Using Integer Programming Techniques in Real-Time Scheduling Analysis

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Abstract—Real-time scheduling theory assists developers of embedded systems in verifying that the timing constraints required by critical software tasks can be feasibly met on a given hardware platform. Fundamental problems in the theory are often formulated as search problems for fixed points of functions and are solved by fixed-point iterations. These fixed-point methods are used widely because they are simple to understand, simple to implement, and seem to work well in practice. These fundamental problems can also be formulated as integer programs and solved with algorithms that are based on theories of linear programming and cutting planes amongst others. However, such algorithms are harder to understand and implement than fixed-point iterations. In this research, we show that ideas like linear programming duality and cutting planes can be used to develop algorithms that are as easy to implement as existing fixed-point iteration schemes but have better convergence properties. We evaluate the algorithms on synthetically generated problem instances to demonstrate that the new algorithms are faster than the existing algorithms.

1 INTRODUCTION

In real-time scheduling theory, a task with hard timing constraints receives an infinite stream of requests and must respond to each request within a fixed amount of time. We consider a system comprising \( n \) independent preemptible sporadic tasks, labeled \( 1, 2, \ldots, n \). Each sporadic task \( i \in [n] \) has the following (integral) characteristics:

| Symbol | Task Characteristic                          |
|--------|----------------------------------------------|
| \( C_i \) | worst-case execution time (wcet)            |
| \( T_i \) | minimum duration between successive request arrivals (period) |
| \( D_i \) | maximum duration between request arrival and response (relative deadline) |

If the \( k \)-request for task \( i \) arrives at time \( t \), then the following statements must be true:

- The request must be completed by time \( t + D_i \) in a feasible schedule. In other words, \( t + D_i \) is the absolute deadline for the completion of the response to the request.
- The \((k+1)\)-th request for task \( i \) cannot arrive before \( t + T_i \).
- If \( D_i = T_i \), then the deadline \( D_i \) is said to be implicit; if \( D_i \leq T_i \), then the deadline \( D_i \) is said to be constrained. If all tasks in a system have implicit deadlines, then the system is an implicit-deadline system. If all tasks in a system have constrained deadlines, then the system is a constrained-deadline system; otherwise, it is an arbitrary-deadline system.

We assume that the tasks run on a single processor and are scheduled using either a fixed-priority (FP) scheduler or an earliest-deadline-first (EDF) scheduler. An FP scheduler assigns static priorities to tasks. An EDF scheduler dynamically assigns the highest priority to any task with the earliest absolute deadline.

Given a collection of tasks and a scheduler, the problem of deciding whether timing constraints are met for all sequences of request arrivals is called the schedulability problem. When the scheduler is an FP (resp., EDF) scheduler, the problem is called FP schedulability (resp., EDF schedulability). An algorithm which solves the problem is called a schedulability test. FP schedulability and EDF schedulability are fundamental problems in real-time scheduling theory, with countless papers devoted to finding faster exact schedulability tests or sufficient schedulability tests with higher accuracy.

A constrained-deadline preemptive uniprocessor system with tasks \( [n] \) is FP schedulable if and only if the subsystem with tasks \( [n-1] \) is schedulable and

\[
\exists t \in (0, D_n) \colon rbf(t) \leq t
\]

holds \([1],[2]\). Here, \( rbf \), the request bound function, is given by

\[
t \mapsto \sum_{j \in [n]} \left\lceil \frac{t}{T_j} \right\rceil C_j. \tag{1}
\]

The above condition is satisfied if and only if the problem

\[
\min \{ t \in (0, D_n] \cap \mathbb{Z} \mid rbf(t) \leq t \} \tag{2}
\]

has an optimal solution. The optimal solution must satisfy the fixed point equation

\[
rbf(t) = t.
\]

The solution to this equation can be found by using fixed-point iteration, i.e., by starting with a safe lower \( t \) for the fixed point and iteratively updating \( t \) to \( rbf(t) \). This algorithm, called Response Time Analysis (RTA), is an exact FP schedulability test for constrained-deadline preemptive uniprocessor systems \([1]\). We show the execution of RTA for a small example in Figure \([1]\)—the up arrows denote the computation of \( rbf(t) \) and the right arrows denote the update.

1. Given a problem instance, a sufficient test either declares that all timing constraints are satisfied or it declares nothing, so the constraints may or may not be violated.
2. For FP schedulability, we assume that tasks are listed in decreasing order of priority.
The solution can be found by starting a safe upper bound \( \mathcal{U} \) for the fixed point equation and iteratively updating its value until it converges. The above update works assuming that all data are integral.

Geometrically, QPA looks like climbing down the steps of \( \text{dbf}(t) \) once to the target instance to find the solution. The reduction from FP schedulability is weak because we may need to create up to \( n \) instances of the target problem to find the solution for the source problem. The strong reduction from FP schedulability is possible because the target, problem (8), is a close generalization of FP schedulability.

We formulate problem (8) as an integer program and solve it by using the cutting plane methodology which involves relaxing the integer program to a linear program, solving the linear program, updating the integer program with a cut if the solution is not integral, and repeating the process (described in more detail in the next section). Two questions must be answered to apply this technique:

- How is the cut generated?
- How is the linear program solved?

We show that for our integer program cuts can be generated so that they only change bounds of variables in the integer program—thus, the number of constraints in the integer program does not increase after incorporating the cut. To solve the linear program, we design a specialized algorithm which uses \( O(n \log(n)) \) arithmetic ops. We can also use the ellipsoidal method, for instance, to solve the program in weak polynomial time but such sophistication is excessive for the problem at hand—our algorithm is conceptually simple and easy to implement and strongly polynomial-time. By composing the strong (resp., weak) reduction with our cutting plane algorithm, we get an iterative algorithm for FP schedulability (resp., EDF schedulability) which does almost linear work in each iteration.

