SOME RESULTS ON THE BEHAVIOUR OF TRANSFER FUNCTIONS AT THE RIGHT HALF PLANE

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Abstract. In this paper, an inequality for a transfer function is obtained assuming that its residues at the poles located on the imaginary axis in the right half plane. In addition, the extremal function of the proposed inequality is obtained by performing sharpness analysis. To interpret the results of analyses in terms of control theory, root-locus curves are plotted. According to the results, marginally and asymptotically stable transfer functions can be determined using the obtained extremal function in the proposed theorem.

1. Introduction. Positive real functions (PRF) have several application fields in electrical engineering [15, 13, 7, 3, 4, 9, 12, 16]. In this study, PRFs will be considered in aspect of control systems theory. Positive real function concept can be considered as one of the fundamental elements in control theory since it is used to represent linear time variant (LTI) control systems [1]. As a fundamental tool used in control systems, several studies are presented in control theory literature on positivity and strictly positivity of control systems. As one of the recent studies, in [6], positive realness and negative imaginary is used to test if a system is state-space symmetric or not. In another study, positive realness is considered in aspect of absolute stability theorem where wide positive realness (WPR) and wide strict positive realness (WSPR) concepts are introduced [17]. In addition to these studies, we can easily increase number of studies on positive realness that can be possibly encountered within control theory [14, 2, 10]. Besides this, there are also many other fields which positive real functions are used [1], e.g.

- Design of passive filters
- Switching systems
- Absolute stability theory
- Circuit synthesis, etc.

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In this study, we aim to understand the behaviour of control systems with PRFs defined in the right half plane. Accordingly, we propose an inequality by assuming that there are \( n \) poles on the imaginary axis as the transfer function \( H(s) \) is in the structure of \( H(s) = \frac{b_0}{s-s_i} + c_0 + c_1 (s-s_i) + c_2 (s-s_i)^2 + \ldots \). The equality case of the inequality presented in the theorem is also derived by carrying out the sharpness analysis. Since the poles of the obtained extremal function exist on the imaginary axis, it is possible to comment out that the proposed analysis results a control system with a marginally and asymptotically stable transfer functions.

Here, the behaviour of the transfer function, \( H(s) \) is considered for the poles which are located on the imaginary axis of the \( s \)-plane. In this consideration, the real part of \( H(s) \) is evaluated from below. At this point, it is worth to note that the obtained transfer function is a natural result of the considered problem. In other words, the obtained transfer function (which is actually a function that gives the equality case of inequality (1.1)) is not an arbitrary function but it is an intuitive result generated by the consider problem. Since the poles of the obtained extremal function are on the imaginary axis of the \( s \)-plane, it is possible to comment that the performed analyses here take us to marginally stable transfer functions. As it is emphasized before, the obtained transfer function has not been arbitrarily selected as increasing the significance of the presented analysis.

This study can be considered as an exemplary application of complex analysis to marginally stable control systems. The contribution of the paper is two-fold such that a novel approach is presented for the design of marginally stable transfer functions of control theory. In addition, presented theorem has significant mathematical inferences in terms of complex analysis. It is possible to consider the presented work as a novel approach for establishment of connection between complex analysis and control theory.

Before passing to main theorem, some preliminary knowledge for PRFs and some fundamental results to be used in proof of the presented theorem will be given [5, 8].

Basically, a function can be qualified as ‘positive real’, if it satisfies the following conditions [11]:

1-) \( H(s) \) is analytic and single valued in \( \Re s \geq 0 \) except possibly for poles on the axis of imaginaries.

2-) \( H(\pi) = H(s) \)

3-) \( \Re H(s) \geq 0 \), in \( \Re s \geq 0 \).

In addition to these conditions, all poles of \( H(s) \) are in the closed left half plane of the \( s \)-plane, i.e. inside the left half plane or on the \( jw \) axis. Accordingly, all \( jw \)-axis poles must be simple, with positive real residues. Since \( s = 0 \) and \( s \to \infty \) lie on the \( jw \) axis, this holds for poles at the origin and at infinity.

