MULTIDIMENSIONAL SMALL DIVISOR FUNCTIONS

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ABSTRACT. This is a short note generalizing the construction from [1], [2] to multi-
indices. We recommend to consider both references first. We obtain polar harmonic
Maaß forms of non-positive integral weight if the dimension is even and greater
than 2. We provide explicit examples in dimension 4, 6, 8, and 10.

1. Introduction - One-dimensional case

In a recent paper [2], Mertens, Ono, and Rolen defined and investigated a new
type of mock modular form, whose coefficients are given by a small divisor function.
We summarize their approach. As usual, we let \( \tau = u + iv \in \mathbb{H} \) and \( q := e^{2\pi i \tau} \). Let \( P_k \left( \frac{n}{d}, d \right) \in \mathbb{Q}[X, Y] \), and \( \psi, \chi \) be Dirichlet characters of moduli \( M_\psi, M_\chi \) respectively.
We denote by \( \chi_{-4} \) the unique odd Dirichlet character of modulus 4, and we define
\[
D_n := \left\{ d \mid n : 1 \leq d \leq \frac{n}{d} \text{ and } d \equiv \frac{n}{d} \pmod{2} \right\},
\]
\[
\sigma^{\text{sm}}_{\ell}(n) := \sum_{d \in D_n} \chi \left( \frac{n}{d} - d \right) \psi \left( \frac{n}{d} + d \right) P_k \left( \frac{n}{d}, d \right).
\]
Additionally, we require Shimura’s theta-function
\[
\theta_\psi(\tau) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) n^{\lambda_\psi} q^n, \quad \lambda_\psi := \frac{1 - \psi(-1)}{2},
\]
and recall that
\[
\theta_\psi \in \begin{cases} 
M_{\frac{1}{2}}(\Gamma_0(4M_\psi^2), \psi) & \text{if } \lambda_\psi = 0, \\
S_{\frac{3}{2}}(\Gamma_0(4M_\psi^2), \psi \cdot \chi_{-4}) & \text{if } \lambda_\psi = 1.
\end{cases}
\]
Furthermore, we recall the definition of a harmonic Maaß form\(^1\).

Definition 1.1 Let \( k \in \frac{1}{2} \mathbb{Z} \), and choose \( N \in \mathbb{N} \) such that \( 4 \mid N \) whenever \( k \notin \mathbb{Z} \).
Let \( \phi \) be a Dirichlet character of modulus \( N \).

(i) A weight \( k \) harmonic Maaß form on a subgroup \( \Gamma_0(N) \) with Nebentypus \( \phi \) is
any smooth function \( f : \mathbb{H} \to \mathbb{C} \) satisfying the following three properties:
(a) For all \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N) \) and all \( \tau \in \mathbb{H} \) we have
\[
f(\tau) = (f|k\gamma)(\tau) := \begin{cases} 
\phi(d)^{-1} (c\tau + d)^{-k} f(\gamma \tau) & \text{if } k \in \mathbb{Z}, \\
\phi(d)^{-1} (\frac{c}{d})^{2k} (c\tau + d)^{-k} f(\gamma \tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z},
\end{cases}
\]

\(^1\)Be aware that there is no overall convention which terminology encodes which growth condition.
where \( \left( \frac{c}{d} \right) \) denotes the extended Legendre symbol, and

\[
\varepsilon_d := \begin{cases} 
1 & \text{if } d \equiv 1 \pmod{4}, \\
i & \text{if } d \equiv 3 \pmod{4}.
\end{cases}
\]

for odd integers \( d \).

(b) The function \( f \) is harmonic with respect to the weight \( k \) hyperbolic Laplacian on \( \mathbb{H} \), especially

\[
0 = \Delta_k f := \left( -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \right) f.
\]

(c) The function \( f \) has at most linear exponential growth at all cusps.

(ii) A polar harmonic Maaß form is a harmonic Maaß form with isolated poles on the upper half plane.

Then the main result of [2] reads as follows.

