OPTIMAL ESTIMATES FOR THE STRESS CONCENTRATION BETWEEN CLOSELY SPACED STIFF $C^{1,\gamma}$ INCLUSIONS IN LINEAR ELASTICITY

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Abstract. This paper concerns the stress concentration in linear elasticity composite materials when the distance $\varepsilon$ between inclusions tends to zero. The problem is to establish the gradient estimate for the Lamé systems with partially infinite coefficients, which models a composite containing a finite number of stiff inclusions. The difficulty introduced in this paper is weakening the smoothness of inclusions from $C^2, \gamma$ to $C^1, \gamma$. However, using a refined analysis, the Campanato’s approach and $W^{1,p}$ estimates for inhomogeneous elliptic system with right hand side in divergence form, we reduce the problem to the framework of Bao, Li and Li (Arch. Ration. Mech. Anal. 215 (2015), 1307-1351). Hence earlier iteration technique applies and we establish the optimal gradient estimates, including upper bounds and lower bounds. Especially, in dimension two, we prove that the blowup rate is $\varepsilon^{-1/(1+\gamma)}$, which is bigger than $\varepsilon^{-1/2}$ obtained before under the assumptions of $C^2, \gamma$ inclusions.

Keywords: Lamé system, Gradient estimates, Blow-up rates.

1. Introduction

1.1. Background. In high-contrast composite materials, when two inclusions are close to touch, the physical field such as the stress or the electric field may be arbitrary large in the narrow region between inclusions. It is quite important to understand such field concentration phenomenon precisely. In this paper, we study the stress concentration phenomenon in high-contrast elastic composite material. We consider the following boundary value problem of Lamé systems with partially degenerated coefficients to model a composite with two stiff inclusions. Let $D$ be a bounded open set in $\mathbb{R}^d$, $d \geq 2$, $D_1$ and $D_2$ be two adjacent convex subdomains, with $\varepsilon$-apart. Let $u = (u^{(1)}, u^{(2)}, \ldots, u^{(d)}): D \to \mathbb{R}^d$ be a vector-valued function, representing the displacement field, and verify

\[
\begin{aligned}
\mathcal{L}_{\lambda, \mu} u &:= \nabla \cdot \left( C^0 e(u) \right) = 0 \quad \text{in } \Omega := D \setminus (D_1 \cup D_2), \\
u|_+ = u|_- &\quad \text{on } \partial D_1 \cup \partial D_2, \\
e(u) &= 0 \quad \text{in } D_1 \cup D_2, \\
\int_{\partial D_i} \frac{\partial u}{\partial \nu_0} + \psi^\alpha &= 0, \quad i = 1, 2, \alpha = 1, 2, \ldots, \frac{d}{2}(d+1), \\
u &\equiv \varphi \quad \text{on } \partial D,
\end{aligned}
\]

(1.1)

where the elastic tensor $C^0$ is

\[
C_{ijkl}^0 = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad i, j, k, l = 1, 2, \ldots, d,
\]

(1.2)
is Kronecker symbol: \( \delta_{ij} = 0 \) for \( i \neq j \), \( \delta_{ij} = 1 \) for \( i = j \), the Lamé pair \( (\lambda, \mu) \) satisfies the strong convexity condition \( \mu > 0 \) and \( d\lambda + 2\mu > 0 \). In addition
\[
e(u) := \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)
\]
is the strain tensor, and the corresponding conormal derivatives on \( \partial D_i \) are defined by
\[
\frac{\partial u}{\partial v_0} |_{+} := (C^0 e(u)) n = \lambda (\nabla \cdot u) n + \mu (\nabla u + (\nabla u)^T) n,
\]
where \( n \) is the unit outer normal of \( \partial D_i, i = 1, 2 \). Here and throughout this paper, the subscript \( \pm \) indicates the limit from outside and inside the domain, respectively.

\( \{ \psi^l \}_{l=1}^{\text{dim}(d+1)} \) is the basis of \( \Psi \), the linear space of rigid displacement in \( \mathbb{R}^d \),
\[
\Psi := \left\{ \psi \in C^1(\mathbb{R}^d; \mathbb{R}^d) : \nabla \psi + (\nabla \psi)^T = 0 \right\}.
\]

The existence, uniqueness and regularity of weak solutions of (1.1), as well as a variational formulation, can be found in the Appendix in [12]. In particular, the \( H^1 \) weak solution is in \( C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^0(\overline{D_1 \cup D_2}; \mathbb{R}^d) \). The solution is also the unique minimizer of the energy functional of (1.1) in appropriate functional space. Moreover, the solution \( u \) of (1.1) is actually a limit of the following isotropic homogeneous linear Lamé systems with piecewise constant coefficients.

\[
\begin{aligned}
\nabla \cdot \left( (\chi \Omega C^0 + \chi D_1 \cup D_2 C^1) e(u) \right) &= 0 \quad \text{in } D, \\
u &= \varphi \quad \text{on } \partial D,
\end{aligned}
\]

where \( \chi_D \) is characteristic function of \( D \). We assume that \( D_1 \cup D_2 \) and \( \Omega \) are occupied by two different homogeneous and isotropic materials with different Lamé constants \( (\lambda_1, \mu_1) \) and \( (\lambda, \mu) \). Then the elasticity tensor for the inclusions and background are, respectively, \( C^1 \) and \( C^0 \), with
\[
C^1_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0,
\]
and \( C^0_{ijkl} \) is defined in [12]. The boundary data is a given vector-valued function \( \varphi \in L^\infty(\partial D, \mathbb{R}^d) \). When \( \min\{\mu_1, d\lambda_1 + 2\mu_1\} \to \infty \), the solution of (1.3) is convergent to the solution of (1.1) in \( H^1(D; \mathbb{R}^d) \). The limit process can be found in [12].

To analyze the initiation and growth of damage in composite materials, Babuška et al. firstly numerically investigated systems [13], with coefficients having jump discontinuities in [3], where they observed that the stress, represented by \( |\nabla u| \), still remains bounded even if the elastic fiber inclusions are densely packed. In order to give a rigorous proof from analysis, Bonnetier and Vogelius [10] firstly studied the scalar equation
\[
\nabla \cdot \left( (1 + (k - 1) \chi D_1 \cup D_2) \nabla u \right) = 0 \quad \text{in } D,
\]
especially, when \( D_1 \) and \( D_2 \) are two touching disks with comparable radii. By using the Möbius transformation and the maximum principle, they proved that the gradient of solutions of (1.4) remains bounded. These results were extended by Li and Vogelius [11] to a large class of divergence form second order elliptic equations with piecewise Hölder continuous coefficients. They established global Lipschitz and piecewise \( C^{1,\alpha} \) estimates. Li and Nirenberg [40] extended to general divergence form elliptic systems including the system of linear elasticity (1.3). As to the high order derivative estimates, we draw the attention of readers to the open
problem on Page 894 of [41]. There are some progress for scalar equation (1.4) in dimension two, see [19,20].

Notice that the estimates in [41] and [40] all depend on the ellipticity of the coefficients. When the coefficients degenerate to 0 or $\infty$, the situation becomes quite different. For the scalar case, we call it perfect conductivity problem when the conductivity constant in fibers degenerates to $\infty$. Keller [34] firstly compute the effective electrical conductivity for a composite containing a dense array of perfectly conducting spheres of cylinders. Since then, the gradient’s blow-up feature has attracted much attention due to its various applications and the difficulties from analysis and computations. Much effort has been devoted to understanding of this blow-up mechanics. It is known that the blow-up rate of $|\nabla u|$ is $\varepsilon^{-1/2}$ in two dimensions [3,6], $\varepsilon \ln \varepsilon |^{-1}$ in three dimensions [9,11], $\varepsilon^{-1}$ in four and higher dimension.

There is a long list of literature in this direction of research, for example, [7,22,23,35,37,39,42,44,45,47–49]. On the other hand, the characterization of the singular behavior of $\nabla u$ for the perfect case was further developed in [4,7,14,15,27,29,30,33,38]. The stress blow-up in the hole case has been characterized by an explicit function in [43]. For more related work on elliptic equations and systems from composites, see [5,11,17–21,28,37,39,46] and the references therein.

For the linear elasticity case (1.1), we are interested in the concentration of the stress (or the gradient) when the distance $\varepsilon$ goes to zero. Because there is significant difficulty in applying the methods for scalar equations to Lamé systems. For instance, the maximum principle does not hold for system. Until recently, Bao, Li and Li [12,13] developed an iteration technique and employed energy method to obtain the pointwise upper bound on the estimates for gradient of solution to (1.1) in all dimensions, while the lower bound estimates are provided in [36] for dimensions 2 and 3. These estimates shows that the blowup rate of $|\nabla u|$ is the same as the scalar case, that is, $\varepsilon^{-1/2}$ in dimension two, $\varepsilon \ln \varepsilon |^{-1}$ in dimension three, and $\varepsilon^{-1}$ in all higher dimensions. All these estimates are established under the assumption that the inclusions are $C^{2,\gamma}$, $0 < \gamma < 1$. Very recently, under the assumption of $C^{4,\gamma}$ inclusions, by using layer potential techniques and variational principle, Kang and Yu [32] consider the characterization of the singular behavior for the gradient to (1.1). They consequently showed the blowup rate of the gradient in $\mathbb{R}^2$ is $\varepsilon^{-1/2}$ as well. Thus, based on the classical partial differential equation theory, a natural question is whether it is possible to further weaken the smoothness of inclusions to $C^{1,\gamma}$ to obtain desirable estimates of solutions of (1.1). The purpose of this paper is to give a definite answer to this question.

The strategy to solve this problem is as follows. We first point out that problem (1.1) has free boundary value feature. Although $e(u) = 0$ implies $u$ is linear combination of $\psi_i$,

$$u = \sum_{l=1}^{d(d+1)/2} C_i^l \psi_i \quad \text{in } \overline{D_i},$$

$C_i^l$ are $d(d+1)$ free constants. This is the biggest difference with the conductivity model [9], where only two free constants need to handle in any dimension. While, in linear elasticity, how to determine these $d(d+1)$ constants $C_i^l$ is one of our main difficulties. To this end, first by continuity of $u$ across the boundaries of $D_i$, we can
decompose the solutions of (1.1), as in [12], as follows:

\[ u = \sum_{l=1}^{d(d+1)} C_l^1 v_l^1 + \sum_{l=1}^{d(d+1)} C_l^2 v_l^2 + v_0 \quad \text{in } \Omega, \quad \text{(1.5)} \]

where \( v_l^i \in C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^2(\Omega; \mathbb{R}^d), \) \( i = 1, 2, \cdots, \frac{d(d+1)}{2} \) satisfy

\[
\begin{aligned}
& \mathcal{L}_{\lambda, \mu} v_l^i = 0 \quad \text{in } \Omega, \\
& v_l^i = \psi_l^i \quad \text{on } \partial D_i, \\
& v_l^i = 0 \quad \text{on } \partial D_j \cup \partial D, \quad j \neq i;
\end{aligned} \quad \text{(1.6)}
\]

\( v_0 \in C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^2(\Omega; \mathbb{R}^d) \) satisfies

\[
\begin{aligned}
& \mathcal{L}_{\lambda, \mu} v_0 = 0 \quad \text{in } \Omega, \\
& v_0 = 0 \quad \text{on } \partial D_1 \cup \partial D_2, \\
& v_0 = \varphi \quad \text{on } \partial D.
\end{aligned} \quad \text{(1.7)}
\]

By the fourth line in (1.1) and the decomposition (1.5) we have a linear system of these free constants \( C_l^i, \)

\[
\sum_{i=1}^{2} \sum_{l=1}^{\frac{d(d+1)}{2}} C_l^i \int_{\partial D_j} \frac{\partial v_l^i}{\partial v_0} \cdot \psi^{k} + \int_{\partial D_j} \frac{\partial v_l^0}{\partial v_0} \cdot \psi^{k} = 0, \quad \text{(1.8)}
\]

where \( j = 1, 2, k = 1, \cdots, \frac{d(d+1)}{2} \). If we had good enough estimates for \( \nabla v_l^i \), then we can solve (1.8). So the hard work is to establish sufficiently good estimates of \( \nabla v_l^i \). However, when the inclusion are of \( C^{1, \gamma} \), new difficulties need to overcome to apply an adapted version of the iteration technique. Here we have the aid of the campanato approach and \( W^{1,p} \) estimates to this end. After we have \( |\nabla v_l^i| \)'s estimates, combining with \( C_l^i \)'s estimates, we finally show that the blowup rate is \( \varepsilon^{-\frac{1}{1-\gamma}} \), which is bigger than \( \varepsilon^{-\frac{1}{2}} \).

Before we state our main results precisely, we first fix our domain and notations. Let \( D_1^0 \) and \( D_2^0 \) be a pair of (touching at the origin) convex subdomains of \( D \), a bounded open set in \( \mathbb{R}^d \). \( D_1^0 \) and \( D_2^0 \) are far away from boundary \( \partial D \) and satisfy

\[
D_1^0 \subset \{ (x', x_d) \in \mathbb{R}^d : x_d > 0 \}, \quad D_2^0 \subset \{ (x', x_d) \in \mathbb{R}^d : x_d < 0 \}.
\]

We use superscripts prime to denote the \((d-1)\)-dimensional variables and domains, such as \( x' \) and \( B' \). Translate \( D_0^i \) (\( i = 1, 2 \)) by \( \pm \frac{\varepsilon}{2} \) along \( x_d \)-axis as follows

\[
D_1^\varepsilon := D_1^0 + (0', \frac{\varepsilon}{2}) \quad \text{and} \quad D_2^\varepsilon := D_2^0 + (0', -\frac{\varepsilon}{2}).
\]

For simplicity of notation, we drop the superscript \( \varepsilon \) and denote

\[
D_i := D_i^\varepsilon (i = 1, 2), \quad \Omega := D \setminus (D_1 \cup D_2),
\]

and \( P_1 := (0', \frac{\varepsilon}{2}), \) \( P_2 := (0', -\frac{\varepsilon}{2}) \) be the two nearest points between \( \partial D_1 \) and \( \partial D_2 \) such that

\[
\text{dist}(P_1, P_2) = \text{dist}(\partial D_1, \partial D_2) = \varepsilon.
\]

We further assume that \( \partial D_1 \) and \( \partial D_2 \) are of \( C^{1, \gamma} \), \( 0 < \gamma < 1 \) and there exists a constant \( R_1 \), independent of \( \varepsilon \), such that the top and bottom boundaries of the narrow region between \( \partial D_1 \) and \( \partial D_2 \) can be represented, respectively, by graphs

\[
x_d = \frac{\varepsilon}{2} + h_1(x') \quad \text{and} \quad x_d = -\frac{\varepsilon}{2} + h_2(x'), \quad \text{for } |x'| \leq 2R_1, \quad \text{(1.9)}
\]
where \( h_1, h_2 \in C^{1, \gamma}(B^2_{2R_1}(0')) \) and satisfy
\[
-\frac{\varepsilon}{2} + h_2(x') < \frac{\varepsilon}{2} + h_1(x'), \quad \text{for } |x'| \leq 2R_1;
\]
(1.10)
\[
h_1(0') = h_2(0') = 0, \quad \nabla' h_1(0') = \nabla' h_2(0') = 0;
\]
(1.11)
\[
\kappa_0|x'|^\gamma \leq |\nabla' h_1(x')|, \quad |\nabla' h_2(x')| \leq \kappa_1|x'|^\gamma, \quad \text{for } |x'| < 2R_1,
\]
(1.12)
and
\[
\|h_1\|_{C^{1, \gamma}(B^2_{R_1})} + \|h_2\|_{C^{1, \gamma}(B^2_{R_1})} \leq \kappa_2,
\]
(1.13)
where the constants \( 0 < \kappa_0 < \kappa_1 < \kappa_2 \). Set
\[
\Omega_r := \{(x', x_d) \in \Omega : -\frac{\varepsilon}{2} + h_2(x') < x_d < \frac{\varepsilon}{2} + h_1(x'), \quad |x'| < r\}.
\]
Assume that for some \( \delta_0 > 0 \),
\[
\delta_0 \leq \mu, \quad d \lambda + 2\mu \leq \frac{1}{\delta_0}.
\]
(1.14)
The first main result in this paper is as follows.

