Abstract. We prove the existence of parabolic arcs with prescribed asymptotic direction for the equation

\[ \ddot{x} = -\frac{x}{|x|^3} + \nabla W(t, x), \quad x \in \mathbb{R}^d, \]

where \( d \geq 2 \) and \( W \) is a (possibly time-dependent) lower order term, for \( |x| \to +\infty \), with respect to the Kepler potential \( 1/|x| \). The result applies to the elliptic restricted three-body problem. The proof relies on a perturbative argument, after an appropriate formulation of the problem in a suitable functional space.

Mathematics Subject Classification (2010). 37J45, 70B05, 70F15.

Keywords. Parabolic solution, Kepler problem, restricted three-body problem.

1. Introduction and statement of the main result

Many significant equations of Celestial Mechanics can be written as

\[ \ddot{x} = -\frac{x}{|x|^3} + \nabla W(t, x), \quad x \in \mathbb{R}^d, \tag{1.1} \]

where \( d \in \{2, 3\} \) and \( W \) is a lower order term, for \( |x| \to +\infty \), with respect to the Kepler potential \( U_0(x) = 1/|x| \). A typical example in this direction is represented by the well-known elliptic restricted three-body problem, modelling the motion of a massless particle \( x \in \mathbb{R}^d \) under the Newtonian attraction of two heavy bodies (the primaries), moving along closed orbits according to Kepler’s laws (see, for instance, [19 Section 2.10]). As shown later in Section 5 the corresponding
equation can be written in the form $[1.1]$, where $W$ depends periodically on time and $W = O(|x|^{-2})$ for $|x| \to +\infty$.

In this paper, we deal with solutions to $[1.1]$ approaching infinity with zero velocity, namely

$$\lim_{t \to +\infty} |x(t)| = +\infty \quad \text{and} \quad \lim_{t \to +\infty} \dot{x}(t) = 0.$$ 

Throughout the paper, we will call such solutions parabolic, according to the terminology used by Chazy in its pioneering paper [8] investigating all the possible final states for a three-body problem as time goes to infinity. Incidentally, let us notice that in the elementary case $W = 0$ such solutions indeed correspond to Keplerian parabolas. We also observe that, when $W$ does not depend explicitly on time, that is $W(t, x) = W(x)$, parabolic solutions to $[1.1]$ could be characterized as solutions going to infinity and having zero-energy, namely

$$\frac{1}{2} |\dot{x}(t)|^2 - U_0(x(t)) - W(x(t)) = 0.$$ 

Under reasonable assumptions on $W$, parabolic solutions to $[1.1]$ are known to exist. Typically, they can be constructed by Dynamical Systems techniques: indeed, via a McGehee-type change of variables, “infinity” (with zero velocity) can be regarded as a fixed point of a suitable Poincaré map associated with $[1.1]$, so that tools from the topological theory of invariant manifolds apply (see [1, 16]). It is however unclear, in principle, how the manifold of parabolic solutions projects on the configuration space.

Our main result provides a contribution in this direction, by showing the existence of parabolic solutions starting from a given point $x_0 \in \mathbb{R}^d$ and having prescribed asymptotic direction (for $t \to +\infty$). Precisely, defining, for $\xi^+ \in \mathbb{S}^d$, $R > 1$ and $\eta \in ]0, 1[$, the set

$$T(\xi^+, R, \eta) = \left\{ x \in \mathbb{R}^d : |x| > R \quad \text{and} \quad \left\langle \frac{x}{|x|}, \xi^+ \right\rangle > \eta \right\},$$

we will prove the following theorem (see the end of the Introduction for some clarification about the notation used).

**Theorem 1.1.** Let $d \in \mathbb{N}$, with $d \geq 2$, and let $W \in C^2([0, +\infty[ \times (\mathbb{R}^d \setminus B_1))$ satisfy

$$|W(t, x)| + |\partial_t W(t, x)| + |x| |\nabla W(t, x)| + |x|^2 |\nabla^2 W(t, x)| = O\left(\frac{1}{|x|^2}\right),$$

for $|x| \to +\infty$, uniformly in $t$. Then, for every $\xi^+ \in \mathbb{S}^d$, there exist $R > 1$ and $\eta \in ]0, 1[$ such that, for every $x_0 \in T(\xi^+, R, \eta)$, there exists a parabolic solution $x : [0, +\infty[ \to \mathbb{R}^d$ of equation $[1.1]$, satisfying $x(0) = x_0$ and

$$\lim_{t \to +\infty} \frac{x(t)}{|x(t)|} = \xi^+.$$ 

Moreover, $|x(t)| \sim \alpha t^{\frac{2}{3}}$ for $t \to +\infty$, where $\alpha = \sqrt[3]{\frac{9}{2}}$.

It is worth noticing that, even if the applications to Celestial Mechanics typically require $d \in \{2, 3\}$, the statement actually holds true for every integer $d \geq 2$. We also emphasize that the above parabolic solutions are locally minimal.
in the sense of Remark 4.2, the variational characterization makes these orbits suitable for the analysis of their Maslov indices as in [2].

Let us remark that parabolic solutions for various equations of Celestial Mechanics (e.g., $N$-body problem, $N$-centre problem, Kepler anisotropic problem) have been investigated in many papers, with different techniques and from different point of views [2] [3] [4] [5] [6] [10] [15] [17] [18] [20] [21]. Generally speaking, their interest mainly comes from the fact that, in spite of the natural intrinsic instability, parabolic orbits can be used as carriers from different regions of the configuration space and, eventually, as building blocks in the construction of solutions with chaotic behavior (see [11] [22] and the references therein). Moreover, parabolic solutions are known to provide precious information on the behavior of general solutions near collisions [9]; finally, they play a role in the applications of weak KAM theory to Celestial Mechanics and can be used to construct weak KAM solutions of the associated Hamilton–Jacobi equation [14].

In the literature about parabolic solutions, the result which appears to be more closely related to Theorem 1.1 is the one by Maderna and Venturelli [15]. In the context of the classical $N$-body problem, they were able to construct a parabolic solution (meaning that the velocity of each body goes to zero as $t \to +\infty$) starting from an arbitrary configuration and approaching (at infinity) any prescribed minimizing normalized central configuration. Such a solution, having infinite action, was obtained as the limit of solutions of a sequence of approximating two-point boundary value problems. The existence of the approximate solutions was ensured by the direct method of the Calculus of Variations (together with Marchal’s lemma). Delicate action level estimates, strongly relying on the homogeneity of the $N$-body problem, were then used to show the convergence to a limit solution and its parabolicity.

Compared to [15], the investigation of equation (1.1) seems to require different arguments. Indeed, besides the failure of the conservation of the angular momentum, due to the fact that equation (1.1) is allowed to be time-dependent, also the Hamiltonian ceases to be a first integral. This gives rise to new phenomena: for instance, while in the autonomous case parabolic solutions to (1.1) exiting a large ball are forced to go to infinity (as a consequence of the Lagrange–Jacobi inequality, see [6 Lemma 2.1]), this dramatically fails to be true when $W$ depends explicitly on time. Indeed, solutions with oscillatory behavior, shadowing parabolic orbits, may exist [11] [12] [13]. For these reasons, it seems to be a hard task to construct parabolic solutions via an approximation argument similar to the one in [15] (however, our result can be seen as an extension of that in [15]).

In order to solve these difficulties, and finally prove Theorem 1.1, we introduce a completely new strategy, which seems to be general enough to be used also in different contexts. Roughly speaking, the crucial idea is to look for solutions to (1.1) having the form

$$x(t) = x_0(t + \lambda) + u_\lambda(t),$$

where $\lambda > 0$ is a (large) parameter, $x_0$ is the homothetic parabolic solution with direction $\xi^+$ of the Kepler problem (that is, $x_0(t) = \alpha t^2 \xi^+$) and $u_\lambda$ is a perturbation term, lying in a suitable functional space $X$. Quite unexpectedly, a
proper choice of $X$ can be made (essentially, $X$ is a space of functions with $L^2$-weak derivative, see Section 2 for more details), in order for the problem in the new unknown $u_\lambda$ to have a standard variational structure (despite the equation being considered on a non-compact time interval) and for $x(t) = x_0(t + \lambda) + u_\lambda(t)$ to satisfy the desired asymptotic properties (namely, $\dot{x}(t) \to 0$ and $x(t)/|x(t)| \to \xi^+$) whenever $u_\lambda \in X$ is a solution. Even more, the problem for $u_\lambda$ can be solved by elementary perturbation arguments: essentially, taking $\lambda \to +\infty$ equation (1.1) reduces to the Kepler problem and, due to the validity of a Hardy-type inequality in $X$, the implicit function theorem can be easily applied. Of course, a major drawback of this procedure lies in its purely perturbative nature: as a consequence, in Theorem 1.1 we need to assume that the initial condition $x_0$ lies in the set $T(\xi^+, R, \eta)$ defined in (1.2), for suitable $R$ and $\eta$.

The plan of the paper is the following. In Section 2 we give the definition of the functional space $X = D^{1,2}_0(1, +\infty)$ and we present some of its properties. In Section 3 we describe in more detail the general strategy (just very briefly sketched in the above discussion), so as to provide an outline of the proof of Theorem 1.1; the complete proof is then given in Section 4. Finally, in Section 5 we discuss some applications of the main result, including the one for the restricted three-body problem.

We end this introductory part by presenting some symbols and some notation used in the present paper. We denote by $|\cdot|$ and by $\langle \cdot, \cdot \rangle$ the Euclidean norm and the standard scalar product in $\mathbb{R}^d$, respectively. The symbol $|\cdot|$ is used also for the operator norm of a matrix $A \in \mathbb{R}^{d \times d}$, that is, $|A| = \sup_{|x| \leq 1} |Ax|$. By $B_r$ (with $r > 0$) we mean the open ball in $\mathbb{R}^d$ centered at 0 with radius $r$, i.e., $B_r = \{x \in \mathbb{R}^d : |x| < r\}$ and, as usual, we set $S^d = \partial B_1$. Finally, for a function $U = U(t, x)$ of time $t$ and position $x$, we denote by $\nabla U$ and by $\nabla^2 U$ the gradient and the Hessian matrix with respect to the variable $x$, respectively.

