ON THE K-THEORY OF Z-CATEGORIES.

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Abstract. We establish connections between the concepts of Noetherian, regular coherent, and regular $n$-coherent categories for $Z$-linear categories with finitely many objects and the corresponding notions for unital rings. These connections enable us to obtain a negative $K$-theory vanishing result, a fundamental theorem, and a homotopy invariance result for the $K$-theory of $Z$-linear categories.

1. Introduction

Let $R$ be an associative ring with a unit. The fundamental theorem in $K$-theory also known as the Bass-Heller-Swan theorem, expresses the $K$-groups of $R[t, t^{-1}]$ in terms of the $K$-groups and Nil-groups of $R$

$$K_i(R[t, t^{-1}]) \simeq K_{i-1}(R) \oplus K_i(R) \oplus \text{Nil}_{i-1}(R) \oplus \text{Nil}_{i-1}(R).$$

The groups $\text{Nil}_i(R)$ for $i \in \mathbb{Z}$, and the $K$-groups $K_i(R)$ for $i < 0$, are known to vanish when $R$ is right regular (i.e. right Noetherian and right regular coherent). Swan [18] proved that $\text{Nil}_i(R)$ also vanishes when $R$ is right regular coherent and $i \geq 0$, using Quillen’s resolution and devissage theorems as the main tools. In [10], we extended the study to $n$-coherent rings, where $n \geq 0$ (here 1-coherent ring is the same as coherent ring, and 0-coherent ring is the same as Noetherian ring). We derived a new expression for $\text{Nil}_i(R)$ for a $n$-regular and $n$-coherent ring $R$, but its vanishing status remains unknown. Our current focus is on exploring various methods for computing these groups.

The algebraic $K$-theory of a ring with a unit can be generalized to categories that have additional structure, and even to non-unital rings. In the context of $K$-theory, it is often more convenient to use additive categories instead of rings. With this motivation in mind, Bartels and Lück extended the notions of regularity and regular coherence to additive categories. In [3] they proved the following result:

Theorem 1.1. [3, Corollary 12.2] Let $C$ be an additive category which is regular. Then $K_i(C) = 0$ for all $i \leq -1$.

The focus of this paper is to extend the notion of regular $n$-coherence and some vanishing results in $K$-theory from rings to $Z$-linear categories. Let $C$ be a $Z$-linear category. We define a right $C$-module as a contravariant $Z$-linear functor $F : C^{op} \to \text{Ab}$. We denote the category of right $C$-modules as $\text{Fun}(C^{op}, \text{Ab})$. Using the Yoneda lemma, we embed $C$ into $\text{Fun}(C^{op}, \text{Ab})$ with the purpose of using homological constructions in $\text{Fun}(C^{op}, \text{Ab})$ which a priori make no sense in $C$. The finiteness conditions for $\text{Fun}(C^{op}, \text{Ab})$ are defined in [5]. As the category of $R$-modules, $\text{Fun}(C^{op}, \text{Ab})$ is a Grothendieck category with a generating set of finitely generated projective objects. A right $C$-module $F$ is said to be of type $\mathcal{FP}_n$ if and
only if there exists an exact sequence

\[ P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow F \rightarrow 0 \]

where \( P_i \) is a finitely generated and projective right \( C \)-module for every \( 0 \leq i \leq n \).

A right \( C \)-module \( F \) is of type \( \mathcal{FP}_\infty \) if it is of type \( \mathcal{FP}_n \) for all \( n \geq 0 \).

We say that \( C \) is right \( n \)-coherent if the category \( \text{Fun}(C^{op}, \text{Ab}) \) is \( n \)-coherent in the sense of \([5, \text{Definition 4.6}]\). In other words, \( C \) is right \( n \)-coherent if and only if the \( C \)-modules of type \( \mathcal{FP}_n \) in \( \text{Fun}(C^{op}, \text{Ab}) \) coincide with those of type \( \mathcal{FP}_\infty \). We say that \( C \) is right \( n \)-coherence if \( C \) is right \( n \)-coherent and every \( C \)-module \( F \) of type \( \mathcal{FP}_n \) has finite projective dimension. In Proposition \([2, \text{Proposition 2.7}]\), we prove that this homological property of \( C \) also holds for \( C_{\text{fin}} \).

In Proposition \([2, \text{Proposition 2.9}]\), we establish necessary and sufficient intrinsic conditions on \( C \) for it to be right regular \( n \)-coherent. Specifically, we demonstrate that an additive category \( C \) is right \( n \)-coherent if and only if the following conditions hold:

1. Every morphism in \( C \) with a pseudo \((n-1)\)-kernel has a pseudo \( n \)-kernel.
2. For every morphism \( f : x \rightarrow y \) in \( C \) with a pseudo \( \infty \)-kernel, there exist \( k \in \mathbb{N} \) and a morphism \( \alpha : x_{k-1} \rightarrow x_{k-1} \) such that the following diagram commutes:

\[
\begin{array}{c}
\xymatrix{
x_k & x_{k-1} & x_{k-2} & \cdots & x_1 & x \ar[rr]^-{f} & & f \\
0 & \alpha & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \ar[rr]^-{f} & & f \\
x_{k-1} & x_{k-1} & x_{k-1} & \cdots & x_{k-1} & x_{k-1} \ar[rr]^-{f} & & f \\
}
\end{array}
\]

The algebraic K-theory of a \( \mathbb{Z} \)-linear category \( C \) is defined using the non-connective spectrum \( K_\infty(C_{\text{fin}}) \), which was introduced in \([16] \). Furthermore, \( C \) is associated with a ring defined as

\[
\mathcal{A}(C) = \bigoplus_{a,b \in \text{ob} C} \text{hom}_C(a, b),
\]

where \( \text{ob} C \) denotes the objects of \( C \). The multiplication and addition in \( \mathcal{A}(C) \) are naturally defined, resulting in a ring with local units. It is important to note that \( \mathcal{A}(C) \) is unital only when \( \text{ob} C \) is finite. Furthermore, it is worth mentioning that there exists an equivalence between the spectrum of the algebraic K-theory of \( C \) and the spectrum of the algebraic K-theory of \( \mathcal{A}(C) \), as shown in \([7, \text{Sec. 4.2}] \). Therefore, the K-theory groups of \( C \) and \( \mathcal{A}(C) \) coincide.

In Section \([3] \), we compare the notions of Noetherianity, regular coherence, and regular \( n \)-coherence of \( \mathcal{A}(C) \) with the corresponding notions for \( C \) when \( C \) is a \( \mathbb{Z} \)-linear category with finitely many objects. This comparison allows us to establish a relationship between the properties of \( C \) and the properties of \( \mathcal{A}(C) \). It is important to note that although some of the results we have used do not require the condition \( C \) having finitely many objects, this condition is necessary to guarantee that the ring
\(\mathcal{A}(\mathcal{C})\) has a unit. A \(\mathbb{Z}\)-linear category \(\mathcal{C}\) is right regular \(n\)-coherent if and only if the additive category \(\mathcal{C}_\oplus\) associated with \(\mathcal{C}\) has this property, as stated in Proposition 2.7. The reason for working with \(\mathbb{Z}\)-linear categories instead of additive categories is that the ring \(\mathcal{A}(\mathcal{C}_\oplus)\) does not have a unit due to the fact that \(\mathcal{C}_\oplus\) has infinitely many objects. By considering \(\mathbb{Z}\)-linear categories, we are able to address this issue and ensure the existence of a unit for the corresponding ring.

We see in Proposition 3.8 that the category \(\text{Fun}(\mathcal{C}^{\text{op}}, \text{Ab})\) is equivalent to \(\text{Mod-}\mathcal{A}(\mathcal{C})\), where \(\text{Mod-}\mathcal{A}(\mathcal{C})\) denotes the category of unital right modules over \(\mathcal{A}(\mathcal{C})\). Furthermore, in Proposition 3.8 we prove that \(\mathcal{C}\) is a \(\mathbb{Z}\)-linear category with finitely many objects then a \(\mathcal{C}\) is right Noetherian \((n\text{-coherent or regular }n\text{-coherent})\) and only if \(\mathcal{A}(\mathcal{C})\) is \((\text{strong }n\text{-coherent or }n\text{-regular and strong }n\text{-coherent})\). We use Proposition 3.8 and results for rings in order to obtain information about the \(K\)-theory of a \(\mathbb{Z}\)-linear category. These results are not completely original but the way to obtain them is.

