Light-like (2,0) Noncommutativity and
Light-Cone Rigid Open Membrane Theory

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Abstract

The six-dimensional (2,0) field theory admits a generalized “noncommutative” deformation associated with turning on a large null 3-form field strength. This theory is studied using its discrete light-cone formulation as quantum mechanics on a blow-up of the ADHM moduli space. We show how to interpret the ADHM manifold as configurations of open membranes, and check our results against basic space-time considerations.
1. Introduction

Quantum theories on non-commutative spaces \cite{1,2} are typically theories of extended objects \cite{3,4,5}. The familiar case is that of a 2-form turned on in a gauge theory, in which case the relevant objects are 1+1 dimensional - either rigid dipoles in the case of space-space non-commutativity \cite{3,4} or fluctuating strings in the time-space case \cite{7,8,9}(see also \cite{10}). A similar mechanism, however, operates when turning on other field strengths, i.e., some higher dimensional extended object (depending on which field strength is turned on) becomes light as the field strength becomes large. In the context of theories decoupled from gravity this was discussed for example in \cite{11,12,13}, and in the context of theories with gravity initially in \cite{14,15}.

In this note we will discuss the deformation of the six-dimensional (2,0) field theory by a 3-form field strength. More precisely we will discuss a deformation by a null 3-form field strength \cite{16}, i.e., a field of the form

\[ H_{+ij} \neq 0 \]  

where \( X^+ \) is a lightcone coordinate, and \( i, j \) are some transverse space-like coordinates. In this case one expects that the extended object that will become light is a open membrane \cite{11,12,13}. If we realize this system as a cluster of M5-branes in 11 dimensional M-theory then the open membrane is the after-decoupling-remnant of an M2-brane ending on the M5-brane \cite{17}. Note that this theory is not OM theory \cite{11,12,13} because the field strength is null and not time like. We will touch on this more at the end of the introduction.

The existence of such a theory has been known for some time now. In \cite{18} a discrete light cone quantization of this theory was suggested, in the spirit of \cite{19,20}. This description is a generalization of the DLCQ description of the (2,0) CFT \cite{21} (or theories with lower susy \cite{22}). Indeed this model was recently used \cite{23} to partially substantiate the extended object nature of the theory by computing leading corrections to the free action of a single tensor multiplets for the case of a single M5-brane. In this paper we will pursue further this description and show how to extract from it information about the configurations and fluctuations of these large membranes.

This is actually only an approximate statement since in this theory there is no truely well defined notion of “the state of a membrane”. As for the (2,0) field theory, this theory has no dimensionless couplings which control the strength of the interactions - all the interactions are of strength one at some energy scale.\footnote{We will be more specific about this later.} Hence the configurations of a single membra...
connected membrane mix strongly with configurations in which there are many smaller membranes. A more precise statement would be to find the wave functional on the space of open membranes which describes an asymptotic scattering state. We will do so later.

Finally, let us comment that turning on a null field strength, as explained in [16], is not the generic field strength which can be turned on [3]. In particular it is not OM theory [11] [12] [13]. In the null theory the particles are replaced by membranes with a small set of dynamical fluctuations. The membranes will have rotational degrees of freedom and their boundary perhaps fluctuates in a limited way, but there are no excitations in the interior of the membrane - i.e., their bulk is rigid. Hence, overall, the number of degrees of freedom relative to the (2,0) CFT is not radically increased. This is to be contrasted with OM theory in which one believes that there is at least some processes in which one will see fully fluctuating open membranes, including fluctuations of the interior. Specializing to this case will enable us on the one hand to have better control of the model, and on the other hand we expect that much of what we say (the open membrane interpretation of the fields in the discrete light cone quantization) will be applicable to the general case.

The organization of the paper is the following. In section 2 we review the set-up and the decoupling limit. We then gauge fix the open membrane to the light cone and extract some basic information about its ground state. We will later compare these results to those of the discrete light cone. In section 3 we review the DLCQ of this theory following [18], show how to derive it from discretizing open membranes, and discuss how to measure the volume of the membrane, its distribution moments, and its ground state wave function. In section 4 we discuss some special examples of points on the ADHM manifold and their open membrane interpretation.

A alternative approach for the quantization of open membranes, and the corresponding derivation of a Matrix model, will be discussed in [31].

2. The NCG (2,0) and its Membranes

2.1. The Decoupled Theory

The kinematical set-up was discussed in [18] [16] and we will only review it briefly here. The background fields that we will turn on are

\[ G^{\mu\nu} = \eta^{\mu\nu}, \]

\[ H_{12+} = H_{34+} \neq 0. \]
As explained in [16] this configuration of $H$ satisfies the self-duality equations for the 3-form field strength on the brane. Next we need to specify the decoupling limit. This is simplest to understand in the case of a single M5-brane. Before turning on the 3-form field, the worldvolume action for a single M5-brane is the free action for a (2,0) tensor multiplet. After turning it on, the action is no longer free but it still has a long wave length approximation. As suggested in [16] the criterion is to keep finite a dimension 9 operator correction to the free action (the operator is discussed in [23]). It is closely related to the extended object nature of the theory. Hence we need to keep finite a dimension -3 combination of the background $H$ field and $M_p$ which will be the coefficient of this operator. This gives us the decoupling limit [18][16]

$$M_p \to \infty, \quad H/M_p^6 = \text{finite.} \quad (2.2)$$

2.2. The Membranes

As mentioned above when one turns on a field strength, what used to be point like particles blows up into extended objects. To which branes they blow up depends on the details of the field strength(s) turned on, but on general grounds these branes are closed and hence only generate multipole moments (generalizing the non-commutative geometry dipoles [4] which appear when turning on a B field). In our case, the ordinary point like particles of the (2,0) multiplet are replaced by open membranes whose boundaries lie on the M5-brane (clearly the theory has been modified from a free theory to an interacting one because, even after decoupling, these open membranes now have dipole charges under the 2-form vector potential on the brane). In this section we will perform a heuristic effective action analysis of these membranes which we will later compare to the discrete light cone predictions.

