Valiron-Titchmarsh Theorem for Positive Temperatures

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Abstract. In this note, we prove an analog of the Valiron-Titchmarsh theorem for positive temperatures.

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1. Introduction and Statement of Results

Let \( f(z) \) be an entire function of order \( \rho < 1 \) with only negative zeros.

\[
\log f(r) \approx \frac{\pi}{\sin \pi \rho} r^\rho \text{ as } r \to \infty.
\]

If the counting function \( n(r) \) of the zeros of \( f \) satisfies \( n(r) \approx r^\rho, r \to \infty \), then, it is immediate that

\[
\log f(r) \approx \frac{\pi}{\sin \pi \rho} r^\rho, r \to \infty.
\]

The converse statement of Tauberian nature was proved independently by Valiron and Titchmarsh [14], and it is often referred as the Valiron-Titchmarsh theorem on entire functions with negative zeros; its current exposition can be found, in [11, Lec 12-13]. For the history and generalizations of the Valiron-Titchmarsh theorem (see, e.g., [10] or [5] and the references therein. In [13], it was shown that, for an entire function of non-integer order with zeros on the negative real half-line, the existence of the asymptotics of a certain form for one of the functions along some ray implies the existence of certain asymptotics for the counting function of the zeros. In [5], the author showed that for an entire function \( f(z) \) of finite order \( \rho \) with all its zeros lie on a finite collection of rays in the interior of a sector \( S: \alpha < \arg z < \beta \), such a function is called completely regular along \( \arg z = \theta(\alpha, \beta) \). If the \( \lim_{r \to \infty} r^{-\rho} \log |f(re^{i\theta})| \) exists, then \( f(z) \) is of completely regular growth on the whole plane. Recently, [6] extended the Drasin complement to the Valiron-Titchmarsh theorem and showed that if \( u \) is
a subharmonic function of this class and order 0 < ρ < 1, then the existence is the lim_{r→∞} log \frac{u(r)}{N(r)}. In [7], the problems under consideration examined the relationship of the initial data $f$ and that of the solution $u$. From the main theorem, we proved the interesting corollary that $u(x, t) = \alpha x^\rho + \bar{\alpha}(r^\alpha)$ for each $t \to \infty$ if and only if $f(y) = \alpha y^\rho + \bar{\alpha}(y^\alpha), y \to \infty$. In [9], we considered two-term analogs of the Valiron-Titchmarsh theorem for the temperatures.

In this note, we prove an analog of the Valiron-Titchmarsh theorem for positive temperatures, i.e for positive solutions of the heat equation.

(1.1) \[ \frac{\partial u(x, t)}{\partial t} = \kappa \frac{\partial^2 u(x, t)}{\partial x^2} \]
in the slab $S^T = \mathbb{R}^2 \times (0, T)$, where the constants $\kappa > 0$ and $0 < T \leq \infty$. Thereafter, these solutions are called temperatures.

It is known [16, p.57] that a positive temperature in $S^T$ has the Gauss-Weierstrass representation

(1.2) \[ u(x, t) = \frac{1}{4\pi\kappa t} \int_{\mathbb{R}^2} e^{-\frac{1}{4\kappa t} ||x-y||^2} \, d\mu(y). \]

where $x = (x_1, x_2), x = (r \cos \theta, r \sin \theta), r \geq 0, 0 \leq \theta < 2\pi$, and $y = (y_1, y_2), y = (s \cos \phi, s \sin \phi), s \geq 0, 0 \leq \phi < 2\pi$ in spherical coordinates respectively. Here, $|| \cdot ||$ is the Euclidean norm in $\mathbb{R}^2$, and $d\mu$ is a non-negative function on $\mathbb{R}^2$.

