MONOPOLE-CATALYSED BARYON DECAY: A BOUNDARY
CONFORMAL FIELD THEORY APPROACH

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Abstract

Monopole-mediated baryon number violation, the Callan-Rubakov effect, is reexamined using boundary conformal field theory techniques. It is shown that the low-energy behaviour is described simply by free fermions with a conformally invariant boundary condition at the dyon location. When the number of fermion flavours is greater than two, this boundary condition is of a non-trivial type which has not been elucidated previously.
I. INTRODUCTION

A spectacular effect of the Adler-Bell-Jackiw anomaly is the catalysis of baryon number violation at strong-interaction rates by dyons, electrically charged magnetic monopoles. Since the important physics occurs in the s-wave channel during scattering of fermions from the dyon, the problem can be mapped into a (1+1)-dimensional one. It was observed that, when the number of massless flavours, $N$ obeys $N \leq 2$, all the complicated physics of the fermion-dyon interaction can be reduced, at low energies, to a simple boundary condition on otherwise free fermions. On the other hand, in the physically interesting case $N > 2$, in which fractional quantum-number production occurs, it was found that the appropriate boundary conditions on the currents did not correspond to any linear boundary condition on the fermions. Since this time, great progress has been made in studying conformally-invariant boundary conditions in (1+1)-dimensional field theories, due to seminal work by Cardy. Cardy’s methods, originally developed to study two-dimensional classical statistical systems with boundaries, have been generalized to the study of quantum-impurity problems. It has become evident that, in a large class of such problems, the low energy physics can be described without explicit reference to the quantum impurity (in this case the dyon) by simply imposing a conformally-invariant boundary condition. Cardy made precise the meaning of “conformally invariant boundary condition”, through the device of modular transformation and the concept of a “boundary state”. In general these “boundary conditions” cannot be written as local, linear boundary conditions on the original fields in the problem. In the case of the Kondo problem, the linear boundary conditions correspond to Fermi liquid fixed points, while the other ones do not.

The purpose of the present work is to re-examine the Callan-Rubakov baryon-number violation effect from a slightly more modern perspective. We will show explicitly that, for all values of $N$, the low-energy physics is completely determined by conformally-invariant boundary conditions, à la Cardy. An essential tool for this construction is non-abelian bosonization; the boundary conditions on the currents, derived earlier, must be supplemented by appropriate “gluing conditions” governing the way that charge and flavour quantum numbers are combined. The topological vacuum angle appears as an explicit free parameter in the boundary state. This leads directly to a calculation of the Green’s functions in the vicinity of the dyon. We hope that our method may serve to clarify the physics of the Callan-Rubakov effect as well as perhaps suggesting analogous phenomena in other areas of physics. The dyon problem requires an extension of Cardy’s boundary conformal field theory techniques to deal with an infinite number of conformal towers. It turns out that the boundary state corresponding to the dyon is closely related to one that occurs in the problem of transmission through barriers in quantum wires.

In the next section we briefly review some results on dyons. In Section III we briefly review Cardy’s boundary formalism and deduce the boundary states and finite-size spectrum for the dyon problem for arbitrary $N$. In Section IV we calculate Green’s functions in the presence of the dyon, directly from the boundary states. These agree exactly with the results of Polchinksi obtained by another method.
II. Dyon-Catalysed Baryon Number Violation

The electrical degree of freedom of the dyon corresponds essentially to a rigid rotator collective co-ordinate associated with gauge transformations. A “stripped down” version of the model, proposed by Polchinski,\textsuperscript{4} involves otherwise free, right-moving fermions interacting near the origin with a rigid rotator. The Hamiltonian is:

\[ H = -i \int_{-\infty}^{\infty} dx \sum_{k=1}^{N} \psi_{k}^{\dagger} \left( \frac{d}{dx} - i\alpha f(x) \right) \psi_{k}(x) + \frac{1}{2I} \Pi^{2} \]  

(2.1)

Here \( \alpha \) is the rigid rotator co-ordinate and \( \Pi \) is the conjugate momentum,

\[ [\Pi, \alpha] = -i. \]  

(2.2)

\( I \) is the moment of inertia of the rotator and \( f(x) \) is an even function which vanishes for \( |x| > r_{0} \) where \( r_{0} \) is of order the dyon core size. It can be assumed to have integral one since we are free to rescale \( \alpha \) and \( \Pi \) and hence \( I \). \( \alpha \) represents a collective co-ordinate of the classical dyon solution and \( \Pi \) corresponds to the electric field or charge of the dyon. Here we work with right-moving fermions on the whole real line. This is obtained from the s-wave problem, originally defined on the positive half-line, by reflecting the incoming wave (left-movers) to the negative axis. A crucial and subtle point in the dimensional reduction of the fermions is that the charge of \( \psi(x) \) is reversed as \( x \) passes through 0. i.e. the conserved electric charge operator is:

\[ Q = 2 \left[ \Pi + \int_{-\infty}^{\infty} dq(x)J_{i}^{j}(x) \right] \]  

(2.3)

where the current operator is

\[ J_{i}^{j}(x) \equiv \psi_{i}^{\dagger}(x)\psi_{j}(x) \]  

(2.4)

and

\[ dq/dx = -f \quad \text{with} \quad q(x) \rightarrow \mp 1/2 \quad \text{as} \quad x \rightarrow \pm \infty \]  

(2.5)

Thus \( \psi(x) \) has charge +1 for \( x > r_{0} \) and charge −1 for \( x < -r_{0} \). [Note the factor of 2 in Eq. (2.3) which does not appear in Ref. (4). We insert it so that the fermions have charge \( \pm 1 \) rather than \( \pm 1/2 \).] Although \( Q \) naively appears to commute with the Hamiltonian it actually fails to do so due to the anomalous Schwinger term in the current commutator:

\[ [J(x), J(y)] = \frac{iN}{2\pi} \delta'(x - y) \]  

(2.6)

Here, and henceforth,

\[ J \equiv J_{i}^{i} \]  

(2.7)

This problem can be corrected by adding an additional term to the Hamiltonian:
where the constant, $C$, is given by:

$$C \equiv \frac{N}{2\pi} \int_{-\infty}^{\infty} dx f^2(x)$$

Despite the fact that the part of $H$ involving $\alpha$ only is of harmonic oscillator type, not invariant under translating $\alpha$, the full Hamiltonian is invariant under the combination of translation of $\alpha$ and phase rotation on the fermions induced by the operator $Q$.

Apart from the electric charge, $Q$ associated with a gauge symmetry, it is also crucial to consider the baryon number density. This is anomalous in the standard electro-weak model and its grand-unified generalizations. This means that there does not exist a conserved gauge-invariant baryon number current. Instead we can define either a current, $J$ which is conserved but not gauge-invariant or else a current $\tilde{J}$ which is gauge invariant but not conserved. It is the latter quantity which corresponds to the observable baryon number and which changes during the process of fermion-dyon scattering. In the reduced model of Polchinski, Eq. (2.1), (2.8), the conserved, but not gauge-invariant baryon number density is simply the current operator $J(x)$ defined above in Eq. (2.4). Including the anomaly,

$$\frac{\partial J}{\partial t} = -\frac{\partial J}{\partial x} + \frac{N}{2\pi} \frac{df}{dx}$$

Thus, $\int_{-\infty}^{\infty} dx J(x)$ is time-independent, assuming that $J(x)$ vanishes at spatial infinity. On the other hand this current does not commute with $Q$:

$$[J(x), Q] = i \frac{N}{\pi} f(x)$$

Temporarily reflecting the incoming wave back to the positive axis, we see that the electric charge density contains $J_L - J_R$ whereas the baryon number density is $J_L + J_R$. The non-commutativity of these operators is the cause of the anomaly in massless (1+1)-dimensional Q.E.D. This problem is solved by adding an extra term to $J(x)$:

$$\tilde{J}(t, x) \equiv J(x) - \frac{N}{2\pi} \alpha(t) f(x)$$

giving a current which commutes with $Q$ (i.e. is “gauge invariant”) but not with $H$ (i.e. is not conserved). Explicitly,

$$\frac{\partial \tilde{J}}{\partial t} = -\frac{\partial \tilde{J}}{\partial x} - \frac{Nf}{2\pi I}\Pi$$

The anomalous commutator of $\tilde{J}$ with itself is also given by Eq. (2.6), unaffected by the additional term. Although the motivation is somewhat different, there is a striking analogy between the shift from $J$ to $\tilde{J}$ in the dyon problem and a similar shift which is used in the conformal field theory analysis of the Kondo effect.\(^6\)

Polchinski has argued\(^4\) that at low energies, we may eliminate the rotor degree of freedom and simply replace it by a boundary condition on the fermions. [This was shown earlier by
Callan and Das\textsuperscript{2,3} from a somewhat different perspective.] It proves simplest to work with the gauge-invariant current, $\tilde{J}$. The equation of motion for $\Pi$ can be expressed in terms of $\tilde{J}$ as:

$$\frac{d\Pi}{dt} = \int_{-\infty}^{\infty} dx f(x) \tilde{J}(x)$$

(2.14)

The pair of linear equations, (2.13) and (2.14) can be integrated to find the time evolution of $\tilde{J}$ and $\Pi$ given their initial values. Integrating Eq. (2.13) gives:

$$\tilde{J}(x,t) = \tilde{J}(-r_0,t-x-r_0) - \frac{N}{2\pi I} \int_{-r_0}^{x} dx' f(x') \Pi(t+x'-x)$$

