HOLOMORPHIC JETS IN SYMPLECTIC MANIFOLDS

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Abstract. We define pointwise partial differential relations for holomorphic discs. Given a relative homotopy class, a relation, and a generic almost complex structure we provide the moduli space of discs which have an injective point with the structure of a smooth manifold. Applications to the local behaviour are given and an adjunction inequality for singularities is derived. Moreover we show that for a coordinate class of a monotone Lagrangian split torus generically the number of non-immersed holomorphic discs is even.

1. Introduction

We consider holomorphic curves $C$ in a symplectic manifold $(M, \omega)$ w.r.t. a compatible almost complex structure $J$. The boundary $\partial C$ is contained in a Lagrangian submanifold $L$. We suppose $C$ to be parametrized by a smooth map $u: (\Sigma, \Gamma) \to (M, L)$ defined on a Riemann surface $\Sigma$ with complex structure $i$ and boundary $\Gamma$ such that $J_u \circ Tu = Tu \circ i$. The map $u$ is called holomorphic, see [10].

Locally holomorphic curves $u$ exists and the partial derivatives
\[ \partial_x u(0), \ldots, \partial_x^r u(0) \]
$(z = x + iy, r = 0, 1, 2 \ldots)$ can take any given value, cf. [20]. This determines the $r$-th Taylor polynomial at $0 \in \mathbb{C}$ because $u$ is a homogeneous solution of the non-linear Cauchy-Riemann equation $u_x + J(u)u_y = 0$. Taking $r$-equivalence classes of germs of holomorphic maps which have the same partial derivatives $\partial_x$ up to order $r$ defines the space of holomorphic $r$-jets on $(\Sigma, \Gamma) \times (M, L)$. This space is a smooth fibration over the source space $(\Sigma, \Gamma)$, see Section 2. We consider the product manifold of jet spaces with the same sources $z_1, \ldots, z_m \in \Sigma$ and $x_1, \ldots, x_m \in \Gamma$ of orders $r_1, \ldots, r_{m_0}, s_1, \ldots, s_{m_0}$. A submanifold $R$ is called a holomorphic jet relation.

Examples are defined by higher order tangency and intersection relations such as holomorphic curves

- intersecting a holomorphic submanifold, see [6],
- intersecting a Lagrangian submanifold (or cycles in there),
- with double points or singularities, see [3, 4, 17, 21, 18, 16].

A holomorphic curve is called somewhere injective if there is an immersed injective point on each connected component. The aim of this second paper of the programme begun in [23] is to show that the moduli space of somewhere injective holomorphic curves which represent a given homology class and jets in $R$ is a manifold provided the almost complex structure is chosen generically, see Section 3.5.
and \[13\] The dimension is determined by the Maslov, resp. first Chern, number and the dimension of \( R \).

In view of the work of Lazzarini \[13, 14\] and McDuff-Salamon \[15\] we study the local behaviour of holomorphic discs for generic almost complex structures. In \[4.1\] we prove that a somewhere injective holomorphic disc has a dense set of injective points in the interior and on the boundary, i.e. is \textbf{simple and simple along the boundary}, see \[23\]. In \[4.2\] we prove Lazzarinis theorem \[14, \text{Theorem B}\] that generically any non-constant holomorphic disc is multiply covered in the remaining dimension 4. In \[4.3\] and \[4.5\] we estimate the number of double points and singularities (counted multiplicity) in terms of topological data. In \[4.6\] we give an example how to define Gromov-Witten type invariants counting discs with singular points. In \[4.7\] we discuss generic existence of immersed and embedded holomorphic curves.

2. Generalized holomorphic tangencies

2.1. \textbf{Definition}. Let \( \Sigma \) be a Riemann surface and \((M, J)\) a \(2n\)-dimensional almost complex manifold. We consider holomorphic maps \( u: \Sigma \rightarrow (M, J) \). A \textbf{local representation} with \textbf{source} \( z \in \Sigma \) and \textbf{target} \( u(z) \in M \) consists of a conformal chart \((U, k)\) about \( z = k(0) \) and a chart \((V, h)\) about \( u(z) = h(0) \) such that w.l.o.g. \((h_*J)(0) = i\) is the standard complex multiplication on \( \mathbb{C}^n \). Usually we will suppress the localization in the notation such that the \textbf{Cauchy-Riemann equation} equals

\[ u_x + J(u)u_y = 0 \]

w.r.t. the conformal coordinates \( z = x + iy \).

Two germs of holomorphic maps \( u, v \) are \textbf{r-equivalent} \((r = 0, 1, 2, \ldots)\) at \( z \in \Sigma \) if \( u(z) = v(z) \) and if in a local representation the partial derivatives \( \partial_x \) at 0 coincide up to order \( r \), i.e. if

\[ \partial_x^\ell u(0) = \partial_x^\ell v(0) \]

for all \( \ell = 1, \ldots, r \).

With the knowledge of \( \partial_x u(0), \ldots, \partial_x^r u(0) \) one can reconstruct the \( r \)-th Taylor polynomial at 0 by taking partial derivatives of the Cauchy-Riemann equation

\[ u_y = J(u)u_x \]

as follows:

\[
\begin{align*}
  u_{xy} &= J(u)u_{xx} + (DJ(u) \cdot u_x)u_x \\
  u_{yy} &= J(u)u_{yx} + (DJ(u) \cdot u_y)u_x \\
  &\vdots
\end{align*}
\]

Therefore, \textbf{all} partial derivatives of \( u \) and \( v \) up to order \( r \) coincide at 0. By the chain rule this implies independence of the chosen local representation which we used in the definition. Therefore, the equivalence relation is well defined. The \( r \)-equivalence class \( j^r_u \) is called the \textbf{holomorphic} \textbf{r-jet}. We denote the space of all \( r \)-jets by

\[ (\Sigma \times M)^r \equiv \text{Jet}^r. \]
2.2. **A representation.** For a small open neighbourhood $V$ of $0 \in \mathbb{R}^{2n}$ the space of holomorphic jets from $\mathbb{C}$ to $(V, J)$ can be identified with

$$(\mathbb{C} \times V)^j \equiv \mathbb{C} \times V \times (\mathbb{R}^{2n})^r.$$ 

