AUGMENTAL HOMOLOGY AND THE
KÜNNETH FORMULA FOR JOINS

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Abstract. The “simplicial complexes” and “join” (∗) today used within combinatorics aren’t the classical concepts, cf. [25] p. 108-9, but, except for ∅, complexes having {∅} as a subcomplex resp. Σ₁∗Σ₂ := \{σᵱ∪σᵳ | σᵱ ∈ Σ₁, σᵳ ∈ Σ₂\}, implying a tacit change of unit element w.r.t. the join operation, from ∅ to {∅}. Extending the classical realization functor to this category of simplicial complexes we end up with a “restricted” category of topological spaces, “containing” the classical and where the classical (co)homology theory, as well as the ad-hoc invented reduced versions, automatically becomes obsolete, in favor of a unifying and more algebraically efficient theory.

This very modest category modification greatly improves the interaction between algebra and topology. E.g. it makes it possible to calculate the homology groups of a topological pair-join, expressed in the relative factor groups, leading up to a truly simple boundary formula for joins of manifolds: Bd(X₁∗X₂) = ((BdX₁)∗X₂)∪(X₁∗(BdX₂)), the “product”-counterpart of which is true also classically. It’s also easily seen that no finite simplicial n-manifold has an (n−2)-dimensional boundary, cf. Cor. 1 p. 26.

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1. Introduction

The importance of the join operation within algebraic topology, becomes apparent for instance through Milnor’s construction of the universal principal fiber bundle in [18] where he also formulate the non-relative Künneth formula for joins as:

\[ \tilde{H}_{q+1}(X \ast Y) \cong \oplus_{a+b=q+1} \left( \tilde{H}_a(X) \otimes \tilde{H}_b(Y) \right) \oplus \operatorname{Tor}^{\mathbb{Z}}(\tilde{H}_q(X), \tilde{H}_q(Y)) \]  

i.e. the “\(X \ast Y = \emptyset\)”-case in our Th. 4 p. 110. Milnor’s results, apparently, inspired G.W. Whitehead to introduce the Augmental Total Chain Complex \(\tilde{S}(\cdot)\) and Augmental Homology, \(\tilde{H}_\ast(\cdot)\), in [29]. This was an attempt of one of the most prominent topologists of our time, to, within the classical frame, extend Milnor’s formula to topological pair joins.

G.W. Whitehead gave the empty space, \(\emptyset\), the status of a \((-1)\)-dimensional standard simplex but, in his pair space theory he never took into account that \(\emptyset\) then would get the identity map, \(\operatorname{Id}_\emptyset\), as a generator for its \((-1)\)-dimensional singular chain group, which, correctly interpreted, actually makes his pair space theory identical to the ordinary relative homology functor. \(\emptyset\) plays a definite role in the Eilenberg-Steenrod formalism, cf. [7] p. 3-4, while the “convention” \(H_\ast(\cdot) := H_\ast(\cdot; \emptyset; \mathbb{Z})\), cp. [7] pp. 3 + 273, is more than a mere “convension” in that it connects the single and the pair space theory and both concepts should be handled with care.

In contemporary combinatorics there is a \((-1)\)-dimensional simplex \(\emptyset\) containing no vertices. A moment of reflexion on the realization functor, \(|\cdot|\), reveals the need for a new topological join unit, \(\{\emptyset\} \neq |\emptyset| = \emptyset\). We’re obviously dealing with a non-classical situation, which one mustn’t try to squeeze into a classical framework.

The new simplex, \(\emptyset\), is contained in every non-empty simplicial complex and induces on its own, a new \((-1)\)-dimensional simplicial sphere \(\{\emptyset\}\). The only homology-apparatus invented by the combinatorialists to handle their ingenious category modification were a jargon like - “We’ll use reduced homology with \(\tilde{H}_i(\{\emptyset\}) = \mathbb{Z}^\ast\).

Classical Simplicial and Singular Homology are accompanied by Reduced Homology Functors in a mishmash that severely cripples Algebraic Topology, e.g. it leaves the reduced functor without influence on the boundary definition w.r.t. manifolds.

To construct a (unifying) homology theory one starts with a uniquely defined category of admissible sets (of pairs) and three naive concepts; homotopy, excision and point. Now, it’s a matter of making these concepts comply with the formalism in [7] p. 114-118. It’s basic knowledge that no two categories are equipped with the same homology theory, since the source category is a part of the definition as is the domain in a function definition. So, our \(\tilde{H}\)-functor(s) doesn’t induce just another homology theory, it’s the first one constructed for the algebraically modernized categories.

Generalizing the boundary concept from triangulable homological manifolds to that of any simplicial quasi-\(n\)-manifold \(\Sigma\), see pp. 11 + 23, the boundary is made up by all simplices having a “link” with zero reduced homology in top dimension i.e. \(\text{Bd} \Sigma := \{ \sigma \in \Sigma | \text{Lk}_\sigma(\sigma) = 0 \} \), where \(\text{Lk}_\sigma(\sigma) := \{ \tau \in \Sigma | [\sigma \cap \tau = \emptyset] \land [\sigma \cup \tau \in \Sigma_0] \} \). Since \(\text{Lk}_\emptyset(\emptyset) = \Sigma\), \(\text{Bd} \Sigma^\ast = \{ \emptyset \} \neq \emptyset\), for the real projective plane \(\mathbb{R}P^2\). Also any point \(\bullet := \{ \emptyset \} \) has the join-unit \(\{\emptyset\}\) as boundary, i.e. the boundary of any 0-ball is the \(-1\)-sphere. \(\bullet\) is the only finite orientable manifold having \(\{\emptyset\}\) as its boundary.

The splitting of homology into a reduced and a relative part, really jams up algebraic topology. Indeed, classically, it’s difficult even to see that the boundary of “the cone of a Möbius band” is the real projective plane, cf. Prop. 1 p. 11 + Ex. 2 p. 28. Moreover, classically it’s a hard-motivated truth that \(\text{Bd}_q\left(\mathbb{R}P^n \mathbb{R}P^m\right) = \mathbb{R}P^n \cup \mathbb{R}P^m\) with \(\mathbb{R}P^n\) the projective n-space and \(\mathbb{Z}_p\) the prime-number field modulo \(p \neq 2\), cf. Ex. 3 p. 28.
2. Augmental Homology Theory

2.1 Notations and Definition of Underlying Categories. The typical
morphisms in the classical category $\mathcal{K}$ of simplicial complexes
with vertices in $W$ are the simplicial maps as defined in [25] p. 109,
implying in particular that; $\text{Mor}(\emptyset, \Sigma) = \{\emptyset\}$, $\text{Mor}(\Sigma, \emptyset) = \emptyset$ if $\Sigma \neq \emptyset$ \text{ and } $\text{Mor}(\emptyset, \emptyset) = \{\emptyset\}$, where $\emptyset_{\Sigma, \Sigma'} = \emptyset$ is the empty function from $\Sigma$ to $\Sigma'$. So; $\emptyset_{\Sigma, \Sigma'} \in \text{Mor}_\chi(\Sigma, \Sigma') \iff \Sigma = \emptyset$. If in a category $\varphi_i \in \text{Mor}(R_i, S_i)$, $i = 1, 2$, we put;

$$\varphi_1 \sqcup \varphi_2 : R_1 \sqcup R_2 \rightarrow S_1 \sqcup S_2 : r \mapsto \begin{cases} \varphi_1(r) & if r \in R_1 \\ \varphi_2(r) & if r \in R_2 \end{cases}$$

**Definition.** (of the objects in $K_\circ$) An (abstract) simplicial complex $\Sigma$ on a vertex set $V_\Sigma$ is a collection (empty or non-empty) of finite (empty or non-empty) subsets $\sigma$ of $V_\Sigma$ satisfying; (a) If $v \in V_\Sigma$, then $\{v\} \in \Sigma$. (b) If $\sigma \in \Sigma$ and $\tau \subset \sigma$ then $\tau \in \Sigma$.

So, $\{\emptyset\}$ is allowed as an object in $K_\circ$. We will write "concept" or "concepts," when we want to stress that a concept is to be related to our modified categories.

If $|\sigma| = \#\sigma := \text{card}(\sigma) = q+1$ then $\text{dim } \sigma := q$ and $\sigma$ is said to be a $q$-simplex, of $\Sigma$, and $\text{dim } \Sigma := \sup \{\text{dim } (\sigma) \mid \sigma \in \Sigma\}$. Writing $\emptyset$ when using $\emptyset$ as a simplex, we get $\text{dim } (\emptyset) = -\infty$ and $\text{dim } (\{\emptyset\}) = 0$.

**Note** that each object in the category $K_\circ$ of simplicial complexes, except for $\emptyset$, includes $\{\emptyset\}$ as a subcomplex. A typical object, in $K_\circ$ is $\Sigma \sqcup \{\emptyset\}$ or $\emptyset$ where $\Sigma \in K_\circ$ and $\psi$ is a morphism in $K_\circ$ if;

(a) $\psi = \varphi \sqcup \text{id}_{\{\emptyset\}}$ for some $\varphi \in \text{Mor}_\chi(\Sigma, \Sigma')$ or

(b) $\psi = 0_{\emptyset, \Sigma_\circ}$. (In particular, $\text{Mor}_\circ(\emptyset_\circ, \emptyset_\circ) = \emptyset$ if and only if $\emptyset_\circ \neq \{\emptyset\}, \emptyset$.)

A functor $E : K_\circ \rightarrow K_\circ$.

Set $E(\Sigma_\circ) = \Sigma_\circ \setminus \{\emptyset\} \in \text{Obj } K_\circ$ and given a morphism $\psi : \Sigma_\circ \rightarrow \Sigma'_\circ$ we put;

$E(\psi) = \varphi$ if $\psi$ fulfills (a) above and

$E(\psi) = 0_{\emptyset, E(\Sigma_\circ)}$ if $\psi$ fulfills (b) above.

A functor $E_\circ : K \rightarrow K_\circ$:

Set $E_\circ(\Sigma) = \Sigma \sqcup \{\emptyset\} \in \text{Obj } K_\circ$ and given simplicial $\varphi : \Sigma \rightarrow \Sigma'$, putting $\psi := \varphi \sqcup \text{id}_{\{\emptyset\}}$, gives;

- $E_\circ \circ = \text{id}_K$
- $\text{im } E_\circ = \text{Obj } (K_\circ) \setminus \{\emptyset\}$
- $E_\circ \circ E = \text{id}_K$ except for $E_\circ E(\emptyset) = \{\emptyset\}$

Similarly, let $C$ be the category of topological spaces and continuous maps. Consider the category $\mathcal{O}_\circ$ with objects: $\emptyset$ together with $X_\emptyset := X + \{\emptyset\}$, for all $X \in \text{Obj } C$, i.e. the set $X_\emptyset := X \sqcup \{\emptyset\}$ equipped with the weak topology, $\tau_\emptyset$, with respect to $X$ and $\{\emptyset\}$, cf. [6] Def. 8.4 p. 132.

$\tilde{f}_\circ \in \text{Mor}_{\emptyset_\circ}(X_\emptyset, Y_\emptyset)$ if; $\tilde{f}_\circ = \begin{cases} a) \text{ f + id}_{\emptyset_\emptyset} (:= f \sqcup \text{id}_{\emptyset_\emptyset}) \text{ with } f \in \text{Mor}(X, Y) \text{ and } f \text{ is on } X \text{ to } Y, \text{ i.e the domain of } f \text{ is the whole of } X \text{ and } X_\emptyset = X + \{\emptyset\}, Y_\emptyset = Y + \{\emptyset\} \text{ or } \\
b) 0_{\emptyset, X_\emptyset} (= \emptyset = \text{ the empty function from } \emptyset \text{ to } Y_\emptyset) \end{cases}$
There are functors \( \mathcal{F} : \{ C \to \mathcal{D} \} \), \( \mathcal{F} : \{ \mathcal{D} \to \mathcal{C} \} \), \( \mathcal{F} : \{ \mathcal{X} \to \mathcal{X} + \{ v \} \} \), \( \mathcal{F} : \{ \emptyset \to \emptyset \} \) resembling \( \mathcal{E}_b \) resp. \( \mathcal{E} \).

**Note.** The “\( \mathcal{F} \)-lift topologies”,

\[
\tau_{\mathcal{X}} = \tau_{\mathcal{X}} \cup \{ X \} = \{ \mathcal{O} | \mathcal{O} = X, \{ N \cup \{ v \} \}; N \text{ closed in } \mathcal{X} \} \cup \{ X \}
\]

and

\[
\tau_{\mathcal{X}} = \mathcal{F}(\tau_{\mathcal{X}}) \cup \{ \emptyset \} = \{ \mathcal{O} = \mathcal{O} \cup \{ v \} | \mathcal{O} \in \tau_{\mathcal{X}} \} \cup \{ \emptyset \}
\]

would also give \( \mathcal{D} \) due to the domain restriction in \( a \), making \( \mathcal{D} \) a link between the two constructions of partial maps treated in [2] pp. 184-6. No extra morphisms, has been allowed into \( \mathcal{D} \) (\( \mathcal{E}_b \)) in the sense that the morphisms, are all targets under \( \mathcal{F} \) (\( \mathcal{E}_b \)) except \( \emptyset, X \) defined through item b, re-establishing \( \emptyset \) as the unique initial object.

The underlying principle for our definitions is that a concept in \( \mathcal{C} (\mathcal{K}) \) is carried over to \( \mathcal{D} \) (\( \mathcal{K} \)) by \( \mathcal{F} \) (\( \mathcal{E}_b \)) with addition of definitions of the concept, for cases that isn’t a proper image under \( \mathcal{F} \) (\( \mathcal{E}_b \)). The definitions of the product/join operations “\( \cdot \)”, “\( \vee \)”, “\( \setminus \)” in page 7 and “\( \setminus \)” below, certainly follows this principle.

**Definition.** \( 0_{\mathcal{X}} := \emptyset \) else; \( X_{\mathcal{X}} / X_{\mathcal{X}_1} := \mathcal{F}_p(\mathcal{F}(X_{\mathcal{X}_1}) / \mathcal{F}(X_{\mathcal{X}_2})) \) if \( X_{\mathcal{X}_1} \neq \emptyset \) in \( \mathcal{D} \) for \( X_{\mathcal{X}_1} \subset X_{\mathcal{X}_2} \).

where “\( / \)” is the classical “quotient” except that \( \mathcal{F}(X_{\mathcal{X}_1}) / \emptyset := \mathcal{F}(X_{\mathcal{X}_1}) \), cp. [2] p. 102.

**Definition.** \( X_{\mathcal{X}_1} +_a X_{\mathcal{X}_2} := \{ \{ X_{\mathcal{X}_1} \} \} \mathcal{F} \mathcal{F}(\mathcal{F}(X_{\mathcal{X}_1}) + \mathcal{F}(X_{\mathcal{X}_2})) \) if \( X_{\mathcal{X}_1} \neq \emptyset \neq X_{\mathcal{X}_2} \)

where “\( + \)” is the classical “topological sum”, defined in [6] p. 127 as the “free union”.

Proposition 1 p. 11 is our key motivation for introducing a topological (1-1) object, which then imposed the following definition of a “setminus”, “\( \setminus \)” in \( \mathcal{D} \).

**Definition.** \( X_{\mathcal{X}} \setminus X_{\mathcal{X}_1} := \{ \emptyset \} \) if \( X_{\mathcal{X}} = \emptyset \), \( X_{\mathcal{X}} \subseteq X_{\mathcal{X}_1} \) or \( X_{\mathcal{X}} = \{ v \} \)

else. (“\( \setminus \):” classical “setminus”)

**Notations:** We have used w.r.t.:=with respect to, and \( \tau \):=the topology of \( \mathcal{X} \). We’ll also use; \( \text{PID} := \text{Principal Ideal Domain}, \text{l.h.s.}(\text{r.h.s.}) := \text{left} (\text{right}) \text{ hand side}, \text{iff} := \text{if and only if}, \text{cp.} := \text{compare}(\text{cf.} := \text{cp.}) . \text{LHS} := \text{Long Homology Sequence and M-Vs} := \text{Mayer-Vietoris sequence}. \) Let, here in Ch. 2, \( \Delta = \{ \emptyset, \delta \} \) be the classical singular chain complex and let “\( \cong \)” denote “homeomorphism” or “chain isomorphism”. \( \bullet (\bullet) \) denotes the one (two) point space \( \{ \bullet, \gamma \} \{ \bullet, \bullet, \gamma \} \).

If \( X_{\mathcal{X}} \neq \emptyset \), \( \{ v \} \) then \( (X_{\mathcal{X}} \tau_{\mathcal{X}}) \) is a non-connected space, and it therefore seems adequate to define \( X_{\mathcal{X}} \neq \emptyset \), \( \{ v \} \) to have a certain point set topological “property”, if \( \mathcal{F}(X_{\mathcal{X}}) \) has the “property” in question, e.g. \( X_{\mathcal{X}} \) is connected, \( \text{iff} X \) is connected.

2.2 Simplicial Augmental Homology Theory and realizations.

\( \text{H} \) denotes the simplicial as well as the singular augmental (co)homology functor \( \mathcal{E}_b \).

Choose oriented \( q \)-simplices to generate \( C^q_{\mathcal{X}}(\Sigma_\mathcal{X}; \mathcal{G}) \), where the coefficient module \( \mathcal{G} \) is a unital \( (\to \mathcal{A} o g = g) \) module over any commutative ring \( A \) with unit. Now;

- \( C^q(\emptyset; \mathcal{G}) \) is identically 0 in all dimensions, where 0 is the additive unit-element.
- \( C^q(\{ \emptyset \}; \mathcal{G}) \) is identically 0 in all dimensions except for \( C^q(\{ \emptyset \}; \mathcal{G}) \equiv \mathcal{G} \).
- \( C^q(\Sigma_\mathcal{X}; \mathcal{G}) \equiv \mathcal{C}(E(\Sigma_\mathcal{X}); \mathcal{G}) \equiv \text{the classical “\( \{ v \} \)”-augmented chain.} \)
By just hanging on to the “{$\emptyset$}-augmented chains”, also when defining relative chains, we get the Relative Simplicial Augmental Homology Functor for $K_\alpha$-pairs, denoted $H_\alpha$, and fulfilling:

$$H_i(\Sigma_{\alpha_1}, \Sigma_{\alpha_2}; G) = \begin{cases} H_i(E(\Sigma_{\alpha_1}), E(\Sigma_{\alpha_2}); G) & \text{if } \Sigma_{\alpha_2} \neq \emptyset \\ H_i(E(\Sigma_{\alpha_1}), G) & \text{if } \Sigma_{\alpha_1} \neq \{\emptyset\}, \emptyset, \text{ and } \Sigma_{\alpha_2} = \emptyset \\ \cong G & \text{if } i = -1 \\ 0 & \text{if } i \neq -1 \\ \text{for all } i \text{ when } \Sigma_{\alpha_1} = \Sigma_{\alpha_2} = \emptyset. \end{cases}$$

$\emptyset \neq \{\emptyset\}$ and both lacks final sub-objects, which under any useful definition of the realization of a simplicial complex implies that $|\emptyset| \neq |\{\emptyset\}|$ demanding the addition of a non-final object $\{\emptyset\}$ into the classical category of topological spaces as join-unit and (-1)-dimensional standard simplex. This approach conforms Homology Theory and considerably simplifies the study of manifolds, cf. p. 13.

We will use Spanier’s definition of the “function space realization” $|\Sigma_o|$ as given in [25] p. 110, unaltered, except for the “$\circ$”s and the underlined addition where $\varphi := \alpha_0$ and $\alpha_0(v) = 0 \forall v \in V_o$:

- “We now define a covariant functor from the category of simplicial complexes, and simplicial maps, to the category of topological spaces, and continuous maps. Given a nonempty simplicial complex, $\Sigma_o$, let $|\Sigma_o|$ be the set of all functions $\alpha$ from the set of vertices of $\Sigma_o$ to $I := [0, 1]$ such that;

(a) For any $\alpha$, $\{v \in V_o | \alpha(v) \neq 0\}$ is a simplex $\sigma_o$ of $\Sigma_o$ (in particular, $\alpha(v) \neq 0$ for only a finite set of vertices).

(b) For any $\alpha \neq \alpha_0$, $\sum_{v \in V_o} \alpha(v) = 1$.

If $\Sigma_o = \emptyset$, we define $|\Sigma_o| = \emptyset$.

The barycentric coordinates $\alpha$, defines a metric

$$d(\alpha, \beta) = \sqrt{\sum_{v \in V_o} |\alpha(v) - \beta(v)|^2}$$

on $|\Sigma_o|$ inducing the topological space $|\Sigma_o|_d$ with the metric topology. We’ll equip $|\Sigma_o|$ with another topology and for this purpose we define the closed simplex $|\sigma_o|$ of $\sigma_o \in \Sigma_o$ i.e. $|\sigma_o| := \{\alpha \in |\Sigma_o| \mid |\alpha(v) = 0 \implies |v \in \sigma_o|\}$.

**Definition.** For $\Sigma_o \neq \emptyset$, $|\Sigma_o|$ is topologized through $|\Sigma_o| := |E(\Sigma_o)| + \{\alpha_0\}$, which is equivalent to give $|\Sigma_o|$ the weak topology w.r.t. the $|\sigma_o|$’s, naturally imbedded in $R^{\Sigma_o} + \{\psi\}$ and we define $\Sigma_o$ to be connected if $|\Sigma_o|$ is, i.e. if $\mathcal{F}(\Sigma_o) \simeq |E(\Sigma_o)|$ is.

