On Center Sets of ODEs Determined by Moments of their Coefficients

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Abstract

The classical H. Poincaré Center-Focus problem asks about the characterization of planar polynomial vector fields such that all their integral trajectories are closed curves whose interiors contain a fixed point, a center. This problem can be reduced to a center problem for some ordinary differential equation whose coefficients are trigonometric polynomials depending polynomially on the coefficients of the field. In this paper we show that the set of centers in the Center-Focus problem can be determined as the set of zeros of some continuous functions from the moments of coefficients of this equation.

1. Introduction.

Consider the ordinary differential equation

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x) v^{i+1}, \quad x \in I_T := [0, T],$$

(1.1)

whose coefficients $a_i$ belong to the Banach space $L^\infty(I_T)$ of bounded measurable complex-valued functions on $I_T$ with the supremum norm. Let $X_i := L^\infty(I_T)$ be the space of coefficients $a_i$ from (1.1), and $X$ be the complex Fréchet space of sequences $a = (a_1, a_2, \ldots) \in \prod_{i \geq 1} X_i$ satisfying

$$\sup_{x \in I_T, i \in \mathbb{N}} \sqrt[2i]{|a_i(x)|} < \infty.$$  

(1.2)

Condition (1.2) implies that for any $a \in X$ the corresponding equation (1.1) has Lipschitz solutions on $I_T$ for all sufficiently small initial values. Such solutions can

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be obtained by Picard iteration. We say that equation (1.1) determines a center if every solution \( v \) of (1.1) with a sufficiently small initial value satisfies \( v(T) = v(0) \). By \( \mathcal{C} \subset X \) we denote the set of centers. The center problem for equation (1.1) is: given \( a \in X \) to determine whether \( a \in \mathcal{C} \). This problem arises naturally in the framework of the geometric theory of ordinary differential equations created by Poincaré. In particular, there is a connection between this problem and the classical Poincaré Center-Focus problem for polynomial vector fields. For a non-degenerate equilibrium point of such a field it can be reduced to the following equivalent form.

Consider the system of ODEs in the plane

\[
\frac{dx}{dt} = -y + F(x, y), \quad \frac{dy}{dt} = x + G(x, y)
\]  

(1.3)

where \( F, G \) are real analytic functions in a neighbourhood of \( 0 \in \mathbb{R}^2 \) whose Taylor expansions at 0 do not contain constant and linear terms. Then for \( F, G \) polynomials of a given degree, the Poincaré Center-Focus Problem asks about conditions on the coefficients of \( F \) and \( G \) under which all trajectories of (1.3) situated in a small neighbourhood of the origin are closed. (A similar problem can be posed for the general case.) Observe that passing to polar coordinates \( x = r \cos \phi, \ y = r \sin \phi \) in (1.3) and expanding the right-hand side of the resulting equation as a series in \( r \) (for \( F \) and \( G \) whose coefficients are sufficiently small) we obtain an equation of the form (1.1) whose coefficients are trigonometric polynomials depending polynomially on the coefficients of (1.3). This reduces the Center-Focus Problem for (1.3) to the center problem for equations (1.1) with coefficients depending polynomially on a parameter. In this paper we will consider a general equation of such form

\[
\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(w, x) v^{i+1}, \quad x \in I_T ,
\]  

(1.4)

where all \( a_i \) are holomorphic functions in \( w \) varying in a Stein domain \( U \subset \mathbb{C}^k \) and \( a_i(w, \cdot) \in L^\infty(I_T) \). We set \( A(w) := (a_1(w, \cdot), a_2(w, \cdot), \ldots) \) and assume that \( A(w) \in X \) for all \( w \in U \). This gives a holomorphic map \( A : U \to X \). Our goal is to study the center problem for (1.4), i.e., the problem of the characterization of the set \( \mathcal{C}_A := A^{-1}(\mathcal{C}) \subset U \).

2. Formulation of Main Results.

2.1. A Characterization of the Center Set. In [Br1] a characterization of the center set \( \mathcal{C} \) is given in terms of the iterated path integrals. For \( a \in X \) let us consider the basic iterated integrals

\[
I_{i_1, \ldots, i_k}(a) := \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq T} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1
\]  

(2.1)

(for \( k = 0 \) we assume that this equals 1). They can be thought of as \( k \)-linear holomorphic functions on \( X \). By the Ree formula [R] the linear space generated by
all such functions is an algebra. A linear combination of iterated integrals of order \( \leq k \) is called an \textit{iterated polynomial of degree} \( k \).

Let \( v(x; r; a), x \in I_T \), be the solution of equation (1.1) corresponding to \( a \in X \) with (sufficiently small) initial value \( v(0; r; a) = r \). Then \( P(a)(r) := v(T; r; a) \) is the \textit{first return map} of this equation.

**Theorem** ([Br1]) For sufficiently small initial values \( r \) the first return map \( P(a) \) is an absolutely convergent power series \( P(a)(r) = r + \sum_{n=1}^{\infty} c_n(a)r^{n+1} \), where

\[
c_n(a) = \sum_{i_1 + \ldots + i_k = n} c_{i_1, \ldots, i_k} I_{i_1, \ldots, i_k}(a), \quad \text{and}
\]

\[
c_{i_1, \ldots, i_k} = (n - i_1 + 1) \cdot (n - i_1 - i_2 + 1) \cdot (n - i_1 - i_2 - i_3 + 1) \ldots 1.
\]

The center set \( C \subset X \) of equation (1.3) is determined by the system of polynomial equations \( c_n(a) = 0, n = 1, 2, \ldots \).

This theorem implies that the center set \( C_A \) for (1.4) is a closed complex subvariety of \( U \) determined as the set of zeros of holomorphic functions \( A^*c_n, 1 \leq n \leq N \); here \( A^*c_n(w) := c_n(A(w)) \), \( w \in U \).

**2.2. Algebraic Structure of the Center Set.** In [Br3] we constructed an algebraic model for the center problem for equation (1.1). Let us briefly describe this construction.

First, from the Theorem of the previous section it follows that any point \( a \in X \) such that all iterated integrals vanish at \( a \), belongs to \( C \). The set \( U \subset C \) of all such points is called the set of \textit{universal centers} of (1.1). In [Br2] we characterized some elements from \( U \). It was related to a certain composition condition whose role and importance for Abel differential equations with polynomial coefficients was studied in [AL], [BFY1], [BFY2], [Y].

Next, given \( a = (a_1, a_2, \ldots) \) and \( b = (b_1, b_2, \ldots) \) from \( X \) we define

\[
a \ast b = (a_1 \ast b_1, a_2 \ast b_2, \ldots) \in X \quad \text{and} \quad a^{-1} = (a_1^{-1}, a_2^{-1}, \ldots) \in X
\]

where for \( i \in \mathbb{N} \),

\[
(a_i \ast b_i)(t) = \begin{cases} 2a_i(2t) & \text{if} \quad 0 < t \leq T/2 \\ 2b_i(2t - T) & \text{if} \quad T/2 < t \leq T \\
\end{cases}
\]

and

\[
a_i^{-1}(t) = -a_i(T - t), \quad 0 < t \leq T.
\]

We say that \( a, b \in X \) are \textit{equivalent} (written, \( a \sim b \)) if \( a \ast b^{-1} \in U \). It was shown in [Br3] that \( \sim \) is an equivalence relation, that is \( X \) partitions into mutually disjoint equivalence classes. Let \( G(X) \) be the set of these classes. Then \( \ast \) induces a multiplication \( \cdot : G(X) \times G(X) \to G(X) \) converting \( G(X) \) to a group. Also, the iterated integrals are constant on any equivalence class. So they can be considered as functions on \( G(X) \). It was proved that these functions separate points on \( G(X) \). Let us equip \( G(X) \) with the weakest topology in which all iterated integrals are continuous. Then \( G(X) \) is a separable topological group. Moreover, it is contractible, arcwise
connected, locally arcwise and simply connected and residually torsion free nilpotent (i.e., finite-dimensional unipotent representations separate elements of \( G(X) \)).

