Why Two Renormalization Groups are Better than One.

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Abstract: The advantages of using more than one renormalization group (RG) in problems with more than one important length scale are discussed. It is shown that: i) using different RG’s can lead to complementary information, i.e. what is very difficult to calculate with an RG based on one flow parameter may be much more accesible using another; ii) using more than one RG requires less physical input in order to describe via RG methods the theory as a function of its parameters; iii) using more than one RG allows one to solve problems with more than one diverging length scale. The above points are illustrated concretely in the context of both particle physics and statistical physics using the techniques of environmentally friendly renormalization. Specifically, finite temperature $\lambda \phi^4$ theory, an Ising-type system in a film geometry, an Ising-type system in a transverse magnetic field, the QCD coupling constant at finite temperature and the crossover between bulk and surface critical behaviour in a semi-infinite system are considered.

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1 Introduction

The renormalization group (RG), through its development in various different guises over the last 50 years, has turned out to be one of the most important and powerful tools available in the study of a multitude of different physical phenomena. The proceedings of the RG conferences \[1\] give testament to the widespread applicability of RG methodology. It may be felt that the RG is a well developed and mature field with nothing fundamentally new to be learned. I feel that such a point of view is mistaken and wish in this article to draw attention to an area of RG usage that has been paid little attention: that of using more than one RG.

The principle area of interest where using more than one RG is particularly useful is that of crossover phenomena, the hallmark of which is the existence of more than one asymptotic region as a function of scale wherein the relevant effective degrees of freedom of the system are qualitatively different. A classic example of such behaviour is that of liquid helium confined between two plates so as to form a film geometry \[2\]. As a function of \(L/\xi_L\), where \(L\) is the film thickness and \(\xi_L\) the correlation length in the transverse dimensions, there are two different fluctuation dominated asymptotic regimes. The first is when \(\xi_L/a\), where \(a\) is the lattice spacing, and \(L/\xi_L\) are both large, the corresponding critical behaviour being that of a three dimensional system. On the other hand when \(\xi_L/a\) is large but \(L/\xi_L\) is small the critical behaviour is that characteristic of a two dimensional system. The difference between this and critical phenomena in an infinite space is that in the latter there is only really one dimensionless ratio of interest, \(\xi_L/a\), whereas in the finite system there are two, the former and \(L/\xi_L\). It is the existence of more than one scaling variable that makes crossovers such a rich phenomenon, both experimentally and theoretically.

Tackling crossover problems with many conventional RG methods is often problematical, if not impossible, the reason being that approximation techniques are more often than not associated with an expansion around a particular fixed point. An intrinsic feature of crossover systems, however, is the existence of more than one fixed point. One thus requires an approximation scheme that is capable of encompassing more than one fixed point. Such an approximation scheme has been developed explicitly for a fairly large class of crossovers (though the general methodology should be applicable to any crossover) under the epithet — “environmentally friendly renormalization” \[3\]-\[7\] — which derives from the fact that many crossovers of interest can be fruitfully thought of as being due to the effects of some “environmental” parameter, a prime example being the film thickness, \(L\), above. The idea is that as the relevant effective degrees of freedom of a system exhibiting a crossover change radically as a function of scale relative to the scales set by the “environment”, i.e. the effective degrees of freedom depend on the environment, then any renormalization scheme that purports to describe the crossover must perform depend on the environment aswell. The aim of this paper is not to discuss at length the formalism of environmentally friendly renormalization but to discuss the advantages of using more than one RG. However, given that more than one RG dictates that there be more than one relevant scale, and therefore the possibility of crossover, it is very natural to discuss multiple environmentally friendly RG’s.

Considerations of more than one RG have been mainly confined to quantum field
theory at finite temperature \cite{5, 6, 7, 8} and to that of an $O(N)$ model below the critical temperature where there exist two correlation lengths — transverse and longitudinal. In \cite{9} the fixed point structure of the coupling was considered. Using two RG’s, one with respect to a fiducial value of the transverse mass and one with respect to the longitudinal mass, it was possible to access the line of fixed points associated with the coexistence curve and the critical point. Recently a more extensive analysis of the Goldstone problem has been carried out in four dimensions using RG’s based on modifications of minimal subtraction \cite{10}.

The format of the paper will be as follows: in section two I will make some general comments, without proof, and observations about some of the generic advantages to be gained from using more than one RG. In section three I consider finite temperature field theory/quantum ferromagnets/finite size effects by considering a RG that uses as running parameter a fiducial value of the finite temperature/size mass. In section four I will consider the same problem but with an RG that uses an arbitrary fiducial temperature as running parameter, comparing the advantages and disadvantages of the two groups. In section five I will consider the use of a momentum and a temperature RG together in the context of the magnetic sector of QCD. In section six the crossover between bulk and surface critical behaviour of an Ising model in a semi-infinite geometry will be studied as it illustrates how more than one RG may be of use in a system where there is more than one diverging length scale. Finally in section seven I will make some conclusions and point to some possible future work.

2 General Considerations

In this section I will make some general observations, without supplying formal derivations, associated with the use of more than one RG. In particular I will try to motivate why using more than one RG may be useful. First let us recall what it means to use a single RG. Consider a field theory described by a Hamiltonian/Lagrangian (the former in the context of statistical mechanics and the latter in the context of quantum field theory), $H = H\{\{g_B\}\}$, which is a function of a set of parameters $\{g_B\}$: momenta, sources, masses, coupling constants, background fields, “environmental” variables etc. One may think of the field theory as being defined by these parameters and their conjugate operators. Of interest is how the theory behaves as a function of “scale”, and more often than not in particular how the relevant effective degrees of freedom of the system change as a function of scale, where by scale one often means as a function of one of the parameters.

