A Primal-Dual Partial Inverse Algorithm for Constrained Monotone Inclusions: Applications to Stochastic Programming and Mean Field Games

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Abstract
In this work, we study a constrained monotone inclusion involving the normal cone to a closed vector subspace and prior information on primal solutions. We model this information by imposing that solutions belong to the fixed point set of an averaged nonexpansive mapping. We characterize the solutions using an auxiliary inclusion that involves the partial inverse operator. Then, we propose the primal-dual partial inverse splitting and we prove its weak convergence to a solution of the inclusion, generalizing several methods in the literature. The efficiency of the proposed method is illustrated in multiple applications including constrained LASSO, stochastic arc capacity expansion problems in transport networks, and variational mean field games with non-local couplings.

Keywords
Constrained convex optimization · Constrained LASSO · Monotone operator theory · Partial inverse method · Primal-dual splitting · Stochastic arc capacity expansion · Mean field games

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1 Introduction

In this paper, we propose a convergent algorithm for solving composite monotone inclusions involving a normal cone to a closed vector subspace and a priori information on the solutions. The precise formulation of the monotone inclusion under study is stated in Problem 3.1 below and models several applications such as evolution inclusions [5, 22, 42], variational inequalities [8, 26], partial differential equations (PDEs) [4, 27, 37], and various optimization problems. In the particular case when monotone operators are subdifferentials of proper lower semicontinuous convex functions, the inclusion covers the optimization problem with a priori information

\[
\text{find } x \in S \cap \arg\min_{x \in V} (F(x) + G(Lx) + H(x)),
\]

where $S$ is a closed convex subset of a real Hilbert space $\mathcal{H}$ modeling the a priori information on the solution, $V \subset \mathcal{H}$ is a closed vector subspace, $L$ is a linear bounded operator from $\mathcal{H}$ to a real Hilbert space $G$, $F: \mathcal{H} \to ]-\infty, +\infty[$, $G: G \to ]-\infty, +\infty[$, and $H: \mathcal{H} \to \mathbb{R}$ are proper lower semicontinuous convex functions, and $H$ is Gâteaux differentiable. This class of problems appears in PDEs [38, Sect. 3], signal and image processing [6, 17, 25], stochastic traffic theory [20, 45], mean field games [10, 16], among other fields. In the aforementioned applications, the vector subspace constraint models intrinsic properties of the solution or non-anticipativity in stochastic problems. In turn, the a priori information can be used to reinforce feasibility in the iterates, resulting in more efficient algorithms, as explored in [15].

In the case when the closed vector subspace is the whole Hilbert space, the monotone inclusion can be solved by the algorithm in [15]. This method uses the a priori information represented by $S$ to improve the efficiency, generalizing the algorithm in [49] for monotone inclusions and in [24] for convex optimization.

In addition, when no a priori information is considered, the methods proposed in [13, 46] solve particular instances of our problem using the partial inverse of a monotone operator introduced in [46]. This mathematical tool exploits the vector subspace structure of the inclusion and has been used, for example, in [2, 12]. Our problem in the more general context, with or without a priori information, can be solved by algorithms in [15] and in [14, 21, 45, 49], respectively, by using product space techniques without special consideration on the vector subspace structure. However, the product space formulation generates methods that include updates of high dimensional dual variables at each iteration, which reduce their performance.

The objective of this paper is to provide an algorithm for solving the inclusion under study by taking advantage of the vector subspace structure and the a priori information of the inclusion. Our method is obtained from the combination of the algorithm in [15] with partial inverse techniques developed in [12, 13, 46]. We illustrate the advantages of the partial inverse approach and the use of the a priori information by means of numerical experiences on particular instances of (1). In this context, the a priori information is modeled by a set formed by some of the constraints of the problem and the additional projections in our method improve the speed of the convergence with respect to existing methods. In order to test the efficiency of our method, we con-
sider the constrained LASSO problem, the arc capacity expansion in traffic networks, and Mean Field Games (MFG) with non-local couplings. The constrained LASSO problem combines sparse minimization and least squares under linear constraints, in which the vector subspace structure appears naturally. This problem appears in portfolio optimization, internet advertising, curve estimation in statistics [28, 30] and covers the generalized LASSO [47], fused LASSO [48], and trend filtering on graphs [50], among others. Our second application is devoted to solve a stochastic arc capacity expansion problem in transport networks. We follow the two stage stochastic programming model described in [19, 20, 51], where an investment decision over arc capacity expansion is made in the first stage, and a traffic assignment problem under uncertain demand is solved in the second stage. Our solution strategy relies on a non-anticipativity approach, exploiting the vector subspace structure of the induced problem. The last application is the numerical approximation of MFG equilibria [29, 33]. The theory of MFGs aims to describe equilibria of symmetric stochastic differential games with a continuum of agents. These, in turn, provide approximate equilibria for the corresponding game with a large, but finite, number of players. In its standard form, MFG equilibria is characterized by a system of two coupled PDEs, which is called MFG system. In some particular cases, the MFG system is the first order optimality condition of a linearly constrained convex optimization problem [11, 33] fitting in our a priori information and vector subspace modeling (see [16]). In the context of ergodic MFGs with non-local couplings [32, 33, 39], we illustrate the efficiency of our method and compare it with some benchmark algorithms.

The paper is organized as follows. In Sect. 2, we set our notation and preliminaries. Section 3 is devoted to the formulation our problem, the characterization of the solutions by using the partial inverse operator, the proof of the weak convergence of our algorithm, and a discussion of the connections with existing methods in the literature. In Sect. 4, we implement the proposed algorithm in the context of constrained convex optimization. More precisely, in Sect. 4.1 we compare the performance of our method with algorithms available in the literature for the constrained LASSO problem, and in Sect. 4.2, we apply our method to the stochastic arc capacity expansion problem in a transport network. In Sect. 4.3, we compare implementations to several formulations of the underlying optimization problem variational ergodic MFG with non-local couplings. Finally, we provide our conclusions and perspectives in Sect. 5.

2 Notation and Background

Throughout this paper, \( \mathcal{H} \) and \( \mathcal{G} \) are real Hilbert spaces. We denote their scalar products by \( \langle \cdot | \cdot \rangle \) and their associated norms by \( \| \cdot \| \). The class of bounded linear operators from \( \mathcal{H} \) to a real Hilbert space \( \mathcal{G} \) is denoted by \( \mathcal{L}(\mathcal{H}, \mathcal{G}) \) and if \( \mathcal{H} = \mathcal{G} \) this class is denoted by \( \mathcal{L}(\mathcal{H}) \). Given \( L \in \mathcal{L}(\mathcal{H}, \mathcal{G}) \), its adjoint operator is denoted by \( L^* \in \mathcal{L}(\mathcal{G}, \mathcal{H}) \). The projection operator onto a nonempty closed convex set \( C \subset \mathcal{H} \) is denoted by \( P_C \) and the normal cone to \( C \) is denoted by \( N_C \). Let \( A : \mathcal{H} \rightrightarrows \mathcal{H} \) be a set-valued operator. We denote by \( \text{gra} \ A \) its graph, by \( A^{-1} : u \mapsto \{ x \in \mathcal{H} \mid u \in Ax \} \) its inverse operator, and by \( J_A := (\text{Id} + A)^{-1} \) its resolvent, where \( \text{Id} \) is the identity operator on \( \mathcal{H} \). Moreover, \( A \) is \( \rho \)-strongly monotone for \( \rho \geq 0 \) iff, for every \( (x, u) \) and \( (y, v) \) in \( \text{gra} \ A \), we have...
\[
\langle x - y \mid u - v \rangle \geq \rho \|x - y\|^2,
\]
it is monotone iff it is 0—strongly monotone, and it is maximally monotone iff it is monotone and there is no monotone operator \(B\) such that \(\text{gra} \ A \subseteq \text{gra} \ B\). Let \(V\) be a closed vector subspace of \(\mathcal{H}\). The partial inverse of \(A\) with respect to \(V\), denoted by \(A_V\), is the operator defined via its graph by \([46]\)

\[
\text{gra} \ A_V := \{(x, u) \in \mathcal{H} \times \mathcal{H} : (P_V x + P_V u, P_V u + P_V x) \in \text{gra} \ A\}. \quad (2)
\]

Let \(T : \mathcal{H} \to \mathcal{H}\). The set of fixed points of \(T\) is denoted by \(\text{Fix} \ T\). The operator \(T\) is \(\alpha\)—averaged nonexpansive for some \(\alpha \in ]0, 1[\) iff

\[
\forall (x, y) \in \mathcal{H}^2, \quad \|Tx - Ty\|^2 \leq \|x - y\|^2 - \left(\frac{1 - \alpha}{\alpha}\right) \|\text{Id} - T\|x - (\text{Id} - T)y\|^2 \quad (3)
\]

and it is \(\beta\)—cocoercive for some \(\beta > 0\) iff

\[
\forall (x, y) \in \mathcal{H}^2, \quad \langle x - y \mid Tx - Ty \rangle \geq \beta \|Tx - Ty\|^2. \quad (4)
\]

The class of lower semicontinuous convex proper functions \(f : \mathcal{H} \to ]-\infty, +\infty]\) is denoted by \(\Gamma_0(\mathcal{H})\). The subdifferential of \(f \in \Gamma_0(\mathcal{H})\) is denoted by \(\partial f\) and the proximity operator of \(f\) at \(x \in \mathcal{H}\) denoted by \(\text{prox}_f \ x\), is the unique minimizer of \(f + \|\cdot - x\|^2/2\). For every nonempty closed convex set \(C \subset \mathcal{H}\), we denote by \(\iota_C \in \Gamma_0(\mathcal{H})\) the indicator function of \(C\). For further information on convex analysis and monotone operator theory, the reader is referred to \([8]\).

### 3 Problem and Main Result

We consider the following composite primal-dual inclusion problem with a priori information.

**Problem 3.1** Let \(\mathcal{H}\) and \(G\) be real Hilbert spaces and let \(V \subset \mathcal{H}\) and \(W \subset G\) be closed vector subspaces. Let \(T : \mathcal{H} \to \mathcal{H}\) be an \(\alpha\)—averaged nonexpansive operator for some \(\alpha \in ]0, 1[\), let \(L \in L(\mathcal{H}, \mathcal{G})\) be such that \(\text{ran} \ L \subset W\), let \(A : \mathcal{H} \rightrightarrows \mathcal{H}\), \(B : \mathcal{G} \rightrightarrows \mathcal{G}\), and \(D : \mathcal{G} \rightrightarrows \mathcal{G}\) be maximally monotone operators, let \(C : \mathcal{H} \to \mathcal{H}\) be a \(\beta\)—cocoercive operator for some \(\beta \in ]0, +\infty]\), and suppose that \(D\) is \(\delta\)—strongly monotone for some \(\delta \in ]0, +\infty]\). The problem is to

find \((x, u) \in (V \cap \text{Fix} \ T) \times W\) such that

\[
\begin{cases}
0 \in Ax + Cx + L^* u + \nabla H x, \\
0 \in B^{-1} u + D^{-1} u - L x,
\end{cases}
\]

under the assumption that \((I)\) admits solutions.

Consider the case when \(W = \mathcal{G}\), \(A = \partial F\), \(B = \partial G\), \(C = \nabla H\), and \(D = \partial \ell\), where \(F \in \Gamma_0(\mathcal{H})\), \(G \in \Gamma_0(\mathcal{G})\), \(H : \mathcal{H} \to \mathbb{R}\) is a differentiable convex function with \(\beta^{-1}\)—Lipschitz gradient, and \(\ell \in \Gamma_0(\mathcal{G})\) is \(\delta\)—strongly convex. By defining \(G \sqcap \ell\) as infimal convolution between \(G\) and \(\ell\), it follows from \(\partial (G \sqcap \ell) = ((\partial G)^{-1} + \partial \ell)^{-1}\)
that Problem 3.1 reduces to the constrained optimization problem

\[
\text{find } x \in \text{Fix } T \cap \arg\min_{x \in V} \left( F(x) + (G \Box \ell)(Lx) + H(x) \right),
\]

(5)

under standard qualification conditions. Note that from [8, Corollary 18.17], \( \nabla H \) is \( \beta \)-cocoercive, \( \ell^* \) is Gâteaux differentiable, and \( \nabla \ell^* \) is \( \delta \)-cocoercive. In the case when \( T = P_C \) for some nonempty closed convex set \( C \), (5) models optimization problems with a priori information on the solution [15]. Moreover, (5) models the problem of finding a common solution to two convex optimization problems when \( T \) is such that \( \text{Fix } T = \arg\min \Phi \), for some convex function \( \Phi \) (e.g., if \( T = \text{prox}_\Phi \), \( \text{Fix } T = \arg\min \Phi \)). More generally, by using the a priori information on solutions with a suitable operator \( T \), we can find common solutions to two monotone inclusions.

In the particular case when \( V = H, W = G \), and \( T = \text{Id} \), [49] solves Problem 3.1, and the corresponding optimization problem can be solved by the method proposed in [24]. Previous methods are generalizations of several classical splitting algorithms in particular instances, such as the proximal-point algorithm [36, 44], the forward-backward splitting [34], and the Chambolle-Pock’s algorithm [18].

In the case when \( V \neq H \), the algorithms proposed in [13, 46] solve Problem 3.1 in particular instances, exploiting the vector subspace structure by using the partial inverse of a monotone operator. A convergent splitting method generalizing the partial inverse algorithm in [46] is proposed in [13] and solves Problem 3.1 when \( B = 0 \). The algorithms proposed in [14, 21, 49] solves Problem 3.1 when \( W = G \) and \( T = \text{Id} \), using product space techniques without special consideration on the vector subspace structure of \( NV \). The resulting method involves higher dimensional dual variables to be updated at each iteration, affecting the performance of the algorithm. In its full generality, Problem 3.1 can be solved by an algorithm proposed in [15], in which \( NV \) is activated as any maximally monotone operator by using product space techniques, which leads to similar drawbacks than those discussed above.

