CARTAN DETERMINANTS AND SHAPOVALOV FORMS

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Abstract. We compute the determinant of the Gram matrix of the Shapovalov form on weight spaces of the basic representation of an affine Kac-Moody algebra of ADE type (possibly twisted). As a consequence, we obtain explicit formulae for the determinants of the Cartan matrices of $p$-blocks of the symmetric group and its double cover, and of the associated Hecke algebras at roots of unity.

1. Introduction

Let $g$ be an affine Kac-Moody algebra of type $X^{(r)}_N$ as in the table:

| $X_N^{(r)}$ | $A^{(1)}_1$ | $D^{(1)}_1$ | $E^{(1)}_6$ | $A^{(2)}_{2l-1}$ | $A^{(2)}_{2l}$ | $D^{(2)}_{l+1}$ | $E^{(2)}_6$ | $D^{(3)}_4$ |
|------------|------------|------------|------------|----------------|----------------|----------------|------------|------------|
| $\ell$     | $\geq 1$   | $\geq 4$   | $6, 7, 8$  | $\geq 3$       | $\geq 1$       | $\geq 2$       | $4$         | $2$         |
| $k$        | $0$        | $0$        | $0$        | $\ell - 1$     | $\ell$         | $1$            | $2$         | $1$         |
| $\alpha$   | $\ell + 1$ | $4$        | $9 - \ell$ | $2$            | $1$            | $2$            | $1$         | $1$         |
| $\beta$    | $1$        | $1$        | $1$        | $\ell$         | $2\ell + 1$    | $2$            | $3$         | $2$         |

We are interested here in the basic representation $V = V(\Lambda_0)$ of $g$, see [11]. Let $|0\rangle$ be a vacuum vector and define the lattice $V_Z := U_Z|0\rangle$ in $V$, where $U_Z$ is the $Z$-subalgebra of the universal enveloping algebra of $g$ generated by the divided powers $e_i^n/n!, \ f_i^n/n! \ (i = 0, 1, \ldots, \ell, \ n \geq 1)$ in the Chevalley generators. Let $(\ldots)_S$ denote the Shapovalov form, the unique Hermitian form on $V$ satisfying $(|0\rangle, |0\rangle)_S = 1$ and $(e_i v, v')_S = (v, f_i v')_S$ for $i = 0, \ldots, \ell$ and all $v, v' \in V$. Its restriction to $V_Z$ gives a symmetric bilinear form

$$(\ldots)_S : V_Z \times V_Z \to \mathbb{Z}.$$ 

Our Main Theorem gives an explicit formula for the determinant of the Gram matrix of this form on each weight space of $V_Z$.

To state the result precisely, recall the description of the weights of $V$ [11, §12.6]: every weight is of the form $w\Lambda_0 - d\delta$ for some $w$ in the Weyl group $W$ associated to $g$ and some integer $d \geq 0$. Also let $\mathcal{P}(d)$ denote the set of all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of $d$. Given $\lambda \in \mathcal{P}(d)$, we can gather together its equal parts to represent it as $\lambda = (1^{r_1}2^{r_2}\ldots)$. Also recall the number $r \in \{1, 2, 3\}$ which comes from the type $X_N^{(r)}$. Then:

Main Theorem. The determinant of the Gram matrix of the Shapovalov form on the $(w\Lambda_0 - d\delta)$-weight space of $V_Z$ is

$$\alpha^{a(d)}\beta^{b(d)}$$

where $a(d) = \sum_{\lambda \in \mathcal{P}(d)} a_\lambda$, $b(d) = \sum_{\lambda \in \mathcal{P}(d)} b_\lambda$

Second author partially supported by the NSF (grant no. DMS-9900134).
and for $\lambda = (1^{r_1} 2^{r_2} \ldots)$,

$$
a_\lambda = \prod_{i \text{ with } r_i} \left( \ell + r_i - 1 \right) \cdot \prod_{i \text{ with } r_i} \left( k + r_i - 1 \right) \cdot \sum_{i \text{ with } r_i} \frac{r_i}{\ell},
$$

$$
b_\lambda = \prod_{i \text{ with } r_i} \left( \ell + r_i - 1 \right) \cdot \prod_{i \text{ with } r_i} \left( k + r_i - 1 \right) \cdot \sum_{i \text{ with } r_i} \frac{r_i}{k},
$$

$\ell, k, \alpha, \beta$ being as in the above table. The generating functions $a(q) = \sum_{d \geq 0} a(d)q^d$ and $b(q) = \sum_{d \geq 0} b(d)q^d$ are given by the formulae

$$
a(q) = T(q^\ell)P(q)^k P(q^r)^{\ell - k},
$$

$$
b(q) = (T(q) - T(q^r))P(q)^k P(q^r)^{\ell - k}
$$

where $P(q) = \prod_{i \geq 1} \frac{1}{1 - q^i}$ is the generating function for the number of partitions of $d$ and $T(q) = \sum_{i \geq 1} q^i$ is the generating function for the number of divisors of $d$.

In [8], De Concini, Kac and Kazhdan constructed the basic representation over $\mathbb{Z}$ (at least in the untwisted cases) using an integral version of the vertex operator construction of [8]. They showed in particular that the basic representation remains irreducible on reduction modulo $p$ if and only if $p \nmid \det X_N$, where $X_N$ is the Cartan matrix of the underlying finite root system; this also follows immediately from our Main Theorem on noting that $\det X_N = \alpha \beta^{n-1}$.

Our interest in the theorem comes instead from modular representation theory. Suppose now that $\mathfrak{g}$ is of type $A_1^{(1)}$ and set $p = (\ell + 1)$. Let $FS_n$ denote the group algebra of the symmetric group over a field $F$ of characteristic $p$ (assuming in this case that $p$ is prime), and let $H_n$ denote the Iwahori-Hecke algebra associated to $S_n$ over an arbitrary field but at a primitive $p$th root of unity (this case making sense for arbitrary $p \geq 2$). By [5, 6], there is an isomorphism between the basic representation $V_\mathbb{Z}$ of $\mathfrak{g}$ and the direct sum $K = \bigoplus_{n \geq 0} K_n$ of the Grothendieck groups $K_n$ of finitely generated projective $FS_n$- (resp. $H_n$-) modules for all $n$. Under the isomorphism, the weight spaces of $V_\mathbb{Z}$ are in 1–1 correspondence with the block components of $K$, a weight space of the form $w\Lambda - d\delta$ corresponding to a block of $p$-weight $d$ (see e.g. [6, §5.3] for the definition of the $p$-weight of a block). Moreover, according to [6, Theorem 14.2], the Shapovalov form corresponds to the usual Cartan pairing $\langle [P], [Q] \rangle = \dim \text{Hom}(P, Q)$ between projective modules $P, Q$. Thus the theorem has the following immediate corollary:

**Corollary 1.** Let $B$ be a block of $p$-weight $d$ of either the group algebra $FS_n$ of the symmetric group over a field of prime characteristic $p$, or the Hecke algebra $H_n$ over an arbitrary field but at a primitive $p$th root of unity, in which case $p \geq 2$ is an arbitrary integer. Then the determinant of the Cartan matrix of $B$ is $p^{N(d)}$ where

$$
N(d) = \sum_{\lambda = (1^{r_1} 2^{r_2} \ldots) \in \mathcal{P}(d)} \frac{r_1 + r_2 + \ldots}{p - 1} \left( \frac{p - 2 + r_1}{r_1} \right) \left( \frac{p - 2 + r_2}{r_2} \right) \ldots.
$$

The generating function $N(q) = \sum_{d \geq 0} N(d)q^d$ equals $T(q)P(q)^{p-1}$. 