The parameters $g_B$ can be related, directly or indirectly, to the correlation functions of the theory at some particular scale, normally a microscopic one associated with the cutoff where a mean-field analysis would be valid. At such scales the correlation functions are very simple functions of the parameters. For instance, in $\lambda \varphi^4$ theory the bare coupling $\lambda_B$ is just the four point vertex function in the mean-field approximation. However, well away from this regime, where fluctuations dominate, the vertex functions will be complicated functions of the $g_B$. Moreover, an approximate calculation of them, for example via perturbation theory in one or more of the $g_B$, in a fluctuation dominated regime leads to extremely poor results (apparently infinitely poor results in the case of relativistic
quantum field theory!)
If one thinks of the $g_i^B$ as coordinates one can ameliorate or sometimes even circumvent completely the above problems via a coordinate change on the space of parameters. One defines a coordinate transformation, often by the use of a set of normalization conditions on a relevant subset of the correlation functions, at a certain, arbitrary scale, $\kappa$. Very often the new parameters are simply related to the correlation functions at the scale $\kappa$. One then rewrites the actual correlation functions of the system of interest in terms of the new “renormalized” coordinates $g^i \equiv g^i(\kappa)$. The result is a one-parameter family of coordinate transformations which have a group structure. Although formally nothing can depend on this change of coordinates, i.e. the underlying physical theory is exactly invariant under this group of reparametrizations, when it comes to implementing an approximation scheme, primarily perturbation theory, it is found, as we shall see, that some coordinate systems are decidedly better than others.

One can think of this reparametrization invariance as a global “gauge” invariance, where gauge now takes on a more literal meaning, as it was thought of originally, in terms of “calibration” invariance. By going to renormalized coordinates one chooses to calibrate the physics of a system of interest in terms of the parameters associated with the same system but at some arbitrary, fiducial scale $\kappa$. The scale may be associated with many different quantities, some being more physical than others. For instance, one might decide to parametrize the physics of interest of a system with correlation length $\xi$ in terms of the parameters associated with a similar system but with correlation length $\kappa^{-1}$. Another example to be treated later would be to describe a system at temperature $T$ in terms of parameters associated with a system at a fiducial temperature $\tau$. Of course, $\kappa$ may enter in a less readily interpretable way such as in minimal subtraction.

Now, as mentioned, the RG is generically a one parameter group of reparametrizations of the parameters that describe a physical system. The infinitesimal generator of this flow is simply the vector field $\kappa(d/d\kappa)$, or in a coordinate basis $\kappa(\partial/\partial\kappa) + \beta^i(\partial/\partial g^i)$. If there are $n$ independent parameters, one of which will be used as flow parameter, and therefore $n$ beta functions, some of which may of course be trivial, then in order to solve the flow equations it is necessary to specify $n - 1$ initial conditions per flow line. Geometrically, once an initial condition has been chosen one is restricted to the flow line corresponding to that initial point, i.e. the RG cannot be used to get from one flow line to another. Hence, to span the space of parameters by the RG flows one needs an $n - 1$-dimensional set of initial conditions. For instance, considering the four point function in $\lambda\varphi^4$ theory at the critical point as a function of two momentum variables $p_1$ and $p_2$; then if one chooses a fiducial value of $p_2$ as flow parameter, $\kappa$, then it is possible to use the RG associated with $\kappa$ to generate the behaviour of the four point function as a function of $p_2$. However, one must use as initial condition the four point coupling at some initial value of $\kappa$, $\kappa_0$, and at some value of $p_1$. The latter will be constant along a particular flow line. This means that in order to generate $\Gamma^{(4)}(p_1, p_2)$ one needs as physical input $\lambda(p_1, \kappa_0)$. One cannot generate $\Gamma^{(4)}(p_1, p_2)$ purely through use of this RG with input $\lambda(p'_1, \kappa_0)$ where $p'_1 \neq p_1$. Of course one can change initial condition but this requires further input, i.e. knowledge of $\lambda(p_1, \kappa_0)$. As we will see in many systems such further information might not be readily available.

Another potential drawback along similar lines stems from using the RG to map to
a place in the space of parameters where a perturbative treatment is more trustworthy. This is often done by choosing the RG scale such that in units of this scale the inverse correlation length is of order one, thereby circumventing infrared difficulties. However, in crossover problems there are generally many length scales of interest, and in particular there may exist more than one relevant correlation length that may diverge. Thus fixing the RG scale such that one correlation length is of order unity will not necessarily help in the regime where the other correlation lengths are large.

The above are potential serious complications that arise due to using an RG analysis based on a single flow parameter. Such complications can in principle be overcome by considering an analysis based on more than one RG. The rest of this paper will give concrete examples of more than one RG in action in certain specific contexts. As already stated the intention here is to point out some general features. If one extends the fundamental relation between bare and renormalized vertex functions to the case of $k$ ($k < n$) RG flows one generates $k$ RG equations for the $N$-point vertex functions based on the reparametrization invariance of the bare theory

$$\kappa_j \frac{d \Gamma^{(N)}}{d \kappa_j}(\{g^i(\{\kappa_j\})\}, \{\kappa_j\}) = \frac{N}{2} \gamma_\phi^j(\{g^i(\{\kappa_j\})\}, \{\kappa_j\}) \Gamma^{(N)}(\{g^i(\{\kappa_j\})\}, \{\kappa_j\}) \tag{1}$$

where $1 \leq j \leq k$, $\gamma_\phi^j = d \ln Z_\phi / d \ln \kappa_j$, and the derivatives are taken at fixed values of the bare parameters. One now has a set of $k$ vector fields which are assumed to be independent. As the bare theory is invariant under a Lie dragging along any of these flows then it is also invariant under the action of the vector field $[\kappa_j d/d\kappa_j, \kappa_i d/d\kappa_i]$. This integrability condition leads to non-trivial relations between the beta functions of the theory. In a coordinate representation each vector field can be decomposed as

$$\kappa_j \frac{d}{d\kappa_j} = \kappa_j \frac{\partial}{\partial \kappa_j} + \beta^j_i \frac{\partial}{\partial g^i} \tag{2}$$

where $\beta^j_i = \kappa_j \frac{d g^i}{d \kappa_j}$. Integrability requires that

$$\kappa_i \frac{d \beta^j_i}{d \kappa_i} = \kappa_j \frac{d \beta^j_i}{d \kappa_j} \tag{3}$$

In the case of two RG’s for example the solution of equation (4) is

$$\Gamma^{(N)}(\{g^i(\kappa_1^0, \kappa_2^0)\}, \kappa_1^0, \kappa_2^0) = e^{\frac{N}{2} \int_{\kappa_1}^{\kappa_2} \gamma_\phi^1(x_1, \kappa_2^0) dx_1} e^{\frac{N}{2} \int_{\kappa_2}^{\infty} \gamma_\phi^2(\kappa_1, x_2) dx_2} \Gamma^{(N)}(\{g^i(\kappa_1, \kappa_2)\}, \kappa_1, \kappa_2) \tag{4}$$

