Condensation phenomena in fat-tailed distributions

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Abstract – We analyze the large-deviations regime of the sample mean of independent and fat-tailed distributed random variables. In this case, a phase transition due to condensation phenomena takes place, yielding failure of standard Large Deviation Theory. In the present work we show how the Density Functional formalism, previously introduced by Dean and Majumdar in the context of Random Matrix Theory, is well suited to tackle this problem. This allows us to characterize the condensation transition in terms of the order parameter, i.e. the Inverse Participation Ratio. By investigating the full distribution of the Inverse Participation Ratio we discover a novel condensation phase transition in the mean vs variance phase diagram. Condensation phenomena addressed above are strongly related to the condensation phenomena in the low temperature phase of disordered systems and are reminiscent of the ones discovered for bipartite entanglement of a random pure state.

Introduction. – Condensation phenomena are ubiquitous in nature. Bose-Einstein condensation \cite{1} is probably the most known example in condensed-matter physics. In disordered systems, the Random Energy Model displays a condensation of the measure at the critical temperature separating paramagnetic and spin-glass phase \cite{2}. Financial correlation matrices \cite{3}, bipartite quantum systems \cite{4}, networks \cite{5}, and non-equilibrium mass transport models \cite{6} are examples of systems in which condensation phenomena take place. In spite of the complexity of this phenomenon, condensation may occur also in very simple systems, e.g. the sum of independent and identically distributed (i.i.d.) random variables with \textit{fat-tailed} distributions.

Fat-tailed distributions have been found to describe events in a variety of frameworks: the magnitude of earthquakes \cite{7}, forest-fires \cite{8}, rain events \cite{9}, cities size \cite{10}, economic wealth \cite{11}, and returns of stocks’ prices \cite{12} among the others. Due to the nature of fat-tailed distributions, extreme events are not so rare, and, depending on the domain, they take the name of financial crashes, hurricanes, billionaires, etc.

In this work, we investigate the appearance of concentration phenomena in the outcome of the sum of fat-tailed distributed random variables. Let us make this statement clearer. Suppose we observe a large daily return in a stock price. Shall we expect that this is generated by the sum of small minute returns with equal size? Or rather that the large daily return is due to a single minute return with very large magnitude? Even further, how does the size of the returns and their probability distribution influence this expectation? Some of these questions have been previously addressed in the literature and a phase transition from a “democratic” to a condensed phase has been recently discovered in connection to low-temperature disordered systems \cite{15}, out-of-equilibrium mass-transport models \cite{6}, and financial applications \cite{16}. By introducing an order parameter, we describe the main features of this phase transition in a statistical-mechanics flavor. By exploiting the formalism introduced by Dean and Majumdar \cite{17} in Random Matrix Theory, we both characterize the phase transition and get an analytical expression for the order

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random variables distributed according to a power-law probability density function (p.d.f.)
\[ p(x) \simeq \frac{A}{x^{\alpha+1}}, \]
with \( \alpha > 0 \). The Central Limit Theorem (CLT) states that the limit p.d.f. of the sum of i.i.d. random variables:
\[ X_N = \sum_{i=1}^{N} x_i \]
converges to the Gaussian distribution if \( \alpha > 2 \) or to a Lévy stable distribution if \( 0 < \alpha \leq 2 \) [15]. Let us consider random variables distributed according to a p.d.f. in the Lévy basin of attraction. As reported by Bouchaud and Georges [19]:

- for \( 0 < \alpha \leq 1 \), both \( x \) and \( \langle X \rangle \) are infinite and \( X_N \) scales as \( N^{1/\alpha} \) (as \( N \ln N \) for \( \alpha = 1 \));
- for \( 1 < \alpha \leq 2 \), \( x \) is finite, whereas \( \langle x^2 \rangle \) and the variance \( \langle X^2 \rangle - \langle X \rangle^2 \) are infinite. The difference between \( X_N \) and \( \langle X \rangle \) scales as \( N^{1/\alpha} \) (as \( \sqrt{N \ln N} \) for \( \alpha = 2 \)).

It is also possible to demonstrate that the largest variable \( x_{\text{max}} \) among all variables \( \{x_1, x_2, \ldots, x_N\} \) for large \( N \) scales as \( N^{1/\alpha} \). Since both \( X_N \) and \( x_{\text{max}} \) scale in the same way for i.i.d. random variables with very broad distributions, the typical outcome of \( X_N \) may yield “condensation”, i.e. it may be dominated by the single variable \( x_{\text{max}} \).

