A REMARK ON THE COX RING OF $\overline{M}_{0,n}$

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Abstract. We present an elementary inductive argument proving that a certain subring of the Cox ring of the moduli space $\overline{M}_{0,n}$ of stable rational curves with $n$ marked points is finitely generated for every $n \geq 3$.

1. Introduction

Let $\overline{M}_{0,n}$ be the moduli space of stable rational curves with $n$ ordered distinct marked points. After [HK], Question 3.2, it has been an intriguing open problem to establish whether $\overline{M}_{0,n}$ is a Mori dream space. Indeed, an affirmative answer is only known for $n \leq 6$ (see [C]), which is just the range of $n$ for which $\overline{M}_{0,n}$ is a (smooth) log-Fano variety. On the other hand, according to the very recent e-print [CT], $\overline{M}_{0,n}$ is not a Mori dream space for $n > 133$ (at least in characteristic zero).

Here instead we present an elementary proof of the following fact:

Theorem 1.1. The graded algebra

\[
\bigoplus_{c_A \geq 0, c_i \geq 0} H^0 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}} \left( \sum_{4 \leq |A| \leq (n+1)/2} c_A D_A - \sum_i c_i \psi_i \right) \right)
\]

is finitely generated for every $n \geq 3$.

As kindly pointed out to us by Maksym Fedorchuk, such a statement cannot hold without any restriction on the index $A$. Indeed, since for every $m_A \in \mathbb{Z}$ and for any $m >> 0$ we have

\[
\sum_A m_A D_A = \sum_A m_A D_A + m \sum_j j(n - j)B_j - m(n - 1) \sum_i \psi_i = \\
= \sum_A c_A D_A - \sum_i c_i \psi_i
\]

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with all \( c_A \geq 0 \) and \( c_i \geq 0 \) (see for instance [M], Lemma 2.9 (2)), it turns out that
\[
\text{Cox} (\mathcal{M}_{0,n}) = \bigoplus_{c_A \geq 0, c_i \geq 0} H^0 \left( \mathcal{M}_{0,n}, \mathcal{O}_{\mathcal{M}_{0,n}} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right).
\]

Our inductive argument works for \( n \geq 5 \) and relies on a vanishing
\[
H^1 \left( \mathcal{M}_{0,n}, \mathcal{O}_{\mathcal{M}_{0,n}} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right) = 0
\]
(see Lemma 2.2 below for a more precise statement).

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We work over an algebraically closed field \( \mathbb{K} \) of arbitrary characteristic.

2. The proofs

The boundary of \( \overline{\mathcal{M}}_{0,n} \) is the union of irreducible divisorial components
\[
D_A = \eta (\overline{\mathcal{M}}_{0,A \cup \{p\}} \times \overline{\mathcal{M}}_{0,P \setminus A \cup \{q\}}),
\]
where \( P = \{1, \ldots, n\} \), \( A \subset P \) and \( \eta \) is the natural morphism obtained by glueing the points \( p \) and \( q \) into an ordinary node. According to [AC], Lemma 3.3 (see also [M], Lemma 2.5), we have:
\[
\psi|_{D_A} = \eta^* (\psi_i) = \begin{cases}
(\psi_i, 0) & \text{if } i \in A \\
(0, \psi_i) & \text{if } i \in P \setminus A
\end{cases}
\]
and
\[
D_A|_{D_A} = \eta^* (D_A) = (-\psi_p, -\psi_q)
\]
\[
D_{P \setminus A}|_{D_A} = \eta^* (D_{P \setminus A}) = (-\psi_p, -\psi_q)
\]
while for every \( B \) different from both \( A \) and \( P \setminus A \):
\[
D_B|_{D_A} = \eta^* (D_B) = \begin{cases}
(D_B, 0) & \text{if } B \subset A \\
(D_{P \setminus B}, 0) & \text{if } P \setminus B \subset A \\
(0, D_B) & \text{if } B \subset P \setminus A \\
(0, D_{P \setminus B}) & \text{if } P \setminus B \subset P \setminus A \\
(0, 0) & \text{otherwise.}
\end{cases}
\]
Lemma 2.1. Let $n \geq 5$. For every $c_A \geq 0$ and every $c_i \geq 0$ we have

$$H^1 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}} \left( \sum_{4 \leq |A| \leq (n+1)/2} c_A D_A - \sum_i c_i \psi_i \right) \right) = 0.$$  

Proof. We argue by induction on $\sum A c_A$.