2. The space $D^{1,2}_0(1, +\infty)$

In our arguments, a crucial role is played by the choice of the functional space. Indeed, after some change of variables we will be led to work in the space $D^{1,2}_0(1, +\infty)$, defined as the subset of continuous functions on $[1, +\infty[$ which vanish at $t = 1$ and can be written as primitives of functions in $L^2(1, +\infty)$. Precisely,

$$D^{1,2}_0(1, +\infty) = \left\{ \varphi \in C([1, +\infty[) : \varphi(t) = \int_1^t v(s) \, ds \text{ for some } v \in L^2(1, +\infty) \right\}.$$ 

Readers familiar with the theory of Sobolev spaces (see, for instance, [7, Chapter 8]) will immediately observe that any $\varphi \in D^{1,2}_0(1, +\infty)$ is differentiable in the sense of distributions, with $\dot{\varphi} = v$; conversely, locally integrable functions with distributional derivatives in $L^2(1, +\infty)$ belong to $D^{1,2}_0(1, +\infty)$ as long as $\varphi(1) = 0$ (recall that continuity up to $t = 1$ is automatically ensured whenever
the weak derivative is in \( L^p(1, +\infty) \) for some \( p \in [1, +\infty] \)). Let us also observe that, by the Cauchy–Schwartz inequality, the pointwise estimate

\[
|\varphi(t)| \leq \|\dot{\varphi}\|_{L^2} \sqrt{t - 1} \leq \|\dot{\varphi}\|_{L^2} \sqrt{t}, \quad \text{for every } t \geq 1,
\]  
holds true, implying that any \( \varphi \in D^{1,2}_0(1, +\infty) \) grows at infinity at most as \( \sqrt{t} \).

We will endow \( D^{1,2}_0(1, +\infty) \) with the norm

\[
\|\varphi\| = \left( \int_1^{+\infty} |\dot{\varphi}(t)|^2 \, dt \right)^{\frac{1}{2}}.
\]

As a consequence, \( (2.1) \) writes as

\[
|\varphi(t)| \leq \|\varphi\| \sqrt{t}, \quad \text{for every } t \geq 1,
\]

implying that \( \varphi_n(t) \to \varphi(t) \) uniformly in \([1, M]\) for any \( M > 1 \), whenever \( \varphi_n \to \varphi \) in \( D^{1,2}_0(1, +\infty) \).

**Proposition 2.1.** The space \( D^{1,2}_0(1, +\infty) \) is a Hilbert space containing the set \( \mathcal{C}_c^\infty(]1, +\infty[) \) as a dense subspace.

**Proof.** We first observe that the norm \( (2.2) \) is induced by the scalar product

\[
(\varphi_1, \varphi_2) \mapsto \int_1^{+\infty} \langle \dot{\varphi}_1(t), \dot{\varphi}_2(t) \rangle \, dt.
\]

Hence, to prove that \( D^{1,2}_0 \) is a Hilbert space (for the rest of the proof, for briefness we omit the domain \((1, +\infty)\) since no ambiguity is possible) we just need to show that Cauchy sequences are convergent. To this end, let us consider a Cauchy sequence \( (\varphi_n)_n \subseteq D^{1,2}_0 \). By definition of the norm, it follows that \( (\dot{\varphi}_n)_n \) is a Cauchy sequence in \( L^2 \); therefore, there exists \( v \in L^2 \) such that \( \dot{\varphi}_n \to v \) in \( L^2 \).

Setting \( \varphi(t) = \int_1^t v(s) \, ds \), we clearly have \( \varphi \in D^{1,2}_0 \) and \( \varphi_n \to \varphi \) in \( D^{1,2}_0 \), thus proving the completeness of \( D^{1,2}_0 \).

We now show the second part of the statement. Given \( \varphi \in D^{1,2}_0 \), let us take \( (v_n)_n \subseteq \mathcal{C}_c^\infty(]1, +\infty[) \) such that \( v_n \to \varphi \) in \( L^2 \) and define \( J_n = \text{supp}(v_n) \). Setting \( T_n = \sup J_n \) and taking \( \gamma \in C^\infty([0, +\infty[) \) such that \( \gamma(t) = 1 \) for \( t \in [0, 1] \) and \( \gamma(t) = 0 \) for \( t \geq 2 \), let us define, for \( t \geq 1 \),

\[
\gamma_n(t) = \gamma\left(\frac{t - 1}{nT_n}\right)
\]

and

\[
\varphi_n(t) = \gamma_n(t) \int_1^t v_n(s) \, ds.
\]

Clearly, \( \varphi_n \in \mathcal{C}_c^\infty(]1, +\infty[) \); we thus need to show that \( \dot{\varphi}_n \to \dot{\varphi} \) in \( L^2 \). To this end, we first observe that

\[
\dot{\varphi}_n(t) = \gamma_n(t)v_n(t) + \frac{1}{nT_n}\gamma\left(\frac{t - 1}{nT_n}\right) \int_1^t v_n(s) \, ds.
\]  

(2.4)
The first term on the right-hand side converges to $\phi$ in $L^2$, since
\[
\|\gamma_n v_n - \phi\|_{L^2} \leq \|\gamma_n (v_n - \phi)\|_{L^2} + \| (\gamma_n - 1) \phi\|_{L^2} \\
\leq \|\gamma\|_{L^\infty} \|v_n - \phi\|_{L^2} + \left( \int_1^{+\infty} |(\gamma_n(t) - 1) \phi(t)|^2 \, dt \right)^{\frac{1}{2}}
\]
and the integral goes to zero by the dominated convergence theorem. On the other hand, the second term on the right-hand side of (2.4) goes to zero in $L^2$; indeed, since by the Cauchy–Schwartz inequality
\[
\text{the integral goes to zero by the dominated convergence theorem.}
\]

Let us first assume that $\phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$, it turns out that $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ is defined as the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to the norm $(\int |\nabla \phi|^p)^{\frac{1}{p}}$ (by the Sobolev–Gagliardo–Nirenberg inequality, it turns out that $\mathcal{D}_0^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$, with $p^*$ the Sobolev critical exponent). Proposition 2.1 thus shows that an analogous characterization can be given for $\mathcal{D}_0^{1,2}(1, +\infty)$ (here, the subscript 0 has been added to mean that functions vanish for $t = 1$); however, a more direct and elementary construction seems to be preferable in the 1-dimensional setting.

As a consequence of Proposition 2.1 we can finally state and prove a further inequality, showing that $\mathcal{D}_0^{1,2}(1, +\infty)$ is continuously embedded in a weighted $L^2$-space. Precisely, we have the following Hardy-type inequality.

**Proposition 2.2.** For every $\phi \in \mathcal{D}_0^{1,2}(1, +\infty)$, it holds that
\[
\int_1^{+\infty} \frac{|\phi(t)|^2}{t^2} \, dt \leq 4 \int_1^{+\infty} |\dot{\phi}(t)|^2 \, dt. \tag{2.5}
\]

**Proof.** Let us first assume that $\phi \in \mathcal{C}_c^\infty([1, +\infty])$. In this case, integrating by parts and using the Cauchy–Schwartz inequality, we directly obtain
\[
\int_1^{+\infty} \frac{|\phi(t)|^2}{t^2} \, dt = -\int_1^{+\infty} \frac{d}{dt} \left( \frac{1}{t} \right) |\phi(t)|^2 \, dt = 2 \int_1^{+\infty} \frac{(\phi(t), \dot{\phi}(t))}{t} \, dt \\
\leq \int_1^{+\infty} 2 |\phi(t)| |\dot{\phi}(t)| \, dt \leq 2 \left( \int_1^{+\infty} \frac{|\phi(t)|^2}{t^2} \, dt \right)^{\frac{1}{2}} \left( \int_1^{+\infty} |\dot{\phi}(t)|^2 \, dt \right)^{\frac{1}{2}},
\]
thus proving (2.5). The general case follows from the density of $\mathcal{C}_c^\infty([1, +\infty])$ in $\mathcal{D}_0^{1,2}(1, +\infty)$, proved in Proposition 2.1. \qed
3. The general strategy

In this section we give an outline of the proof of Theorem 1.1.

3.1. Changing variables

The first step in our proof consists in regarding equation (1.1) as a perturbation at infinity of the Kepler problem, by the introduction of a suitable parameter. This can be done in a standard way using a well-known scale invariance of the Kepler problem; precisely, let us set, for every \( \varepsilon > 0 \), \( t \geq 1 \) and \( y \in \mathbb{R}^d \setminus B_\varepsilon \),

\[
W_\varepsilon(t, y) = \frac{1}{\varepsilon} W \left( \frac{t - 1/\varepsilon^{3/2}}{\varepsilon^{3/2}}, \frac{y}{\varepsilon} \right). \tag{3.1}
\]

It is easy to check that, for all \( x_0 \in \mathbb{R}^d \setminus B_1 \), a function \( x : [0, +\infty[ \to \mathbb{R}^d \) is a solution of (1.1) with \( x(0) = x_0 \) if and only if the function \( y : [1, +\infty[ \to \mathbb{R}^d \) defined as

\[
y(t) = \varepsilon x \left( \frac{t - 1/\varepsilon^{3/2}}{\varepsilon^{3/2}} \right)
\]

is a solution of the equation

\[
\ddot{y} = -\frac{y}{|y|^3} + \nabla W_\varepsilon(t, y), \tag{3.2}
\]

with \( y(1) = \varepsilon x_0 \). Setting, for \( t \geq 1 \) and \( y \in \mathbb{R}^d \setminus B_\varepsilon \),

\[
U_\varepsilon(t, y) = \frac{1}{|y|} + W_\varepsilon(t, y), \tag{3.3}
\]

equation (3.2) reads as

\[
\ddot{y} = \nabla U_\varepsilon(t, y). \tag{3.4}
\]

By the assumptions on \( W \), it is readily checked that \( U_\varepsilon \) can be smoothly extended, when \( \varepsilon \to 0^+ \), to the Kepler potential

\[
U_0(t, y) = U_0(y) = \frac{1}{|y|} \tag{3.5}
\]

(see the beginning of Section 4.1 for more details). In this way, we can consider equation (3.4) even when \( \varepsilon = 0 \), provided that we look for solutions \( y \) which are bounded away from the origin.

