Simplifying monotonicity conditions for entanglement measures

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We show that for a convex function the following, rather modest conditions, are equivalent to monotonicity under local operations and classical communication. The conditions are: 1) invariance under local unitaries, 2) invariance under adding local ancilla in standard pure state 3) on mixtures of states possessing local orthogonal flags the function is equal to its average. The result holds for multipartite systems. It is intriguing that the obtained conditions are equalities. The only inequality is hidden in the condition of convexity.

A basic condition for a function to quantify entanglement is that of nonincreasing under local operations and classical communication (LOCC) [1, 2, 3] (see [4] for review). The condition called LOCC monotonicity is usually not simple to prove for candidates for entanglement measures. The purpose of the paper is to derive conditions equivalent to LOCC monotonicity for convex functions. In other words, we consider a convex function \( f \) and ask what conditions it should satisfy, to be monotone under LOCC. Surprisingly, we obtain that the conditions are rather modest. They are the following: 1) invariance under local unitaries, 2) invariance under adding local ancilla in standard pure state 3) on mixtures of states possessing local orthogonal flags the function is equal to its average:

\[
\sum_i p_i f(\rho_i \otimes |i\rangle \langle i|) = f(\sum_i p_i \rho_i \otimes |i\rangle \langle i|)
\]  

(1)

The last condition can be called affinity on locally orthogonal states. For convex functions, one way inequality follows from convexity. It is rather intuitive that the condition 3) is necessary for LOCC monotonicity. However it is rather surprising that it is also sufficient together with rather modest conditions 1) and 2). Our proofs will be carried out for bipartite states, however they immediately generalize to multipartite case.

To begin with, let us state more precisely what we mean by monotonicity under LOCC. A possible formulation is that for any quantum operation \( \Lambda \) that can be carried out by means of local operations and classical communication we have

\[
f(\rho_{AB}) \geq f(\Lambda(\rho_{AB}))
\]  

(2)

If we treat LOCC operation as measurement with outcomes \( i \), we can rewrite the condition

\[
f(\rho_{AB}) \geq f(\sum_i p_i \sigma_{AB}^i)
\]  

(3)

where \( p_i \) are probabilities of outcomes, and \( \sigma_{AB}^i \) is state given outcome \( i \) was obtained.

One also considers stronger monotonicity condition, which we will adopt in this paper:

**Definition 1** A function \( f \) is LOCC monotone iff it satisfies the following condition

\[
f(\rho_{AB}) \geq \sum_i p_i f(\sigma_{AB}^i)
\]  

(4)

where \( i \) are outcomes of the LOCC operation, \( p_i \) are probabilities of outcomes, and \( \sigma_{AB}^i \) is state given outcome \( i \) was obtained.

We will need the following result of Vidal [3]:

**Theorem 1** A convex function \( f \) is LOCC monotone in the sense of Def. 1 if and only if it does not increase under

a) adding local ancilla

\[
f(\rho_{AB} \otimes \sigma_X) \leq f(\rho_{AB}), \quad X = A', B'
\]  

(5)

b) local partial trace

\[
f(\rho_{AB}) \leq f(\rho_{ABX})
\]  

(6)

c) local unitaries

d) local von Neumann measurements (not necessarily complete),

\[
f(\rho_{AB}) \geq \sum_i p_i f(\sigma_{AB}^i)
\]  

(7)

where \( \sigma_{AB}^i \) is state after obtaining outcome \( i \), and \( p_i \) is probability of such outcome.

**Remark.** From the proof in [3] it is easy to see that the above theorem works also for multipartite systems.

We are now in position to state and prove our new conditions equivalent to LOCC monotonicity for convex functions.

**Theorem 2** For a convex function \( f \) does not increase under LOCC if and only if

1. \( f \) is invariant under local unitary operations

\[
f(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) = f(\rho_{AB})
\]  

(8)

2. \( f \) is invariant under adding local ancilla in arbitrary state at Alice or Bob’s site

\[
f(\rho_{AB} \otimes \sigma_X) = f(\rho_{AB})
\]  

(9)

for \( X = A', B' \).
3. $f$ is affine on locally orthogonal states i.e.
$$f\left(\sum_i p_i \rho_{AB}^i \otimes |i\rangle\langle i|\right) = \sum_i p_i f(\rho^i_{AB} \otimes |i\rangle X (i))$$ \hspace{1cm} (10)
for $X = A', B'$, where $|i\rangle$ are local, orthogonal flags.

