FINITE DIMENSIONAL REPRESENTATIONS OF THE DOUBLE AFFINE HECKE ALGEBRA OF RANK 1

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Abstract. We classify the finite dimensional irreducible representations of the double affine Hecke algebra (DAHA) of type $C^\vee C_1$ in the case when $q$ is not a root of unity.

1. Introduction

The double affine Hecke algebras (DAHA) were introduced by Cherednik at the beginning of nineties [1],[2]. Roughly speaking, DAHA is a nontrivial extension of the affine Hecke algebra corresponding to an affine root system by the group algebra of the coroot lattice. Later, it was discovered that these algebras were connected to many areas of mathematics. For instance, DAHAs lie at the core of the proof of the Macdonald conjectures, given by Cherednik [2]. Subsequently, Etingof and Ginzburg [3] revealed deep connections between the theory of the rational degenerations of DAHA and noncommutative geometry.

In this paper, we study the finite dimensional complex representations of the DAHA of rank 1, namely the algebra $H = H(k_0, k_1, u_0, u_1; q)$ corresponding to the root system $C C_1^\vee$. This algebra was defined by Sahi [4] (see also [5]). As it was shown in [6], it is the most general DAHA of rank 1.

We restrict our attention to the case when $q$ is not a root of unity. The case when $q = 1$ and the parameters $k_0, k_1, u_0, u_1$ are generic was studied in [6]. In the special case when $k_0 = u_0 = u_1 = 1$, the algebra $H$ coincides with DAHA of type $A_1$. For this algebra, the classification of the finite dimensional representations has been done in [7]. We classify the finite dimensional representations of the algebra $H(k_0, k_1, u_0, u_1; q)$ (see theorem [1] in the main body of the paper) and give a description of these representations in terms of the polynomial representations of $H$. In particular, we generalize the result from [8]. The proof of the main result of the paper hinges on results of Crawley-Boevey on Deligne-Simpson problem [9].

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\[1\] Let us remark that the classification from [7] has a small gap fixed in the book [8].
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2. The double affine Hecke algebra of rank 1

Throughout this paper, we will work over the field of complex numbers. Let us consider $q^{1/2}$ a fixed complex number.

**Definition 1.** Let $k_0, k_1, u_0, u_1, q \in \mathbb{C}^*$. The double affine Hecke algebra $H = H(k_0, k_1, u_0, u_1; q)$ of rank 1 of type $CC_1^\vee$ is generated by the elements $T_0, T_1, T_0^\vee, T_1^\vee$ with the relations:

\[
\begin{align*}
(T_0 - k_0)(T_0 + k_0^{-1}) &= 0, \\
(T_1 - k_1)(T_1 + k_1^{-1}) &= 0, \\
(T_0^\vee - u_0)(T_0^\vee + u_0^{-1}) &= 0, \\
(T_1^\vee - u_1)(T_1^\vee + u_1^{-1}) &= 0, \\
T_1^\vee T_1 T_0 T_0^\vee &= q^{-1/2}.
\end{align*}
\]

It is convenient to introduce the following notation:

\[
\mathbf{t} = (t_1, t_2, t_3, t_4) := (k_0, k_1, u_0, u_1).
\]

2.1. Polynomial representations of $H$. We give a description of the finite dimensional irreducible representations of $H$ in terms of the polynomial representations $P_{\pm \delta, \pm \epsilon}$ and $\overline{P}_{\pm \delta, \pm \epsilon}$. Let us describe these representations and their basic properties.

For each of the subalgebras $H_{\text{aff}} = \langle T_1, T_0 \rangle$ and $H_{\text{aff}}^\vee = \langle T_1^\vee, T_0^\vee \rangle$ of $H$, we define the one-dimensional representations $\chi(\epsilon_0, \epsilon_1)$ by

\[
T_i \to \epsilon_i k_i^{\epsilon_i}
\]

for $\epsilon_i = \pm 1$, $i = 0, 1$ and $\chi^\vee(\delta_0, \delta_1)$ by

\[
T_i^\vee \to \delta_i k_i^{\delta_i}
\]

for $\delta_i = \pm 1$ and $i = 0, 1$. We define the following polynomial representations of the algebra $H$:

\[
P^{\epsilon_0, \epsilon_1} = H \otimes H_{\text{aff}} \chi(\epsilon_1, \epsilon_0),
\]

\[
\overline{P}^{\epsilon_0, \epsilon_1} = H \otimes H_{\text{aff}}^\vee \chi^\vee(\delta_1, \delta_0).
\]

