On Standard Concepts Using $ii$-Open Sets

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Abstract
Following Caldas in [1] we introduce and study topological properties of $ii$-derived, $ii$-border, $ii$-frontier, and $ii$-exterior of a set using the concept of $ii$-open sets. Moreover, we prove some further properties of the well-known notions of $ii$-closure and $ii$-interior. We also study a new decomposition of $ii$-continuous functions. Finally, we introduce and study some of the separation axioms specifically $T_{0ii}$, $T_{1ii}$.

Subject Areas
Mathematical Analysis

Keywords
$a$-Open Set, $ii$-Open Set, Separation Axioms

1. Introduction
The notion of $a$-open set was introduced by Njastad in [2]. Caldas in [1] introduced and studied topological properties of $a$-derived, $a$-border, $a$-frontier, $a$-exterior of a set by using the concept of $a$-open sets. In this paper, we introduce and study the same above concepts by using $ii$-open sets. A subset $A$ of $X$ is called $ii$-open set [3] if there exists an open set $G$ in the topology $\tau$ of $X$, such that: $G \neq \emptyset, A \subseteq CL(A \cap G) \text{ and } Int(A) = G$, the complement of an $ii$-open set is an $ii$-closed set. We denote the family of $ii$-open sets in $(X, \tau)$ by $\tau''$. It is shown in [4] that each of $\tau \subseteq \tau''$ and $\tau''$ is a topology on $X$. This property allows us to prove similar properties of $a$-open set. Also, we define $ii$-continuous functions and we study the relation between this type of function and continuous, semi-continuous, $a$-continuous and $i$-continuous functions. Finally, we introduce a new type of separation axioms namely $T_{0ii}$, $T_{1ii}$. We prove similar properties and characterizations of $T_o$ and $T_i$. 

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2. Preliminaries

Throughout this paper, \((X, \tau)\) and \((Y, \sigma)\) (simply \(X\) and \(Y\)) always mean topological spaces. For a subset \(A\) of a space \(X\), \(\text{CL}(A)\) and \(\text{Int}(A)\) denote the closure of \(A\) and the interior of \(A\) respectively. We recall the following definitions, which are useful in the sequel.

**Definition 2.1.** A subset \(A\) of a space \(X\) is called
1) Semi-open set \([5]\) if \(A \subseteq \text{CL}(\text{Int}(A))\).
2) \(\alpha\)-open set \([2]\) if \(A \subseteq \text{Int}(\text{CL}(\text{Int}(A)))\).
3) \(i\)-open set \([3]\) if there exist an open set \(G\) in the topology \(\tau\) of \(X\), such that
   i) \(G \neq \emptyset, X\)
   ii) \(A \subseteq \text{CL}(A \cap G)\)
   The complement of an \(i\)-open set is an \(i\)-closed set.
4) \(ii\)-open set \([4]\) if there exist an open set \(G\) in the topology \(\tau\) of \(X\), such that
   i) \(G \neq \emptyset, X\)
   ii) \(A \subseteq \text{CL}(A \cap G)\)
   iii) \(\text{Int}(A) = G\)
   The complement of an \(ii\)-open set is an \(ii\)-closed set.
5) \(\text{int}\)-open set \([4]\) if there exist an open set \(G\) in the topology \(\tau\) of \(X\) and \(G \neq \emptyset, X\) such that \(\text{Int}(A) = G\). The complement of \(\text{int}\)-open set is \(\text{int}\)-closed set.
6) \(\alpha o\) \((X)\), \(\text{So} \)(\(X)\), \(\text{io} \)(\(X)\), \(\text{ii}o \)(\(X)\), \(\text{into} \)(\(X)\) are family of \(\alpha\)-open, semi-open, \(i\)-open, \(ii\)-open, \(\text{int}\)-open sets respectively.
7) \(\tau', \tau^\alpha\) denote the family of all \(i\)-open sets and \(ii\)-open sets respectively.

**Definition 2.2.** \([3]\) A topological space \(X\) is called
1) \(0\)-\(iT\) if \(a, b\) are two distinct points in \(X\), there exist an \(i\)-open set \(U\) such that either \(a \in U\) and \(b \notin U\), or \(b \in U\) and \(a \notin U\).
2) \(1\)-\(iT\) if \(a, b \in X\) and \(a \neq b\), there exist \(i\)-open sets \(U, V\) containing \(a, b\) respectively, such that \(b \notin U\) and \(a \notin V\).

**Definition 2.3.** A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is called
1) Continuous \([6]\), if \(f^{-1}(G)\) is open in \((X, \tau)\) for every open set \(G\) of \((Y, \sigma)\).
2) \(\alpha\)-continuous \([6]\), if \(f^{-1}(G)\) is \(\alpha\)-open in \((X, \tau)\) for every open set \(G\) of \((Y, \sigma)\).
3) Semi-Continuous \([5]\), if \(f^{-1}(G)\) is semi-open in \((X, \tau)\) for every open set \(G\) of \((Y, \sigma)\).
4) \(i\)-Continuous \([3]\), if \(f^{-1}(G)\) is \(i\)-open in \((X, \tau)\) for every open set \(G\) of \((Y, \sigma)\).