**Proposition 2.1.** Let $J$ be an almost complex structure on $\mathbb{R}^{2n}$ with $J(0) = i$. There exists an open neighbourhood $V$ of $0 \in \mathbb{R}^{2n}$ such that for all $(z_0, p) \in \mathbb{C} \times V$ and $a_1, \ldots, a_r \in \mathbb{R}^{2n}$ there exists a germ of holomorphic maps $u: \mathbb{C} \rightarrow (\mathbb{R}^{2n}, J)$ at $z_0$ satisfying

$$\partial_x u(z_0) = a_1, \ldots, \partial_x^r u(z_0) = a_r.$$ 

**Proof.** Translations in $\mathbb{C}$ are conformal so that it is enough to prove the proposition for $z_0 = 0$. For some $p > 2$ we consider the operator

$$T(\xi) = (\xi_x + i\xi_y; \xi(0), \partial_x \xi(0), \ldots, \partial_x^p \xi(0)).$$

The domain consists of all $\xi \in W^{r+1,p}(D, \mathbb{R}^{2n})$, where $D$ is the unit disc, such that $\xi(e^{2\pi i \theta}) \in C^{(2r+1)p} \mathbb{R}^{2n}$ for all $\theta \in [0, 1)$, with $\mathbb{R}^{2n}$ identified with $(\mathbb{R} \times \{0\})^{2n}$. The target space is $W^{r,p}(D, \mathbb{R}^{2n}) \times (\mathbb{R}^{2n})^{r+1}$. By [15, Chapter C.4] the operator $T$ is invertible.

As on [15, p. 627] (written with Lazzarini) we consider the map

$$F(v) = (v_x + J(v) v_y; v(0), \partial_x v(0), \ldots, \partial_x^p v(0)),$$

whose linearization at zero

$$DF(0) \cdot \xi = T(\xi)$$

is invertible. By the inverse function theorem $F$ is a diffeomorphism near zero so that the equation $F(v) = (0; a_0, \varepsilon a_1, \ldots, \varepsilon^r a_r)$ has a solution $v$ for $\varepsilon > 0$ sufficiently small. The desired holomorphic germ is $u = v \circ 1/\varepsilon$. \hfill \Box

2.3. **Differentiable structure.** Consider local representations $(U, k)$ and $(V, h)$. By Proposition 2.1 and the proof the maps

$$(U \times V)^j \ni j^x_j u \quad \mapsto \quad (kU \times hV)^j \ni j^x_{k(z)} (h \circ u \circ k^{-1})$$

define charts. Therefore, $(\Sigma \times M)^j$ is a manifold of dimension $2(1 + n(r + 1))$. The projection onto $\Sigma \times M$ is an affine fibration, cf. [7].

2.4. **The case with boundary.** Consider a Riemann surface $\Sigma$ with boundary $\Gamma$. The conformal atlas of $\Sigma$ is enriched by boundary preserving conformal maps into the upper half-plane $H^+$. We consider holomorphic maps which take on $\Gamma$ values in a maximally totally real submanifold $L$ of $(M, J)$. By [15, p. 539] there exists charts $h$ of $M$ about points of $L$ which take values in $\mathbb{R}^n$ along $L$ and satisfy $h_* J = i$ on $\mathbb{R}^n$. By a local representation such a choice of charts is understood.

Two germs $u, v: (\Sigma, \Gamma) \rightarrow (M, L)$ of holomorphic maps at $x \in \Gamma$ define the same holomorphic $s$-jet $(s = 0, 1, 2, \ldots)$ provided that $u \mid \Gamma$, $v \mid \Gamma$ are $s$-equivalent in the sense of smooth functions. In other words, the $s$-tangency class $j^x_s u$ is characterized as in [24] w.r.t. local representations preserving the boundary condition. The space of all holomorphic $s$-jets on the boundary is denoted by

$$(\Gamma \times L)^j \equiv \text{Jet}^s.$$ 

**Proposition 2.2.** Any $s$-jet of a smooth function $\Gamma \rightarrow L$ has a holomorphic representative. Moreover, $(\Gamma \times L)^j$ is a manifold of dimension $1 + n(s + 1)$, and an affine fibration over $\Gamma \times L$ induces by the source-target map.
2.5. Holomorphic half-disc representation. We will show that $s$-jets of smooth maps $\Gamma \rightarrow L$ can be represented by the restriction of germs of $J$-holomorphic maps $(\Sigma, \Gamma) \rightarrow (M, L)$ such that the representation depends smoothly on the data. This will prove Proposition 2.2 see [7].

We consider an almost complex structure $J$ on $\mathbb{R}^n$ such that $J = i$ on $\mathbb{R}^n$. We claim that there exists an open neighbourhood $V'$ of 0 in $\mathbb{R}^n$ such that for all $(x_0, p) \in \mathbb{R} \times V'$ and $a_1, \ldots, a_r \in \mathbb{R}^n$ there exists a germ of $J$-holomorphic maps $u: (H^+, \mathbb{R}) \rightarrow (\mathbb{R}^{2n}, \mathbb{R}^n)$ at $x_0$ satisfying

$$\partial_z u(x_0) = a_1, \ldots, \partial_z^r u(x_0) = a_r.$$ 

It is enough to show this for $x_0 = 0$.

