Congruence classes of triangles in $\mathbb{F}_p^2$

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Abstract

In this short note, we give a lower bound on the number of congruence classes of triangles in a small set of points in $\mathbb{F}_p^2$. More precisely, for $A \subset \mathbb{F}_p^2$ with $|A| \leq p^{2/3}$, we prove that the number of congruence classes of triangles determined by points in $A \times A$ is at least $|A|^{7/2}$. This note is not intended for journal publication.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of order $q$ with $q = p^r$. Two $k$-simplices in $\mathbb{F}_q^2$ with vertices $(x_1, \ldots, x_{k+1}), (y_1, \ldots, y_{k+1})$ are in the same congruence class if there exist an orthogonal matrix $\theta$ and a vector $z \in \mathbb{F}_d^q$ such that $z + \theta(x_i) = y_i$ for all $i = 1, 2, \ldots, k+1$. Let $T_{k,d}(E)$ denote the set of congruence classes of $k$-simplices determined by points in $E$. In this paper, we use the notation $X \ll Y$ which means that there exists a positive absolute constant $C$ that does not depend on $X, Y$ and $q$ such that $X \leq CY$.

The first result on the size of $T_{k,d}(E)$ in $\mathbb{F}_q^d$ was established by Hart and Iosevich [4]. They indicated that if $|E| \gg q^{\frac{kd}{2}+\frac{k}{2}}$ with $d \geq \binom{k+1}{2}$, then $|T_{k,d}(E)| \gg q^{\frac{k+1}{2}}$. By employing techniques from spectral graph theory, the second listed author [11] proved that $E$ also contains a copy of all $k$-simplices with non-zero edges under the conditions $d \geq 2k$ and $|E| \gg q^{(d-1)/2+k}$. It follows from the results of Hart and Iosevich [4] and Vinh [11] that we only can get a non-trivial result on the number of congruence classes of triangles when $d \geq 4$. There are several progresses on improving this problem. The result, which we have to mention first, is a result due to Covert et al. [3]. They proved that if $|E| \gg \rho q^2$, then $|T_{2,2}(E)| \gg \rho q^3$. This means that in order to get a constant factor of all congruence classes of triangles in $\mathbb{F}_p^2$, we need the condition $|E| \gg q^2$. In 2014, Bennett, Iosevich and Pakianathan [1], using Elekes and Sharir’s approach in the Erdős distinct distance problem, improved this result, namely, they showed that the condition $|E| \geq q^{7/4}$ is enough to get at least $cq^2$ congruence classes of triangles for some absolute positive constant $c$. Recently, by using Fourier analytic methods and elementary results from group action theory, Bennett, Iosevich, Pakianathan, and Rudnev [2] improved the threshold $q^{7/4}$ to $q^{8/5}$. In the case $E$ is the Cartesian product of sets with different sizes, the authors and Hiep [8] obtained the following improvement.
Theorem 1.1 (Theorem 1.2, [8]). Let $\mathcal{E} = \mathcal{A}_1 \times \mathcal{A}_2$ be a subset in $\mathbb{F}_q^2$. Suppose that 
\[(\min_{1 \leq i \leq 2} |\mathcal{A}_i|)^{-1/2} |\mathcal{E}|^{k+1} \gg q^{2k},\]
then for $1 \leq k \leq 2$, we have
\[|T_{k,2}(\mathcal{E})| \gg q^{(k+1)/2} \cdot |\mathcal{E}|.\]

The following is a consequence of Theorem 1.1.

