On the growth of the $L^p$ norm of the Riemann zeta-function on the line $\text{Re}(s) = 1$

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Abstract

We prove that if $\delta > 0$ and $p$ is real then

$$\sup_T \int_T^{T+\delta} |\zeta(1 + it)|^p dt < \infty,$$

if and only if $-1 < p < 1$. Furthermore, we show the omega estimates

$$\int_T^{T+\delta} |\zeta(1 + it)|^{\pm 1} dt = \Omega(\log \log T),$$

$$\int_T^{T+\delta} |\zeta(1 + it)|^{\pm p} dt = \Omega((\log \log T)^{p-1}), \quad (p > 1)$$

which with the exception of an additional $\log \log \log T$ factor in the second estimate coincides with conditional (under the Riemann hypothesis) order estimates. We also prove weaker unconditional order estimates.

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1 Classical order and omega estimates

The study of the Riemann zeta-function on the line \( \text{Re}(s) = 1 \) has been studied by a lot of authors, starting with the work of Hadamard [8] and de la Vallée-Poussin [6] who proved that \( \zeta(1 + it) \neq 0 \) which implies the prime number theorem. Assuming the Riemann hypothesis, Littlewood [10] showed that

\[
\zeta(1 + it) \ll \log \log t, \quad \zeta(1 + it)^{-1} \ll \log \log t.
\] (1)

Bohr and Landau [3, 4, 5] proved the corresponding omega-estimates

\[
\zeta(1 + it) = \Omega(\log \log t), \quad \zeta(1 + it)^{-1} = \Omega(\log \log t),
\] (2)

unconditionally, so Littlewood's conditional bound is the best possible. The best unconditional bound are the estimates

\[
\zeta(1 + it) \ll (\log t)^{2/3}, \quad \zeta(1 + it)^{-1} \ll (\log t)^{2/3}(\log \log t)^{1/3},
\] (3)

of Vinogradov [13] and Korobov [9]. For a discussion of these results as well as the current record, see the recent paper of Granville-Soundararajan [7].

2 The \( L^p \) norm in short intervals

2.1 Bounds from below

A related question which has been less studied is the question of the \( p' \)th moment of the Riemann zeta-function in short intervals. What can we say about

\[
\int_T^{T+\delta} |\zeta(1 + it)|^p \, dt?
\] (4)

One of our recent results [1, Theorem 7] is the following.
**Theorem 1.** We have the following estimates for the $L^p$ norm, for $p > 0$ of the zeta-function and its inverse in short intervals:

\[
(i) \inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |\zeta(1+it)|^p dt \right)^{1/p} = \frac{\pi^2 e^{-\gamma}}{24} \delta + O(\delta^3),
\]

\[
(ii) \inf_T \left( \frac{1}{\delta} \int_T^{T+\delta} |\zeta(1+it)|^{-p} dt \right)^{1/p} = \frac{e^{-\gamma}}{4} \delta + O(\delta^3),
\]

for $\delta > 0$. Furthermore, both estimates are valid if $\inf$ is replaced by $\lim \inf_{T \to \infty}$, and if $1+it$ is replaced by $\sigma+it$ and the infimum is also taken over $\sigma > 1$.

This result gives lower estimates for this integral. In particular it shows that the infimum is strictly positive and thus gives an analogue of Hadamard and de la Vallée Poussin’s result for the non vanishing of the Riemann zeta-function on the line $\text{Re}(s) = 1$.

As discussed in [1] this can be applied to the question of universality on the 1-line. It should be noted that in a surprise turn of events [2] we recently managed to prove that a Voronin universality type result in fact do hold on the line $\text{Re}(s) = 1$ if we in addition to vertical shift allow scaling in the argument and adding a positive constant in the range.

### 2.2 Bounds from above

Another question is whether we similarly as Littlewood’s and Bohr’s results can obtain omega, and order results for the quantity in (4)? This is the topic for the current paper. The first question regarding this is whether for some $p > 0$ this is bounded. This is answered in the following theorem

**Theorem 2.** We have for real $p$ that

\[
\sup_T \int_T^{T+\delta} |\zeta(1+it)|^p dt < \infty
\]

if and only if $-1 < p < 1$. Furthermore $\sup_T$ can be replaced by $\lim \sup_T$ and the same result holds.