**Step 1:** On the unit disc $D \subset \mathbb{C}$ we consider the operator $u \mapsto u_x + iu_y$ defined on the space $V$ of all $\mathbb{C}^n$-valued functions of Sobolev-class $W^{s+1,p}$, $p > 2$, subject to the boundary condition $u(e^{2\pi i \theta}) \in e^{s\pi i \theta} \mathbb{R}^n$, $\theta \in [0, 1)$. The operator takes values in $W^{s,p}$. By [15, Chapter C.4] this operator is onto whose $(s+1)$-dimensional kernel is generated by

$$p_{\ell,b}(z) = \frac{i^\ell}{\ell!2^{s-\ell}} b(1+z)^{s-\ell}(1-z)^\ell,$$

$b \in \mathbb{R}^n$, $\ell = 0, 1, \ldots, s$. Notice, that

$$j_1^{\ell-1}p_{\ell,b} = 0 \quad \text{and} \quad \partial_\ell^\ell p_{\ell,b}(1) = b.$$ 

Set

$$V_k = \{ u \in V \mid D^\alpha u(1) = 0 \quad \forall |\alpha| \leq k-1 \}$$

and

$$W_k = \{ f \in W^{s,p}(D, \mathbb{C}^n) \mid D^\beta f(1) = 0 \quad \forall |\beta| \leq k-2 \}$$

for $k = 0, 1, \ldots, s$. Due to the pointwise constraints and the boundary condition the operator

$$S_k: V_k \longrightarrow W_k \times \mathbb{R}^n$$

$$u \mapsto (u_x + iu_y, \partial_\ell^\ell u(1))$$

is well defined, onto, and its $n(s - k)$-dimensional kernel is generated by $p_{\ell,b}$ for $\ell = k + 1, \ldots, s$ and $b \in \mathbb{R}^n$.

**Step 2:** Let $\Omega$ be a domain in $H^+$ obtained by smoothing the corners of the unit half-disc such that $\partial \Omega \cap \mathbb{R}$ is an interval $I$ which contains 0. Let $\varphi$ be a conformal diffeomorphism $(\Omega, 0) \rightarrow (D, 1)$ up to the boundary. Let $V_\Omega$ be the space of all $\mathbb{C}^n$-valued functions of Sobolev-class $W^{s+1,p}$ on $\Omega$ subject to the boundary condition

$$u(z) \in e^{\frac{i}{2} \arg \varphi(z)} \mathbb{R}^n,$$

$z \in \partial \Omega$, where the argument is normalized by $\arg(e^{2\pi i \theta}) = 2\pi \theta$. Abbreviate $W^{s,p}(\Omega, \mathbb{C}^n)$ by $W_\Omega$. Define

$$V_{\Omega,k} = \{ u \in V_\Omega \mid D^\alpha u(0) = 0 \quad \forall |\alpha| \leq k-1 \}$$

and

$$W_{\Omega,k} = \{ f \in W_\Omega \mid D^\beta f(0) = 0 \quad \forall |\beta| \leq k-2 \}$$

and

$$S_{\Omega,k}: V_{\Omega,k} \longrightarrow W_{\Omega,k} \times \mathbb{R}^n$$

$$u \mapsto (u_x + iu_y, \partial_\ell^\ell u(0)).$$
The invertible maps $u \mapsto u \circ \varphi$ and 

$$(f, h) \mapsto ((\varphi^1 + i\varphi^2) f \circ \varphi, c^k h)$$

(using $\varphi = \varphi^1 + \varphi^2$ and $\partial_z \varphi(0) = ci$ for a positive constant $c$) conjugate $S_{\Omega, k}$ to $S_k$. Therefore, $S_{\Omega, k}$ is onto and has $n(s-k)$-dimensional kernel.

**Step 3:** Let $L(z)$, $z \in \partial \Omega$, be a loop of totally real subspaces of $\mathbb{C}^n$ with Maslov index $s$ such that $L(x) = \mathbb{R}^n$ for $x \in I$. By Arnol’d’s theorem there exists a smooth function $A: \mathbb{C} \to \text{Gl}(n, \mathbb{C})$ such that $A(0) = 1$ and $A(z)L(z) = e^{\frac{i}{2} \arg z} \mathbb{R}^n$

for all $z \in \partial \Omega$, cf. [15, p. 554]. Consider the operator

$$\hat{Q}: V_\Omega \to W_\Omega \times (\mathbb{R}^n)^{s+1}$$

mapping

$$u \mapsto (u_x + iu_y; u(0), A_x(0)u(0) + u_x(0), \ldots, \partial^s_x(Au)(0)).$$

In order to show surjectivity of $\hat{Q}$ let $f \in W_\Omega$ and $b_0, b_1, \ldots, b_s \in \mathbb{R}^n$ be arbitrarily given. Because $S_{\Omega, 0}$ is onto we find $v^0 \in V_\Omega$ such that $v^0_x + iv^0_y = f$ and $v^0(0) = b_0$. Assume that $v^\ell \in V_\Omega$ is constructed recursively such that $v_x^\ell + iv_y^\ell = f$ and

$$v^0(0) = b_0, \ldots, \partial^s_x(Av^\ell)(0) = b_\ell.$$ 

By an application of $S_{\Omega, \ell+1}$ we find a holomorphic $q \in V_{\Omega, \ell+1}$ such that

$$\partial^s_x q(0) = b_{\ell+1} - \partial^s_x(Av^\ell)(0).$$

Set $v^{\ell+1} = v^\ell + q$ and observe that $v_x^{\ell+1} + iv_y^{\ell+1} = f$ and

$$v_x^{\ell+1}(0) = b_0, \ldots, \partial^s_x(Av^{\ell+1})(0) = b_{\ell+1}.$$ 

Therefore, $u = v^s$ is a solution of $\hat{Q}(u) = (f; b_0, \ldots, b_s)$, i.e. $\hat{Q}$ is onto.

**Step 4:** We show that $\hat{Q}$ has trivial kernel. Consider $u \in \ker \hat{Q}$. If $u$ is not constant so at least one of the coordinate functions $u^j \in \mathbb{C}$. By [8, Lemma 9.5] the sum of the orders of the zeros of $u^j$ on the boundary plus twice the orders of zeros of $u^j$ at the interior equals the Maslov index $s$. But $u^j$ vanishes at least to the $(s+1)$-st order at zero. This implies $u = 0$. Therefore, $\hat{Q}$ is invertible.

