Packing of Rigid Spanning Subgraphs and Spanning Trees

Joseph Cheriyan
Olivier Durand de Gevigney
Zoltán Szigeti

4 November 2013

Abstract

We prove that every \((6k + 2\ell, 2k)\)-connected simple graph contains \(k\) rigid and \(\ell\) connected edge-disjoint spanning subgraphs. This implies a theorem of Jackson and Jordán [6] providing a sufficient condition for the rigidity of a graph and a theorem of Jordán [8] on the packing of rigid spanning subgraphs. Both these results generalize the classic result of Lovász and Yemini [10] saying that every 6-connected graph is rigid. Our approach provides a transparent proof for this theorem.

Our result also gives two improved upper bounds on the connectivity of graphs that have interesting properties: (1) in every 8-connected graph there exists a packing of a spanning tree and a 2-connected spanning subgraph; (2) every 14-connected graph has a 2-connected orientation.

1 Introduction

In this paper, we consider sufficient conditions for the existence of a packing of spanning subgraphs in a given undirected graph \(G = (V, E)\), where by a packing we mean a set of pairwise edge-disjoint subgraphs of \(G\). Let us present a few examples in this area.

A first example is the existence of a packing of \(\ell\) spanning trees in every \(2\ell\)-edge-connected graph. This result is an easy consequence of the classic theorem of Tutte [12] and Nash-Williams [11] that characterizes the existence of such a packing. It is well known that this characterization can be derived from matroid theory as follows. The spanning trees of \(G\) correspond to the bases of the graphic matroid \(\mathcal{C}(G)\) of \(G\). Hence, by matroid union [4], the packings of \(\ell\) spanning trees of \(G\) correspond to the bases of the matroid \(\mathcal{N}_{0,\ell}\) defined as the union of \(\ell\) copies of \(\mathcal{C}(G)\). Thus, the existence of the required packing is characterized by the rank of \(E\) in \(\mathcal{N}_{0,\ell}\). Finally, using the formula of Edmonds [4] for the rank function of \(\mathcal{N}_{0,\ell}\) gives the theorem of Tutte and Nash-Williams.

A more recent example, due to Jordán [8], is the existence of a packing of \(k\) rigid spanning subgraphs in every \(6k\)-connected graph. The definition of rigidity is postponed to the next section but we mention here that the minimally rigid spanning subgraphs of \(G\) correspond to the bases of a matroid, namely the
rigidity matroid $\mathcal{R}(G)$ of $G$. So, as in the previous argument, the existence of a packing of $k$ rigid spanning subgraphs is characterized by the rank of $E$ in the matroid $N_{k,0}$ defined as the union of $k$ copies of $\mathcal{R}(G)$. Jordán [8] used the formula of Edmonds [4] for the rank function of $N_{k,0}$ to prove that $6k$-connectivity implies the desired lower bound on the rank of $E$.

Our main contribution is to provide a new example that gives a sufficient connectivity condition for the existence of a packing of $k$ rigid spanning subgraphs and $\ell$ spanning trees. To prove this result, we naturally introduce the matroid $N_{k,\ell}$ defined as the union of $k$ copies of the rigidity matroid $\mathcal{R}(G)$ and $\ell$ copies of the graphic matroid $\mathcal{C}(G)$.

As a packing of rigid spanning subgraphs turns out to be a packing of spanning 2-connected subgraphs, the packing result of Jordán [8] allowed him to settle the base case of a conjecture of Kriesell (see in [8]) on removable spanning trees and that of a conjecture of Thomassen [13] on orientation of graphs. Our result on the packing of rigid spanning subgraphs and spanning trees enables us to improve the results of Jordán on these conjectures.

2 Definitions

Let $G = (V, E)$ be a graph. For $X \subseteq V$, denote by $d_G(X)$ the degree of $X$, that is, the number of edges of $G$ with one end vertex in $X$ and the other one in $V \setminus X$. We say that $G$ is Eulerian if each vertex of $G$ is of even degree.

A graph $G' = (V', E')$ is a subgraph of $G$ if $V' \subseteq V$ and $E' \subseteq E$. The subgraph $G'$ is called spanning if $V' = V$. A set of pairwise edge-disjoint subgraphs of $G$ is called a packing.