**Theorem 1.2** ([2, Theorem 1.1]) Suppose that \( \psi = \chi \neq 1 \), and that \( P_2 \left( \frac{n}{d}, d \right) = d \). Denote the corresponding small divisor function by \( \sigma_1^{sm} \), and by \( E_2 \) the Eisenstein series

\[
E_2(\tau) := 1 - 24 \sum_{n \geq 1} \left( \sum_{d | n} d \right) q^n.
\]

Define

\[
E^+(\tau) := \frac{1}{\theta_\psi(\tau)} \left( \alpha_\psi E_2(\tau) + \sum_{n \geq 1} \sigma_1^{sm}(n) q^n \right),
\]

\[
E^-(\tau) := (-1)^{\lambda_\psi} \frac{(2\pi)^{\lambda_\psi - \frac{1}{2}} i}{8 \Gamma \left( \frac{1}{2} + \lambda_\psi \right)} \int_{-\infty}^{i \infty} \frac{\theta_\psi(w)}{(-w + \tau)^{\frac{1}{2} - \lambda_\psi}} dw,
\]

where \( \alpha_\psi \) is an implicit constant depending only on \( \psi \) to ensure a certain growth condition. Then the function \( E^+ + E^- \) is a polar harmonic Maaß form of weight \( \frac{3}{2} - \lambda_\psi \) on \( \Gamma_0 \left( 4M_\psi^2 \right) \) with Nebentypus \( \psi \cdot \chi_{-4} \).

In analogy to the classical divisor sums \( \sigma_\psi(n) \), Mertens, Ono, and Rolen called their function \( E^+ \) a mock modular Eisenstein series with Nebentypus. Furthermore, they related their result to partition functions for special choices of \( \psi \), and proved a \( p \)-adic property of \( E^+ \), compare [2, Corollary 1.3, Theorem 1.4].

In [1], Males, Rolen, and the author discovered another example of a polar harmonic Maaß form adapting the construction from [2].
Theorem 1.3 ([1, Theorem 1.1, Theorem 1.3]) Suppose that \( \psi \) is odd, \( \chi \) is even, and that \( P_2 (\frac{n}{d}, d) = d^2 \). Denote the corresponding small divisor function by \( \sigma_{2m}^{sm} \), and define

\[
\begin{align*}
\mathcal{F}^+(\tau) &:= \frac{1}{\theta_{\psi}(\tau)} \left\{ \sum_{n \geq 1} \sigma_{2m}^{sm}(n) q^n + \frac{i}{2} \sum_{n \geq 1} \psi(n) n^2 q^{n^2} + \sum_{n \geq 1} \sigma_{2m}^{sm}(n) q^n \right\} & \text{if } \chi \neq 1, \\
\mathcal{F}^-(\tau) &:= \frac{i}{\pi \sqrt{2}} \int_{-\infty}^{\infty} \frac{\theta_{\chi}(w)}{(-i(w + \tau))^2} dw.
\end{align*}
\]

(i) If \( \chi \neq 1 \) then the function \( \mathcal{F}^+ + \mathcal{F}^- \) is a polar harmonic Maaß form of weight \( \frac{3}{2} \) on \( \Gamma_0 (4M_\psi^2) \cap \Gamma_0 (4M_\chi^2) \) with Nebentypus \( \chi \cdot (\psi \cdot \chi - 4)^{-1} \).

(ii) If \( \chi = 1 \) then the function \( \mathcal{F}^+ + \mathcal{F}^- \) is a polar harmonic Maaß form of weight \( \frac{3}{2} \) on \( \Gamma_0 (4M_\psi^2) \) with Nebentypus \( (\psi \cdot \chi - 4)^{-1} \).

Moreover, if \( \psi = \chi - 4 \), \( \chi = 1 \), Males, Rolen and the author related \( \mathcal{F}^+ \) to Hurwitz class numbers, and proved a \( p \)-adic property of \( \mathcal{F}^+ \) in both cases of \( \chi \) as well, compare [1, Corollary 1.6, Theorem 1.8].

