Kinetic theory of electronic transport in random magnetic fields

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Abstract: We present the theory of quasiparticle transport in perturbatively small inhomogeneous magnetic fields across the ballistic-to-hydrodynamic crossover. In the hydrodynamic limit, the resistivity $\rho$ generically grows proportionally to the rate of momentum-conserving electron-electron collisions at large enough temperatures $T$. In particular, the resulting flow of electrons provides a simple scenario where viscous effects suppress conductance below the ballistic value. This new mechanism for $\rho \propto T^2$ resistivity in a Fermi liquid may describe low $T$ transport in single-band SrTiO$_3$.

1 Introduction

In a translation invariant medium, the electrical resistivity $\rho = 0$ at finite density. This is because part of the current is carried by the momentum, which is a conserved quantity. In a typical metal, $\rho > 0$ because momentum is not conserved. The resistivity is determined by the mechanisms of momentum relaxation. Surprisingly,

$$\rho \propto \frac{1}{\ell_{ee}(T)},$$

(1)
is found experimentally in many correlated electron metals, including Fermi [1, 2] and non-Fermi [3, 4] liquids; here $\ell_{ee}$ is the momentum-conserving mean free path. Umklapp [5] and/or Baber scattering [6] are the conventional explanations for (1); in both cases, electron-electron scattering becomes the limiting/dominant step in momentum relaxation. Yet some metals, such as single-band SrTiO$_3$ [7, 8], exhibit (1) without umklapp or Baber scattering. Perhaps there are novel routes to (1) beyond our conventional transport theory.

A simple breakdown of conventional transport theory occurs when the electrons flow collectively as a classical fluid [9]. One can then solve classical hydrodynamic equations to characterize transport [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Experiments have observed hints of hydrodynamic behavior in electronic transport measurements in some correlated metals [21, 22, 23, 24, 25]. However, a key prediction of the hydrodynamic transport theory is that in a simple Fermi liquid, $\rho \propto \ell_{ee} \propto T^{-2}$ (up to logarithms). Viscous “insulating” behavior is never seen in bulk resistivity measurements; the effect has only been observed in flows through narrow constrictions [26]. The observation of (1) in most correlated metals appears to rule out “textbook” hydrodynamic transport.

In some Fermi liquids, emergent hydrodynamic degrees of freedom are present, immediately short-circuiting viscous effects and leading to (1) [20, 27]. Unfortunately, the simplicity of the Fermi surface in SrTiO$_3$ makes this scenario unlikely. In this letter, we provide another simple hydrodynamic route to (1). Under rather general circumstances, (1) occurs when a viscous fluid flows through inhomogeneous magnetic fields. As many correlated metals, including heavy fermions [28, 29], iron-based superconductors [30], cuprates [31, 32, 33] and SrTiO$_3$ [34, 35, 36] have magnetic disorder, our theory provides a simple way to reconcile (1) with hydrodynamics in unconventional metals including single-band SrTiO$_3$.

One conventionally finds that viscous effects enhance conductance beyond the ballistic limit [10, 11, 17, 18, 19]. However, this is because conventional viscous electron flows move through chemical potential disorder, and not magnetic disorder. This work demonstrates that hydrodynamic transport phenomena are sensitive both to the emergent conservation laws and the nature of disorder. Our computation of the full ballistic-to-hydrodynamic crossover also demonstrates that hydrodynamic effects become visible in transport measurements even when $\ell_{ee}$ is larger than the apparent electron-impurity scattering length.

## 2 Boltzmann Transport

We consider perturbatively small magnetic fields which break translation symmetry. We assume that the resulting many-body Hamiltonian is corrected by a small perturbation

$$H_{\text{imp}} = -\delta \int d^d x \mathbf{J} \cdot \mathbf{a}_{\text{imp}}(x).$$

Here $\delta \ll 1$ is a bookkeeping parameter, $\mathbf{J}$ is the current operator and $\mathbf{a}_{\text{imp}}(x)$ is the magnetic vector potential. Using the memory-matrix formalism, one finds that to leading order in $\delta$, [37, 38, 39]

$$\rho_{ij} = \frac{\delta^2 e^2 n^2}{c^2 \pi^2} \int \frac{d^d k}{(2\pi)^d} k_i k_j a_{\text{imp}}^m(-k) a_{\text{imp}}^n(k) A_{ij}^{mn}(k)$$

with $-en$ the charge density and

$$A_{ij}^{mn}(k) = \lim_{\omega \to 0} \frac{\text{Im} \left( G_{i,j}^{R}(k,\omega) \right)}{\omega}.$$  