Let $F \subseteq E$. We denote by $G_F$ the spanning subgraph of $G$ with edge set $F$, that is, $G_F = (V, F)$. Let us denote by $c(F)$ the number of connected components of $G_F$ and by $K_F$ the set of connected components of $G_F$ of size 1.

Let $T \subseteq V$. We denote by $F(T)$ the set of edges of $G_F$ induced by $T$. We say that $F$ is a $T$-join if the set of odd degree vertices of $G_F$ coincides with $T$. It is well known that if $G_F$ is a connected graph and $T$ is of even cardinality then $G_F$ contains a $T$-join.

For a collection $\mathcal{G}$ of subsets of $V$, we say that $(V, \mathcal{G})$ is a hypergraph. We denote by $V(\mathcal{G})$ the set of vertices that belong to at least one element of $\mathcal{G}$. We will use the following well-known fact:

\[
\text{the sum of the sizes of the elements of } \mathcal{G} \text{ is equal to the sum,}
\]
\[
\text{for each vertex, of the number of elements of } \mathcal{G} \text{ containing it.} \tag{1}
\]

A set $X$ of vertices is called connected in $(V, \mathcal{G})$ if, for any partition of $X$ into two non-empty parts, there exists an element of $\mathcal{G}$ intersecting both parts. In $(V, \mathcal{G})$ a connected component is a maximal connected vertex set. The number of connected components of this hypergraph is denoted by $c(\mathcal{G})$. Let $K_\mathcal{G}$ be the set of connected components of $(V, \mathcal{G})$ of size 1.

For $X \in \mathcal{G}$, we define the border $X_B$ of $X$ as the set of vertices of $X$ that belong to another element of $\mathcal{G}$, that is, $X_B = X \cap (\bigcup_{Y \in \mathcal{G} \setminus \{X\}} Y)$. We also define the inner part $X_I$ of $X$ as the set of vertices of $X$ that belong to no other
element of \( G \), that is, \( X_I = X \setminus X_B \). Let \( I_G \) be the set of elements of \( G \) whose inner part is not empty, that is, \( I_G = \{ X \in G : X_I \neq \emptyset \} \). Since every vertex of \( V(G) \) is contained in at least two elements of \( G \cup \{ X_I : X \in I_G \} \), we have, by (1),

\[
\sum_{X \in G} |X| + \sum_{X \in I_G} |X_I| \geq 2|V(G)|. \tag{2}
\]

A graph \( G = (V, E) \) is called rigid if \( \sum_{X \in G} (2|X| - 3) \geq 2|V| - 3 \) for every collection \( G \) of sets of \( V \) such that \( \{ E(X), X \in G \} \) partitions \( E \). More details about rigid graphs will be given in Section 4.

We will use the following connectivity concepts. The graph \( G \) is called \( p \)-edge-connected if \( d_G(X) \geq p \) for every non-empty proper subset \( X \) of \( V \). We say that \( G \) is \( p \)-connected if \( |V| > p \) and \( G - X \) is connected for all \( X \subset V \) with \( |X| \leq p - 1 \). As in [1], for a pair of positive integers \( (p,q) \), \( G \) is called \( (p,q) \)-connected if \( |V| > \frac{q}{q} \) and \( G - X \) is \( (p-q|X|) \)-edge-connected for all \( X \subset V \), that is, if for every pair of disjoint subsets \( X \) and \( Y \) of \( V \) such that \( Y \neq \emptyset \) and \( X \cup Y \neq V \), we have

\[
d_{G-X}(Y) \geq p - q|X|. \tag{3}
\]

For a better understanding we mention that \( G \) is \((6,2)\)-connected if \( G \) is 6-edge-connected, \( G - v \) is 4-edge-connected for all \( v \in V \) and \( G - \{u,v\} \) is 2-edge-connected for all \( u, v \in V \). It follows from the definitions that \( p \)-edge-connectivity is equivalent to \( (p,p) \)-connectivity. Moreover, since loops and parallel edges do not play any role in vertex connectivity, by the definition of \( (p,q) \)-connectivity, we have the following remark.

**Remark 1.** Every \( p \)-connected graph contains a \( (p,1) \)-connected simple spanning subgraph and \( (p,1) \)-connectivity implies \( (p,q) \)-connectivity for all \( q \geq 1 \).