Further, consider the set \( G_c[[r]] \) of complex power series \( f(r) = r + \sum_{i=1}^{\infty} d_i r^{i+1} \) each convergent in some neighbourhood of \( 0 \in \mathbb{C} \). Let \( d_i : G_c[[r]] \to \mathbb{C} \) be such that \( d_i(f) \) equals the \((i+1)\)-st coefficient of the Taylor expansion of \( f \) at \( 0 \). We equip \( G_c[[r]] \) with the weakest topology in which all \( d_i \) are continuous functions and consider the multiplication \( \circ \) on \( G_c[[r]] \) defined by the composition of series. Then \( G_c[[r]] \) is a separable topological group. Moreover, it is contractible, arcwise connected, locally arcwise and simply connected and residually torsion free nilpotent.

Now, for any \( a \in X \) let \( v(x; r; a) \), \( x \in I_T \), be the Lipschitz solution of equation \((1.3)\) corresponding to \( a \) with initial value \( v(0; r; a) = r \). Clearly for every \( x \) we have \( v(x; r; a) \in G_c[[r]] \). Let \( P(a)(r) := v(T; r; a) \) be the first return map. Then

\[
P(a \ast b) = P(b) \circ P(a).
\]

This together with the fact that \( P(a)(r) \equiv r \) for any \( a \sim 0 (\in X) \) imply that there is a map \( \tilde{P} : G(X) \to G_c[[r]] \) such that \( \tilde{P}([a]) := P(a) \) where \([a]\) denotes the equivalence class containing \( a \in X \). It was proved that \( \tilde{P} \) is a surjective homomorphism of topological groups, the kernel \( \tilde{C} \subset G(X) \) of \( \tilde{P} \) coincides with the image of the center set \( C \subset X \) in \( G(X) \), and \( C \) is contractible, arcwise connected, locally arcwise and simply connected. Also, there is a continuous map \( T : G_c[[r]] \to G(X) \) such that \( \tilde{P} \circ T = id \). In particular, the map \( \tilde{T} : G_c[[r]] \times \tilde{C} \to G(X) \), \( T(f, g) := T(f) \cdot g \), is a homeomorphism.

2.3. Moments. It is natural to try to characterize the center set \( C_A \) of \((1.4)\) as the set of zeros of functions from iterated integrals of a simple form. One of the simplest type of iterated integrals are called moments. Moments play an important role in the study of the center problem for Abel differential equations (see, e.g., [AL], [BFY1], [BFY2], [Y]). For \( a = (a_1, a_2, \ldots) \in X \) they are defined by the formulas

\[
m_{i_1, \ldots, i_{k+1}}(a) := \int_0^T (\tilde{a}_{i_1}(s))^{n_1} \cdots (\tilde{a}_{i_k}(s))^{n_k} a_{i_{k+1}}(s) \, ds
\]

for all possible natural numbers \( n_1, \ldots, n_k \) and \( i_1, \ldots, i_{k+1} \) where

\[
\tilde{a}_i(x) := \int_0^x a_i(s) \, ds.
\]
problem of independent interest: *to describe points $g \in G(X)$ such that if $m(g) = 0$ for all $m \in M$ then $g \in G^1(X_a)$.* Some results in this direction can be obtained from the description of polynomially convex hulls of smooth curves in $\mathbb{C}^n$ (see [W], [A], [HL] and section 3.1 below).

### 2.4. Main Results

We retain the notation of the Introduction. Let us consider the equation (1.4) with all coefficients being either polynomials in $x$ or in $e^{\pm 2\pi i x/T}$.

Then we prove

**Theorem 2.1** For every compact set $K \subset U$ there is a number $N = N(K) \in \mathbb{N}$, moments $m_1, \ldots, m_N \in M$ and continuous functions $f_1, \ldots, f_N$ on $\mathbb{C}^N$ such that the center set $C_A \cap K$ of (1.4) in $K$ is determined as the set of zeros of continuous functions $f_n(A^*m_1, \ldots, A^*m_N)$ on $U$ ($1 \leq n \leq N$). Moreover, the set of universal centers $A^{-1}(U) \cap K$ of (1.4) in $K$ is the set of zeros of the holomorphic functions $A^*m_n, 1 \leq n \leq N$, restricted to $K$.

To formulate some specific versions of this result we need the following

**Definition 2.2** A holomorphic polynomial $p$ on $\mathbb{C}^k$ is called $\alpha$-homogeneous for some $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ if $p(w_1, \ldots, w_k) := p(w_1^{\alpha_1}, \ldots, w_k^{\alpha_k}), (w_1, \ldots, w_k) \in \mathbb{C}^k$, is a homogeneous polynomial.

Assume that all coefficients $a_i$ in (1.4) are $\alpha$-homogeneous polynomials in $w \in \mathbb{C}^k$ for some $\alpha \in \mathbb{N}^k$ whose coefficients are either polynomials in $x$ or in $e^{\pm 2\pi i x/T}$.

Assume also that the set of universal centers $U_A := A^{-1}(U) \subset \mathbb{C}^k$ of (1.4) coincides with $0 \in \mathbb{C}^k$ (this holds, e.g., if there are $a_{i_1}, \ldots, a_{i_l}$ such that the set of zeros of $\tilde{a}_i(w) := \int_0^1 a_i(w, x) dx, 1 \leq s \leq l, w \in \mathbb{C}^k$, coincides with $0$.) Then we prove

**Theorem 2.3** There are moments $m_1, \ldots, m_N \in M$ and holomorphic polynomials $p_1, \ldots, p_N$ on $\mathbb{C}^N$ such that the center set $C_A$ of (1.4) is determined as the set of zeros of holomorphic polynomials $p_n(A^*m_1, \ldots, A^*m_N), 1 \leq n \leq N$.

Assume as before that all coefficients $a_i$ in (1.4) are $\alpha$-homogeneous polynomials in $w \in \mathbb{C}^k$ for some $\alpha \in \mathbb{N}^k$ whose coefficients are either polynomials in $x$ or in $e^{\pm 2\pi i x/T}$. Then the set of universal centers $U_A$ of (1.4) is a complex algebraic subvariety of $\mathbb{C}^k$. By $\mathcal{U}_A$ we denote its closure in $\mathbb{C}P^k := \mathbb{C}^k \cup H_k$, the complex $k$-dimensional projective space (here $H_k$ is the complex hyperplane at infinity). Then $\mathcal{U}_A \subset \mathbb{C}P^k$ is a projective subvariety. Let $V \subset \mathbb{C}^k$ be a complex algebraic subvariety whose closure in $\mathbb{C}P^k$ does not intersect $\mathcal{U}_A$.

