Particular existence of positive solutions for elliptic equations involving regional fractional Laplacian of order \((0, \frac{1}{2}]\)

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Abstract

Our purpose of this paper is to investigate positive solutions of the elliptic equation with regional fractional Laplacian

\[ (-\Delta)_B^s u + u = h(x,u) \quad \text{in} \quad B_1, \quad u \in C_0(B_1), \]

where \((-\Delta)_B^s\) with \(s \in (0, \frac{1}{2}]\) is the regional fractional Laplacian and \(h\) is the nonlinearity.

Ordinarily, positive solutions vanishing at the boundary are not anticipated to be derived for the equations with regional fractional Laplacian of order \(s \in (0, \frac{1}{2}]\). Positive solutions are obtained when the nonlinearity assumes the following two models:

\[ h(x,t) = f(x) \quad \text{or} \quad h(x,t) = h_1(x)t^p + \epsilon h_2(x), \]

where \(p > 1, \epsilon > 0\) small and \(f, h_1, h_2\) are Hölder continuous, radially symmetric and decreasing functions under suitable conditions.

Keywords: Schrödinger equation; Regional Fractional Laplacian; Existence.

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1 Introduction

Let \(s \in (0, 1)\), \(\Omega\) is \(C^2\) domain in \(\mathbb{R}^N\) with \(N \geq 2\), \((-\Delta)_\Omega^s\) be the regional fractional Laplacian defined by

\[ (-\Delta)_\Omega^s u(x) = c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\Omega \setminus B_\varepsilon(x)} \frac{u(z) - u(x)}{|z - x|^{N+2s}} \, dz, \]

where \(B_r(x)\) is the ball with radius \(r\) and the center at \(x\), particularly, denote \(B_r = B_r(0)\), here \(c_{N,s} > 0\) is the normalized constant of fractional Laplacian \((-\Delta)_{\mathbb{R}^N}^s\) (simply we use the notation \((-\Delta)^s\), see \([17]\)). Our interest of this paper is to study the existence of positive solutions to the following Schrödinger equation with regional fractional Laplacian

\[
\begin{cases}
(-\Delta)_B^s u + u = h(x,u) & \text{in} \quad B_1, \\
u \in C_0(B_1),
\end{cases}
\]

where \(h : B_1 \times \mathbb{R} \to \mathbb{R}\) is a measurable function, \(C_0(B_1)\) is the set of function continuous in \(\bar{B}_1\), which vanishes at the boundary \(\partial B_1\).

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In the last few years, nonlocal problems have emerged extensively in physics models, operation research, queuing theory, mathematical finance, risk estimation, and others domains (see [7]). The regional fractional Laplacian serves as a typical operator as the generator of the censored stable process. In fact, from the perspective of Probability, a symmetric $2s$-stable process in $\mathbb{R}^N$ killed upon leaving the set $\Omega$ is called a symmetric $2s$-stable process in $\Omega$. Bogdan, Burdzy and Chen [5] (also see Guan and Ma [14, 22]) extended this type of processes to obtain a version of the strong Markov process, named censored symmetric stable process. A censored stable process in an open set $\Omega \subset \mathbb{R}^N$ is derived from the symmetric stable process by suppressing its jumps from $\Omega$ to $\mathbb{R}^N \setminus \Omega$. We'd like to mention that the censored stable process has some interesting properties, which indicate certain differences between $(\frac{1}{2}, 1)$ and $(0, \frac{1}{2}]$ (see [5, Theorem 1]):

(i) for $s \in (\frac{1}{2}, 1)$, the censored symmetric $2s$-stable process in $\Omega$ has a finite lifetime and will approach $\partial \Omega$;

(ii) for $s \in (0, \frac{1}{2}]$, the censored symmetric $2s$-stable process in $\Omega$ is conservative and will never approach $\partial \Omega$.

On the analysis side, interesting new phenomena occur in relation to elliptic problems involving the regional fractional Laplacian. Let $H^s_0(\Omega)$ be the closure of $C^\infty_c(\Omega)$ under the semi-norm that

$$||u||_{s,\Omega} = \sqrt{\int_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy}.$$ 

The authors in [24] showed that

for $s \in (\frac{1}{2}, 1)$, Hilbert space $H^s_0(\Omega)$ has zero trace;

for $s \in (0, \frac{1}{2}]$, Hilbert space $H^s_0(\Omega)$ has no zero trace.

Let $H^s_c(\Omega)$ be the closure of $C^\infty_c(\Omega)$ under the norm that

$$||u||_{s,\Omega} = \sqrt{\int_{\Omega \times \Omega} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy + \int_{\Omega} u^2 dx}.$$ 

It is worth noting that $1 \in H^s(\Omega)$, $H^s_0(\Omega) = H^s_c(\Omega) \cap L^2(\Omega)$ and it also has no zero trace for $s \in (0, \frac{1}{2}]$. So it is delicate to obtain the solution of (1.1) for $s \in (0, \frac{1}{2}]$ with zero boundary condition.

When $s \in (\frac{1}{2}, 1)$, [13,14] provide estimates on the heat kernel and Green kernel related to the regional fractional Laplacian, [21] builds a formula of integration by part for regional fractional Laplacian, [9] extends this formula to solve regional fractional problem with inhomogeneous terms. Via building the formula of integral by part and related embedding results, [9] obtains the existence of solutions to

$$\begin{cases} 
(\Delta)^s_\Omega u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}$$

(1.2)

for $s \in (\frac{1}{2}, 1)$ when $\Omega$ is a bounded regular domain. For further study of regional fractional Laplacian with $s \in (\frac{1}{2}, 1)$, refer to [2,11] for boundary blowing-up solutions, [18] for boundary regularity, [1] for related Hopf Lemma, [19] for existence of weak solutions with critical semilinear
term. More related study see [3, 4, 15] and references therein on related topics involving the regional fractional Laplacian.

For $s \in (0, \frac{1}{2}]$, the structure of solutions of elliptic equations are very challenging. The authors in [12] showed the nonexistence of solutions to Poisson problem

$$(-\Delta)_{\Omega}^s u = 1 \quad \text{in } \Omega$$

and nonexistence of positive solutions to Lane-Emden equation

$$(-\Delta)_{\Omega}^s u = u^p \quad \text{in } \Omega.$$  

Under the assumption of $\int_{\Omega} f dx = 0$, [29] shows the existence of weak solution to Poisson problem

$$(-\Delta)^s_{\Omega} u = f \quad \text{in } \Omega$$

and show that it is a classical solution when $f$ is regular.

Our first aim of this paper is to show the existence of solution to Poisson problem

$$\begin{cases} (-\Delta)^s_{B_1} u + u = f & \text{in } B_1, \\ u \in C_0(B_1), \end{cases}$$  

when $f$ satisfies some extra condition.

**Theorem 1.1** Assume that $s \in (0, \frac{1}{2}]$ and

($\mathcal{H}_0$) $F \in C^\theta_{\text{loc}}(\overline{B_1})$ with $\theta \in (0, 1)$ is a non-constant nonnegative function, which is radially symmetric and decreasing with respect to $|x|$.

Then there exists $F_0 \in \left[ \inf_{x \in B_1} F(x), \frac{1}{|B_1|} \int_{B_1} F \, dx \right]$ such that for $f = F - F_0$, problem (1.4) has a unique classical positive solution $u_f$, which is radially symmetric and decreasing with respect to $|x|$.

Furthermore, it holds that

$$\int_{B_1} u_f dx = \int_{B_1} f dx.$$  

**Remark 1.1** Note that

$$\int_{B_1} f \, dx = \int_{B_1} F \, dx - F_0 |B_1| > 0.$$  

We emphasize that $f$ can’t be a positive constant when it satisfies assumption ($\mathcal{H}_0$). In fact, note that if $f = 1$ Poisson problem

$$\begin{cases} (-\Delta)^s_{B_1} u + u = 1 & \text{in } B_1, \\ u \in H^s_0(B_1) \end{cases}$$

has a unique classical solution $u_1 \equiv 1 \in H^s_0(B_1)$. However, it isn’t in $C_0(B_1)$. In other words, problem (1.4) has no positive solutions in $C_0(B_1)$ when $f = 1$. 

Now we show the existence of positive solutions to Schrödinger equation (1.1).

**Theorem 1.2** Assume that \( s \in (0, \frac{1}{2}] \) and 

\[
H(d, x, t) = h_1(x)(t - d)^p - d + \epsilon h_2(x),
\]

where \( p > 1, \epsilon > 0 \) and \((H_1)\) functions \( h_1, h_2 \in C^\theta(\overline{B}_1) \) with \( \theta \in (0, 1) \) are radially symmetric and decreasing with respect to \(|x|\), \( h_2 \) is non-constant and \( \inf_{x \in B_1} h_1(x), \inf_{x \in B_1} h_2(x) > 0. \)

Denote 

\[
\epsilon_0 = \left( |B_1| \inf_{x \in B_1} h_2(x) \inf_{x \in B_1} h_1(x) \right)^{-1} \left( \|h_1\|_{L^1(B_1)} \right)^{\frac{1}{p}}.
\]

Then for any \( \epsilon \in (0, \epsilon_0) \), there exists \( d_\epsilon \in \left( \epsilon \inf_{x \in B_1} h_2(x), 1 \right) \) such that for \( h = H(d_\epsilon, \cdot, \cdot) \), problem (1.1) admits a positive solution \( u_\epsilon \in C_0(B_1) \) for \( \epsilon \in (0, \epsilon_0) \), which is radially symmetric and decreasing with respect to \(|x|\).

Note that for \( s \in (\frac{1}{2}, 1) \), the existence could be obtained in \( H^s_0(B_1) \) by the variational method, since the the space \( H^s_0(B_1) \) has the boundary trace in [5]. However, it fails when \( s \in (0, \frac{1}{2}] \). We emphasize from [12] that for \( s \in (0, \frac{1}{2}] \), Lane-Emden equation 

\[
\begin{cases}
(-\Delta)^s_{B_1} u = u^p & \text{in } B_1, \\
u = 0 & \text{on } \partial B_1
\end{cases}
\]

has no positive solutions.

It is worth noting that the solutions of (1.1) are derived via passing to the limit of solutions as \( r_0 \rightarrow 1 \) of 

\[
\begin{cases}
(-\Delta)^s_{B_1} u + u = h(x, u) & \text{in } B_{r_0}, \\
u = 0 & \text{in } \bar{B}_1 \setminus B_{r_0}.
\end{cases}
\]

In order to control the boundary behavior in this approximations, we need the special properties of radial symmetry and the decreasing monotonicity. Our method for these properties is to use the method of moving planes, which requires some properties of symmetries and monotonicities for the nonlinearity \( h \).

**Theorem 1.3** Assume that \( s \in (0, 1) \), 

\[
h(x, t) = h_1(x)h_3(t) + h_2(x),
\]

where \( h_1, h_2 \) verify 

\((H_2)\) \( h_1, h_2 : B_1 \rightarrow [0, +\infty) \) are radially symmetric and decreasing with respect to \(|x|\).
and \( h_3 \) satisfies that

\((\mathcal{H}_3)\) \( h_3 : \mathbb{R}_+ \to \mathbb{R}_+ \) is nondecreasing, locally Lipschitz continuous in \([0, +\infty)\).