For $-1 < p < 1$ this implies similar results on the non universality of the Riemann zeta-function on the 1-line as Theorem 1 as we proved in [1]. More precisely Theorem 2 for $0 < p < 1$ gives an upper bound $M$ for the $L^p$-norm of the functions $\zeta_T(t) := \zeta(1+iT+it)$ on the interval $[0, \delta]$. Thus the zeta-function can not approximate any function $f$ with $L^p$ norm strictly greater than $M$. Thus this gives another
proof of the fact that the usual Voronin universality theorem does not extend to the line Re(s) = 1.

It is reasonable to expect that $T = -\delta/2$ should maximize this integral for $p > 0$. It is clear that this would imply Theorem 1 for positive values of $p$, since the integral with this value of $T$ is divergent exactly when $p \geq 1$. We will not prove this, but rather leave it as an open problem. However we will manage to prove the corresponding result for the following related integral

$$\sup_{T} \int |\zeta(1+it)|^p \theta\left(\frac{t-T}{\delta}\right) dt \tag{6}$$

whenever the Fourier transform

$$\hat{\theta}(\tau) = \int_{-\infty}^{\infty} e^{-2\pi i x} \theta(x) dx \tag{7}$$

is non negative. For the purpose of this paper we will choose the triangular function

$$\theta(x) = \begin{cases} 1 - |x| , & |x| \leq 1 , \\ 0 , & |x| > 1 . \end{cases} \tag{8}$$

For this integral kernel it is well known that its Fourier transform

$$\hat{\theta}(\tau) = \frac{(\sin \pi \tau)^2}{\pi^2 \tau^2} \tag{9}$$

is non negative$^1$.

### 2.2.1 Integral kernels with non negative Fourier transforms and Dirichlet series

Before starting to prove Theorem 1 and Theorem 2 we prove some more general lemmas

**Lemma 1.** Let

$$L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$$

be a Dirichlet series absolutely convergent on Re(s) = $\sigma$, where $a(n) = |a(n)|b(n)$ and where $b(n)$ is a completely multiplicative arithmetical function. Then

$$\limsup_{T \to \infty} \sup_{-\delta \leq t \leq T} \left|\int_{-\delta}^{\delta} |L(\sigma + it)|^2 (\delta - |t|) dtight| = \int_{-\delta}^{\delta} \left|\tilde{L}(\sigma + it)|^2 (\delta - |t|) dt, \tag{10}$$

$^1$This is essentially the Fourier transform of the Fejér kernel
where

\[ \tilde{L}(s) = \sum_{n=1}^{\infty} |a(n)| n^{-s}, \]

and where \( \limsup_{T \to \infty} \) may be replaced by \( \sup_T \).

**Proof.** By using (7), (8), the fact that \( \hat{\theta}(x) \geq 0 \) is non negative and the triangle inequality we see that

\[
\int_{T-\delta}^{T+\delta} |L(\sigma + it)|^2 (\delta - |t|) dt 
\]

\[
= \sum_{m,n=1}^{\infty} \frac{a(n)a(m)}{(nm)^\sigma} \int_{T-\delta}^{T+\delta} \left( \frac{n}{m} \right)^{-it} \delta \theta \left( \frac{t}{\delta} \right) dt, 
\]

\[
= \sum_{m,n=1}^{\infty} \frac{a(n)a(m)}{(nm)^\sigma} \left( \frac{n}{m} \right)^{iT} \int_{-\delta}^{\delta} e^{-i\log \left( \frac{n}{m} \right)} \delta \theta \left( \frac{t}{\delta} \right) dt, 
\]

\[
= \delta^2 \sum_{m,n=1}^{\infty} \frac{|a(n)||a(m)|}{(nm)^\sigma} \delta \left( \frac{\delta}{2\pi} \log \frac{n}{m} \right), 
\]

\[
\leq \delta^2 \sum_{m,n=1}^{\infty} \frac{|a(n)||a(m)|}{(nm)^\sigma} \delta \left( \frac{\delta}{2\pi} \log \frac{n}{m} \right), 
\]

\[
= \int_{-\delta}^{\delta} |L(\sigma + it)|^2 (\delta - |t|) dt. 
\]