In Section 4, we prove that if \(\mathcal{D} = \mathcal{C}, \mathcal{D} = \mathcal{C}_\oplus\) or \(\mathcal{D} = \text{colim}_{f \in F} \mathcal{C}_f\) with \(\mathcal{C}\) and \(\mathcal{C}_f\) regular \(\mathbb{Z}\)-linear categories with finitely many objects, then \(K_i(\mathcal{D}) = 0 \ \forall i < 0\). We also prove that if \(\mathcal{D} = \mathcal{C}, \mathcal{D} = \mathcal{C}_\oplus\) or \(\mathcal{D} = \text{colim}_{f \in F} \mathcal{C}_f\) with \(\mathcal{C}\) and \(\mathcal{C}_f\) regular \(\mathbb{Z}\)-linear categories with finitely many objects, then \(K_{-1}(\mathcal{D}) = 0\),

\[
K_i(\mathcal{D}) \simeq K_i(\mathcal{D}[t]) \quad \text{and} \quad K_{i+1}(\mathcal{D}[t,t^{-1}]) \simeq K_{i+1}(\mathcal{D}) \oplus K_i(\mathcal{D}) \quad \forall i \geq 0.
\]

In Proposition 4.10 we obtain a generalization of [10, Thm 3.2].

2. Modules over \(\mathbb{Z}\)-linear categories

A \(\mathbb{Z}\)-linear category is a category \(\mathcal{C}\) such that for every two objects \(a, b \in \mathcal{C}\), the set of morphisms \(\text{hom}_\mathcal{C}(a, b)\) is an abelian group, and for any other object \(c \in \mathcal{C}\), the composition

\[
\text{hom}_\mathcal{C}(b, c) \times \text{hom}_\mathcal{C}(a, b) \to \text{hom}_\mathcal{C}(a, c)
\]

is a bilinear map. Throughout this paper, we assume that \(\mathbb{Z}\)-linear categories \(\mathcal{C}\) are small, i.e. the collection of objects is a set. A \(\mathbb{Z}\)-linear category is additive if it has an initial object and finite products. We consider the free additive category \(\mathcal{C}_\oplus\) as follow. The objects of \(\mathcal{C}_\oplus\) are finite tuples of objects in \(\mathcal{C}\). A morphism from \(\mathbf{a} = (a_1, \ldots, a_k)\) to \(\mathbf{c} = (c_1, \ldots, c_m)\) for \(a_i, c_j \in \mathcal{C}\) is given by \(m \times k\) matrix of morphisms in \(\mathcal{C}\) (the composition is given by the usual row-by-column multiplication of matrices),

- \(\text{ob}_{\mathcal{C}_\oplus} = \{(c_1, \ldots, c_k) : c_i \in \mathcal{C}, k \in \mathbb{N}\}\)
- \(\text{hom}_{\mathcal{C}_\oplus}(\mathbf{a}, \mathbf{c}) = \prod_{i=1}^{k} \prod_{j=1}^{m} \text{hom}_{\mathcal{C}}(a_i, c_j)\).

There is an obvious embedding \(\mathcal{C} \to \mathcal{C}_\oplus\) which maps objects and morphisms to their associated 1-tuple. If \(\mathcal{C}\) is a \(\mathbb{Z}\)-linear category then \(\mathcal{C}_\oplus\) is a small additive category.

The idempotent completion \(\text{Idem}(\mathcal{C}_\oplus)\) of \(\mathcal{C}_\oplus\) is defined to be the following small additive category.

- \(\text{ob}(\text{Idem}(\mathcal{C}_\oplus)) = \{(c, p) : c \in \text{ob}_{\mathcal{C}_\oplus}, p : c \to c \text{ such that } p^2 = p\}\)
- \(\text{hom}_{\text{Idem}(\mathcal{C}_\oplus)}((\mathbf{c}_1, p_1), (\mathbf{c}_2, p_2)) = \{w : \mathbf{c}_1 \to \mathbf{c}_2 \text{ such that } w = p_2 wp_1\}\).

By construction \(\mathcal{C} \simeq \mathcal{C}_\oplus\) if \(\mathcal{C}\) is additive and \(\mathcal{C}_\oplus \simeq \text{Idem}(\mathcal{C}_\oplus)\) if idempotents split in \(\mathcal{C}_\oplus\). Recall the additive category \(\mathcal{C}_\oplus\) is equivalent to \(\text{Idem}(\mathcal{C}_\oplus)\) if and only if every idempotent has a kernel.

Example 2.1. Given a ring \(R\), consider \(\mathcal{C} = R\) the category which has one object \(*\) and \(\text{hom}_R(*, *) = R\). The multiplication on \(R\) gives the composition on \(R\). The
category $C_{\oplus}$ is the category whose objects are natural numbers $m > 0$ and the morphisms are the matrices with coefficients in $R$, $\text{hom}_{C_{\oplus}}(m, n) = M_{n \times m}(R)$.

Example 2.2. Let $R$ be an associative ring with unity. If $\mathcal{C}$ is the category of finitely generated free $R$-modules, then $\text{Idem}(\mathcal{C})$ is equivalent to the category of finitely generated projective $R$-modules.

2.1. Pseudo $n$-kernels and pseudo $n$-cokernels. Given a $\mathbb{Z}$-linear category $\mathcal{C}$ we recall that a pseudo kernel of a morphism $f : x \to y$ in $\mathcal{C}$ is a morphism $g : k \to x$ with $f \circ g = 0$, such that for any morphism $h : c \to x$ with $f \circ h = 0$, there exists $t : c \to k$ with $g \circ t = h$. Equivalently, a morphism $g : k \to x$ in $\mathcal{C}$ is said to be a pseudo kernel of $f$ if, for any $c \in \text{ob}\mathcal{C}$, the following sequence of abelian groups is exact

$$\text{hom}_\mathcal{C}(c, k) \to \text{hom}_\mathcal{C}(c, x) \to \text{hom}_\mathcal{C}(c, y).$$

Pseudo-kernels have been introduced by Freyd [11] as weak kernels. Pseudo-cokernels are pseudo kernels in $\mathcal{C}^{\text{op}}$. By [13] Corollary 1.1] the categories $\mathcal{C}$, $\mathcal{C}_{\oplus}$ and $\text{Idem}(\mathcal{C}_{\oplus})$ all have pseudo kernels or they don’t. Let us remark that any triangulated or abelian category has pseudo-kernels and pseudo-cokernels.

Let $n \geq 1$ and $f : x \to y$ be a morphism in $\mathcal{C}$. Following [6], we say that $f$ has a pseudo $n$-kernel if there exists a chain of morphisms

$$x_n \xrightarrow{f_n} x_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y$$

such that the following sequence of abelian groups is exact

$$\text{hom}_\mathcal{C}(-, x_n) \xrightarrow{f_{n,2}} \cdots \xrightarrow{f_{2,1}} \text{hom}_\mathcal{C}(-, x_1) \xrightarrow{f_{1,1}} \text{hom}_\mathcal{C}(-, x) \xrightarrow{f} \text{hom}_\mathcal{C}(-, y).$$

We denote the pseudo $n$-kernel by $(f_n, f_{n-1}, \cdots, f_1)$. The case $n = 1$ gives us the classic pseudo-kernels. For convenience, we let $x_0 := x$. Furthermore, any morphism $f$ in $\mathcal{C}$ will be assumed to be a pseudo 0-kernel of itself. We say that $f$ has a pseudo $\infty$-kernel if there exists a chain of morphisms

$$\cdots \xrightarrow{f_{n+1}} x_n \xrightarrow{f_n} x_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y$$

such that the following sequence of abelian groups is exact

$$\cdots \xrightarrow{f_{n+1}} \text{hom}_\mathcal{C}(-, x_{n+1}) \xrightarrow{f_n} \cdots \xrightarrow{f_2} \text{hom}_\mathcal{C}(-, x_1) \xrightarrow{f_1} \text{hom}_\mathcal{C}(-, x) \xrightarrow{f} \text{hom}_\mathcal{C}(-, y).$$

Pseudo $n$-cokernels are defined as pseudo $n$-kernels in $\mathcal{C}^{\text{op}}$.