**Theorem 2.4** There are moments $m_1, \ldots, m_N \in M$ and holomorphic polynomials $p_1, \ldots, p_N$ on $\mathbb{C}^N$ such that the center set $C_A \cap V$ of (1.4) in $V$ is the set of zeros of holomorphic polynomials $p_n(A^*m_1, \ldots, A^*m_N), 1 \leq n \leq N$, restricted to $V$.

**Remark 2.5** (1) All coefficients in (1.4) obtained from complex polynomial vector fields (1.3) of degree $d$ are $\alpha$-homogeneous polynomials on $\mathbb{C}^k$; here $k := d^2 + 3d - 4$ is the number of coefficients of polynomials $F$ and $G$ in (1.3).

(2) For Abel differential equations with polynomial coefficients the result of Theorem 2.3 was conjectured by Yomdin [Y1] without any additional assumption on $U_A$. 

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(3) Under the hypotheses of Theorem 2.1 one can give a description of elements of $\mathcal{U}_A$ in terms of the composition condition, see, e.g., [Br2, Corollaries 1.19, 1.20].

(4) Since $\hat{C} \cap G^1(X) \neq \{1\}$, Theorem 2.1 does not hold for a general equation of the form (1.4).

3. Separation by Moments.

3.1. Auxiliary Results. We require

**Definition 3.1** The polynomially convex hull $\hat{K}$ of a compact set $K \subset \mathbb{C}^n$ is the set of points $z \in \mathbb{C}^n$ such that if $p$ is any holomorphic polynomial in $n$ variables

$$|p(z)| \leq \max_{x \in K} |p(x)| .$$

It is well known (see e.g. [AW]) that $\hat{K}$ is compact, and if $K$ is connected then $\hat{K}$ is connected.

Let $\hat{A} = (\tilde{a}_1, \ldots, \tilde{a}_n) : I_T \to \mathbb{C}^n$ be a Lipschitz curve where $\tilde{a}_i$ are the first integrals of some functions $a_i \in L^\infty(I_T)$, see (2.3). Set $\Gamma := \hat{A}(I_T)$. Then arguments similar (but essentially more easier) to those used in [Br2, Theorem 1.10] give

**Theorem 3.2** Assume that all moments from functions $a_1, \ldots, a_n$ are zeros. Then the path $\hat{A} : I_T \to \mathbb{C}^n$ is closed and represents 0 in the homology groups $H_1(U, \mathbb{C})$ for any connected neighbourhood $U$ of $\hat{\Gamma}$. $\square$

Theorem 3.2 can be sharpened under some additional assumptions on $\hat{\Gamma}$. For instance, similarly to the proof of [Br2, Corollary 1.12] one obtains

**Corollary 3.3** Assume that $\hat{A} : I_T \to \mathbb{C}^n$ satisfies hypotheses of Theorem 3.2 and $\hat{\Gamma}$ is triangulable. Then $\hat{A} : I_T \to \hat{\Gamma}$ represents 0 in $H_1(\hat{\Gamma}, \mathbb{C})$. $\square$

From the result of Alexander [A] it follows that the covering dimension of $\hat{\Gamma}$ is 2. However, it does not imply its triangulability. Some cases when $\hat{\Gamma}$ is triangulable are considered in [W], [HL] (see also [Br2]).

3.2. The proof of Theorem 2.1 is based on a separation result proved in this section.

Let $\mathcal{E}_n$ be the set of closed complex curves in $\mathbb{C}^n$ containing 0 whose normalizations are either $\mathbb{C}$ or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Consider the equation

$$\frac{dv}{dx} = \sum_{i=1}^n a_i(x) v^{i+1}, \quad x \in I_T \ , \ a_i \in L^\infty(I_T) , \ 1 \leq i \leq n . \quad (3.1)$$

For $a = (a_1, \ldots, a_n, 0, \ldots) \in X$ we define the map $\tilde{A}_a : I_T \to \mathbb{C}^n$ by the formula

$$\tilde{A}_a(x) := (\tilde{a}_1(x), \ldots, \tilde{a}_n(x)) \quad (3.2)$$

with $\tilde{a}_i$ given by (2.3). We will assume that

(C_{n1}) There is a curve $C \in \mathcal{E}_n$ such that $\tilde{A}_a(I_T) \subset C$. 

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(C_{n2}) There is a continuous map $A'_a : I_T \to C_n$ such that $\tilde{A}_a = n \circ A'_a$ where $n : C_n \to C$ is the normalization of $C$.

**Remark 3.4** Condition $(C_{n2})$ is valid, for instance, if $\tilde{A}_a$ can be extended to a holomorphic map $U_a \to C$ where $U_a$ is a neighbourhood of $I_T \subset \mathbb{C}$.

Let $D_n$ denote the class of equations (3.1) satisfying conditions $(C_{n1})$ and $(C_{n2})$. By $X(D_n) \subset X$ we denote the set of sequences $a = (a_1, \ldots, a_n, 0, \ldots) \in X$ whose coordinates are coefficients of equations from $D_n$. As before $[a] \in G(X)$ stands for the equivalence class containing $a \in X$.

**Theorem 3.5** Let $a, b \in X(D_n)$ be such that $m([a]) = m([b])$ for every moment $m \in \mathcal{M}$. Then $[a] = [b]$.

**Remark 3.6** Theorem 3.5 implies that if $a \in X(D_n)$ is such that $m(a) = 0$ for every $m \in \mathcal{M}$, then $a \in \mathcal{U}$, the set of universal centers. In this case the structure of $a$ can be described explicitly (see [Br2, Corollary 1.17]).

Since every moment vanishes on the commutant $G^1(X)$, Theorem 3.5 implies

**Corollary 3.7** Let $a, b \in X(D_n)$ satisfy $[a * b^{-1}] \in G^1(X)$. Then $[a] = [b]$. $\square$

Since the center set $\mathcal{C}$ contains $\mathcal{U}$ and is closed under the multiplication $*$, we get

**Corollary 3.8** Let $a, b \in X(D_n)$ be such that $a \in \mathcal{C}$ and $m([a]) = m([b])$ for every moment $m \in \mathcal{M}$. Then $b \in \mathcal{C}$. $\square$

**Proof of Theorem 3.5.** For $a$ and $b$ as in Theorem 3.5 we consider the map $\tilde{A}_{ab^{-1}} : I_T \to \mathbb{C}^n$ defined by (3.2). Then the hypotheses of the theorem imply that

1. $\tilde{A}_{ab^{-1}}(0) = \tilde{A}_{ab^{-1}}(T) = 0$;

2. There are closed complex curves $C_1$ and $C_2$ from $\mathcal{E}_n$ such that

   $\tilde{A}_{ab^{-1}}(I_T) \subset C_1 \cup C_2$ and $\tilde{A}_{ab^{-1}}(I_l) \subset C_l$, $l = 1, 2$,

   where $I_1 := [0, T/2]$ and $I_2 := [T/2, T]$.

