SCATTERING SYSTEMS WITH SEVERAL EVOLUTIONS AND
FORMAL REPRODUCING KERNEL HILBERT SPACES

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Abstract. A Schur-class function in $d$ variables is defined to be an analytic
contractive-operator valued function on the unit polydisk. Such a function is
said to be in the Schur–Agler class if it is contractive when evaluated on any
commutative $d$-tuple of strict contractions on a Hilbert space. It is known
that the Schur–Agler class is a strictly proper subclass of the Schur class if
the number of variables $d$ is more than two. The Schur–Agler class is also
characterized as those functions arising as the transfer function of a certain
type (Givone–Roesser) of conservative multidimensional linear system. Previ-
ous work of the authors identified the Schur–Agler class as those Schur-class
functions which arise as the scattering matrix for a certain type of (not neces-
sarily minimal) Lax–Phillips multievolution scattering system having some ad-
ditional geometric structure. The present paper links this additional geometric
scattering structure directly with a known reproducing-kernel characterization
of the Schur–Agler class. We use extensively the technique of formal repro-
ducing kernel Hilbert spaces that was previously introduced by the authors
and that allows us to manipulate formal power series in several commuting
variables and their inverses (e.g., Fourier series of elements of $L^2$ on a torus)
in the same way as one manipulates analytic functions in the usual setting of
reproducing kernel Hilbert spaces.

Cora Sadosky passed away in December 2010. We lost a collaborator of many years,
and a wonderful friend. Like many others, we miss her.

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1. Introduction

Let $\mathcal{E}$ and $\mathcal{E}_*$ be two Hilbert spaces and let $\mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ denote the Schur class
of holomorphic functions on the unit disk $\mathbb{D}$ in the complex plane with values
equal to contraction operators between $\mathcal{E}$ and $\mathcal{E}_*$. The Schur class was originally
introduced by Schur for the scalar case $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$ and is the natural class
of functions for which one formulates interpolation problems of Nevanlinna–Pick
and Carathéodory–Fejér type. Recently there has been a resurgence of interest in

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matrix- and operator-valued versions of this class due to connections with manifold applications in operator theory, mathematical physics and linear system theory. Specifically, Schur-class functions arise (1) as the characteristic function of a Hilbert-space contraction operator (see [49] as well as [19, 20, 40, 21]), (2) as the scattering matrix of a (discrete-time) Lax-Phillips scattering system (see [39] as well as [41]), and (3) as the transfer function of discrete-time conservative linear system (see [2, 50] as well as [19, 20, 40, 21]). Over the last few decades the connections between these various theories have come be understood (see e.g. [1, 30, 12]). Let us only mention the key structure theorem for Schur-class functions.

**Theorem 1.1.** Let $S : D \rightarrow \mathcal{L}(E, E_*)$ be an operator-valued function defined on the unit disk $D$. Then the following conditions are equivalent:

1. $S$ is in the Schur class $S(E, E_*)$.
2. The kernel $K_S(z, w) = (I - S(z)S(w)^*)/(1 - zw)$ is a positive kernel, i.e. has a factorization
   $$
   \frac{I - S(z)S(w)^*}{1 - zw} = H(z)H(w)^*
   $$
   for some function $H : D \rightarrow \mathcal{L}(H', E_*)$ for some Hilbert space $H'$.
3. There exist a Hilbert space $H$ and a unitary colligation
   $$
   U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} H \\ E \end{bmatrix} \rightarrow \begin{bmatrix} H' \\ E_* \end{bmatrix}
   $$
   so that $S(z)$ can be realized in the form
   $$
   S(z) = D + zC(I - zA)^{-1}B.
   $$

There have recently appeared several types of generalizations of the Schur-class functions to several-variable contexts (see [7, 15] for surveys). We focus here on the generalizations where the unit disk $D$ is replaced by the unit polydisk

$$
D^d = \{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_k| < 1 \text{ for } k = 1, \ldots, d \}
$$

in $d$-dimensional complex space $\mathbb{C}^d$. One can define the $d$-variable Schur class $S_d(E, E_*)$ to consist of functions $S : D^d \rightarrow \mathcal{L}(E, E_*)$ of $d$ complex variables which are analytic on $D^d$ and with values equal to contraction operators from $E$ to $E_*$. However, if $d > 2$ it is known that the analogue of Theorem 1.1 fails. Instead, we define the Schur–Agler class $S_{A_d}(E, E_*)$ (usually shortened to $SA(E, E_*)$ as the number of variables $d$ will be fixed throughout) as the space of functions $S(z) = \sum_{n \in \mathbb{Z}_+^d} S_n z^n$ (with the standard multivariable notation $z^n = z_1^{n_1} \cdots z_d^{n_d}$) holomorphic on $D^d$ with values equal to operators from $E$ to $E_*$ such that

$$
S(T) := \sum_{n \in \mathbb{Z}_+^d} S_n \otimes T^n \in \mathcal{L}(E \otimes K, E_* \otimes K)
$$

is a contraction ($\|S(T)\| \leq 1$) whenever $T = (T_1, \ldots, T_d)$ is a commuting $d$-tuple of strict contractions on some Hilbert space $K$. (Here $T^n = T_1^{n_1} \cdots T_d^{n_d}$ if $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$.) Note that any Schur-Agler-class function is also Schur-class since one can take in particular $K = \mathbb{C}$ and $T = z = (z_1, \ldots, z_d)$ a $d$-tuple of scalar operators. The following analogue of Theorem 1.1 appears in [2, 3] (see [10] for the additional condition $(2')$).
Theorem 1.2. Let \( S: \mathbb{D}^d \to \mathcal{L}(E, E_*) \) be an operator-valued function defined on the unit polydisk \( \mathbb{D}^d \). Then the following are equivalent:

(1) The function \( S \) is in the Schur–Agler class \( SA(E, E_*) \).

(2) There exist auxiliary Hilbert spaces \( \mathcal{H}_1', \ldots, \mathcal{H}_d' \) and operator-valued functions \( H_k: \mathbb{D}^d \to \mathcal{L}(\mathcal{H}_k, E_*) \) for \( k = 1, \ldots, d \) so that \( S \) has the so-called Agler decomposition

\[
I - S(z)S(w)^* = \sum_{k=1}^d (1 - z_k \overline{w}_k) H_k(z) H_k(w)^*. 
\tag{1.1}
\]

(2') There exist auxiliary Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_d \) and operator-valued functions \( H_k = \begin{bmatrix} H_{k1}' & \cdots & H_{kd}' \end{bmatrix} : \mathbb{D}^d \to \mathcal{L}(\mathcal{H}_k, [E]) \) for \( k = 1, \ldots, d \) so that \( S \) has the augmented Agler decomposition

\[
\begin{bmatrix}
I - S(z)S(w)^* & S(z) - S(w)^* \\
S(w) - S(w)^* & I - S(w)^*S(w) 
\end{bmatrix} = \sum_{k=1}^d \begin{bmatrix} H_{k1}(z) & H_{k2}(w)^* \\
H_{k2}(z) & H_{k2}(w)^* 
\end{bmatrix} \circ \begin{bmatrix} 1 - z_k \overline{w}_k & z_k - \overline{w}_k \\
z_k - \overline{w}_k & 1 - z_k \overline{w}_k \end{bmatrix} 
\tag{1.2}
\]

where \( \circ \) is the Schur or entrywise matrix product.

(3) There exist Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_d \) and a unitary colligation \( U \) of the structured form

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1d} & B_1 \\ \vdots & \ddots & \vdots \\ A_{d1} & \cdots & A_{dd} & B_d \\ C_1 & \cdots & C_d & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \end{bmatrix} \to \begin{bmatrix} E \\ \vdots \\ E_* \end{bmatrix} 
\tag{1.3}
\]

so that \( S(z) \) is realized in the form

\[
S(z) = D + C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)B 
\tag{1.4}
\]

where we have set

\[
Z_{\text{diag}}(z) = \begin{bmatrix} z_1 I_{\mathcal{H}_1} & & \\
& \ddots & \\
& & z_d I_{\mathcal{H}_d} \end{bmatrix}.
\]

We remark that the proof of (3) \( \Rightarrow \) (2) or (3) \( \Rightarrow \) (2') is particularly transparent: given that (1.1) holds with \( U \) as in (1.3) unitary, one uses the relations

\[
AA^* + BB^* = I, \quad AC^* + BD^* = 0, \quad CC^* + DD^* = I, \\
A^* A + C^* C = I, \quad A^* B + C^* D = 0, \quad B^* B + D^* D = I 
\tag{1.5}
\]

arising from the unitary property of \( U \) to check that (1.1) holds with

\[
H_k(z) = C(I - Z_{\text{diag}}(z)A)^{-1}P_k
\]

and that (1.2) holds with

\[
H_k^1(z) = C(I - Z_{\text{diag}}(z)A)^{-1}P_k, \quad H_k^2(z) = B^*(I - Z_{\text{diag}}(z)^* A^*)^{-1}P_k 
\tag{1.6}
\]

with \( P_k \) the orthogonal projection of \( \mathcal{H} = \bigoplus_{j=1}^k \mathcal{H}_j \) onto its \( k \)-th coordinate space \( \mathcal{H}_k \) (identified as a subspace of \( \mathcal{H} \)). The original proof of the converse direction (2) \( \Rightarrow \) (3) involves what we call the “lurking isometry argument” (see [7]): from
the given Agler decomposition (1.1) there is associated a natural partially defined isometry \( V \); one can then identify a one-to-one correspondence between unitary extensions \( U \) (possibly outside the original Hilbert space) of \( V \) and realizations of \( S(z) \) in the form (1.4) associated with the structured unitary colligation (1.3). In the sequel, we shall refer to a structured unitary colligation \( U \) of the form (1.3) as a Givone–Roesser unitary (GR-unitary) colligation. (Multidimensional linear systems whose coefficient matrix has a structure of (1.3), though without any norm constraints on its blocks, and whose transfer function has the form (1.4), were introduced in [27, 28]; see (6.2) for the explicit form of the system equations.)

We also note that the equivalence (1) \( \iff \) (3) in Theorem 1.2 for the case of rational inner matrix-valued functions on the bidisk \( D^2 \) was independently proved by Kummert [38] in a stronger form: the spaces \( \mathcal{H}_1, \mathcal{H}_2 \) in (3) are finite-dimensional. This strengthening of Agler’s theorem, which also includes item (2) of Theorem 1.2 with rational matrix-valued functions \( H_1, \ldots, H_d \), appears also in [26, 14, 35]. Moreover, it turns out that this version of Theorem 1.2 for the case of rational inner matrix-valued functions can be extended to any number of variables [36].

A new approach to the issue of unitary realizations of Schur-Agler-class functions was made in [14]. A multievolution Lax–Phillips scattering system \( \mathcal{S} \) (associated with function theory on the polydisk) was defined in [23, 24] (see also [47]). A minimal such scattering system \( \mathcal{S} \) is completely determined (up to unitary equivalence) by its scattering matrix \( S \) which can be any Schur-class function. If the scattering system has some additional geometric structure (“colligation geometry”), one can identify a unitary colligation \( U \) of the form (1.3) which is embedded in \( \mathcal{S} \). When this is the case, the scattering matrix is also the transfer function of the unitary colligation \( U \) embedded in \( \mathcal{S} \) and thus \( S \) is in fact in the Schur–Agler class \( \mathcal{S}(E, E^\ast) \) rather than just the Schur class \( \mathcal{S}(E) \). (Notice that scattering systems associated with a different type of colligation geometry were considered in [33].) In [14] we arrived at a new criterion for a Schur-class function \( S \) to be a Schur-Agler-class function: the Schur-class function \( S \) is in the Schur–Agler class if and only if there is a multievolution scattering system (not necessarily minimal) with scattering function \( S \) possessing the special colligation geometry. By combining the results of [23, 24, 16] with [14], it is clear that, given a Schur-class function \( S \) on the polydisk \( \mathbb{D}^d \), there is a multievolution scattering system \( \mathcal{S} \) having scattering matrix \( S \) along with the special colligation geometry if and only if \( S \) has an Agler decomposition (1.1) (or, equivalently, an augmented Agler decomposition (1.2)). One of the goals of this paper is to construct such a multievolution scattering system \( \mathcal{S} \) (in functional-model form) directly from a given Agler decomposition (1.1) for \( S \).

Given a Schur-class function \( S \), in [14] we presented three functional models (see [44, 45] for the \( d = 1 \) case) for a minimal multievolution scattering system having \( S \) (up to trivial identifications) as its scattering matrix, called the Pavlov model, the de Branges–Rovnyak model and the Sz.-Nagy–Foiaş model. We find that the most convenient one for our purposes here is the de Branges–Rovnyak model. The modification of the de Branges–Rovnyak model introduced in [44, 45] works with two components, one of which is analytic on the disk \( \mathbb{D} \) while the other is conjugate-analytic on \( \mathbb{D} \); it turns out that this modified version of the original de Branges–Rovnyak model is more convenient to work with for many manipulations...
and formulas. However there are difficulties with extending this formalism to the multivariable context; a function may be the product of an analytic function in one variable and a conjugate analytic function in another variable. We here modify the de Branges–Rovnyak functional model once again by making use of the formal power series formalism from [17]; with this formalism the extension to several variables can be done relatively smoothly. In particular, the formal version of the augmented Agler decomposition ([14]) plays a key role for the analysis to follow (see (5.1) below).

The formal reproducing-kernel formalism is the tool which helps clarify the more complicated connections between the various notions of scattering and system minimality (scattering-minimal, closely-connected, strictly closely-connected) in the multivariable situation which were introduced but left mysterious in [16]. These complications are caused by the fact that the colligation geometry may not mesh well with the scattering geometry; specifically, the projections coming from the colligation geometry may not commute with the projection onto the minimal part of the scattering system.

The paper is organized as follows. Section 2 develops the preliminaries on formal reproducing kernel Hilbert spaces. Here we also clarify and enhance some results from [17] on multipliers in the context of formal reproducing kernel Hilbert spaces. Section 3 places the de Branges–Rovnyak model scattering system [14] into a reproducing kernel framework. Section 4, in addition to recalling the admissible-trajectory construction from [14] for embedding a given GR-unitary colligation into a multievolution scattering system having scattering matrix equal to the transfer function of the colligation, also presents a multivariable version of the Schäffer-matrix construction to achieve the same embedding. Here we also recall the converse colligation-geometry characterization of those scattering systems containing an embedded GR-unitary colligation. In Section 5 we obtain a reproducing-kernel analogue of this colligation geometry. Under an additional hypothesis which eliminates the presence of nuisance overlapping spaces, this leads to the true presence of the colligation geometry inside the de Branges–Rovnyak functional-model minimal multievolution scattering system; this is exactly the special case where a minimal multievolution scattering system carries a colligation geometry. Section 6 assumes that we are given a GR-unitary colligation \( U = [A, B] \) and sorts out various notions of minimality in terms of the operators \( A, B, C, D \). Section 7 shows how the original lurking-isometry approach can be carried out in functional-model form; the defect spaces involved in a known parametrization of the set of all the GR-unitary realizations of a given augmented Agler decomposition (see e.g. [8]) can be identified with the overlapping spaces arising from the augmented Agler decomposition. In particular, if the augmented Agler decomposition is strictly closely connected (a weaker form of minimal), then the associated closely-connected GR-unitary colligation realization is unique. Here we also discuss how these canonical functional models for a Schur-Agler-class function with given Agler decomposition relate to those obtained in [9]. The final Section 8 analyzes how to construct a functional-model multievolution scattering system carrying the colligation geometry directly from a given augmented Agler decomposition. Here the Schäffer-matrix construction sheds some insight on the structure of the problem; the results of Section 7 in some sense solve this problem.
Since the paper is quite long we indicate several “pointers” that the reader can use to jump directly to several key locations:

- Remark 6.13 summarizes the various notions of minimality: for (augmented) Agler decompositions, for Givone–Roesser unitary colligations, and for multievolution scattering systems, and the interrelations among these notions.
- Theorem 5.9 identifies explicitly the colligation geometry inside the de Branges–Rovnyak functional-model minimal multievolution scattering system in terms of a minimal (augmented) Agler decomposition, and provides a direct construction of scattering minimal GR-unitary realizations.
- Theorem 7.1 provides a parametrization of all GR-unitary realizations of a given (augmented) Agler decomposition in terms of functional models and overlapping spaces.

We mention that some analogous results concerning de Branges–Rovnyak functional-model realizations for a given Schur-class function, but in the context of Drury–Arveson spaces and coisometric colligations of Fornasini–Marchesini type, have been obtained recently in [4, 10, 11].

We finally note that an analogue of the Agler-decomposition characterization was recently obtained in [29] for the operator-valued $d$-variable Schur class using the methods developed in the present paper (with a reference to its earlier preprint version), namely the formal de Branges–Rovnyak model for a scattering system associated with the given Schur-class function. (This result was later reproved in [37] using standard reproducing-kernel techniques, however only in the case of scalar-valued functions.)

The notation is mostly standard. However there is the potential for confusion between internal and external orthogonal direct sums. We use the notation $\oplus$ and $\bigoplus$ for internal orthogonal direct sums and $\hat{\oplus}$ and $\hat{\bigoplus}$ for external orthogonal direct sums; for a (finite or countably infinite) collection $\{H_j : j \in J\}$ of Hilbert spaces and $h_j \in H_j$, $j \in J$, with $\sum_{j \in J} \|h_j\|_{H_j}^2 < \infty$, we denote by $\hat{\bigoplus} h_j$ the corresponding element of $\hat{\bigoplus} H_j$.

2. Function theory on $L^2(\mathbb{T}^d, \mathcal{F})$: the formal reproducing kernel point of view

2.1. Preliminaries on formal reproducing kernel Hilbert spaces. We recall the notion of a formal reproducing kernel Hilbert space (FRKHS for short) in the commuting indeterminates $z = (z_1, \ldots, z_d)$ from [17]. For $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, we set $z^n = z_1^{n_1} \cdots z_d^{n_d}$ (note that we allow each integer $n_k$ to be positive as well as negative or zero). We say that a Hilbert space whose elements are formal power series $f(z) = \sum_{n \in \mathbb{Z}^d} f_n z^n$ with coefficients in the Hilbert space $\mathcal{F}$ (i.e., $f \in \mathcal{F}[\{z^\pm 1\}]$) where here and throughout for brevity $\mathcal{F}[\{z^\pm 1\}]$ denotes the space of formal power series in the indeterminates $z_1, \ldots, z_n$ and their inverses $z_1^{-1}, \ldots, z_d^{-1}$ that we shall refer to as formal Laurent series is a formal reproducing kernel Hilbert space if the linear operator $\pi_n : f \mapsto f_n$ is continuous for each $n \in \mathbb{Z}^d$. In this case, for each $n \in \mathbb{Z}^d$ there is a formal power series $K_n(z) \in L(\mathcal{F})[\{z^\pm 1\}]$ such that, for each

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2 We emphasize that unlike the usual algebraic definitions we impose no restrictions on the support of our formal Laurent series; as a result the product operation is only partially defined; see Section 2.2 below and especially Remark 2.6.
Here we use, for an arbitrary coefficient Hilbert space $H$, if we then define a pairing be written more compactly in a way suggestive of the classical (non-formal) case

$$
\langle f(z), K(z, w)u \rangle_{H \times H([w \pm 1])} = \langle f(w), u \rangle_{H([w \pm 1]) \times H}.
$$

Here we use, for an arbitrary coefficient Hilbert space $C$, the general notion of a pairing

$$
\langle \cdot, \cdot \rangle_{C \times C[[z^{\pm 1}]]} : C \times C[[z^{\pm 1}]] \to \mathbb{C}[[z^{\pm 1}]]
$$

defined by

$$
\langle c, \sum_{n' \in \mathbb{Z}^d} f_{n'} z^{-n'} \rangle_{C \times C[[z^{\pm 1}]]} = \sum_{n' \in \mathbb{Z}^d} \langle c, f_{n'} \rangle_C z^{n'}.
$$

When convenient we take on occasion the pairing in the reverse order:

$$
\langle \sum_{m' \in \mathbb{Z}^d} f_{m'} z^{m'}, c \rangle_{C[[z^{\pm 1}]] \times C} = \sum_{m' \in \mathbb{Z}^d} \langle f_{m'}, c \rangle_C z^{-m'}.
$$

These are actually special cases of a more general partially defined pairing on $C[[z^{\pm 1}]]$ with values in $C[[z^{\pm 1}]]$ given by

$$
\left\langle \sum_{m' \in \mathbb{Z}^d} f_{m'} z^{m'}, \sum_{n' \in \mathbb{Z}^d} g_{n'} z^{-n'} \right\rangle_{C[[z^{\pm 1}]] \times C[[z^{\pm 1}]]} = \sum_{m' \in \mathbb{Z}^d} \left[ \sum_{n' \in \mathbb{Z}^d} \langle f_{m'}, g_{n'} \rangle_C \right] z^{m'}
$$

where the sum required to compute the coefficient of $z^{m'}$ is finite under the assumption that at least one of $f$ or $g$ is a polynomial in $z_1, \ldots, z_d$ and $z_1^{-1}, \ldots, z_d^{-1}$. When this is the case, we write $H = H(K)$ and we say that $K \in \mathcal{L}(F)[[z^{\pm 1}, w^{\pm 1}]]$ is the reproducing kernel for the formal reproducing kernel Hilbert space $H(K)$.

The following proposition extends the reproducing property (2.1) to the case where a vector $u \in F$ is replaced with a formal power series $u(w) \in F[[w^{\pm 1}]]$.

**Proposition 2.1.** Let $F$ be a Hilbert space, let $K(z, w) = \sum_{n, m \in \mathbb{Z}^d} K_{n, m} z^n w^{-m} \in \mathcal{L}(F)[[z^{\pm 1}, w^{\pm 1}]]$ be the reproducing kernel for a formal reproducing kernel Hilbert space $H(K)$, where we will identify $K(z, w)$ with $\sum_{m \in \mathbb{Z}^d} K_m(z) w^{-m} \in H(K)[w^{\pm 1}]$. Let $u(w) = \sum_{m \in \mathbb{Z}^d} u_m w^{-m} \in F[[w^{\pm 1}]]$ be such that the series $\sum_{\ell \in \mathbb{Z}^d} K_{m-\ell}(z) u_{\ell}$ converges weakly in $H(K)$ for every $m \in \mathbb{Z}^d$. Then $K(z, w) u(w) \in H(K)[w^{\pm 1}]$, for every $f(z) = \sum_{n \in \mathbb{Z}^d} f_n z^n \in H(K)$ and every $m \in \mathbb{Z}^d$ the series $\sum_{\ell \in \mathbb{Z}^d} \langle f_{m-\ell}, u_{\ell} \rangle_F$ converges, and

$$
\langle f(z), K(z, w) u(w) \rangle_{H(K) \times H(K)[[w^{\pm 1}]]} = \langle f(w), u(w) \rangle_{F[[w^{\pm 1}]] \times F[[w^{\pm 1}]]}.
$$

**Proof.** The first statement follows from the identity

$$
K(z, w) u(w) = \sum_{m \in \mathbb{Z}^d} \left( \text{weak } \sum_{\ell \in \mathbb{Z}^d} K_{m-\ell}(z) u_{\ell} \right) w^{-m}.
$$

Next, for every $f \in H(K)$ and $m \in \mathbb{Z}^d$,

$$
\sum_{\ell \in \mathbb{Z}^d} \langle f(z), K_{m-\ell}(z) u_{\ell} \rangle_{H(K)} = \sum_{\ell \in \mathbb{Z}^d} \langle f_{m-\ell}, u_{\ell} \rangle_F,
$$

proving our claim.
and since the sum on the left-hand side converges, so does the sum on the right-hand side. Notice that
\[
\langle f(w), u(w) \rangle_{\mathcal{F}[[w^{\pm 1}]] \times \mathcal{F}[[w^{\pm 1}]]} = \sum_{m \in \mathbb{Z}^d} \left( \sum_{\ell \in \mathbb{Z}^d} \langle f_{m - \ell}, u_\ell \rangle_{\mathcal{F}} \right) w^m.
\]

On the other hand,
\[
\langle f(z), K(z, w)u(w) \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[w^{\pm 1}]]} = \sum_{m \in \mathbb{Z}^d} \left( \sum_{\ell \in \mathbb{Z}^d} \langle f(z), K_{m - \ell}(z)u_\ell \rangle_{\mathcal{H}(K)} \right) w^m
\]
and (2.3) follows. \[\square\]

The following theorem from [17] characterizes which formal power series
\[
K \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]]
\]
arise as the reproducing kernel for some formal reproducing kernel Hilbert space \(\mathcal{H}(K)\).

**Theorem 2.2.** (see [17, Theorem 2.1]) Suppose that \(\mathcal{F}\) is a Hilbert space and that we are given a formal power series \(K \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]]\). Then the following are equivalent.

1. \(K\) is the reproducing kernel for a uniquely determined FRKHS \(\mathcal{H}(K)\) of formal power series in the commuting variables
   \[
z^{\pm 1} = (z_1, \ldots, z_d, z_1^{-1}, \ldots, z_d^{-1})
   \]
with coefficients in \(\mathcal{F}\).

2. There is an auxiliary Hilbert space \(\mathcal{H}'\) and a formal power series \(H(z) \in \mathcal{L}(\mathcal{H}', \mathcal{F})[[z^{\pm 1}]]\) so that
   \[
   K(z, w) = H(z)H(w)^*
   \]
where we use the convention
   \[
   (w^n)^* = w^{-n}, \quad H(w)^* = \sum_{n \in \mathbb{Z}^d} H_n^* w^{-n} \text{ if } H(z) = \sum_{n \in \mathbb{Z}^d} H_n z^n. \tag{2.4}
   \]

(3) \(K(z, w) = \sum_{n, n' \in \mathbb{Z}^d} K_{n, n'} z^n w^{-n'}\) is a positive kernel in the sense that
   \[
   \sum_{n, n' \in \mathbb{Z}^d} \langle K_{n, n'}, u_n, u_{n'} \rangle_{\mathcal{F}} \geq 0
   \]
for all finitely supported \(\mathcal{F}\)-valued functions \(n \mapsto u_n\) on \(\mathbb{Z}^d\).
Moreover, in this case the FRKHS \( \mathcal{H}(K) \) can be defined directly in terms of the formal power series \( H(z) \) appearing in condition (2) by
\[
\mathcal{H}(K) = \{ H(z)h : h \in \mathcal{H}' \}
\]
with norm taken to be the pullback norm
\[
\|H(z)h\|_{\mathcal{H}(K)} = \|Qh\|_{\mathcal{H}'}
\]
where \( Q \) is the orthogonal projection of \( \mathcal{H}' \) onto the orthogonal complement of the kernel of the map \( M_H : \mathcal{H}' \rightarrow F[[z^{\pm 1}]] \) given by
\[
M_H : h \mapsto H(z)h.
\]

**Remark 2.3.** We can always choose a factorization \( K(z, w) = H(z)H(w)^* \) with \( \ker M_H = \{ 0 \} \) so that \( M_H \) is an isometry of \( \mathcal{H}' \) onto \( \mathcal{H}(K) \). (A “canonical” choice is \( \mathcal{H}' = \mathcal{H}(K) \) and \( H_n = \pi_n : \sum_{n \in \mathbb{Z}^d} f_n z^n \mapsto f_n \); it is then easily seen that \( H_n^* u = \sum_{n \in \mathbb{Z}^d} (K_{n,n'} u) z^n \).

When the kernel \( K \in \mathcal{L}(F)[[z^{\pm 1}, w^{\pm 1}]] \) satisfies any of the equivalent conditions in Theorem 2.2, we shall write
\[
K \geq 0.
\]

One can also express the FRKHS \( \mathcal{H}(K) \) directly from the Laurent series coefficients \( K(z, w) = \sum_{n,n' \in \mathbb{Z}^d} K_{n,n'} z^n w^{-n'} \) in a more operator-theoretic way as follows. Suppose that \( [K_{n,n'}]_{n,n'} \) is a matrix of operators on the Hilbert space \( F \) with rows and columns indexed by \( \mathbb{Z}^d \). Note that any such operator-matrix defines a linear operator, denoted by \( K \) with the expectation that this will cause no confusion, from the space of polynomials \( F[z^{\pm 1}] \) into the space of formal power series \( F[[z^{\pm 1}]] \) according to the formula
\[
(Kp)(z) = \sum_{n \in \mathbb{Z}^d} \left( \sum_{n' \in \mathbb{Z}^d} K_{n,n'} p_{n'} \right) z^n \text{ if } p(z) = \sum_{n' \in \mathbb{Z}^d} p_{n'} z^n.
\]
Note that this formula involves only finite sums under the assumption that \( p(z) \) is a polynomial (so \( p_{n'} = 0 \) for all but finitely many \( n' \in \mathbb{Z}^d \)). We also note that there is a natural pairing between \( F[[z^{\pm 1}]] \) and \( F[z^{\pm 1}] \):
\[
\langle f(z), p(z) \rangle_{L^2} = \sum_{n \in \mathbb{Z}^d} \langle f_n, p_n \rangle_F
\]
if \( f(z) = \sum_{n \in \mathbb{Z}^d} f_n z^n \in F[[z^{\pm 1}]] \) and \( p(z) = \sum_{n \in \mathbb{Z}^d} p_n z^n \in F[z^{\pm 1}] \). We say that the operator \( K = [K_{n,n'}]_{n,n' \in \mathcal{L}(F)} \) is *positive-semidefinite* if
\[
\langle Kp, p \rangle_{L^2} \geq 0 \text{ for all } p \in F[z^{\pm 1}].
\]

**Theorem 2.4.** (See [17] Proposition 2.2.) Let \( [K_{n,n'}]_{n,n'} \) be a matrix of operators on the Hilbert space \( F \) with rows and columns indexed by \( \mathbb{Z}^d \), let \( K = [K_{n,n'}]_{n,n' \in \mathbb{Z}^d} \) be the associated operator from \( F[z^{\pm 1}] \) into \( F[[z^{\pm 1}]] \), and define a kernel \( K(z, w) \in \mathcal{L}(F)[[z^{\pm 1}, w^{\pm 1}]] \) by
\[
K(z, w) = \sum_{n,n' \in \mathbb{Z}^d} K_{n,n'} z^n w^{-n'}.
\]
Then the operator \( K \) is positive-semidefinite if and only if the associated kernel \( K(z, w) \) is a positive kernel (i.e., satisfies any one of the three equivalent conditions in Theorem 2.2). In this case, the associated FRKHS \( \mathcal{H}(K) \) can be described as
the closure of the linear manifold $K \mathcal{F}[z^{\pm 1}] \subset \mathcal{F}[z^{\pm 1}]$ in the $\mathcal{H}(K)$-inner product given by

$$
\langle Kp, Kp' \rangle_{\mathcal{H}(K)} = \langle Kp, p' \rangle_{\mathcal{F}[z^{\pm 1}] \times \mathcal{F}[z^{\pm 1}]},
$$

Given any positive formal kernel

$$
K(z, w) = \sum_{n, m \in \mathbb{Z}^d} K_{n, m} z^n w^{-m} \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]],
$$

we have already introduced the notation

$$
K_m(z) = \sum_{n \in \mathbb{Z}^d} K_{n, n} z^n \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}]],
$$

Then, in terms of the associated formal reproducing kernel Hilbert space $\mathcal{H}(K)$ as in Theorem 2.2, we then have

$$
K_m h \in \mathcal{H}(K) \text{ for each } m \in \mathbb{Z}^d \text{ and } h \in \mathcal{F}.
$$

2.2. Multipliers between formal reproducing kernel Hilbert spaces. Let now $K \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]]$ and $K' \in \mathcal{L}(\mathcal{F}')[[z^{\pm 1}, w^{\pm 1}]]$ be two arbitrary positive kernels with associated FRKHSs $\mathcal{H}(K)$ and $\mathcal{H}(K')$. We say that the formal power series

$$
S(z) = \sum_{n \in \mathbb{Z}^d} S_n z^n \in \mathcal{L}(\mathcal{F}, \mathcal{F}')[[z^{\pm 1}]]
$$

is a bounded multiplier from $\mathcal{H}(K)$ into $\mathcal{H}(K')$ (written $S \in \mathcal{M}(K, K')$) if, for each $f \in \mathcal{H}(K)$ the product of formal Laurent series $S(z) \cdot f(z)$ is well-defined, i.e.,

$$
(Sf)_n = \sum_{\ell \in \mathbb{Z}^d} S_{n-\ell} f_{\ell}
$$

converges in the weak topology on $\mathcal{F}'$ for each $n \in \mathbb{Z}^d$, the resulting series $(Sf)(z) = \sum_n (Sf)_n z^n$ is in $\mathcal{H}(K')$, and the associated operator $M_S : \mathcal{H}(K) \to \mathcal{H}(K')$ is bounded as an operator from $\mathcal{H}(K)$ into $\mathcal{H}(K')$.

As explained in [17], many of the results concerning formal reproducing kernel Hilbert spaces can be reduced to the corresponding results for the classical case by observing that any formal reproducing kernel Hilbert space can be viewed as a classical reproducing kernel Hilbert space over $\mathbb{Z}^d$ with reproducing kernel $K(n, m) = K_{n, m}$ corresponding to evaluation of the $n$-th Laurent coefficient $f_n$ (i.e., the value of $f$ at the “point” $n$) of a generic element $f(z) = \sum_{n \in \mathbb{Z}^d} f_n z^n$ of the space. While this technique works well for some results, its applicability for the analysis of bounded multipliers is quite limited since the multipliers of interest in the formal setting act via convolution (rather then pointwise or Schur) multiplication when expressed in terms of Laurent coefficients. Since the convolution multiplication involves a possibly infinite sum, one is necessarily confronted with convergence issues (see Remark 2.6 below) and much more elaborate arguments to verify results whose analogues in the classical setting are straightforward. A first instance of this phenomenon is the following proposition.

**Proposition 2.5.** Suppose that $K$ is positive kernel with coefficients in $\mathcal{L}(\mathcal{F})$, with associated FRKHS $\mathcal{H}(K)$, and suppose that $S$ is a formal power series in $\mathcal{L}(\mathcal{F}, \mathcal{F}')[[z^{\pm 1}]]$. If for each $f \in \mathcal{H}(K)$ the product of formal Laurent series $S(z) \cdot f(z)$...
$f(z)$ is well-defined as in (2.6), then $S(z)K(z,w)S(w)^*$ is a well-defined formal power series

$$S(z)K(z,w)S(w)^* = \sum_{n,m} \Gamma_{n,m} z^n w^{-m}$$

(2.7)

with coefficients $\Gamma_{n,m}$ given by either of the following two iterated sums

$$\Gamma_{n,m} = \text{weak } \sum_{\ell} \left( \text{weak } \sum_{\ell'} S_{n-\ell} K_{\ell,\ell'} S_{m-\ell'}^* \right)$$

$$= \text{weak } \sum_{\ell'} \left( \text{weak } \sum_{\ell} S_{n-\ell} K_{\ell,\ell'} S_{m-\ell'}^* \right), \quad n, m \in \mathbb{Z}^d. \quad (2.8)$$

**Proof.** Indeed, write $K(z,w) = \sum_{n,m \in \mathbb{Z}^d} K_{n,m} z^n w^{-m}$ where $K_{n,m}$ has the factored form

$$K_{n,m} = H_n H_m^*$$

for operators $H_n \in \mathcal{L}(\mathcal{H}', \mathcal{F})$, with $\mathcal{H}'$ some auxiliary Hilbert space. Moreover, as explained in Theorem 2.2, $f(z) = H(z)h = \left( \sum_{n \in \mathbb{Z}^d} H_n z^n \right) h \in \mathcal{H}(K)$ for each $h \in \mathcal{H}'$. The assumption that $S$ is a multiplier for $\mathcal{H}(K)$ tells us that the series $\sum_{n \in \mathbb{Z}^d} S_{n-\ell} H_{\ell}$ converges in the weak operator topology of $\mathcal{L}(\mathcal{H}', \mathcal{F})$. Then $\sum_{\ell' \in \mathbb{Z}^d} H_{\ell'}^* S_{m-\ell'}$ converges in the weak operator topology. We note the following general principle: if $\{A_L\}_{L \in \mathbb{Z}^+}$ and $\{B_{L'}\}_{L' \geq 0}$ are sequences of operators for which the weak limits $\text{weak lim}_{L \to \infty} A_L = A$ and $\text{weak lim}_{L' \to \infty} B_{L'} = B$ exist, then

$$\text{weak lim}_{L \to \infty} \left( \text{weak lim}_{L' \to \infty} A_L B_{L'} \right) = \text{weak lim}_{L \to \infty} A_L B = AB$$

and similarly,

$$\text{weak lim}_{L' \to \infty} \left( \text{weak lim}_{L \to \infty} A_L B_{L'} \right) = \text{weak lim}_{L' \to \infty} A_B L' = AB,$$

i.e.,

$$\text{weak lim}_{L \to \infty} \left( \text{weak lim}_{L' \to \infty} A_L B_{L'} \right) = AB = \text{weak lim}_{L' \to \infty} \left( \text{weak lim}_{L \to \infty} A_L B_{L'} \right).$$

For each fixed $n$ and $m$, we apply this general principle to the operator sequences

$$A_L = \sum_{\ell': |\ell'| \leq L} S_{n-\ell} H_{\ell}, \quad B_{L'} = \sum_{\ell': |\ell'| \leq L'} H_{\ell'}^* S_{m-\ell'}$$

where in general we set

$$|\ell| = \sum_{i=1}^d |\ell_i| \text{ for } \ell = (\ell_1, \ldots, \ell_d) \in \mathbb{Z}^d. \quad (2.9)$$

This enables us to conclude that

$$\text{weak } \sum_{\ell} \left( \text{weak } \sum_{\ell'} S_{n-\ell} H_{\ell} H_{\ell'}^* S_{m-\ell'} \right)$$

$$= \text{weak } \sum_{\ell'} \left( \text{weak } \sum_{\ell} S_{n-\ell} H_{\ell} H_{\ell'}^* S_{m-\ell'} \right) = \Gamma_{n,m},$$
Remark 2.6. While this formalism of formal Laurent series is often convenient for computations, it does have its traps. First of all, the product of two formal power series is not always defined, and thus the associative and distributive properties may be meaningless for some series. On the other hand, even when the relevant products of the series are well defined, they may violate the associative law. As an example, let \( k_{Sz}(z, w) = \sum_{n \in \mathbb{Z}} z^n w^{-n} \) be the bilateral Szegő kernel (see Section 2.4 below). Then it is easily verified that

\[
\left( \sum_{\ell=0}^{\infty} z_1^{\ell} w_1^{-\ell} \right) (1 - z_1^{-1}) = 1 + \sum_{\ell=1}^{\infty} (1 - 1) z_1^{\ell} w_1^{-\ell} = 1
\]

and hence

\[
\left[ \left( \sum_{\ell=0}^{\infty} z_1^{\ell} w_1^{-\ell} \right) (1 - z_1^{-1}) \right] \cdot k_{Sz}(z, w) = 1 \cdot k_{Sz}(z, w) = k_{Sz}(z, w)
\]

while, on the other hand,

\[
\left( \sum_{\ell=0}^{\infty} z_1^{\ell} w_1^{-\ell} \right) \cdot \left[ (1 - z_1^{-1}) k_{Sz}(z, w) \right] = \left( \sum_{\ell=0}^{\infty} z_1^{\ell} w_1^{-\ell} \right) \sum_{n \in \mathbb{Z}} (1 - 1) z^n w^{-n}
\]

\[
= \left( \sum_{\ell=0}^{\infty} z_1^{\ell} w_1^{-\ell} \right) \cdot 0 = 0,
\]

and we have a violation of the associative law.

If \( F, G, \) and \( H \) are formal power series such that \( (F + G)H \) is defined while \( FH \) or \( GH \) are not, then the distributive property does not make sense. E.g., for \( F = -G = k_{Sz} \) and \( H = \sum_{\ell=0}^{\infty} z_1^{\ell} w_1^{-\ell} \), we have \( (F + G)H = 0 \) while the product

\[
FH = \left( \sum_{n \in \mathbb{Z}} z^n w^{-n} \right) \left( \sum_{\ell=0}^{\infty} z_1^{\ell} w_1^{-\ell} \right) = \sum_{m \in \mathbb{Z}} \left( \sum_{\ell=0}^{\infty} \right) z^m w^{-m}
\]

is not defined. On the other hand, if both \( FH \) and \( GH \) are defined, then so is \( (F + G)H \), and \( (F + G)H = FH + GH \). For simplicity, we will show this for the series in \( z^{\pm 1} \) only (the proof for the case of series in \( z^{\pm 1}, w^{\pm 1} \) is analogous).

