BPS STATES IN SUPERSTRINGS WITH SPONTANEOUSLY BROKEN SUSY

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Abstract

We show the existence of a supersymmetry-breaking mechanism in string theory, where $N = 4$ supersymmetry is broken spontaneously to $N = 2$ and $N = 1$ with moduli-dependent gravitino masses. The BPS spectrum of the theory with lower supersymmetry is in one-to-one correspondence with the spectrum of the heterotic $N = 4$ string. The mass splitting of the $N = 4$ spectrum depends on the moduli as well as the three $R$-symmetry charges. In the case of $N = 4 \rightarrow N = 2$, the perturbative $N = 2$ prepotential is determined by the perturbative $N = 4$ BPS states. This observation led us to suggest a method that determines the exact non-perturbative prepotential of the effective $N = 2$ supergravity using the shifted spectrum of the non-perturbative BPS states of the underlying $N = 4$ theory.

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1 Introduction

When a local symmetry is spontaneously broken, the physical states can be classified in terms of the unbroken phase spectrum and in terms of a well-defined mass splitting given in terms of vacuum expectation values of some fields, weighted by the charges of the broken symmetry. In the case of gauge symmetry breaking, the fields with non-zero vev’s are physical scalar fields, while in the case of supersymmetry breaking they are auxiliary fields. In extended supersymmetric theories (local or global), the supersymmetric vacua are degenerate, with zero vacuum energy for any vev of the moduli fields \((S,T)\). For instance, in the case of \(N = 4\) supergravity based on a gauge group \(U(1)^6 \times G\), the space of the moduli fields is given in terms of 2 + 6\(r\) physical scalars, which are coordinates of the coset space \([SU(1,1) \times U(1)]_S \times [SO(6, r) / SO(6) \times SO(r)]_T\). \(r\) is the rank of the gauge group \(G\).

In an arbitrary point of the moduli space the gauge symmetry \(G\) is broken down to \(U(1)^r\) while at some special points of the moduli space the gauge symmetry is extended to some non–Abelian gauge group of the same rank due to the presence of some extra gauge multiplets that become massless at the special points of the moduli space.

In the heterotic \(N = 4\) superstring solution obtained by \(T^6\) compactification of the 10 – \(d\) superstring, the rank of the group \(r\) has a fixed value, \(r = 22\) \([3]-[6]\). In an arbitrary point of the moduli space the gauge group is \(U(1)^r\) and in special points the symmetry is extended as in field theory. There is however a fundamental difference between the field theory Higgs phenomenon and the string theory one. Indeed, if in an \(N = 4\) field theory the gauge group is \(G = U(1)^6 \times SO(32)\) at any given point of the moduli space, then at any other point the remaining gauge symmetry \(G_T\) is always a subgroup of \(G\) with smaller dimensionality \(\text{dim}(G_T) \leq \text{dim}(G)\).

On the contrary, in the string Higgs phenomenon, owing to the existence of winding states, we can connect gauge groups which are not subgroups of a larger gauge group. For instance, it is possible to connect continuously \(G = U(1)^6 \times SO(32)\) with \(G = U(1)^6 \times E_8 \times E_8\), as well as with the most symmetric of same rank, namely \(G = SO(44)\). Indeed, starting from a 10 – \(d\) \(N = 1\) supergravity theory with \(G = SO(32)\) or \(G = E_8 \times E_8\) after compactification in four dimensions the only possible \(N = 4\) supergravity effective theories are based either to \(G = U(1)^6 \times SO(32)\) or \(G = U(1)^6 \times E_8 \times E_8\) (and their subgroups obtained with Higgs phenomenon). In string theory the gauge group can be further extended due to the existence of extra gauge bosons with non-zero winding numbers, which can become massless in special points of the moduli space.

When some auxiliary fields of the supergravity theories have non–vanishing vev, some (or all) of the supersymmetries are spontaneously broken \([11]-[12]\). There is a consistent class of \(N = 1,2\) and \(N = 4\) models defined in flat space-time in which all supersymmetries are broken or partially broken \([1]-[2]\). The most interesting case for our purposes is that in which there is one of the supersymmetries left unbroken. In that case we know that it is possible, in general, to have chiral representations of matter scalar multiplets which can describe the quarks and leptons of the supersymmetric standard model. All previous examples about the partial breaking of \(N = 2\) to \(N = 1\) supersymmetry was done at the field theory level \([13]\). In this work we will first show the extension of the partial spontaneous breaking at the perturbative
string level and then we will generalize our result to the non-perturbative level using as a tool the heterotic–type II string duality of the $N = 4$ 4d-superstrings [14,15].

2 Perturbative $N = 4$ Mass Spectrum

Our starting point is a four-dimensional heterotic $N = 4$ superstring solution. From the world-sheet viewpoint, these theories are constructed by the following left- and right-moving degrees of freedom:

- Four left-moving non-compact supercoordinates, $X^\mu, \Psi^\mu$
- Six left-moving compactified supercoordinates, $\Phi^I, \Psi^I$
- The left-moving super-ghosts, $b,c$ and $\beta, \gamma$
- Four right-moving coordinates, $\bar{X}^\mu$
- Six right-moving compactified coordinates, $\bar{\Phi}^I$
- 32 right-moving fermions, $\bar{\Psi}^A$
- The right-moving ghosts, $\bar{b}, \bar{c}$

In order to obtain a consistent (without ghosts) $N = 4$ solution the left-moving fermions $\Psi^\mu, \Psi^I$ and the $\beta, \gamma$ ghosts must have the same boundary conditions. In that case the global existence of the left-moving spin-3/2 world-sheet supercurrent

$$T_F = \Psi^\mu \partial X^\mu + \Psi^I \partial \Phi^I$$

implies periodic boundary conditions for the compact and non-compact left-moving coordinates, $\Phi^I, X^\mu$. Modular invariance implies the right-moving coordinates $\bar{\Phi}^I, \bar{X}^\mu$ to be periodic as well. The solution with $G = U(1)^6 \times SO(32)$, is when the right-moving fermions $\bar{\Psi}^A$ have the same boundary conditions (periodic or antiperiodic), while the solution with $G = U(1)^6 \times E_8 \times E_8$ is when the $\bar{\Psi}^A = (\bar{\Psi}^A_1, \bar{\Psi}^A_2)$ are in two groups of sixteen with the same boundary conditions. Starting either from the $G = U(1)^6 \times E_8 \times E_8$ solution or from the $G = U(1)^6 \times SO(32)$ we can obtain all others by deforming the momentum lattice of compactified bosons together with the charge lattice of the 32 fermions $\bar{\Psi}^A$.

The partition function of the heterotic $N = 4$ solutions in a generic point of the moduli space is well known and has the following expression [3]:

$$Z(\tau, \bar{\tau}) = \frac{\text{Im} \tau^{-1}}{\eta^2(\tau) \bar{\eta}^2(\bar{\tau})} \times \frac{1}{2} \sum_{\alpha, \beta} (-)^{\alpha+\beta+\alpha\beta} \frac{\eta^4[\beta](\tau)}{\eta^4(\tau)} \frac{\eta^4[\alpha](\bar{\tau})}{\eta^4(\bar{\tau})} \Gamma_{(6,22)}(\tau, \bar{\tau}),$$

where $\Gamma_{(6,22)}(\tau, \bar{\tau})$ denotes the partition function due to the compactified coordinates $\Phi^I, \bar{\Phi}^A$ and due to the sixteen right-moving $U(1)$ currents constructed with the fermions $\bar{\Psi}^I$

$$J^k = \bar{\Psi}^{2k-1} \Psi^{2k}, \quad k = 1, 2, ..., 16.$$  

$\Gamma_{(6,22)}(\tau, \bar{\tau})$ has in total $6 \times 22$ moduli parameters which correspond to (1,1) marginal deformations of the world-sheet action.

$$\delta S^{2d} = \delta T_{IJ} \partial \Phi^I \partial \Phi^J + Y^k_I \partial \Phi^J \ J^k.$$  

2
In terms of the six-dimensional backgrounds of the compactified space, the $T_{I,J}$ moduli are related to the internal background metric $G_{IJ}$ and the internal antisymmetric tensor $B_{I,J}$; $T_{I,J} = G_{I,J} + B_{I,J}$. The $Y^k_I$ moduli are the six-dimensional internal gauge field backgrounds which belong in the Cartan subalgebra of the ten-dimensional gauge group (either $E_8 \times E_8$ or $SO(32)$). From the four-dimensional viewpoint the moduli $T_{I,J}$ and $Y^k_I$ correspond to the vev’s of massless scalar fields, members of the $N = 4$ vector supermultiplets. The explicit form of the $N = 4$ heterotic partition function $\Gamma_{SO(32)}^{T_{I,J}, Y^k_I}$ is:

$$
\Gamma_{SO(32)}^{T_{I,J}, Y^k_I} (T, Y)(\tau, \bar{\tau}) = \text{Im} \tau^{-3} (\det G_{IJ})^{3} \times \sum_{m^I, n^J} \exp \left[ -\pi T_{I,J} \frac{(m^I + \tau n^I)(m^J + \tau n^J)}{\text{Im} \tau} \right] \times \frac{1}{2} \sum_{\gamma, \delta} \prod_{k=1}^{16} \exp \left[ i \frac{\pi}{4} (n^I Y^k_I Y^k_J m^J + 2 \delta Y^k_I n^J) \right] \times \bar{\delta} \left[ \gamma + n^I Y^k_I \right] (\bar{\tau})
$$

(2.5)

When all $Y$-moduli are zero $Y^k_I = 0$ then the gauge group is extended from $G = U(1)^{22}$ to $G = U(1)^6 \times SO(32)$.

An alternative representation of $\Gamma_{(6,22)}$ is the one in which for $Y^k_I = 0$ the extended gauge symmetry is $G = U(1)^6 \times E_8 \times E_8$ instead of $U(1)^6 \times SO(32)$:

$$
\Gamma_{SO(44)}^{E_8 \times E_8} (T, Y)(\tau, \bar{\tau}) = \text{Im} \tau^{-3} (\det G_{IJ})^{3} \times \sum_{m^I, n^J} \exp \left[ -\pi T_{I,J} \frac{(m^I + \tau n^I)(m^J + \tau n^J)}{\text{Im} \tau} \right] \times \frac{1}{2} \sum_{\gamma_1, \delta_1} \prod_{k=1}^{8} \exp \left[ i \frac{\pi}{4} (n^I Y^k_I Y^k_J m^J + 2 \delta_1 Y^k_I n^J) \right] \times \bar{\delta}_1 \left[ \gamma + n^I Y^k_I \right] (\bar{\tau})
$$

$$
\times \frac{1}{2} \sum_{\gamma_2, \delta_2} \prod_{k=9}^{16} \exp \left[ i \frac{\pi}{4} (n^I Y^k_I Y^k_J m^J + 2 \delta_2 Y^k_I n^J) \right] \times \bar{\delta}_2 \left[ \gamma + n^I Y^k_I \right] (\bar{\tau})
$$

(2.6)

Both the $SO(32)$ and $E_8 \times E_8$ representations are connected continuously by marginal deformations with the $G = SO(44)$ maximal gauge symmetry point [3], [4], [5]:

$$
\Gamma_{(6,22)}^{SO(44)} (\tau, \bar{\tau}) = \frac{1}{2} \sum_{\gamma, \delta} \varrho^6 \left[ \frac{\gamma}{\delta} \right] (\tau) \varrho^{22} \left[ \frac{\gamma}{\delta} \right] (\bar{\tau}).
$$

(2.7)

Another useful representation of $\Gamma_{(6,22)}$ is that of the lorentzian left- and right-momentum even self dual lattice. This representation is obtained by performing Poisson resummation on $m^I$, using either $\Gamma_{(6,22)}^{SO(32)} (T, Y)$ or $\Gamma_{(6,22)}^{E_8 \times E_8} (T, Y)$:

$$
\Gamma_{(6,22)}^{SO(32)} (P_I, \bar{P}_I, Q^k) = \sum_{m^I, n^J, Q^k} \exp i \pi \left[ \frac{\tau}{2} P_I g^{IJ} P_J - \frac{\bar{\tau}}{2} \bar{P}_I g^{IJ} \bar{P}_J - \tau \hat{Q}^k \hat{Q}^k \right]
$$

(2.8)

with

$$
\frac{1}{2} P_I g^{IJ} P_J - \frac{1}{2} \bar{P}_I g^{IJ} \bar{P}_J - \hat{Q}^k \hat{Q}^k = 2 m^I n^J - Q^k Q^k = \text{even integer}.
$$

(2.9)
In the above equations $g^{IJ}$ is the inverse of $G_{IJ}$; the lattice momenta $P_I$, $\bar{P}_I$, and the left-charges $\hat{Q}^k$ are given in terms the moduli parameters $G_{IJ}, B_{IJ}$ and $Y^k_I$ and in terms of the charges $(m_I, n^I, Q^k)$:

$$P_I = m_I + Y^k_I Q^k + \frac{1}{2} Y^k_I Y^k_J n^J + B_{IJ} n^J + G_{IJ} n^J$$

$$\bar{P}_I = m_I + Y^k_I Q^k + \frac{1}{2} Y^k_I Y^k_J n^J + B_{IJ} n^J - G_{IJ} n^J$$

$$\hat{Q}^k = Q^k + Y^k_I n^J.$$ (2.10)

All $N = 4$ heterotic solutions are defined in terms of the vev’s of the moduli fields $(T_{IJ}, Y^k_I)$ and thus different solutions are connected to each other by a string-Higgs phenomenon. The heterotic $N = 4$ spectrum is invariant under the target-space duality group $SO(6, 22; \mathbb{Z})$, e.g. the generalization of the $R \rightarrow 1/R$ spectrum symmetry in a compactification on $S^1$. At some special points of the moduli space we have extensions of the gauge group as in the effective $N = 4$ supergravity theories. In string theories further extensions can take place, since due to the non-zero winding charges $(n^I)$ can become massless in special points of the moduli space. Thus, in string theory, a large class of disconnected $N = 4$ supergravities are continuously related among themselves due to the existence of the winding states. This precise fact is the origin of the perturbative string unification of all interactions in string theories.