It is a classical result of Brauer that the determinant of the Cartan matrix of a block of $FS_n$ is a power of $p$ (see [3, 84.17]). Donkin [3] has proved similarly that the determinant of the Cartan matrix of a block of $H_n$ divides a power of $p$. The corollary shows in particular that the determinant is exactly a power of $p$, even in those cases where $p$ is not prime, as had been conjectured by Mathas. We remark that in the case of blocks of $FS_n$, but not of $H_n$, the explicit generating function given in the corollary has also recently been obtained by Bessenrodt and Olsson [3] using methods from block theory.

Finally suppose that $g$ is of type $A^{(2)}_{2\ell}$ and set $p = (2\ell + 1)$. In this case, the Main Theorem can be reinterpreted as a computation of Cartan determinants of the $p$-blocks of the double covers $\hat{S}_n$ of the symmetric group. Following [3, §9-c] for notation, let $S(n)$ be the twisted group algebra of $S_n$ over an algebraically closed field $F$ of characteristic $p$ (assuming $p$ is an odd prime in this case), and let $W(n)$ be the Hecke-Clifford superalgebra over an algebraically closed field of characteristic different from 2 at a primitive $p$th root of unity (for arbitrary odd $p \geq 3$). By [3, 7.16, 8.13, 9.9], there is an isomorphism between the basic representation $V_2$ and the direct sum $K = \bigoplus_{n \geq 0} K_n$ of the Grothendieck groups of finitely generated projective $S(n)$- (resp. $W(n)$-) supermodules, under which a weight space of the form $w\Lambda - d\delta$ maps to a superblock of $p$-bar weight $d$ (see [3, §9-a] for the definition of $p$-bar weight of a superblock), and the Shapovalov form corresponds to the Cartan pairing on projective supermodules (see [3, §7-c]). So:

**Corollary 2.** Let $B$ be a superblock of $p$-bar weight $d$ of either $S(n)$ in odd characteristic $p$, or $W(n)$ at a primitive $p$th root of unity, in which case $p \geq 3$ is an arbitrary odd integer. Then the determinant of the Cartan matrix of $B$ is $p^{N(d)}$ where

$$N(d) = \sum_{\lambda=(1^r2^s...)} \frac{2r_1 + 2r_2 + 2r_3 + \ldots}{p-1} \left( \frac{p-3}{r_1} \right) \left( \frac{p-3}{r_2} \right) \left( \frac{p-3}{r_3} \right) \ldots.$$  

The generating function $N(q) = \sum_{d \geq 0} N(d)q^d$ equals $(T(q) - T(q^2))P(q^{(p-1)/2})$.

It is more natural from the point of view of finite group theory to ask for the Cartan determinant of a block $B$ of the twisted group algebra $S(n)$ in the usual ungraded sense. According to Humphreys’ classification [10], see also [3, 9.16], we can associated to $B$ its $p$-bar weight $d$ and a type $\varepsilon \in \{M, Q\}$. In case $\varepsilon = M$, $B$ coincides with a superblock of $p$-bar weight $d$ and it is immediate that its Cartan determinant is as in Corollary 2. But in the cases when $\varepsilon = Q$ and $d > 0$, the Cartan matrix of $B$ has twice as many rows and columns as the Cartan matrix of the corresponding superblock. Nevertheless, we believe the Cartan determinant is the same, based on explicit computations for small $d$. In other words, we conjecture that Cartan determinants of $p$-blocks of $S(n)$ depend only on the $p$-bar weight $d$, not on the type $\varepsilon$, of the block.

2. The affine algebras

We begin by recalling the construction of the affine Lie algebras from [11, Chapter 8]. Let $X_N^{(r)}$ be an affine Dynkin diagram of ADE type as in the introduction, and let $X_N$ be the underlying finite Dynkin diagram. We use the same numbering of Dynkin diagrams as [11, §4.8] with two exceptions: in the case $X_N^{(r)} = E_6^{(2)}$ we will number the vertices of...
the finite Dynkin diagram $X_N = E_6$ by

![Dynkin diagram](image)

and in the case $X_N^{(r)} = A_{2\ell}^{(2)}$ we will number the vertices of the finite Dynkin diagram $X_N = A_{2\ell}$ by

![Numbered Dynkin diagram](image)

Let $Q'$ denote the root lattice of type $X_N$, with simple roots $\alpha'_i$ and invariant bilinear form $(\cdot, \cdot)'$ normalized so that each $(\alpha'_i, \alpha'_j)' = 2$. Let $\mu : Q' \to Q'$ be a graph automorphism of order $r$, as in e.g. [11, §7.9]. Let

$$\varepsilon : Q' \times Q' \to \{ \pm 1 \}$$

be an asymmetry function as in [11, §7.8] chosen so that $\varepsilon(\mu(\alpha'), \mu(\beta')) = \varepsilon(\alpha', \beta')$. In case $X_N^{(r)} = A_{2\ell}^{(2)}$ this is not possible so we instead require here that $\varepsilon(\mu(\alpha'), \mu(\beta')) = \varepsilon(\beta', \alpha')$. Let $h' = \mathbb{C} \otimes \mathbb{Z} Q'$ viewed as an abelian Lie algebra, and extend $\mu$ and $(\cdot, \cdot)'$ linearly to $h'$. Then we can construct the finite dimensional simple Lie algebra $g'$ of type $X_N$ as the vector space

$$g' = h' \oplus \bigoplus_{\alpha' \text{ a root}} \mathbb{C}E_{\alpha'}$$

viewed as a Lie algebra so that $h'$ is abelian and

$$[\alpha', E_{\beta'}] = (\alpha'|\beta')'E_{\beta'}, \quad [E_{\alpha'}, E_{-\alpha'}] = -\alpha', \quad [E_{\alpha'}, E_{\beta'}] = \begin{cases} 
\varepsilon(\alpha', \beta')E_{\alpha' + \beta'} & \text{if } \alpha' + \beta' \text{ is a root,} \\
0 & \text{otherwise.}
\end{cases}$$

The invariant form on $h'$ extends to $g'$ by $(h'|E_{\alpha'})' = 0$ and $(E_{\alpha'}|E_{\beta'})' = -\delta_{\alpha', -\beta'}$ for all roots $\alpha', \beta'$.