In [5], the action of $T_i$ and $T_i^\vee$ on the representation $P^{\pm 1, \pm 1}$ is presented. The other cases are slight modifications of this one. The action of $T_0, T_1$ on the vectors from $P^{\epsilon_0, \epsilon_1} \simeq \mathbb{C}[z^{\pm 1}]$ is given by the operators

\[
T_0 = \epsilon_0 k_0^{\epsilon_0} s_0 + \frac{(k_0 - k_0^{-1}) + (u_0 - u_0^{-1}) q^{1/2} z^{-1}}{(1 - q z^{-2})},
\]

\[
T_1 = \epsilon_1 k_1^{\epsilon_1} s_1 + \frac{(k_1 - k_1^{-1}) + (u_1 - u_1^{-1}) z}{(1 - z^2)}.
\]
where \( s_0(z^n) = q^n z^{-n} \), \( s_1(z^n) = z^{-n} \) and the element \( q^{1/2} T_0 T_0^\vee \) acts by multiplication by \( z \).

Similarly, the elements \( T_0^\vee \) and \( T_1^\vee \) act on \( \mathcal{P}_{\delta_0, \delta_1} \simeq \mathbb{C}[z^{\pm 1}] \) by the operators

\[
T_0^\vee = \delta_0 u_0^\delta s_0 + \frac{(u_0 - u_0^{-1}) + (k_0 - k_0^{-1}) q^{1/2} z^{-1}}{(1 - q z^{-2})},
\]

\[
T_1^\vee = \delta_1 u_1^\delta s_1 + \frac{(u_1 - u_1^{-1}) + (k_1 - k_1^{-1}) z}{(1 - z^2)},
\]

and the element \( (T_1^\vee T_1)^{-1} \) acts by multiplication by \( z \).

3. Classification of the finite dimensional representations

3.1. The root system \( D_4^{(1)} \). The classification uses the affine root system \( D_4^{(1)} \) \[11\]. The Dynkin graph for \( D_4^{(1)} \) is a star with four legs of length 1. The positive roots correspond to labelings of the vertices of the graph by positive integers with the special condition explained, for example, in \[9\].

Let \( a_1, a_2, a_3, a_4 \) be the labels on the nodes of the four legs, and \( a_0 \) the label on the central node of the graph. Thus, we can think of the positive root \( \alpha \) as a vector \((a_0, a_1, a_2, a_3, a_4)\) in \( \mathbb{N}^5 \).

The positive roots are of two kinds, real and imaginary. We consider only the strict roots, namely those whose labels of the nodes at the legs are not smaller than the label of the central node. The imaginary roots are of the form \( (n, n, n, n, 2n) = n\Delta \), where \( n \) is a natural number. The real roots are of two types. The strict roots of the first type are of the form

\[
r_{i,\epsilon,n} = n\Delta + \epsilon e_i,
\]

where \( i = 0, 1, \epsilon = \pm 1, n > 0 \) and \( e_0 = (0,1,0,0,0) \), \( e_1 = (0,0,1,0,0) \), or

\[
r_{i,\delta,n}^\vee = n\Delta + \delta f_i,
\]

where \( i = 0, 1, \delta = \pm 1, n > 0 \) and \( f_0 = (0,0,0,1,0) \), \( f_1 = (0,0,0,0,1) \). The strict roots of the second type are of the form

\[
r_{c,\delta,n} = c + n\Delta + \sum_{i=0}^{1} \frac{1 - \epsilon_i}{2} e_i + \sum_{i=0}^{1} \frac{1 - \delta_i}{2} f_i,
\]

where \( \delta_i, \epsilon_i = \pm 1, n > 0 \) and \( c = (1,0,0,0,0) \).

3.2. Deligne-Simpson problem. It is easy to see that problem of classification of the representation of \( H \) is closely related to conjecture 1.4 from the paper \[9\] about the Deligne-Simpson problem (they are almost equivalent).