The light cone formulation of the closed membrane was discussed in [24] (as well as its regularization by matrices, which yields the BFSS matrix theory Lagrangian). The light cone gauge fixing condition is defined by

$$X^\pm = \sqrt{\frac{1}{2}}(X^0 \pm X^1) \quad (2.3)$$

$$X^+(\tau, \sigma_1, \sigma_2) = X^+(0) + \tau, \quad (2.4)$$

3 We will refer to the $k$ M5-brane loosely as “the $U(k)$ case”.

3
and the light cone action is
\[ w^{-1} \mathcal{L} = \frac{1}{2}(D_0X)^2 - \frac{1}{4}\{X^a, X^b\}^2, \]
where \( w \) is a 2D measure normalized to 1. The covariant derivative is defined
\[ D_0X = \partial_0X^a - \{\alpha, X\}, \]
where \( \alpha \) is a gauge field for area preserving diffeomorphisms, which is a gauge symmetry of this Lagrangian:
\[ \sigma^r \rightarrow \sigma^r + \beta^r(\sigma), \quad i = 1, 2, \quad \partial_r(w(\sigma)\beta^r(\sigma)) = 0. \]
which we can write locally as
\[ \beta^r(\sigma) = \frac{\epsilon^{rs}}{w(\sigma)}\partial_s\beta(\sigma). \]
The transformation rules of the fields are
\[ \delta X^a = \{\beta, X^a\}, \quad \delta \alpha = \partial_0\beta + \{\beta, \alpha\}, \]
where the Poisson brackets are defined with respect to the measure \( w \)
\[ \{A, B\} = \frac{\epsilon^{rs}}{w(\sigma)}\partial_rA(\sigma)\partial_sB(\sigma). \]

In order to follow the decoupling limit more carefully we would like to re-introduce \( P_- \) and \( M_p \) into the Lagrangian. This is determined by dimensional analysis and longitudinal boost invariance. At this point we will also switch to an open world volume of the form (a 2-disk of area 1) \( \times \) time (we also set the measure \( w \) to 1). The light cone quantization for the open membrane is discussed at greater length in [26]. The action we obtain is:
\[ \mathcal{L}_0 = \frac{P_0}{2}(D_0X)^2 - \frac{M_p^6}{4P_-}(\{X^a, X^b\})^2 \]

Finally we would like to insert the topological term that comes from the coupling of the boundary to the 3-form field strength. Although it is a boundary term it is more convenient to write it as an total derivative integrated over the bulk of the field
\[ \mathcal{L}_1 = \{X^a, X^b\}H_{+ab} \]
This is of course a special case of the topological term 

\[ \int_{\partial M^3} d\tau d\sigma H_{\mu\nu\rho}X^\mu \dot{X}^\nu X'^\rho \]  

after using the lightcone gauge condition (2.4).

One should, however, be very careful how one uses this action. The reasons are familiar:

1. Strictly speaking, this action corresponds to the first quantized theory (like the cylinder in string theory). The problem is that, as mentioned before, the theory is strongly interacting and hence single and multi membrane states mix strongly.

2. There are still the usual IR problems associated with membranes (in the form of thin long-low-energy spikes extending far from the membrane).

3. As a 2+1 field theory, it is not clear how to make sense of it.

These problems are the same as for the closed membrane, and we expect that a matrix regularization, and a proper interpretation of it, will solve them in the same way that the BFSS matrix model “solves” the problems of the membrane theory of [24]. However, even from the perspective of the BFSS model, the action for the membrane is useful for some questions. For example, constructing membranes in Matrix theory essentially generates this Lagrangian from the matrix partons [25]. Hence we can also hope that some qualitative understanding of the dynamics of membranes can be achieved from the Lagrangian (2.8)+(2.9).

We will therefore analyze this Lagrangian semi-classically to obtain some qualitative understanding of its dynamics. A static extremum of this action satisfies

\[ \{X^a, \{X^a, X^b\}\} = 0 \]  

with the boundary condition

\[ -\frac{M_6^6}{P_-} \{X^a, X^b\} \partial_{||}X_b + 2H_{ab}^{ab} \partial_{||}X_b = 0. \]  

The boundary condition is derived by requiring that no energy leaks from the membrane’s boundary. The rate of energy loss is proportional to \(D_0X^a\) times the LHS of equation (2.12). Requiring that it is zero for all values of \(D_0X\) then gives (2.12). Note that the derivatives are parallel to the boundary rather than transverse to it. This is so because of the unusual form of the gradient energy.
We would like to simplify this set of equations, but in a way that still captures the interesting physics. The most relevant degrees of freedom for an extended object with a dipole charge is of course, its area and orientation, and we will focus on these degrees of freedom. We will therefore take the membrane to be planar and of a fixed shape. More precisely we will assume that the 4 coordinates transverse to the light cone are of the form

$$X^i(\tau, \sigma_1, \sigma_2) = \Sigma_{j=1,2} X^i_j(\tau) \sigma^j. \quad (2.13)$$

This is in the spirit of [3] where the degrees of freedom of the dipole in non-commutative geometry are encoded in its length and orientation.

This ansatz automatically satisfies the bulk equation of motion (we will return to the boundary conditions shortly). The issue now is that the minimum of the potential is degenerate. This manifold of degenerate vacua is what interests us the most. In a situation like this the wave function of the ground state of the system is approximately a uniform wavefunction on the manifold of degenerate vacua.$^4$ In our case the remarkable thing is that this submanifold will be closely related to a certain submanifold of the ADHM space, and indeed the ground state of quantum mechanics on the ADHM manifold will be concentrated around the latter.

