Self-Similar Surfaces: Involutions and Perfection

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Abstract

We investigate the problem of when big mapping class groups are generated by involutions. Restricting our attention to the class of self-similar surfaces, which are surfaces with self-similar ends space, as defined by Mann and Rafi, and with 0 or infinite genus, we show that, when the set of maximal ends is infinite, then the mapping class groups of these surfaces are generated by involutions, normally generated by a single involution, and uniformly perfect. In fact, we derive this statement as a corollary of the corresponding statement for the homeomorphism groups of these surfaces. On the other hand, among self-similar surfaces with one maximal end, we produce infinitely many examples in which their big mapping class groups are neither perfect nor generated by torsion elements. These groups also do not have the automatic continuity property.

1 Introduction

Consider a connected and oriented surface \( \Sigma \). We distinguish two types of surfaces, those of finite type, i.e. a closed surface minus finitely many points, or of infinite type otherwise. Let \( G(\Sigma) \) be either the group \( \text{Homeo}^+(\Sigma) \) of orientation preserving self-homeomorphisms of \( \Sigma \) or the mapping class group \( \text{MCG}(\Sigma) \) of \( \Sigma \). We are interested in the algebraic structure of \( G(\Sigma) \), especially when \( \Sigma \) has infinite type.

As a topological group, equipped with the compact open topology, \( \text{Homeo}^+(\Sigma) \) is a non-locally-compact Polish group. \( \text{MCG}(\Sigma) \), being a quotient of \( \text{Homeo}^+(\Sigma) \), inherits a topology. When \( \Sigma \) has finite type, then this topology is discrete and \( \text{MCG}(\Sigma) \) is finitely presented. But when \( \Sigma \) has infinite type, then \( \text{MCG}(\Sigma) \) is also a non-locally-compact Polish group, similar to the homeomorphism group. In particular, \( \text{MCG}(\Sigma) \) is not countably generated, justifying the nomenclature of big mapping class group in the literature.

An obvious group-theoretic problem is to identify canonical generating sets for \( G(\Sigma) \). For any group, a natural choice is its set of involutions, or more broadly, its set of torsion elements. This leads us to ask if \( G(\Sigma) \) is generated by involutions (or torsion elements). (The set of Dehn twists, being countable, can never generate a big mapping class group; and often, they do not even topologically generate \([2]\).) For finite type surfaces, this question is well studied for their mapping class groups; see \([19, 16, 4, 12, 14, 15, 23]\) and the references within for the story on generating by involutions. The goal of this paper is to explore this question for surfaces of infinite type.

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To answer this question for all surfaces of infinite type should be challenging, as \( G(\Sigma) \) is as complicated as the homeomorphism group of the ends space of \( \Sigma \). In trying to tame the world of surfaces of infinite type, Mann and Rafi [18] introduced a preorder on an ends space, and showed that the induced partial order always has maximal elements. They also introduced the notion of self-similar ends spaces. We call a surface self-similar if it has a self-similar ends space and 0 or infinite genus. Among these, we identify a subclass, called *uniformly* self-similar, which are self-similar with infinitely many maximal ends. This subclass, which is uncountable, exhibits additional symmetry, to which the sphere minus a Cantor set belongs. It was observed by Calegari [5] that the mapping class group of the sphere minus a Cantor set is uniformly perfect. We extend this result to all uniformly self-similar surfaces, along with answering the generation problem by involutions for these surfaces. Our main theorem is the following.

**Theorem A.** Let \( \Sigma \) be a uniformly self-similar surface and \( G(\Sigma) \) be either \( \text{Homeo}^+(\Sigma) \) or \( \text{MCG}(\Sigma) \). Then \( G(\Sigma) \) is generated by involutions, normally generated by a single involution, and uniformly perfect. Moreover, each element of \( G(\Sigma) \) is a product of at most 3 commutators and 12 involutions.

Note that the case of \( \text{MCG}(\Sigma) \) follows immediately from that of \( \text{Homeo}^+(\Sigma) \). For the latter case, we use a method akin to *fragmentation*, a well known tool in the study of homeomorphism groups. In more detail, we first observe that a uniformly self-similar ends space \( E \) behaves very much like a Cantor set. Namely, any clopen subset \( U \subseteq E \) containing a proper subset of the maximal ends is homeomorphic to its complement \( U^c \). This gives rise to the notion of a half-space in a uniformly self-similar surface \( \Sigma \), which is a subsurface \( H \subseteq \Sigma \) with a single connected, compact boundary component, such that \( \overline{H}^c \) is also a half-space and homeomorphic to \( H \). (The exact definition is different and appears as Definition 3.2). We then find an \( H \)-translation, that is, a homeomorphism \( \phi \) such that \( \{ \phi^n(H) \}_{n \in \mathbb{Z}} \) are all disjoint. This is a key step in the proof and requires putting the surface \( \Sigma \) into a particular form that reflects its symmetry. By our construction, the \( H \)-translation \( \phi \) is a product of two conjugate involutions. Then, using a standard trick, we write every \( f \in \text{Homeo}(H, \partial H) \) as a commutator of the form \( f = [\hat{f}, \phi] \) for some \( \hat{f} \). The final step is to show \( \text{Homeo}^+(\Sigma) \) is the normal closure of \( \text{Homeo}(H, \partial H) \), and so it is normally generated by \( \phi \) and hence by a single involution. The other statements are achieved by keeping track of the number of commutators or involutions needed at each step.

Many of our steps above carry over to the case of equipping \( \Sigma \) with a marked point \( * \). The key difference is now one can find a curve \( \alpha \subset \Sigma \) which is not contained in any half-space \( H \) of \( \Sigma \). Thus, it is no longer immediate that the Dehn twist about \( \alpha \) can be generated by elements supported on \( H \) and their conjugates. To deal with this issue, we invoke the lantern relation. Using self-similarity, we can find an appropriate lantern, i.e. a 4-holed sphere bounding \( \alpha \) with the other boundary components lying in half-spaces. Once we get all Dehn twists, then combining our previous method together with the fact that mapping class groups of compact surfaces are generated by Dehn twists, we obtain the following theorem.

**Theorem B.** Let \( \Sigma \) be a uniformly self-similar surface with a marked point \( * \in \Sigma \). Then, \( \text{MCG}(\Sigma, *) \) is perfect, generated by involutions, and normally generated by a single involution.

Because we used the lantern relation, our proof does not apply to the homeomorphism group. For a different argument that the mapping class group of the marked sphere minus a Cantor set is perfect, see [22]. Theorem B is sharp in the sense that we cannot expect a statement about uniform
perfection or a bound on the number of involutions, due to the fact that the marked sphere minus a Cantor set provides a counterexample, by [3].

It is not possible for all big mapping class groups to be generated by torsion elements or be perfect, even among the class of self-similar surfaces. One counterexample is the infinite genus surface with one end. This is a self-similar surface, but, by Domat and Dickmann [7], the abelianization of its mapping class group contains $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$ as a summand.

Using their results and a covering trick, we can show the mapping class group of the surface $\mathbb{R}^2 \setminus \mathbb{N}$ has similarly large abelianization. Note that this surface is also self-similar but not uniformly. On the other hand, using a method similar to our proof of Theorem A, we can also show $\text{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ is topologically generated by involutions. Since any homomorphism from a Polish group to $\mathbb{Z}$ is always continuous, this makes its first cohomology group vanish, in contrast with homology.

**Theorem C.** The group $\text{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ surjects onto $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$. In particular, it is not perfect or generated by torsion elements. On the other hand, $\text{MCG}(\mathbb{R}^2 \setminus \mathbb{N})$ is topologically generated by involutions, so $H^1(\text{MCG}(\mathbb{R} \setminus \mathbb{N}), \mathbb{Z}) = 0$.

The statement of topological generation by involutions also extends to the mapping class group of the one-ended infinite genus surface $\Sigma_L$. Additionally, we can get infinitely many examples of surfaces whose mapping class groups have similarly large abelianization, by considering appropriate maps to $\Sigma_L$ or $\mathbb{R}^2 \setminus \mathbb{N}$. Many of these examples are self-similar but not uniformly.

Another application of our result beyond the ones mentioned is the automatic continuity property. Recall a Polish group $G$ has the automatic continuity property if every homomorphism from $G$ to a separable topological group is necessarily continuous. The family of surfaces we construct also gives rise to a large family of homeomorphism groups or big mapping class groups that do not have this property. This gives some progress towards answering [17, Question 2.4]. We highlight the following examples and refer to Theorem 5.3 and Corollary 5.6 for the full technical statement.

**Theorem D.** Let $\Sigma = S^2 \setminus E$, where $S^2$ is the 2-sphere and $E$ is a countable closed subset of the Cantor set homeomorphic to the ordinal $\omega^\alpha + 1$, where $\alpha$ is a countable successor ordinal. Let $G(\Sigma)$ be either $\text{Homeo}^+(\Sigma)$ or $\text{MCG}(\Sigma)$. Then $G(\Sigma)$ is not perfect, is not generated by torsion elements, and does not have the automatic continuity property.

One may wonder what happens in the case of positive genus, rather than 0 or infinite genus. Our methods do not extend to these surfaces. However, for a surface $\Sigma$ obtained by removing a Cantor set from a surface of finite type, Calegari and Chen [6] showed various results for $\text{MCG}(\Sigma)$ including that it is generated by torsion. Additionally, Mann [17] showed $G(\Sigma)$ has the automatic continuity property. It would be interesting to know if their techniques extend to uniformly self-similar ends spaces. We refer to their papers for more details.

