NOTE ON THE NUMBER OF DIVISORS OF REDUCIBLE QUADRATIC POLYNOMIALS

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Abstract

Lapkova [‘On the average number of divisors of reducible quadratic polynomials’, J. Number Theory 180 (2017), 710–729] uses a Tauberian theorem to derive an asymptotic formula for the divisor sum \( \sum_{n \leq x} d(n+\nu) \) where \( \nu \) is a fixed integer and \( d(n) \) denotes the number of divisors of \( n \). We reprove this result with additional terms in the asymptotic formula, by investigating the relationship between this divisor sum and the well-known sum \( \sum_{n \leq x} d(n)d(n+\nu) \).

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1. Introduction

The problem of estimating the average number of divisors of a polynomial was first investigated in the middle of the last century. For example, Erdős [5] proved that for every irreducible polynomial \( P(n) \) with integer coefficients

\[
\sum_{n \leq x} d(P(n)) \asymp x \log x,
\]

where \( d(n) \) counts positive divisors of \( n \). The exact asymptotic formula for the sum \( \sum_{n \leq x} d(P(n)) \) where \( P(n) \) is a polynomial of degree greater than two is still unknown and seems to be a very difficult problem. However, the case of irreducible quadratic polynomials of degree two has been thoroughly investigated and it is known that

\[
\sum_{n \leq x} d(an^2 + bn + c) \sim \lambda x \log x
\]

(1.1)

for any irreducible polynomial \( ax^2 + bx + c \) with integer coefficients, where \( \lambda \) depends on \( a, b, c \). This is an unpublished result (mentioned by Bellman in [1]) due to Bellman and Shapiro, but the first published proof was given by Scourfield in [16]. For the

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case \(a = 1, b = 0\), the precise dependence of \(\lambda\) on \(a, b, c\) was described by Hooley in [8], and for other cases in the series of papers [12–14] by McKee. In general, \(\lambda\) depends on the class number of the quadratic field defined by \(P\), and hence does not admit a completely elementary description.

Bellman [1] also mentioned that there is an unpublished result due to Bellman and Shapiro that (1.1) holds with \(\log x\) replaced by \(\log^2 x\) for reducible quadratic polynomials. The first published proof for \(a = 1, b = 0\) and \(c = -1\) was given by the first author [4], who proved that

\[
\sum_{n \leq x} d((n^2 - 1)) \sim \frac{6}{\pi^2} x \log^2 x.
\]

(1.2)

Moreover, by using a different method, Lapkova extended the recent result of Cipu and Trudgian [2] concerning the case \(-b = c = 1\) and gave the explicit upper bound for the left-hand side of (1.1) with \(-b = c = 4^s, s \geq 0\), where the fastest growing term is exactly \(6/\pi^2\) \(x\log^2 x\) and agrees with (1.2). The explicit upper bounds for these kinds of sums with \(b = -c\) are important in searching for \(D(c^2)-m\)-tuples, namely sets of positive integers \(\{a_1, \ldots, a_m\}\) such that \(a_ia_j + c^2\) is a perfect square for all \(i, j\) with \(1 \leq i < j \leq m\).

In this paper, we will derive a more precise asymptotic formula for the sum \(\sum_{n \leq x} d((n - b)(n - c))\) and prove the following result. We use the standard notation, that is, \(\gamma\) denotes the Euler–Mascheroni constant, \(\zeta(s)\) denotes the Riemann zeta-function, \(\Lambda = \Lambda_1\) and \(\mu\) respectively denote the classical von Mangoldt and Möbius multiplicative functions and

\[
\Lambda_k(n) = \sum_{d|n} \mu(d) \left(\log \frac{n}{d}\right)^k.
\]