Theorem 1.2 ([8]). Let $\mathcal{A}, \mathcal{B}$ be subsets in $\mathbb{F}_q$. If $|\mathcal{A}| \geq q^{1/2+\epsilon}$ and $|\mathcal{B}| \geq q^{1-2\epsilon}$ for some $\epsilon \geq 0$, then we have
\[|T_{2,2}(\mathcal{A} \times \mathcal{B})| \gg q^3.\]

We note that in [2], Bennett et al. gave a construction of $|\mathcal{A}| = q^{1/2-\epsilon'}$ and $|\mathcal{B}| = q$, and the number of congruence classes triangles determined by $\mathcal{A} \times \mathcal{B}$ is at most $c q^{3-\epsilon''}$ for $\epsilon'' > 0$. On the other hand, it follows from Theorem 1.2 that if $|\mathcal{A}| < q^{1/2}$ then we must have $|\mathcal{B}| > q$ to guarantee that $|T_{2,2}(\mathcal{A} \times \mathcal{B})| \gg q^3$. Hence, the condition of $|\mathcal{A}| \cdot |\mathcal{B}|$ in Theorem 1.2 is sharp up to a factor of $q^{1/3}$. From the construction in [2], Iosevich [6] conjectured that the right size of a general set $\mathcal{E} \subseteq \mathbb{F}_q^2$ for getting at least $c q^3$ congruence classes of triangles is around $q^{3/2}$.

The main purpose of this short note is to give a lower bound on the number of congruence classes of triangles in $\mathcal{A} \times \mathcal{A}$ when $\mathcal{A}$ is a small set of points.

Theorem 1.3. Let $\mathbb{F}_p$ be a field of odd prime $p$. For $\mathcal{A} \subseteq \mathbb{F}_p$ with $|\mathcal{A}| \leq p^{2/3}$, we have
\[|T_{2,2}(\mathcal{A} \times \mathcal{A})| \geq |\mathcal{A}|^{7/2}.\]

2 Two proofs of Theorem 1.3

The first proof: Let $\mathbb{F}_p$ be a prime field of order $p$. For $\mathcal{E} \subset \mathbb{F}_p^2$ and $t \in \mathbb{F}_p$, we define $\nu_\mathcal{E}(\lambda)$ as the cardinality of the set \[\{(x, y) \in \mathcal{E} \times \mathcal{E} : ||x - y|| = \lambda\}.\] In order to prove Theorem 1.3, we first need the following lemmas.

Lemma 2.1 ([8]). Let $\mathcal{E}$ be a subset in $\mathbb{F}_p^2$. For a fixed $\lambda \in \mathbb{F}_p$, denote by $H_\lambda(\mathcal{E})$ the number of hinges of the form $(p, q_1, q_2) \in \mathcal{E} \times \mathcal{E} \times \mathcal{E}$ with $||p - q_1|| = ||p - q_2|| = \lambda$. We have the following estimate
\[\sum_{\lambda \in \mathbb{F}_p} \nu_\mathcal{E}(\lambda)^2 \ll \frac{|\mathcal{E}|}{4} \sum_{\lambda \in \mathbb{F}_p} H_\lambda(\mathcal{E}) + |\mathcal{E}|^2.\] (2.1)

By applying the recent Rudnev’s breakthrough on the number of incidences between points and planes in $\mathbb{F}_p^3$ in [9], Petridis [7] obtained the following.

Lemma 2.2 (Petridis, [7]). Let $\mathbb{F}_p$ be a field of order odd prime $p$. For $\mathcal{A} \subseteq \mathbb{F}_p$ with $|\mathcal{A}| \leq p^{2/3}$, we have
\[\sum_{\lambda \in \mathbb{F}_p} H_\lambda(\mathcal{A} \times \mathcal{A}) \ll |\mathcal{A}|^{9/2}.\]
As a consequence of Theorem 2.2, Petridis [7] proved the following result on the number of distinct distances in \( A \times A \).

**Theorem 2.3 (Petridis, [7]).** Let \( \mathbb{F}_p \) be a field of order odd prime \( p \). For \( A \subseteq \mathbb{F}_p \) with \( |A| \leq p^{2/3} \), we have the number of distinct distances determined by points in \( A \times A \) is at least

\[
\min\{p, |A|^{3/2}\}.
\]

Combining Lemma 2.2 and Lemma 2.1, we obtain the following.