The proof of Theorem 1.2 and 1.3 is performed in three main steps. To describe them, we let

\[
\Gamma(s, z) := \int_z^{\infty} t^{s-1} e^{-t} dt,
\]

be the incomplete Gamma function, which is defined for \( \text{Re}(s) > 0 \) and \( z \in \mathbb{C} \). It can be analytically continued in \( s \) via the functional equation

\[
\Gamma(s + 1, z) = s \Gamma(s, z) + z^s e^{-z},
\]

and has the asymptotic behavior

\[
\Gamma(s, v) \sim v^{s-1} e^{-v}, \quad |v| \to \infty
\]

for \( v \in \mathbb{R} \). In addition, let

\[
\xi_\kappa := 2iv^n \frac{\partial}{\partial \kappa} = iv^\kappa \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right)
\]

be the Bruinier–Funke operator of weight \( \kappa \), and

\[
\pi_\kappa f(\tau) := \frac{(\kappa - 1)(2i)^\kappa}{4\pi} \int_{\mathbb{H}} \frac{f(x + iy) y^k dy dx}{(\tau - x + iy)^{\kappa} y^2},
\]

be the weight \( \kappa \) holomorphic projection operator, whenever \( f \) is translation invariant, and the integral converges absolutely.

Moreover, we let

\[
\begin{align*}
g(\tau) &:= \sum_{n \geq 1} \beta(n) q^n, \quad f^+(\tau) := \frac{1}{g(\tau)} \sum_{n \geq 1} \sigma_{2m}^{sm}(n) q^n, \\
f^-(\tau) &:= \sum_{m \geq 1} \alpha(m) m^{k-1} \Gamma(1 - kf, 4\pi mv) q^{-m}, \quad f(\tau) := (f^+ + f^-)(\tau).
\end{align*}
\]

Then we proceed as follows.
(I) Show that
\[ \pi_\kappa (fg)(\tau) = 0. \]
To this end, we rewrite the definition of the given non-holomorphic part (see [1, Lemma 4.1] for instance), and next we utilize the following result. Here and throughout, \( P_r^{(a,b)} \) denotes the Jacobi polynomial of degree \( r \) and parameter \( a, b \), which we introduce in Section 4.1.

**Proposition 1.4** ([1, Proposition 1.7, Corollary 4.2]) Let \( k_f \in \mathbb{R} \setminus \mathbb{N}, k_g \in \mathbb{R} \setminus (-\mathbb{N}) \), such that \( \kappa := k_f + k_g \in \mathbb{Z}_{\geq 2} \). Let \( \alpha(m), \beta(n) \) be two complex sequences, and define the functions \( f, g \) as above. Suppose that

(a) the function \( (fg)(r + iv) \) grows at most polynomially as \( v \searrow 0 \), where \( r \in \mathbb{Q} \), and that

(b) the function \( (fg)(iv) \) grows at most polynomially as \( v \nearrow \infty \).

Then the weight \( \kappa \) holomorphic projection of \( f - g \) is given by
\[ \pi_\kappa (f - g)(\tau) = -\Gamma(1 - k_f) \sum_{m \geq 1} \sum_{n \geq m} \alpha(m) \beta(n) \times \left( n^{k_f - 1} P_{\kappa - 2}^{(1-k_f,1-\kappa)} \left( 1 - \frac{2m}{n} \right) - m^{k_f - 1} q^n \right). \]

Furthermore, it holds that \( \pi_\kappa (f + g)(\tau) = (f + g)(\tau) \).

In addition, the holomorphic part \( f + g \) has to be rewritten as well, see the proof of Theorem 1.2 in [1, Section 4].

(II) We compute
\[ \xi_\kappa (fg)(\tau) = -(4\pi)^{1-k_f-k_g} \left( \sum_{m \geq 1} \overline{\alpha(m)} q^m \right) g(\tau), \]
and choose the coefficients \( \alpha(m), \beta(n) \), such that this function is modular of weight \( 2 - \kappa \).

(III) Conclude that \( fg \) is modular of weight \( \kappa \) by the following result.

