Introduction to Symplectic Topology: corrigenda

Several readers have pointed out to us various small errors and typos in this book. All are minor except for an error in the statement of Theorem 3.17 on p 94 spotted by David Theret. We thank him as well as everyone else who told us of these errors.

The first part of this note is a list of short corrections. The second part contains some longer revised passages.

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A list of Short Corrections

p 10 line 10: "...such as a pendulum or top..."
p 17 line 7: \( J_0 X_{H_t} = \nabla H_t \)
p 22 line 20: "...their solutions." line 28: "...function of the variables..."
p. 24 formula (1.19): for consistency with later formulas there should be a \(-\) sign in this equation
p. 39, line 5 of Proof: for all \( v \in W \)
p. 43 On line 14 the text should read: "To prove this, we choose a positive definite and symmetric matrix \( P \in \text{Sp}(2n) \) such that...." and on line 21/22: "...choosing \( P \) is equivalent to choosing a \( G \)-invariant inner product on \( \mathbb{R}^{2n} \) that is compatible with \( \omega_0 \) in the sense that it has the form \( \omega_0(\cdot, J \cdot) \) for some \( \omega_0 \)-compatible almost complex structure \( J \)."
p. 44 line 1/2: "...sends a matrix \( U \in \text{SU}(n) \) to..."
p. 47 line 18: \( \mu(\Psi) = \frac{1}{2} \sum_t \text{sign} \Gamma(\Psi, t) \)
p. 50 line -3: delete repetition of "intersection"
p. 51 line 10: \(-\langle X(t)u, Y(t)u \rangle \)
p. 53 lines 5 and 6 of Proof: replace \( \omega_0(\Psi^T v, \Psi^T w) \) by \( \omega_0(\Psi^T u, \Psi^T v) \) twice
p. 55 line 10 of Proof: \( A \bar{z}_j = -i \alpha_j z_j \)
p. 60 line 5.6: replace \( E^\omega \) by the orthogonal complement \( E^\perp \)
p. 65 line 10: delete repetition of "that"
p. 71 Exercise 2.66: It would be more clear to say: "there are precisely two" instead of "one nontrivial"
p. 77 last line of Exercise 2.72: \( c_1(\nu_{CP^1}) = 1. \)
p. 79, line -10: "Nondegeneracy..."
p. 82 line -1 of proof: \( \iota(\psi_t^* X) \omega = \psi_t^*(\iota(X) \omega) \)
p. 86, line 2: "By Exercise 2.15, every such..."
p. 89, line 2: "...any vector \( v^* \in T^*_q L \) can..."
p. 94 Statement of Theorem 3.17: Delete the last sentence. As David Theret pointed out, it is easy to find a counterexample to this statement unless one requires that the class \([\omega_t] - [\omega_0] \in H^2(M, Q; \mathbb{R})\) is constant.

p. 96 line -4 of proof: replace \(N\) by \(N\). last 3 lines of Proof: Name the diffeomorphisms \(\chi_t\) instead of \(\phi_t\). Then \(\chi_t^*\omega_t = \omega_0 = \omega\) and the desired extension is \(\rho_t \circ \chi_t\).

p. 100 line 2: \(\phi: \mathcal{N}(L_0) \rightarrow V\)

p. 103 line -16: "...that \(d\alpha\) restricts to..." line -2: Replace Corollary 2.4 by Corollary 2.5

p. 111 line -3: "...symplectization of \(Q\)."

p. 113 line -2 of Proof: \(\psi^*\omega = e^\theta (d\alpha - \alpha \wedge d\theta)\)

p. 115 line 14: "...\(f: \mathbb{C}^{n+1} - \mathbb{C}^n \times \{-1\} \rightarrow \mathbb{C}^{n+1} - \mathbb{C}^n \times \{-i\}\)"

p. 117 line -10: delete the repetition of "of"

p. 215 line 3 in Lemma 6.31: bracket missing in "\(\langle c_1(\nu_\Sigma), [\Sigma] \rangle\)".

p. 219 Exercise 6.38: the curve \(C_3\) should be \(\{z_1 = z_2^3\}\).

p. 240 last line in definition of \(\tilde{\tau}_t(x)\): Replace "\(c - \varepsilon \leq f(x)\)" by "\(f(x) \leq c - \varepsilon\)".

p. 253 lines 8,9: "...that, up to diffeomorphism, there is..."

p. 265 The proof of Lemma 8.2 must be revised (see below).

p. 273 line -3: Replace "\(\mathbb{R}^{2n}\)" by "\(\mathbb{R}^2\)".

p. 274 line 1/2: Replace "\(\mathbb{R}^n\)" by "\(\mathbb{R}\)" and "\(\mathbb{R}^{2n}\)" by "\(\mathbb{R}^2\)".

p. 304 line 2: "...the above action function..."

p. 306 line 12: "...where \(A = A^T = \partial_x \partial_\psi \Phi \in \mathbb{R}^{n \times n}\)"

p. 337 line 14: replace "\(\cup a_k\)" by "\(\cup a_N\)".

