ABSTRACT. We consider \( n \)-folding triangular curves, or \( n \)-folding t-curves, obtained by folding \( n \) times a strip of paper in \( 3 \), each time possibly left then right or right then left, and unfolding it with \( \pi/3 \) angles. An example is the well known terdragon curve. They are self-avoiding like \( n \)-folding curves obtained by folding \( n \) times a strip of paper in two, each time possibly left or right, and unfolding it with \( \pi/2 \) angles.

We also consider complete folding t-curves, which are the curves without endpoint obtained as inductive limits of \( n \)-folding t-curves. We show that each of them can be extended into a unique covering of the plane by disjoint such curves, and this covering satisfies the local isomorphism property introduced to investigate aperiodic tiling systems. Two coverings are locally isomorphic if and only if they are associated to the same sequence of foldings. Each class of locally isomorphic coverings contains exactly \( 2^\omega \) (resp. \( 2^\omega, 2 \) or \( 5, 0 \)) isomorphism classes of coverings by \( 1 \) (resp. \( 2, 3, \geq 4 \)) curves. These properties are partly similar to those of complete folding curves.

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1. Definitions and main results.

We investigate the properties of coverings of the plane by complete folding t-curves. Coverings of the plane by complete folding curves (see [2] and [3]), and by complete Peano-Gosper curves (see [4]) have similar properties. The 3 types of curves were introduced by B. Mandelbrot in [1].

We denote by \( \mathbb{N}^* \) the set of strictly positive integers, and \( \mathbb{R}^+ \) the set of positive real numbers. We consider \( \mathbb{R}^2 \) equipped with a norm \( x \to \| x \| \). For any \( x, y \in \mathbb{R}^2 \), we write \( d(x, y) = \| y - x \| \). For any \( E, F \subset \mathbb{R}^2 \), an isomorphism from \( E \) to \( F \) is a translation \( \tau \) such that \( \tau(E) = F \).

For each \( x \in \mathbb{R}^2 \) and each \( r \in \mathbb{R}^+ \), we denote by \( B(x, r) \) the ball of center \( x \) and radius \( r \). We say that \( E \subset \mathbb{R}^2 \) satisfies the local isomorphism property if, for each \( x \in \mathbb{R}^2 \) and each \( r \in \mathbb{R}^+ \), there exists \( s \in \mathbb{R}^+ \) such that each \( B(y, s) \) contains some \( z \) with \( (B(z, r) \cap E, z) \cong (B(x, r) \cap E, x) \). We say that \( E, F \subset \mathbb{R}^2 \) are locally isomorphic if, for each \( x \in \mathbb{R}^2 \) (resp. \( y \in \mathbb{R}^2 \) and each \( r \in \mathbb{R}^+ \), there exists \( y \in \mathbb{R}^2 \) (resp. \( x \in \mathbb{R}^2 \) such that \( (B(x, r) \cap E, x) \cong (B(y, r) \cap F, y) \). These notions are similar to those considered for aperiodic tilings.
In order to define oriented triangular curves or t-curves, we consider a tiling $P$ of the plane by equilateral triangles. A bounded t-curve (resp. half t-curve, complete t-curve) is a sequence $(A_k)_{0 \leq k \leq n}$ (resp. $(A_k)_{k \in \mathbb{N}}$, $(A_k)_{k \in \mathbb{Z}}$) of segments which are oriented sides of triangles of $P$ such that, for any $A_k, A_{k+1}$, the terminal point of $A_k$ is the initial point of $A_{k+1}$ and $A_k, A_{k+1}$ form a $\mp \pi/3$ angle. We associate to each such curve the sequence $(a_k)_{1 \leq k \leq n}$ (resp. $(a_k)_{k \in \mathbb{N}^*}$, $(a_k)_{k \in \mathbb{Z}}$) with $a_k = +1$ (resp. $a_k = -1$) for each $k$ such that we turn left (resp. right) when we pass from $A_{k-1}$ to $A_k$.

We say that a t-curve $C$ is self-avoiding if each nonoriented side of a triangle is the support of at most one segment of $C$. We represent the curves with slightly rounded angles, so that they do not pass through the vertices of triangles. Then each self-avoiding curve passes at most once through each point of the plane.

We say that a set $C$ of t-curves is a covering of the plane if each nonoriented side is the support of exactly 1 segment of 1 curve of $C$.

We define by induction on $n \in \mathbb{N}$ the sequences $T_{\lambda_1 \cdots \lambda_n}$ for $\lambda_1, \ldots, \lambda_n \in \{-1, +1\}$. We denote by $T$ the empty sequence. For each $n \in \mathbb{N}$ and any $\lambda_1, \ldots, \lambda_{n+1}$, we write $T_{\lambda_1 \cdots \lambda_{n+1}} = (T_{\lambda_1 \cdots \lambda_n}, \lambda_{n+1}, T_{\lambda_1 \cdots \lambda_n}, -\lambda_{n+1}, T_{\lambda_1 \cdots \lambda_n})$.

For each $n \in \mathbb{N}$, a t-curve associated to $T_{\lambda_1 \cdots \lambda_n}$ can be realized as follows: We successively fold $n$ times in three a strip of paper. For each $k \in \{1, \ldots, n\}$, the $k$-th folding is done left then right if $\lambda_{n+1-k} = +1$ and right then left if $\lambda_{n+1-k} = -1$. Then we unfold the strip, keeping a $\pi/3$ angle for each fold.

We call $n$-folding t-curves the curves associated to the sequences $T_{\lambda_1 \cdots \lambda_n}$. The $\infty$-folding t-curves are the half t-curves obtained as inductive limits of $n$-folding t-curves $C_n$ with $C_n$ initial part of $C_{n+1}$ for each $n \in \mathbb{N}$.

We say that a complete t-curve $C$ is a complete folding t-curve if each bounded subcurve of $C$ is contained in an $n$-folding t-curve for some integer $n$. We are going to see that, for each such curve $C$, there exists a unique sequence $\Lambda = (\lambda_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^{\mathbb{N}^*}$ such that each bounded subcurve of $C$ is contained in a t-curve associated to some $T_{\lambda_1 \cdots \lambda_n}$.

Now we state our main results:

**Theorem 1.1.** Each nonoriented complete folding t-curve can be extended in a unique way into a covering of the plane by such curves. This covering satisfies the local isomorphism property and consists of curves associated to the same sequence $\Lambda$.

**Remark.** The first statement of Theorem 1.1 is also true for complete folding curves, except in one special case (see [2, Th. 3.10 and Th. 3.15]).

**Theorem 1.2.** Two coverings of the plane by nonoriented complete folding t-curves are locally isomorphic if and only if they are associated to the same sequence $\Lambda$. 