3. Applications of \(ii\)-Open Sets

**Definition 3.1.** Let \(A\) be a subset of a topological space \((X, \tau)\). A derived set of \(A\) denoted by \(D(A)\) is defined as follows:
\[
D(A) = \{x \in X : (G \setminus A) \setminus \{x\} \neq \emptyset, \forall x \in G\}
\]
A point \(x \in X\) is said to be \(ii\)-limit
point of \( A \) if it satisfies the following assertion: 
\[
(\forall g \in \mathcal{G})(x \in G \Rightarrow (G \cap A) \setminus \{x\} \neq \emptyset).
\]
The set of all \( ii \)-limit points of \( A \) is called the \( ii \)-derived set of \( A \) and is denoted by \( D_{ii}(A) \). Note that \( x \in X \) is not \( ii \)-limit point of \( A \) if and only if there exist an \( ii \)-open set \( G \) in \( X \) such that 
\[
(x \in G \text{ and } (G \cap A) \setminus \{x\} = \emptyset).
\]

**Theorem 3.2.** For subsets \( A, B \) of a space \( X \), the following statements hold:
1) \( D_{ii}(A) \subseteq D(A) \)
2) If \( A \subseteq B \), then \( D_{ii}(A) \subseteq D_{ii}(B) \)
3) \( D_{ii}(A) \cup D_{ii}(B) \subseteq D_{ii}(A \cup B) \) and \( D_{ii}(A \cap B) \subseteq D_{ii}(A) \cap D_{ii}(B) \)
4) \( D_{ii}(D_{ii}(A)) \subseteq A \subseteq D_{ii}(A) \)
5) \( D_{ii}(A \cup D_{ii}(A)) \subseteq A \cup D_{ii}(A) \)

**Proof.** 1) Since every open set is \( ii \)-open \([4]\), it follows that \( D_{ii}(A) \subseteq D(A) \).
2) Let \( x \in D_{ii}(A) \). Then \( G \) is \( ii \)-open set containing \( x \) such that 
\[
(x \in G \text{ and } (G \cap A) \setminus \{x\} \neq \emptyset).
\]
Hence, \( x \in D(A) \).
3) Since \( A \subseteq A \) and \( B \subseteq B \), from (2) we get \( D_{ii}(A) \subseteq D_{ii}(A \cup B) \) and \( D_{ii}(A \cap B) \subseteq D_{ii}(A) \). Therefore \( D_{ii}(A) \) and \( D_{ii}(B) \).
This implies to \( D_{ii}(A) \) and \( D_{ii}(B) \).
4) If \( x \in D_{ii}(A) \setminus A \) and \( G \) is an \( ii \)-open set containing \( x \), then 
\[
G \cap (D_{ii}(A) \setminus \{x\}) \neq \emptyset.
\]
Let \( y \in G \cap (D_{ii}(A) \setminus \{x\}) \). Then, since \( y \in D_{ii}(A) \) and \( y \in G \), \( G \cap (A \setminus \{y\}) \neq \emptyset \). Let \( z \in G \cap (A \setminus \{y\}) \). Then, \( z \neq x \) for \( z \in A \) and \( x \notin A \). Hence, \( G \cap (A \setminus \{x\}) \neq \emptyset \). Therefore, \( x \in D_{ii}(A) \).
5) Let \( x \in D_{ii}(A \cup D_{ii}(A)) \). If \( x \in A \), the result is obvious. So, let, \( x \in D_{ii}(A \cup D_{ii}(A)) \setminus A \) then for \( ii \)-open set \( G \) containing \( x \), 
\[
(G \cap (A \cup D_{ii}(A)) \setminus \{x\}) \neq \emptyset.
\]
Thus, \( G \cap (A \setminus \{x\}) \neq \emptyset \). Hence, \( x \in D_{ii}(A) \).
In general, the converse of (1) may not true and the equality does not hold in (3) of theorem 3.2.

**Example 3.3.** Let \( X = \{a, b, c\} \) and \( r = \{\phi, X, \{b\}\} \). Thus, \( ii o \{x\} = \{\phi, X, \{b\}\} \). Take the following:
1) \( A = \{c\} \). Then, \( D(A) = \{a, b\} \) and \( D_{ii}(A) \neq \emptyset \). Hence, \( D(A) \neq D_{ii}(A) \);
2) \( C = \{a, b\} \) and \( E = \{c\} \). Then \( D_{ii}(C) = \{a, c\} \) and \( D_{ii}(E) = \emptyset \). Hence \( D_{ii}(C \cup E) = D_{ii}(C \bigcup D_{ii}(E)) \).

**Theorem 3.4.** For any subset \( A \) of a space \( X \), \( CL_{ii}(A) = A \cup D_{ii}(A) \).

**Proof.** Since \( D_{ii}(A) \subseteq CL_{ii}(A) \) and \( A \subseteq CL_{ii}(A) \). On the other hand, let \( x \in CL_{ii}(A) \). If \( x \in A \), then the proof is complete. If \( x \notin A \), each \( ii \)-open set
G containing x intersects A at a point distinct from x; so \( x \in D_i(A) \). Thus, \( CL_i(A) \subseteq A \cup D_i(A) \), which completes the proof.

**Definition 3.5.** A point \( x \in X \) is said to be \( ii \)-interior point of A if there exist an \( ii \)-open set G containing x such that \( G \subseteq A \). The set of all \( ii \)-interior points of A is said to be \( ii \)-interior of A and denoted by \( Int_i(A) \).