In the next section we provide our algorithm and main results.

3.1 Algorithm and Convergence

Our algorithm for solving Problem 3.1 is the following.

**Algorithm 3.1** Set \( x^0 \in V, \bar{x}^0 = x^0, y^0 \in V^\perp, u^0 \in G, \tau > 0 \), and set \( \gamma > 0 \).

\[
(\forall k \in \mathbb{N}) \begin{align*}
\eta^{k+1} &= J_{\gamma B^{-1}}(u^k + \gamma (L\bar{x}^k - D^{-1}u^k)) \\
nu^{k+1} &= P_W \eta^{k+1} \\
w^{k+1} &= J_{\tau A}(x^k + \tau y^k - \tau P_V (L^*u^{k+1} + Cx^k)) \\
r^{k+1} &= P_V w^{k+1} \\
x^{k+1} &= P_V Tr^{k+1} \\
y^{k+1} &= y^k + (r^{k+1} - w^{k+1})/\tau \\
\bar{x}^{k+1} &= x^{k+1} + r^{k+1} - x^k.
\end{align*}
\]

(6)
Algorithm 3.1 exploits the vector subspace structure via $P_V$ and $P_W$, and the a priori information on the solution of the inclusion via $T$.

In the following theorem, we prove the weak convergence of the primal-dual sequences generated by Algorithm 3.1 to a solution to Problem 3.1. We first characterize the solutions to Problem 3.1 as solutions to an auxiliary monotone inclusion involving the partial inverse of $A$ with respect to $V$ and we deduce the convergence of the iterates generated by Algorithm 3.1 through a suitable application of the method in [15] to the auxiliary inclusion.

**Theorem 3.1** In the context of Problem 3.1, let $(x^k, u^k)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 3.1, where we assume that $\tau \in ]0, 2\beta[$, $\gamma \in ]0, 2\delta[$, and that

$$\|L\|^2 < \left(\frac{1}{\tau} - \frac{1}{2\beta}\right) \left(\frac{1}{\gamma} - \frac{1}{2\delta}\right).$$

(7)

Then $(x^k, u^k)_{k \in \mathbb{N}}$ converges weakly to a solution $(\widehat{x}, \widehat{u})$ to Problem 3.1.

**Proof** We split the proof in two main parts. For the first part, the key point is to provide an equivalent formulation of Problem 3.1 which satisfies the hypotheses in [15, Problem 3.1] by using the properties of the partial inverse. In the second part, we prove that (6) corresponds to the algorithm proposed in [15, Theorem 3.1] applied to the equivalent formulation.

I. Let $(x, u) \in (V \cap \text{Fix } T) \times W$ be a solution to Problem 3.1. Note that

$$B^{-1} + D^{-1} = ((\tau B)^{-1} + (\tau D)^{-1}) \circ (\tau \text{Id}).$$

Therefore, by using (2), there exists $y \in N_V x = V^\perp$ such that

$$\begin{cases}
y - L^*u - Cx \in Ax \\
Lx \in B^{-1}u + D^{-1}u
\end{cases} \Leftrightarrow \begin{cases}
\tau(y - L^*u - Cx) \in (\tau A)x \\
Lx \in (\tau B)^{-1}(\tau u) + (\tau D)^{-1}(\tau u)
\end{cases} \Leftrightarrow \begin{cases}
-\tau P_V(L^*u + Cx) \in (\tau A)V z \\
Lx \in (\tau B)^{-1}(\tau u) + (\tau D)^{-1}(\tau u)
\end{cases} \Leftrightarrow \begin{cases}
-\tau P_V L^*v \in (\tau A)V z + \tau P_V C_P V z \\
L P_V z \in (\tau B)^{-1}v + (\tau D)^{-1}v,
\end{cases}$$

(8)

where

$$z := x + \tau(y - P_V^\perp(L^*u + Cx)) \quad \text{and} \quad v := \tau u \in W.$$

(10)

In addition, since $P_V z = x \in \text{Fix } T \cap V$, then $T P_V z = P_V z$ and, thus, $P_V T P_V z = P_V^2 z = P_V z = z - P_V^\perp z$, which yields $z \in \text{Fix } M$, where

$$M \equiv P_V T P_V + P_V^\perp.$$

(11)
Thus, by setting
\[
\begin{align*}
\widetilde{A} &= (\tau A)V \\
\widetilde{B} &= \tau B \\
\widetilde{C} &= \tau P_V CP_V \\
\widetilde{D} &= \tau D \\
\widetilde{L} &= LP_V,
\end{align*}
\]
we conclude from (9) that \((z, v)\) defined in (10) solves
\[
\begin{align*}
\text{find } (z, v) \in \text{Fix } M \times W \text{ such that } \begin{cases} \\
-L^*v \in \widetilde{A}z + \widetilde{C}z \\
\widetilde{L}z \in \widetilde{B}^{-1}v + \widetilde{D}^{-1}v.
\end{cases}
\end{align*}
\]
In addition, we have that \(\widetilde{A}\) and \(\widetilde{B}\) are maximally monotone operators [8, Propo-
sition 20.44(v) & Proposition 20.22], \(\widetilde{D}\) is \(\tau \delta\)-strongly monotone, and, since \(P_V\) is a bounded linear operator, \(\widetilde{L}\) is a bounded linear operator such that \(\text{ran } \widetilde{L} \subset \text{ran } L \subset W\). Moreover, since \(P_V\) is nonexpansive and \(P_V^* = P_V\), for every \((x, y) \in H \times H\) we have
\[
\langle \widetilde{C}x - \widetilde{C}y | x - y \rangle = \tau \langle CP_V x - CP_V y | P_V x - P_V y \rangle \\
\geq \tau \beta \|CP_V x - CP_V y\|^2 \\
\geq \tau \beta \|P_V CP_V x - P_V CP_V y\|^2 \\
= \frac{\beta}{\tau} \|\widetilde{C}x - \widetilde{C}y\|^2,
\]
which implies that \(\widetilde{C}\) is \(\frac{\beta}{\tau}\)-cocoercive. Furthermore, we claim that \(M\) defined in (11) is \(\alpha\)-averaged nonexpansive. Indeed, let \((x, z) \in H^2\). Since \(T\) is \(\alpha\)-averaged nonexpansive, \(\text{Id} = P_V + P_V^\perp, P_V\) is nonexpansive, and \(\text{Id} - M = P_V - P_V \circ T \circ P_V = P_V \circ (\text{Id} - T) \circ P_V\), we have
\[
\|Mx - Mz\|^2 = \|P_V TP_V x + P_V^\perp x - P_V TP_V z - P_V^\perp z\|^2 \\
= \|P_V TP_V x - P_V^\perp x - P_V TP_V z\|^2 + \|P_V^\perp x - P_V^\perp z\|^2 \\
\leq \|TP_V x - TP_V z\|^2 + \|P_V^\perp x - P_V^\perp z\|^2 \\
\leq \|P_V x - P_V z\|^2 - \left(1 - \frac{\alpha}{\alpha}\right) \|P_V (\text{Id} - T) P_V x - (\text{Id} - T) P_V z\|^2 \\
+ \|P_V^\perp x - P_V^\perp z\|^2 \\
\leq \|x - z\|^2 - \left(1 - \frac{\alpha}{\alpha}\right) \|P_V (\text{Id} - T) P_V x - P_V (\text{Id} - T) P_V z\|^2 \\
= \|x - z\|^2 - \left(1 - \frac{\alpha}{\alpha}\right) \|(\text{Id} - M)x - (\text{Id} - M)z\|^2.
\]
Altogether, we conclude that (13) reduces to [15, Problem 3.1].

Conversely, suppose that \((\栓, \check{v}) \in \text{Fix } M \times W\) solves (13). By setting \(\check{x} := P_V \栓 \in V, \check{\mu} := \check{v}/\tau \in W,\) and \(\check{y} := P_V \left(\frac{1}{\tau} \栓 + L^* \check{\mu} + C P_V \栓\right) \in V^\perp,\) we get that

\[
\begin{align*}
\栓 &= P_V \栓 + P_V \perp \栓 \\
&= \check{x} + \tau P_V \perp \left(\frac{1}{\tau} \栓\right) \\
&= \check{x} + \tau \left( P_V \perp \left(\frac{1}{\tau} \栓 + L^* \check{\mu} + C P_V \栓\right) - P_V \perp (L^* \check{\mu} + C \check{x}) \right) \\
&= \check{x} + \tau (\check{y} - P_V \perp (L^* \check{\mu} + C \check{x})).
\end{align*}
\]

Hence, \((\check{x}, \check{\mu}, \check{y}, \check{v})\) satisfies (9). Then, since \((\栓, \check{v})\) satisfies (9), we deduce that \((\栓, \check{\mu}, \check{y})\) satisfies (8). In addition, since \(\栓 \in \text{Fix } M,\) then \(\栓 = P_Y T P_V \栓 + \栓 - P_V \栓\) and therefore \(\check{x} \in \text{Fix}(P_Y T).\) Now, since Problem 3.1 has solutions, \(\text{Fix } P_V \cap \text{Fix } T = V \cap \text{Fix } T \neq \emptyset.\) Then, by [8, Proposition 4.49(i)], it follows that \(\text{Fix}(P_Y T) = V \cap \text{Fix } T\) and \(\check{x} \in V \cap \text{Fix } T.\)

That is, \((\栓, \check{\mu}) \in (V \cap \text{Fix } T) \times W\) and therefore \((\栓, \check{\mu})\) is a solution to (I).

II. For every \(k \in \mathbb{N},\) define \(z^k := x^k + \tau y^k, p^k := r^k + \tau y^k, \栓^k := z^{k+1} + p^{k+1} - z^k, \tau^k := z^0,\) and \(\栓^k = z^k - \tau P_V (L^* u^{k+1} + C x^k).\) Note that (6) yields \(\{r^k\}_{k \in \mathbb{N}} \subset V, \{x^k\}_{k \in \mathbb{N}} \subset V,\) and \(\{y^k\}_{k \in \mathbb{N}} \subset V^\perp.\) Hence, since for every \(k \in \mathbb{N},\) \(r^k = P_V p^k\) and \(x^k = P_V z^k,\) it follows from (6) that, for every \(k \in \mathbb{N},\)

\[
\begin{align*}
\begin{cases}
z^{k+1} = P_Y T r^{k+1} + \tau y^{k+1} = P_Y T P_V p^{k+1} + P_V \perp p^{k+1} = M p^{k+1} \\
P_V \栓^{k+1} = x^{k+1} + r^{k+1} - x^k = \栓^{k+1}. 
\end{cases}
\end{align*}
\]

In addition, \(P_V z^0 = P_V x^0 = x^0 = \栓^0.\) Now, from (6) and [12, Proposition 3.1(i)] we deduce

\[
\begin{align*}
p^{k+1} &= r^{k+1} + \tau y^{k+1} \\
&= 2r^{k+1} - u^{k+1} + \tau y^k \\
&= (2P_V - \text{Id}) J_{\tau A} \栓^k + P_V \perp \栓^k \\
&= J_{(\tau A) V} \栓^k. \quad (15)
\end{align*}
\]
Also, note that
\[
\tau J_{\gamma B^{-1}} = \left((\text{Id} + \gamma B^{-1}) \circ (\tau^{-1} \text{Id})\right)^{-1} = \left(\tau^{-1}(\text{Id} + \tau \gamma (\tau B)^{-1})\right)^{-1} = J_{\tau \gamma (\tau B)^{-1} \circ (\tau \text{Id})}.
\]

Let us define \(\sigma = \tau \gamma \in ]0, 2\tau \delta[\) and, for every \(k \in \mathbb{N}\), set \(v^k = \tau u^k\) and \(\zeta^k = \tau \eta^k\). Thus, from \(P_{V^*} = P_V\) and the linearity of \(P_V\) and \(P_W\), we deduce from (12) and (14) that (6) reduces to

\[
(\forall k \in \mathbb{N}) \begin{cases}
\zeta^{k+1} = J_{\sigma \beta^{-1}}(v^k + \sigma(\tilde{\beta}z^k - \tilde{D}^{-1}v^k)) \\
v^{k+1} = P_W z^{k+1} \\
p^{k+1} = J_A(z^k - (\tilde{L}^*v^{k+1} + \tilde{C}z^k)) \\
z^{k+1} = Mp^{k+1} \\
\tilde{z}^{k+1} = z^{k+1} + p^{k+1} - z^k.
\end{cases}
(16)
\]

On the other hand, from (7) we obtain

\[
\|\tilde{L}\|^2 \leq \|L\|^2 < \left(1 - \frac{1}{2\beta/\tau}\right) \left(\frac{1}{\sigma} - \frac{1}{2\tau \delta}\right)
(17)
\]

and \(1 \in ]0, 2\beta/\tau[\). Altogether, it follows that (16) is a particular case of the algorithm in [15, Theorem 3.1]. In addition, from part 3.1, we have that problem (13) satisfies the hypothesis of [15, Problem 3.1]. Then, from [15, Theorem 3.1(ii)] there exists \((\tilde{\xi}, \tilde{\eta}) \in \text{Fix } M \times W\) solution to (13) such that \((z^k, v^k) \to (\tilde{\xi}, \tilde{\eta})\). Therefore, \(u^k = v^k/\tau \to \tilde{\eta}/\tau =: \tilde{u}\) and, since \(P_V\) is weakly continuous, we have \(x^k = P_V z^k \to P_V \tilde{z} =: \tilde{x}\). Furthermore, from part 3.1, we conclude that \((\tilde{\xi}, \tilde{u}) \in (V \cap \text{Fix } T) \times W\) is solution to (I).

\[\square\]

\textbf{Remark 3.1} 1. From the proof of Theorem 3.1, we observe that Problem 3.1 reduces to the auxiliary problem (13), which incorporates the closed vector subspace in all the operators involved by monotonicity preserving operations as the partial inverse [13]. Algorithm 3.1 is obtained by applying a variant of the primaldual splitting [18, 24, 49] including a priori information on the solutions [15] to (13). Hence, our algorithm can be interpreted as an extension of the primaldual splitting, which exploits a priori information and closed vector subspace constraints.