Deligne-Simpson problem poses the question of describing the set of solutions of the system of equations:

\[
A_1 \cdots A_k = 1,
\]

\[
A_i \in C_i, \quad i = 1, \ldots, k,
\]
where $C_i \subset GL(n, \mathbb{C})$ are the conjugacy classes defined by the equations:

$$\alpha_{ij} = \text{rank}((A_i - \xi_{i1}) \cdots (A_i - \xi_{ij})), \quad j = 1, \ldots, w_i,$$

with $\alpha_{i,w_i} = 0$ and $\alpha_{i,0} = n$.

Let $\Gamma_w$ be the star-shaped graph with $k$ legs of length $w_1, \ldots, w_k$. The conjecture 1.4 from [9] claims that there exists an irreducible solution of the Deligne-Simpson problem if and only if $\alpha$ is a positive root of the root system associated to $\Gamma_w$ (see [9] for the definition) and the following equation is satisfied:

$$\xi^{[\alpha]} = 1, \quad \xi^{[\alpha]} := \prod_{i=1}^{k} \prod_{j=1}^{w_i} \xi_{ij}^{\alpha_{i,j-1} - \alpha_{ij}},$$

together with some with some nonresonance conditions (see [9] for more details).

If $V$ is an irreducible representation of $H(t,q)$ then the matrices:

$$A_1 = q^{1/2}T_0, \quad A_2 = T_0^\vee, \quad A_3 = T_1, \quad A_4 = T_1^\vee,$$

is a solution of the Deligne-Simpson problem with $w = (2, 2, 2, 2)$ and the parameters

$$\xi_{11} = k_0 q^{1/2}, \quad \xi_{12} = -k_0^{-1} q^{1/2},$$

$$\xi_{21} = u_0, \quad \xi_{22} = -u_0^{-1},$$

$$\xi_{31} = u_1, \quad \xi_{32} = -u_1^{-1},$$

$$\xi_{41} = k_1, \quad \xi_{42} = -k_1^{-1}.$$

3.3. To state the main theorem of the paper we need to introduce the locally closed subset $\Sigma_\alpha$ ($\alpha$ is a real root of $D_4^{(1)}$) inside the parameter space $\mathbb{C}^4$:

- If $\alpha = r_i^{e_i,n}$ for some $e_i = \pm 1$ and $\delta_i = \pm 1$, $i = 0, 1$ then $t \in \Sigma_\alpha$ iff
  $$\epsilon_1 k_1^{e_i} \epsilon_0 k_0^{e_0} \delta_1 u_1^{\delta_1} \delta_0 u_0^{\delta_0} = q^{-1/2-n}$$
  and $k_1^{2e_i} \neq -q^{m_i}, u_1^{2e_i} \neq -q^{m_i}$ for all $i = 0, 1$, and $(1 + \epsilon_i)/2 \leq m \leq n - 1$.

- If $\alpha = r_{i,\epsilon,n}$ for some $i = 0, 1$ and some $\epsilon = \pm 1$ then $t \in \Sigma_\alpha$ iff
  $$k_1^{2e_i} = -q^n$$
  and $\epsilon_1 k_1^{e_i} \epsilon_0 k_0^{e_0} \delta_1 u_1^{\delta_1} \delta_0 u_0^{\delta_0} \neq q^{-1/2-m}$ for all $m = 0, 1, \ldots, n - 1$.

- If $\alpha = r_i^{\delta,n}$ for some $i = 0, 1$ and some $\delta = \pm 1$ then $t \in \Sigma_\alpha$ iff
  $$k_1^{2e_i} = -q^n$$
  and $\epsilon_1 k_1^{e_i} \epsilon_0 k_0^{e_0} \delta_1 u_1^{\delta_1} \delta_0 u_0^{\delta_0} \neq q^{-1/2-m}$ for all $m = 0, 1, \ldots, n - 1$.