**Theorem 1.1.** Let \( D_1, D_2 \subset D \subset \mathbb{R}^2 \) be two convex bounded \( C^{1, \gamma} \) subdomains with \( \varepsilon \) apart. Suppose (1.9)–(1.11) hold for \( d = 2 \). Let \( u \in H^1(D; \mathbb{R}^2) \cap C^1(\overline{\Omega}; \mathbb{R}^2) \) be the solution to (1.1) with \( \varphi \in L^\infty(\partial D; \mathbb{R}^2) \). Then for small \( \varepsilon > 0 \),
\[
|\nabla u(x_1, x_2)| \leq \frac{C\varepsilon^{-\frac{1}{1+\gamma}}}{\varepsilon + |x_1|^{1+\gamma}} \cdot \|\varphi\|_{L^\infty(\partial D; \mathbb{R}^2)}, \quad \text{for } (x_1, x_2) \in \Omega_{R_1},
\]
(1.15)
and
\[
\|\nabla u\|_{L^\infty(\Omega_{R_1})} \leq \frac{C}{\varepsilon^{-\frac{1}{1+\gamma}}} \cdot \|\varphi\|_{L^\infty(\partial D; \mathbb{R}^2)},
\]
(1.16)
where \( C \) is a positive constant independent of \( \varepsilon \).

Here and throughout this paper, unless otherwise stated, \( C \) denotes a constant, whose value may vary from line to line, depending only on \( n, \delta_0, \kappa_0, \kappa_1, \kappa_2, R_1 \) and an upper bound of the \( C^{1, \gamma} \) norms of \( \partial D_1 \) and \( \partial D_2 \), but not on \( \varepsilon \). We call a constant having such dependence a **universal constant**.

**Remark 1.2.** From the pointwise upper bound estimate (1.15), we have
\[
\|\nabla u\|_{L^\infty(\Omega_{R_1})} = \|\nabla u(0, x_2)\|_{|x_2| < \varepsilon/2} \leq \frac{C\varepsilon^{-\frac{1}{1+\gamma}}}{\varepsilon + |x_1|^{1+\gamma}} \cdot \|\varphi\|_{L^\infty(\partial D)}.
\]
To show that the blow-up rates \( \varepsilon^{-\frac{1}{1+\gamma}} \) is optimal, we also show there are some cases such that the lower bound of \( |\nabla u(x)| \) on the segment \( P_1 P_2 \) is
\[
|\nabla u(0, x_2)| \geq \frac{1}{C\varepsilon^{-\frac{1}{1+\gamma}}}, \quad (0, x_2) \in P_1 P_2.
\]
For more details, see Subsection 3.3.

**Remark 1.3.** The strict assumption (1.12) can be replaced by weaker assumption as follows,
\[
\kappa_0|x'|^{1+\gamma} \leq h_1(x') - h_2(x') \leq \kappa_1|x'|^{1+\gamma}, \quad |\nabla' h_i(x')| \leq \kappa_2|x'|^\gamma, \quad \forall \ |x'| < 2R_1,
\]
for \( \varepsilon \) independent constants \( \kappa_i > 0, \ i = 0, 1, 2 \). In [20], the authors revealed an relationship between the blow up rate of the gradient solution and the order of the relative convexity of inclusions \( m \geq 2 \) in all dimensions. Thus, the result of this paper for \( 1 < m < 2 \) is a supplement to those in [20]. We would like to point out that when \( m = 1 \), for the Lipschitz inclusions, the corner singularity
will be another interesting and challenge topic. For the scalar case, we refer to Kozlov et al’s book [31] and Kang and Yun for bow-tie structure [33].

Following the arguments in the proof of Theorem 1.1, we can also have the pointwise upper bound estimates for higher dimensions \( d \geq 3 \).

**Theorem 1.4.** Let \( D_1, D_2 \subset D \subset \mathbb{R}^d \) be two convex bounded \( C^{1,\gamma} \) subdomains with \( \varepsilon \) apart. Suppose (1.9)–(1.14) hold for \( d \geq 3 \). Let \( u \in H^1(D; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d) \) be the solution to (1.1) with \( \varphi \in L^\infty(\partial D; \mathbb{R}^d) \). Then for small \( \varepsilon > 0 \),

\[
|\nabla u(x', x_d)| \leq \frac{C}{\varepsilon + |x'|^{1+\gamma}} \cdot \|\varphi\|_{L^\infty(\partial D; \mathbb{R}^d)}, \quad \text{for} \ (x', x_d) \in \Omega_{R_1}, \tag{1.17}
\]

and

\[
\|\nabla u\|_{L^\infty(\Omega \setminus \Omega_{R_1})} \leq C\|\varphi\|_{L^\infty(\partial D; \mathbb{R}^2)},
\]

where \( C \) is a positive constant independent of \( \varepsilon \).

The organization of the rest paper is as follows. In Section 2, we first introduce an auxiliary scalar function \( \bar{u} \) to generate a family of vector valued functions, whose gradients will be the major singular terms. Since \( \partial D_1 \) and \( \partial D_2 \) are of \( C^{1,\gamma} \), in order to prove Proposition 2.1, we need to establish the \( C^{1,\gamma} \) estimates and \( W^{1,p} \) estimates for elliptic systems with right hand side in divergence form, with partially zero Dirichlet boundary data, see Theorem 2.2 and 2.3. The proofs are put later in Section 5. Using them to replace the \( W^{2,p} \) estimates used in [12], we adapt the iteration process and obtain \( \nabla v_1 \)'s estimates, see Proposition 2.1. The estimates in Proposition 2.1 for dimension two are proved in Section 3, for higher dimensions in Section 4. Subsection 3.3 is dedicated to the lower bound estimates to show the blowup rate \( \varepsilon^{-1/(1+\gamma)} \) is optimal in dimension two.

2. Main ingredients for the proof of Theorem 1.1 and 1.4

In this section, we shall list the main ingredients to prove Theorem 1.1 and 1.4. Recall that the linear space of rigid displacement, \( \Psi \) in \( \mathbb{R}^d \) is spanned by

\[
\{ e_i, x_j e_k - x_k e_j : 1 \leq i \leq d, 1 \leq j < k \leq d \},
\]

where \( e_1, e_2, \ldots, e_d \) denote the standard basis of \( \mathbb{R}^d \). By the decomposition (1.5), we write

\[
\nabla u = \sum_{l=1}^{d(d+1)} (C_1^l \nabla v_1^l + C_2^l \nabla v_2^l) + \nabla v_0
\]

\[
= \sum_{l=1}^{d} (C_1^l - C_2^l) \nabla v_1^l + \sum_{l=1}^{d} C_2^l \nabla (v_1^l + v_2^l) + \sum_{l=d+1}^{d(d+1)} \sum_{i=1}^{2} C_i^l \nabla v_i^l + \nabla v_0, \text{ in } \Omega.
\]

Thus, in order to prove Theorem 1.1 it suffices to estimate each term in (2.1), one by one. Without loss of generality, we assume that \( \|\varphi\|_{L^\infty(\partial D)} = 1 \) by considering \( u/\|\varphi\|_{L^\infty(\partial D)} \) if \( \|\varphi\|_{L^\infty(\partial D)} > 0 \). If \( \varphi|_{\partial D} = 0 \), then \( u \equiv 0 \).
2.1. **Auxiliary functions.** To estimate $|\nabla v_i^l|$, $i = 1, 2$, $l = 1, 2, \ldots, \frac{d(d+1)}{2}$, we introduce a scalar function $\bar{u} \in C^{1,\gamma}(\mathbb{R}^d)$ such that $\bar{u} = 1$ on $\partial D_1$, $\bar{u} = 0$ on $\partial D_2 \cup \partial D$,

$$\bar{u}(x) = \frac{x_d - h_2(x') + \varepsilon/2}{\varepsilon + h_1(x') - h_2(x')}, \quad x \in \Omega_{R_1},$$

and

$$\|\bar{u}\|_{C^{1,\gamma}(\mathbb{R}^d)} \leq C.$$  

Denoting $\partial_j := \partial/\partial x_j$ and using (1.11), (1.12), a direct calculation yields that

$$|\partial_j \bar{u}(x)| \leq \frac{C|x'|^{\gamma}}{\varepsilon + |x'|^{1+\gamma}}, \quad j = 1, \ldots, d-1, \quad \partial_d \bar{u}(x) = \frac{1}{\delta(x')}, \quad x \in \Omega_{R_1},$$

where

$$\delta(x') := \varepsilon + h_1(x') - h_2(x').$$

Define a family of vector-valued auxiliary functions

$$\bar{u}_1^l = \bar{u}\psi^l, \quad \text{in } \Omega, \quad l = 1, 2, \ldots, \frac{d(d+1)}{2},$$

then $v_i^l = \bar{u}_1^l$ on $\partial \Omega$. Similarly, we define

$$\bar{u}_2^l = \bar{u}\psi^l, \quad \text{in } \Omega, \quad l = 1, 2, \ldots, \frac{d(d+1)}{2},$$

where $\bar{u}$ is a scalar function in $C^{1,\gamma}(\mathbb{R}^d)$ satisfying $\bar{u} = 1$ on $\partial D_2$, $\bar{u} = 0$ on $\partial D_1 \cup \partial D$, $\bar{u} = 1 - \bar{u}, \quad x \in \Omega_{R_1},$

and $\|\bar{u}\|_{C^{1,\gamma}(\mathbb{R}^d)} \leq C$. We shall prove that $\bar{u}_1^l$ are the main singular terms of $v_i^l$ near the origin for $i = 1, 2, l = 1, 2, \ldots, \frac{d(d+1)}{2}$.

2.2. **Estimates of $|\nabla v_i^l|$.** Set

$$w_i^l := v_i^l - \bar{u}_1^l, \quad l = 1, 2, \ldots, \frac{d(d+1)}{2}, \quad i = 1, 2.$$ 

Then

**Proposition 2.1.** Under the hypotheses of Theorem 1.1 and 1.4, let $v_i^l, v_0 \in C^2(\Omega; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$ be the solution of (1.6) and (1.7). Then for small $\varepsilon > 0$, we have

(i) for $l = 1, 2, \ldots, d$, $i = 1, 2$,

$$|\nabla w_i^l(x)| \leq \frac{C}{\varepsilon + |x_1|^{1+\gamma}}, \quad x \in \Omega_{R_1},$$

consequently,

$$\frac{1}{C(\varepsilon + |x_1|^{1+\gamma})} \leq |\nabla w_i^l(x)| \leq \frac{C}{\varepsilon + |x_1|^{1+\gamma}}, \quad x \in \Omega_{R_1}.$$ 

(ii) for $l = d + 1, \ldots, \frac{d(d+1)}{2}$, $i = 1, 2$,

$$|\nabla w_i^l(x)| \leq C, \quad x \in \Omega_{R_1},$$

consequently,

$$|\nabla v_i^l(x)| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^{1+\gamma}}, \quad x \in \Omega_{R_1}. $$

(2.10)
for \( l = 1, 2, \ldots, d, \)
\[
\|\nabla (v^l_1 + v^l_2)\|_{L^\infty(\Omega)} \leq C,  \tag{2.13}
\]
and
\[
\|\nabla v_0\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{L^\infty(\partial D)}, \tag{2.14}
\]
(iv) for \( l = 1, 2, \ldots, \frac{d(d+1)}{2}, i = 1, 2, \)
\[
\|\nabla u^l_i\|_{L^\infty(\Omega \setminus \Omega_{R_1})} \leq C. \tag{2.15}
\]

where \( C \) is a universal constant.

2.3. \( C^{1,\gamma} \) estimates and \( W^{1,p} \) estimates. Since \( h_1 \) and \( h_2 \) here are only of \( C^{1,\gamma} \), now \( \bar{u} \) and \( u \) are not twice continuously differentiable. Thus, we only have the right hand side in divergence form
\[
-\mathcal{L}_{\lambda,\mu} w^l_i = \nabla \cdot (C^0 c(\bar{w}^l_i)).
\]

We are not able to directly follow the iteration approach used in [12] and apply \( W^{2,p} \) estimates to get the estimates of \( w^l_i \). To overcome this difficulty, we here turn to the \( C^{1,\gamma} \) estimates and \( W^{1,p} \) estimates for elliptic system in [24] and adapt it to our setting with partially zero boundary condition, which can be regarded as the analogue of theorem 9.13 in [25] and are of independent interest.

We recall some properties of tensor \( C \). For the isotropic elastic material,
\[
\mathcal{C} := (C_{ijkl}) = (\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl})), \quad \mu > 0, \quad d\lambda + 2\mu > 0. \tag{2.16}
\]
The components \( C_{ijkl} \) satisfy symmetric conditions:
\[
C_{ijkl} = C_{klij}, \quad i, j, k, l = 1, 2, \ldots, d.
\]
Thus \( \mathcal{C} \) satisfies the ellipticity condition: For every \( d \times d \) real symmetric matrix \( A = (A_{ij}) \),
\[
\min_{i,j} \{2\mu, d\lambda + 2\mu\}|A|^2 \leq (\mathcal{C} A, A) \leq \max_{i,j} \{2\mu, d\lambda + 2\mu\}|A|^2, \tag{2.17}
\]
where \( |A|^2 = \sum_{i,j} A_{ij}^2 \). Then we have

**Theorem 2.2** \( (C^{1,\gamma} \) estimates). Let \( Q \) be a bounded domain in \( \mathbb{R}^d \), \( d \geq 2 \), with a \( C^{1,\gamma} \) boundary portion \( \Gamma \subset \partial Q \). Let \( \bar{w} \in H^1(Q; \mathbb{R}^d) \cap C^1(Q \cup \Gamma; \mathbb{R}^d) \) be the solution of
\[
\left\{ \begin{array}{ll}
-\partial_k (C_{ijkl} \partial_l \bar{w}(\cdot)) = \partial_k \bar{f}^k_l & \text{in } Q, \\
\bar{w} = 0 & \text{on } \Gamma,
\end{array} \right. \tag{2.18}
\]
where \( \bar{f}^k_l \in C^\gamma(Q), \) \( 0 < \gamma < 1 \) and \( C_{ijkl} \) is defined in (2.16). Then for any domain \( Q' \subset \subset Q \cup \Gamma \),
\[
\|\bar{w}\|_{C^{1,\gamma}(Q')} \leq C \left(\|\bar{w}\|_{L^\infty(Q)} + \|\bar{F}\|_{\gamma, Q}\right), \tag{2.19}
\]
where \( \bar{F} := (\bar{f}^k_l) \) and \( C = C(n, \gamma, Q', \mathbb{R}) \).