A key property of the RG is that if one is interested in a region, such as near a second order phase transition, where naive perturbation theory is invalid one may use reparametrization invariance to map to a coordinate system where perturbation theory is better behaved by a particular choice of scale $\kappa$. In the case of a simple second order phase transition where the only relevant parameter is temperature this can be achieved by choosing $\kappa$ such that in units of $\kappa$ the correlation length is small. A perturbative expansion of $\Gamma^{(N)}(\kappa)$ is then valid. However, in the case when the correlation functions depend on many parameters it may occur that perturbation theory is badly behaved as a function
of more than one parameter. The use of a single RG scale might be sufficient to map to a subspace of the parameter space which is amenable to a perturbative treatment but generically there will exist other regions that are not perturbatively accessible. By using more than one RG one will have more flexibility in finding a region that is perturbatively accessible. For instance, considering again the case of two momentum variables, $p_1$ and $p_2$; assume that the correlation functions exhibit different singular behaviour in the limits $p_1 \to 0$ and $p_2 \to 0$. A single RG may be used to control the singular behaviour in one limit or the other but not both. Having two RG’s enables one to choose one scale $\kappa_1$ to control the singular $p_1$ behaviour and the other to control the singular $p_2$ behaviour. It is worth emphasizing that this example is not as contrived as it sounds. In deep inelastic scattering in QCD there are logarithms that need to be summed that are functions of the two Bjorken variables $x$ and $Q^2$. A single RG is capable of summing one set but not the other, the use of two RG’s gives the opportunity of controlling both sets of logs.

Another important potential advantage concerns what physical input one needs to specify in an RG treatment. In the case of one RG, for an $n$-dimensional parameter space it was necessary to specify $n - 1$ initial conditions per flow line. With $k$ RG’s in principle one need only specify $n - k$ initial conditions and therefore one has access to a $k$-dimensional subspace of the space of parameters using only one initial condition, as by specifying only one initial condition in this $k$-dimensional space one can flow to any other point purely through solving the RG equations. A concrete example of this will be seen in section five.

### 3 Finite Temperature Field Theory: running the mass

In this section I will consider a RG treatment of finite temperature field theory using the finite temperature mass (inverse correlation length) as an arbitrary RG scale. Attention will be restricted here to $\lambda \varphi^4$ theory considering the Euclidean action

$$S[\varphi_B] = \int_0^\beta dt \int d^{d-1}x \left[ \frac{1}{2} (\nabla \varphi_B)^2 + \frac{1}{2} M_B^2 \varphi_B^2 + \frac{\lambda_B}{4!} \varphi_B^4 \right]$$

in terms of bare quantities and the inverse temperature $\beta = 1/T$. The above action also corresponds to a Hamiltonian in the same universality class as that of a $d$-dimensional Ising model in a transverse magnetic field, $\Gamma$ [4, 11] and of an Ising model in a film geometry with periodic boundary conditions [3, 4]. Any results found in the context of finite temperature field theory are thus easily translatable to that of the transverse field Ising model through the identification $M_B^2 = \Gamma - \Gamma_c^{\text{trans}}$, where $\Gamma_c^{\text{trans}}$ is the critical transverse field in the mean field approximation; and in the context of the Ising model in a film geometry with the identification $T = L^{-1}$.

Near a second order phase transition the finite temperature mass, $M(T) \to 0$, which as is well known leads to severe problems in implementing perturbation theory in the infrared. A “standard” renormalization which emphasizes the ultraviolet, such as minimal subtraction, fails to remedy the situation. As the effective degrees of freedom change qualitatively as a function of temperature, the “environment” here, it is wise to use an
environmentally friendly renormalization that respects this fact \[3,5\]. To specify an environmentally friendly set of coordinates I will use the following normalization conditions

\[
\frac{\partial}{\partial p^2} \Gamma^{(2)}(p, M(T) = \kappa, \lambda(\kappa), T, \kappa) \bigg|_{p=0} = 1, \tag{6}
\]

\[
\Gamma^{(2)}(p = 0, M(T) = \kappa, \lambda(\kappa), T, \kappa) = \kappa^2, \tag{7}
\]

\[
\Gamma^{(4)}(p = 0, M(T) = \kappa, \lambda(\kappa), T, \kappa) = \lambda(\kappa). \tag{8}
\]

Here the renormalization is at a fiducial value, \( \kappa \), of the finite temperature mass. I will concentrate, for the purposes of illustration, on the four point function, which with the above normalization conditions gives the beta function at one loop

\[
\beta(h) = -\varepsilon(T/\kappa)h + h^2 + O(h^3) \tag{9}
\]

in terms of the floating coupling \[3,4\] defined by \( h = -\frac{2}{2} \lambda(\kappa) \kappa \frac{d}{d\kappa} \), i.e. by normalizing the coefficient of \( h^2 \) in \( \beta(h) \) to unity. The symbol \( \circ \) stands for the one-loop diagram with \( k \) propagators, without vertex factors, at zero external momentum. It can be obtained from the following basic diagram in \( d \) dimensions

\[
\bigcirc = T \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \ln(k^2 + (2\pi n \tau)^2 + \kappa^2) \tag{10}
\]

\[
= -\frac{\Gamma(-\frac{d}{2}) \kappa^d}{(4\pi)^{d/2}} - \frac{2T^d}{(4\pi)^{(d-1)/2} \Gamma(d/2)} \int_0^\infty dq \frac{q^d}{\sqrt{q^2 + z^2}} e^{\sqrt{q^2 + z^2} - 1},
\]

where \( z = \kappa/T \), by differentiations with respect to \( \kappa^2 \). The first derivative gives \( d \frac{\bigcirc}{d\kappa^2} = \bigcirc \), whereas for \( k \geq 1 \) we have the general rule that the derivative with respect to \( \kappa^2 \) of the loop with \( k \) propagators gives \( -k \) times the loop with \( k + 1 \) propagators.

