Learning Adversarial Markov Decision Processes with Delayed Feedback

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Abstract

Reinforcement learning typically assumes that agents observe feedback for their actions immediately, but in many real-world applications (like recommendation systems) feedback is observed in delay. This paper studies online learning in episodic Markov decision processes (MDPs) with unknown transitions, adversarially changing costs and unrestricted delayed feedback. That is, the costs and trajectory of episode $k$ are revealed to the learner only in the end of episode $k + d^k$, where the delays $d^k$ are neither identical nor bounded, and are chosen by an oblivious adversary. We present novel algorithms based on policy optimization that achieve near-optimal high-probability regret of $\sqrt{K + D}$ under full-information feedback, where $K$ is the number of episodes and $D = \sum d^k$ is the total delay. Under bandit feedback, we prove similar $\sqrt{K + D}$ regret assuming the costs are stochastic, and $(K + D)^{2/3}$ regret in the general case. We are the first to consider regret minimization in the important setting of MDPs with delayed feedback.

1 Introduction

Delayed feedback is a fundamental challenge in sequential decision making arising in almost all practical applications. For example, recommendation systems learn the utility of a recommendation by detecting occurrence of certain events (e.g., user conversions), which may happen with a variable delay after the recommendation was issued. Other examples include display advertising, autonomous vehicles, video streaming (Changuel, Sayadi, and Kieffer 2012), delays in communication between learning agents (Chen et al. 2020) and system delays in robotics (Mahmood et al. 2018).

Although handling feedback delays is crucial for applying reinforcement learning (RL) in practice, it was only barely studied from a theoretical perspective, as most of the RL literature focuses on the MDP model in which the agent observes feedback regarding her immediate reward and transition to the next state right after performing an action.

This paper makes a substantial step towards closing the major gap on delayed feedback in the RL literature. We consider the challenging adversarial episodic MDP setting where cost functions change arbitrarily between episodes while the transition function remains stationary over time (but unknown to the agent). We present the adversarial MDP with delayed feedback model in which the agent observes feedback for episode $k$ only in the end of episode $k + d^k$, where the delays $d^k$ are unknown and not restricted in any way. This model generalizes standard adversarial MDPs (where $d^k = 0 \ \forall k$), and encompasses great challenges that do not arise in standard RL models, e.g., exploration without feedback and latency in policy updates. Adversarial models are extremely important in practice, as they allow dependencies between costs, unlike stochastic models that assume i.i.d. samples. This is especially important in the presence of delays (that are also adversarial in our model), since it allows dependencies between costs and delays which are well motivated in practice (Lancewicki et al. 2021).

We develop novel policy optimization (PO) algorithms that perform their updates whenever feedback is available and ignore feedback with large delay, and prove that they obtain high-probability regret bounds of order $\sqrt{K + D}$ under full-information feedback and $(K + D)^{2/3}$ under bandit feedback, where $K$ is the number of episodes and $D$ is the sum of delays. Unlike simple reductions that can only handle fixed delay $d$, our algorithms are robust to any kind of variable delays and do not require any prior knowledge. Furthermore, we show that a naive adaptation of existing algorithms suffers from sub-optimal dependence in the number of actions, and present a novel technique that forces exploration in order to achieve tight bounds. To complement our results, we present nearly matching lower bounds of order $\sqrt{K + D}$. See detailed bounds in Table 1.

1.1 Related work

Delays in RL. Although delay is a common challenge in RL algorithms need to face in practice (Schuitena et al. 2010, Liu, Wang, and Liu 2014, Changuel, Sayadi, and Kieffer 2012, Mahmood et al. 2018), the theoretical literature on the subject is very limited. Previous work only studied delayed state observability (Katsikopoulos and Engelbrecht 2003) where the state is observable in delay and the agent picks actions without full knowledge of its current state. This setting is much related to partially observable MDPs (POMDPs) and motivated by scenarios like robotics system delays. Unfortunately, even planning is computationally hard (exponential in the delay $d$) for
delayed state observability \cite{Walsh2009}.

This paper studies a different setting that we call delayed feedback, where the delay only affects the information available to the agent, and not the execution of its policy. Delayed feedback is also an important setting, as it is experienced in recommendation systems and applications where the policy is executed by a different computational unit than the main algorithm (e.g., policy is executed by a robot with limited computational power, while heavy computations are done by the main algorithm on another computer that receives data from the robot in delay). Importantly, unlike delayed state observability, it is not computationally hard to handle delayed feedback, as we show in this paper. The challenges of delayed feedback are very different from the ones of delayed state observability, and include policy updates that occur in delay and exploration without observing feedback.

\textbf{Delays in multi-armed bandit.} Delays were extensively studied in MAB recently as a fundamental issue that arises in many real applications \cite{Vernade2017, Pfister2017, Cesa-Bianchi2018a, ZhouXu2019, Rueckert2020, Lancewicz2021}. Our work is most related to the literature on delays in adversarial MAB, starting with \cite{Cesa-Bianchi2016} that showed the optimal regret for MAB with fixed delay \(d\) is of order \(\sqrt{(A + d)K}\), where \(A\) is the number of actions. Even earlier, variable delays were studied by \cite{Quanrud2015} in online learning with full-information feedback, where they showed optimal \(\sqrt{K + D}\) regret. More recently, \cite{Thune2019, Bistritz2019, Zimmert2020, Gyorgy2020} studied variable delays in MAB, proving optimal \(\sqrt{AK + D}\) regret. Unlike MDPs, in MAB there is no underlying dynamics, and the only challenge is feedback about the cost arriving in delay.

\textbf{Regret minimization in stochastic MDPs.} There is a vast literature on regret minimization in RL that mostly builds on the optimism in face of uncertainty principle. Most literature focuses on the tabular setting, where the number of states is small (see, e.g., \cite{Jaksch2010, Azar2017, Jin2018, Zanette2019}). Recently it was extended to function approximation under various assumptions (see, e.g., \cite{Jin2020}).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 & Known Transition & Unknown Transition & Unknown Transition \\
 & + Delayed Trajectory & + Delayed Cost & + Delayed Trajectory \\
\hline
D-O-REPS (full) & \(H\sqrt{K + D}\) & \(H^{3/2}\sqrt{AK + H\sqrt{D}}\) & \(H^{3/2}\sqrt{AK + H\sqrt{D}}\) \\
D-OPO (full) & \(H\sqrt{K + D}\) & \(H^{3/2}\sqrt{AK + H\sqrt{D}}\) & \(H^{3/2}\sqrt{AK + H\sqrt{D}}\) \\
Lower Bound (full) & \(H\sqrt{K + D}\) & \(H^{3/2}\sqrt{SAK + H\sqrt{D}}\) & \(H^{3/2}\sqrt{SAK + H\sqrt{D}}\) \\
\hline
D-OPO (bandit) & \(HS\sqrt{AK^{2/3} + D^{2/3}}\) & \(HS\sqrt{AK^{2/3} + D^{2/3}}\) & \(HS\sqrt{AK^{2/3} + D^{2/3}}\) \\
Lower Bound (bandit) & \(H\sqrt{SAK + H\sqrt{D}}\) & \(H^{3/2}\sqrt{SAK + H\sqrt{D}}\) & \(H^{3/2}\sqrt{SAK + H\sqrt{D}}\) \\
\hline
\end{tabular}
\caption{Regret bounds comparison (ignoring constant and poly-logarithmic factors) between our algorithms Delayed OPO (D-OPO) and Delayed O-REPS (D-O-REPS), and our lower bound under full-information (full) and bandit feedback (bandit). “Known Transition” assumes costs are observed in delay, while “Unknown Transition” assumes only costs are observed in delay, while in “Delayed Trajectory” the trajectory is also observed in delay, together with the costs.}
\end{table}

Adversarial MDPs. Early works on adversarial MDPs \cite{Even-Dar2009, Neu2013} presented O-REPS -- a reduction to online linear optimization achieving optimal regret bounds with known dynamics. Later, O-REPS was extended to unknown dynamics \cite{Rosenberg2019, Jin2020b, Jin2020a} obtaining near-optimal regret bounds. Recently, \cite{Cai2020, Shani2020, He2021} proved similar regret for PO methods (that are widely used in practice).

\section{Setting}

An episodic adversarial MDP is defined by a tuple \(\mathcal{M} = (S, A, H, p, \{c^k\}_{k=1}^K)\), where \(S\) and \(A\) are finite state and action spaces of sizes \(S\) and \(A\), respectively, \(H\) is the episode length, \(p = \{p_h : S \times A \rightarrow \Delta_{\mathcal{S}}\}_{h=1}^H\) is the transition function, and \(c^k = \{c^k_h : S \times A \rightarrow [0, 1]\}_{h=1}^H\) is the cost function for episode \(k\). For simplicity, \(S \geq \max[A, H^2]\).

The interaction between the learner and the environment proceeds as follows. At the beginning of episode \(k\), the learner starts in a fixed initial state \(s_h^0 = s_{\text{init}} \in S\) and picks a policy \(\pi^k = \{\pi^k_h : S \rightarrow \Delta_{\mathcal{A}}\}_{h=1}^H\) that gives the probability that the agent takes action \(a\) at time \(h\) given that the current state is \(s\). Then, the policy is executed in the MDP generating a trajectory \(U^k = \{(s_h^k, a_h^k)\}_{h=1}^{H_k}\), where \(a_h^k \sim \pi_h^k(\cdot|s_h^k)\) and \(s_{h+1}^k \sim p_h(\cdot|s_h^k, a_h^k)\). With no delays, the learner observes the feedback in the end of the episode, that is, the trajectory \(U^k\) and either the entire cost function \(c^k\) under full-information feedback or the suffered costs \(\{c^k_h(\pi_h^k, a_h^k)\}_{h=1}^{H_k}\) under bandit feedback. In contrast, with delayed feedback, these are revealed to the learner only in the end of episode \(k + d^k\), where the delays \(\{d^k\}_{k=1}^K\) are unknown and chosen by an oblivious adversary before the interaction starts. Denote the total delay by \(D = \sum_k d^k\) and the maximal delay by \(d_{\text{max}} = \max_k d^k\). Note that standard adversarial MDPs are a special case in which \(d^k = 0\ \forall k\).

For a given policy \(\pi\), we define its expected cost with respect to cost function \(c\), when starting from state \(s\) at

\[\text{Regret Minimization in Stochastic MDPs.}
\]
time \( h \), as \( V^\pi_h(s) = \mathbb{E}\left[\sum_{i=h}^\infty c_h(s_{i-h}, a_{i-h})|s_h = s, \pi, p\right] \)
where the expectation is taken over the randomness of the transition function \( p \) and the policy \( \pi \). This is known as the value function of \( \pi \), and we also define the Q-function by
\[
Q^\pi_h(s, a) = \mathbb{E}\left[\sum_{i=h}^\infty c_h(s_{i-h}, a_{i-h})|s_h = s, a_h = a, \pi, p\right].
\]
It is well-known (see Sutton and Barto (2018)) that the value function and Q-function satisfy the Bellman equations:
\[
Q^\pi_h(s, a) = c_h(s, a) + \mathbb{E}_p\left[Q^\pi_{h+1}(s', a')\right]
\]
where \( \langle \cdot, \cdot \rangle \) is the dot product. Let \( V^{k,\pi} \) be the value function of \( \pi \) with respect to \( c^k \). We measure the performance of the learner by the regret – the cumulative difference between the value of the learner’s policies and the value of the best fixed policy in hindsight, i.e.,
\[
\mathcal{R}_K = \sum_{k=1}^{K} V^{k,\pi^*}(s^k_1) - \min_{\pi} \sum_{k=1}^{K} V^{k,\pi}(s^k_1).
\]

### Notations.
Episode indices appear as superscripts and in-episode steps as subscripts. \( \mathcal{F}^k = \{ j : j + d^j = k \} \) denotes the set of episodes that their feedback arrives in the end of episode \( k \), and the number of visits to state-action pair \( (s, a) \) at time \( h \) by the end of episode \( k - 1 \) is denoted by \( m^k_h(s, a) \). Similarly, \( n^k_h(s, a) \) denotes the number of these visits for which feedback was observed until the end of episode \( k - 1 \). \( \mathbb{E}^\pi[\cdot] = \mathbb{E}[\cdot|s^1_1 = s_{\text{init}}, \pi, p] \) denotes the expectation given a policy \( \pi \), the notation \( O(\cdot) \) ignores constant and poly-logarithmic factors and \( x \lor y = \max(x, y) \). We denote the set \( \{1, \ldots, n\} \) by \( [n] \), and the indicator of event \( E \) by \( \mathbb{I}[E] \).

### 3 Warm-up: a black-box reduction
One simple way to deal with delays (adopted in several MAB and online optimization works, e.g., Weinberger and Ordentlich (2002), Joulani, Gyorgy, and Szepesvari (2013)) is to simulate a non-delayed algorithm and use its regret guarantees. Specifically, we can maintain \( d_{\text{max}} + 1 \) instances of the non-delayed algorithm, running the \( i \)-th instance on the episodes \( k \) such that \( k = i \mod (d_{\text{max}} + 1) \). That is, at the first \( d_{\text{max}} + 1 \) episodes, the learner plays the first policy that each instance outputs. By the end of episode \( d_{\text{max}} + 1 \), the feedback for the first episode is observed, allowing the learner to feed it to the first instance. The learner would then play the second output of that instance, and so on. Effectively, each instance plays \( K/(d_{\text{max}} + 1) \) episodes, so we can use the regret of the non-delayed algorithm \( \mathcal{R}_K \) in order to bound \( \mathcal{R}_K \leq (d_{\text{max}} + 1)\mathcal{R}_K/(d_{\text{max}} + 1) \). Plugging in standard adversarial MDP regret bounds (Rosenberg and Mansour (2019a) Jim et al. (2020a), we obtain the following regret for both full-information and bandit feedback:
\[
\mathcal{R}_K = \tilde{O}(H^2S\sqrt{AK(d_{\text{max}} + 1)} + H^2S^2A(d_{\text{max}} + 1)).
\]

While simple in concept, the black-box reduction suffers from many evident shortcomings. First, it is highly non-robust to variable delays as its regret scales with the worst-case delay \( Kd_{\text{max}} \) which becomes very large even if the feedback from just one episode is missing. One of the major challenges that we tackle in the rest of the paper is to achieve regret bounds that are independent of \( d_{\text{max}} \) and scale with the average delay, i.e., the total delay \( D \) which is usually much smaller than worst-case. Second, even if we ignore the problematic dependence in the worst-case delay, this regret bound is still sub-optimal as it suggests a multiplicative relation between \( d_{\text{max}} \) and \( A \) (and \( S^2 \)) which does not appear in the MAB setting. Our analysis focuses on eliminating this sub-optimal dependence through a clever algorithmic feature that forces exploration and ensures tight near-optimal regret. Finally, the reduction is highly inefficient as it requires running \( d_{\text{max}} + 1 \) different algorithms in parallel. Moreover, the \( \sqrt{Kd_{\text{max}}} \) regret under bandit feedback is only achievable using O-REPS algorithms that are extremely inefficient to implement in practice. In contrast, our algorithm is based on efficient and practical PO methods. Its running time is independent of the delays and it does not require any prior knowledge or parameter tuning (unlike the reduction needs to know \( d_{\text{max}} \)). In Section 5 we present experiments showing that our algorithm outperforms generic approaches, such as black-box reduction, not only theoretically but also empirically.

### 4 Delayed OPPO
In this section we present Delayed OPPO (Algorithm 1) and with more details in Appendix A – the first algorithm for regret minimization in adversarial MDPs with delayed feedback. Delayed OPPO is a policy optimization algorithm, and therefore implements a smoother version of Policy Iteration (Sutton and Barto (2018), i.e., it alternates between a policy evaluation step – where an optimistic estimate for the Q-function of the learner’s policy is computed, and a policy improvement step – where the learner’s policy is improved in a “soft” manner regularized by the KL-divergence.

#### Algorithm 1: Delayed OPPO

**Input:** \( S, A, H, \eta > 0, \gamma > 0, \delta > 0 \).

**Initialization:** Set \( \pi_0^k(a|s) = 1/A \) for every \( (s, a, h) \).

For \( k = 1, 2, \ldots, K \) do

- Play episode \( k \) with policy \( \pi^k \).
- Observe feedback from all episodes \( j \in \mathcal{F}^k \).
- Compute cost estimators \( \hat{c}^j \) and confidence set \( \mathcal{P}^k \).

**# Policy Evaluation**

For \( j \in \mathcal{F}^k \) do

- Set \( V^j_{H+1}(s) = 0 \) for every \( s \in S \).
- For \( h = H, \ldots, 1 \) and \( (s, a) \in S \times A \) do

  - \( \hat{p}_h^j(\cdot|s, a) = \arg \min_{p_h(\cdot|s, a) \in \mathcal{P}^k_h(s, a)} \left\{ \hat{p}_h^j(\cdot|s, a), V^j_{h+1}\right\} \).
  - \( Q^k_h(s, a) = \hat{c}_h^j(s, a) + \langle \hat{p}_h^j(\cdot|s, a), V^j_{h+1}\rangle \).
  - \( V^j_h(s) = \langle Q^k_h(s, \cdot), \pi^k(\cdot|s)\rangle \).

End for

**# Policy Improvement**

- \( \pi_{k+1}^k(a|s) = \pi^k_h(a|s) \exp\left( -\eta \sum_{j \in \mathcal{F}^k} Q_{h}^j(s, a) \right) / \sum_{a' \in A} \pi^k_h(a'|s) \exp\left( -\eta \sum_{j \in \mathcal{F}^k} Q_{h}^j(s, a') \right) \).

End for
optimization (OPPO) algorithm (Car et al. [2020], Shani et al. [2020]). As a policy optimization algorithm, it enjoys many merits of practical PO algorithms that have had great empirical success in recent years, e.g., TRPO (Schulman et al. [2015]), PPO (Schulman et al. [2017]) and SAC (Haarnoja et al. [2018]) – It is easy to implement, computationally efficient and readily extends to function approximation.