---

3 The statement of [17, Theorem 2.6] is flawed in not specifying the topology on \( F \) making the product \( S(z)f(z) \) well-defined and in using the strong operator topology and not specifying the order of summation making the product \( S(z)K(z, w)S(w)^* \) well-defined. It is also flawed in asserting that the converse of Proposition 2.5 holds whereas in reality this is false. We notice that the proof of [17, Theorem 2.5] is incorrect as well, since it confuses coefficientwise (Schur) multiplication with the infinite-sum convolution multiplication required in the FRKHS setting here; for a correct proof, see Proposition 2.13 below.
\[ F = F(z), G = G(z), H = H(z): \]

\[
F(z)H(z) + G(z)H(z) = \sum_{n \in \mathbb{Z}^d} \left( \sum_{\ell \in \mathbb{Z}^d} F_{n-\ell} H_{\ell} \right) z^n + \sum_{n \in \mathbb{Z}^d} \left( \sum_{\ell \in \mathbb{Z}^d} G_{n-\ell} H_{\ell} \right) z^n
\]

\[
= \sum_{n \in \mathbb{Z}^d} \left( \sum_{\ell \in \mathbb{Z}^d} F_{n-\ell} H_{\ell} + \sum_{\ell \in \mathbb{Z}^d} G_{n-\ell} H_{\ell} \right) z^n = \sum_{n \in \mathbb{Z}^d} \left( \sum_{\ell \in \mathbb{Z}^d} (F_{n-\ell} + G_{n-\ell}) H_{\ell} \right) z^n
\]

\[ = [F(z) + G(z)]H(z). \]

A well known result in the classical reproducing-kernel-Hilbert-space setting is:

if the \( \mathcal{L}(\mathcal{F}, \mathcal{F}') \)-valued function \( S \) is such that \( S(z)f(z) \in \mathcal{H}(K') \) for each \( f \in \mathcal{H}(K) \) (so \( M_S: f(z) \mapsto S(z) \cdot f(z) \) is well defined as an operator from \( \mathcal{H}(K) \) to \( \mathcal{H}(K') \)), then necessarily \( M_S \) is bounded (i.e., \( \|M_S\|_{\mathcal{L}(\mathcal{H}(K), \mathcal{H}(K'))} < \infty \)). Indeed, it is easily verified that \( M_S \) is closed and then the result follows from the Closed Graph Theorem (see e.g. [16, page 51]). For the FRKHS setting, the parallel result holds, but under the hypothesis that the series \( \sum_{\ell} S_{n-\ell} f_{\ell} \) defining the coefficients of \( S(z) \cdot f(z) \) converges in the norm (not just the weak) topology of \( \mathcal{F}' \), and as a consequence of two applications of the Banach-Stieltjes Theorem rather than of the Closed Graph Theorem. Also it is convenient to assume that the coefficient space \( \mathcal{F}' \) is finite-dimensional, although this hypothesis can be weakened (see Remark 2.9 below). In detail, we have the following result.

**Theorem 2.7.** Let \( K(z, w) \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]] \), \( K'(z, w) \in \mathcal{L}(\mathcal{F}',[[z^{\pm 1}, w^{\pm 1}]] \) be formal positive kernels, and let \( S(z) \in \mathcal{L}(\mathcal{F}, \mathcal{F}')[[z, w]] \). Assume that, for all \( n \) and for all \( f \in \mathcal{H}(K) \), the infinite series \( (Sf)_\ell := \sum_{m \in \mathbb{Z}^d} S_{\ell-m} f_m \) converges in the norm of \( \mathcal{F}' \), and that the resulting formal Laurent series \( Sf := \sum_{\ell \in \mathbb{Z}^d} (Sf)_{\ell} z^\ell \) is in \( \mathcal{H}(K') \). Assume also that \( \dim \mathcal{F}' < \infty \). Then \( M_S: f \mapsto Sf \) is a bounded linear operator form \( \mathcal{H}(K) \) into \( \mathcal{H}(K') \).

For the proof we require the following lemma.

**Lemma 2.8.** Suppose that \( K(z, w) \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]] \) is a formal positive kernel with associated FRKHS \( \mathcal{H}(K) \) and that \( f(z) = \sum_{n \in \mathbb{Z}^d} f_n z^n \) is an element of \( \mathcal{H}(K) \). Then we can recover \( f \) from the collection of coefficients \( \{f_\ell\}_{\ell \in \mathbb{Z}^d} \) according to the following approximation scheme:

\[
f = \mathcal{H}(K) - \lim_{L \to \infty} \text{row}[\pi_\ell^*]_{\ell: |\ell| \leq L}[K_L^{-1}] \text{col}[f_\ell]_{\ell: |\ell| \leq L},
\]

where \( \pi_\ell: f \mapsto f_\ell \) is the coefficient evaluation map with adjoint

\[
\pi_\ell^*: u \mapsto K_\ell(z)u = \sum_{n \in \mathbb{Z}^d} (K_{n,\ell} u) z^n,
\]

where \( K_L \) is the operator on \( \mathcal{F}^N_{\ell} \) (where we set \( N_L = \#\{\ell \in \mathbb{Z}^d: |\ell| \leq L\} \) and the notation \( |\ell| \) is as in (2.3)) with block matrix decomposition given by

\[
K_L := [K_{\ell', \ell}]_{|\ell'|, |\ell| \leq L},
\]

and where \( K_L^{-1} \) is the (possibly unbounded) inverse of the injective selfadjoint operator \( K_L |_{\text{ker} K_L} \).
Proof. Let us set $\mathcal{H}_L = \{ h \in \mathcal{H}(K) : h_\ell = 0 \text{ for } |\ell| \leq L \}$ (where the notation $|\ell|$ is as in (2.9)). Then the orthogonal complement $\mathcal{H}_L^\perp$ of $\mathcal{H}_L$ is given by the span of the kernel functions
\[ \mathcal{H}_L^\perp = \text{span}\{ K_\ell(z)u : |\ell| \leq L, u \in \mathcal{F} \}. \] (2.11)
For $f \in \mathcal{H}(K)$, set $g_L = P_L f$ where $P_L : \mathcal{H}(K) \rightarrow \mathcal{H}_L$ is the orthogonal projection. Then $f - g_L \in \mathcal{H}_L$ and consequently $(f_\ell) = (g_L)_\ell$ for $|\ell| \leq L$. Notice that the sequence of subspaces $\{ \mathcal{H}_L \}_{L=1,2,\ldots}$ is decreasing ($\mathcal{H}_{L+1} \subset \mathcal{H}_L$) with trivial intersection ($\bigcap_{L \geq 1} \mathcal{H}_L = \{0\}$). It follows that the associated sequence of orthogonal projections $P_L : \mathcal{H}(K) \rightarrow \mathcal{H}_L$ is increasing with strong limit equal to the identity operator:
\[ \text{strong lim}_{L \rightarrow \infty} P_L = I_{\mathcal{H}(K)}. \]
Let us set $N_L = \#\{ \ell \in \mathbb{Z}^d : |\ell| \leq L \}$ and define a block $N_L \times N_L$ matrix $\mathbb{K}_L$ by
\[ \mathbb{K}_L := [K_{\ell', \ell}]_{|\ell'|,|\ell| \leq L} : \mathcal{F}^{\oplus N_L} \rightarrow \mathcal{F}^{\oplus N_L}. \]
We know as a consequence of the description of $\mathcal{H}_L^\perp$ in (2.11) that the vector $g_L$ has a presentation of the form
\[ g_L = \sum_{|\ell| \leq L} K_\ell(z)u_\ell \]
for some choice of vectors $u_\ell \in \mathcal{F}$. To solve for the vectors $u_\ell$ (these vectors also depend on the choice of $L$ but we suppress this fact in the notation for simplicity), we note that, for any vector $v \in \mathcal{F}$
\[ \langle f_{\ell'}, v \rangle_{\mathcal{F}} = \langle f, K_{\ell'}(z)v \rangle_{\mathcal{H}(K)} \]
\[ = \langle g_L, K_{\ell'}(z)v \rangle_{\mathcal{H}(K)} \]
\[ = \left\langle \sum_{|\ell| \leq L} K_\ell(z)u_\ell, K_{\ell'}(z)v \right\rangle_{\mathcal{H}(K)} \]
\[ = \sum_{|\ell| \leq L} \langle K_{\ell'}u_\ell, v \rangle_{\mathcal{F}} \]
from which we deduce that
\[ f_{\ell'} = \sum_{|\ell| \leq L} K_{\ell'}u_\ell. \]
It thus follows that $g_L \in \text{im} \mathbb{K}_L \subset \mathcal{F}^{\oplus N_L}$. Let $\mathbb{K}^{-1}_L : \mathcal{F}_L \rightarrow \mathcal{F}_L \subset \mathcal{F}^{\oplus N_L}$ be the inverse of $\mathbb{K}_L |_{(\ker \mathbb{K}_L)^\perp}$. We see that $[u_\ell]_{|\ell| \leq L} = [\mathbb{K}^{-1}_L f_\ell]_{|\ell| \leq L}$. Therefore
\[ f = \lim_{L \rightarrow \infty} P_L f = \lim_{L \rightarrow \infty} \sum_{|\ell| \leq L} K_\ell(z)u_\ell \]
\[ = \text{row} \left[ \pi_\ell \right]_{|\ell| \leq L} \cdot \mathbb{K}^{-1}_L \cdot [f_{\ell}]_{|\ell| \leq L} \]
and the formula (2.10) follows. □

Proof of Theorem 2.7. Notice that
\[ (Sf)_\ell = \lim_{M \rightarrow \infty} \sum_{m : |m| \leq M} S_{\ell-m}f_m = \sum_{m : |m| \leq M} (S_{\ell-m} \circ \pi_m)(f) \]
where $\pi_m: \mathcal{H}(K) \to \mathcal{F}$ is the $m$-th Laurent coefficient evaluation functional, assumed to be norm-continuous. A first application of the Banach-Steinhaus theorem (see [22, Chapter III Section 14]) gives us that the map 

$$\pi_{S,\ell}: f \mapsto (Sf)_\ell$$

is bounded as a linear operator from $\mathcal{H}(K)$ to $\mathcal{F}'$ for each $\ell \in \mathbb{Z}$.

We next apply Lemma 2.8 to $Sf \in \mathcal{H}(K');$ thus we get

$$Sf = \lim_{L \to \infty} \row[[\pi_{S,\ell}^*]_{\ell \leq L} : \mathcal{K}_{L}^{-1}] \cdot \col[(Sf)_\ell]_{\ell \leq L}$$

$$= \lim_{L \to \infty} \row[[\pi_{S,\ell}^*]_{\ell \leq L} : \mathcal{K}_{L}^{-1}] \cdot \col[\pi_{S,\ell}(f)]_{\ell \leq L}$$  (2.12)

with the obvious adaptation of the notation. The next key observation is that the map $f \mapsto [f_\ell]_{\ell \leq L}$ is bounded for each $L = 1, 2, \ldots$; as we are assuming that $\dim \mathcal{F}' < \infty$, we are guaranteed that $\mathcal{K}_{L}^{-1}$ is bounded for each $L$. Hence also the composition $\row[[\pi_{S,\ell}^*]_{\ell \leq L} : \mathcal{K}_{L}^{-1}]$ is bounded from $\mathcal{F}'^{\otimes N_L}$ to $\mathcal{H}(K')$ for each $L$. Finally, by the first application of the Banach-Steinhaus theorem in the first paragraph of the proof, we know that the map $f \mapsto [\pi_{S,\ell}(f)]_{\ell \leq L}$ is bounded from $\mathcal{H}(K)$ to $\mathcal{F}'^{\otimes N_L}$, and hence the composite map in (2.12) is bounded for each $L$. A second application of the Banach-Steinhaus Theorem now guarantees us that the map $M_S: f(z) \mapsto S(z)f(z)$ is bounded from $\mathcal{H}(K)$ into $\mathcal{H}(K')$ as wanted. \hfill \Box

Remark 2.9. A careful analysis of the proof of Theorem 2.7 shows that the conclusion still holds if one replaces the hypothesis that $\mathcal{F}'$ be finite-dimensional with the alternate hypothesis

- the subspace $\mathcal{F}'_L \subset \mathcal{F}'^{\otimes N_L}$ consisting of all vectors $[u'_\ell]_\ell$ in $\mathcal{F}'^{\otimes N_L}$ of the form $[[Sf_\ell]]_{\ell \leq L}$ for some $f \in \mathcal{H}_L$ is a closed subspace of $\mathcal{F}'_L$.

Note that this condition is automatic in case $\mathcal{F}'$ itself is finite-dimensional.

We shall be particularly interested in the case where $\|M_S\| \leq 1$; in this case we write $S \in \mathcal{BM}(K, K')$. We have the following characterization of when a given $S$ is in $\mathcal{BM}(K, K')$ or is a coisometry. We use the convention (2.4) with respect to both sets of variables $z = (z_d^+, z_d^{-1})$ and $w = (w_d^+, \ldots, w_d^{+1})$: if $S(z) = \sum_{n \in \mathbb{Z}^d} S_n z^n \in \mathcal{L}(\mathcal{F}, \mathcal{F})[[z^{\pm 1}]]$, then we define $S(z)^* \in \mathcal{L}(\mathcal{F}', \mathcal{F})[[z^{\pm 1}]]$ by

$$S(z)^* = \sum_{n \in \mathbb{Z}^d} S_n^* z^{-n}.$$

Proposition 2.10. Suppose that $K, K'$ are two positive kernels with coefficients in $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(\mathcal{F}')$ respectively and with respective associated FRKHSs $\mathcal{H}(K)$ and $\mathcal{H}(K')$, and suppose that $S$ is a formal power series in $\mathcal{L}(\mathcal{F}, \mathcal{F})[[z^{\pm 1}]]$. Then the following holds:

1. $S \in \mathcal{BM}(K, K')$ if and only if $S(z)f(z)$ is a well-defined power series (i.e., weak $\sum_{n \in \mathbb{Z}^d} S_{m-n} f_\ell$ exists in $\mathcal{F}'$ for all $m \in \mathbb{Z}^d$) for each $f \in \mathcal{H}(K)$ and hence also $S(z)K(z, w)S(w)^*$ is a well-defined formal power series as in (2.7) - (2.8), and moreover $K_S \geq 0$, where

$$K_S(z, w) := K'(z, w) - S(z)K(z, w)S(w)^* \in \mathcal{L}(\mathcal{F}')[[z^{\pm 1}, w^{\pm 1}]].$$  (2.13)

In this case, $M_S$ is a coisometry if and only if $K_S(z, w) = 0.$
(2) \( S^* \in \mathcal{BM}(K', K) \) if and only if \( S(z)^* f(z) \) is a well-defined power series for each \( f \in \mathcal{H}(K') \) and hence also \( S(z)^* K'(z, w)S(w) \) is a well-defined formal power series similarly to (2.17) - (2.18), and moreover \( K_{S^*} \geq 0 \), where

\[
K_{S^*}(z, w) := K(z, w) - S(z)^* K'(z, w)S(w) \in \mathcal{L}(\mathcal{F})[[z^\pm 1, w^\pm 1]]
\]

In this case, \( M_{S^*} \) is a coisometry if and only if \( K_{S^*}(z, w) = 0 \).

Proof. Suppose that \( S \in \mathcal{M}(K, K') \). By Proposition 2.5 \( S(z)K(z, w)S(w)^* \) is a well-defined formal power series as in (2.7) - (2.8). For all \( f \in \mathcal{H}(K) \)

\[
\langle f(z), (M_{S^*} \otimes I_{C[[u]]})K'(z, w)u' \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[u]]} = \langle S(z)f(z), K'(z, w)u' \rangle_{\mathcal{H}(K') \times \mathcal{H}(K')[[u]]} = \langle S(w)f(w), u' \rangle_{\mathcal{F}[[u]] \times \mathcal{F}}
\]

where we used the reproducing kernel property of \( K' \). Furthermore,

\[
\langle S(w)f(w), u' \rangle_{\mathcal{F}[[u]] \times \mathcal{F}} = \langle f(z), K(z, w)S(w)^* u' \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[u]]}
\]

Here we used Proposition 2.1 for the last equality. Indeed, as we saw in the proof of Proposition 2.5 if \( K(z, w) = H(z)H(w)^* \) as in Theorem 2.2(2), then \( H(z)^* S(w)^* u' \in \mathcal{H}[[u^\pm 1]] \) for every \( u' \in \mathcal{F} \), with the series \( \sum_{m \in \mathbb{Z}^d} H_m^* S_m u' = \sum_{m \in \mathbb{Z}^d} K_{m^{-\epsilon}} S_m^* u' \) convergent weakly in \( \mathcal{H} \) for every \( m \in \mathbb{Z}^d \). Then

\[
\sum_{m \in \mathbb{Z}^d} K_{m^{-\epsilon}} S_m^* u' = \sum_{m \in \mathbb{Z}^d} H(z)H_m^* S_m^* u'
\]

converges weakly in \( \mathcal{H}(K) \) and we may apply Proposition 2.1 with \( u(w) = S(w)^* u' \in \mathcal{F}[[u]] \). It follows that

\[
(M_{S^*} \otimes I_{C[[u]]})K'(z, w)u' = K(z, w)S(w)^* u'
\]

for all \( u' \in \mathcal{F} \). Note that the action of \( M_{S^*} \otimes I_{C[[u]]} \) can be expanded out in terms of coefficients of \( w^m \) as

\[
M_{S^*} : K_n^* (z)u' \mapsto \sum_{n} \left( \sum_{\ell} K_{n^{-\ell}} S_{\ell}^* u' \right) w^{-n}.
\]

and hence we see that

\[
M_{S^*} : K_n^* (z)u' \mapsto \sum_{\ell} K_{n^{-\ell}} S_{\ell}^* u'.
\]

From the formula (2.10) we see that the fact that \( M_{S^*} : \mathcal{H}(K') \rightarrow \mathcal{H}(K) \) has \( \|M_{S^*}\| \leq 1 \) is equivalent to the expression

\[
(K_n u_n, K_m^* u_m')_{\mathcal{F}'} - \sum_{\ell} K_{n^{-\ell}} S_{\ell}^* u_N', \sum_{\ell'} K_{m^{-\ell}} S_{\ell'}^* u_m')_{\mathcal{F}}
\]

being positive as a kernel on \( (\mathbb{Z}^d \times \mathcal{F}) \) (here we view \( (n, u_n') \) and \( (m, u_m') \) as generic elements of \( \mathbb{Z}^d \times \mathcal{F} \)). From the definitions we see that this expression is positive when viewed as a kernel on \( \mathbb{Z}^d \times \mathcal{F} \) if and only if \( K_{S^*}(z, w) = K'(z, w) - S(z)K(z, e)S(w)^* \geq 0 \) is a positive formal kernel. Note that \( M_{S^*} \) being coisometric means that \( M_{S^*} \) is isometric, or (2.17) is the zero kernel; this in turn corresponds
to \( K_S \) being the zero kernel. This completes the proof of necessity in part (1) of Proposition 2.10.

To prove sufficiency in part (1) of Proposition 2.10, we now assume that \( S(z)f(z) \) is well defined for each \( f \in \mathcal{H}(K) \), so weak \( \sum \ell S_{m-\ell}H_\ell \) exists for each \( m \in \mathbb{Z}^d \) (where \( K_{m,n} = H_mH_n^* \) is the Kolmogorov decomposition for \( K(z,w) \)) and hence also weak \( \sum_\ell H_\ell^*S_{m-\ell}^* \) exists. The formula (2.16) for the action of \( M_S^* \) on kernel functions suggests that we define an operator \( T_0 \) mapping kernel functions of \( \mathcal{H}(K') \) into \( \mathcal{H}(K) \) according the formula in (2.10):

\[
T_0: K_S^*(z)u' \mapsto \sum_\ell K_{n-\ell}(z)S_{\ell}^*u'.
\]  

(2.18)

The computation in the first part of the proof (read backwards) tells us that the assumption \( K_S \geq 0 \) implies that \( T_0 \) extends uniquely to a contraction operator from \( \mathcal{H}(K') \) into \( \mathcal{H}(K) \). Furthermore the action of \( T := T_0 \otimes I_{C[[w^{\pm 1}]]} \) on kernel functions \( K'(z,w)u' \in \mathcal{H}(K')[[w^{\pm 1}]] \) is given as in formula (2.19):

\[
T: K'(z,w)u' \mapsto K(z,w)S(w)^*u'.
\]  

(2.19)

From this formula (2.19) for \( T \) it is easy to compute a formula for the action of \( T_0^* \): for \( f \in \mathcal{H}(K) \) and \( u' \in \mathcal{F}' \) we have, using the reproducing kernel property of \( K' \):

\[
\langle (T_0^* f)(w), u' \rangle_{\mathcal{F}'[[w^{\pm 1}]] \times \mathcal{F}} \]
\[
= \langle T_0^* f, K'(z, w)u' \rangle_{\mathcal{H}(K') \times \mathcal{H}(K')[[w^{\pm 1}]]} \]
\[
= \langle f, TK'(z, w)u' \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[w^{\pm 1}]]} \]
\[
= \langle f, K(z, w)S(w)^*u' \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[w^{\pm 1}]]}.
\]

> From (2.14) read backwards, we finally conclude that

\[
\langle (T_0^* f)(w), u' \rangle_{\mathcal{F}'[[w^{\pm 1}]] \times \mathcal{F}} = \langle S(w)f(w), u' \rangle_{\mathcal{F}'[[w^{\pm 1}]] \times \mathcal{F}}
\]

from which we conclude that \( T_0^* = M_S: f(z) \mapsto S(z)f(z) \). As we have already identified \( T_0 \) as mapping \( \mathcal{H}(K') \) contractively into \( \mathcal{H}(K) \), it follows that \( T_0^* = M_S: \mathcal{H}(K) \rightarrow \mathcal{H}(K') \) is well defined with \( \|M_S\| \leq 1 \).

Part (2) amounts to part (1) applied to \( S(z)^* \) in place of \( S(z) \). \( \square \)

Remark 2.11. One might think that the kernel \( K_S(z,w) \) given by (2.13) being well-defined and positive would be sufficient for the results of Proposition 2.10 to be valid. That is not the case can be seen from the following example.

We take as our Hilbert space \( \mathcal{X} = \ell^2 \) with standard orthonormal basis \( \{e_n: n = 0, 1, 2, \ldots \} \) and we set \( P_n \) equal to the orthogonal projection onto the span of the basis vector \( e_n \). We define a formal power series \( S(z) = \sum_{n \in \mathbb{Z}} S_n z^n \) and a formal positive kernel \( K(z,w) = \sum_{n,m \in \mathbb{Z}} H_n H_m^* z^n w^{-m} \) by

\[
S_n = I_{\mathcal{X}}, \quad H_n = \begin{cases} n^2 P_n - (n - 1)^2 P_{n-1} & \text{for } n > 0, \\ 0 & \text{for } n \leq 0. \end{cases}
\]

Then

\[
\sum_{\ell: |\ell| \leq L} H_\ell = \sum_{\ell=1}^L [\ell^2 P_\ell - (\ell - 1)^2 P_{\ell-1}] = L^2 P_L
\]
and hence
\[
\lim_{L \to \infty} \sum_{\ell: |\ell| \leq L} S_{n-\ell} H_{\ell} = \lim_{L \to \infty} L^2 P_L \text{ is not weakly convergent.}
\]

However
\[
\text{weak} \sum_\ell \left( \text{weak} \sum_{\ell'} S_{n-\ell} H_{\ell}^* S_{m-\ell'}^* \right) = \text{weak} \sum_\ell \left( \lim_{L' \to \infty} H_{\ell} (L^2 P_{L'}) \right) = 0
\]
and similarly
\[
\text{weak} \sum_{\ell'} \left( \sum_\ell S_{n-\ell} H_{\ell}^* S_{m-\ell'}^* \right) = 0.
\]

Moreover the kernel
\[
K_S(z, w) = K(z, w) - S(z) K(z, w) S(z)^* = K(z, w) \geq 0.
\]
Thus the kernel \(K_S(z, w)\) is well defined and positive, but the multiplier \(M_S: f(z) \mapsto S(z)f(z)\) is not well defined (much less bounded with norm at most 1) on \(\mathcal{H}(K)\).

2.3. **Formal reproducing kernel Hilbert spaces constructed from others.** There are a couple of ways to construct more complicated FRKHs from a stock of given (simpler) FRKHs. We mention what we shall call *lifted-norm FRKHs* and *pullback FRKHs* described in the next two subsections.

2.3.1. **Lifted-norm FRKHs.** Suppose that \(W = W^*\) is a self-adjoint operator on a FRKH \(\mathcal{H}(K)\) which is positive semidefinite. We define a space \(\mathcal{H}_W^\ell\) (the lifted-norm FRKH associated with \(W\)) via the following recipe. We take a dense subset of \(\mathcal{H}_W^\ell\) to be
\[
\text{im} W = \{Wf: f \in \mathcal{H}(K)\}
\]
with inner product given by
\[
\langle (Wf)(z), (Wg)(z) \rangle_{\mathcal{H}_W^\ell} = \langle (Wf)(z), g(z) \rangle_{\mathcal{H}(K)}
\]
The completion of \(\mathcal{H}_W^\ell\) can be identified as \(\text{im}(W^{1/2})\) with pullback inner product
\[
\langle W^{1/2}f, W^{1/2}g \rangle_{\mathcal{H}_W^\ell} = \langle Qf, g \rangle_{\mathcal{H}(K)} \quad (2.20)
\]
where \(Q: \mathcal{H}(K) \to (\ker W)^\perp\) is the orthogonal projection. The following proposition summarizes the properties of \(\mathcal{H}_W^\ell\) which we shall need in the sequel.

**Proposition 2.12.** Suppose that \(W \in \mathcal{L}(\mathcal{H}(K))\) is positive semidefinite and the space \(\mathcal{H}_W^\ell = \text{im}(W^{1/2})\) is defined with inner product given by (2.20). Then \(\mathcal{H}_W^\ell\) is a FRKH with formal reproducing kernel \(K_W^\ell(z, w) \in \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]]\) given by
\[
K_W^\ell(z, w) = ((W \otimes I_{C[[w^{\pm 1}]]})K)(z, w).
\]

**Proof:** If \(f_n = W^{1/2}g_n\) is a Cauchy sequence in \(\mathcal{H}_W^\ell\), then \(Qg_n\) is Cauchy in \(\mathcal{H}(K)\) and hence converges to a \(g \in \mathcal{H}(K)\). It then follows that \(\{f_n\}\) converges to \(W^{1/2}Qg = W^{1/2}g\) in \(\mathcal{H}_W^\ell\). In this way we see that \(\mathcal{H}_W^\ell\) is complete.
By construction \(((W \otimes I_{\mathcal{C}[|w\pm 1|]})K)(z, w)u \in \mathcal{H}_W^p[[w\pm 1]]\) for each \(u \in \mathcal{F}\) and each \(f \in \mathcal{H}_W^p\) has the form \(W^{\frac{1}{2}}g\) for a \(g \in \mathcal{H}(K)\). We now compute
\[
\langle f(z), ((W \otimes I_{\mathcal{C}[|w\pm 1|]})K)(z, w)u \rangle_{\mathcal{H}_W^p \times \mathcal{H}_W^p[[w\pm 1]]} = \langle (W^{\frac{1}{2}}g)(z), ((W \otimes I_{\mathcal{C}[|w\pm 1|]})K)(z, w)u \rangle_{\mathcal{H}_W^p \times \mathcal{H}_W^p[[w\pm 1]]}
\]
\[
\langle (W^{\frac{1}{2}}g)(z), K(z, w)u \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[w\pm 1]]} = \langle f(z), K(z, w)u \rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[w\pm 1]]}
\]
and we see that indeed \(K_W^p(z, w) := (W \otimes I_{\mathcal{C}[|w\pm 1|]})K)(z, w)\) plays the role of the formal reproducing kernel for the space \(\mathcal{H}_W^p\).

2.3.2. Pullback FRKHSs. Suppose now that \(B \in \mathcal{M}(K, K')\) is a bounded multiplier between two FRKHSs \(\mathcal{H}(K)\) and \(\mathcal{H}(K')\). We define the **pullback RKHS** associated with \(B\) by
\[
\mathcal{H}_B^p = \text{im} M_B = \{B(z)f(z) : f \in \mathcal{H}(K)\} \subset \mathcal{H}(K')
\]
with inner product
\[
\langle B(z)f(z), B(z)g(z) \rangle_{\mathcal{H}_B^p} = \langle (Qf, g)_{\mathcal{H}(K)} \rangle
\]
(2.21)
where \(Q \in \mathcal{L}(\mathcal{H}(K))\) is the orthogonal projection onto \((\ker M_B)^\perp\). The following Proposition summarizes the main features about pullback FRKHSs.

**Proposition 2.13.** Suppose that \(B \in \mathcal{M}(K, K')\) and that \(\mathcal{H}_B^p\) is the associated pullback space with inner product given by (2.21). Then

1. \(\mathcal{H}_B^p\) is a FRKHS with reproducing kernel
\[
K_B^p(z, w) = B(z)K(z, w)B(w)^*.
\]
2. If \(B \in \mathcal{BM}(K, K')\) then \(\mathcal{H}_B^p\) is contractively included in \(\mathcal{H}(K')\):
\[
\|Bf\|_{\mathcal{H}(K')} \leq \|Bf\|_{\mathcal{H}_B^p} \text{ for all } Bf \in \mathcal{H}_B^p
\]
with equality for all \(Bf \in \mathcal{H}_B^p\) if and only if \(M_B : \mathcal{H}(K) \to \mathcal{H}(K')\) is a coisometry.
3. If \(B \in \mathcal{BM}(K, K')\), then the Brangesian complementary space \((\mathcal{H}_B^p)^{1, DB}\) defined as the space of all \(f \in \mathcal{H}(K)\) with finite \((\mathcal{H}_B^p)^{1, DB}\)-norm:
\[
\|f\|^{2}_{(\mathcal{H}_B^p)^{1, DB}} := \sup\{\|fg\|_{\mathcal{H}(K')}^2 - \|g\|_{\mathcal{H}(K)}^2 : g \in \mathcal{H}(K)\} < \infty
\]
is the lifted norm space \(\mathcal{H}_{1-M_B M_B^*}^p\) and is equipped with the reproducing kernel
\[
K_{1-M_B M_B^*}^p(z, w) = K'(z, w) - B(z)K(z, w)B(w)^*.
\]

**Proof.** To prove the first statement (1), notice that by Proposition 2.1 the product \(B(z)K(z, w)B(w)^*u'\) is well defined and is in \(\mathcal{H}_B^p[[w\pm 1]]\). Calculate
\[
\langle B(z)f(z), B(z)K(z, w)B(w)^*u' \rangle_{\mathcal{H}_B^p \times \mathcal{H}_B^p[[w\pm 1]]}
\]
\[
= (Qf(z), K(z, w)B(w)^*u')_{\mathcal{H}(K) \times \mathcal{H}(K)[[w\pm 1]]}
\]
\[
= (B(w)f(w), u')_{\mathcal{F}'[[w\pm 1]] \times \mathcal{F}'}
\]
and the identity \(K_B^p(z, w) = B(z)K(z, w)B(w)^*\) follows.

Notice next that statement (2) is immediate from the definitions.
For the proof of the first part of statement (3), we refer the reader to Sarason’s book [18, Chapter 1] for a general operator-theoretic treatment which handles the FRKHS setup here as a particular case (see also [17, Theorem 2.5]). The statement about the reproducing kernels follows from Proposition 2.12 and the identity (2.15).

Remark 2.14. Proposition 2.13 (1) is essentially equivalent to Proposition 2.10. Indeed, assume Proposition 2.10. Let \( B \in \mathcal{M}(K,K') \). Define a kernel \( \tilde{K} \) by \( \tilde{K}(z,w) = B(z)K(z,w)B(w)^* \). It follows from Proposition 2.10 that \( M_B : f(z) \mapsto B(z)f(z) \) is a coisometry from \( \mathcal{H}(K) \) onto \( \mathcal{H}(\tilde{K}) \), or equivalently, \( \mathcal{H}(\tilde{K}) = \mathcal{H}'_p \). Conversely, assume Proposition 2.13 (1). Let \( S \) be such that \( (z)f(z) \) is well defined for all \( f \in \mathcal{H}(K) \). Set \( \tilde{K}(z,w) = S(z)K(z,w)S(w)^* \). By Proposition 2.14 (1), \( \mathcal{H}(\tilde{K}) = \mathcal{H}'_p \). If \( K_S = K' - \tilde{K} \) (see (2.13)) is a positive kernel, then \( \mathcal{H}(\tilde{K}) \) is contractively included in \( \mathcal{H}(K') \) (see [17, Theorem 2.5]), meaning that \( S \in \mathcal{B}\mathcal{M}(K,K') \).

Remark 2.15. Suppose that \( B \in \mathcal{M}(K) = \mathcal{M}(K,K) \) and \( M_B = M_B^* \geq 0 \). By (2.15), we have

\[
B(z)K(z,w) = \left((M_B \otimes I_{C[[w^{\pm 1}]]})K\right)(z,w) = \left((M_B^* \otimes I_{C[[w^{\pm 1}]]})K\right)(z,w) = K(z,w)B(w)^*, \tag{2.22}
\]

Then \( W := (M_B)^2 = M_B^2 \geq 0 \) and from the discussion above we see that the lifted norm space \( \mathcal{H}_W \) with kernel \( K_W(z,w) = B(z)^2K(z,w) \) is the same as the pullback space \( \mathcal{H}'_p \) with kernel \( K'_p(z,w) = B(z)K(z,w)B(w)^* \). The identity (2.22) explains why these two forms of the kernel agree.

2.3.3. Overlapping spaces. Suppose that we are given a finite or infinite countable collection \( \{\mathcal{H}(K_j) : j \in J\} \) of FRKHSs with the same coefficient space \( \mathcal{F} \); we adapt the convention (2.3) by writing

\[
K_j(z,w) = \sum_{n,m \in \mathbb{Z}^d} [K_j]_{n,m} z^n w^{-m}, \quad [K_j]_{n,m}(z) = \sum_{n \in \mathbb{Z}^d} [K_j]_{n,m} z^n. \tag{2.23}
\]

We let \( K(z,w) = \sum_{j \in J} K_j(z,w) \), where we assume that the series converges in \( \mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]] \) coefficientwise in the weak operator topology, i.e., for every \( n, m \in \mathbb{Z}^d \) the series of coefficients \( \sum_{j \in J} [K_j]_{n,m} \) converges in the weak operator topology of \( \mathcal{L}(\mathcal{F}) \). Then by Theorem 2.2, \( K(z,w) \) is also a positive kernel with associated FRKHS \( \mathcal{H}(K) \); to avoid confusion, for the kernel \( K \) we use the more elaborate notation

\[
K(z,w) = \sum_{n,m \in \mathbb{Z}^d} [K]_{n,m} z^n w^{-m}, \quad [K]_{n,m}(z) = \sum_{n \in \mathbb{Z}^d} [K]_{n,m} z^n. \tag{2.24}
\]

It is often of interest to understand the relation between \( \mathcal{H}(K) \) and the subsidiary spaces \( \mathcal{H}(K_j) (j \in J) \).