Some other fundamental results are given as follows [5, 8]:

If \( f(z) = u(z) + iv(z), \ z = x + iy \) is an arbitrary function, analytic for \( x > 0 \), with a nonnegative real part, then for \( |y| < kx, x \to \infty \), the quotient \( \frac{f(z)}{z} \) has a real, nonnegative limit \( c \), which is equal to

\[
\inf_{x>0} \frac{u(x + iy)}{x}.
\]

In other word if \( f(z) \) is any analytic function taking \( \mathcal{H}^+ = \{ z \in \mathbb{C} : \Re z > 0 \} \) into itself, then \( \frac{f(z)}{z} \) has a non-tangential limit \( c \) as \( z \to \infty \), where furthermore
2. Main results. In this section, we present an inequality for $H(s)$ assuming that it satisfies the positive realness conditions given above with all poles are in the closed left half $s$-plane. It is also assume that transfer function $H(s)$ is in the structure of:

$$H(s) = \frac{\alpha_1}{s-s_i} + c_0 + c_1 (s - s_i) + c_2 (s - s_i)^2 + ...$$

The extremal function of the inequality is obtained by performing sharpness analysis.

**Theorem 2.1.** Let $H(s)$ be a positive real function. Also, let $s_1, s_2, ..., s_n$ be poles on the imaginary axis such that $H(s) = \frac{\alpha_1}{s-s_i} + c_0 + c_1 (s - s_i) + c_2 (s - s_i)^2 + ...$

Then

$$\sum_{i=1}^{n} \frac{\alpha_i}{|s-s_i|^2} \leq \Re H(s). \quad (1.1)$$

This inequality is sharp for the function given by

$$H(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s-s_i} + i\beta,$$

where $\beta$ is positive real number.

**Proof.** Let $s_1, s_2, ..., s_n$ be simple poles on the imaginary axis. Therefore, we have

$$H(s) = \frac{\alpha_1}{s-s_i} + c_0 + c_1 (s - s_i) + c_2 (s - s_i)^2 + ...$$

Assume that $H : \mathcal{H}^+ \to \mathcal{H}^+$. Then, from the definition positive real function $H \left( \frac{1}{z} \right) : \mathcal{H}^+ \to \mathcal{H}^+$ and

$$\lim_{z \to \infty} \frac{H \left( \frac{1}{z} \right)}{z} = \alpha = \inf_{\Re z > 0} \frac{\Re H \left( \frac{1}{z} \right)}{z}.$$

If we denote $s = \frac{1}{z}$, we take

$$\lim_{s \to 0} \frac{H(s)}{s} = \alpha = \inf_{\Re s > 0} \frac{\Re H(s)}{s}$$

and

$$\lim_{s \to 0} s H(s) = \alpha.$$

Therefore, we obtain

$$\frac{\Re H(s)}{\Re \left( \frac{1}{s} \right)} \geq \alpha,$$

$$\Re H(s) \geq \alpha \Re \left( \frac{1}{s} \right)$$

and

$$\Re \left( H(s) - \alpha \frac{1}{s} \right) \geq 0.$$

Now, let $z = \frac{1}{s-s_i}$. If $\Re s > 0$, then we have

$$\Re \left( \frac{1}{s-s_i} \right) = \Re \left( \frac{s - s_i}{|s-s_i|^2} \right) = \frac{\Re s}{|s-s_i|^2} > 0.$$
since $\Re s_i = 0$. Also, from $z = \frac{1}{s-s_i}$,

$$H(s) = H\left(\frac{1}{z} + s_i\right): \mathcal{H}^+ \to \mathcal{H}^+$$

is an analytic function and since

$$|\arg(s-s_i)| < \beta < \frac{\pi}{2},$$

then

$$\left| - \arg\left(\frac{1}{s-s_i}\right) \right| < \beta \text{ and } |\arg z| < \beta.$$

Thus, we obtain

$$\lim_{z \to \infty} \frac{H\left(\frac{1}{z} + s_i\right)}{z} = \alpha = \inf_{\Re z > 0} \frac{\Re H\left(\frac{1}{z} + s_i\right)}{\Re z},$$

$$\lim_{s \to s_i} \frac{H(s)}{s-s_i} = \alpha = \inf_{\Re s > 0} \frac{\Re H(s)}{\Re \left(\frac{1}{s-s_i}\right)}$$

and

$$\lim_{s \to s_i} (s-s_i) H(s) = \alpha = \inf_{\Re s > 0} \frac{\Re H(s)}{\Re \left(\frac{1}{s-s_i}\right)}.$$}

