigma_i|\sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$. Then we say that $G$ is $\sigma$-primary if $G$ is a $\sigma_i$-group for some $\sigma_i \in \sigma$.

A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member of $\mathcal{H} \setminus \{1\}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$. We say that $G$ is $\sigma$-full if $G$ possesses a complete Hall $\sigma$-set. Throughout this paper, $G$ is always supposed to be a $\sigma$-full group.

A subgroup $A$ of $G$ is called a Hall subgroup of $G$ if $\sigma(|A|) \cap \sigma(|G:A|) = \emptyset$; (ii) $\sigma$-subnormal in $G$ if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

such that either $A_{i-1} \unlhd A_i$ or $A_i/(A_{i-1}A_i)$ is $\sigma$-primary for all $i = 1, \ldots, t$; (iii) $\sigma$-quasinormal or $\sigma$-permutably in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$. In particular, $A$ is called $S$-quasinormal or $S$-permutable in $G$ provided $AP = PA$ for all Sylow subgroups $P$ of $G$.

Definition 1.1. We say that a subgroup $A$ of $G$ is $H_\sigma$-subnormally (respectively $H_\sigma$-permutably, $H_\sigma$-normally) embedded in $G$ if $A$ is a $\sigma$-Hall subgroup of some $\sigma$-subnormal (respectively $\sigma$-permutably, normal) subgroup of $G$.

In the special case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, the definition of $H_\sigma$-normally embedded subgroups is equivalent to the concept of Hall normally embedded subgroups in [2, 3], the definition of $H_\sigma$-permutably embedded subgroups is equivalent to the concept of Hall $S$-quasinormally embedded subgroups in [5] and the definition of $H_\sigma$-subnormally embedded subgroups is equivalent to the concept of Hall subnormally embedded subgroups in [7].

Example 1.2. (i) For any $\sigma$, all $\sigma$-Hall subgroups and all $\sigma$-subnormal subgroups of any group $S$ are $H_\sigma$-subnormally embedded in $S$. Now, let $G = (C_7 \times C_3) \times A_5$, where $C_7 \times C_3$ is a non-abelian group of order 21 and $A_5$ is the alternating group of degree 5, and let $H = (C_7 \times C_3)A$, where $A$ is a Sylow 2-subgroup of $A_5$. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{7\}$ and $\sigma_2 = \{7\}'$. Then $H$ is $\sigma$-subnormal in $G$ and $C_3A_5$ is a $\sigma$-Hall subgroup of $G$. In view of Lemma 2.1(1)(5) below, the subgroup $C_3A$ is neither $\sigma$-subnormal in $G$ nor $H_\sigma$-normally embedded in $G$.

(ii) For any $\sigma$, all $\sigma$-Hall subgroups and all $\sigma$-permutably embedded subgroups of any group $S$ are $H_\sigma$-permutably embedded in $S$. Now, let $p > q > r$ be primes, where $r^2$ divides $q - 1$. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{q, r\}$ and $\sigma_2 = \{q, r\}'$. Let $H = Q \times R$ be a group of order $qr^2$, where $C_H(Q) = Q$. Let $P$ be a simple $\mathbb{F}_qH$-module which is faithful for $H$ and $G = P \times H$. Let $R_1$ be a subgroup of $R$ of order $r$. Then the subgroup $V = PR_1$ is $\sigma$-permutably in $G$ and $R_1$ is a $\sigma$-Hall subgroup of $V$. Hence $R_1$ is $H_\sigma$-permutably embedded in $G$. It is also clear that $G$ has no an $S$-permutable subgroup $W$ such that $R_1$ is a Hall subgroup of $W$, so $R_1$ is neither $H_\sigma$-normally embedded nor $S$-permutably
embedded in $G$.

(iii) For any $\sigma$, all $\sigma$-Hall subgroups and all normal subgroups of any group $S$ are $H_\sigma$-normally embedded in $S$. Now, let $P$ be a simple $\mathbb{F}_{11}(C_7 \times C_3)$-module which is faithful for $C_7 \times C_3$. Let $G = (P \times (C_7 \times C_3)) \times A_5$. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{5, 7, 11\}$ and $\sigma_2 = \{5, 7, 11\}'$. Then the subgroup $M = (P \times C_7) \times A_5$ is normal in $G$ and a subgroup $B$ of $A_5$ of order 12 is $\sigma$-Hall subgroup of $M$, so $B$ is $H_\sigma$-normally embedded in $G$. Finally, if $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{7\}$ and $\sigma_2 = \{7\}'$, then $B$ is not $H_\sigma$-normally embedded in $G$.

Recall that $G$ is $\sigma$-nilpotent \cite{10} if $G = H_1 \times \cdots \times H_t$ for some $\sigma$-primary groups $H_1, \ldots, H_t$. The $\sigma$-nilpotent residual $G^{\sigma_{\text{res}}}$ of $G$ is the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$, $G^{\sigma_{\text{res}}}$ denotes the nilpotent residual of $G$. It is clear that every subgroup of a $\sigma$-nilpotent group $G$ is $\sigma$-permutable and $\sigma$-subnormal in $G$.

**Theorem 1.3.** Let $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ be a complete Hall $\sigma$-set of $G$ and $D = G^{\sigma_{\text{res}}}$. Then any two of the following conditions are equivalent:

(i) $G$ has an $H_\sigma$-permutably embedded subgroup of order $|A|$ for each subgroup $A$ of $G$.

(ii) $D$ is a complemented cyclic of square-free order subgroup of $G$ and $|\sigma_i \cap \pi(G)| = 1$ for all $i$ such that $\sigma_i \cap \pi(D) \neq \emptyset$.

(iii) For each set $\{A_1, \ldots, A_t\}$, where $A_i$ is a subgroup (respectively normal subgroup) of $H_i$ for all $i = 1, \ldots, t$, $G$ has an $H_\sigma$-permutably embedded (respectively $H_\sigma$-normally embedded) subgroup of order $|A_1| \cdots |A_t|$.

