Approximation of relaxed Dirichlet problems by boundary value problems in perforated domains

Gianni Dal Maso and Annalisa Malusa
SISSA, Via Beirut 2/4, 34014 Trieste, Italy

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Given an elliptic operator $L$ on a bounded domain $\Omega \subseteq \mathbb{R}^n$, and a positive Radon measure $\mu$ on $\Omega$, not charging polar sets, we discuss an explicit approximation procedure which leads to a sequence of domains $\Omega_h \equiv \Omega$ with the following property: for every $f \in H^{-1}(\Omega)$ the sequence $u_h$ of the solutions of the Dirichlet problems $Lu_h = f$ in $\Omega_h$, $u_h = 0$ on $\partial \Omega_h$, extended to 0 in $\Omega \setminus \Omega_h$, converges to the solution of the "relaxed Dirichlet problem" $Lu + \mu u = f$ in $\Omega$, $u = 0$ on $\partial \Omega$.

1. Introduction

The notion of a "relaxed Dirichlet problem" was introduced in [6] to describe the asymptotic behaviour of the solutions of classical Dirichlet problems in strongly perturbed domains. Given a bounded open subset $\Omega$ of $\mathbb{R}^n$, $n \geq 2$, and an elliptic operator $L$ on $\Omega$, a relaxed Dirichlet problem can be written in the form

$$\begin{cases}
Lu + \mu u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $f \in H^{-1}(\Omega)$ and $\mu$ belongs to the space $\mathcal{M}_0(\Omega)$ of all positive Borel measures on $\Omega$ which do not charge any set of capacity zero.

The main result concerning relaxed Dirichlet problems is the following compactness theorem (see [7, Theorem 4.14]): for every sequence $\{\Omega_h\}$ of open subsets of $\Omega$, there exist a subsequence, still denoted by $\{\Omega_h\}$, and a measure $\mu \in \mathcal{M}_0(\Omega)$, such that for every $f \in H^{-1}(\Omega)$ the solutions $u_h$ of the Dirichlet problems

$$\begin{cases}
Lu_h = f & \text{in } \Omega_h, \\
u_h = 0 & \text{on } \partial \Omega_h,
\end{cases}$$

(1.2)

extended to 0 on $\Omega \setminus \Omega_h$, converge in $L^2(\Omega)$ to the unique solution $u$ of (1.1). Moreover, the following density theorem holds (see [7, Theorem 4.16]): for every $\mu \in \mathcal{M}_0(\Omega)$, there exists a sequence $\{\Omega_h\}$ of open subsets of $\Omega$ such that for every $f \in H^{-1}(\Omega)$ the solution $u$ of (1.1) is the limit in $L^2(\Omega)$ of the sequence $\{u_h\}$ of the solutions of (1.2). The proof of this density theorem provides an explicit approximation only when $\mu$ is the Lebesgue measure, while it is rather indirect in the other cases, and does not suggest any efficient method for the construction of the sets $\Omega_h$.

The aim of this paper is to present an explicit approximation scheme for the relaxed Dirichlet problems (1.1) by means of sequences of classical Dirichlet problems of the form (1.2). We assume that $\mu \in \mathcal{M}_0(\Omega)$ is a Radon measure. The sets $\Omega_h$ will
be obtained by removing an array of small balls from the set $\Omega$. The geometric construction is quite simple. For every $h \in \mathbb{N}$, we fix a partition $\{Q^i_h\}_{i}$ of $\mathbb{R}^n$ composed of cubes with side $1/h$, and we consider the set $I_h$ of all indices $i$ such that $Q^i_h \subset \subset \Omega$. For every $i \in I_h$, let $B^i_h$ be the ball with the same centre as $Q^i_h$ and radius $1/2h$, and let $E^i_h$ be another ball with the same centre such that
\[
\text{cap}^h (E^i_h, B^i_h) = \mu(Q^i_h).
\]
Finally, let $E_h = \bigcup_{i \in I_h} E^i_h$ and $\Omega_h = \Omega \setminus E_h$. Note that the size of the hole $E_h$ contained in the cube $Q^i_h$ depends only on the operator $L$ and on the value of the measure $\mu$ on $Q^i_h$.

By using a very general version of the Poincaré inequality proved by Zamboni [15], we shall show that, if $\mu$ belongs to the Kato space $K^+_n(\Omega)$, i.e. the potential generated by $\mu$ is continuous, then the method introduced by Cioranescu and Murat [4] can be applied, so that for every $f \in H^{-1}(\Omega)$ the solutions $u_h$ of the Dirichlet problems (1.2) converge in $L^2(\Omega)$ to the solution $u$ of the relaxed Dirichlet problem (1.1). To prove that the same result holds also when $\mu$ is an arbitrary Radon measure of the class $\mathcal{M}_0(\Omega)$, we use the method of $\mu$-capacities introduced in [5] and [3].

Finally, if $\mu$ is a Radon measure and $\mu \notin \mathcal{M}_0(\Omega)$, then we prove that our construction leads to the approximation of the solution of the relaxed Dirichlet problem
\[
\begin{cases}
Lu + \mu_0u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where $\mu_0$ is the greatest measure of the class $\mathcal{M}_0(\Omega)$ which is less than or equal to $\mu$.

2. Notation and preliminaries

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. We shall denote by $H^1(\Omega)$ and $H^1_0(\Omega)$ the usual Sobolev spaces, by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$, by $L^p(\Omega)$, $1 \leq p < \infty$, the usual Lebesgue space with respect to the measure $\mu$; if $\mu$ is the Lebesgue measure, we shall use the notation $L^p(\Omega)$.

For every subset $E$ of $\Omega$, the (harmonic) capacity of $E$ with respect to $\Omega$ is defined by
\[
\text{cap} E = \inf \sum_{i} |\nabla u_i|^2 \, dx,
\]
where the infimum is taken over all functions $u_i \in H^1_0(\Omega)$ such that $u_i \geq 1$ a.e. in a neighbourhood of $E$. We say that a property $\mathcal{P}(x)$, depending on a point $x \in \Omega$, holds quasi everywhere (q.e.) in $\Omega$ if there exists a set $E \subset \subset \Omega$ with $\text{cap} (E, \Omega) = 0$, such that $\mathcal{P}$ holds in $\Omega \setminus E$. It is well known that every $u \in H^1(\Omega)$ admits a quasi-continuous representative, which is uniquely defined up to a set of capacity zero (see, e.g. [16, Theorem 3.1.4]). In the sequel, we shall always identify $u$ with its quasi-continuous representative.

By a Borel measure on $\Omega$, we mean a positive, countably additive set function with values in $\mathbb{R}$ defined on the $\sigma$-field of all Borel subsets of $\Omega$; by a Radon measure on $\Omega$ we mean a Borel measure which is finite on every compact subset of $\Omega$. Finally, by $\mathcal{M}_0(\Omega)$ we denote the set of all positive Borel measures $\mu$ on $\Omega$ such that $\mu(E) = 0$ for every Borel set $E \subset \subset \Omega$ with $\text{cap} (E, \Omega) = 0$. If $\mu$ is a Borel measure and $E$ is a
Borel subset of $\Omega$, the Borel measure $\mu|_E$ is defined by $(\mu|_E)(B) = \mu(E \cap B)$ for every Borel set $B \subseteq \Omega$. If $\mu, \nu$ are Radon measures and $\nu$ has a density $f$ with respect to $\mu$, we shall write $\nu = f\mu$. For every $E \subseteq \Omega$, we denote by $\infty_E$ the measure of the class $\mathcal{M}_0(\Omega)$ defined by

$$\infty_E(B) = \begin{cases} 0, & \text{if } \text{cap}(B \cap E, \Omega) = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

We shall see later that these measures are used to express the classical Dirichlet problems (1.2) in the form (1.1). This will allow us to treat problems (1.1) and (1.2) in a unified way.