**Theorem 3.6.** For subset A, B of a space X, the following statements are true:
1. \( Int_i(A) \) is the union of all \( ii \)-open subset of A
2. A is \( ii \)-open if and only if \( A = Int_i(A) \)
3. \( Int_i(\bigcup_{A} G_i) = \bigcup_{A} Int_i(G_i) \)
4. \( Int_i(A) = A \setminus D_i(X \setminus A) \)
5. \( X \setminus Int_i(A) = CL_i(X \setminus A) \)
6. \( X \setminus CL_i(A) = Int_i(X \setminus A) \)
7. If \( A \subseteq B \), then \( Int_i(A) \subseteq Int_i(B) \)
8. \( Int_i(A) \cup Int_i(B) \subseteq Int_i(A \cup B) \)
9. \( Int_i(A) \cap Int_i(B) \supseteq Int_i(A \cap B) \)

**Proof.**
1) Let \( \{ G_i : i \in \Lambda \} \) be a collection of all \( ii \)-open subsets of A. If \( x \in Int_i(A) \), then there exist \( j \in \Lambda \) such that \( x \in G_j \subseteq A \). Hence \( x \in \bigcup_{G_i} G_i \), and so \( Int_i(A) \subseteq \bigcup_{G_i} G_i \). On the other hand, if \( y \in \bigcup_{G_i} G_i \), then \( y \in G_k \subseteq A \) for some \( k \in \Lambda \). Thus \( y \in Int_i(A) \), and \( \bigcup_{G_i} G_i \subseteq Int_i(A) \).

Accordingly, \( \bigcup_{G_i} G_i \subseteq Int_i(A) \).
2) Straightforward.
3) It follows from (1) and (2).
4) If \( x \in A \setminus D_i(X \setminus A) \), then \( x \not\in D_i(X \setminus A) \) and so there exist an \( ii \)-open set G containing x such that \( G \cap (X \setminus A) = \emptyset \). Thus, \( x \in G \subseteq A \) and hence \( x \in Int_i(A) \). This shows that \( A \setminus D_i(X \setminus A) \subseteq Int_i(A) \). Now let \( x \in Int_i(A) \).

Since \( Int_i(A) \in \tau_i \) and \( Int_i(A) \cap (X \setminus A) = \emptyset \). We have \( x \not\in D_i(X \setminus A) \). Therefore, \( Int_i(A) = A \setminus D_i(X \setminus A) \).
5) Using (4) and Theorem (3.4), we have
\[
X \setminus Int_i(A) = X \setminus (A \setminus D_i(X \setminus A)) = (X \setminus A) \cup D_i(X \setminus A) = CL_i(X \setminus A).
\]
6) Using (4) and Theorem (3.4), we get.
\[
Int_i(X \setminus A) = (X \setminus A) \setminus D_i(A) = X \setminus (A \cup D_i(A)) = X \setminus CL_i(A)
\]
7) Since \( A \subseteq B \) and \( Int_i(A) \subseteq A \), \( Int_i(B) \subseteq B \), we get \( Int_i(A) \subseteq Int_i(B) \).
8) Since \( A \subseteq (A \cup B) \) and \( B \subseteq (A \cup B) \), from (7) we get \( Int_i(A) \subseteq Int_i(A \cup B) \), \( Int_i(B) \subseteq Int_i(A \cup B) \). Therefore \( Int_i(A) \cup Int_i(B) \subseteq Int_i(A \cup B) \).
9) Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \), from (7) we get \( Int_i(A \cap B) \subseteq Int_i(A) \), \( Int_i(A \cap B) \subseteq Int_i(B) \). Therefore \( Int_i(A \cap B) \subseteq Int_i(A \cap B) \).

**Definition 3.7.** \( b_i(A) = A \setminus Int_i(A) \) is said to be the \( ii \)-border of A.

**Theorem 3.8.** For a subset A of a space X, the following statements hold:
1. \( b_i(A) \subseteq b(A) \) where \( b(A) \) denotes the border of A
2. \( Int_i(A) \cup b_i(A) = A \)
3) $\text{Int}_\phi (A) \cap b_\phi (A) = \phi$
4) $b_\phi (A) = \phi$ if and only if $A$ is $ii$-open set
5) $b_\phi (\text{Int}_\phi (A)) = \phi$
6) $\text{Int}_\phi (b_\phi (A)) = \phi$
7) $b_\phi (b_\phi (A)) = b_\phi (A)$
8) $b_\phi (A) = A \cap \text{CL}_\phi (X \setminus A)$
9) $b_\phi (A) = A \cap D_\phi (X \setminus A)$

**Proof.**

1) Since $\text{Int}(A) \subseteq \text{Int}_\phi (A)$, we have
$b_\phi (A) = A \setminus \text{Int}_\phi (A) \subseteq A \setminus \text{Int}(A) = b(A)$.
2) and (3). Straightforward.

4) Since $\text{Int}_\phi (A) \subseteq A$, it follows from Theorem 3.6 (2). That $A$ is $ii$-open $\iff A = \text{Int}_\phi (A) \iff b_\phi (A) = A \setminus \text{Int}_\phi (A) = \phi$.
5) Since $\text{Int}_\phi (A)$ is $ii$-open, it follows from (4) that $b_\phi (\text{Int}_\phi (A)) = \phi$.
6) If $x \in \text{Int}_\phi (b_\phi (A))$, then $x \in b_\phi (A)$. On the other hand, since $b_\phi (A) \subseteq A$, $x \in \text{Int}_\phi (b_\phi (A)) \subseteq \text{Int}_\phi (A)$. Hence, $x \in \text{Int}_\phi (A) \cap b_\phi (A)$.
Which contradicts (3). Thus $\text{Int}_\phi (b_\phi (A)) = \phi$.
7) Using (6), we get $b_\phi (b_\phi (A)) = b_\phi (A) \setminus \text{Int}_\phi (b_\phi (A)) = b_\phi (A)$.
8) Using Theorem 3.6 (6), we have
$b_\phi (A) = A \setminus \text{Int}_\phi (A) = A \setminus \text{CL}_\phi (X \setminus A) = A \setminus \text{CL}_\phi (X \setminus A)$
9) Applying (8) and the Theorem (3.4), we have
$b_\phi (A) = A \setminus \text{CL}_\phi (X \setminus A) = A \setminus ((X \setminus A) \cup D_\phi (X \setminus A)) = A \setminus D_\phi (X \setminus A)$.

