Ramanujan type congruences for the Klingen-Eisenstein series

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Abstract

In the case of Siegel modular forms of degree $n$, we prove that, for almost all prime ideals $p$ in any ring of algebraic integers, mod $p^m$ cusp forms are congruent to true cusp forms of the same weight. As an application of this property, we give congruences for the Klingen-Eisenstein series and cusp forms, which can be regarded as a generalization of Ramanujan’s congruence. We will conclude by giving numerical examples.

1 Introduction

Kurokawa [9] found some examples of congruence relations on eigenvalues between the Klingen-Eisenstein series and Hecke eigen cusp forms, in the case of Siegel modular forms of degree 2. Mizumoto [12] and Katsurada-Mizumoto [6] showed some congruence properties of this kind for more general cases. In this paper, we prove congruences on Fourier coefficients between the Klingen-Eisenstein series and cusp forms, in the case of Siegel modular forms of degree $n$. We remark that congruences on Fourier coefficients are stronger properties than congruences on eigenvalues of eigen forms.

In order to show these congruences, we determine all mod $p^m$ cusp forms which are congruent to true cusp forms, where “mod $p^m$ cusp forms” are Siegel modular forms of degree $n$ whose Fourier coefficients of rank $r$ with $0 \leq r \leq n - 1$ vanish modulo $p^m$ (see Definition 3.1). Namely, we can explain our main results as follows:

(1) In the case of Siegel modular forms of degree $n$, for almost all prime ideals $p$ in any ring of algebraic integers, mod $p^m$ cusp forms are congruent to true cusp forms of the same weight (Theorem 3.2).

(2) We take a prime ideal $p$ such that a constant multiple of the Klingen-Eisenstein
series $\alpha[f]_r^n$ attached to a Hecke eigen cusp form $f$ is a mod $p^m$ cusp form. Then there exists a cusp form $F$ such that $\alpha[f]_r^n \equiv F \mod p^m$ (Corollary 3.4).

The congruences we prove can be regarded as a generalization of Ramanujan's congruence which asserts that

$$\sigma_{11}(n) \equiv \tau(n) \mod 691,$$

where $\sigma_m(n)$ is the $m$-th Fourier coefficient of the Eisenstein series of weight 12 (i.e., the sum of $m$-th powers of the divisors of $n$) and $\tau(n)$ is the $n$-th Fourier coefficient of Ramanujan's $\Delta$ function. In the case of degree 2 and of $f = 1$ for the situation (2), we already proved these congruences in [7].

2 Preliminaries

2.1 Notation

First we confirm the notation. For the elementally facts, we refer to Klingen [8]. Let $\Gamma_n = Sp_n(\mathbb{Z})$ be the Siegel modular group of degree $n$ and $\mathbb{H}_n$ the Siegel upper-half space of degree $n$. We denote by $M_k(\Gamma_n)$ the $\mathbb{C}$-vector space of all Siegel modular forms of weight $k$ for $\Gamma_n$, and $S_k(\Gamma_n)$ is the subspace of cusp forms.

Any $f(Z)$ in $M_k(\Gamma_n)$ has a Fourier expansion of the form

$$f(Z) = \sum_{0 \leq T \in \Lambda_n} a(T; F)q^T, \quad q^T := e^{2\pi i \operatorname{tr}(TZ)}, \quad Z \in \mathbb{H}_n,$$

where $T$ runs over all elements of $\Lambda_n$, and

$$\Lambda_n := \{ T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z} \}.$$

For a subring $R$ of $\mathbb{C}$, let $M_k(\Gamma_n)_R \subset M_k(\Gamma_n)$ denote the $R$-module of all modular forms whose Fourier coefficients lie in $R$.