We actually look for solutions to (3.4) with a special form. More precisely, denoting by \( y_0 \) is the homothetic parabolic solution of the Kepler problem having direction \( \xi^+ \), that is

\[
y_0(t) = \alpha t^{2/3} \xi^+, \quad \text{where } \alpha = \sqrt[3]{\frac{9}{2}}, \tag{3.6}
\]

and fixing a function \( w \in C^2([1, +\infty[) \) satisfying

\[
w(1) = 1 \quad \text{and} \quad w(t) = 0, \quad \text{for every } t \geq 2, \tag{3.7}
\]

we look for solutions to (3.4) of the form

\[
y(t) = y_0(t) + \sigma w(t) + \varphi(t), \tag{3.8}
\]
8 A. Boscaggin, W. Dambrosio, G. Feltrin and S. Terracini

where \( \sigma \in \mathbb{R}^d \) is a small parameter and \( \varphi \) is the new unknown function. Setting

\[
y_\sigma(t) = y_0(t) + \sigma w(t),
\]

the equation for \( \varphi \) thus becomes the two-parameter equation

\[
\ddot{\varphi} = \nabla U_\varepsilon(t, y_\sigma(t) + \varphi) - \ddot{y}_\sigma(t),
\]

which we will write as

\[
\ddot{\varphi} = \nabla K_{\varepsilon, \sigma}(t, \varphi) - h_{\varepsilon, \sigma}(t), \tag{3.10}
\]

where

\[
h_{\varepsilon, \sigma}(t) = \ddot{y}_\sigma(t) - \nabla U_\varepsilon(t, y_\sigma(t)) \tag{3.11}
\]

and

\[
K_{\varepsilon, \sigma}(t, \varphi) = U_\varepsilon(t, y_\sigma(t) + \varphi) - U_\varepsilon(t, y_\sigma(t)) - \langle \nabla U_\varepsilon(t, y_\sigma(t)), \varphi \rangle. \tag{3.12}
\]

The crucial point in our approach is that solutions to (3.10) will be required to belong to the functional space \( D_{1,2}^{1,0}(1, +\infty) \). Notice that \( \varphi(1) = 0 \) implies that \( y(1) = \alpha \xi^+ + \sigma \); being

\[
x(t) = \frac{1}{\varepsilon} y_3(t^2 + 1), \tag{3.13}
\]

we thus find

\[
x(0) = \frac{1}{\varepsilon} (\alpha \xi^+ + \sigma).
\]

As proved in Section 4.3 by varying \( \varepsilon \) and \( \sigma \), a set of the form \( \mathcal{T}(\xi^+, R, \eta) \) is thus covered. Moreover, and more remarkably, due to the fact that \( \varphi \in D_{1,2}^{1,0}(1, +\infty) \) is a lower order term, for \( t \to +\infty \), with respect to \( y_0 \), we will prove (see again Section 4.3) that the behavior of \( x \) at infinity is similar to the one of \( y_0 \); in particular, \( x \) is a parabolic solution of \((1.1)\) with prescribed asymptotic direction \( \xi^+ \).

### 3.2. A perturbation argument

In order to find solutions \( \varphi \in D_{1,2}^{1,0}(1, +\infty) \) to equation (3.10), we use a perturbative approach: more precisely, being \( \varphi \equiv 0 \) a solution to (3.10) for \( \varepsilon = 0 \) and \( \sigma = 0 \), solutions for \( \varepsilon \) and \( \sigma \) small enough will be found by an application of the implicit function theorem.

Taking advantage of the variational structure of (3.10), we consider the action functional

\[
A_{\varepsilon, \sigma}(\varphi) = \int_1^{+\infty} \left( \frac{1}{2} |\dot{\varphi}(t)|^2 + K_{\varepsilon, \sigma}(t, \varphi(t)) - \langle h_{\varepsilon, \sigma}(t), \varphi(t) \rangle \right) dt. \tag{3.14}
\]

It will be proved (see Proposition 4.1) that, when \( \varepsilon \) and \( \sigma \) are small enough, such a functional is well-defined and of class \( C^2 \) on a suitable neighborhood of the origin in \( D_{1,2}^{1,0}(1, +\infty) \) and that the three-variable function

\[
F: (\varepsilon, \sigma, \varphi) \mapsto dA_{\varepsilon, \sigma}(\varphi) \in (D_{1,2}^{1,0}(1, +\infty))^*.
\]

is continuous and has a continuous differential with respect to \( \varphi \) (here, the symbol \( (D_{1,2}^{1,0}(1, +\infty))^* \) denotes the dual space of \( D_{1,2}^{1,0}(1, +\infty) \)). By the implicit function theorem, the equation

\[
dA_{\varepsilon, \sigma}(\varphi) = 0
\]
can thus be solved with respect to $\varphi$, for $\varepsilon$ and $\sigma$ small enough, whenever

$$D_\varphi F(0, 0, 0) = d^2 A_{0, 0}(0)$$

is invertible. We will see that this is actually the case (see (4.29) and Proposition 4.2), thus providing critical points $\varphi = \varphi(\varepsilon, \sigma)$ of $A_{\varepsilon, \sigma}(\varphi)$ and, eventually, solutions to equation (3.10) belonging to the space $D_{1,2}^{1,2}(1, +\infty)$.

4. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. More precisely, in Section 4.1 we establish some preliminary estimates, while in Section 4.2 we illustrate the perturbation argument. The proof will be finally described in Section 4.3.

Remark 4.1. Incidentally, let us observe that the choice of $\mathbb{R}^d \setminus B_1$ for the domain of $W(t, \cdot)$ is purely conventional and the conclusion of Theorem 1.1 remains valid for a potential $W \in C^2([0, +\infty[ \times (\mathbb{R}^d \setminus B_2))$, with $\Xi > 0$. Moreover, by a careful inspection of the proof one could check that it is sufficient to assume that $W$ has a first derivative with respect to time. Finally, we remark that the same conclusion of Theorem 1.1 can be proved for the more general equation

$$\ddot{x} = -M \frac{x}{|x|^3} + \nabla W(t, x),$$

where $M > 0$. In this case, the homothetic parabolic solution $y_0$ defined in (3.6) should be replaced by $y_0(t) = \alpha Mt^{\frac{2}{3}} \xi^+$; accordingly, $|x(t)| \sim \alpha Mt^{\frac{2}{3}}$ for $t \to +\infty$.

4.1. Preliminary estimates

Before starting with the proof, we first observe that, from assumption (1.3), we can fix a constant $C > 0$ such that

$$|W(t, x)| \leq \frac{C}{|x|^2}, \quad \text{for every } (t, x) \in [0, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

$$|\nabla W(t, x)| \leq \frac{C}{|x|^3}, \quad \text{for every } (t, x) \in [0, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

$$|\nabla^2 W(t, x)| \leq \frac{C}{|x|^4}, \quad \text{for every } (t, x) \in [0, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

$$|\partial_t W(t, x)| \leq \frac{C}{|x|^2}, \quad \text{for every } (t, x) \in [0, +\infty[ \times (\mathbb{R}^d \setminus B_1).$$

As a consequence, for every $\varepsilon \in ]0, 1[$, the potential $W_\varepsilon$ defined in (3.1) satisfies

$$|W_\varepsilon(t, y)| \leq \frac{C \varepsilon}{|y|^2}, \quad \text{for every } (t, y) \in [1, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

$$|\nabla W_\varepsilon(t, y)| \leq \frac{C \varepsilon}{|y|^3}, \quad \text{for every } (t, y) \in [1, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

$$|\nabla^2 W_\varepsilon(t, y)| \leq \frac{C \varepsilon}{|y|^4}, \quad \text{for every } (t, y) \in [1, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

$$|\partial_t W_\varepsilon(t, y)| \leq \frac{C}{\sqrt{\varepsilon} |y|^2}, \quad \text{for every } (t, y) \in [1, +\infty[ \times (\mathbb{R}^d \setminus B_1).$$
Furthermore, recalling the definitions of $U_\varepsilon$ in (3.3) and of $U_0$ in (3.5), we find $C' > C$ such that, for every $\varepsilon \in [0, 1]$, 

$$|U_\varepsilon(t, y)| \leq \frac{C'}{|y|}, \quad \text{for every } (t, y) \in [1, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

(4.5)

$$|\nabla U_\varepsilon(t, y)| \leq \frac{C'}{|y|^2}, \quad \text{for every } (t, y) \in [1, +\infty[ \times (\mathbb{R}^d \setminus B_1),$$

(4.6)

$$|\nabla^2 U_\varepsilon(t, y)| \leq \frac{C'}{|y|^3}, \quad \text{for every } (t, y) \in [1, +\infty[ \times (\mathbb{R}^d \setminus B_1).$$

(4.7)

Let us notice that from (4.1), (4.2) and (4.3) it easily follows that

$$U_\varepsilon(t, y) \to U_0(y), \quad \text{as } \varepsilon \to 0^+, \quad (4.8)$$

where the above convergence is meant in the sense that $U_\varepsilon(t, y) \to U_0(y)$ and $\nabla^i U_\varepsilon(t, y) \to \nabla^i U_0(y)$, for $i = 1, 2$, uniformly in $[1, +\infty[ \times (\mathbb{R}^d \setminus B_1)$.

We are now ready to start with the proof. As a first step, we are going to fix some constants; precisely let us set

$$r = \frac{\alpha - 1}{4} \max_{t \in [1, 2]} |w(t)| \quad \text{and} \quad \rho = \frac{\alpha - 1}{4},$$

where the function $w$ is as in (3.7). Accordingly, we define the sets

$$B_r = \{ \sigma \in \mathbb{R}^d : |\sigma| < r \} \quad \text{and} \quad \Omega_\rho = \{ \varphi \in \mathcal{D}_0^{1,2}(1, +\infty) : \|\varphi\| < \rho \}.$$  

Then, recalling the definition of $y_\sigma$ given in (3.9), the following preliminary estimate holds true.

**Lemma 4.1.** For every $\sigma \in B_r$ and for every $\varphi \in \Omega_\rho$, it holds that

$$|y_\sigma(t) + \varphi(t)| \geq \frac{\alpha + 1}{2} t^{\frac{\beta}{2}}, \quad \text{for every } t \geq 1. \quad (4.9)$$

**Proof.** Using (2.3) we have

$$|y_\sigma(t) + \varphi(t)| \geq |y_0(t)| - |\sigma w(t)| - |\varphi(t)|$$

$$\geq t^{\frac{\beta}{2}} \left( \alpha - |\sigma| \max_{t \in [1, 2]} |w(t)| t^{-\frac{\beta}{2}} - \|\varphi\| t^{-\frac{1}{6}} \right)$$

$$\geq t^{\frac{\beta}{2}} \left( \alpha - r \max_{t \in [1, 2]} |w(t)| - \rho \right) = \frac{\alpha + 1}{2} t^{\frac{\beta}{2}},$$

for every $t \geq 1$. \qed

Notice in particular that, for every $\sigma \in B_r$ and for every $\varphi \in \Omega_\rho$, it holds

$$|y_\sigma(t) + \varphi(t)| > 1, \quad \text{for every } t \geq 1,$$

so that the functions $h_{\varepsilon, \sigma}$ and $K_{\varepsilon, \sigma}$ (see (3.11) and (3.12)) are well-defined. In particular, due to (4.8), the definitions are meaningful also when $\varepsilon = 0$.