2.2. Categories of Z-linear functors. The category of abelian groups will be denoted by $\text{Ab}$. For any $\mathbb{Z}$-linear category $\mathcal{C}$, we define a left $\mathcal{C}$-module as a $\mathbb{Z}$-linear functor $F : \mathcal{C} \to \text{Ab}$. We consider natural transformations as morphisms of $\mathcal{C}$-modules. Define a right $\mathcal{C}$-module as a $\mathbb{Z}$-linear functor $F : \mathcal{C}^{\text{op}} \to \text{Ab}$. Recall that a $\mathbb{Z}$-linear functor $F : \mathcal{C}^{\text{op}} \to \text{Ab}$ satisfies that $F(f + g) = F(f) + F(g)$ where $f, g \in \text{hom}_{\mathcal{C}^{\text{op}}}(x, y)$. In these categories limits and colimits of functors are defined objectwise. Denote by $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Ab})$ the category of right $\mathcal{C}$-modules. This category is cocomplete and abelian. If $c$ is an object of $\mathcal{C}$ then there is the corresponding representable functor $\text{hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{\text{op}} \to \text{Ab}$.

Lemma 2.3. (Yoneda Lemma) Let $\mathcal{C}$ be any $\mathbb{Z}$-linear category. Take $c \in \mathcal{C}$ and $F$ a right $\mathcal{C}$-module. Then there is a natural identification

$$\text{hom}_{\text{Fun}(\mathcal{C}^{\text{op}}, \text{Ab})}(\text{hom}_{\mathcal{C}}(-, c), F(-)) \cong F(c).$$
By Yoneda Lemma, the family \{\text{hom}_C(-, c)\}_{c \in C} is a generating set of finitely generated projective modules in \text{Fun}(C^{\text{op}}, \text{Ab}). A right \(C\)-module \(M\) is \textit{free} if it is isomorphic to \(\bigoplus_{i \in I} \text{hom}_C(-, a_i)\). It is free and finitely generated if \(I\) is finite.

Let \(R\) be a ring and \(\mathbb{R}\) be the \(\mathbb{Z}\)-linear category defined in Example 2.11. Note that
\[
\text{Mod-}R \cong \text{Fun}(\mathbb{R}, \text{Ab})
\]
\[
\text{R-Mod} \cong \text{Fun}(\mathbb{R}, \text{Ab}).
\]

2.3. \textbf{Finitely \(n\)-presented objects and \(n\)-coherent categories.} Let \(n \geq 1\) be a positive integer. According to [4 Definition 2.1] a right \(C\)-module \(F\) is said to be \textit{finitely \(n\)-presented} or \textit{of type} \(\mathcal{FP}_n\) if the functors \(\text{Ext}^{i}_{\text{Fun}(C^{\text{op}}, \text{Ab})}(F, -)\) preserves direct limits for all \(0 \leq i \leq n - 1\). Denote by \(\mathcal{FP}_0\) to the set of finitely generated objects. Then, a right \(C\)-module \(M\) is of type \(\mathcal{FP}_0\) if there exists a collection of objects \(\{c_j : j \in J\}\) for some finite set \(J\) and an epimorphism \(\bigoplus_{j \in J} \text{hom}_C(-, c_j) \rightarrow M\). Furthermore, a right \(C\)-module \(F\) is said to be of type \(\mathcal{FP}_\infty\) if it is of type \(\mathcal{FP}_n\) for all \(n \geq 0\).

Recall that a \textit{Grothendieck category} is a cocomplete abelian category, with a generating set and with exact direct limits. A Grothendieck category is \textit{locally finitely generated (presented)} if it has a set of finitely generated (presented) generators. In other words, each object is a direct union (limit) of finitely generated (presented) objects. A Grothendieck category is \textit{locally type} \(\mathcal{FP}_n\) \([5\ Definition 2.3]\), if it has a generating set consisting of objects of type \(\mathcal{FP}_n\).

According to [13 Example 3.2] any finitely generated projective right \(C\)-module is of type \(\mathcal{FP}_n\) for all \(n \geq 0\). Then, the functor category \(\text{Fun}(C^{\text{op}}, \text{Ab})\) is a locally type \(\mathcal{FP}_\infty\) Grothendieck category. Therefore, by the [5 Corollary 2.14], a right \(C\)-module \(F\) is of type \(\mathcal{FP}_n\) if and only if there exists an exact sequence
\[
P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow F \rightarrow 0
\]
where \(P_i\) is finitely generated and projective right \(C\)-module for every \(0 \leq i \leq n\).

Recall from [5 Definition 4.1] that a right \(C\)-module \(F\) is \(n\)-\textit{coherent} if satisfies the following conditions:

1. \(F\) is of type \(\mathcal{FP}_n\).
2. If \(S\) is a subobject of \(F\) that is of type \(\mathcal{FP}_{n-1}\) then \(S\) is also of type \(\mathcal{FP}_n\).

\textbf{Definition 2.4.} Let \(C\) be a \(\mathbb{Z}\)-linear category and \(n \geq 0\). We say that \(C\) is right (left) \(n\)-\textit{coherent} if every right (left) \(C\)-module \(F\) of type \(\mathcal{FP}_n\) is \(n\)-coherent.

Note that the \(\mathbb{Z}\)-linear category \(\mathcal{C}\) is right \(n\)-coherent if the category \(\text{Fun}(\mathbb{C}^{\text{op}}, \text{Ab})\) is \(n\)-coherent as a Grothendieck category in the sense of [5 Definition 4.6]. Thus by [5 Theorem 4.7], \(\mathcal{C}\) is right \(n\)-coherent if and only if the \(\mathcal{C}\)-modules of type \(\mathcal{FP}_n\) coincide with the \(\text{C}\)-modules of type \(\mathcal{FP}_\infty\).

In particular, an additive category \(\mathcal{C}\) is Noetherian as defined in [3 Definition 5.2] if and only if it is right 0-coherent \([4]\). Moreover, if \(1 \leq n \leq \infty\) and \(\mathcal{C}\) is any small additive category, then the following conditions are equivalent, as shown in [6 Proposition 5.4]:

1. \(\mathcal{C}\) is right \(n\)-coherent.

\[1\text{In [3], the word right is omitted, but we have chosen to include it in our notation.}\]
If a morphism in $C$ has a pseudo $(n-1)$-kernel, then it has a pseudo $n$-kernel.

**Definition 2.5.** Let $C$ be a $\mathbb{Z}$-linear category and $n \geq 0$. We say that $C$ is **right regular $n$-coherent** if it satisfies the following conditions:

1. $C$ is right $n$-coherent.
2. Every right $C$-module $F$ of type $FP_n$ has a projective dimension.

Let $C$ be a small additive category. Then, according to [3, Definition 5.2] $C$ is regular coherent if and only if it is right regular 1-coherent.

**Example 2.6.** Let $C$ be a small additive category and $n \geq 1$.

I. **Additive category with kernels.** By a result due to Auslander [2, Theorem 2.2.b] a small additive category $C$ with kernels is 1-coherent and every $C$-module of type $FP_1$ has projective dimension at most 2. Then $C$ is right regular 1-coherent.

II. **Von Neumann regular categories.** We recall that $C$ is called von Neumann regular if for any morphism $f : a \to b$ in $C$ there exists a morphism $g : b \to a$ such that $fgf = f$. By [4, Corollary 8.1.3] $C$ is right regular 1-coherent.

III. **Locally finitely presented categories.** An object $c \in C$ is finitely presented if the functor $\text{hom}_C(c, -)$ preserves direct limits. The category $C$ is locally finitely presented if every directed system of objects and morphisms has a direct limit, the class of finitely presented objects of $C$ is skeletally small and every object of $C$ is the direct limit of finitely presented objects. Then by [12, Lemma 2.2], every locally finitely presented category is left 1-coherent.

IV. **n-hereditary categories.** Suppose the following two conditions hold in $C$:

(a) Every morphism in $C$ with a pseudo $(n - 1)$-kernel has a pseudo $n$-kernel.

