Determination of hydraulic fracture parameters using a non-stationary fluid injection

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Abstract. In this paper, one provides a theoretical justification of the possibility of hydraulic fracture parameters determination by using a non-stationary fluid injection. It is assumed that the fluid is pumped into the fractured well with the time-periodic flow rate. It is shown that there is a phase shift between waves of fluid pressure and velocity. For the modelling purposes, the length and width of the fracture are assumed to be fixed. In the case of infinite fracture, one constructs an exact solution that ensures analytical determination of the phase shift in terms of the physical parameters of the problem. In the numerical calculation, the phase shift between pressure and velocity waves is found for a finite fracture. It is shown that the value of the phase shift depends on the physical parameters and on the fracture geometry. This makes it possible to determine parameters of hydraulic fracture, in particular its length, by the experimental measurement of the time shift and comparison with the numerical solution.

1. Introduction
One of the important components of hydraulic fracturing procedure is the control of the fracture consisting of the determination of its basic geometrical parameters: size and position. The control is usually performed during its production and based on the analysis of data of the microseismic monitoring [1]. However, this method is not precise and gives only an estimation of the fracture positioning. Thus, any complementary studies for the control of hydraulic fracture are highly relevant.

In this paper, one provides a theoretical justification of the possibility to use a non-stationary fluid injection to determine the parameters of hydraulic fracture. It is assumed that the fluid is injected into the fractured well with a time-periodic flow rate. It is shown that a phase shift between the waves of fluid pressure and velocity exists in this case. The phase shift depends on the formation parameters as well as the fracture length. Thus, the simultaneous measurement of pressure and flow rate at the borehole provides additional information about the formation and the fracture properties.

Mathematical modelling is based on the Biot’s poroelastic medium model [2]. Boundary conditions on the fracture are adopted from paper [3]. Both the length and the aperture of the fracture are assumed to be fixed. In the case of an infinite fracture, one constructs an exact solution that makes it possible to determine the phase shift explicitly as a function of the problem parameters. For the fracture of a finite length, a numerical solution of the problem is provided. It is shown that the value of the phase shift depends on the physical parameters and on the geometry of the fracture in all cases. This observation indicates a principal possibility for the
determination of physical parameters of the formation and of the fracture by the experimental measurement of the phase shift and comparison with the numerically predicted phase shift.

2. A mathematical model
One considers a planar vertical hydraulic fracture of fixed height $2H$, width $2w$, and length $2L$ extending along the $x$-axis in the poroelastic formation (Figure 1). The fluid is pumped into the fracture through the wellbore that is located along the vertical axis at $x = y = 0$.

![Figure 1. Fracture geometry.](image)

It is assumed that pores are saturated by a single-phase Newtonian fluid with the efficient viscosity $\eta$ and efficient density $\rho$. Also, physical properties of the injected fluid and the formation fluid are assumed to be identical.

Due to the statement of the problem, one uses the plane strain approximation, which implies that there is no deformation along the vertical axis, and all cross-sections by the planes $z = H_1, |H_1| \leq H$ are identical. Hence, the 2D problem in the plane $z = 0$ is solved (Figure 2). The displacement vector $u$ is two-dimensional: $u = (u_1, u_2) = (u, v)$. Fluid pressure $p$ and displacement $u$ depend on time $t$ and coordinates $(x_1, x_2) = (x, y)$. The fracture is located along the line $y = 0$ and occupies the segment $-L \leq x \leq L$.

![Figure 2. Fracture’s cross-section by the plane $z = 0$.](image)

Symmetry with respect to planes $x = 0$ and $y = 0$ makes it possible to solve the problem only in the first quarter of $Oxy$ plane. Consider the domain $\Omega = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$.