Our next two lemmas give some estimates for $h_{\varepsilon, \sigma}$ and $K_{\varepsilon, \sigma}$. 
Lemma 4.2. There exists $C_h > 0$ such that, for every $\varepsilon \in [0,1]$ and $\sigma \in B_r$,
\[ |h_{\varepsilon,\sigma}(t)| \leq \frac{C_h}{t^2}, \quad \text{for every } t \geq 1. \] (4.10)

Proof. In order to prove (4.10), we first suppose that $t \in [1,2]$; notice that in this case we just need to show that $\max_{t \in [1,2]} |h_{\varepsilon,\sigma}(t)|$ is bounded, independently of $\varepsilon$ and $\sigma$. This is easily checked: indeed, on one hand by construction $y_\sigma \to y_0$ in $C^2([1,2])$ as $\sigma \to 0$; on the other hand, since $|y_\sigma(t)| > 1$ we have that $\nabla U_\varepsilon(t,y_\sigma(t))$ is bounded, uniformly in $\varepsilon$ and $\sigma$, by (4.6).

We now suppose that $t \geq 2$; in this case, recalling that $y_\sigma(t) = y_0(t)$, we find
\[ h_{\varepsilon,\sigma}(t) = \dot{y}_0(t) - \nabla U_\varepsilon(t,y_0(t)) = -\nabla W_\varepsilon(t,y_0(t)), \]
for every $t \geq 2$, and the conclusion follows by (4.2). \qed

Lemma 4.3. There exists $C_K > 0$ such that, for every $\varepsilon \in [0,1]$, $\sigma \in B_r$ and $\varphi \in \Omega_\rho$, the following inequalities hold true:
\[ |K_{\varepsilon,\sigma}(t,\varphi(t))| \leq \frac{C_K|\varphi(t)|^2}{t^2}, \quad \text{for every } t \geq 1, \] (4.11)
\[ |\nabla K_{\varepsilon,\sigma}(t,\varphi(t))| \leq \frac{C_K|\varphi(t)|}{t^2}, \quad \text{for every } t \geq 1, \] (4.12)
\[ |\nabla^2 K_{\varepsilon,\sigma}(t,\varphi(t))| \leq \frac{C_K}{t^2}, \quad \text{for every } t \geq 1. \] (4.13)

Proof. We have
\[ \nabla K_{\varepsilon,\sigma}(t,\varphi(t)) = \nabla U_\varepsilon(t,y_\sigma(t) + \varphi(t)) - \nabla U_\varepsilon(t,y_\sigma(t)) \]
and
\[ \nabla^2 K_{\varepsilon,\sigma}(t,\varphi(t)) = \nabla^2 U_\varepsilon(t,y_\sigma(t) + \varphi(t)). \]
for every $t \geq 1$. Moreover, from (4.9), for every $\sigma \in B_r$ and $\varphi \in \Omega_\rho$, we immediately find that
\[ \frac{1}{|y_\sigma(t) + \varphi(t)|^3} \leq \frac{8}{(\alpha + 1)^3 t^2}, \quad \text{for every } t \geq 1. \] (4.14)

Combining (4.7) with (4.14), (4.13) plainly follows. To prove (4.12), we notice that $\nabla K_{\varepsilon,\sigma}(t,0) \equiv 0$ so as to write
\[ \nabla K_{\varepsilon,\sigma}(t,\varphi(t)) = \int_0^1 \frac{d}{ds} \nabla K_{\varepsilon,\sigma}(t,s\varphi(t)) \, ds = \int_0^1 \nabla^2 K_{\varepsilon,\sigma}(t,s\varphi(t))\varphi(t) \, ds, \]
for every $t \geq 1$. Hence, (4.12) directly follows from (4.13) (with $s\varphi$ in place of $\varphi$). In an analogous way (using $K_{\varepsilon,\sigma}(t,0) \equiv 0$) we obtain (4.11) from (4.12). \qed

4.2. The implicit function argument

In this section, we provide the details of the implicit function argument. To this end, we start by recalling the definition of the action functional $A_{\varepsilon,\sigma}$ given in (3.14); notice that we now assume $\sigma \in B_r$ and $\varphi \in \Omega_\rho$ so that all the integrands are well-defined.
Proposition 4.1. For every $\varepsilon \in [0,1]$ and $\sigma \in B_r$, the action functional $A_{\varepsilon, \sigma}$ is of class $C^2$ on the open set $\Omega_{\rho} \subseteq D_0^{1,2}(1, +\infty)$, with
\[
d A_{\varepsilon, \sigma}(\varphi)[\psi] = \int_1^{+\infty} \left( \langle \dot{\varphi}(t), \dot{\psi}(t) \rangle + \langle \nabla K_{\varepsilon, \sigma}(t, \varphi(t)), \psi(t) \rangle + \langle h_{\varepsilon, \sigma}(t), \psi(t) \rangle \right) \, dt
\]
and
\[
d^2 A_{\varepsilon, \sigma}(\varphi)[\psi, \zeta] = \int_1^{+\infty} \left( \langle \dot{\varphi}(t), \dot{\zeta}(t) \rangle + \langle \nabla^2 K_{\varepsilon, \sigma}(t, \varphi(t)) \psi(t), \zeta(t) \rangle \right) \, dt,
\]
for every $\psi, \zeta \in D_0^{1,2}(1, +\infty)$. Moreover, the three-variable function $F: [0,1[ \times B_r \times \Omega_{\rho} \to (D_0^{1,2}(1, +\infty))^*$, $F(\varepsilon, \sigma, \varphi) = d A_{\varepsilon, \sigma}(\varphi)$, is continuous and its differential $D_{\varphi} F$ is continuous.

Proof. We split the action functional as
\[
A_{\varepsilon, \sigma}(\varphi) = A^1(\varphi) + A^2_{\varepsilon, \sigma}(\varphi) + A^3_{\varepsilon, \sigma}(\varphi),
\]
(4.15)
where $A^1(\varphi) = \frac{1}{2} \| \varphi \|^2$,
\[
A^2_{\varepsilon, \sigma}(\varphi) = \int_1^{+\infty} K_{\varepsilon, \sigma}(t, \varphi(t)) \, dt,
\]
$A^3_{\varepsilon, \sigma}(\varphi) = \int_1^{+\infty} \langle h_{\varepsilon, \sigma}(t), \varphi(t) \rangle \, dt$,
and we investigate each term separately.

As for $A^1$, it is well known that it is a smooth functional, with
\[
d A^1(\varphi)[\psi] = \int_1^{+\infty} \langle \dot{\varphi}(t), \dot{\psi}(t) \rangle \, dt \quad \text{and} \quad d^2 A^1(\varphi)[\psi, \zeta] = \int_1^{+\infty} \langle \dot{\psi}(t), \dot{\zeta}(t) \rangle \, dt.
\]
The continuity with respect to the parameters $\varepsilon$ and $\sigma$ is obvious, since they do not appear in the above expressions.

We now focus our attention on the (linear) term $A^3_{\varepsilon, \sigma}$. Using (2.3) and (4.10), we deduce
\[
\left| \int_1^{+\infty} \langle h_{\varepsilon, \sigma}(t), \varphi(t) \rangle \, dt \right| \leq C_h \| \varphi \| \int_1^{+\infty} \frac{dt}{t^\frac{3}{2}} < +\infty,
\]
so that $A^3_{\varepsilon, \sigma}$ is well-defined and continuous. Therefore,
\[
d A^3_{\varepsilon, \sigma}(\varphi)[\psi] = \int_1^{+\infty} \langle h_{\varepsilon, \sigma}(t), \psi(t) \rangle \, dt \quad \text{and} \quad d^2 A^3_{\varepsilon, \sigma}(\varphi)[\psi, \zeta] = 0.
\]
The only thing to check is the continuity of the map $(\varepsilon, \sigma, \varphi) \mapsto d A^3_{\varepsilon, \sigma}(\varphi)$. Precisely, since $d A^3_{\varepsilon, \sigma}(\varphi)$ does not depend on $\varphi$, we need to verify that
\[
(\varepsilon_n, \sigma_n) \to (\varepsilon, \sigma) \quad \Rightarrow \quad \sup_{\| \psi \| \leq 1} \left| \int_1^{+\infty} \langle h_{\varepsilon_n, \sigma_n}(t) - h_{\varepsilon, \sigma}(t), \psi(t) \rangle \, dt \right| \to 0. \quad (4.16)
\]
To prove this, we first use the Cauchy–Schwartz inequality together with (2.5) to get
\[
\sup_{\|\psi\|\leq 1} \left| \int_{1}^{+\infty} \langle h_{\varepsilon,\sigma_n}(t) - h_{\varepsilon,\sigma}(t), \psi(t) \rangle \, dt \right| \leq \sup_{\|\psi\|\leq 1} \int_{1}^{+\infty} \frac{t|\varepsilon_{n,\sigma_n}(t) - \varepsilon_{\sigma}(t)|}{t} \, dt.
\]
\[
\leq \sup_{\|\psi\|\leq 1} \left( \int_{1}^{+\infty} \frac{|\psi(t)|^2}{t^2} \, dt \right)^{1/2} \left( \int_{1}^{+\infty} t^2|h_{\varepsilon,\sigma_n}(t) - h_{\varepsilon,\sigma}(t)|^2 \, dt \right)^{1/2}
\]
\[
\leq 2 \left( \int_{1}^{+\infty} t^2|h_{\varepsilon,\sigma_n}(t) - h_{\varepsilon,\sigma}(t)|^2 \, dt \right)^{1/2}.
\]

We then conclude, using the dominated convergence theorem: indeed, it is easily checked that $h_{\varepsilon,\sigma_n}(t) \to h_{\varepsilon,\sigma}(t)$ pointwise and the integrand is bounded by the integrable function $4C_n^2 t^{-2}$, by (4.10).

We finally deal with the (nonlinear) term $A_{\varepsilon,\sigma}$. Incidentally, observe that the integral is well-defined, as it can be seen by combining (4.11) with (2.5).

