A quantitative version of Catlin-D’Angelo–Quillen theorem

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Abstract Let $f(z, \bar{z})$ be a positive bi-homogeneous hermitian form on $\mathbb{C}^n$, of degree $m$. A theorem proved by Quillen and rediscovered by Catlin and D’Angelo states that for $N$ large enough, $(z, \bar{z})^N f(z, \bar{z})$ can be written as the sum of squares of homogeneous polynomials of degree $m + N$. We show this works for $N \geq C_f((n + m) \log n)^3$ where $C_f$ has a natural expression in terms of coefficients of $f$, inversely proportional to the minimum of $f$ on the sphere. The proof uses a semiclassical point of view on which $1/N$ plays a role of the small parameter $h$.

1 Introduction and main result

Let $f = f(z, \bar{z})$ be a bi-homogeneous form of degree $m \geq 1$ on $\mathbb{C}^n$:

$$f(z, \bar{z}) := \sum_{|\alpha| = |\beta| = m} c_{\alpha \beta} z^\alpha \bar{z}^\beta, \quad z \in \mathbb{C}^n, \quad c_{\alpha \beta} \in \mathbb{C}. \quad (1.1)$$

Here $n \geq 2$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

The following theorem was proved by Quillen [9], and rediscovered by Catlin and D’Angelo [2]:

**Theorem 1** Suppose $f$ is given by (1.1) and that

$$f(z, \bar{z}) > 0, \quad z \neq 0.$$
Then there exists $N_0$ such that for $N > N_0$

\[
\|z\|^{2N} f(z, \bar{z}) = \sum_{j=1}^{d_N} |P_j^N(z)|^2, \quad \|z\|^2 := \sum_{j=1}^{n} |z_j|^2,
\]  

(1.2)

where $P_j^N(z)$ are homogeneous polynomials of degree $m + N$, and $d_N = \binom{n + m + N}{N}$ is the dimension of the space of homogeneous polynomials of degree $m + N$.

This result can be considered as the complex variables analogue of Hilbert’s 17th problem: given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions? The positive answer to this original question was given by Artin [1]. For a survey of recent work on the hermitian case see the review paper by D’Angelo [3].

In this paper we give the following quantitative version of Theorem 1:

**Theorem 2** Let $f$ satisfy the assumptions of Theorem 1 and define

\[
\lambda(f) := \min_{\|z\|=1} f(z, \bar{z}), \quad \Lambda(f) := \left( \sum_{|\alpha|=|\beta|=m} \left( \frac{\alpha!\beta!}{m!^2} \right)^2 |c_{\alpha\beta}|^2 \right)^{1/2}.
\]

(1.3)

Then there exists a universal constant $C$ such that (1.2) holds for

\[
N \geq C \frac{\Lambda(f)}{\lambda(f)} (m + n)^3 \log^3 n.
\]

(1.4)

The proofs of Quillen [9] and Catlin-D’Angelo [2] are based on functional analytic methods related to the study of Toeplitz operators. The existence of $N_0$ such that (1.2) is satisfied is obtained by a non-constructive Fredholm compactness argument—see [7, Section 10] for outlines and comparisons of the two proofs, and also [4] for an elementary introduction to the subject.

Here we take a point of view based on the semiclassical study of Toeplitz operators—see [11, Chapter 13] and references given there. Our proof of Theorem 2 is a quantitative version of the proof of Theorem 1 given in [11, Section 13.5.4]: the compactness argument is replaced by an asymptotic argument with $N = 1/h$, where $h$ is the semiclassical parameter. The symbol calculus for Toeplitz operators allows estimates in terms of $h$ which then translate into a bound on $N$.

Better bounds on $N$ obtained using purely algebraic methods already exist and it is an interesting question if such bounds can be obtained using semiclassical methods.

In the diagonal (real) case, $c_{\alpha\beta} = 0$ if $\alpha \neq \beta$, Theorem 1 is equivalent to a classical theorem of Pólya—see [7, Section 10.1]. In that case a sharp bound on $N$ was given by Powers and Resnick [6]:

\[
N > \frac{m(m-1)}{2} \tilde{\Lambda}(f) - m, \quad \tilde{\Lambda}(f) := \max_{|\alpha|=m} \left\{ \frac{\alpha!}{m!} |c_{\alpha\alpha}| \right\}.
\]

(1.5)
It is remarkable that the bound does not depend on the dimension $n$. To compare this bound to the bound obtained using semiclassical methods, we note that in the diagonal case, the spectral radius used in Lemma (3.1) is given by $\tilde{\Lambda}(f)$. Hence an easy modification of that lemma leads to the bound

$$N \gtrsim \frac{\tilde{\Lambda}(f)}{\lambda(f)} (n + m)^3 \log^3 n,$$

(1.6)

which is weaker than the bound (1.5) from [6], roughly by a factor of $m(1 + n/m)^3$.

In the complex case, To-Yeung [10, Theorem 1] gave an algebraic proof of a better bound than the one provided by our method in Theorem 2. They show that

$$N \geq nm(2m - 1) \frac{\Lambda^*(f)}{\log 2\lambda(f)} - n - m,$$

$$\Lambda^*(f) := \sup_{|z|=1} |f(z, \bar{z})|.$$

The common feature of all these bounds is the denominator $\lambda(f)$ and the standard example of $|z_1|^4 + |z_2|^4 - c|z_1|^2|z_2|^2, 0 < c < 2, z \in \mathbb{C}^2$ (see for instance [11, Section 13.5.4]) shows that the $1/\lambda(f)$ behaviour is optimal.

In Putinar’s generalization of Pólya’s theorem [8], a much larger bound was given by Nie and Schweighofer [5]:

$$N > c \exp\left(m^2 n^m \frac{\Lambda^*(f)}{\lambda(f)}\right)^c,$$

(1.7)

for some $c > 0$.

The paper is organized as follows. In Sect. 2 we recall various basic facts about the Bargmann–Fock space and Toeplitz quantization. Section 3 presents the basic inequality which leads to a bound on $N$. Section 4 provides quantitative estimates on the localization of homogeneous polynomials in the Bargmann–Fock space, with a stationary phase argument given in the appendix. The proof of Theorem 2 is then given in Sect. 5.