**Example 3.9.** Consider the topological space $(X, \tau)$ given in Example (3.3). If $A = \{a, b\}$, then $b_\phi (A) = \phi$ and $b(A) = \{a\}$. Hence, $b(A) \subset b_\phi (A)$, that is, in general, the converse Theorem 3.9 (1) may not be true.

**Definition 3.10.** $\text{Fr}_\phi (A) = \text{CL}_\phi (A) \setminus \text{Int}_\phi (A)$ is said to be the $ii$-frontier of $A$.

**Theorem 3.11.** For a subset $A$ of a space $X$, the following statements hold:

1) $\text{Fr}_\phi (A) \subseteq \text{Fr}(A)$ where $\text{Fr}(A)$ denotes the frontier of $A$
2) $\text{CL}_\phi (A) = \text{Int}_\phi (A) \cup \text{Fr}_\phi (A)$
3) $\text{Int}_\phi (A) \cap \text{Fr}_\phi (A) = \phi$
4) $b_\phi (A) \subseteq \text{Fr}_\phi (A)$
5) $\text{Fr}_\phi (A) = b_\phi (A) \cup D_\phi (A)$
6) $\text{Fr}_\phi (A) = D_\phi (A)$ if and only if $A$ is $ii$-open set
7) $\text{Fr}_\phi (A) = \text{CL}_\phi (A) \cup \text{CL}_\phi (X \setminus A)$
8) $\text{Fr}_\phi (A) = \text{Fr}(X \setminus A)$
9) $\text{Fr}_\phi (A)$ is $ii$-closed
10) $\text{Fr}_\phi (\text{Fr}_\phi (A)) \subseteq \text{Fr}_\phi (A)$
11) $\text{Fr}_\phi (\text{Int}_\phi (A)) \subseteq \text{Fr}_\phi (A)$
12) $\text{Fr}_\phi (\text{CL}_\phi (A)) \subseteq \text{Fr}_\phi (A)$
13) $\text{Int}_\phi (A) = A \setminus \text{Fr}_\phi (A)$

**Proof.**

1) Since $\text{CL}_\phi (A) \subseteq \text{CL}(A)$ and $\text{Int}(A) \subseteq \text{Int}_\phi (A)$, it follows that
$\text{Fr}_\phi (A) = \text{CL}_\phi (A) \setminus \text{Int}_\phi (A) \subseteq \text{CL}(A) \setminus \text{Int}(A) \subseteq \text{CL}(A) \setminus \text{Int}(A) \subseteq \text{Fr}(A)$.
2) $\text{Int}_\phi (A) \cup \text{Fr}_\phi (A) = \text{Int}_\phi (A) \cup (\text{CL}_\phi (A) \setminus \text{Int}_\phi (A)) = \text{CL}_\phi (A)$. 

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3) \( \text{Int}_u(A) \cap \text{Fr}_u(A) = \text{Int}_u(A) \cap \left( \text{CL}_u(A) \setminus \text{Int}_u(A) \right) = \phi \).

4) Since \( A \subseteq \text{CL}_u(A) \), we have
\[
\text{b}_u(A) = A \setminus \text{Int}_u(A) \subseteq \text{CL}_u(A) \setminus \text{Int}_u(A) = \text{Fr}_u(A).
\]

5) Since \( \text{Int}_u(A) \cup \text{Fr}_u(A) = \text{Int}_u(A) \cup \text{b}_u(A) \cup \text{D}_u(A) \),
\[
\text{Fr}_u(A) = \text{b}_u(A) \cup \text{D}_u(A).
\]

6) Assume that \( A \) is \( ii \)-open. Then
\[
\text{Fr}_u(A) = \text{b}_u(A) \cup \text{D}_u(A) \setminus \text{Int}_u(A) = \phi \cup \left( \text{D}_u(A) \setminus A \right) = D_u(A) \setminus A = b_u(X \setminus A),
\]
by using (5), Theorem 3.6 (2), Theorem 3.8 (4) and Theorem 3.8 (9).

Conversely, suppose that \( \text{Fr}_u(A) = b_u(X \setminus A) \). Then
\[
\phi = \text{Fr}_u(A) \setminus b_u(X \setminus A) = \left( \text{CL}_u(A) \setminus \text{Int}_u(A) \right) \setminus \text{Int}_u(X \setminus A) = \text{Int}_u(X \setminus A),
\]
by using (4) and (5) of Theorem 3.6, and so \( A \subseteq \text{Int}_u(A) \). Since \( \text{Int}_u(A) \subseteq A \) in general, it follows that \( \text{Int}_u(A) = A \) so from Theorem 3.6 (2) that \( A \) is \( ii \)-open.

7) \( \text{Fr}_u(A) = \text{CL}_u(A) \setminus \text{Int}_u(A) = \text{CL}_u(A) \cap \left( \text{CL}_u(X \setminus A) \right) \).

8) It follows from (7).

9) \( \text{CL}_u(\text{Fr}_u(A)) = \text{CL}_u(\text{CL}_u(A)) \cap \left( \text{CL}_u(X \setminus A) \right) \). Hence, \( \text{Fr}_u(A) \) is \( ii \)-closed.