We emphasize that $G_{i,j}^{R}$ is evaluated in a homogeneous, disorder-free theory ($\mathbf{a}_{\text{imp}} = 0$).
Now consider a theory of weakly interacting fermionic quasiparticles of dispersion relation $\epsilon(p)$ in $d$ spatial dimensions. For simplicity, we neglect electron-phonon coupling and assume that the chemical potential $\mu$ can be chosen such that (i) umklapp is negligible, and (ii) $\mu \gg k_B T$, so that thermal fluctuations of the Fermi surface are mostly negligible. Henceforth, we will take $\hbar = k_B = 1$ for simplicity.

On long length scales compared to the Fermi wavelength $\lambda_F$, the dynamics of these quasiparticles is well described by (quantum) kinetic theory [40]. One can then use the (semi)classical Boltzmann equation to compute the retarded Green’s function in (4) [27]. It is sufficient to linearize the kinetic equations about an equilibrium distribution function

$$f_{eq}(x, p) \approx \frac{1}{1 + e^{\epsilon(p)/T}}. \quad (5)$$

The distribution function is then

$$f(t, x, p) \approx f_{eq}(x, p) - \frac{\partial f_{eq}}{\partial \epsilon} \Phi(t, x, p). \quad (6)$$

with $\Phi$ infinitesimally small. In this letter, we will often find it useful to employ Dirac notation for $p$-indices of $\Phi$: e.g. if $\Phi(x, p) = \sum_{\alpha} f_{\alpha}(x) g_{\alpha}(p)$, then we write $|\Phi(x)\rangle = \sum_{\alpha} f_{\alpha}(x) |g_{\alpha}\rangle$. The linearized Boltzmann equation can be written in the form

$$\partial_t |\Phi\rangle + L|\Phi\rangle + W|\Phi\rangle = 0, \quad (7)$$

where $L|\Phi\rangle$ corresponds to $v(p) \cdot \partial / \partial x$, and $v(p) = \partial \epsilon / \partial p$. $W$ is the linearized collision integral, which typically takes a complicated form; we will not compute it microscopically here. We assume time-reversal and inversion symmetries, so that $W$ is symmetric. Positivity of entropy production implies that $W$ is positive semi-definite. We define the inner product

$$\langle g_\alpha | g_\beta \rangle \equiv \int \frac{d^d p}{(2\pi)^d} \left( - \frac{\partial f_{eq}}{\partial \epsilon} \right) g_\alpha(p) g_\beta(p). \quad (8)$$

Writing $L(k)$ as the matrix $L$, with $\partial / \partial x \rightarrow i k$ (i.e., in Fourier space), one can show that [27]

$$A_{ij}^{JJ}(k) = \langle J_i | (W + L(k))^{-1} | J_j \rangle \quad (9)$$

where $|J_i\rangle$ refers to $J_i(p) = -e v(p)$. The kinetic limit justifies neglecting the coupling of random magnetic fields to spin in (2). These effects contribute to (7) at subleading order in $\hbar$.

## 3 Circular Fermi Surfaces

We can now compute $\rho_{ij}$ explicitly in any kinetic model where we can perform the matrix inverse in (9). One such “solvable” model is a two dimensional Fermi liquid with a circular Fermi surface, in a relaxation time approximation [17, 18, 19]. Since thermal fluctuations are neglected, $\Phi(x, p)$ can be approximated by $\Phi(x, \theta)$, with $\theta$ an angle on the circular Fermi surface. We expand in harmonics

$$\Phi = \sum_{n \in \mathbb{Z}} a_n(x) e^{in\theta}, \text{ or } |\Phi\rangle = \sum_{n \in \mathbb{Z}} a_n(x) |n\rangle. \quad (10)$$

If the electron-electron interactions are rotationally symmetric,

$$|W|n\rangle = \gamma_n |n\rangle. \quad (11)$$
Charge and momentum conservation enforce
\[ \gamma_{-1} = \gamma_0 = \gamma_1 = 0. \]  
(12)

For certain choices of \( \gamma_n \) for \(|n| \geq 2\), we can explicitly compute (9), using
\[ \langle m|L|n \rangle = \frac{v_F \nu}{2}(ik_x \pm k_y) \delta_{m,n \pm 1}. \]  
(13)

where \( \nu \) is the density of states, which arises due to the inner product \( \langle m|n \rangle = \nu \delta_{m,n} \).