Outside the fracture, functions $u$ and $p$ satisfy the system

$$\Omega : \frac{\partial p}{\partial t} = \text{div} \left( \frac{k_r}{\eta} \nabla p - \alpha \frac{\partial u}{\partial t} \right),$$

(2)

where $\tau$ is the stress tensor, $k_r$ is the formation permeability, $\eta$ is the fluid viscosity, $S_\varepsilon$ is the fluid yielding capacity coefficient, and $\alpha$ is the Biot coefficient. According to the Biot theory [2], the tensor $\tau$ is defined as follows

$$\tau = \lambda \varepsilon(u) \cdot I + 2\mu \mathcal{E}(u) - \alpha p \cdot I, \quad \varepsilon(u) = \text{tr} \mathcal{E}(u) = \mathcal{E}(u)_{ii} = \text{div} u,$$

(3)

where $I_{ij} = \delta_{ij}$, $\mathcal{E}(u)$ is the strain tensor related to the field $u$: $2\mathcal{E}(u)_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$, $\lambda$ and $\mu$ are the elasticity moduli. Given the Young modulus $E$ and the Poisson ratio $\nu$ for the fluid-saturated rock, one can compute the elasticity moduli as

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}.$$
Equation (1) in view of (3) takes the form

\[(\lambda + \mu) \nabla \text{div} u + \mu \Delta u - \alpha \nabla p = 0. \tag{4}\]

Over the outer border \(\Gamma_e = \partial \Omega \cap \{x = a\} \cup \{y = b\}\), a load \(\sigma_\infty\) is applied, and a pore pressure \(p_\infty\) is prescribed:

\[\Gamma_e : \quad p = p_\infty, \quad \tau(n) = -\sigma_\infty n, \tag{5}\]

where \(n\) is the outward normal unit vector to \(\Gamma_e\).

Over the lines \(x = 0\) and \(y = 0\), the symmetry conditions are required

\[
\begin{align*}
\Gamma_l &= \{x = 0, 0 < y < b\} : \quad u = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial p}{\partial x} = 0, \tag{6} \\
\Gamma_s &= \{y = 0, L < x < a\} : \quad \frac{\partial u}{\partial y} = 0, \quad v = 0, \quad \frac{\partial p}{\partial y} = 0. \tag{8} \\
\end{align*}
\]

Within the fracture, the fluid mass conservation law takes the form

\[
\frac{\partial w}{\partial t} + \frac{\partial (wV)}{\partial x} = -q. \tag{9}
\]

Here, the fluid velocity \(V(t,x)\) in \(x\)-direction is obtained by the Darcy’s law

\[
V(t,x) = -\frac{k_c}{\eta} \frac{\partial p}{\partial x}, \tag{10}
\]

where \(k_c\) is the fracture permeability. Exchange of fluid between the fracture and the surrounding medium is governed by the leakoff rate \(q\) that is also given by the Darcy’s law for the formation:

\[
q = -\frac{k_r}{\eta} \frac{\partial p}{\partial y}. \tag{11}
\]

Thus, equations (9), (10), (11) and the requirement that the fracture has a constant width \(w = \text{const}\) yield

\[
\Gamma_e : \quad -\frac{\partial}{\partial x} \left(\frac{w k_c}{\eta} \frac{\partial p}{\partial x}\right) = \frac{k_r}{\eta} \frac{\partial p}{\partial y}. \tag{12}
\]

With \(Q(t)\) being the volumetric flow injection rate at the borehole per unit fracture height (m\(^2\)/s), there are the following injection condition

\[
\Gamma_e : \quad -\frac{w k_c}{\eta} \frac{\partial p}{\partial x} \bigg|_{y=0+,x=0+} = \frac{1}{4} Q(t). \tag{13}
\]

To complete the formulation, the initial data are specified in the following form:

\[t = 0 : \quad u = u_0(x,y), \quad p = p_0(x,y). \]

The mathematical model for a non-fixed fracture of a variable aperture is presented in [3].
3. Exact solution for the infinite fracture

Let one assume that the fracture has infinite length $L = \infty$. In this case, the domain is $\Omega = \mathbb{R}^2$, and the fracture position is described as $\Gamma_c = \{ y = 0 \}$.

The problem (4), (2), (12) is considered. Let one seek a solution in the form of a wave packet

$$u = \varphi e^{i\omega t - (m \cdot x)}$$
(14)

$$p = p_0 e^{i\omega t - (m \cdot x)}.$$  
(15)

Here, $i = \sqrt{-1}$ is the imaginary unit, $\varphi = \left( \varphi_1 \varphi_2 \right)$, $m = \left( m_1 m_2 \right)$ are constant complex vectors, $\omega$ is a real constant. By definition, $(a \cdot b) = a_1 b_1 + a_2 b_2$ for all complex vectors $a = \left( a_1 \right)$, $b = \left( b_1 \right)$.

