Determining normal mode features from numerical simulations using CEOF analysis: I. Test case using transverse oscillations of a magnetic slab

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ABSTRACT

We present a method to determine approximations to a system’s normal mode eigenfrequency and eigenfunctions from time-dependent numerical simulations. The method proceeds iteratively and each step consists of (i) a time-dependent numerical simulation followed by (ii) the Complex Empirical Orthogonal Function (CEOF) analysis of the simulation results. The CEOF analysis provides an approximation to the normal mode eigenfunctions that can be used to set up the initial conditions for the numerical simulation of the following iteration, in which an improved normal mode approximation is obtained. The iterative process is stopped once the global difference between successive approximate eigenfunctions is below a prescribed threshold. In this paper we test the method with a problem for which the normal modes can be computed analytically, namely the transverse oscillations of a magnetic slab, a configuration that has been extensively used in the analysis of transverse coronal loop oscillations. Our equilibrium contains material discontinuities that result in one eigenfunction with a jump across these discontinuities and two eigenfunctions whose normal derivatives are discontinuous there. After 4 iterations, the approximation to the frequency and eigenfunctions are accurate to \( \lesssim 0.7\% \) except for the eigenfunction with discontinuities, which displays a much larger error at these positions.

Subject headings: Methods: numerical — methods: data analysis — Sun: oscillations — techniques: miscellaneous

1. INTRODUCTION

It is always useful to start the study of the small amplitude, oscillatory dynamics of a system with the normal mode analysis, that is, by imposing a time dependence proportional to \( \exp(i \omega t) \) in the governing equations. One reason is that when the system is simple, the mathematical analysis

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can be more easily carried out for the normal modes than for the full time-dependent equations. Moreover, although the results of a numerical simulation often give a more faithful representation of the system dynamics, these results are made of a sum of normal modes with their own amplitudes, so that the knowledge of the normal modes' features can provide us with a physical basis for the understanding of the dynamics of a system.

The problem arises when the equilibrium configuration does not allow a simple solution of the normal mode problem and numerical techniques must be used to determine the normal modes' eigenfunctions and eigenfrequencies. General purpose (i.e., for arbitrary equilibria) numerical codes that provide this information cannot be readily found. On the other hand, general purpose numerical codes to solve time-dependent equations are much more abundant. For this reason, being able to determine the normal modes of a system from time-dependent numerical simulations is a practical effort. A spectral analysis of the variables at different points in the spatial domain do give a good indication of the frequencies present in the results, but the very relevant spatial structure of the associated eigenmodes cannot be easily achieved with such analysis. Hence, a means of extracting from time-dependent simulations the spatial profile of eigenfunctions together with their associated oscillatory frequencies is desirable. The Complex Empirical Orthogonal Function (CEOF) analysis is a tool that satisfies these requirements: it takes as its input the numerical values of one or more variables over a spatial domain and for a given time span, and returns the spatial and temporal information about the main modes of variability contained in the data, which in our case will be the main eigenmodes present in the time-dependent numerical simulations. Thus, the aim of retrieving the normal mode features is feasible with this procedure.

The main advance of this paper is the repeated application of the described combination of time-dependent numerical simulations and CEOF analysis. The later results allow to determine initial conditions (for the numerical simulations) that more accurately resemble those of the normal mode, resulting in a numerical simulation in which the amplitude of all other normal modes is reduced with respect to the previous iteration. Therefore, a repetition of this process leads to better successively approximations to a normal mode and convergence to a prescribed accuracy can be achieved.

Our interest in determining normal mode features from numerical simulations has its origin in the application of a seismological approach to the oscillations of solar coronal structures. A prominent example of this approach is that of transverse coronal loop oscillations, which were observed by, e.g., Aschwanden et al. (1999); Nakariakov et al. (1999). These events have been modeled with the help of straight loop models, whose normal modes can often be obtained either by analytical or numerical means (see Ruderman & Erdélyi 2009 for a review). The effect of curvature on the eigenmodes has been examined for a few equilibrium configurations (van Doorsselaere et al. 2009), but in more general structures the eigenmodes are difficult to calculate and for this reason time-dependent numerical simulations have been carried out to study these transverse oscillations (Selwa et al. 2006, 2007, 2011a,b; Rial et al. 2013). In these simulations, the initial disturbance excites different oscillatory harmonics, whose presence in the results is easily detected by a Fourier
analysis of the variables collected at different points in the numerical domain, but which complicates
the identification of their intrinsic features, that are useful in the seismological application (Andries
et al. 2009). For this reason, we have decided to devise the algorithm described in this paper, which
allows us to isolate the eigenmodes present in a numerical simulation. In a sequel of this work (Rial
et al. 2019) we will apply the technique presented here to the time-dependent numerical simulations
of Rial et al. (2013).

The outline of this paper is as follows: the equilibrium configuration and the equations for small
amplitude perturbations are presented in Section 2. Analytical expressions for the normal modes of
this system are introduced in Section 3. The time-dependent equations are solved in Section 4 for a
prescribed initial condition that resembles the spatial structure of the eigenmode we are interested
in. The CEOF analysis is applied to the results of this simulation (Section 5) and so the first
iteration is complete, which allows us to give an approximation to the normal mode eigenfunctions
and eigenfrequency. We next apply repeatedly the last two steps in an iterative process that
improves the accuracy of the normal mode approximation (Section 6). Our conclusions are finally
discussed in Section 7.

2. EQUILIBRIUM AND LOW-β GOVERNING EQUATIONS

We here use the Cartesian coordinate system shown in Figure 1. The equilibrium is invariant
in the y-direction and consists of a dense plasma slab that extends between \(x = -a\) and \(x = a\), so
that the slab width is \(2a\), embedded in a rarer environment that fills the space \(|x| > a\). The whole
system is bounded by the two planes \(z = \pm L/2\), with \(L\) the slab length. In the equilibrium the
magnetic field is uniform and points in the direction of the slab axis: \(B_0 = B_0 e_z\); in addition, the
plasma is at rest. The equilibrium density is expressed as

\[
\rho_0(x) = \begin{cases} 
\rho_i, & |x| \leq a, \\
\rho_e, & |x| > a.
\end{cases}
\]

The internal (i.e., inside the slab) and external Alfvén velocities are

\[
v_A(x) = \begin{cases} 
v_{Ai} \equiv \frac{B_0}{\sqrt{\mu \rho_i}}, & |x| \leq a, \\
v_{Ae} \equiv \frac{B_0}{\sqrt{\mu \rho_e}}, & |x| > a,
\end{cases}
\]

with \(\mu\) the permeability of free space.

