Upper bound on the rate of convergence and truncation bound for non-homogeneous birth and death processes on $\mathbb{Z}$

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Abstract

We consider the well-known problem of the computation of the (limiting) time-dependent performance characteristics of one-dimensional continuous-time birth and death processes on $\mathbb{Z}$ with time varying and possible state-dependent intensities. First in the literature upper bounds on the rate of convergence along with one new concentration inequality are provided. Upper bounds for the error of truncation are also given. Condition under which a limiting (time-dependent) distribution exists is formulated but relies on the quantities that need to be guessed in each use-case. The developed theory is illustrated by two numerical examples within the queueing theory context.

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1 Introduction

In this paper consideration is given to the random walk on the integers, performed by a particle, which takes only unit steps either to the left or to the right. Its initial position may be arbitrary but fixed. The main quantity under the consideration is the position $X(t) \in \mathbb{Z}$ of the particle at time $t$. Yet meaningful statements related to its average position $E[X(t)]$ given that initially it was at the origin (to be understood here as $X(0) = 0$) will also be given. The particle’s position $X(t)$ at time $t$ is governed by the two independent Poisson processes with possible time-dependent and state-dependent parameters; henceforth if $X(t) = i$ at some time $t$ then $\lambda_i(t)$ and $\mu_i(t)$ denote the motion intensities to the right and left respectively. From the other point of view the $X(t)$ can be viewed as the non-homogeneous birth and death process (BDP) on $\mathbb{Z}$ — a model used for numerous problem instances in finance, genetics, biology, chemistry, physics etc. Just for an example one can refer to the bibliography (up to 1982) in [29], which contains more than 300 papers on the use of BDP in the latter two subjects; a more recent review (up to 2004) can be found in [30]. One intuitively clear example of $X(t)$ (which will be revisited further in the numerical section) is provided by one problem known in the literature as the taxicab problem [31, 32]. There is one queueing point whereto both taxis and passengers arrive one by one in accordance with the two independent Poisson flows possibly with time-dependent and possibly state-dependent arrival intensities. The queue length may take any integer value: negative values mean that there are passengers waiting for taxis, whereas positive values mean that there are taxis waiting for passengers. Whenever the queue length is zero, the queueing point is free from both passengers and taxis. From the given description is can be seen that $X(t)$ can be represented as the difference between the two Poisson variables. If the intensities depend on the state of $X(t)$, it implies that the admission of passengers/taxis to the queueing-point is dynamically controlled. Since the seminal paper [31] such queues and similar to $X(t)$ processes have been the subject of extensive research and now they are usually referred to as double-sided or double-ended queues (see [24, 33]), unrestricted random walks on lattice and bilateral BDPs [34, 35, 36]. Another intuitive but otherwise artificial example (which will also be revisited in the numerical section) is the
system size/queue length in common queueing systems\footnote{Not arbitrary ones, but only those in which the queue-size may change by at most 1 at a time.} at epoch \( t \). If one removes the impenetrable barrier at the origin, which means that the departures are also allowed, when the system size is zero or negative, one arrives at another instance of \( X(t) \) (see \cite{28, 38}). At last, another example can be extracted from the Markov predator-prey models (or other models of species coexistence \cite{14}), in which \( X(t) \) is the difference between the predator and prey populations.

Bilateral non-homogeneous BDPs like \( X(t) \) have already been analyzed in the literature from various perspectives; see, for example, \cite{4, Section 1}. The basic questions under consideration are: the computation of the time-dependent and the limiting probability distribution, methods for the approximation of their transient behaviour, determination of first-passage time densities via analytical and numerical methods. The literature review, which we have been able to make, shows that for one of the general cases i.e. when the state space of \( X(t) \) is \( \mathbb{Z} \) and its transition intensities are allowed to be time- and state-dependent, most of the questions remain open. The only feasible way to deal with such \( X(t) \) seems to be extensive use of numerical schemes for systems of ordinary differential equations (ODEs). For the numerical approaches to be efficient, in the first place one needs to know how to determine a priori the points of convergence and, in the cases when the ODE system is infinite, how to choose the truncation thresholds. In this paper we show that the technique utilizing the notion of the logarithmic norm and already available for the BDPs on the non-negative integers, can be generalized for the BDPs on \( \mathbb{Z} \). The theoretical results which follow are applicable only to those cases when the limiting ergodic distribution exists. The sufficient condition for that is being formulated (see Theorem 1).

The purpose of this paper is two-fold. Firstly we derive first in the literature explicit upper bounds for the rate of convergence of non-homogeneous BDPs on \( \mathbb{Z} \) to the limiting regime (whenever it exists). The class of processes considered includes those with all the transition intensities being possibly time-varying and state-dependent, but bounded (see \cite{1}). Secondly, we derive truncation bounds (see Theorem 2), which allow one to obtain numerical solutions with the desired accuracy. By virtue of two numerical experiments it is demonstrated that this result may be particularly useful for obtaining the limiting values of the time-dependent probabilities.
The questions of convergence of non-homogeneous BDPs (and especially homogeneous) have been considered in many research papers. The approach used here to obtain the results related to the convergence and truncation bounds is, of course, not new. It is based on the theory developed in the series of papers by the authors. Basically it relies on the well-known connection between the transition matrix of a Markov chain and the corresponding ODEs (specifically, Kolmogorov’s forward equations). The main ingredient is the notion of the logarithmic norm of an operator function and those estimates for the differential equations, which are available in the literature. Using this approach in the previous papers it was possible to obtain explicit upper bounds for the distance between two probability distributions (in some special norms) of the BDPs with either finite or countable (in one direction) state space i.e. $\mathbb{Z}^+$. Here we show, that the approach can be generalized to deal with quite general BDPs on the whole set $\mathbb{Z}$. Surprisingly this generalization does not come at price: the upper bounds obtained for the case of $\mathbb{Z}$ are not weaker that in the case of $\mathbb{Z}^+$.