Assuming the magnetic fields come from a density of \( n_{\text{imp}} \) random magnetic dipoles of strength \( m \), oriented normal to and placed a distance \( \xi \) above the electronic plane, we find
\[ \delta^2 \left( a^i_{\text{imp}}(-k)a^j_{\text{imp}}(k) \right)_{\text{dis}} = n_{\text{imp}} \left( \frac{\mu_0 |m|}{2} \right)^2 e^{-2|k| \xi} \delta_{ij}. \]  
(14)

Because of the disorder-averaged rotational invariance, we can replace \( A_{jj} \) in (9) with \( A_{jj} \delta_{ij} \), with
\[ A_{jj}(k) = e^2 v_F^2 \langle 1 | (W + L(k))^{-1} | 1 \rangle. \]  
(15)

Using (3), and assuming isotropy,
\[ \rho = \frac{n_{\text{imp}}}{4\pi} \left( \frac{\mu_0 |m|}{2e\nu} \right)^2 \int dk k^3 e^{-2|k| \xi} A_{jj}(k). \]  
(16)

The simplest model is a “relaxation time” model [41]:
\[ \gamma_n = \frac{v_F}{\ell_{ee}}, \]  
(17)

\( \ell_{ee} \) is the mean free path of momentum-conserving collisions. While it is unlikely that the collision integral takes this form in a 2d Fermi liquid [42, 43], this model correctly reproduces both ballistic and hydrodynamic limits. \( A_{jj} \) has already been computed in this model [17, 18, 27]; the result is
\[ A_{jj}(k) = e^2 v_F^2 \nu \frac{1 + \sqrt{1 + k^2 \ell_{ee}^2}}{k^2 \ell_{ee}}. \]  
(18)

Combining (16) and (18), \( \rho \) can be computed numerically; the result is shown in Figure 1. We emphasize that the qualitative \( \ell_{ee} \) dependence of the resistivity is not sensitive to the choice (14).

When \( \ell_{ee} \gg \xi \), \( A_{jj} \) is approximately independent of \( \ell_{ee} \), and the resulting resistivity is the \( T = 0 \) residual resistivity. The opposite limit \( \ell_{ee} \ll \xi \) is described by classical hydrodynamics. One can compute the resistivity in this limit by solving the textbook Navier-Stokes equations [12, 13, 16] in a slowly varying, perturbatively small inhomogeneous magnetic field:
\[ 0 = \partial_i v_i \]  
(19a)

\[ -enB_{ij} v_j = -en (E_i - \partial_i \mu_i) - \eta \partial_j \partial_i v_i. \]  
(19b)

\( B_{ij} \) is the spatial (magnetic) part of the electromagnetic tensor \( F_{\mu \nu} \), and \( n \) is the electron number density. One perturbatively solves this equation using the ansatz \( v = v_0 + v_1(x) + \cdots, \mu = \mu_1(x) + \cdots \), where \( v_0 \sim \delta^{-2}, v_1 \sim \delta^{-1} \), etc. At \( O(\delta^{-1}) \) these equations imply that
\[ v_{1i}(k) = -\frac{en}{\eta k^2} \left( \delta_{il} - \frac{k_i k_l}{k^2} \right) B_{lj}(k)v_{0j}. \]  
(20)

\[^1\text{The easiest way to derive this result is to note that the dipole field in real space is the curl of the Coulomb potential in electromagnetism in three spatial dimensions. The Fourier transform of the Coulomb potential \( \propto e^{-k\ell_0/k} \).} \]
Averaging (19b) over space at $O(\delta^0)$ we obtain

$$
\rho_{ij} = \frac{1}{\eta} \int \frac{d^d k}{(2\pi)^d} B_{ik}(-k) \left( \frac{\delta_{kl}}{k^2} - \frac{k_k k_l}{k^4} \right) B_{lj}(k).
$$

(21)

In the model (17), one finds $[18, 27]$

$$
\eta = \frac{n^2 \ell_{ee}}{4 \nu v_F}.
$$

(22)

Assuming isotropic $\rho_{ij}$, we observe that (21) and (22) are equivalent to (3) and (18), in the limit $k\ell_{ee} \to 0$.

