The Einstein 3-form $G_\alpha$ and its equivalent 1-form $L_\alpha$ in Riemann-Cartan space

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Abstract
The definition of the Einstein 3-form $G_\alpha$ is motivated by means of the contracted 2nd Bianchi identity. This definition involves first the complete curvature 2-form. The 1-form $L_\alpha$ is defined via $G_\alpha = L^\beta \wedge \ast (\theta_\beta \wedge \theta_\alpha)$ (here $\ast$ is the Hodge-star, $\theta_\alpha$ the coframe). It is equivalent to the Einstein 3-form and represents a certain contraction of the curvature 2-form. A variational formula of Salgado on quadratic invariants of the $L_\alpha$ 1-form is discussed, generalized, and put into proper perspective.

1 Introduction
In a Riemannian space, the curvature can be split into the conformal (Weyl) curvature and a piece which contains a 1-form, here called $L_\alpha$. That 1-form has some very interesting properties: (i) It is closely related to the Einstein 3-form. (ii) It plays an important role in the formulation of the initial value problem which focuses on symmetric-hyperbolic equations for the conformal curvature. (iii) Recently, we learned of a nice formula of Salgado which involves a quadratic invariant of $L_\alpha$. (iv) It plays a role in the context of current investigations of the Cotton 2-form $C_\alpha$. The 1-form $L_\alpha$ appears as the potential of the Cotton 2-form, $C_\alpha := DL_\alpha$. The $C_\alpha$ is related to the conformal properties of space and, in 3 dimensions, substitutes the Weyl curvature in the criterion for conformal flatness.

These points motivate a closer look at $L_\alpha$. We will investigate mainly its algebraic structure and generalize it to an $n$-dimensional Riemann-Cartan space. The differential properties, that is, the Cotton 2-form, will be in the center of interest of a forthcoming article.

In this article, we would like to cast some light on the Einstein $(n-1)$-form and related quantities. Within the framework of the calculus of exterior differential forms these structures will arise quite naturally.
In section 2 we introduce some notation and motivate the definition of the Einstein \((n-1)\)-form. We give the well known “differential argument”, involving the contracted 2nd Bianchi identity, and a less well-known algebraic argument.

In section 3 we derive two quantities equivalent to the Einstein 3-form, the Einstein tensor and the so-called \(L_\alpha\) 1-form.

In section 4 we discuss an invariant containing \(L_\alpha\) and \(G_\alpha\) which was found by Salgado [8] and generalize it to a Riemann-Cartan space.

Section 5 puts the previous results into the context of the irreducible decomposition of the curvature.

We close with a remark concerning the role of the derivative of \(L_\alpha\), \(DL_\alpha\), which also is known as Cotton 2-form \(C_\alpha\).

2 Bianchi identities and the Einstein \((n-1)\)-form

On a differentiable manifold of arbitrary dimension we start with a coframe
\[ \vartheta^\alpha = e_i^\alpha \, dx^i . \]
(1)

The coframe is called natural or holonomic if \(e^\alpha_i = \delta_i^\alpha\). The vector basis or frame which is dual to this particular coframe is denoted by \(e_\alpha\),
\[ e_\alpha = e_i^\alpha \partial_i , \quad e_\alpha \vartheta^\beta = \delta_\alpha^\beta . \]
(2)

We then may introduce a connection 1-form
\[ \Gamma_\alpha^\beta = \Gamma_\alpha^\beta_i \, dx^i . \]
(3)

Thereby we define the exterior covariant derivative of a tensor-valued \(p\)-form \((d\) denotes the exterior derivative)
\[ D\phi_{... \beta \gamma} := d\phi_{... \beta \gamma} - \Gamma_\gamma^\delta \wedge \phi_{... \beta \gamma} + \Gamma_\delta^\beta \wedge \phi_{... \gamma \delta} . \]
(4)

Subsequently, we define the torsion, a vector-valued two-form \(T_\alpha\) by
\[ T^\alpha = \frac{1}{2} T_{ij}^\alpha \, dx^i \wedge dx^j := d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta , \quad \text{1st structure eq.} \]
(5)

and the curvature, an antisymmetric 2-form \(R_{\alpha \beta}\) by
\[ R_{\alpha \beta} = \frac{1}{2} R_{ij\alpha \beta} \, dx^i \wedge dx^j := d\Gamma_\alpha^\beta - \Gamma_\gamma^\beta \wedge \Gamma_\gamma^\beta , \quad \text{2nd structure eq.} \]
(6)

From these definitions, together with that of the covariant exterior derivative, we can deduce the following two identities:
\[ DT^\alpha = R^\alpha_\beta \wedge \vartheta^\beta , \quad \text{1st Bianchi identity} , \]
(7)
\[ DR_{\alpha \beta} = 0 , \quad \text{2nd Bianchi identity} . \]
(8)