One may also wonder whether our results extend to other surfaces of infinite type. In [10], using very similar methods that were developed independently and concurrently, Field, Patel, and Rasmussen proved analogues of some of the above results for other classes of surfaces. Specifically, for their class of surfaces, which are required to have locally CB mapping class group and infinitely many maximal ends among other minor conditions, they show that the commutator lengths of elements in the commutator subgroup are uniformly bounded above and $H_1(\text{MCG}(\Sigma), \mathbb{Z})$ is finitely generated. See [10] for precise statements.
As many cases still remain open, we invite the reader to explore other classes of surfaces of infinite type which may verify the properties in Theorem A or admit an obstruction. It would also be interesting to find other natural generating sets for big mapping class groups or homeomorphism groups. Similar questions can also be asked for the homeomorphism groups of ends spaces.

Here is a brief outline of the paper. In Section 2, we introduce ends spaces and the classification of surfaces of infinite type. Following [18], we define self-similar ends spaces and surfaces and a partial order on ends spaces. We also observe some nice properties about self-similar ends spaces that lead to the definition of half-spaces in uniformly self-similar surfaces. The proof of Theorem A is contained in Section 3, and the proof of Theorem B in Section 4. The two parts of Theorem C appear in Section 5 as Proposition 5.1 and Theorem 5.8. Theorem D follows from Corollary 5.6 as a special case.

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2 Preliminaries

2.1 Partial order on ends spaces

An ends space is a pair \((E, F)\), where \(E\) is a totally disconnected, compact, metrizable space and \(F \subset E\) is a (possibly empty) closed subspace. For simplicity, we will often suppress the notation \(F\), but by convention, all homeomorphisms of \(E\) will be relative to \(F\). For instance, to say \(C \subset E\) is homeomorphic to \(D \subset E\) means there is a homeomorphism from \((C, C \cap F)\) to \((D, D \cap F)\). We denote by \(\text{Homeo}(E, F)\) the group of homeomorphisms of \(E\) preserving \(F\).

The assumptions on \(E\) imply it is homeomorphic to a closed subspace of the standard Cantor set (see [21, Proposition 5]). We will often view \(E\) as this subspace (and \(F\) as a further closed subspace).

Definition 2.1. An ends space \((E, F)\) is called self-similar if for any decomposition of \(E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n\) into pairwise disjoint clopen sets, then there exists some clopen set \(D\) contained in some \(E_i\) such that \((D, D \cap F)\) is homeomorphic to \((E, F)\).

Following [18], given an ends space \((E, F)\), define a preorder \(\preceq\) on \(E\) where for \(x, y \in E\), we say \(x \preceq y\) if every neighborhood \(U\) of \(y\) contains some homeomorphic copy of a neighborhood \(V\) of \(x\). Here and throughout the paper, a neighborhood in an end space will always be a clopen neighborhood. We say \(x\) and \(y\) are equivalent, and write \(x \sim y\), if \(x \preceq y\) and \(y \preceq x\). This defines an equivalence relation on \(E\). For \(x \in E\), denote by \(E(x)\) the equivalence class of \(x\), and denote by \([E]\) the set of equivalence classes. From this we get a partial order \(<\) on \([E]\), defined by \(E(x) < E(y)\) if \(x \preceq y\) and \(x \sim y\). Note that by definition, if \(x \preceq y\), then \(y\) is either locally homeomorphic to \(x\) or an accumulation point of homeomorphic images of \(x\) under \(\text{Homeo}(E, F)\). One easily verifies that, since \(F \subset E\) is closed, either \(E(x) \cap F = \emptyset\) or \(E(x) \cap F = E(x)\). Note additionally that if there is a homeomorphism \((E, F) \rightarrow (E, F)\) which maps \(x \rightarrow y\), then \(x \sim y\). Consequently, self-homeomorphisms of \((E, F)\) preserve each equivalence class.

We say a point \(x \in E\) is maximal if \(E(x)\) is maximal with respect to \(<\). Denote by \(M(E)\) the set of maximal elements in \(E\).

Proposition 2.2 ([18]). Let \((E, F)\) be an ends space. The following statements hold.
• The set \( M(E) \) of maximal elements under the partial order \( < \) is non-empty.
• For every \( x \in M(E) \), its equivalence class \( E(x) \) is either finite or homeomorphic to a Cantor set.
• If \((E, F)\) is self-similar, then \( M(E) \) is a single equivalence class \( E(x) \), and \( E(x) \) is either a single-ton or homeomorphic to a Cantor set.

Observe that when \( M(E) \) is a single equivalence class \( E(x) \) and \( F \neq \emptyset \), then \( E(x) \cap F = E(x) \).

### 2.2 Classification of infinite-type surfaces

By a surface we always mean a connected, orientable 2–manifold. A surface has finite type if its fundamental group is finitely generated; otherwise, it has infinite type. In this paper, we are primarily interested in surfaces of infinite type. We refer to [21] for details.

The collection of compact sets on a surface \( \Sigma \) forms a directed set by inclusion. The space of ends of \( \Sigma \) is

\[
E(\Sigma) = \lim_{\pi_0(\Sigma \setminus K)} \pi_0(\Sigma \setminus K),
\]

where the inverse limit is taken over the collection of compact subsets \( K \subset \Sigma \). Equip each \( \pi_0(\Sigma \setminus K) \) with the discrete topology. Then the limit topology on \( E(\Sigma) \) is a totally disconnected, compact, and metrizable. An element of \( E(\Sigma) \) is called an end of \( \Sigma \).

An end \( e \in E(\Sigma) \) is accumulated by genus if for all subsurface \( S \subset \Sigma \) with \( e \in E(S) \), then \( S \) has infinite genus; otherwise, \( e \) is called planar. Let \( E^g(\Sigma) \) be the subset of \( E(\Sigma) \) consisting of ends accumulated by genus. This is always a closed subset of \( E(\Sigma) \), with \( E(\Sigma) = \emptyset \) if and only if \( \Sigma \) has finite genus. Hence the pair \((E(\Sigma), E^g(\Sigma))\) forms an ends space. Conversely, by [21, Theorem 1], every ends space \((E, F)\) can be realized as the space of ends of some surface \( \Sigma \), with \((E, F) = (E(\Sigma), E^g(\Sigma))\).

Infinite type surfaces are completely classified by the following data: the genus (possibly infinite), and the homeomorphism type of the ends space \((E(\Sigma), E^g(\Sigma))\). More precisely:

**Theorem 2.3** ([13] [21, Theorem 1]). Suppose \( \Sigma \) and \( \Sigma' \) are boundaryless surfaces. Then, \( \Sigma \) and \( \Sigma' \) are homeomorphic if and only if they have the same (possibly infinite) genus and there is a homeomorphism between \((E(\Sigma), E^g(\Sigma))\) and \((E(\Sigma'), E^g(\Sigma'))\).

We remark that although Richards’ classification of infinite type surfaces is only stated for boundaryless surfaces, it easily extends to surfaces with finitely many compact boundary components. That is, two surfaces with the same genus, same number of (finitely many) compact boundary components, and homeomorphic end space pairs \((E, E^g)\) are in fact homeomorphic.

Fix an orientation on a surface \( \Sigma \) and set \((E, F) = (E(\Sigma), E^g(\Sigma))\). Let \( \text{Homeo}^+ (\Sigma) \) be the group of orientation preserving homeomorphisms of \( \Sigma \). This is a topological group equipped with the compact open topology, and moreover it is a Polish group. The connected component of the identity is a closed normal subgroup \( \text{Homeo}_0(\Sigma) \) comprised of homeomorphisms isotopic to the identity. The quotient group

\[
\text{MCG}(\Sigma) = \text{Homeo}^+(\Sigma) / \text{Homeo}_0(\Sigma)
\]

is called the mapping class group of \( \Sigma \). When \( \Sigma \) has finite type, then \( \text{MCG}(\Sigma) \) is discrete and finitely presented. When \( \Sigma \) has infinite type, then \( \text{MCG}(\Sigma) \) is a non locally-compact Polish group.
Every homeomorphism of $\Sigma$ induces a homeomorphism of its ends space $(E, F)$, and two homotopic homeomorphisms of $\Sigma$ induce the same map on $(E, F)$. This gives a continuous homomorphism $\Phi : \text{Homeo}^+(\Sigma) \to \text{Homeo}(E, F)$ that factors through $\text{MCG}(\Sigma)$. By [21], the map $\Phi$ is also surjective.

As noted in [18, Section 4], we also know the preorder $\preceq$ on $E$ is equivalent to: $x \preceq y$ if and only if for every neighborhood $U$ of $y$ there is a neighborhood $V$ of $x$ and $f \in \text{Homeo}^+(\Sigma)$ such that $\Phi(f)(V) \subset U$.

**Definition 2.4.** A surface $\Sigma$ is called self-similar if its space of ends $(E(\Sigma), E^b(\Sigma))$ is self-similar and $\Sigma$ has genus 0 or infinite genus.

Note that when $\Sigma$ is self-similar and has infinite genus, then each maximal end of $E(\Sigma)$ must be accumulated by genus.

**Remark 2.5.** We point out our definition of self-similar surfaces is equivalent to another notion. First, following [18], a subset $A$ of a surface $\Sigma$ is called non-displaceable if $f(A) \cap A \neq \emptyset$ for every $f \in \text{Homeo}(S)$. Then, $\Sigma$ is self-similar if and only if $\Sigma$ has self-similar ends space and no non-displaceable compact subsurfaces. One direction is clear: if $\Sigma$ has finite positive genus, then $\Sigma$ has a compact non-displaceable subsurface. The other direction is observed by [1, Lemma 5.9 and 5.13].

