Chaotic fluctuations in mathematical economics

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Abstract. In this paper we examine a Cournot duopoly model, which expresses the strategic interaction between two firms. We formulate the dynamic adjustment process and investigate the dynamic properties of the stationary point. By introducing a memory mechanism characterized by distributed lag functions, we presuppose that each firm makes production decisions in a cautious manner. This implies that we have to deal with the system of integro-differential equations. By means of numerical simulations we show the occurrence of chaotic fluctuations in the case of fixed delays.

1. Introduction
The study of economic dynamics is one of the most important topics in the field of economics. Economic dynamics is concerned with changes and variations in the economic phenomena which take place in the course of time. In other words, economic dynamics seeks to explain the time paths of economic variables at the company, industry, or macroeconomic level. For this purpose, economists use extensively mathematical models, which are typically represented by difference equations or differential equations.

At an early stage of the development of economic dynamics from the 1950s to the 1970s, a significant portion of research was devoted to resolving the problems on the local stability of steady states and the existence of limit cycles in economic systems. It was not until the late 1970s that economists began to investigate chaotic behavior by using nonlinear dynamical system theory. Such a revolution was induced by the seminal contribution of Li and York [1]. The development of chaos theory in discrete-time dynamical systems has stimulated the publication of a large number of papers in economics (for example, see Rand [2], Benhabib and Day [3, 4], Day [5, 6], Grandmont [7], Puu [8], Nishimura and Yano [9]).

The main purpose of the present paper is to investigate the occurrence of chaotic fluctuations in the dynamic adjustment process by adopting the Cournot duopoly setting. Unlike Puu [8] who shows the emergence of chaotic trajectories in a discrete-time model, we use a continuous-time model of Cournot duopoly dynamics. By introducing a memory mechanism characterized by distributed lag functions, we presuppose that each firm makes production decisions in a cautious manner. This implies that we have to deal with the system of integro-differential equations.

2. The basic framework
As mentioned in Introduction, we consider a Cournot duopoly model, which formulates the strategic interactions between two firms in a market for a single homogeneous commodity. Each firm tries to maximize its profit by taking the output choice of the rival firm as given. The equilibrium point is defined as the intersection of the reaction functions.
Two firms X and Y produce a homogeneous product with output levels $x$ and $y$. The market demand function is assumed to be

$$p = a - b(x + y), \quad a > 0, b > 0,$$

where $p$ is the price of output. Furthermore, the cost of each firm is assumed to be linear, $c_i x_i$ with $c_i > 0 \ (i = X, Y)$. Thus the profit of firm $i$ is given by

$$\pi_X = \left[a - b(x + y)\right] x - c_X x,$$

$$\pi_Y = \left[a - b(x + y)\right] y - c_Y y.$$  

As usual we assume that each firm maximizes its profit taking the quantity supplied by the rival firm as given. Consequently, the first order conditions for profit maximization are

$$-bx + a - b(x + y) - c_X = 0,$$

$$-by + a - b(x + y) - c_Y = 0,$$

which result in the following best-reaction functions:

$$x = R_X(y) = -\frac{1}{2} y + \frac{a - c_X}{2b},$$

$$y = R_Y(x) = -\frac{1}{2} x + \frac{a - c_Y}{2b}.$$  

According to (6), the optimal quantity for firm X depends on the output level of the rival firm. Equivalently, from (7) we obtain the result that the optimal quantity for firm Y depends on the output level of the rival firm. The intersection point of two lines (6)–(7) defines the Cournot-Nash equilibrium:

$$x^* = \frac{a - 2c_X + c_Y}{3b}, \quad y^* = \frac{a + c_X - 2c_Y}{3b}.$$  

In order to guarantee positive values of $x^*$ and $y^*$, we assume that $a - 2c_X + c_Y > 0$ and $a + c_X - 2c_Y > 0$.

Let us now consider the dynamic adjustment process. We adopt the following mechanism:

$$\dot{x} = \alpha_X (R_X(y) - x), \quad \alpha_X > 0,$$

$$\dot{y} = \alpha_Y (R_Y(x) - y), \quad \alpha_Y > 0,$$

where $\alpha_i$ indicates an adjustment speed parameter, which is a positive constant. The adjustment process shows that each firm controls the growth rate of its output according to the difference between its profit maximizing output and its actual output.

Substituting (6) and (7) into (9) leads to the following system:

$$\dot{x}(t) = \alpha_X \left( -\frac{1}{2} y(t) + \frac{a - c_X}{2b} - x(t) \right) x(t),$$

$$\dot{y}(t) = \alpha_Y \left( -\frac{1}{2} x(t) + \frac{a - c_Y}{2b} - y(t) \right) y(t).$$
It is easy to check that the stationary point of system (10) coincides with the Cournot-Nash equilibrium. The Jacobian matrix at the Cournot-Nash equilibrium is

\[
\begin{bmatrix}
-\alpha_X x_* & -(1/2)\alpha_X x_* \\
-(1/2)\alpha_Y y_* & -\alpha_Y y_*
\end{bmatrix}
\]  

By evaluating the Jacobian matrix, we can conclude that the Cournot-Nash equilibrium is locally stable. This is because we can certify that the trace is negative and the determinant is positive. These are the conditions for the local stability of the stationary point.

