Note on Gauge Theories on $M/\Gamma$
and the AdS/CFT Correspondence

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Abstract

It is well known that a weakly coupled $U(N)$ gauge theory on a torus with
sides of length $L$ has extra light states with energies of order $1/(NL)$. We show
that a similar result holds for gauge theories on $M/\Gamma$ where $M$ is any compact
Riemannian manifold and $\Gamma$ is any freely acting discrete isometry group. As in the
toroidal case, this is achieved by adding a suitable nontrivial flat connection. As one application, we consider the AdS/CFT correspondence on spacetimes asymptotic to $AdS_5/\Gamma$. By considering finite size effects at nonzero temperature, we show
that consistency requires these extra light states of the gauge theory on $S^3/\Gamma$.

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1 Introduction

Consider a weakly coupled $U(N)$ gauge theory on an $m$-dimensional torus where the length of each circle is equal to $L$. Naively, one might expect that the lowest energy states will have energy of order $1/L$. However, it is known that this is not the case\cite{1,2,3,4,5}. One can add a flat, but globally nontrivial connection which has the effect of changing the boundary conditions on the fields. Rather than having $N^2$ fields each with period $L$, one can arrange to have $N$ fields which are periodic after transversing one circle $N$ times. The lowest energy states thus have energy of order $1/(NL)$. In string theory, this construction has a simple interpretation. The $U(N)$ gauge theory describes the low energy excitations of $N$ D-branes wrapped on $T^m$. However, it also describes the low energy excitations of one D-brane wrapped $N$ times around a circle. The open strings $(i,j)$ connecting the $i^{th}$ and $j^{th}$ brane in the first case are still present in the second, but after going around the circle they are now identified with the $(i+1,j+1)$ strings. So for the wrapped brane, there are only $N$ different types of open strings labelled by $|i-j|$ on a circle of length $NL$. The latter configuration thus has lower energy states and corresponds to adding the flat connection in the gauge theory.

It is natural to ask if this same construction works for other spaces. Consider a $U(N)$ gauge theory on the quotient space $M/\Gamma$ where $\Gamma$ is any freely acting discrete group of isometries of a compact Riemannian manifold $M$. ($\Gamma$ can be nonabelian.) Since the size of $M/\Gamma$ can be much smaller than the original space $M$, one might have thought that the energy of low lying states is increased. However, by analogy with the case of the torus, one can ask if there exists a flat but globally nontrivial connection such that the energy of states in the presence of this connection is the same as the theory on $M$ without the quotient. We will show that when $N = n|\Gamma|$ (where $|\Gamma|$ is the number of elements in $\Gamma$), the answer is yes. The condition on $N$ is exactly what one would expect from wrapping branes. $U(N)$ gauge theory on $M/\Gamma$ can be obtained by wrapping $N$ branes on $M/\Gamma$, but it can also be obtained by wrapping $N/|\Gamma|$ branes on $M$ and then taking the quotient. In the first case each brane is wrapped once. In the second, each brane is wrapped $|\Gamma|$ times.

Although we have mentioned the energy of states at weak coupling, the result we establish is actually more general. Nontrivial flat connections correspond to different classical vacua. Since our space is compact, one must sum over these vacua in the functional integral. At low temperature, the partition function (even at strong coupling) will be dominated by the sector with the most light states. This is near the connection which makes the space look “as big as possible”.

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Since one often considers branes wrapped on cycles in string theory, this result may have various applications. We consider here its application to the AdS/CFT correspondence \([6, 7]\). In the simplest context, this states that string theory on spacetimes which asymptotically approach \(AdS_5 \times S^5\) is equivalent to a conformal field theory (CFT), namely an \(\mathcal{N} = 4\) super Yang-Mills theory, on \(S^3 \times R\). The gauge group of the Yang-Mills theory is \(U(N)\), where \(N\) is related to the string coupling \(g\), string length \(\ell_s\), and the AdS radius \(R\), by \(N = (4\pi g)^{-1}(R/\ell_s)^4\). The Yang-Mills coupling constant is \(g_{YM}^2 = 4\pi g\). In order for the asymptotic time translation in \(AdS_5\) to agree with Hamiltonian evolution in the gauge theory, one must take the radius of the \(S^3\) to be the same as the radius of the \(AdS\) spacetime. This follows from the fact that if one rescales the AdS metric
\[
 ds^2 = -\left(\frac{r^2}{R^2} + 1\right) dt^2 + \left(\frac{r^2}{R^2} + 1\right)^{-1} dr^2 + r^2 d\Omega_3 \tag{1.1}
\]
by \(R^2/r^2\) so that \(t\) is proper time in the asymptotic metric, the radius of the \(S^3\) is \(R\). In a Hamiltonian formulation \([8, 9]\), the equivalence of these two theories implies that the energies of all states must agree. Of course, in most cases, one cannot compute the energy of states in the strongly coupled field theory. But this is possible for states protected by supersymmetry, which includes the gauge theory analogs of all the supergravity modes.