### 2.3 Stable neighborhoods of ends and self-similarity

We now collect some facts about self-similar ends spaces. The key take away of this section is that self-similar ends spaces with infinitely many maximal ends behave very much like a Cantor set.

**Definition 2.6.** Given $x \in E$, a neighborhood $U$ of $x$ is called stable if any smaller neighborhood $V \subset U$ contains a homeomorphic copy of $U$. (Recall that this means that $(V, V \cap F)$ contains a homeomorphic copy of $(U, U \cap F)$).

**Lemma 2.7 ([1, Lemma 5.4]).** If $(E, F)$ is self-similar, then for all maximal element $x \in E$, the set $E$ is a stable neighborhood of $x$.

The following statement is reminiscent of the statement of [18, Lemma 4.17], but stronger than what the latter implies, though our proof is modeled after theirs.

**Lemma 2.8.** Suppose $(E, F)$ is self-similar. Then for all maximal points $x, y \in M(E)$ and all clopen neighborhoods $U, V$ resp. of $x, y$, there exists a homeomorphism $\varphi : (U, U \cap F) \to (V, V \cap F)$ such that $\varphi(x) = y$.

**Proof.** The proof follows a back-and-forth argument. As usual, we will suppress $F$, so all maps below are maps of pairs relative to $F$.

Let $U_0 = U$ and $V_0 = V$. We define the homeomorphism from $U$ to $V$ inductively on clopen subsets exhausting $U \setminus \{x\}, V \setminus \{y\}$. For convenience, we choose some metric on $E$. We choose $U_1 \subseteq U_0$ to
be a proper neighborhood of \( x \) of diameter less than 1. Since \( E \) is a stable neighborhood of \( y \) by Lemma 2.7, and \( U_0 \setminus U_1 \) is clopen, there is a continuous map

\[
f_0 : U_0 \setminus U_1 \to V_0
\]

which is a homeomorphism onto a clopen image. We can choose \( f_0 \) such that \( \text{im}(f_0) \subseteq V_0 \setminus \{y\} \) for the following reasons. If \( M(E) = \{ y \} \), this is automatic. If \( M(E) \) is a Cantor set, then \( V_0 \setminus \{ y \} \) contains some \( z_0 \in M(E) \), and Lemma 2.7 ensures we can map \( U_0 \setminus U_1 \) homeomorphically into a sufficiently small neighborhood of \( z_0 \) which avoids \( y \). Since \( \text{im}(f_0) \) is clopen, we can choose a proper clopen subset \( V_1 \subseteq V_0 \setminus \text{im}(f_0) \) of \( y \) which has diameter less than 1. By the same token, we can find a map

\[
g_0 : V_0 \setminus (V_1 \cup \text{im}(f_0)) \to U_1 \setminus \{ x \}
\]

which is a homeomorphism onto a proper clopen image. We similarly define a proper clopen neighborhood \( U_2 \subseteq U_1 \setminus \text{im}(g_0) \) of \( x \) which has diameter less than \( \frac{1}{2} \).

Inductively, suppose \( U_0, \ldots, U_{n+1}, V_0, \ldots, V_n \) have been constructed along with maps that are homeomorphic onto their image

\[
f_i : U_i \setminus (U_{i+1} \cup \text{im}(g_{i-1})) \to V_i \setminus V_{i+1}
\]

\[
g_i : V_i \setminus (V_{i+1} \cup \text{im}(f_i)) \to U_{i+1} \setminus U_{i+2}
\]

for \( 0 \leq i \leq n - 1 \). Using Lemma 2.7 as above, we then define a map which is a homeomorphism onto its image

\[
f_n : U_n \setminus (U_{n+1} \cup \text{im}(g_{n-1})) \to V_n \setminus \{ y \}
\]

and choose a proper clopen neighborhood \( V_{n+1} \subseteq V_n \setminus \text{im}(f_n) \) of \( y \), of diameter less than \( \frac{1}{n+1} \). Similarly, we define a map which is a homeomorphism onto its image

\[
g_n : V_n \setminus (V_{n+1} \cup \text{im}(f_n)) \to U_{n+1} \setminus \{ x \}
\]

and choose a proper clopen neighborhood \( U_{n+2} \subseteq U_{n+1} \setminus \text{im}(g_n) \) of \( x \), of diameter less than \( \frac{1}{n+2} \). We thereby inductively construct such a sequence of maps \( f_0, f_1, \ldots \) and \( g_0, g_1, \ldots \).

Now, restrict target spaces of \( f_i, g_i \) to their images. Then, by construction, the domains and images of the \( f_i \) and \( g_i^{-1} \) are disjoint and their respective unions are \( U \setminus \{ x \} \) and \( V \setminus \{ y \} \). Thus, by taking the union of \( f_i \) and \( g_i^{-1} \), we obtain a continuous bijection \( \psi : U \setminus \{ x \} \to V \setminus \{ y \} \) since their domains are open subsets. Similarly, we can define the continuous inverse of \( \psi \) with the \( f_i^{-1} \) and \( g_i \). Moreover, we can extend \( \psi \) to a homeomorphism \( \varphi : U \to V \) by mapping \( x \) to \( y \). \( \square \)

### 3 Generation of the homeomorphism group

Our proof of Theorem A in the case of an unmarked surface proceeds via the following steps. First, we define the notion of a **half-space** of \( \Sigma \) and show that the normal closure of a single involution contains an \( H \)-translation for some half-space \( H \). Formally, if \( H \) is a half-space, we say a homeomorphism \( \varphi \) is an **\( H \)-translation** if \( \varphi^n(H) \) are all pairwise disjoint. We then show that the normal closure of such a \( \varphi \) contains all homeomorphisms supported on \( H \) and that half-space supported homeomorphisms generate \( \text{Homeo}^+(\Sigma) \).
Definition 3.1. A self-similar ends space \((E, F)\) is uniformly self-similar if \(M(E)\) is one equivalence class homeomorphic to a Cantor set. A surface \(\Sigma\) is called uniformly self-similar if \((E(\Sigma), E^\mathbb{R}(\Sigma))\) is uniformly self-similar and \(\Sigma\) has genus 0 or infinity.

Definition 3.2. For a uniformly self-similar surface \(\Sigma\), we will define a half-space to be a subsurface \(H \subseteq \Sigma\) such that

(i) \(H\) is a closed subset of \(\Sigma\).

(ii) \(H\) has a single connected, compact boundary component.

(iii) \(E(H)\) and \(E(H^c)\) both contain a maximal end of \(E(\Sigma)\).

We state the following useful lemma.

Lemma 3.3 (Lemma 2.1 [9]). Let \(\Sigma\) be a surface. Every clopen set \(U\) of \(E(\Sigma)\) is induced by a connected subsurface of \(\Sigma\) with a single boundary circle. Consequently, if \(\Sigma\) is uniformly self-similar, and both \(U, U^c\) contain maximal ends, then this subsurface is a half-space.

The following corollary follows easily from the above Lemma and the fact that \(E(\Sigma)\) is a subspace of a Cantor set.

Corollary 3.4. Let \(\Sigma\) be a uniformly self-similar surface, and let \(x \in M(E)\). There exists a sequence of nested half-spaces \(S_1 \supseteq S_2 \supseteq \ldots\) such that \(\{x\} = \bigcap_i E(S_i)\) and \(\partial S_i\) is compact and connected for all \(i\).

Lemma 3.5. Let \(\Sigma\) be a uniformly self-similar surface. Then, there exists a half-space \(H \subseteq \Sigma\), an involution \(\tau\), and \(\varphi \in \text{Homeo}^+(\Sigma)\) such that \(\{\varphi^n(H)\}_{n \in \mathbb{Z}}\) are all pairwise disjoint and \(\varphi \in \langle \langle \tau \rangle \rangle\). Moreover, \(\varphi\) is a product of two conjugates of \(\tau\), and we can choose \(\tau\) and \(\varphi\) to fix some point in the complement of \(\bigcup_{n \in \mathbb{Z}} \varphi^n(H)\).

Proof. We first construct a somewhat explicit surface which is homeomorphic to \(\Sigma\). Let \(E = E(\Sigma)\) and \(M = M(E)\). Let \(y, z \in M\) be distinct points. Since \(E\) is homeomorphic to a subspace of the Cantor set, we can find disjoint clopen subsets \(\{U_i \mid i \in \mathbb{Z}\}\) such that

- \(U_i \cap M \neq \emptyset\)
- \(E = \{y, z\} \cup \bigcup_i U_i\)
- \(y\) is an accumulation point of \(\{U_i \mid i \leq 0\}\) but not \(\{U_i \mid i \geq 0\}\), and \(z\) is an accumulation point of \(\{U_i \mid i \geq 0\}\) but not \(\{U_i \mid i \leq 0\}\).

By Lemma 3.3, there is a half-space \(\Sigma_0 \subseteq \Sigma\) where \(E(\Sigma_0) = U_0\). We let \(S_i\) be a copy of \(\Sigma_0\) for each \(i \in \mathbb{Z}\), and we let \(S\) be the (oriented) infinite cylinder with countably many disjoint open discs removed in a periodic fashion. (We make sure to choose discs with disjoint closures.) See Figure 1. Let \(\{C_i \mid i \in \mathbb{Z}\}\) be the boundary components of \(S\). Let \(S\) be the surface obtained by gluing \(C_i\) to \(\partial S_i\) via some homeomorphism \(\psi_i : C_i \to \partial S_i\) which respects orientation of the surfaces.

