The classical solutions of two-dimensional gravity

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Abstract

The solutions of two-dimensional gravity following from a non-linear Lagrangian \( \mathcal{L} = f(R) \sqrt{g} \) are classified, and their symmetry and singularity properties are described. Then a conformal transformation is applied to rewrite these solutions as analogous solutions of two-dimensional Einstein-dilaton gravity and vice versa.

KEY : Dilaton gravity in 1+1 dimensions, exact solutions, Birkhoff theorem, conformal transformation

Preprint UNIPO-MATH-98-June-8
to appear 1999 in: Gen. Rel. Grav. 31
rec. June 8, 98, revised February 21, 99

1 Introduction

The classical solutions of gravity theories in one spatial and one temporal dimension and their properties have been discussed recently under the following three points of view:
1. As dimensionally reduced higher-dimensional models with symmetries; the most often used example is the reduction of 3+1-dimensional spherically symmetric space-times to 1+1 dimensions.

2. As model inspired by one of the classes of string or superstring theories.

3. As a toy model for the quantization of 3+1-dimensional gravity, especially for the process of black hole evaporation.

In papers following one of these last two points of view, the emphasis is mainly on the quantization, and the classical behaviour is often only touched. Therefore, many of the classical results are hidden in footnotes or appendices to such papers.

It is the aim of the present paper to concentrate on this classical behaviour independently of an answer to the question to which of these three points of view the results shall apply. We shall start with the following model:

Let $f(R)$ be any given smooth function of the curvature scalar $R$ of a two-dimensional (Pseudo–)Riemannian manifold $V_2$ with metric $g_{ij}$, $(i,j = 1,2)$ and let $g = \det g_{ij}$.

Let

$$
\mathcal{L} = f(R) \sqrt{g}
$$

be the Lagrangian, and for compact spaces $V_2$ the corresponding action is

$$
I = \int_{V_2} \mathcal{L} \, d^2x
$$

Usually, this action integral is also written for noncompact spaces $V_2$; however, for these cases, $I$ need not be well-defined.

The Euler–Lagrange equation (equivalently called: field equation) describes the vanishing of the variational derivative of the Lagrangian eq. (1) with respect to the metric $g_{ij}$. For compact spaces $V_2$ this takes place if and only if the action $I$ eq. (2) has a stationary point there. For noncompact spaces $V_2$ the variation is made for suitably chosen subspaces only; the procedure is done in two steps as follows: First, let $K \subset V_2$ be any compact subset, then a metric $g_{ij}$ is called $K$-stationary if it makes the $K$-action

$$
I_K = \int_K \mathcal{L} \, d^2x
$$

stationary. Second, $g_{ij}$ is called stationary, if it is $K$-stationary for every such compact set $K$.

\footnote{Smooth means $C^\infty$, and if $f$ is only smooth for a certain interval of $R$-values, then we restrict the discussion to just this interval.}
The trace of the Euler–Lagrange equation reads

\[ 0 = G R - f(R) + \Box G \]  

(3)

where

\[ G = \frac{df}{dR} \]  

(4)

and \( \Box \) denotes the D’Alembertian. The remaining part of the field equation is equivalent to the vanishing of the tracefree part\(^3\) of the tensor \( G_{,kl} \), where the semicolon denotes the covariant derivative.

The curvature scalar \( R \) in \( V_2 \) is just the double of the Gaussian curvature, therefore,

\[ \int_{V_2} R \sqrt{g} \, d^2x \]

represents a topological invariant. In fact, it is a multiple of the Euler characteristic if \( V_2 \) is compact and \( g_{ij} \) is positive definite.

If \( f \) is a linear function of \( R \), then \( G \) eq. (4) is a constant and the action \( I \) eq. (2) is simply a linear combination of the volume of \( V_2 \) and its Euler characteristic. For this case, the field equation has either no solution, or every \( V_2 \) represents a solution. Therefore, we assume in the following that

\[ \frac{d^2 f}{d R^2} \neq 0 \]  

(5)

The multiplication of the action by a non–vanishing constant does not alter the set of solutions of the field equation. Together with the above we have now justified to define:

Let \( \alpha \) and \( \beta \) be two constants with \( \alpha \neq 0 \), then the functions \( f(R) \) and \( \alpha f(R) + \beta R \) are considered as equivalent.

The remaining part of this Introduction presents short comments to the cited literature: The classification given below is only a rough one due to the fact that the majority of papers contributes to more than one of the mentioned topics. And, in many cases one should have added “and the references cited there” to get a more complete reference list.

\(^2\)If one looks into the deduction of the field equation one can see: This property is a consequence of the fact that the tracefree part of the Ricci tensor identically vanishes in two dimensions.
Refs. [1-4] consider 2-dimensional gravity from the point of view as dimensionally reduced higher-dimensional gravity models. Ref. [1] compares 2- and 3-dimensional theories, [2] compares with the spherically symmetric solutions of $d$-dimensional Einstein theory ($d > 3$), and [3] with the analogous case in the $d$-dimensional Einstein–dilaton–Maxwell theory. Ref. [4] makes the dimensional reduction from Einstein’s theory in $d$ dimensions for those $d$-dimensional space–times where $n = d - 2$ commuting hypersurface–orthogonal Killing vectors exist, to metric–dilaton gravity in 2 dimensions with $n$ scalar fields.

A slightly different point of view to dimensionally reduced models can be found in [5, 6]; but also in these two papers, cosmological models are dimensionally reduced to 2-dimensional dilaton theories.

Refs. [7-14] are mainly concerned with the higher order theory in two dimensions. [7,8] deal with the discretized version in Regge calculus, [7] with Lagrangian $R^2$, [8] with the more general scale-invariant Lagrangian $R^{k+1}$.

Papers [9-14] consider the classical version of the theories following from nonlinear Lagrangian $f(R)$, and the present paper completes the discussion started in [9], with subsequent papers [10, 12, 14]. In [11], the Lagrangian $R \ln R$ plays a special role, and in [13], for $R^{k+1} + \Lambda$ the solutions have been given in closed form (but not in full generality).