3 $N = 4 \rightarrow N = 2$ spontaneous SUSY breaking

One of the defining characteristics of the $N = 4$ theories is that the states are classified by their transformation properties under the $R$-symmetry group which, for $N = 4$ supersymmetry is $G_R = SU(4) \sim SO(6)$. In the gravitational multiplet the gravitinos are in 4 representation of $G_R$, the graviphotons are in 6, while the graviton, the dilaton and the antisymmetric tensor field are singlets. The degrees of freedom of a massless $N = 4$ vector multiplet are also in definite representations of $G_R$: the scalars are in 6, the gauginos are in 4, while the gauge bosons are singlets. In the heterotic string $G_R$ is constructed in terms of the six-left moving compactified supercoordinates, $(\Phi^I, \Psi^I)$. The world-sheet fermion bilinears $\Psi^I \Psi^J$ form an $SO(6)_{k=1}$ Kac–Moody algebra. In the light-cone picture, the full spectrum of the theory is classified in representations of $SO(6)_{k=1}$ and in terms of the $U(1)_0$ helicity charge $q^0 = \oint j^0$, $j^0 = \Psi^\mu \Psi^\nu$, $\mu, \nu = 3, 4$. In the $N = 4$ spectrum the three internal helicity charges $q^i = \oint j^i$, $j^i = \Psi^{2k-1} \Psi^{2k}$, $k = 1, 2, 3$ and $q^0$ are all simultaneously integers for space-time bosonic states and simultaneously half-integers for the fermionic states:

$$q^i = \text{half integers for space-time fermions}$$

$$q^i = \text{integers for space-time bosons}.$$ (3.1)

Furthermore all physical states have odd total $q^i$ charge (GSO-projection)

$$q^0 + q^1 + q^2 + q^3 = \text{odd integer}.$$ (3.2)
The last condition remains valid for supersymmetric solutions with less than four supersymmetries. In order to have a lower number of supersymmetries, the $q^i$'s must not be simultaneously integers or half-integers. It is then necessary to modify the world-sheet action $S^{2d}$, adding background fields that can change the individual values of the $q^i$'s, keeping however their total $q^i$ charge:

$$
\Delta S^{2d} = \int dz \bar{z} F_{IJ}^a (\Psi^I \Psi^J - \Phi^I \frac{\partial}{\partial z} \Phi^J) \bar{J}^a,
$$

(3.3)

where $\bar{J}^a$ denotes any dimension (0,1) operator. The part of the left-moving operator $(\Phi^I \frac{\partial}{\partial z} \Phi^J)$ is necessary to ensure the $N = (1,0)$ super-reparametrization of the 2-d action. From a higher-dimensional point of view, the $F_{IJ}^a$ denote non-trivial gauge or gravitational $(\mathcal{R}_{IJ}^{(KL)})$ field backgrounds. In four dimensions they give rise to non-vanishing auxiliary fields. The permitted values of $F_{IJ}^a$ $(\mathcal{R}_{IJ}^{(KL)})$ are not arbitrary. Only those for which

$$
U_L(F) = \exp \left[ \int dz F_{IJ}^a (\Psi^I \Psi^J - \Phi^I \frac{\partial}{\partial z} \Phi^J) \right]
$$

(3.4)

commutes with the 2-d super-current $(T_F = \Psi^\mu \partial \Phi^\mu + \Psi^I \partial \Phi^I)$ are allowed. This restriction generates a quantization of the permitted $F_{IJ}^a (\mathcal{R}_{IJ}^{(KL)})$ backgrounds.

A partial $N = 4 \to N = 2$ breaking is possible when $F_{3,4}^3 = -F_{5,6}^5 = H$ is not zero (self-duality condition). Indeed, in that case the $q^2$ and $q^3$ charges are shifted, preserving the total $q^i$ charge. In order to define the full deformation of the spectrum it is necessary to find a representation of the partition function in which the bosonic charges

$$
Q_2^B = \oint dz \Phi^3 \frac{\partial}{\partial z} \Phi^4 \text{ and } Q_3^B = \oint dz \Phi^5 \frac{\partial}{\partial z} \Phi^6
$$

(3.5)

are well defined. As a starting point we fermionize the four internal bosonic coordinates

$$
\partial \Phi^I = y^I w^I, \text{ and } \partial \bar{\Phi}^I = \bar{y}^I \bar{w}^I, \text{ } I = 3, 4, 5, 6.
$$

(3.6)

In this representation the 2d supercurrent is

$$
T_F = \Psi^\mu \partial \Phi^\mu + \sum_{I=1}^2 \Psi^I \partial \Phi^I + \sum_{I=3}^6 \Psi^I y^I w^I.
$$

(3.7)

We will now perform the following $Z_4$ transformation:

$$
\Psi^3 \to \Psi^4, \text{ } y^3 \to y^4, \text{ } \Psi^5 \to -\Psi^6, \text{ } y^5 \to -y^6,
$$

$$
\Psi^4 \to -\Psi^3, \text{ } y^4 \to -y^3, \text{ } \Psi^6 \to \Psi^5, \text{ } y^6 \to y^5,
$$

$$
w^3 \to w^4, \text{ } w^4 \to w^3, \text{ } w^5 \to w^6, \text{ } w^6 \to w^5
$$

(3.8)

which leaves (3.7) invariant. The above transformation corresponds to a $\pi/2$ rotation on the complex fermion basis:

$$
\chi_1 \to e^{2i\pi \phi} \chi_1 \text{ with } \chi_1 = \frac{\Psi^3 + i\Psi^4}{\sqrt{2}}
$$
$\chi_2 \rightarrow e^{-2i\pi\phi} \chi_2$ with $\chi_2 = \frac{\Psi^5 + i\Psi^6}{\sqrt{2}}$

$Y_1 \rightarrow e^{2i\pi\phi} Y_1$ with $Y_1 = \frac{y^3 + iy^4}{\sqrt{2}}$

$Y_2 \rightarrow e^{-2i\pi\phi} Y_2$ with $Y_2 = \frac{y^5 + iy^6}{\sqrt{2}}$

$W_- \rightarrow e^{4i\pi\phi} W_-$ with $W_- = \frac{(w^3 - w^4) + i(w^5 - w^6)}{2}$

$W_+ \rightarrow W_+$ with $W_+ = \frac{(w^3 + w^4) + i(w^5 + w^6)}{2}$

Similarly for the right–moving degrees of freedom ($\bar{\Psi}^I$, $\bar{y}^I$, $\bar{w}^I$, $I = 3, 4, 5, 6$). The above transformation is a symmetry only if the rotation angle is a multiple of $\pi/2$ or $\phi = k/4$, with $k$ integer.

Observe that with the help of the word sheet fermions we can classify the $N = 4$ string spectrum in terms of a left and right $U(1)$ charges $Q_L = j_L$ and $Q_R = j_R$, where

\[
\begin{align*}
j_L &= \chi_1 \chi_1^\dagger - \chi_2 \chi_2^\dagger + Y_1 Y_1^\dagger - Y_2 Y_2^\dagger + 2 W_- W_-^\dagger, \\
Q_L &= q_{\chi_1} - q_{\chi_2} + q_{Y_1} - q_{Y_2} + 2 q_{W_-} \\
\end{align*}
\]

and

\[
\begin{align*}
\bar{j}_R &= \bar{\chi}_1 \bar{\chi}_1^\dagger - \bar{\chi}_2 \bar{\chi}_2^\dagger + \bar{Y}_1 \bar{Y}_1^\dagger - \bar{Y}_2 \bar{Y}_2^\dagger + 2 \bar{W}_- \bar{W}_-^\dagger, \\
\bar{Q}_R &= q_{\bar{\chi}_1} - q_{\bar{\chi}_2} + \bar{q}_{Y_1} - \bar{q}_{Y_2} + 2 \bar{q}_{W_-} \\
\end{align*}
\]

We are now in a position to switch on non-vanishing $F_{ij}^a$ by performing a boost among the fermionic charge lattice and the $\Gamma_{(2,n)}$ lattice:

\[
\begin{align*}
q_{\chi_1} &\rightarrow q_{\chi_1} + h_i n^i, & q_{\chi_2} &\rightarrow q_{\chi_2} - h_i n^i \\
q_{\bar{\chi}_1} &\rightarrow q_{\bar{\chi}_1} + h_i n^i, & q_{\bar{\chi}_2} &\rightarrow q_{\bar{\chi}_2} - h_i n^i \\
q_{Y_1} &\rightarrow q_{Y_1} + h_i n^i, & q_{Y_2} &\rightarrow q_{Y_2} - h_i n^i \\
q_{\bar{Y}_1} &\rightarrow q_{\bar{Y}_1} + h_i n^i, & q_{\bar{Y}_2} &\rightarrow q_{\bar{Y}_2} - h_i n^i \\
q_{W_-} &\rightarrow q_{W_-} + 2h_i n^i, & q_{W_+} &\rightarrow q_{W_+} \\
q_{\bar{W}_-} &\rightarrow q_{\bar{W}_-} + 2h_i n^i, & q_{\bar{W}_+} &\rightarrow q_{\bar{W}_+} \\
\end{align*}
\]

\[
\begin{align*}
P_i^L(h_i) &= P_i^L - h_i (Q_L - Q_R) \\
P_i^R(h_i) &= P_i^R - h_i (Q_L - Q_R) \\
\end{align*}
\]

with

\[
P_i^L = m_i + Y_i^a Q^a + \frac{1}{2} Y_i^a Y_j^a n^i + B_{ij} n^j + G_{ij} n^j
\]
\[ P_i^R = m_i + Y_i^a Q^a + \frac{1}{2} Y_i^a Y_j^a n^j + B_{ij} n^j - G_{ij} n^j \]

\[ (m_{3/2})_{1,2} = 0, \quad (m_{3/2})_{3,4} = \frac{|F|^2}{4 \text{Im} T \text{Im} U}, \]

with \( F = h_1 + U \ h_2 \), \( T \) and \( U \) are usual complex moduli of the \( \Gamma_{(2,2)} \) lattice:

\[ T = i \sqrt{\text{det} G_{ij}} + B_{12}, \]

\[ U = \frac{(i \sqrt{\text{det} G_{ij}} + G_{12})}{G_{22}}, \]

The global existence of the supercurrent implies in this case the quantization condition: \( 4h_i = \text{integer} \). The \( N = 2 \) partition function \( Z^{4 \to 2}(F) \) is obtained from that of \( N = 4 \) by shifting the lattice momenta \( P_i \) and the \( R \)-charges \( q_i \) as above. Performing a Poisson resummation on \( m_i \), we obtain the following expression:

\[ Z^{4 \to 2}(F) = \frac{(\text{Im} \tau)^{-1}}{\eta^2 \bar{\eta}^2} \sum_{m_i,n_i} \frac{1}{2} \sum_{\alpha, \beta} (-)^{\alpha + \beta} \times \frac{\bar{\eta}^{2[\alpha]} \bar{\theta}^{[\alpha + \beta]} \bar{\theta}^{[\alpha - \beta]}}{\eta^{2} \bar{\eta}^{\beta}} \frac{\Gamma_{(2,2)[m_i]}^{(3,1)}}{\eta^4} Z_{(4,4)_{[\beta]}} \]

\[ \times \frac{1}{2} \sum_{\alpha, \beta} \frac{\bar{\eta}^{6[\alpha]} \bar{\theta}^{[\alpha + \beta]} \bar{\theta}^{[\alpha - \beta]}}{\eta^{2} \bar{\eta}^{\beta}} \frac{1}{2} \sum_{\epsilon, \zeta} \frac{\bar{\eta}^{8[\epsilon]} \bar{\theta}^{[\epsilon]}}{\eta^{2} \bar{\eta}^{\epsilon}}, \]

where

\[ \Gamma_{(2,2)[m_i]} = \sqrt{\text{det} G_{ij} (\text{Im} \tau)^{-1}} \times \exp \left[ -\pi G_{ij} (m^i + n^i \tau) (m^j + n^j \bar{\tau}) \right] + 2i \pi B_{ij} m^i n^j \]

and

\[ Z_{(4,4)_{[\beta]}} = \frac{1}{2} \sum_{a,b} \frac{|\bar{\eta}^{[a]}|}{|\eta|^2} \frac{|\bar{\eta}^{[a + 2\delta]}|}{|\eta|^2} \frac{|\bar{\eta}^{[a + \delta]}|}{|\eta|^2} \frac{|\bar{\eta}^{[a - \delta]}|}{|\eta|^2}. \]

When \( h_i = 0 \) (\( \gamma = 0, \delta = 0 \)), \( Z^{4 \to 2}(F = 0) \) corresponds to the \( N = 4 \) heterotic string solution based on a gauge group \( U(1) \times U(1) \times SO(8) \times E_8 \times E_8 \); the \( SO(8) \) gauge group factor corresponds to the extended symmetry of the \( \Gamma_{(4,4)} \) lattice at the fermionic point

\[ Z_{(4,4)_{[0]}} = \frac{1}{2} \sum_{a,b} \frac{|\bar{\eta}^{[a]}|^8}{|\eta|^8}. \]
The sum over \( m^i \) and \( n^i \) gives rise to the \( \Gamma(2,2) \) lattice at an arbitrary point of the moduli space:

\[
\sum_{m^i, n^i} \Gamma_{(2,2)}[n^i_{m^i}] = \Gamma_{(2,2)}[T, U].
\]  

(3.20)

When \( h_i \neq 0 \) (\( \gamma, \delta = (2h_i n^i, 2h_i m^i) \)), then the \( N = 4 \) supersymmetry is spontaneously broken to \( N = 2 \) and the gauge group is reduced to \( U(1)^2 \times E_7 \times E_8 \), as in orbifold models.