---

4 Equivalently, the series converges in the strong operator topology. Indeed, \( [K]_{n,n} \geq 0 \), and it is well known that for a series of nonnegative operators (or for an increasing sequence of selfadjoint operators) on a Hilbert space, weak operator topology and strong operator topology convergence coincide.
We introduce the factorizations
\[ K_j(z, w) = H_j(z)H_j(w)^* \]
for some \( H_j(z) = \sum_{n \in \mathbb{Z}^d} [H_j]_n z^n \in \mathcal{L}(\mathcal{H}', \mathcal{F})[[z^{\pm 1}]] \)
(2.25)
so that as in Theorem 2.2 we know that \( \mathcal{H}(K_j) = \{ H_j(z)h_j : h_j \in \mathcal{H}'_j \} \). Since \( [K_j]_{n, m} = [H_j]_n[H_j]_m^* \), we have that \( \sum_{j \in J} [H_j]_n[H_j]_m^* \) converges in the weak operator topology for all \( n, m \in \mathbb{Z}^d \), and in particular \( \sum_{j \in J} ([H_j]_n[H_j]_m^*v, v) \) converges for all \( n \in \mathbb{Z}^d \) and all \( v \in \mathcal{F} \). It follows that \( \text{col}_{j \in J} \) \([H_j]_n^*\) is a well defined bounded linear operator from \( \mathcal{F} \) to \( \bigoplus_{j \in J} \mathcal{H}_j \) with adjoint
\[ [H]_n = \text{row}_{j \in J} ([H_j]_n) \in \mathcal{L} \left( \bigoplus_{j \in J} \mathcal{H}_j', \mathcal{F} \right) \]
given more precisely by \( \bigoplus_{j \in J} h_j' \mapsto \text{weak} \sum_{j \in J} [H_j]_n h_j' \). From (2.26) we see now that \( K(z, w) \) has the factorization
\[ K(z, w) = H(z)H(w)^* \]
where \( H(z) = \text{row}_{j \in J} [H_j(z)] \). From (2.27) we read off that \( \mathcal{H}(K) \) can be identified as
\[ \mathcal{H}(K) = \left\{ H(z)h : \sum_{j \in J} H_j(z)h_j : h = \bigoplus_{j \in J} h_j \in \bigoplus_{j \in J} \mathcal{H}_j' \right\} \]
\[ = \left\{ \sum_{j \in J} f_j : \bigoplus_{j \in J} f_j \in \bigoplus_{j \in J} \mathcal{H}(K_j) \right\}, \]
where the series converge in \( \mathcal{F}[[z^{\pm 1}]] \) coefficientwise in weak topology. However, only in special circumstances is this sum identifiable with a direct sum since the sum map \( s : \bigoplus_{j \in J} \mathcal{H}(K_j) \rightarrow \mathcal{H}(K) \) given by
\[ s : \bigoplus_{j \in J} f_j(z) \mapsto \sum_{j \in J} f_j(z) \]
may have a kernel. In fact, again as a consequence of Theorem 2.2, we see that the norm on elements of \( \mathcal{H}(K) \) is given by
\[ \left\| \sum_{j \in J} H_j(z)h_j \right\|_{\mathcal{H}(K)} = \left\| \text{P}_{\ker s} + \bigoplus_{j \in J} H_j(z)h_j \right\|_{\bigoplus_{j \in J} \mathcal{H}(K_j)} \]
\[ = \left\| \text{P}_{\ker (H(z))} + \bigoplus_{j \in J} h_j \right\|_{\bigoplus_{j \in J} \mathcal{H}_j'}. \]
To quantify the overlapping of the spaces \( \mathcal{H}(K_j) \) \((j \in J)\), in the spirit of the work of de Branges and Rovnyak [19, 20] (see also [6]), we introduce the overlapping space \( \mathcal{L}((K_j : j \in J)) \) defined by
\[ \mathcal{L}((K_j : j \in J)) = \ker s \]
(2.27)
with norm inherited from \( \bigoplus_{j \in J} \mathcal{H}(K_j) \). Then \( \mathcal{L}((K_j : j \in J)) \) is again a FRKHS whose reproducing kernel \( K_{\mathcal{L}((K_j : j \in J))} \) can be computed explicitly in some special cases (see [19, 20, 6, 18]). In any case we have the following result:
Proposition 2.16. Suppose that \( \{ K_j : j \in J \} \) is a finite or infinite countable collection of positive kernels with the common coefficient space \( F \), and we set \( K(z, w) = \sum_{j \in J} K_j(z, w) \) where we assume that the series converges coefficient-wise in the weak operator topology in \( \mathcal{L}(F)[[z^{\pm 1}, w^{\pm 1}]] \). Then:

1. The sum map \( s \) given by \((2.26)\) is a coisometry from \( \bigoplus_{j \in J} \mathcal{H}(K_j) \) onto \( \mathcal{H}(K) \) with initial space \( D_s \) given by

\[
D_s = \operatorname{span} \left\{ \bigoplus_{j \in J} [K_j]_m(z) u : u \in F, m \in \mathbb{Z}^d \right\}.
\]

2. There is a unitary identification map

\[
\tau : \bigoplus_{j \in J} \mathcal{H}(K_j) \to \mathcal{H}(K) \hat{\oplus} \mathcal{L}(\{ K_j : j \in J \})
\]

given by

\[
\tau : \bigoplus_{j \in J} f_j \mapsto \left[ \begin{array}{c} s \left[ P_{\ker s} \right] \left( \bigoplus_{j \in J} f_j \right) \\ \sum_{j \in J} f_j \end{array} \right] = \left[ \begin{array}{c} \sum_{j \in J} f_j \\ P_{\ker s} \left( \bigoplus_{j \in J} f_j \right) \end{array} \right]
\]

where the overlapping space \( \mathcal{L}(\{ K_j : j \in J \}) \) is given by \((2.27)\). In particular, if \( \mathcal{L}(\{ K_j : j \in J \}) = \{ 0 \} \), then

\[
\mathcal{H}(K) = \bigoplus_{j \in J} \mathcal{H}(K_j).
\]

Proof. In view of the discussion preceding the statement, we only check the assertion concerning the initial space. Note that the computation

\[
\left\langle s \left( \bigoplus_{j \in J} f_j \right), K(\cdot, w) u \right\rangle_{\mathcal{H}(K) \times \mathcal{H}(K)[[w^{\pm 1}]]} = \sum_{j \in J} \langle f_j(w), u \rangle_{F[[w^{\pm 1}]] \times F}
\]

\[
= \sum_{j \in J} \langle f_j, K_j(\cdot, w) u \rangle_{\mathcal{H}(K_j) \times \mathcal{H}(K_j)[[w^{\pm 1}]]}
\]

\[
= \left( \bigoplus_{j \in J} f_j, \bigoplus_{j \in J} K_j(\cdot, w) u \right)_{\bigoplus_{j \in J} \mathcal{H}(K_j) \times \bigoplus_{j \in J} \mathcal{H}(K_j)[[w^{\pm 1}]]}
\]

shows that \( s^* (K(\cdot, w) u) = \bigoplus_{j \in J} K_j(\cdot, w) u \). This in turn amounts to the simultaneous set of equalities

\[
[s^*([K]_m(\cdot) u)]_m(z) = \bigoplus_{j \in J} [K_j]_m(z) u \quad \text{for all } u \in F \text{ and } m \in \mathbb{Z}^d.
\]

Hence \( \ker s = \overline{\operatorname{im} s^*} = D_s \) as asserted. \( \square \)

2.4. The Szegő kernel and associated FRKHSs. We apply these ideas in particular to the case \( k_{Sz}(z, w) = \sum_{n \in \mathbb{Z}^d} z^n w^{-n} \) (the bilateral Szegő kernel) in the space \( C[[z^{\pm 1}, w^{\pm 1}]] \) (here we identify the complex numbers \( C \) with the space \( \mathcal{L}(C) \) of linear operators on \( C \)). Note that \( k_{Sz}(z, w) = h_{Sz}(z) h_{Sz}(w)^* \) if we set

\[
h_{Sz}(z) = \operatorname{row}_{n \in \mathbb{Z}^d} [z^n] \in \mathcal{L}(l^2(\mathbb{Z}^d), C)[[z^{\pm 1}]].
\]

Hence condition (2) in Theorem (2.2) is verified for \( k_{Sz} \) and it follows that \( k_{Sz} \) is the reproducing kernel for a formal reproducing kernel Hilbert space \( \mathcal{H}(k_{Sz}) \). To identify
\(\mathcal{H}(k_{Sz})\) explicitly, note that \(k_{Sz}(z, w) = \sum_{n \in \mathbb{Z}^d} k_{Sz,n}(z) w^{-n}\) where \(k_{Sz,n}(z) = z^n\). Hence, for \(f(z) = \sum_{n \in \mathbb{Z}^d} f_n z^n \in \mathcal{H}(k_{Sz})\) we have the identity

\[
\langle f, z^n \rangle_{\mathcal{H}(k_{Sz})} = f_n \quad \text{for all } n \in \mathbb{Z}^d
\]

from which we see that \(\mathcal{H}(k_{Sz})\) can be identified with the Lebesgue space \(L^2(\mathbb{T}^d)\) of measurable functions on the torus \(\mathbb{T}^d\) with modulus-squared integrable with respect to Lebesgue measure via identification of a function \(f \in L^2(\mathbb{T}^d)\) with its \(d\)-variable Fourier series \(f(z) \sim \sum_{n \in \mathbb{Z}^d} f_n z^n\) (with \(f_n = \int_{\mathbb{T}^d} f(\zeta) \zeta^{-n} |d\zeta|/(2\pi)^d\)). The fact that the operator \(M_{z_k}\) of multiplication by the variable \(z_k\) is unitary on \(L^2(\mathbb{T}^d)\) implies then that \(M_{z_k}\) is unitary on \(\mathcal{H}(k_{Sz})\); this can be seen directly in the FRKHS context as an application of Proposition 2.10 combined with the easily verified identities

\[
k_{Sz}(z, w) = z_k k_{Sz}(z, w) w_k^{-1}, \quad k_{Sz}(z, w) = z_k^{-1} k_{Sz}(z, w) w_k
\]

and the observation that \(M_{z_k}^* = M_{-z_k}\) for \(k = 1, \ldots, d\); this is also a special case of Proposition 2.17 below.

Let \(\mathcal{F}\) be a separable coefficient Hilbert space. Following Sz.-Nagy-Foiaş [49, Chapter V] where the case \(d = 1\) is handled or specializing the very general setting of Hille-Phillips [31, Section 3.1] or the somewhat different general setting of Dixmier [25, Part II], we say that a \(\mathcal{F}\)-valued function \(\zeta \mapsto f(\zeta)\) defined on \(\mathbb{T}^d\) is measurable in the weak sense if

- the scalar-valued function \(\zeta \mapsto \langle f(\zeta)u, \tilde{u} \rangle_\mathcal{F}\) is measurable for each choice of \(u, \tilde{u} \in \mathcal{F}\)

and is measurable in the strong sense if

- the function \(\zeta \mapsto f(\zeta)\) is the almost-everywhere pointwise limits of simple functions

\[
f(\zeta) = \lim_{n \to \infty} f_n(\zeta)
\]

where each \(f_n\) simple means that \(f_n\) is a finite linear combination (with vector coefficients from \(\mathcal{F}\)) of characteristic functions of measurable sets.

The fact that \(\mathcal{F}\) is separable implies that these two notions of measurable are equivalent (see [31, Corollary 2 page 73]) and we simply say that \(f\) is measurable for either of these notions. >From the Parseval relation

\[
\|f(\zeta)\|^2 = \sum_{n} |\langle f(\zeta) e_n, e_n \rangle|^2
\]

where \(\{e_n\}_{n \in \mathbb{Z}^d}\) is any orthonormal basis for \(\mathcal{F}\), we see immediately that \(\zeta \mapsto \|f(\zeta)\|^2\) is measurable in the standard sense for a scalar-valued function \(\tilde{f}\). Hence it makes sense to define a Hilbert space \(L^2(\mathbb{T}^d, \mathcal{F})\) as the space of all measurable \(\mathcal{F}\)-valued functions \(\zeta \mapsto f(\zeta)\) which are norm-squared integrable:

\[
\|f\|_{L^2(\mathbb{T}^d, \mathcal{F})}^2 := \int_{\mathbb{T}^d} \|f(\zeta)\|^2 d\zeta < \infty.
\]

\(^5\)This remains true for Banach-space valued functions \(\zeta \mapsto f(\zeta)\) but by a different argument (see [31, Theorem 3.52 page 72]).
Furthermore, it can be shown (see [19, page 190]) for the case \( d = 1 \) that \( L^2(\mathbb{T}^d, F) \) has an orthogonal decomposition

\[
L^2(\mathbb{T}^d, F) = \bigoplus_{k \in \mathbb{Z}^d} \mathcal{C}_k
\]

where we set

\[
\mathcal{C}_k = \zeta^k F \subset L^2(\mathbb{T}^d, F)
\]

for \( k \in \mathbb{Z}^d \). From this decomposition it is easily verified that we can identify the function space \( L^2(\mathbb{T}^d, F) \) with the formal reproducing kernel Hilbert space \( \mathcal{H}(k_{Sz}I_F) \) by identifying the function \( \zeta^k u \in L^2(\mathbb{T}^d, F) \) with the formal monomial \( z^k u \in \mathcal{H}(k_{Sz}I_F) \) for each \( u \in F \) and then by extending by linearity and taking of limits. What is the same, we identify the function \( f \in L^2(\mathbb{T}^d, F) \) with the formal series \( \hat{f}(z) = \sum_{n \in \mathbb{Z}^d} f_n z^n \) by making use of the Fourier series representation for \( f \)

\[
f(z) \sim \sum_{n \in \mathbb{Z}^d} f_n \zeta^n
\]

where the Fourier coefficients \( f_n \) are given by

\[
f_n = \int_T f(\zeta) \zeta^{-n} d|\zeta|/(2\pi)^d \quad \text{(weak integral),}
\]

just as explained above for the scalar-valued case.

Now let us suppose that we have two separable Hilbert coefficient spaces \( F \) and \( F' \) and suppose that \( \zeta \mapsto S(\zeta) \) is an \( L(F, F') \)-operator-valued function on \( \mathbb{T}^d \). We say that \( S \) is measurable if \( S \) is weakly measurable in the sense that \( \zeta \mapsto \langle S(\zeta)u, u' \rangle_{F'} \) is a measurable scalar-valued function for each \( u \in F \) and \( u' \in F' \). A consequence of Theorem 1 in [25, Part II, Chapter 2, Section 5] is that bounded linear operators \( T \) from \( L^2(\mathbb{T}^d, F) \) to \( L^2(\mathbb{T}^d, F') \) intertwining the scalar multiplication operators \( M_{z_k} \) on \( L^2(\mathbb{T}^d, F) \) and \( L^2(\mathbb{T}^d, F') \)

\[
T(M_{z_k} \otimes I_F) = (M_{z_k} \otimes I_{F'})T
\]

\((k = 1, \ldots, d)\) are characterized as operators of the form \( T = M_S \)

\[
M_S : f(\zeta) \mapsto S(\zeta) f(\zeta)
\]

for an essentially bounded measurable function \( \zeta \mapsto S(\zeta) \in L(F, F') \), with operator norm of \( T \) equal to the essential infinity norm of \( S \)

\[
\|T\|_{op} = \|S\|_{\infty}
\]

(see Proposition 2 in [25, Part II Chapter 2 Section 3]). For convenience we denote the space of all such essentially bounded measurable \( L(F, F') \)-valued functions \( S \) by \( L^\infty(\mathbb{T}^d, L(F, F')) \). Under the isomorphism \( f(\zeta) \sim f(z) \) of \( L^2(\mathbb{T}^d, F) \) with \( \mathcal{H}(k_{Sz}I_F) \) and of \( L^2(\mathbb{T}^d, F') \) with \( \mathcal{H}(k_{Sz}I_{F'}) \), it is clear that such multiplication operators \( M_S \) coincide with the space of multipliers \( \mathcal{M}(k_{Sz}I_F, k_{Sz}I_{F'}) \) and that this correspondence is again given by the Fourier series representation:

\[
S(\zeta) \in L^\infty(\mathbb{T}^d, L(F, F')) \sim S(z) = \sum_{n \in \mathbb{Z}^d} S_n z^n \in \mathcal{M}(k_{Sz}I_F, k_{Sz}I_{F'})
\]

where \( S_n = \int_{\mathbb{T}^d} F(\zeta) \zeta^{-n} d|\zeta|/(2\pi)^d \).

Proposition 2.10 specialized to this situation gives the following.
Proposition 2.17. Suppose that $S$ is an element of $L^\infty(\mathbb{T}^d, \mathcal{L}(\mathcal{F}, \mathcal{F}'))$ with associated Fourier series $S(z) \sim \sum_{n \in \mathbb{Z}^d} S_n z^n$ viewed as a formal Laurent series in $\mathcal{L}(\mathcal{F}, \mathcal{F'})[[z^\pm 1]]$. Then:

1. The function $S$ has $\|S\|_\infty \leq 1$ if and only if its Fourier series $S(z) \sim \sum_{n \in \mathbb{Z}^d} S_n z^n$ is such that any, and hence both, of the kernels

\[ K_S(z, w) = k_{S_1}(z, w)(I_{\mathcal{F}''} - S(z)S(w)^*), \quad K_{S^*}(z, w) = k_{S_2}(z, w)(I_{\mathcal{F}''} - S(z)^*S(w)) \]

are positive kernels (in $\mathcal{L}(\mathcal{F'})[[z^{\pm 1}, w^{\pm 1}]]$ and $\mathcal{L}(\mathcal{F})[[z^{\pm 1}, w^{\pm 1}]]$ respectively).

2. $S$ has isometric values almost everywhere on $\mathbb{T}^d$ $((S(\zeta))^*S(\zeta) = I_{\mathcal{F}''}$ for a.e. $\zeta \in \mathbb{T}^d)$ if and only if its Fourier series $S(z) \sim \sum_{n \in \mathbb{Z}^d} S_n z^n$ satisfies the formal Laurent series identity

\[ k_{S_1}(z, w)(I_{\mathcal{F}''} - S(z)^*S(w)) = 0. \]

3. $S$ has coisometric values almost every on $\mathbb{T}^d$ $(S(\zeta)S(\zeta)^* = I_{\mathcal{F}''}$ for a.e. $\zeta \in \mathbb{T}^d)$ if and only if its Fourier series $S(z) \sim \sum_{n \in \mathbb{Z}^d} S_n z^n$ satisfies the formal Laurent series identity

\[ k_{S_2}(z, w)(I_{\mathcal{F}''} - S(z)S(w)^*) = 0. \]

Remark 2.18. In Proposition 2.17 we use the fact that $M^*_S = M_S^*$ in this case, and it follows from (2.15) that

\[ k_{S_1}(z, w)S(z) = k_{S_2}(z, w)S(w) \]

(notice that formal power series with operator coefficients commute with formal power series with scalar coefficients) i.e.,

\[ \sum_{n, n' \in \mathbb{Z}^d} S_n z^{n+n'} w^{-n'} = \sum_{n, n' \in \mathbb{Z}^d} S_{n'} z^{n-n'} w^{n'} \]

whenever $S(z) \sim \sum_{n \in \mathbb{Z}^d} S_n z^n \in L^\infty(\mathbb{T}^d, \mathcal{L}(\mathcal{F}, \mathcal{F}'))$. Of course this identity can be verified directly by performing the appropriate change of variable in the summation index.

Suppose that $S = S^* \in L^\infty(\mathbb{T}^d, \mathcal{L}(\mathcal{F}))$ with positive-semidefinite values $S(\zeta) \geq 0$ for a.e. $\zeta \in \mathbb{T}^d$. Then in particular the multiplication operator $W = M_S$ on $L^2(\mathbb{T}^d, \mathcal{F}) \sim \mathcal{H}(k_{S_1}I_{\mathcal{F}})$ is positive semidefinite and we may define the lifted norm space $\mathcal{H}_{W}^l$ as in Section 2.3.1. The following proposition lists a few properties of this space.

Proposition 2.19. Suppose that $S \in \mathcal{L}(\mathcal{F})[[z^\pm 1]]$ is such that $W = M_S$ is positive semidefinite on $\mathcal{H}(k_{S_1}I_{\mathcal{F}})$ and the space $\mathcal{H}_{W}^l = \text{im}(W^{1/2})$ with inner product given by (2.20) with $K = k_{S_2}$. Then $\mathcal{H}_{W}^l$ is a FRKHS with formal reproducing kernel $K_W^l(z, w) \in \mathcal{L}(\mathcal{F})[[z^\pm 1, w^\pm 1]]$ given by

\[ K_W^l(z, w) = k_{S_2}(z, w)S(z) = k_{S_2}(z, w)S(w). \]

Proof. Note the equality of the two expressions for $K_W^l(z, w)$ follows from Remark 2.18. The remaining points of the proposition are just specializations of the various results in Proposition 2.12. \qed

Suppose now that $B \in L^\infty(\mathbb{T}^d, \mathcal{L}(\mathcal{F}, \mathcal{F}')) \sim \mathcal{M}(k_{S_2}I_{\mathcal{F}}, k_{S_2}I_{\mathcal{F}'})$. Then we are in the situation of Proposition 2.13 we may define the pullback FRKHS $\mathcal{H}_{B}^l = M_B \cdot (\mathcal{H}(k_{S_1}I_{\mathcal{F}}))$ with inner product given by

\[ \langle B(z)f(z), B(z)g(z) \rangle_{\mathcal{H}_{B}^l} = \langle (Qf)(z), (Qg)(z) \rangle_{\mathcal{H}}(k_{S_1}I_{\mathcal{F}}) \]
where $Q \in \mathcal{L}(\mathcal{H}(k_{sz}I_x))$ is the orthogonal projection onto $(\ker M_B)^\perp$. The following proposition specializes the summary in Proposition 2.13 to this special setting.

**Proposition 2.20.** For $B \in L^\infty(\mathcal{F}, \mathcal{F}')$, the pullback FRKHS $\mathcal{H}_B^p$ has reproducing kernel

$$K_B^p(z, w) = k_{Sz}(z, w)B(z)B(w)^*$$

and the Brangesian complementary space $(\mathcal{H}_B^p)^\perp_B$ has reproducing kernel

$$K_B^{p\perp}(z, w) = k_{Sz}(z, w)(I - B(z)B(w)^*).$$

More generally, following [14] and abusing notation somewhat, for $\Omega$ equal to any subset of $\mathbb{Z}^d$ we let $H^2(\Omega)$ be the subspace of $L^2(\mathbb{T})$ consisting of functions $f \in L^2(\mathbb{T})$ with Fourier coefficients supported on $\Omega$: $f(z) \sim \sum_{n \in \Omega} f_n z^n$. Then it is easily seen that $H^2(\Omega)$ may be viewed (by viewing the Fourier series of $f \in H^2(\Omega)$ as a formal series in $\mathbb{C}[z^{\pm 1}]$) as a FRKHS with reproducing kernel equal to the truncated Szegő kernel

$$k_{Sz, \Omega}(z, w) = \sum_{n \in \Omega} z^n w^{-n}.$$ 

Note that in this notation, the Hardy space over the polydisk, usually written as $H^2(\mathbb{D}^d)$, is now written as $H^2(\mathbb{Z}^d_+)$ and $L^2(\mathbb{T}^d)$ is also written as $H^2(\mathbb{Z}^d)$.

3. **Formal de Branges–Rovnyak model for a scattering system**

We recall the definition of a multievolution scattering system (with polydisk model) from [14]. We define a *d-evolution scattering system* (of Lax–Phillips type with polydisk model) to be a collection

$$\mathcal{S} = (\mathcal{K}; \mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_d); \mathcal{F}, \mathcal{F}_e)$$

where $\mathcal{K}$ is a Hilbert space (the *ambient space*) for the scattering system, $\mathcal{U}_1, \ldots, \mathcal{U}_d$ is a $d$-tuple of commuting unitary operators on $\mathcal{K}$ (the *evolutions* of the system), and such that $\mathcal{F}$ and $\mathcal{F}_e$ are wandering subspaces for $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_d)$, i.e.,

$$\mathcal{U}^n \mathcal{F} \perp \mathcal{F}, \quad \mathcal{U}^n \mathcal{F}_e \perp \mathcal{F}_e$$

for all $n \in \mathbb{Z}^d$ with $n \neq 0$.

We say that a subset $\Omega$ of the integer lattice $\mathbb{Z}^d$ is a *shift-invariant sublattice* if $\Omega$ is invariant for each of the shift operators

$$\sigma_k : n \mapsto n + e_k$$

for $k = 1, \ldots, d$, where we let

$$e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$$

be the $k$-th standard basis vector for $\mathbb{R}^d$ ($k = 1, \ldots, d$). The main examples of shift-invariant sublattices are

- $\Omega = \mathbb{Z}^d$,
- $\Omega = 0$,
- $\Omega = \mathbb{Z}^d_+ := \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : n_k \geq 0 \text{ for } k = 1, \ldots, d\}$,
- $\Omega = \mathbb{Z}^d \setminus \mathbb{Z}^d_+ = \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : n_k < 0 \text{ for at least one } k = 1, \ldots, d\}$,
- $\Omega = \Omega^B := \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : n_1 + \cdots + n_d \geq 0\}$.
where the $B$ in the notation $\Omega^B$ suggests balanced. Given such a shift-invariant sublattice $\Omega$, we define associated outgoing subspace $W^\Omega$ and incoming subspace $W^\Omega_*$ by

$$W^\Omega := \bigoplus_{n \in \Omega} \mathcal{U}^nF, \quad W^\Omega_* := \bigoplus_{n \in \mathbb{Z}^d \setminus \Omega} \mathcal{U}^nF_*.$$

We say that the multievolution scattering system $\mathcal{S}$ is $\Omega$-orthogonal if

$$W^\Omega \perp W^\Omega_*.$$

It turns out (see [14, Proposition 3.1]) that $\mathcal{S}$ is $\Omega$-orthogonal for all shift-invariant sublattices $\Omega$ as soon as it is $\mathbb{Z}^d_+$-orthogonal. We therefore say simply that the $d$-evolution scattering system $\mathcal{S}$ is orthogonal if $\mathcal{S}$ is $\mathbb{Z}^d_+$-orthogonal (and therefore $\Omega$-orthogonal for all shift-invariant sublattices $\Omega$). Thus, in any orthogonal multievolution scattering system $\mathcal{S}$, the ambient space $\mathcal{K}$ has an orthogonal decomposition

$$\mathcal{K} = W^\Omega + \mathcal{V}^\Omega + W^\Omega_*$$

associated with any shift-invariant lattice $\Omega \subset \mathbb{Z}^d$, where

$$\mathcal{V}^\Omega := \mathcal{K} \ominus [W^\Omega \oplus W^\Omega_*]$$

is the scattering space associated with $\Omega$. In the sequel we shall have as a standing assumption that the multievolution scattering system $\mathcal{S}$ is orthogonal.

Another property which often comes up is minimality.

**Definition 3.1.** We shall say that the multievolution scattering system $\mathcal{S}$ is minimal if the smallest subspace of $\mathcal{K}$ containing $\mathcal{F}$ and $\mathcal{F}_*$ and invariant for each of $\mathcal{U}_1, \ldots, \mathcal{U}_d$ and for each of $\mathcal{U}^*_1, \ldots, \mathcal{U}^*_d$ is the whole space $\mathcal{K}$. Equivalently, $\widetilde{W} + \widetilde{W}_*$ is dense in $\mathcal{K}$, where we have set

$$\widetilde{W} := W^{\mathbb{Z}^d} = \bigoplus_{n \in \mathbb{Z}^d} \mathcal{U}^n\mathcal{F}, \quad \widetilde{W}_* := W^\mathbb{Z} = \bigoplus_{n \in \mathbb{Z}^d} \mathcal{U}^n\mathcal{F}_*.$$

Given any such orthogonal scattering system $\mathcal{S}$, we associate Fourier representation operators

$$\Phi : \mathcal{K} \to L^2(T^d, \mathcal{F}) \sim \mathcal{H}(kS_xI_{\mathcal{F}}), \quad \Phi_* : \mathcal{K} \to L^2(T^d, \mathcal{F}_*) \sim \mathcal{H}(kS_xI_{\mathcal{F}_*})$$

by

$$\Phi : k \mapsto \sum_{n \in \mathbb{Z}^d} (P_{\mathcal{F}}U^{-n}k)z^n, \quad \Phi_* : k \mapsto \sum_{n \in \mathbb{Z}^d} (P_{\mathcal{F}_*}U^{-n}k)z^n. \quad (3.3)$$

Then $\Phi$ is a coisometry from $\mathcal{K}$ onto $L^2(T^d, \mathcal{F})$ with initial space equal to $\widetilde{W}$ while $\Phi_*$ is a coisometry form $\mathcal{K}$ onto $L^2(T^d, \mathcal{F}_*)$ with initial space equal to $\widetilde{W}_*$. What is more, if $\Omega$ is any subset of $\mathbb{Z}^d$, then $\Phi|_{W^\Omega}$ is unitary from $W^\Omega$ onto $H^2(\Omega, \mathcal{F})$ and $\Phi_*|_{W^\Omega_*}$ is unitary from $W^\Omega_*$ onto $H^2(\mathbb{Z}^d \setminus \Omega, \mathcal{F}_*)$. A consequence of the orthogonality assumption on $\mathcal{S}$ is that $\text{im} \Phi_*|_{H^2(\mathbb{Z}^d_+, \mathcal{F})} = W^\Omega$ is orthogonal to $W^\Omega_*$. As a consequence

$$\Phi_* \Phi^* : H^2(\mathbb{Z}^d_+, \mathcal{F}) \to H^2(\mathbb{Z}^d_+, \mathcal{F}_*)$$

and commutes with $M_{S_k}$ for $k = 1, \ldots, d$. We conclude that $\Phi_* \Phi^*$ has the form $\Phi_* \Phi^* = M_S$ for a function $S \in H^\infty(\mathbb{Z}^d_+, \mathcal{L}(\mathcal{F}, \mathcal{F}_*))$ (i.e., $S \in H^\infty(\mathbb{D}^d, \mathcal{L}(\mathcal{F}, \mathcal{F}_*))$) in the more standard notation is the boundary-value function of a bounded analytic function on the unit polydisk $\mathbb{D}^d$—see [49, Section V.3]). Moreover, since $M_S = \Phi_* \Phi^*$ is the product of partial isometries, necessarily $M_S$ has norm at most 1 and
$S$ has $\|S\|_{\infty} \leq 1$, i.e., $S \in BH^{\infty}(Z_d^d, L(F, F))$. This polydisk Schur-class function $S$ is called the scattering matrix of the $d$-evolution scattering system $\mathcal{S}$.

We recall the de Branges–Rovnyak model for the multievolution scattering system from [13] associated with any polydisk Schur-class function $S \in \mathcal{S}(F, F_*)$. The de Branges–Rovnyak multievolution scattering system $\mathcal{S}^{dBR}_S$ associated with the polydisk Schur-class function $S \in \mathcal{S}(F, F_*)$ is defined to be

$$\mathcal{S}^{dBR}_S = (\mathcal{K}^{dBR}_S; \mathcal{U}^{dBR}_S = (\mathcal{U}^{dBR}_{S, 1}, \ldots, \mathcal{U}^{dBR}_{S, d}); \mathcal{F}^{dBR}_S, \mathcal{F}^{dBR*}_S)$$

(3.4)

where

$$\mathcal{K}^{dBR}_S = \text{im} \begin{bmatrix} I & M_S \\ M_S & I \end{bmatrix}^{1/2} \subset \begin{bmatrix} L^2(T^d, F_*) \\ L^2(T^d, F) \end{bmatrix}$$

with lifted inner product

$$\bigg \langle \begin{bmatrix} I & M_S \\ M_S & I \end{bmatrix}^{1/2} \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} I & M_S \\ M_S & I \end{bmatrix}^{1/2} \begin{bmatrix} f' \\ g' \end{bmatrix} \bigg \rangle_{\mathcal{K}^{dBR}_S} = \bigg \langle Q \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} f' \\ g' \end{bmatrix} \bigg \rangle_{L^2(T^d, F \oplus F)}$$

where $Q = \text{orthogonal projection onto the orthogonal complement of}$

$$\ker \begin{bmatrix} I & M_S \\ M_S & I \end{bmatrix}^{1/2}; \mathcal{U}^{dBR}_{S, k} = M_{z_k} \otimes I_{\mathcal{K}^{dBR}_S}$ on $\mathcal{K}^{dBR}_S.$ and where we set

$$\mathcal{F}^{dBR}_S = \begin{bmatrix} S \\ I \end{bmatrix} F, \quad \mathcal{F}^{dBR*}_S = \begin{bmatrix} I \\ S^* \end{bmatrix} F.$$  

It is easily checked that $\mathcal{S}^{dBR}_S$ is a minimal scattering system for any polydisk Schur-class function $S$. Conversely, if $\mathcal{S}$ as in (3.1) is any $d$-evolution scattering system with scattering matrix $S$, then it is shown in [14] that the map

$$\Pi^{dBR}_* : \Phi^*_u + \Phi^* g \mapsto \begin{bmatrix} I & M_S \\ M_S & I \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}$$

(3.5)

extends by continuity to define a unitary map from the minimal part $\mathcal{K}_{\text{min}} := \text{clo}(\text{im} \Phi^*_u + \text{im} \Phi^*)$ of the ambient space $\mathcal{K}$ of $\mathcal{S}$ onto $\mathcal{K}^{dBR}_S$ such that

$$\Pi^{dBR}_* \mathcal{U}^{dBR}_{S, k} \Pi^{dBR}_* \text{ for } k = 1, \ldots, d,$$

$$\Pi^{dBR}_* : \mathcal{F} \to \mathcal{F}^{dBR}_S, \quad \Pi^{dBR}_* : \mathcal{F}_* \to \mathcal{F}^{dBR*}_S \text{ unitarily}$$

(3.6)

and the scattering matrix $S_{\mathcal{S}^{dBR}_S}$ associated with the de Branges–Rovnyak model scattering system $\mathcal{S}^{dBR}_S$ coincides with the original scattering function $S$ in the sense that

$$S_{\mathcal{S}^{dBR}_S}(z) = i_{dBR}^S(z)(i_{dBR}^S)^*$$

where $i_{dBR}^S$ and $i_{dBR*}^S$ are the unitary identification maps appearing in (3.6)

$$i_{dBR}^S = \Pi^{dBR}_* |_{\mathcal{F}} : u \mapsto \begin{bmatrix} S \\ I \end{bmatrix} u, \quad i_{dBR*}^S = \Pi^{dBR}_* |_{\mathcal{F}_*} : u* \mapsto \begin{bmatrix} I \\ S^* \end{bmatrix} u*.$$

Let us also mention that $\Pi^{dBR}_*$ can be given more explicitly than the formula (3.5) above given in [14], as given in the following Lemma.

**Lemma 3.2.** Suppose that $\mathcal{S}$ is a multievolution scattering system with map $\Pi^{dBR}_*$ determined on $\mathcal{K}_{\text{min}}$ by (3.5) and extended by linearity to the whole space $\mathcal{K}$ by the requirement that $\Pi^{dBR}_* |_{\mathcal{K} \ominus \mathcal{K}_{\text{min}}} = 0$. Then $\Pi^{dBR}_*$ can be given by the alternative formula

$$\Pi^{dBR}_* = \begin{bmatrix} \Phi \\ \Phi^* \end{bmatrix} : \mathcal{K} \to \begin{bmatrix} L^2(T^d, F_*) \\ L^2(T^d, F) \end{bmatrix}.$$

(3.7)
Proof. For \( f \in L^2(\mathbb{T}^d, \mathcal{F}_s) \) and \( g \in L^2(\mathbb{T}^d, \mathcal{F}) \) we have

\[
\begin{bmatrix}
\Phi_s \\
\Phi
\end{bmatrix}
(\Phi_s f + \Phi^* g) = \begin{bmatrix}
\Phi_s \\
\Phi
\end{bmatrix}
\begin{bmatrix}
\Phi_s^* \\
\Phi^*
\end{bmatrix}
\begin{bmatrix}
f \\
g
\end{bmatrix} = \begin{bmatrix}
\Phi_s \Phi_s^* \\
\Phi \Phi^*
\end{bmatrix}
\begin{bmatrix}
f \\
g
\end{bmatrix} = \begin{bmatrix}
I \\
M_s
\end{bmatrix}
\begin{bmatrix}
f \\
g
\end{bmatrix} = \Pi_{dBR}(\Phi_s f + \Phi^* g)
\]

and it follows that (3.5) and (3.7) agree on \( K_{\text{min}} \). For \( k \perp K_{\text{min}} \), by construction both \( \Phi k = 0 \) and \( \Phi^* k = 0 \). The lemma now follows. \( \square \)

For our purposes here, it will be convenient to use the identifications discussed in Section 2.4 and view Fourier series as formal Laurent series and identify the various spaces \( L^2(\mathbb{T}^d, \mathcal{F}) \), \( L^\infty(\mathbb{T}^d, \mathcal{L}(\mathcal{F}, \mathcal{F}_s)) \), \( H^2(\Omega, \mathcal{F}) \) and \( H^\infty(\Omega, \mathcal{L}(\mathcal{F}, \mathcal{F}_s)) \) as the corresponding spaces of formal Laurent series \( \mathcal{H}(k_{s\mathcal{S}_s I_{\mathcal{F}_s}}, \mathcal{M}(k_{s\mathcal{S}_s I_{\mathcal{F}_s}}, k_{s\mathcal{S}_s I_{\mathcal{F}_s}}), \mathcal{H}(k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}) \) and \( \mathcal{M}(k_{s\mathcal{S}_s I_{\mathcal{F}_s}}, k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}) \) respectively. This is done in the next proposition.

**Proposition 3.3.** After identification \( F(z) \sim \sum_{n \in \mathbb{Z}^d} F_n z^n \) of an \( L^2 \)-function with its Fourier series viewed as a formal Laurent series, the spaces \( K_{s\mathcal{S}_s I_{\mathcal{F}_s}}, F_{s\mathcal{S}_s I_{\mathcal{F}_s}}, F_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}} \), \( W_{s\mathcal{S}_s I_{\mathcal{F}_s}}, W_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}} \) and \( V_{s\mathcal{S}_s I_{\mathcal{F}_s}}, V_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}} \) can be viewed as FRKHS \( \mathcal{H}(\mathcal{K}_{s\mathcal{S}_s I_{\mathcal{F}_s}}), \mathcal{H}(\mathcal{K}_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}), \mathcal{H}(\mathcal{K}_{s\mathcal{S}_s I_{\mathcal{F}_s}}^*), \mathcal{H}(\mathcal{K}_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}^*) \) and \( \mathcal{H}(\mathcal{K}_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}^*) \) where the respective reproducing kernels are given by

\[
K_{s\mathcal{S}_s I_{\mathcal{F}_s}}(z, w) = k_{s\mathcal{S}_s I_{\mathcal{F}_s}}(z, w) = \begin{bmatrix}
I \\
S(z)^* \\
S(z)
\end{bmatrix}
\begin{bmatrix}
S(z)^* \\
S(z)
\end{bmatrix},
\]

\[
K_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}(z, w) = k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}(z, w) = \begin{bmatrix}
I \\
S(z)^* \\
S(z)
\end{bmatrix}
\begin{bmatrix}
S(z)^* \\
S(z)
\end{bmatrix} = \sum_{n \in \Omega} z^n K_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}(z, w)w^{-n},
\]

\[
K_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}^*(z, w) = k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}^*(z, w) = \begin{bmatrix}
I \\
S(z)^* \\
S(z)
\end{bmatrix}
\begin{bmatrix}
S(z)^* \\
S(z)
\end{bmatrix} = \sum_{n \in \Omega} z^n K_{s\mathcal{S}_s I_{\mathcal{F}_s}}^*(z, w)w^{-n},
\]

\[
K_{s\mathcal{S}_s I_{\mathcal{F}_s}}(z, w) = K_{s\mathcal{S}_s I_{\mathcal{F}_s}}(z, w) - K_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}(z, w) - K_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}^*(z, w)
\]

\[
= \left[ k_{s\mathcal{S}_s I_{\mathcal{F}_s}}(z, w)(I - S(z)S(w)^*) - k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}(z, w)(S(z) - S(w)) \right]
\]

\[
= \begin{bmatrix}
k_{s\mathcal{S}_s I_{\mathcal{F}_s}}(z, w)(I - S(z)S(w)^*) \\
k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}(z, w)(I - S(z)S(w)^*) \\
-k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}(z, w)(S(z) - S(w)) \\
k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}^*(z, w)(S(z) - S(w))
\end{bmatrix}
\]

\[
= k_{s\mathcal{S}_s I_{\mathcal{F}_s}}(z, w) \begin{bmatrix}
I - S(z)S(w)^* \\
S(z)^* - S(z)
\end{bmatrix}
\begin{bmatrix}
S(z) - S(w) \\
S(z)^* - S(z)
\end{bmatrix}
\]

**(3.8)**

Proof. By construction \( K_{s\mathcal{S}_s I_{\mathcal{F}_s}}^* \) is the lifted norm space associated with the multiplier \( \begin{bmatrix}
I \\
S(z)^* \\
S(z)
\end{bmatrix} \) acting on \( \mathcal{H}(k_{s\mathcal{S}_s I_{\mathcal{F}_s}}, \mathcal{M}(k_{s\mathcal{S}_s I_{\mathcal{F}_s}}, k_{s\mathcal{S}_s I_{\mathcal{F}_s}}), \mathcal{H}(k_{s\mathcal{S}_s \Omega I_{\mathcal{F}_s}}) \). Hence Proposition 2.12 implies the formula
for \( K_{v}^{s,0}_{dB,R}(z,w) \) in (3.8). Note next that, for \( u, u' \in F \),
\[
\begin{bmatrix} S(z) \\ I \end{bmatrix} \begin{bmatrix} I \\ S(z) \end{bmatrix} u = \begin{bmatrix} I \\ S(z)^* \end{bmatrix} \begin{bmatrix} S(z) \\ I \end{bmatrix} \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ S(z)^* \end{bmatrix} \begin{bmatrix} 0 \\ u' \end{bmatrix} = \langle u, u' \rangle_F
\]
and hence \( F^S_{dB,R} \) is just the pullback space coming from multiplier \( B(z) = \begin{bmatrix} S(z) \\ I \end{bmatrix} \) mapping \( F \) into \( \mathcal{H}(k_{Sz,F,B},\otimes,F) \). It follows from Proposition 2.13 that
\[
K^{S}_{dB,R}(z,w) = k_{Sz,0}(z,w)B(z)B(w)^* = \begin{bmatrix} S(z) \\ I \end{bmatrix} \begin{bmatrix} S(w)^* \\ I \end{bmatrix} = \begin{bmatrix} S(z)S(w)^* \\ S(z) \end{bmatrix} I
\]
in agreement with the formula for \( K^{S}_{dB,R}(z,w) \) given in (3.8). The remaining formulas in (3.8) can be derived similarly. The equivalence of the two expressions for \( K_{v}^{s,0}_{dB,R}(z,w) \) follows from the identity
\[
k_{Sz}(z,w)(S(z) - S(w)) = 0
\]
(see Remark 2.18). This completes the proof of Proposition 3.3. The equivalence of the first two expressions for \( K_{v}^{s,0}_{dB,R}(z,w) \) also enables one to verify directly the selfadjointness property: \( K_{v}^{s,0}_{dB,R}(z,w) = (K_{v}^{s,0}_{dB,R}(w,z))^* \).

\[\square\]

Remark 3.4. It is straightforward to check directly from the definitions that
\[
z^n \cdot F^S_{dB,R} \perp z^n \cdot F^S_{dB,R} \text{ and } z^n \cdot F^S_{dB,R} \perp z^n' \cdot F^S_{dB,R} \text{ in } k^S_{dB,R} \text{ for } n \neq n' \in \mathbb{Z}^d.
\]
By minimal-decomposition theory the identities
\[
K^{W,0}_{dB,R}(z,w) = \sum_{n \in \Omega} z^n K^{S}_{dB,R}(z,w)w^{-k},
\]
\[
K^{W,0}_{dB,R} = \sum_{n \in \mathbb{Z}^d \setminus \Omega} z^n K^{S}_{dB,R}(z,w)w^{-k}
\]
appearing in (3.8) holding for a shift-invariant sublattice \( \Omega \) imply the internal orthogonal-sum decompositions
\[
\mathcal{H}(K^{W,0}_{dB,R}) = \bigoplus_{n \in \Omega} z^n \mathcal{H}(K^{S}_{dB,R}), \quad \mathcal{H}(K^{W,0}_{dB,R}) = \bigoplus_{n \in \mathbb{Z}^d \setminus \Omega} z^n \mathcal{H}(K^{S}_{dB,R}) \quad (3.9)
\]
as is to be expected. We also know from
\[
\mathcal{K} = \mathcal{W}^* \oplus \mathcal{V}^\Omega \oplus \mathcal{W}^\Omega.
\]
that
\[
\mathcal{H}(K^{S}_{dB,R}) = \mathcal{H}(K^{W,0}_{dB,R}) \oplus \mathcal{H}(K^{W,0}_{dB,R}) \oplus \mathcal{H}(K^{W,0}_{dB,R}) \quad (3.10)
\]
We emphasize that the decompositions in (3.9) and (3.10) are internal orthogonal direct sums. The decomposition (3.10) is consistent with the last of the identities in (3.8) with the added information that the spaces
\[
\mathcal{H}(K^{W,0}_{dB,R}), \quad \mathcal{H}(K^{W,0}_{dB,R}), \quad \mathcal{H}(K^{W,0}_{dB,R})
\]
have no overlap.
4. Scattering system containing an embedded unitary colligation

Suppose now that $U = [A \ B\ C \ D]$ as in \([\ref{13}]\) is a Givone–Roesser unitary colligation. As in \([\ref{13}]\) we associate the system of equations

\[
\Sigma = \Sigma(U) \begin{cases} 
  x_1(n + e_1) = A_{11} x_1(n) + \cdots + A_{1d} x_d(n) + B_1 u(n) \\
  \vdots \\
  x_d(n + e_d) = A_{d1} x_1(n) + \cdots + A_{dd} x_d(n) + B_d u(n) \\
  y(n) = C_1 x_1(n) + \cdots + C_d x_d(n) + Du(n)
\end{cases}
\]  

(4.1)

where the basis vectors $e_k$ are as in \([\ref{3.2}]\) and for any $n \in \mathbb{Z}^d$ one has $u(n) \in \mathcal{E}$, $y(n) \in \mathcal{E}_*$, $x_k(n) \in \mathcal{H}_k$, $k = 1, \ldots, d$. Since we are assuming that $U$ is unitary, we may also write the system equations as a backward recursion

\[
\Sigma = \Sigma(U) \begin{cases} 
  x_1(n) = A_{11}^* x_1(n + e_1) + \cdots + A_{1d}^* x_d(n + e_d) + C_1^* y(n) \\
  \vdots \\
  x_d(n) = A_{d1}^* x_1(n + e_1) + \cdots + A_{dd}^* x_d(n + e_d) + C_d^* y(n) \\
  y(n) = B_1^* x_1(n + e_1) + \cdots + B_d^* x_d(n + e_d) + D^* y(n)
\end{cases}
\]  

(4.2)

The form of the system equations here are of Givone–Roesser type (see \([\ref{27}, \ref{28}]\)) with the additional property that the connecting matrix (or colligation) $U$ is unitary—hence we refer to such a system as a conservative Givone–Roesser system (see \([13]\) for more details on the system-theoretic aspects of such systems—where the term Roesser system is used instead).