**Proposition 1.5** ([2, Proposition 2.3]) Let \( h : \mathbb{H} \to \mathbb{C} \) be a translation invariant function such that \( |h(\tau)|e^\delta \) is bounded on \( \mathbb{H} \) for some \( \delta > 0 \). If the weight \( k \) holomorphic projection of \( h \) vanishes identically for some \( k > \delta + 1 \) and \( \xi_\kappa h \) is modular of weight \( 2 - k \) for some subgroup \( \Gamma < \text{SL}_2(\mathbb{Z}) \), then \( h \) is modular of weight \( k \) for \( \Gamma \).

The subtle growth conditions are required to include the case \( \pi_2 \), and are clearly satisfied if we deal with higher weight holomorphic projections, in which case the integral defining \( \pi_k \) converges absolutely.

Lastly, verify harmonicity and the growth property towards the cusps required by the definition of a harmonic Maass form.

Finally, we mention one remark from [1, p. 5], which states that there are more choices of half integral parameters \( k_f, k_g \), which lead to other choices of polynomials \( P_\ell \left( \frac{a}{q}, d \right) \) in the definition of \( \sigma_{\ell} \), such that step (I) above works.
We refer to the first two sections of [1] for more details, and for overall preliminaries introducing the aforementioned objects together with their key properties.

2. Statement of the result

We arrive at the following result by combining the lemmas from the Section 3 as outlined during Section 1. The functions $\sigma_{\ell}^{sm}$ and $f_\ell$ are defined at the beginning of Section 3.

**Theorem 2.1** Let $\psi$ be an odd Dirichlet character, $\chi$ be an even and non-trivial Dirichlet character. Let $\ell \in 2\mathbb{N} + 2$. Define $P_\ell$ as indicated in Corollary 3.2, obtaining the corresponding small divisor function $\sigma_{\ell}^{sm}$. Then the resulting function $f_\ell$ is a polar harmonic Maaß form of weight $2 - \frac{\ell}{2} \in -\mathbb{N}_0$ on $\Gamma_0(4M_\chi^2) \cap \Gamma_0(4M_\psi^2)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi - 4)^{-1}$. Its shadow is given by a non-zero constant multiple of $\theta_{\ell}^\chi$.

In other words, the technique presented in [1], [2] applies straightforward in higher even dimensions, except for dimension two. We plan to find and investigate applications of $f_\ell$ to other areas of number theory, such as combinatorics, as in the one-dimensional case [2, Corollary 1.3].

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3. Multidimensional Case

We fix $\ell \in \mathbb{N}$ throughout. Let $\vec{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$. We recall the usual multi-index conventions

$\vec{n}! := n_1 n_2 \cdots n_\ell, \quad |\vec{n}| := n_1 + \ldots + n_\ell, \quad \|\vec{n}\| := \sqrt{n_1^2 + \ldots + n_\ell^2}.$

We let $\psi \neq 1$, and consider

$\theta_{\psi}(\tau)^\ell = \sum_{\vec{n} \in \mathbb{N}^\ell} \psi(\vec{n}!) \cdot \psi^{\|\vec{n}\|^2} \cdot q^{\|\vec{n}\|^2}.$

Moreover, we relax our assumption to $P_\ell \in \mathbb{Q}(X, Y)$, and we let

$\sigma_{\ell}^{sm}(\vec{n}) := \sum_{\vec{d} \in D_{\vec{n}}} \prod_{j=1}^{\ell} \psi \left( \frac{n_j}{d_j} + \frac{d_j}{2} \right) \left( \frac{n_j}{d_j} - d_j \right) \psi \left( \frac{n_j}{d_j} + \frac{d_j}{2} \right) \left( \frac{n_j}{d_j} - d_j \right) \lambda_{\chi} \lambda_{\psi} \lambda_{\psi} \times P_\ell \left( \| (n_j/d_j)_{1 \leq j \leq \ell} \|^2, \| \vec{d} \|^2 \right).$
Consequently,
\[
f_\ell^+ (\tau) := \frac{1}{\theta_\psi (\tau) \ell} \sum_{\vec{n} \in \mathbb{N}^\ell} \sigma_\ell^{\vec{m}_\ell} (\vec{n}) q^{||\vec{n}||},
\]
\[
f_\ell^- (\tau) := \frac{1}{\Gamma (1 - k_{\ell}\ell)} \sum_{\vec{m}, \vec{n} \in \mathbb{N}^\ell} \chi (\vec{m}) (\vec{m})^{\lambda_x} ||\vec{m}||^{2(k_{\ell}\ell - 1)} \Gamma (1 - k_{\ell}\ell, 4\pi ||\vec{m}||^2) q^{||\vec{m}||^2},
\]
\[
f_\ell (\tau) := (f_\ell^+ + f_\ell^-) (\tau).
\]