Let us consider the linear form
\[
L_{\varepsilon,\sigma,\varphi}[\psi] = \int_{1}^{+\infty} \langle \nabla K_{\varepsilon,\sigma}(t, \varphi(t)), \psi(t) \rangle \, dt, \quad \psi \in D_0^{1,2}(1, +\infty), \quad (4.18)
\]
and the bilinear form
\[
B_{\varepsilon,\sigma,\varphi}[\psi, \zeta] = \int_{1}^{+\infty} \langle \nabla^2 K_{\varepsilon,\sigma}(t, \varphi(t))\psi(t), \zeta(t) \rangle \, dt, \quad \psi, \zeta \in D_0^{1,2}(1, +\infty). \quad (4.19)
\]

Notice that $L_{\varepsilon,\sigma,\varphi}$ and $B_{\varepsilon,\sigma,\varphi}$ are actually well-defined and continuous: indeed, using (4.12) together with the Cauchy–Schwartz inequality and (2.5), we obtain
\[
|L_{\varepsilon,\sigma,\varphi}[\psi]| \leq C_K \int_{1}^{+\infty} \frac{|\varphi(t)||\psi(t)|}{t^2} \, dt \leq C_K \left( \int_{1}^{+\infty} \frac{|\varphi(t)|^2}{t^2} \, dt \right)^{1/2} \left( \int_{1}^{+\infty} \frac{|\psi(t)|^2}{t^2} \, dt \right)^{1/2} \leq 4C_K \|\psi\| \|\varphi\|.
\]
An analogous argument works for $B_{\varepsilon,\sigma,\varphi}$ (using (4.13)).

We now prove that the functions $(\varepsilon, \sigma, \varphi) \mapsto L_{\varepsilon,\sigma,\varphi}$ and $(\varepsilon, \sigma, \varphi) \mapsto B_{\varepsilon,\sigma,\varphi}$ are continuous as functions with values in the space of linear and bilinear forms on $D_0^{1,2}(1, +\infty)$, respectively, that is,
\[
(\varepsilon_n, \sigma_n, \varphi_n) \to (\varepsilon, \sigma, \varphi) \quad \Rightarrow \quad \sup_{\|\psi\|\leq 1} |L_{\varepsilon_n,\sigma_n,\varphi_n}[\psi] - L_{\varepsilon,\sigma,\varphi}[\psi]| \to 0 \quad (4.20)
\]
and
\[
(\varepsilon_n, \sigma_n, \varphi_n) \to (\varepsilon, \sigma, \varphi) \quad \Rightarrow \quad \sup_{\|\psi\|\leq 1} |B_{\varepsilon_n,\sigma_n,\varphi_n}[\psi, \zeta] - B_{\varepsilon,\sigma,\varphi}[\psi, \zeta]| \to 0. \quad (4.21)
\]
We start with (4.20). Arguing as in (4.17), we find

\[
\sup_{\|\psi\| \leq 1} |L_{\epsilon_n, \sigma_n, \varphi_n}[\psi] - L_{\epsilon, \sigma, \varphi}[\psi]| \leq \epsilon_n \sigma_n \phi_n \left[ \psi - L_{\epsilon, \sigma, \varphi}[\psi] \right] \leq 2 \int_1^{+\infty} t^2 |\nabla K_{\epsilon_n, \sigma_n}(t, \varphi_n(t)) - \nabla K_{\epsilon, \sigma}(t, \varphi(t))|^2 \, dt \right)^{1/2}.
\]

We are going to show that the integral goes to zero, using the dominated convergence theorem. To this end, we first observe that the integrand goes to zero pointwise, since \( \varphi_n \to \varphi \) in \( D_0^{1,2}(1, +\infty) \) implies uniform convergence on compact sets (recall the inequality (2.3)). To prove that the integrand is \( L^1 \)-bounded, we use (4.12) and elementary inequalities so as to obtain

\[
t^2 |\nabla K_{\epsilon_n, \sigma_n}(t, \varphi_n(t)) - \nabla K_{\epsilon, \sigma}(t, \varphi(t))|^2 \leq 2C_K \left( \frac{|\varphi_n(t)|^2}{t^2} + \frac{|\varphi(t)|^2}{t^2} \right)
\]

for every \( t \geq 1 \). By Hardy inequality (2.5), the first term on the right-hand side is in \( L^1 \); on the other hand, again by Hardy inequality, the second term goes to zero in \( L^1 \) and thus, up to a subsequence, is \( L^1 \)-dominated. Hence, the dominated convergence theorem applies along a subsequence and a standard argument yields the conclusion for the original sequence.

We now prove (4.21); this will require a more careful analysis. Recalling the definition of \( K_{\epsilon, \sigma} \), we are going to show that

\[
\sup_{\|\psi\| \leq 1} \left| \int_1^{+\infty} \left\langle \left( \nabla^2 U_0 (y_{\sigma_n}(t) + \varphi_n(t)) - \nabla^2 U_0 (y_{\sigma}(t) + \varphi(t)) \right) \psi(t), \zeta(t) \right\rangle \, dt \right| \to 0
\]

and that

\[
\sup_{\|\psi\| \leq 1} \left| \int_1^{+\infty} \left\langle \left( \nabla^2 W_{\epsilon_n}(t, y_{\sigma_n}(t) + \varphi_n(t)) - \nabla^2 W_{\epsilon}(t, y_{\sigma}(t) + \varphi(t)) \right) \psi(t), \zeta(t) \right\rangle \, dt \right| \to 0,
\]

as \( (\epsilon_n, \sigma_n, \varphi_n) \to (\epsilon, \sigma, \varphi) \).

We first deal with (4.22). Preliminarily, we observe that, denoting by \( z_{n, \lambda}(t) \) a generic point along the segment joining \( y_{\sigma}(t) + \varphi(t) \) with \( y_{\sigma_n}(t) + \varphi_n(t) \), that is, for \( \lambda \in [0, 1] \) and \( t \geq 1 \),

\[
z_{n, \lambda}(t) = y_{\sigma}(t) + \varphi(t) + \lambda(y_{\sigma_n}(t) - y_{\sigma}(t) + \varphi_n(t) - \varphi(t)),
\]

the estimate

\[
|z_{n, \lambda}(t)| \geq \frac{\alpha + 1}{4} t^2, \quad \text{for every } t \geq 1,
\]

(4.24)
We are going to show that the integral goes to zero, using the dominated convergence theorem, which goes to zero as $n \to \infty$. For every $n$, when $n$ is large enough, we can use the mean value theorem to obtain

$$|z_{n,\lambda}(t)| \geq |y_\sigma(t) + \varphi(t)| - |y_{\sigma_n}(t) - y_\sigma(t)| - |\varphi_n(t) - \varphi(t)|$$

$$\geq t^{\frac{2}{3}} \left( \frac{\alpha + 1}{2} - |\sigma_n - \sigma| \max_{t \in [1,2]} |w(t)| t^{-\frac{2}{3}} - ||\varphi_n - \varphi|| t^{-\frac{1}{3}} \right)$$

$$\geq t^{\frac{2}{3}} \left( \frac{\alpha + 1}{2} - |\sigma_n - \sigma| \max_{t \in [1,2]} |w(t)| - ||\varphi_n - \varphi|| \right),$$

for every $t \geq 1$, whence the conclusion when $n$ is large enough. In particular we observe that $|z_{n,\lambda}(t)| > \frac{\alpha + 1}{4}$ for every $t \geq 1$, when $n$ is large enough. Therefore, we can use the mean value theorem to obtain

$$|\nabla^2 U_0(y_{\sigma_n}(t) + \varphi_n(t)) - \nabla^2 U_0(y_{\sigma}(t) + \varphi(t))| \leq$$

$$\leq \tilde{C} \sup_{\lambda \in [0,1]} \left| \nabla^2 U_0(z_{n,\lambda}(t)) \right| \left( |y_{\sigma_n}(t) - y_{\sigma}(t)| + |\varphi_n(t) - \varphi(t)| \right),$$

for every $t \geq 1$, where $\tilde{C} > 0$ is a suitable constant. Using the fact that

$$\partial^2_{ijk} U_0(y) = O(|y|^{-4}), \quad \text{as } |y| \to +\infty,$$

for every $i, j, k \in \{1, \ldots, d\}$, together with (4.24) and (2.3), we thus deduce the existence of $\tilde{C} > 0$ such that

$$|\nabla^2 U_0(y_{\sigma_n}(t) + \varphi_n(t)) - \nabla^2 U_0(y_{\sigma}(t) + \varphi(t))| \leq$$

$$\leq \frac{\tilde{C}}{t^{\frac{2}{3}}} \left( |\sigma_n - \sigma| \max_{t \in [1,2]} |w(t)| + ||\varphi_n - \varphi|| t^{\frac{1}{3}} \right),$$

for every $t \geq 1$. Hence, using twice (2.3) (for $\psi$ and $\zeta$) we obtain

$$\sup_{\|\psi\| \leq 1} \left| \int_1^{+\infty} \left\langle \left( \nabla^2 U_0(y_{\sigma_n}(t) + \varphi_n(t)) - \nabla^2 U_0(y_{\sigma}(t) + \varphi(t)) \right) \psi(t), \zeta(t) \right\rangle dt \right| \leq$$

$$\leq \tilde{C} \int_1^{+\infty} \frac{t^{\frac{2}{3}}}{t^{\frac{2}{3}}} \left( |\sigma_n - \sigma| \max_{t \in [1,2]} |w(t)| + ||\varphi_n - \varphi|| t^{\frac{1}{3}} \right) dt$$

$$= \tilde{C} \left( |\sigma_n - \sigma| \max_{t \in [1,2]} |w(t)| \int_1^{+\infty} \frac{dt}{t^{\frac{2}{3}}} + ||\varphi_n - \varphi|| \int_1^{+\infty} \frac{dt}{t^{\frac{1}{3}}} \right),$$

which goes to zero as $n \to \infty$.

As for (4.23), we use again twice (2.3) (for $\psi$ and $\zeta$) to obtain

$$\sup_{\|\psi\| \leq 1} \left| \int_1^{+\infty} \left\langle \left( \nabla^2 W_{\epsilon_n}(t, y_{\sigma_n}(t) + \varphi_n(t)) - \nabla^2 W_{\epsilon}(t, y_{\sigma}(t) + \varphi(t)) \right) \psi(t), \zeta(t) \right\rangle dt \right| \leq$$

$$\leq \int_1^{+\infty} t |\nabla^2 W_{\epsilon_n}(t, y_{\sigma_n}(t) + \varphi_n(t)) - \nabla^2 W_{\epsilon}(t, y_{\sigma}(t) + \varphi(t))| dt.$$

We are going to show that the integral goes to zero, using the dominated convergence theorem. The pointwise convergence of the integrand follows from $\varphi_n \to \varphi$
in $D^{1,\infty}_0(1, +\infty)$; on the other hand, the $L^1$-bound follows from \(4.3\) together with the estimate \(4.9\).