By Kronecker's theorem we may choose \( T \) such that

\[
|b(P) - P^{-iT}| < \epsilon, \quad (P \text{ prime, } P < N_0), 
\]

and by choosing \( \epsilon \) sufficiently small and \( N_0 \) sufficiently large it follows by the fact that \( L(s) \) is absolutely convergent on the line \( \text{Re}(s) = \sigma \) that (10) may be as close to

\[
\delta^2 \sum_{m,n=1}^{\infty} \frac{|a(n)||a(m)|}{(nm)^\sigma} \delta \left( \frac{\delta}{2\pi} \log \frac{n}{m} \right) = \int_{-\delta}^{\delta} |L(\sigma + it)|^2 (\delta - |t|) dt 
\]

as we wish. \( \square \)

From this Lemma we obtain the following result for the Riemann zeta-function.
Lemma 2. Let \( \sigma > 1 \) and \( p \geq 0 \). Then

\[
(i) \quad \sup_T \int_{T-\delta}^{T+\delta} |\zeta(\sigma + it)|^p (\delta - |t|)dt = \int_{-\delta}^{\delta} |\zeta(\sigma + it)|^p (\delta - |t|)dt.
\]

\[
(ii) \quad \sup_T \int_{T-\delta}^{T+\delta} \left| \frac{\zeta(2\sigma + 2it)}{\zeta(\sigma + it)} \right|^p (\delta - |t|)dt = \int_{-\delta}^{\delta} |\zeta(\sigma + it)|^p (\delta - |t|)dt.
\]

Furthermore \( \sup_T \) may be replaced by \( \lim \sup_T \).

Proof. In general we have the following equality for \( \text{Re}(s) > 1 \)

\[
\zeta(s)^p = \prod_{p \text{ prime}} (1 - P^{-s})^{-p} = \sum_{n=1}^{\infty} d_p(n)n^{-s}.
\]

for any real number \( p \), where \( d_p(n) \) denote the generalized divisor function. When \( p \geq 0 \) it follows that \( d_p(n) \) is non negative by the fact that \( d_p(n) \) is a multiplicative function and from the fact that

\[
(1 - P^{-s})^{-p} = \sum_{k=0}^{\infty} \binom{-p}{k} (-1)^k p^{-ks},
\]

where

\[
\binom{-p}{k} (-1)^k = \prod_{j=1}^{k} \frac{p + j - 1}{j} \geq 0.
\]

Thus we may apply Lemma 1 to obtain the first part of Lemma 2. To prove the second part we use the following equality for \( \text{Re}(s) > 1 \)

\[
\left( \frac{\zeta(2s)}{\zeta(s)} \right)^p = \prod_{p \text{ prime}} (1 + P^{-s})^{-p} = \sum_{n=1}^{\infty} \lambda(n)d_p(n)n^{-s}.
\]

where \( \lambda(n) = (-1)^\nu(n) \) and where \( \nu(n) \) counts the number of prime factors (with multiplicity) of \( n \). Since \( a(n) \) is the product of a non negative function \( d_p(n) \) and a completely multiplicative function \( \lambda(n) \), Lemma 2 (ii) follows from Lemma 1. \( \square \)

2.2.2 A stronger upper bound for \(-1 < p < 1\).

We will prove a somewhat stronger result than Theorem 2 in the case \(-1 < p < 1\).
Theorem 3. Assume that \(-1 < p < 1\) and \(\delta > 0\). Then

\[0.3 \frac{\delta^{1-|p|}}{1-|p|} \leq \limsup_{T \to \infty} \int_T^{T+\delta} |\zeta(1+it)|^p dt \leq 12 \frac{\delta^{1-|p|}}{1-|p|} + 6\delta(1+\log(1+\delta)).\]