**Theorem 1.1.** For every positive integer \(v\) and every \(\varepsilon > 0\),

\[
\sum_{n \leq x} d(n(n + v)) = \frac{6}{\pi^2} x(\log^2 x + A_1(v) \log x + A_2(v)) + O(x^{2/3+\varepsilon})
\]

where
\[ A_1(v) = 4\gamma - 2 - 4\frac{\xi''}{\xi}(2) - 2 \sum_{e|v} \frac{\Lambda(e)}{e} \]

\[ A_2(v) = \left(2\gamma - 1 - 2\frac{\xi''}{\xi}(2)\right)^2 + 1 - 4\frac{\xi'''}{\xi}(2) + 4\left(\frac{\xi''}{\xi}(2)\right)^2 \]

\[- \left(4\gamma - 2 - 4\frac{\xi'}{\xi}(2)\right) \sum_{e|v} \frac{\Lambda(e)}{e} + 2 \sum_{e|v} \frac{\Lambda(e) \log e}{e} + \sum_{e|v} \frac{\Lambda_2(e)}{e}. \]

To prove Theorem 1.1, we shall follow Hooley’s suggestion (see [8]) to find the relation between the sums \(\sum_{n \leq x} d(n(n + v))\) and \(\sum_{n \leq x} d(n)d(n + v)\). The latter sum is well investigated and the problem of finding its asymptotic behaviour is known as the binary additive divisor problem. It was first studied by Ingham in relation to the fourth moment of the Riemann zeta function in [9]. Subsequently, in [10], Ingham proved that

\[ \sum_{n \leq x} d(n)d(n + v) = \frac{6}{\pi^2} \sigma_{-1}(v)x \log^2 x + O(x \log x), \]

where \(\sigma_{a}(n) = \sum_{d|n} d^a\). Ingham’s result was improved by Estermann [6], who showed that

\[ \sum_{n \leq x} d(n)d(n + v) = \frac{6}{\pi^2} \sigma_{-1}(v)x (\log^2 x + c_1(v) \log x + c_2(v)) + O(x^{11/12} \log^3 x), \quad (1.3) \]

where

\[ c_1(v) = 4\gamma - 2 - 4\frac{\xi'}{\xi}(2) - 4\frac{\sigma_{-1}^{(1)}}{\sigma_{-1}}(v) \]

\[ c_2(v) = \left(2\gamma - 1 - 2\frac{\xi''}{\xi}(2)\right)^2 + 1 - 4\frac{\xi'''}{\xi}(2) + 4\left(\frac{\xi''}{\xi}(2)\right)^2 \]

\[- 2\left(4\gamma - 2 - 4\frac{\xi'}{\xi}(2)\right)\frac{\sigma_{-1}^{(1)}}{\sigma_{-1}}(v) + 4\frac{\sigma_{-1}^{(2)}}{\sigma_{-1}}(v) \]

and \(\sigma_{a}(n) = \sum_{d|n} d^a \log^k d\). The best estimate for the error term in (1.3) is due to Deshouillers and Iwaniec [3], who showed that the error term is \(O(x^{2/3+\epsilon})\) for every \(\epsilon > 0\). This error term, through the method of this paper, appears in Theorem 1.1, because the proof of this theorem relies essentially on the following crucial lemma.

**Lemma 1.2.** For every \(v > 0\),

\[ \sum_{n \leq x} d(n)d(n + v) = \sum_{e|v} \sum_{n \leq x/e} d(n(n + v/e)) \quad (1.4) \]

and, in consequence,

\[ \sum_{n \leq x} d(n(n + v)) = \sum_{e|v} \mu(e) \sum_{n \leq x/e} d(n)d(n + v/e). \quad (1.5) \]
**Remark 1.3.** Thus, as noted, our proof relies on results from the binary additive divisor problem. Similar problems about the self-correlations of arithmetic functions (such as the left-hand side of (1.4)) can be very difficult. For the Möbius function, Chowla conjectured $\sum_{n \leq x} \mu(n + a_1) \cdots \mu(n + a_t) = o(x)$. The case $t = 1$ is already equivalent to the Prime Number Theorem. Larger $t$ values are related to the recent Möbius disjointness conjecture of Sarnak. For the von Mangoldt function, the asymptotic formula $\sum_{n \leq x} \Lambda(n)\Lambda(n + 2) \sim Ax$ is essentially the twin prime conjecture.

