Two-point correlation functions in inhomogeneous and anisotropic cosmologies

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Abstract. Two-point correlation functions are ubiquitous tools of modern cosmology, appearing in disparate topics ranging from cosmological inflation to late-time astrophysics. When the background spacetime is maximally symmetric, invariance arguments can be used to fix the functional dependence of this function as the invariant distance between any two points. In this paper we introduce a novel formalism which fixes this functional dependence directly from the isometries of the background metric, thus allowing one to quickly assess the overall features of Gaussian correlators without resorting to the full machinery of perturbation theory. As an application we construct the CMB temperature correlation function in one inhomogeneous (namely, an off-center LTB model) and two spatially flat and anisotropic (Bianchi) universes, and derive their covariance matrices in the limit of almost Friedmannian symmetry. We show how the method can be extended to arbitrary N-point correlation functions and illustrate its use by constructing three-point correlation functions in some simple geometries.

Keywords: power spectrum, CMBR theory, non-gaussianity

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1 Introduction

A central assumption of the standard cosmological model is that the universe we observe is a fair sample of an (hypothetical) ensemble of universes. This hypothesis has far reaching consequences, but it also brings along a whole statistical framework from which cosmological observables are to be computed. It follows in particular that cosmological parameters are not deduced directly from physical fields — which in this framework are viewed as one realization of random variables — but rather from their statistical moments, such as the one, two, and higher \( N \)-point correlation functions. When using perturbation theory to describe the clumpy universe, the one-point function is usually defined to be zero, since one is actually interested in the fluctuations of physical fields around their mean values. Thus, the first non-trivial statistical moment is the two-point (or Gaussian) correlation function.

Two-point functions are ubiquitous tools in modern physics. In field theory they are disguised as Green’s functions (or the propagator), whereas in general relativity they could be simply a distance function or a bitensor \([1, 2]\) — just to mention a few examples. In cosmology, two-point \emph{correlation} functions are a cornerstone of the standard ΛCDM model. Once it arises as the quantization of a free field in the early inflationary universe \([3–5]\) (see ref. \([6]\) for an up-to-date review), it propagates to virtually all cosmological and astrophysical computations one might be interested in — most popularly in its Fourier (i.e., the power spectrum) version. The same reasoning holds for higher-order correlation functions in connection with “Beyond-ΛCDM” approaches \([7, 8]\). Therefore, knowledge of the functional dependence of the two-point correlation function (2pcf) is crucial, since it alone can tell a lot about the statistical
properties of cosmological observables, potentially allowing one to disentangle cosmological signals from systematical effects in real data.

There are essentially two independent routes to find the functional dependence of the 2pcf in cosmology. In the first, one uses heuristic symmetry arguments (or its lack thereof) to fix this functional dependence. This idea has been successfully applied in cosmology, mainly in connection with CMB physics, in refs. [9–14]. However straightforward, the phenomenological quality of this approach prevents one to link the resulting 2pcf to the statistics of a field in a well-defined background geometry. Alternatively, one can deploy the full machinery of perturbation theory in the desired spacetime. After dealing with known issues of gauge invariance and mode decomposition, the full set of Einstein equations can be solved and the statistics of the 2pcf can be computed [15–19]. This option is clearly more expensive, but is certain to lead to statistics with known spacetime symmetries.

These considerations lead us to ask whether one can systematically find the functional properties of correlation functions given the spacetime symmetries, and without the need to resort to expensive computations involving perturbation theory. In fact, when metric and fluid perturbations are small, they can be seen as external fields evolving over a fixed background, regardless of their dynamics. By expanding such fields in an appropriate set of basis eigenfunctions, one ensures that their statistical properties will inherit the symmetries of the background metric. Thus, in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, for example, the 2pcf of a random field can only depend on the invariant distance between the two points, since this is the only combination allowed by the symmetries of the FLRW metric. Analogous ideas were explored in refs. [20, 21], where the conformal invariance of the de Sitter spacetime has been used to find the shape of two- and three-point correlation functions in dark-energy dominated universes.

In this work we systematically develop the idea of using the symmetries of the background metric to fix the functional form of the 2pcf. Starting from the definition of a two-point function in a general manifold, we show in section 2 that the imposition of isometric invariance on the 2pcf leads to a set of coupled first order partial differential equations which can be solved by means of well known techniques — but most easily through the method of characteristic curves [22] — to fix the functional form of the 2pcf. We illustrate the method in section 2.1, where we show how it correctly recovers the 2pcf in spatially flat FLRW spacetimes. We end this section by constructing in section 2.2 a formal solution to the aforementioned set of differential equations which holds for any spacetime having at least one Killing vector. In section 3 we apply the formalism to obtain the 2pcf in two different classes of spacetimes. First, we consider the case of an off-center inhomogeneous but spherically symmetric spacetime. Then we show how the 2pcf will appear in a class of homogeneous but spatially flat anisotropic geometries of the Bianchi family. In both cases we derive the CMB temperature covariance matrix in the limit of almost Friedmannian symmetry, and comment on their multipolar signatures. In section 4 we show how the method can be easily generalized to include any N-point correlation function. We conclude with some final remarks in section 5.

2 Formalism

We start with an informal description of what is meant by a two-point correlation function in spacetime. For a rigorous and mathematically complete description of two-point functions in Riemannian spaces, see ref. [23].
A two-point function \( f \) on a manifold \( \mathcal{M} \) is simply a real valued function of a pair of points \( (p, q) \in \mathcal{M} \times \mathcal{M} \). Known examples in physics are Green’s functions, the geodesic distance between two points or Synge’s world function [2]. Here we shall be mainly interested in correlation functions, so we also demand \( f \) to be symmetric

\[
f(p, q) = f(q, p),
\]

since correlation is clearly a pairwise concept. In most interesting situations in cosmology one is dealing with the correlation of random variables in spacetimes with some symmetries. Whenever these variables can be viewed as external fields over a fixed background, their statistical properties will inherit the symmetries of the underlying space. We would thus like to define an invariant correlation function with respect to these symmetries. Suppose that \( \mathcal{M} \) possesses an isometry represented by a one-parameter family of diffeomorphisms \( \phi_\tau \) that maps any point \( p \in \mathcal{M} \) to the point \( \phi_\tau(p) \in \mathcal{M} \) such that \( \phi_0(p) = p \). Clearly, \( f \) will be invariant under this symmetry if

\[
f(p, q) = f(\phi_\tau(p), \phi_\tau(q)).
\]

In practice, though, one is always working in a specific coordinate patch. Suppose that \( \psi \) is a chart on an open interval of \( \mathcal{M} \) and define

\[
f \circ \psi^{-1} = f(\psi^{-1}(x_1^\mu), \psi^{-1}(x_2^\mu)) \equiv \xi(x_1^\mu, x_2^\mu).
\]

Therefore, the components of the curve \( \phi_\tau \) in the coordinate system defined by \( \psi \) are

\[
(\psi \circ \phi_\tau)^\mu|_p = x_1^\mu(\tau), \quad \text{and} \quad (\psi \circ \phi_\tau)^\mu|_q = x_2^\mu(\tau).
\]

Locally, condition (2.2) then reads

\[
\xi(x_1^\mu, x_2^\mu) = \xi(x_1^\mu(\tau), x_2^\mu(\tau))
\]

which, for infinitesimal \( \tau \), is equivalent to

\[
\mathbf{K}(\xi) = 0,
\]

where \( \mathbf{K} = d/d\tau \) is a Killing vector, i.e., the vector tangent to the curves generated by \( \phi_\tau \). This condition is nothing more than

\[
K^\mu \partial_\mu \xi|_p + K^\mu \partial_\mu \xi|_q = 0, \quad K^\mu = \frac{dx^\mu}{d\tau}.
\]

Notice that in deriving this formula we are implicitly assuming that both \( p \) and \( q \) are covered by the same coordinate system. Since in general \( \mathcal{M} \) can have several independent isometries, we generalize the above result to the set of equations

\[
K_a^\mu \partial_\mu \xi|_p + K_a^\mu \partial_\mu \xi|_q = 0, \quad a \in \text{Isom}(\mathcal{M})
\]

where \( \text{Isom}(\mathcal{M}) \) is the set of all isometries of \( \mathcal{M} \). As we will see, this set of equations fully determine the functional dependence of the 2pcf.
2.1 Example: spatially flat FLRW universe

Equations (2.8) form the core of our formalism. They will lead to a set of coupled first order partial differential equations which can be implicitly solved by means of the method of characteristics curves \[22\]. In order to illustrate the method let us consider a two-point function in a spatially flat FLRW universe; it could be, for example, the ensemble average of the gravitational potential at two points on the same time slice. Since FLRW universes are maximally symmetric expanding manifolds they possess six independent Killing vectors: three of translation \((T_i)\) and three of rotation \((R_i)\). In Cartesian coordinates these vectors read

\[
T_i = \partial_i, \quad R_i = \epsilon_{ijk} x^j \partial^k. \tag{2.9}
\]

The two-point function depend on six variables: \(\xi = \xi(x_1, y_1, \ldots, z_2)\). In practice it is easier to work with \((\pm)\)-coordinates defined as

\[
x_{\pm} = x_2 \pm x_1, \quad y_{\pm} = y_2 \pm y_1, \quad z_{\pm} = z_2 \pm z_1, \tag{2.10}
\]

so that \(\xi = \xi(x_-, \ldots, z_+)\). Let us start with the vector \(T_x\). In Cartesian coordinates we have that

\[
T_x = T_x^\mu \partial_\mu = (1, 0, 0),
\]

which implies \(T_x^\mu = \delta^\mu_x\). Thus, for this KV, equations (2.8) give:

\[
2 \frac{\partial \xi}{\partial x_+} = 0. \tag{2.11}
\]