Let $a_i, a_i^\vee$ ($i = 0, \ldots, \ell$) be the numerical labels on the Dynkin diagram $X_N^{(r)}$ and its dual as in [11, §4.8]. We note especially that $a_0 = 1$ if $X_N^{(r)} \neq A_{2\ell}^{(2)}$ and $a_0 = 2$ if $X_N^{(r)} = A_{2\ell}^{(2)}$. It will also be convenient to define

$$c_i = \begin{cases} 
2 & \text{if } X_N^{(r)} = A_{2\ell}^{(2)} \text{ and } i = 0, \\
1 & \text{otherwise;}
\end{cases} \quad d_i = c_ia_i^\vee a_i^{-1} \in \{1, r\}$$

for $i = 0, 1, \ldots, \ell$. Let $m = a_0r$ and fix a primitive $m$th root of unity $\omega \in \mathbb{C}$. In all types other than $A_{2\ell}^{(2)}$, let $\eta : Q' \to \mathbb{C}^\times$ denote the constant function with $\eta(\alpha') = 1$ for all $\alpha' \in Q'$; in type $A_{2\ell}^{(2)}$, define $\eta$ instead by the rules

$$\eta(0) = 1, \quad \eta(\alpha' + \beta') = \eta(\alpha')\eta(\beta')(-1)^{(\alpha'|\beta')}, \quad \eta(\alpha'_j) = \begin{cases} 
1 & j \neq 0, \ell + 1, \\
\omega & j = 0, \ell + 1.
\end{cases}$$

Now extend $\mu$ from $h'$ to $g'$ by declaring that $\mu(E_{\alpha'}) = \eta(\alpha')E_{\mu(\alpha')}$ for all roots $\alpha' \in Q'$. The order of the resulting automorphism $\mu$ of $g'$ is equal to $m$ in all cases.
Decompose
\[ g' = \bigoplus_{n \in \mathbb{Z}/m} g'_n \quad \text{where} \quad g'_n = \{ X \in g' | \mu(X) = \omega^n X \}. \]

Also write \( h'_n = h' \cap g'_n \). Introduce the infinite dimensional Lie algebras
\[ g = \bigoplus_{n \in \mathbb{Z}} g'_n \otimes t^n + \mathbb{C}c \oplus \mathbb{C}d \subseteq g' \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \]
\[ h = h'_0 \oplus 1 \oplus \mathbb{C}c \oplus \mathbb{C}d \subseteq g, \]
\[ t = t^+ \oplus \mathbb{C}c \oplus t^- \subset g \quad \text{where} \quad t^\pm = \bigoplus_{n > 0} h'_n \otimes t^n. \]

Multiplication is defined by the rules
\[ [d, X \otimes t^n] = nX \otimes t^n, \quad [c, g] = 0, \]
\[ [X \otimes t^n, Y \otimes t^k] = [X, Y] \otimes t^{n+k} + \delta_{n,-k} \frac{(X|Y)'r}{m} c. \]

Then \( g \) is the affine Lie algebra of type \( X_N^{(r)} \) with canonical central element \( c \) and scaling element \( d \), and \( h \) is a Cartan subalgebra. As a matter of notation, we will write
\[ X(n) := \sum_{j=0}^{m-1} \omega^{-nj} \mu^j(X) \otimes t^n \in g'_n \otimes t^n \]
for \( X \in g' \) and \( n \in \mathbb{Z} \). The normalized invariant form on \( g \) will be denoted \((.,.)\), and is defined by
\[ (X \otimes t^n|Y \otimes t^k) = \delta_{n,-k} (X|Y)'/r \]
for all \( X \in g_n, Y \in g_k \).

In order to write down a choice of Chevalley generators for \( g \), let \( \ell \) denote the number of \( \mu \)-orbits on the simple roots in \( Q' \). Let
\[ \varepsilon = \begin{cases} 0 & \text{if } X_N^{(r)} \neq A_{2\ell}^{(2)}, \\ \ell & \text{if } X_N^{(r)} = A_{2\ell}^{(2)}, \end{cases} \]
and set
\[ I = \{0, 1, \ldots, \ell\} - \{\varepsilon\}. \]

Then, the \( \alpha_i' \) for \( i \in I \) give a set of representatives for the \( \mu \)-orbits on the simple roots. Define
\[ -\alpha'_\varepsilon = \begin{cases} \text{the longest root in } Q' & \text{if } r = 1 \text{ or } X_N^{(r)} = A_{2\ell}^{(2)}, \\ \alpha'_1 + \cdots + \alpha'_{2\ell-2} & \text{if } X_N^{(r)} = A_{2\ell-1}^{(2)}, \\ \alpha'_1 + \cdots + \alpha'_\ell & \text{if } X_N^{(r)} = D_{\ell+1}^{(2)}, \\ \alpha'_2 + \alpha'_3 + \alpha'_4 & \text{if } X_N^{(r)} = D_{\ell}^{(3)}, \\ \alpha'_1 + 2\alpha'_2 + 2\alpha'_3 + \alpha'_4 + \alpha'_5 + \alpha'_6 & \text{if } X_N^{(r)} = E_6^{(2)}. \end{cases} \]

For \( i = 0, 1, \ldots, \ell \), write
\[ e_i(n) = \frac{\sqrt{m_i}}{a_0 d_i} E_{\alpha'_i}(n) \quad \text{and} \quad f_i(n) = \frac{\sqrt{m_i}}{a_0 d_i} E_{-\alpha'_i}(n). \]
The Chevalley generators of $\mathfrak{g}$ are $e_0 = e_0(1), e_i = e_i(0)$ and $f_0 = f_0(-1), f_i = f_i(0)$ for $i = 1, \ldots, \ell$, as is proved in [11, §8.7] (taking $s_0 = 1, s_1 = \cdots = s_\ell = 0$). We also define $h_i = [e_i, f_i] = \delta_{i,0}c + \frac{c_i}{a_0}d_i^{(0)}$.

Next let $Q \subset \mathfrak{h}^*$ denote the root lattice associated to $\mathfrak{g}$. So following [11, §6.2], $Q = \bigoplus_{i=0}^\ell Z\alpha_i \oplus Z\Lambda_0$ where $\alpha_0, \ldots, \alpha_\ell$ are the simple roots corresponding to $h_0, \ldots, h_\ell$ and $\Lambda_0$ is the zeroth fundamental dominant weight, i.e.

$$\langle h_i, \alpha_j \rangle = \delta_{ij},$$
$$\langle h_i, \Lambda_0 \rangle = \langle d, \alpha_i \rangle = \delta_{i,0},$$
$$\langle d, \Lambda_0 \rangle = 0,$$

for $i, j = 0, \ldots, \ell$. Also as in [11, §6.2], we have the normalized invariant form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ and the element $\delta = \sum_{i=0}^\ell a_i\alpha_i \in Q$.

To conclude, we explain the relationship between the form $(\cdot, \cdot)'$ on $Q'$ and the form $(\cdot, \cdot)$ on $Q$. Introduce the new symmetric bilinear form $(\cdot, \cdot)_\mu$ on $Q'$ defined by

$$(\alpha'|\beta')_\mu = \langle \alpha'| \sum_{j=0}^{r-1} \mu^j(\beta') \rangle'$$

for all $\alpha', \beta' \in Q'$. There is an orthogonal decomposition

$$\mathfrak{h}^* = \mathfrak{h}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0)$$

where $\mathfrak{h}^* = \bigoplus_{i=1}^\ell C\alpha_i$, see [11, §6.2]. As in loc. cit. we write $- : \mathfrak{h}^* \to \mathfrak{h}^*$ for the orthogonal projection, in particular $Q'$ denotes the orthogonal projection of $Q$ onto $\mathfrak{h}^*$.

Define a $\mathbb{Z}$-linear map $\iota : Q' \to \overline{Q}$

by $\iota(\mu^j(\alpha'_i)) = \overline{\alpha}_i$ for each $i \in I$ and $j \geq 0$. The kernel of $\iota$ is the space

$$M' = \{ \alpha' - \mu(\alpha') \mid \alpha' \in Q' \}$$

which is precisely the radical of the bilinear form $(\cdot, \cdot)_\mu$. Moreover, $\iota$ induces an isometry between $Q'/M'$ and $\overline{Q}$ with respect to the forms induced by $(\cdot, \cdot)_\mu$ and $(\cdot, \cdot)$ respectively.