**Proposition.** $|\Sigma_o|$ is always homotopy equivalent to $|\Sigma_o|_d$ ([9] pp. 115, 226.) and $|\Sigma_o|$ is homeomorphic to $|\Sigma_o|_d$ if $\Sigma_o$ is locally finite ([25] p. 119 Th. 8.).
2.3 Singular Augmental Homology Theory.

|σ|, imbedded in $\mathbf{R}^+ + \{0\}$ generates a satisfying set of “standard simplices,” and “singular simplices.” This implies in particular that the “p-standard simplices,” denoted $\Delta^p$, are defined by $\Delta^p := \Delta^p + \{0\}$ where $\Delta^p$ denotes the usual $p$-dimensional standard simplex and $+$ is the topological sum, i.e. $\Delta^p := \Delta^p \sqcup \{0\}$ with the weak topology w.r.t. $\Delta^p$ and $\{0\}$. Now, and most important: $\Delta^{|-1|} := \{0\}$.

Let $T^p$ denote an arbitrary classical singular $p$-simplex ($p \geq 0$). The “singular $p$-simplex,” denoted $\sigma^p$, now stands for a function of the following kind:

$$\sigma^p : \Delta^p = \Delta^p + \{0\} \rightarrow X + \{0\} \text{ where } \sigma^p(0) = 0 \text{ and }$$

$$\sigma^p_{|\Delta^{|-1|}} = T^p \text{ for some ordinary } p \text{-dimensional singular simplex } T^p \text{ for all } p \geq 0.$$

In particular, $$\sigma^{|-1|} : \{0\} \rightarrow X_0 = X + \{0\} : 0 \mapsto 0.$$ The boundary function $\partial_p$ is defined by $\partial_p(\sigma^p) := \mathcal{F}(\partial_p(T^p))$ if $p > 0$ where $\partial_p$ is the ordinary singular boundary function, and $\partial_{p0}(\sigma^p) = \sigma^{|-1|}$ for every singular 0-simplex, $\sigma^0$. Let $\Delta^0 = \{\partial^0, \partial^1\}$ denote the singular augmental chain complex. Observation: $|\Sigma| \neq \emptyset \Rightarrow |\Sigma| = \mathcal{F}_{|\Sigma|}(|E(|\Sigma|)|) \in \mathcal{D}_\Sigma$.

By the strong analogy to classical homology, we omit the proof of the next lemma.

**Lemma.** (Analogously for coHomology.)

$$H_i(X_{\emptyset 1}, X_{\emptyset 2}; G) = \begin{cases} H_i(\mathcal{F}(X_{\emptyset 1}), \mathcal{F}(X_{\emptyset 2}); G) & \text{if } X_{\emptyset 2} \neq \emptyset \\ H_i(\mathcal{F}(X_{\emptyset 1}); G) & \text{if } X_{\emptyset 1} \neq \emptyset, \emptyset \text{ and } X_{\emptyset 2} = \emptyset \\ \emptyset & \text{if } X_{\emptyset 1} = \emptyset, \emptyset \text{ and } X_{\emptyset 2} = \emptyset \end{cases}$$

Note. i. $\Delta(X_1, X_2; G) \Rightarrow \Delta(\mathcal{F}(X_1), \mathcal{F}(X_2); G)$ always. So $H_i(\mathcal{F}(X_1), \mathcal{F}(X_2); G) \neq 0$.

ii. $\Delta(X_{\emptyset 1}, X_{\emptyset 2}) \Rightarrow \Delta(\mathcal{F}(X_{\emptyset 1}), \mathcal{F}(X_{\emptyset 2}); G)$ except if $X_{\emptyset 1} \neq X_{\emptyset 2} = \emptyset$ when the only non-isomorphisms occur for $\Delta^{|-1|}(X_{\emptyset 1}, \emptyset) \Rightarrow \mathbf{Z} \neq \emptyset \Rightarrow \Delta^{|0|}(\mathcal{F}(X_{\emptyset 1}), \emptyset)$ when $H_i(X_{\emptyset 1}, \emptyset) \neq H(\mathcal{F}(X_{\emptyset 1}), \emptyset)$ if $X_{\emptyset 1} \neq \emptyset$.

iii. $C^*(\Sigma_1, \Sigma_2; G) \Rightarrow \Delta(\Sigma_1, |\Sigma_2|; G)$ connects the simplicial and singular functor. ($\Rightarrow \approx$ stands for “chain equivalence”.)

iv. $H_0(X_{\emptyset} + Y_{\emptyset}, \emptyset; G) = H_0(X_{\emptyset}, \emptyset; G) \oplus H_0(Y_{\emptyset}, \emptyset; G)$ but $H_0(X_{\emptyset} + Y_{\emptyset}, \emptyset; G) = = H_0(X_{\emptyset}, \emptyset; G) \oplus H_0(Y_{\emptyset}, \emptyset; G) = H_0(X_{\emptyset}, \emptyset; G) \oplus H_0(Y_{\emptyset}, \emptyset; G)$.

**Definition.** The $p$th Singular Augmental Homology Group of $X_0$ w.r.t. $G = = H_p(X_0; G) := H_p(X_0, \emptyset; G)$. The Coefficient Group $p := H_{|\emptyset|}(\emptyset, \emptyset; G)$.

Using $\mathcal{F}$, (E), we “lift” the concepts of homotopy, excision and point in $C(X)$ into $D_\Sigma$-concepts ($K_\Sigma$-concepts) homotopy, excision and point, respectively.

So; $f_0, g_0 \in D_\Sigma$ are homotopic if and only if $f_0 = g_0 = 0$ or there are homotopic maps $f_1, g_1 \in C$ such that $f_0 = f_1 + \text{Id}_{\{0\}}, g_0 = g_1 + \text{Id}_{\{0\}}$.

An inclusion $(i_0, i_{\alpha}) : (X_0 \setminus U_0, A_0 \setminus U_0) \rightarrow (X_0, A_0), U_0 \neq \emptyset$, is an excision if and only if there is an excision $(i, i_{\alpha}) : (X \setminus U, A \setminus U) \rightarrow (X, A)$ such that $i_0 = i + \text{Id}_{\{0\}}$ and $i_{\alpha} = i_{\alpha} + \text{Id}_{\{0\}}$. 

\{P, \varphi\} \in \mathcal{F} is a \textit{point} of \{P\} + \{\varphi\} = \mathcal{F}_P(\{P\}) and \{P\} \in C is a \textit{point}. So, \{\varphi\} is \textit{not} a point of \varphi.

Conclusion. \mathbf{H}, \partial, is a homotopy theory on the h-category of pairs from \mathcal{P}_0(K_0), c.f. \cite{7} p. 117, i.e. \mathbf{H} fulfills the h-category analogues, given in \cite{7} §88-9 pp. 114-118, of the seven Eilenberg-Steenrod axioms from \cite{7} §3 pp. 10-13. The necessary verifications are either equivalent to the classical or completely trivial. E.g. the \textit{dimension axiom} is fulfilled since \{\varphi\} is not a point of \varphi.

Since the exactness of the relative Mayer-Vietoris sequence of a proper triad, follows from the axioms, cf. \cite{7} p. 43 and, paying proper attention to Note iv, we’ll use it without further motivation.

\[ \mathbf{H}(X) = \mathbf{H}(\mathcal{E}(X), \emptyset) \] explains all the ad-hoc reasoning surrounding the \( \mathbf{H} \)-functor.

3. AUGMENTAL HOMOLOGY MODULES FOR PRODUCTS AND JOINS

3.1 Definitions of the Product and Join Operations. Let \( \nabla \) be one of the classical topological product/join operations \( "\times"/"\shuffle"/"\odot" \), defined in \cite{25} p. 4 \( \nabla \) in \cite{6} p. 128 Ex. 3 including \cite{6} p. 135 Problem 6:1, \cite{21} p. 373 and \cite{29} p. 128 \( \nabla \) in \cite{2} pp. 159-160 and \cite{18} respectively. Recall that; \( X \nabla \emptyset = X = \emptyset \nabla X \) classically.

\[ \text{Definition. } X_{\varphi_1} \nabla X_{\varphi_2} := \left\{ \begin{array}{ll} \emptyset & \text{if } X_{\varphi_1} = \emptyset \text{ or } X_{\varphi_2} = \emptyset \\ \mathcal{F}_P(\mathcal{F}(X_{\varphi_1}) \nabla \mathcal{F}(X_{\varphi_2})) & \text{if } X_{\varphi_1} \neq \emptyset \neq X_{\varphi_2}. \end{array} \right. \]

From now on we’ll delete the \( \varphi/\circ \)-indices. So, e.g. “\( X \) connected” now means “\( \mathcal{F}(X) \) connected”.

\[ \text{Equivalent Join Definition. } \text{Put } \emptyset \nabla X = X \nabla \emptyset := X. \text{ If } X \neq \emptyset; \{\varphi\} \nabla X = \{\varphi\}, \]

\( X \nabla \{\varphi\} = X. \) For \( X, Y \neq \emptyset, \varphi \) let \( X \nabla Y \) denote the set \( X \times Y \times (0,1) \) \textit{pastted} to the set \( X \) by \( \varphi_1 : X \times Y \times \{1\} \rightarrow X; (x, y, 1) \mapsto x \), i.e. the quotient set of \( (X \times Y \times (0,1)) \cup X \), under the equivalence relation \( (x, y, 1) \sim x \) and let \( p_1 : (X \times Y \times (0,1)) \cup X \rightarrow X \nabla Y \) be the quotient function. For \( X, Y = \emptyset \) or \( \{\varphi\} \) let \( X \nabla Y := Y \nabla X \) and else the set \( X \times Y \times (0,1) \) \textit{pastted} to the set \( Y \) by the function \( \varphi_2 : X \times Y \times (0) \rightarrow Y; (x, y, 0) \mapsto y \), and let \( p_2 : (X \times Y \times (0,1)) \cup Y \rightarrow X \nabla Y \) be the quotient function. Put \( X \odot Y := (X \nabla Y) \cup (Y \nabla X). \)

\( (x, y, t) \in X \times Y \times [0,1] \) specifies the point \( (x, y, t) \in X \nabla Y \cap X \odot Y \), one-to-one, if \( 0 < t < 1 \) and the equivalence class containing \( x \) if \( t = 1 \) (\( y \) if \( t = 0 \)), which we denote \( (x, 1) \) \( (y, 0) \). This allows “coordinate functions” \( \xi : X \odot Y \rightarrow [0,1], \eta_i : X \nabla Y \rightarrow X, \eta_i : X \nabla Y \rightarrow Y \) extendable to \( X \odot Y \) through \( \eta_i(y, 0) := x_i \in X \) resp. \( \eta_i(x, 1) := y_i \in Y \) and a projection \( p : X \cup (X \times Y \times [0,1]) \cup Y X \odot Y \).

Let \( X \odot Y \) denote \( X \odot Y \) equipped with the smallest topology making \( \xi, \eta_i \) continuous and \( X \nabla Y, X \odot Y \) with the quotient topology w.r.t. \( p \), i.e. the largest topology making \( p \) continuous \( \Rightarrow \tau_{\odot Y} \subset \tau_{\nabla Y} \).

\[ \text{Pair-definitions. } (X_i ; Y_i) \nabla (Y_i ; Y_i) := (X_i \nabla Y_i, (X_i \nabla Y_i) \odot (X_i \nabla Y_i)), \text{ where } \odot \text{ stand for } "\cup" \text{ or } "\cap" \text{ and if either } X_i \text{ or } Y_i \text{ is not closed (open) cf. } [6] \text{ p. 43 Cor. 1, } (X_i \nabla Y_i, (X_i \nabla Y_i) \odot (X_i \nabla Y_i)) \text{ has to be interpreted as } (X_i \nabla Y_i, (X_i \nabla Y_i) \odot (X_i \nabla Y_i)) \text{ i.e. } (X_i \nabla Y_i) \odot (X_i \nabla Y_i) \text{ with the subspace topology in the 2nd component. Analogously for simplicial complexes with } "\times" \text{ ("\odot")} \text{ from } [7] \text{ p. 67 Def. 8.8 (25) p. 109 Ex. 7.} \]

\[ \text{Note. } i. (X \nabla Y)^{\nabla \odot} \text{ in } X \nabla Y \text{ is homeomorphic to the mapping cylinder w.r.t. the \textit{coordinate} map } q_i : X \times Y \rightarrow X. \text{ ii. } X_i \nabla Y_i \text{ is a subspace of } X_i \nabla Y_i \text{ by } [2] \text{ 5.7.3 p. 163.} \]
$X_1 \ast Y_2$ is a subspace of $X_1 \ast Y_2$ if $X_1, Y_2$ are closed (open).

iii. $(X_1, \{v\}) \times (Y_1, Y_2) = (X_1, \emptyset) \times (Y_1, Y_2)$ if $Y_2 \neq \emptyset$ and $(X_1 \circ Y_2) \cap (X_1 \circ Y_2) = X_1 \circ Y_2$. iv. $\ast$ and $\cdot$ are both commutative but, while $\ast$ is associative by [2] p. 161, $\cdot$ isn’t in general, cf. p 14.

v. “$\ast$” is (still, cf. [5] p. 15,) the categorical product on pairs from $\mathcal{T}$. vi. $\ast := \cdot$.

### 3.2 Augmenting Homology for Products and Joins

Through Lemma + Note ii p. 6 we convert the classical Künneth formula cf. [25] p. 235, mimicking what Milnor did, partially (= line 1), at the end of his proof of [18] p. 431 Lemma 2.1. The ability of a full and clear understanding of Milnor’s proof could be regarded as sufficient prerequisites for our next six pages. The new object $\{v\}$ gives the classical Künneth formula (=4:th line) additional strength but much of the classical beauty is lost - a loss which is regained in the join version i.e. in Theorem 4 p. 10.

**Theorem 1.** For $\{X_1 \times Y_2, X_2 \times Y_1\}$ excusive, $q \geq 0$, $R$ a PID, and assuming Tor$_R^i(G, G') = 0$ then:

$$H_q((X_1 \times Y_2) \times (Y_1, Y_2); G \otimes_R G')$$

\[
\begin{cases}
H_q((X_1; G) \otimes_R H_j(Y_1, G')) \oplus (H_q(Y_1; G') \otimes (G \otimes_R H_0(Y_1, Y_2; G')) \oplus T_1 & \text{if } C_1 \\
\left[H_q((X_1; G) \otimes_R H_j(Y_1, Y_2; G')) \oplus (H_q(Y_1, Y_2; G') \otimes (G \otimes_R H_0(Y_1, Y_2; G')) \oplus T_2 & \text{if } C_2 \\
H_q((X_1, X_2; G) \otimes_R H_j(Y_1, G') \oplus (H_q(X_1, Y_2; G) \otimes_R G') \oplus T_3 & \text{if } C_3 \\
H_q((X_1, Y_2; G) \otimes_R H_j(Y_1, Y_2; G') \oplus T_4 & \text{if } C_4
\end{cases}
\]

The torsion terms, i.e. the $T$-terms, splits as those ahead of them, resp, e.g.

$$T_i = [\text{Tor}^R_i\left(H_q((X_1; G), H_j(Y_1; G'))\right)]_{q-i} \oplus \text{Tor}^R_i\left(H_q-1((X_1; G), G')\right) \oplus \text{Tor}^R_i\left(G, H_{q-1}(Y_1; G')\right),$$

and

$$T_q = [\text{Tor}^R_i\left(H_q((X_1, X_2; G), H_j(Y_1, Y_2; G'))\right)]_{q-i}.$$

$C_1 := "X_1 \times Y_1 \neq \emptyset, \{v\} \text{ and } \emptyset = Y_2", \text{ } C_2 := "X_1 \times Y_1 \neq \emptyset, \{v\} \text{ and } X_2 = \emptyset \neq Y_2", \text{ } C_3 := "X_1 \times Y_1 \neq \emptyset, \{v\} \text{ and } X_2 \neq \emptyset = Y_2", \text{ } C_4 := "X_1 \times Y_1 = \emptyset, \{v\} \text{ or } X_2 \neq \emptyset = Y_2", \text{ and } \text{[...]} \text{ is still, i.e. as in [25] p. 235 Th. 10, to be interpreted as } \bigoplus_{i+j=q, i, j \geq 0}.$

**Lemma.** Let $f: (X, A) \to (Y, B)$ be a relative homeomorphism, i.e., $f: X \to Y$ is continuous and $f: X \setminus A \to Y \setminus B$ is a homeomorphism. If $F: N \times I \to N$ is a (strong) (neighborhood) deformation retraction of $N$ onto $A$ and $B$ and $f(N)$ are closed in $N \setminus A \cup B$, then $B$ is a (strong) (neighborhood) deformation retract of $N'$ through:

$$F': N' \times I \to N'; \begin{cases} (y, t) \mapsto y & \text{if } y \in B, t \in I \\
(y, t) \mapsto f \circ F(f^{-1}(y), t) & \text{if } y \in f(N) \setminus B = f(N \setminus A), t \in I.
\end{cases}$$

**Proof.** $F'$ is continuous as being so when restricted to $f(N) \times I$ resp. $B \times I$, cf. [2] p. 34; 2.5.12.

**Theorem 2.** (Analogously for $\ast$.  $(\Rightarrow \Delta'(\cdot, \cdot) : \text{chain equivalent to } \Delta'(\cdot, \cdot, \cdot)$ )

If $(X_1, X_2) \neq \{(v), \emptyset\} \neq (Y_1, Y_2)$ and $G$ an $A$-module:

$H_q((X_1, X_2) \times (Y_1, Y_2); G) \cong$

\[
\begin{align*}
\bigoplus_{i,j} H_{q-i}((X_1, X_2) \ast (Y_1, Y_2); G) & \oplus H_i((X_1, X_2) \ast (Y_1, Y_2)^{\leq 0.5} + (X_1, X_2) \ast (Y_1, Y_2)^{< 0.5}; G) =
\end{align*}
\]
\[ A \triangleq \begin{cases} 
H_{q+1}(X \ast Y; G) \oplus H_q(X; G) \oplus H_q(Y; G) & \text{if } C_1 \\
H_{q+1}((X, \emptyset) \ast (Y, Y); G) \oplus H_q(Y, Y; G) & \text{if } C_2 \\
H_{q+1}((X, X) \ast (Y, \emptyset); G) \oplus H_q(X, X; G) & \text{if } C_3 \\
H_{q+1}((X, X) \ast (Y, Y); G) & \text{if } C_4 
\end{cases} \]  

(2)

\[ C_1 := "X \times Y \neq \emptyset, \{y\} \text{ and } X_2 = \emptyset \neq X_2", \ C_2 := "X_1 \times Y \neq \emptyset, \{y\} \text{ and } X_2 = \emptyset \neq Y_2", \ C_3 := "X_1 \times Y \neq \emptyset, \{y\} \text{ and } X_2 \neq \emptyset \neq Y_2", \ C_4 := "X_1 \times Y = \emptyset, \{y\} \text{ or } X_2 \neq \emptyset \neq Y_2".\

**Proof.** Split \( X \ast Y \) at \( t = 0.5 \) then; \( X \ (Y) \) is a strong deformation retract of \( (X \ast Y)^{>0.5} \) \( ((X \ast Y)^{<0.5}) \), the mapping cylinder w.r.t. product projection. The relative M-Vs w.r.t. the excisive couple of pairs \( (X, X) \ast (Y, Y)^{>0.5}, (X, X) \ast (Y, Y)^{<0.5} \) splits since the inclusion of their topological sum into \( (X, X)^{>0.5}, (Y, Y)^{<0.5} \) is pair null-homotopic, cf. [21] p. 141 Ex. 6c, and [15] p. 32 Prop. 1.6.8. Since the 1st (2nd) pair is acyclic if \( Y(X,X) \neq \emptyset \) we get Theorem 2. Equivalently for * by the Lemma. □

Milnor finished his proof of [18] Lemma 2.1 p. 431 by simply comparing the r.h.s. of the \( C_1 \)-case in Eq. 1 with that of Eq. 2. Since we are aiming at the stronger result of “natural chain equivalence” in Theorem 3 this isn’t enough and so, we’ll need the following three auxiliary results to prove our next two theorems. We hereby avoid explicit use of “proof by acyclic models”. (“≈” stands for “chain equivalence”.)

5.7.4. ([2] p. 164.) (Here \( E^0 \) is a symbol for a point, i.e. a 0-disc also denoted \( \bullet \).)

There is a homeomorphism:

\[ \nu : X \ast Y \ast E^0 \longrightarrow (X \ast E^0) \times (Y \ast E^0) \]

which restricts to a homeomorphism:

\[ X \ast Y \longrightarrow (X \ast E^0) \times Y \cup X \times (Y \ast E^0). \]

□

**Corollary 5.7.9.** ([15] p. 210.) If \( \phi : C \approx E \) with inverse \( \psi \) and \( \phi' : C' \approx E' \) with inverse \( \psi' \), then

\[ \phi \otimes \phi' : C \otimes C' \approx E \otimes E' \text{ with inverse } \psi \otimes \psi'. \]

□

**Theorem 46.2.** ([21] p. 279.) For free chain complexes \( C, D \) vanishing below a certain dimension and if a chain map \( \lambda : C \rightarrow D \) induces homology isomorphisms in all dimensions, then \( \lambda \) is a chain equivalence. □

**Theorem 3.** (The relative Eilenberg-Zilber theorem for joins.) For an excisive couple \( \{X \ast Y, X \ast Y\} \) from the category of ordered pairs \( ((X, X), (Y, Y)) \) of topological pairs, \( s(\Delta(X, X) \otimes \Delta(Y, Y)) \) is naturally chain equivalent to \( \Delta((X, X) \ast (Y, Y)) \). (“s” stands for suspension i.e. the suspended chain equals the original except that dimension \( i \) in the original is dimension \( i+1 \) in the suspended chain.)