Clearly, \(\xi\) cannot depend on \(x_+\). Since this conclusion will not be straightforward in general, let us illustrate how it follows from the method of characteristics. Let \(\tau\) be the parameter along the integral curves (i.e., the isometry) of \(T_x\). Thus, by definition the tangent vector to this isometry is \(T_x = d/d\tau\), and we have by virtue of eq. (2.6) that

\[
T_x(\xi) = \dot{x}_- \frac{\partial \xi}{\partial x_-} + \dot{x}_+ \frac{\partial \xi}{\partial x_+} + \cdots + \dot{z}_+ \frac{\partial \xi}{\partial z_+} = 0 \tag{2.12}
\]

where a dot means a (partial) derivative with respect to \(\tau\). Comparing (2.11) and (2.12) we see that all coordinates are constant along \(\tau\) except for \(x_+\). Therefore \(\xi\) cannot depend on it. A similar procedure using \(T_y\) and \(T_z\) tell us that \(\xi\) cannot depend on either \(y_+\) or \(z_+\), so that \(\xi = \xi(x_-, y_-, z_-)\). Consider next the vector \(R_z = d/d\rho\). Using \(R_z = (-y, x, 0)\) on (2.8) we find

\[
x_- \frac{\partial \xi}{\partial y_-} - y_- \frac{\partial \xi}{\partial x_-} = 0. \tag{2.13}
\]

On the other hand, we also have that

\[
\frac{d\xi}{d\rho} = \dot{x}_- \frac{\partial \xi}{\partial x_-} + \dot{y}_- \frac{\partial \xi}{\partial y_-} + \dot{z}_- \frac{\partial \xi}{\partial z_-} = 0 \tag{2.14}
\]

where a dot now stands for \(\partial/\partial \rho\). By comparing the last two equations we find

\[
\dot{x}_- = -y_-, \quad \dot{y}_- = x_-, \quad \dot{z}_- = 0. \tag{2.15}
\]

The first pair of equations can be easily decoupled, giving (after an arbitrary choice of phase)

\[
x_- = A \cos \rho, \quad y_- = A \sin \rho, \tag{2.16}
\]
where $A$ is not necessarily a constant, since it can depend on the parameters of other isometries. The most general and $\rho$-independent combination of $x_-$ and $y_-$ is\footnote{The variable $z_-$ is already $\rho$-independent, so it does not enter into this combination.} $x_-^2 + y_-^2 = A^2$, so that $\xi = \xi \left(x_-^2 + y_-^2, z_-, \xi\right)$. Moving on we now consider the vector $R_y$. By the same reasoning we find
\begin{equation}
\frac{z_- x_-}{u_-} \frac{\partial \xi}{\partial x_-} - \frac{z_- \partial \xi}{\partial z_-} = 0.
\end{equation}
However we note that in virtue of (2.16) $x_-$ and $y_-$ are not independent anymore. We thus define $u_2^2 \equiv x_-^2 + y_-^2$ so that equation above becomes
\begin{equation}
\frac{z_- x_-}{u_-} \frac{\partial \xi}{\partial u_-} - \frac{z_- \partial \xi}{\partial z_-} = 0.
\end{equation}
If we now use $R_y = d/dr$ and expand $d\xi/dr$ as a total derivative we find by comparison that
\begin{equation}
u_\xi = \xi(r_2 - r_1) = \sum \frac{2\ell + 1}{4\pi} C_\ell P_\ell(\cos \gamma),
\end{equation}
Figure 1. Schematic representation of two points on the manifold connected by the vector $e = d/ds$ and dragged by the Killing vector $K = d/d\tau$.

Suppose now that $e$ is a vector field commuting with $K$, that is

$$\mathcal{L}_K e^\mu = [K, e]^\mu = 0.$$  \hspace{1cm} (2.24)

Then $e$ is a vector connecting two close points on different curves generated by $K$ (see figure 1). Moreover the quantity $u = g_{\mu\nu} e^\mu e^\nu$ is obviously constant along these isometries, since

$$\mathcal{L}_K u = (\mathcal{L}_K g_{\mu\nu}) e^\mu e^\nu + 2 g_{\mu\nu} (\mathcal{L}_K e^\mu) e^\nu = 0$$  \hspace{1cm} (2.25)

where the first term is zero since $K$ is a Killing vector. This suggests that eq. (2.23) will be solved for any function of $u$. That is

$$\xi = \xi \left( \int_{s_1}^{s_2} \sqrt{g_{\mu\nu} e^\mu e^\nu} ds \right),$$  \hspace{1cm} (2.26)

with $e^\mu = \partial x^\mu / \partial s$, is a solution of (2.23), since $d\xi / d\tau = (d\xi / du)(du / d\tau) = 0$. We have thus found a general solution to eq. (2.23). Since in general we will have more than one Killing vector, the 2pcf we will have more than one argument, provided that we can find a set of independent vectors $\{e_i\}$ commuting with the vectors $\{K_i\}$. When this is the case

$$\xi = \xi \left( \int_{r_1}^{r_2} \sqrt{g_{\mu\nu} e_1^\mu e_1^\nu} dr, \int_{s_1}^{s_2} \sqrt{g_{\mu\nu} e_2^\mu e_2^\nu} ds, \ldots \right)$$  \hspace{1cm} (2.27)

will be a solution to (2.8). This solution is particularly suited to the construction of 2pcf in homogeneous and anisotropic Bianchi geometries where the basis $\{e_i\}$ can always be constructed from the conditions $[K_i, e_j] = 0$. In these cases the metric can be written as [24, 25]

$$ds^2 = -d\tau \otimes d\tau + e^{2\alpha(\tau)} \left(e^{2\beta_{ij}(\tau)}\right)_{ij} e^i \otimes e^j.$$  \hspace{1cm} (2.28)

Here the vectors $\{e^i\}$ are the duals to $\{e_i\}$, $(e^{2\beta})_{ij}$ is a symmetric and traceless $3 \times 3$ matrix whose eigenvalues are the directional scale factors, and $e^{\alpha}$ is the geometrically averaged scale factor. For these vectors we have (no sum over $i$)

$$g_{\mu\nu} e^\mu_i e^\nu_j = e^{2\alpha(\tau)} e^{2\beta_{ij}(\tau)} \delta_{ij}.$$  \hspace{1cm} (2.29)
This implies in particular that
\[ \int_{r_1}^{r_2} \sqrt{g_{\mu\nu} e^\mu_\gamma e^\gamma_\tau} \, dr = e^{\alpha(\tau)} e^{\beta_{11}(\tau)} (r_2 - r_1) \]  
with similar expressions for the other arguments. We have thus arrived at a formal expression for the 2pcf which is valid in any Bianchi spacetime
\[ \xi = \xi \left( e^{\alpha(\tau)} e^{\beta_{11}(\tau)} (r_2 - r_1), e^{\alpha(\tau)} e^{\beta_{22}(\tau)} (s_2 - s_1), e^{\alpha(\tau)} e^{\beta_{33}(\tau)} (t_2 - t_1) \right) . \]  
To convert this function to one valid in a specific coordinate system one have to find the parametric curves of the vectors \( e_i \) in the desired coordinates and invert these relations to obtain the parameters as a function of the coordinates. Of course, the success of this procedure depends on the coordinate system chosen. We will illustrate this method with explicit examples in next section, where we find \( \xi \) for the geometries of Bianchi I and VII\textsubscript{0} universes.

3 Applications

We are now in position to put the above formalism to practical use. We start in section 3.1 with the example of an inhomogeneous universe with an off-center special point around which it is spherically symmetric. This could be seen as an off-center LTB spacetime, though in reality any spherically symmetric solution with a privileged point will lead to the same answer.

Then in section 3.2 we consider two anisotropic spacetimes with spatially flat spatial sections — namely, the models of Bianchi I and VII\textsubscript{0}. We then derive the Friedmannian limit of the 2pcf with first order corrections in both cases, and connect the result with the temperature covariance matrix of CMB fluctuations in section 3.3.