If $\sum A c_A = 0$, then we need to check that

$$(6) \quad H^1 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}} \left( -\sum_i c_i \psi_i \right) \right) = 0.$$  

In order to do so, we make induction on $\sum_i c_i$. Indeed, if $\sum_i c_i = 0$, then we have $H^1(\overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}) = 0$ since $\overline{M}_{0,n}$ is a smooth rational variety and the irregularity is a birational invariant. Assume now $\sum_i c_i > 0$, so that $c_z > 0$ for some $z$. We know that $\psi_z$ is the class of an effective divisor $E$ (see for instance [AC], (3.7)):

$$E = \sum_A D_A$$

with $z \in A$ and $x, y \notin A$ for any choice of distinct elements $x, y \in \{1, \ldots, n\}$. In particular, $E$ has dimension $\dim(E) = n - 4 \geq 1$, is both reduced and connected (indeed, all such $D_A$ intersect $D_{\{x,y\}}$) and by (3) $\psi_z|D_A = (\psi_z, 0)$. Hence from the standard short exact sequence

$$0 \to \mathcal{O}_{\overline{M}_{0,n}}(-E) \to \mathcal{O}_{\overline{M}_{0,n}} \to \mathcal{O}_E \to 0$$

we deduce

$$0 \to \mathcal{O}_{\overline{M}_{0,n}}(-\sum_i c_i \psi_i) \to \mathcal{O}_{\overline{M}_{0,n}}(-\sum_{i \neq z} c_i \psi_i - (c_z - 1)\psi_z) \to \mathcal{O}_E(-\sum_{i \neq z} c_i \psi_i - (c_z - 1)\psi_z) \to 0$$

with

$$H^0 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}} \left( -\sum_{i \neq z} c_i \psi_i - (c_z - 1)\psi_z \right) \right)$$

$$= H^0 \left( E, \mathcal{O}_E(-\sum_{i \neq z} c_i \psi_i - (c_z - 1)\psi_z) \right)$$

both equal to either $\mathbb{K}$ if $\sum_i c_i - 1 = 0$ or to 0 if $\sum_i c_i - 1 > 0$. As a consequence, we obtain an exact sequence

$$0 \to H^1 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(-\sum_i c_i \psi_i) \right) \to$$

$$\to H^1 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}} \left( -\sum_{i \neq z} c_i \psi_i - (c_z - 1)\psi_z \right) \right)$$

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and (6) follows by induction on \( \sum_i c_i \).

Assume now \( \sum_A c_A > 0 \). Among all boundary divisors \( D_A \) with \( 4 \leq |A| \leq (n + 1)/2 \) such that \( c_A > 0 \), choose \( D_B \) minimizing \( |B| \). We may write the short exact sequence

\[
0 \to \mathcal{O}_{M_{0,n}}(\sum_{A \neq B} c_A D_A + (c_B - 1)D_B - \sum_i c_i \psi_i) \to
\]

\[
\mathcal{O}_{M_{0,n}}(\sum_A c_A D_A - \sum_i c_i \psi_i) \to \mathcal{O}_{D_B}(\sum_A c_A D_A - \sum_i c_i \psi_i) \to 0
\]

and deduce

\[
H^1 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(\sum_{A \neq B} c_A D_A + (c_B - 1)D_B - \sum_i c_i \psi_i) \right) \to
\]

\[
H^1 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(\sum_A c_A D_A - \sum_i c_i \psi_i) \right) \to
\]

\[
H^1 \left( D_B, \mathcal{O}_{D_B}(\sum_A c_A D_A - \sum_i c_i \psi_i) \right).
\]

Now, by inductive assumption on \( \sum_A c_A \) we have

\[
H^1 \left( \overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(\sum_{A \neq B} c_A D_A + (c_B - 1)D_B - \sum_i c_i \psi_i) \right) = 0.
\]

On the other hand, according to (2), we have

\[
H^1 \left( \overline{M}_{0,B,P \cup \{p\}} \times \overline{M}_{0,P \setminus B \cup \{q\}}, \eta^* \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right)
\]

where

\[
\eta^* \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) = \]

\[
= \left( -c_B \psi_p - \sum_i c_i \psi_i, -c_B \psi_q + \sum_A c_A D_A - \sum_i c_i \psi_i \right)
\]
by applying 3, 4 and 5 and recalling our choice of $D_B$ such that $B$ minimizes $|B|$. Hence by Künneth formula we obtain

$$H^1 \left( D_B, \mathcal{O}_{D_B} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right) =$$

$$= H^1 \left( \mathcal{M}_{0, B \cup \{p\}}, -c_B \psi_p - \sum_i c_i \psi_i \right) \otimes H^0 \left( \mathcal{M}_{0, p \setminus B \cup \{q\}}, -c_B \psi_q + \sum_A c_A D_A - \sum_i c_i \psi_i \right) \oplus$$

$$\oplus H^0 \left( \mathcal{M}_{0, B \cup \{p\}}, -c_B \psi_p - \sum_i c_i \psi_i \right) \otimes$$

$$\otimes H^1 \left( \mathcal{M}_{0, p \setminus B \cup \{q\}}, -c_B \psi_q + \sum_A c_A D_A - \sum_i c_i \psi_i \right)$$

with

$$H^1 \left( \mathcal{M}_{0, B \cup \{p\}}, -c_B \psi_p - \sum_i c_i \psi_i \right) = 0$$

by 3 and

$$H^0 \left( \mathcal{M}_{0, B \cup \{p\}}, -c_B \psi_p - \sum_i c_i \psi_i \right) = 0,$$

since $c_B > 0$, hence also

$$H^1 \left( D_B, \mathcal{O}_{D_B} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right) = 0$$

and the claim follows.

