The length of a compact extra dimension from shadow

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To explore the possible clues for the extra dimension from the Event Horizon Telescope (EHT) observations, we study the shadow of the rotating 5D black string in General Relativity (GR). Instead of investigating the shadow in the effective 4D theory, we concern the motion of photons along the extra dimension z with a conserved momentum $P_z$, which appears as an effective mass in the geodesic equations of photons. The existence of $P_z$ enlarges the photon regions and the shadow of the rotating 5D black string while it has slight impact on the distortion. Considering the momentum $P_z$ along the extra dimension is same for all the photons, we constrain the dimensionless quantity $cP_z/E_0 \lesssim 0.35$ from the lower bound 39 $\mu$as of the angular diameter in the EHT observations of M87*. However, for an infinite extra dimension, there is no reason to keep all the photons in same $P_z/E_0$, and the scenario permitting all values of $P_z/E_0$ within $0 \sim 1$ is contradictory with the observations. Nevertheless, a compact extra dimension can give a reasonable excuse for the discrete values of $P_z$, and using the upper bound 45 $\mu$as for the ring diameter of M87* we can obtain the range for the length of the compact extra dimension as $2.03125 \text{mm} \lesssim \ell \lesssim 2.6 \text{mm}$ (radius 0.323283 mm $\lesssim R_\ell \lesssim 0.413803$ mm). This is a very interesting result for the first consideration, which supports the hypothesis that the extra dimension is compact to avoid the Gregory-Laflamme (GL) instability. Similarly, from the observations of SgrA*, the constraints $2.28070 \text{mm} \lesssim \ell \lesssim 2.6 \text{mm}$ and $2.13115 \text{mm} \lesssim \ell \lesssim 2.6 \text{mm}$ can be given by the upper bounds of the emission ring and the angular shadow diameter respectively.

Introduction.—The deflection of light near a massive object has been verified since 1919, as a prediction of GR. While in the spacetime near a black hole, the gravity will be so strong that the photons can orbit the black hole and form photon spheres (regions). Such photon orbits are unstable in general, slight deviation will make the photons drop into the black hole or run away to infinity. Therefore a black hole looks like a dark disk surrounded with a shine doughnut, as shown in the first black hole photo of M87* published by the EHT Collaboration in 2019 [1–6], and the second one for SgrA* in 2022 [7]. The observation of the black hole shadow provides direct information of the geometry near the event horizon of the black hole, and develops as a new research focus in recent years.

On the other hand, extra-dimensional theories have always been attractive and people used to pin their hope for detecting extra dimensions on high-energy experiments. After the achievement of the Gravitational Wave (GW) detection [8], physicists tried to study the features of GWs in extra-dimensional theories that can distinguish the effects of extra dimensions from those in other modified gravity theories, mainly the discrete high-frequency spectrum and shortcuts, see review [9]. However, the discrete high-frequency spectrum (about $\geq 300$ GHz) is far beyond the scope of GW detectors at present, and not all GWs in extra-dimensional theories can take shortcuts. Thanks to the accomplishment of the EHT observation, the black hole photos may provide a promising way to detect extra dimensions.

The conception of extra dimensions was first introduced by Gunnar Nordström in 1914, in order to unify electromagnetism and gravity [10, 11]. Then the 5D Kaluza-Klein (KK) theory was proposed with the extra dimension to be a compact circle [12–14], which could recover both the electromagnetism and GR in 4D spacetime. In 1983, the domain wall theory was constructed with an infinite extra dimension and a bulk scalar field [15, 16], where the effective potential well along the extra dimension could localize the energy density of the scalar field on a 3D hypersurface, i.e. the domain wall embedded in the 5D spacetime. However, the zero mode of gravity is difficult to be localized on the domain wall, and there is also hierarchy problem. Finally, Lisa Randall and Raman Sundrum (RS) proposed the well-known RS-I [17] and RS-II [18] models to solve the hierarchy problem, in which a warped structure was introduced to the compact/infinite extra dimension respectively.

Gregory and Laflamme made the pioneering attempt to generalize the 4D Schwarzschild black hole to 5D black string by the extension to an extra dimension with the topology $S^2_{\text{Sch}} \times \mathbb{R}^1$ [19] which can be regarded as an extra hair of black holes. Compared with a hyperspherically symmetric black hole (5D Schwarzschild), a hypercylindrical black hole (5D black string) possesses higher entropy with the same mass when the length of black string is small enough. In other words, an uniform 5D black string with short enough length is thermodynamically preferred than the 5D Schwarzschild black hole, indicating a possible mechanism to trigger dimensional reduction. The 5D black string was thought to be stable under linear perturbations [19], unless the well-known GL instability of black strings/branes [20,
was addressed using the general solution of 10D black strings branes in the low-energy string theory [22].