In what follows $\| \cdot \|$ denotes the $l_1$-norm, i.e. if $x$ is a column vector then $\|x\| = \sum_k |x_k|$. Clearly, $\|x\| = 1$ if $x$ is a probability vector. The operator norm is assumed to be induced by the $l_1$-norm on column vectors i.e. for any linear operator $A$ we have $\|A\| = \sup_j \sum_i |a_{ij}|$.

## 2 Preliminaries

Let $\{X(t), t \geq 0\}$ be the BDP with the state space $\mathbb{Z}$ and the generators $\{Q(t) = (q_{ij}(t)), t \geq 0\}$ defined by

$$q_{i,i+1}(t) = \lambda_i(t), \; q_{i,i-1}(t) = \mu_i(t) \; \text{and} \; q_{ii}(t) = -(\lambda_i(t) + \mu_i(t)), \;$$

In what follows $\lambda_i(t)$ and $\mu_i(t)$ are assumed to be non-random locally integrable for $t \in [0, \infty)$ continuous functions, satisfying

$$0 \leq \lambda_i(t) \leq \bar{\lambda}_i \leq \Delta < \infty, \; 0 \leq \mu_i(t) \leq \bar{\mu}_i \leq \Delta < \infty \; \text{(1)}$$

for all $t \geq 0, \; i \in \mathbb{Z}$ and some constants $\{\bar{\lambda}_i, i \in \mathbb{Z}\}, \{\bar{\mu}_i, i \in \mathbb{Z}\}$ and $\Delta$. The transition diagram of $X(t)$ is shown in the figure below\(^2\).

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\(^2\)In order to keep the figure and the matrices readable, whenever it does not introduce any ambiguity, the argument $t$ of the intensity functions is omitted.
Figure 1: Possible transitions for $X(t)$ and corresponding intensities

Let $p_i(t) = P\{X(t) = i\}$ and $p(t) = (\ldots, p_{-1}(t), p_0(t), p_1(t), \ldots)^T$. For what follows it will be convenient to write the Kolmogorov forward equations for the distribution of $X(t)$ as

$$
\frac{d}{dt}p(t) = A(t)p(t), \quad t \geq 0,
$$

(2)

where $A(t) = (a_{ij}(t))$ is the transposed generator i.e. $a_{ij}(t) = q_{ji}(t)$. Since $\|A(t)\| = 2\sup_{i \in \mathbb{Z}}(\lambda_i(t) + \mu_i(t)) \leq 4\Delta$, the linear operator $A(t)$ is bounded and locally integrable for $t \in [0, \infty)$. Thus (2) is the system of differential equations in the space $l_1$ with the (bounded) linear operator. And thus (see, for instance, [1]) it has the unique solution for arbitrary initial conditions. Moreover, if for some $s \geq 0$ the probabilities $p_i(s)$ are all non-negative for $i \in \mathbb{Z}$ and $\|p(s)\| = 1$, then the same holds for $p(t)$ when $t \geq s$. What follows next relies on the concept of the logarithmic norm of locally integrable operator functions (see [1, 27]) and available from the literature estimates for differential equations; detailed definitions and derivations can be recovered from, for example, [16, Appendix].

3 Basic estimates

**Theorem 1.** Let there exist a doubly infinite sequence of positive numbers $\{d_k, k = \pm 1, \pm 2, \ldots\}$ such that $\inf_{k \in \mathbb{Z} \setminus \{0\}} d_k = 1$ and $\int_0^\infty \beta^*(u)du = \infty$, where $\beta^*(t) = \inf_{k \in \mathbb{Z} \setminus \{0\}} \beta_k^{**}(t)$ and the function $\beta_k^{**}(t)$ is given by

$$
\beta_k^{**}(t) = \begin{cases} 
\lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), & k < -1 \\
\lambda_{-1}(t) + \mu_0(t) - \frac{d_{1}}{d_{-1}}\lambda_0(t) - \frac{d_{-2}}{d_{-1}}\mu_{-1}(t), & k = -1 \\
\lambda_0(t) + \mu_1(t) - \frac{d_{1}}{d_{0}}\lambda_1(t) - \frac{d_{-1}}{d_{0}}\mu_0(t), & k = 1, \\
\lambda_{k-1}(t) + \mu_k(t) - \frac{d_{k+1}}{d_k}\lambda_k(t) - \frac{d_{k-1}}{d_k}\mu_{k-1}(t), & k > 1.
\end{cases}
$$

Special cases of these well-known equations have been the starting point for numerous papers; see, for example, early research on population dynamics in [37, Section 3].
Then \( X(t) \) is weakly ergodic and for all \( t \geq 0 \) and any initial conditions \( p^*(0) \) and \( p^{**}(0) \) it holds that

\[
\|p^*(t) - p^{**}(t)\| \leq e^{-\int_0^t \beta^{**}(u) du} \sum_{k \in \mathbb{Z} \setminus \{0\}}^N (p_k^*(0) - p_k^{**}(0)) \sum_{j = \min(-1,k)}^{\max(-1,k)} d_j. \tag{3}
\]

**Proof.** Since \( p_0(t) = 1 - \sum_{k \in \mathbb{Z} \setminus \{0\}}^\infty p_k(t) \), the Kolmogorov forward equations (2) for the distribution of \( X(t) \) can be re-written as

\[
\frac{d}{dt} z(t) = B(t)z(t) + f(t), \tag{4}
\]

where the vectors \( f(t) \) and \( z(t) \) are

\[
f(t) = (\ldots, 0, \mu_0(t), \lambda_0(t), 0, \ldots)^T, \quad z(t) = (\ldots, p_{-2}(t), p_{-1}(t), p_1(t), p_2(t), \ldots)^T,
\]

and the linear transformation \( B(t) \) is given by the block matrix

\[
B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix},
\]

which entries \( B_{ij}(t) \) are itself matrices of the form

\[
B_{11}(t) = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ \ldots & -\lambda_3 - \mu_3 & \mu_2 & 0 \\ \ldots & \lambda_3 & -\lambda_2 - \mu_2 & \mu_1 \\ \ldots & -\mu_0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \quad B_{12}(t) = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ -\mu_0 & -\mu_0 & -\mu_0 & \ldots \end{pmatrix},
\]

\[
B_{21}(t) = \begin{pmatrix} \ldots & -\lambda_0 & -\lambda_0 & -\lambda_0 \\ \ldots & 0 & 0 & 0 \\ \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \quad B_{22}(t) = \begin{pmatrix} \ldots & -\lambda_1 - \mu_1 - \lambda_0 & \mu_2 - \lambda_0 & -\lambda_0 & \ldots \\ -\mu_1 - \lambda_0 & \lambda_1 & -\mu_2 - \lambda_2 & \mu_3 & \ldots \\ \ldots & 0 & \lambda_2 & -\mu_3 - \lambda_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}
\]