Refs. [15-41] deal with Einstein–dilaton gravity in 2 dimensions mainly from the classical (i.e., non–quantum) point of view. Due to the equivalence of this theory to the above-mentioned nonlinear fourth–order theory (see e.g. [12, 34] and the present paper) many of the results are parallelly developed.

The CGHS-theory [15] was the starting point for several other papers. Further papers on this topic are [16-20]. In [18], the global behaviour of the solutions is discussed from the “kink” point of view: this refers to space–times with twisted causal structure.

Related papers have themes as follows. In Ref. [21]: wormholes; in refs. [22-29]: black holes; in refs. [30,31]: collapse behaviour; in refs. [32-35] the general solutions have been discussed; and a more general discussion about such models can be found in ref. [36-41].

Theories including torsion are discussed in refs. [42-46]; [46] represents a review about this topic.

In refs. [47-60], quantum aspects play the major role, but in all of these papers, the classical aspects are at least mentioned as a byproduct. The main topics of them are as follows: Supergravity [47], entropy [48], gravitational
anomaly [49-51], quantization procedure [52-55], and evaporation of black holes [56-59]. For more details see the review [60].

Refs. [61-68] deal with the differential geometric points of view. In [61], the inequivalence of different definitions for the stationarity of the action is shown; in [62], asymptotic symmetries are considered; in [63], warped products of manifolds and conformal transformations are used to relate several models into each other; and Killing tensors in 2 dimensions are deduced in [64].

Ref. [65] deduced curvature properties and singularity behaviour of several 1+1-dimensional space–times with one Killing vector. It is not directly related to physics, and no field equations are considered. Nevertheless, its results can be directly applied to several of the solutions of 2-dimensional gravity. The geodesics and their completeness has been discussed in refs. [13,65-68].

In [69], a canonical transformation from dilaton gravity into a free field theory is given.

The paper is organized as follows: Sections 2 till 6 deal only with the fourth–order theories according to eqs. (1, 2), Sections 7 till 10 also with dilaton theories.

In more details: Sct. 2 deals with the Birkhoff theorem in 2 dimensions and gives a coordinate–independent proof of it, see the key equation (6). Sct. 3 gives a method to integrate the field equation in Schwarzschild coordinates. The Killing vector from sct. 2 explicitly serves to simplify the deduction, see eq. (12). For the general model eq. (1) the complete solution can be given in closed form, eqs. (10, 14). Sct. 4 concentrates on the scale-invariant case, i.e. \( f(R) = R \ln R \) or \( f(R) = R^{k+1} \). Sct. 5 enumerates the corresponding solutions, and Sct. 6 describes their differential geometric properties.

In Sct. 7 both the transformation from the model eqs. (1, 2) to dilaton gravity and the corresponding back transformation are explicitly given. Sct. 8 applies this transformation to the examples discussed in Sct. 3 and 4, and gives a typical example of a field redefinition. In Sct. 9 a conformal transformation is applied which mediates between different types of dilaton gravity, and in Sct. 10, this conformal transformation is applied to the solution given in Scts. 5 and 6. Section 11 discusses the results.
2 The Killing vector

Let $\varepsilon_{ij}$ be the antisymmetric Levi–Civita pseudo–tensor in $V_2$; it can be defined as follows: In a right-handed locally cartesian coordinate system it holds: $\varepsilon_{12} = 1$. It holds

$$\varepsilon_{ij;k} = 0.$$  

Now we define with $G$ from eq. (4)

$$\xi_l = \varepsilon_{lm} G^m$$  

(6)

The Birkhoff theorem in two dimensions (see e.g. [9,23]) states that locally, every solution of the field equation possesses an isometry.

Proof: If $G$ is constant over a whole region (that means, $\xi_l = 0$ there) then because of inequality (5), $R = \text{const.}$, i.e., locally, the space is of constant curvature and possesses a 3-dimensional isometry group. If $\xi_l$ eq. (6) is a non–vanishing but light–like vector over a whole region then $V_2$ is locally of constant curvature, too. So, apart from some singular points and lines where $\xi_l$ vanishes or is light-like, the vector $\xi_l$ may be assumed to be a non-vanishing time-like or space-like vector and it suffices to show that it represents a Killing vector. To this end we calculate

$$\xi_{l;k} + \xi_{k;l} = \varepsilon_{lm} G^m_{;k} + \varepsilon_{km} G^m_{;l}$$  

(7)

The vanishing of the tracefree part of $G_{ij}$ is equivalent to the existence of a scalar $\Phi$ such that $G_{ij} = \Phi g_{ij}$. We insert this into the r.h.s. of eq. (7) and get

$$= \Phi \varepsilon_{lm} \delta^m_k + \Phi \varepsilon_{km} \delta^m_l = \Phi (\varepsilon_{lk} + \varepsilon_{kl}) = 0$$

q.e.d.

Further, it holds: The field equation if fulfilled iff (= if and only if) the trace equation (3) is fulfilled and the vector defined by eq. (6) represents a Killing vector.

Proof: It remains to show that $\xi_{l;k} + \xi_{k;l} = 0$ implies the vanishing of the tracefree part of $G_{ij}$. To this end we introduce the inverted Levi–Civita pseudo–tensor $\varepsilon^{lm}$ via

$$\varepsilon^{lm} \varepsilon_{mk} = \delta^l_k$$

3Here, a coordinate system is called locally cartesian, if at this point, $g_{ij}$ has diagonal form, and the absolute values of its diagonal terms are all equal to 1. It should be mentioned that the Levi–Civita pseudo–tensor is defined for oriented manifolds only. For the other cases there remains a sign ambiguity; however, in what follows, this ambiguity does not influence the deduction.

4 Here “field equation” is used in the sense “vacuum field equation”; the inclusion of matter would, of course, alter the result, but this is not topic of the present paper.
and get from eq. (6) \[ G^m = \varepsilon^{ml} \xi_l \] (8)

and after applying “;k” we get the requested identity. q.e.d.