We show that RTA and QPA may also be viewed as cutting plane algorithms that generate cuts the same way that we do. However, instead of solving the linear relaxation of the integer program in each iteration, they solve a further relaxation of the linear relaxation. This over-relaxation allows each iteration of RTA and QPA to complete using only \( O(n) \) arithmetic ops but it also leads to slower convergence compared to our algorithm. We measure the number of iterations required by the algorithms in experiments on synthetically generated problem instances for FP schedulability (resp., EDF schedulability) to show that our algorithm is faster and more predictable than RTA (resp., QPA).

## 2 Background

### 2.1 Integer Programs and Cutting Plane Methods

An optimization problem over variable \( x \) that can be expressed as

\[
\min\{c \cdot x \mid x \in \mathbb{R}_{\geq 0}^n, Ax \geq b\},
\]

3. The distinction between strong and weak reductions is similar to the distinction between Karp and Cook reductions for decision problems but we are working with optimization problems.
for some $A \in \mathbb{R}^m \times \mathbb{R}^n, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ is a linear program. Here $c \cdot x$ is the dot product of the vectors. If $x$ is restricted to be integral, then the problem is an (linear) integer program:

$$\min \{ c \cdot x \mid x \in \mathbb{Z}^n_{\geq 0}, Ax \geq b \}. \quad (6)$$

A relaxation of an optimization problem with a maximization (resp., minimization) objective is a simpler optimization problem with optimal value at least as large (resp., small) as the optimal value of the original problem. Problems can be relaxed in many ways but we will primarily be concerned with relaxations of integer programs in which the set of feasible solutions is expanded by

- allowing variables to be continuous instead of integral; and/or
- dropping constraints.

If the relaxation only involves making the variables continuous, then it is called a linear relaxation. Thus, program (6) is a linear relaxation of program (5).

Many optimization problems, including integer programs, can be solved using the cutting-plane methodology which consists of repeating the following steps:

1. Solve a relaxation of the problem.
2. If the relaxation is found to be infeasible, then the original problem must be infeasible and we exit the algorithm.
3. If the optimal solution for the relaxation is feasible for the original problem, then the solution must also be optimal for the original problem and we return the solution.
4. Find one or more valid inequalities that separate the current (nonintegral) solution from the convex hull of the set of feasible solutions to the original problem. Such separating inequalities, or cuts, are guaranteed to exist when the set of feasible solutions of the relaxation contains the convex hull of the set of feasible solutions to the original problem.
5. Add the inequalities to the problem description and go to the first step.

After each iteration, we have a better approximation for the convex hull of the set of feasible solutions to the original problem, but the problem description, in general, is longer. Since the problem is more restricted in each iteration, the optimal values found in each iteration, denoted $Z_1, Z_2, \ldots$, must satisfy

$$Z_1 \leq Z_2 \leq \cdots$$

assuming that the direction of optimization is minimization. These values are sometimes called dual bounds for the optimal value, denoted $Z$. More details about integer programs can be found in standard textbooks [6].

4. An inequality is valid for a problem if it is satisfied by all feasible solutions to the problem.

### 2.2 Linear Programming Duality

The linear program

$$\max \{ b \cdot y \mid y \in \mathbb{R}^m_{\geq 0}, A^T y \leq c \} \quad (7)$$

is called the dual of program (5), which is called the primal program in such a context. Linear programming duality is the idea that for such a pair of programs exactly one of the following statements is true:

- Both programs are infeasible.
- One program is unbounded, and the other program is infeasible.
- Both programs have optimal solutions with the same objective value.

More details about linear programming duality can be found in standard textbooks [7].

### 3 A Generalization of FP Schedulability

In this section, we consider the following problem:

$$\min \left\{ t \in [a, b] \cap \mathbb{Z} \mid \sum_{j \in [n]} \left[ \frac{t + \alpha_j}{T_i} \right] C_j + \beta \leq t \right\}, \quad (8)$$

where

- $n$ is a positive integer.
- For any $i \in [n], C_i, T_i \in \mathbb{Z}_{>0}$ are constants, and $U_i$ denotes the ratio $C_i/T_i$.
- $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}, a \in \mathbb{Z}$, and $b \in \mathbb{Z}$ are constants.

It is useful to view $C_i, T_i$, and $U_i$ as the wcet, period, and utilization, respectively, of task $i$ but our methods do not require us to adhere to this interpretation. If $(\alpha, \beta, a, b)$ is equal to $(0, 0, 1, D_n)$, then problem (8) is identical to problem (2). Thus, problem (8) is a close generalization of problem (2).

If $t = a$ is a feasible solution to the problem, then it is also the optimal solution. We assume that we are not in this trivial case and $t = a$ is infeasible, i.e.,

$$\sum_{j \in [n]} \left[ \frac{a + \alpha_j}{T_i} \right] C_j + \beta > a \quad (9)$$

Now, $t \geq a$ iff for all $i \in [n],$

$$\left[ \frac{t + \alpha_i}{T_i} \right] \geq \left[ \frac{a + \alpha_i}{T_i} \right].$$

The forward direction is trivial; the backward direction uses the inequality in problem (8) and assumption (9). Using the equivalence, the problem can be rewritten as

$$\min \quad t$$

s.t.\begin{align*}
t - \sum_{j \in [n]} C_j \cdot x_j(t) & \leq b \\
x_i(t) & \geq \frac{a + \alpha_i}{T_i}, \quad i \in [n] \\
t & \in \mathbb{Z}
\end{align*}$$

### TABLE 1: Comparing RTA and QPA.

| RTA | QPA |
|-----|-----|
| tasks | constrained-deadline, preemptive | arbitrary-deadline, preemptive |
| processor | single | single |
| scheduler | fixed-priority | earliest-deadline-first |
| functional iteration | update $t$ to $\text{rbd}(t)$ | update $t$ to $\text{dbf}(t) - 1$ |
| final value | least fixed point of $\text{rbd}(t)$, if system is schedulable | greatest fixed point of $\text{dbf}(t) - 1$, if system is unschedulable |
where, for all $i \in [n]$, we have
\[
x_i(t) = \left\lfloor \frac{t + \alpha_i}{T_i} \right\rfloor, \\
x_i = \left\lceil \frac{a + \alpha_i}{T_i} \right\rceil.
\]
x_i(t) is a function of the variable t, and $x_i$ is a constant lower bound for $x_i(t)$.