Denote by \( D_U^* \) and \( D_L^* \) correspondingly the upper and the lower triangular matrix of the form

\[
D_U^* = \begin{pmatrix} 1 & 1 & 1 & \ldots \\ 0 & 1 & 1 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad D_L^* = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ \ldots & -1 & 0 & 0 \\ \ldots & -1 & -1 & 0 \\ \ldots & -1 & -1 & -1 \end{pmatrix}.
\]
Both of these matrices are known as semicirculant matrices. Consider the linear transformation given by the block matrix \( D^* = \begin{pmatrix} D_L & 0 \\ 0 & D_U \end{pmatrix} \). In what follows we will need the inverse linear map of \( D^* \), which is further denoted by \((D^*)^{-1}\). In order to show that it exists for the considered matrix \( D^* \) we will make use of the well-known fact that the mapping of formal power series into the set of infinite semicirculant matrices is an isomorphism. Let us associate with the matrix \( D^*_L \) the formal power series \( P_L(z) = \sum_{i=0}^\infty z^i a_i \) (we write \( P_L(z) \to D^*_L \)). The values of \( a_i \) are in the first bottom row of \( D^*_L \). With the matrix \( D^*_U \) we associate the formal power series \( P_U(z) = \sum_{i=0}^\infty z^i b_i \) (i.e. \( P_U(z) \to D^*_U \)). The values of \( b_i \) are in the first upper row of \( D^*_U \). Consider the matrix \( P(z) = \begin{pmatrix} P_L(z) & 0 \\ 0 & P_U(z) \end{pmatrix} \). Since the mapping \( \to \) is an isomorphism, then \( P_L(z) \to D^*_L \) and \( P_U(z) \to D^*_U \), and thus \( P(z) \to D^* \).

Note now that inverse matrix to \( P(z) \), denote it by \((P(z))^{-1}\), exists since \( P_L(z)P_U(z) \neq 0 \). Denote the formal power series of \( \frac{1}{P_L(z)P_U(z)} \) by \((P_L(z)P_U(z))^{-1}\). Since \( P_L(z)P_U(z) = \sum_{i=0}^\infty z^i(i+1) \), then \((P_L(z)P_U(z))^{-1} = -1 + 2z - z^2 \) (see [25, Theorem 1.2b]). It is straightforward to check that \((P_L(z)P_U(z))^{-1}P_U(z) = -1 + z \) and \((P_L(z)P_U(z))^{-1}P_L(z) = 1 - z \). Thus we have

\[
(P(z))^{-1} = \frac{1}{P_L(z)P_U(z)} \begin{pmatrix} P_U(z) & 0 \\ 0 & P_L(z) \end{pmatrix} = \\
= \begin{pmatrix} (P_L(z)P_U(z))^{-1}P_U(z) & 0 \\ 0 & (P_L(z)P_U(z))^{-1}P_L(z) \end{pmatrix} = \\
= \begin{pmatrix} -1+z & 0 \\ 0 & 1-z \end{pmatrix}. \tag{5}
\]

Both formal power series \(-1 + z \) and \( 1 - z \) have associated semicirculant matrices

\[
1-z \to (D^*_U)^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots \\ 0 & 1 & -1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad -1+z \to (D^*_L)^{-1} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \vdots & -1 & 0 & 0 \\ \vdots & 1 & -1 & 0 \\ \vdots & 0 & 1 & -1 \end{pmatrix}.
\]

Introduce the block matrix \((D^*)^{-1} = \begin{pmatrix} (D^*_L)^{-1} & 0 \\ 0 & (D^*_U)^{-1} \end{pmatrix} \). Thus we have \((P(z))^{-1} \to (D^*)^{-1} \). But since \( P(z)(P(z))^{-1} = I \), then \( D^*(D^*)^{-1} = (D^*)^{-1}D^* = \)
where $I$ is the identity matrix. Thus $(D^*)^{-1}$ is the left and right inverse linear map of $D^*$.

Consider the similarity transformation $D^*B(t)(D^*)^{-1}$, further denoted by $B^*(t)$. It is well-defined and given by the matrix

$$B^*(t) = \begin{pmatrix}
\ddots & \ddots & 0 \\
\ddots & -\left(\lambda_2 + \mu_1\right) & \mu_1 \\
\lambda_1 & -\left(\lambda_1 + \mu_0\right) & 0 \\
0 & \lambda_0 & -\left(\lambda_0 + \mu_1\right) \\
0 & 0 & \lambda_1 \\
0 & 0 & 0 \\
\end{pmatrix}. $$

Note that unlike the matrix $B(t)$ all off-diagonal entries of $B^*(t)$ are non-negative. Choose a double infinite sequence $\{d_k, k = \pm 1, \pm 2, \ldots\}$ of positive numbers and consider the linear transformation $D^{**} = \text{diag}(\ldots, d_{-2}, d_{-1}, d_1, d_2, \ldots)$. It is known (see [26, p. 19]) that $D^{**}$ has a unique right-hand reciprocal, which is the diagonal matrix $\text{diag}(\ldots, 1/d_{-2}, 1/d_{-1}, 1/d_1, 1/d_2, \ldots)0 = (D^{**})^{-1}$. It is straightforward to check, that the linear transformation $B^{**}(t) = D^{**}B^*(t)(D^{**})^{-1}$ is given by the matrix

$$B^{**}(t) = \begin{pmatrix}
\ddots & \ddots & \frac{d_2}{d_1} \mu_1 \\
\ddots & -\left(\lambda_2 + \mu_1\right) & \frac{d_2}{d_1} \mu_1 \\
\frac{d_1}{d_2} \lambda_1 & -\left(\lambda_1 + \mu_0\right) & 0 \\
0 & \lambda_0 \frac{d_1}{d_2} & -\left(\lambda_0 + \mu_1\right) \\
0 & 0 & \frac{d_1}{d_2} \lambda_1 \\
0 & 0 & 0 \\
\end{pmatrix}, $$

which has only non-negative off-diagonal elements.

