Quantum smoother for open quantum systems driven by quantum jump-diffusion processes

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Abstract

In this paper, we develop a quantum smoothing theory to the class of open quantum systems driven by quantum jump–diffusion processes. A quantum smoothing theory, which is derived in a different sense by different researchers, has been applied for diffusive quantum Markovian systems, however, they only deal with quantum Wiener noises. We show how to derive the quantum smoother for quantum dynamical systems driven by quantum jump–diffusion noises.

1 Introduction

This paper deals with a quantum dynamical estimation problem, so-called a quantum smoothing problem, for quantum dynamical systems driven by quantum jump and diffusive noises. Several theories of quantum smoothing have been developed in different ways [1, 2, 3, 4, 5, 6], however, they only deal with open quantum Markovian systems driven by quantum Wiener noises. On the other hand, quantum Poisson processes, which are modeled by quantum counting processes [7], describe phase–diffusion processes, which arise in quantum optics [8, 9] and quantum computing processes [10]. Therefore, it is important to show a formula of the smoother for a given quantum dynamics driven by jump–diffusive quantum noises.

In [6], a quantum smoothing theory based on non-commutative orthogonal principle was introduced. Although the basic technique developed in [6] can apply general quantum Markovian systems, the derivation of the quantum filtering or smoothing equation is a bit tough if the readers are unfamiliar with quantum stochastic calculus. In this paper we show a derivation of the quantum filtering and smoothing equation of the quantum diffusion-jump process.

The rest of the paper is organized as follows. In Section 2, basic notions of the quantum theory are introduced. We show the quantum least mean square estimation method in Section 3. Our main result is in Section 4.

Notation

\( \mathbb{R} \) and \( \mathbb{C} \) are real numbers and complex numbers, respectively, and \( i := \sqrt{-1} \). \( \mathcal{H} \) is a complex Hilbert space and we also denote \( \mathcal{H}_X \) if it is the Hilbert space of the system \( X \). Any linear operator on a Hilbert space \( \mathcal{H} \) is denoted by hat, e.g., \( \hat{X} \). When positive operators \( \hat{X} \) and \( \hat{Y} \) satisfy \( \hat{X} = \hat{Y}^2 \), we denote \( \hat{Y} = \sqrt{\hat{X}} \). The absolute value of operator is defined \( |\hat{X}| := \sqrt{\hat{X}^* \hat{X}} \).\( \mathcal{L}(\mathcal{H}) \) is a set of linear bounded operators on the Hilbert space \( \mathcal{H} \). \( \hat{X} \geq 0 \) means that \( \hat{X} \in \mathcal{L}(\mathcal{H}) \) is a positive operator and \( \hat{X}^* \) implies the conjugate operator of \( \hat{X} \). \( \text{Tr}[\bullet] : \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{C} \) is the trace on linear operators. \( \mathcal{S}(\mathcal{H}) := \{ \hat{\rho} \in \mathcal{L}(\mathcal{H}) \mid \hat{\rho} \geq 0, \text{Tr}[\hat{\rho}] = 1 \} \) is a set of density operators. \( \mathcal{I}_H \) is the identity operator on \( \mathcal{H} \) and we sometimes omit its subscript. Denote \( [\hat{X}, \hat{Y}]_{\pm} := \hat{X}\hat{Y} \pm \hat{Y}\hat{X}, \forall \hat{X}, \hat{Y} \in \mathcal{L}(\mathcal{H}) \). \( \otimes \) represents the Kronecker product for matrices and the tensor product for operators or spaces.

