RELATIVE MATCHED PAIRS OF FINITE GROUPS FROM DEPTH TWO INCLUSIONS OF VON NEUMANN ALGEBRAS TO QUANTUM GROUPOIDS

JEAN-MICHEL VALLIN

Abstract. In this work we give a generalization of matched pairs of (finite) groups to describe a general class of depth two inclusions of factor von Neumann algebras and the C*-quantum groupoids associated with, using double groupoids.

Date: Preliminary version of the 03/27/07.

1991 Mathematics Subject Classification. (2000) 46L37, 20L05, 20G42, 20 99.

Key words and phrases. Subfactors, quantum groupoids, relative matched pairs.
1. Introduction

The first examples of C*-quantum groupoids were discovered by the theoretical physicists Böhm, Szlachanyi and Nill ([BoSz], [BoSzNi], they called them weak Hopf C*-algebras. In the infinite dimensional framework and for a very large class of depth two inclusions of von Neumann algebras, we proved with M.Enock [EV] the existence of such objects, giving a non commutative geometric interpretation of their basic construction. D.Nikshych and L.Vainerman, using general inclusions of depth two subfactors of type $II_1$ with finite index $M_0 \subset M_1$, gave, thanks to a specific bracket, weak Hopf C*-algebra structures to the relative commutants $M_0' \cap M_2$ and $M_1' \cap M_3$, [NV1], M.C. David in [D] gave also a refinement of this property.

In this article, using a generalization of matched pairs of groups inside the theory of groups, we prove the existence of a large class of inclusions of von Neumann algebras in the conditions of Vainerman and Nykshych. As we have no reference, we shall prove, thanks to Dietmar Bisch works ([B1], [BH]), the following property which may be well known by specialists:

Proposition Let $H$ and $K$ be two finite subgroups of a group $G$ such that $G = HK = \{hk/h \in H, k \in K\}$, let $R$ be the type $II_1$ factor, if $G$ acts properly and outerly on $R$ and if one denotes by $R^H$ (resp. $R \rtimes K$) the fixed point algebra (resp. crossed product) of the induced action of $H$ (resp. $K$) then the inclusion $R^H \subset R \rtimes K$ is a depth two inclusion of subfactors of type $II_1$ with finite index.

Thanks to the canonical bracket, we describe simply the C*-quantum groupoids associated with this inclusion, using a double groupoid structure in the sense of Ehresmann [EH] and a pair of C*-quantum groupoid structure associated with $H, K$. These structures are very close to examples due to N. Andruskiewitsch and S.Natale ([AN1] and [AN2]), so we give a precise relation, they ask for, in the introduction of [AN2], between matched pairs and their work. Hence, our aim is to prove the following theorem:

Theorem Let $H$ and $K$ be two finite subgroups of a group $G$ such that $G = HK = \{hk/h \in H, k \in K\}$, then the pair of C*-quantum groupoids in duality associated with the inclusion $R^H \subset R \rtimes K$ is isomorphic to the double groupoid’s pair.

To be clear, the article do not follow the above order (the one of our study).

In the second paragraph we recall the definition of a C*-quantum groupoid and we explain the commutative and symmetric examples.

In the third chapter, we define relative matched pairs of groups, we associate to them two canonical double crossed product and two double groupoid structures in duality; this allows us to define explicit C*-quantum groupoid in duality for this double crossed products. One can see, that the technics we use with sets of representatives, generalize, in a certain way, ideas developped independently in [B], for the particular case of inclusions of groups. Then we prove these pairs give also depth two inclusions of von Neumann algebras and so C*-quantum groupoids in duality in an other way. The fourth chapter proves that these two structures are isomorphic.
So a natural extension of this article will be the generalization of these constructions in the direction of Lesieur’s locally compact groupoids [L]. An other will be a characterization of these objects in terms of cleft extensions in the spirit of S.Vaes and L.Vainerman [VV].

I want to thank a lot L.Vainerman having suggested me this study, D.Bisch for some explanations about the type of inclusion we deal with, and also S.Baaj, M.C.David and M.Enock for the numerous discussions we had.

2. C*-quantum groupoids

Let us recall the definition of a C*-quantum groupoid (or a weak Hopf C*-algebra), one can also see a more synthetic approach (Val1, Val2,...) using a generalization of Baaj and Skandalis’s multiplicative unitaries (BS and BBS). We shall also recall the definition of an action of such a C*-quantum groupoid due to D.Nikshych and L.Vainerman (NV1) (alternative definitions can be found in Val2, Chap.3 or E2, which are suitable for a future generalisation to Lesieur’s measured quantum groupoid theory [L]).

2.1. C*-quantum groupoids.

2.1.1. Definition. (G.Böhm, K.Szlachányi, F.Nill) (BoSz, BoSzNi)

A weak Hopf C*-algebra is a collection $(A, \Gamma, \kappa, \epsilon)$ where: $A$ is a finite-dimensional C*-algebra (or von Neumann algebra), $\Gamma : A \rightarrow A \otimes A$ is a generalized coproduct, which means that: $(\Gamma \otimes i)\Gamma = (i \otimes \Gamma)\Gamma$, $\kappa$ is an antipode on $A$, i.e., a linear map from $A$ to $A$ such that $(\kappa \circ \epsilon)^2 = i$ (where $\epsilon$ is the involution on $A$), $\epsilon(xy) = \kappa(y)\kappa(x)$ for every $x, y$ in $A$ with $(\kappa \otimes \kappa)\Gamma = \varsigma\Gamma \kappa$ (where $\varsigma$ is the usual flip on $A \otimes A$).

We suppose also that $(m(\kappa \otimes i) \otimes i)(\Gamma \otimes i)\Gamma(x) = (1 \otimes x)\Gamma(1)$ (where $m$ is the multiplication of tensors, i.e., $m(a \otimes b) = ab$), and that $\epsilon$ is a counit, i.e., a positive linear form on $A$ such that $(\epsilon \otimes i)\Gamma = (i \otimes \epsilon)\Gamma = i$, and for every $x, y$ in $A$: $(\epsilon \otimes \epsilon)((x \otimes 1)\Gamma(1)(1 \otimes y)) = \epsilon(xy)$.

2.1.2. Results. (cf. NV1, BoSzNi) If $(A, \Gamma, \kappa, \epsilon)$ is a weak Hopf C*-algebra, then the sets $A_t = \{x \in A/\Gamma(x) = \Gamma(1)(x \otimes 1) = (x \otimes 1)\Gamma(1)\}$ and $A_s = \{x \in A/\Gamma(x) = \Gamma(1)(1 \otimes x) = (1 \otimes x)\Gamma(1)\}$ are commuting sub C*-algebras of $A$ and $\kappa(A_t) = A_s$; one calls them respectively target and source Cartan subalgebra of $(A, \Gamma, \kappa, \epsilon)$ or simply basis of $(A, \Gamma, \kappa, \epsilon)$.

In fact, we shall here deal only with the special case of C*-quantum groupoids for which $\kappa$ is involutive, namely weak Kac algebras.

2.2. The commutative and symmetric examples. Let’s recall that a groupoid $\mathcal{G}$ is a small category the morphisms of which are all invertible. In all what follows, $\mathcal{G}$ is finite. Let $\mathcal{G}^0$ be the set of objects, one can identify $\mathcal{G}^0$ to a subset of the morphisms. So a (finite) groupoid can also be viewed as a set $\mathcal{G}$ together with a, not everywhere
defined, multiplication for which there is a set of unities $G^0$, two maps, source denoted by $s$ and target by $t$, from $G$ to $G^0$ so that the product $xy$ of two elements $x, y \in G$ exists if and only if $s(x) = t(y)$; every element $x \in G$ has a unique inverse $x^{-1}$, and one has $x(yz) = (xy)z$ whenever both members make sense. We refer to [R] for the fundamental structures and notations for groupoids.

Let’s denote $H = l^2(G)$, with the usual notations. One can define two $C^*$-quantum groupoids in duality acting on $H = l^2(G)$, namely $(C(G), \Gamma_G, \kappa_G, \epsilon_G)$, the commutative example, and $(\mathcal{R}(G), \hat{\Gamma}_G, \hat{\kappa}_G, \hat{\epsilon}_G)$, the symmetric example, where: $C(G)$ is the commutative involutive algebras of complex valued functions on $G$, $\mathcal{R}(G) = \{\sum_{x \in G} a_x \rho(x)\}$ is the right regular algebra of $G$ and $\rho(x)$ is the partial isometry given by the formula $(\rho(x)\xi)(t) = \xi(tx)$ if $x \in G^{s(t)}$ and $= 0$ otherwise. The two $*$-quantum groupoids structures on $S$ and $\hat{S}$ are given by:

- **Coproducts:**
  \[
  \Gamma_G(f)(x, y) = f(xy) \text{ if } x, y \text{ are composables and } f \in C(G),
  \]
  \[
  = 0 \text{ otherwise}
  \]
  \[
  \hat{\Gamma}_G(\rho(s)) = \rho(s) \otimes \rho(s)
  \]

- **Antipodes:**
  \[
  \kappa_G(f)(x) = f(x^{-1}), \quad \hat{\kappa}_G(\rho(s)) = \rho(s^{-1}) = \rho(s)^* \]

- **Counities:**
  \[
  \epsilon_G(f) = \sum_{u \in G^0} f(u), \quad \hat{\epsilon}_G(\rho(s)) = 1
  \]

To define the symmetric example, one also could consider the left regular representation of $G$: using the $*$-algebra $\mathcal{L}(G) = \{\sum_{s \in G} a_s \lambda(s)\}$ (the left regular algebra of $G$), where $\lambda(s)$ is the partial isometry given by the formula $(\lambda(s)\xi)(t) = \xi(s^{-1}t)$ if $t \in G^{r(s)}$ and $= 0$ otherwise.

### 2.3. Action of a $C^*$-quantum groupoid.

#### 2.3.1. Definition. ([NV1] chap 2) A (left) action of a $C^*$-quantum groupoid $A$ on a von Neuman algebra $M$ is any linear weakly continuous map: $A \otimes M \to M : a \otimes m \mapsto a \triangleright m$ defining on $M$ a left $A$-module structure and such that:

1. $a \triangleright (xy) = (a_{(1)} \triangleright x)(a_{(2)} \triangleright y)$, where $\Gamma(a) = a_{(1)} \otimes a_{(2)}$ with Sweedler notations,
2. $(a \triangleright x)^* = \kappa(a) \triangleright (x^*)$
3. $a \triangleright 1 = \epsilon^l(a) \triangleright 1$, and $a \triangleright 1 = 0$ iff $\epsilon^l(a) = 0$ (where $\epsilon^l(a) = \epsilon(1_{(1)})1_{(2)}$)

Thanks to the antipode, $M$ also becomes a right module ($m.a = \kappa(a) \triangleright m$). When such an action is given one can define the fixed point algebra as the von Neumann $M^A = \{m \in M | \forall a \in A, a \triangleright m = \epsilon^l(a) \triangleright m\}$. Also the von Neumann crossed product $M \rtimes A$ (see [NV1] chap 2) is the $\mathbb{C}$-vector space $M \otimes A$ where $M$ is the right $A_t$ module $M$ obtained via the right multiplication by the elements $a_t \triangleright 1$ ($a_t \in A_t$) and $A$ is the left $A_t$ module obtained by the left $A_t$ multiplication. So elements of $M \otimes A$ are the classes $[m \otimes a]$ with the identification: $m(a_t \triangleright 1) \otimes a \equiv m \otimes a_t a$. The multiplication and
the involution of $M \otimes A$ are given for any $m, m' \in M$ and $a, a' \in A$ by the formulas:

\[
[m \otimes a][m' \otimes a'] = [m(a_{(1)} \triangleright m') \otimes a_{(2)}a']
\]

\[
[m \otimes a]^* = [(a_{(1)}^* \triangleright m^*) \otimes a_{(2)}]
\]

2.4. C*-quantum groupoids in action. We suppose known the Jones s’ tower theory (see [GHJ]) and in this section we recall the tight relation between C*-quantum groupoids actions and depth two inclusions of type $II_1$-factors with finite index $\lambda^{-1} = [M : N]$, (see [NV1] for full detail).