The solution of (9) is

\[
h(z) = e^{-\int_{z_0}^z \varepsilon(x) \frac{dx}{x}} h_0^{-1}(z_0) - \int_{z_0}^z e^{-\int_{x'}^z \varepsilon(x') \frac{dx'}{x'}} \frac{dx}{x}
\]

where

\[
\varepsilon(z) = 5 - d - (7 - d) \sum_{n=-\infty}^{\infty} \frac{4\pi^2 n^2}{z^2} \left(1 + \frac{4\pi^2 n^2}{z^2}\right)^{\frac{d-9}{2}} \sum_{n=-\infty}^{\infty} \left(1 + \frac{4\pi^2 n^2}{z^2}\right)^{\frac{d-7}{2}} \tag{12}
\]

and \( z_0 = T/\kappa_0 \). For \( d < 4 \) one can take the limit \( \kappa_0 \to \infty \) whereupon the dependence on the initial condition drops out completely and one is left with the universal one loop floating coupling

\[
h(z) = (5 - d) \sum_{n=-\infty}^{\infty} (1 + \left(\frac{2\pi n}{z}\right)^2)^{\frac{(d-7)}{2}} \tag{13}
\]

\[
\sum_{n=-\infty}^{\infty} (1 + \left(\frac{2\pi n}{z}\right)^2)^{\frac{(d-5)}{2}}
\]

\[
\sum_{n=-\infty}^{\infty} (1 + \left(\frac{2\pi n}{z}\right)^2)^{\frac{(d+1)}{2}}
\]
There are several important points to be noted here: firstly $h(z)$ displays three fixed points not just two as would be the case for a beta function based on a non-environmentally friendly renormalization scheme such as minimal subtraction. Besides the trivial Gaussian fixed point, in the limit $T/\kappa \to 0$, one finds a $d$-dimensional fixed point $h = (4 - d)$ equivalent to that found using minimal subtraction. However, in the limit $T/\kappa \to \infty$, which physically corresponds to approaching a second order phase transition, one finds a $d - 1$-dimensional fixed point $h = (5 - d)$ which shows that near the phase transition there is a dimensional reduction. In terms of the dimensionful coupling $\lambda$ the above analysis shows that $\lambda \to M(T)/T$ in four dimensions as $M(T) \to 0$. Of course, this does not mean the theory becomes non-interacting; $\lambda \to (T/M(T))$ as $M(T) \to 0$ and hence diverges, however the product goes to a constant which is associated with the three dimensional fixed point. The floating coupling takes this into account in a very natural way.

Thus one sees that environmentally friendly renormalization is capable of describing the entire dimensional crossover between $d$ and $d - 1$ dimensions. One finds that the dimensionally reduced regime can be described in terms of a set of critical exponents characteristic of $d - 1$ dimensions. These critical exponents have been calculated up to two loop order of a Padé-resummed perturbation for the coupling constant $\lambda$ and are in good agreement with experimental values. One also finds that the crossover is completely universal when $T \ll \kappa_0$ and $M(T) \ll \kappa_0$. That is, in terms of the above results, the crossover in the coupling does not depend on $h(z_0)$.

Given that it is possible to describe the complete crossover here what, if any, are the drawbacks? In the context at hand we must think in terms of what parameters are being used to describe the system. As far as the RG is concerned here $T$ is a fixed parameter and the flow is with respect to the finite temperature mass, i.e. $T$ is a constant along any flow line. In the context of a quantum ferromagnet, where $M(T)$ is a function of $\Gamma$, one has access to another parameter, the transverse field, with which $M(T)$ may be changed at fixed $T$. However, for a relativistic field theory, such as the Higgs sector of the standard model, such a parameter, physically at least, does not exist. Thus in this case the flow consists of how the theory varies as a function of bare Higgs mass at fixed temperature. So although one can access the same phase diagram for the quantum ferromagnet and the Higgs model, in the former case varying the finite temperature mass is physically quite intuitive whereas this is not so for the latter where it is much more natural to think of relating the physics at one temperature to another temperature, zero for example. The other input parameter is $h(T, \kappa_0)$. If one is interested in only universal quantities, as in near the second order phase transition above, then there will be no dependence on $h(T, \kappa_0)$, however, there are also non-universal features of interest, such as the critical temperature, which will depend on $h(T, \kappa_0)$. Thus in order to access such quantities at different temperatures one has to supply more information through the couplings $h(T, \kappa_0)$ at different temperatures.

The above shows that using an RG based on the finite temperature mass as running parameter has advantages and disadvantages depending on the physics associated with the particular problem of interest. It is clear though that there exist quantities of interest, such as the critical temperature, that cannot be accessed purely by the above RG. All the above reasons are sufficient motivation for seeking an alternative RG that may overcome some, if not all the above disadvantages.
4 Finite Temperature Field Theory: running the temperature

Given the drawbacks, at least in the context of finite temperature field theory, of running the finite temperature mass one may ask whether one can use another RG to overcome this difficulty. Given that it is of interest to describe what happens at temperature $T$ in terms of zero temperature parameters I will now consider using a fiducial temperature, $\tau$, as the running parameter. Given that $\tau$ is totally arbitrary and at our discretion one generates an RG based on the independence of the bare theory from $\tau$. Once again given that the aim is to describe more than one asymptotic regime wherein the effective degrees of freedom are very different one must find an environmentally friendly coordinate system. This will be specified by the following normalization conditions (I will restrict attention to the disordered phase $T > T_c$, the extension to $T < T_c$ can be found in [7])

\[
\frac{\partial^2}{\partial p^2} \Gamma^{(2)}(p, M(\tau), \lambda(\tau), T = \tau) \bigg|_{p = 0} = 1, \quad (14)
\]

\[
\Gamma^{(2)}(p = 0, M(\tau), \lambda(\tau), T = \tau) = M^2(\tau), \quad (15)
\]

\[
\Gamma^{(4)}(p = 0, M(\tau), \lambda(\tau), T = \tau) = \lambda(\tau). \quad (16)
\]

The flow functions to one loop are

\[
\beta_M = \frac{\lambda}{2} \tau \frac{\partial^1}{\partial \tau} \quad (17)
\]

\[
\beta_\lambda = -\frac{3}{2} \lambda^2 \tau \frac{d}{d\tau} \quad (18)
\]