### 3. The basic representation

Next we recall the construction of the basic representation $V = V(\Lambda_0)$ of $\mathfrak{g}$, following Lepowsky [12]. Let $Z = \langle -1, \omega \rangle \subset \mathbb{C}^\times$ be the multiplicative group generated by $-1$ and $\omega$. Form the central extension

$$1 \to Z \to \tilde{Q} \to Q' \to 1,$$
namely, \( \hat{Q} = \{ e^\alpha_x \mid \alpha' \in Q', x \in Z \} \) with multiplication
\[
e^\alpha_x e^\beta_y = \begin{cases} e^{\alpha + \beta}_{xy}(\alpha', \beta') & \text{if } X_N^{(r)} \neq A_2^{(2)} D_4^{(3)}, \\ e^{\alpha + \beta}_{xy}(\alpha', \beta')(-\omega)^{-(\alpha' \mu(\beta'))'} & \text{if } X_N^{(r)} = A_2^{(2)} \text{ or } D_4^{(3)}, \end{cases}
\]
for \( \alpha', \beta' \in Q', \ x, y \in Z \). The map \( \pi : \hat{Q} \to Q' \) here is defined by \( \pi(e^\alpha_x) = \alpha' \). Let \( \hat{M} = \pi^{-1}(M') \), where \( M' \) is as in [22]. There is a well-defined multiplicative character \( \tau : \hat{M} \to \mathbb{C}^\times \) defined in [12, Proposition 6.1] by
\[
\tau(e^\alpha_x - \mu(\alpha')) = (-1)^{\alpha' \mu(\alpha')/2} x \eta(\alpha') e(\alpha', \mu(\alpha')) \omega^{-a_0^2(\alpha' \mu(\alpha'))/2}.
\]
So we can form the induced \( \hat{Q} \)-module \( C[\hat{Q}] \otimes_{\mathbb{C}[\hat{M}]} \tau \). We note the useful formula
\[
e^\alpha_t \otimes \tau = \eta(\alpha') \omega a_0(\alpha' \mu(\alpha'))/2 e_1^\alpha \otimes \tau \quad (\alpha' \in Q').
\]
View the symmetric algebra \( S(t^-) \) as a \( t^- \)-module in the unique way so that \( c \) acts as 1, elements of \( t^+ \) act by multiplication, and elements of \( t^+ \) annihilate 1. It is \( \mathbb{Z} \)-graded by declaring that
\[
\deg(h \otimes t^{-n}) = \frac{n}{a_0}
\]
for each \( h \in h_-^l, n \geq 1 \). Let
\[
V = S(t^-) \otimes_{\mathbb{C}[\hat{Q}]} \otimes_{\mathbb{C}[\hat{M}]} \tau.
\]
Let \( t \) act on \( S(t^-) \) as given and trivially on \( C[\hat{Q}] \otimes_{\mathbb{C}[\hat{M}]} \tau \), let \( h \otimes t^0 \) for \( h \in h_0 \) act by
\[
(h \otimes t^0)(f \otimes e^\alpha_t \otimes \tau) = (h|\alpha') f \otimes e^\alpha_t \otimes \tau,
\]
and let \( d \) act by
\[
d(f \otimes e^\alpha_t \otimes \tau) = -a_0(\deg(f) + (\alpha' \mu(\alpha'))/2) f \otimes e^\alpha_t \otimes \tau.
\]
We have now defined the action of \( h + t \) on \( V \). To extend the action to all of \( g \), let \( \alpha' \in Q' \) be a root. As in [12, (4.8)], let
\[
\sigma(\alpha') = \begin{cases} 1 & r = 1, \\ \sqrt{2} \omega^{-1}(\alpha' \mu(\alpha'))' & \text{if } X_N^{(r)} = A_2^{(2)} D_4^{(3)} \text{ or } E_6^{(2)}, \\ 2(1 + \omega)^{(\alpha' \mu(\alpha'))'} & \text{if } X_N^{(r)} = A_2^{(2)}.
\end{cases}
\]
Also define
\[
P_{\alpha'}(z) = \exp \left( \sum_{n \geq 1} \frac{\alpha'(-n)z^n}{n} \right), \quad Q_{\alpha'}(z) = \exp \left( -\sum_{n \geq 1} \frac{\alpha(n)z^n}{n} \right),
\]
viewed as elements of \( \operatorname{End}(V)[[z^{\pm 1}]] \). Let
\[
E_{\alpha'}(z) = \sigma(\alpha') P_{\alpha'}(z) Q_{\alpha'}(z^{-1}) e_1^\alpha z^{a_0(\alpha' \mu(\alpha'))/2 - 1}.
\]
Here, \( z^{a_0(\alpha')} \) denotes the operator with
\[
z^{a_0(\alpha')} (f \otimes e^\beta_x \otimes \tau) = z^{(a_0(\alpha'))} f \otimes e^\beta_x \otimes \tau
\]
for each $f \in S(t^-)$ and $\beta \in Q'$, and
\[
e_1'(f \otimes e_2^{\beta} \otimes \tau) = f \otimes (e_1' e_2^{\beta}) \otimes \tau.
\]
Expanding $E_{\alpha'}(z)$ in powers of $z$ we get the required action of $E_{\alpha'}(n) \in g$ on $V$ for each root $\alpha' \in Q'$ and each $n \in \mathbb{Z}$:
\[
E_{\alpha'}(z) = \sum_{n \in \mathbb{Z}} E_{\alpha'}(n) z^{-n-1}.
\]
For a proof that this is a well-defined irreducible representation of $g$ in case $r = 1$ see [11, §14.8]; the general case is due to Lepowsky [12].

Let $C[Q]$ denote the group algebra of $Q$, with natural basis $e^\alpha$ for $\alpha \in Q$ and multiplication $e^\alpha e^\beta = e^{\alpha+\beta}$. Note $C[Q] \otimes C[M]$ has a basis given by the elements $e_1' \otimes \tau$ for all $\alpha' \in \sum_{i \in I} \mathbb{Z} \alpha_i'$. For such an $\alpha'$, let
\[
i(e_1' \otimes \tau) = \begin{cases} 
e_1'(e_1') & \text{if } X_N^{(r)} \neq A_2^{(2)}, D_4^{(3)}, \\ (-\omega)(\alpha'\mu(\alpha'))/2 \ne_1'(e_1') & \text{if } X_N^{(r)} = D_4^{(3)}, \\ (1-\omega)(\alpha'\mu(\alpha')) \ne_1'(e_1') & \text{if } X_N^{(r)} = A_2^{(2)}, 
\end{cases}
\]
recalling the map $i : Q' \rightarrow Q$ defined in (2.1). Extending linearly, we obtain a vector space isomorphism $i : C[Q] \otimes C[M] \rightarrow C[Q]$. For $i = 0, 1, \ldots, \ell$, we define functions $\sigma_i^+ : Q \rightarrow \mathbb{C}^\times$ by the equation
\[
\sigma_i^+(\alpha) e^{\alpha+\epsilon_i} = \pm \frac{\sqrt{c_i}}{a_0 d_i} \sigma(\alpha' \epsilon_1 e_1') (\epsilon_1 e_1')^{-1}(\alpha')
\]
for all $\alpha \in Q$. The choice of the renormalization map $\epsilon_1$ above ensures:

**Lemma 3.1.** For all $i = 0, 1, \ldots, \ell$ and $\alpha \in Q$, $\sigma_i^+(\alpha) \in \{\pm 1\}$. Moreover, for $i \in I$, we have that $\sigma_i^- = -\sigma_i^+$, and $\sigma_i^+ : Q \rightarrow \{\pm 1\}$ is a group homomorphism such that $\sigma_i^+(\alpha_j) = \epsilon(\alpha_i', \alpha_j')$ for each $j \in I$.

