STOCHASTIC HOMOGENIZATION OF A CLASS OF NONCONVEX VISCOUS HJ EQUATIONS IN ONE SPACE DIMENSION

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Abstract. We prove homogenization for a class of nonconvex (possibly degenerate) viscous Hamilton-Jacobi equations in stationary ergodic random environments in one space dimension. The results concern Hamiltonians of the form \( G(p) + V(x, \omega) \), where the nonlinearity \( G \) is a minimum of two or more convex functions with the same absolute minimum, and the potential \( V \) is a bounded stationary process satisfying an additional scaled hill and valley condition. This condition is trivially satisfied in the inviscid case, while it is equivalent to the original hill and valley condition of A. Yilmaz and O. Zeitouni [31] in the uniformly elliptic case. Our approach is based on PDE methods and does not rely on representation formulas for solutions. Using only comparison with suitably constructed super- and sub-solutions, we obtain tight upper and lower bounds for solutions with linear initial data \( u(0, x, \omega) \). Another important ingredient is a general result of P. Cardaliaguet and P. E. Souganidis [13] which guarantees the existence of sublinear correctors for all \( \theta \) outside “flat parts” of effective Hamiltonians associated with the convex functions from which \( G \) is built. We derive crucial derivative estimates for these correctors which allow us to use them as correctors for \( G \).

1. Introduction

We are interested in proving a homogenization result as \( \varepsilon \to 0^+ \) for a viscous Hamilton-Jacobi (HJ) equation of the form

\[
\partial_t u^\varepsilon = \varepsilon a \left( \frac{x}{\varepsilon}, \omega \right) \partial_{xx} u^\varepsilon + G \left( \frac{x}{\varepsilon}, \omega \right) + \beta V \left( \frac{x}{\varepsilon}, \omega \right), \quad (t, x) \in (0, +\infty) \times \mathbb{R}, \tag{1.1}
\]

where \( G : \mathbb{R} \to \mathbb{R} \) belongs to a certain class of continuous, nonconvex and coercive functions. Dependence on a realization of random environment \( \omega \) enters through the diffusion coefficient \( a(x, \omega) \) and potential \( V(x, \omega) \) which are assumed to be stationary with respect to shifts in \( x \) and Lipschitz continuous with a constant independent of \( \omega \). Moreover, we suppose that \( a \) and \( V \) take values in \( [0, 1] \) and with probability 1

\[
\text{ess inf}_{x \in \mathbb{R}} V(x, \omega) = 0 \quad \text{and} \quad \text{ess sup}_{x \in \mathbb{R}} V(x, \omega) = 1. \tag{1.2}
\]

Thus, the parameter \( \beta \geq 0 \) represents the “magnitude” of the potential \( V \). For a complete set of conditions on the coefficients and precise statements of our results, we refer to Section 2.

We shall say that the equation (1.1) homogenizes if there exists a continuous function \( H_\beta(G) : \mathbb{R} \to \mathbb{R} \) called effective Hamiltonian and a set \( \Omega_0 \) of probability 1 such that for every \( \omega \in \Omega_0 \) and every uniformly continuous function \( g \) on \( \mathbb{R} \), the solution \( u^\varepsilon \) of (1.1) satisfying \( u^\varepsilon(0, \cdot, \omega) = g \) converges locally uniformly on \( [0, +\infty) \times \mathbb{R} \) as \( \varepsilon \to 0^+ \) to the
unique solution \( \overline{u} \) of the (deterministic) effective equation

\[
\partial_t \overline{u} + \mathcal{H}_\beta(G)(\partial_x \overline{u}) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}
\]

satisfying \( \overline{u}(0, \cdot) = g \). Solutions to all PDEs considered in this paper are understood in the viscosity sense. We refer the reader to [10, 11, 14] for details on viscosity solution theory.

To put our results in a broader context, we shall first briefly review the existing literature on non-convex homogenization of viscous HJ equations.

1.1. Literature review. Equation (1.1) belongs to a general class of viscous HJ equations of the form

\[
\partial_t u^\varepsilon = \varepsilon \text{tr} \left( A \left( \frac{x}{\varepsilon}, \omega \right) D_{xx}^2 u^\varepsilon \right) + H \left( D_x u^\varepsilon, \frac{x}{\varepsilon}, \omega \right), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d,
\]

where the non-negative definite diffusion matrix \( A(x, \omega) \) and the Hamiltonian \( H(p, x, \omega) \) are stationary under the shifts by \( x \in \mathbb{R}^d \) and satisfy some regularity and growth assumptions.

For homogenization results concerning viscous HJ equation (1.4) with convex (with respect to \( p \)) Hamiltonians in the stationary ergodic setting under various sets of assumptions we refer the reader to [3, 4, 6, 22, 24, 25, 27, 28] and references therein.

Recently it was shown by counterexamples for \( H(p, x, \omega) = G(p) + V(x, \omega) \), first for inviscid (i.e. with \( A \equiv 0 \)) HJ equations, [19], and then also for viscous HJ equations with \( A \equiv \text{const.} \), [18], that in two or more space dimensions a strict local saddle point of \( G \) and a specially “tuned” potential in a very slowly mixing random environment can prevent homogenization. It is not known whether the absence of saddle points and/or fast mixing (or even finite range dependence) conditions on the environment would allow to get a general homogenization result. To date, there exist several classes of examples of homogenization for HJ equations with non-convex Hamiltonians in the stationary ergodic setting for all \( d \geq 1 \), [2, 5, 8, 19, 21, 29], but an overall picture is far from being complete. Among these examples the viscous case is considered only in [2] and [13, Corollary 3.9]. Key assumptions in the last two references which facilitate homogenization are:

- [2]: homogeneity of degree \( \alpha > 1 \) of the Hamiltonian with respect to \( p \);
- [13]: homogeneity of degrees 0 and 1 in \( p \) of the diffusion matrix \( A(p, x, \omega) \) and Hamiltonian \( H(p, x, \omega) \) respectively and radial symmetry of the joint law of \( (A, H) \).

We refer to the original papers for precise statements.

However, for \( d = 1 \), equations of the form (1.4) with \( A \equiv 0 \) in stationary ergodic environments are known to homogenize without any additional mixing conditions, [9, 20]. A cornerstone tool used in these papers is the homogenization result for level-set convex Hamiltonians, [5]. The last result covers all \( d \geq 1 \). Its proof crucially uses the assumption that the original equation is of the first order and does not extend to the viscous case.

Nevertheless it is hard to imagine that addition of a viscous term (especially a uniformly elliptic \( A \)) can turn a homogenizable HJ equation into a non-homogenizable one (under a standard set of assumptions). Thus, further attempts are necessary to resolve the issue even in the one-dimensional case.

For \( d = 1 \), apart from already mentioned works [2, 13], there are other classes of examples of homogenization for viscous HJ, [16, 26, 31]. In [16, Section 4] the authors have shown homogenization of (1.4) with \( H(x, p, \omega) \) which are “pinned” at one or several points on the \( p \)-axis and convex in each interval in between. For example, for every \( \alpha > 1 \) the

\[ \text{polynomially mixing of order 1, [32 Section 3.1]} \]
Hamiltonian \( H(p, x, \omega) = |p|^a - c(x, \omega)|p| \) is pinned at \( p = 0 \) (i.e. \( H(0, x, \omega) \equiv \text{const} \)) and convex in \( p \) on each of the two intervals \((-\infty, 0)\) and \((0, +\infty)\).

Clearly, adding a non-constant potential breaks the pinning property. In particular, homogenization of equation (1.4), where \( d = 1, A \equiv \text{const} > 0, \)

\[
H(p, x, \omega) := \frac{1}{2} |p|^2 - c(x, \omega)|p| + \beta V(x, \omega), \quad 0 < c(x, \omega) \leq C, \quad \beta > 0 \quad (1.5)
\]

remained an open problem even when \( c(x, \omega) \equiv c > 0 \). The authors of [31] introduced a novel hill and valley condition on \( V \) (see (AV) in Subsection [B.1]) and were able to handle the case \( c(x, \omega) \equiv \text{const} > 0 \) in the discrete setting of controlled random walks in a random potential on \( \mathbb{Z} \). This work paved out the way for [26] which gave a proof of homogenization for (1.4) with \( A \equiv 1/2 \) and \( H \) as in (1.5) with \( c(x, \omega) \equiv c > 0 \), retaining the hill and valley condition. The case when both \( c(x, \omega) \) and \( V(x, \omega) \) in (1.5) are non-constant is still open.

While the hill and valley condition clearly excludes the classical periodic case, it holds true for a large and representative class of stationary ergodic environments ranging from those with finite range of dependence or exponentially mixing to very slowly mixing or even non-mixing. In the realm of stationary ergodic media, periodic as well as almost-periodic environments constitute a very important but also a very special sub-class treatable by methods based on compactness. Loss of compactness is considered to be one of the main challenges in dealing with general stationary ergodic media. From this point of view, stationary ergodic potentials which satisfy the hill and valley condition can be considered typical, as we argue in Appendix [B] and to which we refer for further discussion and examples. It would certainly be desirable to drop this condition altogether but, given a relatively slow progress in the viscous case in comparison to the inviscid one, the hill and valley condition, a relaxed version of which we retain in this paper, allows us to move forward without imposing any mixing conditions on the environment (in contrast with the widely accepted in the literature finite range dependence case).

1.2. Discussion of the main results. The current paper presents new results on homogenization of (1.1) with non-convex \( G \) which considerably extend those in [26]. Moreover, it gives a much simpler proof which does not rely on Hopf-Cole transformation or stochastic control representation of solutions and is based solely on PDE techniques. We also replaced the hill and valley condition (AV) on the potential \( V \) (see Appendix [B]) imposed in [26] with a weaker scaled hill and valley condition (V2) (see Section [2]). The two conditions are equivalent in the uniformly elliptic case, i.e. when the diffusion coefficient \( a(x, \omega) \) is bounded away from 0, while in the inviscid case (V2) reduces to (1.2) and, thus, does not add any additional restrictions. We refer to Subsection [B.1] for further details.

Let us recall that [26] considered the equation (1.1) where \( a \equiv 1/2 \) and

\[
G(p) = (G^+ \wedge G^-)(p) = \frac{1}{2} |p|^2 - c|p| = \min \left\{ \frac{1}{2} |p|^2 - cp, \frac{1}{2} |p|^2 + cp \right\} \quad (1.6)
\]

assuming that the potential \( V \) is sufficiently regular, satisfies (1.2) and the already mentioned hill and valley condition. Theorem 2.1 of our paper (see Section [2]) establishes homogenization for (1.1) with a (possibly degenerate) Lipschitz continuous diffusion coefficient \( a: \mathbb{R} \times \Omega \rightarrow [0, 1] \) and \( G = G^+ \wedge G^- \), where \( G^\pm \) are convex and coercive functions with \( \min G^+ = \min G^- \) satisfying fairly general assumptions. Theorem 2.3 extends this result to \( G \) which is the minimum of any finite number of such functions as long as all of them have the same absolute minimum. The assumptions on \( V \) are essentially the same as in [26] except that the hill and valley condition (AV) is replaced with (V2).
Even though our general strategy is analogous to that of [26], the technical realization is different and includes significant shortcuts. Just as in [16, 26], an application of [16, Lemma 4.1] reduces the proof of homogenization to showing that for every $\theta \in \mathbb{R}$,

$$
\mathcal{H}^\beta_{\psi}(G)(\theta) := \liminf_{\varepsilon \to 0^+} u^\varepsilon_{\theta}(1, 0, \omega) = \limsup_{\varepsilon \to 0^+} u^\varepsilon_{\theta}(1, 0, \omega) =: \mathcal{H}^\beta_{\psi}(G)(\theta) \ P\text{-a.s.,} \quad (1.7)
$$

where $u^\varepsilon_{\theta}$ is the solution of (1.1) with initial condition $u^\varepsilon_{\theta}(0, x, \omega) = \theta x$. As in [26], we first establish tight upper and lower bounds for the deterministic functions $\mathcal{H}^\beta_{\psi}(G), \mathcal{H}^\beta_{\psi}(G)$ defined above. This is obtained by constructing suitable sub- and super-solutions for equation (1.1) and by comparing them with the solutions $u^\varepsilon_{\theta}$, where we only exploit well known comparison principles and Lipschitz estimates for solutions of (1.1). The proof does not depend on explicit formulas and does not involve stochastic analysis. It is technically much simpler than that in [26].

The proof of (1.7) for $\theta$ outside the intervals where the effective Hamiltonian is constant depends on construction of sublinear correctors associated with $G^\pm$ and on establishing suitable gradient bounds for these correctors, which allow us to use them as correctors associated with $G$. In [26], such properties were established by direct computation since, due to the special form of the nonlinearity in (1.6), the authors were able to represent the correctors via the Feynman-Kac formula. In our more general setting, the existence of sublinear correctors for $G^\pm$ follows from a recent result of P. Cardaliaguet and P. E. Souganidis [13], while the bounds on their derivatives are consequence of suitable comparison principles for the associated viscous HJ equation that we prove in the Appendix A. The construction in [13] provides sublinear correctors which, in general, are not expected to have stationary gradient. Nevertheless, this is true here and it is due to the fact that sublinear solutions of the corresponding viscous HJ equation are unique up to additive constants, as we show in the Appendix A. This remark is included in the statement of Proposition 5.1 even though this stationarity property is not used in our proof of the homogenization result.

Our second result, Theorem 2.3, extends this homogenization result to $G$ which is the minimum of three or more convex functions with same absolute minimum. The argument is new. It is based on the crucial remark that if $G$ is the minimum of two convex functions with same absolute minimum, then homogenization commutes with convexification, see Section 7.

1.3. Outline of the paper. Precise conditions and statements of the main results, Theorem 2.1 and Theorem 2.3, are given in Section 2. Section 3 presents several basic facts which are used throughout the paper. Upper and lower bounds on the effective Hamiltonian are derived in Section 4. Section 5 is devoted to construction of sublinear correctors and derivative estimates. The proofs of the two main theorems are given in Sections 6 and 7. The necessary PDE results are collected in the Appendix A. Appendix B discusses the original and scaled hill and valley conditions in more detail and shows that they are satisfied for a wide range of typical stationary ergodic environments.

Remark 1.1. Below we sometimes refer to “known results in stationary ergodic homogenization”. The results we have in mind are for convex Hamiltonians. They are contained in many papers cited at the beginning of Section 1.1. However, it is probably most convenient to refer to [6] if necessary, as all our assumptions are satisfied in the setting of [6].

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2. Main results

Let \( \Omega \) be a Polish space, \( \mathcal{F} \) be the \( \sigma \)-algebra of Borel subsets of \( \Omega \), and \( \mathbb{P} \) be a complete probability measure on \((\Omega, \mathcal{F})\). We shall denote by \( \mathcal{B} \) the Borel \( \sigma \)-algebra on \( \mathbb{R} \) and equip the product space \( \mathbb{R} \times \Omega \) with the product \( \sigma \)-algebra \( \mathcal{B} \otimes \mathcal{F} \).

We assume that \( \mathbb{P} \) is invariant under the action of a one-parameter group \((\tau_x)_{x \in \mathbb{R}}\) of transformations \( \tau_x : \Omega \to \Omega \). More precisely, we suppose that the mapping \((x, \omega) \mapsto \tau_x \omega \) from \( \mathbb{R} \times \Omega \) to \( \Omega \) is measurable, \( \tau_0 = id \), \( \tau_{x+y} = \tau_x \circ \tau_y \) for all \( x, y \in \mathbb{R} \), and \( \mathbb{P}(\tau_x(E)) = \mathbb{P}(E) \) for every \( E \in \mathcal{F} \) and \( x \in \mathbb{R} \). We also require that the action by \((\tau_x)_{x \in \mathbb{R}}\) is ergodic, i.e. that any measurable function \( f : \Omega \to \mathbb{R} \) such that \( f(\tau_x \omega) = f(\omega) \) a.s. in \( \Omega \) for every fixed \( x \in \mathbb{R} \) is a.s. constant.

A random process \( f : \mathbb{R} \times \Omega \to \mathbb{R} \) is said to be \textit{stationary with respect to the shifts} \( (\tau_x)_{x \in \mathbb{R}} \) if \( f(x + y, \omega) = f(x, \tau_y \omega) \) for all \( x, y \in \mathbb{R} \) and \( \omega \in \Omega \).