Let \( D = (V, A) \) be a directed graph. We say that \( D \) is strongly connected if for every ordered pair \( (u,v) \in V \times V \) of vertices there is a directed path from \( u \) to \( v \) in \( D \). The digraph \( D \) is called \( p \)-arc-connected if \( D - F \) is strongly connected for all \( F \subseteq A \) with \( |F| \leq p - 1 \). We say that \( D \) is \( p \)-connected if \( |V| > p \) and \( D - X \) is strongly connected for all \( X \subset V \) with \( |X| \leq p - 1 \).

## 3 Results

Lovász and Yemini proved the following sufficient condition for a graph to be rigid.

**Theorem 1** (Lovász and Yemini [10]). Every 6-connected graph is rigid.

The following result of Jackson and Jordán is, by Remark 1, a sharpening of Theorem 1.

**Theorem 2** (Jackson and Jordán [6]). Every \((6,2)\)-connected simple graph is rigid.
Figure 1: A non-rigid $(6, 3)$-connected simple graph $G = (V, E)$. The collection $G$ of the four grey vertex-sets provides a partition of $E$. Hence, since \( \sum_{X \in G} (2|X| - 3) = 4(2 \times 8 - 3) = 52 < 53 = 2 \times 28 - 3 = 2|V| - 3 \), $G$ is not rigid. The reader can easily check that $G$ is $(6, 3)$-connected.

Note that in Theorem 2 the connectivity condition is the best possible since there exist non-rigid $(5, 2)$-connected simple graphs (see [10]) and non-rigid $(6, 3)$-connected simple graphs, for an example see Figure 1.

Jordán generalized Theorem 1 by giving the following sufficient condition for the existence of a packing of rigid spanning subgraphs.

**Theorem 3** (Jordán [8]). Let $k \geq 1$ be an integer. In every $6k$-connected graph there exists a packing of $k$ rigid spanning subgraphs.

The main result of this paper (Theorem 4) contains a common generalization of Theorems 2 and 3. It provides a sufficient condition to have a packing of rigid spanning subgraphs and spanning trees. The proof of Theorem 4 will be given in Section 5.

**Theorem 4.** Let $k \geq 1$ and $\ell \geq 0$ be integers. In every $(6k + 2\ell, 2k)$-connected simple graph there exists a packing of $k$ rigid spanning subgraphs and $\ell$ spanning trees.

Note that Theorem 4 applied for $k = 1$ and $\ell = 0$ provides Theorem 2. By Remark 1, every $6k$-connected graph contains a $(6k, 2k)$-connected simple spanning subgraph, thus Theorem 4 also implies Theorem 3. Let us see some corollaries of the previous results.

One can easily prove that rigid graphs with at least 3 vertices are 2-connected (see Lemma 2.6 in [7]) and so connected. Thus, Theorem 4 gives the following corollary.

**Corollary 1.** Let $k \geq 1$ and $\ell \geq 0$ be integers. In every $(6k + 2\ell, 2k)$-connected simple graph there exists a packing of $k$ 2-connected and $\ell$ connected spanning subgraphs.

Corollary 1 allows us to improve two results of Jordán [8]. The first one deals with the following conjecture of Kriesell, see in [8].
**Conjecture 1** (Kriesell). For every positive integer \( p \), there exists a (smallest) integer \( f(p) \) such that every \( f(p) \)-connected graph \( G \) contains a spanning tree \( T \) for which \( G - E(T) \) is \( p \)-connected.

As Jordán [8] pointed out, Theorem 3 answers this conjecture for \( p = 2 \) by showing that \( f(2) \leq 12 \). Corollary 1 applied for \( k = 1 \) and \( \ell = 1 \) directly implies that \( f(2) \leq 8 \).

**Corollary 2.** Every 8-connected graph \( G \) contains a spanning tree \( T \) such that \( G - E(T) \) is 2-connected.

The other improvement deals with the following conjecture of Thomassen.

**Conjecture 2** (Thomassen [13]). For every positive integer \( p \), there exists a (smallest) integer \( g(p) \) such that every \( g(p) \)-connected graph \( G \) has a \( p \)-connected orientation.

By applying Theorem 3 and an orientation result of Berg and Jordán [2], Jordán [8] proved the conjecture for \( p = 2 \) by showing that \( g(2) \leq 18 \). Applying the same approach, that is, using a packing theorem (Corollary 1) and an orientation theorem (Theorem 5), we can prove a more general result (Corollary 3) that, in turn, implies \( g(2) \leq 14 \).