Another class of measures in which we are interested is the Kato space.

**Definition 2.1.** The Kato space $K_n^+(\Omega)$ is the cone of all positive Radon measures $\mu$ on $\Omega$ such that

$$\lim_{r \to 0^+} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} G_n(y-x) \, d\mu(y) = 0,$$

where $G_n$ is the fundamental solution of the Laplace operator $-\Delta$ in $\mathbb{R}^n$, and $B_r(x)$ denotes the open ball with centre $x$ and radius $r$.

For every $\mu \in K_n^+(\Omega)$ and for every Borel set $A \subseteq \Omega$, we define

$$\|\mu\|_{K_n^+(A)} = \sup_{x \in A} \int_A |y-x|^{2-n} \, d\mu(y), \quad \text{if } n \geq 3,$$

$$\|\mu\|_{K_n^+(A)} = \sup_{x \in A} \log \left( \frac{\text{diam}(A)}{|y-x|} \right) \, d\mu(y) + \mu(A), \quad \text{if } n = 2.$$

For every $\mu \in K_n^+(\Omega)$, it is easy to see that $\|\mu\|_{K_n^+(A)} < +\infty$ and $\|\mu\|_{K_n^+(A)}$ tends to zero as $\text{diam}(A)$ tends to zero. We recall that every measure in $K_n^+(\Omega)$ is bounded and belongs to $H^{-1}(\Omega)$. For more details about this subject, we refer to [1, 6, 10, 13]. We shall use in the following a Poincaré inequality involving Kato measures.

**Lemma 2.2.** Let $A$ be a Borel subset of a ball $B_R = B_R(x_0)$ such that $\text{diam}(A) \geq qR$ for some $q \in (0, 1)$, and let $\mu \in K_n^+(A)$. Then there exists a positive constant $c$, depending only on $q$ and on the dimension $n$ of the space, such that

$$\int_A u^2 \, d\mu \leq c \|\mu\|_{K_n^+(A)} \int_{B_R} |\nabla u|^2 \, dx$$

for every $u \in H_0^1(B_R)$.

**Proof.** An inequality of this kind was proved by Zamboni [15] in the case $n \geq 3$, $A = B_R$, and $\mu$ absolutely continuous with respect to the Lebesgue measure. The same arguments can be adapted, up to minor modifications, also to the general case. The main change in the case $n = 2$ is the use of the inequality

$$\int_{B_R} \frac{1}{|x-y||z-y|} \, dy \leq c_q \left( \log \left( \frac{\text{diam}(A)}{|x-z|} \right) + 1 \right) \quad \forall x, z \in A,$$

which can be proved by direct computation. \qed
Finally we need a type of dominated convergence theorem for measures in $H^{-1}(\Omega)$. 

**Lemma 2.3.** Let $\{\mu_h\}$ be a sequence of positive measures belonging to $H^{-1}(\Omega)$ that converges to $0$ in the weak* topology of measures and suppose that there exists $\mu \in H^{-1}(\Omega)$ such that $\mu_h \leq \mu$. Then the sequence $\{\mu_h\}$ converges to $0$ strongly in $H^{-1}(\Omega)$.

**Proof.** This result can be obtained easily by using the strong compactness of the order intervals in $H^{-1}(\Omega)$. However, we give here self-contained elementary proof. Let us define $v_h = \mu - \mu_h$. Clearly $\|v_h\|_{H^{-1}(\Omega)} \leq \|\mu\|_{H^{-1}(\Omega)}$ and so, up to a subsequence, $\{v_h\}$ converges to $\mu$ weakly in $H^{-1}(\Omega)$. The previous inequality, together with the lower semicontinuity of the norm, implies that $\|v_h\|_{H^{-1}(\Omega)}$ converges to $\|\mu\|_{H^{-1}(\Omega)}$. This shows that $\{v_h\}$ converges to $\mu$ strongly in $H^{-1}(\Omega)$ and concludes the proof of the lemma. \qed

Let $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ be a linear elliptic operator in divergence form

$$Lu = -\text{div}(Av),$$

where $A = A(x) = (a_{ij}(x))$ is a symmetric $n \times n$ matrix of bounded measurable functions satisfying, for a suitable constant $\alpha > 0$, the ellipticity condition

$$\alpha |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \leq \alpha^{-1} |\xi|^2$$

for a.e. $x$ in $\Omega$, and for every $\xi \in \mathbb{R}^n$.

A set function $\text{cap}_L$ can be associated with every measure $\mu$ in the class $\mathcal{M}_0(\Omega)$.

**Definition 2.4.** Let $\mu \in \mathcal{M}_0(\Omega)$. For every open set $A \subseteq \Omega$ and for every Borel set $E \subseteq A$, we define the $\mu$-capacity of $E$ in $A$ corresponding to the operator $L$ as

$$\text{cap}^L_{\mu}(E, A) = \min \left\{ \langle Lu, u \rangle + \int_E u^2 \, d\mu : u \in H_0^1(A) \right\},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

The $\mu$-capacity corresponding to $L = -\Delta$ will be denoted by $\text{cap}_\mu$, while the $\mu$-capacity with respect to $\mu = \infty$ will be denoted by $\text{cap}^L$. The latter coincides with the classical capacity relative to the operator $L$ according to the definition of [12] and [14]. If $L = -\Delta$ and $\mu = \infty$, then $\text{cap}^L$ coincides with the harmonic capacity introduced at the beginning of this section. If $\mu = \infty_F$ for some $F \subseteq \Omega$, and $L$ is any elliptic operator, then $\text{cap}^L(E, A) = \text{cap}^L(E \cap F, A)$ for every $E \subseteq A$.

Some of the properties of $\text{cap}^L_{\mu}$ are stated in the following proposition:

**Proposition 2.5.** Let $\mu \in \mathcal{M}_0(\Omega)$, $A, B$ be open subsets of $\Omega$ and $E, F$ subsets of $A$. Then

(i) $\text{cap}^L_{\mu}(\emptyset, A) = 0$;

(ii) $E \subseteq F \Rightarrow \text{cap}^L_{\mu}(E, A) \leq \text{cap}^L_{\mu}(F, A)$;

(iii) $\text{cap}^L_{\mu}(E \cup F, A) \leq \text{cap}^L_{\mu}(E, A) + \text{cap}^L_{\mu}(F, A)$;

(iv) $A \subseteq B \Rightarrow \text{cap}^L_{\mu}(E, A) \leq \text{cap}^L_{\mu}(E, B)$;

(v) $\alpha \text{cap}^L_{\mu}(E, A) \leq \text{cap}^L_{\mu}(E, A) \leq \alpha^{-1} \text{cap}_{\mu}(E, A) \leq \alpha^{-1} \text{cap}(E, A)$;

(vi) if $\{E_n\}$ is an increasing sequence of subsets of $A$ and $E = \cup_n E_n$, then $\text{cap}^L_{\mu}(E, A) = \sup_n \text{cap}^L_{\mu}(E_n, A)$.