Let $M_0 \subset M_1$ be such an inclusion.

Let $M_0 \subset M_1 \subset M_2 = < M_1, e_1 > \subset M_3 = < M_2, e_2 > \subset ...$ be the Jones construction. Let $A = M_0' \cap M_2$, $B = M_1' \cap M_3$, and let $\tau$ be the Markov trace of the inclusion; if one defines for every $a \in A$ and $b \in B$ the bracket:

\[
< a, b > = d\lambda^{-2}\tau(ae_2e_1b)
\]

where $d = \text{dim}(M_0' \cap M_1)$, this defines a non degenerate duality between $A$ and $B$, the following theorem is true:

2.4.1. Theorem. ([NV1]th 4.17 and chap 6), [D] 3.8.4

For every $b \in B$, let’s define $\Gamma_B(b), \epsilon_B(b), \kappa_B(b)$, such that for any $a, a' \in A$ one has:

\[
< aa', b > = < a \otimes a', \Gamma_B(b) >
\]

\[
\epsilon_B(b) = < 1, b >
\]

\[
< a, \kappa_B(b) > = < a^*, b^* >
\]

and let’s suppose that $\Gamma_B$ is multiplicative, then:

i) $(B, \Gamma_B, \kappa_B, \epsilon_B)$ is a C*-quantum groupoid,

ii) the map $\triangleright : B \otimes M_2 \hookrightarrow M_2 : b \triangleright x = \lambda^{-1}E_{M_2}(bxe_2)$ defines a left action of $B$ on $M_2$,

$M_1$ is the fixed point subalgebra $M_2^B$ and the map $\theta : [x \otimes b] \mapsto xb$ is an isomorphism between the crossed product $M_2 \rtimes B$ and $M_3$.

iii) using the above bracket leads to define a dual C*-quantum groupoid on $A$ and an action of $B$ on $A$ which is the restriction of the action $\triangleright$ to $A$.

2.4.2. Corollary. The map $[a \otimes b] \mapsto ab$ is an isomorphism between the crossed product $A \rtimes B$ and $M_0' \cap M_3$.

Proof: Clearly the map $[a \otimes b] \mapsto ab$ is an isomorphism, its image is included in $M_0' \cap M_3$; conversely, due to the depth two condition, the Jones projection $e_2$ is also the first Jones projection for the derived tower: $M_0' \cap M_1 \subset M_0' \cap M_2 \subset M_0' \cap M_3$, hence $M_0' \cap M_3$ is generated by $M_0' \cap M_2$ and $e_2$ so it is included in (and equal to) the image of the isomorphism.

3. Relative matched pairs of groups

Let’s now explain what we mean by a relative matched pair.
3.1. Relative matched pairs of groups.

3.1.1. Definition. Let $G$ be a group, two any subgroups $H, K$ of $G$ are said to be a relative matched pair if and only if $G = HK = \{hk/h \in H, k \in K\}$.

3.1.2. Remark and notations. A relative matched pair $H, K$ is a matched pair if and only if $H \cap K = \{e\}$ where $e$ is the unit of $G$. For the sake of simplicity, let’s denote $S = H \cap K$. Of course one can construct a lot of examples of relative matched pairs: if $H$ is any subgroup of $G$ then $H, G$ is a relative matched pair different from a matched pair if $H \neq \{e\}$. Let’s give a nice machinery to obtain examples:

3.1.3. Lemma (Frattini Argument, th 1.11.8 of [G]). Let $G$ be a finite group, let $N$ be a normal subgroup of $G$, let $P$ be a Sylow $p$-subgroup of $N$, then $G = NN_G(P)$, where $N_G(P)$ is the normalizer of $P$ in $G$.

3.1.4. Lemma and notations. Let $H, K$ be a relative matched pair in $G$, for any $(h, k)$ in $H \times K$ let’s denote $p_1(hk) = hS$ and $p_2(hk) = Sk$, this defines two maps $p_1 : G \to H|S$ and $p_2 : G \to S|K$ such that:

i) for any $g$ in $G$ and any $(h, k)$ in $p_1(g) \times p_2(g)$ there exists a unique $(h', k')$ in $p_1(g) \times p_2(g)$ verifying $g = hk' = h'k$

ii) for any $g, g'$ in $G$, one has: $p_1(g) = p_1(g')$ (resp.$p_2(g) = p_2(g')$) if and only if there exists $k \in K$ (resp.$h \in H$) such that $g' = gk$ (resp.$g' = hg$).

Proof: For any $g \in G$, there exists $(h, k)$ in $H \times K$ such that $g = hk$ and if $(h', k')$ in $H \times K$ verifies $hk = h'k'$, this exactly means there exists $t \in S$ such that $h' = ht$ and $k' = t^{-1}k$, the existence of $p_1$ and $p_2$ and the first assertion of the lemma follow. For any $g, g'$ in $G$, let $(h', k') \in H \times K$ such that $g = h'k'$, so if $p_1(g') = p_1(g)$ then there exists $k \in K$ such that $g' = h'k$, so $g' = (h'k^{-1})g$, conversely, if there exists $k \in K$ such that $g' = gk$ then $g' = h'k'k$, hence $h' \in p_1(g')$ so $p_1(g) = p_1(g')$. \qed

3.1.5. Remark and notation. As $K, H$ is also a relative matched pair, there exists also two maps $p_1' : G \to K|S$ and $p_2' : G \to S|H$, defined for any $(h, k)$ in $H \times K$ by $p_1'(kh) = kS$ and $p_2'(kh) = Sh$

3.2. Double crossed products associated with relative matched pairs.

3.2.1. Lemma. For any $(h, k)$ in $H \times K$, let $k \triangleright hS$ (resp.$h \triangleright kS$) be equal to $p_1(hk)$ (resp.$p_1'(hk)$), this defines a left action of group $K$ on the set $H|S$ and a left action of group $H$ on the set $K|S$.

Proof: Easy. \qed
3.2.2. Notations. Till the end of this article we shall consider an exhaustive family 
$I = \{k_1, k_2, ...\}$ (resp. $J = \{h_1, h_2, ...\}$) of representatives of $K|S$ (resp. $H|S$). We shall 
now extend the actions above:

3.2.3. Proposition. Let $\triangleright_i'$ be defined for any $h \in H$, $k_i \in I$ and $s \in S$ by:

$$h \triangleright_i' k_i s = k_j s \quad \text{where} \quad \{k_j\} = I \cap h \triangleright_i k_i S$$

then $\triangleright_i'$ is an action of $H$ on $K$ and for all $k \in K$, $s \in S$, one has: $h \triangleright_i'(ks) = (h \triangleright_i' k) s$
and $h \triangleright_i' s = s$, if $I'$ is an other exhaustive family, the actions $\triangleright_i'$ and $\triangleright_i$ are conjugate.

Proof: Of course, $\triangleright_i'$ is well defined, for any $k \in K$, $e \triangleright_i' k = k$, and for any $h' \in H$
and $s \in S$, let $k_i \in I$ and $\sigma \in S$, such that $k = k_i \sigma$, one has: $h \triangleright_i'(ks) = h \triangleright_i'(k_i \sigma s) = k_j s$
where $\{k_j\} = I \cap h \triangleright_i k_i S$, hence $h \triangleright_i'(ks) = (k_j \sigma s) = (h \triangleright_i' k) s$. If $s_0 = I \cap e S$
then $h \triangleright_i' s = h \triangleright_i'(s_0(s_0^{-1} s)) = s_0(s_0^{-1} s) = s$.

Now let $I^1, I^2$ be two exhaustive families, for $i = 1, 2$, and any $c \in K|S$, let $k_i^c = I^i \cap C$.
We can define a permutation $\phi$ of $K$ such that for all $c \in K|S$ and $s \in S$, one has:

$$\phi(k_i^c s) = k_i^c s$$

and $\phi$ realizes a conjugation between the two actions.

\[\square\]

3.2.4. Notations. i) By reversing $H$ and $K$, one can also extend the action $\triangleright_i$ to an 
action $\triangleright_j$, of $K$ on $H$, with the same properties as in proposition 3.2.3.

ii) For any $h \in H$ and $k \in K$ let $h \vartriangleleft_i k$ and $k \vartriangleright_j h$ be the unique element in $K$ and $H$
respectively such that: $hk = (h \vartriangleleft_i k) (h \vartriangleright_j k)$ and $kh = (k \vartriangleright_j h) (k \vartriangleleft_i h)$.

One must keep in mind that in general $\vartriangleleft_i$ and $\vartriangleright_j$ are not ( right) actions.

Let’s define a double crossed product with a relative matched pair. In fact one can extend the action $\triangleright_i'$ to the crossed product $C(K) \rtimes S$ of $K$ by the right action of $S$, 
which is the $\ast$-algebra generated by a group of unitaries $(\rho(s))_{s \in S}$ and a partition of
the unity $(\chi_k)_{k \in K}$ with the commutation relations: $\rho(s)\chi_k = \chi_{ks^{-1}}\rho(s)$.

3.2.5. Proposition. For any $h \in H$, $k \in K$ and $s \in S$, let’s define:
\[ \sigma^I_h(\rho(s)\chi_k) = \rho(s)\chi_{h\circ' k} \]

then \((\sigma^I_h)_{h \in H}\), is an action of \(H\) on the crossed product \(C(K) \rtimes S\), if \(I'\) is an other exhaustive family, the actions \((\sigma^I_h)_{h \in H}\) and \((\sigma^{I'}_h)_{h \in H}\) are conjugate.

Proof:

Obviously for any \(h \in H\), \(\sigma^I_h\) is a well defined linear endomorphism on \(C(K) \rtimes \rho S\). For any \(k \in K\) and \(s \in S\), due to 1), one has: \(\sigma^I_h(\rho(s)\chi_k \rho(s)^*) = \sigma^I_h(\chi_{ks^{-1}}) = \chi_{(h_s'k_{s^{-1}})h} = \rho(s)\sigma^I_h(\chi_k)\rho(s)^*\), the proposition follows.

\[\square\]

So the double crossed product \((C(K) \rtimes S) \rtimes H\) can be viewed as the \(*\)-algebra generated by the families \((\rho(s))_{s \in S}\), \((\chi_k)_{k \in K}\) and a group of unitaries \((V_h)_{h \in H}\) with the additional commutation relations: \(\rho(s)V_h = V_h\rho(s)\) and \(V_h\chi_k = \chi_{h_s'k_{s^{-1}}}V_h\). Also one can extend the action \(\bowtie\) to an action \((\sigma^I_k)_{k \in K}\) of \(K\) on the crossed product \(C(H) \rtimes \rho S\), with the same properties.

3.3. Double groupoid structures and quantum groupoids structures associated with relative matched pairs.

3.3.1. Definition. Let \(\mathcal{T}\) be the set : \(\{k | h, h' \in H, k, k' \in K \ k h = k'h'\}\) and let \(\mathcal{T}'\) be the set : \(\{h | h, h' \in H, k, k' \in K \ k h = h'k'\}\)

Following N. Andruskiewitsch and S.Natale’s work \([AN2]\), we are able to define two double groupoid structures \(\mathcal{T}\) and \(\mathcal{T}'\). Let’s define ”horizontal” and ”vertical” products on the squares of \(\mathcal{T}\):

let \(\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}\) and \(\begin{array}{|c|c|} \hline a' & b' \\ \hline c' & d' \\ \hline \end{array}\) be in \(\mathcal{T}\), they will be composable for the horizontal product \(h^*_{K}\) if and only if \(b = c'\) and \(\begin{array}{|c|c|} \hline a & b \\ \hline h & k \\ \hline \end{array}\) \(b' = \begin{array}{|c|c|} \hline a & a' \\ \hline b & b' \\ \hline \end{array}\), they will be composable for the vertical product \(v^*_{H}\) if and only if \(d = a'\) and

\(\begin{array}{|c|c|} \hline c & b \\ \hline d & d' \\ \hline \end{array}\) \(b' = \begin{array}{|c|c|} \hline c & c' \\ \hline d & d' \\ \hline \end{array}\) \(bb'\)

One easily sees that \(\mathcal{T}\) is a groupoid with basis \(H\) for the horizontal product and a groupoid with basis \(K\) for the vertical one, of course there is analogue properties for
\( T' \), but its structures also comes from the transpose map: \( T \rightarrow T' \) defined by:

\[
\begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ d & b \end{pmatrix}
\]

3.3.2. **Remark.** Due to lemma 3.1.4, the corner maps associated with these double groupoids (paragraph 1.4 of [AN2]) are all constant and equal to \( |S| \).