**Remark 1.4.** There is no serious obstacle to make the implied constant in (1.3) explicit. One can easily obtain the following explicit upper bound

$$\sum_{n \leq x} d(n(n + v)) \leq \frac{6}{\pi^2} x (\log^2 x + A_1(v) \log x + A_2(v)) + A_3(v) x^{11/12} \log^3 x,$$

where the constants $A_1$ and $A_2$ are defined as before and the constant $A_3$ can be explicitly computed from Estermann’s proof of (1.3). The above inequality improves known upper bounds in the sense that it holds for general polynomials and, what is more important, the first three leading terms agree with our asymptotic formula in Theorem 1.1, whereas known results (see [2, Lemma 5.2], [11, Theorem 3 and Corollary 4]) give an explicit upper bound where only the first leading term agrees with the asymptotic formula.

**2. The proof of Lemma 1.2**

First, notice that for every multiplicative function $f(n)$ and for all integers $a, b$,

$$f(a)f(b) = f(\gcd(a, b))f(\text{lcm}(a, b)).$$

To see this, it suffices to consider the case when $a = p^\alpha$, $b = p^\beta$ and then use the fact that $\{\min(\alpha, \beta), \max(\alpha, \beta)\} = \{\alpha, \beta\}$.

Now let us assume that $f(n)$ is a multiplicative function such that, for every prime $p$ and every positive integer $n$,

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) - f(p^{\alpha-1}).$$

(2.1)

We now prove by induction on $\alpha$ that for such a multiplicative function,

$$\sum_{m=0}^\beta f(p^{\alpha+\beta-2m}) = f(p^{\alpha})f(p^{\beta})$$

(2.2)

for all integers $\alpha, \beta$ with $0 \leq \beta \leq \alpha$. If $\alpha = 0$, then $\beta = 0$ and (2.2) holds trivially. If $\alpha = 1$, then $\beta = 0$ or $\beta = 1$. In the former case, (2.2) holds trivially, whereas the latter case needs (2.1) with $n = 2$.

Now, let us assume that (2.2) holds for $\alpha \leq A$ and all nonnegative integers $\beta \leq \alpha$, and consider $\alpha = A + 1$. If $\beta \leq A - 1$, then our assertion is implied by using (2.1) twice along with our inductive hypothesis. So it remains to consider the cases $(\alpha, \beta) = (A + 1, A)$ and $(\alpha, \beta) = (A + 1, A + 1)$. In the first case, it suffices to write the left-hand
side of (2.2) as $\sum_{m=0}^{A-1} f(p^{2A+1-2m})$ and apply (2.1). In the second case, we firstly apply (2.1) for $n = 1$ to write the left-hand side of (2.2) as $f(p)^2 + \sum_{m=0}^{A-1} f(p^{2A+2-2m})$, and then apply (2.1) again for $n = 2A + 1$.

Next, let us observe that for every multiplicative function satisfying (2.1),

$$f(a)f(b) = \sum_{e \mid \gcd(a,b)} f\left(\frac{ab}{e^2}\right).$$

Indeed, the above equation holds trivially when $\gcd(a, b) = 1$, so let us assume that $\gcd(a, b) = \prod_{j=1}^{k} p_j^{\alpha_j}$ for some positive integers $\alpha_j$, and $ab = q \prod_{j=1}^{k} p_j^{\beta_j}$ for some $\beta_j \geq 2\alpha_j$ and some integer $q$ coprime with the $p_j$. Then (2.2) gives

$$\sum_{e \mid \gcd(a,b)} f\left(\frac{ab}{e^2}\right) = f(q) \sum_{\{a_1, \ldots, a_k\} \subseteq \{0 \leq \alpha_j \leq \beta_j\}} f\left(p_j^{\alpha_j - 2a_j}\right) = f(q) \prod_{j=1}^{k} \sum_{a_j=0}^{\alpha_j} f\left(p_j^{\beta_j - 2a_j}\right)$$

$$= f(q) \prod_{j=1}^{k} f\left(p_j^{\alpha_j}\right) f\left(p_j^{\beta_j - \alpha_j}\right) = f\left(\gcd(a, b)\right) f\left(\text{lcm}(a, b)\right)$$

$$= f(a)f(b).$$

Since the multiplicative function $d(n)$ satisfies (2.1), one obtains the following lemma.