We first claim that \(S \cong \Sigma\). By Theorem 2.3, we need only prove that \(S, \Sigma\) have the same genus and that there is a homeomorphism of end spaces mapping \(E^\mathbb{R}(S)\) to \(E^\mathbb{R}(\Sigma)\). We will implicitly use some results from [21] without referencing them. Recall that \(\Sigma\) has genus 0 or \(\infty\), and in the latter case,
a maximal end must be accumulated by genus. Thus $S_i$ and $S$ have infinite genus if and only if $\Sigma$ does. By Lemma 2.8, we have that $E(S_i) \cong U_0 \cong U_i$ (respecting genus accumulation). By construction of $S$, all of $E(S_i)$ are clopen subsets of $E(S)$. Moreover, for $i \in \mathbb{Z}$, let $D_i$ be disjoint curves in the cylinder $S$ such that they are all translates of each other, separate the two ends of the cylinder, and the two sides of $D_i$ contain $\{C_j \mid j < i\}$ and $\{C_j \mid j \geq i\}$. See Figure 1. Let $P_i^+, P_i^-$ be the subsurfaces of $S$ on either side of $D_i$. Then, $P_i^+$ for $i \geq 0$ (resp. $P_i^-$ for $i \leq 0$) defines an end $z'$ (resp. $y'$) of $S$. Then, for $n \in \mathbb{N}$,

$$E(S) = E(P_n^-) \cup E(P_n^+) \cup \bigcup_{i=-n}^{n-1} E(S_i),$$

and since only $y', z'$ are in all $E(P_n^-)$ and $E(P_n^+)$ resp., we have $E(S) = \{y', z'\} \cup \bigcup_i E(S_i)$. Moreover, it’s clear that $E(S_i)$ accumulate to $y'$ but not $z'$ as $i \to -\infty$ and $E(S_i)$ accumulates to $z'$ but not $y'$ as $i \to \infty$. Therefore, we may define a homeomorphism $E(S) \to E(\Sigma)$ mapping $E(S_i) \to U_i$ and $\{y', z'\} \to \{y, z\}$.

Recall that the homeomorphism $E(S_i) \cong U_i$ maps ends accumulated by genus to ends accumulated by genus. If $S$ has infinite genus, then every $S_i$ has infinite genus and so $y', z'$ are accumulated by genus (as must $y, z$ as they are maximal in $E(\Sigma)$). Consequently, $E(S) \cong E(\Sigma)$ maps $E\Sigma(S)$ to $E\Sigma(\Sigma)$ and only $E\Sigma(S)$ to $E\Sigma(\Sigma)$. Consequently, $S \cong \Sigma$.

We now construct an explicit involution $\tau$ of $S$ which normally generates the desired $\varphi$. First, we define an involution $\tau$ on $S$. We simply take the “rotation” about an axis piercing $D_0$ and interchanging the ends $y', z'$. This induces a homeomorphism between pairs of curves $C_i$ and $C_j$, where $j \geq 0$ and $i = -(j + 1)$. For such a pair $i < j$ related by $\tau(C_i) = C_j$, define a homeomorphism $\tau_{i,j} : S_i \to S_j$ such that

$$\tau_{i,j}|_{S_i} = \psi_j \circ \tau \circ \psi_i^{-1}.$$

For the same pair, define $\tau_{j,i} : S_j \to S_i$ as the inverse of $\tau_{i,j}$. Note that

$$\tau_{j,i}|_{S_j} = \psi_i \circ \tau^{-1} \circ \psi_j^{-1} = \psi_i \circ \tau \circ \psi_j^{-1}$$

since $\tau$ has order 2. Thus, the $\tau_{i,j}$ agree on the overlap with $\tau$, and so we extend $\tau$ to a homeomorphism $\tau$ on all of $S$ via the $\tau_{i,j}$. It is clear that $\tau$ has order 2.

We can similarly define an involution $\sigma$ which is a “rotation” with axis piercing $D_1$. Then, $\sigma(\tau(S_i)) = S_{i+2}$. I.e. $\varphi = \sigma \circ \tau$ is the desired $H$-translation where $H = S_0$. This establishes the lemma.

To show that we can choose $\tau$ and $\varphi$ to also fix a point outside the $S_i$, we do the following. We homotope $D_0$ and $D_1$ within $S$ towards each other until they meet tangentially at one point. One
can choose an involution $\tau$ that permutes the $C_i$ in the same manner as above, maps $D_0$ to itself and fixes the common point of $D_0 \cap D_1$. Similarly, $\sigma$ may be chosen to map $D_1$ to itself and fix the common point of $D_0 \cap D_1$. Since the new $\tau$ and $\sigma$ each permute the $C_i$ in the same manner as before, the rest of the argument goes through, and $\varphi$ will fix the same point.

**Remark 3.6.** Note that in the above proof, the translation $\varphi$ (in the version without a fixed point) verifies that a surface $\Sigma$ with uniformly self-similar ends space with 0 or infinite genus has no non-displaceable surfaces. This also follows from [1, Lemma 5.9, Lemma 5.13] which prove it in the case where $E(\Sigma)$ is merely self-similar, but our construction gives a different perspective.

We now show that an $H$-translation normally generates $\text{Homeo}(H, \partial H) < \text{Homeo}^+(\Sigma)$ for some half-space $H$. The proof technique is sometimes referred to as a “swindle”.

**Lemma 3.7.** Let $\Sigma$ be uniformly self-similar, and let $\varphi$ have the properties as described in Lemma 3.5. Then, $\langle \langle \varphi \rangle \rangle$ contains $\text{Homeo}(H, \partial H)$. Moreover, every element of $\text{Homeo}(H, \partial H)$ is a product of $\varphi$ and a conjugate of $\varphi^{-1}$.

**Proof.** Let $f \in \text{Homeo}(H, \partial H) \subseteq \text{Homeo}^+(\Sigma)$. We let $\hat{f} = \prod_{i=0}^{\infty} \varphi^{-i} f \varphi^i$. This is well-defined since $\varphi^{-i} f \varphi^i$ is supported on $\varphi^{-i}(H)$ and these are pairwise disjoint for all $i \geq 0$ by assumption. Then, we have the following computation, which, again, is valid because of disjoint supports.

$$[\hat{f}, \varphi^{-1}] = \hat{f} \varphi^{-1} \hat{f}^{-1} \varphi = \left( \prod_{i=0}^{\infty} \varphi^{-i} f \varphi^i \right) \varphi^{-1} \left( \prod_{i=0}^{\infty} \varphi^{-i} f^{-1} \varphi^i \right) \varphi = \left( \prod_{i=0}^{\infty} \varphi^{-i} f \varphi^i \right) \left( \prod_{i=1}^{\infty} \varphi^{-i} f^{-1} \varphi^i \right) = f. \quad \Box$$

### 3.1 Half-space homeomorphisms generate

Our proof that homeomorphisms of half-spaces generate $\text{Homeo}^+(\Sigma)$ relies on a few key facts about half-spaces in uniformly self-similar surfaces. We record these as lemmas which we will prove below.

**Lemma 3.8.** Let $H_1, H_2 \subset \Sigma$ be two half-spaces. Then, one of $H_1 \cap H_2^c, H_2 \cap H_1^c$ contains a half-space.

**Lemma 3.9.** If $H_1, H_2$ are two distinct half-spaces contained in a third distinct half-space $H_3$ and both are disjoint from a fourth half-space $H_4 \subset H_3$, then there exists $\varphi \in \text{Homeo}^+(\Sigma)$ supported on $H_3$ such that $\varphi(H_1) = H_2$.

**Lemma 3.10.** If $H \subset \Sigma$ is a half-space, then so is $\overline{H}$. All half-spaces are homeomorphic via an ambient homeomorphism of $\Sigma$. Every half-space contains two disjoint half-spaces.

Using these three lemmas we can prove one of our main results, namely, that half-space homeomorphisms generate $\text{Homeo}^+(\Sigma)$.

**Theorem 3.11.** Let $\Sigma$ be a uniformly self-similar surface, and let $H \subset \Sigma$ be a half-space. Then, $\text{Homeo}^+(\Sigma)$ is the normal closure of the subgroup $\text{Homeo}(H, \partial H)$. Furthermore, every element of $\text{Homeo}^+(\Sigma)$ is a product of at most 3 homeomorphisms, each of which is a conjugate to an element of $\text{Homeo}(H, \partial H)$.
Proof. Let \( f \in \text{Homeo}^+(\Sigma) \). First, note that by Lemma 3.10, any half-space supported homeomorphism is conjugate into \( \text{Homeo}(H, \partial H) \). Thus, it suffices to show \( f \) is a product of at most 3 half-space supported homeomorphisms.

Let \( H_1 = H \) and \( H_2 = f(H) \). We now apply Lemma 3.8, and first consider the case where \( H_1^c \cap H_2^c \) contains a half-space. By Lemma 3.10, \( H_1^c \cap H_2^c \) contains two disjoint half-spaces \( H_3, H_4 \). Applying Lemma 3.9 to \( H_1, H_2, H_3^c, \) and \( H_4 \), we see that there is a homeomorphism \( \varphi_1 \), supported on \( H_3^c \), such that \( \varphi_1(H_2) = \varphi_1(f(H_1)) = H_1 \). By further composing by some \( \varphi_2 \) supported on \( H_1 \), we can ensure \( \varphi_2 \circ \varphi_1 \circ f \) restricts to the identity on \( H_1 \). Finally, composing by an appropriate third homeomorphism \( \varphi_3 \) supported on \( H_1^c \), we obtain \( \varphi_3 \circ \varphi_2 \circ \varphi_1 \circ f = \text{Id} \). Note that \( \varphi_2 \circ \varphi_1 \) is supported on \( H_3^c \), so in this case we only require two half-space supported homeomorphisms.