3.1 Universe with a special point

The 2pcf in an universe with a special point was studied from a phenomenological standpoint in ref. [11]. More recently, the effect of an off-center spherically symmetric void on the frequency and polarization of CMB photons was investigated by the authors of ref. [26]. Here we shall model an off-center spherically symmetric universe by its Killing vectors. Let \( w = (a, b, c) \) represent the spatial coordinates of this point with respect to our frame (see figure 2). Then the only isometries are rotations about \( w \). These are represented by the following KVs:
\[ R_i = \epsilon_{ijk} (x^j - w^j) \, \partial^k . \]  
Let us start with rotations around the \( z \)-axis. Applying \( R_z = (-y + b, x - a, 0) \) to eq. (2.8) leads to
\[ (x_+ - 2a) \, \frac{\partial \xi}{\partial y_-} + x_- \, \frac{\partial \xi}{\partial y_-} - (y_+ - 2b) \, \frac{\partial \xi}{\partial x_+} - y_- \, \frac{\partial \xi}{\partial x_-} = 0 . \]  
Let \( \rho \) be the parameter along the integral curve of \( R_z \), such that \( R_z = d/d\rho \). Comparing the above with \( d\xi/d\rho = 0 \) gives
\[ \dot{x}_+ = -y_+ + 2b , \quad \dot{x}_- = -y_- , \quad \dot{y}_+ = x_+ - 2a , \quad \dot{y}_- = x_- , \quad \dot{z}_\pm = 0 . \]  
After decoupling and solving these equations we find that the combinations \( u^2 \equiv x_-^2 + y_-^2 \) and \( v_+ \equiv (x_+ - 2a)^2 + (y_+ - 2b)^2 \) are constants with respect to \( \rho \), so that \( \xi = \xi (u_-, v_+, z_-, z_+) \). We next consider \( R_y = (z, 0, -x) \) and change variables from \( (x_-, x_+) \) to \( (u_-, v_+) \). This gives
\[ \frac{z_+ x_-}{u_-} \, \frac{\partial \xi}{\partial u_-} + \frac{(z_+ - 2c) (x_+ - 2a)}{v_+} \, \frac{\partial \xi}{\partial v_+} - (x_+ - 2a) \, \frac{\partial \xi}{\partial z_+} - x_- \, \frac{\partial \xi}{\partial z_-} = 0 . \]
Figure 2. Schematic representation of a universe with a special point $Q$ at a distance $w$ from our position at $P$. In such universe the correlation between two photons coming from positions $A$ and $B$ can only depend on $|r_2 - r_1|$ and $|r_2 + r_1 - 2w|$.

We now compare this to $d\xi/dr = 0$, where $r$ is such that $R_y = d/dr$. This gives

$$u_- \dot{u}_- = z_- x_- , \quad v_+ \dot{v}_+ = (z_+ - 2c)(x_+ - 2a) , \quad \dot{z}_+ = -x_+ + 2a , \quad \dot{z}_- = -x_- . \quad (3.5)$$

Combining the last equation with the first and the third with the second we find two constant combinations of variables: $u^2 + z^2$ and $v^2 + (z+ - 2c)^2$. Thus, $\xi = \xi \left( u^2 + z^2 , v^2 + (z_+ - 2c)^2 \right)$. After a little algebra on the second argument, the final solution can be written as

$$\xi_w = \xi_w \left( |r_2 - r_1| , |r_2 - w|^2 + |r_1 - w|^2 + 2 (r_2 - w) \cdot (r_1 - w) \right) . \quad (3.6)$$

This result is compatible with the one found heuristically by the authors of ref. [11]. Note however that the above solution is more restrictive than theirs, since here we can obviously write $\xi_w = \xi_w \left( |r_2 - r_1| , |r_2 + r_1 - 2w| \right)$ whereas in [11] this is not possible.\(^3\)

Before continuing we would like to make two comments about solution (3.7). First, the casual reader could be worried that the above solution does not seem to recover (2.21) when $w = 0$. This happens because (3.7) is only invariant under rotations, whereas (2.21) also obeys translation symmetry. If the universe is homogeneous then $w = 0$ and we can further impose translation invariance through the condition $T_r(\xi) = \nabla_{r_2} \xi + \nabla_{r_1} \xi = 0$, thus eliminating the dependence on $|r_2 + r_1|$. As a corollary of this result we find that the 2pcf in a inhomogeneous but spherically symmetric universe about its origin — such as in Lemaître-Tolman-Bondi (LTB) universes — is

$$\xi_0 = \xi_0 \left( |r_2 - r_1| , |r_2 + r_1| \right) \quad (3.8)$$

\(^3\)This difference does not affect the conclusions found by those authors, since most of their results are actually extracted from an ansatz of the power spectrum, and not from $\xi$. 

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which is functionally equivalent to \( \xi_0 (r_1, r_2, \hat{r}_1 \cdot \hat{r}_2) \), since the only angle entering eq. (3.8) is that between \( r_1 \) and \( r_2 \). The second remark is that, as stressed in [11], the 2pcf (3.7) possess a global shift symmetry of the form

\[
\mathbf{r}_{1,2} \rightarrow \mathbf{r}_{1,2} + \mathbf{a}, \quad \mathbf{w} \rightarrow \mathbf{w} + \mathbf{a}
\]

(3.9)

for any vector \( \mathbf{a} \). This corresponds to the freedom in placing a special point in an otherwise homogeneous universe, which is only defined up to a global translation. We will come back to this issue in section 4.

Equation (3.7) concludes our task of finding the 2pcf in an universe with a special point. The question of whether we actually live close to an off-center spherically symmetric universe can be tested by measuring off-diagonal terms in the covariance matrix of CMB temperature fluctuations. Since we know that our universe is very close to FLRW, we can test this hypothesis by deriving the FLRW limit of eq. (3.7), including leading order corrections. To do that we first introduce the variables

\[
\mathbf{r}_\pm = \mathbf{r}_2 \pm \mathbf{r}_1.
\]

(3.10)

Then we note that the desired limit of (3.7) involves two independent expansions, namely, one in \(|\mathbf{w}|\) and another one in powers of \( r_+ = |\mathbf{r}_+| \). Let us start with the former. Assuming \(|\mathbf{w}| \ll 1\) we have

\[
|r_+ - 2w| = r_+ - 2\mathbf{n}_+ \cdot \mathbf{w} + O(|\mathbf{w}|^2).
\]

(3.11)

Thus

\[
\xi_w \approx \xi_w (r_-, r_+ - 2\mathbf{n}_+ \cdot \mathbf{w}) = \xi_0 (r_-, r_+) - 2\partial_0 (r_-, r_+) \mathbf{n}_+ \cdot \mathbf{w} + \cdots.
\]

(3.12)

Next we assume that \( \xi_0 (r_-, r_+) \) varies weakly with \( r_+ \) and write

\[
\xi_0 (r_-, r_+) = \xi_0 (r_-, 0) + \partial_0 (r_-, 0) \mathbf{n}_+ \cdot \mathbf{w} + \cdots.
\]

(3.13)

It is important to note that we are not treating \( r_+ \) as a small parameter. Indeed, this will hardly be the case, since for coincident points on the CMB sphere we have \( r_+ = 2\Delta \eta \), which is not assumed as small. On the other hand, the assumption that \( \xi_0 \) varies weakly with \( r_+ \) implies that its translational invariance is only slightly broken. That is

\[
\mathbf{T}_r(\xi_0 (r_-, r_+)) = \nabla_{r_+} \xi_0 (r_-, r_+) = \partial_0 (r_-, 0) \mathbf{n}_+ \ll 1.
\]

(3.14)

Since \( \xi_0 (r_-, 0) = \xi_{FL} (r_-) \), we finally find that

\[
\xi_w = \xi_{FL} (r_-) + \frac{\partial_0 (r_-, 0)}{\partial_0 (r_+, 0)} (r_+ - 2\mathbf{n}_+ \cdot \mathbf{w}),
\]

(3.15)

which is the desired result.\footnote{Rigorously speaking (3.15) should also include second order terms since \( \partial_0 / \partial_0 (\mathbf{n}_+ \cdot \mathbf{w}) \) is formed from the product of two small quantities. However, second order corrections from (3.11) will multiply \( \partial_0 / \partial_0 (\mathbf{n}_+ \cdot \mathbf{w}) \) in (3.12), producing a third order term. The only remaining second order term is a correction to (3.13), but this does not induce any angular dependence on \( \xi_w \).}
In order to extract the amplitude of the leading corrections one still needs the specific shape of the function $\xi_0 (r_-, r_+)$, which at this point can only be fixed from first physical principles [16, 26, 27]. Note however that, as far as the angular dependence is concerned, there is no new information in $\xi_0 (r_-, r_+)$ as compared to (2.22), since its angular dependence also comes from the angle between $r_2$ and $r_1$. This means that the middle term in (3.15) will not induce off-diagonal correlations in the CMB covariance matrix, although it will surely alter the amplitude of the isotropic temperature spectrum, i.e., the $C_\ell$s. On the other hand, the last term will induce a dipole coming from the angle between $\hat{n}_+$ and $w$. We show in section 3.3 how these multipolar coefficients can be directly linked to the temperature covariance matrix.

3.2 Anisotropic universes

In order to derive the 2pcf in Bianchi universes we start from the general solution (2.31), which we have already proven to solve (2.8). One can check that the same results follow instead from the direct application of eq. (2.8), up to the dependence on the directional scale factors. We will here focus on two spatially flat anisotropic solutions and postpone a complete analysis with other Bianchi metrics to a future work.

3.2.1 Bianchi I

We start with the simple Bianchi-I metric, which admits three translational KVs:

$$T_x = \partial_x, \quad T_y = \partial_y, \quad T_z = \partial_z.$$  \hfill (3.16)

The set of triad $\{e_i\}$ vectors which are invariant under the action of these isometries are [28, 29]

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$  \hfill (3.17)

Let us solve for the integral curves of the first vector. Putting $e^i_1 = dx^i/d\tau$ we find that $x = r_-, y = z = \text{constant}$. Thus, we can invert the relation between the parameter and the coordinates to find

$$r_2 - r_1 = x_2 - x_1 = x_-.$$  \hfill (3.18)

Solving for $e_2$ and $e_3$ leads to $s_2 - s_1 = y_-$ and $t_2 - t_1 = z_-$. Plugging these results back into (2.31) then gives

$$\xi_I = \xi_I \left( e^{\alpha(\tau)} e^{\beta_{11}(\tau)} x_-, e^{\alpha(\tau)} e^{\beta_{22}(\tau)} y_-, e^{\alpha(\tau)} e^{\beta_{33}(\tau)} z_- \right),$$  \hfill (3.19)

which is the desired solution.