For now let us proceed to study the manifold of degenerate vacua. It is easy to see why there are degenerate minima. Completing to a square the potential is

$$\frac{M^6}{4P_-} \Sigma_{a,b}(X^a_1 X^b_2 - X^a_2 X^b_1 - \frac{2P}{M^6} H^{ab})^2 \quad (2.14)$$

For the potential to have a single non-degenerate minimum, then all the terms vanish independently. It is easy to see that this is impossible to achieve. For example if we pick (without loss of generality) $H_{+12} = H_{+34} \neq 0$ then to satisfy (2.12) we need to set, for example,

$$\partial_{\sigma_1} X^1, \partial_{\sigma_2} X^2 \neq 0 \rightarrow \{X^1, X^2\} \neq 0$$

$$\partial_{\sigma_1} X^3, \partial_{\sigma_2} X^4 \neq 0 \rightarrow \{X^3, X^4\} \neq 0$$

but then we run into the problem that

$$\{X^1, X^4\}, \{X^2, X^3\} \neq 0$$

$^4$ Because we are in finite volume, the system is quantum mechanics rather then a quantum field theory with different superselection sectors.
as well.

Characterizing the manifold of degenerate vacua is not difficult. Switching to an
\( SU(2)_L \times SU(2)_R \sim SO(4) \) transverse notation \( X^{ai} \rightarrow X^{\alpha \dot{\alpha} i}, \ i = 1, 2 \) and regarding the
pair of \( X^i \) as \( 2 \times 2 \) matrices (in the \( \alpha \dot{\alpha} \) indices) the potential is

\[
|X^1 X^{2\dagger} - X^2 X^{1\dagger} - H_+ \sigma^2|^2 + |X^{1\dagger} X^2 - X^{2\dagger} X^1|^2. \tag{2.15}
\]

(the \( a, b \) indices of \( H_+ \) are encoded in the matrix \( \sigma^2 \)). It is now easy to minimize the potential. Using an \( SU(2)_R \) (the one broken by \( H_{+ab} \)) we can minimize the first term by
bringing the matrix \( X^1 X^{2\dagger} - X^2 X^{1\dagger} \) to be parallel to \( H_{+ab} = H_+ \sigma^2 \). \( SU(2)_L \) then spans
the degenerate vacua. Up to irrelevant constants the vacuum manifold is therefore given
by

\[
\{X^a, X^b\}_{self-dual} \propto H_{+ab} \tag{2.16}
\]

\[
(\{X^a, X^b\}_{anti-self-dual})^2 \propto |H_+|^2
\]

Finally, in regards to the boundary conditions, one can show that every configuration along
the minima satisfies (2.12), in fact the equations for the minima (in terms of \( X^{1,2} \)) are
precisely (2.12). We will verify the predictions (2.16) in the discrete light cone model in
sections 3 and 4.

Another way of understanding this degeneration is the following. In the case of NCYM
the topological Lagrangian is \( \dot{X}^i B_{ij} \delta^j \) where \( \delta^j \) is the size of the dipole in the \( j \) direction.
This gives the relation \( P_i = B_{ij} \delta^j \), which can be inverted (if \( B \) is degenerate
we can not restrict ourselves to the topological term) to yield a specific dipole direction
as a function of the momentum. In the case of a 3 form the topological Lagrangian is
\( \dot{X}^i H_{ijk} V^{jk} \) where \( V^{jk} \) is the volume in the \( j-k \) plane. This can not be inverted to yield a
specific membrane orientation as a function of the momentum, i.e., there is a degeneracy.

Two comments are in order. The first is that we have dropped a significant amount of
information in this approximation. It should not, however, be difficult to reinstate it. The
2nd is another precursor to what is to come. The readers familiar with the D0-D4 system
will realize that the self-dual part of the potential (2.14) closely resembles the structure of
the F/D-term constraints of the D0-D4 system to which we turn in section 3.
2.3. Another Approach to Open Membranes

We have discussed so far an approach to open membranes in which one takes the world volume to be a disk. This disk replaces the sphere in the case of the closed membrane. One then needs to consider reparametrizations which keep the boundary of the disk fixed, i.e., only a subgroup remains. This approach is discussed in \cite{26,31}. However, in the DLCQ description that we will use \cite{18} (which we will review in the next section) the entire $U(N)$ gauge symmetry is kept, i.e., the reparametrization group of the sphere is retained. We would therefore like to keep the topology of a closed sphere even when discussing the world sheet of open membranes.

This may sound impossible but is actually very familiar - a similar thing happens in string theory. There the insertion of any vertex operators on the world sheet is equivalent to adding a boundary to the world sheet. By conformal invariance such an insertion is equivalent to an infinite tube, and the particle content of the vertex operator is described by boundary conditions at the end of that tube. But as is clear, even though the vertex operator is an implicit boundary, the correct symmetry group is the conformal symmetries of the sphere, except that the vertex operator transforms under these symmetries.

A very similar thing will happen here. One can make the closed sphere into an open sphere by inserting an “impurity” at some point along the sphere. This impurity will allow the fields to be non-smooth around this point, which will in effect make it into an open disk. As in the case of string theory, these impurities will transform under the reparametrizations of the sphere, i.e., under the $U(N)$ gauge symmetry. The readers already familiar with the model in \cite{18}, will guess correctly that these are fundamental hypermultiplets.

In the remainder of this subsection we will try and quantify how many “impurity” degrees of freedom are required in order to make a closed surface into an open one. This is done in preparation for the next section where we will present the discrete light cone description of this model, and where it will become clear what these impurities are.

Instead of 4 real coordinates, we use 2 complex coordinates $X, \tilde{X}^*$, which are $SU(2)_R$ doublet ($SU(2)_L$ mixes $X$ and $\tilde{X}$). The potential is

$$\int d^2\sigma Tr \left( \frac{\{X, X^*\} + \{\tilde{X}, \tilde{X}^*\}}{2\{X^*, \tilde{X}^*\}} - 2\{X, \tilde{X}\} \right)^2$$

(2.17)
Suppose we allow the fields to be discontinuous at one point, i.e., as one approaches the point from different directions, the limiting value of the fields may be different. One typically does not allow this because of energetic reasons. Suppose we cut off a small circle of radius $\epsilon$ around the point, and we allow the fields to be arbitrary around the hole. In this case, the generic behavior of $X$, $\tilde{X}$ around a singularity of the type which interests us is

$$X = f_0(\theta) + f_1(\theta)r + f_2(\theta)r^2 + \ldots$$
$$\tilde{X} = g_0(\theta) + g_1(\theta)r + g_2(\theta)r^2 + \ldots$$

which gives a Poisson bracket which diverges as $\frac{1}{r}$. Hence the divergence behaves like

$$\int_\epsilon r dr \frac{1}{r^2} d\theta f(\theta),$$

where $f$ is some functional of $f_0, g_0, f_1, g_1$. In order to allow for the fields to fluctuate in a generic fashion we would like to eliminate this divergence. This can easily be achieved by inserting some dynamical degrees of freedom at the singularity, such that these exactly cancel the singular part of the Poisson bracket. The action will therefore now be

$$\int d^2\sigma Tr \left( \{X, X^*\} + \{\tilde{X}, \tilde{X}^*\} - 2\{X, \tilde{X}\} \right) + \Delta(\sigma) \frac{1}{\epsilon} L_{2 \times 2}^2,$$

where $\Delta$ is some distribution localized around the specific point which we excise from the worldvolume.