Let $r$ be a non-negative integer with $0 \leq r \leq n - 1$. Let $\Delta_{n,r}$ be the (Klingen) parabolic subgroup of $\Gamma_n$ defined by

$$\Delta_{n,r} := \left\{ \left( \begin{array}{cc} * & * \\ 0_{n-r,n+r} & * \end{array} \right) \in \Gamma_n \right\}.$$

Let $k$ a positive even integer with $k > n + r + 1$ and $f \in S_k(\Gamma_r)$ a Hecke eigen form. Then the Klingen-Eisenstein series attached to $f$ is defined by

$$[f]_r^n(Z) := \sum_{M = (\begin{array}{cc} A & B \\ C & D \end{array}) \in \Delta_{n,r} \setminus \Gamma_n} \det(CZ + D)^{-k}f((MZ)^*) \quad (Z \in \mathbb{H}_n);$$

here $Z^*$ denotes the $r \times r$-submatrix in the upper left corner of $Z$. This series $[f]_r^n$ defines a Hecke eigen form which belongs to $M_k(\Gamma_n)$. Let $K_f$ be the number field
generated over \( \mathbb{Q} \) by the eigenvalues of the Hecke operators over \( \mathbb{Q} \) on \( f \). Then it is known that \([f]_n^p \in M_k(\Gamma_n)_{K_f}\) by [10] [11] [15].

Let \( \Phi : M_k(\Gamma_n) \to M_k(\Gamma_{n-1}) \) be the Siegel \( \Phi \)-operator. Then we have

\[
\Phi([f]_n^p) = \begin{cases} 
[f]_r^{n-1} & \text{if } n > r + 1, \\
 f & \text{if } n = r + 1.
\end{cases}
\] (2.1)

3 Main results and their proofs

3.1 Main results

Let \( K \) be an algebraic number field and \( \mathcal{O} = \mathcal{O}_K \) the ring of integers in \( K \). For a prime ideal \( \mathfrak{p} \) in \( \mathcal{O} \), we denote by \( \mathcal{O}_\mathfrak{p} \) the localization of \( \mathcal{O} \) at \( \mathfrak{p} \). First our main result concerns “mod \( \mathfrak{p}^m \) cusp forms” defined as

Definition 3.1. Let \( f \in M_k(\Gamma_n)_{\mathcal{O}_\mathfrak{p}} \). We call \( f \) a mod \( \mathfrak{p}^m \) cusp form if \( \Phi(f) \equiv 0 \mod \mathfrak{p}^m \).

Theorem 3.2. For a finite set \( S_n(K) \) of prime ideals in \( K \) depends on \( n \), we have the following: Let \( k > 2n \) and \( \mathfrak{p} \) be a prime ideal of \( \mathcal{O} \) with \( \mathfrak{p} \not\in S_n(K) \). Let \( f \in M_k(\Gamma_n)_{\mathcal{O}_\mathfrak{p}} \) be a mod \( \mathfrak{p}^m \) cusp form. In other words, we assume that \( f \in M_k(\Gamma_n)_{\mathcal{O}_\mathfrak{p}} \) satisfies \( \Phi(f) \equiv 0 \mod \mathfrak{p}^m \). Then there exists \( g \in S_k(\Gamma_n)_{\mathcal{O}_\mathfrak{p}} \) such that \( f \equiv g \mod \mathfrak{p}^m \).

Remark 3.3. Since there does not exist non-cusp form of odd weight, the statement for the case where \( k \) is odd in Theorem [3.2] is trivial.

We will see how to determine the exceptional set \( S_n(K) \) in the later section (Definition [3.9]). As an application of this theorem, we obtain congruences between the Klingen-Eisenstein series and cusp forms:

Let \( v_\mathfrak{p} \) be the normalized additive valuation with respect to \( \mathfrak{p} \). We define two values \( v_\mathfrak{p}(f) \) and \( v_\mathfrak{p}^{(n')}(f) \) for \( f \in M_k(\Gamma_n)_K \) by

\[
v_\mathfrak{p}(f) := \min\{v_\mathfrak{p}(a_f(T)) \mid T \in \Lambda_n\},
\]

\[
v_\mathfrak{p}^{(n')}(f) := \min\{v_\mathfrak{p}(a_f(T)) \mid T \in \Lambda_n, \text{ rank}(T) = n' \} \quad (0 \leq n' \leq n).
\]

Then we have

Corollary 3.4. Let \( k > 2n \) be even and \( f \in S_k(\Gamma_n)_{K_f} \) (\( n > r \)) a Hecke eigen form. For the Klingen-Eisenstein series \([f]_n^p \) attached to \( f \), we choose a prime ideal \( \mathfrak{p} \) in \( \mathcal{O}_{K_f} \) with \( \mathfrak{p} \not\in S_n(K_f) \) such that \( v_\mathfrak{p}^{(n)}([f]_n^p) = v_\mathfrak{p}(\Phi([f]_n^p)) - m \) (\( m \in \mathbb{Z}_{\geq 1} \)). Then there exists \( F \in S_k(\Gamma_n)_{\mathcal{O}_\mathfrak{p}} \) such that \( \alpha[f]_n^p \equiv F \mod \mathfrak{p}^m \) for some \( 0 \neq \alpha \in \mathfrak{p}^m \).