Let $\mathfrak{F}$ be a class of groups. A subgroup $H$ of $G$ is said to be an $\mathfrak{F}$-covering subgroup of $G$ \cite{10} VI, Definition 7.8 if $H \in \mathfrak{F}$ and for every subgroup $E$ of $G$ such that $H \leq E$ and $E/N \in \mathfrak{F}$ it follows that $E = NH$. We say that a subgroup $H$ of $G$ is a $\sigma$-Carter subgroup of $G$ if $H$ is an $\mathfrak{N}_\sigma$-covering subgroup of $G$, where $\mathfrak{N}_\sigma$ is the class of all $\sigma$-nilpotent groups.

A group $G$ is said to have a Sylow tower if $G$ has a normal series $1 = G_0 < G_1 < \cdots < G_{t-1} < G_t = G$, where $|G_{i}/G_{i-1}|$ is the order of some Sylow subgroup of $G$ for each $i \in \{1, \ldots, t\}$. A chief factor of $G$ is said to be $\sigma$-central (in $G$) \cite{11} if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is $\sigma$-primary. Otherwise, $H/K$ is called $\sigma$-eccentric (in $G$).

We say that $G$ is a $H\sigma E$-group if the following conditions are hold: (i) $G = D \rtimes M$, where $D = G^{\sigma_{\text{res}}}$ is a $\sigma$-Hall subgroup of $G$ and $|\sigma(D)| = |\pi(D)|$; (ii) $D$ has a Sylow tower and every chief factor of $G$ below $D$ is $\sigma$-eccentric; (iii) $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$.

We do not still know the structure of a group $G$ having a $H_\sigma$-subnormally embedded subgroup of order $|A|$ for each subgroup $A$ of $G$. Nevertheless, the following fact is true.

**Theorem 1.4.** Any two of the following conditions are equivalent:

(i) Every subgroup of $G$ is $H_\sigma$-subnormally embedded in $G$. 

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(ii) Every $\sigma$-subnormal subgroup $H$ of $G$ is an $H\sigma E$-group of the form $H = D \times M$, where $D = H^{\sigma_{E}}$ and $M$ is a $\sigma$-Carter subgroup of $H$.

(iii) Every $\sigma$-subnormal subgroup of $G$ is an $H\sigma E$-group.

Now, let us consider some corollaries of Theorems 1.3 and 1.4. First note that since a nilpotent group $G$ possesses a normal subgroup of order $n$ for each integer $n$ dividing $|G|$, in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, Theorem 1.3 covers Theorem 11 in [6], Theorem 2.7 in [8] and Theorems 3.1 and 3.2 in [7].

From Theorem 1.3 we also get the following result.

**Corollary 1.5.** Suppose that $G$ possesses a complete Hall $\sigma$-set $\mathfrak{H} = \{1, H_1, \ldots, H_t\}$ such that $H_i$ is nilpotent for all $i = 1, \ldots, t$. Then $G$ has an $H_\sigma$-normally embedded subgroup of order $|H|$ for each subgroup $H$ of $G$ if and only if the nilpotent residual $D = G^{\sigma_{I}}$ of $G$ is cyclic of square-free order and $|\sigma_i \cap \pi(G)| = 1$ for all $i$ such that $\sigma_i \cap \pi(D) \neq \emptyset$.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Corollary 1.5 the following known result.

**Corollary 1.6** (Ballester-Bolinches, Qiao [11]). $G$ has a Hall normally embedded subgroup of order $|H|$ for each subgroup $H$ of $G$ if and only if the nilpotent residual $G^{\sigma_{I}}$ of $G$ is cyclic of square-free order.

On the basis of Theorems 1.3 and 1.4 we prove also the next two theorems.

**Theorem 1.7.** Any two of the following conditions are equivalent:

(i) Every subgroup of $G$ is $H_\sigma$-normally embedded in $G$.

(ii) $G = D \times M$ is a $H\sigma E$-group, where $D$ is a cyclic group of square-free order and $M$ is a Dedekind group.

(iii) $G = D \times M$, where $D$ is a $\sigma$-Hall cyclic subgroup of $G$ of square-free order with $|\sigma(D)| = |\pi(D)|$ and $M$ is a Dedekind group.

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Theorem 1.7 the following known result.

**Corollary 1.8** (Li, Liu [8]). Every subgroup of $G$ is a Hall normally embedded subgroup of $G$ if and only if $G = D \times M$, where $D = G^{\sigma_{I}}$ is a cyclic Hall subgroup of $G$ of square-free order and $M$ is a Dedekind group.

**Theorem 1.9.** Any two of the following conditions are equivalent:

(i) Every subgroup of $G$ is $H_\sigma$-permutably embedded in $G$.

(ii) $G = D \times M$ is a $H\sigma E$-group, where $D = G^{\sigma_{I}}$ is a cyclic group of square-free order.

(iii) $G = D \times M$, where $D$ is a $\sigma$-Hall cyclic subgroup of $G$ of square-free order with $|\sigma(D)| = |\pi(D)|$ and $M$ is $\sigma$-nilpotent.

**Corollary 1.10.** Every subgroup of $G$ is a Hall $S$-quasinormally embedded subgroup of $G$ if and only if $G = D \times M$, where $D = G^{\sigma_{I}}$ is a cyclic Hall subgroup of $G$ of square-free order and $M$ is a
Carter subgroup of $G$.

In conclusion of this section, consider the following example.