(2.15)

We are interested in incoming particles of energy $E << 1/r_0 \approx 1/I$. As we shall see, this implies that $\tilde{J}(x,t)$ and $\Pi(t)$ vary slowly in time (apart from a transient part of $\Pi$ which can be ignored). Thus we may approximate Eq. (2.15) by:

$$\tilde{J}(x,t) \approx \tilde{J}(-r_0,t) + \frac{N}{2\pi I} \Pi(t)[q(x) - q(-r_0)]$$

(2.16)

for $x$ of $O(r_0)$. Plugging this into Eq. (2.14) gives:

$$\frac{d\Pi}{dt} \approx \tilde{J}(-r_0,t) - \frac{N}{4\pi I} \Pi$$

(2.17)

This gives:

$$\Pi(t) = e^{-\frac{N}{4\pi I} t} \Pi(0) + \int_{0}^{t} dt' e^{-\frac{N}{4\pi I} (t-t')} \tilde{J}(-r_0,t')$$

(2.18)

Assuming that $\tilde{J}(-r_0,t)$ varies slowly in time, and assuming $t >> 1/I$ so that the initial value of $\Pi(0)$ has decayed to essentially 0, we may approximate:

$$\Pi(t) \approx \frac{4\pi I}{N} \tilde{J}(-r_0,t)$$

(2.19)

Thus we see that $\Pi(t)$ and hence $\tilde{J}(x,t)$ vary slowly in time as assumed. Finally plugging Eq. (2.19) into Eq. (2.16) we obtain:

$$\tilde{J}(r_0,t) \approx -\tilde{J}(-r_0,t)$$

(2.20)

We see that $\tilde{J}(x,t)$ flips sign as it passes by the dyon! This implies violation of baryon number conservation.\textsuperscript{3,4} On the other hand, the $SU(N)$ currents,

$$J^A \equiv \psi^{\dagger i}(T^A)^i_j \psi_j$$

(2.21)

with $tr T^A = 0$, and $tr T^A T^B = (1/2)\delta^{AB}$, do not couple to the rotor and suffer no anomaly. They are both conserved and gauge-invariant. Consequently, they vary slowly in space near the dyon:

$$J^A(r_0,t) \approx J^A(-r_0,t)$$

(2.22)
It appears that it may be possible to ignore the dyon completely, at low energies and long distances, and simply impose the boundary conditions of Eq. (2.20) and (2.22) on the fermions. i.e., we formally let \( r_0 \to 0 \) and require that

\[
\tilde{J}(0^+) = -\tilde{J}(0^-)
\] (2.23)

while requiring \( J^A(x) \) to be continuous at the origin. As was observed by Callan and Das\(^3\) and by Polchinski\(^4\), this only works straightforwardly for \( N = 1 \) or 2. i.e., only in these cases are the boundary conditions on the currents equivalent to linear boundary conditions on the fermion fields. For \( N = 1 \) this is achieved by the condition:

\[
\psi(0^-) = e^{i\theta} \psi^\dagger(0^+)
\] (2.24)

Here \( \theta \) is an arbitrary angle, which, as Polchinski argued, should be identified with the topological angle of the non-abelian gauge theory. For \( N = 2 \) the boundary conditions are:

\[
\psi_i(0^-) = i e^{i\theta} \epsilon_{ij} \psi_j^\dagger(0^+)
\] (2.25)

Note that the latter conserves the \( SU(2) \) symmetry.

For \( N > 2 \) no linear boundary condition on the fermions is equivalent to the boundary conditions of Eq. (2.20) and (2.22) on the currents.\(^3,\(^4\) This can be easily seen because such a linear condition would have to take the form:

\[
\psi_i(0^-) = U_{ij} \psi_j^\dagger(0^+)
\] (2.26)

for some unitary matrix, \( U \). While this automatically gives the correct condition on the charge current, Eq. (2.20), in order for it to give the correct condition on the flavour currents, Eq. (2.22), the matrix \( U \) must satisfy the condition:

\[
U^\dagger T^A U = - (T^A)^t = -(T^A)^* \] (2.27)

for all \( A \). This would imply that the fundamental representation of \( SU(N) \) is real which is untrue for \( N > 2 \). Instead the problem was solved by other means. Essentially, instead of eliminating the dyon, Polchinski integrated out the fermions. This could be done exactly using an identity due to Schwinger for the fermion determinant. Note that the boundary conditions on the currents (and, for that matter the boundary conditions on the fermions for the \( N = 1 \) or 2 cases) are time-independent and scale-invariant. At low energies and long distances compared to the dyon size and energy-level spacing, \( 1/I \), no scale enters into the behaviour of the massless fermions. We will show in the next section that it is possible to solve the \( N > 2 \) problem in the same way as for \( N \leq 2 \), to wit by simply imposing conformally invariant boundary conditions on non-interacting fermions. However, the notion of “boundary condition” must be generalized somewhat, following Cardy.\(^5\)

**III. BOUNDARY STATES**

In this section we briefly review Cardy’s\(^5\) boundary conformal field theory and then deduce the boundary states corresponding to the dyon, for arbitrary number of flavours, \( N \).
This determines all low-energy properties of the system including the low energy excitation spectrum for a dyon in a finite spherical box and the Green’s functions, which are discussed in the next section.

A system defined on the half-space \( x \geq 0 \) with a boundary condition at \( x = 0 \) can still be invariant under the infinite-dimensional subgroup of the conformal group which leaves the boundary invariant. i.e., writing \( z = \tau + ix \), where \( \tau \) is imaginary time, we require invariance under analytic transformations \( z \to w(z) \) such that \( w(\tau)^* = w(\tau) \). The boundary condition is assumed to imply that no momentum flows across the boundary, i.e.,

\[
T_L(0, \tau) = T_R(0, \tau)
\]  

(3.1)

where \( T_L \) and \( T_R \) are the left and right components of the energy-momentum tensor. Since \( T_L(x, t) = T_L(t + x) \) and \( T_R(x, t) = T_R(t - x) \), this implies that we may regard \( T_L \) as the analytic continuation of \( T_R \) to the negative \( x \)-axis; i.e.,

\[
T_L(x, t) = T_R(-x, t).
\]  

(3.2)

This formulation was implicit in Section II.

Cardy’s formalism is based on modular invariance; i.e. on the possibility of exchanging space and imaginary time in a \((1+1)\) dimensional conformal field theory with boundaries. Thus, we consider a conformal field theory defined on an interval of length \( l \) with (in general different) conformally invariant boundary conditions at \( x = 0 \) and \( x = l \) which we denote generically by \( A \) and \( B \). It is convenient to consider the system at finite temperature, \( T \), so that the fields are defined on an interval of length \( \beta \equiv 1/T \) in the imaginary time direction with periodic boundary conditions (in the case of bosonic fields). The Hamiltonian, including the boundary conditions is denoted \( H^{l}_{AB} \) and the corresponding partition function:

\[
Z_{AB} \equiv \text{tr} e^{-\beta H^{l}_{AB}}
\]  

(3.3)

Using Eq. (3.2), at \( x = 0 \) and the equivalent condition at \( x = l \), we may regard the system as consisting of right-movers only on an interval of length \( 2l \). Thus \( Z_{AB} \) must be a sum of characters corresponding to the various conformal towers in the right-moving sector of the theory. i.e.,

\[
Z_{AB} = \sum_p n^p_{AB} \chi_p(e^{-\pi\beta/l})
\]  

(3.4)

where \( p \) labels the various conformal towers, the multiplicities, \( n^p_{AB} \) are non-negative integers and

\[
\chi_p(q) \equiv \sum_m d^m_p q^{x_p + m - c/24}
\]  

(3.5)

where \( x_p \) is the scaling dimension of the corresponding primary field and the \( d^m_p \)'s are non-negative integer degeneracies. \( c \) is the Virasoro central charge, the conformal anomaly parameter. For convenience, we define:

\[
q \equiv e^{-\pi\beta/l}.
\]  

(3.6)
Cardy made the fundamental observation that we should be able to reinterpret $Z_{AB}$ by regarding the periodic direction as being space and the other one as being time. Now the Hamiltonian is defined on a periodic interval of length $\beta$; we denote it as $H^\beta_P$. The imaginary time interval is $l$. However, it is no longer appropriate to take an operator trace since the system is not periodic in the new “time” direction. Rather, we must consider matrix elements between some states $\langle A|$ and $|B >$. ie.

$$Z_{AB} = \langle A|e^{-lH^\beta_P}|B >$$ (3.7)

The boundary states, $|A >$ and $|B >$ correspond to the boundary conditions in the original formulation. The zero momentum condition of Eq. (3.1) implies that all consistent boundary states must obey:

$$[T_L(x) - T_R(x)]|A >= 0$$ (3.8)

Thus, in particular, expanding in eigenstates of $H^\beta_P$, all states will have equal left and right energies: $x \equiv x_L + x_R = 2x_R$. In fact, Eq. (3.8) implies that all boundary states must contain infinite sums over all descendents of a given primary state of the form:

$$|p > = \sum_m |p; m >_L \otimes |p; m >_R$$ (3.9)

Here the integer $m$ schematically labels all descendents of the primary states. The primary state $|p; 0 >$ has unit normalization. The left and right conformal towers occurring must be equivalent. Such states, corresponding to a particular conformal tower, are known as Ishibashi states. Thus we again obtain the characters of the conformal towers but this time $e^{(-4\pi l/\beta)x}$ occurs. Defining

$$\tilde{q} \equiv e^{-4\pi l/\beta},$$ (3.10)

we obtain:

$$Z_{AB} = \sum_p \langle A|p><p|B > \chi_p(\tilde{q})$$ (3.11)

Equating the expressions of Eq. (3.4) and (3.11) gives a set of powerful consistency conditions on possible boundary states $|A >$ and spectra, $n_{AB}$, that we refer to as the Cardy conditions. In principle, one wishes to find a complete set of solutions of these equations in order to enumerate all possible conformally invariant boundary conditions, or equivalently boundary states, for a given problem, ie. a given $H^\beta_P$. Given any two consistent boundary states $|A >$ and $|B >$ the linear combination $n|A > + m|B >$ where $n$ and $m$ are non-negative integers is always another consistent state. Note that the partition function involving this state and another one, $C$ is:

$$Z_{nA+mB,C} = nZ_{AC} + mZ_{BC}$$ (3.12)

We are interested in finding a complete basis of states from which all solutions can be constructed in this way. In the case of the Ising model, for example, there are only three
such basis boundary states corresponding to spin-up, spin-down or free boundary conditions on the Ising spins. Once we have found the boundary states, we can directly obtain not only the finite-size spectrum but also the Green’s functions as is explained in the next section.