It is known [16, p.57, Theor 2.10] that if $u$ is real-valued and continuous up to the boundary $\mathbb{R}^2 \times [0]$, then $d\mu(y) = u(y, 0)dy$. It is also known [16, p.8], that under this assumption the measures $\mu$ are absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^2$. To derive an analog of the Valiron-Titchmarsh theorem for the positive temperatures, we assume that the function $d\mu(y)$ is supported on the ray $\arg y = \theta_0$. Thus, in the case under consideration, (1.2) can be represented as

(1.3) \[ u(x, t) = \frac{1}{4\pi\kappa t} \int_{0}^{\infty} e^{-\frac{1}{4\kappa t} ||x-y||^2} \, dn(s), \]

where now $y = (s \cos \theta_0, s \sin \theta_0)$ and $n(s)$ denotes the number of zeros in the circle $|z| \leq r$ and $n(s) = \mu(\{|y| \leq s\})$ is the total $\mu$-measure of the disk $|y| \leq s$. Since we are interested in asymptotic properties of temperatures, we can assume without any loss of generality that some vicinity at the origin is $\mathbb{R}^2$ is free of the measure $d\mu$, i.e. $n(s_0) = 0$ for some $s_0 > 0$, in particular, $n(0) = 0$. Moreover, since $u(0, t) < \infty$, we
have from (1.3)

\[(1.4) \quad \int_0^\infty e^{-\frac{1}{4}t} s^2 dn(s) < \infty.\]

We also assume for the rest of the note that for any \(t > 0\),

\[(1.5) \quad \lim_{s \to \infty} n(s)e^{-s^2/4t} = 0.\]

Integrating (1.4) by parts and using (1.5), we derive the representation

\[(1.6) \quad u(0, t) = \frac{1}{4\pi \kappa t} \int_0^\infty sn(s)e^{-\frac{1}{4}s^2} ds.\]

Suppose the measure \(u\) is supported on the ray \((s, \theta_0), s > 0, 0 \leq \theta_0 < 2\pi\). Thus, in (1.3) the measure \(n(s) = n((0, s])\) of the semi-interval \((0, s]\) for all \(s > 0\). Integrating (1.3) by parts, we have

\[(1.7) \quad u(x, t) = \frac{1}{2t} \int_0^\infty (s - r\cos(\theta - \theta_0)) e^{-\frac{r^2 - 2r\cos(\theta - \theta_0)}{4t}} n(s) ds \]

\[= \frac{1}{2t} e^{-\frac{r^2}{4t}} \int_0^\infty (s - r\cos(\theta - \theta_0)) e^{-\frac{s^2 - 2r\cos(\theta - \theta_0)}{4t}} n(s) ds.\]

**Remark 1.1.** Since we are interested in temperatures with power growth of the measure, when (1.5) is clearly valid, (1.5) is not an essential restriction for us.

**Theorem 1.1.** Let the temperature \(u\) have a representation

\[u(x, t) = Ax^\alpha + v(x, t)\]

where

\[\lim_{x \to \infty} x^{-\alpha} \int_0^x v(y, t) dy = 0\]

is uniform in \(t \in [0, t_0]\) for some \(0 < t_0 < a\). Then

\[f(r) = Ar^\alpha + o(r^\alpha), r \to \infty.\]

**Proof.** For a function \(f\), denote \([f](x) = \frac{1}{2}((f(x^+) + f(x^-))\). By assumption \(f(y) = 0\) in some neighborhood of \(y = 0\), thus \([f](0) = 0\) and by [17, p. 69, Theor. 6],

\[\lim_{t \to 0} \int_0^r u(y, t) dy = [f](r)\]

therefore

\[\lim_{t \to 0} \int_0^r f(y) dy + \lim_{t \to 0} \int_0^r v(y, t) dy = Ar^\alpha + \lim_{t \to 0} \int_0^r v(y, t) dy.\]
The limits as \( r \to +\infty \) and as \( t \to 0^+ \) can be interchanged due to the uniformity assumption, thus

\[
\lim_{r \to \infty} r^{-\alpha} \lim_{t \to 0^+} \int_0^r v(y, t) dy = \lim_{t \to 0^+} \lim_{r \to \infty} r^{-\alpha} \int_0^r v(y, t) dy = 0.
\]

