Classical and quantum exact solutions for a FRW in chiral like cosmology

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In this work, first, we study a flat Friedmann-Robertson-Walker Universe with two scalar fields but only one potential term, which can be thought as a simple quintessence plus a K-essence model. Employing the Hamiltonian formalism we are able to obtain the classical and quantum solutions. The second model studied, is also a flat Friedmann-Robertson-Walker Universe with two scalar fields, with the difference that the two potentials are considered as well as the standard kinetic energy and the mixed term (chiral field approach). Regarding this second model, it is shown that setting to zero the coefficient accompanying the mixed momenta term, two possible cases can be studied: a quintom like case ($m_{12}^2$) and a quintessence like case ($m_{12}^2$). For both scenarios classical and quantum solutions are presented.

I. INTRODUCTION

One of the main goals of modern cosmology is to be able to adequately describe the early Universe. In this sense, the first proposal for such a description was the Big-Bang theory; unfortunately this theory suffered from two problems: that of flatness and that of the horizon. At the beginning of the 80’s of the last century, the idea of inflation was introduced [1–3], healing the problems that the Big-Bang theory had. Boldly speaking, the inflation process is a period of exponential growth in our Universe. During this process, in addition to solving the problems already mentioned, the inflation mechanism also explains the homogeneity and isotropy currently observed in the Universe. Another important aspect of inflation is that the fluctuations generated during this period give rise to a primordial spectrum of density perturbations [5–8], which is nearly scale invariant, adiabatic and Gaussian, and is in agreement with cosmological observations [9, 10].

On the other hand, scalar fields have been extensively used in the past three decades as the possible major matter components for the evolution of the Universe. They can describe various phenomena of our Universe such as the inflationary era, the late time acceleration, the dark matter component of the Universe and the unification of early inflation to late acceleration, to mention a few [2, 13–25] (in the sense that what determines the inflationary model is the form of the potential). From a phenomenological point of view, the most successful models have been those that have incorporated quintessence scalar fields and slow-roll inflation [10, 18, 26–35], chiral cosmology connected to $f(R)$ theories or nonlinear sigma model [36–39, 48]. In general, scalar field theories use a single scalar field with its respective potential; however, in recent years there have been proposals where scalar field theories consist of multiple scalar fields and which have yielded quite interesting results within the context of the evolution of the Universe. For example, it has been found that a system consisting of two scalar fields can describe the crossing of the cosmological constant boundary “−1” (known as quintom models [43, 52]), which for a single scalar field model is impossible since the single scalar field models can only describe either the quintessence or the phantom regime. Furthermore, these multi-field models can also explain the early inflationary era of the Universe, known as hybrid inflation [53–57] and gives a different graceful exit in comparison with the standard inflationary paradigm [58–60]. In addition, the dynamical possibilities in multi-field inflationary scenarios are considerably richer than in single-field

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models, such as in the primordial inflation perturbations analysis [61, 62] or the assisted inflation as discussed in [63, 64].

In these models, as in the single field ones, the potential associated to the scalar fields plays a very important role. In several cases the potential that is employed is a simple exponential product of the scalar fields or a series of linear sum exponentials [65]. For example, in [12, 65] a potential of the form \( V(\phi, \sigma) = V_0 e^{-\lambda_1 \phi - \lambda_2 \sigma} \) was employed. This potential was found under the connection between the time derivatives of the momenta, namely, \( \Pi_\phi \propto \Pi_\sigma \), provided that \( \partial V / \partial \phi = \alpha \partial V / \partial \sigma \), (this type of potentials have also been found under other considerations [10, 57]). Following this line of thought, first, we shall consider the case when both potentials are proportional between them, leaving the theory with only one potential but both kinetic terms, and show that the exact analytical solution is obtained employing the mathematical tools of Hamilton’s formalism. Then, the other scenario that we will investigate is when both potentials come into play, that is, \( V(\phi, \sigma) = V_1 e^{-\lambda_1 \phi} + V_2 e^{-\lambda_2 \sigma} \) (this class of potential in the scalar fields yields the so called chiral cosmology (nonlinear sigma model) [66]); in this setting, because the two potentials are considered a mixed kinetic term has to be introduced with a coupling parameter which allows us to obtain a constraint in order to separate the equations and be able to solve the problem analytically. The advantage of this scenario is that we can reproduce the quintessence or quintom cosmologies choosing appropriately the sign of this mixed kinetic term. These types of scalar fields models have been introduced in the literature in order to provide an alternative description to primordial inflation [12, 43, 50, 67, 73].

For the quantum cosmological cases we implement a basic formulation by means of the Wheeler-DeWitt (WDW) equation. In order of being able to obtain solutions to the WDW equation, several approaches have been studied, such is the case of [74], where a debate of what a typical wave function of the Universe is presented. In [75], a review on quantum cosmology where the problem of how the Universe emerged from Big Bang singularity can no longer be neglected in the GUT epoch is discussed. Moreover, the best candidates for quantum solutions are those that have a damping behavior with respect to the scale factor, since only such wave functions allow for good solutions when using a Wentzel-Kramers-Brillouin (WKB) approximation for any scenario in the evolution of our Universe [76, 77]. Furthermore, in the context of a single scalar field a family of scalar potentials is obtained in the Bohmian formalism [57, 78], or supersymmetric quantum cosmology [79–81], where among others a general potential of the form \( V(\phi) = V_0 e^{-\lambda \phi} \) is examined.

This work is arranged as follows. In section II we will introduce the model with two scalar fields, where both kinetic terms are taken into account but only one term of the scalar potential is present. From the corresponding Einstein-Klein-Gordon (EKG) equations and the Hamiltonian density, we are able to obtain three different solutions for the model. Also, into this same model we do the analysis for different types of standard matter (stiff matter and radiation) obtaining their corresponding solutions and present the dust scenario for arbitrary scalar potential, whose particular form obtained in the solution, plays an important role in the behavior of the volume function. In section III, we analyze a two scalar field cosmological model but where both scalar potentials come into play. As will be shown, within this model we can distinguish between two scenarios: a quintom like case and a quintessence like case. For both scenarios we will present the Hamilton equations and their corresponding solutions. Next, in section IV we consider the quantum versions of the previous cosmological models calculating their corresponding WDW equations and its solutions. Finally, section V we give our final remarks.

II. FIRST MODEL

Let’s start by introducing the multi-field Lagrangian density that we will be working with, and can be thought of as standard quintessence plus a simple K-essence model, including the standard matter, thus we have

\[
\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi_1 \nabla_\nu \phi_1 - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi_2 \nabla_\nu \phi_2 + V(\phi_1) \right) + \sqrt{-g} \mathcal{L}_{\text{matter}},
\]

(1)

where \( R \) is the Ricci scalar, \( V(\phi_1) \) is the corresponding scalar field potential and \( \mathcal{L}_{\text{matter}} \) corresponds at the contribution of ordinary matter for barotropic perfect fluid, \( P = \gamma \rho \), with \( \rho \) the energy density, \( P \) is the pressure of the fluid in the co-moving frame, \( \gamma \) is the barotropic constant, the equation of state to scalar field is \( P_\phi = \omega_\phi \rho_\phi \), with the pressure and energy density of the scalar field are defined in the standard way in the gauge \( N=1 \), \( 16\pi G P_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \) and \( 16\pi G \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \), however in the gauge \( N \neq 1 \), this is defined with the modification in \( V(\phi) \rightarrow N^2 V(\phi) \) and \( \rho \rightarrow N^2 \rho \), and the reduced Planck mass \( M_P^2 = 1/8\pi G = 1 \). Before we continue, it is important to mention that in principle the potential in (1) should be of the form \( V(\phi_1, \phi_2) \), but as has already been expressed, this form of the Lagrangian density is obtained when one considers that the potentials are proportional between them, leaving the theory with only one dynamical potential, and two kinetic terms. To obtain the corresponding EKG field equations we
must perform the variations of Eq. (1) with respect to the metric and the scalar fields, also we include the conservation law of the energy-momentum tensor of a perfect fluid of ordinary matter, giving

\[ G_{\alpha\beta} = -T_{\alpha\beta} - \frac{1}{2} \left( \nabla_\alpha \phi_1 \nabla_\beta \phi_1 - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi_1 \nabla_\nu \phi_1 \right) + \frac{1}{2} g_{\alpha\beta} V(\phi_1) \]

\[ -\frac{1}{2} \left( \nabla_\alpha \phi_2 \nabla_\beta \phi_2 - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi_2 \nabla_\nu \phi_2 \right), \]

\[ \Box \phi_1 - \frac{\partial V}{\partial \phi_1} = g^{\mu\nu} \phi_{1,\mu\nu} - g^{\alpha\beta} \Gamma^\nu_{\alpha\beta} \nabla_\nu \phi_1 - \frac{\partial V}{\partial \phi_1} = 0, \]

\[ g^{\mu\nu} \phi_{2,\mu\nu} - g^{\alpha\beta} \Gamma^\nu_{\alpha\beta} \nabla_\nu \phi_2 = 0, \]

\[ \nabla_\nu T^{\mu\nu} = 0, \quad T_{\mu\nu} = (\rho + P) u_\mu u_\nu + g_{\mu\nu} P. \]

As we are considering a flat FRW Universe, the line element to be used in this work is

\[ ds^2 = -N(t)^2 dt^2 + e^{2\Omega(t)} \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \]

where \( N \) is the lapse function, \( A(t) = e^{\Omega(t)} \) is the scale factor in the Misner parametrization and \( \Omega \) a scalar function with interval from \(-\infty \) to \( \infty \). The Einstein-Klein-Gordon field equations and conservation law of the energy-momentum tensor of a perfect fluid of ordinary matter, are

\[ \frac{3\dot{\Omega}^2}{N^2} - \frac{\dot{\phi}_1^2}{4N^2} - \frac{\dot{\phi}_2^2}{4N^2} - \frac{1}{2} V(\phi_1) - \rho = 0, \]

\[ \frac{2\dot{\Omega}^2}{N^2} + \frac{3\dot{\phi}_2^2}{N^2} - \frac{2\dot{\phi}_1^2}{4N^2} + \frac{\dot{\phi}_2^2}{4N^2} - \frac{1}{2} V(\phi_1) + P = 0, \]

\[ \frac{\dot{\phi}_1^2}{N^2} + \frac{3\dot{\phi}_2^2}{N^2} - \frac{\dot{\phi}_1^2}{N^3} + \dot{V}(\phi_1) = 0, \]

\[ \frac{\dot{\phi}_2^2}{N^2} + \frac{3\dot{\phi}_2^2}{N^2} - \frac{\dot{\phi}_2^2}{N^3} = 0, \]

\[ \rho = \rho_\gamma e^{-3(\gamma + 1)\Omega}, \]

where \( \rho_\gamma \) is an integration constant that will depend on the epoch of the Universe being analysed. From the last equation we can obtain the solution for the scalar field \( \phi_2 \) (in quadrature form), giving

\[ \phi_2 = \phi_{20} + k_1 \int \frac{e^{-3\Omega}}{N} dt, \]

where \( \phi_{20} \) and \( k_1 \) are integration constants.

Taking the metric (10), the Ricci scalar takes the form \( R = -6 \frac{\ddot{\Omega}}{N^2} - 12 \frac{\dot{\phi}_2^2}{N^2} + \frac{\dot{\Omega}^2}{N^2} \) and after plugging it into (11), where we take the particular scalar potential \( V(\phi_1) = V_1 e^{-\lambda_1 \phi_1} \), the Lagrangian density becomes (we dropped a total time derivative, where a second time derivative of scale factor \( \Omega \) was included)

\[ \mathcal{L} = e^{3\Omega} \left( \frac{6\dot{\Omega}^2}{N} - \frac{\dot{\phi}_1^2}{2N} - \frac{\dot{\phi}_2^2}{2N} + NV_1 e^{-\lambda_1 \phi_1} + 2N\rho \right), \]

where a “.” represents a time derivative.

