Schubert Calculus via Fermionic Variables

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Abstract

Imanishi, Jinzenji and Kuwata provided a recipe for computing Euler number of Grassmann manifold $G(k, N)$ using physical model and its path-integral [S. Imanishi, M. Jinzenji and K. Kuwata, Journal of Geometry and Physics, Volume 180, October 2022, 104623]. They demonstrated that the cohomology ring of $G(k, N)$ is represented by fermionic variables. In this study, using only fermionic variables, we computed an integral of the Chern classes of the dual bundle of the tautological bundle on $G(k, N)$. In other words, the intersection number of the Schubert cycles is obtained using the fermion integral.

1 Introduction

1.1 Background

In this study, we aim to compute the intersection numbers of Schubert cycles. We used fermionic variables and their integrals in [9]. In this section, we explain the background of the study. The complex Grassmann manifold $G(k, N)$ is the space parameterizing all $k$-dimensional linear subspaces of $N$-dimensional complex vector space $\mathbb{C}^N$. Because the elements of its cohomology ring are represented by the Poincaré dual of some Schubert cycles of $G(k, N)$, their integral provides the intersection number of Schubert cycles. This research is called Schubert calculus, and has been studied in combinatorics, representation theory, and other fields [10]. The integral of these cohomology classes can be computed using localization theory or the Landau-Ginzburg formulation. In the localization theory, a fixed-point theorem for a compact manifold with torus action is used. In particular, the formula for the intersection number is provided using the localization theory [1, 2, 4]. However, the Landau-Ginzburg formulation [3, 5] uses a potential function provided by the total Chern class of the tautological bundle of $G(k, N)$ and residue. However, we do not use these theories. We employed the theory of [9]. Imanishi et al. constructed a physical toy model for computing the Euler number of $G(k, N)$. The model was constructed using two types of variables. One is a commutative variable called a bosonic variable, while the other is an anticommutative variable, called a fermionic variable. In [9], it was found that the cohomology ring of $G(k, N)$ can be represented by fermionic variables, and that the Euler number is provided by their integral. Therefore, the intersection number of Schubert cycles can be obtained using fermion integrals. Generally, it is difficult to perform this calculation. However, in some cases, the number of intersections can be calculated using this method. In this study, we demonstrated the use of the method of [9].
1.2 Organization of the paper

This paper is divided into two sections.

In Section 1, we describe our background and theorem. In addition to the background described above, we introduce the relationship between Chern classes and Schubert cycles, our theorem in this paper, and the theory in [7]. First, we remark on Chern classes and Schubert cycles. Next, we introduce the theorem. Finally, we introduce the relation between the Chern classes and fermionic variables in [9].

In Section 2, we provide the proof of our theorem. We computed the fermion integral to prove the theorem. We also summarize the important results of the fermion integrals.

1.3 Chern classes and Schubert cycles

In this section, we explain the relation between the Chern classes and Schubert cycles and our theorem. We also summarize the important results of the fermion integrals.
Second, we introduce the Schubert cycle and explain the relationship between Chern classes and Schubert cycles. For a more detailed discussion, please refer to [5]. For any flag $V : 0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = \mathbb{C}^N$, Schubert manifold $\sigma_a(V)$ is defined as follows:

$$
\sigma_a(V) := \{ \lambda \in G(k, N) | \dim(\lambda \cap V_{N-k+i-a_i}) \geq i (1 \leq i \leq k) \},
$$

where $a = (a_1, \cdots, a_k)$ denotes a sequence of natural numbers that satisfies $0 \leq a_k \leq a_{k-1} \leq \cdots \leq a_1 \leq N - k$. $\sigma_a(V)$ is a subvariety of $G(k, N)$ of dimension $\sum_{i=1}^{k} a_i$. The homology class of $\sigma_a(V)$ is independent of the chosen flag. Therefore, let $\sigma_a(V)$ as the homology class be denoted by $\sigma_a$. Let $\sigma_a^\vee$ be the Poincaré dual of the cycle $\sigma_a$. For simplicity of notation, we omit 0 from $a$. For example, $\sigma_{a_1, a_2, \cdots, a_n}$ denotes $\sigma_{(a_1, a_2, \cdots, a_n, 0, \cdots, 0)}$. The relationship between $i$-th Chern class of a vector bundle $E$ and that of its dual bundle $E^*$ is provided by $c_i(E^*) = (-1)^i c_i(E)$. From this formula and the Gauss-Bonnet theorem, we obtain:

$$
c_i(S^*) = (-1)^i c_i(S) = \sigma_{1, \cdots, 1}^\vee =: \sigma_{1(i)}^\vee.
$$

Finally, we introduce.