Observational evidences tell us that our universe is very close to isotropic [30–32]. We can thus obtain the FLRW limit of $\xi_I$ by Taylor expanding this function around $\beta_{ii} = 0$. We find

$$\xi_I = \xi_I (x_-, y_-, z_-) + \left[ \beta_{11} x_- \frac{\partial}{\partial x_-} + \beta_{22} y_- \frac{\partial}{\partial y_-} + \beta_{33} z_- \frac{\partial}{\partial z_-} \right] \xi_I (x_-, y_-, z_-) + \cdots$$  \hfill (3.20)

where we have omitted the the functional dependence on $\alpha$ for simplicity. In order to proceed, we note that the failure of the above expression to be rotationally invariant is proportional to $\beta_{ii}$, in the sense that full rotational isotropy should be exactly recovered if $\beta_{ii} = 0$. In fact, by applying, say, $R_z$ to the above expression we find

$$R_z (\xi_I) = R_z (\xi_I (x_-, y_-, z_-)) + \beta_{11} (\cdots) + \beta_{33} (\cdots) + \beta_{33} (\cdots)$$  \hfill (3.21)
where the ellipses contain terms like $x_\perp \partial_{y_\perp} (x_\perp \partial_{x_\perp} \xi)$ and so on, but which are not relevant for this discussion. The important point is that the right hand side is linear in $\beta_{ii}$. Therefore, for $\xi_I$ to be rotationally invariant at zero-order in $\beta_{ii}$, it is necessary that $R_z (\xi_I (x_\perp, y_\perp, z_\perp)) = 0$ at this order. Repeating the analysis with $R_y$ or $R_x$ then leads to the isotropic condition

$$\xi_I (x_\perp, y_\perp, z_\perp) = \xi_{FL} (r_\perp).$$

(3.22)

Thus, the FLRW limit of (3.19) including first order anisotropic corrections is

$$\xi_I = \xi_{FL} + \frac{1}{r_\perp} \frac{\partial \xi_{FL}}{\partial r_\perp} \left[ \beta_{11} x_\perp^2 + \beta_{22} y_\perp^2 + \beta_{33} z_\perp^2 \right].$$

(3.23)

Clearly, this function will induce quadrupolar corrections in the statistics of CMB, as is already known [15, 29, 33, 34]. Interestingly, though, it does not alter the isotropic spectrum, as we will see in section 3.3.

### 3.2.2 Bianchi VII$$_0$$

The spatial topology of Bianchi-VII$$_0$$ solution is $\mathbb{R}^3$, so that it also has a flat FLRW limit when $\beta_{ii} = 0$. The isometries of this space can be seen as two orthogonal displacements in the $xy$-plane, and a displacement in the $z$-axis followed by a rotation in the $xy$-plane [28, 29] (see figure 3). In Cartesian coordinates the three KVs are

$$T_x = \partial_x, \quad T_y = \partial_y, \quad T_z = \partial_z + x \partial_y - y \partial_x.$$  

(3.24)

The set of of invariant triad vectors are [28, 29]

$$\mathbf{e}_1 = (\cos z, \sin z, 0), \quad \mathbf{e}_2 = (-\sin z, \cos z, 0), \quad \mathbf{e}_3 = (0, 0, 1).$$

(3.25)

\[^5\text{Note that we chose a different orientation of the axis as compared to refs. [28, 29].}\]
Let us consider the first vector with components $e_1^i = dx^i/dr$. Its integral curves are

$$x(r) = x_0 + (\cos z_0) r, \quad y(r) = y_0 + (\sin z_0) r, \quad z(r) = z_0.$$  

(3.26)

These relations can be easily inverted to give one of the curve segments between the two points as a function of their coordinates:

$$r_2 - r_1 = \sqrt{x_2^2 + y_2^2}.$$  

(3.27)

A similar computation involving $e_2^i = dx^i/ds$ and $e_3^i = dx^i/dt$ then gives the other two segments of the curve

$$s_2 - s_1 = \sqrt{x_2^2 + y_2^2}, \quad t_2 - t_1 = z_-.$$  

(3.28)

This completes the task of finding the parameters of the integral curves of the vectors $\{e_i\}$ as a function of the coordinates. Inserting the above expressions in (2.31) finally gives

$$\xi_{\text{VII}0} = \xi_{\text{VII}0} \left( e^\alpha(\tau) e^{\beta_{11}}(\tau) \sqrt{x_-^2 + y_-^2}, e^\alpha(\tau) e^{\beta_{22}}(\tau) \sqrt{x_-^2 + y_-^2}, e^\alpha(\tau) e^{\beta_{33}}(\tau) z_- \right).$$  

(3.29)

The isotropic limit of $\xi_{\text{VII}0}$ follows the same discussion of the last section. The only difference is that, at zero-order in $\beta_{ii}$, $\xi_{\text{VII}0}$ is automatically invariant under $R_z$, as follows from its isometries. Thus we just require that $R_x (\xi_{\text{VII}0}) = 0$ at zero order, which then gives

$$\xi_{\text{VII}0} \left( \sqrt{x_-^2 + y_-^2}, \sqrt{x_-^2 + y_-^2}, z_- \right) = \xi_{\text{FL}} (r_-).$$  

(3.30)

Thus

$$\xi_{\text{VII}0} = \xi_{\text{FL}}(r_-) + \frac{1}{r_-} \frac{\partial \xi_{\text{FL}}}{\partial r_-} \left[ (\beta_{11} + \beta_{22}) (x_-^2 + y_-^2) + \beta_{33} z_-^2 \right]$$  

(3.31)

completes the desired expansion.

### 3.3 CMB covariance matrix

Equations (3.7), (3.19), and (3.29), together with their FLRW expansions, are the main results of last section. We emphasize that these results are completely general and can be equally applied to large scale structures as well as to CMB physics. Here we are interested in the latter, so that we shall now derive the multipolar expansion of functions (3.15), (3.23), and (3.31) and relate them to the corresponding CMB covariance matrix in the limit of large angles, i.e., assuming only the Sachs-Wolfe effect.

We start by recalling that all the 2pcf’s that we are considering have the generic form

$$\xi(r_2, r_1) = \begin{cases} 
\xi(r_-) & \text{for Bianchi I and VII0}, \\
\xi(r_+, w) & \text{for a universe with a special point}, 
\end{cases}$$  

(3.32)

where $(r_+, r_-)$ were defined in (3.10) and $w$ is the special point introduced in section 3.1. For simplicity, we have omitted any extra dependence on $(r_+, r_-)$, since these will not lead to anisotropies. We will also omit the dependency of $\xi(r_+, w)$ on $w$, thus calling both 2pcf’s above generically as $\xi(r_\pm)$. This allows us to expand the $\xi(r_2, r_1)$ collectively as

$$\xi(r_\pm) = \sum_{\ell, m} \xi_{\ell m}(r_\pm) Y_{\ell m}(\hat{n}_\pm), \quad r_\pm = r_\mp \hat{n}_\pm.$$  

(3.33)
assumed to be isotropic. Under this assumption the CMB covariance matrix reads

\[ \xi_{LL} = \xi_{0} + \xi_{0} r_{+} + 4w\sqrt{\frac{T}{\pi}} \xi_{t} + \xi_{t} r_{-} \]

where \( r_{+} = \arccos(\hat{n}_{+} \cdot \hat{z}) \), and \( r_{-} = \arccos(\hat{n}_{-} \cdot \hat{z}) \). Notice that for \( r_{+} \) we have defined the \( z \) axis along \( w \). The resulting expressions are collected in table 1, and can be directly related to the CMB temperature covariance matrix, as we now show.

At large scales the Sachs-Wolfe effect (\( \Delta T/T = \Phi/3 \)) gives the main contribution to the temperature fluctuations. In order to compute the full effect of inhomogeneous or anisotropic geometries in a real CMB map, gravitational evolution and re-ionization effects should be taken into account. Clearly, such effects will not be provided by our formalism, which is geometric in nature. On the other hand, we can picture a scenario in which the asymmetries of the early universe are washed out by inflation, but where quantum fluctuations preserve such asymmetries on the statistics of the primordial gravitational potential. This is the approach followed in, e.g., refs. [10, 15, 34]. In this scenario, primordial inhomogeneities and anisotropies are contained in the statistics of CMB, and subsequent evolutionary effects are assumed to be isotropic. Under this assumption the CMB covariance matrix reads

\[ \langle a_{\ell_{1}m_{1}}a_{\ell_{2}m_{2}}^{*} \rangle = \frac{1}{9} \int d^{2}n_{1} d^{2}n_{2} \langle \Phi_{1}(r_{1})\Phi_{2}(r_{2}) \rangle_{\pm} Y_{\ell_{1}m_{1}}(\hat{n}_{1})Y_{\ell_{2}m_{2}}^{*}(\hat{n}_{2}) . \]  

The two-point correlation function in this case is the ensemble average of the gravitational potential:

\[ \xi(r_{\pm}) = \langle \Phi_{2}(r_{2})\Phi_{1}(r_{1}) \rangle_{\pm} . \]