We insert this setting into the constructive method described in the first section, and devote a subsection to each step.

3.1. First step. We verify that the first step continues to hold due to exactly the same proofs as in [1, Section 3]. We have to be careful regarding the summation conditions, which are determined one step after the application of the Lipschitz summation formula. Explicitly, we obtain
\[
\pi_\ell (f_\ell^- \theta_\psi^\ell (\tau)) = - \sum_{r \geq 1} \sum_{\vec{m}, \vec{n} \in \mathbb{N}^\ell, ||\vec{n}||^2 - ||\vec{m}||^2 = r} \chi (\vec{m}) (\vec{m})^{\lambda_x} \psi (\vec{n}) (\vec{n})^{\lambda_y} \times \left( ||\vec{m}||^{2(k_{\ell}\ell - 1)} P_{\kappa - 2} (1 - 2 ||\vec{m}||^2) - ||\vec{m}||^{2(k_{\ell}\ell - 1)} \right) q^r.
\]

To match this expression with \( f_\ell^+ \theta_\psi^\ell \), we rewrite the small divisor function. We substitute
\[
\vec{a} := \left( \frac{n_1}{d_1} + \frac{a_1}{2}, \ldots, \frac{n_\ell}{d_\ell} + \frac{a_\ell}{2} \right), \quad \vec{b} := \left( \frac{n_1}{d_1} - \frac{a_1}{2}, \ldots, \frac{n_\ell}{d_\ell} - \frac{a_\ell}{2} \right),
\]
from which we deduce
\[
\vec{a} = \vec{a} - \vec{b}, \quad \vec{a} + \vec{b} = (n_j/d_j)_{1 \leq j \leq \ell}, \quad ||\vec{a}|| = ||\vec{a}||^2 - ||\vec{b}||^2.
\]

Thus,
\[
f_\ell^+ \theta_\psi^\ell (\tau) = \sum_{\vec{b} \in \mathbb{N}^\ell} \sum_{\vec{a} \in \mathbb{N}^\ell} \chi (\vec{b}) (\vec{b})^{\lambda_x} \psi (\vec{a}) (\vec{a})^{\lambda_y} P_{\ell} (||\vec{a} + \vec{b}||, ||\vec{a} - \vec{b}||) q^{||\vec{a}||^2 - ||\vec{b}||^2}.
\]

We transform the summation condition.

**Lemma 3.1** We have
\[
f_\ell^+ \theta_\psi^\ell (\tau) = \sum_{r \geq 1} \sum_{\vec{m}, \vec{n} \in \mathbb{N}^\ell, ||\vec{n}||^2 - ||\vec{m}||^2 = r} \chi (\vec{m}) (\vec{m})^{\lambda_x} \psi (\vec{n}) (\vec{n})^{\lambda_y} P_{\ell} (||\vec{m} + \vec{n}||, ||\vec{m} - \vec{n}||) q^r.
\]

**Proof:** Note that if \( \vec{a} - \vec{b} \in \mathbb{N}^\ell \), then
\[
||\vec{a}||^2 - ||\vec{b}||^2 = \sum_{j=1}^{\ell} (a_j + b_j)(a_j - b_j) \geq 1.
\]
Conversely, suppose $\|\vec{a}\|^2 - \|\vec{b}\|^2 \geq 1$. Recall that $n_j = (a_j + b_j)(a_j - b_j) \in \mathbb{N}$ for every $1 \leq j \leq \ell$ by definition of $f^+$, and $a_j + b_j$ is always positive. Thus, $(a_j - b_j) \geq 1$ for every $1 \leq j \leq \ell$, which proves the lemma.