**Proof.** The second isomorphism is the key and is induced by the pair homeomorphism in [2] 5.7.4 p. 164. For the 2nd last isomorphism we use [15] p. 210 Corollary 5.7.9 and that LHS-homomorphisms are “chain map”-induced. Note that the second component in the third module is an excissive union.

\[ H_\ast(X \ast Y) \xrightarrow{\cong} H_\ast(X \ast Y \ast \{v, y\}, X \ast Y) \xrightarrow{\cong} \]

\[ \cong H_{q+1}((X \ast \{u, y\}) \times (Y \ast \{v, y\}), ((X \ast \{u, y\}) \times Y) \cup (X \times (Y \ast \{v, y\}))) \]
\[
\mathbb{H}_*(X;G) \cong \mathbb{H}_*(Y;G) \cong \mathbb{H}_*(Z;G)
\]

Motivation: The underlying chains on the l.h.s. and r.h.s. are, by Note II p. 6, isomorphic to their classical counterparts on which we use the classical Eilenberg-Zilber Theorem.

\[
\mathbb{H}_{*+i}(\Delta(X) \otimes \Delta(Y)) \cong \mathbb{H}_*(\Delta(X) \otimes \Delta(Y)).
\]

Now the non-relative Eilenberg-Zilber Theorem for joins follows from [21] p. 279 Th. 46.2 above.

Substituting, in the \(x\)-original proof [25] p. 234, “\(\hat{\Delta}\)”, “\(\Delta/\Delta^0\)”, “Theorem 3, 1st part” for “\(\Delta\)”, “\(\Delta\)”, “Theorem 6” resp. will do since;

\[
s(\Delta(X_i) \otimes \Delta(Y_i)) / (s(\Delta(X_i) \otimes \Delta(Y_i)) + s(\Delta(X_i) \otimes \Delta(Y_i))) =
\]

\[
= s(\Delta(X_i) \otimes \Delta(Y_i)) / (\Delta(X_i) \otimes \Delta(Y_i) + \Delta(X_i) \otimes \Delta(Y_i))
\]

\[
= s(\Delta(X_i) / \Delta(Y_i)) \otimes (\Delta(Y_i) / \Delta(Y_i)).
\]

Theorem 5. (The Künneth Formula for Topological Joins; cp. [25] p. 235.)

If \(X\) is an excisive couple in \(X\) and \(R\) a PID, \(G, G'\) \(R\)-modules and \(\text{Tor}_1^R(G, G') = 0\), then the functorial sequences below are (non-naturals) split exact;

\[
0 \rightarrow \bigoplus_{i+j=q} \mathbb{H}_i(X_1, X_2; G) \otimes_R \mathbb{H}_j(Y_1, Y_2; G') \rightarrow (3)
\]

\[
\rightarrow \mathbb{H}_{*+i}(X_1, X_2) \otimes (Y_1, Y_2; G) \otimes_R \mathbb{H}_j(Y_1, Y_2; G') \rightarrow 0
\]

Analogously with “\(\ast\)” substituted for “\(\hat{\ast}\)” and [25] p. 247 Th. 11 gives the cohomology-analog.

Putting \((X_1, X_2) = (\{\varphi\}, \emptyset)\) in Theorem 4, our next theorem immediately follows.

Theorem 6. (The Universal Coefficient Theorem for (co)homology.)

\[
\mathbb{H}_*(Y_1, Y_2; G) \cong \mathbb{H}_*(Y_1, Y_2; R \otimes_R G) \cong
\]

\[
\mathbb{H}_*(Y_1, Y_2; R \otimes_R G) \cong \mathbb{H}_*(Y_1, Y_2; R \otimes_R G) \otimes \text{Tor}_1^R(\mathbb{H}_{*+i}(Y_1, Y_2; R), G),
\]

for any \(R\)-PID module \(G\).

If all \(\mathbb{H}_*(Y_1, Y_2; R)\) are of finite type or \(G\) is finitely generated, then;

\[
\mathbb{H}_*(Y_1, Y_2; G) \cong \mathbb{H}_*(Y_1, Y_2; R \otimes_R G) \cong (\mathbb{H}_*(Y_1, Y_2; R) \otimes_R G) \otimes \text{Tor}_1^R(\mathbb{H}_{*+i}(Y_1, Y_2; R), G).
\]
3.3. Local Augmental Homology Groups for Products and Joins.

Proposition 1 is our key motivation for introducing a topological (-1)-object, which then imposed the definition p. 4, of a “setminus”, “\”, in $D_\sigma$, revealing the true implication of boundary definitions w.r.t. manifolds as given in pp. 13 + 23. Somewhat specialized, it’s found in [10] p. 162 and partially also in [22] p. 116 Lemma 3.3. “$X\backslash x$” usually stands for “$X\backslash \{x\}$” and we’ll write $x$ for $\{x, \emptyset\}$ as a notational convention. Recall the definition of $\alpha_\sigma$, p. 5 and that $\dim Lk_\sigma = \dim S - \#\sigma$.

**Proposition 1.** Let $G$ be any module over a commutative ring $A$ with unit. With $\alpha \in \text{Int}_\sigma$ and $\alpha = \alpha_\sigma \iff \sigma = \emptyset$, the following module isomorphisms are all induced by chain equivalences, cf. [21] p. 279 Th. 46.2 quoted here in p. 9.

$$H_{-\star}(Lk_\sigma; G) \cong H_\alpha(\Sigma, \text{cost}_\sigma; G) \cong H'_\alpha(\Sigma, \text{cost}_\sigma; G) \cong H'_\alpha(\Sigma, \text{cost}_\sigma; G) \cong H'_\alpha(\Sigma, \text{cost}_\sigma; G).$$

**Proof.** (Cf. definitions p. 30-1.) The “\"-definition p. 4 and [21] Th. 46.2 p. 279+pp. 194-199 Lemma 35.1-35.2 + Lemma 63.1 p. 374 gives the two ending isomorphisms since $\text{cost}_\sigma$ is a deformation retract of $\Sigma \backslash \alpha$, while already on the chain level; $C^0(\Sigma, \text{cost}_\sigma) = C^0(\text{cost}_\sigma, \sigma \ast Lk_\sigma) = C^0(\Sigma \ast Lk_\sigma, \sigma \ast Lk_\sigma) = C^0(\Sigma \ast Lk_\sigma).$ $\square$

**Lemma.** If $x \not\in (y \in Y)$ is a closed point in $X \times Y$, then $(X \times (Y \backslash y), (X \backslash x) \times Y)$ and $(X \times Y \backslash y, (X \times X) \times Y)$ are both excusive pairs.

**Proof.** [25] p. 188 Th. 3, since $X \times (Y \backslash y)$ $(X \ast (Y \backslash y))$ is open in $(X \times (Y \backslash y)) \cup ((X \backslash x) \times Y) (X \ast (Y \backslash y)) \cup ((X \backslash x) \times Y)$, which proves the excusivity. $\square$

**Theorem 6.** For $x \in X \not\in Y)$ closed and $(t, x \ast y, t) := \{(x, y, t) \mid 0 < t \leq t \leq t < 1\}$

i. $H_{p,\#}_\alpha(X \ast Y, X \ast Y \backslash x, x, y, t); G) \cong H_{p,\#}_\alpha(X \ast Y, X \ast Y \backslash x, (t, x \ast y, t); G) \cong [\text{A simple calculation}] \cong H_{p,\#}_\alpha((X, X \times Y) \times Y, Y ; G) \cong$

ii. $H_{p,\#}_\alpha(X \ast Y, X \ast Y \backslash x, (y, 0); G) \cong H_{p,\#}_\alpha((X, 0) \times Y \backslash y, Y ; G)$

and equivalently for the $(x, 1)$-points.

All isomorphisms are induced by chain equivalences, cf. [21] p. 279 Th. 46.2 quoted here in p. 9. Analogously for “*” substituted for “#” and for cohomology.

**Proof.** i. \[
A := X \times Y \backslash \{(x, y, t) \mid t \leq t \leq t\} \quad \implies \quad \{ A \cup B = X \ast Y \backslash x, (t, x \ast y, t) \\
A \cap B = X \times Y \times (0, 1) \times \{y\} \times \{0, 1\}, \text{ with } x \times y \times (0, 1) := \{x\} \times \{y\} \times \{0, 1\} \cup \{\emptyset\} \text{ and } (x, y, t) := \{x, y, t\} \}
\]

Now, using the null-homotopy in the relative $\text{M-}V$s w.r.t. $\{(X \times Y, A), (X \times Y, B)\}$ and the resulting splitting of it and the involved pair deformation retractions as in the proof of Th. 2, we get;
3.4. Singular Homology Manifolds under Products and Joins.

\( H_{i+1}(X \bowtie Y, X \bowtie Y \setminus \{(t_1, (x_0, y_0), t_2)\}) \cong H_i(\{X \times Y \setminus (0, 1), X \times (0, 1) \setminus \{x_0\} \times (y_0) \}) \cong \)

\[ \cong \left[ \text{Motivation: The underlying pair on the r.h.s.} \right] \]

\[ \cong \text{is a pair deformation retract of that on the l.h.s.} \]

\[ \cong \text{H}_i(X \times Y \times \{t_0, \varphi\}, X \times Y \times \{t_0, \varphi\} \setminus \{(x_0, y_0, t_0)\}) \cong H_i(X \times Y, X \times Y \setminus \{x_0, y_0\}) =
\]

\[ = H_i(X, Y, (X \setminus \{y_0\}) \cup ((X \setminus \{x_0\}) \times Y) = H_i((X, X \setminus \{y_0\}) \times (Y, \setminus \{y_0\}). \] □

\[ i. \quad \begin{cases} A := X \cup Y, \\
B := X \cup Y, X \times \{y_0\} \times [0, 1] \end{cases} \quad \Rightarrow \quad \begin{cases} A \cup B = X \bowtie Y \setminus \{y_0, 0\} \\
A \cap B = X \times (Y \setminus \{y_0\}) \times (0, 1) \end{cases} \quad \text{where}
\]

\((x_0, y_0, t) \in X \times \{y_0\} \times [0, 1]\) is independent of \(x_0\) and \((x_0, y_0, t_0) := \{(x_0, y_0, t_0, \varphi)\}.
\]

Now use Th. 1 p. 8 line 2 and that the r.h.s. is a pair deformation retract of the l.h.s.; \((X \times Y \setminus (0, 1), X \times (0, 1) \setminus \{x_0, y_0\}) - (X \times Y, X \times (0, 1)) = (X, \emptyset) \setminus (Y, \setminus \{y_0\}). \] □

3.4. Singular Homology Manifolds under Products and Joins.

Definition. \( \emptyset \) is a weak homology manifold. Else, a \( T_p \)-space (\( \Leftrightarrow \) all points are closed) \( X \in D_\infty \) is a weak homology manifold \((n, w)-\text{manifold}) if for some \( A \)-module \( \mathcal{R}; \)

\[ H_i(X, X \setminus \{x\}; G) = 0 \quad \text{if} \quad i \neq n \quad \text{for all} \quad \varphi \neq x \in X, \] \hspace{5cm} \( (4.i) \)

\[ H_i(X, X \setminus \{x\}; G) \cong G \oplus \mathcal{R} \quad \text{for some} \quad \varphi \neq x \in X \quad \text{if} \quad X \neq \{\varphi\}. \] \( (4.ii') \)

An \( n \)-w-hmw \( X \) is joinable \((n, jw)-\text{manifold}) if \((4.i)\) holds also for \( x = \varphi. \)

An \( n \)-jw-hmw \( X \) is a weak homology \( n \)-sphere \((n, w)-\text{space}) if \( H_i(X \setminus \{x\}; G) = 0 \forall x \in X. \)

Definitions of technical nature. \( X \) is acyclic if \( H_i(X, \emptyset; G) = 0 \) for all \( i \in \mathbb{Z}. \)

So, \( \{\varphi\} \) (\( = \{\emptyset\} \)) isn't acyclic. \( X \) is weakly direct if \( H_i(X; G) \cong G \oplus \mathcal{P} \) for some \( i \) and some \( A \)-module \( \mathcal{P}. \)

\( X \) is locally weakly direct if \( H_i(X, X \setminus \{x\}; G) \cong G \oplus \mathcal{Q} \) for some \( i \), some \( A \)-module \( \mathcal{Q} \) and some \( \varphi \neq x \in X. \)

An \( n \)-w-hmw \( X \) is ordinary \((\forall i \geq n \quad \text{and} \quad \forall x \in X) \)

Corollary. (to Th. 6). For locally weakly direct \((\Rightarrow X_i \neq \emptyset, \{\varphi\}) \) \( \mathcal{T}_p \)-spaces \( X_i, X_j. \)

i. \( X \times X_i (n_i + n_j) \)-w-hmw \( \Leftrightarrow \) \( X_i, X_j \) both \( n_i \)-w-hmw \( \Leftrightarrow \) \( X_i \times X_j \) jw-hmw.

ii. \( X_i \times X_j (n_i + n_j) \)-w-hmw \( \Leftrightarrow \) \( X_i, X_j \) both w-hmw.

iii. \( X_i \times X_j (n_i + n_j) \)-w-hmw if \( X_i \times X_j (n_i + n_j) \)-w-hmw if \( X_i \neq X_j \) both \( n_i \)-w-hmw and acyclic.\( \]

\[ \Leftrightarrow \left[ X_i, X_j \right] \neq \emptyset \quad \Rightarrow \text{acyclic}. \)

\[ \Rightarrow [X_i, X_j \bowtie X_i \times X_j \bowtie X_i \times X_j] \Rightarrow \left[ X_i \neq X_j \right. \bowtie \text{acyclic}. \]

\[ \text{So,} \quad X_i \times X_j \text{ is never a w-hmw.} \)

iv. \( X_i \times X_j \) are weakly direct then \( X_i, X_j \) both w-hmw \( \iff \) \( X_i \times X_j \) is.

Proof. Augmentual Homology, like classical, isn't sensitive to base ring changes.

So, ignore \( A \) and instead use the integers \( \mathbb{Z}; \) \((i-iii). \) Use Th. 1, 4-6 and the weak directness \( \text{to transpose non-zeros from one side to the other, using Th. 6.}\) for only joins, i.e., in particular, with \( \epsilon = 0 \) or 1 depending on wether \( \vee = \times \) or \( * \) resp., use:

\[ (\text{H}_{p+\epsilon})(X \vee X, X_i \vee X_j \setminus \{x_j, y_2\}; G) \cong \text{H}_{i+\epsilon}(X_1, X_j \setminus x_j \times (X_2, X_j \setminus y_j); G \cong \}

\[ \text{[Lemma p. 11]} \cong \bigoplus_{\epsilon \in \{0, 1\}} \text{H}_i(X_1, X_1 \setminus x_1 \times Z \otimes \text{H}_j(X_2, X_2 \setminus x_2; G) \cong \}

\[ \text{[Eq. 1 p. 8]} \cong \oplus \text{H}_i(X_1, X_1 \setminus x_1 \setminus Z, H_j(X_2, X_2 \setminus x_2); G)). \]

iv. Use, by the Five Lemma, the chain equivalence of the second component in the first and the last item of Th. 6.\) and the \( \mathcal{M} \)-Vs w.r.t. \( \{(X \times \{y_0\}), ((Y \setminus x_0) \times Y) \}. \] □
Definition. \( \emptyset \) is a homology_{\infty} - \infty-manifold and \( X = \bullet \bullet \) is a homology_{\infty} 0-manifold. Else, a connected, locally compact Hausdorff space \( X \in D_{g} \) is a (singular) homology_{X} n-manifold (n-hm_{X}) if:

\[
\begin{align*}
H_{i}(X, X \setminus x; G) &= 0 \quad \text{if} \quad i \neq n \quad \text{for all} \quad \emptyset \neq x \in X, \\
H_{n}(X, X \setminus x; G) &= 0 \quad \text{or} \quad G \quad \text{for all} \quad \emptyset \neq x \in X \quad \text{and} \quad G \quad \text{for some} \quad x \in X. \quad (4.i)
\end{align*}
\]

The boundary: \( \text{Bd}_{g}X := \{ x \in X | H_{n}(X, X \setminus x; G) = 0 \} \). If \( \text{Bd}_{g}X \neq \emptyset \) (\( \text{Bd}_{g}X = \emptyset \)), \( X \) is called a homology_{X} n-manifold with (without) boundary.

A compact \( n \)-manifold \( S \) is orientable_{g} if \( H_{1}(S, \text{Bd}S; G) = G \). An \( n \)-manifold is orientable_{g} if all its compact \( n \)-submanifolds are orientable - else non-orientable_{g}.

Orientability is left undefined for \( \emptyset \). An \( n \)-hm_{X} \( X \) is joinable if (4) holds also for \( x = \emptyset \). An \( n \)-hm_{X} \( X \neq \emptyset \) is a homology_{X} n-sphere (n-hsp_{X}) if for all \( x \in X \), \( H_{i}(X, X \setminus x; G) = G \) if \( i = n \) and 0 else.

Note 1. Triangulable manifolds \( \neq \emptyset \) are ordinary, by Note 1 p. 25, and locally weakly direct_{g} since \( H_{\dim \Sigma}((\Sigma |, | \Sigma \setminus \alpha; G) \cong G \) for any \( \alpha \in \text{Int} \sigma \) if \( \sigma \) is a maximndimensional simplex i.e. if \( \# \sigma - 1 = \dim \sigma = \dim \Sigma \), since now \( \text{Lk}_{\Sigma} \sigma = \{ \emptyset \} \). Prop. 1. p. 11 and Lemma p. 5 now gives the claim. If \( \alpha \in \text{Int} \sigma \), Proposition 1. p. 11 also implies that:

\[
H_{\dim \Sigma}(\Sigma, \text{cost}_{\Sigma} \sigma; Z) \cong H_{\dim \Sigma}((\Sigma |, | \Sigma \setminus \alpha; Z) \cong H_{\dim \Sigma}(\Sigma |, | \Sigma \setminus \alpha; Z) \cong \text{H} \text{(}1k_{\Sigma} \sigma; Z) \quad \text{is a direct sum of} \quad Z\text{-terms.}
\]

When \( v \) in Th. 7, all through, is interpreted as \( \ast \), the word “manifold(s)” (on the r.h.s.) temporarily excludes \( \emptyset, \{ \emptyset \} \) and \( \bullet \bullet \), and we assume \( \varepsilon := 0 \). When \( v \), all through, is interpreted as \( \ast \), put \( \varepsilon := 1 \), and let the word “manifold(s)” on the right hand side be limited to “any compact joinable homology_{X} n-manifold”.

Theorem 7. For locally weakly direct_{g} \( T_{i} \)-spaces \( X_{1}, X_{2} \) and any A-module \( G \):

1. \( X_{i} \vee X_{2} \) is a homology_{X_{1}} \( n_{i} + n_{2} + \varepsilon \)-manifold \iff \( X_{i} \) is a n-hm_{X_{1}} \( i = 1,2 \).

2. \( \text{Bd}_{g}(\bullet \times X) = \bullet \times (\text{Bd}_{g}X) \). Else; \( \text{Bd}_{g}(X \vee X_{2}) = ((\text{Bd}_{g}X_{1}) \vee X_{2}) \cup (X_{1} \vee (\text{Bd}_{g}X_{2})) \).

3. \( X_{i} \vee X_{2} \) is orientable_{g} \iff \( X_{i}, X_{2} \) are both orientable_{g}.

Proof. Th. 7 is trivially true for \( X \vee 1 \) and \( X_{i} \vee \{ \emptyset \} \). Else, exactly as for the above Corollary, adding for 7.1. that for Hausdorff-like spaces (\( = \) all compact subsets are locally compact), in particular for Hausdorff spaces, \( X_{i} \vee X_{2} \) is locally compact (Hausdorff) \iff \( X_{i}, X_{2} \) both are compact (Hausdorff), cf. [4] p. 224. \( \square \)

Note 2. \( (X_{i} \vee X_{2}, \text{Bd}(X_{i} \vee X_{2})) = \{ 7.2 \} = (X_{i} \vee X_{2}, X_{1} \vee \text{Bd}X_{2} \cup \text{Bd}X_{1} \vee X_{2}) [\text{Def. p. 7} = (X_{i}, \text{Bd}X_{1}) \vee (X_{2}, \text{Bd}X_{2}). \]

Equivalently; \( \text{H}_{g}(X_{i} \vee X_{2}) = X_{i} \vee \text{H}_{g}X_{2} \cup \text{H}_{g}X_{1} \vee X_{2} \), where for any space \( X \):

\( \text{H}_{g}X = \text{set of Homologically}_{g} \text{ unstable points} := \{ x \in X | H_{i}(X, X \setminus x; G) = 0 \ \forall i \in \mathbb{Z} \} \).

Example. \( \bullet, \bullet \bullet \) and \( \bullet \bullet \bullet \) are all 0-whm_{g} but \( \bullet \bullet \bullet \bullet \bullet \) isn’t a 0-hm_{g}. As follows from above, \( \emptyset \simeq (\bullet \bullet) \ast (\bullet \bullet \bullet) \) i.e. the join of two 0-hsp_{g} is a 1-hsp_{g}, cf. Ex. 4 p. 28.

\( \emptyset \simeq (\bullet \bullet \bullet) \ast (\bullet \bullet \bullet) \) and \( \emptyset \ast (\bullet \bullet \bullet \bullet \bullet) \) are both 1-jwhm_{g} but nighter is a 1-hm_{g}, which actually contradicts the statement in [27] p. 122 Corollary 2.12(ii).