(b) For every morphism $f : x \to y$ in $C$ with pseudo $n$-kernel $(f_n, \cdots, f_1)$, there exists an endomorphism $\alpha : x_{n-1} \to x_{n-1}$ making the following diagram commute:

$\begin{array}{cccccccccc}
 & & f_k & & f_{k-1} & & f_{k-2} & & \cdots & & f_2 & & f_1 & & f & & y \\
\downarrow & & & & & & & & & & & & & & & & & & \\
x_k & \xrightarrow{f_k} & x_{k-1} & \xrightarrow{f_{k-1}} & x_{k-2} & \xrightarrow{f_{k-2}} & \cdots & \xrightarrow{f_2} & x_1 & \xrightarrow{f_1} & x & \xrightarrow{f} & y \\
\downarrow & & & & & & & & & & & & & & & & & & \\
0 & \xrightarrow{\alpha} & x_{k-1} & & f_{k-1} & & f_{k-2} & & \cdots & & f_2 & & f_1 & & f & & \\
\end{array}$

By [6, Theorem 5.5], $C$ is right $n$-coherent and every $C$-module of type $FP_n$ has projective dimension less than or equal 1. Therefore, $C$ is right regular $n$-coherent.

Due to [15, Lemma 1.1, 1.2] we have the following equivalences of categories

$\text{Fun}(C^{\text{op}}, \text{Ab}) \simeq \text{Fun}(C_{\oplus}^{\text{op}}, \text{Ab}) \simeq \text{Fun}((\text{Idem}(C_{\oplus})^{\text{op}}, \text{Ab})$

In other words $C, C_{\oplus}$ and $\text{Idem}(C_{\oplus})$ are Morita equivalents. In particular, we obtain the following result.

**Proposition 2.7.** Let $C$ be a $\mathbb{Z}$-linear category. The following are equivalent:

1. $C$ is right regular $n$-coherent.
(2) \( C_n \) is right regular \( n \)-coherent.

(3) \( \text{Idem}(C_n) \) is right regular \( n \)-coherent.

Let \( R \) be a ring with unity. A finitely \( n \)-presented right \( R \)-module \( M \) is \( n \)-coherent if every finitely \((n-1)\)-presented submodule \( N \subseteq M \) is finitely \( n \)-presented. The ring \( R \) is right \( n \)-coherent if \( R \) is \( n \)-coherent as a right \( R \)-module (i.e. if each finitely \((n-1)\)-presented right ideal of \( R \) is finitely \( n \)-presented). We say that \( R \) is strong right \( n \)-coherent if each finitely \( n \)-presented right \( R \)-module is finitely \((n+1)\)-presented. A strong \( n \)-coherence ring is equivalent to a \( n \)-coherence ring for \( n = 1 \), but it is an open question for \( n \geq 2 \). A coherent ring is a \( 1 \)-coherent ring (strong \( 1 \)-coherent ring) and it is regular if and only if every finitely presented module has finite projective dimension. Motivated by this we introduce in [10, Definition 2.9] the definition of \( n \)-regular ring. Let \( n \geq 1 \), a ring \( R \) is called right \( n \)-regular if each finitely \( n \)-presented right \( R \)-module has finite projective dimension.

**Corollary 2.8.** Let \( R \) be a ring with unity and \( n \geq 1 \). Then the following are equivalent.

1. The ring \( R \) is strong right \( n \)-coherent or right \( n \)-regular and strong right \( n \)-coherent respectively;
2. The additive category \( R_n \) is right \( n \)-coherent or right regular \( n \)-coherent respectively;
3. The additive category \( \text{Idem}(R_n) \) is right \( n \)-coherent or right regular \( n \)-coherent respectively.

Let \( C \) be a small additive category. By [3] Lemma 6.8], \( C \) is right Noetherian if and only if each object \( c \) has the following property. Consider any directed set \( I \) and collections of morphisms \( \{f_i : a_i \to c\}_{i \in I} \) with \( c \) as target such that \( f_i \subseteq f_j \) holds for \( i \leq j \), then there exists \( i_0 \in I \) with \( f_i \subseteq f_{i_0} \) for all \( i \in I \). Our aim is to find out intrinsic condition of \( C \) which guarantees that Fun(\( C^{op}, \text{Ab} \)) is regular \( n \)-coherent.

**Proposition 2.9.** Let \( C \) be a small additive category and \( n \geq 1 \). The following are equivalent:

1. \( C \) is right regular \( n \)-coherent.
2. The following conditions hold in \( C \):
   i) Every morphism in \( C \) with a pseudo \((n-1)\)-kernel has a pseudo \( n \)-kernel.
   ii) For every morphism \( f : x \to y \) in \( C \) with pseudo \( \infty \)-kernel there exists \( k \in \mathbb{N} \) and \( \alpha : x_{k-1} \to x_{k-1} \) making the following diagram commute:
   
   \[
   \begin{array}{c}
   x_k \xrightarrow{f_k} x_{k-1} \xrightarrow{f_{k-1}} x_{k-2} \rightarrow \cdots \xrightarrow{\cdots f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y \\
   \downarrow \alpha \\
   x_{k-1} \xrightarrow{f_{k-1}}
   \end{array}
   \]

**Proof.** \((1 \Rightarrow 2)\) Suppose that \( C \) is right regular \( n \)-coherent. First, we note that \((i)\) is clear by [3] Prop 5.4. Now suppose that \( f : x \to y \) is a morphism in \( C \) with a pseudo \( \infty \)-kernel \((\cdots, f_3, f_2, f_1)\). Here, we let \( f_0 := f \). Thus \( \text{coker}(f_*) \) is of type \( \mathcal{FP}_\infty \) in Fun(\( C^{op}, \text{Ab} \)) because there exists an exact sequence of the form

\[
\cdots \xrightarrow{f_*} \text{hom}_C(-, x_1) \xrightarrow{f_*} \text{hom}_C(-, x) \xrightarrow{f_*} \text{hom}_C(-, y) \to \text{coker}(f_*) \to 0.
\]
There exists \( k \in \mathbb{N} \) such that \( \ker(f_k) \) has projective dimension \( \leq k \). It implies that \( \ker(f_{k-2*}) = \ker(f_{k-1*}) = \ker(f_k) \) is projective too. Consider

\[
\cdots \to \text{hom}_C(-, x_k) \xrightarrow{f_{k*}} \text{hom}_C(-, x_{k-1}) \to \cdots
\]

where \( \iota : \text{im}(f_{k*}) \hookrightarrow \text{hom}_C(-, x_{k-1}) \) and \( \sigma : \text{hom}_C(-, x_k) \to \text{im}(f_{k*}) \) are the canonical morphisms. There exists \( \iota' : \text{im}(f_{k*}) \to \text{hom}_C(-, x_k) \) and \( \sigma' : \text{hom}_C(-, x_{k-1}) \to \text{im}(f_{k*}) \) such that \( \sigma \circ \iota' = \text{id}_{\text{im}(f_{k*})} \) and \( \iota' \circ \iota = \text{id}_{\text{im}(f_{k*})} \). By Yoneda Lemma and using the same techniques [6, Theorem 5.5] there exists \( h : x_{k-1} \to x_k \) in \( C \) such that \( h_* : \text{hom}_C(-, x_{k-1}) \to \text{hom}_C(-, x_k) \) satisfy \( h_* f_{k*} = \iota' \circ \sigma \). The morphism \( \alpha := \text{id}_{x_{k-1}} - f_k \circ h \) satisfies the desired condition.