Intriguingly, the GL instability can be evaded by the compactification of the extra dimensions where the wavelength along the circle is required to be smaller than a critical value given by the numerical results and in agreement with the entropy argument [20]. In this work, we are devoted to explore the topology/length of the extra dimension using the EHT observations of the black hole shadow. We choose the simplest rotating 5D black strings in GR, $M_5^{Kerr} \times R^1$ and $M_5^{Kerr} \times S^1$, the latter is also a fundamental black hole solution in KK theory. Recently, the observational appearance of the most general black hole solution in KK theory has been studied elaborately in 4D, where small electric/magnetic charges of the black holes can meet the EHT observations while they are still indistinguishable from the Kerr case [24]. This result manifests that the model we choose can be regarded as a proper approximation for the realistic situation. As we shall see later, the reduction of the parameters is conductive to give an independent constraint for the length of the compact extra dimension.

Geodesic Equations.—In general, astrophysical black holes can be described by Kerr metric in four dimensions, if an extra spatial dimension is introduced in the simplest (uniform) way, then the constructed spacetime is still a solution of the vacuum Einstein equations of GR [25]

$$ds^2 = -\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \vartheta) dt^2 - \frac{4aMr}{\Sigma} \sin^2 \vartheta dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\vartheta^2 + \frac{1}{\Sigma} (\rho^4 - \Delta a^2 \sin^2 \vartheta) \sin^2 \vartheta d\varphi^2 + dz^2,$$  

where $\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2$. This solution describes a rotating uniform black string, where $M$ is the mass density proportional to the mass of the black string $M = M\ell$ ($\ell$ is the length of the black string), and $a$ is associated to the angular momentum $J = Ma$. When the last term in the metric disappears, the dimensions of all the quantities reduce to the normal ones in Kerr case.

If we consider the extra spatial dimension $z$ as a compact circle to avoid the GL instability, then periodic conditions along the extra dimension will be required (we suppose the circumference of the compact extra dimension is the same as the length of black string $\ell$). Throughout the paper we use the units $c = G = \hbar = 1$, unless the units are specifically mentioned.

The radius of event horizon and Cauchy horizon $r_{\pm} = M \pm \sqrt{M^2 - a^2}$ can be obtained from $\Delta = 0$, where $a^2 \leq a_{\text{max}}^2 = M^2$. The ring singularity is located at $r = 0$ and $\vartheta = \pi/2$, solved from $\Sigma = 0$. Due to the rotation, the ergoregion and causality violation region appear with $g_{tt} > 0$ and $g_{\varphi\varphi} < 0$ respectively, as in 4D Kerr spacetime.

The geodesic equations of a test particle with mass $m$ can be obtained from the the Hamilton-Jacobi equation considering conserved energy $E_0$, angular momentum $L_\varphi$

along $\varphi$ and momentum $P_z$ in $z$ direction

$$i = \frac{E_0 \rho^3 - a^2 E_0 \sin^2 \vartheta \Delta - 2aML_z r}{\Delta \Sigma},$$

$$\dot{\varphi} = \frac{L_\varphi \csc^2 \vartheta \Delta + 2aME_0 r - a^2 L_z}{\Delta \Sigma},$$

$$\dot{r} = \sqrt{R} \frac{\rho^4}{\Sigma} \sin^2 \vartheta, \quad \dot{\vartheta} = \frac{\rho^4}{\Sigma} \sin^2 \vartheta, \quad \dot{z} = P_z,$$

$$R = (E_0 \rho^2 - aL_\varphi^2) - K\Delta - r^2 (m^2 + P_z^2) \Delta, \quad \Theta = K - a^2 \cos^2 \vartheta (m^2 + P_z^2) - \frac{(aE_0 \sin^2 \vartheta - L_\varphi)^2}{\sin^2 \vartheta}$$

with
dots represent the derivatives with respect to the affine parameter $\tau = \tau/m$ (\tau is the proper time) and $K$ is Carter constant [26].

