EXISTENCE AND CLASSIFICATION OF RADIAL SOLUTIONS OF
A NONLINEAR NONAUTONOMOUS DIRICHLET PROBLEM

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Abstract. This paper generalizes a classification of solutions of a superlinear Dirichlet problem given in [13] to a nonautonomous case. In [12] the increasing of \( f(t) \) was used to prove the classification and in [13] the unicity of the solution of the Cauchy problem was used. Here the classification appears as a consequence of the a priori estimates. It results that existence classificarion remain true for a class of nonautonomous problems.

1. Introduction

We are interested by radial solutions of the nonautonomous problem
\[
- \Delta u = g(u) - \lambda - f(x), \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial \Omega
\]
where \( \Omega \) denotes the unit ball in \( \mathbb{R}^n \), \( \lambda > 0 \), \( f \) is a \( C^1 \) radial function on \( \Omega \), \( g \in C^{0,\alpha}(\mathbb{R}, \mathbb{R}) \) and there exists \( A > 0 \) such that \( g_+ = g|_{[A,\infty[} \) is positive, increasing, differentiable and convex, \( g_- = g|_{]-\infty,-A[} \) is positive, convex and decreasing. In addition
\[
\lim \frac{g(x)}{x} = \pm \infty, \quad x \to \pm \infty
\]
\[
\lim \sqrt{\frac{R(x)}{x} \frac{g_+^{-1}(x)}{g_-^{-1}(x)}} = \pm \infty, \quad x \to \pm \infty
\]

A classical problem of the existence of radial solutions still interesting in superlinear case see [6] and [10].

For the positone problem different methods have been used [10], and for the nonpositone problem, radial solutions have been considered using the shooting method [6,3]. Here we deal with the nonpositone problem using the homotopy of the topological degree [8].

P. L. Lions in [9] notes that many existence results of nodal solutions have been obtained but no classification of solutions have been given.

Remark that a classification of solutions set was introduced by P.H. Rabinowitz [11] based on the number of zeros of the solution \( u(t) \) to prove existence results for a semilinear Sturm-liouville problem.

In this paper we use the homotopy of the topological degree and a classification of solutions based on the number of zeros of the second hand side of Eq.(1) \( g(u(t)) - \lambda - f(t) = 0, \ t \in \mathbb{R} \). This approach represents an alternative for the shooting method and have been used in [12,13].

This paper generalize the existence result given in [13] for a nonautonomous case. The main result is the Proposition(3) in which the classification of solutions set appears as a consequence of the a priori estimates. Indeed in [12] the classification was given by the increasing property of \( f(t) \), see proof of Proposition(3) Eq.(2.19), and in [13] the unicity of the solution of the Cauchy problem was used, see Proposition(4) [13].
Remark that the topological method is not limited by critic Pohozaev-Sobolev exponent but only by a priori estimates. Hence, the existence result given in Theorem (1) [13] depends only on conditions (2) and (3) and stills valid for $\mathbb{R}^n$, $n \geq 1$. To our knowledge the most general existence results known at this time for nodal solutions of nonpositone Elliptic problems are subject to the limite of critic Pohozaev-Sobolev exponent.

A remarkable a priori estimates for positive solutions of elliptic problems was given in [7] and used to get existence result with the topological degree.

Here, properties of the nonpositone problem and nodal solutions have been exploited to get an a priori estimates which is not limited by the critic Pohozaev-Sobolev exponent.

The plane of the proof is similar to [13] and most arguments of proofs remain true for (1). So we will give details just for the proof of Proposition(3) which generalizes Proposition(4) in [13].

2. Existence and classification of solutions

We consider the problem

\[ -u''(t) - \frac{n-1}{t}u'(t) = g(u(t)) - \lambda - f(t) \]
\[ u'(0) = 0, u(1) = 0 \]

$u$ having a local minimum in zero. This is a non autonomous problem related to (5) in [13]. In addition suppose that $f \in C^1([0,1],\mathbb{R})$.

Recall that $\lambda > 0, g \in C(\mathbb{R},\mathbb{R})$ and there exists $A > 0$ such that $g_+ = g|_{[A,\infty]}$ is positive, increasing, differentiable and convex, $g_- = g|_{[-\infty,-A]}$ is positive, convex and decreasing. In addition

\[ \lim_{x \to \pm \infty} g(x) = \pm \infty \]

Let $k \in \mathbb{N}, \lambda > A, E = \{ u \in C^1([0,1],\mathbb{R}) : u'(0) \leq 0, u(1) = 0 \}$ and $Z_k(\lambda)$ a subset of $E$ defined by

$Z_k(\lambda) = \{ u \in E : u(t) - g_+^{-1}(\lambda + f(t)) \text{ has } k \text{ simple zeros in } [0,1] \}$

We denote $M = \|f\|_{C^1}$.