**Example 1.11.** Let $5 < p_1 < p_2 < \cdots < p_n$ be a set of primes and $p$ a prime such that either $p > p_n$ or $p$ divides $p_i - 1$ for all $i = 1, \ldots, n$. Let $A$ be a group of order $p$ and $P_i$ a simple $F_{p_i}$-module which is faithful for $A$. Let $L_i = P_i \rtimes A$ and

$$B = (\ldots ((L_1 \rtimes L_2) \rtimes L_3) \rtimes \cdots) \rtimes L_n$$

(see [10, p. 50]). We can assume without loss of generality that $L_i \leq B$ for all $i = 1, \ldots, n$. Let $G = B \rtimes A_5$, where $A_5$ is the alternating group of degree 5, and let $\sigma$ be a partition of $\mathbb{P}$ such that for some different indices $i, j, i_1, \ldots, i_n \in I$ we have $p \in \sigma_i$, $\{2, 3, 5\} \subseteq \sigma_j$ and $p_k \in \sigma_{i_k}$ for all $k = 1, \ldots, n$. Then

$$D = P_1 P_2 \cdots P_n = G^{\sigma_i}$$

is a $\sigma$-Hall subgroup of $G$ and $G = D \rtimes (A \times A_5)$.

We show that every subnormal subgroup $H$ of $G$ satisfies Condition (ii) in Theorem 1.4. If $H^{\sigma_i} = 1$, it is evident. Hence we can assume without loss of generality $A \leq H$ since every $p'$-subgroup of $G$ is $\sigma$-nilpotent. But then

$$H = (H \cap D) \rtimes (A \times (H \cap A_5))$$

by Lemma 2.1(4), where $H \cap D$ is a normal $\sigma$-Hall subgroup of $H$ and $M = A \times (H \cap A_5)$ is a $\sigma$-nilpotent subgroup of $H$. Moreover, $H \cap A_5$ induces on every non-identity Sylow subgroup of $H \cap D$ a non-trivial irreducible module of automorphisms. Therefore $H^{\sigma_i} = H \cap D$ and $|\sigma(H^{\sigma_i})| = |\pi(H^{\sigma_i})|$.

It is also clear that $M$ is a $\sigma$-Carter subgroup of $H$ and every chief factor of $H$ below $H^{\sigma_i}$ is $\sigma$-eccentric in $H$. Thus $G$ satisfies Condition (ii) in Theorem 1.4, and so every subgroup $H$ of $G$ is $H_{\sigma}$-subnormally embedded in $G$. On the other hand, the subgroup $DAC_2$, where $C_2$ is a subgroup of order 2 of $G$, is not Hall subnormally embedded in $G$ since $C_2$ is not a Sylow subgroup of any subnormal subgroup of $G$.

Finally, if $p$ divides $p_i - 1$ for all $i = 1, \ldots, n$, then $|P_i| = p_i$ for all $i = 1, \ldots, n$, so $G$ satisfies Condition (ii) in Theorem 1.9 and hence satisfies Condition (ii) in Theorem 1.3.

### 2 Basic lemmas

An integer $n$ is called a $\Pi$-number if $\sigma(n) \subseteq \Pi$. A subgroup $H$ of $G$ is called a Hall $\Pi$-subgroup of $G$ if $|H|$ is a $\Pi$-number and $|G:H|$ is a $\Pi'$-number.

**Lemma 2.1** (See Lemma 2.6 in [1]). Let $A, K$ and $N$ be subgroups of $G$, where $A$ is $\sigma$-subnormal in $G$ and $N$ is normal in $G$.

1. $A \cap K$ is $\sigma$-subnormal in $K$. 

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If $K$ is $\sigma$-subnormal in $G$, then $A \cap K$ and $\langle A, K \rangle$ are $\sigma$-subnormal in $G$.

(3) $AN/N$ is $\sigma$-subnormal in $G/N$.

(4) If $H \neq 1$ is a Hall $\Pi$-subgroup of $G$ and $A$ is not a $\Pi'$-group, then $A \cap H \neq 1$ is a Hall $\Pi$-subgroup of $A$.

(5) If $|G : A|$ is a $\sigma_i$-number, then $O^{\sigma_i}(A) = O^{\sigma_i}(G)$.

(6) If $V/N$ is a $\sigma$-subnormal subgroup of $G/N$, then $V$ is $\sigma$-subnormal in $G$.

(7) If $K$ is a $\sigma$-subnormal subgroup of $A$, then $K$ is $\sigma$-subnormal in $G$.

A group $G$ is said to be $\sigma$-soluble if every chief factor of $G$ is $\sigma$-primary.

**Lemma 2.2** (See Lemmas 2.8 and 3.2 and Theorems B and C in [1]). Let $A$, $K$ and $N$ be subgroups of $G$, where $A$ is $\sigma$-permutable in $G$ and $N$ is normal in $G$.

(1) $AN/N$ is $\sigma$-permutable in $G/N$.

(2) If $G$ is $\sigma$-soluble, then $A \cap K$ is $\sigma$-permutable in $K$.

(3) If $N \leq K$, $K/N$ is $\sigma$-permutable in $G/N$ and $G$ is $\sigma$-soluble, then $K$ is $\sigma$-permutable in $G$.

(4) $A$ is $\sigma$-subnormal in $G$.

(5) If $G$ is $\sigma$-soluble and $K$ is $\sigma$-permutable in $G$, then $K \cap A$ is $\sigma$-permutable in $G$.

**Lemma 2.3.** Let $H$ be a normal subgroup of $G$. If $H/H \cap \Phi(G)$ is a $\Pi$-group, then $H$ has a $\Pi$-subgroup, say $E$, and $E$ is normal in $G$. Hence, if $H/H \cap \Phi(G)$ is $\sigma$-nilpotent, then $H$ is $\sigma$-nilpotent.