To obtain the dual description of the gauge theory on \(S^3/\Gamma\), we must consider spacetimes which asymptotically approach the quotient of \(AdS_5\) by a freely acting discrete subgroup \(\Gamma\) of \(SO(4)\) acting on the three-spheres of spherical symmetry. If we start with global \(AdS_5\), this produces an orbifold singularity at \(r = 0\). We will make some comments about this orbifold in section 5, but it is difficult to analyze exactly due to the Ramond-Ramond background. However we can confirm the existence of extra light states and avoid the orbifold singularity by going to finite temperature. At temperatures \(T > T_{\text{crit}} \sim 1/R\), the thermal state of the gauge theory is described by the Schwarzschild-AdS solution. One can now take the quotient by \(\Gamma\) without producing any additional singularities. Using this dual description, one can ask when finite size effects become important. We will see from the gravity theory that the answer is \(T \sim 1/R\) for all \(\Gamma\) (provided \(|\Gamma| < R/\ell_s\)). Since the volume of \(S^3/\Gamma\) is much smaller than \(S^3\), this is possible in the gauge theory only if there are extra light states.

In the next section, we show the existence of a nontrivial flat connection on \(M/\Gamma\) with the desired properties. In section 3 we review the case of gauge theory on \(T^3\) and show that the existence of extra light states in this case can be seen from the AdS/CFT correspondence by considering nonzero temperature. A similar argument allows one to infer the existence of extra light states of the gauge theory on \(S^3/\Gamma\). This is discussed
in section 4, and a final section contains some remarks about the zero temperature case (where one has an orbifold of AdS). For a discussion of bound states of branes wrapped on $M/\Gamma$ with nontrivial flat connections see \[10\].

2 Nontrivial flat connection on $M/\Gamma$

In this section we consider a $U(N)$ gauge theory on $M/\Gamma$, where $\Gamma$ is any freely acting discrete isometry group of $M$. We want to find a flat connection so that the density of modes is essentially indistinguishable from the same theory on $M$. More precisely, consider a scalar field $\phi : M/\Gamma \to \mathbb{C}^N \otimes \mathbb{C}^N$ transforming under the adjoint representation of $U(N)$. (The gauge theory also has spinors and gauge fields in the adjoint, and the following argument will apply \textit{mutatis mutandis} for them as well.) For clarity we will focus on the eigenmodes of the gauge covariant Laplacian. At weak coupling, the corresponding eigenvalues are directly related to the energy of states. Normally, one expects the eigenvalues to be larger on $M/\Gamma$, but we will show the following:

There exists a flat connection $A$ such that for every eigenvalue $\lambda$ of the Laplacian on $M$ (with zero connection) there exist solutions to $D^2_A \phi = \lambda \phi$ on $M/\Gamma$. Furthermore, the number of such modes is $1/|\Gamma|$ times the number of corresponding modes on $M$.

Since the volume of $M/\Gamma$ is also reduced by $|\Gamma|$, the number of modes per unit volume is the same as on $M$.

We shall first consider the case $N = |\Gamma|$. The argument is then easily generalized to the case when $N = n|\Gamma|$ for some integer $n$. In the case where $N = |\Gamma|$ the gauge group is $U(|\Gamma|)$. The required nontrivial flat $U(|\Gamma|)$ connection is defined as follows. $M \to M/\Gamma$ is a nontrivial principal $\Gamma$-bundle \[11\]. The regular representation $\mathcal{R} : \Gamma \to U(|\Gamma|)$ allows us to define an associated principal $U(|\Gamma|)$-bundle, using the left action of $\mathcal{R}(\Gamma)$ on $U(|\Gamma|)$. This associated bundle is reducible to an $\mathcal{R}(\Gamma)$ bundle that can be identified with the original $\Gamma$-bundle $M$. The original bundle $M \to M/\Gamma$ carries a natural flat $\Gamma$ connection, since the fiber is discrete: the horizontal subspace at a point $u$ in $M$ is just the full tangent space to $M$ at $u$. This connection induces the required connection $A$ on the associated bundle, with holonomy group $\mathcal{R}(\Gamma)$.