In the first four equations, the momentum $P_z$ always appears as square $P_z^2$ together with $m^2$, indicating that the momentum along the extra dimension participates as an effective mass $\sqrt{m^2 + P_z^2}$ for the particle. While for photons with a momentum $P_z$ along the extra dimension, the motions along $t, r, \vartheta, \varphi$ directions are exactly the same as the motions of a massive particle $m = P_z$ without the extra dimension. When the extra dimension disappears or the motion of the particle is within the 4D hypersurface ($P_z = 0$), the equations reduce to those for 4D Kerr spacetime, which implies that if a particle can not move along the extra dimension then we can not recognize the existence of extra dimensions via the motion of particle or shadow.

Photon Regions.—For the spherical orbits, the conditions $\dot{r} = 0$ and $\dot{\tau} = 0$ are required, which are equivalent to $R(r) = 0$ and $R'(r) = 0$. Using these conditions we can obtain the constants of motion $K_E \equiv K/E_0$ and $L_\varphi \equiv L_\varphi/E_0$ selected by the radius $r_p$ of photon orbits

$$K_E = \frac{2r_p \Delta_a}{\Delta_p^2} \left(2\sqrt{2r_p - P_E^2 \Delta_p} - P_E^2 \Delta_p' + 4r_p\right) - r_p^2 P_E^2,$$

$$L_\varphi = a + \frac{r_p^2}{a} - \frac{\Delta_p}{a \Delta_p'} \left(2\sqrt{2r_p - P_E^2 \Delta_p} + 2r_p\right),$$

where $P_E \equiv P_z/E_0$, $\Delta_p \equiv (\Delta(p), \Delta_p' \equiv \Delta'(p)$ and the primes represent the derivatives with respect to $r$. Here we have set $m = 0$ for photons. The condition $\Theta \geq 0$ gives the photon region

$$\left(r_x \Delta_p - \Sigma_p \Delta_p' \right)^2 \leq 4a^2 r_p^2 r_x \Delta_p \sin^2(\vartheta)\left(\Sigma_p \Delta_p' + 2r_p \Delta_p\right),$$

$$-a^2 P_E^2 \sin^2(\vartheta) \Delta_p' (\Sigma_p \Delta_p' + 2r_p \Delta_p),$$

where $\Sigma_p \equiv \Sigma(r_p)$ and $r_x = 2r_p + \sqrt{2r_p \sqrt{2r_p - P_E^2 \Delta_p}}$.

For each value of $r_p$, a range of $\vartheta$ for photon orbits can be obtained from the above inequality (8). For the static case $a = 0$, the photon region degenerates a photon sphere, the radius of which can be solved from $r_x \Delta_p - r_p^2 \Delta_p' = 0$.
With relations (7), the stable condition \(R''(r_p) < 0\) for the photon orbits under radial perturbations can be expressed as
\[
\frac{\Delta_p^2}{2E_0^2} R''(r_p) = 2r_p \Delta_p^2 (2r_p - P_E^2 \Delta_p') - \Delta_p (\Delta_p' - r_p \Delta'_p) \tag{9}
\]
\[
\times \left( P_E^2 \Delta_p' - 2 \sqrt{2r_p (2r_p - P_E^2 \Delta_p')} - 4r_p \right) < 0.
\]

When \(P_E \to 0\) all the relations (7)–(9) will reduce back to those in Kerr case. In FIG. 1, we plot the photon regions and the stable region for the spherical photon orbits in the 5D rotating black string spacetime, compared with the Kerr case \((P_z/E_0 = 0)\). (For simplicity here we do not show the photon regions and the causality violation regions with negative \(r\)). It shows that with the increase of \(P_E\), the photon regions outside the event horizon move out with larger radius, while the photon regions inside the Cauchy horizon move in with smaller radius. Similar with the Kerr case, the photon regions outside the event horizons are unstable while the photon regions inside the Cauchy horizons are divided into stable parts and unstable parts \((P_z/E_0 = 0.1)\). However, if we further raise \(P_z/E_0\), for example \(P_z/E_0 = 0.5\), the photon regions inside the Cauchy horizon will disappear.

**FIG. 1.** The photon regions outside the event horizon (left panel) and inside the Cauchy horizon (right panel) are plotted with \(M = 1\) and \(a = \frac{1}{2}a_{\text{max}}\). The parameter \(P_E\) does not change the positions of horizons (red dashed circles) and ergoregion (cyan region). Besides, the stable regions (right panel) for the spherical photon orbits are filled with oblique lines, where the solid boundary is for \(P_z/E_0 = 0\) and the dashed boundary is for \(P_z/E_0 = 0.1\).