We now consider the boundary states corresponding to the dyon. The periodic Hamiltonian, $H^P$, for our problem is simply the Hamiltonian for $N$ flavours of free fermions (left and right movers). We will also find the boundary state corresponding to a trivial boundary condition on the fermions. For instance, we might wish to impose a vanishing boundary condition in the three-dimensional problem on the surface of a sphere of radius $l$, with the dyon at the centre. In principle there could be many possible boundary states for such a $c = N$ theory for large $N$. Our search is considerably simplified by the boundary conditions on the currents of Eq. (2.20) and (2.22). As Cardy argued, because the currents have conformal spin 1 these conditions pick up a relative minus sign under the modular transformation. Thus the flavour current boundary condition becomes:

$$[J^E_L(x) + J^E_R(x)]|A> = 0 \quad (3.13)$$

(We hope the reader is not confused by our notation. The superscript $E$ labels the generators of $SU(N)$.). As remarked above, this condition holds both for the free boundary condition and also for the dyon. Likewise we have conditions on the boundary states coming from the charge current boundary condition. This condition has a different form for the free and dyon boundary condition:

$$[J^L_L(x) \pm J^R_R(x)]|A> = 0 \quad (3.14)$$

where the upper and lower sign refers to the free and dyon case respectively. This change in sign for the dyon case follows from Eq. (2.20). These conditions allow us to enormously reduce the set of Ishibashi states from which we build our boundary states.

To take advantage of this simplification we use non-abelian bosonization.\textsuperscript{13,14} This means that we represent the $N$ flavours of free fermions in terms of a level $k = 1$ $SU(N)$ Wess-Zumino-Witten field, $g$ containing the flavour degrees of freedom together with a free boson, $\phi$, representing the charge degrees of freedom. The free fermion energy momentum tensor for right-movers can be written entirely in terms of the flavour and charge currents as:

$$T^R = \frac{\pi}{N} J^R_R J^R_R + \frac{2\pi}{N+1} J^A_R J^A_R \quad (3.15)$$

Here,

$$J^R_R = \sqrt{N/\pi \partial \phi / \partial x} \quad (3.16)$$

and $J^A_R$ can be expressed in terms of $g$. This already suggests a simplification in our treatment of the dyon problem because the entire interacting Hamiltonian can be written as:

$$H = \int^{\infty}_{-\infty} dx \left[ \frac{\pi}{N} J^R_R J^R_R + \frac{2\pi}{N+1} J^A_R J^A_R - \alpha f J^R_R \right] + \frac{\Pi^2}{2l} + \frac{C}{2} \alpha^2 \quad (3.17)$$

Note that the interaction only involves the charge current. The flavour degrees of freedom appear to play a completely passive role. This is the opposite of what happens in the Kondo
problem where it is the spin currents which interact with the impurity and the charge current remains non-interacting.\(^6\)

Let us first consider the properties of \(H_\beta^P\) in more detail. Because \(\text{tre}^{-\beta H_{A\beta}}\) for fermions involves \textit{antiperiodic} boundary conditions in the imaginary time direction, in order to maintain modular invariance, we must also impose anti-periodic boundary conditions in the space direction, in \(H_\beta^P\). [Actually, only one generator of the modular group is used; switching space and imaginary time.] The eigenstates of \(H_\beta^P\) then consists of a direct product of left and right eigenstates. The allowed momenta are:

\[
k = \pm (2\pi/\beta)(n + 1/2)
\]

where \(n\) is a positive integer and the + or - sign occurs for right-movers or left-movers respectively. The spectrum of \(H_\beta^P\) for one flavour of right-movers can be seen to be \(E = (2\pi/\beta)x\) where

\[
x = -1/24 + Q^2/2 + \sum_{m=1}^{\infty} n_m m
\]

Here \(Q\), an integer, is the fermion-number of the state and the \(n_m\)’s are non-negative integers. We have included the universal \(O(1/\beta)\) term in the groundstate energy.\(^15\) A derivation of this formula is given in Ref. (6). It can be understood from abelian bosonization. We write the right-moving fermion in terms of a right-moving boson:

\[
\psi_R \propto e^{i\sqrt{4\pi} \phi_R}
\]

Taking into account that \(\phi_R\) does not commute with itself at different spatial points we conclude that \textit{periodic} boundary conditions should be imposed on it. It then has the mode expansion:

\[
\phi_R(x, t) = \phi_{0R} + \frac{\sqrt{\pi} Q(x - t)}{\beta} + \sum_{m=1}^{\infty} \frac{1}{\sqrt{4\pi m}} \left( e^{-2\pi i (t-x)/\beta} a_{mR} + \text{h.c.} \right)
\]

where \(Q\) is an integer, the soliton number. Plugging this mode expansion into the Hamiltonian:

\[
H_R = (1/2) \int_0^\beta dx [(\partial \phi_R/\partial t)^2 + (\partial \phi_R/\partial x)^2]
\]

we obtain the spectrum of Eq. (3.19). \(n_m\) labels the occupation number of the \(m\)th harmonic mode, i.e. the eigenvalue of \(a_{mR}^\dagger a_{mR}\). With \(N\) flavours we may write the spectrum as \(N\) copies of the above. Alternatively, using non-abelian bosonization, we separate it into a charge and flavour part. [See, for example Ref. (6).] The charge boson, \(\phi\) can be regarded as the canonically normalized sum of bosons for each flavour:

\[
\phi_R \equiv (1/\sqrt{N}) \sum_{i=1}^{N} \phi_{iR}
\]
It has an identical mode expansion to Eq. (3.21) except that $Q$ is replaced by $Q/\sqrt{N}$. Now $Q$ has the interpretation of the total fermion number. Consequently, the spectrum of right-moving charge excitations is:

$$x = -1/24 + Q^2/2N + \sum_{m=1}^{\infty} n_m m$$

(3.24)

If we classify states into conformal towers using the $U(1)$ current algebra as well as the Virasoro algebra, then we obtain one conformal tower for each integer value of $Q$, all states in a given conformal tower having the same charge, $Q$. The corresponding character is:

$$\chi_Q(q) \equiv \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots q^{-1/24+Q^2/2N+\sum_{m=1}^{\infty} n_m m} = [\eta(q)]^{-1} q^{Q^2/2N}$$

(3.25)

where

$$\eta(q) \equiv q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$$

(3.26)

There are $N$ different conformal towers of $SU(N)_1$; the highest weight states transform under the antisymmetric $p$-index tensor representation of $SU(N)$ (ie. the representation whose Young tableau consists of a single column of length $p$), for $p = 0, 1, \ldots, N - 1$. The corresponding scaled energies are:

$$x_p = -(N - 1)/24 + p(N - p)/2N$$

(3.27)

Certain “gluing conditions” determine which combinations of flavour and charge conformal towers occur in the spectrum with periodic boundary conditions. These can be shown to be:

$$Q = p \mod N$$

(3.28)

[Recall that $p$ labels the global $SU(N)$ representation of the highest weight state in the conformal tower; descendents can transform under other representations of global $SU(N)$.] Note, for example, that the lowest energy excitation has $p = Q = 1$. It has $x = (N - 1)/2N + 1/2N = 1/2$ and corresponds to a single fermion state which does indeed have these quantum numbers and energy [fundamental representation, charge 1, energy $(\pi/\beta)$]. Clearly a state with $Q = 1, p = 0$, with $x = 1/2N$ should not be permitted since it does not correspond to any free fermion state.