**Step 5:** Let $V_L$ be the space of all $\mathbb{C}^n$-valued functions of Sobolev-class $W^{s+1, p}$ on $\Omega$ such that $u(z) \in L(z)$ for all $z \in \partial \Omega$. Consider the operator

$$T: V_L \to W_\Omega \times (\mathbb{R}^n)^{s+1}$$

mapping

$$u \mapsto (u_x + iu_y; u(0), \partial_x u(0), \ldots, \partial^s_x u(0)).$$

The operator

$$Q = (A^{-1} \times 1) \circ T \circ A$$

equals

$$Q(u) = \hat{Q}(u) + (A^{-1}(A_x + iA_y)u; 0, \ldots, 0).$$

The second term is a compact perturbation. Because $\hat{Q}$ is invertible the operator $Q$ is Fredholm of index zero. Hence, $T$ is a Fredholm operator of index zero. Because the Maslov index of $L$ is $s$ an application of [8, Lemma 9.5] as in Step 4 shows triviality of $\ker T$. Hence, the operator $T$ is invertible.

**Step 6:** Taking the derivative of the map $F$ at $v = 0$ as in Proposition [21] proves the claim. Q.E.D.
3. The universal jet space

We consider a 2n-dimensional symplectic manifold \((M, \omega)\), a Lagrangian submanifold \(L\), and a compatible almost complex structure \(J_0\) with the induced metric \(\omega(., J_0.)\).

3.1. Almost complex structures. The space \(\mathcal{J}\) of all compatible almost complex structures \(J\) is a Fréchet manifold. The model is the space of symmetric endomorphism fields \(Y\) on \(M\) which anti-commute with \(J_0\). The homeomorphism

\[ Y \mapsto J_0(1 + Y)(1 - Y)^{-1}, \]

\[ \|Y\|_{C^0} < 1, \]

serves as a global chart for \(\mathcal{J}\). The inverse is

\[ H: J \mapsto (J + J_0)^{-1}(J - J_0), \]

cf. [2, Proposition 1.1.6].

For global considerations additionally we require the almost complex structures to coincide with \(J_0\) in the complement of a relative compact subset of \(M\), the so-called perturbation domain.

For a sequence \((\varepsilon_j)\) of positive real numbers we consider infinitesimal almost complex structures which are finite in the Floer-norm

\[ \sum_{j=0}^{\infty} \varepsilon_j \|Y\|_{C^0}. \]

We choose \((\varepsilon_j)\) such that the resulting open subset is separable and dense in \(C^\infty\), cf. [19]. The image under \(H^{-1}\) is the separable Banach manifold \(\mathcal{I}\) of compatible almost complex structures with the induced Floer-topology.

3.2. Definition. We consider points \(z_1, \ldots, z_{m_0}\) on \(\Sigma\) and \(x_1, \ldots, x_{m_1}\) on the boundary \(\Gamma\). A source is a tuple

\[ (z, x) = (z_1, \ldots, z_{m_0}, x_1, \ldots, x_{m_1}) \]

of pairwise distinct points. The space of all sources is denoted by

\[ \mathcal{S} \subset \Sigma^{m_0} \times \Gamma^{m_1} \equiv (\Sigma \times \Gamma)^m, \]

where we wrote \(m = (m_0, m_1)\) for the number of source points. The length by definition is \(\|m\| = 2m_0 + m_1\).

A germ of maps \(u: (\Sigma, \Gamma) \to (M, L)\) at \((z, x)\) is a germ of product maps

\[ u^m = (u^1, \ldots, u^{m_0}, v^1, \ldots, v^{m_1}): (\Sigma, \Gamma)^m \to (M, L)^m \]

at \((z, x) \in \mathcal{S}\), where we think of the \(v^d\) to be extended to a neighbourhood of the sources. Similarly, a germ of almost complex structures \(J \in \mathcal{J}\) at

\[ u(z, x) = (u^1(z_1), \ldots, v^{m_1}(x_{m_1})) \]

is understood. We say that \((u, J)\) is a holomorphic multi-germ of maps \(u^m\) at \((z, x)\) if \(J \in \mathcal{J}\) is a germ at \(u(z, x)\) and \(u\) is a germ of \(J\)-holomorphic maps at \((z, x)\). Notice, that about equal targets of \(u\) for different sources the almost complex structure may differ.

On the set of holomorphic multi-germs \((u, J)\) we define a \((r, s)\)-tangency relation: Consider a vector of non-negative integers

\[ t = (r, s) = (r_1, \ldots, r_{m_0}, s_1, \ldots, s_{m_1}). \]
The length is
\[ ||t|| = 2(r_1, \ldots, r_m) + (s_1, \ldots, s_m). \]

Two holomorphic multi-germs \((u_1, J_1)\) and \((u_2, J_2)\) are said to be \(t\)-equivalent at \((z, x)\) if \(u_1(z, x) = u_2(z, x) = p\), the \((t - 1)\)-jets of \(J_1\) and \(J_2\) at \(p\) coincide, and \(j^t_{(z, x)}u_1 = j^t_{(z, x)}u_2\), where we used the notation
\[ j^t_{(z, x)}u = (j^1_{z, x}u, j^2_{z, x}u) = (j^{r_{m_0}}_{z, x}u, j^{s_1}_{z, x}u, \ldots, j^{s_m}_{z, x}u) \]
of holomorphic jets. As in [2.1] one verifies that this is a correct definition. The equivalence class of a holomorphic multi-germ \((u, J)\) is denoted by \(j^t_{(z, x)}(u, J)\); the space of all \(t\)-tangency classes holomorphic for some \(J \in \mathcal{J}\) by \(\mathcal{J}^t\).

We denote by \(E^{(t-1)}\) the space of jets of smooth sections into the bundle \(E\) over \(M\) whose fibres consists of \(\omega\)-compatible complex structures on the corresponding tangent spaces of \(M\).

**Proposition 3.1.** The natural map
\[ j^t_{(z, x)}(u, J) \mapsto ((z, x), j^{t-1}_{u(z, x)} J) \]
is a locally trivial fibration
\[ \mathcal{J}^t \rightarrow S \times E^{(t-1)} \]
with affine fibre of dimension \(n||t||\).