As a corollary from this proof we get

\[ \Box G = 2 \varepsilon^{12} \xi_{2;1} \] (9)

### 3 Schwarzschild coordinates

In this section, we consider solutions of the field equation in such a region where the Killing vector \( \xi_l \) eq. (6) is a non–vanishing time–like or space–like vector. Then, locally, we may use Schwarzschild coordinates

\[ ds^2 = g_{ij} dx^i dx^j = \frac{dw^2}{A(w)} \pm A(w) dy^2 \] (10)

The overall change \( ds^2 \rightarrow -ds^2 \) does not represent an essential change, so we have to deal with two signatures: the upper sign in eq. (10) gives the Euclidean, the lower sign gives the Lorentzian signature. Here we concentrate on the Euclidian signature case only, but locally, an imaginary transformation \( y \rightarrow iy \) gives all the corresponding Lorentz signature solutions, too.

We assume \((w = x^1, y = x^2)\) to represent a right–handed system. Therefore, in the coordinates of metric (10) we have \( \varepsilon_{12} = 1, \varepsilon^{12} = -1 \).

From metric (10) we get

\[ R = -\frac{d^2 A}{dw^2}, \] (11)

The constant curvature cases are already excluded, so we have to assume that \( A(w) \) is not a polynomial of degree 2 or less. Under these circumstances, metric (10) has exactly one isometry, a translation into the \( y \)–direction reflecting the fact, that \( g_{ij} \) does not depend on \( y \), i.e. \( \xi^i = \alpha(0, 1) \), where \( \alpha \) is a non-essential non-vanishing constant. We get

\[ \xi_i = (0, \alpha A(w)) \] (12)

We insert eq. (12) into eq. (8) and get \( G_{;1} = -\alpha \), i.e., \( G \) is a linear but not constant function of \( w \). By a linear transformation of \( w \) which does not alter the ansatz eq. (10) we can achieve \( G(R) = w \). This equation can be, at least locally, inverted to

\[ R = \psi(w). \]

\[ ^5 \text{with } \alpha = -1 \text{ this is consistent with eq. (6)} \]
From eq. (9) we get \( \Box G = \frac{dA}{dw} \), and then the trace (3) of the field equation reads

\[
0 = w \cdot \psi(w) - f(\psi(w)) + \frac{dA}{dw}
\]  \hspace{1cm} (13)

We introduce the integration constant \( C \) and get

\[
A(w) = C + \int f(\psi(w)) - w \cdot \psi(w) dw
\]  \hspace{1cm} (14)

which represents the general solution to the field equation.

**Example:**

Let us take \( f(R) = e^R \) and apply the above procedure to this Lagrangian.

From eq. (4) we get \( G = e^R = w > 0 \), i.e., \( R = \psi(w) = \ln w \) and \( \dot{f}(\psi(w)) = w \). From eq. (14) we get

\[
A(w) = C - \frac{w^2}{2} \ln w + \frac{3w^2}{4}
\]

via \( \frac{dA}{dw} = w - w \ln w \), so \( A(w) \) has a local extremum at \( w = e \), i.e. \( R = 1 \). It turns out to be a maximum. In dependence on the value of \( C \), different types of solutions can be constructed. \( \Box \) In contrast to the models to be discussed in the next sections, \( R = 0 \) does not play a special role here.

## 4 Scale-invariant fourth-order gravity

A field equation is scale invariant if the following holds: If a metric \( g_{ij} \) is a solution and \( c \neq 0 \) a constant, then the homothetically equivalent metric \( c^2 g_{ij} \) is a solution, too. It holds, cf. [12]: \( f(R) = R \ln |R| \) and \( |R|^{k+1} \) represent the only cases that lead to scale-invariant field equations. We define for an arbitrary real \( k \neq -1 \)

\[
f(R) = \begin{cases} 
R \ln |R| - R & k = 0 \\
\frac{R^{1+k}}{1+k} R^{|R|^k} & k \neq 0 
\end{cases}
\]  \hspace{1cm} (15)

This covers all cases with a scale-invariant field equation. In most cases, \( R = 0 \) represents a singular point of the field equation, and for those cases

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\( ^6 \)This is a quite general property of these models: Adding a constant \( C \) to \( A \) eqs. (10,14), then these two space-times are no more isometric in general, but, according to eq. (11), they have the same curvature. This is also interesting from the purely differential geometric point of view: under which circumstances the curvature uniquely determines the metric, cf. e.g. [65]
we restrict to the range $R \neq 0$ and allow $R \to 0$ in the solutions only afterwards. From eq. (15) we get with (4)

$$G(R) = \frac{df}{dR} = \begin{cases} \ln |R| & k = 0 \\ |R|^k & k \neq 0 \end{cases}$$

(16)

The trace (3) of the field equation reads

$$0 = R + \Box (\ln |R|), \; k = 0$$

(17)

$$0 = R + \left(1 + \frac{1}{k}\right) |R|^{-k} \Box (|R|^k), \; k \neq 0$$

(18)

Instead of explicitly solving equation (14) it proves useful to insert (10) into (17), (18), and we get

$$\frac{dA}{dw} \cdot \frac{d^3A}{dw^3} = \frac{1}{k + 1} \left(\frac{d^2A}{dw^2}\right)^2$$

(19)

This equation is valid for all values $k$ and for both signs of $R$. Even the limit $k \to \pm \infty$ makes sense: In this limit we get $d^3A/dw^3 = 0$ which is equivalent to the requirement that the 2-space is of constant curvature.

It is remarkable that the case $k = 0$ from equation (15) is now smoothly incorporated in equation (19).

The limit $k \to -1$ leads to $d^2A/dw^2 = 0$, i.e. flat space.