Now, suppose we are in the case where \( H_1 \cap H_2^c \) contains a half-space. By Lemma 3.10, we can assume \( H_1 \cap H_2^c \) contains three disjoint half-spaces \( H_3, H_4, H_5 \). By Lemma 3.9 applied to \( H_2, H_4, H_5^c \), and \( H_5 \), there exists a homeomorphism \( \psi \) supported on \( H_5^c \) such that \( \psi(H_2) = H_5 \). Since \( H_5 \subset H_1 \), the subsurface \( H_1^c \cap \psi(f(H_1))^c = H_1^c \) contains a half-space, and we are reduced to the previous case. In this case, we see that \( f \) is a product of three half-space supported homeomorphisms.

We now prove the required lemmas.

Proof of Lemma 3.8. By definition, \( E(H_2^c) \) contains some maximal end \( x \). By Corollary 3.4, there exist nested half-spaces \( S_1 \supset S_2 \supset \ldots \) such that \( \cap_i E(S_i) = \{x\} \). Since these half-spaces are compact sets, eventually some \( S_i \) does not intersect \( \partial H_1 \cup \partial H_2 \). Since \( x \in E(H_2^c) \) and the \( S_i \) are connected, either \( S_i \subseteq H_1 \cap H_2^c \) or \( S_i \subseteq H_1^c \cap H_2 \).

Proof of Lemma 3.10. The first statement follows immediately from the definition of half-space. Since a half-space has a maximal end of \( \Sigma \), it has the same genus as \( \Sigma \). Thus, by the classification of surfaces and Lemma 2.8, any two half-spaces are homeomorphic. Since the (closures of) the complements are half-spaces too, we can map the complement to the complement and extend the homeomorphism to all of \( \Sigma \).

By assumption, \( E(H) \) has some maximal end \( x \) of \( E(\Sigma) \). Since \( E(H) \) is a clopen in \( E(\Sigma) \) and the set of maximal ends is a Cantor set, \( E(H) \) contains another distinct maximal end \( y \). Using Corollary 3.4 for \( x \) and \( y \) and compact of boundaries of half spaces, we can easily deduce the existence of the required half-spaces.

Proof of Lemma 3.9. The presence of the half-space \( H_4 \) guarantees that \( E(H_3 \setminus H_1) \) and \( E(H_3 \setminus H_2) \) both contain a maximal end. Thus, by Lemma 2.8,

\[
(E(H_3 \setminus H_1), E(H_3 \setminus H_2)) \cong E(H_3 \setminus H_2) \cong (E(H_3 \setminus H_2), E(H_3 \setminus H_2)).
\]

Clearly, the two subsurfaces have the same genus and finite number of boundary components, and so \( H_3 \setminus H_4 \cong H_3 \setminus H_2 \). Similarly, by Lemma 3.10, \( H_1 \cong H_2 \). By arranging the homeomorphisms to be identical on the overlapping boundary component, we produce a homeomorphism \( H_3 \rightarrow H_3 \) mapping \( H_1 \rightarrow H_2 \) and \( H_3 \setminus H_1 \rightarrow H_3 \setminus H_2 \).

Theorem 3.12. If \( \Sigma \) is uniformly self-similar, then \( \text{Homeo}^+(\Sigma) \)
• is normally generated by a single involution,
• is normally generated by an $H$-translation,
• is uniformly perfect.

Moreover, each element of $\text{Homeo}^+(\Sigma)$ is a product of at most 3 commutators, 6 $H$-translations, and 12 involutions.

**Proof.** Combine Lemma 3.5, Lemma 3.7, and Theorem 3.11.

By considering quotients of $\text{Homeo}^+(\Sigma)$ onto the mapping class group $\text{MCG}(\Sigma)$ and the homeomorphism group of its ends space $(E(\Sigma), E^e(\Sigma))$, we also derive the following corollaries. Note that for a half-space $H \subset \Sigma$, $E(H)$ is a clopen set containing a non-empty proper subset of $M(E(\Sigma))$.

**Corollary 3.13.** If $\Sigma$ is uniformly self-similar, then the statements of Theorem 3.12 also hold for $\text{MCG}(\Sigma)$.

**Corollary 3.14.** If $(E, F)$ is uniformly self-similar, then the statements of Theorem 3.12 also hold for $\text{Homeo}(E, F)$, where a half-space $H \subset E$ is a clopen set containing a non-empty proper subset of $M(E)$.

**Proof.** By [21, Theorem 2], there is a surface $\Sigma$ such that $(E(\Sigma), E^e(\Sigma)) \cong (E, F)$ and the genus is 0 if $F = \emptyset$ and infinite if $F \neq \emptyset$. Thus $\Sigma$ is uniformly self-similar when $(E, F)$ is. The corollary then follows from Theorem 3.12 and surjectivity of $\text{Homeo}^+(\Sigma) \to \text{Homeo}(E(\Sigma), E^e(\Sigma))$.

## 4 Surfaces with a marked point

The proof in the case of a marked surface is very similar to the unmarked case, and we will use some of the same lemmas. Let $\Sigma$ be a uniformly self-similar surface with a fixed basepoint $* \in \Sigma$. We define half-space exactly as before, but distinguish between marked half-spaces which contain $*$ and unmarked half-spaces which don't. The main new lemma we require is the following.

**Lemma 4.1.** Let $\Sigma$ be a uniformly self-similar surface with a marked point $* \in \Sigma$. Let $H \subseteq \Sigma$ be an unmarked half-space. Then, every Dehn twist in $\text{MCG}(\Sigma, \ast)$ is contained in the normal closure of $\text{MCG}(H, \partial H)$.

**Remark 4.2.** For convenience and simplicity, we will conflate half-spaces and simple closed curves with their ambient isotopy classes rel $*$ throughout this section.

**Proof.** Let $T_\gamma \in \text{MCG}(\Sigma, \ast)$ be the Dehn twist about a simple closed curve $\gamma$ (which avoids $*$). First, we consider the case where $\gamma$ is nonseparating. Then, $\Sigma$ has infinite genus. Since $\gamma$ is compact, Corollary 3.4 implies that $\gamma$ is contained in some half-space (or the closure of its complement which is also a half-space) which we denote by $H$. This case is concluded if $H$ is unmarked. Suppose instead $H$ is marked. Then, since $\gamma$ is nonseparating, we can find a path from $\partial H$ to $*$ which avoids $\gamma$. Deleting some small regular neighborhood of this path from $H$, we obtain an unmarked half-space containing $\gamma$. 

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Note: The content of the image appears to be a continuation of the previous discussion, focusing on the properties of uniformly self-similar surfaces and their implications on the structure of the mapping class group and homeomorphism group of the ends space. The provided text captures the essence of these discussions, emphasizing the role of involutions, $H$-translations, and the uniform perfection of the groups involved.
Now, suppose \( \gamma \) is a separating curve, and let \( S_1, S_2 \subseteq \Sigma \) be the two surfaces on either side of \( \gamma \). If both \( E(S_1), E(S_2) \) contain a maximal end of \( \Sigma \), then they are both half-spaces whose mapping class groups contain \( T_{\gamma} \), and one must be unmarked. Suppose, w.l.o.g., that \( E(S_1) \) contains no maximal ends. If \( S_1 \) is also unmarked, then we can connect it by some strip (avoiding \( * \)) to an unmarked half-space in \( S_2 \) to create a new unmarked half-space \( H' \) which contains \( S_1 \). Then \( \text{Homeo}(H', \partial H') \) contains \( T_{\gamma} \).

The difficult case is when \( S_1 \) contains no maximal ends but is marked. Using Corollary 3.4 repeatedly, we can find three disjoint half-spaces \( H_1, H_2, H_3 \) contained in \( S_2 \). Since half-spaces have connected boundary, the complement of \( H_1 \cup H_2 \cup H_3 \cup S_1 \) is connected, and we may choose disjoint paths \( \alpha_1, \alpha_2 \) in this complement connecting \( \gamma = \partial S_1 \) to \( \partial H_1, \partial H_2 \) respectively. Let \( L \) be a regular neighborhood of \( \gamma \cup \partial H_1 \cup \partial H_2 \cup \alpha_1 \cup \alpha_2 \) in this complement. Then, \( L \) is a sphere with 4 boundary components, i.e. a lantern, where three boundary curves are \( \gamma, \partial H_1, \) and \( \partial H_2 \) and the fourth is some simple closed curve \( \beta \) bounding a half-space \( H_4 \) containing \( H_3 \).

We seek to use the lantern relation to show that \( f \) is a product of homeomorphisms supported on an unmarked half-space. The lantern relation implies that \( T_{\gamma} \) is equal to a word in the Dehn twists about \( \partial H_1, \partial H_2, \beta \) and three other simple closed curves \( \delta_1, \delta_2, \delta_3 \) each of which separates \( L \) into two three-holed spheres. Thus for all \( i = 1, 2, 3 \), each side of \( \delta_i \) must contain at least one of \( H_1, H_2, H_3 \), i.e. each \( \delta_i \) separates \( \Sigma \) into one marked and one unmarked half-space, and thus \( \delta_i \) lies in an unmarked half-space. Consequently, the twists about \( \partial H_1, \partial H_2, \beta \), and the \( \delta_i \) are all supported on an unmarked half-space. The lemma follows.