2. When \(T\) is weakly continuous, we have \(Tr^k \to \tilde{x}\), where \((r^k)_{k\in\mathbb{N}}\) is defined in (6). Indeed, by the proof of [15, Theorem 3.1(ii)], the sequence \((p^k)_{k\in\mathbb{N}}\) in the algorithm (16) satisfies that \(p^k - z^k \to 0\). Then \(p^k = p^k - z^k + z^k - \tilde{z}\) and, since \(P_V\) is weakly continuous, it follows that \(r^k = P_V p^k \to P_V \tilde{z} =: \tilde{x}\). Thus, since \(\tilde{x} \in \text{Fix } T\), \(Tr^k \to T\tilde{x} = \tilde{x}\). This fact helps to obtain a faster convergence
in the context of convex optimization with affine linear constraints, as we will see in our numerical experiences. Indeed, in this case we take $T = P_C$, where $C$ represents some selection of the affine linear constraints ($T$ is weakly continuous by [8, Proposition 4.19(i)]). Therefore, $\{T_r^k\}_{k \in \mathbb{N}} \subset C$, which impose feasibility of the converging iterates to the primal solution.

In particular, if $\mathcal{H}$ is finite dimensional and $T = P_C$ for a nonempty closed convex $C \subset \mathcal{H}$, we deduce, from the fact that $P_C$ is continuous, that $T_r^k \rightarrow \hat{x}$.

3. When $V = \mathcal{H}$, we have that $V^\perp = \{0\}$ and $P_V = \text{Id}$. Thus, the algorithm (6) reduces to

\[
(\forall k \in \mathbb{N}) \begin{align*}
\eta^{k+1} &= J_{\gamma B^{-1}}(u^k + \gamma(Lx^k - D^{-1}u^k)) \\
u^{k+1} &= PW\eta^{k+1} \\
u^{k+1} &= J_{\tau A}(x^k - \tau(L^*u^{k+1} + Cx^k)) \\
x^{k+1} &= Tw^{k+1} \\
x^{k+1} &= x^{k+1} + w^{k+1} - x^k,
\end{align*}
\] (18)

which is the algorithm proposed in [15, Theorem 3.1] when the stepsizes $\tau$ and $\gamma$ are fixed.

4. When $T = \text{Id}$, $W = G$, and $B = D^{-1} = 0$, we have that for every $\lambda > 0$, $J_{\lambda B} = \text{Id}$. Hence $J_{\gamma B^{-1}} = 0$ by [8, Proposition 23.7(ii)]. Then, the algorithm (6) reduces to

\[
(\forall k \in \mathbb{N}) \begin{align*}
\tilde{z}^k &= x^k + \tau y^k - \tau P_V Cx^k \\
w^{k+1} &= J_{\tau A}\tilde{z}^k \\
x^{k+1} &= P_V w^{k+1} \\
y^{k+1} &= y^k + (x^{k+1} - w^{k+1})/\tau,
\end{align*}
\] (19)

which is the algorithm proposed in [13, Corollary 5.3] without relaxation steps ($\lambda_n \equiv 1$).

5. In the context of the convex optimization problem (5), the proposed algorithm (6) reduces to

\[
(\forall k \in \mathbb{N}) \begin{align*}
\eta^{k+1} &= \text{prox}_{\gamma G^*}(u^k + \gamma(Lx^k - \nabla e^n(u^k))) \\
u^{k+1} &= PW\eta^{k+1} \\
\tilde{z}^k &= x^k + \tau y^k - \tau P_V (L^*u^{k+1} + \nabla H(x^k)) \\
u^{k+1} &= \text{prox}_{\tau F}\tilde{z}^k \\
r^{k+1} &= P_V u^{k+1} \\
x^{k+1} &= P_V T_r^k \\
y^{k+1} &= y^k + (r^{k+1} - w^{k+1})/\tau \\
x^{k+1} &= x^{k+1} + r^{k+1} - x^k.
\end{align*}
\] (20)
In particular, when $T = \text{Id}$, $V = \mathcal{H}$, $W = \mathcal{G}$, and $\ell = \ell_{\{0\}}$, we deduce that the algorithm (20) is equivalent to

\[
(\forall k \in \mathbb{N}) \begin{cases} 
    u^{k+1} = \text{prox}_{\gamma G^*} \left( u^k + \gamma L \tilde{x}^k \right) \\
    \tilde{z}^k = x^k - \tau (L^* u^{k+1} + \nabla H(x^k)) \\
    x^{k+1} = \text{prox}_{\tau F} \tilde{z}^k \\
    x^k = 2x^{k+1} - x^k,
\end{cases}
\]  

(21)

which is a version of the algorithm proposed in [24, Algorithm 3.1] without considering summable errors or relaxation steps. If additionally $H = 0$, the method (21) reduces to [18, Algorithm 1].

4 Applications and Numerical Experiences

In this section, we illustrate the efficiency of the proposed method in three applications. First, we consider a sparse constrained convex optimization problem without including a priori information on the solution ($T = \text{Id}$), called constrained LASSO [28, 30]. The constraint is given by the kernel of a linear operator and we apply our primal-dual method exploiting the vector subspace structure of the problem. The second numerical experiment is the application of the proposed method to the arc capacity expansion problem in transport networks [20]. The problem is to find the optimal investment decision in arc capacity and network flow operation under an uncertain environment. We solve the two-stage stochastic arc capacity expansion problem over a directed graph using our primal-dual partial inverse method in which the closed vector subspace includes the non-anticipativity constraint. In our last application, we consider the finite difference approximation introduced in [1] of a second order ergodic variational MFGs with non-local couplings. This discretization preserves the variational structure of the resulting system of equations, which is solved using our primal-dual partial inverse method. In this framework, the underlying vector subspace includes the linear discrete ergodic Fokker–Planck equation appearing in the associated optimization problem.

All proposed algorithms are implemented in MATLAB 2020A and run in a computer with MacOS 11.6, 3 GHz 6-Core Intel Core i5 8GB RAM.

4.1 Constrained LASSO

We consider the following problem

\[
\min_{x \in \mathbb{R}^n} \alpha \|x\|_1 + \frac{1}{2} \|Ax - b\|_2^2.
\]  

(22)

where $\alpha > 0$, $A \in \mathbb{R}^{p \times n}$, $R \in \mathbb{R}^{m \times n}$ satisfies $\ker R^T = \{0\}$, and $b \in \mathbb{R}^p$. Note that, by setting $f = \alpha \| \cdot \|_1$, the problem in (22) can be written in at least the following

\[
\min_{x \in \mathbb{R}^n} \alpha \|x\|_1 + \frac{1}{2} \|Ax - b\|_2^2.
\]  

(22)
three equivalent manners:

\[
\begin{align*}
\text{minimize } & f(x) + \frac{1}{2} g_2(Ax) + \frac{1}{2} \tau_{|0|}(Rx), \quad \text{where } g_2 = \| \cdot - b \|_2^2; \\
\text{minimize } & f(x) + h(x), \quad \text{where } h = \frac{1}{2} \| A(\cdot) - b \|_2^2,
\end{align*}
\]

and

\[
\begin{align*}
\text{minimize } & f(x) + g_1(Ax), \quad \text{where } g_1 = \frac{1}{2} \| \cdot - b \|_2^2.
\end{align*}
\]

Observe that \( f \in \mathcal{I}_0(\mathbb{R}^n), g_1 \in \mathcal{I}_0(\mathbb{R}^p), \) and \( g_2 = 2g_1 \in \mathcal{I}_0(\mathbb{R}^p). \) Therefore, the problem in (23) satisfies the hypotheses in [49, Corollary 4.2(i)]. Thus, since [8, Proposition 24.8(i)& Theorem 14.3(ii)] yields \( \forall \gamma > 0 \) \( \text{prox}_{\gamma g^*} : x \mapsto 2(x - \gamma b)/(\gamma + 2), \) the primal-dual method proposed in [49, Corollary 4.2(i)] without relaxation steps reduces to

\[
(\forall k \in \mathbb{N}) \begin{cases} \\
\chi^{k+1} = \text{prox}_{\tau \alpha \| \cdot \|_1}(x^k - \frac{\tau}{\alpha} A^\top v_1^k - \frac{\tau}{2} R^\top v_2^k) \\
v_1^{k+1} = 2(v_1^k + \sigma_1 A \chi^k - \sigma_1 b)/(\sigma_1 + 2) \\
v_2^{k+1} = v_2^k + \sigma_2 R \chi^k.
\end{cases}
\]

where \((\chi^0, v_1^0, v_2^0) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m\) and the strictly positive constants \( \tau, \sigma_1, \) and \( \sigma_2 \) satisfy the condition \( \frac{\tau}{\alpha} \sigma_1 \| A \|_2^2 + \frac{\tau}{2} \sigma_2 \| R \|_2^2 < 1. \) On the other hand, we have that \( h : \mathbb{R}^n \to \mathbb{R} \) is a differentiable convex function, and \( \forall h : x \mapsto A^\top (Ax - b) = \| A^\top A \| - \text{Lipschitz continuous}. \) Then, by setting \( V = \ker R, \) the problem in (24) can be solved by the algorithm in [13, Proposition 6.7] detailed in (19), which reduces to

\[
(\forall k \in \mathbb{N}) \begin{cases} \\
w^{k+1} = \text{prox}_{\lambda \alpha \| \cdot \|_1}(x^k + \lambda y^k - \lambda P_{\ker R} A^\top (Ax^k - b)) \\
x^{k+1} = P_{\ker R} w^{k+1} \\
y^{k+1} = y^k + (x^{k+1} - w^{k+1})/\lambda.
\end{cases}
\]

where \( x^0 \in \ker R, y^0 \in (\ker R)^\perp, \) and \( \lambda \in ]0, 2/\| A^\top A \|]. \) Moreover, from [8, Proposition 24.8(i)& Theorem 14.3(ii)] we have \( \forall \gamma > 0 \) \( \text{prox}_{\gamma g^*} : x \mapsto (x - \gamma b)/(\gamma + 1). \) By setting \( \mathcal{H} = \mathbb{R}^n, \mathcal{G} = \mathbb{R}^p, \) and \( V = \ker R, \) the problem in (25) satisfies the hypotheses in (5), which is a particular instance of Problem 3.1. Therefore, (20) in the case \( T = \text{Id} \) reduces to

\[
(\forall k \in \mathbb{N}) \begin{cases} \\
u^{k+1} = (u^k + \gamma (Ax^k - b))/(\gamma + 1) \\
\tilde{z}^k = x^k + \tau \chi^k - \tau P_{\ker R} A^\top u^{k+1} \\
u^{k+1} = \text{prox}_{\tau \alpha \| \cdot \|_1}\tilde{z}^k \\
x^{k+1} = P_{\ker R} u^{k+1} \\
y^{k+1} = y^k + (x^{k+1} - w^{k+1})/\tau \\
\chi^{k+1} = 2x^{k+1} - x^k.
\end{cases}
\]
Table 1  Average execution time (number of iterations) with relative error tolerance $e = 10^{-6}$

| $(n, p, m)$ | PD with subspaces | FB with subspaces | PD generalized |
|------------|------------------|------------------|----------------|
| (500, 250, 25) | 0.639 (3284) | 3.909 (21,059) | 1.317 (4469) |
| (500, 250, 50) | 0.822 (3565) | 5.014 (22,145) | 1.254 (5081) |
| (500, 250, 100) | 1.289 (3523) | 7.956 (21,374) | 1.817 (5527) |
| (500, 750, 25) | 0.488 (2184) | 3.615 (16,577) | 1.012 (2991) |
| (500, 750, 50) | 0.579 (2229) | 4.197 (16,445) | 0.862 (3063) |
| (500, 750, 100) | 0.854 (2117) | 6.345 (15,853) | 1.129 (3066) |
| (1000, 500, 50) | 3.032 (8910) | 16.781 (49,199) | 4.665 (11,125) |
| (1000, 500, 100) | 5.278 (9716) | 27.937 (51,287) | 5.591 (12,615) |
| (1000, 500, 200) | 10.830 (9036) | 57.976 (48,314) | 7.283 (13,014) |
| (1000, 1500, 50) | 6.252 (4869) | 44.335 (34,553) | 7.610 (6378) |
| (1000, 1500, 100) | 7.911 (4992) | 54.217 (34,507) | 8.691 (6484) |
| (1000, 1500, 200) | 11.570 (4642) | 79.844 (32,110) | 9.882 (6169) |

where $x^0 \in \ker R, \bar{x}^0 = x^0, y^0 \in (\ker R)^\perp, u^0 \in \mathbb{R}^p$, and $(\tau, \gamma) \in ]0, +\infty[$ are such that $\tau \gamma \|A\|_2 < 1$.