3.3.3. **Notation.** For any \( t \in T \) (resp. \( t' \in T' \)) let’s denote by \( t^-h \) (resp. \( t'^-h \)) and \( t^-v \) (resp. \( t'^-v \)) the inverse of \( t \) (resp.\( t' \)) for the horizontal and vertical product respectively. Let’s also denote \( t^-hv \) the double inverse that is \( t'^hv = (t^-h)^-v = (t^-v)^-h \) and let’s denote \( t'^hv \) the similar object for \( t' \).

Let \( \mathbb{C}T \) (resp. \( \mathbb{C}T' \)) be the \( \mathbb{C} \) vector space with canonical basis \( T \) (resp. \( T' \)) then the horizontal product on \( T \) gives a natural structure of \( \ast \)-algebra to \( \mathbb{C}T \) (resp. \( \mathbb{C}T' \)) with a canonical duality bracket given for any \( x \in T \) and \( x' \in T' \) by:

\[
<x, x'> = \begin{cases} 
|S| & \text{if } x' = x^t \\
0 & \text{otherwise} 
\end{cases}
\]

3.3.4. **Lemma.** For any \( h, h', h_1, h'_1 \in H \) and \( k, k', k_1, k'_1 \in K \), one has:

\[
\begin{pmatrix} k_1 \\ k'_1 \end{pmatrix} h_1 = \begin{pmatrix} k \\ k' \end{pmatrix} h \quad \text{if and only if} \quad \begin{cases} 
hk = k_1h_1 \\
h = h'_1 \\
k' = k_1 
\end{cases}
\]

**Proof:** If \( \begin{pmatrix} k_1 \\ k'_1 \end{pmatrix} h_1 = \begin{pmatrix} k \\ k' \end{pmatrix} h \), then \( \begin{cases} 
hk = k_1h_1 \\
h = h'_1 \\
k' = k_1 
\end{cases} \)

Conversely if \( \begin{cases} 
hk = k_1h_1 \\
h = h'_1 \\
k' = k_1 
\end{cases} \) then \( \begin{cases} 
hk = k_1h_1 = h'_1k'_1 \\
k'k' = hk = k_1h_1 \\
h = h'_1 \\
k' = k_1 
\end{cases} \)

so \( \begin{cases} 
k = k'_1 \\
h' = h_1 \\
h = h'_1 \\
k' = k_1 
\end{cases} \) that means \( \begin{pmatrix} k \\ k' \end{pmatrix} h_1 = \begin{pmatrix} k_1 \\ k'_1 \end{pmatrix} h \) . \( \blacksquare \)
3.3.5. Theorem (see also [AN2]). The bracket below gives to $\mathbb{C}T$ and $\mathbb{C}T'$ structures of $\ast$- quantum groupoids in duality, for any $t \in T$, one has:

\[
\Gamma(t) = \frac{1}{|H \cap K|} \sum_{t_1^{h_1} t_2^{h_2} = t} t_1 \otimes t_2
\]

\[
\kappa(t) = t^{-h
}
\]

\[
\epsilon(t) = \begin{cases} 
|H \cap K| \text{ if } t \text{ is of the form } e \begin{array}{c}
h
\end{array} e \\
0 \text{ otherwise }
\end{cases}
\]

Proof: This is simple calculations. \(\square\)

3.3.6. Remarks. Due to remark 3.3.2, these structures are in tight relation with [AN2] 2.1. The bracket between $\mathbb{C}T$ and $\mathbb{C}T'$ allows us to define a canonical left action of the quantum groupoid $\mathbb{C}T$ on the von Neumann algebra $\mathbb{C}T'$.

For any $x \in T$ and any $x_1', x_2' \in T'$, let's define $x_1' \triangleright x$ to be the unique element in $T$ such that:

\[
< \Gamma(x), x_2' \otimes x_1' >= < x, x_2' \begin{array}{c}
h
\end{array} x_1' >= x_1' \triangleright x, x_2'
\]

It's very easy to see that for any $y \in T'$ and $x \in T$:

\[
y \triangleright x = \begin{cases} 
(y')^{-v_1} \begin{array}{c}
h
\end{array} x \text{ if this product exists }\\
0 \text{ otherwise }
\end{cases}
\]

We can give an operator algebra interpretation of these structures.

3.3.7. Proposition. The $\ast$-algebras $(C(K) \rtimes S) \rtimes H$ and $(C(H) \rtimes S) \rtimes K$ are respectively isomorphic to the $\ast$-algebras $\mathbb{C}T$ and $\mathbb{C}T'$.

Proof: For any $(h, k, s) \in H \times K \times S$, let $t(V_h \chi_{k}\rho(s)) = h \begin{array}{c}
(k \triangleright k)
\end{array} s$, for any $(h', k', s') \in H \times K \times S$, one has:

\[
V_h \chi_{k}\rho(s)V_{h'} \chi_{k'}\rho(s') = \delta_{k s, h' h' k'}V_{h h'} \chi_{k' s^{-1}} \rho(s s'), \text{ and } (V_h \chi_{k}\rho(s))^* = \rho(s^{-1}) \chi_{k'} V_{h^{-1}} = V_{h^{-1}} \chi_{(h' k' k)s} \rho(s^{-1}),
\]

from this we can deduce that:

\[
t((V_h \chi_{k}\rho(s))(V_{h'} \chi_{k'}\rho(s'))) = t(V_h \chi_{k}\rho(s)) \begin{array}{c}
h
\end{array} t(V_{h'} \chi_{k'}\rho(s')) \text{ and also:}
\]

\[
\]
\[
t((V_\chi k \rho(s))^*) = t(\rho(s^{-1})\chi_{h' \vartriangleright ks} V_{h^{-1}}) = t((V_\chi k \rho(s))^*)^{h^{-1}} \]
\[
= k s^{\square (h' \vartriangleright \kappa s)^{-1} i} \]
\[
= t((V_\chi k \rho(s))^*)^{h}.
\]

The proposition follows \[\square\]

We shall see in next chapter, using a suitable inclusion of von Neumann algebras, that the von Neumann algebra crossed product \(\mathbb{C} T \rtimes \mathbb{C} T'\) is isomorphic to \(\mathbb{C}[H \cap K] \otimes \mathcal{L}(\mathbb{C}^{[H \cap K]})\).

3.4. ([BH] chap 4 or [HS]) Quantum groupoids associated with inclusions of von Neumann algebras coming from relative matched pairs \(H, K\). In [BH] is given a very deep study of inclusions of the form \(R^H \subset R \rtimes K\) where \(H\) and \(K\) are any subgroups of a group \(G\) acting properly and outerly on the hyperfinite type \(II_1\) factor \(R\), in such a way that we can identify \(G\) with a subgroup of \(Out R\), in particular there is here no ambiguity for the inclusion \(H \cap K \subset Out R\); let’s call \(\alpha\) the action of \(K\) and \(\beta\) the one of \(H\) here these actions coincide on \(S = H \cap K\). In [BH], it is proved that this inclusion is finite depth if and only if the group generated by \(H\) and \(K\) in \(Out R\) is finite, and, in that situation, it is irreducible and depth two when \(H, K\) is a matched pair. In this section, using any relative matched pair, we obtain still depth two but no more irreducible inclusions, so using [NV2] or [EV] or [D], these inclusions come from quantum groupoids actions.

Let’s give some facts about these inclusions of the form \(R^H \subset R \rtimes K\). First, one can observe that \(Out R\), can be identified, using Sauvageot Connes fusion multiplication and the contragredient procedure, to a group of \(R - R\) bimodules over \(L^2(R)\); and using the sum operation on bimodules, \(Out R\) has also a second operation with a distributivity property. We shall follow the same notations as in [BH] and we identify any element of \(Out R\) to the bimodule associated with this element. Let \(\gamma =_{R^H} L^2(R)_{R \times K}\) and \(\chi =_{R} L^2(R \times K)_{R \times K}\), then for any \(h \in H\) and \(k \in K\) one has:

\[
(1) \quad \gamma h = \gamma, \quad k\chi = \chi
\]
\[
(2) \quad \overline{\gamma} = \bigoplus_{h \in H} h, \quad \chi\overline{\chi} = \bigoplus_{k \in K} k.
\]

3.4.1. Lemma. Let \(H, K\) be two finite subgroups of \(Out R\) such that \(HK = \{hk/h \in H, k \in K\}\) is a subgroup of \(Out R\), then for any \(g\) in \(G = HK\), one has \(\gamma g\chi = \gamma\chi\).

Proof: For any \(g\) in \(G = HK\), there exist \(h \in H\) and \(k \in K\) such that \(g = hk\), hence due to (1), one has \(\gamma g\chi = \gamma hk\chi = \gamma\chi\). \[\square\]
3.4.2. Lemma. Let $M_0 \subset M_1 \subset M_2 \subset \ldots$ a Jones tower of type $II_1$ factors such that the bimodule $M_0L^2(M_2)_{M_1}$ is an ampliation of $M_0L^2(M_1)_{M_1}$, i.e there exists an integer $n$ such that $M_0L^2(M_2)_{M_1}$ is isomorphic to $M_0 \otimes_1 (L^2(M_1) \otimes \mathbb{C}^n)_{M_1} \otimes_1$ then the inclusion $M_0 \subset M_1$ is depth two.

**Proof:** In this lemma’s conditions, let $x$ be any irreducible $M_0'$-$M_1$ subbimodule of $\rho =_{M_0} L^2(M_2)_{M_1}$: it appears $n$ times as an irreducible subbimodule of $\rho \rho \rho =_{M_0} L^2(M_1)_{M_1}$, let $x_1, x_2, \ldots x_n$ these subbimodules. In Bratelli’s diagramm, there exists an irreducible subbimodule $y$ of $\rho \rho =_{M_0} L^2(M_1)_{M_0}$ which has to be deleted in the principal graph and is connected to $x$. Due to the Frobenius reciprocity theorem, $y$ is connected to $x_1, x_2, \ldots x_n$, hence any of these has to be deleted in the principal graph; but any irreducible subbimodule of $M_0L^2(M_1)_{M_1}$ is of this form for a good choice of $x$. So the principal graph stops at level two, which is the definition of depth two. \(\square\)

3.4.3. Theorem. Let $H, K$ be two finite subgroups of $\text{Out} R$ such that $HK = \{hk/h \in H, k \in K\}$ is a subgroup of $\text{Out} R$, then the inclusion $R^H \subset R \rtimes K$ is depth two. Let $M_2$ be the third element of Jones’s tower of inclusion $R^H \subset R \rtimes K$, then there exists a quantum groupoid structure on $(R^H)' \cap M_2$ over the basis $(R^H)' \cap R \rtimes K$ and an action $\gamma$ of $(R^H)' \cap M_2$ on $R \rtimes K$ in such a way that the inclusion $R^H \subset R \rtimes K \subset M_2$ is isomorphic to $(R \rtimes K)_{\gamma} \subset R \rtimes K \subset (R \rtimes K) \rtimes ((R^H)' \cap M_2)$.