5 Solutions of scale-invariant gravity

Equation (19) has a 3-parameter set of solutions. The integration constants are $C, D, E$, and we get

$$A(w) = C + D \cdot \ln |w - E|, \; k = -\frac{1}{2}$$

(20)

$$A(w) = C + D \cdot e^{Ew}, \; k = 0$$

(21)

$$A(w) = C + D \cdot |w - E|^{2+\frac{k}{2}} \text{ else}$$

(22)

Now, the two signatures of the metric have to be distinguished.
5.1 The Euclidean signature

The following examples of solutions are especially interesting:

\[ k = -\frac{1}{2} : \quad ds^2 = \frac{dw^2}{\ln w} + \ln w \, dy^2, \quad w > 1, \quad R = \frac{1}{w^2} \quad (23) \]

\[ k = 0 : \quad ds^2 = \frac{dw^2}{e^w} + e^w \, dy^2, \quad R = -e^w \quad (24) \]

and for the other \( k \)-values:

\[ ds^2 = \frac{dw^2}{w^{2+\frac{1}{k}}} + w^{2+\frac{1}{k}} \, dy^2, \quad w > 0, \quad R = -w^{\frac{1}{k}} \left( 1 + \frac{1}{k} \right) \left( 2 + \frac{1}{k} \right). \quad (25) \]

Equations (24), (25) can together be written in conformal coordinates as follows

\[ ds^2 = x^{\frac{1}{k+1}} \cdot \frac{dx^2 + dy^2}{x^2}, \quad x > 0 \quad (26) \]

In synchronized coordinates this reads

\[ ds^2 = dr^2 + r^{-4k-2} \, dy^2, \quad r > 0 \quad (27) \]

However, in both cases, the limit \( k \to -1/2 \) does not lead to the solution eq. (23). A direct calculation shows the following: eq. (23) can be written in conformal or synchronized coordinates only by the use of logarithmic integrals.

With these examples eqs. (23 - 25), however, the set of solutions is not exhaustet. Let us complete their enumeration as follows: \( D = 0 \) gives flat space only, so we exclude this case. A translation of \( w \) can be used to get \( E = 0 \) in (20, 22), and to get \( C/D \in \{-1, 0, 1\} \) in (21). Then \( w \) can be multiplied by a suitable factor to get \( C/D \in \{-1, 0, 1\} \) in (20, 22) and to get \( E = 1 \) in (21).

Finally, we apply the following point of view: homothetically equivalent metrics are considered as equivalent. So, \( A \) can be multiplied by any non-vanishing constant (and \( y \) has to be divided by the same constant). So, we always get \( D \in \{-1, 1\} \), and we get a finite list of solutions as follows:

For \( k = -1/2 \): With \( C = 0 \) we get (23) and its counterpart

\[ ds^2 = \frac{dw^2}{|\ln w|} + |\ln w| \, dy^2, \quad 0 < w < 1, \quad R = -\frac{1}{w^2} \quad (28) \]

With \( C \in \{-1, 1\} \) we get 4 further metrics, however: all of them are isometric to (23) or (28) which can be seen by multiplying or dividing \( w \) by \( e \).
For \( k = 0 \): With \( C = 1 \) we get
\[
d s^2 = \frac{d w^2}{1 \pm e^w} + (1 \pm e^w) d y^2, \quad (w < 0 \text{ for sign}), \quad R = \mp e^w, \tag{29}
\]
With \( C = -1 \) we get
\[
d s^2 = \frac{d w^2}{e^w - 1} + (e^w - 1) d y^2, \quad w > 0, \quad R = -e^w, \tag{30}
\]
i.e., together with metric (24) we have 4 cases.

For all other values \( k \neq -1 \): With \( C = 1 \) we get
\[
d s^2 = \frac{d w^2}{1 \pm w^{2+\frac{1}{k}}} + \left(1 \pm w^{2+\frac{1}{k}}\right) d y^2, \tag{31}
\]
and with \( C = -1 \)
\[
d s^2 = \frac{d w^2}{w^{2+\frac{1}{k}} - 1} + \left(w^{2+\frac{1}{k}} - 1\right) d y^2, \tag{32}
\]
i.e., for every value \( k \) we have together with metric (25) again 4 cases. The range of \( w \) has to be chosen within the interval \( w > 0 \) such that \( A > 0 \). \( R \) is, up to the sign, always the same as in eq. (25).

If \( 1/k \) represents an odd natural number, then the two signs of eq. (31) give solutions which can be smoothly pasted together at the line \( w = 0 \) (after redefinition \( w \rightarrow -w \) in one of the parts), so the number of solutions is reduced by one for these cases.

5.2 The Lorentzian signature

As already mentioned, the corresponding Lorentzian solutions can be obtained from the Euclidean ones by an imaginary rotation of the coordinate \( y \). However, after this rotation, we have more cases where solutions can be pasted together. This is connected with regular zeroes of the function \( A \), where the character of the Killing vector \( \xi_k \) changes from time-like to space-like. The line where this happens is called horizon. Let us summarize these solutions:

\[
k = \frac{-1}{2}, \quad d s^2 = \frac{d w^2}{\ln w} - \ln w \, d y^2, \quad w > 0 \tag{33}
\]
represents the Wick rotated solutions (23) and (28) pasted together at \( w = 1 \).
\[ k = 0 : \quad ds^2 = \frac{dw^2}{e^w} - e^w dy^2, \quad (34) \]

is taken from (24), and

\[ k = 0 : \quad ds^2 = \frac{dw^2}{1 \pm e^w} - (1 \pm e^w) dy^2, \quad (35) \]

puts together solutions (29, 30).

For the other values \( k \neq -1 \) we get:

\[ ds^2 = \frac{dw^2}{w^{2+\frac{1}{k}}} - w^{2+\frac{1}{k}} dy^2, \quad (36) \]

from (25) and (31, 32) can be pasted together as follows:

\[ ds^2 = \frac{dw^2}{1 \pm w^{2+\frac{1}{k}}} - \left(1 \pm w^{2+\frac{1}{k}}\right) dy^2. \quad (37) \]

The remark from the end of subsection 5.1. applies to this solution, too.

Finally, it should be mentioned, that we considered solutions being related by the transformation \( ds^2 \rightarrow -ds^2 \) as equivalent ones. (This transformation does not change the signature of the solutions, but it changes the sign of the curvature scalar.)

## 6 Properties of these solutions

In this section, we discuss the properties of the solutions found in sect. 5.