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Theorem 1.3. For each $\Lambda = (\lambda_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^{\mathbb{N}^*}$, the class of coverings of the plane by nonoriented complete folding t-curves associated to $\Lambda$ is the union of:

1) $2^{\omega}$ isomorphism classes of coverings by 1 curve;
2) $2^{\omega}$ isomorphism classes of coverings by 2 curves;
3) $2^{\omega}$ isomorphism classes of coverings by 3 curves having 1 vertex in common, where each of the 3 curves is the union of 2 $\infty$-folding t-curves starting from that vertex;
4) only if there exists $k \in \mathbb{N}^*$ such that $\lambda_{n+1} = -\lambda_n$ for each $n \geq k$, $3^{\omega}$ isomorphism classes of coverings by 3 curves, with 1 curve separating the 2 others (Figure 1 gives an example with $k = 1$ and $\lambda_1 = -1$).

In each of the cases 3) and 4), the coverings are equivalent up to isometry.

We denote by (P) the following property of a set $E$ of oriented sides of triangles of $\mathcal{P}$: If $A, B \in E$ are sides of the same triangle, then they define the same direction of rotation around its center.

There are 2 opposite sets $E_1, E_2$ which satisfy (P) and such that each nonoriented side is the support of 1 element of $E_1$ and 1 element of $E_2$. The segments of any oriented t-curve are all contained in $E_1$, or all contained in $E_2$, and therefore satisfy (P).

We say that a covering $\mathcal{C}$ by oriented t-curves satisfies (P) if the set of segments of curves of $\mathcal{C}$ satisfies (P), or equivalently if it is equal to $E_1$ or $E_2$. 
Theorem 1.4. Each covering of the plane by oriented complete folding t-curves satisfies the local isomorphism property if and only if it satisfies \((P)\). Any coverings with these properties are locally isomorphic if and only if they are associated to the same sequence \(\Lambda\) and have the same orientation. Exactly 2 coverings with these properties are associated to each covering of the plane by nonoriented complete folding t-curves and their orientations are opposite.

2. Detailed results and proofs.

Unless otherwise specified, all the curves that we consider are oriented.

For any t-curves \(C, D\), if the terminal point of \(C\) is the initial point of \(D\) and if the terminal segment of \(C\) and the initial segment of \(D\) form a \(\pm \pi/3\) angle, then we denote by \(CD\) the t-curve obtained by connecting them.

For each \(n \in \mathbb{N}^*\), any \(\lambda_1, \ldots, \lambda_n \in \{-1, +1\}\) and each \(n\)-folding t-curve \(C\) associated to \(T_{\lambda_1} \cdots \lambda_n\), we consider the three \((n-1)\)-folding t-curves \(C^I, C^M, C^S\) associated to \(T_{\lambda_1} \cdots \lambda_{n-1}\) such that \(C = C^I C^M C^S\).

For each \(k \in \mathbb{N}\) and each sequence \(S = (\alpha_1, \ldots, \alpha_k) \in \{-1, +1\}^k\), we write \(\overline{S} = (-\alpha_k, \ldots, -\alpha_1)\). For each sequence \(S = (\alpha_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^\mathbb{N}^*\), we write \(\overline{S} = (-\alpha_n)_{n \in \mathbb{N}^*}\). In both cases, the reverse of a curve associated to \(S\) is associated to \(\overline{S}\).

For each \(n \in \mathbb{N}\) and any \(\lambda_1, \ldots, \lambda_n \in \{-1, +1\}\), we have \(T_{\lambda_1} \cdots \lambda_n = \overline{T_{\lambda_1} \cdots \lambda_n}\). Consequently, the reverse of a curve associated to \(T_{\lambda_1} \cdots \lambda_n\) is also associated to \(T_{\lambda_1} \cdots \lambda_n\).

For each \(\Lambda = (\lambda_k)_{k \in \mathbb{N}^*} \in \{-1, +1\}^\mathbb{N}^*\) and each \(n \in \mathbb{N}\), we write \(\Lambda_n = (\lambda_1, \ldots, \lambda_n)\) and \(T_{\Lambda_n} = T_{\lambda_1} \cdots \lambda_n\). We denote by \(T_\Lambda\) the inductive limit of the sequences \(T_{\Lambda_n}\) with \(T_{\Lambda_0}\) initial segment of \(T_{\Lambda_{n+1}}\) for each \(n \in \mathbb{N}\).

Proposition 2.1. Let \(C\) be a complete t-curve associated to a sequence \(S = (s_h)_{h \in \mathbb{Z}} \in \{-1, +1\}^\mathbb{Z}\). Then the properties 1), 2), 3) below are equivalent:

1) \(C\) is a complete folding t-curve.
2) For each \(k \in \mathbb{N}\), there exists \(h \in \mathbb{Z}\) such that \(s_{h+3^k+3^{k+1}i} = -s_{h+3^k+3^{k+1}j}\) for any \(i, j \in \mathbb{Z}\).
3) There exists a unique sequence \(\Lambda = (\lambda_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^\mathbb{N}^*\) such that exactly 1 of the 2 following properties is true:
   a) \(S\) is equivalent to \((T_{\Lambda}, +1, T_{\Lambda})\) or \((T_{\Lambda}, -1, T_{\Lambda})\) modulo a translation of \(\mathbb{Z}\);
   b) \(C = \cup_{n \in \mathbb{N}} C_n\) for a sequence \((C_n)_{n \in \mathbb{N}}\) such that, for each \(n \in \mathbb{N}\), \(C_n\) is an \(n\)-folding t-curve associated to \(T_{\Lambda_n}\) and \(C_n \in \{C_n^I, C_n^M, C_n^S\}\).

Proof. 1) or 3) implies 2) because we have \(s_{3^k+3^{k+1}i} = -s_{3^k+3^{k+1}j}\) for each \(n \in \mathbb{N}\), each \(n\)-folding t-curve associated to a sequence \((s_h)_{1 \leq h \leq 3^n - 1}\) each \(k \in \{0, \ldots, n - 1\}\) and any integers \(0 \leq i, j \leq 3^n - k - 1\). If the case a) of 3) is realized, then, in 2), we can take the same \(h\) for each \(k \in \mathbb{N}\).
Now we show that 2) implies 1) and 3). For each \( g \in \mathbb{Z} \), we consider the point \( z_g \) of the plane which is associated to \( s_g \) in the correspondence between \( C \) and \( S \). For \( g \in \mathbb{Z} \) and \( n \in \mathbb{N} \), we denote by \( \alpha_n(g) \) the largest integer \( h \leq g \) such that \( s_{h+3k+3} = -s_{h+3k+2} \) for each \( k \in \{0, \ldots, n-1\} \) and any \( i, j \in \mathbb{Z} \). We have \( g < h \) for \( \beta_n(g) = \alpha_n(g) + 3^n \). The part \( C_n(g) \) of \( C \) between \( z_{\alpha_n(g)} \) and \( z_{\beta_n(g)} \) is an \( n \)-folding t-curve. We have \( C_n(g) \in \{ C_{n+1}(g), C_{n+1}(g)^M, C_{n+1}(g)^S \} \).