Let us consider the unscaled version of (1.1) (i.e. with \( \varepsilon = 1 \))

\[
\partial_t u = a(x, \omega) \partial^2_{xx} u + G(\partial_x u) + \beta V(x, \omega) \quad \text{in } (0, +\infty) \times \mathbb{R},
\]

(2.1)

where \( a, V : \mathbb{R} \times \Omega \to [0, 1] \) are continuous stationary random processes and \( V \) satisfies (V1) and (V2). We shall also assume that for some \( \kappa \in (0, +\infty) \),

\[
(A) \quad \sqrt{a(\cdot, \omega)} : \mathbb{R} \to [0, 1] \quad \text{is } \kappa-\text{Lipschitz continuous for all } \omega \in \Omega;
\]

\[
(V1) \quad V(\cdot, \omega) : \mathbb{R} \to [0, 1] \quad \text{is } \kappa-\text{Lipschitz continuous for all } \omega \in \Omega.
\]

In addition, we shall suppose that \( V \) under \( \mathbb{P} \) satisfies the \textit{scaled hill} (respectively, \textit{scaled valley}) condition:

(V2) for every \( h \in (0, 1) \) and \( y > 0 \) there exists a set \( \Omega(h, y) \) of probability 1 such that, for every \( \omega \in \Omega(h, y) \), there exists \( \ell_1 < \ell_2 \in \mathbb{R} \) and \( \delta \in (0, 1) \) such that

\[
- (a) \quad \int_{\ell_1}^{\ell_2} \frac{1}{a(x, \omega) \wedge \delta} \, dx = 2y;
\]

and

\[
- (h) \quad \text{(hill)} \quad V(\cdot, \omega) \geq h \quad \text{on } [\ell_1, \ell_2];
\]

(respectively,

\[
- (v) \quad \text{(valley)} \quad V(\cdot, \omega) \leq h \quad \text{on } [\ell_1, \ell_2].
\]

Following [26], an interval \( I \) will be called \( h \)-hill (resp. \( h \)-valley) if \( V(x, \omega) \geq h \) (resp. \( V(x, \omega) \leq h \)) for every \( x \in I \).

Next, we introduce the family \( \mathcal{M}(\gamma, \alpha_0, \alpha_1) \) of continuous functions \( G : \mathbb{R} \to \mathbb{R} \) satisfying the following conditions, for fixed constants \( \alpha_0, \alpha_1 > 0 \) and \( \gamma > 1 \):

\[
(G1) \quad \alpha_0 |p|^{\gamma} - 1/\alpha_0 \leq G(p) \leq \alpha_1 (|p|^{\gamma} + 1) \quad \text{for all } x, p \in \mathbb{R};
\]

\[
(G2) \quad |G(p) - G(q)| \leq \alpha_1 (|p| + |q| + 1)^{\gamma - 1} |p - q| \quad \text{for all } p, q \in \mathbb{R}.
\]

The above assumptions guarantee well posedness in \( UC([0, +\infty) \times \mathbb{R}] \) of the Cauchy problem for parabolic equation (2.1) as well as suitable Lipschitz estimates for solutions of (2.1) with linear initial data, see Theorem A.2 and Proposition A.3 in Section A.2. They will be also used to show that condition (H) in [13] is fulfilled, see the proof of Proposition 5.1.
We stress that our results hold (with the same proofs) under any other set of assumptions apt to ensure the same kind of PDE results.

Since functions from $\mathcal{H}(\gamma, \alpha_0, \alpha_1)$ are bounded from below in view of (G1), in the sequel without loss of generality we shall always assume that $G$ is non-negative.

As stated in the introduction, we shall prove homogenization for the rescaled version (1.1) of equation (2.1) for a class of nonconvex functions $G$ in $\mathcal{H}(\gamma, \alpha_0, \alpha_1)$. With a slight abuse of terminology, in the sequel we shall say that equation (2.1) homogenizes if the rescaled equation (1.1) homogenizes.

For given $c_+ \geq c_- \in \mathbb{R}$, let $G^+, G^- : \mathbb{R} \to [0, +\infty)$ be convex functions from $\mathcal{H}(\gamma, \alpha_0, \alpha_1)$ with $G^+(c_+) = G^-(c_-) = 0$. Let us furthermore assume that there exists $\hat{p} \in [c_-, c_+]$ such that

$$(G^- \wedge G^+)(p) = G^-(p) \text{ if } p < \hat{p}, \quad (G^- \wedge G^+)(p) = G^+(p) \text{ if } p \geq \hat{p}.$$ 

By well-known results in stationary ergodic homogenization, the equation (2.1) with $G := G^\pm$ homogenizes and the effective Hamiltonian $\mathcal{H}_\beta(G^\pm)$ is convex. We shall prove that equation (2.1) homogenizes for $G := G^- \wedge G^+$ as well. The precise statement is given in the next theorem.

**Theorem 2.1.** Let $a, V : \mathbb{R} \times \Omega \to [0, 1]$ be continuous stationary processes satisfying (A), (1.2), (V1)–(V2) and $G^+, G^- : \mathbb{R} \to [0, +\infty)$ be convex functions as above. Then the viscous HJ equation (2.1) with $G := G^- \wedge G^+$ homogenizes and the effective Hamiltonian $\mathcal{H}_\beta(G^- \wedge G^+)$ can be characterized as follows:

(a) (Strong potential) if $\beta \geq (G^- \wedge G^+)(\hat{p})$, then

$$\mathcal{H}_\beta(G^- \wedge G^+)(\theta) = \begin{cases} \mathcal{H}_\beta(G^+)(\theta) & \text{if } \theta > c_+ \\ \beta & \text{if } c_- \leq \theta \leq c_+ \\ \mathcal{H}_\beta(G^-)(\theta) & \text{if } \theta < c_-; \end{cases}$$

(b) (Weak potential) if $\beta < (G^- \wedge G^+)(\hat{p})$, then

$$\mathcal{H}_\beta(G^- \wedge G^+)(\theta) = \begin{cases} \mathcal{H}_\beta(G^+)(\theta) & \text{if } \theta > \theta_+ \\ (G^- \wedge G^+)(\hat{p}) & \text{if } \theta_- \leq \theta \leq \theta_+ \\ \mathcal{H}_\beta(G^-)(\theta) & \text{if } \theta < \theta_-; \end{cases}$$

where $\theta_+$ (resp. $\theta_-$) is the unique solution in $[\hat{p}, c_+]$ (resp. $[c_-, \hat{p}]$) of the equation

$$\mathcal{H}_\beta(G^+)(\theta) = (G^- \wedge G^+)(\hat{p}) \quad \text{(resp. } \mathcal{H}_\beta(G^-)(\theta) = (G^- \wedge G^+)(\hat{p}) \text{).}$$

**Remark 2.2.** As we shall see, $\mathcal{H}_\beta(G^\pm) \geq \beta$ on $\mathbb{R}$ and $\mathcal{H}_\beta(G^-)(c_-) = \mathcal{H}_\beta(G^+)(c_+) = \beta$, see Proposition 3.1. Hence, item (a) above amounts to saying that $\mathcal{H}_\beta(G^- \wedge G^+)$ is the lower convex envelope of the functions $\mathcal{H}_\beta(G^+)$ and $\mathcal{H}_\beta(G^-)$. “Convexification” of the effective Hamiltonian in the strong potential case has been already observed in the non-viscous case, see [3][0][29].

Our second result generalizes Theorem 2.1 to Hamiltonians which can be represented as a minimum of more than two convex Hamiltonians. More precisely, let $n \in \mathbb{N}$ with $n \geq 2$ and $G_0, G_1, \ldots, G_n \in \mathcal{H}(\gamma, \alpha_0, \alpha_1)$ be convex non-negative functions such that $G_0(c_0) = G_1(c_1) = \cdots = G_n(c_n) = 0$ for some $c_0 = c_1 < \cdots < c_n$ and, for each $i \in \{0, 1, \ldots, n-1\}$,

$$(G_i \wedge G_{i+1})(p) = G_i(p) \text{ if } p < \tilde{p}_{i,i+1}, \quad (G_i \wedge G_{i+1})(p) = G_{i+1}(p) \text{ if } p \geq \tilde{p}_{i,i+1}$$

for some $\tilde{p}_{i,i+1} \in (c_i, c_{i+1})$. 


Theorem 2.3. Let \( a, V : \mathbb{R} \times \Omega \to [0, 1] \) be continuous stationary processes satisfying (A), (1.2), (V1)-(V2), \( n \geq 2 \), and \( G_0, G_1, \ldots, G_n : \mathbb{R} \to [0, +\infty) \) be convex functions as above. Then the viscous HJ equation (2.1) with \( G := G_0 \land G_1 \land \cdots \land G_n \) homogenizes and the effective Hamiltonian \( \mathcal{H}_\beta(G_0 \land G_1 \land \cdots \land G_n) \) is given by the following formula:

\[
\mathcal{H}_\beta(G_0 \land G_1 \land \cdots \land G_n)(\theta) = \min_{i \in \{1, 2, \ldots, n\}} \mathcal{H}_\beta(G_{i-1} \land G_i)(\theta)
\]

\[
= \begin{cases} 
\mathcal{H}_\beta(G_0 \land G_1)(\theta), & \text{if } \theta \leq c_1; \\
\mathcal{H}_\beta(G_{i-1} \land G_i)(\theta), & \text{if } c_{i-1} < \theta \leq c_i, \quad i \in \{2, 3, \ldots, n-1\}; \\
\mathcal{H}_\beta(G_{n-1} \land G_n)(\theta), & \text{if } \theta > c_{n-1}.
\end{cases}
\]

Remark 2.4. To avoid repetition, we assume throughout the paper without further mention that \( a, V : \mathbb{R} \times \Omega \to [0, 1] \) are continuous stationary processes satisfying (A), (1.2), and (V1). Condition (V2) will be imposed only as needed.

3. Preliminaries

For a given \( G \in \mathcal{H}(\gamma, \alpha_0, \alpha_1) \), let us denote by \( u_\theta \) the unique Lipschitz solution to (2.1) with initial condition \( u_0(0, x) = \theta x \) on \( \mathbb{R} \), and define the following deterministic quantities, defined almost surely in \( \Omega \), see Proposition 3.1 below for the details:

\[
\mathcal{H}_\beta^U(G)(\theta) := \limsup_{t \to +\infty} \frac{u_\theta(t, 0, \omega)}{t}, \quad \mathcal{H}_\beta^L(G)(\theta) := \liminf_{t \to +\infty} \frac{u_\theta(t, 0, \omega)}{t}.
\]

Observe that, if we denote by \( u_\theta^0 \) the solution of (1.1) with initial condition \( u_\theta^0(0, x, \omega) = \theta x \) then we have \( u_\theta^0(t, x, \omega) = \varepsilon u_\theta(t/\varepsilon, x/\varepsilon, \omega) \). Thus, the above definition of \( \mathcal{H}_\beta^U(G)(\theta) \) and \( \mathcal{H}_\beta^L(G)(\theta) \) is consistent with the one given in (1.7).

In view of [16, Lemma 4.1] and Proposition A.3 in order to prove homogenization it is enough to show that \( \mathcal{H}_\beta^U(G)(\theta) = \mathcal{H}_\beta^L(G)(\theta) \) for every \( \theta \in \mathbb{R} \). In this instance, their common value will be denoted by \( \mathcal{H}_\beta(G)(\theta) \). The function \( \mathcal{H}_\beta(G) : \mathbb{R} \to \mathbb{R} \) is the effective Hamiltonian associated to \( G \).

The following holds:

Proposition 3.1. Let \( G \in \mathcal{H}(\gamma, \alpha_0, \alpha_1) \). Then the limits in (3.1) above are almost surely constant and, moreover,

(i) \( \mathcal{H}_\beta^U(G)(\theta) \geq \mathcal{H}_\beta^L(G)(\theta) \geq \alpha_0 |\theta|^{\gamma} - 1/\alpha_0 \) for all \( \theta \in \mathbb{R} \);

(ii) if \( V \) satisfies the scaled hill condition (V2)(a),(h) then \( \mathcal{H}_\beta^L(G)(\theta) \geq \beta \) for every \( \theta \in \mathbb{R} \);

(iii) for every \( \theta \in \mathbb{R} \), the functions \( \beta \mapsto \mathcal{H}_\beta^L(G)(\theta) \) and \( \beta \mapsto \mathcal{H}_\beta^U(G)(\theta) \) are nondecreasing and Lipschitz continuous with respect to \( \beta > 0 \);

(iv) if \( G(0) = 0 \), then \( \mathcal{H}_\beta^L(G)(0) = \mathcal{H}_\beta^U(G)(0) = \beta \).

If, in addition, \( G \) is convex, then \( \mathcal{H}_\beta^L(G)(\theta) = \mathcal{H}_\beta^U(G)(\theta) =: \mathcal{H}_\beta(G)(\theta) \) for all \( \theta \in \mathbb{R} \), and the function \( \mathcal{H}_\beta(G) : \mathbb{R} \to \mathbb{R} \) is convex.

Proof. Recall that \( u_\theta \) denotes the unique solution of (2.1) with initial condition \( u_\theta(0, x, \omega) = \theta x \).

To prove the first assertion, we temporarily denote by \( \mathcal{H}_\beta^U(G)(\theta, \omega), \mathcal{H}_\beta^L(G)(\theta, \omega) \) the limsup and liminf appearing in (3.1), respectively. Fix \( z \in \mathbb{R}, \omega \in \Omega \) and set \( w(t, x) := u_\theta(t, x + z, \omega) - z \theta \). Then \( w \) satisfies \( w(0, x) = \theta x \) and

\[
\partial_t w = a(z + x, \omega)c_x^2 w + G(c_x w) + \beta V(z + x, \omega) \quad \text{in } (0, +\infty) \times \mathbb{R}.
\]
By stationarity of $a$ and $V$ and uniqueness, it follows that $w = u(\cdot,\cdot,\tau z \omega)$. Hence

$$H_{\beta}^U(G)(\theta, \tau z \omega) = \limsup_{t \to +\infty} \frac{u_\theta(t, z, \omega) - z \theta}{t} = \limsup_{t \to +\infty} \frac{u_\theta(t, 0, \omega)}{t} = H_{\beta}^U(G)(\theta, \omega),$$

where for the second equality we have used the fact that $u_\theta$ is Lipschitz, see Proposition A.3. By ergodicity, we conclude that the map $\omega \mapsto H_{\beta}^U(G)(\theta, \omega)$ is almost surely constant. Similar argument applies to $\omega \mapsto H_{\beta}^L(G)(\theta, \omega)$.

(i) The first inequality follows by the very definition of $H_{\beta}^U(G)$ and $H_{\beta}^L(G)$. To prove the second inequality, set $\alpha(h) := \alpha_0|h|^\gamma - 1/\alpha_0$ and note that the function $v_\theta(t, x) := \theta x + \alpha(|\theta|) t$ is a subsolution of $(2.1)$ with $v_\theta(0, x) = \theta x$. By applying the comparison principle stated in Proposition A.1 to the functions $v_\theta(t, x) - \theta x$ and $u_\theta(t, x) - \theta x$ we get

$$H_{\beta}^U(G)(\theta) = \liminf_{t \to +\infty} \frac{u_\theta(t, 0, \omega)}{t} \geq \liminf_{t \to +\infty} \frac{v_\theta(t, 0, \omega)}{t} = \alpha(|\theta|).$$

(ii) The assertion is a direct consequence of Proposition A.1 below.

(iii) We prove the assertion for $H_{\beta}^L(G)$ only, the argument for $H_{\beta}^U(G)$ being analogous.

