Normal forms of two-dimensional metrics admitting exactly one essential projective vector field

Gianni Manno; Andreas Vollmer†

November 28, 2017

Abstract

We give a complete list of mutually non-isometric normal forms for the two-dimensional metrics that admit one projective vector field, which is essential, i.e. non-homothetic. This extends the results of [6, 13], solving a problem posed by Sophus Lie in 1882 [11].

1 Basic definitions and description of the main results

Let $(M,g)$ be a smooth Riemannian or pseudo-Riemannian manifold of dimension 2. In the following, the abbreviation “metric” is used for both Riemannian and pseudo-Riemannian metrics, unless otherwise specified.

Definition 1. A projective transformation is a (local) diffeomorphism of $M$ sending geodesics into geodesics (where we view geodesics as unparametrized curves). A vector field on $M$ is called projective if its (local) flow acts by projective transformations.

The set of projective vector fields of a metric $g$ forms a Lie algebra ([10], denoted by $\mathfrak{p}(g)$ in the following. Infinitesimal homotheties, i.e. vector fields $w$ such that $L_wg = \lambda g$ for some $\lambda \in \mathbb{R}$, are examples of projective vector fields.

Definition 2. A projective vector field that is not an infinitesimal homothety is called essential.

Definition 3. Two metrics are called projectively equivalent if they have the same geodesics (as unparametrized curves). The collection of all metrics projectively equivalent to a given metric $g$ is called the projective class of $g$. We denote it by $\mathfrak{P}(g)$.

It turns out that metrics belonging to the same projective class admit the same algebra of projective vector fields. A natural question is to characterize the projective classes of metrics with a prescribed Lie algebra of projective symmetries. A more thorough investigation is to find a complete list of mutually non-isometric normal forms of metrics within the same projective class. In these directions, in his 1882 paper [11], Sophus Lie formulated the following problem: Find the metrics that describe surfaces whose geodesic curves admit an infinitesimal transformation (i.e. with $\mathfrak{p}(g) \geq 1$). Fubini referred to this problem as the “Lie problem”, see [1, 8]. An overview of the history of the problem can be found, for instance, in [1, 2]. Important works in the field, with respect to the considerations that follow below, are for instance [7, 12, 6, 13, 5].

In [6], mutually non-isometric normal forms of 2-dimensional metrics $g$ with $\dim(\mathfrak{p}(g)) \geq 2$ were found around generic points. Metrics admitting an infinitesimal homothety $w$ were described in [11]; locally around a generic point there is a system of coordinates $(x, y)$ such that $w = \partial_x$ and the metric is of the form

$$g = e^{\lambda x} \begin{pmatrix} E(y) & F(y) \\ F(y) & G(y) \end{pmatrix},$$

where the matrix on the right hand side is a non-degenerate matrix. On the other hand, metrics admitting one and only one essential projective vector field have been found in [13], around generic points, via an explicit description of all projective classes. However, no normal forms in terms of mutually non-isometric metrics are provided in [13]. The main outcome of the present paper is to close this gap: Theorem 1 below provides

---

* Dipartimento di Scienze Matematiche (DISMA), Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, giovanni.manno@polito.it
† Istituto Nazionale di Alta Matematica – Dipartimento di Scienze Matematiche (DISMA), Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, andreasdvollmer@gmail.com
a complete list of mutually non-isometric normal forms for the 2-dimensional metrics admitting an essential projective vector field. Moreover, [13] discusses the projective classes of metrics with one essential projective vector field, without specifying exactly which metrics admit an essential projective vector field. In Proposition 4 it is clarified how to distinguish metrics with an essential projective vector field from those with a homothetic one. For the formulation of the main result, we need the following proposition.

**Proposition 1** ([3, 4, 7]). Let \( g_1 \) and \( g_2 \) be projectively equivalent, non-proportional metrics. Then, in a neighborhood of almost every point, there are coordinates \((x, y)\) such that the metrics assume one of the following three normal forms:

|       | A (Liouville case) | B (complex Liouville case) | C (Jordan block case) |
|-------|-------------------|---------------------------|-----------------------|
| \( g_1 \) | \((X - Y)(dx^2 \pm dy^2)\) | \((h(z) - h(z)) (dz^2 - dz^2)\) | \((1 + xY')dx^2\) |
| \( g_2 \) | \((\frac{1}{2} z \pm \frac{1}{2} i)(\frac{dz^2 \pm dy^2}{z})\) | \((\frac{e^z - h(z)}{1 + h(z)})(\frac{dz^2 - dz^2}{h(z)}\) | \(\frac{1}{4} x^2 \left( -2Y dxdy \right) + (1 + xY')dy^2\) |

Here, \( X = X(x) \) and \( Y = Y(y) \) are functions of one variable only, and in the complex Liouville case we use coordinates \( z = x + iy \), \( \overline{z} = x - iy \).