We next introduce perturbations whose evolution is described by the ideal MHD equations,
that in the low-\(\beta\) limit (i.e., negligible plasma pressure) and in the absence of gravity read (Priest
2014).
Fig. 1.— Sketch of the equilibrium configuration, made of a plasma slab of width $2a$, length $L$, and density $\rho_i$ embedded in an environment with density $\rho_e$.

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}),
\]
(3)

\[
\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B},
\]
(4)

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).
\]
(5)

Here $\rho$, $\mathbf{v}$, and $\mathbf{B}$ are the total (equilibrium plus perturbed) density, velocity, and magnetic field. Assuming small amplitude perturbations, Equations (3)–(5) can be linearized. The density perturbation is only present in the first of these equations, so that it is a secondary quantity that can be obtained once the velocity ($\mathbf{v}$) and magnetic field ($\mathbf{b}$) perturbations are known. The linearized momentum and induction equations can be expressed as follows:

\[
\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \frac{1}{\mu} (\nabla \times \mathbf{b}) \times \mathbf{B}_0,
\]
(6)

\[
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0),
\]
(7)

where $\mathbf{v}$ and $\mathbf{b}$ are both functions of position and time.

Now, perturbations are assumed to propagate in the $y$-direction with wavenumber $k_y$ and so the $y$-dependence of $\mathbf{v}$ and $\mathbf{b}$ is of the form $\exp(-ik_y y)$. The Cartesian components of Equations (6)
and (7) then reduce to

\[
\begin{align*}
\frac{\partial v_x}{\partial t} &= \frac{B_0}{\mu \rho_0} \left( \frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x} \right), \\
\frac{\partial v_y}{\partial t} &= \frac{B_0}{\mu \rho_0} \left( \frac{\partial b_y}{\partial z} + i k_y b_z \right), \\
\frac{\partial b_x}{\partial t} &= B_0 \frac{\partial v_x}{\partial z}, \\
\frac{\partial b_y}{\partial t} &= B_0 \frac{\partial v_y}{\partial z}, \\
\frac{\partial b_z}{\partial t} &= -B_0 \left( \frac{\partial v_x}{\partial x} - i k_y v_y \right).
\end{align*}
\]

The velocity and magnetic field perturbations in these expressions are \( \mathbf{v}(x, z, t) = v_x(x, z, t) \hat{e}_x + v_y(x, z, t) \hat{e}_y \) and \( \mathbf{b}(x, z, t) = b_x(x, z, t) \hat{e}_x + b_y(x, z, t) \hat{e}_y + b_z(x, z, t) \hat{e}_z. \)

In this work we impose that the slab has a finite length, \( L \), in the \( z \)-direction (Figure 1) and that its ends are line-tied, that is, that the velocity perturbations are zero there. Given that the plasma properties are uniform along the slab, the \( z \)-dependence of \( v_x \) and \( v_y \) is \( \cos(k_z z) \). Equations (8)–(12) then reveal that the \( z \)-dependence of \( b_z \) is \( \cos(k_z z) \), while that of \( b_x \) and \( b_y \) is \( \sin(k_z z) \). \( k_z \) must be suitably selected so as to satisfy the boundary conditions at the slab ends; here we are interested in the longitudinally fundamental mode and thus chose \( k_z = \pi/L \).

3. NORMAL MODES

To study normal modes a temporal dependence of the form \( \exp(i \omega t) \) is also imposed and so the perturbed velocity and magnetic field components are

\[
\begin{align*}
v_x(x, z, t) &= \hat{v}_x(x) \cos(k_z z) e^{i \omega t}, & v_y(x, z, t) &= \hat{v}_y(x) \cos(k_z z) e^{i \omega t}, \\
b_x(x, z, t) &= \hat{b}_x(x) \sin(k_z z) e^{i \omega t}, & b_y(x, z, t) &= \hat{b}_y(x) \sin(k_z z) e^{i \omega t}, \\
b_z(x, z, t) &= \hat{b}_z(x) \cos(k_z z) e^{i \omega t}.
\end{align*}
\]

Equations (8)–(12) now reduce to

\footnote{The right-hand side of the \( z \)-component of Equation (6) is equal to zero and so it leads to \( v_z = 0 \).}
\[ \omega \hat{v}_x = -\frac{B_0}{\mu_0} \left[ k_z (i \hat{b}_x) - \frac{d}{dx} (i \hat{b}_z) \right], \quad (16) \]

\[ \omega (i \hat{v}_y) = \frac{B_0}{\mu_0} \left[ k_z \hat{b}_y + k_y (i \hat{b}_z) \right], \quad (17) \]

\[ \omega (i \hat{b}_z) = -B_0 k_z \hat{v}_x, \quad (18) \]

\[ \omega \hat{b}_y = B_0 k_z (i \hat{v}_y), \quad (19) \]

\[ \omega (i \hat{b}_z) = -B_0 \left[ \frac{d \hat{v}_x}{dx} - k_y (i \hat{v}_y) \right]. \quad (20) \]

Now, the problem is to compute the \( x \)-dependence of the eigenfunctions \( \hat{v}_x, i \hat{v}_y, i \hat{b}_x, \hat{b}_y, \) and \( i \hat{b}_z, \) which are all real, and the eigenvalue \( \omega \).

It is straightforward to eliminate all variables in favor of \( \hat{v}_x \), which leads to the following ordinary differential equation,

\[ \frac{d^2 \hat{v}_x}{dx^2} = m^2 \hat{v}_x, \quad (21) \]

with

\[ m^2 = k_y^2 + k_z^2 - \frac{\omega^2}{v_A^2}. \quad (22) \]

The parameter \( m \) takes the value \( m_{i,e} \) when the Alfvén speed is substituted by its value \( v_{Ai,e} \) inside and outside the slab, respectively. After determining \( \hat{v}_x \) one can obtain \( i \hat{v}_y \) and \( i \hat{b}_z \) from

\[ i \hat{v}_y = \frac{k_y}{m^2} \frac{d \hat{v}_x}{dx}, \quad (23) \]

\[ \frac{i \hat{b}_z}{B_0} = -\frac{1}{\omega m^2} \frac{d \hat{v}_x}{dx}, \quad (24) \]

where

\[ \kappa^2 = k_z^2 - \frac{\omega^2}{v_A^2} \equiv m^2 - k_y^2. \quad (25) \]
Again, \( \kappa \) takes the values \( \kappa_{i,e} \) inside and outside the slab, respectively. The eigenfunctions \( \hat{b}_x \) and \( \hat{b}_y \) follow from Equations (18) and (19), and are just proportional to \( \hat{v}_x \) and \( i\hat{v}_y \), respectively.

