Calculation of three-layer rotation shells for an axisymmetric load in conditions of nonlinear creep

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Abstract. The article is devoted to the calculation of three-layer shells of rotation with a light filler under the creep conditions. Unlike previous works of the authors, the hypotheses of the theory of shallow shells are not used. General geometric and physical equations are presented, as well as a system of resolving equations for axisymmetrically loaded structures. An example of calculating a spherical dome is given. The features of changes in the stress-strain state of the considered structure during creep are revealed.

Introduction

When deriving the equations, we will rely on the technical theory of three-layer structures, according to which it is assumed that the thickness of the sheathings δ is small compared to the total thickness of the plate h, bending moments and torques, as well as longitudinal and shear forces are completely perceived by the sheathings, and the filler only works on transverse shear, perceiving the forces $Q_{\alpha}$ and $Q_{\beta}$. In addition to these assumptions, the hypothesis of the incompressibility of the middle layer in the direction normal to the median surface of the shell is introduced. The element of a three-layer shell with a lightweight filler is shown in figure 1.

![Figure 1. Three-layer shell with lightweight filler element](image-url)
We obtain the geometric equations from the relations between displacements and deformations for a three-dimensional body in the curvilinear coordinates $\alpha, \beta, \eta$, having the form:

$$
\varepsilon_\alpha = \frac{1}{H_1} \frac{\partial u}{\partial \alpha} + \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial \beta} v + \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial \eta} w; \\
\varepsilon_\beta = \frac{1}{H_2} \frac{\partial v}{\partial \beta} + \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial \eta} w + \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial \alpha} u; \\
\varepsilon_\eta = \frac{1}{H_3} \frac{\partial w}{\partial \eta} + \frac{1}{H_3 H_1} \frac{\partial H_3}{\partial \alpha} u + \frac{1}{H_3 H_2} \frac{\partial H_3}{\partial \beta} v; \\
\gamma_{\alpha\beta} = \frac{H_1}{H_2} \frac{\partial}{\partial \beta} \left( \frac{1}{H_1} u \right) + \frac{H_2}{H_1} \frac{\partial}{\partial \alpha} \left( \frac{1}{H_2} v \right); \\
\gamma_{\beta\eta} = \frac{H_2}{H_3} \frac{\partial}{\partial \eta} \left( \frac{1}{H_2} v \right) + \frac{H_3}{H_2} \frac{\partial}{\partial \beta} \left( \frac{1}{H_3} w \right); \\
\gamma_{\eta\alpha} = \frac{H_3}{H_1} \frac{\partial}{\partial \alpha} \left( \frac{1}{H_3} w \right) + \frac{H_1}{H_3} \frac{\partial}{\partial \eta} \left( \frac{1}{H_1} u \right),
$$

(1)

where $H_1, H_2, H_3$ - Lame coefficients.

The curvilinear coordinate $\eta$ is replaced by the rectilinear coordinate $z$. In this case, the relationship between the Lame coefficients and the coefficients of the first quadratic form will be carried out according to the formulas:

$$
H_1 = A(1 + k_1 z); H_2 = B(1 + k_2 z); H_3 = 1,
$$

(2)

where $k_1$ and $k_2$ are the main curvatures.

Substituting (1) into (2), we obtain:

$$
\varepsilon_\alpha = \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{1}{AB} \frac{\partial A}{\partial \beta} v + k_1 w; \\
\varepsilon_\beta = \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{1}{AB} \frac{\partial B}{\partial \alpha} u + k_2 w; \\
\varepsilon_\eta = \frac{1}{A} \frac{\partial w}{\partial \eta}; \\
\gamma_{\alpha\beta} = \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{A} u \right) + \frac{B}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{B} v \right); \\
\gamma_{\beta\eta} = \frac{B}{A} \frac{\partial}{\partial \eta} \left( \frac{1}{B} v \right) + \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{A} u \right); \\
\gamma_{\eta\alpha} = \frac{A}{B} \frac{\partial}{\partial \alpha} \left( \frac{1}{A} w \right) + \frac{B}{A} \frac{\partial}{\partial \eta} \left( \frac{1}{B} w \right).
$$

(3)