Note that, since $\ker R^\top = \{0\}$, $RR^\top$ is invertible. Then, by [8, Example 29.17(iii)], we have $P_{\ker R} = \text{Id} - R^\top (RR^\top)^{-1} R$. On the other hand, by [8, Proposition 24.11 & Example 24.20], we have

$$(\forall \tau > 0) \quad \text{prox}_{\tau \|\cdot\|_1} : x \mapsto \left( \text{sign}(x_i) \max\{|x_i| - \tau, 0\}\right)_{1 \leq i \leq n},$$

where sign is 1 when the argument is positive and $-1$ if it is strictly negative.

For each method, we obtain the average execution time and the average number of iterations from 20 random instances for the matrices $A, R,$ and $b$, using $\alpha = 1$. We measure the efficiency for different values of $m, n,$ and $p$. We label the algorithm in (26) as PD generalized, algorithm in (27) as FB with subspaces, and algorithm in (28) as PD with subspaces. For every algorithm, we obtain the values of $\tau, \gamma, \lambda, \sigma_1,$ and $\sigma_2$ by discretizing the parameter set in which the algorithm converges and selecting the parameters such that the method stops in a minimum number of iterations. This procedure is repeated for every dimension of matrices and vectors. In particular, we fix $\sigma_1 = \sigma_2$ for the method in (26). The results are shown in Table 1.

We observe a significant gain in efficiency when we use PD with subspaces with respect to the other two methods. The number of iterations is reduced in $25-30\%$ with respect to PD generalized. The construction of PD with subspaces exploiting the vector subspace and primal-dual structure of the problem explains these benefits. However, the computational time used by PD with subspaces is larger than that of PD generalized when the vector subspace is smaller ($m = 200$). The presence of two projections onto ker $R$ at each iteration of PD with subspaces explains this behaviour since the matrices to be inverted are of larger dimension.
4.2 Capacity Expansion Problem in Transport Networks

In this section we aim at solving the traffic assignment problem with arc-capacity expansion at minimal cost on a network under uncertainty. Let \( \mathcal{A} \) be the set of arcs and let \( \mathcal{O} \) and \( \mathcal{D} \) be the sets of origin and destination nodes of the network, respectively. The set of routes from \( o \in \mathcal{O} \) to \( d \in \mathcal{D} \) is denoted by \( R_{od} \) and \( R := \bigcup_{(o, d) \in \mathcal{O} \times \mathcal{D}} R_{od} \) is the set of all routes. The arc-route incidence matrix \( N \in \mathbb{R}^{\mid \mathcal{A} \mid \times \mid \mathcal{R} \mid} \) is defined by \( N_{ar} = 1 \), if arc \( a \) belongs to the route \( r \), and \( N_{ar} = 0 \), otherwise.

The uncertainty is modeled by a finite set \( \Xi \) of possible scenarios. For every scenario \( \xi \in \Xi \), \( p_\xi \in [0, 1] \) is its probability of occurrence, \( h_{od, \xi} \in \mathbb{R}_+ \) is the forecasted demand from \( o \in \mathcal{O} \) to \( d \in \mathcal{D} \), \( c_{a, \xi} \in \mathbb{R}_+ \) is the corresponding capacity of the arc \( a \in \mathcal{A} \), \( t_{a, \xi} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is an increasing and \( \chi_{a, \xi} \)-Lipschitz continuous travel time function on arc \( a \in \mathcal{A} \), for some \( \chi_{a, \xi} > 0 \), and the variable \( f_{r, \xi} \in \mathbb{R}_+ \) stands for the flow in route \( r \in R \).

In the problem of this section, we consider the expansion of flow capacity at each arc in order to improve the efficiency of the network operation. We model this decision making process in a two-stage stochastic problem. The first stage reflects the investment in capacity and the second corresponds to the operation of the network in an uncertain environment.

In order to solve this problem, we take a non-anticipativity approach [20], letting our first stage decision variable depend on the scenario and imposing a non-anticipativity constraint. We denote by \( x_{a, \xi} \in \mathbb{R}_+ \) the variable of capacity expansion on arc \( a \in \mathcal{A} \) in scenario \( \xi \in \Xi \) and the non-anticipativity condition is defined by the constraint

\[
\mathcal{N} := \{ x \in \mathbb{R}^{\mid \mathcal{A} \mid \times \mid \Xi \mid} : (\forall (\xi, \xi') \in \Xi^2) \ x_\xi = x_{\xi'} \},
\]

where \( x_\xi \in \mathbb{R}^{\mid \mathcal{A} \mid} \) is the vector of capacity expansion for scenario \( \xi \in \Xi \) and we denote \( f_\xi \in \mathbb{R}^{\mid \mathcal{R} \mid} \) analogously. We restrict the capacity expansion variables by imposing, for every \( a \in \mathcal{A} \) and \( \xi \in \Xi \), \( x_{a, \xi} \in [0, M_a] \), where \( M_a > 0 \) represents the upper bound of capacity expansion on arc \( a \in \mathcal{A} \). Additionally, we model the expansion investment cost via a quadratic function defined by a symmetric positive definite matrix \( Q \in \mathbb{R}^{\mid \mathcal{A} \mid \times \mid \mathcal{A} \mid} \).

**Problem 4.1** The problem is to

\[
\begin{align*}
\text{minimize} \quad & \sum_{\xi \in \Xi} p_\xi \left[ \sum_{a \in \mathcal{A}} \int_0^{(Nf_\xi)_a} t_{a, \xi}(z)dz + \frac{1}{2} x_\xi^\top Q x_\xi \right] \\
\text{s.t.} \quad & (\forall \xi \in \Xi)(\forall a \in \mathcal{A}) \quad (Nf_\xi)_a - x_{a, \xi} \leq c_{a, \xi}, \quad (29) \\
& (\forall \xi \in \Xi)(\forall (o, d) \in \mathcal{O} \times \mathcal{D}) \quad \sum_{r \in R_{od}} f_{r, \xi} = h_{od, \xi}, \quad (30)
\end{align*}
\]

where \( D := \chi_{a \in \mathcal{A}}[0, M_a] \), and we assume the existence of solutions.
The first term of the objective function in Problem 4.1 represents the expected operational cost of the network and only depends on traffic flows. This term leads to optimality conditions defining a Wardrop equilibrium \cite{9}. The second term in the objective function is the expansion investment cost. Constraints in (29) represent that, for every arc \( a \in \mathcal{A} \), the flow cannot exceed the expanded capacity \( c_a,\xi + x_a,\xi \) at each scenario \( \xi \in \Xi \), while (30) are the demand constraints.

We solve Problem 4.1 following the structure of the problem in (5) with \( T = \text{Id} \).

We consider the following two equivalent formulations.

### 4.2.1 Primal-Dual Formulation

Note that Problem 4.1 can be equivalently written as

\[
\begin{align*}
\min_{(x, f) \in \mathbb{R}^{\mathcal{A} \times \Xi} \times \mathbb{R}^{\mathcal{R} \times \Xi}} & \quad F(x, f) + G(L(x, f)) + H(x, f), \\
\text{subject to} & \quad (\forall \xi \in \Xi) V_\xi^+ := \left\{ f \in \mathbb{R}^{\mathcal{R}} : \left( \forall (o, d) \in \mathcal{O} \times \mathcal{D} \right) \sum_{r \in \mathcal{R}_{od}} f_r = h_{od,\xi} \right\}, \\
& \quad A := (D^{\mathcal{O}} \cap \mathcal{D}) \times \left( \bigtimes_{\mathcal{X} \in \Xi} V_\xi^+ \right), \\
& \quad F := \iota_A, \\
& \quad (\forall \xi \in \Xi) \quad \Theta_\xi := \left\{ (x, u) \in \mathbb{R}^{\mathcal{A} \times \Xi} : \left( \forall a \in \mathcal{A} \right) u_a - x_a \leq c_a,\xi \right\} \\
& \quad G := \iota \times \Theta_\xi, \\
& \quad L : (x, f) \mapsto (x, Nf_\xi)_{\xi \in \Xi}, \\
& \quad H : (x, f) \mapsto \sum_{\xi \in \Xi} p_\xi \left[ \sum_{a \in \mathcal{A}} \int_0^{(Nf_\xi)_a} t_{a,\xi}(z)dz + \frac{1}{2} x_\xi^T Q x_\xi \right].
\end{align*}
\]

Observe that \( F \) and \( G \) are lower semicontinuous convex proper functions, and \( L \) is linear and bounded with \( \|L\| \leq \max\{1, \|N\|\} \). Moreover, note that since \( (t_{a,\xi})_{a \in \mathcal{A}, \xi \in \Xi} \) are increasing, \( N \) is linear, and \( Q \) is definite positive, \( H \) is a separable convex function. In addition, by defining

\[
\psi : f \mapsto (p_\xi N^T (t_{a,\xi} ((Nf_\xi)_a)))_{a \in \mathcal{A}, \xi \in \Xi},
\]

simple computations yield

\[
\nabla H : (x, f) \mapsto \left( (p_\xi Q x_\xi)_{\xi \in \Xi} \cdot \psi(f) \right),
\]

which is Lipschitz continuous with constant

\[
\beta^{-1} := \max_{\xi \in \Xi} \left( p_\xi \max \left\{ \|Q\|, \|N\|^2 \max_{a \in \mathcal{A}} x_{a,\xi} \right\} \right).
\]
Altogether, (P) is a particular instance of problem in (5) with \( V = \mathbb{R}^{|A||\Xi|} \times \mathbb{R}^{|R||\Xi|} \) and \( \ell = \ell(0) \). Therefore, this formulation satisfies the hypotheses in [24] and its algorithm, detailed in (21), reduces to

\[
(\forall k \in \mathbb{N}) \quad \begin{cases}
\tilde{x}^k = x^k - \tau (p^{k+1} + (p_k Q x^k)_{\xi \in \Xi}) \\
\tilde{f}^k = f^k - \tau (N^T u^{k+1} + \psi(f^k)) \\
x^{k+1} = P_{\mathbb{D}|\Xi \cap \mathcal{N}} \tilde{x}^k \\
f^k = P_{\mathbb{V}^+} \tilde{f}^k \\
x^{k+1} = 2x^{k+1} - x^k \\
f^{k+1} = 2f^{k+1} - f^k,
\end{cases}
\] (32)

where \((x^0, f^0) \in \mathbb{R}^{|A||\Xi|} \times \mathbb{R}^{|R||\Xi|}\), \((x^0, f^0) = (x^0, f^0), (p^0, u^0) \in \mathbb{R}^{|A||\Xi|} \times \mathbb{R}^{|A||\Xi|} \), and \( \tau \in [0, 2\beta] \) and \( \gamma \in [0, +\infty] \) are such that

\[
\tau \gamma \max \left\{ 1, \|N\|^2 \right\} < 1 - \frac{\tau}{2\beta}.
\] (33)

Remark 4.1 The projections \( (P_{\theta_{\xi}})_{\xi \in \Xi}, P_{\mathbb{D}|\Xi \cap \mathcal{N}}, \) and \( (P_{\mathbb{V}^+})_{\xi \in \Xi} \) appearing in (32), can be computed efficiently, as we detail below.

(i) Let \( \xi \in \Xi \) and let \((x, u) \in \mathbb{R}^{|A|} \times \mathbb{R}^{|A|}\). We deduce from [8, Proposition 29.3 & Example 29.20] that \( P_{\theta_{\xi}}(x, u) = (P_{a, \xi}(x, u))_{a \in \mathcal{A}} \), where

\[
(\forall a \in A) \quad P_{a, \xi}(x, u) = \begin{cases}
\left(\frac{x_a + u_a - c_{a, \xi}}{2}, \frac{x_a + u_a + c_{a, \xi}}{2}\right), & \text{if } u_a - x_a - c_{a, \xi} > 0; \\
(x_a, u_a), & \text{otherwise}.
\end{cases}
\]

(ii) Let \( \xi \in \Xi \) and note that

\[
V^+_\xi = \bigotimes_{(o, d) \in \mathcal{O} \times \mathcal{D}} V_{o d, \xi},
\]

where, for every \((o, d) \in \mathcal{O} \times \mathcal{D}\), \( V_{o d, \xi} := \{ f \in \mathbb{R}^{|R_{o d}|} : \sum_{r \in R_{o d}} f_r = h_{o d, \xi}\} \). It follows from [8, Proposition 29.3] that

\[
P_{\mathbb{V}^+_{\xi}}: f = (f_{o d})_{o \in \mathcal{O}} \mapsto (P_{V_{o d, \xi}} f_{o d})_{o \in \mathcal{O}} \cdot d \in \mathcal{D}.
\]

For every \((o, d) \in \mathcal{O} \times \mathcal{D}\), the projection \( P_{V_{o d, \xi}} \) can be computed efficiently by using the quasi-Newton algorithm developed in [23].
(iii) Note that
\[ D^{||\mathcal{Y}||} \cap \mathcal{N} = \bigtimes_{a \in A} C_a, \]
where, for every \( a \in A \), \( C_a = \{ y \in [0, M_a]^{||\mathcal{Y}||} : (\forall (\xi, \xi') \in \mathcal{Y}^2) \ y_\xi = y_{\xi'} \} \). It follows from [8, Proposition 29.3] that \( P_{D^{||\mathcal{Y}||} \cap \mathcal{N}} : x = (x_a)_{a \in A} \mapsto (P_{C_a} x_a)_{a \in A} \) and
\[
(\forall a \in A) \quad P_{C_a} : y \mapsto \text{mid}(0, \bar{y}, M_a) 1,
\] (34)
where \( 1 = (1, \ldots, 1)^T \in \mathbb{R}^{||\mathcal{Y}||}, \bar{y} = \frac{1}{||\mathcal{Y}||} \sum_{\xi \in \mathcal{Y}} y_\xi, \) and \( \text{mid}(a, b, c) \) is the middle value among \( a, b, \) and \( c \). In order to prove (34), let \( a \in A, y \in \mathbb{R}^{||\mathcal{Y}||}, \) and set \( \bar{\theta} := \text{mid}(0, \bar{y}, M_a) \in [0, M_a] \). For every \( x \in C_a \), we have \( x = \eta 1 \) for some \( \eta \in [0, M_a] \) and
\[
(y - \hat{\theta} 1)^T (x - \hat{\theta} 1) = (\eta - \hat{\theta}) ||\mathcal{Y}|| (\bar{y} - \hat{\theta})
\]
\[
= \begin{cases} 
\eta ||\mathcal{Y}|| \bar{y}, & \text{if } \bar{y} < 0; \\
0, & \text{if } \bar{y} \in [0, M_a]; \\
(\eta - M_a) ||\mathcal{Y}|| (\bar{y} - M_a), & \text{if } \bar{y} > M_a,
\end{cases}
\]
which yields \( (y - \hat{\theta} 1)^T (x - \hat{\theta} 1) \leq 0 \) and obtain from [8, Theorem 3.16] that \( P_{C_a} y = \hat{\theta} 1 \).