### 6.1 The curvature invariants

First of all, let us calculate the curvature invariants using metric (10) and \( R \) from eq. (11). We get

\[ R_{ij} R^{ij} = A \left(\frac{d^3 A}{dw^3}\right)^2 \quad (38) \]

and

\[ \Box R = -A \frac{d^4 A}{dw^4} - \frac{dA}{dw} \frac{d^3 A}{dw^3} \quad (39) \]
By use of eq. (19) we eliminate $d^2A/dw^3$ and get

$$R;_i R^{;i} = \frac{A}{(dA/dw)^2} \cdot \frac{R^4}{(k+1)^2}$$

(40)

Applying $\frac{d}{dw}$ to eq. (19) we can also eliminate $d^4A/dw^4$ and get by use of eq. (40)

$$R \square R = - \frac{R^3}{k+1} - (k-1) R;_i R^{;i}$$

(41)

Neither $A$ nor $dA/dw$ have an invariant meaning because they can be changed by a coordinate transformation $w \rightarrow \alpha w$. However, as can be seen from eq. (40), for $R \neq 0$ the quotient $A/(dA/dw)^2$ possesses an invariant meaning after $k$ has been fixed.

For the higher order curvature invariants we get a result analogous to eq. (41): After $k$ has been fixed, all of them can be expressed as a function of $R$ and $R;_i R^{;i}$. For $k \neq 1$, we can also say: Every invariant can be expressed by use of $R$ and $\square R$. So, within these models we get: If one of the curvature invariants diverges, then already $R$ diverges. In other words: $R$ alone is sufficient to decide whether a curvature singularity exists.

### 6.2 Selfsimilarity of solutions

A solution $ds^2$ of the field equation is called selfsimilar, if for every constant $\alpha > 0$ the homothetically equivalent metric $\alpha ds^2$ is isometric to $ds^2$. It holds: A space of constant curvature is selfsimilar iff $R = 0$. From the solutions of sct. 5 the following ones are selfsimilar: (24), (25), consequently also (26), (27), (34) and (36). The remaining ones – which include all solutions with a horizon – are not selfsimilar: (23), (28 - 33), (35) and (37).

### 6.3 Geodesics

To get a better knowledge about these solutions it proves useful to calculate their geodesics. This is necessary, because in many cases, a coordinate tends to infinity without describing an infinite distance. For the Lorentzian case, one has also to distinguish between completeness of the 3 types of geodesics, the result may be different, cf. e.g. [13,44,66-68].

#### 6.3.1 The case $k = -1/2$

Metric (23) represents a geodesically complete surface of infinite surface area and topology $E^2$ of the Euclidean plane if one point $[w = 1]$ is added as
symmetry center. This can be seen as follows: Starting from an arbitrary point, a geodesic with finite natural parameter ends always at a finite value of the coordinate $w$, so $w \to \infty$ makes no problem.

To analyze the neighbourhood of $w = 1$ we introduce new coordinates $z = 2\sqrt{w - 1}$ and a $2\pi$-cyclic coordinate $\phi = y/2$. The eq. (23) reads

$$ds^2 = \frac{dz^2}{1 - z^2/8 + \Sigma} + z^2(1 - z^2/8 + \Sigma)d\phi^2$$

where

$$\Sigma = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1} \cdot \frac{z^{2n}}{4^n}$$

which is regular as $z \to 0$ and rotationally symmetric with $[z = 0]$ as center of symmetry.

Metric (28) can be analyzed quite similarly: If we put $v = 1/w$ then we get

$$ds^2 = \frac{dv^2}{v^4 \ln v} + \ln v dy^2, \quad v > 1, \quad R = -v^2$$

and the behaviour for $v \to 1$ is the same as for $w \to 1$ in the above example from eq. (23). But for $v \to \infty$ we have a curvature singularity in a finite invariant distance which can be seen by calculating the length of the geodesic $[y = 0]$. The surface has topology $E^2$ and surface area $4\pi$.

The Lorentzian signature solution (33) can be analyzed by introducing Bondi coordinates, i.e., for $w > 1$ we define

$$u = y - \int \frac{dw}{\ln w}$$

instead of $y$, and then we apply an analytic continuation to the whole interval $[w > 0]$ afterwards. We get (cf. the remark at the end of sct. 5.2.)

$$ds^2 = \ln w du^2 + 2du dw, \quad R = \frac{1}{w^2}$$

The line $[w = 1]$ represents a regular horizon; $[w = 0]$ is a curvature singularity, it will be reached after finite invariant length of a geodesic, so it represents a true curvature singularity; and for $w \to \infty$ the space is asymptotically uncurved but not asymptotically flat.
6.3.2 The Euclidean cases $k \neq -1/2, C = 0$

The cases with $C = 0$, where $k \neq -1/2$ and $k \neq -1$, can be analyzed in synchronized coordinates. For

$$ds^2 = dr^2 + a^2(r)dy^2$$

we have

$$R = -\frac{2}{a} \cdot \frac{d^2a}{dr^2}$$

For metric (27) we have $a(r) = r^{-2k-1}$, i.e.

$$R = -\frac{4}{r^2}(k+1)(2k+1)$$

leading to a true curvature singularity as $r \to 0$ which can be reached at finite invariant geodesic distance. For $r \to \infty$ the space is asymptotically uncurved.

6.3.3 The Lorentzian cases $k \neq -1/2, C = 0$

These cases are described by eq. (36), but the properties can easier be analyzed by writing it in synchronized coordinates, which leads to a Wick-rotated form of metric (27). The curvature behaves exactly as in subsection 6.3.2.