From (3) deformations of the sheathings are written in the form:

$$
\varepsilon_\alpha^{(+)} = \frac{1}{A} \frac{\partial u^{(+)}}{\partial \alpha} + \frac{1}{AB} \frac{\partial A}{\partial \beta} v^{(+)} + k_1 w^{(+)}; \\
\varepsilon_\beta^{(+)} = \frac{1}{B} \frac{\partial v^{(+)}}{\partial \beta} + \frac{1}{AB} \frac{\partial B}{\partial \alpha} u^{(+)}; \\
\varepsilon_\eta^{(+)} = \frac{1}{A} \frac{\partial w^{(+)}}{\partial \eta}; \\
\gamma_{\alpha\beta}^{(+)} = \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{A} u^{(+)} \right) + \frac{B}{A} \frac{\partial}{\partial \alpha} \left( \frac{1}{B} v^{(+)} \right); \\
\gamma_{\beta\eta}^{(+)} = \frac{B}{A} \frac{\partial}{\partial \eta} \left( \frac{1}{B} v^{(+)} \right) + \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{1}{A} u^{(+)} \right); \\
\gamma_{\eta\alpha}^{(+)} = \frac{A}{B} \frac{\partial}{\partial \alpha} \left( \frac{1}{A} w^{(+)} \right) + \frac{B}{A} \frac{\partial}{\partial \eta} \left( \frac{1}{B} w^{(+)} \right).
$$

(4)

Index "+" in formula (4) corresponds to the lower sheathing and "-" corresponds to the upper sheathing. For the displacements of the filler $u^m$ and $v^m$ along the thickness of the shell, we take the linear distribution:

$$
u^m = \frac{u^+ - u^-}{2} + \frac{u^+ u^-}{h} z = u + \alpha_h z; \\
v^m = \frac{v^+ - v^-}{2} + \frac{v^+ v^-}{h} z = v + \beta_h z.
$$

(5)

Substituting (5) into (3), we obtain the relations between displacements and strains for the filler:

$$
\gamma_{\alpha\beta}^m = \alpha_h - k_1 (u + \alpha_h z) + \frac{1}{A} \frac{\partial w}{\partial \alpha}; \\
\gamma_{\beta\eta}^m = \beta_h - k_2 (v + \beta_h z) + \frac{1}{B} \frac{\partial v}{\partial \beta}.
$$

(6)

**Methods**

When obtaining the physical equations, we assume that the sheathing’s material is orthotropic. The relationship between stresses and strains in taking into account forced deformations, can be written as:

$$
\sigma_\alpha^{(+)} = \frac{E_1}{1 - \nu_1 \nu_2} \left( \varepsilon_\alpha^{(+)} + \nu_2 \varepsilon_\beta^{(+)} - \left( \varepsilon_\alpha^{(+)(\nu)} + \nu_2 \varepsilon_\beta^{(+)(\nu)} \right) \right); \\
\sigma_\beta^{(+)} = \frac{E_2}{1 - \nu_1 \nu_2} \left( \varepsilon_\beta^{(+)} + \nu_1 \varepsilon_\alpha^{(+)} - \left( \varepsilon_\beta^{(+)(\nu)} + \nu_1 \varepsilon_\alpha^{(+)(\nu)} \right) \right); \\
\varepsilon^{(+)} = G \left( \gamma_{\alpha\beta}^{(+)} - \gamma_{\alpha\beta}^{(+)(\nu)} \right).
$$

(7)
where $e^{+(-)}_a$, $e^{+(-)}_\beta$, $\gamma^{+(-)}_{\alpha\beta}$ - forced deformations, which include creep strains, thermal expansion, chemical shrinkage, etc.

Bending moments and torque are defined as follows:

$$M_\alpha = \delta\left(\sigma^+_{\alpha} - \sigma^-_{\alpha}\right) + \frac{1}{2} \left[ A \frac{\partial A}{\partial \alpha} + B \frac{\partial B}{\partial \beta} + C \frac{\partial C}{\partial \gamma} \right] - M^*_\alpha;$$

$$M_\beta = \delta\left(\sigma^+_{\beta} - \sigma^-_{\beta}\right) + \frac{1}{2} \left[ A \frac{\partial A}{\partial \beta} + B \frac{\partial B}{\partial \alpha} + C \frac{\partial C}{\partial \gamma} \right] - M^*_\beta;$$