The Hölder semi-norm of matrix-valued function \( \bar{F} = (\bar{f}^k_l) \) is defined as follows:
\[
[\bar{F}]_{\gamma, Q} := \max_{1 \leq k, l \leq d} [\bar{f}^k_l]_{\gamma, Q} \quad \text{and} \quad [\bar{f}^k_l]_{\gamma, Q} = \sup_{x, y \in Q} \frac{|\bar{f}^k_l(x) - \bar{f}^k_l(y)|}{|x - y|^\gamma}. \tag{2.20}
\]

For elliptic equations, the famous De Giorgi-Nash approach or Moser’s iteration are usually used to get the estimates in \( L^\infty \). But these approaches are unable to be applied for the lamé system. Here, we need the following \( W^{1,p} \) estimates for (2.18).
Theorem 2.3 (W^{1,p} estimates). Let Q and $\Gamma$ be defined as in Theorem 2.2 for $d \geq 2$. Let $\bar{w} \in H^1(Q; \mathbb{R}^d)$ be the weak solution of \ref{eq:2.14} with $\bar{f}_k^i \in C^r(Q)$, $0 < \gamma < 1$ and $k, i = 1, 2, \cdots, d$. Then, for any $2 \leq p < \infty$ and any domain $Q' \subset \subset Q \cup \Gamma$,
\[
\|\bar{w}\|_{W^{1,p}(Q')} \leq C(\|\bar{w}\|_{H^1(Q)} + \|\bar{F}\|_{L^p(Q)}),
\]
where $C = C(\lambda, \mu, p, Q')$ and $\bar{F} := (\bar{f}_k^i)$. In particular, if $p > d$, it holds that
\[
\|\bar{w}\|_{C^r(Q')} \leq C(\|\bar{w}\|_{H^1(Q)} + \|\bar{F}\|_{\gamma, Q}),
\]
where $0 < \gamma \leq 1 - d/p$ and $C = C(\lambda, \mu, \gamma, p, Q')$.

For readers’ convenience, the proofs of Theorem 2.2 and 2.3 are given later in Section 6.

2.4. Estimates of $C^1_l$. For the estimates of $C^1_l$, we have

Proposition 2.4. Let $C^1_l$ be defined in \ref{eq:1.15}. Then
\[
|C^1_l| \leq C, \quad \text{for } i = 1, 2, \quad \text{for } l = 1, 2, \cdots, \frac{d(d + 1)}{2}.
\]
where $C$ is independent of $\varepsilon$. In particular, for $d = 2$, one has
\[
|C^1_l - C^1_l| \leq C\varepsilon^{1/\gamma}, \quad \text{for } l = 1, 2.
\]

2.5. Proof of Theorem 1.1 and 1.4. We are in position to prove Theorem 1.1 and 1.4 by using Proposition 2.4 and 2.4.

Proof of Theorem 1.1. Noticing that when $d = 2$, by using \ref{eq:2.14}, \ref{eq:2.12} and \ref{eq:2.14}-\ref{eq:2.22}, one has for $x \in \Omega_{R_1}$,
\[
|\nabla \bar{u}(x)| \leq \sum_{i=1}^{2} |C^1_i - C^1_i| |\nabla \bar{v}_i(x)| + C \sum_{i=1}^{2} |\nabla \bar{v}_i^1(x)| + C \leq \frac{C\varepsilon^{1/\gamma}}{\varepsilon + |x;|^{1+\gamma}}.
\]
Thus, \ref{eq:1.15} is proved. Moreover, \ref{eq:1.16} follows from \ref{eq:2.15}, \ref{eq:2.15} and \ref{eq:2.23}.

Proof of Theorem 1.4. By using \ref{eq:2.10}, \ref{eq:2.12}, \ref{eq:2.14} and \ref{eq:2.23} for $d \geq 3$, one has for $x \in \Omega_{R_1}$
\[
|\nabla \bar{u}| \leq \sum_{i=1}^{2} \sum_{l=1}^{\frac{d(d + 1)}{2}} |C^1_l| |\nabla \bar{v}_l^1| + |\nabla \bar{v}_0| \leq \frac{C}{\varepsilon + |x;|^{1+\gamma}}.
\]
The proof of \ref{eq:1.17} is completed.

3. Estimates for Theorem 1.1

This section is devoted to the proof of Proposition 2.4 for dimensions two. In subsection 3.3, the lower bound estimates imply the optimality of the blowup rate obtained in Theorem 1.1.

Since $\partial D_1$ and $\partial D_2$ are of $C^1, \gamma$, we are not able to directly follow the iteration technique developed in \ref{12}. In this end, we first calculate the Hölder semi-norm of $\nabla \bar{u}$,
\[
|\nabla \bar{u}|_{\gamma, \Omega_{(z_1)}} \leq C\delta(z_1)^{-\gamma - \gamma - 1} + C\delta(z_1)^{-1 + \gamma + \gamma - 1}.
\]
where $\delta(z_1)$ is defined in (2.3) and $s \leq C\delta(z_1)$,

$$\hat{\Omega}_s(z_1) := \left\{ x \in \mathbb{R}^2 : -\frac{\varepsilon}{2} + h_2(x_1) < x_2 < \frac{\varepsilon}{2} + h_1(x_1), |x_1 - z_1| < s \right\},$$

for $0 < s < \frac{1}{2\kappa_1}\delta(z_1)^{1/(1+\gamma)} \leq R_1$, $\kappa_1$ is defined in (1.12).

Indeed, for any $(x_1, x_2) \in \hat{\Omega}_s(z_1)$, $s \leq \delta(z_1)$,

$$|x_1| \leq |x_1 - z_1| + |z_1| < s + |z_1| \leq C\delta(z_1)^{1/(1+\gamma)}. \quad (3.2)$$

This, together with mean value theorem and (1.12), implies that for any $x, \bar{x} \in \hat{\Omega}_s(z')$ with $x_1 \neq \bar{x}_1$,

$$|h_i(x_1) - h_i(\bar{x}_1)| = |\nabla h_i(x_\theta_i)||x_1 - \bar{x}_1| \leq C\delta(z_1)^{\frac{\gamma}{1+\gamma}}|x_1 - \bar{x}_1|, \quad i = 1, 2, \quad (3.3)$$

and

$$\varepsilon + (h_1 - h_2)(\bar{x}_1) \geq \delta(z_1) - C\delta(z_1)^{\frac{\gamma}{1+\gamma}}s \geq \frac{1}{2}\delta(z_1), \quad \varepsilon + (h_1 - h_2)(x_1) \geq \frac{1}{2}\delta(z_1). \quad (3.4)$$

Thus, for

$$\partial_2 \bar{u}(x) = \frac{1}{\varepsilon + h_1(x_1) - h_2(x_1)},$$

we have

$$\left| \frac{\partial_2 \bar{u}(x) - \partial_2 \bar{u}(\bar{x})}{|x - \bar{x}|^\gamma} \right| \leq \frac{C\delta(z_1)^{\frac{\gamma}{1+\gamma}}s^{1-\gamma}}{\delta(z_1)^2} \leq C\delta(z_1)^{-\frac{2+\gamma}{1+\gamma}}s^{1-\gamma}. \quad (3.5)$$

While,

$$\partial_1 \bar{u}(x) = \frac{-\partial h_2(x_1)}{\delta(x_1)} + \frac{(x_2 - h_2(x_1) + \varepsilon/2)(\partial_1 h_2(x_1) - \partial_1 h_1(x_1))}{\delta(x_1)}$$

$$:= \Phi_1(x) + \Phi_2(x).$$

By virtue of (1.12) and (3.2)–(3.4), a direct calculation yields

$$\left| \frac{\Phi_1(x) - \Phi_1(\bar{x})}{|x - \bar{x}|^\gamma} \right| \leq \frac{C}{\delta(z_1)} + \frac{\delta(z_1)^{\frac{\gamma}{1+\gamma}}s^{1-\gamma}}{\delta(z_1)^2} \leq C\delta(z_1)^{-1} + C\delta(z_1)^{-\frac{2+\gamma}{1+\gamma}}s^{1-\gamma},$$

and

$$\left| \frac{\Phi_2(x) - \Phi_2(\bar{x})}{|x - \bar{x}|^\gamma} \right| \leq C\delta(z_1)^{-1 - \frac{1}{1+\gamma}}s^{1-\gamma} + C\delta(z_1)^{-\gamma - \frac{1}{1+\gamma}}.$$

Noting that $\gamma + \frac{1}{1+\gamma} > 1$, we have

$$\left| \frac{\partial_1 \bar{u}(x) - \partial_1 \bar{u}(\bar{x})}{|x - \bar{x}|^\gamma} \right| \leq C\delta(z_1)^{-\frac{2+\gamma}{1+\gamma}}s^{1-\gamma} + C\delta(z_1)^{-\gamma - \frac{1}{1+\gamma}}. \quad (3.6)$$

Thus, (3.1) immediately follows from (3.5) and (3.6).

### 3.1. Proof of Proposition 2.1 in two dimensions.
3.1.1. Estimates $|\nabla v^l|$, $i, l = 1, 2$.

Proof of Proposition 2.1 when $d = 2$. We only prove (2.9) for $i = l = 1$, since the same proof applies to the other cases. For simplicity, we denote $w := w^1$. Recall $w$ satisfies
\begin{equation}
\begin{aligned}
- \mathcal{L}_{\lambda, \mu} w &= \nabla \cdot ( \mathcal{C}^0 e(\bar{u}^1) ) \\
 0 &= w \quad \text{on } \partial \Omega.
\end{aligned}
\end{equation}

Clearly, $w$ still satisfies
\begin{equation}
- \mathcal{L}_{\lambda, \mu} w = \nabla \cdot ( \mathcal{C}^0 e(\bar{u}^1) - \mathcal{M} ) \quad \text{in } \Omega,
\end{equation}
for any constant matrix $\mathcal{M} = (a_{ij})$. We will take full advantage of this main difference with that in [12, 13]. The proof is divided into three steps.

**STEP 1.** The boundedness of the total energy:
\begin{equation}
\int_{\Omega} |\nabla w|^2 \, dx \leq C.
\end{equation}

In fact, we multiply (3.7) by $w$, make use of integration by parts, to obtain
\begin{equation}
\int_{\Omega} (\mathcal{C}^0 e(w), e(w)) \, dx = \int_{\Omega} (\nabla \cdot (\mathcal{C}^0 e(\bar{u}^1))) \cdot w \, dx.
\end{equation}

For the left hand side, it follows from (2.17) and the First Korn inequality that
\begin{equation}
\int_{\Omega} (\mathcal{C}^0 e(w), e(w)) \, dx \geq C \int_{\Omega} |e(w)|^2 \, dx \geq C \int_{\Omega} |\nabla w|^2 \, dx.
\end{equation}

For the right hand of (3.10),
\begin{align*}
\int_{\Omega} (\nabla \cdot (\mathcal{C}^0 e(\bar{u}^1))) \cdot w \, dx &= \int \mu \, \text{div}(\nabla \bar{u}) w^{(1)} + (\lambda + \mu) \left( \partial_1 (\partial_1 \bar{u}) w^{(1)} + \partial_2 (\partial_1 \bar{u}) w^{(2)} \right) \, dx.
\end{align*}

It follows from integration by parts and Hölder inequality that
\begin{equation}
\int_{\Omega} \left( \partial_1 (\partial_1 \bar{u}) w^{(1)} + \partial_2 (\partial_1 \bar{u}) w^{(2)} \right) \, dx \leq C \left( \int_{\Omega} |\partial_1 \bar{u}|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}.
\end{equation}

By the mean value theorem, there exists $r_0 \in (\frac{R_1}{2}, \frac{2R_1}{3})$ such that
\begin{align*}
\int_{\Omega_{2R_1/3} \setminus \Omega_{R_1/2}} |\nabla w| \, dx &\leq C \int_{\Omega_{2R_1/3} \setminus \Omega_{R_1/2}} |\nabla w| \, dx \\
&\leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^{\frac{1}{2}}.
\end{align*}

Then, noticing that $\partial_2 \bar{u} = 0$ in $\Omega_{R_1}$ and [23], one has
\begin{equation}
\left| \int_{\Omega} \text{div}(\nabla \bar{u}) w^{(1)} \, dx \right| \leq \int_{\Omega_{r_0}} |\partial_1 (\partial_1 \bar{u}) w^{(1)}| \, dx + \int_{\Omega_{1-r_0}} |\partial_1 (\partial_1 \bar{u}) w^{(1)}| \, dx
\end{equation}
First,
\[ \int_{\Omega_{\varepsilon_0}} \partial_1 (\partial_1 \bar{u}) w^{(1)} \, dx = - \int_{\Omega_{\varepsilon_0}} \partial_1 \bar{u} \partial_1 w^{(1)} \, dx + \int_{|x_1| = \varepsilon_0} w^{(1)} \partial_1 \bar{u} \, dx \]
\[ := I_1 + I_2. \]

By using (2.4) and (3.14), we have
\[ |I_1| \leq C \left( \int_{\Omega_{\varepsilon_0}} |\partial_1 \bar{u}|^2 \, dx \right)^\frac{1}{2} \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^\frac{1}{2} \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^\frac{1}{2}, \]
and
\[ |I_2| \leq C \int_{\Omega \cap \{ |x_1| = \varepsilon_0, -\frac{\varepsilon_0}{2} + h_2(x_1) < x_2 < \frac{\varepsilon_0}{2} + h_1(x_1) \}} |w| \, dx \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^\frac{1}{2}. \]

For the second term in the right hand side of (3.17), using Young’s inequality, we have for any \( \varepsilon > 0 \)
\[ \int_{\Omega_{\varepsilon_0}} \text{div}(\nabla \bar{u}) w^{(1)} \, dx \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^\frac{1}{2}. \]

These, combining with (3.10)–(3.13), yield
\[ \int_{\Omega} |\nabla w|^2 \, dx \leq C \left( \int_{\Omega} |\nabla w|^2 \, dx \right)^\frac{1}{2}. \]

So that (3.9) is proved.

**STEP 2. The local energy estimates:**
\[ \int_{\Omega_{\delta(z_1)}} |\nabla w|^2 \, dx \leq C \delta(z_1)^{-\frac{2\varepsilon}{h_1(z_1) - h_2(z_1)}}, \quad (3.16) \]

where \( \delta(z_1) = \varepsilon + h_1(z_1) - h_2(z_1) \).

Indeed, for \( 0 < t < s < R_1 \), let \( \eta \) be a cutoff function satisfying
\[ \eta(x_1) = \begin{cases} 1 & \text{if } |x_1 - z_1| < t, \\ 0 & \text{if } |x_1 - z_1| > s, \end{cases} \quad \text{and} \quad |\eta'(x_1)| \leq \frac{2}{s-t}. \]

Multiplying (3.8) by \( \eta^2 w \) and using integration by parts, one has
\[ \int_{\Omega_{\delta(z_1)}} \left( \mathcal{C}_0 e(w), e(\eta^2 w) \right) \, dx = - \int_{\Omega_{\delta(z_1)}} \left( \mathcal{C}_0 e(\bar{u}_1) - \mathcal{M}, \nabla(\eta^2 w) \right) \, dx. \quad (3.17) \]

For the left hand side of (3.17), using the first Korn inequality and standard arguments, one has
\[ \int_{\Omega_{\delta(z_1)}} \left( \mathcal{C}_0 e(w), e(\eta^2 w) \right) \, dx \geq \frac{1}{C} \int_{\Omega_{\delta(z_1)}} |\nabla(\eta w)|^2 \, dx - C \int_{\Omega_{\delta(z_1)}} |w|^2 |\eta'|^2 \, dx. \]

For the right hand side of (3.17), using Young’s inequality, we have for any \( \zeta > 0 \)
\[ \left| \int_{\Omega_{\delta(z_1)}} \left( \mathcal{C}_0 e(\bar{u}_1) - \mathcal{M}, \nabla(\eta^2 w) \right) \, dx \right| \leq \zeta \int_{\Omega_{\delta(z_1)}} \eta^2 |\nabla w|^2 \, dx \]
\[ + \frac{C}{\zeta} \int_{\Omega_{\delta(z_1)}} |\eta'|^2 |w|^2 \, dx + \frac{C}{\zeta} \int_{\Omega_{\delta(z_1)}} |\mathcal{C}_0 e(\bar{u}_1) - \mathcal{M}|^2 \, dx. \]
It follows from (3.18), (3.19) and (3.20) that
\[
\int_{\Omega_s(z_1)} |\nabla w|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\Omega_s(z_1)} |w|^2 \, dx + C \int_{\Omega_s(z_1)} |C^0e(\bar{u}_1^1) - M_1|^2 \, dx. \tag{3.18}
\]