10) \[
\text{Fr}_u(\text{Fr}_u(A)) = \text{CL}_u(\text{Fr}_u(A)) \cap \text{CL}_u(X \setminus \text{Fr}_u(A)) \subseteq \text{CL}_u(\text{Fr}_u(A)) = \text{Fr}_u(A).
\]

11) Using Theorem 3.6 (3), we get
\[
\text{Fr}_u(\text{Int}_u(A)) = \text{CL}_u(\text{Int}_u(A)) \setminus \text{Int}_u(\text{Int}_u(A)) \subseteq \text{CL}_u(\text{Int}_u(A)) \setminus \text{Int}_u(A) = \text{Fr}_u(A).
\]

12) \[
\text{Fr}_u(\text{CL}_u(A)) = \text{CL}_u(\text{CL}_u(A)) \setminus \text{Int}_u(\text{CL}_u(A)) = \text{CL}_u(\text{CL}_u(A) \setminus \text{Int}_u(\text{CL}_u(A)) = \text{Fr}_u(A).
\]

13) \( A \setminus \text{Fr}_u(A) = (A \setminus \text{CL}_u(A)) \setminus \text{Int}_u(A) = \text{Int}_u(A) \).

The converses of (1) and (4) of Theorem 3.11 are not true in general, as shown by Example

**Example 3.12.** Consider the topological space \((X, \tau)\) given in Example 3.3. If \( A = \{ c \} \), then \( \text{Fr}_u(A) = \{ a, c \} \not\subseteq \{ c \} = \text{Fr}_u(A) \), and if \( B = \{ a, b \} \), then \( \text{Fr}_u(B) = \{ c \} \not\subseteq \text{Fr}_u(B) \).

**Definition 3.13.** \( \text{Ext}_u(A) = \text{Int}_u(X \setminus A) \) is said to be an \( ii \)-exterior of \( A \).

**Theorem 3.14.** For a subset \( A \) of a space \( X \), the following statements hold:

1) \( \text{Ext}(A) \subseteq \text{Ext}_u(A) \) where \( \text{Ext}(A) \) denotes the exterior of \( A \)
2) \( \text{Ext}_u(A) \) is \( ii \)-open
3) \( \text{Ext}_u(A) = \text{Int}_u(X \setminus A) = X \setminus \text{CL}_u(A) \)
4) \( \text{Ext}_u(\text{Ext}_u(A)) = \text{Int}_u(\text{CL}_u(A)) \)
5) If \( A \subseteq B \), then \( \text{Ext}_u(A) \supseteq \text{Ext}_u(B) \)
6) \( \text{Ext}_u(A \cup B) \subseteq \text{Ext}_u(A) \cup \text{Ext}_u(B) \)
7) \( \text{Ext}_u(A \cap B) \supseteq \text{Ext}_u(A) \cap \text{Ext}_u(B) \)
8) \( \text{Ext}_u(X) = \phi \)
9) \( \text{Ext}_u(\phi) = X \)
10) \( \text{Ext}_u(A) = \text{Ext}_u(X \setminus \text{Ext}_u(A)) \)
11) \( \text{Int}_i(A) \subseteq \text{Ext}_i(\text{Ext}_i(A)) \)

12) \( X = \text{Ext}_i(A) \cup \text{Ext}_i(A) \cup \text{Fr}_i(A) \)

**Proof.** 1) It follows from Theorem 3.6 (1).

2) It is straightforward by Theorem 3.6 (6).

3) \( \text{Ext}_i(A) = \text{Ext}_i(X \setminus \text{CL}_i(A)) \)

4) Assume that \( A \subset B \). Then \( \text{Ext}_i(B) = \text{Ext}_i(X \setminus B) \subseteq \text{Ext}_i(X \setminus A) = \text{Ext}_i(A) \), by using Theorem 3.6 (7).

5) Applying Theorem 3.6 (8), we get

\[
\text{Ext}_i(A \cup B) = \text{Int}_i(X \setminus (A \cup B)) = \text{Int}_i((X \setminus A) \cup (X \setminus B))
\]

\[
\subseteq \text{Int}_i(X \setminus A) \cup \text{Int}_i(X \setminus B) = \text{Ext}_i(A) \cup \text{Ext}_i(B).
\]

6) Applying Theorem 3.6 (9), we obtain

\[
\text{Ext}_i(A \cap B) = \text{Int}_i(X \setminus (A \cap B)) = \text{Int}_i((X \setminus A) \cap (X \setminus B))
\]

\[
\supset \text{Int}_i(X \setminus A) \cap \text{Int}_i(X \setminus B) = \text{Ext}_i(A) \cap \text{Ext}_i(B).
\]

7) Straightforward.

8) Straightforward.

9) \( \text{Ext}_i(X \setminus \text{Ext}_i(A)) = \text{Ext}_i(X \setminus \text{Int}_i(X \setminus A)) = \text{Int}_i(X \setminus \text{Int}_i(X \setminus A)) \)

\[
= \text{Int}_i(X \setminus A) = \text{Ext}_i(A).
\]

10) \( \text{Int}_i(X \setminus \text{Ext}_i(A)) = \text{Ext}_i(\text{Ext}_i(A)) \).

**Example 3.15.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, \{c, d\}\} \). Thus, \( \text{Int}(X) = \{\emptyset, X, \{c, d\}, \{b, c, d\}, \{a, c, d\}\} \). If \( A = \{a\} \) and \( B = \{b\} \). Then \( \text{Ext}_i(A) \not\subseteq \text{Ext}(A) \). \( \text{Ext}_i(A \cap B) \neq \text{Ext}_i(A) \cap \text{Ext}_i(B) \) and \( \text{Ext}_i(A \cup B) \neq \text{Ext}_i(A) \cup \text{Ext}_i(B) \).

**4. A New Decomposition of \( ii \)-Continuity**

We begin by the following definition:

**Definition 4.1.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called \( ii \)-continuous if \( f^{-1}(G) \) is \( ii \)-open set in \( (X, \tau) \) for any open set \( G \) of \( (Y, \sigma) \).

