A note on partially-greedy bases in quasi-Banach spaces

by

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Abstract. We continue the study of greedy-type bases in quasi-Banach spaces started by Albiac et al. [arXiv:1903.11651]. In the present paper, we study partially-greedy bases, focusing on two main results:

• Characterization of partially-greedy bases in quasi-Banach spaces in terms of quasi-greediness and various conservative-like properties.
• Given a $C$-partially-greedy basis in a quasi-Banach space, there exists a “renorming” such that the basis is 1-partially-greedy.

1. Introduction. For several years now, it has been considered an important problem of mathematical analysis, how to represent a function $f$ in a particular space $X$, in the form

$$f = \sum_{n=1}^{\infty} a_n e_n$$

for given basic functions $(e_n)_{n=1}^{\infty}$ and for suitable scalars $a_n$ (coefficients). In the literature we can find various examples of such representations, for instance Fourier series of functions or Taylor expansions. In functional analysis, one considers expansions with regard to a basis, that is, $B = (e_n)_{n=1}^{\infty}$ is a Schauder basis or a Markushevich basis.

One of the main goals of approximation theory is to find good approximations of $f$ in terms of finite sums. Concretely, we want to find algorithms of approximation $(T_m)_{m=1}^{\infty}$, where

$$T_m(f) = \sum_{n \in A} b_n e_n,$$

$A$ is a finite set of cardinality $m$ and the $b_n$ could be different from the...
original coefficients of $f$. Moreover, we would like to ensure that $(T_m)_{m=1}^\infty$ produces a “good approximation” (which could be interpreted according to one’s preferences).

Since 1999, in the field of non-linear approximation, one of the most important algorithms is the greedy algorithm $(G_m)_{m=1}^\infty$, where for an element $f$ in $X$, the algorithm selects the largest coefficients of $f$ in modulus. This algorithm was introduced by S. V. Konyagin and V. N. Temlyakov [10] and various properties of it have been analyzed by several authors, among them S. J. Dilworth, N. J. Kalton, D. Kutzarova and P. Wojtaszczyk (see [7, 8, 12]).

In this paper, we focus on the following property that was introduced in [8]: there exists a positive constant $C$ such that

$$
\|f - G_m(f)\| \leq C\|f - S_m(f)\|, \quad \forall f \in X, \forall m \in \mathbb{N},
$$

where $S_m$ denotes the $m$th partial sum. The importance of this property lies in the fact that we want to compare if a non-linear approximation is better than linear approximation. Here, we extend the main known results about (1.1) in Banach spaces to the case of quasi-Banach spaces, using recent results proved in [1].

The structure of the paper is the following: In Section 2 we give the basic definitions about bases in quasi-Banach spaces and some operators. In Section 3 we give the definition of the greedy algorithm, we talk about some greedy-type bases and we analyze some properties of conservativeness. In Section 4 we give the main characterization of partially-greedy bases in quasi-Banach spaces, and finally, in Section 5, we discuss renorming of quasi-Banach spaces using greedy-type bases.

2. Preliminaries on quasi-Banach spaces. We say that a map $\| \cdot \| : X \to [0, +\infty)$ defined on a vector space $X$ over $F = \mathbb{R}$ or $\mathbb{C}$ is a quasi-norm if

(a) $\|f\| > 0$ for all $f \neq 0$,
(b) $\|tf\| = |t| \|f\|$ for all $t \in F$ and $f \in X$,
(c) there exists a positive constant $k$ such that for all $f, g \in X$,

$$
\|f + g\| \leq k(\|f\| + \|g\|).
$$

Given $p \in (0, 1]$, a $p$-norm is a map $\| \cdot \| : X \to [0, +\infty)$ satisfying (a), (b) and

(d) $\|f + g\|^p \leq \|f\|^p + \|g\|^p$ for all $f, g \in X$.

Of course, (d) implies (c) with $k = 2^{1/p-1}$. If $\| \cdot \|$ is a quasi-norm (resp. $p$-norm) on $X$ that defines a complete metrizable topology, then $X$ is called a quasi-Banach space (resp. $p$-Banach space). Thanks to the Aoki–Rolewicz
Theorem (see [4], [11]), any quasi-Banach space $X$ is $p$-convex, that is,

$$\left\| \sum_{j=1}^{n} x_j \right\| \leq C \left( \sum_{j=1}^{n} \|x_j\|^p \right)^{1/p}, \quad n \in \mathbb{N}, x_j \in X.$$ 

This way, $X$ becomes $p$-Banach under a suitable renorming, for some $0 < p \leq 1$.

2.1. Bases. Throughout this paper, a basis in a quasi-Banach space $X$ is a semi-normalized and total Markushhevich basis, i.e., $B = (e_n)_{n=1}^{\infty} \subset X$ satisfies the following conditions:

(i) $[e_n : n \in \mathbb{N}] = X$ (completion),
(ii) there exists a unique sequence $(e^*_n)_{n=1}^{\infty} \subset X^*$, called biorthogonal functionals, such that $e^*_n(e_m) = \delta_{n,m},$
(iii) if $e^*_n(f) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ (totality),
(iv) there exist $c_1, c_2 > 0$ such that $0 < c_1 \leq \{\|e_n\|, \|e^*_n\|\} \leq c_2 < \infty$ for all $n \in \mathbb{N}$ (semi-normalization).