Our task is to quantify what is the minimal set of degrees of freedom in $L$ such that we will remove the $\log(\epsilon)$ singularity in the energy. The matrix of Poisson brackets is clearly a 2 by 2 anti-hermitian matrix in the adjoint of $SU(2)_R$, hence we will require the same from $L$. Such matrices can be parameterized by a single vector which is a doublet of $SU(2)_R$, where $L$ is their bi-linear. Hence we see that in order to open a hole in the worldsheet we need a single doublet of $SU(2)_R$. If we denote this vector as $(Q_1, Q_2)$ then

$$L = \begin{pmatrix} Q_1^* \\ Q_2^* \end{pmatrix} (Q_1, Q_2) - \begin{pmatrix} Q_2 \\ -Q_1 \end{pmatrix} (Q_2^*, -Q_1^*)$$

The values that the $Q$ impurity fields will take in order to obtain a finite energy are given by

$$\lim_{\sigma \to \text{excised point}} \left( \{X, X^*\} - \{\tilde{X}, \tilde{X}^*\} + Q_1 Q_1^* - Q_2 Q_2^* \right) = 0$$

Note again these these look precisely like F/D-terms for a $\mathcal{N} = 8$ system. After discretizing by matrices, these will in fact become exactly that. The fact that these are point like impurities leads to these becoming vectors of $U(N)$ after discretization. We will encounter precisely such fields in the discrete light cone quantization, to which we move next.
3. The Resolved (2,0) Matrix Model

3.1. Review of the Model

The discrete light cone quantization of the (2,0) field theory with a large 3-form turned on is given in [15], where it appeared as a natural generalization of the DLCQ of the (2,0) field theory [21]. The model for the latter is quantum mechanics on the moduli space of Instantons, and turning on a 3-form field (in the decoupling scaling described above) corresponds to resolving this manifold. The model when the field strength is turned on is therefore actually under better control than that of the (2,0) SCFT.

For $k$ M5-branes and momentum $p_- = N/R$, the moduli space of Instantons in question, $\mathcal{M}_{N,k}$, is the moduli space of $N$ instantons in a $U(k)$ group. The deformation which takes us from the (2,0) CFT to the light-like “non-commutative” theory corresponds to turning on a non-zero FI term, which makes the space smooth. We will begin by discussing the unresolved model (of the (2,0) CFT) and then discuss the resolution.

3.1.1 The “commutative” model

A simple concrete description of our theory is as the Higgs branch of a $U(N)$ gauge theory with 8 supercharges, an adjoint hypermultiplet (consisting of 2 complex adjoint field $X, \tilde{X}$), and $k$ fundamental hypermultiplets (consisting of $Q_i, \tilde{Q}_i^\dagger$, $i = 1..k$ in the fundamental (antifundamental) of $U(N)$). We will also denote these occasionally as $Q_\alpha^i$, $i = 1,..k$, $\alpha = 1, 2$). The Higgs branch is parameterized by the values of these fields, subject to the vanishing of the F/D-terms:

\[
[X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger] + Q_i Q_i^\dagger - (\tilde{Q}_i^\dagger)(\tilde{Q}_i) = 0
\] (3.1)

and

\[
[X, \tilde{X}] + Q_i \tilde{Q}_i = 0,
\] (3.2)

modulo $U(N)$ gauge transformation. The total (real) dimension of this space is $4Nk$, and it is a hyperKähler manifold. The space is also equipped with a natural hyperKähler metric which is the restriction of the flat metric on the linear space to $\mathcal{M}_{N,k}$.

The global symmetries of the theory are $SU(2)_R \times SU(2)_L \times Spin(5) \times U(k)$. The first two factors form an $SO(4)$ which correspond to the rotation of the 4 direction transverse to the lightcone coordinates (inside the 5-brane). $Spin(5)$ is an R-symmetry of the (2,0) CFT (which can be understood geometrically as rotations transverse to the 5-brane). The last factor, which is a global flavor symmetry in the quantum mechanics, is believed to
have a 6D dimensional interpretation as a global remnant of the gauge symmetry of the (2,0) CFT, after some gauge fixing when going to the light cone. The supercharges in the quantum mechanics are in the \((2,1,4,1)\) representation of this group, and the fields described above transform as follows:

\[
\begin{array}{cccccc}
U(N) & SU(2)_R & SU(2)_L & Spin(5) & U(k) \\
X_H & N^2 & 2 & 2 & 1 & 1 \\
\Theta_X & N^2 & 1 & 2 & 4 & 1 \\
Q_H & N & 2 & 1 & 1 & k \\
\psi_Q & N & 1 & 1 & 4 & k,
\end{array}
\]

(3.3)

where \(X_H\) denotes the scalars \(X\) and \(\tilde{X}\), and \(Q_H\) denote the scalars in the fundamental \(Q\) and \(\tilde{Q}^\ast\) (and \(\Theta_X\) and \(\psi_Q\) denote their superpartners).

3.1.2 The “Non-commutative” case

We pass from the ordinary (2,0) CFT to the theory with light like (2,0) “non-commutativity” by turning on a field strength \(H_{+ab}\) where \(a, b\) are indices in the 4 coordinates transverse to the light cone coordinates. The self-duality relation then amounts to requiring that \(H\) is self-dual in the indices \(a, b\). Its quantum numbers are therefore such that it transforms under \(SU(2)_R\) and is invariant under all the other symmetries of the model.