**Proof.** Let $D = O_{\Pi'}(H)$. Then, since $H \cap \Phi(G)$ is nilpotent, $D$ is a Hall $\Pi'$-subgroup of $H$. Hence by the Schur-Zassenhaus theorem, $H$ has a Hall $\Pi$-subgroup, say $E$. It is clear that $H$ is $\pi'$-soluble where $\pi' = \cup_{\sigma_i \in \Pi} \sigma_i$, so any two Hall $\Pi$-subgroups of $H$ are conjugate. By the Frattini argument,

$G = HN_G(E) = (E(H \cap \Phi(G)))N_G(E) = N_G(E)$.

Therefore $E$ is normal in $G$. The lemma is proved.

**Lemma 2.4.** If every chief factor of $G$ below $D = G^{\sigma_n}$ is cyclic, then $D$ is nilpotent.

**Proof.** Assume that this is false and let $G$ be a counterexample of minimal order. Let $R$ be a minimal normal subgroup of $G$. Then from the $G$-isomorphism

$D/D \cap R \simeq DR/R = (G/R)^{\sigma_n}$

we know that every chief factor of $G/R$ below $DR/R$ is cyclic, so the choice of $G$ implies that $D/D \cap R \simeq DR/R$ is nilpotent. Hence $R \leq D$ and $R$ is the unique minimal normal subgroup of $G$. In view of Lemma 2.3, $R \not\leq \Phi(G)$ and so $R = C_R(R)$ by [13, A, 15.2]. But by hypothesis, $|R|$ is a prime, hence $G/R = G/C_G(R)$ is cyclic, so $G$ is supersoluble and so $G^{\sigma_n}$ is nilpotent since $G^{\sigma_n} \leq G^{\sigma_l}$. The lemma is proved.
The following lemma is evident.

**Lemma 2.5.** The class of all σ-soluble groups is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the σ-soluble group by a σ-soluble group is a σ-soluble group as well.

Let $A$, $B$ and $R$ be subgroups of $G$. Then $A$ is said to $R$-permute with $B$ [12] if for some $x \in R$ we have $AB^x = B^xA$.

If $G$ has a complete Hall σ-set $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ such that $H_iH_j = H_jH_i$ for all $i, j$, then we say that $\{H_1, \ldots, H_t\}$ is a σ-basis of $G$.

**Lemma 2.6** (See Theorems A and B in [2]). Assume that $G$ is σ-soluble.

(i) $G$ has a σ-basis $\{H_1, \ldots, H_t\}$ such that for each $i \neq j$ every Sylow subgroup of $H_i$ $G$-permutes with every Sylow subgroup of $H_j$.

(ii) For any $\Pi$, the following hold: $G$ has a Hall $\Pi$-subgroup $E$, every $\Pi$-subgroup of $G$ is contained in some conjugate of $E$ and $E$ $G$-permutes with every Sylow subgroup of $G$.

**Lemma 2.7.** Let $H$, $E$ and $R$ be subgroups of $G$. Suppose that $H$ is $H_\sigma$-subnormally embedded in $G$ and $R$ is normal in $G$.

1. If $H \leq E$, then $H$ is $H_\sigma$-subnormally embedded in $E$.
2. $HR/R$ is $H_\sigma$-subnormally embedded in $G/R$.
3. If $S$ is a σ-subnormal subgroup of $G$, then $H \cap S$ is $H_\sigma$-subnormally embedded in $G$.
4. If $|G : H|$ is σ-primary, then $H$ is either a $\sigma$-Hall subgroup of $G$ or $\sigma$-subnormal in $G$.

**Proof.** Let $V$ be a σ-subnormal subgroup of $G$ such that $H$ is a $\sigma$-Hall subgroup of $V$.

1. This assertion is a corollary of Lemma 2.1(1).
2. In view of Lemma 2.1(3), $VR/R$ is $\sigma$-subnormal subgroup of $G/R$. It is also clear that $HR/R$ is a $\sigma$-Hall subgroup of $VR/R$. Hence $HR/R$ is $H_\sigma$-subnormally embedded in $G/R$.
3. By Lemma 2.1(1)(2), $V \cap S$ is $\sigma$-subnormal both in $V$ and in $G$ and so $H \cap (V \cap S) = H \cap S$ is a $\sigma$-Hall subgroup of $V \cap S$ by Lemma 2.1(4), as required.
4. Assume that $H$ is not $\sigma$-subnormal in $G$. Then $H < V$. By hypothesis, $|G : H|$ is $\sigma$-primary, say $|G : H|$ is a $\sigma_i$-number. Then $|V : H|$ is a $\sigma_i$-number. But $H$ is a $\sigma$-Hall subgroup of $V$. Hence $H$ is a $\sigma$-Hall subgroup of $G$.

The lemma is proved.

**Lemma 2.8.** Let $H$ be a $\sigma$-subnormal subgroup of a $\sigma$-soluble group $G$. If $|G : H|$ is a $\sigma_i$-number and $B$ is a $\sigma_i$-complement of $H$, then $G = HN_G(B)$.

**Proof.** Assume that this lemma is false and let $G$ be a counterexample of minimal order. Then $H < G$, so $G$ has a proper subgroup $M$ such that $H \leq M$, $|G : M_G|$ is a $\sigma_i$-number and $H$ is $\sigma$-subnormal in $M$. The choice of $G$ implies that $M = HN_M(B)$. On the other hand, clearly that $B$
is a $\sigma_i$-complement of $M_G$. Since $G$ is $\sigma$-soluble, Lemma 2.6 and the Frattini argument imply that

$$G = M_G N_G(B) = M N_G(B) = H N_M(B) N_G(B) = H N_G(B).$$

The lemma is proved.

The following lemma is well-known (see for example [14, Lemma 3.29] or [15, 1.10.10]).

**Lemma 2.9.** Let $H/K$ be an abelian chief factor of $G$ and $V$ a maximal subgroup of $G$ such that $K \leq V$ and $HV = G$. Then

$$G/V_G \simeq (H/K) \times (G/C_G(H/K)).$$

### 3 Proofs of the results

**Proof of Theorem 1.3.** Without loss of generality we may assume that $H_i$ is a $\sigma_i$-group for all $i = 1, \ldots, t$.