**Lemma 2.4.** Let \( \mathbb{F}_p \) be a field of order odd prime \( p \). For \( A \subseteq \mathbb{F}_p \) with \( |A| \leq p^{2/3} \), we have

\[
\sum_{\lambda \in \mathbb{F}_p} \nu_{A \times A}(\lambda)^2 \ll |A|^{13/2}. \tag{2.2}
\]

**Proof of Theorem 1.3.** On one hand, by applying the Cauchy-Schwarz inequality, we have

\[
|T_{2,2}(A \times A)| \geq \frac{|A|^{12}}{N},
\]

where \( N \) is the number of pairs of congruence triangles. On the other hand, two triangles, which are denoted by \( \Delta(a_1, a_2, a_3) \) and \( \Delta(b_1, b_2, b_3) \), are in the same congruence class if there exist an orthogonal matrix \( M \) and a vector \( z \in \mathbb{F}_p^2 \) such that

\[
Ma_i + z = b_i, \quad 1 \leq i \leq 3.
\]

This implies that

\[
N \leq |A \times A| \sum_{\lambda \in \mathbb{F}_p} \nu_{A \times A}(\lambda)^2.
\]

Thus, it follows from Lemma 2.4 that

\[
N \leq |A \times A| \sum_{\lambda \in \mathbb{F}_p} \nu_{A \times A}(\lambda)^2 = |A|^{7/2}.
\]

In other words,

\[
|T_{2,2}(A \times A)| \geq |A|^{7/2},
\]

and the theorem follows. \( \square \)

**The second proof:** The following lemma was suggested by Frank de Zeeuw.

**Lemma 2.5.** Let \( \mathbb{F}_p \) be a field of order odd prime \( p \). For \( A \subseteq \mathbb{F}_p \), we have

\[
|T_{2,2}(A \times A)| \geq \frac{1}{6}(|A|^2 - 2) \cdot |\Delta_{\mathbb{F}_p}(A \times A)|,
\]

where \( \Delta_{\mathbb{F}_p}(A \times A) \) is the set of distinct distances determined by points in \( A \times A \).
Proof. Suppose that $m = |\Delta_{F_p}(A \times A)|$, then there exist $m$ segments with distinct distances. For each segment, it is easy to check that there are at least $((|A|^2 - 2)/2)$ congruence classes of triangles with the same base. On the other hand, when we count congruence classes of triangles with the bases from the set of $m$ segments, each class will be repeated at most $3$ times. This implies that

$$|T_{2,2}(A \times A)| \geq \frac{1}{6}(|A|^2 - 2) \cdot |\Delta_{F_p}(A \times A)|,$$

which concludes the proof of the lemma.

From Theorem 2.3 and Lemma 2.5 we are able to obtain a slight weaker version of Theorem 1.3 as follows.

**Theorem 2.6.** Let $F_p$ be a field of odd prime $p$. For $A \subseteq F_p$ with $|A| \leq p^{2/3}$, we have

$$|T_{2,2}(A \times A)| \geq \frac{1}{6} |A|^{7/2}.$$

Let $F_q$ be a finite field of order $q$. For $E \subseteq F_q^2$ with $|E| \geq \rho q^2$ with $q^{-1/2} \ll \rho < 1$. Iosevich and Rudnev [5] proved that the number of distinct distances determined by points in $E$ is at least $q - 1$. It follows from Lemma 2.5 that the number of congruence classes of triangles in $E$ is at least $\rho(q - 1)q^2$. In other words, we have recovered the result due to Covert et al. [3] and Vinh [11] for the case $d = 2$.

**Theorem 2.7 (Covert et al., [3]).** Let $F_q$ be a finite field of order $q$. For $E \subseteq F_q^2$ with $|E| \geq \rho q^2$ with $q^{-1/2} \ll \rho < 1$, then the number of congruence classes of triangles in $E$ is at least $c\rho q^3$, for some positive constant $c$.

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