In order to make the previous statements more rigorous, following [20], we can define the weight of the \( i \)-th term of the sum \( X_N \):
\[ w_i = \frac{x_i}{X_N} \]
and the \( k \)-th (non-centered) sample moment of the weights:
\[ Y_k = \frac{1}{N} \sum_{i=1}^{N} (w_i)^k, \]
where \( k > 1 \). The variable \( Y_k \) can be used to quantify the degree of condensation of \( X_N \), and therefore is a good candidate as an order parameter. Let us consider, for example, the second moment \( Y_2 \), which is called Inverse Participation Ratio (IPR). If all the \( w_i \) are of order \( 1/N \) then \( Y_2 \sim 1/N \) and tends to zero for large \( N \). On the other hand, if at least one \( w_i \) remains finite when \( N \to \infty \), then \( Y_2 \) will also be finite. We will refer to the former as a democratic outcome and to the latter as a condensed outcome. In this case, the variables carrying a finite weight \( w_i \) of \( X_N \) are called condensates. The average value \( \langle Y_k \rangle \) can be analytically computed in the large \( N \) limit [20].

\[ \langle Y_k \rangle \simeq \frac{\Gamma(k-\alpha)}{\Gamma(k)\Gamma(1-\alpha)}, \]
whereas it vanishes for \( \alpha > 1 \). In the case of \( k = 2 \), the average value of the IPR reads: \( \langle Y_2 \rangle = \max\{1-\alpha,0\} \) (see Fig. 1). The critical value \( \alpha_c = 1 \), separates a democratic phase (\( \alpha > \alpha_c \)) from a condensed phase (\( \alpha < \alpha_c \)). We can therefore describe the phase transition in a statistical-mechanics flavor, using the random variable \( Y_k \) as an order parameter to characterize the phase transition.

**Large deviations.** - While in the previous section we have shown that, for extremely fat-tailed distributions (\( \alpha < 1 \)), the typical outcomes of \( X_N \) is condensed, in the following we investigate the appearance of condensation phenomena in the large-deviations regime of fat-tailed distributions where \( X_N \) is not typically condensed (\( \alpha > 1 \)). It is well known that the CLT, when applicable, provides a good approximation only for the center of the p.d.f. of \( X_N \), leaving its tails subject to further investigation. In this scenario, we can invoke the LDT as an extension or refinement of the Law of Large Numbers and of the CLT [21]. Without loss of generality let us consider the rescaled random variable \( S_N = X_N/N \), corresponding to the sample mean of the i.i.d. random variables \( \{x_1, x_2, \ldots, x_N\} \). In order to analyze the large-deviations regime we consider the p.d.f. \( P_N(S_N = m) \), denoting the probability that \( S_N \) assumes a value in the infinitesimal interval \([m,m+dm])\:

\[ P_N(S_N = m) = \int_{m}^{m+dm} \prod_{i=1}^{N} dx_i p(x_i) \, \delta \left( \frac{1}{N} \sum_{i=1}^{N} x_i - m \right). \]
sub-exponential p.d.f. $p(x)$ and finite expectation value $\langle x \rangle$, the LDT leads to a well-defined rate function for $m < \langle x \rangle$, whereas for $m > \langle x \rangle$ the large-deviations principle does not hold anymore and the rate function vanishes. This suggests that a phase transition is occurring, where the mean $m$ plays the role of a control parameter, and its critical value is $m_c = \langle x \rangle$. It has been argued [16] that condensation phenomena in the outcomes of the sample mean $S_N$ are at the very basis of this phase transition. Unfortunately, the “failure” of the LDT in the condensed regime does not allow to have a quantitative prediction on the way in which the condensed sample mean is typically obtained. We will overcome some of the limitations imposed by the standard LDT by exploiting the Density Functional Method, introduced in the context of Random Matrix Theory [17].