**Theorem 5** (Király and Szigeti [9]). An Eulerian graph \( G = (V, E) \) has an orientation \( D \) such that \( D - v \) is \( p \)-arc-connected for all \( v \in V \) if and only if \( G - v \) is \( 2p \)-edge-connected for all \( v \in V \).

Corollary 1 and Theorem 5 imply the following corollary which, specialized for \( p = 1 \), gives, by Remark 1, the claimed upper bound for \( g(2) \).

**Corollary 3.** Every simple \((12p + 2, 4p)\)-connected graph \( G \) has an orientation \( D \) such that \( D - v \) is \( p \)-arc-connected for all \( v \in V \).

**Proof.** Let \( G = (V, E) \) be a simple \((12p + 2, 4p)\)-connected graph. By Theorem 5 it suffices to prove that \( G \) contains an Eulerian spanning subgraph \( H \) such that \( H - v \) is \( 2p \)-edge-connected for all \( v \in V \). By Corollary 1, in \( G \) there exists a packing of \( 2p \) 2-connected spanning subgraphs \( H_i = (V, E_i) \) \((i = 1, \ldots , 2p)\) and a spanning tree \( F \). Define \( H' = (V, \bigcup_{i=1}^{2p} E_i) \). For all \( i = 1, \ldots , 2p \), since \( H_i \) is 2-connected, \( H_i - v \) is connected; hence \( H' - v \) is \( 2p \)-edge-connected for all \( v \in V \). Let \( T \) be the set of vertices of odd degree in \( H' \) and \( F' \) a \( T \)-join in the tree \( F \). Now, adding the edges of this \( T \)-join \( F' \) to \( H' \) provides the required spanning subgraph of \( G \).

Finally, we mention the following conjecture of Frank that would imply \( g(2) = 4 \).

**Conjecture 3** (Frank [5]). A graph has a 2-connected orientation if and only if it is \((4, 2)\)-connected.

### 4 Preliminaries

Let \( G = (V, E) \) be a graph. In this section we present some simple facts about the graphic matroid \( \mathcal{C}(G) \), the rigidity matroid \( \mathcal{R}(G) \) and the matroid \( \mathcal{N}_{k,\ell}(G) \).
introduced in the Introduction.

We will denote by $C(G)$ the graphic matroid of $G$ on ground-set $E$, that is an edge set $F$ of $G$ is independent in $C(G)$ if and only if $G_F$ is a forest. Let $n = |V|$ be the number of vertices in $G$. It is well known that the rank function $r_C$ of $C(G)$ satisfies the following:

$$r_C(F) = n - c(F). \quad (4)$$

We will denote by $R(G)$ the rigidity matroid of $G$ on ground-set $E$ with rank function $r_R$ (for a definition we refer the reader to [10]). For $F \subseteq E$, by a theorem of Lovász and Yemini [10], we have

$$r_R(F) = \min \sum_{X \in G} (2|X| - 3), \quad (5)$$

where the minimum is taken over all collections $G$ of subsets of $V$ such that $\{F(X), X \in G\}$ partitions $F$. Note that

$$r_R(E) \leq 2|V| - 3 \quad (6)$$

and equality holds if and only if $G$ is rigid.

For a subset $F$ of $E$, let $G$ be a collection of subsets of $V$ such that $\{F(X), X \in G\}$ partitions $F$ that minimizes the right hand side of (5). It is well known that each element of $G$ induces a rigid subgraph of $G_F$. (For example, see the proof of Lemma 2.4 in [7].) Note also that, if $G$ is simple, then every element of $G$ of size 2 induces at most one (in fact exactly one) edge and contributes exactly one to the sum. So we have the following simple but very useful observation.

**Remark 2.** If $G$ is simple, then

$$r_R(F) = \min \sum_{X \in H} (2|X| - 3) + |F \setminus H|, \quad (7)$$

where the minimum is taken over all subsets $H \subseteq F$ and all collections $H$ of subsets of $V$ such that $\{F(X), X \in H\}$ partitions $H$ and each element of $H$ induces a rigid subgraph of $G_H$ of size at least 3.

The following claim provides insight into the structure of the minimizers of (7).

**Claim 1.** Let $G = (V, E)$ be a simple graph and $F \subseteq E$. Let $H \subseteq F$ and $H$ be a collection of subsets of $V$ that minimize the right hand side of (7).

(i) For every $H^* \subseteq H$, $r_R(\cup_{X \in H^*} F(X)) = \sum_{X \in H^*} (2|X| - 3)$.