Extending $\tilde{A}_{ab^{-1}}$ to a $T$-periodic map $\mathbb{R} \to \mathbb{C}^n$ and identifying the quotient $\mathbb{R}/(T \cdot \mathbb{Z})$ with a circle $S$ we think of $\tilde{A}_{ab^{-1}}$ as a closed path $S \to \mathbb{C}^n$. Set $\Gamma := \tilde{A}_{ab^{-1}}(I_T)$ and $\Gamma_l := \tilde{A}_{ab^{-1}}(I_l), l = 1, 2$. Let $\widehat{\Gamma} \subset \mathbb{C}^n$ be the polynomially convex hull of $\Gamma$. Then from Theorem 3.2 by the hypotheses of Theorem 3.5 we get that for any connected neighbourhood $U$ of $\widehat{\Gamma}$ the path $\tilde{A}_{ab^{-1}} : S \to U$ represents 0 in $H_1(U, \mathbb{C})$. Next, since $C_1 \cup C_2$ is a closed complex subvariety of $\mathbb{C}^n$, an argument used in [Br2, Lemma 5.1] shows that $\widehat{\Gamma} \subset C_1 \cup C_2$. Since $C_1 \cup C_2$ is triangulable, the previous implies easily that $\tilde{A}_{ab^{-1}} : S \to C_1 \cup C_2$ represents 0 in $H_1(C_1 \cup C_2, \mathbb{C})$. We will consider two cases.
(A) $C_1 \neq C_2$.

Let us prove

Lemma 3.9

$$\tilde{A}_{a\vee b}^{-1}(T/2) = 0.$$  

**Proof.** Suppose, to the contrary, that $\tilde{A}_{a\vee b}^{-1}(T/2) := z \neq 0$. Let $f$ be a function on $C_1$ constant in some neighbourhoods of singular points of $C_1 \cup C_2$ in $C_1$, smooth outside these singular points, equals 0 in a neighbourhood of $0 \in \mathbb{C}^n$ and 1 in a neighbourhood of $z$. Then $df$ is well defined smooth 1-form on $C_1$. We can extend it by 0 to $C_2$ to get a smooth $d$-closed 1-form $\omega$ on $C_1 \cup C_2$. Since $C_1 \cup C_2$ is a complex space, it is Lipschitz triangulable. This and the fact that $\tilde{A}_{a\vee b}^{-1} : S \to C_1 \cup C_2$ represents 0 in $H_1(C_1 \cup C_2, \mathbb{C})$ imply that $\int_0^T \tilde{A}_{a\vee b}^{-1}(\omega) = 0$. On the other hand,

$$\int_0^T \tilde{A}_{a\vee b}^{-1}(\omega) = \int_0^{T/2} \tilde{A}_{a\vee b}^{-1}(df) = f(z) - f(0) = 1.$$  

This contradiction proves the lemma. \(\square\)

From this lemma we obtain that $\tilde{A}_{a\vee b}^{-1} : I_l \to C_1$, $l = 1, 2$, are closed paths.

**Lemma 3.10** $\tilde{A}_{a\vee b}^{-1} : I_l \to C_1$ represents 0 in $H_1(C_1, \mathbb{C})$, $l = 1, 2$.

**Proof.** Let $U_l$ be a neighbourhood of $C_l$ such that $C_l$ is a deformation retract of $U_l$. (Such $U_l$ exists, e.g., by the Lojasiewicz triangulation theorem [L].) Then $H_1(U_l, \mathbb{C}) \cong H_1(C_l, \mathbb{C})$. Also we can define $H_1(U_l, \mathbb{C})$ by the de Rham theorem (i.e., by integration of smooth $d$-closed 1-forms on $U_l$). Let $S_l \subset C_l$ be the (discrete) set of singular points of $C_1 \cup C_2$ in $C_l$. Then it is easy to see that for every smooth $d$-closed 1-form $\omega$ on $U_l$ there exists a smooth $d$-closed 1-form $\omega'$ on $U_l$, equals 0 in a neighbourhood of $S_l$, such that $\omega - \omega' = df$ for some smooth function $f$ on $U_l$. Thus one can define $H_1(U_l, \mathbb{C})$ be means of such forms $\omega'$. (The class of such forms is denoted by $E^1(S_l)$.) Next, for $\omega \in E^1(S_l)$ consider its restriction $\omega|_{C_l}$ to $C_l$. Then $\omega|_{C_l}$ can be extended by 0 to $C_2$ to get a smooth $d$-closed 1-form $\bar{\omega}$ on $C_1 \cup C_2$. Since $\tilde{A}_{a\vee b}^{-1} : S \to C_1 \cup C_2$ represents 0 in $H_1(C_1 \cup C_2, \mathbb{C})$, as in the proof of Lemma 3.9 we obtain

$$0 = \int_0^T \tilde{A}_{a\vee b}^{-1}(\bar{\omega}) = \int_{I_l} \tilde{A}_{a\vee b}^{-1}(\omega).$$  

This completes the proof of the lemma. \(\square\)

Note that after a reparametrization the path $\tilde{A}_{a\vee b}^{-1} : I_1 \to C_1$ coincides with $\tilde{A}_a$ and the path $\tilde{A}_{a\vee b}^{-1} : I_2 \to C_2$ with $-\tilde{A}_b$.

**Lemma 3.11**

$$[a] = [b] = 1.$$  

**Proof.** Let $n_l : C_{l} \to C_1$ be the normalization of $C_l$. According to condition $(C_{n2})$ in the definition of the class $X(D_n)$ and the above remark there are paths $A_l : I_l \to C_{l}$ such that $n_l \circ A_l = \tilde{A}_{a\vee b}^{-1}|_{I_l}$. Since $\tilde{A}_{a\vee b}^{-1} : I_l \to C_l$ represents 0 in $H_1(C_l, \mathbb{C})$, the path $A_l$ is closed. (For otherwise, as before we can construct a
smooth $d$-closed 1-form on $C_l$ whose integral over $\tilde{A}_{a\ast b^{-1}}|_{I_l}$ is not zero.) This implies easily that $A_l : I_l \to C_{ln}$ represents 0 in $H_1(C_{ln}, \mathbb{C})$. Then by the definition of the class $\mathcal{E}_n$ we obtain that $A_l : I_l \to C_{ln}$ is homotopically trivial. Hence the same is valid for $\tilde{A}_a$ and $\tilde{A}_b$. From here we deduce easily that all iterated integrals vanish at $[a]$ and $[b]$, i.e., $[a] = [b] = 1$ (see [Br2, Corollary 1.17] for details).

This completes the proof of the theorem in case (A).

(B) $C_1 = C_2$.

In this case we apply condition $(C_{n2})$ and the arguments from the proof of Lemma 3.11 to get $[a \ast b^{-1}] = 1$. We leave the details to the readers.

The proof of the theorem is complete. \hfill \Box

4. Proof of Theorem 2.1.

Let $A(w) := (a_1(w, \cdot), \ldots) \in X$, $w \in U$, be sequences of coefficients of equation (1.4). We set $\mathcal{A} :=\{[A(w)] \in G(X) : w \in U\}$. Our proof is based on

**Proposition 4.1** Suppose that the coefficients of (1.4) satisfy the hypotheses of Theorem 2.1. Then moments separate points on $\mathcal{A}$.