\((2 \Rightarrow 1)\) Suppose that the affirmation (2) is satisfied for \( n \geq 1 \). Using the condition (2-i), we deduce that \( C \) is right \( n \)-coherent [6 Prop 5.4] and thus \( FP_n = FP_\infty \). Now, for each \( F : C^op \to Ab \) of type \( FP_n \) we get an exact sequence of the form

\[
\cdots \to \text{hom}_C(-, x_n) \to \cdots \to \text{hom}_C(-, x_1) \xrightarrow{f_1} \text{hom}_C(-, x) \xrightarrow{f} \text{hom}_C(-, y) \to F \to 0
\]

where \( f : x \to y \) is a morphism in \( C \). It implies that \( f \) has a pseudo \( \infty \)-kernel, and therefore, there is \( k \in \mathbb{N} \) and an endomorphism \( \alpha : x_{k-1} \to x_{k-1} \) making the following diagram commute:

\[
\begin{array}{ccc}
 & x_k & \\
 f_k & \downarrow & f_{k-1} \\
 & x_{k-1} & \\
0 & \alpha & f_{k-1}
\end{array}
\]

\[
\begin{array}{ccc}
 x_{k-1} & \alpha & x_{k-1} \\
 f_{k-1} & \downarrow & f_{k-1}
\end{array}
\]

Next, we show that \( \text{im}(f_{k-1*}) = \ker(f_{k-2*}) \) is a projective functor. Consider

\[
\begin{array}{ccc}
\text{hom}_C(-, x_k) & \xrightarrow{f_{k*}} & \text{hom}_C(-, x_{k-1}) \\
\downarrow & & \downarrow \\
\text{hom}_C(-, x_{k-1}) & \xrightarrow{f_{k-1*}} & \text{hom}_C(-, x_{k-2}) \\
\downarrow & & \downarrow \\
\text{im}(f_{k-1*}) & \xrightarrow{\iota} & \text{hom}_C(-, x_{k-2})
\end{array}
\]

where \( \sigma : \text{hom}_C(-, x_{k-1}) \to \text{im}(f_{k-1*}) \) and \( \iota : \text{im}(f_{k-1*}) \to \text{hom}_C(-, x_{k-2}) \) are the canonical natural transformations. Note that \( \text{im}(f_{k-1*}) = \ker(f_{k*}) \), then there exists unique natural transformation \( t : \text{im}(f_{k-1*}) \to \text{hom}_C(-, x_{k-1}) \) such that \( t \circ \sigma = \alpha_* \). Moreover, applying the same techniques [6 Theorem 5.5] we have

\[
\iota \circ \text{id}_{\text{im}(f_{k-1*})} \circ \sigma = \iota \circ \sigma = f_{k-1*} \circ (f_{k-1*} \circ \alpha) = f_{k-1*} \circ \alpha = f_{k-1*} \circ t \circ \sigma = \iota \circ \sigma \circ \iota \circ \sigma
\]

which implies that

\[
\text{id}_{\text{im}(f_{k-1*})} = \sigma \circ t.
\]

Then \( \sigma \) is a split epimorphism, and therefore, \( \text{im}(f_{k-1*}) \) is projective. \( \square \)
According to [3], a small additive category $C$ is considered to be right regular if it satisfies two conditions: it is both right Noetherian and right regular 1-coherent. It’s important to note that this usage of regular should not be confused with the concept of von Neumann regular.

**Corollary 2.10.** Let $C$ be a small additive category. The following are equivalent

1. $C$ is right regular.
2. The following conditions hold in $C$:
   i) Every object $c$ in $C$ has the following property. Consider any directed set $I$ and collections of morphisms $\{f_i : a_i \to c\}_{i \in I}$ with $c$ as target such that $f_i \subseteq f_j$ holds for $i \leq j$. Then there exists $i_0 \in I$ with $f_i \subseteq f_{i_0}$ for all $i \in I$.
   ii) For every morphism $f : x \to y$ in $C$ with pseudo $\infty$-kernel there exists $k \in \mathbb{N}$ and $\alpha : x_{k-1} \to x_k$ making the following diagram commute:

   ![Diagram](image)

In [3] another type of regularity is introduced due to bad behavior of regularity with respect to infinity products. Let $R$ be a ring with unity. Specifically, $R$ is right $l$-uniformly regular coherent, if every finitely presented right $R$-module $M$ admits a $l$-dimensional finite projective resolution, i.e. there exists an exact sequence

$$0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$$

where each $P_i$ is finitely generated and projective right $R$-module. This concept is extended to additive categories in [3, Section 6]. Let $C$ be a $\mathbb{Z}$-linear category and $l \geq 1$. We say that $C$ is right $l$-uniformly regular coherent, if every right $C$-module $F$ of type $FP_1$ admits a $l$-dimensional finite projective resolution, i.e. there exists an exact sequence

$$0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to F \to 0$$

where each $P_i$ is finitely generated and projective right $C$-module.

The equivalence $\text{Fun}(C^{\text{op}}, \text{Ab}) \simeq \text{Fun}(C_{\diamond}^{\text{op}}, \text{Ab})$ implies that $C$ is right $l$-uniformly regular coherent if and only if $C_{\diamond}$ is right $l$-uniformly regular coherent. Note that, if $C$ is right 1-coherent and every right $C$-module $F$ of type $FP_1$ has a projective dimension $\leq l$ then $C$ is right $l$-uniformly regular coherent.

**Corollary 2.11.** Let $l \geq 1$ and let $C$ be a small additive category. Suppose that $C$ is right 1-coherent. Then, the following are equivalent:

1. $C$ is right $l$-uniformly regular coherent.
2. For every morphism $f : x \to y$ in $C$ there exists $l \in \mathbb{N}$, an pseudo $l$-kernel $(f_1, f_{l-1}, \cdots, f_1)$ of $f$ and $\alpha : x_{l-1} \to x_{l-1}$ making the following diagram commute:
3. The ring $\mathcal{A}(C)$ and the $\mathbb{Z}$-linear category $C$

In this section we study the relation between some properties of a $\mathbb{Z}$-linear category $C$ with the properties of a ring $\mathcal{A}(C)$ associated with it. We prove the categories $\text{Fun}(C^{\text{op}}, \text{Ab})$ and $\text{Mod-}\mathcal{A}(C)$ are equivalent.

3.1. The ring $\mathcal{A}(C)$. Let $C$ be a $\mathbb{Z}$-linear category. Recall from [7]

\[(3.1) \quad \mathcal{A}(C) = \bigoplus_{a,b \in \text{ob} C} \text{hom}_C(a, b).\]

If $f \in \mathcal{A}(C)$ write $f_{a,b}$ for the component in $\text{hom}_C(b, a)$. The following multiplication law

\[(3.2) \quad (fg)_{a,b} = \sum_{c \in \text{ob} C} f_{a,c}g_{c,b}\]

makes $\mathcal{A}(C)$ into an associative ring, which is unital if and only if $\text{ob} C$ is finite. Whatever the cardinal of $\text{ob} C$ is, $\mathcal{A}(C)$ is always a ring with local units, i.e. a filtering colimit of unital rings.

3.2. The $\mathbb{Z}$-modules. Recall that $M$ is a unital right $\mathcal{A}(C)$-module if $M \cdot \mathcal{A}(C) = M$. Consider $\text{Mod-}\mathcal{A}(C)$ the category of unital right $\mathcal{A}(C)$-modules. Let us define functors

\[S(\cdot) : \text{Fun}(C^{\text{op}}, \text{Ab}) \to \text{Mod-}\mathcal{A}(C) \quad (-)_C : \text{Mod-}\mathcal{A}(C) \to \text{Fun}(C^{\text{op}}, \text{Ab})\]

Let $M \in \text{Fun}(C^{\text{op}}, \text{Ab})$

\[S(M) = \bigoplus_{a \in \text{ob} C} M(a)\]

Let $N \in \text{Mod-}\mathcal{A}(C)$

\[N_C : C^{\text{op}} \to \text{Ab} \quad a \mapsto N \cdot \text{id}_a.\]

\[\textbf{Lemma 3.3.} \quad \text{If } N \text{ is a unital right } \mathcal{A}(C)\text{-module then}\]

\[\bigoplus_{a \in \text{ob} C} N \cdot \text{id}_a = N.\]

\[\text{Proof.} \quad \text{For every } a \in \text{ob} C \text{ we have } N \cdot \text{id}_a \subseteq N \text{ then } \bigoplus_{a \in \text{ob} C} N \cdot \text{id}_a \subseteq N. \quad \text{Let } n \in N, \text{ because } N \text{ is unital } N = N \cdot \mathcal{A}(C) \text{ then } n = \sum_{i=1}^{m} n_i \cdot f_i \text{ with } n_i \in N \text{ and } f_i \in \text{hom}_C(a_i, b_i). \quad \text{Let } I = \{a \in \text{ob} C : a = a_i, \text{for some } i = 1, \ldots, m\} \text{ then}\]

\[n = \sum_{i=1}^{m} n_i \cdot f_i = (\sum_{i=1}^{m} n_i \cdot f_i) \cdot (\sum_{a \in I} \text{id}_a) = n \cdot \sum_{a \in I} \text{id}_a\]

We conclude $N \subseteq \bigoplus_{a \in \text{ob} C} N \cdot \text{id}_a$. \qed
Proposition 3.4. Let \( C \) be a \( \mathbb{Z} \)-linear category then
\[
S(-) : \text{Fun}(C^{op}, \text{Ab}) \to \text{Mod}\cdot\mathcal{A}(C) \quad (-)_C : \text{Mod}\cdot\mathcal{A}(C) \to \text{Fun}(C^{op}, \text{Ab})
\]
are an equivalence of categories.