We now recall from \([14]\) how one can associate a multievolution Lax–Phillips scattering system

\[
\mathcal{S}(\Sigma(U)) = (\mathcal{T}; \mathcal{U} = (U_1, \ldots, U_d); \mathcal{F}, \mathcal{F}_*)
\]

with any conservative Givone–Roesser system $\Sigma(U)$. The ambient space for the scattering system $\mathcal{S}(\Sigma(U))$ is the space $\mathcal{T}$ of admissible trajectories of $\Sigma(U)$ defined as follows. By a trajectory of the system $\Sigma(U)$ we mean a $\mathcal{E} \times \mathcal{H}_1 \times \cdots \times \mathcal{H}_d \times \mathcal{E}_*$-valued function

\[
 n \mapsto (u(n), x_1(n), \ldots, x_d(n), y(n))
\]

on $\mathbb{Z}^d$ which satisfies the system of equations \((4.1)\) (or equivalently \((4.2)\)) for all $n \in \mathbb{Z}^d$. A given trajectory $(u, x, y)$ is said to be admissible if it has finite energy:

\[
\| (u, x, y) \|^2_{\mathcal{T}} := \| u \|^2_{Z(\Omega, \mathcal{E})} + \| x \|^2_{Z(\Omega, \mathcal{E}_*)} + \| y \|^2_{Z(\Omega, \mathcal{E}_*)} < \infty.
\]  

(4.3)

In general we use the notation $l^2(\Omega, \mathcal{F})$ to indicate the space of $\mathcal{F}$-valued functions on the set $\Omega$ which are norm-square integrable with respect to the discrete measure on $\Omega$:

\[
l^2(\Omega, \mathcal{F}) = \left\{ \{ x(\omega) \}_{\omega \in \Omega} : x(\omega) \in \mathcal{F} \text{ with } \sum_{\omega \in \Omega} | x(\omega) |^2_{\mathcal{F}} < \infty \right\}.
\]

Of the three terms occurring in the definition \((4.3)\) of $\| (u, x, y) \|^2_{\mathcal{T}}$ the first and last are fairly evident

\[
\| u \|^2_{Z(\Omega, \mathcal{E})} = \sum_{n \in \Omega} \| u(n) \|^2_{\mathcal{E}}, \quad \| y \|^2_{Z(\Omega, \mathcal{E}_*)} = \sum_{n \in \mathbb{Z}^d \setminus \Omega} \| y(n) \|^2_{\mathcal{E}_*}
\]

while the precise meaning of the middle term is explained below. The space of all admissible trajectories is denoted by $\mathcal{T}$ and is a Hilbert space in the $\mathcal{T}$-norm given by \((4.3)\).
To explain the meaning of $\|x|_{\partial\Omega}\|_{2,\Omega}$, we need to recall the decomposition of boundary of a shift-invariant sublattice $\Omega$ in finite and infinite parts, as introduced in [13]. The finite boundary of $\Omega$ is given by

$$\partial\Omega_{\text{fin}} = \bigcup_{k=1}^{d} \partial_{k}\Omega_{\text{fin}}$$

where

$$\partial_{k}\Omega_{\text{fin}} = \{(n_1, \ldots, n_d) \in \mathbb{Z}^d : n_k = \min\{m : (n_1, \ldots, n_{k-1}, m, n_{k+1}, \ldots, n_d) \in \Omega\}\}$$

while the infinite boundary $\partial\Omega_{\infty}$ is the union of the boundaries at plus and minus infinity:

$$\partial\Omega_{\infty} = \partial\Omega_{+\infty} \cup \partial\Omega_{-\infty}.$$  

The boundary of $\Omega$ at plus infinity is in turn given by the union of plus-infinity $k$-boundaries

$$\partial\Omega_{+\infty} = \bigcup_{k=1}^{d} \partial_{k}\Omega_{+\infty}$$

where an element of the plus-infinity $k$-boundary is defined to be the collection of lines of the form

$$\ell_{n,k} = n + \mathbb{Z} \cdot e_k$$

for some $n \in \mathbb{Z}^d$ and $k \in \{1, \ldots, d\}$ (4.4)

which have trivial intersection with $\Omega$:

$$\partial_{k}\Omega_{+\infty} = \{\ell_{n,k} : n \in \mathbb{Z}^d \text{ such that } \ell_{n,k} \cap \Omega = \emptyset\}.$$  

Similarly, the boundary of $\Omega$ at minus infinity is defined to be the union of minus-infinity $k$-boundaries

$$\partial\Omega_{-\infty} = \bigcup_{k=1}^{d} \partial_{k}\Omega_{-\infty}$$

with the minus-infinity $k$-boundary in turn defined to be the set of all lines $\ell_{n,k}$ (for some $n \in \mathbb{Z}^d$) which are completely contained in $\Omega$:

$$\partial_{k}\Omega_{-\infty} = \{\ell_{n,k} : \ell_{n,k} \subset \Omega\}.$$  

For $x(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_d(n) \end{bmatrix}$ an $\mathcal{H}$-valued function on $\mathbb{Z}^d$, where $\mathcal{H} = \begin{bmatrix} \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_d \end{bmatrix}$ as in the state space for a Givone–Roesser system, and $\Omega$ is a shift-invariant sublattice, we

\footnote{Note that a line $\ell_{n,k}$ as in [1.3] is parametrized by the pair $n \in \mathbb{Z}^d$ and $k \in \{1, \ldots, d\}$ but the resulting line $\ell_{n,k}$ is independent of the particular value $n_k$ of the $k$-th coordinate of the $d$-tuple $n \in \mathbb{Z}^d$; such lines are really determined by the $(k-1)$-tuple $\mathring{n}^k = (n_1, \ldots, n_{k-1}, n_{k+1}, \ldots, n_d)$ together with the choice of $k \in \{1, \ldots, d\}$. The oversimplified notation $\ell_{n,k}$ nevertheless is convenient for the manipulations to follow. In [13, 14] the $(d-1)$-tuple $\mathring{n}^k$ was used as the index rather the the $d$-tuple $n$.}
now are able to define the middle term on the right-hand side of (4.3) by
\[ \| \cdot | \Omega \|^2_{\partial \Omega} = \sum_{k=1}^d \left\{ \sum_{n \in \partial \Omega_{\text{fin}}} \| x_k(n) \|^2 + \sum_{n \in \partial \Omega_{\text{inf}}} \lim_{m \to -\infty} \| (x_k | e_{n,k})(m) \|^2 \right\}, \]

where we have set in general
\[ (x_k | e_{n,k})(m) = x_k(n + m - n_k) \in \partial \Omega, \]

Note that this quantity \( \| \cdot | \Omega \|^2_{\partial \Omega} \) is always defined (finite or infinite), and that by

the Monotone Convergence Theorem, the order of the summation and limit signs

in the last two terms is immaterial. When this quantity is finite, we say that \( x \)

is norm square-summable over \( \partial \Omega \). Note that the quantity \( \| \cdot | \Omega \|^2_{\partial \Omega} \)

is the sum of more elementary pieces:

\[ \| \cdot | \Omega \|^2_{\partial \Omega} = \| \cdot | \Omega_{\text{fin}} \|^2_{\partial \Omega_{\text{fin}}} + \| \cdot | \Omega_{-\infty} \|^2_{\partial \Omega_{-\infty}} + \| \cdot | \Omega_{+\infty} \|^2_{\partial \Omega_{+\infty}} \]

where

\[ \| \cdot | \Omega_{\text{fin}} \|^2_{\partial \Omega_{\text{fin}}} = \sum_{k=1}^d \| x_k | \partial \Omega_{\text{fin}} \|^2_{\partial \Omega_{\text{fin}}}, \]

\[ \| \cdot | \Omega_{-\infty} \|^2_{\partial \Omega_{-\infty}} = \sum_{n \in \partial \Omega_{-\infty}} \sum_{k=1}^d \| x_k | e_{n,k} \|^2_{\partial \Omega_{-\infty}}, \]

\[ \| \cdot | \Omega_{+\infty} \|^2_{\partial \Omega_{+\infty}} = \sum_{n \in \partial \Omega_{+\infty}} \sum_{k=1}^d \| x_k | e_{n,k} \|^2_{\partial \Omega_{+\infty}}. \]

and finally,

\[ \| x_k | \partial \Omega_{\text{fin}} \|^2_{\partial \Omega_{\text{fin}}} = \sum_{n \in \partial \Omega_{\text{fin}}} \| x_k(n) \|^2_{\partial \Omega_{k}}, \]

\[ \| x_k | e_{n,k} \|^2_{\partial \Omega_{-\infty}} = \lim_{m \to -\infty} \| (x_k | e_{n,k})(m) \|^2_{\partial \Omega_{k}}, \]

\[ \| x_k | e_{n,k} \|^2_{\partial \Omega_{+\infty}} = \lim_{m \to +\infty} \| (x_k | e_{n,k})(m) \|^2_{\partial \Omega_{k}}. \]

It is shown in [13, Proposition 3.4] that admissibility and in fact the \( \mathcal{T} \)-norm given

by (4.3) is independent of the choice of shift-invariant sublattice. Then the space \( \mathcal{T} = \mathcal{T}^2 \)

consisting of all admissible system trajectories \( (u, x, y) \) is a Hilbert space in the \( \mathcal{T} \)-norm.

To make sense of initial conditions at minus infinity or of final conditions at plus infinity for system trajectories, it is convenient to introduce so-called residual spaces \( \mathcal{R}_k \) and \( \mathcal{R}_{+k} \) by

\[ \mathcal{R}_k = \{ \bar{h} = \{ h(t) \}_{t=-\infty}^\infty : h(t) \in \mathcal{H}_k \text{ with } h(t) = A_{kk}^* h(t + 1) \text{ for all } t \in \mathbb{Z} \} \]

and

\[ \mathcal{R}_{+k} = \{ \bar{h} = \{ h(t) \}_{t=-\infty}^\infty : h(t) \in \mathcal{H}_k \text{ with } h(t + 1) = A_{kk} h(t) \text{ for all } t \in \mathbb{Z} \}, \]

and

\[ \| \bar{h} \|^2_{\mathcal{R}_k} = \lim_{t \to -\infty} \| h(t) \|^2_{\mathcal{H}_k}, \]

\[ \| \bar{h} \|^2_{\mathcal{R}_{+k}} = \lim_{t \to +\infty} \| h(t) \|^2_{\mathcal{H}_k}. \]
We also need a formal definition of the notion of two vector-valued sequences having the same asymptotics at plus or minus infinity: given two \( \mathcal{H} \)-valued sequences \( x \) and \( x' \) we say that \( x \) is asymptotic to \( x' \) at plus-infinity, denoted as \( x \sim x' \) if \( \lim_{t \to +\infty} \| x(t) - x'(t) \|_H = 0 \). Similarly, we say that \( x \) is asymptotic to \( x' \) at minus-infinity, denoted as \( x \sim x' \), if \( \lim_{t \to -\infty} \| x(t) - x'(t) \|_H = 0 \). Then, as is also shown in [13], given a GR-unitary colligation \( U \) and a shift-invariant sublattice \( \Omega \) as above, then, corresponding to any \( \mathcal{H}_k \)-valued sequence of the form \( x_k|_{\ell_{n,k}} \) for an admissible trajectory \((u, x, y)\) and a line \( \ell_{n,k} \in \partial \Omega_{+\infty} \), there is a sequence \( \tilde{h}_k \in R_k \) so that \( x_k|_{\ell_{n,k}} \sim \tilde{h}_k \). Similarly, given any line \( \ell_{n,k} \in \partial \Omega_{-\infty} \), there is a sequence \( \tilde{h}_{nk} \in R_{nk} \) so that \( x_k|_{\ell_{n,k}} \sim \tilde{h}_{nk} \). Conversely, one can solve the initial value problem for any square-summable initial data: given initial/final state data, future input string and past output string relative to a given invariant sublattice \( \Omega \) of the form

\[
\begin{bmatrix}
\hat{d} \\
d_{k=1} \hat{d} x_k \\
d_{k=1} \hat{d} x_k (0, +\infty)_0 \\
d_{k=1} \hat{d} x_k (-\infty, 0)_0
\end{bmatrix} \in \begin{bmatrix}
\hat{d} \\
d_{k=1} \hat{d} (\partial_k \Omega_{\ell \Omega_{+\infty}}, \mathcal{H}_k) \\
d_{k=1} \hat{d} (\partial_k \Omega_{+\infty}, \mathcal{R}_k) \\
d_{k=1} \hat{d} (\partial_k \Omega_{-\infty}, \mathcal{R}_{nk})
\end{bmatrix}
\]

\( u_0^0 \in \ell^2(\Omega, \mathcal{U}) \), \( y_0^0 \in \ell^2(\mathbb{Z}^d \setminus \Omega, \mathcal{Y}) \),

there is a unique admissible system trajectory \((u, x, y)\) so that

\[
\begin{align*}
\vert u \vert \Omega &= u_0^0, & y \vert _{\mathbb{Z}^d \setminus \Omega} &= y_0^0, & x \vert _{\partial \Omega_{\ell \Omega_{+\infty}}} &= \hat{x}_k \partial \Omega_{\ell \Omega_{+\infty}} 0, \\
x \vert _{\ell_{n,k} + \infty} &\sim ([x_k (\ell_{n,k})]_0^0)_{\ell_{n,k}} \text{ for each } \ell_{n,k} \in \partial \Omega_{+\infty}, \\
x \vert _{\ell_{n,k} - \infty} &\sim ([x_k (\ell_{n,k})]_0^0)_{\ell_{n,k}} \text{ for each } \ell_{n,k} \in \partial \Omega_{-\infty}.
\end{align*}
\]

Hence we see that the space \( \mathcal{T} \) is isometrically isomorphic to the space its \( \Omega \)-coordinate version

\[
\mathcal{T}_\Omega := \ell^2(\Omega, \mathcal{E}) \oplus \bigoplus_{k=1}^d \ell^2(\partial_k \Omega_{\ell \Omega_{+\infty}}, \mathcal{H}_k) \oplus \bigoplus_{k=1}^d \ell^2(\partial_k \Omega_{+\infty}, \mathcal{R}_k)
\]

under the map \( \Gamma: \mathcal{T} \to \mathcal{T}_\Omega \) given by

\[
\begin{align*}
\Gamma: (u, x, y) &\mapsto \begin{bmatrix}
\hat{d} u_0^0 \\
\hat{d} x_k |_{\partial \Omega_{\ell \Omega_{+\infty}}} \\
\hat{d} x_k (0, +\infty)_0 \\
\hat{d} x_k (-\infty, 0)_0
\end{bmatrix}
\end{align*}
\]

with the inverse \( \Gamma^{-1} \) of \( \Gamma \) computed by solving the initial value problem described above.

Moreover, by using the independence of the \( \mathcal{T} \)-norm on the choice of shift-invariant sublattice \( \Omega \), it is not difficult to show that the shift operators

\[
\mathcal{U}_k: (u(\cdot), x(\cdot), y(\cdot)) \mapsto (u'(\cdot), x'(\cdot), y'(\cdot))
\]

(4.6)
where
\[ u'(n) = u(n - e_k) = u(\sigma_k^{-1}(n)) \]
\[ x'(n) = x(n - e_k) = x(\sigma_k^{-1}(n)) \]
\[ y'(n) = y(n - e_k) = y(\sigma_k^{-1}(n)). \]

for \( k = 1, \ldots, d \) are well-defined and unitary on \( \mathcal{T} \) with inverses given by
\[ U_k^{-1}: (u, x, y) \mapsto (u'', x'', y'') \]
with
\[ u''(n) = u(n + e_k) = u(\sigma_k(n)) \]
\[ x''(n) = x(n + e_k) = x(\sigma_k(n)) \]
\[ y''(n) = y(n + e_k) = y(\sigma_k(n)). \]

In addition we define subspaces \( \mathcal{F} \) and \( \mathcal{F}_* \) of \( \mathcal{T} \) by
\[ \mathcal{F} = \{(u, x, y): u|_{\Omega \setminus \{0\}} = 0, \|x|_{\partial \Omega}\|_{\partial \Omega}^2 = 0, y|_{\Omega \setminus \{0\}} = 0\} \]
where \( \Omega \) is any shift-invariant sublattice with \( 0 \in \Omega \)
\[ \mathcal{F}_* = \{(u, x, y): u|_{\Omega' \setminus \{0\}} = 0, \|x|_{\partial \Omega'}\|_{\partial \Omega'}^2 = 0, y|_{\Omega' \setminus \{0\}} = 0\} \]
where \( \Omega' \) is any shift-invariant sublattice with \( 0 \notin \Omega' \).

Then, by [24, Theorem 4.4],
\[ \mathcal{S}(\Sigma(U)) = (\mathcal{T}: \mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_d); \mathcal{F}, \mathcal{F}_*) \]
is a multievolution Lax–Phillips scattering system in the sense defined in Section 8 above. Moreover, the scattering matrix \( S(z) \) of the scattering system \( \mathcal{S}(\Sigma(U)) \) can be identified with the transfer function \( T_{\Sigma(U)}(z) \) of the conservative Givone–Roesser system \( \Sigma(U) \) after some trivial modifications: if we let
\[ \Omega = \text{any shift-invariant sublattice with } 0 \in \Omega, \]
\[ \Omega' = \text{any shift-invariant sublattice with } 0 \notin \Omega', \]
then
\[ S_{\mathcal{S}(\Sigma(U))}(z)i = i_* T_{\Sigma(U)}(z) \text{ for almost all } z \in \mathbb{T}^d \]
(4.8)
where \( i: \mathcal{E} \rightarrow \mathcal{F} \) and \( i_*: \mathcal{E}_* \rightarrow \mathcal{F}_* \) are the unitary identification maps\(^7\)
\[ i: e \mapsto (u(\cdot), x(\cdot), y(\cdot)) \in \mathcal{T} \text{ such that } u|_{\Omega} = \{\delta_n, e \} \in \mathcal{F}, x|_{\partial \Omega} = 0, y|_{\Omega \setminus \{0\}} = 0 \]
(4.9)
\[ i_*: e_* \mapsto (u(\cdot), x(\cdot), y(\cdot)) \in \mathcal{T} \text{ such that } u|_{\Omega'} = 0, x|_{\partial \Omega'} = 0, y|_{\Omega' \setminus \{0\}} = \{\delta_n, e_* \} \in \mathcal{F}_*. \]
(4.10)

Let us say that a multievolution scattering system \( \mathcal{S} \) of the form \( \mathcal{S}(\Sigma(U)) \) for a conservative Givone–Roesser system \( \Sigma(U) \) is a multievolution scattering system with embedded unitary colligation \( U \); one of the main connections between \( U \) and \( \mathcal{S}(\Sigma(U)) \) is the property \( [18] \): the scattering matrix for \( \mathcal{S}(\Sigma(U)) \) coincides with the transfer function for \( \Sigma(U) \).

\(^7\)It turns out that the operators \( i \) and \( i_* \) defined via (4.9) and (4.10) are independent of the particular choice of shift-invariant sublattices \( \Omega \) and \( \Omega' \) satisfying (4.4).
Remark 4.1. The above analysis shows that a unitarily equivalent version of the multievolution scattering system is determined by using the unitary map $\Gamma$ given by \[(4.5)\] to map the space of admissible trajectories $\mathcal{T}$ to the $\Omega$-coordinatized version $\mathcal{T}_\Omega$; specifically, we may define $\mathcal{G}_\Omega(\Sigma(U))$ by

$$\mathcal{G}_\Omega(\Sigma(U)) = (\mathcal{T}_\Omega; U_\Omega = (U_{\Omega,1}, \ldots, U_{\Omega,d}); \mathcal{F}_\Omega, \mathcal{F}_{\Omega*})$$

where, for $k = 1, \ldots, d$, $U_{\Omega,k}$ on $\mathcal{T}_\Omega$ is given by

$$U_{\Omega,k} = \Gamma U_k^{*}, \mathcal{F}_\Omega = \Gamma \mathcal{F}, \mathcal{F}_{\Omega*} = \Gamma \mathcal{F},$$

(with $U_k$ on $\mathcal{T}$ given by \[(4.6)\]). Of course, $\mathcal{T}$ depends on $U$ and $\Gamma$ and $\Gamma^{*}$ depend on $U$ and $\Omega$; as $U$ and $\Omega$ are considered fixed, we suppress this dependence from the notation. In the preceding discussion (as well as in \[(14)\]) we avoided working out these operators $U_{\Omega,k}$ explicitly. A convenient special case is the balanced shift-invariant sublattice $\Omega^B$ (see Section 4.4.1 of \[(14)\] and also Section 1.5 of \[(33)\]) given by

$$\Omega^B = \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : n_1 + \cdots + n_d \geq 0\}$$

as there are no infinite boundary components in this case. For each $k$ the finite boundary is equal to the subset $\Xi$ of $\mathbb{Z}^d$ given by

$$\Xi = \{n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : n_1 + \cdots + n_d = 0\}.$$ 

Hence the $\Omega^B$-coordinatized version of $\mathcal{T}$ reduces to

$$\mathcal{T}_{\Omega^B} = \left[\ell^2(\mathbb{Z}^d \setminus \Omega^B, \mathcal{E}_*) \bigoplus_{k=1}^d \ell^2(\Xi, \mathcal{H}_k) \right].$$

We view elements of $\ell^2(\mathbb{Z}^d \setminus \Omega^B, \mathcal{E}_*)$ as functions $n \mapsto \vec{e}_*(n)$ of $n \in \mathbb{Z}^d \setminus \Omega^B$ with values $\vec{e}_*(n) \in \mathcal{E}_*$, and similarly for $\ell^2(\Omega^B, \mathcal{E})$. We view elements of $\bigoplus_{k=1}^d \ell^2(\Xi, \mathcal{H}_k)$ as functions $(n, j) \mapsto \vec{x}(n, j)$ of $(n, j) \in \Xi \times \{1, \ldots, d\}$ such that the value $\vec{x}(n, j)$ is in $\mathcal{H}_j$. For the computation to follow, we drop the subscript $\Omega^B$ and write $U_k$ rather than $U_{\Omega^B,k}$. For a fixed $k \in \{1, \ldots, d\}$, the operator $U_k^{*}$ on $\mathcal{T}_{\Omega^B}$ can be viewed as a $3 \times 3$-block matrix of the form

$$U_k^{*} = \begin{bmatrix} [U_k^{*}]_{11} & [U_k^{*}]_{12} & [U_k^{*}]_{13} \\ 0 & [U_k^{*}]_{22} & [U_k^{*}]_{23} \\ 0 & 0 & [U_k^{*}]_{33} \end{bmatrix}.$$ 

The various matrix entries can be given explicitly as follows:

$$([U_k^{*}]_{11} \vec{e}_*) (n) = \begin{cases} \vec{e}_*(n + e_k) & \text{if } n + e_k \in \mathbb{Z}^d \setminus \Omega^B, \\ 0 & \text{otherwise,} \end{cases}$$

$$([U_k^{*}]_{12} \vec{x}) (n) = \begin{cases} \sum_{j=1}^d C_j \vec{x}(n + e_k, j) & \text{if } n + e_k \in \Xi, \\ 0 & \text{otherwise,} \end{cases}$$

where $C_j$ is the sum over $j = 1, \ldots, d$ of the coefficients $C_j$ associated with $\vec{x}$. 

Note that $C_j$ is the sum over $j = 1, \ldots, d$ of the coefficients $C_j$ associated with $\vec{x}$.
Moreover, the last action (4.13) can be decomposed further; for each \( m \neq n \) that (4.13) is unitary from the fact that the mapping 

\[
\delta_{n^0} \left( \bigoplus_{j=1}^{d} \ell^2(\Xi, H_{n^0}) \right) \mapsto \left[ \bigoplus_{j=1}^{d} \delta_{n^0 + e_k} \ell^2(\Xi, H_{n^0 + e_k}) \right], \tag{4.14}
\]

unitarily (where, for \( m \in \Xi \), \( \delta_m \) denotes the delta function on \( \Xi \): \( \delta_m(n) = 0 \) for \( n \neq m \) and \( \delta_m(n) = 1 \) for \( n = m \) for \( n \in \Xi \)). Given that (4.14) is unitary, it follows that (4.13) is unitary from the fact that the mapping

\[
(n, j) \mapsto (n + e_j - e_k, j)
\]

is bijective on \( \Xi \times \{1, \ldots, d\} \).
Explicitly, the mapping (4.11) is given by

\[
\mathcal{U}_k : \left[ \frac{\delta_{n^0} \left( \bigoplus_{j=1}^{d} x_j(\cdot) \right)}{\delta_n e(\cdot)} \right] \mapsto \left[ \begin{array}{cccc}
\delta_{n^0 + e_1 - e_k} & \cdots & \delta_{n^0 + e_d - e_k} \\
A & B \\
C & D
\end{array} \right] \left[ \bigoplus_{j=1}^{d} x_j(\cdot) \right] e(\cdot),
\]

from which the unitarity of the mapping (4.14) follows immediately from the unitarity of the colligation matrix \( U = [A B] \). With a little more work one can verify explicitly that the \( d \)-tuple \( (\mathcal{U}_1, \ldots, \mathcal{U}_d) \) is commutative. This representation for the scattering system with given embedded GR-unitary colligation reduces to the well-known Schäffer matrix for the classical \( d = 1 \) case (see [49]).

It is also shown in [14] that the multievolution scattering system \( \mathfrak{S}(\Sigma(U)) \) with given embedded unitary colligation \( U \) carries some additional geometric structure. Given such a scattering system \( \mathfrak{S}(\Sigma(U)) \), define subspaces \( \mathcal{L}_k, \mathcal{M}_k \) and \( \mathcal{M}_{**} \) by

\[
\mathcal{L}_k = \{(u, x, y) \in \mathcal{T} : u|_{\Omega} = 0, x|_{\partial \Omega_{\infty}} = 0, x|_{\partial_j \Omega_{\infty}} = 0 \text{ for } j \neq k, x|_{\partial_k \Omega_{\infty}} \\text{and } y|_{\partial \Omega_{\infty}} = 0 \}
\]

where \( \Omega \) is any shift-invariant sublattice with \( 0 \in \partial_k \Omega_{\infty} \),

\[
\mathcal{M}_k = \{(u, x, y) \in \mathcal{T} : u|_{\Omega} = 0, x|_{\partial \Omega_{\infty}} = 0, x|_{\partial_j \Omega_{\infty}} = 0, x|_{\partial_j \Omega_{\infty}} = 0 \text{ for } j \neq k, x|_{\partial_k \Omega_{\infty}} \}
\]

where \( \Omega \) is any shift-invariant sublattice with \( \ell_{0,k} \in \partial_k \Omega_{\infty} \),

\[
\mathcal{M}_{**} = \{(u, x, y) \in \mathcal{T} : u|_{\Omega} = 0, x|_{\partial \Omega_{\infty}} = 0, x|_{\partial_j \Omega_{\infty}} = 0, x|_{\partial_j \Omega_{\infty}} = 0 \text{ for } j \neq k, x|_{\partial_k \Omega_{\infty}} \}
\]

where \( \Omega \) is any shift-invariant sublattice with \( \ell_{0,k} \in \partial_k \Omega_{\infty} \).

It is shown in [14] that \( \mathcal{L}_1, \ldots, \mathcal{L}_d \) are mutually orthogonal and that \( \mathcal{L}_k \) is wandering for the commuting unitary \((d-1)\)-tuple \( \mathcal{U}_k \) (equal to the tuple \( \mathcal{U} \) with \( \mathcal{U}_k \) left out), i.e.,

\[
(\mathcal{U}_k)^n \mathcal{L}_{k} \perp (\mathcal{U}_k)^{n'} \mathcal{L}_{k} \text{ if } n \neq n' \text{ in } \mathbb{Z}^{d-1};
\]

where we use the notation

\[
(\mathcal{U}_k)^n \mathcal{L}_{k} = \mathcal{U}_1^n \cdots \mathcal{U}_{k-1}^{n_{k-1}} \mathcal{U}_k^{n_k+1} \cdots \mathcal{U}_d^{n_d}
\]

for \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \) and \( k \in \{1, \ldots, d\} \). Also shown in [14] is that fact that \( \mathcal{M}_k \) and \( \mathcal{M}_{**} \) are doubly invariant for \( \mathcal{U}_k \) for \( k = 1, \ldots, d \). Moreover, if we let \( \Omega \) be any shift-invariant sublattice, then the scattering subspace \( \mathcal{V}^\Omega \) of \( \mathfrak{S}(\Sigma(U)) \) associated with \( \Omega \) given by

\[
\mathcal{V}^\Omega = \mathcal{T} \oplus \left( \bigoplus_{n \in \mathbb{Z}^d(\Omega)} \mathcal{U}^n \mathcal{F} \right) \oplus \left( \bigoplus_{n \in \Omega} \mathcal{U}^n \mathcal{F} \right)
\]
has the orthogonal decomposition
\[
\mathcal{V}_\Omega = \left( \bigoplus_{k=1}^{d} \bigoplus_{n \in \partial \Omega_{\text{fin}}} \mathcal{U}^n \mathcal{L}_k \right) \\
\oplus \left( \bigoplus_{k=1}^{d} \bigoplus_{n' : \ell_{n',k} \in \partial \Omega_{\infty}} (\hat{\mathcal{U}}^k)^{n''} \mathcal{M}_k \right) \\
\oplus \left( \bigoplus_{k=1}^{d} \bigoplus_{n'' : \ell_{n'',k} \in \partial \Omega_{-\infty}} (\hat{\mathcal{U}}^k)^{n'''} \mathcal{M}_{s+k} \right).
\] (OD)

In addition, the subspaces \( \mathcal{F}, \mathcal{F}_s, \mathcal{L}_k, \mathcal{M}_k \) and \( \mathcal{M}_{s+k} \) satisfy the Compatible Decomposition Property
\[
\left( \bigoplus_{k=1}^{d} \mathcal{L}_k \right) \oplus \mathcal{F} = \mathcal{F}_s \oplus \left( \bigoplus_{j=1}^{d} \mathcal{U}_j \mathcal{L}_j \right),
\] (CDP)

and the strong limit properties
\[
\text{strong limit}_{m \to \infty} P_{\mathcal{U}^m \mathcal{L}_k} = P_{\mathcal{M}_k}, \quad \text{(SL)}
\]
\[
\text{strong limit}_{m \to \infty} P_{\mathcal{U}^m \mathcal{L}_k} = P_{\mathcal{M}_{s+k}}, \quad \text{(SLs)}
\]
(see [14] Theorem 4.6). One of the main results from [14] is the converse.

**Theorem 4.2.** (See [14] Theorem 4.8.) Let \( \mathcal{S} = (\mathcal{K}; \mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_d); \mathcal{F}, \mathcal{F}_s) \) be a \( d \)-evolution unitary scattering system and let \( \Omega \) be a shift-invariant sublattice. Assume:

1. **There exist subspaces** \( \mathcal{L}_1, \ldots, \mathcal{L}_d, \mathcal{M}_1, \ldots, \mathcal{M}_d \) \( \text{and} \) \( \mathcal{M}_{s1}, \ldots, \mathcal{M}_{sd} \) of \( \mathcal{K} \) such that
   a. \( \mathcal{M}_k \) \( \text{and} \) \( \mathcal{M}_{s+k} \) \( \text{are} \) doubly invariant for \( \mathcal{U}_k \) \( \text{for each} \) \( k = 1, \ldots, d \), and
   b. the scattering subspace
   \[
   
   \mathcal{V}_\Omega := \mathcal{K} \ominus \left[ \bigoplus_{n \in \Omega} \left( \bigoplus_{n \in \mathbb{Z}^d \setminus \Omega} \mathcal{U}^n \mathcal{F} \right) \right.
   \left. \bigoplus_{n \in \mathbb{Z}^d \setminus \Omega} \mathcal{U}^n \mathcal{F}_s \right]
   \]
   has the orthogonal decomposition
   \[
   \mathcal{V}_\Omega = \left( \bigoplus_{k=1}^{d} \bigoplus_{n \in \partial \Omega_{\text{fin}}} \mathcal{U}^n \mathcal{F}_s \right) \\
   \oplus \left( \bigoplus_{k=1}^{d} \bigoplus_{n' : \ell_{n',k} \in \partial \Omega_{\infty}} (\hat{\mathcal{U}}^k)^{n''} \mathcal{M}_k \right) \\
   \oplus \left( \bigoplus_{k=1}^{d} \bigoplus_{n'' : \ell_{n'',k} \in \partial \Omega_{-\infty}} (\hat{\mathcal{U}}^k)^{n'''} \mathcal{M}_{s+k} \right).
   \] (OD)

2. \( (\mathcal{S}, \mathcal{L}_1, \ldots, \mathcal{L}_d) \) has the compatible decomposition property
\[
\left( \bigoplus_{k=1}^{d} \mathcal{L}_k \right) \oplus \mathcal{F} = \mathcal{F}_s \oplus \left( \bigoplus_{k=1}^{d} \mathcal{U}_k \mathcal{L}_k \right).
\] (CDP)

3. **The subspaces** \( \mathcal{L}_k \) \( \text{and} \) \( \mathcal{M}_k \) \( \text{are} \) connected via the formula
\[
\text{strong limit}_{m \to \infty} P_{\mathcal{U}^m \mathcal{L}_k} = P_{\mathcal{M}_k}. \quad \text{(S)}
\]
Similarly the subspaces \( \mathcal{L}_k \) \( \text{and} \) \( \mathcal{M}_{s+k} \) \( \text{are} \) connected via the formula
\[
\text{strong limit}_{m \to \infty} P_{\mathcal{U}^m \mathcal{L}_k} = P_{\mathcal{M}_{s+k}}. \quad \text{(S)}
\]
Let $\mathcal{H}_k, \mathcal{E}, \mathcal{E}_* $ be copies of $\mathcal{L}_k, \mathcal{F}$ an $\mathcal{F}_*$ respectively with associated unitary identification maps

$$i_{\mathcal{H}_k}: \mathcal{H}_k \to \mathcal{L}_k \text{ for } k = 1, \ldots, d, \quad i_\mathcal{E}: \mathcal{E} \to \mathcal{F}, \quad i_{\mathcal{E}_*}: \mathcal{E}_* \to \mathcal{F}_*.$$ 

Define the operator $U: \left( \bigoplus_{k=1}^d \mathcal{H}_k \right) \widehat{\otimes} \mathcal{E} \to \left( \bigoplus_{k=1}^d \mathcal{H}_k \right) \widehat{\otimes} \mathcal{E}_*$ by

$$U = \left[ \begin{array}{c} (i_{\mathcal{H}_1})^* U_1 \\ \vdots \\ (i_{\mathcal{H}_d})^* U_d \\ (i_\mathcal{E})^* \end{array} \right]$$

$$= \left[ \begin{array}{c} H_1 \\ \vdots \\ H_d \\ \mathcal{E} \end{array} \right] \to \left[ \begin{array}{c} H_1 \\ \vdots \\ H_d \\ \mathcal{E}_* \end{array} \right]. \quad (4.15)$$

Then $U$ is a $d$-variable unitary colligation such that $\mathcal{S}$ is unitarily equivalent to $\mathcal{G}(\Sigma(U))$ under the map $J: \mathcal{K} \to \mathcal{T}$ given by

$$J: \xi \mapsto (u(\cdot), x_1(\cdot), \ldots, x_d(\cdot), y(\cdot)) \quad (4.16)$$

with

$$u(n) = i_\mathcal{E} U^n \xi, \quad x_k(n) = i_{\mathcal{H}_k} U^n \xi \text{ for } k = 1, \ldots, d, \quad y(n) = i_\mathcal{E} U^n \xi \quad (4.17)$$

for $n \in \mathbb{Z}^d$. Moreover, if $\Omega'$ is any other shift-invariant sublattice, then property (OD) holds with $\Omega'$ in place of $\Omega$ as well.