**Theorem 1.2.** Under some assumptions on a temperature \( u \)

\[
u(r) = \lim_{t \to 0^+} \int_{|y| \leq r} v(y, t) dy
\]

if \( u(x, t) \approx A(\theta_0)|x|^\rho \). Then

(1.8)

\[
f|\chi| = \alpha|x|^\rho + \overline{\sigma}(x^\rho)
\]

**Proof.** By [16, Theor. 7.2]

(1.9)

\[
u(r) = \lim_{t \to 0^+} \int_{|y| \leq r} v(y, t) dy
\]

\[
= \lim_{t \to 0^+} \int_{|y| \leq r} A(\theta)|x|^\rho dy + \lim_{t \to 0^+} \int_{|y| \leq r} u(y, t) dy, y = (y_1, y_2) = (B, \theta).
\]

Suppose

\[
\lim_{t \to 0^+} r^{-s-2} \int_{|y| \leq r} V(s, \theta, t) dy = 0.
\]

If under some assumptions on a temperature \( u \),

\[
u(r) = \lim_{t \to 0^+} \int_{|y| \leq r} u(y, t) dy.
\]

Then,

\[
\lim_{r = |x| \to \infty} \frac{u(x, t)}{|x|^\rho+2} = \frac{1}{\rho+2} \int_0^{2\pi} A(\theta) d\theta.
\]

Since

\[
u(x, t) = A(t, \theta)|x|^\rho + V(x, t), y = (y_1, y_2) = (B, \theta).
\]

It follows,

\[
\int_{|y| \leq r} A(t, \theta) s^\rho dy = \int_0^r \int_{0}^{2\pi} s A(t, \theta) s^\rho d\theta ds
\]
\[
= \int_0^{2\pi} A(t, \theta) \int_0^r s^{\rho+1} ds
\]
\[
= \frac{r^{\rho+2}}{\rho + 2} \int_0^{2\pi} A(t, \theta) d\theta
\]

Now, we can state our result.

**Theorem 1.3.** Let \( u(x, t) \) be a positive temperature given by (1.2) with the measure \( d\mu \) supported at the ray, \( \arg z = \theta_0 \). If \( n(s_0) = 0 \) and

\[
(1.10) \quad n(s) = a_0 s^{\alpha(s)} + n_1(s), \quad s > s_0,
\]

where the constants \( a_0 \) and \( \alpha \) satisfy \( a_0 \geq 0, \alpha > -1 \), and the remainder

\[
\lim_{s \to \infty} s^{-\alpha(s)} n_1(s) = 0,
\]

then

\[
(1.11) \quad u(x, t) = \begin{cases} 
\frac{a_0}{2^{\frac{\alpha}{2}}\kappa t} \left( r \cos(\theta - \theta_0) \right)^{\alpha} e^{-\frac{r^2 \sin^2(\theta - \theta_0)}{4t}}, & \cos(\theta - \theta_0) > 0 \\
\frac{a_0}{\pi} \Gamma\left( \frac{\alpha+1}{2} \right) 2^{\alpha-2} t^{\frac{\alpha-1}{2}} e^{-\frac{t}{4}}, & \cos(\theta - \theta_0) = 0 \\
\frac{a_0}{\pi} \Gamma(\alpha + 1) 2^{\alpha-1} t^{\alpha} e^{-\frac{t}{4}}, & \cos(\theta - \theta_0) < 0.
\end{cases}
\]

**Proof.** We write (1.3) as

\[
(1.12) \quad u(x, t) = u_0(x, t) + u_1(x, t)
\]

\[
\equiv \frac{a_0}{8\pi t^2} \int_0^\infty e^{-\frac{4}{t} ||x-y||^2} s^{\alpha(s)} ds + \frac{b}{8\pi t^2} \int_0^\infty e^{-\frac{4}{t} ||x-y||^2} n_1(s) ds.
\]