There exists a unique sequence \( \Lambda = (\lambda_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^{\mathbb{N}^*} \) such that, for each \( n \in \mathbb{N} \) and each \( g \in \mathbb{Z} \), \( C_n(g) \) is associated to \( T_{\lambda_n} \). For any \( g, h \in \mathbb{Z} \) such that \( g < h \), we have \( C_n(g) = C_n(h) \) for \( n \) large enough, or \( \beta_n(g) = \alpha_n(h) \) for \( n \) large enough.

If there exists \( g \in \mathbb{Z} \) such that \( \cup_{n \in \mathbb{N}} C_n(g) = C \), then the case b) of 3) is realized and \( C \) is clearly a complete folding t-curve.

Otherwise, there exist \( m \in \mathbb{N}^* \) and \( g, h \in \mathbb{Z} \) with \( g < h \) such that \( \beta_n(g) = \alpha_n(h) \) for each integer \( n \geq m \). Then the case a) of 3) is realized. Moreover, \( C \) is a complete folding t-curve since, for each integer \( n \geq m \), \( C_n(g)^M C_n(g)^S C_n(h)^I \) or \( C_n(g)^S C_n(h)^I C_n(h)^M \) is an \( n \)-folding t-curve contained in \( C \), and we obtain in that way an increasing sequence \( (D_n)_{n \in \mathbb{N}} \) such that \( D = \cup_{n \in \mathbb{N}} D_n \), with \( D_n \) an \( n \)-folding t-curve for each \( n \in \mathbb{N} \). \( \blacksquare \)

Similarly to the case of folding curves in [2] and [3] and Peano-Gosper curves in [4], we define a derivation \( \Delta \) on folding t-curves.

For each \( n \in \mathbb{N}^* \), we divide each \( n \)-folding t-curve into sequences of 3 consecutive segments. We replace each sequence with a unique segment whose initial and terminal points are the initial point of the first segment and the terminal point of the third segment.

If \( C \) is an \( n \)-folding t-curve associated to a sequence \( (\alpha_i)_{1 \leq i \leq 3^n-1} \in \{-1, +1\}^{3^n-1} \), then \( \Delta(C) \) is an \((n-1)\)-folding t-curve associated to \((\alpha_3i)_{1 \leq i \leq 3^{n-1}-1} \).

The definition of \( \Delta \) naturally extends to \( \infty \)-folding t-curves. Now we see that it also extends to complete folding t-curves.

Consider any such curve \( C = (A_k)_{k \in \mathbb{Z}} \), and the associated sequence \( (s_k)_{k \in \mathbb{Z}} \in \{-1, +1\}^{\mathbb{Z}} \). Then, by Proposition 2.1, there exist \( \varepsilon \in \{-1, +1\} \) and \( h \in \mathbb{Z} \), whose remainder modulo 3 is completely determined by \( C \), such that \( s_{h+3k+1} = \varepsilon \) and \( s_{h+3k+2} = -\varepsilon \) for each \( k \in \mathbb{Z} \). The curve \( \Delta(C) \) is obtained by replacing each \( A_{h+3k} A_{h+3k+1} A_{h+3k+2} \) with a unique segment; it is associated to the sequence \( (s_{h+3k})_{k \in \mathbb{Z}} \).

If an \( \infty \)-folding or a complete folding t-curve \( C \) is the inductive limit of \( n \)-folding t-curves \( C_n \subset C \), then \( \Delta(C) \) is the inductive limit of the curves \( \Delta(C_n) \) for \( n \in \mathbb{N}^* \).

For each \( k \in \mathbb{N}^* \) and each \( n \)-folding for \( n \geq k \), \( \infty \)-folding or complete folding t-curve \( C \), \( \Delta^k(C) \) is obtained in the same way by replacing sequences of \( 3^k \) consecutive segments with 1 segment.
Now we begin to show the plane-filling properties of folding t-curves. For each \( n \in \mathbb{N}^* \), we call an \( n \)-triangle any equilateral triangle such that each side consists of \( n \) sides of triangles of \( \mathcal{P} \). We say that a set of curves \( \mathcal{C} \) covers an \( n \)-triangle \( U \) if each side of a triangle of \( \mathcal{P} \) contained in \( U \) is the support of at least one segment of a curve of \( \mathcal{C} \).

**Proposition 2.2.** For each \( n \in \mathbb{N}^* \), each \( (2^n) \)-folding t-curve covers a \( 2^{n-1} \)-triangle.

\[
\text{Proof.} \quad \text{For each } n \in \mathbb{N}^*, \text{ we denote by } k_n \text{ the largest integer } k \text{ such that each } (2^n) \text{-folding t-curve covers a } k \text{-triangle.}
\]

First we observe that \( k_1 = 1 \). Figure 2 shows two possible cases. Any other case is equivalent to one of them up to isometry.

Now it is enough to prove that \( k_2 \geq 3 \) and \( k_n \geq 3k_{n-1} - 3 \) for each \( n \geq 3 \). If \( C \) is a \( (2n) \)-folding t-curve for an integer \( n \geq 2 \), we consider the tiling \( \mathcal{Q} \) of the plane by equilateral triangles which is associated to \( \Delta^2(C) \). Each triangle of \( \mathcal{Q} \) is the union of 9 nonoverlapping triangles of \( \mathcal{P} \).

For any segments \( R_1, R_2, R_3 \) of \( \Delta^2(C) \), if their supports are the sides of a \( 1 \)-triangle \( U \) of \( \mathcal{Q} \), then their orientations define the same direction of rotation around the center of \( U \) since \( \Delta^2(C) \) satisfies (P). Figure 3 shows two possible configurations for \( \Delta^{-2}(R_1), \Delta^{-2}(R_2), \Delta^{-2}(R_3) \). Any other configuration is equivalent to one of them up to isometry.
We see from Figure 3 that \( E = \Delta^{-2}(R_1) \cup \Delta^{-2}(R_2) \cup \Delta^{-2}(R_3) \) necessarily covers a 3-triangle of \( \mathcal{P} \). In particular, we have \( k_2 \geq 3 \). We also see that each side of a 1-triangle of \( \mathcal{P} \) contained in \( U \) is the support of a segment of \( E \), except possibly if one of its endpoints is a vertex of \( U \).

Now we show that, for each integer \( k \geq 2 \), if \( \Delta^2(C) \) covers a \( k \)-triangle \( X \) of \( \mathcal{Q} \), then \( C \) covers the \((3k - 3)\)-triangle \( W \) of \( \mathcal{P} \) contained in the interior of \( X \).

This property is a consequence of the two following facts: First, by the argument above, for each 1-triangle \( U \) of \( \mathcal{Q} \), if \( \Delta^2(C) \) covers \( U \), then each side of a 1-triangle of \( \mathcal{P} \) contained in \( U \) is the support of a segment of \( C \) if its endpoints are not vertices of \( U \). Second, for each vertex \( z \) of a 1-triangle of \( \mathcal{Q} \), if \( z \) belongs to \( W \), then \( z \) is an endpoint of 6 segments of \( C \) since it is an endpoint of 6 segments of \( \Delta^2(C) \).