Let $\beta_1, \beta_2 \in (0, +\infty)$ and denote by $u_i$ the solution of $(2.1)$ with $\beta = \beta_i$ satisfying $u_i(0, x) = \theta x$ in $\mathbb{R}$. Then

$$\partial_t u_1 \leq a(x, \omega) \partial_x^2 u_1 + G(\partial_x u_1) + \beta_2 V(x, \omega) + |\beta_1 - \beta_2| \text{ in } (0, +\infty) \times \mathbb{R}.$$ 

This means that $u_1 - |\beta_1 - \beta_2| t$ is a subsolution of $(2.1)$ with $\beta := \beta_2$ and initial condition $\theta x$. By comparison we infer $u_2 \geq u_1 - |\beta_1 - \beta_2| t$, hence

$$H_{\beta_2}^L(G)(\theta) = \liminf_{t \to +\infty} \frac{u_2(t, 0, \omega)}{t} \geq \liminf_{t \to +\infty} \frac{u_1(t, 0, \omega) - |\beta_1 - \beta_2| t}{t} = H_{\beta_1}^L(G)(\theta) - |\beta_1 - \beta_2|.$$ 

By interchanging the role of $\beta_1$ and $\beta_2$ we infer $|H_{\beta_1}^L(G)(\theta) - H_{\beta_2}^L(G)(\theta)| \leq |\beta_1 - \beta_2|$. If $\beta_1 \geq \beta_2$, we furthermore have

$$\partial_t u_1 \geq a(x, \omega) \partial_x^2 u_1 + G(\partial_x u_1) + \beta_2 V(x, \omega), \text{ in } (0, +\infty) \times \mathbb{R},$$

meaning that $u_1$ is a supersolution of $(2.1)$ with $\beta := \beta_2$. By comparison we infer $u_2 \leq u_1$, hence

$$H_{\beta_2}^L(G)(\theta) = \liminf_{t \to +\infty} \frac{u_2(t, 0, \omega)}{t} \leq \liminf_{t \to +\infty} \frac{u_1(t, 0, \omega)}{t} = H_{\beta_1}^L(G)(\theta),$$

yielding the claimed monotonicity of $\beta \mapsto H_{\beta}^L(G)(\theta)$.

(iv) It suffices to show that $H_{\beta}^U(G)(0) \leq \beta$. This follows from the fact that the function $w(t, x) = \beta t$ is a supersolution of $(2.1)$ satisfying $w(0, x) = 0$, as it can be easily seen. By comparison, we get $u_0(t, x) \leq \beta t$, yielding $H_{\beta}^U(G)(0) \leq \beta$.

The last assertion follows by well known results in stationary ergodic homogenization. 

We now return to the setting of Section 2. The next proposition shows that without loss of generality we can assume that $c_+ = -c_-$.

**Proposition 3.2.** For given $c_+ \geq c_- \in \mathbb{R}$, let $G^+, G^- : \mathbb{R} \to \mathbb{R}$ be functions satisfying (G1)-(G2) with $G^+(c_+) = G^-(c_-) = 0$ and set $G_{c_+} := G^- \wedge G^+$. Let

$$\tilde{G}^\pm(p) := G^\pm \left(p + \frac{c_+ + c_-}{2}\right), \quad \tilde{G}(p) := G_{c_+} \left(p + \frac{c_+ + c_-}{2}\right) = (\tilde{G}^+ \wedge \tilde{G}^-)(p),$$

where for the second equality we have used the fact that $G^\pm$ is Lipschitz, see Proposition A.3. By ergodicity, we conclude that the map $\omega \mapsto \tilde{H}_{\beta}^U(G)(\theta, \omega)$ is almost surely constant. Similar argument applies to $\omega \mapsto \tilde{H}_{\beta}^L(G)(\theta, \omega)$.
for every $p \in \mathbb{R}$. If (2.1) homogenizes with $G := \tilde{G}$, then the same holds with $G := G_{c^+}$. Furthermore, the associated effective Hamiltonians satisfy the following relation:

$$
\mathcal{H}_\beta(G_{c^\pm})(\theta) = \mathcal{H}_\beta(\tilde{G}) \left( \theta - \frac{c_+ + c_-}{2} \right) \quad \text{for all } \theta \in \mathbb{R}.
$$

(3.2)

Note that $\tilde{G}^+(c) = \tilde{G}^-(c) = 0$ with $c = (c_+ - c_-)/2$.

Proof. Let us set $k := -(c_+ + c_-)/2$. For every fixed $\theta \in \mathbb{R}$, let us denote by $v_\theta$ the solution of (2.1) with $G := G_{c^\pm}$ and initial condition $v_\theta(0, x) = \theta x$. The function $u(t, x) = v_\theta(t, x) + k x$ solves equation (2.1) with $G := \tilde{G}$ and initial condition $u(0, x) = (\theta + k)x$. Since the latter equation homogenizes by hypothesis, we get

$$
\mathcal{H}_\beta(\tilde{G})(\theta + k) = \lim_{t \to +\infty} \frac{u(t, 0, \omega)}{t} = \lim_{t \to +\infty} \frac{v_\theta(t, 0, \omega)}{t},
$$

yielding (3.2). □

We shall therefore restrict our attention to the case $c_+ = -c_- =: c$ and set $G_c := G^- \wedge G^+$. Up to replacing $G_c$ with $G_c(p) := G_c(-p)$, $p \in \mathbb{R}$, we can furthermore assume, without loss of generality, that $\bar{p} \geq 0$. Note that

$$
G^-(\bar{p}) = G^+(\bar{p}) = G_c(\bar{p}) = \max_{p \in [-c, c]} G_c(p).
$$

(3.3)

4. UPPER AND LOWER BOUNDS

Our goal is to show that $\mathcal{H}_\beta^L(G_c) = \mathcal{H}_\beta^U(G_c)$, which is a necessary and sufficient condition for homogenization of (2.1) with $G := G_c$, as remarked above. We start by proving suitable lower and upper bounds for these lower and upper limits.

4.1. Lower bound. We aim at proving the following lower bound:

$$
\mathcal{H}_\beta^L(G_c)(\theta) \geq \beta \quad \text{for every } \theta \in \mathbb{R}.
$$

(4.1)

This follows from the following more general result:

Proposition 4.1. Let $G \in \mathcal{H}(\gamma, \alpha_0, \alpha_1)$ and $V$ satisfy the scaled hill condition (V2)(a),(h). Then

$$
\mathcal{H}_\beta^L(G)(\theta) \geq \beta \quad \text{for every } \theta \in \mathbb{R}.
$$

(4.2)

Proof. Let us fix $\theta \in \mathbb{R}$. We want to find a subsolution $v$ to (2.1) satisfying $v(0, x) \leq \theta x$. Pick $\varepsilon > 0$ and $h \in (0,1)$. Choose $y > 0$ big enough so that

$$
G(\theta + \varepsilon p) \geq \beta h \quad \text{for every } |p| \geq y.
$$

(4.3)

Choose $\Omega(h, y) \subseteq \Omega$ of probability 1 as in the scaled hill condition, see (V2). Pick $\omega \in \Omega(h, y)$ and choose $\ell_1 < \ell_2$ and $\delta$ such that (a),(h) hold in (V2). Pick $x_0 \in (\ell_1, \ell_2)$ such that

$$
\int_{x_0}^{x_0} \frac{1}{a(r, \omega) \vee \delta} \, dr = \int_{x_0}^{\ell_2} \frac{1}{a(r, \omega) \vee \delta} \, dr = y
$$

(4.4)

and set

$$
\chi(x) := \int_{x_0}^{x} \frac{1}{a(r, \omega) \vee \delta} \, dr, \quad x \in \mathbb{R}
$$

$$
v^\varepsilon(t, x) = \theta x - \varepsilon \int_{0}^{x} \chi(s) \, ds + (\beta h - \varepsilon)t, \quad (t, x) \in [0, +\infty) \times \mathbb{R}.
$$
Note that $\partial_x v^\varepsilon(t,x) = \theta - \varepsilon \chi(x)$, $a(x,\omega)\partial_{xx}^2 v^\varepsilon(t,x) \geq -\varepsilon$ in $(0,\infty) \times \mathbb{R}$. We are going to show that $v^\varepsilon$ is a subsolution of (2.1). Indeed, for every $t > 0$ and $x \in \mathbb{R}$ we have
\[
a(x,\omega)\partial_{xx}^2 v^\varepsilon + G(\partial_x v^\varepsilon) + \beta V(x,\omega) \geq -\varepsilon + G(\theta - \varepsilon \chi(x)) + \beta V(x,\omega) \geq -\varepsilon + \beta h
\]
For $x \in [\ell_1, \ell_2]$, the above inequality holds true for $V(\cdot,\omega) \geq h$ in $[\ell_1, \ell_2]$ by (V2)-(h) and $G \geq 0$ in $\mathbb{R}$. For $x \in \mathbb{R}[\ell_1, \ell_2]$, it holds true for $G(\theta - \varepsilon \chi(x)) \geq \beta h$ in $(-\infty, \ell_1) \cup (\ell_2, +\infty)$ in view of (4.3), (4.4) and $V(\cdot,\omega) \geq 0$ in $\mathbb{R}$. In either case, $v^\varepsilon$ is a subsolution of (2.1) satisfying
\[
v^\varepsilon(0,\omega) = \theta x - \varepsilon \int_{x_0}^x \chi(s) \, ds \leq \theta x.
\]
Let $u_\theta$ be the solution of (2.1) satisfying $u_\theta(0,\omega) = \theta x$. Since $u_\theta$ is Lipschitz on $(0,\infty) \times \mathbb{R}$, see Proposition A.3, the function $u_\theta(t,x,\omega) - \theta x$ is bounded in $[0,T] \times \mathbb{R}$, for every fixed $T > 0$. We can therefore apply the comparison principle stated in Proposition A.1 to $u_\theta(t,x,\omega) - \theta x$ and $v^\varepsilon(t,x) - \theta x$ with $G(\theta + \cdot)$ in place of $G$ and get $u_\theta(t,x,\omega) \geq v^\varepsilon(t,x)$ for every $(t,x) \in (0,\infty) \times \mathbb{R}$ and $\omega \in \Omega_x$. We conclude that
\[
\liminf_{t \to +\infty} \frac{u_\theta(t,0,\omega)}{t} \geq \liminf_{t \to +\infty} \frac{v^\varepsilon(t,0)}{t} = \beta h - \varepsilon.
\]
Since this holds for every $\omega \in \Omega(h,\gamma)$ and $\mathbb{P}(\Omega(h,\gamma)) > 0$, we infer that
\[
\mathcal{H}_{\beta}(G)(\theta) \geq \beta h - \varepsilon.
\]
Now let $\varepsilon \to 0^+$ and then $h \to 1^-$ to get the desired lower bound (4.2). \hfill \Box

4.2. General upper bound. We aim at proving the following general upper bound
\[
\mathcal{H}_{\beta}^U(G_c)(\theta) \leq \min \{ \mathcal{H}_{\beta}(G^-)(\theta), \mathcal{H}_{\beta}(G^+)(\theta) \} \quad \text{for all } \theta \in \mathbb{R}.
\]
Since the function $G^\pm$ are convex, equation (2.1) with $G := G^\pm$ homogenizes with effective Hamiltonian $\theta \mapsto \mathcal{H}_{\beta}(G^\pm)(\theta)$. For every fixed $\theta \in \mathbb{R}$, let us denote by $u_\theta^\pm$ the solution to (2.1) with $G := G^\pm$ and initial condition $u_\theta^\pm(0,\omega) = \theta x$. Since $G_c \leq G^\pm$, by the comparison principle we infer
\[
\mathcal{H}_{\beta}^U(G_c)(\theta) = \limsup_{t \to +\infty} \frac{u_\theta(t,0,\omega)}{t} \leq \limsup_{t \to +\infty} \frac{u_\theta^\pm(t,0,\omega)}{t} = \mathcal{H}_{\beta}(G^\pm)(\theta),
\]
yielding the sought general upper bound. \hfill \Box

4.3. Upper bound when $|\theta| \leq c$. We aim at proving the following upper bound
\[
\mathcal{H}_{\beta}^U(G_c)(\theta) \leq \max\{\beta, G_c(\hat{\rho})\} \quad \text{for } |\theta| \leq c.
\]
This bound follows from the next proposition by recalling (3.3), i.e.
\[
G_c(\hat{\rho}) = \max_{p \in [-c,c]} G_c(p).
\]
Proposition 4.2. Let $G \in \mathcal{H}(\gamma,\alpha_0,\alpha_1)$ be such that $G(\pm c) = 0$ and $V$ satisfy the scaled valley condition (V2)(a),(v). Then
\[
\mathcal{H}_{\beta}^U(G)(\theta) \leq \max\{\beta, \max_{[-c,c]} G(\cdot)\} \quad \text{for every } |\theta| \leq c.
\]
Proof. Let us denote the right-hand side of (4.7) by $\eta$. We would like to find a supersolution $w$ to (2.1) of the form $w(t,x) := \tilde{w}(x) + \eta t$ with $\tilde{w}(x) \geq \theta x$. The naive idea is to set $\tilde{w}(x) := c|x|$. An easy computation shows that $\tilde{w}$ satisfies, for $x \neq 0$,
\[
a(x,\omega)\partial_{xx}^2 \tilde{w} + G(\partial_x \tilde{w}) + \beta V(x,\omega) = G(\pm c) + \beta V(x,\omega) \leq \beta \leq \eta,
\]
so \( w(t, x) \) is a supersolution to (2.1) in \( \mathbb{R} \setminus \{0\} \times (0, +\infty) \). The problem is that \( w(t, x) \) is not a supersolution at \( x = 0 \). Note that \( \dot{w}(x) = |x| \geq |\theta||x| \geq \theta x \).

We need to modify the definition of \( w(t, x) = c|x| + \eta \). We begin by replacing the function \( x \mapsto c|x| \) with a smooth one, whose derivative is equal to \( c \) or \( -c \) outside a compact interval. To this aim, let us fix \( h \in (0, 1) \) and \( y > 0 \). Choose \( \Omega(h, y) \subseteq \Omega \) of probability 1 as in the scaled valley condition (V2). Pick \( \omega \in \Omega(h, y) \) and choose \( \ell_1 < \ell_2 \) and \( \delta \) such that (a),(v) hold in (V2). Set \( a_\delta(x) := a(x, \omega) \vee \delta \) and pick \( x_0 \in (\ell_1, \ell_2) \) such that

\[
\int_{\ell_1}^{x_0} \frac{1}{a_\delta(r)} dr = \int_{x_0}^{\ell_2} \frac{1}{a_\delta(r)} dr = y
\]

(4.8)

We define a function \( s: \mathbb{R} \to \mathbb{R} \) by setting

\[
s(x) := \begin{cases} 
-c & \text{if } x < \ell_1 \\
\frac{c}{2} \left( \frac{1}{y} \int_{x_0}^{x} \frac{1}{a_\delta(r)} dr \right) \left( 3 - \left( \frac{1}{y} \int_{x_0}^{x} \frac{1}{a_\delta(r)} dr \right)^2 \right) & \text{if } x \in (\ell_1, \ell_2) \\
\frac{c}{2} & \text{if } x > \ell_2.
\end{cases}
\]

First notice that \( s(\ell_1) = -c, s(\ell_2) = c \), yielding that \( s \) is continuous. Furthermore,

\[
s'(x) = \frac{3c}{2y a_\delta(x)} \left( 1 - \left( \frac{1}{y} \int_{x_0}^{x} \frac{1}{a_\delta(r)} dr \right)^2 \right)
\]

for \( x \in (\ell_1, \ell_2) \), hence \( s'(\ell_1) = s'(\ell_2) = 0 \), showing that \( s \) is actually of class \( C^1 \). Also notice that \( s' > 0 \) in \((\ell_1, \ell_2)\), in particular

\[
-c = s(\ell_1) < s(x) < s(\ell_2) = c \quad \text{for all } x \in (\ell_1, \ell_2).
\]

(4.9)

For \( x \in [\ell_1, \ell_2] \), we get

\[
a(x, \omega) s'(x) + G(s(x)) + \beta V(x, \omega) \leq \frac{3c}{y} + G(s(x)) + \beta V(x, \omega) \leq \frac{3c}{2y} + \eta + \beta h
\]

(4.10)

in view of (4.10) and of the fact that \( V(\cdot, \omega) \leq h \) in \( [\ell_1, \ell_2] \) by (V2)-(v).