\(^1\text{We use Latin letters for holonomic and Greek letters for anholonomic indices.}\)
Supplied with a metric $g$ and the corresponding Hodge-star duality operator $^*$, we can define the $\eta$-basis:

\[
\begin{align*}
\eta & := 1^*, \quad \text{basis of n-forms,} \\
\eta_{\alpha_1} & := \theta_{\alpha_1}^*, \quad \text{basis of (n-1)-forms,} \\
\eta_{\alpha_1 \alpha_2} & := (\theta_{\alpha_1} \wedge \theta_{\alpha_2})^*, \quad \text{basis of (n-2)-forms,} \\
\vdots & \quad \vdots \\
\eta_{\alpha_1 \alpha_2 \cdots \alpha_n} & := (\theta_{\alpha_1} \wedge \theta_{\alpha_2} \wedge \cdots \wedge \theta_{\alpha_n})^*, \quad \text{basis of (n-n)-forms,}
\end{align*}
\]

If we require metric-compatibility of the connection, i.e., $Dg_{\alpha\beta} = 0$, we arrive at a Riemann-Cartan space. In orthonormal frames, we find $\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha}$.

By contracting the second Bianchi identity (8) twice, we find

\[
e_\beta e_\alpha D R^{\alpha\beta} = 0 \quad \Rightarrow \quad \theta_\alpha \wedge \theta_\beta \wedge ^*DR^{\alpha\beta} = 0.
\]

This corresponds to an irreducible piece of the second Bianchi identity, see \[8\]. For $n > 3$, we obtain another differential identity of the curvature $2$-form by taking the exterior products of eq.(10) and $\theta_\beta$. By using rule H5 for the Hodge-star (see appendix) we find

\[
DR^{\alpha\beta} \wedge \eta_{\alpha\beta\gamma} = 0.
\]

By differentiation,

\[
D \left( R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} \right) = \left( DR^{\beta\gamma} \right) \wedge \eta_{\alpha\beta\gamma} + R^{\beta\gamma} \wedge D \eta_{\alpha\beta\gamma},
\]

or, by (11) and $D\eta_{\alpha\beta\gamma} = T^\delta \wedge \eta_{\alpha\beta\delta}$ (see \[3\], eq.(3.8.5)),

\[
D \left( \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} \right) = \frac{1}{2} \eta_{\alpha\beta\gamma} \wedge T^\delta \wedge R^{\beta\gamma}.
\]

In a Riemann space, where the torsion is zero, and in a Weitzenböck space, where the curvature is zero, the term on the right hand side vanishes.

Another interesting property of the $(n-1)$-form $\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma}$ should be mentioned here. From special relativity we know that the energy-momentum current density has to be represented by a vector-valued $(n-1)$-form. Suppose one has the idea to link energy-momentum to curvature. We then notice that the expression $\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma}$ is one of the two most obvious vector valued $(n-1)$-forms linear in the curvature. The other one is $\theta_\beta \wedge ^*R^{\alpha\beta}$. However, in a Riemannian space, only the first quantity is automatically conserved which perfectly matches the conserved energy-momentum.\[2\]

\[2\] The $\eta$-basis seemingly was introduced by Trautman, see \[11\].

\[3\] In a Riemannian space we find for the covariant derivative of $\theta_\beta \wedge ^*R^{\alpha\beta}$, using the notation introduced in section \[3\], $D \left( \theta_\beta \wedge ^*R^{\alpha\beta} \right) = -D^* \left( e_\beta \wedge ^*R^{\alpha\beta} \right) = -D^* \left( e_\beta \wedge ^*R^{\alpha\beta} \right) = -\left( \nabla_\beta R^{\alpha\beta} \right) \theta^{\alpha\beta} \wedge \eta^\gamma = -\left( \nabla_\beta R^{\alpha\beta} \right) \theta^{\alpha\beta} \wedge \eta^\gamma = -\frac{1}{2} \left( \nabla_\beta R \right) \eta$. The last expression, which is non-zero in general, is obtained by means of the 2nd Bianchi identity.

\[4\] In a Riemann-Cartan space appear additional forces on the right-hand side of the energy-momentum law, such as the Mathisson-Papapetrou force. This is consistent with the non-vanishing right-hand side of eq.(13), see \[5\].
These considerations motivate the definition

\[ G_\alpha := \frac{1}{2} \eta_{\alpha \beta \gamma} \wedge R^{\beta \gamma}, \quad \textit{Einstein \,(n-1)-form.} \quad (14) \]