The important difference between the \( N = 2 \) model described above and the orbifold models \( \mathbb{N} \) of order \( \mathbb{N} \) is in the parameters \( \gamma \) and \( \delta \), which appear as arguments in \( \vartheta \)-functions. In the model in which some of the \( N = 4 \) the supersymmetries are broken spontaneously, \( \gamma = 2h_i n^i \) and \( \delta = 2h_i m^i \) are not independent but are given in terms of the \( h_i \) and in terms of the charges \( n^i, m^i \) of the \( \Gamma(2,2)[n^i_{m^i}] \) lattice. In the standard symmetric orbifolds of order \( \mathbb{N} \), the arguments \( \gamma \) and \( \delta \), (\( \gamma = 2l/\mathbb{N} \) and \( \delta = 2k/\mathbb{N} \) with \( l, k = 0,1,\ldots,\mathbb{N} - 1 \)), are independent arguments; their summation gives rise to the orbifold projections and to some additional states in the twisted sector:

\[
Z_{\text{orb}}^{N=2} = \left( \frac{\text{Im}\tau}{|\eta(\tau)|^4} \right) \frac{1}{\mathbb{N}} \sum_{\gamma, \delta = 0}^{1} \sum_{\alpha, \beta} \left(-\right)^{\alpha+\beta+\alpha\beta} \frac{\vartheta^2[\gamma\beta](\tau)}{\eta(\tau)} \frac{\vartheta^2[\delta\beta](\tau)}{\eta(\tau)} \frac{\Gamma_{(2,2)}}{|\eta(\tau)|^4} Z_{(4,4)[\beta]}.
\]  

(3.21)

In the language of orbifolds, the spontaneously broken theory, \( Z^{4\to2} \), corresponds to a freely acting orbifold. Indeed, using the quantization condition,

\[
\mathbb{N} h_i = \text{integer}
\]

and the mod 2 periodicity properties of the \( \vartheta \)-functions in the arguments

\[
\vartheta^{[a+2k]} = \vartheta^a | e^{i\pi a},
\]

(3.22)

(3.23)

it is possible to write the \( Z^{4\to2} \) theory in orbifold language. In order to make this correspondence explicit, we must first redefine the lattice charges \( n^i = \mathbb{N}\hat{n}^i + \gamma^i \) and \( m^i = \mathbb{N}\hat{m}^i + \delta^i \). Thanks to the property (3.23), the above lattice charge redefinition makes the arguments of the \( \vartheta \)-functions independent of \( \hat{n}^i \) and \( \hat{m}^i \); they depend only on \( \hat{\gamma} = 2h_i\gamma^i \) and \( \hat{\delta} = 2h_i\delta^i \). Performing now a Poisson resummation on \( \hat{n}^i \), we obtain the orbifold representation for \( Z^{4\to2} \) theory:

\[
Z^{4\to2}(F) = \left( \frac{\text{Im}\tau}{\eta(\tau)|\eta(\tau)|^4} \right) \sum_{\gamma, \delta = 0}^{1} \sum_{\alpha, \beta} \left(-\right)^{\alpha+\beta+\alpha\beta} \frac{\vartheta^2[\gamma\beta](\tau)}{\eta(\tau)} \frac{\vartheta^2[\delta\beta](\tau)}{\eta(\tau)} \frac{\Gamma_{(2,2)}}{|\eta(\tau)|^4} Z_{(4,4)[\beta]}.
\]

(3.24)
where
\[ \Gamma_{(2,2)} \left[ \gamma_i^j \right] = \sum \exp i\pi \left[ \frac{2\delta_i^j\hat{m}_i}{N} + \frac{1}{2} P^L_i \gamma^{ij} P^L_j - \frac{1}{2} P^R_i \gamma^{ij} P^R_j \right], \quad (3.25) \]
with
\[ P^L_i = \hat{m}_i + \left( \hat{n}_i + \frac{\gamma^j}{N} \right) G_{ij} \quad \text{and} \quad P^R_i = \hat{m}_i - \left( \hat{n}_i + \frac{\gamma^j}{N} \right) G_{ij}. \quad (3.26) \]
The connection of the Z^{4-2} with the freely acting orbifolds gives us the way to switch all the moduli of Z_{(4,4)} and so to move out of the SO(8) extended symmetry point. This extension can be done by replacing the SO(8) characters of Z^{SO(8)}_{(4,4)}[\gamma], which are defined at the fermionic point
\[ Z^{SO(8)}_{(4,4)}[\gamma] = \frac{1}{2} \sum_{a,b} \frac{\left| \vartheta_{[a]}^{[a]} \right|^2 \left| \vartheta_{[a+2]}^{[a+2]} \right|^2 \left| \vartheta_{[b+2]}^{[b+2]} \right|^2 \left| \vartheta_{[b-2]}^{[b-2]} \right|^2}{|\eta(\tau)|^2 \left| \eta(\tau) \right|^2 \left| \eta(\tau) \right|^2 \left| \eta(\tau) \right|^2}, \quad (3.27) \]
by the characters of the Z_N-orbifold, Z^{(twist)}_{(4,4)}[\gamma].

- When (\gamma, \delta) = (0, 0), Z^{SO(8)}_{(4,4)}[0] must be replaced by the “untwisted” orbifold partition function, which depends on the T_{IJ} moduli
\[ Z^{SO(8)}_{(4,4)}[0] \rightarrow Z^{(4,4)}[0] = T_{IJ} = \frac{\Gamma_{(4,4)}[T_{IJ}]}{|\eta(\tau)|^8}. \quad (3.28) \]

- When (\gamma, \delta) \neq (0, 0), Z^{SO(8)}_{(4,4)}[\gamma] no modification is necessary since the “twisted” orbifold partition function remains the same at any point of the moduli space:
\[ Z^{SO(8)}_{(4,4)}[\gamma] \rightarrow Z^{(twist)}_{(4,4)}[\gamma] = Z^{SO(8)}_{(4,4)}[\gamma] \quad (\gamma, \delta) \neq (0, 0). \quad (3.29) \]
The models described above are special cases of a general class of models having the interpretation of freely acting orbifolds of the N = 4 heterotic string theory. They are obtained in the following way. Consider \( \Gamma(6,22) \) and set the appropriate moduli to special values, so that it factorizes as
\[ \Gamma_{(6,22)} \rightarrow \Gamma_{(2,18)} \Gamma_{(4,4)} \quad (3.30) \]
Now consider the orbifold that acts as a Z_N rotation on \( \Gamma_{(4,4)} \) and as a translation by an N-th lattice vector \( \vec{v}/N \) with \( \vec{v} = (\vec{e}_L; \vec{e}_R; \vec{\zeta}) \), on \( \Gamma_{(2,18)} \). \( \vec{e}_{L,R} \) are two-dimensional vectors while \( \vec{\zeta} \) is a sixteen dimensional vector. Owing to the accompanying translation on \( \Gamma_{(2,18)} \), this is a freely acting orbifold.

The two types of constructions of N = 4 → N = 2 theories we have presented above, have complementary features. In the first approach of using a specific generalized boost at the fermionic point, it is evident that there is a one-to-one correspondence of states between the original N = 4 supersymmetric theory and the final spontaneously broken N = 4 → N = 2 theory. This is what should be expected during spontaneous symmetry breaking. In the second, freely acting orbifold approach, we have a clear geometrical intuition about the spontaneously broken theory, which will be very useful for the identification of the heterotic and type II dual theories.

Inspection of the standard N = 4 gravitino vertex operators shows that two of them are invariant while the other two transform, one with a phase \( e^{2\pi i/N} \) and the other with \( e^{-2\pi i/N} \). In
order for them to survive in the spectrum they have to pair up with a state of the (2, 18) lattice carrying momentum \( p = (\vec{m}; \vec{n}, \vec{Q}) \) but no oscillators (these will shift the mass to the Planck scale). Since such a lattice state picks up a phase \( e^{2\pi i p/|N|} \), one of the two massive gravitinos will have momentum \( p_1 \) with the property that \( p_1 \cdot \epsilon = 1 \mod N \) while the other \( p_2 \) with \( p_2 \cdot \epsilon = -1 \mod N \). The mass formulae given in (3.14) are special cases of the above.

There is an essential difference between the models with spontaneous breaking of the \( N = 4 \to N = 2 \) and the standard \( N = 2 \) orbifold models.

- First, in the spontaneously broken case, one expects an effective restoration of the \( N = 4 \) supersymmetry in a corner of the moduli space \( T, U \), where the two massive gravitinos become light, \( m_{3/2} \to 0 \).
- Second, in the standard obifolds there is no restoration of the \( N = 4 \) supersymmetry at any point of the moduli space.

If there is an effective restoration of the \( N = 4 \) supersymmetry in the spontaneously broken case, then one must find zero higher-genus corrections to the coupling constants of the theory in the \( N = 4 \) restoration limit \( m_{3/2} \to 0 \). This restoration phenomenon has been checked in ref. [17, 18] where the one-loop corrections of the coupling constants were performed for a class of \( Z_2 \) models based on \( E_8 \times E_7 \times SU(2) \times U(1)^2 \) gauge group. A more detailed discussion of the general heterotic models and their type II duals will appear in ref. [19]. Here I will restrict myself to the case of \( Z_2 \) freely acting orbifolds with \( F = h_1 + U h_2 = 1/2 \). The \( m_{3/2} \to 0 \) limit in this class of models corresponds to the corner of the moduli space \( \text{Im} T \text{Im} U \to \infty \), which implies an effective decompactification of one of the two coordinates of \( \Gamma_{2,2}(T, U) \), \( (R_1 \to \infty \) and \( R_2 \) arbitrary; \( \text{Im} T \sim R_1 R_2 \), \( \text{Im} U \sim R_1/R_2 \)). In this limit, \( T, U \to \infty \), one expects vanishing corrections to the coupling constants due to the effective \( N = 4 \) restoration. Using the explicit results of ref. [17],

\[
\Delta_{(8,7)}^{\text{free yield}} = \frac{16\pi^2}{g_{E_8}^2} - \frac{16\pi^2}{g_{E_7}^2} = \delta b \log \left[ |\mu|^2 \text{Im} \, T\text{Im} \, U |\vartheta_4(T) \vartheta_4(U)|^4 \right],
\]

where \( \delta b = b_8 - b_7 \) and \( b_i \) are the \( \beta \)-function coefficients due to massless particles. When \( T \) and \( U \) are large, \( \text{Im} T \text{Im} U \gg 1 \), due to the asymptotic behaviour of \( \vartheta_4(T) = 1 + O(e^{-\pi T}) \):

\[
\Delta_{(8,7)}^{\text{free yield}} \to \delta b \log |\mu|^2 \text{Im} T \text{Im} U. \tag{3.32}
\]

The logarithmic contribution is an artefact due to the infrared divergences. In fact by turning on Wilson lines appropriately (e.g. small Higgs vev’s of the vector multiplets), we can arrange that there are no charged states with masses \( \mu_W^2 \sim |W|^2/\text{Im} T \text{Im} U \) below \( m_{3/2}^2 \). In this case the logarithmic term becomes:

\[
\delta b \log |\mu|^2 \text{Im} T \text{Im} U \to \delta b \log \frac{\mu_W^2}{m_{3/2}^2 + \mu_W^2} \sim O\left( \frac{m_{3/2}^2}{\mu_W^2} \right); \tag{3.33}
\]

the logarithmic divergence thus disappears and the thresholds vanish, which shows the restoration of the \( N = 4 \) supersymmetry in the light massive gravitino limit as expected. In the calculation of individual couplings, there is an extra contribution \( Y(T, U) \), which is “universal” for \( g_{E_8} \) and \( g_{E_7} \); the explicit calculation in ref. [17]–[19] shows that \( Y(T, U) \) behaves like

\[
Y(T, U) \to \frac{m_{3/2}^2}{M_s^2} \quad \text{as} \quad m_{3/2} \to 0. \tag{3.34}
\]
Thus individual couplings also vanish in the limit $m_{3/2} \to 0$.

In the standard orbifold with $N = 2$ space-time supersymmetry, the corrections to the coupling constants have a different behaviour for $T, U \gg 1$ [20, 18]:

$$\Delta_{orb}^{(8,7)} = \delta b \log \left[ \mu^2 \text{Im} T \text{Im} U |\eta(T)\eta(U)|^4 \right]. \quad (3.35)$$

When $T, U$ is large, $\text{Im} T \text{Im} U \gg 1$,

$$\Delta_{orb}^{(8,7)} \to \delta b \left[ \frac{\pi}{3} (\text{Im} T + \text{Im} U) + \log \frac{|W'|^2}{M_s^2} \right] + \text{finite terms}. \quad (3.36)$$

So, in the standard orbifolds, the correction to the coupling constants grows linearly with the five-dimensional volume. This shows that the $N = 2$ supersymmetry is “not extended” in the decompactification limit $R_1 \to \infty$. On the other hand there is an extension of the supersymmetry in the freely acting orbifold case.