The Fourier representation operators $\Phi$ and $\Phi_*$ (see (3.3)) are given by

$$\Phi(\xi)(z) = \sum_{n \in \mathbb{Z}^d} i\xi \frac{\partial}{\partial z} \left( \sum_{n \in \mathbb{Z}^d} i\xi \frac{\partial}{\partial z} \right) \xi(n) z^n, \quad (\Phi_* \xi)(z) = \sum_{n \in \mathbb{Z}^d} i\xi \frac{\partial}{\partial z} \left( \sum_{n \in \mathbb{Z}^d} i\xi \frac{\partial}{\partial z} \right) \xi(n) z^n \text{ if } J\xi = (u(\cdot), x(\cdot), y(\cdot)).$$

In addition, the scattering system $\mathcal{G}(\Sigma(U))$ is minimal (see Definition 2.7) if and only if the GR-unitary colligation $U$ is scattering-minimal in the sense that for some (or equivalently, for any) shift-invariant sublattice $\Omega$ the map

$$\Pi^{\Omega, \Omega}_U: \left. \left( \bigoplus_{k=1}^d \mathcal{H}_k \right) \right|_{\Omega} \to (\mathcal{E}_* \oplus \mathcal{E})[[z^{\pm 1}]] \text{ is injective} \quad (4.19)$$

where $\Pi^{\Omega, \Omega}_U$ is given by

$$\Pi^{\Omega, \Omega}_U: \left[ \begin{array}{c} \bigoplus_{k=1}^d \mathcal{H}_k \\ \bigoplus_{k=1}^d \mathcal{H}_k^{\mathcal{H}_k} \\ \bigoplus_{k=1}^d \mathcal{H}_k^{\mathcal{H}_k} \end{array} \right] (z) \to \left[ \begin{array}{c} \bigoplus_{k=1}^d \mathcal{H}_k^{\mathcal{H}_k} \\ \bigoplus_{k=1}^d \mathcal{H}_k^{\mathcal{H}_k} \\ \bigoplus_{k=1}^d \mathcal{H}_k^{\mathcal{H}_k} \end{array} \right] (z)$$

where

$$\bar{y}(z) = C(I - Z_{\text{diag}}(z) A)^{-1} \left[ \begin{array}{c} x_1^{0, \infty}(z) \\ \vdots \\ x_d^{0, \infty}(z) \end{array} \right] - C(I - Z_{\text{diag}}(z) A)^{-1} \left[ \begin{array}{c} x_1^{0, \infty}(z) \\ \vdots \\ x_d^{0, \infty}(z) \end{array} \right] - Z_{\text{diag}}(z) A \left[ \begin{array}{c} x_1^{0, \infty}(z) \\ \vdots \\ x_d^{0, \infty}(z) \end{array} \right].$$
\[ \hat{u}(z) = B^*Z_{\text{diag}}(z)^{-1}(I - A^*Z_{\text{diag}}(z)^{-1})^{-1} \left[ \begin{array}{c} x^0_{\Omega} + 1,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \\ \vdots \\ x^0_{\Omega} + d,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \end{array} \right] \]
\[ + B^*Z_{\text{diag}}(z)^{-1} \left[ \begin{array}{c} x^1_{\Omega} + 1,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \\ \vdots \\ x^1_{\Omega} + d,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \end{array} \right] \]
\[ \cdot \left( A^*Z_{\text{diag}}(z)^{-1} \left[ \begin{array}{c} x^0_{\Omega} + 1,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \\ \vdots \\ x^0_{\Omega} + d,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \end{array} \right] - \left[ \begin{array}{c} x^0_{\Omega} + 1,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \\ \vdots \\ x^0_{\Omega} + d,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \end{array} \right] \right). \] (4.20)

where
\[ \hat{x}^0_{\Omega}(z) = \sum_{m: \ell_m,k \in \partial \Omega_{\text{fin}} \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}} \sum_{t=-\infty}^{\infty} \left( x^0_{\Omega}(\ell) \right) \left( t \right) x^t_{\Omega}(z^\ell) \hat{m}^k \]

and where
\[ \hat{x}^0_{\Omega}(z) = \sum_{m: \ell_m,k \in \partial \Omega_{\text{fin}} \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}} \sum_{t=-\infty}^{\infty} \left( x^0_{\Omega}(\ell) \right) \left( t \right) x^t_{\Omega}(z^\ell) \hat{m}^k. \]

Proof. The only new observation beyond [13] Theorem 4.8 is the criterion (4.19) for minimality of the scattering system \( \mathfrak{S}(\Sigma(U)) \) expressed directly in terms of the GR-unitary colligation \( U \). In general, a vector \( \xi \) in the ambient space \( K \) for the scattering system \( \mathfrak{S} \) is orthogonal to the minimal subspace equal to the closure of \( \mathcal{W} + \mathcal{W} \subset \kappa \) for \( \mathfrak{S} \) (see Definition 3.1) if and only if both \( \Phi\xi = 0 \) and \( \Phi_\lambda \xi = 0 \). From (4.18) we see that this is equivalent to the element \( \mathcal{F}\xi \) having the form \((0, x(\cdot), 0)\). Hence minimality of the scattering system \( \mathfrak{S}(\Sigma(U)) \) translates to:
\[ (0, x(\cdot), 0) \in \mathcal{T} \Rightarrow x(\cdot) = 0. \]

If we use the isometric isomorphism \( \Gamma: \mathcal{T} \to \mathcal{T}_{\partial \Omega} \) as in (4.15), we see that the condition is:
\[ \Gamma^{-1} \left[ \begin{array}{c} 0 \\ \bigoplus_{k=1}^{d} x^0_{\Omega} + 1,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \\ \vdots \\ \bigoplus_{k=1}^{d} x^0_{\Omega} + d,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \end{array} \right] \]
\[ = \bigoplus_{k=1}^{d} x^0_{\Omega} = (0, 0, 0) \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}} \]

has the form \((0, x(\cdot), 0) \Rightarrow x(\cdot) = 0. \)

The formula (4.20) is just the formula from [13] Theorem 4.15 for the input string \( u \) and output string \( y \) associated with the unique trajectory \((u, x, y)\) which solves the initial value problem
\[ u|_{\partial \Omega} = 0, \quad x|_{\partial \Omega} = \left[ \begin{array}{c} \bigoplus_{k=1}^{d} x^0_{\Omega} + 1,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \\ \vdots \\ \bigoplus_{k=1}^{d} x^0_{\Omega} + d,0 \ni \Omega_{\text{fin}} \ni \partial \Omega_{\text{fin}}(z) \end{array} \right], \quad y|_{\partial \Omega} = 0; \]

for more complete details on how to make rigorous sense of the formulas in (4.20), we refer to [13]. Hence minimality of the scattering system \( \mathfrak{S}(\Sigma(U)) \) is equivalent to injectivity of the map \( \Pi_{U}^{dBR, \partial \Omega} \) as wanted. \( \square \)
Remark 4.3. Note that the formula for  in (4.20) involves only \( x|_{\partial \Omega_{\text{fin}} \cup \partial \Omega_{-\infty}} \) while the formula for \( \hat{u} \) in (4.20) involves only \( x|_{\partial \Omega_{\text{fin}} \cup \partial \Omega_{+\infty}} \). In the spirit of the terminology introduced in [34], given a position \( n \in \mathbb{Z}^d \) and a shift-invariant sublattice with \( n \in \cap_{k=0}^d \partial_k \Omega_{\text{fin}} \), it makes sense to view \( x(n) \) as the local state at \( n \), \( x|_{\partial \Omega_{\text{fin}} \cup \partial \Omega_{-\infty}} \) as the global state for the forward system (4.1) at \( n \), and \( x|_{\partial \Omega_{\text{fin}} \cup \partial \Omega_{+\infty}} \) as the global state for the backward system (4.2) at position \( n \).

As a corollary, we have the following result.

**Corollary 4.4.** Suppose that \( S \in S(\mathcal{E}, \mathcal{E}_*) \) is in the \( d \)-variable Schur–Agler class. Then the following conditions are equivalent.

1. \( S(z) \) has an Agler decomposition

\[
I - S(z)S(w)^* = \sum_{k=1}^d (1 - z_k \overline{w_k}) H_k(z) H_k(w)^*
\]

for some operator-valued functions \( H_k : \mathbb{D}^d \to \mathcal{L}(\mathcal{H}_k', \mathcal{E}) \) and some Hilbert spaces \( \mathcal{H}_1', \ldots, \mathcal{H}_d' \).

2. There exists a multievolution scattering system \( \mathcal{S}(\Sigma(U)) \) with scattering matrix \( S_{\mathcal{E}}(z) \) equal to \( S(z) \) which satisfies conditions (\text{OD}), (\text{CDP}), (\text{SL}) and (\text{SL*}).

**Proof.** For the direction (2) \( \Rightarrow \) (1), use formula (4.13) to define \( U = [A \ B] \) and then set \( H_k(z) = C(I - Z_{\text{diag}}(z) A)^{-1} P_k \) for \( k = 1, \ldots, d \). For the direction (1) \( \Rightarrow \) (2), use the result of [2] to deduce that \( S(z) \) has a realization \( S(z) = T_{\Sigma(U)}(z) \) as the transfer function of a conservative Givone–Roesser system \( \Sigma(U) \) and then use the analysis in [14] to deduce that the ambient space \( \mathcal{K} \) of the associated scattering system \( \mathcal{S}(\Sigma(U)) \) contains subspaces \( \mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_{k,k} (k = 1, \ldots, d) \) meeting all the conditions (\text{OD}), (\text{CDP}), (\text{SL}) and (\text{SL*}).

While the proof of (2) \( \Rightarrow \) (1) is quite explicit, the proof of (1) \( \Rightarrow \) (2) in Corollary 4.4 is rather indirect. A goal of the present paper is to provide a direct proof of the implication (1) \( \Rightarrow \) (2) in Corollary 4.4. In detail, given a Schur-class function \( S \), it is always possible to form the model scattering system \( \mathcal{S}_{dBR}^S \) (see (3.1)) which has \( S \) as its scattering function. If \( S \) has an Agler decomposition, then we know that there is a possibly nonminimal scattering system \( \mathcal{S} \) having \( S \) as its scattering matrix for which there are subspaces \( \mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_{k,k} (k = 1, \ldots, d) \) satisfying (\text{OD}), (\text{CDP}), (\text{SL}), (\text{SL*}). As a goal for the present paper, we seek to construct such a scattering system with ambient space \( \mathcal{K}_{dBR}^S \oplus \mathcal{K}_{\text{non-min}} \) and identify these spaces explicitly inside the functional model space \( \mathcal{K}_{dBR}^S \) in terms of reproducing kernel representations. In case the various kernels in (3.8) have no overlapping, the associated scattering system is minimal and the identification is relatively straightforward. In the end, one can explicitly construct subspaces \( \mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_{k,k} (k = 1, \ldots, d) \) of \( \mathcal{K}_{dBR}^S \) from the Agler decomposition data functions \( \{ H_1(z), \ldots, H_d(z) \} \). From these subspaces, by using the results of [14] one is led to a unitary colligation \( U = [A \ B] \) so that \( S = S_{\mathcal{S}_{dBR}^S} = T_{\Sigma(U)} \). In this way we arrive at a new proof of (1) \( \Rightarrow \) (2) in Corollary 4.4 which bypasses Theorem 1.2. More precisely, we carry out this construction under the assumption that we have the data functions \( \{ H_1(z), \ldots, H_d(z) \} \) for an augmented Agler decomposition, and then obtain an explicit function-theoretic proof for (2') \( \Rightarrow \) (3) in Theorem 1.2.
5. Functional models for scattering systems containing an embedded unitary colligation

In this section we assume that we are given a unitary colligation \( U = [A \ B] \) as in (1.3) to which we associate the multievolution scattering system \( \mathcal{S}(\Sigma(U)) \) as in Section 4. When viewed in a coordinate-free way, \( \mathcal{S}(\Sigma(U)) \) is a multievolution scattering system \( \mathcal{S} \) as in (4.1) with the additional structure that there exist subspaces \( \mathcal{L}_1, \ldots, \mathcal{L}_d, \mathcal{M}_1, \ldots, \mathcal{M}_d \) and \( \mathcal{M}_{-1}, \ldots, \mathcal{M}_{-d} \) so that \((\text{OD}), (\text{CDP}), \text{SL} \) and \((\text{SL})\) all hold. As explained in Section 3 (see Lemma 3.2), the map

\[
\Pi_{\text{dBR}}: k \mapsto \Phi_k = \left[ \frac{\sum_{n \in \mathbb{Z}^d} (P_{\mathcal{F}}U^* n_k) z^n}{\sum_{n \in \mathbb{Z}^d} (P_{\mathcal{F}}U^* n_k) z^n} \right]
\]

is a coisometry from the ambient space \( \mathcal{K} \) onto the ambient space \( \mathcal{K}_{dBR}^S \) for the de Branges–Rovnyak model scattering system \( \mathcal{S}_{\text{dBR}} \). When expressed in terms of trajectory coordinates, \( \Pi_{\text{dBR}} \) assumes the form

\[
\Pi_{\text{dBR}}: (u(\cdot), x(\cdot), y(\cdot)) \mapsto \left[ \frac{\sum_{n \in \mathbb{Z}^d} y(n) z^n}{\sum_{n \in \mathbb{Z}^d} u(n) z^n} \right] := \left[ \hat{y}(z) \hat{u}(z) \right].
\]

If \( S(z) = \sum_{n \in \mathbb{Z}^d} S_n z^n \) is in the Schur–Agler class \( \mathcal{S}(\mathcal{F}, \mathcal{F}_*) \), then we may view \( S(z) \) either in the classical sense as an analytic \( \mathcal{L}(\mathcal{F}, \mathcal{F}_*) \)-valued function on the unit polydisk, or purely formally as in Section 2 as a formal power series in the indeterminates \( z_1, \ldots, z_d \). In the former case we view \( S(w)^* \) as the conjugate analytic function defined on \( \mathbb{D}^d \) given by \( S(w)^* = \sum_{n \in \mathbb{Z}^d} S_n^* w^n \) while in the latter we follow the convention (2.2) and define \( S(w)^* \in \mathcal{L}(\mathcal{F}_*, \mathcal{F})[[z^{\pm 1}]] \) as \( S(w)^* = \sum_{n \in \mathbb{Z}^d} S_n^* w^{-n} \). It is relatively clear how to view an Agler decomposition (1.1) or an augmented Agler decomposition (1.2) either in the sesquianalytic sense or in the formal power series sense. However the Schur matrix product (i.e., entrywise matrix product) in (1.2) is not so convenient. The next lemma gives a more convenient form of the augmented Agler decomposition which avoids the Schur–matrix product.

**Lemma 5.1.** Suppose that \( S(z) \) is an analytic \( \mathcal{L}(\mathcal{F}, \mathcal{F}_*) \)-valued function on the unit polydisk \( \mathbb{D}^d \) which has a (classical) sesquianalytic augmented Agler decomposition as in (1.2). Then:

1. If we view \( S(z) = \sum_{n \in \mathbb{Z}^d} S_n z^n \) as a formal power series in \( \mathcal{L}(\mathcal{F}, \mathcal{F}_*)[[z^{\pm 1}]] \), the augmented Agler decomposition (1.2) can be reexpressed as the following formal augmented Agler-decomposition:

\[
\begin{bmatrix}
I - S(z)S(w)^* & S(w) - S(z) \\
S(z)^* - S(w)^* & S(z)^* S(w) - I
\end{bmatrix} = \sum_{k=1}^d (1 - \bar{z}_k w_k^{-1}) \tilde{K}_k(z, w)
\]

where

\[
\tilde{K}_k(z, w) = \begin{bmatrix}
\tilde{H}_k^1(z) & \tilde{H}_k^2(z^{-1}) \\
\tilde{H}_k^2(z) & \tilde{H}_k^1(w^{-1})
\end{bmatrix}
\]

where we have set

\[
\tilde{H}_k^1(z) = H_k^1(z), \quad \tilde{H}_k^2(z) = z_k^{-1} H_k^2(z^{-1}).
\]
Hence, if we set
\[\tilde{H}_k^1(z) = C(I - Z_{\text{diag}}(z)A)^{-1}P_k\]
and \(\tilde{H}_k^2(z^{-1}) = z_k^{-1}B^*(I - Z_{\text{diag}}(z^{-1})A^*)^{-1}P_k\)
\[(5.2)\]
in (5.1), and the kernel \(\tilde{K}_k(z, w)\) appearing in (5.1) is given explicitly as
\[\tilde{K}_k(z, w) = \left[ \begin{array}{c} C(I - Z_{\text{diag}}(z)A)^{-1} \\ z_k^{-1}B^*(I - Z_{\text{diag}}(z^{-1})A^*)^{-1} \end{array} \right] P_k.\]
\[(5.2)\]

Theorem 1.2 we know that (1.2) holds with
\[\text{Remark}\]
\[\text{Remark}\]

Proof. Given that (1.2) holds, a change of sign in the second column gives
\[\text{We then replace } w \text{ by } \overline{w} \text{ in the second column and } z \text{ by } \overline{z} \text{ in the second row to get}\]
\[\text{We then replace } \overline{w}_k \text{ by } w_k^{-1}, \overline{z}_k \text{ by } z_k^{-1} \text{ and interpret } S(z) \text{ and } H_k^1(z) (j = 1, 2, k = 1, \ldots, d) \text{ in the formal sense with the convention (2.4) in force to arrive at}\]
\[\text{Next observe that}\]
\[\text{Hence, if we set}\]
\[\text{we arrive at (5.1) as wanted.}\]
\[\text{If we know a realization (1.4) for } S(z), \text{ we can arrive at the formulas for } \tilde{H}_k^1(z) \text{ via direct substitution using the unitary relations (1.5) for } U. \text{ Alternatively, by Theorem 1.2 we know that (1.2) holds with } H_k^1(z) \text{ and } H_k^2(z) \text{ as in (1.6). Setting } \tilde{H}_k^1(z) \text{ as in (5.3) then gives us the formulas for } \tilde{H}_k^1(z) \text{ given in (5.2).}\]

Remark 5.2. Part of the content of Theorem 1.2 is that \(S\) has an augmented Agler decomposition (and hence the formal augmented Agler decomposition (5.1)) whenever \(S\) has the simpler Agler decomposition (1.1). The usual proof for this fact goes through the implication (2) \(\Rightarrow\) (3) in Theorem 1.2 (achieved through the “lurking isometry” argument—see (16)) to arrive at a GR-unitary realization for \(S\); one then
arrives at (2′) (or the amended form (5.1)) via the formulas (1.6) or (5.2). For the completeness of the function-theoretic approach of this paper, it would be nice to have a direct function-theoretic proof of (2) ⇒ (2′) in Theorem 1.2 which does not pass through a conservative-GR realization for $S$.

In the sequel we assume that we are given a Schur-class function $S \in \mathcal{S}(E, E_*)$ together with an augmented Agler decomposition; to lighten the notation, we drop the tildes and write simply

$$
\begin{bmatrix}
I - S(z)S(w)^* & S(w) - S(z) \\
S(z)^* - S(w)^* & S(z)^*S(w) - I
\end{bmatrix} = \sum_{k=1}^{d} (1 - z_k w_k^{-1}) K_k(z, w)
$$

(5.4)

for the (formal) augmented Agler decomposition. In case we are given a realization $S(z) = D + C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)B$ for $S$ with $U = [D B]$ unitary, then by part (3) of Lemma 5.1 one can take $K_k(z, w)$ to be given by

$$
K_k(z, w) = \left[ -\frac{C(I - Z_{\text{diag}}(z)A)^{-1}}{z_k^{-1}B^* (I - Z_{\text{diag}}(z)^*A^{-1})} \bigg] P_k \cdot \left[ (I - A^*Z_{\text{diag}}(w)A)^{-1}C^* - I - AZ_{\text{diag}}(w)B w_k \right].
$$

(5.5)

The augmented Agler decomposition (5.4) leads to the following kernel identities.

**Proposition 5.3.** Suppose that $S \in \mathcal{S}(E, E_*)$ has augmented Agler decomposition (5.4) and that $\Omega \subset \mathbb{Z}^d$ is a shift-invariant sublattice. We shall use the decomposition

$$
K_k(z, w) = \begin{bmatrix}
K_{11}^{11}(z, w) & K_{11}^{12}(z, w) \\
K_{12}^{11}(z, w) & K_{12}^{22}(z, w)
\end{bmatrix}
$$

(5.6)

of the kernels appearing in (5.8) for $k = 1, \ldots, d$. Then:

1. The kernel function $K_{\overline{\partial} \Omega, \omega}(z, w)$ appearing in (5.8) has the decomposition

$$
K_{\overline{\partial} \Omega, \omega}(z, w) = \sum_{k=1}^{d} \left( \sum_{n \in \partial \Omega, \Omega_{\infty}} z^n w^{-n} \right) K_k(z, w) + K_{\overline{\partial} \Omega, \omega}(z, w)
$$

(5.7)

where $K_{\overline{\partial} \Omega, \omega}(z, w)$ has the form

$$
K_{\overline{\partial} \Omega, \omega}(z, w) = \begin{bmatrix}
K_{11}^{11}(z, w) & 0 \\
0 & K_{22}^{22}(z, w)
\end{bmatrix}
$$

with

$$
K_{11}^{11}(z, w) = \sum_{\ell \in \partial \Omega, \Omega_{\infty}} (\hat{z}^k)^{\ell} (\hat{w}^k)^{-\hat{n}^k} K_{-\infty, k}^{11}(z, w),
$$

$$
K_{22}^{22}(z, w) = \sum_{\ell \in \partial \Omega, \Omega_{\infty}} (\hat{z}^k)^{\ell} (\hat{w}^k)^{-\hat{n}^k} K_{+\infty, k}^{22}(z, w)
$$

where we set

$$
K_{-\infty, k}^{11}(z, w) = \lim_{t \to -\infty} z_k^t w_k^{-t} K_k^{11}(z, w),
$$

$$
K_{+\infty, k}^{22}(z, w) = \lim_{t \to +\infty} z_k^t w_k^{-t} K_k^{22}(z, w)
$$

(5.8)

with the limit interpreted in the sense of strong convergence of power-series coefficients, and where

$$(\hat{z}^k)^{\ell} = z_1^{n_1} \cdots z_k^{n_k} \cdots z_d^{n_d} \quad \text{and} \quad \hat{n}^k = (n_1', \ldots, \hat{n}_k', \ldots, n_d'),$$

and $\Omega_{\infty}$ is a shift-invariant sublattice.
i.e., the $k$-th term is omitted. 

(2) We have the following kernel analogue of (CDP):

$$
\sum_{k=1}^{d} K_k(z, w) + \left[ S(z) I \right] \left[ (S(w))^* I \right] = \sum_{k=1}^{d} z_k K_k(z, w) w_k^{-1} + \left[ I (S(z))^* I S(w) \right]. 
$$

(5.9)

**Remark 5.4.** We view conditions (5.7), (5.8) and (5.9) as kernel function analogues of the subspace conditions (OD), (SL) and (SL$^*$), and (CDP) in Theorem 4.2. We indicate below (see Theorems [5.9] and [S.1]) how, under certain conditions, these kernel decompositions can be translated to the validity of the subspace conditions (OD), (SL) and (SL$^*$), and (CDP) for a functional-model multievolution scattering system having scattering matrix $S$.

**Proof of Proposition 5.3.** In addition to (5.6), let us decompose $K_{\gamma_{dBR}}(z, w)$ as

$$
K_{\gamma_{dBR}}(z, w) = \begin{bmatrix}
K^{11}_{\gamma_{dBR}}(z, w) & K^{12}_{\gamma_{dBR}}(z, w) \\
K^{21}_{\gamma_{dBR}}(z, w) & K^{22}_{\gamma_{dBR}}(z, w)
\end{bmatrix},
$$

We shall first verify the (1,1)-entry in the equation (5.7). The computations must be regularized properly in order to avoid the traps illustrated in Remark 2.6. To perform the regularization we introduce the approximate shift-invariant sublattice

$$
\Omega^M = \{ n = (n_1, \ldots, n_d) \in \Omega : n_k \geq M \text{ for all } k = 1, \ldots, d \}.
$$

We assert that

$$
k_{\gamma_{dBR}}(z, w)(I - S(z)S(w)^*) = \lim_{M \to -\infty} k_{\gamma_{dBR}}(z, w)(I - S(z)S(w)^*)
$$

(5.10)

(with convergence taken to be strong coefficientwise). Indeed, the coefficient of $z^m w^{-m}$ on the left-hand side of (5.10) is

$$
[k_{\gamma_{dBR}}(z, w)(I - S(z)S(w)^*)]_{n,m} = \chi_{\Omega}(n) \delta_{n,m} I_{\mathbb{E}} + \text{weak} \sum_{n' \in \mathbb{Z}_d^+} \chi_{\Omega}(n - n') \chi_{\Omega}(m - m') \delta_{n - n', m - m'} S_{n'} S_{n'}^*
$$

(5.11)

(with $\chi_{\Omega}$ equal to the characteristic function of the set $\Omega$ and with $(k, \ell) \mapsto \delta_{k, \ell}$ equal to the Kronecker delta-function); note that the infinite series in (5.11) converges weakly since $S(z)$ is a bounded multiplier from $\mathcal{H}(k_{\gamma_{dBR}})$ to $\mathcal{H}(k_{\gamma_{dBR}})$ and, hence, is a bounded multiplier $\mathcal{H}(k_{\gamma_{dBR}})$ to $\mathcal{H}(k_{\gamma_{dBR}})$ (see Section 2.2). The coefficient of $z^m w^{-m}$ inside the limit on the right-hand side of (5.10) is

$$
[k_{\gamma_{dBR}}(z, w)(I - S(z)S(w)^*)]_{n,m} = \chi_{\Omega}(n) \delta_{n,m} I_{\mathbb{E}} + \text{weak} \sum_{n' \in \mathbb{Z}_d^+ : n - n' \in \Omega^M} S_{n'} S_{n'}^*
$$

(5.12)

Note that the series in (5.12) is just a finite truncation of the series in (5.11). Since the series (5.11) defining the $(n, m)$-coefficient of $k_{\gamma_{dBR}}(z, w)(I - S(z)S(w)^*)$ converges, the assertion (5.10) follows as wanted.
Combining (5.10) with (5.11) then gives
\[
K_{\nu,n}^{11}(z,w) = k_{\Omega}(z,w)(I - S(z)S(w)^*)
= \lim_{M \to -\infty} k_{\Omega}(z,w)(I - S(z)S(w)^*). \tag{5.13}
\]
If we plug the augmented Agler decomposition (5.4) ((1,1)-component only for the moment) into (5.13) we arrive at
\[
K_{\nu,n}^{11}(z,w) = \lim_{M \to -\infty} k_{\Omega}(z,w)\left(\sum_{k=1}^{d}(1 - z_kw_k^{-1})K_{k}^{11}(z,w)\right). \tag{5.14}
\]
Since \(K_{k}^{11}(z,w) = \sum_{n,m \in \mathbb{Z}^d} [K_{k}^{11}]_{n,m}z^n w^{-m}\) (i.e., the coefficients are supported on \(\mathbb{Z}^d \times \mathbb{Z}^d\)) and since \(\Omega^M\) has no boundary at \(-\infty\), only finite sums are involved in the expressions defining the coefficients of \(k_{\Omega}(z,w)K_{k}^{11}(z,w)\) and of \(z_kw_k^{-1}k_{\Omega}(z,w)K_{k}^{11}(z,w)\), and the associativity is valid\(^8\). We are therefore able to continue (5.14) in the form
\[
K_{\nu,n}^{11}(z,w) = \lim_{M \to -\infty} k_{\Omega}(z,w)\left(\sum_{k=1}^{d}(1 - z_kw_k^{-1})K_{k}^{11}(z,w)\right) = \lim_{M \to -\infty} \sum_{k=1}^{d}\left(k_{\Omega}(z,w)(1 - z_kw_k^{-1})\right)K_{k}^{11}(z,w). \tag{5.15}
\]
Since (as has already been pointed out and used) \(\partial_k \Omega^M = \emptyset\) for each \(k = 1, \ldots, d\), one can verify the useful identity
\[
k_{\Omega}(z,w)(1 - z_kw_k^{-1}) = k_{\partial_k \Omega^M}(z,w). \tag{5.16}
\]
It is convenient to decompose the finite boundary components \(\partial_k \Omega^M\) of \(\Omega^M\) as
\[
\partial_k^{M}_{\text{fin}} = (\partial_k \Omega_{\text{fin}} \cap \partial_k \Omega^M) \cup \left[\partial_k \Omega^M \setminus (\partial_k \Omega_{\text{fin}} \cap \partial \Omega^M)\right]
=: \partial_k^{M}_{\text{fin}}(\Omega) \cup \partial_k^{M}_{\text{II}}(\Omega).
\]
Note that the asymptotics of \(\partial_k^{M}_{\text{II}}(\Omega)\) can be understood as follows: for a given \(n' \in \mathbb{Z}^d\), we have
\[
n' = \partial_k^{M}_{\text{II}}(\Omega) \Rightarrow n_k' = M,
\ell_{n',k} \in \partial_k \Omega_{\text{fin}} \Rightarrow \ell_{n',k}|_{n_k' = M} \in \partial_k^{M}_{\text{II}}(\Omega),
\ell_{n',k} \notin \partial_k \Omega_{\text{fin}} \Rightarrow \text{there exists } N_{n'} > -\infty \text{ so that } \ell_{n',k}|_{n_k' = M} \notin \partial_k^{M}_{\text{II}}(\Omega) \text{ once } M < N_{n'} \tag{5.17}
\]
where we have used the (presumably transparent) notation
\[
\ell_{n',k}|_{n_k' = M} = (n_1', \ldots, n_{k-1}', M, n_{k+1}', \ldots, n_d') \in \mathbb{Z}^d.
\]
\(^8\)Note in particular that the starting point of the example in Remark 2.6 is the observation that the collections of kernels \(K_1(z,w) = \sum_{l=0}^{\infty} s_1^l w_1^{-l}\), \(K_j(z,w) = 0\) for \(2 \leq j \leq d\) form an Agler decomposition for the scalar-valued Schur-Agler-class function \(S(z) = 0\).
From (5.15) combined with (5.16) we write

\[ K_{11}^{11}(z, w) = \lim_{M \to -\infty} \sum_{k=1}^{d} \left[ k_{\Omega,0}^{M,0} \Omega(z, w) K_{k}^{11}(z, w) + k_{\Omega,0}^{M,1} \Omega(z, w) K_{k}^{11}(z, w) \right]. \]  

Note that the term \( k_{\Omega,0}^{M,0} \Omega(z, w) K_{k}^{11}(z, w) \), as a function of \( M \), is an increasing sequence of positive kernels as \( M \to -\infty \) (i.e., \( k_{\Omega,0}^{M,0} \Omega(z, w) K_{k}^{11}(z, w) \) is a positive kernel for each fixed \( M \) and the difference

\[
\lim_{M \to -\infty} k_{\Omega,0}^{M,0}(z, w) K_{k}^{11}(z, w) - k_{\Omega,0}^{M,1}(z, w) K_{k}^{11}(z, w)
\]

is a positive kernel for \( M' < M \). As each term \( k_{\Omega,0}^{M,0} \Omega(z, w) K_{k}^{11}(z, w) \) is also a positive kernel, we see that the sequence \( \{k_{\Omega,0}^{M,0} \Omega(z, w) K_{k}^{11}(z, w)\}_{M=-1,-2,...} \) is bounded above by the positive kernel \( K_{11}^{11}(z, w) \). We conclude that the limit

\[ \lim_{M \to -\infty} k_{\Omega,0}^{M,0}(z, w) K_{k}^{11}(z, w) \]

exists (strongly coefficientwise). Moreover, since the sequence

\[ \{k_{\Omega,0}^{M,0}(z, w) K_{k}^{11}(z, w)\}_{M=-1,-2,...} \]

can be identified as the sequence of partial sums for \( k_{\Omega,0}^{M} \Omega(z, w) K_{k}^{11}(z, w) \) (see [13] page 173), we can identify the limit explicitly as

\[ \lim_{M \to -\infty} k_{\Omega,0}^{M,0}(z, w) K_{k}^{11}(z, w) = k_{\Omega,0}^{M} \Omega(z, w) K_{k}(z, w). \]

From (5.18) we conclude that \( \lim_{M \to -\infty} \sum_{k=1}^{d} k_{\Omega,0}^{M,1}(z, w) K_{k}^{11}(z, w) \) with value

\[ \lim_{M \to -\infty} \sum_{k=1}^{d} k_{\Omega,0}^{M,1}(z, w) K_{k}^{11}(z, w) = K_{11}^{11}(z, w) - \sum_{k=1}^{d} k_{\Omega,0}^{M} \Omega(z, w) K_{k}^{11}(z, w). \]  

(5.19)

Our next goal is to verify that the limit

\[ \lim_{M \to -\infty} z_{k}^{M} w_{k}^{M} K_{k}^{11}(z, w) =: K_{-\infty,k}(z, w) \]  

(5.20)

exists for each \( k = 1, \ldots, d \). To this end, let us fix an index \( k_{0} \in \{1, \ldots, d\} \) and define the shift-invariant sublattice \( \Omega_{k_{0}} \) by

\[ \Omega_{k_{0}} = \{ n \in \mathbb{Z}_{+}^{d} : n_{j} \geq 0 \text{ for } j \neq k_{0} \}. \]

We compute

\[
\partial \Omega_{k_{0}} = \emptyset,
\]

\[
\partial \Omega_{k_{0}} = \{ n \in \mathbb{Z}^{d} : n_{k} = 0, n_{j} \geq 0 \text{ for } j \neq k_{0} \text{ for } k \neq k_{0}. \}
\]

Similarly, under the assumption that \( M < 0 \), we have

\[
\partial \Omega_{k_{0}} = \{ n \in \mathbb{Z}^{d} : n_{j} \geq 0 \text{ for } j \neq k_{0} \text{ and } n_{k_{0}} = M \},
\]

\[
\partial \Omega_{k_{0}} = \{ n \in \mathbb{Z}^{d} : n_{k} = 0, n_{j} \geq 0 \text{ for } j \neq k_{0}, n_{k_{0}} \geq M \text{ for } k \neq k_{0}. \}
\]

We conclude that, for \( M < 0 \), we have

\[
\partial \Omega_{k_{0}} = \{ n \in \mathbb{Z}^{d} : n_{j} \geq 0 \text{ for } j \neq k_{0}, n_{k_{0}} = M \},
\]

\[
\partial \Omega_{k_{0}} = 0 \text{ for } k \neq k_{0}. \]  

(5.21)
From (5.19) applied to the case \( \Omega = \Omega_{k_0} \) we conclude that
\[
\lim_{M \to -\infty} \sum_{n \in \partial_{k_0, I}^M(\Omega_{k_0})} z_{k_0}^n w_{k_0}^{-n} K_{k_0}^{11}(z, w) \text{ exists.} \tag{5.22}
\]

We now do a similar analysis for the shift-invariant sublattice \( \Omega_{k_0}^0 \) given by
\[
\Omega_{k_0}^0 = \left\{ n \in \mathbb{Z}^d : n_j \geq 0 \text{ for } j \neq k_0, \sum_{j : j \neq k_0} n_j > 0 \right\}.
\]

We again compute
\[
\partial_{k_0}(\Omega_{k_0}^0)_{\text{fin}} = \emptyset,
\]
\[
\partial_k(\Omega_{k_0}^0)_{\text{fin}} = \left\{ n \in \mathbb{Z}^d : n_j = 0 \text{ for } j \notin \{k_0\} \text{ and } n_k = 1 \right\} \
\cup \left\{ n \in \mathbb{Z}^d : n_k = 0, n_j \geq 0 \text{ for } j \neq k_0, \sum_{j : j \neq k_0} n_j > 0 \right\} \text{ for } k \neq k_0,
\]
while, under the assumption that \( M < 0 \), we have
\[
\partial_{k_0}(\Omega_{k_0}^0)_M = \left\{ n \in \mathbb{Z}^d : n_j \geq 0 \text{ for } j \neq k_0, \sum_{j : j \neq k_0} n_j > 0, n_{k_0} = M \right\},
\]
\[
\partial_k(\Omega_{k_0}^0)_M = \left\{ n \in \mathbb{Z}^d : n_j = 0 \text{ for } j \notin \{k_0\}, n_k = 1, n_{k_0} \geq M \right\} \
\cup \left\{ n \in \mathbb{Z} : n_k = 0, n_j \geq 0 \text{ for } j \neq k_0, \sum_{j : j \neq k_0} n_j > 0, n_{k_0} \geq M \right\} \text{ for } k \neq k_0.
\]
It then follows that
\[
\partial_{k_0, I}^M(\Omega_{k_0}^0) = \left\{ n \in \mathbb{Z}^d : n_j \geq 0 \text{ for } j \neq k_0, \sum_{j : j \neq k_0} n_j > 0, n_{k_0} = M \right\},
\]
\[
\partial_{k, I}^M(\Omega_{k_0}^0) = \emptyset \text{ for } k \neq k_0. \tag{5.23}
\]

If we apply (5.19) with \( \Omega = \Omega_{k_0}^0 \) we then get
\[
\lim_{M \to -\infty} \sum_{n \in \partial_{k_0, I}^M(\Omega_{k_0}^0)} z_{k_0}^n w_{k_0}^{-n} K_{k_0}^{11}(z, w) \text{ exists.} \tag{5.24}
\]

If we observe from (5.21) and (5.23) that \( \partial_{k_0, I}^M(\Omega_{k_0}^0) \subset \partial_{k_0, I}^M(\Omega_{k_0}) \) with
\[
\partial_{k_0, I}^M(\Omega_{k_0}) \setminus \partial_{k_0, I}^M(\Omega_{k_0}^0) = \{(n_1, \ldots, n_d) : n_j = 0 \text{ for } j \neq k_0, n_{k_0} = M \}
\]
and then subtract (5.24) from (5.22), we finally arrive at the existence of the limit
\[
\lim_{M \to -\infty} \sum_{n \in \partial_{k_0, I}^M(\Omega_{k_0}) \setminus \partial_{k_0, I}^M(\Omega_{k_0}^0)} z_{k_0}^n w_{k_0}^{-n} K_{k_0}^{11}(z, w) = \lim_{M \to -\infty} z_{k_0}^M w_{k_0}^{-M} K_{k_0}^{11}(z, w)
\]
and (5.20) follows. From the observation that
\[
z_{k_0}^M w_{k_0}^{-M} K_{k_0}^{11}(z, w) = k_{S_k, \Xi_{M, k}}(z)(I - S(z)S(w)^*) \tag{5.25}
\]
where the subset
\[
\Xi_{M, k} = \{ n \in \mathbb{Z}^d : n_j \geq 0 \text{ for } j \neq k_0, n_{k_0} \geq M \}
\]
is increasing as \( M \) decreases to \(-\infty\), we see that the sequence \( \{z_{k_0}^M w_{k_0}^{-M} K_{k_0}^{11}(z, w)\} \) is an increasing sequence of positive kernels as \( M \to -\infty \).
We now return to the setting where $\Omega$ is a general shift-invariant sublattice. To check the validity of the $(1,1)$-entry of (5.7), it remains now only to verify that
\[
\sum_{k=1}^{d} k_{S,\partial_{k,II}(\Omega)}(z, w) K_{k}^{11}(z, w) \to \sum_{k} \sum_{n: \ell_{n,k} \in \partial_{k,\Omega}^{-\infty}} (z^{k})^{\hat{n}^{k}} (w^{k})^{-\hat{n}^{k}} K_{-\infty,k}^{11}(z, w)
\]
as $M \to -\infty$. To do this it suffices to show that the identity holds termwise:
\[
\lim_{M \to -\infty} k_{S,\partial_{k,II}(\Omega)}(z, w) K_{k}^{11}(z, w) = \sum_{n: \ell_{n,k} \in \partial_{k,\Omega}^{-\infty}} z^{n} w^{-n},
\]
for each fixed $k = 1, \ldots, d$. From (5.17) we see that
\[
k_{S,\partial_{k,II}(\Omega)}(z, w) = \sum_{n: \ell_{n,k} = M} z^{n} w^{-n},
\]
and, for a fixed $n' \in \mathbb{Z}^{d}$, if we set $n'(M) = \ell_{n',k} = M$ we have
\[
\lim_{M \to -\infty} X_{\partial_{k,II}(\Omega)}(M) z^{n'(M)} w^{-n'(M)} K_{k}^{11}(z, w) = \begin{cases}
0 & \text{if } \ell_{n',k} \in \partial_{k,\Omega}^{-\infty}, \\
(z^{k})^{\hat{n}^{k}} (w^{k})^{-\hat{n}^{k}} K_{-\infty,k}^{11}(z, w) & \text{if } \ell_{n',k} \notin \partial_{k,\Omega}^{-\infty}.
\end{cases}
\]
We conclude that (5.26) holds up to an interchange of the limit and summation signs. However, this interchange is justified by the monotone convergence theorem since, as we have observed in (5.25), the sequence $\{z^{M} w^{-M} K_{k}^{11}(z, w)\}$ converges monotonically (with positivity of a difference measured as kernel positivity) to $K_{-\infty,k}^{11}(z, w)$ as $M \to -\infty$. This completes the proof of the validity of the $(1,1)$-block entry in (5.7).