**Notation** We denote $\langle x, y \rangle$ for $x, y \in \mathbb{C}^n$ the euclidean quadratic form on $\mathbb{C}^n$ (not the hermitian scalar product): $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we define $\|z\|$ as the standard hermitian norm: $\|z\|^2 := \sum_{i=1}^n z_i \bar{z}_i = \langle z, \bar{z} \rangle$. The measure $dm(z)$ denotes the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$. The space of homogeneous polynomials of degree $M$ is denoted $\mathcal{P}_M$. Finally, for two quantities $A, B$, we write $A \gtrsim B$, if there exist a (large, universal) constant $C$, such that $A \geq CB$.

## 2 Preliminaries: Bargmann–Fock space and Toeplitz quantization

Quillen’s original proof of Theorem 1 used the Bargmann–Fock space—see [7, Section 10], [9] and [11, Section 13.5.4]. We modify it slightly by introducing a semiclassical parameter $h$ and considering the subspace of homogeneous polynomials of degree $M, \mathcal{P}_M$. 
A Hilbert space Bargmann–Fock norm on $\mathcal{P}_M$ is given by
\[
\|u\|_{\mathcal{P}_M}^2 = \int_{\mathbb{C}^n} |u(z)|^2 e^{-\|z\|^2/h} dm(z)
\]
and we can extend this norm to any function $u$ such that
\[
\int_{\mathbb{C}^n} |u(z, \bar{z})|^2 e^{-\|z\|^2/h} dm(z) < \infty.
\]
We denote the resulting space by $L^2_{\Phi}$.

The closed subspace of holomorphic functions is denoted by $H_{\Phi}$. The measure $\exp(-\|z\|^2/h)dm(z)$ will sometimes be written as $dG(z)$.

The Bergman projector $\Pi_{\Phi}$, is the orthogonal projector $L^2_{\Phi} \rightarrow H_{\Phi}$ and to compute it we identify an orthonormal basis of $H_{\Phi}$. The following standard lemma is a rephrasing of [11, Theorem 13.16]:

**Lemma 2.1** Let us define
\[
f_{\alpha}(z) := \frac{1}{(\pi h)^{n/2}} \left( \frac{1}{h^{\|\alpha\|} \alpha!} \right)^{1/2} z^\alpha.
\]

Then
(i) The set of $f_{\alpha}$’s is an orthonormal basis on $H_{\Phi}$.
(ii) The Bergman projector $\Pi_{\Phi}$ can be written
\[
\Pi_{\Phi} u(z) = \int_{\mathbb{C}^n} \Pi(z, w) u(w) dm(w)
\]
where
\[
\Pi(z, w) := \frac{1}{(\pi h)^{n}} \exp \left( \frac{1}{h} (\langle z, \bar{w} \rangle - |w|^2) \right).
\]

To connect the study of positive bi-homogeneous forms to Bargmann–Fock space, we recall the standard result (see [11, Lemma 13.17]):

**Lemma 2.2** A bi-homogeneous form of degree $m$ can be written as a sum of squares of homogeneous polynomials,
\[
f(z, \bar{z}) = \sum_{j=1}^k |P_j(z)|^2, \quad P_j(z) = \sum_{|\alpha|=m} p_{j\alpha} z^\alpha,
\]
if and only if the matrix $(c_{\alpha\beta})_{|\alpha|=|\beta|=m}$ is positive semidefinite.
Thus to prove Theorem 1 we need to show that the matrix of the hermitian form 
$$\langle z, \bar{z} \rangle^N f(z, \bar{z})$$ is positive for $N$ large enough. Let us compute this matrix. Since
$$\langle z, \bar{z} \rangle^N = \sum_{|\mu|=N} \frac{z^\mu \bar{z}^\mu}{\mu!},$$
$$\langle z, \bar{z} \rangle^N f(z, \bar{z}) = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^\alpha \bar{z}^\beta + \mu = \sum_{|\gamma|=|\rho|=m+N} c_{\gamma\mu}^N z^\gamma \bar{z}^\rho,$$
where
$$c_{\gamma\mu}^N = \sum_{\alpha+\mu=\rho, \beta+\mu=\gamma, |\mu|=N} \frac{c_{\alpha\beta}}{\mu!}, \quad |\rho| = |\gamma| = N + m. \quad (2.2)$$
The following essential idea comes from the work of Quillen in [9]. It relates the
positivity of the matrix (2.2) to the positivity of a differential operator.
Let $P_f$ be the following differential operator
$$P_f = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^\alpha (h \partial z)^\beta : H_\Phi \longrightarrow H_\Phi. \quad (2.3)$$
Since $f$ is real, $c_{\alpha\beta} = c_{\beta\alpha}$. Thus the formula (2.5) shows that $P_f$ is self adjoint. Let us explain now how the positivity condition and the operator $P_f$ are related.
A simple calculation (see [11, Section 13.5.5]) based on the definition and (2.4) shows that for all $u, v \in P_{m+N}$,
$$\langle P_f u, v \rangle_{P_{m+N}} = \pi^n h^{n+N+2m} \sum_{|\gamma|=|\rho|=m+N} \rho! \gamma! c_{\gamma\mu}^N u_\rho v_\gamma \bar{v}_\rho$$
where $u_\rho \in \mathbb{C}$, $v_\gamma \in \mathbb{C}$, are given in
$$u = \sum_{|\rho|=m+N} u_\rho z^\rho, \quad v = \sum_{|\gamma|=m+N} v_\gamma z^\gamma.$$
Thus proving that the matrix (2.2) is positive definite is equivalent to proving that $P_f$
is a positive operator on $P_{m+N}$. To make this quantitative we use the following lemma
which is an application of a more general formula given in [11, Theorem 13.10]:
Lemma 2.3 Let $\Pi_\Phi$ be the orthogonal projector from $L^2_\Phi$ to $H_\Phi$. Then
$$P_f |_{P_{m+N}} = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^\alpha \Pi_\Phi (z^\beta \cdot), \quad (2.4)$$
and
$$P_f |_{P_{m+N}} = \Pi_\Phi q(z, \bar{z}) \Pi_\Phi \quad (2.5)$$
where

\[ q(z, \bar{z}) = \sum_{j=0}^{m} \frac{h^j}{j!} (-\frac{1}{4} \Delta)^j f(z, \bar{z}). \quad (2.6) \]

Using (2.5), positivity of \( P_f \) on \( P_{N+m} \) follows from inequality

\[ \langle \Pi_\Phi q \Pi_\Phi u, u \rangle_{P_{m+N}} \geq c \| u \|_{L_\Phi^2}^2, \quad u \in P_{N+m}, \]

for some constant \( c > 0 \). But since \( \Pi_\Phi u = u \) and \( \Pi_\Phi^* = \Pi_\Phi \), it suffices to prove that for all \( u \in P_{N+m} \), with \( L_\Phi^2 \)-norm equal to 1,

\[ \langle q(z, \bar{z})u, u \rangle_{L_\Phi^2} \geq c, \quad (2.7) \]

and (2.7) is the starting point of our work.