The following proposition recalls the a priori estimate given in proposition (2) in [13].

**Proposition.** There exist $C > 0$ and $K(\lambda)$ a continuous function defined on $[C,\infty]$ such that, for each solution $(u, \lambda)$ of (1) satisfying $\lambda > C$ and $u'(0) \leq 0$, we have $\|u\| < K(\lambda)$.

For a local maximum $\beta$

$u(\beta) < 2R(4(\lambda + M)), \quad R(x) = \max\{|g_+^{-1}(x)|, |g_-^{-1}(x)|\}$

and for a local minimum $\alpha$

$|u(\alpha)| \leq R(\lambda + M)$

The proof of the propostion is the same as proof of Proposition(2) in [13].

**Some general formulas.**

— The mean theorem gives

\[
\left| \int_{g_+^{-1}(\lambda + m_2)}^{g_+^{-1}(\lambda + m_1)} (g(u) - \lambda) du \right| \leq m_2 |g_+^{-1}(\lambda + m_2) - g_+^{-1}(\lambda + m_1)|
\]
and gives $\mu \in [m_1, m_2]$

$$g_+^{-1}(\lambda + m_2) - g_+^{-1}(\lambda + m_1) = \frac{m_2 - m_1}{g'(g_+^{-1}(\lambda + \mu))}$$

(6) $$\left| \int_{g_+^{-1}(\lambda + m_2)}^{g_+^{-1}(\lambda + m_1)} g(u) - \lambda \, du \right| \leq m_2 \left| \frac{m_2 - m_1}{g'(g_+^{-1}(\lambda + \mu))} \right|$$

— for $x > a$ large enough $g_+$ is convex then

$$g'(x) > \frac{g(x) - g(a)}{x - a}$$

for $x$ large enough there exists $\gamma > 0$ such that

$$g'(x) > \gamma g(x)/x$$

set $x = g^{-1}(\lambda + \mu)$ to get

(7) $$\frac{1}{g'(g_+^{-1}(\lambda + \mu)))} \to 0, \lambda \to +\infty$$

— Let $a, b \in [0, 1]$

$$\left| \int_a^b fudt \right| = \left( f(b)u(b) - f(a)u(a) \right) + \int_a^b f'udt$$

(8) $$\left| \int_a^b f'udt \right| \leq 3M \max |u(t)| \leq 6M R(4(\lambda + M))$$

— The concavity of $g_+^{-1}$ implies that for $x > \alpha$ large enough

$$\frac{g_+^{-1}(x) - g_+^{-1}(\alpha)}{x - \alpha}$$

is decreasing, then for $b > a > 0$ and $\lambda$ large enough

$$\frac{g_+^{-1}(b\lambda) - g_+^{-1}(\alpha)}{b\lambda - \alpha} < \frac{g_+^{-1}(a\lambda) - g_+^{-1}(\alpha)}{a\lambda - \alpha}$$

we deduce that there exists $\gamma > 0$ such that

(9) $$g_+^{-1}(b\lambda) < \gamma g_+^{-1}(a\lambda)$$

**Remark 1.** Increasing of $g$ gives, for $\lambda$ large enough, $u > g_+^{-1}(\lambda + f)$ implies $g(u) - (\lambda + f) > 0$, and $u = g_+^{-1}(\lambda + f)$ implies $g(u) - (\lambda + f) = 0$, hence $0 \leq u < g_+^{-1}(\lambda + f)$ implies $g(u) - (\lambda + f) < 0$.

For $\beta$ a local maximum $g(u(\beta)) - (\lambda + f(\beta)) \geq 0$, from which $g_+^{-1}(\lambda + f(\beta)) \leq u(\beta)$. (contrapositive of the last implication)

For $\alpha$ a positive local minimum $g_+^{-1}(\lambda + f(\alpha)) \geq u(\alpha)$.

The following lemma generalizes Lemma(5) in [13] which is used in the following to prove Proposition(3).

Estimation of the derivative at zeros of $u(t) - g_+^{-1}(\lambda + f(t))$.

**Lemma 2.** There exists a sequence $(A_k)$ $(k \geq 1)$ of positive numbers such that, for each solution $(u, \lambda)$ of (1) satisfying $u'(0) \leq 0, \lambda > A_k$ and $u - g_+^{-1}(\lambda + f)$ having at least $k$ zeros, there exist $B > 0$ satisfying for the $k$ largest zeros

$$|u'(\tau)| > B \sqrt{\lambda g_+^{-1}(\lambda/2)}$$
Proof. The $k$ largest zeros of $u - g_+^{-1}(\lambda + f)$ are denoted by $\tau_1 < \tau_2 < \ldots < \tau_k < 1$. $\tau_k$ represents the largest zero.

Estimation of $u'(\tau_k)$.