To solve Equation (21) one must impose boundary conditions in the \( x \)-direction, together with the proper jump conditions at the \( x = \pm a \) interfaces, which according to, e.g., Goedbloed & Poedts (2004) are the continuity of the normal velocity component \( (\hat{v}_x) \) and of the total pressure, which in turn leads to the continuity of \( i\hat{b}_z \).

Because of the symmetry of the equilibrium and of Equations (16)–(20) with respect to \( x = 0 \), eigenfunctions are either even or odd: for kink modes \( \hat{v}_x \) and \( i\hat{b}_x \) are even about the slab axis, while \( i\hat{v}_y \), \( \hat{b}_y \), and \( i\hat{b}_z \) are odd; for sausage modes, the parity of the 5 eigenfunctions is the opposite.

### 3.1. Laterally evanescent normal modes

Arregui et al. (2007) solved the eigenproblem of Equations (16)–(20) for solutions that are laterally evanescent, that is, for which the perturbations vanish as \( x \to \pm \infty \); the interested reader is referred to their Section 3. The kink solution that satisfies these constraints has the following \( x \)-velocity component:

\[
\hat{v}_x(x) = \begin{cases} 
C \exp(m_e x), & \text{for } x < -a, \\
A \cosh(m_i x), & \text{for } -a \leq x \leq a, \\
C \exp(-m_e x), & \text{for } x > a,
\end{cases}
\]

where the positive value of \( m_e \) is taken and

\[
C = A \exp(m_e a) \cosh(m_i a).
\]

The constant \( A \) can be arbitrarily chosen and so we set \( A = 1 \). The eigenfrequency is the solution to the dispersion relation

\[
\tanh(m_i a) = -\frac{\kappa_e^2}{\kappa_i^2} \frac{m_i}{m_e}.
\]

Figure 2 displays the eigenfunctions \( \hat{v}_x \), \( i\hat{v}_y \), and \( i\hat{b}_z \) for the fundamental kink mode. They possess the parity and continuity properties described above: \( \hat{v}_x \) is even and \( i\hat{v}_y \) and \( i\hat{b}_z \) are odd, and \( \hat{v}_x \) and \( i\hat{b}_z \) are continuous at the interfaces \( x = \pm a \). On the other hand, \( i\hat{v}_y \) jumps abruptly at these boundaries. In addition, these functions decay exponentially with \( x \), as described by Equation (26).

\(^2\) The imposed boundary conditions in the \( x \)-direction are also symmetric: see Sections 3.1 and 3.2...
3.2. Laterally confined normal modes

In Section 4 we solve numerically the initial value problem made of Equations (8)–(12) with suitable initial and boundary conditions. We consider the spatial domain $-20a \leq x \leq 20a$. Given that the boundaries are sufficiently far from the slab, the evanescent eigensolution of Figure 2 is almost zero at the edges of the numerical domain and so it is, in practice, a solution to the initial
value problem. Placing the boundaries at a finite distance from the slab, however, adds new, non-
evanescent eigensolutions that can be excited by the initial perturbation. The kink solution such
that $\hat{v}_x$ vanishes at $x = \pm L_x$ is:

$$
\hat{v}_x(x) = \begin{cases} 
  C \sin n_e (L_x + x), & \text{for } x < -a, \\
  A \cos(n_i x), & \text{for } -a \leq x \leq a, \\
  C \sin n_e (L_x - x), & \text{for } x > a,
\end{cases}
$$

where the positive value of $n_e$ is taken and

$${n_{i,e}^2} = -m_{i,e}^2,$$  \hspace{1cm} (30)

$$C = A \frac{\cos(n_i a)}{\sin n_e (L_x - a)}. \hspace{1cm} (31)$$

As before, we set $A = 1$ with no loss of generality. In the present case the dispersion relation reads

$$\tan(n_i a) = -\frac{\kappa_i^2}{\kappa_e^2} \frac{n_i}{n_e} \cot n_e (L_x - a). \hspace{1cm} (32)$$

The eigenfunctions of the fundamental confined kink solution are shown in Figure 3. This
solution also satisfies all the parity and boundary restrictions, but shows a remarkable difference in
the frequency and spatial structure compared to those of the evanescent case. In addition to this,
we also observe that while the amplitudes of $\hat{v}_x$ and $i\hat{v}_y$ are similar in the evanescent mode, and 10
times larger than that of $i\hat{b}_z/B_0$, the confined eigensolution has much larger amplitudes of $i\hat{v}_y$ and
$i\hat{b}_z/B_0$ compared to $\hat{v}_x$.

4. TIME-DEPENDENT NUMERICAL SIMULATIONS

4.1. Simulation setup and numerical method

We next drop the assumption that perturbations vary as $\exp(i\omega t)$ and are sinusoidal in the
$z$-direction, and solve Equations (8)–(12) in the region $-L_x \leq x \leq L_x, -L/2 \leq z \leq L/2$; see
Figure 1. The coefficients of the system of partial differential equations can be made real by using
the independent variables $v_x, iv_y, b_x, ib_y$, and $b_z$. Our initial disturbance is such that the full slab
is subject to an initial transverse forcing given by

$$v_x(x, z, t = 0) = v_0 \cos(k_z z) \exp \left( -\frac{x^2}{a^2} \right), \hspace{1cm} (33)$$
while all other variables are initially zero. We must emphasize that the imposed \( \cos(kz) \) dependence of \( v_x \) in this expression leads to the solution of Equations (8)–(12) having the \( z \)-dependence given by Equations (13)–(15). Furthermore, the \( v_x \) perturbation is even about \( x = 0 \) and so can only excite kink modes. Since the transverse profile of \( v_x \) resembles that of the laterally evanescent fundamental kink mode (top panel of Figure 2), a large portion of the energy in the initial disturbance will go to this mode. But one can expect that the laterally confined mode of Figure 3 will also be excited.

In the numerical simulation we use the same parameter values of Section 3, namely, \( \rho_i/\rho_e = 10 \)
and $k_z a = \pi/50$. The simulation box is determined by the lengths $L_x = 20a$ and $L = 50a$, and a uniform grid of $4001 \times 51$ points in the $x$- and $z$-directions is used. The grid is coarser along the slab because it is sufficient to capture well the smooth sinusoidal dependence of variables in the $z$-direction; on the other hand, the grid is much finer across the slab because the normal modes have much more structure in this direction. The numerical simulation is stopped at $t \simeq 280\tau_{Ai}$, which is $\sim 4.5$ and $\sim 75$ periods of the laterally evanescent and confined modes, respectively. The time step is $\Delta t = 0.704\tau_{Ai}$ and so $\simeq 5.3$ simulation results are available during each period of the laterally confined mode.