3 The basic estimate

We define the ring \( \Omega_\varepsilon \) as

\[ \Omega_\varepsilon := \{ z \in \mathbb{C}^n, 1 - \varepsilon \leq \| z \|^2 \leq 1 + \varepsilon \}. \]

For \( u \in P_{N+m} \) with \( L_\Phi^2 \)-norm equal to 1 we have

\[ \langle qu, u \rangle_{L_\Phi^2} = \int_{\mathbb{C}^n} q(z, \bar{z})|u(z)|^2 e^{-\| z \|^2/h} \, dm(z) \]

\[ = \int_{\mathbb{C}^n \setminus \Omega_\varepsilon} q(z, \bar{z})|u(z)|^2 e^{-\| z \|^2/h} \, dm(z) + \int_{\Omega_\varepsilon} q(z, \bar{z})|u(z)|^2 e^{-\| z \|^2/h} \, dm(z) \]

\[ \geq \min_{\mathbb{C}^n \setminus \Omega_\varepsilon} q \left( \| u \|_{L_\Phi^2}^2 - \| u \|_{L_\Phi^2(\Omega_\varepsilon)}^2 \right) + \int_{\Omega_\varepsilon} q(z, \bar{z})|u(z)|^2 e^{-\| z \|^2/h} \, dm(z) \]

\[ = \min_{\mathbb{C}^n \setminus \Omega_\varepsilon} q \left( 1 - \| u \|_{L_\Phi^2(\Omega_\varepsilon)}^2 \right) + \langle qu, u \rangle_{L_\Phi^2(\Omega_\varepsilon)}. \]
Recalling (2.6) we see that
\[
\langle qu, u \rangle_{L^2_\Phi(\Omega_\varepsilon)} = \sum_{j=0}^{m} \frac{h^j}{j!} \int_{\Omega_\varepsilon} \left( \frac{1}{\Phi_1(\Omega_\varepsilon)} \right)^{2(m-j)} \frac{\|z\|^2}{\|z\|^2 h} \ dm(z)
\]
\[
\geq - \sum_{j=0}^{m} \frac{h^j}{j!} \max_{\Omega_\varepsilon} \left( \frac{1}{\|z\|^2} \right) \left| \left( \frac{1}{4\Delta} \right)^j f(z, \bar{z}) \right| \|z\|^2 \|u(z)\|^2 / \Phi_1(\Omega_\varepsilon)
\]
\[
\geq - \sum_{j=0}^{m} \frac{h^j}{j!} E_\varepsilon(h, m + N, m - j) \max_{\|z\|=1} \left| \left( \frac{1}{4\Delta} \right)^j f(z, \bar{z}) \right| ,
\]
(3.1)
where the quantity \( E_\varepsilon(h, M, k) \) is defined as
\[
E_\varepsilon(h, M, k) := \sup_{u \in P_M, \|u\|_{P_M} = 1} \|z\|^k \|u\|_{L^2_\Phi(\Omega_\varepsilon)},
\]
(3.2)
and where we used the homogeneity of \( \Delta^j f \) of degree \( 2(m - j) \).

Rearranging the terms we obtain
\[
\langle qu, u \rangle_{L^2_\Phi(\Omega_\varepsilon)} \geq ((1 - E_\varepsilon(h, m + N, 0)) \ min q) - \sum_{j=0}^{m} \frac{h^j}{j!} E_\varepsilon(h, m + N, m - j) \max_{\|z\|=1} \left| \left( \frac{1}{4\Delta} \right)^j f(z, \bar{z}) \right| .
\]
(3.3)
Moreover,
\[
\min_{1-2\varepsilon \leq \|z\|^2 \leq 1+2\varepsilon} q(z, \bar{z}) = \min_{1-2\varepsilon \leq \|z\|^2 \leq 1+2\varepsilon} \sum_{j=0}^{m} \frac{h^j}{j!} \left( -\frac{1}{4\Delta} \right)^j f(z, \bar{z})
\]
\[
\geq (1 - 2\varepsilon)^m \lambda(f) - \sum_{j=1}^{m} \frac{h^j}{j!} (1+2\varepsilon)^{m-j} \max_{\|z\|=1} \left| \left( \frac{1}{4\Delta} \right)^j f(z, \bar{z}) \right| .
\]

We see that we need an upper bound for \( \max_{\|z\|=1} \left| \left( \frac{1}{4\Delta} \right)^j f(z, \bar{z}) \right| \) and that is given in the following

**Lemma 3.1** We have the estimate:
\[
\max_{\|z\|=1} \left| \left( \frac{1}{4\Delta} \right)^j f(z, \bar{z}) \right| \leq (nm^2)^j \Lambda(f),
\]
(3.4)
where \( \Lambda(f) \) is defined in (1.3).