**Remark.** Since $f$ is assumed to be convex, in condition 3 it is enough to check inequality in one direction. One can formulate the conditions in a more elegant way as follows

**Theorem 3** For a convex function $f$ does not increase under LOCC if and only if
[LUI] $f$ satisfies local unitary invariance (LUI)
$$f(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) = f(\rho_{AB})$$ \hspace{1cm} (11)
[FLAGS] $f$ satisfies
$$f\left(\sum_i p_i \rho_{AB}^i \otimes |i\rangle X (i)\right) = \sum_i p_i f(\rho_{AB}^i)$$ \hspace{1cm} (12)
for $X = A', B'$ where $|i\rangle$ are local, orthogonal flags.

The condition FLAGS is very intuitive: if we have a mixture of states with local flags, then it is very reasonable to assume that the mixture has entanglement equal to average entanglement of the states.

**Remark.** In the proof we will not use the fact that the system is bipartite, so that the theorem holds also for multiparty systems. It is also worth mentioning that the condition LUI is usually immediate to verify, so that for convex functions monotonicity is in a sense reduced just to the condition FLAGS.

**Proof of equivalence.** Let us argue that the theorems are equivalent. Since the condition LUI is a restatement of condition 1) of Theorem 2 we need to show that FLAGS is equivalent to conditions 2) and 3). First let us see that the condition FLAGS implies condition 2) of Theorem 2. To this end, we consider spectral decomposition of the state $\sigma_X = \sum_k q_k |\phi_k\rangle \langle \phi_k|$. Now in condition FLAGS, we take probabilities to be equal to $q_k$, flags to be $\phi_k$ and the states $\rho_{AB}^i = \rho_{AB}$, and get condition 2). Now let us derive condition 3). To get it from FLAGS we need the following equality
$$f(\rho^i_{AB}) = f(\rho^i_{AB} \otimes |i\rangle X (i))$$ \hspace{1cm} (13)
This however is a consequence of condition 2), which as we have just shown follows from FLAGS.

Now we need to prove that conditions 2) and 3) imply FLAGS. Obviously condition 2) implies 4). which together with 3) gives FLAGS. This ends the proof of equivalence of theorems.

**Proof of Theorem 2** Let us first show that $f$ does not increase under LOCC if and only if
[LUI] $f$ satisfies local unitary invariance (LUI)
$$f(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) = f(\rho_{AB})$$ \hspace{1cm} (14)
[LUI] Let us now prove the converse. Thus we assume that a function is convex and it is LOCC monotone. The condition LUI is is simply condition c), and the conditions a) and b) together imply condition 2), so that it is enough to show that 3) is implied. The inequality $\leq$ in the condition follows from the fact that function is convex. Let us now prove inequality $\geq$. This follows immediately from condition d), if one takes the (incomplete) measurement to be measurement of the flags. This ends the proof.
We will illustrate our theorem by several examples.

Example 1. A well known measure of entanglement is negativity given by
\[ E_N(\rho) = \|\rho^{TA}\| = \|\rho^{\bar{U}}\| \] (17)
where \( T_X \) is partial transpose performed on subsystem \( X \), and \( \|\cdot\| \) is trace norm. It was shown to be LOCC monotone in the sense of definition \( \text{U} \) in \( \text{U} \). A simple proof of monotonicity in the sense of (2) was provided in \( \text{U} \). Now, using our result, we are able to provide equally simple proof of stronger monotonicity of Def. \( \text{U} \). Of course the function \( E_N \) is convex, because partial transpose is linear, and norm is convex. We have to prove the conditions LUI and FLAGS. The condition LUI follows from the fact that partial transpose in a sense commutes with local unitaries. Namely for unitaries \( U_A \) and \( W_B \) we have
\[ U_A \otimes W_B T_{AB} U_A^{\dagger} W_B^{\dagger} = (U_A \otimes W_B) \rho_{AB} (U_A^{\dagger} \otimes W_B^{\dagger}) T_{A} \] (18)
where \( W_B \) is again some unitary. Let us now pass to condition FLAGS. First of all it is easy to see that for operators \( A_i \) of disjoint supports we have
\[ \| \sum_i A_i \| = \sum_i \| A_i \| \] (19)
We will now take \( A_i = p_i (\rho_{AB}^{i})^{TA} \otimes |i\rangle B' \langle i| \). Because of orthogonal flags, the operators have disjoint supports, so that we get
\[ \left\| \sum_i p_i (\rho_{AB}^{i})^{TA} \otimes |i\rangle B' \langle i| \right\| = \sum_i p_i \left\| (\rho_{AB}^{i})^{TA} \otimes |i\rangle B' \langle i| \right\| = \sum_i p_i \left\| (\rho_{AB}^{i})^{TA} \right\| \] (20)
the last inequality follows from the property of trace norm \( \| A \otimes B \| = \| A \| \otimes \| B \| \).