2 Basics of quantum theory

Here we briefly review the quantum theory. For details, see, e.g., [11, 12]. Every quantum system is described by a suitably defined Hilbert space \( \mathcal{H} \). All of the quantum physical quantities are denoted by self-adjoint operators on \( \mathcal{H} \). In this paper, we only consider linear bounded operators except quantum noise operators. We denote a set of linear bounded operators on \( \mathcal{H} \) by \( \mathcal{L}(\mathcal{H}) \). The observation of any quantum physical quantity is a randomly chosen number from the spectrum of the corresponding self-adjoint operator. Random outcomes of all bounded operators make the quantum statistics and the quantum expectation \( \mathbb{P}_\rho \) is defined as \( \mathbb{P}_\rho[X] := \text{Tr}[\hat{\rho}X], \hat{\rho} \in \mathcal{S}(\mathcal{H}) \). The quantum version of \( \sigma \)-measurable functions is von Neumann algebra, which is, roughly speaking, an algebra generated by projection operators with algebraic operations [13]. Let \( \mathcal{A} \subseteq \mathcal{L}(\mathcal{H}) \) be a von Neumann subalgebra. A pair \( (\mathcal{A}, \mathbb{P}_\rho) \) is called the quantum probability space. For a given quantum probability space \( (\mathcal{A}, \mathbb{P}_\rho) \), a subalgebra \( \mathcal{N}_\rho := \{ \hat{X} \in \mathcal{A} \mid \mathbb{P}_\rho[\hat{X}^* \hat{X}] = 0 \} \) of \( \mathcal{A} \) is a quantum version of the measure zero set with respect to \( \mathbb{P}_\rho \), called...
the left kernel of \( P_\rho \) [14]. The left kernel \( N_\rho' \) is not empty because it always includes 0. Moreover, \( N_\rho' \) is a left ideal and satisfies \( P_\rho \left( (\hat{X} + \hat{Z}_1)^* (\hat{Y} + \hat{Z}_2) \right) = P_\rho \left[ \hat{X}^* \hat{Y} \right] \) for any \( \hat{X}, \hat{Y} \in A \) and \( \hat{Z}_1, \hat{Z}_2 \in N_\rho' \) (see, e.g., [14, Lemma 9.6 in Chap. 1]). If for \( \hat{X}, \hat{Y} \in A \), there exists \( \hat{Z} \in N_\rho' \) s.t. \( \hat{X} = \hat{Y} + \hat{Z} \), then we denote \( \hat{X} = \hat{Y}, P_\rho-\text{a.s.} \) or \( \hat{\rho} -\text{a.s.} \) for short.

3 Least mean square estimation

In this section, we introduce an pre-inner product which is a key to derive a quantum smoother [6]. Let \( \mathcal{Y} \) be a commutative \( * \)-subalgebra of \( \mathcal{L}(H) \). We introduce another \( * \)-subalgebra whose elements commute with all of the elements in \( \mathcal{Y} \);

\[
\mathcal{Y}' := \{ \hat{X} \in \mathcal{L}(H) \mid \hat{X} \hat{Y} = \hat{Y} \hat{X}, \forall \hat{Y} \in \mathcal{Y} \}.
\]

\( \mathcal{Y}' \) is called a commutant of \( \mathcal{Y} \) in \( \mathcal{L}(H) \). Hereafter we assume \( \mathcal{Y} = (\mathcal{Y}')' \), i.e., \( \mathcal{Y} \) is a commutative von Neumann subalgebra [15, 14]. Every von Neumann algebra is a generalization of the set of the \( \sigma \)-measurable bounded functions and especially every commutative von Neumann algebra is isomorphic to the corresponding set of the \( \sigma \)-measurable bounded functions. Note that \( \mathcal{Y}' \) is generally non-commutative \( * \)-subalgebra.

3.1 Definitions

We introduce three approximations of a given \( \hat{X} \in \mathcal{L}(H) \) here and show that two of them provides the best estimations in different measures in following subsection.

Definition 1. For given \( \hat{\rho} \in \mathcal{S}(H) \), the pre-inner product \( \left( \bullet, \bullet \right)_\hat{\rho} : \mathcal{L}(H) \times \mathcal{L}(H) \to \mathbb{C} \) is defined by

\[
\left( \hat{X}, \hat{Y} \right)_\hat{\rho} := P_\hat{\rho} \left[ \hat{X}^* \hat{Y} \right].
\]

This pre-inner product satisfies the Cauchy-Schwarz inequality (see, e.g., [14, Prop. 9.5 in Chap. 1]). \( \left( \hat{X}, \hat{X} \right)_\hat{\rho} = 0 \) is the necessary and sufficient condition for \( \hat{X} \in N_\rho' \). We use the pre-inner product to find the best approximation in \( \mathcal{Y} \). Hereafter, we call \( \left( \bullet, \bullet \right)_\hat{\rho} \) the inner product for short.

Let us define the quantum conditional expectation (see, e.g., [15, Sec. 3] or [14, Prop. 2.36]).