In the unmarked case, we replace Lemma 3.9 with the following.

**Lemma 4.3.** If \( H_1, H_2, \) and \( H_3 \) are disjoint unmarked half-spaces, then there is a homeomorphism \( \varphi \in \text{Homeo}(\Sigma) \) supported on some unmarked half-space \( H_4 \) such that \( \varphi(H_1) = H_2 \).

**Proof.** By Lemma 3.10, \( H_3 \) contains two unmarked disjoint half-spaces \( H'_3, H''_3 \). Since half-spaces have single boundary components, the complement of \( H_1 \cup H_2 \cup H'_3 \cup H''_3 \) is connected, and we can attach \( H_1 \) to \( H_2 \) and \( H'_3 \) by two strips disjoint from \( H''_3 \) and the marked point to create a subsurface \( H_4 \) with a single boundary circle that contains \( H_1, H_2, \) and \( H'_3 \). Since both \( E(H_4) \supset E(H_1) \) and \( E(H'_3) \supset E(H''_3) \) contain a maximal end, \( H_4 \) is a half-space. We can now apply Lemma 3.9 to \( H_1, H_2, H_4, \) and \( H'_3 \).

We can now prove the analogous theorem that half-space supported homeomorphisms generate \( \text{MCG}(\Sigma, *) \).

**Theorem 4.4.** Let \( \Sigma \) be a uniformly self-similar surface with a marked point \( * \in \Sigma \), and let \( H \subseteq \Sigma \) be an unmarked half-space. Then, \( \text{MCG}(\Sigma, *) \) is generated by the normal closure of \( \text{MCG}(H, \partial H) \).

**Proof.** Let \( f \in \text{MCG}(\Sigma, *) \). All unmarked half-spaces are the same up to \( \text{Homeo}(\Sigma, *) \) by an argument nearly identical to that in the proof of Lemma 3.10. Therefore, it suffices to show \( f \) is a product of mapping classes supported on unmarked half-spaces.

Let \( H_1 \) be an unmarked half-space, and let \( C \) be a simple closed curve such that \( C \) and \( \partial H_1 \) bound an annulus containing \( * \) (in the interior). Let \( H_2 = f(H_1) \). Then, by Lemma 3.8, either \( H_1 \cap H'_2 \)
or $H^c_1 \cap H^c_2$ contains a half-space, which we can choose to be unmarked, by passing to a deeper half-space if necessary.

We first show that there is some mapping class $g$ in the normal closure of $\text{MCG}(H, \partial H)$ such that $g_1(f(H_1)) = H_1$ and $g_1 \circ f|_{H_1} = id|_{H_1}$. Let's first consider the case where $H^c_1 \cap H^c_2$ contains an unmarked half-space. Then, by Lemma 3.10, $H^c_1 \cap H^c_2$ contains two disjoint unmarked half-spaces $H_3, H_4$. By Lemma 4.3, there are two mapping classes supported on some unmarked half-spaces, one which maps $H_2$ to $H_3$ and another which maps $H_3$ to $H_1$. By composing these maps with some appropriate third mapping class supported on $H_1$, we obtain the desired $g_1$. If instead $H_1 \cap H^c_2$ contains an unmarked half-space, then $H_1 \cap H^c_2$ contains two disjoint unmarked half-spaces $H_3, H_4$. By Lemma 4.3, there is some mapping class $h$ supported on an unmarked half-space such that $h(f(H_1)) = h(H_2) = H_3 \subset H_1$. Thus $H^c_1 \cap H^c_2 = H^c_1$ contains an unmarked half-space, the one bounded by $C$, and we are reduced to the first case.

Let $C' = g_1(f(C))$. Then $C'$ and $\partial H_1$ bound an annulus containing *. (Note that $C'$ need not be $C$ up to ambient isotopy fixing *.) Let $S \subset \Sigma$ be a compact subsurface with the following properties.

- $S$ contains the annulus bounded by $\partial H_1$ and $C$ and the annulus bounded by $\partial H_1$ and $C'$.
- $S$ does not intersect the interior of $H_1$
- no boundary component bounds a disc in $\Sigma$.

Within $S$, each of $C, C'$ is a separating curve which bounds an annulus with $\partial H_1 \subset \partial S$ containing a marked point. Consequently, the genus of the separating curves $C, C'$ must be identical and they partition the boundary of $S$ identically. Thus, there is some mapping class in $\text{MCG}(S, \partial S \cup *)$ mapping $C'$ to $C$. Since $\text{MCG}(S, \partial S \cup *)$ is generated by Dehn twists, by Lemma 4.1, there is some $g_2$ in the normal closure of $\text{MCG}(H, \partial H)$ such that $g_2(g_1(f(C)) = C$ and $g_2 \circ g_1 \circ f|_{H_1} = id|_{H_1}$.

Let $H_0$ be the unmarked half-space bounded by $C$. Clearly, $g_2(g_1(f(H_0)) = H_0$. Since the mapping class group of the annulus with a marked point between $H_0$ and $H_1$ is generated by Dehn twists, by Lemma 4.1, we can compose by some third element $g_3$ in the normal closure of $\text{MCG}(H, \partial H)$ such that $g_3 \circ g_2 \circ g_1 \circ f = id$.

We can now easily prove the analogous theorem for the mapping class group of a marked uniformly self-similar surface. Note that we have no statements about uniform perfection, or about a bound on the word length of an element as a product of involutions, or about the homeomorphism group. The first two are impossible by a result of J. Bavard [3] in the case of $S^2$ minus a Cantor set. The proof fails to show every element is a word of uniformly bounded length in involutions and half-space supported homeomorphisms only because of the step where we map $C'$ to $C$. The theorem is only proven for the mapping class group and not the homeomorphism group because we use the lantern relation in the proof of Lemma 4.1.

**Theorem 4.5.** Let $\Sigma$ be a uniformly self-similar surface with a marked point * $\in \Sigma$. Then, $\text{MCG}(\Sigma, *)$ is generated by involutions. Moreover, $\text{MCG}(\Sigma, *)$ is normally generated by a single involution and is a perfect group.

**Proof.** Let $\tau$ and $\varphi$ be as in Lemma 3.5. Lemma 3.7 applies equally to $\text{MCG}(\Sigma, *)$ (with an identical proof), and so $\langle \langle \tau \rangle \rangle$ contains $\text{MCG}(H, \partial H)$ for some unmarked half-space and all elements of
MCG(H, ∂H) are a single commutator in MCG(Σ). Theorem 4.4 finishes the proof.

5 Self-similar but not uniformly

It is natural to wonder whether surfaces with a self-similar ends space and with genus 0 or ∞ are generated by involutions, are perfect, etc. It is already known that the mapping class group of the one-ended, infinite genus surface has abelianization containing an uncountable direct sum of Q’s [7]. This surface fits into this category, but perhaps is not a particularly compelling example since the results of [7] are for pure mapping class groups of infinite-type surfaces, and for the one-ended infinite genus surface, the mapping class group happens to coincide with the pure mapping class group. However, using a covering trick and some of the results of [7], we can prove that the abelianization of MCG(R² \ N) is similarly large.

Proposition 5.1. MCG(R² \ N) surjects onto ⊔_2ℵ₀ Q.

For the proof of the proposition, we need the following fact about abelian groups, which follows from [11, Theorem 21.3 and 23.1].

Lemma 5.2. Let A be an abelian group. Suppose A contains ⊔_I Q for some non-empty set I. Then A surjects onto ⊔_I Q.

Proof of Proposition 5.1. Let Σ_L be the infinite genus surface with one end. This admits a 2-fold branched cover of R² where R² = Σ_L/D and D = Z/2Z acts by an involution. See Figure 2. More formally, one can construct this from gluing infinitely many copies of a 2-fold branch cover of an annulus by a 2-holed torus and one copy of a 2-fold branched cover of a disc by a 1-holed torus. Let Σ_F be R² with the branch points removed, i.e. Σ_F ≅ R² \ N, and let Σ_PL be Σ_L with the branch points removed. Then, Σ_PL → Σ_F is a regular degree 2 cover with deck group D.

Choose marked points ˜* ∈ Σ_PL and * ∈ Σ_F. We first show that there is a lifting homomorphism MCG(Σ_F, *) → MCG(Σ_PL, ˜*), defined by lifting representative homeomorphisms. Since these are mapping class groups fixing marked points, it is a straightforward consequence of covering space theory that such a homomorphism exists and is unique provided the action of MCG(Σ_F, *) preserves the subgroup K = ker(π₁(Σ_F, *) → D). Since the punctures of Σ_F came from branched points of degree 2, any simple loop in π₁(Σ_F, *) which encloses a single puncture does not lift to a closed curve in Σ_PL and so must map to the nontrivial element of D. We can choose a basis {β_n}n∈N of

Figure 2: The surface Σ_L admits an involution which gives a degree 2 branched cover of R² branched at the red points.
the free group $\pi_1(\Sigma_F, \ast)$ consisting entirely of simple loops each enclosing a single puncture. Consequently, $K$ consists precisely of those even length words in this generating set. For any mapping class $f \in \text{MCG}(\Sigma_F, \ast)$, the set $\{f(\beta_n)\}_{n \in \mathbb{N}}$ is also another generating set of simple loops enclosing single punctures, and for the same reasons $K$ consists of words of even length in these generators. It is clear then that $f(K) = K$.

