FINITE-DIMENSIONAL SUBALGEBRAS
IN POLYNOMIAL LIE ALGEBRAS OF RANK ONE*

Let $W_n(K)$ be the Lie algebra of derivations of the polynomial algebra $K[[X]] := K[x_1, \ldots, x_n]$ over an algebraically closed field $K$ of characteristic zero. A subalgebra $L \subseteq W_n(K)$ is called polynomial if it is a submodule of the $K[[X]]$-module $W_n(K)$. We prove that the centralizer of every nonzero element in $L$ is abelian provided that $L$ is of rank one. This fact allows to classify finite-dimensional subalgebras in polynomial Lie algebras of rank one.

Introduction. Let $K$ be an algebraically closed field of characteristic zero and $K[[X]] := K[x_1, \ldots, x_n]$ the polynomial algebra over $K$. Recall that a derivation of $K[[X]]$ is a linear operator $D: K[[X]] \to K[[X]]$ such that

$$D(fg) = D(f)g + fD(g)$$

for all $f, g \in K[[X]]$.

Every derivation of the algebra $K[[X]]$ has the form

$$P_1 \frac{\partial}{\partial x_1} + \ldots + P_n \frac{\partial}{\partial x_n}$$

for some $P_1, \ldots, P_n \in K[[X]]$.

A derivation $D$ may be extended to the derivation $\overline{D}$ of the field of rational functions $K((X)) := K((x_1, \ldots, x_n))$ by

$$\overline{D} \left( \frac{f}{g} \right) := \frac{D(f)g - fD(g)}{g^2}.$$

The kernel $S$ of $\overline{D}$ is an algebraically closed subfield of $K((X))$, cf. [6] (Lemma 2.1).

Denote by $W_n(K)$ the Lie algebra of all derivations of $K[[X]]$ with respect to the standard commutator. The study of the structure of the Lie algebra $W_n(K)$ and of its subalgebras is an important problem appearing in various contexts (note that in case $K = \mathbb{R}$ or $K = \mathbb{C}$ we have the Lie algebra $W_n(K)$ of all vector fields with polynomial coefficients on $\mathbb{R}^n$ or $\mathbb{C}^n$). Since $W_n(K)$ is a free $K[[X]]$-module (with the basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$), it is natural to consider the subalgebras $L \subseteq W_n(K)$ which are

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Let $L$ be a subalgebra of the Lie algebra $W_n(\mathbb{K})$. Assume that $L$ is a submodule of rank one in the $\mathbb{K}[X]$-module $W_n(\mathbb{K})$. Then the centralizer of any nonzero element in $L$ is abelian.
Proof. By Lemma 2, the subalgebra \( L \) has the form \( ID_0 \) for some reduced derivation \( D_0 \in W_n(\mathbb{K}) \). Denote by \( \overline{D_0} \) the extension of \( D_0 \) to the field \( \mathbb{K}(X) \), and let \( S \) be the kernel of \( \overline{D_0} \). Take any nonzero element \( fD_0 \in L, f \in I \), and consider its centralizer \( C = C_L(fD_0) \). For every nonzero element \( gD_0 \in C \) one has

\[
[fD_0, gD_0] = (fD_0(g) - gD_0(f))D_0 = 0.
\]

This implies \( D_0(f)g - fD_0(g) = 0 \), thus \( \overline{D_0}(f/g) = 0 \) and \( f/g \in S \). Take another nonzero element \( hD_0 \in C \). By the same arguments we get \( f/h \in S \). This shows that \( g/h \in S \). The latter condition is equivalent to \( [gD_0, hD_0] = 0 \), so the subalgebra \( C \) is abelian.

Proposition 1 is proved.

The next proposition seems to be known, but having no precise reference we supply it with a complete proof. By \( Z(F) \) we denote the center of a Lie algebra \( F \).

Proposition 2. Let \( F \) be a finite-dimensional Lie algebra over an algebraically closed field \( \mathbb{K} \) of characteristic zero. Assume that the centralizers of all nonzero elements in \( F \) are abelian. Then either \( F \) is abelian, or \( F \cong A \ltimes \langle b \rangle \), where \( b \in F, A \subset F \) is an abelian ideal and \( Z(F) = 0 \), or \( F \cong \mathfrak{sl}_2(\mathbb{K}) \).

Proof. If the centralizers of all nonzero elements of a Lie algebra \( F \) are abelian, then the same property holds for every subalgebra of \( F \). Assume that \( F \) is not abelian and the centralizers of all elements of \( F \) are abelian. Then the center \( Z(F) \) is trivial.

Case 1. \( F \) is solvable. Then \( F \) contains a non-central one-dimensional ideal \( \langle a \rangle \), see [3] (II.4.1, Corollary B). Let \( A \) be the centralizer of \( a \) in \( F \). Clearly, \( A \) is an abelian ideal of codimension one in \( F \). Then \( F \cong A \ltimes \langle b \rangle \) for any \( b \in F \setminus A \).