### 4.2.2 Vector Subspace Primal-Dual Formulation

For the second equivalent formulation of Problem 4.1 consider the closed vector subspace
\[ S = \left\{ f \in \mathbb{R}^{||\mathcal{Y}||} : (\forall \xi \in \mathcal{Y}) (\forall (o, d) \in \mathcal{O} \times \mathcal{D}) \sum_{r \in R_{od}} f_{r,\xi} = 0 \right\} \]
and let \( \tilde{f}^0 \) defined by
\[
(\forall \xi \in \mathcal{Y})(\forall (o, d) \in \mathcal{O} \times \mathcal{D})(\forall r \in R_{od}) \quad \tilde{f}^0_{r,\xi} = h_{od,\xi}/|R_{od}|,
\]
which satisfies (30). Then, under the notation in (31), Problem 4.1 is equivalent to
\[
\begin{align*}
\text{minimize} & \quad \hat{F}(x, f + \tilde{f}^0) + G(L(x, f + \tilde{f}^0)) + H(x, f + \tilde{f}^0), \\
\text{subject to} & \quad (x, f) \in \mathcal{N} \times S
\end{align*}
\] (PV)
where \( \hat{F} = I_{D^{||\mathcal{Y}||} \times \mathbb{R}^{||\mathcal{Y}||}} \). Note that, the difference with respect to (P) is that in (PV) we propose a vector subspace splitting on function \( F \) defined in (31).
In addition, observe that $\tilde{F}(\cdot + (0, \tilde{f}^0))$ and $G(\cdot + L(0, \tilde{f}^0))$ are lower semi-continuous, convex, and proper, and $H(\cdot + (0, \tilde{f}^0))$ is convex differentiable with $\beta^{-1}$-Lipschitz gradient. Thus, $(P_V)$ satisfies the hypotheses of problem (5) with $V = \mathcal{N} \times \mathcal{S}$ and $\ell = \iota_{(0)}$. Hence, by using [8, Proposition 29.1(i)], the algorithm in (20) with $T = \text{Id}$ reduces to

\[
\begin{align*}
\tilde{p}^k &= p^k + \gamma x^k \\
\tilde{u}^k &= u^k + \gamma N(f^k + \tilde{f}^0) \\
(\forall \xi \in \Xi) \quad (p^{k+1}_\xi, u^{k+1}_\xi) &= (\tilde{p}^{k}_\xi, \tilde{u}^{k}_\xi) - \gamma P_{\Theta}[\psi]^{-1}(\tilde{p}^{k}_\xi, \tilde{u}^{k}_\xi) \\
\hat{x}^k &= x^k + \tau y^k - \tau P_N(p^{k+1} + (p_\xi Q_{\xi}^{k})_{\xi \in \Xi}) \\
f^k &= f^k + \tau g^k - \tau P_S(N^T u^{k+1} + \psi(f^k + \tilde{f}^0)) \\
z^{k+1} &= P_P[\gamma \hat{x}^{k}] \\
\ell^{k+1} &= P_P[(\tilde{f}^k + \tilde{f}^0) - f^0] \\
x^{k+1} &= P_N z^{k+1} \\
f^{k+1} &= P_S \ell^{k+1} \\
y^{k+1} &= y^k + (x^{k+1} - z^{k+1})/\tau \\
g^{k+1} &= g^k + (f^{k+1} - \ell^{k+1})/\tau \\
\hat{x}^{k+1} &= 2x^{k+1} - x^k \\
f^{k+1} &= 2f^{k+1} - f^k,
\end{align*}
\]

where $(x^0, f^0) = (\bar{x}^0, \bar{f}^0) \in \mathcal{N} \times \mathcal{S}$, $(p^0, u^0, f^0) \in \mathbb{R}^{\mid A\mid \mid \Xi \mid} \times \mathbb{R}^{\mid A\mid \mid \Xi \mid}$, $(y^0, g^0) \in \mathcal{N}^{\perp} \times \mathcal{S}^{\perp}$, and $\tau \in [0, 2\beta[ \text{ and } \gamma \in ]0, +\infty[ \text{ satisfy (33).}$

**Remark 4.2** The projections appearing in (35) are explicit. Indeed, for every $x \in \mathbb{R}^{\mid A\mid \mid \Xi \mid}$ and $\xi \in \Xi$, we have that $(P_N x)^{\xi} = 1/|\Xi| \sum_{\xi' \in \Xi} x^{\xi'}$ and $(P_P[\gamma \hat{x}])^{\xi} = (\text{mid}(0, x_d, \xi, M_d))_{d \in A}$. Moreover, for every $f \in \mathbb{R}^{\mid R\mid \mid \Xi \mid}$ we have, for every $(o, d) \in \mathcal{O} \times \mathcal{D}$ and $r \in R_{od}$, $(P_S f)^{r} = (fr, \xi - 1/|R_{od}| \sum_{r' \in R_{od}} fr', \xi)^{\xi} \in \Xi$ and $(P_P[\gamma \hat{x}])^{\xi} = (\max\{0, fr, \xi\})^{\xi} \in \Xi$.

### 4.2.3 Numerical Experiences

In this subsection we compare the efficiency of the algorithms in (32) and (35) to solve the arc capacity expansion problem. We consider two networks used in [20]. Network 1, represented in Fig. 1, has 7 arcs and 6 paths and Network 2, in Fig. 2, has 19 arcs and 25 paths.

In our numerical experiences we set $p_\xi \equiv 1/|\Xi|$, $(c_\xi)_{\xi \in \Xi}$ as a sample of the random variable $100 \cdot b + d \cdot \beta(2, 2)$, where

\[ \beta(2, 2) \]
Fig. 1 Network 1 [20]

Fig. 2 Network 2 [20]

\[ b = \begin{cases} 
(1, 1, 2, 2, 1, 1, 1) & \text{in Network 1} \\
(10, 4.4, 1.4, 10, 3, 4.4, 10, 2, 2, 4, 7, 7, 7, 4, 3.5, 2.2, 4.4, 7) & \text{in Network 2}
\end{cases} \]

and

\[ d = \begin{cases} 
(15, 15, 30, 30, 15, 15, 15) & \text{in Network 1} \\
(15, 6.6, 2.1, 15, 4.5, 6.6, 15, 3, 3, 6, 10.5, 10.5, 10.5, 6, 5.25, 3.3, 6.6, 10.5) & \text{in Network 2}
\end{cases} \]
Table 2  Average execution time (number of iterations) with relative error tolerance $e = 10^{-10}$

| Network 1 | $|\mathcal{S}| = 1$ | $|\mathcal{S}| = 3$ | $|\mathcal{S}| = 5$ | $|\mathcal{S}| = 10$ |
|-----------|-------------------|-------------------|-------------------|-------------------|
| Algorithm (32) | 0.082 (1143) | 0.731 (3217) | 1.363 (4199) | 4.388 (5698) |
| Algorithm (35) | 0.075 (1160) | 0.607 (3284) | 1.098 (4294) | 3.485 (5804) |
| % improvement of time | 8.54% | 16.96% | 19.44% | 20.58% |

| Network 2 | $|\mathcal{S}| = 1$ | $|\mathcal{S}| = 3$ | $|\mathcal{S}| = 5$ | $|\mathcal{S}| = 10$ |
|-----------|-------------------|-------------------|-------------------|-------------------|
| Algorithm (32) | 0.864 (4801) | 10.195 (27,285) | 16.166 (27,660) | 45.327 (39,790) |
| Algorithm (35) | 0.637 (4816) | 7.627 (28,147) | 12.069 (28,885) | 33.204 (40,848) |
| % improvement of time | 26.27% | 25.19% | 25.34% | 26.75% |

and $(h_\xi)_{\xi \in \mathcal{E}}$ as a sample of the random variable

$$h = \begin{cases} (h_{1,4}, h_{1,5}) &\sim (150, 180) + (120, 96) \cdot \text{beta}(5, 1) \quad \text{in Network 1} \\ (h_{1,2}, h_{1,3}, h_{4,2}, h_{4,3}) &\sim (300, 700, 500, 350) \\
 &+ (120, 120, 120, 120) \cdot \text{beta}(50, 10) \quad \text{in Network 2.} \end{cases}$$

We set the capacity limits $(M_a)_{a \in \mathcal{A}} = \theta d$, where $\theta = 40$ in Network 1 and $\theta = 200$ in Network 2. We also set $Q = \text{Id} \in \mathbb{R}^{||\mathcal{A}|| \times ||\mathcal{A}||}$ and, for every $a \in \mathcal{A}$ and $\xi \in \mathcal{E}$, the travel time function is $t_{a,\xi} : u \mapsto \eta_a (1 + 0.15 u/c_{a,\xi})$, where

$$\eta = \begin{cases} (6, 4, 3, 5, 6, 4, 1) &\quad \text{in Network 1} \\ (7, 9, 9, 12, 3, 9, 5, 13, 5, 9, 9, 10, 9, 6, 9, 8, 7, 14, 11) &\quad \text{in Network 2.} \end{cases}$$

We implement the algorithms in (32) and (35) for different values of $|\mathcal{S}|$. We obtain the following results by considering 20 random realizations of $(c_\xi)_{\xi \in \mathcal{E}}$ and $(h_\xi)_{\xi \in \mathcal{E}}$ (Table 2).

Note that the algorithm with vector subspaces in (35) is more time efficient compared to the classical primal-dual algorithm in (32). Indeed, the percentage of improvement reaches up to 26.75%, for the larger dimensional case of Network 2 and $\mathcal{S} = 10$. It is worth to notice that the number of iterations is lower in average for the approach without vector subspaces, but it is explained by the subroutines that compute the projections onto $(V_\xi^+)_{\xi \in \mathcal{E}}$, which lead to a larger computational time by iteration. In order to show the difference of both algorithms, in Figs. 3 and 4 we illustrate the relative error depending on the execution time for the best and the worst instance respect to convergence time.
4.3 Stationary MFG with Non-local Couplings

Let us consider the following second order ergodic MFG system \([32, 33]\)

\[-\nu \Delta u + H(x, \nabla u) + \lambda = \phi(x, m) \quad \text{in } \mathbb{T}^d,\]

\[-\nu \Delta m - \text{div}(\partial_p H(x, \nabla u)m) = 0 \quad \text{in } \mathbb{T}^d,\]

\[\int_{\mathbb{T}^d} u(x)dx = 0, \quad m \geq 0, \quad \int_{\mathbb{T}^d} m(x)dx = 1.\] (36)

In the system above, \(\mathbb{T}^d\) denotes the \(d\)-dimensional torus and the unknowns are \(u : \mathbb{T}^d \to \mathbb{R}, m : \mathbb{T}^d \to \mathbb{R}, \) and \(\lambda \in \mathbb{R}.\) The function \(H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}\) is the so-called Hamiltonian function, \(\phi : \mathbb{T}^d \times L^1(\mathbb{T}^d) \to \mathbb{R},\) and \(\nu \in [0, +\infty[.\) We assume that, for all \(x \in \mathbb{T}^d,\) the function \(p \mapsto H(x, p)\) is convex. Existence and uniqueness results for solutions \((u, m)\) to system (36) have been shown, under suitable assumptions, in \([7, 32, 33, 39].\) The function \(\phi\) is called coupling or interaction term and system (36) is called a MFG system with non-local coupling. In contrast, the local coupling case corresponds to (36) when \(\phi : \mathbb{T}^d \times [0, +\infty[ \to \mathbb{R}\) and the right-hand-side of the first equation is replaced by \(\phi(x, m(x)).\)
For the sake of simplicity, we will assume that

\[ \mathcal{H} : (x, p) \mapsto \frac{|p|^2}{2} \quad \text{and} \quad \phi : (x, m) \mapsto \int_{\mathbb{T}^d} k(x, y)m(y)dy + k_0(x), \tag{37} \]

where \( k_0 : \mathbb{T}^d \to \mathbb{R} \) is a Lipschitz function and \( k : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R} \) is a smooth function satisfying that, for every finite selection of points \((x_r)_{1 \leq r \leq \ell}\) in \( \mathbb{T}^d \) and for every \( m : \mathbb{T}^d \to \mathbb{R} \), we have

\[ \sum_{r=1}^{\ell} \sum_{s=1}^{\ell} k(x_r, x_s) m(x_r) m(x_s) \geq 0 \tag{38} \]

and \( k(x_r, x_s) = k(x_s, x_r) \) for every \( r \) and \( s \) in \( \{1, \ldots, \ell\} \). The condition (38) is known as the Positive Definite Symmetric (PDS) property \([40, \text{Sect. 6.2}] \) and implies that \( k \) satisfies the monotonicity condition

\[ \int_{\mathbb{T}^d \times \mathbb{T}^d} k(x, y)(m_1(x) - m_2(x))(m_1(y) - m_2(y))dx dy \geq 0, \tag{39} \]

for all \( m_1 \) and \( m_2 \) in \( L^1(\mathbb{T}^d) \) such that \( \int_{\mathbb{T}^d} m_1(x)dx = \int_{\mathbb{T}^d} m_2(x)dx = 1 \). Under the previous assumptions, the results in \([32, 33]\) ensure the existence of an unique classical solution \((u, m)\) to (36). Moreover, it follows from (37) and (38) that (36) admits a variational formulation, i.e., it corresponds to the optimality condition of an optimization problem \([33]\).