**Remark 1.** Proposition 1 provides normal forms for a pair of non-proportional, projectively equivalent metrics. These have been used in [13] to arrive at a description of all projective classes of metrics with precisely one, essential projective vector field, organized into 10 cases belonging to 3 different types according to Proposition 1, see Proposition 3 which is taken from [13]. In Theorem 1, we keep this organization of the projective classes.

Now we can formulate our main result. The following theorem provides a list of normal forms of metrics admitting exactly one essential projective vector field, indicating the type of the metric according to Proposition 1.

**Notation.** From now on we adopt, for brevity of exposition, the following convention: Whenever there is a non-integer real exponent, the base expression is to be understood as the absolute value if it is real-valued, unless explicitly stated otherwise. The same convention will apply if the exponent is real, but non-specified. For instance, \( y^k \) with \( k \in (0, 1) \) shall be understood as \(|y|^k\), for real \( y \), and \( \det(g)^{3/2} \) is to be understood as identical to \(|\det(g)|^{3/2}\). This convention shall not be applied if the base expression is complex-valued. For instance, for \( z = \rho e^{i\theta} \in \mathbb{C} \) with \( \rho > 0 \) and \( \theta \in [0, 2\pi) \), we have that \( z^k = (\rho e^{i\theta})^k = \rho^k e^{i\theta k} \).

The indefinite integral of a function \( f(y) \) is denoted by \( \int f(\xi) d\xi \). Note that in Theorem 1 below the constant of integration is absorbed by a translation of the coordinate \( x \).

**Theorem 1.** Let \( g \) be a Riemannian or pseudo-Riemannian metric on a 2-dimensional manifold \( M \). Let us assume that \( \dim(p(g)) = 1 \) in any neighborhood of \( M \) and that \( p(g) \) is generated by an essential projective vector field. Then, in a neighborhood of almost every point there exists a local coordinate system \((x, y)\) such that \( g \) assumes one of the following mutually non-isometric normal forms (organized according to their type, see Proposition 1).

(A) Liouville case

1. For \( k \in \mathbb{R} \setminus \{0\} \), \( \varepsilon_2 \in \{\pm 1\} \) and in the projective class determined by \( \xi = 2 \frac{1 - e^{2x}}{1 + e^{2x}} \in (0, 1) \cup (1, 4) \), \( \varepsilon \in \{\pm 1\} \) and \( h \in \mathbb{R} \setminus \{0\} \) (if \( \xi = 2 \), \( h \neq -\varepsilon_1 \))

\[
g = k \left( \frac{(e^{2x} - h e^{\xi y}) e^{2x} + \varepsilon (e^{\xi x} - h e^{\xi y}) e^{2y}}{(1 + e^{2x} h) (1 + \frac{2 e^{2x} \varepsilon}{1 + e^{2x}}) dy^2} \right).
\]

Special cases: If \( h = -1 \), we require \( \varepsilon_2 = 1 \). If \( h = +1 \) and \( \varepsilon = 1 \), the parameter \( k \) may only assume positive values, \( k > 0 \).

2. For \( k \in \mathbb{R} \setminus \{0\} \) and the projective class determined by \( h \in \mathbb{R} \setminus \{0\} \):

\[
g = k \left( \frac{(y - x) e^{-3x}}{x^2 y} dx^2 + \frac{h(y - x) e^{-3y}}{xy^2} dy^2 \right).
\]

In case \( h = 1 \), \( k > 0 \) is positive.

2In a sense, the pair of metrics of Proposition 1 are the “generators”, via Formula (5) below, of their projective class. What could (and actually does) happen, is that the projective class of a metric needs to be described with more than two (actually three) generators (this is the “superintegrable” case).
3. For $\theta \in [0, 2\pi)$, $\kappa > 0$ and the projective class determined by $h \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$ (if $\lambda = 0$: $h \neq \pm 1$):

For $\lambda \neq 0$:

$$g = \frac{\sin(y - x)}{\sin(y + \theta)} \left( e^{-3\lambda x} \frac{dx^2}{\sin(x + \theta)} + h e^{-3\lambda y} \frac{dy^2}{\sin(y + \theta)} \right),$$

for $\lambda = 0$:

$$g = \frac{\sin(y - x)}{\sin(y)} \left( \frac{dx^2}{\sin(x)} + h \frac{dy^2}{\sin(y)} \right).$$

Special cases: If $\lambda \neq 0$ and $h = e^{-6\lambda n \pi}$ for some $n \in \mathbb{Z}$, we require $\theta \in [0, \pi)$.

(B) Complex–Liouville case

Below, we use the notation $\mathbb{P}^1 = e^{i\alpha}$ where $0 \leq \alpha < \pi$, in the complex plane $\mathbb{C} \sim \mathbb{R}^2$.