**Corollary 2.3.** For each covering of the plane by oriented complete folding \( t \)-curves, the local isomorphism property implies \( (P) \).

**Proof.** If \( \mathcal{C} \) is such a covering, then, by Proposition 2.2, each curve \( C \in \mathcal{C} \) covers arbitrarily large balls. The covering \( \mathcal{C} \) satisfies \( (P) \) on any such ball, and therefore on the whole plane by the local isomorphism property.

Now we fix a sequence \( \Lambda = (\lambda_n)_{n \in \mathbb{N}^*} \in \{-1, +1\}^{\mathbb{N}^*} \) and we investigate the coverings of the plane by folding \( t \)-curves associated to \( \Lambda \).

If \( U \) is the set of vertices of a tiling of the plane by equilateral triangles, then, for each \( x \in U \), there exists a unique partition \( U = G(U, x) \cup H(U, x) \) such that:
1) \( H(U, x) \) is the set of vertices of a tiling of the plane by regular hexagons;
2) \( x \in G(U, x) \) and \( G(U, x) \) is the set of centers of hexagons.
Moreover, \( G(U, x) \) is also the set of vertices of a tiling of the plane by equilateral triangles.

We consider the set \( V \) of vertices of triangles of \( \mathcal{P} \). For each \( x \in V \), we write \( V_0(x) = V \) and, for each \( n \in \mathbb{N} \), \( V_{n+1}(x) = G(V_n(x), x) \) and \( U_{n+1}(x) = H(V_n(x), x) \).

We define by induction on \( n \in \mathbb{N} \) a covering \( \mathcal{C}_n(\Lambda_n, x) \) of the plane by \( n \)-folding \( t \)-curves associated to \( T_{\Lambda_n} \). We take for \( \mathcal{C}_0(\Lambda_0, x) \) one of the sets \( E_1, E_2 \) defined above.

For each \( n \in \mathbb{N} \), supposing that \( \mathcal{C}_n(\Lambda_n, x) \) is already defined, we define \( \mathcal{C}_{n+1}(\Lambda_{n+1}, x) \) as follows: If \( \lambda_{n+1} = +1 \) (resp. \( -1 \)), then, for each \( y \in V_{n+1}(x) \) and each curve \( A \in \mathcal{C}_n(\Lambda_n, x) \) starting from \( y \), we put in \( \mathcal{C}_{n+1}(\Lambda_{n+1}, x) \) the curve \( ABC \), where:
1) \( B \) is the curve of \( \mathcal{C}_n(\Lambda_n, x) \) starting from the endpoint of \( A \) and such that its first segment is just at the left (resp. right) of the last segment of \( A \);
2) \( C \) is the curve of \( C_n(\Lambda_n, x) \) starting from the endpoint of \( B \) and such that its first segment is just at the right (resp. left) of the last segment of \( B \).

For each \( n \in \mathbb{N} \), each nonoriented side of a triangle of \( \mathcal{P} \) is the support of exactly 1 segment of 1 curve of \( C_n(\Lambda_n, x) \). Each curve of \( C_n(\Lambda_n, x) \) connects a pair of points of \( V_n(x) \) with minimal distance. Each such pair \((y, z)\) is connected by a unique curve of \( C_n(\Lambda_n, x) \). If \( n \geq 1 \), then this curve contains the 2 points of \( W_n(x) \) which are between \( y \) and \( z \).

For each \( n \in \mathbb{N} \), if \( C \) is a curve associated to \( T_{\Lambda_n} \), then \( C \) or its reverse belongs to \( C_n(\Lambda_n, x) = C_n(\Lambda_n, y) \), where \( x \) and \( y \) are the endpoints of \( C \). In particular, we have:

**Proposition 2.4.** Each folding t-curve is self-avoiding.

Now let us consider a sequence \( X = (x_n)_{n \in \mathbb{N}} \) with \( x_{n+1} \in V_n(x_n) \) for each \( n \in \mathbb{N} \). Then, for each \( n \in \mathbb{N} \), each curve of \( C_{n+1}(\Lambda_{n+1}, x_{n+1}) \) is obtained by concatenation of 3 curves of \( C_n(\Lambda_n, x_n) \). We denote by \( \mathcal{C}(\Lambda, X) \) the set of inductive limits of curves \( C_n \in C_n(\Lambda_n, x_n) \).

If \( \cap_{n \in \mathbb{N}} V_n(x_n) = \emptyset \), then \( \mathcal{C}(\Lambda, X) \) is a covering of the plane by complete folding t-curves. We write \( \mathcal{C}^0(\Lambda, X) = \mathcal{C}(\Lambda, X) \).

Otherwise, there exists \( x \in V \) such that \( \cap_{n \in \mathbb{N}} V_n(x_n) = \{x\} \). Then \( \mathcal{C}(\Lambda, X) \) is the union of 3 \( \infty \)-folding t-curves starting at \( x \) and 3 reversed such curves ending at \( x \) since, for each \( n \in \mathbb{N} \), each pair of points in \( W_{n+1}(x_{n+1}) = W_{n+1}(x) \) which forms a \( 3^n \)-triangle with \( x \) is contained in one of the 6 curves with endpoint \( x \) in \( C_{n+1}(\Lambda_{n+1}, x_{n+1}) = C_{n+1}(\Lambda_{n+1}, x) \).

Then we denote by \( \mathcal{C}^+(\Lambda, X) \) (resp. \( \mathcal{C}^-(\Lambda, X) \)) the set of curves obtained from \( \mathcal{C}(\Lambda, X) \) by connecting each terminal segment of reversed half curve with the initial segment of half curve just at its left (resp. right). By Proposition 2.1, each of these sets is a covering of the plane by 3 complete folding t-curves.

It follows from Proposition 2.1 that each complete folding t-curve can be extended into a covering \( \mathcal{C}^0(\Lambda, X) \). We shall prove later that, actually, it can only be extended into 1 covering. Consequently, all the coverings by complete folding t-curves are of the form \( \mathcal{C}^0(\Lambda, X) \).

**Proposition 2.5** below implies that:

1) the coverings \( \mathcal{C}^0(\Lambda, X) \) satisfy the local isomorphism property in a strong form, similarly to coverings by complete folding curves considered in [2] and [3], or coverings by complete Peano-Gosper curves considered in [4];

2) any coverings \( \mathcal{C}^0(\Lambda, X), \mathcal{C}^0(\Lambda, Y) \) are locally isomorphic.

In order to make proofs simpler, we do not consider the balls \( B(x, r) \), but “hexagonal balls”: for each vertex \( y \) of a triangle of \( \mathcal{P} \) and each \( k \in \mathbb{N}^* \), we denote by \( H(y, k) \) the regular hexagon of center \( y \) such that each of its sides is the union of \( k \) sides of triangles of \( \mathcal{P} \), and \( H^*(y, k) \) its interior.