**Proof.** See [6, Proposition 3.11, Theorem 3.10] and [5, Theorem 2.9]. \qed

Now we introduce the notion of relaxed Dirichlet problems.
DEFINITION 2.6. Given $\mu \in \mathcal{M}_0(\Omega)$ and $f \in H^{-1}(\Omega)$, we say that a function $u$ is a solution of the relaxed Dirichlet problem

\[
\begin{align*}
\begin{cases}
Lu + \mu u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\tag{2.2}
\]

if $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ and

\[
\langle Lu, v \rangle + \int_{\Omega} uv \, d\mu = \langle f, v \rangle
\]

for every $v \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$.

We recall that for every $f \in H^{-1}(\Omega)$ there exists a unique solution $u$ of problem (2.2) (see [6, Theorem 2.4]). It is easy to see that, if $E$ is a closed set, then $u$ is a solution of

\[
\begin{align*}
\begin{cases}
Lu + \infty_{E} u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

if and only if $u = 0$ q.e. in $E \cap \Omega$ and $u|_{\Omega \setminus E}$ is a weak solution of the classical boundary value problem

\[
\begin{align*}
\begin{cases}
Lu = f \quad \text{in } \Omega \setminus E, \\
u \in H^1_0(\Omega \setminus E).
\end{cases}
\end{align*}
\]

DEFINITION 2.7. A sequence $\{\mu_h\}$ in $\mathcal{M}_0(\Omega)$ $\gamma'$-converges to $\mu \in \mathcal{M}_0(\Omega)$ if, for every $f \in H^{-1}(\Omega)$, the sequence $\{u_h\}$ of the solutions of the problems

\[
\begin{align*}
\begin{cases}
Lu_h + \mu_h u_h &= f \quad \text{in } \Omega, \\
u_h &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

converges strongly in $L^2(\Omega)$ to the solution $u$ of the problem

\[
\begin{align*}
\begin{cases}
Lu + \mu u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\end{align*}
\]

With every $\mu \in \mathcal{M}_0(\Omega)$, we associate the lower semicontinuous quadratic functional on $H^1_0(\Omega)$ defined by

\[
F_{\mu}(u) = \langle Lu, u \rangle + \int_{\Omega} u^2 \, d\mu.
\]

The following theorem shows the connection between $\gamma'$-convergence of the measures $\mu_h$ and $\Gamma$-convergence of the corresponding functionals $F_{\mu_h}$.

THEOREM 2.8. A sequence $\{\mu_h\}$ in $\mathcal{M}_0(\Omega)$ $\gamma'$-converges to the measure $\mu \in \mathcal{M}_0(\Omega)$, if and only if the following conditions are satisfied for every $u \in H^1_0(\Omega)$:

(a) for every sequence $\{u_h\}$ in $H^1_0(\Omega)$ converging to $u$ in $L^2(\Omega)$

\[
F_{\mu}(u) \leq \lim \inf_{h \to \infty} F_{\mu_h}(u_h);
\]
there exists a sequence \( \{u_h\} \) in \( H^1_0(\Omega) \) converging to \( u \) in \( L^2(\Omega) \) such that

\[
F_p(u) = \lim_{h \to \infty} F_{p_h}(u_h).
\]

**Proof.** See [2, Proposition 2.9]. □

Our definition of \( \gamma^L \)-convergence coincides with the definition considered in [5]. As shown in [2, Proposition 2.8], if properties (a) and (b) hold on \( \Omega \), then they also hold for every open set \( \Omega' \subseteq \Omega \). Conversely, if (a) and (b) hold for every open set \( \Omega' \subseteq \Omega \), then they hold on \( \Omega \). So our definition of \( \gamma^L \)-convergence differs from the definition given in [3] only in the fact that now the ambient space is \( \Omega \) instead of \( \mathbb{R}^n \). When \( L = -\Delta \), our definition coincides with the definition given in [7].

**Remark 2.9.** Let \( \{\lambda_h\} \) and \( \{\mu_h\} \) be two sequences in \( M_0(\Omega) \) which \( \gamma^L \)-converge to \( \lambda \) and \( \mu \), respectively. If \( \lambda_h \leq \mu_h \) for every \( h \), by Theorem 2.8 we have

\[
\int_{\Omega} u^2 \, d\lambda \leq \int_{\Omega} u^2 \, d\mu
\]

for every \( u \in H^1_0(\Omega) \). In particular, if \( \mu \) is a Radon measure, then \( \lambda \leq \mu \).

We briefly recall some properties of the \( \gamma^L \)-convergence of measures in \( M_0(\Omega) \).

**Theorem 2.10.** For every sequence \( \{\mu_h\} \) in \( M_0(\Omega) \) there exists a subsequence \( \{\mu_{h_k}\} \) which \( \gamma^L \)-converges to a measure \( \mu \) in \( M_0(\Omega) \).

**Proof.** The proof for the case \( L = -\Delta \), can be found in [7, Theorem 4.14]. The proof in the general case is similar. □

**Theorem 2.11.** Let \( \{\mu_h\} \) be a sequence in \( M_0(\Omega) \) which \( \gamma^L \)-converges to a measure \( \mu \) in \( M_0(\Omega) \). Then

\[
\cap^L_{\mu}(A, B) \leq \liminf_{h \to \infty} \cap^L_{\mu_h}(A, B),
\]

for every pair of open sets \( A, B \), with \( A \subseteq B \subseteq \Omega \).

**Proof.** See [5, Proposition 5.7]. □

We consider now a sufficient condition for the \( \gamma^L \)-convergence of a sequence of measures of the form \( \{\infty_{E_h}\} \), where \( \{E_h\} \) is a sequence of compact subsets of \( \Omega \). In this case, if \( \Omega_h = \Omega \setminus E_h \), the solution \( u_h \) coincides with the solution of the classical problem

\[
\begin{cases}
Lu_h = f & \text{in } \Omega_h, \\
u_h = 0 & \text{on } \partial \Omega,
\end{cases}
\]

prolonged to zero outside \( \Omega_h \).

Assume that \( \{E_h\} \) satisfies the following hypotheses, studied by Cioranescu and Murat: there exist a measure \( \mu \in W^{-1,\infty}(\Omega) \), a sequence \( \{w_h\} \) in \( H^1(\Omega) \), and two
sequences of positive measures of $H^{-1}(\Omega)$, $\{v_h\}$ and $\{\lambda_h\}$, such that

\[
\begin{align*}
    w_h &\rightharpoonup 1 \text{ weakly in } H^1(\Omega), \\
    w_h &\rightarrow 0 \text{ q.e. in } E_h, \\
    Lw_h &= v_h - \lambda_h, \\
    v_h &\rightarrow \mu \text{ strongly in } H^{-1}(\Omega), \\
    \lambda_h &\rightarrow \mu \text{ weakly in } H^{-1}(\Omega),
\end{align*}
\]

and $\langle \lambda_h, v \rangle = 0$ for every $h \in \mathbb{N}$ and for every $v \in H^1_0(\Omega)$, with $v = 0$ q.e. in $E_h$.