Note that in $\beta(\lambda)$ it is a total derivative $d/d\tau = \partial/\partial \tau + \beta_M \partial/\partial M$ that appears rather than a partial derivative. This is essential because as the critical temperature is approached both $\lambda$ and $M \sim \tau - T_c$ (see below) which implies that

\[
\frac{\beta_\lambda - \beta_\lambda^R}{\beta_\lambda^R} = \frac{4\pi \tau}{9M} - \frac{2}{3\pi} + \ldots, \quad (19)
\]

diverges at the critical temperature, where $\beta_\lambda^R$ is the beta function found from using $\partial/\partial \tau$. The additional term (19) is therefore the dominant contribution and cannot be neglected. Thus treating $\tau dM^2/d\tau$ as being of higher order will not be consistent. If one had dropped this term, one would then find that the coupling rather than going to zero approaches a constant. However as is known from previous work [3, 4, 5], and as was seen in the last section, the coupling $\lambda \to 0$ as one approaches the critical point from either above or below. In the present approach this is a direct consequence of the total derivative in the flow function $\beta_\lambda$. Of course, as also mentioned in the previous section the vanishing of $\lambda$ does not imply the theory becomes non-interacting. In terms of the floating coupling using $M$ as a parameter measuring the distance from $T_c$ we obtain for $h$ as $M \to 0$ the flow equation

\[
M \frac{\partial h}{\partial M} = -(5 - d)h + h^2 + O \left( \frac{M}{\tau} \right). \quad (20)
\]
This equation has, in the limit $M \to 0$, a familiar fixed point structure, with a stable, non-trivial fixed point at $h^* = (5 - d)$. Moreover, it has the added advantage of providing a coupling that remains “small” for all temperatures, and is proportional to the zero-temperature coupling in the zero temperature limit.

The differential equation for the coupling (18) is easy to solve, since it contains a total derivative, and to the order we are working takes the same form in both phases. The solution is

$$\lambda^{-1}(\tau) = \lambda^{-1}(\tau_0) + \frac{3}{2} \left[ \mathcal{O} (M(\tau), \tau) - \mathcal{O} (M(\tau_0), \tau_0) \right],$$  \hspace{1cm} (21)

After solving the flow equations one is free to choose the reference temperature $\tau$ equal to the actual temperature $T$ of interest. In fact this is essential if one wishes to obtain perturbatively sensible results for physical quantities. The renormalization conditions \[(15), (16)\] show that the parameters $M(T)$ and $\lambda(T)$ describe the behaviour of the vertex functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ at zero momentum. With these equations one is also able to determine the critical temperature in terms of the zero-temperature parameters $M(0)$ and $\lambda(0)$. As may be expected on dimensional grounds $T_c$ is proportional to $M(0)$, the constant of proportionality being a function of $\lambda(0)$. If one takes values for $\lambda(0)$ and $M(0)$ to be those associated with estimates for the equivalent parameters in the Higgs sector of the standard model one finds, for $\lambda(0) = 1.98$ and $M(0) = 200 \text{GeV}$ that $T_c = 613 \text{GeV}$.

Since $M(T) \ll T$ near $T_c$ the finite-temperature four-dimensional theory reduces there to a three-dimensional Landau-Ginzburg model. In the neighbourhood of the critical temperature the general vertex functions have the form

$$\Gamma^{(n)}_{\pm} = \gamma^{(n)}_{\pm} (T - T_c)^{\nu \left( d_c - \frac{2 + \eta}{2} \right)},$$ \hspace{1cm} (22)

where $d_c = d - 1$ is the reduced dimension at the critical point, $\nu$ and $\eta$ are critical exponents and $\gamma^{(n)}_{\pm}$ are amplitudes. The appearance of the critical exponent $\nu$ is unusual in particle physics. It reflects the need for composite operator renormalization and the physical dependence of $M(T)$ on temperature. This ensures that the exponent $\nu$ is physically accessible in finite temperature field theory whereas it is usually not experimentally observable in a particle physics context as the dependence of the renormalized mass on the bare mass is experimentally inaccessible. As we see here, however, the exponent $\nu$ in fact plays a highly significant role near the critical point.

Near the critical temperature one finds $M^2_{\pm} = (C^{\pm})^{-1} |T - T_c|^\gamma$ with $\lambda_{\pm} = l^{\pm} |T - T_c|^\nu$ and $m_{\pm} = (f_{l^{\pm}}^{-1})^{-1} |T - T_c|^\eta$, where the notation of Liu and Fisher \[(12)\] for the amplitudes is being used. The temperature flow equations give values for the exponents $\nu = 1$ and $\eta = 0$ at one loop, which are not as good as those obtained by flowing the mass parameter. In terms of amplitude ratios one finds

$$\frac{C^+}{C^-} = 4, \quad \frac{f^+_{l^+}}{f^-_{l^-}} = \sqrt{\frac{12}{13}} \approx 1.92, \quad \frac{l^+}{l^-} = \frac{1}{2}. \hspace{1cm} (23)$$

The fact that these are not one is indicative of a cusp in the graphs of mass and coupling versus temperature as the theory passes through the critical temperature. These ratios are universal numbers analogous to the critical exponents. The best estimates for the
amplitude ratios are the high- and low-temperature series expansion results of Liu and Fisher [12] who find

\[
\frac{C^+}{C^-} = 4.95 \pm 0.15, \quad \frac{f_1^+}{f_1^-} = 1.96 \pm 0.01, \tag{24}
\]

which are in good agreement with the above RG results. By comparison: at tree level (mean field theory) \( C^+/C^- = 2 \) and \( f_1^+/f_1^- = 1.41 \), whilst in the \( \varepsilon \) expansion at order \( \varepsilon^2 \), assuming dimensional reduction, \( C^+/C^- = 4.8 \) and \( f_1^+/f_1^- = 1.91 \).

One can see that the values found for the amplitude ratios are substantially better than the corresponding results for critical exponents. This indicates a complimentarity between the current approach of flowing the environment, temperature, and that of the previous section where the flow parameter was the finite temperature mass. At one loop the latter group gives better results for exponents whereas the former gives better results for amplitudes, but both schemes should converge to the same results as one goes to higher orders. The crucial point here is that by using a second different RG it has been possible to access information that would have been very difficult to obtain by running the RG of section 3. Here with this group one has access to non-universal quantities such as the critical temperature and critical amplitudes. It is clear then that there are distinct advantages in being able to implement more than one RG. In the next section we will consider a situation where two groups are used but this time at the same time.