Therefore, we take

$$\frac{\Re H(s)}{\Re \left(\frac{1}{s-s_i}\right)} \geq \alpha,$$

$$\Re H(s) \geq \alpha \Re \left(\frac{1}{s-s_i}\right)$$

and

$$\Re \left( H(s) - \frac{\alpha}{s-s_i} \right) \geq 0.$$}

Let $H : \mathcal{H}^+ \to \mathcal{H}^+$ be an analytic function and $\lim H(x+s_i) = \infty$ at the points $s_i, i = 1, 2, .., n$. If $\lim_{x \to 0} H(x+s_i)x = \alpha_i$, then $\alpha_i \geq 0$ and

$$\Re \left( H(s) - \sum_{i=1}^{n} \alpha_i \frac{1}{s-s_i} \right) \geq 0, \forall s \in \mathcal{H}^+. \quad (1.2)$$

It is clear that we obtain the inequality (1.2) by induction, since the function $H(s) - \alpha_i \frac{1}{s-s_i}$ has the poles at $s_m, m \neq i$. From the inequality (1.2), we obtain

$$\sum_{i=1}^{n} \frac{\alpha_i \Re s}{|s-s_i|^2} \leq \Re H(s).$$

If the equality is satisfied at any point $s_0 \in \mathcal{H}^+$, then according to the maximum principle,

$$\Re \left( H(s) - \sum_{i=1}^{n} \alpha_i \frac{1}{s-s_i} \right) \equiv 0.$$}

From here, we get

$$H(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s-s_i} + i\beta,$$

where $\beta$ is positive real number.
Some Results on the Behaviour of Transfer Functions

Figure 1. Root-locus curves for the transfer function $H(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s - s_i} + i\beta$. It is assumed that $\alpha_i$’s equal to 1 and $\beta$ is zero.

The figures are presented for different $n$ values: (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, (d) $n = 4$.

For simulation analysis, the obtained general transfer function in the theorem, $H(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s - s_i} + i\beta$, can be subjected to root-locus analysis. For simplicity, assume that $\beta = 0$. Then, the general transfer function takes the following form:

$$H(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s - s_i}.$$ 

Without loss of generality, assume that $\alpha_i = 1$, $i = 1, \ldots, n$. $s_i$ coefficients can be assigned to make all the poles to be located on the imaginary axis.

As an exemplary application of the proposed theorem, root-locus curves of transfer functions for $n = 1, \ldots, 4$ are considered. For different $n$ values, the corresponding transfer functions can be determined one-by-one.

Firstly, it can be easily seen that $H_{n=1}(s) = \frac{1}{s}$ for $n = 1$ when it is assumed that $s_1 = 0$ and $\alpha_1 = 1$. For $n = 2$ case, assuming that $s_1 = i$ and $s_2 = -i$ with $\alpha_1 = \alpha_2 = 1$, following transfer function is obtained:

$$H_{n=2}(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s - s_i} = \frac{1}{s + i} + \frac{1}{s - i} = \frac{2s}{s^2 + 1}.$$
Similar analyses can be carried out for $n = 3$ and $n = 4$. Assuming that $s_1 = 0$, $s_2 = i$, $s_3 = -i$, and $s_1 = i$, $s_2 = -i$, $s_3 = 2i$, $s_4 = -2i$ for $n = 3$ and $n = 4$, respectively, following transfer functions are obtained:

$$H_{n=3}(s) = \frac{3s^2 + 1}{s^3 + s}, n = 3$$

$$H_{n=4}(s) = \frac{4s^3 + 10s}{s^4 + 5s^2 + 4}, n = 4$$

Root-locus curves are given in Fig. 1a through 1d for each transfer function $H_{n=1}(s)$ through $H_{n=4}(s)$, respectively. Without examining the figures, we can say that all systems are marginally stable since all of them have their poles on imaginary axis. It is possible to infer from these sub-figures that the system complexity increased with higher values of $n$ as expected. It is possible to say that all transfer functions are asymptotically stable since all of them have a zero for $s \to \infty$. Accordingly, it is possible to conclude as the system transfer functions resulted with the proposed theorem show marginally and asymptotically stable properties.

3. Conclusion. In this study, positive real functions are considered in aspect of control theory. Assuming that the transfer function $H(s)$ is in the structure of $H(s) = \frac{c_0 + c_1(s-s_1) + c_2(s-s_1)^2 + \cdots}{s^n}$ an inequality is proposed and the extremal function is obtained via sharpness analysis. Using this extremal function as the transfer function $H(s)$, root-locus analysis is carried out. It is observed that marginally and asymptotically stable control systems can be designed using the results of the theorem presented in this study.

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