To explain the proof we note that since \( f \) is a bihomogeneous form of degree \( m \), \( \Delta^k f \) is a bihomogeneous form of degree \( m - k \). If we have estimates on \( f \), and if
we find an explicit relation between estimates on $f$ and $\Delta f$, related to the bound on \[
abla \max_{\|z\|=1} \|f(z, \bar{z})\|, \]

Proof For $z \in \mathbb{C}^n$ satisfying $\|z\| = 1$, put $z = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$, with $\sum r_i^2 = 1$. Then

$$|f(z, \bar{z})| \leq \sum_{|\alpha| = |\beta| = m} |c_{\alpha\beta}| r^\alpha r^\beta \leq \sum_{|\alpha| = |\beta| = m} \sqrt{\frac{\alpha! \beta!}{m!}} |c_{\alpha\beta}| \sqrt{\frac{m!}{\alpha!}} \sqrt{\frac{m!}{\beta!}} r^\alpha r^\beta = \langle \tilde{C} R, R \rangle,$$

$$\tilde{C} := \left( \frac{\sqrt{\alpha! \beta!}}{m!} |c_{\alpha\beta}| \right)_{|\alpha| = |\beta| = m}, \quad R := \left( \sqrt{\frac{m!}{\alpha!}} r^\alpha \right)_{|\alpha| = m}.$$

Since

$$\|R\|^2 = \sum_{|\alpha| = m} \frac{m!}{\alpha!} r^{2\alpha} = \|r\|^{2m} = 1$$

we have

$$|f(z, \bar{z})| \leq \langle \tilde{C} R, R \rangle \leq \rho(\tilde{C}),$$

where $\rho(\tilde{C})$ is the spectral radius of $\tilde{C}$. The spectral radius can be estimated by $\Lambda(f)$ given in (1.3); we write $\tilde{C} = U D U^{-1}$, where $D$ and $U$ are diagonal and orthogonal matrix, respectively. Then

$$\Lambda(f)^2 = \text{tr}(\tilde{C} \tilde{C}^*) = \text{tr}(U D^2 U^{-1}) = \text{tr}(D)^2 \geq \rho(\tilde{C})^2$$

and hence

$$\max_{\|z\|=1} |f(z, \bar{z})| \leq \Lambda(f). \quad (3.5)$$

We now need to find a relation between $\Lambda(f)$ and $\Lambda \left( \frac{1}{4} \Delta f \right)$. Let $D := (d_{\gamma\rho})$ be the matrix of the bi-homogeneous form $\frac{1}{4} \Delta f$, and let us chose $\gamma$, $\rho$ with $|\gamma| = |\rho| = m - 1$. Denoting by $\tilde{c}_{\alpha\beta}$ the entries of $\tilde{C}$ we obtain

$$d_{\gamma\rho} = \frac{1}{\gamma! \rho!} \frac{\partial^\gamma}{\partial z^\gamma} \frac{\partial^\rho}{\partial \bar{z}^\rho} \sum_{i=1}^n \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^i} f(0, 0) = \sum_{i=0}^n (\gamma_i + 1)(\rho_i + 1) c_{\gamma+e_i, \rho+e_i}$$

$$= \sum_{i=0}^n (\gamma_i + 1)(\rho_i + 1) \frac{m!}{(\gamma + e_i)(\rho + e_i)} \tilde{c}_{\gamma+e_i, \rho+e_i}, \quad (3.6)$$

$$= \frac{m!}{\sqrt{\gamma! \rho!}} \sum_{i=1}^n \sqrt{(\gamma_i + 1)(\rho_i + 1)} \tilde{c}_{\gamma+e_i, \rho+e_i}, \quad (3.7)$$

$$= \frac{m!}{\sqrt{\gamma! \rho!}} \sum_{i=1}^n \sqrt{(\gamma_i + 1)(\rho_i + 1)} \tilde{c}_{\gamma+e_i, \rho+e_i}, \quad (3.8)$$
\[ d_{\gamma \rho} = \frac{(m-1)!}{\sqrt{\gamma! \rho!}} m^2 \sum_{i=1}^{n} \tilde{c}_{\gamma+e_i, \rho+e_i}. \]  

(3.9)

If we put \( \tilde{d}_{\gamma \rho} := \frac{\sqrt{\gamma! \rho!} d_{\gamma \rho}}{(m-1)!} \), and denote the corresponding matrix by \( \tilde{D} \), then

\[ \tilde{d}_{\gamma \rho} \leq m^2 \sum_{i=1}^{n} \tilde{c}_{\gamma+e_i, \rho+e_i}, \]  

(3.10)

and

\[ \Lambda \left( \frac{1}{4} \Delta f \right)^2 = \sum_{|\gamma|=|\rho|=m-1}^{} d_{\gamma \rho}^2 \leq \sum_{|\gamma|=|\rho|=m-1} m^4 \left( \sum_{i=1}^{n} \tilde{c}_{\gamma+e_i, \rho+e_i} \right)^2 \]  

(3.11)

\[ \leq m^4 n \sum_{|\gamma|=|\rho|=m-1} \sum_{i=1}^{n} \tilde{c}_{\gamma+e_i, \rho+e_i}^2 \]  

(3.12)

\[ \leq n m^4 \cdot n \Lambda (f)^2. \]  

(3.13)

An easy recursion then gives

\[ \Lambda \left( \left( \frac{1}{4} \Delta \right)^j f \right) \leq (nm^2)^j \Lambda (f). \]

and inequality (3.5) applied to \( \left( \frac{1}{4} \Delta \right)^j f \) instead of \( f \) proves the lemma.

The lemma and the lower bound stated after the inequality (3.3) imply

\[ \min_{1-2\varepsilon \leq \|z\|^2 \leq 1+2\varepsilon} q(z, \bar{z}) \geq \lambda (f) (1-2\varepsilon)^m - \Lambda (f) \sum_{j=1}^{m} \frac{1}{j!} (nm^2 h)^j (1+2\varepsilon)^{m-j}. \]  

(3.14)

This combined with (3.3) leads to the basic inequality:

\[ \langle qu, u \rangle_{L^2_{\Phi}} \geq (1 - E_{\varepsilon} (h, m + N, 0)) \]

\[ (\lambda (f) (1-2\varepsilon)^m - \Lambda (f) \sum_{j=1}^{m} \frac{1}{j!} (nm^2 h)^j (1+2\varepsilon)^{m-j}) \]

\[ - \Lambda (f) \sum_{j=0}^{m} \frac{1}{j!} (nm^2 h)^j E_{\varepsilon} (h, m + N, m - j). \]  

(3.15)

All the work that follows is aimed at finding \( h_0 \) such that for \( h < h_0 \) the right hand side of (3.15) is positive.
4 Estimates on \( E_\varepsilon \)