**Proposition 2.5.** Consider two sequences \( X = (x_k)_{k \in \mathbb{N}} \) and \( Y = (y_k)_{k \in \mathbb{N}} \)
such that \( x_{k+1} \in V_k(x_k) \) and \( y_{k+1} \in V_k(y_k) \) for each \( k \in \mathbb{N} \). Let \( \alpha, \beta \in \{0, -, +\} \) be such that \( \mathcal{C}^\alpha(\Lambda, X) \) and \( \mathcal{C}^\beta(\Lambda, Y) \) exist. Then, for each \( n \in \mathbb{N}^* \) and any \( x, y \in V \), there exists \( z \in V \cap H(y, 5.3^n) \) such that \( \mathcal{C}^\alpha(\Lambda, X) \upharpoonright H^*(x, 3^n) \cong \mathcal{C}^\beta(\Lambda, Y) \upharpoonright H^*(z, 3^n) \).

**Proof.** There exists \( t \in V_{2n+2}(x_{2n+2}) \) such that \( x \in H(t, 2.3^n) \), and therefore \( H^*(x, 3^n) \subset H^*(t, 3^{n+1}) \). For each \( u \in V_{2n+2}(y_{2n+2}) \), the sets \( \mathcal{C}^\alpha(\Lambda, X) \upharpoonright H^*(t, 3^{n+1}) \) and \( \mathcal{C}^\beta(\Lambda, Y) \upharpoonright H^*(u, 3^{n+1}) \) are equivalent up to translation, except possibly concerning the way to connect the 6 segments with endpoint \( t \) (resp. \( u \)).

There exist \( u, v, w \in V_{2n+2}(y_{2n+2}) \) which form a \( 3^{n+1} \)-triangle containing \( y \), which implies \( u, v, w \in H(y, 3^{n+1}) \). One of these points, for instance \( w \), belongs to \( V_{2n+2}(x_{2n+3}) \), while the 2 others, in that case \( u \) and \( v \), belong to \( W_{2n+3}(y_{2n+3}) \). Then the connexions of the 6 segments of \( \mathcal{C}^\beta(\Lambda, Y) \) in \( u \) and \( v \) are different.

It follows that the connexions of \( \mathcal{C}^\alpha(\Lambda, X) \) in \( t \) are the same as the connexions of \( \mathcal{C}^\beta(\Lambda, Y) \) in \( u \), which implies \( \mathcal{C}^\alpha(\Lambda, X) \upharpoonright H^*(t, 3^{n+1}) \cong \mathcal{C}^\beta(\Lambda, Y) \upharpoonright H^*(u, 3^{n+1}) \), or the same as the connexions of \( \mathcal{C}^\beta(\Lambda, Y) \) in \( v \), which implies \( \mathcal{C}^\alpha(\Lambda, X) \upharpoonright H^*(t, 3^{n+1}) \cong \mathcal{C}^\beta(\Lambda, Y) \upharpoonright H^*(v, 3^{n+1}) \). Consequently, there exists \( z \in V \cap H(u, 2.3^n) \), or \( z \in V \cap H(v, 2.3^n) \), such that \( \mathcal{C}^\alpha(\Lambda, X) \upharpoonright H^*(x, 3^n) \cong \mathcal{C}^\beta(\Lambda, Y) \upharpoonright H^*(z, 3^n) \). In both cases, we have \( z \in V \cap H(y, 5.3^n) \).

For each folding t-curve \( C \) with at least 3 segments, we say that \( x \in V \) belongs to \( C \) and we write \( x \in C \) if \( x \) is an endpoint of segments of \( C \). We denote by \( F_L(C) \) (resp. \( F_R(C) \)) the union of the sides \([x, y]\) of triangles of \( \mathcal{P} \) such that \( x, y \in C \), such that \( C \) contains exactly 1 of the 2 points which form equilateral triangles with \( x \) and \( y \), and such that the second point is at the left (resp. right) of \( C \). We write \( F(C) = F_L(C) \cup F_R(C) \).

Now we state several properties of \( F(C) \). The properties for an \( n \)-folding t-curve are true because it belongs to a covering of the plane by \( n \)-folding t-curves which are equivalent up to rotation or translation. The properties for \( \infty \)-folding and complete folding t-curves follow since, if such a curve \( C \) is the inductive limit of some \( n \)-folding t-curves \( C_n \), then any segment \([x, y]\) is contained in \( F(C) \) if and only if it is contained in \( F(C_n) \) for \( n \) large enough.

First, let \( n \geq 1 \) be an integer and let \( C \) be an \( n \)-folding t-curve associated to a sequence \( T_{\lambda_1, \ldots, \lambda_n} \). Consider the initial points \( w, x \) of \( C^1, C^M \) and the terminal points \( y, z \) of \( C^M, C^S \). Then \( F_L(C) \cap F_R(C) = \{w, z\} \).

If \( \lambda_n = +1 \) (resp. \(-1\)), then \( y \in F_L(C) \) and \( x \in F_R(C) \) (resp. \( x \in F_L(C) \) and \( y \in F_R(C) \)). We denote by \( F_{LI}(C) \) the part of \( F_L(C) \) between \( w \) and \( y \) (resp. \( x \)), \( F_{LS}(C) \) the part of \( F_L(C) \) between \( y \) (resp. \( x \)) and \( z \), \( F_{RI}(C) \) the part of \( F_R(C) \) between \( w \) and \( x \) (resp. \( y \)), \( F_{RS}(C) \) the part of \( F_R(C) \) between \( x \) (resp. \( y \)) and \( z \).
Now suppose $n \geq 2$. If $\lambda_n = +1$, then we have $F_{LI}(C) = F_{LI}(C^1) F_{LS}(C^M)$, $F_{LS}(C) = F_L(C^S)$, $F_{RI}(C) = F_{R}(C^1)$, $F_{RS}(C) = F_{RI}(C^M) F_{RS}(C^S)$. If $\lambda_n = -1$, then we have $F_{LI}(C) = F_{L}(C^1)$, $F_{LS}(C) = F_{LI}(C^M) F_{LS}(C^S)$, $F_{RI}(C) = F_{RI}(C^1) F_{RS}(C^M)$, $F_{RS}(C) = F_{R}(C^S)$.

Now consider an integer $n \geq 1$ and 6 nonoverlapping $n$-folding t-curves $C_1, \ldots, C_6$ associated to the same sequence $T_{\lambda_1, \ldots, \lambda_n}$ and having the common endpoint $x$. Suppose that $C_1, \ldots, C_6$ are disposed consecutively clockwise around $x$ and that $x$ is the terminal point of $C_1, C_3, C_5$ and the initial point of $C_2, C_4, C_6$. Write $C_7 = C_1$. Then we have $F(C_i) \cap F(C_{i+1}) = F_{LS}(C_i) = F_{LI}(C_{i+1})$ for $i = 1, 3, 5$ and $F(C_i) \cap F(C_{i+1}) = F_{RI}(C_i) = F_{RS}(C_{i+1})$ for $i = 2, 4, 6$.