The following is a trivial example of how to “eliminate” neighborhood retract-subspaces. \( H_{i}((\emptyset; G) = H_{i}((\emptyset \ast (\emptyset \ast G) = [21] \text{Ex. p. 2.30} = H_{i}((\ast \emptyset); G) = \bigoplus H_{i}((\bullet \ast (\emptyset \ast G) \ast (\emptyset \ast G) = \bigoplus H_{i}((\emptyset \ast (\emptyset \ast G) \ast (\emptyset \ast G) = [\text{Eq. p. 10} = R \otimes (G \ast (G \ast G) = R \ast (G \ast (G \ast G) = [\text{Lemma p. 6} = R \otimes G = G \ast G. \quad (H_{i}(\emptyset; G) = 0 \text{ if } i \neq 1) \)
I: General Topological Properties for Realizations of Simplicial Complexes

I:1 Realizations and Local Homology Groups Related to Simplicial Products and Joins. Only fairly recent the first examples of non-triangulable topological (i.e. 0-differentiable) manifolds have been successfully constructed, cp. [23] §5, and since all differentiable manifolds are triangulable, cf. [20] p. 103 Th. 10.6, essentially all spaces within mathematical physics are triangulable. The realization, p. 5, of any simplicial complex, is a CW-complex. Our CW-complexes, will have, as the so called “relative CW-complexes” defined in [9] p. 326, a (−1)-cell \( \{ y \} \) (“an ideal cell”), but their topology is that of the spaces in the category \( \mathcal{D} \) defined in p. 3. CW-complexes are compactly generated, perfectly normal spaces, [9] pp. 22, 112, 242, that are locally contractible in a strong sense and (hereditarily) paracompact, cf. [9] pp. 28-29 Th. 1.3.2 + Th. 1.3.5 (Ex. 1 p. 33). A topological space has the homotopy type of a CW-complex iff it has the type of the realization of a simplicial complex, which it has iff it has the type of an ANR, cf. [9] p. 226 Th. 5.2.1.

The \( \mathbf{k} \)-ification \( \mathbf{k}(X) \) of \( X \) is \( X \) with its topology enlarged to the weak topology w.r.t. its compact subspaces, cf. [4]. Put \( X \times Y := k(X \times Y) \). If \( X \) and \( Y \) are CW-complexes, this is a proper topology-enlargement only if none of the two underlying complexes are locally finite and at least one is uncountable. Let \( X \times Y \) be the quotient space w.r.t. \( p: (X \times Y) \times \mathbf{I} \rightarrow X \circ Y \) from p. 7. See [16] p. 214 for relevant distinctions. Now, simplicial \( \times, * \) “commute” with realization by turning into \( \times, * \) respectively. Unlike * defined in p. 7, \( * \) is actually associative for arbitrary topological spaces.

Definition. (cf. [7] Def. 8.8 p. 67.) Given ordered simplicial complexes \( \Delta \) and \( \Delta'' \), i.e. the vertex sets \( V_\Delta \) and \( V_{\Delta''} \) are partially ordered so that each simplex becomes lefwardly ordered resp. The Ordered Simplicial Cartesian Product \( \Delta \times \Delta'' \) of \( \Delta \) and \( \Delta'' \) (triangulates \( |\Delta| \times |\Delta''| \) and) is defined through \( V_{\Delta \times \Delta''} := \{(v', v'') \} = V_\Delta \times V_{\Delta''} \). Put \( w_{i,j} := (v', v'') \). Now, simplices in \( \Delta \times \Delta'' \) are sets \( \{w_{i,j} : w_{i,j} \in \Delta, w_{i,j} \in \Delta'' \} \), with \( w_{i,j} \neq w_{i',j}, w_{i,j} \neq w_{i,j'} \), and \( v_i' \leq v_i'' \leq v_i''' \) where \( v_i', v_i'', v_i''' \) is a sequence of vertices, with repetitions possible, constituting a simplex in \( \Delta \times \Delta'' \).

Lemma. (cp. [7] p. 68.) \( \eta: ([p_1],[p_2]), \Sigma_1 \times \Sigma_2 \rightarrow \eta \) the simplicial projection, triangulates \( |\Sigma_1| \times |\Sigma_2| \). If \( L_1 \) and \( L_2 \) are subcomplexes of \( \Sigma_1 \) and \( \Sigma_2 \), then \( \eta \) carries \( [L_1 \times L_2] \) onto \( [L_1] \times [L_2] \). Furthermore, this triangulation has the property that, for each vertex \( B \) of \( \Sigma_2 \), say, the correspondence \( x \rightarrow (x, B) \) is a simplicial map of \( \Sigma_1 \) into \( \Sigma_1 \times \Sigma_2 \). Similarly for joins, cp. [28] p. 99.

Proof. (\( \times \).) The simplicial projections \( p_i: \Sigma_i \times \Sigma_2 \rightarrow \Sigma_i \) gives realized continuous maps \( |p_i|, i = 1, 2. \eta := ([p_1],[p_2]): |\Sigma_1 \times \Sigma_2| \rightarrow |\Sigma_1| \times |\Sigma_2| \) is bijective and continuous cf. [2] 2.5.6 p. 32 + Ex. 12, 14 p. 106-7. \( 2_{\Sigma_1 \times \Sigma_2} (\tau_{\Sigma_1 \times \Sigma_2}) \) is the weak topology w.r.t. the compact subspaces \( \{[\Sigma_1] \times [\Sigma_2]\}_{\Sigma_1 \in \Sigma_2}, \{[\Sigma_1] \times [\Sigma_2]\}_{\Sigma_2 \in \Sigma_2} \), cf. [9] p. 246 Prop. A.2.1. \( ([p_1],[p_2]) ([\Sigma_1] \times [\Sigma_2]) \cap A = ([\Sigma_1] \times [\Sigma_2]) \cap ([p_1],[p_2]) (\{A\}) \) i.e. \( ([p_1],[p_2]) \) is continuous. (\( \ast \).) As for \( * \), cp. [29] (3.3) p. 59, with \( \Sigma_1 + \Sigma_2 := \{\sigma \cup \sigma | \sigma \in \Sigma_1 (i = 1, 2)\} \), using:

\[
\eta : |\Sigma_1 + \Sigma_2| = |\{\sigma \cup \sigma | \sigma \in \Sigma_1 (i = 1, 2)\}| \rightarrow
\]

\[
\rightarrow |\Sigma_1| \uparrow |\Sigma_2| : t_1 v_1' + ... + t_q v_q' + t_{q+1} v_{q+1}' + ... + t_{n+v} v_{n+v}' \mapsto
\]

\[
\rightarrow (t_1 \frac{v_1'}{\Sigma_1'} + ... + t_q \frac{v_q'}{\Sigma_1'}) + (t_{q+1} \frac{v_{q+1}'}{\Sigma_1'} + ... + t_{n+v} \frac{v_{n+v}'}{\Sigma_1'}). \square
\]
So: \( \eta: (|\Sigma\cup\Sigma|,|\Sigma\cap\Sigma|,\{(\alpha_i,\alpha_j)\}) \cong (|\Sigma\cap\Sigma|,|\Sigma\cap\Sigma|,\{(\alpha_i,\alpha_j)\}) \) is a homeomorphism if \( \eta(\alpha_i,\alpha_j) = (\alpha_i,\alpha_j) \) and it's easily seen that \( k(|\Sigma|*|\Delta|) = |\Sigma|*|\Delta| = k(|\Sigma|*|\Delta|) \), since these spaces have the same topology on their compact subsets. Moreover, if \( \Delta \subset \Sigma, \Delta' \subset \Sigma' \) then \( |\Delta|*|\Delta'| \) is a subspace of \( |\Sigma|*|\Sigma'| \).

**Example.** Let \( Z^* \) be the positive integers i.e. 1,2,..., regarded as a simplicial complex and let \( |Z^*| \) be its realization according to p. 5. Imbed \( |Z^*| \subset |Z^*| \) into the positive \( R^+ \)-axis in the real plane \( R^2 := R^+ \times R^- \), and connect, with a straight line, each of its points with \( (0,1) \in R^2 \) and denote the result \( Z^* \circ \bullet \) and let \( (\cdot)^\tau \) denote the topology of \( \cdot \), then, from right to left, we have natural one-one maps implying:

\[
(\tau|Z^*| \circ \bullet)^\tau \subset (\tau|Z^*| \circ \bullet)^\tau \subset (\tau|Z^*| \circ \bullet)^\tau \subseteq (\tau|Z^*| \circ \bullet)^\tau.
\]

The first “\( \subset \)” follows from a simple check of metric topologies. The equality follows since the topology of both these non-metrizable \( k \)-spaces, i.e. also of the first, is determined as the identification topology w.r.t. the canonical quotient map from \( |Z^*| \times I \) where \( I \) is the unit interval. Let \( V_n \) be all points on the segment from the cone point \( \omega_0 := (0,1) \) within \( 1/n \) from \( \omega_0 \), and \( V := \bigcup V_n \). \( V \) isn't open in \( (Z^* \circ \bullet)^\tau \) but it is open in the final topology \( (Z^* \circ \bullet)^\tau \). p. 127. The last paragraph prior to this example gives the rest since \( k \)-iffactions preserve the subset-relations w.r.t. topologies, implying that the “\( \subseteq \)” is an equality if \( (Z^* \circ \bullet)^\tau \) is a \( k \)-space.

\[
dim(\Sigma \times \Delta) = \dim \Sigma + \dim \Delta \quad \text{and} \quad \dim(\Sigma \ast \Delta) = \dim \Sigma + \dim \Delta + 1.
\]

If \( \alpha_i \in \text{Int} \sigma \subseteq |\Sigma|, \alpha_i := \eta(\alpha_i,\alpha_j) \in \text{Int} \sigma \subset |\Sigma| \times |\Sigma| \) and \( c_{\sigma} := \dim \sigma + \dim \sigma - \dim \sigma \) then \( c_{\sigma} \geq 0 \) and \( c_{\sigma} = 0 \) iff \( \sigma \) is a maximal simplex in \( \sigma_1 \cup \sigma_2 \subset \Sigma_1 \times \Sigma_2 \).

**Corollary.** (to Th. 6) Let \( G, G' \) be arbitrary modules over a \( PID R \) such that \( Tor_1^R(G, G') = 0 \), then, for any \( \emptyset \neq \sigma \subseteq \Sigma_1 \times \Sigma_2 \) with \( \eta(\text{Int}(\sigma)) \subset \text{Int}(\sigma_1) \times \text{Int}(\sigma_2); \)

\[
\begin{align*}
\text{H}_{n+1}(Lk_{\Sigma_1 \times \Sigma_2} (\sigma_1 \cup \sigma_2); \bigotimes_{R} G') & \cong \bigoplus_{p+q = 0} \text{H}_{p}(\text{Lk}_{\Sigma_1} (\sigma_1; \bigotimes_{R} G), \bigotimes_{R} \text{H}_{q}(\text{Lk}_{\Sigma_1} (\sigma_1; \bigotimes_{R} G'), \bigotimes_{R} G') \bigotimes_{R} \text{H}_{n+1}(Lk_{\Sigma_1 \times \Sigma_2} (\sigma_1 \cup \sigma_2); \bigotimes_{R} G')
\end{align*}
\]

So, if \( \emptyset \neq \sigma \) and \( c_{\sigma} = 0 \) then \( \text{H}_{n}(Lk_{\Sigma_1 \times \Sigma_2} (\sigma_1 \cup \sigma_2); \bigotimes_{R} G') \cong \bigoplus_{p+q = 0} \text{H}_{p}(Lk_{\Sigma_1} (\sigma_1; \bigotimes_{R} G), \bigotimes_{R} \text{H}_{q}(Lk_{\Sigma_1} (\sigma_1; \bigotimes_{R} G'), \bigotimes_{R} \text{H}_{n}(Lk_{\Sigma_1 \times \Sigma_2} (\sigma_1 \cup \sigma_2); \bigotimes_{R} G')
\]

**Proof.** Note that \( \sigma \neq \emptyset \Rightarrow \sigma \neq \emptyset, j = 1, 2 \). The isomorphisms of the underlined modules are, by Proposition 1 p. 11, Theorem 6i p. 11 in simplicial disguise, and holds even without the \( PID \)-assumption. Prop. 1 p. 30 and Theorem 4 p. 10 gives the second isomorphism even for \( \sigma_j = 0 \) and/or \( \sigma_j = \emptyset \). \( \square \)

### I:2 Simplicial Connectedness Properties under Products and Joins.

Any \( \Sigma \) is representable as \( \Sigma = \bigcup_{\sigma \in \Sigma} \sigma^m \), where \( \sigma^m \) denotes “maximal simplex”.

**Definition 1.** Two maximal faces \( \sigma, \tau \in \Sigma \) are strongly connected if they can be connected by a finite sequence \( \sigma = \delta_1, \delta_2, ..., \delta_s = \tau \) of maximal faces with \( \#(\delta_j \cap \delta_{j+1}) = \max \#(\delta_j \cap \delta_{j+1}) - 1 \) for consecutives. Strong connectedness imposes an equivalence relation among the maximal faces, the equivalence classes of which defines the maximal strongly connected components of \( \Sigma \), cp. [2] p. 419ff. \( \Sigma \) is said to be strongly connected if each pair of its maximal simplices are strongly connected.

A submaximal face has exactly one vertex less then some maximal face containing it.

**Note.** Strongly connected complexes are pure, i.e. \( \sigma \in \Sigma \) maximal \( \Rightarrow \dim \sigma = \dim \Sigma \).
Definition 2. (cp. [2] p. 419ff.) $\Delta - \Delta := \{ \delta \in \Delta \mid \delta \notin \Delta \}$ is connected as a poset (partially ordered set) w.r.t. simplex inclusion if for every pair $\sigma, \tau \in \Delta - \Delta$ there is a chain $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \tau$ where $\sigma_i \in \Delta - \Delta$ and $\sigma_i \leq \sigma_{i+1}$ or $\sigma_i \geq \sigma_{i+1}$.

Note. (cf. [10] p. 162.) $\Delta - \Delta, \{ \emptyset \} \subseteq \Delta$ is connected as a poset iff $|\Delta| \setminus \emptyset \subseteq |\Delta|$ is pathwise connected. When $\Delta = \{ \emptyset \}$, then the notion of connectedness as a poset is equivalent to the usual one for $\Delta$. $|\Delta|$ is connected iff $|\Delta^0|$ is. $\Delta^0 := \{ \sigma \in \Delta \mid \# \sigma \leq p + 1 \}$.

Lemma 1. ([10] p. 163) $\Delta - \Delta$ is connected as a poset iff to each pair of maximal simplices $\sigma, \tau \in \Delta - \Delta$ there is a chain in $\Delta - \Delta$, $\sigma = \sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_m = \tau$, where the $\sigma_i$ are maximal faces and $\sigma_i, \sigma_{i+1}$, and $\sigma_{i+2}, \sigma_{i+1}$ are situated in different components of $\operatorname{Lk} \sigma_{i+1}, (i = 0, 1, \ldots, m - 1)$. □

Lemma 1 gives Lemma 2 found in [1] p. 1856:

Lemma 2. Let $\Sigma$ be a finite dimensional simplicial complex, and assume that $\operatorname{Lk}_p \sigma$ is connected for all $\sigma \in \Sigma$, i.e. including $\emptyset, \in \Sigma$, such that $\dim \operatorname{Lk}_p \sigma \geq 1$. Then $\Sigma$ is pure and strongly connected. □

Lemma 3 (4) is related to the defining properties for quasi- (pseudo-) manifolds.

Read $v$ in Lemma 3 as "x" or all through as "*" when it's trivially true if any $\Sigma_i := \{ \emptyset \}$ and else for * $\Sigma$, are assumed to be connected or 0-dimensional, "codima $\geq 2" (\Rightarrow \dim \operatorname{Lk}_p \sigma \geq 1)$ means that a maximal simplex, $\tau$ say, containing $\sigma$ always fulfills $\dim \geq \dim + 2$. $G_i, G_2$ are A-modules. For definitions of Int and $\sigma$ see p. 30.

Lemma 3. If $\dim \Sigma_i \geq 0$ and $v_i := \dim \sigma_i (i = 1, 2)$ then $D_1, D_2$ are all equivalent:

$D_1$) $H_0(\operatorname{Lk} \Sigma_1 \cup \Sigma_2; \sigma; G_1 \otimes G_2) = 0$ for $\emptyset, \neq \sigma \in \Sigma_1 \cup \Sigma_2$, whenever codima $\geq 2$.

$D_1'$) $H_0(\operatorname{Lk} \Sigma_1 \cup \Sigma_2; G_i) = 0$ for $\emptyset, \neq \sigma \in \Sigma_i$, whenever codima $\geq 2 (i = 1, 2)$.

$D_2$) $H_{v+1}(\Sigma_1 \cup \Sigma_2; G_1 \otimes G_2) = 0$ for $\emptyset, \neq \sigma \in \Sigma_1 \cup \Sigma_2$, if codima $\geq 2$.

$D_2'$) $H_{v+1}(\Sigma_1 \cup \Sigma_2; G_i) = 0$ for $\emptyset, \neq \sigma \in \Sigma_i$, whenever codima $\geq 2 (i = 1, 2)$.

$D_3$) $H_{v+1}(\| \Sigma_1 \cup \Sigma_2 \|, \Sigma_1 \cup \Sigma_2; \alpha; G_1 \otimes G_2) = 0$ for all $\alpha \neq \alpha \in \text{Int} \sigma$, if codima $\geq 2$.

$D_3'$) $H_{v+1}(\| \Sigma_1 \cup \Sigma_2 \|, \Sigma_1 \cup \Sigma_2; G_i) = 0$ for $\alpha_0 \neq \alpha \in \text{Int} \sigma_i$, if codima $\geq 2 (i = 1, 2)$.

$D_4$) $H(v+1)(\| \Sigma_1 \cup \Sigma_2 \|, \Sigma_1 \cup \Sigma_2; \alpha, \alpha_2; G_i \otimes G_2) = 0$ for all $\alpha \neq \alpha, \alpha_2 \in \text{Int} \sigma \subseteq |\Sigma_1 \cup \Sigma_2 |$, if codima $\geq 2$, where $|\Sigma_1 \cup \Sigma_2 | \xrightarrow{\cong} |\Sigma_1 \cup \Sigma_2 |$ and $\eta(\alpha, \alpha_2) = (\alpha_1, \alpha_2)$, (K-filaktion never effect the homology modules.)

Proof. By the homogeneity of the interior of $[\sigma_i, \bar{\sigma}_j]$ we only need to deal with simplices $\sigma$ fulfilling $c_\sigma = 0$. Proposition 1 p. 11 + p. 15 top, implies that all non-primed resp. primed items are equivalent among themselves. The above connectedness conditions are not coefficient sensitive, so suppose $G_i := k$, a field. $D_1 \Leftrightarrow D_1'$ by Cor. p. 15. For joins of finite complexes, this is done explicitly in [10] p. 172. □

Lemma 4. ([8] p. 81 gives a proof, valid for any finite-dimensional complexes.)

A) If $d := \dim \Sigma_i \geq 0$ then; $\Sigma_i \times \Sigma_2$ is pure $\iff \Sigma_1$ and $\Sigma_2$ are both pure.

B) If $\dim \sigma_{i}^m \geq 1$ for each maximal simplex $\sigma_{i}^m \in \Sigma_i$ then;

Any submaximal face in $\Sigma_i \times \Sigma_2$ lies in at most (exactly) two maximal faces $\iff$

Any submaximal face in $\Sigma_i$ lies in at most (exactly) two maximal faces of $\Sigma_i, i = 1, 2$.

C) If $d > 0$ then; $\Sigma_i \times \Sigma_2$ strongly connected $\iff \Sigma_i, \Sigma_2$ both strongly connected. □

Note. Lemma 4 is true also for * with exactly the same reading but now with no other restriction than that $\Sigma_i \neq \emptyset$ and this includes in particular item B.
II: Concepts Related to Combinatorics and Commutative Algebra

II:1 Definition of Stanley-Reisner rings.

Stanley-Reisner (St-Re) ring theory is a basic tool within combinatorics, where it supports the use of commutative algebra. The definition of $m_{\delta}$, in $I_\Delta = \{m_{\delta} \mid \delta \not\in \Delta\}$, in existing literature is so vague that it allows you to state nothing but $A[\emptyset] = A[\emptyset]$. Our definition of $m_{\delta}$ below rectifies this and we conclude that $A[\emptyset] \cong A[\emptyset] = \{\text{The trivial ring} = A[\emptyset]\}$.

Definition. A subset $s \subseteq W \supset V_\Delta$ is said to be a non-simplex (w.r.t. $W$) of a simplicial complex $\Delta$, denoted $s \notin \Delta$, if $s \not\in \Delta$ but $s = (s)(\dim s) \subseteq \Delta$ (i.e. the $(\dim s - 1)$-dimensional skeleton of $s$, consisting of all proper subsets of $s$, is a subcomplex of $\Delta$). For a simplex $\delta = \{v_1, \ldots, v_k\}$ we define $m_\delta$ to be the squarefree monomial $m_\delta := 1_A \cdot v_{i_1} \cdot \ldots \cdot v_{i_k}$ where $A[W]$ is the graded polynomial algebra on the variable set $W$ over the commutative ring $A$ with unit $1_A$. So, $m_\emptyset = 1_A$.

Let $A[\Delta] := A[W] / I_\Delta$ where $I_\Delta$ is the ideal generated by $\{m_{\delta} \mid \delta \notin \Delta\}$. $A[\Delta]$ is called the “face ring” or “Stanley-Reisner (St-Re) ring” of $\Delta$ over $A$. Frequently $A = k$, a field.

Note. i. Let $P$ be the set of finite subsets of the set $P$ then: $A[P] = A[P]$. $P$ is known as “the full simplicial complex on $P$” and the natural numbers $N$ gives $N$ as “the infinite simplex” $\Delta = \bigcap_{s \notin \Delta} \mathcal{P}s$. See pp. 30-1 for definitions.

ii. $A[\Delta] \cong \left( \frac{A[V_\Delta]}{(m_{\delta} \mid \delta \notin \Delta)} \right)$, if $\Delta \neq \emptyset, \{\emptyset\}$. So, the choice of the universe $W$ isn’t all that critical. If $\Delta = \emptyset$, then the set of non-simplices equals $\{\emptyset\}$, since $\emptyset \not\in \emptyset$, and $V_\emptyset^{-1} = \emptyset^{-1} = \emptyset \subset \emptyset$, implying: $A[\emptyset] = 0 = \text{The trivial ring}$, since $m_\emptyset = 1_A$.