The associated conjugate momenta can be calculated in ordinary fashion, that is \( \partial \mathcal{L}/\partial \dot{q}_i \), which gives

\[ \Pi_{\dot{\Omega}} = \frac{12 e^{3\Omega}}{N} - \frac{\dot{\Omega}}{12} - \Pi_{\Omega}, \]

\[ \Pi_{\dot{\phi}_1} = -\frac{e^{3\Omega}}{N} \phi_1, \]

\[ \Pi_{\dot{\phi}_2} = -e^{3\Omega} \phi_2, \]

\[ \Pi_{\dot{\phi}_2} = -Ne^{-3\Omega} \Pi_{\phi_2}, \]

\[ \Pi_{\phi_1} = -Ne^{-3\Omega} \Pi_{\phi_1}, \]

\[ \Pi_{\phi_2} = -Ne^{-3\Omega} \Pi_{\phi_2}, \]

\[ \Pi_{\phi_1} = -Ne^{-3\Omega} \Pi_{\phi_1}, \]

\[ \Pi_{\phi_2} = -Ne^{-3\Omega} \Pi_{\phi_2}. \]
Writing (13) in a canonical form, i.e. \( L_{\text{can}} = \Pi_{\Omega} \dot{\Omega} - N \mathcal{H} \), we can perform the variation of this canonical Lagrangian with respect to the lapse function \( N \). \( \delta L_{\text{can}}/\delta N = 0 \), resulting in the constraint \( \mathcal{H} = 0 \), then, the lapse function acts as a Lagrange multiplier, hence the Hamiltonian density is

\[
\mathcal{H} = \frac{e^{-3\Omega}}{24} \left[ \Pi_{\Omega}^2 - 12 \Pi_{\phi_1}^2 - 12 \Pi_{\phi_2}^2 - U(\phi_1, \Omega) - 48 \rho_\gamma e^{-3(\gamma-1)\Omega} \right],
\]

where \( U(\phi_1, \Omega) = 24V_1 e^{-\lambda_1 \phi_1 + 6\Omega} \). The corresponding Hamilton equations are

\[
\begin{align*}
\dot{\Omega} &= \frac{e^{-3\Omega}}{24} 2\Pi_{\Omega}, \\
\dot{\phi}_1 &= -\frac{e^{-3\Omega}}{24} 24\Pi_{\phi_1}, \\
\dot{\phi}_2 &= -\frac{e^{-3\Omega}}{24} 24\Pi_{\phi_2}, \\
\end{align*}
\]

\[
\begin{align*}
\ddot{\Omega} &= \frac{e^{-3\Omega}}{24} 6U(\phi_1, \Omega) - 144 \rho_\gamma (\gamma - 1) e^{-3(\gamma-1)\Omega}, \\
\ddot{\phi}_1 &= -\frac{e^{-3\Omega}}{24} \lambda_1 U(\phi_1, \Omega), \\
\ddot{\phi}_2 &= 0.
\end{align*}
\]

This last set of equations cannot be decoupled due to the presence of the factor \( e^{-3\Omega} \). Taking the advantage that the gauge of \( N \) can be fixed, dictated by the form of the Hamiltonian density Eq. (15), we can set \( N = 24 e^{3\Omega} \), enabling us to find solutions to the problem at hand. Now the metric \( (6) \) takes the form

\[
ds^2 = e^{2\Omega} \left[ -576 e^{4\Omega} dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

It is worth mentioning that applying the transformation \( dt = 24 e^{2\Omega} d\tau \) in the FRW metric (in Minkowskian coordinates) one obtains a metric in conformal coordinates given by \( (17) \).

Working with the Hamilton’s equations of motion we have that the canonical velocities and momenta are

\[
\begin{align*}
\dot{\Omega} &= 2\Pi_{\Omega}, \\
\dot{\phi}_1 &= -24\Pi_{\phi_1}, \\
\dot{\phi}_2 &= -24\Pi_{\phi_2}, \\
\end{align*}
\]

\[
\begin{align*}
\ddot{\Omega} &= 6U(\phi_1, \Omega) - 144 \rho_\gamma (\gamma - 1) e^{-3(\gamma-1)\Omega}, \\
\ddot{\phi}_1 &= -\lambda_1 U(\phi_1, \Omega), \\
\ddot{\phi}_2 &= 0. \\
\end{align*}
\]

In the following sections we will solve this set of equations for particular values of \( \gamma \).

### A. Master equation for \( \rho_\gamma = 0 \) or \( \rho_\gamma \neq 0 \) (\( \gamma = 1 \))

In this section we construct a master equation from Eqs. (18) that will allow us to obtain the different solutions for the model under study. We start by realizing that from the last equation of (18) it follows that \( \Pi_{\phi_2} = p_{\phi_2} = \text{constant} \). Also, a relation between \( \Pi_{\Omega} \) and \( \Pi_{\phi_1} \) can be achieved noticing that

\[
\begin{align*}
\dot{\Pi}_{\phi_1} &= -\frac{6}{\lambda_1}, \\
\Pi_{\phi_1} &= -\frac{\lambda_1}{6} \Pi_{\Omega} + p_{\phi_1},
\end{align*}
\]

yielding

\[
\Pi_{\phi_1} = -\frac{\lambda_1}{6} \Pi_{\Omega} + p_{\phi_1},
\]

where \( p_{\phi_1} \) is an integration constant and remains a free parameter of the model to be adjusted with the data collected from the cosmological observations. Also from Eqs. (18) we find the following general relation between the coordinates fields \( (\Omega, \phi_1) \) as follow: substituting equation (20) into \( \dot{\phi}_1 \) we have \( \dot{\phi}_1 = 4\lambda_1 \Pi_{\Omega} - 24 p_{\phi_1} \), reinserting the equation for \( \dot{\Omega} \) and integrating, we obtain

\[
\Delta \phi_1 = 2\lambda_1 \Delta \Omega - 24 p_{\phi_1} \Delta t.
\]

On the other hand, for the case considered here, we can rewrite the Hamiltonian (15) in the following form

\[
\Pi_{\Omega}^2 - 12 \Pi_{\phi_1}^2 - 12 \Pi_{\phi_2}^2 - 48 \rho_1 = U(\phi_1, \Omega),
\]

and after replacing Eq. (20), the expression for the momenta \( \Pi_{\Omega} \) and \( \Pi_{\phi_2} \), we have

\[
2 (3 - \lambda_1^2) \Pi_{\Omega}^2 + 24 \lambda_1 p_{\phi_1} \Pi_{\Omega} - 72 [p_{\phi_1}^2 + p_{\phi_2}^2 + 4 \rho_1] = \dot{\Pi}_{\Omega},
\]
enabling us to obtain a temporal dependence for $\Pi_\Omega(t)$ which allows us to construct a master equation:

$$\frac{d\Pi_\Omega}{a_1\Pi_\Omega^2 + a_2\Pi_\Omega - a_3} = dt,$$

(24)

where the parameters $a_i$, $i = 1, 2, 3$, are

$$a_1 = 2(3 - \lambda_1^2), \quad a_2 = 24\lambda_1 p_{\phi_1}, \quad a_3 = 72 \left[ p_{\phi_1}^2 + p_{\phi_2}^2 + 4p_1 \right].$$

(25)

Subsequently by analyzing the parameter $\lambda_1^2$ we will obtain three different solutions. The first of them will be considering $\lambda_1^2 < 3$, where in [10] according to Planck data [9], it has been pointed out that this value gives an inflationary period for a single scalar field cosmology. Secondly, we are going to consider the value $\lambda_1^2 > 3$, although the references mention this choice does not exhibit an inflationary epoch for a single scalar field, but things might change under the considered model, in light that two scalar fields are taken into account. Lastly, the value for $\lambda_1 = \sqrt{3}$ will lead us to the third solution. The particular case that $p_{\phi_2} = 0$ was presented in [10], for $\lambda_1 < \sqrt{3}$, where the model was tested with the Planck data and the analysis for the case of $\lambda_1 = \sqrt{3}$ was addressed in [11]. Also, in [12], using a different class of scalar potential (a product of exponential functions) the authors obtain exact solutions for a FRW multi-field cosmological model.

1. Solution for $\lambda_1^2 < 3$.

For this particular case, we have the following solution

$$\frac{1}{24\omega} \ln \left[ \frac{\eta\Pi_\Omega + 6\lambda_1 p_{\phi_1} - 6\omega}{\eta\Pi_\Omega + 6\lambda_1 p_{\phi_1} + 6\omega} \right] = t - t_0,$$

(26)

where $\eta = 3 - \lambda_1^2 > 0$, $\omega = \sqrt{3p_{\phi_1}^2 + \eta(p_{\phi_2}^2 + 4p_1)}$, and $t_0$ is a time-like integration constant. With Eq.(26), the canonical momentum $\Pi_{\phi_1}$ and the rest of variables can be solved, hence the solutions are

$$\Omega(t) = \Omega_0 - \frac{12\lambda_1 p_{\phi_1}(t - t_0)}{\eta} - \frac{1}{\eta} \ln \left[ \sinh \left[ 12\omega(t - t_0) \right] \right],$$

(27)

$$\phi_1(t) = \phi_{10} - \frac{72\lambda_1 p_{\phi_1}(t - t_0)}{\eta} - \frac{2\lambda_1}{\eta} \ln \left[ \sinh \left[ 12\omega(t - t_0) \right] \right],$$

(28)

$$\phi_2(t) = \phi_{20} - 24p_{\phi_2}(t - t_0),$$

(29)

$$\Pi_{\phi_1}(t) = -\frac{6\lambda_1 p_{\phi_1}}{\eta} - \frac{6\omega}{\eta} \coth \left[ 12\omega(t - t_0) \right],$$

(30)

$$\Pi_\phi(t) = \frac{3p_{\phi_1}}{\eta} + \frac{\lambda_1 \omega}{\eta} \coth \left[ 12\omega(t - t_0) \right],$$

(31)

where $(\Omega_0, \phi_{10}, \phi_{20})$ are integration constants and the constraint $2\eta V_0 = \omega^2 e^{\lambda_1 \phi_{10} - 6t_0}$, which is obtained when we introduce all solutions in the set of EKG equations [3,10]. This set of solutions is a complete and exact classical representation of a canonical scalar field with exponential potential in a flat FRW metric, then the scale factor becomes

$$A(t) = A_0 \exp \left[ -\frac{12\lambda_1 p_{\phi_1}(t - t_0)}{\eta} \right] \left( \text{csch} \left[ 12\omega(t - t_0) \right] \right)^{\frac{1}{2}},$$

(32)

being $A_0 = e^{\Omega_0}$. From [32] it is evident that the scale factor has a decreasing behavior, therefore, in order to have a growing volume function in the inflationary epoch we must impose that $p_{\phi_1} < 0$ (which must be taken into account in all the equations in this section). As we will see below, the new sign will be reflected in the deceleration parameter. One may question if there is a procedure to rewrite Eq. (32) in terms of $t_{\text{phys}}$, where $t_{\text{phys}} = \int dt N(t)$, however, a forthright relation between $t_{\text{phys}}$ and $t$ is far from being determined, since one must first compute such integral, if possible, and then obtain $t = t(t_{\text{phys}})$ which can be a nontrivial endeavor. However, all observable parameters must be evaluated at $t_{\text{phys}}$, or in terms of an equivalent evolution variable, yet ascertaining an appropriate manipulation of the gauge.
2. Solution for $\lambda_1^2 > 3$.

For this case $\eta < 0$ and $a_1 = 2(3 - \lambda_1^2) < 0$, so the master equation (24) can be casted as

$$\frac{d\Pi}{-m_1 \Pi^2 + a_2 \Pi - a_3} = dt$$  \hspace{1cm} (33)

where we have included the minus sign such that the constant $m_1 = 2(\lambda_1^2 - 3) = 2\beta > 0$. Then, defining $\omega_1^2 = a_2^2 - 8\beta a_3 = 576\omega_2^2$ with $\omega_2^2 = 3p_{\phi_1} - \beta(p_{\phi_2} + 4\rho_1)$, we can rewrite (33) as

$$\frac{8\beta d\Pi}{\omega_1^2 - (4\beta \Pi - 24\lambda_1 p_{\phi_1})^2} = dt,$$

(34)

where the constraint over the parameters $p_{\phi_1} > \sqrt{(p_{\phi_2} + 4\rho_1) \left[ (\lambda_1^2)^2 - 1\right]}$ must be fulfilled. In order to be able to integrate Eq.(34), as a final step, we resort to the change of variables $z = 4\beta \Pi^2 - 24\lambda_1 p_{\phi_1}$, thus, the solution for the momenta $\Pi(t)$ becomes

$$\Pi(t) = \frac{6\lambda_1 p_{\phi_1}}{\beta} + \frac{6\omega_2}{\beta} \tanh (12\omega_2(t - t_0)) .$$

(35)

Using the relations from Eq.(18) and after some algebra, the solutions for the set of variables $(\Omega, \phi_1, \phi_2)$ and $(\Pi_{\phi_1}, \Pi_{\phi_2})$ are:

$$\Omega = \Omega_0 + \frac{12\lambda_1 p_{\phi_1}}{\beta} (t - t_0) + \frac{1}{\beta} \ln \left[ \cosh (12\omega_2(t - t_0)) \right],$$

(36)

$$\phi_1 = \phi_{1_0} + \frac{72 p_{\phi_1}}{\beta} (t - t_0) + \frac{2\lambda_1}{\beta} \ln \left[ \cosh (12\omega_2(t - t_0)) \right],$$

(37)

$$\phi_2 = \phi_{2_0} - 24 p_{\phi_2} (t - t_0),$$

(38)

$$\Pi_{\phi_1} = -\frac{3 p_{\phi_1}}{\beta} - \frac{2\lambda_1 \omega_2}{\beta} \tanh (12\omega_2(t - t_0)),$$

(39)

$$\Pi_{\phi_2} = p_{\phi_2},$$

(40)

where $(\Omega_0, \phi_{1_0}, \phi_{2_0})$ are all integration constants. In order that the above solutions fulfill the EKG Eqs.(2-4), all constants must satisfy that $2\beta V_0 = \omega_2^2 e^{\lambda_1 \phi_{1_0} - 6\Omega_0}$. Finally the scale factor becomes

$$A(t) = A_0 \exp \left[ \frac{12\lambda_1 p_{\phi_1}}{\beta} (t - t_0) \right] \cosh \frac{\pi}{2} (12\omega_2(t - t_0)),$$

(41)

here, as before, $A_0 = e^{\Omega_0}$.