If $C$ is an $\infty$-folding t-curve with initial point $x$, then $F_L(C), F_R(C)$ are half curves and $F_L(C) \cap F_R(C) = \{x\}$. If $C$ is a complete folding t-curve, then each of the curves $F_L(C), F_R(C)$ is complete or empty; we have $F_L(C) \cap F_R(C) = \emptyset$.

For each folding t-curve $C$ with at least 3 segments, each side $S$ of a triangle of $P$ is the support of a segment of $C$ if it is inside $F(C)$. It is not the support of a segment of $C$ if it is outside $F(C)$. It is possibly the support or not the support of a segment of $C$ if it is on $F(C)$.

If $C = (C_i)_{i \in I}$ is a covering of the plane by folding t-curves with at least 3 segments, then the regions $R_i$ containing the curves $C_i$ and limited by the curves $F(C_i)$ are nonoverlapping and cover the plane.

**Lemma 2.6.** Consider $n \in \mathbb{N}^*$ and $\lambda_1, \ldots, \lambda_n \in \{-1, +1\}$. Let $C$ be an $n$-folding t-curve associated to $T_{\lambda_1, \ldots, \lambda_n}$. Then there exist some sequences $(x_i)_{0 \leq i \leq 2^n}$ and $(y_i)_{0 \leq i \leq 2^n}$, with $x_0 = y_0 = 0$ initial point of $C$ and $x_{2^n} = y_{2^n}$ terminal point of $C$, such that:

1) each segment $[x_i, x_{i+1}]$ or $[y_i, y_{i+1}]$ is a side of a triangle of $P$;

2) each angle $\angle x_{i-1}x_i x_{i+1}$ or $\angle y_{i-1}y_i y_{i+1}$ is equal to $\pm 2\pi/3$;

3) $F_{LI}(C) = \cup_{1 \leq i \leq 2^n-1} [x_{i-1}, x_i]$, $F_{LS}(C) = \cup_{2^n-1+1 \leq i \leq 2^n} [x_{i-1}, x_i]$, $F_{RI}(C) = \cup_{1 \leq i \leq 2^n-1} [y_{i-1}, y_i]$, $F_{RS}(C) = \cup_{2^n-1+1 \leq i \leq 2^n} [y_{i-1}, y_i]$;

4) for $0 \leq i \leq 2^n$ and $1 \leq k \leq n$, each point $x_i, y_i$ belongs to $V_k(C)$ if and only if $i$ is divisible by $2^k$.

Now associate to $F_{LI}(C)$ and $F_{RI}(C)$ the sequences $(\alpha_i)_{1 \leq i \leq 2^n-1}$ and $(\beta_i)_{1 \leq i \leq 2^n-1}$ with $\alpha_i = +1$ (resp. $-1$) if $x_{i-1}x_i x_{i+1} = +2\pi/3$ (resp. $-2\pi/3$), and $\beta_i = +1$ (resp. $-1$) if $y_{i-1}y_i y_{i+1} = +2\pi/3$ (resp. $-2\pi/3$). Then we have $\alpha_{2^n-1} = -1$, $\beta_{2^n-1} = +1$ and $\alpha_{2^k+2^k+1+i} = \beta_{2^k+2^k+i} = (-1)^i \lambda_{k+2}$ for $0 \leq k \leq n-2$ and $0 \leq i \leq 2^{n-1} - 1$.

**Proof.** Lemma 2.6 is clearly true for $n = 1$. Now suppose that it is true for an integer $n \geq 1$ and consider an $(n + 1)$-folding t-curve $C$ associated to a sequence $T_{\lambda_1, \ldots, \lambda_{n+1}}$. We can assume $\lambda_{n+1} = +1$ since the case $\lambda_{n+1} = -1$ is
similar. Also, we only show the results for $F_L(C)$ since the proof for $F_R(C)$ is similar.

The curves $C^I, C^M, C^S$ are associated to $T_{\lambda_1, \ldots, \lambda_n}$. We have $F_{LI}(C) = F_{LI}(C^I)F_{LS}(C^M)$ and $F_{LS}(C) = F_L(C^S) = F_{LI}(C^S)F_{LS}(C^S)$. The sequences associated to $F_{LI}(C^I)$ and $F_{LI}(C^S)$ are equal. The sequences associated to $F_{LS}(C^M)$ and $F_{LS}(C^S)$ are also equal. We turn with a $+2\pi/3$ (resp. $-2\pi/3, -2\pi/3$) angle when we pass from $F_{LI}(C^I)$ to $F_{LS}(C^M)$ (resp. from $F_{LS}(C^M)$ to $F_{LI}(C^S)$, from $F_{LI}(C^S)$ to $F_{LS}(C^S)$). Consequently, the results follow from the induction hypothesis. $\blacksquare$

**Corollary 2.7.** Let $C$ be a complete folding t-curve such that $F_L(C)$ (resp. $F_R(C)$) is nonempty. Consider the sequence $(x_n)_{n \in \mathbb{Z}}$ of endpoints of segments of $F_L(C)$ (resp. $F_R(C)$) and the sequence $(\alpha_n)_{n \in \mathbb{Z}}$, with $\alpha_n = +1$ (resp. $-1$) if $x_{n-1}x_nx_{n+1} = +2\pi/3$ (resp. $-2\pi/3$). Then there exists a sequence $(n_k)_{k \in \mathbb{N}^*}$ such that $x_{n_k+2^{k+1}} \in W_k(C)$ and $\alpha_{n_k+2^k} = (-1)^i\alpha_{n_k}$ for each $k \in \mathbb{N}$ and each $i \in \mathbb{Z}$.

**Remark.** If $C$ is obtained from an $\infty$-folding t-curve, then there exists a unique $n \in \mathbb{Z} - \bigcup_{k \in \mathbb{N}^*}(n_k + 2^k\mathbb{Z})$; we have $x_n \in \bigcap_{k \in \mathbb{N}^*}V_k(C)$. Otherwise, we have $\mathbb{Z} = \bigcup_{k \in \mathbb{N}^*}(n_k + 2^k\mathbb{Z})$.

**Lemma 2.8.** Any set of disjoint complete folding t-curves is finite.

**Proof.** We assume that the sides of triangles of $\mathcal{P}$ have length 1 and we show by induction on $n \in \mathbb{N}^*$ that $d(x, y) \leq \rho_n = \left\lfloor (\sqrt{3})^{n-1}(4 - \sqrt{3}) - 1 \right\rfloor / (\sqrt{3} - 1)$ for any points $x, y$ of an $n$-folding t-curve. We have $\rho_1 = \sqrt{3}$ and the property is clearly true for $n = 1$.