Definition 2 (Quantum conditional expectation). Let \( (\mathcal{L}(H), P_\rho) \) be a quantum probability space and \( \mathcal{Y} \) be a commutative von Neumann sub-algebra of \( \mathcal{L}(H) \). A linear operator \( \hat{Q} \in \mathcal{Y} \) is called a version of the quantum conditional expectation if there exists \( \hat{Q} \in \mathcal{Y} \) satisfies

\[
\left( \hat{Z}, \hat{X} - \hat{Q} \right)_\hat{\rho} = 0, \quad \forall \hat{Z} \in \mathcal{Y} \tag{1}
\]

for arbitrary fixed \( \hat{X} \in \mathcal{Y}' \). Then we denote \( \hat{Q} = P_\hat{\rho} \left[ \hat{X} \mid \mathcal{Y} \right] \).

Some properties of the quantum conditional expectation are shown in, for example, [15]. The definition of the quantum conditional expectation implies that the \( \hat{X} - P_\hat{\rho} \left[ \hat{X} \mid \mathcal{Y} \right] \) and the commutative sub-algebra \( \mathcal{Y} \) are orthogonal under state \( P_\rho \). We extend the definition of orthogonality to non-commutative regime.

Definition 3. Let \( (\mathcal{L}(H), P_\rho) \) be a quantum probability space and \( \mathcal{Y} \) be a commutative von Neumann sub-algebra of \( \mathcal{L}(H) \). For arbitrary fixed \( \hat{X} \in \mathcal{L}(H) \), we define following operators: A linear operator \( \hat{Q} \in \mathcal{Y} \) is called a version of quantum least mean square estimate if there exists \( \hat{Q} \in \mathcal{Y} \) that satisfies

\[
\left( \hat{Z}, \hat{X} - \hat{Q} \right)_\hat{\rho} = 0, \quad \forall \hat{Z} \in \mathcal{Y}. \tag{2}
\]

Then we denote \( \hat{Q} = Q_\rho \left[ \hat{X} \mid \mathcal{Y} \right] \).

We call Eq. (2) the orthogonal condition. If \( N_\rho' \cap \mathcal{Y} \neq \{0\} \), then there are many operators that satisfy above conditions. This is why we use “a version of.” Obviously, \( P_\rho \left[ \hat{X} \right] = P_\rho Q_\rho \left[ \hat{X} \mid \mathcal{Y} \right] \) holds, i.e., this is an unbiased estimate. The name “the quantum least mean square estimate” is originated from its statistical property shown in the following subsection. Note that even if \( \hat{X} = \hat{X}^* Q_\rho \left[ \hat{X} \mid \mathcal{Y} \right] \) is not self-adjoint in general.

3.2 Basic properties

We also introduce some important properties of the quantum least mean square estimate. We omit the proofs, so if the readers would like to confirm how to prove them, see [6] for the details. From the definitions, following properties hold.

1. (linearity) \( Q_\rho [\bullet \mid \mathcal{Y}] : \mathcal{L}(H) \to \mathcal{Y} \) is linear.

2. (uniqueness) For any \( \hat{X} \in \mathcal{L}(H) \), \( Q_\rho \left[ \hat{X} \mid \mathcal{Y} \right] \) is uniquely determined \( P_\rho \)-a.s.

3. (least mean squareness) For arbitrary \( \hat{X} \in \mathcal{L}(H) \),

\[
\left( \hat{X} - Q_\rho \left[ \hat{X} \mid \mathcal{Y} \right], \hat{X} - Q_\rho \left[ \hat{X} \mid \mathcal{Y} \right] \right)_\hat{\rho} \leq \left( \hat{X} - \hat{Z}, \hat{X} - \hat{Z} \right)_\hat{\rho}, \quad \forall \hat{Z} \in \mathcal{Y}.
\]