Remark 3.5. (1) The assumption \( v_\mathfrak{p}^{(n)}([f]_n^p) = v_\mathfrak{p}(\Phi([f]_n^p)) - m \) is equivalent to the fact that \( \alpha[f]_n^p \) is a non-zero mod \( \mathfrak{p}^m \) cusp form for some \( \alpha \in \mathfrak{p}^m \) satisfying \( v_\mathfrak{p}(\alpha[f]_n^p) = 0 \).
(2) For a prime $l$ and $1 \leq i \leq n$, we define Hecke operators $T(l)$ and $T_i(l^2)$ by $T(l) = \Gamma_n \text{diag}(1_1, l_1) \Gamma_n$ and $T_i(l^2) = \Gamma_n \text{diag}(1_i, l_{1-i}, l_i, l_{1-n-i}) \Gamma_n$. For an eigen form $F$ and a Hecke operator $T$, we denote by $\lambda(T, F)$ the Hecke eigenvalue of $T$. By Deligne-Serre lifting lemma ([3] Lemma 6.11), we can take an eigen form $G \in S_k(\Gamma_n)$ such that $\lambda(T, [f]^n_i) \equiv \lambda(T, G)$ mod $\mathfrak{p}$ for $T = T(l)$, $T_i(l^2)$, $l \neq p$ and $1 \leq i \leq n$.

(3) If $r = 0$, $[f]^n_i$ is the ordinary Siegel-Eisenstein series. In particular, if $n = 2$, this was proved by [7].

Using the integrality theorem obtained by Mizumoto [14], we can give conditions on $\mathfrak{p}$ to find congruences for the Klingen-Eisenstein series and cusp forms as in Corollary 3.4. We shall introduce an example:

To apply his theorem, we assume that
(i) $f \in M_k(\Gamma_r) \mathcal{O}_{K_f}$, and one of the Fourier coefficients of $f$ is equal to 1,
(ii) $L(k - r, f, St) \neq 0$, where $L(s, f, St)$ is the standard $L$-function of $f$.

Then Mizumoto’s result states that

$$a(T; [f]^n_i) \in c_k(r, n) \mu_k(r) \prod_{i=r+1}^{n} \text{Num} \left( \frac{B_{2k-2i}}{k-i} \right)^{-1} \cdot L^*(k - r, f, St)^{-1} \mathcal{A}(f)^{-1}$$

for some $c_k(r, n) \in \mathbb{Q}^\times$ and $\mu_k(r) \in \mathbb{Z}$ which are computable. Here $\mathcal{A}(f)$ is an integral ideal of $\mathcal{O}_{K_f}$, $\text{Num}(\ast)$ is the numerator,

$$L^*(k - r, f, St) := \frac{L(k - r, f, St)}{\pi^{(2r+1)k-\frac{3(r+1)}{2}}} \in K_f$$

and $(f, f)$ is the Petersson norm of $f$. For the precise definitions of these numbers, see [14]. This property tells us all possible primes appearing in denominators of all Fourier coefficients of $[f]^n_i$, since the property (2.1).

For example, we consider a simple case where $r = n - 1$. We choose $\mathfrak{p}$ satisfying

$$v_{\mathfrak{p}} \left( c_k(r, n) \mu_k(r)^{-1} \prod_{i=r+1}^{n} \text{Num} \left( \frac{B_{2k-2i}}{k-i} \right)^{-1} \cdot L^*(k - r, f, St)^{-1} \right) = -m.$$ 

Then $[f]^n_i$ is a mod $\mathfrak{p}^m$ cusp form for any $\alpha \in \mathfrak{p}^m$. Applying Theorem 3.2, we can find $F \in S_k(\Gamma_n)_{\mathcal{O}_{\mathfrak{p}}}$ such that $[f]^n_i \equiv F$ mod $\mathfrak{p}^m$. Remark that it may become $[f]^n_i \equiv F \equiv 0$ mod $\mathfrak{p}^m$ for this choice of $\mathfrak{p}$, compared with Corollary 3.4.