Hence, we achieve the following result by virtue of Proposition 1.4.

**Corollary 3.2** If $P_\ell$ is defined by the condition

$$\|\vec{b}\|^2(kf_\ell - 1)P_{n-2} \left(1 - 2\frac{\|\vec{a}\|^2}{\|\vec{b}\|^2}\right) - \|\vec{a}\|^2(2kf_\ell - 1) = P_\ell \left(\|\vec{a} + \vec{b}\|, \|\vec{a} - \vec{b}\|\right),$$

then we have $\pi_\kappa(f_\ell \theta_\ell^\psi)(\tau) = 0$.

3.2. **Second step.** We summarize the result of a standard calculation.

**Lemma 3.3** We have

$$\xi_\kappa(f_\ell \theta_\ell^\psi)(\tau) = -\frac{(4\pi)^{1-kf_\ell}}{\Gamma(1 - kf_\ell)} v^{k_\ell \psi} \theta_\chi(\tau)^{k_\ell} \frac{|\theta_\psi(\tau)|^{2k_\ell}}{\theta_\psi(\tau)^{k_\ell}}$$

away from the zeros of $\theta_\psi$.

**Proof:** By definition and linearity of $\xi_\kappa$, it holds that

$$\xi_\kappa(f_\ell \theta_\ell^\psi)(\tau) = (\xi_\kappa f_\ell^-)(\tau) \cdot \theta_\psi(\tau)^{k_\ell} + f_\ell^- \left(\xi_\kappa \theta_\ell^\psi\right)(\tau) = (\xi_\kappa f_\ell^-)(\tau) \cdot \theta_\psi(\tau)^{k_\ell},$$

where the last step uses that $\theta_\ell^\psi$ is holomorphic. Next, one computes

$$\left(\xi_\kappa f_\ell^-\right)(\tau) = -\frac{(4\pi)^{1-kf_\ell}}{\Gamma(1 - kf_\ell)} v^{k_\ell \psi} \sum_{\vec{m} \in \mathbb{N}^\ell} \chi(\vec{m}) \chi^{\ell}(\vec{m})^{\lambda_\chi} q^{\|\vec{m}\|^2},$$

from which we infer the claim.

Combining the previous result with the modularity of Shimura’s theta function (see equation (1)), and the fact that

$$\text{Im}(\gamma \tau) = \frac{v}{|c\tau + d|^2}$$

for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{Z})$ and every $\tau \in \mathbb{H}$, we obtain the following corollary.

**Corollary 3.4** If $\chi \neq 1$ then $\xi_\kappa(f_\ell \theta_\ell^\psi)$ is modular of weight

$$\ell \left(\frac{1}{2} + \lambda_\chi\right) - \ell \left(\frac{1}{2} + \lambda_\psi\right)$$

on $\Gamma_0(4M_\chi^2) \cap \Gamma_0(4M_\psi^2)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi^{-4})^{-1}$.

Thus, we stipulate $\psi$ to be odd, and $\chi$ to be even and non-trivial, getting

$$\kappa = 2 - (-\ell) \in \mathbb{Z}_{\geq 2}, \quad k_\ell = 2 - \frac{\ell}{2},$$

as desired.

\[\text{\footnotesize{\textsuperscript{2}Compare the proof of [1, Lemma 2.12] for some intermediate steps.}}\]
3.3. **Third step.** We verify the two remaining conditions of a polar harmonic Maaß form.

**Lemma 3.5** Let \( \tau \in \mathbb{H} \) with \( \theta_\psi(\tau) \neq 0 \). Then, the function \( f_\ell = f_\ell^+ + f_\ell^- \) satisfies

\[
0 = \Delta_{k_\ell} f_\ell,
\]

and has the required growth property of a polar harmonic Maaß form.