We find the principal term of the asymptotic formula by estimating \( u_0 \),

\[
(1.13) \quad u_0(x, t) = \frac{a_0}{4\pi kt} \int_0^\infty e^{-\frac{4}{t} ||x-y||^2} s^{\alpha(s)} ds
\]

\[
= \frac{a_0}{4\pi kt} \int_0^\infty e^{-\frac{s^2 - 2rs \cos(\theta - \theta_0) + r^2}{4t}} s^{\alpha(s)} ds,
\]

\[
= \frac{a_0 e^{-\frac{r^2}{4t}}}{4\pi kt} \int_0^\infty e^{-\frac{s^2 - 2rs \cos(\theta - \theta_0)}{4t}} s^{\alpha(s)} ds.
\]
To simplify the integral \( u_0 \), set \( s = \sqrt{2t}w \), \( \alpha = -\nu - 1 \), while letting \( z = -\frac{r \cos(\theta - \theta_0)}{\sqrt{2t}} \),

\[
u_0(x,t) = \frac{a_0}{4\pi kt} e^{\frac{z^2}{2}} \left( \sqrt{2t} \right)^{\alpha(s)+1} \int_0^\infty w^\alpha e^{-\frac{w^2}{2} - zw} \, dw.
\]

The above integral \( u_0 \) can be expressed through and estimated by making use of the parabolic cylinder functions, where \( D_\nu(z) \) is the Weber function [1, Section 8.3,(3)],

\[
D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)} \int_0^\infty e^{-zt-t^2/2} t^{-\nu-1} \, dt, \ \Re \nu < 0.
\]

Using (2.2) and (2.3), \( u_0 \) can be written as

\[
u_0(x,t) = \frac{a_0 \Gamma(\alpha(s)+1)}{\pi} \frac{2^{\alpha(s) - 1} \alpha(s)}{\Gamma(1+\sin^2(\theta - \theta_0))} D_{-\alpha-1}(z),
\]

We consider here two cases, \( \cos(\theta - \theta_0) > 0 \) and \( \cos(\theta - \theta_0) < 0 \). We cut the \( z \) - plane along the negative \( x \) - axis, \( x = \text{Re}z \), thus \( -\pi < \theta \leq \pi \) and fix the value \( \text{arg}(1) = 0 \). Then in the half-plane, \( \theta_0 - \pi/2 < \theta < \theta_0 + \pi/2, z = -\frac{r \cos(\theta - \theta_0)}{\sqrt{2t}} < 0, \text{arg}z = \pi \).

Using the known asymptotic formula [2, p. 307] for a fixed value of \( \nu \) and \( \pi/4 < |\text{arg}z| < 5/4\pi \), as \( z \to \infty \), we have

\[
D_\nu(z) = z^\nu e^{-\frac{1}{4}z^2} \left[ \sum_{n=0}^N \frac{(-1/2\nu)_n (1/2 - 1/2\nu)_n}{n!(-1/2z^2)^n} + O(|z^2|^{-N-1}) \right]
\]

\[- \frac{(2\pi)^{1/2}}{\Gamma(-\nu)} e^{\nu\pi i} z^{-\nu-1} e^{\frac{1}{4}z^2} \left[ \sum_{n=0}^N \frac{(1/2\nu)_n (1/2 + 1/2\nu)_n}{n!(1/2z^2)^n} + O(|z^2|^{-N-1}) \right].
\]

For \( \cos(\theta - \theta_0) > 0, e^{\frac{1}{4}z^2} \gg e^{-\frac{1}{4}z^2}, n = 0 \),

\[
D_\nu(z) \approx - \frac{(2\pi)^{1/2}}{\Gamma(-\nu)} e^{\nu\pi i} z^{-\nu-1} e^{\frac{1}{4}z^2}
\]

Thus,

\[
u_0(x,t) \approx \frac{a_0}{\pi} \frac{2^{\alpha(s)-1} \alpha(s-1)}{\Gamma(\alpha(s))} \left( \frac{r \cos(\theta - \theta_0)}{\sqrt{2\pi}} \right)^{\alpha(s)} e^{-\frac{r^2}{2\pi} \sin^2(\theta - \theta_0)}
\]