Proof. Let \( N \in \text{Mod}\cdot\mathcal{A}(C) \) and \( M \in \text{Fun}(C^{op}, \text{Ab}) \) then
\[
S(N) = \bigoplus_{a \in \text{ob}C} N_c(a) = \bigoplus_{a \in \text{ob}C} N \cdot \text{id}_a = N
\]
\[
(S(M))_C(c) = S(M) \cdot \text{id}_c = \bigoplus_{a \in \text{ob}C} M(a) \cdot \text{id}_c = M(c) \quad \forall c \in \text{ob}C.
\]
\[\Box\]

The abelian structure of \( \text{Fun}(C^{op}, \text{Ab}) \) comes from the abelian structure in \( \text{Ab} \). A sequence \( M \xrightarrow{f} N \xrightarrow{g} R \) is exact in \( \text{Fun}(C^{op}, \text{Ab}) \) if for each object \( c \in C \) the sequence \( M(c) \xrightarrow{f(c)} N(c) \xrightarrow{g(c)} R(c) \) is exact in \( \text{Ab} \).

Proposition 3.5. Let \( C \) be a \( \mathbb{Z} \)-linear category then
\[
S(-) : \text{Fun}(C^{op}, \text{Ab}) \to \text{Mod}\cdot\mathcal{A}(C) \quad (-)_C : \text{Mod}\cdot\mathcal{A}(C) \to \text{Fun}(C^{op}, \text{Ab})
\]
are exact functors.

Proof. Let \( M \xrightarrow{f} N \xrightarrow{g} R \) be an exact sequence in \( \text{Mod}\cdot\mathcal{A}(C) \). Let us prove \( M_c \xrightarrow{f_c} N_c \xrightarrow{g_c} R_c \) is exact in \( \text{Fun}(C^{op}, \text{Ab}) \) showing \( M_c(a) \xrightarrow{f_c(a)} N_c(a) \xrightarrow{g_c(a)} R_c(a) \) is exact for every object \( a \in C \). By functoriality \( \text{im}(f_c(a)) \subseteq \ker(g_c(a)) \). Let \( n \cdot \text{id}_a \in \ker(g_c(a)) \) then
\[
g_c(a)(n \cdot \text{id}_a) = g(n) \cdot \text{id}_a = g(n \cdot \text{id}_a) = 0
\]
then \( n \cdot \text{id}_a \in \ker(g) = \text{im}(f) \). There exists \( m \in M \) such that \( f(m) = n \cdot \text{id}_a \) then
\[
f_c(a)(m \cdot \text{id}_a) = f(m) \cdot \text{id}_a = f(m) \cdot \text{id}_a = (n \cdot \text{id}_a) \cdot \text{id}_a = n \cdot \text{id}_a
\]
then \( n \cdot \text{id}_a \in \text{im}(f_c(a)) \). We conclude \( (-)_C \) is exact.

We proceed to show \( S \) is exact. Let \( M \xrightarrow{f} N \xrightarrow{g} R \) be an exact sequence in \( \text{Fun}(C^{op}, \text{Ab}) \). Consider
\[
S(M) = \bigoplus_{a \in \text{ob}C} M(a) \xrightarrow{S(f)} S(N) = \bigoplus_{a \in \text{ob}C} N(a) \xrightarrow{S(g)} S(R) = \bigoplus_{a \in \text{ob}C} R(a)
\]
Similarly as above, let \( \sum_{a \in C} x_a \in \ker S(g) \) then
\[
S(g)(\sum_{a \in C} x_a) = \sum_{a \in C} g(a)(x_a) = 0 \quad \forall x_{a} \in N(a) \\
x_a \in \ker g(a) = \text{im} f(a) \quad \forall x_{a} \in N(a) \\
\exists y_a \in M(a) \text{ such that } f(a)(y_a) = x_a
\]
\[\Box\]

Corollary 3.6. Let \( C \) be a \( \mathbb{Z} \)-linear category.

1. If \( p : M \to N \) is an epimorphism in \( \text{Mod}\cdot\mathcal{A}(C) \) then \( p_C : M_C \to N_C \) is an epimorphism in \( \text{Fun}(C^{op}, \text{Ab}) \).
2. If \( \pi : M \to N \) is an epimorphism in \( \text{Fun}(C^{op}, \text{Ab}) \) then \( S(\pi) : S(M) \to S(N) \) is an epimorphism in \( \text{Mod}\cdot\mathcal{A}(C) \).
3. \( (M \oplus N)_C = M_C \oplus N_C \) in \( \text{Fun}(C^{op}, \text{Ab}) \).
4. \( S(M \oplus N) = S(M) \oplus S(N) \) in \( \text{Mod}\cdot\mathcal{A}(C) \).
Let $A$ be a ring with local units. From [19] we recall that an $A$-module is quasi-free if it is isomorphic to a direct sum of modules of the form $e \cdot A$ with $e^2 = e$, $e \in A$. Quasi-free modules over a ring with local units play the same role as free modules over a ring with unity. Also recall that $M$ is a finitely generated module if and only if it is an image of a finitely generated quasi-free module. A finitely generated module $M$ is projective if and only if it is a direct summand of a finitely generated quasi-free modules. In this paper we work with $A = \mathcal{A}(C)$ and we say that $M$ is a quasi-free right $\mathcal{A}(C)$-module if it is isomorphic to a finite sum of modules $\text{id}_a \cdot \mathcal{A}(C)$.

**Lemma 3.7.** Let $C$ be a $\mathbb{Z}$-linear category.

1. If $F$ is a free finitely generated right $C$-module, then $S(F)$ is a quasi-free finitely generated right $\mathcal{A}(C)$-module.
2. If $P$ is a projective finitely generated right $C$-module, then $S(P)$ is projective and finitely generated right $\mathcal{A}(C)$-module.
3. If $M$ is a quasi-free finitely generated right $\mathcal{A}(C)$-module then $M_C$ is a finitely generated free right $C$-module.
4. If $P$ is a projective finitely generated right $\mathcal{A}(C)$-module then $P_C$ is projective and finitely generated right $C$-module.

**Proof.**

1. Let $I$ be a finite subset of objects in $C$ such that $F = \bigoplus_{b \in I} \text{hom}_C(-, b)$. Then we have

$$S(F) = \bigoplus_{b \in I} S(\text{hom}_C(-, b)) = \bigoplus_{b \in I} \text{id}_b \cdot \mathcal{A}(C)$$

This shows that $S(F)$ is a quasi-free finitely generated right $\mathcal{A}(C)$-module.

2. Suppose $P$ is a finitely generated projective right $C$-module. Then there exists a module $Q$ such that $P \oplus Q = F$, where $F$ is a free module. Moreover, we have $S(P) \oplus S(Q) = S(F)$, where $S(F)$ is quasi-free and finitely generated. Therefore, $S(P)$ is also projective.

3. Suppose $M$ is a quasi-free finitely generated right $\mathcal{A}(C)$-module. Then there exists a finite set $I$ such that $M = \bigoplus_{b \in I} \text{id}_b \cdot \mathcal{A}(C)$. Note that for any object $a$ in $C$,

$$M_C(a) = M \cdot \text{id}_a = (\bigoplus_{b \in I} \text{id}_b \cdot \mathcal{A}(C)) \cdot \text{id}_a = \bigoplus_{b \in I} \text{hom}_C(a, b)$$

Therefore, we have

$$M_C = \bigoplus_{b \in I} \text{hom}_C(-, b)$$

which is a free finitely generated module in $\text{Fun}(C^{op}, \text{Ab})$.