Under these hypotheses, the sequence $\{u_h\}$ converges weakly in $H^1_0(\Omega)$ to the weak solution $u$ of the problem

\[
\begin{align*}
    \left\{ \begin{array}{ll}
    Lu + \mu u = f & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega
    \end{array} \right.
\end{align*}
\]

(see [4, Théorème 1.2]). Later, Kacimi and Murat pointed out that the hypotheses $\mu \in W^{-1,\infty}(\Omega)$ can be replaced by $\mu \in H^{-1}(\Omega)$ (see [9, Remarque 2.4]). In conclusion, using the language introduced by Definition 2.7, the following theorem holds:

**Theorem 2.12.** If $\{E_h\}$ satisfies the hypotheses considered above, with $\mu \in H^{-1}(\Omega)$, then the sequence of measures $\{\omega E_h\} \gamma^L$-converges to the measure $\mu$.

### 3. The main results

In this section, we prove that for every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ the general approximation rule outlined in the Introduction provides a sequence of measures of the form $\{\omega E_h\}$ which $\gamma^L$-converges to $\mu$ according to Definition 2.7.

To deal with the case $\mu \in K_+^*(\Omega)$, we need the following lemmas:

**Lemma 3.1.** Let $U$ and $V$ be open subsets of $\Omega$, with $V \subset U \subset \Omega$, and let $w$ be the $L$-capacitary potential of $V$ with respect to $U$, i.e. the unique solution of

\[
\left\{ \begin{array}{ll}
    w \in H^1_0(U), & w \geq 1 \text{ q.e. on } V, \\
    \langle Lw, v - w \rangle \geq 0, & \forall v \in H^1_0(U), v \geq 1 \text{ q.e. on } V.
\end{array} \right.
\]

Let us extend $w$ to $\Omega$ by setting $w = 0$ on $\Omega \setminus U$. Then $w \in H^1_0(\Omega)$ and $w = 1$ q.e. on $V$. Moreover, there exist two positive Radon measures $\gamma$ and $v$ belonging to $H^{-1}(\Omega)$ such that $\text{supp } \gamma \subseteq \partial V$, $\text{supp } v \subseteq \partial U$, $Lw = \gamma - v$ in $\Omega$, and $v(\Omega) = \gamma(\Omega) = \text{cap}^L(V, U)$.

We call $\gamma$ (respectively $v$) the inner (respectively outer) $L$-capacitary distribution of $V$ with respect to $U$.

**Proof of Lemma 3.1.** It is well known (see [14, Section 3]) that there exists a positive Radon measure $\gamma \in H^{-1}(U)$, with $\text{supp } \gamma \subseteq \partial V$, such that $Lw = \gamma$ in $\Omega$ and $\gamma(\Omega) = \text{cap}^L(V, U)$. Let us consider now the following obstacle problem:

\[
\left\{ \begin{array}{ll}
    z \in H^1_0(\Omega), & z \geq 0 \text{ q.e. in } \Omega \setminus U, \\
    \langle Lz + \gamma, v - z \rangle \geq 0 & \forall v \in H^1_0(\Omega), v \geq 0 \text{ q.e. in } \Omega \setminus U.
\end{array} \right.
\]

It is well known that there exists a unique solution $z$ of this problem, that $z$ is a
supersolution of $L + \gamma$, i.e. $Lz + \gamma = \nu \geq 0$ for some positive measure $\nu \in H^{-1}(\Omega)$, and that $z \leq \zeta$ for every supersolution $\zeta \in H^1(\Omega)$ of $L + \gamma$ with $\zeta \geq 0$ q.e. in $\Omega \setminus U$ (see [11, Section II.6]). Since $\gamma$ is a positive measure, 0 is a supersolution of $L + \gamma$. Consequently $z \leq 0$ q.e. in $\Omega$. As $z \geq 0$ q.e. in $\Omega \setminus U$, we conclude that $z = 0$ q.e. in $\Omega \setminus U$, hence $z \in H^1_0(U)$. On the other hand, $Lz + \gamma = 0$ in $U$. As $Lw = \gamma$ on $U$, by uniqueness we can conclude that $z = -w$ in $U$, hence in $\Omega$. This implies $Lw = \gamma - \nu$ in $\Omega$. As $Lw - \gamma = 0$ in $U$ and in $\Omega \setminus U$, we conclude that $\text{supp } v \subseteq \partial U$. Since $Lw = \gamma - \nu$ in $\Omega$, we have

$$
\int_{\Omega} A \nabla w \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\gamma - \int_{\Omega} \varphi \, d\nu \quad \forall \varphi \in H^1_0(\Omega).
$$

Let $\psi$ be a cut-off function of class $C^\infty(\Omega)$ such that $\psi(x) = 1$ in $\overline{U}$. Choosing $\varphi = \psi(w - 1)$ as test function, we obtain

$$
\int_{\Omega} A \nabla w \cdot \nabla w \, \psi \, dx + \int_{\Omega} A \nabla w \cdot \nabla \psi \, (w - 1) \, dx = \int_{\Omega} \psi(w - 1) \, d\gamma + \int_{\Omega} \psi(1 - w) \, d\nu
$$

and, using the fact that $w = 1$ $\gamma$-a.e. in $\Omega$ and $\psi(1 - w) = 1$ q.e. on $\text{supp } v$, we obtain

$$
\int_{\Omega} A \nabla w \cdot \nabla w \, dx = v(\Omega). \quad \text{As } \gamma(\Omega) = \text{cap}_L(V, U) = \int_{\Omega} A \nabla w \cdot \nabla w \, dx, \text{ we conclude that } v(\Omega) = \gamma(\Omega) = \text{cap}_L(V, U). \quad \Box
$$

Let us fix $x^0 \in \Omega$. For every $\rho > 0$, let $B_\rho = B_\rho(x^0)$ and let $Q_\rho$ be the open cube $\{x \in \mathbb{R}^n : -\rho < x_k - x_k^0 < \rho \text{ for } k = 1, \ldots, n\}$. If $0 < \rho < r$ and $B_\rho \subset \subset \Omega$, let $w_\rho^\star$ be the $L$-capacitary potential of $B_\rho$ with respect to $B_r$, and let $v_\rho^\star$ be the corresponding outer $L$-capacitary distribution.

**Lemma 3.2.** For every $q \in (0, 1)$ there exists a constant $c = c(q, a, n)$, independent of the operator $L$, such that, if $B_r \subset \subset \Omega$ and $0 < \rho \leq qr$, then

$$
\frac{1}{v_\rho^\star(\partial B_r)} \int_{\partial B_r} \varphi \, dv_\rho^\star \leq c \frac{1}{v_r^\gamma(\partial B_r)} \int_{\partial B_r} \varphi \, dv_r^\gamma
$$

for every $\varphi \in H^1(Q_r)$ with $\varphi \geq 0$ q.e. in $Q_r$.