Let \( u \in C_0(B_1) \) be a nonnegative, nonzero solution of \((1.6)\), then \( u \) is radially symmetric and strictly decreasing in \( r = |x| \) for \( r \in (0, r_0) \).

The rest of this paper is organized as follows. In Section 2, we recall the connection of regional fractional Laplacian and fractional Laplacian, properties of viscosity solutions and regularity estimates. In Section 3, we prove Theorem 1.3 by the method of moving planes. Section 4 and Section 5 are devoted to solve the solutions to the related Poisson problem and Schrödinger equations, respectively. Finally, we annex properties of Green kernel of the fractional Laplacian.

2 Preliminary

2.1 Connections of regional fractional Laplacian and fractional Laplacian

Note that by the zero extension of the function in \( \mathbb{R}^N \setminus \Omega \), we can build the connection between regional fractional Laplacian and fractional Laplacian.

Given \( u \in C_0(\Omega) \), we denote

\[
\tilde{u}(x) = \begin{cases} 
    u(x), & x \in \Omega, \\
    0, & x \notin \Omega.
\end{cases}
\]  

Then for \( x \in \Omega \), it holds that

\[
(-\Delta)^s \tilde{u}(x) = \text{p.v.} \int_{\mathbb{R}^N} \frac{\tilde{u}(x) - \tilde{u}(y)}{|x - y|^{N + 2s}} \, dy
\]

\[
= \text{p.v.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} \, dy + \int_{\Omega^c} \frac{u(x)}{|x - y|^{N + 2s}} \, dy
\]

\[
= (-\Delta)^s_{\Omega} u(x) + u(x) \varphi_\Omega(x),
\]

where

\[
\varphi_\Omega(x) = \int_{\Omega^c} \frac{dy}{|x - y|^{N + 2s}}.
\]  

Proposition 2.1 Let \( \varphi_\Omega \) be defined by (2.2).

(i) If \( \Omega \) is \( C^2 \), then \( \varphi_\Omega \) is locally Lipschitz continuous.

(ii) If \( \Omega = B_1 \), then \( \varphi_{B_1} \) is radially symmetric, decreasing and there exists \( c_1 > 0 \) such that

\[
\lim_{x \to \partial B_1} \varphi_{B_1}(x)(1 - |x|)^{2s} = c_1.
\]  

The proof is addressed in the Appendix.
2.2 Viscosity solution

We start with the definition of viscosity solutions, inspired by the definition of viscosity sense for nonlocal problems in [8].

Definition 2.1 (i) We say that a function \( u \in C(\overline{\Omega}) \) is a viscosity super-solution (sub-solution) of

\[
\begin{cases}
(\Delta)^s_{\Omega} u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

if \( u \geq 0 \) (resp. \( u \leq 0 \)) on \( \partial \Omega \) and for every point \( x_0 \in \Omega \) and some neighborhood \( V \) of \( x_0 \) with \( \overline{V} \subset \Omega \) and for any \( \varphi \in C^2(\overline{V}) \) such that \( u(x_0) = \varphi(x_0) \) and \( x_0 \) is the minimum (resp. maximum) point of \( u - \varphi \) in \( V \), let

\[
\tilde{u} = \begin{cases}
\varphi & \text{in } V, \\
u & \text{in } \Omega \setminus V,
\end{cases}
\]

we have that

\[
(\Delta)^s_{\Omega} \tilde{u}(x_0) \geq f(x_0) \quad \text{(resp. } (\Delta)^s_{\Omega} \tilde{u}(x_0) \leq f(x_0)).\]

(ii) We say that \( u \) is a viscosity solution of (2.4) if it is a viscosity super-solution and also a viscosity sub-solution of (2.4).

Theorem 2.1 Assume that the functions \( f : \Omega \to \mathbb{R}, h : \partial \Omega \to \mathbb{R} \) are continuous. Let \( u \) and \( v \) be a viscosity super-solution and sub-solution of (2.4), respectively. Then

\[
v \leq u \quad \text{in } \Omega.
\]

Proof. Let us define \( w = u - v \), then

\[
\begin{cases}
(\Delta)^s_{\Omega} w \geq 0 & \text{in } \Omega, \\
w \geq 0 & \text{on } \partial \Omega.
\end{cases}
\]

If (2.5) fails, then there exists \( x_0 \in \Omega \) such that

\[
w(x_0) = u(x_0) - v(x_0) = \min_{x \in \Omega} w(x) < 0,
\]

then in the viscosity sense,

\[
(\Delta)^s_{\Omega} w(x_0) \geq 0.
\]

Since \( w \) is a viscosity super solution, \( x_0 \) is the minimum point in \( \Omega \) and \( w \geq 0 \) on \( \partial \Omega \), then we can take a small neighborhood \( V_0 \) of \( x_0 \) such that \( \tilde{w} = w(x_0) \) in \( V_0 \), \( \tilde{w} = w \) in \( \Omega \setminus V_0 \). From (2.7), we have that

\[
(\Delta)^s_{\Omega} \tilde{w}(x_0) \geq 0.
\]

But the definition of regional fractional Laplacian implies that

\[
(\Delta)^s_{\Omega} \tilde{w}(x_0) = \int_{\Omega \setminus V_0} \frac{w(x_0) - w(y)}{|x_0 - y|^{N+2s}} dy < 0,
\]

which is impossible.
Remark 2.1 Let $u$ be a continuous function in $\Omega$ and $x_0$ be a minimum point of $u$, then $(-\Delta)^s_{\Omega} u(x_0) \geq 0$ in the viscosity sense, where the equality holds if and only if $u$ is a constant.

We recall the stability property for viscosity solutions in our setting.

Theorem 2.2 [11, Theorem 2.2] Assume that the function $g : \Omega \to \mathbb{R}$ is continuous. Let $u_n$, $(n \in \mathbb{N})$ be a sequence of functions in $C(\Omega)$, uniformly bounded in $L^1(\Omega)$, $g_n$ and $g$ be continuous in $\Omega$ such that

$(-\Delta)^s_{\Omega} u_n \geq g_n$ (resp. $(-\Delta)^s_{\Omega} u_n \leq g_n$) in $\Omega$ in viscosity sense, $u_n \geq g_n$ (resp. $u_n \geq g_n$) on $\partial \Omega$.

$u_n \to u$ locally uniformly in $\Omega$,

$u_n \to u$ in $L^1(\Omega)$,

$g_n \to g$ locally uniformly in $\Omega$.

Then $(-\Delta)^s_{\Omega} u \geq g$ (resp. $(-\Delta)^s_{\Omega} u \leq g$) in $\Omega$ in the viscosity sense.

Next we have an interior regularity result. For simplicity, we denote by $\mathcal{O}$ the space $C^t_{0, t-t_0}$ for $t \in (t_0, t_0 + 1)$, $t_0$ is a positive integer.

Proposition 2.2 Assume that $s \in (0, 1)$, $g \in C^\theta_{\text{loc}}(\Omega)$ with $\theta > 0$, $w \in C^2_{\text{loc}}(\Omega) \cap L^1(\Omega)$ with $\epsilon > 0$ and $2s + \epsilon$ not being an integer, is a solution of

$$(-\Delta)^s_{\Omega} w = g \quad \text{in} \quad \mathcal{O}. \quad (2.8)$$

Let $\mathcal{O}_1, \mathcal{O}_2$ be open $C^2$ sets such that

$$\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_2 \subset \mathcal{O} \subset \Omega.$$

Then (i) for any $\gamma \in (0, 2s)$ not an integer, there exists $c_6 > 0$ such that

$$\|w\|_{C^\gamma(\mathcal{O}_1)} \leq c_6 (\|w\|_{L^\infty(\mathcal{O}_2)} + \|w\|_{L^1(\Omega)} + \|g\|_{L^\infty(\mathcal{O}_2)}); \quad (2.9)$$

(ii) for any $\gamma' \in (0, \min\{\theta, \epsilon\})$, $2s + \gamma'$ not an integer, there exists $c_7 > 0$ such that

$$\|w\|_{C^{2s+\gamma'}(\mathcal{O}_1)} \leq c_8 \left(\|w\|_{L^\infty(\mathcal{O}_2)} + \|w\|_{L^1(\Omega)} + \|g\|_{C^{\gamma'}(\mathcal{O}_2)}\right). \quad (2.10)$$

Proof. The proof is similar to [11 Proposition 2.1] for $s \in (\frac{1}{2}, 1)$. For the reader’s convenience, we give the details. Let $\tilde{w} = w$ in $\Omega$, $\tilde{w} = 0$ in $\mathbb{R}^N \setminus \Omega$, we have that

$$(-\Delta)^s \tilde{w}(x) = (-\Delta)^s_{\Omega} w(x) + w(x)\varphi_\Omega(x), \quad \forall x \in \mathcal{O},$$

where $\varphi_\Omega$ is defined as (2.2). Note that $\varphi_\Omega \in C^{0, 1}_{\text{loc}}(\Omega)$. Combining with (2.8), we have that

$$(-\Delta)^s \tilde{w}(x) = g(x) + w(x)\varphi_\Omega(x), \quad \forall x \in \mathcal{O}.$$

By [?, Lemma 3.1], for any $\gamma \in (0, 2s)$, we have that

$$\|w\|_{C^\gamma(\mathcal{O}_1)} \leq c_7 \left(\|w\|_{L^\infty(\mathcal{O}_2)} + \|w\|_{L^1(\Omega)} + \|g + w\varphi_\Omega\|_{L^\infty(\mathcal{O}_2)}\right) \leq c_8 \left(\|w\|_{L^\infty(\mathcal{O}_2)} + \|w\|_{L^1(\Omega)} + \|g\|_{L^\infty(\mathcal{O}_2)}\right).$$
and by [26, Lemma 2.10], for any \( \epsilon' \in (0, \min\{\theta, \epsilon\}) \), we have that
\[
\|w\|_{C^{2+\epsilon'}(\Omega_1)} \leq c_9 \left( \|w\|_{C^{\epsilon'}(\Omega_2)} + \|g + w\phi\|_{C^{\epsilon'}(\Omega_2)} \right)
\leq c_{10} \left( \|w\|_{L^\infty(\Omega_2)} + \|w\|_{L^1(\Omega)} + \|g\|_{C^{\epsilon'}(\Omega_2)} \right).
\]

\[\square\]

3 Radial symmetry and decreasing monotonocity

3.1 Maximum Principle for small domain

The essential tool for the moving planes in a ball is the Maximum Principle for small domains:

**Proposition 3.1** Let \( \Omega \) be an open set in \( B_1 \) such that \( |\Omega| \leq 2^{-N}|B_1| \). Suppose that \( \phi: \Omega \to \mathbb{R} \) is in \( L^\infty(\Omega) \) satisfying
\[
\|\phi\|_{L^\infty(\Omega)} < +\infty,
\]
\( w \in L^1(B_1) \cap C(\bar{\Omega}) \) is a solution of
\[
\begin{cases}
(-\Delta)^s_{B_1} w \geq \phi w & \text{in } \Omega, \\
w \geq 0 & \text{in } B_1 \setminus \Omega.
\end{cases}
\]

Then there exists \( \delta > 0 \) such that for \( |\Omega| < \delta \), \( w \geq 0 \) in \( \Omega \).

In order to prove Proposition 3.1, we need the following estimate.