Notice
\[
C^0e(\bar{u}_1^1) = \begin{pmatrix}
(2\mu + \lambda)\partial_1 \bar{u} & \mu \partial_2 \bar{u} \\
\mu \partial_2 \bar{u} & \lambda \partial_1 \bar{u}
\end{pmatrix},
\]
we define the constant matrix \(M_1 = (a_{ij}), \ i, j = 1, 2\) by
\[
M_1 = \int_{\Omega_s(z_1)} C^0e(\bar{u}_1^1(y)) \, dy := \frac{1}{|\Omega_s(z_1)|} \int_{\Omega_s(z_1)} C^0e(\bar{u}_1^1(y)) \, dy.
\]

**Case 1.** For \(|z_1| \leq \frac{e}{s-t}, 0 < s < \frac{1}{e^{1/\gamma}}\), then \(\varepsilon \leq \delta(z_1) \leq C\varepsilon\). By a direct calculation, we have
\[
\int_{\Omega_s(z_1)} |w|^2 \, dx = \int_{|x_1-x|<s} \int_{-\varepsilon^{2+2h_1(z_1)}}^{\varepsilon^{2+2h_1(z_1)}} \left( \int_{-\frac{s}{2}+h_2(x_1)}^{\frac{s}{2}+h_2(x_1)} \partial_2 w \, dx_2 \right)^2 \, dx_2 \, dx_1 
\leq C\varepsilon^2 \int_{\Omega_s(z_1)} |\nabla w|^2 \, dx, \tag{3.19}
\]
and by the definition of semi-norm \([\cdot]_{\gamma, \Omega_s(z_1)}\) in (2.20),
\[
|C^0e(\bar{u}_1^1) - M_1|^2 \leq |(2\mu + \lambda)\partial_1 \bar{u} - a_{11}|^2 + 2|\mu \partial_2 \bar{u} - a_{12}|^2 + |\lambda \partial_1 \bar{u} - a_{22}|^2
\leq \frac{C}{|\Omega_s(z_1)|} \int_{\Omega_s(z_1)} (|\partial_1 \bar{u}(x) - \partial_1 \bar{u}(y)|^2 + |\partial_2 \bar{u}(x) - \partial_2 \bar{u}(y)|^2) \, dy
\leq \frac{C|\nabla \bar{u}|^2_{\gamma, \Omega_s(z_1)}}{|\Omega_s(z_1)|} \int_{\Omega_s(z_1)} |x - y|^{2\gamma} \, dy
\leq C|\nabla \bar{u}|^2_{\gamma, \Omega_s(z_1)} (s^{2\gamma} + \delta(z_1)^{2\gamma}).
\]

Using (3.1) and by direct calculation,
\[
\int_{\Omega_s(z_1)} |C^0e(\bar{u}_1^1) - M_1|^2 \, dx
\leq \int_{\Omega_s(z_1)} \left( |(2\mu + \lambda)\partial_1 \bar{u} - a_{11}|^2 + 2|\mu \partial_2 \bar{u} - a_{12}|^2 + |\lambda \partial_1 \bar{u} - a_{22}|^2 \right) \, dx
\leq C|\nabla \bar{u}|^2_{\gamma, \Omega_s(z_1)} \int_{\Omega_s(z_1)} (s^{2\gamma} + \delta(z_1)^{2\gamma}) \, dx
\leq C \left( \frac{s^{\gamma}}{s-\varepsilon^{1/\gamma}} + \frac{s^{3-2\gamma}}{\varepsilon^{1-2s-2\gamma}} \right) := G(s). \tag{3.20}
\]

It follows from (3.18), (3.19) and (3.20) that
\[
F(t) \leq \left( \frac{c_1\varepsilon}{s-t} \right)^2 F(s) + CG(s), \quad \forall \ 0 < t < \varepsilon^{1/\gamma}, \tag{3.21}
\]
here \(c_1\) is a fixed constant, and
\[
F(t) := \int_{\Omega_s(z_1)} |\nabla w|^2 \, dx. \tag{3.22}
\]
Let $k = \left[ \frac{1}{4c_1 \varepsilon^{1+\gamma}} \right]$ and $t_i = \delta + 2c_1 i \varepsilon$, $i = 0, 1, 2, \ldots, k$. It is easy to see from (3.21) that
\[ G(t_{i+1}) \leq C(i + 1)^3 \varepsilon^{2+\gamma}. \]
Taking $s = t_{i+1}$ and $t = t_i$ in (3.21), we have the following iteration formula
\[ F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^3 \varepsilon^{2+\gamma}. \]
After $k$ iterations, and by virtue of (3.19), we have
\[ F(t_0) \leq \left( \frac{1}{4} \right)^k F(t_k) + C \varepsilon^{2+\gamma} \sum_{i=0}^{k-1} \left( \frac{1}{4} \right)^i (i + 1)^3 \leq C \varepsilon^{2+\gamma}. \]
This is (3.10) with $\delta(z_1) \leq C \varepsilon$.

**Case 2.** For $\varepsilon^{1+\gamma} \leq |z_1| \leq R_1$, $0 < s < |z_1|$, then $|z_1|^{1+\gamma} \leq \delta(z_1) \leq C |z_1|^{1+\gamma}$. The estimates (3.19) and (3.20) become, respectively,
\[ \int_{\Omega_s(z_1)} |w|^2 \, dx \leq C |z_1|^{2(1+\gamma)} \int_{\Omega_s(z_1)} |\nabla w|^2 \, dx, \quad \text{if } 0 < s < \frac{2}{3} |z_1|, \tag{3.23} \]
and
\[ \int_{\Omega_s(z_1)} |\mathcal{C}^0 \mathcal{e}(u_1') - \mathcal{M}|^2 \, dx \leq C \left( \frac{s}{|z_1|^{1-\gamma}} + \frac{s^{3-2\gamma}}{|z_1|^{3-\gamma-2\gamma}} \right) := H(s). \tag{3.24} \]
In view of (3.18), and (3.23), estimate (3.21) becomes,
\[ F(t) \leq \left( \frac{c_2 |z_1|^{1+\gamma}}{s - t} \right)^2 F(s) + CH(s), \quad \forall \ 0 < t < s < \frac{2}{3} |z_1|, \tag{3.25} \]
where $c_2$ is another fixed constant. Let $k = \left[ \frac{1}{4c_2 |z_1|^{1+\gamma}} \right]$ and $t_i = \delta + 2c_2 i |z_1|^{1+\gamma}$, $i = 0, 1, 2, \ldots, k$. From (3.21), one has
\[ H(t_{i+1}) \leq C(i + 1)^3 |z_1|^{2\gamma}. \]
Then, taking $s = t_{i+1}$ and $t = t_i$ in (3.26), the iteration formula is
\[ F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^3 |z_1|^{2\gamma}. \]
After $k$ iterations, and using (3.26) again,
\[ F(t_0) \leq \left( \frac{1}{4} \right)^k F(t_k) + C |z_1|^{2\gamma} \sum_{i=0}^{k-1} \left( \frac{1}{4} \right)^i (i + 1)^3 \leq C |z_1|^{2\gamma}. \]
Thus, (3.10) is proved.

**STEP 3. Rescaling and $L^\infty$ estimates of $|\nabla w|$**.

Making the following change of variables on $\Omega_s(z_1)$ as in (12)
\[ \begin{cases} x_1 = z_1 = \delta y_1, \\ x_2 = \delta y_2, \end{cases} \]
then $\Omega_s(z_1)$ becomes $Q_1$ of nearly unit size, where
\[ Q_r = \left\{ y \in \mathbb{R}^2 : -\frac{\varepsilon}{2\delta} + \frac{1}{\delta} h_2(\delta y_1 + z_1) < y_2 < \frac{\varepsilon}{2\delta} + \frac{1}{\delta} h_1(\delta y_1 + z_1), |y_1| < r \right\}, \]
for \( r \leq 1 \), and the top and bottom boundaries become

\[
\Gamma^+_r = \left\{ y \in \mathbb{R}^2 : y_2 = \frac{\varepsilon}{2\delta} + \frac{1}{\delta} h_1(\delta y_1 + z_1), \ |y_1| < r \right\},
\]

and

\[
\Gamma^-_r = \left\{ y \in \mathbb{R}^2 : y_2 = -\frac{\varepsilon}{2\delta} + \frac{1}{\delta} h_2(\delta y_1 + z_1), \ |y_1| < r \right\}.
\]

We denote

\[
\bar{w}(y_1, y_2) := w(\delta y_1 + z_1, \delta y_2), \quad \bar{u}(y_1, y_2) := \bar{u}_1(\delta y_1 + z_1, \delta y_2), \quad (y_1, y_2) \in Q_1.
\]

From (3.7), we see that \( \bar{w} \) satisfies

\[
\begin{cases}
-\partial_k (C_{ijkl}\partial_l \bar{w}^{(j)}) = \partial_k (C_{ijkl}\partial_l \bar{u}^{(j)}) & \text{in } Q_1, \\
\bar{w} = 0 & \text{on } \Gamma^+_1.
\end{cases}
\]

(3.26)

Applying Theorem 2.3 for (3.26) with \( \hat{T}_k = C_{ijkl}\partial_l \bar{u}^{(j)} \), the \( L^\infty \) estimates, and noticing that

\[
[C_{ijkl}\partial_l \bar{u}^{(j)}]_{\gamma, Q_1} \leq C[\nabla \bar{u}]_{\gamma, Q_1},
\]

we obtain

\[
\|\bar{w}\|_{L^\infty(Q_{1/2})} \leq C \left( \|\bar{w}\|_{H^1(Q_1)} + [\nabla \bar{u}]_{\gamma, Q_1} \right).
\]

(3.27)

By using the \( C^{1, \gamma} \) estimates, Theorem 2.4 for (3.26) with \( \hat{T}_k = C_{ijkl}\partial_l \bar{u}^{(j)} \) on \( Q_{1/2} \), we have

\[
\|\bar{w}\|_{C^{1, \gamma}(Q_{1/4})} \leq C \left( \|\bar{w}\|_{L^\infty(Q_{1/2})} + [\nabla \bar{u}]_{\gamma, Q_{1/2}} \right).
\]

Combining with (3.28), one has

\[
\|\nabla \bar{w}\|_{L^\infty(Q_{1/4})} \leq C \left( \|\nabla \bar{w}\|_{L^2(Q_1)} + [\nabla \bar{u}]_{\gamma, Q_1} \right).
\]

Rescaling back to the original region \( \hat{\Omega}_3(z_1) \),

\[
\|\nabla w\|_{L^\infty(\hat{\Omega}_{3/4}(z_1))} \leq \frac{C}{\delta} \left( \|\nabla w\|_{L^2(\hat{\Omega}_3(z_1))} + \delta^{1+\gamma}[\nabla \bar{u}_1]_{\gamma, \hat{\Omega}_3(z_1)} \right).
\]

(3.28)

Here, combining with (3.11) and (2.6), one has

\[
[\nabla \bar{u}_1]_{\gamma, \hat{\Omega}_3(z_1)} \leq [\nabla \bar{u}]_{\gamma, \hat{\Omega}_3(z_1)} \leq C\delta^{-\gamma - \frac{1}{4\delta}}.
\]

(3.29)

By virtue of (3.29) and (3.16), we have for \( (z_1, x_2) \in \hat{\Omega}_{3/4}(z_1) \) and \( |z_1| \leq R_1 \),

\[
|\nabla w(z_1, x_2)| \leq \|\nabla w\|_{L^\infty(\hat{\Omega}_{3/4}(z_1))} \leq C \left( \delta^{-1} \cdot \delta^{1/\delta} + \delta^\gamma \cdot \delta^{-\gamma - \frac{1}{4\delta}} \right) \leq C\delta^{-\frac{1}{4\delta}}.
\]

Thus, we finish the proof of Proposition 2.1.

3.1.2. Estimates \( |\nabla v_i^3|, i = 1, 2 \).

Proof of 2.12 when \( d = 2 \). Before we provide the proof, we recall that the solution \( u \) of (1.1) is the unique function which has the least energy in appropriate functional space (see [12, theorem 6.6]), characterized by,

\[
E[u] = \min_{v \in S} E[v], \quad E[v] = \frac{1}{2} \int_{\Omega} \left( C_0 e(v), e(v) \right) dx,
\]

(3.30)

where

\[
S := \{ u \in H^1_0(D; \mathbb{R}^d) \ | \ e(u) = 0 \text{ in } D_1 \cup D_2 \}.
\]

Moreover, noticing that

\[
\bar{u}_1 = (x_2 \bar{u}, -x_1 \bar{u})^T, \quad \bar{u}_2 = (x_2 \bar{u}, -x_1 \bar{u})^T,
\]

and

\[
\bar{u}_3 \sim (x_2 \bar{u}, -x_1 \bar{u})^T,
\]
from (1.11)-(1.13) and (2.4), we obtain
\[
|\nabla \bar{u}_i^3(x)| \leq \frac{C(\varepsilon + |x_1|)}{\varepsilon + |x_1|^{1+\gamma}} \quad x \in \Omega_{R_1} \quad \text{and} \quad |\nabla \bar{u}_i^3(x)| \leq C \quad x \in \Omega \setminus \Omega_{R_1}.
\] (3.31)

Since the proof is similar to (2.9), we only give the key differences in the proof and take \(i = 1\) for instance. For simplicity, denote \(w := \bar{u}_1^3\), then \(w\) satisfies
\[
\left\{ \begin{array}{ll}
-\mathcal{L}_{\lambda, \mu} w = \nabla \cdot (C^0 e(\bar{u}_1) - \mathcal{M}) & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{array} \right.
\]
for any constant matrix \(\mathcal{M} = (a_{ij})\), \(i, j = 1, 2\).