**Proof.** Let us fix $q, r, \rho, \varphi$ as required, and let $u \in H^1_0(\Omega)$ be a function whose restriction to $B_r$ is a solution of the Dirichlet problem

$$
\begin{cases}
Lu = 0 & \text{in } B_r, \\
u - \varphi \in H^1_0(B_r).
\end{cases}
$$

We may assume that $u = \varphi$ q.e. on the annulus $B_R \setminus \overline{B_r}$, for some $R > r$, so that $u = \varphi$ q.e. on $B_R \setminus B_r$. By De Giorgi's theorem, we have $u \in C^0(B_r)$. For every $s \in (0, r)$, we want to prove that

$$
\frac{1}{v_s^\gamma(\partial B_r)} \int_{\partial B_s} u \, d\gamma_s^\star = \frac{1}{v_r^\gamma(\partial B_r)} \int_{\partial B_r} \varphi \, dv_r^\gamma,
$$

where $\gamma_s^\star$ is the inner $L$-capacitary distribution associated with $w_s^\star$. Using the symmetry of the operator $L$, we get

$$
0 = \int_{B_r} A \nabla u \cdot \nabla w_s^\star \, dx = \int_{\Omega} A \nabla w_s^\star \cdot \nabla u \, dx
$$
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\[ v_r^\tau(\partial B_r) = \text{cap}^L(B_\tau, B_r) = \gamma_r^\tau(\partial B_r), \]

we obtain (3.1).

Now we remark that, by the maximum principle, \( u \geq 0 \) on \( B_r \). On the other hand, by Harnack’s inequality,

\[ \sup_{B_{qr}} u \leq c \inf_{B_{qr}} u, \]

where the constant \( c \) depends only on \( n, q, \alpha \), (see [14, Theorem 8.1]). If we apply (3.1) with \( s = p \) and \( s = qr \), we obtain

\[ \frac{1}{V_r^\tau(\partial B_r)} \int_{\partial B_r} \varphi \, dv_r^\tau = \frac{1}{\gamma_r^\tau(\partial B_p)} \int_{\partial B_p} u \, dv_r^\tau \leq \sup_{B_{qr}} u \]

\[ \leq c \inf_{B_{qr}} u \leq c \frac{1}{\gamma_r^\tau(\partial B_{qr})} \int_{\partial B_{qr}} u \, dv_r^{qr} \]

\[ = c \frac{1}{v_r^{qr}(\partial B_r)} \int_{\partial B_r} \varphi \, dv_r^{qr}, \]

and the lemma is proved. \( \square \)

For every \( 0 < p < r \), with \( B_\tau \subset \subset \Omega \), let \( M^\rho_r : H^1(Q_r) \rightarrow \mathbb{R} \) be the linear function defined by

\[ M^\rho_r u = \frac{1}{v_r^\rho(\partial B_r)} \int_{\partial B_r} u \, dv_r^\rho, \tag{3.2} \]

where \( v_r^\rho \) is the outer \( L \)-capacitary distribution of \( B_\rho \) with respect to \( B_r \).

**Lemma 3.3.** For every \( q \in (0, 1) \), there exists a constant \( c = c(q, \alpha, n) \) such that, if \( Q_\tau \subset \subset \Omega \) and \( 0 < p \leq qr \), then

\[ \| u - M^\rho_r u \|_{L^2(Q_r)} \leq cr \| \nabla u \|_{L^2(Q_\tau)}, \]

for every \( u \in H^1(Q_r) \).

**Proof.** Let us fix \( q, p, r \) as required. It is not restrictive to assume \( x^0 = 0 \). Let \( Q = Q_1 \) and \( B = B_1 \). Let us consider the operator \( L_r \) defined by \( L_r u = -\text{div}(A_r \nabla u) \), where \( A_r(y) = A(ry) \). It is easy to check that, if \( w_r^\rho(x) \) is the \( L \)-capacitary potential of \( B_\rho \) with respect to \( B_r \), then \( v_r^\tau(y) = w_r^\sigma(ry) \) is the \( L_r \)-capacitary potential of \( B_{\rho r} \) with respect to \( B \). By Lemma 3.1 we can write \( L_r v_r^\tau = \lambda_r^\tau - \mu_r^\tau \), with supp \( \lambda_r^\tau \subseteq \partial B_{\rho r} \) and supp \( \mu_r^\tau \subseteq \partial B \). We want to prove that, for every \( u \in H^1(Q_\tau) \), we have

\[ \int_{\partial B_r} u \, dv_r^\tau = r^{n-2} \int_{\partial B} u_r \, d\mu_r^\tau, \tag{3.3} \]

where \( u_r(y) = u(ry) \). Let us fix \( u \in H^1(Q_r) \) and let \( \psi \in C_0^\infty(\Omega) \) be a cut-off function such that \( \psi = 1 \) on \( \partial B_\tau \) and \( \psi = 0 \) on \( \overline{B}_r \). If \( \psi_r(y) = \psi(ry) \), then

\[ \int_{\partial B_r} u \, dv_r^\tau = \int_{\partial B_r} u \psi \, dv_r^\tau = - \int_{B_r} A \nabla w_r^\tau \cdot \nabla (u \psi) \, dx \]
which proves (3.3). Taking \( u = 1 \), we get \( v r^2 / r^2 (\partial B_r) = r^{n-2} \mu^r (\partial B) \), so that the previous equality gives

\[
\frac{1}{v r^2 (\partial B_r)} \int_{\partial B_r} u \, dv^r = \frac{1}{\mu^r (\partial B)} \int_{\partial B} u \, d\mu^r
\]

(3.4)

for every \( u \in H^1 (Q_r) \). Finally, we recall that, if \( P \) is a projection from \( H^1 (Q) \) into \( \mathbb{R} \), then the following Poincaré inequality holds for every \( u \) in \( H^1 (Q) \):

\[
\| u - P(u) \|_{L^2 (Q)} \leq \beta \| (H^1 (Q))' \| \| \nabla u \|_{L^2 (Q)},
\]

where \((H^1 (Q))'\) is the dual space of \( H^1 (Q) \) and the constant \( \beta \) depends only on the dimension \( n \) of the space (see [16, Theorem 4.2.1]). Applying this result to

\[
P^r (u) = \frac{1}{\mu^r (\partial B)} \int_{\partial B} u \, d\mu^r,
\]

and using (3.4), we obtain

\[
\| u - M^r u \|_{L^2 (Q_r)}^2 = r^n \int_{Q} (u_r - P^r (u_r))^2 \, dy
\]

\[
\leq \beta^2 r^n \left( \frac{1}{\mu^r (\partial B)} \| \mu^r (H^1 (Q))' \| \right)^2 \int_{Q_r} |\nabla u_r|^2 \, dy
\]

\[
= \beta^2 r^n \left( \frac{1}{\mu^r (\partial B)} \| \mu^r (H^1 (Q))' \| \right)^2 \int_{Q_r} |\nabla u|^2 \, dx.
\]

(3.5)

It remains to estimate

\[
\frac{1}{\mu^r (\partial B)} \| \mu^r (H^1 (Q))' \|
\]

By Lemma 3.2, applied to \( L_r \), we obtain

\[
\left| \frac{1}{\mu^r (\partial B)} \int_{\partial B} \phi \, d\mu^r \right| \leq c \frac{1}{\mu^r (\partial B)} \int_{\partial B} |\phi| \, d\mu^r
\]

for every \( \phi \in H^1 (Q) \), so that

\[
\frac{1}{\mu^r (\partial B)} \| \mu^r (H^1 (Q))' \| \leq c \frac{1}{\mu^r (\partial B)} \| \mu^r (H^1 (Q))' \|
\]