The above problem can be formulated as an integer program:
\[
\begin{align*}
\min & \quad t \\
\text{s.t.} & \quad t \leq b \\
& \quad t - \sum_{j \in [n]} C_j x_j \geq \beta \\
& \quad T_i x_i - t \geq \alpha_i, \quad i \in [n] \\
& \quad x_i \geq x_i^*, \quad i \in [n] \\
& \quad t \in \mathbb{Z} \\
& \quad x \in \mathbb{Z}^n
\end{align*}
\]
(10)

Here, $x_i$ is an integer variable (not a function), but the relationship between $x_i$ and $t$ is preserved in this transformation from the previous problem because any optimal solution $(t, x)$ to the integer program satisfies
\[
x_i = \max \left\{ \left\lfloor \frac{t + \alpha_i}{T_i} \right\rfloor, \frac{a + \alpha_i}{T_i} \right\} = \frac{t + \alpha_i}{T_i}
\]
\[
x = \sum_{j \in [n]} C_j x_j
\]
(11)

In the second equality on the first line, we use the definition of $x_i$ and the fact that $t \geq a$.

To apply the cutting plane methodology on program (10), we relax it by making the variables continuous and dropping the upper bound for $b$ for $t$:
\[
\begin{align*}
\min & \quad t \\
\text{s.t.} & \quad t - \sum_{j \in [n]} C_j x_j \geq \beta \\
& \quad T_i x_i - t \geq \alpha_i, \quad i \in [n] \\
& \quad x_i \geq x_i^*, \quad i \in [n] \\
& \quad t \in \mathbb{R} \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]
(11)

This is almost a linear relaxation of program (10). The decision to drop $t \leq b$ from the program is not consequential because one of the following three conditions must be true:

- The linear relaxation and program (11) are both feasible.
- Both programs are feasible and have the same optimal value.
- Linear relaxation is infeasible, but program (11) is feasible and has optimal value greater than $b$.

The last condition can be checked after solving program (11). So, from hereon, we refer to program (11) as a linear relaxation. In any optimal solution $(t^*, x^*)$ to this linear program, we must have
\[
t^* = \sum_{j \in [n]} C_j x_j^* \\
\forall i \in [n] : x_i^* = \max \left\{ \frac{t^* + \alpha_i}{T_i}, x_i \right\}
\]
If $x^*$ is integral, then $t^*$ is also integral and we have found an optimal solution to program (10). Otherwise, there must exist a $j \in [n]$ where $x_j^*$ is fractional. $x_j^*$ cannot equal $x_j$, which is integral; therefore, it must equal $(t^* + \alpha_j)/T_j$. Using the third constraint in program (10) and the fact that $t^*$ is a strict lower bound for an optimal $t$, any optimal solution $(t, x)$ to program (10) must satisfy
\[
x_j \geq \frac{t + \alpha_j}{T_j} > \frac{t^* + \alpha_j}{T_j} = x_j^*.
\]
Since $x_j$ is integral, the stronger inequality
\[
x_j \geq \frac{t^* + \alpha_j}{T_j}
\]
is also valid. Moreover, $x_j^*$ does not satisfy the inequality $x_j^* \geq \frac{t^* + \alpha_j}{T_j}$ because $x_j^* = (t^* + \alpha_j)/T_j$ is fractional. Therefore, $x_j \geq \frac{t^* + \alpha_j}{T_j}$ is a cut. This cut increases the lower bound for $x_j$ in the original program but does not increase the number of constraints. There are at most $n$ such cuts all of which are incorporated into program (10) by updating $x_i$ to
\[
\left\lceil \frac{t^* + \alpha_i}{T_i} \right\rceil
\]
for all $i \in [n]$. These updates maintain the integrality of $x$ which is utilized in generating cuts in the next iteration.

Algorithm 1 shows the cutting plane algorithm that we have developed for problem (8). For any $i \in [n]$, the minimum (resp., maximum) value of $x_i$ is $\lfloor (a + \alpha_i)/T_i \rfloor$ (resp., $\lceil (b + \alpha_i)/T_i \rceil$), and $x_i$ can assume $O(\min_j \lfloor (b - a)/T_j \rfloor)$ values. Since at least one element in $x$ is incremented in each iteration, the loop runs $O(\min_j \lfloor (b - a)/T_j \rfloor n)$ times. Inside each loop, the linear program can be solved in time polynomial in the size of the representation of the program (8). The size of the representation of the program is linearly bounded by $|I|$, the size of the representation of the original instance of problem (8). Thus, the algorithm has $O(\min_j \lfloor (b - a)/T_j \rfloor \text{poly}(|I|))$ running time.

**Theorem 1.** Algorithm 1 is a cutting plane algorithm for problem (8) which uses
\[
O \left( \left\lceil \frac{b - a}{\min_j \lfloor I \rfloor \text{poly}(|I|)} \right\rceil \right)
\]
5. Similar logic is used in generating Gomory cuts [6, Ch. 8].
We introduce a map \( f : [n] \to \mathbb{Q} \) given by
\[
k \mapsto \frac{\beta + \sum_{j \in [n]\setminus[k]} \alpha_j U_j + \sum_{j \in [k]} z_j C_j}{1 - \sum_{j \in [n]\setminus[k]} U_j}.
\] (14)

It may be verified that the above optimal value for \( k = 1 \) and \( \sum_{j \in [n]} U_j = 1 \) is equal to \( f(1) \). If \( k > 1 \) or \( \sum_{j \in [n]} U_j < 1 \), then \( z_k \) can be eliminated from the above program to get
\[
\max \quad (1 - \sum_{j \in [n]\setminus[k]} U_j)(f(k) - (\xi_k T_k - \alpha_k)) + v + \xi_k T_k - \alpha_k
\begin{align*}
\text{s.t.} \quad & v \in [1 - \sum_{j \in [n]\setminus[k]} U_j, (1 - \sum_{j \in [n]\setminus[k-1]} U_j)^{-1}]
\end{align*}
\]