The above discussion was given for the right-moving sector. The left-moving sector has an identical spectrum with corresponding conformal towers labelled by $Q_L$ and $p_L$. The gluing conditions are imposed separately on left and right sectors:

$$Q_R = p_R \mod N$$

$$Q_L = p_L \mod N$$

(3.29)

Let us now consider the relevant Ishibashi states. We first discuss the charge states in some detail. Considering first the condition $[J_L(x) - J_R(x)] A >= 0$, this clearly implies
and

\[ Q_L = Q_R. \]

Furthermore, when we sum over the occupation number of all finite-momentum harmonic modes, created by the operators, \( a_{Lm}^\dagger \) and \( a_{Rm}^\dagger \) where \( m \) labels momentum, we must ensure that all corresponding left and right occupation numbers are equal. This means that the Ishibashi states take the form:

\[ |Q >_{c\pm} \equiv |Q >_R \otimes |Q >_L \otimes \exp[\sum_{m=1}^\infty a_{Rm}^\dagger a_{Lm}^\dagger] |0 > \]  

(3.30)

The first two kets represent the soliton modes of the right and left moving bosons respectively and \( |0 > \) represents the direct product of vacuum states for left and right harmonic modes. The other set of Ishibashi states obeying the condition \( [J_L(x) + J_R(x)] A >= 0 \) can be obtained from this one by simply making a unitary transformation that flips the sign of \( J_L \) while leaving \( J_R \) invariant. This corresponds to taking \( Q_L \rightarrow -Q_L \) and \( a_{Lm}^\dagger \rightarrow -a_{Lm}^\dagger \). This state is:

\[ |Q >_{c+} \equiv |Q >_R \otimes -Q >_L \exp[-\sum_{m=1}^\infty a_{Rm}^\dagger a_{Lm}^\dagger] |0 > \]  

(3.31)

The Ishibashi states in the flavour sector take the form:

\[ |p >_f \equiv \sum_{m=1}^\infty |p, m >_R \otimes U |p, m >_L . \]  

(3.32)

Here \( p \) labels the \( N - 1 \) flavour conformal towers and \( m \) generically labels all descendents. The anti-unitary operator \( U \) has the property \( J_L^A U = -U J_L^A \). In particular, it takes \( p_L \rightarrow N - p_L \).

All boundary states that we need will be linear combinations of products of the form \(|Q >_{c\pm} \otimes |p >_f \). An immediate restriction on the possibilities arises from the gluing conditions inherent in \( H^p \), namely \( Q_L = p_L \) (mod \( N \)), \( Q_R = p_R \) (mod \( N \)). The only products of charge and flavour Ishibashi states involving \( |Q >_{c+} \), which satisfy these conditions obey \( Q = p \) (mod \( N \)). Note that \( |Q >_{c+} \) contains charge \( Q_L = -Q_R = -Q \) and \( |p >_f \) contains the states in the conformal tower \( p_L = N - p_R = N - p \), satisfying the gluing condition. If we use the charge states \(|Q >_{c-}\), the possibilities are even more restricted. Since \(|Q >_{c-}\) contains only states with \( Q_L = Q_R = Q \) while \(|p >_f \) contains only states in the conformal towers with \( p_R = p \), \( p_L = N - p \) we see that there is only one way of satisfying the gluing conditions for odd \( N \); we must choose \( p = 0, Q = 0 \) (mod \( N \)). In the case of even \( N \) there is one other possibility; \( p = N/2, Q = N/2 \) (mod \( N \)) since then \( Q_L = N/2 \) (mod \( N \)), \( Q_R = N/2 \) (mod \( N \)) and \( p_L = p_R = N/2 \). The gluing conditions inherent in \( H^p \) and the boundary conditions on the currents have restricted the possible boundary states to such an extent that it is now straightforward to construct the needed boundary states corresponding to free fermions or a dyon.

We begin with the case of free fermions. A convenient set of conformally invariant boundary conditions is:

\[ \psi_{Ri}(0) = e^{-i\theta} \psi_{Li}(0) \]  

(3.33)

Thus we may analytically continue to \( x < 0 \), defining:

\[ \psi_{Ri}(-x, t) \equiv e^{-i\theta} \psi_{Li}(x, t) \]  

(3.34)
If we also impose the boundary condition at \( l \):

\[
\psi_{R_l}(l) = -e^{-i\theta'}\psi_{L_l}(l),
\]

(3.35)

then the problem is equivalent to having right-movers only on a circle of circumference \( 2l \) with the twisted boundary condition:

\[
\psi_{R_l}(-l) = -e^{i(\theta'-\theta)}\psi_{R_l}(l)
\]

(3.36)

The parameter \( \theta \) corresponds to a phase shift. The allowed momenta of the fermions are

\[
k = (\pi/l)[n + 1/2 \pm (\theta - \theta')/2\pi]
\]

(3.37)

where the upper (lower) sign is for particles (anti-particles) and \( n \) must be an integer such that \( k \geq 0 \). The spectrum can be written as: [See ref. (6).]

\[
E = \pi l \left\{-N/24 + \frac{1}{2N} \left[ Q + \frac{(\theta' - \theta)}{2\pi} N \right]^2 + \frac{p(N-p)}{2N} + n_c + n_f \right\}
\]

(3.38)

Here \( n_c \) and \( n_f \) are integers labelling descendents in the charge and flavour sector respectively; \( Q \) and \( p \) label the charge and flavour conformal towers to which the state belongs. The gluing condition \( Q = p \mod N \) must be imposed. We have included the \( O(1/l) \) contribution to the groundstate energy. The corresponding partition function is:

\[
Z_{F\theta,F\theta'} = \sum_{p=0}^{N-1} \chi_p^f(q) \sum_{m=-\infty}^{\infty} \chi_{p+mN}^c(q,\theta' - \theta)
\]

(3.39)

Here \( \chi_p^f(q) \) is the character of the \( p^{th} \) flavour conformal tower. \( \chi_Q^c(q,\theta) \) is the character of the charge \( Q \) conformal tower with the phase shift \( \theta \) included. Explicitly:

\[
\chi_Q^c(q,\theta) \equiv \frac{1}{\eta(q)} q^{(Q+\theta N/2\pi)^2/2N},
\]

(3.40)

Note that

\[
\chi_Q^c(q,\theta + 2\pi/N) = \chi_Q^c(q,\theta)
\]

(3.41)

so a relative phase shift of \( 2\pi/N \) is equivalent to shifting the gluing conditions to \( Q = p + 1 \mod N \). This observation is basic to the conformal field theory treatment of the Kondo problem.\(^6\)

We wish to find the corresponding boundary states \( |F\theta> \), such that:

\[
Z_{F\theta,F\theta'} = <F\theta|e^{-iH_p^3}|F\theta'>
\]

(3.42)

Since Eq. (3.42) can be expressed as a sum over conformal dimensions, \( x \) of \( \tilde{q}^x \), it is convenient to express \( Z_{F\theta,F\theta'} \) in Eq. (3.39) in terms of \( \tilde{q} \). The needed modular transformation involves the so-called modular S-matrix. We work this out explicitly in the charge case. The modular transformation of \( \eta \) is:
\[ \eta(q) = \sqrt{2l/\beta \eta(\bar{q})} \]  
\hspace{1cm} (3.43)

[See for example Ref. (16)]. While \( \chi_{Q}^{c}(q, \theta) \) does not itself have nice modular transformation properties, it turns out that the sum of this quantity over \( Q \) in Eq. (3.39) does. We use the Poisson sum formula, i.e. the Fourier transform of the periodic \( \delta \)-function, \( \delta_{P}(x) \):

\[ \delta_{P}(x) \equiv \sum_{m=-\infty}^{\infty} \delta(x - m) = \sum_{Q=-\infty}^{\infty} e^{2\pi i Qx} \]  
\hspace{1cm} (3.44)

This gives:

\[ \sum_{m=-\infty}^{\infty} e^{(p + mN + \theta N)/2N} = \sqrt{2l/\beta N} \sum_{Q=-\infty}^{\infty} e^{-iQ(2\pi p/N + \theta)} q^{Q^2/2N} \]  
\hspace{1cm} (3.45)

Hence,

\[ \sum_{m=-\infty}^{\infty} \chi_{P+mN}^{c}(q, \theta) = \frac{1}{\sqrt{N}} \sum_{Q=-\infty}^{\infty} e^{-i(2\pi p/N + \theta)Q} \chi_{Q}^{c}(\bar{q}, 0) \]  
\hspace{1cm} (3.46)

We also need the modular transformation property of the flavour characters. (The explicit values of these characters is not needed.) It is found that,\(^{17}\)

\[ \chi_{p}^{f}(q) = \sum_{p' = 0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi ipp'/N} \chi_{p'}^{f}(\bar{q}) \]  
\hspace{1cm} (3.47)

We now substitute Eq. (3.46) and (3.47) into Eq. (3.39). Using the identity:

\[ \frac{1}{N} \sum_{p=0}^{N-1} e^{2\pi ip(\theta - Q)/N} = \sum_{m'=-\infty}^{\infty} \delta_{Q,m'N+p'}, \]  
\hspace{1cm} (3.48)

we obtain finally:

\[ Z_{F\theta,F\theta'} = \sum_{p=0}^{N-1} \sum_{m=-\infty}^{\infty} e^{i(\theta' - \theta)(mN+p)} \chi_{p}^{f}(\bar{q}) \chi_{p+mN}^{c}(\bar{q}, 0) \]  
\hspace{1cm} (3.49)

We now look for the boundary states \(|F\theta>\) using Eq. (3.11). Note that we must use the charge Ishibashi states \(|Q>_{c+}\) of Eq. (3.31) in this case in order to satisfy the condition of Eq. (3.14) on the boundary states with the positive sign. Let us write the state as:

\[ |F\theta> = \sum_{p=0}^{N-1} \sum_{m=-\infty}^{\infty} a_{mN+p}(\theta) |mN + p >_{c+} \otimes |p >_{f} \]  
\hspace{1cm} (3.50)