**Shadow.**—The unstable photon regions outside the event horizon make the direct observation of black holes possible. The photons that escape the spherical photon orbits of a black hole due to the instability and are received by an observer in the domain of outer communication, form the boundary of the dark silhouette of the black hole. This dark silhouette is the so-called black hole shadow from the view of the observer.

For an observer \((r_o, \vartheta_o)\) at spatial infinity, the observed shadow boundary can be described in Cartesian coordinates on the projected plane
\[
X = \lim_{r_o \to \infty} \left( -r_o^2 \sin \vartheta_o \frac{d\varphi}{dr} \right) = -\frac{L_E \csc \vartheta_o}{\sqrt{1 - P_E^2}}, \tag{10}
\]
\[
Y = \lim_{r_o \to \infty} \left( r_o^2 \frac{d\vartheta}{dr} \right) = \pm \frac{\sqrt{K_E + 2aL_E - \Xi}}{\sqrt{1 - P_E^2}}, \tag{11}
\]
with \(\Xi = L_E^2 \csc^2 \vartheta_o + a^2 P_E^2 \cos^2 \vartheta_o + a^2 \sin^2 \vartheta_o\), where we have substituted into the geodesic equations.

**FIG. 2.** The shadow boundary of 5D rotating black string being observed at spatial infinity and \(\vartheta_o = \pi/2\) is depicted with \(a/M = 0.2\) (left panel) and \(a/M = 0.8\) (right panel) respectively. The black dashed curve represents the Kerr case.

The shadow boundary of 5D rotating black string being observed at spatial infinity and \(\vartheta_o = \pi/2\) is depicted with various values of \(P_z/E_0\) in FIG. 2. It shows that with the increase of \(P_z/E_0\), the shadow region expands larger in all the directions, while the growth of \(a\) only translate and distort the shadow towards the right direction. It is known that the existence of electric/magnetic charge will make the shadow region shrinks for Kerr-Newman case, therefore the effect of the extra dimension is easily to be distinguished from the effects from other parameters like \(a\) and \(c\).

Similar effects of enlargement can be found in other extra-dimensional theories. For instance, in RS scenario, a negative tidal charge enlarges the shadow of a rotating braneworld black hole and reduces its deformation with respect to Kerr spacetime [27]. It is intriguing that the shadows of black holes in most modified theories of gravity, in Loop Quantum Gravity (LQG) [28] and also black holes with additional sources surrounding [29] or hairy black holes [30] have been studied to be smaller and more distorted compared with the Kerr black hole case.

The boundary of shadow can be approximated by a reference circle and estimated by the radius \(R_s\) of this circle and the deviation \(\delta_s\) of the left edge of the shadow from the circle, proposed by Hioki and Maeda [31]. Using the top \((X_t, Y_t)\), right \((X_r, 0)\), left \((X_l, 0)\) edges of the shadow and the leftmost edge \((X_l', 0)\) of the reference circle, the size and distortion of the black hole shadow can be characterized as
\[
R_s = \frac{(X_t - X_r)^2 + Y_t^2}{2|X_r - X_t|}, \quad \delta_s = \frac{|X_l - X_l'|}{R_s}. \tag{12}
\]

In FIG. 3, we present the influences of \(a/M\) and \(P_z/E_0\) on the size \(R_s/M\) and distortion \(\delta_s\) of the black hole shadow at equatorial plane \(\vartheta_o = \pi/2\). The size \(R_s/M\) becomes apparently larger with the increase of \(P_z/E_0\) while it grows slightly as \(a/M\) rises. In addition, the distortion \(\delta_s\) gets larger with the increase of \(a/M\) while it raises slightly as \(P_z/E_0\) rises. In a word, the effects of \(a/M\) and \(P_z/E_0\) are mainly on the distortion \(\delta_s\) and the size \(R_s/M\) respectively.
Later, Kumar and Ghosh defined the area $A$ and oblateness $D$ to estimate shadows with haphazard shapes

$$A = 2 \int Y(r_p) dX(r_p) = 2 \int_{r_p, \text{max}}^{r_p, \text{min}} \left( Y(r_p) \frac{dX(r_p)}{dr_p} \right) dr_p,$$

$$D = \frac{X_o - X_l}{Y_o - Y_b},$$

where $Y_b = -Y_l$ in our case. For an equatorial observer in Kerr spacetime, the oblateness $D$ varies from $D = 1$ (static case) to $D = \frac{x^2}{2}$ (extremal case) [32]. Moreover, the average shadow radius $\bar{R}$ and the circularity deviation $\Delta C$ have been defined in [33]