Once again, the \( \pm \) labels correspond to the off-center LTB and Bianchi models, respectively. Likewise, the \( \pm \) notation in \( \langle a_{\ell_{1}m_{1}}a_{\ell_{2}m_{2}}^{*} \rangle_{\pm} \) indicates that each covariance matrix corresponds to one of each correlation function in (3.32). Usually, deviations from isotropy and homogeneity are quantified directly in terms of the power spectrum

\[ \pm P(k) = \int d^{3}r_{\pm} e^{-ik\cdot r_{\pm}} \xi(r_{\pm}) . \]

For example, by expanding \( \pm P(k) \) in harmonics one can show that [7, 10]

\[ \langle a_{\ell_{1}m_{1}}a_{\ell_{2}m_{2}}^{*} \rangle_{\pm} = i^{\ell_{1}+\ell_{2}} \frac{2}{9\pi} \sum_{\ell_{m},m} \int k^{2}dk \pm P_{\ell_{m}}(k)j_{\ell_{1}}(k\Delta)j_{\ell_{2}}(k\Delta)(-1)^{m}G_{\ell_{m}1,\ell_{1}m_{1},m_{2}}^{\ell_{2}1} \],

where

\[ G_{\ell_{m}1,\ell_{1}m_{1},m_{2}}^{\ell_{2}1} = \int d^{2}\hat{n} Y_{\ell_{1}m_{1}}(\hat{n})Y_{\ell_{2}m_{2}}^{*}(\hat{n})Y_{\ell_{m}3}(\hat{n}) \]
are the Gaunt coefficients (see appendix A.3) and \( \pm P_{\ell m}(k) \) are the multipolar coefficients of the power spectrum. Then, from (3.38) and the coupling properties of the Gaunt coefficients, a feature in the power spectrum can be directly converted into a feature in the covariance matrix. In fact it is easy to extract \( \pm P_{\ell m}(k) \) from the coefficients in table 1 by means of the so-called Hankel transform (see appendix A.1):

\[
\pm P_{\ell m} = 4\pi i^{-\ell} \int_0^\infty r_\pm dr_\pm j_\ell(kr_\pm)\xi_{\ell m}(r_\pm).
\]

(3.40)

However, it is interesting to have an expression for the covariance matrix directly in terms of \( \xi_{\ell m} \). This can be obtained by inserting the above expression into (3.38), which gives

\[
(a_{\ell_1 m_1}a_{\ell_2 m_2})_\pm = \frac{8}{9} \sum_{\ell_3,m_3} \int_0^{2\Delta \eta} r_\pm^2 dr_\pm \xi_{\ell_3 m_3}^* (r_\pm) J_{\ell_1 \ell_2 \ell_3}^{(\pm)}(r_\pm) G_{\ell_1 \ell_2 \ell_3}^{\ell_1 \ell_2 \ell_3} \, .
\]

(3.41)

where the coefficients \( J_{\ell_1 \ell_2 \ell_3}^{(\pm)} \) are implicitly defined in terms of the following integral

\[
J_{\ell_1 \ell_2 \ell_3}^{(\pm)}(R, r_1, r_2) \equiv i^{\ell_2 \mp \ell_1 - \ell_3} \int_0^\infty k^2 dk j_\ell_1(kr_1) j_\ell_2(kr_2) j_\ell_3(kR) \, .
\]

(3.42)

This integral can be analytically solved and the result can be expressed in terms of Wigner 6-J symbols [35]. Parity symmetries of the Wigner 6-J symbols then result in several properties of the coefficients \( J_{\ell_1 \ell_2 \ell_3}^{(\pm)} \) [35]. For our discussion, the most relevant property is that these coefficients vanish whenever \( R \) lies outside the range

\[
|r_1 - r_2| \leq R \leq r_1 + r_2 \, .
\]

(3.43)

In the context of CMB, \( r_1 = r_2 = \Delta \eta \) and \( r_\pm = \sqrt{2} \Delta \eta \sqrt{1 \pm \hat{n}_2 \cdot \hat{n}_1} \), so that

\[
0 \leq r_\pm \leq 2\Delta \eta \, .
\]

(3.44)

This explains why the domain of the integral in (3.41) is limited. This result makes perfect sense since it is impossible to consider points in the CMB sphere whose separation is larger than \( 2\Delta \eta \).

Returning to table 1 we see that each geometry leaves its own fingerprint on the temperature spectrum. To the lowest order in \( \beta_{ii} \) in this formalism both Bianchi-I and VIIo models produce quadrupolar anisotropies whereas only the latter will alter the isotropic temperature spectrum (i.e., the \( C_\ell \)'s). Higher multipoles come from higher order corrections in \( \beta_{ii} \), but we will not consider those here. Parity symmetry of these two models prevent even-odd couplings of the harmonic coefficients [36]. For an off-center LTB universe, on the other hand, there will be dipolar couplings as well as a change of the angular spectrum at low \( \ell s \), which depends on the derivatives of the function \( \xi_0(r_-, r_+) \) evaluated at \( r_+ = 0 \). Although one can obtain these features by inserting the coefficients \( \xi_\ell m \) directly in (3.41), it is easier to relate them to the Bipolar Spherical Harmonics (BipoSH) coefficients [9, 37] (see the appendix A.4)

\[
(\pm) A_{\ell_1 \ell_2}^{LM} = \frac{8}{9} \int r_\pm^2 dr_\pm \xi_{LM}(r_\pm) J_{\ell_1 \ell_2}^{(\pm)}(r_\pm) F_{\ell_1 \ell_2 L} \, .
\]

(3.45)

where the set of coefficients \( F_{\ell_1 \ell_2 L} \) were defined in (A.13), and relate these to the covariance matrix using (A.11). Thus, Bianchi-I and VIIo geometries lead to a quadrupolar BipoSH \( A_{\ell_1 \ell_2}^{2LM} \), whereas an off-center LTB produces a dipolar BipoSH \( A_{\ell_1 \ell_2}^{1LM} \).
As a final remark, we emphasize that expression (3.41) should be seen as containing anisotropies and inhomogeneities only from the initial conditions, after which we assume the universe to be pure FLRW. In particular, contributions resulting from integrated effects from the last scattering surface to us — like the effect induced by the lensing potential in anisotropic [18, 38] and inhomogeneous [27, 39] universes — cannot be extracted from this formalism in its present form. On the other hand, the multipolar features resulting from the coefficients in table 1 would still be preserved — perhaps in an integrated version — as long as perturbations are functions of the background coordinates. Indeed this is corroborated by the results of [18, 38], where quadrupolar corrections in the correlation of weak-lensing convergence of large-scale structure in a Bianchi-I spacetime was found. Furthermore, since (2.8) was designed to work on scalar functions, it cannot be directly applied to the cross-correlations between scalar, vector and tensor perturbations, which are known to couple dynamically through the evolution of the background shear [40, 41]. Nevertheless, since tensor fields are still seen as external fields in a fixed background, tensor correlators should be expected to obey a similar formalism as the one presented here (see also [1]). We postpone a deeper investigation of these issues to a future work.

4 Non-Gaussian correlations

It is straightforward to extend this formalism to non-Gaussian correlation functions. Let \( \varphi \) be any \( N \)-point \( (N > 2) \) correlation function. Repeating the arguments leading to condition (2.8) then gives

\[
\sum_{j=1}^{N} R_{\mu}^{j} \partial_{\mu} \varphi_{j} = 0. \tag{4.1}
\]

The first non-trivial non-Gaussian statistical moment is the three-point correlation function (3pcf). Let us consider this function in an FLRW universe, where there are both translational and rotational symmetries. Imposing invariance under the vector \( T_{\mu}^a = (1, 0, 0) \) gives the following condition on \( \varphi \):\[
\frac{\partial \varphi}{\partial x_1} + \frac{\partial \varphi}{\partial x_2} + \frac{\partial \varphi}{\partial x_3} = 0. \tag{4.2}
\]

This is solved by any \( \varphi \) with an arbitrary dependence on the variable \( X \) defined as

\[
X = lx_1 + mx_2 + nx_3, \quad l + m + n = 0, \tag{4.3}
\]

with constants \( (l, m, n) \). However, the constraint on these constants allows us to write

\[
X = l(x_1 - x_3) + m(x_2 - x_3), \tag{4.4}
\]

which shows that \( \varphi \) can actually depend on the two “base” combinations \( (x_1 - x_3) \) and \( (x_2 - x_3) \). Since we have no more constraints, these are the simplest combinations of \( x \)-coordinates on which \( \varphi \) can depend. Applying the same reasoning for translations along \( y \)- and \( z \)-directions then gives

\[
\varphi_{\text{homog.}} (r_1, r_2, r_3) = \varphi_{\text{homog.}} (r_1 - r_3, r_2 - r_3) \tag{4.5}
\]

which is the most general homogeneous three-point function [7, 42]. To obtain an expression which is also invariant under rotations we introduce \( u = r_1 - r_3 \) and \( v = r_2 - r_3 \) and simply
note that the task of finding \( \varphi(\mathbf{u}, \mathbf{v}) \) invariant under rotations has already been solved in section 3.1. The solution is simply a function depending on the modulus of \( \mathbf{u} \pm \mathbf{v} \) (see eqs. (3.8)). In terms of the original variables this becomes\(^6\)

\[
\varphi_{FL} = \varphi_{FL}(|\mathbf{r}_1 - \mathbf{r}_2|, |\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3|). \tag{4.6}
\]

Since the reasoning we used to arrive at this result might not be entirely obvious, we note that rotations around the \( z \)-axis of the vectors \( \mathbf{r}_1, \mathbf{r}_2 \) and \( \mathbf{r}_3 \) are equal to rotations around the \( z \)-axis of \( \mathbf{u} \) and \( \mathbf{v} \):

\[
\mathbf{R}_z = \left( x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1} \right) + \left( x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2} \right) + \left( x_3 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial x_3} \right) = x_1 \frac{\partial}{\partial u_y} - y_1 \frac{\partial}{\partial u_x} + x_2 \frac{\partial}{\partial v_y} - y_2 \frac{\partial}{\partial v_x} + x_3 \left( \frac{\partial}{\partial u_y} - \frac{\partial}{\partial v_y} \right) - y_3 \left( \frac{\partial}{\partial u_x} - \frac{\partial}{\partial v_x} \right)
\]

where we have introduced a (hopefully obvious) new notation for the components of \( \mathbf{u} \) and \( \mathbf{v} \). An equivalent result holds for \( \mathbf{R}_y \) and \( \mathbf{R}_x \), as one can easily check. Then, by repeating the analysis of section 3.1 we find \( \varphi = \varphi(|\mathbf{u} - \mathbf{v}|, |\mathbf{u} + \mathbf{v}|) \) which gives (4.6) upon replacing \( \mathbf{u} \) and \( \mathbf{v} \) by their definitions.