First, the total energy is bounded, that is,
\[
\int_{\Omega} |\nabla w|^2 \, dx \leq C. \tag{3.32}
\]

In fact, by virtue of (1.6), (3.30) and (3.31), one has
\[
E(v_3^3) \leq |\nabla \bar{u}_3^3|_{L^2(\Omega)} \leq C,
\]
then, it follows from (1.6), (2.17) and the first Kohn inequality that
\[
\|\nabla v_3^3\|_{L^2(\Omega)} \leq \|\nabla (v_3^3 - \bar{u}_3^3)\|_{L^2(\Omega)} + \|\nabla \bar{u}_3^3\|_{L^2(\Omega)} \leq \sqrt{2}\|e(v_3^3 - \bar{u}_3^3)\|_{L^2(\Omega)} + C
\]
\[
\leq C\|e(v_3^3)\|_{L^2(\Omega)} + C \leq CE(v_3^3) + C \leq C.
\]

Thus
\[
\int_{\Omega} |\nabla w|^2 \, dx \leq 2 \int_{\Omega} |\nabla v_3^3|^2 \, dx + 2 \int_{\Omega} |\nabla \bar{u}_3^3|^2 \, dx \leq C.
\]

Next, we estimate the local energy estimates:
\[
\int_{\tilde{\Omega}_{s(t_1)}(z_1)} |\nabla w|^2 \, dx \leq C\delta(z_1)^2. \tag{3.33}
\]

As in the proof of (3.16), we have, instead of (3.18),
\[
\int_{\tilde{\Omega}_{s(t_1)}(z_1)} |\nabla w|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\tilde{\Omega}_{s(t_1)}} |w|^2 \, dx + C \int_{\tilde{\Omega}_{s(t_1)}} |C^0 e(\bar{u}_1^3) - \mathcal{M}|^2 \, dx, \tag{3.34}
\]
where
\[
C^0 e(\bar{u}_1^3) = \begin{pmatrix}
2\mu x_2 \partial_1 \bar{u} + \lambda (x_2 \partial_1 \bar{u} - x_1 \partial_2 \bar{u}) & \mu (x_2 \partial_2 \bar{u} - x_1 \partial_1 \bar{u}) \\
\mu (x_2 \partial_2 \bar{u} - x_1 \partial_1 \bar{u}) & -2\mu x_1 \partial_2 \bar{u} + \lambda (x_2 \partial_1 \bar{u} - x_1 \partial_2 \bar{u})
\end{pmatrix}.
\]

Then we take \(\mathcal{M} = \mathcal{M}_2\) in (3.34),
\[
\mathcal{M}_2 = \int_{\tilde{\Omega}_{s(t_1)}} C^0 e(\bar{u}_1^3(y)) \, dy.
\]

**Case 1.** For \(|z_1| \leq \varepsilon \frac{1}{s-t}\).

We still have (3.19) for \(0 < s < \varepsilon \frac{1}{s-t}\). By using (3.1), a direct calculation leads to
\[
\int_{\tilde{\Omega}_{s(t_1)}(z_1)} |C^0 e(\bar{u}_1^3) - \mathcal{M}_2|^2 \, dx \leq C(\varepsilon^{2\gamma} + \frac{1}{s-t}) s^{2\gamma} + \varepsilon s := \tilde{G}(s). \tag{3.35}
\]

Instead of (3.21), we have
\[
F(t) \leq \left(\frac{c_1 \varepsilon}{s-t}\right)^2 F(s) + C \tilde{G}(s), \quad \forall \ 0 < t < \varepsilon \frac{1}{s-t}.
\]
We define \( \{t_i\} \), \( k \) and iterate as in the proof of (3.16), to obtain that
\[
F(t_0) \leq \left( \frac{1}{4} \right)^k F(t_k) + C \varepsilon \sum_{i=1}^{k-1} \left( \frac{1}{4} \right)^i (i+1)^2 \leq C \varepsilon^2.
\]
This implies that
\[
\int_{\Omega_\delta(z_1)} |\nabla w|^2 \, dx \leq C \varepsilon^2.
\]

**Case 2.** For \( \varepsilon \frac{1}{2} \leq |z_1| \leq R_1 \).

Estimate (3.23) remains the same. Estimate (3.35) becomes
\[
\int_{\Omega_\delta(z_1)} \left| C^0 e(\bar{u}_1^3) - \mathcal{M}_2 \right|^2 \, dx \leq C \left( |z_1|^{1+\gamma} s + |z_1|^{1+2\gamma+2\gamma^2} s^{-2\gamma} \right) := \bar{H}(s). \tag{3.36}
\]

Estimate (3.25) becomes
\[
F(t) \leq \left( \frac{c_2 |z_1|^{1+\gamma}}{s-t} \right)^2 F(s) + C \bar{H}(s), \quad \forall \ 0 < t < s < \frac{2}{3} |z_1|.
\tag{3.37}
\]
Define \( \{t_i\} \), \( k \) and iterate as in the proof of (3.16), to obtain
\[
F(t_0) \leq \left( \frac{1}{4} \right)^k F(t_k) + C |z_1|^{2(1+\gamma)} \sum_{i=1}^{k-1} \left( \frac{1}{4} \right)^i (i+1)^2 \leq C |z_1|^{2(1+\gamma)}.
\]

Thus, (3.33) is proved.

Similar to the calculation of (3.1), one has
\[
\left[ C^0 e(\bar{u}_1^3) \right]_{\gamma, \Omega_\delta(z_1)} \leq C \left| \nabla \bar{u}_1^3 \right|_{\gamma, \Omega_\delta(z_1)} \leq C \delta^{-\gamma}.
\]

Same to the proof of (2.9), by using (3.33), Theorem 2.2 and Lemma 2.3,
\[
\|\nabla w\|_{L^\infty(\Omega_\delta(z_1))} \leq C \left( \|\nabla w\|_{L^2(\Omega_\delta(z_1))} + \delta^{1+\gamma} [C^0 e(\bar{u}_1^3)]_{\gamma, \Omega_\delta(z_1)} \right) \leq C \delta + C \leq C.
\]

Thus, (2.11) is proved. \( \square \)

**Proof of (2.13)-(2.15).** The proof of (2.13) and (2.14) follow from theorem 1.1 in [37].

Moreover, due to (3.32) and (3.33), one has
\[
\int_{\Omega_\delta(z_1)} |\nabla v_1|^2 \, dx \leq 2 \int_{\Omega_\delta(z_1)} \left( |\nabla \bar{u}_1^l|^2 + |\nabla w_1|^2 \right) \, dx \leq C.
\]

Then (2.15) follows from the classical elliptic estimates (see [1] and [2]). \( \square \)

3.2. Proof of Proposition 2.4

**Proof.** By trace theorem and a minor modification of the proof of Lemma 4.1 in [12], we can obtain that (2.23).

Next, we prove (2.24). Denote
\[
\alpha_{ijkl} = -\int_{\partial D_j} \frac{\partial v^k}{\partial n_i} \cdot \psi^l, \quad \beta_{ij} = \int_{\partial D_j} \frac{\partial u_0}{\partial n_i} \cdot \psi^l \quad i, j = 1, 2, \quad k, l = 1, 2, 3.
\tag{3.38}
\]
Integrating by parts and using (1.16) and (1.17), one has

\[ a_{ij}^{kl} = \int_{\Omega} \left( \nabla^0 e(v_i^k), e(v_j^l) \right) dx, \quad b_j^l = -\int_{\Omega} \left( \nabla^0 e(v_0), e(v_j^l) \right) dx. \]

Then (1.18) becomes

\[
\begin{aligned}
3 & \sum_{k=1}^3 C_{1i}^k a_{11}^k + \sum_{k=1}^3 C_{2i}^k a_{21}^k - b_1^i = 0 \\
3 & \sum_{k=1}^3 C_{1i}^k a_{12}^k + \sum_{k=1}^3 C_{2i}^k a_{22}^k - b_1^2 = 0
\end{aligned}
\quad (3.39)
\]

To estimate \(|C_1^k - C_2^k|, k = 1, 2,\) we just use the first equation in (3.39). For simplicity, we denote the 3 × 3 matrix \((a_{ij}^{kl})\) by \(A_{ij}\), then

\[ A_{11} C_1 + A_{21} C_2 = b_1, \quad (3.40) \]

where \(C_i = (C_i^1, C_i^2, C_i^3)^T, i = 1, 2\) and \(b_1 = (b_1^1, b_1^2, b_1^3)^T\). Then, we rewrite (3.40)

\[ A_{11} (C_1 - C_2) = p := b_1 - (A_{11} + A_{21}) C_2, \]

that is

\[ A_{11} (C_1 - C_2) = \begin{pmatrix}
(a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\
a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\
a_{11}^{31} & a_{11}^{32} & a_{11}^{33}
\end{pmatrix}
\begin{pmatrix}
(C_1^1 - C_2^1) \\
(C_1^2 - C_2^2) \\
(C_1^3 - C_2^3)
\end{pmatrix}
= \begin{pmatrix}
p^1 \\
p^2 \\
p^3
\end{pmatrix}, \quad (3.41) \]

Since \(A_{11}\) is positive definite (see lemma 4.4 in [12]), it follows from (3.41) and Cramer’s rule that

\[ C_1^1 - C_2^1 = \frac{1}{\det A_{11}} \begin{vmatrix}
a_{11}^{22} & a_{11}^{23} \\
a_{11}^{32} & a_{11}^{33}
\end{vmatrix} p^1 + \begin{vmatrix}
a_{11}^{12} & a_{11}^{13} \\
a_{11}^{32} & a_{11}^{33}
\end{vmatrix} p^2 + \begin{vmatrix}
1 & 1 \\
2 & 3
\end{vmatrix} p^3, \quad (3.42) \]

and

\[ C_1^2 - C_2^2 = \frac{1}{\det A_{11}} \begin{vmatrix}
a_{11}^{21} & a_{11}^{23} \\
a_{11}^{31} & a_{11}^{33}
\end{vmatrix} - p^1 + \begin{vmatrix}
a_{11}^{11} & a_{11}^{13} \\
a_{11}^{31} & a_{11}^{33}
\end{vmatrix} p^2 - \begin{vmatrix}
1 & 1 \\
2 & 3
\end{vmatrix} p^3, \quad (3.43) \]

We first claim that

\[
\frac{1}{C} \varepsilon^{-\frac{2\pi}{\sqrt{2}}} \leq a_{1k}^{kl} \leq C \varepsilon^{-\frac{2\pi}{\sqrt{2}}}, \quad k = 1, 2; \quad (3.44)
\]

\[
\frac{1}{C} \leq a_{ii}^{33} \leq C; \quad (3.45)
\]

\[
|a_{11}^{12}| = |a_{11}^{21}| \leq C |\ln \varepsilon|; \quad (3.46)
\]

\[
|a_{11}^{4k}| = |a_{11}^{5k}| \leq C \quad k = 1, 2; \quad (3.47)
\]

and consequently,

\[
\frac{1}{C} \varepsilon^{-\frac{2\pi}{\sqrt{2}}} \leq \det A_{11} \leq C \varepsilon^{-\frac{2\pi}{\sqrt{2}}}. \quad (3.48)
\]

In fact, for (3.44), by using (2.17) and (2.10), one has for \(k = 1, 2,\)

\[
a_{1k}^{kl} = \int_{\Omega} \left( \nabla^0 e(v_1^k), e(v_1^l) \right) dx \leq C \int_{\Omega} |\nabla v_1^k|^2 dx \leq C \varepsilon^{-\frac{2\pi}{\sqrt{2}}}. \quad (3.47)
\]
Moreover, by using the first Korn inequality and (3.4),
\[ a_{11}^1 = \int_{\Omega} \left( C_0^0 e(v_1^1), e(v_1^1) \right) dx \geq \frac{1}{C} \int_{\Omega} |e(v_1^1)|^2 dx \geq \frac{1}{C} \int_{\Omega} |\nabla v_1^1|^2 \ dx \]
\[ \geq \frac{1}{C} \int_{\Omega} |\nabla u_1^1|^2 dx - C \int_{\Omega} |\nabla w_1|^2 dx \]
\[ \geq \frac{1}{C} \int_{\Omega_{\delta_1}} \frac{1}{\varepsilon + |x_1|^{1+\gamma}} dx - C \geq \frac{1}{C} \varepsilon^{-\frac{1}{1+\gamma}}. \]

Similarly, we also have
\[ a_{11}^2 \geq \frac{1}{C} \varepsilon^{-\frac{1}{1+\gamma}}. \]

Thus, (3.44) is proved.

For (3.45), by using (2.12), we have
\[ a_{11}^3 = \int_{\Omega} \left( C_0^0 e(v_1^3), e(v_1^3) \right) dx \leq C \int_{\Omega} |e(v_1^3)|^2 dx \leq C \int_{\Omega} |\nabla v_1^3|^2 dx \leq C. \]

Moreover, by argument in contradiction (For more details, see (4.18) in [12]), we can see that it holds
\[ \|\nabla v_1^3\|_{L^2(\Omega_{\delta_1} \setminus \Omega_{\delta_1/2})} \leq C \|e(v_1^3)\|_{L^2(\Omega_{\delta_1} \setminus \Omega_{\delta_1/2})}. \]

Then
\[ a_{11}^3 = \int_{\Omega} \left( C_0^0 e(v_1^3), e(v_1^3) \right) dx \geq \frac{1}{C} \int_{\Omega_{\delta_1} \setminus \Omega_{\delta_1/2}} |e(v_1^3)|^2 dx \]
\[ \geq \frac{1}{C} \int_{\Omega_{\delta_1} \setminus \Omega_{\delta_1/2}} |\nabla v_1^3|^2 dx \geq \frac{1}{C}. \]

Hence, we have (3.45).

For \( a_{11}^2, a_{11}^3 \), we firstly notice that
\[ a_{11}^2 = \int_{\Omega} \left( C_0^0 e(v_1^2), e(v_1^2) \right) dx = \int_{\Omega} \left( C_0^0 e(v_1^2), \nabla e(v_1^2) \right) dx = a_{11}^1. \]

Moreover
\[ a_{11}^2 = - \int_{\partial D_1} \frac{\partial v_1^2}{\partial \nu} \cdot \psi^2 - \int_{\partial D_1} \frac{\partial u_1^1}{\partial \nu} \cdot \psi^2 - \int_{\partial D_1} \frac{\partial (v_1^1 - u_1^1)}{\partial \nu} \cdot \psi^2 = -I_1 - I_2. \]

We divide \( I_1 \) into two parts
\[ I_1 = \int_{\partial D_1 \cap C_{\delta_1}} \frac{\partial v_1^1}{\partial \nu} \cdot \psi^2 + \int_{\partial D_1 \setminus C_{\delta_1}} \frac{\partial v_1^1}{\partial \nu} \cdot \psi^2, \]
where the cylinder \( C_r \) is defined as
\[ C_r := \left\{ x \in \mathbb{R}^d : -\frac{\varepsilon}{2} + 2 \min_{|x_1| = r} h_2(x_1) \leq x_2 \leq \frac{\varepsilon}{2} + 2 \max_{|x_1| = r} h_1(x_1), \quad |x_1| < r \right\}. \]

Noticing that on the boundary \( \partial D_1 \cap C_{\delta_1} \),
\[ n_1 = \frac{\partial h_1(x_1)}{\sqrt{1 + |\partial h_1(x_1)|^2}}, \quad n_2 = \frac{1}{\sqrt{1 + |\partial h_1(x_1)|^2}}. \]
Then, we have
\[
\frac{\partial \bar{u}_1}{\partial \nu} \cdot \psi^2 \bigg|_{\partial D_1 \cap C_{R_1}} = \left( (\lambda(\nabla \cdot \bar{u}_1)\mathbf{n} + \mu(\nabla \bar{u}_1^2 + (\nabla \bar{u}_1)^T)\mathbf{n} \right) \cdot \psi^2 = \lambda \partial_1 \bar{u}_2 + \mu \partial_2 \bar{u}_1.
\]
Combining with (1.12), (2.3) and (2.4), we have
\[
|I_1| \leq \int_{\partial D_1 \cap C_{R_1}} \frac{\partial v_1^1}{\partial \nu} \cdot \psi^2 + \int_{\partial D_1 \setminus C_{R_1}} \frac{\partial v_1^1}{\partial \nu} \cdot \psi^2 \leq \int_{\partial D_1 \cap C_{R_1}} |\lambda \partial_1 \bar{u}_2 + \mu \partial_2 \bar{u}_1| dS + C \\
\leq \int_{|x_1| \leq R_1} \frac{C|x_1|^\gamma}{\varepsilon + |x_1|^{1+\gamma}} dx_1 + C|\ln \varepsilon|.
\]
For $I_2$, by using (2.9), (2.15) and (2.3), we can obtain
\[
|I_2| \leq \int_{\partial D_1 \cap C_{R_1}} \left( (\lambda(\nabla \cdot (v_1^1 - \bar{u}_1))\mathbf{n} + \mu((v_1^1 - \bar{u}_1)^2 + (\nabla (v_1^1 - \bar{u}_1))^T)\mathbf{n} \right) \cdot \psi^2 + C \\
\leq \int_{|x_1| \leq R_1} \frac{C}{(\varepsilon + |x_1|^{1+\gamma})^{1+\gamma}} dx_1 + C|\ln \varepsilon|.
\]
Thus, (3.46) is proved.
For 3.47,
\[
a_{11}^{3k} = a_{11}^{3k} = \int_{\Omega} \left( C^0 e(v_1^k), e(v_1^3) \right) dx = \int_{\Omega} \left( C^0 \nabla v_1^k, \nabla v_1^3 \right) dx, \quad k = 1, 2.
\]
Similar to the proof of (3.46), using (3.44) and (3.45), we have for $k = 1$
\[
a_{11}^{11} = \int_{\Omega_{R_1}} \left( C^0 \nabla v_1^1, \nabla v_1^3 \right) dx + O(1) \\
= \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^1, \nabla \bar{u}_1^3 \right) dx + \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^1, \nabla w_1^3 \right) dx + \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^3, \nabla w_1^1 \right) dx \\
+ \int_{\Omega_{R_1}} \left( C^0 \nabla w_1^1, \nabla w_1^3 \right) dx + O(1) \\
= \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^1, \nabla \bar{u}_1^3 \right) dx + \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^1, \nabla w_1^3 \right) dx \\
+ \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^3, \nabla w_1^1 \right) dx + O(1) := I_1 + I_2 + I_3 + O(1).
\]
Since
\[
\left( C^0 \nabla \bar{u}_1^1, \nabla \bar{u}_1^3 \right) = \left( (\lambda + 2\mu)\partial_1 \bar{u} + \mu \partial_2 \bar{u} \right) \left( \begin{array}{c} -x_2 \partial_1 \bar{u} - \lambda \partial_1 \bar{u} \\ x_2 \partial_1 \bar{u} + \mu \partial_2 \bar{u} + \lambda \partial_1 \bar{u} \\ x_2 \partial_1 \bar{u} + \mu \partial_2 \bar{u} - \lambda \partial_1 \bar{u} \end{array} \right)
\]
\[
= (\lambda + 2\mu)x_2 (\partial_1 \bar{u})^2 + \mu x_2 (\partial_2 \bar{u})^2 - (\lambda + \mu) x_1 \partial_1 \bar{u} \partial_2 \bar{u},
\]
Hence, by using (2.11), one has

\[ |I_1| \leq C \left( \int_{\Omega_{R_1}} \frac{|x_2||x_1|^{2\gamma}}{\varepsilon + |x_1|^{1+\gamma}} \, dx \right. \]
\[ + \int_{\Omega_{R_1}} \frac{|x_2|}{\varepsilon + |x_1|^{1+\gamma}} \, dx \) \leq C. \]