The main difference that Delayed OPPO introduces is performing updates using all the available feedback at the current time step. Furthermore, in Sections 4.1 and 4.2 we equip our algorithm with novel mechanisms that make it robust to all kinds of variable delays without any prior knowledge and enable us to prove tight regret bounds. Importantly, these mechanisms improve existing results even for the fundamental problem of delayed MAB. Even with these algorithmic mechanisms, proving our regret bounds requires careful analysis and new ideas that do not appear in the MAB with delays literature, as we tackle the much more complex MDP environment.

In the beginning of episode $k$, the algorithm computes an optimistic estimate $Q^k$ of $Q^π$ for all the episodes $j$ that their feedback just arrived. To that end, we maintain confidence sets that contain the true transition function $p$ with high probability, and are built using all the trajectories available at the moment. That is, for every $(s, a, h)$, we compute the empirical transition function $\hat{p}_h^k(s' \mid s, a)$ and define the confidence set $P_h^k(s, a)$ as the set of transition functions $p_h^k(\cdot \mid s, a)$ such that, for every $s' \in S$,

$$|p_h^k(s' \mid s, a) - \hat{p}_h^k(s' \mid s, a)| \leq c_h^k(s' \mid s, a),$$

where $c_h^k(s' \mid s, a) = \Theta(\sqrt{\hat{p}_h^k(s' \mid s, a)/n_h^k(s, a)} + 1/n_h^k(s, a))$ is the confidence set radius. Then, the confidence set for episode $k$ is defined by $\tilde{P}_h^k = \{P_h^k(s, a)\}_{s, a, h}$. Under bandit feedback, the computation of $Q^k$ also requires including the cost function $c^j$ in state-action pairs that were not visited in that episode. For building these estimates, we utilize optimistic importance-sampling estimators (Jin et al. [2020a]) that first optimistically estimate the probability to visit each state $s$ in each time $h$ of episode $j$ by $u_h^j(s) = \max_{a \in A} \tilde{P}_h^k(s, a)$, then set the estimator to be $c_h^j(s, a) = \tilde{c}_h^j(s, a) = \max_{a \in A} \tilde{P}_h^k(s, a)$ with an exploration parameter $\gamma > 0$.

After the optimistic $Q$-functions are computed, we use them to improve the policy via a softmax update, i.e., we update $\pi_{k+1}^h(a \mid s) \propto \pi_h^k(a \mid s) \exp(-\eta \sum_{j \in F_h} Q^k_h(s, a))$ for learning rate $\eta > 0$. This update form, which may be characterized as an online mirror descent (Beck and Teboulle [2003]) step with KL-regularization, stands in the heart of the following regret analysis (full proofs in Appendix B). We note that [Theorem 1] handles only delayed feedback regarding the costs, while assuming that feedback regarding the learner’s trajectory arrives without delay.

**Theorem 1.** Running Delayed OPPO with delayed cost feedback and non-delayed trajectory feedback guarantees, with probability $1 - \delta$, under full-information feedback:

$$R_K = O(HS/\sqrt{AK}^2 + H^2 D^2/3 + H^2 d_{max}).$$

and under bandit feedback:

$$R_K = O(HS/\sqrt{AK}^2 + H^2 D^2/3 + H^2 d_{max}).$$

**Proof sketch.** With standard regret decomposition (based on the difference value lemma), we can show that the regret scales with two main terms: ($A = \sum_k V_h^k(s_k)$) − $V_h^k(s_k)$ is the bias between the estimated and true value of $\pi^k$; and ($B = \sum_k E^k[Q_h^k(s_k^*) - \pi_h^k(\cdot \mid s_k^*)]$) which, for a fixed $(s, h) \in S \times [H]$, can be viewed as a regret of a delayed MAB algorithm with full-information feedback, where the losses are the estimated $Q$-functions.

Since the trajectories are not observed in delay, we can bound term (A) similarly to Shani et al. (2020) using our confidence sets that shrink over time. To bound term (B), we fix $(s, h)$ and follow a “cheating” algorithm technique (György and Joulani [2020]). To that end, we define the “cheating” algorithm that does not experience delay and sees one step into the future, i.e., in episode $k$ it plays policy $\pi_{k+1}^h(a \mid s) \propto e^{-\eta \sum_{j=1}^k Q^k_h(s, a)}$. Then, we can break term (B) into two terms: (i) The regret of the “cheating” algorithm which is bounded by $\log Q_k + 1$ using a Be-The-Leader argument (see, e.g., Joulani, György, and Szepesvári [2020]), and (ii) The difference between $\pi_{k+1}^h$ and $\pi_k^h$ which can be bound by looking at the exponential weights update form. Specifically, we bound the ratio $\pi_{k+1}^h(s, a)/\pi_k^h(s, a)$ from below by $1 - \eta \sum_{j \in F_h} \pi_{k+1}^h(s, a)$, and this bounds term (ii) in terms of the missing feedback, i.e.,

$$\sum_{k=1}^K \sum_{a \in A} \pi_h^k(a \mid s) Q_h^k(s, a) \leq \sum_{k=1}^K \sum_{a \in A} \pi_h^k(a \mid s) Q_h^k(s, a) - \eta \sum_{k=1}^K \sum_{a \in A} \pi_h^k(a \mid s) Q_h^k(s, a).$$

Under full-information feedback, our estimates of the $Q$-function are always bounded by $H$, which leads to

$$\sum_{k=1}^K \sum_{a \in A} \pi_h^k(a \mid s) Q_h^k(s, a) \leq \sum_{k=1}^K \sum_{a \in A} \pi_h^k(a \mid s) Q_h^k(s, a) - \eta H^2 \sum_{k=1}^K \sum_{a \in A} \pi_h^k(a \mid s) Q_h^k(s, a).$$

To finish the proof we set $\eta = 1/H/\sqrt{K+D}$. Under bandit feedback, this argument becomes a lot more delicate because the $Q$-function estimates are naively bounded only by $H/\gamma$. Thus, we need to prove concentration of $\sum_k V_h^k(s)$ around $\sum_k V_h^\pi(s)$ (which is indeed bounded by $HK$). \qed

Notice that the regret bound in [Theorem 1] overcomes the major problems that we had with the black-box reduction
approach. Namely, the regret scales with the total delay $D$ and not the worst-case delay $Kd_{\text{max}}$ (the extra additive dependence in $d_{\text{max}}$ is avoided altogether in Section 4.2, and $D$ is not multiplied by neither $S$ nor $A$. Finally, as a direct corollary of Theorem 1 we deduce the regret bound for the known transitions case, in which term (A) does not appear (at least under full-information feedback). Notice that with known transitions, there is no need to handle delays in the trajectory feedback since dynamics are known.

**Theorem 2.** Running Delayed OPPO with known transition function guarantees, with probability $1 - \delta$, under full-information feedback: $R_K = \tilde{O}(H^2S\sqrt{AK} + D/\delta)$, and bandit feedback: $R_K = \tilde{O}(HS\sqrt{AK}^2/3 + H^2D^{2/3} + H^2d_{\text{max}})$.

### 4.1 Handling delayed trajectories

Previously, we analyzed the Delayed OPPO algorithm in the setting where only cost is observed in delay. In this section, we face the delayed trajectory feedback setting in which the trajectory of episode $k$ is observed only in the end of episode $k + d_k$ together with the cost. We emphasize that, while the trajectory from episode $k$ is observed in delay, the policy $\pi^k$ is executed regularly (see discussion in Section 1.1).

Delayed trajectory feedback is a unique challenge in MDPs that does not arise in MAB, as no underlying dynamics exist. With this technique we are able to isolate the first $d_{\text{max}}$ visits to each state-action pair, and for other visits use the fact that some knowledge of the transition function is already evident. This with the technique we are able to get the improved bound $(A) \lesssim H\sqrt{S} \sum_{s \in S} \sum_{a \in A} \sum_{h=1}^K \frac{\mu^k_h(s, a)}{\sqrt{n^k_h(s, a)}}$. (2)

Now we address the delays. Fix $(s, a, h)$ and denote the number of unobserved visits by $N^k_h(s, a) = (m^k_h(s, a) - n^k_h(s, a))$. Next, we decouple the statistical estimation error and the effect of the delays in the following way,  

$$ \sum_k \frac{\mu^k_h(s, a)}{\sqrt{n^k_h(s, a)}} \leq \sum_k \frac{\mu^k_h(s, a)}{\sqrt{m^k_h(s, a)}} \sqrt{1 + \frac{N^k_h(s, a)}{n^k_h(s, a)}} \leq \sum_k \frac{\mu^k_h(s, a)}{\sqrt{m^k_h(s, a)}} + \sum_k \frac{\mu^k_h(s, a)}{\sqrt{m^k_h(s, a)}} \sqrt{\frac{N^k_h(s, a)}{n^k_h(s, a)}}. $$ (3)

The first term is unaffected by delays and bounded by $H^2S\sqrt{AK}$. For the second term, we utilize explicit exploration in the sense that $n^k_h(s, a) \geq d_{\text{max}}$. Combine this with the observation that $N^k_h(s, a) \leq d_{\text{max}}$ (since $d_{\text{max}}$ is the maximal delay), to obtain the bound $H^2SA\sqrt{K}$. Finally, to get the tight bound (i.e., eliminate the extra $\sqrt{A}$),
we split the second sum into: (1) episodes with \( n_k^j(s, a) \geq d_{\text{max}} \) where \( N_k^j(s, a) \) is tightly bounded by 1 (and not \( A \)), and (2) episodes with \( n_k^j(s, a) < d_{\text{max}} \) in which the regret scales as \( \sqrt{d_{\text{max}}} \) (which is at most \( \sqrt{K} \)).

### 4.2 Large delays and unknown total delay

In this section we address two final issues with our Delayed OPPO algorithm. First, we eliminate the dependence in the maximal delay \( d_{\text{max}} \) that may be as large as \( K \) even when the total delay is relatively small. Second, we avoid the need for any prior knowledge regarding the delays which is hardly ever available, making the algorithm parameter-free.

To handle large delays, we use a skipping technique (Thune, Cesa-Bianchi, and Seldin 2019). That is, if some feedback arrives in delay larger than \( \beta \) (where \( \beta > 0 \) is a skipping parameter), we just ignore it. Thus, effectively, the maximal delay experienced by the algorithm is \( \beta \), but we also need to bound the number of skipped episodes. To that end, let \( K_\beta \) be the set of skipped episodes and note that \( D = \sum_{k=1}^K d_k \geq |K_\beta|/\beta \), implying that the number of skipped episodes is bounded by \( |K_\beta| \leq D/\beta \). In Appendix C we apply the skipping technique to all the settings considered in the paper to obtain the final regret bounds in Table 1. Here, we take the unknown transitions case with delayed trajectory feedback and under full-information feedback as an example.

Setting \( \beta = \sqrt{H/\max S} \) yields the following bound that is independent of the maximal delay \( d_{\text{max}} \):

\[
R_K = \tilde{O}(H^2S\sqrt{AK} + H^2\sqrt{D} + H^2S\beta + HD/\beta) = \tilde{O}(H^2S\sqrt{AK} + H^{3/2}\sqrt{SD}).
\]

To address unknown number of episodes and total delay, we design a new doubling scheme. Unlike Bistritz et al. (2019) that end up with a worse bound in delayed MAB due to doubling, we carefully tuned mechanism obtains the same regret bounds (as if \( K \) and \( D \) were known). Moreover, when applied to MAB, our technique confirms the conjecture of Bistritz et al. (2019) that optimal regret with unknown \( K \) and \( D \) is achievable using a doubling scheme (due to lack of space, we defer the details to Appendix D). Note that \( K \) and \( D \) are the only parameters that the algorithm requires, since the skipping scheme replaces the need to know \( d_{\text{max}} \) with the parameter \( \beta \) (which is tuned using \( D \)). The doubling scheme manages the tuning of the algorithm’s parameters \( \eta, \gamma, \beta \), making it completely parameter-free and eliminating the need for any prior knowledge regarding the delays.

The doubling scheme maintains an optimistic estimate of \( D \) and uses it to tune the algorithm’s parameters. Every time the estimate doubles, the algorithm is restarted with the new doubled estimate. This ensures that our optimistic estimate is always relatively close to the true value of \( D \) and that the number of restarts is only logarithmic, allowing us to keep the same regret bounds. The optimistic estimate of \( D \) is computed as follows. Let \( M^k \) be the number of episodes with missing feedback at the end of episode \( k \). Notice that \( \sum_{k=1}^K M^k \leq D \) because the feedback from episode \( j \) was missing in exactly \( d_j \) episodes. Thus, at the end of episode \( k \) our optimistic estimate is \( \sum_{j=1}^k M^j \). So for every episode \( j \) with observed feedback, its delay is estimated by exactly \( d_j \), and if its feedback was not observed, then we estimate it as if feedback will be observed in the next episode.

In Appendix D we give the full pseudo-code of Delayed OPPO when combined with doubling, and formally prove that our regret bounds are not damaged by doubling.

### 5 Additional results and empirical evaluation

#### Lower bound.

For episodic stochastic MDPs, the optimal minimax regret bound is \( \Theta(H^{3/2}\sqrt{SAK}) \) (Azar, Osband, and Munos 2017; Jin et al. 2018). As adversarial MDPs generalize the stochastic MDP model, this lower bound also applies to our setting. The lower bound for multi-arm bandits with delays is based on a simple reduction to non-delayed MAB with full-information feedback. Namely, we can construct a non-delayed algorithm for full-information feedback using an algorithm \( A \) for fixed delay \( d \) by simply feeding \( A \) with the same cost function for \( d \) consecutive rounds. Using the known lower bound for full-information MAB, this yields a \( \Omega(\sqrt{dK}) = \Omega(\sqrt{D}) \) lower bound which easily translates to a \( \Omega(H\sqrt{D}) \) lower bound in adversarial MDPs. Combining these two bounds gives a lower bound of \( \Omega(H^{3/2}\sqrt{SAK} + H\sqrt{D}) \) for all settings, except for full-information feedback with known dynamics where the lower bound is \( \Omega(H\sqrt{K + D}) \). In light of this lower bound, we discuss the regret of Delayed OPPO and open problems.

For bandit feedback, our \((K + D)^{3/2}\) regret bounds are still far from the lower bound. However, it is important to emphasize that we cannot expect more from PO methods. Our bounds match state-of-the-art regret bounds for policy optimization under bandit feedback (Shani et al. 2020). It is an open problem whether PO methods can obtain \( \sqrt{K} \) regret in adversarial MDPs under bandit feedback (even with known dynamics). Currently, the only algorithm with \( \sqrt{K} \) regret for this setting is O-REPS (Jin et al. 2020a). It remains an important and interesting open problem to extend it to delayed feedback in the bandit case (see next paragraph).

Under full-information feedback, our regret bounds match the lower bound up to a factor of \( \sqrt{S} \) (there is also sub-optimal dependence in \( H \) but it can be avoided with Delayed O-REPS as discussed in the next paragraph). However, this extra \( \sqrt{S} \) factor already appears in the regret bounds for adversarial MDPs without delays (Rosenberg and Mansour 2019; Jin et al. 2020a). Determining the correct dependence in \( S \) for adversarial MDPs is an important open problem that must be solved without delays first. We note that if only cost feedback is delayed (and not trajectory feedback), then the delays are not entangled in the estimation of the transition function, and therefore the \( \sqrt{D} \) term in our regret is optimal!

Another important note: even with delayed trajectory feedback, our \( \sqrt{D} \) term is still optimal for a wide class of delays – monotonic delays. That is, if the sequence of delays is monotonic, i.e., \( d_j \leq d_k \) for \( j < k \), then the \( \sqrt{D} \) term of our regret bound for delayed trajectory feedback is not multiplied by \( \sqrt{S} \). This follows because in this case term \( (A) \) that handles estimation error of \( p \) can be analysed with respect to the actual number of visitation, since by the time
we estimate \( Q^k \) at the end of episode \( k + d^k \) we already have all the feedback for \( j < k \). Monotonic delays include the fundamental setting of a fixed delay \( d \).

**O-REPS vs OPPO.** PO methods directly optimize the policy. Practically, this translates to estimating the \( Q \)-function and then applying a closed-form update to the policy in each state. Alternatively, O-REPS methods (Zimin and Neu [2013]) optimize over the state-action occupancy measures instead of directly on policies. This requires solving a global convex optimization problem of size \( H^2 S A \) (Rosenberg and Mansour [2019a]) in the beginning of each episode, which has no closed-form solution and is extremely inefficient computationally. Another significant shortcoming of O-REPS is the difficulty to scale it up to function approximation, since the constrained optimization problem becomes non-convex. On the other hand, PO methods extend naturally to function approximation and enjoy great empirical success (e.g., Haarnoja et al. [2018]).

Other than their practical merits, this paper reveals an important theoretical advantage of PO methods over O-REPS – simple update form. We utilize the exponential weights update form of Delayed OPPO in order to investigate the propagation of delayed feedback through the episodes. This results in an intuitive analysis that achieves the best available PO regret bounds even when feedback is delayed. On the other hand, there is very limited understanding regarding the solution for the O-REPS optimization problem, making it very hard to extend beyond its current scope. Specifically, studying the effect of delays on this optimization problem is extremely challenging and takes involved analysis. While we were able to analyze Delayed O-REPS under full-information feedback (Appendix E) and give tight regret bounds (Theorem 4), we were not able to extend our analysis to bandit feedback because it involves a complicated in-depth investigation of the difference between any two consecutive occupancy measures chosen by the algorithm. Our analysis bounds this difference under full-information feedback, but in order to bound the regret under bandit feedback its ratio (and the high variance of importance-sampling estimators) must also be bounded. Extending Delayed O-REPS to bandit feedback remains an important open problem, for which our analysis lays the foundations, and is currently the only way that can achieve \( \sqrt{K} \) regret in the presence of delays.