**Proof.** With help of a Hermitian trivialization the fibre of \(E \rightarrow M\) is diffeomorphic to the unite ball \(W\) in the \(n(n+1)\)-dimensional vector space of symmetric \((2n \times 2n)\)-matrices which anti-commute with \(i\). Hence, \(E^{r-1}\) is modeled on \(V \times W \times \mathbb{R}^{d_r-1}\) with \(d_r = \frac{(2n + r - 1)!}{(2n)! (r-1)!^n (n+1)}\), where \(V \subset \mathbb{R}^{2n}\) (resp. \(V' \subset \mathbb{R}^n\)) is an open subset. As in Section 2 the space of holomorphic \(r\)-jets (for fixed \(J\)) can be described locally by \(U \times V \times \mathbb{R}^{2n}\) (resp. \(U' \times V' \times \mathbb{R}^n\)) for open subsets \(U \subset \Sigma\) (resp. \(U' \subset \Gamma\)).

We claim that the local model for \(\mathcal{J}^t\) is
\[ (U, U')^m \times (V, V')^m \times (\mathbb{R}^{2n}, \mathbb{R}^n)^t \times (W \times \mathbb{R}^{d_{r-1}}) \times \ldots \times (W \times \mathbb{R}^{d_{r-m_1}}) \].

Considering each \(m\)-coordinate separately we represent jets of \(J \in \mathcal{J}\) by local \(\omega\)-compatible almost complex structures. Represent a holomorphic jet by a germ of holomorphic maps w.r.t. the constructed local \(J\)'s as in Section 2. Notice that a smooth variation of jets of \(J \in \mathcal{J}\) is followed by a smooth variation of local \(J\)'s. A smooth variation of \(a = (a_0, a_1, \ldots, a_r)\) induces a smooth variation of local holomorphic maps:

In view of Propositions [2.1] and [2.2] we define
\[ \mathcal{F}(b, v, J) = (v_x + J(v)v_y; v(0) - b_0, \partial_x v(0) - b_1, \ldots, \partial_x^r v(0) - b_r) \]
The partial derivative
\[ \frac{\partial \mathcal{F}}{\partial v}(b, 0, J) \cdot \xi = T(\xi) \]
is invertible. By the implicit function theorem there exists a local smooth map $v = v(b, J)$ such that
\[ \mathcal{F}(b, v(b, J), J) = 0. \]

The local map
\[ u(a_0, a_1, \ldots, a_r, J)(z) = v(a_0, \varepsilon a_1, \ldots, \varepsilon^r a_r, J)(z/\varepsilon) \]

is $J$-holomorphic with $j_0^u = a$, which varies smoothly with the data (including the variation given by translations of the source $z_0$ (resp. $x_0$)).

The argument requires $J(0) = i$ (resp. $J = i$ on $\mathbb{R}^n$) for $J \in J(V)$ and open subsets $V \subset \mathbb{R}^n \equiv M$. In order to achieve this we take a symplectic Darboux chart (resp. Weinstein chart which is a symplectic embedding $(T^*\mathbb{R}^n, \mathbb{R}^n) \to (T^*L, \mathcal{O}_{T^*L})$) induced by a chart $\mathbb{R}^n \to L$ followed by a local symplectic embedding $(T^*L, \mathcal{O}_{T^*L}) \to (M, L)$ induced by a Weinstein neighbourhood.) Set
\[ \varphi_J(x, y) = x^i \partial_{x^i} + y^j J(x, y) \partial_{x^j}. \]

Because $J$ is compatible with $dx \wedge dy$ and the $\partial_{x^j}$’s span the Lagrangian $\mathbb{R}^n$ about 0 (resp. on $\mathbb{R}^n$) $\partial_{x^j}, J(x, y) \partial_{x^j}, j = 1, \ldots, n$, is a base. The differential is $(i, J)$-complex and invertible at 0 (resp. on $\mathbb{R}^n$). Transforming the position and the almost complex structures via
\[ (x, y, J) \mapsto (\varphi_J(x, y), (\varphi_J)_* J) \]

allows the use of the map $\mathcal{F}$. This shows smoothness of the transition function which are affine, see [7].

**Remark 3.2.** The proof allows a description of a tangent vector of $\text{Jet}^t$ at a point $j^t_{(z, x)}(u, J)$. Let $Y$ be a local infinitesimal almost complex structure in $T_1 \mathcal{F}$ and $\xi$ a variation of local holomorphic maps. I.e. $\xi$ is a local vector field along $u$ such that $(v, w) \in T_{(z, x)} \mathcal{S}$, $\hat{\xi}_m = \xi_m + Du_m \cdot (v, w)$, and $\hat{Y} = Y + DJ(u) \cdot \hat{\xi}$ solve
\[ \hat{\xi}_x + J(u) \hat{\xi}_y + \hat{Y}(u) u_y = 0. \]

Hence, a tangent vector can be represented by $j^t_{(v, w)}(\hat{\xi}, Y)$ (sic) understood analogously to the case of $(u, J)$’s.

### 3.3. Universal relations.

Denote by $\left( (\Sigma, \Gamma) \times (M, L) \right)^{(t)}$ the smooth $t$-jet space of maps $(\Sigma, \Gamma) \to (M, L)$. The jet with sources in $\Gamma$ are computed via relative charts. Because the $r$-th Taylor polynomial of a $J$-holomorphic map is completely determined by the holomorphic $r$-jet and the $(r-1)$-jet of $J$ there is a well defined map
\[ j^t_{(z, x)}(u, J) \mapsto (j^t_{(z, x)} u, j^{t-1}_{u(z, x)} J). \]

Because of Proposition 3.1 (local description) and Remark 3.2 this map is an embedding
\[ \text{Jet}^t \hookrightarrow \left( (\Sigma, \Gamma) \times (M, L) \right)^{(t)} \times E^{(t-1)}, \]

whose projection to the second factor is a surjective submersion.