Thanks to [11, Lemma 1.8], we know that if $f \in X$ then $f = \sum_{n=1}^{\infty} e^*_n(f)e_n,$ where $\lim_{n \to +\infty} e^*_n(f) = 0$ and the expansion is only formal but the assignment of coefficients is still unique. Also, we set $\text{supp}(f) = \{n \in \mathbb{N} : e^*_n(f) \neq 0\}.$

Associated with a basis, we can consider the projection operator $P_A,$ where for a finite set $A \subset \mathbb{N},$

$$P_A(f) = \sum_{n \in A} e^*_n(f)e_n.$$ 

It is well known that if $P_A$ is uniformly bounded, then $B$ is called unconditional. We write $S_k := P_{\{1, \ldots, k\}}$ for the $m$th partial sum. Also, if there is a positive constant $C$ such that

$$\|S_k(f)\| \leq C\|f\|, \quad \forall f \in X, \forall k \in \mathbb{N},$$

we say that $B$ is a Schauder basis. Finally, if $A$ is a finite set, we denote by $\Psi_A$ the collection of signs $\varepsilon$:

$$\Psi_A := \{\varepsilon = (\varepsilon_n)_{n \in A} : |\varepsilon_n| = 1, n \in A\},$$

and $1_{\varepsilon A}[B, X] := 1_{\varepsilon A}$ is the indicator sum:

$$1_{\varepsilon A} = \sum_{n \in A} \varepsilon_n e_n,$$

where $\varepsilon \in \Psi_A.$ If $\varepsilon \equiv 1$, we write $1_A.$ As usual, if $A, B \subset \mathbb{N}$ are finite sets, we write $A < B$ if $\max A < \min B.$
2.2. $p$-convexity. Consider, for $0 < p \leq 1$, the following geometrical constants as in [1]:

\[
A_p = \frac{1}{(2^p - 1)^{1/p}}, \quad B_p = \begin{cases} 2^{1/p}A_p & \text{if } F = \mathbb{R}, \\ 4^{1/p}A_p & \text{if } F = \mathbb{C}. \end{cases}
\]

The following result is a combination of two corollaries of [1, Theorem 1.2].

**Proposition 2.1 ([1, Corollaries 1.3 and 1.4])**. Let $\mathcal{B} = (e_n)_{n=1}^{\infty}$ be a basis in a $p$-Banach space and $J$ a finite set. Then:

(a) For any scalars $(a_n)_{n \in J}$ with $0 \leq a_n \leq 1$ and any $g \in X$, we have

\[
\left\| g + \sum_{n \in J} a_n e_n \right\| \leq A_p \sup \left\{ \left\| g + \sum_{n \in A} e_n \right\| : A \subseteq J \right\}.
\]

(b) For any scalars $(a_n)_{n \in J}$ with $|a_n| \leq 1$ and any $g \in X$, we have

\[
\left\| g + \sum_{n \in J} a_n e_n \right\| \leq A_p \sup_{A \subseteq J} \left\{ \left\| g + \sum_{n \in J} \varepsilon_n e_n \right\| : |\varepsilon_n| = 1 \right\}.
\]

(c) For any scalars $(a_n)_{n \in J}$ with $|a_n| \leq 1$, we have

\[
\left\| \sum_{n \in J} a_n e_n \right\| \leq B_p \sup_{A \subseteq J} \left\| \sum_{n \in A} e_n \right\|.
\]

3. The greedy algorithm and greedy-type bases. For each $f \in X$ and $m \in \mathbb{N}$, S. V. Konyagin and V. N. Temlyakov defined in [10] a greedy sum of $f$ of order $m$ by

\[
G_m(f) = \sum_{n=1}^{m} e_{\pi(n)}^*(f) e_{\pi(n)},
\]

where $\pi$ is a greedy ordering, that is, $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation such that supp($f$) $\subseteq$ $\pi(\mathbb{N})$ and $|e_{\pi(i)}^*(f)| \geq |e_{\pi(j)}^*(f)|$ for $i \leq j$. The series $\sum_{n=1}^{\infty} e_{\pi(n)}^*(f) e_{\pi(n)}$ is called the greedy series. Also, $G_m(f) = P_A(f)$, where the set $A = \text{supp}(G_m(f))$ is called a greedy set of $f$ and satisfies $|A| = m$ and

\[
\min_{n \in A} |e_n^*(f)| \geq \max_{n \notin A} |e_n^*(f)|.
\]

3.1. Quasi-greedy and partially-greedy bases. To study the convergence of the greedy algorithm, we consider the following types of bases introduced by S. V. Konyagin and V. N. Temlyakov.