(i), (iii) $\Rightarrow$ (ii) Assume that this is false. Then $D \neq 1$ and so $t > 1$.

First we prove the following claim.

(*) If $p \in \sigma_i \cap \pi(G)$, then $G$ has a $\sigma$-permutable subgroup $E$ with $|E| = |G|_{\sigma_i'}p$.

We can assume without loss of generality that $i = 1$. In fact, to prove Claim (*), we consistently build the $\sigma$-permutable subgroups $E_2, \ldots, E_t$ such that $|H_2| \cdots |H_t|$ divides $|E_j|$ and $|E_j|_{\sigma_i} = p$ for all $j = 2, \ldots, t$.

By hypothesis, $G$ has an $H_{\sigma_i}$-permutably embedded subgroup $X$ of order $p$. Let $V$ be a $\sigma$-permutable subgroup of $G$ such that $X$ is a $\sigma$-Hall subgroup of $V$. Then $|V|_{\sigma_i} = p$ and $G$ has a complete Hall $\sigma$-set $\{K_1, \ldots, K_t\}$, where $K_i$ is a $\sigma_i$-group for all $i = 1, \ldots, t$, such that $VK_i = K_i V$ for all $i = 1, \ldots, t$. Let $W = V K_2$. Then $|W|_{\sigma_i} = p$.

Next we show that there is an $H_{\sigma_i}$-permutably embedded subgroup $Y$ of $G$ such that $|Y| = |W|$. It is enough to consider the case when Condition (iii) holds. Let $A_1$ be a subgroup of $H_1$ of order $p$, $A_2 = H_2$ and $A_i = H_i \cap V$ for all $i > 2$. Then

$$|A_2| = |H_2| = |K_2|.$$  

On the other hand, $V \cap K_1$ and $V \cap H_i$ are Hall $\sigma_i$-subgroups of $V$ by Lemmas 2.1(4) and 2.2(4) and so $|V \cap K_i| = |V \cap H_i|$. Also, for every $i > 2$ we have

$$|W : V \cap K_i| = |V K_2 : V \cap K_i| = |V| |K_2| : |V \cap K_2||V \cap K_i|$$

is a $\sigma_i'$-number and hence $V \cap K_i = W \cap K_i$ is a Hall $\sigma_i$-subgroup of $W$. Therefore,

$$|W| = p|H_2||V \cap H_3| \cdots |V \cap H_t|$$
and so $G$ has an $H_\sigma$-permutably embedded subgroup $Y$ of order

$$|W| = |A_1| \cdots |A_t|$$

by hypothesis.

Let $E_2$ be a $\sigma$-permutable subgroup of $G$ such that $Y$ is a $\sigma$-Hall subgroup of $E_2$. Then $|H_2| = |K_2|$ divides $|E_2|$ and $E_2|_{\sigma_1} = p$. Now, arguing by induction, assume that $G$ has a $\sigma$-permutable subgroup $E_i$ of $A$ such that $|H_2| \cdots |H_{i-1}|$ divides $|E_{i-1}|$ and $E_{i-1}|_{\sigma_1} = p$. Then for some Hall $\sigma_t$-group $L$ we have $E_{i-1}L = L E_{i-1}$, and if $E_t = E_{i-1}L$, then $|E_t| = |G|_{\sigma' t} p$ and $E_t$ clearly is $\sigma$-permutable in $G$, as required.

Now, let $p \in \sigma_i \cap \pi(D)$ and let $P$ be a Sylow $p$-subgroup of $D$. Then, by Claim (*), $G$ possesses a $\sigma$-permutable subgroup $E$ such that $|E| = |G|_{\sigma' t} p$. Lemma 2.2(4) implies that $E$ is $\sigma$-subnormal in $G$. Let $j \neq i$. Then $H_j^z \cap E$ is a Hall $\sigma_j$-subgroup of $E$ by Lemma 2.1(4), so $|E : H_j^z \cap E|$ is a $\sigma_j'$-number. But $|H_j^z|$ divides $|E|$ and hence $|H_j^z|$ divides $|H_j \cap E|$. Thererore $H_j^z \leq E$ for all $x \in E$. Thus $H_j^z \leq E$ and so $G/E_G$ is a $\sigma_t$-group. Hence $D \leq E_G \leq E$, so $|P| = p$. Therefore $G$ is supersoluble by [10] IV, 2.9 and so every chief factor of $G$ below $D$ is cyclic. Hence $D$ is nilpotent by Lemma 2.4, so $D$ is cyclic of square-free order. Hence $D$ is complemented in $G$ by Theorem 11.8 in [17].

Finally, assume that $|\sigma_i \cap \pi(G)| > 1$ and let $q \in \sigma_i \cap \pi(G) \setminus \{p\}$. Then $G$ possesses a $\sigma$-permutable subgroup $F$ such that $|F| = |G|_{\sigma' t} q$. Then $D \leq F_G \leq F$. Therefore $D \leq E \cap F$ and so $p$ does not divide $|D|$. This contradiction completes the proof of the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) First we show that for every $i$ and for every subgroup (respectively normal subgroup) $A_i$ of $H_i$, there is an $H_\sigma$-permutably embedded (respectively $H_\sigma$-normally embedded) subgroup $E_i$ of $G$ such that $|E_i| = |A_i||G|_{\sigma' i}$. Since $G$ evidently is $\sigma$-soluble, it has a $\sigma_i$-complement $E$ by Lemma 2.6. Therefore, it is enough to consider the case when $A_i \neq 1$ since every $\sigma$-Hall subgroup of $G$ is an $H_\sigma$-normally embedded in $G$.