4. For any \( \hat{X} \in \mathcal{Y} \), \( Q_\rho \left[ \hat{X} \mid \mathcal{Y} \right] = \hat{X} \).

5. Let \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) be commutative von Neumann sub-algebras of \( \mathcal{L}(H) \) and assume \( \mathcal{Y}_1 \subseteq \mathcal{Y}_2 \). Then, for any \( \hat{X} = \hat{X}^* \in \mathcal{L}(H) \),

\[
\left( \hat{X} - Q_\rho \left[ \hat{X} \mid \mathcal{Y}_1 \right], \hat{X} - Q_\rho \left[ \hat{X} \mid \mathcal{Y}_2 \right] \right)_\hat{\rho} \leq \left( \hat{X} - Q_\rho \left[ \hat{X} \mid \mathcal{Y}_1 \right], \hat{X} - Q_\rho \left[ \hat{X} \mid \mathcal{Y}_1 \right] \right)_\hat{\rho}.
\]
4 Quantum dynamics, a measurement process, and dynamical estimators

4.1 A quantum dynamical system driven by quantum jump–diffusion noises

We consider an open quantum system interacting with vacuum environment and denote \( H_S \) and \( H_E \) for their Hilbert space. Since the time evolution of the the system and environment is described by a unitary operator on \( \mathcal{H} := H_S \otimes H_E \), the evolution equation of the system is described by the following quantum stochastic differential equation [16]:

\[
d\hat{U}(t) = \left( -i\hat{H} - \frac{1}{2} \hat{L}^* \hat{L} \right) dt + \hat{L} d\hat{A}^* - \hat{L}^* d\hat{S}(t) + (\hat{S} - 1) d\hat{\Lambda}(t) \right) \hat{U}(t) \tag{3}
\]

where \( \hat{H} = \hat{H}^* \), \( \hat{L}, \hat{S} \in \mathcal{L}(H_S) \), \( \hat{S} \) is a unitary operator. \( \hat{L} \) and \( \hat{S} \) imply the interaction between the quantum system and its environment, and \( \hat{H} \) is the system’s Hamiltonian. \( \hat{A}(t), \hat{A}^*(t), \hat{\Lambda}(t) \in \mathcal{L}(H_E) \) are an annihilation, creation, and quantum counting processes, respectively. Note that, e.g., \( \hat{L} d\hat{A}^*(t) \) in Eq. (3) implies \((\hat{L} \otimes 1_{H_S})(1_S \otimes d\hat{A}^*(t))\), but we omit the identity operators for simplicity. Each of the triplet \((\hat{A}, \hat{A}^*, \hat{\Lambda})\) is called a quantum noise, and \( \hat{A} \) and \( \hat{A}^* \) describe quantum Wiener processes and \( \hat{\Lambda} \) describes a quantum Poisson process under adequate quantum probability law, called a state [7]. See, e.g., [17] for physical interpretation of the quantum noises. Using the following quantum noise multiplication formulas [16]

\[
\begin{align*}
d\hat{A}(t)d\hat{A}^*(t) &= dt, \quad d\hat{A}(t)d\hat{A}(t) = d\hat{A}(t), \\
d\hat{A}(t)d\hat{L}(t) &= d\hat{L}(t), \quad d\hat{A}(t)d\hat{L}^*(t) = d\hat{L}^*(t), \\
dtd\hat{A}(t) &= dt d\hat{A}(t) = dt, \quad d\hat{A}^*(t) = \hat{d} d\hat{A}(t) = dt^2 = 0,
\end{align*}
\]

the quantum evolution equation of the open quantum system is described as follows: let \( \hat{X}(t) := U(t)^* X U(t), \hat{H}(t) := U(t)^* H U(t), \hat{L}(t) := U(t)^* L U(t) \), and \( \hat{S}(t) := U(t)^* SU(t) \), then

\[
d\hat{X}(t) = i[\hat{X}(t), \hat{H}(t)]_ \ldots dt + \frac{1}{2} \left( \hat{L}^*(t)[\hat{X}(t), \hat{L}(t)]_ \ldots + [\hat{L}^*(t), \hat{X}(t)]_ \ldots \hat{L}(t) \right) dt + \hat{S}^*(t)[\hat{X}(t), \hat{L}(t)]_ \ldots d\hat{A}^*(t) + [\hat{L}^*(t), \hat{X}(t)]_ \ldots \hat{S}(t) d\hat{A}(t) + \left( \hat{S}^*(t)\hat{X}(t)\hat{S}(t) - \hat{X}(t) \right) d\hat{\Lambda}(t) \tag{4}
\]