3 Alternative representations of the Einstein \,(n-1)-form

The \,(n-1)-form \,(14) naturally appears as a piece of the identically vanishing \,(n+1)-form \( R^{\beta \gamma} \wedge \eta_{\alpha} \). We try to extract \( G_\alpha \) from \( R^{\beta \gamma} \wedge \eta_{\alpha} \) by contracting the latter twice:

\[
0 = e_\gamma \lnot e_\beta \left( R^{\beta \gamma} \wedge \eta_{\alpha} \right) = e_\gamma \lnot \left( e_\beta \lnot R^{\beta \gamma} \right) \wedge \eta_{\alpha} + R^{\beta \gamma} \wedge \eta_{\alpha \beta} \\
= \left( e_\gamma \lnot e_\beta \right) R^{\beta \gamma} \wedge \eta_{\alpha} - \left( e_\beta \lnot \right) R^{\beta \gamma} \wedge \eta_{\alpha \gamma} + \left( e_\gamma \lnot \right) R^{\beta \gamma} \wedge \eta_{\alpha \beta} + R^{\beta \gamma} \wedge \eta_{\alpha \beta \gamma} \\
= -\text{Ric}^{\beta} \wedge \eta_{\alpha \beta} + \frac{1}{2} \text{R} \eta_{\alpha} + 2G_\alpha, \\
(15)\]

\[ \Rightarrow G_\alpha = -\text{Ric}^{\beta} \wedge \eta_{\alpha \beta} + \frac{1}{2} \text{R} \eta_{\alpha}, \quad (16) \]

where we introduced the Ricci 1-form \( \text{Ric}_\alpha := e_\beta \lnot R_\alpha^{\beta} \), and its trace, the curvature scalar \( \text{R} := e_\alpha \lnot \text{Ric}_\alpha \). In a Riemann-Cartan space we have \( R^{\beta \gamma} = -R^{\beta \gamma} \).

By means of the definition of the curvature 2-form, eq. \((1)\), we find \( \text{Ric}_\alpha = \text{Ric}_{\nu \alpha} \vartheta^\nu \), where \( \text{Ric}_{\nu \alpha} := R_{\mu \nu \alpha} \mu \) denotes the Ricci tensor. It is symmetric if the torsion is covariantly constant as, for instance, in a Riemann space, where \( T^\alpha \equiv 0 \).

Often the Ricci tensor is defined by contraction of the 2nd and 4th index in our Schouten notation of the curvature tensor. Because of the antisymmetry of the curvature tensor, this definition of the Ricci tensor differs from our convention by a sign. This applies also to quantities which are derived from the Ricci tensor, like the Einstein tensor and the \( L_{\alpha \beta} \) tensor.

Einstein tensor

Since \( G_\alpha \) is a \,(n-1)-form, we can decompose it with respect to the basis of \,(n-1)-forms \( \eta_\alpha \):

\[ G_\alpha = G_\alpha^{\beta} \eta_\beta. \quad (17) \]

By convention, we contract here the 2nd index of \( G_\alpha^{\beta} \). In order to determine the components \( G_\alpha^{\beta} \), we just have to rewrite the first term of the right hand side of eq. \((15)\):

\[
\text{Ric}^{\beta} \wedge \eta_{\alpha \beta} = \text{Ric}_\mu^{\beta} \vartheta^\mu \wedge \eta_{\alpha \beta} = \text{Ric}_\mu^{\beta} \vartheta^\mu \wedge (\vartheta_\alpha \wedge \vartheta_\beta) \\
= -\text{Ric}_\mu^{\beta} \times (e^\mu \lnot (\vartheta_\alpha \wedge \vartheta_\beta)) = -\text{Ric}_\mu^{\beta} \times (\delta_\alpha^{\mu} \vartheta_\beta - \vartheta_\alpha \delta_\beta^{\mu}) \\
= -\text{Ric}_\alpha^{\beta} \eta_\beta + \text{R} \eta_\alpha. \quad (18)\]

4
By substituting (18) into (15) we find

\[ G_{\alpha\beta} = \text{Ric}_{\alpha\beta} - \frac{1}{2} \, R \, g_{\alpha\beta}. \]  

(19)

Thereby we recover the usual definition of the Einstein tensor.

**The \( L_\alpha \) 1-form**

Using the identity \( \vartheta^\alpha \wedge (e_\alpha \mathcal{J} \Phi) = (\text{rank } \Phi) \, \Phi \), we can rewrite \( \eta_\alpha \) according to

\[ (n - 1) \, \eta_\alpha = \vartheta^\beta \wedge (e_\beta \mathcal{J} \eta_\alpha) = \vartheta^\beta \wedge \eta_{\alpha\beta} \, . \]

(20)

Substituting this into (16), we arrive at

\[ G_\alpha = \text{Ric}^\beta \wedge \eta_{\beta\alpha} - \frac{1}{2(n - 1)} \, R \, \vartheta^\beta \wedge \eta_{\beta\alpha} \, . \]

(21)

This suggests the definition

\[ L^\beta := \text{Ric}^\beta - \frac{1}{2(n - 1)} \, R \, \vartheta^\beta, \]  

(22)

such that

\[ G_\alpha = L^\beta \wedge \eta_{\beta\alpha} \, . \]  

(23)

The decomposition of \( L_\alpha \) in components reads

\[ L^\alpha =: L_\mu^\alpha \, \vartheta^\mu \, . \]  

(24)

In a Riemann-Cartan space, the tensor

\[ L_\mu^\alpha = \text{Ric}_\mu^\alpha - \frac{1}{2(n - 1)} \, R \delta_\mu^\alpha, \]  

(25)

if \( \alpha \) is lowered, is not symmetric in general.