Coming back to (4), note that any upper bound on the convergence rate to the limiting regime for $X(t)$, corresponds to the same bound for the solutions of the system

$$\frac{d}{dt}y(t) = B(t)y(t),$$

(6)
without the free term \( f(t) \). Here the vector \( \mathbf{y}(t) = (\ldots, y_{-2}(t), y_1(t), y_1(t), y_2(t), \ldots)^T \) and its elements can either positive or negative. Denote \( D = D^{**}D^* \) and \( \mathbf{u}(t) = D\mathbf{u}(t) \). By left-multiplying both parts of (6) by \( D \), we get

\[
\frac{d}{dt} \mathbf{u}(t) = B^{**}(t)\mathbf{u}(t),
\]

where \( \mathbf{u}(t) = (\ldots, u_{-2}(t), u_1(t), u_1(t), u_2(t), \ldots)^T \) is, as well as \( \mathbf{y}(t) \), the vector with the elements of arbitrary signs. Let us estimate the logarithmic norm of \( B^{**}(t) \). It is well-known that in the \( l_1 \)-norm the logarithmic norm of a (locally integrable) operator \( F(t) = (f_{ij}(t)) \) is equal to \( \sup_i \left( f_{ii}(t) + \sum_{j \neq i} |f_{ji}(t)| \right) = \gamma(F(t)) \) (see, for example, [16, Appendix]). By direct inspection it can be instantly seen that the \( k \)th column sum of \( B^{**}(t) \) is equal to \( -\beta_k^{**}(t) \), where

\[
\beta_k^{**}(t) = \begin{cases} 
\lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), & k < -1 \\
\lambda_{-1}(t) + \mu_0(t) - \frac{d_1}{d_{-1}}\lambda_0(t) - \frac{d_0}{d_{-1}}\mu_{-1}(t), & k = -1 \\
\lambda_0(t) + \mu_1(t) - \frac{d_2}{d_1}\lambda_1(t) - \frac{d_1}{d_1}\mu_0(t), & k = 1 \\
\lambda_{k-1}(t) + \mu_k(t) - \frac{d_{k+1}}{d_k}\lambda_k(t) - \frac{d_{k-1}}{d_k}\mu_{k-1}(t), & k > 1.
\end{cases}
\]

Thus \( \gamma(B^{**}(t)) = \sup_{k \in \mathbb{Z} \setminus \{0\}} (-\beta_k^{**}(t)) = -\inf_{k \in \mathbb{Z} \setminus \{0\}} \beta_k^{**}(t) \). Let \( \{d_k, k = \pm 1, \pm 2, \ldots \} \) be such a doubly infinite sequence, that \( \inf_{k \in \mathbb{Z} \setminus \{0\}} \beta_k^{**}(t) < \infty \) for \( t \geq 0 \). Denote \( \inf_{k \in \mathbb{Z} \setminus \{0\}} \beta_k^{**}(t) = \beta^{**}(t) \). Then

\[
\|B^{**}(t)\| \leq 4\Delta - \beta^{**}(t)
\]

and thus \( B^{**}(t) \) is the bounded operator. Now, if \( V(t, z) \) is the Cauchy operator of the equation (7), then for any \( t \) and \( s \) the following bound holds (for the justification see, for example, [16, Theorem A2]):

\[
\|V(t, s)\| \leq e^{-\int_s^t \beta^{**}(u) du}, \quad 0 \leq s \leq t.
\]

(8)

Now let \( \mathbf{p}^*(t) \) and \( \mathbf{p}^{**}(t) \) be such that the corresponding \( D\mathbf{z}^*(t) \) and \( D\mathbf{z}^{**}(t) \) exist. Then for any \( t \geq 0 \) we have

\[
\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 2\|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\| \leq \|D\mathbf{z}^*(t) - D\mathbf{z}^{**}(t)\| \leq e^{-\int_0^t \beta^{**}(u) du} \|D\mathbf{z}^*(0) - D\mathbf{z}^{**}(0)\|.
\]
The inequality (3) holds even if \( \int_0^\infty \beta^{**}(u) du < \infty \). This happens only if the intensities approach 0 as \( t \) becomes infinite and thus the limiting ergodic distribution cannot not exist (cf. [32, Example 3]). It is also worth noticing here that \( \beta^{**(t)} \) is not necessarily an everywhere positive function.

**Corollary 1.** Assume that under the assumptions of the Theorem 1 there exist positive constants \( M \) and \( \beta^{**} \) such that \( e^{-\int_s^t \beta^{**}(\tau) d\tau} \leq M e^{-\beta^{**(t-s)}} \) for any \( 0 \leq s \leq t \). Then for any positive integer \( N \) and all \( t \geq 0 \) it holds that

\[
Pr(|X(t)| \geq N) \leq M \left( e^{-\beta^{**} t} \sum_{k=-N}^{N} \sum_{j=\min(1,k)}^{\max(-1,k)} p_k(0) d_j + \frac{d_{-1} \bar{\mu}_0 + d_1 \bar{\lambda}_0}{\beta^{**}} \left( \sum_{j=-N}^{N} d_j - 1 \sum_{j=1}^{N} d_j \right) \right).
\]

(9)