Case 2. \( F \) is semisimple. Then \( F = F_1 \oplus \ldots \oplus F_k \) is the sum of simple ideals. Since the centralizer of every element \( x \in F_1 \) contains \( F_2 \oplus \ldots \oplus F_k \), we conclude that \( F \) is simple. Let \( H \) be a Cartan subalgebra in \( F \) and \( F = N_- \oplus H \oplus N_+ \) the Cartan decomposition with opposite maximal nilpotent subalgebras \( N_- \) and \( N_+ \) in \( F \), see [3] (II.8.1). Since the centralizer of every element in \( N_+ \) is abelian, either the subalgebra \( N_+ \) is abelian or \( Z(N_+) = 0 \). The second possibility is excluded because \( N_+ \) is nilpotent. Thus \( N_+ \) is abelian. This is the case if and only if the root system of the Lie algebra \( F \) has rank one, or, equivalently, \( F \cong \mathfrak{sl}_2(\mathbb{K}) \).

Case 3. \( F \) is neither solvable nor semisimple. Consider the Levi decomposition \( F = R \ltimes G \), where \( G \) is a maximal semisimple subalgebra and \( R \) is the radical of \( F \). By Case 2, the algebra \( G \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{K}) \). Denote by \( A \) the ideal of \( R \) which coincides with \( R \) if \( R \) is abelian, and \( A = [R, R] \) otherwise. By Case 1, the ideal \( A \) is abelian. Consider the decomposition \( A = A_1 \oplus \ldots \oplus A_\ell \) into simple \( G \)-modules with respect to the adjoint representation. If \( \dim A_1 = 1 \), then the centralizer of a nonzero element in \( A_1 \) contains \( G \), a contradiction. Suppose that \( \dim A_1 \geq 2 \). Fix an \( \mathfrak{sl}_2 \)-triple \( \{e, h, f\} \) in \( G \) and take a highest vector \( x \in A_1 \) with respect to the Borel subalgebra \( \langle e, h \rangle \). Then \( [e, x] = 0 \) and the centralizer \( C_F(x) \) contains the subalgebra \( A \ltimes \langle e \rangle \). The latter is not abelian because the adjoint action of the element \( e \) on \( A_1 \) is not trivial. This contradiction concludes the proof.

2. Main results. In this section we get a classification of finite-dimensional subalgebras in polynomial Lie algebras of rank one.

Theorem 1. Let \( L \) be a polynomial Lie algebra of rank one in \( W_n(\mathbb{K}) \), where \( \mathbb{K} \) is an algebraically closed field of characteristic zero, and \( F \subset L \) a finite-dimensional subalgebra. Then one of the following conditions holds:

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(1) $F$ is abelian;
(2) $F \cong A \times \langle b \rangle$, where $A \subset F$ is an abelian ideal and $[b, a] = a$ for every $a \in A$;
(3) $F$ is a three-dimensional simple Lie algebra, i.e., $F \cong sl_2(K)$.

**Proof.** By Propositions 1 and 2, every finite-dimensional subalgebra $F \subset L$ is either abelian, or has the form $A \times \langle b \rangle$, or is isomorphic to $sl_2(K)$. It remains to prove that in the second case we may find $b \in F$ with $[b, a] = a$ for every $a \in A$. Take any element $b$ with $F = A \times \langle b \rangle$.

Let us prove that the operator $ad(b)$ is diagonalizable. Assuming the converse, let $a_0, a_1 \in A$ be nonzero elements with $[b, a_1] = \lambda a_1 + a_0$, $[b, a_0] = \lambda a_0$ for some $\lambda \in K$. By Lemma 2, the subalgebra $L$ has the form $ID_0$ for some ideal $I \subseteq K[X]$ and some reduced derivation $D_0 \in W_n(K)$. Set $a_0 = fD_0$, $a_1 = gD_0$, $b = hD_0$, $f, g, h \in I$. The relations $[b, a_1] = \lambda a_1 + a_0$, $[b, a_0] = \lambda a_0$, and $[a_0, a_1] = 0$ are equivalent to

$$hD_0(g) - gD_0(h) = \lambda g + f, \quad hD_0(f) - fD_0(h) = \lambda f, \quad fD_0(g) - gD_0(f) = 0.$$ 

Multiplying the second relation by $g$, we get $hD_0(f) - fD_0(h) = \lambda f g$. This and the third relation imply $hD_0(g) - gD_0(h) = \lambda f g = hD_0(g) - gD_0(h) = \lambda g$. Together with the first relation it gives $f = 0$, a contradiction.

Now assume that $[b, a_1] = \lambda a_1$ and $[b, a_2] = \lambda a_2$ for some $\lambda_1, \lambda_2 \in K$. If $a_1 = fD_0$, $a_2 = gD_0$, $b = hD_0$, then we obtain the relations

$$hD_0(f) - fD_0(h) = \lambda_1 f, \quad hD_0(g) - gD_0(h) = \lambda_2 g, \quad fD_0(g) - gD_0(f) = 0.$$ 

Consequently,

$$ghD_0(f) = gf(\lambda_1 + D_0(h)) = fhD_0(g) = fg(\lambda_2 + D_0(h)).$$

This proves that $\lambda_1 = \lambda_2$ and hence $ad(b)$ is a scalar operator. Since $F$ is not abelian, $ad(b)$ is nonzero and, multiplying by a suitable scalar, we may assume that $ad(b)$ is the identical operator.