As a consequence of Lemma 2.1, from 7 we obtain a short exact sequence

$$0 \rightarrow H^0 \left( \mathcal{M}_{0, n}, \mathcal{O}_{\mathcal{M}_{0, n}} \left( \sum_{A \neq B} c_A D_A + (c_B - 1) D_B - \sum_i c_i \psi_i \right) \right) \rightarrow$$

$$\rightarrow H^0 \left( \mathcal{M}_{0, n}, \mathcal{O}_{\mathcal{M}_{0, n}} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right) \rightarrow$$

$$\rightarrow H^0 \left( D_B, \mathcal{O}_{D_B} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right) \rightarrow 0$$
for every boundary divisor $D_B$ with $4 \leq |B| \leq (n + 1)/2$, so that all restriction maps

\begin{equation}
\rho_B : H^0 \left( \overline{\mathcal{M}}_{0,n}, \mathcal{O}_{\overline{\mathcal{M}}_{0,n}} \left( \sum_{4 \leq |A| \leq (n+1)/2} c_A D_A - \sum_i c_i \psi_i \right) \right) \to H^0 \left( D_B, \mathcal{O}_{D_B} \left( \sum_{4 \leq |A| \leq (n+1)/2} c_A D_A - \sum_i c_i \psi_i \right) \right)
\end{equation}

are surjective with kernel

\begin{equation}
\ker(\rho_B) = H^0 \left( \overline{\mathcal{M}}_{0,n}, \mathcal{O}_{\overline{\mathcal{M}}_{0,n}} \left( \sum_{A \neq B} c_A D_A + (c_B - 1) D_B - \sum_i c_i \psi_i \right) \right).
\end{equation}

The idea of the next crucial technical result has been inspired by [F], Chapter I., Theorem (2.3).

**Proposition 2.2.** If for every $B \subseteq \{1, \ldots, n\}$ such that $4 \leq |B| \leq (n + 1)/2$ the graded algebras

\[ A_B = \bigoplus_{c_A \geq 0, c_i \geq 0} H^0 \left( D_B, \mathcal{O}_{D_B} \left( \sum_{4 \leq |A| \leq (n+1)/2} c_A D_A - \sum_i c_i \psi_i \right) \right) \]

are finitely generated by homogeneous elements $\rho_B(\gamma_j^B)$ for $j = 1, \ldots, N(B)$, then the graded algebra

\[ A = \bigoplus_{c_A \geq 0, c_i \geq 0} H^0 \left( \overline{\mathcal{M}}_{0,n}, \mathcal{O}_{\overline{\mathcal{M}}_{0,n}} \left( \sum_{4 \leq |A| \leq (n+1)/2} c_A D_A - \sum_i c_i \psi_i \right) \right) \]

is finitely generated by (the sections corresponding to) the boundary divisors $D_B$ together with the elements $\gamma_j^B$ for every $B$ and $j$.

**Proof.** Let $\mathcal{S} \subseteq A$ be the subalgebra generated by all sections $s_B$ with $D_B = \{s_B = 0\}$ and by all $\gamma_j^B$. In order to prove that $\mathcal{S} = A$, we are going to check that the following equality holds for all homogeneous components:

\[ S \cap H^0 \left( \overline{\mathcal{M}}_{0,n}, \mathcal{O}_{\overline{\mathcal{M}}_{0,n}} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right) = H^0 \left( \overline{\mathcal{M}}_{0,n}, \mathcal{O}_{\overline{\mathcal{M}}_{0,n}} \left( \sum_A c_A D_A - \sum_i c_i \psi_i \right) \right). \]

We argue by induction on $\sum_A c_A$.

If $\sum_A c_A = 0$, then $H^0 \left( \overline{\mathcal{M}}_{0,n}, \mathcal{O}_{\overline{\mathcal{M}}_{0,n}} \left( - \sum_i c_i \psi_i \right) \right)$ equals either $\mathbb{K}$ if $\sum_i c_i = 0$ or $0$ if $\sum_i c_i > 0$ and in both cases there is nothing to prove.
Thus assume $\sum A c_A > 0$, in particular there is $B$ with $4 \leq |B| \leq (n + 1)/2$ such that $c_B > 0$. By (8) and (9), the vector space

$$H^0\left(\overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(\sum A c_A D_A - \sum i c_i \psi_i)\right)$$

is generated by

$$s_B \otimes H^0\left(\overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(\sum_{A \neq B} c_A D_A + (c_B - 1)D_B - \sum c_i \psi_i)\right)$$

and by a set of elements $\gamma_j^B$ such that $\rho_B(\gamma_j^B)$ generate

$$H^0\left(D_B, \mathcal{O}_{D_B}(\sum A c_A D_A - \sum c_i \psi_i)\right),$$

hence the claim follows by induction on $\sum A c_A$. 

Proof of Theorem 1.1. We argue by induction on $n$, starting from the trivial cases $n \leq 6$ and then applying Proposition 2.2 for $n \geq 7$. 

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