A field $\phi$ on $M/\Gamma$ transforming under the adjoint action of the gauge group $U(|\Gamma|)$ can be characterized as the pullback $s^* \tilde{\phi}$, by a local section $s : M/\Gamma \to M$, of a map

\footnote{We are actually interested in fields valued in the hermitian matrices. This restriction will not lead to any complications however.}
\[ \tilde{\varphi} : M \to C^{[\Gamma]} \otimes C^{[\Gamma]} \] transforming as

\[ \tilde{\varphi}(u \cdot \gamma) = \mathcal{R}(\gamma^{-1})\tilde{\varphi}(u)\mathcal{R}(\gamma) \quad (2.1) \]

under the right action of \( \Gamma \) on \( M \). That is, \( \tilde{\varphi} \) is “equivariant” under the right action of \( \Gamma \) in the adjoint of the regular representation of \( \Gamma \). This global characterization of \( \varphi \) as the pullback of an equivariant map \( \tilde{\varphi} \) on \( M \) encodes the effect of the flat connection. It is equivalent to specifying “twisted boundary conditions” for \( \varphi \) on \( M/\Gamma \) in the following sense. If \( M/\Gamma \) is covered by a collection of horizontal local sections \( \{ s_\alpha : U_\alpha \to M \} \), then in each of the local gauge patches the connection \( s_\alpha^* A \) vanishes, but on the overlaps \( U_\alpha \cap U_\beta \) there is generally a nontrivial gauge transformation relating \( s_\alpha^* \tilde{\varphi} \) to \( s_\beta^* \tilde{\varphi} \). Because of the nontrivial holonomy of \( A \) there is no global horizontal section, so not all of these non-trivial overlaps can be removed. On a circle, there would be just one non-trivial periodicity condition, while on \( S^3/\Gamma \) these “twisted boundary conditions” are generally more complicated.

Viewed in this way, the fields \( \varphi \) on \( M/\Gamma \) in the presence of the connection \( A \) are just the equivariant subclass of all the fields on \( M \). For each gauge orbit \( u \cdot \Gamma \) in \( M \), \( \tilde{\varphi} \) can be specified freely only at one point, its value at the rest of the points being determined by the equivariance condition \( (2.1) \). Since there are \( |\Gamma| \) points in an orbit, it is plausible that the number of equivariant \( \tilde{\varphi} \)-modes is \( 1/|\Gamma| \) times the number of unrestricted modes on \( M \). We now show that this is indeed the case.

The solutions to the mode equation \( D_0^2 \psi = \lambda \psi \) on \( M \) can be decomposed according to unitary irreducible representations of the action of \( \Gamma \). The case of a general group will be treated below using some results from representation theory, but it is perhaps helpful to the reader to first treat the more elementary case where \( \Gamma \) is a cyclic group generated by some group element \( \gamma \). This is basically the case that has been previously treated in the literature \([5]\) for the quotient \( S^1/\Gamma \). The unitary irreducible representations are then one-dimensional and characterized by a root of unity \( \exp(i2\pi k/|\Gamma|) \):

\[ \psi(u \cdot \gamma) = e^{i2\pi k/|\Gamma|} \psi(u) \quad (2.2) \]

for some integer \( k < |\Gamma| \) (since \( \gamma^{|\Gamma|} = 1 \)). For such a \( \psi \) the equivariance condition \( (2.1) \) becomes a local condition

\[ \mathcal{R}(\gamma^{-1})\psi(u)\mathcal{R}(\gamma) = e^{i2\pi k/|\Gamma|} \psi(u) \quad (2.3) \]

on the matrix \( \psi(u) \). We need to determine the dimension of the subspace of \( |\Gamma| \times |\Gamma| \) matrices satisfying this condition.
To this end let us use the eigen-basis \{ |p \rangle \} of \( R(\gamma) \):

\[
R(\gamma)|p\rangle = e^{i2\pi p/|\Gamma|}|p\rangle, \quad p = 1, \ldots, |\Gamma|.
\] (2.4)

A basis for the space of \(|\Gamma| \times |\Gamma|\) matrices is \{ |p\rangle\langle q| \}. Under the adjoint action of \( R(\gamma) \), these basis matrices transform as

\[
R(\gamma^{-1})|p\rangle\langle q|R(\gamma) = e^{-i2\pi(p+q)/|\Gamma|}|p\rangle\langle q|.
\] (2.5)

The matrix \(|p\rangle\langle q|\) thus satisfies the equivariance relation (2.3) provided \( p+q = -k \) modulo \(|\Gamma|\). The number of such \((p,q)\) pairs is \(|\Gamma|\), so there is a \(|\Gamma|\)-dimensional equivariant subspace of the \(|\Gamma|^2\)-dimensional space of matrices. That is, there are \(1/|\Gamma|\) times as many equivariant modes as there are unconstrained modes, which is what we set out to prove.