Our goal in this section is to prove that the quantity \( E_\varepsilon(h, M, m) \) roughly decreases like \( \exp(-M\varepsilon^2) \), under some assumptions relating \( \varepsilon, h, M, m, n \). It is essentially due to the fact that the homogeneous polynomials are localised in \( L^2/\Phi_1 \)-norm around the sphere \( S^{2n-1} \subset \mathbb{C}^n \), with \( 1/h \sim M \) – see [11, Theorem 13.16, (ii)] for an explanation of this using the harmonic oscillator. Here we prove

**Lemma 4.1** Let \( \varepsilon, h, m, n, M > 0 \) and let us call

\[
\sigma := h(M + m + n - 1). \tag{4.1}
\]

**Assume that**

\[
\frac{3}{2} > \sigma > 1, \quad 1 \geq \varepsilon \geq 4(\sigma - 1). \tag{4.2}
\]

**Then for** \( E_\varepsilon \) **defined in (3.2) we have**

\[
E_\varepsilon(h, M, m) \lesssim h^m (M + m + n)^{2n + m} \frac{1}{\varepsilon^2} \exp \left( -\frac{M\varepsilon^2}{16} \right). \tag{4.3}
\]

**Proof** Let \( \Pi^M_\Phi \) be the projection from \( L^2_\Phi \) to \( \mathcal{P}_M \). For \( u \in \mathcal{P}_M \). To estimate the right hand side in (3.2) we note that

\[
\| z \|^m u \|_{L^2_\Phi(\Omega_\varepsilon)}^2 = \langle u, \Pi^M_\Phi z \langle 2m \Omega_\varepsilon \Pi^M_\Phi u \rangle_{L^2_\Phi} \leq \| \Pi^M_\Phi z \langle 2m \Omega_\varepsilon \Pi^M_\Phi \|_{L^2_\Phi} \| u \|_{L^2_\Phi}^2.
\]

Hence it suffices to estimate the norm operator \( \| \Pi^M_\Phi z \langle 2m \Omega_\varepsilon \Pi^M_\Phi \| \), and for that we will use the following standard variant of Schur’s Lemma:

**Lemma 4.2** Let \( (X, \mu) \) be a measure space, \( K : L^2(X) \rightarrow L^2(X) \) a selfadjoint operator with kernel \( k \), that is

\[
Ku(x) = \int_X k(x, y)u(y)d\mu(y).
\]

**Assume that** there exists an almost everywhere positive function \( p \) on \( X \) and \( \lambda > 0 \) such that

\[
\int_X |k(x, y)|p(y)d\mu(y) \leq \lambda p(x). \tag{4.4}
\]

**Then** \( \| K \| \leq \lambda \).
To apply the lemma we first construct the kernel of the projector $\Pi^M_\Phi = \sum_{|\alpha|=M} f_\alpha f_\alpha^*$, where $f_\alpha$ was defined in (2.1), and $f_\alpha^*$ is the linear form $\langle f_\alpha, \cdot \rangle_{L^2_\Phi}$. Writing

$$\Pi^M_\Phi u(z) := \int_{\mathbb{C}^n} \Pi^M(z, w)u(w)e^{-\|w\|^2/h} dm(w),$$

we have

$$\Pi^M(z, w) = \sum_{|\alpha|=M} f_\alpha(z)\overline{f_\alpha(w)} = \sum_{|\alpha|=M} \frac{1}{(\pi h)^n} \left( \frac{1}{h^M\alpha!} \right) z^\alpha \overline{w}^\alpha$$

$$= \frac{1}{\pi^h h^{n+M}} \sum_{|\alpha|=M} \frac{1}{\alpha!} z^\alpha \overline{w}^\alpha = \frac{\langle z, w \rangle^M}{M!\pi^h h^{n+M}}.$$

It follows that the integral kernel of $K = \Pi^M_\Phi \|z\|^{2m} \Omega_\Phi$, $\Pi^M_\Phi$ with respect to the Gaussian measure $dG(z) := \exp(-\|z\|^2/h)dm(z)$, $k$, is given by

$$k(z, w) = \int_{\Omega_\varepsilon} \frac{\langle z, \xi \rangle^M}{M!\pi^h h^{n+M}} \frac{\langle \xi, \overline{w} \rangle^M}{M!\pi^h h^{n+M}} \|\xi\|^{2m} dG(\xi).$$

This suggests natural choice of the weight $p = \|z\|^M$ in lemma 4.2, and we need to estimate the corresponding parameter $\lambda$ in (4.4). For that, we need an upper bound on the integral

$$\int_{\mathbb{C}^n} |k(z, w)| \|w\|^M dG(w).$$

An application of the Cauchy–Schwarz inequality gives

$$\int_{\mathbb{C}^n} |k(z, w)| \|w\|^M dG(w) \leq \int_{\mathbb{C}^n} \int_{\Omega_\varepsilon} \|w\|^M \frac{\|z\|^M \|\xi\|^M}{M!\pi^h h^{n+M}} \frac{\|\xi\|^M \|w\|^M}{M!\pi^h h^{n+M}} \|\xi\|^{2m} dG(\xi) dG(w)$$

$$\leq \|z\|^M \left( \int_{\mathbb{C}^n} \frac{\|w\|^{2M}}{M!\pi^h h^{n+M}} dG(w) \right) \left( \int_{\Omega_\varepsilon} \frac{\|\xi\|^{2M+2m}}{M!\pi^h h^{n+M}} dG(\xi) \right).$$

Thus it is sufficient to estimate the following integrals:

$$I_1 = \int_{\mathbb{C}^n} \frac{\|w\|^{2M}}{M!\pi^h h^{n+M}} dG(w), \quad I_2 = \int_{\Omega_\varepsilon} \frac{\|\xi\|^{2M+2m}}{M!\pi^h h^{n+M}} dG(\xi). \quad (4.5)$$
A polar coordinates change of variables, followed by a substitution $t = r^2/h$, gives

$$I_1 = \frac{|S^{2n-1}|}{M! \pi^n h^{n+M}} \int_0^\infty r^{2M+2n-1} e^{-r^2/h} dr = \frac{|S^{2n-1}|}{2M! \pi^n h^{n+M}} \int_0^\infty t^{M+n-1} e^{-t} dt$$

$$= \frac{(M + n - 1)!}{M!(n-1)!} \leq \binom{M + n}{n} \leq (M + n)^n,$$

where $|S^{2n-1}| = 2\pi^n/(n-1)!$ denotes the volume of the $2n-1$ dimensional sphere.