Substituting (14) and (15) to the system of equations (4), (2), (12) the system of algebraic equations with four complex unknowns $\varphi_1$, $\varphi_2$, $m_1$, $m_2$ is obtained:

$$(\lambda + \mu) (\varphi \cdot m) m + \mu (m \cdot m) \varphi + \alpha_p \varphi m = 0,$$  
(16)

$$i \omega p_0 = \frac{k_r}{S_{E \eta}} (m \cdot m) p_0 + \frac{\alpha}{S_{E}} i \omega (\varphi \cdot m),$$  
(17)

$$m_1^2 = \sigma m_2,$$  
(18)

where $\sigma = k_r/(\omega k)$. One can show that the unique solution of (16)–(18) has the following form

$$m = \left( \sigma \sqrt{\frac{-\cos \theta}{\cos 2\theta}} e^{\frac{\sigma}{2}} \right), \quad \varphi = \frac{i \alpha p_0}{A(\lambda + 2\mu)} m,$$  
(19)

where

$$\theta = \frac{1}{2} \left( \pi - \arcsin \left( \sqrt{\frac{\sigma^4}{16 A^2} + 1 - \frac{\sigma^2}{4 A}} \right) \right), \quad A = \omega \frac{k_r}{S_{E \eta}} \left( \frac{\alpha^2}{S_{E}} \frac{(\lambda + 2\mu)}{1} + 1 \right).$$

Since $m$ is a complex vector, it can be represented in the form $m = m_1 + im_2$, where $m_1$ and $m_2$ are real and imaginary parts of $m$, respectively. Both the real and the imaginary parts of (4), (2), (12) give the solution of the initial problem:

$$\text{Re } u = C(m_2 \cos(\omega t - m_2 \cdot x) + m_1 \sin(\omega t - m_2 \cdot x)) e^{-m_1 \cdot x},$$
\text{Re } p = p_0 \cos(\omega t - m_2 \cdot x) e^{-m_1 \cdot x},$$
(20)

$$\text{Im } u = -C(m_1 \cos(\omega t - m_2 \cdot x) - m_2 \sin(\omega t - m_2 \cdot x)) e^{-m_1 \cdot x},$$
\text{Im } p = p_0 \sin(\omega t - m_2 \cdot x) e^{-m_1 \cdot x},$$
(21)

where $C = \frac{-\alpha p_0}{A(\lambda + 2\mu)}$.

$$m_1 = \left( \sigma \sqrt{\frac{-\cos \theta}{\cos 2\theta}} \cos \frac{\theta}{2} \right), \quad m_2 = \left( \sigma \sqrt{\frac{-\cos \theta}{\cos 2\theta}} \sin \frac{\theta}{2} \right)$$

are the real vectors; $\omega$ and $p_0$ are free parameters.
4. The phase shift
An analysis of exact solutions (20), (21) shows that there is a phase shift between fluid pressure and velocity waves. Indeed, according to the Darcy’s law, the fluid velocity is proportional to the pressure gradient:

\[ \mathbf{V} \sim -\nabla p = m_0 e^{i\omega t - \mathbf{m} \cdot \mathbf{x}}. \]

The fluid velocity within the fracture can be represented as

\[ V_1 \sim m_1 p_0 e^{i\omega t - m_1 x_1} = |m_1| p_0 e^{i(\omega t + \arg m_1 - m_1 x_1)}. \]

Hence, the number \( \arg m_1 / \omega \) determines the phase shift between velocity and pressure waves. In view of \( \arg m_2 = \theta \) by introduction of the dimensionless complex \( \xi = \sigma^2 / A \), one can find \( \arg m_1 \) as a function of \( \xi \):

\[
\arg m_1 = \frac{1}{2} \theta = \frac{1}{4} \left( \pi - \arcsin \left( \sqrt{\frac{\sigma^4}{16A^2} + 1 - \frac{\sigma^2}{4A}} \right) \right) = \\
\frac{1}{4} \left( \pi - \arcsin \left( \sqrt{\frac{\xi^2}{16} + 1 - \frac{\xi}{4}} \right) \right) = \arg m_1(\xi).
\]

This dependence is presented in Figure 3. Thus, in the case of an infinite fracture, the phase shift is fully determined by the value of \( \xi \).