The numerical method used to solve the linearized wave equations is based on the method of lines (MOL). Time and space are treated independently, using a third order Runge-Kutta method and a six order finite difference method, respectively. Artificial dissipation is included to avoid oscillations on the grid scale. This method has been used successfully in the past (e.g., Bona et al. 2009) and has a weak effect on the attenuation of the physical oscillations reported in the simulations. Since the linear hyperbolic MHD equations are solved explicitly, the time step is subject to the CFL condition. Note that in the linearized MHD equations there are terms proportional to $k_y$, these terms are incorporated to the code as simple source terms.

Although we solve the linearized MHD equations, there are jumps in the perturbed variables (in $iv_y$ and $b_y$) due to the discontinuities in the equilibrium variables. We have decided to use a simple numerical scheme that is not shock-capturing (better suited for discontinuities) since the effect of the jump is rather small in the temporal evolution of the different quantities.

Line-tying conditions are applied at $z = \pm L/2$, meaning that the velocities are zero while for the rest of the variables the derivatives with respect to $z$ are zero. At $x = \pm L_x$ we impose that the derivatives with respect to $x$ of all the variables are zero. This condition does not allow a perfect transmission of the waves and some reflections are produced. A direct consequence of these reflections is the presence of the laterally confined normal mode in our simulations.

4.2. Results

We have already mentioned that our initial disturbance leads to the perturbed variables having the $z$-dependence of Equations (13)–(15), and for this reason we concentrate in the $x$-dependence of variables by taking a cut along $z = 0$. Two snapshots are presented in Figure 4; see also the accompanying movie. The first row corresponds to a time in which the contribution of the confined mode is small, but non-negligible. This makes $v_x$ and $iv_y$ quite similar to the $x$-dependence of the evanescent normal mode (compare with Figure 2). The $b_z$ function, however, lacks such strong similarity because this eigenfunction has a large amplitude for the confined mode and so a small contribution from this solution can dominate over that of the evanescent mode. At $t = 199.3\tau_{Ai}$, bottom row of Figure 4, both the evanescent and confined modes add to the perturbed variables, which do not resemble the eigenfunctions of Figures 2 and 3. In particular, note the absence of the
two extrema of \( v_x \) at \( x = \pm a \) and the presence of two spikes in \( iv_y \) at the same location.

Fig. 4.— Numerical simulation: perturbed variables \( v_x \), \( iv_y \), \( b_z \) across the slab middle point (\( z = 0 \)) for two times: \( t = 129.6\tau_{Ai} \) (top row) and \( t = 199.3\tau_{Ai} \) (bottom row). The two dotted lines correspond to the slab boundaries. (An animation of this figure is available in the online journal.)

We next examine the animation of Figure 4. The emission of waves that propagate away from the system is clear until \( t \approx 50\tau_{Ai} \), after which a fast and a slow oscillation are evident; we will later give evidence that they are caused by the confined and evanescent kink modes, respectively. Regarding the jump conditions we imposed to derive the normal mode eigenfunctions, namely the continuity of \( v_x \) and \( b_z \) at the slab boundary, they are satisfied during the whole simulation, whereas \( iv_y \) displays a strong jump at this interface. An additional agreement between the numerical solutions and the evanescent and confined kink modes are the boundary conditions: \( v_x \) is essentially zero at the domain limits, but this is not the case of the other two eigenfunctions because of the contribution of the confined mode, for which \( v_y \) and \( b_z \) are maximum there (Figure 3).

More convincing evidence of the simultaneous excitation of the two kink normal modes comes from the spectral analysis of \( v_x \), \( iv_y \), and \( b_z \) at a given location, which is selected so that both normal modes have a non-negligible amplitude. Thus, for the transverse velocity component we choose the point \( x = 0, z = 0 \), while for the other two variables the position \( x = a, z = 0 \) is preferred. The Lomb-Scargle periodograms \( \text{[Lomb 1976 Scargle 1982]} \) at these points are shown in Figure 5. The three panels display the largest power peak at \( \nu = 0.01601/\tau_{Ai} \) (i.e., \( \omega = 0.1006/\tau_{Ai} \)) and the second highest peak at \( \nu = 0.2544/\tau_{Ai} \) (i.e., \( \omega = 1.598/\tau_{Ai} \)). The two frequencies agree very well with the normal mode frequencies \( \omega = 0.1011/\tau_{Ai} \) and \( \omega = 1.676/\tau_{Ai} \) of the evanescent and confined modes, respectively. The power spectra show their strongest peaks at these two frequencies regardless of the position of the selected point (as far as the two normal modes do not vanish at this position).
It is worth noting that the power at $\omega = 0.1011/\tau_{Ai}$ is 2–3 orders of magnitude higher than that at $\omega = 1.676/\tau_{Ai}$ for $v_x$ and $iv_y$, although in the case of $b_z$ the two peaks are of similar magnitude. The reason for this is that the height of a power peak comes from the combination of the energy deposited by the initial disturbance in each normal mode (which is much larger for the evanescent one) and the amplitude of each variable (which in the case of $b_z$ is much smaller for the evanescent normal mode than for the confined one; see Figures 2 and 3, respectively). The combination of these two factors results in the function $b_z$ containing similar power for the two normal modes in this numerical simulation.

5. COMPLEX EMPIRICAL ORTHOGONAL FUNCTION ANALYSIS

In this section we go beyond the normal mode frequency we just obtained and attempt to determine the normal mode eigenfunction structure using the Complex Empirical Orthogonal Function (CEOF) analysis. We will here give a very brief summary of this method; a detailed description can be found in Horel (1984); Wallace & Dickinson (1972); von Storch & Zwiers (1999) and an application to the study of coronal oscillations in Terradas et al. (2004).

The CEOF analysis is a numerical method that takes as its input a field $U(\vec{r}, t_l)$ discretized over a spatial mesh of points $\vec{r} = (x_i, y_j, z_k)$ and evaluated at the discrete times $t_l$. Its output is a set of CEOF modes, which are not necessarily associated to physical modes of the system under study, each of them described by four measures called the temporal amplitude and phase and the spatial amplitude and phase. Together with these measures, the CEOF analysis associates to each mode a fraction of the total field variance. Once the CEOF code is fed with the input field, the “highest contributing” CEOF mode, that is, the one associated to the largest fraction of the total field variance, is retrieved first and other CEOF modes are obtained next in decreasing order of their fraction of the total field variance. The sum of the modes’ fraction of the total field variance tends to 1 as the number of CEOF modes is increased. The execution is stopped when the percentage of the total field variance accounted for by all the retrieved CEOF modes exceeds a pre-established value, here taken as 99.9%.