**Theorem 1.**

$$
\int_{G(k, N)} (\sigma_{1(i)}^\vee)^{kN-k^2} = (kN - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=k}^{N-k} j!}.
$$

$$
\int_{G(k, N)} (\sigma_{1(1)}^\vee)^{kN-k^2-2}(\sigma_{1(2)}^\vee)^2 = \frac{(kN - k^2 - 2)!((N-k)(N-k+1)k(k-1))}{2} \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}.
$$

$$
\int_{G(k, N)} (\sigma_{1(1)}^\vee)^{kN-k^2-4}(\sigma_{1(2)}^\vee)^4 \cdot \left[ k(k-1)(N-k)(N-k-1) + 2(k-2)(k-3)(N-k) + 4(k-2)(N-k-1) \right].
$$

Here, we assume that $N$ and $k$ in (1.11) and (1.12) satisfy $kN - k^2 - 2 \geq 0$ and $kN - k^2 - 4 \geq 0$, respectively.

Note that these are the intersection numbers of $\sigma_{1(1)}$ and $\sigma_{1(2)}$. However, the results of (1.10) are already well known [5, 11]. When $k = 2$, the intersection numbers of $\sigma_{1(1)}^\vee$ and $\sigma_{1(2)}^\vee$ in $G(2, N)$ are known [5].

$$
\int_{G(2, N)} (\sigma_{1(1)}^\vee)^{2N-4-2l}(\sigma_{1(2)}^\vee)^l = \frac{(2(N-2-l))!}{(N-2-l)!(N-1-l)!}.
$$

(It is also derived by S. Imanishi’s Masters thesis using fermionic variables [8].) In particular, we obtain the following results from (1.10), (1.11), and (1.12):

$$
\int_{G(2, N)} (\sigma_{1(1)}^\vee)^{2N-4} = \frac{(2N-4)!}{(N-2)!(N-1)!},
$$

$$
\int_{G(2, N)} (\sigma_{1(2)}^\vee)^{2N-6}(\sigma_{1(1)}^\vee)^2 = \frac{(2N-6)!}{(N-3)!(N-2)!},
$$

$$
\int_{G(2, N)} (\sigma_{1(1)}^\vee)^{2N-8}(\sigma_{1(2)}^\vee)^2 = \frac{(2N-8)!}{(N-2)!(N-3)!}.
$$
1.4 Fermionic variables and Cohomology ring of $G(k, N)$ (Review of [9])

We summarize the representation of the cohomology ring of $G(k, N)$ using fermionic variables [9]. We introduce the fermionic variables $\psi_1^j, \psi_2^j (s = 1, \cdots, N - k, \ j = 1, \cdots k)$ and $(k \times k)$ matrix

$$
\Phi := \sum_{s=1}^{N-k} \left( \begin{array}{cccc}
\psi_1^s \psi_1^s & \cdots & \psi_1^s \psi_k^s \\
\vdots & \ddots & \vdots \\
\psi_k^s \psi_1^s & \cdots & \psi_k^s \psi_k^s
\end{array} \right).
$$

(1.17)

The fermionic variables $\psi_s^j, \bar{\psi}_s^j$ satisfy the following conditions.

$$
\psi_s^j \bar{\psi}_s^j = \psi_s^j \bar{\psi}_s^j = 0, \quad \psi_s^j \psi_s^j = -\psi_s^j \psi_s^j, \quad \psi_s^j \psi_s^j = -\psi_s^j \psi_s^j, \quad \psi_s^j \psi_s^j = -\psi_s^j \psi_s^j
$$

(1.18)

where $D \psi := \prod_{s=1}^{N-k} d \psi_s^1 d \psi_s^2 \cdots d \psi_s^k d \bar{\psi}_s^k$. We define $\tau_j (j = 1, 2, \cdots, k)$ as

$$
1 + \tau_j t + \cdots + \tau_k t^k := \text{det}(I_k + t \Phi) = \prod_{j=1}^{k} (1 + \lambda_j t).
$$