3. Solution for $\lambda_1^2 = 3$.

For completeness, we include the exotic case for $\lambda_1^2 = 3$ (and $\eta = 0$) which emerge as consequence of SUSY Quantum Mechanics applied to cosmological models [79]; for this case the coefficient $a_1 = 0$ and the master equation to solve is reduced to

$$\int \frac{d\Pi}{a_2 \Pi - a_3} = \int dt,$$

(42)

thus $\Pi(t)$ becomes

$$\Pi(t) = \frac{a_3}{a_2} + p e^{a_2(t - t_0)},$$

(43)
where $p$ is an integration constant. As before, we can use relations from Eq. (18) and after some manipulation, the solutions for $(\Omega, \phi_1, \phi_2)$ and $(\Pi_{\phi_1}, \Pi_{\phi_2})$ are:

$$
\Omega = \Omega_0 + 2\sqrt{3}\left(\frac{p_{\phi_1}^2 + p_{\phi_2}^2 + 4\rho_1}{p_{\phi_1}}(t - t_0) + \frac{\sqrt{3}p}{36p_{\phi_1}}e^{24\sqrt{3}p_{\phi_1}(t-t_0)}\right),
$$

(44)

$$
\phi_1 = \phi_{10} + \left(\frac{p_{\phi_2}^2 + 4\rho_1}{p_{\phi_1}} - p_{\phi_1}^2\right)(t - t_0) + \frac{p}{6} e^{24\sqrt{3}p_{\phi_1}(t-t_0)},
$$

(45)

$$
\phi_2 = \phi_{20} - 24p_{\phi_2}(t - t_0),
$$

(46)

$$
\Pi_{\phi_1} = \frac{1}{2}\left[p_{\phi_1} - (p_{\phi_2}^2 + 4\rho_1)\right] - \frac{\sqrt{3}p}{6} e^{24\sqrt{3}p_{\phi_1}(t-t_0)},
$$

(47)

$$
\Pi_{\phi_2} = p_{\phi_2},
$$

(48)

again $(\Omega_0, \phi_{10}, \phi_{20})$ are all integration constants. If we want the equations (44)-(48) satisfy the EKG Eqs. (2)-(4), all constants must fulfill that $V_0 = \frac{\sqrt{3}p_{\phi_1}}{6}e^{-6\Omega_0 + \lambda_1 \phi_{10}}$. Finally, the scale factor $A(t)$ for this case is

$$
A(t) = A_0 \exp\left[2\sqrt{3}\left(\frac{p_{\phi_1}^2 + p_{\phi_2}^2 + 4\rho_1}{p_{\phi_1}}(t - t_0)\right)\right] \exp\left[\frac{\sqrt{3}p}{36p_{\phi_1}}e^{24\sqrt{3}p_{\phi_1}(t-t_0)}\right],
$$

(49)

where $A_0 = e^{\Omega_0}$.

4. Deceleration and barotropic parameters.

One way that we can analyze the dynamical behavior of the scale factor for each of the three solutions of the model under consideration, is to calculate the deceleration parameter, which is defined as

$$
q = -\frac{\dot{A}\ddot{A}}{A^2}.
$$

(50)

The corresponding deceleration parameters are given by

$$
q_1 = -1 - \frac{\eta\omega^2}{[\lambda_1 p_{\phi_1}\sinh(12\omega t) - \omega\cosh(12\omega t)]^2},
$$

(51)

$$
q_2 = -1 - \frac{\beta\omega^2}{[\lambda_1^2 p_{\phi_2} \cosh(12\omega t) + \omega'\sinh(12\omega t)]^2},
$$

(52)

$$
q_3 = -1 - \frac{12\sqrt{3}p_{\phi_1}^2 \exp(24\sqrt{3}p_{\phi_1} t)}{[pp_{\phi_1}\exp(24\sqrt{3}p_{\phi_1} t) + \sqrt{3}(p_{\phi_1}^2 + p_{\phi_2}^2 + 4\rho_1)]^2},
$$

(53)

where $q_1, q_2$ and $q_3$ stand for the solutions for $\lambda_1^2 < 3$, $\lambda_1^2 > 3$ and $\lambda_1^2 = 3$, respectively; also $\omega' = \sqrt{3}p_{\phi_1} - \eta'(p_{\phi_2}^2 + 4\rho_1)$ and $\eta' = (\lambda_1')^2 - 3$, ($\lambda_1'$ being a constant). In Fig. (1) we can see the behavior of each of the deceleration parameters when matter is absent (left panel) and when matter is incorporated (right panel). For the scenario $\rho_1 = 0$ it is observed that $q_1$ exhibits a deceleration and acceleration behavior. $q_3$ also presents a deceleration/acceleration stage, being the former only for a short period of time (compared to $q_1$) to then accelerate. When matter is incorporated things change for $q_1$ and $q_3$, from the right panel of Fig. (1), we can see that the former, again, presents a deceleration period to later have a sudden growth. In this setup, $q_1$ starts in a slow deceleration stage to then gradually accelerate. Lastly, in contrast to $q_1$ and $q_3$, $q_2$ has only an accelerated stage with the similarity that in both scenarios the growth is slow. After a sufficient period of time the three solutions (in both scenarios) converge to the same value -1. Also, we can write the barotropic parameter $\omega_T = \frac{(Pressure)_{total}}{(energy\ density)_{total}}$ that is related to the deceleration parameter $q$ under the Einstein equations Equations (7) and (8), for the gauge $N \neq 1$, as

$$
\omega_T = \frac{2}{3}q + \frac{5}{3}.
$$

(54)
Figure 1: Deceleration parameter for the three classical solutions. Here we have taken (for both figures) $\lambda_1 = 0.5$, $\lambda'_1 = 2$, $p_{\phi_1} = 0.4$, $p_{\phi_2} = 0.2$ and $p = 0.7$. For the left figure we set $\rho_1 = 0$, whereas for the right one $\rho_1 = 0.04$. And taking the deceleration parameters given by equations (51)-(53) the barotropic parameter for each of the solutions are given by

$$\omega_{T1} = 1 - \frac{2}{3} \left[ \lambda_1 p_{\phi_1} \text{Sinh}(12 \omega t) - \omega \text{Cosh}(12 \omega t) \right]^2,$$

$$\omega_{T2} = 1 - \frac{2}{3} \left[ \lambda'_1 p_{\phi_1} \text{Cosh}(12 \omega' t) + \omega' \text{Sinh}(12 \omega' t) \right]^2,$$

$$\omega_{T3} = 1 - \frac{2}{3} \left[ pp_{\phi_1} \text{Exp}(24 \sqrt{3} p_{\phi_1} t) \right]^{1/2} \left[ pp_{\phi_1} \text{Exp}(24 \sqrt{3} p_{\phi_1} t) + \sqrt{3} (p_{\phi_1}^2 + p_{\phi_2}^2 + 4 \rho_1) \right]^{1/2}. $$

(55)  
(56)  
(57)

Analyzing the behavior $\omega_{Ti}$ from the last set of equations it can be observed that for late times all three values tend

Figure 2: Barotropic parameters for the three different solutions without standard matter. For the three $\omega_T$ we have taken $\lambda_1 = 0.5$, $\lambda'_1 = 2$, $p_{\phi_1} = 0.4$, $p_{\phi_2} = 0.2$ and $p = 0.7$. 

to 1. It is a well known fact that for a quintessence model the lower and upper bounds are given by $-1 \leq \omega_T \leq 1$, from Fig (2) we can see that our model fits within this parameters.

For completeness, the barotropic parameter in the gauge $N = 1$ becomes

$$\omega_T = \frac{2}{3} q - \frac{1}{3}. \quad (58)$$

B. Standard matter scenario ($\gamma \neq 1$)

Trying to solve the set of equations (18) for values of $\gamma \neq 1$ under Hamilton’s approach is a dead end. To sort out this setback, we must implement another method. For this particular case we start with arbitrary scalar potential field $V(\phi_1)$ employing the gauge $N = 1$ and the EKG equations. As we are considering a flat FRW Universe, the line element can be written as

$$ds^2 = -dt^2 + A^2(t) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (59)$$

where $A(t)$ is the scale factor of the model. The corresponding EKG equations are

$$3 \left( \frac{\dot{A}}{A} \right)^2 - \frac{1}{4} \dot{\phi}_1^2 - \frac{1}{4} \dot{\phi}_2^2 - 8\pi G_N \rho - \frac{1}{2} V(\phi_1) = 0, \quad (60)$$

$$2 \frac{\ddot{A}}{A} + \left( \frac{\dot{A}}{A} \right)^2 + \frac{1}{4} \dot{\phi}_1^2 + \frac{1}{4} \dot{\phi}_2^2 + 8\pi G_N \gamma \rho - \frac{1}{2} V(\phi_1) = 0, \quad (61)$$

$$-3 \frac{\dot{A}}{A} \dot{\phi}_1 - \dot{\phi}_1 \dot{\phi}_1 - \frac{\partial V(\phi_1)}{\partial \phi} = 0, \quad \rightarrow \quad \frac{d}{dt} \ln \left( A^3 \dot{\phi}_1 \right) = -\frac{\dot{V}}{\dot{\phi}_1^2}, \quad (62)$$

$$\frac{3}{A} \dot{\phi}_2 + \ddot{\phi}_2 = 0, \quad \rightarrow \quad \Delta \phi_2 = c_2 \int \frac{dt}{A^3}. \quad (63)$$

Considering a barotropic equation of state for the scalar field of the form $P_\phi = \omega_\phi \rho_\phi$ we can obtain the kinetic energy $K_\phi = \frac{1 + \omega_\phi}{1 - \omega_\phi} V(\phi)$, therefore, equation (62) can be written in general way as

$$\frac{d}{dt} \left[ \ln \left( A^6 V^{\frac{2}{1-\omega_\phi}} \right) \right] = 0 \quad \rightarrow \quad V = c_\omega A^{-3(1+\omega_\phi)}, \quad (64)$$

hence, solutions to the scalar field can be expressed in quadrature form as

$$\Delta \phi_1 = c_\omega \int \frac{dt}{A^{\frac{1}{3(1+\omega_\phi)}}}. \quad (65)$$

For the attractor case $\omega_\phi = \gamma \neq 1$, Eq. (60) can be casted as

$$3 \left( \frac{\dot{A}}{A} \right)^2 = \frac{c_1^2 + 8\pi G \rho_\gamma}{A^{3(1+\gamma)}} + \frac{c_2^2}{A^6}, \quad \rightarrow \quad \frac{A^2 dA}{\sqrt{b_\gamma A^{3(1-\gamma)} + b_2}} = dt, \quad (66)$$

where we have identified the scalar fields contributions as $\frac{1}{4} \dot{\phi}_1^2 = \frac{c_1^2}{A^6}$ and $\frac{1}{4} \dot{\phi}_2^2 = \frac{c_2^2}{A^{3(1+\omega_\phi)}}$ with $c_1$ and $c_2$ integration constants (see equations (63) and, (64)); and $b_\gamma = c_1^2 + \frac{8}{3} \pi G \rho_\gamma$ and $b_2 = c_2^2$. The general solution of (66) is given in terms of a Hypergeometric function,

$$\Delta t = \frac{A^3}{\sqrt{3} c_2} \frac{2F_1}{2} \left[ \frac{1}{2}, 1 - \gamma, 2 - \gamma, \frac{8\pi G \rho_\gamma + c_2^2}{c_1^2 A^{3(1-\gamma)}} \right], \quad (\gamma \neq 1). \quad (67)$$
1. Dust Epoch.

For the dust scenario in the standard matter we choose $\gamma = 0$, therefore equation (66) reduces to

$$\frac{A^2 dA}{\sqrt{b_0 A^3 + b_2}} = dt,$$

(68)

being $b_0 = \frac{c_1^2}{4} + \frac{8}{3} \pi G \rho_0$ and $b_2 = \frac{c_2^2}{4}$. Considering the change of variables $z = b_0 A^3 + b_2$ we arrived to the solution for the scale factor, which is

$$A^3(t) = a_0 \left[(a_1 t + a_2)^2 - 1\right],$$

(69)

here $a_0 = \frac{b_2}{b_0} = \frac{c_2^2}{c_1^2 + 8 \pi G \rho_0}$, and $a_1 = \frac{3b_0}{2\sqrt{b_2}} = \frac{\sqrt{3} c_1^2 + 8 \pi G \rho_0}{c_2}$, and its dynamical behavior can be observed in Fig.(3). Now with this solution at hand, we can write the scalar fields as

$$\phi_1(t) = \phi_{10} + \frac{c_1}{a_1} \sqrt{\frac{2}{a_0}} \ln \left[u + \sqrt{u^2 - 1}\right]$$

$$= \phi_{10} + \frac{4}{\sqrt{6(c_1^2 + 8 \pi G \rho_0)}} \ln \left[u + \sqrt{u^2 - 1}\right],$$

(70)

$$\phi_2(t) = \phi_{20} + \frac{2}{\sqrt{3}} \ln \left(1 - \frac{2}{u+1}\right),$$

(71)

where $u(t) = a_1 t + a_2$. From (70) is possible to rewrite the variable $u$ in terms of the $\phi_1$ as $u = \text{Cosh} \left( \frac{\sqrt{6}}{4} \sqrt{c_1^2 + 8 \pi G \rho_0} \Delta \phi_1 \right)$, so the corresponding scalar potential becomes

$$V(\phi_1) = \frac{c_1^2 (c_2^2 + 8 \pi G \rho_0)}{c_2^2} \text{Csch}^2 \left( \frac{\sqrt{6}}{4} \sqrt{c_1^2 + 8 \pi G \rho_0} \Delta \phi_1 \right).$$

(72)

At this point it is worth mentioning that this kind of potential is consistent with a (volume) accelerated expansion in the dust scenario. Also, from (71) we can see that the scalar field $\phi_2$ acquires a constant value of $\phi_{20}$ for late times. In a previous work [82], this type of scalar potential was found for an anisotropic cosmological model were the anisotropic functions vanish for late times.