In our case the field $U$ can be, for example, the velocity component $v_x(x_i, z_k, t_l)$ obtained in the numerical simulation of Section 4. This means that the input field is a three-dimensional data cube. Such as mentioned above, each of the obtained CEOF modes has empirically computed temporal and spatial measures, called the temporal amplitude, $R(t_l)$, the temporal phase, $\phi(t_l)$, the spatial amplitude, $S(x_i, z_k)$, and the spatial phase, $\theta(x_i, z_k)$. The spatial and temporal variability of the field described by this CEOF mode is

$$\text{Re} \left\{ R(t_l) e^{i\phi(t_l)} S(x_i, z_k) e^{-i\theta(x_i, z_k)} \right\}, \quad (34)$$

where $\text{Re}$ denotes the real part. A CEOF mode that, for example, represents a propagating wave
Fig. 5.— Numerical simulation: Lomb-Scargle periodogram of $v_x$ at position $x = 0, z = 0$ (top), $iv_y$ at position $x = a, z = 0$ (middle), and $b_z$ at position $x = a, z = 0$ (bottom). To compute the power spectra only data for $t \geq 50\tau_{Ai}$ are kept so as to remove the effect of the transient in the frequency estimation.

has a temporal phase that varies linearly with $t_l$ and a spatial phase that varies linearly with $x_i$ and $z_k$. The CEOF modes that we expect to find when analyzing the results of the numerical simulation, however, are standing waves. In this case, the temporal phase also varies linearly with $t_l$, but the spatial phase is such that two regions in which the difference of $\theta(x_i, z_k)$ is an integer multiple of $2\pi$ correspond to in-phase oscillations, while oscillations that are in anti-phase display a phase difference that is an odd multiple of $\pi$. In our results we will also find that a standing wave
can have a phase that slowly varies in space, which is nothing but a modulation of $S(x_i, z_k)$ by the factor $\exp[-i\theta(x_i, z_k)]$. Section 3 of Terradas et al. (2004) gives simple two-dimensional examples of the outcome of the CEOF analysis when applied to a synthetic signal made of the sum of a propagating and a standing wave.

Our hypothesis is that the CEOF analysis applied to the results of the numerical simulation of Section 4 will provide an approximation, by means of Equation (34), to the evanescent normal mode eigenfunctions. Given that the eigenfunctions do not depend on time, we will ignore the temporal variation given by the measures $R(t)$ and $\phi(t)$ in Equation (34) and will only retain the real part of the spatial measures. Let us first consider the variable $v_x$, which for the evanescent normal mode has the eigenfunction $\hat{v}_x(x) \cos(k_z z)$, where $\hat{v}_x(x)$ is given by Equation (26). The CEOF approximation to this eigenfunction is

$$\tilde{v}_x(x_i, z_k) = S_{v_x}(x_i, z_k) \cos \theta_{v_x}(x_i, z_k),$$

(35)

where $S_{v_x}$ and $\theta_{v_x}$ are the spatial amplitude and phase of $v_x$ and the tilde in $\tilde{v}_x$ indicates that this is an approximation to the normal mode $v_x$. Regarding $i\hat{v}_y$, its eigenfunction for the confined mode is $i\hat{v}_y(x) \cos(k_z z)$ with $i\hat{v}_y(x)$ given by Equations (23) and (26). The CEOF approximation to this eigenfunction is

$$i\tilde{v}_y(x_i, z_k) = S_{i\hat{v}_y}(x_i, z_k) \cos \theta_{i\hat{v}_y}(x_i, z_k),$$

(36)

with $S_{i\hat{v}_y}$ and $\theta_{i\hat{v}_y}$ the spatial amplitude and phase of $i\hat{v}_y$. The case of $b_z$ requires special attention. Its eigenfunction is $i\hat{b}_z(x) \cos(k_z z)$, where $i\hat{b}_z(x)$ can be obtained from Equations (24) and (26). In the numerical simulation, however, we have not used the variable $i\hat{b}_z$ but $b_z$. For this reason, the CEOF approximation to $i\hat{b}_z$ requires inserting a factor $i$ inside $\text{Re} \{\ldots \}$ of Equation (34). We then have that the CEOF approximation to $i\hat{b}_z$ is

$$i\tilde{b}_z(x_i, z_k) = S_{b_z}(x_i, z_k) \sin \theta_{b_z}(x_i, z_k),$$

(37)

where $S_{b_z}$ and $\theta_{b_z}$ are the spatial amplitude and phase of $b_z$. Regarding $b_x$ and $b_y$, Equations (18) and (19) tell us that their respective CEOF approximations can be obtained from those of $v_x$ and $i\hat{v}_y$. Finally, approximations to $\hat{v}_x(x)$, $i\hat{v}_y(x)$, and $i\hat{b}_z(x)$ can be derived by taking a cut along $z = z_k$ of Equations (35), (36), and (37).

Before applying the CEOF method to the results of our simulation, two more comments are in order. First, the transient phase is excluded from the analysis by considering $t \geq 50\tau_{Ai}$ only. Second, to reduce the memory requirements and speed up the computation of the CEOF modes, the values of $v_x$, $i\hat{v}_y$ and $b_z$ are interpolated from the 4001×51 numerical grid to a grid of $N_x \times N_z$ points (here we use $N_x = 201$, $N_z = 25$). To do so, in the $x$- and $z$-directions only 1 every 20 points and 1 every 2 points, respectively, from the numerical simulation are kept for the CEOF analysis.
5.1. **CEOF analysis of \( v_x \)**

We ignore for the moment the variables \( iv_y \) and \( b_z \) and feed the CEOF code with the field \( v_x \). The input information of the CEOF code then consists of a data cube with \( 201 \times 25 \times 330 \) points in the \( x-, z-, \) and \( t- \) directions.

The first CEOF mode, i.e., the one that has the highest contribution to the field \( v_x(x_i, z_k, t_l) \), has the four measures represented in Figure 6. This mode accounts for 99.4% of the variance in \( v_x \). The temporal amplitude is more or less constant in time but, in addition to the expected edge effects, it displays very high frequency, small amplitude oscillations that are caused by the influence of the confined mode, an influence that the CEOF analysis has not been able to filter out. Regarding the temporal phase, the empirical results are very well fitted by the straight line \( \phi(t) = \omega t + \phi_0 \) with \( \omega = 0.1015/\tau_{Ai} \). This value is in excellent agreement with the normal mode frequency \( \omega = 0.1011/\tau_{Ai} \). The spatial amplitude and phase (see bottom panels of Figure 6) are a bit harder to interpret, but the former is clearly reminiscent of the \( v_x \) velocity component of the evanescent normal mode: the variation in the \( z- \) direction is similar to \( \cos k z \), while that in the \( x- \) direction resembles the eigenfunction \( \hat{v}_x \); see top panel of Figure 2. A numerical comparison between the normal mode eigenfunction, \( v_x(x) \cos k z \), and its approximation from the CEOF analysis, \( \tilde{v}_x \), is presented in Figure 7 with the help of the difference

\[
\Delta v_x(x_i, z_k) = \hat{v}_x(x_i) \cos k z_k - \tilde{v}_x(x_i, z_k), \tag{38}
\]

between the two functions. To make this figure, the eigenfunction is normalized to a maximum value of 1 and the CEOF function of Equation (35) is also normalized to 1 at the position where the eigenfunction is maximum. The conclusion from this figure is that the CEOF analysis of the numerical simulation results allows us to recover the normal mode eigenfunction \( \hat{v}_x \) with an error below 3%.