4. Suppose $P$ is a projective finitely generated $\mathcal{A}(C)$-module. Then there exists a module $Q$ such that $P \oplus Q = F$, where $F$ is a quasi-free finitely generated $\mathcal{A}(C)$-module. We have

$$P_C \oplus Q_C = F_C.$$ 

Therefore, $P_C$ is also projective and finitely generated. □

**Proposition 3.8.** Let $C$ be a $\mathbb{Z}$-linear category with finitely many objects and $n \geq 1$.

1. The category $C$ is right Noetherian if and only if $\mathcal{A}(C)$ is a right Noetherian ring.
The category \( \mathcal{C} \) is right \( n \)-coherent if and only if \( \mathcal{A}(\mathcal{C}) \) is a strong right \( n \)-coherent ring.

The category \( \mathcal{C} \) is regular \( n \)-coherent if and only if \( \mathcal{A}(\mathcal{C}) \) is a right \( n \)-regular and strong right \( n \)-coherent ring.

Proof. (1) Let \( M \) be a finitely generated right \( \mathcal{A}(\mathcal{C}) \)-module and let \( N \) be a submodule. Consider the epimorphism

\[
\mathcal{A}(\mathcal{C}) \oplus \ldots \oplus \mathcal{A}(\mathcal{C}) \to M,
\]

and let us apply Corollary 3.6 to obtain the following epimorphism:

\[
\mathcal{A}(\mathcal{C}) C \oplus \ldots \oplus \mathcal{A}(\mathcal{C}) C \to M C.
\]

As \( \mathcal{A}(\mathcal{C}) C = \bigoplus_{b \in \text{ob} \mathcal{C}} \text{hom}_{\mathcal{C}}(-, b) \) we obtain that \( M C \) is finitely generated.

Since \( \mathcal{C} \) is right Noetherian, we can conclude that \( N C \) is also finitely generated. Moreover, there exists an epimorphism

\[
\bigoplus_{i \in I} \text{hom}_{\mathcal{C}}(-, a_i) \to N C
\]

then

\[
\bigoplus_{i \in I} S(\text{hom}_{\mathcal{C}}(-, a_i)) \to S(N C) = N.
\]

Consider the projection

\[
p_i : \mathcal{A}(\mathcal{C}) \to S(\text{hom}_{\mathcal{C}}(-, a_i)) = \bigoplus_{c \in \text{ob} \mathcal{C}} \text{hom}_{\mathcal{C}}(c, a_i)
\]

Taking \( n = \# I \) we obtain an epimorphism

\[
\mathcal{A}(\mathcal{C})^n \to \bigoplus_{i \in I} S(\text{hom}_{\mathcal{C}}(-, a_i)) \to N,
\]

then \( N \) is finitely generated.

Conversely if \( M \in \text{Fun}(\mathcal{C}^{op}, \text{Ab}) \) is finitely generated let us show that every subobject is also finitely generated. Take \( N \) as a submodule of \( M \). There is an epimorphism

\[
\bigoplus_{i \in I} \text{hom}_{\mathcal{C}}(-, a_i) \to M
\]

then we have an epimorphism

\[
\bigoplus_{i \in I, c \in \text{ob} \mathcal{C}} \text{hom}_{\mathcal{C}}(c, a_i) = \bigoplus_{i \in I} S(\text{hom}_{\mathcal{C}}(-, a_i)) \to S(M).
\]

We obtain that \( S(N) \) is a submodule of \( S(M) \) which is finitely generated, then \( S(N) \) is also finitely generated and \( S(N) C = N \) is finitely generated.

(2) Let \( M \) be a finitely \( n \)-presented right \( \mathcal{A}(\mathcal{C}) \)-module. Consider \( m_0, m_1, \ldots, m_n \in \mathbb{N} \) such that

\[
\mathcal{A}(\mathcal{C})^{m_n} \to \mathcal{A}(\mathcal{C})^{m_{n-1}} \to \ldots \to \mathcal{A}(\mathcal{C})^{m_1} \to \mathcal{A}(\mathcal{C})^{m_0} \to M \to 0
\]

is exact. By Proposition 3.5 the following is also an exact sequence

\[
\mathcal{A}(\mathcal{C}) C^{m_n} \to \mathcal{A}(\mathcal{C}) C^{m_{n-1}} \to \ldots \to \mathcal{A}(\mathcal{C}) C^{m_1} \to \mathcal{A}(\mathcal{C}) C^{m_0} \to M C \to 0
\]
As \( \mathcal{A}(\mathcal{C})_g = \bigoplus_{b \in \text{Ob}\mathcal{C}} \text{hom}_\mathcal{C}(-, b) \) we obtain that \( M\mathcal{C} \) is of type \( \mathcal{FP}_n \). Because \( \mathcal{C} \) is right \( n \)-coherent there exists an exact sequence
\[
\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to M\mathcal{C} \to 0
\]
where each \( P_i \) is both projective and finitely generated. Then,
\[
\cdots \to S(P_{n+1}) \to S(P_n) \to \cdots \to S(P_1) \to S(P_0) \to M \to 0
\]
is exact and by Lemma 3.7, \( S(P_i) \) is projective and finitely generated. Therefore, \( \mathcal{A}(\mathcal{C}) \) is a strong right \( n \)-coherent ring.

Conversely, if \( F \in \text{Fun}(\mathcal{C}^{\text{op}}, \text{Ab}) \) is of type \( \mathcal{FP}_n \) then \( S(F) \) is an \( \mathcal{A}(\mathcal{C}) \)-module of the type \( \mathcal{FP}_n \). As \( \mathcal{A}(\mathcal{C}) \) is a strong right \( n \)-coherent ring there exists \( P_1 \) projective finitely generated \( \mathcal{A}(\mathcal{C}) \)-modules such that
\[
\cdots \to P_n \to \cdots \to P_1 \to P_0 \to S(F) \to 0
\]
Then
\[
\cdots \to (P_n)_\mathcal{C} \to \cdots \to (P_1)_\mathcal{C} \to (P_0)_\mathcal{C} \to F \to 0
\]
where \( (P_i)_\mathcal{C} \) are projective and finitely generated by Lemma 3.7. (4).

(3) Let \( M \) be a finitely \( n \)-presented right \( \mathcal{A}(\mathcal{C}) \)-module. By the previous item, we know that \( M\mathcal{C} \) is of type \( \mathcal{FP}_n \).

Since \( \mathcal{C} \) is right \( n \)-regular, there exists an exact sequence
\[
0 \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M\mathcal{C} \to 0
\]
where each \( P_i \) is a finitely generated projective module. Then, the sequence
\[
0 \to S(P_k) \to S(P_{k-1}) \to \cdots \to S(P_1) \to S(P_0) \to M \to 0
\]
is exact, and by Lemma 3.7, \( S(P_i) \) is also finitely generated and projective. Therefore, \( \mathcal{A}(\mathcal{C}) \) is a right \( n \)-regular and strong right \( n \)-coherent ring. The conversely is similar.

\[\Box\]

Example 3.9. Let us consider some examples of \( \mathbb{Z} \)-linear categories with finitely many objects.

(1) Let \( R \) be a ring and \( G = \mathbb{Z}_n \). Consider \( \tilde{R} = \frac{R[t]}{<t^n>} \). The category \( \mathcal{C}_{\tilde{R}} \) is the category with \( n \) objects and
\[
\text{hom}_{\mathcal{C}_{\tilde{R}}}(p, q) = \tilde{R}_{q-p} = R
\]
Note \( \mathcal{A}(\mathcal{C}_{\tilde{R}}) = M_{n \times n}(R) \). If \( R \) is a Noetherian ring, then \( \mathcal{A}(\mathcal{C}_{\tilde{R}}) \) is also Noetherian. By Proposition 3.8 then \( \mathcal{C}_{\tilde{R}} \) is Noetherian.

(2) We recall from [8] that a ring \( R \) is said to be \((n, d)\)-ring if every \( n \)-presented \( R \)-module has projective dimension at most \( d \). Remark that if \( n \leq n' \) and \( d \leq d' \), then every \((n, d)\)-ring is also a \((n', d')\)-ring.