We finally claim that the linear form $L_{\varepsilon, \sigma, \varphi}$ defined in \(4.18\) and the bilinear form $B_{\varepsilon, \sigma, \varphi}$ defined in \(4.19\) are, respectively, the first and second Gateaux differential of $A^{2,\sigma}_{\varepsilon, \sigma}$ at the point $\varphi$, that is, for every $\psi \in D^{1,\infty}_0(1, +\infty)$,

\[
\lim_{\vartheta \to 0} \int_1^{+\infty} \left( K_{\varepsilon, \sigma}(t, \varphi(t) + \vartheta \psi(t)) - K_{\varepsilon, \sigma}(t, \varphi(t)) \right) dt = 0 \tag{4.26}
\]

and

\[
\lim_{\vartheta \to 0} \sup_{\|\zeta\| \leq 1} \left| \int_1^{+\infty} \left( \frac{\nabla K_{\varepsilon, \sigma}(t, \varphi(t) + \vartheta \psi(t)) - \nabla K_{\varepsilon, \sigma}(t, \varphi(t))}{\vartheta} \right) dt \right| = 0. \tag{4.27}
\]

We begin by verifying \(4.26\). Defining, for $\vartheta \in [-1, 1]$ and $t \geq 1$,

\[
g_\vartheta(t) = \int_0^1 \left( \nabla K_{\varepsilon, \sigma}(t, \varphi(t) + s \vartheta \psi(t)) - \nabla K_{\varepsilon, \sigma}(t, \varphi(t)) \right) ds,
\]

it can be readily checked that \(4.26\) equivalently reads as

\[
\lim_{\vartheta \to 0} \int_1^{+\infty} \langle g_\vartheta(t), \psi(t) \rangle dt = 0. \tag{4.28}
\]

It is easy to see that $g_\vartheta(t) \to 0$ as $\vartheta \to 0$ for every $t \geq 1$. On the other hand, using \(4.12\) we find

\[
|g_\vartheta(t)| \leq \frac{C_K}{t^2} \int_0^1 \left( |\varphi(t) + s \vartheta \psi(t)| + |\varphi(t)| \right) ds \leq \frac{C_K}{t^2} \left( 2|\varphi(t)| + |\psi(t)| \right),
\]

implying

\[
|\langle g_\vartheta(t), \psi(t) \rangle| \leq C_K \left( \frac{2|\varphi(t)||\psi(t)|}{t^2} + \frac{|\psi(t)|^2}{t^2} \right),
\]

for every $t \geq 1$. By Hardy inequality \(2.5\), the right-hand side is an $L^1$-function; therefore the dominated convergence theorem applies yielding \(4.28\).

We now focus on \(4.27\). Similarly as before, we are led to verify that

\[
\lim_{\vartheta \to 0} \sup_{\|\zeta\| \leq 1} \int_1^{+\infty} \langle G_\vartheta(t) \psi(t), \zeta(t) \rangle dt = 0,
\]

where, for $\vartheta \in [-1, 1]$ and $t \geq 1$,

\[
G_\vartheta(t) = \int_0^1 \left( \nabla^2 K_{\varepsilon, \sigma}(t, \varphi(t) + s \vartheta \psi(t)) - \nabla^2 K_{\varepsilon, \sigma}(t, \varphi(t)) \right) ds.
\]

Again, $G_\vartheta(t) \to 0$ as $\vartheta \to 0$ for every $t \geq 1$. Using twice \(2.3\) (for $\psi$ and $\zeta$), we find

\[
\sup_{\|\zeta\| \leq 1} \int_1^{+\infty} \langle G_\vartheta(t) \psi(t), \zeta(t) \rangle dt \leq \|\psi\| \int_1^{+\infty} t |G_\vartheta(t)| dt
\]
and we can thus conclude by showing that the function $t|G_\theta(t)|$ is $L^1$-bounded. To this end, (4.13) is not enough and we have to use the same strategy as for the proof of (4.19). Precisely, we first write

$$G_\theta(t) = \int_0^1 \left( \nabla^2 U_0(y_\sigma(t) + \varphi(t) + s\psi(t)) - \nabla^2 U_0(y_\sigma(t) + \varphi(t)) \right) \, ds$$

$$+ \int_0^1 \left( \nabla^2 W_\varepsilon(t, y_\sigma(t) + \varphi(t) + s\psi(t)) - \nabla^2 W_\varepsilon(t, y_\sigma(t) + \varphi(t)) \right) \, ds.$$  

Arguing exactly as in the proof of (4.25) (with $\sigma_n = \sigma$ and $\varphi + s\psi$ in place of $\varphi_n$), we find on one hand that, when $|\psi|$ is small enough,

$$\left| \int_0^1 \left( \nabla^2 U_0(y_\sigma(t) + \varphi(t) + s\psi(t)) - \nabla^2 U_0(y_\sigma(t) + \varphi(t)) \right) \, ds \right| \leq \frac{\bar{C}}{t^{\frac{3}{2}}} \|\psi\| t^{\frac{1}{2}},$$

for every $t \geq 1$. On the other hand, using twice (4.9) (the first time with $\varphi + s\psi$ in place of $\varphi$) together with (4.3) we find a constant $\bar{C} > 0$ such that, when $|\psi|$ is small enough,

$$\left| \int_0^1 \left( \nabla^2 W_\varepsilon(t, y_\sigma(t) + \varphi(t) + s\psi(t)) - \nabla^2 W_\varepsilon(t, y_\sigma(t) + \varphi(t)) \right) \, ds \right| \leq \frac{\bar{C}}{t^{\frac{3}{2}},}$$

for every $t \geq 1$. Summing up, for $|\psi|$ small enough, we find

$$t|G_\theta(t)| \leq \frac{\bar{C} \|\psi\|}{t^{\frac{3}{2}}} + \frac{\bar{C}}{t^{\frac{3}{2}}},$$

for every $t \geq 1$, proving the desired $L^1$-bound.

We are finally ready to summarize and conclude. The existence of the limit in (4.26) implies that the linear form $L_{\varepsilon,\sigma,\varphi}$ defined in (4.18) is the Gateaux differential of $A_{\varepsilon,\sigma}$ at the point $\varphi$. Since such a form is continuous (in $\varphi$), as proved in (4.20), we infer that $L_{\varepsilon,\sigma,\varphi}$ is the (Fréchet) differential of $A_{\varepsilon,\sigma}$ at the point $\varphi$ (and, moreover, $A_{\varepsilon,\sigma}$ is of class $C^1$ on the open set $\Omega_\rho$).

Similarly, the existence of the limit in (4.27) implies that the bilinear form $B_{\varepsilon,\sigma,\varphi}$ defined in (4.19) is the Gateaux differential of $dA_{\varepsilon,\sigma}$ at the point $\varphi$. Since such a form is continuous (in $\varphi$) as proved in (4.21), we infer that $B_{\varepsilon,\sigma,\varphi}$ is the (Fréchet) differential of $dA_{\varepsilon,\sigma}$ at the point $\varphi$, that is, the second differential of $A_{\varepsilon,\sigma}$ at the point $\varphi$. Hence, the functional $A_{\varepsilon,\sigma}$ is of class $C^2$ on the open set $\Omega_\rho$.

Recalling (4.15) and the discussion about $A^1$ and $A^3_{\varepsilon,\sigma}$ at the beginning of the proof, we conclude that the functional $A_{\varepsilon,\sigma}$ is of class $C^2$ on the open set $\Omega_\rho$.

All this implies that $F$ is differentiable in $\varphi$, with differential

$$D_\varphi F(\varepsilon, \sigma, \varphi) = d^2 A_{\varepsilon,\sigma}(\varphi) = d^2 A^1(\varphi) + d^2 A^2_{\varepsilon,\sigma}(\varphi).$$

The continuity of $F$ and $D_\varphi F$ with respect to the three variables $(\varepsilon, \sigma, \varphi)$ thus follows from (4.16), (4.20) and (4.21). \qed

Our goal now is to apply the implicit function theorem to the function $F$. To this end, we first observe that

$$h_{0,0}(t) \equiv 0, \quad \nabla K_{0,0}(t, 0) \equiv 0,$$
implying $F(0, 0, 0) = 0$. On the other hand,

$$\nabla^2 K_{0, 0}(t, 0) = \nabla^2 U_0(y_0(t)), \quad \text{for every } t \geq 1,$$

so that

$$D_\varphi F(0, 0, 0)[\psi, \zeta] = \int_1^{+\infty} \left( \langle \dot{\psi}(t), \dot{\zeta}(t) \rangle + \langle \nabla^2 U_0(y_0(t)) \psi(t), \zeta(t) \rangle \right) \, dt, \quad (4.29)$$

for every $\psi, \zeta \in D_0^{1,2}(1, +\infty)$. In the above formula, we have meant the differential $D_\varphi F(0, 0, 0)$ as a continuous bilinear form on $D_0^{1,2}(1, +\infty)$, using the canonical isomorphism between bilinear forms on a Banach space $X$ and linear operators from $X$ to $X^*$. Notice that the invertibility of $D_\varphi F(0, 0, 0)$ (as a linear operator from $D_0^{1,2}(1, +\infty)$ to its dual) is equivalent to the fact that for every $T \in (D_0^{1,2}(1, +\infty))^*$ there exists $\psi_T \in D_0^{1,2}(1, +\infty)$ such that

$$D_\varphi F(0, 0, 0)[\psi_T, \zeta] = T[\zeta], \quad \text{for every } \zeta \in D_0^{1,2}(1, +\infty).$$

To show that this is true, we are going to use the Lax–Milgram theorem, by proving that the quadratic form associated to $D_\varphi F(0, 0, 0)$ is coercive. This is the content of the next proposition.