(1.20)

Here, $\lambda_j (j = 1, \cdots, k)$ are eigenvalues of $\Phi$. Specifically, $\lambda_j$ is the degree $j$ elementary symmetric polynomial of $\lambda_1, \cdots, \lambda_k$. Note that $\tau_k$ is identical with $\det(\Phi)$ and $\tau_1$ is identical with $\text{tr}(\Phi)$. In [9], the following theorems were proved:

**Theorem 2.** [9]

$$
\prod_{j=0}^{k-1} \frac{j!}{j!} \int D \psi (\det(\Phi))^{N-k} = 1.
$$

(1.21)

**Theorem 3.** [9]

$$
H^*(G(k, N)) = \mathbb{R}[c_1(S^*), \cdots, c_k(S^*), (a_i = 0 \ (i > N - k))] \simeq \mathbb{R}[\tau_1, \cdots, \tau_k].
$$

(1.22)

Theorem 3 is provided by ring homomorphism $f : \mathbb{R}[c_1(S^*), \cdots, c_k(S^*)] \rightarrow \mathbb{R}[\tau_1, \cdots, \tau_k]$, which is defined as

$$
f(c_j(S^*)) = \tau_j \ (j = 1, 2, \cdots, k).
$$

(1.23)

From the isomorphism $H^*(G(k, N)) \simeq \mathbb{R}[\tau_1, \cdots, \tau_k]$, $x_j$ is identified as $\lambda_j$. Theorem 2 corresponds to the normalization condition of the integration on $G(k, N)$ given by

$$
\int_{G(k,N)} (\sigma^j_{1(\psi)})^{N-k} = 1.
$$

(1.24)

Therefore, we obtain the following formula:

$$
\int_{G(k,N)} g(x_1, \cdots, x_k) = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=0}^{N-k} j!} \int D \psi g(\lambda_1, \cdots, \lambda_k),
$$

(1.25)

where $g(x_1, \cdots, x_k)$ is a symmetric polynomial of $x_1, \cdots, x_k$ that represents an element of $H^*(G(k, N))$. 

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2 Proof of our theorem

2.1 Proof of Theorem

Proof. (Theorem 1)
First, we prove (1.10). From (1.4), (1.20), and (1.25), we have

\[
\int G(k, N) \left( \sigma_{1(i)}^\vee \right)^{kN - k^2} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi (\Phi)^{kN - k^2} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi \left( \sum_{s=1}^{N-k} \sum_{j=1}^{k} \psi_s^\vee \psi_s \right)^{kN - k^2}.
\]

From the multinomial theorem and the conditions of the fermionic variables \( \psi_s^\vee \psi_s^\vee = \psi_s^\vee \psi_s = 0 \), we obtain:

\[
\int G(k, N) \left( \sigma_{1(i)}^\vee \right)^{kN - k^2} = (Nk - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi \prod_{s=1}^{N-k} \prod_{j=1}^{k} \psi_s^\vee \psi_s = (kN - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}.
\]

Second, we show that (1.11) and (1.12). In the same way as in (1.10),

\[
\int G(k, N) \left( \sigma_{1(i)}^\vee \right)^{kN - k^2 - 2l(\sigma_{1(i)}^\vee)} = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi (\tau_1)^{kN - k^2 - 2l(\tau_1)} (l = 1, 2).
\]

As \( \tau_2 = \frac{1}{2} \left\{ (\text{tr} (\Phi))^2 - (\text{tr} (\Phi)^2) \right\} \),

\[
\int D\psi (\tau_1)^{kN - k^2 - 2l(\tau_1)} = \frac{1}{2^l} \int D\psi \left\{ (\text{tr} (\Phi))^2 - (\text{tr} (\Phi)^2) \right\}^l
\]

\[
= \frac{1}{2^l} \sum_{m=0}^{l} \binom{l}{m} (-1)^m \int D\psi (\text{tr} (\Phi))^{kN - k^2 - 2m} (\text{tr} (\Phi)^2)^m.
\]

Let us define

\[
P_m := \int D\psi (\text{tr} (\Phi))^{kN - k^2 - 2m} (\text{tr} (\Phi)^2)^m \ (m = 0, 1, 2).
\]

As can be observed from the calculation in (1.10), \( P_0 = (kN - k^2)! \). We can obtain the following result for \( P_1 \) and \( P_2 \).