The $(1,2)$-entry of (5.7) can be verified by a completely parallel argument. In this case, however, it is automatic that
\[
\lim_{M \to -\infty} z^{M} w^{-M} K_{k}^{12}(z, w) = 0
\]
since $K_{k}^{12}(z, w)$ has the form
\[
K_{k}^{12}(z, w) = \sum_{n \in \mathbb{Z}_{+}^{d}, m \in -\mathbb{Z}_{+}^{d}} [K^{12}]_{n,m} z^{n} w^{-m}.
\]

For the $(2,1)$ and $(2,2)$ terms, one should define $\Omega_{b}^{M}$ to be the approximating backward shift-invariant sublattice
\[
\Omega_{b}^{M} = \{ n = (n_1, \ldots, n_d) \in \Omega: n_k \leq M \text{ for all } k = 1, \ldots, d \}
\]
and then take a limit as $M \to +\infty$. With these adjustments, the arguments for the $(2,2)$ and $(2,1)$ cases are exactly the same as the arguments for the $(1,1)$ and $(1,2)$ cases respectively. This completes the verification of all four entries of (5.7).
To verify (5.9), spell out (5.1) as
\[
\sum_{k=1}^{d} (1 - z_k w_k^{-1}) K_{k}^{11}(z, w) = I - S(z) S(w)^{*},
\]
\[
\sum_{k=1}^{d} (1 - z_k w_k^{-1}) K_{k}^{12}(z, w) = S(w) - S(z),
\]
\[
\sum_{k=1}^{d} (1 - z_k w_k^{-1}) K_{k}^{21}(z, w) = S(z)^{*} - S(w)^{*},
\]
\[
\sum_{k=1}^{d} (1 - z_k w_k^{-1}) K_{k}^{22}(z, w) = S(z)^{*} S(w) - 1.
\]
Then rearrange to get
\[
\sum_{k=1}^{d} K_{k}^{11}(z, w) + S(z) S(w)^{*} = \sum_{k=1}^{d} z_k w_k^{-1} K_{k}^{11}(z, w) + I
\]
\[
\sum_{k=1}^{d} K_{k}^{12}(z, w) + S(z) = \sum_{k=1}^{d} z_k w_k^{-1} K_{k}^{12}(z, w) + S(w),
\]
\[
\sum_{k=1}^{d} K_{k}^{21}(z, w) + S(w)^{*} = \sum_{k=1}^{d} z_k w_k^{-1} K_{k}^{21}(z, w) + S(z)^{*},
\]
\[
\sum_{k=1}^{d} K_{k}^{22}(z, w) + I = \sum_{k=1}^{d} z_k w_k^{-1} K_{k}^{22}(z, w) + S(z)^{*} S(w).
\]
This amounts to the spelling out of (5.9) as wanted. \(\square\)

Remark 5.5. For the case \(d = 1\), the limit (5.20) can be evaluated explicitly as follows. We note that \(K_{k}^{11}(z, w) = K^{11}(z, w)\) where
\[
K^{11}(z, w) = (1 - z w^{-1})^{-1} (I - S(z) S(w)^{*}) = \sum_{n=0}^{\infty} z^n w^{-n} - \left( \sum_{n=0}^{\infty} z^n w^{-n} \right) S(z) S(w)^{*}
\]
\[
= \sum_{n \geq 0} z^n w^{-n} - \sum_{\ell \geq j, i \geq 0} S_i S_j^* z^{i+j} w^{-j-\ell}
\]
\[
= \sum_{n \geq 0} z^n w^{-n} - \sum_{n, m \geq 0} \sum_{\ell = 0}^{\min \{n, m\}} S_{n-\ell} S_{m-\ell}^* z^n w^{-m}.
\]
We write
\[
K^{11}(z, w) = \sum_{n, m \geq 0} [K^{11}]_{n, m} z^n w^{-m} \text{ where } [K^{11}]_{n, m} = \delta_{n, m} - \sum_{\ell = 0}^{\min \{n, m\}} S_{n-\ell} S_{m-\ell}^*.
\]
Hence
\[
z^M K^{11}(z, w) w^{-M} = \sum_{n, m \geq M} K_{n-M, m-M}^{11} z^n w^{-m}
\]
where,
\[
[K^{11}]_{n-M,m-M} = \delta_{n-M,m-M} - \sum_{\ell=0}^{\min\{n-M,m-M\}} S_{n-M-\ell} S_{m-M-\ell}^*.
\]
For \(n \leq m\) we can then write
\[
[K^{11}]_{n-M,m-M} = \delta_{n,m} - \sum_{\ell=0}^{n-M} S_{\ell} S_{m-n+\ell}^*
\]
and we have
\[
\lim_{M \to -\infty} [K^{11}]_{n-M,m-M} = \delta_{n,m} - \sum_{\ell=0}^{\infty} S_{\ell} S_{m-n+\ell}^* \text{ if } n \leq m.
\]
Similarly we see that
\[
\lim_{M \to -\infty} [z^M K^{11}(z,w) w^{-M}]_{\alpha,\beta} = \delta_{\alpha,\beta} - \sum_{\ell=0}^{\infty} S_{n-m+\ell} S_{\ell}^* \text{ if } n > m.
\]
We note that the limiting matrix \([X]_{n,m} := \lim_{M \to -\infty} [K^{11}]_{n-M,m-M}\) is necessarily Toeplitz, i.e., the matrix entry \([X]_{n,m}\) depends only on the difference \(n - m\). One can say in general that the existence of the limit \(\lim_{M \to -\infty} [K^{11}]_{n-M,m-M}\) means that \([K^{11}]\) is asymptotically Toeplitz. The content of the assertion (5.20) is that the matrices \([K^{11}]_{n,m}\) associated with the kernels
\[
K^{11}_k(z,w) = \sum_{n,m \in \mathbb{Z}^d_+} [K^{11}]_{n,m} z^n w^{-m}
\]
are asymptotically Toeplitz in direction \(k\) for \(k = 1, \ldots, d\).

Our next goal is to show how the kernel decompositions (5.7), (5.8) and (5.9) associated with an augmented Agler decomposition (5.4) can be turned in the subspace decompositions (OD), (SL) and (SL∗), and (CDP) for a functional-model multievolution scattering system having scattering matrix \(S\). The formula (4.15) in Theorem 4.2 then gives rise to a GR-unitary realization for \(S\) and we have an alternative functional-model proof of \((2') \Rightarrow (3)\) in Theorem 1.2. We carry out the details of this procedure only for a more tractable special case. For this purpose we introduce the following notion of strictly closely connected augmented Agler decomposition. Section 6 below explores the meaning of a strictly closely connected Agler decomposition in terms of a GR-unitary realization \(U = [A B; C D]\) for the given augmented Agler decomposition.

**Definition 5.6.** Suppose that we are given an augmented Agler decomposition (5.4) for a given Schur-Agler-class function \(S(z)\). We say that this augmented Agler decomposition is strictly closely connected if both collections of reproducing kernel Hilbert spaces \((\mathcal{H}(K_1), \ldots, \mathcal{H}(K_d))\) and \((z_1 \mathcal{H}(K_1), \ldots, z_d \mathcal{H}(K_d))\) have no overlap, i.e., if both associated overlapping spaces (see Section 2.3.3) are trivial:
\[
\mathcal{L} := \mathcal{L}(K_1, \ldots, K_d) = \{0\} \quad \text{and} \quad \mathcal{L} := \mathcal{L}(K'_1, \ldots, K'_d) = \{0\}
\]
where we set \(K'_k(z,w) = z_k K(z,w) w^{-1}\) for \(k = 1, \ldots, d\).

The connection between the kernel CDP property (5.9) and the subspace CDP property (CDP) is particularly transparent for the case of strictly closely connected augmented Agler decompositions.
Lemma 5.7. Suppose that we are given a strictly closely connected augmented Agler decomposition (5.4) for a Schur–Agler class function $S \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$. Then the functional-model version of the property (CDP) holds, i.e., (CDP) holds with $L = \mathcal{H}(K_k), \mathcal{F} = \mathcal{H}(\left[ \begin{bmatrix} S(z) \end{bmatrix} \right] \left[ S(w)^* I \right])$, $\mathcal{F}_* = \mathcal{H}(\left[ \begin{bmatrix} I \\ S(z)^* \end{bmatrix} \right] \left[ I S(w)^* \right])$ and with $\mathcal{U}_k$ equal to the multiplication operator $M_\Omega$:

$$
\bigoplus_{k=1}^{d} \mathcal{H}(K_k) \oplus \mathcal{H}(\left[ \begin{bmatrix} I \\ S(z) \end{bmatrix} \right] \left[ S(w)^* I \right]) = \bigoplus_{k=1}^{d} z_k \mathcal{H}(K_k) \oplus \mathcal{H}(\left[ \begin{bmatrix} I \\ S(z)^* \end{bmatrix} \right] \left[ I S(w)^* \right]).
$$

(5.27)

Proof. From the decomposition (5.7) we read off that

$$
\text{span}_{1 \leq k \leq d} \mathcal{H}(K_k) \subset \mathcal{H}(K_{\text{fin}}^\mathcal{V}_{\text{fin}}) \cap \mathcal{H}(K_{\text{fin}}^\mathcal{V}_{\text{fin}})
$$

if $\Omega$ is a shift-invariant sublattice with $0 \in \partial_k \Omega_{\text{fin}}$ for each $k = 1, \ldots, d$. From (5.10) combined with the fourth of the identities (3.8), we see that it is always the case that

$$
\text{span}_{1 \leq k \leq d} \mathcal{H}(K_k) \cap \mathcal{H}(\left[ \begin{bmatrix} S(z) \end{bmatrix} \right] \left[ S(w)^* I \right]) = \{0\}.
$$

Similarly, by choosing $\Omega'$ with $e_k \in \partial_k \Omega_{\text{fin}}$ for $k = 1, \ldots, d$, one can read off that it is always the case that

$$
\text{span}_{1 \leq k \leq d} z_k \mathcal{H}(K_k) \cap \mathcal{H}(\left[ \begin{bmatrix} I \\ S(z)^* \end{bmatrix} \right] \left[ I S(w)^* \right]) = \{0\}.
$$

Hence if $\{K_k(z, w); k = 1, \ldots, d\}$ is a strictly closely connected augmented Agler decomposition, we conclude that (5.27) holds as wanted.

We next introduce the notion of minimal augmented Agler decomposition.

Definition 5.8. Let $\Omega$ be a given shift-invariant sublattice of $\mathbb{Z}^d$. We say that the augmented Agler decomposition (5.1) is minimal if both of the following conditions hold:

1. $\{K_k\}$ is strictly closely connected (see Definition 5.6).
2. The overlapping space $\mathcal{L}_{\partial_k \mathcal{V}_{\text{fin}}}^\mathcal{V}_{\text{fin}} \mathcal{V}_{\text{fin}}$ associated with the sum-decomposition of $K_{\mathcal{V}_{\text{fin}}^\mathcal{V}_{\text{fin}}}$ in (5.4) is trivial, i.e., the collection of subspaces $z^n \mathcal{H}(K_k)$ for $n \in \partial_k \Omega_{\text{fin}}$,

$$
(\mathbb{Z}^k)^n \mathcal{H}(\left[ \begin{bmatrix} 0 \\ 0 \\ K_{\mathcal{V}_{\text{fin}}^\mathcal{V}_{\text{fin}}} \end{bmatrix} \right])
$$

for $\ell_{n', k} \in \partial_k \Omega_{\text{fin}}$.

$$(\mathbb{Z}^k)^n \mathcal{H}(\left[ \begin{bmatrix} K_{\mathcal{V}_{\text{fin}}^\mathcal{V}_{\text{fin}}} \end{bmatrix} \right])
$$

for $\ell_{n', k} \in \partial_k \Omega_{\text{fin}}$ form a direct-sum decomposition of $\mathcal{H}(K_{\mathcal{V}_{\text{fin}}^\mathcal{V}_{\text{fin}}})$.

We shall see from Corollary 5.11 below that condition (1) in Definition 5.8 follows automatically from condition (2) in case all finite components $\partial_k \Omega_{\text{fin}}$ of the boundary of $\Omega$ are nonempty. On the other hand, we shall see as a consequence of Example 6.9 and Remark 6.12 below that condition (1) in general does not imply condition (2). At this stage it is possible to give the following explicit reproducing-kernel-space construction for a scattering system realizing a given Agler decomposition under this minimality assumption.

Theorem 5.9. (1) The notion of minimal augmented Agler decomposition is independent of the choice of shift-invariant sublattice.
(2) Assume that $S \in \mathcal{S}(\mathcal{E}, \mathcal{E}_*)$ is a Schur-Agler-class function with augmented Agler decomposition which is minimal (see Definition 5.8) with respect to the shift-invariant sublattice $\Omega$. Let $\mathcal{S}_{dBR}^S$ be the associated de Branges–Rovnyak model multievolution scattering system with scattering matrix $S$ (see (3.3)). Define subspaces of $K_{dBR}^S$ according to

$$\mathcal{L}_k = \mathcal{H}(K_k) \text{ for } k = 1, \ldots, d,$$

$$\mathcal{M}_k = \mathcal{H}\left(\begin{bmatrix} 0 & 0 \\ 0 & K_{+\infty,k} \end{bmatrix}\right)$$

$$\mathcal{M}_{+k} = \mathcal{H}\left(\begin{bmatrix} [K_{-\infty,k}] & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Then the collection of spaces $\mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_{+k}$ ($k = 1, \ldots, d$) satisfies properties (OD), (SL), (SL)* and (CDP). The associated GR-unitary colligation $U$ given by (4.11) can be written explicitly as

$$U_{dBR}^S = \begin{bmatrix} A_{dBR,11}^S & \cdots & A_{dBR,1d}^S & B_{dBR,1}^S \\ \vdots & \ddots & \vdots & \vdots \\ A_{dBR,d1}^S & \cdots & A_{dBR,dd}^S & B_{dBR,d}^S \\ C_{dBR,1}^S & \cdots & C_{dBR,d}^S & D \end{bmatrix} \begin{bmatrix} \mathcal{H}(K_1) \\ \vdots \\ \mathcal{H}(K_d) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}(K_1) \\ \vdots \\ \mathcal{H}(K_d) \end{bmatrix}$$

where

$$A_{ij} \begin{bmatrix} f_j \\ g_j \end{bmatrix} = P_{\mathcal{H}(K_i)} \begin{bmatrix} z_i^{-1} f_j(z) \\ z_i^{-1} g_j(z) \end{bmatrix}, \quad B_i e = P_{\mathcal{H}(K_i)} \begin{bmatrix} z_i^{-1} S(z) e \\ z_i^{-1} e \end{bmatrix},$$

$$C_j \begin{bmatrix} f_j \\ g_j \end{bmatrix} = f_{j,0}, \quad D e = S(0) e$$

for $i, j = 1, \ldots, d$, $\begin{bmatrix} f_j \\ g_j \end{bmatrix} \in \mathcal{H}(K_j)$ and $e \in \mathcal{E}$.

**Proof.** Let $\mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_{+k}$ be defined as in (5.28). By Lemma 5.7 we know that the property (CDP) holds. Then the import of the kernel identity (5.7) combined with condition (2) in Definition 5.8 is that the space $\mathcal{H}(K_{\Omega,dBR}^S)$ has the internal orthogonal decomposition

$$\mathcal{H}(K_{\Omega,dBR}^S) = \bigoplus_{k=1}^d \bigoplus_{n \in \partial_K \Omega_{f,n}} z^n \mathcal{H}(K_k) \oplus \bigoplus_{k=1}^d \bigoplus_{\ell, n, \tilde{\ell} \in \partial_K \Omega_{+\infty}} \begin{bmatrix} 0 & 0 \\ 0 & (z^{k})^{n-\tilde{\ell}} \mathcal{H}(K_{+\infty,k}) \end{bmatrix}$$

$$\oplus \bigoplus_{k=1}^d \bigoplus_{\ell, n, \tilde{\ell} \in \partial_K \Omega_{-\infty}} \begin{bmatrix} (z^{k})^{n-\tilde{\ell}} \mathcal{H}(K_{-\infty,k}) \\ 0 \\ 0 \end{bmatrix}.$$
Finally, the fact that the notion of minimal Agler decomposition is independent of the choice of shift-invariant sublattice follows from the general fact from [14] that the property (OD) is independent of the choice of shift-invariant sublattice \( \Omega \) once it is established that the multievolution scattering system arises from an embedded GR-unitary colligation. \( \square \)

Remark 5.10. We note that the functional-model unitary colligation (5.29) constructed from a minimal Agler decomposition \( \{ K_k(z, w): k = 1, \ldots, d \} \) as in Theorem 5.9 is scattering-minimal (see the end of Theorem 4.2) since the de Branges–Rovnyak model multievolution scattering system \( \mathcal{S}_{\text{dBR}}^S \) is a minimal scattering system. We do not know in general if every Schur-Agler-class function has a minimal augmented Agler decomposition.

The next two subsections give independent approaches for computing explicitly the kernels at infinity \( K_{11}^{-\infty, k}(z, w) \) and \( K_{22}^{+\infty, k}(z, w) \) in terms of the coefficients \( A, B, C, D \) of a conservative Givone–Roesser realization of \( S(z) \). These two subsections (Sections 5.1 and 5.2 below) can be omitted on first reading without loss of continuity.

5.1. **Direct computation of \( K_{11}^{-\infty, k} \) in terms of \( A, B, C, D \).** Assuming a conservative Givone–Roesser realization for \( S(z) \), we can identify the kernels at infinity \( K_{11}^{-\infty, k}(z, w) \) and \( K_{22}^{+\infty, k}(z, w) \) more explicitly in terms of \( A, B, C, D \) as follows. We consider only the case \( K_{11}^{-\infty, k}(z, w) \) as \( K_{22}^{+\infty, k}(z, w) \) is similar. A useful notation is the Givone–Roesser functional calculus on a block matrix

\[
A = \begin{bmatrix}
A_{11} & \cdots & A_{1d} \\
\vdots & \ddots & \vdots \\
A_{d1} & \cdots & A_{dd}
\end{bmatrix}
\]

given by

\[
A^w = P_{i_N}A P_{i_{N-1}}A \cdots P_{i_1}A \text{ if } w = i_N i_{N-1} \cdots i_1 \in \mathcal{F}_d
\]

where \( \mathcal{F}_d \) denotes the free semigroup consisting of all finite words \( i_N \cdots i_1 \) in the letters \( \{1, \ldots, d\} \) (so \( i_\ell \in \{1, \ldots, d\} \) for each \( \ell = 1, \ldots, N \)). It is useful to introduce the abelianization map \( a: \mathcal{F}_d \to \mathbb{Z}_d^+ \) given by

\[
a(i_N \cdots i_1) = (n_1, \ldots, n_d) \text{ if } n_k = \#\{ \ell: i_\ell = k \} \text{ for } k = 1, \ldots, d
\]

(5.31)

together with the abelianized Givone–Roesser functional calculus

\[
(A^a)^n = \sum_{w \in \mathcal{F}_d: a(w) = n} A^w.
\]

(5.32)

Then we have the formal power series expansion

\[
C(I - Z_{\text{diag}}(z)A)^{-1} = \sum_{n \in \mathbb{Z}_d^+} (A^a)^n z^n.
\]

(5.5)

From (5.5) we know that

\[
K_{11}^{\pm \infty, k}(z, w) = C(I - Z_{\text{diag}}(z)A)^{-1} P_k(I - A^* Z_{\text{diag}}(w)^{-1})^{-1} C^*.
\]
Therefore
\[
K_{-\infty,k}^{11}(z,w) = \lim_{t \to -\infty} z_k^t w_k^{-t} K_k^{11}(z,w)
\]
\[
= \lim_{t \to -\infty} z_k^t w_k^{-t} C(I - Z_{\text{diag}}(z)A)^{-1} P_k(I - A^* Z_{\text{diag}}(z)^{-1}) C^*
\]
\[
= \lim_{t \to -\infty} \sum_{n,m \in \mathbb{Z}^d_+} C(A^n) P_k(A^{sn})^m C^* z_k^n w_k^{-m-t}
\]
\[
= \sum_{\alpha,\beta \in \mathbb{Z}^d: (\hat{\alpha}^k), (\hat{\beta}^k) \in \mathbb{Z}^d_{+-1}} \left[ \lim_{t \to -\infty} C(A^k)^{\alpha-\text{te}_k} P_k(A^{sn})^{\beta-\text{te}_k} C^* \right] z^{\alpha} w^{-\beta}.
\]
i.e.,
\[
K_{-\infty,k}^{11}(z,w) = \sum_{\alpha,\beta \in \mathbb{Z}^d: (\hat{\alpha}^k), (\hat{\beta}^k) \in \mathbb{Z}^d_{+-1}} [K_{-\infty,k}^{11}]_{\alpha,\beta} z^{\alpha} w^{-\beta}
\]
where
\[
[K_{-\infty,k}^{11}]_{\alpha,\beta} = \lim_{t \to -\infty} C(A^k)^{\alpha-\text{te}_k} P_k(A^{sn})^{\beta-\text{te}_k} C^*.
\]
(5.33)

In case \((\hat{\alpha}^k) = (\hat{\beta}^k) = 0 \in \mathbb{Z}^d_{+-1}\), formula (5.33) can be simplified somewhat:
\[
[K_{-\infty,k}^{11}]_{\alpha,\beta} = \begin{cases} 
C_k \Delta A^s_{kk} A^{\beta-\alpha} C_k & \text{if } \alpha_k \leq \beta_k \\
C_k A^{\alpha-\beta} \Delta A^s_{kk} C_k & \text{if } \alpha_k > \beta_k
\end{cases}
\]
where we have set
\[
\Delta A^s_{kk} = \text{strong limit } s \to +\infty A^s_{kk} A^{s*}_{kk}.
\]
In particular, this formula applies for the case \(d = 1\) and gives the kernel associated with the Sz.-Nagy–Foiaş defect space \(R_\ast\) (see [13] page 139).

5.2. Computation of \(K_{-\infty,k}^{11}\) via solution of initial value problem. Suppose that \(\Omega\) is a shift-invariant sublattice such that \(\ell_{0,k}\) is part of \(\partial_k \Omega_{-\infty}\). Theorem 4.15 in [13] indicates how to construct a trajectory \(\xi = (u, x, y)\) by specifying initial condition \(x^0\) for \(x\) on \(\partial \Omega\), a future input \(u^+ \in \ell^2(\Omega, E)\) and a past output \(y^- \in \ell^2(\mathbb{Z}^d \setminus \Omega, E)\). In particular, if we specify
\[
u^+ = 0, \quad y^- = 0, \quad x^0 |_{\partial_j \Omega_{tt}} = 0 \quad \text{and} \quad x^0 |_{\partial_j \Omega_{+\infty}} = 0 \quad \text{for all } j = 1, \ldots, d,
\]
\[
u^0 |_{\partial_j \Omega_{\infty}} = 0 \quad \text{for } j \neq k,
\]
\[
u^0 |_{\partial_n, k} = 0 \quad \text{for } n, k \in \partial_k \Omega_{-\infty} \text{ with } n' \neq 0,
\]
\[
u^0 |_{\partial_0, k} = \bar{h}
\]
where \(\bar{h}\) is a prespecified element of the defect space
\[
R_{\ast k} = \{ \bar{h} = (h(t))_{t \in \mathbb{Z}} \in \ell(\mathbb{Z}, H_k): h(t + 1) = A_{kk} h(t) \}
\]
and \(\|\bar{h}\|^2_{R_{\ast k}} := \lim_{t \to -\infty} \|h(t)\|^2_{H_k} < \infty\),
then the solution of the initial value problem is given by \(\xi = (u, x, y)\) where
\[
\hat{u}(z) = 0,
\]
\[
\hat{x}(z) = i_k \hat{h}^k(z) - (I - Z_{\text{diag}}(z)A)^{-1}(i_k \hat{h}^k(z_k) - Z_{\text{diag}}(z) A_{kk} \hat{h}^k(z_k),
\]
\[
\hat{y}(z) = C_k \hat{h}^k(z) - C(I - Z_{\text{diag}}(z)A)^{-1}(i_k \hat{h}^k(z_k) - Z_{\text{diag}}(z) A_{kk} \hat{h}^k(z_k).
where \( i_k \) is the injection

\[
i_k : h_k \mapsto \begin{bmatrix} 0 \\ \vdots \\ h_k \\ \vdots \\ 0 \end{bmatrix}
\]

of \( H_k \) into the \( k \)-th component of \( \bigoplus_{j=1}^d H_d \) and where we have set

\[
\hat{h}^k(z_k) = \sum_{t=-\infty}^{+\infty} h(t)z_k^t.
\]

The space \( \mathcal{H}(K_{-\infty,k}^{11}) \) is the first component of the image under the de Branges-Rovnyak map \( \Pi_{dBR,1}^{\mathcal{H}} \) of all such trajectories, where, in general, the de Branges–Rovnyak map is given by

\[
\Pi_{dBR,1}^{\mathcal{H}} : (u, x, y) \mapsto \begin{bmatrix} \hat{y}(z) \\ \hat{u}(z) \end{bmatrix}.
\]

We thus see that \( \mathcal{H}(K_{-\infty,k}^{11}) \) is the image of the map giving rise to the \( Z \)-transformed output signal \( \hat{y}(z) \) from an initial condition \( \vec{h} \in \mathbb{R}^k \), i.e.,

\[
\mathcal{H}(K_{-\infty,k}^{11}) = \text{im} \Pi_{-\infty,k}^{dBR,1},
\]

where \( \Pi_{-\infty,k}^{dBR,1} : \mathbb{R}^k \rightarrow \mathbb{E} \) is given by

\[
\Pi_{-\infty,k}^{dBR,1} : \vec{h} \mapsto C_{k} \hat{h}^k(z_k) - C(I - Z_{\text{diag}}(z)A)^{-1}(i_k \hat{h}^k(z_k) - Z_{\text{diag}}(z)A i_k \hat{h}^k(z_k)) \quad (5.34)
\]

and where the norm on \( \mathcal{H}(K_{-\infty,k}^{11}) \) is equal to the lifted norm from \( \mathbb{R}^k \). If we define an element \( \Pi_{-\infty,k}^{dBR,1} (z) \in \mathcal{L}(\mathcal{R}_k, \mathcal{E}_*)[[z^{\pm 1}]] \) by

\[
\Pi_{-\infty,k}^{dBR,1} (z)(\vec{h}) = \left( \Pi_{-\infty,k}^{dBR,1} (\vec{h}) \right) (z).
\]

it then follows from the results in Section 2.3.1 that the kernel \( K_{-\infty,k}^{11} \) can alternatively be computed as

\[
K_{-\infty,k}^{11}(z, w) = \Pi_{-\infty,k}^{dBR,1}(z) \left( \Pi_{-\infty,k}^{dBR,1}(w) \right)^*.
\]

In this subsection we show how this formula for \( K_{-\infty,k}^{11}(z, w) \) is consistent with (5.33). We shall use the notation \((A^a)^m\) introduced in Section 5.1 (see displays (5.31) and (5.32)).

We first verify the following.

**Proposition 5.11.** The coefficients \([\Pi_{-\infty,k}^{dBR,1}]_m\) of the formal power series

\[
\Pi_{-\infty,k}^{dBR,1}(z) = \sum_{m \in \mathbb{Z}^d} [\Pi_{-\infty,k}^{dBR,1}]_m z^m
\]

defined in (5.34) and (5.35) are given by

\[
[\Pi_{-\infty,k}^{dBR,1}]_m = \begin{cases} 0 & \text{if } \hat{m}^k \notin \mathbb{Z}^{d-1}, \\
\lim_{T \to -\infty} C(A^a)^{m - T e_k} i_k h(T) & \text{if } \hat{m}^k \in \mathbb{Z}^{d-1}.
\end{cases}
\]

(5.37)
with, for given \(e_s \in \mathcal{E}_s\), adjoint given by

\[
\Pi^{dBR,1}_{-\infty,k}^*e_s = 0 \text{ if } \bar{m}_k \notin \mathbb{Z}^{d-1},
\]

\[
\Pi^{dBR,1}_{-\infty,k}^*e_s = \bar{h} \text{ where } \bar{h} \text{ is the unique element of } \mathcal{R}_{sk} \text{ such that }
\]

\[
\bar{h} \sim \{P_k(A^*)^{m_{-t}e_k}C^*e_s\}_{t \leq m_k} \text{ if } \bar{m}_k \in \mathbb{Z}^{d-1}.
\]

(5.38)

Here \(\bar{h} \sim \{g(t)\}_{t \leq m_k}\) means that \(\bar{h}\) and \(\{g(t)\}_{t \leq m_k}\) have the same asymptotics at \(-\infty\) in the sense that

\[
\lim_{t \to -\infty} (f(t), h(t) - g(t))_{\mathcal{H}_k} = 0 \text{ for all } f = \{f(t)\}_{t \in \mathbb{Z}} \in \mathcal{R}_{sk}.
\]

In case \(\bar{m}_k = 0\), the formula for \(\Pi^{dBR,1}_{-\infty,k}^*e_s\) can be given explicitly as

\[
\Pi^{dBR,1}_{-\infty,k}^*e_s = \bar{h} = \{h(t)\}_{t \in \mathbb{Z}} \in \mathcal{R}_{sk} \text{ where }
\]

\[
h(t) = \begin{cases} 
\Delta A_{kk}^s A_{kk}^{m_k-t} C_k^*e_s & \text{if } t \leq m_k \\
A_{kk}^{t-m_k} \Delta A_{kk}^s C_k^*e_s & \text{if } t > m_k.
\end{cases}
\]

(5.39)

where we have set

\[
\Delta A_{kk}^s = \text{strong limit}_{s \to +\infty} A_{kk}^s A_{kk}^{*s}.
\]

Proof. We compute

\[
i_k \bar{h}(z_k) - Z_{\text{diag}}(z) \mathcal{A} \bar{h}(z_k) = \sum_{t = -\infty}^{+\infty} i_k h(t) z_k^t = \sum_{j = 1}^{d} z_j P_j A \left( \sum_{t = -\infty}^{+\infty} i_k h(t) z_k^t \right)
\]

\[
= \sum_{t = -\infty}^{+\infty} i_k h(t) z_k^t - \sum_{j \neq k} z_j P_j A \left( \sum_{t = -\infty}^{+\infty} i_k h(t) z_k^t \right)
\]

\[
= \sum_{t = -\infty}^{+\infty} i_k h(t) z_k^t - \sum_{t = -\infty}^{+\infty} i_k h(t+1) z_k^{t+1} - \sum_{j \neq k} z_j P_j A \left( \sum_{t = -\infty}^{+\infty} i_k h(t) z_k^t \right)
\]

(5.40)

(where we use that \(h(t+1) = A_{kk} h(t)\) since \(\bar{h} \in \mathcal{R}_{sk}\))

\[
= - \sum_{j \neq k} \sum_{t = -\infty}^{+\infty} P_j A i_k h(t) z_j z_k^t.
\]

Thus

\[
\Pi^{dBR,1}_{-\infty,k}(z) = C_k \bar{h}(z_k) - (I - Z_{\text{diag}}(z) \mathcal{A})^{-1}(i_k \bar{h}(z_k) - Z_{\text{diag}}(z) \mathcal{A} \bar{h}(z_k))
\]

\[
= \sum_{t = -\infty}^{+\infty} C_k h(t) z_k^t + \sum_{n \in \mathbb{Z}_d^+} \sum_{j \neq k} \sum_{t = -\infty}^{+\infty} C(A^n)^n P_j A i_k h(t) z^n z_j z_k^t
\]

\[
= \sum_{m \in \mathbb{Z}_d^+} \Pi^{dBR,1}_{-\infty,k}^m \bar{h}^m.
\]
where we have set $[\Pi_{-\infty,k}^{dBR}]_m \vec{h}$ equal to
\[
0 \text{ if } \hat{m}^k \notin \mathbb{Z}_+^{d-1}, \quad C_k h(m_k) \text{ if } \hat{m}^k = 0,
\]
\[
\sum_{t=-\infty}^{m_k} \sum_{j: j \neq k, m_j \geq 1} C(A^a)^{m-e_j} P_j A_i k h(t) \quad \text{if } \hat{m}^k \in \mathbb{Z}_+^{d-1} \setminus \{0\}. \tag{5.40}
\]
For the case where $\hat{m}^k \in \mathbb{Z}_+^{d-1} \setminus \{0\}$, we may simplify the formula as follows:
\[
[\Pi_{-\infty,k}^{dBR}]_m = \lim_{T \to -\infty} \sum_{t=T}^{m_k} \sum_{j: j \neq k, m_j \geq 1} C(A^a)^{m-e_j} P_j A_i k h(t)
\]
\[
= \lim_{T \to -\infty} \left[ \sum_{t=T}^{m_k} C(A^a)^{m-e_k} i_k h(t) - \sum_{t=T}^{m_k-1} C(A^a)^{m-(t+1)e_k} P_k A_i k h(t) \right]
\]
\[
= \lim_{T \to -\infty} \left[ \sum_{t=T}^{m_k} C(A^a)^{m-te_k} i_k h(t) - \sum_{t=T}^{m_k-1} C(A^a)^{m-(t+1)e_k} i_k h(t + 1) \right]
\]
\[
= \lim_{T \to -\infty} C(A^a)^{m-te_k} i_k h(T). \tag{5.41}
\]
For the case where $\hat{m}^k = 0$, this formula makes sense and can be simplified further as follows. Since
\[
(A^a)^{m-te_k} i_k h(T) = (P_k A)^{m_k-te_k} i_k h(T) = i_k h(m_k - T + T) = i_k h(m_k),
\]
the formula (5.41) in this case collapses to $C_k h(m_k)$ agreeing with the formula for $[\Pi_{-\infty,k}^{dBR}]_m$ in (5.40) for this case. Thus the formula (5.40) can be simplified to
\[
[\Pi_{-\infty,k}^{dBR}]_m = \begin{cases} 0 & \text{if } \hat{m}^k \notin \mathbb{Z}_+^{d-1}, \\ \lim_{T \to -\infty} C(A^a)^{m-te_k} i_k h(T) & \text{if } \hat{m}^k \in \mathbb{Z}_+^{d-1}. \end{cases}
\]
agreeing with the formula (5.39) in the statement of the proposition. (If we interpret $(A^a)^n = 0$ whenever $n \notin \mathbb{Z}_+$, then the second formula in (5.39) in fact suffices for all cases.)

If $\hat{m}^k \notin \mathbb{Z}_+^{d-1}$, then trivially $[\Pi_{-\infty,k}^{dBR}]_m = 0$. For the case where $\hat{m}^k \in \mathbb{Z}_+^{d-1}$, the element $\vec{h} = [\Pi_{-\infty,k}^{dBR}]_m e_s$ is by definition the necessarily unique element of $\mathcal{R}_{s_k}$ such that
\[
\langle [\Pi_{-\infty,k}^{dBR}]_m \vec{g}, e_s \rangle_{\mathcal{E}_s} = \langle \vec{g}, \vec{h} \rangle_{\mathcal{R}_{s_k}},
\]
or, equivalently, such that
\[
\left\langle \lim_{T \to -\infty} C(A^a)^{m-te_k} P_k g(T), e_s \right\rangle_{\mathcal{E}_s} = \lim_{T \to -\infty} \langle g(T), h(T) \rangle_{\mathcal{H}_k}
\]
which in turn is equivalent to
\[
\lim_{T \to -\infty} \langle g(T), P_k (A^a)^{m-te_k} C^* e_s \rangle = \lim_{T \to -\infty} \langle g(T), h(T) \rangle_{\mathcal{H}_k}
\]
for all $\vec{g} = \{g(t)\}_{t \in \mathbb{Z}} \in \mathcal{R}_{s_k}$. We conclude that $\vec{h} = \{h(t)\}_{t \in \mathbb{Z}} \in \mathcal{R}_{s_k}$ is uniquely determined by the constraint that
\[
\vec{h} \sim \{P_k (A^a)^{m-te_k} C^* e_s \}_{t \leq m_k}
\]
as asserted in (5.38).
We compute this case we have the simplification

\[ \langle A^{*n} \rangle^{m-T} e_s C^* e_s = i_k A_k^{*m_k-t} C_k^* e_s \text{ for } t \leq m_k. \]

For convenience, let us set

\[ h_{\text{pre}}(t) = \begin{cases} 
A_k^{*m_k-t} C_k^* e_s & \text{if } t \leq m_k, \\
A_k^{-m_k} C_k^* e_s & \text{if } t > m_k.
\end{cases} \]

This candidate for \( [\Pi_{\text{pre}}] \) satisfies the recursion

\[ h_{\text{pre}}(t+1) = A_k h(t) \text{ for all } t \in \mathbb{Z}. \]

We check that \( h = \{h(t)\}_{t \in \mathbb{Z}} \) given as in (5.43)

\[ h(t) = \begin{cases} 
\Delta A_k^{*m_k-t} C_k^* e_s & \text{if } t \leq m_k, \\
A_k^{-m_k} \Delta A_k^{*m} & \text{if } t > m_k,
\end{cases} \]

maintains the same asymptotics at \(-\infty\) and satisfies the recurrence relation. Indeed, from the fundamental identity

\[ A_k \Delta A_k^{*m} A_k^{*m} = \Delta A_k^{*m} \] (5.42)

it is easily checked that \( h \) satisfies the recursion

\[ h(t + 1) = A_k h(t). \]

As \( 0 \leq \Delta A_k^{*m} \leq I, \Delta A_k^{*m} \) has a nonnegative square root \( \langle \Delta A_k^{*m} \rangle^{1/2} \). Moreover, the computation

\[ \|h(t)\|^2 \leq \|\Delta A_k^{*m} \|^2 \cdot \|\Delta A_k^{*m-t} C_k^* e_s \|^2 \]

\[ \leq \|\Delta A_k^{*m} \| \cdot \sup_{s \in \mathbb{Z}^+} \|A_k^{*m} \Delta A_k^{*m} C_k^* e_s, C_k^* e_s \| \]

\[ = \|\Delta A_k^{*m} \| \cdot \langle \Delta A_k^{*m} C_k^* e_s, C_k^* e_s \rangle \] (using (5.42))

\[ \leq \|\Delta A_k^{*m} \| \cdot \|\Delta A_k^{*m} C_k^* e_s \|^2 < \infty \]

shows that \( \hat{h} \in \mathcal{R}_{\star,k} \).

It remains to show that \( \hat{h} \sim \hat{h}_{\text{pre}}, \) i.e., that, for all \( f \in \mathcal{R}_{\star,k} \) we have

\[ \lim_{t \to -\infty} \langle h_{\text{pre}}(t) - h(t), f(t) \rangle_{\mathcal{H}_k} := \lim_{t \to -\infty} \langle (I - \Delta A_k^{*m}) A_k^{*m-t} C_k^* e_s, f(t) \rangle_{\mathcal{H}_k} = 0. \] (5.43)

As \( \|f(t)\| \) is bounded as \( t \to -\infty \), so also is \( \|(I - \Delta A_k^{*m})^{1/2} f(t)\| \). By the Cauchy-Schwarz inequality, (5.43) follows if we show

\[ \lim_{s \to +\infty} \|(I - \Delta A_k^{*m})^{1/2} A_k^{*s} C_k^* e_s \| = 0. \] (5.44)

We compute

\[ \lim_{s \to +\infty} \|(I - \Delta A_k^{*m})^{1/2} A_k^{*s} C_k^* e_s \|^2 = \lim_{s \to +\infty} \left\langle A_k^{*s} (I - \Delta A_k^{*m}) A_k^{*s} C_k^* e_s, C_k^* e_s \right\rangle_{\mathcal{H}_k} \]

\[ = \lim_{s \to +\infty} \left\langle A_k^{*s} (I - \Delta A_k^{*m}) A_k^{*s} C_k^* e_s, C_k^* e_s \right\rangle_{\mathcal{H}_k} - \left\langle \Delta A_k^{*m} C_k^* e_s, C_k^* e_s \right\rangle_{\mathcal{H}_k} \] (using (5.42) again)

\[ = 0 \]

and (5.44) follows as needed. This completes the proof of all parts of Proposition 5.11. \(\square\)
Remark 5.12. The expressions \((5.40)\) and \((5.37)\) represent the two distinct regularization methods used in \([13]\). The proof above that \((5.40) \Rightarrow (5.37)\) gives the connection between these two regularization methods which was left implicit in \([13]\).

Remark 5.13. We conjecture that an explicit formula for \([\Pi_dBR,1]_{\alpha,\beta}\) is
\[
[\Pi_dBR,1]_{\alpha,\beta} e_\ast = \lim_{T \to -\infty} \left( P_k A \right)^{t-T} P_k (A^* a)^{m-Te_k} C^* e_\ast
\]
but we have not been able to verify that this expression has the required asymptotics at \(-\infty\).