**Proof.** Assume to the contrary that there are $w_1, w_2 \in U$ with $[A(w_1)] \neq [A(w_2)]$ such that $m([A(w_1)]) = m([A(w_2)])$ for all moments $m \in \mathcal{M}$. We set $A_n(w) := (a_1(w, \cdot), \ldots, a_n(w, \cdot), 0, \ldots) \in X$. Then it is easy to see that the previous condition is equivalent to

$$m([A_n(w_1)]) = m([A_n(w_2)]) \quad \text{for all } n \in \mathbb{N}, \ m \in \mathcal{M}.$$  \hspace{1cm} (4.1)

We will show that $A_n(w) \in X(D_n)$ for all $w \in U$. Then from (4.1) and Theorem 3.5 we will get $[A_n(w_1)] = [A_n(w_2)]$ for all $n$. Clearly this is equivalent to the fact that $[A(w_1)] = [A(w_2)]$ in contradiction to our assumption. This will complete the proof of the proposition.

So, let us show that $A_n(w) \in X(D_n)$. We consider several cases.

(A) All coefficients $a_i(w, x)$ in the definition of $A_n(w)$ are polynomials in $x$.

Replacing $x$ by $z \in \mathbb{C}$ we obtain holomorphic polynomials $a_i(w, z)$ on $\mathbb{C}$. We can extend definitions (2.3) and (3.2) in this case:

$$\tilde{a}_i(w, x) := \int_0^x a_i(w, z) \, dz \quad \text{and} \quad \tilde{A}_{A_n(w)}(z) := (\tilde{a}_1(w, z), \ldots, \tilde{a}_n(w, z)),$$  \hspace{1cm} $z \in \mathbb{C}$.  

Since $\tilde{a}_i(w, \cdot)$ are holomorphic polynomials, the map $\tilde{A}_{A_n(w)}$ is holomorphic and polynomial. In particular, $C := \tilde{A}_{A_n(w)}(\mathbb{C})$ is a (possibly singular) rational curve in $\mathbb{C}^n$. It is easy to check that in this case the normalization $C_n$ of $C$ is $\mathbb{C}$. Thus conditions $(C_{n1})$ and $(C_{n2})$ are valid in this case, cf. Remark 3.4.

(B) All coefficients $a_i(w, x)$ in the definition of $A_n(w)$ are polynomials in $e^{\pm 2\pi i x/T}$ such that $\tilde{a}_i(w, \cdot)$ defined by (2.3) are $T$-periodic functions.
In this case as above we can extend $\tilde{A}_{A_n(w)}$ to a holomorphic map $\mathbb{C} \to \mathbb{C}^n$. The coordinates of this map are polynomials in $e^{\pm 2\pi iz/T}$. Let $\mathbb{C} \to \mathbb{C}^*, z \mapsto e^{2\pi iz/T}$, be the covering map; here $\mathbb{C}^*$ is the quotient of $\mathbb{C}$ by the action of the group $T \cdot \mathbb{Z}$ by translations along the $x$-axis. The map $\tilde{A}_{A_n(w)}$ is invariant with respect to this action and therefore it determines a holomorphic map $A'_{A_n(w)} : \mathbb{C}^* \to \mathbb{C}^n$ whose pullback to $\mathbb{C}$ coincides with $\tilde{A}_{A_n(w)}$. By the definition the coordinates of $A'_{A_n(w)}$ are Laurent polynomials. In particular, $A'_{A_n(w)}$ is algebraic and the Zariski closure of the image $A'_{A_n(w)}(\mathbb{C}^*)$ in $\mathbb{C}^n$ is a (possibly singular) rational curve $C$. It is easy to see that in this case the normalization $C_n$ of $C$ is either $\mathbb{C}$ or $\mathbb{C}^*$. Thus according to Remark 3.4, $A_n(w) \in X(D_n)$ in this case.

(C) All coefficients $a_i(w, x)$ in the definition of $A_n(w)$ are polynomials in $e^{\pm 2\pi ix/T}$ and at least one of $\tilde{a}_i(w, \cdot)$ in the definition of $\tilde{A}_{A_n(w)}$ is not $T$-periodic.

In this case we have

$$\tilde{a}_i(w, z) = c_i(w)z + b_i(w, z)$$

where $b_i(w, \cdot)$ are polynomials in $e^{\pm 2\pi iz/T}$ and at least one of $c_i(w)$ is not zero.

**Lemma 4.2** $C = \tilde{A}_{A_n(w)}(\mathbb{C})$ belongs to a closed complex curve in $\mathbb{C}^n$.

**Proof.** Assume without loss of generality that $c_1(w) \neq 0$. Let us define functions $a'_1$ by the formulas

$$a'_1(w, z) := \frac{\tilde{a}_1(w, z)}{c_1(w)} \quad \text{and} \quad a'_i(w, z) := \tilde{a}_i(w, z) - c_i(w) a'_1(w, z), \quad 2 \leq i \leq n.$$ 

Next, define $A : \mathbb{C} \to \mathbb{C}^n$ by

$$A'(z) := (a'_1(w, z), \ldots, a'_n(w, z)), \quad z \in \mathbb{C}.$$ 

By the definition the (invertible) linear transformation $L : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$L(z) := (c_1(w)z_1, z_2 + c_2(w)z_1, \ldots, z_n + c_n(w)z_1), \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

maps $C' := A'(\mathbb{C})$ onto $C$. Thus in order to prove that $C$ belongs to a closed complex curve in $\mathbb{C}^n$ it suffices to check the same for $C'$. So without loss of generality we may assume that $\tilde{a}_i = a'_i$, that is,

$$\tilde{a}_1(w, z) = z + b_1(w, z) \quad \text{and} \quad \tilde{a}_i(w, z) = b_i(w, z), \quad 2 \leq i \leq n.$$ 

Representing $\mathbb{C}^n$ as $\mathbb{C} \times \mathbb{C}^{n-1}$ we write $\tilde{A}_{A_n(w)} = (A_1, A_2) : \mathbb{C} \to \mathbb{C} \times \mathbb{C}^{n-1}$ where $A_1 := \tilde{a}_1(w, \cdot)$ and $A_2 : \mathbb{C} \to \mathbb{C}^{n-1}$ is a holomorphic $T$-periodic map. Let $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the natural projection. We will consider three cases.

(1) Assume that at least one of $b_i(w, z)$ in the definition of $A_2$ has nonzero terms involving $e^{2\pi iz/T}$ and $e^{-2\pi iz/T}$ for some positive integers $d$ and $m$. 

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Let $r : \mathbb{C} \to \mathbb{C}^*$, $r(z) := e^{2\pi i z/T}$, be the covering map, and $A'_2 : \mathbb{C}^* \to \mathbb{C}^{n-1}$ be the holomorphic map such that $A_2 := A'_2 \circ r$. The above assumption implies that $A'_2$ is proper. We prove that $\tilde{A}_{A_n(w)} : \mathbb{C} \to \mathbb{C}^n$ is also proper.