**Definition 3.1 ([10])**. We say that a basis $\mathcal{B}$ in a quasi-Banach space $X$ is quasi-greedy if there is a positive constant $C$ such that for all $f \in X$,

\[
\|P_A(f)\| \leq C\|f\|
\]
whenever \( A \) is a finite greedy set of \( f \). The smallest such \( C \) is called the \textit{quasi-greedy constant} of the basis, it is denoted by \( C_{\text{qg}} = C_{\text{qg}}[\mathcal{B}, X] \) and we say that \( \mathcal{B} \) is \( C_{\text{qg}} \)-quasi-greedy.

The following result was proved in \([12]\) and \([1]\):

**Theorem 3.2.** Let \( \mathcal{B} \) be a basis in a quasi-Banach space \( X \). The following are equivalent:

- \( \mathcal{B} \) is quasi-greedy.
- For every \( f \in X \), the greedy series of \( f \in X \) converges.

Consider the following weaker version of quasi-greediness that we need for our purposes.

**Definition 3.3 ([1]).** We say that a basis \( \mathcal{B} \) in a quasi-Banach space \( X \) is \textit{quasi-greedy for largest coefficients} if there exists a positive constant \( C \) such that

\[
\|1_{\varepsilon A}\| \leq C\|f + 1_{\varepsilon A}\| \tag{3.2}
\]

for any finite set \( A \subset \mathbb{N} \), any \( \varepsilon \in \Psi_A \) and any \( f \in X \) such that \( \text{supp}(f) \cap A = \emptyset \) and \( \max_{n \in \text{supp}(f)} |e^*_n(f)| \leq 1 \). The smallest such \( C \) is denoted by \( C_{\text{ql}} = C_{\text{ql}}[\mathcal{B}, X] \) and we say that \( \mathcal{B} \) is \( C_{\text{ql}} \)-quasi-greedy for largest coefficients. Of course, \( C_{\text{ql}} \leq C_{\text{qg}} \).

Since 1999, various types of convergence of the greedy algorithm have been studied, and various greedy-type bases have been introduced, such as greedy bases and almost-greedy bases. These bases were introduced and characterized (in the context of Banach spaces) in \([10, 8]\). Recently, in \([1]\), the authors characterized the same bases in the context of quasi-Banach spaces analyzing the lack of convexity in the results. Now, we study the characterization of partially-greedy bases.

**Definition 3.4.** We say that a basis \( \mathcal{B} \) in a quasi-Banach space \( X \) is \textit{partially-greedy} if there is a positive constant \( C \) such that, for all \( f \in X \) and all finite greedy sets \( A \) of \( f \),

\[
\|f - P_A(f)\| \leq C \inf_{k \leq |A|} \|f - S_k(f)\|. \tag{3.3}
\]

The smallest such \( C \) is called the \textit{partially-greedy constant} of the basis, it is denoted by \( C_{\text{pg}} = C_{\text{pg}}[\mathcal{B}, X] \) and we say that \( \mathcal{B} \) is \( C_{\text{pg}} \)-partially-greedy.

**Remark 3.5.** In \([8]\), the authors defined partially-greedy bases as those for which there exists a positive constant \( C \) such that

\[
\|f - P_A(f)\| \leq C \|f - S_m(f)\|, \quad \forall f \in X, \forall \text{ greedy set } A \text{ with } |A| = m.
\]

When \( \mathcal{B} \) is a Schauder basis, both definitions are equivalent. We work with \((3.3)\) inspired by the results obtained recently in \([5]\).
In the following subsections, we introduce and analyze the main tools that we need to characterize partially-greediness in the context of quasi-Banach spaces.

3.2. The truncation operator. The following definitions were introduced in [7]. Take \( f \in X, A \subseteq \mathbb{N} \) finite and \( \varepsilon \equiv \{ \text{sign}(e_n^*(f)) \} \). Define, for \( f \in X \) and \( A \) a finite greedy set of \( f \),

\[
U(f, A) = \min_{n \in A} |e_n^*(f)|1_{\varepsilon A},
\]

\[
T(f, A) = U(f, A) + P_{A^c}(f).
\]

These operators are called the restricted truncation operator and the truncation operator, respectively. We set

\[
\Gamma_u = \Gamma_u[B, X] = \sup\{ \|U(f, A)\| : \|f\| \leq 1, A \text{ a greedy set of } f \},
\]

\[
\Gamma_t = \Gamma_t[B, X] = \sup\{ \|T(f, A)\| : \|f\| \leq 1, A \text{ a greedy set of } f \}.
\]

**Theorem 3.6 ([1, Theorem 3.13, Proposition 3.14]).** Let \( B \) be a quasi-greedy basis in a quasi-Banach space \( X \). Then:
- The restricted truncation operator \( U \) is uniformly bounded, that is, \( \Gamma_u < \infty \). Also, if \( X \) is a \( p \)-Banach space then \( \Gamma_u \leq C \eta_p(C_{qg}) \), where, for \( u > 0 \),

\[
\eta_p(u) = \min_{0 < t < 1} (1 - t^p)^{-1/p} (1 - (1 + A_p^{-1}u^{-1}t)^{-p})^{-1/p},
\]

and \( \eta_p(C_{qg}) \lesssim C_{qg}^{1+1/p} \).
- The truncation operator \( T \) is uniformly bounded, that is, \( \Gamma_t < \infty \). Also, if \( X \) is a \( p \)-Banach space, then \( \Gamma_t \leq C_{qg} (1 + C_{qg} \eta_p(C_{qg}))^{1/p} \).