There is one interesting remark we would like to make about eq. (4.6). Notice that if we make the identification \( \mathbf{r}_3 = \mathbf{w} \) the 3pcf will have exactly the same functional dependence as the 2pcf in eq. (3.7) — namely, a Gaussian correlation in an universe with a special point. This suggests that the bispectrum (the Fourier transform of the 3pcf) in a FLRW universe could mimic the power spectrum in an off-center LTB universe. Interestingly, it has been argued that a (statistically homogeneous and isotropic) bispectrum in the strong squeezed limit will induce statistical anisotropies in the power spectrum [43]. In Fourier space the power spectrum and bispectrum have the form (assuming statistical homogeneity and isotropy)

\[
\xi(\mathbf{k}_1, \mathbf{k}_2) = P(k_1)\delta(\mathbf{k}_1 + \mathbf{k}_2), \tag{4.7}
\]
\[
\varphi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = B(k_1, k_2, k_3 \cdot \mathbf{k}_2)\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \tag{4.8}
\]

In the squeezed limit \( k_1 \approx -k_2 \) the wave vector \( k_3 \approx 0 \) corresponds to a long wavelength perturbation which is equivalent to a spatial gradient. This gradient modulates the lower order statistics leading to an effective power spectrum which is now anisotropic: \( P(k_1) \to P_{\text{eff}}(k_1) \).

We add to these the fact that the delta \( \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \) in the bispectrum breaks the statistical independence previously existing between \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) in (4.7). Thus, in the presence of a bispectrum \( \varphi(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \), \( \xi(\mathbf{k}_1, \mathbf{k}_2) \) is no longer translational invariant.\(^7\) In real space, the similarity between (4.6) and (3.7) is just reflecting the fact that the third point in the 3pcf could itself be seen as a “special” point. Analogously, a special point of an LTB universe will itself correlate with any two points previously correlated.

\(^6\)Note that since \( \varphi(|\mathbf{u} - \mathbf{v}|, |\mathbf{u} + \mathbf{v}|) \) is equivalent to \( \varphi(u, v, \mathbf{u} \cdot \mathbf{v}) \), eq. (4.6) is also equivalent to \( \varphi(|\mathbf{r}_1 - \mathbf{r}_3|, |\mathbf{r}_2 - \mathbf{r}_3|, (\mathbf{r}_1 - \mathbf{r}_3) \cdot (\mathbf{r}_2 - \mathbf{r}_3)) \), which appears to be more common in the literature [42].

\(^7\)Note that this holds for any \( \mathbf{k}_1, \mathbf{k}_2 \) and \( \mathbf{k}_3 \), regardless of the squeezed limit.
As one last application let us consider the 3pcf in a LTB universe. Rotational invariance around $R^2_x = (-y, x, 0)$ gives
\[
\left( x_1 \frac{\partial \varphi}{\partial y_1} + x_2 \frac{\partial \varphi}{\partial y_2} + x_3 \frac{\partial \varphi}{\partial y_3} \right) - \left( y_1 \frac{\partial \varphi}{\partial x_1} + y_2 \frac{\partial \varphi}{\partial x_2} + y_3 \frac{\partial \varphi}{\partial x_3} \right) = 0. \tag{4.9}
\]
We could try solving this equation with the introduction of two new variables $X = lx_1 + mx_2 + nx_3$ and $Y = ly_1 + my_2 + ny_3$. This would give
\[
X \frac{\partial \varphi}{\partial Y} - Y \frac{\partial \varphi}{\partial X} = 0. \tag{4.10}
\]
The use of characteristics would then tell us that $\dot{X} = -Y$ and $\dot{Y} = X$, which implies that $\varphi$ is a function of the constant combination $X^2 + Y^2$. This solution however is not the most general one. To see that, note that in the absence of translational invariance the constraint in (4.3) no longer holds. In this case we have
\[
l + m + n = 2p \tag{4.11}
\]
for some constant $p$. We can thus rewrite the variable $X$ as
\[
X = l (x_1 - x_3) + m (x_2 - x_3) + p (x_1 + x_3) + p (x_2 + x_3) - p (x_1 + x_2) \tag{4.12}
\]
with an analogous expression for $Y$. This tells us that there are actually five “base” combinations on which $\varphi$ will depend, i.e., $\varphi = \varphi(x_1 - x_3, x_2 - x_3, \ldots, x_1 + x_2, \ldots)$. Repeating the analysis for $R_y$ and $R_x$, which we hope by now has become clear, we find
\[
\varphi_0 = \varphi_0 (|r_1 - r_3|, |r_2 - r_3|, |r_1 + r_3|, |r_2 + r_3|, |r_1 + r_2|). \tag{4.13}
\]
Note in particular that the combination $|r_1 + r_2|$ cannot be neglected, as one could have expected from a naive comparison with (3.8). The reason is that while $r_1 - r_3$ is linearly dependent on $r_1 - r_3$ and $r_2 - r_3$ (thus eliminating the need to include the former), the vectors $r_1 + r_2, r_1 + r_3$ and $r_2 + r_3$ are linearly independent (the plane made by any two of them will not contain the third), and thus should all be included.

Finally, the 3pcf in an off-center LTB universe is
\[
\varphi_w = \varphi_w (|r_1 - r_3|, |r_2 - r_3|, |r_1 + r_3 - 2w|, |r_2 + r_3 - 2w|, |r_1 + r_2 - 2w|). \tag{4.14}
\]
This can be obtained from (4.13) as follows: since the location of the special point $w$ is arbitrary, the 3pcf should satisfy a shift symmetry analogous to (3.9). We thus shift all points in (4.13) by an arbitrary amount $a$ and $w$ so as to make the result shift invariant. This gives the above result.

5 Final remarks

Correlation functions belong to the core of modern cosmology. The perspective of extending the ΛCDM model to inhomogeneous, anisotropic, and non-Gaussian universes depends crucially on our abilities to model and measure such functions with increasing levels of sophistication. In this work we have introduced a novel formalism which allows us to fix the functional dependence of correlation functions given the underlying spacetime (continuous) symmetries. Given a set of Killing vectors, we have found a set of first order partial differential equation which can be solved for the functional dependence of the correlation function.
The method works for arbitrary $N$-point correlators as long as one stays in the Born approximation — that is, as long as cosmological perturbations can be treated as external fields in a fixed background. We have also provided a general solution to the two-point correlation function which naturally introduces the time dependence, provided one finds a set of triad vectors commuting the Killing vector fields. This solution is particularly useful in applications to Bianchi cosmologies, where such triad of vectors can always be found [28, 29].

We have successfully applied the formalism to the two-point function in three different cosmological spacetimes, namely, the anisotropic and spatially flat solutions of Bianchi type I and VII$_0$, and to the case of an off-center LTB universe, which includes the standard LTB model as a special case. Specializing to the case of CMB temperature fluctuations, we have provided asymptotic expansions of these correlation functions around the known Friedmannian case. Each spacetime leaves its own multipolar fingerprint on the CMB covariance matrix $\langle a_{\ell_1 m_1} a_{\ell_2 m_2} \rangle$. To the lowest order in the expansion parameters, we have found that Bianchi-I spacetimes lead to quadrupolar couplings $\langle a_{\ell_1 m_1} a_{\ell_1 \pm 2, m_2} \rangle$ while preserving the isotropic angular spectrum $C_\ell$. Bianchi VII$_0$ models, on the other hand, lead to quadrupole couplings as well as suppression of the $C_\ell$s, whereas an off-center LTB metric leads to dipolar couplings and a modification of the $C_\ell$s — the latter depending on a free function which has to be fixed by solving the photon transport equations in this geometry.

We have also applied the method to infer the functional dependence of (non-Gaussian) three point correlation functions to the (well-known) case of a FLRW universe, and also to the case of an off-center LTB universe. As a byproduct we have found a formal link between the three-point correlation function in an FLRW universe and a Gaussian 2pcf in an off-center LTB universe. This link results from the fact that a universe with a strong dependence on the three-point correlation function is geometrically degenerate to a Gaussian universe with a special point.