**Lemma 3.1** Let \( \Omega \subset B_1 \) be an open set such that \( |\Omega| \leq 2^{-N}|B_1| \). Suppose that \( g: \Omega \to \mathbb{R} \) is in \( L^\infty(\Omega) \), \( w \in L^1(B_1) \cap C(\bar{\Omega}) \) is a solution of
\[
\begin{cases}
(-\Delta)^s_{B_1} w \geq g & \text{in } \Omega, \\
w \geq 0 & \text{in } B_1 \setminus \Omega.
\end{cases}
\]

Then there exists \( c_{11} > 0 \) such that
\[
-\inf_{\Omega} w \leq c_{11}\|g\|_{L^\infty(\Omega)} |\Omega|^{\frac{2s}{N}}.
\]

**Proof.** The result is obvious if \( \inf_{\Omega} w \geq 0 \). Now we assume that \( \inf_{\Omega} w < 0 \), then there exists \( x_0 \in \Omega \) such that
\[
w(x_0) = \inf_{x \in \Omega} w(x) < 0.
\]

Combining with (3.3), we have that
\[
-\|g\|_{L^\infty(\Omega)} \leq g(x_0) \leq (-\Delta)^s_{B_1} w(x_0).
\]

(3.5)
By the definition of \((-\Delta)^s_{B_1}\), we have that
\[
(-\Delta)^s_{B_1} w(x_0) = \text{p.v.} \int_{B_1} \frac{w(x_0) - w(y)}{|x_0 - y|^{N+2s}} dy
\]
\[
= \text{p.v.} \int_{\mathcal{O}} \frac{w(x_0) - w(y)}{|x_0 - y|^{N+2s}} dy + \int_{B_1 \setminus \mathcal{O}} \frac{w(x_0) - w(y)}{|x_0 - y|^{N+2s}} dy
\]
\[
\leq \int_{B_1 \setminus \mathcal{O}} \frac{w(x_0)}{|x_0 - y|^{N+2s}} dy.
\]
Let
\[
r = (|\omega_N|^{-1}|\mathcal{O}|)^{\frac{1}{s}} \leq \frac{1}{2},
\]
by the fact that \(|\mathcal{O}| \leq 2^{-N}|B_1|\), it holds that \(|\mathcal{O}| = |B_r(x_0)|\). We let the vertical plane with respect to \(x_0\)
\[
\mathcal{P}(x_0) = \{ z \in \mathbb{R}^N \mid z \cdot x_0 = 0 \}.
\]
Thanks to the decreasing monotonicity of the kernel \(\frac{1}{|x-y|^{N+2s}}\), we obtain that
\[
\int_{B_1 \setminus B_r(x_0)} \frac{1}{|x_0 - y|^{N+2s}} dy \leq \int_{B_1 \setminus \mathcal{O}} \frac{1}{|x_0 - y|^{N+2s}} dy,
\]
since \(|B_r(x_0) \setminus \mathcal{O}| = |\mathcal{O} \setminus B_r(x_0)|\). Thus, we derive that
\[
(-\Delta)^s_{B_1} w(x_0) \leq w(0) \int_{B_1 \setminus \mathcal{O}} \frac{1}{|x_0 - y|^{N+2s}} dy \leq w(0) \int_{B_1 \setminus B_r(x_0)} \frac{1}{|x_0 - y|^{N+2s}} dy.
\]
(3.6)
Observe that for \(x_0 = 0\), we have that
\[
\int_{B_1 \setminus B_r(x_0)} \frac{1}{|x_0 - y|^{N+2s}} dy = \frac{1}{2s} \omega_N (r^{-2s} - 1) \geq \frac{1}{2s} \omega_N r^{-2s}.
\]
When \(x_0 \in B_1 \setminus \{0\}\), let \(r_0 = \sqrt{1 + |x_0|^2} \in (1, \sqrt{2})\),
\[
\mathcal{P}(x_0) = B_1 \cap \partial B_{r_0}(x_0)
\]
and the cone
\[
\mathcal{C}(x_0) = \{ tx_0 + (1-t)z : \forall z \in \mathcal{P}(x_0), \forall t \in (0, 1) \}.
\]
Then \(\mathcal{C}(x_0) \subset B_1, |\mathcal{C}(x_0)| > 14|B_{r_0}(x_0)|\) and
\[
\int_{B_1 \setminus B_r(x_0)} \frac{1}{|x_0 - y|^{N+2s}} dy > \int_{\mathcal{C}(x_0) \setminus B_r(x_0)} \frac{1}{|x_0 - y|^{N+2s}} dy
\]
\[
\geq \frac{1}{4} \int_{B_{r_0}(x_0) \setminus B_r(x_0)} \frac{1}{|x_0 - y|^{N+2s}} dy
\]
\[
= \frac{1}{8s} \omega_N (r^{-2s} - r_0^{-2s}) \geq \frac{1}{8s} \omega_N r^{-2s}.
\]
As a consequence, we derive that
\[
(-\Delta)^s_{B_1} w(x_0) \leq w(0) \int_{B_1 \setminus \Omega} \frac{1}{|x_0 - y|^{N+2s}} dy
\leq w(0) \int_{B_1 \setminus B_r(x_0)} \frac{1}{|x_0 - y|^{N+2s}} dy \leq \frac{1}{c_{11}} |\Omega|^\frac{2s}{N} w(0),
\]
where \(c_{11} = 8s|\partial B_1|^\frac{2s}{N} - 1\).

Finally, together with (3.5), we have that
\[
-\|g\|_{L^\infty(\Omega)} \leq (-\Delta)^s_{B_1} w(x_0) \leq \frac{1}{c_{11}} w(x_0) |\Omega|^\frac{2s}{N},
\]
which implies that
\[
w(x_0) \geq -c_{11} \|g\|_{L^\infty(\Omega)} |\Omega|^\frac{2s}{N},
\]
that is,
\[
-\inf_{\Omega} w \leq c_{11} \|g\|_{L^\infty(\Omega)} |\Omega|^\frac{2s}{N}.
\]
We complete the proof. \(\square\)

**Proof of Proposition 3.1.** Let us define \(\Omega^- = \{ x \in \Omega \mid w(x) < 0 \} \), then we observe that
\[
\begin{cases}
(-\Delta)^s_{B_1} w(x) \geq \phi(x) w(x), & x \in \Omega^-,
\vspace{1em}
w(x) \geq 0, & x \in B_1 \setminus \Omega^-.
\end{cases}
\]
(3.7)

Using Lemma 3.1 with \(g = \phi w\), we have that
\[
\|w\|_{L^\infty(\Omega^-)} = -\inf_{\Omega^-} w \leq c_{11} \|\phi\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega^-)} |\Omega|^\frac{2s}{N}.
\]

Then there exists \(\delta > 0\) such that for \(|\Omega| \leq \delta\), we have that
\[
c_{9} \|\phi\|_{L^\infty(\Omega)} |\Omega|^\frac{2s}{N} \leq c_{11} \|\phi\|_{L^\infty(\Omega)} \delta^\frac{2s}{N} < 1,
\]
then \(\|w\|_{L^\infty(\Omega^-)} = 0\), that is, \(\Omega^-\) is empty, which ends the proof. \(\square\)

### 3.2 Moving planes

**Proof of Theorem 1.3.** Given \(\lambda \in (0, r_0)\), let us define
\[
\Sigma_\lambda = \{ x = (x^1, x') \in B_1 \mid x^1 > \lambda \}, \quad T_\lambda = \{ x = (x^1, x') \in B_1 \mid x^1 = \lambda \},
\]
\[
\Sigma = \Sigma_\lambda \cup (\Sigma_\lambda)_\lambda, \quad w_\lambda(x) = u_\lambda(x) - u(x)
\]
and
\[
u_\lambda(x) = \begin{cases}
u(x_\lambda), & x \in \Sigma, \\
u(x), & x \in B_1 \setminus \Sigma,
\end{cases}
\]
where \( x_\lambda = (2\lambda - x^1, x') \) for \( x = (x^1, x') \in B_1 \). For any subset \( A \) of \( B_1 \), we write \( A_\lambda = \{ x_\lambda : x \in A \} \).

**Step 1:** We prove that \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \) if \( \lambda \in (0, r_0) \) is close to \( r_0 \). Indeed, let
\[
\Sigma^-_\lambda = \{ x \in \Sigma_\lambda \mid w_\lambda(x) < 0 \}
\]
and
\[
w^-_\lambda(x) = \begin{cases} w_\lambda(x), & x \in \Sigma^-_\lambda, \\ 0, & x \in B_1 \setminus \Sigma^-_\lambda, \end{cases} \quad w^+_\lambda(x) = \begin{cases} 0, & x \in \Sigma^-_\lambda, \\ w_\lambda(x), & x \in B_1 \setminus \Sigma^-_\lambda. \end{cases}
\]
By the linearity of the regional fractional Laplacian, we have that for all \( 0 < \lambda < 1 \),
\[
(-\Delta)^{s}_{B_1} w^+_\lambda(x) \leq 0, \quad \forall x \in \Sigma^-_\lambda.
\]
In fact, for \( x \in \Sigma^-_\lambda \), \( w^+_\lambda(x) = 0 \) and
\[
(-\Delta)^{s}_{B_1} w^+_\lambda(x) = -\int_{B_1 \setminus \Sigma^-_\lambda} \frac{w_\lambda(y)}{|x-y|^{N+2s}} dy = -\int_{B_1 \setminus \Sigma} \frac{w_\lambda(y)}{|x-y|^{N+2s}} dy - \int_{\Sigma \setminus \Sigma^-_\lambda} \frac{w_\lambda(y)}{|x-y|^{N+2s}} dy = -\int_{\Sigma \setminus \Sigma^-_\lambda} \frac{w_\lambda(y)}{|x-y|^{N+2s}} dy =: -I_1 - I_2 - I_3.
\]
Note that
\[
I_1 = \int_{B_1 \setminus \Sigma} (u_\lambda(y) - u(y)) \frac{1}{|x-y|^{N+2s}} dy = 0.
\]
Since \( w_\lambda(y^\lambda) = -w_\lambda(y) \) for any \( y \in B_1 \), then
\[
I_2 = \int_{\Sigma \setminus \Sigma^-_\lambda} w_\lambda(y) \frac{1}{|x-y|^{N+2s}} dy \quad I_3 = \int_{\Sigma \setminus \Sigma^-_\lambda} w_\lambda(y) \frac{1}{|x-y^\lambda|^{N+2s}} dy.
\]
For \( x \in \Sigma^-_\lambda \) and \( y \in \Sigma \setminus \Sigma^-_\lambda \), we have \( x - y = (x^1 - y^1, x' - y') \), \( x - y^\lambda = (x^1 + y^1 - 2\lambda, x' - y') \), \( |x^1 + y^1 - 2\lambda| > |x^1 - y^1| \), then
\[
\frac{1}{|x-y|^{N+2s}} \geq \frac{1}{|x-y^\lambda|^{N+2s}}.
\]
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Combing with \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \setminus \Sigma^-_\lambda \), we have that

\[
I_2 \geq 0.
\]