Let us begin with the case \( \theta = 0 \). It can easily be shown that the products of characters, \( \chi_{p}^{f}(\bar{q}) \chi_{p+mN}^{c}(\bar{q}) \) are linearly independent for different values of \( p, N \), apart from the equality:

\[ \chi_{p}^{f}(\bar{q}) \chi_{p+mN}^{c}(\bar{q}) = \chi_{N-p}^{f}(\bar{q}) \chi_{-p-mN}^{c}(\bar{q}) \]  
\hspace{1cm} (3.51)
Thus, from considering $Z_{F_0,F_0}$ we see that:

$$|a_Q(0)|^2 + |a_{-Q}(0)|^2 = 2,$$

(3.52)

for all $Q$. Since the boundary condition of Eq. (3.33), (3.35) is charge conjugation invar-ant, for $\theta = 0$, we should impose this symmetry on the corresponding state, $|F_0>$. Thus, $a_Q = a_{-Q}$. Thus we obtain $|a_Q(0)| = 1$, for all $Q$. We may then redefine the phases of the Ishibashi states so that:

$$a_Q(0) = 1$$

(3.53)

We pause to comment on the significance of charge conjugation in the underlying three-
dimensional theory. The symmetry which we are interested in changes the sign of the electric
charge, but not the magnetic charge of the dyon. This symmetry is CP. Note that
parity, $P$, *does not* take $x \rightarrow -x$ in the dimensionally reduced theory. The reason is that
we are dealing with s-wave components of three-dimensional fields; $x$ actually corresponds
to the radial variable, $r$. ($x$ can be negative as well as positive because we have reflected
the incoming wave to the negative axis.) Thus three-dimensional CP corresponds to one-
dimensional C. We shall hencforth refer to this symmetry by its three-dimensional name, CP.

A simple solution, consistent with Eq. (3.49), for general $\theta$ is:

$$a_Q(\theta) = e^{iQ\theta}$$

(3.54)

This can be seen to be the unique solution which reproduces all the Green’s functions
correctly, by the procedure explained in the next section. We note that it can be obtained
from the $\theta = 0$ state by a unitary transformation:

$$|F\theta> = e^{i\hat{Q}\theta/2}|F_0>$$

(3.55)

where $\hat{Q}$ is the charge operator defined in Eq. (2.3).

We now wish to determine the boundary state, $|D>$, corresponding to the dyon. In this
case, we don’t know, a priori, the spectrum so we must proceed somewhat differently. In fact
we will find a one-parameter family of solutions, analogous to the free fermion states, $|F\theta>$
found above. We will argue that this parameter should be identified with the gauge theory
topological angle. Fortunately, the arguments given above drastically restrict the possible
form of $|D>$. The most general possible form which is equally valid for $N$ even or odd is:

$$|D> = \sum_{m=-\infty}^{\infty} a_m |mN>_c \otimes |0>_f$$

(3.56)

where the $a_m$ are free parameters. Note that we must use the $|Q>_c$ states in this case
to satisfy the reversed sign charge current boundary condition of Eq. (3.14). We restrict
the $a_m$’s by considering the matrix element $<D|e^{-iH_0^f}|D>$. This formally corresponds to
having “dyon boundary conditions” at both ends of the line interval. It is difficult to imagine
a physical situation in three dimensions that corresponds to this, but it is certainly possible
in the reduced one-dimensional theory. We insist, following Cardy, that this matrix element
correspond to a well-behaved partition function; ie. it should have the general form of Eq. (3.4), (3.5) for some integers $d^m_p$ and $n^p_{AB}$. The matrix element is given by:

$$Z_{DD} = \chi^f_0(\tilde{q}) \sum_{m=0}^{\infty} |a_m|^2 \chi^c_{mN}(\tilde{q})$$  \hfill (3.57)

In order to ensure that this gives a well-behaved partition function, we need to modular transform. However, the modular transform of a single charge character, $\chi_Q(\tilde{q})$ involves an integral over a continuous range of charge conformal towers. ie.:

$$\frac{1}{\eta(q)} q^{m^2 N/2} = \frac{1}{\sqrt{N} \eta(q)} \int_{-\infty}^{\infty} dx q^{x^2/2N} e^{2\pi i mx}$$  \hfill (3.58)

Thus, we obtain:

$$Z_{DD} = \frac{1}{N} \sum_{p=0}^{N-1} \chi^f_p(\tilde{q}) \frac{1}{\eta(q)} \int_{-\infty}^{\infty} dx q^{x^2/2N} \sum_{m=-\infty}^{\infty} |a_m|^2 e^{2\pi i mx}$$  \hfill (3.59)

At this point we also require CP invariance. This holds provided that the topological angle is 0 or $\pi$ in the four-dimensional non-abelian gauge theory. Hence, $a_m = a_{-m}$. If we also require that the groundstate be unique, then:

$$\sum_{m=-\infty}^{\infty} |a_m|^2 e^{2\pi i mx} = N \delta_P(x)$$  \hfill (3.60)

The spectrum is then:

$$Z_{DD} = \sum_{p=0}^{N-1} \chi^f_p(\tilde{q}) \chi^c_{Q}(q, 0)$$  \hfill (3.61)

Eq. (3.60) then determines $|a_m|^2 = N$. We may choose the phase of the $|Q >_{c-}$ Ishibashi states so that $a_m = \sqrt{N}$.

We should consider the matrix elements between free and dyon states. This corresponds to a situation where a dyon, corresponding to boundary state $|D >$, given in Eq. (3.56), is at the centre of a spherical box of radius $l$. At the surface of the box we impose the free boundary condition with phase shift $\theta$:

$$\psi_R(l) = -e^{-i\theta} \psi_L(l)$$  \hfill (3.62)

We need the matrix element $c_+ < Q| e^{-iH^0_p} |Q' >_{c-}$. We see from the definitions that this vanishes unless $Q = Q' = 0$. In this case the overlap of the coherent states involving the harmonic modes gives:

$$c_+ < 0| e^{-iH^0_p} |0 >_{c-} = \tilde{q}^{-1/24} \prod_{n=1}^{\infty} (1 + \tilde{q}^n)^{-1} \equiv W(\tilde{q})$$  \hfill (3.63)

To proceed, we need the modular transform of $W(\tilde{q})$. This can be obtained from the Euler identity, the Jacobi triple product formula and Eq. (3.46):
\[ W(\bar{q}) = \sqrt{2} W_+ (q) \equiv \frac{1}{\eta(q) \sqrt{2}} \sum_{Q = -\infty}^{\infty} q^{(Q+1/2)^2/4} \] (3.64)

[See, for example, Ref. (16).] Also modular transforming the \( \chi_p^f \), using Eq. (3.47), we obtain:

\[ Z_{F\theta,D} \equiv \langle \theta| e^{-iH_{\beta}\theta}| D \rangle = \sum_{p=0}^{N-1} \chi_p^f(q) a_0 \sqrt{2/N} \sqrt{W_+(q)} \] (3.65)

We see that we must require \( a_0 = \sqrt{N/2} \) to obtain a unique groundstate. But from considering \( Z_{DD} \), above, we concluded that \( a_0 = \sqrt{N} \). It is impossible to find values for the \( a_m \)'s such that the groundstate is non-degenerate in both \( Z_{DD} \) and \( Z_{F\theta,D} \). We see that the solution with the minimal possible groundstate degeneracy in \( Z_{DD} \) and \( Z_{F\theta,D} \), is \( a_0 = \sqrt{2N} \), giving a two-fold degenerate groundstate in both partition functions.

For even \( N \) a more general boundary state of dyon type exists. We will find that the peculiar degeneracy mentioned above can be avoided in this case. Indeed, the odd \( N \) problem does not arise from a well-defined (3+1)-dimensional field theory and is believed to be pathological.\(^4\) Polchinski\(^4\) showed that the Green’s functions exhibit unphysical behaviour for odd \( N \). The most general solution for even \( N \), consistent with the current boundary conditions and the gluing conditions is:

\[ | D \rangle = \sum_{m = -\infty}^{\infty} [a_m | mN >_{c-} \otimes | 0 >_f + b_m | (m + 1/2)N >_{c-} \otimes | N/2 >_f \] (3.66)

Modular transforming we obtain:

\[ Z_{DD} = \frac{1}{N} \sum_{p=0}^{N-1} \chi_p^f(q) \frac{1}{\eta(q)} \int_{-\infty}^{\infty} dq \frac{x^2}{2N} \sum_{m = -\infty}^{\infty} \left[ | a_m |^2 e^{2\pi i m x} + (-1)^p | b_m |^2 e^{2\pi i (m+1/2) x} \right] \] (3.67)

Now the choice, \( a_m = b_m = \sqrt{N/2} \) gives the partition function:

\[ Z_{DD'} = \sum_{p=0}^{N-1} \chi_p^f(q) \sum_{Q = -\infty}^{\infty} \frac{1}{2} [1 + (-1)^p + Q] \chi_Q^c(q, 0) \] (3.68)

This is the unique choice giving a non-degenerate, CP invariant groundstate, using our phase freedom to make all the \( a_m \)'s and \( b_m \)'s real and positive. The dyon-free fermion partition function is again given by Eq. (3.65) with \( a_0 = \sqrt{N/2} \). In this case, we have well-behaved partition functions with unique groundstates in both dyon-dyon and dyon-free cases. We refer to this CP invariant state as \( |D0\rangle \).