We would like to end with some remarks on the limitations and possible extensions of the formalism. First we stress that, although the method can be used to quickly give the CMB multipolar couplings in a given geometry, it cannot be expected to give more information than that. The case of Bianchi-I is a clear example. While the quadrupolar couplings we found here are compatible with the result of more in-depth analysis, the present formalism cannot predict the oscillations in the power spectrum resulting from linear perturbation theory [34, 40] nor the correlation between scalar and tensor modes arising from the dynamical couplings with the shear [15, 41]. Second, we have not considered the case of spin functions, which are of central importance to the physics of polarization and weak-lensing of the CMB. The case of vector two-point functions in de Sitter spacetimes have been addressed in [1] using a different formalism, where it was found that it also has the same symmetries of the background space. In the present formalism this conclusion is not immediate since equation (2.8), when applied to more general tensor correlators, will introduce new terms coming from the Lie derivative of the tensor. We postpone such analysis to future publications. Nonetheless, we emphasize that the method developed here is general, and can be equally useful in applications to quantum field theory in curved spacetime.

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A Miscellanea

We gather here some useful formulae and results which were used in the main text.

A.1 Power spectrum and Hankel transform

In the examples considered in this work, the correlation function lacks global rotation symmetry, so that it depends on the vector connecting two points in the following manner

$$\xi(r_2, r_1) = \xi(r_\pm), \quad r_\pm = r_2 \pm r_1. \quad (A.1)$$

In this case the power spectrum also becomes a direction-dependent function of the Fourier vector

$$\pm P(k) = \int d^3r_\pm e^{-i k \cdot r_\pm} \xi(r_\pm). \quad (A.2)$$

To relate the multipolar coefficients of $P_\pm$ to those of $\xi$ we first use Rayleigh’s expansion

$$e^{-i k \cdot r_\pm} = 4\pi \sum_{\ell, m} i^{-\ell} j_\ell(kr_\pm) Y_{\ell m}(\hat{k}) Y_{\ell m}^*(\hat{n}_\pm), \quad r_\pm = r_\pm \hat{n}_\pm. \quad (A.3)$$

Next we decompose both $P_\pm$ and $\xi$ into spherical harmonics and use their orthogonality relation to express the multipolar coefficients of each function. The result is the Hankel transform of the power spectrum (see ref. [44] for its use in cosmology)

$$\pm P_{\ell m} = 4\pi i^{-\ell} \int_0^\infty r_\pm dr_\pm j_\ell(kr_\pm) \xi_{\ell m}(r_\pm). \quad (A.4)$$

A.2 Covariance matrix

Since expression (3.41) is not very popular, we show here that it does lead to the correct results when the universe is homogeneous and isotropic. For a FLRW universe we have

$$\xi_{\ell_3 m_3}(r_-) = \xi_{0 0}(r_-) \delta_{\ell_3 0} \delta_{m_3 0}. \quad (A.5)$$

For this multipolar combination the Gaunt factor becomes

$$G_{m_1 m_2 0}^{\ell_1 \ell_2 0} = \frac{(-1)^{m_1}}{\sqrt{4\pi}} \delta_{\ell_3 0} \delta_{m_3 0}. \quad (A.6)$$

Moreover

$$J_{\ell_1, \ell_2 0}^{(-)}(r_-) = \int_0^\infty k^2 dk j_{\ell_1}(k\Delta) j_{\ell_2}(k\Delta) j_0(kr_-)$$

$$= \frac{\pi}{2 (\Delta)^2 r_-} \int_0^\infty dx J_{\ell_1+1/2}^2(x) \sin(2ax), \quad a \equiv r_-/(2\Delta)$$

$$= \frac{\pi}{4 (\Delta)^2 r_-} P_{\ell_1} \left(1 - 2a^2\right)$$

where in the last step we have used integral 6.672.5 of ref. [45]. Next we recall that

$$r_-^2 = 2 (\Delta)^2 (1 - \cos \gamma) = 4 (\Delta)^2 a^2 \quad (A.7)$$
which gives
\[ J^{(-)}_{\ell_{1}\ell_{1}0}(r_{-}) = \frac{\pi}{4(\Delta \eta)^2 r_{-}} P_{\ell_{1}}(\cos \gamma). \]  

(A.8)

Bringing everything together in expression (3.41) we find
\[
\langle a_{\ell_{1}m_{1}} a_{\ell_{2}m_{2}} \rangle = \frac{8}{9} \int_{0}^{2\Delta \eta} r_{-}^2 \, dr_{-} \, \xi_{00}^{*}(r_{-}) J^{(-)}_{\ell_{1}\ell_{1}0}(r_{-}) \left( -\frac{1}{4\pi} \right)^{\frac{1}{2}} \delta_{\ell_{1}\ell_{2}} \delta_{m_{1},-m_{2}} ,
\]
\[
= \frac{2\pi}{9(\Delta \eta)^2} \left( -\frac{1}{4\pi} \right)^{\frac{1}{2}} \int_{0}^{2\Delta \eta} r_{-} \, dr_{-} \, \xi_{00}^{*}(r_{-}) P_{\ell_{1}}(\cos \gamma) \delta_{\ell_{1}\ell_{2}} \delta_{m_{1},-m_{2}} ,
\]

We now note that \( r_{-} \, dr_{-} = (\Delta \eta)^2 \, d(-\cos \gamma) \) so that
\[
\langle a_{\ell_{1}m_{1}} a_{\ell_{2}m_{2}} \rangle = (-1)^{m_{1}} \left[ \frac{2\pi}{9} \int_{-1}^{1} d(\cos \gamma) \xi_{FL}(\gamma) P_{\ell_{1}}(\cos \gamma) \right] \delta_{\ell_{1}\ell_{2}} \delta_{m_{1},-m_{2}} ,
\]
\[
= (-1)^{m_{1}} C_{\ell_{1}\ell_{2}\ell_{3}} \delta_{\ell_{1}\ell_{2}} \delta_{m_{1},-m_{2}} ,
\]

where in the last line we have used eq. (2.22) and \( \xi_{FL}(\gamma) = \xi_{00}/\sqrt{4\pi} \).

A.3 Gaunt coefficients

The Gaunt coefficients result from the integral of three spherical harmonics over the sphere. They are given by [46]
\[
C_{\ell_{1}\ell_{2}\ell_{3}} = \int d^3 \hat{n} \, Y_{\ell_{1}m_{1}}(\hat{n}) Y_{\ell_{2}m_{2}}(\hat{n}) Y_{\ell_{3}m_{3}}(\hat{n})
\]
\[
= \sqrt{\frac{(2\ell_{1}+1)(2\ell_{2}+1)(2\ell_{3}+1)}{4\pi}} \left( \begin{array}{ccc} \ell_{1} & \ell_{2} & \ell_{3} \\ m_{1} & m_{2} & m_{3} \end{array} \right) .
\]

where the \( 3 \times 2 \) matrices are the Wigner 3-J symbols. The Gaunt coefficients are identically zero whenever the sum \( \ell_{1} + \ell_{2} + \ell_{3} \) is an odd number, and whenever \( m_{1} + m_{2} + m_{3} \neq 0 \).

A.4 Bipolar power spectrum

The bipolar power spectrum [9, 37] are the harmonic coefficients of the correlation function when expanded in a basis of bipolar spherical harmonics [47]. They are related to the covariance matrix as
\[
A^{LM}_{\ell_{1}\ell_{2}} = \sum_{m_{1},m_{2}} \langle a_{\ell_{1}m_{1}} a_{\ell_{2}m_{2}} \rangle (-1)^{M+\ell_{1}-\ell_{2}} \sqrt{2L+1} \left( \begin{array}{ccc} \ell_{1} & \ell_{2} & L \\ m_{1} & m_{2} & -M \end{array} \right) .
\]

(A.9)

Using the identity [46]
\[
\sum_{L,M} (2L+1) \left( \begin{array}{ccc} \ell_{1} & \ell_{2} & L \\ m_{1} & m_{2} & M \end{array} \right) \left( \begin{array}{ccc} \ell_{1} & \ell_{2} & L \\ m_{1} & m_{2} & M \end{array} \right) = \delta_{m_{1}m_{1}^{'}} \delta_{m_{2}m_{2}^{'}} ,
\]

(A.10)

the inverse relation is found to be
\[
\langle a_{\ell_{1}m_{1}} a_{\ell_{2}m_{2}}^{*} \rangle = (-1)^{m_{1}+\ell_{1}-\ell_{2}} \sum_{L,M} \sqrt{2L+1} A^{LM}_{\ell_{1}\ell_{2}} \left( \begin{array}{ccc} \ell_{1} & \ell_{2} & L \\ m_{1} & m_{2} & -M \end{array} \right) .
\]

(A.11)

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By inserting (3.41) into (A.9) and using [46]

\[
\sum_{m_1,m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix} = \frac{\delta_{L\ell_3} \delta_{M,-m_3}}{\sqrt{2L+1}}
\]

one arrives at (3.45), where the coefficients \( \mathcal{F}_{\ell_1 \ell_2 L} \) were defined by

\[
\mathcal{F}_{\ell_1 \ell_2 L} = (-1)^{\ell_1 - \ell_2} \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2L + 1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix}.
\]