Note that the constructed exact solution (20), (21) has a disadvantage, because it does not describe the fluid injection at the downhole at \( x = y = 0 \) (the condition (13) is not satisfied). Therefore, the obtained result regarding the phase shift can be observed as a motivation for the further analysis of the correct statement of the problem for the finite-length fracture.

5. Numerical solution
In this section, one constructs a numerical solution for the problem with a finite-length fracture as formulated in Section 2. The problem is solved by the finite-element method. To this end, one writes a variational formulation of the model developed above. First, one homogenises the boundary conditions over the outer boundary \( \Gamma_e \). By denoting,

\[ \tilde{u} = u - \kappa x, \quad \tilde{p} = p - p_\infty, \quad \kappa = \frac{\alpha p_\infty - \sigma_\infty}{2(\lambda + \mu)}, \]

Figure 3. Graph of \( \arg m_1(\xi) \)
it is found that
\[ \mathcal{E}(\mathbf{u}) = \mathcal{E}(\mathbf{u}) + \mathbf{x} \cdot \mathbf{I}, \quad \tau = \tilde{\tau} - \sigma_{\infty} \cdot \mathbf{I}, \quad \tilde{\tau} = \lambda \epsilon(\mathbf{u}) \cdot \mathbf{I} + 2\mu \mathcal{E}(\mathbf{u}) - \alpha \tilde{\mathbf{p}} \cdot \mathbf{I}. \]

Functions \( \hat{\mathbf{u}} \) and \( \hat{\mathbf{p}} \) solve the boundary-value problem
\[ \Omega : \ \text{div} \ \tilde{\tau} = 0, \quad S_c \frac{\partial \tilde{\mathbf{p}}}{\partial t} = \text{div} \left( \frac{k_r}{\eta} \nabla \tilde{\mathbf{p}} - \alpha \frac{\partial \hat{\mathbf{u}}}{\partial t} \right), \quad (22) \]
\[ \Gamma_e : \ \hat{\mathbf{p}} = 0, \quad \tilde{\tau}(\mathbf{n}) = 0, \]
\[ \Gamma_l : \ \frac{\partial \tilde{\mathbf{v}}}{\partial x} = 0, \quad \hat{\mathbf{u}} = 0, \quad \frac{\partial \tilde{\mathbf{p}}}{\partial x} = 0, \]
\[ \Gamma_s : \ \frac{\partial \hat{\mathbf{u}}}{\partial y} = 0, \quad \hat{\mathbf{v}} = 0, \quad \frac{\partial \tilde{\mathbf{p}}}{\partial y} = 0, \]
\[ \Gamma_e : \ \hat{\mathbf{v}} = 0, \quad -\frac{\partial}{\partial x} \left( \frac{w k_c}{\eta} \frac{\partial \tilde{\mathbf{p}}}{\partial x} \right) = \frac{k_r}{\eta} \frac{\partial \tilde{\mathbf{p}}}{\partial y}, \quad (23) \]
\[ \Gamma_e : \ -\frac{w k_c \frac{\partial \tilde{\mathbf{p}}}{\partial x}}{\eta} \big|_{y=0,x=0+} = \frac{1}{4} Q(t). \quad (24) \]

For simplicity, the initial data are chosen as follows:
\[ t = 0 : \ \hat{\mathbf{u}} = 0, \quad \tilde{\mathbf{p}} = 0. \quad (25) \]

Let one denote by \( \psi(x, y) \) and \( \varphi(x, y) \) any smooth scalar and vector test functions such that
\[ \psi|_{\Gamma_s} = 0, \quad \varphi \cdot \mathbf{n}|_{\Gamma_s} = 0. \quad (26) \]