Implicit in (19) is the assumption that only charge and momentum are independent long-lived quantities. Even in the presence of additional conserved quantities, there is often a contribution to $\rho$ of the form (21). An explicit example of this is a generalized version of (17), consisting of two disconnected Fermi surfaces of identical $\nu$, but differing Fermi velocities $v_{F,1}$ and $v_{F,2}$ [27]. The distribution function is now described by $|nA\rangle$, where $A = 1, 2$ denotes the pocket. Adding a term to $W$ to ensure that only total (but not relative) momentum is conserved [27], one can numerically compute $A_{J,j}(k)$ and $\rho$ for any value of $\ell_{ee}$. The results are plotted in Figure 1. In the hydrodynamic limit, one finds

$$
A_{J,j}(k) = \frac{2v_{F,2}^2 (v_{F,1}^2 + v_{F,2}^2)}{\ell_{ee} (v_{F,1}^4 + v_{F,2}^4) k^2}
$$

(23)
which is consistent with numerical data.

A final solvable model has [27]

\[
\gamma_n = \frac{v_F}{\ell_{ee}} \times \left\{ \begin{array}{ll}
    b & |n| = 2 \\
    1 & |n| > 2
\end{array} \right.
\]  

(24) generalizes to

\[
\mathcal{A}_{JJ}(k) = \frac{\nu}{v_F} \frac{2b - 1 + \sqrt{1 + k^2 \ell_{ee}^2}}{k^2 \ell_{ee}}.
\]  

(25)

Figure 1 again shows \( \rho \) as a function of \( b \) and \( \ell_{ee} \) in this model. When \( b = 1 \), we observe that \( \mathcal{A}_{JJ}(k) \) is a strictly increasing function of \( \ell_{ee} \), and so decreasing \( \ell_{ee} \) now decreases \( \rho \). For any \( 0 < b < 1 \), interactions ultimately suppress transport in the hydrodynamic limit, though \( \rho \propto 1/\eta \propto 2b/\ell_{ee} \) can have a very small coefficient relative to \( b = 1 \), which reduces to (17).

The above examples make clear that typically – but not always – \( \mathcal{A}_{JJ}(k) \) is a decreasing function of \( \ell_{ee} \). The single/double Fermi surface models both exhibit \( \partial \mathcal{A}_{JJ}/\partial \ell_{ee} < 0 \) in the presence of magnetic disorder, while (24) has \( \partial \mathcal{A}_{JJ}/\partial \ell_{ee} > 0 \). In contrast, in the presence of random potential disorder [27], \( \rho \) is an increasing function of \( \ell_{ee} \) in the simplest models of a single Fermi surface, while a decreasing function in the model with a long lived \( n = 2 \) harmonic, and the model of two Fermi surfaces.

4 General Principles

We now show that these examples are illustrative of a more general phenomenon. Intuitively, in a typical Fermi liquid, the current “overlaps” with the momentum operator:

\[
\mathbf{J} = \kappa \mathbf{P} + \cdots
\]

For a given wave number \( k \), we write \( \mathbf{P} = \cos \theta \mathbf{P}_\parallel + \sin \theta \mathbf{P}_\perp \) with \( \mathbf{P}_\parallel \cdot \mathbf{k} \neq 0, \mathbf{P}_\perp \cdot \mathbf{k} = 0 \). In a textbook fluid, this transverse momentum can only diffuse. As \( k \to 0 \), the spectral weight \( \mathcal{A}_{JJ} \) is dominated by the spectral weight of transverse momentum, which scales as [44, 45]

\[
\mathcal{A}_{JJ} \approx \kappa^2 \sin^2 \theta \mathcal{A}_{P_\perp P_\perp} \sim \frac{\sin^2 \theta}{\eta k^2} \sim \frac{\sin^2 \theta}{\ell_{ee} k^2}.
\]  

(26)

(In contrast, one typically finds \( \mathcal{A}_{P_\parallel P_\parallel} \sim \eta k^0 \).) (21) is a direct consequence of (26).