Since \( w_\lambda(y) < 0 \) for \( y \in \Sigma^-_\lambda \) and \( w_\lambda(y^\lambda) = -w_\lambda(y) \) for any \( y \in B_1 \), we have that

\[
I_3 = \int_{(\Sigma^-_\lambda)^\lambda} w_\lambda(y) \frac{1}{|x - y|^{N + 2s}} dy \quad = \quad \int_{\Sigma^-_\lambda} \frac{w_\lambda(y^\lambda)}{|x - y^\lambda|^{N + 2s}} dy
\]

\[
\quad = -\int_{\Sigma^-_\lambda} \frac{w_\lambda(y)}{|x - y^\lambda|^{N + 2s}} dy \geq 0.
\]

Hence, we obtain that for \( \lambda \in (0, 1) \),

\[
(-\Delta)^\lambda B_1 w_\lambda^+(x) \leq 0, \quad \forall x \in \Sigma^-_\lambda,
\]

that is,

\[
(-\Delta)^\lambda B_1 [w_\lambda - w^-_\lambda](x) \leq 0, \quad \forall x \in \Sigma^-_\lambda.
\]

Then for \( x \in \Sigma^-_\lambda \), it holds that

\[
(-\Delta)^\lambda B_1 w_\lambda(x) \leq (-\Delta)^\lambda B_1 [w^-_\lambda](x)
\]

and

\[
(-\Delta)^\lambda B_1 w^-_\lambda(x) \geq (-\Delta)^\lambda B_1 w_\lambda(x)
\]

\[
= (-\Delta)^\lambda B_1 u_\lambda(x) - (-\Delta)^\lambda B_1 u(x)
\]

\[
= h(x_\lambda, u_\lambda(x)) - h(x, u(x)) + u(x) - u_\lambda(x)
\]

\[
= \left( h_1(x_\lambda) - h_1(x)\right) h_3(u_\lambda(x)) + h_1(x) \left( h_3(u_\lambda(x)) - h_3(u(x))\right)
\]

\[
+ h_2(x_\lambda(x)) - h_2(x) + u(x) - u_\lambda(x)
\]

\[
\geq \left( h_1(x)\psi(x) + 1\right) (u_\lambda(x) - u(x)),
\]

where the last inequality holds by the assumption \((\mathcal{H}_2)\) and \((\mathcal{H}_3)\),

\[
\psi(x) = \frac{h_3(u_\lambda(x)) - h_3(u)}{u_\lambda(x) - u(x)},
\]

which is bounded for \( x \in \Sigma^-_\lambda \).

Choosing \( \lambda \in (0, r_0) \) close enough to \( r_0 \), then \( |\Sigma^-_\lambda| \) is small enough, by \( w^-_\lambda = 0 \) in \((\Sigma^-_\lambda)^c\), it follows by Proposition \([11]\) that

\[
w_\lambda \geq 0 \quad \text{in} \quad \Sigma^-_\lambda.
\]

Then \( \Sigma^-_\lambda \) is empty, that is,

\[
w_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda.
\]
Step 2: We claim that for $0 < \lambda < 1$, if $w_\lambda \geq 0$ and $w_\lambda \not\equiv 0$ in $\Sigma_\lambda$, then $w_\lambda > 0$ in $\Sigma_\lambda$.

If this is not true, then there exists $x_0 \in \Sigma_\lambda$ such that $w_\lambda(x_0) = 0$ and then $u_\lambda(x_0) = u(x_0)$ and

\[
(-\Delta)^s_{B_1} w_\lambda(x_0) = (-\Delta)^s_{B_1} u_\lambda(x_0) - (-\Delta)^s_{B_1} u(x_0)
\]

\[=
 h((x_0)_\lambda, u_\lambda(x_0)) - h(x_0, u(x_0))
\]

\[\geq (h_1((x_0)_\lambda - h_1(x_0)) h_3(u_\lambda(x_0))
\]

\[\geq 0.
\]

(3.8)

However, since $x_0$ is the minimal of $w_\lambda$ and by the definition of the regional fractional Laplacian

\[
(-\Delta)^s_{B_1}[w_\lambda](x_0) = -\int_{B_1} \frac{w_\lambda(y)}{|x_o - y|^{N + 2s}} dy
\]

\[= -\int_{\Sigma^-_\lambda} \frac{w_\lambda^-(y)}{|x_o - y|^{N + 2s}} dy - \int_{B_1 \setminus \Sigma^-_\lambda} \frac{w_\lambda^+(y)}{|x_o - y|^{N + 2s}} dy
\]

\[\leq -\int_{B_1 \setminus \Sigma^-_\lambda} \frac{w_\lambda^+(y)}{|x_o - y|^{N + 2s}} dy
\]

\[< 0.
\]

By the fact that $w_\lambda^+ \geq 0$ and $w_\lambda^+ \not\equiv 0$ in $\Sigma^-_\lambda$. So we obtain a contradiction from (3.8). Thus, $w_\lambda > 0$ in $\Sigma_\lambda$ if $\lambda \in (0, 1)$ is close to $r_0$.

Step 3: We show $\lambda_0 = 0$, where

$\lambda_0 = \inf\{\lambda \in (0, r_0) \mid w_\lambda > 0 \text{ in } \Sigma_\lambda\}.$

If it is not true, i.e. $\lambda_0 > 0$, by the definition of $\lambda_0$, we have that $w_{\lambda_0} \geq 0$ in $\Sigma_{\lambda_0}$ and $w_{\lambda_0} \not\equiv 0$ in $\Sigma_{\lambda_0}$. By Step 2, we have $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$.

Claim 1. If $w_\lambda > 0$ in $\Sigma_\lambda$ for $\lambda \in (0, 1)$, then there exists $\epsilon \in (0, \lambda)$ such that $w_{\lambda \epsilon} > 0$ in $\Sigma_{\lambda \epsilon}$, where $\lambda_\epsilon = \lambda - \epsilon$.

Assume that Claim 1 is true, then there exists some $\epsilon \in (0, \lambda_0)$ such that $w_{\lambda_0 - \epsilon} > 0$ in $\Sigma_{\lambda_0 - \epsilon}$, which implies that

$\lambda_0 - \epsilon \geq \lambda_0$,

which is impossible. Then we obtain $\lambda_0 = 0$.

Now we only need to prove Claim 1 to complete Step 3.

Proof of Claim 1. Let $D_\mu = \{x \in \Sigma_\lambda \mid dist(x, \partial \Sigma_\lambda) \geq \mu\}$ for $\mu > 0$ small. Since $w_\lambda > 0$ in $\Sigma_\lambda$ and $D_\mu$ is compact, then there exists $\mu_0 > 0$ such that $w_\lambda \geq \mu_0$ in $D_\mu$. By continuity of $w_\lambda(x)$, for $\epsilon > 0$ small enough, we denote $\lambda_\epsilon = \lambda - \epsilon$, then

$w_\lambda(\epsilon)(x) \geq 0$ in $D_\mu$.

As a consequence,

$\Sigma^-_{\lambda_\epsilon} \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu$.
and $|\Sigma_{\lambda}|$ small if $\epsilon$ and $\mu$ small.

By Step 1, $(-\Delta)^{s}_{B_{1}} w_{\lambda}(x) \leq 0$ in $x \in \Sigma_{\lambda}$, Since $w_{\lambda}^{+} = 0$ in $(\Sigma_{\lambda})^{C}$ with $|\Sigma_{\lambda}|$ small for $\epsilon$ and $\mu$ small, $\varphi(x) = \frac{u_{\lambda}^{+}(x) - u^{+}(x)}{u_{\lambda}(x) - u(x)}$, similar with Step 1, then we have $w_{\lambda, u} \geq 0$ in $\Sigma_{\lambda}$. And since $\lambda_{e} > 0$, $w_{\lambda, u} \neq 0$ in $\Sigma_{\lambda}$, we have that $w_{\lambda, u} > 0$ in $\Sigma_{\lambda}$. Thus, Claim 1 is true.

We conclude from the fact of $\lambda_{0} = 0$ that

$$u(-x^{1}, x') \geq u(x^{1}, x') \quad \text{for} \quad x^{1} \geq 0.$$  

Using the same way, do moving plane from left side to 0, we have

$$u(-x^{1}, x') \leq u(x^{1}, x') \quad \text{for} \quad x^{1} \geq 0.$$  

Then

$$u(-x^{1}, x') = u(x^{1}, x') \quad \text{for} \quad x^{1} \geq 0.$$  

Step 4: we prove $u(x)$ is strictly decreasing in the $x_{1}$ direction for $x = (x^{1}, x') \in B_{r, 0}$, $x_{1} > 0$. By contradiction, if there exists $(x^{1}, x'), (\tilde{x}^{1}, x') \in \Omega$, $0 < x^{1} < \tilde{x}^{1}$ such that

$$u(x^{1}, x') \leq u(\tilde{x}^{1}, x').$$  

(3.9)

Let $\lambda = \frac{x^{1} + \tilde{x}^{1}}{2}$ and by arguments above, we have

$$w_{\lambda}(x) > 0 \quad \text{for} \quad x \in \Sigma_{\lambda}.$$  

Since $(\tilde{x}^{1}, x') \in \Sigma_{\lambda}$, then

$$0 < w_{\lambda}(\tilde{x}^{1}, x') = u_{\lambda}(\tilde{x}^{1}, x') - u(\tilde{x}^{1}, x') = u(x^{1}, x') - u(\tilde{x}^{1}, x'),$$  

i.e.

$$u((x^{1}, x')) > u((\tilde{x}^{1}, x')),$$

which is impossible with (3.9). Hence, $u(x)$ is strictly decreasing in the $x_{1}$ direction for $x = (x^{1}, x') \in \Omega$ and $x^{1} > 0$. \hfill $\square$

### 4 Poisson problems

In order to prove Theorem 1.1, we need the following existence results.

**Lemma 4.1** Let $s \in (0, 1)$, $r \in (0, 1)$, $F : B_{1} \to [0, +\infty)$ be Hölder continuous, then

$$(-\Delta)^{s}_{B_{r}} u + u = F \quad \text{in} \quad B_{r},$$

$$u = 0 \quad \text{in} \quad \bar{B}_{1} \setminus B_{r},$$  

has a unique positive solution $u_{r, F} \in C_{0}(B_{1})$.

Moreover, (i) if $F$ is radially symmetric function decreasing with respect to $|x|$, then $u_{r, F}$ is radially symmetric and decreasing with respect to $|x|$;

(ii) $r \to u_{r, F}$ is non-decreasing, i.e.

$$u_{r_{1}, F} \leq u_{r_{2}, F} \quad \text{if} \quad 0 < r_{1} < r_{2} < 1.$$
Proof. Let $H^s_0(B_r)$ be the closure of $C^\infty_0(B_r)$, with zero value in $\mathbb{R}^N \setminus B_r$, under the norm that
\[ \|u\|_{s,r} = \sqrt{\int_{B_1 \times B_1} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dxdy + \int_{B_1} u^2 dx}, \]
which is a Hilbert space with the inner product
\[ \langle u, v \rangle_{s,r} = \int_{B_1 \times B_1} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy + \int_{B_1} uv dx. \]
Note that $H^s_0(B_r) \subset H^s_0(B_1)$ and from [17, Corollary 7.2], the embedding $H^s_0(B_1) \hookrightarrow L^q(B_1)$ is compact for $q \in [2, \frac{2N}{N-2s})$. Since $F$ is Hölder continuous, it follows by the standard argument of variational methods to find the critical point of
\[ J_{s,r} : H^s_0(B_r) \to \mathbb{R}, \quad J_s(u) = \frac{1}{2} \|u\|_{s,r}^2 - \int_{B_1} Fudx, \]
which has a unique critical point $u_{r,F} \in H^s_0(B_r)$. Note that the critical point is the weak solution of (4.1) in the sense that
\[ \langle u_{r,F}, \xi \rangle_{s,r} = \int_{B_r} F\xi dx, \quad \forall \xi \in H^s_0(B_r) \quad (4.2) \]
and taking $\xi = u_{r,F}$, the Hölder inequality and fractional Sobolev embedding [17] Theorem 6.7, Remark 6.8 implies that
\[
\|u_{r,F}\|_{s,r}^2 = \int_{B_r} F u_{r,F} dx 
\leq \|u_{r,F}\|_{L^{2^*_s}(B_r)} \|F\|_{L^{p^*_s}(B_r)} \leq c_{12} \|u_{r,F}\|_{s,r} \|F\|_{L^{p^*_s}(B_1)},
\]
that is,
\[ \|u_{r,F}\|_{s,r} \leq c_{12} \|F\|_{L^{p^*_s}(B_1)}, \quad (4.3) \]
where $2^*_s = \frac{2N}{N-2s}$, $p^*_s = \frac{2N}{N+2s}$ and $c_{12} > 0$ is independent of $r$.