From our assumption \( \sum_{j \in [n]} U_j \leq 1 \), it follows that the left end of the interval is positive—the constraint \( v \geq 0 \), which was present in program (13), is redundant and not included above. From the assumption that \( k > 1 \) or \( \sum_{j \in [n]} U_j < 1 \), it follows that the right end of the interval is well defined and greater than the left end of the interval. Thus, program (13) is feasible and bounded. Using the assumption that \( k > 1 \) or \( \sum_{j \in [n]} U_j < 1 \), we can also infer that \( f(k - 1) \) is well defined. Then, \( f \) satisfies the following identity:
\[
f(k)(1 - \sum_{j \in [n]\setminus[k]} U_j) = f(k - 1)(1 - \sum_{j \in [n]\setminus[k-1]} U_j) + (\xi_k T_k - \alpha_k)U_k.
\]

Using the identity, the objective can also be written as
\[
(1 - \sum_{j \in [n]\setminus[k-1]} U_j)(f(k - 1) - (\xi_k T_k - \alpha_k)) + v + \xi_k T_k - \alpha_k.
\]

Substituting the two ends of the interval for \( v \) in the above expressions for the objective, we get that the optimal value is \( f(k) \) or \( f(k - 1) \). The optimal values from the \( n \) linear programs can be combined to get the optimal value for the linear program (12) as follows: if \( k = 1 \) and \( \sum_{j \in [n]} U_j = 1 \), then the optimal value is \( \max_{k \in [n]} f(k) \), otherwise the optimal value is \( \max_{k \in [0]\cup[n]} f(k) \).

The above analysis is summarized in Algorithm 2. We assert that the utilization does not exceed 1 on line 1. We check unboundedness of the dual program on line 2; if the condition is true, then we simply return a string “infeasible” on line 3. We enforce the order required by the analysis on line 5. We compute \( \max_k f(k) \) on lines 6–9, taking the well-definedness of \( f(0) \) into account. Sorting on line 5 takes \( O(n \log(n)) \) arithmetic ops. Checking infeasibility on line 2, and finding the maximum values on lines 6 and 8 use \( O(n) \) arithmetic ops. Other lines use a constant number of arithmetic ops. Therefore, Algorithm 2 uses \( O(n \log(n)) \) arithmetic ops. The next theorems follow.

**Theorem 2.** The linear program (11) can be solved by Algorithm 2 using \( O(n \log(n)) \) arithmetic ops.
Theorem 3. If Algorithm 1 calls Algorithm 2 on line 6, then it solves problem (8) using
\[ O\left(\frac{b-a}{\min_{j\in[n]} T_j}\right)n^2 \log(n) \]
arithmetic ops.

3.2 An efficient algorithm for finding max f

By refining our previous analysis, we can find the maximum value of f without necessarily computing f for all n or (n + 1) values in its domain. Previously, we observed that when f(k − 1) is well-defined, the objective of the k-th subproblem is to maximize
\[(1 - \sum_{j\in[n]\setminus[k]} U_j)(f(k) - (x_k T_k - \alpha_k)v + x_k T_k - \alpha_k)\]
and the only constraint is that v is restricted to
\[(1 - \sum_{j\in[n]\setminus[k]} U_j)^{-1}, (1 - \sum_{j\in[n]\setminus[k+1]} U_j)^{-1}].\]
The coefficient of the objective in the objective is nonnegative iff f(k) ≥ x_k T_k - \alpha_k if f(k − 1) ≥ x_k T_k - \alpha_k. If the coefficient is nonnegative, then v must equal the right end of the interval, i.e., (1 − \sum_{j\in[n]\setminus[k]} U_j)^{-1}, and, hence, the optimal value is f(k − 1). Similarly, the coefficient of v in the objective is nonpositive iff f(k) ≤ x_k T_k - \alpha_k if f(k − 1) ≤ x_k T_k - \alpha_k, and if the coefficient is nonpositive then the optimal value is f(k). Therefore, at least one of the following conditions must be true:
\[f(k - 1) \leq f(k) \leq x_k T_k - \alpha_k, \quad x_k T_k - \alpha_k \leq f(k) \leq f(k - 1)\]
(15)
f has a local maximum point at j iff f(j) ≥ f(j − 1) and f(j) ≥ f(j + 1). The first condition defaults to true if f(j − 1) is not well defined — this can happen at j = 0, or j = 1 if \(\sum_{j\in[n]} U_j = 1\). The second condition defaults to true if j = n because f(n + 1) is not well defined. Assuming that these corner cases are handled appropriately, f has a local maximum point at j iff
\[x_{j+1} T_{j+1} - \alpha_{j+1} \leq f(j) \leq x_j T_j - \alpha_j.\]
Assume, for the sake of contradiction, that k_1 and k_2 are two distinct local maxima of f with k_1 < k_2 and f(k_2) > f(k_1). Then, we immediately get a contradiction:
\[f(k_2) \leq x_k T_k - \alpha_k \leq x_{k+1} T_{k+1} - \alpha_{k+1} \leq f(k_1).\]
The first and third inequalities follow from the above characterization of local maxima, and the middle inequality uses the fact that the tasks are sorted in a nondecreasing order using the key \(x_j T_j - \alpha_j\) for all \(j \in [n]\). Therefore, the first local maximum point must be the global maximum point. Lines 6–9 in Algorithm 2 can be replaced by a call to Algorithm 3. This optimization has no effect on the upper bound for the worst-case running time of Algorithm 4 but it reduces the average running time of Algorithm 2 which is the main workhorse in Algorithm 1.