(ii) For every non-empty $H^* \subseteq H$, there exists a vertex in $V(H^*)$ that is contained in a single element of $H^*$.

(iii) $|I_H| + |K_H| \geq c(H)$.

(iv) The connected components of $(V, H)$ and those of $G_H$ coincide.
Proof. (i) Since $\{F(X), X \in \mathcal{H}\}$ partitions $\mathcal{H}$, we have, by (7) and subadditivity of $r_R$, 
\[
\sum_{X \in \mathcal{H}} (2|X| - 3) + |F \setminus \mathcal{H}| = r_R(F)
\]
\[
\leq r_R(\cup_{X \in \mathcal{H}} F(X)) + r_R(\cup_{X \in \mathcal{H} \setminus \mathcal{H}^*} F(X)) + r_R(F \setminus \mathcal{H})
\]
\[
\leq \sum_{X \in \mathcal{H}^*} r_R(F(X)) + \sum_{X \in \mathcal{H} \setminus \mathcal{H}^*} r_R(F(X)) + |F \setminus \mathcal{H}|
\]
\[
\leq \sum_{X \in \mathcal{H}^*} (2|X| - 3) + \sum_{X \in \mathcal{H} \setminus \mathcal{H}^*} (2|X| - 3) + |F \setminus \mathcal{H}|.
\]
So equality holds everywhere and (i) follows.

(ii) By contradiction, suppose that every vertex of $V(\mathcal{H}^*)$ is contained in at least two elements of $\mathcal{H}^*$. Hence, by (5), (i), since the size of each element of $\mathcal{H}^*$ is at least 3 and by (1), we have $2|V(\mathcal{H}^*)| - 3 \geq r_R(\cup_{X \in \mathcal{H}^*} F(X)) = \sum_{X \in \mathcal{H}^*} (2|X| - 3) = \sum_{X \in \mathcal{H}^*} |X| + \sum_{X \in \mathcal{H}^*}. (|X| - 3) \geq 2|V(\mathcal{H}^*)| + 0$, a contradiction.

(iii) Let $C$ be a connected component of $(V, \mathcal{H})$ that is not in $\mathcal{K}_H$ and $\mathcal{H}^*$ the elements of $\mathcal{H}$ contained in $C$. By (ii), there exists in $C$ a vertex $v$ contained in a single element $X$ of $\mathcal{H}^*$. Hence, by definition of $\mathcal{H}^*$, $v \in X_I$ and so $X \in \mathcal{I}_H$. Thus we proved that $C$ contains an element of $\mathcal{I}_H$. Since the connected components of $(V, \mathcal{H})$ are disjoint, (iii) follows.

(iv) Let $U$ be a connected component of $G_H$ and $\emptyset \neq W \subseteq U$. Then, there exists an edge of $H$ with one end in $W$ and the other end in $U \setminus W$. Since $\{F(X), X \in \mathcal{H}\}$ partitions $\mathcal{H}$, this edge is contained in an element of $\mathcal{H}$ that intersects both $W$ and $U \setminus W$. So $U$ is connected in $(V, \mathcal{H})$.

Let $U$ be a connected component of $(V, \mathcal{H})$ and $W \subseteq U$. Then, there exists an element $X$ of $\mathcal{H}$ intersecting both $W$ and $U \setminus W$. Since $X \subseteq U$ and $X$ induces a rigid, and so connected, subgraph of $G_H$, there exists an edge of $H$ with one end in $X \setminus W \subseteq W$ and the other in $X \setminus W \subseteq U \setminus W$. So $U$ is connected in $G_H$. This ends the proof of (iv). $\blacksquare$