References

[1] B. Allen and T. Jacobson, Vector Two Point Functions in Maximally Symmetric Spaces, Commun. Math. Phys. 103 (1986) 669 [arXiv:1102.0529] [SPIRE].
[2] E. Poisson, A. Pound and I. Vega, The Motion of point particles in curved spacetime, Living Rev. Rel. 14 (2011) 7 [arXiv:1102.0529] [SPIRE].
[3] A.D. Linde, Particle physics and inflationary cosmology, Contemp. Concepts Phys. 5 (1990) 1 [hep-th/0503203] [SPIRE].
[4] A.R. Liddle and D.H. Lyth, Cosmological inflation and large scale structure, Cambridge University Press (2000).
[5] V. Mukhanov, Physical Foundations of Cosmology, Cambridge University Press (2005).
[6] J.-P. Uzan, Inflation in the standard cosmological model, Compt. Rendus Phys. 16 (2015) 875.
[7] L.R. Abramo and T.S. Pereira, Testing Gaussianity, homogeneity and isotropy with the cosmic microwave background, Adv. Astron. 2010 (2010) 378203 [arXiv:1002.3173] [SPIRE].
[8] P. Bull et al., Beyond \( \Lambda \)CDM: Problems, solutions and the road ahead, Phys. Dark Univ. 12 (2016) 56 [arXiv:1512.05356] [SPIRE].
[9] A. Hajian and T. Souradeep, Measuring statistical isotropy of the CMB anisotropy, Astophys. J. 597 (2003) L5 [astro-ph/0308001] [SPIRE].
[10] A.R. Pullen and M. Kamionkowski, Cosmic Microwave Background Statistics for a Direction-Dependent Primordial Power Spectrum, Phys. Rev. D76 (2007) 103529 [arXiv:0709.1144] [SPIRE].
[11] S.M. Carroll, C.-Y. Tseng and M.B. Wise, Translational Invariance and the Anisotropy of the Cosmic Microwave Background, Phys. Rev. D81 (2010) 083501 [arXiv:0811.1086] [SPIRE].
[12] T.S. Pereira and L.R. Abramo, Angular-planar CMB power spectrum, Phys. Rev. D80 (2009) 063525 [arXiv:0907.2340] [SPIRE].
[13] L.R. Abramo, A. Bernui and T.S. Pereira, Searching for planar signatures in WMAP, JCAP 12 (2009) 013 [arXiv:0909.5395] [SPIRE].
[14] A.L. Froes, T.S. Pereira, A. Bernui and G.D. Starkman, New geometric representations of the CMB two-point correlation function, Phys. Rev. D92 (2015) 043508 [arXiv:1506.00705] [SPIRE].
[15] T.S. Pereira, C. Pitrou and J.-P. Uzan, Theory of cosmological perturbations in an anisotropic universe, JCAP 09 (2007) 006 [arXiv:0707.0736] [SPIRE].
[16] C. Clarkson, T. Clifton and S. February, Perturbation Theory in Lemaitre-Tolman-Bondi Cosmology, JCAP 06 (2009) 025 [arXiv:0903.5040] [SPIRE].
[17] T.S. Pereira, S. Carneiro and G.A.M. Marugan, Inflationary Perturbations in Anisotropic, Shear-Free Universes, JCAP 05 (2012) 040 [arXiv:1203.2072] [SPIRE].
[18] C. Pitrou, T.S. Pereira and J.-P. Uzan, “Weak-lensing by the large scale structure in a spatially anisotropic universe: theory and predictions,” *Phys. Rev.* **D92** (2015) 023501 [arXiv:1503.01125] [arXiv:1503.01125] [IN:SPIRE].

[19] T.S. Pereira, G.A.M. Marugán and S. Carneiro, “Cosmological Signatures of Anisotropic Spatial Curvature,” *JCAP* **07** (2015) 029 [arXiv:1505.00794] [arXiv:1505.00794] [IN:SPIRE].

[20] I. Antoniadis, P.O. Mazur and E. Mottola, “Conformal invariance and cosmic background radiation,” *Phys. Rev. Lett.* **79** (1997) 14 [astro-ph/9611208] [astro-ph/9611208] [IN:SPIRE].

[21] I. Antoniadis, P.O. Mazur and E. Mottola, “Conformal Invariance, Dark Energy and CMB Non-Gaussianity,” *JCAP* **09** (2012) 024 [arXiv:1103.4164] [arXiv:1103.4164] [IN:SPIRE].

[22] Courant and D. Hilbert, *Methods of mathematical physics*, volume II, Wiley (1966).

[23] O. Kowalski and L. Vanhecke, “Two-point functions on riemannian manifolds,” *Ann. Global Anal. Geom.* **3** (1985) 95.

[24] S.W. Hawking, *On the Rotation of the universe*, *Mon. Not. Roy. Astron. Soc.* **142** (1969) 129 [IN:SPIRE].

[25] R. Courant and D. Hilbert, *Exact solutions of Einstein’s field equations*. Cambridge University Press (2009).

[26] G. Cusin, C. Pitrou and J.-P. Uzan, “Are we living near the center of a local void?”, *arXiv:1609.02061* [arXiv:1609.02061] [IN:SPIRE].

[27] I. Masina and A. Notari, “Detecting the Cold Spot as a Void with the Non-Diagonal Two-Point Function,” *JCAP* **09** (2010) 028 [arXiv:1007.0204] [arXiv:1007.0204] [IN:SPIRE].

[28] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, “Exact solutions of Einstein’s field equations,” Cambridge University Press (2009).

[29] A. Pontzen and A. Challinor, “Linearization of homogeneous, nearly-isotropic cosmological models,” *Class. Quant. Grav.* **28** (2011) 185007 [arXiv:1009.3935] [arXiv:1009.3935] [IN:SPIRE].

[30] PLANck collaboration, P.A.R. Ade et al., “Planck 2013 results. XXIII. Isotropy and statistics of the CMB,” *Astron. Astrophys.* **571** (2014) A23 [arXiv:1303.5083] [arXiv:1303.5083] [IN:SPIRE].

[31] PLANck collaboration, P.A.R. Ade et al., “Planck 2015 results. XVI. Isotropy and statistics of the CMB,” *Astron. Astrophys.* **594** (2016) A16 [arXiv:1506.07135] [arXiv:1506.07135] [IN:SPIRE].

[32] D. Saadeh, S.M. Feeney, A. Pontzen, H.V. Peiris and J.D. McEwen, “How isotropic is the Universe?”, *Phys. Rev. Lett.* **117** (2016) 131302 [arXiv:1605.07178] [arXiv:1605.07178] [IN:SPIRE].

[33] J.D. Barrow, D.H. Sonoda and R. Juszkiewicz, “Structure of the cosmic microwave background,” *Nature* **305** (1983) 397.

[34] A.E. Gumrukcuoglu, C.R. Contaldi and M. Peloso, “Inflationary perturbations in anisotropic backgrounds and their imprint on the CMB,” *JCAP* **11** (2007) 005 [arXiv:0707.4179] [arXiv:0707.4179] [IN:SPIRE].

[35] R. Mehrem, J.T. Londergan and M.H. Macfarlane, “Analytic expressions for integrals of products of spherical Bessel functions,” *J. Phys.* **A24** (1991) 1435 [arXiv:1509.09166] [arXiv:1509.09166] [IN:SPIRE].

[36] T. Pereira and C. Pitrou, “Isotropization of the universe during inflation,” *Comptes Rendus Physique* **16** (2015) 1027 [arXiv:1509.09166] [arXiv:1509.09166] [IN:SPIRE].

[37] A. Hajian and T. Souradeep, “The cosmic microwave background bipolar power spectrum: Basic formalism and applications,” *astro-ph/0501001* [astro-ph/0501001] [IN:SPIRE].

[38] T.S. Pereira, C. Pitrou and J.-P. Uzan, “Weak-lensing $B$-modes as a probe of the isotropy of the universe,” *Astron. Astrophys.* **585** (2016) L3 [arXiv:1503.01127] [arXiv:1503.01127] [IN:SPIRE].

[39] I. Masina and A. Notari, “The Cold Spot as a Large Void: Lensing Effect on CMB Two and Three Point Correlation Functions,” *JCAP* **07** (2009) 035 [arXiv:0905.1073] [arXiv:0905.1073] [IN:SPIRE].
[40] C. Pitrou, T.S. Pereira and J.-P. Uzan, Predictions from an anisotropic inflationary era, JCAP 04 (2008) 004 [arXiv:0801.3596] [InSPIRE].

[41] A.E. Gumrukcuoglu, B. Himmetoglu and M. Peloso, Scalar-Scalar, Scalar-Tensor and Tensor-Tensor Correlators from Anisotropic Inflation, Phys. Rev. D81 (2010) 063528 [arXiv:1001.4088] [InSPIRE].

[42] G.W. Pettinari, The intrinsic bispectrum of the Cosmic Microwave Background, Ph.D. Thesis, Portsmouth U., ICG, 2013-09, Springer Theses (2016) [arXiv:1405.2280] [InSPIRE].

[43] F. Schmidt and L. Hui, Cosmic Microwave Background Power Asymmetry from Non-Gaussian Modulation, Phys. Rev. Lett. 110 (2013) 011301 [Erratum ibid. 110 (2013) 059902] [arXiv:1210.2965] [InSPIRE].

[44] L.R. Abramo, P.H. Reimberg and H.S. Xavier, CMB in a box: causal structure and the Fourier-Bessel expansion, Phys. Rev. D82 (2010) 043510 [arXiv:1005.0563] [InSPIRE].

[45] A. Jeffrey and D. Zwillinger, Table of integrals, series, and products, Academic Press (2007).

[46] A.R. Edmonds, Angular momentum in quantum mechanics, Princeton University Press (1996).

[47] D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii, Quantum theory of angular momentum, World Scientific (1988).