p. 340 line -11: "...every \(t\). A compact invariant set..."

p. 360 line -3: \(\psi\) is a symplectic embedding not a symplectomorphism

p. 361 line 16: Replace \(\overline{T}_G\) by \(\overline{w}_G\)
p 362 line 1: \( c(E) = \pi r_1^2 = w_L(E) \)

p 364 line 5 of Proof: the \( \psi_t \) are diffeomorphisms not symplectomorphisms

p. 367 lines -7, -3: \( \mathcal{L}(\{\phi_t\}) \)

p 374 line -12: \( f \) should be a smooth embedding rather than a diffeomorphism

p. 376 line 6: the restriction of \( \psi_H \) to \( Z_{2e} \) is called \( \Psi_H \)

page 72: Remark 2.68 (ii)

Several students have pointed out that the sentence “The axioms imply that this integer depends only on the homology class of \( f \)” is hard to substantiate.

Change this and the rest of Remark 2.68 (ii) to:

“We will see in Exercise 2.75 this integer depends only on the homology class of \( f \). Thus the first Chern number generalizes to a homomorphism \( H_2(M, \mathbb{Z}) \rightarrow \mathbb{Z} \). This gives rise a cohomology class \( c_1(E) \in H^2(M, \mathbb{Z})/\text{torsion} \). There is in fact a natural choice of a lift of this class to \( H^2(M, \mathbb{Z}) \), also denoted by \( c_1(E) \), which is called the first Chern class. We shall not discuss this lift in detail, but only remark that in the case of a line bundle \( L \rightarrow M \) the class \( c_1(L) \in H^2(M, \mathbb{Z}) \) is Poincaré dual to the homology class determined by the zero set of a generic section.”

Then replace the current exercises 2.75 and 2.76 by the following new version:

**Exercise 2.75 (i)** Prove that every symplectic vector bundle \( E \) over a Riemann surface \( \Sigma \) decomposes as a direct sum of 2-dimensional symplectic vector bundles. **Hint:** Show that any such vector bundle of rank \( > 2 \) has a nonvanishing section.

(ii) Suppose that \( \Sigma \) is oriented and that the bundle \( E \) above extends over a compact oriented 3-manifold \( X \) with boundary \( \partial X = \Sigma \). Prove that the restriction \( E|\Sigma \) has Chern class zero. **Hint:** Use (i) above and look at a section \( s \) as in Theorem 2.71.

(iii) Use (i) and (ii) above to substantiate the claim made in Remark 2.68 above that the Chern class \( c_1(f^*E) \) depends only on the homology class of \( f \). Here the
main problem is that when \( f_\ast([\Sigma]) \) is null-homologous the 3-chain that bounds it need not be representable by a 3-manifold. However its singularities can be assumed to have codimension 2 and so the proof of (ii) goes through. 

This is a new exercise that should go at the very end of Chapter 2.

**Exercise** Prove that every symplectic vector bundle \( E \to \Sigma \) over a Riemann surface \( \Sigma \) which admits a Lagrangian subbundle can be symplectically trivialized. **Hint:** Use the proof of Theorem 2.67 to show that \( c_1(E) = 0 \).

This is a revised version of Lemma 8.2.

**Lemma 8.2** The Poincaré section \( \Sigma \cap U \) is a symplectic submanifold of \( M \) and the Poincaré section map \( \psi : \Sigma \cap U \to \Sigma \) is a symplectomorphism.

**Proof:** The hypersurface \( \Sigma \) is of dimension \( 2n - 2 \) and the tangent space at \( p \) is

\[ T_p\Sigma = \{ v \in T_pM \mid dG(p)v = dH(p)v = 0 \}. \]

The condition \( \{ G, H \} = \omega(X_G, X_H) \neq 0 \) shows that the 2-dimensional subspace spanned by \( X_G(p) \) and \( X_H(p) \) is a complement of \( T_p\Sigma \). Now let \( v \in T_p\Sigma \) and suppose that \( \omega(v, w) = 0 \) for all \( w \in T_p\Sigma \). Then \( \omega(X_H(p), v) = dH(p)v = 0 \) and \( \omega(X_G(p), v) = dG(p)v = 0 \) and hence \( v = 0 \). Thus the 2-form \( \omega \) is nondegenerate on the subspace \( T_p\Sigma \subset \mathbb{R}^{2n} \).