The numerical approximation of solutions to variational MFGs with local couplings has been addressed in \([3, 10, 11, 16, 31]\). We focus in the case when \( \phi(x, \cdot) \) is non-local, in which the methods in the aforementioned references are costly to implement. In this context, \([35]\) solves (36) via approximated solutions to the analogous time-dependent MFG system as the time horizon goes to infinity. In contrast, we propose algorithms following a direct approach based on the variational formulation of (36), in which the vector subspace structure arises.

### 4.3.1 Finite Difference Approximation

We now introduce a discretization of system (36) in the two-dimensional case, following \([1]\). Let \( N \in \mathbb{N} \), set \( h = 1/N \), set \( \mathcal{I}_N = \{0, \ldots, N - 1\} \), and consider the uniform grid \( \mathbb{T}^2_h = \{ (hi, hj) \mid (i, j) \in \mathcal{I}_N^2 \} \) on \( \mathbb{T}^2 \). Let us define \( \mathcal{M}_h \) as the set of real valued functions defined on \( \mathbb{T}^2_h \), \( \mathcal{W}_h = \mathcal{M}_h^4 \), \( 1 \in \mathcal{M}_h \) as the constant function equals to 1, and set \( \mathcal{Y}_h = \{ z \in \mathcal{M}_h \mid \sum_{i,j \in \mathcal{I}_N^2} z_{i,j} = 0 \} \), where we denote, for every \((i, j) \in \mathcal{I}_N^2 \) and \( z \in \mathcal{M}_h \), \( z_{i,j} = z(x_{i,j}) \). We define \( D_1 : \mathcal{M}_h \to \mathcal{Y}_h \), \( D_2 : \mathcal{M}_h \to \mathcal{Y}_h \),
$D_h: \mathcal{M}_h \to \mathcal{Y}_h^4$, $\Delta_h: \mathcal{M}_h \to \mathcal{Y}_h$, and $\text{div}_h: \mathcal{W}_h \to \mathcal{Y}_h$ by

$$(D_1 z)_{i,j} = \frac{z_{i+1,j} - z_{i,j}}{h}, \quad (D_2 z)_{i,j} = \frac{z_{i,j+1} - z_{i,j}}{h},$$

$$[D_h z]_{i,j} = \frac{((D_1 z)_{i,j}, (D_1 z)_{i-1,j}, (D_2 z)_{i,j}, (D_2 z)_{i,j-1}),}{h^2}$$

$$(\Delta_h z)_{i,j} = \frac{z_{i-1,j} + z_{i+1,j} + z_{i,j-1} + z_{i,j+1} - 4z_{i,j}}{h^2},$$

$$(\text{div}_h(w))_{i,j} = (D_1 w^1)_{i-1,j} + (D_1 w^2)_{i,j} + (D_2 w^3)_{i,j-1} + (D_2 w^4)_{i,j}, \quad (40)$$

where previous definitions hold for every $i$ and $j$ in $\mathcal{I}_N$, $z \in \mathcal{M}_h$, $w \in \mathcal{W}_h$, and the sums between the indexes are taken modulo $N$. We also define $K_h: \mathcal{M}_h \to \mathcal{M}_h$ and $K_0 \in \mathcal{M}_h$ by

$$(\forall m \in \mathcal{M}_h)(\forall (i,j) \in \mathcal{I}_N^2) \begin{cases} (K_h m)_{i,j} = h^2 \sum_{i',j'} k(x_{i,j}, x_{i',j'}) m_{i',j'}, \\ (K_0)_{i,j} = k_0(x_{i,j}). \end{cases} \quad (41)$$

Observe that condition in (38) implies that $K_h$ is positive semidefinite. Let us consider the cone $C = [0, +\infty[ \times ]-\infty, 0] \times [0, +\infty[ \times ]-\infty, 0]$. The orthogonal projection onto $C^{N^2}$ is given by

$$(\forall w \in \mathcal{W}_h)(\forall (i,j) \in \mathcal{I}_N^2) \quad (P_{C^{N^2}}(w))_{i,j} = ([w^1_{i,j}]_+, [w^2_{i,j}]_-, [w^3_{i,j}]_+, [w^4_{i,j}]_-), \quad (42)$$

where, for every $\xi \in \mathbb{R}, [\xi]_+ = \max\{0, \xi\}$ and $[\xi]_- = \min\{0, \xi\}$. The finite difference scheme proposed in [1] to approximate (36) is the following: for every $i$ and $j$ in $\mathcal{I}_N$,
\[-\nu(\Delta h u)_{i,j} + \frac{1}{2} |PC( - [D_h u]_{i,j})|^2 + \lambda = (K_h m)_{i,j} + (K_0)_{i,j}, \]
\[-\nu(\Delta h m)_{i,j} + (\text{div}_h (mP_C^2( - [D_h u])))_{i,j} = 0, \]
\[m_{i,j} \geq 0, \quad h^2 \sum_{i,j} m_{i,j} = 1, \quad \sum_{i,j} u_{i,j} = 0, \quad \text{(43)}\]

where \((m, u) \in \mathcal{M}_h^2\) and \(\lambda \in \mathbb{R}\) are the unknowns.

In order to obtain a variational interpretation of (43), note that \((\mathcal{M}_h, \langle \cdot | \cdot \rangle)\), \((\mathcal{W}_h, \langle \cdot | \cdot \rangle_{\mathcal{W}_h})\), and \((\mathcal{Y}_h, \langle \cdot | \cdot \rangle)\) are Hilbert spaces, where
\[
\langle \cdot | \cdot \rangle : (m_1, m_2) \mapsto \sum_{i,j=0}^{N-1} (m_1)_{i,j} (m_2)_{i,j},
\]
\[
\langle \cdot | \cdot \rangle_{\mathcal{W}_h} : (w_1, w_2) \mapsto \sum_{\ell=1}^{4} \langle w_1^\ell | w_2^\ell \rangle.
\]

The adjoint operators \((-\Delta_h)^* : \mathcal{Y}_h \to \mathcal{M}_h\) and \((\text{div}_h)^* : \mathcal{Y}_h \to \mathcal{W}_h\) are given by
\[
((i, j) \in \mathcal{T}_N^2) \quad ((-\Delta_h)^* u)_{i,j} = -(\Delta_h u)_{i,j}, \quad ((\text{div}_h)^* u)_{i,j} = -[D_h u]_{i,j}. \quad \text{(44)}
\]

**Remark 4.1** From the definition of \([D_h u]\) and the identity
\[
\sum_{i,j=0}^{N-1} u_{i,j} (-\Delta_h u)_{i,j} = \sum_{i,j=0}^{N-1} \left[ (D_1 u)_{i,j}^2 + (D_2 u)_{i,j}^2 \right],
\]
we have that both, \((\text{div}_h)^*\) and \((-\Delta_h)^*\) are injective operators. Thus, we also have that both, \(\text{div}_h\) and \(-\Delta_h\), are surjective operators.

### 4.3.2 Variational Formulation, Existence, and Uniqueness

Consider the function \(b : \mathbb{R} \times \mathbb{R}^4 \to ]-\infty, +\infty]\) defined by
\[
b : (\eta, \omega) \mapsto \begin{cases} 
\frac{|\omega|^2}{2\eta}, & \text{if } \eta > 0 \text{ and } \omega \in C; \\
0, & \text{if } (\eta, \omega) = (0, 0) \text{ and } \omega \in C; \\
+\infty, & \text{otherwise.}
\end{cases} \quad \text{(45)}
\]

Define the functions \(B_h : \mathcal{M}_h \times \mathcal{W}_h \to ]-\infty, +\infty]\) and \(\Phi_h : \mathcal{M}_h \to \mathbb{R}\) by
\[
B_h : (m, w) \mapsto \sum_{i,j=0}^{N-1} b(m_{i,j}, w_{i,j})
\]
\[
\Phi_h : m \mapsto \frac{1}{2} \langle m | K_h m \rangle + \langle K_0 | m \rangle.
\]
where $K_0$ and $K_h$ are defined in (41). Note that, since $k$ is a PDS kernel, the function $\Phi_h$ is convex. We consider now the optimization problem

$$\min_{(m, w) \in \mathcal{M}_h \times \mathcal{V}_h} B_h(m, w) + \Phi_h(m)$$

$$-v\Delta_h m + \text{div}_h w = 0,$$

$$h^2 \sum_{i,j} m_{i,j} = 1. \quad (46)$$

We first provide existence and uniqueness of the solution to (46) and to (43), by assuming $v > 0$ and without any strict convexity assumption. This result is interesting in its own right and the proof of the uniqueness does not follow the standard Lasry-Lions monotonicity argument [33, Theorem 2.4] used in [1, Proposition 3] for the finite difference discretization.

**Proposition 4.1** Let $v > 0$. Then, there exists a unique solution $(\hat{m}, \hat{u}, \hat{\lambda})$ to system (43). Moreover, $\hat{m}$ is strictly positive, and $(\hat{m}, \hat{m} P_C(-[D_h\hat{u}]))$ is the unique solution to (46).

**Proof** Denote by $\mathcal{S}$ the set of solutions to (46). The proofs that $\mathcal{S} \neq \emptyset$ and that if $(\hat{m}, \hat{w}) \in \mathcal{S}$ then $\hat{m}$ is strictly positive, follow exactly the same arguments than those in [16, Theorem 2.1] and in [16, Corollary 2.1], respectively. Fix $(\hat{m}, \hat{w}) \in \mathcal{S}$ and denote by $\Lambda(\hat{m}, \hat{w})$ the set of Lagrange multipliers at $(\hat{m}, \hat{w})$, i.e., the set of $(u, \lambda) \in \mathcal{Y}_h \times \mathbb{R}$ such that

$$v(-\Delta_h)^*u + \lambda 1 - \nabla_m B_h(\hat{m}, \hat{w}) = \nabla \Phi_h(\hat{m}),$$

$$(\text{div}_h)^*u = \nabla_u B_h(\hat{m}, \hat{w}), \quad (47)$$

where we observe that $B_h(\hat{m}, \cdot)$ and $B_h(\cdot, \hat{w})$ are differentiable since $\hat{m} > 0$.

The constraints of problem (46) being affine, it follows from [16, Lemma 2.2] and [8, Fact 15.25(i)] that $\Lambda(\hat{m}, \hat{w}) \neq \emptyset$. Moreover, we deduce from Remark 4.1 that, for every $u \in \mathcal{Y}_h$ and $\lambda \in \mathbb{R}$ such that $(-\Delta_h)^*(u) + \lambda 1 = 0$, we have

$$0 = \langle u \mid (-\Delta_h)^*(u) + \lambda 1 \rangle = \sum_{i,j=0}^{N-1} \left[(D_1 u)^2_{i,j} + (D_2 u)^2_{i,j}\right],$$

and hence $u = 0$ and $\lambda = 0$. Therefore, $\mathcal{Y}_h \times \mathbb{R} \ni (u, \lambda) \mapsto v(-\Delta_h)^*(u) + \lambda 1 \in \mathcal{M}_h$ is injective and, from the first equation of (47), we obtain that $\Lambda(\hat{m}, \hat{w})$ is a singleton, say $\Lambda(\hat{m}, \hat{w}) = \{(\hat{u}, \hat{\lambda})\}$. Hence, the convexity of the problem in (46), [8, Theorem 19.1], and [16, Lemma 2.2] imply that

$$(\forall (m', w') \in \mathcal{S}) \quad \Lambda(m', w') = \Lambda(\hat{m}, \hat{w}) = \{(\hat{u}, \hat{\lambda})\}. \quad (49)$$

In addition, arguing as in the proof of [16, Theorem 2.1(ii)], since $(\hat{u}, \hat{\lambda})$ is a solution to (47), we obtain that $(\hat{m}, \hat{u}, \hat{\lambda})$ solves (43). Conversely, if $(m', u', \lambda')$ solves (43),
by setting \( w' = m' P_{C_{N^2}}( - \lfloor D_h u' \rfloor ) \), we deduce that \( (m', w', u', \lambda') \) solves (47) and \( (m', w') \) satisfies the constraints in (46). Since the latter is a convex optimization problem, we obtain from [8, Theorem 19.1] and (49) that \( (m', w') \in S \) and \( (u', \lambda') \in \Lambda(m', w') = \{ (\hat{u}, \hat{\lambda}) \} \). Altogether, we have proved the uniqueness of \( (\hat{u}, \hat{\lambda}) \) in (43). It only remains to prove the uniqueness of \( \hat{m} \), since it implies the uniqueness of \( \hat{w} \). For this purpose, define the linear operator \( E : M_h \to M_h \) as

\[
(\forall (i, j) \in I_N^2) \quad (Em)_{i,j} = -v(\Delta_h m)_{i,j} + \left( \text{div}_h \left( m P_{C_{N^2}}(- [D_h \hat{u}]) \right) \right)_{i,j}.
\]