6.3.4 The Euclidean cases $k \neq -1/2, C \neq 0$

For metric (29-32) the properties are as follows:

Metric (29) (i.e. $k = 0, C = 1$):

upper sign: $w \to \infty$ represents a true curvature singularity in finite invariant distance.

lower sign: $w \to -\infty$ gives an asymptotically flat surface, whereas the limit $w \to 0$ gives a regular surface if one point $[w = 0]$ will be attached as center of symmetry and the coordinate $y$ will be considered to be a cyclic one. This is analogous to the case with $k = -1/2$ discussed above, the only difference is that we have now $(n+1)!$ instead of $n+1$ in the denominator of $\Sigma$:

$$ds^2 = \frac{dz^2}{1 - z^2/8 + \Sigma} + z^2(1 - z^2/8 + \Sigma)d\phi^2$$

where

$$\Sigma = \sum_{n=2}^{\infty} \frac{(-1)^n}{(n+1)!} \cdot \frac{z^{2n}}{4^n}$$
which is regular as \( z \to 0 \) and rotationally symmetric with \( [z = 0] \) as center of symmetry.

Metric (30) (i.e. \( k = 0, C = -1 \)):

has a true curvature singularity as \( w \to \infty \) which can be reached at finite geodesic distance. The behaviour near \( w = 0 \) is similar as for eq. (29).

Metrics (31) and (32) (i.e. \( k \neq 0, C \neq 1 \)):

They have the curvature scalar

\[
R = \pm w^{1/k} \left( 2 + \frac{1}{k} \right) \left( 1 + \frac{1}{k} \right)
\]

For \( k > 0 \) we get: \( R \) diverges as \( w \to \infty \), and this will be reached at finite geodesic distance, so we have a true curvature singularity. This applies to (31), upper sign, and to (32). For (31), lower sign, we have to restrict to the interval \( w < 1 \). To get a regular behaviour there, we have again to attach one additional point \( [w = 1] \) as center of symmetry, and to make \( y \) a cyclic coordinate. For these cases, the solution is regular as \( w \to 0 \). The analytic continuation to negative values \( w \) is possible if \( 1/k \) is an integer. Let us present two typical examples:

\( k = 1 \), then metric (31), lower sign reads

\[
ds^2 = \frac{dw^2}{1 - w^3} + \left( 1 - w^3 \right) dy^2, \quad w < 1
\]

which has a singularity at finite distance as \( w \to -\infty \). The other cases with odd \( 1/k \) are similar.

Even \( 1/k \), \( k = \frac{1}{2(n-1)} \) with an integer \( n \geq 2 \), then metric (31), lower sign reads

\[
ds^2 = \frac{dw^2}{1 - w^{2n}} + \left( 1 - w^{2n} \right) dy^2, \quad -1 < w < 1
\]

This is a metric with mirror symmetry at \( w = 1 \), so we have to attach two points, \( w = 1 \) and \( w = -1 \), as the two centers of the rotational symmetry. This represents a regular solution with finite volume and spherical topology \( S^2 \). The coordinate \( y \) has to be cyclic with period \( 2\pi/n \) to ensure local regularity at \( w = 1 \), therefore the total volume equals \( 4\pi/n = \frac{8\pi k}{2k+1} \). As a test one can calculate that really \( \int R \sqrt{\gamma} d^2x = 8\pi \) as requested from the Gauss-Bonnet theorem. (The case \( n = 1 \) would give the standard \( S^2 \) of constant curvature.) For these values \( k \), the field equation has a singular point at \( R = 0 \); the regular solution given here has always \( R \geq 0 \), and this singular point is only touched at the line \([w = 0]\).
If $1/k$ is not an integer, then no smooth continuation to negative values $w$ is possible.

For $k < 0$ it holds: $R$ diverges as $w \to 0$ which can be reached after finite invariant distance. At $R = 0$, there is a singular point of the field equation.

### 6.3.5 The Lorentzian cases $k \neq -1/2, C \neq 0$

These cases are described by eq. (337). With the upper sign, the analysis is exactly as in sect. 6.3.4., for the lower sign we have additionally to analyze the horizons at $w = 1$. Let again $2 + \frac{1}{k} = 2n$, then

$$ds^2 = \frac{dw^2}{1-w^{2n}} - (1-w^{2n})dy^2$$

goes over to

$$ds^2 = (1-w^{2n})du^2 + 2du
dv$$

via Bondi coordinates

$$u = y - \int \frac{dw}{1-w^{2n}}$$

which proves regularity via crossing the horizon at $w = 1$ for positive integers $n$.

### 7 Transformations relating to dilaton gravity

Now we give a relation of fourth-order gravity, section 1 to 6, to dilaton gravity. This relation is possible for those regions where $G \neq 0$, cf. eqs. (1,4). Due to inequality (5) the equation $G = 0$ can be fulfilled at singular lines only. Therefore, the transformation to be deduced below will be valid “almost everywhere”.

#### 7.1 From fourth–order to dilaton gravity

Without loss of generality let $G > 0$, otherwise we simply change $f(R)$ to $-f(R)$, cf. eq. (4). Then we define $\phi$ by

$$e^{-2\phi} = G(R) .$$

We invert this relation (which can locally be done because of inequality (5)) to $R = R(\varphi)$ and define

$$V(\varphi) = e^{-2\varphi}R(\varphi) - f(R(\varphi))$$

(42)
Then the Lagrangian eq. (1) can be written as

\[ L(\varphi, g_{ij}) = [e^{-2\varphi} R - V(\varphi)] \sqrt{g} \]  \hspace{1cm} (43)

Now we forget for a moment how we deduced eq. (43) and take it as given Lagrangian. The variation of \( L \) eq. (43) with respect to \( \varphi \) gives

\[ 0 = 2e^{-2\varphi} R + \frac{dV}{d\varphi} \]  \hspace{1cm} (44)

The variation of this \( L \) with respect to \( g_{ij} \) has the trace

\[ 0 = V(\varphi) + \Box(e^{-2\varphi}) \]  \hspace{1cm} (45)

and the traceless part of \( (e^{-2\varphi})_{lm} \) has to vanish.

This is the transformation from fourth-order gravity to dilaton gravity, which can already be found in several of the cited papers, e.g. [11][7].