**Proof:** Let $(M_k)_{k \in K}$ be Jones tower of inclusion $R^H \subset R \rtimes K$, due to chap.2 and 3 of [BH], the bimodule $M_0L^2(M_2)_{M_1}$ is equal to $\gamma(\chi \chi \gamma)\chi$, but using (4) and lemma 3.4.1, one has

$$\gamma(\chi \chi \gamma)\chi = \gamma(\bigoplus_{h \in H, k \in K} hk)\chi = |S|\gamma(\bigoplus_{g \in HK} g)\chi = |S||HK|\gamma \chi = |H||K|\gamma \chi$$

So, due to lemma 3.4.2, the inclusion $R^H \subset R \rtimes K$ is depth two and one can apply theorem 2.4.1 or [D] chap 3 or [E1] theorem 9.2 to conclude. \(\square\)

3.4.4. Corollary. In the conditions of theorem 3.4.3 the von Neumann algebra $M_0 \cap M_3$ is isomorphic to $\mathcal{L}(H \cap K) \otimes \mathcal{L}(\mathbb{C}^{|H|\cap |K|})$.

**Proof:** We have seen that $\gamma(\chi \chi \gamma)\chi = |H||K|\gamma \chi$, which proves the corollary. \(\square\)

4. The $C^*$-Quantum Groupoid Structure Associated with Inclusions of the Form $R^H \subset R \rtimes K$

Till the end of this section, we deal with a finite relative matched pair $H, K$ acting properly and outerly on $R$. In [HS], a complete description of Jones tower for the inclusion $R^H \subset R \rtimes K$ is given in the particular case when $H, K$ is a finite matched
pair of groups, a good part of this work remains true for a relative matched pair, but one
must keep in mind that the actions considered in [HS], are not exactly the one we use
but are tightly related: the action of $H$ on $K$ is for example given by $h.k = (h \triangleright k)^{-1}$.
To be short, we shall denote $M_0 = R^H$, $M_1 = R \rtimes K$, and $M_2$ will be the third element
of the basic construction $M_0 \subset M_1 \subset M_2$. All the crossed product here will be defined
"in the right manner", for example $R \rtimes K$ is the vector space (which is a $*$-algebra) in
$R \otimes \mathcal{L}(l^2(K))$ generated by the products $\alpha(r)(1 \otimes \rho(k))$, where $\alpha(r)$ is $r$ viewed in
the crossed product as the fonction: $k' \mapsto \alpha_k(r)$ i.e $\alpha(r) = (\sum_{k' \in K} \alpha_k(r) \otimes \chi_{k'})$, where $\chi_{k'}$ is
the characteristic fonction of $\{k'\}$.

4.1. The $*$-algebra structure of $M'_0 \cap M_2$. One can easily see what are the basis
$(R^H)' \cap R \rtimes K$ of quantum groupoid $M'_0 \cap M_2$.

4.1.1. Lemma(BH or HS). The algebra $(R^H)' \cap R \rtimes K$ is isomorphic to the group
algebra $\mathcal{L}(S)$.

Proof: If $u_k$ and $v_h$ are canonical implementations of $\alpha$ and $\beta$ on $L^2(R)$, one can
suppose $u_x = v_x$ for any $x$ in $S$, these $u_x$ generate a $*$-algebra isomorphic to $\mathbb{C}[S]$ and
are clearly in $(R^H)' \cap R \rtimes K$, on the other hand, using the computation in the proof
of 4.1 in [BH], one has: $\dim((R^H)' \cap R \rtimes K) = \text{card}S$. The lemma follows. \(\square\)

4.1.2. Remark. The basis $(R^H)' \cap R \rtimes K$ of quantum groupoid $(R^H)' \cap R \rtimes K$ are
commutative if and only if $S$ is abelian. Hence, when $S$ is non abelian, this quantum
groupoid structure does not come from a matched pair of groupoids (see [Val2]).

Now let’s give a description of the two first steps of the basic construction for the
inclusion $R^H \subset R \rtimes K$. For our computations, it will be more convenient to express
as soon as possible all the algebras in $M_3$. As there is two actions, namely $\alpha$ and $\beta$
of respectively $K$ and $H$ on $R$, there is also two actions $\alpha^1$ and $\beta^1$ of $K$ and $H$ on
$R^1 = R \otimes \mathcal{L}(l^2(K))$. As well known (see [Val3] th 5.3 for a very old reference...), one
can identify $R^1$ with the double crossed product $R \rtimes K \rtimes \hat{K}$, $\alpha^1$ is the bidual action
$\hat{\alpha}$, so $\alpha^1 = \alpha \otimes \text{Ad}\lambda$ the fixed points algebra of which is $M^1$, and $\beta^1$ is the action of
$H$ on $R^1 = R \otimes \mathcal{L}(l^2(K))$ defined by $\beta^1 = \beta \otimes 1$. Let $(w_h)_{h \in H}$ be a group of unitaries
of $R^1 \rtimes H$ implementing $\beta^1$, hence $R^1$ is the set of sums: $\sum_{k,k' \in K} x_{k,k'} \otimes \rho(k)\chi_{k'}$, where
$x_{k,k'}$ is in $R$, and $R^1 \rtimes H$ (viewed in $L^2(R^1)$) is the set of sums: $\sum_{h \in H; k,k' \in K} (x_{k,k'},h \otimes
\rho(k)\chi_{k'})w_h$, where $x_{k,k'}$ is a non zero element in $R$, $\chi_k$ is the multiplication operator by the caracteristic fonction of $\{k\}$. In fact, for any algebraic basis $(x_i)_{i \in I}$ of $R$
the family $((x_i \otimes \rho(k)\chi_{k'})w_h)_{i \in I, h \in H; k,k' \in K}$ is an algebraic basis for $R^1 \rtimes H$, also the
family $((x_i \otimes \lambda(k)\chi_{k'})w_h)_{i \in I, h \in H; k,k' \in K}$ is an algebraic basis for $R^1 \rtimes H$. Let $\tau$ be
the normalized tracial state for the Jones ’s tower, and $\tilde{\tau}$ the one of $R^1 \rtimes H$, let
$E : R^1 \rtimes H \to R \rtimes K$ be the $\tilde{\tau}$ preserving conditional expectation, then by routine calculations, for any $k, k' \in K$, $x \in R$ and $h \in H$, one has:

\[
\tilde{\tau}(\alpha(x)(1 \otimes \rho(k)\chi_{k'})w_h) = \frac{1}{|K|} \tau(x)\delta_{k,e}\delta_{h,e}
\]

\[
E(\alpha(x)(1 \otimes \rho(k)\chi_{k'})w_h) = \frac{1}{|K|} \alpha(x)(1 \otimes \rho(k))\delta_{h,e}, \text{ where } \alpha(x) \text{ is } x \text{ viewed in } M_1 = R \rtimes K, \text{ i.e } \alpha(x) = \sum_{k_0} \alpha_{k_0}(x) \otimes \chi_{k_0}.
\]

4.1.3. Proposition (HS). With Jones’s notations, $(M_2, e_1)$ can be identified with $(R \otimes \mathcal{L}(l^2(K)) \rtimes H, \frac{1}{|H|}(1 \otimes \chi_e) \sum_{h \in H} w_h)).$

Proof: Let $e$ be the projection in $M_0 (= R^H)$ equal to $\frac{1}{|H|}(1 \otimes \chi_e) \sum_{h \in H} w_h$. For any $r$ in $M_0$, let $\alpha(r)$ be $r$ viewed in $R^1 \rtimes H$ (as a von Neumann subalgebra of $R \otimes \mathcal{L}(l^2(K))$ i.e: $\alpha(r) = \sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0}$.

As $r$ is a fixed point for $\beta$, one has:

\[
e \alpha(r) - \alpha(r)e = \frac{1}{|H|}(1 \otimes \chi_e) \sum_{h \in H} w_h \sum_{k_0} (\alpha_{k_0}(r) \otimes \chi_{k_0}) - (\sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0}) \frac{1}{|H|}(1 \otimes \chi_e) \sum_{h \in H} w_h
\]

\[
= \frac{1}{|H|} \sum_{h \in H} w_h (r \otimes \chi_e) - (r \otimes \chi_e) w_h
\]

So we have $[e, \alpha(r)] = 0$, and:

\[
E(e) = \frac{1}{|H||K|} 1 = [R^1 \rtimes H : R^1]^{-1}[R \rtimes K \rtimes \hat{K} : R \rtimes \hat{K}]^{-1} 1
\]

\[
= [R^1 \rtimes H : M_1]^{-1} 1
\]

and also:

\[
E(e) = \frac{1}{|H||K|} 1 = [R : R^H]^{-1}[M_1 : R]^{-1} 1 = [M_1 : M_0]^{-1} 1
\]

\[
= [M_2 : M_1]^{-1} 1
\]

Hence, one can apply the equivalence $1^0$ and $2^0$ of Proposition 1.2 in [PP] to conclude that $(M_2, e_1)$ can be identified with $(R \otimes \mathcal{L}(l^2(K)) \rtimes H, \frac{1}{|H|}(1 \otimes \chi_e) \sum_{h \in H} w_h))$. \qed
Let’s give a description of the second step of the basic construction for the inclusion $R^H \subset R \rtimes K$. One can consider the double crossed product $(R^1 \rtimes H) \rtimes \hat{H} (\ R^1 = R \otimes \mathcal{L}(l^2(H))))$ which can be identified with $R^2 = R^1 \otimes \mathcal{L}(l^2(H)) = R \otimes \mathcal{L}(l^2(K)) \otimes \mathcal{L}(l^2(H))$, we can define an action on $R^2$ by $\alpha^2 = \alpha_1 \otimes 1 = \alpha \otimes Ad(\lambda) \otimes 1$ and consider a group of unitaries $(v_k)_{k \in K}$ implementing $\alpha^2$ such that the crossed product $R^2 \rtimes K = (R \otimes \mathcal{L}(l^2(K)) \otimes \mathcal{L}(l^2(H))) \rtimes K$ has a canonical basis of the form $((y_j \otimes \rho(h)\chi_{h'})v_k)_{i \in J, h,h' \in H; k \in K}$ for any basis $(y_j)_{j \in J}$ of $R^1 = R \otimes \mathcal{L}(l^2(H))$.

4.1.4. Proposition(HS). With Jones’s notations, $(M_3, e_2)$ can be identified with $(R \otimes \mathcal{L}(l^2(K)) \otimes \mathcal{L}(l^2(H)) \rtimes K, \frac{1}{|K|} (1 \otimes 1 \otimes \chi_e) \sum_{k \in K} v_k)$.

Proof: As $M_1$ is just the fixed points algebra of $\alpha^1$, the demonstration is the same as proposition 4.1.3.

Let’s see, with our identifications, how $M_2$ is included in $M_3$: any $z \in M_2$, belonging to $R \otimes \mathcal{L}(l^2(H))$, must be viewed in $M_3$ as the operator $\sum_{h \in H} \beta^1_h(z) \otimes \chi_h$ in $R \otimes \mathcal{L}(K) \otimes \mathcal{L}(H)$, so any element of $M_2$ of the form $1 \otimes x$ where $x$ is in $\mathcal{L}(l^2(K))$, is seen as $1 \otimes x \otimes 1$ in $M_3$ and $w_s$ is just the operator $1 \otimes 1 \otimes \rho(s)$. One can identify $\tau$ with the "Markov trace" $\tau$. If $M_3$ is viewed as a triple crossed product $((R^1 \rtimes H) \rtimes \hat{H}) \rtimes \hat{K}$, for any element $r^1 \in R^1 h, h' \in H$ and $k \in K$: $\tau(\beta^1(r^1)(1 \otimes 1 \otimes \rho(h)\chi_{h'})v_k) = \frac{1}{|M|} \tau(r^1)\delta_{h,e}\delta_{k,e}$.