To prove that \( \psi \) is a symplectomorphism we consider the 2-form

\[ \omega_H = \omega + dH \wedge dt \]

on \( \mathbb{R} \times M \). This is the **differential form of Cartan**. It has a 1-dimensional kernel consisting of those pairs \((\theta, v) \in \mathbb{R} \times T_pM\) which satisfy

\[ v = \theta X_H(p). \]

Now let \( D \subset \mathbb{C} \) denote the unit disc in the complex plane and let \( u : D \to \Sigma \) be a 2-dimensional surface in \( \Sigma \). We must prove that

\[ \int_D u^* \psi^* \omega = \int_D u^* \omega. \]

To see this consider the manifold with corners

\[ \Omega = \{(t, z) \mid z \in D, 0 \leq t \leq \tau(u(z))\} \]

and define \( v : \Omega \to \mathbb{R} \times M \) by

\[ v(t, z) = \left( t, \phi^t(u(z)) \right). \]
Denote $v_0(z) = v(0, z)$ and $v_1(z) = v(\tau(u(z)), z)$. Then $v_0^* \omega_H = u^* \omega$ and $v_1^* \omega_H = u^* \psi^* \omega$. Moreover, the tangent plane to the surface $v(R \times \partial D)$ contains the kernel of $\omega_H$. Hence the 2-form $v^* \omega_H$ vanishes on the surface $R \times \partial D$. Since $\omega_H$ is closed it follows from Stokes’ theorem that

$$0 = \int_\Omega v^* d\omega_H = \int_{\partial \Omega} \omega_H = \int_D u^* \psi^* \omega - \int_D u^* \omega.$$ 

Hence $\psi$ is a symplectomorphism. ✷

This is a revised version of Lemma 12.37 and Exercise 12.38.

**Lemma 12.37** Let $H$ be any Hamiltonian which equals

$$H_\infty(z) = (\pi + \varepsilon)|z_1|^2 + \frac{1}{2}\pi|z_r|^2.$$ 

for large $|z|$. Then the functional $\Phi^*_H : R^{2nN} \to R$ satisfies the Palais–Smale condition.

**Proof:** The Palais–Smale condition asserts that for every sequence $z^\nu$ in $R^{2nN}$

$$\|\text{grad} \Phi^*_H(z^\nu)\|_\tau \to 0 \quad \implies \quad \sup_\nu \|z^\nu\|_\tau < \infty.$$ 

Suppose otherwise that $\|z^\nu\|_\tau \to \infty$. Then, since $\text{grad} \Phi^*_H(z^\nu)$ converges to zero, we claim that all components $z_j^{\nu}$ of $z^\nu$ must diverge to infinity. Clearly, this will follow if we prove the inequality

$$\min_j |z_j^{\nu}| \geq \frac{1}{c} \max_j |z_j^{\nu}| - 1$$

for some constant $c \geq 1$ which is independent of $\nu$. A proof of this is sketched in Exercise 12.38 below. Hence we may assume that the $z_j^{\nu}$ all lie in a region in which $H(z) = H_\infty(z)$. Now consider the sequence

$$w^\nu = \frac{z^\nu}{\|z^\nu\|_\tau}.$$ 

This sequence has a convergent subsequence, and it is easy to check that the limit has norm 1 and is a critical point of $\Phi^*_{H_\infty}$. But, because the flow of $H_\infty$ has no nonconstant periodic orbits of period 1, the fixed point 0 is the only critical point of this functional. This contradiction proves the lemma. ✷

**Exercise 12.38** This exercise fills in a missing detail in the proof of the above Lemma. Given a vector $z$ in $R^{2nN}$ with components $z_j = (x_j, y_j) \in R^{2n}$ denote by $\zeta_j = (\xi_j, \eta_j) \in R^{2n}$ the $j$th component of $\text{grad} \Phi^*_H(z)$. Then $x_{j+1}$ is the unique solution of the equation

$$x_{j+1} = F_j(x_j + 1, \eta_j)$$
where

\[ F_j(x_{j+1}, \eta_j) = x_j + \tau \frac{\partial V_\tau}{\partial y}(x_{j+1}, y_j) + \tau \eta_j. \]

Prove that for sufficiently small \( \tau \) and any \( \eta_j \) the map \( x_{j+1} \mapsto F_j(x_{j+1}, \eta_j) \) is a contraction with Lipschitz constant \( \alpha = \tau \sup_x |\partial^2 V_\tau/\partial x \partial y(x, y_j)| < 1 \). Deduce that

\[ |x_{j+1} - x_j| \leq \frac{\tau}{1 - \alpha} \left| \frac{\partial V_\tau}{\partial y}(x_j, y_j) + \eta_j \right|. \]

Use this and the inequality

\[ |y_{j+1} - y_j| \leq \tau \left| \frac{\partial V_\tau}{\partial x}(x_{j+1}, y_j) + \xi_{j+1} \right| \]

to conclude that, if \( \|\text{grad } \Phi_H^\tau(z)\|_\tau \leq 1 \) and \( \tau \) is sufficiently small then

\[ |z_{j+1} - z_j| \leq \frac{1}{2}(|z_j| + 1). \]

This implies

\[ 2^{-j}(|z_0| + 1) \leq |z_j| + 1 \leq 2^j(|z_0| + 1) \]

for \( j = 0, \ldots, N \). Hence, if \( z_\nu \in \mathbb{R}^{2nN} \) is a sequence with \( \|\text{grad } \Phi_H^\tau(z_\nu)\|_\tau \leq 1 \) and \( \tau \) sufficiently small such that \( \|z_\nu\|_\tau \to \infty \), then \( \|z_{\nu_j}\|_{\mathbb{R}^{2n}} \to \infty \) for all \( j \). ∎

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