Let us make this intuition precise. We decompose the set of all \( |\Phi(p)\rangle \) into three groups such that

\[
\mathbf{W} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mathbf{W}_0
\end{pmatrix},
\]  

(27a)

\[
\mathbf{L} = \begin{pmatrix}
\mathbf{L}_1 & 0 & -\mathbf{L}_1^T \\
0 & 0 & \mathbf{L}_3 \\
\mathbf{L}_2 & \mathbf{L}_3 & \mathbf{L}_4
\end{pmatrix},
\]  

(27b)

with \( \mathbf{L}_1 \) invertible. The first row of \( |\Phi\rangle \) contains conserved quantities which mix among themselves under streaming (analogous to sound waves), the second row of \( |\Phi\rangle \) contains conserved quantities which only couple via streaming to non-conserved modes (purely diffusive hydrodynamic modes), and the third row of \( |\Phi\rangle \) corresponds to non-conserved modes. In the hydrodynamic limit \( k\ell_{ee} \ll 1 \), using the scalings \( \mathbf{L} \sim k \) and \( \mathbf{W}_0 \sim \ell_{ee}^{-1} \), along with block matrix inversion identities, we find

\[
(W + \mathbf{L})^{-1} \approx \begin{pmatrix}
\ell_{ee} & k^{-1} & k^{-1} \\
k^{-1} & \ell_{ee}^{-1} & k^{-1} \\
k^{-1} & k^{-1} & \ell_{ee}
\end{pmatrix}.
\]  

(28)

The notion of overlap can be made precise in the memory matrix formalism: see [39].
A_{jj} \sim \ell_{ee}^{-1}k^{-2}$ if and only if $|J_i\rangle$ has any overlap with a conserved quantity that only couples to non-conserved modes.

In both the single/two Fermi surface models, transverse momentum is conserved and only couples to higher order non-conserved even harmonics; thus $\rho \sim 1/\ell_{ee}$. For example, in the model (17), when $k_y = 0$ the three blocks are $\{|0\rangle, \{1\} + \{-1\} \}, \{|1\} - \{-1\} \}$, and $\{|j\rangle$ for all $|j| \ge 2\}$ respectively. In the model of an exactly conserved $j = 2$ harmonic, momentum and current are identical, and only couple to conserved modes through streaming: thus, with magnetic disorder, $\rho \propto \ell_{ee}$ in the hydrodynamic limit. We emphasize, however, that the $j = 2$ model is not generic.

In fact, transverse momentum $|P_i\rangle$ generically only couples to non-conserved modes. Suppose that all inversion-even conserved quantities $|\rho\rangle$ transform in the trivial representation of the point group $G$ of the crystal, while $|P_i\rangle$ and $|J_i\rangle$ transform under representations $R_J$ and $R_P$. Note that $\epsilon(p)$, and thus $f_{eq}$ and the inner product (8) are invariant under the action of $G$ on $p$. Suppose that $R_J \otimes R_P = 1 \oplus \cdots$ contains exactly one copy of the trivial representation $1$. An explicit computation [27] shows that

$$\langle J_i|P_j \rangle = -\epsilon d_{ij},$$

and so there is at least one copy. Schur’s orthogonality relation [46] then implies

$$\langle \rho|L(k_i)|P_j \rangle \sim \delta_{ij},$$

as we know from (29) that the trivial representation of the function $p_{vj}$ is found by taking the trace. $\langle P_k|L(k_i)|P_j \rangle = 0$ follows from inversion symmetry. We conclude that whenever $R_J \otimes R_P = 1 \oplus \cdots$, the streaming terms $L$ couple transverse momenta only to non-conserved inversion-even modes. Conservation of transverse momentum, together with (28), implies (1).

5 Application to SrTiO$_3$

Excellent $\rho \propto T^2$ scaling has recently been observed in SrTiO$_3$, even at carrier densities where there is a single small Fermi surface [7, 8], with no appreciable umklapp, Baber scattering or obvious “imbalance mode”. What we have shown in this letter is that this scaling can result from the interplay of electron-electron interactions and the magnetic impurities whose presence in SrTiO$_3$ is well-established [34, 35, 36].

In the weak disorder limit, contributions to $\rho(T)$ from magnetic vs. potential disorder add via Matthiesen’s rule. In simple models such as (17), $\rho_{\text{potential}}(T)$ is a decreasing function of $T$ and may fix the residual resistivity, which appears unrelated to the coefficient of $T^2$ [7, 8]. We predict that changing the density and/or strength of magnetic/non-magnetic impurities would most strongly affect the measured coefficients of $T^2/T^0$ in $\rho(T)$, respectively.

The resistivity of SrTiO$_3$ can also be measured to temperature $T \gtrsim T_F$, where thermal diffusion will also become important. A microscopic computation of the kinetic coefficients of SrTiO$_3$ could determine the quantitative impact of this additional hydrodynamic mode on $\rho(T)$.

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