The method. – Let us consider again $N$ non-negative i.i.d. random variables $\{x_1, x_2, \ldots, x_N\}$ with sub-exponential distribution $p(x)$. We are interested in investigating how the large deviations of $S_N$ are obtained in terms of the set $\{x_1, x_2, \ldots, x_N\}$, namely, if condensation phenomena take place. Given the constraint $S_N = m$, we can write the conditional joint p.d.f. of the variables:

$$P_N(x_1, x_2, \ldots, x_N|m) = \frac{\prod_i p(x_i)}{P_N(S_N = m)} \delta\left(\frac{1}{N} \sum_i x_i - m\right). \quad (7)$$

In a statistical-mechanics interpretation, the distribution $\rho(x)$ can be thought as a Boltzmann weight at unitary temperature:

$$P_N(x_1, x_2, \ldots, x_N|m) \propto \exp\{-E([x_i])\} \quad (8)$$

where the effective energy is given by:

$$E([x_i]) = -\sum_{i=1}^N \ln p(x_i) \quad \text{with} \quad \frac{1}{N} \sum_{i=1}^N x_i = m. \quad (9)$$

Therefore, we are dealing with a system of $N$ independent particles in a potential $V(x) = -\ln p(x)$ interacting through a global constraint. The normalization constant $P_N(S_N = m)$ plays the role of a partition function and can be evaluated through the integral [10]. In the large $N$ limit, we can make a change of variables and trade the multiple integral over the $N$ variables $\{x_1, x_2, \ldots, x_N\}$ with a functional integral over the density function:

$$\rho(x) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i), \quad (10)$$

with the constraint: $\int dx \rho(x) = 1$. This leads to:

$$P_N(S_N = m) \propto \int \mathcal{D}\rho \, e^{-NE[\rho]} \quad (11)$$

where $E[\rho]$ is the (intensive) functional form of the effective energy [9]. At the leading order in $N$, it reads:

$$E[\rho] = \int_0^\infty dx \rho(x) \ln \frac{\rho(x)}{p(x)} + \rho_0 \left( \int_0^\infty dx \rho(x) - 1 \right) + \mu_1 \left( \int_0^\infty dx \delta \rho(x) \, m \right). \quad (12)$$

The first term is the Kullback-Leibler divergence $D_{KL}(\rho(x) || p(x))$, measuring the distance between the two distributions $\rho(x)$ and $p(x)$ [22]. It is the difference of the energetic term $- \int dx \rho(x) \ln p(x)$ and the entropic term $- \int dx \rho(x) \ln \rho(x)$, coming from the Jacobian of the change of variables. The Lagrange multipliers $\rho_0$ and $\mu_1$ enforce the constraints on the normalization of $\rho(x)$ and on the sample mean $S_N$. The functional integral [11] can be evaluated through the saddle-point approximation, leading to $P_N(S_N = m) \sim e^{-NE[\rho^*]}$. The density $\rho^*(x)$ which minimizes the effective energy $E[\rho]$ can be found via the saddle-point equation:

$$\frac{\delta E}{\delta \rho} \bigg|_{\rho = \rho^*} = 0. \quad (13)$$

Using the expression (12) we get:

$$\rho^*(x) = \frac{p(x) e^{\mu_1 x}}{\int dx p(x) e^{\mu_1 x}} \quad (14)$$

where the Lagrange multiplier $\mu_1$ is fixed by the constraint:

$$\int dx \frac{x p(x) e^{\mu_1 x}}{\int dx p(x) e^{\mu_1 x}} = m. \quad (15)$$

Equation (15) admits a solution only for $m < \langle x \rangle$. In this case, we obtain for sub-exponential distributions the same results imposed by Sanov’s theorem to discrete random variables [2,22]. If we use expression (12) to define the effective energy of the system, we are implicitly assuming that the large deviations of $S_N$ are obtained through a democratic outcome, since all variables $\{x_1, x_2, \ldots, x_N\}$
have a unique scaling behaviour determined by the saddle-point density \( \rho^\ast (x) \). Expression \( \text{(12)} \), then, relies on a democratic ansatz. In order to explore the unaccessible region \( m > \langle x \rangle \), we must break down this ansatz and turn to a condensed ansatz, imposing that variables could have different scaling behaviours. This can be achieved through the substitution:

\[
N \rho(x) = (N - 1) \rho_c(x) + \delta(x - x_c), \tag{16}
\]