As we mentioned in the Introduction, to have a packing of $k$ rigid spanning subgraphs and $\ell$ spanning trees in $G$, we must find $k$ bases in the rigidity matroid $\mathcal{R}(G)$ and $\ell$ bases in the graphic matroid $\mathcal{C}(G)$ all pairwise disjoint. To do that we will need the following matroid. For $k \geq 0$ and $\ell \geq 0$, define $\mathcal{N}_{k,\ell}(G)$ as the matroid on ground-set $E$, obtained by taking the matroid union of $k$ copies of the rigidity matroid $\mathcal{R}(G)$ and $\ell$ copies of the graphic matroid $\mathcal{C}(G)$. Let $r_{k,\ell}$ be the rank function of $\mathcal{N}_{k,\ell}(G)$. By a theorem of Edmonds [4], for the rank of matroid unions,
\[
r_{k,\ell}(E) = \min_{F \subseteq E} kr_{\mathcal{R}}(F) + \ell r_{\mathcal{C}}(F) + |E \setminus F|.
\]
Observe that
\[
r_{k,\ell}(E) \leq k r_{\mathcal{R}}(E) + \ell r_{\mathcal{C}}(E) \leq k(2n-3) + \ell(n-1).
\]
Jordán [8] used the matroid $N_{k,0}(G)$ to prove Theorem 3 and pointed out that using $N_{k,\ell}(G)$ one could prove a theorem on the packing of rigid spanning subgraphs and spanning trees. We tried to fulfill this gap by following the proof of [8] but we failed. To achieve this aim we had to find a new proof technique.

5 Proofs

In this section we provide the proofs of our results. Let us first demonstrate our proof technique by giving a transparent proof for Theorems 1 and 2. We emphasize that in the first two proofs we use only Remark 2 from the Preliminaries.

Proof of Theorem 1. By Remark 1, we may assume that $G$ is simple. Then, by (7), there exist a subset $H \subseteq E$ and a collection $\mathcal{H}$ of subsets of $V$ of sizes at least 3 such that $\{ E(X), X \in \mathcal{H} \}$ partitions $H$ and $r_R(E) = \sum_{X \in \mathcal{H}} (2|X| - 3) + |E \setminus H|$. If $V \in \mathcal{H}$, then $r_R(E) \geq 2|V| - 3$, hence, by (6), $G$ is rigid. So in the following we assume that $V \notin \mathcal{H}$ and find a contradiction.

Recall that, for $X \in \mathcal{H}$,

\[
X_B = X \cap (\cup_{Y \in \mathcal{H} - X} Y), \quad X_I = X \setminus X_B \quad \text{and} \quad I_H = \{ X \in \mathcal{H} : X_I \neq \emptyset \}.
\]

Each edge of $H$ being induced by an element of $\mathcal{H}$, it contributes neither to $d_G - X_B(X_I)$ for $X \in I_H$ nor to $d_G(v)$ for $v \in V \setminus V(\mathcal{H})$. Thus, since for $X \in I_H$, $\emptyset \neq X_I \neq V \setminus X_B$, we have, by 6-connectivity of $G$,

\[
|E \setminus H| \geq \frac{1}{2} \left( \sum_{X \in I_H} d_{G - X_B}(X_I) + \sum_{v \in V \setminus V(\mathcal{H})} d_G(v) \right) \\
\geq \frac{1}{2} \left( \sum_{X \in I_H} (6 - |X_B|) + \sum_{v \in V \setminus V(\mathcal{H})} 6 \right) \quad (\ast) \\
\geq \sum_{X \in I_H} (3 - |X_B|) + 2(|V| - |V(\mathcal{H})|).
\]

By $|X| \geq 3$ for $X \in \mathcal{H} \setminus I_H$, (10) and (2), we have

\[
r_R(E) = \sum_{X \in \mathcal{H}} (2|X| - 3) + |E \setminus H| \\
\geq \left( \sum_{X \in \mathcal{H}} |X| + \sum_{X \in I_H} (|X| - 3) \right) + \left( \sum_{X \in I_H} (3 - |X_B|) + 2(|V| - |V(\mathcal{H})|) \right) \\
\geq \sum_{X \in \mathcal{H}} |X| + \sum_{X \in I_H} |X_I| + 2(|V| - |V(\mathcal{H})|) \\
\geq 2|V|.
\]

Hence, by (6), we have $2|V| - 3 \geq r_R(E) \geq 2|V|$, a contradiction.

Proof of Theorem 2. The proof of Theorem 2 is obtained from the proof of Theorem 1 by replacing $d_{G - X_B}(X_I) \geq 6 - |X_B|$ by $d_{G - X_B}(X_I) \geq 6 - 2|X_B|$ in the inequality $(\ast)$. This means that in the proof of Theorem 1 we used $(6,2)$-connectivity instead of 6-connectivity.
Here comes the proof of the main result.