Figure 1 shows the trajectories resulting from four different initial conditions: \(A_1(\ln 10, \ln 10), A_2(\ln 10, \ln 0.01), A_3(\ln 0.01, \ln 0.01), A_4(\ln 0.01, \ln 10)\). All trajectories of the system converge to the unique Cournot-Nash equilibrium. This result suggests that the equilibrium point is globally stable in the economically meaningful region. Note that this figure plots the natural logarithm of the output level of firm Y on the vertical axis versus the natural logarithm of the output level of firm X on the horizontal axis.

3. Extended model

In this section we shall extend the previous analysis by introducing a memory mechanism characterized by distributed lag functions. We assume that each firm adjusts its output by considering the past values as well as the current value of its own output. In this sense each firm makes production decisions in a cautious manner. We specify the following system of integro-differential equations:

\[
\frac{\dot{x}}{x} = \alpha_X \left( R_X(y(t)) - \int_{-\infty}^{t} w_X(s)x(s)ds \right), \quad (12a)
\]

\[
\frac{\dot{y}}{y} = \alpha_Y \left( R_Y(x(t)) - \int_{-\infty}^{t} w_Y(s)y(s)ds \right). \quad (12b)
\]

Here we assume that \(w_i(s)\) indicates the following distributed lag function, i.e.,

\[
w_i(s) = \left( \frac{n_i}{\tau_i} \right)^{n_i} \frac{(t-s)^{n_i-1}}{(n_i-1)!} e^{-(n_i/\tau_i)(t-s)}, \quad \tau_i > 0,
\]
where \( n_i \) is a positive integer \((i = X, Y)\). On the account of the fact that \( \int_{-\infty}^{t} w_i(s)ds = 1 \), the function \( w_i(s) \) can be interpreted as a weighting function, which is equivalent to a density function with the mean \( \tau_i \) and the variance \( \tau_i^2/n_i \).

It is important to note that we can classify the weighting function in three possible cases according to the values of \( n_i \). First, we consider the case \( n_i = 1 \). In this case we obtain \( w_i(s) = (1/\tau_i)e^{-(t-s)/\tau_i} \), which corresponds to the exponentially declining weighting function. This case implies that the firm evaluates the most recent event as being the most relevant, since the function \( w_i(s) \) places more weight on more recent data. Secondly, we explain the case of \( n_i \geq 2 \). In this case the function \( w_i(s) \) has a one-hump shape, which shows that a sharp peak emerges around time \( t - \tau_i \) as \( n_i \) increases. Thirdly, we discuss the case \( n_i \to +\infty \). When \( n_i \) becomes indefinitely large, \( w_i(s) \) leads to a Dirac delta function that appears as a sharp peak at \( t = \tau_i \). To put it simply, we consider a fixed delay \( \tau_i(>0) \):

\[
\lim_{n_i \to \infty} \int_{-\infty}^{t} w(s)x(s)ds = x(t - \tau_i).
\]

The important point to note is that MacDonald’s linear chain trick transforms system (12) into an expanded system of ordinary differential equations. We can therefore obtain the analytical results about the Hopf bifurcation phenomena by using the linear chain trick. In this sense, the linear chain trick is a useful tool for investigating the systems of integro-differential equations. On this point, see MacDonald [10].

In the present paper we concentrate on the case of fixed time delays: \( n_X \to \infty, \tau_X > 0, n_Y \to \infty, \) and \( \tau_Y > 0 \). Since the weighting function \( w_i(s) \) tends to a Dirac delta function as \( n_i \to \infty \), we obtain the following system of delay differential equations:

\[
\begin{align*}
\dot{x} &= \alpha_X(R_X(y(t)) - x(t - \tau_1)), \\
\dot{y} &= \alpha_Y(R_Y(x(t)) - y(t - \tau_2)).
\end{align*}
\]  

The dynamic equations mean that at time \( t \) firm \( i \) adjusts its output at a rate proportional to the difference between its profit maximizing output at time \( t \) and its actual output at some preceding time \( t - \tau_i \) \((i = X, Y)\).

With (13) and the reaction functions, we obtain the following dynamical system:

\[
\begin{align*}
\dot{x}(t) &= \alpha_X \left(-\frac{1}{2}y(t) + \frac{a - cX}{2b} - x(t - \tau_1)\right) x(t), \\
\dot{y}(t) &= \alpha_Y \left(-\frac{1}{2}x(t) + \frac{a - cY}{2b} - y(t - \tau_2)\right) y(t).
\end{align*}
\]

Let us now start our analysis of system (14). The model to consider here is precisely identical to a model developed by Shibata and Saito [11]. Their model is an interesting contribution to the field of mathematical biology. They investigated the population dynamics of two competing species with fixed time lags and showed the appearance of chaotic dynamic trajectories by means of numerical simulations.

Following Shibata and Saito, we hereafter resort to numerical experiments to examine the dynamic properties of our system. We set the following parameter values: \( a = 3, b = 1, c_X = c_Y = 1, \tau_X = 1.6, \tau_Y = 0.9, \alpha_X = \alpha_Y = 2 \).

Figure 2 illustrates the emergence of a chaotic attractor in system (14). The trajectories never converge to the Cournot-Nash equilibrium, but exhibit perpetual fluctuations, while remaining forever in a bounded region of the state space. Although they are governed by the deterministic system, the behavior of phase trajectories looks random. This result implies that we can observe complex endogenous fluctuations in a particular market even if exogenous shocks are absent.
4. Conclusion
We have investigated the dynamic properties of Cournot duopoly models, which express the strategic interaction between two firms. By means of numerical simulations we have shown the occurrence of chaotic oscillations in the case of fixed delays.

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