**Theorem 4.** Running Delayed O-REPS under full-information feedback guarantees, with probability \( 1 - \delta \), with known transitions: \( R_K = \tilde{O}(H^{3/2} K + D) \), and with unknown dynamics, delayed cost feedback and non-delayed trajectory feedback: \( R_K = \tilde{O}(H^3 S \sqrt{AK} + H^\sqrt{D}) \).

**Stochastic MDP with delayed feedback.** Most of the RL literature has focused on stochastic MDPs – a special case of adversarial MDPs where cost \( c_k^j(s, a) \) of episode \( k \) is sampled i.i.d from a fixed distribution \( C_h(s, a) \). Thus, studying the effects of delayed feedback on stochastic MDPs is a natural question. With stochastic costs, OPPO obtains \( \sqrt{K} \) regret even under bandit feedback, since we can replace importance-sampling estimators with an empirical average.

This means that with stochastic costs and bandit feedback, our Delayed OPPO algorithm obtains the same near-optimal regret bounds as under full-information feedback. However, the \( \sqrt{D} \) lower bound heavily relies on adversarial costs, as it uses a sequence of costs that change every \( d \) episodes, suggesting that \( \sqrt{D} \) dependence might not be necessary.

Indeed, for stochastic cost, delayed versions of optimistic algorithms (e.g., Zanette and Brunskill [2018]) have regret scaling as the estimation error (term (A) in Eq. 12), which means that our analysis (Appendix D) proves regret that does not scale with \( \sqrt{D} \) but only with \( H^2 S d_{\text{max}} \). Again, this can be improved to \( H^2 S d_{\text{max}} \) using explicit exploration.

**Theorem 5.** Running an optimistic algorithm with explicit exploration, with delayed bandit cost feedback and delayed trajectory feedback guarantees, with probability \( 1 - \delta \), regret bound of \( \tilde{O}(H^2 S \sqrt{AK} + H^2 S d_{\text{max}}) \) in stochastic MDPs.

This contribution is important, even for the rich literature on delayed stochastic MAB. Lancewicki et al. [2021] show that the (optimistic) UCB algorithm may suffer sub-optimal regret of \( A d_{\text{max}} \). Furthermore, they were able to remove the \( A \) factor by an action-elimination algorithm which explores active arms equally. Since optimism is currently the only approach for handling unknown transitions in adversarial MDPs, it was crucial for us to find a novel solution to handle delays in optimistic algorithms. Theorem 5 shows that optimistic algorithms (like UCB) can indeed be “fixed” to handle delays optimally, using explicit exploration.

**Empirical evaluation.** We used synthetic experiments to compare the performance of Delayed OPPO to two other generic approaches for handling delays: Parallel-OPPO – running in parallel \( d_{\text{max}} \) online algorithms, as described in Section 3, and Pipeline-OPPO – another simple approach for turning a non-delayed algorithm to an algorithm that handles delays by simply waiting for the first \( d_{\text{max}} \) episodes and then feeding the feedback always with delay \( d_{\text{max}} \). We used a simple \( 10 \times 10 \) grid world (with \( H = 50, K = 500 \)) where the agent starts in one corner and needs to reach the opposite corner, which is the goal state. The cost is 1 in all states except for \( 0 \) cost in the goal state. Delays are drawn i.i.d from a geometric distribution with mean 10, and the maximum delay \( d_{\text{max}} \) is computed on the sequence of realized delays (it is roughly \( 10 \log K \approx 60 \)).

Fig. 1 shows Delayed OPPO significantly outperforms the other approaches, thus highlighting the importance of handling variable delays and not simply considering the worst-case delay \( d_{\text{max}} \). An important note is that, apart from its very high cost, Parallel-OPPO also requires much more memory (factor \( d_{\text{max}} \) more). For more implementation details and additional experiments, see Appendix F.

![Figure 1: Average cost of delayed algorithms in grid world with geometrically distributed delays.](image-url)
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A  The Delayed OPPO Algorithm

Algorithm 2: Delayed OPPO with known transition function

**Input:** State space \( \mathcal{S} \), Action space \( \mathcal{A} \), Horizon \( H \), Transition function \( p \), Learning rate \( \eta > 0 \), Exploration parameter \( \gamma > 0 \).

**Initialization:** Set \( \pi^1_h(a | s) = 1/A \) for every \( (s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H] \).

for \( k = 1, 2, \ldots, K \) do

Play episode \( k \) with policy \( \pi^k \).

Observe feedback from all episodes \( j \in \mathcal{F}^k \).

# Policy Evaluation

for \( j \in \mathcal{F}^k \) do

\( \forall s \in \mathcal{S} : V^j_{H+1}(s) = 0. \)

for \( h = H, \ldots, 1 \) and \( (s, a) \in \mathcal{S} \times \mathcal{A} \) do

if bandit feedback then

\( \hat{c}^j_h(s, a) = \frac{\hat{c}^j_h(s, a) \cdot q^j_h(s, a)}{\gamma^j_h} \).

else if full-information feedback then

\( \hat{c}^j_h(s, a) = c^j_h(s, a). \)

end if

\( Q^j_h(s, a) = \hat{c}^j_h(s, a) + \langle p_h(\cdot | s, a), V^1_h \rangle. \)

\( V^j_h(s) = \langle Q^j_h(\cdot, \cdot), \pi^j_h(\cdot | s) \rangle. \)

end for

end for

# Policy Improvement

for \( (s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H] \) do

\( \pi^{k+1}_h(a | s) = \frac{\pi^k_h(a | s) \exp(-\eta \sum_{j \in \mathcal{F}^k} Q^j_h(s, a))}{\sum_{a' \in \mathcal{A}} \pi^k_h(a' | s) \exp(-\eta \sum_{j \in \mathcal{F}^k} Q^j_h(s, a'))}. \)

end for

end for
Algorithm 3: Delayed OPPO with unknown transition function

Input: State space \( \mathcal{S} \), Action space \( \mathcal{A} \), Horizon \( H \), Learning rate \( \eta > 0 \), Exploration parameter \( \gamma > 0 \), Confidence parameter \( \delta > 0 \), Explicit exploration parameter \( \text{UseExplicitExploration} \in \{ \text{true}, \text{false} \} \), maximal delay \( d_{max} \).

Initialization: Set \( \pi^1_h(a \mid s) = 1/|A| \) for every \((s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H] \), \( K_{exp} = 0 \).

for \( k = 1, 2, \ldots, K \) do
  for \( s \in \mathcal{S} \) do
    if \( \text{UseExplicitExploration} = \text{true} \) and \( n^k_h(s) \leq 2d_{max} \log \frac{HSA \delta}{\delta} \) then
      \( \forall a \in \mathcal{A}: \hat{\pi}^k_h(a \mid s) = 1/|A| \).
    else
      \( \forall a \in \mathcal{A}: \hat{\pi}^k_h(a \mid s) = \pi^k_h(a \mid s) \).
    end if
  end for
  Play episode \( k \) with policy \( \hat{\pi}^k \).
  if trajectory feedback is not delayed then
    Observe trajectory \( U^k = \{(x^k_h, a^k_h)\}_{h=1}^H \).
  end if
  Observe feedback from all episodes \( j \in \mathcal{F}^k \) and update \( h^{k+1} \) and \( \hat{p}^{k+1} \).
  if explicit exploration was used in some \( j \in \mathcal{F}^k \), then add \( j \) to \( K_{exp} \).
  Compute confidence set \( \mathcal{P}^k = \{ P^k_h(s, a) \}_{s,a,h} \), where \( P^k_h(s, a) \) contains all transition functions \( p^k_h(\cdot \mid s, a) \) such that for every \( s' \in \mathcal{S} \),
  \[
  |p^k_h(s'|s, a) - \hat{p}^k_h(s'|s, a)| \leq \frac{c^k_h(s'|s, a)}{n^k_h(s, a)} \leq 4\sqrt{\frac{\hat{p}^k_h(s'|s, a)(1 - \hat{p}^k_h(s'|s, a))}{n^k_h(s, a) + 1} + 10 \frac{\ln HSA \delta}{n^k_h(s, a)}}.
  \]

# Policy Evaluation

for \( j \in \mathcal{F}^k \setminus K_{exp} \) do
  \( \forall s \in \mathcal{S}: V^j_{H+1}(s) = 0. \)
  for \( h = H, \ldots, 1 \) and \((s, a) \in \mathcal{S} \times \mathcal{A} \) do
    if bandit feedback then
      \( u^j_h(s) = \max_{s', \pi'} q^j_h(\pi') (s) = \max_{s', \pi'} \Pr[s_h = s \mid s_1 = s_{init}, \pi, p'] \).
      \( c^j_h(s, a) = c^j_h(s, a) = c^j_h(s, a) = c^j_h(s, a) = c^j_h(s, a) \).
      \( \hat{p}^j_h(\cdot \mid s, a) \subset \arg \min_{p^j_h(\cdot \mid s, a) \in \mathcal{P}^j_h(\cdot \mid s, a)} \langle p^j_h(\cdot \mid s, a), V^j_{h+1} \rangle. \)
    else if full-information feedback then
      \( c^j_h(s, a) = c^j_h(s, a) \).
      \( \hat{p}^j_h(\cdot \mid s, a) \subset \arg \min_{p^j_h(\cdot \mid s, a) \in \mathcal{P}^j_h(\cdot \mid s, a)} \langle p^j_h(\cdot \mid s, a), V^j_{h+1} \rangle. \)
    end if
  end for
end for

# Policy Improvement

for \((s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H] \) do
  \( \pi^{k+1}_h(a \mid s) = \frac{\pi^k_h(a \mid s) \exp(-\eta \sum_{j \in \mathcal{F}^k \setminus K_{exp}} Q^j_h(s, a))}{\sum_{a' \in \mathcal{A}} \pi^k_h(a' \mid s) \exp(-\eta \sum_{j \in \mathcal{F}^k \setminus K_{exp}} Q^j_h(s, a'))}. \)
end for
**B Full proofs of main theorems**

We start by defining failure events. The rest of the analysis focuses on the good event: the event in which none of the failure events occur. We show that in order to guarantee a regret bound that holds with probability of $1 - \delta$, we only need to pay a factor which is logarithmic in $1/\delta$. In Appendix B.1, we define the failure events and bound their probability in Lemmas 1 and 3.

In Appendix B.2, we prove the regret bound for the most challenging case: unknown transition function + delayed trajectory feedback + bandit feedback, that is, Theorem 3 with bandit feedback. Then, the rest of the proofs follow easily as corollaries.

In Appendix B.3, we prove the regret for the case of unknown transition function + delayed trajectory feedback + full-information feedback, that is, Theorem 3 with full-information feedback.

In Appendix B.4, we prove the regret for the case of unknown transition function but with non-delayed trajectory feedback, that is, Theorem 1. Finally, in Appendix B.5, we prove the regret for the case of known transition function, that is, Theorem 2.

**Remark 1.** The analysis for bandit feedback uses estimated $Q$-functions $Q^j$ that were computed with the confidence set $\mathcal{P}^j$ and not $\mathcal{P}^j+d$, as in the full-information case. This simplifies concentration arguments but ignores a lot of data. It also means that under bandit feedback the algorithm has worse space complexity, since we may need to keep up to $d_{max}$ empirical transition functions. However, the space complexity can be easily reduced by re-computing the confidence sets only when the number of visits to some state-action pair is doubled, and not in the end of every episode.

### B.1 Failure events

Fix some probability $\delta'$. We now define basic failure events:

- $F_{1}^{\text{basic}} = \{\exists k, s', s, a, h : |p_h(s' \mid s, a) - \hat{p}_k^h(s' \mid s, a)| > \epsilon_k^h(s' \mid s, a)\}$
  - where $\epsilon_k^h(s' \mid s, a) = 4\sqrt{\frac{\hat{p}_k^h(s'|s,a)(1-\hat{p}_k^h(s'|s,a)) \ln \frac{HSAK}{n_k^h(s,a)} + 1}{n_k^h(s,a)}}$.

- $F_{2}^{\text{basic}} = \{\exists k, s, a, h : |p_h(\cdot \mid s, a) - \hat{p}_k^h(\cdot \mid s, a)| > \sqrt{\frac{14S \ln \frac{HSAK}{n_k^h(s,a)}}{n_k^h(s,a)}}\}$

- $F_{3}^{\text{basic}} = \{\sum_{k,s,a,h} (q_k^h(s,a) - I(s_h = s, a_h = a) \min\{2, r_k^h(s,a)\}) > 6\sqrt{K \ln \frac{1}{\delta'}}\}$
  - where $r_k^h(s,a) = 8\sqrt{\frac{S \ln \frac{HSAK}{n_k^h(s,a)}}{n_k^h(s,a)}} + 2005 \ln \frac{HSAK}{n_k^h(s,a)} \cdot q_k^h(s,a) = q_k^{P,\pi}(s) n_k^h(a|s)$, and $q_k^{P,\pi}(s) = \Pr[s_h = s \mid s_1 = s_{\text{init}}, \pi, p]$.

- $F_{4}^{\text{basic}} = \{\exists k, s, a, h : \sum_{k'=1}^{K} c_{k'}^h(s,a) > \frac{\ln HSAK}{2\delta'}\}$

- $F_{5}^{\text{basic}} = \{\exists k, s, a, h : n_k^h(s,a) > d_{\text{max}} \ln \frac{HSAK}{n_k^h(s,a)} \wedge n_k^h(s,a) - d_{\text{max}} \leq \frac{d_{\text{max}}}{2\delta}\}$
  - where $n_k^h(s) = \sum_{a'} n_k^h(s,a')$.

We define the basic good by $G^{\text{basic}} = \bigcap_{i=1}^{6} \neg F_{i}^{\text{basic}}$.

**Lemma 1.** The basic good event $G^{\text{basic}}$, occurs with probability of at least $1 - 5\delta'$.

**Proof.**

- By (Maurer and Pontil 2009, Theorem 4), $\Pr (F_{3}^{\text{basic}}) \leq \delta'$.
- By (Jaksch, Ortner, and Auer 2010, Lemma 17) and union bounds, $\Pr (F_{4}^{\text{basic}}) \leq \delta'$.
- Let $Y_k = \sum_{s,a,h} (q_k^h(s,a) - I(s_h = s, a_h = a)) \min\{2, r_k^h(s,a)\}$. Note that $r_k^h$ depends on the history up to the end of episode $k-1, H_k-1$. Therefore, $\mathbb{E}[Y_k | H_{k-1}] = 0$ (as $\mathbb{E}[q_k^h(s_h = s, a_h = a) | H_{k-1}] = q_k^h(s,a)$). That is, $\sum_{k} Y_k$ is a martingale. Also, $|Y_k| \leq 4$. By Azuma-Hoeffding inequality,

$$\Pr (F_{3}^{\text{basic}}) = \Pr \left( \sum_{k} Y_k > 6\sqrt{K \ln 1/\delta'} \right) \leq \delta'.$$

- By (Shani et al. 2020, Lemma 6), $\Pr (F_{4}^{\text{basic}}) \leq \delta'$.\footnote{We have invoked Lemma 6 of (Shani et al. 2020) with $\alpha_k^h(s',a') = I(s' = s, a' = a)$ and then take union bound over all $s, a$ and $h$.}
Fix $h, s, a$ and $k$ such that $n_h^k(s) \geq d_{\text{max}} \log \frac{HSA}{\delta}$, and let $k_0$ be the first episode such that $n_h^{k_0}(s) \geq d_{\text{max}} \log \frac{HSA}{\delta}$. Since, for at least the first $d_{\text{max}} \log \frac{HSA}{\delta}$ visits in $s$, we play a uniform policy in $s$, $E \left[ n_h^{k_0}(s, a) \right] \geq \frac{d_{\text{max}}}{2} \log \frac{HSA}{\delta}$. By Chernoff bound,

$$
\Pr \left( n_h^{k_0}(s, a) \geq \frac{1}{2} d_{\text{max}} \right) \leq e^{-\frac{1}{2} \frac{d_{\text{max}}}{A} \log \frac{HSA}{\delta} \leq \frac{\delta'}{HSA},
$$

where the last holds whenever $d_{\text{max}} \geq 8A$ (if not, we can actually get better regret bounds). Taking the union bound over $h, s, a$ and noting that it is sufficient to show the claim for the first $k$ that satisfies $n_h^k(s) \geq d_{\text{max}} \log \frac{HSA}{\delta}$, gives us

$$
\Pr(-F_{\text{basic}}^k) \leq \delta'.
$$

Using the union bound on the above failure events and taking the complement gives us the desired result.

Define $\hat{c}_h^k(s' \mid s, a) = 8\sqrt{\frac{p_h(s'|s, a) (1 - p_h(s'|s, a)) \ln \frac{HSAK}{\delta}}{n_h(s, a) \sqrt{1}}} + 100 \ln \frac{\ln \frac{HSAK}{\delta}}{n_h(s, a) \sqrt{1}}$.

Lemma 2. Given the basic good event, the following relations holds:

1. $|p_h(s' \mid s, a) - \hat{p}_h(s' \mid s, a)| \leq \hat{c}_h^k(s' \mid s, a)$.
2. $\sum_{s'} |p_h(s' \mid s, a) - \hat{p}_h(s' \mid s, a)| \leq r_h^k(s, a)$.

Proof.

1. Under bandit feedback, the first inequality now holds by $\hat{F}_{\text{basic}}^k$ and (Rosenberg et al. 2020, Lemma B.13). Under full information, recall that $\hat{p}_h^k$ is computed after episode $k + d^k$. That is, $\hat{p}_h^k \in P^{k + d^k}$. Similar to the bandit case,

$$
|p_h(s' \mid s, a) - \hat{p}_h(s' \mid s, a)| \leq \hat{c}_h^{k+d^k}(s' \mid s, a) \leq \hat{c}_h^k(s' \mid s, a),
$$

where the second inequality is since $\hat{c}_h^k(s' \mid s, a)$ is decreasing in $k$.