Proof. Let \(0 \leq q < 1\). From Lemma 2, by letting \(\sigma \to 1^+\) and using continuity, it follows that

\[I\left(\frac{\delta}{2}, q\right) \leq \limsup_{T \to \infty} \int_T^{T+\delta} |\zeta(1+it)|^q dt \leq 2I(\delta, q), \tag{13}\]

and

\[I\left(\frac{\delta}{2}, q\right) \leq \limsup_{T \to \infty} \int_T^{T+\delta} \left|\frac{\zeta(2+2it)}{\zeta(1+it)}\right|^q dt \leq 2I(\delta, q), \tag{14}\]

where

\[I(\delta, q) := \int_{-\delta}^{\delta} |\zeta(1+it)|^q \left(1 - \frac{|t|}{\delta}\right) dt.\]

By the Laurent expansion at \(s = 1\) of the Riemann zeta-function and the bound \(|\zeta(1+it)| \leq 1 + \log(|t|+1)\) valid for \(|t| \geq 1\) we have that

\[|t|^{-q} \leq |\zeta(1+it)|^q \leq |t|^{-q} + 1 + \log(|t|+1), \quad (0 \leq q \leq 1),\]

and it follows by integrating this inequality that

\[\frac{2\delta^{1-q}}{(1-q)(2-q)} \leq I(\delta, q) \leq \frac{2\delta^{1-q}}{(1-q)(2-q)} + \delta(1+\log(\delta+1)). \tag{15}\]

Our lemma follows for \(p = q \geq 0\) from (13) and (15) with the somewhat stronger lower constant \(0.5\) (rather than 0.3) and upper constant 4 (rather than 12). If \(-1 < p < 0\) we let \(q = -p\) and our results follows from (14), (15) and the inequality

\[\frac{1}{3} < |\zeta(2+2it)| < \frac{5}{3}, \quad (t \in \mathbb{R}). \tag{16}\]

3 Order estimates and Omega estimates

For the final proof of Theorem 2 we need that the \(p\)th moment of the Riemann zeta-function in short intervals on the 1-line is unbounded for \(|p| \geq 1\). While this in fact follows from Theorem 3 when \(p \to 1^-\) we are interested in obtaining more precise Omega-estimates and Order estimates that answers the question on how fast such integrals might grow.
3.1 Order estimates

**Theorem 4.** Assuming the Riemann hypothesis then for any $\delta > 0$ we have

$$\int_T^{T+\delta} |\zeta(1 + it)|^{\pm 1} dt = O(\log \log \log T).$$

**Proof.** We have that

$$\int_T^{T+\delta} |\zeta(1 + it)|^{\pm 1} dt \leq \sup_{t \in [T, T+\delta]} |\zeta(1 + it)|^{1-p} \int_T^{T+\delta} |\zeta(1 + it)|^{\pm p} dt.$$  

The result follows from choosing $p = 1 - 1/\log \log T$, invoking Littlewood’s bound (1) on the first part and using Theorem 3 on the remaining integral.

**Theorem 5.** For any $\delta > 0$ we have

$$\int_T^{T+\delta} |\zeta(1 + it)|^{\pm 1} dt = O(\log \log T).$$

**Proof.** This follows from Theorem 3 in the same way as Theorem 4, but by choosing $p = 1 - 1/\log \log T$ and invoking Vinogradov-Korobov’s estimate (3) in view of Littlewood’s.

**Theorem 6.** Assuming the Riemann hypothesis, then for any $\delta > 0$ and $p > 1$ the following bound holds true

$$\int_T^{T+\delta} |\zeta(1 + it)|^{\pm p} dt = O((\log \log T)(\log \log T)^{p-1}).$$

**Proof.** We have that

$$\int_T^{T+\delta} |\zeta(1 + it)|^{p} dt \leq \sup_{t \in [T, T+\delta]} |\zeta(1 + it)|^{p-1} \int_T^{T+\delta} |\zeta(1 + it)|^{\pm p} dt.$$  

The result follows from Theorem 4 and Littlewood’s bound (1).