**Theorem 4.2.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function then:

1) Every continuous function is an \( ii \)-continuous,
2) Every \( ii \)-continuous function is an \( i \)-continuous,
3) Every \( a \)-continuous function is an \( ii \)-continuous.

**Proof.** 1) Let \( G \) be open set in \( (Y, \sigma) \). Since \( f \) is continuous, it follows that \( f^{-1}(G) \) is open set in \( (X, \tau) \). But every open set is \( ii \)-open set [4]. Hence \( f^{-1}(G) \) is \( ii \)-open set in \( (X, \tau) \). Thus \( f \) is \( ii \)-continuous.

2) Let \( G \) be open set in \( (Y, \sigma) \). Since \( f \) is an \( ii \)-continuous, it follows that \( f^{-1}(G) \) is an \( ii \)-open set in \( (X, \tau) \). But every \( ii \)-open set is \( i \)-open set [4]. Hence \( f^{-1}(G) \) is \( i \)-open set in \( (X, \tau) \). Thus \( f \) is \( i \)-continuous.

3) Let \( G \) be open set in \( (Y, \sigma) \). Since \( f \) is \( a \)-continuous, it follows that
$f^{-1}(G)$ is $a$-open set in $(X, \tau)$. But every $a$-open set is $ii$-open set [4]. Hence $f^{-1}(G)$ is $ii$-open set in $(X, \tau)$. Thus $f$ is an $ii$-continuous.

The converse need not be true by the following example.

**Example 4.3.** Let
$$X = \{a, b, c, d\}, \quad \tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}\}$$

and
$$Y = \{a, b, c, d\}, \quad \sigma = \{\phi, Y, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b, d\}\}$$

and
$$ii o(x) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\},$$

$$io(x) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\},$$

$$ao(x) = \{\phi, X, \{a\}, \{b\}, \{a, c\}, \{a, b, d\}, \{a, b, c\}, \{a, b, c, d\}\}.$$ Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function then $f^{-1}(\{a\}) = \{a\}$, $f^{-1}(\{b\}) = \{b\}$, $f^{-1}(\{c\}) = \{c\}$, $f^{-1}(\{d\}) = \{d\}$. Then $f$ is $ii$-continuous, but $f$ is not $a$-continuous, since for the open set $\{a, d\}$ in $(Y, \sigma)$, $f^{-1}(\{a, d\}) = \{a, d\}$ is not $a$-open in $(X, \tau)$ and $f$ is not continuous, since for the open set $\{a, d\}$ in $(Y, \sigma)$, $f^{-1}(\{a, d\}) = \{a, d\}$ is not open in $(X, \tau)$.

Now when $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f^{-1}(\{a\}) = \{b\}$, $f^{-1}(\{b\}) = \{a\}$, $f^{-1}(\{c\}) = \{d\}$, $f^{-1}(\{d\}) = \{c\}$ we get $f$ is $i$-continuous, but $f$ is not $ii$-continuous, since for the open set $\{a, d\}$ in $(Y, \sigma)$, $f^{-1}(\{a, d\}) = \{b, d\}$ is not $ii$-open in $(X, \tau)$.

**Theorem 4.4.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function then every semi-continuous function is an $ii$-continuous.

**Proof.** Let $G$ be open set in $(Y, \sigma)$. Since $f$ is semi-continuous, it follows that $f^{-1}(G)$ is semi-open set in $(X, \tau)$. But every semi-open set is $ii$-open set [4]. Hence $f^{-1}(G)$ is $ii$-open set in $(X, \tau)$. Thus $f$ is an $ii$-continuous.

**Definition 4.5.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $int$-continuous if $f^{-1}(G)$ is int-open set in $(X, \tau)$ for any open set $G$ in $(Y, \sigma)$.

**Theorem 4.6.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function then:

1) Every continuous function is $int$-continuous,
2) Every $ii$-continuous function is $int$-continuous,
3) Every $a$-continuous function is $int$-continuous.

**Proof.**
1) Let $G$ be open set in $(Y, \sigma)$. Since $f$ is continuous, it follows that $f^{-1}(G)$ is open set in $(X, \tau)$. But every open set is $int$-open set [4]. Hence $f^{-1}(G)$ is $int$-open set in $(X, \tau)$. Thus $f$ is $int$-continuous.

2) Let $G$ be open set in $(Y, \sigma)$. Since $f$ is $ii$-continuous, it follows that $f^{-1}(G)$ is an $ii$-open set in $(X, \tau)$. But every $ii$-open set is $int$-open set [4]. Hence $f^{-1}(G)$ is $int$-open set in $(X, \tau)$. Thus $f$ is $int$-continuous.

3) Let $G$ be open set in $(Y, \sigma)$. Since $f$ is $a$-continuous, it follows that $f^{-1}(G)$ is $a$-open set in $(X, \tau)$. But every $a$-open set is $int$-open set [4]. Hence $f^{-1}(G)$ is $int$-open set in $(X, \tau)$. Thus $f$ is $int$-continuous.

The converse need not be true by the following example.
Example 4.7. Let \( X = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}, \{b, c\}\} \) and \( Y = \{a, b, c\} \), \( \sigma = \{\phi, Y, \{a\}, \{a, c\}\} \) and \( \text{into}(x) = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}, \{a, c\}\} \), \( \text{ii}(x) = \alpha(x) = \{\phi, X, \{a\}, \{b, c\}\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be the identity function then \( f^{-1}(\{a\}) = \{a\} \), \( f^{-1}(\{b\}) = \{b\} \), \( f^{-1}(\{c\}) = \{c\} \). Then \( f \) is int-continuous, but \( f \) is not ii-continuous, since for the open set \( \{a, c\} \) in \( (Y, \sigma) \), \( f^{-1}(\{a, c\}) = \{a, c\} \) is not \( ii \)-open in \( (X, \tau) \) and \( f \) is not continuous, since for the open set \( \{a, c\} \) in \( (Y, \sigma) \), \( f^{-1}(\{a, c\}) = \{a, c\} \) is not open in \( (X, \tau) \) and \( f \) is not \( a \)-continuous, since for the open set \( \{a, c\} \) in \( (Y, \sigma) \), \( f^{-1}(\{a, c\}) = \{a, c\} \) is not \( a \)-open.