Let $\tilde{H}^s_0(B_r)$ be the closure of $C^\infty_0(B_r)$, with zero value in $\mathbb{R}^N \setminus B_r$, under the norm that
\[ \|u\|_{s,r} = \sqrt{\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dxdy + \int_{B_r} u^2 dx}, \]
with the inner product
\[ \langle u, v \rangle_{s,r} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy + \int_{B_r} uv dx. \]

Direct computation shows that
\[ \|u\|_{s,r}^2 = \|u\|_{s,r}^2 + 2 \int_{B_1} \varphi_{B_1}(x) u^2(x) dx, \]
where \( \varphi_{B_1} \) is defined in (2.2) with \( \Omega = B_1 \), i.e.

\[
\varphi_{B_1}(x) = \int_{B_1^c} \frac{dy}{|x-y|^{N+2s}}.
\]

which by Proposition 2.1 that \( \varphi_{B_1} \) is Lipschitz in \( B_r \). Thus, \( u_{r,F} \) verifies that

\[
\langle u_{r,F}, \xi \rangle_{s,r} = \int_{B_r} F \xi dx + 2 \int_{B_1} \varphi_{B_1} u_{r,F} \xi dx, \quad \forall \xi \in H^s_0(B_r),
\]

which means that \( u_{r,F} \) is a weak solution of

\[
\begin{cases}
(-\Delta)^s u + u = F + \varphi_{B_1} u & \text{in } B_r, \\
u = 0 & \text{in } B_r^c.
\end{cases}
\]

(4.4)

Note that the solution could be expressed by

\[
u_{r,F} = \Phi_r * F,
\]

where \( F = F + (\varphi_{B_1} - 1) u_{r,F} \) and \( \Phi_r \) is the Green kernel of \( B_r \) under the zero condition \( \Phi_r = 0 \) in \( \mathbb{R}^N \setminus B_r \). Since \( \varphi_{B_1} - 1 \) is uniformly bounded in \( B_r \) and \( u_{r,F} \in L^{2^*_s} \), then \( F \in L^{2^*_s} \) and if it follows by Lemma A.1 that \( u_{r,F} \in L^{\infty}(B_r) \) if \( 2^*_s > \frac{N}{2s} \) and we are done; or \( u_{r,F} \in L^{q_1}(B_r) \) if

\[
q_1 = \frac{Nq_0}{2sq_0-N} = q_0 \quad \text{if} \quad q_0 := 2^*_s > \frac{N}{2s}, \quad \text{in this case,} \quad F \in L^{q_1}.
\]

Repeat the procedure, we can find \( i_0 \geq 1 \) such that \( u_{r,F} \in L^{\infty}(B_r) \) and

\[
\|u_{r,F}\|_{L^{\infty}} \leq c_{13}\|F\|_{L^{\infty}(B_r)},
\]

where \( c_{13} > 0 \) depends on \( r \). From (1.3), we have that \( u_{r,F} \) has a uniform bound in \( H^s_0(B_r) \) from Proposition 2.2 we obtain that for any \( \mathcal{O} \subset \bar{\mathcal{O}} \subset B_r \),

\[
\|u_{r,F}\|_{C^\gamma(\mathcal{O})} \leq c_{14} \left( \|u_{r,F}\|_{L^{\infty}(B_r)} + \|u_{r,F}\|_{L^1(B_r)} + \|F\|_{L^\infty(B_r)} \right)
\]

\[
\leq c_{15}\|F\|_{L^{\infty}(B_r)}
\]

and then by (2.10), we have that

\[
\|u_{r,F}\|_{C^{2s+\epsilon}(\mathcal{O})} \leq c_{16} \left( \|u_{r,F}\|_{L^{\infty}(B_r)} + \|u_{r,F}\|_{L^1(B_r)} + \|F\|_{C^\theta(B_r)} \right)
\]

\[
\leq c_{17}\|F\|_{C^\theta(B_r)}.
\]

It follows by boundary regularity in [26] that \( u_{r,F} \) is a classical solution of (4.4), then it is the solution of (4.1.1). Thus, for \( \xi \in C^{2s+\theta}_{loc}(B_r) \cap C^s_0(B_r) \), it holds that

\[
\int_{B_r} F \xi dx = \int_{B_r} (\xi(-\Delta)^s_{B_1} u_{r,F} + u_{r,F} \xi) dx
\]

\[
= \int_{B_r} (\xi(-\Delta)^s u_{r,F} + u_{r,F} \xi - \varphi_{B_1}(x) u_{r,F} \xi) dx
\]

\[
= \int_{B_r} (u_{r,F}(-\Delta)^s \xi + u_{r,F} \xi - \varphi_{B_1}(x) u_{r,F} \xi) dx
\]

\[
= \int_{B_r} (u_{r,F}(-\Delta)^s_{B_1} \xi + u_{r,F} \xi) dx,
\]
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that is,
\[ \int_{B_r} u_{r,F}((−Δ)^s_{B_1}\xi + \xi)\,dx = \int_{B_r} F\xi\,dx. \]  (4.5)

By the method of moving planes in Theorem 1.3 with \( h_1 = 0 \) and \( h_2 = F \), the solution \( u_{r,F} \) is radially symmetric and decreasing to \(|x|\).

For \( 0 < r_1 < r_2 < 1 \), we see that the solution \( u_{r_2,F} \) is a super solution of (4.1) with \( r = r_1 \), then the maximum principle shows that \( u_{r_2,F} \geq u_{r_1,F} \) in \( B_1 \).

\[ \square \]

Proof of Theorem 1.1. It follows by Lemma 4.1, problem (4.1) has a unique classical solution \( u_{r,F} \), which is positive, radially symmetric and decreasing with respect to \(|x|\) and \( r \):

the function \( u_{1,F} \) is the critical point of the functional
\[ J_s: H^s_0(B_1) \rightarrow \mathbb{R}, \quad J_s(u) = \frac{1}{2}\|u\|_{s,1}^2 - \int_{B_1} Fudx, \]
whose critical point is unique. Then \( \int_{B_1} u_{1,F}dx = \int_{B_1} Fdx \) by taking \( \xi \equiv 1 \in H^s_0(B_1) \) in (4.6) for \( s \in (0, \frac{1}{2}] \).

Moreover, \( u_{1,F} \) inherits the positivity, the symmetry property and decreasing monotonicity of \( u_{r,F} \), then \( u_{1,F} \) is locally bounded in \( B_1 \setminus \{0\} \).

To prove \( u_{1,F} \in L^\infty(B_1) \). Let \( t > 1 \) and
\[ w_t = (u_{1,F} - t)_+ \text{ in } B_1, \]
which is $H^s_0(B_1)$. For $\sigma \in (0, \frac{1}{2}]$, we have that
\[
\int_{B_1} F w_t \, dx = \langle u_{1,F}, w_t \rangle_s
\]
\[
= \int \int_{\{x,y \in B_1: |x-y|<\sigma\}} \frac{(u_{1,F}(x) - u_{1,F}(y))(w_t(x) - w_t(y))}{|x-y|^{N+2s}} \, dxdy
\]
\[
+ \int \int_{\{x,y \in B_1: |x-y|\geq \sigma\}} \frac{(u_{1,F}(x) - u_{1,F}(y))(w_t(x) - w_t(y))}{|x-y|^{N+2s}} \, dxdy
\]
\[
\geq \int \int_{\{x,y \in B_1: |x-y|<\sigma\}} \frac{(w_t(x) - w_t(y))^2}{|x-y|^{N+2s}} \, dxdy
\]
\[
+ 2c_{18} \left( \sigma^{-2s} \int_{B_1} u_{1,F}w_t \, dx - \int_{B_1} (\kappa_\sigma * u_{1,F})w_t \, dx \right)
\]
\[
:= F_\sigma + 2c_{18}(\mathcal{E}_{1,\sigma} - \mathcal{E}_{2,\sigma}),
\]
where $c_{18} = \frac{\omega_N}{2s}$ and $\kappa_\sigma = \chi_{[0,1)} \cdot |\cdot|^{N-2s}$, the last inequality holds by the fact that
\[
(u_{1,F}(x) - u_{1,F}(y))(w_t(x) - w_t(y)) = ((u_{1,F}(x) - t)(u_{1,F}(y) - t))(w_t(x) - w_t(y))
\]
\[
= (u_{1,F}(x) - t)w_t(x) + (u_{1,F}(y) - t)w_t(y) - (u_{1,F}(x) - t)w_t(y) - (u_{1,F}(y) - t)w_t(x)
\]
\[
= w_t^2(x) + w_t^2(y) - 2w_t(x)w_t(y) + (u_{1,F}(x) - t)w_t(x) + (u_{1,F}(y) - t)w_t(y)
\]
\[
\geq (w_t(x) - w_t(y))^2
\]
for $t_- = \max\{ -t, 0\}$ and $x, y \in B_1$.

Direct computations show that
\[
\|\kappa_\sigma * u_{1,F}\|_{L^\infty(B_1)} \leq \|u_{1,F}\|_{L^2(B_1)} \left( \int_{\mathbb{R}^N \setminus B_\sigma} |y|^{-(N+2s)p^*} \, ds \right)^{\frac{1}{p^*}}
\]
\[
\leq c_{10}\|F\|_{L^{p^*}(B_1)} \sigma^{-\frac{N+2s}{2}}
\]
\[
\leq c_{20}\|F\|_{L^\infty(B_1)} \sigma^{-\frac{N+2s}{2}},
\]
which implies that
\[
\mathcal{E}_{2,\sigma} \leq c_{20}\|F\|_{L^\infty(B_1)} \sigma^{-\frac{N+2s}{2}} \int_{B_1} w_t \, dx.
\]
Moreover, it holds that
\[
\int_{B_1} F w_t \, dx \leq \|F\|_{L^\infty(B_1)} \int_{B_1} w_t \, dx, \quad \mathcal{E}_{1,\sigma} \geq c_{21}\sigma^{-2s}t \int_{B_1} w_t \, dx.
\]
As a consequence, if \( t \geq t_0 \) for some \( t_0 > 0 \) large enough, we obtain that
\[
F_\sigma \leq (\|F\|_{L^\infty(B_1)}(c\sigma^{-\frac{N+2s}{2}} + 1) - c_{21}\sigma^{-2s}t) \int_{B_1} w_t dx \leq 0,
\]
where
\[
\|F\|_{L^\infty(B_1)}(c\sigma^{-\frac{N+2s}{2}} + 1) - c_{21}\sigma^{-2s}t < 0 \quad \text{if} \ t \text{ is large}.
\]
So we have that \( w_t = 0 \) a.e. in \( \Omega \), which means \( u_{1,F} \leq t_0 \). The \( L^\infty \) bound of \( u_{1,F} \) is obtained.