5.2. **Joint CEOF analysis of \( v_x, iv_y, \) and \( b_z \)**

As we said before, this work aims at obtaining the normal modes frequency and eigenfunctions by the combined application of time-dependent numerical simulations and the CEOF analysis. We have just seen that \( \hat{v}_x \) can be approximated with a reasonable accuracy, but we have no hint about the spatial profile of \( iv_y(x) \) and \( \hat{b}_z(x) \). The CEOF method has the possibility of analyzing several fields simultaneously, which allows a better characterization of the physical modes because more restrictions are imposed by the higher complexity of the combined fields. Then, instead of using the data cube of \( v_x \) as the input of the CEOF analysis, we now run the CEOF code on the fields \( v_x(x_i, z_k, t_l) \), \( iv_y(x_i, z_k, t_l) \), and \( b_z(x_i, z_k, t_l) \) together. To do this, the three data cubes are put next to each other and a larger data cube is created. We choose to join the three \( 201 \times 25 \times 330 \) cubes by attaching their \( xt- \) faces, so that the CEOF input is a cube of \( 201 \times 75 \times 330 \) data values. After the
CEO analysis is complete, we again obtain a collection of CEOF modes, each of them characterized by its temporal amplitude and phase, \( R(t) \) and \( \phi(t) \), and its spatial amplitude and phase, \( S(x,z) \) and \( \theta(x,z) \), that can be split into the spatial measures \( S_{v_x}(x,z) \) and \( \theta_{v_x}(x,z) \) of the field \( v_x(x_i,z_k,t_l) \), \( S_{iv_y} \) and \( \theta_{iv_y} \) of the field \( iv_y(x_i,z_k,t_l) \), and \( S_{b_z} \) and \( \theta_{b_z} \) of the field \( b_z(x_i,z_k,t_l) \). These measures can in turn be inserted into Equations (35)–(37) to obtain the approximate CEOF eigenfunctions.

First, we compare the performance of the CEOF method when applied only to \( v_x \) and to the combination of three fields. The first CEOF mode again corresponds to the evanescent normal mode. It accounts for 86.0% of the total field variance, which is clearly smaller than the 99.4% obtained when the CEOF analysis was applied to \( v_x \) only, the reason being that the combination of the three fields \( v_x, iv_y, \) and \( b_z \) contains higher variability and so a single CEOF mode represents a smaller percentage of that variability. The temporal amplitude of this mode displays similar high-frequency oscillations to those found in top left panel of Figure 6. The frequency of the first

Fig. 6.— CEOF analysis of \( v_x \): temporal amplitude \( (R(t)) \), temporal phase \( (\phi(t)) \), represented by a dashed line, and linear least-squares fit, represented by a solid line), spatial amplitude \( (S_{v_x}(x,z)) \), and spatial phase \( (\theta_{v_x}(x,z)) \) of the first CEOF mode. Dotted lines are plotted at the slab boundaries.
Fig. 7.— CEOF analysis of $v_x$: difference, $\Delta v_x$, between the normal mode eigenfunction and its approximation from the first CEOF mode. Left: two dimensional distribution of $\Delta v_x$. Right: cut of $\Delta v_x$ along $z = 0$. Dotted lines are plotted at the slab boundaries.

CEOF mode, obtained as in the top right panel of Figure 6, is $\omega = \frac{0.1016}{\tau_{Ai}}$, a slightly worse approximation to the normal mode frequency $\omega = \frac{0.1011}{\tau_{Ai}}$ than that achieved in Section 5.1. Furthermore, the error of $v_x$, which is displayed in the top row of Figure 8 reveals that the present CEOF approximation to the evanescent normal mode has a larger error than that of the CEOF analysis applied exclusively to $v_x$; compare with Figure 7. We therefore conclude that, for the present numerical simulation, performing the CEOF analysis with $v_x$ provides better estimates for the evanescent mode frequency and the $\hat{v}_x$ velocity component than when both $v_x$, $iv_y$, and $b_z$ are jointly analyzed. However, our purpose is to obtain all normal mode features, and this includes all physical variables associated with the evanescent normal mode. Thus, the CEOF analysis of $\{v_x, iv_y, b_z\}$ provides us with an approximation to the normal mode $\hat{v}_y$ that cannot be achieved with the analysis of Section 5.1 unless Equation (23) is used once the derivative of $\hat{v}_x$ is numerically obtained. The same applies to $\hat{b}_z$. We prefer not to follow this path and to use the CEOF analysis only.

We next turn our attention to the error of $iv_y$ and $ib_z$. The middle row of Figure 8 gives the difference between $\hat{v}_y(x) \cos k_z z$ and the $iv_y$ variable obtained with the CEOF analysis, $\hat{iv}_y$. The definition of this error is analogous to that of Equation (38). Except for the points on the boundaries, $x = \pm a$, the error is small but not negligible inside the slab ($|x| < a$) and practically zero outside the slab ($|x| > a$). The bottom row of Figure 8 gives the difference between $\hat{b}_z(x) \cos k_z z$ and $ib_z$ from the CEOF analysis, $\hat{ib}_z$. This variable also attains at the slab boundary its largest error, of the order of 15% the eigenfunction value.

The conclusion of this section is that, while the CEOF approximation to $\hat{v}_x$ is acceptable, those to $\hat{iv}_y$ and $\hat{ib}_z$ are not too good. This situation will be improved by the application of an iterative method (Section 6).
Fig. 8.— CEOF analysis of \( \{v_x, iv_y, b_z\} \): same as Figure 7 for \( v_x \) (top row), \( iv_y \) (middle row), and \( b_z \) (bottom row).
5.3. Joint CEOF analysis of \( v_x \) and \( iv_y \)

As an exercise, we next proceed to do a combined CEOF analysis of \( \{ v_x, iv_y \} \). Given the small amplitude of \( b_z \), it turns out that the first CEOF mode has almost identical properties to that of the combined CEOF analysis of \( \{ v_x, iv_y, b_z \} \), namely the same frequency and spatial distribution of \( v_x \) and \( iv_y \). Hence, concerning these two variables, there is no advantage in including \( b_z \) in the CEOF analysis.