**Theorem 7.** For any $\delta > 0$ and $p > 1$ the following bound holds true

$$\int_T^{T+\delta} |\zeta(1 + it)|^{\pm p} dt = O((\log \log T)(\log T)^{2/3(p-1)}).$$

**Proof.** This follows from Theorem 5 in the same way as Theorem 6 follows by Theorem 4 by invoking Vinogradov-Korobov’s estimate (3) in view of Littlewood’s bound (1).
3.2 Omega estimates

We have the following Omega estimates

**Theorem 8.** We have for any fixed \( \delta > 0 \) that

(i) \[ \int_{T}^{T+\delta} |\zeta(1 + it)|^{\pm 1} \, dt = \Omega(\log \log T), \]

(ii) \[ \int_{T}^{T+\delta} |\zeta(1 + it)|^{\pm p} \, dt = \Omega((\log \log T)^{p-1}), \quad (p > 1). \]

We would like to remark that Theorem 8 with \( p = 2 \) answers a question of Weber [12, Problem 6.4] in the case \( \sigma = 1 \) in the affirmative.\(^2\)

**Proof.** Again we are going to use a convolution. Since it is more convenient to have the compact support on the sum side we will consider convolution by the Fourier transform of the triangular function (and higher order convolutions of the triangular function). Define recursively

\[ \theta_1(x) = \theta(x) \] (17)

\[ \theta_n(x) = (\theta_{n-1} \ast \theta)(x) = \int_{-\infty}^{\infty} \theta_{n-1}(t) \theta(x - t) \, dt \] (18)

It is clear that \( 0 \leq \theta_n(x) \leq 1 \) is a continuous function with support on \([-n, n]\) and by (9) it is clear that its Fourier transform satisfies

\[ \hat{\theta}_n(t) = (\hat{\theta}(t))^n = \left( \frac{\sin \pi t}{\pi t} \right)^{2n}. \] (19)

Consider

\[ \zeta_{n,N}^p(s) := \int_{-\infty}^{\infty} \left( \zeta \left( s + i \frac{x}{N} \right) \right)^p \hat{\theta}_n(x) \, dx, \quad \text{Re}(s) > 1 \] (20)

and

\[ Z_{n,N}^p(s) := \int_{-\infty}^{\infty} \left( \frac{\zeta(2s + 2i \frac{x}{N})}{\zeta(s + i \frac{x}{N})} \right)^p \hat{\theta}_n(x) \, dx, \quad \text{Re}(s) > 1 \] (21)

where the functions are defined by continuous extension when \( \text{Re}(s) = 1 \) and where \( \hat{\theta}_n(x) \) is given by (19). In particular \( \zeta_{n,N}^p(s) \) is a smoothed version of the \( p \)'th power.

\(^2\)Ramūnas Garunkščis remarked that the case \( 1/2 < \sigma < 1 \) in Weber’s problem follows as a direct consequence of the Voronin universality theorem.
of usual Riemann zeta-function. From now on choose $n > p/2$ to be an integer. It follows from the convolution (20) and the Laurent expansion of the zeta-function at $s = 1$ that

$$\left| \zeta_{n,N}^p(1 + it) \right| = t^{-p} \left( 1 + O(t) + O((Nt)^{p-2n}) \right), \quad (N^{-1} \leq t \leq 1).$$

Thus, since $p < 2n$, by calculus, we have for fixed $\delta > 0$ and $p \geq 1$ that

$$\int_{0}^{\delta} \left| \zeta_{n,N}^p(1 + it) \right| dt \gg \begin{cases} \log N + O(1), & p = 1, \\ N^{p-1}, & p > 1. \end{cases} \tag{22}$$

By (20), (21) and (8) we have the Dirichlet series expansions

$$\zeta_{n,N}^p(s) = \sum_{j=1}^{[\exp(nN)]} \frac{d_p(j)}{j^s} \theta_n \left( \frac{\log j}{N} \right), \tag{23}$$

and

$$Z_{n,N}^p(s) = \sum_{j=1}^{[\exp(nN)]} \frac{d_p(j) \lambda(j)}{j^s} \theta_n \left( \frac{\log j}{N} \right), \tag{24}$$