$$H = \delta\left(\tau^+_{\alpha\beta} - \tau^-_{\alpha\beta}\right) + \frac{1}{2} \left[ A \frac{\partial A}{\partial \beta} + B \frac{\partial B}{\partial \alpha} + C \frac{\partial C}{\partial \gamma} \right] - H^*;$$

where

$$D_1 = \frac{E_\delta h^2}{2(1-\nu_1 v_2)}; D_2 = \frac{E_2 \delta h^2}{2(1-v_1 v_2)}; D_K = \frac{G \delta h^2}{2}; M^*_\alpha = \frac{D_1}{h} \left(\epsilon^{+\alpha}_a - \epsilon^{-\alpha}_a + v_1 \left(\epsilon^{+\alpha}_a - \epsilon^{-\alpha}_a\right)\right);$$

$$M^*_\beta = \frac{D_2}{h} \left(\epsilon^{+\beta}_a - \epsilon^{-\beta}_a + v_1 \left(\epsilon^{+\beta}_a - \epsilon^{-\beta}_a\right)\right); H^* = \frac{D_K}{h} \left(\gamma^{+\alpha\beta} - \gamma^{-\alpha\beta}\right).$$

Longitudinal and shear forces are written as:

$$N_\alpha = \delta\left(\sigma^+_{\alpha} + \sigma^-_{\alpha}\right) - \frac{2E_\delta}{1-\nu_1 v_2} \left[ \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{1}{B} \frac{\partial B}{\partial \alpha} + v_1 \left(\frac{\partial u}{\partial \beta} + \frac{1}{A} \frac{\partial A}{\partial \alpha}\right) \right] + \left(k_1 + v_1 k_2\right)w = N^*_\alpha;$$

$$N_\beta = \delta\left(\sigma^+_{\beta} + \sigma^-_{\beta}\right) - \frac{2E_\delta}{1-\nu_1 v_2} \left[ \frac{1}{B} \frac{\partial u}{\partial \beta} + \frac{1}{A} \frac{\partial A}{\partial \beta} + v_1 \left(\frac{\partial u}{\partial \alpha} + \frac{1}{B} \frac{\partial B}{\partial \beta}\right) \right] + \left(k_2 + v_1 k_1\right)w = N^*_\beta;$$

$$S = \delta\left(\tau^+_{\alpha\beta} + \tau^-_{\alpha\beta}\right) = 2G \delta\left[ \frac{1}{A} \frac{\partial u}{\partial \beta} + \frac{1}{B} \frac{\partial \beta}{\partial \alpha} \right] - S^*;$$

where

$$N^*_\alpha = \frac{E_\delta}{1-\nu_1 v_2} \left(\epsilon^{+\alpha}_a + \epsilon^{-\alpha}_a + v_1 \left(\epsilon^{+\alpha}_a + \epsilon^{-\alpha}_a\right)\right); N^*_\beta = \frac{E_\delta}{1-\nu_1 v_2} \left(\epsilon^{+\beta}_a + \epsilon^{-\beta}_a + v_1 \left(\epsilon^{+\beta}_a + \epsilon^{-\beta}_a\right)\right);$$

$$S^* = G \delta\left(\gamma^{+\alpha\beta} + \gamma^{-\alpha\beta}\right).$$

For the shear forces, the following relationships will be valid:

$$Q_\alpha = \frac{h_i^2}{h_i^2} \tau^m_{\alpha \gamma}dz = \frac{h_i^2}{h_i^2} G_m(z)\left(\gamma^m_{\alpha \gamma} - \gamma^m_{\alpha \gamma}\right)dz =$$

$$= \left(\alpha_h + \frac{1}{A} \frac{\partial \alpha}{\partial \alpha} - k_1 \alpha_h\right) \int_{-h_i^2}^{h_i^2} G_m(z)dz - k_2 \alpha_h \int_{-h_i^2}^{h_i^2} G_m(z)dz - Q^*_\alpha;$$

$$Q_\beta = \frac{h_i^2}{h_i^2} \tau^m_{\beta \gamma}dz = \frac{h_i^2}{h_i^2} G_m(z)\left(\gamma^m_{\beta \gamma} - \gamma^m_{\beta \gamma}\right)dz =$$

$$= \left(\beta_h + \frac{1}{B} \frac{\partial \beta}{\partial \beta} - k_2 \beta_h\right) \int_{-h_i^2}^{h_i^2} G_m(z)dz - k_2 \beta_h \int_{-h_i^2}^{h_i^2} G_m(z)dz - Q^*_\beta,$$

where $G_m$ - middle layer shear modulus.