$$\bar{R} = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} R^2(\omega) d\omega},$$

$$\Delta C = \frac{1}{\bar{R}} \sqrt{\frac{1}{2\pi} \int_0^{2\pi} (R(\omega) - \bar{R})^2 d\omega},$$

where $R(\omega)$ and $\omega$ are the polar coordinates describing the shadow boundary

$$R(\omega) = \sqrt{(X - X_c)^2 + (Y - Y_c)^2},$$

$$\omega = \tan^{-1} \left( \frac{Y - Y_c}{X - X_c} \right),$$

and $(X_c, Y_c)$ is the origin of the polar coordinates with $X_c = (X_l + X_o) / 2$ and $Y_c = 0$. Most importantly, the angular diameter $\theta_d$ of the shadow can be obtained as [34]

$$\theta_d = \frac{2R_a}{d} = \frac{2R_a / M}{d / M}, \quad R_a = \sqrt{A / \pi}.$$

Now we consider the realistic parameter values for the supermassive black hole in M87*. Firstly the inclination angle $\vartheta_o = 17^\circ$ can be given by the orientation of the jets in M87* [35]. Then the distance $d = 16.8 \pm 0.8 \text{Mpc}$ is adopted based on the three recent stellar population measurements [6]. However, the mass estimates from stellar and gas dynamics does not agree with each other, here we use the average value $M = (6.5 \pm 0.7) \times 10^9 M_\odot$ inferred from the geometric models, General Relativistic Magnetohydrodynamic (GRMHD) models and image domain ring extraction, where the distance $d = 16.8 \pm 0.8 \text{Mpc}$ has been applied. In fact, only a ratio $d / M$ is required for the calculations

$$d / M = \frac{16.8 \text{Mpc}}{6.5 \times 10^9 M_\odot} \simeq 5.40573 \times 10^{10}.$$  

(19)

It is worth mentioning that here we assign the value of mass to $M$ instead of $M = M_\odot$, because this value is inferred from the Kerr case, in which $M$ is the mass. Using these realistic parameter values, we plot the angular diameter $\theta_d$ and the circularity deviation $\Delta C$ in FIG. 4, which satisfies the constraint $\Delta C \lesssim 0.1$ from the EHT observations of M87*.

Similarly, we also plot the angular diameter $\theta_d$ of the 5D rotating black string shadow using the EHT observations of SgrA* in FIG. 5, where $\vartheta_o = 5^\circ$ and $d / M = 4.24628 \times 10^{10}$ have been applied.

Constrain the Extra Dimension.—Not as we anticipate before the calculations, the length of the extra dimension does not appear in the expressions of the shadow boundary
(10)(11), but we still expect to find some clues of the extra dimension from the present results.

Firstly, for fixed $P_z/E_0$, we can constrain the dimensionless quantity $cP_z/E_0$ from the lower bound $\theta_d = 39 \mu$as of the EHT observation of M87*

$$\frac{P_z}{E_0} = \frac{cP_z}{E_0} = \frac{cP_z}{pe} = \frac{v_z}{c} \lesssim 0.35,$$

where 0.35 is the possible maximum value of the lower bound within the spin measurement in FIG. 4.

However, this constraint could not be proper for an infinite extra dimension, where in principle the value of $P_z/E_0$ can be arbitrary within $0 \sim 1$. Especially for large enough $P_z/E_0$, the shadow boundary lies outside the outer boundary $\theta_d = 45 \mu$as as of EHT observations, which indicates that the luminosity distribution produced by these photons is beyond the observed bright region and hence it contradicts with the EHT observations.

Interestingly, a compact extra dimension can exactly give a reasonable excuse for the particular choices of $P_z/E_0$. For a compact extra dimension with a length $\ell$, the momentum $P_z$ of the photons is limited to be box normalized $P_z = 2\pi \hbar n / \ell$, $n = 0, \pm 1, \pm 2, \ldots$, in this way we can relate the length of the extra dimension to the momentum $P_z$ as

$$\frac{v_z}{c} = \frac{cP_z}{E_0} = \frac{2\pi \hbar n c}{E_0 \ell} = \frac{n\lambda_0}{\ell},$$

where $\lambda_0$ is the wavelength of the photons. Note that for a given length $\ell$ of the extra dimension, there are infinite choices of $n$ (i.e. choices of $P_z$), but an upper limit for the velocity $v_z/c \leq 1$ can result in truncation of the larger values of $n$.