By using (2.11) and (2.4), one has

\[ |I_2| \leq \left| \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^1, \nabla w_3^1 \right) \, dx \right| \leq C \int_{\Omega_{R_1}} |\nabla \bar{u}_1^1| \, dx \leq C. \]

By using (3.31) and (2.4),

\[ |I_3| \leq \left| \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^2, \nabla u_1^3 \right) \, dx \right| \leq C \left( \int_{\Omega_{R_1}} |\nabla \bar{u}_1^2|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_{R_1}} |\nabla u_1^3|^2 \, dx \right)^{\frac{1}{2}} \leq C. \]

Therefore, we have

\[ |a_{11}^3| \leq C. \]

Similarly, by using (3.42), (3.31), (3.33), (2.4) and (3.32), one has

\[ a_{11}^{32} = \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^2, \nabla u_1^3 \right) \, dx + O(1) \]
\[ = \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^2, \nabla u_1^3 \right) \, dx + \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^2, \nabla w_3^1 \right) \, dx + \int_{\Omega_{R_1}} \left( C^0 \nabla w_1^2, \nabla u_1^3 \right) \, dx + O(1) \]
\[ = \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^2, \nabla u_1^3 \right) \, dx + O(1). \]

Since

\[ \left( C^0 \nabla \bar{u}_1^2, \nabla u_1^3 \right) = \left( \frac{\lambda \partial_2 \bar{u}}{\mu \partial_1 \bar{u}} \right) \left( \frac{\mu \partial_1 \bar{u}}{(\lambda + 2\mu) \partial_2 \bar{u}} \right) \]
\[ = \left( \lambda + \mu \right) x_2 \partial_1 \bar{u} \partial_2 \bar{u} - \mu x_1 (\partial_1 \bar{u})^2 - (\lambda + 2\mu) x_1 (\partial_2 \bar{u})^2, \]

then, by using (2.4), one has

\[ \int_{\Omega_{R_1}} \left( C^0 \nabla \bar{u}_1^2, \nabla u_1^3 \right) \, dx \leq C \int_{\Omega_{R_1}} \frac{|x_2||x_1|^{\gamma}}{\varepsilon + |x_1|^{1+\gamma}} \, dx + C \int_{\Omega_{R_1}} \frac{|x_1|^{1+2\gamma}}{\varepsilon + |x_1|^{1+\gamma}} \, dx \leq C. \]

Thus, we have (3.47). Combing with (3.44), (3.47), we obtain (3.48).

Moreover, it follows from lemma 4.3 in [12] that

\[ |p| = |b_1 - (a_{11} + a_{21}) C_2| \leq C. \]

Then, combining with (3.32), (3.44), (3.47) and (3.49), we have

\[ C_1^1 - C_2^1 = \frac{1}{\det A_{11}} \left( (p_{1}^{1} a_{11}^{22} a_{11}^{33} - p_{3}^{3} a_{11}^{22} a_{11}^{33}) + O(\ln \varepsilon) \right). \]

Hence

\[ |C_1^1 - C_2^1| \leq C \varepsilon^{\frac{1}{1+\gamma}}. \]
Similarly, from (3.43), one has
\[ C_1^2 - C_2^2 = \frac{1}{\det A_{11}} \left( (p^2 a_{11}^{11} a_{33}^{33} - p^3 a_{11}^{11} a_{11}^{21}) + O(|\ln \varepsilon|) \right). \]
Thus
\[ |C_1^2 - C_2^2| \leq C \varepsilon \gamma. \]
The proof is completed. \( \Box \)

3.3. The lower bounds. From the decomposition (2.41), we rewrite
\[ \nabla u = \sum_{k=1}^{2} (C_k^1 - C_k^2) \nabla v_k^1 + \sum_{i=1}^{2} C_i^3 \nabla v_i^3 + \nabla u_b, \]
where
\[ u_b = \sum_{k=1}^{2} C_k^2 (v_k^1 + v_k^2) + v_0. \tag{3.50} \]
It follows from the fourth line of (1.1) that
\[ \sum_{k=1}^{2} (C_k^1 - C_k^2) \int_{\partial D_j} \frac{\partial v_k^1}{\partial \nu} \cdot \psi^l + \sum_{i=1}^{2} C_i^3 \int_{\partial D_j} \frac{\partial v_i^3}{\partial \nu} \cdot \psi^l + \int_{\partial D_j} \frac{\partial u_b}{\partial \nu} \cdot \psi^l = 0, \tag{3.51} \]
where \( j = 1, 2, l = 1, 2, 3. \) Denote
\[ b^k_j := b^k_j[\varphi] = \int_{\partial D_j} \frac{\partial u_b}{\partial \nu} \cdot \psi^k, \quad k = 1, 2, 3. \tag{3.52} \]
By the same argument in \( [36], \) we can obtain
\[ b^k_1 \to b^k_{*1} \text{ as } \varepsilon \to 0. \tag{3.53} \]
Here
\[ b^k_{*1}[\varphi] = \int_{\partial D^*_1} \frac{\partial u^*_b}{\partial \nu} \cdot \psi^k, \quad k = 1, 2, \]
and \( u^*_b \) satisfies the following boundary value problem:
\[ \begin{cases} -\mathcal{L}_{\lambda, \mu} u^*_b = 0 & \text{in } \Omega^*, \\ u^*_b = \sum_{k=1}^{2} C_k^3 \psi^k & \text{on } \partial D^*_1 \cup \partial D^*_2, \\ u^*_b = \varphi & \text{on } \partial D, \end{cases} \]
where \( C_k^* = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (C_k^1 + C_k^2) \) and \( D^*_i = \{ x \in \mathbb{R}^n : x + P_i \in D_i \}, \ i = 1, 2, \Omega^* = \bar{D} \setminus D^*_1 \cap D^*_2. \)
Moreover, by using the denotation \( 3.38 \) and \( 3.52, 3.51 \) becomes
\[ \begin{cases} \sum_{k=1}^{2} (C_k^1 - C_k^2) a_{11}^{kl} + \sum_{i=1}^{2} C_i^3 a_{11}^{kl} - b_k^1 = 0 & \text{in } \Omega^*, \\ \sum_{k=1}^{2} (C_k^1 - C_k^2) a_{12}^{kl} + \sum_{i=1}^{2} C_i^3 a_{12}^{kl} - b_k^2 = 0 & \text{in } \Omega^*. \end{cases} \tag{3.54} \]
In order to estimate the lower bound of $C_1^1 - C_2^1$ and $C_1^2 - C_2^2$ from (3.54), we choose $k = 1, 2, 3$ for $j = 1$ and $k = 3$ for $j = 2$. Then

$$AX := \begin{pmatrix}
    a_{11}^{11} & a_{11}^{12} & a_{11}^{13} & a_{12}^{13} \\
    a_{11}^{21} & a_{11}^{22} & a_{11}^{23} & a_{12}^{23} \\
    a_{11}^{31} & a_{11}^{32} & a_{11}^{33} & a_{12}^{33} \\
    a_{21}^{31} & a_{21}^{32} & a_{21}^{33} & a_{22}^{33}
\end{pmatrix} \begin{pmatrix}
    C_1^1 - C_2^1 \\
    C_1^2 - C_2^2 \\
    C_1^3 \\
    C_2^3
\end{pmatrix} = \begin{pmatrix}
    b_1^1 \\
    b_1^2 \\
    b_1^3 \\
    b_2^3
\end{pmatrix}.$$  

We first claim that

$$|b_k^i| \leq C. \quad (3.55)$$

Indeed, from the definition (3.50),

$$b_k^i = \int_{\partial D_1} \frac{\partial u_k}{\partial \nu} \cdot \psi^k = \sum_{k=1}^{2} C_2^k \int_{\partial D_1} \frac{\partial (v_k + v_k^3)}{\partial \nu} \cdot \psi^k + \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi^k.$$

Integrating by parts and using (2.12) and (2.11), one has

$$\left| \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi^k \right| \leq \int_{\Omega} \left( C^0 e(v_0), e(v_1^k) \right) dx \leq C.$$

Similarly, by using (2.13), we have

$$\left| \int_{\partial D_1} \frac{\partial (v_k + v_k^3)}{\partial \nu} \cdot \psi^k \right| \leq C.$$

Then, combining with (2.23), one has (3.55).

Moreover, for $|a_{12}^{31}| = |a_{21}^{33}|$,

$$|a_{12}^{31}| = \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi^3 = \int_{\Omega} \left( C^0 e(v_2^3), e(v_1^3) \right) dx \leq C.$$

Similarly, by using integration by parts and Proposition 2.1, we can obtain (3.56).

Thus, combining with (3.44)-(3.47), (3.55) and (3.56), one has $A$ is invertible and

$$\frac{\varepsilon}{C} \leq \det A \leq C \varepsilon^{-\frac{2}{1+\gamma}}.$$

Further, denoting

$$B_1 := \begin{pmatrix}
    b_1^1 & a_{11}^{12} & a_{11}^{13} & a_{12}^{13} \\
    b_2^1 & a_{11}^{22} & a_{11}^{23} & a_{12}^{23} \\
    b_1^3 & a_{11}^{32} & a_{11}^{33} & a_{12}^{33} \\
    b_2^3 & a_{21}^{32} & a_{21}^{33} & a_{22}^{33}
\end{pmatrix}, \quad B_2 := \begin{pmatrix}
    a_{11}^{11} & b_1^1 & a_{11}^{13} & a_{12}^{13} \\
    a_{11}^{21} & b_2^1 & a_{11}^{23} & a_{12}^{23} \\
    a_{11}^{31} & b_1^3 & a_{11}^{33} & a_{12}^{33} \\
    a_{21}^{31} & b_2^3 & a_{21}^{33} & a_{22}^{33}
\end{pmatrix}.$$
and using Cramer’s rule, we have
\[ C_1^2 - C_2^2 = a_{11}^2 b_1^2 \frac{\det B_2(34 : 34) - \det B_2(24 : 34) + \det B_2(23 : 34)}{\det A} \]
\[ + O(\varepsilon^{\frac{2}{1 + \gamma}} |\ln \varepsilon|) \]
\[ = a_{11}^2 b_1^2 \frac{\det B_2(34 : 34)}{\det A} + o(\varepsilon^{\frac{2}{1 + \gamma}}), \]

and
\[ C_1^2 - C_2^2 = a_{11}^2 b_2^2 \frac{\det B_2(34 : 34) - \det B_2(13 : 34) + \det B_2(12 : 34)}{\det A} \]
\[ + O(\varepsilon^{\frac{2}{1 + \gamma}} |\ln \varepsilon|) \]
\[ = a_{11}^2 b_2^2 \frac{\det B_2(34 : 34)}{\det A} + o(\varepsilon^{\frac{2}{1 + \gamma}}), \]

where \( B_i(kl; mn) \) denotes the line \( k, l \) and column \( m, n \) of the matrix \( B_i \), \( i = 1, 2, \ldots, n \), \( n = 1, 2, 3, 4 \).

From lemma 4.4 in [36], we can see that \( \det B_1(34 : 34) = \det B_2(34 : 34) = a_{11}^3 a_{33}^3 - a_{13}^3 a_{31}^3 \geq \frac{1}{C} \), then, if \( b_1^{k_0} [\varphi] \neq 0 \) for some \( \varphi \in L^\infty(\partial D, \mathbb{R}^2) \), \( k = 1, 2 \), one has
\[ |C_1^k - C_2^k| = \frac{\varepsilon^{\frac{2}{1 + \gamma}}}{C} |b_1^k| + o(\varepsilon^{\frac{2}{1 + \gamma}}). \]

Therefore, noticing that (3.53)
\[ |C_1^k - C_2^k| \geq \frac{\varepsilon^{\frac{2}{1 + \gamma}}}{C} |b_1^k[\varphi]|. \]

Hence, if there exists \( k_0 \in \{1, 2\} \) such that \( b_1^{k_0} \neq 0 \), one has for small \( \varepsilon > 0 \)
\[ |\nabla u(x)| \geq \sum_{k=1}^{2} |C_1^k - C_2^k| \cdot |\nabla v_1^k(x)| - C \geq \frac{|b_1^{k_0}[\varphi]|}{C} \varepsilon^{-\frac{2}{1 + \gamma}}, \quad \text{for } x \in P_1P_2. \]

The proof is completed.

4. Estimates in Higher dimensions \( d \geq 3 \)

Recall that \( v_1^d \) and \( v_0 \) is defined by (1.6) and (1.7). From the first line of the decomposition (2.1),
\[ |\nabla u| \leq \sum_{i=1}^{2} \sum_{l=1}^{d(d+1)/2} |C_i^l||\nabla v_i^l| + |\nabla v_0| \quad \text{in } \Omega. \]

The notations \( \hat{\Omega}_s(\varepsilon') \) and \( \Omega_R_1 \) are defined accordingly. The auxiliary functions \( \hat{u} \) and \( \underline{u} \) are defined as in (2.2) and (2.3). Define \( \hat{u}_1^d = \hat{u} \psi^d \) and \( \underline{u}_2^d = \underline{u} \psi^d \) as (2.6) and (2.7) with \( x_1, x_2 \) replaced by \( x', x_d, l = 1, 2, \ldots, d(d+1)/2 \).