In the above discussion we worked with all the matrices in \( C|\Gamma| \otimes C|\Gamma| \), but we are really interested in fields taking values in the hermitian matrices, for which the set \{ \(|p\rangle\langle q| + |q\rangle\langle p|\) \} forms a basis. Since the transformation rule (2.5) is symmetric in \( p \) and \( q \), these basis matrices transform the same way as the non-hermitian ones, so the previous analysis restricts without modification to the hermitian subspace.

Now let us generalize to an arbitrary non-abelian subgroup \( \Gamma \). In this case the irreducible representations are no longer one-dimensional, but fortunately representation theory easily yields the result as follows. To begin with let us reexpress the equivariance condition (2.1) as

\[
R(\gamma)\tilde{\varphi}(u \cdot \gamma)R(\gamma^{-1}) = \tilde{\varphi}(u),
\] (2.6)

that is, the matrix-valued function must transform under the trivial representation of \( \Gamma \) when both the argument (the point on \( M \)) and the matrix are simultaneously transformed. The counting problem then reduces to this: how many singlets are there among the matrix-valued functions on \( M \)?

The matrix valued functions carry the representation \( S \otimes (R \otimes R^*) \), where \( S \) is the vector space of scalar-valued functions and here \( R \) stands for the vector space of the regular representation and \( R^* \simeq R \) is its dual. Let \( W \) be one of the irreducible representations into which \( S \) decomposes, and consider the matrix-valued functions transforming under \( W \) in the argument. These carry the representation \( W \otimes (R \otimes R) \), where \( R \) has now been identified with its dual.

To find the number of singlets we need three results from representation theory [12]. First, all irreducible representations of a finite group are finite dimensional. Second, the singlet occurs once in \( R \). Third, if \( \mathcal{V} \) is any irreducible representation then \( \mathcal{V} \otimes R = \).
\((\dim V)R\). The first two results follow from the fact that each irreducible representation occurs in \(R\) with multiplicity equal to its dimension. The third result can be seen easily using the multiplicative property of characters: \(\chi_{V \otimes R} = \chi_V \cdot \chi_R\). The character of the regular representation vanishes everywhere except at the identity, so the same is true for \(V \otimes R\). At the identity we have \(\chi_{V \otimes R}(e) = \chi_V(e) \cdot \chi_R(e) = (\dim V)|\Gamma|\), hence \(\chi_{V \otimes R} = \chi(\dim V)R\). Since the character determines the representation, it follows that \(V \otimes R = (\dim V)R\).

Using the first and third results, we have \(W \otimes (R \otimes R) = (|\Gamma|\dim W)R\). Using the second result, there are therefore \(|\Gamma|\dim W\) singlets. This number is indeed \(1/|\Gamma|\) times the dimension of the full space of matrix-valued functions transforming in the argument via \(W\), namely \(|\Gamma|^2\dim W\).

The counting is now easily generalized to the case where \(N = n|\Gamma|\) for some integer \(n\). As in the case \(N = |\Gamma|\), there is a principal \(U(n|\Gamma|)\)-bundle associated to the \(\Gamma\)-bundle \(M \to M/\Gamma\), now by the left action of \(R(\Gamma)\) on \(U(n|\Gamma|)\) via the \(n\)-fold direct sum \(nR(\Gamma)\) of the regular representation. As before, this associated bundle is reducible to a \(\Gamma\)-bundle equivalent to \(M \to M/\Gamma\), and the nontrivial flat \(\Gamma\)-connection on \(M \to M/\Gamma\) induces the required flat connection on the \(U(n|\Gamma|)\)-bundle. The fields \(\varphi : M/\Gamma \to C^n|\Gamma| \otimes C^n|\Gamma|\) transforming under the adjoint representation of \(U(n|\Gamma|)\) are now realized as maps \(\tilde{\varphi} : M \to C^n|\Gamma| \otimes C^n|\Gamma|\) that are equivariant under the representation \(nR(\Gamma) \subset U(n|\Gamma|)\) of \(\Gamma\), i.e. as before they are singlets under the combined action of \(\Gamma\) on the argument and the matrix value. The \(N \times N\) matrices carry the representation \(nR \otimes nR = (n^2|\Gamma|)R\), and the matrix-valued functions transforming under \(W\) in the argument carry \((n^2|\Gamma|)W \otimes R = (n^2|\Gamma|\dim W)R\). The dimension of the space of singlet functions is therefore \(n^2|\Gamma|\dim W\), which is again \(1/|\Gamma|\) times the dimension of the full space of \(N \times N\) matrix-valued functions transforming in the argument via \(W\).