**Proposition 4.2.** For every $\psi \in D_0^{1,2}(1, +\infty)$, it holds that

$$\int_1^{+\infty} \left( |\dot{\psi}(t)|^2 + \langle \nabla^2 U_0(y_0(t)) \psi(t), \psi(t) \rangle \right) \, dt \geq \frac{1}{9} \|\psi\|^2.$$

**Proof.** A standard computation provides

$$\nabla^2 U_0(y) = -\frac{\Id}{|y|^2} + 3 \frac{y \otimes y}{|y|^5}, \quad \text{for every } y \neq 0,$$

where $y \otimes y$ denotes the symmetric square matrix with components $(y \otimes y)_{ij} = y_i y_j$ for $i, j \in \{1, \ldots, d\}$. Using the well-known fact that this matrix is positive semidefinite\(^1\) and recalling (3.6), we find

$$\langle \nabla^2 U_0(y_0(t)) \psi(t), \psi(t) \rangle \geq -\frac{|\psi(t)|^2}{|y_0(t)|^3} = -\frac{2}{9} \frac{|\psi(t)|^2}{t^2},$$

for every $t \geq 1$. By Hardy inequality (2.5), we finally obtain

$$\int_1^{+\infty} \left( |\dot{\psi}(t)|^2 + \langle \nabla^2 U_0(y_0(t)) \psi(t), \psi(t) \rangle \right) \, dt \geq \left( 1 - \frac{8}{9} \right) \int_1^{+\infty} |\dot{\psi}(t)|^2 \, dt,$$

as desired. \qed

Summing up, by the implicit function theorem, there exist $\varepsilon^* \in ]0, 1[$ and $r^* \in ]0, r[$ such that, for every $\varepsilon \in ]0, \varepsilon^*[\text{ and for every } \sigma \in B_{r^*}$, there exists a solution $\varphi \in \Omega_\sigma$ of the equation

$$F(\varepsilon, \sigma, \varphi) = 0.$$

\(^1\)Indeed, meaning vectors in $\mathbb{R}^d$ as column vectors, it holds that $y \otimes y = yy^\top$, where $y^\top$ is the transpose of $y$. Therefore, for every $x \in \mathbb{R}^d$

$$\langle (y \otimes y)x, x \rangle = x^\top (y \otimes y)x = x^\top yy^\top x = (x, y)^2 \geq 0.$$
This means that $\varphi$ is a critical point of the action functional $A_{\varepsilon,\sigma}$. Since the space $C_c^\infty([1, +\infty[)$ is contained in $D_0^{1,2}(1, +\infty)$, we deduce that $\varphi$ is a solution of equation \((3.10)\) in the sense of distributions. By a standard regularity argument, $\varphi \in C^2([1, +\infty[)$ and solves the equation in the classical sense.

We summarize the above discussion in the final proposition of this section.

**Theorem 4.1.** There exist $\varepsilon^* \in ]0, 1[$ and $r^* \in ]0, r[\), such that, for every $\varepsilon \in ]0, \varepsilon^*[\) and $\sigma \in B_{r^*}$, there exists $\varphi \in C^2([1, +\infty[) \cap D_0^{1,2}(1, +\infty)$ solution of \((3.10)\).

**Remark 4.2.** It is worth noticing that, due to Proposition 4.2, it easily follows that $\varphi$ is a non-degenerate local minimum for the corresponding action functional $A_{\varepsilon,\sigma}$.

### 4.3. Conclusion of the proof

We are now in a position to conclude the proof of Theorem 1.1. To this end, let us consider the numbers $\varepsilon^*$ and $r^*$ given in Theorem 4.1 and set

$$R = \frac{\sqrt{\alpha^2 + (r^*)^2}}{\varepsilon^*}, \quad \eta = \frac{\alpha}{\sqrt{\alpha^2 + (r^*)^2}}.$$  

Notice that $R > 1$ and $\eta \in ]0, 1[$. Accordingly, we consider the set $T(\xi^+, R, \eta)$ defined in \((1.2)\).

Let us fix $x_0 \in T(\xi^+, R, \eta)$. We claim that there exist $\varepsilon \in ]0, \varepsilon^*[\) and $\sigma \in B_{r^*}$ such that

$$x_0 = \frac{1}{\varepsilon} \left( \alpha \xi^+ + \sigma \right). \quad (4.30)$$

To see this, we first observe that, without loss of generality, we can assume that $\xi^+ = (1, 0_{\mathbb{R}^{d-1}}) \in \mathbb{R}^d$. Then, we can look for $\sigma$ of the form $\sigma = (0, \sigma')$, with $\sigma' \in \mathbb{R}^{d-1}$. With this position, and setting $x_0 = \left( (x_0)_1, x_0' \right) \in \mathbb{R} \times \mathbb{R}^{d-1}$, from \((4.30)\) we obtain

$$(x_0)_1 = \frac{\alpha}{\varepsilon}, \quad x_0' = \frac{\sigma'}{\varepsilon}. \quad (4.31)$$

Observing that, since $x_0 \in T(\xi^+, R, \eta)$,

$$(x_0)_1 = \langle x_0, \xi^+ \rangle > \frac{\alpha}{\sqrt{\alpha^2 + (r^*)^2}} |x_0| > \frac{\alpha}{\sqrt{\alpha^2 + (r^*)^2}} \frac{\sqrt{\alpha^2 + (r^*)^2}}{\varepsilon^*} = \frac{\alpha}{\varepsilon^*} > 0,$$

from \((4.31)\) we infer that

$$\varepsilon = \frac{\alpha}{(x_0)_1}, \quad \sigma' = \frac{\alpha}{(x_0)_1} x_0'.$$  

From \((4.32)\), we immediately deduce that $\varepsilon \in ]0, \varepsilon^*[\) and $\sigma < r^*$. To this end, we first notice that, from \((4.32)\), we deduce

$$(x_0)_1^2 > \frac{\alpha^2}{\alpha^2 + (r^*)^2} \left( (x_0)_1^2 + |x_0'|^2 \right),$$

implying

$$|x_0'| < \frac{r^*}{\alpha} (x_0)_1.$$
and finally that
\[
|\sigma| = |\sigma'| = \frac{\alpha}{(x_0)_{1}} |x'_{0}| < \frac{\alpha}{(x_0)_{1}} r^{*} (x_0)_{1} = r^{*},
\]
as desired.

Now, given \( \varepsilon \) and \( \sigma \) as in (4.30), let us consider the solution \( \varphi \) of (3.10) as in Theorem 4.1 and define

\[
y(t) = y_{\sigma}(t) + \varphi(t), \quad \text{for every } t \geq 1,
\]
as in (3.8). As already observed in Section 3.1, since \( \varphi \) is a solution of (3.10) on \([1, +\infty]\), the function \( y \) is a solution of (3.2) on \([1, +\infty]\) and the function \( x \) defined by

\[
x(t) = \frac{1}{\varepsilon} y(\varepsilon^{\frac{3}{2}} t + 1), \quad \text{for every } t \geq 0,
\]
as in (3.13), is a solution of (1.1) on \([0, +\infty]\). We also notice that

\[
x(0) = \frac{1}{\varepsilon} y(1) = \frac{1}{\varepsilon} (\alpha \xi^{+} + \sigma) = x_{0},
\]
by (4.30). Moreover, recalling that \( y_{\sigma}(t) = y_{0}(t) \) for \( t \geq 2 \) and using inequality (2.3), it holds that \( |y(t)| \sim \alpha t^{\frac{2}{3}} \) for \( t \to +\infty \), implying \( |x(t)| \sim \alpha t^{\frac{2}{3}} \) for \( t \to +\infty \), as well.

In order to conclude the proof we thus need to show that \( x \) has asymptotic direction \( \xi^{+} \) and that \( \dot{x}(t) \to 0 \) for \( t \to +\infty \). Recalling the change of variables (4.33), it is immediate to see that this is the case if and only if the corresponding properties are verified by \( y \), that is,

\[
\lim_{t \to +\infty} \frac{y(t)}{|y(t)|} = \xi^{+} \tag{4.34}
\]
and

\[
\lim_{t \to +\infty} \dot{y}(t) = 0. \tag{4.35}
\]

As far as (4.34) is concerned, we recall again that \( y_{\sigma}(t) = y_{0}(t) \) for \( t \geq 2 \), implying

\[
\frac{y(t)}{|y(t)|} = \frac{y_{0}(t) + \varphi(t)}{|y_{0}(t) + \varphi(t)|} = \frac{y_{0}(t) + \varphi(t)}{|y_{0}(t)| \left(1 + \frac{|\varphi(t)|^{2}}{|y_{0}(t)|^{2}} + \frac{2(y_{0}(t), \varphi(t))}{|y_{0}(t)|^{2}}\right)^{-\frac{1}{2}}}
\]
\[
= \frac{y_{0}(t) + \varphi(t)}{|y_{0}(t)| \left(1 + \frac{|\varphi(t)|^{2}}{|y_{0}(t)|^{2}} + \frac{2(y_{0}(t), \varphi(t))}{|y_{0}(t)|^{2}}\right)^{-\frac{1}{2}}}
\]
\[
= \left(\frac{y_{0}(t)}{|y_{0}(t)|} + \frac{\varphi(t)}{|y_{0}(t)|}\right) \left(1 + \frac{|\varphi(t)|^{2}}{|y_{0}(t)|^{2}} + \frac{2(y_{0}(t), \varphi(t))}{|y_{0}(t)|^{2}}\right)^{-\frac{1}{2}},
\]
for every \( t \geq 2 \). Recalling that \( y_{0}(t) = \alpha t^{\frac{2}{3}} \xi^{+} \) and using (2.3), we infer that

\[
\frac{y_{0}(t)}{|y_{0}(t)|} = \xi^{+} \quad \text{and} \quad \lim_{t \to +\infty} \frac{\varphi(t)}{|y_{0}(t)|} = 0,
\]
thus concluding that
\[ \lim_{t \to +\infty} \frac{y(t)}{|y(t)|} = \xi^+ . \]

Finally, we prove the validity of (4.35). We are going to use an energy argument, based on the function \( E: [1, +\infty] \to \mathbb{R} \) defined by
\[ E(t) = \frac{1}{2} |\dot{y}(t)|^2 - U_\varepsilon(t, y(t)). \]

Recalling (3.3), a simple computation shows that
\[ \dot{E}(t) = \partial_t U_\varepsilon(t, y(t)) = \partial_t W_\varepsilon(t, y(t)), \]
for every \( t \geq 1 \).