Similar to the free case, we may find a simple one-parameter generalization of this state by making a unitary transformation:

\[ | D\theta \rangle \equiv e^{i\bar{B}\theta/N} | D0 \rangle \] (3.69)

Here
\[
\hat{B} \equiv \int_{-\infty}^{\infty} dx \tilde{J}(x)
\]  

(3.70)

is the baryon number. [Note that we must use the baryon number, \( \hat{B} \), here rather than the electric charge, \( \hat{Q} \) as in Eq. (3.55) since the \( |Q >_{c_+} \) states occur in this case.] The unitary transformation of Eq. (3.70) is precisely the one that connects the various “\( \theta \)-vacua” in the (3+1)-dimensional gauge theory. ie. the full Hilbert Space separates into sectors labelled by the angle, \( \theta \). Making this unitary transformation is equivalent to introducing the topological angle into the Lagrangian. [See, for example the discussion in Ref. (4) where \( B \) is referred to as \( Q_5 \), the chiral charge.] Thus we may identify the CP non-invariant state, \( |D\theta > \), as the dyon boundary state in the theory with topological angle, \( \theta \). We will confirm this in the next section by computing the Green’s functions and comparing to the results of Polchinski.

The corresponding coefficients, defined in Eq. (3.66) are:

\[
a_m = \sqrt{\frac{N}{2}} e^{im2\theta}, \quad b_m = \sqrt{\frac{N}{2}} e^{i(2m+1)\theta}
\]

(3.71)

Explicitly,

\[
|D\theta > = \sqrt{\frac{N}{2}} \sum_{m=\infty}^{\infty} [e^{2im\theta}|mN >_{c_-} \otimes|0 >_f + e^{i(2m+1)\theta}((m + 1/2)N >_{c_-} \otimes|N/2 >_f]
\]

(3.72)

The dyon boundary state, \( |D\theta > \) of Eq. (3.72), is the main result of this paper.

Note that the linear combination of boundary states:

\[
|D\theta > + |D\theta + \pi> = \sqrt{2N} \sum_{m=-\infty}^{\infty} e^{2im\theta}|mN >_{c_-} \otimes|0 >_f,
\]

(3.73)

has the form of Eq. (3.56). In other words, for even \( N \), that state is actually a linear combination of two basis boundary states, which explains the factor of two degeneracy. On the other hand, for odd \( N \), it cannot be written as a linear combination and the degeneracy cannot be eliminated.

The corresponding spectrum for two states \( |D\theta > \) and \( |D\theta' > \) is:

\[
Z_{D\theta,D\theta'} = \sum_{p=0}^{N-1} \sum_{Q=-\infty}^{\infty} \chi^c_Q[q, 2(\theta' - \theta)/N] \frac{1}{2}[1 + (-1)^{p+Q}] = \sum_{p=0}^{N-1} \sum_{Q=p \mod 2}^{\infty} \chi^c_Q[q, 2(\theta' - \theta)/N]
\]

(3.74)

Note that the restriction on combining charge and flavour excitations, \( p = Q \mod 2 \) is less restrictive than the gluing condition in the free case, \( p = Q \mod N \) for even \( N \geq 4 \). It leads to the occurrence of states with exotic quantum numbers. It can be shown5 by a conformal transformation that the primary states in the spectrum with \( D\theta - D\theta' \) boundary conditions are in one-to-one correspondence with the boundary operators present with a dyon boundary condition. Hence these also have exotic quantum numbers and scaling dimension, as we shall
see explicitly in the next section. Note also that shifting $\theta$ by $\theta \to \theta + \pi$ is equivalent to changing the gluing conditions to $p = Q + 1 \mod 2$ by Eq. (3.41).

The matrix element between the dyon state of Eq. (3.66) and (3.71) and the free fermion state of Eq. (3.50) and (3.54) is given by:

$$Z_{F\theta D\theta'} \equiv \langle F\theta | e^{-iH_{\theta'}^D} | D\theta' \rangle = \frac{\sqrt{N}}{2} \chi_0^f(\tilde{q}) W(\tilde{q}) = \sum_{p=0}^{N-1} \chi_p^f(q) W_+(q)$$  \hspace{1cm} (3.75)

This is again a consistent partition function with a non-degenerate groundstate. This partition function for “a dyon in a box” has not, to our knowledge, been obtained before.

The occurrence of $W_+(q)$ can be understood by considering the effect of imposing the charge boundary conditions:

$$J_R = \pm J_L$$  \hspace{1cm} (3.76)

with opposite sign at the two ends of the system. These conditions essentially imply:

$$\phi_R(0) = \phi_L(0)$$

$$\phi_R(l) = -\phi_L(l)$$  \hspace{1cm} (3.77)

Equivalently, working with a right-mover only:

$$\phi_R(l) = -\phi_R(-l)$$  \hspace{1cm} (3.78)

This twisted boundary condition determines the mode expansion:

$$\phi_R(x-t) = \sum_{m=0}^{\infty} \frac{1}{\sqrt{4\pi(m+1/2)}} \left[ e^{-i2\pi(m+1/2)(t-x)/2l} a_{m+1/2} + h.c. \right]$$  \hspace{1cm} (3.79)

Note that unlike the periodic case, there are no soliton modes and the harmonic modes have half-integer, rather than integer frequencies. The spectrum is now $E = (\pi/l)x$ with

$$x = 1/48 + \sum_{m=0}^{\infty} (m + 1/2) n_{m+1/2}$$  \hspace{1cm} (3.80)

The partition function is thus:

$$Z_{\text{twisted}}(q) = q^{1/48} \prod_{m=0}^{\infty} [1 - q^{m+1/2}]^{-1}$$  \hspace{1cm} (3.81)

from Euler’s identity and the Jacobi triple product formula we find:

$$Z_{\text{twisted}}(q) = W_+(q)$$  \hspace{1cm} (3.82)

On the other hand, the dyon-dyon partition function, like the free-free one, involves the charge characters, $\chi_\theta^Q(q, \theta)$ rather than $W(q)$. This follows because with the charge boundary conditions of Eq. (3.76) with the same sign at the two ends, the boundary condition in the right-moving formulation is simply periodic:
\[ \phi_R(l) = \phi_R(-l) \pmod{\sqrt{\pi}} \] (3.83)

To summarize the results of this section, we have studied the most general conformally invariant boundary states consistent with the boundary conditions on the currents corresponding to the dyon. We found it necessary to impose extra conditions, uniqueness of the groundstate and CP invariance, in order to find the dyon boundary state for \( \theta = 0 \). The states for general \( \theta \) were then found by making the appropriate unitary transformation which maps between the \( \theta \)-vacua in the bulk gauge theory. We note that the philosophy used here to determine the needed boundary states is quite different than in the work on the Kondo effect. In that case physical considerations led to the "fusion rule hypothesis" which led to a unique boundary state. In the dyon problem fusion has not played a role. Instead the current boundary conditions, together with a groundstate uniqueness condition led to a one-parameter family of basis states of dyon type. These involve Ishibashi states of a different type than those that occur in the free or Kondo case due to the reversed sign in the current boundary conditions.

**IV. GREEN'S FUNCTIONS**

Now that we have determined the dyon boundary state, we can calculate arbitrary Green’s functions at long distances and times compared to \( r_0 \) and \( I \) using the boundary conformal field theory techniques developed earlier.\(^5\) We shall see that this method gives identical results to those obtained earlier by Polchinski\(^4\) by integrating out the fermions using the Schwinger determinant formula. This provides a useful check on the validity of the boundary conformal field theory technique for this problem. We recall the basic ingredients of the boundary conformal field theory calculation of Green’s functions.\(^5,19,9,10\) The first, and crucial point is that left and right Virasoro algebras are identified. Furthermore, any Green’s functions involving only right-moving fields is unaffected by the boundary. In the formulation of Section I this means that any Green’s functions where all points, \( x_i \) are positive (or all negative) are unchanged. In general, Green’s functions involving both left and right moving fields are treated as if all fields were right-moving from the point of view of applying the Virasoro and Kac-Moody algebras to simplify the form of the Green’s functions. However, these algebras alone never completely determined the Green’s functions; certain additional conditions are needed. Let us first consider two-point functions. These must have the form:\(^5\)

\[ < \Phi_R(z) \Phi_L^\dagger(z') > \propto 1/(z - z')^{2x} \] (4.1)

The only undetermined feature is the overall normalization. This is determined by the boundary state. In the case where \( \Phi_R \) is a Virasoro primary field with unit normalized two-point function in the bulk:

\[ < \Phi_R(z) \Phi_R^\dagger(z') > = 1/(z - z')^{2x}, \] (4.2)

the boundary normalization is:\(^{19}\)

\[ < \Phi_R(z) \Phi_L^\dagger(z') > = < \Phi; 0 | A > < 0; 0 | A > 1/[i(z'^* - z)]^{2x} \] (4.3)
Here |A⟩ is the boundary state and |Φ; 0⟩ is the product of left and right primary states corresponding to the conformal tower of Φ, which occurs in the Ishibashi state of Eq. (3.9). |0; 0⟩ is the vacuum state. In the case of higher n-point Green’s functions the algebras in general only determine the Green’s functions up to a larger number of parameters related to the monodromy matrix. These parameters can all be fixed by using the operator product expansion together with Eq. (4.3).