With the information from Proposition \([5.11]\) in hand combined with the formula \((5.36)\), we may compute the coefficients \([K_{-\infty,k}]_{\alpha,\beta}\) for the kernel \(K_{-\infty,k}(z, w)\) as follows. For \(e_\ast \in \mathcal{E}_\ast\) and \(\beta \in \mathbb{Z}^d\) with \((\beta^k) \in \mathbb{Z}_+^{d-1}\), set
\[
h(t) = [\Pi_{-\infty,k}]_{\beta} e_\ast, \quad h_{\text{pre}}(t) = P_k (A^* a)^{m-t e_k} C^* e_\ast
\]
so \(h \sim h_{\text{pre}}\). Then, at least at a formal level, using that
\[
\lim_{T \to -\infty} h(T) = \lim_{T \to -\infty} h_{\text{pre}}(T),
\]
we then have, for \(\alpha, \beta \in \mathbb{Z}^d\) with \((\alpha^k), (\beta^k) \in \mathbb{Z}_+^{d-1}\),
\[
[K_{-\infty,k}]_{\alpha,\beta} e_\ast = [\Pi_{-\infty,k}]_{\alpha,\alpha} [\Pi_{-\infty,k}]_{\beta} e_\ast
\]
\[
= \lim_{T \to -\infty} C(A^* a)^{T e_k} h(T)
\]
\[
= \lim_{T \to -\infty} C(A^* a)^{T e_k} h_{\text{pre}}(T)
\]
\[
= \lim_{T \to -\infty} C(A^* a)^{T e_k} P_k (A^* a)^{\beta - T e_k} C^* e_\ast
\]
which agrees with the formula \((5.33)\) for \([K_{-\infty,k}]_{\alpha,\beta}\). This concludes our computation of \(K_{-\infty,k}(z, w)\) via the formula for solution of the initial value problem from \([13]\).

6. Finer structure of GR-unitary colligations

In this section it will be convenient to work with a Givone–Roesser unitary colligation in more coordinate-free form. By a Givone–Roesser unitary colligation in coordinate-free form, we mean a unitary operator \(U\) of the form
\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : [\mathcal{H}] \to [\mathcal{H}]
\]
(6.1)
together with a collection \(\{P_1, \ldots, P_d\}\) of orthogonal projections on \(\mathcal{H}\) with pairwise-orthogonal ranges having span equal to the whole space \(\mathcal{H}\). Given such a coordinate-free Givone–Roesser unitary colligation, the associated transfer function is
\[
S(z) = D + C(I_{\mathcal{H}} - Z_{\text{diag}}(z) A)^{-1} Z_{\text{diag}}(z) B
\]
where now \(Z_{\text{diag}}(z)\) stands for the operator-pencil
\[
Z_{\text{diag}}(z) = z_1 P_1 + \cdots + z_d P_d.
\]
with associated system equations
\[
\Sigma(U) : \begin{cases} P_k x(s_k(n)) = P_k A x(n) + P_k B u(n) & \text{for } k = 1, \ldots, d, \\
y(n) = C x(n) + D u(n). \end{cases}
\]
(6.2)
We consider here only the conservative case corresponding to the condition that $U$ be unitary. This of course is equivalent to our original definition (1.3) of Givone–Roesser unitary colligation after specification of the block basis for $\mathcal{H}$ associated with the family of orthogonal projections $\{P_1, \ldots, P_d\}$. This coordinate-free version gives a little more flexibility in the construction of examples.

Given a Givone–Roesser unitary colligation $U$, we already introduced the notion of scattering-minimal at the end of Theorem 4.2. We have the following additional notions of minimality for a GR-unitary colligation.

**Definition 6.1.**

1. We say that the GR-unitary colligation $U$ as in (6.1) is closely connected if $\mathcal{H}$ is equal to the smallest subspace of $\mathcal{H}$ containing $\text{im} B$ and $\text{im} C^*$ which is invariant for $A$, $A^*$ and $P_k$ for all $k = 1, \ldots, d$.

2. We say that $U$ is strictly closely connected if the restriction of the de Branges–Rovnyak identification map to $\text{im} i_\mathcal{H} = \mathcal{L}$ in the multievolution scattering system in which $U$ is embedded (see Section 4) is injective, or, more concretely, the map $\Pi^{dBR}_U: \mathcal{H} \to (\mathcal{E}_s \oplus \mathcal{E})[[z^{\pm 1}]]$ given by

$$\Pi^{dBR}_U: h \mapsto \begin{bmatrix} C(I - Z_{\text{diag}}(z)A)^{-1} \\ B^* (I - Z_{\text{diag}}(z)^{-1}A^*)^{-1} Z_{\text{diag}}(z)^{-1} \end{bmatrix} h$$

is injective.

3. We say that the GR-unitary colligation $U$ is shifted strictly closely connected if the map $\Pi^{dBR}_U: \mathcal{H} \to (\mathcal{E}_s \oplus cE)(\langle z^{\pm 1} \rangle)$ given by

$$\Pi^{dBR}_U: h \mapsto \begin{bmatrix} C(I - Z_{\text{diag}}(z)A)^{-1} Z_{\text{diag}}(z) \\ B^* (I - Z_{\text{diag}}(z)^{-1}A^*)^{-1} \end{bmatrix} h$$

is injective.

If we introduce subspaces

$$\mathcal{H}_{cc} = \text{span}\{p(A, A^*, P_1, \ldots, P_d)(\text{im} B + \text{im} C^*) : p = \text{polynomial in } d + 2 \text{ noncommuting variables}\}$$

$$\mathcal{H}_{succ} = \text{span}\{\text{im}(A^a)^n C^*, \text{im}(A^n) P_k B : n \in \mathbb{Z}^d_+, k = 1, \ldots, d\}$$

$$\mathcal{H}_{ssucc} = \text{span}\{\text{im}(A^a)^n P_k C^*, \text{im}(A^n) B : n \in \mathbb{Z}^d_+, k = 1, \ldots, d\}$$

Then the following follows easily from the definitions.

**Proposition 6.2.** Let $U$ be a GR-unitary colligation. Then:

1. $U$ is closely connected $\iff \mathcal{H}_{cc} = \mathcal{H}$,
2. $U$ is strictly closely connected $\iff \mathcal{H}_{succ} = \mathcal{H}$, and
3. $U$ is shifted strictly closely connected $\iff \mathcal{H}_{ssucc} = \mathcal{H}$.

We may also view a GR-unitary colligation $U$ as embedded in a multievolution Lax–Phillips scattering system $\mathcal{S}(\Sigma(U))$ via the admissible-trajectory-space construction form $[14]$ described in Section 4. This scattering system $\mathcal{S}(\Sigma(U))$ then has the additional geometric structure described in Theorem 4.2, the ambient space $\mathcal{K}$ for $\mathcal{S}(\Sigma(U))$ contains subspaces $\mathcal{L}_1, \ldots, \mathcal{L}_d$, $\mathcal{M}_1, \ldots, \mathcal{M}_d$ and $\mathcal{M}_1, \ldots, \mathcal{M}_d$, so that properties (OD), (CDP), (SL) and (SLs) are satisfied for any shift-invariant sublattice $\Omega$. In this situation there are natural unitary identification maps

$$i_\mathcal{H}_k: \mathcal{H}_k \to \mathcal{L}_k, \quad i_\mathcal{E}: \mathcal{E} \to \mathcal{F}, \quad i_\mathcal{E}_s: \mathcal{E}_s \to \mathcal{F}_s$$
and the multievolution $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_d)$ on $\mathcal{K}$ is connected with the GR-unitary colligation $U = \begin{bmatrix} [A_{ij}] & [B_i] \\ [C_j] & D \end{bmatrix}$ via equation (6.15). As the unitary colligation $U$ uniquely determines the multievolution scattering system $\mathcal{S}(\Sigma(U))$, properties of the GR-unitary colligation translate to properties of the scattering system $\mathcal{S}(\Sigma(U))$. Note that the last statement of Theorem 4.2 is an instance of the reverse direction: the property of minimal for the scattering system $\mathcal{S}(\Sigma(U))$ is translated to the property of scattering-minimal for the colligation $U$. The next proposition does such an analysis for the colligation properties mentioned above. Here we use the notation $\mathcal{K}_{\text{scat-min}}$ to denote the ambient space for the minimal part of the scattering system $\mathcal{S}(\Sigma(U))$, namely in the notation of Definition 3.1.

$$\mathcal{K}_{\text{scat-min}} = \text{Span}[\hat{\mathcal{W}} + \hat{\mathcal{W}}_s].$$

**Proposition 6.3.** Let $U$ be a GR-unitary colligation embedded in the multievolution scattering system $\mathcal{S}(\Sigma(U))$ as described in Section 4. Then:

1. $U$ is scattering minimal if and only if $(\mathcal{K}_{\text{scat-min}})^\perp = \{0\}$, or, equivalently, for any (or, equivalently, for some) shift-invariant sublattice $\Omega$, we have

$$\bigcap \gamma^d \cap (\mathcal{K}_{\text{scat-min}})^\perp = \{0\}.$$  

2. $U$ is strictly closely connected if and only if

$$\mathcal{L} \cap (\mathcal{K}_{\text{scat-min}})^\perp = \{0\}.$$  

3. $U$ is shifted strictly closely connected if and only if the space $\mathcal{L}' := \mathcal{U}_1\mathcal{L}_1 \oplus \cdots \oplus \mathcal{U}_d\mathcal{L}_d$ has the property

$$\mathcal{L}' \cap (\mathcal{K}_{\text{scat-min}})^\perp = \{0\}.$$  

4. $U$ is closely connected if and only if $\mathcal{V}^\Omega = \mathcal{V}^\Omega_{\text{cc}}$ where

$$\gamma^d = \bigoplus_{k=1}^d \bigoplus_{n \in \Omega_{kn}} z^n \mathcal{L}_{kk,\text{cc}} \oplus \bigoplus_{k=1}^d \bigoplus_{n' : f_{\nu_1,\hat{s},k} \in \partial \Omega_{kn}} (\hat{\mathcal{U}}^k)^{n' \times k} \mathcal{M}_{cc,k}$$

$$\oplus \bigoplus_{k=1}^d \bigoplus_{n' : f_{\nu_2,\hat{s},k} \in \partial \Omega_{kn}} (\hat{\mathcal{U}}^k)^{n' \times k} \mathcal{M}_{cc,k}$$

where $\mathcal{L}_{cc,k} \subset \mathcal{L}_k$ and $\mathcal{L}_cc := \mathcal{L}_{cc,1} \oplus \cdots \oplus \mathcal{L}_{cc,d}$ is given by $\mathcal{L}_{cc}$ is the smallest subspace of $\mathcal{L}$ which contains $P_{\mathcal{L}^j,\mathcal{F}_j}^\bigcap$ $P_{\mathcal{F}_j}$ for each $j = 1, \ldots, d$ and which is invariant under $P_{\mathcal{L}^j,\mathcal{F}_j}^\bigcap P_{\mathcal{L}^k}$ for each pair of indices $j, k = 1, \ldots, d$.

while $\mathcal{M}_{cc,k}$ and $\mathcal{M}_{acc,k}$ are given by

$$P_{\mathcal{M}_{cc,k}} = \text{strong limit}_{m \to \infty} P_{\mathcal{U}^m_{\text{cc},k}}.$$  

Equivalently, $U$ is closely connected if and only if

$$\mathcal{L} = \mathcal{L}_{cc}.$$  

In addition, $\mathcal{L}_{cc} = i_{\mathcal{H}}(\mathcal{H}_{cc})$ where $\mathcal{H}_{cc}$ is given by the first line of (6.15).

**Proof:** The first assertion follows from the first of the general identities (6.23) given below.

As for the second assertion, note that the map $\Pi_{dbR}^d$, after using the identification $i_{\mathcal{H}} : \mathcal{H} \to \mathcal{L}$ between $\mathcal{H}$ and $\mathcal{L}$, can be viewed as the restriction (to the state space $\mathcal{L}$) of the de Branges–Rovnyak model map $\Pi_{dbR}^d : \mathcal{K} \to \mathcal{K}_{dbR}^d$ to the
de Branges-Rovnyak-model multievolution scattering system given in (3.5). As discussed in Section 3 above, we know that \( \Pi_{dBR} \) is a partial isometry with initial space equal to \( K_{\text{scat-min}} \) and hence kernel equal to \( (K_{\text{scat-min}})^\perp \). Hence the condition that \( U \) is strictly closely-connected translates to the assertion that \( \Pi_{dBR}|_L \) has trivial kernel, i.e., to \( L \cap (K_{\text{scat-min}})^\perp = \{0\} \).

In a similar way, the map \( \Pi_{dBR}' \), after appropriate identifications, amounts to the restriction of \( \Pi_{dBR} \) to \( U_1L_1 \oplus \cdots \oplus U_dL_d \). The third assertion now follows in the same way as the second assertion.

The fourth assertion follows immediately from the definitions and from the connection (4.15) between \( U \) and \( U = (U_1, \ldots, U_d) \).

\[ \square \]

We now obtain the following relations among these various notions as a corollary to Proposition 6.3.

**Corollary 6.4.** Let \( U \) be a GR-unitary colligation. Then:

1. \( U \) scattering minimal \( \Rightarrow \) \( U \) strictly closely connected \( \Rightarrow \) \( U \) closely connected.
2. \( U \) scattering minimal \( \Rightarrow \) \( U \) shifted strictly closely connected \( \Rightarrow \) \( U \) closely connected.

**Proof.** If \( U \) is scattering minimal, by definition \( (K_{\text{scat-min}})^\perp = \{0\} \); hence \( U \) is also strictly closely connected and shifted strictly closely connected by the criteria given in statements (2) and (3) of Proposition 6.3.

To verify the second assertions in (1) and (2), we note that \( L_{cc\perp} := L \ominus L_{cc} \) decomposes as

\[ L_{cc\perp} = L_{cc\perp,1} \oplus \cdots \oplus L_{cc\perp,d} \]

with \( L_{cc\perp,k} \subset L_{cc\perp} \cap L_k \), reducing for \( U_k \) and wandering for \( \widehat{U}^k = \{U_j : j \neq k\} \). One can then check that

\[ \mathcal{V}^\Omega \ominus \mathcal{V}_{cc}^\Omega = \bigoplus_{k=1}^{d} \bigoplus_{n \in \mathbb{Z}^d} (\widehat{U}^k)^{\hat{n}} L_{cc\perp,k} \subset (K_{\text{scat-min}})^\perp ; \quad (6.6) \]

indeed, a nice exercise is to determine how the orthogonal decomposition in (6.6) fits with the orthogonal decomposition of \( \mathcal{V}^\Omega \) itself in (OD). In particular, the condition \( L_{cc\perp} \neq \{0\} \) forces also \( L \cap (K_{\text{scat-min}})^\perp \neq 0 \) and \( L' \cap (K_{\text{scat-min}})^\perp \neq \{0\} \). Reading off from the criteria (2), (3) and (4) in Proposition 6.3 we see that \( U \) not closely connected forces both \( U \) not strictly closely connected and \( U \) not shifted strictly closely connected. This completes the proof of the second implications in both (1) and (2) of the corollary.

\[ \square \]

We now give a couple of examples to show that the converse of each implication in Corollary 6.4 in general fails.

**Example 6.5.** The 2-variable Schur–Agler class function \( S(z_1, z_2) = z_1z_2 \) has a closely connected GR-unitary realization \( U \) which is not strictly closely connected and not shifted strictly closely connected. This example is based on Example 3.8 in
We take $\mathcal{H} = \mathbb{C}^4$, $d = 2$, $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$ with

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and hence $Z(z) := z_1 P_1 + z_2 P_2$ is given by

$$Z(z) = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & \frac{z_1 + z_2}{2} & \frac{z_1 - z_2}{2} \\ 0 & 0 & \frac{z_1 - z_2}{2} & \frac{z_1 + z_2}{2} \end{bmatrix}$$

We take $U = [A \ B \ C \ D]$ where

$$A = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \ 0 \ 0 \ 1], \quad D = 0.$$

We check that $U$ is closely connected as follows. Note that

$$\text{im} \ P_1 B = \begin{bmatrix} C \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{im} \ P_2 B = \begin{bmatrix} 0 \\ C \\ 0 \\ 0 \end{bmatrix},$$

$$\text{im} \ A P_1 B = \text{im} \ A|_{\mathbb{C}^4 \oplus \mathbb{C}^4} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \text{im} \ C* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ C \end{bmatrix}.$$

As the span of these images is already the whole space $\mathbb{C}^4$, we see that $U$ is indeed closely connected.

We check that $U$ is not strictly closely connected as follows. First note that $U$ being strictly closely connected can be equivalently expressed as

$$\bigcap_{n \geq 0} \ker C(Z(z)A)^n \cap \bigcap_{n \geq 0} \ker B^* Z(z)_{\text{diag}}(z) (A^* Z(z)^{-1})^n = \{0\} \quad (6.7)$$

(where we set $Z(z) = z_1 P_1 + z_2 P_2$ and where the operators involved are considered as acting on constant vectors in $\mathcal{H}$). For the case here, we compute

$$Z(z)A = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} z_1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} z_2 & 0 \\ -\frac{2}{\sqrt{2}} & \frac{z_1}{\sqrt{2}} & 0 & 0 \\ \frac{z_2}{\sqrt{2}} & \frac{z_1}{\sqrt{2}} & 0 & 0 \end{bmatrix},$$

$$CZ(z)A = \begin{bmatrix} \frac{z_2}{\sqrt{2}} & \frac{z_1}{\sqrt{2}} & 0 & 0 \end{bmatrix}, \quad C(Z(z)A)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. $$
We conclude that
\[ \bigcap_{n \geq 0} \ker C(Z(z)A)^n = \ker C \cap \ker CZ(z)A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \] (6.8)

Similarly, we compute
\[ Z(z)B = \begin{bmatrix} \frac{z_1}{\sqrt{2}} \\ \frac{z_2}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \quad Z(z)AZ(z)B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (Z(z)A)^2Z(z)B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \]

Hence
\[ \bigcap_{n \geq 0} \ker B^*Z(z)^{-1}(A^*Z(z)^{-1})^n = \ker B^*Z(z)^{-1} \cap \ker B^*Z(z)^{-1}A^*Z(z)^{-1} \]
\[ = \ker \begin{bmatrix} \frac{z_1}{\sqrt{2}} & \frac{z_2}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cap \ker \begin{bmatrix} 0 & 0 & 0 & z_1^{-1}z_2^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \] (6.9)

Intersecting (6.8) and (6.9) gives
\[ \bigcap_{n \geq 0} \ker C(Z(z)A)^n \bigcap_{n \geq 0} \ker B^*Z(z)^{-1}(A^*Z(z)^{-1})^n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]
and we conclude from the criterion (6.7) that \( U \) for this example is not strictly closely connected as claimed.

To show that \( U \) is not shifted strictly closely connected, it suffices to show that
\[ \bigcap_{n \geq 0} \ker C(Z(z)A)^n Z(z) \cap \bigcap_{n \geq 0} \ker B^*(Z(z)A^*)^n = \{ 0 \}. \]

Straightforward computations show that
\[ CZ(z) = \begin{bmatrix} 0 & 0 & \frac{z_1}{\sqrt{2}} & \frac{z_2}{\sqrt{2}} \\ 0 & 0 & \frac{z_1}{\sqrt{2}} & \frac{z_2}{\sqrt{2}} \end{bmatrix}, \quad CZ(z)AZ(z) = \begin{bmatrix} z_1z_2/\sqrt{2} & z_1z_2/\sqrt{2} & 0 & 0 \end{bmatrix}, \]
\[ C(Z(z)A)^nZ(z) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for } n \geq 2 \]
and we conclude that
\[ \bigcap_{n \geq 0} \ker C(Z(z)A)^nZ(z) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \]

Similarly, from
\[ AZ(z)B = \begin{bmatrix} 0 & 0 \\ 0 & (-z_1+z_2)/\sqrt{2} \end{bmatrix}, \quad (AZ(z))^nB = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } n \geq 2, \]
we see that
\[ \bigcap_{n \geq 0} \ker B^*(Z(z)A^*)^n = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \]
and it follows that \( U \) also is not shifted strictly closely connected.
Finally, we compute
\[ S(z_1, z_2) := D + C(I - Z(z)A)^{-1}Z(z)B \]
\[ = D + \sum_{n=0}^{\infty} (C(Z(z)A)^n Z(z)B \]
\[ = 0 + CZ(z)B + CZ(z)AZ(z)B \]
\[ = 0 + 0 + \begin{bmatrix} \frac{z_1}{\sqrt{2}} & \frac{z_1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{z_2}{\sqrt{2}} \end{bmatrix} \]
\[ = z_1z_2. \]

**Example 6.6.** There exists a GR-unitary colligation matrix $U$ which is both strictly closely connected and shifted strictly closely connected but not scattering minimal. We take $\mathcal{H} = \mathbb{C}^8$, $\mathcal{E} = \mathcal{E}_s = \mathbb{C}^3$ with

\[
A = \begin{bmatrix}
0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\
\end{bmatrix}, \\
B = \begin{bmatrix}
0 & 0 & -\sqrt{\frac{2}{3}} \\
0 & -\frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 \\
-\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 \\
-\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \\
C = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \\
D = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \\
P_1 = \begin{bmatrix}
\frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \\
P_2 = \begin{bmatrix}
\frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \\
\]

and we set $Z(z) = z_1P_1 + z_2P_2$. It is straightforward to see that $P_1$ and $P_2$ are complementary orthogonal projections on $\mathbb{C}^8$ and that $U = [A C] B$ is a unitary matrix. Hence the associated system $\Sigma(U, P_1, P_2)$ is a GR-conservative linear system. Since

\[
P_1B = \begin{bmatrix}
0 & 0 & -\sqrt{\frac{2}{3}} \\
0 & 0 & 0 \\
0 & -\sqrt{\frac{2}{3}} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad P_2B = \begin{bmatrix}
0 & 0 & 0 \\
-\sqrt{\frac{2}{3}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
\]

and $C^*C^3 = \{0\} \oplus \mathbb{C}^3$, we have

\[
\mathbb{C}^8 = P_1BC^3 + P_2BC^3 + C^*C^3 \subset \mathcal{H}_{sc} \subset \mathcal{H} = \mathbb{C}^8,
\]

thus $\Sigma(U, P_1, P_2)$ is strictly closely connected. One can also verify that

\[
\mathbb{C}^8 = BC^3 + P_1C^3 + P_2C^3 \subset \mathcal{H}_{sc} \subset \mathcal{H} = \mathbb{C}^8
\]

and hence $U$ is also shifted strictly closely connected.
Consider now the subspace $\mathcal{H}_0$ in $L^2_{\mathbb{C}}$ consisting of vectors of the form

$$x(z_1, z_2) = \begin{bmatrix} z_1 \\ z_2 \\ z_1 \\ z_2 \\ f(z_1, z_2) \\ g(z_1, z_2) \end{bmatrix}, \quad f, g \in L^2_{\mathbb{C}}. \quad (6.10)$$

It is routine to check that $\mathcal{H}_0 \subset \ker C$, $\mathcal{H}_0 \subset \ker B^* Z(z)^{-1}$, and that $\mathcal{H}_0$ reduces $Z(z)A$. Therefore

$$\{0\} \neq \mathcal{H}_0 \subset \left( \bigcap_{n \in \mathbb{Z}_+} \ker C(Z(z)A)^n \right) \cap \left( \bigcap_{n \in \mathbb{Z}_+} \ker B^* (Z(z))^{-1} (A^*(Z(z))^{-1})^n \right).$$

If we choose $\Omega$ to be the so-called balanced shift-invariant sublattice

$$\Omega^B = \{(n_1, n_2) \in \mathbb{Z}^2 : n_1 + n_2 \geq 0\}$$

then the infinite boundary of $\Omega^B$ is empty ($\partial_k \Omega^B_{\pm \infty} = \emptyset$ for $k = 1, 2$) while the finite boundary is such that

$$(H^2(\partial_1 \Omega^B_{fin}, \text{im } P_1) \oplus (H^2(\partial_2 \Omega^B_{fin}, \text{im } P_2)) \cong H^2(\Xi, \mathbb{C}^8)$$

where we set

$$\Xi = \{(n_1, n_2) \in \mathbb{Z}^2 : n_1 + n_2 = 0\}.$$

Hence, if we take $f(z_1, z_2) = z_2^{-1} f_0(z_1, z_2)$ and $g(z_1, z_2) = z_2^{-1} g_0(z_1, z_2)$ where $f_0$ and $g_0$ are in $H^2(\Xi)$ and not both zero, then the resulting function $x(z_1, z_2)$ in (6.10) is the $Z$-transform of a nonzero element of $\bigoplus_{k=1}^2 \ell^2(\partial_k \Omega^B_{fin}, \text{im } P_k)$ which is in the kernel $\Pi^B_{\text{fin}}$. We conclude that $\Sigma(U, P_1, P_2)$ is not scattering minimal.

Let us now introduce the subspaces

$$\mathcal{L}_{\text{sc}} := i_R(\mathcal{H}_{\text{sc}}), \quad \mathcal{L}_{\text{ssc}} := i_R(\mathcal{H}_{\text{ssc}}). \quad (6.11)$$

Then it is not difficult to verify that

$$\mathcal{L}_{\text{sc}} = \mathcal{L} \ominus (\mathcal{L} \cap (K_{\text{scat-min}}^+) \ominus (\mathcal{L}^\prime \cap (K_{\text{scat-min}}^+)$$

where we use the notation

$$\mathcal{L}^\prime := U_1 \mathcal{L}_1 \oplus \cdots \oplus U_d \mathcal{L}_d.$$

From Proposition 6.2 we know that the condition $\mathcal{L}_{\text{sc}} = \mathcal{L}$ characterizes the GR-unitary colligation $U$ being strictly closely connected and we know that $\mathcal{L}_{\text{ssc}} = \mathcal{L}^\prime$ characterizes $U$ being shifted strictly closely connected. We now give connections between these subspaces and the various formal reproducing kernel Hilbert spaces arising from the kernels in the associated augmented Agler decomposition.

**Proposition 6.7.** Suppose that $U$ is a GR-unitary colligation with transfer function $S(z) = D + C(I - Z_{\text{diag}}(z)A)^{-1} Z_{\text{diag}}(z)B$ and induced augmented Agler decomposition $[5.4]$ with

$$K_k(z, w) = H_k(z) H_k(w)^* \quad \text{with} \quad H_k(z) = \left[ \begin{array}{c} C(I - Z_{\text{diag}}(z)A)^{-1} \\ z_k^{-1} B^*(I - Z(z)^{-1} A^*)^{-1} \end{array} \right] P_k.$$
Let the subspaces $L_{scc}$ and $L_{sscc}$ be defined by (6.11) and (6.3). Set

\[ K(z, w) := \sum_{k=1}^{d} K_k(z, w), \quad K'_k(z, w) := z_k K_k(z, w) w_k^{-1}, \]

\[ K'(z, w) := \sum_{k=1}^{d} K'(z, w) = \sum_{k=1}^{d} z_k K_k(z, w) w_k^{-1}. \]

Then:

1. The mappings
   \[ \Pi_{dBR}|_{L_{scc}} : L_{scc} \to \mathcal{H}(K), \]
   \[ \Pi_{dBR}|_{L_{scat}} \cap (K_{scat-min})^\perp : L_k \cap (K_{scat-min})^\perp \to \mathcal{H}(K), \]
   \[ \Pi_{dBR}|_{L_{sscc}} : L_{sscc} \to \mathcal{H}(K'), \]
   \[ \Pi_{dBR}|_{U_k L_k \cap (K_{scat-min})^\perp} : U_k L_k \cap (K_{scat-min})^\perp \to \mathcal{H}(K'_k) \]

are all unitary between the indicated spaces.

2. The overlapping space (see [2.2] above)
   \[ \mathcal{L} := \mathcal{L}(K_1, \ldots, K_d) \]

is trivial (equal to \{0\}) if and only if one (or, equivalently, both) of the subspaces $\mathcal{L} \cap (K_{scat-min})^\perp$ and $L_{scc}$ is/are invariant under each of the orthogonal projection operators $P_{L_k} : \mathcal{L} \to L_k$.

3. The overlapping space
   \[ \mathcal{L}' := \mathcal{L}(K'_1, \ldots, K'_d) \]

is trivial if and only if one (or, equivalently, both) of the subspaces $\mathcal{L}' \cap (K_{scat-min})^\perp$ and $L_{sscc}$ is/are invariant under each of the orthogonal projection operators $P_{L_k} : \mathcal{L}' \to U_k L_k$ for $k = 1, \ldots, d$.

**Proof.** The unitarity of each of the four maps (6.12) can be seen as a direct consequence of the last assertion in Theorem [2.2]

By definition (see [2.2]), the overlapping space $\mathcal{L} := \mathcal{L}(K_1, \ldots, K_d)$ is trivial if and only if

\[ h_k \in \mathcal{H}(K_k), h_1 + \cdots + h_d = 0 \Rightarrow h_k = 0 \text{ for } k = 1, \ldots, d. \quad (6.13) \]

Using that $\mathcal{H}(K_k) = \text{im} \Pi_{dBR}|_{L_k}$ and that $\mathcal{H}(K) = \text{im} \Pi_{dBR}|_{L}$ with lifted norm in each case, we see that (6.13) translates to

\[ \ell_k \in L_k, \ell_1 + \cdots + \ell_d \in L \cap (K_{scat-min})^\perp \Rightarrow \ell_k \in L_k \cap (K_{scat-min})^\perp \text{ for } k = 1, \ldots, d. \]

This in turn is just the assertion that $\mathcal{L} \cap (K_{scat-min})^\perp$ is invariant under each $P_{L_k}$, i.e., for $Q = P_{L \cap (K_{scat-min})^\perp}$, we have

\[ P_{L_k} Q = Q P_{L_k} Q. \quad (6.14) \]

Note that $Q = I - P$ where we set $P = P_{L_{scc}}$ (and all operators are considered as acting on $L$). Substituting this expression into (6.14) gives

\[ P_{L_k} (I - P) = (I - P) P_{L_k} (I - P) \]

which simplifies to

\[ PP_{L_k} = PP_{L_k} P. \]
Taking adjoints finally gives
\[ P_L P = PP_L P, \]
i.e., that \( L_{sc} \) is invariant under each \( P_L \) as well. The steps are reversible, so we see that one of \( L_{sc} \) and \( L \oplus L_{sc} = L \cap (K_{scat-min})^\perp \) is invariant under each \( P_L \) if and only if the other is as well.

The third assertion in Proposition 6.7 is proved in a completely analogous way by noting the triviality of the overlapping space \( \mathcal{L}' := \mathcal{L}(K_1', \ldots, K_d') \) is characterized by
\[ \ell_k' \in U_k L_k' \Leftrightarrow \ell_k' \in U_k L_k \cap (K_{scat-minus})^\perp \quad \text{for } k = 1, \ldots, d. \]

□

A consequence of the next result is that we may drop the term “shifted strictly closely connected” for GR-unitary colligations since it is equivalent to “strictly closely connected”.

**Theorem 6.8.** Suppose that we are given a GR-unitary colligation \( \mathcal{L}(z) \) with associated transfer function \( S(z) \) and augmented Agler decomposition as in Proposition 6.7. Then the following are equivalent:

1. \( H_{sc} = H_{cc} \)
2. \( H_{ssc} = H_{cc} \)
3. \( H_{sc} \) and \( H_{ssc} \) are invariant under \( P_k \) for \( k = 1, \ldots, d \).
4. The augmented Agler decomposition is strictly closely connected (see Definition 5.6).

**Proof.** We first note that (3) \( \Leftrightarrow \) (4) amounts to the content of Proposition 6.7. Thus it remains only to prove (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3).

**Proof of (1) \( \Leftrightarrow \) (2):** Without loss of generality we may assume that the unitary colligation \( U = [A B; C D] \) is closely connected, as cutting down to the closely connected part has no effect on the Agler decomposition or on the strictly closely connected and shifted strictly closely connected parts. Then we have the equivalences
\[ H_{sc} = H_{cc} \Leftrightarrow \ker \Pi_{U}^{dBR} = \{0\}, \quad H_{ssc} = H_{cc} \Leftrightarrow \ker \Pi_{U}^{dBR'} = \{0\} \]
where \( \Pi_{U}^{dBR} \) and \( \Pi_{U}^{dBR'} \) are as in (6.3) and (6.4). As \( \text{im} \Pi_{U}^{dBR} = \mathcal{L}_{dBR}^S \subset K_{scat-min}^{S(z), dBR} \) is orthogonal to \( \mathcal{F}_{dBR}^S = [S(z)] E, \) we see that
\[ \ker \Pi_{U}^{dBR} \{0\} = \ker \Pi_{U}^{dBR} [S(z)] \subset \mathcal{H} \]
and similarly
\[ \ker \Pi_{U}^{dBR'} \{0\} = \ker \Pi_{U}^{dBR'} [I S(z)^*] \subset \mathcal{H} \]
On the other hand a direct calculation shows that
\[ \Pi_{U}^{dBR} [S(z)] = \Pi_{U}^{dBR'} [I S(z)^*] U, \]
Similarly, the condition (6.16) can be rewritten as
\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} & S(z) \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}Z_{\text{diag}}(z)^{-1} & I
\end{bmatrix}
\]
i.e.,
\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} & 1 \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1} & I
\end{bmatrix}
\]
In fact, this identity amounts to a coordinate form of the property (CDP)

We conclude that
\[
\ker H\oplus H\text{ is invariant under each } P_k\text{ while from (2) we see in particular that } H_{\text{scc}} \text{ is invariant under each } P_k\text{ and (3) follows.}
\]

Conversely, assume (3) or equivalently, both \( H \oplus H_{\text{scc}} \) and \( H \oplus H_{\text{scc}} \) are invariant under \( P_k \) for \( k = 1, \ldots, d \). This version of condition (3) can be expressed as the simultaneous validity of the two conditions:
\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}Z_{\text{diag}}(z)^{-1}
\end{bmatrix}
\begin{bmatrix} h \end{bmatrix} = 0 \Rightarrow
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}Z_{\text{diag}}(z)^{-1}
\end{bmatrix}
P_k h = 0 \text{ for each } k = 1, \ldots, d, \quad (6.15)
\]
along with
\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z) \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}
\end{bmatrix}
\begin{bmatrix} h \end{bmatrix} = 0 \Rightarrow
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z) \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}
\end{bmatrix}
P_k h = 0 \text{ for each } k = 1, \ldots, d. \quad (6.16)
\]
Note that the second expression in (6.15) can be reexpressed as
\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}Z_{\text{diag}}(z)^{-1}
\end{bmatrix}
P_k h = \begin{bmatrix} z_k^{-1} & C(I - Z_{\text{diag}}(z)A)^{-1} \end{bmatrix} P_k h
\]
and hence (6.15) can be rewritten as
\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}Z_{\text{diag}}(z)^{-1}
\end{bmatrix}
\begin{bmatrix} h \end{bmatrix} = 0 \Rightarrow
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}
\end{bmatrix}
P_k h = 0 \text{ for each } k = 1, \ldots, d. \quad (6.17)
\]
Similarly, the condition (6.10) can be rewritten as
\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z) \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}
\end{bmatrix}
\begin{bmatrix} h \end{bmatrix} = 0 \Rightarrow
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*)^{-1}
\end{bmatrix}
P_k h = 0 \text{ for each } k = 1, \ldots, d. \quad (6.18)
To prove (1), it suffices to show that $\mathcal{H} \ominus \mathcal{H}_{scc}$ is invariant also under $A$ and $A^*$, or in detail

\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1}Z_{\text{diag}}(z)^{-1})
\end{bmatrix} h = 0 \Rightarrow
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1}Z_{\text{diag}}(z)^{-1})
\end{bmatrix} x = 0 \text{ for } x = Ah \text{ and } x = A^*h. \tag{6.19}
\]

By (6.17) we can assume instead that

\[
\begin{bmatrix}
C(I - Z_{\text{diag}}(z)A)^{-1} \\
B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1})
\end{bmatrix} P_k h = 0 \text{ for } k = 1, \ldots, d. \tag{6.20}
\]

Note that

\[
C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)Ah = -Ch + C(I - Z_{\text{diag}}(z)A)^{-1}h. \tag{6.21}
\]

From (6.20) we have in particular that $C(I - Z_{\text{diag}}(z)A)^{-1}h = 0$ and hence also the constant term $Ch$ vanishes. These observations combined with the identity (6.21) tell us that

\[
C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)Ah = 0.
\]

We now have the hypothesis for the implication (6.18) with $Ah$ in place of $h$. From (6.13) we therefore conclude that $C(I - Z_{\text{diag}}(z)A)^{-1}Ah = 0$ and the top component of (6.19) is verified for the case $x = Ah$.

We next note that

\[
B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1})Ah = B^*Ah + B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1}Z_{\text{diag}}(z)^{-1}A^*Ah
\]

\[
= -D^*Ch + B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1}Z_{\text{diag}}(z)^{-1}(I - C^*C)h
\]

\[
= -D^*Ch - B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1}Z_{\text{diag}}(z)^{-1}C^*Ch
\]

\[
= 0
\]

where we used the fact that $U$ is unitary and again the top component of (6.20).

We have now verified the bottom component of (6.19) for the case $x = Ah$.

We next observe, again using that $U$ is unitary,

\[
C(I - Z_{\text{diag}}(z)A)^{-1}A^*h = CA^*h + C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)AA^*h
\]

\[
= -DB^*h + C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)(I - BB^*)h
\]

\[
= [-D - C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)B]B^*h + C(I - Z_{\text{diag}}(z)A)^{-1}Z_{\text{diag}}(z)h. \tag{6.22}
\]

From the bottom component of (6.20) we read off that $B^*h = 0$ and hence the first term of (6.22) vanishes. The vanishing of the second term follows from the top component of (6.20). In this way we have verified the validity of the top component of (6.19) for the case $x = A^*h$.

Next observe that

\[
B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1})Z_{\text{diag}}(z)^{-1}A^*h = -B^*h + B^*(I - Z_{\text{diag}}(z)^{-1}A^*^{-1})h.
\]

The vanishing of this quantity follows from the bottom component of (6.20). We have now verified the bottom component of (6.19) for the case $x = A^*h$. We now have completed the proof of Theorem 6.8.
Remark 6.9. An open question is whether it can happen that \( \mathcal{H}_{\text{sc}} \) is invariant under the projections \( P_k \) \((k = 1, \ldots, d)\) without also \( \mathcal{H}_{\text{ssc}} \) being invariant under each \( P_k \). An equivalent version of the question is whether it can happen, for a given augmented Agler decomposition \((5.4)\), that the collection of subspaces \( \{ \mathcal{H}(K_1), \ldots, \mathcal{H}(K_d) \} \) has no overlapping while \( \{ z_1 \mathcal{H}(K_1), \ldots, z_d \mathcal{H}(K_d) \} \) has nontrivial overlapping, or vice versa. Note that we have defined the augmented Agler decomposition \((5.4)\) to be strictly closely connected when both collections \( \{ \mathcal{H}(K_1), \ldots, \mathcal{H}(K_d) \} \) and \( \{ z_1 \mathcal{H}(K_1), \ldots, z_d \mathcal{H}(K_d) \} \) have trivial overlapping. Another equivalent version of the open question is whether \( \mathcal{H}_{\text{sc}} \) being invariant under each \( P_k \) implies that \( \mathcal{H}_{\text{sc}} = \mathcal{H}_{\text{cc}} \); the work in the proof of Theorem 6.8 shows that \( \mathcal{H}_{\text{sc}} = \mathcal{H}_{\text{cc}} \) if we assume that both \( \mathcal{H}_{\text{sc}} \) and \( \mathcal{H}_{\text{ssc}} \) are invariant under each \( P_k \).

The analysis in Proposition 6.7 dissects the interplay between the two decompositions \( \mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_d \) and the decomposition \( \mathcal{K} = \mathcal{K}_{\text{scat-min}} \oplus (\mathcal{K}_{\text{scat-min}})_{\perp} \) of the ambient space of the scattering system. A similar analysis can be done for the decomposition \((6.1)\) of the scattering space \( \mathbf{V}^\Omega \) associated with any shift-invariant sublattice \( \Omega \). One distinguishing feature of this situation versus that in Proposition 6.7 is that we always have that \( (\mathcal{K}_{\text{scat-min}})_{\perp} \subset \mathbf{V}^\Omega \); indeed from

\[
\hat{W}_s + \hat{W} \supset \bigoplus_{n \in \mathbb{Z}^d \setminus \Omega} \mathcal{U}^n \mathcal{F}_s \oplus \bigoplus_{n \in \Omega} \mathcal{U}^n \mathcal{F} = (\mathbf{V}^\Omega)^{\perp},
\]

we see that

\[
(\mathcal{K}_{\text{scat-min}})^{\perp} := \left( \hat{W}_s + \hat{W} \right)^{\perp} \subset \mathbf{V}^\Omega.
\]

Hence we have the simplifications

\[
\mathbf{V}^\Omega \cap (\mathcal{K}_{\text{scat-min}})^{\perp} = (\mathcal{K}_{\text{scat-min}})^{\perp}, \quad \mathbf{V}^\Omega \ominus (\mathbf{V}^\Omega \cap (\mathcal{K}_{\text{scat-min}})^{\perp}) = \mathbf{V}^\Omega \cap \mathcal{K}_{\text{scat-min}}.
\]

The following summarizes the situation.