5 Finite Temperature Field Theory: running temperature and momentum

In this section we will consider running two RG’s in the context of investigating the QCD coupling constant in the magnetic sector as a function of momentum and temperature (see [3] for more details). We use as a renormalization condition that the static (i.e. zero energy), spatial three-gluon vertex equals the tree-level vertex in the symmetric momentum configuration

\[
\Gamma_{\text{symm}}^{abc}(p_i^0 = 0, \vec{p}, g_{\kappa, \tau}, T = \tau) = g_{\kappa, \tau} f^{abc}_{ijk} [g_{ij}(p_1 - p_2)_k + \text{cycl.}] . \tag{25}
\]

In contradiction to the previously discussed cases this chosen renormalization condition depends now on two arbitrary parameters, the momentum scale \( \kappa \), and the temperature scale \( \tau \). Therefore we can perform an RG analysis with respect to both parameters, i.e. we can run more than one environmental parameter at the same time.

In order to get rid of ambiguities arising from gauge dependence the Landau gauge Background Field Feynman rules resulting from the Vilkovisky-de Witt effective action are used. Due to the corresponding Ward Identities the calculation is simplified in that one only has to calculate the transverse gluon self energy function \( \Pi^{\text{Tr}} \) in the static limit. In terms of the coupling \( \alpha_{\kappa, \tau} := g_{\kappa, \tau}^2 / 4\pi^2 \) the \( \beta \) functions are then

\[
\kappa \frac{d \alpha_{\kappa, \tau}}{d \kappa} = \alpha_{\kappa, \tau} \left| \vec{p} \right| \left| \frac{d \Pi^{\text{Tr}}}{d \left| \vec{p} \right| \left| \vec{p} \right|} \right|_{\left| \vec{p} \right| = \kappa, T = \tau} ,
\]

\[
\tau \frac{d \alpha_{\kappa, \tau}}{d \tau} = \alpha_{\kappa, \tau} T \left| \frac{d \Pi^{\text{Tr}}}{dT} \right|_{\left| \vec{p} \right| = \kappa, T = \tau} . \tag{26}
\]
The $\tau$ RG is needed to draw conclusions about the temperature dependence of the coupling. This cannot be done using the $\kappa$-scheme alone without assuming something about the temperature dependence of the initial value of the coupling used in solving the differential equation as was seen in section 3 in the context of $\lambda\varphi^4$.

The resulting two beta functions are

$$\kappa \frac{d\alpha_{\kappa,\tau}}{d\kappa} = \beta_{\text{vac}} + \beta_{\text{th}},$$
$$\tau \frac{d\alpha_{\kappa,\tau}}{d\tau} = -\beta_{\text{th}},$$

where the vacuum contribution is, as usual,

$$\beta_{\text{vac}} = \alpha^2_{\kappa,\tau} \left( -\frac{11}{6} N_c + \frac{1}{3} N_f \right),$$

and where, in terms of the IR and UV convergent integrals

$$F^n_n = \int_0^\infty dx \frac{x^n}{e^{\kappa x/2\tau} - \eta} \left[ \log \left| \frac{x + 1}{x - 1} \right| + \frac{\beta_{\text{th}}}{\beta_{\kappa,\tau}} \right],$$

and

$$G^n_n = \int_0^\infty dx \frac{1}{e^{\kappa x/2\tau} - \eta} \left( \frac{x}{x^2 - 1} \right)^n,$$

the thermal contribution is given by

$$\beta_{\text{th}} = \alpha^2_{\kappa,\tau} \left[ \left( \frac{21}{16} F_0^1 + \frac{3}{4} F_2^1 - \frac{3}{2} G_1^1 - \frac{25}{8} G_1^1 + G_2^1 \right) N_c + \left( \frac{1}{4} F_{0}^{-1} + \frac{3}{4} F_{2}^{-1} - \frac{3}{2} G_{0}^{-1} - G_{1}^{-1} \right) N_f \right].$$

Because the two beta functions (27) are not exactly each other’s opposite the RG improved coupling is not just a function of the ratio $\kappa/\tau$. There is another dimensionful scale (such as $\Lambda_{\text{QCD}}$) that comes from an initial condition for these differential equations. The solution of the set of coupled differential equations can be written in the form

$$\alpha_{\kappa,\tau} = \frac{1}{\left( \frac{11}{6} N_c - \frac{1}{3} N_f \right) \ln \frac{\kappa}{\Lambda_{\text{QCD}}} - f \left( \frac{ \kappa }{ \tau } \right)},$$

where the function $f$ satisfies $\beta_{\text{th}} = \alpha^2_{\kappa,\tau} \kappa \frac{df}{d\kappa}$ with the initial condition $\lim_{\kappa \downarrow 0} f = 0$ so that one can identify $\Lambda_{\text{QCD}}$ with the usual zero-temperature QCD scale. Actually this function $f$ can be found in terms of the functions $F$ and $G$:

$$f = \left( \frac{21}{16} F_0^1 + \frac{1}{4} F_2^1 + \frac{7}{8} G_1^1 \right) N_c + \left( \frac{1}{4} F_0^{-1} + \frac{1}{4} F_2^{-1} \right) N_f.$$

The high-temperature behaviour (i.e. for $\tau \gg \kappa$) is determined by

$$f \longrightarrow N_c \frac{11\pi^2}{16} + \left( \frac{11}{6} N_c - \frac{1}{3} N_f \right) \ln \frac{\kappa}{\tau} + O(1).$$

The sign of this coefficient is such that for increasing temperatures at fixed momentum scale one enters a strong coupling regime and the coupling grows without bound. The
opposite sign would lead to asymptotic freedom in this limit as originally suggested in \cite{13}. The limit $\tau/\kappa \to \infty$ is an IR limit where confinement takes place, so unless at higher loop order the magnetic mass increases quickly enough with temperature in order to act as an effective IR cutoff, one apparently cannot circumvent this problem without actually solving confinement. This is an important consideration when considering phase transitions which involve non-abelian gauge fields.

In the regime $\tau \gg \kappa$ the beta functions behave as in a three-dimensional theory so this is the region where dimensional reduction occurs. Here it is natural, as for $\lambda \phi^4$, to use a different dimensionless coupling $u = \alpha_{\kappa, \tau} \frac{\tau}{\kappa}$ since then fixed points may turn up more clearly. However in this case such a reparametrization cannot avoid the strong coupling region.