We prove that, if the property is true for an integer $n \geq 1$, then it is also true for $n + 1$. We consider an $(n + 1)$-folding t-curve $C$. We represent $D = \Delta(C)$ in such a way that $C$ and $D$ have the same endpoints, which gives the length $\sqrt{3}$ to the segments of $D$.

By the induction hypothesis, for $D$ represented in that way, we have $d(x, y) \leq \rho_n\sqrt{3}$ for any $x, y \in D$. Moreover, for each $z \in C$, there exists $y \in D$ such that $d(y, z) \leq 1/2$. It follows that, for any $v, w \in C$, we have $d(v, w) \leq \rho_n\sqrt{3} + 1 = \rho_{n+1}$.

Now we see that, for $x \in \mathbb{R}^2$ and $k \in \mathbb{N}$ large enough, there exists no $r \in \mathbb{R}^+$ such that $B(x, r)$ contains points of $k$ disjoint $n$-folding t-curves for each $n \in \mathbb{N}^*$. This follows since each $n$-folding t-curve contains $3^n$ sides of triangles of $P$, and any such curve containing a point of $B(x, r)$ is necessarily contained in $B(x, r + \rho_n)$ by the property above. $\blacksquare$

**Lemma 2.9.** Let $\mathcal{C}$ be a covering of the plane by complete folding t-curves. Then we have $V_k(C) = V_k(D)$ for each $k \in \mathbb{N}^*$ and any $C, D \in \mathcal{C}$, and all the curves of $\mathcal{C}$ are associated to the same sequence $(\lambda_n)_{n \in \mathbb{N}^*}$. 

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**Proof.** For each such \( C \) and each \( C \in \mathcal{C} \), we consider the sequence \( \Lambda(C) = (\lambda_n(C))_{n \in \mathbb{N}^*} \) associated to \( C \). It suffices to show that each such \( C \) satisfies \( V_1(C) = V_1(D) \) and \( \lambda_1(C) = \lambda_1(D) \) for any \( C, D \in \mathcal{C} \), since these properties imply that \( \{\Delta(C) \mid C \in \mathcal{C}\} \) is also a covering of the plane by complete folding \( t \)-curves. By Lemma 2.8, it is enough to prove \( V_1(C) = V_1(D) \) and \( \lambda_1(C) = \lambda_1(D) \) when \( F(C) \cap F(D) \) is unbounded.

We consider some consecutive segments \([x_0, x_1], \ldots, [x_8, x_9]\) of \( F(C) \cap F(D) \), and the integers \( \alpha_1, \ldots, \alpha_8 \in \{-1, +1\} \) associated to the vertices \( x_1, \ldots, x_8 \). According to Corollary 2.7, we have \( \alpha_1 = -\alpha_3 = \alpha_5 = -\alpha_7 \) or \( \alpha_2 = -\alpha_4 = \alpha_6 = -\alpha_8 \), and the 2 properties cannot be simultaneously true since the first one implies \( \alpha_2 = -\alpha_6 \) or \( \alpha_4 = -\alpha_8 \).

Let us suppose for instance that \( \alpha_1 = -\alpha_3 = \alpha_5 = -\alpha_7 \). Then we have \( x_2 \in V_1(C) \) and \( x_2 \in V_1(D) \), whence \( V_1(C) = V_1(D) \). It follows \( \lambda_1(C) = \lambda_1(D) \) because each side of the hexagon \( H(x_2, 1) \) is the support of a segment which belongs to 1 of the 6 sequences of 3 segments starting or ending at \( x_2 \), and each of these sequences is contained in \( C \) or in \( D \) and has its other endpoint in \( V_1(C) = V_1(D) \). ■

It follows from Lemma 2.9 that each covering of the plane by nonoriented complete folding \( t \)-curves has exactly 2 representatives of the form \( \mathcal{C}^\alpha(\Lambda, X) \), and their orientations are opposite. Consequently, for the proofs of Theorems 1.1, 1.2 and 1.3, it suffices to consider the coverings of the plane by oriented complete folding \( t \)-curves which are of that form.

When we defined the coverings \( \mathcal{C}^\alpha(\Lambda, X) \) with \( \alpha \in \{-, +\} \), we observed that any such covering \( \mathcal{C} \) consists of 3 curves whose common point is the unique point \( x \in \cap_{n \in \mathbb{N}} V_n(C) \).

**Proof of Theorems 1.1 and 1.2.** The first statement of Theorem 1.1 is true because, if \( \mathcal{C} \) is such a covering, then the unique point \( x \in \cap_{n \in \mathbb{N}} V_n(C) \), if it exists, belongs to each curve \( C \in \mathcal{C} \), so that the connections in \( x \) are determined by \( C \). The statements concerning local isomorphism follow from Proposition 2.5 and Lemma 2.9. ■

**Proof of Theorem 1.3.** First consider a covering \( \mathcal{C} \) associated to \( \Lambda \) such that there exists \( x \in \cap_{n \in \mathbb{N}} V_n(C) \). Then \( \mathcal{C} \) realizes the case 3) and we obtain a representative of the other isomorphism class of coverings of that type by changing the connections in \( x \). We observe that \( \mathcal{C} \) is invariant through a rotation of center \( x \) and angle \( 2\pi/3 \).

Now we consider the coverings \( \mathcal{C} \) associated to \( \Lambda \) which are not of that type. Then, for each \( C \in \mathcal{C} \), there exist some sequences \((C_n)_{n \in \mathbb{N}}\) and \((P_n)_{n \in \mathbb{N}}\) such that \( C = \cup_{n \in \mathbb{N}} C_n \) and, for each \( n \in \mathbb{N}, C_n \) is an \( n \)-folding \( t \)-curve, \( P_n \in \{I, M, S\} \) and \( C_n = C_{n+1}^{P_n} \).
If \( C \) contains at least 3 curves, then there exists 3 different curves \( C, D, E \) such that \( F(C) \cap F(D) \neq \emptyset \) and \( F(C) \cap F(E) \neq \emptyset \). Consider a segment \([u, v] \subset F(C) \cap F(D)\) and suppose for instance that \([u, v] \subset F_L(C)\). Then we also have \([u, v] \subset F_L(D)\).

Consider the sequences \((C_n)_{n \in \mathbb{N}}\) and \((P_n)_{n \in \mathbb{N}}\) defined as above. Then there exist \( h \in \mathbb{N}^* \) such that \([u, v] \subset F_L(C_n)\) for each \( n \geq h \), and \((\alpha_n)_{n \geq h}\) such that \( \alpha_n \in \{I, S\} \) and \([u, v] \subset F_{L\alpha_n}(C_n)\) for each \( n \geq h \). It follows \( F_{L\alpha_n}(C_n) \subset F_{L\alpha_n+1}(C_{n+1}) \) for \( n \geq h \). We have \( F_L(C) = \bigcup_{n \geq h} F_{L\alpha_n}(C_n) \) since, for each \( n \geq h \), \( \{\alpha_n, \alpha_n'\} = \{I, S\} \) implies \( F_{\alpha_n'}(C_n) \subset F_{L\alpha_n+1}(C_{n+1}) \) or \( F_{\alpha_n'}(C_n) \cap F_L(C_{n+1}) = \emptyset \).