For otherwise, there exists a bounded convergent sequence $\{z_i\} \subset C \subset \mathbb{C}^n$ such that $\tilde{A}_{A_n(w)}^{-1}(\{z_i\}) \subset \mathbb{C}$ is unbounded. By the definition of $A_2$
\[
\tilde{A}_{A_n(w)}^{-1}(\{z_i\}) \subset A_2^{-1}(\{p(z_i)\}) := \{y_i\} + T \cdot \mathbb{Z} \subset \mathbb{C}
\]

where $\{y_i\}$ is a sequence of points in the strip $S_T = \{z \in \mathbb{C} : \text{Re} \, z \in [0, T)\}$. Since $A'_2$ is proper, $\{y_i\} \subset S_T$ is bounded. We write $A_1(z) = z + h_1(z) + h_2(z)$ where $h_1$ is a polynomial in $e^{2\pi iz/T}$ and $h_2$ is a polynomial in $e^{-2\pi iz/T}$. Then $h_1 + h_2$ is bounded on $\{y_i\} + T \cdot \mathbb{Z}$. This and the boundedness of $A_1(\tilde{A}_{A_n(w)}^{-1}(\{z_i\}))$ imply that $z$ is bounded on $\tilde{A}_{A_n(w)}^{-1}(\{z_i\})$. Therefore $\tilde{A}_{A_n(w)}^{-1}(\{z_i\})$ is bounded, a contradiction.

Since $\tilde{A}_{A_n(w)} : \mathbb{C} \to \mathbb{C}^n$ is proper, by the Remmert proper map theorem (see, e.g., [GH]) $C = \tilde{A}_{A_n(w)}(\mathbb{C}) \subset \mathbb{C}^n$ is a closed complex subvariety. This completes the proof of the lemma in case (1).

(2) Assume that all $b_1(w, z)$ in the definition of $A_2$ are either polynomials in $e^{2\pi iz/T}$ or in $e^{-2\pi iz/T}$ and $A_2$ is not constant.

Assume, e.g., that all $b_1(w, z)$ are polynomials in $e^{2\pi iz/T}$ (the proof in the case $b_1(w, z)$ are polynomials in $e^{-2\pi iz/T}$ is similar). Let $A'_2 : \mathbb{C}^* \to \mathbb{C}^{n-1}$ be such that $A_2 = A'_2 \circ r$. It is extended to a polynomial map $\mathbb{C} \to \mathbb{C}^{n-1}$ (denoted also by $A'_2$). We set $o := A'_2(0) \in \mathbb{C}^{n-1}$ and $M := A'_2^{-1}(o) \subset \mathbb{C}$. Then $M := \{s_i\} + T \cdot \mathbb{Z}$ where $\{s_i\} \subset S_T$ is finite. Let us consider the complex line $V := p^{-1}(o) \subset \mathbb{C}^n$. Since the map $A'_2 : \mathbb{C} \setminus (A'_2)^{-1} \{o\} \to \mathbb{C}^{n-1} \setminus \{o\}$ is proper, the same argument as in the proof of case (1) implies that $\tilde{A}_{A_n(w)} : \mathbb{C} \setminus M \to \mathbb{C}^n \setminus V$ is proper. Therefore by the proper map theorem $\tilde{A}_{A_n(w)}(\mathbb{C} \setminus M)$ is a closed complex curve in $\mathbb{C}^n \setminus V$.

Further, $\tilde{A}_{A_n(w)}$ maps $M$ onto $V \cap C$. Note that the definition of $M$ and $A_1$ imply that $A_1$ is unbounded on every infinite subset of $M$. Thus $V \cap C \subset \mathbb{C}^n$ is a discrete subset. In particular, $V \cap C \subset \mathbb{C}^n$ is a complex analytic subset and $\tilde{A}_{A_n(w)}(\mathbb{C} \setminus M)$ is a closed complex curve in $\mathbb{C}^n \setminus (V \cap C)$. Therefore by the Levi extension theorem (see, e.g., [GH, Chapter 3]) the closure of $\tilde{A}_{A_n(w)}(\mathbb{C} \setminus M)$ is a closed complex curve in $\mathbb{C}^n$. To complete the proof of the lemma in this case observe that this closure coincides with $C$.

(3) $A_2$ is constant.

Let $o := A_2(\mathbb{C}) \subset \mathbb{C}^{n-1}$. Then $C$ is an open everywhere dense subset of the complex line $\{(z, o) : z \in \mathbb{C}\} \subset \mathbb{C}^n$.

The proof of the lemma is complete. \hfill \qed

**Lemma 4.3** Let $\tilde{C} \subset \mathbb{C}^n$ be the closed complex curve containing $C = \tilde{A}_{A_n(w)}(\mathbb{C})$ from Lemma 4.2. Then its normalization $\tilde{C}$ is biholomorphic to $\mathbb{C}$.

**Proof.** Clearly it suffices to prove the lemma for $C$ satisfying conditions (1) or (2). In this case we have $C = \tilde{C}$. Let $n : C_n \to C$ be the normalization of $C$. Then there
is a holomorphic map \( \tilde{A}_n : C \to C_n \) such that \( \tilde{A}_{A_n(w)} = n \circ \tilde{A}_n \). Recall that in the proof of Lemma 4.2 we established that \( \tilde{A}_{A_n(w)} : C \setminus \tilde{A}_{A_n(w)}^{-1}(S) \to C \setminus S \) is proper for a discrete set \( S \subset C \). This implies easily that \( A_n : C \to C_n \) is a proper surjective map. In particular, the induced homomorphism \( \tilde{A}_n^* : H^1(C_n, \mathbb{C}) \to H^1(C, \mathbb{C}) \) is injective. Thus \( H^1(C_n, \mathbb{C}) = 0 \) and, since \( C_n \) is non-compact, \( C_n \cong \mathbb{C} \). □

Lemma 4.2 and 4.3 show that \( A_n(w) \in X(D) \) in case (C).

(D) At least one of the coefficients \( a_i(w, x) \) in the definition of \( A_n(w) \) is a non-constant polynomial in \( x \).

It is easy to see that in this case the extended map \( \tilde{A}_{A_n(w)} : C \to \mathbb{C}^n \) is proper.

Then by the Remmert theorem \( \tilde{A}_{A_n(w)}(C) \subset \mathbb{C}^n \) is a closed complex subvariety.

Similarly to the proof of Lemma 4.3 we get that the normalization of \( C \) is \( \mathbb{C} \).

Thus we have proved that for all possible cases \( A_n(w) \in X(D_n) \). As we explained above this implies the required statement of the proposition. □

Using this proposition let us complete the proof of the theorem.

By the definition for every \( m \in \mathcal{M} \) its pullback \( A^*m \) is a holomorphic function on \( U \). Let \( K \subset U \) be a compact subset. Since \( U \) is Stein, without loss of generality we will assume that \( K \) is holomorphically convex in \( U \). Then we can choose a relatively compact Stein neighbourhood \( V \subset U \) of \( K \). Consider a closed complex subvariety \( Z \) of \( U \times U \) given by equations \( A^*m(z) - A^*m(w) = 0 \), \( z \times w \in U \times U \), \( m \in \mathcal{M} \). By so-called Cartan’s Theorem B, and by our choice of \( V \) we obtain that there is a number \( l = l(K) \in \mathbb{N} \) such that the set \( Z \cap (V \times V) \) coincides with \( \{ z \times w \in V \times V : A^*m(z) - A^*m(w) = 0 \, , \, 1 \leq i \leq l \} \). Further, according to Remark 3.6 the set of universal centers \( U_A \) of equation (1.4) satisfying the hypotheses of Theorem 2.1 is determined as the set of zeros of holomorphic functions \( A^*m \) on \( U \) (\( m \in \mathcal{M} \)). As above we can find a number \( s = s(K) \in \mathbb{N} \) such that \( U_A \cap V \) coincides with the set of zeros of \( A^*m_i \), \( 1 \leq i \leq s \), in \( V \). Without loss of generality we will assume that \( l = s \).

