Going Higher in the First-Order Quantifier Alternation Hierarchy on Words

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Abstract. We investigate the quantifier alternation hierarchy in first-order logic on finite words. Levels in this hierarchy are defined by counting the number of quantifier alternations in formulas. We prove that one can decide membership of a regular language to the levels $B\Sigma_2$ (boolean combination of formulas having only 1 alternation) and $\Sigma_3$ (formulas having only 2 alternations beginning with an existential block). Our proof works by considering a deeper problem, called separation, which, once solved for lower levels, allows us to solve membership for higher levels.

The connection between logic and automata theory is well known and has a fruitful history in computer science. It was first observed when Büchi, Elgot and Trakhtenbrot proved independently that the regular languages are exactly those that can be defined using a monadic second-order logic (MSO) formula. Since then, many efforts have been made to investigate and understand the expressive power of relevant fragments of MSO. In this field, the yardstick result is often to prove decidable characterizations, i.e., to design an algorithm which, given as input a regular language, decides whether it can be defined in the fragment under investigation. More than the algorithm itself, the main motivation is the insight given by its proof. Indeed, in order to prove a decidable characterization, one has to consider and understand all properties that can be expressed in the fragment.

The most prominent fragment of MSO is first-order logic (FO) equipped with a predicate "<" for the linear-order. The expressive power of FO is now well-understood over words and a decidable characterization has been obtained. The result, Schützenberger’s Theorem [19,9], states that a regular language is definable in FO if and only if its syntactic monoid is aperiodic. The syntactic monoid is a finite algebraic structure that can effectively be computed from any representation of the language. Moreover, aperiodicity can be rephrased as an equation that needs to be satisfied by all elements of the monoid. Therefore, Schützenberger’s Theorem can indeed be used to decide definability in FO.

In this paper, we investigate an important hierarchy inside FO, obtained by classifying formulas according to the number of quantifier alternations in their prenex normal form. More precisely, an FO formula is $\Sigma_i$ if its prenex normal form has at most $(i - 1)$ quantifier alternations and starts with a block of existential quantifiers. The hierarchy also involves the classes $B\Sigma_i$ of boolean combinations of $\Sigma_i$ formulas, and the classes $\Delta_i$ of languages that can be defined by both a $\Sigma_i$ and

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the negation of a $\Sigma_i$ formula. The quantifier alternation hierarchy was proved to be strict \cite{[29]}: $\Delta_i \not\subseteq \Sigma_i \not\subseteq B\Sigma_i \not\subseteq \Delta_{i+1}$. In the literature, many efforts have been made to find decidable characterizations of levels of this well-known hierarchy.

Despite these efforts, only the lower levels are known to be decidable. The class $B\Sigma_1$ consists exactly of all piecewise testable languages, \textit{i.e.}, such that membership of a word only depends on its subwords up to a fixed size. These languages were characterized by Simon \cite{[20]} as those whose syntactic monoid is $J$-trivial. A decidable characterization of $\Sigma_2$ (and hence of $\Delta_2$ as well) was proven in \cite{[3]}. For $\Delta_2$, the literature is very rich \cite{[25]}. For example, these are exactly the languages definable by the two variable restriction of FO \cite{[27]}. These are also those whose syntactic monoid is in the class $DA$ \cite{[13]}. For higher levels in the hierarchy, getting decidable characterizations remained an important open problem. In particular, the case of $B\Sigma_2$ has a very rich history and a series of combinatorial, logical, and algebraic conjectures have been proposed over the years. We refer to \cite{[11][12][10][12]} for an exhaustive bibliography. So far, the only known effective result was partial, working only when the alphabet is of size 2 \cite{[24]}. One of the main motivations for investigating this class in formal language theory is its ties with two other famous hierarchies defined in terms of regular expressions. In the first one, the Straubing-Thérentier hierarchy \cite{[22][26]}, level $i$ corresponds exactly to the class $B\Sigma_i$ \cite{[28]}. In the second one, the dot-depth hierarchy \cite{[7]}, level $i$ corresponds to adding a predicate for the successor relation in $B\Sigma_i$ \cite{[28]}. Proving decidability for $B\Sigma_2$ immediately proves decidability of level 2 in the Straubing-Thérentier hierarchy, but also in the dot-depth hierarchy using a reduction by Straubing \cite{[23]}. In this paper, we prove decidability for $B\Sigma_2$, $\Delta_3$ and $\Sigma_3$. These new results are based on a deeper decision problem than decidable characterizations: the separation problem. Fix a class $\text{Sep}$ of languages. The $\text{Sep}$-separation problem amounts to decide whether, given two input regular languages, there exists a third language in $\text{Sep}$ containing the first language while being disjoint from the second one. This problem generalizes decidable characterizations. Indeed, since regular languages are closed under complement, testing membership in $\text{Sep}$ can be achieved by testing whether the input is $\text{Sep}$-separable from its complement. Historically, the separation problem was first investigated as a special case of a deep problem in semigroup theory, see \cite{[1]}. This line of research gave solutions to the problem for several classes. However, the motivations are disconnected from our own, and the proofs rely on deep, purely algebraic arguments. Recently, a research effort has been made to investigate this problem from a different perspective, with the aim of finding new and self-contained proofs relying on elementary ideas and notions from language theory only \cite{[8][15][13][16]}. This paper is a continuation of this effort: we solve the separation problem for $\Sigma_2$, and use our solution as a basis to obtain decidable characterizations for $B\Sigma_2$, $\Delta_3$ and $\Sigma_3$. Our solution works as follows: given two regular languages, one can easily construct a monoid morphism $\alpha : A^* \to M$ that recognizes both of them. We then design an algorithm that computes, inside the monoid $M$, enough $\Sigma_2$-related information to answer the $\Sigma_2$-separation question for \textit{any} pair of languages that are recognized by $\alpha$. It turns out that it is also possible (though much more