3.2 Proof of the theorem

In order to define $S_n(K)$ and to prove the theorem, we start with introducing some basic properties.

**Lemma 3.6.** Let $\bigoplus_k M_k(\Gamma_n)_{\mathbb{Z}(p)} = \mathbb{Z}(p)[f_1, \ldots, f_s]/C$ with a relation $C$ among the generators. Then we have $\bigoplus_k M_k(\Gamma_n)_{\mathcal{O}_p} = \mathcal{O}_p[f_1, \ldots, f_s]/C$. 


**Proof.** By the same argument of Mizumoto [13] Lemma A.4, we have $M_k(\Gamma_n)_\mathbb{Z(p)} \otimes \mathbb{Z(p)} = \mathcal{O}_p = M_k(\Gamma_n)_\mathcal{O}_p$. Thus we have

$$\bigoplus_k M_k(\Gamma_n)_\mathcal{O}_p = \left( \bigoplus_k M_k(\Gamma_n)_\mathbb{Z(p)} \right) \otimes \mathbb{Z(p)} \mathcal{O}_p = (\mathbb{Z(p)}[f_1, \ldots, f_s]/C) \otimes \mathbb{Z(p)} \mathcal{O}_p = \mathcal{O}_p[f_1, \ldots, f_s]/C.$$  

The finite generation of \( \bigoplus_k M_k(\Gamma_n)_\mathbb{Z} \) is known by Faltings-Chai [4]. Namely, we always assume that \( \bigoplus_k M_k(\Gamma_n)_\mathbb{Z(p)} = \mathbb{Z(p)}[f_1, \ldots, f_s]/C \) for any prime \( p \) and hence also that \( \bigoplus_k M_k(\Gamma_n)_\mathcal{O}_p = \mathcal{O}_p[f_1, \ldots, f_s]/C \) for any prime ideal \( p \).

**Lemma 3.7.** Assume that \( \bigoplus_k M_k(\Gamma_n)_\mathcal{O}_p = \mathcal{O}_p[f_1, \ldots, f_s]/C \) with \( f_i \in M_k(\Gamma_n)_\mathcal{O}_p \). Let \( M \) be a natural number. We take the minimum of integers \( \alpha_i \in \mathbb{Z}_{\geq 0} \) such that, the weight of \( f_i^{\alpha_i} \) is strictly greater than \( M \). Then the graded algebra \( \bigoplus_{M < k} M_k(\Gamma_n)_\mathcal{O}_p \) is generated over \( \mathcal{O}_p \) by the following finitely many monomials:

\[
\begin{align*}
  f_1^{\alpha_1}, \ldots, f_s^{\alpha_s}, \\
  f_1^{i_1} \cdots f_s^{i_s} \quad (i_1k_1 + \cdots + i_sk_s > M, \ 0 \leq i_j < 2\alpha_j).
\end{align*}
\]

**Proof.** First, we remark that any \( g \in M_k(\Gamma_n)_\mathcal{O}_p \) can be written by a linear combination of monomials of the form \( f_1^{\alpha_1} \cdots f_s^{\alpha_s} \). Hence we may consider only the case \( g = f_1^{\alpha_1} \cdots f_s^{\alpha_s} \).

Let \( k_0 := \alpha_1k_1 + \cdots + \alpha_sk_s \). If \( 2k_0 \geq k > M \), then the assertion is trivial. Hence, we assume that \( k > 2k_0 \). Now we consider \( \alpha_i = \alpha_iq_i + r_i \ (0 \leq r_i < \alpha_i) \). Then there exists \( j_0 \) such that \( q_{j_0} \geq 1 \) because of \( k > 2k_0 \). In this case, we may consider the following decomposition:

\[
\begin{align*}
  g &= h_1 \cdot h_2, \\
  h_1 &:= f_1^{r_1} \cdots f_{j_0-1}^{r_{j_0-1}} f_{j_0}^{r_{j_0} + \alpha_j_{j_0}} f_{j_0+1}^{r_{j_0+1}} \cdots f_s^{r_s}, \\
  h_2 &:= f_1^{\alpha_1q_1} \cdots f_{j_0-1}^{\alpha_{j_0-1}q_{j_0-1}} f_{j_0}^{\alpha_{j_0}(q_{j_0}-1)} f_{j_0+1}^{\alpha_{j_0+1}q_{j_0+1}} \cdots f_s^{\alpha_sq_s}.
\end{align*}
\]

Then, both \( h_1 \) and \( h_2 \) are written by the monomials of (3.1) and (3.2). This completes the proof.