4.1.5. Notations. For any $h \in H, k \in K, k' \in p_2(k^{-1}h^{-1}k)$, let’s define $w_{k,h,k'} = 1 \otimes \rho(k')\chi_k \otimes \rho(h)$.

4.1.6. Lemma. i) $\{w_{k,h,k'}/h \in H, k \in K, k' \in p_2(k^{-1}h^{-1}k)\}$ defines a basis of $M_0 \cap M_2$.

ii) for any $h \in H, k \in K, k' \in p_2(k^{-1}h^{-1}k)$, $w_{k,h,k'}$ is a partial isometry with initial support $1 \otimes \chi_k \otimes 1$ and final support $1 \otimes \chi_{kk^{-1}} \otimes 1$.

Proof: Let’s make the computations in $M_2$. Let $r$ be any element in $M_0 = R^H$ and $y = \sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0}$ the same viewed in $M_2 = R^1 \rtimes H$, for any element $x = \sum_{h \in H; k,k' \in K} (x_{k,k',h} \otimes \rho(k)\chi_{k'} \otimes \chi_k \otimes \rho(h))w_h$, $x$ commutes with $M_0$ means that for any $r \in M_0$, one has:
\[
\left( \sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0} \right) \sum_{h \in H; k, k' \in K} (x_{k, k', h} \otimes \rho(k) \chi_{k'}) w_h = \\
\sum_{h \in H; k, k' \in K} (x_{k, k', h} \otimes \rho(k) \chi_{k'}) w_h \left( \sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0} \right)
\]

On the one hand, one has:
\[
\left( \sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0} \right) \sum_{h \in H; k, k' \in K} (x_{k, k', h} \otimes \rho(k) \chi_{k'}) w_h = \\
\sum_{h \in H; k, k' \in K} (\alpha_{k_0}(r)x_{k, k', h} \otimes \chi_{k_0}\rho(k)\chi_{k'}) w_h \\
= \sum_{h \in H; k, k' \in K} (\alpha_{k_0}(r)x_{k, k', h} \otimes \rho(k)\chi_{k_0}\chi_{k'}) w_h \\
= \sum_{h \in H; k, k' \in K} (\alpha_{k_0}(r)x_{k, k_0k, k'} \otimes \rho(k)\chi_{k_0}) w_h \\
= \sum_{h \in H; k, k' \in K} (\alpha_{k_0\alpha^{-1}}(r)x_{k, k_0, h} \otimes \rho(k)\chi_{k_0}) w_h
\]

On the other hand, one has:
\[
\sum_{h \in H; k, k' \in K} (x_{k, k', h} \otimes \rho(k)\chi_{k'}) w_h \left( \sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0} \right) = \\
\sum_{h \in H; k, k' \in K} (x_{k, k', h} \otimes \rho(k)\chi_{k'}) (\sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0}) w_h \\
= \sum_{h \in H; k, k' \in K} (x_{k, k', h} \otimes \rho(k)\chi_{k'}) (\sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0}) w_h \\
= \sum_{h \in H; k, k' \in K} (x_{k, k_0h, h}\beta_h(\alpha_{k_0}(r)) \otimes \rho(k)\chi_{k_0}) w_h \\
= \sum_{h \in H; k, k_0, k' \in K} (x_{k, k_0h, h}\beta_h(\alpha_{k_0}(r)) \otimes \rho(k)\chi_{k_0}) w_h
\]

So \( x \) is in \( M'_0 \cap M_2 \) if and only if for any \( h \in H; k_0, k \in K \) and any \( r \in R^H \), one has:
\( \alpha_{k_0\alpha^{-1}}(r)x_{k, k_0, h} = x_{k, k_0h, h}\beta_h(\alpha_{k_0}(r)) \).
Applying \( \alpha_{k_0\alpha^{-1}} \), this is equivalent to:
\( r\alpha_{k_0\alpha^{-1}}(x_{k, k_0, h}) = \alpha_{k_0\alpha^{-1}}(x_{k, k_0, h})(\alpha_{k_0\alpha^{-1}}\beta_h\alpha_{k_0})(r) \).

So, using lemma 3.1 of [HS], for any \( h \in H; k_0, k \in K \) such that \( x_{k, k_0, h} \neq 0 \), there exist \( \lambda \in \mathbb{C} - \{0\} \), \( v_{h, k_0, k} \) in \( \mathcal{U}(R) \) (the set of unitaries of \( R \)) and \( h' \) in \( H \) such that: \( \alpha_{k_0\alpha^{-1}}(x_{k, k_0, h}) = \lambda v_{h, k_0, k} \) and \( Ad(v_{h, k_0, k})\alpha_{k_0\alpha^{-1}}\beta_h\alpha_{k_0} = \beta_{h'} \). Hence one has:
\( Ad(v_{h, k_0, k}) = \beta_{h'}\alpha_{k_0\alpha^{-1}}\beta_h\alpha_{k_0}\alpha^{-1} \), this equality in \( G \) means:
\( v_{h, k_0, k} \in Z(R)(=\mathbb{C}) \) and \( h'k_0^{-1}h^{-1}k_0 k^{-1} = e \), which are exactly the two conditions:
\( x_{h, k_0, k} \in \mathbb{C} \) and \( k \in p_2(k_0^{-1}h^{-1}k_0) \).

ii) This is an easy computation.
4.1.7. **Corollary.** i) The family \((1 \otimes \rho(s) \chi_k \otimes 1)_{s \in H \cap K, k \in K}\) is a linear basis for a \(*\)-subalgebra of \(M'_0 \cap M_2\) isomorphic to the crossed product \(C(K) \rtimes (H \cap K)\),

ii) the family \((1 \otimes \rho(s) \otimes 1)_{s \in H \cap K}\) is a group of unitaries and a linear basis for \(M'_0 \cap M_1\),

iii) the family \((1 \otimes \lambda(s) \otimes \rho(s))_{s \in H \cap K}\) is a group of unitaries and a linear basis for \(M'_1 \cap M_2\).

**Proof:** The assertion i) is trivial. For any \(s \in S\) and \(k \in K\), one has: \(1 \otimes \rho(s) = \sum_{k_1 \in K} w_{k_1, e, s} \) and \(1 \otimes \chi_k = w_{k, e, e}\). For any \(s \in S\), one has: \(\{1 \otimes \rho(s) / s \in S\} \subset M_1 \cap (M'_0 \cap M_2) = M'_0 \cap M_1\) and for dimension reasons ii) follows. As the families \((1 \otimes \lambda(s))_{s \in S}\) and \((w_s)_{s \in S}\) are commuting groups of unitaries, so \(((1 \otimes \lambda(s))w_s)_{s \in S}\) is a group of unitaries, and one has: \(\lambda(s) = \sum_{k \in K} \rho(k^{-1}s^{-1}k)\chi_k\), hence: \((1 \otimes \lambda(s))w_s = \sum_{k \in K} w_{k,k^{-1}s^{-1}k}\) hence it is in \(M_2\). Let \(r\) be any element in \(R\) and \(y = \sum_{k_0} \alpha_{k_0}(r) \otimes \chi_{k_0}\) the same viewed in \(M_1 = R \rtimes H\), for any \(\sigma \in S\) one has:

\[
y(1 \otimes \rho(\sigma))w_s(1 \otimes \lambda(s)) = \sum_{k_0} (\alpha_{k_0}(r) \otimes \chi_{k_0}\rho(\sigma))w_s(1 \otimes \lambda(s))
\]

\[
= w_s(1 \otimes \lambda(s)) \left( \sum_{k_0} (\beta_{s^{-1}}\alpha_{k_0})(r) \otimes \chi_{s^{-1}k_0}\rho(\sigma) \right)
\]

\[
= w_s(1 \otimes \lambda(s)) \left( \sum_{k_0} (\alpha_{s^{-1}k_0})(r) \otimes \chi_{s^{-1}k_0}\rho(\sigma) \right)
\]

\[
= w_s(1 \otimes \lambda(s)) \left( \sum_{k_0} (\alpha_{k_0})(r) \otimes \chi_{k_0}\rho(\sigma) \right)
\]

\[
= w_s(1 \otimes \lambda(s))y(1 \otimes \rho(\sigma))
\]

So, the unitary \(w_s(1 \otimes \lambda(s))\) is in \(M'_1 \cap M_2\) and for dimension reasons iii) follows. □

4.1.8. **Lemma.** Using the notations of paragraph 3.1, for any \(h \in H\) and \(k \in K\), let \(k'_I(k, h) = (h \lor h')^{-1}k\), then \(k'_I(k, h)\) is in \(p_2(k^{-1}h^{-1}k)\) and for any \(h' \in H\) and \(k' \in K\), one has:

\[k'_I(h \lor h', k')k'_I(k', h') = k'_I(k', hh')\]
4.1.6. Proof: For any \( h \in H \) and \( k \in K \), one has: \( k^{-1}h^{-1}k = (hk)^{-1}k = (h \triangleright_k)^{-1}k \). For any \( k' \in K \), one has:
\[
k'(k'(h' \triangleright_I k', h)k'(k', h'))^{-1} = k'(k'(k', h'))^{-1}(k'(h' \triangleright_I k', h))^{-1} = (h' \triangleright_I k'')(h' \triangleright_I k', h) = h' \triangleright_I (h' \triangleright_I k') = (kh') \triangleright_I k'.
\]
So \( k'(h' \triangleright_I k', h)k'(k', h') = k'(k', hh') \).

4.1.9. Corollary and notations. For any \( h \in H \) and \( k \in K \), the element \( W_h^I = \sum_{k \in K} 1 \otimes \rho((h \triangleright_I k)^{-1}) k \chi_k \otimes \rho(h) \) is in \( M_0' \cap M_2 \).

Proof: This comes from lemmas 4.1.6 and 4.1.8.

4.1.10. Theorem. The family \( (W_h^I)_{h \in H} \) is a one parameter group of unitaries in \( M_0' \cap M_2 \), and it implements an action of \( H \) on the \( * \)-subalgebra of \( M_0' \cap M_2 \) generated by the family \( (1 \otimes \rho(s) \chi_k \otimes 1)_{s \in S, k \in K} \) which is equivalent to the action \( \sigma_f \) defined in theorem 3.2.5. The \( * \)-algebra \( M_0' \cap M_2 \) is isomorphic to the double crossed product \( (C(K) \rtimes (H \cap K)) \rtimes H \) and to \( \mathbb{C} \mathcal{T} \).

Proof: For any \( k_0 \in K \), and \( h \in K \), due to 4.1.6 ii), one has:
\[
W_h^I(1 \otimes \chi_{k_0} \otimes 1)W_h^{I*} = \sum_{k, k' \in H} w_{k, h, k'}(k, h)(1 \otimes \chi_{k_0}) w_{k, h, k'}^*(k', h) = w_{k_0, h, k'}(k_0, h) w_{k_0, h, k'}^*(k_0, h) = 1 \otimes \chi_{h'} \otimes 1
\]
Suming this equality for all \( k_0 \in K \) gives that \( W_h^I \) is a unitary.