Now we can rewrite the construction of the basic representation $V$ in terms of the Chevalley generators. We will identify
\[
V = S(t^-) \otimes C[Q] \otimes C[M] \tau = S(t^-) \otimes C[Q]
\]
via the map $id \otimes i$. Then, the actions of $h_i$ for $i = 0, \ldots, \ell$ and of $d$ are as
\[
h_i(f \otimes e^\alpha) = (\delta_{i,0} + \langle h_i, \alpha \rangle) f \otimes e^\alpha,
\]
\[
d(f \otimes e^\alpha) = -a_0 (\deg(f) + (\alpha|\alpha)/2) f \otimes e^\alpha
\]
for all $\alpha \in Q$. In particular, we note from this that
\[
\text{wt}(f \otimes e^\alpha) = \Lambda_0 + \alpha - (\deg(f) + (\alpha|\alpha)/2) \delta
\]
for each homogeneous $f \in S(t^-)$ and $\alpha \in Q$. This shows that $1 \otimes e^\alpha$ is a highest weight vector in $V$ of highest weight $\Lambda_0$ (cf. [11, Lemma 12.6]), identifying $V$ with the irreducible
highest weight module \( V(\Lambda_0) \). Finally, for \( i = 0, \ldots, \ell \),

\[
e_i(z) = \sum_{n \in \mathbb{Z}} e_i(n) \otimes z^{-n-1} = P_{\alpha'_i}(z)Q_{\alpha'_i}(z^{-1})e^{a_0(zomo_a\alpha_i/2-1)s_i^+},
\]

\[
f_i(z) = \sum_{n \in \mathbb{Z}} f_i(n) \otimes z^{-n-1} = P_{-\alpha'_i}(z)Q_{-\alpha'_i}(z^{-1})e^{-a_0(zomo_a\alpha_i/2-1)s_i^-},
\]

where

\[
z^{a_0\alpha_i}(f \otimes e^\beta) = z^{(a_0\alpha_i/\beta)} f \otimes e^\beta,
\]

\[
s_i^\pm(f \otimes e^\beta) = \sigma_i^\pm(\beta)f \otimes e^\beta,
\]

\[
e^{\pmz}(f \otimes e^\beta) = f \otimes e^{\beta \pm z}.
\]

The following lemma will be needed later on:

**Lemma 3.5.** For \( i_1, \ldots, i_s \in \{0, \ldots, \ell\} \), roots \( \beta'_1, \ldots, \beta'_s \in Q' \) and \( \gamma \in \overline{Q} \), we have that

\[
e_{i_1}(z_1)e_{i_2}(z_2) \ldots e_{i_s}(z_s)P_{\beta'_1}(w_1) \ldots P_{\beta'_s}(w_s) \otimes e^\gamma =
\]

\[
\pm \prod_{1 \leq u \leq s} z_u^{a_0(\alpha_{i_u} | \alpha_{i_u})-1+a_0(\alpha_{i_u} | \gamma)}
\]

\[
\times \prod_{1 \leq u \leq s} z_u^{a_0(\alpha_{i_u} | \alpha_{i_u})} \prod_{k \in \mathbb{Z}/m} \left( 1 - \omega^{-k\frac{z_u}{z_u}} \right)^{(\mu^k(\alpha_{i_u}' | \alpha_{i_u}'))}
\]

\[
\times \prod_{1 \leq u \leq s} \prod_{1 \leq e \leq r} \prod_{k \in \mathbb{Z}/m} \left( 1 - \omega^{-k\frac{w_e}{z_u}} \right)^{(\mu^k(\alpha_{i_u}' | \beta'_e')}
\]

\[
\times P_{\alpha'_1}(z_1) \ldots P_{\alpha'_s}(z_s)P_{\beta'_1}(w_1) \ldots P_{\beta'_s}(w_s) \otimes e^{\gamma+z_{i_1}+\ldots+z_{i_s}}.
\]

A similar formula holds for \( f_{i_1}(z_1) \ldots f_{i_s}(z_s)P_{\beta'_1}(w_1) \ldots P_{\beta'_s}(w_s) \otimes e^\gamma \), replacing \( \alpha_{i_u} \) by \(-\alpha_{i_u} \), \( \alpha_{i_u}' \) by \(-\alpha_{i_u}' \), and \( \bar{\alpha}_{i_u} \) by \(-\bar{\alpha}_{i_u} \) everywhere.

**Proof.** This follows from the following commutation relation obtained in [12, 3.4]: for \( \alpha', \beta' \in Q' \),

\[
Q_{\alpha'}(z^{-1})P_{\beta'}(w) = P_{\beta'}(w)Q_{\alpha'}(z^{-1}) \prod_{k \in \mathbb{Z}/m} (1 - \omega^{-k\frac{w}{z}})^{\mu^k(\alpha') | \beta')},
\]

which is a consequence of the Campbell-Hausdorff formula, cf. [12, (14.8.12)].

\[\square\]

4. THE INTEGRAL FORM

As in the introduction, let \( U_\mathbb{Z} \) denote the \( \mathbb{Z} \)-subalgebra of the universal enveloping algebra of \( \mathfrak{g} \) generated by the elements \( e_i^r/r! \), \( f_i^r/r! \) for \( i = 0, \ldots, \ell \) and \( r \geq 0 \), and let

\[
V_\mathbb{Z} := U_\mathbb{Z}(1 \otimes e^0) \subset V.
\]

In this section, we will give an explicit description of \( V_\mathbb{Z} \).

To start with, let \( \tau : \mathfrak{g} \to \mathfrak{g} \) be the antilinear Chevalley antiautomorphism, so

\[
\tau(d) = d, \quad \tau(e_{i}(n)) = f_{i}(-n), \quad \tau(f_{i}(n)) = e_{i}(-n).
\]
for each $i = 0, \ldots, \ell$ and $n \in \mathbb{Z}$, cf. [4, §§7.6, 8.3]. The Shapovalov form $\langle \cdot, \cdot \rangle_S$ on $V$ is the unique Hermitian form such that

$$ (1 \otimes e^0, 1 \otimes e^0)_S = 1 \quad \text{and} \quad (xv, w)_S = (v, \tau(x)w)_S $$

for all $v, w \in V$, $x \in g$. The restriction of $\tau$ to $t$ gives the antilinear Chevalley antiautomorphism of $t$, and we can also consider the Shapovalov form on $S(t^-)$, satisfying $(1, 1)_S = 1$ and $(xf, g)_S = (f, \tau(x)g)_S$ for all $f, g \in S(t^-)$, $x \in t$.

**Lemma 4.1.** For all $f, g \in S(t^-)$ and $\alpha, \beta \in \mathcal{Q}$, $(f \otimes e^\alpha, g \otimes e^\beta)_S = (f, g)_S$.

**Proof.** Since different weight spaces are orthogonal and in view of (3.2), this reduces to checking that $(1 \otimes e^\alpha, 1 \otimes e^\alpha)_S = 1$ for all $\alpha \in \mathcal{Q}$. Proceeding by induction, we may assume that there is some $\beta \in \mathcal{Q}$ and $i \in I$ such that $(1 \otimes e^\beta, 1 \otimes e^\beta)_S = 1$ and either $\alpha = \beta + \varpi_i$ or $\alpha = \beta - \varpi_i$.