Figure 3: Volume function for the dust epoch in the standard matter, for $a_0 = 0.001, a_1 = a_2 = 10$.  

In the dust scenario. Also, from (71) we can see that the scalar field $\phi_2$ acquires a constant value of $\phi_{20}$ for late times. In a previous work [82], this type of scalar potential was found for an anisotropic cosmological model were the anisotropic functions vanish for late times.
With this results we can analyze the dynamical behavior of the volume \( v = A^3 \) calculating the deceleration parameter, which for the present setup takes the following form

\[
q = -\frac{\ddot{v}}{v^2},
\]

resulting in

\[
q(t) = -\frac{1}{2} + \frac{1}{2u^2}.
\]

We can easily check that taken the limit \( t \to \infty \) in (74) the deceleration parameter is \( q(t) = -\frac{1}{2} \) (remember that \( u \) is a function that depends linearly with respect to \( t \)), indicating us that the Universe undergoes a volume accelerated expansion, supporting the above results. In [82] we found similar behavior for the volume function but for an anisotropic cosmology. Also, for this case the barotropic parameter \( \omega_T \), is

\[
\omega_T = \frac{2}{3}q - \frac{1}{3} = \frac{1}{3u^2} - \frac{2}{3},
\]

we can observe that the asymptotic behavior of \( \omega_T \) (with respect to time) tends to \(-\frac{2}{3}\) which is a signal that the volume function has a big expansion.

C. Radiation epoch \((\gamma = \frac{1}{3})\)

Finally, we address the radiation epoch where \( \gamma = \frac{1}{3} \). For this scenario the Friedmann equation (66) displays the following form

\[
\frac{A^2 dA}{\sqrt{b_0 A^2 + b_2}} = dt,
\]

and whose solution is given by

\[
2b_0 \Delta t = A\sqrt{b_0 A^2 + b_2} - \frac{b_2}{\sqrt{b_0}} \ln \left[ A + \sqrt{A^2 + \frac{b_2}{b_0}} \right].
\]

Unfortunately we don’t have enough information to be able to write the scale factor as a function of time, however, one would expect that by inverting equation (77) the functional form of \( A^3(t) \) would be an increasing function.

III. SECOND MODEL

The next cosmological model to consider is one where in addition to considering the two scalar fields, two potential terms also come into play. The action for such a Universe is [36–38, 40, 42]

\[
\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2}g^{\mu\nu}m^{ab}\nabla_\mu \phi_a \nabla_\nu \phi_b + V(\phi_1, \phi_2) \right),
\]

where \( R \) is the Ricci scalar, \( V(\phi_1, \phi_2) \) is the corresponding scalar field potential, and \( m^{ab} \) is a \( 2 \times 2 \) constant matrix and \( m^{12} = m^{21} \). The corresponding variations of Eq. (78), with respect to the metric and the scalar fields gives the EKG field equations

\[
G_{\alpha\beta} = -\frac{1}{2}m^{ab} \left( \nabla_\alpha \phi_a \nabla_\beta \phi_b - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}\nabla_\mu \phi_a \nabla_\nu \phi_b \right) + \frac{1}{2}g_{\alpha\beta} V(\phi_1, \phi_2),
\]

\[
m^{ab} \Box \phi_b - \frac{\partial V}{\partial \phi_a} = m^{ab}g^{\mu\nu}\phi_b_{,\mu\nu} - m^{ab}g^{\alpha\beta}g_{\alpha\beta} \nabla_\nu \phi_b - \frac{\partial V}{\partial \phi_a} = 0,
\]

where \( a, b = 1, 2 \). The line element to be considered for this two-field cosmological model is the flat FRW

\[
ds^2 = -N(t)^2 dt^2 + e^{2\Omega(t)} \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]
here, as in the previous case, \( N \) represents the lapse function, \( \Lambda(t) = e^{\Omega(t)} \) the scale factor in the Misner parametrization and \( \Omega \) a scalar function whose interval is \(( -\infty, \infty )\). Consequently the Klein-Gordon equations are (here the \( t = \frac{d}{dt}, d\tau = Ndt \))

\[
m^{11}\phi''_1\phi'_1 + m^{12}\phi''_2\phi'_1 + 3\Omega' \left( m^{11}\phi''_1 + m^{12}\phi''_2 \right) + \left( \dot{V} \right)_{\phi_1} = 0, \tag{82}
\]

\[
m^{22}\phi''_2\phi'_2 + m^{12}\phi''_1\phi'_2 + 3\Omega' \left( m^{22}\phi''_2 + m^{12}\phi''_1 \right) + \left( \dot{V} \right)_{\phi_2} = 0. \tag{83}
\]

where \( \left( \dot{V} \right)_{\phi_i} \) means that the time derivative is calculated maintaining \( \phi_i \) constant. Building the corresponding Lagrangian and Hamiltonian densities for this cosmological model, classical solutions to EKG Eqs. (79-80) can be found using the Hamilton's approach; and also the quantum formalism can be determined and solved, as we will show below. In this line of thought, plugging the metric Eq. (81) into Eq. (78) and where we have taken the particular scalar potential \( V(\phi_1, \phi_2) = V_1 e^{-\lambda_1 \phi_1} + V_2 e^{-\lambda_2 \phi_2} \), which is appropriate for this model, now the Lagrangian density reads

\[
\mathcal{L} = e^{3\Omega} \left( \frac{6\dot{\phi}_1^2}{N} - \frac{m^{11}\dot{\phi}_1^2}{2N} - \frac{m^{22}\dot{\phi}_2^2}{2N} - \frac{m^{12}\dot{\phi}_1 \phi_2}{N} + NV_1 e^{-\lambda_1 \phi_1} + NV_2 e^{-\lambda_2 \phi_2} \right), \tag{84}
\]

here, as before, an upper “..” represents a time derivative. In (84) we can also include a contribution term \( 2N\rho \), regarding the standard matter content, as it was done in the Lagrangian density \([13]\); for the case of stiff matter (\( \gamma = 1 \)), this contribution in the Hamiltonian density would be reflected with the addition of a constant term and the treatment could be carried out in the same manner as in section \([\text{II}]\) however, in the following we will perform an analysis without standard matter. From (84) the resulting momenta are given by

\[
\Pi_{\Omega} = \frac{12}{N} e^{3\Omega} \dot{\Omega}, \quad \dot{\Omega} = \frac{Ne^{-3\Omega}}{12} \Pi_{\Omega}, \tag{85}
\]

\[
\Pi_{\phi_1} = -\frac{e^{3\Omega}}{N} \left( m^{11} \dot{\phi}_1 + m^{12} \phi_2 \right), \quad \dot{\phi}_1 = \frac{Ne^{-3\Omega}}{\triangle} \left( -m^{22} \Pi_{\phi_1} + m^{12} \Pi_{\phi_2} \right),
\]

\[
\Pi_{\phi_2} = -\frac{e^{3\Omega}}{N} \left( m^{12} \dot{\phi}_1 + m^{22} \phi_2 \right), \quad \phi_2 = \frac{Ne^{-3\Omega}}{\triangle} \left( m^{12} \Pi_{\phi_1} - m^{11} \Pi_{\phi_2} \right),
\]

where \( \triangle = m^{11}m^{22} - (m^{12})^2 \). Writing (84) in a canonical form, i.e. \( \partial \mathcal{L}_{can} = \Pi_i \theta_i - N\mathcal{H} \), we can perform the variation of this canonical Lagrangian with respect to the lapse function \( N \), \( \delta \mathcal{L}_{can} / \delta N = 0 \), resulting in the constraint \( \mathcal{H} = 0 \), hence the Hamiltonian density is

\[
\mathcal{H} = \frac{e^{-3\Omega}}{24} \left[ \Pi_{\Omega}^2 - \frac{12m^{22}}{\triangle} \Pi_{\phi_1}^2 - \frac{12m^{11}}{\triangle} \Pi_{\phi_2}^2 + \frac{24m^{12}}{\triangle} \Pi_{\phi_1} \Pi_{\phi_2} 
- 24V_1 e^{-\lambda_1 \phi_1 + 6\Omega} - 24V_2 e^{-\lambda_2 \phi_2 + 6\Omega} \right]. \tag{86}
\]

Proposing the following canonical transformation on the variables \((\Omega, \phi_1, \phi_2) \leftrightarrow (\xi_1, \xi_2, \xi_3)\)

\[
\xi_1 = -6\Omega + \lambda_1 \phi_1, \quad \xi_2 = -6\Omega + \lambda_2 \phi_2, \quad \xi_3 = -4\Omega + \frac{\lambda_1}{6} \phi_1 + \frac{\lambda_2}{6} \phi_2,
\]

\[
\Omega = \frac{\xi_1 + \xi_2 - 6\xi_3}{12}, \quad \phi_1 = \frac{3\xi_1 + \xi_2 - 6\xi_3}{2\lambda_1}, \quad \phi_2 = \frac{\xi_1 + 3\xi_2 - 6\xi_3}{2\lambda_2}, \tag{87}
\]

and setting the gauge \( N = 24e^{3\Omega} \), allows us to find a new set of conjugate momenta \((P_1, P_2, P_3)\)

\[
\Pi_{\Omega} = -6P_1 - 6P_2 - 4P_3, \quad \Pi_{\phi_1} = \lambda_1 P_1 + \frac{\lambda_1}{6} P_3, \quad \Pi_{\phi_2} = \lambda_2 P_2 + \frac{\lambda_2}{6} P_3, \tag{88}
\]
which finally leads us to the Hamiltonian density

\[
\mathcal{H} = 12 \left(3 - \frac{\lambda_1^2 m^{22}}{\Delta}\right) P_1^2 + 12 \left(3 - \frac{\lambda_2^2 m^{11}}{\Delta}\right) P_2^2 + \left(16 + \frac{-\lambda_1^2 m^{22} + 2 \lambda_1 \lambda_2 m^{12} - \lambda_2^2 m^{11}}{3\Delta}\right) P_3^2
\]

\[
+ 12 \left[\left(4 + \frac{\lambda_1 \lambda_2 m^{12} - \lambda_1^2 m^{22}}{3\Delta}\right) P_1 + \left(4 + \frac{\lambda_1 \lambda_2 m^{12} - \lambda_2^2 m^{11}}{3\Delta}\right) P_2\right] P_3
\]

\[
+ 24 \left(3 + \frac{\lambda_1 \lambda_2 m^{12}}{\Delta}\right) P_1 P_2 - 24 \left(V_1 e^{-\xi_1} + V_2 e^{-\xi_2}\right), \tag{89}
\]

the parameter \(\Delta\) is the same that was defined after equations (88). The form that the Hamiltonian density (89) acquires after applying the transformation (87) into (86) will, in the end, allows us to obtain the solutions for this model. First, let’s compute Hamilton’s equations, which read

\[
\dot{\xi}_1 = 24 \left(3 - \frac{\lambda_1^2 m^{22}}{\Delta}\right) P_1 + 24 \left(3 + \frac{\lambda_1 \lambda_2 m^{12}}{\Delta}\right) P_2 + 12 \left(4 + \frac{\lambda_1 \lambda_2 m^{12} - \lambda_2^2 m^{11}}{3\Delta}\right) P_3,
\]

\[
\dot{\xi}_2 = 24 \left(3 - \frac{\lambda_2^2 m^{11}}{\Delta}\right) P_1 + 24 \left(3 + \frac{\lambda_1 \lambda_2 m^{12}}{\Delta}\right) P_2 + 12 \left(4 + \frac{\lambda_1 \lambda_2 m^{12} - \lambda_1^2 m^{22}}{3\Delta}\right) P_3,
\]

\[
\dot{\xi}_3 = 12 \left[\left(4 + \frac{\lambda_1 \lambda_2 m^{12} - \lambda_1^2 m^{22}}{3\Delta}\right) P_1 + \left(4 + \frac{\lambda_1 \lambda_2 m^{12} - \lambda_2^2 m^{11}}{3\Delta}\right) P_2\right]
\]

\[
+ 2 \left(16 + \frac{-\lambda_1^2 m^{22} + 2 \lambda_1 \lambda_2 m^{12} - \lambda_2^2 m^{11}}{3\Delta}\right) P_3,
\]

\[
\dot{P}_1 = -24 V_1 e^{-\xi_1},
\]

\[
\dot{P}_2 = -24 V_2 e^{-\xi_2},
\]

\[
\dot{P}_3 = 0, \tag{90}
\]

from this last set of equations is straightforward to see that \(P_3\) is a constant. Taking the time derivative of the first equation in (90), we obtain

\[
\dot{\xi}_1 = -576 V_1 \left(3 - \frac{\lambda_1^2 m^{22}}{\Delta}\right) e^{-\xi_1} - 576 V_2 \left(3 + \frac{\lambda_1 \lambda_2 m^{12}}{\Delta}\right) e^{-\xi_2}. \tag{91}
\]

The main purpose of introducing the transformation (87) was to be able to separate the set of equations arising from the Hamiltonian density (89). To reach a solution for our problem we set to zero the coefficient that is multiplying the mixed momenta term in (89), which yield the following constraint on the matrix element \(m^{12}\)

\[
m^{12} = \frac{\lambda_1 \lambda_2}{6} \left(1 \pm \sqrt{1 + \frac{36 m^{11} m^{22}}{\lambda_1^2 \lambda_2^2}}\right), \tag{92}
\]

which implies that the second term in the square root of (92) is a real number, say \(\ell = 36(m^{11} m^{22}/\lambda_1^2 \lambda_2^2) \in \mathbb{R}^+\), giving the same weight to the matrix elements \(m^{11}\) and \(m^{22}\), whose values are \(m^{11} = (\sqrt{\ell}/6)\lambda_1^2\) and \(m^{22} = (\sqrt{\ell}/6)\lambda_2^2\). We are going to distinguish two possible scenarios for \(m^{12}\) as: \(m^{12}_+ = (\lambda_1 \lambda_2/6) \left(1 + \sqrt{1 + \ell}\right) > 0\) and \(m^{12}_- = -\lambda_1 \lambda_2/6 \left(\sqrt{1 + \ell} - 1\right) < 0\). This two choices of \(m^{12}\) enables us to have (what we called) a quintom like case and quintessence like case, (however, the stiff matter scenario is dominant in all cases). With these two possible values for the matrix element \(m^{12}\) we can see that \(\Delta_+ = -\lambda_1^2 \lambda_2^2 \left(1 + \sqrt{1 + \ell}\right) < 0\) for \(m^{12}_+\) and \(\Delta_- = \frac{\lambda_1^2 \lambda_2^2}{18} \left(\sqrt{1 + \ell} - 1\right) > 0\) for \(m^{12}_-\).