Given a root $\alpha = (\alpha_0, a_1, a_2, a_3, a_4)$ we can set $a_{i1} = a_1$ and $a_{i0} = a_0$ then the the equation for the closure $\Sigma_\alpha$ is of the form $\xi^{[\alpha]} = 1$.\]
3.4. Classification. Let us define the polynomials
\[
E_n(a; z) = z^{-n} \prod_{i=-n}^{-1} (z - q^i/a) \prod_{i=0}^{n} (z - aq^i),
\]
\[
E_{-n}(a; z) = z^{-n} \prod_{i=-n}^{-1} (z - q^i/a) \prod_{i=0}^{n-1} (z - aq^i).
\]

Suppose that \( V \) is a finite dimensional representation of \( H(\mathfrak{t}, q) \). Then we define \( \dim(V) = \bar{d} = (d_0, d_1, d_2, d_3, d_4) \) where \( d_0 = \dim(V), d_1 = \dim(\text{Im}(T_0 - k_0)), d_2 = \dim(\text{Im}(T_1 - k_1)), d_3 = \dim(\text{Im}(T_0^\vee - u_0)), d_4 = \dim(\text{Im}(T_1^\vee - u_1)) \).

**Theorem 1.** Suppose that \( q \) is not a root of 1. Then
- The algebra \( H = H(\mathfrak{t}; q) \) has a finite dimensional irreducible representation of dimension \( \bar{d} \) iff \( \alpha = d_0c + d_1e_0 + d_2e_1 + d_3f_0 + d_4f_1 \) is a strict real root for the \( D_4^{(1)} \) root system and \( \mathfrak{t} \in \Sigma_\alpha \).
- The algebra \( H \) could have only one irreducible representation of dimension \( \alpha \). This representation \( T_\alpha \) has an explicit description as a quotient of the polynomial representation:
\[
\begin{align*}
V_\alpha &= \mathcal{P}^{\alpha,0,1}/(E_n(\epsilon_0k_0^\alpha \delta_0u_0^\alpha, z)), & \text{when } \alpha = r_{\pm,0,n} \\
V_\alpha &= \mathcal{P}^{\alpha,1}/(E_n(q^{1/2}\epsilon k_0^\alpha u_0; z)), & \text{when } \alpha = r_{0,\pm,n} \\
V_\alpha &= \mathcal{P}^{\alpha,1}/(E_n(q^{1/2}\epsilon k_1^\alpha u_1; z)), & \text{when } \alpha = r_{1,\pm,n} \\
V_\alpha &= \tilde{\mathcal{P}}^{\alpha,1}/(E_n(\delta u_0^\alpha k_0^\alpha; z)), & \text{when } \alpha = r_{0,\pm,n} \\
V_\alpha &= \tilde{\mathcal{P}}^{\alpha,1}/(E_n(\delta u_1^\alpha k_0^\alpha; z)), & \text{when } \alpha = r_{1,\pm,n}.
\end{align*}
\]

**Proof.** In the next section we prove that all finite dimensional representations of \( H \) are rigid (see Lemma 11). Hence the first part of the theorem is equivalent to theorem 1.5 from [9].

Let us give the proof for the first case of the second half of the theorem. If we have a representation \( V \) of dimension \( \alpha = r_{\pm,0,n} \) then from the definition of dimension we get that there exist vectors \( w, v \in V \) such that
\[
\begin{align*}
T_0 w &= \epsilon_0 k_0^{\alpha_0} w, & T_1 w &= \epsilon_1 k_1^{\alpha_1} w, \\
T_0^\vee v &= \delta_0 u_0^{\alpha_0} v, & T_0 v &= \epsilon_0 k_0^{\alpha_0} v.
\end{align*}
\]

Indeed, if we know that \( k_i \neq -k_i^{-1}, u_i \neq -u_i^{-1}, i = 0, 1 \) then \( T_i, T_i^\vee \) are semisimple and the statement follows from the first part. If say \( k_i = -k_i^{-1} \) for some \( i = 0, 1 \) then the first part implies \( \mathfrak{t} \in \Sigma_\alpha \). Hence \( \epsilon_i = 1 \) and \( \dim(\text{ker}(T_i - \epsilon_i k_i)) = n + 1 \).