Since this model has 8 supercharges, corrections to it are very restricted. Fortunately, there is a simple deformation with precisely the right quantum numbers, which is a deformation of the model by turning on a FI term in the \(U(1)\) part of \(U(N)\). This changes the F/D-term constraints (and breaks the \(SU(2)_R\)) but is otherwise a fairly mild deformation of the model above. By an \(SU(2)_R\) rotation we can bring the F/D-terms to the form

\[
[X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger] + Q_i Q_i^\dagger - (\tilde{Q}_i^\dagger)(\tilde{Q}_i) = \zeta
\]

\[
[X, \tilde{X}] + Q_i \tilde{Q}_i = 0.
\]

(3.4)

It is also easy to figure out the precise relation between \(\zeta\) and \(H\). This relation is uniquely determined by dimensional analysis and longitudinal boost invariance to be (up to a constant which will not be important to us)

\[
\zeta \propto \frac{H}{RM_p^6},
\]

(3.5)

For a more detailed discussion the reader is referred to [18] [16].
3.2. The Open Membrane Interpretation

We have discussed before how to generate open membranes from closed ones by the addition of impurity degrees of freedom on the worldvolume. In this section we will relate this discussion to the discrete light cone model above. Roughly, keeping the topology of a closed sphere corresponds to keeping $U(N)$ as a gauge symmetry and keeping the adjoint matter fields. The addition of impurity degrees of freedom corresponds to the addition of the hypermultiplets $Q$. For the purposes of this section we will work prior to decoupling, i.e., we will not impose that the F/D-terms are precisely zero, but rather one pays a finite amount energy for their violation. This will have the effect of allowing the membranes to fluctuate more freely and it will be simpler to analyze their structure. We will also not be turning on an $H$ field.

It is known that the procedure \cite{24} for discretizing closed membranes actually results in the BFSS matrix model.

\begin{align*}
X(\tau, \sigma) &\rightarrow X(\tau)_{N \times N} \\
\{A, B\} &\rightarrow \frac{1}{N}[A, B] \\
\int d^2 \sigma A &\rightarrow \frac{1}{N} Tr(A)
\end{align*}

reparam. $\rightarrow U(N)$ gauge trans.

These relations continue to hold pretty much as they are in our case.

Next we would like to substantiate the role of the hypermultiplets $Q$ as the impurities which open the surface:

1. We examined before what quantum numbers these impurities are required to have, and it turned out that a doublet of $SU(2)_R$ did the job. These are precisely the quantum numbers of the $Q$’s.

2. We can examine related occurrences of the hypermultiplets $Q$ in matrix models, and see if they correspond to making closed world volumes into open ones. This is indeed the case. Such impurities are responsible, for example, for making closed strings into open strings. For our purposes any Matrix model of a background with a D-brane would do, and we would focus on the case of the D4-brane, which is discussed in \cite{27}. This case is of direct relevance to the M5-brane case since it is its double dimensional reduction, and turning closed strings into open ones is precisely turning closed membranes into open ones.
The Matrix model for IIA in the presence of a D4-brane is readily derived from that of the Matrix model of M-theory in the presence of a 5-brane. One goes to the IIA limit by compactifying on a circle, and the model is now the 1+1 $\mathcal{N} = 16$ SYM on a circle with the addition of $\mathcal{N} = 8$ hypermultiplets localized at discrete points along the circle, $\sigma_i$. The theory away from the impurities has 8 scalar fields out of which 3 parameterize fluctuations parallel to the D4-brane, and the $SU(2) \sim SO(3)$ symmetry that rotates them is nothing but the R-symmetry of the model. The F and D-term constraints of this system are therefore:

$$D_\sigma X^a - \delta(\sigma - \sigma_i)Q^\alpha \sigma_\alpha^{a\beta} Q^*_{\beta} = 0 \quad (3.7)$$

Open strings are now generated when the hypermultiplets $Q$ obtain a VEV. In this case the $X$ variables can jump between the two sides of the impurity. In fact, if we solve for $Q$’s in terms of the $X$’s then any jumps are allowed. The value of the coordinate $X$ as one approaches the impurities from above or below corresponds to the position of the end of the string, and the process by which the value of $Q$ changes from being zero to being non-zero correspond to a closed string splitting into open strings. We are less interested here in the details of this process but the main lesson that we would like to draw is that the generation of holes in the worldsheet is a dynamical process written in terms of worldsheet variables, i.e., the hypermultiplets get a VEV and “cut the world sheet”. Since we have similar hypermultiplets in the ADHM sigma model, the interpretation is clearly similar.

3. Finally, one can examine the form of the $Q$’s within the context of fuzzy spheres. Let us focus on one of the elements in the $SU(2)_R$ doublet, say $Q^1$, which we will denote by $Q$ and take to be a column vector of $SU(N)$. The schematic form of the D-term constraints is

$$[X_i, X_j] + QQ^\dagger + \ldots = 0 \quad (3.8)$$

We are accustomed to thinking about the commutator $[X, X]$ as measuring the volume of the membrane, and this suggests $QQ^\dagger$ is also a volume. More precisely, we would like to identify it as a function on the sphere which has a support of volume $1/N$, i.e., the smallest allowed volume (in this sense it should be considered as a “point” on the fuzzy sphere). Furthermore, since we are not allowed to have any more detailed information on such a small volume, it should be interpreted as a fixed function on such a volume.
As supporting evidence for this interpretation, one can consider the following simple example. Suppose we are interested in a function that is 1 on a strip of width $1/\sqrt{N}$ around the equator. When integrating such a function with any other function which vanishes close to the equator, then the result is zero. Embedding the fuzzy sphere in $R^3$ with coordinates $z_{1,2,3}$, then any function that vanishes near the equator is of the form $z_3 \times P(z_1, z_2, z_3)$ where $P$ is an arbitrary function of the $z$ variables. Hence we require that the matrix corresponding to the equator indicator function $M$ will satisfy

$$Tr(z_3 P(z) M) = 0 \text{ for all } P \to Z_3 M = 0$$

(3.9)