Keeping in mind the boundary conditions over \( \Gamma_e, \Gamma_l, \Gamma_s, \Gamma_c \), one multiplies the first equation in (22) and the second equation in (22) by \( \varphi \) and \( \psi \), respectively, and integrates over the domain \( \Omega \). In this way by using equation (23) and (24), one arrives at the equalities
\[ \int_{\Omega} \lambda \text{div} \hat{\mathbf{u}} \text{div} \varphi + 2\mu \mathcal{E}(\hat{\mathbf{u}}) : \mathcal{E}(\varphi) - \alpha \hat{\mathbf{p}} \text{div} \varphi \text{d}x \text{d}y = 0, \]
\[ \int_{\Omega} \left[ \left( S_c \frac{\partial \tilde{\mathbf{p}}}{\partial t} + \alpha \frac{\partial \tilde{\mathbf{p}}}{\partial t} \right) \psi + \frac{k_r}{\eta} \nabla \tilde{\mathbf{p}} \cdot \nabla \psi \right] \text{d}x \text{d}y = \int_{\Gamma_e} \frac{w k_c \frac{\partial \psi}{\partial x} \frac{\partial \tilde{\mathbf{p}}}{\partial x}}{\eta} \text{d}x + \psi(0, 0)Q(t)/2, \]

Next, the dimensionless variables are introduced
\[ x' = \frac{x}{L}, \quad t' = \frac{t}{t_s}, \quad \hat{\mathbf{u}}' = \frac{\hat{\mathbf{u}}}{w}, \quad \tilde{\mathbf{p}}' = \frac{\tilde{\mathbf{p}}}{p_s}, \quad Q_a' = \frac{Q}{Q_a}, \]

where \( t_s, p_s, \) and \( Q_a \) are reference values; \( L' = 1 \) is the dimensionless length of fracture. In the new variables,
\[ \Omega' = \{(x', y') : 0 \leq x' \leq \frac{a}{L}, 0 \leq y' \leq \frac{b}{L}\}, \]
\[ \Gamma_e' = \{(x', y') : y' = 0, 0 \leq x' \leq 1\}, \quad \Gamma_s' = \{y' = 0, 1 \leq x' \leq \frac{a}{L}\}, \]
\[ \Gamma_l' = \partial \Omega' \cap \{(x' = \frac{a}{L}) \cup \{y' = \frac{b}{L}\}\}, \quad \Gamma_l' = \{x' = 0, 0 \leq y' \leq \frac{b}{L}\}. \]
Omitting primes for simplicity, one arrives at the following dimensionless variational formulation

\[
\int_{\Omega} a_1 \text{div } u \text{ div } \varphi + a_2 \mathcal{E}(u) : \mathcal{E}(\varphi) - \alpha p \text{ div } \varphi \, dx dy = 0, \tag{27}
\]

\[
\int_{\Omega} \left( b_1 \frac{\partial p}{\partial t} + \alpha \text{ div } \frac{\partial u}{\partial t} \right) \psi + b_2 \nabla p \cdot \nabla \psi \right) \, dx dy = \int_{\Gamma_c} b_3 \frac{\partial \psi}{\partial x} \frac{\partial p}{\partial x} \, dx + b_4 \psi(0,0)Q. \tag{28}
\]

The test functions satisfy the conditions (26), and the dimensionless numbers are defined as follows:

\[
a_1 = \frac{\lambda w}{L \mu}, \quad a_2 = \frac{2 \mu w}{L \mu}, \quad b_1 = \frac{S_k p_* L}{w}, \quad b_2 = \frac{t_* p_* k_r}{\eta w L}, \quad b_3 = \frac{p_* t_* k}{\eta L^2}, \quad b_4 = \frac{Q_* t_*}{2 w L H},
\]

where \( H \) is the half of the fracture height.

The calculations are performed with the help of the freely accessible finite element PDE solver FreeFEM++ [4]. The time derivatives are approximated by finite differences: \( \partial f/\partial t \approx (f^{n+1} - f^n)/\Delta t \) where \( f \) is one of functions \( p \) or \( u \), and \( \Delta t \) is the time step. The upper index denotes the time instant: \( f^n = f(t_n, x) \), \( t_n = n \Delta t \).