**Lemma 3.8.** For \( k > 2n \), the restricted Siegel \( \Phi \)-operator \( \Phi_K : M_k(\Gamma_n)_K \to M_k(\Gamma_{n-1})_K \) is surjective.

**Proof.** By Shimura [17], we have \( M_k(\Gamma_n)_K = M_k(\Gamma_n)_\mathbb{Q} \otimes \mathbb{Q} K \). Since \( \mathbb{C} \) is faithfully flat over \( K \), the surjectivity of \( \Phi_K \) is equivalent to that of \( \Phi : M_k(\Gamma_n)_\mathbb{C} \to M_k(\Gamma_{n-1})_\mathbb{C} \). The surjectivity of \( \Phi \) was proved by Klingenberg [8]. Therefore, we obtain the assertion of the lemma.

\[\square\]
In order to prove the theorem, it suffices to consider the case where the weight is even (see Remark 3.3). From Lemma 3.7, we may assume that $\bigoplus_{2n<k\in\mathbb{Z}} M_k(\Gamma_{n-1})\mathcal{O}_p = \mathcal{O}_p[f_1, \ldots, f_s]/C$. Applying Lemma 3.8, we have $\Phi^{-1}_K(f_i) \neq \phi$ for any $i$ with $1 \leq i \leq s$.

We are now in a position to define the set $S_n(K)$ and to prove Theorem 3.2.

**Definition 3.9.** Let $S_n(K)$ be the set of all prime ideals $p$ in $\mathcal{O}$ such that, there exists $i$ which satisfies that for all $F_i \in \Phi^{-1}_K(f_i)$ we have $v_p(F_i) < 0$. Note that $S_n(K)$ is a finite set depends on $n$ not depends on generators of $\bigoplus_{2n<k\in\mathbb{Z}} M_k(\Gamma_{n-1})\mathcal{O}_p$ (Remark 3.10 in Subsection 3.3).

**Proof of Theorem 3.2.** We choose a polynomial $P \in \mathcal{O}_p[x_1, \ldots, x_s]$ such that $\Phi(F) = P(f_1, \ldots, f_s)$. Since $\Phi(F) = P(f_1, \ldots, f_s) \equiv 0$ mod $p^n$, there exists $\gamma \in p^n$ such that $\gamma^{-1}\Phi(F) \in M_k(\Gamma_{n-1})\mathcal{O}_p$. In fact, we may choose $\gamma$ as $\gamma := a(T_0; \Phi(f))$ for some $T_0$ which satisfies $v_p(a(T_0; \Phi(f))) = v_p(\Phi(f))$. Hence we can find $Q \in \mathcal{O}_p[x_1, \ldots, x_s]$ such that $\gamma^{-1}\Phi(F) = Q(f_1, \ldots, f_s)$. Since $p \notin S_n(K)$, there exists $F_i \in \Phi^{-1}_K(f_i)$ such that $v_p(F_i) \geq 0$ for each $i$ with $1 \leq i \leq s$. Then $F_i \in \gamma Q(F_1, \ldots, F_s) + \text{Ker}\Phi$. Hence there exists $G \in \text{Ker}\Phi$ such that $F = \gamma Q(F_1, \ldots, F_s) + G$. Note that $Q(F_1, \ldots, F_s) \in M_k(\Gamma_n)\mathcal{O}_p$ because of $v_p(F_i) \geq 0$ and hence $G \in S_k(\Gamma_n)\mathcal{O}_p$. This implies $F \equiv G$ mod $p^n$ because of $\gamma \in p^n$. This completes the proof of Theorem 3.2. \hfill \Box