Turning to $I_2$ in (4.5) we make two changes of variables, $z = r\theta$, then $r^2 = t$, so that

$$I_2 = |S^{2n-1}| \int_{t^2 \in [1 \pm \varepsilon]} \frac{r^{2M+2m+2n-1}}{M! \pi^n h^{n+M}} \exp \left( -\frac{r^2}{h} \right) dr$$

$$= \frac{|S^{2n-1}|}{2M! \pi^n h^{n+M}} \int_{t \in [1 \pm \varepsilon]} t^{M+m+n-1} e^{-t/h} dt. \quad (4.7)$$

The last integral is very close to the integral appearing in the following lemma which will be proved in the appendix:

**Lemma 4.3** Let $\rho > 0$, $\delta < 1$. We define

$$J(\rho, \delta) := \int_{t \in [1-\delta, 1+\delta]} t^\rho e^{-\rho t} dt.$$

Then

$$J(\rho, \delta) \lesssim \frac{1}{\rho \delta^2} \exp \left( -\rho \left( 1 + \frac{\delta^2}{4} \right) \right). \quad (4.8)$$

To apply this lemma to the last integral in (4.7) we make the change of variable $t/h = (M + m + n - 1)s$. To assure that the interval of integration does not change much, we claim that under assumptions of Lemma 4.1 we have,

$$[1 \pm \varepsilon/2] \subset \frac{1}{h(M + m + n - 1)}[1 \pm \varepsilon] = \frac{1}{\sigma}[1 \pm \varepsilon]. \quad (4.9)$$

Indeed, (4.2) implies the following inequalities:

$$1 - \frac{\varepsilon}{2} \geq \frac{1}{\sigma}(1 - \varepsilon), \quad 1 + \frac{\varepsilon}{2} \leq \frac{1}{\sigma}(1 + \varepsilon). \quad (4.10)$$

The first one is straightforward, since it is equivalent to $2\sigma - 2 \geq (\sigma - 2)\varepsilon$, and $\sigma - 2 < 0$. The second inequality in (4.10) is equivalent to $(2\sigma - 2)/(2 - \sigma) \leq \varepsilon$, and $2 - \sigma > 0$, so it is also true.
so that in view of (4.2) we need to check that \((2\sigma - 2)/(2 - \sigma) \leq 4(\sigma - 1)\) which follows from the assumption \(\sigma < 3/2\).

Returning to (4.7) we have

\[
\int_{t \notin [1\pm \varepsilon]} t^{n+m+M-1} e^{-t/h} dt \leq [h(M+m+n-1)]^{M+m+n} \int_{s \notin [1\pm \varepsilon/2]} (te^{-t})^{M+m+n-1} dt.
\]

Applying Lemma 4.3 gives

\[
\int_{t \notin [1\pm \varepsilon/2]} t^{n+m+M-1} e^{-t/h} dt \lesssim \frac{[h(M+m+n-1)]^{M+m+n}}{(M+m+n-1)\varepsilon^2} e^{-(M+n+m-1)(1+\varepsilon^2/16)}
\]

\[
\lesssim \frac{[h(M+m+n)]^{M+m+n}}{\varepsilon^2} e^{-(M+n+m)(1+\varepsilon^2/16)}.
\]

Hence

\[
I_2 \lesssim [h(M+m+n)]^{M+m+n} \frac{S^{2n-1}}{2\pi^n} \frac{1}{hM!} \frac{1}{h^n \varepsilon^2} e^{-M\varepsilon^2/16}
\]

\[
\lesssim h^m (M+m+n)^{M+m+n} \frac{e^{-M-n-m}}{M!(n-1)!} \frac{1}{\varepsilon^2} e^{-M\varepsilon^2/16}.
\]

To simplify the upper bound in (4.11) we first use Stirling’s formula to obtain (with a small irrelevant loss since \(k^k \lesssim k!e^k/\sqrt{k}\))

\[(M+m+n)^{M+m+n} \lesssim (M+m+n)! e^{M+m+n}.
\]

Thus the bound in (4.11) can be replaced by

\[
I_2 \lesssim h^m (M+m+n)! \frac{1}{M!} \frac{1}{\varepsilon^2} e^{-M\varepsilon^2/16} \lesssim h^m (M+m+n)^{m+n} \frac{1}{\varepsilon^2} e^{-M\varepsilon^2/16}.
\]

Combining this with the bound (4.6), and applying Lemma 4.2 gives

\[
\|K\| \lesssim h^m (M+n)^n (M+m+n)^{m+n} \frac{1}{\varepsilon^2} e^{-h^{-1/3}/16}
\]

\[
\lesssim h^m (M+n+m)^{2n+m} \frac{1}{\varepsilon^2} e^{-h^{-1/3}/16}.
\]

This completes the proof of Lemma 4.1. \(\square\)

5 Proof of Theorem 2

We now combine the basic inequality (3.15) with the estimate on \(E_\varepsilon\) given in Lemma 4.1. We split (3.15) into four terms:
(i) $A_0 = \lambda(f)(1 - 2\varepsilon)^m$ which is the leading term;
(ii) $A_1 = \lambda(f)E_\varepsilon(h, m + N, 0)(1 - 2\varepsilon)^m$ decreases exponentially to 0 as $h \to 0$;
(iii) $A_2 = \Lambda(f) \sum_{j=1}^{m} \frac{1}{j!}(nm^2 h)^j (1 + 2\varepsilon)^{m-j}$ will be estimated by noting that it is dominated by its first term;
(iv) $A_3 = \Lambda(f) \sum_{j=0}^{m} \frac{1}{j!}(nm^2 h)^j E_\varepsilon(h, m + N, m - j)$ will require more care but decreases exponentially to 0 as $h \to 0$.

We want to optimize the parameters $h, M, \varepsilon$ as functions of the order of $f, m,$ and the dimension $n$. We aim to show that $A_0 \gg A_1, A_2, A_3$, using Lemma 4.1. For this we need to check that the assumption (4.2) is satisfied.