From (4.4) and (4.9) we deduce that
\[ |\dot{E}(t)| \leq \frac{C}{\sqrt{\varepsilon}|y(t)|^2} \leq \frac{4C}{\sqrt{\varepsilon}(\alpha + 1)^2 t^{\frac{3}{2}}} , \]
for every \( t \geq 1 \), implying that \( \int_1^{+\infty} |\dot{E}(t)| \, dt < +\infty \). As a consequence, writing
\[ E(t) = E(1) + \int_1^t \dot{E}(s) \, ds , \]
for every \( t \geq 1 \), we infer that the limit \( \lim_{t \to +\infty} E(t) \) exists and it is finite. Since by (4.5) we have
\[ \lim_{t \to +\infty} U_\varepsilon(t, y(t)) = 0 , \]
we deduce that there exists \( \ell \in \mathbb{R} \) such that
\[ \lim_{t \to +\infty} |\dot{y}(t)|^2 = \ell . \]

On the other hand, since \( \varphi \in \mathcal{D}_{0}^{1, 2}(1, +\infty) \) we have \( \int_1^{+\infty} |\dot{\varphi}(t)|^2 \, dt < +\infty \), implying
\[ \liminf_{t \to +\infty} |\dot{\varphi}(t)| = 0 . \]

From this, since
\[ \lim_{t \to +\infty} \dot{y}_\sigma(t) = \lim_{t \to +\infty} \dot{y}_0(t) = 0 , \]
we obtain
\[ \liminf_{t \to +\infty} |\dot{y}(t)| = 0 . \]

Then, we deduce that \( \ell = 0 \), thus proving (4.35).

5. Some applications

An application of Theorem 1.1 can be given for the equation
\[ \ddot{x} = - \sum_{i=1}^{N} \frac{m_i (x - c_i(t))}{|x - c_i(t)|^3} , \quad x \in \mathbb{R}^d , \quad (5.1) \]
where, for \( i = 1, \ldots, N \), \( m_i > 0 \) and \( c_i: \mathbb{R} \to \mathbb{R}^d \) are functions of class \( \mathcal{C}^1 \).

Equation (5.1) models the motion of zero-mass particle \( x \) under the Newtonian attraction of \( N \) moving bodies \( c_i \) of mass \( m_i \). The case when all the bodies \( c_i \) are
fixed, namely \( c_i(t) \equiv c_i \) for every \( i = 1, \ldots, N \), is usually referred to as \( N \)-centre problem. For general moving bodies \( c_i \), we have the following result.

**Corollary 5.1.** Let us suppose that
\[
\sup_{t \in \mathbb{R}} |c_i(t)| + \sup_{t \in \mathbb{R}} |\dot{c}_i(t)| < +\infty, \quad \text{for every } i = 1, \ldots, N.
\]
(5.2)

Then, for every \( \xi^+ \in S^d \), there exist \( R > \sup_{t \in \mathbb{R}} |c_i(t)| \) and \( \eta \in ]0, 1[ \) such that, for every \( x_0 \in T(\xi^+, R, \eta) \), there exists a parabolic solution \( x: [0, +\infty[ \to \mathbb{R}^d \) of equation (5.1), satisfying
\[
x(0) = x_0 \text{ and } \lim_{t \to +\infty} \frac{x(t)}{|x(t)|} = \xi^+.
\]
Moreover, \( |x(t)| \sim \alpha Mt^{\frac{3}{2}} \) for \( t \to +\infty \), where \( \alpha = \frac{3\sqrt{9}}{2} \) and \( M = \sum_{i=1}^{N} m_i \).

**Remark 5.1.** Of course, a symmetric statement holds yielding a parabolic solution with prescribed asymptotic direction for \( t \to -\infty \).

**Proof.** We are going to check that the assumptions of the more general version of Theorem 1.1 described in Remark 4.1 are satisfied.

To this end, we first observe that equation (5.1) can be written as \( \ddot{x}(t) = \nabla V(t, x) \), where
\[
V(t, x) = \sum_{i=1}^{N} \frac{m_i}{|x - c_i(t)|}.
\]
Defining, for \( |x| > \Xi = \max_{i} \sup_{t \in \mathbb{R}} |c_i(t)| + 1 \),
\[
W(t, x) = V(t, x) - \frac{M}{|x|}
\]
and setting \( g(t) = \sum_{i=1}^{N} m_i c_i(t) \), it can be checked that the following asymptotic expansions hold true:
\[
W(t, x) = \frac{\langle g(t), x \rangle}{|x|^3} + O\left( \frac{1}{|x|^3} \right),
\]
\[
\partial_t W(t, x) = \frac{\langle \dot{g}(t), x \rangle}{|x|^3} + O\left( \frac{1}{|x|^3} \right),
\]
\[
\nabla W(t, x) = \frac{g(t)}{|x|^3} - 3 \frac{\langle g(t), x \rangle x}{|x|^5} + O\left( \frac{1}{|x|^4} \right),
\]
\[
\nabla^2 W(t, x) = -6 \frac{g(t) \otimes x}{|x|^5} - 3 \frac{\langle g(t), x \rangle x}{|x|^5} 1_d + 15 \frac{\langle g(t), x \rangle x \otimes x}{|x|^7} + O\left( \frac{1}{|x|^5} \right),
\]
for \( |x| \to +\infty \), uniformly in \( t \in \mathbb{R} \) due to assumption (5.2). Here, we have used the notation \( x \otimes x \) for the square matrix of components \( (x \otimes x)_{ij} = x_i x_j \).

Using once more assumption (5.2) together with elementary linear algebra inequalities, we see that condition (1.3) is satisfied. Therefore, Theorem 1.1 can be applied, yielding the conclusion. \qed
The elliptic restricted (planar) three-body problem is a particular case of equation (5.1), where $d = 2, N = 2$ and

$$m_1 = \mu, \quad m_2 = 1 - \mu, \quad c_1(t) = -\mu q_0(t), \quad c_2(t) = (1 - \mu) q_0(t),$$

with $\mu \in ]0, 1[$ and $q_0$ a $2\pi$-periodic function (see [12] for more details). Of course, condition (5.2) is satisfied due to the periodicity so that Corollary 5.1 straightly applies. The elliptic restricted spatial three-body problem could be treated in the same manner (simply, $d = 3$).

References

[1] I. Baldomá, E. Fontich, P. Martín, Whiskered parabolic tori in the planar $(n + 1)$-body problem, arXiv:1812.01286.
[2] V. Barutello, X. Hu, A. Portaluri, S. Terracini, An index theory for asymptotic motions under singular potentials, arXiv:1705.01291.
[3] V. Barutello, S. Terracini, G. Verzini, Entire minimal parabolic trajectories: the planar anisotropic Kepler problem, Arch. Ration. Mech. Anal. 207 (2013) 583–609.
[4] V. Barutello, S. Terracini, G. Verzini, Entire parabolic trajectories as minimal phase transitions, Calc. Var. Partial Differential Equations 49 (2014) 391–429.
[5] A. Boscaggin, W. Dambrosio, D. Papini, Parabolic solutions for the planar $N$-centre problem: multiplicity and scattering, Ann. Mat. Pura Appl. 197 (2018) 869–882.
[6] A. Boscaggin, W. Dambrosio, S. Terracini, Scattering parabolic solutions for the spatial $N$-centre problem, Arch. Ration. Mech. Anal. 223 (2017) 1269–1306.
[7] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York, 2011.
[8] J. Chazy, Sur l’allure du mouvement dans le problème des trois corps quand le temps croît indéfiniment, Ann. Sci. École Norm. Sup. 39 (1922) 29–130.
[9] A. Chenciner, Collisions totales, mouvements complètement paraboliques et réduction des homothéties dans le problème des $n$ corps, Regul. Chaotic Dyn. 3 (1998) 93–106.
[10] A. da Luz, E. Maderna, On the free time minimizers of the Newtonian $N$-body problem, Math. Proc. Cambridge Philos. Soc. 156 (2014) 209–227.
[11] M. Guardia, P. Martín, T. M. Seara, Oscillatory motions for the restricted planar circular three body problem, Invent. Math. 203 (2016) 417–492.
[12] M. Guardia, T. M. Seara, P. Martín, L. Sabbagh, Oscillatory orbits in the restricted elliptic planar three body problem, Discrete Contin. Dyn. Syst. 37 (2017) 229–256.
[13] J. Llibre, C. Simó, Oscillatory solutions in the planar restricted three-body problem, Math. Ann. 248 (1980) 153–184.
[14] E. Maderna, On weak KAM theory for $N$-body problems, Ergodic Theory Dynam. Systems 32 (2012) 1019–1041.
[15] E. Maderna, A. Venturelli, Globally minimizing parabolic motions in the Newtonian $N$-body problem, Arch. Ration. Mech. Anal. 194 (2009) 283–313.
[16] R. McGehee, A stable manifold theorem for degenerate fixed points with applications to celestial mechanics, J. Differential Equations 14 (1973) 70–88.
[17] R. Moeckel, R. Montgomery, H. Sánchez Morgado, Free time minimizers for the three-body problem, Celestial Mech. Dynam. Astronom. 130 (2018) Art. 28, 28 pp.
[18] R. Montgomery, Metric cones, \( N \)-body collisions, and Marchal’s lemma, arXiv:1804.03059.

[19] H. Pollard, Celestial mechanics, Carus Mathematical Monographs, No. 18, Mathematical Association of America, Washington, D. C., 1976.

[20] D. G. Saari, The manifold structure for collision and for hyperbolic-parabolic orbits in the \( n \)-body problem, J. Differential Equations 55 (1984) 300–329.

[21] D. G. Saari, N. D. Hulkower, On the manifolds of total collapse orbits and of completely parabolic orbits for the \( n \)-body problem, J. Differential Equations 41 (1981) 27–43.

[22] N. Soave, S. Terracini, Symbolic dynamics for the \( N \)-centre problem at negative energies, Discrete Contin. Dyn. Syst. 32 (2012) 3245–3301.

Alberto Boscaggin
Dipartimento di Matematica “Giuseppe Peano”, Università di Torino
Via Carlo Alberto, 10 - 10123 Torino, Italy
e-mail: alberto.boscaggin@unito.it

Walter Dambrosio
Dipartimento di Matematica “Giuseppe Peano”, Università di Torino
Via Carlo Alberto, 10 - 10123 Torino, Italy
e-mail: walter.dambrosio@unito.it

Guglielmo Feltrin
Dipartimento di Scienze Matematiche “Giuseppe Luigi Lagrange”, Politecnico di Torino
Corso Duca degli Abruzzi, 24 - 10129 Torino, Italy
e-mail: guglielmo.feltrin@polito.it

Susanna Terracini
Dipartimento di Matematica “Giuseppe Peano”, Università di Torino
Via Carlo Alberto, 10 - 10123 Torino, Italy
e-mail: susanna.terracini@unito.it