**Proof.** Consider (4) and note that its solution is

\[
z(t) = V(t,0)z(0) + \int_0^t V(t,\tau)f(\tau) d\tau.
\]

(10)

Let us left-multiply the left and the right part of the previous relation by \( D \). Using the estimates obtained in the Theorem 1 and assuming that there exist constants \( M > 0 \) and \( \beta^{**} > 0 \) such that \( e^{-\int_s^t \beta^{**}(u) du} \leq M e^{-\beta^{**(t-s)}} \) for any \( 0 \leq s \leq t \), we get

\[
\|Dz(t)\| \leq \|V(t,0)\| \|Dz(0)\| + \int_0^t \|V(t,s)\| \|Df(s)\| ds \leq \leq Me^{-\beta^{**} t} \|Dz(0)\| + \int_0^t Me^{-\beta^{**(t-s)}} \|Df(s)\| ds \leq Me^{-\beta^{**} t} \left( \sum_{k=-N}^{-1} p_k(t) \sum_{j=k} d_j + \sum_{k=1}^{N} p_k(t) \sum_{j=1}^k d_j \right) + M \frac{d_{-1} \bar{\mu}_0 + d_1 \bar{\lambda}_0}{\beta^{**}}.
\]

(11)

Indeed, \( \|Df(t)\| = d_{-1} \mu_0(t) + d_1 \lambda_0(t) \leq (d_{-1} \bar{\mu}_0 + d_1 \bar{\lambda}_0) \). Note that since all
$d_k$ are positive, then for any positive integer $N$ we have

$$
\|Dz(t)\| = \cdots + (d_{-2} + d_{-1}) p_{-2}(t) + d_{-1} p_{-1}(t) + d_1 p_1(t) + (d_1 + d_2) p_2(t) + \cdots =
$$

$$
= \sum_{k=-\infty}^{-1} p_k(t) \sum_{j=k}^{-1} d_j + \sum_{k=1}^{\infty} p_k(t) \sum_{j=1}^{k} d_j 
\geq \sum_{k=-\infty}^{-N} p_k(t) \sum_{j=k}^{-1} d_j + \sum_{k=N}^{\infty} p_k(t) \sum_{j=1}^{k} d_j.
$$

From here we get two concentration inequalities for $X(t)$, which are valid for any integer $N > 0$:

$$
\sum_{k=-\infty}^{-N} p_k(t) \leq \frac{\|Dz(t)\|}{\sum_{j=-N}^{-1} d_j}, \quad \sum_{k=N}^{\infty} p_k(t) \leq \frac{\|Dz(t)\|}{\sum_{j=1}^{N} d_j}.
$$

Combining this with the upper bound for $\|Dz(t)\|$ we get for any positive integer $N$:

$$
Pr(X(t) \leq -N) \leq M \left( e^{-\beta^{**} t} \sum_{k=-N}^{N} p_k(0) \sum_{j=\min(1,k)}^{\max(-1,k)} d_j + \frac{d_{-1} \bar{\mu}_0 + d_1 \bar{\lambda}_0}{\beta^{**}} \right) \left( \sum_{j=-N}^{-1} d_j \right)^{-1},
$$

$$
Pr(X(t) \geq N) \leq M \left( e^{-\beta^{**} t} \sum_{k=-N}^{N} p_k(0) \sum_{j=\min(1,k)}^{\max(-1,k)} d_j + \frac{d_{-1} \bar{\mu}_0 + d_1 \bar{\lambda}_0}{\beta^{**}} \right) \left( \sum_{j=1}^{N} d_j \right)^{-1}.
$$

Note that since the bound in the Corollary 1 is valid for any $t$, it is valid for the limiting probabilities as well (if they exist). One can simplify the bound (9) by fixing the initial condition. For example, if $p_0(0) = 1$, which implies that $z(0) = 0$, then the first term in the brackets in the right hand-side of (9) is zero.

**Theorem 2.** Let $X(t)$ be a BDP on $\mathbb{Z}$ for which the Corollary 1 holds. Let $X^*(t)$ be its truncated version with the state space $\{N_1, \ldots, 0, \ldots, N_2\}$,
\[ N_1 < 0, \ N_2 > 0. \] If there exist constants \( \beta^*, \ M^* \) and \( \{d^*_k, k \in \mathbb{Z} \setminus \{0\}\} \), such that the Corollary 1 holds for \( X^*(t) \), then the following upper bound for the difference between the probability distributions of \( X(t) \) and \( X^*(t) \)

\[
\|P(t) - P^*(t)\| \leq \frac{4M^* (\bar{\mu} d_{-1}^* + \bar{\lambda}_0 d_1^*)}{d\beta^* \beta^{**}} \left( \frac{\sum_{j=N_1-1}^{-1} d_j \bar{\mu}_{N_1}}{\sum_{j=N_1}^{-1} d_j} + \frac{\sum_{j=1}^{N_2+1} d_j \bar{\lambda}_{N_2}}{\sum_{j=1}^{N_2} d_j} \right) \tag{12}
\]

holds for any \( t \geq 0 \) if \( X(0) = X^*(0) = 0 \).

**Proof.** Consider the BDP \( X^*(t) \) with the state space \( \mathbb{Z} \) and the intensities

\[
\lambda^*_k(t) = \lambda_k(t) \text{ if } N_1 \leq k < N_2, \quad \text{and } \mu^*_k(t) = \mu_k(t) \text{ if } N_1 < k \leq N_2 \text{ and other intensities equal to zero. Thus the linear operator } A^*(t) \text{ is still given by the bi-infinite matrix. The Kolmogorov forward equations for the distribution of } X^*(t), \text{ being the truncated } X(t), \text{ are}
\]

\[
\frac{d}{dt}P^*(t) = A^*(t) P^*(t), \tag{13}
\]

where

\[
P^*(t) = (\ldots, 0, p_{N_1}(t), \ldots, p_{-2}(t), p_{-1}(t), p_{0}(t), p_1(t), p_2(t), \ldots, p_{N_2}(t), 0, \ldots)^T.
\]