Now for any \( s \in S \), \( k \in K \) and \( h \in H \), one easily sees that \( k'(ks^{-1}, h) = sk'(h, k)s^{-1} \), from this one deduces that:
\[
(1 \otimes \rho(s) \otimes 1)w_{k, h, k'}(k, h) = 1 \otimes \rho(sk'(k, h)) \chi_k \otimes \rho(h) = 1 \otimes \rho(sk'(k, h)s^{-1}s) \chi_k \otimes \rho(h) = 1 \otimes \rho(h'(ks^{-1}, h) \rho(s) \chi_k \otimes \rho(h) = (1 \otimes \rho(k'(ks^{-1}, h) \chi_{ks^{-1}} \otimes \rho(h))(1 \otimes \rho(s) \otimes 1) = w_{ks^{-1}, h, k'}(ks, h)(1 \otimes \rho(s) \otimes 1)
\]
From this one deduces that:
\[
W_h^I(1 \otimes \rho(s) \otimes 1)W_h^{I*} = 1 \otimes \rho(s) \otimes 1
\]
Also, for any \( h, h' \) in \( H \), due to lemma 4.1.6 ii) and lemma 4.1.8, we have:
\[ W_h^I W_{h'}^I = \sum_{k,k' \in K} w_{k,h,k'_I(k,h)} w_{k',h',k'_I(k',h')} = \sum_{k' \in K} w_{h'v'_I(k',h')} \]

\[ \sum_{k' \in K} (1 \otimes \rho(k'_I(h'v'_I k', h))) \rho(h) \rho(h') \]

\[ = \sum_{k' \in K} (1 \otimes \rho(k'_I(h'v'_I k', h)) k'_I(k', h')) \chi_{h'} \rho(hh') \]

\[ = \sum_{k' \in K} (1 \otimes \rho(k'_I(h', hh'))) \chi_{h'} \rho(hh') = W_{hh'}^I \]

So the group of unitaries \((W_h^I)_{h \in I}\) implements an action on the \(\ast\)-algebra generated by the families \((1 \otimes \rho(s) \otimes 1)\) and \((1 \otimes \chi_k \otimes 1)\) which is clearly isomorphic to \(C(K) \times \rho \sigma^l\) and this action is equivalent to \(\sigma^l\).

For any \(s \in S, h \in H\) and \(k \in K\), one has: \((1 \otimes \rho(s) \otimes 1)W_h^I(1 \otimes \chi_k \otimes 1) = w_{h,h,sk'_I(k,h)}\), this proves that the crossed product is exactly \(M'_0 \cap M_2\) and so, by theorem 3.2.5, \(M'_0 \cap M_2\) is isomorphic to the crossed product \((C(K) \times \rho \sigma^l \times H)\), the theorem follows.

**4.1.11. Corollary and notations.** For any \(t \in \mathcal{T}\), let \((h,k,s)\) be the unique element in \(H \times K \times S\) such that \(t = h v'_I k \bigwedge_{(h'v'_I k')s}(h'v'_{k'}k)\), let \(\theta^I_t\) be the element in \(M'_0 \cap M_2\) equal to \(W^I_t(1 \otimes \chi_k \rho(s) \otimes 1)\), then the family \((\theta^I_t)_{t \in \mathcal{T}}\) is a basis of \(M'_0 \cap M_2\) such that for any \(t' \in \mathcal{T}\), one has:

\[ \theta^I_t \theta^I_{t'} = \begin{cases} 
\theta^I_{tt'} & \text{if } t \text{ and } t' \text{ are composable for } h \bigstar H \\
0 & \text{otherwise}
\end{cases} \]

\[ (\theta^I_t)^* = \theta^I_{t^{-1}} \]

**Proof:** Using theorem 4.1.10, this is just a reformulation of proposition 3.3.7.

**4.2. The \(\ast\)-algebra structure of \(M'_1 \cap M_3\).**

**4.2.1. Notations.** For any \(h \in H, k \in K\) and \(h' \in p'_2(h^{-1}k^{-1}h)\), let’s define \(k' = h'h^{-1}kh\) and \(v_{h,k,h'} = (1 \otimes \lambda(k'k^{-1}) \otimes \rho(h'))\chi_h v_k\).
4.2. Lemma. i) \( \{v_{h,k,h}/h \in H, k \in K, h' \in p_2(h^{-1}k^{-1}h)\} \) defines a basis of \( M'_1 \cap M_3 \).

ii) for any \( h \in H, k \in K, h' \in p_2(h^{-1}k^{-1}h) \), \( v_{h,k,h'} \) is a partial isometry with initial support \( 1 \otimes 1 \otimes \chi_h \) and final support \( 1 \otimes 1 \otimes \chi_{hh'}^{-1} \).

Proof: Let \( y \) be any element in \( M_1 \) and \( \sum_{h_1}^h \beta_{h_1}(y) \otimes \chi_{h_1} \) the same viewed in \( M_3 \). Let

\[
x = \sum_{k \in K; h,h' \in H} (x_{h,h',k} \otimes \rho(h)\chi_{h'}) v_k
\]

be any element of \( M_3 \), \( x \) commutes with \( M_1 \) means that for any \( y \in M_1 \), one has:

\[
\left( \sum_{h_1}^h \beta_{h_1}(y) \otimes \chi_{h_1} \right) \sum_{k \in K; h,h' \in H} (x_{h,h',k} \otimes \rho(h)\chi_{h'}) v_k = \\
\sum_{k \in K; h,h' \in H} (x_{h,h',k} \otimes \rho(h)\chi_{h'}) v_k \left( \sum_{h_1}^h \beta_{h_1}(y) \otimes \chi_{h_1} \right)
\]

On the one hand, one has:

\[
\left( \sum_{h_1}^h \beta_{h_1}(y) \otimes \chi_{h_1} \right) \sum_{k \in K; h,h' \in H} (x_{h,h',k} \otimes \rho(h)\chi_{h'}) v_k = \\
= \sum_{h_1,h,h',h'' \in H; k \in K} (\beta_{h_1}(y)x_{h,h',k} \otimes \chi_{h_1}\rho(h)\chi_{h'}) v_k \\
= \sum_{h_1,h,h' \in H; k \in K} (\beta_{h_1}(y)x_{h,h',k} \otimes \rho(h)\chi_{h_1}\chi_{h'}) v_k \\
= \sum_{h, h_0 \in H; k \in K} (\beta_{h_1}^{-1}(y)x_{h,h_0,k} \otimes \rho(h)\chi_{h_0}) v_k
\]

On the other hand, one has:

\[
\sum_{k \in K; h,h' \in H} (x_{h,h',k} \otimes \rho(h)\chi_{h'}) v_k \left( \sum_{h_1}^h \beta_{h_1}(y) \otimes \chi_{h_1} \right) = \\
\sum_{k \in K; h,h' \in H} (x_{h,h',k} \otimes \rho(h)\chi_{h'}) \alpha_k^1 \left( \sum_{h_1}^h \beta_{h_1}(y) \otimes \chi_{h_1} \right) v_k \\
= \sum_{k \in K; h,h' \in H} (x_{h,h',k} \otimes \rho(h)\chi_{h'}) \alpha_k^1 \left( \sum_{h_1}^h (\alpha_k^1 \beta_{h_1}^1)(y) \otimes \chi_{h_1} \right) v_k \\
= \sum_{k \in K; h,h' \in H} (x_{h,h',k} (\alpha_k^1 \beta_{h_1}^1)(y) \otimes \rho(h)\chi_{h'\chi_{h_1}}) v_k \\
= \sum_{k \in K; h,h_1 \in H} (x_{h,h_1,k} (\alpha_k^1 \beta_{h_1}^1)(y) \otimes \rho(h)\chi_{h_1}) v_k
\]
So $x$ is in $M'_1 \cap M_3$ if and only if for any $h, h_1 \in H; k \in K$ and any $y \in M_1$, one has: 

$$\beta^1_{h_1h^{-1}}(y)x_{h,h_1} = x_{h,h_1}(\alpha_k^1 \beta^1_{h_1})(y).$$

Applying $\beta^1_{hh_1^{-1}}$, this is equivalent to:

$$y\beta^1_{hh_1^{-1}}(x_{h,h_1}) = \beta^1_{hh_1^{-1}}(x_{h,h_1})(\beta^1_{h_11} \alpha_k^1 \beta^1_{h_1})(y)$$

As $\alpha^1 = \alpha \otimes \text{Ad} \lambda$, $\alpha^1$ is outer and using lemma 3.1 of [HS], for any $h, h_1 \in H; k \in K$ such that $x_{h,h_1,k} \neq 0$, there exist $\mu$ in $\mathbb{C} - \{0\}$, $\nu_{h,h_1,k}$ in $U(R \otimes \mathcal{L}(l^2(K)))$ and $k'$ in $K$ such that: 

$$\beta^1_{hh_1^{-1}}(x_{h,h_1,k}) = \mu \nu_{h,h_1,k} \text{ and } \text{Ad}(v_{h,h_1,k})\beta^1_{h_11} \alpha_k^1 = \alpha_k^1.$$ Hence one has:

$$\text{Ad}(v_{h,h_1,k})(r \otimes 1) = \alpha_k \beta^1_{h_1^{-1}} \alpha_k^{-1} \beta^1_{h_1} \otimes 1$$

this equality implies there exists an element $v_{h,h_1,k}^1 \in \mathcal{L}(l^2(K))$ such that $v_{h,h_1,k} = 1 \otimes v_{h,h_1,k}^1; k' h_1^{-1} k^{-1} h_1 = e$, and $\text{Ad}(v_{h,h_1,k}^1) = \text{Ad}(k'k^{-1})$ which are exactly the two conditions: $x_{h,h_1,k} \in \mathbb{C}(1 \otimes \lambda(k'k^{-1}))$ and $h \in p'_2(h_1^{-1} k^{-1} h_1)$.

ii) This is an easy computation.

4.2.4. Lemma. For any $h \in H$ and $k, k' \in K$, one has:

$$h'_j(k \triangleright j, k')h'_j(h, k) = h'_j(h, kk')$$

$$\langle k' \triangleright j (k \triangleright j) \rangle (k \triangleright j) = k' k \triangleright j$$

Proof: The first identity is just lemma 4.1.8 where one flips $H$ and $K$. Also, for any $h \in H$ and $k, k' \in K$, one has:

$$k' k \triangleright j = (k' k \triangleright j)^{-1} k' kh = (k' k \triangleright j)^{-1} k' (k \triangleright j)(k \triangleright j)$$

$$= (k' \triangleright j)^{-1} (k' \triangleright j)(k \triangleright j)(k' \triangleright j)(k \triangleright j)$$

$$= (k' \triangleright j)(k \triangleright j)(k \triangleright j)$$

□
4.2.5. Theorem. The family \((W^J_k)^*\) is a group of unitaries in \(M'_1 \cap M_3\), it implements an action of \(H\) on the sub \(*\)-algebra of \(M'_1 \cap M_3\) generated by the family \((1 \otimes \lambda(s) \otimes \rho(s))\chi_h)\) for \(s \in H \cap K, h \in K\) which is equivalent to the action \(\sigma^f\) defined in 4.2.3. The \(*\)-algebra \(M'_1 \cap M_3\) is isomorphic to the crossed product \((C(H) \rtimes^\rho (H \cap K)) \rtimes K\) and to \(\mathbb{C}T\).

Proof: For any \(h_0 \in H\), and \(k \in K\), due to 4.2.2 ii), one has:

\[
W^J_k (1 \otimes 1 \otimes \chi_{h_0}) W^J_k^* = \sum_{h,h^1 \in H} v_{h,k,h^1}(h,k)(1 \otimes 1 \otimes \chi_{h_0}) v^*_{h^1,k,h^1}(h,k) = v_{h_0,k,h^1}(h_0,k) v^*_{h_0,k,h^1}(h_0,k) = 1 \otimes 1 \otimes \chi_{kh_0}
\]

Suming this equality for all \(h_0 \in H\) gives that \(W^J_k\) is a unitary.