### A. Quintom like case

We begin analyzing the quintom like case, for which the matrix element \(m^{12}_- = -b_\ell \lambda_1 \lambda_2\), and \(b_\ell = (\sqrt{1 + \ell} - 1)/6\) has been defined in order to simplified the calculations. Taking into account all the above, the Hamiltonian density is rewritten as,

\[
\mathcal{H} = -\frac{P_1^2}{\mu_1} - \frac{P_2^2}{\mu_2} + \left(48 - \frac{1}{3c_\ell}\right) (P_1 + P_2) P_3 + \left(16 - \frac{1}{18c_\ell}\right) P_3^2 - 24 V_1 e^{-\xi_1} - 24 V_2 e^{-\xi_2}. \tag{93}
\]
where we have define the parameters \( \mu_i = \frac{\sqrt{7}}{36[(1+\sqrt{1+\tau})-\sqrt{\tau}]} \) and \( c_i = \frac{\sqrt{7}}{36[(1+\sqrt{1+\tau})+\sqrt{\tau}]} \). Thus, Hamilton equations for the new simplified coordinate \( \xi_i \) are

\[
\begin{align*}
\dot{\xi}_1 &= -\frac{2P_1}{\mu_i} + \left(48 - \frac{1}{3c_i}\right)P_3, \\
\dot{\xi}_2 &= -\frac{2P_2}{\mu_i} + \left(48 - \frac{1}{3c_i}\right)P_3, \\
\dot{\xi}_3 &= \left(48 - \frac{1}{3c_i}\right)(P_1 + P_2) + 2\left(16 - \frac{1}{18c_i}\right)P_3,
\end{align*}
\]

(94)

the equations for \( \dot{P}_1, \dot{P}_2 \) and \( \dot{P}_3 \) remain the same as in [90]. Taking the derivative of the first equation of (94) yields

\[
\dot{\xi}_1 = \frac{48V_1}{\mu_i} e^{-\xi_1},
\]

(95)

which has a solution of the form

\[
e^{-\xi_1} = \frac{\mu_i r_i^2}{24V_1} \text{Sech}^2 (r_1 t - q_1).
\]

(96)

From [94] we can see that \( \dot{\xi}_2 \) has the same functional structure as \( \dot{\xi}_1 \), therefore its solution will be of the same form as (96), so we have

\[
e^{-\xi_2} = \frac{\mu_i r_i^2}{24V_2} \text{Sech}^2 (r_2 t - q_2),
\]

(97)

where \( r_i \) and \( q_i \) (with \( i = 1, 2 \)) are integration constants, both at [96] and [97]. Reinserting these solutions into Hamilton equations for the momenta, we obtain

\[
P_1 = \alpha_1 - \mu_i r_1 \text{Tanh} (r_1 t - q_1),
\]

(98)

\[
P_2 = \alpha_2 - \mu_i r_2 \text{Tanh} (r_2 t - q_2).
\]

(99)

With [98] and [99], it can be easily check that the Hamiltonian is identically null when

\[
\alpha_1 = \alpha_2 = \frac{72\mu_i - 1}{6} p_3, \quad p_3^2 = \frac{\mu_i (r_1^2 + r_2^2)}{4(72\mu_i + 1)}.
\]

(100)

Now we are in position write the solutions for the \( \xi_i \) coordinates, which read

\[
\begin{align*}
\xi_1 &= \beta_1 + \ln \left[\cosh^2 (r_1 t - q_1)\right], \\
\xi_2 &= \beta_2 + \ln \left[\cosh^2 (r_2 t - q_2)\right], \\
\xi_3 &= \beta_3 + p_3 \left[16 (1 + 72\mu_i) - 8 \frac{\mu_i}{c_i}\right] \Delta t \\
&\quad - \left(48 - \frac{1}{3c_i}\right) \mu_i \ln \left[\cosh (r_1 t - q_1) \cosh (r_2 t - q_2)\right],
\end{align*}
\]

(101)

(102)

(103)

(104)

here the \( \beta_i \), with \( i = 1, 2, 3 \), terms are constants coming from integration. Applying the inverse canonical transformation we obtain the solutions in the original variables \( (\Omega, \phi_1, \phi_2) \) as

\[
\begin{align*}
\Omega &= \Omega_0 + \frac{1}{12} \ln \left[\cosh^2 (r_1 t - q_1) \cosh^2 (r_2 t - q_2)\right] - \frac{1}{2} p_3 \left[16 (1 + 72\mu_i) - 8 \frac{\mu_i}{c_i}\right] \Delta t \\
&\quad + \frac{1}{2} \left(48 - \frac{1}{3c_i}\right) \mu_i \ln \left[\cosh (r_1 t - q_1) \cosh (r_2 t - q_2)\right],
\end{align*}
\]

(105)

\[
\begin{align*}
\phi_1 &= \phi_{10} + \frac{1}{2\lambda_1} \ln \left[\cosh^6 (r_1 t - q_1) \cosh^2 (r_2 t - q_2)\right] + 6\mu_i \left(48 - \frac{1}{3c_i}\right) \times \\
&\quad \ln \left[\cosh (r_1 t - q_1) \cosh (r_2 t - q_2)\right] - \frac{3}{\lambda_1} p_3 \left[16 (1 + 72\mu_i) - 8 \frac{\mu_i}{c_i}\right] \Delta t,
\end{align*}
\]

(106)

\[
\begin{align*}
\phi_2 &= \phi_{20} + \frac{1}{2\lambda_2} \ln \left[\cosh^2 (r_1 t - q_1) \cosh^6 (r_2 t - q_2)\right] + 6\mu_i \left(48 - \frac{1}{3c_i}\right) \times \\
&\quad \ln \left[\cosh (r_1 t - q_1) \cosh (r_2 t - q_2)\right] - \frac{3}{\lambda_2} p_3 \left[16 (1 + 72\mu_i) - 8 \frac{\mu_i}{c_i}\right] \Delta t,
\end{align*}
\]

(107)
Thus, the scale factor is given by

$$\Omega_0 = \frac{\beta_1 + \beta_2 - 6\beta_3}{12}, \quad \phi_{10} = \frac{3\beta_1 + \beta_2 - 6\beta_3}{2\lambda_1}, \quad \phi_{20} = \frac{\beta_1 + 3\beta_2 - 6\beta_3}{2\lambda_2}. \quad (106)$$

Thus, the scale factor is given by

$$A(t) = A_0 \cosh^{\frac{1}{2} + \alpha_0} (r_1 t - q_1) \cosh^{\frac{1}{2} + \alpha_0} (r_2 t - q_2) \ e^{-\beta_q \Delta t}, \quad (107)$$

where \(\beta_q = \frac{P_2}{T} \left(16(1 + 72\mu_\ell) - 8\mu_\ell^2\right)\) and \(\alpha_0 = \frac{1}{2} \left(48 - \frac{1}{\mu_\ell}\right) \mu_\ell\). The dynamical behavior of the volume function is presented in Fig. [4]. The deceleration parameter for this quintom cosmological model becomes

$$q_{\text{quintom}} = -1 - \alpha_0 \frac{(r_1^2 \cosh^2(r_2 t - q_2) + r_2^2 \cosh^2(r_1 t - q_1))}{T}, \quad (108)$$

with

$$T = \alpha_0^2 (r_1^2 \sinh^2(r_1 t - q_1) \cosh^2(r_2 t - q_2) + r_2^2 \sinh^2(r_2 t - q_2) \cosh^2(r_1 t - q_1)) + 2\alpha_0 \sinh(r_1 t - q_1) \sinh(r_2 t - q_2) \{\alpha_0 r_1 r_2 \cosh(r_1 t - q_1) \cosh(r_2 t - q_2) + \beta_q \{r_2 \cosh(r_2 t - q_2) \sinh(r_1 t - q_1) + r_1 \sinh(r_2 t - q_2) \cosh(r_1 t - q_1)\}\} + \beta_q^2 \cosh^2(r_1 t - q_1) \cosh^2(r_2 t - q_2), \quad (109)$$

and \(\alpha_0 = \frac{1}{6} + \alpha_q\). For this model, the barotropic parameter \(\omega_T\) takes the form

$$\omega_{\text{quintom}} = 1 - \frac{2}{3} \alpha_0 \frac{(r_1^2 \cosh^2(r_2 t - q_2) + r_2^2 \cosh^2(r_1 t - q_1))}{T}. \quad (110)$$

The corresponding behavior of the deceleration and the barotropic parameters are also shown in the figure [4] below.

### B. Quintessence like case

Now we turn our attention to the quintessence like case, for which the matrix element \(m_{\ell 2} = q_\ell \lambda_1 \lambda_2\), and \(q_\ell = (1 + \sqrt{1 + t}) / 6\) has been defined to make the calculations simpler. The Hamiltonian density describing this quintessence model is rewritten as

$$\mathcal{H} = \frac{P_1^2}{\nu_\ell} + \frac{P_2^2}{\nu_\ell} + \left(48 - \frac{1}{3c'_\ell}\right) (P_1 + P_2) P_3 + \left(16 - \frac{1}{18c'_\ell}\right) P_3^2 - 24V_1 e^{-\xi_1} - 24V_2 e^{-\xi_2}, \quad (111)$$

here we define the parameters \(\nu_\ell = \frac{\sqrt{7}}{36(\sqrt{1 + t} + \sqrt{1 - t})}\) and \(c'_\ell = \frac{\sqrt{7}}{36(\sqrt{1 + t} + \sqrt{1 - t})}\).

From (111) we can calculate Hamilton equations for the phase space spanned by \((\xi_1, P_1)\), given by

$$\dot{\xi}_1 = \frac{2P_1}{\nu_\ell} + \left(48 - \frac{1}{3c'_\ell}\right) P_3, \quad (112)$$

as in the quintom case \(\dot{P}_1, \dot{P}_2\) and \(\dot{P}_3\) remain the same as in (90). Proceeding in a similar way as in the previous case, we take the derivative of the first equation in (112), obtaining

$$\dot{\xi}_1 = -\frac{48V_1}{\nu_\ell} e^{-\xi_1}, \quad (113)$$

which the corresponding solution is

$$e^{-\xi_1} = \frac{\nu_\ell r_1^2}{24V_1} \cosh^2(r_1 t - q_1). \quad (114)$$
Also in this quintessence like setting, the functional form of \( \dot{\xi}_2 \) is the same as \( \dot{\xi}_1 \), indicating that the solution is of the same type as (114), that is

\[
e^{-\xi_2} = \frac{\nu_1 r_2^2}{24V_2} \text{Csch}^2 (r_2 t - q_2), \tag{115}\]

in (114) and (115) the \( r_i \) and \( q_i \) (with \( i = 1, 2 \)) are constants coming from integration. With (114) and (115) at hand, we can reinsert them into Hamilton equations for the momenta, giving

\[
P_1 = -a_1 + \nu_r r_1 \text{Coth} (r_1 t - q_1), \tag{116}\]
\[
P_2 = -a_2 + \nu_r r_2 \text{Coth} (r_2 t - q_2), \tag{117}\]

where it can be easily verify that with (116) and (117) at hand the Hamiltonian is identically zero when

\[
a_1 = a_2 = \frac{72\nu_r + 1}{6} p_3, \quad p_3^2 = \frac{\nu_r (r_1^2 + r_2^2)}{4(72\nu_r - 1)}, \tag{118}\]