This implies that the map \( P_{\mathfrak{t}} \ni 1 \leftrightarrow v \in V \) extends to an \( H \)-homomorphism \( \varphi: \mathcal{P}_{\mathfrak{t}} \to V \). The kernel of \( \varphi \) is the ideal \( (E) \), where \( E \in \mathbb{C}[z^\pm 1] = \mathcal{P}_{\mathfrak{t}} \) is a Laurent polynomial.
To identify $E$ we need to find the spectrum of $z = q^{1/2}T_0T_0'$ on $V$. It is easy because for the same reason as before we have the $H$-homomorphism 
\[ \psi: V \to H \otimes H' \chi = \mathcal{P}' \] 
where $H'$ is a subalgebra of $H$ generated by $T_0, T_0'$:
\[ \chi(T_0) = \epsilon_0 k_0, \quad \chi(T_0') = \delta_0 u_0. \]

Because of the PBW theorem we have a canonical identification $\mathcal{P}' \simeq \mathbb{C}[y^{\pm 1}]$ with $y = T_1 T_0$. Analogously to the polynomial representations $\mathcal{P}_L$ (see for example [5] section 3.5) we can introduce a complete ordering on the monomials:
\[ 1 < y^{-1} < y < y^{-2} < y^2 < \ldots. \]
The action of the operator $z = q^{1/2}T_0T_0'$ is upper triangular with respect to this ordering:
\[ z(y^i) = \rho_i y^i + \text{lower order terms}, \]
where $\rho_i = \epsilon_0 \delta_0 k_0^0 u_0^{-\delta_0} q^{1/2+i}$ for $i \geq 0$ and $\rho_i = \epsilon_0 \delta_0 k_0^{-\epsilon_0} u_0^{-\delta_0} q^{1/2+i}$ for $i < 0$.

Thus we have the identification $V \simeq \mathcal{P}'/(E')$ where $E' \in \mathbb{C}[y^{\pm 1}]$. We know that $\dim(V) = 2n + 1$, hence we can choose $E'$ in the form
\[ E' = y^n + cy^{-n} + \text{lower order terms}, \]
with $c \neq 0$. This implies that the monomials $1, y^{-1}, y, \ldots, y^{-n}, y^n$ project onto a basis in $\mathcal{P}'/(E')$. The upper triangularity of $z$ implies that the spectrum of $z$ is $\rho_{-n}, \ldots, \rho_n$. This implies $E = E_n(\epsilon_0 k_0^0 \delta_0 k_0^{\delta_0}, z)$. It can be shown by a direct computation that the ideal $(E_n(\epsilon_0 k_0^0 \delta_0 k_0^{\delta_0}, z))$ is invariant under the action of $T_0, T_1$ if $\mathfrak{L} \in \Sigma_\alpha$.

\[ \square \]

3.5. Rigidity of the finite dimensional representations. In this section we prove

**Lemma 1.** Suppose that $q$ is not a root of unity and the algebra $H = H(\mathfrak{L}, q)$ has a finite dimensional irreducible representation $V$. Then this representation is rigid, i.e. it does not admit any deformations.

The proof of Lemma 1 goes along the lines of the calculation from theorem 5.7 in [10].

**Proposition 1.** Let $C_i \subset GL(n), i = 1, \ldots, 4$ be the conjugacy classes in $GL(n)$ (i.e. subset of matrices with fixed Jordan normal form) and $\mathcal{C} = C_1 \times C_2 \times C_3 \times C_4$ be the subset of the irreducible 4-tuples. Let $\Delta \subset \mathcal{C}$ be the subset of 4-tuples $(a, b, c, d)$ of matrices with the property $abcd = q^{1/2}$. If
\[ \det(C_1) \det(C_2) \det(C_3) \det(C_4) = q^{n/2} \]
then
\[ \dim \Delta = \dim C_1 + \dim C_2 + \dim C_3 + \dim C_4 - n^2 + 1, \]
when RHS is nonnegative and $\Delta = \emptyset$ if RHS is negative.

**Proof.** Obviously we have the equality
\[ \dim \mathcal{C} = \dim C_1 + \dim C_2 + \dim C_3 + \dim C_4, \]
and inequality

\[ \dim \Delta \geq \dim \mathcal{C} - (n^2 - 1). \]

To prove the opposite inequality we need to show that the value \( q^{1/2}d^{-1} \) is regular for the map \( \nu : C_1 \times C_2 \times C_3 \to GL(n) \) given by \( \nu(a, b, c) = abc \). Let us recall that in our case the value \( q^{1/2}d^{-1} \) is regular if and only if the dimension of the image of the linear map \( d\nu \) is equal to \( n^2 - 1 \). Let us explain why this is true.