Note that for \( O \subset \bar{O} \subset B_1 \), then for \( r \) close to 1 such that \( O \subset \bar{O} \subset B_r \),
\[
\|u_{r,F}\|_{C^s(O)} \leq c_{22} \left( \|u_{1,F}\|_{L^\infty(B_1)} + \|u_{1,F}\|_{L^1(B_r)} + \|F\|_{L^\infty(B_1)} \right)
\]
and by (2.10), we have that
\[
\|u_{r,F}\|_{C^{2s+e'}(O)} \leq c_{24} \left( \|u_{1,F}\|_{L^\infty(B_1)} + \|w\|_{L^1(B_r)} + \|F\|_{C^\theta(B_r)} \right)
\]
and \( c_{25} \|F\|_{C^\theta(B_1)} \).

Then \( u_{r,F} \to u_{1,F} \) as \( r \to 1^- \) locally in \( C^{2s+e'}(B_1) \). By the stability results of Theorem 2.2 we obtain that \( u_{1,F} \in C^{2s+e}(B_1) \) and it verifies that
\[
(-\Delta)^s_{B_1} u_{1,F} + u_{1,F} = F \quad \text{in} \ B_1.
\]

As proved above, \( u_{1,F} \) is radially symmetric function decreasing with respect to \( |x| \), then we can denote
\[
d_{1,F} = \lim_{|x| \to 1^-} u_{1,F}(x) \geq 0.
\]
Let
\[
u_f(x) = u_{1,F} - d_{1,F},
\]
then \( u_f \in C_0(B_1) \) is nonnegative and verifies that
\[
\begin{aligned}
\begin{cases}
(-\Delta)^s_{B_1} u_f + u_f = F - d_{1,F} & \text{in} \ B_1, \\
u_f = 0 & \text{on} \ \partial B_1.
\end{cases}
\end{aligned}
\]
and
\[
0 \leq \int_{B_1} u_f dx = \int_{B_1} (F - d_{1,F}) dx.
\]
If \( F = f_0 \) is a constant, then \( f_0 - d_{1,F} \) is a unique solution of
\[
\begin{aligned}
\begin{cases}
(-\Delta)^s_{B_1} u + u = F - d_{1,F} & \text{in} \ B_1, \\
u \in H^s_0(B_1)
\end{cases}
\end{aligned}
\]
and by the uniqueness, \( u_f \equiv f_0 - d_{1,F} \) and by the zero boundary, we have that \( f_0 = d_{1,F} \).
If $F$ is not a constant, $u_f$ is no longer a constant, $\int_{B_1} u_f dx > 0$, then $\int_{B_1} (F - d_{1,F}) dx > 0$ which implies $d_{1,F} < \frac{1}{|B_1|} \int_{B_1} F(x) dx$.

Finally, we claim that $d_{1,F} \geq \inf_{x \in B_1} F(x)$. In fact, Letting $d_0 := \inf_{x \in B_1} F(x) > 0$, then by comparison principle

$$u_{r,d_0} \leq u_{r,F}$$

where $u_{r,d_0}, u_{r,F}$ are the solutions of (4.1) with non-homogeneous term $d_0$ and $F$ respectively.

By the convergence, we obtain that

$$u_{1,F} \geq u_{1,d_0} \equiv d_0 \quad \text{in} \ B_1,$$

which implies that $d_{1,F} \geq \inf_{x \in B_1} F(x)$. \hfill \Box

From the the proof of Theorem 1.1, we conclude that

**Corollary 4.1** Assume that $s \in \left(0, \frac{1}{2}\right)$, $F \in C^\theta(\bar{B}_1)$ with $\theta \in (0, 1)$, is a nonnegative function, radially symmetric and decreasing with respect to $|x|$.

Then problem

$$\begin{cases}
(-\Delta)^s_{B_1} u + u = F & \text{in} \ B_1, \\
u \in H^s_0(B_1)
\end{cases}$$

(4.9)

has a unique positive solution $u_{1,F} \in C(\bar{B}_1)$, which is radially symmetric and decreasing with respect to $|x|$.

Moreover, (i) the mapping: $F \mapsto u_{1,F}$ is increasing and

$$d_{1,F} \in \left[ \inf_{x \in B_1} F(x), \frac{1}{|B_1|} \int_{B_1} F dx \right];$$

(ii) if $F$ is a positive constant, we derive that $u_{1,F} = F$.

**Proof.** For $F_1 \leq F_2$, $u_{r,F_1} \leq u_{r,F_2}$ by the previous proof, passing to the limit we get the mapping: $F \mapsto u_{1,F}$ is increasing.

If $F$ is a constant, then $w := u_{1,F} - F$ is a solution of

$$\begin{cases}
(-\Delta)^s_{B_1} u + u = 0 & \text{in} \ B_1, \\
u \in C_0(B_1)
\end{cases}$$

which only has a zero solution by the maximum principle. Then $u_{1,F} = F$. \hfill \Box

## 5 Schrödinger equation

Under the assumption of Theorem 1.2 Schrödinger equation (1.1) could be written as

$$\begin{cases}
(-\Delta)^s_{B_1} u + u = h_1 u^p + \epsilon h_2 & \text{in} \ B_1, \\
u \in H^s_0(B_1),
\end{cases}$$

(5.1)

where $p > 1$ and $\epsilon > 0$. 
Proof of Theorem 1.2. Let $u_{h_2}$ be the unique solution of

$$( -\Delta )_{B_1}^s u + u = h_2 \quad \text{in} \quad B_1, \quad u \in H_0^s(B_1).$$

Now we define the iterating sequence

$$v_0 := \epsilon u_{h_2} > 0,$$

and by Corollary 4.1, $v_n$ with $n = 1, 2, \cdots$ is the unique solution of

$$( -\Delta )_{B_1}^s u + u = h_1 v_{n-1} + \epsilon h_2 \quad \text{in} \quad B_1, \quad u \in H_0^s(B_1). \quad (5.2)$$

and we have that $v_1 \geq v_0$. Assuming that $v_{n-1} \geq v_{n-2}$ in $B_1$, we apply Corollary 4.1 to obtain that $v_n \geq v_{n-1}$ in $B_1$.

We next build an upper bound for the sequence $\{v_n\}_n$. For $t > 0$, denote

$$w_t = t,$$

then

$$( -\Delta )_{B_1}^s w_t + w_t - h_1 w_t^p = t - \epsilon h_1 L(t),$$

and letting

$$L(t) = t - \epsilon h_1 \| h_1 \|_{L^\infty(B_1)},$$

note that $L(\cdot)$ has maximum $\frac{p-1}{p} (p \| h_1 \|_{L^\infty(B_1)})^{-\frac{1}{p-1}}$ in at $t_p = (p \| h_1 \|_{L^\infty(B_1)})^{-\frac{1}{p-1}}$.

In order to find the upper solution, we take $t = t_p$ and if

$$\epsilon \| h_2 \|_{L^\infty(B_1)} \leq \frac{p-1}{p} (p \| h_1 \|_{L^\infty(B_1)})^{-\frac{1}{p-1}}. \quad (5.3)$$

then

$$( -\Delta )_{B_1}^s w_{t_p} + w_{t_p} \geq h_1 w_{t_p}^p + \epsilon h_2. \quad (5.4)$$

Note that (5.3) holds if

$$\epsilon \leq \epsilon_p := \frac{p-1}{p} (p \| h_1 \|_{L^\infty(B_1)})^{-\frac{1}{p-1}} \| h_2 \|_{L^\infty(B_1)}^{-1}.$$

Obviously, we have that $w_{t_p} \geq v_0$. Inductively, we obtain

$$v_n \leq w_{t_p}. \quad (5.5)$$
for all \( n \in \mathbb{N} \). Therefore, the sequence \( \{v_n\}_n \) converges. Let \( u_\epsilon := \lim_{n \to \infty} v_n \) in \( B_1 \). By the regularity results, \( u_\epsilon \) is a solution of (5.1).

We claim that \( u_\epsilon \) is the minimal solution of (1.1), that is, for any nonnegative solution \( u \) of (1.3), we always have \( u \leq u_\epsilon \). Indeed, there holds

\[
(-\Delta)^s_{B_1} u + u = h_1 w^p + \epsilon h_2 \geq (-\Delta)^s_{B_1} v_0 + v_0 \quad \text{in } B_1, \quad u_\epsilon = u \quad \text{on } \partial B_1
\]

then \( u \geq v_0 \), \( u_\epsilon = u \) on \( \partial B_1 \) and

\[
(-\Delta)^s_{B_1} u + u = h_1 w^p + \epsilon h_2 \geq h_1 v_0^p + \epsilon h_2 = (-\Delta)^s_{B_1} v_1 + v_1 \quad \text{in } B_1,
\]

which implies that \( u \geq v_1 \) in \( B_1 \). We may show inductively that

\[
u \geq v_n
\]

for all \( n \in \mathbb{N} \). The claim follows.

From above argument, if problem (5.1) has a nonnegative solution \( u_\epsilon_1 \) for \( \epsilon_1 > 0 \), then (5.1) admits a minimal solution \( u_\epsilon \) for all \( \epsilon \in (0, \epsilon_1] \). As a result, the mapping \( \epsilon \mapsto u_\epsilon \) is increasing. So we may define

\[
\epsilon^* = \sup \{ \epsilon > 0 : (5.1) \text{ has minimal solution for } \epsilon \}
\]

and we have that

\[
\epsilon^* \geq \epsilon_p.
\]

Finally, we prove that \( \epsilon^* < +\infty \). Assume that (5.1) has a positive solution for \( \epsilon > 0 \). Our above proof shows that (5.1) has a minimal solution \( u_\epsilon \). Let \( u_{h_1} \) be the solution of

\[
\begin{cases}
(-\Delta)^s_{B_1} u + u = h_1 & \text{in } B_1, \\
u \in H^s_0(B_1).
\end{cases}
\]

If \( h_0 := \inf_{x \in B_1} h_1(x) > 0 \), then \( u_{h_1} \geq h_0 \) by Corollary 4.1