Theorem 1 is proved.

Let us show that all three possibilities indicated in Theorem 1 are realizable. Take a derivation $D_0 \in W_n(K)$ such that there exist non-constant polynomials $p, q \in K[X]$ with $D_0(p) = 0$ and $D_0(q) = 1$. For example, one may take $D_0 = \frac{\partial}{\partial x_2} + P_1 \frac{\partial}{\partial x_3} + \ldots + P_n \frac{\partial}{\partial x_n}$ with arbitrary $P_1, \ldots, P_n \in K[X]$, and $p = x_1, q = x_2$.

The subalgebra $\langle D_0, pD_0, \ldots, p^{m-1}D_0 \rangle$ is an $m$-dimensional abelian subalgebra in $K[X]D_0$ for every positive integer $m$.

The subalgebra $A \times \langle b \rangle$ with $\dim A = m$ may be obtained by setting $A = \langle D_0, pD_0, \ldots, p^{m-1}D_0 \rangle$ and $b = -qD_0$. Indeed,

$$[-qD_0, f(p)D_0] = (-D_0(f(p)) + f(p)D_0(q))D_0 = f(p)D_0$$

for every $f(p) \in K[p]$.

Finally, the derivations $e = q^2D_0$, $h = 2qD_0$ and $f = -D_0$ form an $sl_2$-triple in $K[X]D_0$.

**Remark 1.** The structure of finite-dimensional subalgebras in a polynomial Lie algebra $L = ID_0$ depends on properties of the derivation $D_0$. In particular, if $\text{Ker}(D_0) = K$, then all abelian subalgebras in $K[X]D_0$ are one-dimensional.
Our last result concerns finite-dimensional subalgebras in the Lie algebra $W_1(\mathbb{K})$.

By Lemma 2, every polynomial Lie algebra in $W_1(\mathbb{K})$ has the form $L = q(x)\mathbb{K}[x] \frac{\partial}{\partial x}$
with some polynomial $q(x) \in \mathbb{K}[x]$.

**Proposition 3.** Let $L = q(x)\mathbb{K}[x] \frac{\partial}{\partial x}$ be a polynomial algebra.

1. If $\deg q(x) \geq 2$, then every finite dimensional Lie subalgebra in $L$ is one-dimensional.

2. If $\deg q(x) = 1$, then every finite dimensional Lie subalgebra in $L$ is either one-dimensional or coincides with $F_k = \left\langle q(x)\frac{\partial}{\partial x}, q(x)k \frac{\partial}{\partial x} \right\rangle$ for some $k \geq 2$.

3. If $q(x) = \text{const} \neq 0$ (i.e., $L = W_1(\mathbb{K})$), then every finite dimensional Lie subalgebra in $L$ is either one-dimensional, or coincides with $F_{k,\beta} = \left\langle (x + \beta)^k \frac{\partial}{\partial x}, (x + \beta)^{k+1} \frac{\partial}{\partial x} \right\rangle$ for some $\beta \in \mathbb{K}$ and $k = 0, 2, 3, \ldots$, or is a three-dimensional subalgebra
$$\left\langle \frac{\partial}{\partial x}, (x + \beta) \frac{\partial}{\partial x}, (x + \beta)^2 \frac{\partial}{\partial x} \right\rangle,$$
where $\beta \in \mathbb{K}$.

**Proof.** Let us describe all two-dimensional subalgebras in $W_1(\mathbb{K})$. Every such subalgebra has the form
$$\left\langle f(x) \frac{\partial}{\partial x}, g(x) \frac{\partial}{\partial x} \right\rangle \quad \text{with} \quad f(x), g(x) \in \mathbb{K}[x] \quad \text{and} \quad fg' - f'g = g. \quad (\ast)$$
If $\deg(f) \geq 2$, then looking at the highest terms of $fg'$ and $f'g$, we get $\deg(f) = \deg(g)$. But the polynomials $(f + \lambda g, g)$ satisfy relation $(\ast)$ for every $\lambda \in \mathbb{K}$, and thus we may assume that $f$ is linear. Each root of $g$ is also a root of $f$, so $g$ is proportional to $f^k$ for some $k = 0, 2, 3, \ldots$. This observation together with Theorem 1 and Remark 1 proves all the assertions.

Proposition 3 is proved.

If we consider obtained in Proposition 3 realizations up to automorphisms of the polynomial ring $\mathbb{K}[x]$, then in case $\deg q(x) = 1$ for the Lie algebra $F_k$ one can take $q(x) = x$, and in case $q(x) = \text{const} \neq 0$ one can take $\beta = 0$.

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