Let $\eta \in ]\tau_k, 1[$ be the smallest zero of $u(t)$. Since $g_+^{-1}(\lambda + f) > u$, from remark (1) $u$ has no local maximum in $[\tau_k, \eta]$, then it is decreasing on $[\tau_k, \eta]$ and from (1) it is convex.

Let $a$ be the unique element of $[\tau_k, \eta]$ such that $u(a) = g_+^{-1}(\lambda/2)$. Denoting by $h(t)$ the segment joining $u(a)$ and $u(\eta) = 0$, and setting $v(t) = h(t) - u(t)$ on $[a, \eta]$, then $-v'' = \lambda + f - g(u) - pu'$. Since $u < g_+^{-1}(\lambda/2)$ and is decreasing, $-v'' > \lambda/2$, since $v < g_+^{-1}(\lambda/2)$

$$-v'' > \frac{\lambda}{2g_+^{-1}(\lambda/2)}v,$$ on $]a, \eta[$

$v(\eta) = v(a) = 0$

setting $t = s(\eta - a) + a, s \in [0, 1]$ and $w(s) = v(s(\eta - a) + a)$

$$-w'' > (\eta - a)^2 \frac{\lambda}{2g_+^{-1}(\lambda/2)}w,$$ on $]0, 1[$

$w(0) = w(1) = 0$

the comparison theorem of Sturm gives $(\eta - a) < \sqrt{2\pi} \sqrt{g_+^{-1}(\lambda/2)}/\lambda$.

Since $u$ is convex on $]\eta, \tau_k[,$ $|u'(\tau_k)| > |u'(a)| > u(a)/|\eta - a|$, hence $|u'(\tau_k)| > \frac{\sqrt{\pi}}{2\sqrt{\lambda}} g_+^{-1}(\lambda/2)$.

We shall use the recurrence argument.

Let $B > 0, \delta > 0, \tau_i$ and $\tau_{i+1}$ two consecutive zeros such that $|u'(\tau_{i+1})| > B\sqrt{\lambda g_+^{-1}(\lambda/2)}$, then for $\lambda$ large enough we have $|u'(\tau_i)| > (B - \delta)\sqrt{\lambda g_+^{-1}(\lambda/2)}$.

Indeed, multiplying (1) by $u'$ and integrating to get

$$\frac{u^2(\tau_i)}{2} - \frac{u^2(\tau_{i+1})}{2} + \int_{u(\tau_i)}^{u(\tau_{i+1})} (g(u) - \lambda)du - \int_{\tau_i}^{\tau_{i+1}} fu'dt$$

then (3.5) give

$$\frac{u^2(\tau_i)}{2} > \frac{B^2}{2} \sqrt{\lambda g_+^{-1}(\lambda/2)} - M \left| \frac{f(\tau_{i+1}) - f(\tau_i)}{g'(g_+^{-1}(\lambda))} \right| - 6M R(4(\lambda + M))$$

(2.6) gives $\frac{R(4(\lambda + M))}{\sqrt{\lambda g_+^{-1}(\lambda/2)}} \rightarrow 0$ and (4) gives $\frac{f(\tau_{i+1}) - f(\tau_i)}{g'(g_+^{-1}(\lambda))} \rightarrow 0$. 

The following proposition generalizes the Proposition (4) in [13].

**Proposition 3.** There exists a sequence $(B_k)(k \geq 0)$ of positive numbers such that, for each $\lambda > B_k$, (1) has no solution $u \in \partial Z_{2k}(\lambda)$ satisfying $u'(0) \leq 0$.

**Proof.** By contradiction, let $u \in \partial Z_{2k}(\lambda)$ be a solution of (1).

Case $k = 0$: Let $v_0$ a sequence of solutions in $Z_0(\lambda)$ such that $v_n \rightarrow u$. From remark (1) $v_0 < 0$ on $[0, 1]$ then $u \leq 0$, hence $u - g_+^{-1}(\lambda + f)$ has no zero from which $u \in Z_0(\lambda)$ thus $u \notin \partial Z_0(\lambda)$, contradiction.

Case $k \geq 1$: First, we shall prove that $u - g_+^{-1}(\lambda + f)$ has at most $2k$ simple zeros.