Note that the second equation in (43) with \( u = \hat{u} \) is equivalent to \( m \in \ker E \). Then, since [8, Fact 2.25] yields \( M_h = \text{ran}(E^*) \oplus \ker(E) \), the rank-nullity theorem implies

\[
\dim(\text{ran}(E^*)) + \dim(\ker(E)) = \dim(\ker(E^*)) + \dim(\text{ran}(E^*))
\]

and, hence,

\[
\dim(\ker(E)) = \dim(\ker(E^*)�)
\]

We claim that

\[
\ker(E^*) = \{ \alpha 1 \mid \alpha \in \mathbb{R} \}, \tag{52}
\]

which implies that there exists \( \hat{z} \in M_h \) such that \( \ker(E) = \{ \alpha \hat{z} \mid \alpha \in \mathbb{R} \} \) in view of (51). Thus, there exists \( \hat{\alpha} \in \mathbb{R} \) such that \( \hat{m} = \hat{\alpha} \hat{z} \) and, since \( h^2 \sum_{i,j} \hat{m}_{i,j} = 1 \), we deduce \( \sum_{i,j} \hat{z}_{i,j} \neq 0 \). Now, if \( \hat{m} \in \ker(E) \) and \( h^2 \sum_{i,j} \hat{m}_{i,j} = 0 \), there exists \( \hat{\alpha} \in \mathbb{R} \) such that \( \hat{m} = \hat{\alpha} \hat{z} \). Hence, \( 0 = \sum_{i,j} (\hat{m}_{i,j} - \hat{m}_{i,j}) = (\hat{\alpha} - \hat{\alpha}) \sum_{i,j} \hat{z}_{i,j} \), which yields \( \hat{\alpha} = \hat{\alpha} \), implying the uniqueness of \( \hat{m} \).

It remains to prove (52). Note that it follows from (50) and (44) that

\[
(\forall (i, j) \in I_N^2) \quad (E^*z)_{i,j} = -v(\Delta_h z)_{i,j} - \langle P_{C_{N^2}}(- [D_h \hat{u}]) \rangle_{i,j} \mid [D_h z]_{i,j}, \tag{53}
\]

and, therefore, we deduce that \( \{ \alpha 1 \mid \alpha \in \mathbb{R} \} \subset \ker(E^*) \). Conversely, suppose that there exists a nonconstant \( z \in M_h \) such that \( E^*z = 0 \), which, from (53) and (40), is equivalent to

\[
(\forall (i, j) \in I_N^2) \quad 0 = -v(\Delta_h z)_{i,j} - \langle P_{C_{N^2}}(- [D_h \hat{u}]) \rangle_{i,j} \mid [D_h z]_{i,j} \tag{54}
\]

For every \( x_{i,j} \in \mathbb{T}_h^2 \), set \( N(x_{i,j}) = \{ x_{i-1,j}, x_{i+1,j}, x_{i,j-1}, x_{i,j+1} \} \). Since \( \mathbb{T}_h^2 \) is a finite set, there exist \( x_{i_0,j_0} \in \mathbb{T}_h^2 \), \( n \in \mathbb{N} \setminus \{ 0 \} \), and \( \{ x_{i_k,j_k} \}_{k=1}^n \subset \mathbb{T}_h^2 \) such that for all \( k = 0, \ldots, n - 1 \) we have \( x_{i_{k+1},j_{k+1}} \in N(x_{i_k,j_k}) \), \( z(x_{i_k,j_k}) < z(x_{i_{k+1},j_{k+1}}) \), and \( z(x_{i_n,j_n}) \geq z(x) \) for all \( x \in N(x_{i_n,j_n}) \). Hence, \( (D_1 z)_{i_n,j_n} \leq 0, (D_2 z)_{i_n,j_n} \leq 0, (D_1 z)_{i_{n-1},j_n} \geq 0, \) and \( (D_2 z)_{i_n,j_{n-1}} \geq 0 \), and one of previous inequalities is strict.
Altogether, since (54) in \((i_n, j_n)\) can be written as

\[
(D_1 z)_{i_n,j_n}(v - h[(D_1 \hat{u})_{i_n,j_n}] - ) + (D_2 z)_{i_n,j_n}(v - h[(D_2 \hat{u})_{i_n,j_n}] - )
= (D_1 z)_{i_{n-1},j_n}(v + h[(D_1 \hat{u})_{i_{n-1},j_n}] + ) + (D_2 z)_{i_n,j_{n-1}}(v + h[(D_2 \hat{u})_{i_n,j_{n-1}] + },
\]
we obtain a contradiction, and the proof is complete. \(\square\)

**Remark 4.2** Note that Proposition 4.1 also holds for every convex differentiable non-local coupling \(\Phi_h\).

### 4.3.3 Algorithms

Now we focus on numerical approaches to solve (46) for several formulations of the problem. We start with numerical approaches in [13, 24] and we compare their efficiency with the vector subspace technique introduced in this paper.

Note that (46) is equivalent to

\[
\min_{(m, w) \in \mathcal{M}_h \times \mathcal{W}_h} F(m, w) + G_1(L_1(m, w)) + H(m, w), \tag{55}
\]

where

\[
\begin{align*}
F &: (m, w) \mapsto B_h(m, w) + (K_0 | m) \\
G_1 &: \iota_{(0,1)} \\
L_1 &: (m, w) \mapsto (-v \Delta_h m + \text{div}_h(w), h^2(1 | m)) \\
H &: (m, w) \mapsto \frac{1}{2}(m | K_h m).
\end{align*}
\]

Observe that

\[
\nabla H : (m, w) \mapsto (K_h m, 0) \tag{57}
\]
is \(\|K_h\|\)-Lipschitz. Then the algorithm proposed in [15], detailed in (18), with \(T = P_S\) solves (55), which reduces to, for every \(k \in \mathbb{N}\),

\[
\begin{align*}
(p^{k+1}, v^{k+1}) &= (p^k + \gamma (-v \Delta_h \bar{m}^k + \text{div}_h(\bar{m}^k)), v^k + \gamma h^2(1 | \bar{m}^k) - \gamma) \\
(n^{k+1}, z^{k+1}) &= (n^k - \tau (-v \Delta_h)^* p^{k+1} + h^2 v^{k+1} 1 + K_h m^k), w^k - \tau (\text{div}_h)^* p^{k+1}) \\
(m^{k+1}, \tilde{w}^{k+1}) &= \text{prox}_F(n^{k+1}, z^{k+1}) \\
(\tilde{m}^{k+1}, \tilde{w}^{k+1}) &= P_S(m^{k+1}, \tilde{w}^{k+1}) \\
(\tilde{m}^{k+1}, \tilde{w}^{k+1}) &= (m^{k+1} + m^{k+1} - m^k, w^{k+1} + \tilde{w}^{k+1} - w^k),
\end{align*}
\]

where \((m^0, w^0) \in \mathcal{M}_h \times \mathcal{W}_h\), \((\bar{m}^0, \bar{w}^0) = (m^0, w^0)\), \((\bar{m}^0, \bar{w}^0) \in \mathcal{M}_h \times \mathbb{R}\), and \(S \supset \arg \min (F + G \circ L_1 + H)\). This method converges if \(\gamma > 0\) and \(\tau > 0\) satisfy

\[
\|L_1\|^2 < \frac{1}{\gamma} \left( \frac{1}{\tau} - \frac{\|K_h\|}{2} \right). \tag{59}
\]
When $S = M_h \times W_h$, (58) reduces to the method proposed in [24], detailed in (21). As noticed in [16, Remark 4.1], this algorithm generates unfeasible primal sequences leading to slow convergence and it will be not considered in our comparisons in Sect. 4.3.4. To reinforce feasibility, we consider

$$S = \{ (m, w) \in M_h \times W_h \mid h^2(1 \mid m) = 1 \}$$

as in [16].

An equivalent formulation to (46) is

$$\min_{(m, w) \in M_h \times W_h} F(m, w) + G_2(m, w) + H(m, w),$$

where $G_2 = \iota_{\{0,1\}} \circ L_1$. The formulation in (60) can also be solved by [24] in the case when the linear operator is $Id$ (see the detailed implementation in (21)). Note that $1 \in ker(-\nu \Delta_h)$ and $h^2(1 \mid 1) = 1$, which yields $G_2 = \iota_{ker L_1 + (1,0)}$ and $\text{prox}_{\gamma G_2^*:} (m, w) \mapsto (Id - P_{ker L_1})(m - \gamma 1, w)$ [8, Theorem 14.3(ii)]. Hence, it follows from (57) that algorithm in [24] reduces to

$$(\forall k \in \mathbb{N}) \begin{cases} (p^{k+1}, \ell^{k+1}) = (Id - P_{ker L_1})(p^k + \gamma \bar{m}^k - \gamma 1, \ell^k + \gamma \bar{w}^k) \\ (m^{k+1}, w^{k+1}) = \text{prox}_{\tau F}(m^k - \tau(p^{k+1} + K_h m^k), w^k - \tau \ell^{k+1}) \\ (\bar{m}^{k+1}, \bar{w}^{k+1}) = (2m^{k+1} - m^k, 2w^{k+1} - w^k), \end{cases}$$

where $(m^0, w^0) \in M_h \times W_h$, $(\bar{m}^0, \bar{w}^0) = (m^0, w^0)$ and $(p^0, \ell^0) \in M_h \times W_h$. In this case, the algorithm converges if $\gamma > 0$ and $\tau > 0$ satisfy

$$\gamma < \frac{1}{\tau} - \frac{\|K_h\|}{2}.$$ 

On the other hand, by defining the closed vector subspace

$$V = ker L_1,$$

we have $G_2 = \iota_{V + (1,0)}$ and, by setting $\rho = m - 1$, (60) is equivalent to

$$\min_{(\rho, w) \in V} F(\rho + 1, w) + H(\rho + 1, w).$$

This problem can be solved by using the algorithm in [13, Corollary 5.5] detailed in (19). Note that $\nabla H(\cdot + (1,0))$: $(\rho, w) \mapsto (K_h(\rho + 1), 0)$ is $\|K_h\|$–Lipschitz and the algorithm in (19) reduces to, for every $k \in \mathbb{N},

$$\begin{cases} (s^{k+1}, \tau^{k+1}) = \text{prox}_{\tau F}((\rho^k + 1 + \tau z^k, w^k + \tau v^k) - \tau P_{ker L_1}(K_h(\rho^k + 1), 0)) \\ (\rho^{k+1}, w^{k+1}) = P_{ker L_1}(s^{k+1} - 1, \tau^{k+1}) \\ (z^{k+1}, v^{k+1}) = (z^k + (\rho^{k+1} - s^{k+1} + 1)/\tau, v^k + (w^{k+1} - \tau^{k+1})/\tau), \end{cases}$$

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where $(\rho^0, w^0) \in \ker L_1$ and $(z^0, v^0) \in (\ker L_1)\perp$. The algorithm converges under the condition $0 < \tau < 2/\|K_h\|$. 

An alternative method for solving (64) is our algorithm when the linear operator is $\text{Id}$. For every $\gamma > 0$, [8, Proposition 24.8(ii)] yields $\text{prox}_{H(+1,0))/\gamma} : (\rho, w) \mapsto ((\text{Id} + K_h/\gamma)^{-1}(\rho + 1), w)$ and from [8, Theorem 14.3(ii)] we obtain

$$
\text{prox}_{\gamma H(+1,0))} : (\rho, w) \mapsto (\rho, w - \gamma \text{prox}_{H(+1,0)/\gamma}(\rho/\gamma, w/\gamma)) = (\rho + \gamma 1 - \gamma ((\text{Id} + K_h/\gamma)^{-1}(\rho/\gamma + 1), 0).
$$

(66)

Hence, by using a similar translation for $\text{prox}_{T^F(-1,0)}, (20)$ in the case when $W$ is the whole space and $T = \text{Id}$ reduces to

$$
(\forall k \in \mathbb{N}) \begin{cases}
    p^{k+1} = p^k + \gamma \bar{p}^k + \gamma 1 - \gamma ((\text{Id} + K_h/\gamma)^{-1}(p^k/\gamma + \bar{p}^k + 1) \\
    (s^{k+1}, t^{k+1}) = \text{prox}_{T^F}((p^k + 1 + \tau z^k, w^k + \tau v^k) - \tau P_{\ker L_1}(p^{k+1}, 0)) \\
    (\rho^{k+1}, u^{k+1}) = P_{\ker L_1}((s^{k+1} - 1, t^{k+1}) \\
    (z^{k+1}, v^{k+1}) = (z^k + (\rho^{k+1} - s^{k+1} + 1)/\tau, v^k + (w^{k+1} - t^{k+1})/\tau) \\
    \bar{p}^{k+1} = 2\rho^{k+1} - \rho^k.
\end{cases}
$$

(67)

where $(\rho^0, w^0) \in \ker L_1$, $\bar{p}^0 = \rho_0$, $(z^0, v^0) \in (\ker L_1)\perp$, and $\rho^0 \in M_h$. In this context, the algorithm converges for every $\tau > 0$ and $\gamma > 0$ satisfying $\tau \gamma < 1$, in view of (7). Note that $P_{\ker L_1}$ can be computed by using [8, Example 29.17(iii)].