### 3.3 Remark on $S_n(K)$

**Remark 3.10.** For each prime ideal $p$, it does not depend on the choice of generators of $\bigoplus_{2n<k\in\mathbb{Z}} M_k(\Gamma_{n-1})\mathcal{O}_p$ whether $p$ belongs to the exceptional set $S_n(K)$ or not. Namely, we get the following property: Assume that $\bigoplus_{2n<k\in\mathbb{Z}} M_k(\Gamma_{n-1})\mathcal{O}_p = \mathcal{O}_p[f_1, \ldots, f_s]/C = \mathcal{O}_p[f'_1, \ldots, f'_s]/C'$. If we can take $F_i \in M_k(\Gamma_n)\mathcal{O}_p$ such that $\Phi(F_i) = f_i$ for each $i$ with $1 \leq i \leq s$, then we can take $F'_i \in M_k(\Gamma_n)\mathcal{O}_p$ such that $\Phi(F'_i) = f'_i$ for each $j$ with $1 \leq j \leq t$.

**Proof.** For each $1 \leq j \leq t$, we can write as $f'_i = P(f_1, \ldots, f_s)$ for some polynomial $P \in \mathcal{O}_p[x_1, \ldots, x_s]$. If we put $F'_i := P(F_1, \ldots, F_s)$, then $F'_i \in M_k(\Gamma_n)\mathcal{O}_p$ and $\Phi(F'_i) = f'_i$. \hfill \Box

**Remark 3.11.** (1) We have $S_n(K) \subset \{p \mid p \cap \mathbb{Z} \in S_n(\mathbb{Q})\}$. Hence, to obtain the congruences as in Corollary 3.7, it suffices to except the prime ideals above $p$ with $(p) \in S_n(\mathbb{Q})$.

(2) Let $g := [K: \mathbb{Q}] < \infty$. Then $S_n(K) \supset \{p \mid p \cap \mathbb{Z} = (p) \in S_n(\mathbb{Q}), {p} \nmid g\}$.

**Proof.** (1) Let $p \in S_n(K)$. If we assume that $\bigoplus_k M_k(\Gamma_n)\mathcal{Z}(p) = \mathcal{Z}(p)[f_1, \ldots, f_s]/C$, then $\bigoplus_k M_k(\Gamma_n)\mathcal{O}_p = \mathcal{O}_p[f_1, \ldots, f_s]/C$ by Lemma 3.6. Since $p \in S_n(K)$, there exists $i$ with $1 \leq i \leq s$ such that for all $F_i \in \Phi^{-1}_K(f_i)$, we have $v_p(F_i) < 0$. In particular, for all $F_i \in \Phi^{-1}_K(f_i)$, we have $v_p(F_i) < 0$.

(2) Let $p \cap \mathbb{Z} = (p) \in S_n(\mathbb{Q}), {p} \nmid g$ and $\bigoplus_k M_k(\Gamma_n)\mathcal{Z}(p) = \mathcal{Z}(p)[f_1, \ldots, f_s]/C$. Seeking
a contradiction, we suppose that, for each $i$ with $1 \leq i \leq s$, there exists $F_i \in \Phi^{-1}_k(f_i)$ such that $v_p(F_i) \geq 0$. We consider $G_i := \sum_{\sigma \in \text{Emb}(K, \mathbb{C})} F_i^\sigma \in M_k(\Gamma_n)_{\mathbb{Q}}$. Note that $G_i \in M_k(\Gamma_n)_{\mathbb{Z}(p)}$ because of $v_p(F_i) \geq 0$ and that $\Phi(G_i) = g f_i$ since $\Phi(F_i^\sigma) = f_i \in M_k(\Gamma_n)_{\mathbb{Q}}$ for any $\sigma \in \text{Emb}(K, \mathbb{C})$. By the assumption $p \nmid g$, we have $v_p(g^{-1} G_i) \geq 0$ and $g^{-1} G_i \in \Phi^{-1}_k(f_i)$. This contradicts for $p \cap \mathbb{Z} = \langle p \rangle \in S_n(\mathbb{Q})$.