Since \( p^*_0(t) = 1 - \sum_{j \neq 0} p^*_j(t) \), then we obtain from (13)

\[
\frac{d}{dt}z^*(t) = B^*(t) z^*(t) + f^*(t). \tag{14}
\]

Rewrite (14) in the form:

\[
\frac{d}{dt}z^*(t) = B(t) z^*(t) + (B^*(t) - B(t)) z^*(t) + f^*(t). \tag{15}
\]

Then we have the following relations between the solutions of (14) and (15):

\[
z(t) - z^*(t) = V(t, 0) (z(0) - z^*(0)) + \\
+ \int_0^t V(t, s) (B(s) - B^*(s)) z^*(s) \, ds + \\
+ \int_0^t V(t, s) (f(s) - f^*(s)) \, ds. \tag{16}
\]
where
\[
(B(s) - B^*(s))z^*(s) = (\cdots, 0, \mu_{N_1}p_{N_1}^*, -\mu_{N_1}p_{N_1}^*, 0, \cdots, 0, -\lambda_{N_2}p_{N_2}^*, \lambda_{N_2}p_{N_2}^*, 0, \cdots)^T.
\]

For simplicity we assume further, that \(z(0) = z^*(0) = 0\) (i.e. \(X(0) = X^*(0) = 0\) with the probability 1) Then \(f(s) = f^*(s)\) for any \(s\). Next, it is clear that the first and the third terms in the (16) are equal to zero and the difference between \(z(t)\) and \(z^*(t)\) is just
\[
z(t) - z^*(t) = \int_0^t V(t, s)(B(s) - B^*(s))z^*(s)\, ds. \tag{17}
\]

Let \(\{d^*_k, k = \pm 1, \pm 2, \ldots\}\) be an double infinite sequence of positive numbers such that there exist positive \(M^*\) and \(\alpha^*\) such that
\[
e^{-\int_s^t \beta^*(\tau)\, d\tau} \leq M^* e^{-(t-s)\beta^*}, \tag{18}
\]
for any \(0 \leq s \leq t\), where \(\beta^*(t) = \inf_{k \in \mathbb{Z}\setminus\{0\}} \beta^*_k(t)\) and the functions \(\beta^*_k(t)\) is given by
\[
\beta^*_k(t) = \begin{cases} \lambda^*_k(t) + \mu^*_k(t) - \frac{d^*_k}{d^*_k + 1} \lambda^*_{k+1}(t) - \frac{d^*_k}{d^*_k - 1} \mu^*_{k-1}(t), & k < -1, \\
\lambda^*_{k-1}(t) + \mu^*_0(t) - \frac{d^*_k}{d^*_k + 1} \lambda^*_{k+1}(t) - \frac{d^*_k}{d^*_k - 1} \mu^*_{k-1}(t), & k = -1, \\
\lambda^*_0(t) + \mu^*_1(t) - \frac{d^*_k}{d^*_k + 1} \lambda^*_{k+1}(t) - \frac{d^*_k}{d^*_k - 1} \mu^*_{k-1}(t), & k = 1, \\
\lambda^*_k(t) + \mu^*_k(t) - \frac{d^*_k}{d^*_k + 1} \lambda^*_{k+1}(t) - \frac{d^*_k}{d^*_k - 1} \mu^*_{k-1}(t), & k > 1.
\end{cases}
\]

By left-multiplying both parts of (17) by the matrix \(D\), introduced above, and using the estimates obtained above we get:
\[
\|D(B(s) - B^*(s))z^*(s)\| \leq \sum_{j=N_1-1}^{N_1} d_j + \sum_{j=N_1}^{N_2+1} d_j \left| \mu_{N_1}(s)p_{N_1}^*(s) + \right| \sum_{j=1}^{N_2+1} d_j \left| \lambda_{N_2}(s)p_{N_2}^*(s) \right| \leq 2 \sum_{j=N_1-1}^{N_1} d_j \lambda_{N_1}(s)p_{N_1}^*(s) + 2 \sum_{j=1}^{N_2+1} d_j \lambda_{N_2}(s)p_{N_2}^*(s). \tag{19}
\]
Since \( p_k^* (t) \leq \frac{M^*}{\beta^*} (\mu_0 d_{-1}^* + \lambda_0 d_1^*) (\sum_{j=\max(-1,k)}^{\min(1,k)} d_j^*)^{-1} \) for any \( k \neq 0 \), then under the assumption that both BDPs start in the 0th state, relations (17) and (19) imply the bound

\[
\| D(z(t) - z^*(t)) \| \leq \frac{2 M M^* (\mu_0 d_{-1}^* + \lambda_0 d_1^*)}{\beta^* \beta^{**}} \left( \sum_{j=\min(k,1)}^{N_1} d_j \mu_{N_1} + \sum_{j=1}^{N_2} d_j \lambda_{N_2} \right).
\]

Put \( d = \min (d_{-1}, d_1) \). The following sequence of inequalities completes the proof:

\[
\| P(t) - P^*(t) \| \leq \cdots + |p_{-1}(t) - p^*_{-1}(t)| + |p_0(t) - p_0^*(t)| + |p_1(t) - p_1^*(t)| + \cdots \leq \\
\leq \cdots + \frac{d_{-1} + d_{-2}}{d_{-1} + d_{-2}} |p_{-2}(t) - p^*_{-2}(t)| + \frac{d_{-1}}{d_{-1}} |p_{-1}(t) - p^*_{-1}(t)| + \\
+ \frac{d_1}{d_1} |p_1(t) - p_1^*(t)| + \frac{d_1 + d_2}{d_1 + d_2} |p_2(t) - p_2^*(t)| + \cdots \leq \\
\cdots + \frac{d_{-1} + d_{-2}}{d} |p_{-2}(t) - p^*_{-2}(t)| + \frac{d_{-1}}{d} |p_{-1}(t) - p^*_{-1}(t)| + \\
+ \frac{d_1}{d} |p_1(t) - p_1^*(t)| + \frac{d_1 + d_2}{d} |p_2(t) - p_2^*(t)| + \cdots \leq \frac{1}{d} \| D(z(t) - z^*(t)) \|. 
\]