Since $\emptyset \in \Delta$ for every simplicial complex $\Delta \neq \emptyset, \{\emptyset\}$ is a non-simplex of $\Delta$ for every $v \in W \setminus V_\Delta$, i.e. $[v \notin V_\Delta] \leftrightarrow [v \notin \Delta \neq \emptyset]$. So $A[\emptyset] = A$ since $\{\emptyset \notin \Delta\} = W$.

iii. $k[\Delta, \Delta] \cong k[\Delta] \otimes k[\Delta]$, with $I_{\Delta, \Delta} = \{(m_{\delta} \mid \delta \notin \Delta \lor \delta \notin \Delta \land [\delta \notin \Delta \lor \Delta])\}$ by [8] Example 1 p. 70.

iv. If $\Delta_i \neq \emptyset i = 1, 2$, it’s well known that

a. $I_{\Delta, 1 \cup \Delta, 2} = I_{\Delta, 1} \cap I_{\Delta, 2} = \{(m = \text{Lcm}(m_{\delta_1}, m_{\delta_2}) \mid \delta \notin \Delta_i, i = 1, 2)\}$

b. $I_{\Delta, 1 \cap \Delta, 2} = I_{\Delta, 1} + I_{\Delta, 2} = \{(m_{\delta} \mid \delta \notin \Delta_1 \lor \delta \notin \Delta_2)\}$ in $A[W]$

$I_{\Delta, 1 \cap \Delta, 2}$ and $I_{\Delta, 1} + I_{\Delta, 2}$ are generated by a set (no restrictions on its cardinality) of squarefree monomials, if both $I_{\Delta, 1}$ and $I_{\Delta, 2}$ are. These squarefree monomially generated ideals form a distributive sublattice $(\mathcal{F}; \cap, +, A[W])$, of the ordinary lattice structure on the set of ideals of the polynomial ring $A[W]$, with a counter-part, with reversed lattice order, called the squarefree monomial rings with unit, denoted $(A^*[W]; \cap, +, A)$, $A[W]$ is the unique homogeneous maximal ideal and zero element. We can use the ordinary subset structure to define a distributive lattice structure on $\Sigma^*_W := \text{The set of non-empty simplicial complexes over } W$, with $\{\emptyset\}$ as zero element and denoted $(\Sigma^*_W; \cup, \cap, \{\emptyset\})$. The Weyman/Fr"oberg/Schwartzau Construction eliminates the above squarefree-demand, cf. [27] p. 107.

Proposition. The Stanley-Reisner Ring Assignment Functor defines a monomorphism on distributive lattices from $(\Sigma^*_W; \cup, \cap, \{\emptyset\})$ to $(A^*[W], \cap, +, A)$, which is an isomorphism for finite $W$. □
II:2 Buchsbaum, Cohen-Macaulay and 2-Cohen-Macaulay Complexes.

We’re now in position to give combinatorial/algebraic counterparts of the weak homology manifolds defined in §3.4 p. 12. Prop. 1 and Th. 8 below, together with Th. 11 p 25, show how simplicial homology manifolds can be inductively generated.

Combinatorialists call a finite simplicial complex a Buchsbaum (Bbm) complex if its Stanley-Reisner Ring is a Buchsbaum (Cohen-Macaulay) ring. We won’t use the ring theoretic definitions of Bbm or C-M and therefore we won’t write them out. Instead we’ll use some homology theoretical characterizations found in [26] pp. 73, 60-1 resp. 94 to deduce, through Prop. 1 p. 11, the following consistent definitions for arbitrary modules and topological spaces.

**Definition.** X is “Bbm” (“CM”, “2-CM”) if X is an n-whelm (n-jwhelm, n-whelm).

A simplicial complex Σ is “Bbm”, “CM”, resp. 2-“CM” if |Σ| is. In particular, Σ is 2-“CM” iff “CM” and \( H_i(\text{cost}_\delta; G) = 0 \), ∀ \( \delta \in \Delta \), cp. [26] Prop. 3.7 p. 94.

The n in “n-Bbm” (“n-CM”) resp. 2-“n-CM” is deleted since any interior point α of a realization of a maximal simplex gives; \( H_i(\Delta, |\Delta| \setminus \alpha; G) = G. \) \( \sigma \in \Sigma \) is maximal-dimensional if \( \dim \sigma = \dim \Sigma \) and “\( \setminus \)” is defined in p. 4.

So we’re simply renaming n-whelm, n-jwhelm and n-whelm to “Bbm”, “CM” resp. 2-“CM”, where the quotation marks indicate that we’re not limited to compact spaces nor to just Z or k as coefficient modules. N.B., the definition p. 31 of \( [\mathcal{X}(X)] \), through which each topological space X can be provided with a Stanley-Reisner ring, which, w.r.t. “Bbm”, “CM”- and 2-“CM”-ness is triangulation invariant.

**Proposition 1.** The following conditions are equivalent: (We assume \( \dim \Delta = n \).)

a. \( \Delta \) is “Bbm”;

b. (Schenzel) \( \Delta \) is pure and \( L_k \delta \) is “CM” \( \forall \emptyset \neq \delta \in \Delta, \)

c. (Reisner) \( \Delta \) is pure and \( L_k \delta \) is “CM” \( \forall v \in \delta \).

**Proof.** Use Proposition 1 p. 11 and Lemma 2 p. 16 and then use Eq. I p. 30. □

**Example.** When limited to compact polytopes and k as coefficient module, we add, from [26] p. 73, the following Buchsbaum-equivalence using local cohomology;

d. (Schenzel) \( \Delta \) is Buchsbaum iff \( \dim H_{\text{h,reg}}(k[\Sigma]) \leq \infty \) if \( 0 < i < \dim k[\Sigma] \), in which case \( H_{\text{h,reg}}(k[\Sigma]) \equiv H_{\text{reg}}(k[\Sigma]; k) \), cf. [27] p. 144 for proof. Here, “\( \dim \)” is Krull dimension, which for Stanley-Reisner Rings is simply 1+ the simplicial dimension.

For \( \Gamma, \bar{\Gamma}, \Gamma \) finite and CM we get the following K"unneth formula for ring theoretical local cohomology; (“\( ,+ \)” indicates the unique homogeneous maximal ideal of “\( \cdot + \)”.)

1. \( H_\Gamma(k[\Gamma]\otimes k[1]) \equiv \frac{k[\Gamma]\otimes k[1]}{(k[\Gamma]\otimes k[1])^+}, \) cf. p. 17 + Eq. 3 p. 10 and Corollary i. p. 12 \( \equiv \bigoplus_{i+j=0} H_i(k[1]) \otimes H_j(k[1]) \).

2. Put: \( \beta_{\delta}(X) := \inf \{ j \mid \exists x \in X \wedge H_j(X, X \setminus x; G) \neq 0 \} \). For a finite \( \Delta \), \( \beta_{\delta}(\Delta) \) is related to the concept “depth of the ring \( k[\Delta] \)” and “CM-ness of \( k[\Delta] \)” through \( \beta_{\delta}(\Delta) = \text{depth}(k[\Delta]) - 1 \), in [3] Ex. 5.1.23 and [26] p. 142 Ex. 34. See also [22].

**Proposition 2.** a. \( \Delta \) is “CM” \( \iff \) \( [H_i(\Delta, \text{cost}_\delta; G) = 0 \forall \delta \in \Delta \) and \( \forall i \leq n - 1]. \)

b. \( \Delta \) is 2-“CM” \( \iff \) \( [H_i(\text{cost}_\delta; G) = 0 \forall \delta \in \Delta \) and \( \forall i \leq n - 1]. \)

**Proof.** Use the LHS w.r.t. (\( \Delta, \text{cost}_\delta \)), Prop. 1 p. 11 and the fact that \( \text{cost}_\emptyset = \emptyset \) respectively \( \text{cost}_\delta = \Delta \) if \( \delta \not\in \Delta \). □
Put $\Delta^{(p)} := \{ \delta \in \Delta \mid \# \delta \leq p + 1 \}$, $\Delta' := \Delta^{(n-1)} \setminus \Delta^{(n)}$, $\Delta' := \Delta^{(n-1)}$, $n := \dim \Delta$. So, $\Delta^{(n)} = \Delta$.

**Theorem 8.** a. $\Delta$ is “$CM_2$” if and only if $\Delta'$ is 2-“$CM_G$” and $H_f(\Delta, \delta, G) = 0 \forall \delta \in \Delta$.

b. $\Delta$ is 2-“$CM_G$” if and only if $\Delta'$ is 2-“$CM_G$” and $H_f(\delta, G) = 0 \forall \delta \in \Gamma$.

**Proof.** Proposition 2 together with the fact that adding or deleting $n$-simplices does not effect homology groups of degree $\leq n-2$. See Proposition 3, e. p. 31. □

**Lemma 1.** $\Delta$ “$CM_2$” $\iff$

\[
\begin{cases}
(a) & H_i(\delta, G) = 0 \forall \delta \in \Delta \quad \text{if } i \leq n - 2 \\
(b) & H_i(\delta, G) = 0 \forall \delta, \delta' \in \Delta \quad \text{if } i \leq n - 3.
\end{cases}
\]

**Proof.** (a) Use Proposition 1 p. 11, the definition of “$CM$”-ness and the LHS w.r.t. $\langle \Delta, \delta \rangle$ which reads:

$$\cdots \rightarrow H_i(\Delta, \delta \in \Delta) \rightarrow H_i(\delta, G) \rightarrow H_i(\Delta, \delta \in \Delta) \rightarrow \cdots$$

(b) Apply Prop. 3, p. 31 a+b to the M-Vs w.r.t. $\langle \delta, \delta \rangle$, then use a, i.e;

$$\cdots \rightarrow H_i(\delta, \delta \in \Delta) \rightarrow H_i(\delta, G) \rightarrow H_i(\delta, \delta \in \Delta) \rightarrow \cdots$$

**Observation.** To turn the implication in lemma 1 into an equivalence we just have to add $H_i(\Delta, G) = 0$ for $i \leq n - 1$, giving us the equivalence in:

$$\Delta$ “$CM_2$” $\iff$

\[
\begin{cases}
(i) & H_i(\Delta, G) = 0 \quad \text{if } i \leq n - 1 \\
(ii) & H_i(\delta, G) = 0 \forall \delta \in \Delta \quad \text{if } i \leq n - 2 \\
(iii) & H_i(\delta, G) = 0 \forall \delta, \delta' \in \Delta \quad \text{if } i \leq n - 3
\end{cases}
\]  (1)

$H_f(\delta, G) = 0 \forall \delta \in \Delta$ allows one more step in the proof of Lemma 1b, i.e.;

$$\Delta$ “$CM_2$” $\iff$

\[
\begin{cases}
(i) & H_i(\Delta, G) = 0 \quad \text{if } i \leq n - 1 \\
(ii) & H_i(\delta, G) = 0 \forall \delta \in \Delta \quad \text{if } i \leq n - 1 \\
(iii) & H_i(\delta, G) = 0 \forall \delta, \delta' \in \Delta \quad \text{if } i \leq n - 2
\end{cases}
\]  (2)

Note that a: i) in Eq. 2 is a consequence of ii) and iii) by the M-Vs above and that b: the l.h.s. is by definition equivalent to $\Delta$ being 2-“$CM$”.

Since, $[\Delta$ “$CM_2$” $\iff H_i(\delta, G) = 0 \forall \delta \in \Delta$ and $i \leq n - 1]$, item iii in Eq. 1 is, by the LHS w.r.t. $\langle \delta, \delta \rangle$, totally superfluous as far as the equivalence is concerned but never the less it becomes quite useful when substituting $\delta_1 \in \delta$ for every occurrence of $\Delta$ and using that $\delta_1 \in \delta$ $\iff \delta_1 \notin \delta$, for $\delta_1 \notin \delta$, we get:

$$\text{cost}_i \delta$ “$CM_2$” $\forall \delta \in \Delta$ $\iff$

\[
\begin{cases}
(i) & H_i(\delta, G) = 0 \quad \text{if } i \leq n - 1 \forall \delta \in \Delta \\
(ii) & H_i(\delta, G) = 0 \quad \text{if } i \leq n - 2 \forall \delta, \delta' \in \Delta.
\end{cases}
\]  (3)

**Remark 1.** The LHS w.r.t. $\langle \delta, \delta \rangle$, Corollary 3, p. 12 and Note 1, p. 25 gives;

If $\Delta$ is 2-“$CM_2$” then Note 1, p. 13 plus Prop. 1, p. 11 implies that $H_f(\Delta, G) \neq 0$, i.e. $\text{Hip}(\delta)$, def. p. 12, is empty if it’s a subcomplex, which it is for manifolds, cf. p. 25.
Remark 2. Set \( \bullet \mapsto := \{\emptyset, \{v_i\}, \{v_i, v_j\}, \{v_i, v_j, v_k\}\} \), \( n_\bullet := \dim \text{cost}_\bullet \varphi \) and \( n_\Delta := \dim \Delta \).

Note that it’s always true that: \( n_\bullet - 1 \leq n_\bullet \leq n_\Delta \) if \( \tau \subset \delta \).

Now, \( [n_\bullet = n_\bullet \text{ and costy pure } \forall v \in V_\Delta] \iff [n_\bullet = n_\bullet \text{ and cost}_\bullet \varphi \text{ pure } \forall \emptyset \neq \delta \in \Delta] \iff \]
\( \iff [\text{cost}_\bullet \left( \bigcup_{v \in \bullet} \text{cost}_\bullet \varphi \right) \text{ pure } \forall \emptyset \neq \delta \in \Delta \neq \bullet \iff \Delta \text{ is pure} \iff \]
\( \iff [\forall v \in V_\Delta \text{ is a cone point (p. 22)] } \iff [n_\bullet = n_\bullet - 1 \iff \emptyset \neq \delta \in \Delta \Rightarrow \varphi \in \delta \in \Delta \text{ with } \dim \delta_i = n_\bullet] \).

Since, \( n_\bullet := \dim \text{cost}_\bullet \varphi = n_\bullet - 1 \iff \emptyset \neq \varphi \subset \delta \in \Delta \text{ and } \dim \delta_i = n_\bullet \), we conclude that: If \( \Delta \) is pure then \( \varphi \) consists of nothing but cone points, cf. p. 22.

So, \( [\Delta \text{ pure and } n_\bullet = \dim \Delta \forall \delta \in \Delta] \iff [\Delta \text{ pure and has no cone points}] \).

\( [n_\bullet = n_\bullet - 1 \forall v \in V_\Delta] \iff V_\delta \text{ is finite and } \Delta = \bigcup_{i \in I} (:= \text{the full complex w.r.t. } V_\Delta). \)

Theorem 9. The following two conditions are equivalent to “\( \Delta \) is \( 2\text{-CM}_\bullet \)”:

- \( \text{a. } \text{cost}_\bullet \varphi \text{ is } \text{CM}_\bullet^n \text{, } \forall \delta \in \Delta \) and \( \text{ii. } n_\bullet := \dim \text{cost}_\bullet \varphi = \dim \Delta =: n_\bullet \forall \emptyset \neq \delta \in \Delta. \)
- \( \text{b. } \text{cost}_\bullet \varphi \text{ is } \text{CM}_\bullet^n \text{, } \forall \delta \in \Delta \) and \( \text{ii. } \{\cdot \mapsto \bullet \} \neq \Delta \text{ has no cone points}. \)

Proof. If there is no dimension collapse in Eq. 3 it is equivalent to Eq. 2. \( \square \)

Our next corollary was, originally ring theoretically proven by T. Hibi. We’ll essentially keep Hibi’s formulation, though using that \( \Delta := \Delta \setminus \{\tau \in \Delta \mid \tau \supset \delta_i \text{ for some } \in I\} = \bigcup_{i \in I} \text{cost}_\delta \).

Corollary. ([12 Corollary p. 95-6] Let \( \Delta \) be a pure simplicial complex of dimension \( n \) and \( \{\delta_i\}_{i \in I} \) a finite set of faces in \( \Delta \) satisfying \( \delta_i \cup \delta_j \notin \Delta \forall i \neq j \). Set, \( \Delta := \bigcap_{i \in I} \text{cost}_\delta \).

- \( \text{a. } \) If \( \Delta \) is \( \text{CM}_\bullet^n \) and \( \dim \Delta < n \), then \( \dim \Delta' = n - 1 \) and \( \Delta' \) is \( \text{CM}_\bullet^n \).
- \( \text{b. } \) If \( \text{st}_\Delta \delta \) is \( \text{CM}_\bullet^n \forall i \in I \) and \( \Delta \) is \( \text{CM}_\bullet^n \) of dimension \( n \), then \( \Delta \) is also \( \text{CM}_\bullet^n \).

Proof. \( \text{a. }[\delta \cup \delta \notin \Delta \forall i \neq j \in I] \iff [\text{cost}_\delta \cup \bigcap_{i \in \sigma} \text{cost}_\delta) = \Delta] \iff [\text{cost}_\delta \cup \bigcap_{i \in \sigma} \text{cost}_\delta) \cap \Delta = \Delta] \iff \dim \Delta = n - 1 \iff \Delta' = \Delta \cap \Delta' = \bigcup_{i \in I} \text{cost}_\delta \). By Th. 8 we know that \( \Delta \) is \( 2\text{-CM}_\bullet^n \), implying that \( \text{cost}_\Delta \delta \) is \( \text{CM}_\bullet^n \forall i \in I \).

Induction using the M-Vs w.r.t. \( \text{cost}_\delta \cup \bigcap_{i \in \sigma} \text{cost}_\delta) \) gives \( \text{H}(\Delta'; \mathbf{G}) = 0 \forall i < n_\bullet - 1 \).

For links, use Prop. 2a + b, p. 30. E.g. \( \text{Lk}_\delta \delta = \text{Lk}_\delta \delta \cap \bigcup_{i \in \sigma} \text{cost}_\delta) \) = \( \bigcup_{i \in I} \text{Lk}_\delta \delta \) = \( \text{CM}_\bullet^n \).

Definition. \( \Delta \setminus \{\{v_i, \ldots, v_p\}\} := \{\delta \in \Delta \mid \delta \cap \{v_i, \ldots, v_p\} = \emptyset\} \). \( (\Delta \setminus \{\emptyset\} = \text{cost}_\Delta \).

Permutations and partitions within \( \{v_i, \ldots, v_p\} \) doesn’t effect the result, i.e:

**Lemma 2.** \( \Delta \setminus \{\{v_i, \ldots, v_p\}\} = (\Delta \setminus \{\{v_i, \ldots, v_p\}\}) \setminus \{\{v_i', \ldots, v_p'\}\} \) and \( \Delta \setminus \{\{v_i, \ldots, v_p\}\} = \bigcap_{i \in I} \text{cost}_\delta \).

**Alternative Definition.** For \( k \in \mathbb{N} \), \( \Delta \) is \( k\text{-CM}_\bullet^n \) if for every subset \( T \subset V_\Delta \) such that \( \#T = k - 1 \), we have:

- \( \text{i. } \Delta \setminus \{T\} \) is \( \text{CM}_\bullet^n \)
- \( \text{ii. } \dim \Delta \setminus \{T\} = \dim \Delta =: n_k =: n \).

Changing “\( \#T = k - 1 \)” to “\( \#T < k \)” doesn’t alter the extension of the definition. (Iterate in Th. 9b mutatis mutandis.) So, for \( \mathbf{G} = \mathbf{k} \), a field, it’s equivalent to Kenneth Baclawski’s original definition in Europ. J. Combinatorics 3 (1982) p. 295.
II.3 Segre Products. The St-Re ring for a simplicial product is a Segre product.

Definition. The Segre product of the graded $A$-algebras $R_1$ and $R_2$, denoted $R = \sigma_{\alpha}(R_1, R_2)$ or $R = \sigma(R_1, R_2)$, is defined through: $[R]_p = [R_1]_p \otimes_A [R_2]_p$, $\forall p \in \mathbb{N}$.

Example 1. The trivial Segre product, $R_1 \otimes R_2$, is equipped with the trivial product, i.e. every product of elements, both of which lacks ring term, equals 0.

2. The “canonical” Segre product, $R_1 \otimes R_2$, is equipped with a product induced by extending (linearly and distributively) the componentwise multiplication on simple homogeneous elements: If $m_1^i \otimes m_2^j \in [R_1 \otimes R_2]_\alpha$ and $m_1^k \otimes m_2^\ell \in [R_1 \otimes R_2]_\beta$ then $(m_1^i \otimes m_2^j)(m_1^k \otimes m_2^\ell) := m_1^{i+k} m_2^{j+\ell} \in [R_1 \otimes R_2]_{\alpha+\beta}$.

3. The “canonical” generator-order sensitive Segre product, $R_1 \otimes R_2$, of two graded $k$-standard algebras $R_1$ and $R_2$ presupposes the existence of a uniquely defined partially ordered minimal set of generators for $R_1$ $(R_2)$ in $[R_1]_i$ $([R_2]_i)$ and is equipped with a product induced by extending (distributively and linearly) the following operation defined on simple homogeneous elements, each of which, now are presumed to be written, in product form, as an increasing chain of the specified linearly ordered generators: If $m_1^{11} \otimes m_2^{21} \in [R_1 \otimes R_2]_\alpha$ and $m_1^{22} \otimes m_2^{12} \in [R_1 \otimes R_2]_\beta$ then $(m_1^{11} \otimes m_2^{21})(m_1^{22} \otimes m_2^{12}) := (m_1^{11}m_2^{21} \otimes m_2^{12}m_2^{22}) \in [R_1 \otimes R_2]_{\alpha+\beta}$ if by “pairwise” permutations, $(m_1^{11}m_2^{21} \otimes m_2^{12}m_2^{22})$ can be made into a chain in the product ordering, and 0 otherwise. Here, $(x, y)$ is a pair in $(m_1^{11}m_2^{12}, m_2^{21}m_2^{22})$ if $x$ occupy the same position as $y$ counting from left to right in $m_1^{11}m_2^{12}$ and $m_2^{21}m_2^{22}$ respectively.