4 NEW Schedulability Tests

4.1 FP Scheduling

Problem 2 can be reduced to problem (8) by initializing \((\alpha, \beta, a, b)\) in the target instance to \((0, 0, 1, D_n)\). However, we consider a subtly different reduction which, when composed with Algorithm 1, yields a simpler algorithm. Problem 2 is a question about the behavior of rbf in \([0, D_n]\) for constrained-deadline systems, but in this interval \([t/T_n]\) must equal 1. Thus, the condition \(rbf(t) \leq t\) may be written as
\[\sum_{j\in[n-1]} \left[\frac{t}{T_j}\right] C_j + C_n \leq t.\]
(16)
If \(\sum_{j\in[n-1]} U_j \geq 1\) then the problem is infeasible. So, we assume that this condition is checked initially, and now \(\sum_{j\in[n-1]} U_j < 1\) holds. Using inequality 16, problem 2 can be reduced to problem (8) by initializing \((n, \alpha, \beta, a, b)\) in the target instance to equal \((n - 1, 0, C_n, C_n, D_n)\). We solve the target instance by using Algorithm 1 and Algorithm 2 (see Theorem 3). Since the total utilization is less than one, only lines 5 and 9 in Algorithm 2 do meaningful work and \(f(0)\) is well defined. \(f\) is given by
\[k \rightarrow \frac{C_n + \sum_{j\in[n]} x_j C_j}{1 - \sum_{j\in[n-1]} U_j},\]
(17)
and we have
\[f(0) = \frac{C_n}{1 - \sum_{j\in[n-1]} U_j}.\]
\(f(0)\) is independent of \(x\) and since line 9 in Algorithm 2 is bound to run at least once \(f(0)\) is a lower bound for \(t\). Note that \(f(0)\) has been proposed as an initial value for RTA before using a different analysis method in our case, the bound pops out of Algorithm 2 when it is simplified. Since \(f(0)\) is a lower bound for \(t\), when reducing problem 2 to problem (8), \((n, \alpha, \beta, a, b)\) in the target instance can be initialized to \((n - 1, 0, C_n, f(0), D_n)\) and \(f(0)\) can be ignored on line 9 in Algorithm 2. Algorithm 1 and Algorithm 2 can be combined and simplified, as described above, to get Algorithm 4. The next theorem follows from Theorem 3.

Theorem 4. Algorithm 4 solves problem 2 using
\[O\left(\frac{D_n - C_n}{\min_{j\in[n-1]} T_j}\right)n^2 \log(n)\]
arithmetic ops.

RTA starts with a lower bound \(t\) for the optimal \(t\), and iteratively updates \(t\) to \(rbf(t)\) until \(t\) stabilizes or \(t > D_n\). This seems vaguely similar to the sequence of lower bounds produced by cutting-plane methods. RTA uses the same cut generation technique as Algorithm 1 but it does not
A cutting plane algorithm for problem (2).

1. if \( \sum_{j \in [n]} U_j \geq 1 \) then
   2. return “infeasible”
3. end if
4. Initialize \( t = \lceil f(0) \rceil \) (see Def. (17)).
5. repeat
6. Update \( x_k \) to \( \lceil t/T_j \rceil \) for all \( i \in [n-1] \).
7. Sort the tasks and \( x \) in a nonincreasing order using the key \( x_j T_j \) for each \( j \in [n-1] \).
8. Update \( t \) to \( \max_{k \in [n-1]} \{ f(k) \} \)
9. until \( x \) stabilizes or \( t > D_n \).
10. if \( t > D_n \) then
11. return “infeasible”
12. else
13. return \( t \)
14. end if

solve the linear program (11). Instead, it solves the further relaxation of program (11) created by omitting the second constraint in the program:

\[
\begin{align*}
\min \quad & t \\
\text{s.t.} \quad & t - \sum_{j \in [n]} C_j x_j \geq 0 \\
& x_i \geq \frac{1}{k_i}, \quad i \in [n] \\
& t \in \mathbb{R} \\
& x \in \mathbb{R}^n
\end{align*}
\]

The optimal value for this simple program simply equals \( \sum_{j \in [n]} C_j x_j = \sum_{j \in [n]} C_j \left[ \frac{t}{T_j} \right] = rbf(t) \) and can be computed by using \( O(n) \) arithmetic ops but this efficiency inside the loop comes at the cost of over-relaxing the original problem, finding smaller dual bounds in each iteration, and eventually using a large number of iterations.

In contrast, Algorithm 4 does not over-relax the problem and finds larger dual bounds at each step. The reduction in the number of iterations comes at the cost of an increase in the worst-case number of arithmetic ops in the loop by a factor of \( \log(n) \). We believe that in this trade-off, Algorithm 4 is the winner because \( \log(n) \) is not a high cost to pay for faster convergence using the proper dual bounds implied by linear relaxation.

A function like \( f \) is used by Lu et al. in a heuristic method to solve problem (2) [10]. However, their method, being heuristic, tries to guess a good \( k \in [n] \) and tries \( f(k) \) as the next dual bound. If the guess turns out to be lower than the current dual bound, they backtrack to using RTA. In contrast, Theorem 1 guarantees that Algorithm 4 always finds the best dual bound implied by linear relaxation. Nguyen et al. also use a similar function in a linear-time algorithm for problem (2) in the special case of harmonic periods [11]. Algorithm 4 may be viewed as a generalization of their algorithm to arbitrary periods.

4.2 EDF Scheduling

4.2.1 The upper bound \( L \)

The interval of interest in problem (4) is \( [D_1, L] \). The hyperperiod of the system, denoted \( T \), equals \( \text{lcm}\{T_i | i \in [n]\} \) and can be used as \( L \). A stronger bound

\[
L_a = \min\{t \in (0, T) | \text{rbf}(t) \leq t\}.
\]

can be used as \( L \) but it is harder to compute. Finally, if \( \sum_{j \in [n]} U_j < 1 \), then

\[
L_b = \max \left\{ \frac{\max_{j \in [n]} (T_j - D_j) U_j}{1 - \sum_{j \in [n]} U_j} \right\}
\]

can also be used as \( L \). Many researchers have contributed to these bounds—more details are provided, for instance, by George et al. [12]. The next theorem follows from the definition of \( L_a \) and Theorem 3.