Letting \( u_{h_1} \) as test function, we have that

\[
\int_{B_1} u_{h_1}^p h_1 u_{h_1} dx + \epsilon \int_{B_1} h_2 u_{h_1} dx = \int_{B_1} [(-\Delta)^s_{B_1} u_\epsilon + u_\epsilon] u_{h_1} dx
\]

\[
= \int_{B_1} u_\epsilon [(-\Delta)^s_{B_1} u_{h_1} + u_{h_1}] dx
\]

\[
= \int_{B_1} u_\epsilon h_1 dx
\]

\[
\leq \left( \int_{B_1} u_{h_1}^p h_1 u_{h_1} dx \right)^{\frac{1}{p}} \left( \int_{B_1} h_1(u_{h_1})^{-\frac{1}{p-1}} dx \right)^{\frac{1}{p-1}}
\]

\[
\leq c_{26} \left( \int_{B_1} u_{h_1}^p h_1 u_{h_1} dx \right)^{\frac{1}{p}},
\]
where
\[ c_{26} = \left( \int_{B_1} h_1(u_{h_1})^{-\frac{1}{p-1}} \, dx \right)^{1-\frac{1}{p}} \leq h_0^{-\frac{1}{p}} \|h_1\|_{L^1(B_1)}^{1-\frac{1}{p}} < +\infty. \]
Thus, we have that
\[ \int_{B_1} u_p^p h_1 u_{h_1} \, dx \leq c_{26}^{\frac{p}{p-1}} \tag{5.6} \]
and we have that
\[ \epsilon \leq \frac{c_{26}^{\frac{p}{p-1}}}{\int_{B_1} h_2 u_{h_1} \, dx} = \frac{\int_{B_1} h_1(u_{h_1})^{-\frac{1}{p-1}} \, dx}{\int_{B_1} h_2 u_{h_1} \, dx}, \tag{5.7} \]
which means
\[ \epsilon^* \leq \frac{\int_{B_1} h_1(u_{h_1})^{-\frac{1}{p-1}} \, dx}{\int_{B_1} h_2 u_{h_1} \, dx} < +\infty. \]
Finally, \( u_\epsilon \) is radially symmetric function decreasing with respect to \( |x| \), then we can denote
\[ d_\epsilon = \lim_{|x| \to 1^-} u_\epsilon(x) \geq 0. \]
By by Corollary 4.1, we have that
\[ d_\epsilon \geq \epsilon \lim_{|x| \to 1^-} h_{h_2} \geq \epsilon \inf_{x \in B_1} h_2(x). \]
By (5.6), we have that
\[ h_0^2 |B_1|^p \leq d_\epsilon \int_{B_1} h_1 u_{h_1} \, dx \]
\[ \leq \int_{B_1} u_p^p h_1 u_{h_1} \, dx \leq c_{26}^{\frac{p}{p-1}} \leq h_0 \|h_1\|_{L^1(B_1)}, \]
that is
\[ d_\epsilon \leq \left( \frac{1}{h_0 |B_1|} \|h_1\|_{L^1(B_1)} \right)^{\frac{1}{p}}. \]
Let
\[ w_\epsilon(x) = u_\epsilon - d_\epsilon, \]
then \( w_\epsilon \in C_0(B_1) \) is nonnegative and verifies that
\[
\begin{cases}
(\Delta)^{\delta} B_1 w_\epsilon + w_\epsilon = h_1(w_\epsilon + d_\epsilon)^p - d_\epsilon + \epsilon h_2 & \text{in } B_1, \\
w_\epsilon = 0 & \text{on } \partial B_1,
\end{cases}
\tag{5.8}
\]
and
\[ 0 \leq \int_{B_1} w_\epsilon \, dx = \int_{B_1} \left( h_1(w_\epsilon + d_\epsilon)^p - d_\epsilon + \epsilon h_2 \right) \, dx. \]
The proof ends.
A Some estimates

A.1 Proof of Proposition 2.1

(i) For \( x_1, x_2 \in \Omega \) and any \( z \in \mathbb{R}^N \setminus \Omega \), we have that

\[
|z - x_1| \geq \rho(x_1) + \rho(z), \quad |z - x_2| \geq \rho(x_2) + \rho(z)
\]

and

\[
||z - x_1|^{N+2s} - |z - x_2|^{N+2s}| \leq c_2 |x_1 - x_2| (|z - x_1|^{N+2s} - 1 + |z - x_2|^{N+2s} - 1),
\]

where \( \rho(x) = \text{dist}(x, \partial \Omega), \) \( c_2 > 0 \) is independent of \( x_1 \) and \( x_2 \). Then

\[
|\varphi_\Omega(x_1) - \varphi_\Omega(x_2)| \leq \int_{\Omega^c} \frac{|z - x_2|^{N + 2s} - |z - x_1|^{N + 2s}}{|z - x_1|^{N + 2s} |z - x_2|^{N + 2s}} \, dz
\]

\[
\leq c_2 |x_1 - x_2| \left[ \int_{\Omega^c} \frac{dz}{|z - x_1| |z - x_2|^{|N + 2s|}} + \int_{\Omega^c} \frac{dz}{|z - x_1|^{N + 2s} |z - x_2|} \right].
\]

By direct computation, we have that

\[
\int_{\Omega^c} \frac{1}{|z - x_1| |z - x_2|^{|N + 2s|}} \, dz \leq \int_{\mathbb{R}^N \setminus B_{\rho(x_1)}(x_1)} \frac{1}{|z - x_1|^{N + 2s + 1}} \, dz
\]

\[
+ \int_{\mathbb{R}^N \setminus B_{\rho(x_2)}(x_2)} \frac{1}{|z - x_2|^{N + 2s + 1}} \, dz
\]

\[
\leq c_3 (\rho(x_1)^{-1 - 2s} + \rho(x_2)^{-1 - 2s})
\]

and similar to obtain that

\[
\int_{\Omega^c} \frac{1}{|z - x_1|^{N + 2s} |z - x_2|} \, dz \leq c_4 (\rho(x_1)^{-1 - 2s} + \rho(x_2)^{-1 - 2s}),
\]

where \( c_3, c_4 > 0 \) are independent of \( x_1 \) and \( x_2 \). Then

\[
|\varphi_\Omega(x_1) - \varphi_\Omega(x_2)| \leq c_5 (\rho(x_1)^{-1 - 2s} + \rho(x_2)^{-1 - 2s}) |x_1 - x_2|,
\]

where \( c_5 = c_2 (c_3 + c_4) \), it implies that \( \varphi_\Omega \) is locally Lipschitz continuous.

(ii) Firstly, we claim that \( \varphi_{B_1}(x) = \varphi_{B_1}(z) \) if \( |x| = |z| \). In fact, denote \( A \) a matrix with \( |A| = 1 \) and \( z = Ax \), we have that

\[
\varphi_{B_1}(z) = \varphi_{B_1}(Ax) = \int_{B_1^c} \frac{dy}{|Ax - y|^{N + 2s}} = \int_{B_1^c} \frac{dy}{|x - \tilde{y}|^{N + 2s}} = \varphi_{B_1}(x),
\]

where \( \tilde{y} = A^{-1}y \).
Now we show the monotonicity. By the radial symmetry of $\varphi$, we let $\varphi(r) = \varphi_{B_1}(x)$, $r = |x| \in (0, 1)$.

Fixed $x_1 = t_1e_1$, $x_2 = t_2e_1$, $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^N$, $0 < t_1 < t_2 < 1$, by direct computation, it yields that

$$
\varphi(t_1) - \varphi(t_2) = \int_{B_1^c} \frac{1}{|t_1e_1 - y|^{N+2s}} - \frac{1}{|t_2e_1 - y|^{N+2s}} dy
$$

$$
= \int_{A_1 \cup A_2} \frac{1}{|t_1e_1 - y|^{N+2s}} - \frac{1}{|t_2e_1 - y|^{N+2s}} dy
$$

$$
+ \int_{A_0} \frac{1}{|t_1e_1 - y|^{N+2s}} - \frac{1}{|t_2e_1 - y|^{N+2s}} dy,
$$

where $A_0 = B_1((t_1 + t_2)e_1) \setminus B_1$,

$$
A_1 = \left\{ (x^1, x^\prime) \mid (x^1, x^\prime) \in (-\infty, \frac{t_1 + t_2}{2}) \times \mathbb{R}^{N-1} \setminus B_1 \right\}
$$

and

$$
A_2 = \left\{ (x^1, x^\prime) \mid (x^1, x^\prime) \in \left(\frac{t_1 + t_2}{2}, +\infty\right) \times \mathbb{R}^{N-1} \setminus B_1((t_1 + t_2)e_1) \right\}.
$$

Observe that

$$
\int_{A_1 \cup A_2} \frac{1}{|t_1e_1 - y|^{N+2s}} - \frac{1}{|t_2e_1 - y|^{N+2s}} dy = 0.
$$

Since $|t_1e_1 - y| > |t_2e_1 - y|$ for any $y \in A_0$, then it deduces that

$$
\varphi(t_1) - \varphi(t_2) = \int_{A_0} \frac{1}{|t_1e_1 - y|^{N+2s}} - \frac{1}{|t_2e_1 - y|^{N+2s}} dy < 0.
$$

and then

$$
\varphi_{B_1}(x) = \int_{B_1^c} \frac{dy}{|x - y|^{N+2s}} = \int_{B_1^c} \frac{dz}{|\tilde{z}|^{N+2s}}
$$

$$
= \int_{B_1^c} \frac{(1 - |x|)^N \tilde{z}^{N+2s}}{(1 - |x|)^{N+2s} |\tilde{z}|^{N+2s}}
$$

$$
= \frac{1}{(1 - |x|)^{2s}} \int_{B_1^c} \frac{d\tilde{z}}{|\tilde{z}|^{N+2s}}.
$$
Combining with \( \bigcap B_{\frac{1}{(1-|x|)}}^c \) = \((-\infty, -1) \times \mathbb{R}^{N-1}\) and

\[
\int_{(-\infty, -1) \times \mathbb{R}^{N-1}} \frac{d\tilde{z}}{|\tilde{z}|^{N+2s}} = \int_{-\infty}^{-1} d\tilde{z}_1 \int_{\mathbb{R}^{N-1}} \frac{d\tilde{z}'}{(|\tilde{z}_1|^2 + |\tilde{z}'|^2)^{\frac{N+2s}{2}}} = \int_{-\infty}^{-1} \frac{d\tilde{z}_1}{|\tilde{z}_1|^{N+2s}(1+|t'|^2)^{\frac{N+2s}{2}}} = \int_{\mathbb{R}^{N-1}} \frac{dt'}{(1+|t'|^2)^{\frac{N+2s}{2}}} \int_{-\infty}^{-1} \frac{d\tilde{z}_1}{\tilde{z}_1^{2s+1}} =: c_1,
\]

it deduces (2.3). \(\square\)

### A.2 Potential inequalities

For \( r_0 > 0 \), denote \( \Phi_{r_0} \) the Green kernel of \((-\Delta)^s\) in \( B_{r_0} \) with the zero Dirichlet boundary condition in \( \mathbb{R}^N \times \mathbb{R}^N \setminus (B_{r_0} \times B_{r_0}) \), observe that

\[
\Phi_{r_0}(x, y) \leq c_{27} |x - y|^{2s-N}
\]

for some \( c_{27} > 0 \) independent of \( r_0 \).

**Lemma A.1** Assume that \( s \in (0, 1) \) and integer \( N \geq 2 \).

1. If \( \frac{1}{q} < \frac{2s}{N} \), then there exists some \( c_{28} > 0 \) such that

\[
\|\Phi_{r_0} * h\|_{\infty} \leq c_{28} \|h\|_q; \quad (A.2)
\]

2. If \( \frac{1}{q} \leq \frac{1}{r} + \frac{2s}{N}, \quad q > 1 \), then there exists some \( c_{29} > 0 \) such that

\[
\|\Phi_{r_0} * h\|_r \leq c_{29} \|h\|_q. \quad (A.3)
\]

3. If \( 1 < \frac{1}{r} + \frac{2s}{N} \), then there exists some \( c_{30} > 0 \) such that

\[
\|\Phi_{r_0} * h\|_r \leq c_{30} \|h\|_1. \quad (A.4)
\]
Proof. Together with \((A.1)\), we apply Hardy-Littlewood-Sobolev theorem for the fractional integration \([28,\text{Chapter 5, section 1}]\). For the convenience of the readers, we provides the details of the proof.