We have so far shown that the nontrivial flat connection \(A\) makes the space \(M/\Gamma\) “look bigger” for fields valued in the adjoint of \(U(N)\). The argument can also be applied to fields valued in the fundamental representation. These are maps \(\psi : M/\Gamma \to C^N\), which can be characterized as local pullbacks of an equivariant map \(\tilde{\psi} : M \to C^N\). Let us consider here the case \(N = |\Gamma|\). The generalization to \(N = n|\Gamma|\) follows that given above for fields in the adjoint. The equivariance condition is then \(\tilde{\psi}(u \cdot \gamma) = R(\gamma^{-1})\tilde{\psi}(u)\), which implies that \(\tilde{\psi}\) is a singlet in \(S \otimes R\). Those \(\tilde{\psi}\) transforming in the argument under the irreducible representation \(W\) carry the representation \(W \otimes R = (\dim W)R\). This contains \((\dim W)\) singlets, which is again \(1/|\Gamma|\) times the dimension of the full space of fields in the fundamental transforming under \(W\) in the argument.
3 Gauge theory on $T^3$ at nonzero temperature

We now turn to applications of the above result using the AdS/CFT correspondence. Before discussing the case of gauge theory on $S^3/\Gamma$, we consider the simpler case of gauge theory on $T^3$. The dual supergravity description of $\mathcal{N} = 4$ super Yang-Mills at temperature $T$ on a three torus is given by the near horizon limit of a near extremal black three brane. This is the product of $S^5$ and

$$ds^2 = \frac{r^2}{R^2} \left[ -\left( 1 - \frac{r_0^4}{r^4} \right) dt^2 + dx_idx^i \right] + \left( 1 - \frac{r_0^4}{r^4} \right)^{-1} \frac{R^2}{r^2} dr^2$$

(3.1)

where the three coordinates $x_i$ are periodically identified. The Hawking temperature is $T = r_0/(\pi R^2)$ and the total mass is

$$M = \frac{3\pi^2 N^2 T^4 V_3}{8}$$

(3.2)

where $V_3$ is the volume of the three torus and we have used the fact that $G_5 = \pi R^3/2N^2$. This is known to agree with the weakly coupled gauge theory up to an overall factor of $3/4$ \cite{13}.

If the coordinates $x_i$ each have length $L$, one would expect finite size effects to show up in the gauge theory when $T \lesssim 1/L$. However as long as the metric (3.1) is valid, supergravity predicts (3.2) with no finite size corrections. The supergravity solution is expected to break down when either the curvature at the horizon or the proper lengths of the compact directions at the horizon are of order the string scale $\ell_s$. The curvature at the horizon is always of order $1/R^2$, so the curvature is never a problem as long as the AdS radius is much larger than the string scale, $R \gg \ell_s$. The second condition is satisfied when $r_0 L/R \sim \ell_s$ or $TLR \sim \ell_s$. So when $T \sim 1$, the compact directions at the horizon are still much larger than the string scale, and there should be no corrections. This means that super Yang-Mills on $T^3$ must have states lighter than $1/L$ so the finite size corrections are suppressed.

As discussed in the introduction, it has already been shown in other contexts that this is indeed the case. Since the potential energy is positive definite, the energy of each state should increase as the coupling increases\footnote{One can show that the potential energy is of the same order as the kinetic energy \cite{13}, so the energy of each state increases by only a factor of two or so. This provides a qualitative explanation of the factor of $3/4$ discrepancy between the weak and strong coupling results.}. So evidence for light states at strong coupling implies that there must be light states at weak coupling. These light states arise
by introducing a flat, but nontrivial, connection on $T^3$. If one views the gauge theory as describing the low energy excitations of 3-branes, the standard description without a background connection corresponds to $N$ branes each wrapped once around the torus. The flat connection describes multiply wrapped branes. If one brane is wrapped $N$ times around one circle, there will be very light states with energies of order $1/(NL)$, but the effective theory of these states will be one dimensional, and their contribution to the energy will be relatively small. To maintain a three dimensional theory (as indicated by (3.2)) one considers instead a single brane wrapped $N^{1/3}$ times around each of the three circles. Then the excitations have energy in multiples of $1/(N^{1/3}L)$, so we would not expect to see finite size corrections until $TL \sim N^{-1/3}$. We have seen that string theory allows finite size corrections to supergravity when $TL \sim \ell_s/R = (4\pi gN)^{-1/4}$. Since $(gN)^{1/4} \ll N^{1/3}$ (for large $N$ and $g < 1$) one sees that finite size effects in the weakly coupled gauge theory do not become important until a temperature well below that at which they are allowed by the correspondence with string theory. One might even conclude that the string winding mode corrections on the gravity side should not affect the relation between energy and temperature until $T \sim 1/(N^{1/3}L)$. On the other hand, it might be that at strong coupling in the gauge theory, finite size effects already become important at the higher temperature $T \sim (4\pi gN)^{-1/4}L^{-1}$.