6. ITERATIVE METHOD

6.1. Description of the method

The scheme we have used so far consists of two steps: (i) a time-dependent numerical simulation of Equations (8)–(12) followed by (ii) the CEOF analysis of the obtained results. If we imagine that an eigenmode is perfectly described by a CEOF mode, then one could run a numerical simulation with initial conditions given by the eigenfunctions and so the obtained temporal evolution would be that of the eigenmode. At this point, this is not the case, but we have seen that the CEOF analysis produces an approximation to a normal mode eigenfunctions. We thus devise an iterative method that is made of the repeated application of steps (i) and (ii), in which the initial conditions of the numerical simulation of a given iteration are taken from the CEOF method of the previous iteration. The iterations will be stopped once a given measure of goodness is reached.

We first need to determine which information is required from the CEOF analysis to fix the initial conditions. Rather than using the time dependence \( \exp(i\omega t) \) of Equations (13)–(15) we assume that \( v_x(x, z, t) \) is maximum at \( t = 0 \) and so it has the form \( v_x(x, z, t) = \hat{v}_x(x) \cos(k_z z) \cos(\omega t) \). Now, Equations (8)–(12) tell us that \( v_x \) and \( iv_y \) are in phase (in time) and that they are a quarter of a period out of phase with respect to \( b_x \), \( ib_y \), and \( b_z \). This implies that \( iv_y(x, z, t) \) is also maximum at \( t = 0 \) and that the perturbed magnetic field components vanish at the start of the numerical simulation. Hence, the information that the CEOF analysis must provide to repeat step (i) is an approximation to \( \hat{v}_x \) and \( i\hat{v}_y \). As we have seen in Sections 5.2 and 5.3, this can be achieved by the application of the CEOF method to \( \{ v_x, iv_y, b_z \} \) or to \( \{ v_x, iv_y \} \).

6.2. Results

We are then ready to carry out the iterative process. Iteration #1 consists of the numerical simulation of Section 4 and the CEOF analysis of Section 5.2. The results we present now are a summary of the performance of 4 iterations. The numerical simulation of iteration #4 is almost free from the “contaminating” short-period oscillations of the confined mode, that are only visually appreciable outside the slab \( (|x| > a) \) in \( iv_y \) and \( b_z \), but not in \( v_x \); see Figure 9 and the accompanying movie. The spatial profile of the normal mode eigenfunctions (Figures 2 and 3) is such that the
confined mode $i\hat{v}_y$ and $i\hat{b}_z$ are maximum near the $x$-boundaries, where the evanescent mode has a small amplitude. This explains why the influence of the confined mode in the numerical simulation is strong in $iv_y$ and $b_z$ outside the slab. There is only another undesirable feature in these numerical simulation results, namely the two spikes of $iv_y$ at $x = \pm a$.

![Graphs showing confined and evanescent modes](image)

Fig. 9.— Iterative method: same as Figure 4 for the numerical simulation of iteration #4. (An animation of this figure is available in the online journal.)

The power spectra of $v_x$, $iv_y$, and $b_z$, shown in Figure 10, corroborate our statement about the minor role of the confined mode in these results: the power peak at the confined normal mode frequency ($\nu = 0.2544/\tau_{AI}$) is absent in the Lomb-Scagle periodogram of $v_x$ and $iv_y$, and only a small trace of it is present in the periodogram of $b_z$. The frequency of the evanescent normal mode is identical to that obtained from the power spectra in the first iteration: $\nu = 0.01601/\tau_{AI}$, or $\omega = 0.1006/\tau_{AI}$. One can also see that the power of this peak of $v_x$ and $iv_y$ remains almost identical from iteration #1 to iteration #4, but that in $b_y$ has increased by an order of magnitude.

The iterative method yields an approximate frequency that comes from the CEOF analysis of iteration #4. A linear least-squares fit to the temporal phase, which is shown together with the temporal amplitude in Figure 11, gives the frequency $\omega = 0.1016/\tau_{AI}$, which is a bit worse than that given by the CEOF analysis of $v_x$ in iteration #1 ($\omega = 0.1015/\tau_{AI}$), but still in good agreement with the normal mode frequency ($\omega = 0.1011/\tau_{AI}$). The error associated to the approximate value $\omega = 0.1016/\tau_{AI}$ is 0.5%, so that the obtained accuracy is excellent. It is worth to mention that the temporal amplitude is free from the short-period oscillations caused by the confined mode that were found in the CEOF analysis of iteration #1: see Figure 6.
6.3. Error and stopping criterion

We next examine in detail the error of the CEOF approximation to the eigenfunctions. Again we consider the results of iteration #4 and show these errors in Figure 12. Although their maximum values are reduced by a factor of 2, the errors display some of the patterns of iteration #1: the error of $v_x$ is maximum in the slab, the error of $iv_y$ has a dominant component at the slab boundaries, but has been strongly reduced inside the slab during the iterative process, and the error of $b_z$ is maximum at the slab boundaries, but has become much more confined to the slab neighborhood.
Fig. 11.— Iterative method: temporal amplitude (left panel) and temporal phase (right panel) for the first CEOF mode of iteration #4. On the right panel the temporal phase and the linear least-squares fit are represented by a dashed line and a solid line, respectively.

We finally analyze the evolution of the error with the iterations\(^3\). At the end of iteration \(n\), with \(n = 1, 2, \ldots\), Equations (35)–(37) provide us with approximations for the three main eigenfunctions; we denote these approximations with the superscript \(n\), i.e., \(\tilde{v}_{x}^{n}\), \(i\tilde{v}_{y}^{n}\), \(i\tilde{b}_{z}^{n}\). For each eigenfunction, we define a global measure of the error, \(\varepsilon\), by summing over the spatial domain the squares of the difference between the normal mode eigenfunction and the CEOF approximation. For example, for the iteration \(n\) and the variable \(v_{x}\) this global error is:

\[
\varepsilon_{v_{x}}^{n} = \frac{1}{N_{x}N_{z} \max_{i,k} |\tilde{v}_{x}(x_{i}) \cos k_{z}z_{k}|} \left\{ \sum_{i,k} [\tilde{v}_{x}(x_{i}) \cos k_{z}z_{k} - \tilde{v}_{x}^{n}(x_{i}, z_{k})]^{2} \right\}^{1/2}, \quad n = 1, 2, \ldots, (39)
\]

where the factor \(\max_{i,k} |\tilde{v}_{x}(x_{i}) \cos k_{z}z_{k}|\) in the denominator provides the right normalization that enables us to compare the error of different eigenfunctions. The additional factors \(N_{x}\) and \(N_{z}\) give an additional normalization that removes the dependence of \(\varepsilon_{v_{x}}^{n}\) on the number of points in the CEOF analysis. The definitions of \(\varepsilon_{v_{y}}^{n}\) and \(\varepsilon_{b_{z}}^{n}\) are done in a similar way.