Indeed, let $\tau$ be a simple zero, then there exist $\epsilon_0 > 0, \epsilon_1 > 0$ and $\delta > 0$ such that $\tau$ is the unique zero on $]\tau - \epsilon_0, \tau + \epsilon_0[$, (one assume that $u - g_+^{-1}(\lambda + f)$ is
increasing. If it is decreasing the inequalities are inverse and the proof is similar)

\[
\begin{align*}
& u(\tau - \epsilon_0) - g_+^{-1}(\lambda + f(\tau - \epsilon_0)) < -\epsilon_1 \\
& u(\tau + \epsilon_0) - g_+^{-1}(\lambda + f(\tau + \epsilon_0)) > \epsilon_1 \\
& u' - \left[ g_+^{-1}(\lambda + f) \right]' > \delta
\end{align*}
\]

Let \((v_n)\) be a sequence of \(Z_{2k}(\lambda)\) such that \(v_n \to u\) in \(E\), there exists \(n(\epsilon_0, \epsilon_1, \delta) \in N\) such that for \(n > n(\epsilon_0, \epsilon_1, \delta)\)

\[
\begin{align*}
& u(\tau - \epsilon_0) - v_n(\tau - \epsilon_0) < \epsilon_1/2 \\
& u(\tau + \epsilon_0) - v_n(\tau + \epsilon_0) > \epsilon_1/2 \\
& \|u' - v_n'\|_\infty < \delta/2
\end{align*}
\]

from which

\[
\begin{align*}
& v_n(\tau - \epsilon_0) - g_+^{-1}(\lambda + f(\tau - \epsilon_0)) < -\epsilon_1/2 \\
& v_n(\tau + \epsilon_0) - g_+^{-1}(\lambda + f(\tau + \epsilon_0)) > \epsilon_1/2 \\
& v_n' - \left[ g_+^{-1}(\lambda + f) \right]' > \delta/2
\end{align*}
\]

which implies that \(v_n - g_+^{-1}(\lambda + f)\) has a unique simple zero on \([\tau - \epsilon_0, \tau + \epsilon_0]\). Since \(v_n \in Z_{2k}(\lambda)\), \(v_n - g_+^{-1}(\lambda + f)\) has exactly 2k simple zeros, then \(u - g_+^{-1}(\lambda + f)\) has at most 2k simple zeros.

There are not exactly \(m\) simple zeros with \(m < 2k\).

Indeed, by contradiction assume that \(u \in Z_m(\lambda)\). Since \(Z_{2k}(\lambda)\) and \(Z_m(\lambda)\) are open sets of \(E\) and \(Z_m(\lambda) \cap Z_{2k}(\lambda) \neq \emptyset\), then \(Z_m(\lambda) \cap \partial Z_{2k}(\lambda) = \emptyset\), contradiction.

Last, since there exist at most 2k simple zeros of \(u - g_+^{-1}(\lambda + f)\) there exists \(\tau_j\) a zero which is not simple \(j \leq 2k + 1\). From the lemma (2), there exists \(A_{2k+1} > 0\) such that for \(\lambda > A_{2k+1}\)

\[
|u'(\tau_j)| > B \sqrt{\lambda} g_+^{-1}(\lambda/2)
\]

On the other hand

\[
\left[ g_+^{-1}(\lambda + f) \right]' = \frac{f'}{g_+^{-1}(\lambda + f)},
\]

(4) implies that \(|u'(\tau_j)| > \left| \left( g_+^{-1}(\lambda + f(\tau_j)) \right)' \right| \) for \(\lambda\) large enough, then \(\tau_j\) is a simple zero of \(u - g_+^{-1}(\lambda + f)\), contradiction.

\[
\]

References

[1] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.

[2] A. Capozzi, D. Fortunato, G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985) 463–470.

[3] A. Castro and R. Shivaji, Multiple Solutions for a Dirichlet Problem with Jumping Nonlinearities, II J. Math. Analysis and Appl 133, 505-528 (1988).

[4] G. Cerami, D. Fortunato, M. Struwe, Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 341–350.

[5] G. Cerami, S. Solimini, M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, J. Funct. Anal. 69 (1986) 289–306.

[6] W. Dambrosio: On the multiplicity of radial solutions to superlinear Dirichlet problems in bounded domains, J. Diff. Equat. 196 (2004) 91-118.

[7] D.G.de Figueiredo, P.L. Lions, R.D. Nussbaum : A priori estimates and existence results of positive solutions of semilinear elliptic equations, J. Math. Pures. Appl.

[8] J.Leray et J.Schauder: Topologie et équations fonctionnelles. Ann Sci. Ecole Norm Sup Vol. 3, 51, 1934, p 45-414.

[9] P. L. Lions: On the existence of positive solutions of semilinear elliptic equations, SIAM review, Vol 24, (1982)

[10] Massimo Grossi : Nodal solutions for an elliptic problem involving large nonlinearities. J. Differential Equations 245 (2008) 2917–2938.

[11] P. H. Rabinowitz: Nonlinear Sturm-Liouville Problems for Second order ordinary differential equations, Communications on pure and applied mathematics, vol XXIII (1970), 939-961.

[12] M. Rouaki: Topological Degree and a nonlinear Dirichlet problem. Nonlinear Analysis MTA 54 (2003) 801-817.
[13] M. Rouaki: *Nodal radial solutions for a superlinear problem*. Nonlinear Analysis RWA 8 (2007) 563-571.

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