Now for any \(s \in S, k \in K\) and \(h \in H\), one easily sees that \(h^J(hs^{-1},k) = sh^J(h,k)s^{-1}\) and \(k \triangleleft (hs^{-1}) = s(k \triangleleft h)s^{-1}\), from this one deduces that:

\[
(1 \otimes \lambda(s) \otimes \rho(s))v_{h,k,h^1}(h,k) = (1 \otimes \lambda(s(k \triangleleft h)k^{-1}) \otimes \rho(sh^J(h,k))v_k
\]

\[
= (1 \otimes \lambda(s(k \triangleleft h)s^{-1}k^{-1}k^{-1}k^{-1}) \otimes \rho(sh^J(h,k)s^{-1}k^{-1}k^{-1})v_k
\]

\[
= (1 \otimes \lambda((k \triangleleft h)s^{-1}k^{-1}k^{-1}k^{-1}) \otimes \rho(h^J(hs^{-1},k)\rho(s))v_k
\]

\[
= (1 \otimes \lambda((k \triangleleft h)s^{-1}k^{-1}) \otimes \rho(h^J(hs^{-1},k)\chi_{hs^{-1}})(1 \otimes \lambda(ksk^{-1}) \otimes \rho(s))v_k
\]

\[
v_{hs^{-1},k,h^1}(hs^{-1},k)(1 \otimes \lambda(s) \otimes \rho(s))
\]

From this one deduces that:

\[
W^J_k (1 \otimes \lambda(s) \otimes \rho(s)) W^J_k^* = 1 \otimes \lambda(s) \otimes \rho(s)
\]

Now, for any \(k, k'\) in \(K\), due to 4.2.2 and 4.2.4, we have:

\[
W^J_{k'} W^J_k = \sum_{h,h_1 \in H} v_{h_1,k',h^1}(h_1,k') v_{h,k,h^1}(h,k) = \sum_{h \in H} v_{kh,k',h^1(kh,k')(h,k)} v_{h,k,h^1}(h,k)
\]
But for any $h \in H$, one has:

\[ v_{k \triangleright h, k', j}(k \triangleright j) v_{h, k, h', j}(h, k) = \]

\[ (1 \otimes \lambda((k' \triangleleft j) \triangleright_k h) k^{-1}) \otimes \rho(h'_j(k \triangleright_j h, k')) v_k (1 \otimes \lambda((k \triangleleft_j h) k^{-1}) \otimes \rho(h'_j(h, k)) \chi_h) v_k \]

\[ = (1 \otimes \lambda((k' \triangleleft j) \triangleright_k h) k^{-1}) \otimes \rho(h'_j(k \triangleright_j h, k')) v_k (1 \otimes \lambda((k \triangleleft_j h) k^{-1}) \otimes \rho(h'_j(h, k)) \chi_h) v_k \]

\[ = (1 \otimes \lambda((k' \triangleleft j) \triangleright_k h) k^{-1}) \otimes \rho(h'_j(k \triangleright_j h, k'))(1 \otimes \lambda((k' \triangleleft j) k^{-1}) \otimes \rho(h'_j(h, k)) \chi_h) v_{kk'} \]

\[ = (1 \otimes \lambda((k' \triangleleft j) \triangleright_k h) (k \triangleleft h)(k' k^{-1}) \otimes \rho(h'_j(k \triangleright_j h, k') h'_j(h, k)) \chi_h) v_{kk'} \]

One deduces that:

\[ W^J_k W^J_k = \sum_{h \in H} (1 \otimes \lambda((k' k \triangleleft h)(k' k^{-1}) \otimes \rho(h'_j(h, k' k^{-1}) \chi_h)) v_{kk'} = W^J_{kk'} \]

So the family $(W^J_k)_{k \in K}$ is a group of unitaries, which implements an action of $H$ on the sub $*$-algebra of $M'_1 \cap M_3$ generated by the family $(1 \otimes \lambda(s) \otimes \rho(s))_{s \in S, h \in K}$, as for any $s \in S$, $h \in H$ and $kK$, one has: $(1 \otimes \lambda(s) \otimes \rho(s)) W^J_k (1 \otimes 1 \otimes \chi_h) = v_{h, k, s h, k}$, this proves that the crossed product is exactly $M'_1 \cap M_3$ and so, by theorem 3.2.5 where $H$ and $K$ are flipped, $M'_1 \cap M_3$ is isomorphic to the crossed product $(C(K) \rtimes S) \rtimes H$. \qed

4.2.6. Corollary and notations. For any $t \in T'$, let $(k, h, s)$ be the unique element

\[ \begin{array}{c}
    k \\
    \triangleright_j h
  \end{array} \]

\[ s \]

in $K \times H \times S$ such that $t = \begin{array}{c}
    k \\
    \triangleright_j h
  \end{array} \]

\[ s \]

Let $\theta^J_t$ be the element in $M'_1 \cap M_3$

equal to $W^J_{k} (1 \otimes \lambda(s) \otimes \chi_h \rho(s))$, then the family $(\theta^J_t)_{t \in T'}$ is a basis of $M'_1 \cap M_3$ such that for any $t' \in T'$, one has:

\[ \theta^J_t \theta^J_{t'} = \begin{cases}
    \theta^J_{t \cdot h} & \text{if } t \text{ and } t' \text{ are composable for } k \\
    0 & \text{otherwise}
  \end{cases} \]

\[ (\theta_t)^* = \theta_{t^*} \]

\[ \theta^J_t \theta^J_{t'} = \begin{cases}
    \theta^J_{t \cdot h} & \text{if } t \text{ and } t' \text{ are composable for } k \\
    0 & \text{otherwise}
  \end{cases} \]

\[ (\theta_t)^* = \theta_{t^*} \]

Proof: Using theorem 4.2.5, this is just a reformulation of proposition 3.3.7 \qed

4.3. The co-algebras structures of $M'_0 \cap M_2$ and $M'_1 \cap M_3$. Let’s apply the results given in paragraph 2.4 to find co-algebras structures on $M'_0 \cap M_2$ and $M'_1 \cap M_3$. With
notations of paragraph 4.1, one can use the pairing with $M'_1 \cap M_3$ defined for any $a \in M'_0 \cap M_2$ and $b \in M'_1 \cap M_3$ by:

$$< a, b > = |H \cap K||H|^2|K|^2 \tau(ae_2e_1 b).$$

For $M'_0 \cap M_2$, the coproduct $\Gamma$, the antipod $\kappa$ and the counit $\epsilon$ are given by the following formulas:

$$\epsilon(a) = < a, 1 >$$
$$< \Gamma(a), b \otimes b' > = < a, bb' >$$
$$< \kappa(a), b > = < a^*, b^* >$$

One has equivalent formulas for $(M'_1 \cap M_3, \hat{\Gamma}, \hat{\kappa}, \hat{\epsilon})$.

4.3.1. **Remark.** The general bracket given in [NV1] or [D], uses a slightly more complicated formula, as we shall see later we are here in a simpler situation for which $\Gamma$ is multiplicative and one can apply theorem 4.17 of [NV1].

4.3.2. **Lemma.** For any $h, h' \in H$, $k, \in K$, $s, s' \in S$, one has:

$$(1 \otimes \rho(s) \chi_k \otimes 1)W_h W_{k_1}^{\epsilon} = (1 \otimes \lambda(s') \otimes \rho(s') \chi_{h'}) =$$

$$= |H|^{-1}|K|^{-1}v_{h^{-1}k_1^{-1}h'}(1 \otimes \rho(sk^{-1}(h^{-1} k_1^{-1} k) \chi_{h^{-1} h'} \lambda(s') \otimes \rho(hh') \chi_{h'}).$$

**Proof:** For any $h \in H$, $k, \in K$, $s \in S$, one has:

$$(1 \otimes \rho(s) \chi_k \otimes 1)W_h W_{k_1}^{\epsilon} =$$

$$= |K|^{-1}(1 \otimes \rho(s) \otimes 1)(1 \otimes \rho(k^{-1}(h^{-1} k_1^{-1} k) \chi_{h^{-1} h'} \lambda(s) \otimes \rho(h))(1 \otimes 1 \otimes \chi_{e}) (\sum_{k_1 \in K} v_{k_1} e_{1})$$

$$= |H|^{-1}|K|^{-1} \sum_{k_1 \in K} v_{k_1} \alpha_{k_1}^{2}(1 \otimes \rho(sk^{-1}(h^{-1} k_1^{-1} k) \chi_{h^{-1} h'} \chi_{h_1} \otimes \rho(h) \chi_{e}) (1 \otimes 1 \otimes \sum_{h_1 \in H} \rho(h_1))$$

$$= |H|^{-1}|K|^{-1} \sum_{k_1 \in K, h_1 \in H} v_{k_1} (1 \otimes \rho(sk^{-1}(h^{-1} k_1^{-1} k) \chi_{h_1} \otimes \rho(h) \chi_{e}) (1 \otimes 1 \otimes \sum_{h_1 \in H} \rho(h_1)))$$

Hence, for any $h' \in H$ and $s' \in S$, one has:
Lemma 4.3.3. \( \rho \) where \( \Theta \)

Hence, using lemma 4.2.3, we have:

\[
(1 \otimes \rho(s) \chi_k \otimes 1)W^I_h e_2 e_1 (1 \otimes \lambda(s') \otimes \rho(s') \chi_{h'}) =
\]

\[
= |H|^{-1} |K|^{-1} v_{h^{-1}k}(1 \otimes \rho(sk^{-1}(h^{-1}k)) \lambda(s') \otimes \rho(h) \chi_e \lambda(s') \otimes \rho(s') \chi_{h'})
\]

\[
= |H|^{-1} |K|^{-1} v_{h^{-1}k}(1 \otimes \rho(sk^{-1}(h^{-1}k)) \chi_e \lambda(s') \otimes \rho(h) \chi_e \lambda(s') \otimes \rho(s') \chi_{h'})
\]

\[
= |H|^{-1} |K|^{-1} v_{h^{-1}k}(1 \otimes \rho(sk^{-1}(h^{-1}k)) \chi_e \lambda(s') \otimes \rho(h) \chi_e (\sum_{h_1 \in H} \rho(h_1)) \chi_{h'})
\]

4.3.3. Lemma. For any \( h, h' \in H, k, k' \in K, s, s' \in S \), one has:

\[
< (1 \otimes \rho(s) \chi_k \otimes 1)W^I_h, (1 \otimes \lambda(s') \otimes \rho(s') \chi_{h'})W^{J}_{k'} > =
\]

\[
= |S| \delta_{k' h^{-1}k, k, e} \delta_{h(k^{-1}j h')} e \delta_{(k'' g(k^{-1}j h'))k, s''} ^{s''}
\]

Proof: For any \( h' \in H, k' \in K \), due to 4.2.3 and 4.1.10, one has:

\[
(1 \otimes 1 \otimes \delta_{h'})W^J_{k'} = (1 \otimes 1 \otimes \delta_{h'})W^J_{k'} (1 \otimes 1 \otimes \delta_{k''})
\]

\[
= (1 \otimes 1 \otimes \delta_{h'})(1 \otimes \Theta_1 \otimes \Theta_2 \chi_{k''} v_{k'})
\]

where \( \Theta_1 = \lambda((k' g (k'' j h')) k'' \lambda) \) and \( \Theta_2 = \rho(h^{-1} (k'' j h')) \).