For \( x \in (-\infty, \ell_1) \cup (\ell_2, +\infty) \), we have

\[
a(x, \omega) s'(x) + G(s(x)) + \beta V(x, \omega) \leq G(\pm c) + \beta = \beta \leq \eta.
\]

(4.11)

Now set

\[
w(t, x) := k(\theta) + \int_{x_0}^{x} s(r) dr + \left( \eta + \frac{3c}{2y} + \beta h \right) t, \quad (t, x) \in [0, +\infty) \times \mathbb{R},
\]

with \( k(\theta) \) chosen big enough so that \( w(0, x) \geq \theta x \). For instance, take

\[
-k(\theta) := \theta x_0 + \min_{\ell_1 \leq x \leq \ell_2} \int_{x_0}^{x} (s(r) - \theta) dr.
\]

From what proved above we infer that \( w \) is a \( C^2 \) (classical) supersolution of (2.1) satisfying \( w(0, x) \geq \theta x \). Let \( u_\theta \) be the solution of (2.1) satisfying \( u_\theta(0, x) = \theta x \). Since \( u_\theta \) is Lipschitz on \((0, +\infty) \times \mathbb{R}\) by Proposition 3.3, the function \( u_\theta(t, x) - \theta x \) is bounded in \([0, T] \times \mathbb{R}\) for every fixed \( T > 0 \). We can therefore apply the comparison principle stated in Proposition 3.3 to \( u_\theta(t, x) - \theta x \) and \( w(t, x) - \theta x \) with \( G(\theta + \cdot) \) in place of \( G \) and get

\[
u_\theta(t, x, \omega) \leq w(t, x)
\]

for every \( (t, x) \in (0, +\infty) \times \mathbb{R}, \)
in particular
\[
\limsup_{t \to +\infty} \frac{u_\theta(t, 0, \omega)}{t} \leq \limsup_{t \to +\infty} \frac{w(t, 0)}{t} = \eta + \frac{3c}{2y} + \beta h.
\]
Since this holds for every \( \omega \in \Omega(h, y) \) and \( \mathbb{P}(\Omega(h, y)) > 0 \), we infer that
\[
\mathcal{H}_\beta^U(G)(\theta) \leq \eta + \frac{3c}{2y} + \beta h.
\]
Now we send \( h \to 0^+ \) and \( y \to +\infty \) to get the upper bound \((4.6)\).

5. Existence of correctors

The goal of the present section is to single out conditions on \( \theta \in \mathbb{R} \) under which we have correctors for \((2.1)\). In the sequel, we will say that a function \( u : \mathbb{R} \to \mathbb{R} \) is sublinear or has sublinear growth to mean that
\[
\lim_{|x| \to +\infty} \frac{u(x)}{1 + |x|} = 0.
\]

5.1. Correctors. In this subsection, we collect and prove some key results we shall need for our analysis. We shall assume that \( G : \mathbb{R} \to [0, +\infty) \) is a function in \( \mathcal{H}(\gamma, \alpha_0, \alpha_1) \) satisfying the following additional assumption:

(G3) \( G(0) = 0; \)

(G4) \( G \) is convex.

Notice that conditions (G3)-(G4) and the fact that \( G \geq 0 \) in \( \mathbb{R} \) imply that \( G \) is nonincreasing in \((-\infty, 0]\) and nondecreasing in \([0, +\infty)\). By known results in stationary ergodic homogenization, the equation \((2.1)\) homogenizes. We shall denote by \( \mathcal{H}_\beta(G) \) the corresponding effective Hamiltonian. Since \( V \geq 0 \), we get
\[
\mathcal{H}_\beta(G)(\theta) \geq G(\theta) \quad \text{for all } \theta \in \mathbb{R}. \quad (5.1)
\]
We know that \( \mathcal{H}_\beta(G) \) is convex and coercive and has a minimum at 0 with \( \mathcal{H}_\beta(G)(0) = \beta \), see Proposition \(3.1\). The following proposition shows the existence of a Lipschitz continuous corrector with stationary gradient for every \( \theta \) satisfying \( \mathcal{H}_\beta(G)(\theta) > \beta \).

Proposition 5.1. Let \( \theta \in \mathbb{R} \) be such that \( \mathcal{H}_\beta(G)(\theta) > \beta \). Then there exists a random variable \( \Omega \ni \omega \mapsto F_\theta(\cdot, \omega) \in C(\mathbb{R}) \) such that, for every \( \omega \) in a set \( \Omega_\theta \) of probability 1, \( F_\theta(\cdot, \omega) \) is the unique sublinear viscosity solution of the stationary viscous Hamilton–Jacobi equation
\[
a(x, \omega)u'' + G(\theta + u') + \beta V(x, \omega) = \mathcal{H}_\beta(G)(\theta) \quad \text{in } \mathbb{R} \quad (5.2)
\]
satisfying \( F_\theta(0, \omega) = 0 \) for every \( \omega \in \Omega_\theta \). The set \( \Omega_\theta \) is invariant under the action of \((\tau_z)_{z \in \mathbb{R}}\), i.e. \( \tau_z(\Omega_\theta) = \Omega_\theta \) for every \( z \in \mathbb{R} \). Furthermore, the function \( F_\theta(\cdot, \omega) \) is \( \kappa(\theta) \)-Lipschitz continuous on \( \mathbb{R} \) for \( \mathbb{P}\)-a.e. \( \omega \in \Omega_\theta \), where \( \kappa : \mathbb{R} \to [0, +\infty) \) is a locally bounded function, and has stationary gradient, i.e. for every \( \omega \) in a set of probability 1 we have
\[
F_\theta(\cdot + z, \omega) = F_\theta(\cdot, \tau_z \omega) \quad \text{a.e. on } \mathbb{R} \quad \text{for every } z \in \mathbb{R}.
\]

Proof. Let us set \( \tilde{G}(p) := G(-p) \) for all \( p \in \mathbb{R} \). Then equation \((2.1)\) with \( \tilde{G} \) in place of \( G \) also homogenizes, with effective Hamiltonian \( \mathcal{H}_\beta(\tilde{G}) \) satisfying \( \mathcal{H}_\beta(\tilde{G})(-\theta) = \mathcal{H}_\beta(G)(\theta) \). The function \( u(x, \omega) := -F_\theta(x, \omega) \) is a viscosity solution to
\[
-a(x, \omega)u'' + \tilde{G}(-\theta + u') + \beta V(x, \omega) = \mathcal{H}_\beta(\tilde{G})(-\theta) \quad \text{in } \mathbb{R}. \quad (5.3)
\]
Hence, it will be enough to prove the assertion for \( u \). We want to apply Theorem 2.1 in [13], which was proved under the following
Assumption (H): for any \( \theta \in \mathbb{R} \), the approximate corrector equation
\[
\lambda v_{\lambda, \theta} - a(x, \omega)v_{\lambda, \theta}^\mu + \tilde{G}(\theta + v_{\lambda, \theta}^\mu) + \beta V(x, \omega) = 0 \quad \text{in } \mathbb{R}
\]  
(5.4)
satisfies a comparison principle in \( C_b(\mathbb{R}) \), and, for any \( R > 0 \), there exists a constant \( \kappa(R) > 0 \) such that, if \( |\theta| \leq R \), then the unique bounded solution \( v_{\lambda, \theta} \) of (5.4) satisfies
\[
\|\lambda v_{\lambda, \theta}\|_{\infty} + \|v_{\lambda, \theta}'\|_{\infty} \leq \kappa(R) \quad \text{for all } \lambda > 0.
\]  
(5.5)

Let us check that assumption (H) holds in our framework. The validity of the required comparison principle for (5.4) is guaranteed by [7, Theorem 2.1]. Since the functions \( \pm C(R)/\lambda \) with \( C(R) := \beta + \sup_{|\theta| \leq R} \tilde{G}(\theta) \) are a bounded super- and sub- solution to (5.4), respectively, we immediately derive by comparison that \( \|\lambda v_{\lambda, \theta}\|_{\infty} \leq C(R) \). This bound, together with the quantitative Lipschitz bounds for \( v_{\lambda, \theta} \) provided by [7, Theorem 3.1], imply that (5.5) holds for a suitable nondecreasing function \( \kappa : \mathbb{R} \to [0, +\infty) \).

Following [13], we choose \( R > |\theta| \) and denote by
\[
\Theta := \{ v \in \text{Lip}(\mathbb{R}) \mid v(0) = 0, \|v'\|_{\infty} \leq \kappa(R) \}
\]
the metric subspace of \( C(\mathbb{R}) \). It is easily seen that \( \Theta \) is a compact metric space. The inequality \( \mathcal{H}_\beta(\tilde{G})(-\theta) > \beta \) implies \( \theta \neq 0 \), so, according to Corollary A.3 for each fixed \( \omega \in \Omega \) there is at most one sublinear solution of (5.3) in \( \Theta \), let us call it \( \tilde{u}(\cdot, \omega) \). Now note that \(-\theta\) is an extremal point of the closed interval \{ \( \theta \in \mathbb{R} \mid \mathcal{H}_\beta(\tilde{G})(\theta) \leq \mathcal{H}_\beta(\tilde{G})(-\theta) \} \), for \( \mathcal{H}_\beta(\tilde{G})(-\theta) > \beta = \min \mathcal{H}_\beta(\tilde{G}) \) and \( \mathcal{H}_\beta(\tilde{G}) \) is convex. In [13, Theorem 2.1] the authors have obtained a probability measure \( \mu \) on \( \Omega \times \Theta \) (we can forget about the third coordinate in \( \tilde{\Omega} \) as, in our setting, the restriction of \( \mu \) on the third coordinate is a Dirac mass at \( \mathcal{H}_\beta(\tilde{G})(-\theta) \) such that \( \mu(E_\theta) = 1 \), where
\[
E_\theta := \{ (\omega, v) \in \Omega \times \Theta \mid v \text{ is a sublinear solution of (5.3)} \}.
\]
Furthermore, the set \( E_\theta \) is invariant under the shifts \( \tau_\omega : (\omega, v) \mapsto (\tau_\omega \omega, v(\cdot + z) - v(z)) \).

Indeed, if \( v \in \Theta \) is a sublinear solution of (5.3) for some \( \omega \), then \( v(\cdot + z) - v(z) \) belongs to \( \Theta \) and is a sublinear solution of (5.3) with \( \tau_\omega \omega \) in place of \( \omega \), since \( V(\cdot + z, \omega) = V(\cdot, \tau_\omega \omega) \) in \( \mathbb{R} \). In particular, we get that \((\omega, v) \in E_\theta \) implies \( v = \tilde{u}(\cdot, \omega) \). Let \( \Omega_\theta := \{ \tau_1(E_\theta) \} \), where \( \tau_1 : \Omega \times \Theta \to \Omega \) denotes the standard projection, and recall that the first marginal of the measure \( \mu \) is \( \mathbb{P} \). Then \( \Omega_\theta \in \mathcal{F} \) and \( \tau_\omega(\Omega_\theta) = \Omega_\theta \) for all \( \omega \in \mathbb{R} \), in the light of what previously remarked.

By making use of the disintegration theorem (see [17, Theorem 10.2.2]) we get that there exists a family of random probability measures \( \mu_\omega \) on \( \Theta \) such that \( \mu = \mu_\omega \otimes \mathbb{P} \), i.e.
\[
\int_{\Omega \times \Theta} \phi(\omega, v) \, d\mu(\omega, v) = \int_{\Omega} \left( \int_{\Theta} \phi(\omega, v) \, d\mu_\omega(v) \right) \, d\mathbb{P}(\omega) \quad \text{for all } \phi \in C(\Omega \times \Theta).
\]

By what observed above, for every \( \omega \in \Omega_\theta \) the measure \( \mu_\omega \) is the Dirac measure concentrated at \( \tilde{u}(\cdot, \omega) \), hence the map \( \Omega_\theta \ni \omega \mapsto \tilde{u}(\cdot, \omega) \in \Theta \) is a random variable. The sought random variable \( u : \Omega \to C(\mathbb{R}) \) is thus obtained by setting
\[
u(\cdot, \omega) = \tilde{u}(\cdot, \omega) \quad \text{if } \omega \in \Omega_\theta, \quad u(\cdot, \omega) = 0 \quad \text{otherwise}.
\]

Lastly, for every \( \omega \in \Omega_\theta \) and \( z \in \mathbb{R} \), we have \( u(\cdot + z, \omega) - u(\cdot, \omega) = u(\cdot, \tau_\omega \omega) \) in \( \mathbb{R} \) in view of Corollary A.3 since both are sublinear solutions of (5.3) with \( \tau_\omega \omega \) in place of \( \omega \). By differentiating this identity we get \( u'(\cdot + z, \omega) = u'(\cdot, \tau_\omega \omega) \) a.e. in \( \mathbb{R} \), for every \( z \in \mathbb{R} \) and \( \omega \in \Omega_\theta \).

\[\square\]
From now on, when we say that a random variable $\Omega \ni \omega \mapsto F_{\theta}(\cdot, \omega) \in C(\mathbb{R})$ is a corrector for (5.2) we will mean that $F_{\theta}(\cdot, \omega)$ is a sublinear, Lipschitz continuous viscosity solution of (5.2) satisfying $F_{\theta}(0, \omega) = 0$ for every $\omega \in \Omega_{\theta}$, where $\Omega_{\theta}$ is a set of probability 1 which is invariant under the action of $(\tau_{z})_{z \in \mathbb{R}}$, with no further specification. In view of what remarked above, a corrector automatically possesses stationary gradient. We point out that our arguments below do not use this property.

We are interested in obtaining suitable upper and lower bounds for $F'_{\theta}$ depending on $\theta$. We start with the following lemma.

**Lemma 5.2.** Let us consider the following viscous Hamilton–Jacobi equation

$$-a(x, \omega)u'' + \tilde{G}(u') + \beta V(x, \omega) = \lambda \quad \text{in } I,$$

(5.6) where $\lambda > \beta$ and $I$ is either $(-\infty, y)$ or $(y, +\infty)$ for a fixed $y \in \mathbb{R}$.

(i) Let $I = (-\infty, y)$ and $a^{-}_{\lambda}, b^{-}_{\lambda} > 0$ such that $G(a^{-}_{\lambda}) = \lambda - \beta$, $G(b^{-}_{\lambda}) = \lambda$. Then the functions

$$v_{-}(x) := a^{-}_{\lambda}|x - y| = -a^{-}_{\lambda}(x - y), \quad w_{-}(x) := b^{-}_{\lambda}|x - y| = -b^{-}_{\lambda}(x - y)$$

are, respectively, a sub- and a super-solution of (5.6) in $I = (-\infty, y)$.

(ii) Let $I = (y, +\infty)$ and $a^{+}_{\lambda}, b^{+}_{\lambda} > 0$ such that $G(-a^{+}_{\lambda}) = \lambda - \beta$, $G(-b^{+}_{\lambda}) = \lambda$. Then the functions

$$v_{+}(x) := a^{+}_{\lambda}|x - y| = a^{+}_{\lambda}(x - y), \quad w_{+}(x) := b^{+}_{\lambda}|x - y| = b^{+}_{\lambda}(x - y)$$

are, respectively, a sub- and a super-solution of (5.6) in $I = (y, +\infty)$.

**Proof.** Let us prove (i). We have

$$-a(x, \omega)(v_{-})''(x) + \tilde{G}(v'_{-}(x)) + \beta V(x, \omega) \leq \tilde{G}(-a^{-}_{\lambda}) + \beta = \lambda - \beta + \beta = \lambda \quad \text{for all } x < y,$$

showing that $v_{-}$ is a subsolution of (5.6) in $I = (-\infty, y)$. Analogously,

$$-a(x, \omega)(w_{-})''(x) + \tilde{G}(w'_{-}(x)) + \beta V(x, \omega) \geq \tilde{G}(-b^{-}_{\lambda}) = \lambda \quad \text{for all } x < y,$$

showing that $w_{-}$ is a supersolution of (5.6) in $I = (-\infty, y)$. The proof of (ii) is similar and is omitted. \hfill \Box

By comparison, we get the following statement.

**Proposition 5.3.** Let $\theta \in \mathbb{R}$ such that $\mathcal{H}_{\beta}(G)(\theta) > \beta$. Set $\lambda := \mathcal{H}_{\beta}(G)(\theta)$. For every $y \in \mathbb{R}$ and $\omega \in \Omega_{\theta}$, the following holds:

(i) if $\theta > 0$, then

$$a^{-}_{\lambda}(x - y) \geq \theta(x - y) + F_{\theta}(x, \omega) - F_{\theta}(y, \omega) \geq b^{-}_{\lambda}(x - y) \quad \text{for all } x \in (-\infty, y),$$

with $b^{-}_{\lambda} > a^{-}_{\lambda} > 0$ such that $G(a^{-}_{\lambda}) = \lambda - \beta$, $G(b^{-}_{\lambda}) = \lambda$;

(ii) if $\theta < 0$, then

$$-a^{-}_{\lambda}(x - y) \geq \theta(x - y) + F_{\theta}(x, \omega) - F_{\theta}(y, \omega) \geq -b^{-}_{\lambda}(x - y) \quad \text{for all } x \in (y, +\infty),$$

with $b^{-}_{\lambda} > a^{-}_{\lambda} > 0$ such that $G(-a^{-}_{\lambda}) = \lambda - \beta$, $G(-b^{-}_{\lambda}) = \lambda$.