We now have a lifting homomorphism $\text{MCG}(\Sigma_F, \ast) \to \text{MCG}(\Sigma_{PL}, \hat{\ast})$. Since the points deleted from $\Sigma_L$ are isolated, $\text{MCG}(\Sigma_{PL}, \hat{\ast})$ preserves that set of ends, and we have a well-defined forgetful map $\text{MCG}(\Sigma_{PL}, \hat{\ast}) \to \text{MCG}(\Sigma_L, \hat{\ast})$. In [7], explicit mapping classes are constructed which project to non-trivial elements in the abelianization of $\text{MCG}(\Sigma_L, \hat{\ast})$. (See [7, Theorem 6.1].) Specifically, if $\{\gamma_n\}_{n \in \mathbb{N}}$ is a sequence of distinct, pairwise disjoint, separating, simple closed curves where each $\gamma_n$ separates the marked point from the single end of $\Sigma_L$, then the subgroup topologically generated by the twists $\{T_{\gamma_n}\}_{n \in \mathbb{N}}$ projects to a group containing a $\bigoplus_{\aleph_0} \mathbb{Q}$. One can easily find such $\gamma_n$ which double cover simple closed curves $\alpha_n$ in $\Sigma_F$, and so $T_{\gamma_n}$ is the lift of $T^{2\alpha_n}_n$. (E.g. one can choose $\alpha_1$ to be a simple closed curve bounding a disc with the marked point and three punctures and then choose $\alpha_i, \alpha_{i+1}$ to always bound an annulus with two punctures. Then $\gamma_i$ are the preimages of the $\alpha_i$ under the covering map.) Thus, $\text{MCG}(\Sigma_F, \hat{\ast})$ maps onto the same abelian group (generated by the $T_{\gamma_n}$). I.e. $H_1(\text{MCG}(\Sigma_F, \hat{\ast}); \mathbb{Z})$ has a quotient $A$ containing $\bigoplus_{\aleph_0} \mathbb{Q}$. By Lemma 5.2, $A$ maps onto $\bigoplus_{\aleph_0} \mathbb{Q}$, so we also get a surjection $\varphi : H_1(\text{MCG}(\Sigma_F, \ast); \mathbb{Z}) \to \bigoplus_{\aleph_0} \mathbb{Q}$.

To pass to $\text{MCG}(\Sigma_F)$, we borrow a technique from [7]. Consider the Birman short exact sequence (see [7])

$$1 \longrightarrow \pi_1(\Sigma_F, \ast) \longrightarrow \text{MCG}(\Sigma_F, \hat{\ast}) \longrightarrow \text{MCG}(\Sigma_F) \longrightarrow 1.$$ 

Abelianization is right exact, so we get the commutative diagram

$$\bigoplus_{\aleph_0} \mathbb{Z} \longrightarrow H_1(\text{MCG}(\Sigma_F, \hat{\ast}); \mathbb{Z}) \longrightarrow H_1(\text{MCG}(\Sigma_F); \mathbb{Z}) \longrightarrow 0$$

$$\downarrow \text{id} \quad \downarrow \varphi \quad \downarrow \hat{\varphi}$$

$$\bigoplus_{\aleph_0} \mathbb{Z} \longrightarrow \bigoplus_{\aleph_0} \mathbb{Q} \longrightarrow P \longrightarrow 0$$

The image of $\bigoplus_{\aleph_0} \mathbb{Z}$ in $\bigoplus_{\aleph_0} \mathbb{Q}$ still misses a copy of $\bigoplus_{\aleph_0} \mathbb{Q}$, so the quotient $P$ still contains a copy of $\bigoplus_{\aleph_0} \mathbb{Q}$. The map $\hat{\varphi}$ is surjective, so we can conclude $H_1(\text{MCG}(\Sigma_F); \mathbb{Z})$ surjects onto $\bigoplus_{\aleph_0} \mathbb{Q}$, again by Lemma 5.2.

We now produce many more classes of examples by building surfaces that naturally map onto the one-ended infinite genus surface $\Sigma_L$ or $\Sigma_F = \mathbb{R}^2 \setminus \mathbb{N}$.

**Theorem 5.3.** Suppose $\Sigma$ is a surface of one of the following two types.

(1) $E(\Sigma)$ has exactly one end accumulated by genus.

(2) $\Sigma$ has genus 0 and $E(\Sigma)$ has one maximal end $y$, such that in the partial order on $[E]$, the class of $y$ has an immediate predecessor $E(x)$ with countably infinite cardinality.

Then $\text{MCG}(\Sigma)$ maps onto $\bigoplus_{\aleph_0} \mathbb{Q}$. 

Proof. Choose a marked point $*$ on $\Sigma$. Note that by the same trick of using the Birman short exact sequence

$$1 \longrightarrow \pi_1(\Sigma, \ast) \longrightarrow \text{MCG}(\Sigma, \ast) \longrightarrow \text{MCG}(\Sigma) \longrightarrow 1,$$

it is enough to show the abelianization of $\text{MCG}(\Sigma, \ast)$ maps onto $\bigoplus_{\aleph_0} \mathbb{Q}$.

The statement for the one-ended infinite genus surface $\Sigma_L$ is by [7]. The statement for $\Sigma_F = \mathbb{R}^2 \setminus \mathbb{N}$ is Proposition 5.1. For all other case, we will consider an appropriate map to one of these two surfaces.

On $\Sigma_L$, we will say a sequence of simple closed curves $\{\gamma_n\}_{n \in \mathbb{N}}$ is good if the curves are distinct, pairwise disjoint, separating, and each curve separates the maximal end of $\Sigma_L$ from the marked point. On $\Sigma_F$, a sequence of curves $\{\alpha_n\}_{n \in \mathbb{N}}$ is good if under the covering map $(\Sigma_L, \ast) \rightarrow (\Sigma_F, \ast)$, each $\alpha_n$ is double covered by a curve $\gamma_n$ and the sequence $\{\gamma_n\}$ is good. By [7] and the proof of Proposition 5.1, the subgroup topologically generated by Dehn twists about a good sequence of curves maps onto $\bigoplus_{\aleph_0} \mathbb{Q}$ under the map to the abelianization of the mapping class group. First, assume $\Sigma$ is of the first type. The proof in the other case will be similar. The assumption on $\Sigma$ means we have a map $(\Sigma, \ast) \rightarrow (\Sigma_L, \ast)$ by forgetting all but the only end accumulated by genus. This induces a well-defined map $\text{MCG}(\Sigma, \ast) \rightarrow \text{MCG}(\Sigma_L, \ast)$, since this end is invariant under $\text{MCG}(\Sigma, \ast)$. By the previous paragraph, it is enough to exhibit a sequence of pairwise disjoint curves $\{\alpha_n\}_{n \in \mathbb{N}}$ on $\Sigma$ whose image under the forgetful map forms a good sequence on $\Sigma_L$. To do this we will represent $\Sigma$ in an explicit way as described below.

Identify $S^2 = \mathbb{R}^2 \cup \{\infty\}$ with base point $\infty$. We will construct $\Sigma$ from $S^2$ by removing points from $\mathbb{R}^2 \subset S^2$ and attaching handles appropriately. Let $K \subset [0,1]$ be the standard Cantor set. Recall $E(\Sigma)$ is homeomorphic to a closed subset of $K$. Since the homeomorphism group of $K$ acts transitively, we can realize $E(\Sigma)$ as a closed subset $E \subset K$ with the only end accumulated by genus at 0. By [21], the ends space of $S^2 - E$ is homeomorphic to $E$. It remains to attach handles to $\mathbb{R}^2 - E$ so that the handles will only accumulate onto the origin. To this end, choose a sequence $\{y_n\}_{n \in \mathbb{N}} \subset [0,1] - K$ such that $y_n \rightarrow 0$ monotonically. In particular, $\{y_n\} \cap E = \emptyset$. Let $d_n = y_n - y_{n+1}$. For each $n$, let $T_n$ be a torus with one boundary component. Let $z_n$ be the midpoint of $[y_{n+1}, y_n]$. In $\mathbb{R}^2$, let $p_n = (-z_n, 0)$, and $B_n$ be the open ball of diameter $d_n/2$ centered at $p_n$. Now remove each $B_n$ from $\mathbb{R}^2$ and attach $T_n$ by gluing $\partial T_n$ to $\partial B_n$. Let $\Sigma'$ be the resulting surface with marked point.

Figure 3: Building $\Sigma$ of the first type.
∞. By construction, the tori accumulate only onto the origin. It follows then by the classification
of surfaces, Σ is homeomorphic to Σ, and we can make this homeomorphism take ∞ to *. By
filling in all of E except the origin, we get a marked surface (Σ', ∞) homeomorphic to (Σ_L, *), and
a representation of the forgetful map (Σ, *) → (Σ_L, *). Using this picture, it is now easy to find
the curves α_n which we take to be the circle of radius y_n centered at the origin. By construction, the
circles |α_n| avoid E, are pairwise disjoint, and each separates the origin from ∞. Furthermore,
since there is a handle between two consecutive circles α_n and α_{n+1}, namely T_{n+1}, these circles
remain topologically distinct after filling in all of E − {0}. This finishes the proof in this case.