The trace of \( L_\alpha \) is proportional to the curvature scalar

\[ L := e_\alpha \mathcal{J} L^\alpha = \frac{n - 2}{2(n - 1)} \, R \, . \]  

(26)

We collect the results of this section for a Riemann-Cartan space in

\[ G_\alpha = \frac{1}{2} \, R^\beta_\gamma \wedge \eta_{\alpha\beta\gamma}, \]

\[ = L^\beta \wedge \eta_{\beta\alpha}, \]  

(27)

\[ = G_\alpha^\beta \wedge \eta_\beta \, . \]
4 On Salgado’s formula

Since $G_\alpha$ is a $(n-1)$-form and $L^\alpha$ a 1-form, $L^\alpha \wedge G_\alpha$ is a scalar-valued $n$-form and thus a possible candidate for a Lagrange $n$-form. Moreover, we can guess that the variation of $L^\alpha \wedge G_\alpha$ with respect to $L^\alpha$ should yield $G_\alpha$, and vice versa. However, because $L^\alpha$ and $G_\alpha$ are not independent, we have to check this explicitly. By means of the results of the last section we find

$$
\frac{1}{2} \left( L^\alpha \wedge G_\alpha \right) = \frac{1}{2} \left( \delta L^\alpha \wedge G_\alpha + L^\alpha \wedge \delta G_\alpha \right)
= \frac{1}{2} \left( \delta L^\alpha \wedge G_\alpha + L^\alpha \wedge \delta \left( L^\gamma \wedge \eta_{\gamma \alpha} \right) \right)
= \frac{1}{2} \left( \delta L^\alpha \wedge G_\alpha + L^\alpha \wedge \delta L^\gamma \wedge \eta_{\gamma \alpha} + L^\alpha \wedge L^\gamma \wedge \delta \eta_{\gamma \alpha} \right)
= \frac{1}{2} \left( \delta L^\alpha \wedge G_\alpha = \delta L^\gamma \wedge L^\alpha \wedge \eta_{\gamma \alpha} + L^\alpha \wedge L^\gamma \wedge \delta \eta_{\gamma \alpha} \right)
= \frac{1}{2} \delta L^\alpha \wedge G_\alpha = \frac{1}{2} L^\alpha \wedge L^\beta \wedge \delta \eta_{\alpha \beta}.
$$

(28)

Thus we have

$$
\frac{1}{2} \frac{\delta \left( L^\alpha \wedge G_\alpha \right)}{\delta L^\beta} = G_\beta.
$$

(29)

This formula also becomes apparent by noticing that

$$
\frac{1}{2} L^\alpha \wedge G_\alpha = - \frac{1}{2} L^\alpha \wedge L^\beta \wedge \eta_{\alpha \beta}.
$$

(30)

For the variation of $L^\alpha \wedge G_\alpha$ with respect to $G_\alpha$ we have to express $L^\alpha$ in terms of $G_\alpha$. We start from

$$
G_\alpha = L^\beta \wedge \eta_{\beta \alpha} = L^\beta \wedge \eta_{\beta \alpha}.
$$

(31)

The term $\partial^\mu \wedge \eta_{\beta \alpha}$ can be rewritten as in eq. (18) yielding

$$
G_\alpha = L^\alpha \wedge \eta_{\beta \alpha} - L \eta_\alpha = \left( L^\alpha \wedge L^\beta - L^\beta \wedge \delta \eta_\alpha \right) \eta_{\beta \alpha}.
$$

(32)

From this equation we infer for the traces $L$ and $\hat{G}$

$$
\hat{G} := e_\alpha \star G^\alpha = \star \left( G^\alpha \wedge \partial_\alpha \right) = (-1)^{(n-1+\text{ind})} (1 - n) L.
$$

(33)

The last two equations lead to

$$
L^\alpha \wedge = \left( L^\alpha \wedge \partial_\beta \right) = e^\mu \wedge \eta_{\beta \alpha} = e^\mu \wedge \left( (-1)^{(n-1+\text{ind})} \star \eta_{\beta \alpha} \right)
= (-1)^{(n-1+\text{ind})} \left( e^\mu \wedge G_\alpha - \frac{1}{n-1} \hat{G} \delta^\mu_{\alpha \beta} \right).
$$

(34)
or, by multiplying with $\vartheta^\alpha$ and using the rules for the Hodge-dual,
\[
L^\mu = (-1)^{\text{ind}} \ast \left[ e^\alpha \ast (G_\alpha \wedge \vartheta^\mu) - \frac{1}{n-1} e^\mu \ast (G_\alpha \wedge \vartheta^\alpha) \right].
\]
(35)