A submanifold $\mathcal{R}$ of $\text{Jet}^t$ is called a **higher order intersection relation** if the image under the natural map is a product manifold with second factor $E^{(t-1)}$, i.e. $\mathcal{R}$ is $J$-independent. $\mathcal{R}$ is called **source-free** if the relation puts no restriction to the marked points, i.e. in a local fibre chart (as in the prove of Proposition 3.1) $\mathcal{R}$ is a product manifold with first factor $(U, U')^m$ which corresponds to the source space $\mathcal{S}$. 
Proposition 3.3. The intersection $R$ of $\mathcal{R}$ with any fibre over $E^{(t-1)}$ is a submanifold and the codimension of $R$ is

$$||m|| + n(||m|| + ||t||) - \dim R.$$ 

The intersection $R$ of a source-free relation $\mathcal{R}$ with any fibre over $S \times E^{(t-1)}$ is a submanifold and the codimension of $R$ is

$$n(||m|| + ||t||) - \dim R.$$ 

3.4. Universal moduli space. Let $\Sigma$ be a Riemann surface which is closed or compact with boundary $\Gamma$. Let $R$ be an intersection relation of order $t$ and let $A$ be a relative singular integer 2-homology class in $(M, L)$, and $u$ is a $J$-holomorphic map $(\Sigma, \Gamma) \to (M, L)$ representing $A$ such that

- $j^t_{(\mathbf{z}, \mathbf{x})}(u, J) \in R$,
- $u$ is simple (the open set of injective points is dense) and simple along the boundary (the open set of injective points of $u|_{\Gamma}$ is dense in $\Gamma$), see [23, Corollary 8.5],
- $u(\Sigma)$ is contained in the perturbation domain, see 3.1.

Proposition 3.4. The universal moduli space $U$ is a separable Banach manifold.

Proof. The universal moduli space for the empty relation $U_0$ (i.e. $m = (0, 0)$) is a separable Banach manifold, see [15, Chapter 3]. In the presence of a relation $R$ we consider the jet extension map

$$(u, J, \mathbf{z}, \mathbf{x}) \mapsto j^t_{(\mathbf{z}, \mathbf{x})}(u, J).$$

The preimage of $R$ under

$$j^t : U_0 \times S \to \text{Jet}^t$$

is $U$. We consider a tangent vector of $\text{Jet}^t$ at $j^t_{(\mathbf{z}, \mathbf{x})}(u, J)$ which is transverse to $R$. By $J$-independence of $R$ we are free to assume that the tangent vector is taken w.r.t. the trivial infinitesimal almost complex structure, see Remark 3.2. I.e. for $(v, w) \in T_{(\mathbf{z}, \mathbf{x})}S$ and a local holomorphic section $\xi$ of $(u^*TM, u^*TL)$ near $(\mathbf{z}, \mathbf{x})$ the tangent vector is given by $j^t_{(\mathbf{v}, \mathbf{w})}(\xi, 0)$. The holomorphic curve $u$ has the annulus property (see [23, Theorem 1.2]) and the half-annulus property, see [23, Theorem 1.3 and Corollary 9.5]. Therefore, as on [15, p. 63] $\xi$ extends to a smooth global section $\xi$ such that $(\xi, Y)$ is tangent to $U_0$ for an infinitesimal almost complex structure $Y \in I$, see [15, Exercise 3.4.5]. Because the universal $t$-jet space has finite dimension the $t$-jet extension map is transverse to $R$. The claim follows with [12, Section II.2].

Remark 3.5. If $\Gamma = \emptyset$, Proposition 3.4 follows from [6, Lemmata 6.5, 6.6] and [15, Chapter 3]. Moreover, it is enough to require the holomorphic curves to intersect the perturbation domain non-trivially. If $\Gamma \neq \emptyset$ and $||t|| = 0$, Proposition 3.4 follows with [15, Chapter 3]. In that case the assumptions can be relaxed: Each connected component of $\Sigma$ has an injective point of $u$ which is mapped to the perturbation domain.
3.5. Generic perturbation. By the argument in 3.4 $U$ is a submanifold in $U_0 \times S$ with co-dimension given by Proposition 3.3. Moreover, the projection $U_0 \to \mathcal{I}$ is a smooth Fredholm map of index $\mu(A) + n\chi(\Sigma)$, where $\mu$ denotes the Maslov index and $\chi$ the Euler characteristic, see [15] Chapter 3 and Appendix C]. Therefore, the induced map $U_0 \times S \to \mathcal{I}$ is Fredholm of index $\mu(A) + n\chi(\Sigma)$, see [15, Appendix A]. We call $U_0$ the $R$-moduli space. By Sard-Smale theorem the set of regular values is of second Baire category in $\mathcal{I}$ so that by the implicit function theorem the preimage $U_{0, J}$ of a regular value $J \in \mathcal{I}$ is a manifold of dimension given by the index, cf. [15, Appendix A]. We call $U_{0, J}$ the $R$-regular or generic.

Theorem. The set of all $R$-regular almost complex structures is of second Baire category in $\mathcal{I}$ (and therefore dense in $\mathcal{J}$). For all $R$-regular $J \in \mathcal{I}$ the $R$-moduli space is a manifold of dimension

$$\mu(A) + n(\chi(\Sigma) - \|m\| - \|t\|) + \dim R.$$

If $R$ is source-free the dimension is

$$\mu(A) + n(\chi(\Sigma) - \|t\|) + (1 - n)\|m\| + \dim R.$$

After an additional perturbation of $J \in \mathcal{I}$ smoothness of $R$-moduli spaces holds for holomorphic maps which have an injective point on each connected component of $\Sigma$, see Corollary 3 below.

4. Enumerative relations

We assume the dimension of $M$ to be greater or equal than 4.

4.1. A priori perturbation and local behaviour. We consider holomorphic curves $u$ such that on each connected component there exists an injective point which is mapped by $u$ into the perturbation domain. By Remark 3.5 the following moduli problem can be assumed to be transverse for a generic choice of $J \in \mathcal{I}$:

- $[u] = A$,
- $u$ has $n_0$ interior, $n_1$ boundary, and $\ell$ mixed double points, i.e. $u$ is subject to the intersection relation
  $$(\Delta_M)^{n_0} \times (\{(p, q) \in M \times L \mid p = q\})^\ell \times (\Delta_L)^{n_1}$$
  of order $\|t\| = 0$ and length $\|m\| = (2n_0 + \ell, 2n_1 + \ell)$.

Taking the dimension formula into account this leads as in [5, 14] to the adjunction inequality

$$(n - 2)(n_0, n_1) + (2n - 3)\ell \leq \mu(A) + n\chi(\Sigma).$$

Therefore, $u$ has finitely many double points if $n \geq 3$. In particular, $u$ is simple and simple along the boundary. For $n = 2$ there are finitely many mixed double points. With [13, 14] $u$ is simple, cf. Remark 4.1 for an alternative argument.