First suppose that $D \leq E$. Then $E/D$ is normal in $G$ since $G/D$ is $\sigma$-nilpotent. Therefore

$$(E/D) \times (A_iD/D) = EA_i/D$$

is $\sigma$-permutable (respectively normal) in

$$G/D = (E/D) \times (H_iD/D).$$

Hence $E_i = EA_i$ is $\sigma$-permutable (respectively normal) in $G$ by Lemma 2.2(3) and $|E_i| = |A_i||G|_{\sigma' i}$.

Now suppose that $D \nleq E$. Then $D \cap H_i \neq 1$, so $H_i$ is a $p$-group for some prime $p$ since for each $\sigma_i \in \sigma(D)$ we have $|\sigma_i \cap \pi(G)| = 1$ by hypothesis. Hence $H_2$ has a normal subgroup $A$ such that $D_p \leq A$ and $|A| = |A_i|$, where $D_p$ is a Sylow $p$-subgroup of $D$. Then $D \leq AE$. Moreover,

$$AE/D = (DA/D) \times (ED/D)$$
since $ED/D$ is a Hall $\sigma_i$-subgroup of $G/D$. Therefore $E_i = AE$ is $\sigma$-permutable (respectively normal) in $G$ by Lemma 2.2(3) and $|E_i| = |A_j||G|_{\sigma_i}$.

Let $E = E_1 \cap \cdots \cap E_t$. Then $|E| = |A_1| \cdots |A_t|$ since

$$((G : E_i), (G : E_j)) = 1$$

for all $i \neq j$. Note that $E_i$ is either a $\sigma$-Hall subgroup of $G$ or $\sigma$-permutable (respectively normal) in $G$. Indeed, let $V$ be a $\sigma$-permutable (respectively normal) subgroup of $G$ such that $E_i$ is a $\sigma$-Hall subgroup of $V$. Assume that $E_i$ is not $\sigma$-permutable (respectively not normal) in $G$. Then $E_i < V$. Since $|G : E_i|$ is $\sigma$-number, $|V : E_i|$ is a $\sigma$-number. But $E_i$ is a $\sigma$-Hall subgroup of $V$. Hence $E_i = V$ is a $\sigma$-Hall subgroup of $G$.

Assume that $E_1, \ldots, E_r$ are $\sigma$-permutable (respectively normal) in $G$ and $E_i$ is a $\sigma$-Hall subgroup of $G$ for all $i > r$. Then $E^0 = E_1 \cap \cdots \cap E_r$ is $\sigma$-permutable (respectively normal) in $G$ by Lemma 2.2(5) and $E^* = E_{r+1} \cap \cdots \cap E_t$ is a $\sigma$-Hall subgroup of $G$. Now, $E = E^0 \cap E^*$ is a $\sigma$-Hall subgroup of $E^0$ by Lemmas 2.1(4) and 2.2(4), so $E$ is $H_\sigma$-permutably (respectively $H_\sigma$-normally) embedded in $G$. Hence (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i) Since $G$ is $\sigma$-soluble, $H$ is $\sigma$-soluble. Hence $H$ has a $\sigma$-basis $\{L_1, \ldots, L_r\}$ such that $L_i \leq H_i$ for all $i = 1, \ldots, r$ by Lemma 2.6. Therefore from the implication (ii) $\Rightarrow$ (iii) we get that $G$ has an $H_\sigma$-permutably embedded subgroup of order $|L_1| \cdots |L_r| = |H|$.

The theorem is proved.

**Proof of Theorem 1.4.** (i) $\Rightarrow$ (ii) Assume that this is false and let $G$ be a counterexample of minimal order. Then some $\sigma$-subnormal subgroup $V$ of $G$ is not an $H\sigma E$-group. Moreover, $D = G^{\sigma_r} \neq 1$, so $|\sigma(G)| > 1$.

(1) Condition (ii) is true on every proper section $H/K$ of $G$, that is, $K \neq 1$ or $H \neq G$. Hence $V = G$ (This directly follows from Lemma 2.7(1)(2) and the choice of $G$).

(2) $G$ is $\sigma$-soluble.

In view of Claim (1) and Lemma 2.5, it is enough to show that $G$ is not simple. Assume that this is false. Then 1 is the only proper $\sigma$-subnormal subgroup of $G$ since $|\sigma(G)| > 1$. Hence every subgroup of $G$ is a $\sigma$-Hall subgroup of $G$. Therefore for a Sylow $p$-subgroup $P$ of $G$, where $p$ is the smallest prime divisor of $|G|$, we have $|P| = p$ and so $|G| = p$ by [10] IV, 2.8. This contradiction shows that we have (2).

(3) If $|G : H|$ is a $\sigma$-number and $H$ is not a $\sigma$-Hall subgroup of $G$, then $H$ is $\sigma$-subnormal in $G$ and a $\sigma$-complement $E$ of $H$ is normal in $G$ (This follows from Lemmas 2.7(4) and 2.8).

(4) $D$ is a Hall subgroup of $G$. Hence $D$ has a complement $M$ in $G$.

Suppose that this is false and let $P$ be a Sylow $p$-subgroup of $D$ such that $1 < P < G_p$, where $G_p \in \text{Syl}_p(G)$. We can assume without loss of generality that $G_p \leq H_1$. Let $R$ be a minimal normal subgroup of $G$ contained in $D$. 


Since $D$ is soluble by Claim (2), $R$ is a $q$-group for some prime $q$. Moreover, $D/R = (G/R)^{p_{R}}$ is a Hall subgroup of $G/R$ by Claim (1) and Proposition 2.2.8 in [15]. Suppose that $PR/R \neq 1$. Then $PR/R \in \text{Syl}_{p}(G/R)$. If $q \neq p$, then $P \in \text{Syl}_{p}(G)$. This contradicts the fact that $P < G_{p}$. Hence $q = p$, so $R \leq P$ and therefore $P/R \in \text{Syl}_{p}(G/R)$. It follows that $P \in \text{Syl}_{p}(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is a Sylow $p$-subgroup of $D$. Therefore $R$ is a unique minimal normal subgroup of $G$ contained in $D$. It is also clear that a $p$-complement of $D$ is a Hall subgroup of $G$.