Let us consider the holomorphic map

\[ M_l = (A^*m_1, \ldots, A^*m_l) : U \to \mathbb{C}^l. \]

By \( [\, ] : X \to G(X), a \mapsto [a] \), we denote the canonical surjection, see section 2.2. Also, by \( A_K \subset A \) we denote the set \( \{ [A(w)] \in G(X) : w \in K \} \).

**Lemma 4.4** There is a continuous map \( \tilde{M}_l : A \to \mathbb{C}^l \) which embeds \( A_K \) into \( \mathbb{C}^l \) and satisfies \( M_l = \tilde{M}_l \circ [\, ] \circ A \).

**Proof.** Since every moment is the pullback with respect to \( [\, ] \) of a continuous function on \( G(X) \), the map \( M_l : A \to \mathbb{C}^l, M_l([a]) := M_l(a), a \in A(U) \), is well defined and continuous. Let us show that \( \tilde{M}_l \) separates points on \( A_K \). Let \( g_1, g_2 \in A_K \) satisfy \( \tilde{M}_l(g_1) = \tilde{M}_l(g_2) \). Assume that \( w_i \in K \) is such that \([A(w_i)] = g_i, i = 1, 2\). Then by our assumption \( m_i(A(w_1)) = m_i(A(w_2)) \) for all \( 1 \leq i \leq l \). From here by the choice of \( l \) we get \( m(A(w_1)) = m(A(w_2)) \) for all \( m \in \mathcal{M} \). Then Proposition 4.1 implies that \( g_1 = g_2 \). □
Remark 4.5 Since \( K \subset U \) is compact, \( A_K \subset G(X) \) is a compact subset by the definition of the topology on \( G(X) \), see [Br3, Theorem 2.4]. Thus from the lemma we get \( \tilde{M}_t : A_K \to M_t(K) \) is a homeomorphism.

Let \( c_n, n \in \mathbb{N} \), be iterated polynomials on \( X \) whose set of zeros is the center set \( C \), see section 2.1. By the definition, the set of zeros of holomorphic functions \( A^*c_n \) on \( U \) is \( C_A \). Next, by our choice of \( V \subset U \), there is a number \( N = N(K) \in \mathbb{N} \) such that \( C_A \cap V \) coincides with the set of zeros of the family \( \{A^*c_i\}_{1 \leq i \leq N} \), in \( U \). Let \( \tilde{c}_i \) be a continuous function on \( G(X) \) whose pullback to \( X \) coincides with \( c_i \). According to Lemma 4.4 and the Tietze-Urysohn extension theorem applied to the continuous functions \( (\tilde{M}_t^{-1})^*|_{A_K}(\tilde{c}_i|_{A_K}) \) on the compact set \( A_t(K) \subset \mathbb{C}_t \), there are continuous functions \( f_i \) on \( \mathbb{C}_t \) such that \( \tilde{M}_t^*f_i|_{A_K} = \tilde{c}_i|_{A_K}, 1 \leq i \leq N \). Finally, the application of Lemma 4.4 leads to the required statement. Namely, under the hypotheses of the theorem the center set \( C_A \cap K \) of (1.4) in \( K \) coincides with the set of zeros of continuous functions \( f_n(A^*m_1, \ldots, A^*m_l) \) on \( U \) \((1 \leq n \leq N)\). Also, by our choice of \( l \) the set of universal centers \( U_A \cap K \) of (1.4) in \( K \) is determined as the set of zeros of the holomorphic functions \( A^*m_n, 1 \leq n \leq l \), restricted to \( K \). \( \square \)

5. Proofs of Theorems 2.3 and 2.4.

5.1. Proof of Theorem 2.3. Assume that all coefficients \( a_i \) in (1.4) are \( \alpha \)-homogeneous polynomials in \( w \in \mathbb{C}^k \) for some \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \) whose coefficients are either polynomials in \( x \) or in \( e^{\pm 2\pi i x/T} \). Let us consider the holomorphic map \( P_\alpha : \mathbb{C}^k \to \mathbb{C}^k \),

\[
P_\alpha(w_1, \ldots, w_k) := (w_1^{\alpha_1}, \ldots, w_k^{\alpha_k}), \quad (w_1, \ldots, w_k) \in \mathbb{C}^k.
\]

By \( \tilde{a}_i(\cdot, x) \) we denote the pullback of \( a_i(\cdot, x) \) to \( \mathbb{C}^k \) with respect to \( P_\alpha \). Then by the definition all \( \tilde{a}_i \) are homogeneous. Let us consider the equation (1.4) whose coefficients are \( \tilde{a}_i \). By \( \hat{A} := (\tilde{a}_i(w, \cdot), \ldots) \) we denote the corresponding map of \( \mathbb{C}^k \) into \( X \), and by \( U_{A_{\lambda}} \) and \( C_{\lambda} \) the sets of universal centers and centers for this equation. Now the hypotheses of the theorem imply that \( U_{\hat{A}} = \{0\} \subset \mathbb{C}^k \). Also, we clearly have

\[
C_{\hat{A}} = P_\alpha^{-1}(C_A) \quad \text{and} \quad U_{\hat{A}} = P_\alpha^{-1}(U_A). \tag{5.1}
\]

Suppose that we will prove that \( C_{\hat{A}} \subset \mathbb{C}^k \) is determined as the set of zeros of holomorphic polynomials \( p_n(\hat{A}^*m_1, \ldots, \hat{A}^*m_N), 1 \leq n \leq N \), on \( \mathbb{C}^k \), where \( p_n \) are some holomorphic polynomials on \( \mathbb{C}^N \). By the definition, see (2.2), \( \hat{A}^*m_i \) is the pullback with respect to \( P_\alpha \) of the holomorphic polynomial \( A^*m_i, 1 \leq i \leq N \). Thus \( p_n(\hat{A}^*m_1, \ldots, \hat{A}^*m_N) \) is the pullback with respect to \( P_\alpha \) of \( p_n(A^*m_1, \ldots, A^*m_N), 1 \leq n \leq N \). From here and (5.1) we get that \( C_{\hat{A}} \) is determined as the set of zeros of holomorphic polynomials \( p_n(A^*m_1, \ldots, A^*m_N), 1 \leq n \leq N \), on \( \mathbb{C}^k \). This will complete the proof of the theorem.

Thus without loss of generality we may assume that the coefficients \( a_i \) in the original equation (1.4) are homogeneous polynomials in \( w \in \mathbb{C}^k \).
Further, let us consider a complex algebraic subvariety $Z$ of $\mathbb{C}^k \times \mathbb{C}^k$ given by polynomial equations $A^*m(z) - A^*m(w) = 0$, $z \times w \in \mathbb{C}^k \times \mathbb{C}^k$, $m \in \mathcal{M}$. By the Hilbert finiteness theorem there is a number $l \in \mathbb{N}$ such that $Z = \{z \times v \in \mathbb{C}^k \times \mathbb{C}^k : A^*m_i(z) - A^*m_i(w) = 0, 1 \leq i \leq l\}$. Also, according to Remark 3.6 the set of universal centers $\mathcal{U}_A$ satisfying the hypotheses of Theorem 2.1 is determined as the set of zeros of homogeneous holomorphic polynomials $A^*m$ on $\mathbb{C}^k$ ($m \in \mathcal{M}$). As above we can find a number $s \in \mathbb{N}$ such that $\mathcal{U}_A$ coincides with the set of zeros of $A^*m_i$, $1 \leq i \leq s$. Without loss of generality we will assume that $l = s$.