So, the solutions for the \( \xi_i \) coordinates become

\[
\xi_1 = \beta_1 + \ln \left[ \text{Sinh}^2 (r_1 t - q_1) \right], \tag{119}\]
\[
\xi_2 = \beta_2 + \ln \left[ \text{Sinh}^2 (r_2 t - q_2) \right], \tag{120}\]
\[
\xi_3 = \beta_3 - p_3 \left[ 16 (72\nu_r - 1) - 8 \frac{\nu_r}{c_f^2} \Delta t \right]
+ \left( 48 - \frac{1}{3c_f^2} \right) \nu_r \ln \left[ \text{Sinh} (r_1 t - q_1) \text{Sinh} (r_2 t - q_2) \right], \tag{121}\]

where \( \beta_i \) are integration constants. After applying the inverse canonical transformation we get the solutions in terms of the original variables (\( \Omega, \phi_1, \phi_2 \)) as

\[
\Omega = \Omega_0 + \frac{1}{12} \ln \left[ \text{Sinh}^2 (r_1 t - q_1) \text{Sinh}^2 (r_2 t - q_2) \right] + \frac{1}{2} p_3 \left[ 16 (72\nu_r - 1) - 8 \frac{\nu_r}{c_f^2} \right] \Delta t

- \frac{1}{2} \left( 48 - \frac{1}{3c_f^2} \right) \nu_r \ln \left[ \text{Sinh} (r_1 t - q_1) \text{Sinh} (r_2 t - q_2) \right], \tag{123}\]
\[
\phi_1 = \phi_{10} + \frac{1}{2\lambda_1} \ln \left[ \text{Sinh}^6 (r_1 t - q_1) \text{Sinh}^2 (r_2 t - q_2) \right] - 6 \left( 48 - \frac{1}{3c_f^2} \right) \nu_r \times 
\ln \left[ \text{Sinh} (r_1 t - q_1) \text{Sinh} (r_2 t - q_2) \right] + \frac{3}{\lambda_1} p_3 \left[ 16 (72\nu_r - 1) - 8 \frac{\nu_r}{c_f^2} \right] \Delta t,
\phi_2 = \phi_{20} + \frac{1}{2\lambda_2} \ln \left[ \text{Sinh}^2 (r_1 t - q_1) \text{Sinh}^6 (r_2 t - q_2) \right] - 6 \left( 48 - \frac{1}{3c_f^2} \right) \nu_r \times 
\ln \left[ \text{Sinh} (r_1 t - q_1) \text{Sinh} (r_2 t - q_2) \right] + \frac{3}{\lambda_2} p_3 \left[ 16 (72\nu_r - 1) - 8 \frac{\nu_r}{c_f^2} \right] \Delta t, \tag{124}\]

where \( \Omega_0, \phi_{10} \) and \( \phi_{20} \) are given in terms of the \( \beta_i \) constants as

\[
\Omega_0 = \frac{\beta_1 + \beta_2 - 6\beta_3}{12}, \quad \phi_{10} = \frac{3\beta_1 + \beta_2 - 6\beta_3}{2\lambda_1}, \quad \phi_{20} = \frac{\beta_1 + 3\beta_2 - 6\beta_3}{2\lambda_2}. \tag{125}\]

In this case, the scale factor acquires the form

\[
A(t) = A_0 \text{Sinh}^{\frac{\beta_q}{2}} (r_1 t - q_1) \text{Sinh}^{\frac{\beta_q}{2}} (r_2 t - q_2) e^{\nu_r t} \Delta t, \tag{126}\]

where \( \beta_q = \frac{p_3}{2} \left( 16(72\nu_r - 1) - 8 \frac{\nu_r}{c_f^2} \right) \) and \( \alpha_q = \frac{1}{2} \left( 48 - \frac{1}{3c_f^2} \right) \nu_r \).

The deceleration parameter for this case is

\[
q_{\text{quintessence}} = -1 - \frac{\alpha_q (r_1^2 \text{Sinh}^2 (r_2 t - q_2) + r_2^2 \text{Sinh}^2 (r_1 t - q_1))}{Q}, \tag{127}\]
with
\[
Q = \alpha_q^2 (r_2 \sinh^2(r_2 t - q_2) \cosh^2(r_1 t - q_1) + r_1 \sinh^2(r_1 t - q_1) \cosh^2(r_2 t - q_2)) + 2\alpha_q \sinh(r_1 t - q_1) \sinh(r_2 t - q_2) \{+\alpha' q r_1 r_2 \cosh(r_1 t - q_1) \cosh(r_2 t - q_2) + \beta_q' \sinh(r_2 t - q_2) \sinh(r_1 t - q_1) + r_1 \sinh(r_2 t - q_2) \cosh(r_1 t - q_1)\} + \beta_q^2 \sinh^2(r_1 t - q_1) \sinh^2(r_2 t - q_2).
\]

(127)

where the constant \(\alpha' = \alpha_q - \frac{1}{6}\). We are able to calculate the barotropic parameter employing the equation (54), giving
\[
\omega_{\text{quintessence}} = 1 - \frac{1}{3} \frac{2 \alpha'_q (r_1^2 \sinh^2(r_2 t - q_2) + r_2^2 \sinh^2(r_1 t - q_1))}{Q},
\]

(128)
in the lower right panel of the figure shown below we can see the behavior of the barotropic parameter.

\[\text{Figure 4:}\] In the top panel we can see the behavior the dynamical evolution of the volume for the quintom model (left) and the quintessence model (right). In the bottom left panel its shown the deceleration parameter for both models and in the bottom right panel we show the barotropic parameter for both models. The values taken for this figures were \(\ell = 1, p_3 = 1, r_1 = r_2 = 1, q_1 = q_2 = 1, \alpha_q = 0.069, \beta_q = 9.656, \alpha_q' = 0.402, \beta_q' = 1.656.\)

It is clear that the standard quintessence model with two scalar fields cannot be reproduce under this approach, because when we set \(m^{12} = 0\), this imply that parameter \(\ell\) is equal to zero, then, the matrix elements \(m^{11} = m^{22}\) are also zero, this was the challenge to resolve.

IV. QUANTUM APPROACH

In this section we present the quantum version of the classical cosmological models studied above along with its solutions. Since we already have the classical Hamiltonian density, the quantum counterpart can be obtained making the usual replacement \(\Pi_{q^\mu} = -i\hbar \partial_{q^\mu}\). First we modified the classical Hamiltonian density (15) in order to
consider the factor ordering problem between the function $e^{-3\Omega}$ and its moment $\Pi_\Omega$, introducing the linear term as $e^{-3\Omega} \Pi_\Omega^2 \rightarrow e^{-3\Omega} [\Pi_\Omega^2 + Q i \hbar \Pi_\Omega]$ where Q is a real number that measure the ambiguity in the factor ordering.

A. Quantum Quintessence-K-essence standard case

The quantum version for the first cosmological model we employ the modified Hamiltonian density,

$$H = \Pi_\Omega^2 + Q i \hbar \Pi_\Omega - 12 \Pi_{\phi_1}^2 - 12 \Pi_{\phi_2}^2 - 24 V_1 e^{6\Omega - \lambda_1 \phi_1}, \tag{129}$$

at this point, in order to obtain the Wheeler-DeWitt equation, we implement the following change of variables $(\Omega, \phi_1, \phi_2) \leftrightarrow (\xi_1, \xi_2, \xi_3)$

$$\begin{align*}
\xi_1 &= 6 \Omega - \lambda_1 \phi_1, \\
\Omega &= \xi_2, \\
\xi_2 &= \Omega, \\
\xi_3 &= \phi_2, \\
\phi_1 &= -\xi_1 + 6 \xi_2, \\
\phi_2 &= \xi_3,
\end{align*} \tag{130}$$

and also, obtaining a new set of conjugate momenta (in the same manner as \([14]\)), of the variables $(\xi_1, \xi_2, \xi_3)$, namely $(\Pi_{\phi_1}, P_1, P_2)$, which read

$$\Pi_{\phi_1} = -\Pi_\Omega = -\lambda_1 P_1, \quad P_1 = P_2 = 3, \quad (P_1, P_2, P_3), \tag{131}$$

which in turn transform the Hamiltonian density \([129]\) as

$$H = 12 \left( 3 - \lambda_1^2 \right) P_1^2 + P_2^2 + 12 P_1 P_2 - 12 a_2^2 + i \hbar Q (6 \phi_1 + \phi_2) - 24 V_1 e^{6\xi_1}. \tag{132}$$

Introducing the replacement $\Pi_{q^\nu} = -i \hbar \partial_{q^\nu}$, the WDW equation becomes

$$H \Psi = -12 \hbar^2 \left( 3 - \lambda_1^2 \right) \frac{\partial^2 \Psi}{\partial \xi_1^2} - \hbar^2 \frac{\partial^2 \Psi}{\partial \xi_2^2} - 12 \hbar^2 \frac{\partial^2 \Psi}{\partial \xi_1 \partial \xi_2} + 12 \hbar^2 \frac{\partial^2 \Psi}{\partial \xi_3^2} + Q \hbar^2 \left( 6 \frac{\partial \Psi}{\partial \xi_1} + \frac{\partial \Psi}{\partial \xi_2} \right) - 24 V_1 e^{6\xi_1} \Psi = 0, \tag{133}$$

due that the scalar potential does not depend on the coordinates $(\xi_2, \xi_3)$, we propose the following ansatz for the wave function $\Psi(\xi_1, \xi_2, \xi_3) = e^{-i a_2 \xi_2 + a_3 \xi_3} / h G(\xi_1)$ where $a_2$ and $a_3$ are arbitrary constants. Introducing the mentioned ansatz in \([133]\) we have that

$$-12 \hbar^2 \left( 3 - \lambda_1^2 \right) \frac{d^2 G}{G \, d \xi_1^2} + 6 \hbar (2 a_2 + \hbar Q) \frac{d G}{G \, d \xi_1} - a_2 (a_2 + h Q) + 12 a_3^2 - 24 V_1 e^{6\xi_1} = 0,$$

where we also divided the whole equation by the ansatz; in turn leads us to the following differential equation

$$\frac{d^2 G}{d \xi_1^2} - 2 a_2 + h Q \frac{d G}{d \xi_1} + \frac{1}{12 \hbar^2 (3 - \lambda_1^2)} \left( 24 V_1 e^{6\xi_1} + \eta \right) G = 0, \tag{134}$$

here $\eta = a_2 (a_2 + h Q) - 12 a_3^2$. The last equation can be casted as $y'' + ay' + (be^{cx} + c) y = 0$ (and whose solutions will depend on the value of $\lambda_1$) \([38]\), where

$$y = \text{Exp} \left( -\frac{ax}{2} \right) Z_\nu \left( 2 \sqrt{b} \text{e}^{\frac{cx}{2}} \right), \tag{135}$$
here $Z_\nu$ is the Bessel function and $\nu = \sqrt{a^2 - 4c/\kappa}$ being the order. The corresponding relations between the coefficients of (134) and $a, b, c$ and $\kappa$ are

$$a = \begin{cases} 
\frac{2\alpha_2 + hQ}{2\lambda_1^2}, & \text{when } \lambda_1^2 > 3 \\
-\frac{2\alpha_2 + hQ}{2\lambda_1^2}, & \text{when } \lambda_1^2 < 3
\end{cases} \quad (136)$$

$$b = \begin{cases} 
-\frac{2\nu_0}{\kappa^2(\lambda_1^2 - 3)}, & \text{when } \lambda_1^2 > 3 \\
\frac{2\nu_0}{\kappa^2(3 - \lambda_1^2)}, & \text{when } \lambda_1^2 < 3
\end{cases} \quad (137)$$

$$c = \begin{cases} 
-\frac{\nu}{12\kappa^2(\lambda_1^2 - 3)}, & \text{when } \lambda_1^2 > 3 \\
\frac{\nu}{12\kappa^2(3 - \lambda_1^2)}, & \text{when } \lambda_1^2 < 3
\end{cases} \quad (138)$$

$$\kappa = 1, \quad (139)$$

according to the constant $b$, the solution to the function $G$ becomes

$$G(\xi_1) = \text{Exp} \left( \frac{-2\alpha_2 + hQ}{4\lambda_1^2(\lambda_1^2 - 3)} \xi_1 \right) K_{\nu_1} \left( \frac{2}{\hbar} \sqrt{\frac{2\nu_0}{\lambda_1^2 - 3}} \, e^{\frac{\xi_1}{\lambda_1^2 - 3}} \right), \quad \lambda_1^2 > 3 \quad (140)$$

$$G(\xi_1) = \text{Exp} \left( \frac{2\alpha_2 + hQ}{4\lambda_1^2(3 - \lambda_1^2)} \xi_1 \right) J_{\nu_2} \left( \frac{2}{\hbar} \sqrt{\frac{2\nu_0}{3 - \lambda_1^2}} \, e^{\frac{\xi_1}{3 - \lambda_1^2}} \right), \quad \lambda_1^2 < 3 \quad (141)$$

and the wavefunction takes the form

$$\Psi_{\nu_1} = \text{Exp} \left( -\frac{2\alpha_2 + hQ}{4\lambda_1^2(\lambda_1^2 - 3)} \xi_1 - \frac{a_2\xi_2 + a_3\xi_3}{\hbar} \right) K_{\nu_1} \left( \frac{2}{\hbar} \sqrt{\frac{2\nu_0}{\lambda_1^2 - 3}} \, e^{\frac{\xi_1}{\lambda_1^2 - 3}} \right), \quad \lambda_1^2 > 3 \quad (142)$$