Since Hodge-star, interior and exterior products are linear, eq.(35) is linear in $G_\alpha$. Consequently, the variation of $L_\alpha$ with respect to $G_\alpha$ reads
\[
\delta L_\alpha = L_\alpha (G_\beta + \delta G_\beta) - L_\alpha (G_\beta) = L_\alpha (\delta G_\beta).
\]
(36)

A simple, but somewhat lengthy, calculation shows
\[
L_\alpha (\delta G_\beta) \wedge G_\alpha = L_\alpha \wedge (\delta G_\alpha).
\]
(37)

Then the variation of $L_\alpha \wedge G_\alpha$ with respect to $G_\alpha$ turns out to be
\[
\frac{1}{2} \delta (L_\alpha \wedge G_\alpha) = \frac{1}{2} \left[ (\delta L_\alpha) \wedge G_\alpha + L_\alpha \wedge (\delta G_\alpha) \right] = L_\alpha \wedge (\delta G_\alpha).
\]
(38)

\[
\frac{1}{2} \frac{\delta (L_\alpha \wedge G_\alpha)}{\delta G_\beta} = (-1)^{(n-1)} L_\beta.
\]
(39)

### Component representation

Evaluating $e_\beta \ast e_\alpha \ast (L_\alpha \wedge L_\beta \wedge \eta) = 0$ yields
\[
\frac{1}{2} L_\alpha \wedge G_\alpha = -\frac{1}{2} L_\alpha \wedge L_\beta \wedge \eta_{\alpha\beta} = -\frac{1}{2} \left( L_\alpha \wedge L_\beta \wedge (L_\alpha \wedge L_\beta) \right) \eta
\]
\[
= - (L_{\alpha \beta} \wedge L_{\alpha \beta}) \eta.
\]
(40)

Eq.(29) corresponds to the Salgado formula 8
\[
d \left( L_{\alpha \beta} \wedge L_{\alpha \beta} \right) \wedge L_\mu = -L_\mu \wedge \delta_\mu \wedge L = -G_\mu \wedge L,
\]
(41)

which was found by Salgado in a Riemannian context. It remains valid in a Riemann-Cartan space. Eqs.(19, 25) yield
\[
L_{\alpha \beta} = G_{\alpha \beta} - \frac{1}{n-1} G_\mu \wedge G^\mu \wedge \delta_\beta.
\]
(42)

Differentiating the last equation we get
\[
\frac{\partial L_{\alpha \beta}}{\partial G_\mu \wedge} = \delta_\mu \wedge \delta_\beta - \frac{1}{n-1} G_\mu \wedge \delta_\beta.
\]
(43)

---

5If $D T^\alpha = 0$, $L_{\alpha \beta}$ is symmetric and we simply have
\[
L_{\alpha \beta} = (L_\mu \wedge \vartheta^\mu) = L_\mu \wedge \vartheta^\mu = L_\mu \wedge \eta^\mu = L_\mu \wedge \eta^\mu,
\]
or, by substituting this into eq.(32),
\[
G_\alpha = \ast L_\alpha - \eta_\alpha.
\]
6See appendix.
Substituting eq.(42) into $L_\alpha^\beta \bar{L}_\beta^\alpha$, we find

$$L_\alpha^\beta \bar{L}_\beta^\alpha = -\frac{1}{2} L_\alpha^\beta \bar{G}^\alpha_\beta .$$

(44)

From the last two equation we derive

$$\frac{d}{d\bar{G}_\mu^\nu} (L_\alpha^\beta \bar{L}_\beta^\alpha) = -L_\nu^\mu .$$

(45)

5 The 1-form $L_\alpha$ and the irreducible decomposition of the curvature

The 1-form $L_\alpha$ represents the trace-part (that is, the second rank pieces of a fourth rank quantity) $e_\beta \bar{R}^\alpha_\beta = L_\alpha + \frac{1}{n-2} L_\alpha^\beta \bar{G}^\alpha_\beta$ of the curvature. This property seems to be nothing special because it also belongs to other contractions of the curvature (like the Ricci- and the Einstein-tensor). However, the Einstein tensor, which is a trace-modified Ricci-tensor, is an interesting quantity because of a property not shared by the Ricci-tensor, namely to be divergence-free. What are the properties peculiar to $L_\alpha$?