(3.6)

By proposition 2.5(v) and by Lemma 3.1, we have

\[
\mu^r (\partial B) = \text{cap}^{L_r} (B_q, B) \geq c \text{cap} (B_q, B).
\]

(3.7)

Let \( \zeta \in C_0^\infty (\mathbb{R}^n) \) be a cut-off function such that \( \zeta = 1 \) on \( \partial B \), \( \zeta = 0 \) on \( B_q \), \( 0 \leq \zeta \leq 1 \)
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Let $\phi \in H^1(Q)$ we obtain
\[
\int_Q \phi \, d\mu^\rho = \int_{\partial B} \phi \, d\mu^\rho = - \int_B A \nabla \psi^\rho \cdot \nabla (\phi \zeta) \, dy
\]
\[
\leq c_q \alpha^{-1} (\text{cap}^{L^p}(B, B))^2 \|\phi\|_{H^1(Q)} \leq c_q \alpha^{-1} (\text{cap} (B_q, B))^2 \|\phi\|_{H^1(Q)}. \tag{3.8}
\]
From (3.6), (3.7), (3.8) we obtain
\[
\frac{1}{\mu^\rho (\partial B)} \|\mu^\rho\|_{(H^1(Q))'} \leq k(q, \alpha, n),
\]
which, together with (3.5), concludes the proof of the lemma. $\Box$

For every $r > 0$, let $\hat{Q}_r$ be the cube $\{x \in \mathbb{R}^n: -r \leq x_k - x_k^0 < r \text{ for } k = 1, \ldots, n\}$, so that $Q_r$ is the interior of $\hat{Q}_r$.

**Lemma 3.4.** Let $\mu$ be a measure of $K_n^+ (\Omega)$. For every $r > 0$, with $Q_r \subset \subset \Omega$, let $\rho = \rho (r) \in (0, r)$ be the radius such that $\text{cap}^L(B_q, B_r) = \mu (\hat{Q}_r)$, and let $M_r = M_r^\mu (r)$, where $M^\mu_r$ is the average defined in (3.2). Then there exists a function $\omega_{\mu}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $\lim_{r \rightarrow 0^+} \omega_{\mu} (r) = 0$, such that
\[
\|u - M_r u\|_{L^2(Q_r)} \leq \omega_{\mu} (r) \|\nabla u\|_{L^2(Q_r)} \tag{3.9}
\]
for every $u \in H^1(\Omega)$.

**Proof.** First of all, we prove that for every $q \in (0, 1)$ there exists $r_q > 0$ such that $\rho (r) \leq qr$ for $r \leq r_q$. We consider only the case $n \geq 3$; the case $n = 2$ is analogous. Since $\mu$ is a Kato measure, for every $r > 0$ we have
\[
\mu (\Omega \cap B_r)^2 \leq \int_{\Omega \cap B_r} |y - x^0|^2 \, d\mu (y) \leq \psi (r),
\]
where $\psi$ is an increasing function with $\lim_{r \rightarrow 0^+} \psi (r) = 0$. If $\rho = \rho (r) > qr$, then, recalling that $\text{cap} (B_{qr}, B_r) = c_q r^{n-2}$, and using Proposition 2.5(v), we obtain
\[
\alpha c_q r^{n-2} \leq \alpha \text{ cap} (B_q, B_r) \leq \text{cap}^L (B_q, B_r) = \mu (\hat{Q}_r).
\]
So we can write $\alpha c_q r^{n-2} \leq \mu (\hat{Q}_r) \leq \mu (\Omega \cap B_{nr}) \leq \beta_n \psi (nr) r^{n-2}$. Choosing $r_q$ such that $\psi (nr_q) < \alpha c_q / \beta_n$, we obtain a contradiction for $r \leq r_q$. Therefore, there exists $r_q > 0$, with $Q_{r_q} \subset \subset \Omega$, such that $\rho (r) \leq qr$ for every $r \leq r_q$. Since $c_q \rightarrow + \infty$ as $q \rightarrow 1$, we can choose $r_q$ so that for every $r > 0$, with $Q_r \subset \subset \Omega$, there exists $q \in (0, 1)$, with $r \leq r_q$.

Let us fix $q \in (0, 1)$. It is clearly enough to prove (3.9) for every $r \leq r_q$. As $\mu \in K_n^+ (\Omega)$, by Lemma 2.2, there exists a constant $c_n > 0$ such that, if $Q_r \subset \subset \Omega$, then
\[
\int_{\hat{Q}_r} u^2 \, d\mu \leq c_n \|\mu\|_{K^+ (\hat{Q}_r)} \int_{B_{nr}} |\nabla u|^2 \, dx \tag{3.10}
\]
for every $u \in H^1_0(B_{nr})$.

Let us fix a bounded extension operator $\Pi: H^1 (Q, \Omega) \rightarrow H^1_0 (B_n)$, and for every $r > 0$ let us define the extension operator $\Pi_r: H^1 (Q_r) \rightarrow H^1_0 (B_{nr})$ by $(\Pi_r u)(x) = (\Pi u)(x/r)$, where $u_r (y) = u (ry)$. It is easily seen that the boundedness of $\Pi$ implies the existence
of a constant \( k_n > 0 \) such that

\[
\int_{B_{nr}} \left| \nabla (\Pi_r v) \right|^2 \, dx \leq k_n \left( \int_{Q_r} \left| \nabla v \right|^2 \, dx + \frac{1}{r^2} \int_{Q_r} v^2 \, dx \right)
\]

for every \( v \in H^1(Q_r) \). Note that, if \( v \in H^1(\Omega) \) and \( Q_r \subset \subset \Omega \), then \( v = \Pi_r v \) q.e. on \( Q_r \), since both functions are quasi-continuous and coincide on \( Q_r \). Using (3.10) and (3.11), for every \( u \in H^1(\Omega) \) we obtain

\[
\int_{Q_r} (u - M_r u)^2 \, d\mu \leq c_n \| \mu \|_{K_+} (Q_r) \int_{B_{nr}} \left( \nabla (\Pi_r (u - M_r u)) \right)^2 \, dx
\]

\[
\leq \frac{c_n k_n \| \mu \|_{K_+} (Q_r)}{r^2} \left( \int_{Q_r} |\nabla u|^2 \, dx + \frac{1}{r^2} \int_{Q_r} (u - M_r u)^2 \, dx \right).
\]

As \( r \leq r_q \), we have \( \rho = \rho(r) \leq q r \), so that Lemma 3.3 implies that

\[
\frac{1}{r^2} \int_{Q_r} (u - M_r u)^2 \, dx \leq c^2 \int_{Q_r} |\nabla u|^2 \, dx,
\]

hence

\[
\int_{Q_r} (u - M_r u)^2 \, d\mu \leq c_n k_n (1 + c^2) \| \mu \|_{K_+} (Q_r) \int_{Q_r} |\nabla u|^2 \, dx,
\]

for every \( r \leq r_q \) and for every \( u \in H^1(\Omega) \). Since \( \| \mu \|_{K_+} (Q_r) \) tends to zero as \( r \) tends to zero, the statement is proved. \( \square \)

We are now in a position to prove our result for Kato measures. Let \( \{Q_h^i\}_{i \in \mathbb{Z}^n} \) be the partition of \( \mathbb{R}^n \) composed of the cubes

\[
Q_h^i = \{ x \in \mathbb{R}^n : i_k/h \leq x_k < (i_k + 1)/h \text{ for } k = 1, \ldots, n \}.
\]

**Theorem 3.5.** Let \( \mu \in K_+^c (\Omega) \). Let \( I_h \) be the set of all indices \( i \) such that \( Q_h^i \subset \subset \Omega \). For every \( i \in I_h \), let \( B_h^i \) be the ball with the same centre as \( Q_h^i \) and radius \( 1/2h \), and let \( E_h^i \) be another ball with the same centre such that

\[
\text{cap}^L (E_h^i, B_h^i) = \mu(Q_h^i).
\]

Define \( E_h = \bigcup_{i \in I_h} E_h^i \). Then the measures \( \mu_{E_h} \) \( \gamma^L \)-converge to \( \mu \) as \( h \to \infty \).