$$\Psi_{\nu_2} = \text{Exp} \left( \frac{2\alpha_2 + hQ}{4\lambda_1^2(3 - \lambda_1^2)} \xi_1 - \frac{a_2\xi_2 + a_3\xi_3}{\hbar} \right) J_{\nu_2} \left( \frac{2}{\hbar} \sqrt{\frac{2\nu_0}{3 - \lambda_1^2}} \, e^{\frac{\xi_1}{3 - \lambda_1^2}} \right), \quad \lambda_1^2 < 3. \quad (143)$$

where $\nu_1 = \sqrt{-\frac{2\alpha_2 + hQ}{4\lambda_1^2(\lambda_1^2 - 3)}} + \frac{\nu_0}{12\kappa^2(\lambda_1^2 - 3)}$ and $\nu_2 = \sqrt{\frac{2\alpha_2 + hQ}{4\lambda_1^2(3 - \lambda_1^2)}} - \frac{\nu_0}{12\kappa^2(3 - \lambda_1^2)}$ are the corresponding orders of the wave function. Applying the inverse transformation on the variables $\xi_i$, we can write the wave function in terms of the original variables $(A = e^{i\Omega}, \phi_i)$, which read

$$\Psi_{\nu_1} = A^{-\alpha_1} \text{Exp} \left( \frac{2\alpha_2 + hQ}{4\lambda_1^2(\lambda_1^2 - 3)} \lambda_1 \phi_1 - \frac{a_3\phi_2}{\hbar} \right) K_{\nu_1} \left( \frac{2}{\hbar} \sqrt{\frac{2\nu_0}{\lambda_1^2 - 3}} \, A^3 e^{\frac{\lambda_1^2 \phi_1}{\lambda_1^2 - 3}} \right), \quad \lambda_1^2 > 3 \quad (144)$$

$$\Psi_{\nu_2} = A^{-\alpha_2} \text{Exp} \left( -\frac{2\alpha_2 + hQ}{4\lambda_1^2(\lambda_1^2 - 3)} \lambda_1 \phi_1 - \frac{a_3\phi_2}{\hbar} \right) J_{\nu_2} \left( \frac{2}{\hbar} \sqrt{\frac{2\nu_0}{3 - \lambda_1^2}} \, A^3 e^{\frac{\lambda_2 \phi_1}{3 - \lambda_1^2}} \right), \quad \lambda_1^2 < 3. \quad (145)$$

with $\alpha_1 = \frac{1}{\pi} \left( a_2 + \frac{3}{2} \frac{2\alpha_2 + hQ}{\lambda_1^2 - 3} \right)$ and $\alpha_2 = \frac{1}{\pi} \left( a_2 - \frac{3}{2} \frac{2\alpha_2 + hQ}{3 - \lambda_1^2} \right)$. The behavior of the wave function of this model, when $\lambda_1^2 < 3$, can be seen in the following figures. In Fig. [6] it can be observed that the probability density has a damped behavior, which is a good characteristic in a wave function and this kind of demeanor has been reported in [12]. In Figure [7] a 2D view of the probability density for different values of $\phi_1$ is shown, we can also see the importance of the existence scalar fields during primordial inflation.

The behavior of the wave function for $\lambda_1^2 > 3$ can be seen in Fig. [7], where the probability density presents a damping behavior with respect to the scale factor, which as mention before, is a desire characteristic in a wave function. Here, the parameter $Q$, for negative values, plays the role of a retarder of the wave function and compresses the length on the axis where the field evolves; then the inflation epoch should also be retarded as time evolves.
Figure 5: Behaviour of density probability, with $Q = 1$, $\lambda_1 = 0.21$, $a_2 = 0.6$, $a_3 = 1$, $\nu_2$.

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**B. Quantum quintom like case**

For the second cosmological model, the quintom like case, the quantum version of this model is obtained applying, again, the recipe $\Pi_{\nu\mu} = -i\hbar \partial_{\nu\mu}$ to the Hamiltonian density (93), hence

$$\frac{\hbar^2}{\mu_i} \frac{\partial^2}{\partial \xi_1^2} + \frac{\hbar^2}{\mu_i} \frac{\partial^2}{\partial \xi_2^2} - \hbar^2 \left( 48 - \frac{1}{3c_i} \right) \left( \frac{\partial^2}{\partial \xi_3 \partial \xi_1} + \frac{\partial^2}{\partial \xi_3 \partial \xi_2} \right) - \hbar^2 \left( 16 - \frac{1}{18c_i} \right) \frac{\partial^2}{\partial \xi_3^2} - 24V_1 e^{-\xi_1} - 24V_2 e^{-\xi_2} \right) \Psi = 0,$$

(146)

because the scalar potential does not depend on the coordinate $\xi_3$, we propose the following ansatz for the wave function $\Psi(\xi_1, \xi_2, \xi_3) = e^{(a_3/\hbar)\xi_3} A(\xi_1) B(\xi_2)$ where $a_3$ is an arbitrary constant. Substituting and dividing by the ansatz in (146), we obtain

$$\frac{\hbar^2}{\mu_i} \frac{d^2 A}{d \xi_1^2} + \frac{\hbar^2}{\mu_i} \frac{d^2 B}{d \xi_2^2} - a_3 \hbar \left( 48 - \frac{1}{3c_i} \right) \left( \frac{d A}{d \xi_1} + \frac{d B}{d \xi_2} \right) - a_3^2 \left( 16 - \frac{1}{18c_i} \right) - 24V_1 e^{-\xi_1} - 24V_2 e^{-\xi_2} = 0,$$

(147)
where we can separate the equations as
\[
\begin{align*}
\frac{d^2 A}{d\xi_1^2} - \frac{a_3 \mu_e}{\hbar} \left( 48 - \frac{1}{3c_i} \right) \frac{dA}{d\xi_1} - \frac{\mu_e}{\hbar^2} \left( \frac{a_3^2}{2} \left( 16 - \frac{1}{18c_i} \right) - \alpha^2 + 24V_1 e^{-\xi_1} \right) A &= 0, \\
\frac{d^2 B}{d\xi_2^2} - \frac{a_3 \mu_e}{\hbar} \left( 48 - \frac{1}{3c_i} \right) \frac{dB}{d\xi_2} - \frac{\mu_e}{\hbar^2} \left( \frac{a_3^2}{2} \left( 16 - \frac{1}{18c_i} \right) + \alpha^2 + 24V_2 e^{-\xi_2} \right) B &= 0,
\end{align*}
\]
(148)
(149)
with \(\alpha^2\) being the separation constant. The corresponding solutions of (148) and (149) have the following form\[83\]
\[
Y(x) = \text{Exp} \left( -\frac{ax}{2} \right) Z_{\nu} \left( \frac{2\sqrt{b}}{\lambda} e^{\frac{\lambda x}{2}} \right),
\]
(150)
here \(Z_{\nu}\) are the generic Bessel function with order \(\nu = \sqrt{a^2 - 4c}/\lambda\). If \(\sqrt{b}\) is real, \(Z_{\nu}\) are the ordinary Bessel function, otherwise the solution will be given by the modified Bessel function. Making the following identifications
\[
\begin{align*}
\lambda &= -1, \\
a &= \frac{a_3 \mu_e}{\hbar} \left( 48 - \frac{1}{3c_i} \right), \\
b_{1,2} &= -\frac{\mu_e}{\hbar^2} 24V_{1,2}, \\
c_\pm &= -\frac{\mu_e}{\hbar^2} \left( a_3 \left( 8 - \frac{1}{36c_i} \right) \mp \alpha^2 \right), \\
\nu_\pm &= \sqrt{\frac{a^2}{\mu_e} + 4c},
\end{align*}
\]
(151)
(152)
(153)
(154)
(155)
we can check that the value for \(\sqrt{b}\) is imaginary, which as already mentioned, gives a solution in terms of the modified Bessel function \(Z_{\nu} = K_{\nu}\) whose order lies in the reals. Thus, the wave function is
\[
\Psi_{\nu_\pm} = \text{Exp} \left( \left( \frac{\mu_e}{2\hbar} \left( 48 - \frac{1}{3c_i} \right) \left( \xi_1 + \xi_2 \right) + \frac{\xi_3}{\hbar} \right) a_3 \right) K_{\nu_-} \left( \frac{4}{\hbar} \sqrt{6V_1 \mu_e} e^{-\frac{\lambda_1}{2}} \right) \times K_{\nu_+} \left( \frac{4}{\hbar} \sqrt{6V_2 \mu_e} e^{-\frac{\lambda_2}{2}} \right),
\]
(156)
so, the wave function in the original variables becomes
\[
\Psi_{\nu_\pm} = A \frac{\beta a_3}{\sqrt{6\hbar}} \text{Exp} \left( \frac{\beta \xi_3}{6\hbar} \left( \lambda_1 \phi_1 + \lambda_2 \phi_2 \right) \right) K_{\nu_-} \left( \frac{4}{\hbar} \sqrt{6V_1 \mu_e} A^3 e^{-\frac{\lambda_1}{2} \phi_1} \right) \times K_{\nu_+} \left( \frac{4}{\hbar} \sqrt{6V_2 \mu_e} A^3 e^{-\frac{\lambda_2}{2} \phi_2} \right),
\]
(157)
where $\delta_\ell = \frac{3}{2} \mu_\ell \left( 48 - \frac{1}{3c_\ell} \right) + 1$ and $\beta_\ell = 3 \mu_\ell \left( 48 - \frac{1}{3c_\ell} \right) + 1$. In Fig. 8 the behavior of the probability density with respect of the scale factor and the scalar field $\phi_1$ is depicted; we can note that as the scale factor evolves the probability density has a decaying behavior and has a moderate growth in the axis where the scalar field develops (a similar behavior was found for the quantum solution $\lambda_1^2 > 3$ when $Q = -10$).

**Figure 8**: Probability density for the quintom like cosmology; with $\ell = 1$, $a_3 = 1$, $\alpha = 1$, $\delta_\ell = 1.207$, $\beta_\ell = 1.414$, $\lambda_1 = 0.1$, $\mu_\ell = 0.019$, $c_\ell = 0.08$.

C. Quantum quintessence like case

Lastly, we are going to consider the quantum version of the quintessence like case. As in the previous two models, what we want is to obtain an equation of the form $\mathcal{H}\Psi(\xi) = 0$, to achieve this we introduce the standard prescription

$$\Pi^\mu_q = -ih\partial_q^\mu$$

in (111), obtaining

$$\begin{aligned}
-\frac{\hbar^2}{\nu_1} \frac{d^2 A}{d\xi_1^2} - \frac{\hbar^2}{\nu_2} \frac{d^2 B}{d\xi_2^2} - b_3 h \left( 48 - \frac{1}{3c_\ell} \right) \left( \frac{d^2}{d\xi_3 d\xi_1} + \frac{d^2}{d\xi_3 d\xi_2} \right) - \hbar^2 \left( 16 - \frac{1}{18c_\ell} \right) \frac{\partial^2}{\partial\xi_3^2} - 24V_1 e^{-\xi_1} - 24V_2 e^{-\xi_2} \right) \Psi = 0,
\end{aligned}$$

(158)

we can see that the scalar potential does not depend on the coordinate $\xi_3$, consequently we propose the following ansatz for the wave function $\Psi(\xi_1, \xi_2, \xi_3) = e^{(b_3 h/\hbar)\xi_3} A(\xi_1) B(\xi_2)$ where $b_3$ is an arbitrary constant. Applying and dividing by the ansatz in (158) we get

$$\begin{aligned}
-\frac{\hbar^2}{\nu_1} \frac{d^2 A}{d\xi_1^2} - \frac{\hbar^2}{\nu_2} \frac{d^2 B}{d\xi_2^2} - b_3 h \left( 48 - \frac{1}{3c_\ell} \right) \left( \frac{1}{A} \frac{dA}{d\xi_1} + \frac{1}{B} \frac{dB}{d\xi_2} \right) - b_3^2 \left( 16 - \frac{1}{18c_\ell} \right) - 24V_1 e^{-\xi_1} - 24V_2 e^{-\xi_2} = 0,
\end{aligned}$$

(159)

separating the equations we have that

$$\begin{aligned}
\frac{d^2 A}{d\xi_1^2} + \frac{b_3 h}{h} \left( 48 - \frac{1}{3c_\ell} \right) \frac{dA}{d\xi_1} + \frac{\nu_1}{h^2} b_3^2 \left( 8 - \frac{1}{36c_\ell} \right) - \alpha^2 + 24V_1 e^{-\xi_1} \right) A = 0,
\end{aligned}$$

(160)

$$\begin{aligned}
\frac{d^2 B}{d\xi_2^2} + \frac{b_3 h}{h} \left( 48 - \frac{1}{3c_\ell} \right) \frac{dB}{d\xi_2} + \frac{\nu_2}{h^2} b_3^2 \left( 8 - \frac{1}{36c_\ell} \right) + \alpha^2 + 24V_2 e^{-\xi_2} \right) B = 0,
\end{aligned}$$