Proof of \((A.3)\). For any \(x \in \Omega\) and \(q' = \frac{q}{q-1}\), by Hölder inequality and \((A.1)\), it holds that
\[
\|\Phi_{r_0} * h\|_{\infty} \leq \left\| \left( \int_{B_{r_0}} \Phi_{q}' \, dy \right)^{\frac{q}{2}} \left( \int_{B_{r_0}(x)} \abs{h(y)}^{q} \, dy \right)^{\frac{1}{2}} \right\|_{L^\infty(\Omega)}
\leq c_{31} \|h\|_{q} \left( \int_{B_{r_0}(x)} \frac{1}{\abs{x-y}^{(N-2s)q'}} \, dy \right)^{\frac{1}{2}}
\leq c_{32} \|h\|_{q},
\]
by the fact that
\[
\frac{1}{q} < \frac{2s}{N}, \quad (N-2s)q' < N
\]
and
\[
\int_{B_{r_0}(x)} \frac{1}{\abs{x-y}^{(N-2s)q'}} \, dy < +\infty.
\]

Proof of \((A.3)\) and \((A.4)\) with \(r \leq q\). By Minkowski inequality, we have that
\[
\|\Phi_{r_0} * h\|_{r} = \|h * \Phi_{r_0}\|_{r}
\leq c_{33} \left( \int_{\mathbb{R}^N} \left( \int_{B_{r_0}} \frac{\abs{h(x-y)\chi_{B_{r_0}(y)}}}{\abs{y}^{N-2s}} \, dx \right)^{r} \, dy \right)^{\frac{1}{r}}
\leq c_{34} \left[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\abs{h(x-y)\chi_{B_{r_0}(y)}}}{\abs{y}^{N-2s}} \, dx \, dy \right]^{\frac{1}{r}}
\leq c_{35} \left[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \abs{h(x-y)\chi_{B_{r_0}(y)}} \, dx \frac{\chi_{B_{r_0}(y)}}{\abs{y}^{N-2s}} \, dy \right]^{\frac{1}{r}}
\leq c_{36} \|h\|_{L^r(\mathbb{R}^N)}.
\]

Proof of \((A.3)\) and \((A.4)\) with \(r > q\). We claim that if \(r > s\) and \(\frac{1}{r} = \frac{1}{q} - \frac{2s}{N}\), the mapping \(h \to (\Phi_{r_0}h_0) * h\) is weak-type \((q,r^*)\) in the sense that
\[
\{|x \in \mathbb{R}^N : |(\Phi_{r_0}h_0) * h(x)| > t\} \lesssim \left( A_{q,r^*} \frac{\|h\|_{L^q(\Omega)}^{r^*}}{t} \right)^{r^*}, \quad h \in L^q(B_{r_0}), \quad (A.5)
\]
for all \(t > 0\), where \(A_{q,r^*} > 0\).

For \(\nu > 0\), we denote
\[
G_1 = \Phi_{r_0}h_0\chi_{B_r}, \quad G_2 = \Phi_{r_0}h_0\chi_{\mathbb{R}^N \setminus B_r}.
\]

It deduces that
\[
\left\{x \in B_{r_0} : |(\Phi_{r_0}h_0) * h(x)| > 2t\right\}
\leq \left\{x \in \mathbb{R}^N : |G_1 * h(x)| > t\right\} + \left\{x \in \mathbb{R}^N : |G_2 * h(x)| > t\right\}.
\]
One hand, by Minkowski inequality, we have that
\[
\left\{ x \in \mathbb{R}^N : |G_1 \ast h(x)| > t \right\} \leq \frac{\|G_1 \ast h\|^s}{t^s} = \frac{\|h \ast G_1\|^s}{t^s} \\
\leq \frac{\left[ \int_{\mathbb{R}^N} (\int_{\mathbb{R}^N} |h(x-y)|^s dx)^\frac{1}{s} \frac{1}{s^s} \int_{B_{\nu}} |y|^{2\alpha-N} \chi_{B_{\nu}}(y) dy \right]^s}{t^s} \\
\leq \frac{\|h\|^s}{t^s} \frac{1}{s^s} \int_{B_{\nu}} |y|^{2\alpha-N} dy = c_{37} \nu^{2s} \frac{\|h\|^s}{t^s}.
\]

On the other hand, direct computation shows that
\[
\|G_2 \ast h\|_{\infty} \leq c_{38} \left\| \int_{\mathbb{R}^N} \chi_{B_{\nu}}(x-y)(\Phi_{r_0} \eta_0) |h(y)| dy \right\|_{\infty} \\
\leq \left( \int_{\mathbb{R}^N} |h(y)|^q dy \right)^\frac{1}{q} \left( \int_{B_{2}(x)} \chi_{B_{\nu}}(\Phi_{r_0} \eta_0)^q dy \right)^\frac{1}{q'} \|h\|_{q'} \left\| \Phi_{r_0} \eta_0 \chi_{B_{\nu}} \right\|_{q'},
\]
where \( q' = \frac{q}{q-1} \) if \( q > 1 \), if not, \( q' = \infty \).

Since
\[
\| \Phi_{r_0} \eta_0 \chi_{B_{\nu}} \|_{L^q'({\mathbb{R}^N})} = \left( \int_{B_{2}(x) \setminus B_{\nu}} |x|^{(2s-N)q'} d\nu \right)^\frac{1}{q'} = c_{39} \nu^{2s-N} q',
\]
letting \( \nu = \left( \frac{t}{c_{39} \|h\|_q} \right)^{\frac{1}{s-N}} \), we have that
\[
\|G_2 \ast h\|_{\infty} \leq t,
\]
that is,
\[
\left\{ x \in \mathbb{R}^N : |G_2 \ast h(x)| > t \right\} = 0.
\]
Then
\[
\left\{ x \in \mathbb{R}^N : |(\Phi_{r_0} \eta_0) \ast h| > 2t \right\} \leq c_{40} \|h\|_q \frac{\nu^{2s q}}{t^q} \leq c_{41} \|h\|_{q^r} \frac{\nu^{r^s q}}{t^{r^s}}.
\]
The case (ii) and (iii) with \( r > s \) follows by Marcinkiewicz Interpolation Theorem.

Conflicts of interest: The authors declare that they have no conflicts of interest regarding this work.

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References

[1] N. Abatangelo, M. Fall, R. Temgoua: A Hopf lemma for the regional fractional Laplacian. *Ann. Mat. Pura Appl. (4) 202*(1), 95–113 (2023).

[2] N. Abatangelo, D. Gómez-Castro, J. Vázquez: Singular boundary behaviour and large solutions for fractional elliptic equations. *J. Lond. Math. Soc. (2) 107*(2), 568–615 (2023).

[3] C. Alves, C. Torres: Existence and multiplicity of solutions for a non-linear Schrödinger equation with non-local regional diffusion. *J. Math. Phys., 58*(11), 111507, 19 pp. (2017).

[4] S. Bhattacharyya, T. Ghosh, G. Uhlmann: Inverse problems for the fractional-Laplacian with lower order non-local perturbations. *Trans. Amer. Math. Soc., 374*(5), 3053–3075 (2021).

[5] K. Bogdan, K. Burdzy, Z. Chen: Censored stable processes. *Probability theory and related fields, 127*, 89–152 (2003).

[6] K. Bogdan, B. Dyda: The best constant in a fractional Hardy inequality. *Math. Nachr., 284*, 629–638 (2011).

[7] L. Caffarelli, S. Salsa, L. Silvestre: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math., 171*(2), 425–461 (2008).

[8] L. Caffarelli, L. Silvestre: Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math., 62*, 597–638 (2009).

[9] H. Chen: The Dirichlet problem involving regional fractional Laplacian. *J. Math. Phys., 59*(7), 071504 (2018).

[10] H. Chen, P. Felmer, A. Quaas: Large solutions to elliptic equations involving fractional Laplacian. *Ann. Inst. H. Poincaré C-AN, 32*(6), 1199–1228 (2015).

[11] H. Chen, H. Hajaiej: Boundary blow-up solutions of elliptic equations involving regional fractional Laplacian. *Comm. Math. Sci., 17*(4), 989–1004 (2019).

[12] H. Chen, Y. Wei: Non-existence of Poisson problem involving regional fractional Laplacian with order in $(0, \frac{1}{2}]$. *Accepted by Diff. Int. Eq., ArXiv: 2007.05775* (2020).

[13] Z. Chen, P. Kim: Green function estimate for censored stable processes. *Probability theory and related fields, 124*, 595–610 (2002).

[14] Z. Chen, P. Kim, R. Song: Two-sided heat kernel estimates for censored stable-like processes. *Probability theory and related fields, 146*, 361–399 (2010).

[15] S. Djitte, M. Fall, T. Weth: A fractional Hadamard formula and applications. *Calc. Var. PDE 60*(6), No. 231, 31 pp. (2021).

[16] B. Dyda: A fractional order Hardy inequality. *Illinois J. Math., 48*, 575–588 (2004).
Equations with regional fractional Laplacian

[17] E. Di Nezza, G., Palatucci, E. Valdinoci: Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136 (5), 521–573 (2012).

[18] M. Fall: Regional fractional Laplacians: boundary regularity. *J. Diff. Eq.*, 320, 598–658 (2022).

[19] M. Fall R. Temgoua: Existence results for nonlocal problems governed by the regional fractional Laplacian *NoDEA Nonlinear Differential Equations Appl.* 30(2), Paper No. 18, 32 pp.(2023)

[20] P. Gressman: Fractional Poincaré and logarithmic Sobolev inequalities for measure spaces. *J. Funct. Anal.*, 265(6), 867–889 (2003).

[21] Q. Guan: Integration by parts formula for regional fractional Laplacian. *Comm. Math. Phys.*, 266, 289–329 (2006).

[22] Q. Guan, Z. Ma: Reflected symmetric A-stable processes and regional fractional Laplacian. *Probability theory and related fields*, 134, 649–694 (2006).

[23] P-T. Huynh, P-T. Nguyen: Semilinear nonlocal elliptic equations with source term and measure data. *J. Anal. Math.* 149(1), 49–111 (2023).

[24] S. Mikhailov: Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Anal. Appl.*, 378, 324–342 (2011).

[25] R. Servadei, E. Valdinoci: Weak and viscosity solutions of the fractional Laplace equation. *Publ. Mat.*, 58(1), 133–154 (2014).

[26] X. Ros-Oton, J. Serra: The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl.*, 101, 275–302 (2014).

[27] Y. Sire, E. Valdinoci: Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result. *J. Funct. Anal.*, 256(6), 1842–1864 (2009).

[28] E. Stein: Singular Integrals and Differentiability Properties of Functions. *Princeton University Press* (1970).

[29] R. Temgoua: On the s-derivative of weak solutions of the Poisson problem for the regional fractional Laplacian. *Comm. Pure Appl. Anal.*, 22(1), 228–248 (2023).