If one allows the momentum-scale to change with temperature simultaneously, the high-temperature limit can be taken in many ways. In the region $\tau \gg \kappa$ the shape of the constant coupling contours is given by $\tau \sim \kappa \ln \frac{\kappa}{\Lambda_{QCD}}$. This characterizes exactly along which paths in the $(\tau, \kappa)$-plane the coupling increases or decreases. For example at a fixed ratio $\tau/\kappa$ (no matter what this ratio is) one eventually finds a coupling that decreases like $1/\ln \kappa$, much in the same way as at zero temperature. This is a natural contour to consider for a weak-coupling regime where one could treat the quark-gluon plasma as a perfect gas, as then the thermal average of the momentum of massless quanta at temperature $T$ is proportional to the temperature. However at low momenta the assumption of weak coupling breaks down. Furthermore, instead of considering quantities at the average momentum it is more appropriate to use thermal averages of the quantities themselves as a weighted integral over all momenta. But once again one runs into problems at the low-momentum end as long as we cannot treat the strong-coupling regime.

In this section then we see the advantages of using two independent RG’s simultaneously. Using only one initial condition, i.e. one point in the two dimensional $p, T$ parameter space, it is possible to reach any other point purely through RG flows.

6 Bulk/Surface Crossover

In this section I illustrate the use of more than one RG in a case where there exists more than one diverging length scale — the crossover between surface and bulk critical behaviour in a semi-infinite system. The basics of this subject have been covered by Hans Diehl \cite{14} in another article in this volume so here I will give only a cursory treatment. Consider a $d$-dimensional ($d > 2$) semi-infinite Ising-like system described by the continuum Hamiltonian

$$H = \int_0^{\infty} dz \int d^{d-1}x \left[ \frac{1}{2} (\nabla \varphi_B)^2 + \frac{1}{2} (T - T_{cB}^{bm}) \varphi_B^2 + \frac{1}{2} C_B \delta(z) \varphi_B^2 + \frac{\lambda_B}{4!} \varphi_B^4 - H_B \varphi_B \right]$$  \hspace{1cm} (35)

where $C_B$ is the bare surface enhancement parameter, $T_{cB}^{bm}$ is the bulk mean field critical temperature and $H_B$ is a position dependent magnetic field that we can restrict to having only surface support if required. The free field propagator in a mixed representation where we Fourier transform in the infinite transverse directions is

$$G(z, z', p) = \frac{1}{2\kappa_p} \left( e^{-\kappa_p |z - z'|} + C_p e^{-\kappa_p (z + z')} \right)$$  \hspace{1cm} (36)
where $\kappa_p = (p^2 + T - T_{cB}^m)^{\frac{1}{2}}$ and $C_p = (\kappa_p - C_p)/(\kappa_p + C_p)$. The important observation here is that there are two types of pole associated with the propagator for $p = 0$: one where $T = T_{cB}^m$ and one where $(T - T_{cB}^m)^{\frac{1}{2}} + C_B = 0$. The former corresponds to the bulk critical point while the second, for $C_B < 0$ corresponds to a surface phase transition at a mean field surface critical temperature $T_{csm}^m = T_{cB}^m + |C_B|^2$. Note that in this case where the surface interactions are enhanced relative to those of the bulk the surface critical temperature is higher than that of the bulk and therefore the surface will order at a temperature $T_{csm}^m$ in the presence of a disordered bulk.

Note that it is explicitly $z$ dependent \cite{15}. As there are two RG scales one can naturally introduce two dimensionless couplings $\lambda_1 = \lambda \kappa_1^{d-4}$ and $\lambda_2 = \lambda \kappa_2^{d-4}$ with beta functions

$$\kappa_i \frac{d \lambda_i}{d \kappa_i} = -(4 - d) \lambda_i + 2 \gamma_i \lambda_i - 3 \lambda_i^2 \kappa_i^{4-d} F_i(z, \kappa_1, \kappa_2)$$

where

$$F_i = \int_0^\infty dz' \frac{d^{d-1}p}{(2\pi)^{d-1}} G^{(2)}(z, z', p = 0, \kappa_1, \kappa_2) \kappa_i \frac{d G^{(2)}(z, z', p = 0, \kappa_1, \kappa_2)}{d \kappa_i}$$

The conditions \cite{32} and \cite{14} serve to fix the shifts in the bulk critical temperature and the surface enhancement respectively. The second condition determines the wavefunction renormalization to be

$$Z_\varphi(z, \kappa_1, \kappa_2) = 1 + \frac{\lambda}{8} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{C_p}{\kappa^2} e^{-2\kappa p z}$$

As there are two possible, independent diverging length scales in this problem --- the bulk correlation length and the surface correlation length --- it is not clear that a RG based on one parameter will be sufficient to give a reliable perturbative treatment of both phase transitions. That is to say that if one uses an RG to map out from the bulk critical region to a mean field type region where a perturbative treatment should be adequate it is not clear that this will be sufficient to cope with the badly behaved surface fluctuations. It is also clear that due to the inhomogeneity fluctuations are position dependent and therefore to implement an environmentally friendly renormalization one should have a set of normalization conditions that are capable of capturing the relevant position dependence.

Given the existence of two diverging length scales and position dependence the best course of action would seem to be to implement two environmentally friendly RG’s. To show how this works I will consider here only the case of the four point vertex function at one loop. Near the surface phase transition one ought to find a $d-1$ dimensional fixed point for the coupling and near the bulk transition a $d$-dimensional one. For $c < 0$, where $c$ is the renormalized surface parameter, the normalization conditions I will use are the following:

$$\Gamma^{(4)}(z, p_0 = 0, \kappa_1, \kappa_2) \equiv \int dz_1 dz_2 dz_3 \Gamma^{(4)}(z, z_1, z_2, z_3, p_0 = 0, \kappa_1, \kappa_2) = \lambda(z, \kappa_1, \kappa_2)$$

$$\frac{\partial}{\partial z} G^{(2)}(z, z' = 0, p = 0, \kappa_1, \kappa_2) = -\kappa_1 G^{(2)}(z, z' = 0, p = 0, \kappa_1, \kappa_2)$$

$$\Gamma^{(2)}(z = 0, z' = 0, p = 0, \kappa_1 = 0, \kappa_2) = 0$$

The conditions \cite{33} and \cite{14} serve to fix the shifts in the bulk critical temperature and the surface enhancement respectively. The second condition determines the wavefunction renormalization to be

$$Z_\varphi(z, \kappa_1, \kappa_2) = 1 + \frac{\lambda}{8} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{C_p}{\kappa^2} e^{-2\kappa p z}$$