Note 1. ([27] p. 39-40) Every Segre product of $R_1$ and $R_2$ is module-isomorphic by definition and so, they all have the same Hilbert series. The Hilbert series of a graded $k$-algebra $R = \bigoplus_{i \geq 0} R_i$ is $\text{Hilb}_R(t) := \sum_{i \geq 0} (\dim_k R_i) t^i := \sum_{i \geq 0} (\dim_k R_i) t^i$.

It is called the discrete algebra if it is isomorphic to the moving (Gröbner) basis.

2. If $R_1$ and $R_2$ are graded algebras finitely generated (over $k$) by $x_1, \ldots, x_n \in [R_1]_1, y_1, \ldots, y_m \in [R_2]_1$, resp., then $R_1 \otimes R_2$ and $R_1 \otimes R_2$ are generated by $(x_1 \otimes y_1), \ldots, (x_n \otimes y_m)$, and $\dim R_1 \otimes R_2 = \dim R_1 \otimes R_2 = \dim R_1 + \dim R_2 - 1$.

3. The generator-order sensitive case covers all cases above. In the theory of Hodges Algebras and in particularly in its specialization to Algebras with Straightening Laws (ASLs), the generator-order is the main issue, cf. [3] § 7.1 and [14] p. 123 ff.

8. p. 72 Lemma gives a reduced (Gröbner) basis, $C' \cup D$, for $\mathbb{T}$ in $k[\Delta \times \Delta]$ $\cong k[V_1 \times V_2]/I$ with $C' := \{ u_{\lambda,\nu} \mid \lambda < \nu \wedge \mu > \xi \}$, $u_{\lambda,\nu} := (u_{\lambda,\nu}, u_{\lambda,\nu}) \in V_1 \times V_2$, where the subindices reflect the assumed linear ordering on the factor simplices and with $\mathbb{T}$ as the projection down onto the $i$th factor:

$$D := \left\{ w = u_{\lambda,\nu} \cdots u_{\lambda,\nu} \mid \left[ (\mathbb{T} \Lambda) \notin \Delta \right] \wedge \left[ (\mathbb{T} \Lambda) \notin \Delta \right] \wedge \left[ \mathbb{T} \Lambda \notin \Delta \right] \wedge \left[ \mathbb{T} \Lambda \notin \Delta \right] \right\}.$$

$C' \cup D = \{ m_i \mid \delta \notin \Delta \times \Delta \}$ and the identification $v_i \otimes v_i \leftrightarrow (v_i, v_i)$ gives, see [8] Theorem 1.71, the following graded $k$-algebra isomorphism of degree zero; $k[\Delta \times \Delta] = k[\Delta] \otimes k[\Delta]$, which, in the Hodges Algebra terminology, is the discrete algebra with the same data as $k[\Delta] \otimes k[\Delta]$, cf. [3] § 7.1. If the discrete algebra is “C-M” or Gorenstein (Definition p. 22), so is the original by [3] Corollary 7.1.6. Any finitely generated graded $k$-algebra has a Hodges Algebra structure, see [14] p. 145.
II:4 Gorenstein Complexes.

Definition 1. \( v \in V_\Sigma \) is a cone point if \( v \) is a vertex in every maximal simplex in \( \Sigma \).

Definition 2. (cp. [26] Prop 5.1) Let \( \Sigma \) be an arbitrary (finite) complex and put; 
\[ \Gamma := \text{core} \Sigma := \{ \sigma \in \Sigma \mid \sigma \text{ contains no cone points} \}. \]
Then; \( \emptyset \) isn’t \( \text{Gorenstein}_\Sigma \), while \( \emptyset \neq \Sigma \) is \( \text{Gorenstein}_\Sigma \) (Gor) if \( H_i(\Gamma; G) \neq 0 \) if \( i \neq \dim \Gamma \) and \( \emptyset \neq \Gamma \) if \( i = \dim \Gamma \) \( \forall \alpha \in \Gamma \).

Note. \( \Sigma \text{ Gorenstein}_\Sigma \iff \Sigma \text{ finite and } |\Gamma| \) is a homology sphere as defined in section 13. 
\[ \delta_\Sigma := \{ v \in V_\Sigma \mid v \text{ is a cone point} \} \subseteq \Sigma. \] Now; \( v \) is a cone point \( \iff \) \( \forall \tau \subseteq \Sigma \) \( v \in \tau \) and so, 
\[ \text{core} \Sigma := \{ \tau \in \Sigma \mid |\delta_\Sigma \cap \tau = 0 \} \cap (\delta_\Sigma \cup \tau \in \Sigma \} \} \text{.} \]

Proposition 1. (Cf. [8] p. 77.) \( \Sigma_1 \ast \Sigma_2 \text{ Gorenstein}_\Sigma \iff \Sigma_1, \Sigma_2 \text{ both Gorenstein}_\Sigma. \) \( \square \)

Gorensteinness is, unlike “Bbm"-,”CM", and 2- “CM"-ness, triangulation-sensitive and in particular, the Gorensteinness for products is sensitive to the partial orders, assumed in the definition, given to the vertex sets of the factors. See p. 21 for \( \{ m_3 \} \delta_\Sigma \ast_\Sigma \text{ for } \Sigma_1 \times \Sigma_2 \text{.} \] In [8] p. 80, the product is represented in the form of matrices, one for each pair \( (\delta_\Sigma \ast_\Sigma, \text{maximal simplices } \delta \Sigma \ast_\Sigma, i = 1, 2). \) It is then easily seen that a cone point must occupy the upper left corner in each matrix or the lower right corner in each matrix. So a product \( (\dim \Sigma_1 \geq 1) \) can never have more than \( 2 \) cone points. For Gorensteinness to be preserved under product the factors must have at least one cone point to preserve even “CM"-ness, by Corollary iii p. 12.

Proposition 2. (Cf. [8] p. 83ff. for proof.) Let \( \Delta_1, \Delta_2 \text{ be two arbitrary finite simplicial complexes with } \dim \Delta_i \geq 1, (i = 1, 2) \) and a linear order defined on their vertex sets \( V_{\Delta_1}, V_{\Delta_2} \), respectively, then; 
\[ \Delta_1 \times \Delta_2 \text{ Gorenstein}_\Sigma \iff \Delta_1, \Delta_2 \text{ both Gorenstein}_\Sigma \text{ and condition I or II holds, where;} \]
(I) \( \Delta_1, \Delta_2 \text{ has exactly one cone point each - either both minimal or both maximal.} \)
(II) \( \Delta \text{ has exactly two cone points, one minimal and the other maximal, } i=1,2. \) \( \square \)

Example. Gorensteinness is character sensitive! Let \( \Gamma := \text{core} \Sigma = \Sigma \) be a \( 3 \)-dimensional \( \text{Gorenstein}_\Sigma \) complex where \( k \) is the prime field \( \mathbb{Z}_p \) of characteristic \( p \). This implies, in particular, that \( \Gamma \) is a homology \( 3 \)-manifold. Put \( H_i := H_i(\Gamma; \mathbb{Z} \Sigma) \), then; 
\[ H_i = 0, H_1 = \mathbb{Z} \Sigma, \] and \( H_2 \) has no torsion by Lemma 1.1 p. 24. Poincare’ duality and [25] p. 244 Corollary 4 gives \( H_i = H_i \oplus \mathbb{Z}^{E_i} \), where \( t \) := torsion-submodule of \( o \).
So a \( \Sigma = \Gamma \) with a pure torsion \( H_1 = \mathbb{Z}^{E_i} \), say, gives us an example of a presumptive character sensitive Gorenstein complex. Examples of such orientable compact combinatorial manifold without boundary is given by the projective space of dimension \( 3, \mathbb{P}^3 \) and the lens space \( L(n, k) \text{ where } H_1(\mathbb{P}^3; \mathbb{Z}) = \mathbb{Z} \Sigma, \) and \( H_1(L(n, k); \mathbb{Z}) = \mathbb{Z} \Sigma \). So, \( \mathbb{P}^3 \) is \( \text{Gorenstein}_\Sigma \) for char \( k \neq 2 \text{ (char } k \neq n) \) while it is not even Buchsbaum for char \( k = 2 \text{ (char } k = n) \), cf. [21] p. 231-243 for details on \( \mathbb{P}^3 \) and \( L(n, k) \). Cf. [26] Prop. 5.1 p. 65 or [8] p. 75 for Gorenstein equivalences. A Gorenstein \( \Delta \) isn’t in general shellable, since if so, \( \Delta \) would be \( \text{CM}_\Sigma \) but \( L(n, k) \) and \( \mathbb{P}^3 \) isn’t. Indeed, in 1958 M.E. Rudin published An Unshellable Triangulation of the Tetrahedron.

Other examples are given by Jeff Weeks’ computer program “SnapPean hosted at http://thames.northnet.org/weeks/index/SnapPea.htm”, e.g. \( H_1(S^3(5, 1); \mathbb{Z}) = \mathbb{Z} \) for the old tutorial example of SnapPean \( S^3(5, 1) \), i.e. the Dehn surgery filling w.r.t. a figure eight complement with diffeomorphism kernel generated by \( (5,1) \). \( S^3(5, 1) \) is \( \text{Gorenstein}_\Sigma \) if char \( k \neq 5 \) but not even Bbm\( k \) if char \( k = 5 \), cf. [24] Ch. 9 for more on surgery.
III: Simplicial Manifolds

III:1 Definitions.

We will make extensive use of Proposition 1 p. 11 without explicit notification.

Definition 1. An n-dimensional pseudomanifold is a locally finite n-complex Σ such that;
(a) Σ is pure, i.e. the maximal simplices in Σ are all n-dimensional.
(b) Every (n - 1)-simplex of Σ is the face of at most two n-simplices of Σ.
(c) If s and s' are n-simplices in Σ, there is a finite sequence s = s_0, s_1, ..., s_m = s' of n-simplices in Σ such that s_i ∩ s_{i+1} is an (n - 1)-simplex for 0 ≤ i < m.

The boundary, BdΣ, of an n-dimensional pseudomanifold Σ, is the subcomplex generated by those (n - 1)-simplices which are faces of exactly one n-simplex in Σ.

Definition 2. Σ = •• is a quasi-0-manifold. Else, Σ is a quasi-n-manifold if it’s an n-dimensional, locally finite complex fulfilling;
(a) Σ is pure. (a is redundant since it is a consequence of γ by Lemma 2 p. 16.)
(b) Every (n - 1)-simplex of Σ is the face of at most two n-simplices of Σ.
(c) Lk_σ Σ is connected i.e. H_0(Lk_σ Σ; G) = 0 for all σ ∈ Σ, s.a. dim σ < n - 1.

The boundary with respect to G, denoted Bd_Σ Σ := σ ∈ Σ | H_n(Σ, cost_σ G; G) = 0, where G is a unital module over a commutative ring A. (β in Def. 1-2 ⇒ • and •• are the only 0-manifolds.)

Note 1. Σ is (locally) finite ⇐⇒ |Σ| is (locally) compact. By Th. 5 p. 10; if X is a homology n-manifold (n-hm) then X is a n-hm for any R-PID module G.

[25] p. 207-8 treats the classical standard case R = G = Z = {the integers}, which in our exposition more represents a particularly straightforward extreme case. A simplicial complex Σ is called a hm if |Σ| is, – now n = dim Σ.

From a purely technical point of view we really don’t need the “locally finiteness”-assumption, as is seen from Corollary p. 12.

Definition 3. (Let “manifold” stand for pseudo-, quasi- or homology manifold.) A compact n-manifold, S, is orientable, if H_i(S; G) = G. An n-manifold is orientable if all its compact n-submanifolds are orientable – else, non-orientable. Orientability is left undefined for ∅.

Definition 4. {B_j}^j_j is the set of strongly connected boundary components of Σ if {B_j}^j_j is the maximal strongly connected components of Bd_Σ Σ, from Definition 1 p. 15. (⇒ R_j^j pure and if σ is a maximal simplex in B_j, then Lk_σ Σ = Lk_σ = {∅_j}).

Note 2. ∅, {∅_j}, and 0-dimensional complexes with either one, •, or two, ••, vertices are the only manifolds in dimensions ≤ 0, and the [1-manifolds] are finite/infinite 1-circles and (half)lines, while [Σ is a quasi-2-manifold] ⇐⇒ [Σ is a homology 2-manifold]. Def. 1 γ is paraphrased by “Σ is strongly connected” and ••-complexes, though strongly connected, are the only non-connected manifolds. Note also that S^1 := {∅_j} is the boundary of the 0-ball, •, the double of which is the 0-sphere, ••. Both the (-1)-sphere {∅_j} and the 0-sphere •• has, as preferred, empty boundary.

• is the only compact orientable manifold with its boundary equal to {∅} ({∅}).

---

1For a classical manifold X ≠ • s.a. BdX = ∅; Bd_∅ X = ∅ if X is compact and orientable and Bd_∅ X = {∅} else.
III:2  Auxiliaries.

Lemma 1. For a finite n-pseudomanifold $\Sigma$:

i. (cf. [25] p. 206 Ex. E2.)

$H_n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}) = \mathbb{Z}$ and $H_{n-1}(\Sigma, \text{Bd}\Sigma; \mathbb{Z})$ has no torsion, or $H_n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}) = 0$ and the torsion submodule of $H_{n-1}(\Sigma, \text{Bd}\Sigma; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$.

ii. $\Sigma$ non-orientable $\iff$ $\Sigma$ non-orientable and $\text{Tor}(\mathbb{Z}, G) \neq G$.

So in particular: Manifolds are orientable w.r.t. $\mathbb{Z}_2$.

iii. $H^n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}) = \mathbb{Z} (\mathbb{Z}_2)$ when $\Sigma$ is (non-)orientable.

Proof. By conditions $\alpha$ and $\beta$ in Def. 1, a possible relative n-cycle in $C_\mathbb{Z}(\Sigma, \text{Bd}\Sigma; \mathbb{Z}_m)$ must include all oriented n-simplices all of which with coefficients of one and the same value. When the boundary function is applied to such a possible n-cycle the result is an $(n-1)$-chain that includes all oriented $(n-1)$-simplices, not supported by the boundary, all of which with coefficients 0 or $ \pm 2c \in \mathbb{Z}_m$. So, $H_n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}_m) = \mathbb{Z}$ for all $m$, and $H_n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}_m) = 0 \iff m \neq 2$. The Universal Coefficient Theorem (= Th. 5 p. 10) now gives:

\[
\mathbb{Z}_2 = H_n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}_2) \cong H_n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \oplus \text{Tor}_1(\mathbb{Z}, (\Sigma, \text{Bd}\Sigma; \mathbb{Z}), \mathbb{Z}_2)
\]

where the last homology module in each formula, by [25] p. 225 Cor. 11, can be substituted by its torsion submodule. Since $\Sigma$ is finite, $H_n(\Sigma, \text{Bd}\Sigma; \mathbb{Z}) = c_1 + c_2 + \ldots + c_n$ by The Structure Theorem for Finitely Generated Modules over PID, cf. [25] p. 9. Now, a simple check, using [25] p. 221 Example 4, gives i, which gives iii by [25] p. 244 Corollary. Theorem 5 p. 10 and i implies ii. $\square$

Proposition 1 p. 18 together with Proposition 1 p. 11 gives the next Lemma.

Lemma 2.i. $\Sigma$ is a n-hm $\iff$ $\Sigma$ is a quasi-manifold $\iff$ $\Sigma$ is an n-pseudomanifold.

ii. $\Sigma$ is a n-hm $\iff$ it’s a “Bbm” pseudomanifold and $H_n(\text{lk}_\Sigma G) = 0$ or $G \forall \sigma \neq \emptyset$. $\square$

The “only if”-part of Th. 10 was given for finite quasi-manifolds in [10] p. 166. Def. 1 p. 15 makes perfect sense even for non-simplex posets like $\Gamma \setminus \Delta$ (Def. 2 p. 16) which allow us to say that $\Gamma \setminus \Delta$ is or is not strongly connected (as a poset) depending on whether $\Gamma \setminus \Delta$ fulfills Def. 1 p. 15 or not. Now, for quasi-manifolds $\Gamma \setminus \Delta$ connected as a poset (Def. 2 p. 16) is equivalent to $\Gamma \setminus \Delta$ strongly connected which is a simple consequence of Lemma 1 p. 16 and the definition of quasi-manifolds, cf. [10] p. 165 Lemma 4. $\Sigma$ connected as a poset for any simplicial complex $\Sigma$ and any $\sigma \neq \emptyset$ and $\Sigma \text{Bd}\Sigma$ connected for any pseudomanifold $\Sigma$. The “if”-part of Theorem 10 can fail for an infinite $\Sigma$.

Theorem 10. If $G$ is a module over a commutative ring $A$ with unit, $\Sigma$ a finite n-pseudomanifold, and $\Delta \subset \Gamma \subset \Sigma$ and $\text{dim } \Gamma = \text{dim } \Sigma$ (Injectivity otherwise trivial!) then; $\Sigma \setminus \Delta$ is strongly connected $\iff H_n(\Sigma, \Delta; G) \rightarrow H_n(\Sigma, \Gamma; G)$ is an injection.

Proof. Each strongly connected n-component $\Gamma$, of $\Gamma$ is an n-pseudomanifold with $(\text{Bd}\Gamma)^{-1} \neq \emptyset$ since $\Gamma \subset \Sigma$. Now; $\text{Bd}\Gamma$ is all imbedded in $\Delta := \Gamma \setminus \Delta$ $\iff H_n(\Gamma, \Delta; G) \neq 0$, i.e. if every sequence from $\gamma \in (\Gamma \setminus \Delta)^+$ to $\sigma \in (\Sigma \setminus \Gamma)^+$ connects via $\Delta$. Now; the relative LHS w.r.t. $(\Sigma, \Gamma, \Delta)$ gives our claim. (Cp. the Jordan Curve Theorem.) $\square$
\[ \Gamma = \operatorname{cost}_\tau \tau \text{ and } \Delta = \operatorname{cost}_\tau \emptyset \text{ resp. } \operatorname{cost}_\tau \text{ gives } b \text{ in the next Corollary.} \]

**Corollary 1.** If \( \Sigma \) is an \( n \)-pseudomanifold and \( \sigma \subset \tau \in \Sigma \), then:

a. \( H_\ast (\Sigma, \operatorname{cost}_\tau \Sigma; G) \rightarrow H_n (\Sigma, \operatorname{cost}_\tau \Sigma; G) \) injective \iff \( \Sigma \setminus \operatorname{cost}_\tau \Sigma \) strongly connected.

b. \( H_n (\Sigma \setminus \alpha; G) \) is a finite \( n \)-manifold, \( \Delta = \operatorname{cost}_\tau \operatorname{cost}_\tau \Sigma = 0 \), if \( \alpha \in \operatorname{Int}(\tau) \) for any \( \tau \in \Sigma \). (Cp. the proof of Proposition 1 p. 11.) \( \square \)

**Note 1.** Corollary 1.a implies that the boundary of a any manifold is a subcomplex.

b implies that any simplicial manifold is "ordinary", def. p. 12, and that \( \operatorname{Bd}_G \Sigma \neq \emptyset \iff H_n (\Sigma; G) = 0 \).

**Corollary 2.** i. If \( \Sigma \) is an \( n \)-pseudomanifold with \( \# I \geq 2 \) then;

\[ H (\Sigma, \bigcup_B G) = 0 \text{ and } H (\Sigma, B; G) = 0. \]

ii. Both \( H (\Sigma; Z) \) and \( H (\Sigma, \operatorname{Bd}_G \Sigma; Z) \) equals 0 or \( Z \).

**Proof.** i. \( \bigcup_B C \subset \operatorname{cost}_\tau \sigma \) for some \( B \)-maxidimensional \( \sigma \in B \) and vice versa. \( \triangleright \)

ii. \( \dim \tau = n \Rightarrow [H (\Sigma, \operatorname{cost}_\tau \Sigma; G) = 0] \land [\operatorname{Bd}_G \Sigma \subset \operatorname{cost}_\tau \Sigma] \) and \( \operatorname{Bd}_G \Sigma \) strongly connected. Th. 10 and the LHS gives the injections \( H (\Sigma; G) \rightarrow H (\Sigma, \operatorname{Bd}_G \Sigma; G) \rightarrow G \). \( \square \)

**Note 2.** Gorenstein \( \Rightarrow \Sigma \) finite. \( [\Sigma, \Sigma \not\subset \Sigma \text{ locally finite}] \iff [\Sigma, \Sigma \not\subset \emptyset, \{\emptyset\} \text{ both finite}] \).

**Theorem 11.** i.a. (cp. [10] p. 168, [11] p. 32.) \( \Sigma \) is a quasi-manifold \iff \( \Sigma = \bullet \) or \( \Sigma \text{ is connected and } \operatorname{Lk}_G \Sigma \) is a finite quasi-manifold for all \( \emptyset \neq \sigma \in \Sigma \).