2. The second relation simply holds by the first relation and Jensen’s inequality.

We define the following bad events that will not occur with high probability, given that the basic good event occurs:

- $F_{1,\text{cond}} = \left\{ \sum_{k, s, a, h} \hat{g}_h^k(s) \pi_h^k(s \mid a) (E[c_h^k(s, a) \mid H^{k-1}] - \hat{c}_h^k(s, a)) > H \sqrt{K \ln \frac{H}{\delta}} \right\}$

where $H^{k-1}$ denotes the history up to episode $k$.

- $F_{2,\text{cond}} = \left\{ \exists h, s : \sum_{k=1}^K V_h^k(s) - V_h^0(s) > \frac{H}{2} \ln \frac{HSAK}{\delta} \right\}$

- $F_{3,\text{cond}} = \left\{ \exists s, a : \sum_{k=1}^K \sum_h E[s, h] \left[ \hat{c}_h^k(s' \mid h, s, a) (V_{h+1}^k(s) - V_{h+1}^0(s)) \right] > \frac{H^2 S \ln \frac{HSAK}{\delta}}{2} \right\}$

- $F_{4,\text{cond}} = \left\{ \exists h, s : \sum_k \left\{ |j : j \leq k, j + d^j \geq k| \right\} \left( V_h^k(s) - V_h^0(s) \right) > d_{\text{max}} H \frac{1}{2} \ln \frac{HSAK}{\delta} \right\}$

We define the conditioned good event by $G_{\text{cond}} = -F_{1,\text{cond}} \cap -F_{2,\text{cond}} \cap -F_{3,\text{cond}} \cap -F_{4,\text{cond}}$.

Lemma 3. Conditioned on $G_{\text{basic}}$, the conditioned good event $G_{\text{cond}}$, occurs with probability of at least $1 - 4\delta'$. That is,

$$
\Pr (G_{\text{cond}} \mid G_{\text{basic}}) \geq 1 - 4\delta'.
$$

Proof.

- Following the proof in (Shani et al. 2020, appendix C.1.3), $\Pr (F_{i,\text{cond}} \mid G_{\text{basic}}) \leq \delta'$ for $i = 1, 2, 3$.

- By (Shani et al. 2020, Lemmas 7 and 10) and the fact that $|\{ j : j \leq k, j + d^j \geq k \}| \leq d_{\text{max}}$, $\Pr (F_{4,\text{cond}} \mid G_{\text{basic}}) \leq \delta'$.

Using the union bound on the above failure events and taking the complement gives us the desired result.

Finally, we define the global good event, $G := G_{\text{basic}} \cap G_{\text{cond}}$. Using the union bound, for $\delta > 0$ and $\delta' = \frac{\delta}{2 n}$, $\Pr(G) \geq 1 - \delta$. 

B.2 Proof of Theorem 3 with Bandit Feedback

With probability at least $1 - \delta$ the good event holds. For now on, we assume we are outside the failure event, and then our regret bound holds with probability at least $1 - \delta$.

According to the value difference lemma of [Shani et al. (2020)],

$$R_K = \sum_{k=1}^{K} V_1^\pi(k) - V_1^\pi(1)$$

$$= \sum_{k=1}^{K} V_1^\pi(k) - V_1^k(1) + \sum_{k=1}^{K} V_1^k(1) - V_1^\pi(1)$$

$$= \sum_{k=1}^{K} V_1^\pi(k) - V_1^k(1)$$

(A)

$$+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}[\{Q^k(s_h, \cdot), \pi^k(\cdot | s_h) - \pi_h(\cdot | s_h)\}]$$

(B)

$$+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}[Q^k(s_h, a_h) - c^k(s_h, a_h) - \langle p_h(\cdot | s_h, a_h), V_k^k(V_{k+1}) \rangle].$$

(C)  \hspace{1cm} (4)

We continue bounding each of these terms separately. Term (A) is bounded in Appendix B.2.1 by $O(H^2S\sqrt{AK} + H^2S\text{Ad}_{\text{max}} + \gamma KHSA + \frac{H^2S}{\gamma} + H^2S^2A)$, Term (B) is bounded in Appendix B.2.2 by $O(\frac{H}{\eta} + \frac{n}{\gamma}H^3(K + D) + \frac{n}{\gamma^2}d_{\text{max}}H^3)$ and Term (C) is bounded in Appendix B.2.3 by $O(\frac{H}{\gamma})$. This gives a total regret bound of

$$R_K = O\left(H^2S\text{Ad}_{\text{max}} + H^2S\sqrt{AK} + \frac{H^2S}{\gamma} + \gamma KHSA + \frac{H}{\eta} + \frac{n}{\gamma}H^3(K + D) + \frac{n}{\gamma^2}d_{\text{max}}H^3 + \frac{H}{\gamma} + H^2S^2A\right).$$

Choosing $\eta = \frac{1}{2(A^{\frac{1}{2}}(K+D)^{\frac{1}{2}})^{\frac{1}{2}}}$ and $\gamma = \frac{1}{2(A^{\frac{1}{2}}(K+D)^{\frac{1}{2}})^{\frac{1}{2}}}$ gives the theorem’s statement.

Remark 2 (Delayed OPPO with stochastic costs under bandit feedback). As shown by [Shani et al. (2020)], when the costs are stochastic, we can replace the importance-sampling estimator with a simple empirical average. This means that our estimator is now bounded by $H$ and eliminates all the terms that depend on $\gamma$. Thus, we obtain a regret of $O(H^2S\sqrt{AK} + H^2\sqrt{D} + H^2S^2A + H^2S\text{Ad}_{\text{max}})$ that is similar to the full-information feedback case. Moreover, in this case we can again use explicit exploration to reduce the last term to $O(H^2\text{Ad}_{\text{max}})$

B.2.1 Bounding Term (A)

Lemma 4. Conditioned on the good event $G$,

$$\sum_{k=1}^{K} V_1^\pi(k) - V_1^k(1) = O\left(H^2S\sqrt{AK} + H^2S^2A + \frac{H^2S}{\gamma} + \gamma KHSA + H^2S\text{Ad}_{\text{max}}\right).$$
Proof. We start with a value difference lemma [Shani et al. 2020].

\[
\sum_{k=1}^{K} V_1^n(s_1^k) - V_1^k(s_1^k) = \sum_{k=1}^{K} \sum_{h=1}^{H} E^{\pi^k}[c_h^k(s_h^k, a_h^k) - \hat{c}_h^k(s_h^k, a_h^k)] \\
+ E^{\pi^k}[\langle p_h(\cdot \mid s_h^k, a_h^k) - \hat{p}_h(\cdot \mid s_h^k, a_h^k), V_{h+1}^k \rangle] \\
\leq \sum_{k=1}^{K} \sum_{h=1}^{H} E^{\pi^k}[c_h^k(s_h^k, a_h^k) - \hat{c}_h^k(s_h^k, a_h^k)] \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} E^{\pi^k}[\|p_h(s' \mid s_h^k, a_h^k) - \hat{p}_h(s' \mid s_h^k, a_h^k)\|V_{h+1}^k(s')] \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} E^{\pi^k}[\langle c_h^k(\cdot \mid s_h^k, a_h^k), V_{h+1}^k - V_{h+1}^k \rangle],
\]

(A.1)

where we have used [Lemma 2] for the inequality. For any \(k, h, s, a\) we have

\[
c_h^k(s, a) - \hat{c}_h^k(s, a) = c_h^k(s, a) - E[c_h^k(s, a) \mid \mathcal{H}^{k-1}] + E[c_h^k(s, a) \mid \mathcal{H}^{k-1}] - \hat{c}_h^k(s, a) \\
= c_h^k(s, a) \left(1 - \frac{q_h^k(s)\pi_h^k(a \mid s)}{u_h^k(s)\pi_h^k(a \mid s) + \gamma}\right) + E[\hat{c}_h^k(s, a) \mid \mathcal{H}^{k-1}] - \hat{c}_h^k(s, a) \\
= c_h^k(s, a) \left(\frac{(u_h^k(s) - q_h^k(s))\pi_h^k(a \mid s) + \gamma}{u_h^k(s)\pi_h^k(a \mid s) + \gamma}\right) + E[\hat{c}_h^k(s, a) \mid \mathcal{H}^{k-1}] - \hat{c}_h^k(s, a).
\]

Therefore (denoting \(q_h^k(s) = q_h^{\pi^k}(s)\)).

\[
(A.1) = \sum_{k=1}^{K} \sum_{h=1}^{H} E^{\pi^k}\left[c_h^k(s_h^k, a_h^k) \left(\frac{(u_h^k(s_h^k) - q_h^k(s_h^k))\pi_h^k(a_h \mid s_h^k) + \gamma}{u_h^k(s_h^k)\pi_h^k(a_h \mid s_h^k) + \gamma}\right)\right] \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} E^{\pi^k}\left[\mathcal{H}^{k-1} - c_h^k(s_h^k, a_h^k)\right] \\
= \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s,a} q_h^k(s)\pi_h^k(s \mid a)c_h^k(s, a) \left(\frac{(u_h^k(s) - q_h^k(s))\pi_h^k(a \mid s) + \gamma}{u_h^k(s)\pi_h^k(a \mid s) + \gamma}\right) \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s,a} q_h^k(s)\pi_h^k(s \mid a) \left(E[c_h^k(s, a) \mid \mathcal{H}^{k-1}] - \hat{c}_h^k(s, a)\right).
\]

(A.1.1)
Under the good event $p \in \mathcal{P}^{k-1}$. Hence by definition $u_k^h(s) \geq q_k^h(s)$. Therefore,

\[(A.1.1) \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s,a} q_k^k(s) \pi_k^k(s | a) c_k^k(s, a) \left( \frac{(u_k^h(s) - q_k^h(s)) \pi_k^h(a | s) + \gamma}{q_k^h(s) \pi_k^h(a | s)} \right) \]

\[= \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s,a} c_k^k(s, a) \left( (u_k^h(s) - q_k^h(s)) \pi_k^h(a | s) + \gamma \right) \]

\[\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} (u_k^h(s) - q_k^h(s)) + \gamma KHSA \]

\[\leq HS\sqrt{AK} + HSA_{d_{max}} + \gamma KHSA,\]

where the last inequality follows similarly to the bound of Term (A.2) when combined with Lemma 20 of Shani et al. (2020).

Under the good event $G$ (in particular, $\neg F_{cond}$),

\[(A.1.2) \leq H \sqrt{K \frac{\ln H}{2\delta'}}.\]

By Lemma 5 and the fact that $V_{h+1}^{\pi_h} \leq H$,

\[(A.2) \leq H^2S\sqrt{AK} + H^2SA_{d_{max}} + H^2S^2A.\]

Finally, conditioned on the good event ($\neg F_{cond}$),

\[(A.3) \leq \frac{H^2S}{2\gamma} \ln \frac{H^2SK}{\delta'}.\]

We get that term (A) can be bounded by,

\[\sum_{k=1}^{K} V_1^{\pi_h^k}(s_1^k) - V_1^k(s_1^k) \leq (A.1.1) + (A.1.2) + (A.2) + (A.3) \]

\[\lesssim H^2S\sqrt{AK} + H^2SA_{d_{max}} + \gamma KHSA + \frac{H^2S}{\gamma},\]

where $\lesssim$ ignores poly-logarithmic factors.

\[\square\]

Lemma 5. Under the good event,

\[\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s'} \mathbb{E}[p_h(s' | s_h^k, a_h^k) - \hat{p}_h(s' | s_h^k, a_h^k)] \lesssim HS\sqrt{AK} + HSA_{d_{max}} + HS^2A.\]

Proof. Define,

\[\mathcal{K}(s, a, h) = \{ k \in [K] : s_h^k = s, a_h^k = a, u_h^k(s, a) \leq d_{max} \},\]

which are the episodes in which we visited $(s, a, h)$ but haven’t observe haven’t observed more than $d_{max}$ samples. For episodes in $\mathcal{K}(s, a, h)$, we bound the estimation error of $p$ by a constant. For the rest of the episodes we utilize the fact that the amount unobserved feedback is smaller than the observed feedback (see (6) below).
Note that $|K(s, a, h)| \leq 2d_{\text{max}}$. Thus,

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s' \in S} E^{n^k} \left[ p_h(s' \mid s_h^k, a_h^k) - \hat{p}_h^k(s' \mid s_h^k, a_h^k) \right] = \sum_{k=1}^{K} \sum_{s, a, h} a_h^k(s, a) \sum_{s' \in S} \left| p_h(s' \mid s, a) - \hat{p}_h^k(s' \mid s, a) \right| \leq \sum_{k=1}^{K} \sum_{s, a, h} a_h^k(s, a) \min\{2, r_h^k(s, a)\} \leq \sum_{k=1}^{K} \sum_{s, a, h} (g_h^k(s, a) - \mathbb{I}\{s_h^k = s, a_h^k = a\}) \min\{2, r_h^k(s, a)\} + 2 \sum_{s, a, h, k \in K(s, a, h)} \mathbb{I}\{s_h^k = s, a_h^k = a\} + \sum_{s, a, h, k \notin K(s, a, h)} \mathbb{I}\{s_h^k = s, a_h^k = a\} r_h^k(s, a) \leq \sqrt{K} + \text{HSAd}_{\text{max}} + \sum_{s, a, h, k \notin K(s, a, h)} 1 \mathbb{I}\{s_h^k = s, a_h^k = a\} r_h^k(s, a) \leq \sqrt{K} + \text{HSAd}_{\text{max}} + \sqrt{S} \sum_{s, a, h, k \notin K(s, a, h)} \frac{\mathbb{I}\{s_h^k = s, a_h^k = a\}}{\sqrt{n_h^k(s, a) \vee 1}}.$$

(5)

The first inequality follows the fact that $||p_h(\cdot \mid s, a) - \hat{p}_h^k(\cdot \mid s, a)||_1 \leq 2$ and Lemma 2 And the third inequality is by $F_{3, \text{basic}}$. and $|K(s, a, h)| \leq d_{\text{max}}$. Now,

$$\sum_{k \notin K(s, a, h)} \mathbb{I}\{s_h^k = s, a_h^k = a\} \leq \sum_{k \notin K(s, a, h)} \frac{\mathbb{I}\{s_h^k = s, a_h^k = a\}}{\sqrt{n_h^k(s, a) \vee 1}} \leq \sum_{k \notin K(s, a, h)} \frac{1 \vee \sum_{j=1}^{k-1} \mathbb{I}\{s_h^j = s, a_h^j = a\}}{1 \vee \sum_{j^1 + d^1 \leq k-1} \mathbb{I}\{s_h^j = s, a_h^j = a\}} \leq \sum_{k \notin K(s, a, h)} \frac{1 \vee \sum_{j^1 + d^1 \leq k-1} \mathbb{I}\{s_h^j = s, a_h^j = a\}}{1 \vee \sum_{j^1 + d^1 \leq k-1} \mathbb{I}\{s_h^j = s, a_h^j = a\}}.$$

Since for any $k \notin K(s, a, h),$

$$\sum_{j^1 + d^1 \geq k} \mathbb{I}\{s_h^j = s, a_h^j = a\} \leq d_{\text{max}} \leq \sum_{j^1 + d^1 \leq k-1} \mathbb{I}\{s_h^j = s, a_h^j = a\}$$

we have

$$\sum_{k \notin K(s, a, h)} \frac{\mathbb{I}\{s_h^k = s, a_h^k = a\}}{\sqrt{n_h^k(s, a) \vee 1}} \leq \sum_{k \notin K(s, a, h)} \frac{\mathbb{I}\{s_h^k = s, a_h^k = a\}}{\sqrt{1 \vee \sum_{j=1}^{k-1} \mathbb{I}\{s_h^j = s, a_h^j = a\}}} \leq \sum_{k=1}^{K} \frac{\mathbb{I}\{s_h^k = s, a_h^k = a\}}{\sqrt{1 \vee \sum_{j=1}^{k-1} \mathbb{I}\{s_h^j = s, a_h^j = a\}}}.$$
Finally, (7) and (8) appear in the non-delayed setting and can be bounded when summing over \((s, a, h)\) by \(\hat{O}(H \sqrt{SAK})\) and \(\hat{O}(HSA)\), respectively (see for example [Jin et al., 2020a] Lemma 4). Plugging back in (5) completes the proof. 

\[ \sum \] 

\[ O \] 

B.2.2 Bounding Term (B)

**Lemma 6.** Let \(s \in S\). Conditioned on the good event \(G\),

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \langle Q^k_h(s, \cdot), \pi^k_h(\cdot \mid s) - \pi_h(\cdot \mid s) \rangle \leq \frac{H \log(A)}{\eta} + \eta \frac{H^3}{\gamma}(K + D) + \eta \sum_{d_{max}} H^3 \ln \frac{H}{\delta}.
\]

**Proof.** We adopt the technique presented in [György and Joulani, 2020] for MAB, in order to bound the term \(\sum_k \langle Q^k_h(s, \cdot), \pi^k_h(\cdot \mid s) - \pi_h(\cdot \mid s) \rangle\), for each \(s\) and \(h\) separately.