For a shell with constant mechanical characteristics in thickness, formulas (10) are simplified and take the form:

$$Q_\alpha = q_m(h) \left(\alpha_h + \frac{1}{A} \frac{\partial \alpha}{\partial \alpha} - k_1 \alpha_h\right) - Q^*_\alpha; \quad Q_\beta = q_m(h) \left(\beta_h + \frac{1}{B} \frac{\partial \beta}{\partial \beta} - k_2 \beta_h\right) - Q^*_\beta. $$

(11)
Let us now proceed to the case of an axially symmetric loading. The element of an axisymmetrically loaded rotation shell is shown in figure 2. Putting in the general equations (8), (9), (11) \( \alpha = \varphi, \beta = \theta, A = R_1, B = r = R_2 \sin \varphi, k_1 = 1/R_1, k_2 = 1/R_2, \) and also taking into account axial symmetry, we obtain the following relationships for internal forces:

\[
M_\varphi = D_1 \left( \frac{1}{R_1} \frac{d \alpha_h}{d \varphi} + \frac{v_2}{R_2} \operatorname{ctg} \varphi \alpha_h \right) - M^*_\varphi; \quad M_\theta = D_2 \left( \frac{\operatorname{ctg} \varphi \alpha_h + v_1}{R_2} \frac{d \alpha_h}{d \varphi} \right) - M^*_\theta;
\]

\[
Q = G_m h \left( \alpha_h - \frac{u}{R_1} + \frac{1}{R_1} \frac{dw}{d \varphi} \right) - Q^*;
\]

\[
N_\varphi = \frac{2E_1 \delta}{1 - \nu_1 \nu_2} \left( \frac{1}{R_1} \frac{du}{d \varphi} + \frac{v_2}{R_2} \operatorname{ctg} \varphi u + \left[ \frac{1}{R_1} + \frac{v_2}{R_2} \right] w \right) - N^*_\varphi;
\]

\[
N_\theta = \frac{2E_2 \delta}{1 - \nu_1 \nu_2} \left( \frac{1}{R_2} \operatorname{ctg} \varphi u + \frac{v_1}{R_1} \frac{du}{d \varphi} + \left[ \frac{v_1}{R_1} + \frac{1}{R_1} \right] w \right) - N^*_\theta.
\]

**Figure 2.** The equilibrium of the rotation shell element

The equilibrium equations for the shell under consideration can be written as [1]:

\[
\frac{1}{r R_1} \frac{d}{d \varphi} \left[ (-Q \cos \varphi + N_\varphi \sin \varphi) r \right] = -p_z;
\]

\[
- \left( \frac{N_\varphi + N_\theta}{R_1 R_2} \right) + \frac{1}{r R_1} \frac{d}{d \varphi} (Q r) + p = 0;
\]

\[
\frac{d}{d \varphi} (M_\varphi r) - Q r R_1 - M_\theta \frac{dr}{d \varphi} = 0,
\]

where \( p_z = -p \cos \varphi + t \sin \varphi - \) vertical component of external load, \( p \) is the normal component of the surface load, \( t \) is the tangent component of the external load.

Omitting the intermediate calculations, we present for the problem under consideration the final system of resolving equations:
\[
\frac{R_2}{R_1} \frac{d^2 \alpha_h}{d\varphi^2} + \frac{d \alpha_h}{d\varphi} \left[ \frac{d}{d\varphi} \left( \frac{R_2}{R_1} + \frac{R_2}{R_1} \cotg \varphi \right) \right] + \alpha_h \left( -\frac{v_2 - \frac{v_1}{R_1} R_1}{v_1} \cotg \varphi - \frac{R_2}{R_1} \right) - \frac{RV}{D_1} = 0
\]

\[
= \frac{1}{D_1 \sin \varphi} \left( \frac{d}{d\varphi} \left( M_\varphi \right) - R_1 \cos \varphi M_\varphi \right); \tag{14}
\]