**Proof of Theorem 4.** Let \( k \geq 1 \) and \( \ell \geq 0 \) be integers and \( G = (V, E) \) a \((6k + 2\ell, 2k)\)-connected simple graph. To prove the theorem we use the matroid \( \mathcal{N}_{k,\ell} \) defined in Section 4 and show that

\[
r_{k,\ell}(E) = k(2n - 3) + \ell(n - 1). \tag{11}
\]

Choose \( F \) a smallest-size set of edges that gives the rank of \( E \) in \( \mathcal{N}_{k,\ell} \), that is, which minimizes the right hand side of (8). By (7), there exist a subset \( H \subseteq F \) and a collection \( \mathcal{H} \) of subsets of \( V \) of sizes at least 3 such that \( \{F(X), X \in \mathcal{H}\} \) partitions \( H \) and

\[
r_H(F) = \sum_{X \in \mathcal{H}} (2|X| - 3) + |F \setminus H|. \tag{12}
\]

**Claim 2.** \( H = F \).

**Proof.** Since \( \mathcal{H} \) is a collection of subsets of \( V \) of sizes at least 3 such that \( \{H(X), X \in \mathcal{H}\} \) partitions \( H \), we have, by (12),

\[
r_H(H) \leq \sum_{X \in \mathcal{H}} (2|X| - 3) = r_H(F) - |F \setminus H|. \]

Hence, since the rank function \( r_C \) is non-decreasing and \( k \geq 1 \), we have

\[
kr_H(H) + \ell r_C(H) + |E \setminus H| \leq kr_H(F) + \ell r_C(F) + |E \setminus H| - k|F \setminus H|.
\]

Thus \( H \) also minimizes the right hand side of (8) and, by \( H \subseteq F \) and the minimality of \( F, H = F \). \( \blacksquare \)

If \( V \in \mathcal{H} \), then, by (12),

\[
r_H(F) \geq \sum_{X \in \mathcal{H}} (2|X| - 3) \geq 2n - 3 \quad \text{and, by Claim 2 and Remark 2,} \quad G_F \quad \text{is connected, that is,} \quad r_C(F) = n - 1. \quad \text{Hence, by (9), we have (11) and the theorem is proved. From now on, we assume that} \quad V \notin \mathcal{H} \quad \text{and we will show a contradiction.}
\]

Recall the definitions of the border \( X_B = X \cap (\bigcup_{Y \in \mathcal{H}} X_Y)^c \), the inner part \( X_I = X \setminus X_B \) for \( X \in \mathcal{H} \), \( \mathcal{I}_\mathcal{H} = \{X \in \mathcal{H} : X_I \neq \emptyset\} \) and the sets \( \mathcal{K}_F \) and \( \mathcal{K}_I \) of connected components of \( G_F \) and \( (V, \mathcal{H}) \) of size 1. By Claim 1 (iv), \( \mathcal{K}_F = \mathcal{K}_I \).

Let us use the connectivity condition on \( G \) to show a lower bound on \( |E \setminus F| \).

**Claim 3.** \( |E \setminus F| \geq k \left( \sum_{X \in \mathcal{I}_\mathcal{H}} (3 - |X_B|) + 3|\mathcal{K}_F| \right) + \ell \left( |\mathcal{I}_\mathcal{H}| + |\mathcal{K}_F| \right) \).

**Proof.** By \( V \notin \mathcal{H} \), for \( X \in \mathcal{I}_\mathcal{H}, \emptyset \neq X_I \neq V \setminus X_B \). Then, for \( X \in \mathcal{I}_\mathcal{H} \) and for \( v \in \mathcal{K}_F \) we have, by \((6k + 2\ell, 2k)\)-connectivity of \( G \),

\[
d_{G-X_B}(X_I) \geq (6k + 2\ell) - 2k|X_B|, \tag{13}
\]

\[
d_G(v) \geq 6k + 2\ell. \tag{14}
\]

Since, by Claim 2, every edge of \( F \) is induced by an element of \( \mathcal{H} \) and by definition of \( X_I \), for \( X \in \mathcal{I}_\mathcal{H} \), no edge of \( F \) contributes to \( d_{G-X_B}(X_I) \). Each \( v \in \mathcal{K}_F \) is a connected component of the graph \( G_F \), thus no edge of \( F \) contributes
to $d_G(v)$. Hence, by (13), (14) and $\ell \geq 0$, we obtain the required lower bound on $|E \setminus F|$, 
\begin{align*}
|E \setminus F| & \geq \frac{1}{2} \left( \sum_{X \in \mathcal{I}_H} d_{G - X_H} (X_I) + \sum_{v \in K_F} d_G(v) \right) \\
& \geq \frac{1}{2} \left( (6k + 2\ell)|I_H| - 2k \sum_{X \in \mathcal{I}_H} |X_B| + (6k + 2\ell)|K_F| \right) \\
& \geq k \left( \sum_{X \in \mathcal{I}_H} (3 - |X_B|) + 3|K_F| \right) + \ell \left( |I_H| + |K_F| \right). \hspace{1cm} \blacksquare
\end{align*}