**Remark 3.7.** The estimate \( \eta(C_{qg}) \lesssim C_{qg}^{1+1/p} \) comes from [1, Remark 3.9].

3.3. Properties about conservativeness. Consider the following property introduced recently in [5].

**Definition 3.8 ([5]).** We say that a basis \( B \) in a quasi-Banach space \( X \) is partially-symmetric for largest coefficients if there exists a positive constant \( C \) such that

\[
\|f + 1_{\varepsilon A}\| \leq C\|f + 1_{\varepsilon' B}\|
\]

for any pair of sets \( A, B \), any \( \varepsilon \in \Psi_A, \varepsilon' \in \Psi_B \) and any \( f \in X \) such that \( |A| \leq |B| \), \( \max_{n \in \text{supp}(f)} |e_n^*(f)| \leq 1 \), \( A < \text{supp}(f) \cup B \) and \( B \cap \text{supp}(f) = \emptyset \). The smallest such \( C \) is denoted by \( \Delta_{pl} = \Delta_{pl}[B, X] \) and we say that \( B \) is \( \Delta_{pl} \)-partially-symmetric for largest coefficients.

**Definition 3.9 ([8]).** We say that a basis \( B \) in a quasi-Banach space \( X \) is super-conservative if there exists a positive constant \( C \) such that

\[
\|1_{\varepsilon A}\| \leq C\|1_{\varepsilon' B}\|
\]
for any pair of sets $A, B$ with $|A| \leq |B|$ and $A < B$, and any choice of signs $\varepsilon \in \Psi_A$, $\varepsilon' \in \Psi_B$. The smallest such $C$ is denoted by $\Delta_\varepsilon = \Delta_\varepsilon[B, X]$ and we say that $B$ is $\Delta_\varepsilon$-super-conservative.

Also, we say that $B$ is conservative if (3.5) is satisfied with $\varepsilon \equiv \varepsilon' \equiv 1$, that is,

$$\|1_A\| \leq C\|1_B\|$$

for any pair of sets $A, B$ with $|A| \leq |B|$ and $A < B$. The smallest such $C$ is denoted by $\Delta = \Delta[B, X]$ and we say that $B$ is $\Delta$-conservative.

Of course, $\Delta \leq \Delta_\varepsilon$. In the following result, we characterize when $B$ is partially-symmetric for largest coefficients.

**Proposition 3.10.** Let $B$ be a basis in a quasi-Banach space $X$.

(a) $B$ is partially-symmetric for largest coefficients if and only if there exists a positive constant $C$ such that

$$\|f\| \leq C\|f - S_k(f) + 1_{\varepsilon B}\|$$

for any finite set $B$, any sign $\varepsilon \in \Psi_B$, any element $f \in X$ and any natural number $k$ such that $B \cap \text{supp}(f) = \emptyset$, $k < \min B$ and $\max_{n \in \text{supp}(f)} |e_n^*(f)| \leq 1$. Moreover, if $X$ is a $p$-Banach space and $C$ is the smallest constant satisfying (3.7), then $\Delta_{pl} \leq C \leq A_p \Delta_{pl}$.

(b) $B$ is partially-symmetric for largest coefficients if and only if $B$ is super-conservative and quasi-greedy for largest coefficients. Moreover, if $X$ is a $p$-Banach space, then

$$\Delta_\varepsilon \leq \Delta_{pl},$$

$$C_{ql} \leq (1 + \Delta_{pl}^p)^{1/p},$$

$$\Delta_{pl} \leq (1 + (1 + \Delta_\varepsilon^p)C_{ql}^p)^{1/p}.$$

**Proof.** (a) Assume (3.7) and to show that $B$ is partially-symmetric for largest coefficients, take $A, B, f, \varepsilon$ and $\varepsilon'$ as in (3.4). Define $f' := f + 1_{\varepsilon A}$ and take $k := \max A$. Thus,

$$S_k(f + 1_{\varepsilon A}) = 1_{\varepsilon A}.$$

Hence, applying (3.7), we get

$$\|f + 1_{\varepsilon A}\| = \|f'\| \leq C\|f' - S_k(f') + 1_{\varepsilon' B}\| = C\|f + 1_{\varepsilon A} - S_k(f + 1_{\varepsilon A}) + 1_{\varepsilon' B}\| = C\|f + 1_{\varepsilon' B}\|.$$

Thus, $B$ is $\Delta_{pl}$-partially-symmetric for largest coefficients with $\Delta_{pl} \leq C$.

Conversely, assume that $B$ is $\Delta_{pl}$-partially-symmetric for largest coefficients. Take $B, f, k$ and $\varepsilon$ as in (3.7), and define $A = \{1, \ldots, k\}$ with
\( k < \min B \). Hence, by Proposition 2.1,
\[
\|f\| = \|f - S_k(f) + S_k(f)\| \leq \mathbf{A}_p \sup \{\|f - S_k(f) + \mathbf{1}_E\| : \varepsilon \in \Psi_A\}
\]
\[
\leq \mathbf{A}_p \Delta_{pl} \|f - S_k(f) + \mathbf{1}_{\varepsilon'B}\|,
\]
which proves (a).