Multiplication of Eq. (28) by the time step \( \Delta t \) and subtraction from Eq. (27) provides a symmetrical formulation with respect to the unknown function \((u, p)\) and the test functions \((\varphi, \psi)\). This assures the symmetry of the stiffness matrix, which is important for the correctness of the numerical algorithm.

To perform computations, one takes 30 mesh vertices over the outer boundary \( \Gamma_c \), 80 over \( \Gamma_t \), 40 over \( \Gamma_f \) and 17 over \( \Gamma_s \). The total number of mesh vertices is 1743. Dimensionless time step is equal to 0.0005 which is equivalent to 1.8 s. Since we start from zero initial data (25), at the beginning of calculations the process non-periodic. However, after some initial period of time (about 2 s), the system comes to a periodic state where the pressure waves repeat at every pulsation of the input data.

6. Numerical simulation

One performs calculations using the following data: \( L = 150 \text{ m}, w = 0.5 \text{ cm}, H = 21 \text{ m}, \alpha = 0.7, k_r = 10 \text{ mD}, k_c = 10^4 \text{ D}, 0.3 \text{ cp}, E = 15 \text{ GPa}, \nu = 0.18, S_{k_*}^{-1} = 1.754 \text{ GPa}, Q_* = 100 \text{ m}^3/\text{day}, a = 200 \text{ m}, b = 200 \text{ m}, t_* = 1 \text{ h}, p_* = 100 \text{ MPa}.

In calculations, one uses a periodic function of the flow injection rate at the downhole \( Q(t) = Q_* \sin(\omega t) \). By using the numerically calculated pressure \( p \), the flow velocity at the downhole is computed via the Darcy’s law (10). Now, one plots the \( vp \)-diagram of the process by drawing a curve \( \{(V(t), p(t)), t = [t_1, t_2]\} \) in the plane \( O vp \). Here, \( t_1 \) is considered to be great enough for the system to come to the periodic state.

In order to find the dependence of the phase shift on the problem’s parameters, calculations for \( \omega t_* = 3, 5, 10 \) and \( L = 10^3, 2.5 \cdot 10^3, 1.5 \cdot 10^4 \) are done. The resulting \( vp \)-diagrams are shown in Figure 4. One can see the elliptical shape of the diagrams, which implies the existence of the phase shift between pressure and velocity waves. In case of zero time shift the diagrams would be flat. The dependence of the time shift on the physical parameters of the problem is also noticed for variation of \( w, \eta, k_r, k_c, S_k \).

Explicit numerical calculation of the time shift based on the calculations with a denser set of variation of parameters \( L \) and \( \omega \) makes it possible to represent the graphs of the dependence of the time shift on the parameters as shown in Figure 5.

One can see the strongly pronounced dependence of the phase shift and the shape of \( vp \)-diagram to length of the fracture \( L \) and frequency of fluid injection \( \omega \). These dependences allow solving the parametric optimization problem: determining formation parameters on the base of the experimental determination of the phase shift between pressure and velocity waves.
Figure 4. $vp$-diagrams for different length of fracture $L$ and frequency $\omega$.

Figure 5. Phase shift as a function of length of the fracture $L$ and frequency of fluid injection $\omega$.

7. Conclusion
Suppose, we can organize a periodic injection/suction of fluid to the fracture at the downhole according to some periodic law. Doing this experiment with different frequencies $\omega_1, \ldots, \omega_k$ we measure simultaneously pressure and fluid velocity at the downhole. Then, knowing the pressure and velocity as functions of time, we can calculate the values of the phase shifts between pressure and velocity waves for different $\omega$. Using the calculations based on the presented model and methods of mathematical optimization we can determine a parameter of interest, for example length of fracture $L$ with a reasonable accuracy.

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References
[1] Rothert E and Shapiro S 2003 Geophysics 68 685–9
[2] Biot M 1956 J. Acoust. Soc. Amer. 28 168–78
[3] Shelukhin V, Baikov V, Golovin S, Davletbaev A and Starovoitov V 2014 Int. J. Sol. Struct. 51 2116–22
[4] Hecht F 2012 J. Numer. Math. 20 251–65