Proposition 1.

\[
P_1 = (kN - k^2 - 2)!k(N - k)(N - k).
\]

\[
P_2 = (kN - k^2 - 4)!k(N - k)\left[ k(N - k)^3 - 2(N - k)(k^2 + 2) + (N - k)(k^3 + 10k) - 4k^2 - 2 \right].
\]

We will prove these results later in this paper. From Proposition we have

\[
\int D\psi (\tau_1)^{kN - k^2 - 2(\tau_2)} = \frac{1}{2}(P_0 - P_1) = \frac{1}{2}(kN - k^2 - 2)!k(N - k)\left\{ (kN - k^2 - 1) - (N - 2k) \right\}
\]

\[
= \frac{1}{2} (kN - k^2 - 2)!((N - k)(N - k + 1)k(k - 1)).
\]

We obtain (1.11). Similarly, we obtain (1.12) from \( \int D\psi (\tau_1)^{kN - k^2 - 4(\tau_2)} = \frac{1}{4}(P_0 - 2P_1 + P_2) \). We have proved Theorem.
2.2 Proof of Proposition 1

\textbf{Proof.} (Proposition 1).

We compute \( P_1 \). Let \( \omega^{ij} \) be \( \sum_{s=1}^{N-k} \psi^i_s \psi^j_s \bar{\psi}^i_s \bar{\psi}^j_s \). By definition,

\[
\text{tr} \left( \Phi^2 \right) = \sum_{i,j=1}^{k} \omega^{ij} \omega^{ji} = \sum_{i=1}^{k} (\omega^{ii})^2 + \sum_{i \neq j} \omega^{ij} \omega^{ji},
\]

(2.35)

\[
P_1 = \int D\psi \left( \text{tr} \left( \Phi \right) \right)^{kN-k^2-2} \left( \text{tr} \left( \Phi^2 \right) \right)
\]

(2.36)

\[
\begin{align*}
&= \sum_{i=1}^{k} \int D\psi \left( \sum_{n=1}^{kN-k^2-2} (\omega^{ii})^2 + \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{kN-k^2-2} \omega^{ij} \omega^{ji} \right) \\
&= \sum_{i=1}^{k} \sum_{p_n} \frac{(kN-k^2-2)!}{\Pi_{n=1}^{k} p_n!} \int D\psi \left( \prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) (\omega^{ii})^2 \\
&+ \sum_{i \neq j} \sum_{p_n} \frac{(kN-k^2-2)!}{\Pi_{n=1}^{k} p_n!} \int D\psi \left( \prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) \omega^{ij} \omega^{ji}.
\end{align*}
\]

(2.37)

(2.38)

Here, \( \sum_{p_n} \) indicates that the sum includes all combinations from 0 to \( kN - k - 2 \) indices \( p_1 \) through \( p_k \), such that the sum of all \( p_n(n = 1, \ldots, k) \) is \( kN - k^2 - 2 \). In the first term, because each \( \omega^{ii} \) \((i = 1, \ldots, k)\) must be \( N - k \) for the fermion integral to be non-zero, \( p_n = N - k(n \neq i) \) and \( p_i = N - k - 1 \). In the second term, \( p_n = N - k(n \neq i, j) \) and \( p_i = p_j = N - k - 1 \).

\[
P_1 = \sum_{i=1}^{k} \frac{(kN-k^2-2)!}{((N-k)!)^{k-1}(N-k)!} \int D\psi \left( \prod_{n=1}^{k} (\omega^{nn})^{N-k} \right) \\
+ \sum_{i \neq j} \frac{(kN-k^2-2)!}{((N-k)!)^{k-2}(N-k-1)!^2} \int D\psi \left( \prod_{n=1}^{k} (\omega^{nn})^{N-k} \right) (\omega^{ii} \omega^{jj})^{N-k-1} \omega^{ij} \omega^{ji}.
\]

(2.39)

From \( \omega^{ii} = \sum_{s=1}^{N-k} \psi^i_s \psi^i_s \bar{\psi}^i_s \bar{\psi}^i_s \), the multinomial theorem and conditions of the fermionic variables \( \psi^i_s \psi^i_s = \psi^i_s \bar{\psi}^i_s = 0 \).