The proof uses a comparison to a “cheating” algorithm that does not experience delay and sees one step into the future. Define

\[
\tilde{\pi}^k_h(a \mid s) = \frac{\exp \left( -\eta \sum_{j:j \leq k-1} Q^j_h(s, a) \right)}{\sum_{a' \in A} \exp \left( -\eta \sum_{j:j \leq k-1} Q^j_h(s, a') \right)}.
\]

We break the sum term in the following way:

\[
\sum_k \langle Q^k_h(s, \cdot), \pi^k_h(\cdot \mid s) - \pi_h(\cdot \mid s) \rangle = \sum_k \langle Q^k_h(s, \cdot), \tilde{\pi}^{k+1}_h(\cdot \mid s) - \pi_h(\cdot \mid s) \rangle \tag{B.1}
\]

\[
+ \sum_k \langle Q^k_h(s, \cdot), \pi^k_h(\cdot \mid s) - \tilde{\pi}^{k+1}_h(\cdot \mid s) \rangle . \tag{B.2}
\]

Term (B.1) is the regret of the “cheating” algorithm. Using [Joulani, György, and Szepesvári, 2020] Theorem 3\(^{[1]}\)

\[
(B.1) \leq \frac{\ln(A)}{\eta} . \tag{9}
\]

Now, using the definition of \(\tilde{\pi}^k_h\),

\[
\frac{\tilde{\pi}^{k+1}_h(a \mid s)}{\pi^k_h(a \mid s)} = \frac{\exp \left( -\eta \sum_{j:j \leq k} Q^j_h(s, a) \right)}{\sum_{a' \in A} \exp \left( -\eta \sum_{j:j \leq k} Q^j_h(s, a') \right)} \cdot \frac{\exp \left( -\eta \sum_{j:j+d \leq k-1} Q^j_h(s, a') \right)}{\exp \left( -\eta \sum_{j:j+d \leq k-1} Q^j_h(s, a) \right)} \cdot \frac{\sum_{a'} \exp \left( -\eta \sum_{j:j+d \leq k-1} Q^j_h(s, a') \right)}{\exp \left( -\eta \sum_{j:j+d \leq k-1} Q^j_h(s, a) \right)}
\]

\[
\geq \exp \left( -\eta \sum_{j:j \leq k, j+d \geq k} Q^j_h(s, a) \right) \cdot \frac{\sum_{a'} \exp \left( -\eta \sum_{j:j+d \leq k-1} Q^j_h(s, a') \right)}{\exp \left( -\eta \sum_{j:j+d \leq k-1} Q^j_h(s, a) \right)} \geq 1 - \eta \sum_{j:j \leq k, j+d \geq k} Q^j_h(s, a)
\]

where in the first inequality we have used \(\sum_{j:j+d \leq k-1} Q^j_h(s, a) \leq \sum_{j:j \leq k} Q^j_h(s, a')\), and for the second inequality we have

[1] We choose the regularizers in [Joulani, György, and Szepesvári, 2020] to be \(q_0(x) = r_1(x) = \frac{1}{\eta} \sum_i x_i \log x_i\) and the rest are zero, which makes the update of their ADA-MD algorithm as in our policy improvement step. The statement now follows from [Joulani, György, and Szepesvári, 2020] Theorem 3, the fact that the Bregman divergence is positive and that entropy is bounded by \(\log A\).
used the fact that $e^x \geq 1 + x$ for any $x$. Using the above,

\[
(B.2) = \sum_k \langle Q_h^k(s, \cdot), \pi_h^k(\cdot | s) - \bar{\pi}_h^{k+1}(\cdot | s) \rangle \\
= \sum_k \sum_{a \in A} Q_h^k(s, a) \left( \pi_h^k(a | s) - \bar{\pi}_h^{k+1}(a | s) \right) \\
= \sum_k \sum_{a \in A} Q_h^k(s, a) \pi_h^k(a | s) \left( 1 - \frac{\bar{\pi}_h^{k+1}(a | s)}{\pi_h^k(a | s)} \right) \\
\leq \sum_k \sum_{a \in A} \pi_h^k(a | s) Q_h^k(s, a) \sum_{j : j \leq k, j + d^j \geq k} Q^j_h(s, a) \\
\leq \sum_k \sum_{a \in A} \pi_h^k(a | s) Q_h^k(s, a) \sum_{j : j \leq k, j + d^j \geq k} \frac{H}{\gamma} \\
\leq \sum_k \pi_h^k(a | s) Q_h^k(s, a) \sum_{j : j \leq k, j + d^j \geq k} \frac{H}{\gamma} \\
= \sum_k \pi_h^k(a | s) Q_h^k(s, a) \\
\leq \frac{H}{\gamma} \sum_k \{\{j : j \leq k, j + d^j \geq k\} \sum_{a \in A} \pi_h^k(a | s) Q_h^k(s, a) \\
\leq \frac{H}{\gamma} \sum_k \{\{j : j \leq k, j + d^j \geq k\} |V^k_h(s)\}.
\]

Under the good event (in particular, $-F_{\text{cond}}^G$),

\[
\sum_k \{\{j : j \leq k, j + d^j \geq k\} \left( V^k_h(s) - V^h_\pi(s) \right) \} \leq d_{\max} \frac{H}{\gamma} \ln \frac{H}{\delta}.
\]

Therefore,

\[
(B.2) \leq \eta \frac{H}{\gamma} \sum_k \{\{j : j \leq k, j + d^j \geq k\} |V^k_h(s)\} + \frac{\eta}{\gamma^2} d_{\max} H^2 \ln \frac{H}{\delta} \\
\leq \eta \frac{H^2}{\gamma} \sum_k \{\{j : j \leq k, j + d^j \geq k\} + \frac{\eta}{\gamma^2} d_{\max} H^2 \ln \frac{H}{\delta} \\
\leq \eta \frac{H^2}{\gamma} (K + D) + \frac{\eta}{\gamma^2} d_{\max} H^2 \ln \frac{H}{\delta}.
\]

where the last inequality holds since,

\[
\sum_k \{\{j : j \leq k, j + d^j \geq k\} = \sum_j \sum_k \{j \leq k \leq j + d^j \} \leq \sum_j (1 + d^j) \leq K + D.
\]

Combining (9) and (10) and summing over $h$, completes the proof. \hfill \Box

### B.2.3 Bounding Term (C)

**Lemma 7.** Conditioned on the good event $G$,

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}^\pi \left[ Q_h^k(s_h^k, a_h^k) - c_h^k(s_h^k, a_h^k) - \langle p_h(\cdot | s_h^k, a_h^k), V_{h+1}^k \rangle \right] \leq \frac{H}{2\gamma} \log \frac{SAHK}{\delta'}.
\]
Proof. Using Bellman equations,

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}^{\pi} \left[ Q^k_h(s^k_h, a^k_h) - c^k_h(s^k_h, a^k_h) - \langle p_h(\cdot \mid s^k_h, a^k_h), V^k_{h+1} \rangle \right] = \\
= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}^{\pi} \left[ c^k_h(s^k_h, a^k_h) - c^k_h(s^k_h, a^k_h) \right] \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}^{\pi} \left[ \langle \hat{p}^k_h(\cdot \mid s^k_h, a^k_h), V^k_{h+1} \rangle - \langle p_h(\cdot \mid s^k_h, a^k_h), V^k_{h+1} \rangle \right].
\]  

(C.1)

For any \( h, s \) and \( a \), under the good event,

\[
\sum_{k=1}^{K} c^k_h(s, a) \leq \sum_{k=1}^{K} c^k_h(s, a) - \frac{q^k_h(s)}{u^k_h(s)} c^k_h(s, a) \leq \frac{\log(\frac{SAHK}{\delta})}{2\gamma},
\]

where the first inequality is due to the fact that under the good event \( p \in \mathcal{P}^{k-1} \) and so \( u^k_h(s) = \max_{\hat{p} \in \mathcal{P}^{k-1}} \hat{p} \cdot \pi^h \geq q^k_h(s) \). The second inequality follows directly from \( -F^4_{\text{basic}} \). Therefore,

\[(C.1) \leq \frac{H}{2\gamma} \log \frac{SAHK}{\delta}.\]

Once again, since under the good event \( p \in \mathcal{P}^k \), then for all \( h, s \) and \( a \),

\[
\langle \hat{p}^k_h(\cdot \mid s, a), V^k_{h+1} \rangle = \min_{p_h(\cdot \mid s, a) \in \mathcal{P}^k} \langle \hat{p}^k_h(\cdot \mid s, a), V^k_{h+1} \rangle \leq \langle p_h(\cdot \mid s, a), V^k_{h+1} \rangle.
\]

Therefore \((C.2) \leq 0\), which completes the proof of the lemma. \( \square \)

**B.3 Proof of Theorem 3 under full-information feedback**

The proof follows almost immediately from the proof in Appendix B.2 by noting that some of the terms become zero since we use the actual cost function and not an estimated one. In addition, in this setting we use the explicit exploration which yield a better bound on term \((A)\).

Let \( K_{exp}(s, h) \) be the episodes in which we used the uniform policy in state \( s \) at time \( h \), because we did not receive enough feedback from that state. That is,

\[
K_{exp}(s, h) = \left\{ k \in [K] : s^k_h = s, n^k_h(s) \leq d_{max} \log \frac{HSA}{\delta} \right\}.
\]

Also, define \( K_{exp} = \bigcup_{s,h} K_{exp}(s, h) \). Recall that the algorithm keeps track of \( K_{exp} \), and preforms the policy improvement step only with respect to rounds that are not in \( K_{exp} \). For any \( (k, s, h) \) we have that \( m^k_h(s) - n^k_h(s) \leq d_{max} \) and thus,

\[
|K_{exp}(s, h)| = \left| \left\{ k : s^k_h = s, n^k_h(s) \leq d_{max} \log \frac{HSA}{\delta} \right\} \right| \\
\leq \left| \left\{ k : s^k_h = s, m^k_h(s) \leq 2d_{max} \log \frac{HSA}{\delta} \right\} \right| \lesssim d_{max}.
\]

By taking the union over \( s \) and \( h \) we have, \( |K_{exp}| \lesssim HSD_{max} \).

Similarly to the proof in Appendix B.2 we use the value difference lemma of Shani et al. (2020), on episodes that are not in
\[ \mathcal{R}_K \leq H^2 S d_{\text{max}} + \sum_{k \notin \mathcal{K}_{\text{exp}}} V^\pi_k(s^k_t) - V^\pi_1(s^k_1) \]

\[ = H^2 S d_{\text{max}} + \sum_{k \notin \mathcal{K}_{\text{exp}}} V^\pi_k(s^k_t) - V^\pi_1(s^k_1) \]

\[ \text{(A)} \]

\[ + \sum_{k \notin \mathcal{K}_{\text{exp}}} \sum_{h=1}^H \mathbb{E}^\pi \left[ (Q^k_h(s^k_h, \cdot), \pi^k_h(\cdot | s^k_h) - \pi^k_h(\cdot | s^k_h)) \right] \]

\[ \text{(B)} \]

\[ + \sum_{k \notin \mathcal{K}_{\text{exp}}} \sum_{h=1}^H \mathbb{E}^\pi \left[ (\tilde{p}^k_h(\cdot | s^k_h, a^k_h) - p_h(\cdot | s^k_h, a^k_h), V^k_{h+1}) \right]. \]

\[ \text{(C)} \]

We continue bounding each of these terms separately. Term (A) is bounded in Appendix B.3.1 by \( \tilde{O}\left(H^2 S \sqrt{\frac{A}{K}} + H^2 S^{1/2} A^{3/2} \sqrt{d_{\text{max}}} + H^2 S^2 A^2 + H^2 S d_{\text{max}}\right)\). Terms (B) and (C) are bounded in Appendix B.3.2 by \( O\left(\frac{H}{\eta} + \eta H^3(D + K)\right)\).

With that being said, if \( k + d^k \) is strictly monotone in \( k \) (e.g., under fixed delay), then all feedback of episodes \( j < k \) are available at time \( k + d^k \). In that case we can essentially achieve the bounds of Theorem 2 even when trajectory feedback is delayed, and even without explicit exploration.

**B.3.1 Bounding Term (A) under full-information feedback**

Term (A) under full-information can be written as,

\[ (A) = \sum_{k \notin \mathcal{K}_{\text{exp}}} V^\pi_k(s^k_t) - V^\pi_1(s^k_1) \]

\[ = \sum_{k \notin \mathcal{K}_{\text{exp}}} \sum_{h=1}^H \mathbb{E}^\pi \left[ (p_h(\cdot | s^k_h, a^k_h) - \tilde{p}^k_h(\cdot | s^k_h, a^k_h), V^k_{h+1}) \right] \]

\[ \leq H \sum_{k=1}^K \sum_{h=1}^H \sum_{s'} \mathbb{E}^\pi \left[ (p_h(s' | s^k_h, a^k_h) - \tilde{p}^k_h(s' | s^k_h, a^k_h)) \right]. \]

The last is bounded by Lemma 8 which is analogous to Lemma 5. This gives us the following bound on term (A):

\[ (A) \leq H^2 S \sqrt{\frac{A}{K}} + H^2 S^{1/2} A^{3/2} \sqrt{d_{\text{max}}} + H^2 S^2 A^2 + H^2 S d_{\text{max}}. \]

**Lemma 8.** Under the good event, with explicit exploration (UserExplicitExploration = true),

\[ \sum_{k=1}^K \sum_{h=1}^H \sum_{s' \in S} \mathbb{E}^\pi \left[ (p_h(s' | s^k_h, a^k_h) - \tilde{p}^k_h(s' | s^k_h, a^k_h)) \right] \leq H S \sqrt{\frac{A}{K}} + H S d_{\text{max}} \]

\[ + H S^2 A^2 + H S^{1/2} A^{3/2} \sqrt{d_{\text{max}}}. \]
Proof. Similarly to the proof of Lemma 5,

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s \in S} E^{s^k} \left[ \left| p_h(s' \mid s^k, a^k_h) - \hat{p}_h(s' \mid s^k, a^k_h) \right| \right] = \\
= \sum_{k=1}^{K} \sum_{s,a,h} q_k^k(s,a) \sum_{s' \in S} |p_h(s' \mid s, a) - \hat{p}_h(s' \mid s, a)| \\
\leq \sum_{k=1}^{K} \sum_{s,a,h} q_k^k(s,a) \min \{2, r_k^k(s,h) \} \\
= \sum_{k=1}^{K} \sum_{s,a,h} \left( q_k^k(s,a) - \mathbb{I}\{ s^k_h = s, a^k_h = a \} \right) \min \{2, r_k^k(s,h) \} \\
\quad + \sum_{k=1}^{K} \sum_{s,a,h} \mathbb{I}\{ s^k_h = s, a^k_h = a \} \left( 2 - r_k^k(s,h) \right) \\
\leq \sqrt{K} + \sum_{k=1}^{K} \sum_{s,a,h} \mathbb{I}\{ s^k_h = s, a^k_h = a \} \min \{2, r_k^k(s,h) \} \\
\leq \sqrt{K} + 2 \sum_{s,h} \sum_{k \in K} \sum_{a} \mathbb{I}\{ s^k_h = s, a^k_h = a \} \min \{1, \frac{1}{r_k^k(s,h)} \} \\
\quad + \sum_{s,a,h} \sum_{k \in K} \mathbb{I}\{ s^k_h = s, a^k_h = a \} r_k^k(s,h) \\
\leq \sqrt{K} + HSD_{\max} + \sqrt{S} \sum_{s,a,h} \sum_{k \in K} \frac{1}{n_k^h(s,a) \vee 1} + S \sum_{s,a,h} \frac{1}{n_k^h(s,a)} \sum_{k \in K} \mathbb{I}\{ s^k_h = s, a^k_h = a \}. 
\]

The last two terms are bounded using Lemmas 9 and 10 which completes the proof. □

Lemma 9. It holds that

\[
\sum_{s \in S} \sum_{a \in A} \sum_{h=1}^{H} \sum_{k \notin K_{\text{exp}}} \frac{\mathbb{I}\{ s^k_h = s, a^k_h = a \}}{\sqrt{n_k^h(s,a) \vee 1}} \lesssim H \sqrt{SAK} + HSA^{3/2} \sqrt{d_{\max}}.
\]
Proof. For any $s, a$ and $h$,

$$\sum_{k \in K_{exp}} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{n_h^k(s, a) \vee 1}} \leq$$

$$\leq \sum_{k \in K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

$$\leq \sum_{k \in K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

$$\leq \sum_{k \in K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

$$\leq \sum_{k \in K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

$$\leq \sum_{k=1}^{K} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

(D.1)

$$+ \sum_{k \in K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

(D.2)

As mentioned before, (D.1) appears in the non-delayed setting and can be bounded when summing over $(s, a, h)$ by $\tilde{O}(H\sqrt{SAK})$. For bounding (D.2) we’ll use the notion of $K(s, a, h)$ defined in the proof of Lemma 9 and the fact that outside $K(s, a, h)$ the inequality in (4) holds. Recall that $|K(s, a, h)| \leq 2d_{max}$ since every visit is observable after $d_{max}$ episodes.