Next, let us introduce a measurement process in order to construct a quantum dynamical estimator. In quantum filtering theory, two types of measurement processes are commonly used: a homodyne measurement and counting measurement. Roughly speaking, they correspond to the measurements which are corrupted by Wiener noise and Poisson noise, respectively [15]. We only consider a homodyne measurement in this paper. The measurement process is represented by \( \hat{Y}(t) := \hat{U}^*(t)(\hat{A}(t) + \hat{A}(t)^*)\hat{U}(t) \) and its quantum stochastic differential equation is

\[
d\hat{Y}(t) = \hat{S}(t)d\hat{A}(t) + \hat{S}^*(t)d\hat{L}(t) + (\hat{L}(t) + \hat{L}^*(t)) dt \tag{5}
\]

Since \( \{\hat{Y}(s)\}_{s \in [0, t]} \) gives a commutative subalgebra \( \mathcal{Y}_t \) of the von Neuman algebra \( \mathcal{L}(H) \) [15], there is a set of classical random variables that corresponds to \( \mathcal{Y}_t \). Therefore, the quantum filter and smoother can be constructed as classical stochastic systems.

4.2 Quantum filtering

Let \( \mathbb{P} : \mathcal{L}(H) \rightarrow \mathbb{C} \) be a state, \( \hat{\pi}_t(\hat{X}) := \mathbb{P}(\hat{X}(t) | \mathcal{Y}_t) \) be the quantum conditional expectation of \( \hat{X}(t) \). Then the quantum filter is

\[
d\hat{\pi}_t(\hat{X}) = \hat{\pi}_t \left( i[\hat{H}, \hat{X}]_ \ldots \right) + \frac{1}{2} \hat{\pi}_t \left( \hat{L}^*[\hat{X}, \hat{L}]_ \ldots + [\hat{L}, \hat{X}][\hat{L}]_ \ldots \right) dt + \hat{\pi}_t \left( \hat{L}^* \hat{X} + \hat{X} \hat{L} - \hat{\pi}_t(\hat{X})(\hat{L} + \hat{L}^*) \right) \times \left( d\hat{Y}_t - \hat{\pi}_t(\hat{L} + \hat{L}^*) dt \right). \tag{6}
\]

4.3 Quantum smoothing

Let \( \mathbb{Q}_t(\hat{X}) := \mathbb{Q}_t[\hat{X} | \mathcal{Y}_t] \) be the quantum least mean estimator of \( \hat{X} \) on \( \mathcal{Y}_t \) [6]. As in the classical smoothing theory [18], the innovation process \( d\hat{Y}_t - \hat{\pi}_t(\hat{L} + \hat{L}^*) dt \) is the essential information update. Since \( \mathbb{Q}_t(\hat{X}) \) can be regarded as a classical stochastic process, we can exploit the Fujisaki–Kallianpur–Kunita’s theorem. Besides, the estimand \( \hat{X}(0) = \hat{X} \) does not vary in time; therefore, the dynamics of the estimator should be found in the following form.

\[
d\mathbb{Q}_t(\hat{X}) = \hat{\Gamma}_t \left( d\hat{Y}_t - \hat{\pi}_t(\hat{L} + \hat{L}^*) dt \right), \tag{7}
\]

where \( \hat{\Gamma}_t \in \mathcal{Y}_t \) is the unknown coefficient. Note that the process (7) is martingale in classical sense [18]. The rest problem is how to determine the coefficient.

Theorem 4.

\[
d\mathbb{Q}_t(\hat{X}) = \hat{\mathbb{Q}}_t \left( \hat{X}(\hat{L}(t) + \hat{L}^*(t)) - \mathbb{Q}_t(\hat{X}) \hat{\pi}_t(\hat{L} + \hat{L}^*) \right) \times \left( d\hat{Y}_t - \hat{\pi}_t(\hat{L} + \hat{L}^*) dt \right), \tag{8}
\]
Proof.