1.b. \( \Sigma \) is a homology\( n \)-manifold \iff \( \Sigma = \bullet \bullet \) or \( H_0 (\Sigma; G) = 0 \) and \( \operatorname{Lk}_G \sigma \) is a finite "CM\( n \)-homology\( n \)-manifold \( \forall \emptyset \neq \sigma \in \Sigma. \)

ii. \( \Sigma \) quasi-manifold \( \iff \operatorname{Bd}_G (\operatorname{Lk}_G \Sigma) = \operatorname{Lk}_G \sigma \) if \( \sigma \in \operatorname{Bd}_G \Sigma \) and \( \emptyset \) else.

iii. \( \Sigma \) is a quasi-manifold \iff \( \operatorname{Lk}_G \sigma \) is a pseudomanifold \( \forall \sigma \in \Sigma. \)

**Proof.** i. A simple check confirms all our claims for \( \dim \Sigma \leq 1 \), cf. Note 2 p. 23.

So, assume \( \dim \Sigma \geq 2 \) and note that \( \sigma \in \operatorname{Bd}_G \Sigma \iff \operatorname{Bd}_G (\operatorname{Lk}_G \sigma) \neq \emptyset. \)

i.a. (\( \Leftarrow \)) That \( \operatorname{Lk}_G \sigma \), with \( \dim \operatorname{Lk}_G \sigma = 0 \), is a quasi-0-manifold implies condition definition \( 2 \beta \) p. 23 and since the other "links" are all \( \text{connected condition } 2 \gamma \) follows. \( \triangleright \)

(\( \Rightarrow \)) Definition condition \( 2 \beta \) p. 23 implies that 0-dimensional links are \( \bullet \) or \( \bullet \bullet \) while Eq. I p. 30 gives the necessary connectedness of 'links of links', cp. Lemma 2 p. 16. \( \triangleright \)

i.b. Lemma 2. ii above plus Proposition 1 p. 18 and Eq. I p. 30

ii. Pureness is a local property, i.e. \( \Sigma \text{ pure } \iff \operatorname{Lk}_G \sigma \text{ pure.} \) Put \( n := \dim \Sigma. \) Now;

\[ \epsilon \in \operatorname{Bd}_G (\operatorname{Lk}_G \sigma) \iff 0 = H (\Sigma, \operatorname{Lk}_G \sigma; G) = \left[ \begin{align*}
& \text{Eq. I p. 30} \quad 0 = H (\Sigma, \operatorname{Lk}_G \sigma; G) = H (\Sigma, \operatorname{Lk}_G \sigma; G) \quad \text{and } \epsilon \in \operatorname{Lk}_G \sigma. \\
& \text{So;} \quad \epsilon \in \operatorname{Bd}_G (\operatorname{Lk}_G \sigma) \iff [\sigma \cup \epsilon] \in \operatorname{Bd}_G \Sigma \text{ and } \epsilon \in \operatorname{Lk}_G \sigma] \iff [\sigma \cup \epsilon] \in \operatorname{Bd}_G \Sigma \text{ and } \sigma \cap \epsilon = \emptyset] \iff [\epsilon \in \operatorname{Lk}_G \sigma].
\]

iii. (\( \Rightarrow \)) Lemma 2. i and i.a above. (\( \Leftarrow \)) All links are connected, except for \( \bullet \bullet \). \( \square \)
Corollary 1. For any quasi-n-manifold \( \Sigma \) except infinite 1-circles;
\[
\dim B_j \geq n - 2 \implies \dim B_j = n - 1.
\]

Proof. Check \( n \leq 1 \). Now; assume \( n \geq 2 \). If \( \dim \sigma = \dim B_i = n - 2 \) and \( \sigma \in B_i \) then;
\[
\text{Lk}_\sigma = \text{Lk}_B = \{\emptyset\}.
\]
By Th. 11; \( \text{Lk}_\sigma = \text{Bd}_G(\text{Lk}_\sigma) = \{\text{Lk}_B \text{is}, by \text{Th. 11}, \text{a finite quasi-}\}
\]
\( \text{n-manifold i.e. (a circle or) a line.} \) = (\( \emptyset \) or) \( \bullet \bullet \).
Contradiction! \( \square \)

Denote \( \Sigma \) by \( \Sigma_{ps}, \Sigma_i \) and \( \Sigma_n \) when it’s assumed to be a pseudo-, quasi- resp. a homology manifold. Note also that; \( \sigma \in \text{Bd}_{\Sigma_{ps}} \iff \text{Bd}(\text{Lk}_\sigma) = \text{Lk}_B \neq \emptyset \) by Th. 11.ii.

Corollary 2.i. Each boundary component \( B_j, j \in \mathbf{I} \), of \( \Sigma_n \) is a pseudomanifold.

ii. If \( \Sigma \) is finite with \( \text{Bd}\Sigma_{ps} = \bigcup B_i \) and \( -1 \leq \dim B_i < \dim \Sigma - 1 \) for some \( i \in \mathbf{I} \) then \( \Sigma_n \) is nonorientable.

iii. For any orientable quasi-n-manifold \( \Sigma \), each boundary component \( B_i := \text{Bd}_{\Sigma_{ps}} \neq \emptyset \) is an orientable \((n-1)\)-pseudomanifold without boundary.

iv. \( \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) = \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) = \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) \) i.e. even if \( G \neq G \).
Orientability is independent of \( G \), as long as \( \text{Tor}_1^G(\mathbf{Z}, G') \neq G' \) (Lemma 1.ii p. 24).
Moreover, \( \text{Bd}_\Sigma \) is always, while \( (\text{Bd}_\Sigma)^\alpha = (\text{Bd}_\Sigma)^\alpha \) and \( (\text{Bd}_\Sigma)^\alpha \) is \( (\text{Bd}_\Sigma)^\alpha \) except for infinite 1-circles in which case \( (\text{Bd}_\Sigma)^\alpha = \emptyset \).

v. \( \text{Tor}_1^G(\mathbf{Z}, G) = 0 \implies \text{Bd}_\Sigma = \text{Bd}_\Sigma \).

vi. \( \text{Bd}_\Sigma \) and \( \text{Bd}_\Sigma \) are equal with equality if \( \text{Bd}_\Sigma = 0 \) or \( \text{dim} \Sigma = n - 1 \) \( \forall j \in \mathbf{I}, \)
except if \( \Sigma \) is infinite and \( \text{Bd}_\Sigma \) and \( \{\emptyset\} \neq \emptyset = \text{Bd}_\Sigma \) (by Lemma 1.i+ii p. plus Th. 5 p. 10 since \( \emptyset \in \text{Bd}_\Sigma \neq \emptyset \) if \( \Sigma_n \) is infinite.). If \( \Sigma_n \text{Gorenstein}_2 \), then; \( \text{Bd}_\Sigma = \text{Bd}_\Sigma \).

Proof. i. The claim is true if \( \dim \Sigma \leq 1 \) and assume it’s true for dimensions \( \leq n - 1 \).
\( \alpha \) and \( \gamma \) are true by definition of \( B \), so only \( \beta \) remains. If \( \dim B = m \) and \( \sigma \in B \) with \( \dim \sigma = m - 1 \), then \( \text{Bd}(\text{Lk}_\sigma) = \text{Lk}_B \neq \text{Lk}_B \) where the r.h.s. is zero dimensional and so, strongly connected, implying, by the induction assumption, that the sole component on the l.h.s. is a 0-pseudomanifold i.e. \( \bullet \bullet \) or \( \bullet \bullet \).

\( \implies \dim B_j < n - 2 \) by Cor. 1 giving the 2:nd equality and Th. 10a gives the arrow in;
\[
\text{H}_*(\Sigma, \text{Bd}_\Sigma; G) = \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) = \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) \implies \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) = 0 \text{ if } \sigma \in \text{Bd}_\Sigma \wedge \text{Bd}_j.
\]

iii. We can w.l.o.g. assume that \( \Sigma_n \) is finite. If \( n - 2 = \dim \sigma \) and \( \sigma \in B_i \) then \( \dim \text{Lk}_\sigma = 1 \) and so, \( \text{Lk}_B \sigma = \text{Lk}_B(\text{Lk}_B(\sigma)) = \bullet \bullet \) gives \( \text{Bd}(B_j) = \emptyset \).

Now, from the boundary component definition we learn that, \( \dim B_j = n - 1 \forall j \in \mathbf{I} \implies \dim B_j \cap B_j < n - 2 \forall j \neq i \), which gives;
\[
\cdots \implies \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) \implies \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) \implies \text{H}_*(\Sigma, \text{Bd}_\Sigma; G) \implies \cdots.
\]
Corollary 1 and \[ \sigma \in \Sigma_1 \cap \text{Bd} \Sigma \text{ iff } \text{Lk} \sigma = \emptyset. \]

\[ \text{v. Bd}_1 \Sigma \ni \sigma \in \text{Bd}_2 \Sigma \Leftrightarrow H((\text{Lk} \sigma; \mathbb{G})) \neq 0 = H((\text{Lk} \sigma; \mathbb{Z})) \Leftrightarrow \text{Bd}_1 \text{Lk} \sigma = \emptyset \neq \text{Bd}_2 \text{Lk} \sigma, \]

iv \implies n - \# \sigma - 3 \geq m : \dim \text{Bd}_2 \text{Lk} \sigma \implies

\[ \implies 0 \neq H((\text{Lk} \sigma; \mathbb{Z})) = H((\text{Lk} \sigma; \mathbb{Z})) \otimes \mathbb{G} \oplus \text{Tor}_q^\mathbb{Z} H((\text{Lk} \sigma; \mathbb{Z}), \mathbb{G}) = 0 \] by assumption.

\[ = [\text{dimensional reasons}] = \text{Tor}_q^\mathbb{Z} H((\text{Lk} \sigma; \mathbb{Z}), \text{Bd}_2 \text{Lk} \sigma; \mathbb{Z}), \mathbb{G}) = 0. \text{ Contradiction! - since the torsion module of } H((\text{Lk} \sigma; \mathbb{Z}), \text{Bd}_2 \text{Lk} \sigma; \mathbb{Z}) \text{ is either } 0 \text{ or homomorphic to } \mathbb{Z}. \]

By Lemma 1 i p. 24. Now, use that only the torsion sub-modules matters in the torsion product by [25] Corollary 11 p. 225.

vi. iii and \text{Bd}_2 \Sigma \subseteq \text{Bd}_2 \Sigma, \text{ by Corollary 2 p. 25 plus Theorem 5 p. 10.} \quad \square

### III.3 Products and Joins of Simplicial Manifolds.

Let in the next theorem, when \( \forall, \) all through, is interpreted as \( \times, \) the word "manifold(s)" in 12.1 temporarily excludes \( \emptyset, \{\emptyset\}, \) and \( \bullet, \)

When \( \forall, \) all through, is interpreted as \( \ast \) let the word "manifold(s)" in 12.1 stand for finite "pseudo-manifold(s)", ("quasi-manifold(s)")'s, cf. [10] 4.2 pp. 171-2. We conclude, w.r.t. joins, that Th. 12 is trivial if \( \Sigma_1 \) or \( \Sigma_2 = \{\emptyset\} \) and else, \( \Sigma_1, \Sigma_2 \) must be finite since otherwise, there join isn’t locally finite, \( \epsilon = 0/1 \) if \( \forall = \times \ast \).

**Theorem 12.** If \( \mathbb{G} \) is an \( \mathbb{A} \)-module, \( \mathbb{A} \) commutative with unit, and \( V_i \neq \emptyset \) then:

12.1. \( \Sigma_1 \vee \Sigma_2 \) is a \( (n_1 + n_2 + \epsilon) \)-manifold \iff \( \Sigma_i \) is a \( n_i \)-manifold.

12.2. \( \text{Bd}(\bullet \times \Sigma) = \bullet \times (\text{Bd} \Sigma). \) Else; \( \text{Bd}(\Sigma_1 \vee \Sigma_2) = ((\text{Bd} \Sigma_1) \vee \Sigma_2) \cup (\Sigma_1 \vee (\text{Bd} \Sigma_2)). \)

12.3. If any side of 12.1 holds; \( \Sigma_1 \vee \Sigma_2 \) is orientable \iff \( \Sigma_1, \Sigma_2 \) are both orientable.

**Proof.** (12.1) [Pseudomanifolds] Lemma 4 p. 16. [Quasi-~] Lemma 3 + 4 p. 16. >

The rest of this proof could be substituted for a reference to the proof of Theorem 7 p. 13, Proposition 1 p. 11 and Corollary 2.iv p. 26.

12.2) [Quasi-manifolds] Put \( n := \dim \Sigma_1 \times \Sigma_2 = \dim \Sigma_1 + \dim \Sigma_2 = n_1 + n_2. \) The invariance of local \( \text{Homology} \) within \( \text{Int} \sigma \times \text{Int} \sigma \) implies, through Prop 1 p. 11, that w.l.o.g. we’ll only need to study simplices with \( c_{v} = 0 \) (Def. p. 15). Put \( v := \dim \sigma = \dim \sigma_{\alpha} + \dim \sigma_{\beta} = v_1 + v_2. \) We need to prove that; \( \sigma \in \text{Bd}_2 \Sigma_1 \times \Sigma_2 \iff \sigma_{\alpha} \in \text{Bd}_2 \Sigma_{1} \) or \( \sigma_{\beta} \in \text{Bd}_2 \Sigma_{2}. \) This follows from Lemma 2.ii p. 25, Th. 11 and Corollary 15 p. 14 with \( \mathbb{G}' := \mathbb{Z} \) which, after simplification through Note 1 p. 3, gives;

\[ \text{H}_{v_1}(\Sigma_1 ; \Sigma_2; \mathbb{G}) \cong \text{H}_{v_1}(\Sigma_1 ; \Sigma_2; \mathbb{Z}) \otimes \mathbb{Z} \text{H}_{v_2}(\Sigma_2; \mathbb{G}) \oplus \text{Tor}^{\mathbb{Z}}(\text{H}_{v_1}(\Sigma_1 ; \Sigma_2; \mathbb{Z}), \text{H}_{v_2}(\Sigma_2; \mathbb{Z}), \mathbb{G}). \] A similar reasoning holds also for pseudomanifolds with \( \sigma \) restricted to the submaximal simplices. Use Corollary 15 also for joins.

12.3) \( (\Sigma_1 \vee \Sigma_2, \text{Bd}(\Sigma_1 \vee \Sigma_2)) = (\Sigma_1 \vee \Sigma_2, \Sigma_1 \vee \text{Bd} \Sigma_2 \cup \text{Bd} \Sigma_1 \vee \Sigma_2) = \)

= (Def. p. 5) = (\( \Sigma_1, \text{Bd} \Sigma_2 \vee \Sigma_1 \).

By Cor. 2.iv p. 26 we can w.l.o.g. confine our study to pseudomanifolds, and choose the coefficient module to be, say, a field \( \mathbb{k} \) (\( \text{char} \mathbb{k} \neq 2 \)) or \( \mathbb{Z}. \) Since any finite max-dimensional submanifold, i.e. a submanifold of maximal dimension, in \( \Sigma_1 \times \Sigma_2 \) (\( \Sigma_1 \times \Sigma_2 \), cp. [29] (3.3) p. 59) can be embedded in the product (join) of two finite max-dimensional submanifolds, we confine, w.l.o.g., our attention to finite max-dimensional submanifolds \( \Sigma_1, \Sigma_2 \) of \( \Sigma_1, \Sigma_2 \). Now, use Eq. 1 p. 8 (Eq. 3 p. 10). \( \square \)
Example 1. For a triangulation \( \Gamma \) of a two-dimensional cylinder \( \text{Bd}_2 \Gamma = \text{two circles} \).
\[ \text{Bd}_2 \bullet = \{ \emptyset \} \). By Th. 12 \( \text{Bd}_2 (\Gamma \ast \bullet) = \Gamma \cup \{ \{ \text{two circles} \} \ast \bullet \} \). So, \( \text{Bd}_2 (\Gamma \ast \bullet) \), \( \mathbb{R}^1 \)-realizable as a pinched torus, is a 2-pseudomanifold but not a quasi-manifold.

2. “The boundary w.r.t. \( Z \) of the cone of the M"obius band” = \( \text{Bd}_2 (\mathcal{M} \ast \bullet) = (\mathcal{M} \ast \emptyset) \cup \{ \{ \text{circle} \} \ast \bullet \} = \mathcal{M} \cup \{ \text{2-disk} \} \) which is a well-known representation of the real projective plane \( \mathcal{P}^2 \). So, \( \text{Bd}_2 (\mathcal{M} \ast \bullet) \) is a homology 2-manifold with boundary \( \{ \emptyset \} \neq \emptyset \) if \( p \neq 2 \). \( \mathbb{Z}_p := \) the Prime-number field modulo \( p \).

\( \mathcal{P} \) # \( \mathcal{S}^2 = \mathcal{P} \cup \{ \{ \text{2-disk} \} \} \) confirms the obvious, i.e. that the n-sphere is the unit element w.r.t. the connected sum of two n-manifolds, cf. [21] p. 38ff + Ex. 3 p. 366. “\( \cup \)” is “union through homeomorphic identification of boundaries”.

\( \text{Bd}_2 (\mathcal{M} \ast \mathcal{S}^1) = (\mathcal{M} \ast \emptyset) \cup \{ \{ \text{circle} \} \ast \mathcal{S}^1 \} = \mathcal{S}^1 \). So, \( \text{Bd}_2 (\mathcal{M} \ast \mathcal{S}^1) = (\mathcal{S}^1 \ast \mathcal{M} \ast \emptyset) \cup \{ \mathcal{M} \ast \mathcal{S}^1 \} \) is a quasi-4-manifold without boundary, represented as the connected sum of two copies of \( (\mathcal{M} \ast \mathcal{S}^1) \).

3. Let \( \mathcal{P}^2 (\mathcal{P}^2) \) be a triangulation of the projective plane (projective space with \( \text{dim} \mathcal{P} = 4 \)) implying \( \text{Bd}_2 \mathcal{P}^2 \mathcal{P}^2 = \{ \emptyset \} \). So, by Th. 12, \( \text{Bd}_2 (\mathcal{P}^2 \ast \mathcal{P}^2) = \mathcal{P}^2 \cup \mathcal{P}^2 \).

\( p \neq 2 \) (\( \text{Bd}_2 (\mathcal{P}^2 \ast \mathcal{P}^2) = \emptyset \) if \( p = 2 \)), and \( \text{dim} \text{Bd}_2 (\mathcal{P}^2 \ast \mathcal{P}^2) = 4 \) while \( \text{dim} (\mathcal{P}^2 \ast \mathcal{P}^2) = 7 \), cf. Corollary 1 p. 26. \( \text{Bd}_2 (\mathcal{P}^2 \ast \bullet) = \bullet \) and \( \text{Bd}_2 (\mathcal{P}^2 \ast \bullet) = \mathcal{P}^2 \cup \bullet \).

4. \( E^n := \) the \( m \)-unit ball. With \( n := p + q, p, q \geq 0, \mathcal{S}^n = \text{Bd} E^{n-1} \simeq \text{Bd} (E^n \ast E^n) \simeq E^n \ast \mathcal{S}^{n-1} \ast \mathcal{S}^{n-1} \ast E^n \simeq \text{Bd} (E^{n+1} \ast E^n) \simeq E^{n+1} \ast E^n \ast \mathcal{S}^{n-1} \ast \mathcal{S}^{n-1} \ast E^n \) by Th. 12 and Lemma p. 14. Cp. [17] p. 198 Ex. 16 on surgery. \( S^n \ast \mathcal{S}^{n-1} \ast \mathcal{S}^{n-1} \ast E^n \) also hold.

5. \( \mathcal{P}^2 \) w.r.t. \( G := \mathbb{Z}_p \ast \mathbb{Z}_p \) is a non-orientable homology 2-manifold without boundary.

See also [21] p. 376 for some non-intuitive manifold examples. Also [27] pp. 123-131 gives insights on different aspects of different kinds of simplicial manifolds.

Proposition. If \( \Sigma_i \) is finite and \( -1 \leq \text{dim} B_{\Sigma_i} < \text{dim} \Sigma - 1 \) then \( \text{Lk}_{B_{\Sigma_i}} \) is non-orientable for all \( \delta \in B_{\Sigma_i} \). (Note that Cor. 2. ii p. 26 is the special case \( \text{Lk}_{\emptyset} \Sigma_i = \Sigma \).)

Proof. Use the proof of Cor. 2. ii p. 26 plus the end of Note 3 p. 25. \( \square \)

III:4 Simplicial Homology \( \mathbb{G} \) Manifolds and Their Boundaries.

In this section we’ll work mainly with finite simplicial complexes and though we’re still working with arbitrary coefficient modules we’ll delete those annoying quotation marks surrounding “CM\( \mathbb{G} \)”. The coefficient module plays, through the St-R ring functor, a more delicate role in commutative ring theory than it does here in our Homology theory, so when it isn’t a Cohen-Macaulay ring we can not be sure that a CM complex gives rise to a CM St-R ring.

Lemma 1. i. \( \Sigma \) is a homology \( \mathbb{G} \) manifold iff \( ([\Sigma = \bullet] \text{ or } \Sigma \text{ is connected } \text{ and } \text{Lk}_V \text{ is a finite CM\( \mathbb{G} \) pseudo manifold for all vertices } V \in V_\Sigma) \) and \( \text{Bd}_2 \Sigma_i = \text{Bd}_2 \Sigma_i \text{ or else; } \text{Bd}_2 \Sigma_i = \emptyset \neq \emptyset \neq \emptyset \neq \emptyset \text{ or } \text{Lk}_V \text{ is a CM\( \mathbb{G} \) pseudo manifold for all vertices } V \in V_\Sigma) \)\( \text{Bd}_2 \Sigma_i = \emptyset \neq \emptyset \text{ or } \text{Lk}_V \text{ is a CM\( \mathbb{G} \) pseudo manifold for all vertices } V \in V_\Sigma) \)\( \text{Bd}_2 \Sigma_i = \emptyset \neq \emptyset \text{ or } \text{Lk}_V \text{ is a CM\( \mathbb{G} \) pseudo manifold for all vertices } V \in V_\Sigma) \)\( \text{Bd}_2 \Sigma_i = \emptyset \neq \emptyset \text{ or } \text{Lk}_V \text{ is a CM\( \mathbb{G} \) pseudo manifold for all vertices } V \in V_\Sigma) \).

ii. If \( \Delta \) is a CM\( \mathbb{G} \) homology \( \mathbb{G} \) manifold then \( \text{Bd}_2 \Delta = \text{Bd}_2 \Delta \) for any module \( \mathbb{G} \), and so, for a homology \( \mathbb{G} \) manifold \( \Sigma, \text{Bd}_2 \Sigma = \emptyset \neq \emptyset \neq \emptyset \neq \emptyset \text{ or dim } B_{\Sigma} = (n-1) \) for each boundary component.