In a Riemann-Cartan space, the 1-form $L_\alpha$ represents that part of the curvature 2-form which has the structure $\bar{\nu}_\beta \langle 1 - \text{form} \rangle_\beta$. To see this, one has to perform an irreducible decomposition of the curvature. We use the results obtained in [7] and find

$$R_\alpha^\beta = (1) R_\alpha^\beta + (2) R_\alpha^\beta + (3) R_\alpha^\beta + (4) R_\alpha^\beta + (5) R_\alpha^\beta + (6) R_\alpha^\beta$$

$$= (1) R_\alpha^\beta + (2) R_\alpha^\beta + (3) R_\alpha^\beta + (4) R_\alpha^\beta + (5) R_\alpha^\beta + (6) R_\alpha^\beta$$

$$= \text{Weyl}_{\alpha\beta} + \bar{R}_{\alpha\beta} + \bar{R}_{\alpha\beta}$$

$$= \text{irreducible} + \text{pseudo-trace piece} + \text{trace piece}$$

(46)

The $(i) R_\alpha^\beta$ denote the 6 irreducible pieces of the curvature in a Riemann-Cartan space. For their precise definition we refer the reader to the literature, see [6], e.g., because, in this context, the main result is contained in the second line of (46). The curvature decomposes into the conformal curvature Weyl$_{\alpha\beta}$, a trace piece piece $\bar{R}_{\alpha\beta}$, determined by $L_\alpha$, and a pseudo-trace piece $\bar{R}_{\alpha\beta}$, which is determined by a $(n-3)$-form$P_\alpha$. If $DT^\alpha \equiv 0$, i.e., in particular in a Riemannian space, we have, due to the 1st Bianchi identity (6), $(2) R_\alpha^\beta = (3) R_\alpha^\beta = (5) R_\alpha^\beta = 0$ and thus

$$R_\alpha^\beta = \text{Weyl}_{\alpha\beta} - \frac{2}{n-2} \bar{\nu}_{\alpha \beta} , \quad \text{if } T^\alpha \equiv 0 .$$

(47)

For the sake of completeness we display its explicit form:

$$P_\alpha := \star (R^\beta_\alpha \wedge \bar{\nu}_\beta) - \frac{1}{n-2} \bar{\nu}_\alpha \wedge \star (R^\beta_\alpha \wedge \bar{\nu}_\beta \wedge \bar{\nu}_\gamma) - \frac{1}{n-2} \bar{\nu}_\alpha \star \left[ \bar{\nu}_\beta \wedge \star (R^\gamma_\beta \wedge \bar{\nu}_\gamma) \right]$$

8
The corresponding formula in Ricci calculus is often used for defining the Weyl tensor $C_{\alpha\beta\gamma\delta}$. Eq. (47), decomposed into components reads

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + 4\frac{n-2}{n} g[\alpha[\gamma] L_{\delta][\beta]]$$, \quad \text{if} \quad T_{\alpha\beta} \equiv 0. \quad (48)$$

In a Riemannian space, Weyl $\alpha\beta$ transforms like a conformal density. Thus, $L_{\alpha}$ represents that piece of the curvature which does not transform like a conformal density. This properties of $L_{\alpha}$ are well known, compare [2] and [9].

For the number of independent components we have

$$\text{Weyl}_{\alpha\beta} \rightarrow \frac{1}{12}(n+2)(n+1)n(n-3), \quad (49)$$
$$\hat{R}_{\alpha\beta} \rightarrow \frac{1}{6}(n+1)(n-1)n(n-3), \quad (50)$$
$$\hat{R}_{\alpha\beta} \rightarrow n^2. \quad (51)$$

These pieces have characteristic trace properties

$$e_{\alpha} \text{Weyl}_{\alpha\beta} = e_{\alpha} \hat{R}_{\alpha\beta} = 0, \quad e_{\alpha} \hat{\hat{R}}_{\alpha\beta} = e_{\alpha} \hat{R}_{\alpha\beta} = 0. \quad (52)$$

By means of those we find $\text{Weyl}_{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = \hat{R}_{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = 0$. Thus, only the piece $\hat{R}_{\alpha\beta}$ contributes to the Einstein 3-form. This is even more apparent by substituting $\vartheta_{\gamma} \wedge \eta_{\alpha\beta\gamma} = (n-2)\eta_{\alpha\beta\gamma}$ into eq. (27).

$$G_{\alpha} = L_{\beta} \wedge \eta_{\beta\alpha} = \frac{1}{n-2} L_{\beta} \wedge \vartheta_{\gamma} \wedge \eta_{\beta\alpha\gamma} = -\frac{1}{n-2} \vartheta[\beta \wedge L_{\gamma}] \wedge \eta_{\alpha\beta\gamma},$$

or, by using (46),

$$G_{\alpha} = \frac{1}{2} \hat{R}_{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}. \quad (53)$$

We can use this relation in order to obtain another well motivated representation of the invariant $L^\alpha \wedge G_{\alpha}$ by rewriting it according to [8]

$$I_{\text{S}} := -L_{\alpha}^\alpha \eta_{\beta\gamma} = \frac{1}{2} L_{\beta} \wedge \eta_{\beta\alpha} = \frac{1}{4} L_{\alpha}^\alpha \wedge \eta_{\alpha\beta\gamma} \wedge \hat{R}_{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}$$

$$= -\frac{n-2}{8(n-3)} \hat{R}_{\alpha\beta} \wedge \hat{R}_{\beta\gamma} \wedge \hat{R}_{\gamma\delta} \eta_{\alpha\beta\gamma\delta}. \quad (54)$$

In this way, $I_{\text{S}}$ turned out to be one of the basic quadratic invariants [9] (scalar-valued $n$-forms) of $\hat{R}_{\alpha\beta}$.  