**Proof.** By Proposition 5.1, we have that the function

$$u(x) := - (\theta x + F_{\theta}(x, \omega)) + \theta y + F_{\theta}(y, \omega)$$

is a Lipschitz continuous solution to (5.6) with $I := \mathbb{R}$ satisfying $u(y) = 0$.

Let us first consider the case $\theta > 0$. By sublinearity of $F_{\theta}$, the function $u$ is bounded from below in $I = (-\infty, y)$. By Theorem A.16 and Lemma A.2, we have

$$-a^{-}_{\lambda}(x - y) = v_{-}(x) \leq u(x) = - (\theta x + F_{\theta}(x, \omega)) + \theta y + F_{\theta}(y, \omega) \quad \text{for all } x < y,$$
proving the first inequality of assertion (i). To prove the second one, note that the functions \( \tilde{u}(x) := u(x) + \theta x \) and \( \tilde{w}(x) := w_-(x) + \theta x = -b_\lambda^-(x - y) + \theta x \) are, respectively, a sub- and a super-solution of

\[
-a(x, \omega)u'' + \tilde{G}(-\theta + u') + \beta V(x, \omega) = \lambda \quad \text{in} \ I = (-\infty, y).
\]

Furthermore, \( G(b^-_\lambda) = \lambda = \mathcal{H}_\beta(G)(\theta) \geq G(\theta) \) in view of (5.1), so \( b^-_\lambda \geq \theta > 0 \) by monotonicity of \( G \) on \([0, +\infty)\). Then the sub- and super-solution \( \tilde{u} \) and \( \tilde{w} \) satisfy the assumption of Theorem A.4, which gives

\[-F_\theta(x, \omega) + \theta y + F_\theta(y, \omega) = \tilde{u}(x) \leq \tilde{w}(x) = -b^-_\lambda(x - y) + \theta x \quad \text{for all} \ x < y,
\]
yielding the second inequality in assertion (i).

Let us now consider the case \( \theta < 0 \). By sublinearity of \( F_\theta \), the function \( u \) is bounded from below in \( I = (y, +\infty) \). By Theorem A.6 and Lemma 5.2 we have

\[a^+_\lambda(x - y) = \psi(x) \leq u(x) = -\theta x + F_\theta(x, \omega) \leq \theta y + F_\theta(y, \omega) \quad \text{for all} \ x > y,
\]
proving the first inequality of assertion (ii). To prove the second one, we argue as above with \( \tilde{u}(x) := u(x) + \theta x \) and \( \tilde{w}(x) := w_+(x) + \theta x \) for \( x \in I = (y, +\infty) \). Analogously, we have \( G(-b^+_\lambda) = \lambda = \mathcal{H}_\beta(G)(\theta) \geq G(\theta) \), so \( -b^+_\lambda \leq \theta < 0 \), i.e. \( b^+_\lambda + \theta \geq 0 \). Again, via a direct application of Theorem A.4 we get

\[-F_\theta(x, \omega) + \theta y + F_\theta(y, \omega) = \tilde{u}(x) \leq \tilde{w}(x) = b^+_\lambda(x - y) + \theta x \quad \text{for all} \ x > y,
\]
yielding the second inequality in assertion (ii). \( \square \)

From the previous proposition we infer the following result.

**Proposition 5.4.** Let \( \theta \in \mathbb{R} \) such that \( \mathcal{H}_\beta(G)(\theta) > \beta \). Set \( \lambda := \mathcal{H}_\beta(G)(\theta) \). For every \( \omega \in \Omega_\theta \), the following holds:

(i) if \( \theta > 0 \), then

\[a^-_\lambda \leq \theta + F_\theta(y, \omega) \leq b^-_\lambda \quad \text{for a.e.} \ y \in \mathbb{R},
\]
with \( b^-_\lambda > a^-_\lambda > 0 \) such that \( G(a^-_\lambda) = \lambda - \beta \), \( G(b^-_\lambda) = \lambda \);

(ii) if \( \theta < 0 \), then

\[-b^+_\lambda \leq \theta + F_\theta(y, \omega) \leq -a^+_\lambda \quad \text{for a.e.} \ y \in \mathbb{R},
\]
with \( b^+_\lambda > a^+_\lambda > 0 \) such that \( G(-a^+_\lambda) = \lambda - \beta \), \( G(-b^+_\lambda) = \lambda \).

**Proof.** Let \( y \) be a differentiability point of \( F_\theta(\cdot, \omega) \). If \( \theta > 0 \), then from Proposition 5.3(i) we get

\[a^-_\lambda \leq \lim_{h \to 0^-} \frac{\theta h + F_\theta(y + h, \omega) - F_\theta(y, \omega)}{h} \leq b^-_\lambda
\]
yielding assertion (i). If \( \theta < 0 \), we make use of Proposition 5.3(ii) and get

\[-a^+_\lambda \geq \lim_{h \to 0^+} \frac{\theta h + F_\theta(y + h, \omega) - F_\theta(y, \omega)}{h} \geq -b^+_\lambda,
\]
yielding assertion (ii). \( \square \)
5.2. Outside the flat part. In this subsection, we shall prove the following theorem.

**Theorem 5.5.**

(a) Assume either one of the following conditions:
   (i) \( \theta < -c \) and \( \mathcal{H}_\beta(G^-)(\theta) > \beta \); 
   (ii) \( -c < \theta \leq \hat{p} \) and \( \beta < \mathcal{H}_\beta(G^-)(\theta) \leq G_c(\hat{p}) \).

Then
\[
\mathcal{H}_\beta^L(G_c)(\theta) = \mathcal{H}_\beta^U(G_c)(\theta) = \mathcal{H}_\beta(G^-)(\theta) = \min\{ \mathcal{H}_\beta(G^-)(\theta), \mathcal{H}_\beta(G^+)(\theta) \}.
\]

(b) Assume either one of the following conditions:
   (i) \( \theta > c \) and \( \mathcal{H}_\beta(G^+)(\theta) > \beta \); 
   (ii) \( \hat{p} \leq \theta < c \) and \( \beta < \mathcal{H}_\beta(G^+)(\theta) \leq G_c(\hat{p}) \).

Then
\[
\mathcal{H}_\beta^L(G_c)(\theta) = \mathcal{H}_\beta^U(G_c)(\theta) = \mathcal{H}_\beta(G^+)(\theta) = \min\{ \mathcal{H}_\beta(G^-)(\theta), \mathcal{H}_\beta(G^+)(\theta) \}.
\]

The proof of this result is based on a series of lemmas, which we shall prove first.

**Lemma 5.6.** Let \( F_\theta : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) be a corrector of the equation
\[
a(x, \omega)u'' + G_c(\theta + u') + \beta V(x, \omega) = \lambda \quad \text{in } \mathbb{R}
\]
for some \( \theta \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \). Then \( \mathcal{H}_\beta^L(G_c)(\theta) = \mathcal{H}_\beta^U(G_c)(\theta) = \lambda \).

**Proof.** According to Proposition 5.1 we know that \( F_\theta \) is globally Lipschitz on \( \mathbb{R} \). Then the function

\[
v(t, x) := \theta x + F_\theta(x, \omega) + \lambda t
\]

is a solution to (2.1) with \( G := G_c \) and initial condition \( v(0, x) = F_\theta(x, \omega) + \theta x \). Fix \( \varepsilon > 0 \) and choose a constant \( k_\varepsilon > 0 \) large enough so that the function

\[
v^\varepsilon(t, x) = v(t, x) - \varepsilon x - k_\varepsilon \quad \text{where} \quad \langle x \rangle := \sqrt{1 + |x|^2}
\]

satisfies \( v^\varepsilon(0, x) = F_\theta(x, \omega) + \theta x - \varepsilon x - k_\varepsilon \leq \theta x \) in \( \mathbb{R} \). This is possible since the function \( F_\theta \) has sublinear growth. Now

\[
\partial_x v^\varepsilon(t, x) = F_\theta' + \theta - \varepsilon \frac{x}{\langle x \rangle}, \quad \partial_{xx}^2 v^\varepsilon(t, x) = F_\theta'' - \frac{\varepsilon}{\langle x \rangle^3}
\]

and

\[
a(x, \omega) \left( \partial_{xx}^2 v^\varepsilon \right) + G_c(\partial_x v^\varepsilon) + \beta V(x, \omega)
\]

\[
= \left( -\frac{\varepsilon}{\langle x \rangle^3} + F_\theta'' \right) a(x, \omega) + G_c \left( \theta + F_\theta' - \frac{\varepsilon x}{\langle x \rangle} \right) + \beta V(x, \omega) =: A.
\]

From the fact that \( |F_\theta'| \) is bounded on \( \mathbb{R} \) we infer that there exists a constant \( C(\theta) \) such that

\[
A \geq -C(\theta)\varepsilon + a(x, \omega)F_\theta'' + G_c(\theta + F_\theta') + \beta V(x, \omega) = -C(\theta)\varepsilon + \lambda.
\]

This means that the function \( \tilde{v}^\varepsilon(t, x) = v^\varepsilon(t, x) - C(\theta)\varepsilon t \) is a subsolution of (2.1) with \( \tilde{v}^\varepsilon(0, x) \leq \theta x \), hence by comparison we infer

\[
u_\theta(t, x) \geq \tilde{v}^\varepsilon(t, x) \quad \text{for all } (t, x) \in [0, +\infty) \times \mathbb{R}.
\]

So

\[
\mathcal{H}_\beta^L(G_c)(\theta) = \liminf_{t \to +\infty} \frac{u_\theta(t, 0, \omega)}{t} \geq \liminf_{t \to +\infty} \frac{\tilde{v}^\varepsilon(t, 0, \omega)}{t} = \lambda - C(\theta)\varepsilon.
\]

By letting \( \varepsilon \to 0^+ \) we obtain the lower bound \( \mathcal{H}_\beta^L(G_c)(\theta) \geq \lambda \). A similar argument gives the upper bound \( \mathcal{H}_\beta^U(G_c)(\theta) \leq \lambda \), thus proving the assertion. \( \square \)
Lemma 5.7. Let $F^-_g : \mathbb{R} \times \Omega \to \mathbb{R}$ be a corrector of the equation
\[ a(x, \omega)u'' + G^-(\theta + u') + \beta V(x, \omega) = \mathcal{H}_\beta(G^-)(\theta) \quad \text{in } \mathbb{R}. \] (5.8)
Assume either one of the following conditions:
(i) $\theta < -c$ and $\mathcal{H}_\beta(G^-)(\theta) > \beta$;
(ii) $\theta > -c$ and $\beta < \mathcal{H}_\beta(G^-)(\theta) \leq G^- (\hat{p})$.
Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have
\[ \theta + (F^-_g)'(x, \omega) \leq \hat{p} \quad \text{for a.e. } x \in \mathbb{R}. \]

Remark 5.8. Note that the inequality $\mathcal{H}_\beta(G^-)(\theta) \leq G^- (\hat{p})$ for $\theta > -c$ implies $\theta \leq \hat{p}$. This follows from the fact that $\mathcal{H}_\beta(G^-) \geq G^-$ on $\mathbb{R}$ and $G^-$ is nondecreasing on $[-c, +\infty)$.

Proof. We will make use of Proposition 5.4 with $G'(\cdot) := G^- (-c), \mathcal{H}_\beta(G)(\cdot) := \mathcal{H}_\beta(G^-)(\cdot - c)$ and $\theta + c$ in place of $\theta$. Consequently, we will have $\lambda := \mathcal{H}_\beta(G^-)(\theta)$ and $F_{\theta+c} = F^-_g$.

(i) The inequality $\theta + c < 0$ means $\theta + c + F'_{\theta+c}(x, \omega) \leq -a^-_\lambda$ for a.e. $x \in \mathbb{R}$ with $a^+_\lambda > 0$ such that $G^- (-a^+_\lambda - c) = \lambda - \beta > 0$, so
\[ \theta + F'_{\theta+c}(x, \omega) \leq -a^-_\lambda - c < 0 \leq \hat{p} \quad \text{for a.e. } x \in \mathbb{R}. \]

(ii) The inequality $\theta + c > 0$ means $\theta + c + F'_{\theta+c}(x, \omega) \leq b^+_\lambda$ for a.e. $x \in \mathbb{R}$ with $b^+_\lambda > 0$ such that $G^- (b^-_\lambda - c) = \lambda$. Now $G^- (\hat{p}) \geq \lambda = G^- (b^-_\lambda - c)$, so $b^+_\lambda \leq \hat{p} + c$ by monotonicity of $G^-$ on $[-c, +\infty)$, yielding
\[ \theta + F'_{\theta+c}(x, \omega) \leq \hat{p} \quad \text{for a.e. } x \in \mathbb{R}. \]

Lemma 5.9. Let $F^+_g : \mathbb{R} \times \Omega \to \mathbb{R}$ be a corrector of the equation
\[ a(x, \omega)u'' + G^+(\theta + u') + \beta V(x, \omega) = \mathcal{H}_\beta(G^+)(\theta) \quad \text{in } \mathbb{R}. \] (5.9)
Assume either one of the following conditions:
(i) $\theta > c$ and $\mathcal{H}_\beta(G^+)(\theta) > \beta$;
(ii) $\theta < c$ and $\beta < \mathcal{H}_\beta(G^+)(\theta) \leq G^+(\hat{p})$.
Then for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have
\[ \theta + (F^+_g)'(x, \omega) \geq \hat{p} \quad \text{for a.e. } x \in \mathbb{R}. \]

Remark 5.10. Note that the inequality $\mathcal{H}_\beta(G^+)(\theta) \leq G^+(\hat{p})$ for $\theta < c$ implies $\theta \geq \hat{p}$. This follows from the fact that $\mathcal{H}_\beta(G^+) \geq G^+$ on $\mathbb{R}$ and $G^+$ is nonincreasing on $(-\infty, c]$.

Proof. We will make use of Proposition 5.4 with $G'(\cdot) := G^+(\cdot + c), \mathcal{H}_\beta(G)(\cdot) := \mathcal{H}_\beta(G^+(\cdot + c)$ and $\theta - c$ in place of $\theta$. Consequently, we will have $\lambda := \mathcal{H}_\beta(G^+)(\theta)$ and $F_{\theta-c} = F^+_g$.

(i) The inequality $\theta - c > 0$ means $\theta - c + F'_{\theta-c}(x, \omega) \geq a^-_\lambda$ for a.e. $x \in \mathbb{R}$ with $a^-_\lambda > 0$ such that $G^+(a^-_\lambda + c) = \lambda - \beta > 0$, so
\[ \theta + F'_{\theta-c}(x, \omega) \geq c + a^-_\lambda > c \geq \hat{p} \quad \text{for a.e. } x \in \mathbb{R}. \]

(ii) The inequality $\theta - c < 0$ means $\theta - c + F'_{\theta-c}(x, \omega) \geq -b^+_\lambda$ for a.e. $x \in \mathbb{R}$ with $b^+_\lambda > 0$ such that $G^+(-b^+_\lambda + c) = \lambda > 0$. Now $G^+(\hat{p}) \geq \lambda = G^+(-b^+_\lambda + c)$, so $0 \geq -b^+_\lambda + c \geq \hat{p}$ by monotonicity of $G^+$ on $(-\infty, c]$, yielding
\[ \theta + F'_{\theta-c}(x, \omega) \geq \hat{p} \quad \text{for a.e. } x \in \mathbb{R}. \]

We are now ready to prove Theorem 5.5.
Proof of Theorem 5.5. (a) Let $F^-_\theta$ be a corrector of equation (5.8). According to Lemma 5.4
\[
\theta + (F^-_\theta)' \leq \tilde{\rho} \quad \text{for a.e. } x \in \mathbb{R}.
\]
This implies that any $C^2$ sub or supertangent $\varphi$ to $F^-_\theta$ at some $x_0 \in \mathbb{R}$ will satisfy \( \theta + \varphi'(x_0) \leq \tilde{\rho} \), hence $G^-(\theta + \varphi'(x_0)) = G_c(\theta + \varphi'(x_0))$. We derive from this that $F^-_\theta$ is a corrector of equation (5.7) with $\lambda := H_\beta(G^-)(\theta)$. In view of Lemma 5.6 and of the upper bound (4.5), we get the assertion. The proof of item (b) is similar. \( \square \)

6. Proof of Theorem 2.1

In this section we prove Theorem 2.1. When $\theta \leq -c$, we have
\[
\mathcal{H}^L_\beta(G_c)(\theta) = \mathcal{H}^U_\beta(G_c)(\theta) = H_\beta(G^-)(\theta).
\]
This is a direct consequence of Theorem 5.5 when $H_\beta(G^-)(\theta) > \beta$, but it is also true when $H_\beta(G^-)(\theta) = \beta$ in view of the lower bound (4.1) and the general upper bound (4.5). When $\theta \geq c$, a similar argument yields
\[
\mathcal{H}^L_\beta(G_c)(\theta) = \mathcal{H}^U_\beta(G_c)(\theta) = H_\beta(G^+)(\theta).
\]
When $\theta \in (-c, c)$, we have to proceed differently according to whether $\beta \geq G_c(\tilde{\rho})$ or $\beta < G_c(\tilde{\rho})$.