In the opposite limit $\text{Im} T \text{Im} U \to 0$, the situation is different:

i) In the freely acting orbifolds the two massive gravitinos becomes superheavy: $m_{3/2} \to \infty$ in the limit $\text{Im} T \text{Im} U \to 0$.

ii) In the standard orbifolds, thanks to the duality symmetry $R_i \to 1/R_i$ the behaviour $T, U \to 0$ is identical to the dual model with $T' = -1/T, U' = -1/U \to \infty$ and thus

$$\Delta_{orb}^{(8,7)}(T, U, W) = \Delta_{orb}^{(8,7)}(T', U', W') \to \delta b' \left[ \frac{\pi}{3} (\text{Im} T' + \text{Im} U') + \log \frac{|W'|^2}{M_s^2} \right] + \text{finite terms} \quad (3.37)$$

In the freely acting orbifolds, the $SO(2,2; \mathbb{Z})$ duality symmetry is reduced to a smaller subgroup due to the $Z_2$ action on the lattice. Thus one expects non-restoration of the $N = 4$ supersymmetry in this limit ($T, U \to 0 m_{3/2} \to \infty$):

$$\Delta_{freely}^{(8,7)} = \delta b' \left[ \log |\vartheta_2(T')|^{\vartheta_2(T')/4} + \log \frac{|W'|^2}{M_s^2} \right]. \quad (3.38)$$

In the above equation we have used the $\vartheta$-identity

$$\text{Im} T |\vartheta_4(T)|^4 = \text{Im} T' |\vartheta_2(T')|^{4}, \quad T' = -\frac{1}{T}. \quad (3.39)$$

Using the asymptotic behaviour for $T', U' \gg 1$ of $\log |\vartheta_2(T')|^{\vartheta_2(T')/4}$ one obtains:

$$\Delta_{freely}^{(8,7)} \to \delta b' \left[ \frac{\pi}{3} (\text{Im} T' + \text{Im} U') + \log \frac{|W'|^2}{M_s^2} \right] + \text{finite terms}. \quad (3.40)$$

It is interesting to observe that the $m_{3/2} \to \infty$ limit [17] of the freely acting orbifolds corresponds to a corner in the moduli space of $T, U$ where the two classes of theories (the freely and non-freely acting orbifolds) “touch” each other. Both theories are effectively five-dimensional. Thus the five-dimensional standard $N = 2$ orbifolds can be viewed as an $m_{3/2} \to \infty$ limit of some spontaneously broken $N = 4$ models.
4 \quad \text{N = 4 \rightarrow N = 1 spontaneous SUSY breaking}

Using the connection between the freely acting orbifolds and the spontaneous breaking \( N = 4 \rightarrow N = 2 \), we can proceed to further break the supersymmetry to \( N = 1 \). We will restrict ourselves to the case where the possible quantized parameters are of order \( N=2 \). In that case the spontaneously broken \( N = 4 \rightarrow N = 1 \) theory is strongly connected to \( Z^2 \times Z^2 \) freely acting orbifolds; the \( Z^2 \times Z^2 \) acts simultaneously as a rotation on the coordinates \( \Phi^I, \Phi^J \) and \( \Psi^I, \Psi^J \) of the two complex planes and as a translation on the third complex plane \( \Phi^L \).

Denoting by \( \Phi_A \), \( A = 1, 2, 3 \), the complex internal coordinates and by \( \chi_A \), \( A = 1, 2, 3 \), the three complex fermionic world-sheet superpartners, the non-trivial actions of the orbifold are:

1) \( \Phi_1 \rightarrow \Phi_1 + 2\pi h_1, \quad (\Phi_2, \chi_2) \rightarrow e^{i2\pi h_1}(\Phi_2, \chi_2), \quad (\Phi_3, \chi_3) \rightarrow e^{-i2\pi h_1}(\Phi_3, \chi_3). \)

2) \( \Phi_2 \rightarrow \Phi_2 + 2\pi h_2, \quad (\Phi_1, \chi_1) \rightarrow e^{i2\pi h_2}(\Phi_1, \chi_1), \quad (\Phi_3, \chi_3) \rightarrow e^{-i2\pi h_2}(\Phi_3, \chi_3). \)

3) \( \Phi_3 \rightarrow \Phi_3 + 2\pi h_3, \quad (\Phi_1, \chi_1) \rightarrow e^{i2\pi h_3}(\Phi_1, \chi_1), \quad (\Phi_2, \chi_2) \rightarrow e^{-i2\pi h_3}(\Phi_2, \chi_2). \)

In order to obtain the partition function and define the theory, we need to introduce the “shifted” and “twisted” characters of the three complex coordinates. We denote by \( \gamma_A \) the translation shifts and by \( H_A, G_A \) the rotation twists. When the “twist” is zero \( (H_A, G_A) = (0, 0) \):

\[
Z_{A, \gamma_A : \delta_A ; 0} = \frac{\Gamma_{(2,2)}[\gamma_A]}{|\eta|^4} \delta(H_A)\delta(G_A). \tag{4.1}
\]

When the twist is non-zero \( (H_A, G_A) \neq (0, 0) \):

\[
Z_{A, \gamma_A : \delta_A ; G_A} = \frac{1}{2} Z_{\text{twist}, (2,2)}^{H_A} [G_A] \times [\delta(\gamma_A) \delta(\delta_A) + \delta(\gamma_A + H_A) \delta(\delta_A + G_A)]. \tag{4.2}
\]

In the above equation, the \( Z_{\text{twist}, (2,2)}^{H_A} [G_A] \) can be written either in terms of a twisted boson or in terms of 2d-fermionic characters with shifted boundary conditions:

\[
Z_{\text{twist}, (2,2)}^{H_A} [G_A] = \left. \frac{4|\eta|^2}{\psi(1 + H_A, G_A)\psi(1 - H_A, G_A)} \right| = \frac{1}{2} \sum_{a,b} \frac{|\vartheta^2(\chi^A - H_A)\vartheta((\chi^A + H_A, \delta_A - G_A))|}{|\eta|^4} \quad \text{if} \quad (H_A, G_A) \neq (0,0). \tag{4.3}
\]

The world-sheet modular properties of \( Z_{A, \gamma_A : \delta_A ; G_A} \) are the same for any point of the moduli space and thus at the \( SO(4)_A \) fermionic point, which takes the following form:

\[
Z_{A, \gamma_A ; \delta_A ; G_A} |_{T_0^A, U_0^A} = \frac{1}{2} \sum_{a,b} \frac{|\vartheta^2(\chi^A - H_A)\vartheta((\chi^A + H_A, \delta_A - G_A))|}{|\eta|^4} e^{i\pi(a\delta_A + b\gamma_A + \gamma_A\delta_A)}. \tag{4.4}
\]

The above expression makes the world-sheet modular properties of \( Z_{A, \gamma_A ; \delta_A ; G_A} \) more transparent under \( SL(2, Z) \); it also makes clear the connection to the fermionic models. The role of the phase factor \( e^{i\pi(a\delta_A + b\gamma_A + \gamma_A\delta_A)} \) is of main importance, since it clarifies the way we had to choose the coefficient of the fermionic characters.

We are now in a position to construct consistent \( N = 4 \rightarrow N = 1 \) models using the fermionic construction algorithm \[4]. Although these constructions are at special points of the
moduli space \((T^A_0, U^A_0)\), the generalization of them for arbitrary moduli is automatic by a simple replacement of the fermionic “twisted” characters with the characters of the ”shifted” and ”twisted” bosonic coordinates:

\[
Z_A \left[ \gamma_A ; H_A \right] |_{T^A_0, U^A_0} \rightarrow Z_A \left[ \gamma_A ; H_A \right] |_{T, U}.
\]

Many models can be constructed in this way. We will display below the partition function of a model with one unbroken and three spontaneously broken supersymmetries, \(N = 4 \rightarrow N = 1\) (the unbroken gauge group of this example is \(E_8 \times E_6 \times U(1)^2\)).

\[
Z^{4\rightarrow1}(F_i) = \frac{(\text{Im} \tau)^{-1}}{\eta^2 \bar{\eta}^2} \times \frac{1}{4} \sum_{h_1, g_1} Z_1 \left[ h_1 ; h_2, g_1 ; g_2 \right] Z_2 \left[ h_2 ; h_3, g_2 ; g_3 \right] Z_3 \left[ h_3 ; h_1, g_3 ; g_1 \right] \\
\times \left[ \frac{\bar{\eta}^{g_1}}{8} \right] \left[ \frac{\bar{\eta}^{g_2}}{8} \right] \left[ \frac{\bar{\eta}^{g_3}}{8} \right] \delta(h_1 + h_2 + h_3) \delta(g_1 + g_2 + g_3) \frac{1}{2} \sum_{\alpha, \beta} \frac{\bar{\eta}^{\alpha + \beta}}{\eta^{\alpha + \beta}}.
\]

The existence of one unbroken supersymmetry is ensured because of the relations \(h_1 + h_2 + h_3 = 0\) and \(g_1 + g_2 + g_3 = 0\); these relations guarantee the existence of an \(N = 2\) superconformal symmetry on the world-sheet and thus the existence of \(N = 1\) space-time supersymmetry [21].

It is easy to see that the partition function \(Z^{4\rightarrow1}\) can be decomposed into four sectors:

- **The \(N = 4\) sector**, with no rotations or translations in all three complex planes \(((h_A, g_A) = (0, 0))\)
- **Three \(N = 2\) sectors**, with a non-zero translation in one of the complex planes and opposite non-zero translations in the remaining two complex planes.

The contribution to the partition function of the \(N = 4\) sector is one quarter of the \(N = 4\) partition function with lattice momenta in the reduced \(\Gamma(2, 2)^3\) lattice. The contribution of the other three \(N = 2\) sectors are equal sector by sector to the corresponding \(N = 4 \rightarrow N = 2\) partition function divided by a factor of 2. The untwisted complex plane lattice momenta correspond to the shifted \(\Gamma_{(2, 2)} \tilde{\gamma}^A_A\) lattice. The moduli-dependent corrections to the gauge couplings can be easily determined by combining the results of the individual \(N = 2\) sectors.

\[
\frac{16\pi^2}{g^2_{E_8}} - \frac{16\pi^2}{g^2_{E_6}} = \Delta(8, 6) = \frac{1}{2} \sum_{A=1}^{3} \Delta^A(8, 7).
\]

where the expressions of the \(\Delta^A(8, 7)\) are given in (4.31).

As we mentioned in the \(N = 4 \rightarrow N = 2\) spontaneous breaking, one expects a restoration of the \(N = 4\) supersymmetry in the limit in which the massive gravitinos become massless; in order to prove the \(N = 4\) restoration in the \(N = 4 \rightarrow N = 1\) defined above as a \(Z^2 \times Z^2\) freely acting orbifold, we need to identify the three massive gravitinos and express their masses in terms of the moduli fields and the three \(R\)-symmetry charges \(q_i\) \((i = 1, 2, 3)\):

\[
m^2_{3/2}(q_i) = \frac{|q_2 - q_3|^2}{4 \text{ Im } T_1 \text{ Im } U_1} + \frac{|q_3 - q_1|^2}{4 \text{ Im } T_2 \text{ Im } U_2} + \frac{|q_1 - q_2|^2}{4 \text{ Im } T_3 \text{ Im } U_3}.
\]
with \(|q_0 + q_1 + q_2 + q_3| = 1\) and \(|q_i| = |q_0| = \frac{1}{2}\) where \(q_0\) is the left-helicity charge. Using the above expression, one finds the desired result:

\[
\begin{align*}
(m_{3/2})_1 &= \frac{1}{4 \, \text{Im} \, T_2 \, \text{Im} \, U_2} + \frac{1}{4 \, \text{Im} \, T_3 \, \text{Im} \, U_3}, \\
(m_{3/2})_2 &= \frac{1}{4 \, \text{Im} \, T_3 \, \text{Im} \, U_3} + \frac{1}{4 \, \text{Im} \, T_1 \, \text{Im} \, U_1}, \\
(m_{3/2})_3 &= \frac{1}{4 \, \text{Im} \, T_1 \, \text{Im} \, U_1} + \frac{1}{4 \, \text{Im} \, T_2 \, \text{Im} \, U_2}, \\
(m_{3/2})_0 &= 0. 
\end{align*}
\]

The three massive gravitinos become massless in the decompactification limit \(\text{Im} \, T_I \, \text{Im} \, U_I \to \infty, \ I = 1, 2, 3\), with ratios \(\text{Im} \, T_I / \text{Im} \, U_I\) fixed. Thus the full restoration of the \(N = 4\) supersymmetry effectively takes place in seven dimensions. Partial restoration of an \(N = 2\) supersymmetry can happen in six dimensions when \(\text{Im} T_I \, \text{Im} U_I \to \infty, \ I = 1, 2\); in this limit \((m_{3/2})_0 = 0\) and \((m_{3/2})_3 \to 0\).