Let us first consider the fermion two-point functions. We can immediately restrict the possible non-zero two-point functions by using the U(1) gauge symmetry corresponding to Q and the SU(N) flavour symmetry. In the formulation of Section I, the charge of ψ changes sign as it passes through the origin. Thus <ψ†(z)ψ(z*)⟩ vanishes due to charge conservation. (Throughout this section we use the convention that z = τ + iσ with r ≥ 0.)

<ψi(z1)ψj(z2*)⟩ is allowed to be non-zero by charge conservation but vanishes by flavour symmetry for all N > 2. In the N = 2 case it may have the form:

<ψi(z1)ψj(z2*)⟩ = Cℓij(z1 − z2*)

(4.4)

To determine the normalization constant, C, we use the boundary state. In this case the operator ΦR in Eq. (4.3) is ψi(z1), the p = Q = 1 primary field. Φ†L corresponds to ψj(−z2*). The corresponding state does indeed occur in the dyon boundary state |Dθ⟩ of Eq. (3.66) and (3.71). The needed amplitude is:

<Φ; 0|A⟩

(4.5)

Using the standard bulk normalization for the fermion two-point function we obtain

<ψi(z1)ψj(z2*)⟩ = ieθεij(z1 − z2*)

(4.6)

Imposing time-reversal symmetry requires θ = 0 or π. Note that time reversal interchanges left and right-movers. In the present formulation it takes ψi(t, r) → ψi(−t, −r). Taking into account that time-reversal is anti-unitary and setting the time co-ordinates to zero, we conclude:

<ψ1(r)ψ2(−r)⟩ = <ψ1(−r)ψ2(r)>*

(4.7)

This is only true for θ = 0 or π from Eq. (4.6). It was shown earlier that the phase θ appearing in this Green’s function should be identified with the topological angle in the (3+1)-dimensional gauge theory. [See the discussion in Ref. (4)]. This topological term breaks time-reversal symmetry except for these two values of θ. We note that this Green’s function is that of the free fermion theory with the boundary condition of Eq. (2.25).

Let us now turn to the four-point functions. These are determined by bulk conformal field theory techniques together with Eq. (4.3).19 [The corresponding calculations for the Kondo problem appear in Ref. (9) and (10).] We use non-abelian bosonization to represent the fermion fields in terms of an SU(N) level k = 1 Kac-Moody field gi and a free charge boson φ. We work entirely with right-movers on the whole real line. gi can be thought of
as the right-moving part of an $SU(N)$ matrix field, $g^j_i$. $\phi$ corresponds to the right-moving part of a free boson field. The basic non-abelian bosonization formula is:

$$\psi_i(r) \propto g_i(r)e^{i\sqrt{4\pi/N}\phi(r)}$$
$$\psi_i(-r) \propto g_i(-r)e^{-i\sqrt{4\pi/N}\phi(-r)}$$

(Here $r > 0$.) The change in sign in the exponent for the charge boson corresponds to the reversal of the charge of the fermion field upon passing through the origin. With a simple free fermion boundary condition, $\psi R(0) = \psi L(0)$, the bosonization formulas are the same but without this sign change.

Green’s functions in which all fields are on the same side of the origin are unaffected by the impurity. This is a general property of boundary conformal field theory.\(^5\) Taking into account charge and flavour conservation there are only two non-trivial four-point Green’s functions, $\langle \psi_i(z_1)\psi_j(z_2)\psi_k(z_3)\psi_l(z_4) \rangle$ and $\langle \psi_i(z_1)\psi_j(z_2)\psi_k(z_3)\psi_l(z_4) \rangle$. (The latter is only non-zero for $N = 4$.) Using the non-abelian bosonization formula of Eq.(4.8), the first Green’s function becomes:

$$\langle \psi_i(z_1)\psi_j(z_2)\psi_k(z_3)\psi_l(z_4) \rangle = \langle \exp\{i\sqrt{4\pi/N}[\phi(z_1) - \phi(z_2) \mp \phi(z_3) \pm \phi(z_4)]\} \rangle > \cdot g_l(z_1)g_j(z_2)g_k(z_3)g_l(z_4)$$

(4.9)

Here the upper or lower signs refer to the case of free or dyon boundary conditions respectively. The free boson charge Green’s function is given by:

$$\langle \psi_i(z_1)\psi_j(z_2)\psi_k(z_3)\psi_l(z_4) \rangle > \langle \exp\{i\sqrt{4\pi/N}[\phi(z_1) - \phi(z_2) \mp \phi(z_3) \pm \phi(z_4)]\} \rangle \propto [(z_1 - z_2)(z_3 - z_4)]^{-1/N}
\cdot [(z_1 - z_4^*)(z_2 - z_3^*)/(z_1 - z_3^*)(z_4^* - z_2)]^\pm 1/N$$

(4.10)

where the upper and lower signs refer to the free and dyon case respectively. The WZW flavour Green’s function is determined by the Knizhnik-Zamolodchikov equations\(^14\) up to two free parameters, $U_p$, $p = 0, 1$:

$$\langle g_l(z_1)g_j(z_2)g_k(z_3)g_l(z_4) \rangle = [(z_1 - z_4^*)(z_2 - z_3^*)]^{-1+1/N} \sum_{p=0}^1 U_p[\delta_i^j\delta_k^l\mathcal{F}_1^{(p)}(x) + \delta_i^j\delta_k^l\mathcal{F}_2^{(p)}(x)]$$

(4.11)

Here $x$ is the “cross-ratio”

$$x \equiv \frac{(z_1 - z_2)(z_3 - z_4^*)}{(z_1 - z_4^*)(z_3 - z_2)}$$

(4.12)

The functions $\mathcal{F}_A^{(0)}$ are given by:\(^14\)

$$\mathcal{F}_1^{(0)}(x) = [x(1-x)]^{1/N-1}(1-x)$$
$$\mathcal{F}_2^{(0)}(x) = [x(1-x)]^{1/N-1}x$$

(4.13)

The other two functions can be expressed in terms of hypergeometric functions.\(^14\) As $x \to 0$ they both behave as:
\[ \mathcal{F}_A^{(1)} \rightarrow x^{1/N-1+N/(N+1)} \]  

(4.14)

We may take the proportionality in Eq. (4.10) to be an equality by redefining the constants \( U_p \). By choosing \( U_0 = -1/(2\pi)^2, \) \( U_1 = 0 \) we obtain the correct free fermion result:

\[
< \psi_i(z_1) \psi^j(z_2) \psi^k(z_3^*) \psi_l(z_4^*) > = \frac{\delta^i_j \delta^k_l}{(2\pi)^2(z_1 - z_2)(z_3^* - z_4^*)} + \frac{\delta^i_j \delta^k_l}{(2\pi)^2(z_1 - z_3^*)(z_4^* - z_2)} 
\]  

(4.15)

Now let us consider the dyon case. All that remains to be determined is the two coefficients \( U_p \). These can be fixed by considering the bulk limit, \( z_1 \rightarrow z_2, z_3^* \rightarrow z_4^* \). Note that the second factor in Eq. (4.10), whose exponent takes the opposite sign in the dyon case, goes to \((-1)^{\pm 1/N}\) in this limit. Therefore, in order to obtain the free result in this bulk limit, we must choose the same values for the parameters, \( U_p \). In particular we must again choose \( U_1 = 0 \) in order to avoid a singularity of the form \( [(z_1 - z_2)(z_3^* - z_4^*)]^{N/(N+1)-1} \) since this does not correspond to the dimension of any bulk operator. Thus we obtain:

\[
< \psi_i(z_1) \psi^j(z_2) \psi^k(z_3^*) \psi_l(z_4^*) > = \left[ \frac{(z_1 - z_2^*)(z_4^* - z_2)}{(z_1 - z_3^*)(z_2 - z_3^*)} \right]^{2/N} 
\cdot \left[ \frac{\delta^i_j \delta^k_l}{(2\pi)^2(z_1 - z_2)(z_3^* - z_4^*)} + \frac{\delta^i_j \delta^k_l}{(2\pi)^2(z_1 - z_3^*)(z_4^* - z_2)} \right] 
\]  

(4.16)

By now considering the boundary limits of this expression we can deduce the presence of various boundary operators in the operator product expansion. These operators must correspond to the states in the spectrum with dyon boundary conditions at both ends, \( Z_{D\theta,D\theta}^{5} \). Choosing equal angles \( \theta \) at both ends the spectrum of primary states corresponds to scaling dimensions:

\[ x = Q^2/2N + p(N - p)/2N \]  

(4.17)

with the restriction:

\[ p = Q \pmod{2} \]  

(4.18)

Let us now consider the surface limit \( z_1 \rightarrow z_3^*, z_2 \rightarrow z_4^* \) of Eq. (4.16). (In this limit all four points approach the boundary.) This gives:

\[
< \psi_i(z_1) \psi^j(z_2) \psi^k(z_3^*) \psi_l(z_4^*) > \rightarrow \frac{\delta^i_j \delta^k_l}{(2\pi)^2[(z_1 - z_3^*)(z_4^* - z_2)]^{1-2/N}|\tau_1 - \tau_2|^{4/N}} 
\]  

(4.19)

This leading singular behaviour results from the boundary OPE:

\[ \psi_i(z_1) \psi^k(z_3^*) \rightarrow \frac{\delta^k_i}{(2\pi)(z_1 - z_3^*)^{1-2/N}} \mathcal{O}_{2,0}(\tau_1) \]  