6.1. The case $\beta \geq G_c(\tilde{\rho})$. When $|\theta| < c$, the lower bound (4.1) matches with the upper bound (4.6), hence we get
\[
\mathcal{H}^L_\beta(G_c)(\theta) = \mathcal{H}^U_\beta(G_c)(\theta) = \beta.
\]
We have thus shown that
\[
H_\beta(G_c)(\theta) = \begin{cases} 
H_\beta(G^+)(\theta) & \text{if } \theta > c \\
\beta & \text{if } -c \leq \theta \leq c \\
H_\beta(G^-)(\theta) & \text{if } \theta < -c.
\end{cases}
\]
In other words, $H_\beta(G_c)(\theta)$ is the (lower) convex envelope of $H_\beta(G^-)(\theta)$ and $H_\beta(G^+)(\theta)$.

6.2. The case $\beta < G_c(\tilde{\rho})$. Let $\theta \in (-c, \tilde{\rho}]$. If $\beta < H_\beta(G^-)(\theta) \leq G_c(\tilde{\rho})$, Theorem 5.5 yields
\[
\mathcal{H}^L_\beta(G_c)(\theta) = \mathcal{H}^U_\beta(G_c)(\theta) = H_\beta(G^-)(\theta).
\]
Let us now consider the case $H_\beta(G^-)(\theta) > G_c(\tilde{\rho})$. We first remark that
\[
H_0(G^-)(\theta) = G^-(\theta) = G_c(\theta) \leq G_c(\tilde{\rho}).
\]
By Proposition 3.1 we know that the map $\tilde{\beta} \mapsto H^L_\tilde{\beta}(G)(\theta)$ is continuous and nondecreasing on $[0, +\infty)$ with $G := G^-$ or $G := G_c$. We infer that there exists a $\beta_- \in [0, \beta)$ such that $H_{\beta_-}(G^-)(\theta) = G_c(\tilde{\rho})$, hence by the previous step we get that (6.1) holds with $\beta_-$ in place of $\beta$. By monotonicity we get
\[
H^L_{\beta_-}(G_c)(\theta) \geq H^L_{\beta_+}(G_c)(\theta) = H_{\beta_+}(G^-)(\theta) = G_c(\tilde{\rho}).
\]
By taking into account the upper bound (4.6), we conclude that
\[
H^L_\beta(G_c)(\theta) = H^U_\beta(G_c)(\theta) = G_c(\tilde{\rho}).
\]
When $\theta \in [\tilde{\rho}, c)$, arguing analogously we get
\[
H^U_\beta(G_c)(\theta) = H^L_\beta(G_c)(\theta) = \begin{cases} 
H_\beta(G^+)(\theta) & \text{if } \beta < H_\beta(G^+)(\theta) \leq G_c(\tilde{\rho}) \\
G_c(\tilde{\rho}) & \text{if } G_c(\tilde{\rho}) < H_\beta(G^+)(\theta).
\end{cases}
\]
We have thus shown that
\[ \mathcal{H}_\beta(G_s)(\theta) = \begin{cases} 
\mathcal{H}_\beta(G^+(\theta)) & \text{if } \theta > \theta_+ \\
G_s(\bar{p}) & \text{if } \theta_- \leq \theta \leq \theta_+ \\
\mathcal{H}_\beta(G^-(\theta)) & \text{if } \theta < \theta_-, 
\end{cases} \]
where \( \theta_+ \) (resp. \( \theta_- \)) is the unique solution in \([\bar{p}, c]\) (resp. \((-c, \bar{c})\)) of the equation
\[ \mathcal{H}_\beta(G^+(\theta)) = G_s(\bar{p}) \quad \text{(resp. } \mathcal{H}_\beta(G^-(\theta)) = G_s(\bar{p}) \text{)}. \]
Indeed, \( \mathcal{H}_\beta(G^+(\bar{p}) \geq G^+(\bar{p}) = G_s(\bar{p}) > \beta = \mathcal{H}_\beta(G^+(c)) \), hence the existence and uniqueness of such a \( \theta_+ \) follows from the convexity of \( \theta \mapsto \mathcal{H}_\beta(G^+(\theta)) \). The reasoning for \( \theta_- \) is analogous.

7. **Proof of Theorem 2.3**

Throughout this section we assume that \( V : \mathbb{R} \times \Omega \to [0, +\infty) \) is a stationary potential satisfying (V1)-(V2). We start with a proposition, which is the key observation needed for the proof of Theorem 2.3. This proposition states that in our setting homogenization commutes with convexification (i.e. taking the convex envelope of the momentum part of the original Hamiltonian).

Given a function \( h : \mathbb{R} \to \mathbb{R} \), we shall denote by \( \text{conv}(h) \) its (lower) convex envelope
\[ \text{conv}(h)(p) := \sup\{g(p) : g \text{ is convex and } \forall x \in \mathbb{R}, \ g(x) \leq h(x)\}, \ \forall p \in \mathbb{R}. \]

**Proposition 7.1.** Let \( c_+ \geq c_- \) and \( G^\pm \in \mathcal{H}(\gamma, \alpha_0, \alpha_1) \) be non-negative convex functions such that \( G^-(c_-) = G^+(c_+ = 0 \) and
\[ (G^+ \land G^-)(p) = G^-(p) \quad \text{for } p \leq c_-, \quad (G^+ \land G^-)(p) = G^+(p) \quad \text{for } p \geq c_+. \]
Then
\[ \mathcal{H}_\beta(\text{conv}(G^+ \land G^-)) = \text{conv}(\mathcal{H}_\beta(G^+) \land \mathcal{H}_\beta(G^-)). \]

In turn, the above proposition is a simple consequence of the following observation.

**Lemma 7.2.** Let \( G, G^+ \in \mathcal{H}(\gamma, \alpha_0, \alpha_1) \) be non-negative convex functions such that \( G(0) = G^+(0) = 0 \).

(i) If \( G(p) = G^+(p) \) for all \( p \geq 0 \), then \( \mathcal{H}_\beta(G^+(\theta)) = \mathcal{H}_\beta(G(\theta)) \) for all \( \theta \geq 0 \).

(ii) If \( G(p) = G^-(p) \) for all \( p \leq 0 \), then \( \mathcal{H}_\beta(G^-(\theta)) = \mathcal{H}_\beta(G(\theta)) \) for all \( \theta \leq 0 \).

**Proof.** We shall prove only item (i), since the argument for (ii) is symmetric. Fix an arbitrary \( \theta \geq 0 \) such that \( \lambda := \mathcal{H}_\beta(G(\theta)) \). Then there is a corrector \( F_\theta(x, \omega) \) for
\[ a(x, \omega)u'' + G(\theta + u') + \beta V(x, \omega) = \lambda \quad \text{in } \mathbb{R}. \]

We claim that \( F_\theta(x, \omega) \) is also a corrector for
\[ a(x, \omega)u'' + G^+(\theta + u') + \beta V(x, \omega) = \lambda \quad \text{in } \mathbb{R}. \quad (7.1) \]
This follows immediately from derivative estimates of Proposition 5.3. Indeed, by this proposition,
\[ a^-_\lambda \leq \theta + F_\theta'(x, \omega) \leq b^-_\lambda, \]
where \( a^-_\lambda, b^-_\lambda > 0, G(a^-_\lambda) = \lambda - \beta, \) and \( G(b^-_\lambda) = \lambda. \) Since \( G^+(p) = G(p) \) for all \( p \geq 0, \) we conclude that
\[ G^+(\theta + F_\theta(x, \omega)) = G(\theta + F_\theta(x, \omega)) \quad \text{in } \mathbb{R}. \]
in the viscosity sense. The existence of a corrector for equation (7.1) with \( \lambda = H_\beta(G)(\theta) \) implies that \( H_\beta(G^+)(\theta) = \lambda = H_\beta(G)(\theta) \). We conclude that

\[
H_\beta(G^+)(\theta) = H_\beta(G)(\theta)
\]

on the set \( \{ \theta \geq 0 | H_\beta(G)(\theta) > \beta \} \).

Exchanging the roles of \( G \) and \( G^+ \) we also have that

\[
H_\beta(G)(\theta) = H_\beta(G^+)(\theta)
\]

on the set \( \{ \theta \geq 0 | H_\beta(G^+)(\theta) > \beta \} \).

The last two statements in combination with the fact that \( (H_\beta(G) \wedge H_\beta(G^+))(\theta) \geq \beta \) for all \( \theta \in \mathbb{R} \) complete the proof of the lemma.

Proof of Proposition 7.1. Since \( \beta = \min_{\theta \in \mathbb{R}} H_\beta(G^+)(\theta) \), we have that

\[
\text{conv}(H_\beta(G^+) \wedge H_\beta(G^-))(\theta) = \begin{cases} 
H_\beta(G^+)(\theta), & \text{if } \theta \geq c_+; \\
\beta, & \text{if } -c_- \leq \theta \leq c_+; \\
H_\beta(G^-)(\theta), & \text{if } \theta \leq -c_-.
\end{cases}
\]

(7.2)

On the other hand,

\[
G(p) := \text{conv}(G^+ \wedge G^-)(p) = \begin{cases} 
G^+(p), & \text{if } p \geq c_+; \\
0, & \text{if } -c_- \leq p \leq c_+; \\
G^-(p), & \text{if } p \leq -c_-
\end{cases}
\]

Applying the first part of Lemma 7.2 to functions \( G(\cdot + c_+), G^+(\cdot + c_+) \) and the second to functions \( G(\cdot - c_-), G^- (\cdot - c_-) \) we infer that

\[
H_\beta(G)(\theta) = \begin{cases} 
H_\beta(G^+)(\theta), & \text{if } \theta \geq c_+; \\
H_\beta(G^-)(\theta), & \text{if } \theta \leq -c_-
\end{cases}
\]

By Proposition 3.1(iv), we know that \( H_\beta(G)(c_+) = H_\beta(G)(c_-) = \beta \). Combining this with the fact that \( H_\beta(G) \) is convex and \( H_\beta(G)(\theta) \geq \beta \) for all \( \theta \in \mathbb{R} \), see Proposition 4.1 we get that \( H_\beta(G) \) coincides with the right hand side of (7.2). This finishes the proof.

Proof of Theorem 2.3. First of all, we note that by Theorem 1.1 the right hand side of (2.2) is equal to (2.3).

Upper bound. Since \( (G_0 \wedge G_1 \wedge \cdots \wedge G_n)(p) \leq G_{i-1,i}(p) \) for all \( p \in \mathbb{R} \) and \( i \in \{1,2,\ldots,n\} \), by comparison, the left hand side of (2.2) does not exceed the right hand side.

Lower bound. We introduce a piece of notation first. For all \( i < j \), let us set \( G_{ij} := G_i \wedge G_j \) and denote by \( \hat{p}_{ij} \in (c_i, c_j) \) a solution of the equation \( G_i(p) = G_j(p) \). Note that

\[
(G_i \wedge G_j)(p) = G_i(p) \quad \text{if } p \leq \hat{p}_{ij}, \quad (G_i \wedge G_j)(p) = G_j(p) \quad \text{if } p \geq \hat{p}_{ij},
\]

and

\[
G_{ij}(\hat{p}_{ij}) = G_i(\hat{p}_{ij}) = G_j(\hat{p}_{ij}) = \max_{p \in [c_i, c_j]} G_{ij}(p).
\]

Set \( G_{00} := G_0 \) and \( G_{nn} := G_n \). By comparison, for each \( i \in \{1,2,\ldots,n\} \)

\[
H_\beta(G_0 \wedge G_1 \wedge \cdots \wedge G_n) \geq H_\beta(\text{conv}(G_{0,i-1}) \wedge \text{conv}(G_{in})).
\]

(7.3)

Next we shall write the formula for \( H_\beta(G_{i-1,i}) \) from Theorem 1.1 in a way which covers both weak and strong potential cases. For \( i \in \{1,2,\ldots,n\} \)

\[
H_\beta(G_{i-1,i})(\theta) = \begin{cases} 
H_\beta(G_{i-1})(\theta), & \text{if } \theta < \theta_{i-1,i}^-; \\
G_{i-1,i}(\hat{p}_{i-1,i}) \lor \beta, & \text{if } \theta_{i-1,i}^- \leq \theta \leq \theta_{i-1,i}^+; \\
H_\beta(G_i)(\theta), & \text{if } \theta > \theta_{i-1,i}^+.
\end{cases}
\]

(7.4)
where $\theta^{-}_{i-1,i}$ (resp. $\theta^{+}_{i-1,i}$) is the smallest (resp. largest) solution in $[c_{i-1}, \hat{p}_{i-1,i}]$ (resp. $[\hat{p}_{i-1,i}, c_i]$) of the equation

$$\mathcal{H}_\beta(G_{i-1})(\theta) = G_{i-1,i}(\hat{p}_{i-1,i}) \vee \beta \quad \text{(resp. } \mathcal{H}_\beta(G_i)(\theta) = G_{i-1,i}(\hat{p}_{i-1,i}) \vee \beta).$$

In the strong potential case we simply have $\theta^{-}_{i-1,i} = c_{i-1}$ and $\theta^{+}_{i-1,i} = c_i$ (see Figure 1).

![Figure 1](image_url)

**Figure 1.** The original Hamiltonian $G_0(p) \land G_1(p) \land G_2(p)$ is depicted in blue and the effective Hamiltonian is in black. Note that $\theta^{0}_{01} = c_0$ and $\theta^{+}_{01} = c_1$.

In the same way, for each $i \in \{1, 2, \ldots, n\}$ and $\theta \in [c_{i-1}, c_i]$ we get that

$$\mathcal{H}_\beta(\text{conv}(G_{0,i-1}) \land \text{conv}(G_{im})) = \begin{cases} \mathcal{H}_\beta(\text{conv}(G_{0,i-1}))(\theta), & \text{if } \theta < \theta^{-}_{i-1,i}; \\ G_{i-1,i}(\hat{p}_{i-1,i}) \vee \beta, & \text{if } \theta^{-}_{i-1,i} \leq \theta \leq \theta^{+}_{i-1,i}; \\ \mathcal{H}_\beta(\text{conv}(G_{im}))(\theta), & \text{if } \theta > \theta^{+}_{i-1,i}. \end{cases} \quad (7.5)$$

We emphasize that $\theta^{\pm}_{i-1,i}$ which appear in (7.5) are the same as in (7.4). Indeed, by the definition of $\theta^{-}_{i-1,i}$ and Proposition (7.1), $\theta^{-}_{i-1,i} \geq c_{i-1}$ and

$$\mathcal{H}_\beta(\text{conv}(G_{0,i-1}))(\theta) = \text{conv}(\mathcal{H}_\beta(G_0) \land \mathcal{H}_\beta(G_{i-1}))(\theta) = \mathcal{H}_\beta(G_{i-1})(\theta) \quad \text{for } \theta \geq c_{i-1}.$$ 

Similarly, $\theta^{+}_{i-1,i} \leq c_i$ and

$$\mathcal{H}_\beta(\text{conv}(G_{im}))(\theta) = \text{conv}(\mathcal{H}_\beta(G_1) \land \mathcal{H}_\beta(G_n))(\theta) = \mathcal{H}_\beta(G_i)(\theta) \quad \text{for } \theta \leq c_i.$$ 

These formulas together with (7.4) and (7.5) imply that for all $\theta \in [c_{i-1}, c_i]$, $i \in \{1, 2, \ldots, n\}$

$$\mathcal{H}_\beta(\text{conv}(G_{0,i-1}) \land \text{conv}(G_{im}))(\theta) = \mathcal{H}_\beta(G_{i-1,i})(\theta). \quad (7.6)$$

From (7.3), (7.5), and (7.6) we conclude that

$$\mathcal{H}_\beta(G_0 \land G_1 \land \ldots \land G_n)(\theta) \geq \max_{j \in \{1,2,\ldots,n\}} \mathcal{H}_\beta(\text{conv}(G_{0,j-1}) \land \text{conv}(G_{jn}))(\theta) \quad (7.7)$$

$$= \begin{cases} \mathcal{H}_\beta(G_{00}), & \text{if } \theta < c_0; \\ \mathcal{H}_\beta(G_{i-1,i}), & \text{if } c_{i-1} \leq \theta \leq c_i, \quad i \in \{1,2,\ldots,n\}; \\ \mathcal{H}_\beta(G_{nn}), & \text{if } \theta > c_n. \end{cases} \quad (7.8)$$

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Indeed, for all $\theta \leq c_0 < \theta_{01}$ and all $j \in \{1, 2, \ldots, n\}$

$$\mathcal{H}_\beta(\text{conv}(G_{0,j-1}) \land \text{conv}(G_{j,n}))(\theta) \overset{\text{Prop. A.1}}{=} \mathcal{H}_\beta(\text{conv}(G_{0,j-1}))(\theta) \overset{(7.9)}{=} \mathcal{H}_\beta(G_{0,j-1}))(\theta) = \mathcal{H}_\beta(G_0)(\theta).$$

This proves that the right hand side of (7.7) is equal to the first line of (7.8) when $\theta \leq c_0$. Similar argument establishes the equality for $\theta \geq c_n$. Next, combining (7.6) with the fact that for all $i, j \in \{1, 2, \ldots, n\}$ such that $j \neq i$

$$\mathcal{H}_\beta(\text{conv}(G_{0,j-1}) \land \text{conv}(G_{j,n}))(\theta) = \beta \leq \mathcal{H}_\beta(G_{i-1,i}))(\theta), \quad \theta \in [c_{i-1}, c_i],$$

we obtain the equality between the right hand side of (7.7) and (7.8). Finally, we notice that (7.8) coincides with (2.3). This completes the proof.