The basic strategy is outlined as follows:

- (4.2) is satisfied if for all $0 \leq j \leq m$, $h^{-1} \sim N + 2m + n - j$ and $h(N + 2m + n - j) \leq 1$. Thus we need $h^{-1} \sim N \gg m, n$.
- $A_0 \gtrsim A_1$: we want to apply Lemma 4.1 and thus we need $\varepsilon^2/h \geq -n \log(h)$;
- $A_0 \gtrsim A_2$: for this to hold $A_2$ has to be greater than the first term of the sum in $A_2$, $nm^2(1 + 2\varepsilon)^{m-1}h$; thus the term $(1 + 2\varepsilon)^m$ has to remain bounded as $m \to \infty$: we need $\varepsilon \lesssim 1/m$.
- $A_0 \gtrsim A_3$: the term $A_0$ has – at least – to be greater than the first term of the sum in $A_3$; thus we need to have $A_0 \gtrsim E_\varepsilon(h, N + m, m)$; using Lemma 4.1, this holds when $\varepsilon^2/h \geq -(n + m) \log(h)$.

We define $\varepsilon$ as $\varepsilon = h^a$, where $a$ will be determined. From the considerations above we need

$$h^{2a-1} \gtrsim (n + m) \log \frac{1}{h} \quad \text{and} \quad h^a \lesssim 1/m.$$ 

To express this as one condition, we demand $a = 1 - 2a$, that is, $a = 1/3$. This leads to the necessary relations:

$$\varepsilon = h^{-1/3}/16, \quad h \lesssim (n + m)^{-3}, \quad N = h^{-1}. \quad (5.1)$$

Application of the estimates on $E_\varepsilon$.

To use estimates on $E_\varepsilon(h, m + N, m - j)$ for $0 \leq j \leq m$ we need the assumption (4.2) to hold. That means that

$$1 < h(N + 2m + n - j - 1) < \frac{3}{2}, \quad 1 \geq \varepsilon \geq 4(h(N + 2m - j + n) - 1). \quad (5.2)$$

Since $N = 1/h$, both inequalities are satisfied for all $0 \leq j \leq m$ if they are satisfied for $j = 0$. Recallingl that $\varepsilon = h^{-1/3}/16 \leq 1$, this in turn follows from

$$4h(2m + n) \leq \varepsilon, \quad h(2m + n) \leq \frac{1}{2}. \quad (5.3)$$
If \( h \leq \frac{1}{64}(m + n)^{-3} \), then

\[
\frac{8\delta}{(m + n)^2} \leq \frac{\delta^{1/3}}{(m + n)},
\]

which implies (5.3). We conclude that (5.2) holds, hence also (4.2), and hence we can apply Lemma 4.1 to \( E_\varepsilon(h, m + N, m - j), 0 \leq j \leq m \).

Final estimate on \( h \).

We first start by simplifying \( A_1 \). Lemma 4.1 shows that

\[
E_\varepsilon(h, m + N, 0) \lesssim (n + m + N)^{2n}e^{-2}e^{-N\varepsilon^2/16} \lesssim t(3/h)^{2n+1}e^{-h^{-1/3}/16}.
\]

Thus

\[
A_1 \lesssim \lambda(f)(1 - 2\varepsilon)^m(3/h)^{2n+1}e^{-h^{-1/3}/16}.
\]

(5.4)

To treat \( A_2 \) we note that

\[
A_2 = \Lambda(f) \sum_{j=1}^{m} \frac{(nm^2h)^j}{j!}(1 + 2\varepsilon)^{m-j} \leq \Lambda(f)(1 + 2\varepsilon)^m(e^{nm^2h} - 1).
\]

But since \( h \leq (n + m)^{-3}, nm^2h \leq 1 \), and thus \( (nm^2h) - 1 \lesssim nm^2h \), and

\[
A_2 \lesssim \Lambda(f)(1 + 2\varepsilon)^m nm^2h.
\]

(5.5)

We finally treat \( A_3 \). For that, we need the estimate on \( E_\varepsilon(h, m + N, m - j) \) proved in Lemma 4.1:

\[
E_\varepsilon(h, m + N, m - j) \lesssim h^{m-j}(N + n + 2m - j)^{2n+m-j}e^{-2}\varepsilon^2e^{-(m+N)\varepsilon^2/16}
\lesssim (3N)^{2n}e^{-2}(3hN)^{m-j}e^{-(m+N)\varepsilon^2/16}
\lesssim (3/h)^{2n+1}3m e^{-h^{-1/3}/16}.
\]

Inserting this in the definition of \( A_3 \),

\[
A_3 = \Lambda(f) \sum_{j=0}^{m} \frac{(nm^2h)^j}{j!}E_\varepsilon(h, m + N, m - j).
\]
this gives

\[ A_3 \lesssim \Lambda(f) \sum_{j=0}^{m} (nm^2 h)^j \frac{(3/h)^{2n+1} 3^m e^{-h^{-1/3}/16}}{j!} \]

\[ \lesssim \Lambda(f)(3/h)^{2n+1} 3^m e^{-h^{-1/3}/16}. \]

Here we used again \( nm^2 h \leq 1 \). Thus we get:

\[ A_3 \lesssim 3^m (3/h)^{2n+1} e^{-h^{-1/3}/16}. \] (5.6)

We recall that we are looking for \( h_0 \) such that for \( h < h_0 \),

\[ \lambda(f)(1 - 2\varepsilon)^m \geq A_1 + A_2 + A_3 \] (5.7)

is satisfied. In view of (5.4), (5.5), (5.6), to obtain (5.7) it is sufficient to have

\[ \lambda(f)(1 - 2\varepsilon)^m \geq 3\lambda(f)(1 - 2\varepsilon)^m (3/h)^{2n+1} e^{-h^{-1/3}/16}, \] (5.8)

\[ \lambda(f)(1 - 2\varepsilon)^m \geq 3\Lambda(f)(1 + 2\varepsilon)^m nm^2 h, \] (5.9)

\[ \lambda(f)(1 - 2\varepsilon)^m \geq 3\Lambda(f)3^m (3/h)^{2n+1} e^{-h^{-1/3}/16}. \] (5.10)

Since \( h \leq \delta = 1/64 \), \( \varepsilon \leq 1/4 \) and then \( (1 - 2\varepsilon)^m \geq 10^{-m} \); moreover

\[ \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^m \leq (1 + 8\varepsilon)^m \leq \left( 1 + \frac{8}{m} \right)^m \lesssim 1. \]