4 Gauge theory on $S^3/\Gamma$ at nonzero temperature

We turn now to the case of a gauge theory on $S^3/\Gamma$. Let us first ask how one sees finite size effects due to the $S^3$ (with no quotient) using the AdS/CFT correspondence. A thermal state in the gauge theory at high temperature is described on the supergravity side by a large Schwarzschild AdS black hole

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_3$$

(4.1)

with

$$f(r) = \frac{r^2}{R^2} + 1 - \frac{r_0^2}{r^2}$$

(4.2)

The Schwarzschild radius $r_+$ is defined by $f(r_+) = 0$. The ADM mass (neglecting the constant $AdS_5$ contribution discussed in the next section) is

$$M = \frac{3\pi r_0^2}{8G_5}$$

(4.3)
where $G_5$ is the five dimensional Newton’s constant. The Hawking temperature is

$$T = \frac{R^2 + 2r_+^2}{2\pi r_+ R^2} \quad (4.4)$$

It follows that black holes have a minimum temperature $T_{\text{min}} = \sqrt{2}/\pi R$. At low temperature, the corresponding supergravity solution is just a gas of particles in $AdS_5$. The temperature at which the description supergravity solution can be computed by comparing the action for the euclidean black hole and $AdS_5$ with imaginary time periodically identified. This was done in [15] (following [16]) with the result that the black hole dominates for $r_+ > R$ corresponding to $T > T_{\text{crit}} = 3/(2\pi R)$. We assume this is the case in the following.

We wish to compute how the mass changes as a function of temperature. (One could equally well compute the entropy as a function of temperature, but that will be determined in terms of the mass since $dM = TdS$.) For convenience, we set $R = 1$ and measure all quantities in units of the AdS radius. First note that since $f(r_+) = 0$,

$$r_0^2 = r_+^2 + r_+^4 \quad (4.5)$$

Using (4.4) we solve for $r_+$ in terms of $T$:

$$2r_+ = \pi T + \pi T \left( 1 - \frac{2}{\pi^2 T^2} \right)^{1/2} \quad (4.6)$$

Substituting this into (4.3) yields

$$r_0^2 = \frac{1}{2} \pi^4 T^4 \left[ 1 + \left( 1 - \frac{2}{\pi^2 T^2} \right)^{1/2} \right] - \frac{\pi^2 T^2}{2} - \frac{1}{4} \quad (4.7)$$

To leading order for large temperature, the mass is therefore

$$M = \frac{3\pi}{8G_5} \left[ \pi^4 T^4 - \pi^2 T^2 + O(T^0) \right] \quad (4.8)$$

We now want to compare this with the energy of a weakly coupled $U(N)$ SYM. This calculation was done in [17, 18]. It was found that finite size effects alter the factor of $3/4$ that relates the energies in the high temperature limit. Here we recap the result.

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4Given $T$, there are two solutions to this equation for the Schwarzschild radius $r_+$. We are interested in the larger one, $r_+ > R$, since small black holes have negative specific heat.

5This calculation of the supergravity energy at given temperature is actually an underestimate, since we have neglected the energy in the Hawking radiation outside the black hole. However, as long as the AdS radius is large compared to the Planck scale, this contribution is negligible.
We first translate the supergravity result (4.8) into field theory language. Since the volume of a unit three sphere is \( V = 2\pi^2 \) and \( G_5 = \pi/2N^2 \), this becomes

\[
M = \frac{3N^2V}{8}(\pi^2T^4 - T^2) \tag{4.9}
\]

The first term agrees with the torus result (4.9). The second corresponds to finite size effects in the gauge theory. It is important only when \( T \) is of order one in units of the AdS radius, i.e., \( T \sim 1/R \).