Now summing (D.2) over $s$ and $a$,

$$\sum_{s,a} \sum_{k \in K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

$$= \sum_{s,a} \sum_{k \in K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

(D.2.1)

$$+ \sum_{s,a} \sum_{k \notin K_{exp}(s, h)} \frac{I\{s_k^h = s, a_k^h = a\}}{\sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}} \times \sqrt{1 + \sum_{j=1}^{k-1} I\{s_j^h = s, a_j^h = a\}}$$

(D.2.2)

Now, under the good event $(-F_5^{basic})$ we have that for any $k \notin K_{exp}(s, h),$

$$\sum_{j:j+d \leq k-1} I\{s_j^h = s, a_j^h = a\} = n_h^k(s, a) \geq \Omega \left( \frac{d_{max}}{A} \right).$$
Also, deterministically we have \( \sum_{j:j<k,j+d' \geq k} \mathbb{P}\{s^k_h = s, a^k_h = a\} \leq d_{\max} \). Hence,

\[
(D.2.1) \leq \sqrt{A} \sum_{s,a} \sum_{k \notin \mathcal{K}_{exp}(s,h), k \notin \mathcal{K}(s,a,h)} \frac{\mathbb{I}\{s^k_h = s, a^k_h = a\}}{1 / \sqrt{\sum_{j=1}^{k-1} \mathbb{I}\{s^j_h = s, a^j_h = a\}}} \leq SA^{3/2} \sqrt{d_{\max}},
\]

where the last inequality follows from the fact that for any time the nominator is 1, the sum in the denominator has increased by 1 as well. Hence it is bounded by the sum \( \sum_{i=1}^{2d_{\max}} \frac{1}{\sqrt{i}} \leq O(\sqrt{d_{\max}}) \). For last, recall that (6) holds for all \( k \notin \mathcal{K}(s,a,h) \). Hence,

\[
(D.2.2) \leq \sum_{s,a} \sum_{k \notin \mathcal{K}_{exp}(s,h), k \notin \mathcal{K}(s,a,h)} \frac{\mathbb{I}\{s^k_h = s, a^k_h = a\}}{1 \lor \sum_{j=1}^{k-1} \mathbb{I}\{s^j_h = s, a^j_h = a\}} \leq \sqrt{SA^2}.
\]

\[\Box\]

**Lemma 10.** It holds that

\[
\sum_{s} \sum_{a} \sum_{h=1}^{H} \frac{\mathbb{I}\{s^h_s = s, a^h_s = a\}}{n^h_s(s,a) \lor 1} \leq HSA^2.
\]

**Proof.** For any \( s, a \) and \( h \),

\[
\sum_{k \notin \mathcal{K}_{exp}} \frac{\mathbb{I}\{s^k_h = s, a^k_h = a\}}{n^k_h(s,a) \lor 1} \leq \sum_{k \notin \mathcal{K}_{exp}(s,h)} \frac{\mathbb{I}\{s^k_h = s, a^k_h = a\}}{1 \lor \sum_{j=1}^{k-1} \mathbb{I}\{s^j_h = s, a^j_h = a\}} \cdot \frac{1 \lor \sum_{j:j+d' \leq k} \mathbb{I}\{s^j_h = s, a^j_h = a\}}{1 \lor \sum_{j:j+d' \leq k-1} \mathbb{I}\{s^j_h = s, a^j_h = a\}} \leq \sum_{k \notin \mathcal{K}_{exp}(s,h)} \frac{\mathbb{I}\{s^k_h = s, a^k_h = a\}}{1 \lor \sum_{j=1}^{k-1} \mathbb{I}\{s^j_h = s, a^j_h = a\}} \cdot (2 + \frac{\sum_{j:j<k,j+d' \geq k} \mathbb{I}\{s^j_h = s, a^j_h = a\}}{1 \lor \sum_{j:j+d' \leq k-1} \mathbb{I}\{s^j_h = s, a^j_h = a\}}) \leq 3A \sum_{k \notin \mathcal{K}_{exp}(s,h)} \frac{\mathbb{I}\{s^k_h = s, a^k_h = a\}}{1 \lor \sum_{j=1}^{k-1} \mathbb{I}\{s^j_h = s, a^j_h = a\}} \leq A \log(KH),
\]

where we used the fact that \( \sum_{j:j<k,j+d' \geq k} \mathbb{P}\{s^j_h = s, a^j_h = a\} \) is always bounded by \( d_{\max} \), and that \( \sum_{j:j+d' \leq k-1} \mathbb{P}\{s^j_h = s, a^j_h = a\} \) is at least \( \Omega(d_{\max}/A) \) for episodes that are not in \( \mathcal{K}_{exp} \) under the good event. The last inequality follows from standard arguments ([Rosenberg et al. 2020](#) Lemma B.18), since there is no delay involved in this sum. \(\Box\)

### B.3.2 Bounding Terms (B) and (C) under full-information feedback

Following the analysis of term (B.2) in the proof of Lemma 6, since we run the policy improvement step only over \( [K] \setminus \mathcal{K}_{exp} \), and since under full-information \( V^h_k \leq H \) and the costs are bounded by 1,

\[
(B.2) \leq \eta H^2(K + D).
\]

Hence, term (B) can be bounded by,

\[
(B) \leq \frac{H \log A}{\eta} + \eta H^3(D + K).
\]

For last, term (C.1) is now zero, and so

\[
(C) \leq 0.
\]
B.4 Proof of Theorem 1

When the trajectories are observed without delay, Term (A) is bounded similarly to Shani et al. (2020) since it is no longer affected by the delay. Moreover, since there is no explicit exploration we also do not have the extra $H^2Sd_{max}$ factor. Terms (B) and (C) remain unchanged.

Thus, with bandit feedback, we obtain the regret bound

$$R_K = \tilde{O} \left( H^2S\sqrt{AK} + H^2S^2A + \gamma K H S A + \frac{H}{\eta} + \frac{\eta}{\gamma} H^3(K + D) + \frac{H}{\gamma} + \frac{\eta}{\gamma^2} H^3d_{max} \right),$$

and choosing $\eta = \frac{1}{H(A^{1/2}(K + D)^{3/2})}$ and $\gamma = \frac{1}{(A^{3/2}(K + D)^{1/2})}$ gives the theorem’s statement.

Similarly, with full-information feedback, we obtain the regret bound

$$R_K = \tilde{O} \left( H^{3/2}S\sqrt{AK} + H^2S^2A + \frac{H}{\eta} + \eta H^3(K + D) \right)$$

and choosing $\eta = \frac{1}{H\sqrt{K + D}}$ gives the theorem’s statement.

We note that Shani et al. (2020) bound Term (A) by $\tilde{O}(H^2S\sqrt{AK})$, but with full-information feedback it can actually be bounded by $\tilde{O}(H^{3/2}S\sqrt{AK})$. This is obtained by known Bernstein-based confidence bounds analysis (Azar, Osband, and Munos 2017; Zanette and Brunskill 2019). For example, one can follow Lemmas 4.6,4.7,4.8 of Rosenberg et al. (2020) which are more general.

The reason that this bound (that improves by a factor of $\sqrt{H}$) does not hold with delayed trajectory feedback is Lemma 10 where we bound $\sum_{s \in S} \sum_{a \in A} \sum_{h=1}^{H} \sum_{k \notin \mathcal{K}_{kexp}} \frac{H_{s,a}^k}{n_k^s(s,a)\geq 1}$ by $\tilde{O}(HSA^2)$ instead of $\tilde{O}(HSA)$ when the trajectory feedback is not delayed. Thus, the analysis of Rosenberg et al. (2020) gets a bound of $\tilde{O}(H^{3/2}SA\sqrt{K})$ instead of $\tilde{O}(H^{3/2}S\sqrt{AK})$, since it bounds Term (A) roughly by

$$H\sqrt{SK} \sum_{s \in S} \sum_{a \in A} \sum_{h=1}^{H} \sum_{k \notin \mathcal{K}_{kexp}} \frac{\mathbb{I}\{s_h^k = s, a_h^k = a\}}{n_k^s(s,a)\vee 1}.$$

B.5 Proof of Theorem 2

When dynamics are known, we use the actual transition function instead of the estimated one. Under the bandit-feedback, the terms (A.2), (A.3) in Appendix B.2 become zero. Since we use the actual occupancy measure of the policy (and do not compute it using some transition function from the confidence set), Term (A.1.1) is now bounded by $\gamma K H S A$. Term (A.2.1) remains unchanged. Therefore,

$$(A) \lesssim \gamma K H S A + H\sqrt{K}.$$

Term (B) remains unchanged and Term (C) is now bounded by $\tilde{O}(H/\gamma)$, as (C.2) zeroes.

Thus, with known transition function and bandit feedback, we obtain the regret bound

$$R_K = \tilde{O} \left( H\sqrt{K} + \gamma K H S A + \frac{H}{\eta} + \frac{\eta}{\gamma} H^3(K + D) + \frac{H}{\gamma} + \frac{\eta}{\gamma^2} H^3d_{max} \right),$$

and choosing $\eta = \frac{1}{H(A^{1/2}(K + D)^{3/2})}$ and $\gamma = \frac{1}{(A^{3/2}(K + D)^{1/2})}$ gives the theorem’s statement.

Similarly, with known transition function and full-information feedback (the only non-zero term now is Term (B)), we obtain the regret bound

$$R_K = \tilde{O} \left( H + \eta H^3(K + D) \right)$$

and choosing $\eta = \frac{1}{H\sqrt{K + D}}$ gives the theorem’s statement.
**C Skipping scheme for handling large delays**

In this section we show that by skipping episodes with large delays, we can substitute the $d_{\text{max}}$ term in Theorems 1 to 3 by $\sqrt{D}$. This was presented by Thune, Cesa-Bianchi, and Seldin (2019) for MAB with delays and can be easily applied to our setting as well. The idea is to skip episodes with delay larger than $\sqrt{D}$ and bound the regret on skipped episodes trivially by $H$. That way, effectively the maximum delay is $\sqrt{D}$ and the number of skipped episodes is at most $\sqrt{D}$ as well. The skipping scheme can be generalized for arbitrary threshold as presented in **Algorithm 4**.

**Lemma 11.** Assume that we have a regret bound for ALG when simulating ALG with a threshold $\beta > 0$.

Observe feedback from all episodes in $F^k = \{j : j + d^j = k\}$.

Feed ALG all episodes $j \in F^k$ such that $d^j \leq \beta$.

The proof of Lemma 11 relies on the fact that the algorithm does not observe feedback outside of $\mathcal{K}_\beta$, and the total delay on those rounds is $D_\beta$, the first sum is bounded by $R^{\text{ALG}}(|\mathcal{K}_\beta|, D_\beta, \beta)$.

Since the the algorithm policies $\pi^k$ are affected only by feedback from $\mathcal{K}_\beta$, and the total delay on those rounds is $D_\beta$, the first sum is bounded by $R^{\text{ALG}}(|\mathcal{K}_\beta|, D_\beta, \beta)$.

Since the value function is bounded by $H$, the second sum is bounded by $H(K - |\mathcal{K}_\beta|)$.

**Remark 4.** The proof of Lemma 11 relies on the fact that the algorithm does not observe feedback outside of $\mathcal{K}_\beta$. However, if the trajectory feedback is available immediately at the end of the episode, we can also feed the algorithm with trajectory feedback outside of $\mathcal{K}_\beta$. This can only shrink the confidence intervals and reduce the regret.

**Lemma 12.** For any threshold $\beta > 0$, the number of skipped rounds under **Algorithm 4** is bounded by $K - \mathcal{K}_\beta < \frac{D}{\beta}$.

**Proof.** First note that,

$$K - \mathcal{K}_\beta = \sum_{k=1}^{K} (1 - \mathbb{I}\{d^k \leq \beta\}) = \sum_{k=1}^{K} \mathbb{I}\{d^k > \beta\}.$$  

Now, we can bound the sum of delays by,

$$D \geq \sum_{k=1}^{K} d^k \mathbb{I}\{d^k > \beta\} > \sum_{k=1}^{K} \beta \mathbb{I}\{d^k > \beta\} = \beta (K - \mathcal{K}_\beta).$$  

Dividing both sides by $\beta$ completes the proof.

Using the last two lemmas and the regret guarantees that we show in previous sections, we can now deduce regret bounds for delayed OPPO, when simulated by **Algorithm 4**. In some settings there was no dependence on $d_{\text{max}}$, and thus no skipping is needed.

**Corollary 1.** Running Delayed OPPO, results in the following regret bounds (with probability at least $1 - \delta$):

- **Under bandit feedback, known dynamics**, and with threshold $\beta = \sqrt{D/HS}$,

  $$\mathcal{R}_K = \widetilde{O}\left(HS\sqrt{A\beta^{2/3}} + H^2 D^{2/3}\right).$$  

• Under bandit feedback, unknown dynamics, non-delayed trajectory feedback, and with threshold \( \beta = \sqrt{D/H_S} \),
\[
    R_K = \tilde{O}\left( HS\sqrt{AK}^{2/3} + H^2 D^{2/3} + H^2 S^2 A \right).
\]

• Under full-information feedback, unknown dynamics delayed trajectory feedback, and with threshold \( \beta = \sqrt{D/H_S} \),
\[
    R_K = \tilde{O}\left( H^2 S\sqrt{AK} + H^{3/2}\sqrt{SD} + H^2 S^2 A^3 \right).
\]

• Under bandit feedback, unknown dynamics, delayed trajectory feedback, and with threshold \( \beta = \sqrt{D/H_S A} \),
\[
    R_K = \tilde{O}\left( HS\sqrt{AK}^{2/3} + H^2 D^{2/3} + H^3 S^2 A^3 + S^3 A^3 \right).
\]

Proof. The first three regret bounds follow immediately from the regret bounds we show for Delayed OPPO, Lemma 11 and Lemma 12. For the last bound, we directly get a bound of,
\[
    \tilde{O}\left( HS\sqrt{AK}^{2/3} + H^2 D^{2/3} + H^{3/2} SAD + H^2 S^2 A^3 + S^3 A^3 \right).
\]

Note that if the third term dominates over the second, then \( D \leq \left(\frac{SA}{H}\right)^3 \), which implies that
\[
    H^{2/3} S\sqrt{SD} \leq S^3 A^3.
\]
\[\square\]
D Doubling trick for handling unknown number of episodes and total delay

Algorithm 5 for unknown $D$ and $K$, uses the well-known doubling trick. Unlike (Bistritz et al. 2019) that aim to estimate $K + D$, we aim to estimate the value $(A^{3/2}K + D)$ under bandit feedback which is the core difference that allows us to obtain the exact same bounds as the case where $K$ and $D$ are known. In fact, using the proper tuning and our analysis, one can achieve optimal regret in Delayed MAB using a doubling trick (see Remark 6). The second difference is that we incorporate the skipping scheme and tune the threshold using our estimate which allows us to preserve the merits of the skipping procedure.

On the technical side, the proof of Theorem 6 adopts some ideas from (Bistritz et al. 2019), but requires more delicate analysis, especially when handling term (i) in the proof.

Denote by $M^k$ the number of missing samples at the end of episode $k$. That is, $M^k = k - \sum_{j=1}^{k} |F^j|$. At time $k$ we estimate the value of $(A^{3/2}K + D)$ by $A^{3/2}k + \sum_{j=1}^{k} M^j$, and initializes a new phase whenever the estimation doubles itself. Note that this is an optimistic estimation of $(A^{3/2}K + D)$. If the feedback from episode $j$ arrived, then we estimate its delay exactly by $d^j$. However, if the feedback did not arrive, we estimate it as if the feedback will arrive in the next episode.

Algorithm 5: Delayed OPPO with known transition function, bandit feedback and unknown $D$ and $K$

Input: State space $\mathcal{S}$, Action space $\mathcal{A}$, Horizon $H$, Transition function $p$.
Initialization: Set $\pi^1_h(a | s) = \frac{1}{|A|}$ for every $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$, $e = 1$, $\eta_e = H^{-1/2} - 2e/3$, $\gamma_e = 2^e/3$, $\beta_e = 2^{e/2}$.
for $k = 1, 2, \ldots, K$ do
  Play episode $k$ with policy $\pi^k$.
  Observe feedback from all episodes $j \in F^k$.
  # Policy Evaluation
  for $j \in F^k$ such that $d^j \leq \beta_e$ do
    $\forall s \in \mathcal{S} : V^j_{H+1}(s) = 0$.
    for $h = H, \ldots, 1$ and $(s, a) \in \mathcal{S} \times \mathcal{A}$ do
      $\tilde{c}^j_h(s, a) = \frac{Q^j_h(s, a)}{q^{\pi^j_h}(s) + }_{\gamma_e}$.
      $Q^j_h(s, a) = \tilde{c}^j_h(s, a) + (p_h(\cdot | s, a), V^j_{h+1})$.
      $V^j_h(s) = (Q^j_h(s, \cdot), \pi^j_h(\cdot | s))$.
    end for
  end for
  # Policy Improvement
  for $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$ do
    $\pi^{k+1}_h(a | s) = \frac{\pi_k^h(a | s) \exp(-\eta_k \sum_{j \in F^k, d^j \leq \beta_e} Q^j_h(s, a))}{\sum_{a' \in \mathcal{A}} \pi^k_h(a' | s) \exp(-\eta_k \sum_{j \in F^k, d^j \leq \beta_e} Q^j_h(s, a'))}$.
  end for
  # Doubling
  if $A^{3/2}k + \sum_{j=1}^{k} M^j = A^{3/2}k + \sum_{j=1}^{\gamma_e} (j - \sum_{i=1}^{j} |F^j|) > 2^e$ then
    Set $\pi^k_h(a | s) = \frac{1}{|A|}$ for every $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$, $e = e + 1$, $\eta_e = H^{-1/2} - 2e/3$, $\gamma_e = 2^e/3$, $\beta_e = 2^{e/2}$.
  end if
end for

Theorem 6. Under bandit feedback, if the transition function is known, then (with probability at least $1 - \delta$), the regret of Algorithm 5 is bounded by,

$$R_K \leq \tilde{O}(HS\sqrt{A}K^{2/3} + H^2D^{2/3}).$$

Proof. Let $K_e$ be the episodes in phase $e$, $\mathcal{N}'_e = \{k \in K_e : k + d^k \notin K_e\}$, $\mathcal{N}_{\beta_e} = \{k \in K_e : d^k \leq \beta_e\}$, and $\mathcal{N}_e = \mathcal{N}'_e \cup \mathcal{N}_{\beta_e}$ which is the set of episodes with missing feedback in phase $e$. Also, denote the last episode of phase $e$ by $K_e$. Using Theorem 2 and Lemma 11,

$$\sum_{k \in K_e} V^k_1(s^1_k) - V^\pi_1(s^1_k) = \tilde{O}(\gamma_e |K_e| HSA + \frac{H}{\gamma_e} + \frac{\eta_e}{\gamma_e} H^3 \sum_{k \in K_e \setminus \mathcal{N}_e} (1 + d^k) + \frac{H}{\gamma_e} + H^3 \frac{\eta_e}{\gamma_e} \beta_e + H \sqrt{|K_e|} + H |\mathcal{N}_e|).$$