which allows the variable \( x_c \) to have a different scaling behaviour with respect to the bulk composed of the other \((N - 1)\) variables and described by the density \( \rho_c(x) \). The new ansatz accounts for the spontaneous symmetry break \( S_N \rightarrow S_{N-1} \times 1 \), where \( S_N \) is the permutation group of \( N \) elements. Indeed, in the democratic phase the system is invariant under a generic permutation of the random variables, whereas in the condensed phase this is not the case. The new density functional, expressed in terms of the density \( \rho_c(x) \), leads to the effective energy:

\[
E[\rho_c, x_c] = \int_0^\infty dx \, \rho_c(x) \ln \frac{\rho_c(x)}{p(x)} - \frac{1}{N} \ln p(x) + \mu_0 \left( \int_0^\infty \! dx \, \rho_c(x) - 1 \right) + \mu_1 \left( \int_0^\infty \! dx \, x \, \rho_c(x) + x_c - m \right). \tag{17}
\]

In order to have the same scaling in \( N \) for all the leading terms of equation \( \text{(17)} \) we impose \( x_c = Nt \). By neglecting lower order terms in \( N \) we get:

\[
E[\rho_c, Nt] \simeq \int_0^\infty \! dx \, \rho_c(x) \ln \frac{\rho_c(x)}{p(x)} + \mu_0 \left( \int_0^\infty \! dx \, \rho_c(x) - 1 \right) + \mu_1 \left( \int_0^\infty \! dx \, x \, \rho_c(x) + t - m \right). \tag{18}
\]

The saddle-point density \( \rho^\ast_c(x) \) and value \( t^\ast \) that minimize the effective energy must verify:

\[
\frac{\delta E}{\delta \rho_c} \bigg|_{\rho_c = \rho^\ast_c, t = t^\ast} = 0 \quad \text{and} \quad \frac{\partial E}{\partial t} \bigg|_{\rho_c = \rho^\ast_c, t = t^\ast} = 0. \tag{19}
\]

The minimization brings to \( \rho^\ast_c(x) = p(x) \) and \( t^\ast = m - \langle x \rangle \). Therefore, for large \( N \), the \((N - 1)\) non-condensed random variables behave as independent and distributed according to the original p.d.f. \( p(x) \). At variance, the condensed variable \( x_c^\ast = Nt^\ast \) behaves as:

\[
x_c^\ast \simeq N(m - \langle x \rangle) \tag{20}
\]

and carries a finite fraction of \( S_N \). Notice that, at the leading order in \( N \), the saddle-point energy \( E[\rho^\ast_c, x_c^\ast] \) vanishes and leads to a vanishing rate function \( f(m) \) for all \( m > \langle x \rangle \). This is in full agreement with the standard results of the LDT.

In conclusion, the Density Functional Method allows to identify the critical point \( m_c = \langle x \rangle \) separating a democratic phase \( (m < m_c) \) from a condensed phase \( (m > m_c) \). The condensed phase exists only if the expected value \( \langle x \rangle \) is finite. These results, in the case of power-law distributions \( \text{(1)} \), are represented in Fig. 2 and are in full agreement with standard LDT results \( \text{(11)} \) and with the grand-canonical analysis of the mass-transport model performed in \( \text{(6)} \).

We can further investigate the phase transition from the democratic to the condensed phase by studying the behaviour of the order parameter \( Y_k \) in the two regions. We will denote by \( \langle \cdot \rangle_m \) the average of a generic random variable according to the constrained measure \( \text{(7)} \). With some simple algebra we can thus express the constrained average of the order parameter \( Y_k \) as:

\[
\langle Y_k \rangle_m = \frac{1}{N^{k-1} m^k} \int_0^\infty \! dx \, x^k \langle \rho(x) \rangle_m, \tag{21}
\]

where \( \langle \rho(x) \rangle_m = \left( \frac{1}{N} \sum_{i=1}^N \delta(x - x_i) \right)_m \) is the marginal distribution of a random variable \( x \) with respect to the constrained measure \( \text{(7)} \). In the large \( N \) limit, the marginal distribution \( \langle \rho(x) \rangle_m \) is asymptotically equal to the saddle-point density \( \rho^\ast(x) \) (democratic phase) and to \( \frac{N-1}{N} \rho_c^\ast(x) + \frac{1}{N} \delta(x - x_c^\ast) \) (condensed phase). We can therefore compute \( \langle Y_k \rangle_m \), obtaining that \( \langle Y_k \rangle_m \) vanishes in the region \( m < m_c \) and:

\[
\langle Y_k \rangle_m \simeq \left( 1 - \frac{m - m_c}{m} \right)^k \tag{22}
\]

in the region \( m > m_c \). The behaviour of the IPR \( \langle Y^2 \rangle_m \) is shown in Fig. 1.