Let us consider the polynomial map

$$M_l = (A^*m_1, \ldots, A^*m_l) : \mathbb{C}^k \to \mathbb{C}^l.$$ (5.2)

We prove

**Lemma 5.1** Under the hypotheses of the theorem $M_l$ is a proper map.

**Proof.** Assume, to the contrary, that $M_l$ is not proper. Then there exists an unbounded sequence $\{w_j\} \subset \mathbb{C}^k$ such that $\{M_l(w_j)\}$ converges to some $w \in \mathbb{C}^l$. We write $w_j = |w_j| \cdot v_j$ where $|\cdot|$ is the Euclidean norm on $\mathbb{C}^k$ and $v_j$ belongs to the unit sphere $S^{2k-1} \subset \mathbb{C}^k$. Without loss of generality we will assume that $\{v_j\}$ converges to a point $v \in S^{2k-1}$. Now, since every $A^*m_i$ is a homogeneous polynomial, we have $A^*m_i(w_j) = |w_j|^{d_i} A^*m_i(v_j)$, where $d_i$ is the degree of $A^*m_i$. According to our assumptions $\{A^*m_i(w_j)\} \subset \mathbb{C}$ converges and $|w_j| \to \infty$ as $j \to \infty$. This implies easily that $A^*m_i(v) = 0$ for all $1 \leq i \leq l$. But then by the choice of $l$ we have $A^*m(v) = 0$ for all $m \in \mathcal{M}$. Thus by Remark 3.6 we get from here that $v \in \mathcal{U}_A$ which contradicts to $\mathcal{U}_A = \{0\}$. This contradiction proves the lemma. $\blacksquare$

Since $\mathcal{C}_A \subset \mathbb{C}^k$ is a complex algebraic subvariety, from this lemma and the fact that $M_l$ is polynomial it follows that $M_l(\mathcal{C}_A) \subset \mathbb{C}^l$ is a complex algebraic subvariety, as well, see, e.g., [GH]. Hence, by the Chow theorem there are holomorphic polynomials $p_1, \ldots, p_N$ on $\mathbb{C}^l$ whose set of zeros is $M_l(\mathcal{C}_A)$. Then the set of zeros of their pullbacks $p_n(A^*m_1, \ldots, A^*m_N)$ with respect to $M_l$ to $\mathbb{C}^k$ determines the set $M_l^{-1}(M_l(\mathcal{C}_A))$. Let us show now that

$$M_l^{-1}(M_l(\mathcal{C}_A)) = \mathcal{C}_A.$$ (5.3)

Assume, to the contrary, that there is $w \in M_l^{-1}(M_l(\mathcal{C}_A)) \setminus \mathcal{C}_A$. By the definition there exists some $v \in \mathcal{C}_A$ such that $M_l(w) = M_l(v)$. This implies that $A^*m_i(w) = A^*m_i(v)$ for all $1 \leq i \leq l$. But then by our choice of $l$ the similar identity holds for every $m \in \mathcal{M}$. Now according to Proposition 4.1 we get from here $[A(w)] = [A(v)]$ in $G(X)$. Then from Corollary 3.8 we obtain that $w \in \mathcal{C}_A$. This contradiction proves (5.3) and completes the proof of the theorem. $\blacksquare$

5.2. **Proof of Theorem 2.4.** As in the proof of Theorem 2.3 we may assume without loss of generality that all coefficients $a_i$ in (1.4) are homogeneous polynomials in $w \in \mathbb{C}^k$ whose coefficients are either polynomials in $x$ or in $e^{\pm 2\pi ix/T}$. Let us consider the polynomial map $M_l : \mathbb{C}^k \to \mathbb{C}^l$ constructed in the same way as in the proof of Theorem 2.3, see (5.2). Let $V \subset \mathbb{C}^k$ be a complex algebraic subvariety whose closure in $\mathbb{C}^k$ does not meet $\bar{\mathcal{U}}_A$. Then we prove
Lemma 5.2 Under the hypotheses of the theorem $M_l|_V$ is a proper map.

Proof. Let $p : \mathbb{C}^k \setminus \{0\} \to \mathbb{CP}^{k-1}, (w_1, \ldots, w_k) \mapsto (w_1 : \ldots : w_k), (w_1, \ldots, w_k) \in \mathbb{C}^k$, be the canonical projection determining $\mathbb{CP}^{k-1}$. By the definition $p$ can be extended to a holomorphic map $\hat{p} : \mathbb{CP}^k \setminus \{0\} \to \mathbb{CP}^{k-1}$ such that $\hat{p}|_{H_k} : H_k \to \mathbb{CP}^{k-1}$ is biholomorphic. (Here $H_k \subset \mathbb{CP}^k$ is the hyperplane at infinity.) Further, since $U_A$ is determined as the set of zeros of homogeneous polynomials $A^*m_i, 1 \leq i \leq l$, see Remark 3.6, $\hat{p} (U_A \setminus \{0\})$ is a projective variety isomorphic to $\overline{U_A} \cap H_k$.

Assume, to the contrary, that $M_l|_V$ is not proper. Then there is an unbounded sequence $\{w_j\} \subset V$ such that $\{M_l(w_j)\}$ converges in $\mathbb{C}^l$. Without loss of generality we will assume that $\{w_j\}$ converges to a point $\overline{w} \in H_k \cap \nabla$. By our assumption $\overline{w} \notin \overline{U_A}$. We write $w_j = |w_j| \cdot v_j$ where $|\cdot|$ is the Euclidean norm on $\mathbb{C}^k$ and $v_j$ belongs to the unit sphere $S^{2k-1} \subset \mathbb{C}^k$. Without loss of generality we will assume that $\{v_j\}$ converges to a point $v \in S^{2k-1}$. Then as in the proof of Lemma 5.1 we get $v \in U_A$. Further, by our construction, $\hat{p}(v) = \hat{p}(\overline{w})$. Hence, we have $\hat{p}(\overline{w}) \in \hat{p} (U_A \setminus \{0\})$. From here and the facts that $\hat{p} : H_k \to \mathbb{CP}^{k-1}$ and $\hat{p} : U_A \cap H_k \to \hat{p} (U_A \setminus \{0\})$ are isomorphisms we derive $\overline{w} \in \overline{U_A}$. This contradiction proves the lemma. \hfill \Box

Since $C_A \cap V \subset \mathbb{C}^k$ is a complex algebraic subvariety, from this Lemma we get that $M_l(C_A \cap V) \subset \mathbb{C}^l$ is a complex algebraic subvariety, as well. Thus by the Chow theorem $M_l(C_A \cap V)$ is determined as the set of zeros of holomorphic polynomials on $\mathbb{C}^l$. Finally, an argument similar to that used in the proof of Theorem 2.3 shows that the zero locus of the pullbacks of these polynomials to $V$ with respect to $M_l$ coincides with $C_A \cap V$. \hfill \Box

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