(15)
By definition of our doubling scheme,

\[ 2^e \geq A^{3/2} K_e + \sum_{k=1}^{K_e} M^k \]

\[ = A^{3/2} K_e + \sum_{k=1}^{K_e} \sum_{j=1}^K \mathbb{I}\{j \leq k, j + d^j > k\} \]

\[ = A^{3/2} K_e + \sum_{j=1}^{K_e} \sum_{k=1}^K \mathbb{I}\{j \leq k, j + d^j > k\} \]

\[ \geq K_e + \sum_{j \in \mathcal{K}_e \setminus \mathcal{N}_e} \sum_{k=1}^{K_e} \mathbb{I}\{j \leq k, j + d^j > k\} \]

\[ \geq |\mathcal{K}_e| + \sum_{j \in \mathcal{K}_e \setminus \mathcal{N}_e} d^j \]

\[ \geq \sum_{j \in \mathcal{K}_e \setminus \mathcal{N}_e} (d^j + 1), \quad (16) \]

where (\*) holds since \( j + d^j \leq K_e \) for any \( j \in \mathcal{K}_e \setminus \mathcal{N}_e \). We now bound \( |\mathcal{N}_e'| \). Similarly to the above,

\[ 2^e \geq \sum_{j=1}^{K_e} \sum_{k=1}^{K_e} \mathbb{I}\{j \leq k, j + d^j > k\} \]

\[ \geq \sum_{j \in \mathcal{N}_e'} \sum_{k=K_e-j+1}^{K_e} \mathbb{I}\{j \leq k, j + d^j > k\} \]

\[ = \sum_{j \in \mathcal{N}_e'} K_e - j + 1 \]

\[ \geq \sum_{j=|\mathcal{N}_e'|}^{K_e} K_e - j + 1 \]

\[ \geq \sum_{j=1}^{|\mathcal{N}_e'|+1} j \]

\[ = \frac{1}{2}(|\mathcal{N}_e'| + 1)(|\mathcal{N}_e'| + 2) \]

\[ \geq \frac{1}{2} |\mathcal{N}_e'|^2, \]

where (\*) follows from the fact that for any \( k \in \mathcal{K}_e \) we have that \( j + d^j > k \), and (\**) follows by choosing the largest possible \(|\mathcal{N}_e'|\) indices. The above implies that,

\[ |\mathcal{N}_e'| \leq 2^{\frac{e+1}{2}}. \quad (17) \]
For last, we bound \( |\mathcal{N}_{\beta_k} \setminus \mathcal{N}_e'| \),

\[
2^e \geq \sum_{j=1}^{K} \sum_{k=1}^{K} \mathbb{I}\{j \leq k, j + d^j > k\}
\]

\[
\geq \sum_{j \in \mathcal{N}_{\beta_k} \setminus \mathcal{N}_e'} \sum_{k=1}^{K} \mathbb{I}\{j \leq k, j + d^j > k\}
\]

\[
= \sum_{j \in \mathcal{N}_{\beta_k} \setminus \mathcal{N}_e'} d^j
\]

\[
\geq (\ast) \sum_{j \in \mathcal{N}_{\beta_k} \setminus \mathcal{N}_e'} \beta_e
\]

\[
= |\mathcal{N}_{\beta_k} \setminus \mathcal{N}_e'| \beta_e,
\]

where (\ast) follows because \( j \notin \mathcal{N}_e' \) and so \( j + d^j \leq K_e \). And (\ast\ast) follows the definition of \( \mathcal{N}_{\beta_k} \). This implies that,

\[
|\mathcal{N}_{\beta_k} \setminus \mathcal{N}_e'| \leq 2^\frac{e}{2}.
\]

Combining with 17 gives us,

\[
|\mathcal{N}_e| = |\mathcal{N}_e' \cup \mathcal{N}_{\beta_k}| \leq 2^\frac{e+1}{2}.
\]

Plugging the bounds of 16 and 18 into 15, noting that \( |\mathcal{K}_e| \leq 2^e \), plugging the values of \( \eta_e, \gamma_e \) and \( \beta_e \), and summing over all phases gives us,

\[
\mathcal{R}_K \leq \hat{O}\left(HSA \sum_{e=1}^{E} |\mathcal{K}_e|2^{-\frac{e}{2}} + H^2 \sum_{e=1}^{E} 2^{\frac{e}{2}}\right),
\]

where \( E \leq \log(A^{3/2}K + \Delta) \) is the number of phases. Term (ii) above is a sum of geometric series and can be bounded by \( O(D + A^{3/2}K)^{2/3} \leq O(D^{2/3} + SKK^{2/3}) \) since \( A \leq S \). Term (i) is bounded by the value of the following maximization problem,

\[
\max \sum_{e=1}^{E} |\mathcal{K}_e|2^{-\frac{e}{2}}
\]

subject to \( A^{3/2} |\mathcal{K}_e| \leq 2^e \), \( \sum_{e=1}^{E} |\mathcal{K}_e| = K \),

which is necessarily bounded by

\[
\max \sum_{e=1}^{\infty} x_e 2^{-\frac{e}{2}}
\]

subject to \( 0 \leq x_e \leq \frac{2^e}{A^{3/2}}, \sum_{e=1}^{\infty} x_e = K \).

Since \( 2^{-\frac{e}{2}} \) is decreasing, this is maximized whenever the first \( x_e \)'s are at maximum value. There are at most \( \lceil \log(K A^{3/2}) \rceil \) non-zero \( x_e \)'s and so,

\[
\sum_{e=1}^{E} |\mathcal{K}_e|2^{-\frac{e}{2}} \leq \frac{1}{A^{3/2}} \sum_{e=1}^{\lceil \log(K A^{3/2}) \rceil} 2^{\frac{e}{2}} \leq \hat{O} \left(\frac{1}{A^{3/2}}(A^{3/2}K)^{\frac{e}{2}}\right) \leq \hat{O} \left(\frac{1}{\sqrt{A}}K^{\frac{e}{2}}\right),
\]

which gives us the desired bound.

\[\square\]

**Remark 5.** The exact same proof holds for the rest of settings, in which we get the exact same regret as in Corollary 1 and Theorem 2. For full information, one should set the doubling condition to be \( k + \sum_{j=1}^{k} M^j > 2^e \), and \( \eta_e = H^{-1}2^{-\frac{e}{2}} \).

**Remark 6.** Bistritz et al. (2019) set, for delayed MAB, the doubling condition to be \( k + \sum_{j=1}^{k} M^j > 2^e \) which results in sub-optimal \( O(A\sqrt{K} + \sqrt{D}) \) regret, instead of \( \hat{O} (\sqrt{AK} + \sqrt{D}) \). Changing the condition to be \( Ak + \sum_{j=1}^{k} M^j > 2^e \) and following the new ideas in our analysis leads to the optimal \( \hat{O} (\sqrt{AK} + \sqrt{D}) \) regret. Interestingly, our analysis shows that the doubling condition of Bistritz et al. (2019) is appropriate for full-information feedback but not to bandit feedback, where our new condition is crucial in order to get optimal regret.
E Delayed O-REPS

Given an MDP $\mathcal{M}$, a policy $\pi$ induces an occupancy measure $q = q^\pi$, which satisfies the following:

$$
\sum_{a, s'} q_h(s_{init}, a, s') = 1 \quad (19)
$$

$$
\sum_{s, a, s'} q_h(s, a, s') = 1 \quad \forall h = 1, \ldots, H \quad (20)
$$

$$
\sum_{s', a} q_h(s, a, s') = \sum_{s'} q_h-1(s', a, s) \quad \forall s \in \mathcal{S} \text{ and } h = 2, \ldots, H \quad (21)
$$

If $q$ satisfies (19), (20) and (21), then it induces a policy and a transition function in the following way:

$$
\pi_h^q(a \mid s) = \frac{\sum_{a'} q_h(s, a, a')}{\sum_{a'} q_h(s', a', s')}
$$

$$
p_h^q(s' \mid s, a) = \frac{q(s', a, s)}{\sum_{a} q(s', a, s)}
$$

The next lemma characterize the occupancy measures induced by some policy $\pi$.

**Lemma 13** (Lemma 3.1, Rosenberg and Mansour (2019a)). An occupancy measure $q$ that satisfies (19), (20) and (21) is induced by some policy $\pi$ if and only if $\bar{p} = p$.

**Definition 1.** Given an MDP $\mathcal{M}$, we define $\Delta(\mathcal{M})$ to be the set of all $q \in [0, 1]^{H \times S \times A \times S}$ that satisfies (19), (20) and (21) such that $\bar{p} = p$ (where $p$ is the transition function of $\mathcal{M}$).

For convenience, in this section we let the cost functions to be a function of the current state, the action taken and the next state: $c_h(s, a, s')$. So the value of a policy, $\pi$ is given by

$$
V^\pi(s_{init}) = \langle q^\pi, c \rangle = \sum_h \langle q^\pi_h, c_h \rangle
$$

and the regret with respect to a policy $\pi$ is given by

$$
\mathcal{R}_K = \sum_{k=1}^{K} \langle q^{p_k} - q^\pi, c^k \rangle.
$$

The above can be treated as an online linear optimization problem. Indeed, O-REPS (Zimin and Neu 2013) treats it as such by running Online Mirror Decent (OMD) on the set of occupancy measures. That is, at each episode the algorithm plays the policy induced by the occupancy measure $q^k$ and updates the occupancy measure for the next episode by,

$$
q^{k+1} = \arg \min_{q \in \Delta(\mathcal{M})} \left\{ \eta(q, c^k) + D_R(q\|q^k) \right\},
$$

where is $R$ the unnormalized negative entropy. That is,

$$
R(q) = \sum_h \sum_{s, a, s'} q_h(s, a, s') \log q_h(s, a, s') - q_h(s, a, s')
$$

and $D_R$ is the Bregman divergence associated with $R$ (Har is also known as the Kullback-Leibler divergence). If the feedback is delayed we would update the occupancy measure using all the feedback that arrives at the end of the current episode, i.e.,

$$
q^{k+1} = \arg \min_{q \in \Delta(\mathcal{M})} \left\{ \eta(q, c^k) + D_R(q\|q^k) \right\},
$$

Whenever the transition function is unknown, $\Delta(\mathcal{M})$ can not be computed. In this case we adopt the method of Rosenberg and Mansour (2019a) and extend $\Delta(\mathcal{M})$ by the next definition.

**Definition 2.** For any $k \in [K]$, we define $\Delta(\mathcal{M}, k)$ to be the set of all $q \in [0, 1]^{H \times S \times A \times S}$ that satisfies (19), (20), (21) and

$$
\forall h, s', a, s : |p_h^k(s' \mid s, a) - \bar{p}_h^k(s' \mid s, a)| \leq c^k_h(s' \mid s, a).
$$
Algorithm 6: Delayed O-REPS

**Input:** State space $S$, Action space $A$, Learning rate $\eta > 0$, Confidence parameter $\delta > 0$.

**Initialization:** Set $\pi_1^h(a | s) = \mathbb{1}/A$, $q_h^k(s, a, s') = 1/S^2A$ for every $(s, a, s', h) \in S \times A \times S \times [H]$.

**for** $k = 1, 2, ..., K$ **do**

- Play episode $k$ with policy $\pi_k^h$.
- Observe feedback from all episodes $j \in F_k$, and last trajectory $U_k = \{(s_h^k, a_k^j)\}_{h=1}^H$.
- Update transition function estimation.
  - # Update Occupancy Measure
    - if transition function is known then
      $$ q^{k+1} = \arg\min_{q \in \Delta(M)} \{ \eta \langle q, \sum_{j \in F_k} c_j \rangle + D_R(q\|q^k) \} $$
    - else
      $$ q^{k+1} = \arg\min_{q \in \Delta(M,k)} \{ \eta \langle q, \sum_{j \in F_k} c_j \rangle + D_R(q\|q^k) \} $$
  - end if
  - # Update Policy
    - Set $\pi_{h+1}^k(a | s) = \frac{\sum_{s', a, s''} q_h^{k+1}(s, a, s')}{\sum_{s', a, s''} q_h^k(s, a, s'')}$ for every $(s, a, h) \in S \times A \times [H]$.

**end for**

The update step will now be with respect to $\Delta(M, k)$. We have that with high probability $\Delta(M, k)$ contain $\Delta(M)$ for all $k$, and so the estimation of the value function is again optimistic. Delayed O-REPS for unknown dynamics is presented in Algorithm 6.

The update step can be implemented by first solving the unconstrained convex optimization problem,

$$ \tilde{q}^{k+1} = \arg\min \left\{ \eta \left\langle q, \sum_{j \in F_k} c_j \right\rangle + D_R(q\|q^k) \right\}, \tag{22} $$

and then projecting onto the set $\Delta(M, k)$ with respect to $D_R(\cdot\|\tilde{q}^{k+1})$. That is,

$$ q^{k+1} = \arg\min_{q \in \Delta(M, k)} D_R(q\|\tilde{q}^{k+1}). $$

The solution for (22) is simply given by,

$$ q^{k+1}_h(s, a, s') = q^k_h(s, a, s') e^{-\eta \sum_{j \in F_k} c_j(s, a, s')}. $$

**Theorem 7.** Running Delayed O-REPS under full-information feedback and delayed cost feedback guarantees the following regret, with probability at least $1 - \delta$,

$$ R_K = \tilde{O}(H^{3/2}S\sqrt{AK} + H\sqrt{D} + H^2S^2A). $$

Moreover, if the transition function is known, we obtain regret of

$$ R_K = \tilde{O}(H\sqrt{K} + D). $$

**E.1 Proof of Theorem 7**

Given a policy $\pi$ and a transition $p$, we denote the occupancy measure of $\pi$ with respect to $p$, by $q^{p, \pi}$. That is, $q^{p, \pi}_h(s, a, s') = \Pr[s_h^k = s, a_h^k = a, s_{h+1}^k = s' | s_1^k = s_{init}, \pi, p]$. Also, denote $p^k = p^{\pi}$, and note that by definition $q^{p_{\pi}, \pi}_h = q^k$. We define the following good event $G = \neg P_{\text{basic}}$ where $P_{\text{basic}}$ defined in Appendix B. As shown in Lemma 1, $G$ occurs with probability of at least $1 - \delta$. As consequence we have that for all episodes, $\Delta(M) \subseteq \Delta(M, k)$. From this point we analyse the regret given that $G$ occurred.

We break the regret in the following way:

$$ \sum_{k=1}^K \langle q^{p, \pi} - q^{p, \pi}, c^k \rangle = \sum_{k=1}^K \langle q^{p, \pi} - q^{p_{\pi}, \pi}, c^k \rangle + \sum_{k=1}^K \langle q^{p_{\pi}, \pi} - q^{p, \pi}, c^k \rangle \\
= \sum_{k=1}^K \langle q^{p, \pi} - q^{p_{\pi}, \pi}, c^k \rangle + \sum_{k=1}^K \langle q^k - q^{\pi}, c^k \rangle. \tag{23} $$
The first term, under the good event, is bounded similarly as in the proof of Theorem 1 by,
\[
\sum_{k=1}^{K} (q^k, q^{k+1}, c^k) \leq H^{3/2} S \sqrt{AK} + H^2 S^2 A.
\] (24)

For the second term, we adopt the approach of (Thune, Cesa-Bianchi, and Seldin 2019; Bistritz et al. 2019), and modify the standard analysis of OMD. We start with Lemma 14 which bounds the regret of playing \( \pi_{k+d} \) at episode \( k \).

**Lemma 14.** If \( \Delta(\mathcal{M}) \subseteq \Delta(\mathcal{M}, k) \) for all \( k \), then for any \( q \in \Delta(\mathcal{M}) \), delayed O-REPS satisfies
\[
\sum_{k=1}^{K} (c^k, q^{k+1} - q^k) \leq \frac{2H \log(HSA)}{\eta} + \eta HK.
\]

**Proof.** Note that \( \tilde{q}_h^{k+1}(s, a, s') = q_h^k(s, a, s') e^{-\eta \sum_{j \in F_k} c_h^j(s, a, s')} \). Taking the log,
\[
\eta \sum_{j \in F_k} c_h^j(s, a, s') = \log q_h^k(s, a, s') - \log \tilde{q}_h^{k+1}(s, a, s').
\]
Hence for any \( q \)
\[
\eta \left( \sum_{j \in F_k} c_h^j, q^k - q \right) = \langle \log q^k - \log \tilde{q}_h^{k+1}, q^k - q \rangle
\]
\[
= DR(q||q^k) - DR(q||\tilde{q}_h^{k+1}) + DR(q^k||\tilde{q}_h^{k+1})
\]
\[
\leq DR(q||q^k) - DR(q||\tilde{q}_h^{k+1}) + DR(q^k||\tilde{q}_h^{k+1}) + DR(q^k||q^{k+1})
\]
\[
\leq DR(q||q^k) - DR(q||\tilde{q}_h^{k+1}) + DR(q^k||q^{k+1}),
\]
where the first equality follows directly the definition of Bregman divergence. The first inequality is by (Zimin 2013 Lemma 1.2) and the assumption that \( \Delta(\mathcal{M}) \subseteq \Delta(\mathcal{M}, k) \). The second inequality is since Bregman divergence is non-negative. Now, the last term is bounded by,
\[
DR(q^k||q^{k+1}) \leq DR(q^k||\tilde{q}_h^{k+1}) + DR(\tilde{q}_h^{k+1}||q^k)
\]
\[
= \sum_h \sum_{s,a,s'} \tilde{q}_h^{k+1}(s, a, s') \log \frac{\tilde{q}_h^{k+1}(s, a, s')}{q_h^k(s, a, s')}
\]
\[
+ \sum_h \sum_{s,a,s'} q_h^k(s, a, s') \log \frac{q_h^k(s, a, s')}{\tilde{q}_h^{k+1}(s, a, s')}
\]
\[
= \langle q^k - \tilde{q}_h^{k+1}, \log q^k - \log \tilde{q}_h^{k+1} \rangle
\]
\[
= \eta \left\langle q^k - \tilde{q}_h^{k+1}, \sum_{j \in F_k} c^j \right\rangle.
\]
We get that
\[
\eta \left( \sum_{j \in F_k} c^j, q^k - q \right) \leq DR(q||q^k) - DR(q||\tilde{q}_h^{k+1}) + \eta \left\langle q^k - \tilde{q}_h^{k+1}, \sum_{j \in F_k} c^j \right\rangle.
\]
Summing over \( k \) and dividing by \( \eta \), we get
\[
\sum_{k=1}^{K} \left( c^j, q^k - q \right) \leq \frac{DR(q||q^1) - DR(q||q^{K+1})}{\eta} + \sum_{k=1}^{K} \left\langle q^k - \tilde{q}_h^{k+1}, \sum_{j \in F_k} c^j \right\rangle.
\]
\[
\leq \frac{DR(q||q^1)}{\eta} + \sum_{k=1}^{K} \left\langle q^k - \tilde{q}_h^{k+1}, \sum_{j \in F_k} c^j \right\rangle
\]
\[
\leq \frac{2H \log(SA)}{\eta} + \sum_{k=1}^{K} \left\langle q^k - \tilde{q}_h^{k+1}, \sum_{j \in F_k} c^j \right\rangle.
\] (**)
where the last inequality is a standard argument (see [Zimin 2013; Hazan 2019]). We now rearrange (*) and (**):

\[
(*) = \sum_{k=1}^{K} \sum_{j=1}^{K} I\{j + d^j = k\} \langle c^j, q^k - q \rangle \\
= \sum_{j=1}^{K} \sum_{k=1}^{K} I\{j + d^j = k\} \langle c^j, q^k - q \rangle \\
= \sum_{j=1}^{K} (c^j, q^{j+d^j} - q) \\
= \sum_{k=1}^{K} (c^k, q^{k+d^k} - q).
\]