\[
P_1 = \sum_{i=1}^{k} \frac{(kN-k^2-2)!}{(N-k)!} \left( \omega^{ii} \right)^{N-k} \\
+ \sum_{i \neq j} \frac{(kN-k^2-2)!}{((N-k)!)^{k-2}(N-k-1)!^2} \left( \omega^{ii} \omega^{jj} \right)^{N-k-1} \left( \sum_{s,t=1}^{N-k} \psi^i_s \psi^j_t \psi^j_t \psi^i_s \right).
\]

(2.40)

In the second term, \( (\omega^{ii} \omega^{jj})^{N-k-1} \) contains \( N - k - 1 \) \( \psi^i_s \psi^j_t \psi^j_t \psi^i_s \), and \( \psi^i_s \psi^j_t \psi^i_s \). Therefore, it must be \( s = t \) based on the conditions of the fermionic variables.

\[
P_1 = (kN-k^2-2)!k(N-k)(N-k-1) \\
- \sum_{i \neq j} \sum_{s=1}^{N-k} \frac{(kN-k^2-2)!}{((N-k-1)!)^2} \int D\psi \left( \prod_{n=1}^{N-k} \psi^i_s \psi^j_t \right) (\omega^{ii} \omega^{jj})^{N-k-1} \left( \psi^i_s \psi^j_t \psi^i_s \psi^j_t \right) \\
= (kN-k^2-2)!k(N-k)(N-k-1) \\
- \sum_{i \neq j} \sum_{s=1}^{N-k} \frac{(kN-k^2-2)!}{((N-k-1)!)^2} \int D\psi \left( \prod_{n=1}^{N-k} \psi^i_s \psi^j_t \right) \left( \psi^i_s \psi^j_t \psi^i_s \psi^j_t \right).
\]

(2.41)

(2.42)
Therefore, we obtain
\[ P = (kN - k^2 - 2)!k(N - k)(N - k - 1) - \sum_{i \neq j} \sum_{s=1}^{N-k} (kN - k^2 - 2)! \] (2.43)

Subsequently, we calculate \( Q \)
\[ Q = (kN - k^2 - 2)!\{k(N - k)(N - k - 1) - (N - k)k(k - 1)\} = (kN - k^2 - 2)!k(N - k)(N - 2k). \] (2.44)

Therefore, we obtain \( P_1 \). We compute \( P_2 \).
\[
P_2 = \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} \left( \sum_{i=1}^{k} (\omega^{ii})^2 + \sum_{i \neq j} \omega^{ij} \omega^{ji} \right)^2
\] (2.45)
\[
= \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} \left[ \sum_{i,j} (\omega^{ii} \omega^{jj})^2 + 2 \sum_{m=1}^{k} \sum_{i \neq j} (\omega^{mm})^2 \omega^{ij} \omega^{ji} + \sum_{a \neq b} \sum_{i \neq j} \omega^{ab} \omega^{ba} \omega^{ij} \omega^{ji} \right].
\] (2.46)

We define \( Q_1, Q_2 \) and \( Q_3 \) as follows.

\[ Q_1 := \sum_{i,j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii} \omega^{jj})^2, \quad Q_2 := 2 \sum_{m=1}^{k} \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{mm})^2 \omega^{ij} \omega^{ji}, \] (2.47)

\[ Q_3 := \sum_{a \neq b} \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} \omega^{ab} \omega^{ba} \omega^{ij} \omega^{ji}. \] (2.48)

Thereafter, \( P_2 = Q_1 + Q_2 + Q_3 \). We consider \( Q_1 \):
\[
Q_1 = \sum_{i=1}^{k} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii})^4 + \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ij})^2. \] (2.49)

We can compute the above equation in the same manner as \( P_1 \). Consequently,
\[ Q_1 = (kN - k^2 - 4)!k(N - k)\{(N - k - 1)(N - k - 2)(N - k - 3) + (k - 1)(N - k)(N - k - 1)^2\}. \] (2.50)

Subsequently, we calculate \( Q_2 \).
\[
Q_2 = 2 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ii})^2 \omega^{ij} \omega^{ji} + 2 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{ij})^2 \omega^{ij} \omega^{ji}
\] + \[ 2 \sum_{i \neq j} \sum_{m \neq i,j} \int D\psi \left( \sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-4} (\omega^{mm})^2 \omega^{ij} \omega^{ji}. \] (2.51)