Taking $Z_3$ to be the diagonal ($\sqrt{N}, ..., -\sqrt{N}$) we see that $M$ is of the form $QQ^\dagger$ with

$$Q^\dagger = (0, 0, ... 1_{\frac{N}{2}}, 0, ... 0)$$

(3.10)

Hence we see that the $Q$ is a degree of freedom associated with a minimal area insertion on the sphere - a slightly thickened point or a line. We have chosen here a matrix $M$ which corresponded to the equator but it is not difficult to see that there are choices which correspond to, for example, the poles of the sphere. More precisely, since the smallest volume on the fuzzy sphere is $1/N$ of the volume of the sphere, and we have at most $k$ $Q$ vectors, the maximal excised volume if $k/N$ times the volume of the sphere, which is negligible in the $k << N$ matrix theory limit. We should think about it as an “anti-surface”. By the insertion of such an impurity we allow the surface to have a boundary, which is no other then the boundary between the minimal impurity surface and the rest of the sphere. Hence we are able to generate open membranes from closed ones.

### 3.3. Measuring Open Membrane

This also leads us to the more precise way of measuring the surface of the membrane. We have use the fact before that $[X^i, X^j]$ provides us with information about the volume in the $i - j$ plane. The problem is, of course, that if we want to get a number for the volume the we face the obvious problem that $Tr([X^i, X^j]) = 0$. In the usual applications of Matrix theory this does not bother us, since the volume comes with a sign (i.e, its the source for a charge) and the two orientations of the closed compact membrane cancel each

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6 The choices of different geometries for the minimal area patches probably corresponds to different orbits of the $SU(2)$ isometry of the sphere in the $N$’th dimensional representation.
other, yielding the correct result that the signed volume is zero. However, for the case of
the open membrane it is a problem since an open stretched compact membrane can have
a net signed volume.

One way out is, as for closed membranes, to work with dipoles and higher moments.
This, however, is unsatisfactory since the volume is a well defined concept and we would
like to be able to measure it. It also turns out that in specific cases (discussed later) naively
computing the moments gives results which are difficult to interpret. There is fortunately a
simple resolution, along the lines of the discussion above, which matches our expectations
(in the large $N$ limit).

Recall that we have divided our sphere into two parts where the smaller part is actually
excised, and is indicated by where the $Q, \tilde{Q}$ vectors are supported. By the volume of the
membrane in spacetime we actually mean the volume that is covered by the rest of the
sphere. Therefore if we want to measure the volume we actually need to integrate over
the sphere except the excised patch. Denoting by $A$ the excised patch on the sphere, and
by $\text{ind}(A)$ a function that is 1 on that patch and 0 elsewhere, then the integration we are
interested in is actually

$$\int d^2\sigma \{X_1, X_2\}(1 - \text{ind}(A)).$$

(3.11)

which has the immediate generalization to the non-commutative case

$$\text{Tr}([X^1, X^2](1 - P))$$

(3.12)

where $P$ is the matrix that corresponds to the indicator function, i.e., it is a projection
operator which projects out the subspace spanned by $Q, \tilde{Q}$. Note that typically this is a
two dimensional space, but it may degenerate to a one dimensional space in special points
on the moduli space. In the non-degenerate case the matrix is given by

$$P = \frac{QQ^\dagger}{|Q|^2} + \frac{\tilde{Q}^\dagger \tilde{Q}}{|\tilde{Q}|^2}.$$  (3.13)

This is the first step in the definition but it is not complete because we still need to specify
how to compute

$$\int d^2\sigma \{X_1, X_2\}(1 - \text{ind}(A)) f(\sigma_1, \sigma_2)$$

(3.14)

for an arbitrary $f$. The issue is to specify the insertions of $1 - P$ for the general case. To
solve this we would like to find a matrix form for $\{X^1, X^2\}$, which already includes the
$1 - P$ projection, and would be suitable for integration at with any function. At this point
we will motivate such a guess, and will check it in a later section. There it will be clear that in order to obtain sensible results it is necessary to insert the projection operator $1 - P$.

Recall that in [25] for the case of a closed membrane, the charge density $[X^1, X^2]$ was obtained by a computation analogous to a computation of the central charge density in BPS formulas via commutators of charges and currents. In order for the central charges to be real, then $[X^1, X^2]$ has to be anti-hermitian (the analogue of imaginary for matrices). The charge density that we will define will still have this property. Hence we would like to modify the expression $[X^1, X^2]$ by $1 - P$ in a way which preserved anti-hermiticity. The natural guess is

\[
\text{charge density} = (1 - P)[X^1, X^2](1 - P). \tag{3.15}
\]

We will shortly provide some checks on this expression for specific discretized open membrane configurations.

Finally let us examine the volume given by this prescription and compare them to the volumes we estimated before.

\[
\text{Tr}([X, \tilde{X}](1 - P)) = 0 \tag{3.16}
\]
\[
\text{Tr}(([X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger])(1 - P)) \propto (N - 2)\zeta \tag{3.17}
\]
i.e, these are the same volumes as we suggested before in (2.16), in the large N limit.

3.4. An Example: Equations of motion

We have seen before that the finite energy configuration space for the membrane is

\[
\{X^a, \{X^a, X^b\}\} = 0. \tag{3.18}
\]

For all other configurations one has to pay an energy of proportional to $M_p$, i.e., they are not allowed. We would like to verify this relation in the Matrix model.

This will also serve to check the prescription (3.15), that the membrane is actually described by the $X$ matrices projected by $1 - P$. We are interested in the product of $\{X^a, \{X^a, X^b\}\}$ restricted to the non-excised volume of the sphere. Hence a guess for this product would be the symmetric product

\[
(1 - P)[X^a, [X^a, X^b]](1 - P) \tag{3.19}
\]

More tests are clearly needed since we will check only a few very special cases.
on the ADHM moduli space. We will evaluate

\[ [X^a, [X^a, X^1]] = \]

\[ = -\frac{1}{4} [X - X^\dagger, -QQ^\dagger + \tilde{Q}^\dagger \tilde{Q}] + \frac{1}{2} [\tilde{X}^\dagger, Q\tilde{Q}] - \frac{1}{2} [\tilde{X}, \tilde{Q}^\dagger \tilde{Q}] \]

where we have used the F/D-term. This clearly satisfies

\[ (1 - P)[X^a, [X^a, X^1]](1 - P) = 0 \]

which is the expected result.