**Lemma 2.1.** Let $v$ be a positive integer. Then

$$d(n)d(n + v) = \sum_{e \mid \gcd(n,v)} d\left(\frac{n(n + v)}{e^2}\right).$$

Now we are ready to prove (1.4). Lemma 2.1 implies (1.4), since

$$\sum_{f \mid v} \sum_{n \leq \frac{v}{f}} d\left(n + \frac{v}{f}\right) = \sum_{f \mid v} \sum_{n \leq \frac{v}{f}} d\left(\frac{n(n + v)}{f^2}\right)$$

$$= \sum_{e \mid v} \sum_{n \leq \frac{v}{\gcd(n,v)}} \sum_{f \mid e} d\left(\frac{n(n + v)}{f^2}\right)$$

$$= \sum_{e \mid v} \sum_{n \leq \frac{v}{\gcd(n,v)}} d(n)d(n + v)$$

$$= \sum_{n \leq x} d(n)d(n + v).$$
On the other hand, one can easily deduce from (1.4) that

\[ \sum_{e \mid v} \mu(e) \sum_{n \leq x/e} d(n)d(n + v/e) = \sum_{e \mid v} \mu(e) \sum_{f \mid v/e} \sum_{n \leq x/ef} d(n(n + v/ef)) \]

\[ = \sum_{e \mid v} \sum_{n \leq x/e'} d(n(n + v/e')) \sum_{e \mid e'} \mu(e) \]

\[ = \sum_{n \leq x} d(n(n + v)). \]

**Remark 2.2.** The crucial property of a multiplicative function \( f(n) \) for the above reasoning is that it satisfies (2.1). In the literature there are many well-known multiplicative functions satisfying the similar identity

\[ f(p^{n+1}) = f(p)f(p^n) - g(p)f(p^{n-1}) \] (2.3)

for a suitable completely multiplicative function \( g \). Obviously, from our point of view, the case \( g \equiv 0 \) is not interesting as it implies that \( f \) is completely multiplicative, so let us assume that \( g \neq 0 \). Then, for example, \( \sigma_{\alpha} \) satisfies the above identity with \( g(p) = p^{\alpha} \). Moreover, it was noticed by Ramanujan, and proved by Mordell [15], that Ramanujan’s \( \tau \) function satisfies this identity with \( g(p) = p^{11} \), and more generally, (2.3) is true for normalised eigenforms of weight 2k with \( g(p) = p^{2k-1} \).

Using a similar argument as above, one can easily show that for every multiplicative function \( f(n) \) satisfying (2.3),

\[ \sum_{n \leq x} f(n)f(n + v) = \sum_{e \mid v} \mu(e) \sum_{n \leq x/e} f(n(n + v/e)) \]

and, in consequence, since every nonzero completely multiplicative function \( g \) is inverse to \( \mu_g \) with respect to the Dirichlet convolution,

\[ \sum_{n \leq x} f(n(n + v)) = \sum_{e \mid v} \mu(e)g(e) \sum_{n \leq x/e} f(n)f(n + v/e). \]

Hence, the asymptotic behaviour of \( \sum_{n \leq x} f(n(n + v)) \) can be deduced from the behaviour of \( \sum_{n \leq x} f(n)f(n + v) \) and vice-versa. For example, one can easily deduce from [7] that for \( \alpha > 0 \),

\[ \sum_{n \leq x} \sigma_{\alpha}(n(n + v)) = \frac{1}{2\alpha + 1} \frac{\zeta(\alpha + 1)^2}{\zeta(2\alpha + 2)} x^{2\alpha+1} \sum_{d \mid v} d^{-2\alpha-1} \sum_{e \mid d} \mu(e)e^{\alpha} + O(x^\omega \log^c x), \]

where \( \omega = 2\alpha + 1 - \min(\alpha, 1) \) and \( c = \begin{cases} 0, & \alpha > 1, \\ 1, & \alpha < 1, \\ 2, & \alpha = 1. \end{cases} \)
3. The proof of Theorem 1.1