From \( \omega^{ij} \omega^{ji} = \omega^{ji} \omega^{ij} \), if we replace \( i \) with \( j \) and \( j \) with \( i \) in the second term, it is the same as in the
first term.

\[ Q_2 = 4 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega_n^{i} \right)^{kN-k^2-4} (\omega^{ii})^2 \omega^{ij} \omega^{ji} + 2 \sum_{i \neq j} \sum_{m \neq i, j} \int D\psi \left( \sum_{n=1}^{k} \omega_n^{m} \right)^{kN-k^2-4} (\omega^{mn})^2 \omega^{ij} \omega^{ji} \]

\[ = 4 \sum_{i \neq j} \sum_{p_n} \frac{(kN-k^2-4)!}{\prod_{q=1}^{k} p_q!} \int D\psi \left( \prod_{n=1}^{k} (\omega^{mn})^{p_n} \right) (\omega^{ii})^2 \omega^{ij} \omega^{ji} + 2 \sum_{i \neq j} \sum_{m \neq i, j} \frac{(kN-k^2-4)!}{(N-k-2)!(N-k-1)!} \int D\psi \left( \prod_{n=1}^{k} (\omega^{mn})^{p_n} \right) (\omega^{mn})^2 \omega^{ij} \omega^{ji}. \]

(2.52)

Thereafter, \( \sum_{p_n} \) is the sum of all combinations from 0 to \( kN-k^2-4 \) indices \( p_i \) through \( p_k \) such that the sum of \( p_n(n=1, \cdots, k) \) is \( kN-k^2-4 \). From the condition of fermionic integration and the condition of fermionic variables \( \psi_i^a \psi_i^a = 0 \), in the first term, \( p_n = N-k(n \neq i, j) \) and \( p_i = N-k-3 \), \( p_j = N-k-1 \). In the second term, \( p_n = N-k(n \neq i, j, m) \) and \( p_i = p_j = N-k-1 \), \( p_m = N-k-2 \). Therefore,

\[ Q_2 = 4 \sum_{i \neq j} \frac{(kN-k^2-4)!}{(N-k-3)!(N-k-1)!} \int D\psi \left( \prod_{n=1}^{N-k} \prod_{l=1}^{N} \psi_i^n \bar{\psi}_l^n \right) (\omega^{ii})^{N-k-1} \omega^{ij} \omega^{ji} + 2 \sum_{i \neq j} \sum_{m \neq i, j} \frac{(kN-k^2-4)!}{(N-k-2)!} \int D\psi \left( \prod_{n=1}^{N-k} \prod_{l=1}^{N} \psi_i^n \bar{\psi}_l^n \right) (\omega^{ij})^{N-k-1} \omega^{ij} \omega^{ji}. \]

(2.53)

Here, we can calculate the fermion integral in the same manner as \( P_1 \). We obtain

\[ \int D\psi \left( \prod_{n \neq i, j} \prod_{l=1}^{N-k} \psi_i^n \bar{\psi}_l^n \right) (\omega^{ii})^{N-k-1} \omega^{ij} \omega^{ji} = -(N-k)(N-k-1)!^2. \]

(2.55)

\[ Q_2 = -4 \sum_{i \neq j} \frac{(kN-k^2-4)!(N-k)!}{(N-k-3)!} - 2 \sum_{i \neq j} \sum_{m \neq i, j} \frac{(kN-k^2-4)!(N-k)!}{(N-k-2)!} \]

\[ = -4 \frac{(kN-k^2-4)!(N-k)!}{(N-k-3)!} k(k-1) - 2 \frac{(kN-k^2-4)!(N-k)!}{(N-k-2)!} k(k-1)(k-2) \]

\[ = (kN-k^2-4)!k(N-k)(k-1)[-4(N-k-1)(N-k-2)-2(N-k)(N-k-1)(k-2)]. \]

(2.56)

Finally, we compute \( Q_3 \).

\[ Q_3 = \sum_{a \neq b} \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} \omega_n^{a} \right)^{kN-k^2-4} \omega^{ab} \omega^{ij} \omega^{ji}. \]

(2.57)