4. Some examples

We would now like to discuss the open membrane interpretation of some points in \( M_{N,k} \), as a check on the formulas we derived above. For most of the \( M_{N,k} \) explicit solution for the ADHM constraints are not known, but some special cases are known. The case that we will focus on is the one that corresponds to a single 5-branes, i.e., instantons in \( U(1) \). The manifold of two such instantons is very familiar and we will discuss it at length as our basic check. Then we check the validity of the formulation for an arbitrary number of instantons on a configuration of a large round open disk, and finally we will briefly discuss well separated multi-membrane configurations as a check that the model factorizes properly.

4.1. The \( P_- = 2/R \)

We begin by discussing the case of two instantons in a \( U(1) \) gauge field. Before turning on the blow-up parameter the moduli space is

\[ M_{2,1,\zeta=0} = R^4 \times (R^4/Z_2) \]

The FI term acts as a blow-up of the \( Z_2 \) singularity and the 2nd component is replaced by a smooth ALE space.

The \( R^4 \) part has the obvious interpretation of the center of mass coordinate for the entire system. The \( R^4/Z_2 \) part, before blowing up, is also easy to interpret - it describes the relative position of two identical particles (each carrying one unit of momentum). The metric on the relative position component is flat at any non-zero separation of the particle,
which corresponds to the fact that the single (2,0) multiplet theory, without an $H$ field, is a free theory.

The interpretation as two particles is not the entire story. Even before blowing up one can ask where is the single particle state with two units of momenta. There is a natural candidate for it, which is the harmonic wave function dual to the shrunken cycle at the origin of (defined by a limiting process as we shrink the cycle, and defines a singular form at the end). But because the theory is free, it is up to us whether or not we include such a state. Since the two $p_- = 1/R$ particles can not dynamically combine to make the $p_- = 2/R$ state, there is no dynamical necessity for including it at this point.

By turning on a non-commutativity parameter we introduce interactions into the theory. It is now necessary that the Hilbert space includes the single particle $p_- = 2/R$ state. Indeed, $\mathcal{M}_{2,1}$ has now been smoothed out, and the aforementioned harmonic form is a smooth well behaved form on this space. Hence it is automatically included in the Hilbert space of the theory (which the Hilbert space of $L^2$ harmonic forms on the space) and there is in fact no natural way of separating it out. The state which interests us the most is precisely this single particle $p_- = 2/R$ state. The reason is that it is this type of states which are relevant in the limit $N \to \infty$, $E_{DLCQ} \propto 1/N$ (which is necessary in order to obtain non-compact space-time results from a discrete light cone quantization procedure). This is similar to the statement that in the BFSS model, the relevant excitations are the bound states of D0-branes with a finite fraction of the null momentum.

After blow-up this form is perfectly regular and manageable. Since it corresponds to a single particle state it is actually as close as we can get to the state of a single membrane (up to the caveats mentioned before). Because we have put all the momentum in a single membrane, it is also the largest membrane in the $p_- = 2/R$ sector, which simplifies the analysis.

Fortunately the Harmonic form corresponding to this states is very familiar. In the resolved space we have blown-up the singular point into a sphere of some finite radius set by $\zeta$. The harmonic form is localized near this two sphere and decays (like a power law) as we move away from the sphere (and from what used to be the origin of the space before blow-up). To have an interpretation of a single open membrane we clearly need to stay close to this sphere because as we go away the interpretation as two separate $p_- = 1/R$ particles becomes better. The fact that there is a smooth transition from one interpretation to another, and that the wave function is, strictly speaking, supported at both is nothing but a manifestation of the single-membrane/multi-membrane mixing.
Therefore, we would like to study configurations along this minimal sphere and see whether they agree with our expectations from the previous sections. The form of the ADHM matrices on this cycle are given in [28], and are

\[
X = \begin{pmatrix} 0 & p_1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & p_2 \\ 0 & 0 \end{pmatrix}, \quad Q = \sqrt{2\zeta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{Q} = 0
\] (4.2)

where \(p_1\) and \(p_2\) are complex numbers satisfying the constraint

\[|p_1|^2 + |p_2|^2 = \zeta.\]

The 4 real coordinate matrices are then easy to derive \((X = X_1 + iX_2, \ \tilde{X} = X_3 + iX_4)\) and, using formula (3.12), give us the following volumes:

- The self-dual volume:
  \[
  V_{12} + V_{34} = \frac{1}{2} \zeta, \\
  V_{13} - V_{24} = 0, \\
  V_{14} + V_{23} = 0
  \] (4.3)

- and the anti-self-dual volume:
  \[
  V_{12} - V_{34} = \frac{1}{2} (|p_1|^2 - |p_2|^2), \\
  V_{13} + V_{24} = \frac{1}{2} (p_1p_2^* - p_1^*p_2), \\
  V_{14} - V_{23} = -\frac{1}{2} (p_1p_2^* + p_2p_1^*)
  \] (4.4)

The right hand quantities can be written as

\[\propto (p_1, p_2)\sigma^i(p_1, p_2)^\dagger\]

Both of these match precisely our expectations from section 2.

Next we would like to check the situation of a pair of well separated particles, each with momentum \(p_- = 1/R\). As we explained above, this is the useful approximate description far away from the origin of the \(R^4/Z_2\). We will see that in this regime the insertion of the projection operator \(1 - P\) (as in (3.12)) is necessary in order to compute some quantities, (although it does not solve all the puzzles in other quantities).