To compute the weakly coupled gauge theory energy directly, we need to calculate the modes for scalars, spinors and vectors on \( S^3 \), and evaluate the sums which yield the average energy at a given temperature. One way to do so is to use the results of this calculation from the ancient Ref. [19]. The result is that for large temperature, the energy density for a single scalar, Weyl fermion, or vector is

\[
e_0 = \frac{1}{30}\pi^2T^4, \quad e_{1/2} = \frac{7}{120}\pi^2T^4 - \frac{T^2}{48}, \quad e_1 = \frac{1}{15}\pi^2T^4 - \frac{T^2}{6} \tag{4.10}
\]

The first term in each case is the usual blackbody expression, and the second term is the leading finite size correction. (For the scalar field, there is no correction of order \( T^2 \).)

The total energy for the weakly coupled SYM theory, which has six scalars, four Weyl fermions, and one vector, is therefore

\[
E = N^2V(6e_0 + 4e_{1/2} + e_1), \tag{4.11}
\]

Comparing (4.9) and (4.11) we see that the leading terms show the well known factor of \( 3/4 \) difference between the weakly coupled gauge theory and the supergravity prediction for the strongly coupled regime. The next terms show that this factor of \( 3/4 \) discrepancy is not preserved as we lower the temperature. For a given temperature, the supergravity energy starts out smaller than the weakly coupled field theory energy, and decreases more quickly as we lower the temperature. This implies that the interactions, which are omitted in the weakly coupled gauge theory calculations, must become relatively more important when the temperature is lowered. This makes sense since, at sufficiently low temperature, there is a critical point at which the gauge theory becomes confining. This critical point corresponds to the transition where the entropy drops precipitously and the black hole goes away.

\[\text{At the minimum black hole temperature, numerical evaluation of the mode sums shows that the ratio of black hole mass to the weakly coupled SYM energy is } \sim 0.17.\]
The Schwarzschild-AdS metric (4.1) has spatial slices with topology $S^3 \times R$. Hence one can take the quotient of $S^3$ by any freely acting discrete symmetry group $\Gamma \subset SO(4)$ without producing additional singularities. The temperature is unaffected by the quotient. The only effect on the mass (4.9) is the reduction of the volume $V$ by a factor $\Gamma$. Since the energy density is unchanged, the finite size effects do not arise until $T \sim 1/R$ as before. Similarly, in comparing the euclidean action for Schwarzschild-AdS metric and periodically identified AdS, the quotient affects both of them equally by changing an overall factor corresponding to the volume of a unit $S^3$. Thus the critical temperature at which the black hole geometry dominates is also unchanged. The supergravity solution can be trusted as long as string corrections are not important. Since $r_+ > R$ and $R \gg \ell_s$, the Schwarzschild radius is much larger than the string scale. The other condition to check is that all noncontractible loops have size much larger than $\ell_s$. This will be true as long as $|\Gamma| \ll R/\ell_s$. (This condition is sufficient, but not necessary. Depending on how $\Gamma$ acts on the sphere, we could have $|\Gamma|$ larger than $R/\ell_s$ – but certainly no larger than $(R/\ell_s)^3$ – and still have no noncontractible curve with length less than $\ell_s$.) Now consider the gauge theory side. Even though the volume of space is drastically reduced (from $S^3$ to $S^3/\Gamma$), the AdS/CFT correspondence implies that the density of low energy states is not affected. This is possible only if one adds the nontrivial flat connection discussed in section 2.

The result in section 2 assumed that $N/|\Gamma|$ is an integer. This was needed in order to construct a connection with holonomy in the direct sum of $N/|\Gamma|$ copies of the regular representation of $\Gamma$. In terms of D-branes, this corresponds to assuming that each brane is wrapped $\Gamma$ times, yielding a total of $N$ local branes. If $N/|\Gamma|$ is not an integer then the result cannot hold precisely, and one would expect the relative size of the corrections will be of the order of $|\Gamma|/N$. These corrections will be small unless $|\Gamma|$ is comparable to $N$. As noted above, however, validity of the supergravity solution outside the black hole horizon at finite temperature requires that the length of any noncontractible curves be much longer than $\ell_s$. This implies that $|\Gamma| < (R/\ell_s)^3 = (4\pi gN)^{3/4} \ll N$. Hence string corrections to supergravity are already expected before $\Gamma$ becomes large enough for non-integer $N/|\Gamma|$ to be an issue.

The orbifold singularity does not contribute to the gravitational action since if one removes a small tubular neighborhood of the singularity with radius $\epsilon$, the extra surface term at this inner boundary goes to zero as $\epsilon \to 0$.  