Now suppose Σ is of the second type. We first claim that, by forgetting all but the maximal end of
E(Σ) and the class E(x) of its immediate predecessor, we get a map (Σ, *) → (Σ_F, *). Taking the same
approach as above, realize E(Σ) as a closed subset E ⊂ K with the maximal end at the origin. By
Richards, the surface (S^2 \ E, ∞) is homeomorphic to (Σ, *). We claim the origin is the only accumu-
lation point of E(x). Since E(x) has no successor except the origin, any other accumulation point
of E(x) must be equivalent to x. But then every point in E(x) is an accumulation point of E(x). This
makes E(x) ∪ {0) a closed and perfect subset of K, so it is homeomorphic to K, contradicting our
assumption that E(x) has countable cardinality. Thus, for any compact interval I ⊂ (0, 1], I ∩ E(x)
has finite cardinality, so we can enumerate E(x) as a decreasing sequence {x_n}_{n ∈ N} ⊂ E converging
to 0. This shows (S^2 \ (E(x) ∪ {0}), ∞) ∼= (Σ_F, *). Since mapping classes induce homeomorphisms of
the ends space which, as noted in Section 2, preserve the equivalence class of each end, we obtain
a well-defined map MCG(Σ, *) → MCG(Σ_F, *). To finish, take any point y_n ∈ [x_{n+1}, x_n] such that
{y_n} ∩ E = ∅. Then the circles |α_n| of radius y_n centered at the origin are pairwise disjoint and
we can extract from them a subsequence that project to a good sequence of curves on Σ_F. This
finishes the proof. 

Remark 5.4. Note that for Σ of the second type in Theorem 5.3, we do not need the maximal end y
to have a unique immediate predecessor. This is because the mapping class group always preserves
equivalence classes of ends, so even if y has other immediate predecessors, the map forgetting all
ends except y and E(x) still induces a well-defined homomorphism on the level of mapping class
groups.

Remark 5.5. In our setting above, it seems plausible that the forgetful map from MCG(Σ) to either
MCG(Σ_L) or MCG(Σ_F) is surjective, but we will not pursue that statement here.

We record some consequences of Theorem 5.3.

Corollary 5.6. Suppose Σ is a surface that satisfies one of the descriptions in Theorem 5.3. Let * ∈ Σ
be a marked point. Let G be either Homeo^+(Σ), MCG(Σ), Homeo^+(Σ, *), or MCG(Σ, *). Then G is not
perfect, is not generated by torsion elements, and does not have the automatic continuity property.

Proof. Since a Polish group is separable, it can have at most c = 2^<ω continuous epimorphisms to Q.
But ⊕ 2^<ω Q has 2^c epimorphisms to Q. So, by Theorem 5.3, MCG(Σ) is not perfect, is not generated
by torsion elements, and does not have the automatic continuity property. These three properties
are inherited by quotients, so Homeo^+(Σ) also cannot have any of these properties. The same
argument applies to a marked Σ. 

□
Remark 5.7. If $E$ is a countable ends space homeomorphic to $\omega^\alpha + 1$, for some countable successor ordinal $\alpha$, then $\Sigma = S^2 \setminus E$ is a surface of type 2 of Theorem 5.3, by [18, Proposition 4.3]. This gives Theorem D of the introduction.

5.1 Topological generation by involutions

Theorem 5.8. Let $\Sigma$ be either $\mathbb{R}^2 \setminus \mathbb{N}$ or the infinite genus surface with one end. Then $\text{MCG}(\Sigma)$ is topologically generated by involutions and is the topological closure of the normal closure of a single involution. Consequentially, $H^1(\text{MCG}(\Sigma), \mathbb{Z}) = 0$.

Proof. We first focus on $\Sigma = \mathbb{R}^2 \setminus \mathbb{N}$. The beginning of the proof is very similar to that of Theorem 3.12. Note that $\mathbb{R}^2 \setminus \mathbb{N}$ is homeomorphic to $\mathbb{R}^2 \setminus \mathbb{Z}^2$. This is because both surfaces have genus 0, and their ends spaces are homeomorphic. The advantage of viewing the surface as $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is as follows.

Let $\tau$ be the rotation in the plane by angle $\pi$ centered at the origin, i.e. $\tau(x + iy) = e^{i\pi}(x + iy)$. We also have the translation $\phi(x + iy) = (x + 1) + iy$. Both maps preserve $\mathbb{Z}^2$, so they induce homeomorphisms of $\Sigma$, where $\tau$ has order 2. One checks that $\phi \tau \phi^{-1} \tau = \phi^2$.

We define a half-space of $\Sigma$ to be a closed subset $H \subset \Sigma$, such that $\partial H$ is a properly embedded simple arc joining infinity to itself, and both $H$ and $H^c$ contain infinitely many punctures (isolated ends) of $\Sigma$. We will consider an explicit half-space in $\Sigma$. Let $h(x) = \sec(\pi x) - .5$ with domain $(-.5, .5)$. The graph of $h(x)$ is a convex curve that misses all of $\mathbb{Z}^2$ and is contained in the vertical strip $\{(x, y): -.5 \leq x \leq .5\}$. See figure 4. The set $H = \{(x, y) \in \Sigma: y \geq h(x)\}$ is a half-space, and $\phi^2$ is an $H$–free translation, in the sense that $\langle \phi^2n H \rangle_{n \in \mathbb{Z}}$ are pairwise disjoint. Therefore, with the same swindle as before, we obtain

$\text{Homeo}(H, \partial H) \leq \langle \langle \phi^2 \rangle \rangle \leq \langle \langle \tau \rangle \rangle$.

While the swindle still works, the rest of the proof for Theorem 3.12 does (and should) not work. The only statement that seems to fail is Lemma 3.8, (and so Theorem 3.11 also fails in this case).

Figure 4: The half-space $H$ in $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

We now move to the mapping class group. As before, to simplify the discussion, we will conflate half-spaces and simple closed curves with their ambient isotopy classes. We will keep on denoting $\phi$ and $\tau$ for their mapping classes.

Consider the short exact sequence

$$1 \to \text{PMCG}(\Sigma) \to \text{MCG}(\Sigma) \to \text{Homeo}(E(\Sigma)) \to 1,$$

where $\text{PMCG}(\Sigma)$ is called the pure mapping class group, i.e. the subgroup fixing each end of $\Sigma$. Since $\Sigma$ has no genus, by [20], $\text{PMCG}(\Sigma) = \text{PMCG}_c(\Sigma)$, where $\text{PMCG}_c(\Sigma)$ is the subgroup of compactly supported mapping classes. Since Dehn twists generate the pure mapping class group of
any compact surface, PMCG(Σ) is topologically generated by Dehn twists. The goal now is to show every Dehn twist in MCG(Σ) is contained in the normal closure of MCG(H, ∂H), and that the normal closure of MCG(H, ∂H) surjects onto Homeo(E(Σ))

We first deal with the Dehn twists. Let α ∈ Σ be any simple closed curve. Then α bounds a topological disk containing finitely many points of Z^2. Choose a simple closed curve β ⊂ H that bounds an equal number of points of Z^2. We can find a homeomorphism f ∈ Homeo^+(Σ), such that f(α) = β. This is simply the change-of-coordinate principle made possible by the classification of surfaces. We now have

\[ T_α = T_{f^{-1}(β)} = f^{-1}T_β f \in \langle \langle \text{MCG}(H, ∂H) \rangle \rangle . \]

To show \langle \langle \text{MCG}(H, ∂H) \rangle \rangle surjects onto Homeo(E(Σ)), we produce sufficiently many permutations of non-maximal ends. First note that E(Σ) has exactly one maximal end, represented by ∞, which must be invariant under any homeomorphism. Every other end is isolated, so Homeo(E(Σ)) is nothing other than the permutation group Sym(Z^2) on Z^2. Within H, pair off infinitely many punctures/ends \{(x_{i,1}, x_{i,2})\}_{i \in I} such that x_{i,2} is directly above x_{i,1} and all the pairs are pairwise disjoint. It is clear that MCG(H, ∂H) contains a mapping class f which transposes all pairs simultaneously. Note that \( \bigcup_{i \in I} \{(x_{i,1}, x_{i,2})\} \) is both infinite and co-infinite in E(Σ). Since by [21], MCG(Σ) surjects onto Homeo(E(Σ)) = Sym(Z^2), the image of \langle \langle \text{MCG}(H, ∂H) \rangle \rangle in Sym(Z^2) contains all order 2 permutations supported on infinite, co-infinite subsets. It is straightforward to show this set generates Sym(Z^2). In summary, we have shown \langle \langle \text{MCG}(H, ∂H) \rangle \rangle topologically generates PMCG(Σ) and surjects onto Homeo(E(Σ)). This yields

\[ \text{MCG}(Σ) = \langle \langle \text{MCG}(H, ∂H) \rangle \rangle = \langle \langle φ^2 \rangle \rangle = \langle \langle τ \rangle \rangle . \]

To go from \( \mathbb{R}^2 \setminus Z^2 \) to the one-ended infinite genus surface Σ_L we observe that instead of removing the integer lattice points from \( \mathbb{R}^2 \), we can remove a small disk from each lattice point and glue on a handle to get a surface Σ’ homeomorphic to Σ_L. Furthermore, we can make sure τ and φ preserve Σ’. A half-space in Σ’ is simply a closed component of a dividing arc that cuts off two component of infinite genus. The explicit half-space H we defined for \( \mathbb{R}^2 \setminus Z^2 \) can also be made into a half-space here. Then running the same argument as above and observing that PMCG(Σ’) = MCG(Σ’) completes the proof.

The last statement about the cohomology of these groups follows from the fact that any homomorphism from a Polish group to Z is automatically continuous [8].

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