Proposition 6.10. Suppose that \( U \) is a GR-unitary colligation embedded in a multievolution scattering system as in Theorem 4.2 with associated transfer function \( S \) and augmented Agler decomposition \((5.4)\).

1. The mappings

\[
\Pi_{dBR}|_{\mathbf{V}^\Omega} : \mathbf{V}^\Omega \cap \mathcal{K}_{\text{scat-min}} \to \mathcal{H}(\mathcal{K}_{\mathbf{V}^\Omega_{dBR}}),
\]

\[
\Pi_{dBR}|_{\chi} : \chi \ominus (\chi \cap (\mathcal{K}_{\text{scat-min}})^{\perp}) \to \mathcal{H}(\mathcal{K}_{\mathbf{\chi}_{dBR}})
\]

are all unitary between the indicated spaces. Here \( \chi \) is any one of the direct summand spaces

\[
\chi = \mathcal{U}^n \mathcal{L}_k \text{ for } n \in \partial_k \Omega_{\text{fin}},
\]

\[
\chi = (\hat{U}^k)^{\hat{n}_k} \mathcal{M}_k \text{ for } \ell_{n^\prime, k} \in \partial_k \Omega_{+\infty},
\]

\[
\chi = (\hat{U}^k)^{\hat{n}_k} \mathcal{M}_k \text{ for } \ell_{n^\prime, k} \in \partial_k \Omega_{-\infty}
\]

(6.23)
and that \( \Omega \) is any shift-invariant sublattice. Since \( \Omega \) is also invariant under \( \perp \), for any \( n \),

\[
\hat{\chi} \in K_{\hat{\ell} \ell}^{\perp} \cap V_n = (\hat{w}^k)^{\perp} \subset \hat{w}^{\perp} \text{ for each of the}
\]



Proof. Given the validity of the identities in (6.23), the proofs of statements (1) and (2) proceed exactly as the proof of Proposition 1.4, given above; we leave the precise details to the reader.

Let us suppose now that \( \Omega \) is a shift-invariant sublattice such that \( \partial \ell \Omega_{\text{fin}} \neq \emptyset \) for each \( \ell = 1, \ldots, d \) and that \( \mathcal{L}_{\Omega^{\text{fin}}} = \{0\} \). Fix \( \ell \in \{1, \ldots, d\} \). By assumption we can find at least one \( n \in \partial \ell \Omega_{\text{fin}} \). Then \( \chi = U^n \mathcal{L}_\ell \) is one of the direct-summand spaces in the list (6.24). Since by assumption \( \mathcal{L}_{\Omega^{\text{fin}}} = \{0\} \), by the equivalence of (a) and (b) in statement (2) of the proposition, we know that \( (K_{\text{scat-min}})^\perp \) is invariant under \( P_\mathcal{L}_\ell \) for each \( k = 1, \ldots, d \). Conversely, if \( (K_{\text{scat-min}})^\perp \) is invariant under \( P_\mathcal{L}_\ell \) for each \( k = 1, \ldots, d \) and \( \Omega \) is any shift-invariant sublattice, then \( \mathcal{L}_{\Omega^{\text{fin}}} = \{0\} \).

(3) Suppose that \( \partial \ell \Omega_{\text{fin}} \neq \emptyset \) for each \( \ell = 1, \ldots, d \). Then \( (K_{\text{scat-min}})^\perp \) is invariant under \( P_\mathcal{L}_\ell \) for each \( k = 1, \ldots, d \). Conversely, if \( (K_{\text{scat-min}})^\perp \) is invariant under \( P_\mathcal{L}_\ell \) for each \( k = 1, \ldots, d \) and \( \Omega \) is any shift-invariant sublattice, then \( \mathcal{L}_{\Omega^{\text{fin}}} = \{0\} \).

(4) The augmented Agler decomposition \( \{K_k; k = 1, \ldots, d\} \) associated with \( U \) is minimal (in the sense of Definition 5.8) if and only if \( (K_{\text{scat-min}})^\perp \) is invariant under \( P_\mathcal{L}_\ell \) for each \( k = 1, \ldots, d \). In this case we have the identity

\[
\mathcal{V}^{\Omega} \cap \Omega^{\text{fin}} = (K_{\text{scat-min}})^\perp
\]

and \( U \) is closely connected if and only if \( \mathcal{S}(\Sigma(U)) \) is scattering-minimal.

\[
K_{\hat{\chi} \hat{\ell} \ell}^{\perp} = \left(\hat{\chi} \cdot \hat{w}^k\right)^{\perp} = \left(\hat{w}^k\right)^{\perp} \cap \Omega^{\text{fin}}
\]

and \( (K_{\text{scat-min}})^\perp \) is reducing for \( \hat{w}^{\perp} \). It follows that \( (K_{\text{scat-min}})^\perp \) is reducing for \( U_1, \ldots, U_d \) and \( P_{\mathcal{L}_k} = U^{\perp} P_\mathcal{L}_k U^{\perp} \) for any \( n \in \mathbb{Z}^d \); we see that \( (K_{\text{scat-min}})^\perp \) is invariant under \( P_{\mathcal{L}_k} \) for each \( n \in \mathbb{Z}^d \), in particular, for \( \ell = 1, \ldots, d \). From the characterization of \( P_{\mathcal{M}_k} \) via the strong limit (SL) in Theorem 1.4, it follows that \( (K_{\text{scat-min}})^\perp \) is also invariant under \( P_{\mathcal{M}_k} \). Since, for any \( n \in \mathbb{Z}^d \),

\[
P_{\hat{\chi} \hat{w}^k} = (\hat{\chi} \cdot \hat{w}^k)^{\perp} P_{\mathcal{M}_k} (\hat{\chi} \cdot \hat{w}^k)^{\perp}
\]

and \( (K_{\text{scat-min}})^\perp \) is reducing for \( (\hat{\chi} \cdot \hat{w}^k)^{\perp} \), it follows that \( (K_{\text{scat-min}})^\perp \) is invariant for \( P_{\hat{\chi} \hat{w}^k} \) as well for any \( n \in \mathbb{Z}^d \), in particular, for \( n \) such that \( \ell n_k \in \partial \ell \Omega_{\infty} \). That \( (K_{\text{scat-min}})^\perp \) is invariant for \( P_{\hat{\chi} \hat{w}^k} \) for each \( n \) such that \( \ell n_k \in \partial \Omega_{\infty} \).
follows in a similar way by starting with $\{S^*_L\}$ in place of $\{S_L\}$. We have now verified condition (b) in statement (2); by statement (2) of the proposition it follows that $\mathcal{L}_{\text{fin}} = \{0\}$ as wanted. This completes the proof of statement (3) in the proposition.

Suppose that the augmented Agler decomposition $\{K_k: k = 1, \ldots, d\}$ associated with $U$ is minimal in the sense of Definition 5.8. By statement (1) in Theorem 5.9 we know that $\mathcal{L}_{\text{fin}} = \{0\}$ for any shift-invariant sublattice $\Omega$; in particular we may choose $\Omega$ having nonempty finite boundary components. Then from statement (3) of the proposition already proved, we see that $(K_{\text{scat-min}})^\perp$ is invariant under $P_{L_k}$ for each $k = 1, \ldots, d$. Conversely, if $(K_{\text{scat-min}})^\perp$ is invariant under each $P_{L_k}$, then $\mathcal{L}_{\text{fin}} = \{0\}$ by the converse side of statement (3) above, and both $\mathcal{L}$ and $\mathcal{L}'$ are trivial by statements (2) and (3) in Proposition 6.7 from which it follows that both conditions in Definition 5.8 are satisfied and $\{K_k: k = 1, \ldots, d\}$ is minimal.

As noted in (6.10), the containment $\subset$ in (6.24) always holds; we prove the reverse containment under the hypothesis that $(K_{\text{scat-min}})^\perp$ is invariant under $P_{L_k}$ for $k = 1, \ldots, d$. Equivalently, the assumption is that $K_{\text{scat-min}}$ is invariant under each $P_{L_k}$. From the formula for $L_{cc}$ in statement (4) of Proposition 6.3 we see that $L_{cc}$ is then contained in $K_{\text{scat-min}}$. It then follows from the way that $V_{K_k}^{\Omega}$ is constructed that $V_{cc}^{\Omega} \subset K_{\text{scat-min}}$. Taking orthogonal complements inside $\Omega$, then gives $\Omega \otimes V_{cc}^{\Omega} \supset (K_{\text{scat-min}})^\perp$ and (6.24) follows. It then follows immediately that $U$ closely connected $\left(V_{cc}^{\Omega} = V_{cc}^{\Omega_{\text{fin}}}\right)$ if and only if $\mathcal{S}(\Sigma(U))$ is scattering-minimal $\left((K_{\text{scat-min}})^\perp = \{0\}\right)$.

As a corollary we obtain the following complement to Definition 5.8.

**Corollary 6.11.** Suppose that we are given an augmented Agler decomposition (5.1) for a Schur class formal power series $S \in \mathcal{L}(E, E)[[z^{\pm 1}]]$ and suppose that $\Omega$ is a shift-invariant sublattice such that each finite boundary component $\partial_k \Omega_{\text{fin}} (k = 1, \ldots, d)$ is nonempty. Assume that $\mathcal{L}_{\text{fin}} = \{0\}$. Then also the overlapping spaces $\mathcal{L} = \mathcal{L}(K_1, \ldots, K_d)$ and $\mathcal{L}' = \mathcal{L}(K'_1, \ldots, K'_d)$ are trivial, i.e., for the case that all finite boundary components of $\Omega$ are nonempty, condition (1) is a consequence of condition (2) in Definition 5.8.

**Proof.** Simply combine statements (2) and (3) of Proposition 6.7 with statement (3) of Proposition 6.10. □

**Remark 6.12.** We note that Example 6.6 can be used to illustrate an additional point: there exists a strictly closely connected Agler decomposition which is not minimal, i.e., condition (1) does not in general imply condition (2) in Definition 5.8. Indeed, consider the GR-unitary colligation presented in Example 6.6 and its associated Agler decomposition $\{K_k: k = 1, \ldots, d\}$, say. As $U$ is both strictly closely connected and shifted strictly closely connected, the associated Agler decomposition is strictly closely connected. We also verified for this example that $U$ is closely connected. Were it the case that the Agler decomposition were minimal, then as a consequence of statement (4) in Proposition 6.10 it would follow that $U$ is also scattering-minimal, contrary to the main point of the example. Of course it should also be possible to check directly that certain subspaces in the list (6.24) for the balanced cut in this example have nontrivial intersection in order to verify that the augmented Agler decomposition for this example is not minimal.

**Remark 6.13.** For the reader’s convenience, we summarize here the various notions of minimality: for augmented Agler decompositions, for Givone–Roesser unitary
colligations, and for multievolution scattering systems, and the interrelations among these notions.

**I: Augmented Agler decompositions**

1.1. *Strictly closely connected* — see Definition 5.6.

1.2. *Minimal* — see Definition 5.8.

**II: GR unitary colligations**

II.0. *Closely connected* — see Definition 6.1(1).

II.1a. *Strictly closely connected* — see Definition 6.1(2).

II.1b. *Shifted strictly closely connected* — see Definition 6.1(3).

II.2. *Kernel minimal*: the augmented Agler decomposition induced by the colligation has property I.2.

II.3. *Scattering minimal* — see (4.19)–(4.20).

**III: Multievolution scattering systems**

III.3. *Minimal* — see Definition 3.1.

First, we note that II.1a and II.1b are equivalent by Theorem 6.8. Second, we remark that every unitary colligation can be replaced by its closely connected part by compressing the state space $H$ to $H_{cc}$, which does not affect the transfer function, the induced Agler decomposition, or the minimal part of the associated scattering system. Assume now that the GR unitary colligation is closely connected, i.e., II.0 holds. Then (1) by Theorem 6.8 conditions II.1a and II.1b are each equivalent to condition I.1 for the induced Agler decomposition; (2) by Theorem 4.2, condition II.3 is equivalent to condition III.3 for the associated scattering system; (3) by Definition 5.8 I.2 implies I.1, and thus II.2 implies both II.1a and II.1b; (4) by Proposition 6.10, II.2 implies II.3, and II.3 together with either II.1a or II.1b imply II.2. Thus we conclude that, under assumption II.0, we have II.3&II.1a ⇔ II.3&II.1b ⇔ II.2.

### 7. Closely connected GR-unitary colligations compatible with given augmented Agler decomposition

By the results of [2, 3, 16] we know that GR-unitary realizations of a given Schur-Agler-class function (viewed here as a formal power series) $S(z) \in SA(\mathcal{E}, \mathcal{E}_*)$ can be constructed from a given augmented (or even nonaugmented) Agler decomposition (5.4). Conversely, a given GR-unitary realization for $S$ picks out a particular augmented Agler decomposition via (5.5).

When this is the case (i.e., $U$ and $\{K_k\}$ are connected via (5.5)), we shall say that the GR-unitary colligation $U = (\begin{bmatrix} A & B \\ C & D \end{bmatrix}, P_1, \ldots, P_d)$ is *compatible* with the augmented Agler decomposition $\{K_k: k = 1, \ldots, d\}$ for $S$. We next present a description of the set of all GR-unitary colligations which realize $S(z)$ and are compatible with a given augmented Agler decomposition $K_1(z, w), \ldots, K_d(z, w)$ for $S(z)$. The statement of the result requires a few preliminaries.

We start with an augmented Agler decomposition for $S(z)$:

$$
\begin{bmatrix}
I - S(z)S(w)^* & S(w) - S(z) \\
S(z)^* - S(w)^* & S(z)^*S(w) - I
\end{bmatrix} = \sum_{k=1}^{d} (1 - z_k w_k^{-1}) K_k(z, w).
$$
Let us introduce the notation

\[ K'_k(z, w) = z_k K_k(z, w) w_k^{-1}, \]
\[ K_F(z, w) = \begin{bmatrix} S(z) & I \\ S(w)^* & I \end{bmatrix}, \quad K_{F,*}(z, w) = \begin{bmatrix} I & S(w) \\ S(z)^* \end{bmatrix} \]

so we may rewrite (5.9) as

\[ \sum_{k=1}^{d} K_k(z, w) + K_F(z, w) = \sum_{k=1}^{d} K'_k(z, w) + K_{F,*}(z, w) =: K(z, w). \quad (7.1) \]

As a consequence of the reproducing kernel property, this in turn means that, for each \( e, e' \in \mathcal{E} \) and \( e_*, e'_* \in \mathcal{E}_* \),

\[ \sum_{k=1}^{d} \left( K_k(\cdot, w) \begin{bmatrix} e_* \\ e' \end{bmatrix}, K_k(\cdot, z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right) + \left( K_F(\cdot, w) \begin{bmatrix} e_* \\ e' \end{bmatrix}, K_F(\cdot, z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right) = \sum_{k=1}^{d} \left( K'_k(\cdot, w) \begin{bmatrix} e_* \\ e' \end{bmatrix}, K'_k(\cdot, z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right) + \left( K_{F,*}(\cdot, w) \begin{bmatrix} e_* \\ e' \end{bmatrix}, K_{F,*}(\cdot, z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right) \]

(7.2)

where the inner products are of the form

\[ \mathcal{H}(K_k)[[w^{\pm 1}]] \times \mathcal{H}(K_k)[[z^{\pm 1}]], \quad \mathcal{H}(K_F)[[w^{\pm 1}]] \times \mathcal{H}(K_F)[[z^{\pm 1}]], \]
\[ \mathcal{H}(K'_k)[[w^{\pm 1}]] \times \mathcal{H}(K'_k)[[z^{\pm 1}]], \quad \mathcal{H}(K_{F,*})[[w^{\pm 1}]] \times \mathcal{H}(K_{F,*})[[z^{\pm 1}]] \]

with values in \( \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] \). As (7.2) is an identity between formal power series in \( \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] \), the identity must hold coefficientwise for coefficients of \( z^{-n} w^m \). Hence, if we use the notation (2.23) and (2.24), we have, for all \( e, e' \in \mathcal{E}, e_*, e'_* \in \mathcal{E}_* \) and \( m, n \in \mathbb{Z}^d \),

\[ \sum_{k=1}^{d} \left( [K_k]_n(z) \begin{bmatrix} e_* \\ e' \end{bmatrix}, [K_k]_m(z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right)_{\mathcal{H}(K_k)} + \left( [K_F]_n(z) \begin{bmatrix} e_* \\ e' \end{bmatrix}, [K_F]_m(z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right)_{\mathcal{H}(K_F)} = \sum_{k=1}^{d} \left( [K'_k]_n(z) \begin{bmatrix} e_* \\ e' \end{bmatrix}, [K'_k]_m(z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right)_{\mathcal{H}(K'_k)} + \left( [K_{F,*}]_n(z) \begin{bmatrix} e_* \\ e' \end{bmatrix}, [K_{F,*}]_m(z) \begin{bmatrix} e'_* \\ e' \end{bmatrix} \right)_{\mathcal{H}(K_{F,*})}. \]
If we now define subspaces $\hat{D}$ and $\hat{R}$ according to

$$\hat{D} = \text{span}\left\{ \begin{bmatrix} [K_1]_m(z) \\ \vdots \\ [K_d]_m(z) \\ [K_F]_m(z) \end{bmatrix} \middle| \begin{bmatrix} e \\ e_\ast \end{bmatrix} : e_\ast \in E, e \in E, m \in \mathbb{Z}^d \right\} \subseteq \begin{bmatrix} \mathcal{H}(K_1) \\ \vdots \\ \mathcal{H}(K_d) \\ \mathcal{H}(K_F) \end{bmatrix},$$

$$\hat{R} = \text{span}\left\{ \begin{bmatrix} [K'_1]_m(z) \\ \vdots \\ [K'_d]_m(z) \\ [K_{F_\ast}]_m(z) \end{bmatrix} \middle| \begin{bmatrix} e \\ e_\ast \end{bmatrix} : e_\ast \in E, e \in E, m \in \mathbb{Z}^d \right\} \subseteq \begin{bmatrix} \mathcal{H}(K'_1) \\ \vdots \\ \mathcal{H}(K'_d) \\ \mathcal{H}(K_{F_\ast}) \end{bmatrix},$$

we see that the map $\hat{V}$ given by

$$\hat{V} : \begin{bmatrix} [K_1]_m(z) \\ \vdots \\ [K_d]_m(z) \\ [K_F]_m(z) \end{bmatrix} \mapsto \begin{bmatrix} [K'_1]_m(z) \\ \vdots \\ [K'_d]_m(z) \\ [K_{F_\ast}]_m(z) \end{bmatrix}$$

extends by linearity and continuity to define a unitary operator from $\hat{D}$ onto $\hat{R}$. Alternatively, to see that $\hat{V} : \hat{D} \to \hat{R}$ extends to a unitary, one can note that $\hat{V}$ extends to

$$\hat{V} = (s')^\ast s|_{\hat{D}}$$

where

$$s : \bigoplus_{k=1}^d \mathcal{H}(K_k) \rightarrow \mathcal{H}(K), \quad s' : \bigoplus_{k=1}^d \mathcal{H}(K'_k) \rightarrow \mathcal{H}(K)$$

are the coisometric sum maps as in Proposition 2.16 with respective initial spaces $\hat{D}$ and $\hat{R}$. From Proposition 2.16 we know that we can identify the defect spaces as overlapping spaces:

$$\mathcal{L} := \bigoplus_{k=1}^d \mathcal{H}(K_k) \mathcal{H}(K_F) \mathcal{L} = \left[ \begin{array}{c} \mathcal{L}(K_1, \ldots, K_d) \\ 0 \end{array} \right],$$

$$\mathcal{L}' := \bigoplus_{k=1}^d \mathcal{H}(K'_k) \mathcal{H}(K_{F_\ast}) \mathcal{L} = \left[ \begin{array}{c} \mathcal{L}(K'_1, \ldots, K'_d) \\ 0 \end{array} \right].$$

Using the identification maps explained in Proposition 2.16 we have the following unitary identifications, where $K$ denotes the common value of the two expressions in (7.1):

$$\left( \bigoplus_{k=1}^d \mathcal{H}(K_k) \mathcal{H}(K_F) \right) \oplus \mathcal{L} \cong (\mathcal{H}(K) \oplus \mathcal{L}) \oplus \mathcal{L}'$$

$$\cong (\mathcal{H}(K) \oplus \mathcal{L}') \oplus \mathcal{L}$$

$$\cong \left( \bigoplus_{k=1}^d \mathcal{H}(K'_k) \mathcal{H}(K_{F_\ast}) \right) \oplus \mathcal{L}.$$
The composition of these identification operators gives us a unitary map

$$
\hat{U}_0: \left[ \bigoplus_{k=1}^{d} \mathcal{H}(K_k) \right]_{\mathcal{L}'} \rightarrow \left[ \bigoplus_{k=1}^{d} \mathcal{H}(K'_k) \right]_{\mathcal{L}}
$$

such that

$$\hat{U}_0 \begin{bmatrix} d \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{V}d \\ 0 \end{bmatrix} \text{ if } \hat{d} \in \hat{D},$$

$$\hat{U}_0: \begin{bmatrix} \ell \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \ell \end{bmatrix} \text{ if } \ell \in \mathcal{L},$$

$$\hat{U}_0: \begin{bmatrix} 0 \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \rho' \end{bmatrix} \text{ if } \rho' \in \mathcal{L}'. $$

In particular, \( \hat{U}_0 \) can be viewed as a unitary extension of \( \hat{V} \). Note that \( e \mapsto \begin{bmatrix} S(z) \end{bmatrix} e \) is an isometry from \( \mathcal{E} \) onto \( \mathcal{H}(K_{\mathcal{F}}) \) and that \( e_* \mapsto \begin{bmatrix} I \end{bmatrix} e_* \) is an isometry from \( \mathcal{E}_* \) onto \( \mathcal{H}(K_{\mathcal{F}_*}) \). In addition, multiplication by \( \begin{bmatrix} z_1^{-1} & \cdots & z_d^{-1} \end{bmatrix} \) maps \( \bigoplus_{k=1}^{d} \mathcal{H}(K'_k) \) unitarily onto \( \bigoplus_{k=1}^{d} \mathcal{H}(K_k) \). In addition one can check directly the following formulas:

$$[K_k(\cdot, w)w_k^{-1}]_m(z) = [K_k]_{m-e_{k}}(z),$$

$$\begin{bmatrix} I \\ S(z)^* \end{bmatrix} \begin{bmatrix} S(w)^* & I \end{bmatrix}_m(z) = \begin{bmatrix} I \\ S(z)^* \end{bmatrix} \begin{bmatrix} S_m^* & \delta_{m,0}I \end{bmatrix},$$

$$\begin{bmatrix} I \\ S(z)^* \end{bmatrix} \begin{bmatrix} I & S(w) \end{bmatrix}_m(z) = \begin{bmatrix} I \\ S(z)^* \end{bmatrix} \begin{bmatrix} \delta_{m,0}I & S_{-m} \end{bmatrix}. $$

If we incorporate the identifications

$$\mathcal{E} \sim \mathcal{H}(K_{\mathcal{F}}), \quad \mathcal{E}_* \sim \mathcal{H}(K_{\mathcal{F}_*}), \quad \mathcal{H}(K'_k) \sim \mathcal{H}(K_k)$$

described above with these observations, then the modified versions \( \mathcal{D} \) and \( \mathcal{R} \) of \( \hat{D} \) and \( \hat{R} \) respectively are given by

$$\mathcal{D} = \text{span} \left\{ \begin{bmatrix} [K_1]_{m}(z) \\ \vdots \\ [K_d]_{m}(z) \\ [S_m] \end{bmatrix} : \begin{bmatrix} e_* \\ e \end{bmatrix} : e_* \in \mathcal{E}_*, e \in \mathcal{E}, m \in \mathbb{Z}^d \right\} \subset \mathcal{H}(K_1)$$

$$\mathcal{R} = \text{span} \left\{ \begin{bmatrix} [K_1]_{m-e_{k}}(z) \\ \vdots \\ [K_d]_{m-e_{k}}(z) \\ [\delta_{m,0}I \ S_{-m}] \end{bmatrix} : \begin{bmatrix} e_* \\ e \end{bmatrix} : e_* \in \mathcal{E}_*, e \in \mathcal{E}, m \in \mathbb{Z}^d \right\} \subset \mathcal{H}(K_1)$$

and the modified version \( U_0 \) of \( \hat{U}_0 \),

$$U_0: \left[ \bigoplus_{k=1}^{d} \mathcal{H}(K_k) \right]_{\mathcal{E}} \rightarrow \left[ \bigoplus_{k=1}^{d} \mathcal{H}(K_k) \right]_{\mathcal{E}_*} $$
is an extension of the unitary \( V : \mathcal{D} \rightarrow \mathcal{R} \) given by

\[
V : \begin{bmatrix}
[K_1]_m(z) \\
\vdots \\
[K_d]_m(z) \\
\begin{bmatrix}
S_m^* & \delta_{m,0}I
\end{bmatrix}
\end{bmatrix} \begin{bmatrix}
e_∗ \\
e
\end{bmatrix} : e_* \mapsto \begin{bmatrix}
[K_1]_{m-e_1}(z) \\
\vdots \\
[K_d]_{m-e_d}(z) \\
\begin{bmatrix}
\delta_{m,0}I & S_{-m}
\end{bmatrix}
\end{bmatrix} \begin{bmatrix}
e_∗ \\
e
\end{bmatrix}.
\tag{7.5}
\]

Let us write \( U_0 \) in matrix form

\[
U_0 = \begin{bmatrix}
U_{0,11} & U_{0,12} & U_{0,13} \\
U_{0,21} & U_{0,22} & U_{0,23} \\
U_{0,31} & U_{0,32} & 0
\end{bmatrix} : \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}(K_k) \\
\mathcal{E} \\
\mathcal{L}'
\end{bmatrix} \rightarrow \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}(K_k) \\
\mathcal{E}_s \\
\mathcal{L}
\end{bmatrix}.
\]

Then we have the following result.

**Theorem 7.1.** Suppose that \( S(z) \in \mathcal{L}(\mathcal{E}, \mathcal{E}_s)[[z]] \) is in \( \mathcal{SA}(\mathcal{E}, \mathcal{E}_s) \) and that we are given an augmented Agler decomposition \( K_1(z,w), \ldots, K_d(z,w) \) for \( S(z) \) as in (5.3). Then:

1. Any GR-unitary colligation \( U \) which gives a realization of \( S \) compatible with the given augmented Agler decomposition \( K_1(z,w), \ldots, K_d(z,w) \) is unitarily equivalent to a unitary extension

\[
U = \begin{bmatrix}
A_{11} & A_{12} & B_1 \\
A_{21} & A_{22} & B_2 \\
C_{12} & C_2 & D
\end{bmatrix} : \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}(K_k) \\
\mathcal{E} \\
\mathcal{L}'
\end{bmatrix} \rightarrow \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}(K_k) \\
\mathcal{E}_s \\
\mathcal{L}
\end{bmatrix}
\]

of the partially defined unitary operator \( V \).

2. Such unitary extensions \( U \) are parametrized by GR-unitary colligations of the form

\[
\tilde{U} = \begin{bmatrix}
\tilde{U}_{11} & \tilde{U}_{12} \\
\tilde{U}_{21} & \tilde{U}_{22}
\end{bmatrix} : \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k \\
\mathcal{L}
\end{bmatrix} \rightarrow \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k \\
\mathcal{L}'
\end{bmatrix}
\]

via the feedback connection: the system of equations

\[
U = \begin{bmatrix}
A_{11} & A_{12} & B_1 \\
A_{21} & A_{22} & B_2 \\
C_1 & C_2 & D
\end{bmatrix} : \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k \\
\mathcal{E} \\
\mathcal{L}'
\end{bmatrix} \rightarrow \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k \\
\mathcal{E}_s \\
\mathcal{L}
\end{bmatrix}
\tag{7.6}
\]

means that there are uniquely determined vectors \( \ell \in \mathcal{L} \) and \( \ell' \in \mathcal{L}' \) so that the following system of equations is verified:

\[
U_0 : \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k \\
\mathcal{E}_s
\end{bmatrix} \rightarrow \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k' \\
\mathcal{E}_s
\end{bmatrix},
\tag{7.7}
\]

\[
\tilde{U} : \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k \\
\mathcal{E}_s
\end{bmatrix} \rightarrow \begin{bmatrix}
\bigoplus_{k=1}^d \mathcal{H}_k' \\
\mathcal{E}_s
\end{bmatrix}.
\tag{7.8}
\]
Explicitly, $U = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}$ is given by

$$
\begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix} = \mathcal{F}_\ell \begin{pmatrix} U_{0,11} & 0 & 0 & 0 & U_{0,13} \\ 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{bmatrix} \tilde{U}_{11} \\ \tilde{U}_{21} \\ \tilde{U}_{22} \end{bmatrix}
$$

$$
= \begin{bmatrix} U_{0,11} + U_{0,13} \tilde{U}_{22} U_{0,31} & U_{0,13} \tilde{U}_{21} & U_{0,12} + U_{0,13} \tilde{U}_{22} U_{0,32} \\ \tilde{U}_{12} U_{0,31} & \tilde{U}_{11} & \tilde{U}_{12} U_{0,32} \\ U_{0,21} + U_{0,23} \tilde{U}_{22} U_{0,31} & U_{0,23} \tilde{U}_{21} & U_{0,22} + U_{0,23} \tilde{U}_{22} U_{0,32} \end{bmatrix}. \tag{7.9}
$$

Here we use the general notation $\mathcal{F}_\ell ([A B \begin{bmatrix} C & D \end{bmatrix}], X) = A + B(I - XD)^{-1}XC$ for the result of the lower Redheffer linear-fractional map induced by $[A B \begin{bmatrix} C & D \end{bmatrix}]$ acting on $X$.

(3) The colligation $\tilde{U}$ is closely connected if and only if there is no reducing subspace of $U$ (that is, a subspace that $U$ maps onto itself) contained in $\tilde{H} = \bigoplus_{k=1}^d \tilde{H}_k$ and invariant under the projections $P_k$, $k = 1, \ldots, d$.

(4) $\tilde{U}$ is unitaly equivalent to $\tilde{U}'$ (as colligations) if and only if $U$ is unitarily equivalent to $U'$ (as colligations) by a unitary equivalence which is the identity on $\bigoplus_{k=1}^d H(K)$.

Proof. This result is the reproducing-kernel version of a result essentially contained in [8, Theorem 5.1]—one must make the identification between formal power series kernels $K(z, w)$ and sesquianalytic kernels $K(z, w)$. In addition one should use the factorization $K_k(z, w) = H_k(z)H_k(w)^*$ for an $K_k(z) = \sum_{n \in \mathbb{Z}_+^d} H_{k,n,z^n}$ where $H_{k,n} \in \mathcal{L}(\mathcal{H}_k, \mathcal{F}_+ \mathcal{F})$ and the fact that (under the assumption that multiplication by $H_k(z)$ acting on $H_k$ is injective) the map $x \mapsto H_k(z)$ is an isometric isomorphism of $H_k$ onto $\mathcal{H}(K_k)$. In addition, one should specialize the result in [8] to the case where (1) $Q(z) = \begin{bmatrix} z_1 \\ \vdots \\ z_d \end{bmatrix}$ so that the associated domain $D_0$ is the unit polydisk $\mathbb{D}^d$ and where (2) the interpolation problem is taken to be the full-matrix-value (rather than tangential) interpolation problem with set of prescribed interpolation nodes equal to the whole polydisk $\mathbb{D}^d$. The result also appears in [16, Theorem 2.1 (3) and Theorem 6.1] in more implicit form. In addition, we are here parametrizing unitary colligations rather than the associated characteristic functions; formulas of the type (7.9) for the result of the lower feedback loop of the type (7.8) have been known in the control literature for some time (see e.g. [32, Section 2.1]). \hfill \Box

Remark 7.2. Our ultimate goal here would be to obtain a complete description of closely connected GR-unitary realizations of $S$ which are compatible with a given augmented Agler decomposition up to a unitary equivalence. The description in Theorem 7.1 falls short of achieving this goal in two ways:
• letting the load $\tilde{U}$ run over all closely connected GR-unitary colligations with input space $\mathcal{L}$ and output space $\mathcal{L}'$ may give us some non-closely-connected GR-unitary colligations $U$;

• unitarily non-equivalent loads $\tilde{U}$ can give unitarily equivalent GR-unitary colligations $U$.

What is added here (as opposed to what already appears in [16, 8]) to the results on parametrization of unitary colligations realizing a given augmented Agler decomposition is the result coming out of Proposition 2.10 that the defect spaces

$$\bigoplus_{k=1}^{d} \mathcal{H}(K_k) \ominus \mathcal{D}, \quad \bigoplus_{k=1}^{d} \mathcal{H}(K_k) \ominus \mathcal{E}_*$$

can be identified with the overlapping spaces $\mathcal{L}(K_1, \ldots, K_d)$ and $\mathcal{L}(K'_1, \ldots, K'_d)$ respectively. As a result we have the following corollary.

**Corollary 7.3.** Suppose that the augmented Agler decomposition for the Schur-Agler-class function $S(z)$ is strictly closely connected, i.e., such that both the corresponding overlapping spaces $\mathcal{L}(K_1, \ldots, K_d)$ and $\mathcal{L}(K'_1, \ldots, K'_d)$ are trivial. Then there is a unique, up to a unitary equivalence of colligations, GR-unitary closely-connected colligation realizing $S$ compatible with this given augmented Agler decomposition. Moreover, this colligation is necessarily strictly closely connected, and if, in addition, the augmented Agler decomposition is minimal, then this colligation is scattering minimal (see Remark 6.13).

**Remark 7.4.** Canonical functional models for a Schur-Agler-class function with are also studied in [9] using the classical (rather than formal) reproducing kernel Hilbert spaces associated with the kernels in an augmented Agler decomposition. In these models the state space is taken to be $\bigoplus_{k=1}^{d} \mathcal{H}(K_k)$ (no overlapping spaces). To make this possible, the class of colligations considered is “weakly unitary” (whereby the colligation operator is required only to be contractive with it and its adjoint isometric only when restricted to certain canonical subspaces). Then it is shown that any weakly unitary “closely connected” realization is unitarily equivalent to such a (two-component) canonical functional-model realization. The meaning of “closely connected” as defined in [9] is closely related to but not quite the same as “strictly closely connected” or “shifted strictly closely connected” as given in Definition 6.1 above: namely, the map $\bar{\Pi}_U^{BR}$ given by

$$\bar{\Pi}_U^{BR}: h \mapsto \left[ \begin{array}{c} C(I - Z_{\text{diag}}(z)A)^{-1} \\ B^*(I - Z_{\text{diag}}(z)A^*)^{-1} \end{array} \right] h$$

should be injective. Also the work in [9] makes no contact with multievolution scattering systems and the associated scattering geometry.

**8. Functional-model colligations and scattering systems realizing a given augmented Agler decomposition**

Suppose that we are given a Schur-class function $S \in S(\mathcal{E}, \mathcal{E}_*)$ with given augmented Agler decomposition $\{K_k: k = 1, \ldots, d\}$ (so (5.4) holds). Proposition 3.3 presents a functional-model minimal scattering system having $S$ as its scattering matrix. As a consequence of Proposition 6.10 we see that this model has the
extra geometric structure (i.e., the existence of auxiliary subspaces \( \mathcal{L}_k, \mathcal{M}_k, \mathcal{M}_{\ast k} \) satisfying conditions (OD), (CDP), (SL) and (SL∗) to lead to a functional-model GR-unitary colligation compatible with the given augmented Agler decomposition if and only if the augmented Agler decomposition is minimal, and then this extra structure (and the associated compatible GR-unitary colligation) can be presented in functional form as in Theorem 5.9. Another situation where a (not necessarily minimal) multievolution scattering system with embedded GR-unitary colligation compatible with a preassigned augmented Agler decomposition can be presented in functional-model form is the following. The result is immediate from Remark 4.1 and Corollary 7.3.

**Theorem 8.1.** Suppose that we are given an augmented Agler decomposition for the Schur-class function \( S \in \mathcal{S}(\mathcal{E}, \mathcal{E}_s) \) which is strictly closely connected (see Definition 7.6). Then there is an essentially unique closely connected GR-unitary colligation compatible with \( \{ K_k : k = 1, \ldots, d \} \) and an essentially unique (not necessarily minimal) associated scattering system \( \mathfrak{G}(\Sigma(U)) \) which can be constructed in functional model form as follows. Note that the subspaces \( \mathcal{D} \) and \( \mathcal{R} \) given by (7.3) and (7.4) are equal to all of \( \bigoplus_{k=1}^d \mathcal{H}(K_k) \oplus \mathcal{E} \) and \( \bigoplus_{k=1}^d \mathcal{H}(K_k) \oplus \mathcal{E}_s \) respectively and the operator \( V \) given by (7.5) is unitary. Define an operator

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \left( \bigoplus_{k=1}^d \mathcal{H}(K_k) \right) \rightarrow \left( \bigoplus_{k=1}^d \mathcal{H}(K_k) \right)
\]

implicitly by \( U = V \) where \( V \) is as in (7.5). Construct a scattering system, denoted as \( \mathfrak{G}_{\Omega B}(\Sigma(U)) \), with scattering space

\[
\mathcal{V}^B := \bigoplus_{k=1}^d \bigoplus_{n \in \Xi} z^n \mathcal{H}(K_k)
\]

as in Remark 4.1. Then \( \mathfrak{G} \) is a multievolution scattering system with embedded GR-unitary colligation \( U \) compatible with the given augmented Agler decomposition \( \{ K_k : k = 1, \ldots, d \} \).

Note that the main difference between Theorem 5.9 and Theorem 8.1 is that the construction of the scattering space in Theorem 5.9 involved internal orthogonal direct sums while the construction in Theorem 8.1 necessarily calls for external direct sums. We note also that the orthogonal complement of the minimal-scattering subspace \( K_{\text{scat-min}}^\perp \) in Theorem 8.1 can be identified with the overlapping space \( \mathcal{L}_{V^B} \).

For the general case, by Theorem 7.1 we know that the amount of information determined by a given augmented Agler decomposition \( \{ K_k : k = 1, \ldots, d \} \) in the construction of a compatible GR-unitary colligation is precisely the following: we may assume that \( \mathcal{H}(K_k) \subset \mathcal{L}_k \) and that \( U|_{\mathcal{D}} \) agrees with \( V \) given by (7.5). Following the construction in Remark 4.1 this translates to the following concerning the construction of the associated multievolution scattering system: we know that \( \mathcal{H}(z^n K_k(z, w^{-n})) = z^n \mathcal{H}(K_k(z, w)) \) may be assumed to be a subspace of \( \mathcal{U}^n\mathcal{L}_k \) for each \( n \in \Xi \) and, for each \( k = 1, \ldots, d \), we know how each \( \mathcal{U}_k \) is defined on \( \mathcal{H}(K_{W, z, 0}) \bigoplus \bigoplus_{k=1}^d \bigoplus_{n \in \Xi} z^n \mathcal{D}_0 \bigoplus \mathcal{H}(K_{W, z, 0}) \) (where we write \( \mathcal{D} = \mathcal{D}_0 \oplus \mathcal{F} \)). The problem of constructing a multievolution scattering system compatible with the given augmented Agler decomposition then reduces to the following: find a simultaneous commuting unitary extension of partially defined isometries \( V = (V_1, \ldots, V_d) \)
(defined on spaces in functional-model form) while maintaining the additional geometric structure \((OD), (CDP), (SL), (SL^\ast)\). Remarkably, due to the constructions in Section 4 (i.e., the general trajectory-space construction or, more concretely, the Schäffer-matrix construction in Remark 4.1), this a priori hard problem comes down to finding a single unitary extension \(U\) of the partially defined unitary colligation \(V\); this latter problem in turn can be handled as in Theorem 7.1.

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