Note that it is explicitly $z$ dependent \cite{15}. As there are two RG scales one can naturally introduce two dimensionless couplings $\lambda_1 = \lambda \kappa_1^{d-4}$ and $\lambda_2 = \lambda \kappa_2^{d-4}$ with beta functions

$$\kappa_i \frac{d \lambda_i}{d \kappa_i} = -(4 - d) \lambda_i + 2 \gamma_i \lambda_i - 3 \lambda_i^2 \kappa_i^{4-d} F_i(z, \kappa_1, \kappa_2)$$

where

$$F_i = \int_0^\infty dz' \frac{d^{d-1}p}{(2\pi)^{d-1}} G^{(2)}(z, z', p = 0, \kappa_1, \kappa_2) \kappa_i \frac{d G^{(2)}(z, z', p = 0, \kappa_1, \kappa_2)}{d \kappa_i}$$

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and $\gamma \varphi^i = d \ln Z_{\varphi}/d \ln \kappa_i$.

As I am considering a situation where the surface orders before the bulk I will be interested mainly in how the four point function varies as a function of the surface correlation length. In the limit $z \to \infty$, $\beta_2 \to -(4 - d)\lambda_2$ which just shows that $\lambda$ scales with its canonical dimension with respect to $\kappa_2$ as one would expect. In the limit $z \to 0$, $\kappa_2 \to 0$ for $\kappa_1$ fixed, i.e. as the surface transition is approached one finds

$$\kappa_2 \frac{d\lambda_2}{d\kappa_2} = -(4 - d)\lambda_2 + 3\lambda_2^2 S_{d-1} \Gamma\left(\frac{d - 1}{2}\right) \Gamma\left(\frac{7 - d}{2}\right) \frac{\kappa_1}{\kappa_2}$$

(44)

where $S_{d-1}$ is the volume of the $d - 1$-dimensional sphere. Bearing in mind that the surface exhibits $d - 1$-dimensional behaviour the natural $d - 1$-dimensional dimensionless coupling is $u_2 = \lambda_2(\kappa_1/\kappa_2)$ which satisfies

$$\kappa_2 \frac{du_2}{d\kappa_2} = -(5 - d)u_2 + 3u_2^2 S_{d-1} \Gamma\left(\frac{d - 1}{2}\right) \Gamma\left(\frac{7 - d}{2}\right) \frac{\kappa_1}{\kappa_2}$$

(45)

which yields precisely the $d - 1$-dimensional Wilson-Fisher fixed point. In terms of the floating coupling one finds

$$\kappa_2 \frac{dh}{d\kappa_2} = -(4 - d_{\text{eff}}^2)h + h^2$$

(46)

where the effective dimension $d_{\text{eff}}^2$ is

$$d_{\text{eff}}^2 = d + \kappa_2 \frac{d}{d\kappa_2} \ln F_2(z, \kappa_1, \kappa_2)$$

(47)

which in the limit $(\kappa_2/\kappa_1) \to 0$, $\kappa_1 z \to 0$ yields $d_{\text{eff}}^2 \to d - 1$. The anomalous dimension $\gamma_\varphi^2$ is

$$\gamma_\varphi^2 = \lambda(\kappa_1, \kappa_2) \frac{\kappa_2^{d-d}}{2(\kappa_1^2 - \kappa_2^2)^{\frac{d-1}{2}}} S_{d-1} \int_0^\infty dp \frac{p^{d-2} e^{-\kappa p z}}{(\kappa p + \kappa_1)(\kappa p + (\kappa_1^2 - \kappa_2^2)^{\frac{d}{2}})^2}$$

(48)

Near the surface phase transition, $z \to 0$, $\kappa_2 \to 0$ one finds $\gamma_\varphi^2 \to \lambda_2(\kappa_1, \kappa_2)(\kappa_2/\kappa_1)$. As $\lambda_2(\kappa_1, \kappa_2)(\kappa_2/\kappa_1) \sim (\kappa_2/\kappa_1)$ one sees that the anomalous dimension of the field at the surface goes to zero as we know it must at the one loop level in a $d - 1$-dimensional $\lambda\varphi^4$ theory.

To describe the extraordinary transition one can use the above techniques but with the added complication that the bulk is ordering in the presence of an already ordered surface and therefore one must use a propagator associated with a $z$-dependent magnetization. Conceptually this is not difficult but makes for much more complicated calculations. For $c > 0$ there is only one diverging length scale, the bulk correlation length, therefore a single RG will suffice. One finds that the normal $d$-dimensional Wilson-Fisher fixed point is the only non-trivial one. More details will be given in a future publication.
7 Conclusions

In this paper I have tried to give a flavour for why one should consider using more than one RG, especially in considering crossover systems. The principal advantages are: i) access to complementary information, as shown in the example of comparing finite temperature field theory with a running finite temperature mass RG and with a running temperature RG; ii) less physical input and therefore more predictive power. This was illustrated in the case of finite temperature QCD where with two RG’s specification of the coupling at one momentum and one temperature was sufficient for the RG to be able to calculate the coupling at any other momentum or temperature. In the case of one RG it would be neccessary to have a line of initial conditions; iii) more flexibility in finding a perturbatively treatable region of parameter space in problems with more than one diverging length scale as was seen in section six.

Without doubt there are very many problems where the use of more than one RG would facilitate a solution, here I will mention just a few of relevance. Certainly the case of the crossover between bulk and surface critical behaviour deserves a much more detaiiled analysis to be able to access full crossover scaling functions for both the surface and extraordinary transitions. A more detailed analysis of the $O(N)$ theory would also be useful. In terms of particle physics the standard model could be examined as a function of the Higgs mass and the mass of the vector bosons for instance; also deep inelastic scattering could be examined as a function of $Q^2$ and $x$, the use of two RG’s being sufficient to resum the logs associated with both variables. Finally, there are many interesting questions to be answered about the methodology of using more than one RG such as: are the beta functions always integrable in perturbation theory? are there any other consistency requirements for using more than one RG etc.?

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