The top panel of Figure 13 presents the global errors for the first 4 iterations. In each iteration, \(v_{x}\) has the smallest error (possibly because it is the eigenfunction with less “contamination” from the confined normal mode) and \(i v_{y}\) displays the largest global error (because of the large contributions at the slab boundaries, that do not disappear with the iterative process). We see that the biggest improvement in the global error is obtained in iteration #2, for which a remarkable reduction in

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\(^3\) Before computing the errors described here we normalize the normal mode eigenfunctions and their approximation from the CEOF analysis so that \(v_{x}\) equals zero at \(x = 0, z = 0\).
Fig. 12.— Same as Figure 8 for the CEOF analysis of iteration #4.

\( \varepsilon_{v_z} \) and \( \varepsilon_{ib_z} \) is found. The subsequent variation of the three global errors is much more moderate and so for this problem two iterations give a good compromise between the error associated to the CEOF approximations and the computer time spent.
Fig. 13.— Global error (top panel) and global difference (bottom panel) as a function of the iteration number.

The case studied in this work allows us to compute the global error because of our knowledge of the exact eigenfunctions. In a general case, in which the eigenfunctions are unknown and our aim is just to obtain them, a proxy for the global error can be computed by comparing the approximate eigenfunctions of two successive iterations. Thus, in Equation (39) we replace the normal mode eigenfunction \( \tilde{v}_x(x_i) \cos k_z z_k \) by its CEOF approximation in the iteration \( n + 1 \). We also rewrite the iteration indices and substitute \( n + 1 \) by \( n \). This gives the following definition for the global difference between the \( v_x \) eigenfunction of iterations \#n and \#n − 1:

\[
\delta_{v_x}^n = \frac{1}{N_x N_z \max_{i,k} |\tilde{v}_x^n(x_i, z_k)|} \left\{ \sum_{i,k} \left[ \tilde{v}_x^n(x_i, z_k) - \tilde{v}_x^{n-1}(x_i, z_k) \right]^2 \right\}^{1/2}, \quad n = 2, 3, \ldots
\] (40)

Analogous expressions can be written for \( \delta_{iv_y} \) and \( \delta_{ib_z} \).
The variation of the global difference with the iterations is displayed in the bottom panel of Figure 13. The convergence is quite fast, with $\delta_{ib_z}$ showing an improvement of roughly two orders of magnitude per iteration, the highest convergence rate of all variables. $\delta_{iv_y}$ and $\delta_{iv_y}$ are reduced at a slower pace, namely, almost an order of magnitude per iteration. We see that all variables attain a global difference smaller than $10^{-5}$ in iteration #4, a value that we use here as our iteration stopping criterion.

7. CONCLUSIONS

In this paper we have devised a method to determine a physical system normal modes, i.e., their eigenfunctions and eigenfrequencies, by the iterative application of time-dependent numerical simulations of the equations that govern the system dynamics and the CEOF analysis of the simulation results. We have illustrated how the CEOF method can be applied to all the non-redundant variables: in our case, in particular, this means that we can avoid including $b_x$ and $ib_y$ in the CEOF computation because their eigenfunctions can be readily computed from the other three ($v_x$, $iv_y$, and $b_z$). At the end of each iteration, the CEOF approximations to the eigenfunctions are used as the initial conditions for the time-dependent numerical simulation of the next iteration. Finally, we have examined the global error of the approximate eigenfunctions as a function of the iteration number and have established a convergence criterion based on the global difference between the approximate eigenfunctions of consecutive time steps.

The particular problem studied here presents some advantages and disadvantages for the performance of the numerical simulation + CEOF technique we have described. In the first group, the main advantage is that our initial condition for iteration #1 (see Equation [33]) has the exact sinusoidal dependence of normal modes in the $z$-direction, which prevents harmonics with different number of nodes in this direction from being present in the simulation results. This in turn reduces the number of CEOF modes that need to be sifted with the CEOF analysis. So one obviously should use all information available about the eigenfunctions to impose initial conditions that resemble the most the spatial profile of the eigenfunctions being searched. The main disadvantage of our test case is the presence of sharp boundaries in the equilibrium structure, which leads to abrupt jumps of the eigenfunction $i\hat{v}_y$ and non-derivable eigenfunctions $\hat{v}_x$ and $i\hat{b}_z$ at these positions (see Figure 2). We have found that these normal mode features result in the presence of large errors at the slab boundary, which are quite substantial for the approximation to $i\hat{v}_y$.

We have obtained an approximation to the normal mode of interest (the evanescent mode) with great accuracy: after 4 iterations, the frequency is wrong by only 0.5% and the eigenfunctions $\hat{v}_x$ and $i\hat{b}_z$ have maximum errors of the order of 0.6% and 0.7%, respectively. The case of $i\hat{v}_y$ is worse because of the difficulties of recovering a function that jumps at $x = \pm a$. If these two lines are ignored, the maximum error of $i\hat{v}_y$ is also of the order of 0.6%.

A possible improvement of the method that would allow us to avoid the complications at
$x = \pm a$ would be to substitute the CEOF analysis in the full spatial domain by three analyses in the domains $L_x \leq x < -a$, $-a \leq x \leq a$, and $a < x \leq L_x$. All eigenfunctions and their derivatives are continuous in each of the three domains and so we may obtain an approximation to $i\hat{v}_y$ that is free from the large errors at the slab boundaries. Splitting the spatial domain in several pieces when doing the CEOF analysis is somehow similar to removing the initial time to get rid of the wave transients, such as we have done here. We have not yet tried this improvement and leave it for a future work.

All our efforts have been directed towards obtaining an approximation to the evanescent eigenmode, but the confined mode can also be iteratively approximated. By doing this, however, we would learn nothing new because the working of the iterative method has been sufficiently illustrated with the evanescent mode.

In the second paper of this series [Rial et al. 2019] we will apply the technique presented here to the time-dependent numerical simulations of a loop embedded in a coronal arcade carried out by [Rial et al. 2013]. The equilibrium structure is similar to the one used in the present paper but includes a curved slab. The initial condition used by [Rial et al. 2013] is such that various longitudinal harmonics $^4$ are excited. This makes the application of our technique more challenging because of the presence of more normal modes in the numerical simulation, but for this reason it is also more attractive.

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