Definition 4.8. A subset \( A \) of \( X \) is called weakly \( ii \)-open set if \( A \) is \( ii \)-open set and \( \text{Int} A \subseteq \text{CL} \text{Int} A \). A subset \( A \) of a space \( X \) is \( a \)-open set if and only if \( A \) is weakly \( ii \)-open.

Proof. Let \( A \) be \( a \)-open set. Since \( \text{Int} \text{CL} \text{Int} (A) \subseteq A \subseteq \text{CL} (A) \). Therefore \( A \subseteq \text{CL} (\text{Int} (A)) \cap \text{CL} (A) \), this implies that \( A \subseteq \text{CL} (\text{Int} (A) \cap A) \). Now, put \( G = \text{Int} (A) \) where \( G \neq \phi, X \), then \( A \) is \( ii \)-open set. Therefore, \( A \) is weakly \( ii \)-open set.

Conversely, Let \( A \) be weakly \( ii \)-open set, then there exist an open set \( G \neq \phi, X \), such that \( G = \text{Int} (A) \) satisfying \( A \subseteq \text{CL} (\text{Int} (A) \cap A) \) and \( A \) is \( ii \)-open set. Since \( A \subseteq \text{CL} (\text{Int} (A) \cap A) \), this implies that \( A \subseteq \text{CL} (\text{Int} (A)) \) and \( \text{Int} (A) \subseteq \text{CL} (\text{Int} (A)) \). Since \( A \) is \( ii \)-open set, using (2) from Theorem (3.6), we get \( A = \text{Int} (A) \). Therefore \( A \subseteq \text{Int} (\text{CL} (\text{Int} (A))) \). Thus \( A \) is \( a \)-open set.

As a summary the following Figure 1 shows the relations among semi-continuous, \( ii \)-continuous, \( i \)-continuous, int-continuous, \( a \)-continuous and continuous.

Figure 1. Relations among semi-continuous, \( ii \)-continuous, \( i \)-continuous, int-continuous, \( a \)-continuous and continuous.
**Corollary 4.10.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \alpha \)-continuous if and only if it is weakly \( ii \)-continuous.

Proof. Clear from Theorem 4.9.

5. \( ii \)-Separating Axioms

In this section we define \( T_{0ii} \) and \( T_{1ii} \) spaces for \( ii \)-open sets and we determine them by giving many examples. Specially, we define \( T_{1ii} \) spaces to compare them with \( T_{0ii} \) space.

**Definition 5.1.** A topological space \( X \) is called

1) \( T_{0ii} \) if \( a, b \) are two distinct points in \( X \), there exist \( ii \)-open set \( U \) such that either \( a \in U \) and \( b \notin U \), and \( b \in U \) and \( a \notin U \).

2) \( T_{1ii} \) if \( a, b \in X \) and \( a \neq b \), there exist \( ii \)-open sets \( U, V \) containing \( a, b \) respectively, such that \( b \notin U \) and \( a \notin V \).

**Example 5.2.** Let \( X = \{a, b\}, \tau = \tau = \phi, X, \{a\}, \{b\} \) \( (X, \tau) \) and \( (X, \tau) \) are topological spaces.

1) \( a, b \in X \) \( (a \neq b) \) there exists \( \{a\} \in \tau \) such that \( a \in \{a\}, b \notin \{a\} \).

Therefore \( (X, \tau) \) is \( T_{0ii} \).

2) \( a, b \in X \) \( (a \neq b) \) there exists \( \{a\}, \{b\} \in \tau \) such that \( a \in \{a\}, b \in \{b\} \).

Therefore \( (X, \tau) \) is \( T_{1ii} \).

**Theorem 5.3.**

1) Every \( T_{0} \) -space is \( T_{0ii} \) -space,
2) Every \( T_{1} \) -space is \( T_{0ii} \) -space,
3) Every \( T_{ii} \) -space is \( T_{1ii} \) -space,
4) Every \( T_{1ii} \) -space is \( T_{0ii} \) -space.

Proof. (1), (2), (3) and (4) follow using the fact that every open set is \( ii \)-open [4]. The converse needs not to be true by the following example.

**Example 5.4.** Let

\[ X = \{a, b, c\}, \tau = \{\phi, X, \{a\}\} \] and \( \tau' = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\} \).

\( (X, \tau) \) and \( (X, \tau') \) are topological spaces.

\( (X, \tau) \) is not \( T_{0ii} \) -space because, \( b, c \in X \) \( (b \neq c) \) there is no open set \( G \) such that \( b \in G, c \notin G \).

\( (X, \tau) \) is \( T_{0ii} \) -space because, \( a, b \in X \) \( (a \neq b) \) there exists \( \{a\} \in \tau' \) such that \( a \in \{a\}, b \notin \{a\} \).

\( a, c \in X \) \( (a \neq c) \) there exists \( \{a\} \in \tau' \) such that \( a \in \{a\}, c \notin \{a\} \).

\( b, c \in X \) \( (b \neq c) \) there exists \( \{a, b\} \in \tau' \) such that \( b \in \{a, b\}, c \notin \{a, b\} \).