Corollary 1. There exists a subset $\mathcal{I}_\infty$ of second Baire category in $\mathcal{I}$ such that for all $J \in \mathcal{I}_\infty$ any somewhere injective $J$-holomorphic curve is simple and simple along the boundary.
Proof. \( I_\infty \) is the intersection of sets of second category, where the intersection is taken over all relative integral homology 2-classes \( A \) in \((M, L)\) (which is known to be countable) and all intersection relations as above counted by \( n_0, n_1, \) and \( \ell \). This proves the corollary in dimension \( 2n \geq 6 \).

For \( 2n = 4 \) it suffices to establish simplicity along the boundary. We consider the following moduli problem: Let \( N \) be a natural number. Let \( P_N \) be a partition of \( \Gamma \) into \( N \) segments (i.e. connected open subsets) of equal length. We equip each segment \( S \in P_N \) with pairwise distinct points \( y_1, \ldots, y_N \in S \). The holomorphic curves \( u \) are assumed to be somewhere injective in the sense of Remark 3.5. We require

1. \( [u] = A \),
2. there are pairwise distinct points \( x_1, \ldots, x_N \in \Gamma \) each different form the \( y_j \)'s such that \( u(x_j) = u(y_j) \) for all \( j = 1, \ldots, N \).

Let \( I_\infty \) be the intersection of all regular values in \( I \) taken over all \( A \) and \( N \) corresponding to the moduli problems. I.e. the moduli spaces \( \{(u, x_1, \ldots, x_N)\} \) are cut out transversely for \( J \in I_\infty \) and are of dimension \( \mu(A) + 2\chi(\Sigma) - N \).

Arguing by contradiction we suppose that \( u \) is a somewhere injective holomorphic curve which is not simple along the boundary. By [23, Proposition 9.3] there exists a diffeomorphism \( \varphi : S_1 \to S_2 \) between two disjoint segments of \( \Gamma \) such that \( u(x) = u(\varphi(x)) \) for all \( x \in S_1 \). Let \( N > \mu([u]) + 2\chi(\Sigma) \) be a natural number such that the length of a segment in \( P_N \) is smaller than \( 1/2 \) times the length of \( S_2 \). Let \( S \in P_N \) be a segment which is contained in \( S_2 \). In particular, with \( x_j := \varphi^{-1}(y_j) \) we obtain \( u(x_j) = u(y_j) \) for all \( j = 1, \ldots, N \). Therefore, \( (u, x_1, \ldots, x_N) \) is an element of one of the above moduli spaces which has a negative dimension. This contradiction proves the claim.

\( \square \)

Remark 4.1. A similar argument shows simplicity of \( u \). Replace \( P_N \) by a covering by open balls of radius \( 1/N \) each equipped with \( N \) distinct points. Simplicity fails if there is a diffeomorphism \( \varphi : U_1 \to U_2 \) between two open disjoint subsets of \( \Sigma \) such that \( u(z) = u(\varphi(z)) \) for all \( z \in U_1 \), see [23, Proposition 2.7]. Choose \( N > \frac{1}{2}(\mu([u]) + 2\chi(\Sigma)) \) such that there is a ball in \( P_N \) which is a subset of \( U_2 \).

For reasons of beauty we give one more argument. Because there are finitely many mixed double points the weak and the strong variant of simplicity along the boundary considered in [23] are the same. By [23, Proposition 6.4] the argument in the above proof, which shows simplicity along the boundary, suffices to show simplicity.

4.2. Generic multiply covered discs. A non-constant holomorphic disc \( u \) (i.e. \( \Sigma = D \)) is called multiply covered if there exists a simple holomorphic disc \( v \) and a holomorphic map \( \pi : (D, \partial D) \to (D, \partial D) \) continuous up to the boundary with \( \pi^{-1}(\partial D) = \partial D \) such that \( u = v \circ \pi \). In [14] Theorem B] Lazzarini proved the remarkable fact that generically all non-constant holomorphic discs attached to \( L \) are multiply covered provided \( \dim M \geq 6 \). We drop the restriction to the dimension.

Corollary 2. For generic \( J \in I \) each non-constant holomorphic discs which is contained in the perturbation domain is multiply covered.

\( \hspace{1cm} \)

Proof. Lazzarini's proof of [14, Theorem B] is based on his decomposition theorem [14, Theorem A]. Each non-constant holomorphic disc \( u \) can be cut along an embedded graph in \( D \) into finitely many simple holomorphic discs whose union has
the same image as \( u \) and whose homology classes weighted with positive multiples add up to \([u]\). On \([14\text{, p. 254/5}]\) he shows that the graph equals \( \partial D \) using (\([14\text{, Proposition 5.15}]\), which can replaced by) the following two results:

- Let \( v \) be a simple holomorphic disc. Then there exists no diffeomorphism \( \varphi : S_1 \to S_2 \) between two disjoint open segments on \( \partial D \) such that \( v = v \circ \varphi \).
- Let \( v_1, v_2 \) be simple holomorphic discs with \( v_1(D) \not\subset v_2(D) \) and \( v_2(D) \not\subset v_1(D) \). Then there exists no diffeomorphism \( \varphi : S_1 \to S_2 \) between two disjoint open segments on \( \partial D \) such that \( v_1 = v_2 \circ \varphi \).

Notice that \( v_1, v_2 \) define a somewhere injective holomorphic map on \( D \sqcup D \) in the sense that each connected component has an injective point. An application of the argument from Corollary \([1]\) proves both items. This establishes Lazzarini’s theorem for \( \dim M = 4 \).

\[ \square \]

4.3. **Addendum to the Theorem.** Using Corollary \([1]\) the Theorem can be extended in the case the relation puts conditions on the holomorphic jets of order \( \geq 1 \) along the boundary.

**Corollary 3.** For any \( \mathcal{R}\)-regular \( J \in \mathcal{I}_\infty \) the conclusion of the Theorem holds for \( J \)-holomorphic curves which are contained in the perturbation domain and are somewhere injective.