In a similar way,

\[
(**) = \sum_{k=1}^{K} \sum_{j \in \mathcal{F}^k} \langle q^k - \tilde{q}^{k+1}, c^j \rangle \\
= \sum_{j=1}^{K} \sum_{k=1}^{K} I\{j \in \mathcal{F}^k\} \langle q^k - \tilde{q}^{k+1}, c^j \rangle \\
= \sum_{j=1}^{K} (q^j - \tilde{q}^{j+1}, c^j) \\
= \sum_{k=1}^{K} (q^{k+d^k} - \tilde{q}^{k+d^k+1}, c^k).
\]

This gives us,

\[
\sum_{k=1}^{K} (c^k, q^{k+d^k} - q) \leq \frac{2H \log(SA)}{\eta} + \sum_{k=1}^{K} (q^{k+d^k} - \tilde{q}^{k+d^k+1}, c^k).
\]

It remains to bound the second term on the right hand side:

\[
\sum_{k} (q^{k+d^k} - \tilde{q}^{k+d^k+1}, c^k) = \sum_{k} \sum_{h} c_h^{k}(s, a, s') \sum_{s, a, s'} (q_h^{k+d^k}(s, a, s') - \tilde{q}_h^{k+d^k+1}(s, a, s')) \\
= \sum_{k} \sum_{h} \sum_{s, a, s'} c_h^{k}(s, a, s') \left( q_h^{k+d^k}(s, a, s') - \tilde{q}_h^{k+d^k+1}(s, a, s') \right) e^{-\eta \sum_{j \in \mathcal{F}^k+d^k+1} c_h^{j}(s, a, s')} \\
\leq \sum_{k} \sum_{h} \sum_{s, a, s'} q_h^{k+d^k}(s, a, s') \left( 1 - e^{-\eta \sum_{j \in \mathcal{F}^k+d^k+1} c_h^{j}(s, a, s')} \right) \\
\leq \eta H \sum_{k} \sum_{h} \sum_{s, a, s'} q_h^{k+d^k}(s, a, s') \left( \sum_{j \in \mathcal{F}^k+d^k+1} c_h^{j}(s, a, s') \right) \\
\leq \eta H \sum_{k} \left| \mathcal{F}^{k+d^k+1} \right| \\
\leq \eta H K.
\]

This completes the proof of the lemma. 

Using Lemma 14 we can bounds the regret as,
\[
\sum_{k=1}^{K} (e^k, q^k - q) \leq \frac{2H \log(\eta HSA)}{\eta} + \eta H K + \sum_{k=1}^{K} (e^k, q^k - q^{k+d^k}).
\]

(25)

The next lemma is a generalization of Lemma 1 in [Zimin and Neuf2013], which allows us to bound the distance between two consecutive occupancy measures.

**Lemma 15.** For \(q^k_h\) that are generated by delayed O-REPS, we have that,

\[
\sum_h D_R(q^k_h \| q^{k+1}_h) \leq \sum_h \sum_{s,a,s'} q^k_h(s,a,s') \left( \eta \sum_{j \in \mathcal{X}^k} \frac{c^j_h(s,a,s')^2}{2} \right).
\]

**Proof.** First we present some notations. Given \(v_h, c_h : S \times A \times S \rightarrow \mathbb{R}\), define

\[
B^k_h(s,a,s' | v,e) = e_h(s,a,s') + v_h(s,a,s') - \eta \sum_{j \in \mathcal{X}^k} c^j_h(s,a,s') - \sum_{s''} \bar{p}^k_h(s'' | s,a) v_{h+1}(s,a,s'').
\]

Given \(\beta_h : \mathbb{R} \rightarrow \mathbb{R}\) and \(\mu^+_h, \mu^-_h : S \times A \times S \rightarrow \mathbb{R}_{\geq 0}\), define

\[
v^\beta_h(s,a,s') = \mu^+_h(s,a,s') - \mu^-_h(s,a,s')
\]

\[
e^{\mu^+_h, \beta_h}_h(s,a,s') = (\mu^-_h(s,a,s') + \mu^+_h(s,a,s')) e^h(s' | s,a) + \beta_h(s) - \beta_{h+1}(s'),
\]

where we always set \(\beta_1 = \beta_H = 0\). For last, define

\[
Z^k_h(v,e) = \sum_{s,a,s'} q^k_h(s,a,s') e^{B^k_h(s,a,s'|v,e)}.
\]

By [Rosenberg and Mansour2019a] Theorem 4.2), we have that

\[
q^{k+1}_h(s,a,s') = \frac{q^k_h(s,a,s') e^{B^k_h(s,a,s'|v^k_h,e^{\mu^+_h, \beta^k}_h)}}{Z^k_h(v^{\mu^+_h, e^{\mu^+_h, \beta^k}}_h)}.
\]

where

\[
\mu^k, \beta^k = \arg \min_{\beta, \mu \geq 0} \sum_{h=1}^{H} \log Z^k_h(v^{\mu^k_h, e^{\mu^k_h, \beta^k}_h}).
\]

Now, we have that

\[
\sum_h D_R(q^k_h \| q^{k+1}_h) = \sum_h \sum_{s,a,s'} q^k_h(s,a,s') \log \frac{q^k_h(s,a,s')}{q^{k+1}_h(s,a,s')}
\]

\[
= \sum_h \sum_{s,a,s'} q^k_h(s,a,s') \log \frac{Z^k_h(v^{\mu^k_h, e^{\mu^k_h, \beta^k}_h})}{e^{B^k_h(s,a,s'|v^{\mu^k_h, e^{\mu^k_h, \beta^k}_h})}}
\]

\[
= \log Z^k_h(v^{\mu^k_h, e^{\mu^k_h, \beta^k}_h}) - \sum_h \sum_{s,a,s'} q^k_h(s,a,s') B^k_h(s,a,s' | v^{\mu^k_h, e^{\mu^k_h, \beta^k}_h}).
\]
By definition of \( \mu^k_h, \beta^k_h \), term \( A \) can be bounded by

\[
(A) \leq \sum_h \log Z_h^k(0, 0) \\
= \sum_h \log \left( \sum_{s, a, s'} q_h^k(s, a, s') e^{B_h^k(s, a, s')[0, 0]} \right) \\
= \sum_h \log \left( \sum_{s, a, s'} q_h^k(s, a, s') e^{-\eta \sum_{j \in \mathcal{F}^k} c_h^j(s, a, s')} \right) \\
\leq \sum_h \log \left( \sum_{s, a, s'} q_h^k(s, a, s') \left( 1 - \eta \sum_{j \in \mathcal{F}^k} c_h^j(s, a, s') + \frac{(\eta \sum_{j \in \mathcal{F}^k} c_h^j(s, a, s'))^2}{2} \right) \right) \\
= \log \left( 1 - \eta \sum_{s, a, s'} q_h^k(s, a, s') c_h^j(s, a, s') + \frac{(\eta \sum_{j \in \mathcal{F}^k} c_h^j(s, a, s'))^2}{2} \right) \\
\leq -\eta \sum_{s, a, s'} \sum_{j \in \mathcal{F}^k} q_h^k(s, a, s') c_h^j(s, a, s') + \frac{(\eta \sum_{j \in \mathcal{F}^k} c_h^j(s, a, s'))^2}{2}. \\
\end{equation}

Term \( B \) can be rewritten as

\[
(B) = \sum_h \sum_{s, a, s'} q_h^k(s, a, s') e^{\mu_h^k, \beta_h^k(s, a, s')} + \nu_h^k(s, a, s') \\
- \eta \sum_{s, s'} c_h^j(s, a, s') - \sum_{s, s'} p_h^k(s'' | s, a) \nu_h^k(s, a, s'') \\
= \sum_h \sum_{s, a, s'} q_h^k(s, a, s') e^{\mu_h^k, \beta_h^k(s, a, s')} + \sum_h \sum_{s, a, s'} q_h^k(s, a, s') \nu_h^k(s, a, s') \\
- \eta \sum_{s, a, s'} \sum_{j \in \mathcal{F}^k} q_h^k(s, a, s') c_h^j(s, a, s') \\
- \sum_{s, a, s'} \sum_{s''} q_h^k(s, a, s') p_h^k(s'' | s, a) \nu_h^k(s, a, s'') \\
= \sum_h \sum_{s, a, s'} q_h^k(s, a, s') e^{\mu_h^k, \beta_h^k(s, a, s')} + \sum_h \sum_{s, a, s'} q_h^k(s, a, s') \nu_h^k(s, a, s') \\
- \eta \sum_{s, a, s'} \sum_{j \in \mathcal{F}^k} q_h^k(s, a, s') c_h^j(s, a, s') \\
- \sum_{s, a, s'} \sum_{s''} q_h^k(s, a, s') p_h^k(s'' | s, a) \nu_h^k(s, a, s'') \\
= \sum_h \sum_{s, a, s'} q_h^k(s, a, s') e^{\mu_h^k, \beta_h^k(s, a, s')} + \sum_h \sum_{s, a, s'} q_h^k(s, a, s') \nu_h^k(s, a, s') \\
- \eta \sum_{s, a, s'} \sum_{j \in \mathcal{F}^k} q_h^k(s, a, s') c_h^j(s, a, s') - \sum_{s, a, s'} \sum_{j \in \mathcal{F}^k} q_h^k(s, a, s') \nu_h^k(s, a, s'') \\
= \sum_h \sum_{s, a, s'} q_h^k(s, a, s') e^{\mu_h^k, \beta_h^k(s, a, s')} - \eta \sum_{s, a, s'} \sum_{j \in \mathcal{F}^k} q_h^k(s, a, s') c_h^j(s, a, s').
\]
Overall we get

\[
\sum_h D_R(q_h^k \| q_h^{k+1}) \leq \sum_h \sum_{s,a,s'} q_h^k(s,a,s') \frac{(\eta \sum_{j \in \mathcal{F}_k} c_h^j(s,a,s'))^2}{2} - \sum_h \sum_{s,a,s'} q_h^k(s,a,s') \mu_h \beta_h^k(s,a,s') \\
\leq \sum_h \sum_{s,a,s'} q_h^k(s,a,s') \frac{(\eta \sum_{j \in \mathcal{F}_k} c_h^j(s,a,s'))^2}{2} - \sum_h \sum_{s,a,s'} q_h^k(s,a,s') (\beta_h^k(s) - \beta_{h+1}^k(s')).
\]

For last, we show that the second term is 0, which completes the proof

\[
\sum_h \sum_{s,a,s'} q_h^k(s,a,s') (\beta_h^k(s) - \beta_{h+1}^k(s')) =
\sum_h \left[ \sum_{s,a,s'} q_h^k(s,a,s') \beta_h^k(s) - \sum_{s,a,s'} q_h^k(s,a,s') \beta_{h+1}^k(s') \right] =
\sum_s \left[ \sum_h q_h^k(s) \beta_h^k(s) - \sum_{s'} q_{h+1}^k(s') \beta_{h+1}^k(s') \right] =
\sum_s \left[ q_h^k(s) \beta_h^k(s) - q_h^k(s) \beta_{h+1}^k(s) \right] = 0,
\]

where the last equality follows since \( \beta_1^k = \beta_H^k = 0 \). \qed

We now use the lemma above to bound the last term in (25):

\[
\sum_{k=1}^{K} \langle c_k^k, q_k^k - q_{k+1}^k \rangle \leq \sum_{k=1}^{K} \sum_h \sum_{s,a,s'} \left| q_h^k(s,a,s') - q_{h+1}^k(s,a,s') \right| \\
\leq \sum_{k=1}^{K} \sum_{j=k}^{k+d_h^k-1} \sum_{h} \sum_{s,a,s'} \left| q_h^j(s,a,s') - q_{h+1}^j(s,a,s') \right| \\
\leq 2 \sum_{k=1}^{K} \sum_{j=k}^{k+d_h^k-1} \sum_{h} \sqrt{2D_R(q_h^j \| q_h^{j+1})} \quad \text{(by Pinsker’s inequality)} \\
\leq 2 \sum_{k=1}^{K} \sum_{j=k}^{k+d_h^k-1} \sqrt{2H \sum_{h} D_R(q_h^j \| q_h^{j+1})} \quad \text{(by Jensen’s inequality)} \\
\approx \sum_{k=1}^{K} \sum_{j=k}^{k+d_h^k-1} \sqrt{H \sum_{h} \sum_{s,a,s'} q_h^j(s,a,s') \left( \eta \sum_{i \in \mathcal{F}_j} c_h^i(s,a,s') \right)^2} \quad \text{(by Lemma 15)} \\
\leq \sum_{k=1}^{K} \sum_{j=k}^{k+d_h^k-1} \sqrt{H \sum_{h} \sum_{s,a,s'} q_h^j(s,a,s') (|\mathcal{F}_j|)^2} \quad \text{(by Lemma 15)} \\
= \eta H \sum_{k=1}^{K} \sum_{j=k}^{k+d_h^k-1} |\mathcal{F}_j| \leq \eta H D,
\]

where the last inequality is shown in the proof of Theorem 1 in (Thune, Cesa-Bianchi, and Seldin 2019). Combining the above with (23), (24) and (25) completes the proof.
Figure 2: On the left: Average cost of delayed algorithms in grid world with walls with geometrically distributed delays with mean 10. On the right: the grid environment with the wall states, where green is $s_{\text{init}}$, yellow is $s_{\text{goal}}$, dark blue are regular states and blue are the wall states.

F  Full the details of the empirical evaluation and more experiments

We conducted our experiments on a grid-world of size $10 \times 10$ i.e., $S = 100$ and horizon $H = 50$. There are four types of states: Initial state, $s_{\text{init}}$, which is always the top-left corner, goal state, $s_{\text{goal}}$, which is always the bottom-right corner, wall states which are not reachable and regular states which are the rest of the states on the grid. There are four actions $A = \{\text{up}, \text{down}, \text{right}, \text{left}\}$. After taking an action, the agent moves with probability 0.9 towards the adjacent state in the corresponding direction, provided that this is not a wall state or falls outside the grid. With probability 0.1 the direction is perturbed uniformly. The cost function is defined as $c(s, a) = I\{s \neq s_{\text{goal}}\}$.

Implementation of the algorithms. As presented by (Shani et al. 2020), under stochastic MDP, the estimated transition $\hat{p}_h^j$ can be replaced with the observed empirical transitions, and instead reduce the cost by order of $1/\sqrt{n_h^j(s, a)}$ during the policy evaluation step. This forces the algorithm to explore states that were not visited enough and keeps the estimated Q-function optimistic as our algorithm does. For better computational efficiency we implemented the policy evaluation step in our algorithm with these kind of estimates. All algorithms where run under full-information cost feedback. All algorithms where tested with a fixed learning rate $\eta = 0.1$. Reduction maintains $d_{\text{max}} + 1$ copies of OPPO where $d_{\text{max}}$ is realized maximal delay. Effectively each copy suffers no delay so this is reduced to standard OPPO (Shani et al. 2020). Each of our experiment take 2-5 hours of computation time on a CPU.

Adding wall states. The results of Fig. 1 where tested on a simple grid without wall states. In Fig. 2 we added wall states so that a more complex dynamics needs to be learned.

Note that the convergence time of all algorithms increases compared to Fig. 1 as a more complex policy need to be learned. However, Delayed OPPO still keeps its great advantage over the other two alternatives.

Convergence time of Parallel-OPPO. In all the experiments we presented so far, Parallel-OPPO have not shown an improvement over time. In order to show the difference on convergence time, we changed the delay distribution to be geometric with mean 2 and increased the number of episodes to $K = 2000$. The results are averaged over 10 runs and appear in Fig. 3.

The maximal delay scale approximately as $2 \log(K) \approx 15$. While each copy of OPPO that Parallel-OPPO maintains suffers effectively no delay, this is insignificant compared to the fact that Delayed OPPO observe approximately 15 times more observation then each copy. Pipeline-OPPO performs quite well in this case, as the maximal delay is quite small and approximately after 15 episodes it has a pipeline of observations. With that being said, note that even when the maximal delay quite small, Delayed OPPO definitely outperforms Pipeline-OPPO. In addition, it is sufficient to have a single large delay in order to have a major reduction in the performance of Pipeline-OPPO. In real-world application it is quite common to have few large delays. In fact, few delays might be infinite, for example due to packet loss over a network. This would make Pipeline-OPPO and Parallel-OPPO completely degenerate, while the effect of few missing observations on Delayed-OPPO is only minor.
Figure 3: Average cost of delayed algorithms in grid world with walls with geometrically distributed delays with mean 2.