Suppose that $\alpha = \beta + \varpi_i$. Letting $n = -a_0(\alpha_i|\beta) - a_0(\alpha_i|\alpha_i)/2$, one checks easily using (3.3) to (3.4) that

$$ e_i(n)(1 \otimes e^\beta) = \sigma_i^+(\alpha)(1 \otimes e^\alpha), \quad f_i(-n)(1 \otimes e^\alpha) = \sigma_i^-(\beta)(1 \otimes e^\beta). \quad (4.2) $$

Hence,

$$ (1 \otimes e^\alpha, 1 \otimes e^\alpha)_S = \sigma_i^+(\beta)(e_i(n)(1 \otimes e^\beta), 1 \otimes e^\alpha)_S $$

$$ = \sigma_i^+(\beta)(1 \otimes e^\beta, f_i(-n)(1 \otimes e^\alpha))_S $$

$$ = \sigma_i^-(\alpha)\sigma_i^+(\beta)(1 \otimes e^\beta, 1 \otimes e^\beta)_S = 1, $$

since $\sigma_i^-(\alpha) = \sigma_i^+(\beta)$ by Lemma 3.1.

A similar argument in the case that $\alpha = \beta - \varpi_i$ completes the proof. \hfill $\Box$

**Lemma 4.3.** For all $i = 0, 1, \ldots, \ell$ and $n \in \mathbb{Z}$, the elements $e_i(n)$ and $f_i(n)$ belong to $U_Z$.

**Proof.** Suppose that $e_i(n) \neq 0$. Then, $\text{wt}(e_i(n)) = \varpi_i + \frac{n}{a_0} \delta$ is a real root, hence is conjugate under the Weyl group $W$ associated to $g$ to some simple root $\alpha_j$. So we can find simple reflections $s_{i_1}, \ldots, s_{i_\ell} \in W$ such that $\varpi_i + \frac{n}{a_0} \delta = s_{i_1} \cdots s_{i_\ell} \alpha_j$. Let $r_i^{ad}$ be the automorphism of $g$ defined by $r_i^{ad} = \exp(\text{ad}f_i)\exp(-\text{ad}e_i)\exp(\text{ad}f_i)$, for $i = 0, 1, \ldots, \ell$. Since real root spaces of $g$ are one dimensional,

$$ r_i^{ad} \cdots r_i^{ad} e_j = ce_i(n) $$

for some non-zero scalar $c$. Now, $\tau(\exp(\text{ad}y)(x)) = \exp(-\text{ad}\tau(y)(\tau(x)))$, whence by an $SL_2$-calculation we have $r_i^{ad}(x) = \tau(r_i^{ad}(\tau(x)))$ for all $x \in g$, we also get that

$$ r_i^{ad} \cdots r_i^{ad} f_j = cf_i(-n). $$

But the $r_i^{ad}$ preserve the normalized invariant form on $g$, so

$$ a_j(a_j^\vee)^{-1} = (e_j|f_j) = (ce_i(n)|cf_i(-n)) = c^2 a_i(a_i^\vee)^{-1}. $$

Clearly, $\alpha_i$ and $\alpha_j$ are roots of the same length, i.e. $a_j(a_j^\vee)^{-1} = a_i(a_i^\vee)^{-1}$, so this gives that $c = \pm 1$. Finally, the action of $r_i^{ad}$ on $g$ leaves $U_Z \cap g$ invariant, so

$$ e_i(n) = \pm r_i^{ad} \cdots r_i^{ad} e_j \in U_Z, $$

and similarly $f_i(n) \in U_Z$ too. \hfill $\Box$
For \( n \geq 1 \) and \( i = 0, 1, \ldots, \ell \), define
\[
y^{(i)}_{nd_i} = \frac{\alpha^{(i)}_t(-a_{0nd_i})}{a_{0nd_i}}, \quad x^{(i)}_{nd_i} = \sum_{k_1 + 2k_2 + \cdots = n} \frac{y^{(i)k_1}_{nd_i} y^{(i)k_2}_{2nd_i} y^{(i)k_3}_{3nd_i}}{k_1! k_2! k_3!} \ldots.
\]
Observe that
\[
P_{\alpha^{(i)}_t}(z) = \exp \left( \sum_{n \geq 1} y^{(i)}_{nd_i} z^{a_{0nd_i}} \right) = 1 + \sum_{n \geq 1} x^{(i)}_{nd_i} z^{a_{0nd_i}}.
\] (4.4)
The \( y^{(i)}_{nd_i} \) for \( n \geq 1 \) and \( i \in I \) give a basis for \( t^- \). So \( S(t^-) \) is equal to the free polynomial algebra
\[
B := \mathbb{C}[y^{(i)}_{nd_i} \mid n \geq 1, i \in I].
\]
Since the \( x^{(i)}_{nd_i} \) are related to the \( y^{(i)}_{nd_i} \) in a unitriangular way, we obtain a \( \mathbb{Z} \)-form
\[
B_{\mathbb{Z}} := \mathbb{Z}[x^{(i)}_{nd_i} \mid n \geq 1, i \in I] \subset B
\]
for \( B \). As \( \alpha^{(i)}_t \) is an integral linear combination of the \( \alpha^{(i)}_t \) with \( i \in I \), it follows from (4.4) that the elements \( x^{(i)}_{nd_i} \) also belong to the lattice \( B_{\mathbb{Z}} \). The \( \mathbb{Z} \)-grading on \( B_{\mathbb{Z}} \) induced by the grading on \( S(t^-) \) is determined by \( \deg(y^{(i)}_{nd_i}) = \deg(x^{(i)}_{nd_i}) = n \).

The following theorem (or rather its \( q \)-analogue) for the non-twisted case has been proved in [4]. Our argument for the general case is similar.

**Theorem 4.5.** \( V_{\mathbb{Z}} = B_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{Q}] \).

**Proof.** Let us first show that \( B_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{Q}] \subseteq V_{\mathbb{Z}} \). Fix \( i_1, \ldots, i_s \in I \), and let
\[
M(i_1, \ldots, i_s) = \{ (n_1, \ldots, n_s) \mid n_1 \geq \cdots \geq n_s \geq 0 \text{ and } d_{ij}|n_j \text{ for all } j = 1, \ldots, s \},
\]
Denote by \( > \) the dominance ordering on partitions belonging to \( M(i_1, \ldots, i_s) \). We will show that \( x^{(i_1)}_{n_1} \cdots x^{(i_s)}_{n_s} \otimes e^\beta \in V_{\mathbb{Z}} \) for all \( (n_1, \ldots, n_s) \in M(i_1, \ldots, i_s) \) and each \( \beta \in \mathcal{Q} \).

Clearly every monomial in \( B_{\mathbb{Z}} \) is of the form \( x^{(i_1)}_{n_1} \cdots x^{(i_s)}_{n_s} \) for some choice of \( i_1, \ldots, i_s \) and \( (n_1, \ldots, n_s) \in M(i_1, \ldots, i_s) \), so this is good enough.