Hence, using lemma 4.3.2, one has:

\[
(1 \otimes \rho(s) \chi_k \otimes 1)W^I_h e_2 e_1 (1 \otimes \lambda(s') \otimes \rho(s') \chi_{h'})W^{J}_{k'} =
\]

\[
= |H|^{-1} |K|^{-1} v_{h^{-1}k}(1 \otimes \rho(sk^{-1}(h^{-1}k)) \chi_e \lambda(s') \Theta_1 \otimes \rho(h h') \chi_{h'} \Theta_2 \chi_{k''} v_{k'}
\]

\[
= |H|^{-1} |K|^{-1} v_{h^{-1}k}(1 \otimes \rho(sk^{-1}(h^{-1}k)) \chi_e \lambda(s') \Theta_1 \otimes \rho(h(k'' j h') \chi_{k''} v_{k'})
\]

25
As \( \tau \) is a trace, one has:

\[
(\|H\|K||H \cap K\rangle)^{-1} < (1 \otimes \rho(s)\chi_k \otimes 1)W_h^J, (1 \otimes \lambda(s') \otimes \rho(s')\chi_{k'}W_{k'}^J > = \\
= \tau((1 \otimes \rho(sk^{-1}(h^{-1} \triangleright_l k))\chi_k\lambda(s')\Theta_1 \otimes \rho(h(k'^{-1} \triangleright_j h'))\chi_{k'^{-1}j}h')\nu_k\nu_{h'^{-1}j/k}, \\
= \tau(\beta^1(1 \otimes \rho(sk^{-1}(h^{-1} \triangleright_l k))\chi_k\lambda(s')\Theta_1)(1 \otimes 1 \otimes \rho(h(k'^{-1} \triangleright_j h'))\chi_{k'^{-1}j}h')\nu_k\nu_{h'^{-1}j/k}, \\
= |K|^{-1}\delta_{k'h^{-1}j'k',e_\delta h(k'^{-1}j'h')},e_\tau(\beta^1(1 \otimes \rho(sk^{-1}(h^{-1} \triangleright_l k))\chi_k\lambda(s')\Theta_1))
\]

Using the fact that for any \( k_1 \in K \), \( \chi_e\lambda(k_1) = \chi_e\rho(k_1^{-1}) = \rho(k_1^{-1})\chi_{k_1^{-1}}, \) one also has:

\[
(\|H\|K||H \cap K\rangle)^{-1} < (1 \otimes \rho(s)\chi_k \otimes 1)W_h^J, (1 \otimes \lambda(s') \otimes \rho(s')\chi_{k'}W_{k'}^J > = \\
= |K|^{-1}\delta_{k'h^{-1}j'k',e_\delta h(k'^{-1}j'h')},e_\tau(\beta^1(1 \otimes \rho(sk^{-1}(h^{-1} \triangleright_l k))k'(k'^{-1}j'h')^{-1} s^{-1})\chi_z))
\]

for a certain \( z\), so we have:

\[
(\|H\|K||H \cap K\rangle)^{-1} < (1 \otimes \rho(s)\chi_k \otimes 1)W_h^J, (1 \otimes \lambda(s') \otimes \rho(s')\chi_{k'}W_{k'}^J > = \\
= |K|^{-1}|H|^{-1}\delta_{k'h^{-1}j'k',e_\delta h(k'^{-1}j'h')},e_\delta s^{-1}k'^{-1}j'h')^{-1} s^{-1})\chi_z)
\]

\[
= |K|^{-1}|H|^{-1}\delta_{k'h^{-1}j'k',e_\delta h(k'^{-1}j'h')},e_\delta s^{-1}k'^{-1}j'h')^{-1} s^{-1}
\]

The lemma follows. \( \square \)

4.3.4. **Lemma.** For all \( h, h' \in H \) and \( k, k' \in K \) such that \( k' = h \triangleright_j k \) and \( h = k' \triangleright_j h' \), then there exists a unique \( \sigma \in S \) such that: \( \sigma = (k'h')^{-1}hk = (k'^{-1}j')h')^{-1}j'k'k \)

**Proof:** For all \( h, h' \in H \) and \( k, k' \in K \) such that \( k' = h \triangleright_j k \) and \( h = k' \triangleright_j h' \), then there exist \( h_1 \in H \) and \( k_1 \in K \) such that \( hk_1 = k'h' \) and \( hk = k'h_1 \). One deduces that \( h'^{-1}k'^{-1}hk = k_1^{-1}k' = h'^{-1}h_1 \) and the existence (and uniqueness) of \( \sigma \) such that \( \sigma = (k'h')^{-1}hk \). One also has:

\[
(k'^{-1}j' \triangleright_j h')k = (k'^{-1}j' \triangleright_j h)k = (k'^{-1}j' \triangleright_j h)^{-1}(k'^{-1}j'k) = (k'^{-1}j'kh) = (k'h')^{-1}hk
\]

The lemma follows. \( \square \)

4.3.5. **Proposition.** Using notations 3.3.3, 4.1.11 and 4.2.6, for any \( x \in T \) and \( x' \in T' \), one has:

\[
< \theta^I_x, \theta^J_{x'} > = |H \cap K|\delta_{x',x'} = < x, x' >
\]

26
Proof: For any \( x \in T \) and \( x' \in T' \) let \((h, k, s) \) in \( H \times K \times S \) and \((k', h', s') \) in \( K \times H \times S \) such that \( x = h \triangleright k \quad \begin{array}{c} h \downarrow k \\ (h \downarrow k) s \end{array} \) and \( x' = k' \triangleright h' \quad \begin{array}{c} k' \downarrow h' \\ (k' \downarrow h') s' \end{array} \), then due to lemmas 4.3.3, 4.3.4 and 3.3.4 one has:

\[
\langle \theta^I_x, \theta^J_{x'} \rangle = |S| \delta_{k', h'^{-1}k} \delta_{h, k'^{-1}h} \delta_{h'^{-1}h, s'^{-1}}
\]

\[
= |S| \delta_{k', h'^{-1}k} \delta_{h, k'^{-1}h} \delta_{h'^{-1}h, s'^{-1}}
\]

\[
= |S| \delta_{k', h'^{-1}k} \delta_{h, k'^{-1}h} \delta_{hks, k'h's'}
\]

\[
= \begin{cases} 
|S| & \text{if } \begin{array}{c} hks = k'h's' \\
    k' = h \triangleright_I k \\
    h = k' \triangleright_J h' 
\end{array} \\
0 & \text{otherwise}
\end{cases}
\]

\[
= |S| \delta_{x', x'}
\]

\[
= \langle x, x' \rangle
\]

\[\square\]

4.3.6. **Theorem.** The pair of C*-quantum groupoids in duality \((M'_0 \cap M_2, \Gamma, \kappa, \epsilon)\) and \((M'_1 \cap M_3, \hat{\Gamma}, \hat{\kappa}, \hat{\epsilon})\) is isomorphic to the pair of C*-quantum groupoids in duality associated with \((\mathbb{C}T, \mathbb{C}T')\). For any \( x \in T \), and \( x' \in T' \), one has:

\[
\Gamma(\theta^I_x) = \frac{1}{|H \cap K|} \sum_{x' \in T'} \theta^I_{x'} \otimes \theta^I_{x^\prime} ; \quad \hat{\Gamma}(\theta^I_x) = \frac{1}{|H \cap K|} \sum_{x' \in T'} \theta^J_{x'} \otimes \theta^J_{x^\prime}
\]

\[
\kappa(\theta^I_x) = \theta^I_{x-hv} ; \quad \hat{\kappa}(\theta^J_x) = \theta^J_{x-hv}
\]

\[
e(\theta^I_x) = \begin{cases} 
|H \cap K| & \text{if } x \text{ of the form } \begin{array}{c} e \\
    e \downarrow \begin{array}{c} h \\
    h
\end{array} \\
\end{array} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\hat{e}(\theta^J_x) = \begin{cases} 
|H \cap K| & \text{if } x' \text{ of the form } \begin{array}{c} e \\
    e \downarrow \begin{array}{c} k \\
    k
\end{array} \\
\end{array} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\langle \theta^I_x, \theta^I_{x'} \rangle = |H \cap K| \delta_{x', x'} = \langle x, x' \rangle
\]

\[\square\]

4.3.7. **Corollary.** The von Neumann algebra crossed product \(\mathbb{C}T \times \mathbb{C}T'\) is isomorphic to \(\mathbb{C}[H \cap K] \otimes \mathcal{L}(\mathbb{C}^{H||K||})\).

\[\square\]

Proof: This is a direct consequence of corollary 2.4.2 and corollary 3.4.4. \[\square\]
REFERENCES

[AN1]  N. Andruskiewitsch & S. Natale: Double categories and quantum groupoids *Publ. Mat. Urug.* **10** (2005) 11-51; 2

[AN2]  N. Andruskiewitsch & S. Natale: Tensor categories attached to double groupoids (math.QA. 0408045) to appear in Adv. Math.; 2, 8, 9, 10

[B]  E.J. Beggs: Making non-trivially associated tensor categories from left coset representatives *Journal of pure and Applied Mathematics* **177** (2003), 5-41; 2

[B1]  D. Bisch: Higher Relative Commutants and the Fusion Algebra Associated to a Subfactor *Fields Institute Communications* **13**, (1997), 13-63; 2

[BS]  S. Baaj & G. Skandalis: Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres. *Ann. Sci. ENS* **26** (1993), 425-488; 3

[BBS]  S. Baaj & E. Blanchard & G. Skandalis: Unitaires multiplicatifs en dimension finie et leurs sous-objets. *Ann. Inst. Fourier* **49** (1999), 1305-1344; 3

[BH]  D. Bisch & U. Haagerup: Composition of subfactors: new examples of infinite depth subfactors, *Ann. Sci. ENS* 4ème série **29** n°3 (1996), 329-383; 2, 11, 12, 13

[BoSz]  G. Böhm & K. Szlachányi: Weak C*-Hopf algebras: the coassociative symmetry of non integral dimensions, *Quantum groups and quantum spaces. Banach Center Publications* **40** (1997), 9-19; 2, 3

[BoSzNi]  G. Böhm & K. Szlachányi & F. Nill: Weak Hopf Algebras I. Integral Theory and C*-structure. *Journal of Algebra* **221** (1999), 385-438; 2, 3

[D]  M.C. David: C*-groupoides quantiques et inclusions de facteurs: structure symétrique et autodualit actions sur le facteur hyperfini de type $II_1$. *Journal of operator theory* **54**:1 (2005), 27-68; 2, 5, 11, 12, 24

[D2]  M.C. David: Private communication

[E1]  M. Enock: Inclusions of von Neumann algebras and quantum groupoids III. *Journal of Functional Analysis* **223** (2005), 311-364; 12

[E2]  M. Enock: Measured quantum groupoids in action; 3

[EH]  C. Ehresmann: Catégories structurées. *Ann. Sci. École Norm. Sup.* (3), **80** (1963) 349-426; 2

[EV]  M. Enock & J.M. Vallin: Inclusions of von Neumann algebras and quantum groupoids, *Journal of Functional Analysis* **172** (2000), 243-300.; 2, 11

[G]  C.F. Gardiner, Algebraic structures: *Ellis Horwood limited* (1986) John Wiley & Sons; 6

[GHJ]  F.M. Goodman, P. de la Harpe, V.F.R. Jones: Coxeter graphs and towers of algebras *M.S.R.I. publications* 14: 5

[HS]  J.H. Hong, W. Szymbanski: Composition of subfactors and twisted crossed products.*Journal of operator theory* **37**(1997), 281-302.; 11, 12, 13, 16, 21

[L]  F. Lesieur: thesis, http://tel.ccsd.cnrs.fr/documents/archives0/00/00/55/05; 3

[NV1]  D. Nikshych & L. Vainerman: A characterization of depth 2 subfactors of $II_1$ factors. *JFA* **171** (2000), 278-307; 2, 3, 4, 5, 24

[NV2]  D. Nikshych & L. Vainerman: A Galois correspondence for $II_1$-factors and quantum groupoids. *JFA* **178** (2000), 113-142; 11

[PP]  M. Pimsner & S. Popa: Iterating the basic construction, Trans. Amer. Math. Soc. **310**(1988), 127-133; 14

[R]  J. Renault: *A groupoid approach to C*-algebras* Lect. Notes in Mah. **793** Springer-Verlag 1980; 4

[Val1]  J.M. Vallin: Groupoïdes quantiques finis. *Journal of Algebra* **26** (2001), 425-488; 3

[Val2]  J.M. Vallin: Actions and coactions of finite quantum groupoids on von Neumann algebras, extensions of the matched pair procedure (submitted to Journal of Algebra); 3, 13

[Val3]  J.M. Vallin: C*-algèbres de Hopf et C*-algèbres de Kac. *Pro. London. Math. Soc.* **50**, No.3 (1985), 131-174; 13

[VV]  S. Vaes & L. Vainerman: Extensions of locally compact quantum groups and the bicrossed product construction *Advances in Mathematics* **175** (2003), 1-101;
UMR CNRS 6628 Université d’Orléans, Institut de Mathématiques de Chevaleret, Plateau 7D, 175 rue du Chevaleret 75013 Paris, e-mail: jmva @ math.jussieu.fr