□

APPENDIX A. PDE results

In this appendix we state and prove some PDE results we need for our study. We introduce the following list of conditions on the ingredients of the parabolic and stationary Hamilton–Jacobi equations we will consider, for fixed constants $\alpha_0, \alpha_1, \kappa > 0$ and $\gamma > 1$:

(AV) $\sqrt{a}, V : \mathbb{R} \to [0, 1]$ are $\kappa$–Lipschitz continuous;

(G1) $\alpha_0|p|^\gamma - 1/\alpha_0 \leq G(p) \leq \alpha_1(|p|^\gamma + 1)$ for all $x, p \in \mathbb{R}$;

(G2) $|G(p) - G(q)| \leq \alpha_1 (|p| + |q| + 1)^{\gamma-1}|p - q|$ for all $p, q \in \mathbb{R}$;

(G3) $G(0) = 0$;

(G4) $G(\cdot)$ is convex;

(G5) $G(p) \geq 0$ for every $p \in \mathbb{R}$.

In what follows, we will denote by LSC($X$) and USC($X$) the space of lower and upper semi-continuous real functions on the topological space $X$, respectively.

A.1. Parabolic equation. We consider the parabolic equation

$$\partial_t u = a(x)\partial_{xx}^2 u + G(\partial_x u) + \beta V(x, \omega) \quad \text{in} \ (0, +\infty) \times \mathbb{R}. \quad (A.1)$$

We have the following comparison result.

Proposition A.1. Assume condition (AV) and let $G \in C(\mathbb{R})$. Let $v \in \text{USC}([0, T] \times \mathbb{R})$, $w \in \text{LSC}([0, T] \times \mathbb{R})$ be, respectively, a sub- and a super-solutions of (A.1) satisfying

$$\limsup_{|x| \to +\infty} \sup_{t \in [0, T]} \frac{v(t, x)}{1 + |x|} \leq 0 \leq \liminf_{|x| \to +\infty} \inf_{t \in [0, T]} \frac{w(t, x)}{1 + |x|}. \quad (A.2)$$

Let us furthermore assume that either $\partial_x v$ or $\partial_x w$ belongs to $L^\infty ((0, T) \times \mathbb{R})$. Then

$$v(t, x) - w(t, x) \leq \sup_{\mathbb{R}} \left( v(0, \cdot) - w(0, \cdot) \right) \quad \text{for every} \ (t, x) \in [0, T] \times \mathbb{R}.$$

The proof is standard, see for instance [16] Proposition 2.3 and [15] Appendix A. The next result shows that equation (A.1) is well posed in $\text{UC}([0, +\infty) \times \mathbb{R})$.

Theorem A.2. Assume conditions (AV) and (G1)-(G2). Then for every $g \in \text{UC}(\mathbb{R})$ there exists a unique solution $u \in \text{UC}([0, +\infty) \times \mathbb{R})$ of (A.1) satisfying $u(0, \cdot) = g$ on $\mathbb{R}$.

We also need the following Lipschitz bounds for solutions to (A.1) with linear initial data. We refer to [16] Theorem 2.8 for proofs.
Proposition A.3. Assume conditions (AV) and (G1)-(G2). For every \( \theta \in \mathbb{R} \), the unique solution \( u_0 \) of (A.1) in \( UC([0, +\infty) \times \mathbb{R}) \) with initial condition \( u(0, x) = \theta x \) is \( \kappa(\theta) \)-Lipschitz continuous in \( [0, +\infty) \times \mathbb{R} \), for some locally bounded functions \( \kappa : \mathbb{R} \to [0, +\infty) \).

A.2. Stationary viscous Hamilton–Jacobi equation. Let us consider the equation
\[-a(x)u'' + G'(u') + \beta V(x) = \lambda \quad \text{in} \quad I,
\]
where \( I \) is an open subset of \( \mathbb{R} \) and \( \lambda > \beta > 0 \). We will be interested in the cases when \( I = \mathbb{R} \setminus \{y\} \), \( I = (-\infty, y) \), \( I = (y, +\infty) \).

The following comparison principle holds:

Theorem A.4. Assume conditions (AV) and (G1)–(G4). Let \( y, \theta \in \mathbb{R} \), \( \lambda > \beta > 0 \) and \( u \in \text{LSC}(\mathbb{R}), \tilde{v} \in \text{USC}(\mathbb{R}) \) be, respectively, a super- and sub- solution of
\[-a(x)u'' + G(\theta + u') + \beta V(x) = \lambda \quad \text{in} \quad \mathbb{R} \setminus \{y\},
\]
satisfying
\[\limsup_{|x| \to +\infty} \frac{v(x)}{1 + |x|} \leq \liminf_{|x| \to +\infty} \frac{u(x)}{1 + |x|}.\]
The following statements hold:

(i) if \( \theta > 0 \), then \( (v - u)(x) \leq (v - u)(y) \) for every \( x \geq y \);

(ii) if \( \theta < 0 \), then \( (v - u)(x) \leq (v - u)(y) \) for every \( x \leq y \);

(iii) if \( \theta = 0 \) and \( v \in \text{Lip}(\mathbb{R}) \), then \( (v - u)(x) \leq (v - u)(y) \) for every \( x \in \mathbb{R} \).

Proof. Without loss of generality, we can assume \( y = 0 \) and \( v(0) = u(0) = 0 \). Let us set \( \tilde{v}(x) := \theta x + v(x) \), \( \tilde{u}(x) := \theta x + u(x) \) and, for \( \mu \in (0, 1) \), \( \tilde{v}_\mu(x) := \mu \tilde{v}(x) = \mu v(x) + (1 - \mu)0 \).

Since the function \( v_0 \equiv 0 \) is a strict subsolution of (A.3) in \( \mathbb{R} \) (due to the fact that \( \lambda > \beta > 0 \) and \( G(0) = 0 \)), by convexity of \( G \) we infer that \( \tilde{v}_\mu \) is a strict subsolution to (A.3) in \( \mathbb{R} \setminus \{0\} \), see [7, Lemma 2.4], i.e. \( \tilde{v}_\mu \) satisfies the following inequality in the viscosity sense for some \( \delta > 0 \):
\[-a(x)v''_\mu + G(v'_\mu) + \beta V(x) < \lambda - \delta \quad \text{in} \quad \mathbb{R} \setminus \{0\}.
\]

Now, if \( \theta > 0 \), we have
\[\limsup_{x \to +\infty} \frac{\tilde{v}_\mu(x) - \tilde{u}(x)}{1 + |x|} \leq \limsup_{x \to +\infty} \frac{-\mu v(x) - u(x)}{1 + |x|} \leq \lim_{x \to +\infty} \frac{-\mu v(x) + \mu \theta x}{1 + |x|} - \lim_{x \to +\infty} \frac{u(x)}{1 + |x|} \leq \lim_{x \to +\infty} \frac{-\mu v(x)}{1 + |x|} = -(1 - \mu)\theta < 0,
\]
in particular \( (\tilde{v}_\mu - \tilde{u})(x) \to -\infty \) as \( x \to +\infty \). This means that the open set \( I_\mu := \{x > 0 \mid \tilde{v}_\mu - \tilde{u} > 0\} \) is bounded, so we can apply [7, Theorem 2.2] to get
\[\sup_{I_\mu} (\tilde{v}_\mu - \tilde{u}) \leq \sup_{\partial I_\mu} (\tilde{v}_\mu - \tilde{u}) = 0,
\]
where in the last equality we have also used the fact that \( v_\mu(0) - u(0) = 0 \). From this we infer that
\[\tilde{v}_\mu(x) - \tilde{u}(x) = (\mu v(x) - u(x)) - (1 - \mu)\theta x \leq 0 \quad \text{for all} \quad x \geq 0.
\]
By sending \( \mu \to 1 \) we get
\[v(x) - u(x) \leq 0 = v(0) - u(0) \quad \text{for all} \quad x \geq 0, \text{ as asserted.} \]
If \( \theta < 0 \), then, arguing as above, we get
\[
\limsup_{x \to -\infty} \frac{\tilde{v}_\mu(x) - \tilde{u}(x)}{1 + |x|} \leq \lim_{x \to -\infty} -(1 - \mu)\theta \frac{x}{|x|} = (1 - \mu)\theta < 0,
\]
in particular \((\tilde{v}_\mu - u)(x) \to -\infty \) as \( x \to -\infty \). This means that the open set \( I_\mu := \{ x < 0 \mid \tilde{v}_\mu - u > 0 \} \) is bounded. By arguing as in the previous case, we conclude that \( v(x) - u(x) \leq 0 = v(0) - u(0) \) for all \( x \leq 0 \).

If \( \theta = 0 \), then \( \tilde{v} = v \) and \( \tilde{u} = u \). Let us write \( v_\mu \) in place of \( \tilde{v}_\mu \) and set \( v_\mu^\varepsilon(x) := v_\mu(x) - \varepsilon \sqrt{1 + x^2} \) for every \( x \in \mathbb{R} \). Because of (A.3) and the fact that \( v_\mu \in \text{Lip}(\mathbb{R}) \), an easy computation shows that for \( \varepsilon > 0 \) small enough \( v_\mu^\varepsilon \) is a strict subsolution to (A.3) in \( \mathbb{R}\setminus\{0\} \), i.e. satisfies (A.5). We have
\[
\limsup_{|x| \to +\infty} \frac{v_\mu^\varepsilon(x) - u(x)}{1 + |x|} \leq \limsup_{|x| \to +\infty} \frac{\mu v(x) - \varepsilon \sqrt{1 + x^2}}{1 + |x|} - \liminf_{|x| \to +\infty} \frac{u(x)}{1 + |x|} \leq -\varepsilon < 0,
\]
in particular \((v_\mu^\varepsilon - u)(x) \to -\infty \) as \(|x| \to +\infty \). This means that the open set \( I_\mu := \{ x \in \mathbb{R} \mid v_\mu^\varepsilon - u > 0 \} \) is bounded, so we can apply [7, Theorem 2.2] and argue as above to infer
\[
v_\mu^\varepsilon(x) - u(x) = (\mu v(x) - u(x)) - \varepsilon \sqrt{1 + x^2} \leq 0 \quad \text{for all} \ x \in \mathbb{R}.
\]
By sending \( \varepsilon \searrow 0 \) and \( \mu \nearrow 1 \) we conclude that \( v(x) - u(x) \leq 0 = v(0) - u(0) \) for all \( x \in \mathbb{R} \).

As a corollary we infer

**Corollary A.5.** Let \( \theta \in \mathbb{R}\setminus\{0\} \) and \( u_1, u_2 \) be sublinear solutions of
\[
-a(x)u'' + G(\theta + u') + \beta V(x) = \lambda \quad \text{in} \ \mathbb{R},
\]
where \( \lambda > \beta > 0 \). Then \( u_1 - u_2 \) is constant on \( \mathbb{R} \).

**Proof.** To fix ideas, let us assume \( \theta > 0 \). Let us fix \( y \in \mathbb{R} \). By applying Theorem A.4 we get
\[
(u_1 - u_2)(x) \leq (u_1 - u_2)(y) \quad \text{for all} \ x \geq y,
\]
and, symmetrically,
\[
(u_2 - u_1)(x) \leq (u_2 - u_1)(y) \quad \text{for all} \ x \geq y.
\]
We conclude that, for every \( y \in \mathbb{R} \), we have
\[
(u_1 - u_2)(x) = (u_1 - u_2)(y) \quad \text{for all} \ x \geq y.
\]
This readily implies that \( u_1 - u_2 \) is constant on \( \mathbb{R} \). The argument in the case \( \theta < 0 \) is analogous. \( \square \)

We also need the following version of the comparison principle.

**Theorem A.6.** Assume conditions (AV) and (G1)–(G5). Let \( \lambda > \beta > 0 \), \( y \in \mathbb{R} \) and \( I \) be either \( I = (-\infty, y) \) or \( I = (y, +\infty) \). Let \( u \in \text{Lip}(I) \) and \( v(x) := \kappa |x - y|, \ \kappa > 0 \), be, respectively, a super- and sub- solution of
\[
-a(x)u'' + G(u') + \beta V(x) = \lambda \quad \text{in} \ I \quad \text{(A.6)}
\]
with
\[
\liminf_{x \in I, |x| \to +\infty} \frac{u(x)}{1 + |x|} \geq 0.
\]
Then
\[
(v - u)(x) \leq (v - u)(y) \quad \text{for all} \ x \in I.
\]
This comparison principle can be easily proved arguing as in the proof of Theorem A.3 (iii) with the aid of the following lemma.

**Lemma A.7.** Let \( y \in \mathbb{R} \) and let \( I \) be either \( I = (-\infty, y) \) or \( I = (y, +\infty) \). Let \( v(x) := \kappa|x-y|, \kappa > 0, \) be a subsolution to (A.6) where \( \lambda > \beta > 0 \). Then \( v(x) = \sup_{w \in \mathcal{S}^-}(v) w(x), x \in I, \) where we have denoted by \( \mathcal{S}^- \) the set of bounded subsolutions \( w : I \to \mathbb{R} \) of (A.6) satisfying \( w \leq v \) in \( I \).