**Theorem 5.** If \( (\alpha, \beta, a, b) \) in Algorithm 4 is initialized to \( (0, 0, 0, T) \), then Algorithm 4 combined with Algorithm 2 computes the upper bound \( L_a \) using

\[
O \left( \frac{T}{\min_{j \in [n]} \{T_j\}} \right) n^2 \log(n)
\]

arithmetic ops.

4.2.2 Solving EDF Schedulability

Baruah et al. [13, Thm. 3.5] propose the following integer program for finding a feasible solution for the problem (4) if \( L \) is ignored:

\[
\begin{align*}
\max \quad & 0 \\
\text{s.t.} \quad & T_i x_i \leq t - D_i, \quad i \in [n] \\
& \sum_{j \in [n]} C_j x_j \geq t + 1 \\
& t \in \mathbb{Z}_{\geq 0} \\
& x \in \mathbb{Z}^n
\end{align*}
\]

The single task \( (C_1, D_1, T_1) = (2, 1, 100) \) misses its first deadline but the integer program

\[
\begin{align*}
\max \quad & 0 \\
\text{s.t.} \quad & 100 x \leq t - 1 \\
& 2 x \geq t + 1 \\
& t \in \mathbb{Z}_{\geq 0} \\
& x \in \mathbb{Z}
\end{align*}
\]

is infeasible and does not find any deadline misses. This example demonstrates that the formulation contains an off-by-one error: if we want \( x_i \) to model \( \lceil (t + T_i - D_i)/T_i \rceil \) in the integer program, then we must have \( T_i x_i \leq t + T_i - D_i \).

However, the new formulation is still incorrect. The task set

| \( i \) | \( C_i \) | \( D_i \) | \( T_i \) |
|---|---|---|---|
| 1 | 5 | 10 | 13 |
| 2 | 6 | 10 | 17 |
| 3 | 1 | 31 | 20 |

6. The integer program has been adapted for synchronous systems by setting \( t_1 \) and \( k_i \) to 0. The notation has been changed to match our notation.
misses a deadline at time 10 but the integer program
\[
\begin{align*}
\max & \quad 0 \\
\text{s.t.} & \quad 13x_1 \leq t + 3 \\
& \quad 17x_2 \leq t + 7 \\
& \quad 20x_3 \leq t - 11 \\
& \quad 5x_1 + 6x_2 + x_3 \geq t + 1 \\
& \quad t \in \mathbb{Z}_{\geq 0} \\
& \quad x \in \mathbb{Z}^3
\end{align*}
\]
and its linear relaxation is infeasible. Intuitively, we expect \((t, x_1, x_2, x_3) = (10, 1, 1, 0)\) to be feasible, but it is not feasible because it violates the third inequality. The cause for the error is that the condition \(t \geq D_j - T_j\) in the subscript in the definition of \(dbf\) (definition (6)) is not accounted for in the formulation. Therefore, a little more care is needed to formulate the problem as an integer program.

Instead of trying to formulate problem (4) as a single integer program, we divide it into \(O(n)\) subproblems with simple integer programming formulations. We assume that task 1 has the smallest deadline. We assume that tasks \([n] \setminus \{1\}\) are listed in a nondecreasing order using the key \(D_j - T_j\) for each task \(j\). Now, we consider \(n\) subproblems of problem (4) where \(t\) is restricted to the following \(n\) intervals:
\[
\begin{align*}
&[D_1, \max\{D_1, D_2 - T_2\}], \\
&[\max\{D_1, D_2 - T_2\}, \max\{D_1, D_3 - T_3\}], \ldots, \\
&[\max\{D_1, D_{n-1} - T_{n-1}\}, \max\{D_1, D_n - T_n\}], \\
&[\max\{D_1, D_n - T_n\}, L).
\end{align*}
\]
Note that some of these intervals may be empty—for instance, if all deadlines are constrained, then only the last interval is nonempty. In the \(k\)-th interval, the \(dbf\) function can be simplified to
\[
\sum_{i \in [k]} \left[ \frac{t + T_i - D_i}{T_i} \right] C_i
\]
because \(t < D_{k+1} - T_{k+1} \leq D_{k+2} - T_{k+2} \leq \cdots \leq L\). Thus, the \(k\)-th subproblem is given by:
\[
\max \{ t \in [a_k, b_k] \cap \mathbb{Z} \mid t < dbf_k(t) \}
\]
where
\[
dbf_k(t) = \sum_{i \in [k]} \left[ \frac{t + T_i - D_i}{T_i} \right] C_i,
\]
\[
a_k = \max\{D_1, D_k - T_k\}, \\
b_k = \begin{cases} \max\{D_1, D_{k+1} - T_{k+1}\} - 1 & k < n \\ L - 1 & k = n \end{cases}
\]
By replacing \(t\) with \(-t\), problem (20) can be rewritten as
\[
\min \{ t \in [-b_k, -a_k] \cap \mathbb{Z} \mid t + dbf_k(-t) \geq 1 \}.
\]
If an optimal value is found, then it must be negated to get the value of the original instance. Using the identity \([-x] = -[x]\), we have
\[
\begin{align*}
dbf_k(-t) &= \sum_{i \in [k]} \left[ \frac{-t + T_i - D_i}{T_i} \right] C_i \\
&= -\sum_{i \in [k]} \left[ \frac{t + D_i - T_i}{T_i} \right] C_i.
\end{align*}
\]
Thus, the problem can be rewritten again as
\[
\min \{ t \in [-b_k, -a_k] \cap \mathbb{Z} \mid \sum_{i \in [k]} \left[ \frac{t + D_i - T_i}{T_i} \right] C_i + 1 \leq t \}.
\]