**Proof:** The first assertion follows by construction of \( f_\ell \). Since \( \theta_\psi^{\ell} \) is of exponential decay towards all cusps, the function \( f_\ell^+ \) admits at most linear exponential growth towards all cusps. In particular, the cusp \( i\infty \) is a removable singularity of \( f^+ \), because both numerator and denominator vanish at \( i\infty \) of order \( \ell \). In addition, the function \( f_\ell^- \) decays exponentially towards \( i\infty \), since the incomplete Gamma function does (and it dominates the powers of \( q \)). The transformation behaviour of \( \theta_\chi \) under the full modular group \( \text{SL}_2(\mathbb{Z}) \) implies that \( f_\ell^- \) is of at most moderate growth towards all cusps. Indeed, choosing suitable scaling matrices yields additional factors of polynomial growth inside the Fourier expansion of \( f_\ell^- \). This establishes the second assertion. \( \square \)

3.4. **Conclusion.** We justify the application of Proposition 1.4, which proves Theorem 2.1.

**Proof of Theorem 2.1:** By definition, the Fourier coefficients of \( \theta_\psi f_\ell^+ \) expanded at \( i\infty \) are of moderate growth, whence the growth of \( \theta_\psi f_\ell^- \) towards any cusp has to be moderate. Consequently, the growth of \( \theta_\psi f_\ell \) towards any cusp is moderate according to the proof of Lemma 3.5. Thus, the assumptions in Proposition 1.4 are satisfied by \( \theta_\psi f_\ell \). Performing the outlined steps concludes the proof of Theorem 2.1. \( \square \)

4. **Numerical examples**

4.1. **An interlude on Jacobi polynomials.** The Jacobi polynomials \( P^{(a,b)}_r \) admit a representation in terms of Gauß’ hypergeometric function \(_2F_1\), namely

\[
P^{(a,b)}_r(z) = \frac{\Gamma(a+r+1)}{r! \Gamma(a+1)} _2F_1 \left( -r, a+b+r+1, a+1, \frac{1-z}{2} \right),
\]

for any \( r \in \mathbb{N} \). This yields many identities between Jacobi polynomials of “neighboring” degree \( r \) and parameters \( a, b \), that is \( r \in \{r-1, r, r+1\} \) and analogously for \( a, b \). For instance, one could use Gauß contiguous relations, to obtain such identities.

In particular, this leads to a recursive characterization of the Jacobi polynomials. More precisely, we have

\[
P^{(a,b)}_0(z) = 1, \quad P^{(a,b)}_1(z) = \frac{1}{2} (a - b + (a + b + 2)z),
\]

\[
c_1(j) P^{(a,b)}_{j+1}(z) = (c_2(j) + c_3(j)z) P^{(a,b)}_j(z) - c_4(j) P^{(a,b)}_{j-1}(z),
\]
where
\[ c_1(j) = 2(j + 1)(j + a + b + 1)(2j + a + b), \quad c_2(j) = (2j + a + b + 1) \left( a^2 - b^2 \right), \]
\[ c_3(j) = (2j + a + b)(2j + a + b + 1)(2j + a + b + 2), \]
\[ c_4(j) = 2(j + a)(j + b)(2j + a + b + 2). \]

4.2. **Explicit examples.** Note that the parallelogram law and the fact \(|n| = \|\vec{a} + \vec{b}\||\vec{a} - \vec{b}\|\) yield
\[
\|\vec{a}\|^2 = \frac{\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2}{4} + \frac{\|\vec{a} + \vec{b}\||\vec{a} - \vec{b}\|}{2}, \\
\|\vec{b}\|^2 = \frac{\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2}{4} - \frac{\|\vec{a} + \vec{b}\||\vec{a} - \vec{b}\|}{2}.
\]
The case \(\ell = 2\) has to be excluded since \(k_{f,\ell} \neq 1\).