Proof. i. Prop. 1 p. 18, Lemma 2. ii p. 24 and the proof of Corollary 2. v p. 27.

ii. Use Theorem 5 p. 10 to prove the boundary equality and then, put \( \mathbb{G} = \mathbb{Z}_\Sigma \) in Corollary 2. iii p. 26. \( \square \)

Lemma 2. For a finite \( \Sigma, \delta \Sigma (\Sigma, B_\Sigma \Sigma; \mathbb{G}) \rightarrow (\Sigma, B_\Sigma \Sigma \Sigma; \mathbb{G}) \) in the relative M-Vs. w.r.t. \{ \{ \Sigma, B_\Sigma \}, (\Sigma, \Sigma \cup B_\Sigma) \} \) is injective if \( \text{#I} \geq 2 \) in Definition 4 p. 23.
So, \([H^i_\Sigma(\Sigma, \{\emptyset\}; G) = 0] \implies [H^i_\Sigma(\Sigma, Bd\Sigma; G) = 0\text{ or } Bd\Sigma \text{ is strongly connected}],\)
eq \] e.g., if \(\Sigma \neq \bullet, \bullet\) is a CM\(_n\) quasi-n-manifold. If \(\Sigma\) is a finite CM\(_n\) quasi-n-manifold then \(Bd\Sigma = \emptyset\) or it’s strongly connected and \(\dim (Bd\Sigma) = n_\Sigma - 1\) (by Lemma 1).

**Proof.** Use the relative M-Vs, w.r.t. \(\{(\Sigma, B_i), (\Sigma, \cup B_i)\}\), \(\dim (B_i \cap \cup B_i) \leq n_\Sigma - 3\) and Corollary 2.1 p. 25. □

**Theorem 13.i.** If \(\Sigma\) is a finite orientable\(_\Sigma\) CM\(_n\) homology\(_\Sigma\) \(-1\)-manifold with boundary then, \(Bd\Sigma\) is an orientable\(_\Sigma\) homology\(_\Sigma\) \((-1\)-manifold without boundary.

**ii.** Moreover; \(Bd\Sigma\) is Gorenstein\(_\Sigma\).

**Proof.** i. Induction over the dimension, using Th. 11 i,ii, once the connectedness of the boundary is established through Lemma 2, while orientability\(_\Sigma\) resp.
\(\text{dim}(Bd\Sigma) = 1\) follows from Cor. 2.iii-iv p. 26.

**ii.** \(\Sigma \ast \bullet\) is a finite orientable\(_\Sigma\) CM\(_n\) homology\(_\Sigma\) \((-1\)-manifold with boundary by Th. 12 + Cor. i p. 12.
\(Bd\Sigma(\Sigma \ast \bullet) = \{\emptyset\circ \text{or } \text{Th. 12, 2}\} = (\Sigma \ast Bd\Sigma(\bullet, \bullet)) \cup ((Bd\Sigma) \ast (\bullet, \bullet)) =
\Sigma \ast \emptyset \cup (Bd\Sigma) \ast (\bullet, \bullet) = (Bd\Sigma) \ast (\bullet, \bullet)\) where the l.h.s. is an orientable\(_\Sigma\) homology\(_\Sigma\) \(-n\)-manifold without boundary by the first part. So, \(Bd\Sigma\) is an orientable\(_\Sigma\) CM\(_n\) homology\(_\Sigma\) \((-1\)-manifold without boundary by Th. 12 i.e. it’s Gorenstein\(_\Sigma\). □

**Note.** \(\emptyset \neq \Delta\) is a \(2\)-CM\(_n\) homology\(_\Sigma\) manifold \(\iff \Delta = \text{core}\Delta\) is Gorenstein\(_\Sigma\) \(\iff \Delta\) is a homology\(_\Sigma\) sphere.

**Corollary 1.** (Cp. [17] p. 190.) If \(\Sigma\) is a finite orientable\(_\Sigma\) homology\(_\Sigma\) \(-1\)-manifold with boundary, so is \(2\Sigma\) except that \(Bd_2(2\Sigma) = \emptyset\). \(2\Sigma = "\text{the double of } \Sigma" := \Sigma \setminus \Sigma\) where \(\Sigma\) is a disjoint mirrored copy of \(\Sigma\) and "\(\ast \)" is "the union through identification of the boundary vertices". If \(\Sigma\) is CM\(_n\) then \(2\Sigma = 2\text{-CM}\(_n\).\)

**Proof.** Apply the M-Vs to the pair \((\text{lk}_\Sigma, \text{lk}_\Sigma)\) using Prop. 2. a p. 30 and then to \((\Sigma, \Sigma)\) for the CM\(_n\) case, or even simpler, use [14] p. 57 (23.6) Lemma, where also the non-relative augmental M-Vs is used. □

**Theorem 14.** If \(\Sigma\) is a finite CM\(_n\)-homology\(_\Sigma\) \(-1\)-manifold, then \(\Sigma\) is orientable\(_\Sigma\).

**Proof.** \(\Sigma\) finite CM\(_n\) \(\iff \Sigma\) CM\(_n\) for all prime fields \(\mathbb{Z}\) of characteristic \(\mathfrak{p}\), by (M.A. Reisner, 1976) induction over \(\dim \Sigma\), Theorem 5 p. 10 and the Structure Theorem for Finitely Generated Modules over PIDs. So, \(\Sigma\) is a finite CM\(_n\)-homology\(_\Sigma\) \(-n\)-manifold for any prime number \(\mathfrak{p}\) by Lemma 1.1, since \(Bd_2 \Sigma = Bd_2 \Sigma\) by Lemma 1.ii above. In particular, \(Bd_2 \Sigma\) is Gorenstein\(_\Sigma\) by Lemma 1.ii p. 24 and Theorem 13 above. If \(Bd_2 \Sigma = \emptyset\) then \(\Sigma\) is orientable\(_\Sigma\). Now, if \(Bd_2 \Sigma \neq \emptyset\) then \(\dim Bd_2 \Sigma = n_\Sigma - 1\) by Cor. 2.iii+iv p. 26 and, in particular, \(Bd_2 \Sigma = Bd_2 \Sigma\) is a quasi-\((-n-1\))-manifold.

\(Bd_2 (Bd_2 \Sigma) = \emptyset\) since \(Bd_2 \Sigma\) is Gorenstein\(_\Sigma\) and so, \(\dim Bd_2 (Bd_2 \Sigma) = n_\Sigma - 4\) by Cor. 1 p. 26. So if \(Bd_2(Bd_2 \Sigma) \neq \emptyset\) then, by Cor. 2 ii p. 26, \(Bd_2 \Sigma\) is nonorientable\(_\Sigma\) i.e. \(H_i (Bd_2 \Sigma; B(Hd_2 \Sigma; \mathbb{Z})) = H_i (Bd_2 \Sigma; \mathbb{Z}) = 0\) and the torsion submodule of \(H_i (Bd_2 \Sigma; B(Hd_2 \Sigma; \mathbb{Z})) = [\text{the dimensions reasons.}] = H_i (Bd_2 \Sigma; \mathbb{Z})\) is isomorphic to \(\mathbb{Z}\) by Lemma 1.1 p. 24. In particular, \(H_i (Bd_2 \Sigma; \mathbb{Z}) \otimes \mathbb{Z} \neq 0\) implying, by Th. 5 p. 10, that \(H_i (Bd_2 \Sigma; \mathbb{Z}) = H_i (Bd_2 \Sigma; \mathbb{Z}) = H_i (Bd_2 \Sigma; \mathbb{Z}) \otimes \mathbb{Z} \otimes \text{Tor}_i (H_i (Bd_2 \Sigma; \mathbb{Z}), \mathbb{Z}_p) \neq 0\) contradicting the Gorenstein\(_\Sigma\)-ness of \(Bd_2 \Sigma\). □

By Proposition 1 p. 18 we now get.

**Corollary 2.** Each simplicial homology\(_\Sigma\) \(-n\)-manifold \(\Sigma\) is locally orientable. □
APPENDIX: SIMPLICIAL CALCULUS AND SIMPLICIAL SETS

The complex, of all subsets of a simplex, is denoted $\hat{\sigma}$, while the boundary of $\sigma$, $\partial$, is the set of all proper subsets. $(\partial := \{ \tau \mid \tau \subsetneq \sigma \}) = \hat{\sigma} \setminus \{ \emptyset \}$ and $\emptyset = \emptyset$.

The closed star of $\sigma$ w.r.t. $\Sigma^\ast = \{ \tau \in \Sigma \mid \tau \subsetneq \emptyset \}$. The open star of $\sigma$ w.r.t. $\Sigma^\ast = \{ \alpha \in [\Sigma] \mid [v \in \sigma] \implies [\alpha(v) \neq 0] \}$. So, $\alpha \notin \text{st}_\Sigma \sigma$ except for $\text{st}_\emptyset \emptyset = [\Sigma]$. ($\text{st}_\emptyset (\sigma) = \{ \alpha \in [\Sigma] \mid \exists \epsilon \in \text{st}_\emptyset (\{ v \}) \}$.

The closure of $\sigma$ w.r.t. $\Sigma^\ast = \{ \alpha \in [\Sigma] \mid [\alpha(v) \neq 0] \implies [v \in \sigma] \}$. So, $|\emptyset| := \{ \alpha \}$. The interior of $\sigma$ w.r.t. $\Sigma^\ast = \{ \alpha \in [\Sigma] \mid [\alpha(v) \neq 0] \}$.

Int$(\sigma)$ is an open subspace of $[\Sigma]$. If $\text{st}_\Sigma \sigma$ is a maximal simplex in $\Sigma$ and $\text{Int}(\emptyset) := \{ \alpha \}$.

The barycenter $\hat{\sigma}$ of $\sigma$ is the $\alpha \in \text{Int}(\sigma)$ in $[\Sigma]$ fulfilling $\forall \in \sigma \Rightarrow \alpha(v) = \frac{1}{|\Sigma|}$ while $\emptyset := \alpha_0$.

The link of $\sigma$ w.r.t. $\Sigma^\ast = \text{Lk}_{\Sigma^\ast} := \{ \tau \in [\Sigma] \mid [\tau \cap \sigma = \emptyset ] \wedge [\tau \cap \sigma \in [\Sigma] \}$. So, $\text{Lk}(\emptyset) = [\Sigma]$, $\in \text{Lk}_k[\tau] \iff \tau \in \text{Lk}_k[\Sigma] \& \text{Lk}_k[\sigma] \notin [\Sigma]$, while $\text{Lk}_k[\tau] = [\emptyset \{ \#{ }^\ast \\text{maximal} ]$.

$\Sigma_1 \cup \Sigma_2 := \{ \sigma_1 \cup \sigma_2 | \sigma_1 \in \Sigma_i \} \ (i = 1, 2)$. In particular $\Sigma \cup \{ \emptyset \} = \{ \emptyset \} \ast [\Sigma] = [\Sigma]$.

**Proposition 1.** If $V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset$, then $[\tau \in \Sigma_1 \ast \Sigma_2] \iff [\exists! \sigma_i \in \Sigma_i \text{ so that } \tau = \sigma_1 \cup \sigma_2 ]$, (Direct from definition.) and $\text{Lk}_{\Sigma_1 \ast \Sigma_2}(\sigma_1 \cup \sigma_2) = (\text{Lk}\sigma_1 \ast (\text{Lk}\sigma_2))$. (Proved by bracket juggling.)

Any link is an iterated link of vertices and $\text{Lk}_k[\tau] = \{ \emptyset \} \ast \text{Lk}_k[\tau] = \text{Lk}_k[\sigma] \ast \text{Lk}_k[\tau] = \text{Lk}_k[\sigma \cup \tau] = \text{Lk}_k[\sigma \cup \tau] = \text{Lk}_k[\sigma \cup \tau]$. So, $\tau \notin \text{Lk}_k[\sigma] \Rightarrow \text{Lk}_k[\tau] = \emptyset$ while $\tau \in \text{Lk}_k[\sigma] \Rightarrow \text{Lk}_k[\tau] = \text{Lk}_k[\sigma \cup \tau] = \text{Lk}_k[\sigma \cup \tau] \ (\text{Lk}_k[\sigma \cap \text{Lk}_k \tau])$. (I).

$\tau \in \text{Lk}_k[\sigma] \iff \exists \sigma \in \text{Lk}_k[\sigma] = \exists \sigma \in \text{Lk}_k[\sigma] = \exists \sigma \in \text{Lk}_k[\sigma] = \text{Lk}_k[\tau] = \text{Lk}_k[\tau] = \text{Lk}_k[\tau] = \text{Lk}_k[\tau]$. (Proposition 1).

Put: $n := \dim \Delta$, $\Delta^{(n)} := \{ \delta \in \Delta \mid \# \delta \leq p + 1 \}$, $\Delta^{(n)} = \Delta^{(n)}$, $\Delta^{(n)} := \Delta^{(n)} \& \Delta^{(n)}$. $\Gamma \subset \Sigma$ is full in $\Sigma$ if for all $\sigma \in \Sigma$; $\sigma \subset V_\tau \Rightarrow \sigma \in \Gamma$.

**Proposition 2.** a) $\text{Lk}_k[\delta] = \text{Lk}_k[\delta] \cup \text{Lk}_k[\delta]$ , b) $\text{Lk}_k[\delta] = \text{Lk}_k[\delta] \cap \text{Lk}_k[\delta]$, c) $\text{Lk}_k[\delta] = \text{Lk}_k[\delta] \cup \text{Lk}_k[\delta]$, d) $\text{Lk}_k[\delta] = \text{Lk}_k[\delta] \cup \text{Lk}_k[\delta]$, e) $\text{Lk}_k[\delta] = \text{Lk}_k[\delta] \cup \text{Lk}_k[\delta]$. (iv holds also for topological spaces under the $\hat{\delta}$-join.)

$\Delta$ pure $\iff \text{Lk}_k[\delta]$ $\text{Lk}_k[\delta]$ $\emptyset \neq \delta \in \Delta$.

$\Gamma \cap \text{Lk}_k[\delta] = \text{Lk}_k[\delta] \forall \delta \in \Gamma \iff \Gamma$ is full in $\Delta$ $\iff \Gamma \cap \text{Lk}_k[\delta] = \text{Lk}_k[\delta] \forall \delta \in \Gamma$. (II)

**Proof.** a. Associativity and distributivity of the logical connectives. $\Rightarrow$ b. $(\implies)$ $\Delta$ pure $\implies \Delta$ pure, so $\text{Lk}_k[\delta] = \{ \tau \in \Delta \mid \tau \cap \delta = \emptyset \} \wedge (\text{Lk}_k[\delta]) = \{ \tau \in \Delta \mid \tau \cap \delta = \emptyset \} \wedge (\text{Lk}_k[\delta]) = \{ \tau \in \Delta \mid \# \tau \leq n \} = \{ \tau \in \Delta \mid \# \tau \leq n \} = \{ \tau \in \Delta \mid \# \tau \leq n \} = \{ \tau \in \Delta \mid \# \tau \leq n \} = \{ \tau \in \Delta \mid \# \tau \leq n \}$. $(\iff)$ If $\Delta$ non-pure, then $\exists \delta \in \Delta$ maximal in both $\Delta'$ and $\Delta$ i.e., $\text{Lk}_k[\delta] = \{ \emptyset \}$ $\emptyset \neq \{ \emptyset \} = \text{Lk}_k[\delta]$.
The contrast of $\sigma \in \Sigma = \text{cost}_x \sigma := \{ \tau \in \Sigma \mid \tau \not\supseteq \sigma \}$, cost$_x \| = \emptyset$ and cost$_x \sigma = \Sigma$ iff $\sigma \notin \Sigma$.

**Proposition 3.** Without assumptions whether $\delta, \delta \in \Delta$, or not, the following holds:

a. cost$_x (\delta \cup \delta) = \text{cost}_x \delta \cup \text{cost}_x \delta$ and $\delta = \{ \nu, \ldots, \nu \} \implies \text{cost}_x \delta = \bigcup_{\nu} \text{cost}_x \nu$.

b. $\text{cost}_x \delta \lambda = \text{cost}_x \delta \cap \text{cost}_x \lambda = \text{cost}_x \delta \lambda$ with $\lambda \subseteq \delta$. (Note: $\delta \subseteq \Delta \iff [\text{cost}_x \delta \lambda = \text{cost}_x \lambda = \emptyset = \emptyset \subseteq \text{cost}_x \delta \lambda]$.)

d. $\delta, \tau \in \Delta \implies [\delta \in \text{cost}_x \tau \iff \emptyset \subseteq \text{cost}_x \delta \subseteq \text{cost}_x \tau \iff \text{cost}_x \delta = \emptyset \subseteq \text{cost}_x \tau \implies \text{cost}_x \delta = \text{cost}_x \tau]$. (Note: $\delta \subseteq \Delta \iff [\emptyset \subseteq \text{cost}_x \delta \subseteq \text{cost}_x \tau \iff \text{cost}_x \delta \subseteq \text{cost}_x \tau \implies \text{cost}_x \delta = \text{cost}_x \tau]$.)

e. If $\delta \neq \emptyset$ then:

1. $(([\text{cost}_x \delta]') = \text{cost}_x \delta] \iff [n_\delta = n_\delta]$

2. $[\text{cost}_x \delta = \text{cost}_x \delta] \iff [n_\delta = n_\delta - 1]$

f. $\text{st}_x \delta \cap \text{cost}_x \delta = \text{cost}_x \delta = \delta \ast \text{Lk}_x \delta$.

**Proof.** If $\Gamma \subseteq \Delta$ then cost$_x \Gamma = \text{cost}_x \Gamma \cap \Gamma \forall \gamma$ and cost$_x (\delta \cup \delta) = \{ \tau \in \Delta \mid [\delta \cup \delta \subset \tau] \} = \{ \tau \in \Delta \mid [\delta \subset \tau] \cup [\delta \setminus \tau] \} = \text{cost}_\delta \cup \text{cost}_\delta \cap \text{cost}_x \delta$, giving a and b. A “brute force”-check gives c, the “$\tau = \{ \nu \}$”-case of d, while $\tau \in \text{Lk}_x \delta \implies [\nu \in \text{Lk}_x \delta \forall \nu \in \tau \in \Delta \iff \delta \in \text{Lk}_x \nu (\forall \text{cost}_x \nu) \forall \nu \in \tau \in \Delta$ gives d from a, c and Proposition 2a i above.

With $n_\delta := \dim \text{cost}_x \delta$ and $n_\delta := \dim \Delta < \infty$, we get: $\delta \subseteq \text{cost}_x \delta \Rightarrow n_\delta - 1 \leq n_\delta \leq n_\delta$ and $\forall \emptyset \neq \tau, \delta$. Now, for e; cost$_x \delta' = [\text{iff } n_\delta = n_\delta - 1] = (\text{cost}_x \delta) \cap \tau' = \text{cost}_x \delta \cap \delta \ast (\text{cost}_x \delta)' = = [\text{iff } n_\delta = n_\delta] = (\text{cost}_x \delta) \cap \tau'$, and f; $\text{st}_x \delta = \delta \ast \text{Lk}_x \delta = \delta \ast \text{Lk}_x \delta \cup \{ \tau \in \Delta \mid \delta \subset \tau \}$. (Note: $\delta \subseteq \Delta \iff [\text{cost}_x \delta \lambda \subseteq \text{cost}_x \lambda \iff \text{cost}_x \delta \lambda = \text{cost}_x \lambda \cup \{ \nu \} \subset \text{cost}_x \delta \lambda \implies \text{cost}_x \delta \lambda = \text{cost}_x \lambda]$.)

With $\delta, \tau \in \Sigma; \delta \cup \tau \notin \Sigma \iff \delta \notin \text{st}_x \Sigma \iff \text{st}_x \delta \cap \text{st}_x \tau = \{ \alpha \} \iff \text{st}_x \tau \cap \text{st}_x \delta = \{ \alpha \}$ and $\text{cost}_x \sigma \setminus \text{st}_x \sigma = [\delta \ast \text{Lk}_x \sigma] = [\text{st}_x \sigma \cap \text{cost}_x \sigma]$, by [21] p. 372, 62.6 and Proposition 3.f above.

(III) $\text{st}_x \sigma = \{ \tau \in \Sigma \mid \sigma \cup \tau \in \Sigma \} = \sigma \ast \text{Lk}_x \sigma$ and identifying $\text{cost}_x \sigma$ with its homeomorphic image in $[\Sigma]$ through; $\text{cost}_x \sigma \simeq [\Sigma] \setminus \text{st}_x \sigma$, we get $\text{st}_x \sigma \simeq [\Sigma] \setminus \text{cost}_x \sigma$. (The same is true also for joins.)

Definitions of the category of simplicial sets, former semi-simplicial complexes, usually uses the category of non-empty ordinals but not to comply with the introduction of the categories $D_\nu \left( K_\nu \right)$ in Ch. 2, we have to use the category of ordinals i.e. we include the empty ordinal $\Delta (\emptyset)$ in a consistent way. Denote an ordered simplicial complex $\Sigma$ when regarded as a simplicial set by $\tilde{\Sigma}$, and let $\tilde{\Sigma} \simeq \Sigma$ be the semi-simplicial product of $\Sigma$ and $\Sigma$, while $\uparrow \Sigma$ is the Milnor realization of $\Sigma$. [9] p. 160 Prop. 4.3.15 + p. 165 Ex. 1+2 gives; (The same is true also for joins.)

$\Sigma \times \Sigma \simeq \Sigma \times \Sigma \simeq \Sigma \times \Sigma \simeq \Sigma \times \Sigma \simeq \Sigma \times \Sigma$.

The Milnor realization $\uparrow \Sigma$ of any simplicial set $\Sigma$ is triangulable by [9] p. 209 Cor. 4.6.12. E.g: the augmental complex $\Delta (X)$ w.r.t. any topological space $X$, is a simplicial set and, cf. [19] p. 362 Th. 4, the map $j: \uparrow \Delta (X) \to X$ is a weak homotopy equivalence i.e. induces isomorphisms in homotopy groups, and $j$ is a true homotopy equivalence if $X$ is of homotopy CW-type, cf. [9] pp. 76, 189ff, 221-2.
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