5 A divide and conquer approach for problem (4).
\begin{algorithm}
1: Let task 1 denote the task with the smallest deadline.
2: Sort the remaining tasks in a nondecreasing order using the key \(D_j - T_j\) for each \(j\).
3: \(k \leftarrow n\)
4: \textbf{while} \(k > 0\) \textbf{do}
5: \quad \(a_k \leftarrow \max\{D_1, D_k - T_k\}\)
6: \quad \textbf{if} \(k = n\) \textbf{then}
7: \quad \quad \(b_k \leftarrow (L - \alpha - 1 - \beta)\) \textbf{if} \(\sum_{j \in [k]} U_j = 1\) \textbf{else} \([-f_k(0)])
8: \quad \textbf{else}
9: \quad \quad \(b_k \leftarrow (D_1 - 1)\) \textbf{if} \(D_{k+1} - T_{k+1} < D_1\) \textbf{else} \([-f_k(0)])
10: \textbf{end if}
11: \quad \textbf{if} \(b_k < a_k\) \textbf{then}
12: \quad \quad \(k \leftarrow k - 1\)
13: \quad \textbf{continue}
14: \textbf{end if}
15: \textbf{if} \(b_k < a_k\) \textbf{then}
16: \quad \textbf{if} \(b_k < a_k\) \textbf{then}
17: \quad \quad \(k \leftarrow k - 1\)
18: \textbf{end if}
19: \textbf{end while}
20: \textbf{return} “infeasible”
\end{algorithm}

This problem can be reduced to problem (3) by choosing \((n, \alpha, \beta, a, b)\) in the target of the reduction to equal \((k, [D_1 - T_1, \ldots, D_k - T_k], 1, -b_k, -a_k)\). After the reduction, \(f\) is given by
\[
\ell \mapsto 1 + \sum_{j \in [k] \setminus [\ell]} (D_j - T_j) U_j + \sum_{j \in [k] \setminus [\ell]} \mathbb{Z} C_j 1 - \sum_{j \in [k] \setminus [\ell]} U_j
\]
and denoted \(f_k\). For any \(k \in [n]\) if \(\sum_{j \in [k]} U_j < 1\) then Algorithm 2 can be simplified to just lines 5 and 9, and the bound
\[
-f_k(0) = \sum_{j \in [k]} (T_j - D_j) U_j - 1 \leq \sum_{j \in [k]} U_j
\]
pops out of the algorithm (we provide fewer details here because the simplification steps are identical to the ones described in Section 4.1). Since \(\sum_{j \in [k]} U_j < 1\) for all \(k \in [n - 1]\), we may modify \(b_k\) to
\[
\begin{align*}
D_1 - 1 & \quad k < n, D_{k+1} - T_{k+1} \leq D_1 \\
-f_k(0) & \quad k < n, D_{k+1} - T_{k+1} > D_1 \\
L_\alpha - 1 & \quad k = n, \sum_{j \in [k]} U_j < 1 \\
L_\alpha - 1 & \quad k = n, \sum_{j \in [k]} U_j = 1.
\end{align*}
\]
The bound \(L_k\) is implicitly present in this definition of \(b_k\).

The next theorems follow from the above discussion and from Theorem 6.

Theorem 6. If \((n, \alpha, \beta, a, b)\) in Algorithm 1 is initialized with \((k, [D_1 - T_1, \ldots, D_k - T_k], 1, -b_k, -a_k)\), then Algorithm 2 combined with Algorithm 2 solves problem (20) using
\[
O\left(\frac{b_k - a_k}{\min_{j \in [k]} \{T_k\}}\right) k^2 \log(k)
\]
arithmetic ops.
5.1 FP Scheduling

We compare the number of iterations used by RTA to the number of iterations used by Algorithm 4 for synthetically generated instances of problem (4). For both algorithms, we use the initial value

\[ C_n/(1 - \sum_{j\in[n-1]} U_j). \]

The number of tasks in the system, \( n \), is fixed to one of the values

25, 50, 75, 100

and the total utilization of the subsystem \([n-1]\) is fixed to one of the values

0.70, 0.80, 0.90, 0.99

For each total utilization value, we sample \( m \) utilization vectors uniformly using the Randfiedsum algorithm [14], where \( m = 10000 \). Each utilization vector contains the utilization of the first \( n - 1 \) tasks of a system. Task \( n \) has a very small utilization because we choose an arbitrarily large value for \( D_n = T_n \) so that the generated instances are not too easy for Algorithm 1—note that \( D_n \) is in the numerator in the characterization of running time in Theorem 4. To complete the specification of the system, we sample \( n \) wcets uniformly in \([1, 1000] \cap \mathbb{N}\). The fixed interval reflects the idea that we do not expect a single reaction to an event to be too long in real systems.

For \( n = 25 \) and variable utilization, we show normalized histograms of the running times of Algorithm 4 and RTA in Figure 3. Similar to the previous paragraph, the histogram for Algorithm 4 is to the left of the histogram for RTA—this is true for both the peak of the histogram and the right end of the spread of the histogram where the frequency is nearly zero. Thus, Algorithm 4 is faster on average and in the worst-case than RTA. For each utilization, the histogram for Algorithm 4 is thinner and taller than the histogram for RTA. Therefore, the running times of Algorithm 4 are more predictable than those of RTA. We provide precise values for average running times, standard deviations, and maximum running times in Table 2.

For utilization 0.90 and variable \( n \), we show normalized histograms of the running times of Algorithm 4 and RTA in Figure 3. Similar to the previous paragraph, the histogram for Algorithm 4 is to the left of the histogram for RTA in all cases. The histograms do not vary as much as they did in the previous paragraph—while the total utilization is bounded from above by \( 1 \), \( n \) is not, and harder instances may be generated more frequently for much higher \( n \) but we decided that 100 is a reasonable upper bound for \( n \) in our experiments. From the figure, it is evident that Algorithm 4 is faster on average and in the worst case than RTA for all values of \( n \).

Although we show results for two slices of our experiments, similar trends were observed for all other slices.