(b) Assume that \( B \) is \( \Delta_{pl} \)-partially-symmetric for largest coefficients. Take \( f, A, \varepsilon \in \Psi_A \) as in (3.2). Then
\[
\|1_{\varepsilon A}\|^p \leq \|f + 1_{\varepsilon A}\|^p + \|f\|^p \leq \|f + 1_{\varepsilon A}\|^p + \Delta_{pl}^p \|f + 1_{\varepsilon A}\|^p
\]
\[
= (1 + \Delta_{pl}^p) \|f + 1_{\varepsilon A}\|^p.
\]
Thus, \( B \) is \( C_{ql} \)-quasi-greedy for largest coefficients with \( C_{ql} \leq (1 + \Delta_{pl}^p)^{1/p} \).
The fact that \( B \) is super-conservative with \( \Delta_s \leq \Delta_{pl} \) follows from the definition.

Conversely, assume that \( B \) is \( \Delta_s \)-super-conservative and \( C_{ql} \)-quasi-greedy for largest coefficients. Take \( f, A, B, \varepsilon, \varepsilon' \) as in (3.4). Then
\[
\|f + 1_{\varepsilon A}\|^p \leq \|f + 1_{\varepsilon'B}\|^p + \|1_{\varepsilon A}\|^p + \|1_{\varepsilon'B}\|^p
\]
\[
\leq \|f + 1_{\varepsilon'B}\|^p + (1 + \Delta_{pl}^p) \|1_{\varepsilon'B}\|^p
\]
\[
\leq \|f + 1_{\varepsilon'B}\|^p + (1 + \Delta_{pl}^p) C_{ql}^{pq} \|f + 1_{\varepsilon'B}\|^p
\]
\[
= (1 + (1 + \Delta_{pl}^p) C_{ql}^{pq}) \|f + 1_{\varepsilon'B}\|^p
\]
Consequently, \( B \) is \( \Delta_{pl} \)-partially-symmetric for largest coefficients with \( \Delta_{pl} \leq (1 + (1 + \Delta_{pl}^p) C_{ql}^{pq})^{1/p} \). 

**Question.** Is it possible to characterize super-conservativeness using conservativeness? That is, is there any property \( X \) such that if \( B \) is conservative with the Property \( X \) then \( B \) is super-conservative? This question is still open in the case of Banach spaces.

4. **Characterization of partially-greediness.** The following characterization will be useful in the next section about renormings. Also, a property similar to that appearing in this theorem can be found in [6] and [10].

**Theorem 4.1.** Let \( B \) be a basis in a quasi-Banach space \( X \). Then \( B \) is partially-greedy if and only if there exists a positive constant \( D \) such that
\[
\|f\| \leq D \|f - S_k(f) + z\|
\]
for all \( f, z \in X \) and \( k \in \mathbb{N} \) such that \( |\text{supp}(z)| < \infty, k < \min \text{supp}(z), \)
\( k \leq |\text{supp}(z)|, \text{supp}(f) \cap \text{supp}(z) = \emptyset \) and
\[
\max_{n \in \text{supp}(f)} |e_n^*(f)| \leq \min_{n \in \text{supp}(z)} |e_n^*(z)|.
\]
Moreover, \( \inf D = C_{pg} \).
Proof. Assume \([4.1]\). Let \(f \in \mathbb{X}\), let \(A\) be a greedy set of \(f\) with cardinality \(m \in \mathbb{N}\) and take \(k \leq m\). Define the elements \(f' := f - P_A(f)\) and \(z = P_A(f) - S_k(P_A(f))\). Of course, \(z\) and \(f'\) satisfy the conditions of \([4.1]\). Hence,

\[
\|f - P_A(f)\| = \|f'\| \leq D \|f' - S_k(f') + z\|
\]

\[
= D \|f - P_A(f) - S_k(f - P_A(f)) + P_A(f) - P_A(S_k(f))\|
\]

\[
= D \|f - P_A(f) - S_k(f) + S_k(P_A(f)) + P_A(f) - P_A(S_k(f))\|
\]

\[
= D \|f - S_k(f)\|.
\]

Thus, since this estimate holds for any greedy set \(A\) and any \(k \leq |A|\), the basis is \(C_{pg}\)-partially-greedy with \(C_{pg} \leq D\).

Conversely, assume \(\mathcal{B}\) is a \(C_{pg}\)-partially-greedy basis; we will show \([4.1]\). Take \(f, z\) and \(k\) as in \([4.1]\). Considering the element \(g := f + z\), \(\text{supp}(z)\) is a greedy set of \(g\) with \(m = |\text{supp}(z)|\). Hence,

\[
\|f\| = \|g - z\| \leq C_{pg} \inf_{j \leq m} \|g - S_j(f + z)\|
\]

\[
\leq C_{pg} \|g - S_k(f + z)\| \leq C_{pg} \|f - S_k(f) + z\|,
\]

where the last inequality holds for any \(k\) as in \([4.1]\). Hence, \([4.1]\) is proved. 