Proof of Proposition 2.1 in dimensions \( d \geq 3 \). Since the proof is similar to that in dimensions two, we only prove the main difference. The proof of (3.39), (3.32) are the same as that in dimensions two.
To prove (2.11), by virtue of (2.4), one has for $i = 1, 2, l = 1, 2, \cdots, d$,
\[
|\nabla \tilde{u}_i(x)| \leq \frac{C}{\varepsilon + |x'|^{1+\gamma}} \quad x \in \Omega_{R_1}.
\]
And instead of (3.20), by taking
\[
F \text{ Denote and taking k iterations, we have}
\]
\[
\int_{\Omega_{(z')}} \left| \nabla^{0} \hat{e}(\hat{u}^l_i(y)) \right| \, dy,
\]
we have for $0 \leq |z'| \leq \varepsilon^{1/\gamma}, 0 < s < \varepsilon^{1/\gamma}$,
\[
\int_{\Omega_{(z')}} \int_{\Omega_{(z')}} \left| \nabla^{0} \hat{e}(\hat{u}^l_i) - M_3 \right|^2 \, dx \leq C \left( \varepsilon^{-1+1/\gamma} s^{d-1} + \varepsilon^{1+1/\gamma} s^{d-1} \right) := G(s).
\]
Denote $F(t) := \int_{\Omega_{(z')}} \nabla(v^l_i - \hat{u}^l_i)^2 \, dx$,
\[
F(t) \leq \left( \frac{C_1 \varepsilon}{s - t} \right)^2 F(s) + CG(s), \quad \forall 0 < t < s < \varepsilon^{1/\gamma}. \quad (4.1)
\]
Similar to case 1 of step 2 in Subsection 3.1.1, set $t_i = \delta + 2C_1i\varepsilon, i = 0, 1, 2, \cdots$, and let $k = \left[ \frac{1}{4C_2\varepsilon^{1/\gamma}} \right]$. By using (4.1) with $s = t_{i+1}$ and $t = t_i$, we have
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^{d+1} \varepsilon^{d-1}, \quad i = 1, 2, \cdots.
\]
After $k$ iterations, we obtain
\[
\int_{\Omega_{(z')}} \left| \nabla(v^l_i - \hat{u}^l_i) \right|^2 \, dx \leq C \varepsilon^{d-1+1/\gamma}.
\]
Instead of (3.21), for $\varepsilon^{1/\gamma} < |z'| < R_1, 0 < s < \frac{2}{3} |z'|$,
\[
\int_{\Omega_{(z')}} \int_{\Omega_{(z')}} \left| \nabla^{0} \hat{e}(\hat{u}^l_i) - M_3 \right|^2 \, dx \leq C \left( \frac{s^{d-1+2\gamma}}{|z'|^{1+\gamma+2\gamma^2}} + \frac{s^{d-1}}{|z'|^{1-\gamma}} \right) := H(s).
\]
and
\[
F(t) \leq \left( \frac{C_2 |z'|^{1+\gamma}}{s - t} \right)^2 F(s) + CH(s), \quad \forall 0 < t < s < \frac{2}{3} |z'|. \quad (4.2)
\]
Let $t_i = \delta + 2C_2i|z'|^{1+\gamma}, i = 0, 1, 2, \cdots$ and $k = \left[ \frac{1}{4C_2|z'|} \right]$. By (4.2) with $t = t_i$ and $s = t_{i+1}$, we have
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^{d+1} |z'|^{(1+\gamma)(d-\frac{1}{1+\gamma})}, \quad i = 1, 2, \cdots.
\]
After $k$ iteration, we have
\[
\int_{\Omega_{(z')}} \left| \nabla(v^l_i - \hat{u}^l_i) \right|^2 \, dx \leq C |z'|^{(1+\gamma)(d-\frac{2}{1+\gamma})}.
\]
Thus, instead of (3.10), we have
\[
\int_{\Omega_{(z')}} \left| \nabla(v^l_i - \hat{u}^l_i) \right|^2 \, dx \leq C \delta^{d-\frac{2}{1+\gamma}}. \quad (4.3)
\]
As in step 3 in Subsection 3.1.1, by using Theorem 2.23 and 2.24, we have, instead of (3.28)
\[
\| \nabla(v^l_i - \hat{u}^l_i) \|_{L^\infty(\hat{\Omega}_{t_A/4}(z'))} \leq \frac{C}{\delta} \left( \delta^{1-\frac{d}{2}} \| \nabla(v^l_i - \hat{u}^l_i) \|_{L^2(\hat{\Omega}_{t_A}(z'))} + \delta^{3+\gamma} |\nabla \hat{u}_i|_{\gamma, \hat{\Omega}_{t_A}(z')} \right).
\]
By using (4.13) and (5.1), one has (2.10) holds for $d \geq 3$. 

In order to prove (2.12), from the definition of \( \bar{u}_l^i \) and (2.13), one has for \( i = 1, 2, l = d + 1, \cdots, \frac{d(d+1)}{2} \),
\[
|\nabla \bar{u}_l^i| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^{1+\gamma}} \quad x \in \Omega_{R_1} \quad \text{and} \quad |\nabla \bar{u}_l^i(x)| \leq C \quad x \in \Omega \setminus \Omega_{R_1}.
\]
Instead of (3.35), we have for \( 0 < |z'| < \frac{\varepsilon}{1+\gamma} \), \( 0 < s < \varepsilon^\frac{1}{1+\gamma} \),
\[
\int_{\hat{\Omega}_{s}(z')} |\mathcal{C}^0(\bar{u}_l^i) - \mathcal{M}_3|^2 \, dx \leq C(\varepsilon^{1-2\gamma}s^{d+2\gamma-1} + \varepsilon^{s^{d-1}}) =: \bar{G}(s).
\]
Denoting \( F(t) = \int_{\hat{\Omega}_{s}(z')} |\nabla (v_l^i - \bar{u}_l^i)|^2 \, dx \),
\[
F(t) \leq \left( \frac{C_1 \varepsilon}{s-t} \right)^2 F(s) + C \bar{G}(s), \quad \forall \ 0 < t < s < \varepsilon^\frac{1}{1+\gamma}.
\]
Similar to case 1 of step 2 in subsection 3.1.2, set \( t_i = \delta + 2C_1 \varepsilon, i = 1, 2, \cdots \) and let \( k = \left[ \frac{1}{4C_1 \varepsilon^\frac{1}{1+\gamma}} \right] \). By taking \( s = t_{i+1} \) and \( t = t_i \) in (4.4), one has
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^{d-1} \varepsilon^d.
\]
After \( k \) iteration, one has
\[
\int_{\hat{\Omega}_{s}(z')} |\nabla (v_l^i - \bar{u}_l^i)|^2 \, dx \leq C \varepsilon^d.
\]
For \( \varepsilon^\frac{1}{1+\gamma} < |z'| < R_1, 0 < s < \frac{2}{3} |z'| \), instead of (3.36) and (3.37), one has
\[
\int_{\hat{\Omega}_{s}(z')} |\mathcal{C}^0(\bar{u}_l^i) - \mathcal{M}_3|^2 \, dx \leq C(|z'|^{(1+\gamma)(1-2\gamma)}s^{d+2\gamma-1} + |z'|^{(1+\gamma)\frac{d}{2}}) =: \bar{H}(s),
\]
and
\[
F(t) \leq \left( \frac{C_2 |z'|^{1+\gamma}}{s-t} \right)^2 F(s) + C \bar{H}(s), \quad \forall \ 0 < t < s < \frac{2}{3} |z'|.
\]
Let \( t_i = \delta + 2C_2 |z'|^{1+\gamma}, i = 0, 1, 2, \cdots \) and \( k = \left[ \frac{1}{4C_2 |z'|^{1+\gamma}} \right] \), by taking \( s = t_{i+1} \) and \( t = t_i \) in (4.4). After \( k \) iteration, we have
\[
\int_{\hat{\Omega}_{s}(z')} |\nabla (v_l^i - \bar{u}_l^i)|^2 \, dx \leq C |z'|^{(1+\gamma)d}.
\]
Thus, we have
\[
\int_{\hat{\Omega}_{s}(z')} |\nabla (v_l^i - \bar{u}_l^i)|^2 \, dx \leq C \delta^d, \quad i = 1, 2, \ l = d + 1, \cdots, \frac{d(d+1)}{2}.
\]
As in step 3 of subsection 3.1.2, one has
\[
|\mathcal{C}^0(\bar{u}_l^i)_{\gamma,\hat{\Omega}_{s}(z')}| \leq C|\nabla \bar{u}_l^i|_{\gamma,\hat{\Omega}_{s}(z')} \leq C \delta^{-\gamma},
\]
and by using (4.7), one has
\[
\|\nabla (v_l^i - \bar{u}_l^i)\|_{L^\infty(\hat{\Omega}_{s}(z'))} \leq \frac{C}{\delta} \left( \delta^{1-\gamma} \|\nabla (v_l^i - \bar{u}_l^i)\|_{L^2(\hat{\Omega}_{s}(z'))} + \delta^{1+\gamma} |\nabla \bar{u}_l^i|_{\gamma,\hat{\Omega}_{s}(z')} \right) \leq C.
\]
Thus, (2.12) also holds for \( d \geq 3 \). The proof is completed. \( \square \)
5. $C^{1,\gamma}$ estimates and $W^{1,p}$ Estimates

5.1. $C^{1,\gamma}$ Estimates. In this section, we shall use the Campanato’s approach, see e.g. [24], to prove Theorem 2.2. Let $Q$ be a Lipschitz domain in $\mathbb{R}^d$, the Campanato space $L^{2,\lambda}(Q)$, $\lambda \geq 0$, is defined as follows

$$L^{2,\lambda}(Q) := \left\{ u \in L^2(Q) : \sup_{x_0 \in Q} \int_{B_r(x_0) \cap Q} \frac{|u - u_{x_0,\rho}|^2}{\rho^2} dx < +\infty \right\},$$

where $u_{x_0,\rho} := \frac{1}{|Q \cap B_r(x_0)|} \int_{Q \cap B_r(x_0)} u(x) dx$. It is endowed with the norm

$$\|u\|_{L^{2,\lambda}(Q)} := \|u\|_{L^2(Q)} + [u]_{L^{2,\lambda}(Q)},$$

where the semi-norm $[\cdot]_{L^{2,\lambda}(Q)}$ is defined by

$$[u]_{L^{2,\lambda}(Q)} := \sup_{x_0 \in Q} \int_{B_r(x_0) \cap Q} |u - u_{x_0,\rho}|^2 \rho^{\lambda} dx.$$

It is clear that if $d < \lambda \leq d + 2$ and $\gamma = \frac{\lambda d}{2}$, the Campanato space $L^{2,\lambda}(Q)$ is equivalent to the Hölder space $C^{0,\gamma}(Q)$.

We first recall a classical result in [24].

**Theorem 5.1.** (Theorem 5.14 in [24]) Let $Q$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $\tilde{w} \in H^1(Q, \mathbb{R}^d)$ be a solution for

$$-\partial_k (C_{ijkl} \partial_l \tilde{w}^j) = \partial_k \tilde{f}^i \quad \text{in} \quad Q \quad (5.1)$$

with $\tilde{f}^k \in C^\gamma(Q)$, $0 < \gamma < 1$ and $C_{ijkl}$ constant and satisfying (2.10) and (2.11). Then $\nabla \tilde{w} \in L^{2,d+2\gamma}(Q)$ and for $B_R := B_R(x_0) \subset Q$,

$$\|\nabla \tilde{w}\|_{L^{2,d+2\gamma}(B_R/2)} \leq C \left( \|\nabla \tilde{w}\|_{L^2(B_R)} + [\tilde{F}]_{L^{2,d+2\gamma}(B_R)} \right),$$

where $\tilde{F} := (\tilde{f}^k)$ and $C = C(d, \gamma, R)$.

From the proof of Theorem 5.1 and the equivalence between Hölder space and Campanato space, we have the following interior estimates.

**Corollary 5.2.** Under the hypotheses of Theorem 5.1. Let $\tilde{w}$ be the solution of (2.18). Then for $B_R := B_R(x_0) \subset Q$,

$$[\nabla \tilde{w}]_{\gamma,B_R/2} \leq C \left( \frac{1}{R^{1+\gamma}} \|\tilde{w}\|_{L^\infty(B_R)} + [\tilde{F}]_{\gamma,B_R} \right), \quad (5.2)$$

where $C = C(d, \gamma)$.

**Proof of Theorem 5.2.** Since $\Gamma \in C^{1,\gamma}$, then for any $x_0 \in \Gamma$, there exists a neighbourhood $U$ of $x_0$ and a homeomorphism $\Psi \in C^{1,\gamma}(U)$ such that

$$\Psi(U \cap Q) = B_1^+ = \{ y \in B_1(0) : y_d > 0 \},$$

$$\Psi(U \cap \Gamma) = \partial B_1^+ \cap \{ y \in \mathbb{R}^d : y_d = 0 \},$$

where $B_1(0) := \{ y \in \mathbb{R}^d : |y| < 1 \}$. Under the transformation $y = \Psi(x) = (\Psi^1(x), \cdots, \Psi^d(x))$, we denote

$$\mathcal{W}(y) := \tilde{w}(\Psi^{-1}(y)), \quad \mathcal{J}(y) := \frac{\partial(\Psi^1, \cdots, \Psi^d)}{\partial(x^1, \cdots, x^d)} \circ \Psi^{-1}(y), \quad |\mathcal{J}(y)| := \det \mathcal{J}(y),$$

where $\mathcal{J}(y)$ is the Jacobian determinant of the transformation. Then

$$\int_{B_R} |\nabla \tilde{w}|^2 dx \leq C \int_{B_R} |\nabla \tilde{w}|^2 + \int_{B_R} |\nabla \tilde{w}|^2.$$
and
\[ A_{ijkl}(y) := C_{ijkl} \left| J(y) \right| \partial_i \Psi^j (\Psi^{-1}(y)) \partial_k \Psi^k (\Psi^{-1}(y)), \]
\[ F^k_i(y) := \left| J(y) \right| \partial_i \Psi^k (\Psi^{-1}(y)) \tilde{f}^j_i (\Psi^{-1}(y)). \]
Then (2.18) becomes
\[ -\partial_t (A_{ijkl}(y) \partial_i \mathcal{W}^j) = \partial_t F^k_i \quad \text{in} \quad B^+_R, \quad (5.3) \]
and \( \mathcal{W} = 0 \) on \( \partial B^+_R \cap \partial \mathbb{R}^d_+ \). Let \( y_0 = \Psi(x_0) \), freeze the coefficients, and rewrite (5.3) in the form
\[ -\partial_t (A_{ijkl}(y_0) \partial_i \mathcal{W}^j) = \partial_t ((A_{ijkl}(y) - A_{ijkl}(y_0)) \partial_i \mathcal{W}^j) + \partial_t F^k_i. \]
Then, from the proof of Theorem 5.1 and the equivalence between Hölder space and Campanato space, we have that for \( 0 < R \leq 1, \)
\[ \left\| \nabla \mathcal{W} \right\|_{\gamma, B^+_R} \leq C \left( \frac{1}{R^{1+\gamma}} \| \mathcal{W} \|_{L^\infty(B^+_R)} + \left[ \mathcal{F} \right]_{\gamma, B^+_R} \right) + C \left( (A_{ijkl}(y) - A_{ijkl}(y_0)) \partial_i \mathcal{W}^j \right)_{\gamma, B^+_R}, \]
where \( \mathcal{F} := (F^k_i) \). Since \( A_{ijkl}(y) \in C^\gamma, \)
\[ \left[ (A_{ijkl}(y) - A_{ijkl}(y_0)) \partial_i \mathcal{W}^j \right]_{\gamma, B^+_R} \leq C \left( R^\gamma \left\| \nabla \mathcal{W} \right\|_{\gamma, B^+_R} + \| \nabla \mathcal{W} \|_{L^\infty(B^+_R)} \right). \]
By the interpolation inequality, one has
\[ \| \nabla \mathcal{W} \|_{L^\infty(B^+_R)} \leq R^\gamma \left\| \nabla \mathcal{W} \right\|_{\gamma, B^+_R} + \frac{C}{R} \| \mathcal{W} \|_{L^\infty(B^+_R)}, \]
where \( C = C(n) \). Hence,
\[ \left\| \nabla \mathcal{W} \right\|_{\gamma, B^+_R} \leq C \left( \frac{1}{R^{1+\gamma}} \| \mathcal{W} \|_{L^\infty(B^+_R)} + R^\gamma \left\| \nabla \mathcal{W} \right\|_{\gamma, B^+_R} + \left[ \mathcal{F} \right]_{\gamma, B^+_R} \right). \quad (5.4) \]
Since \( \Psi \) is a homeomorphism, it follows that the norms in (5.4) defined on \( B^+_R \) are equivalent to those on \( \mathcal{N} = \Psi^{-1}(B^+_R) \), respectively. Thus, rescaling back to the variable \( x \), we obtain
\[ \left\| \nabla \tilde{w} \right\|_{\gamma, \mathcal{N}'} \leq C \left( \frac{1}{R^{1+\gamma}} \| \tilde{w} \|_{L^\infty(\mathcal{N})} + R^\gamma \left\| \nabla \tilde{w} \right\|_{\gamma, \mathcal{N}'} + \left[ \tilde{\mathcal{F}} \right]_{\gamma, \mathcal{N}'} \right). \]
where \( \mathcal{N}' = \Psi^{-1}(B^+_{R/2}) \) and \( C = C(n, \gamma, \Psi) \). Furthermore, there exists a constant \( 0 < \sigma < 1 \) independent on \( R \) such that \( B_{\sigma R}(x_0) \cap Q \subset \mathcal{N}' \).
Therefore, recalling that \( \Gamma \subset \partial Q \) is a boundary portion, for any domain \( Q' \subset \subset Q \) and for each \( x_0 \in Q' \cap \Gamma \), there exist \( R_0 := R_0(x_0) \) and \( C_0 = C_0(n, \gamma, x_0) \) such that
\[ \left\| \nabla \tilde{w} \right\|_{\gamma, B_{R_0}(x_0) \cap Q'} \leq C_0 \left( R_0^\gamma \left\| \nabla \tilde{w} \right\|_{\gamma, Q'} + \frac{1}{R_0^{1+\gamma}} \| \tilde{w} \|_{L^\infty(Q)} + \left[ \tilde{\mathcal{F}} \right]_{\gamma, Q} \right). \quad (5.5) \]
Applying the finite covering theorem to the collection of \( B_{R_0/2}(x_0) \) for all \( x_0 \in \Gamma \cap Q' \), there exist finite \( B_{R_j/2}(x_j) \), \( j = 1, 2, \ldots, K \), covering \( \Gamma \cap Q' \). Let \( C_j \) be the constant in (5.5) corresponding to \( x_j \). Set
\[ \widehat{C} := \max_{1 \leq j \leq K} \{ C_j \}, \quad \widehat{R} := \min_{1 \leq j \leq K} \left\{ \frac{R_j}{2} \right\}. \]
Thus, for any $x_0 \in \Gamma \cap Q'$, there exists $j_0 \in \{1, 2, \ldots, K\}$ such that $B_{R}(x_0) \subset B_{R_j}(x_{j_0})$ and