Similarly, for \( \cos(\theta - \theta_0) < 0 \)

\[
D_\nu(z) \approx \frac{(2\pi)^{1/2}}{\Gamma(-\nu)} e^{-\frac{1}{4}z^2}
\]

\[1\text{The corresponding formulas in [1, Sect. 8.4] contain misprints - missing brackets.}\]
\[ u_0(x, t) = \frac{a_0}{\pi} 2^{\frac{\alpha-3}{2}} t^{\frac{\alpha-1}{2}} (r \cos(\theta - \theta_0))^{\alpha} e^{-\frac{r^2}{4t}} \]

Now, we estimate \( u_1 \) by (1.6),
\[ u_1(x, t) = \frac{b}{4\pi \kappa t} \int_0^\infty e^{-\frac{1}{4\kappa t}||x-y||^2} n_1(s) ds. \]

Since \( \lim_{s \to \infty} s^{-\alpha(s)} n_1(s) = 0 \). Therefore,
\[ Ae^{-\frac{1}{4\kappa t}||x-y||^2} \to 0, s \to \infty, \]
uniformly in \( t \).

References

[1] Bateman, H., Erdelyi, A., *Higher Transcendental Functions*, Vol. 2. McGraw-Hill, New York, Toronto, London, 1953.

[2] Bleistein, N., Handelsman, R.A., *Asymptotic Expansions of Integrals*. Dover, New York, 1986.

[3] Cannon, J.R., *The One-Dimensional Heat Equation*. Addison-Wesley Publ. Co., 1984.

[4] Evgrafov, M.A., *Asymptotic Estimates and Entire Functions*. 3rd Ed. (Russian), "Nauka", Moscow, 1979.

[5] Kheyfits, A.I., A generalization of the E. Titchmarsh theorem on entire functions with negative zeros, *Izv. VUZov. Math.* 1973, No. 2 (129), 99-105.

[6] Kheyfits, A.I., A Complement to the Valiron-Titchmarsh Theorem for Subharmonics Functions, *Anal. Theory Appl.* 30 (2014), no. 1, 136–140.

[7] Kheyfits, A.I., Lacay, J.B., *Asymptotic Behavior of the One-Dimensional Heat Equation* International Journal of Evolution Equations, 2010, Vol. 5, pp.103-108.

[8] Lacay, J.B., *Asymptotic Behavior of Positive Solutions of the Heat Equation*. PanAmerican Mathematics International Journal 2011, Vol. 5, pp.99-103.

[9] Lacay, J.B., Abelian and Tauberian Theorems for Solutions of the One-Dimensional Heat Equation with Respect to Proximate Order, *International Journal of Mathematical Analysis*, 2017, Vol. 11, pp.173-188.

[10] Levin, B.Y., *The Distribution of Zeros of Entire Functions* Transl. Math. Monographs, Vol 5, AMS, Providence, RI, 1980.
[11] Levin, B.Y., Lectures on Entire Functions. Amer. Math. Soc., Providence, Rhode Island, 1996.

[12] Seneta, E., Regularly Varying Functions, Lecture Notes in Mathematics, Vol. 508, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[13] Strochik, N. N. Some Tauberian theorems for entire functions with negative zeros. Dynamical systems and complex analysis (Russian), 42–54, “Naukova Dumka”, Kiev, 1992.

[14] Titchmarsh, E.C., On integral functions with real negative zeros. Proc. London Math. Soc. 26, p.185-200. 1927.

[15] Valiron, G., Sur les fonctions entières d’ordre nul et d’ordre fini et en particulier les fonctions à correspondance régulière Ann. Fac. Sci. Univ. Toulouse,5, 117–257 (1914). MathSciNet Google Scholar

[16] Watson, E., Parabolic Equations on an Infinite Strip. M. Dekker, New York, Basel, 1989.

[17] Widder, D.V., The Heat Equation, Acad. Press, New York, San Francisco, London, 1975.

[18] Zakharov, S.V., Heat distribution in an infinite rod, Mathematical Notes, 2006, Vol. 80, 366-371.