The sum \( \sum_{a \neq b} \sum_{i \neq j} \) can be divided into the following seven cases.

| Sum patterns of \((i, j)\) and \((a, b)\) |
|---|
| (1) \(i = a, j = b\); (2) \(i = b, j = a\); (3) \(i = a, j \neq b\); (4) \(i = b, j \neq a\); (5) \(i \neq a, j = b\); (6) \(i \neq b, j = a\); (7) \(i \neq a, b \neq j \neq a, b\). |

From the symmetry of \(a, b\) and \(i, j\), (1) and (2) have the same form: Similarly, (3), (4), (5), and (6)
have the same form: Therefore,

\[
Q_3 = 2 \sum_{i \neq j} \int D\psi \left( \sum_{n=1}^{k} i^{n} \right)^{kN-k^2-4} (\omega^{ij} \omega^{ji})^2 + 4 \sum_{i \neq j} \sum_{b \neq i, j} \int D\psi \left( \sum_{n=1}^{k} i^{n} \right)^{kN-k^2-4} \omega^{ib} \omega^{bi} \omega^{ij} \omega^{ji}
\]

\[
+ \sum_{(i,j,a,b)}' \int D\psi \left( \sum_{n=1}^{k} i^{n} \right)^{kN-k^2-4} \omega^{ab} \omega^{ba} \omega^{ij} \omega^{ji}.
\]

Here, \(\sum_{(i,j,a,b)}'\) implies that \(i, j, a, b\) are different from each other in the summation.

\[
Q_3 = 2 \sum_{i \neq j} \frac{(kN-k^2-4)!}{((N-k)!)^{k-2}((N-k-2)!)^2} \int D\psi \left( \prod_{n \neq i,j} (\omega^{nn})^{N-k} \right) (\omega^{ij} \omega^{ji})^{N-k-2} (\omega^{ij} \omega^{ji})^2
\]

\[
+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \frac{(kN-k^2-4)!}{((N-k)!)^{k-3}((N-k-2)!)^2} \prod_{n \neq i,j,b} \omega^{nn} N-k \omega^{ij} \omega^{ji}
\]

\[
= 2 \sum_{i \neq j} \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \int D\psi \left( \prod_{n \neq i,j} \psi_{i}^{n} \psi_{j}^{n} \right) (\omega^{ij} \omega^{ji})^{N-k-2} \left( \sum_{s_1, s_2, t_1, t_2} \psi_{s_1}^{i} \psi_{s_1}^{j} \psi_{t_1}^{i} \psi_{t_1}^{j} \psi_{s_2}^{i} \psi_{s_2}^{j} \psi_{t_2}^{i} \psi_{t_2}^{j} \right)
\]

\[
+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \frac{(kN-k^2-4)!}{((N-k-1)!)^2} \prod_{n \neq i,j,b} \omega^{nn} N-k \omega^{ij} \omega^{ji}
\]

\[
+ \sum_{(i,j,a,b)}' \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji}
\]

\[
= 2 \sum_{i \neq j} \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \int D\psi \left( \prod_{n \neq i,j} \psi_{i}^{n} \psi_{j}^{n} \right) (\omega^{ij} \omega^{ji})^{N-k-2} \left( \sum_{s_1, s_2, t_1, t_2} \psi_{s_1}^{i} \psi_{s_1}^{j} \psi_{t_1}^{i} \psi_{t_1}^{j} \psi_{s_2}^{i} \psi_{s_2}^{j} \psi_{t_2}^{i} \psi_{t_2}^{j} \right)
\]

\[
+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \frac{(kN-k^2-4)!}{((N-k-1)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji}
\]

\[
+ \sum_{(i,j,a,b)}' \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji} + \sum_{(i,j,a,b)}' \frac{(kN-k^2-4)!}{((N-k-1)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji}
\]

\[
= 2 \sum_{i \neq j} \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \int D\psi \left( \prod_{n \neq i,j} \psi_{i}^{n} \psi_{j}^{n} \right) (\omega^{ij} \omega^{ji})^{N-k-2} \left( \sum_{s_1, s_2, t_1, t_2} \psi_{s_1}^{i} \psi_{s_1}^{j} \psi_{t_1}^{i} \psi_{t_1}^{j} \psi_{s_2}^{i} \psi_{s_2}^{j} \psi_{t_2}^{i} \psi_{t_2}^{j} \right)
\]

\[
+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \frac{(kN-k^2-4)!}{((N-k-1)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji}
\]

\[
+ \sum_{(i,j,a,b)}' \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji} + \sum_{(i,j,a,b)}' \frac{(kN-k^2-4)!}{((N-k-1)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji}.
\]