$$\left[\nabla \tilde{w}\right]_{\gamma, B_{R}(x_0) \cap Q'} \leq \bar{C} \left( \bar{R}^{\gamma} \left[\nabla \tilde{w}\right]_{\gamma, Q'} + \frac{1}{\bar{R}^{1+\gamma}} \| \tilde{w} \|_{L^\infty(Q)} + [\bar{F}]_{\gamma, Q} \right).$$

(5.6)

Finally, we give the estimates on $Q'$. Let $C$ be the constant in (5.2) from Corollary [5.2]. Let

$$\overline{C} := \max\{\bar{C}, C\} \quad \text{and} \quad \overline{R} := \min\{3\overline{C}^{-1/\gamma}, \bar{R}\}.$$

For any $x^1, x^2 \in Q'$, there are three cases to occur:

(i) $|x^1 - x^2| \geq \frac{\overline{R}}{2}$;
(ii) there exists $1 \leq j_0 \leq K$ such that $x^1, x^2 \in B_{\bar{R}/2}(x_{j_0}) \cap Q'$;
(iii) $x^1, x^2 \in B_{\bar{R}/2} \subset Q'$.

For case (i), we have

$$\frac{|\nabla \tilde{w}(x^1) - \nabla \tilde{w}(x^2)|}{|x^1 - x^2|^{\gamma}} \leq \frac{4}{\overline{R}} \| \nabla \tilde{w} \|_{L^\infty(Q')}.$$

For case (ii), it follows from (5.6) that

$$\frac{|\nabla \tilde{w}(x^1) - \nabla \tilde{w}(x^2)|}{|x^1 - x^2|^{\gamma}} \leq C \left[\nabla \tilde{w}\right]_{\gamma, B_{\bar{R}/2}(x_{j_0}) \cap Q'} \leq C \left[\nabla \tilde{w}\right]_{\gamma, B_{\bar{R}/2}(x_{j_0}) \cap Q'}$$

$$\leq \overline{C} \left( \bar{R}^{\gamma} \left[\nabla \tilde{w}\right]_{\gamma, Q'} + \frac{1}{\bar{R}^{1+\gamma}} \| \tilde{w} \|_{L^\infty(Q)} + [\bar{F}]_{\gamma, Q} \right).$$

For case (iii), by using Corollary [5.2] one has

$$\frac{|\nabla \tilde{w}(x^1) - \nabla \tilde{w}(x^2)|}{|x^1 - x^2|^{\gamma}} \leq C \left[\nabla \tilde{w}\right]_{\gamma, B_{\bar{R}/2}} \leq \overline{C} \left( \frac{1}{\bar{R}^{1+\gamma}} \| \tilde{w} \|_{L^\infty(Q)} + [\bar{F}]_{\gamma, Q} \right).$$

Hence, in either case, we obtain

$$\left[\nabla \tilde{w}\right]_{\gamma, Q'} \leq \overline{C} \left( \bar{R}^{\gamma} \left[\nabla \tilde{w}\right]_{\gamma, Q'} + \frac{1}{\bar{R}^{1+\gamma}} \| \tilde{w} \|_{L^\infty(Q)} + [\bar{F}]_{\gamma, Q} \right) + \frac{4}{\overline{R}} \| \nabla \tilde{w} \|_{L^\infty(Q')}.$$

By the interpolation inequality, see e.g. [25] Lemma 6.32,

$$\frac{4}{\overline{R}} \| \nabla \tilde{w} \|_{L^\infty(Q')} \leq \frac{1}{3} \left[\nabla \tilde{w}\right]_{\gamma, Q'} + \frac{C}{\overline{R}^{1+\gamma}} \| \tilde{w} \|_{L^\infty(Q')}$$

$$\leq \frac{1}{3} \left[\nabla \tilde{w}\right]_{\gamma, Q'} + \frac{C}{\overline{R}^{1+\gamma}} \| \tilde{w} \|_{L^\infty(Q)},$$

where $C = C(n, \gamma)$. Since $\overline{R} \leq (3\overline{C})^{-1/\gamma}$, we get

$$\left[\nabla \tilde{w}\right]_{\gamma, Q'} \leq C \left( \| \tilde{w} \|_{L^\infty(Q)} + [\bar{F}]_{\gamma, Q} \right),$$

where $C = C(n, \gamma, Q', Q)$. By using the interpolation inequality, we obtain [25].

□
5.2. $W^{1,p}$ estimates.

Proof of Theorem 2.3 First, we give the $W^{1,p}$ interior estimates. For any ball $B_R := B_R(x_0) \subset Q$ with $R \leq 1$, since $\bar{w} \neq 0$ on $\partial B_R$, we choose a cut-off function $\eta \in C_0^\infty(B_R)$ such that

$$0 \leq \eta \leq 1 \quad \text{in } B_\rho, \quad |\nabla \eta| \leq \frac{C}{R - \rho}.$$  

One easily computes that $\eta \bar{w}$ satisfies

$$\int_{B_R} C_{ijkl} \partial_l (\eta \bar{w}^{(j)}) \partial_k \varphi^{(i)} \, dx = \int_{B_R} G_i \varphi^{(i)} \, dx + \int_{B_R} \tilde{F}^k_i \partial_k \varphi^{(i)} \, dx, \quad \forall \varphi \in C_0^\infty(B_R; \mathbb{R}^d),$$

where

$$G_i := \tilde{f}^k_i \partial_k \eta - C_{ijkl} \partial_l \bar{w}^{(j)} \partial_k \eta,$$

$$\tilde{F}^k_i := \tilde{f}^k_i \eta + C_{ijkl} \bar{w}^{(j)} \partial_i \eta.$$ 

Let $v \in H^1_0(B_R; \mathbb{R}^d)$ be the weak solution of

$$- \Delta v^{(i)} = G_i.$$  

(5.7)

We conclude that $\eta \bar{w}$ satisfies

$$\int_{B_R} C_{ijkl} \partial_l (\eta \bar{w}^{(j)}) \partial_k \varphi^{(i)} \, dx = \int_{B_R} \tilde{F}^k_i \partial_k \varphi^{(i)} \, dx, \quad \forall \varphi \in C_0^\infty(B_R; \mathbb{R}^d),$$

where $\tilde{F}^k_i := \tilde{f}^k_i + \partial_k v^{(i)}$. Since $\tilde{f}^k_i \in C^\gamma$, then $\tilde{F}^k_i \in L^p(B_R)$ for any $d \leq p < \infty$. We firstly assume that $\bar{w} \in W^{1,q}(B_R; \mathbb{R}^d)$, $q \geq 2$. Then we have

$$G_i \in L^{p \wedge q}(B_R), \quad \text{where } p \wedge q := \min\{p, q\},$$

(5.8)

$$\tilde{F}^k_i \in L^{p \wedge q^*}(B_R), \quad \text{where } q^* = \frac{dq}{d - q}.$$  

(5.9)

On the account of (5.7) and $L^2$ theory, $\partial^2 v \in L^2(B_R)$ and

$$-\Delta (\partial_k v^{(i)}) = \partial_k G_i.$$  

Then, combing with (5.8), theorem 7.1 in [24] yields $\nabla (\partial_k v^{(i)}) \in L^{p \wedge q}(B_R)$ and it follows from Sobolev embedding that $\partial_k v^{(i)} \in L^{(p \wedge q)^*}(B_R)$. Thus, from (5.2), we have $\tilde{F}^k_i \in L^{p \wedge q^*}(B_R)$. Furthermore, by using theorem 7.1 in [24] again, we have

$$\|\nabla (\eta \bar{w})\|_{L^{p \wedge q^*}(B_R)} \leq C \|\tilde{F}\|_{L^{p \wedge q^*}(B_R)},$$

where $C = C(d, \lambda, \mu, p, q)$ and $\tilde{F} := (\tilde{F}^k_i)$, $i, k = 1, \cdots, d$. Thus, from (5.8), (5.9) and the definition of $G_i$ and $\tilde{F}^k_i$, one has

$$\|\nabla \bar{w}\|_{L^{p \wedge q^*}(B_R)} \leq \frac{C}{R - \rho} (\|\tilde{w}\|_{W^{1,q}(B_R)} + \|\tilde{F}\|_{L^p(B_R)}),$$

(5.10)

where $C = C(d, \lambda, \mu, p, q)$.

Next, we prove that $\nabla \bar{w} \in L^p(B_{R/2})$. Choose a series of balls with radius

$$\frac{R}{2} < \cdots < R_k < \cdots < R_2 < R_1 < R.$$ 

In (5.10), we firstly take $\rho = R_1$ and $q = 2$, then we have

$$\|\nabla \bar{w}\|_{L^{p \wedge 2^*}(B_{R_1})} \leq \frac{C}{R - R_1} (\|\tilde{w}\|_{W^{1,2}(B_R)} + \|\tilde{F}\|_{L^p(B_R)}).$$
If \( p \leq 2^* \), the proof is completed. If \( p > 2^* \), then \( \nabla \tilde{w} \in L^{2^*}(B_{R_1}) \) and

\[
\|\nabla \tilde{w}\|_{L^{2^*}(B_{R_1})} \leq \frac{C}{R - R_1} (\|\tilde{w}\|_{W^{1,2^*}(B_R)} + \|\tilde{F}\|_{L^p(B_R)}). \tag{5.11}
\]

By taking \( R = R_1, \rho = R_2 \) and \( q = 2^* \) in [5.10] and combing with [5.11], one has

\[
\|\nabla \tilde{w}\|_{L^{p^*2^*}(B_{R_2})} \leq \frac{C}{R_2 - R_1} (\|\tilde{w}\|_{W^{1,2^*}(B_{R_1})} + \|\tilde{F}\|_{L^p(B_{R_1})})
\leq \frac{C}{(R - R_1)(R_1 - R_2)} (\|\tilde{w}\|_{W^{1,2^*}(B_R)} + \|\tilde{F}\|_{L^p(B_R)}).
\]

If \( p \leq 2^{**} \), then the proof is completed. If \( p > 2^{**} \), continuing the above argument within finite steps, one has \( \nabla \tilde{w} \in L^p(B_{R/2}) \) and

\[
\|\nabla \tilde{w}\|_{L^p(B_{R/2})} \leq C (\|\tilde{w}\|_{H^1(B_R)} + \|\tilde{F}\|_{L^p(B_{R/2})}), \tag{5.12}
\]

where \( C = C(d, \lambda, \mu, p, \text{dist}(B_R, \partial Q)) \).

Now, we give the \( W^{1,p} \) estimates near the boundary \( \Gamma \). By using the technology of locally flattening the boundary, which is same to the proof in Theorem 2.2. For simplicity, we use the same notation. Hence, we have that \( \mathcal{W}(y) := \tilde{w}(\Psi^{-1}(y)) \in H^1(B_{R_1}^+, \mathbb{R}^2) \) satisfies

\[
\int_{B_{R_1}^+} A_{ijkl}(y) \partial_i \mathcal{W}^j \partial_\nu^l \, dy = \int_{B_{R_1}^+} F_i \partial_k \varphi^l \, dy, \quad \forall \varphi \in H_0^1(B_{R_1}^+, \mathbb{R}^2).
\]

Following the proof of theorem 7.2 in [24], we obtain that for any \( d \leq p < \infty \),

\[
\|\nabla \mathcal{W}\|_{L^p(B_{R_1}^+)} \leq C (\|\mathcal{W}\|_{H^1(B_{R_1}^+)} + \|F\|_{L^p(B_{R_1}^+)})
\]

where \( C = C(\lambda, \mu, p, R, \Psi) \). Rescaling back to the original variable \( x \), we obtain

\[
\|\nabla \tilde{w}\|_{L^p(N')} \leq C (\|\tilde{w}\|_{H^1(N')} + \|\tilde{F}\|_{L^p(N')}),
\]

where \( N' = \Psi^{-1}(B_{R_1}^+), N = \Psi^{-1}(B_R) \) and \( C = C(\lambda, \mu, p, R, \Psi) \). Furthermore, there exists a constant \( 0 < \sigma < 1 \) independent on \( R \) such that \( B_{\sigma R}(x_0) \cap Q \subset N' \).

Therefore, for any \( x_0 \in Q' \cap \Gamma \), there exists \( R_0 := R_0(x_0) > 0 \) such that

\[
\|\nabla \tilde{w}\|_{L^p(B_{\sigma R_0}(x_0) \cap Q')} \leq C (\|\tilde{w}\|_{H^1(Q)} + \|\tilde{F}\|_{L^p(Q)}), \tag{5.13}
\]

where \( C = C(\lambda, \mu, p, x_0, R) \). Combining with (5.12) and (5.13) and making use of the finite covering theorem, we obtain that

\[
\|\nabla \tilde{w}\|_{L^p(Q')} \leq C (\|\tilde{w}\|_{H^1(Q)} + \|\tilde{F}\|_{L^p(Q)}),
\]

where \( C = C(\lambda, \mu, p, Q') \). Thus (2.21) follows from the interpolation inequality.

In particular, since \( \tilde{w} \) still satisfies (2.18) with \( \tilde{F} \) replacing by \( \tilde{F} - \mathcal{M} \), for any constant matrix \( \mathcal{M} = (a_{ik}), i, k = 1, 2, \cdots, d \), then, noticing that \( W^{1,p} \hookrightarrow C^{0,\tau}_{\mathcal{M}} \) for \( 0 < \tau \leq 1 - \frac{d}{p} \), (2.22) is proved. \( \square \)

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