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8 We use $(n-3)\eta_{\alpha\beta\gamma} = \vartheta^\delta \wedge \eta_{\alpha\beta\gamma}$.

9 In the case $n = 4$ we can define the Lie-dual of $\hat{R}_{\alpha\beta}$ by $\hat{R}_{\alpha\beta}^* := \frac{1}{2} \hat{R}_{\gamma\delta} \eta_{\alpha\beta\gamma\delta}$. Then $I_{\text{S}}$ reads $I_{\text{S}} = -\frac{1}{8} \hat{R}_{\alpha\beta} \wedge \hat{R}_{\alpha\beta}^*$. Using the irreducible decomposition [40], $I_{\text{S}}$ can also be expressed in terms of the Hodge-dual $I_{\text{S}} \propto R_{\alpha\beta} \wedge \star (4) R_{\alpha\beta} - (5) R_{\alpha\beta} - (6) R_{\alpha\beta}$. If $DT_{\alpha} \equiv 0$, we can use $G_{\alpha} = *L_{\alpha} = L_{\eta_{\alpha}}$ and obtain $I_{\text{S}} = \frac{1}{2} \left[L_{\alpha}^\alpha \wedge L_{\alpha} \wedge (L)^2 \eta \right] = \frac{1}{2} \left[L_{\alpha}^\alpha \wedge L_{\alpha} \wedge (L)^2 \right] \eta$. The positions of the indices differ from those in eq. (40)!
We collect the various representation of the invariant $I_S$ in

$$I_S = \frac{1}{2} L^\alpha \wedge G_\alpha = -\frac{1}{2} L^\alpha \wedge L^\beta \wedge \eta_{\alpha\beta} = -\frac{n-2}{8(n-3)} \hat{R}^{\alpha\beta} \wedge \hat{R}^{\gamma\delta} \eta_{\alpha\beta\gamma\delta}. \quad (55)$$

6 Discussion

In view of our observations, we may put the main result of our investigations as follows. The basic quantity here is the “trace-part” $\hat{R}_{\alpha\beta}$ of the curvature with its $n^2$ independent components. A vector-valued 1-form, a vector-valued $(n-1)$-form, and a 2nd rank tensor valued 0-form, respectively, have the same number of independent components. Thus $\hat{R}_{\alpha\beta}$, as displayed in eq.(53) and in eq.(27), can be mapped into a $(n-1)$-form by means of the $\eta$-basis, yielding the Einstein 3-form $G_\alpha$, into a 1-form, yielding the $L_\alpha$ 1-form, and into an $n \times n$ matrix, yielding the Einstein tensor $G_{\alpha\beta}$. Realizing this, makes the algebraic relations between the stated quantities quite clear. These results are represented by Eqs.(27, 53), and (46).

We also would like to mention that eq.(23) which expresses $G_\alpha$ in terms of $L_\beta$ and eq.(35) which expresses $L_\alpha$ in terms of $G_\alpha$ hold for arbitrary $(n-1)$ forms $G_\alpha$ and 1-forms $L_\beta$, respectively. Hence, eq.(23) and eq.(35) establish a duality relation between $(n-1)$-forms and 1-forms.

The invariant $L^\alpha [\alpha L^\beta]$, which was derived by Salgado as second principal invariant of $L^\alpha [\alpha L^\beta]$ (arising in connection with the characteristic polynomial), in our context emerges (i) as one of the the most obvious invariants constructed from $L_\alpha$ and $G_\alpha$, (ii) as a basic quadratic invariant of $L_\alpha$, and (iii) as a basic quadratic invariant of the curvature piece $\hat{R}_{\alpha\beta}$. These results are displayed in (53) and (23, 53).

We have extensively studied the algebraic properties of $L_\alpha$. It was quite helpful to check all formulas by means of the Excalc package of the computer algebra system Reduce, see [10].

Also the differential properties of $L_\alpha$ are very interesting. In a Riemannian space we have $D\vartheta^{\alpha} = 0$. By using (47), we can represent the second Bianchi Identity as follows

$$D \text{Weyl}_{\alpha\beta} = -\frac{2}{n-2} \vartheta_{[\alpha} \wedge C_{\beta]}, \quad (56)$$

where we defined the Cotton 2-form by

$$C_\alpha := DL_\alpha. \quad (57)$$

In this way, $L_\alpha$ appears as potential of the Cotton 2-form. Since the conformal (Weyl-) curvature is tracefree, the (twice) contracted 2nd Bianchi identity reads

$$0 = e_\alpha \mathcal{J} C^\alpha = e_\alpha \mathcal{J} DL^\alpha. \quad (58)$$
The Cotton 2-form, especially its relation to the conformal properties of space-time, is subject of a current project \cite{4}. The definition \cite{57} of the Cotton 2-form can be transferred to a Riemann-Cartan space.