4.2.1. **Higher even dimensions.** On one hand, if \(\ell = 4\) for instance, we have
\[ \kappa = 6, \quad k_{f,4} = 0, \quad P_{4} \left( \frac{1}{4} - 5 \right) \left( 1 - 2 \frac{\|\vec{a}\|^2}{\|\vec{b}\|^2} \right) - \frac{1}{\|\vec{a}\|^2} = \frac{\left( \|\vec{a}\|^2 - \|\vec{b}\|^2 \right)^5}{\|\vec{a}\|^2 \|\vec{b}\|^{10}}, \]
and thus, we choose the function \(P_4\) as
\[ P_4 \left( \|\vec{a} + \vec{b}\|, \|\vec{a} - \vec{b}\| \right) = \frac{\|\vec{a} - \vec{b}\|^5 \|\vec{a} + \vec{b}\|^5}{\left( \frac{\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2}{4} - \|\vec{a} + \vec{b}\||\vec{a} - \vec{b}\| \right)^5}. \]

Similarly, we compute (with \(x := \|\vec{a}\|, y := \|\vec{b}\|\))
\[ y^{-4} P_{6} \left( 2^{2} - 7 \right) \left( 1 - 2 \frac{x^2}{y^2} \right) - x^{-4} = \left( \frac{x^2 - y^2}{x^4 y^2} \right)^2 7 x^2 + y^2, \]
\[ y^{-6} P_{8} \left( 3^{2} - 9 \right) \left( 1 - 2 \frac{x^2}{y^2} \right) - x^{-6} = \left( \frac{x^2 - y^2}{x^6 y^2} \right)^3 45 x^4 + 9 x^2 y^2 + y^4, \]
\[ y^{-8} P_{10} \left( 4^{2} - 11 \right) \left( 1 - 2 \frac{x^2}{y^2} \right) - x^{-8} = \left( \frac{x^2 - y^2}{x^8 y^2} \right)^5 286 x^6 + 66 x^4 y^2 + 11 x^2 y^4 + y^6, \]
from which we read off the corresponding definitions of \(P_\ell\).

Because of the aforementioned recursive nature of the Jacobi polynomials, the indicated pattern continues to hold for every even dimension \(\ell \in 2\mathbb{N}+2\) by induction.

4.2.2. **Higher odd dimensions.** On the other hand, the case of dimension \(\ell \in 2\mathbb{N}+1\) produces more complicated functions \(P_\ell\). For example, if \(\ell = 3\) we have
\[ \kappa = 5, \quad k_{f,3} = \frac{1}{2}, \]
\[ P_{3} \left( \frac{1}{3} - 1 \right) \left( 1 - 2 \frac{x^2}{y^2} \right) - \frac{1}{\|\vec{a}\|^2} = \frac{\left( \|\vec{a}\| - \|\vec{b}\| \right)^4 \left( \|\vec{a}\|^3 + 20 \|\vec{a}\|^2 \|\vec{b}\| + 29 \|\vec{a}\| \|\vec{b}\|^2 + 16 \|\vec{b}\|^3 \right)}{16 \|\vec{a}\| \|\vec{b}\|^4}. \]
and if $\ell = 5$, we have
\[ \kappa = 7, \quad k_{f_5} = -\frac{1}{2}, \]
\[
\begin{align*}
P_{\ell}(\frac{3}{2}, -6) & \left( 1 - \frac{\|\vec{a}\|^2}{1024\|\vec{b}\|^2} \right) - \frac{1}{104} \\
& = -\frac{693}{256}\|\vec{a}\|^{13} + \frac{4095}{256}\|\vec{b}\|^{11} - \frac{10010}{256}\|\vec{a}\|^9\|\vec{b}\|^4 + 12870\|\vec{a}\|^7\|\vec{b}\|^6 - 9009\|\vec{a}\|^5\|\vec{b}\|^8 + 3003\|\vec{a}\|^3\|\vec{b}\|^{10} - \frac{256}{104}\|\vec{b}\|^{13}.
\end{align*}
\]
We observe that we are left with odd powers of $\|\vec{a}\|$, $\|\vec{b}\|$ in both odd-dimensional cases. If we keep the dependence of $P_\ell$ on $\|\vec{a} \pm \vec{b}\|$, which ultimately justifies the terminology “divisor function”, then odd powers obstruct a definition of $P_\ell$ via the parallelogram law in these cases of $\ell$. Once more, an inductive argument via the recursive characterization of the Jacobi polynomials extends this phenomenon to all odd dimensions $\ell \in 2\mathbb{N} + 1$.

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