**Proof.** Without any loss of generality, we can assume \( y = 0 \). For every \( \mu \in [0, 1) \) we set \( v_\mu(x) := \mu v(x) \). Since the function \( v_0 := 0 \) is a strict subsolution of (A.6) in \( I \) (due to the fact that \( \lambda > \beta > 0 \) and \( G(0) = 0 \)), by convexity of \( G \) we infer that \( v_\mu = \mu v + (1 - \mu)v_0 \) is a strict subsolution to (A.6) in \( I \), i.e. \( v_\mu \) satisfies the following inequality in the viscosity sense for some \( \delta > 0 \):

\[
-a(x)v''_\mu + G(v'_\mu) + \beta V(x) = G(v'_\mu) + \beta V(x) < \lambda - \delta \quad \text{in } I. \tag{A.7}
\]

Since \( v(x) = \sup_{\mu \in (0,1)} v_\mu(x) \), we infer that it is sufficient to prove the assertion by additionally assuming that \( v(x) = \kappa|x| \) satisfies (A.7) for some \( \delta > 0 \). For fixed \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), we define \( \varphi_{\varepsilon,n}(u) := \int_0^u g_{\varepsilon,n}(t) \, dt \) for all \( u \geq 0 \), where

\[
g_{\varepsilon,n}(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \kappa n, \\ f(\varepsilon(t-n)) & \text{if } t > \kappa n, \end{cases}
\]

with \( f(s) := e^{-s^2} \). Let \( C > 0 \) be such that \(-C \leq f'(s) < 0 \) for every \( s \geq 0 \). An easy check shows that \( \varphi_{\varepsilon,n} \) is bounded and of class \( C^1 \) on \([0, +\infty)\), \( 0 < \varphi'_{\varepsilon,n} \leq 1 \) and \( \varphi''_{\varepsilon,n} \geq -\varepsilon C \) in \([0, +\infty)\). Let us set \( v_{\varepsilon,n}(x) := \varphi_{\varepsilon,n}(v(x)) \).

\[
v_{\varepsilon,n} \leq v \quad \text{in } I, \quad v_n = v \quad \text{in } I \cap [-n,n], \quad v''_{\varepsilon,n}(x) = \kappa^2 \varphi''_{\varepsilon,n}(v(x)) \geq -C \kappa^2 \varepsilon \quad \text{in } I. \tag{A.8}
\]

Also notice that \( v'_{\varepsilon,n}(x) = \varphi'_{\varepsilon,n}(v(x))v'(x) \) and \( v'(x) \) have the same sign (either positive or negative) and \( |v'_{\varepsilon,n}| \leq |v'| \) in \( I \). Since \( G \) is non-increasing in \((0, +\infty)\) and non-decreasing on \([0, +\infty)\), we infer that \( G(v'_{\varepsilon,n}(x)) \leq G(v'(x)) \) for every \( x \in I \). So

\[
-a(x)v''_{\varepsilon,n} + G(v'_{\varepsilon,n}) + \beta V(x) \leq C \kappa^2 \varepsilon + G(v') + \beta V(x) \leq \lambda - \delta + C \kappa^2 \varepsilon \quad \text{in } I,
\]

hence by choosing \( \varepsilon < \delta/(C \kappa^2) \) we get that \( v_n := v_{\varepsilon,n} \in \mathcal{S}^-(v) \). By taking into account (A.8), we conclude that \( v(x) = \sup_{n \in \mathbb{N}} v_n(x) \), which clearly implies the assertion. \( \square \)

**APPENDIX B. HILLS AND VALLEYS**

In this section we discuss the relationship between the original hill and valley condition of [25] (see (AV) below) and its scaled version (V2), give examples of potentials which satisfy conditions (1.2), (V1) and (AV) and argue that potentials satisfying (AV) can be thought of as “typical” in the general stationary ergodic setting.

**B.1. Comparison of (AV) and (V2).** In [25] the authors considered the case \( a = 1/2 \) and imposed the following hill and valley condition on \( V \):

\[
(V) \quad \mathbb{P}(\forall x \in [-y,y], V(x,\omega) \geq h) > 0; \\
(V') \quad \mathbb{P}(\forall x \in [-y,y], V(x,\omega) \leq h) > 0.
\]

First of all we note that (V) can be stated equivalently as follows.

\[
(V') \quad \text{for all } h \in (0,1) \text{ and } y > 0 \text{ there is a set } \Omega(h,y) \text{ of probability 1 such that for each } \omega \in \Omega(h,y) \text{ there is an } \ell \in \mathbb{R} \text{ such that } V(x,\omega) \geq h \text{ for all } x \in \ell, \ell + 2y.
\]
Indeed, if we let \( A(h, y) = \bigcup_{\ell \in \mathbb{R}} \{ \forall x \in [\ell, \ell + 2y], V(x, \omega) \geq h \} \) (“there is an \( h \)-hill of length \( 2y \)”) then \((\wedge)\) implies that \( \mathbb{P}(A(h, y)) > 0 \). But the event \( A(h, y) \) is invariant under translations. Therefore, by the ergodicity assumption its probability is equal to 1, and \((\wedge')\) follows. Conversely, note that

\[
(\wedge') \iff \forall h \in (0, 1), \forall y > 0, \mathbb{P}(A(h, y)) = 1
\]

\[
\iff \forall h \in (0, 1), \forall y > 0, \mathbb{P}\left( \bigcup_{\ell \in \mathbb{Z}} \{ \forall x \in [\ell, \ell + 2y], V(x, \omega) \geq h \} \right) = 1.
\]

Since

\[
\mathbb{P}\left( \bigcup_{\ell \in \mathbb{Z}} \{ \forall x \in [\ell, \ell + 2y], V(x, \omega) \geq h \} \right) \leq \sum_{\ell \in \mathbb{Z}} \mathbb{P}(\forall x \in [\ell, \ell + 2y], V(x, \omega) \geq h)
\]

and, by stationarity,

\[
\forall \ell \in \mathbb{R}, \quad \mathbb{P}(\forall x \in [\ell, \ell + 2y], V(x, \omega) \geq h) = \mathbb{P}(\forall x \in [-y, y], V(x, \omega) \geq h),
\]

we conclude that \((\wedge')\) implies \((\wedge)\). The valley condition \((\lor)\) admits a similar equivalent formulation.

Now it is easily seen that in the uniformly elliptic case, i.e. when \( a(x, \omega) \geq a_0 > 0 \) for all \( x \in \mathbb{R} \) and \( \omega \in \Omega \), \((AV)\) is equivalent to \((V2)\). We also point out that if \( a(x_0, \omega) = 0 \) for some \( x_0 \in \mathbb{R} \), then \((A)\) implies that \( \int_I 1/a(x, \omega) \, dx = +\infty \) for every interval \( I \) containing \( x_0 \). In particular, if \( a \equiv 0 \) and \( V \) is continuous then \((V2)\) reduces to \((1.2)\) and, thus, can be dropped altogether.

**B.2. Examples.** We start with probably the simplest class of examples. They are based on stationary renewal processes and i.i.d. sequences.

**Example B.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \( F \) be a distribution function on \( \mathbb{R} \) with \( F(0) = 0 \) and a finite mean \( m := \int_0^{+\infty} (1 - F(t)) \, dt \). Let \((X_k)_{k \in \mathbb{Z} \setminus \{-1, 0\}}\) be a sequence of i.i.d. random variables with a common distribution \( F \) and choose \((X_{-1}, X_0)\) to be independent from this sequence and distributed as follows:

\[
\mathbb{P}(X_{-1} \geq x, X_0 \geq y) = \frac{1}{m} \int_{x+y}^{+\infty} (1 - F(t)) \, dt \quad \text{for all } x, y \geq 0.
\]

Define \( S_0 = X_0, S_n = \sum_{k=0}^n X_k, S_{-n} = -\sum_{k=-1}^n X_k \) for all \( n \in \mathbb{N} \). The process \((S_k)_{k \in \mathbb{Z}}\) is a stationary renewal process on \( \mathbb{R} \) (see [23, Theorem 9.1, Chapter 9])\footnote{If \( F(x) = \mathbb{1}_{(1, +\infty)}(x) \) then we get \( S_k = X_0 + k, k \in \mathbb{Z} \), where \( X_0 \) is uniform on \([0, 1]\) and \( X_{-1} = 1 - X_0 \).} Next we take an i.i.d. sequence \((\xi_k)_{k \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z}\) of Bernoulli random variables with parameter \( p \in (0, 1) \) which is independent of \((X_k)_{k \in \mathbb{Z}}\), and define

\[
V(S_k, \omega) = \xi_k, \quad V(x, \omega) = \xi_k + (x - S_k) \frac{\xi_{k+1} - \xi_k}{S_{k+1} - S_k}, \quad \text{for } S_k < x < S_{k+1}, \ k \in \mathbb{Z}.
\]

In words, we let \( V(S_k, \omega) = \xi_k \) and then linearly interpolate between the neighboring points. This stationary ergodic potential satisfies \((AV)\) since for every \( y > 0 \) there is an integer \( n \geq 0 \) such that

(i) the probability that \([-y, y]\) contains exactly \( n \) renewal points is strictly positive and

(ii) \( \mathbb{P}(\xi_0 = \xi_1 = \cdots = \xi_{n+1} = j) > 0 \) for \( j \in \{0, 1\} \) (we require require that at all \( n \) points of the renewal process as well as at the points immediately preceeding and following these \( n \) points the potential takes the same value \( j \)).
If, in addition, we assume that \( F(x) = 0 \) for all \( x < 1/\kappa \) then \( V \) also satisfies (V1).

Both properties (V1) and (AV) are preserved if we replace Bernoulli distribution for \( \xi_0 \) with any probability distribution on \([0, 1]\) as long as \( \mathbb{P}(\xi_0 \in [0, h]) > 0 \) and \( \mathbb{P}(\xi_0 \in [1 - h, 1]) > 0 \) for all \( h \in (0, 1/2) \).

**Remark B.2.** We also note that if we have a stationary ergodic potential \( V_0 \) with values in \([0, 1]\) which satisfies (AV) but not (V1), then we can take a \( \kappa \)-Lipschitz mollifier \( f \) supported in \([0, 1]\) and set \( V(x, \omega) = \int \mathbb{R} V_0(y, \omega) f(y - x) \, dx \). The resulting process \( V \) will be stationary ergodic and will satisfy both (V1) and (AV).

**Example B.3.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \( U \) be a uniform random variable on \([0, 1]\) and \( B_x = (B_x(t, \omega))_{t \geq 0} \) be independent standard Brownian motions which are independent from \( U \). Define a two-sided Brownian motion \( B = (B(x, \omega))_{x \in \mathbb{R}} \) starting from \( U \) by

\[
B(x, \omega) = U(\omega) + \begin{cases} B_+(x, \omega), & \text{if } x \geq 0; \\ B_-(x, \omega), & \text{if } x < 0. \end{cases}
\]

We define \( V(x, \omega) \) to be a Brownian motion with two-sided reflection, at 0 and at 1. The following informal description gives a pathwise construction of \( V \) from \( B \) which avoids unsightly formulas. Imagine a fully transparent plane \( \mathbb{R}^2 \) and draw on it horizontal lines \( y = k \) for all \( k \in \mathbb{Z} \). Given a path of \( B \), draw its graph \( (x, B(x, \omega)) \subseteq \mathbb{R}^2 \), fold \( \mathbb{R}^2 \) as if it were a sheet of paper along the line \( y = 0 \), then fold back along \( y = 1 \) and \( y = -1 \), and continue folding back and forth until every line \( y = k \), \( k \in \mathbb{Z} \), of the original plane has been used. The resulting curve contained in \( \mathbb{R} \times [0, 1] \) is the graph of \( V(x, \omega), x \in \mathbb{R} \).

The stationary ergodic process \( V \) satisfies (AV), since (see, for example, [12, Theorem (6.6), p. 60]) for every \( h, R > 0 \)

\[
\mathbb{P} \left( \max_{x \in [-R, R]} |B(x, \omega) - 1| < h \right) > 0 \quad \text{and} \quad \mathbb{P} \left( \max_{x \in [-R, R]} |B(x, \omega)| < h \right) > 0.
\]

The paths \( V(x, \omega), x \in \mathbb{R} \), are almost surely not Lipschitz, but, as we pointed out in Remark B.2 a mollification will fix this problem while preserving all other relevant properties.

In this example Brownian motion with two-sided reflection can be replaced by a more general \( \text{Lévy} \) process with two-sided reflection\(^3\) under suitable conditions on the support of its \( \text{Lévy} \) measure.

A very different class of examples was given in [26, Example 1.3]. Stochastic processes in this class have finite range of dependence and, thus, are mixing with any rate. An example which is not mixing can be constructed in the same way as in [31, Example 1.3] by using points of a renewal process in Example B.1 instead of points of \( \mathbb{Z} \).

**B.3. Discussion.** Suppose that \( V \) is stationary ergodic and satisfies (V1) and (AV). When does such process satisfy (AV)? On \( \mathbb{Z} \), an argument which shows that a very broad and natural class of stationary ergodic processes satisfies (AV) was already put forward in [31, Example 1.2]. We state the following simple sufficient condition for processes on \( \mathbb{R} \) which can be checked in many cases. Let \((\Omega, (\mathcal{F}(x))_{x \in \mathbb{R}}, \mathcal{F}, \mathbb{P})\) be a probability space, \( \mathcal{F}(x_1) \subseteq \mathcal{F}(x_2) \subseteq \mathcal{F} \) for all \( x_1 < x_2 \) be a filtration, and \( V \) be a stationary ergodic process which satisfies (V1) and (AV) and is adapted to this filtration. Observe that if for each

\(^3\)A rigorous construction and properties of \( \text{Lévy} \) processes with two-sided reflection can be found in [1].
\[ y > 0, \ h \in (0, 1/2) \] there is an \( n \in \mathbb{N} \) and \( 0 < x_{1} < x_{2} < \cdots < x_{n} < x_{0} + 2y \leq x_{n+1} \) such that \( \max_{1 \leq i \leq n} |x_{i+1} - x_{i}| < 1/(2\kappa) \),

\[
P(|V(x_{i}, \omega)| < h/2, \ \forall i \in \{1, \ldots, n\} \mid F(0)) > 0 \quad \text{and} \quad \P(|1 - V(x_{i}, \omega)| < h/2, \ \forall i \in \{1, \ldots, n\} \mid F(0)) > 0, \quad (B.1)
\]

then \( (AV) \) holds. All that \( (B.1) \) is asking for is that given the past of the process up to some fixed time (which by stationarity can be taken to be 0), the probability that it finds itself “on a hill” (resp. “in a valley”) at some future time \( x_{1} > 0 \) and then happens to be there also at sufficiently many times \( x_{2}, \ldots, x_{n} \) within \( 1/(2\kappa) \) of each other is not equal to zero. Many natural stochastic processes will satisfy \( (B.1) \). We remark that for Examples \( B.1 \) and \( B.3 \) the property \( (B.1) \) holds.

In conclusion, we mention an example of a process \( V \) based on an i.i.d. sequence, satisfying (V1) and (L2) for which (AV) fails. It is discussed in detail in \( [30] \) where the author shows, in particular, that relative to the probability measure on \( C(\mathbb{R}) \) corresponding to this process the sets of periodic and almost periodic functions are negligible. For reader’s convenience we reproduce a version of this example here.

**Example B.4.** Without loss of generality we assume that \( \kappa \geq 1 \) and let \( \varphi(x) \) be a \( 2\kappa \)-Lipschitz function such that (a) \( 0 \leq \varphi(x) \leq 1 \) for all \( x \in \mathbb{R} \), (b) \( \text{supp} \varphi(x) \subseteq [-1/2, 1/2] \), and (c) \( \varphi(x) = 1 \) iff \( x = 0 \). In the setting of Example \( B.1 \) take \( F(x) = \mathbb{1}_{[1, +\infty)}(x) \) so that \( S_{k+1} - S_{k} = 1 \) with probability 1 and let \( \zeta_{k} = 2\xi_{k} - 1 \in \{-1, 1\} \). Define

\[
V(x, \omega) = \frac{1}{2} \left( 1 + \sum_{k \in \mathbb{Z}} \zeta_{k} \varphi(x - S_{k}) \right).
\]

The process \( V \) clearly satisfies (V1) and (L2) but not (AV). On the other hand, no matter which \( \varphi(x) \) as above we fix, it is enough to take any \( F(x) \) such that \( F(x) = 0 \) for all \( x < 1 \), \( 1 - F(x) > 0 \) for all \( x > 0 \) and set

\[
V(x, \omega) = \frac{1}{2} \left( 1 + \sum_{k \in \mathbb{Z}} \zeta_{k} \left( \varphi\left( \frac{x - S_{k}}{S_{k+1} - S_{k}} \right) \mathbb{1}_{[0, +\infty)}(x - S_{k}) + \varphi\left( \frac{x - S_{k}}{S_{k} - S_{k-1}} \right) \mathbb{1}_{(-\infty, 0)}(x - S_{k}) \right) \right)
\]

to get the process which satisfies (V1) and (AV)\(^{5}\).

Example \( B.4 \) as well as the sufficient condition \( (B.1) \) indicate that essentially the only obstacle for the validity of (AV) is the “rigidity” of trajectories of \( V \) which is atypical for many classes of stationary ergodic processes, and even this “rigidity” can sometimes be rectified by making the ingredients “more random”.

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4\(^{\text{Function } \varphi \text{ can have any Lipschitz constant as long as it is at least } 2. \text{ This will lead to an obvious adjustment of other parameters. Our choice gives the simplest formulas.}}\)

5\(^{\text{(AV) is satisfied because } \varphi \text{ is continuous and } \P(S_{k} - S_{k-1} > \ell, S_{k+1} - S_{k} > \ell) = (1 - F(\ell))^{2} > 0 \text{ for all } k \in \mathbb{Z} \setminus \{-1, 0\} \text{ no matter how large } \ell > 1. \text{ is.}}\)
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