Thus, by (12), Claims 2, 3, $|X| \geq 3$ $(X \in \mathcal{H} \setminus \mathcal{I}_H)$, Claim 1 (iv), (iii) and (2), we get 
\begin{align*}
r_{k,\ell}(E) &= k \sum_{X \in \mathcal{H}} (2|X| - 3) + |E \setminus F| + \ell(n - c(F)) \\
& \geq k \left( \sum_{X \in \mathcal{H}} |X| + \sum_{X \in \mathcal{I}_H} (|X| - 3) \right) + k \left( \sum_{X \in \mathcal{I}_H} (3 - |X_B|) + 3|K_F| \right) \\
& \quad + \ell \left( |I_H| + |K_F| \right) + \ell(n - c(F)) \\
& \geq k \left( \sum_{X \in \mathcal{H}} |X| + \sum_{X \in \mathcal{I}_H} |X_I| + 2|K_H| \right) + \ell \left( c(H) + n - c(F) \right) \\
& \geq 2kn + \ell n.
\end{align*}

By $k \geq 1$ and $\ell \geq 0$, this contradicts (9). \hspace{1cm} \blacksquare

Remark that the proof actually shows that if $G$ is simple and $(6k + 2\ell, 2k)$-connected and if $F \subseteq E$ is such that $|F| \leq 3k + \ell$, then in $G' = (V, E \setminus F)$ there exists a packing of $k$ rigid spanning subgraphs and $\ell$ spanning trees.

We mention that Theorem 4 was slightly generalized by Durand de Gevigney and Nguyen [3] for finding bases of a particular count matroid and spanning trees pairwise edge-disjoint. Their proof applies the discharging method.

## References

[1] A. R. Berg and T. Jordán. Sparse certificates and removable cycles in $l$-mixed $p$-connected graphs *Operations Research Letters*, 33(2):111-114, 2005.

[2] A. R. Berg and T. Jordán. Two-connected orientations of Eulerian graphs. *Journal of Graph Theory*, 52(3):230–242, 2006.

[3] O. Durand de Gevigney. Orientations of graphs: structures and algorithms. PhD Thesis, 2013.

[4] J. Edmonds. Matroid partition. In *Mathematics of the Decision Sciences Part I*, volume 11, pages 335–345. AMS, Providence, RI, 1968.
[5] A. Frank. Connectivity and network flows. In Handbook of Combinatorics, pages 117–177. Elsevier, Amsterdam, 1995.

[6] B. Jackson and T. Jordán. A sufficient connectivity condition for generic rigidity in the plane. Discrete Applied Mathematics, 157(8):1965–1968, 2009.

[7] B. Jackson and T. Jordán. Connected rigidity matroids and unique realizations of graphs. Journal of Combinatorial Theory, Series B, 94(1):1 – 29, 2005.

[8] T. Jordán. On the existence of $k$ edge-disjoint 2-connected spanning subgraphs. Journal of Combinatorial Theory, Series B, 95(2):257–262, 2005.

[9] Z. Király and Z. Szigeti. Simultaneous well-balanced orientations of graphs. Journal of Combinatorial Theory, Series B, 96(5):684–692, 2006.

[10] L. Lovász and Y. Yemini. On generic rigidity in the plane. SIAM Journal on Algebraic and Discrete Methods, 3(1):91–98, 1982.

[11] C. St. J. A. Nash-Williams. Edge-disjoint spanning trees of finite graphs. Journal of the London Mathematical Society, 36:445–450, 1961.

[12] W. T. Tutte. On the problem of decomposing a graph into $n$ connected factors. Journal of the London Mathematical Society, 36(1):221–230, 1961.

[13] C. Thomassen. Configurations in graphs of large minimum degree, connectivity, or chromatic number. Annals of the New York Academy of Sciences, 555(1):402–412, 1989.