The argumentation of the Theorem 2 allows one also to obtain the upper bound for the truncation error, when computing the average value \( E X(t) \) given that initially the process was in the 0th state. Let \( W = \inf_{k \geq 1} \left( \frac{\sum_{j=k}^{k-1} d_j}{k}, \frac{\sum_{j=k}^{k-1} d_j}{k} \right) \).
Then
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} k|p_k(t) - p_k^*(t)| = \cdots + 1|p_{-1}(t) - p_{-1}^*(t)| + 0|p_0(t) - p_0^*(t)| + 1|p_1(t) - p_1^*(t)| + \cdots \leq \cdots + 2 \frac{d_{-1} + d_{-2}}{d_{-1} + d_{-2}} |p_{-2}(t) - p_{-2}^*(t)| + 1 \frac{d_{-1}}{d_{-1}} |p_{-1}(t) - p_{-1}^*(t)| + \cdots + \frac{d_{-1} + d_{-2}}{W} |p_{-2}(t) - p_{-2}^*(t)| + \frac{d_{-1}}{W} |p_{-1}(t) - p_{-1}^*(t)| + \frac{d_1}{W} |p_1(t) - p_1^*(t)| + \frac{d_1 + d_2}{W} |p_2(t) - p_2^*(t)| \cdots \leq \frac{1}{W} \| D(z(t) - z^*(t)) \|.
\]

Using now the upper bound for $\| D(z(t) - z^*(t)) \|$ from the Theorem 2, we get the upper bound for the $\sum_{k \in \mathbb{Z} \setminus \{0\}} k|p_k(t) - p_k^*(t)|$. In what follows we show, how the developed theory can be used to obtain explicit results.

4 Numerical examples

Two examples are considered in this section. Their main purpose is to illustrate that the developed theory indeed allows one to study numerically arbitrary bilateral BDP $X(t)$ with uniformly bounded and state-dependent intensity functions. Specific forms of the intensity functions have been chosen for convenience of computation. In each case it is assumed that $X(0) = 0$.

In the first example we consider the randomized random walk on the integers, say $X(t)$, which represents the position at time $t$ of a particle moving along, say $x$-axis, according to the following rules. Its position $X(t)$ can be shifted by at most 1 to the right or left, and it is assumed that these changes are governed by the two Poisson processes with the time-varying parameters. Specifically, when the particle is in a position $i$ on the positive part of the $x$-axis, it will move to the position $j$ in the infinitesimal time $h > 0$ with the probability

\[
Pr \{X(t + h) = j | X(t) = i\} = \begin{cases} 
\lambda(t)h, & \text{if } j - i = 1, \\
\mu_i(t)h, & j - i = 1, \\
-(\lambda(t) + \mu_i(t))h, & j = i, \\
0, & \text{otherwise}.
\end{cases}
\]
The next position \( j \) in the infinitesimal time \( h > 0 \) of the particle residing in the position \( i \) on the negative part of the \( x \)-axis is governed by the probability 
\[
Pr \{ X(t + h) = -j | X(t) = -i \}.
\]
If the particle enters the state 0 then its next state is 1 or \(-1\) with the probability \( \lambda(t)h \).

We make further simplifications. Let us assume that \( \mu_i(t) = min(i; S)\mu(t) \). Then when the particle is in the non-negative part of the \( x \)-axis, then \( X(t) \) represents the number of customers present in the classic \( M/M/S/\infty \) queue at epoch \( t \). This example is somewhat artificial one and is due to [28].

From the Theorem 1 one can obtain the upper bound for the rate of convergence, if a double infinite sequence, say \( \{d_k, k = \pm 1, \pm 2, \ldots \} \), can be found such that \( \int_0^\infty \beta^{**}(u)du = \infty \). Let us put \( d_1 = 1 \) and \( d_k = d^{k-1} \) for \( k \geq 2 \), where \( d > 1 \). Then we have:
\[
\beta^{**}_k(t) = \begin{cases} 
\mu(t) - d\lambda(t), & k = 1 \\
\mu(t) - (d - 1)\lambda(t) + \frac{(k-1)(d-1)}{d}\mu(t), & 2 \leq k \leq S \\
(1 - \frac{1}{d})(S\mu(t) - d\lambda(t)), & k > S.
\end{cases} \tag{20}
\]

Assume for now that there exists \( \theta(t) \) such that \( (S\mu(t) - d\lambda(t)) \geq \theta(t) \). Then \( \beta^{**}(t) = min(\mu(t) - d\lambda(t), (1 - \frac{1}{d})\theta(t)) \) and the upper bound follows from [3]. Further insight can be gained if one fixes exact values of \( S, \lambda_k(t) \) and \( \mu_k(t) \). So let us assume that \( S = 2, \lambda(t) = 1 + \sin(2\pi t), \mu_k(t) = 3 \min(k, S) \).

Then if one puts \( d = \frac{8}{7} \) and \( d_k = \left(\frac{8}{7}\right)^{k-1} \) for \( k \geq 1 \), then the constants \( \beta^{**} \) and \( M \) from the Theorem 1 and Corollary 1 are equal to \( \beta^{**} = \frac{13}{28} \) and \( M = 1 \).