**Proof.** Let \( v_h^i \) be the \( L \)-capacitary potential of \( E_h^i \) with respect to \( B_h^i \), extended to 0 on \( \Omega \), and let \( w_h^i = 1 - v_h^i \). By Lemma 3.1, we obtain \( Lw_h^i = \lambda_h^i - \lambda_h^i \) in \( \Omega \), with \( v_h^i, \lambda_h^i \in H^{-1}(\Omega) \), \( v_h^i \geq 0 \), \( \lambda_h^i \geq 0 \), \( \text{supp } v_h^i \subseteq \partial B_h^i \), \( \text{supp } \lambda_h^i \subseteq \partial E_h^i \), and

\[
v_h^i (Q_h^i) = \lambda_h^i (Q_h^i) = \text{cap}^L (E_h^i, B_h^i) = \mu(Q_h^i).
\]

Let us define \( w_h \in H^1(\Omega) \) as

\[
w_h = \begin{cases} w_h^i & \text{in } B_h^i \setminus E_h^i, \\ 0 & \text{in } E_h^i, \\ 1 & \text{elsewhere} \end{cases}
\]

(3.13)
and the measures $\nu_h$ and $\lambda_h$ as

$$\nu_h = \sum_{i \in I_h} \nu^i_h, \quad \lambda_h = \sum_{i \in I_h} \lambda^i_h. \quad (3.14)$$

We want to prove that all hypotheses of Theorem 2.12 hold for $w_h$ and $v_h$.

First of all, we prove that $w_h$ converges weakly to 1 in $H^1(\Omega)$.

Since, by the maximum principle, $0 \leq w_h \leq 1$ in $\Omega$, we have that $\{w_h\}$ is bounded in $L^2(\Omega)$. On the other hand,

$$\alpha \int_{\Omega} |\nabla w_h|^2 \, dx \leq \sum_{i \in I_h} \text{cap}_L(E^i_h, B^i_h) = \sum_{i \in I_h} \mu(Q^i_h) \leq \mu(\Omega).$$

Thus $\{w_h\}$ is bounded in $H^1(\Omega)$ so that there exist a subsequence (still denoted $\{w_h\}$) and a function $w \in H^1(\Omega)$, such that $\{w_h\}$ converges to $w$ weakly in $H^1(\Omega)$, and hence strongly in $L^2(\Omega)$. We are going to show that $w = 1$ a.e. in $\Omega$, using the arguments of Cioranescu and Murat (see [4, Théorème 2.2]). Let us consider the family

\[ \{C_h^i\}_{i \in \mathbb{Z}^n} \]

of all open balls with radius $\left(\sqrt{n} - 1\right)/2h$ and centres in the vertices $i/h$ of the cubes $Q^i_h$. In these balls we have $w_h = 1$. Therefore, if we define $C_h$ as the union of the balls $C_h^i$ contained in $\Omega$, we have $w_h \chi_{C_h} = \chi_{C_h}$, where $\chi_{C_h}$ is the characteristic function of $C_h$. Since $\{\chi_{C_h}\}$ converges to a positive constant in the weak* topology of $L^\infty(\Omega)$, passing to the limit in the equality $w_h \chi_{C_h} = \chi_{C_h}$ we obtain $w = 1$ a.e. in $\Omega$.

It remains to prove that the measures $\nu_h$ defined in (3.14) converge to $\mu$ in the strong topology $H^{-1}(\Omega)$. Indeed, since $w_h$ converges to 1 weakly in $H^1(\Omega)$, this implies also that $\lambda_h$ converges weakly to $\mu$ in $H^{-1}(\Omega)$.

For every $h \in \mathbb{N}$, we introduce the polyrectangle $P_h = \bigcup_{i \in I_h} Q^i_h$ and we define $S_h = \Omega \setminus P_h$. Moreover, for every $\varphi \in H_0^1(\Omega)$, we consider the function

$$\varphi_h = \sum_{i \in I_h} (M^i_h \varphi) \chi_{Q^i_h},$$

where, according to (3.2),

$$M^i_h \varphi = \frac{1}{\nu^i_h(\partial B^i_h)} \int_{\partial B^i_h} \varphi \, dv^i_h,$$

and we define $\varepsilon_h = \|\mu\|_{S_h} \|H^{1-}(\Omega)}$. Note that $\{\varepsilon_h\}$ tends to zero by Lemma 2.3. Recalling that $\mu(Q^i_h) = \nu^i_h(\partial B^i_h)$ and using the Poincaré inequality (3.9), we have that,

\[
\begin{align*}
|\langle \nu_h, \varphi \rangle - \langle \mu, \varphi \rangle | &= \left| \sum_{i \in I_h} \frac{\mu(Q^i_h)}{\nu^i_h(\partial B^i_h)} \int_{\partial B^i_h} \varphi \, dv^i_h - \sum_{i \in I_h} \int_{Q^i_h} \varphi \, d\mu - \int_{S_h} \varphi \, d\mu \right| \\
&\leq \int_{P_h} |\varphi - \varphi_h| \, d\mu + \int_{S_h} |\varphi| \, d\mu \\
&\leq \left( \mu(\Omega) \int_{P_h} (\varphi - \varphi_h)^2 \, d\mu \right)^{\frac{1}{2}} + \|\mu\|_{S_h} \|H^{1-}(\Omega)} \|\varphi\|_{H_0^1(\Omega)} \\
&= \left( \mu(\Omega) \sum_{i \in I_h} \|\varphi - M^i_h \varphi\|_{L^2(Q^i_h)}^2 \right)^{\frac{1}{2}} + \varepsilon_h \|\varphi\|_{H_0^1(\Omega)}
\end{align*}
\]
Thus we obtain
\[ \| v_h - \mu \|_{H^{-1}(\Omega)} \leq \mu(\Omega)^\frac{1}{2} (1/h^2) + \varepsilon_h \| \phi \|_{H^1(\Omega)}. \]

hence \( \{v_h\} \) converges to \( \mu \) strongly in \( H^{-1}(\Omega) \). Therefore \( \{\infty_Eh\} \) \( \gamma^L \)-converges to \( \mu \) by Theorem 2.12. \( \square \)

In order to generalise this result to every Radon measure, we need the following results:

**Proposition 3.6.** For every Radon measure \( \mu \in \mathcal{M}_0(\Omega) \), there exist a measure \( v \in K^+_n(\Omega) \) and a positive Borel function \( g : \Omega \to [0, +\infty] \) such that \( \mu = gv \).