Now we show that $R \notin \Phi(G)$. Indeed, assume that $R \leq \Phi(G)$. Then $D \neq R$ by Lemma 2.3 since $D = G^{\Phi(G)}$. On the other hand, $D/R$ is a $p'$-group. Hence $O_{p'}(D) \neq 1$ by Lemma 2.3. But $O_{p'}(D)$ is characteristic in $D$ and so it is normal $G$. Therefore $G$ has a minimal normal subgroup $L$ such that $L \neq R$ and $L \leq D$. This contradiction shows that $R \notin \Phi(G)$.

Let $S$ be a maximal subgroup of $G$ such that $RS = G$. Then $|G:S|$ is a $p$-number. Hence, since $R$ is not a Sylow $p$-subgroup of $G$, $p$ divides $|S|$. Then $S$ is not a Hall subgroup of $G$ and so $S$ is not a $\sigma$-Hall subgroup of $G$. Therefore $S$ is $\sigma$-subnormal in $G$ by Claim (3) and so $G/S_{G}$ is a $\sigma_{i}$-group, which implies that

$$R \leq D \leq S_{G} \leq S$$

and so $G = RS = S$. This contradiction completes the proof of (4).

(5) If $M \leq E < G$, then $E$ is not $\sigma$-permutable in $G$ and so $E$ a $\sigma$-Hall subgroup of $G$.

Assume that $E$ is $\sigma$-permutable in $G$. Then $E$ is $\sigma$-subnormal in $G$ by Lemma 2.2(4). Then there is a subgroup chain

$$E = E_{0} \leq E_{1} \leq \cdots \leq E_{r} = G$$

such that either $E_{i-1}$ is normal in $E_{i}$ or $E_{i}/(E_{i-1} E_{i})_{E_{i}}$ is $\sigma$-primary for all $i = 1, \ldots, r$. Let $V = E_{r-1}$. We can assume without loss of generality that $V \neq G$. Therefore, since $G$ is $\sigma$-soluble by Claim (2), for some $\sigma$-primary chief factor $G/W$ of $G$ we have $E \leq V \leq W$. Also we have $D \leq W$ and so $G = DE \leq W$, a contradiction. Hence $E$ is not $\sigma$-permutable in $G$.

By hypothesis, $G$ has a $\sigma$-permutable subgroup $S$ such that $E$ is a $\sigma$-Hall subgroup of $S$. But then $S = G$, by the above argument, so $E$ is a $\sigma$-Hall subgroup of $G$. In particular, $M$ is a $\sigma$-Hall subgroup of $G$ and so $D$ is a $\sigma$-Hall subgroup of $G$.

(6) $D$ is soluble, $|\sigma(D)| = |\pi(D)|$ and $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$.

Let $p \in \sigma \in \sigma(D)$. Lemma 2.6 and Claims (2) and (4) imply that for some Sylow $p$-subgroup $P$ of $G$ we have $PM = MP$. Moreover, $MP$ is a $\sigma$-Hall subgroup of $G$ by Claim (5). Hence $|\sigma_{i} \cap \pi(G)| = 1$ for all $i$ such that $\sigma_{i} \cap \pi(D) \neq \emptyset$ and so $|\sigma(D)| = |\pi(D)|$. Therefore, since $D$ is soluble by Claim (2), $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$ by Claim (5).

(7) $M$ is a $\sigma$-Carter subgroup of $G$.

Let $R$ be a minimal normal subgroup of $G$ contained in $D$ and $E$ a subgroup of $G$ containing $M$. 

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We need to show that $E = E^{\mathfrak{m}}_M$. Claim (1) implies that $RM/R$ is a $\sigma$-Carter subgroup of $G/R$, so 

$$ER/R = (ER/R)^{\mathfrak{m}}_M (RM/R).$$

Hence $ER = E^{\mathfrak{m}}_M R$ since 

$$(ER/R)^{\mathfrak{m}}_M = E^{\mathfrak{m}}_M R/R.$$ 

Claim (6) implies that $R$ is a $p$-group for some prime $p$. Claims (4), (5) and (6) imply that $R$, $E$ and $E^{\mathfrak{m}}_M$ are $\sigma$-Hall subgroups of $G$. Therefore, if $R \not\leq E$, then $E$ and $E^{\mathfrak{m}}_M$ are Hall $p'$-subgroups of $ER = E^{\mathfrak{m}}_M R$, so $E = E^{\mathfrak{m}}_M$. Finally, assume that $R \leq E$ but $R \not\leq E^{\mathfrak{m}}_M$. Then $R \cap E^{\mathfrak{m}}_M = 1$. On the other hand, since 

$$DE/D \cong E/D \cap E$$

is $\sigma$-nilpotent, $E^{\mathfrak{m}}_M \leq D$ and so $M \cap E^{\mathfrak{m}}_M = 1$. Therefore 

$$E^{\mathfrak{m}}_M \cap RM = (E^{\mathfrak{m}}_M \cap R)(E^{\mathfrak{m}}_M \cap M) = 1.$$ 

Then 

$$E/E^{\mathfrak{m}}_M = E^{\mathfrak{m}}_M R/E^{\mathfrak{m}}_M \cong MR$$

is $\sigma$-nilpotent. Hence $M \leq C_G(R)$. Suppose that $C_G(R) < G$ and let $C_G(R) \leq W < G$, where $G/W$ is a chief factor of $G$. Claim (2) implies that $G/W$ is $\sigma$-primary, so $D \leq W$. But then $G = DM \leq W < G$, a contradiction. Therefore $C_G(R) = G$, that is, $R \leq Z(G)$. Let $V$ be a complement to $R$ in $D$. Then $V$ is a Hall normal subgroup of $D$, so it is characteristic in $D$. Hence $V$ is normal in $G$ and $G/V \cong RM$ is $\sigma$-nilpotent, so $D \leq V < D$. This contradiction completes the proof of (7).