(161)
where $\alpha^2$ is the separation constant. These last two equations are similar to the quantum quintom like case (148) and (149). Proceeding in a similar fashion as the previous subsection IV B, we make the following identifications

$$\lambda = -1,$$

$$a = \frac{b_3 \nu_t}{\hbar} \left(48 - \frac{1}{3c'_t}\right),$$

$$b_{1,2} = \frac{\nu_t}{\hbar^2} 24 V_{1,2},$$

$$c_\mp = \frac{\nu_t}{\hbar^2} \left(b_3^2 \left(8 - \frac{1}{36c'_t}\right) \mp \alpha^2\right),$$

and conclude that the solutions are given by the ordinary Bessel function $J_\nu$ with order $\nu = \sqrt{(a^2/\nu_t) + 4c_\mp}$. Thus, the wave function becomes

$$\Psi_{\nu \pm} = \exp \left[\left(\frac{\nu_t}{2\hbar} \left(48 - \frac{1}{3c'_t}\right) (-\xi_1 - \xi_2) + \frac{\xi_3}{\hbar}\right) b_3\right] J_{\nu_-} \left(\frac{4}{\hbar} \sqrt{6V_1 \nu_t} e^{-\frac{\xi_1}{\hbar}}\right) \times$$

$$J_{\nu_+} \left(\frac{4}{\hbar} \sqrt{6V_2 \nu_t} e^{-\frac{\xi_2}{\hbar}}\right).$$

written in the original variables, become

$$\Psi_{\nu \pm} = A^{\frac{\delta_\ell}{\nu_t}} \exp \left(-\frac{\beta_\ell b_3}{6\hbar} (\lambda_1 \phi_1 + \lambda_2 2 \phi_2)\right) J_{\nu_-} \left(\frac{4}{\hbar} \sqrt{6V_1 \nu_t} A^3 e^{-\frac{\lambda_1}{2} \phi_1}\right) \times$$

$$J_{\nu_+} \left(\frac{4}{\hbar} \sqrt{6V_2 \nu_t} A^3 e^{-\frac{\lambda_2}{2} \phi_2}\right).$$

where $\delta_\ell = \frac{3}{2} \nu_t \left(48 - \frac{1}{3c'_t}\right) - 1$ and $\beta_\ell = -3 \nu_t \left(48 - \frac{1}{36c'_t}\right) + 1$. Fig. (9) shows the probability density for the quintessence cosmological model in terms of the scale factor and the scalar field $\phi_1$; we can see that the probability density dies away in a muffled manner with respect of the scale factor, which is a good characteristic of a wave function, as pointed out in section IV A. Another thing we can notice is that the probability density has a moderate increase in the direction were the scalar field evolves (as in the quantum quintom case).

**Figure 9:** Probability density for the quintessence like cosmological model, where $\ell = 1$, $a_3 = -1$, $\alpha = 0.1$, $\delta'_\ell = -0.792$, $\beta'_\ell = 0.585$, $\lambda_1 = 0.3$, $\mu'_\ell = 0.019$, $c'_t = 0.08$. 
V. FINAL REMARKS

In this work we have studied two multi-field cosmological models, for which classical and quantum solutions were found. For both setups we work with a flat FRW cosmology. In the first model, we consider two scalar fields but only one potential term and standard matter in the stiff scenario, which can be seen as a simple quintessence plus a K-essence model. In the second one, we also considered two scalar fields with the difference that the two potential terms are taken into account, as well as the standard kinetic energy and the mixed term, which are present in chiral field approach and when standard matter is included it can be thought of as a stiff matter scenario. Regarding this second model, it is shown that two possible cases can be studied: a quintom like model and a quintessence like model considering the mixed term in the scalar fields.

For the first flat FRW model, applying the Hamiltonian approach, we where able to find three different classical solutions depending on the value of the parameter $\lambda^2_i$. In each of the three cases, to analyze the dynamical behavior of the model under consideration, we calculate the deceleration parameter $q_1$ (where $q_1$, $q_2$ and $q_3$ stand for the solutions for $\lambda^2_i < 3$, $\lambda^2_i > 3$ and $\lambda^2_i = 3$, respectively) when no matter is present and when standard matter is included. In Fig. (3) we can observe the temporal evolution of the deceleration parameters, in the case for $\rho_\gamma = 0$ it is observed that $q_1$ and $q_3$ have a deceleration/acceleration phase, where for both parameters the former is for a short period of time to then accelerate, whereas in the case when standard matter is included ($\rho_\gamma \neq 0$), the behavior of deceleration/acceleration persist, but $q_3$ gets outgrown by $q_1$. For the case of $q_2$, in contrast to $q_1$ and $q_3$, an accelerated behavior is only present and a slow growth in both scenarios with no significant difference is shown. Ultimately, in both scenarios, the asymptotic behavior of the three solutions goes to the same value, $-1$. In Fig. (4) we can observe the behavior of the barotropic parameter (in the gauge $N = 1$) for the three different solutions of $\lambda^2_i$, for each of the three parameters the asymptotic behavior approaches $1$. For completeness, we also calculate the barotropic parameter for the gauge $N = 1$, given by Eq. (58). In this model, multiplying the kinetic energy term associated to the second scalar field by an arbitrary function $F(\phi_2)$, would be an interesting exercise to see if exact solutions can be found; research in this direction has been performed in [34,55]. Another avenue that can be explored is the one presented in [84], as well the ideas pursued in [47,48]. In section II B we investigate the case when standard matter is included, in this particular case we found the scale factor of the Universe [69] has an accelerated growth and whose dynamical demeanor can be observed in Fig. (3), this characteristic is also corroborated with the deceleration parameter given by (74). We also found that the scalar potential, given by (72), is consistent with a (volume) accelerated expansion in the dust scenario. In addition, we found that the scale field $\phi_2$ acquires a constant value for late times, this feature was also obtained for an anisotropic cosmological model, where the anisotropic parameters vanished for late times [82]. To round off this analysis we calculate the barotropic parameter, given by (77). Finally, we study the case for $\gamma = \frac{1}{3}$, which corresponds to the radiation era. The solution of the master equation for this case is given by (77), unfortunately the expression for the scale factor is not given explicitly in terms of $t$, given that the solution obtained is not invertible, nevertheless, one would expect (in light of [67]) that if we could get $A(t)$ it would be a function with an accelerated growth.

The quantum version of these model were obtained making the usual replacement $\Pi_{q\nu} = -i\hbar \partial_{q\nu}$ in the classical Hamiltonian density, where the linear term $e^{-3i\Pi_{\phi}^{\phi}} \rightarrow e^{-3i\Pi_{\phi}^{\phi}} + Q_i \Pi_{\phi}^{\phi}$ was introduced, in order to account for the factor ordering problem, where $Q$ is a real number that measures the ambiguity in the factor ordering. In this set up we found that, for $\lambda^2_i < 3$, the wave function has a damping behavior, which is a good characteristic that has also been reported in [12,66], features that can be seen in Fig. (5) and Fig. (6). The quantum solution for $\lambda^2_i > 3$ is given Eq. (142), and the behavior of the wave function is presented in Fig. (7), for which the damping behavior remains, with the difference that for negative values, the parameter $Q$ plays the role of a retarder of the wave function and the length of the scalar field is compressed, signaling that the inflation period should also be retarded over time.

For the second model, in addition to considering the scalar fields, the two terms of the potential were also considered. First, we were able to find classical solutions to the EKG equations (79-80) using the Hamiltonian formalism. In this model we were able to distinguish two types of solutions: a quintom type and a quintessence type. For the first, the solutions are given by the equations (105) while for the second are given by (123), with these two sets of solutions the scale factor, deceleration parameter and the barotropic parameter could be found. In Fig. (4) we can see the behavior of the volume function, the $q$-parameters and the barotropic parameters for the models. The volume function for both models has an accelerated growth. The deceleration parameter for the quintom case increases more rapidly than quintessence counterpart to then stabilizing at $-1$; finally the barotropic parameter for the quintom model acquires faster the asymptotic value of $1$. Quantum solutions for to this model were also found. The solution for the quantum quintom like model is given by (157); in Fig. (8) can be appreciated that the probability density drops as the scale factor develops while in the direction of the scalar field it has a steady increment. The solution for the quantum quintessence like case, is given by (168) and the probability density its shown in Fig. (9), where the probability density dies away in a damped manner as the scale factor evolves, also a slight increase in the the direction of the scalar function is observed. Finally, we can say that this work has already been done considering the anisotropic bianchi
type I and the solutions found are generalization of the solutions to this work \cite{87}.

In the page below, three tables are presented where our results are included.

Acknowledgments

This work was partially supported by PROMEP grants UGTO-CA-3. J.S. was partially supported SNI-CONACYT. This work is part of the collaboration within the Instituto Avanzado de Cosmología and Red PROMEP: Gravitation and Mathematical Physics under project Quantum aspects of gravity in cosmological models, phenomenology and geometry of space-time. Many calculations where done by Symbolic Program REDUCE 3.8. We also want to thank the anonymous referees for their valuable recommendations.

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\end{thebibliography}
| Case | Scale Factor | $q_1$ - parameter | $\omega_T$ |
|------|--------------|------------------|-----------|
| $\lambda_1^2 < 3$ | $A(t) = A_0 \exp \left[ \frac{-12\lambda_1 p_{\phi 1}}{q} (t - t_0) \right] \left( \cosh \left[ 12(2\omega t - t_0) \right] \right)^{\frac{q_1}{2}}$ | $q_1 = -1 - \frac{\eta_0^2}{\lambda_1 p_{\phi 1} \sinh(12\omega t) + \cosh(12\omega t)}$ | $\omega_T = 1 - \frac{2\eta_0^2}{3 \lambda_1 p_{\phi 1} \sinh(12\omega t) + \cosh(12\omega t)}$ |
| $\lambda_1^2 > 3$ | $A(t) = A_0 \exp \left[ \frac{-12\lambda_1 p_{\phi 2}}{q} (t - t_0) \right] \cosh \left[ 12(\omega t - t_0) \right]$ | $q_2 = -1 - \frac{\eta_0^2}{\lambda_1 p_{\phi 2} \cosh(12\omega t') + \sinh(12\omega t')} \frac{1}{\omega'}$ | $\omega_T = 1 - \frac{2\eta_0^2}{3 \lambda_1 p_{\phi 2} \cosh(12\omega t') + \sinh(12\omega t')}$ |
| $\lambda_2^2 = 3$ | $A(t) = A_0 \exp \left[ \frac{2\sqrt{3} p_{\phi 1} + p_{\phi 2} + 4\rho_1}{p_{\phi 1}} (t - t_0) \right] \exp \left[ \frac{-\sqrt{3} p_{\phi 1} \ln \left( A + \sqrt{A^2 + b^2} \right)}{b_0} \right]$ | $q_3 = -1 - \frac{12\sqrt{3} p_{\phi 1} \exp(2\sqrt{3} p_{\phi 1} t)}{p_{\phi 1} \exp(2\sqrt{3} p_{\phi 1} t) + \sqrt{A^2 + b^2} + 4\rho_1} \frac{1}{\omega'}$ | $\omega_T = 1 - \frac{2\eta_0^2}{3 \lambda_1 p_{\phi 1} \cosh(12\omega t') + \sinh(12\omega t')} \frac{1}{\omega'}$ |
| $\gamma = 0$ | $A^3(t) = 0 \left( A_1 + A_2 \right)^2 - 1$ | $q(t) = -\frac{1}{3} \frac{1}{\sqrt{1 + 2A_1 + A_2}}$ | $\omega_T = 1 - \frac{2\eta_0^2}{3 \lambda_1 p_{\phi 1} \cosh(12\omega t') + \sinh(12\omega t')} \frac{1}{\omega'}$ |
| $\gamma = \frac{1}{3}$ | $2b_0 \Delta t = A_1 \sqrt{A^2 + b_2} - \frac{1}{\sqrt{1 + 2A_1 + A_2}} \ln A + \sqrt{A^2 + b^2}$ | | |
Table II: Table 2: Second Model (classical)

| Case       | Scale Factor | $q_{\text{scale}}$ | $q_{\text{quintom}}$ | $q_{\text{quintessence}}$ |
|------------|--------------|---------------------|-----------------------|-----------------------------|
| quintom    | $A(t) = A_0 \cosh \left( \frac{1}{6} t - \alpha q (r - r_1) \cosh \left( \frac{1}{6} t - \alpha q (r - r_2) e^{-\beta q \Delta t} \right) \right)$ | $q_{\text{quintom}} = -\frac{1}{2}$ | $q_{\text{quintessence}} = 1$ |
| quintessence | $A(t) = A_0 \sinh \left( \frac{1}{6} t - \alpha q (r - r_1) \sinh \left( \frac{1}{6} t - \alpha q (r - r_2) e^{\beta q \Delta t} \right) \right)$ | $q_{\text{quintessence}} = -\frac{2}{3}$ | $q_{\text{quintessence}} = 1$ |

Note: $r_1$ and $r_2$ are constants related to the case.
Table III: Table 3: Quantum solutions (first two lines correspond to the first model and the second two lines to the second model)

| Case  | Wave Function                      |
|-------|-----------------------------------|
| $\lambda^2 < 3$ | $\Psi_{\nu} = A - \alpha_2 \exp \left(\frac{2}{\hbar} \sqrt{2V_0 \lambda^2} \beta_1 \phi_1 - \alpha_2 \phi_2 \right)$ |
| $\lambda^2 > 3$ | $\Psi_{\nu} = A - \alpha_1 \exp \left(\frac{2}{\hbar} \sqrt{2V_0 \lambda^2} \beta_1 \phi_1 - \alpha_1 \phi_2 \right)$ |

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