Thus (5.9), (5.10) can be changed in

\[ \lambda(f) \geq 3\Lambda(f) nm^2 h, \] (5.11)

\[ \lambda(f) \geq 3\Lambda(f) 30^m (3/h)^{2n+1} e^{-h^{-1/3}/16}. \] (5.12)

Since \( \lambda(f) \leq \Lambda(f) \), (5.8) and (5.12) are both implied by

\[ \frac{\lambda(f)}{\Lambda(f)} \geq 3 \cdot 30^m (30/h)^{2n+1} e^{-h^{-1/3}/16}. \] (5.13)

The logarithmic version of this inequality is

\[ \log \left( \frac{\lambda(f)}{\Lambda(f)} \right) \geq \log(3) + (m + 2n + 1) \log(30) - (2n + 1) \log(h) - h^{-1/3}/16, \]
and thus taking $h \lesssim \log (\Lambda(f)/\lambda(f))^{-3} (m + n)^{-3} \log(n)^{-3}$ assures its validity. Indeed,

$$h \lesssim \log \left(\frac{\Lambda(f)}{\lambda(f)}\right)^{-3} \implies \log \left(\frac{\lambda(f)}{\Lambda(f)}\right) \gtrsim -h^{-1/3} \quad (5.14)$$

and

$$h \lesssim (n \log(n))^{-3} \Rightarrow n \log(h) \gtrsim -h^{-1/3}. \quad (5.15)$$

The estimate (5.11) is straightforward: we need

$$h \lesssim \frac{\lambda(f)}{\Lambda(f)} n^{-1} m^{-2}. \quad (5.16)$$

Let us chose

$$h \lesssim \min \left(\frac{\lambda(f)}{\Lambda(f)}, \log \left(\frac{\Lambda(f)}{\lambda(f)}\right)^{-3}\right) (m + n)^{-3} \log(n)^{-3}.$$

Then $h$ satisfies the three necessary conditions for Theorem 2 to hold: (5.14), (5.15), and (5.16). The bound on $N = 1/h$ is then given by

$$N \gtrsim \max \left(\log \left(\frac{\Lambda(f)}{\lambda(f)}\right)^{3}, \frac{\Lambda(f)}{\lambda(f)}\right) (m + n)^{3} \log^{3} n$$

which is the same as

$$N \gtrsim \frac{\Lambda(f)}{\lambda(f)} (m + n)^{3} \log^{3} n.$$

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**Appendix: a non-stationary phase lemma**

We prove Lemma 4.3. Let $\varphi(t) = -\log(t) + t$. Then $\varphi$ is a one to one mapping on $(0, 1]$ and on $[1, \infty)$. Let us then consider the following integrals:

$$J^{-}(\rho, \delta) = \int_{0}^{1-\delta} e^{\rho(\log(t)-t)} dt, \quad J^{+}(\rho, \delta) = \int_{1+\delta}^{\infty} e^{\rho(\log(t)-t)} dt.$$
The change of variable $\varphi(t) = x$ gives

$$J^{-}(\rho, \delta) = \int_{c^{-}}^{\infty} e^{-\rho x} \left( \frac{1}{\varphi^{-1}(x)} - 1 \right)^{-1} dx,$$

with $c^{-} = \varphi(1 - \delta)$. Thus we need estimates on $\varphi^{-1}(x)$. But on $(0, 1 - \delta]$, we have $\varphi(t) \leq 1 - \delta - \log(t)$. It implies $\varphi^{-1}(x) \leq e^{1-\delta-x}$. This gives

$$J^{-}(\rho, \delta) \leq \int_{c^{-}}^{\infty} \frac{e^{-\rho x}}{e^{x-1+\delta} - 1} dx.$$ 

A lower bound for $e^{x-1+\delta} - 1$ is

$$e^{x-1+\delta} - 1 \geq (e^{-1+\delta} - e^{-c^{-}})e^{x} \geq \delta e^{-1+\delta+x},$$

and hence

$$J^{-}(\rho, \delta) \leq \int_{c^{-}}^{\infty} e^{1-\delta} \frac{e^{-(\rho+1)x}}{\delta} dx = \frac{1 - \delta}{\delta(\rho + 1)}((1 - \delta)e^{-1+\delta})^{\rho}$$

$$\leq \frac{1}{\rho \delta}((1 - \delta)e^{-1+\delta})^{\rho}. \quad (A.1)$$

The same change of variable applied to $J^{+}$ gives

$$J^{+}(\rho, \delta) = \int_{c^{+}}^{\infty} e^{-\rho x} \left( 1 - \frac{1}{\varphi^{-1}(x)} \right)^{-1} dx$$

with $c^{+} = \varphi(1 + \delta)$. On $(1 + \delta, \infty)$, we have $\varphi(t) \leq t$ and then $\varphi^{-1}(x) \geq x$.

$$J^{+}(\rho, \delta) \leq \int_{c^{+}}^{\infty} e^{-\rho x} \left( 1 - \frac{1}{x} \right)^{-1} dx \leq \frac{c^{+}}{c^{+} - 1} \int_{c^{+}}^{\infty} e^{-\rho x} dx.$$ 

Since $\delta < 1$,

$$\frac{c^{+}}{c^{+} - 1} = \frac{\varphi(1 + \delta)}{\varphi(1 + \delta) - 1} \lesssim \frac{1}{\delta^{2}}$$

and thus

$$J^{+}(\rho, \delta) \lesssim \frac{1}{\rho \delta^{2}}(1 + \delta)e^{-1-\delta})^{\rho}. \quad (A.2)$$
Now,
\[(1 - \delta)e^{-1+\delta} \leq (1 + \delta)e^{-1-\delta}, \quad \delta^2 \leq \delta,\]
and hence the estimates (A.1) and (A.2) give
\[J(\rho, \delta) = J_-(\rho, \delta) + J_+(\rho, \delta) \lesssim \frac{1}{\rho \delta^2}((1 + \delta)e^{-1-\delta})^\rho.\]
Also,
\[(1 + \delta)e^{-\delta} \leq e^{-\delta^2/4},\]
so that finally
\[J(\rho, \delta) \lesssim \frac{1}{\rho \delta^2} \exp\left(-\rho \left(1 + \frac{\delta^2}{4}\right)\right).\]

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