Theorem 4.2. Let \(\mathcal{B}\) be a basis in a quasi-Banach space \(\mathbb{X}\). The following are equivalent:

(i) \(\mathcal{B}\) is partially-greedy.

(ii) \(\mathcal{B}\) is quasi-greedy and partially-symmetric for largest coefficients.

(iii) \(\mathcal{B}\) is quasi-greedy and super-conservative.

(iv) \(\mathcal{B}\) is partially-symmetric for largest coefficients and the truncation operator is uniformly bounded.

(v) \(\mathcal{B}\) is quasi-greedy and conservative.

Moreover, \(\Delta_{pl} \leq C_{pg}\), and when \(\mathbb{X}\) is a \(p\)-Banach space, we also have

\[
C_{pg} \leq A_p \Delta_{pl} \Gamma_{\ell}, \quad C_{qg} \leq 2^{1/p} C_{pg}, \quad \Delta_{s} \leq B_p^2 \Delta_{C_{qg}}.
\]

Proof. (i)⇒(ii) The argument to show that the basis is partially-symmetric for largest coefficients comes from \([5\), Proposition 1.13\]. Indeed, take \(|A| \leq |B| < \infty\), \(f \in \mathbb{X}\) such that \(|e_n^*(f)| \leq 1\) for all \(n \in \mathbb{N}\), \(A < \text{supp}(f) \cup B\), \(B \cap \text{supp}(f) = \emptyset\), \(\varepsilon \in \Psi_A\) and \(\varepsilon' \in \Psi_B\). Define

\[
x := f + 1_{\varepsilon A} + 1_{\varepsilon' B} + 1_D,
\]

where \(D = [1, \ldots, m] \setminus A\) with \(m := \max A\). Then, since \(m = |A \cup D| \leq |B \cup D|\) and \(B \cup D\) is a greedy set for \(x\),

\[
\|f + 1_{\varepsilon A}\| = \|x - P_{B \cup D}(x)\| \leq C_{pg} \inf_{k \leq |B \cup D|} \|x - S_k(x)\|
\]

\[
\leq C_{pg} \|x - S_m(x)\| = C_{pg} \|f + 1_{\varepsilon' B}\|.
\]
Hence, $\mathcal{B}$ is $\Delta_{pl}$-partially-symmetric for largest coefficients with $\Delta_{pl} \leq C_{pg}$.
To show that $\mathcal{B}$ is quasi-greedy, taking $k = 0$ in the definition of partially-greediness, we obtain
\[
\|f - PA(f)\| \leq C_{pg}\|f\|
\]
for any finite greedy set $A$ of $f$. Hence, $\mathcal{B}$ is quasi-greedy with $C_{qg} \leq 2^{1/p} C_{pg}$.

(ii)$\Rightarrow$(iii) follows from Proposition 3.10(b).

(iii)$\Rightarrow$(iv) follows from Theorem 3.6 and Proposition 3.10(b).

(iv)$\Rightarrow$(i) Assume that $\mathcal{B}$ is $\Delta_{pl}$-partially-symmetric for largest coefficients and the truncation operator is uniformly bounded with constant $\Gamma_t$.

Take $f \in X$, $P_B(f)$ with $B$ a greedy set of $f$ of cardinality $m$, and let $A = \{1, \ldots, k\}$ with $k \leq m$. Then
\[
f - P_B(f) = P_{(A \cup B)^c}(f - S_k(f)) + P_{A \setminus B}(f).
\]
Thus, taking $t := \min_{n \in B}|e_n^*(f)|$ and applying Proposition 2.1, we get
\[
\|f - P_B(f)\| \leq A_p \sup_{\eta \in \Psi_{A \setminus B}} \|P_{(A \cup B)^c}(f - S_k(f)) + t \eta(B \setminus A)\|.
\]

If we select $\varepsilon \equiv \{\text{sign}(e_n^*(f))\}$, applying the fact that $\mathcal{B}$ is $\Delta_{pl}$-partially-symmetric for largest coefficients in combination with (4.2) we obtain
\[
\|f - P_B(f)\| \leq A_p \Delta_{pl} \|P_{(A \cup B)^c}(f - S_k(f)) + t \varepsilon(B \setminus A)\|.
\]
Since this holds for any $k \leq |B|$ and any finite greedy set $B$, it follows that $\mathcal{B}$ is $C_{pg}$-partially-greedy with $C_{pg} \leq A_p \Delta_{pl} \Gamma_t$.

(iii)$\Rightarrow$(v) This implication is trivial by definitions.

(v)$\Rightarrow$(iii) We only have to prove that $\mathcal{B}$ is super-conservative. For that, consider $|A| \leq |B|$, $A < B$, $\varepsilon \in \Psi_A$ and $\varepsilon' \in \Psi_B$. Then
\[
\|1_{\varepsilon A}\| \leq B_p \sup_{D \subseteq A} \|1_D\| \leq B_p \Delta \|1_B\|.
\]
Now, applying Lemma 2.2(ii), we obtain
\[
\|1_B\| \leq B_p C_{qg} \|1_{\varepsilon'B}\|.
\]
Thus, by (4.3) and (4.4), the proof is complete.