\section{N = 2 \rightarrow N = 1 spontaneous SUSY breaking}

Using similar techniques as before, it is possible to construct \(N = 2\) models with one of the supersymmetries to be spontaneously broken, \(N = 2 \rightarrow N = 1\). In this class of models the restoration of \(N = 2\) takes place in six dimensions. No further restoration of supersymmetry is possible. Examples can be obtained as in \((T^2 \otimes K_3)/Z_f^3\) orbifold compactification in which the \(Z_f^3\) is freely acting. A representative example of this class of models is the one in which the \(T_{3/2}\) compactification is chosen to be at the orbifold point \(T^4/Z_f^2 \sim K_3\) (we denote by \(Z_f^3\) the orbifold group and by \(Z_f^3\) that which corresponds to the freely acting orbifold). We will give below the exact partition function that corresponds to this construction. From the explicit expression we can directly verify the effective restoration of \(N = 2\) supersymmetry in the large-volume limit of \(K_3\). Using the \(Z_f^3 \otimes Z_f^3\) orbifold notation, the partition function of the \((T^2 \otimes T^4/Z_f^2)/Z_f^3\) model is:

\[
\begin{align*}
Z^{2 \rightarrow 1}(F_i) &= \frac{(\text{Im} \tau)^{-1}}{\eta^2 \bar{\eta}^2} \times \frac{1}{2} \sum_{h_f, g_f} \frac{1}{2} \sum_{h_o, g_o} Z_1 [0; h_f, 0; g_f] Z_2 [h_f; h_o] Z_3 [h_f; -h_f - h_o] \\
&\quad \times \frac{1}{2} \sum_{\alpha, \beta} (-)^{\alpha + \beta + \alpha \beta} \frac{\theta_{[\alpha]} \theta_{[\alpha]} \theta_{[\beta + g_f]} \theta_{[\beta + h_o]} \theta_{[\beta - h_f - h_o]} \theta_{[\beta - h_f - h_o]}}{\eta \bar{\eta} \eta \bar{\eta} \eta \bar{\eta}} \\
&\quad \times \frac{1}{2} \sum_{\alpha, \beta} \frac{\tilde{\theta}_{[\bar{\alpha}]} \tilde{\theta}_{[\bar{\alpha}]} \tilde{\theta}_{[\bar{\beta} + g_f]} \tilde{\theta}_{[\bar{\beta} + h_o]} \tilde{\theta}_{[\bar{\beta} - h_f - h_o]} (\bar{\tau})}{\bar{\eta} \eta \bar{\eta} \eta \bar{\eta}} \times \frac{1}{2} \sum_{\epsilon, \zeta} \frac{1}{\bar{\eta}^8}. 
\end{align*}
\]

In the above expression, the parameters \((h_f, g_f)\) and \((h_o, g_o)\) correspond to \(Z_f^2\) and \(Z_f^2\) respectively. The unbroken gauge group of this model is the \(E_8 \otimes E_6 \otimes U(1)^2\). Switching on continuous or discrete Wilson lines, we can construct a large class of models with different gauge group but with a universal behaviour with respect to the \(N = 2\) restoration at the large moduli limit; the
massive gravitino of the broken $N = 2$ becomes massless when $(\text{Im } T_2 \text{ Im } U_2$ and $\text{Im } T_3 \text{ Im } U_3$ large).

$$\left( m_{3/2}^2 \right)_1 = \frac{1}{4 \text{ Im } T_2 \text{ Im } U_2} + \frac{1}{4 \text{ Im } T_3 \text{ Im } U_3}, (m_{3/2}^2)_0 = 0. \quad (5.2)$$

An easy way to view this ground-state is as an orbifold of the original $N = 4$ theory by the following non-trivial $Z_2 \times Z_2$ elements: $(1, r, r)$, $(r, r, t)$, $(r, t, r)$, $(r$ stands for “$\pi$-rotation” and $t$ for $1/2$–lattice translation); $(1, r, r)$ has four fixed planes while the others have none. Because of the $N = 2$ restoration phenomenon, we expect that the only non-vanishing corrections to the gauge coupling constants are those that correspond to the $N = 2$ sector with $(h_o, g_o) = (0, 0)$ and $(h_o, g_o) \neq (0, 0)$. Indeed in this sector the $Z^2_o \times Z^2_o$ acts trivially on the $\Gamma_{(2,2)}(T_1, U_1)$ lattice as in the usual orbifolds. On the other hand, in the remaining two $N = 2$ sectors,

1. $(h_o, g_o) = (0, 0)$, $(h_o, g_o) \neq (0, 0)$
2. $(h_o, g_o) + (h_f, g_f) = (0, 0)$, $(h_o, g_o) \neq (0, 0)$.

In both sectors the corresponding $Z^2$ acts without fixed points because of the simultaneous non-trivial shift $(h_f, g_f)$ on the corresponding $\Gamma_{(2,2)}(T_A, U_A)$, $A = 2, 3$, lattice.

The moduli-dependent corrections to the gauge couplings can be easily determined by combining the results of the individual $N = 2$ sectors.

$$\Delta_{(8,6)} = \frac{16\pi^2}{g^2_{E_8}} - \frac{16\pi^2}{g^2_{E_6}} = \frac{1}{2} \left( \Delta_{(8,7)}^1 + \Delta_{(8,7)}^2 + \Delta_{(8,7)}^3 \right), \quad (5.3)$$

where the $\Delta_{(8,7)}^A$ are the threshold corrections of the three $N = 2$ sectors:

$$\Delta_{(8,7)}^1 = (b_8^1 - b_7^1) \log \left| \mu^2 \text{Im } T_1 \text{Im } U_1 | \eta(T_1) \eta(U_1) \right|^4$$

$$\rightarrow (b_8^1 - b_7^1) \left[ \frac{\pi}{3} (\text{Im } T_1 + \text{Im } U_1) + \log |\mu|^2 \text{Im } T_1 \text{Im } U_1 \right] \quad (5.4)$$

which corresponds to the threshold corrections of the standard orbifolds.

On the other hand $\Delta_{(8,7)}^A$ for $A = 2, 3$ will correspond to the threshold corrections of freely acting orbifolds which have different behaviour in the large-moduli limit:

$$\Delta_{(8,7)}^A = (b_8^A - b_7^A) \log \left| \mu^2 \text{Im } T_A \text{Im } U_A \right| + (b_8^A - b_7^A) \log \left| \vartheta_A(T_A) \vartheta_A(U_A) \right|^4$$

$$\rightarrow (b_8 - b_7) \log |\mu|^2 \text{Im } T_A \text{Im } U_A. \quad (5.5)$$

Modulo the artificial sub-leading logarithmic contribution (due to the infrared divergences), the moduli contribution of the second and third plane $T_A$, $U_A$, $A = 2, 3$, is exponentially suppressed due to the asymptotic behaviour of $\vartheta_A(T_A)$, $\vartheta_A(U_A)$ for large $T_A$ and $U_A$, $\vartheta_A(T_A) = 1 + \mathcal{O}(e^{-i\pi T_A})$.

There is a large class of such models obtained from $N = 2$ $Z_2$ orbifold compactifications by using $D_4$ type symmetries that act on the twist fields as well as the lattice.
6 Non-Perturbative BPS $N = 4$ Mass Formula

The low-energy effective $N = 4$ supergravity is manifestly invariant under the $SO(6, 22; R)$ group. The full string theory, however, is only invariant under the discrete subgroup $O(6, 22, Z)$. Furthermore the equations of motion and Bianchi identities are invariant under the $SL(2, R)$ transformation

$$S \rightarrow \frac{aS + b}{cS + d} \quad \text{with} \quad ad - bc = 1, \quad (6.1)$$

provided we perform an $SL(2, R)$ transformation on the “electric” and “magnetic” fields and charges. In particular, the transformation $S \rightarrow -1/S$ interchanges electric and magnetic charges. It has been conjectured [22]–[24] that a discrete subgroup $SL(2, Z)$ of this continuous symmetry of the equations of motion of the effective theory is a (non-perturbative) symmetry of the full theory. In the heterotic theory, only electrically charged states exist; these charges are the six quantized momenta and windings as well as the 16 $U(1)$ charges of $E_8 \times E_8$ or $SO(32)$, $(m_I, n^I, Q^k)$. Obviously the spectrum of the heterotic theory is not invariant under $SL(2, Z)$. For this to be true it is necessary to include in the theory non-perturbative states that carry both electric ($m_I, n^I, Q^k$) and magnetic ($\tilde{m}_I, \tilde{n}^I, \tilde{Q}^k$) charges [23], [25]–[28]. Thanks to the $N = 4$ supersymmetric algebra and its central extension, one can write down an exact mass formula for all stable perturbative and non-perturbative states, which preserves at least one of the four supersymmetries (BPS-states).

$$M^2_{BPS} = \frac{(P_I + S \Pi_I) g^{IJ} (P_J + S \Pi_J) - 1}{4\text{Im}S}$$

where the “electric” and the “magnetic” momenta $P_I$ and $\Pi_I$ are given in terms of the “electric” $(m_I, n^I, Q^k)$ and “magnetic” $(\tilde{m}_I, \tilde{n}^I, \tilde{Q}^k)$ charges:

$$P_I = m_I + Y^k_I Q^k + \frac{1}{2} Y^k_I Y^j_I n^J + B_{IJ} n^J$$

$$\Pi_I = \tilde{m}_I + Y^k_I \tilde{Q}^k + \frac{1}{2} Y^k_I Y^j_I \tilde{n}^J + B_{IJ} \tilde{n}^J$$

(6.3)

The square-root factor in the BPS mass formula is proportional to the difference of the two central charges squared: depending on whether this vanishes or not, the representation preserves 1/2 or 1/4 of the supersymmetries, (either short or intermediate supermultiplets). For the perturbative BPS states of the heterotic string $(\tilde{m}_I, \tilde{n}^I, \tilde{Q}^k) = 0$, and thus belong to short supermultiplets. Their mass reads

$$M^2_{BPS, pert} = \frac{1}{4\text{Im}S} P_I g^{IJ} P_J.$$

(6.4)

The factor of $\text{Im}S$ is there because masses are measured in units of $M_{\text{Planck}}$. The BPS mass formula is manifestly invariant under $SL(2, Z)_S$.

$$S \rightarrow S + 1; \quad P_I \rightarrow P_I + \Pi_I, \quad \Pi_I \rightarrow \Pi_I$$
\[ S \rightarrow -\frac{1}{S}; \quad \Pi_I \rightarrow P_I, \quad P_I \rightarrow -\Pi_I. \quad (6.5) \]

Although the mass formula for non-perturbative BPS states is understood, we do not know a priori the multiplicities of all these states. From the \( N = 4 \) heterotic string we know the multiplicities when \( \Pi_I = 0 \). Using \( SL(2, \mathbb{Z}) \) we also know the multiplicities of all states with \( P_I \, g^{IJ} \Pi_J = 0 \). To go further and learn more about the states with \( P_I \, g^{IJ} \Pi_J \neq 0 \) (namely intermediate multiplets) it is necessary to go beyond the string picture and learn more about the non-perturbative structure of the theory. The heterotic string on \( T^6 \) is supposed to be equivalent, in the strong coupling limit to the type II theory compactified on \( K^3 \times T^2 \). Moreover, there is a hypothetical 11-d theory (\( M \)-theory) that includes the non-perturbative dynamics of type IIA theory [15]. Thus compactification of \( M \)-theory on \( K^3 \times T^3 \) contains all the relevant non-perturbative information about the heterotic \( N = 4 \) theory. This idea led to a conjecture on the multiplicities of dyonic BPS states in the 4-d \( N = 4 \) theory [31]. This will be an important input, for our non-perturbative analysis of the spontaneously broken \( N = 4 \) theory.

7 BPS states in models with partial SUSY breaking \( N = 4 \rightarrow N = 2 \)

Let us consider an interesting question concerning the BPS spectrum of the theories where \( N = 4 \) is spontaneously broken to \( N = 2 \). In the original heterotic \( N = 4 \) theory, there are only short BPS multiplets in the perturbative spectrum. Their multiplicities can be easily counted by using helicity supertrace formulae [30]. In particular, the supertrace of helicity to the power 4 counts the multiplicities of \( N = 4 \) short (massless or massive) multiplets. From the partition function of the heterotic \( N = 4 \), we can construct the helicity-generating partition function:

\[
Z_{\text{het}}^{N=4}(v, \bar{v}) = \text{Str}[q^{L_0} \bar{q}^{L_0} e^{2\pi i v \lambda_R - 2\pi i \lambda_L}] = \frac{1}{2} \sum_{\alpha, \beta} (-1)^{\alpha + \beta + \alpha \beta} \frac{\hat{\partial}^{[\alpha]}(v) \hat{\partial}^{[\beta]}(\bar{v})}{\eta^{12} \bar{\eta}^{24}} \xi(v) \bar{\xi}(\bar{v}) \frac{\Gamma(6,22)}{\text{Im} \tau}. \quad (7.1)
\]

The physical helicity in closed string theory \( \lambda \) is the sum of the left helicity \( \lambda_L \) and the right helicity \( \lambda_R \):

\[
\xi(v) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^n e^{2\pi i v})(1 - q^n e^{-2\pi i v})} = \frac{\sin \pi v}{\pi} \frac{\vartheta_1'(v)}{\vartheta_1(v)} \quad (\xi(v) = \xi(-v)), \quad (7.2)
\]

which counts the contributions to the helicity due to the world-sheet bosons. If we define

\[
Q = \frac{1}{2\pi i} \frac{\partial}{\partial v}, \quad \bar{Q} = -\frac{1}{2\pi i} \frac{\partial}{\partial \bar{v}}, \quad (7.3)
\]

then

\[
B_4 = \langle \lambda^4 \rangle = (Q + \bar{Q})^4 Z_{N=4}^{\text{het}}(v, \bar{v})|_{v=\bar{v}=0} = \frac{3 \Gamma_{6,22}}{2 \bar{\eta}^{24}}.
\]

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The numerator provides the mass formula while the denominator $1/\eta^{24}$ provides the multiplicities. More precisely defined:

$$\frac{1}{\eta^{24}} = \sum_{N=-1}^{\infty} d(N)q^N = \frac{1}{q} + 24 + 324q + O(q^2). \quad (7.5)$$

Then at the mass levels $M^2 = \frac{1}{4}P_L^2$, with

$$q_e^2 \equiv 2\vec{m} \cdot \vec{n} - \vec{Q} \cdot \vec{Q}, \quad (7.6)$$

the multiplicity is $d(q_e^2/2)$. The generalization of the string “electric” multiplicity to the non-perturbative dyonic states needs to assume a genus-2 generating function $\Phi(\tau_{ij}) = \Phi(\tau^{ij})$:

$$\Phi(\tau_{ij}) = \sum_{N_{ij}} d(N_{ij}) \exp \left[ 2i\pi N_{ij} \tau^{ij} \right], \quad (7.7)$$

where the levels $N_{ij}$ of a dyonic state are characterized by the electric and magnetic charges

$$\vec{q}_1 \equiv \vec{q}_e = (\vec{m}, \vec{n}, \vec{Q}), \quad \vec{q}_2 \equiv \vec{q}_m = (\vec{\tilde{m}}, \vec{\tilde{n}}, \vec{\tilde{Q}})$$

$$2N_{ij} = \vec{q}_i \cdot \vec{q}_i ; \quad (7.8)$$

the above equation generalizes the “electric” matching condition for the dyons and magnetic monopoles. The non-perturbative multiplicities are determined in terms of the electric and magnetic charges $d(q_e^2/2))$. Thus the knowledge of the generating function $\Phi(\tau_{ij})$ determines the full spectrum of the perturbative and non-perturbative BPS states in terms of the moduli fields $(S, T_{IJ}, Y_k^I)$ and the charges. In ref. [31], it was conjectured that the generating function $\Phi(\tau_{ij})$ is the genus-2 determinant of 24 bosons:

$$\Phi(\tau_{ij}) = \eta[\tau_{ij}]^{-24} = \left( \prod_{\text{even}} \vartheta[\tau_{ij}] \right)^{-2}. \quad (7.9)$$

Using the genus-2 interpretation of the non-perturbative multiplicities, we will present in section 9 an algorithm that determines the non-perturbative BPS spectrum of the $N = 2$ theories in terms of the shifted $N = 4$ spectrum. Here we will restrict ourselves to the perturbative heterotic and type II cases. Some general facts, valid in $N = 2$ theories, are in order:

• The $N = 2$ massless multiplets $M_0^\lambda$ have the following helicity content:

$$\pm \left( \lambda \pm \frac{1}{2} \right) + 2(\pm \lambda); \quad (7.10)$$

$M_0^0$ is the hypermultiplet, $M_0^{1/2}$ is the vector multiplet, while $M_0^{3/2}$ is the supergravity multiplet.