\( (X, \tau) \) is not \( T_{1ii} \) -space because, \( a, b \in X \) \( (a \neq b) \) there exists \( X \in \tau \) such that \( a \in X, b \in X \).

\( (X, \tau) \) is not \( T_{1ii} \) -space because, \( b, a \in X \) \( (a \neq b) \) there exists \( \{a, b\} \in \tau' \) such that \( a \in \{a, b\}, b \notin \{a, b\} \).

**Theorem 5.5.** Every \( T_{1ii} \) -space is \( T_{1ii} \) -space.

Proof. Let \( X \) be \( T_{1ii} \) -space. Let \( a, b \) be two distinct points in \( X \). Since \( X \) is \( T_{1ii} \) -space there exist two \( \alpha \)-open sets \( U, V \) in \( X \) such that \( a \in U, b \notin U, a \notin V \),
Since every \(a\)-open set is \(ii\)-open set [4], \(U, V\) is an \(ii\)-open set in \(X\). Hence \(X\) is \(T_{ii}\)-space.

**Theorem 5.6.** Every \(T_{ii}\)-space is \(T_i\)-space.

**Proof.** Let \(X\) be a \(T_{ii}\)-space. Let \(a, b\) be two distinct points in \(X\). Since \(X\) is \(T_{ii}\)-space there exist two \(ii\)-open sets \(U, V\) in \(X\) such that \(a \in U\), \(b \notin U\), \(a \notin V\), \(b \in V\). Since every \(ii\)-open set is \(i\)-open set [4], \(U, V\) is an \(i\)-open set in \(X\). This is contradiction. Therefore \(X\) is \(T_{ii}\)-space.

Similarly, \(X\) is \(T_{ii}\)-space. Let \(a, b\) be two distinct points in \(X\). Since \(X\) is \(T_{ii}\)-space because, \(a, b \in X\) (\(a \neq b\)) there exists \(\{a\}, \{b\} \in \tau'\) such that \(a \in \{a\}\), \(b \notin \{a\}\) and \(a \notin \{b\}\). \(a, c \in X\) (\(a \neq c\)) there exists \(\{a\}, \{c\} \in \tau'\) such that \(a \in \{a\}\), \(c \notin \{a\}\) and \(c \notin \{c\}\).

\(b, c \in X\) (\(b \neq c\)) there exists \(\{c\}, \{b\} \in \tau'\) such that \(c \in \{c\}\), \(b \notin \{c\}\) and \(b \notin \{b\}\), \(c \in \{b\}\).

\((X, \tau)\) is not \(T_{ii}\)-space because, \(b, c \in X\) (\(c \neq b\)) there exists \(\{b\}, \{c\} \in \tau^a\) such that \(c \in \{b\}, b \notin \{c\}\).

**Theorem 5.8.** A space \(X\) is \(T_{oi}\)-space if and only if \(CL_0(\{x\}) \neq CL_0(\{y\})\) for every pair of distinct points \(x, y\) of \(X\).

**Proof.** Let \(X\) be a \(T_{oi}\)-space. Let \(x, y \in X\) such that \(x \neq y\), then there exists an \(ii\)-open set \(U\) containing one of the points but not the other, then \(x \in U\) and \(y \notin U\). Then \(X \setminus U\) is \(ii\)-closed set containing \(y\) but not \(x\). But \(CL_0(\{y\})\) is the smallest \(ii\)-closed set containing \(y\); Therefore \(CL_0(\{y\}) \subset X \setminus U\) and hence \(x \notin CL_0(\{y\})\). Thus \(CL_0(\{x\}) \neq CL_0(\{y\})\).

Conversely, Suppose for any \(x, y \in X\) with \(x \neq y\), \(CL_0(\{x\}) \neq CL_0(\{y\})\). Let \(z \in X\) such that \(z \in CL_0(\{x\})\) but \(z \notin CL_0(\{y\})\). If \(x \in CL_0(\{y\})\) then \(CL_0(\{x\}) \subset CL_0(\{y\})\) and hence \(z \in CL_0(\{y\})\). This is contradiction. Therefore \(X \notin CL_0(\{y\})\). Therefore \(X \setminus CL_0(\{y\})\) is \(ii\)-open set containing \(x\) but not \(y\); Hence \(X\) is an \(T_{oi}\)-space.

**Theorem 5.9.** A space \((X, \tau)\) is \(T_{ii}\)-space if and only if the singletons are \(ii\)-closed sets.

**Proof.** Let \(X\) be \(T_{ii}\)-space and let \(x \in X\), to prove that \(\{x\}\) is \(ii\)-closed set. We will prove \(X \setminus \{x\}\) is \(ii\)-open set in \(X\). Let \(y \in X \setminus \{x\}\), implies \(x \neq y\) and since \(X\) is \(T_{ii}\)-space then their exist two \(ii\)-open sets \(U, V\) such that \(x \notin U\), \(y \in V \subset X \setminus \{x\}\). Since \(y \in V \subset X \setminus \{x\}\), then \(X \setminus \{x\}\) is \(ii\)-open set. Hence \(\{x\}\) is \(ii\)-closed set.

Conversely, Let \(x \neq y \in X\) then \(\{x\}, \{y\}\) are \(ii\)-closed sets. That is \(X \setminus \{x\}\) is \(ii\)-open set clearly, \(x \notin X \setminus \{x\}\) and \(y \in X \setminus \{x\}\). Similarly \(X \setminus \{y\}\) is \(ii\)-open set, \(y \notin X \setminus \{y\}\) and \(x \in X \setminus \{y\}\). Hence \(X\) is an \(T_{ii}\)-space.

As a consequence the following Figure 2 shows the relations among \(T_0, T_{oi}\), \(T_i, T_{ii}\) and \(T_{ii}\).
Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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