Remark 4.3. The last theorem, concretely the equivalence (i)$\Leftrightarrow$(v), is a generalization of [8, Theorem 3.4] from the setting of Banach spaces with a Schauder basis to the setting of quasi-Banach spaces with a Markushevich basis.

The above theorem gives estimates for the partially-greedy constant using the partially-symmetry for largest coefficients constant and the truncation operator constant. We complete that result providing estimates of $C_{pg}$ in...
terms of the quasi-greedy constant of the basis and some constants related to conservative-like properties.

Theorem 4.4. Let $\mathcal{B}$ be a basis of a $p$-Banach space $X$. Assume that $\mathcal{B}$ is $C_{qg}$-quasi-greedy. Then:

(i) If $\mathcal{B}$ is $\Delta$-conservative, then $\mathcal{B}$ is $C_{pg}$-partially-greedy with

$$C_{pg} \leq C_{qg} \left(2 + (A_pB_p\Delta C_{qg}^2\eta_p(C_{qg}))^p\right)^{1/p}.$$  

(ii) If $\mathcal{B}$ is $\Delta_s$-super-conservative, then $\mathcal{B}$ is $C_{pg}$-partially-greedy with

$$C_{pg} \leq C_{qg} \left(2 + (A_p\Delta_s\eta_p(C_{qg}))^p\right)^{1/p}.$$  

(iii) If $\mathcal{B}$ is $\Delta_{pl}$-partially-symmetric for largest coefficients, then $\mathcal{B}$ is $C_{pg}$-partially-greedy with

$$C_{pg} \leq A_p\Delta_{pl}C_{qg}(1 + C_{pg}^p\eta_p(C_{qg}))^{1/p}.$$  

Proof. As in Theorem 4.2, take $f \in X$, $P_B(f)$ with $B$ a finite greedy set of cardinality $m$, and $A = \{1, \ldots, k\}$ with $k \leq m$. Then

$$f - P_B(f) = P_{(A \cup B)^c}(f - S_k(f)) + P_{A \setminus B}(f).$$  

Of course, since $B \setminus A$ is a greedy set for $f - S_k(f)$, under the hypothesis of quasi-greediness we have

$$\|P_{(A \cup B)^c}(f - S_k(f))\|^p = \|f - S_k(f) - P_{B \setminus A}(f - S_k(f))\|^p \leq \|f - S_k(f)\|^p + \|P_{B \setminus A}(f - S_k(f))\|^p \leq 2C_{qg}^p\|f - S_k(f)\|^p.$$  

Hence,

$$\|P_{(A \cup B)^c}(f - S_k(f))\| \leq 2^{1/p}C_{qg}\|f - S_k(f)\|. \tag{4.5}$$

(i) By (4.5), we only need to control $\|P_{A \setminus B}(f)\|$.

(6) $\|P_{A \setminus B}(f)\| \leq B_p\Delta \max_{n \in A \setminus B} |e_n^*(f)| \|1_{B \setminus A}\| \leq B_p\Delta \min_{n \in B \setminus A} |e_n^*(f)| \|1_{B \setminus A}\| \leq A_pB_p\Delta C_{qg}^2\eta_p(C_{qg}) \|P_{B \setminus A}(f)\|$, where the last inequality is due to $[1, \text{Theorem } 3.10]$. Now, taking into account that $B \setminus A$ is a greedy set for $f - S_k(f)$, since

$$\|P_{B \setminus A}(f)\| = \|P_{B \setminus A}(f - S_k(f))\| \leq C_{qg}\|f - S_k(f)\|,$$

by (4.5) and (4.6) we obtain the result.
(ii) Taking \( \varepsilon = \{ \text{sign}(e_n^*(f)) \} \), we get

\[
\| P_{A \setminus B}(f) \| \leq A_p \Delta_s \max_{n \in A \setminus B} |e_n^*(f)| \| 1_{\varepsilon(B \setminus A)} \|
\]

\[
\leq A_p \Delta_s \min_{n \in B \setminus A} |e_n^*(f)| \| 1_{\varepsilon(B \setminus A)} \|
\]

\[
\leq A_p \Delta_s C_{qg} \eta_p(C_{qg}) \| f - S_k(f) \|.
\]

By (4.5) and (4.7), we obtain the desired estimate.

(iii) follows from Theorem 4.2(iv) and Theorem 3.6.

5. Renormings of partially-greedy bases. In [9], the authors dealt with one of the most difficult problems in greedy approximation theory: renorming Banach spaces with greedy bases. In that paper, the authors characterized 1-greedy bases (generalizing the result proved in [3]) and also proved that, for a fixed \( \varepsilon > 0 \), it is possible to find a renorming in \( L_p, 1 < p < \infty \), such that the Haar system is \((1 + \varepsilon)\)-greedy, but it is open whether one can get the constant 1. Renorming Banach spaces with greedy bases is also recently discussed in [2]. One of the most important tools in all of these papers is convexity! Also, in Banach spaces, it is well known that a renorming \( \| \cdot \|_0 \) of \((X, \| \cdot \|)\) has the form

\[
\| f \|_0 = \max\{ a \| f \|, \| T(f) \|_Y \}
\]

for some \( 0 < a < \infty \) and some bounded linear operator \( T \) from \( X \) into a Banach space \( Y \).