On the space of matrices there is a natural nondegenerate pairing \( x, y \mapsto tr(xy) \). Let us use this pairing to describe the orthogonal complement to the image of the differential of the map \( \nu \). It is easy to see that \( t \) belongs to the orthogonal complement if and only if it satisfies equations

\[ (1) \quad [a, bct] = 0, \quad [cta, b] = 0, \quad [tab, c] = 0. \]

Indeed let \( u = [x, a], v = [y, b], w = [z, c] \) be the tangent vectors to \( C_1, C_2, C_3 \) at the points \( a, b, c \). Then \( t \) satisfies the equation \( tr(t([x, a]bc + a[y, b]c + ab[z, c])) \) for any \( x, y, z \). Now the cyclic invariance of the trace implies the equations (1).

Set \( r = bcta \). Then \( [a, r] = [b, r] = [c, r] = 0 \). Since \( a, b, c \) are irreducible, \( r \) is a scalar, and \( t = \lambda a^{-1}b^{-1}c^{-1} \) for some \( \lambda \in \mathbb{C} \). That is, the value \( q^{1/2}d^{-1} \) is regular.

**Proof of Lemma** Suppose \( V \) is a representation of the algebra \( H(\mathfrak{L}; q) \) of dimension \( n \). Let \( C_1, C_2, C_3, C_4 \) be the conjugacy classes of the elements \( T_0, T_0^\vee, T_1^\vee, T_1 \in \text{End}(V) \). The quadratic relations for the elements \( T_0, T_0^\vee, T_1^\vee, T_1 \) give the restrictions on the possible type of the Jordan normal form of the elements from \( C_i \), \( i = 1, \ldots, 4 \). Namely there are two cases.

The first case is when \( t_i^2 \neq -1 \). In this case all matrices from \( C_i \) are diagonalizable. That is, there is a number \( 1 \leq d_i \leq n \) such that for all matrices \( X \in C_i \) we have \( \dim Ker(X - t_i) = d_i \) and \( \dim Ker(X + t_i^{-1}) = n - d_i \). In this case \( \dim C_i = 2d_i(n - d_i) \).

The second case is when \( t_i^2 = -1 \). In this case there is a number \( 1 \leq d_i \leq n \) such that \( \dim Ker(X - t_i) = d_i \) for all \( X \in C_i \). That is, the Jordan normal form of \( X \) has \( n - d_i \) Jordan block of size 2. In this case we have, again, \( \dim C_i = 2d_i(n - d_i) \).

Now let us remark that the subset \( \Delta \) from the previous lemma is the subset of \( n \)-dimensional irreducible representations of \( H(\mathfrak{L}; q) \) with fixed Jordan normal forms of elements \( T_0, T_0^\vee, T_1^\vee, T_1 \). The action of \( PGL(n) \) on the space of \( n \)-dimensional irreducible representations is free, hence the space of equivalence classes of irreducible representations \( \Delta/PGL(n) \) has the dimension:

\[ D = D(d_1, d_2, d_3, d_4) = 2(1 - n^2 + \sum_{i=1}^{4} d_i(n - d_i)), \]

where \( d_1 = \dim(Ker(T_0 - t_1)), d_2 = \dim(Ker(T_0^\vee - t_2)), d_3 = \dim(Ker(T_1^\vee - t_3)), d_4 = \dim(Ker(T_1 - t_4)) \).
It is easy to see that if $n = 2N + 1$ then $D \geq 0$ if and only if $d_i = N + \delta_i$, $\delta_i = 0, 1$. In this case $D = 0$ and the calculation of the determinants gives $\prod_{i=1}^{4} (\epsilon_i e_i^{\epsilon_i}) = q^{1/2+N}$, $\epsilon_i = (2d_i - 1)/2$, $i = 1, \ldots, 4$. That is, this irreducible representation is rigid.

If $n = 2N$, there are two possibilities for $D$ to be positive. The first case is when $d_i = N$. In this case we have $D = 2$ and calculation of the determinants gives $q^N = 1$. The second case is when there exists $1 \leq i \leq 4$ and $\epsilon = \pm 1$ such that $d_i = N + \epsilon$ and $d_j = N$ for $j \neq i$. In this case $D = 0$ and $t_i^\epsilon = -q^N$. □

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