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7The orbifold singularity does not contribute to the gravitational action since if one removes a small tubular neighborhood of the singularity with radius $\epsilon$, the extra surface term at this inner boundary goes to zero as $\epsilon \to 0$.  

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5 Gauge theory on $S^3/\Gamma$ at zero temperature

We have discussed nonzero temperature above, but it is natural to ask whether one can also see these extra light states at zero temperature. In this case, the spacetime is just $AdS_5$, and we want to consider the quotient by a discrete subgroup $\Gamma$ of $SO(4)$ acting on the three-spheres of spherical symmetry. This produces an orbifold with fixed points at $r = 0$. Starting with [20], there has been considerable interest in the effect on the AdS/CFT correspondence of taking quotients $S^5/\Gamma$. These orbifolds reduce the amount of supersymmetry and allow new phenomena. There has been relatively less investigation of orbifolds of $AdS_5$ (although see e.g. [21, 22, 23]). It is difficult to calculate the precise spectrum of twisted sector states due to the Ramond-Ramond background. However the following argument suggests that the extra light states in the gauge theory are necessary for consistency of the AdS/CFT correspondence in this case also.

Consider the low energy excitations on the gravity side. There are the usual supergravity modes which are invariant under $\Gamma$. Before the quotient, these modes correspond to states on $S^3$ which are invariant under $\Gamma$. So they exist in the theory on $S^3/\Gamma$ without an extra flat connection. However, there are also modes in the twisted sector localized in $AdS/\Gamma$ near the fixed point. These should be similar to the modes in the twisted sector of a flat space orbifold. If $\Gamma$ breaks all the supersymmetry, then there are tachyons [24] representing an instability in the strongly coupled gauge theory. It was shown in [21] that if $\Gamma$ lies in an $SU(2)$ subgroup of $SO(4)$, then half the AdS supersymmetry is preserved. (This is the same condition for orbifolds in flat space.) In this case, it is plausible that there are massless modes. These modes are localized near the origin of $AdS_5$, so they correspond to fields on $S^5$ cross time. Since the $S^5$ has radius $R$ and is unaffected by the quotient, these fields produce states with energy of order $1/R$. In the gauge theory, these light states can be reproduced only with the nontrivial flat connection.

A less direct way to see the existence of the extra light states is by looking at the ground state energy density. A boundary stress-energy tensor for asymptotically AdS spacetimes was proposed in [25]. For $AdS_5$, this gives a finite nonzero answer for the total energy

$$E = 3\pi R^2/32G_5.$$  

This turns out to agree exactly with the Casimir energy of $\mathcal{N} = 4, U(N)$ super Yang-Mills on a three-sphere of radius $R$. (The Casimir energy can be computed even at strong coupling since it is determined by the trace anomaly which, due to supersymmetry, does not receive corrections.) Now imagine taking the

$^8$It was pointed out in [20] that one cannot assign a nonzero energy to pure AdS without breaking the symmetry. However, as explained in [27], the AdS symmetry is indeed broken by the need to regulate infrared divergences.
quotient by $\Gamma$. On the gravity side, since the local geometry is unchanged and the volume of spheres are reduced, it is clear that the total energy will be reduced by a factor of $|\Gamma|$. If this is identified with the ground state energy of the gauge theory on $S^3/\Gamma$, then the energy density is unchanged by the quotient. This is possible only due to the nontrivial flat connection.

Before one can conclude that the AdS/CFT correspondence predicts that the ground state energy density of the gauge theory on $S^3/\Gamma$ is the same as on $S^3$, one needs to answer the following open question: Do there exist solutions of Einstein's equation with negative cosmological constant which approach $AdS_5/\Gamma$ asymptotically and have less energy than $AdS_5/\Gamma$? If so, AdS/CFT would predict that the ground state energy density of the strongly coupled gauge theory on $S^3/\Gamma$ is even smaller than that of the gauge theory on $S^3$. Although this seems unlikely, it has been shown that with zero cosmological constant, there are smooth 4 + 1 dimensional vacuum solutions which are asymptotically locally euclidean and have negative total energy [28]. In fact the energy is unbounded from below. In the case of Kaluza-Klein boundary conditions (one direction compactified on a circle), it is also true that there are asymptotically flat vacuum solutions with arbitrarily negative energy. However, in this case, there is evidence that when the cosmological constant is negative, the energy is bounded from below [29]. So if there exist solutions asymptotic to $AdS_5/\Gamma$ with energy less than $AdS_5/\Gamma$, it is plausible that the energy will at least be bounded from below.

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