5.2 EDF Scheduling

We compare the number of iterations used by QPA to the number of iterations used by Algorithm 5 for synthetically generated instances of problem (4). Recall that Algorithm 4 uses a divide-and-conquer approach to create \( n \) subproblems where the \( k \)-th subproblem is the same as the original problem except that \( t \) is restricted to \([a_k, b_k] \subseteq [D_t, L]\). Although we present Algorithm 4 as a sequential algorithm with subproblems solved in the order \( n, n-1, \ldots \), the algorithm can also proceed in parallel. Moreover, the \( k \)-th subproblems can be further divided into smaller subproblems by considering subintervals of \([a_k, b_k]\). If the sole concern is to find a feasible (not necessarily optimal) solution to problem (4), then once a feasible solution is found in one

7. The synchrony hypothesis, used by reactive languages like Esterel, assumes that reactions are small enough to fit within a tick so that they appear to be instantaneous to the programmer [15].
branch, then other branches may be aborted. Such branching strategies are also applicable to QPA, and some branching heuristics have been proposed for the feasibility variant of problem \[4] by Zhang and Burns [16]. In our evaluation, we use our sequential divide-and-conquer approach for QPA too so that the comparison is really between Algorithm \[1] and QPA. We measure the running time by adding the number of iterations for each subproblem.

The number of tasks in the system, \( n \), is fixed to one of the values 

\[
25, 50, 75, 100, 
\]

the total utilization of the system is fixed to one of the values 

\[
0.70, 0.80, 0.90, 0.99, 
\]

and the total density \( n \) of the system is fixed to one of the values 

\[
1.25, 1.50, 1.75, 2.00. 
\]

Since the total utilization is less than one in all our configurations, we use the initial value \( L_0 \) for both algorithms. For each total utilization (resp., density) value, we sample \( n \) utilization vectors (resp., density vectors) of length \( n \) uniformly using the Randfixedsum algorithm. To complete the specification of the system, we sample \( n \) wcets uniformly in \([1, 1000] \cap \mathbb{Z}\).

For \( n = 50 \), \( \sum_{j \in [n]} \delta_j = 0.95 \) and variable utilization, we show normalized histograms of the running times of QPA and Algorithm \( \[1] \) in Figure 4 and provide summary statistics in Table 3. Like the histograms in the previous section, histograms have large spreads at higher utilizations suggesting that harder instances are generated more frequently at higher utilizations. However, the x-axis extends beyond the points where the tails of the histograms seem to disappear—unlike the histograms in the previous section, the histograms in this section contains outliers which consume a very large number of iterations for at least one of the algorithms. From the maximum running time columns in Table 3, we can see that the outliers belong to QPA. This suggests that the worst-case running times of Algorithm \( \[1] \) may be bounded by a smaller function than the equivalent function for QPA. In the analysis of Algorithm \( \[1] \) in this paper, the worst-case bound is the same as that of RTA/QPA, but we conjecture that tighter bounds may be found by better analysis. For each utilization, the histogram for Algorithm \( \[1] \) is taller, thinner and to the left of the histogram for QPA (note the mean and standard deviation values in Table 3). Therefore, the running times of Algorithm \( \[1] \) are smaller and more predictable than those of QPA.

For \( n = 50 \), \( \sum_{j \in [n]} U_j = 0.90 \), and variable density, we show normalized histograms of the running times of QPA and Algorithm \( \[1] \) in Figure 5. Similar to the previous paragraph, the histogram for Algorithm \( \[1] \) is to the left of the histogram for QPA in all cases. The peaks of the histograms for Algorithm \( \[1] \) are more pronounced for lower densities. It is reasonable to expect easier instances to be generated when density is closer to 1 because when density is less than or equal to one the instances are always schedulable and solved by both algorithms within a constant number

---

8. The ratio \( \delta_i = C_i / D_i \) is called the density of task \( i \).

---

**Fig. 2:** Normalized histograms of #iterations required by RTA and Algorithm \[4\] for \( n = 25 \) and variable utilization.

**Fig. 3:** Normalized histograms of #iterations required by RTA and Algorithm \[4\] for \( \sum_{j \in [n]} U_j = 0.90 \) and variable \( n \).

| \( \sum_{j \in [n]} \delta_j \) | Mean QPA | Alg [1] | Mean QPA | Alg [1] | Mean QPA | Alg [1] |
|------------------------------|---------|---------|---------|---------|---------|---------|
| 0.65                         | 20.72   | 10.98   | 7.53    | 4.68    | 73      | 45      |
| 0.75                         | 24.58   | 12.61   | 9.21    | 5.61    | 81      | 51      |
| 0.85                         | 29.19   | 14.44   | 12.14   | 6.99    | 137     | 76      |
| 0.95                         | 35.76   | 16.81   | 18.96   | 9.60    | 231     | 112     |

**Table 3:** Statistics for \( n = 50 \), \( \sum_{j \in [n]} \delta_j = 0.95 \), and variable utilization.
of steps. Although we show results for a density between 1 and 2, we observe similar results when the total density is greater than 2.

For \[ \sum_{j \in [n]} U_j = 0.90, \sum_{j \in [n]} \delta_j = 1.75, \] and variable \( n \), we show normalized histograms of the running times of QPA and Algorithm I in Figure 3. Similar to the previous paragraphs, the histogram for Algorithm I is taller, thinner and to the left of the histogram for QPA in all cases.

We have shown results for three slices of our experiment. For all other slices, we found Algorithm I to be faster and more predictable than QPA.

6 CONCLUSION

In this paper, we reduced FP schedulability and EDF schedulability to a common target problem. Both source problems are difficult problems, and the reductions are polynomial-time. Therefore, the target problem is a good abstraction of the source problems that captures the essence of their hardness. Faster algorithms for the abstract problem immediately yield faster algorithms for FP and EDF schedulability—we hope that this fact will encourage researchers to study the abstract problem in greater detail and from different perspectives.

We used techniques from the theory of integer programming to develop a cutting plane algorithm for the abstract problem and showed that fixed-point iteration algorithms like RTA and QPA are suboptimal special cases of this algorithm. We hope that these newly discovered connections between fixed-point iteration methods and cutting plane methods will stimulate work on incorporating more advanced theories from integer programming into real-time scheduling theory.

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Fig. 6: Normalized histograms of #iterations required by QPA and Algorithm [1] for \( \sum_{j \in [n]} U_j = 0.90 \), \( \sum_{j \in [n]} \delta_j = 1.75 \), and variable \( n \).

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