Renorming quasi-Banach spaces with greedy bases has recently been studied in [1], where one of the most important tools is the following lemma:

**Lemma 5.1** ([1, Lemma 11.1]). Let \((X, \| \cdot \|)\) be a quasi-Banach space. Assume that \( \| \cdot \|_0 : X \to [0, +\infty) \) is such that, for every \( t \in F \) and for every \( f \in X \),

- \( \| tf \|_0 = |t| \| f \|_0 \),
- \( \| f \|_0 \approx \| f \| \).

Then \( \| \cdot \|_0 \) is a renorming of \( \| \cdot \| \).

This lemma allows us to have renormings of quasi-Banach spaces based on non-linear operators! Here, we follow the ideas of [1, Theorem 11.3] to give a renorming such that \( C_{pg} = 1 \).

**Theorem 5.2.** Let \( B \) be a partially-greedy basis of a quasi-Banach space \( X \). Then there is a renorming of \( X \) with respect to which \( C_{pg} = 1 \).

**Proof.** Based on Theorem 4.1 we introduce the following quantity:

\[
\| f \|_a = \inf \{ \| f - S_k(f) + z \| : (k, z) \in D(f) \},
\]
where $(k, z) \in D(f)$ if $|\supp(z)| < \infty$, $k < \min \supp(z)$, $\supp(z) \cap \supp(f) = \emptyset$, $k \leq |\supp(z)|$ and $\max_{n \in \supp(f)} |e_n^*(f)| \leq \min_{n \in \supp(z)} |e_n^*(z)|$. By Theorem 4.1 $\|f\|_a \approx \|f\|$ for $f \in X$, so in view of Lemma 5.1 $\|\cdot\|_a$ is a renorming of $\|\cdot\|$.

Take now $(k, z) \in D(f)$ and write $g := f - S_k(f) + z$ and $A = \{1, \ldots, k\}$. Take $(m, y) \in D(g)$ and define $B = \{1, \ldots, m\}$ and

$$B_1 = B \cap \supp(f - S_k(f)), \quad B_2 = B \cap \supp(z).$$

Then

$$g - S_m(g) = f - P_{A \cup B_1}(f) + z - P_{B_2}(z).$$

It is clear that $\supp(z - P_{B_2}(z)) \cap \supp(y) = \emptyset$ and

$$p = |A \cup B_1| = k + m - |B_2|$$

$$\leq |\supp(z)| + |\supp(y)| - |B_2|$$

$$= |\supp(z - P_{B_2}(z))| + |\supp(y)|$$

$$= |\supp(z - P_{B_2}(z) + y)|.$$

We infer that $(p, z - P_{B_2}(z) + y) \in D(f)$ and

$$\|f\|_a \leq \|f - S_p(f) + z - P_{B_2}(z) + y\| = \|g - S_m(g) + y\|.$$

Taking the infimum over $(m, y)$ we get $\|f\|_a \leq \|g\|_a$, and in view of the estimates of Theorem 4.1 the proof is complete.

Final comments and open problems. One of the main results in this paper is the characterization of partially-greediness in the world of quasi-Banach spaces by quasi-greediness and conservativeness (and other close properties). This result is a generalization of the characterization of partially-greedy bases presented in [8] and [5] for Banach spaces. The main techniques to show our result are given in the recent paper [1]. One remark here is that, in [8], the authors introduced partially-greediness as in Remark 3.4 and characterized those bases under the assumption that the basis $B$ is Schauder. In [5], the authors introduced, for Banach spaces, the definition of partially-greediness that we have used here and gave the characterization using Markushevich bases, noting that, under the Schauder basis condition, both definitions are equivalent.

In view of this comment, the main question is the following: if $B$ is a semi-normalized Markushevich basis in a quasi-Banach (or Banach) space, are both definitions equivalent? For Banach spaces, one step in this direction is given in [5], but we do not have a general answer.

Finally, note that the result concerning renormings is the continuation of some results of [1] and, for Banach spaces, it is unknown if it is possible or not to have a similar result.
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Index of the most important constants

| Symbol | Name of constant | Equation |
|--------|------------------|----------|
| $\Delta_{pl}$ | Partially-symmetry for largest coeffs. constant | (3.4) |
| $\Delta_s$ | Super-conservativeness constant | (3.5) |
| $\Delta$ | Conservativeness constant | (3.5) |
| $C_{qg}$ | Quasi-greedy constant | (3.1) |
| $C_{qg}$ | Quasi-greedy for largest coeffs. constant | (3.2) |
| $C_{pg}$ | Partially-greedy constant | (3.3) |
| $\Gamma_u$ | Restricted truncation operator constant | Section 3.2 |
| $\Gamma_t$ | Truncation operator constant | Section 3.2 |

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Partially-greedy bases in quasi-Banach spaces

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