To start with, each \( e_i(n), f_i(-n) \in U_{\mathbb{Z}} \) by Lemma 4.3. So an obvious inductive argument using (1.2) gives that \( 1 \otimes e^\gamma \in V_{\mathbb{Z}} \) for each \( \gamma \in \mathcal{Q} \). Hence, letting \( \gamma = \beta - \alpha_{i_1} - \cdots - \alpha_{i_s} \), Lemma 4.3 implies that all coefficients of \( e_i(z_1) \cdots e_i(z_s) \otimes e^\beta \) belong to \( V_{\mathbb{Z}} \). Applying Lemma 4.5, we deduce that all the coefficients of
\[
X := \left( \prod_{1 \leq u < v \leq s} \prod_{k \in \mathbb{Z}/m} \left( 1 - \omega^{-k} \frac{z_v}{z_u} \right)^{\mu_k(\alpha^{(i_1)}_u)\alpha^{(i_1)}_u'} \right) P_{\alpha^{(i_1)}_1}(z_1) \cdots P_{\alpha^{(i_s)}_s}(z_s) \otimes e^\beta
\]
belong to \( V_{\mathbb{Z}} \). One checks that in all cases,
\[
\prod_{k \in \mathbb{Z}/m} \left( 1 - \omega^{-k} \frac{z_v}{z_u} \right)^{\mu(\alpha^{(i_1)}_u)\alpha^{(i_1)}_u'} = 1 + (*),
\]
where \((*)\) is a \(\mathbb{Z}\)-linear combination of \(\left(\frac{z_n}{x_n}\right)^p\) for \(p \geq 1\). It follows that the \(z_1^{a_{n_1}} \ldots z_s^{a_{n_s}}\)-coefficient of \(X\) equals

\[
x^{(i_1)}_{n_1} \ldots x^{(i_s)}_{n_s} \otimes e^\beta + (**),
\]

where \((**)\) is a \(\mathbb{Z}\)-linear combination of \(x^{(i_1)}_{n_1} \ldots x^{(i_s)}_{n_s} \otimes e^\beta\) for \((n'_1, \ldots, n'_s) > (n_1, \ldots, n_s)\).

Using downward induction on this ordering, we deduce that \(x^{(i_1)}_{n_1} \ldots x^{(i_s)}_{n_s} \in V_Z\).

Finally, we prove that \(B \otimes \mathbb{Z} Z[\mathcal{Q}] \supseteq V_Z\). As the high weight vector \(1 \otimes e^0\) belongs to \(B \otimes \mathbb{Z} Z[\mathcal{Q}] \supseteq V_Z\), it suffices to show that \(B \otimes \mathbb{Z} Z[\mathcal{Q}]\) is invariant under each of the operators \(f_i(n)\) for \(n \in \mathbb{Z}, s \geq 1\) and \(i = 0, 1, \ldots, \ell\). Fix \(i \in \{0, 1, \ldots, \ell\}\) and consider

\[
Y := f_i(z_1) \ldots f_i(z_s) P_{a_{i_1}}(w_1) \ldots P_{a_{i_\ell}}(w_\ell) \otimes e^{\gamma + s\sigma_i}
\]

for \(i_1, \ldots, i_\ell \in I\) and \(\gamma \in \mathcal{Q}\). Applying Lemma \([15]\) and simplifying,

\[
Y = \pm (z_1 \ldots z_s)^{a_{\alpha_0} + (-a_{\alpha_0}(\alpha_0) - 1 - a_{\alpha_0}(\alpha_0) + s\sigma_i)} \prod_{1 \leq u < v \leq s} \prod_{k \in \mathbb{Z}/m} \left( z_u - \omega^{-k} z_v \right)^{(\mu^k(\alpha_0')|\alpha_0')} \times \prod_{1 \leq u \leq s} \prod_{k \in \mathbb{Z}/m} \left( 1 - \omega^{-k} \frac{w_u}{z_u} \right)^{-(\mu^k(\alpha_0')|\alpha_0')} \times P_{-\alpha_0'}(z_1) \ldots P_{-\alpha_0'}(z_s) P_{a_{i_1}}(w_1) \ldots P_{a_{i_\ell}}(w_\ell) \otimes e^\gamma.
\]

Certainly, each coefficient of \(P_{-\alpha_0'}(z_1) \ldots P_{-\alpha_0'}(z_s) P_{a_{i_1}}(w_1) \ldots P_{a_{i_\ell}}(w_\ell) \otimes e^\gamma\) belongs to \(V_Z\).

One checks that

\[
\prod_{k \in \mathbb{Z}/m} \left( z_u - \omega^{-k} z_v \right)^{(\mu^k(\alpha_0')|\alpha_0')} = \left\{ \begin{array}{ll}
\left( z_u - z_v \right)^{2(z_u + z_v)^2} & \text{if } X_N^{(i)} = A_2^{(2)} \text{ and } i = 0, \\
\left( z_u - z_v \right)^{2(z_u d_i - z_v d_i)^2} & \text{otherwise}.
\end{array} \right.
\]

Hence in all cases, \(Y\) looks like \(\prod_{1 \leq u < v \leq s} (z_u - z_v)^2\) times an expression that is symmetric in \(z_1, \ldots, z_s\). Hence, by [15], Lemma 2.5(ii), the coefficient of \((z_1 \ldots z_s)^{-n-1}\) in \(Y\) is divisible by \(s!\). Hence, all coefficients of \((f_i(n)^k/s!) P_{a_{i_1}}(w_1) \ldots P_{a_{i_\ell}}(w_s) \otimes e^{\gamma - s\sigma_i}\) belong to \(V_Z\), which completes the proof. \(\Box\)

5. THE DETERMINANT

Fix now some \(d \geq 0\). Lemma \([4.3]\) and Theorem \([4.5]\) reduce the problem of computing the determinant of the Shapovalov form on the \((w\Lambda_0 - d\delta)\) weight space of \(V_Z\) for any \(w \in W\) to the problem of computing the determinant of the Shapovalov form on the degree \(d\) component of \(B_Z\). To tackle the latter question, observe

\[
B = \bigotimes_{i \in I} \mathbb{C}[y_{d_i}, y_{2d_i}, \ldots], \quad B_Z = \bigotimes_{i \in I} \mathbb{Z}[x_{d_i}, x_{2d_i}, \ldots].
\]
Lemma 5.2. For all $i \in I, n \geq 1$ and $f, g \in B$. Moreover, there is a homogeneous basis for the lattice $B_Z$ (given by Schur polynomials) that is orthonormal with respect to the form $(\ldots)_K$. In particular, the determinant of the form $(\ldots)_K$ on the degree $d$ component of $B_Z$ is equal to 1. Our strategy will therefore be to relate the Shapovalov form $(\ldots)_S$ to the form $(\ldots)_K$.

Define $I(n) = \{i \in I \mid d_i | n\}$. Introduce the matrices $A^{(n)} = (a_{i,j}^{(n)})_{i,j \in I(n)}$ with

$$a_{i,j}^{(n)} = \frac{1}{d_i} (\alpha'_i \sum_{k=0}^{r-1} \omega^{a_0 nk} \mu_k(\alpha'_j))'.$$

Recall $\alpha, \beta$ from the introduction. One verifies:

**Lemma 5.1.** For any $n \geq 0$, we have $\det A^{(n)} = \begin{cases} \alpha & \text{if } r \mid n, \\ \beta & \text{if } r \nmid n, \end{cases}$

The significance of the matrices $A^{(n)}$ is that for $i \in I(n)$, the element $\tau(ny_n^{(i)}/d_i) = \alpha'_i(a_0n)/a_0d_i$ acts on $B$ as the operator $\sum_{j \in I(n)} a_{i,j}^{(n)} \frac{\partial}{\partial y_n^{(j)}}$. We set

$$z_n^{(i)} = \sum_{j \in I(n)} a_{i,j}^{(n)} y_n^{(j)}.$$

For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ we let $I(\lambda) = I(\lambda_1) \times \cdots \times I(\lambda_h)$. Given $\ul{i} = (i_1, \ldots, i_h) \in I(\lambda)$, let

$$x_\lambda^{(1)} = x_{\lambda_{i_1}} \cdots x_{\lambda_{i_h}}, \quad y_\lambda^{(1)} = y_{\lambda_{i_1}} \cdots y_{\lambda_{i_h}}, \quad \ul{z}_\lambda = z_{\lambda_{i_1}} \cdots z_{\lambda_{i_h}},$$

all elements of $B$ of degree $|\lambda|$.