• The massive BPS multiplets have the following $SO(3)$ spin content

$$M^j : [j] \otimes ([1/2] + [2]) \quad (7.11)$$

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and contain $2(2j + 1)$ bosonic states and an equal number of fermionic ones.

- Finally the generic long-massive multiplet has the following $SO(3)$ content

$$L^j : \, [j] \otimes ([1] + 4[1/2] + 5[0]) \quad (7.12)$$

- The $N = 2$ BPS states correspond to the short multiplets and are picked up by the supertrace of helicity squared, $B_2 = \langle \lambda^2 \rangle$. We have

$$B_2(M^0_0) = (-1)^{2\lambda} \cdot B_2(L^j) = 0 \, , \, B_2(M^j) = (-1)^{2j+1}(2j + 1)/2. \quad (7.13)$$

For the perturbative heterotic theory, a direct computation determines $B_2$ in terms of the characters of the shifted $\Gamma_{2,18}^{[h]}$ and of the four twisted (right-moving) bosons $Z_{4,4}^{[h]}$. In what follows we will assume for simplicity the two heterotic theories $N = 2$ theories.

$$\tau_2 B_2 = \tau_2 \langle \lambda^2 \rangle = \Gamma_{2,18}^{[0]} + \frac{\bar{\vartheta}_2 \vartheta_2}{\eta^2} - \frac{\Gamma_{2,18}^{[1]} \bar{\vartheta}_2 \vartheta_2}{\eta^2} - \frac{\Gamma_{2,18}^{[1]} \bar{\vartheta}_2 \vartheta_2}{\eta^2}$$

$$= \frac{\Gamma_{2,18}^{[0]} + \Gamma_{2,18}^{[0]}}{2} F_1 - \frac{\Gamma_{2,18}^{[0]} - \Gamma_{2,18}^{[0]}}{2} F_1$$

$$- \frac{\Gamma_{2,18}^{[1]} + \Gamma_{2,18}^{[1]}}{2} F_+ - \frac{\Gamma_{2,18}^{[1]} - \Gamma_{2,18}^{[1]}}{2} F_- \quad (7.14)$$

with

$$\bar{F}_1 = \frac{\bar{\vartheta}_3 \vartheta_4}{\eta^2}, \quad \bar{F}_\pm = \frac{\bar{\vartheta}_3 (\vartheta_3 \pm \vartheta_4)}{\eta^2}. \quad (7.15)$$

For all $N = 2$ heterotic theories $B_2$ has universal modular properties under $SL(2, Z)_\tau$:

$$\tau \rightarrow \tau + 1 \quad B_2 \rightarrow B_2 \quad \tau \rightarrow -\frac{1}{\tau} \quad B_2 \rightarrow \tau^2 B_2. \quad (7.16)$$

All functions $\bar{F}_i$ have positive coefficients and have the generic expansions

$$F_1 = \frac{1}{q} + \sum_{n=0}^{\infty} d_1(n)q^n = \frac{1}{q} + 16 + 156q + O(q^2) \quad (7.17)$$

$$F_+ = \frac{8}{q^{3/4}} + q^{1/4} \sum_{n=0}^{\infty} d_+(n)q^n = \frac{8}{q^{3/4}} + 8q^{1/4}(30 + 481q + O(q^2)) \quad (7.18)$$

$$F_- = \frac{32}{q^{1/4}} + q^{3/4} \sum_{n=0}^{\infty} d_-(n)q^n = \frac{32}{q^{1/4}} + 32q^{3/4}(26 + 375q + O(q^2)). \quad (7.19)$$

Also the lattice sums $(\Gamma_{2,18}^{[0]} \pm \Gamma_{2,18}^{[1]})/2$ have positive multiplicities.

The contribution of the generic massless multiplets is given by the constant coefficient of $F_1$ and agrees with the expectation, $16 = 20 - 4$, since we have the supergravity multiplet and 19 vector multiplets contributing 20, and 4 hypermultiplets contributing $-4$. Turning off all the Wilson lines and restoring the $E_7 \times E_8$ group, the above result becomes

$$\langle \lambda^2 \rangle = \Gamma_{2,2}^{[0]} \frac{\bar{\vartheta}_3 \vartheta_4 (\bar{\vartheta}_3 + \vartheta_4)}{2\eta^2} \bar{E}_4 + \Gamma_{2,2}^{[1]} \frac{\bar{\vartheta}_3 \vartheta_4 (\bar{\vartheta}_2 + \vartheta_2)}{2\eta^2} \bar{E}_4 - \Gamma_{2,2}^{[1]} \frac{\bar{\vartheta}_3 \vartheta_4 (\bar{\vartheta}_2 + \vartheta_2)}{2\eta^2} \bar{E}_4 \quad (7.20)$$
Let us analyse the $N = 4 \rightarrow N = 2$ BPS mass formulae; we denote by $\varepsilon' = (\tilde{\varepsilon}_L, \tilde{\varepsilon}_R, \tilde{\zeta})$ the shift vector of the $\Gamma_{(2,18)}$; $\varepsilon'$ must satisfy the modular-invariant constraint $\varepsilon' \cdot \varepsilon' \equiv 2\tilde{\varepsilon}_L \cdot \tilde{\varepsilon}_R - \tilde{\zeta} \cdot \tilde{\zeta} = 2(-1 + \text{mod } 4)$. The mass formula for the BPS states is:

- In the “untwisted” sector $h = 0$, $\varepsilon = 0$

$$M^2(h = 0) = \frac{|-m_1U + m_2 + Tn_1 + (TU - \frac{1}{2}\tilde{W}^2)n_2 + \tilde{W} \cdot \tilde{Q}|^2}{4S_2\left(T_2U_2 - \frac{1}{2}\text{Im}\tilde{W}^2\right)}, \quad (7.21)$$

where $\tilde{W}$ is the 16-dimensional complex vector of Wilson lines. When the integer

$$\rho = \tilde{m} \cdot \tilde{n} - \frac{1}{2}\tilde{Q} \cdot \tilde{Q}, \quad (7.22)$$

is even, these states are vector-like multiplets with multiplicity function $d_1(s)$ of (7.17) where $s$ is:

$$s = \tilde{m} \cdot \tilde{n} - \frac{1}{2}\tilde{Q} \cdot \tilde{Q}. \quad (7.23)$$

When $\rho$ is odd, these states are hypermultiplets-like with multiplicities $d_1(s)$.

- In the “twisted” sector $h = 1$, $\varepsilon \neq 0$

$$M^2(h = 1) = \frac{|-m'_1U + m'_2 + Tn'_1 + (TU - \frac{1}{2}\tilde{W}^2)n'_2 + \tilde{W} \cdot \tilde{Q}'|^2}{4S_2\left(T_2U_2 - \frac{1}{2}\text{Im}\tilde{W}^2\right)} \quad (7.24)$$

with

$$\tilde{m}' \equiv \tilde{m} + \frac{\tilde{\varepsilon}_L}{2}, \quad \tilde{n}' \equiv \tilde{n} + \frac{\tilde{\varepsilon}_R}{2}, \quad \tilde{Q}' \equiv \tilde{Q} + \frac{\tilde{\zeta}}{2}. \quad (7.25)$$

When $\rho$ is even ($\rho'$ odd) the states are hypermultiplet-like with multiplicities $d_+(s')$, with

$$s' = \tilde{m}' \cdot \tilde{n}' - \frac{1}{2}\tilde{Q}' \cdot \tilde{Q}' \quad (7.26)$$

and

$$\rho' = \tilde{m}' \cdot \tilde{\varepsilon}_R + \tilde{n}' \cdot \varepsilon_L - \tilde{Q}' \cdot \tilde{\zeta}. \quad (7.27)$$

When $\rho$ is odd ($\rho'$ even) the states are hypermultiplet-like with multiplicities $d_-(s')$.

Let us discuss here the gauge-symmetry enhancements in the presence of shift vectors. For simplicity we will ignore the charged sector coupled to the Wilson lines and focus on the $\gamma_{(2,2)}$ part. Let us first consider the untwisted sector ($h = 0$). According to the above analysis, the masses are given by the unshifted mass formula (7.21) and they are vector multiplets when $\rho$ is even and hypermultiplets when $\rho$ is odd. Now the points where the standard $\Gamma_{(2,2)}$ mass vanishes are well known. At $T = U$, there are two configurations with zero mass, given by $m_1 = n_1 = \pm 1$, all the rest being zero. For both states, $|\rho| = |\varepsilon^1_L + \varepsilon^1_R|$. Depending on it being even or odd, these states are either vector multiplets that enhance the gauge group $U(1)^2 \rightarrow SU(2) \times U(1)$ or hypermultiplets charged under one of the $U(1)$’s.
Let us now look for states becoming massless in the twisted \((h = 1)\) sector at \(T = U\). Again we obtain \(m_1 + \varepsilon^1_L/2 = n_1 + \varepsilon^1_R/2\). Since \(\varepsilon^1_L, \varepsilon^1_R\) are either 0 or 1, the previous condition can be satisfied only if both \(\varepsilon^1_L = \varepsilon^1_R = \psi\), with \(\psi = 0, 1\). Then \(\rho = 2m_1\psi\) and is always even. The matching condition here for such a state becomes \(s = 3/4\) when \(\rho\) is even (see (7.18)). Thus the condition on \(m_1\) becomes
\[
m_1^2 + \psi m_1 = \frac{3}{4} - \varepsilon^2/8. \tag{7.28}
\]
From modular invariance we have \(\varepsilon^2/2 = -1 \mod 4 = 4k - 1\ k \in \mathbb{Z}\); then eq. (7.28) becomes
\[
m_1^2 + \psi m_1 + k - 1 = 0 \tag{7.29}
\]
and has either two solutions or none in the field of integers, depending on \(\psi\) and \(k\). All such potentially massless states are hypermultiplets, come with multiplicity 8 and have equal and opposite charge under one of the 2-torus \(U(1)\)'s.

8 Heterotic-Type II dual pairs with partially broken SUSY

\(N = 4 \rightarrow N = 2\)

The heterotic string compactified on \(T^4\), with \(N = 2\) in \((6 - d)\) space-time supersymmetry, has been conjectured to be dual to type II theory compactified on \(K_3\) \([29, 14]\). This duality changes the sign of the dilaton, dualizes the field strength of the antisymmetric tensor and leaves the \((4,20)\) gauge fields \(A^i_{\mu}\), the \(SO(4, 20)\) moduli and the Einstein metric invariant. Obviously this duality descends in four dimensions by compactifying both theories on an extra \(T^2\). In four dimensions there are four extra gauge fields, two coming from the metric \(A^i_{\mu}\) whose charges are the momenta of the \(T^2\) and two coming from the antisymmetric tensor \(B_{i\mu}\), whose charges are the winding numbers of the \(T^2\). Also, we have three extra scalars from the components of the metric on \(T^2\), \(G_{ij}\) and one from the antisymmetric tensor \(B_{ij}\). There are also \(2 \times 24\) extra scalars, \(Y^i_I\) coming from the 6-d gauge bosons plus one more \(A\), which is the four-dimensional dual of the antisymmetric tensor. If we denote heterotic variables by unprimed names and type II ones by primed names, then the heterotic-type II duality in four dimensions implies that
\[
e^{-\phi} = \sqrt{\det G_{ij}}, \quad e^{-\phi'} = \sqrt{\det G'_{ij}} \tag{8.1}
\]
\[
\frac{G_{ij}}{\sqrt{\det G_{ij}}} = \frac{G'_{ij}}{\sqrt{\det G'_{ij}}}, \quad A^i_{\mu} = A^i_{\mu}' \tag{8.2}
\]
\[
e^{-\phi} g_{\mu\nu} = e^{-\phi'} g'_{\mu\nu} \rightarrow g^E_{\mu\nu} = g'^E_{\mu\nu} \tag{8.3}
\]
\[
M_{4,20}' = M_{4,20}, \quad A^i_{\mu}' = A^i_{\mu}, \quad Y^i_I = Y^i_I' \tag{8.4}
\]
\[
A = \frac{1}{2} \varepsilon^{ij} B'_{ij}, \quad A' = \frac{1}{2} \varepsilon^{ij} B_{ij}. \tag{8.5}
\]
Moreover, it effects an electric–magnetic duality transformation on the $B^i_\mu$ gauge fields
\[
\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{ij} F^B_{j,\rho\sigma} = \frac{\delta S^{\text{het}}}{\delta F^B_{i,\mu\nu}}.
\] (8.6)

On the electric and magnetic charges it acts on the $T^2$ charges and leaves the rest invariant.