(4.20)

where \( \mathcal{O}_{2,0}(\tau_1) \) is the \( Q = 2, p = 0 \) charge 2 flavour singlet boundary operator of dimension \( x = Q^2/2N = 2/N \) and correlation function:




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The other boundary limit is $z_1 \rightarrow z_1^\ast$, $z_2 \rightarrow z_3^\ast$. This gives:

$$<\psi_1(z_1)\psi^j(z_2)\psi^{k}\psi_1(z_3^\ast)> \rightarrow \frac{(-\delta_i^j\delta_k^l + \delta_i^j\delta_l^k)}{(2\pi)^2|\tau_1 - \tau_2|^{2-4/N}}$$

In this case the boundary OPE is:

$$\psi_1(z_1)\psi_1(z_3^\ast) \rightarrow \mathcal{O}_{0,2,\delta}(\tau_1)/(2\pi)(z_1 - z_4^\ast)^{2/N}$$

where $\mathcal{O}_{0,2,\delta}(\tau_1)$ is the charge 0 $SU(N)$ antisymmetric tensor, $Q = 0$, $p = 2$, field of dimension $x = p(N - p)/2N = 1 - 2/N$ and correlation function:

$$<\mathcal{O}_{0,2,\delta}(\tau_1)\mathcal{O}_{0,2}^j(\tau_2)> = (\delta_i^j\delta_k^l - \delta_i^j\delta_l^k)/|\tau_1 - \tau_2|^{2-4/N}$$

Now we turn to the other non-trivial four-point function, $<\psi_1(z_1)\psi_j(z_2)\psi_k(z_3^\ast)\psi_1(z_4^\ast)>$. This must vanish by $SU(N)$ symmetry except for $N = 4$ where it may be proportional to the antisymmetric tensor invariant $\epsilon_{ijkl}$. Note that this operator has charge $Q = 0$ since two points are on the positive axis and two on the negative axis. Unlike the previous case, this Green’s function vanishes in the free fermion case due to charge conservation. In the dyon case, with $N = 4$, using non-abelian bosonization it becomes:

$$<\psi_1(z_1)\psi_j(z_2)\psi_k(z_3^\ast)\psi_1(z_4^\ast)> = <\exp\{i\sqrt{\pi}[\phi(z_1) + \phi(z_2) - \phi(z_3^\ast) - \phi(z_4^\ast)]}\>\n< g_1(z_1)g_j(z_2)g_k(z_3^\ast)g_l(z_4^\ast)>$$

The charge correlation function is:

$$<\exp\{i\sqrt{\pi}[\phi(z_1) + \phi(z_2) - \phi(z_3^\ast) - \phi(z_4^\ast)]\}> \propto \left[\frac{(z_1 - z_2)(z_3^\ast - z_4^\ast)}{(z_1 - z_3^\ast)(z_1 - z_4^\ast)(z_2 - z_3^\ast)(z_2 - z_4^\ast)}\right]^{1/4}$$

The flavour correlation function must be a solution of the appropriate Knizhnik-Zamolodchikov (KZ) equation. In this case there is only a single allowed tensor structure, $\epsilon_{ijkl}$ rather than two as in the previous case. Consequently the KZ equation should only have one solution. This solution can be determined by dimensional analysis and the symmetry between the four co-ordinates to be:

$$< g_1(z_1)g_j(z_2)g_k(z_3^\ast)g_l(z_4^\ast)> \propto \epsilon_{ijkl}/[(z_1 - z_2)(z_3^\ast - z_4^\ast)(z_1 - z_3^\ast)(z_1 - z_4^\ast)(z_2 - z_3^\ast)(z_2 - z_4^\ast)]^{1/4}$$

Combining these two factors we obtain:

$$<\psi_1(z_1)\psi_j(z_2)\psi_k(z_3^\ast)\psi_1(z_4^\ast)> = C\epsilon_{ijkl}/[(z_1 - z_2)(z_3^\ast - z_4^\ast)(z_1 - z_3^\ast)(z_2 - z_3^\ast)]^{1/2}$$

where $C$ is a constant, to be determined. Note that we have determined the form of the Green’s function by symmetry and scaling considerations up to a single constant, $C$, which
can be determined from the boundary state. This is done by taking the bulk limit, \( z_1 \to z_2, z_3^* \to z_4^* \) in which:

\[
< \psi_i(z_1)\psi_j(z_2)\psi_k(z_3^*)\psi_l(z_4^*) > = C \epsilon_{ijkl}/(z_1 - z_3^*)^2
\]  (4.29)

Note that this limit is given by the bulk OPE:

\[
\psi_i(z_1)\psi_j(z_2) \to (1/2\pi)O_{2,2,ij}(z_1)
\]  (4.30)

where \( O_{2,2,ij} \) is the charge 2 antisymmetric tensor \( Q = p = 2 \) operator of dimension \( x = 1 \) with bulk correlation function:

\[
< O_{2,2,ij}(z_1)O_{2,2,kl}^{\dagger}(z_2) >_{\text{bulk}} = (\delta^k_i \delta^l_j - \delta^k_j \delta^l_i)/(z_1 - z_2)^2
\]  (4.31)

To determine the constant, \( C \), we need the two-point function of \( O_{2,2,ij} \) when the two points straddle the boundary. This depends on the boundary state, \( |D\theta> \). The state corresponding to the operator \( O_{2,2,ij}(z)O_{2,2,kl}(z^*) \) is contained in the Ishibashi state \( |2 >_c \otimes |2 >_f \). Thus the constant \( C \) is determined from \( b_0/a_0 = e^{i\theta} \) from Eq. (3.66) and (3.71).

\[
< \psi_i(z_1)\psi_j(z_2)\psi_k(z_3^*)\psi_l(z_4^*) > = \frac{e^{i\theta} \epsilon_{ijkl}}{(2\pi)^2[(z_1 - z_3^*)(z_1 - z_1^*)(z_2 - z_3^*)(z_2 - z_4^*)]^{1/2}}
\]  (4.32)

Again, time-reversal invariance requires \( \theta = 0 \) or \( \pi \); \( \theta \) is identified as the topological angle of the bulk gauge theory.

The boundary limit, \( z_1 \to z_3^* \), is determined by the boundary OPE:

\[
\psi_i(z_1)\psi_k(z_3^*) \to \frac{e^{i\theta/2}O_{02,ik}(\tau_1)}{2\pi(z_1 - z_3)^{1/2}}
\]  (4.33)

where \( O_{02,ik} \) is the \( Q = 0, p = 2 \) operator of \( x = 1/2 \), and correlation function:

\[
< O_{02,ik}(\tau_1)O_{02,jl}(\tau_2) > = \frac{\epsilon_{ijkl}}{\tau_1 - \tau_2}
\]  (4.34)

Both results, Eq. (4.16) and (4.32) agree exactly with those obtained by Polchinski by integrating out the fermions. It seems clear that higher n-point correlation functions can also be obtained using the present methods. In particular, consider non-zero Green’s functions with a total charge of \( nN/2 \) at \( x > 0 \) (and hence charge \( -nN/2 \) at \( x < 0 \)). The corresponding boundary matrix element is proportional to \( a_{n/2} \) for \( n \) even or \( b_{(n-1)/2} \) for \( n \) odd. We see from Eq. (3.71) that this Green’s function has the \( \theta \)-dependence \( e^{in\theta} \) in agreement with Polchinski.4

V. CONCLUSIONS

We conclude by comparing the Kondo and Callan-Rubakov effects, from the viewpoint of boundary conformal field theory. Both problems involve free fermions interacting with a localized quantum mechanical degree of freedom. Both problems involve singular s-wave
scattering effects; since higher spherical harmonics are unimportant, they can be reduced to one-dimensional problems involving incoming and outgoing waves on the half-line or equivalently right-movers only on the entire line. In both problems it is convenient to use non-abelian bosonization to separate charge and spin (or more generally flavour) degrees of freedom. In the Kondo problem, the interaction only involves spin (ie. flavour) fields; in the dyon problem it only involves charge fields. In both problems the local quantum-mechanical degree of freedom can be eliminated from the low-energy, long-wavelength effective theory, leaving behind only a boundary condition. In the Kondo problem this elimination can be affected by redefining the spin current so as to “adsorb” the impurity spin. In the dyon problem, a new gauge-invariant charge current is defined which involves the dyon as well as the fermions. (This is the analogue of the adsorption process.) In the Kondo problem the effective boundary condition can be found from the fusion rules. This doesn’t seem to be possible for the dyon problem where a different class of Ishibashi states, dictated by the reversed sign of the current boundary conditions, occurs.

Renormalization group ideas play a crucial role in our understanding of the Kondo effect. The effective theory with the impurity eliminated and an effective boundary condition should be regarded as an infrared stable fixed point to which the system renormalizes. The dyon problem is effectively harmonic in bosonic variables and there is not really any coupling constant to renormalize. As far as we can see, renormalization group ideas don’t seem to be relevant. Nonetheless, there is an effective description, valid at sufficiently long distances and low energies, with the impurity eliminated and replaced by an effective conformally invariant boundary condition.

There are versions of both problems for which the effective boundary conditions can be expressed linearly in the fermion fields. These may be thought of as “Fermi liquid boundary conditions”. (In the dyon problem, this corresponds to $N \leq 2$.) In other cases the boundary conditions are of “non-Fermi liquid” type and lead to exotic behaviour such as fractional scaling dimensions and fractional particle production.

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