We have $S_2(\mathbb{Q}) \subset \{2, 3\}$ by [7]. We shall consider $S_3(\mathbb{Q})$. Let $E_k^{(n)} \in M_k(\Gamma_n)_{\mathbb{Q}}$ be the normalized Siegel-Eisenstein series of weight $k$ and degree $n$. Let $X_k \in S_k(\Gamma_2)_{\mathbb{Z}}$ ($k = 10, 12$) be Igusa’s cusp forms normalized as $a \left( \left( \begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix} \right) ; X_k \right) = 1$ in [5]. Then $\bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}(p)} = \mathbb{Z}(p) [E_4^{(2)}, E_6^{(2)}, X_{10}, X_{12}]$ holds for any prime $p \geq 5$ (cf. Nagaoka [10]). Note that $E_k^{(3)} \in \Phi^{-1}_k(E^{(2)}_k)$ for any even $k$. We can construct $F_k \in \Phi^{-1}_k(X_k)$ ($k = 10, 12$) by

$$F_{10} : = -\frac{43867}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 53} (E_{10}^{(3)} - E_4^{(3)} E_6^{(3)}),$$

$$F_{12} : = \frac{131 \cdot 593}{2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 337} (3^2 \cdot 7^2 E_4^{(3)} E_6^{(3)} + 2 \cdot 5^3 E_6^{(3)} E_4^{(3)} - 691 E_{12}^{(3)}).$$

Moreover, we know all possible primes which appear in the denominators of $E_k^{(3)}$ by Böcherer’s results [1]. Hence, it suffices to except all primes in the denominators of the constant factors in (3.3) and all possible primes appearing the denominators of $E_k^{(3)}$ for $k = 4, 6, 10, 12$. In this way, we get

$$S_3(\mathbb{Q}) \subset \{2, 3, 5, 7, 53, 131, 337, 593, 43867\}$$

**Problem 3.12.** For the general degree cases, give an explicit bound $C_n$ such that

$$\max S_n(\mathbb{Q}) < C_n.$$

## 4 Numerical examples

We give some numerical examples of Corollary 3.4 for the case of degree 2. For simplicity, we put $E_k := E_k^{(2)}$. Let $\Delta \in S_{12}(\Gamma_1)$ be Ramanujan’s delta function. We write simply $(m, r, n)$ for $\left( \begin{smallmatrix} n & r \\ \frac{1}{2} & m \end{smallmatrix} \right) \in \Lambda_2$. In the following construction of examples, we apply Sturm type theorem obtained by [2]. In order to prove a congruence between two modular forms of even weight $k$ of degree 2 by using the theorem in [2], it suffices to check the congruences for Fourier coefficients for

$$T = (1, 0, 1), (1, 1, 1) \quad \text{if} \quad 10 \leq k \leq 18,$$

$$T = (1, 0, 1), (1, 0, 2), (1, 1, 1), (1, 1, 2), (2, 0, 2), (2, 1, 2), (2, 2, 2) \quad \text{if} \quad 20 \leq k \leq 28.$$  

The reason is that all Fourier coefficients corresponding to $(n, r, m), (m, r, n), (n, -r, m), (m, -r, n)$ are the same in the case of even weight.
Weight 12

We consider a Hecke eigen form \( f_{12} := 7\Delta \in S_{12}(\Gamma_1) \). Then the Klingen-Eisenstein series \( [f_{12}]_2^1 \) is a mod 7 cusps form. Hence, there exists a cusps form \( F_{12} \in S_{12}(\Gamma_2) \) such that \( [f_{12}]_1^2 \equiv F_{12} \mod 7 \) by Corollary 3.4. In fact, we can confirm this congruence as follows: We set \( F_{12} := X_{12} \in S_{12}(\Gamma_2) \). The following table is of the Fourier coefficients modulo 7 of \([f_{12}]_2^1\) and \(F_{12}\):

| \( T = (m, r, n) \) | \( a(T; [f_{12}]_2^1) \) | \( a(T; F_{12}) \) | modulo 7 |
|-----------------|-----------------|-----------------|----------|
| (1, 0, 1)       | 1242            | 10              | 3        |
| (1, 1, 1)       | 92              | 1               | 1        |

Applying Sturm type theorem mentioned above, we have \([f_{12}]_1^2 \equiv F_{12} \mod 7\).