For the configurations of moduli we are interested in, namely the factorization $(6, 22) \rightarrow (2, 18) \times (4, 4)$, we proceed as follows. In the case of the heterotic string the complex moduli $T, U, \vec{W}$ are defined in terms of $G_{ij}, B_{ij}$ and $Y^k_i$, $i, j = (1, 2)$. However, for the type II string the situation is different. A careful analysis of the tree-level action shows that there is an analogue of the Green–Schwarz term $B \wedge F \wedge F$ at tree level; this appears at one loop at the heterotic side for 4-d descendants of both $B \wedge F^4$ and $B \wedge R^4$; the $B \wedge R \wedge R$ term appears at one loop in the type II side \[32\]. This term changes at tree level the definition of the type II $S'$ field. There is an analogous phenomenon, which changes also at tree level the definition of the $T'$ field. The correct formulae read:
\[
S' = A' - \frac{1}{2} Y^I_1 Y^I_2 + \frac{U_1}{2} Y^I_2 Y^I_2 + i \left( e^{-\phi'} + \frac{U_2}{2} Y^I_2 Y^I_2 \right)
\] (8.7)
\[
T' = \sqrt{\det G'_{ij}} + i B'
\] (8.8)
where as usual
\[
\frac{1}{\sqrt{\det G'_{ij}}} G'_{ij} = \frac{1}{U_2} \left( \begin{array}{cc} 1 & U_1 \\ U_1 & |U|^2 \end{array} \right).
\] (8.9)

Thus (8.1)–(8.5) translate to
\[
U = U' , \quad \vec{W} = \vec{W}' , \quad S = T' , \quad T = S'.
\] (8.10)

Let us indicate how the $N = 4$ heterotic-type II duality works at the level the restricted $SO(2, 18)$ BPS formula:
\[
M^2_{BPS} = \frac{|P + S \Pi|^2}{\text{Im} S (\text{Im} T \text{Im} U - \frac{1}{2} \text{Im} \vec{W} \cdot \text{Im} \vec{W})}
\] (8.11)
where $P$ and $\Pi$ are given in terms of the “electric” and “magnetic” charges and in terms of the complex moduli $T, U, \vec{W}$:
\[
P = -m_1 + n_1 T + m_2 U + n_2 (TU - \frac{1}{2} \vec{W} \cdot \vec{W}) + Q \cdot \vec{W}
\]
and
\[
\Pi = -\tilde{m}_1 + \tilde{n}_1 T + \tilde{m}_2 U + \tilde{n}_2 (TU - \frac{1}{2} \vec{W} \cdot \vec{W}) + \bar{Q} \cdot \vec{W}
\] (8.12)
We start first from the heterotic string not necessarily weakly coupled. We would like, however, to end up and compare with the weakly coupled type II string. Thus we must take the limit \( T_2 \) large in the mass formula and keep light states:

\[
M_{\text{het}}^2 = \left| -m_1 + m_2 U + \bar{W} \cdot Q + S(\bar{m}_2 - \bar{m}_1 U + \bar{W} \cdot \bar{Q}) \right|^2 / 4 \left( S_2(T_2 U_2 - (\bar{W}_2)^2/2) \right) \tag{8.13}
\]

However the terms containing the charged states are really absent; \( \bar{m} \cdot \bar{n} = 0 \) and \( \bar{q} \cdot \bar{q} = 0 \) and thus, there are no physical states with non-trivial \( \vec{Q}, \bar{\vec{Q}} \). Taking this into account, then using type II variables from (8.10), we can write (8.13) as

\[
M_{\text{pert-II}}^2 = \left| -m_1 + m_2 U' + T'(\bar{m}_2) - \bar{m}_1 U' \right|^2 / 4 \left( S'_{2} - \frac{\bar{W}'_{2}}{2T_{2}} \right) T'_{2}U'_{2} \tag{8.14}
\]

which gives the correct tree-level type II mass formula in the large \( T'_2 \) limit, taking into account (8.7) and the duality map.

Owing to the adiabatic argument of ref. [33], we can obtain new dual heterotic–type II pairs by orbifolding both the \( N = 4 \) heterotic and \( N = 4 \) type II strings, by the same freely acting symmetry. Thus we would like to identify the duals of the heterotic models constructed in the previous sections with spontaneously broken supersymmetry.

For concreteness we will go to the \( \mathbb{Z}_2 \) sub-manifold of \( K_3 \), where the conformal field theory is explicit, and we will map directly the heterotic to the type II string. The type II partition function on \( K_3 \times T^2 \) at the orbifold point is

\[
Z_{N=4}^{II} = \frac{1}{\text{Im} \tau |\eta|^4} \left\{ \frac{1}{2} \sum_{h,g=0}^{1} \frac{\Gamma_{(2,2)}(h)}{\eta^4} \frac{Z^{\text{twist}}_{(4,4)}(4\eta)}{\eta^{2(h)}} \right\} \times \frac{1}{2} \sum_{\alpha, \beta=0}^{1} (-1)^{\alpha+\beta+\alpha \beta} \frac{\vartheta_2(\eta) \vartheta_{[\beta+g]}(\eta \vartheta_{[\beta-g]}(\eta \vartheta_{[\alpha-h]}(\eta)))}{\eta^4} \tag{8.15}
\]

Let us examine the massless bosonic spectrum of the \( N = 4 \) type II, and try to match it to that of the \( N = 4 \) heterotic string [32, 4].

- In the NS–NS (\( \alpha = \bar{\alpha} = 0 \)) untwisted sector (\( h = 0 \)), there are 32 degrees of freedom, corresponding to the graviton, 2 scalars (axion-dilaton), 4 vectors, and another 20 scalars (the \( \Gamma_{2,2} \) and \( Z^{\text{twist}}_{(4,4)} \) moduli). Two of the gauge bosons are graviphotons while the other two belong to \( U(1) \) vector multiplets. Thus these four gauge bosons have lattice signature \( (2,2) \). Similarly the \( (2,2) \) moduli belong to these two vector multiplets while the \( (4,4) \) moduli are in multiplets with vectors coming from the R–R untwisted sector.

- In the NS–NS (\( \alpha = \bar{\alpha} = 0 \)) twisted sector (\( h = 1 \)), there are 16 \( \mathbb{Z}_2 \) invariant states in the \( T^4/\mathbb{Z}_2 \) part: \( H^T \). There are in total \( 4 \times 16 \) massless states; all of them are scalars in multiplets with vectors coming from the R–R twisted sector.

- In the R–R (\( \alpha = \bar{\alpha} = 1 \)) untwisted sector there are 32 physical degrees of freedom. These correspond to 8 vectors and 16 scalars. The vectors have lattice signature \( (4,4) \) and four of them
are graviphotons while the other four are in vector multiplets. The sixteen scalars complete the six vector multiplets.

- In the R-R, twisted sector, there are $4 \times 16$ massless states corresponding to 16 vectors and 32 scalars.

Here the gauge group is composed of $U(1)$'s, which implies that we are sitting at a generic point in the space of Wilson lines. The perturbative spectrum is charged under two of the graviphotons and two of the other gauge bosons with charges given by $P_L, P_R$ of the $T^2$.

Consider now the freely acting orbifold on the heterotic side as a $\pi$ rotation on the $(4,4)$ part of the lattice and as a translation $\vec{\varepsilon}$ on the $\Gamma_{(2,18)}$ lattice. Again for simplicity we focus on the $Z_2$ case. On the type II side the $Z_2$ rotation on the $(4,4)$ part changes the sign of the massless states coming from the untwisted R–R sector as well as the scalars coming from the twisted NS–NS sector. The effect of the $(2,18)$ translation $\vec{\varepsilon} = (\vec{\varepsilon}_L; \vec{\varepsilon}_R, \vec{\zeta})$ is to give phases to massive charged states, but has no effect on the massless spectrum. Thus at the massless level the NS–NS twisted and R–R untwisted sectors have to be projected out. The projection in the type II case, which has the same effect as the $(4,4)$ rotation in the heterotic side, is a combination of the right fermion number operator $(-1)^F_R$, which changes the sign of the right-moving Ramond sector, and the symmetry transformation $e = (-1)^h$, which acts on the twisted states of the orbifold with a minus sign and is inert on anything else.

The $\vec{\zeta}$ translation vector does not act in the perturbative type II string since the perturbative spectrum does not contain states charged under the 16 gauge bosons coming from the R-R twisted sector. However it will act on non-perturbative $D$-brane states carrying R–R charges.

Finally the phase coming from the translation of the $(2,2)$ piece is

$$(-1)^{\vec{m} \cdot \vec{\varepsilon}_R} + (-1)^{\vec{\eta} \cdot \vec{\varepsilon}_L}$$

in the heterotic side. Under the type II–heterotic map, this becomes, in the type II side:

$$(-1)^{\vec{m} \cdot \vec{\varepsilon}_R + \vec{\eta} \cdot \vec{\varepsilon}_L}$$

where $\vec{a} \times \vec{b} = a_1b_2 - a_2b_1$. Thus the $\varepsilon_L$ translation acts on the type II side on the magnetically charged states of the momentum-gauge fields of the two-torus; it is thus, not visible in type II perturbation theory.

The type II duals have 20 vector-multiplets and 4 hypermultiplets; thus they are “mirrors” of the type II models discussed in ref. [33] with 4 vector multiplets and 20 hypermultiplets. Therefore, the perturbative partition function of the type II models dual to the heterotic ones is

$$Z_{II}^{4\rightarrow 2} = \frac{1}{\text{Im} \tau |\eta|^2} \sum_{h,g,h_g=0} \frac{\Gamma^{\varepsilon_R[h]}_{2,2|g}}{|\eta|^2} Z_{\text{twist}}^{(4,4)|g} \times \frac{1}{2} \sum_{\alpha,\beta=0} (-1)^{\alpha+\beta+\alpha\beta} \frac{\partial^2[\alpha][\beta+g]}{\eta^4} \frac{\partial[\alpha-h][\beta-g]}{\bar{\eta}^4} \times (-1)^{(\bar{\alpha}+h)\bar{g}+(\bar{h}+g)\bar{h}}.$$  (8.18)

Here the reader might have noticed a potential puzzle. Consider a heterotic model defined by a translation vector with $\vec{\varepsilon}_R = \vec{0}$. In this model, in the limit $\text{Im} \tau \rightarrow 0$ $N = 2$ supersymmetry
is restored to $N = 4$. Alternatively speaking, $m_{3/2} \sim \text{Im } T$. Thus in weakly-coupled heterotic string we take $S \to \infty$ and also $T \to 0$. According to our duality map described above, there is no perturbative shift of the $T^2$ in the type II side. Thus, at the perturbative level, the type II $N = 2$ theory does not look like a spontaneously broken $N = 4$. However a look at (8.17) is sufficient to convince us that there are two gravitinos, with $m_{3/2} \sim \text{Im } S'$, which are light in the strong coupling region of the type II theory and certainly not visible in the weak coupling type II perturbation theory.

A similar phenomenon can happen in reverse. Consider a freely acting orbifold of the type II ($N = 4$) side, as in (8.18), where the (2,2) lattice translation acts on the windings of the two-torus with the phase $(-1)^{\vec{e}_L \cdot \vec{n}}$. This is modular-invariant on the type II side. On the heterotic side the shift of the two-torus becomes non-perturbative via the heterotic-type II map, $(-1)^{\vec{m} \times \vec{e}_L}$. Thus, in heterotic perturbation theory, we only see the $Z_2$ rotation of the $(4,4)$ torus. As it stands the heterotic $N = 2$ model is not modular-invariant. An extra shift in the gauge lattice is needed (not visible on the type II side). Thus the perturbative heterotic ground state has a $K_3 \times T^2$ structure (at the $Z_2$ orbifold point) and the supersymmetry $N = 4 \to N = 2$ looks explicitly broken in perturbation theory. Turning on all Wilson lines we find that the generic massless spectrum has 19 vector multiplets (including the dilaton) and 4 hypermultiplets. Moreover the $SL(2, Z)_S$ is broken to $\Gamma^-(2)_S$ as can easily be seen by following the fate of $T$-duality of the type II dual.

Another comment concerns the fate of the $SL(2, Z)_S$ electric–magnetic duality symmetry of the original $N=4$ theory, in the spontaneously broken phase. It is known that in the $N = 4$ case $SL(2, Z)_S$ is a corollary of heterotic–type II duality, since the $T$-duality of type II translates into the $S$-duality of the heterotic theory. Let us investigate what remains of the perturbative $T$ duality in the broken type-II theory. We have argued above that the two-torus on the type II side gets a (perturbative) shift $(\vec{0}; \vec{e}_R)$ that amounts to the phase $(-1)^{\vec{m} \cdot \vec{e}_R}$. The $SL(2, Z)_T$ acts on the two-torus charges as the set of matrices

$$SL(2, Z)_T : \left( \begin{array}{c} \vec{m} \\ \vec{n} \end{array} \right) \to \left( \begin{array}{cc} a & 1 \\ -c & d \end{array} \right) \left( \begin{array}{c} \vec{m} \\ \vec{n} \end{array} \right);$$

$$ad - bc = 1 , \quad a, b, c, d \in \mathbb{Z} \quad (8.19)$$

There are two subgroups of $SL(2, Z)$ that are relevant here; one is $\Gamma^+(2)$ defined by $b$ even in (8.19); the other one is $\Gamma^-(2)$ defined by $c$ even in (8.19). Thus when $\vec{e}_R \neq \vec{0}$, $SL(2, Z)_T$ is broken to $\Gamma^+(2)_T$. Thus, the $S$-duality group is reduced to $\Gamma^+(2)_S$.