The general case. — In order to fully characterize the transition from the democratic to the condensed phase in the large-deviations regime of \( S_N \), we can study the full p.d.f. of \( Y_k \) with respect to the constrained measure \( \text{(7)} \). Since \( S_N = m \), we can consider the rescaled random variable \( R_k = N^{k-1} m^k Y_k \), which is equal to the \( k \)-th sample moment \( \frac{1}{N} \sum_{i=1}^N x_i^k \). We can write the constrained p.d.f. of \( R_k \) as:

\[
P_N(R_k = r_k | S_N = m) = \frac{P_N(R_k = r_k, S_N = m)}{P_N(S_N = m)} \tag{23}
\]

where the joint probability \( P_N(R_k = r_k, S_N = m) \) reads:

\[
P_N(R_k = r_k, S_N = m) = \prod_{i=1}^N \int \! dx_i \, p(x_i) \times \delta \left( \frac{1}{N} \sum_{i=1}^N x_i - m \right) \delta \left( \frac{1}{N} \sum_{i=1}^N x_i^k - r_k \right). \tag{24}
\]

The denominator \( P_N(S_N = m) \) in eq. \( \text{(28)} \) has been already studied in the previous section and plays the role of a normalization constant, so let us focus on the numerator \( P_N(R_k = r_k, S_N = m) \). As for \( P_N(S_N = m) \), the integral
can be analyzed with the Density Functional Method according to the previous sections. The democratic ansatz yields the effective energy:

\[ E[p] = \int_0^\infty dx \rho(x) \log \frac{\rho(x)}{p(x)} + \mu_0 \left( \int_0^\infty dx \rho(x) - 1 \right) + \mu_1 \left( \int_0^\infty dx x \rho(x) - m \right) + \mu_k \left( \int_0^\infty dx x^k \rho(x) - r_k \right). \]  

(25)

The latter is similar to equation (12), but with the additional constraint on the moment \( R_k \). The minimization of the functional \( E[p] \) leads to the saddle-point solution:

\[ \rho^*(x) = \frac{p(x) e^{\mu_1 x + \mu_k x^k}}{\int dx p(x) e^{\mu_1 x + \mu_k x^k}}. \]  

(26)

The Lagrange multipliers \( \mu_1 \) and \( \mu_k \) are functions of \( m \) and \( r_k \) and are implicitly defined by imposing the constraints:

\[ \int dx x p(x) e^{\mu_1 x + \mu_k x^k} = m, \]  

(27)

\[ \int dx x^k p(x) e^{\mu_1 x + \mu_k x^k} = r_k. \]  

(28)

As in the previous case, the democratic ansatz holds as long as equations (27) and (28) have a solution in terms of \( \mu_1 \) and \( \mu_k \). If the values \( m \) and \( r_k \) do not admit a solution for equations (27) and (28), then we must turn from the democratic to the condensed ansatz (see eq. (16)). In this case, the effective energy becomes:

\[ E[\rho_c, x_c] = \int_0^\infty dx \rho_c(x) \log \frac{\rho_c(x)}{p(x)} - \frac{1}{N} \ln p(x) + \mu_0 \left( \int_0^\infty dx \rho_c(x) - 1 \right) + \mu_1 \left( \int_0^\infty dx x \rho_c(x) + \frac{x_c}{N} - m \right) + \mu_k \left( \int_0^\infty dx x^k \rho_c(x) + \frac{k}{N} - r_k \right). \]  

(29)

Once again, we must rescale the variables in equation (29) in order to have the same scaling in \( N \) for all the leading terms. Assuming that \( k > 1 \), at the leading order in \( N \) we can write:

\[ E[\rho_c, Nt] \simeq \int_0^\infty dx \rho_c(x) \log \frac{\rho_c(x)}{p(x)} + \mu_0 \left( \int_0^\infty dx \rho_c(x) - 1 \right) + \mu_1 \left( \int_0^\infty dx x \rho_c(x) + t - m \right) + \mu_k \left( \int_0^\infty dx x^k \rho_c(x) + t^k - r_k \right). \]  