Consider \((D_n)_{n \in \mathbb{N}}\) and \((Q_n)_{n \in \mathbb{N}}\) such that \( D = \bigcup_{n \in \mathbb{N}} D_n \) and, for each \( n \in \mathbb{N} \), \( D_n \) is an \( n \)-folding t-curve, \( Q_n \in \{I, M, S\} \) and \( D_n = D_n^{Q_n+1} \). Then there exist \( k \in \mathbb{N}^* \) such that \([u, v] \subset F_L(D_n)\) for each \( n \geq k \), and \((\beta_n)_{n \geq k}\) such that \( \beta_n \in \{I, S\} \) and \([u, v] \subset F_{L\beta_n}(D_n)\) for each \( n \geq k \). We have \( F_{L\beta_n}(C_n) = F_{L\beta_n}(D_n) \) for \( n \geq \sup(h, k) \). It follows \( F_L(C) = F_L(D) \) since we also have \( F_{L\beta_n}(D_n) \subset F_{L\beta_n+1}(D_{n+1}) \) for \( n \geq k \) and \( F_L(D) = \bigcup_{n \geq k} F_{L\beta_n}(D_n) \).

By the same arguments as above, we have \( F_R(C) = F_R(E) \) and there exist an integer \( l \) and a sequence \((\gamma_n)_{n \geq l}\) such that \( F_R(C) = \bigcup_{n \geq l} F_{R\gamma_n}(C_n) \), with \( \gamma_n \in \{I, S\} \) and \( F_{R\gamma_n}(C_n) \subset F_{R\gamma_n+1}(C_{n+1}) \) for \( n \geq l \).

For each \( n \geq \sup(h, l) \), one of the 4 cases below is realized:

- \( P_n = I \) and \((\alpha_n, \gamma_n) = (I, I)\);
- \( P_n = S \) and \((\alpha_n, \gamma_n) = (S, S)\);
- \( P_n = M \) and \((\alpha_n, \gamma_n) = (I, S)\), \( \lambda_{n+1} = +1 \);
- \( P_n = M \) and \((\alpha_n, \gamma_n) = (S, I)\), \( \lambda_{n+1} = -1 \).

Consequently, there exists an integer \( m \) such that one of the properties below is true for all the integers \( n \geq m \):

- a) \( P_n = I \) and \((\alpha_n, \gamma_n) = (I, I)\);
- b) \( P_n = S \) and \((\alpha_n, \gamma_n) = (S, S)\);
- c) \( P_n = M \) and \((\alpha_n, \gamma_n) = (I, S)\) and \( \lambda_n = +1 \) for \( n \) even; \((\alpha_n, \gamma_n) = (S, I)\) and \( \lambda_n = -1 \) for \( n \) odd;
- d) \( P_n = M \) and \((\alpha_n, \gamma_n) = (I, S)\) and \( \lambda_n = +1 \) for \( n \) odd; \((\alpha_n, \gamma_n) = (S, I)\) and \( \lambda_n = -1 \) for \( n \) even.

The cases a) and b) imply \( \bigcup_{n \in \mathbb{N}} C_n \neq C \), whence a contradiction.

On the other hand, each of the cases c), d), for each of the values of \( \Lambda \) which realize it, gives a covering of the plane by 3 complete folding t-curves, with 1 of them separating the 2 others. This covering is determined up to isometry by \( \Lambda \), since we have \( P_n = M \) for each \( n \geq m \).

In this situation, we obtain representatives of the 2 other isomorphism classes of coverings by applying a rotation of angle \( \pi/3 \).

Each covering of the plane by at least 3 complete folding t-curves has one of the forms considered above. In particular, it cannot contain more than 3 curves.
It remains to be proved that there exist $2^\omega$ isomorphism classes of coverings of the plane by 1 (resp. 2) complete folding t-curves associated to $\Lambda$. Each curve $C$ in such a covering is determined up to isometry by the sequence $(P_n)_{n \in \mathbb{N}}$ defined as above. There are countably many sequences which give $C$, since any 2 such sequences only differ by a finite number of terms. Consequently, it suffices to show that $2^\omega$ choices of $(P_n)_{n \in \mathbb{N}}$ give coverings by 1 (resp. 2) complete folding t-curves.

In order to obtain a curve $C$ which covers the plane, it suffices to have $F_L(C_{8n}) \cap F_L(C_{8n+2}) = \emptyset$ and $F_R(C_{8n+2}) \cap F_R(C_{8n+4}) = \emptyset$ for each $n \in \mathbb{N}$. The first property is realized for $P_{8n} = I$ and $P_{8n+1} = M$ (resp. $S$) if $\lambda_{8n+2} = +1$ (resp. $-1$). The second property is realized for $P_{8n+2} = I$ and $P_{8n+3} = S$ (resp. $M$) if $\lambda_{8n+4} = +1$ (resp. $-1$). For each $n \in \mathbb{N}$, $P_{8n+4}, \ldots, P_{8n+7}$ can be chosen arbitrarily.

Now we prove that $2^\omega$ choices of $(P_n)_{n \in \mathbb{N}}$ give coverings by 2 curves. As only countably many choices give coverings by 3 curves, it suffices to show that $2^\omega$ choices give a curve $C$ with $F_L(C) \neq \emptyset$.

For each $n \in \mathbb{N}^*$, we consider the sequences $(P_1, \ldots, P_{n-1})$, $(C_1, \ldots, C_n)$, $(\alpha_1, \ldots, \alpha_n)$, with $C_i$ an $i$-folding t-curve for $1 \leq i \leq n$, $C_i = C_{i+1}^P$ and $F_{L\alpha_i}(C_i) \subset F_{L\alpha_{i+1}}(C_{i+1})$ for $1 \leq i \leq n-1$. We observe that, for any such sequences, there are 2 different ways to choose $P_{n+1}$ so that the $(n+1)$-folding t-curve $C_{n+1}$ defined by $C_n = C_{n+1}^P$ satisfies $F_{L\alpha_n}(C_n) \subset F_L(C_{n+1})$, and therefore $F_{L\alpha_n}(C_n) \subset F_{L\alpha_{n+1}}(C_{n+1})$ for the appropriate $\alpha_{n+1}$. ■

**Remark.** Each covering of the plane by 3 nonoriented complete folding t-curves is invariant through a central symmetry.

**Proof of Theorem 1.4.** If $C$ is a covering of the plane by oriented complete folding t-curves which satisfies (P), then, for any $x, y \in V$, the orientation of the 6 segments with endpoint $x$ and the orientation of the 6 segments with endpoint $y$ are equivalent up to translation.

Using this fact, we see that the first statement of Theorem 1.4 follows from Corollary 2.3 and Proposition 2.5, the second statement follows from Theorem 1.2 and the third statement is clear. ■

**References**

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