Let \( R, S \) be a finite direct sum of fields and \( \mathcal{C} \) be the \( \mathbb{Z} \)-linear category with two objects \( a \) and \( b \) such that \( \text{hom}_\mathcal{C}(a, b) = \text{hom}_\mathcal{C}(b, a) = 0 \), \( \text{hom}_\mathcal{C}(a, a) = R \) and \( \text{hom}_\mathcal{C}(b, b) = S \). Notice \( \mathcal{A}(\mathcal{C}) = R \oplus S \). by [8] Theorem 1.3 (i) \( \mathcal{A}(\mathcal{C}) \) is a \((0, 0)\)-ring and hence a Noetherian and regular coherent ring.

(3) Let \( G \) be a finite commutative group. An associative ring \( R \) graded by \( G \) is
\[
R = \bigoplus_{g \in G} R_g
\]
such that the multiplication satisfies $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. A (left) graded module over $R$ is an $R$-module $M$ together with a decomposition $M = \bigoplus_{g \in G} M_g$ such that $R_g M_h \subseteq M_{g+h}$. We denote by $R$-GrMod the category of graded $R$-modules. The category $C_R$ is the $Z$-linear category whose set of objects is $\{g : g \in G\}$ and whose morphism groups are given by $\text{hom}_{C_R}(g, h) = R_{h-g}$. By [9, Lemma 2.2] there is an equivalence between $R$-GrMod and the additive functor category $\text{Fun}(C_R, \text{Ab})$.

4. K-theory of $Z$-linear categories

4.1. Vanishing negative $K$-theory. In this section, we have a result of vanishing negative $K$-theory of $Z$-linear categories. Recall from [7, Section 4] the definition of the $K$-theory spectrum of a $Z$-linear category $\mathcal{C}$, the $K$-theory spectrum of the ring $A(\mathcal{C})$ and the map

$$\varphi : K(\mathcal{C}) \to K(A(\mathcal{C}))$$

which is a natural equivalence in $\mathcal{C}$, see [7, Proposition 4.2.8].

**Theorem 4.2.** Let $\mathcal{C}$ be a $Z$-linear category with finitely many objects.

1. If $\mathcal{C}$ is right regular, then $K_i(\mathcal{C}) = 0$ for all $i < 0$.
2. If $\mathcal{C}$ is right regular coherent, then $K_{-1}(\mathcal{C}) = 0$.

**Proof.** Assume that $\mathcal{C}$ is a right regular category. Then, by Proposition 3.8, $A(\mathcal{C})$ is a right regular ring. By the fundamental theorem of $K$-theory, we have $K_i(A(\mathcal{C})) = 0$ for all $i < 0$. It follows that $K_i(\mathcal{C}) \simeq K_i(A(\mathcal{C})) = 0 \quad \forall i < 0$.

Now, assume that $\mathcal{C}$ is a right regular coherent category. Then $A(\mathcal{C})$ is a right regular coherent ring, by Proposition 3.8. By [1, Theorem 3.30], we have $K_{-1}(A(\mathcal{C})) = 0$, and $K_{-1}(\mathcal{C}) \simeq K_{-1}(A(\mathcal{C})) = 0$.

**Corollary 4.3.** Let $\mathcal{D} = \mathcal{C} \oplus \mathcal{C}$ with $\mathcal{C}$ be a $Z$-linear category with finitely many objects.

1. If $\mathcal{C}$ is right regular, then $K_i(\mathcal{D}) = 0$ for all $i < 0$.
2. If $\mathcal{C}$ is right regular coherent, then $K_{-1}(\mathcal{D}) = 0$.

**Definition 4.4.** A $Z$-linear category $\mathcal{C}$ is right $AF$-regular if there is $\{\mathcal{C}_f\}_{f \in F}$ a direct system of right regular $Z$-linear categories with finitely many objects such that

$$\mathcal{C} = \text{colim}_{f \in F} \mathcal{C}_f$$

Similarly, we say that $\mathcal{C}$ is right $AF$-Noetherian (AF-regular coherent) if

$$\mathcal{C} = \text{colim}_{f \in F} \mathcal{C}_f$$

with $\mathcal{C}_f$ being directed systems of right Noetherian (regular coherent) $Z$-linear categories with finitely many objects.

**Theorem 4.5.** Let $\mathcal{C}$ be a $Z$-linear category.

1. If $\mathcal{C}$ is right $AF$-regular, then $K_i(\mathcal{C}) = 0 \forall i < 0$.
2. If $\mathcal{C}$ is right $AF$-regular coherent, then $K_{-1}(\mathcal{C}) = 0$. 
Corollary 4.9. Let $C$ be an isomorphism for every categories. Then $K_i(C) = \lim_{t \in F} K_i(C_f)$ for all $i < 0$. The rest of the proof follows from Theorem 4.2. □

4.2. Fundamental Theorem and homotopy invariance. Let $C$ be a $\mathbb{Z}$-linear category. We consider the category $C[t]$ with the same objects of $C$ and morphisms are

$$\hom_{C[t]}(a, b) = \{ \sum_{i=0}^{n} f_i t^i : n \in \mathbb{N} \quad f_i \in \hom_C(a, b) \}.$$

Let us also consider the category $C[t^{-1}]$ with the same objects of $C$ and morphisms are

$$\hom_{C[t^{-1}]}(a, b) = \{ \sum_{i=-n}^{n} f_i t^i : n \in \mathbb{N} \quad f_i \in \hom_C(a, b) \}.$$

Remark 4.6. If $C$ is a $\mathbb{Z}$-linear category then $C[t]$ and $C[t^{-1}]$ are $\mathbb{Z}$-linear categories and

$$\mathcal{A}(C[t]) \cong \mathcal{A}(C)[t] \quad \mathcal{A}(C[t^{-1}]) \cong \mathcal{A}(C)[t^{-1}].$$

Theorem 4.7. Let $C$ be a right regular coherent $\mathbb{Z}$-linear category with finitely many objects then

$$K_i(C) \cong K_i(C[t]) \quad K_{i+1}(C[t^{-1}]) \cong K_{i+1}(C) \oplus K_i(C) \quad i \geq 0.$$

Proof. Because [4.1] is a weak equivalence then $K_i(C) \cong K_i(A(C))$. Observe that $A(C)$ is a right regular coherent ring, by Proposition [3.8] Using [18 Cor 5.3] and [18] Thm 6.1] we obtain

$$K_i(A(C)) \cong K_i(A(C)[t])$$

for $i \geq 0$. By Remark [4.9] and using again that [4.1] is a weak equivalence we obtain

$$K_i(C) \cong K_i(A(C)) \cong K_i(A(C)[t]) \cong K_i(A(C[t])) \cong K_i(C[t]) \quad i \geq 0$$

Similarly

$$K_{i+1}(C[t^{-1}]) \cong K_{i+1}(A(C[t^{-1}]))$$

$$\cong K_{i+1}(A(C)[t^{-1}])$$

$$\cong K_{i+1}(A(C)) \oplus K_i(A(C)) \quad \text{by [18] Thm 6.1] }$$

$$\cong K_{i+1}(C) \oplus K_i(C) \quad i \geq 0$$

□

Corollary 4.8. Let $C$ be a right AF-regular coherent $\mathbb{Z}$-linear category then

$$K_i(C) \cong K_i(C[t]) \quad K_{i+1}(C[t^{-1}]) \cong K_{i+1}(C) \oplus K_i(C) \quad i \geq 0$$

Denote by $\mathbb{Z}$-Cat to the category of $\mathbb{Z}$-linear categories. Consider $\mathcal{F}$ a full subcategory of $\mathbb{Z}$-Cat. A functor $F : \mathbb{Z}$-Cat $\to \mathcal{D}$ is $\mathcal{F}$-homotopy invariant if

$$F(i) : F(C) \to F(C[t])$$

is an isomorphism for every $C$ in $\mathcal{F}$.

Corollary 4.9. Let $\mathcal{F}$ the full subcategory of right AF-regular coherent $\mathbb{Z}$-linear categories. Then $K_i$ is $\mathcal{F}$-homotopy invariant for $i \geq 0$. 

Using [10, Thm 3.2] and Proposition [18] we obtain the following result.

**Proposition 4.10.** Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category with finitely many objects. Suppose that $\mathcal{C}$ is right regular $n$-coherent. Then

$$K_i(\mathcal{C}) \cong K_i(\text{FP}_n(A(\mathcal{C}))) \quad i \geq 0.$$

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