In the above discussion, it is obvious that there are non-perturbative ambiguities in the translation-related projections. The most general projection conceivable is determined by the “electric” translation vector $\vec{e}_l$ but simultaneously by a “magnetic” translation vector $\vec{e}_R$ whose effects are not visible in the perturbative spectrum. Parts of these translations are never perturbatively visible either in the heterotic nor in the type II side. We will comment more on this issue in the next section.

One more remark is in order about the type II duals described above. Inspection shows that all of the $N = 2$ spacetime supersymmetry comes from the left side. Consequently, in these models the $S$ field is in a vector multiplet [33]. Thus, as in the heterotic side, the vector-moduli space gets corrections while the hypermultiplet moduli space does not. At generic Wilson
lines this class of models has a massless spectrum, which consists, apart from the supergravity and the dilaton vector multiplet, of 18 vector multiplets and 4 neutral hypermultiplets (the moduli of the four-torus). The non-perturbatively exact hypermultiplet quaternionic manifold is $SO(4,4)/SO(4) \times SO(4)$. The exactness of the hypermultiplet moduli space restricts the orbifolding possibilities on the type II side to the ones described in (8.18).

9 Non-perturbative BPS spectrum in partially broken SUSY $N = 4 \rightarrow N = 2$

Our conjecture for the non-perturbative multiplicities consists in generalizing the perturbative multiplicity functions (7.14)–(7.19) $F_i$ in genus-2. First we rewrite $F_i$ in a more convenient form:

$$F_1 = \frac{1}{\eta^{24}} \bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right], \quad F_{\pm} = \frac{1}{\eta^{24}} \left(\bar{\chi}\left[\begin{matrix} h \\ 0 \end{matrix}\right] \pm \bar{\chi}\left[\begin{matrix} 0 \\ h \end{matrix}\right]\right)$$

(9.1)

where $\bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right]$ are given in terms of the characters of four twisted 2d right-moving bosons:

$$\bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right] = \frac{4(-)^h \theta^h}{\bar{\eta}[1+h] \bar{\eta}[1-h]},$$

(9.2)

where in the above equation $(h, g) \neq (0,0)$.

We can extend the validity of $\bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right]$ for all $(h, g)$ sectors using identities between right-moving, bosonic and fermionic, “twisted” characters:

$$\bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right] = \frac{1}{8 \eta^{24}} \sum_{a,b} (-)^h \bar{\eta}^{4[a+h]} \bar{\eta}^{4[a-h]} \bar{\eta}[1+h] \bar{\eta}[1-h].$$

(9.3)

In this expression, the absence of the $(h, g) = (0,0)$ sector is due to the vanishing of the odd-spin structures ($\bar{\eta}[\tau]$ terms). In genus-2 $h$ and $g$ become $\bar{h} = (h, \bar{h})$ and $\bar{g} = (g, \bar{g})$ in correspondence with the “electric” and “magnetic” charge shifts. The generalization in genus-2 of the twisted characters consists in promoting the various $\vartheta$-functions with characteristics in genus-2

$$\bar{\vartheta}[a + h + g](\tau) \rightarrow \bar{\vartheta}\left[\begin{matrix} a + h \\ b + g \end{matrix}\right](\bar{\tau}ij).$$

(9.4)

Then, the proposed non-perturbative multiplicities will be generated by the genus-2 functions:

$$F\left[\begin{matrix} h \\ g \end{matrix}\right] = \Phi(\bar{\tau}ij) \bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right](\bar{\tau}ij),$$

(9.5)

where $\Phi(\bar{\tau}ij)$ is the $N = 4$ multiplicity function and $\bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right](\bar{\tau}ij)$ are the genus–2 analogues of the genus-1 “twisted” characters $\bar{\chi}\left[\begin{matrix} h \\ g \end{matrix}\right](\bar{\tau})$ defined above.

Using the genus-2 multiplicity functions, we can construct weighted free-energy super-traces, which extend at the non-perturbative level the same perturbative quantities, e.g. the moduli dependence of the gauge and gravitational couplings. We define by $L^D$ the following quantity:

$$L^D = \int_c [dt \prod dX^{ij}] \sum_{h_i, g_i} \sum_{q_i} D(\bar{\tau}ij) F\left[\begin{matrix} h \\ g \end{matrix}\right](\bar{\tau}ij) \times \exp \left[-2i\pi \Re \tau^{ij} \left(\bar{q}_i + \bar{\varepsilon}_i\right) \cdot \left(\bar{q}_j + \bar{\varepsilon}_j\right)\right]$$

26
\begin{equation}
\times \exp \left[ -\pi t M^2_{BPS} (S; \vec{q}, \vec{\varepsilon}) \right],
\end{equation}

where \( M^2_{BPS} (S; \vec{q}, \vec{\varepsilon}) \) stands for the non-perturbative mass formula \( \text{(8.11,8.12)} \) with shifted charges; \( M^2_{BPS} \) depends on the shifted “electric” and “magnetic” charges, the moduli \( T, U, \) and \( \vec{W} \) as well as the dilaton–axion moduli field \( S \). The period matrix \( \tau_{ij} \) of genus-2 in eq.\( (9.7) \), is constructed in terms of the parameters \( t, X^{ij} \) and \( S \) in the following way:

\begin{equation}
t = \sqrt{\det(\tau_{ij})}, \quad X^{ij} = \Re \tau^{ij}, \quad \text{and} \quad \frac{\tau^{ij}}{\sqrt{\det \tau^{ij}}} = \frac{1}{\Im S} \left( \begin{array}{cc} \Re S & |S|^2 \\
\end{array} \right).
\end{equation}

The integration on \( X^{ij} \) in the domain \([-1/2, +1/2]\) would give rise to the non-perturbative matching conditions \( \text{(7.8)} \). The relevant multiplicities are generated by the functions \( F_{\vec{h}, \vec{g}} \). This is a suggestive argument, and stands in a similar footing with the analogous \( \tau_1 \) integration in the perturbative string. However we suggest that, like in the string case, the correct integration domain is the genus-two fundamental region. Thus we expect that the integration over \( t \) (in the fundamental domain of genus-2 with \( S \) fixed) gives rise to the non-perturbative quantity \( L^D[S; T, U, \vec{W}] \) in terms of all moduli, \( S \) included.

The kernel \( D \) is the genus-2 analogue of a product of charge operators. In the perturbative string, this is given by a product of right-moving lattice vectors and contains also a “back-reaction” term \( [35] \). There is an analogue of “right-moving” charges in the non-perturbative case when we also include the magnetic charges. The charge sum for the overall trace can be written in the perturbative case as a \( \bar{\tau} \) derivative, which generalizes in the non-perturbative treatment to the \( \partial_{\tau_{11}} + \partial_{\tau_{22}} \). The “back-reaction” term can be fixed since it has to restore the modular properties of the \( \bar{\tau}^{i}j \) derivatives.

The physical interpretation of the summation over the “magnetic” charges reproduces the Euclidean space-time instanton corrections to the couplings.

The determination of the non-perturbative effective couplings constants (the gravitational one included) defines without any ambiguity the non-perturbative prepotential of the \( N = 2 \) effective theory. Therefore, the knowledge of \( L^D \) determines at the non-perturbative level the \( N = 2 \) low-energy effective supergravity, which includes terms up to two derivatives.

\section{10 Outlook}

We have demonstrated the existence of partial spontaneous supersymmetry breaking in string theory, and gave several concrete examples in both the heterotic and type II theories. We have studied the issue of restoration of supersymmetry, at the classical and perturbative level. We have further analysed the consequences of heterotic–type II duality valid for the \( N = 2 \) models we presented. We have pointed out that in the dual theories the \( N = 4 \rightarrow N = 2 \) supersymmetries may look explicitly broken in their perturbation theory. This was also corroborated by our conjecture on the full non-perturbative structure of their effective theories. In some cases we can predict some novel non-perturbative (non-geometric) transitions between vacua of the type II string with \((2,0)\) and \((1,1)\) space-time supersymmetry.

An analysis of the perturbative BPS states of strings, with supersymmetry spontaneously broken \( N = 4 \rightarrow N = 2 \), and the underlying duality structure permit us to conjecture the full
non-perturbative form of the effective field theory. This conjecture needs to be elaborated and tested in the context of explicit models. This will be the subject of future analysis.

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References

[1] M. de Roo, Nucl. Phys B255 (1985) 515; Phys. Lett. 156 (1985) 331; E. Bergshoeff, I.G. Koh and E. Sezgin, Phys. Lett. B155 (1985) 71.

[2] S. Ferrara, L. Girardello, C. Kounnas and M. Porrati, Phys. Lett. B192 (1987) 368.

[3] K.S. Narain, Phys. Lett. B169 (1986) 41; K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[4] I. Antoniadis, C. Bachas C. Kounnas and P. Windey, Phys. Lett. B171 (1986) 51; I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. B289 (1987) 87.

[5] H. Kawai, D.C. Lewellen and S.-H.H. Tye, Nucl. Phys. B288 (1987) 1.

[6] W. Lerche, D. Lüst and A.N. Schellekens, Nucl. Phys. B287 (1987) 477.

[7] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46;

[8] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678; K.S. Narain, M.H. Sarmadi and C. Vafa, Nucl. Phys. B288 (1987) 551.

[9] D. Gepner, Phys. Lett. B199 (1987) 370; Nucl. Phys. B296 (1988) 757; L.E. Ibanez, H. P. Nilles and F. Quevedo, Phys. Lett. B187 (1987) 25.

[10] C. Kounnas and M. Porrati, Nucl. Phys. B310 (1988) 355, S. Ferrara, C. Kounnas, M. Porrati and F. Zirwer, Nucl. Phys. B318 (1989) 75.

[11] I. Antoniadis, Phys. Lett. B246 (1990) 377; I. Antoniadis and C. Kounnas, Phys. Lett. B261 (1991) 369.

[12] C. Kounnas and B. Rostand, Nucl. Phys. B341 (1990) 641.

[13] I. Antoniadis, H. Partouche and T.R. Taylor, Phys. Lett. B372 (1996) 155; S. Ferrara, L. Girardello and M. Porrati, Phys. Lett. B376 (1996) 275; J. Bagger and A. Galperin, [hep-th/9608177].

[14] C. Hull and P. Townsend, Nucl. Phys. B438 (1995) 109, [hep-th/9410167].

[15] E. Witten, Nucl. Phys. B443 (1995) 85, [hep-th/9503124].
[16] K. Kikkawa and M. Yamasaki, Phys. Lett. B149 (1984) 357; N. Sakai and I. Senda, Progr. Theor. Phys. 75 (1986) 692; V.P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. B287 (1987) 402; B. Sathiapalan, Phys. Rev. Lett. 58 (1987) 1597; R. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. 115 (1988) 649; A.Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167; A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 301; M. Dine, P. Huet and N. Seiberg, Nucl. Phys. B322 (1989) 301.

[17] E. Kiritsis, C. Kounnas, M. Petropoulos and J. Rizos, Phys. Lett. B385 (1996) 87, hep-th/9606087.

[18] E. Kiritsis, C. Kounnas, M. Petropoulos and J. Rizos, hep-th/9605011; hep-th/9608034.

[19] E. Kiritsis, C. Kounnas, M. Petropoulos and J. Rizos, CERN/TH.96/...to appear

[20] L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1991) 649.

[21] T. Banks and L. Dixon, Nucl. Phys. B307 (1988) 93.

[22] A. Font, L. Ibanez, D. Lüst and F. Quevedo, Phys. Lett. B249 (1990) 35; S.J. Rey, Phys. Rev. D43 (1991) 256.

[23] A. Sen, Phys. Lett. B303 (1993) 22, B329 (1994) 217 and Nucl. Phys. B329 (1994) 217.

[24] J. Schwarz and A. Sen, Phys. Lett. B312 (1993) 105, and Nucl. Phys. B411 (1994) 35.

[25] M. Cvetic and D. Youm, Phys. Rev. D53 (1996) 584; Phys. Lett. B359 (1995) 87; hep-th/9508058; hep-th/9510098; hep-th/9512127; Kwan-Leung Chan and M. Cvetic, Phys. Lett. B375 (1996) 98.

[26] M. Duff, J.T. Liu and J. Rahmfeld, Nucl. Phys. B459 (1996) 125, hep-th/9508094.

[27] M. Cvetic and A. Tseytlin, Phys. Lett. B366 (1996) 95, hep-th/9510097; Phys. Rev. D53 (1996) 5619, hep-th/9512031.

[28] G. Lopes-Cardoso, G. Curio, D. Lüst, T. Mohaupt and S. J. Rey, Nucl. Phys. B464 (1996) 18, hep-th/9512129.

[29] M. Duff and R. Khuri, Nucl. Phys. B411 (1994) 473, hep-th/9305142; M. Duff and R. Minasian, Nucl. Phys. B436 (1995) 507, hep-th/9406198.

[30] C. Bachas and E. Kiritsis, hep-th/9611203.

[31] R. Dijkgraaf, E. Verlinde and H. Verlinde, hep-th/9607020.

[32] C. Vafa and E. Witten, Nucl. Phys. B447 (1995) 261.

[33] C. Vafa and E. Witten, hep-th/9507050.
[34] S. Ferrara and C. Kounnas, Nucl. Phys. B328 (1989) 406.

[35] E. Kiritsis and C. Kounnas, Nucl. Phys. B442 (1995) 472.