Weight 16

Let \( a \) be a root of the polynomial \( x^2 - x - 12837 \) and put \( K = \mathbb{Q}(a) \). Since \( \dim S_{16}(\Gamma_1) = 1 \), we can find a unique cusps form \( f_{16} \in S_{16}(\Gamma_1) \) such that \( a(1; f_{16}) = 7^2 \cdot 11 \). If we put \( p = (7, a+4) \), then \([f_{16}]_2^1\) is a mod \( p^2 \) cusps form. There exists a unique normalized Hecke eigen form \( g_{30} \in S_{30}(\Gamma_1) \) such that the eigenvalue is \(-192a + 4416\) for the Hecke operator \( T(2) \). Let \( F_{16} \in S_{16}(\Gamma_2) \) be the Saito-Kurokawa lift of \( g_{30} \) normalized as the table below. Then we have \([f_{16}]_1^2 \equiv F_{16} \mod p^2\). In fact, their Fourier coefficients are given in the following table:

| \( T = (m, r, n) \) | \( a(T; [f_{16}]_2^1) \) | \( a(T; F_{16}) \) | modulo \( p^2 \) |
|-----------------|-----------------|-----------------|----------|
| (1, 0, 1)       | 5394            | \( 80a + 3600 \) | 4        |
| (1, 1, 1)       | 124             | \( 8a + 1248 \)  | 26       |

Applying Sturm type theorem repeatedly, we have \([f_{16}]_1^2 \equiv F_{16} \mod p^2\).

Weight 20

In this case also \( \dim S_{20}(\Gamma_1) = 1 \). Thus there exists a unique cusps form \( f_{20} \in S_{20}(\Gamma_1) \) such that \( a(1; f_{20}) = 11 \cdot 71^2 \). Then \([f_{20}]_1^2\) is a mod \( 71^2 \) cusps form. Let \( F_{20} \in S_{20}(\Gamma_2) \) be the unique Hecke eigen form such that \( F_{20} \) is not Saito-Kurokawa lift. Explicitly, we can write as

\[
F_{20} = 38(E_4 E_6 X_{10} + E_4^2 X_{12} - 1785600 X_{10}^2). 
\]

Then we have \([f_{20}]_1^2 \equiv F_{20} \mod 71^2\). In fact, we can confirm this by the following table and an application of Sturm type theorem:
\[ T = (m, r, n) \quad a(T; [f_{20}]^2) \quad a(T; F_{20}) \quad \text{modulo } 71^2 \]

|           | \( a(T; [f_{20}]^2) \) | \( a(T; F_{20}) \) |
|-----------|----------------------|------------------|
| \( 1, 0, 1 \) | 10386                | 304              |
| \( 1, 0, 2 \) | 1925356716           | 198816           |
| \( 1, 1, 1 \) | 76                   | 76               |
| \( 1, 1, 2 \) | 162929376            | 4256             |
| \( 2, 0, 2 \) | 1238800286736        | -335343616       |
| \( 2, 1, 2 \) | 385264596000         | 278989920        |
| \( 2, 2, 2 \) | 9084897120           | -63912960        |

Weight 22

Since \( \dim S_{22}(\Gamma_1) = 1 \), there exists a unique cusp form \( f_{22} \in S_{22}(\Gamma_1) \) such that \( a(1; f_{22}) = 7 \cdot 13 \cdot 17 \cdot 61 \cdot 103 \). Then \([f_{22}]^2\) is a mod 61 cusp form. Let \( F_{22} \in S_{22}(\Gamma_2) \) be the unique Hecke eigen form such that \( F_{22} \) is not Saito-Kurokawa lift. Explicitly, we can write as

\[
F_{22} = 2 \cdot 3^{-2}(-61E_4^3X_{10} - 5E_6^2X_{10} + 30E_4E_6X_{12} + 80870400X_{10}X_{12}).
\]

Then we have \([f_{22}]^2 \equiv F_{22} \text{ mod } 61\). In fact, we can confirm this by the following table and Sturm type theorem:

|           | \( a(T; [f_{22}]^2) \) | \( a(T; F_{22}) \) |
|-----------|----------------------|------------------|
| \( 1, 0, 1 \) | -179610              | 96               |
| \( 1, 0, 2 \) | -133169475780        | -1728            |
| \( 1, 1, 1 \) | -740                 | -8               |
| \( 1, 1, 2 \) | -8620265280          | -10752           |
| \( 2, 0, 2 \) | 54428790246720       | -313368576       |
| \( 2, 1, 2 \) | 15093047985984       | 142287360        |
| \( 2, 2, 2 \) | 223472730240         | 17725440         |

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