**Lemma 5.2.** For $\ul{i} \in I(\lambda)$ and any $f \in B$, we have $(y_\lambda^{(i)}, f)_S = (\ul{z}_\lambda, f)_K$.

**Proof.** Proceed by induction on the number of non-zero parts of $\lambda$, starting induction from the obvious fact that $(1, f)_S = (1, f)_K$. For the induction step, note that for $i \in I(n)$,

$$(y_n^{(i)}, y_\lambda^{(i)}, f)_S = \left(\frac{d_i}{n} y_\lambda^{(i)}, \sum_{j \in I(n)} a_{i,j}^{(n)} \frac{\partial}{\partial y_n^{(j)}} f\right)_S$$

$$= \left(\frac{d_i}{n} \ul{z}_\lambda, \sum_{j \in I(n)} a_{i,j}^{(n)} \frac{\partial}{\partial y_n^{(j)}} f\right)_K = (\ul{z}_\lambda, f)_K.$$

Let

$$\Omega(\lambda) = \{\ul{i} \in I(\lambda) \mid i_j \leq i_{j+1} \text{ whenever } \lambda_j = \lambda_{j+1}\}.$$
Then \( \{x^{(i)}_\lambda | \lambda \in \mathcal{P}(d), \lambda \in \Omega(\lambda) \} \), \( \{y^{(i)}_\lambda | \lambda \in \mathcal{P}(d), \lambda \in \Omega(\lambda) \} \) and \( \{z^{(i)}_\lambda | \lambda \in \mathcal{P}(d), \lambda \in \Omega(\lambda) \} \) give three different bases for the degree \( d \) component of \( B \). Consider the transition matrices \( P = (p^{(i)}_{\lambda,\mu}) \) and \( Q = (q^{(i)}_{\lambda,\mu}) \) where \( \lambda, \mu \in \mathcal{P}(d) \), \( \lambda \in \Omega(\lambda) \), \( \mu \in \Omega(\mu) \) defined from

\[
x^{(i)}_\lambda = \sum_{\mu \in \mathcal{P}(d), \mu \in \Omega(\mu)} p^{(i)}_{\lambda,\mu} \lambda^{(i)}_\mu, \quad z^{(i)}_\lambda = \sum_{\mu \in \mathcal{P}(d), \mu \in \Omega(\mu)} q^{(i)}_{\lambda,\mu} \lambda^{(i)}_\mu.
\]

**Lemma 5.3.** The matrix \( Q \) is block diagonal, i.e. \( q^{(i)}_{\lambda,\mu} = 0 \) for \( \lambda \neq \mu \). Moreover, the determinant of the \( \lambda \)-block \( Q_\lambda = (q^{(i)}_{\lambda,\mu})_{\lambda, \mu \in \Omega(\lambda)} \) of \( Q \) is \( \alpha^{\lambda_1} \beta^{\lambda_2} \), notation as in the introduction.

**Proof.** It is immediate from the definition that \( q^{(i)}_{\lambda,\mu} = 0 \) for \( \lambda \neq \mu \). So consider the \( \lambda \)-block \( Q_\lambda \) of \( Q \). Represent \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0) \) instead as \( (1^{\alpha_1} 2^{\alpha_2} \cdots s^{\alpha_s}) \). By definition,

\[
z^{(i)}_\lambda = \sum_{\lambda \in \Omega(\lambda)} a^{(i)}_{\lambda, j_1} \cdots a^{(i)}_{\lambda, j_h} \mu^{(i)}_\lambda.
\]

Thus \( Q_\lambda \) is the matrix \( S^{\alpha_1}(A^{(1)}) \otimes \cdots \otimes S^{\alpha_s}(A^{(n)}) \), a tensor product of symmetric powers of the matrices \( A^{(n)} \). Now, note that for an \( n \times n \) matrix \( A \),

\[
\det S^m(A) = (\det A)^{\binom{n+m-1}{m}},
\]

while for an \( n \times n \) matrix \( B \) and an \( m \times m \) matrix \( C \),

\[
\det(B \otimes C) = (\det B)^m (\det C)^n.
\]

These are both proved by reducing to the case that the matrices are diagonal. Combining the formulae with Lemma 5.1, one computes \( \det Q_\lambda = \alpha^{\lambda_1} \beta^{\lambda_2} \). \( \square \)

Now we can prove the main theorem:

**Theorem 5.4.** The determinant of the restriction of the Shapovalov form to the degree \( d \) part of \( B \) is \( \prod_{\lambda \in \mathcal{P}(d)} \alpha^{\lambda_1} \beta^{\lambda_2} \).

**Proof.** Consider the matrices \( M = (m^{(i)}_{\lambda,\mu}) \) and \( N = (n^{(i)}_{\lambda,\mu}) \) for \( \lambda, \mu \in \mathcal{P}(d) \), \( \lambda \in \Omega(\lambda) \), \( \mu \in \Omega(\mu) \) defined from

\[
m^{(i)}_{\lambda,\mu} = (x^{(i)}_\lambda, x^{(i)}_\mu)_S, \quad n^{(i)}_{\lambda,\mu} = (x^{(i)}_\lambda, x^{(i)}_\mu)_K.
\]

Recalling the transition matrices \( P \) and \( Q \) introduced above, Lemma 5.2 gives at once that \( M = PQP^{-1}N \). On the other hand, \( N \) has determinant 1, since as we observed above the degree \( d \) component of \( B \) admits an orthonormal basis with respect to the contravariant form \( (\ldots)_K \) (see [3, Corollary 2.1]). So we can compute \( \det M \) at once using Lemma 5.3. \( \square \)

We finally indicate how to deduce the generating functions \( a(q) \) and \( b(q) \) stated in the introduction. By definition, \( a_\lambda \) is

\[
h \times \left( \frac{1}{\ell} \right)^{\text{the number of ways of coloring the parts } \lambda_i \equiv 0 (r) \text{ with } \ell \text{ colors}} \times \left( \frac{1}{r} \right)^{\text{the number of ways of coloring the parts } \lambda_i \not\equiv 0 (r) \text{ with } k \text{ colors}} \right)
\]
where \( h \) is the number of \( \lambda_i \equiv 0 \) (\( r \)). Consider

\[
G(q, t, u) = \left( \prod_{n \geq 1} \frac{1}{1 - q^n t} \right)^{\ell} \left( \prod_{n \geq 1} \frac{1 - q^n u}{1 - q^n u} \right)^k.
\]

The coefficient of \( q^d t^h u^i \) is equal to the number of partitions of \( d \) with \( h \) parts divisible by \( r \) colored by \( \ell \) different colors and with \( i \) parts not divisible by \( r \) colored by \( k \) different colors. Hence the generating function \( a(q) \) for \( a(d) = \sum_{\lambda \in \mathcal{P}(d)} a_{\lambda} \) is equal to \( \frac{1}{\ell \, d!} G(q, t, u)|_{t=u=1} \).

Similarly, \( b(q) = \frac{1}{k \, d!} G(q, t, u)|_{t=u=1} \).

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