7 This paper by Solodukhin is the published version of the preprint from 1994 cited in ref. [12].

7.2 From dilaton gravity to fourth-order

To go the other direction – which seems not to be worked out so explicitly up to now – let us start from dilaton gravity eq. (43) and calculate by eq. (44)

\[ R = -\frac{1}{2} e^{2\varphi} \frac{dV}{d\varphi} \]  \hspace{1cm} (46)

Now we have to distinguish two cases: If this \( R \) is constant, i.e., eq. (43) reads

\[ L = e^{-2\varphi}(R - \Lambda) \sqrt{g} \]

which is the Jackiw–Teitelboim theory, then the transformation to fourth-order gravity is impossible. In all other cases, we can locally invert eq. (46) to \( \varphi = \varphi(R) \), and then we insert this into eq. (43) and get

\[ \mathcal{L} = f(R) \sqrt{g} = L(\varphi(R), g_{ij}) \]  \hspace{1cm} (47)

which is the equivalent fourth-order theory eq. (1).

This transformation transforms not only the actions into each other, but also the solutions of the field equation – unless they belong to the mentioned singular exceptions – are transformed into each other.
**Example**: Let us start from dilaton gravity eq. (43) with a potential

\[ V(\varphi) = \frac{1}{2} \sin(2e^{-2\varphi}) \]

From eq. (46) we get \( R = \cos(2e^{-2\varphi}) \), i.e.

\[ \varphi = -\frac{1}{2} \ln\left(\frac{1}{2} \arccos R\right) \]

leading via eq. (47) to

\[ f(R) = \frac{R}{2} \arccos R - \frac{1}{2} \sqrt{1 - R^2} \]

### 8 Examples for dilaton gravity

Before we turn to the really interesting cases, we try to elucidate the procedure by applying it to the example \( f(R) = e^R \) given at the end of sect. 3: On gets

\[ G = e^R = e^{-2\varphi}, \text{ i.e. } R = -2\varphi \]

With eq. (42) we get \( V(\varphi) = e^{-2\varphi}(-1 - 2\varphi) \) and then eq. (43) reads

\[ L(\varphi, g_{ij}) = e^{-2\varphi}[R + 1 + 2\varphi]\sqrt{g}. \]

#### 8.1 The scale-invariant case

Let us start from the fourth-order theory defined by eqs. (1, 15) with \( k \neq -1 \). We get from \( \varphi = -\frac{1}{2} \ln G \) and eq. (16)

\[ R(\varphi) = \begin{cases} 
\pm \exp\left(e^{-2\varphi}\right) & \text{if } k = 0 \\
\pm e^{-2\varphi/k} & \text{if } k \neq 0
\end{cases} \quad (48) \]

where the upper sign corresponds to the case \( R > 0 \), and for \( k = 0 \) we restrict to the range \( |R| > 1 \).

Application of equation (42) gives

\[ V(\varphi) = \begin{cases} 
\pm \exp\left(e^{-2\varphi}\right) & \text{if } k = 0 \\
\pm \frac{k}{1+k} \exp\left(-2\varphi\left(1 + \frac{1}{k}\right)\right) & \text{if } k \neq 0
\end{cases} \quad (49) \]

*The factors “2” are inserted for convenience only, and we discuss the range \( 0 < \exp(-2\varphi) < \pi/4 \), i.e. \( 0 < R < 1 \); putting \( e^{-2\varphi} = \Phi \), as will be done in section 8.3., the potential is just \( \sin \Phi \).
By a suitable translation of $\varphi$, the factor $\frac{k}{k+1}$ can be made vanish, and so we get from eqs. (43) and (49)

$$L(\varphi, g_{ij}) = [e^{-2\varphi} R \mp \exp(e^{-2\varphi})] \sqrt{g}, \; k = 0$$

(50)

and

$$L(\varphi, g_{ij}) = e^{-2\varphi} \left(R \mp e^{-2\varphi/k}\right) \sqrt{g}, \; k \neq 0$$

(51)

This is the well-known dilaton gravity in an exponential potential, whereas equation (50) is the tree-level string action.

### 8.2 Addition of a divergence

Let us now look what happens if we apply the transformations mentioned at the end of sct. 1 before the conformal transformation of sct. 7 is carried out: The factor $\alpha$ gives nothing but an irrelevant translation of $\varphi$. However, $\beta$ will influence the potential as follows: We get $G + \beta$ instead of $G$ and

$$-\frac{1}{2} \ln(e^{-2\varphi} + \beta)$$

instead of $\varphi$ which represents the transformation called “new non-conformal extra symmetry” in J. Crux et al., Phys. Lett. B 402 (1997) 270.

### 8.3 A field redefinition for $\varphi$

Sometimes, the dilaton is written as $\Phi = e^{-2\varphi}$; this represents only a field redefinition, so all other properties remain unchanged. Instead of (43) we get

$$\dot{L}(\Phi, g_{ij}) = [\Phi R - \dot{V}(\Phi)] \sqrt{g}$$

where $\dot{V}(\Phi) = V(\varphi)$ at $\Phi = e^{-2\varphi}$. Variation of this Lagrangian with respect to $\Phi$ gives

$$R = \frac{d\dot{V}}{d\Phi}$$

being equivalent to (44). From the variation with respect to the metric we get that the traceless part of $\Phi_{,lm}$ has to vanish, and its trace eq. (45) now simply reads

$$0 = \dot{V}(\Phi) + \Box \Phi$$

so, from eq. (50) we get the $k = 0$-result

$$\dot{L}(\Phi, g_{ij}) = [\Phi R \mp e^\Phi] \sqrt{g}, \; \Phi = \ln |R|$$
and from eq. (51) the remaining cases
\[ \hat{L}(\Phi, g_{ij}) = \Phi(R \mp \Phi^{1/k})\sqrt{g}, \quad \Phi = |R|^k \]
Again, \( k \to \pm\infty \) gives the Jackiw-Teitelboim theory.