By (19), (20), (21) and the triangle inequality it follows for $T \geq T_0$ sufficiently large and $N \geq 1$ that

$$0.1 \int_{T}^{T + \delta} \left| \zeta_{n,N}^p(1 + it) \right| dt \leq \max_{T/2 \leq X \leq 2T} \int_{X}^{X + \delta} \left| \zeta(1 + it) \right|^p dt, \tag{25}$$

and

$$0.1 \int_{T}^{T + \delta} \left| Z_{n,N}^p(1 + it) \right| dt \leq \max_{T/2 \leq X \leq 2T} \int_{X}^{X + \delta} \left| \frac{\zeta(2 + 2it)}{\zeta(1 + it)} \right|^p dt. \tag{26}$$

Thus it is sufficient to bound the left hand side of (25). By Dirichlet’s approximation theorem there exists for each $N > 0$ some

$$0 \leq T_N \leq N^{4\pi(\exp(nN))}, \tag{27}$$

where $\pi(\exp(nN))$ here denote the number of primes less than $\exp(nN)$, such that

$$\text{dist} \left( \frac{T_N \log P}{2\pi}, Z \right) < N^{-4}, \quad (P \text{ prime}, 2 \leq P \leq \exp(nN)). \tag{28}$$

The integrals (20) and (21) should here be interpreted as the limit when $s \to 1^+$. 
It follows from (28) that
\[ |j^{iT_N} - 1| < N^{-2}, \quad (1 \leq j \leq \exp(nN)) \] (29)

For such a \( T_N \) it follows from (23) that
\[ \left| \frac{\zeta_{n,N}^p(1 + iT_N + it)}{\zeta_{n,N}^p(1 + it)} \right| \leq N^{-1}, \quad (t \in \mathbb{R}) \] (30)

By noticing that it follows from (27) that \( \log \log T_N \ll N \) and the inequalities (22), (25) and (30) it follows that
\[ \max_{T_N/2 \leq X \leq 2T_N} \int_X^{X+\delta} |\zeta(1 + it)|^p dt \gg \begin{cases} \log \log \log T_N, & p = 1, \\ (\log \log T_N)^{p-1}, & p > 1, \end{cases} \]

which gives our result for \( p \geq 1 \). For negative moments \( p \leq -1 \) we need some corresponding result for \( Z_{n,N}^p(s) \). It is not sufficient to use the Dirichlet approximation theorem directly and we need some effective variant of the Kronecker approximation theorem. Bohr and Landau \([3, 5]\) proved that
\[ |P^{iT} + 1| < \frac{1}{M}, \quad (1 \leq P \leq M, P \text{ prime}), \] (31)

holds for some \( 0 < T < \exp(M^6) \). By using \( M = \lfloor \exp(nN) \rfloor \) and \( T_N = T \) it follows from (24) that
\[ \left| \frac{Z_{n,N}^p(1 + iT_N + it)}{\zeta_{n,N}^{-p}(1 + it)} \right| \leq 1, \quad (t \in \mathbb{R}) \] (32)

holds for some \( 0 \leq T_N \leq \exp(6nN) \). Thus by combining (22), (26), (27), (16) and (32) it follows that
\[ \max_{T_N/2 \leq X \leq 2T_N} \int_X^{X+\delta} |\zeta(1 + it)|^{-p} dt \gg \begin{cases} \log \log \log T_N, & p = 1, \\ (\log \log T_N)^{p-1}, & p > 1. \end{cases} \]

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4 Proof of Theorem 2

Theorem 2 follows from Theorem 3 and Theorem 8.

\footnote{The worst case is when \( n \) is a power of 2}

\footnote{see also the discussion in \([11, p.182]\)}

\footnote{Most of the results of this paper were developed in 2011-2013 in an unfinished manuscript.}
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