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7 Appendix

7.1 Some relations for the exterior and interior products

In order to avoid dimension-dependent signs, it is of special importance to take care of the order of the forms in the exterior products. We would like to remind the reader of the following relations which hold for a $p$-form $\phi$ and a $q$-form $\psi$:

\[
\phi \wedge \psi = (-1)^{pq} \psi \wedge \phi, \quad (59)
\]

\[
e_{\mu} J(\phi \wedge \psi) = (e_{\mu} J\phi) \wedge \psi + (-1)^p \phi \wedge (e_{\mu} J\psi). \quad (60)
\]

7.2 The variational derivative with respect to $p$-forms

The variation of a function $F$ which depends on a $p$-form $\psi$ is defined to be

\[
\delta F := F(\psi + \delta \psi) - F(\psi), \quad (61)
\]

where the $p$-form $\delta \psi$ is supposed to be an arbitrary “small” deviation. With given $F$, we can elementary evaluate the right-hand side of eq.(61). We then neglect all terms of quadratic and higher order in $\delta \psi$ and bring the result into the form

\[
\delta F = \delta \psi \wedge (\ldots). \quad (62)
\]

The expression in the parentheses is defined to be the partial derivative with respect to $\psi$. This prescription especially fixes the sign. The generalization to an arbitrary number of forms or tensor-valued forms is straightforward.

Due to the definition, the variation obeys a Leibniz-rule

\[
\delta(\phi \wedge \psi) \overset{(61)}{=} ((\phi + \delta \phi) \wedge (\psi + \delta \psi)) - \phi \wedge \psi = \phi \wedge \delta \psi + \delta \phi \wedge \psi + \delta \phi \wedge \delta \psi \leftarrow 0 = \phi \wedge \delta \psi + \delta \phi \wedge \psi. \quad (63)
\]

The variational derivative can be introduced in the usual way. However, in this context we just note that in the case in which $F$ does not depend on the derivatives $d\psi$, the partial and the variational derivative coincide.
7.3 Some relations for the Hodge-star

We frequently made use of the following relations for the Hodge-star. \( \psi \) and \( \phi \) are two \( p \)-forms of the same degree, \( a, b \in \mathbb{R} \) are numbers.

\[
\star(a\psi + b\phi) = a\star\psi + b\star\phi, \quad \text{H1.} \tag{64}
\]

\[
\star \star \psi = (-1)^{p(n-p)+\text{ind}} \psi, \quad \text{H2.} \tag{65}
\]

where \( \text{ind} \) denotes the number of negative Eigenvalues of the metric, 3 in the case of a \((3+1)\)-dimensional spacetime.

\[
\star(e_\alpha \psi) = (-1)^{p-1} \vartheta_a \wedge \star\psi, \quad \text{H3.} \tag{66}
\]

\[
e_\alpha \star \psi = \star(\psi \wedge \vartheta_\alpha), \quad \text{H4.} \tag{67}
\]

\[
\star \psi \wedge \phi = \star \phi \wedge \psi, \quad \text{H5.} \tag{68}
\]

7.4 Variation of \( L_\alpha \)

We substitute eq.(35) in eq.(36): \( \delta L_\alpha = L_\alpha (\delta G_\beta) \)

\[
0 = e_\beta \star \left[ *G_\alpha \wedge \delta(G_\beta \wedge \vartheta^\alpha) \right] = \left( e_\beta \star *G_\alpha \right) \delta(G_\beta \wedge \vartheta^\alpha) - *G_\alpha \wedge \left[ e_\beta \star (\delta G_\beta \wedge \vartheta^\alpha) \right]. \tag{70}
\]

\( *G_\alpha \) is a 1-form and \( \delta G_\beta \wedge \vartheta^\beta, \delta G_\beta \wedge \vartheta^\beta \) are \( n \)-forms. Thus

\[
0 = e_\beta \star \left[ *G_\alpha \wedge (\delta G_\beta \wedge \vartheta^\alpha) \right] = \left( e_\beta \star *G_\alpha \right) \delta G_\beta \wedge \vartheta^\alpha - *G_\alpha \wedge \left[ e_\beta \star (\delta G_\beta \wedge \vartheta^\alpha) \right]. \tag{69}
\]

Substituting this into eq.(70) yields

\[
\delta L_\alpha \wedge G_\alpha = (-1)^{n-1} \left[ (e_\beta \star *G_\alpha) \vartheta^\alpha - \frac{1}{n-1} (e_\alpha \star G_\alpha) \vartheta^\beta \right] \wedge \delta G_\beta \tag{71}
\]
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