For the truncated process \( X^*(t) \) with the truncation threshold \( N = 150 \) one can put \( d^* = \frac{4}{7} \) and \( d_k^* = \left(\frac{4}{7}\right)^{k-1} \) for \( k \geq 1 \). Then the constants \( \beta^* \) and \( M^* \) from the Theorem 2 are equal to \( \beta^* = \frac{1}{3} \) and \( M^* = 1 \). Thus, since \( \|D(z(t) - z^*(t))\| \leq 2 \times 10^{-8} \) from (12) one gets
\[
\|p(t) - p^*(t)\| \leq 2 \times 10^{-8}.
\]

and from the comments, following the Theorem 2, one obtains
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \infty |k| |p_k(t) - p_k^*(t)| \leq 2 \times 10^{-8}.
\]

As expected in this example, the limiting average position \( EX(t) \) fluctuates around 0, whereas the limiting variance \( \text{Var}X(t) \) is not (see Fig. 2). It remains finite as the time becomes infinite.
As the second example we consider the double-ended queueing system with the state space $\mathcal{X} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. Let $X(t)$ be the queue length of taxi or passenger at time $t$. If $X(t) > 0$, the number of passengers in the system is $X(t)$ and there is no taxi queue. If $X(t) < 0$, the
number of taxis in the system is \(-X(t)\) and there is no passenger queue. If \(X(t) = 0\), there is no taxi nor passenger. Passengers and taxis arrive according to Poisson process. Passengers (one to four passengers traveling together are considered as one passenger) arrive to the queueing system according to a Poisson process with rate \(\lambda(t)\). Obviously, \(\{X(t), t \geq 0\}\) is a one-dimensional continuous time Markov chain. The dynamic control of taxi depends on the state \(X(t)\) of the system. If there is no passenger (i.e. \(X(t) \leq 0\)) waiting in the system, the taxi arrival rate is \(\mu_1(t)\), otherwise (i.e. \(X(t) \geq 0\)) the taxi arrival rate is \(\mu_2(t)\). Obviously, the arrival rate of taxis with passengers is higher than that without passengers, i.e. \(\mu_1(t) \leq \mu_2(t)\). Passengers and taxis match according to the first-in-first-out discipline and matching is instantaneous. The transposed intensity matrix \(A(t)\) for the considered problem has the following structure:

\[
A(t) = \begin{pmatrix}
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-3 & \cdots & -\mu_1 - \lambda & \mu_1 & 0 & 0 & 0 & 0 \\
-2 & \cdots & \lambda & -\mu_1 - \lambda & \mu_1 & 0 & 0 & 0 \\
-1 & \cdots & 0 & \lambda & -\mu_1 - \lambda & \mu_1 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & \lambda & -\mu_1 - \lambda & \mu_2 & 0 & 0 \\
1 & \cdots & 0 & 0 & 0 & \lambda & -\mu_2 - \lambda & \mu_2 & 0 \\
2 & \cdots & 0 & 0 & 0 & 0 & \lambda & -\mu_2 - \lambda & \mu_2 \\
3 & \cdots & 0 & 0 & 0 & 0 & 0 & \lambda & -\mu_2 - \lambda \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]

The \(\text{Theorem 1}\) yields the upper bound for the convergence rate, if a double infinite sequence, say \(\{d_k, k = \pm 1, \pm 2, \ldots\}\), can be found such that \(\int_0^\infty \beta^{**}(u) du = \infty\). Let us put \(d_1 = 1, d_{-1} = c\) and \(d_k = \delta^{k-1}, d_{-k} = c \cdot d_k\) for \(k \geq 2\), where \(\delta < 1\). Then we have:

\[
\beta^{**}(t) = \begin{cases} 
\left(\frac{1}{5} - 1\right) (\delta \mu_1(t) - \lambda(t)), & k < -1, \\
(1 - \delta) \mu_1(t) + \left(1 - \frac{1}{5}\right) \lambda(t), & k = -1, \\
(1 - \delta) \lambda(t) + \mu_2(t) - c \mu_1(t), & k = 1, \\
\left(\frac{1}{5} - 1\right) (\delta \lambda(t) - \mu_2(t)), & k > 1.
\end{cases} \tag{21}
\]

The value of \(\beta^{**}(t) = \min_{k \in \mathbb{Z} \setminus \{0\}} \beta_k^{**}(t)\) cannot be written out unless the exact values of \(\lambda(t)\) and \(\mu_k(t)\) are assumed. Let us fix \(\lambda(t) = 2 + \frac{\sin(2\pi t)}{4}\), \(\mu_1(t) = 1 + \frac{\sin(2\pi t)}{8}\) and \(\mu_2(t) = 4 + \frac{\cos(2\pi t)}{4}\). Then if one puts \(d = \frac{8}{7}, c = 2\)
and $d_k = \left(\frac{3}{2}\right)^{k-1}$ for $k \geq 1$, then the constants $\beta^{**}$ and $M$ from the Theorem 1 and Corollary 1 are equal to $\beta^{**} = 0.09375$ and $M = 1$. For the truncated process $X^*(t)$ with the truncation threshold $N = 150$ one can put $d^* = \sqrt{2}$, $c^* = 2$ and $d_k^* = \sqrt{2}^{k-1}$ for $k \geq 1$. Then the constants $\beta^*$ and $M^*$ from the Theorem 2 are equal to $\beta^* = 0.09375$ and $M^* = 1$. Thus, since $\|D(z(t) - z^*(t))\| \leq 10^{-7}$ from (12) one gets

$$\|p(t) - p^*(t)\| \leq 10^{-7}.$$ 

and from the comments, following the Theorem 2, one obtains

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} |k| \|p_k(t) - p_k^*(t)\| \leq 10^{-7}.$$ 

Fig. 3 shows the variation of $p_k(t)$ with $t$ for five different values $k$.

Figure 4: Limiting probability of the process $X(t)$ at time $t$, showing variation with $t$ for given positions $(-5, -2, 0, 2, 5)$.

In this example, as in the previous one, the limiting average position $\mathbb{E}X(t)$ fluctuates around 0 (see Fig. 4).

The limiting variance $\text{Var}X(t)$ is not around zero (see Fig. 5) and remains finite as the time becomes infinite.
5 Conclusion

The developed theory for bilateral BDPs facilitates their numerical analysis by providing upper ergodicity and truncation bounds. The latter can be used to understand when the limiting regime is reached and show to properly
truncate the bi-infinite state space. The weak point of the obtained results is the unknown bi-infinite sequence of positive numbers \( \{d_k\} \), for which no rule of thumb can be suggested and in each new use-case is has to be guessed. Having no probabilistic meaning this sequence can be considered as the analogue of Lyapunov functions.

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