Example 2. Consider relative entropy of entanglement given by
\[ E_R(\rho) = \inf_{\sigma \in S} S(\rho|\sigma) \] (21)
where \( S \) is the set of separable states, and \( S(\rho|\sigma) = \text{Tr} \rho \log \rho - \text{Tr} \rho \log \sigma \). The proof of weaker monotonicity is immediate from definition of the measure \( \text{U} \). The proof of stronger monotonicity is somewhat more involved. However, due to double convexity of relative entropy, we know that relative entropy of entanglement is convex. Then we can apply our criteria. LUI follows immediately from invariance of set \( S \) under local unitary operations. Let us prove that also FLAGS is satisfied. Again, we have to prove inequality "\( \geq \)". To see it, consider arbitrary separable states \( \sigma_{ABB'} \). Since relative entropy does not increase under dephasing, and set of separable states is closed under local dephasing, we can dephase the state on subsystem \( B' \) in basis given by flags \( |i\rangle \) and the obtained state can be only better candidate for infimum on the l.h.s of (12). The new state is of the form \( \sum_i p_i \sigma_{AB}^i \otimes |i\rangle B \langle i| \) where \( \sigma_{AB}^i \) are again separable states. Because of orthogonality of flags we have
\[ S(\sum_i p_i \rho_{AB}^i \otimes |i\rangle B \langle i|) \sum_i p_i \sigma_{AB}^i \otimes |i\rangle B \langle i|) = \sum_i p_i S(\rho_{AB}^i |\sigma_{AB}^i) \] (22)
Thus for any candidate for infimum of left-hand-side, we get a candidate for infimum of right-hand-side, which proves the inequality.

To summarize, we have shown, that for a convex function, the LOCC monotonicity is equivalent to two simple conditions, local unitary invariance and the condition called FLAGS, which roughly speaking means, that the measure should not go down, if we mix states that can be locally distinguished without disturbance. It is rather obvious that that the condition is necessary for monotonicity. However it might be surprising that for convex functions satisfying LUI it is also sufficient. It is also interesting, that the conditions are not inequalities, as might be expected from the nature of monotonicity. The only inequality is hidden in the condition of convexity.

Since the condition FLAGS turned out to be so powerful for functions that are convex and satisfy local unitary invariance, we think it can be also considered on its own, as an important property in the context of distant lab paradigm. However we should remember that in presence of activation effects we would expect that it does not hold for distillable entanglement. (The reason is the same for which the latter measure is expected to be nonconvex.) Finally, we hope that the conditions we derived will simplify proofs of monotonicity for new entanglement measures, in particular, it may help to determine, if candidates for entanglement measures proposed in \( \text{U} \) satisfy LOCC monotonicity (though we do not know whether the proposed functions are convex).

We also hope, that the result may increase our understanding of what is actually hidden behind the postulate of nonincreasing under local operations and classical communication.

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[1] C. H. Bennett, D. P. DiVincenzo, J. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1997), quant-ph/9604024.
[2] V. Vedral and M. B. Plenio, Phys. Rev. A 57, 1619 (1998), quant-ph/9707035.
[3] G. Vidal, J. Mod. Opt. 47, 355 (2000), quant-ph/9807077.
[4] M. Horodecki, QIC 1, 3 (2001).
[5] G. Vidal and R. Werner, Phys. Rev. A 65, 032314 (2002), quant-ph/0102117.
[6] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A 58, 883 (1998), quant-ph/9804024.
[7] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[8] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 84, 4260 (2000).
[9] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 86, 5844 (2001), erratum, quant-ph/9912076.
[10] P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. Lett 82, 1056 (1999), quant-ph/9806058.
[11] P. W. Shor, J. Smolin, and B. Terhal, PRL 86, 2681 (2001), quant-ph/0106052.
[12] M. Horodecki, A. Sen(De), and U. Sen, Phys. Rev. A 70, 052326 (2004), quant-ph/0403169.