For the last formulation of this section, observe that, since $k$ is a PDS kernel, it follows from (38) that the operator $K_h$ defined in (41) is positive semidefinite and, thus, there exists a symmetric positive semidefinite linear operator $K_h^{1/2} : M_h \to M_h$ such that, for every $m \in M_h$, $(m | K_h m) = \sum_{i,j} |(K_h^{1/2} m)_{i,j}|^2$ [43, Theorem VI.9]. Hence, (64) can be written equivalently as

$$
\min_{(\rho, w) \in V} F(\rho + 1, w) + G_3(L_2(\rho + 1, w)),
$$

(68)

where

$$
\begin{align*}
G_3 &= \frac{1}{2} \| \cdot \|^2, \\
L_2 : (m, w) &\mapsto K_h^{1/2} m.
\end{align*}
$$

(69)

From [8, Proposition 24.8(i) & Theorem 14.3(ii)], we deduce that, for every $\gamma > 0$,

$$
\text{prox}_{\gamma G_3(-1/2)}, 1) : \rho \mapsto (\rho - \gamma \text{prox}_{\|1\|^2/(2\gamma)}(\rho/\gamma)) = (\rho + \gamma K_h^{1/2} 1)/(1 + \gamma).
$$

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Moreover, since $L^*_2 : m \mapsto (K^{1/2}_h m, 0)$, the algorithm in (20) in the case when $W$ is the whole space and $T = I_d$ reduces to, for every $k \in \mathbb{N}$,

\[
\begin{align*}
\rho^{k+1} &= (\rho^k + \gamma K^{1/2}_h (\rho^k + 1))/(1 + \gamma) \\
(s^{k+1}, t^{k+1}) &= \text{prox}_{\tau F} \left( (\rho^k + 1 + \tau z^k, w^k + \tau v^k) - \tau P_{\ker L_1}(K^{1/2}_h p^{k+1}, 0) \right) \\
(\rho^{k+1}, w^{k+1}) &= P_{\ker L_1}(s^{k+1} - 1, t^{k+1}) \\
(z^{k+1}, v^{k+1}) &= (z^k + (\rho^{k+1} - s^{k+1} + 1)/\tau, v^k + (w^{k+1} - t^{k+1})/\tau)
\end{align*}
\] (70)

where $(\rho^0, w^0) \in \ker L_1, \rho^0 = \rho^0, (z^0, v^0) \in (\ker L_1)^\perp$, and $p^0 \in \mathcal{M}_h$. In this context, the algorithm converges for every $\tau > 0$ and $\gamma > 0$ satisfying $\tau \gamma K^{1/2}_h \| \tau < 1$, in view of (7).

Remark 4.3 Note that, when $\| K_h \|$ is large, algorithm (70) allows for a larger set of admissible step-sizes than the algorithm in (61) in view of condition (62). Indeed, (62) imposes condition $\tau < 2/\| K_h \|$ on the primal step-size of (61), which affects its efficiency as we will see in Sect. 4.3.4.

In all previous methods we need to compute $\text{prox}_F$, where $F$ is defined in (56). Note that

\[ F : (m, w) \mapsto \sum_{i,j=0}^{N-1} f_{i,j}(m_{i,j}, w_{i,j}), \] (71)

where $f_{i,j} : (\eta, \omega) \mapsto b(\eta, \omega) + (K_0)_{i,j} \eta$. We deduce from [16, Corollary 3.1] that $F$ is convex, proper, and lower semicontinuous, for every $\gamma > 0$, $\text{prox}_{\gamma F} : (m, w) \mapsto (\text{prox}_{\gamma f_{i,j}}(m_{i,j}, w_{i,j}))_{0 \leq i, j \leq N-1}$, and, for every $0 \leq i, j \leq N - 1$,

\[ \text{prox}_{\gamma f_{i,j}} : (\eta, \omega) \mapsto \begin{cases} (0, 0), & \text{if } \gamma (K_0)_{i,j} \geq \eta + |P_C \omega|^2/(2\gamma); \\ (p^*, P^*), & \text{otherwise,} \end{cases} \] (72)

where $p^* > 0$ is the unique solution to

\[ (p + \gamma (K_0)_{i,j} - \eta) (p + \gamma)^2 - \frac{\gamma}{2} |P_C \omega|^2 = 0. \] (73)

This computation is also obtained in [41, Proposition 1] in the case $C = \mathbb{R}^4$.

Remark 4.4 Observe that algorithms in (58), (61), (65), (67), and (70) can also be used in the presence of local couplings as those studied in [16].
Table 3 Execution time, number of iterations with error tolerance $5h^3$, value of the objective function in solution, and residuals of solution for the case $h = 1/20$ and $\nu = 0.05$

| Algorithm | Time (s) | Iter. | Obj. value | $\|L_1(m^*, w^*) - (0, 1)\|$ | $d_{CN}^2(w^*)$ |
|-----------|---------|-------|------------|-----------------------------|-----------------|
| (58)      | 0.67    | 55    | 20,000     | 3.469                       | 0               |
| (61)      | 32.03   | 833   | 20,148     | 0.00135                     | 0               |
| (65)      | 39.46   | 658   | 20,158     | $5.467 \times 10^{-13}$     | $1.518 \times 10^{-9}$ |
| (67)      | 43.77   | 774   | 20,172     | $1.059 \times 10^{-12}$     | $1.379 \times 10^{-5}$ |
| (70)      | 12.88   | 260   | 20,169     | $3.058 \times 10^{-12}$     | $2.150 \times 10^{-8}$ |

Table 4 Execution time, number of iterations with error tolerance $5h^3$, value of the objective function in solution, and residuals of solution for the case $h = 1/20$ and $\nu = 0.2$

| Algorithm | Time (s) | Iter. | Obj. value | $\|L_1(m^*, w^*) - (0, 1)\|$ | $d_{CN}^2(w^*)$ |
|-----------|---------|-------|------------|-----------------------------|-----------------|
| (58)      | 1.03    | 81    | 20,000     | 15.365                      | 0               |
| (61)      | 14.10   | 312   | 20,076     | 0.00309                     | 0               |
| (65)      | 13.84   | 226   | 20,083     | $5.373 \times 10^{-13}$     | $2.688 \times 10^{-9}$ |
| (67)      | 47.04   | 741   | 20,089     | $1.403 \times 10^{-12}$     | $2.779 \times 10^{-12}$ |
| (70)      | 5.17    | 80    | 20,089     | $5.183 \times 10^{-13}$     | $1.555 \times 10^{-8}$ |

4.3.4 Numerical Experiments

We consider $h \in \{1/20, 1/40\}$, the positive definite non-local coupling in [1] given by

$$K_h = \mu (\text{Id} - \Delta_h)^{-p}$$

for $\mu = 10$ and $p = 1$, and

$$(\forall (i, j) \in \mathcal{I}_N^2) \quad (K_0)_{i,j} = -\sin(2\pi hj) + \sin(2\pi hi) + \cos(4\pi hi).$$

We vary $\nu \in \{0.05, 0.2, 0.5\}$ and, for every $h \in \{1/20, 1/40\}$, we choose to stop every algorithm when the $L^2$ norm of the difference between two consecutive iterations is less than $5h^3$ or the number of iterations exceeds 3000. In Tables 3, 4, 5, 6, 7, and 8, we report computational time, number of iterations, value of the objective function, and residuals of the constraints at the resulting vector $(m^*, w^*)$.

Observe that, in all cases, Algorithms (58) and (61) stop at iterates which are far from the solution, since the residual $\|L_1(m^*, w^*) - (0, 1)\|$ is far away from 0 for the chosen tolerance. This residual is larger as viscosity increases. In contrast, for the same tolerance, the vector subspace based Algorithms (65), (67), and (70) achieve iterates with negligible residuals. Among the latter, our proposed Algorithm (70) is the most efficient in terms of computational time and number of iterations. We explain this good behavior by the fact that (70) takes full advantage of the convex quadratic cost by
Table 5 Execution time, number of iterations with error tolerance $5h^3$, value of the objective function in solution, and residuals of solution for the case $h = 1/20$ and $\nu = 0.5$

| Algorithm | Time (s) | Iter. | Obj. value | $\|L_1(\mathbf{m}^*, \mathbf{w}^*) - (0, 1)\|$ | $d^2_{CN^2}(\mathbf{w}^*)$ |
|----------|---------|-------|------------|---------------------------------|-------------------------------|
| (58)     | 1.01    | 87    | 20,000     | 34.964                          | 0                            |
| (61)     | 7.24    | 161   | 20,014     | 0.00528                          | 0                            |
| (65)     | 7.18    | 128   | 20,017     | $1.063 \times 10^{-12}$         | $2.176 \times 10^{-10}$      |
| (67)     | 43.22   | 741   | 20,021     | $1.127 \times 10^{-12}$         | $1.536 \times 10^{-13}$      |
| (70)     | 4.38    | 78    | 20,021     | $1.108 \times 10^{-12}$         | $1.395 \times 10^{-11}$      |

Table 6 Execution time, number of iterations with error tolerance $5h^3$, value of the objective function in solution, and residuals of solution for the case $h = 1/40$ and $\nu = 0.05$

| Algorithm | Time (s) | Iter. | Obj. value | $\|L_1(\mathbf{m}^*, \mathbf{w}^*) - (0, 1)\|$ | $d^2_{CN^2}(\mathbf{w}^*)$ |
|----------|---------|-------|------------|---------------------------------|-------------------------------|
| (58)     | 22.32   | 462   | 80,000     | 7.365                           | 0                            |
| (61)     | 2389.47 | 3000  | 80,705     | 0.00786                         | 0                            |
| (65)     | 2776.92 | 1915  | 80,710     | $3.152 \times 10^{-12}$         | $5.686 \times 10^{-11}$      |
| (67)     | 4189.86 | 3000  | 80,717     | $4.061 \times 10^{-12}$         | $3.258 \times 10^{-6}$       |
| (70)     | 751.18  | 695   | 80,717     | $3.757 \times 10^{-12}$         | $9.748 \times 10^{-9}$       |

Table 7 Execution time, number of iterations with error tolerance $5h^3$, value of the objective function in solution, and residuals of solution for the case $h = 1/40$ and $\nu = 0.2$

| Algorithm | Time (s) | Iter. | Obj. value | $\|L_1(\mathbf{m}^*, \mathbf{w}^*) - (0, 1)\|$ | $d^2_{CN^2}(\mathbf{w}^*)$ |
|----------|---------|-------|------------|---------------------------------|-------------------------------|
| (58)     | 38.55   | 727   | 80,000     | 27.414                          | 0                            |
| (61)     | 533.00  | 727   | 80,369     | 0.00770                         | 0                            |
| (65)     | 636.59  | 461   | 80,372     | $4.219 \times 10^{-12}$         | $1.298 \times 10^{-10}$      |
| (67)     | 1387.21 | 950   | 80,376     | $4.708 \times 10^{-12}$         | $5.260 \times 10^{-12}$      |
| (70)     | 136.66  | 119   | 80,375     | $4.491 \times 10^{-12}$         | $2.492 \times 10^{-9}$       |

Table 8 Execution time, number of iterations with error tolerance $5h^3$, value of the objective function in solution, and residuals of solution for the case $h = 1/40$ and $\nu = 0.5$

| Algorithm | Time (s) | Iter. | Obj. value | $\|L_1(\mathbf{m}^*, \mathbf{w}^*) - (0, 1)\|$ | $d^2_{CN^2}(\mathbf{w}^*)$ |
|----------|---------|-------|------------|---------------------------------|-------------------------------|
| (58)     | 37.06   | 724   | 80,000     | 70.955                          | 0                            |
| (61)     | 286.23  | 387   | 80,082     | 0.0296                          | 0                            |
| (65)     | 428.79  | 251   | 80,083     | $8.392 \times 10^{-12}$         | $2.099 \times 10^{-11}$      |
| (67)     | 2036.42 | 950   | 80,085     | $8.279 \times 10^{-12}$         | $1.335 \times 10^{-12}$      |
| (70)     | 111.39  | 100   | 80,085     | $8.408 \times 10^{-12}$         | $2.984 \times 10^{-12}$      |
splitting $K_h^{1/2}$ from $\|\cdot\|^2/2$ in its architecture (see (69)). A reason for this improvement is the larger step-sizes that this algorithm can take, as stated in Remark 4.3. In the case of more general couplings, Algorithms (65) and (67) are also efficient alternatives to solve (46).

In Figs. 5, 6, and 7 we illustrate the solution obtained from algorithm (70) for different values of $\nu$ and $h$.

Fig. 5 Solution $m^* \in \mathcal{M}_h$ to (46) for $\nu = 0.05$ with $h = 1/20$ (left) and $h = 1/40$ (right).

Fig. 6 Solution $m^* \in \mathcal{M}_h$ to (46) for $\nu = 0.2$ with $h = 1/20$ (left) and $h = 1/40$ (right).

Fig. 7 Solution $m^* \in \mathcal{M}_h$ to (46) for $\nu = 0.5$ with $h = 1/20$ (left) and $h = 1/40$ (right).
5 Conclusions

We propose a primal-dual method with partial inverse for solving constrained composite monotone inclusions involving a normal cone to a closed vector subspace. When the monotone operators are subdifferentials of convex functions, our method solves composite convex optimization problems over closed vector subspaces. We also incorporate a priori information on the solution of the monotone inclusion, which produces an additional projection step in the primal-dual algorithm. Either this projection or our vector subspace approach produces significant gains in numerical efficiency with respect to the available methods in the literature.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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