The other example, \( f(R) = e^R \), leads to
\[ \hat{L}(\Phi, g_{ij}) = \Phi(R + 1 - \ln \Phi)\sqrt{g}. \]

9 A conformal transformation

The theory defined by the Lagrangian (43) is often rewritten in a conformally related metric
\[ \tilde{g}_{ij} = e^{-2\varphi}g_{ij} \tag{52} \]
One should observe that the conformal factor uniquely depends on the curvature scalar \( R \) of the metric \( g_{ij} \) because of \( e^{-2\varphi} = G(R) \), cf. sct. 7. This conformal relation is globally defined. \[ \]

We use the abbreviation \( (\tilde{\nabla}\varphi)^2 = \tilde{g}^{lm}\varphi_l\varphi_m \) and get from equation eq. (43) via the conditions \( L = \tilde{L}, \sqrt{\tilde{g}} = e^{-2\varphi}\sqrt{g} \) and
\[ R = e^{-2\varphi}[\tilde{R} + 4(\tilde{\nabla}\varphi)^2] \tag{53} \]
which follows from (52) now
\[ \tilde{L} = \left(e^{-2\varphi}[\tilde{R} + 4(\tilde{\nabla}\varphi)^2] - e^{2\varphi}V(\varphi)\right)\sqrt{\tilde{g}} \tag{54} \]
The examples discussed above are transformed to:

\[ k = 0 : \quad \tilde{L} = \left(e^{-2\varphi}[\tilde{R} + 4(\tilde{\nabla}\varphi)^2] \mp \exp(2\varphi + e^{-2\varphi})\right)\sqrt{\tilde{g}} \]
and
\[ k \neq 0 : \quad \tilde{L} = \left(e^{-2\varphi}[\tilde{R} + 4(\tilde{\nabla}\varphi)^2] \mp \exp(-2\varphi/k)\right)\sqrt{\tilde{g}} \]
The case \( k = 0 \) corresponds to Liouville gravity, and \( k = 1 \), which has \( f(R) = \frac{1}{2}R^2 \), to the CQHS-model [15].

To transform the example from the end of sct. 3 \( \mathcal{L} = e^R\sqrt{g} \) we get
\[ \tilde{L} = \left(e^{-2\varphi}[\tilde{R} + 4(\tilde{\nabla}\varphi)^2] + 2\varphi + 1\right)\sqrt{\tilde{g}} \]
i.e., simply a linear potential in this conformal picture.

\[ \text{Therefore, this conformal relation is not only different but also different in character to the conformal relation following from the property that all two-spaces are locally conformally flat – because in general, the conformal factor for the latter is not globally defined.} \]

21
10 Conformal transformation of solutions of scale-invariant gravity

In this section, we transform some of the solutions of sections 5 and 6 to the form \( ds^2 \) according to section 9.

\[
d\tilde{s}^2 = e^{-2\varphi} ds^2, \quad e^{-2\varphi} = G(R) = \begin{cases} 
\ln |R| & k = 0 \\
|R|^k & k \neq 0
\end{cases}
\]

according to equation (16). From eq. (23) we get (always up to irrelevant constant factors)

\[
d\tilde{s}^2 = \frac{w}{\ln w} dw^2 + w \ln w dy^2, \quad \varphi = \frac{1}{2} \ln w, \quad k = -\frac{1}{2}
\]

from eq. (24) we get

\[
d\tilde{s}^2 = \frac{w}{e^w} dw^2 + we^w dy^2, \quad k = 0
\]

and from eq. (25) we get

\[
d\tilde{s}^2 = \frac{dw^2}{w^{1+\frac{1}{k}}} + w^{3+\frac{1}{k}} dy^2
\]

for the other values \( k \neq -1 \).

The black hole solution given at the end of sct. 6.3.1. has with \( k = -1/2 \) now \( e^{-2\varphi} = |R|^{-1/2} = w \), i.e.

\[
d\tilde{s}^2 = w \ln w \ du^2 + 2w \ du \ dw, \quad \varphi = -\frac{1}{2} \ln w
\]

keeping the regularity at the horizon \( w = 1 \).

The regular Euclidean solution given at the end of sct. 6.3.4. has \( k = 1/2 \) \( R = 12 w^2 \), hence \( e^{-2\varphi} = \sqrt{R} = 2\sqrt{3} |w| \), so the conformal transformation breaks down at \( w = 0 \), just that line, where the fourth–order field equation had its singular points.

In the metrics of this section, \( w \) is no more a coordinate giving the metric in Schwarzschild form, however, a coordinate change \( \tilde{w} = \tilde{w}(w) \) can simply be calculated to get that form.
11 Discussion

Let us show some unexpected relations of space-times discussed in this paper to models discussed from other points of view:

1. From the example at the end of sct. 7 (i.e., that one which has in the dilaton version simply the potential $\sin \Phi$) we can find the solution with the method of sct. 3 as follows: We get $G = \frac{1}{2} \arccos R$, i.e., $R = \cos(2w)$, and by eq. (14) $A(w) = C + \cos(2w)$. For the case $C = 0$, the metric reads

$$ds^2 = \frac{dw^2}{\cos(2w)} - \cos(2w)dy^2$$

Replacing $w$ by $z$ according to $2w = \arctan \sinh(2z)$ we get

$$ds^2 = dy^2 - dz^2 \cosh(2z)$$

In a next coordinate transformation we change from the hyperbolic coordinates $y, z$ ($z$ is the parameter for the hyperbolic rotation), to conformal cartesian coordiantes via

$$t = e^y \cosh z, \quad x = e^y \sinh z$$

and get the metric extensively discussed in ref. 18 from a totally different origin:

$$ds^2 = \frac{dt^2 - dx^2}{t^2 + x^2}.$$ 

2. The metric (35), here deduced as solution for the Lagrangian $R \ln R$, has several seemingly unrelated origins: from $c = 1$ Liouville gravity, from non-critical string theory, from a bosonic sigma model, and from $k = 9/4$ gauged $SO(2,1)/SO(1,1)$ WZW model see e.g. refs. [22,24,34] for further details.

3. Finally, one should note that also 2-dimensional gravity with torsion is equivalent to special types of generalized 2-dimensional dilaton gravity, cf. [44,45].

Acknowledgement

I thank Claudia Bernutat for independently checking the essential calculations and Miguel Sanchez for useful comments. Financial supports from DFG and HSP III are gratefully acknowledged.
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