\[
\begin{align*}
&d[(\hat{X} - Q_{t}(\hat{X}))\hat{Y}_{t}] \\
&= -dQ_{t}(\hat{X})\hat{Y}_{t} + (\hat{X} - Q_{t}(\hat{X}))d\hat{Y}_{t} - dQ_{t}(\hat{X})d\hat{Y}_{t} \\
&= -\hat{\Gamma}_{t}\left(d\hat{Y}_{t} - \hat{\pi}_{t}(\hat{L} + \hat{L}^{*})dt\right)\hat{Y}_{t} \\
&\quad + (\hat{X} - Q_{t}(\hat{X}))\left(\hat{S}(t)d\hat{A}(t) + \hat{S}^{*}(t)d\hat{A}^{*}(t)\right) \\
&\quad + (\hat{L}(t) + \hat{L}^{*}(t))dt \\
&\quad - \hat{\Gamma}_{t}\left(d\hat{Y}_{t} - \hat{\pi}_{t}(\hat{L} + \hat{L}^{*})dt\right)d\hat{Y}_{t}.
\end{align*}
\]

From the quantum noise multiplication formulas,

\[
\left(d\hat{Y}_{t} - \hat{\pi}_{t}(\hat{L} + \hat{L}^{*})dt\right)d\hat{Y}_{t} = dt.
\]

Since \(Q_{t}[d[(\hat{X} - Q_{t}(\hat{X}))\hat{Y}_{t}]|\mathcal{Y}_{t}] = 0\), taking this operation to the both side of (9), then we obtain

\[
\hat{\Gamma}_{t} = Q_{t}\left(\hat{X}(\hat{L}(t) + \hat{L}^{*}(t)) - Q_{t}(\hat{X})\hat{\pi}_{t}(\hat{L} + \hat{L}^{*})\right).
\]

\[
\square
\]

5 Conclusion

We introduced a quantum smoother for quantum Markovian systems driven by quantum jump–diffusion noises. The derivation of the quantum smoother is shown.

References

[1] Masahiro Yanagisawa: Quantum smoothing. e-print arXiv:0711.3885, 2007.

[2] Mankei Tsang: Time-symmetric quantum theory of smoothing. Physical Review Letters, 102(25):250403, 2009.

[3] Mankei Tsang: A Bayesian quasi-probability approach to inferring the past of quantum observables. e-print arXiv:1403.3353, 2014.

[4] Ivonne Guevara and Howard Wiseman: Quantum State Smoothing. Physical Review Letters, 115:180407, Oct 2015.

[5] Søren Gammelmark, Brian Julsgaard, and Klaus Mølmer: Past Quantum States of a Monitored System. Physical Review Letters, 111:160401, Oct 2013.

[6] Kentaro Ohki: A smoothing theory for open quantum systems: The least mean square approach. In 2015 54th IEEE Conference on Decision and Control, pages 4350 – 4355, 2015.

[7] K. R. Parthasarathy: An Introduction to Quantum Stochastic Calculus, volume 85 of Monographs in Mathematics. Birkhauser Verlag, 1992.

[8] H. A. Bachor and T. C. Ralph: A Guide to Experiments in Quantum Optics. Wiley–VCH, 2004.

[9] D. F. Walls and G. J. Milburn: Quantum Optics. Springer Verlag, 2 edition, 2008.

[10] Michael A. Nielsen and Isaac L. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000.

[11] Howard M. Wiseman and Gerard J. Milburn: Quantum Measurement and Control. Cambridge University Press, 2009.

[12] Alexander S. Holevo: Probability and Statistical Aspects of Quantum Theory. New York: North-Holland, 1982.

[13] Luc Bouten, Ramon van Handel, and Matthew R. James: A Discrete Invitation to Quantum Filtering and Feedback Control. SIAM Review, 51(2):239–316, 2009.

[14] Masamichi Takesaki: Theory of Operator Algebras I. Springer, 2003.

[15] Luc Bouten, Ramon van Handel, and Matthew R. James: An Introduction to Quantum Filtering. SIAM Journal on Control and Optimization, 46(6):2199–2241, 2007.

[16] R. L. Hudson and K. R. Parthasarathy: Quantum Ito’s Formula and Stochastic Evolutions. Communications in Mathematical Physics, 93(3):301–323, 1984.

[17] Crispin W. Gardiner and Peter Zoller: Quantum Noise : A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods With Applications to Quantum Optics. Springer–Verlag, 3 edition, 2004.

[18] Venkatarama Krishnan: Nonlinear Filtering and Smoothing : An Introduction to Martingales, Stochastic Integrals and Estimation. John Wiley & Sons, 1984.