First let us note that
\[
\sum_{e|v} \frac{\mu(e)}{e} \sigma_{(k)}(-v/e) = \sum_{d|e} \frac{\Lambda_k(d)}{d} \tag{3.1}
\]
and
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{e|v} \frac{\mu(e)}{e} \sigma_{-(k)} \left( \frac{v}{e} \right) (\log e)^{n-k} = \sum_{d|e} \left( \frac{\log d}{d} \right)^n \sum_{e|d} \mu(e) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases} \tag{3.2}
\]

Note that (1.3) together with (1.5) and (3.2) for \( n = 0 \) gives
\[
\sum_{n \leq x} d(n(n + v)) \sim \frac{6}{\pi^2} \sum_{e|v} \mu(e) \sigma_1 \left( \frac{v}{e} \right) \log^2 x = \frac{6}{\pi^2} x \log^2 x.
\]

Next, combining (1.3) with (1.5) yields
\[
A_1(v) = \frac{6}{\pi^2} \sum_{e|v} \frac{\mu(e)}{e} \left( c_1 \left( \frac{v}{e} \right) - 2 \log e \right)
\]
\[
= \frac{6}{\pi^2} \left( 4\gamma - 2 - 4 \frac{\zeta'}{\zeta}(2) - 4 \sum_{e|v} \frac{\mu(e)}{e} \sigma_1 \left( \frac{v}{e} \right) \right) - 2 \sum_{e|v} \frac{\mu(e)}{e} \sigma_1 \left( \frac{v}{e} \right) \log e.
\]

Thus, (3.1) and (3.2) for \( n = 1 \) give
\[
A_1(v) = \frac{6}{\pi^2} \left( 4\gamma - 2 - 4 \frac{\zeta'}{\zeta}(2) - 2 \sum_{e|v} \frac{\mu(e)}{e} \sigma_1 \left( \frac{v}{e} \right) \right)
\]
\[
= \frac{6}{\pi^2} \left( 4\gamma - 2 - 4 \frac{\zeta'}{\zeta}(2) - 2 \sum_{e|v} \frac{\Lambda(e)}{e} \right).
\]

Similarly, one can compute \( A_2(v) \). First let us note that
\[
A_2(v) = \frac{6}{\pi^2} \sum_{e|v} \frac{\mu(e)}{e} \sigma_{-1} \left( \frac{v}{e} \right) \left( \log^2 e - c_1 \left( \frac{v}{e} \right) \log e + c_2 \left( \frac{v}{e} \right) \right)
\]
\[
= \frac{6}{\pi^2} \left( \left( 2\gamma - 1 - 2 \frac{\zeta'}{\zeta}(2) \right)^2 + 1 - 4 \frac{\zeta''}{\zeta}(2) + 4 \left( \frac{\zeta'}{\zeta}(2) \right)^2 \right)
\]
\[
+ \sum_{e|v} \frac{\mu(e)}{e} \sigma_{-1} \left( \frac{v}{e} \right) \log^2 e + 4 \sum_{e|v} \frac{\mu(e)}{e} \sigma_1 \left( \frac{v}{e} \right) \log e + 4 \sum_{e|v} \frac{\mu(e)}{e} \sigma_{1} \left( \frac{v}{e} \right)
\]
\[
- \left( 4\gamma - 2 - 4 \frac{\zeta'}{\zeta}(2) \right) \sum_{e|v} \frac{\mu(e)}{e} \sigma_{-1} \left( \frac{v}{e} \right) \log e
\]
\[
- 2 \left( 4\gamma - 2 - 4 \frac{\zeta'}{\zeta}(2) \right) \sum_{e|v} \frac{\mu(e)}{e} \sigma_{1} \left( \frac{v}{e} \right). \]
Then, as in the case of $A_1(v)$, we see that the last two summands give

$$-\left(4\gamma - 2 - 4\frac{\zeta'}{\zeta}(2)\right) \sum_{v | e} \frac{\Lambda(e)}{e}.$$

Finally, (3.2) for $n = 2$ together with (3.1) and the fact that

$$\sum_{e | v} \frac{\mu(e)}{e} \sigma_{-1}(\frac{v}{e}) \log^2 e + \sum_{e | v} \frac{\mu(e)}{e} \sigma_{-1}(\frac{v}{e}) \log e = -\sum_{e | v} \frac{\Lambda(e) \log e}{e}$$

completes the proof.

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