4.4. **Adjunction inequality.** The group of conformal automorphisms \( G \) of \( (\Sigma, \Gamma) \) acts on the \( \mathcal{R}\)-moduli space for any source-free relation \( \mathcal{R} \) by

\[
g \cdot (u, z, x) = (u \circ g^{-1}, g(z), g(x))
\]

such that the quotient

\[ M_R = \mathcal{U}_J/G \]

is a smooth manifold of dimension

\[
\mu(A) + n(\chi(\Sigma) - \|t\|) + (1 - n)\|m\| + \dim R - d,
\]

where \( d \) is the dimension of \( G \).

A consequence of the discussion in \([3,1]\) we obtain:

**Corollary 4.** For a generic choice of \( J \in \mathcal{I} \) and all \( J \)-holomorphic curves \( u \) which have an injective point on each connected component of \( \Sigma \) which are mapped into the perturbation domain by \( u \) we have:

\[
(n - 2)\|(n_0, n_1)\| + (2n - 3)\ell \leq \mu([u]) + n\chi(\Sigma) - d.
\]

Here \( n_0 \) (resp. \( n_1, \ell \)) is the number of interior (resp. boundary, mixed) double points.

In particular, if the Maslov index vanishes and \( n \geq 3 \) generically somewhere injective holomorphic disc maps are injective.

4.5. **Singularities.** A non-constant holomorphic map \( u \) has a **singularity of order** \( k \) at \( z \) if \( j^k_z u = 0 \) and \( D^{k+1} u(z) \neq 0 \). By Carleman’s similarity principle (cf. \([11\text{, Chapter A.6}]\) and \([1\text{, Theorem A.2}]\)) \( k \) is the greatest natural number such that all partial derivatives of \( u \) at \( z \) vanish up to order \( k \).

For a generic choice of compatible almost complex structure the moduli space of unparametrized somewhere injective (in the sense of Remark \([3,5]\) holomorphic curves with vanishing derivatives at \( m \) points up to order \( k \) has dimension

\[
\mu(A) + n(\chi(\Sigma) - \|k\|) + \|m\| - d.
\]
As in Corollary 3 we obtain, cf. [21, Corollary 3.17] and [17, 16]:

**Corollary 5.** For generic $J \in \mathcal{I}$ and all somewhere injective $J$-holomorphic curves $u$ which are contained in the perturbation domain we have:

$$n\|k\| - \|m\| \leq \mu([u]) + n\chi(\Sigma) - d.$$

Here $k$ is the order of singularities of $u$ at $m$ points.

For example, generically, all somewhere injective holomorphic discs with Maslov index less or equal than 1 are immersed.

**Remark 4.2.** The map $(u, z, x) \mapsto u$ which is defined on the moduli space of curves as in Corollary 5 is an immersion. This is because the kernel of the linearization $(\xi, v, w) \mapsto \xi$ is given by all $(v, w) \in T\mathcal{S}$ satisfying $j^*_{(z, x)} Tu(v, w) = 0$.

**4.6. Example.** Consider a rational split Lagrangian submanifold $L = S^1 \times L'$ in $\mathbb{R}^2 \times \mathbb{R}^{2n-2}$ with area spectrum $\pi\mathbb{Z}$. For generic compatible almost complex structures which equal $i$ outside a large ball we consider holomorphic discs representing the class $[D \times \{\ast\}]$, which are simple, see [13, 14]. By Corollary 5 there is at most one singularity on each disc, which is simple and on the boundary. Hence the zero-dimensional moduli space of discs with a simple boundary singularity $\mathcal{M}_{\partial, \text{sing}}$ embeds via $[u, z, x] \mapsto [u]$ into the moduli space of all discs $\mathcal{M}$. By Gromov compactness $\mathcal{M}$ is compact, see [9, 10]. Therefore, there are finitely many discs with singularities. The number is even as an argument using a cobordism with boundary $\mathcal{M}_{\partial, \text{sing}}$ which corresponds to a generic path of almost complex structures with starting point $i$ shows, cf. [15, p. 43 and Remark 3.2.8].

**4.7. Embeddings.** We consider somewhere injective holomorphic curves contained in the perturbation domain which represent a homology class $A$. Generically, the corresponding moduli space $\mathcal{M}$ is a manifolds of dimension $\mu(A) + n\chi(\Sigma) - d$. We can assume that the moduli spaces of curves which additionally have a singularity at the interior or on the boundary are manifolds of dimension

- $\dim \mathcal{M}_{\text{sing}} = \dim \mathcal{M} + 2(1 - n)$
- $\dim \mathcal{M}_{\partial, \text{sing}} = \dim \mathcal{M} + 1 - n$

these with interior, mixed, and boundary double points are manifolds of dimension

- $\dim \mathcal{M}_{\text{inter}} = \dim \mathcal{M} + 2(2 - n)$
- $\dim \mathcal{M}_{\text{mix}} = \dim \mathcal{M} + 3 - 2n$
- $\dim \mathcal{M}_{\partial, \text{inter}} = \dim \mathcal{M} + 2 - n$

and that the moduli space of curves intersection $L$ at an interior point is a manifold of dimension

- $\dim \mathcal{M}_L = \dim \mathcal{M} + 2 - n$.

Each of the above moduli spaces admits a smooth map into $\mathcal{M}$ induced by forgetting the marked points. The complement of the images is the subset of perfectly embedded curves. By Sards theorem the union of the critical values has measure zero. Because the closure of the images of the moduli spaces subject to a pure intersection relation is contained in the union with the images of $\mathcal{M}_{\text{sing}}$ and $\mathcal{M}_{\partial, \text{sing}}$ we obtain:

**Corollary 6.** Generically, the set of all perfectly embedded holomorphic curves in $\mathcal{M}$ is open and dense provided $\dim \mathcal{M} \geq 6$. 
An immersed holomorphic curve is called **clean** if there is no mixed double point and all double points and all interior intersections with $L$ are simple and transverse. With similar arguments we get:

**Corollary 7.** If $\dim M = 4$ the set of all clean immersions in $\mathcal{M}$ generically is open and dense.

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