**Proof.** See [2, Proposition 2.5]. \( \square \)

**Proposition 3.7.** Let \( \lambda \in \mathcal{M}_0(\Omega) \), let \( \mu \) be a Radon measure in \( \mathcal{M}_0(\Omega) \); for every \( x \in \Omega \) let
\[ f(x) = \liminf_{r \to 0} \frac{\text{cap}_{\lambda}^L(B_r(x), B_{2r}(x))}{\mu(B_r(x))}. \]

Assume that \( f \) is bounded. Then \( \lambda \) is a Radon measure and we have \( \lambda = f\mu \).

**Proof.** See [3, Theorem 2.3]. \( \square \)

**Proposition 3.8.** Let \( \mu \) be a positive Radon measure on \( \Omega \). Then there exists a unique pair \( (\mu_0, \mu_1) \) of Radon measures on \( \Omega \) such that:

(i) \( \mu = \mu_0 + \mu_1 \);

(ii) \( \mu_0 \in \mathcal{M}_0(\Omega) \);

(iii) \( \mu_1 = \mu \mathbb{1}_N \), for some Borel set \( N \) with \( \text{cap}(N, \Omega) = 0 \).

**Proof.** See [8, Lemma 2.1]. \( \square \)

We are now in a position to prove our main result in its most general form.

**Theorem 3.9.** Let \( \mu \) be a positive Radon measure on \( \Omega \). Let \( \{Q_h\} \) and \( \{E_h\} \) be defined as in Theorem 3.5. Then the following results hold:

(i) if \( \mu \) belongs to \( \mathcal{M}_0(\Omega) \), then \( \{\infty_{E_h}\} \) \( \gamma^L \)-converges to \( \mu \);

(ii) if \( \mu = \mu_0 + \mu_1 \), with \( \mu_0 \) and \( \mu_1 \) as in Proposition 3.8, then \( \{\infty_{E_h}\} \) \( \gamma^L \)-converges to \( \mu_0 \).

**Proof.** If \( \mu \) is a Radon measure in \( \mathcal{M}_0(\Omega) \), then, by Proposition 3.6, \( \mu = gv \), where \( v \in K^+_n(\Omega) \) and \( g \) is a positive Borel function. By Theorem 2.10, there exists a subsequence, still denoted by \( \{E_h\} \), and a measure \( \lambda \in \mathcal{M}_0(\Omega) \), such that \( \{\infty_{E_h}\} \) \( \gamma^L \)-converges to \( \lambda \). Let \( x \in \Omega \) and let \( r > 0 \) such that \( B_{2r}(x) \subseteq \Omega \). We want to prove that, for every Borel set \( E \subseteq B_{2r} \),
\[ \text{cap}_{\lambda}^L(E, B_{2r}(x)) \leq \mu(E). \] (3.15)

If \( A \) and \( A' \) are two open sets such that \( A' \subseteq A \subseteq B_{2r}(x) \) and \( h \) is small enough, we
have
\[ \bigcup_{E_h \cap A' \neq \emptyset} Q_h \subseteq A, \]
hence, by Proposition 2.5,
\[ \text{cap}^L (E_h \cap A', B_{2r}(x)) \leq \sum_{E_h \cap A' \neq \emptyset} \text{cap}^L (E_h, B_{2r}(x)) \]
\[ \leq \sum_{E_h \cap A' \neq \emptyset} \text{cap}^L (E_h, B'_h) = \sum_{E_h \cap A' \neq \emptyset} \mu(Q_h) \leq \mu(A). \]
Using Theorem 2.11, we obtain
\[ \text{cap}^L (A', B_{2r}(x)) \leq \liminf_{h \to \infty} \text{cap}^L (E_h \cap A', B_{2r}(x)) \leq \mu(A) \]
and, as \( A' \supseteq A \), we obtain \( \text{cap}^L (A, B_{2r}(x)) \leq \mu(A) \) for every open set \( A \subseteq B_{2r}(x) \) (see Proposition 2.5(vi)). Since \( \mu \) is a Radon measure, this inequality can be easily extended to all Borel subsets of \( B_{2r}(x) \). So (3.15) is proved. Choosing \( E = B_{r}(x) \) in (3.15) and applying Proposition 3.7, we obtain that \( \lambda \) is a Radon measure and that \( \lambda \leq \mu \).

Define, for \( k \in \mathbb{N} \), the measures \( \mu^k = g^k \nu \), where \( g^k(x) = \min (g(x), k) \). As \( \mu^k \in K^+ \mathcal{M}_0 (\Omega) \), by Theorem 3.5 for every \( k \) there exists a sequence \( \{E_{k,h}\} \) such that \( \{\infty_{E_{k,h}}\} \) \( \gamma^L \)-converges to \( \mu^k \). Since \( \mu^k \leq \mu \), the construction of Theorem 3.5 implies that \( E_{k,h} \subseteq E_h \) for every \( h \) and \( k \). By Remark 2.9 this implies \( \lambda \geq \mu^k \) for every \( k \), hence \( \lambda \geq \mu \). Since the \( \gamma^L \)-limit does not depend on the subsequence, the whole sequence \( \{\infty_{E_h}\} \) \( \gamma^L \)-converges to \( \mu \).

Now let \( \mu \) be any Radon measure on \( \Omega \). By Proposition 3.8, we can write \( \mu = \mu_0 + \mu_1 \), with \( \mu_0 \in \mathcal{M}_0 (\Omega) \) and \( \mu_1 = \mu \bigcap \overline{N} \), where \( N \) is a Borel set with \( \text{cap} (N, \Omega) = 0 \). Arguing as before, let \( \lambda \) be the \( \gamma^L \)-limit of a subsequence of \( \{\infty_{E_h}\} \). If \( x \in \Omega \) and \( r > 0 \) is such that \( B_{2r}(x) \subseteq \Omega \), we have
\[ \text{cap}^L (B_r(x), B_{2r}(x)) = \text{cap}^L (B_r(x) \setminus N, B_{2r}(x)), \]
since \( \text{cap} (N, B_{2r}(x)) = 0 \) (see Proposition 2.5). Therefore (3.15), applied with \( E = B_r(x) \setminus N \), gives
\[ \text{cap}^L (B_r(x), B_{2r}(x)) \leq \mu(B_r(x) \setminus N) = \mu_0(B_r(x)). \]
By applying again Proposition 3.7, we obtain \( \lambda \leq \mu_0 \).

Since \( \mu_0 \) is a Radon measure of \( \mathcal{M}_0 (\Omega) \), by the first part of this theorem we can construct the holes \( E_{0,h} \) such that \( \{\infty_{E_{0,h}}\} \) \( \gamma^L \)-converges to \( \mu_0 \). Since \( \mu(Q_h) \geq \mu_0(Q_h) \), we have \( E_{0,h} \subseteq E_h \), hence, by Remark 2.9, \( \lambda \geq \mu_0 \). As the opposite inequality has already been proved, we obtain \( \lambda = \mu_0 \). As before, this implies that the whole sequence \( \{\infty_{E_h}\} \) \( \gamma^L \)-converges to \( \mu_0 \). \( \square \)

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