(30)

where \( x_c = Nt, \hat{r}_k = N^{1-k} r_k \) and \( \hat{\mu}_k = N^{k-1} \mu_k \). Searching for the saddle-point values \( \rho^*_c(x) \) and \( t^* \) that minimize the effective energy, we notice that the last term of (30) determines \( t^* = (\hat{r}_k)^{1/k} \). Therefore, the asymptotic value of the condensate for large \( N \) is:

\[ x_c^* \simeq (N r_k)^{1/k}. \]  

(31)

The condensate scales differently with respect to the previous case (20) and is driven by the constraint on the moment rather then by the constraint on the mean. The remaining terms of (30) determine the value of the density \( \rho^*(x) \):

\[ \rho^*(x) = \frac{p(x) e^{\mu_1 x}}{\int dx p(x) e^{\mu_1 x}} \]  

(32)

where \( \mu_1 \) is defined by:

\[ \int dx x p(x) e^{\mu_1 x} = m - N^{1/k} r_k^{1/k}. \]  

(33)

So, for large \( N \), we recover for \( \rho^*(x) \) the same expression of the previous section (see eqs. (11) and (15)).

We can use (24) and (25) to study the phase diagram of the system in the space of the control parameters \( m \) and \( r_k \). The two equations should be inverted in order to write \( \mu_1 \) and \( \mu_k \) as functions of \( m \) and \( r_k \). For \( k > 1 \), the phase boundary between the democratic and the condensed phases is defined by the constraint \( \mu_k(m, r_k) = 0 \). Indeed, for \( \mu_k < 0 \) the integrals in (24) and (25) diverge and the democratic ansatz does not hold anymore. For the sake of simplicity, let us consider the case of a fat-tailed p.d.f. \( p(x) \) with finite expectation values \( \langle x \rangle \) and \( \langle x^k \rangle \). For \( \mu_1 = 0 \) and \( \mu_k = 0 \) we find the critical point \( (m, r_k) = (\langle x \rangle, \langle x^k \rangle) \), which lies on the phase boundary. For \( \mu_1 > 0 \) and \( \mu_k = 0 \) we find a regular curve which lies in the region of the space with \( m < \langle x \rangle \). For \( \mu_k < 0 \) we cannot set \( \mu_k = 0 \) directly, but we can set \( \mu_k < 0 \) and than take the limit \( \mu_k \to 0 \). We studied the last curve numerically and the results show a straight line going from \( (\langle x \rangle, \langle x^k \rangle) \) to \( (\infty, \infty) \). The obtained phase diagram is represented in Fig. 3. We recall that these results are valid for the case of sub-exponential distributions with finite expected values \( \langle x \rangle \) and \( \langle x^k \rangle \). These results are in agreement with the ones independently obtained by J. Szavits-Nossan, M.R. Evans and S.N. Majumdar in (29).

Conclusions. – In this work we have investigated the large-deviations regime of the sample mean of independent and fat-tailed distributed random variables. The standard Large Deviation Theory fails in the case of condensation phenomena. We have reported a thorough characterization of the phase transition in terms of the order parameter i.e. the \( k \)-th moment of the weight or, specifically, the Inverse Participation Ratio. It is non-vanishing in the condensed phase and it presents a non-analyticity at the critical point. The investigation of the full distribution of the order parameter leads to perform calculations with an extra-constraint on the sample moment. New condensation phenomena take place in the mean vs moment phase space.
Phase transitions due to condensation phenomena play an important role in different models. The obtained results present strong connections with the low temperature phase of disordered systems. In particular, the entropy-vanishing phase transition in the Random Energy Model is strictly related to the condensation phenomena of fat-tailed random variables, as put forward in several works [15,20,24]. The phase transition from a democratic to a condensed regime due to the extra constraint on the Inverse Participation Ratio is reminiscent to the one taking place in the distribution of Renyi entropies in bipartite quantum systems [1].

The present work opens several perspectives. As shown in [20], the Inverse Participation Ratio could present a strong non-self averaging nature. A complete characterization of the full probability distribution of the Inverse Participation Ration may deserve some attention. It could be interesting to investigate the case of distributions showing fat-tails on both sides of the real axes. Finally, it could be tempting to compare our results with experimental measures. Stocks returns could be an interesting testing ground [25].

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