We consider sums of \(s_1, s_2, t_1\) and \(t_2\). In the first term, the sum can be divided into two ways, \((s_1 = t_1, s_2 = t_2, s_1 \neq s_2)\) and \((s_1 = t_2, s_2 = t_1, s_1 \neq s_2)\). In the second term, it must be \((s_1 = t_1, s_2 = t_2, s_1 \neq s_2)\). In the third term, it must be \((s_1 = t_1, s_2 = t_2)\). Because the first term is symmetric for \(s_1\) and \(s_2\) and \(t_1\) and \(t_2\),

\[
Q_3 = 4 \sum_{i \neq j} \sum_{s_1, s_2} \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \int D\psi \left( \prod_{n \neq i,j} \psi_{i}^{n} \psi_{j}^{n} \right) (\omega^{ij} \omega^{ji})^{N-k-2} (\psi_{s_1}^{i} \psi_{s_1}^{j} \psi_{s_1}^{i} \psi_{s_1}^{j} \psi_{s_2}^{i} \psi_{s_2}^{j} \psi_{s_2}^{i} \psi_{s_2}^{j})
\]

\[
+ 4 \sum_{i \neq j} \sum_{b \neq i, j} \sum_{s_1, s_2} \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \prod_{n \neq a, b, i,j} \omega^{nn} N-k \omega^{ij} \omega^{ji}.
\]

\[9\]
Therefore, we obtain (2.33) from Proposition 1.

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References

[1] A. Weber. Equivariant Chern classes and localization theorem. *J. Singul.*, 5:153–176, 2012. DOI:10.5427/jsing.2012.5k, URL: https://doi.org/10.5427/jsing.2012.5k. arXiv:1110.5515.

[2] D. T. Hiep. Identities involving (doubly) symmetric polynomials and integrals over Grassmannians, 2016. arXiv, DOI: 10.48550/ARXIV.1607.04850 URL: https://arxiv.org/abs/1607.04850.

[3] E. Witten. The Verlinde algebra and the cohomology of the Grassmannian. In Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 357–422. Int. Press, Cambridge, MA, 1995. arXiv:hep-th/9312104.

[4] M. Zielenkiewicz. Integration over homogeneous spaces for classical Lie groups using iterated residues at infinity. *Centr. Eur. J. Math.*, 12:574–583, 2014. DOI:10.2478/s11533-013-0372-z, URL: https://doi.org/10.2478/s11533-013-0372-z. arXiv:1212.6623.

[5] N. Chair. Intersection numbers on Grassmannians, and on the space of holomorphic maps from $CP^1$ into $G(r,C^n)$. *J. Geom. Phys.*, 38(2):170–182, 2001. DOI:10.1016/S0393-0440(00)00059-0, URL:https://doi.org/10.1016/S0393-0440(00)00059-0. arXiv:hep-th/9808170.

[6] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. Wiley–Interscience, New York, 1978.

[7] R. Bott and L. W. Tu. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics. Springer New York, NY, 1982.

[8] S. Imanishi. Computation of Euler number for the Grassmann manifold $G(2,N)$ via Mathai-Quillen formalism (in Japanese, printed in Japan). Master’s thesis, Hokkaido University, 2019.
[9] S. Imanishi, M. Jinzenji, and K. Kuwata. Evaluation of Euler number of complex Grassmann manifold $G(k,N)$ via Mathai-Quillen formalism. *Journal of Geometry and Physics*, 180:104623, 2022. DOI:https://doi.org/10.1016/j.geomphys.2022.104623.

[10] T. Ikeda and H. Naruse. Modern Schubert calculus, from the special polynomial theory’s point of view (in Japanese, printed in Japan). *Sugaku*, 63(3):313–337, 2011. DOI:10.11429/sugaku.0633313, URL:https://doi.org/10.11429/sugaku.0633313.

[11] W. Fulton. *Intersection Theory*. Springer-Verlag Berlin Hidelberg, 1998.