Now let us go through all the possibilities. Primarily for $n = 0$, the ground state $P_z = 0$ (Kerr case) is within the range of the observation. Subsequently if the choice $n = 1$ is also permitted by the observation, then the shadow boundary for the corresponding $P_z/E_0$ should be smaller than the outer boundary of the bright annulus

$$\frac{P_z}{E_0} = \frac{\lambda_0}{\ell} \lesssim 0.64, \quad \Rightarrow \quad \ell \gtrsim \frac{\lambda_0}{0.64} \approx 2.03125 \text{ mm},$$

here we have applied the EHT observing wavelength $\lambda_0 = 1.3 \text{ mm}$. If the series requires to be cut off at $n = 2$, i.e.

$$\frac{v_z}{c} = \frac{cP_z}{E_0} = \frac{2\lambda_0}{\ell} > 1, \quad \Rightarrow \quad \ell < 2\lambda_0 \approx 2.6 \text{ mm},$$

then the only allowed values of $P_z$ are $P_z = 0$ and $P_z = 2\pi \hbar / \ell$, in agreement with the observations. Finally if only choices $n = 0, 1, 2$ are permitted, similar calculations will lead to invalid scale $4.0625 \text{ mm} \lesssim \ell \lesssim 3.9 \text{ mm}$.

For the general case, if the permitted choices are $n = 0, 1, 2, \ldots, k$, then we have $k\lambda_0/0.64 \lesssim \ell < (k + 1)\lambda_0$, which is only valid for $k \leq 1$. Therefore the only possibility is the case $n = 0, 1$, and the corresponding constraint on the length of the extra dimension is then

$$2.03125 \text{ mm} \lesssim \ell \lesssim 2.6 \text{ mm}.$$

Similarly, from the observations of SgrA*, the constraints $2.28070 \text{ mm} \lesssim \ell \lesssim 2.6 \text{ mm}$ and $2.13115 \text{ mm} \lesssim \ell \lesssim 2.6 \text{ mm}$ can be given by the upper bounds of the emission ring and the angular shadow diameter respectively.

The above analysis indicates that we could give a possible constraint on the length of the extra dimension from the EHT observations, which is surprising for the first consideration since it supports the hypothesis that the extra dimension is compact avoiding the GL instability. Nevertheless, the possibility that there is no extra dimension or the photons can not move in the extra spatial direction is still hard to rule out. Besides, it is noticed that if the length of the compact extra dimension is smaller than the wavelength of the photons, then the photons can not move along the extra dimension even if the extra dimension exists. This conclusion can be inferred from the equation (21). Therefore such constraint is based on the EHT observing wavelength and only valid when the length of the compact extra dimension is larger than the EHT observing wavelength.

It is worth noting that the critical length of the 5D static black string to avoid the GL instability is $\ell_{GL} = 2\pi r_c / 0.88$ [37], which is much larger than the constraints we obtained.

Conclusions.—In this paper, we study the photon regions and shadow of the 5D rotating black string in GR, with a conserved momentum $P_z$ of the photons along the extra dimension. We find that the conserved momentum $P_z$ appears as an effective mass in the geodesic equations of the photons, and enlarges the size of the shadow, while it almost has no impact on the distortion of shadow. To compare with the EHT observations, we calculate the observables of the shadow like angular diameter $\theta_d$ and circularity deviation $\Delta C$, which suggests that the black string model with an infinite extra dimension could be ruled out.

More significantly, the length of the compact extra dimension can be constrained as $2.03125 \text{ mm} \lesssim \ell \lesssim 2.6 \text{ mm}$ via the box normalization from the observations of M87*. Similarly, from the observations of SgrA*, the constraints $2.28070 \text{ mm} \lesssim \ell \lesssim 2.6 \text{ mm}$ and $2.13115 \text{ mm} \lesssim \ell \lesssim 2.6 \text{ mm}$ can be given by the upper bounds of the emission ring and the angular shadow diameter respectively.

Acknowledgments.—We thank Rong-Gen Cai, Yong-Shun Hu and Dong-Chao Zheng for beneficial discussions. This work is supported by National Natural Science Foundation of China under Grant No. 12147119 and China Postdoctoral Science Foundation under Grant No. 2021M700142. X.M. Kuang is partly supported by Fok Ying Tung Education Foundation under Grant No. 171006 and Natural Science Foundation of Jiangsu Province under Grant No.BK20211601.

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