\[
\frac{R_2}{R_1} \frac{d^2 V}{d\varphi^2} + \frac{dV}{d\varphi} \left[ \frac{R_2}{R_1} \cotg \varphi + \frac{d}{d\varphi} \left( \frac{R_2}{R_1} \right) \right] + V \left( \frac{v_2 - \frac{v_1}{R_1} R_1}{v_1} \cotg \varphi - \frac{R_2}{R_1} \right) - \frac{2E_2 \delta}{G_\varphi h} = 0
\]

\[
= -2E_2 \delta \alpha h R_1 + \Phi(\varphi) + \Phi^*(\varphi),
\]

where

\[
\Phi(\varphi) = (1 + v_2) \left[ \frac{F(\varphi)}{\sin^2 \varphi} - pR_1 \frac{E_2}{E_1} \right] \cotg \varphi + \left( \frac{1 + v_1}{1 + v_2} \right) \frac{E_2}{E_1} \frac{R_1}{R_2 \sin^2 \varphi} \frac{d}{d\varphi} \left( -pR_2 + \frac{F(\varphi)}{\sin^2 \varphi} \left( \frac{1}{R_1} + \frac{v_2}{R_2} \right) \right);
\]

\[
\Phi^*(\varphi) = \frac{2Q E_2 \delta R_1}{G_\varphi h} - R_2 \frac{d}{d\varphi} \left( \frac{N_\varphi - v_2 N_\varphi^*}{\sin^2 \varphi} \frac{d\varphi}{d\varphi} \right) + R_1 \cotg \varphi \left( \frac{N_\varphi - v_2 N_\varphi^*}{\sin^2 \varphi} \frac{1}{1 + v_2} \frac{E_2}{E_1} \right) - (1 + v_2) \frac{N_\varphi^*}{\sin^2 \varphi} ;
\]

\[
V = R_2 Q; F(\varphi) = - \int_{\varphi_0}^{\varphi} pR_1 \frac{d\varphi}{d\varphi}.
\]

**Results and Discussion**

A three-layer dome in the form of a hemisphere rigidly clamped at the base and loaded with a uniformly distributed load of intensity \( q \) on the horizontal projection of the shell was calculated. Sheathings were accepted as elastic and isotropic.

The boundary conditions at the vertex (\( \varphi = 0 \)) for the structure under consideration are:

\[
\alpha_h = 0; \quad V = 0. \tag{15}
\]

At the base (\( \varphi = \pi / 2 \)), the boundary conditions are written in the form:

\[
\alpha_h = 0; \quad \varepsilon_\varphi = \frac{u \cos \varphi + w \sin \varphi}{r} = \frac{1}{2E_\delta} \left[ \frac{1}{R_1} \frac{dV}{d\varphi} + pR_2 - \frac{F(\varphi)}{\sin^2 \varphi} \left( \frac{1}{R_1} + \frac{v}{R_2} \right) - \frac{v \cotg \varphi}{R_2} V \right] = 0. \tag{16}
\]

The calculations were performed at \( R = 6 \) m, \( h = 8 \) cm, \( \delta = 1 \) mm, \( q = 1 \) kPa. As the material of the sheathings and the filler, steel and polyurethane foam were adopted, respectively. To describe the creep of the middle layer material, the nonlinear Maxwell-Gurevich equation was used. The creep law is not given here, it can be found in works [2-3], as well as the rheological characteristics of polyurethane foam.

The solution of system (17) was carried out by the Euler method in combination with the finite difference method.

As a result of the calculation, it was found that the maximum displacements in time are constant, and the tangential stresses in the filler and bending moments decrease. Graphs of changes in the maximum values of \( \tau_{\varphi \varphi}^m \) and \( M_\varphi \) are shown respectively in figure 3 and 4. Similar results were obtained earlier in papers [3-5] for three-layer shallow shells. Note that in three-layer beams and plates, in contrast to shells, an increase in displacements occurs at constant stresses and internal forces [6-10].
Summary

The general geometric and physical equations are obtained for three-layer shells with a lightweight filler taking into account forced deformations, which include creep, shrinkage, and temperature deformations. For an axisymmetrically loaded shell of revolution, the problem is reduced to a system of two differential equations and a technique for its solving is proposed. Using a spherical shell as an example, it is shown that displacements during creep do not change, and bending moments and tangential stresses in the filler decrease. Thus, the creep of the filler has a positive effect on the stress-strain state of the structure under consideration.

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