(8) $D$ possesses a Sylow tower.

Let $R$ be a minimal normal subgroup of $G$ contained in $D$. Then $R$ is a $p$-group for some prime $p$ by Claim (6). Moreover, the Frattini argument implies that for some Sylow $p$-subgroup $P$ of $D$ we have $M \leq N_G(P)$ and so $R = P$ since $M$ acts irreducible on $P$ by Claim (6). On the other hand, by Claim (1), $D/R$ possesses a Sylow tower. Hence we have (8).

(9) Every chief factor of $G$ below $D$ is $\sigma$-eccentric.

Let $H/K$ be a chief factor of $G$ below $D$. Then $H/K$ is a $p$-group for some prime $p$ since $D$ is soluble by Claim (6). By the Frattini argument, there exist a Sylow $p$-subgroup $P$ and a $p$-complement $E$ of $D$ such that $M \leq N_G(P)$ and $M \leq N_G(E)$. Then $M \leq N_G(P \cap K)$ and $M \leq N_G(P \cap H)$. Hence $P \cap K = 1$ and $P \cap H = P$ by Claim (6), so $H = K \rtimes P$. Let $V = EM$. Then $K \leq V$ and $HV = G$, so $V$ is a maximal subgroup of $G$. Hence 

$$G/V \cong (H/K) \rtimes (G/C_G(H/K))$$

by Lemma 2.9. Therefore, if $H/K$ is $\sigma$-central in $G$, then $D \leq V_G$, which is impossible since evidently $p$ does not divide $|V|$. Thus we have (9).
From Claims (4)–(9) it follows that $G$ is a $H\sigma E$-group, contrary to our assumption on $G = V$. Hence (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) By hypothesis, $G = D \times M$, where $D = G^{\sigma x}$ is a $\sigma$-Hall subgroup of $G$, $|\sigma(D)| = |\pi(D)|$ and $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$.

(*) Every subgroup $A$ of $G$ containing $M$ is a $\sigma$-Hall subgroup of $G$.

Let $D_0 = D \cap A$. Then $A = D_0 \times M$ and $D_0 \neq 1$. Let $p \in \pi(D_0)$. The Frattini argument and Lemma 2.6 imply that for some Sylow $p$-subgroup $P_0$ of $D_0$ and some Sylow $p$-subgroup $P$ of $D$ we have $M \leq N_G(P_0)$, $M \leq N_G(P)$ and $P_0M \leq PM$. Hence, since $M$ acts irreducibly on every $M$-invariant Sylow subgroup of $D$, $P_0 = P$. Therefore every Sylow subgroup of $A$ is a Sylow subgroup of $G$. Hence $A$ is a $\sigma$-Hall subgroup of $G$ since $|\sigma(D)| = |\pi(D)|$ and $M$ is a $\sigma$-Hall subgroup of $G$.

Now, let $A$ be a subgroup of $G$. First assume that $DA < G$. By Lemma 2.1(6), $DA$ is $\sigma$-subnormal in $G$. Therefore every $\sigma$-subnormal subgroup of $DA$ is also $\sigma$-subnormal in $G$. Hence Condition (iii) holds for $DA$, so $A$ is $H_\sigma$-subnormally embedded in $DA$ by induction. But then $A$ is $H_\sigma$-subnormally embedded in $G$ by Lemma 2.1(7).

Finally, suppose that $DA = G$. Then, since $G$ is $\sigma$-soluble, for some $x$ we have $M \leq A^x$ by Lemma 2.6. Hence $A^x$ is a $\sigma$-Hall subgroup of $G$ by Claim (*), so $A^x$ is an $H_\sigma$-subnormally embedded subgroup of $G$. But then $A$ is an $H_\sigma$-subnormally embedded subgroup of $G$. Therefore the implication (iii) $\Rightarrow$ (i) is true.

The theorem is proved.

Proof of Theorem 1.9. (i) $\Rightarrow$ (ii) This follows from Lemma 2.2(4) and Theorems 1.3 and 1.4.

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) Let $A$ be any subgroup of $G$. Then $DA$ is $\sigma$-permutable in $G$ by Lemma 2.2(3) since $G$ is $\sigma$-soluble. On the other hand, since $|\sigma(D)| = |\pi(D)|$ and $D$ is a cyclic $\sigma$-Hall subgroup of $G$ of square-free order, $A$ is a $\sigma$-Hall subgroup of $DA$. Hence $A$ is $H_\sigma$-permutably embedded in $G$. Therefore the implication (iii) $\Rightarrow$ (i) is true.

The theorem is proved.

Proof of Theorem 1.7. (i) $\Rightarrow$ (ii) In view of Theorem 1.9, it is enough to show that if $D \leq L \leq G$ and $L$ is a $\sigma$-Hall subgroup of some normal subgroup $V$ of $G$, then $L$ is normal in $G$. But since $G/D$ is $\sigma$-nilpotent, $L/D$ is $\sigma$-subnormal in $G/D$, so $L$ is $\sigma$-subnormal in $G$ by Lemma 2.1(6). Hence $L$ is $\sigma$-subnormal in $V$ by Lemma 2.1(1). But then $L$ is a normal in $V$ by Lemma 2.1(4) and so $L$ is a characteristic subgroup of $V$. It follows that $L$ is normal in $G$.

(ii) $\Rightarrow$ (iii) This implication is evident.

(iii) $\Rightarrow$ (i) See the proof of the implication (iii) $\Rightarrow$ (i) in Theorem 1.9.

The theorem is proved.
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