The general solution for 2 instantons in \(U(1)\) was given in [29]. The solution is (neglecting the center of mass)

\[
X = \frac{z_0}{2} \begin{pmatrix} 1 & \sqrt{2b/a} \\ 0 & -1 \end{pmatrix}, \quad \tilde{X} = \frac{z_1}{2} \begin{pmatrix} 1 & \sqrt{2b/a} \\ 0 & -1 \end{pmatrix}, \quad Q = \sqrt{\zeta} \begin{pmatrix} \sqrt{1-b/a} \\ \sqrt{1+b/a} \end{pmatrix}, \quad \tilde{Q} = 0
\] (4.5)
where
\[ a = \frac{|z_0|^2 + |z_1|^2}{2\zeta}, \quad b = \frac{1}{a + \sqrt{1 + a^2}} \]  
(4.6)

When \( z_0 \) and \( z_1 \) are taken to infinity this corresponds to two well separated particles (each blown up into a small membrane).

We would like to check how our procedure fares in computing the position of the particles. We will compare the results with and without the insertion of the projection operator, i.e, for example,

\[ \text{Tr}((X, X^\dagger) + [\Xi, \tilde{X}^\dagger])X^{2l}) \quad \text{vs.} \]

\[ \text{Tr}((1 - P)([X, X^\dagger] + [\Xi, \tilde{X}^\dagger])(1 - P)X^{2l}) \]

where we have used the volume elements (the commutator) parallel to \( H_+ \) to indicate where the particles are (the commutator is non-zero there, indicating that there is a small piece of open membrane there). Computing the trace without the projection operator might also appear to be a natural extension of the computation for the closed membrane.

Evaluating these expression to the leading order \( 1/|z|^2 = 1/(|z_0|^2 + |z_1|^2) \) we obtain that the first expression is

\[ \text{Tr}\left(\left(\frac{4\zeta^2}{|z|^2} - \frac{4\zeta}{|z|^2}\right)\left(\frac{z_0}{2}\right)^{2l}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \sim \frac{|\zeta|^2}{|z_0|^2 + |z_1|^2}\left(\frac{z_0}{2}\right)^{2l} \]

and the 2nd is

\[ \text{Tr}\left(\left(\begin{array}{cc} 1/2 & -1/2 \\ -1/2 & 1/2 \end{array}\right)\left(\begin{array}{cc} 4\zeta^2/|z|^2 & -4\zeta \\ -4\zeta & -4\zeta^2/|z|^2 \end{array}\right)\left(\frac{z_0}{2}\right)^{2l}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \sim \left(\frac{z_0}{2}\right)^{2l} \]

where the 1st matrix in 2nd expression is the projection operator

\[ 1 - P \rightarrow 1 - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \]

using (4.5). It is clear that without inserting the projection operators we obtain results which are difficult to interpret, whereas with the insertion of the projection operators we precisely get the expected results.

On the down side, even with the projection operator not all expression are easy to interpret. For example one can evaluate the volumes in the different directions. Of the self-dual volumes, only \( \text{Tr}((1 - P)([X_1, X_2] + [X_3, X_4])) \) is non-zero, as it should be. However, the anti-self-dual volumes are also not zero. Unlike when discussing a single membrane,
this result is not what one expects, although only applying some indirect arguments. When the two membranes are far apart their anti-self-dual volumes are uncorrelated (the self-
dual-volume are correlated with the external field, and therefore parallel between the two remote membranes). However, for a tiny membrane corresponding to \( p_- = 1/R \) (say, at the lowest level of the DLCQ) there are no degrees of freedom to store the information on the direction of the membrane, hence the \( p_- = 1/R \) describes the wave function of the tiny membrane after averaging over all anti-self-dual orientations. Hence the average signed volumes for the \( p_- = 1/R \) particles should vanish. In particular it should vanish for two remote membranes at the \( p_- = 2/R \) level, which is not what we find. We would like to suggest, however, that this is an artifact of the fact that we are working at small \( p_- \). Clearly the lack of ability to describe independent orientations of the two remote membranes has to do with the fact that the manifold is of very low dimension, i.e., low \( p_- \).

4.2. A Large Round Disk Solution

We are going to discuss the simplest coplanar arbitrary \( N \) solution which corresponds to an open membrane whose image in spacetime looks like a planar disk. The solution is

\[
X = \begin{pmatrix}
0 & \sqrt{N-1} \\
0 & \sqrt{N-2} \\
0 & 1 \\
0 & 0
\end{pmatrix},
Q \propto \begin{pmatrix}
0 \\
0 \\
1 \\
1
\end{pmatrix},
\tilde{Q} = \tilde{X} = 0 \quad (4.8)
\]

This configuration has a \( U(1) \) symmetry which mixes \( X \) and \( X^\dagger \), suggesting that this membrane is the round disk. We would like to compute the size of the membrane in the large \( N \) limit and we will do so by computing its moments \(< r^l >\) which are, to leading order in \( N \) (and neglecting overall coefficients)

\[
Tr((1-P)[X,X^\dagger](1-P)(XX^\dagger)^l) \sim \frac{1}{l+1}N^{l+1} \quad (at \ large \ N) \quad (4.9)
\]

which indicates a uniform disk of radius \( R \propto \sqrt{N} \) (ordering ambiguities in the translation from \( r^2 \) to \( XX^\dagger \) are subleading in \( N \)).

This is also a good test case to examine our procedure for inserting the projection operator \( 1 - P \). Without inserting it, the results for the moments would be

\[-N^{l+1} + \frac{1}{l+1}N^{l+1}\]

which corresponds to a disk of membrane with a ring of anti-membrane charge along its perimeter. Again is a configuration which is difficult to interpret, whereas the insertion of the projection operators yields precisely the expected results.

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5. Discussion

The D0-D4 system deformed by a FI term is the discrete light cone quantization of the 6-dimensional (2,0) theory with a large $H_3$ field turned on (in a specific way), i.e., a theory which generalizes non-commutative geometry to 3-forms. We showed how to naturally interpret this model as configurations of open fluctuating membranes, how to evaluate the membrane ground state wave function and suggested how to measure some aspects of membrane size and shape. The key to this identification is the addition of the matter fields in the fundamental representation of the $U(N)$ gauge symmetry - the reparametrization of the sphere - which correspond to point like impurities on the surface on which the surface can rip open and become an open membrane.

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