4. For $\kappa \in \mathbb{R} \setminus \{0\}$ and the projective class determined by $h \in \mathbb{P}^1$, $\xi = 2 \frac{\lambda - 1}{2\alpha + 1} \in (0, 1) \cup (1, 4)$ (if $\xi = 2$: $h \neq \pm 1$):

$$g = \kappa \left( \frac{h z \xi - \xi z}{(1 + h z \xi)(1 + h z^2)} d\xi^2 - \frac{h z \xi - \xi z}{(1 + h z^2)^2(1 + h z \xi^2)} d\xi^2 \right).$$

5. For $\kappa \in \mathbb{R} \setminus \{0\}$ and the projective class determined by $h \in \mathbb{P}^1$:

$$g = \kappa (\overline{z} - z) \left( h e^{-3z} \frac{dx^2}{x^2} - \frac{h e^{-3\overline{z}}}{\overline{z}^2} d\overline{z}^2 \right).$$

6. For $\theta \in [0, 2\pi)$, $\kappa > 0$ and the projective class determined by $h \in \mathbb{P}^1$ and $\lambda \in \mathbb{R}$ (if $\lambda = 0$: $h \neq \pm 1$):

For $\lambda \neq 0$:

$$g = \frac{\sin(\overline{z} - z)}{\sin(\overline{z} + \theta) \sin(z + \theta)} \left( \frac{h e^{-3\lambda z}}{\sin(z + \theta)} d\xi^2 - \frac{h e^{-3\lambda \overline{z}}}{\sin(\overline{z} + \theta)} d\overline{z}^2 \right),$$

for $\lambda = 0$:

$$g = \frac{\sin(\overline{z} - z)}{\sin(\overline{z}) \sin(z)} \left( h \frac{dx^2}{\sin(z)} - \frac{h}{\sin(\overline{z})} d\overline{z}^2 \right).$$

(C) Jordan block case

7. For $\varepsilon \in \{\pm 1\}$, $\kappa \in \mathbb{R} \setminus \{0\}$ and the projective class determined by $\xi = 2 \frac{\lambda - 1}{2\alpha + 1} \in (0, \frac{1}{\varepsilon}) \cup (\frac{1}{\varepsilon}, 1) \cup (1, 4)$:

$$g = \kappa \left( -2 \frac{y^2}{(y - \varepsilon)^3} dx dy + \frac{(y + \varepsilon)^2}{(y - \varepsilon)^4} dy^2 \right)$$

8. For $\kappa \in \mathbb{R} \setminus \{0\}$:

$$g = \kappa (Y(y) + x) dx dy \quad \text{with } Y(y) = \int_{y}^{\infty} \frac{e^{-3\lambda s}}{\sqrt{s^2 + 1}} \, ds,$$

9. For the class determined by $\lambda \in \mathbb{R}$:

$$g = \kappa (Y(\lambda, y) + x) dx dy \quad \text{with } Y(\lambda, y) = \int_{y}^{\infty} \frac{e^{-3\lambda s}}{\sqrt{s^2 + 1}} \, ds,$$

where $\kappa > 0$ if $\lambda = 0$, and $\kappa = e^{\alpha \lambda}$ with $\alpha \in [0, 2\pi)$ otherwise.

10. Superintegrable case: For $\kappa \in \mathbb{R} \setminus \{0\}$, $\varepsilon \in \{\pm 1\}$ and $c \in \mathbb{R}$, respectively:

(a) $g = \kappa \left( -2 \frac{y^2 + x}{(y - \varepsilon)^3} dx dy + \frac{(y^2 + x)^2}{(y - \varepsilon)^4} dy^2 \right)$

(b) $g = \frac{\kappa}{F(0, \varepsilon; y; y^2)} \left( 9(y^2 + x)^2 dx^2 - 2(y^3 + 9xy - 2c)(y^2 + x) dx dy + 12x(y^2 + x^2) dy^2 \right)$

(c) $g = \frac{\kappa}{F(\varepsilon; c; y; y^2)} \left( 9(y^2 + x)^2 dx^2 - 2(y^3 + 2\varepsilon y + 9xy - 2c)(y^2 + x) dx dy + 4(\varepsilon + 3x)(y^2 + x^2) dy^2 \right)$

with $F(\zeta; c; x, y) = y^6 - 9xy^4 + 27x^2y^2 - 27x^3 + 4c^2 - (36xy + 9y^3) c + (18xy^2 - 5y^4 - 9x^2 - 8cy) \zeta + 4y^2 \zeta^2$. 
Structure of the paper. The paper is organized as follows: In Section 2, we discuss our approach and the techniques that are made use of in the proof of Theorem 1. First, in Section 2.1, we present results that are needed in the proof, then in Section 2.2 we discuss the general procedure followed in the proof of Theorem 1. This proof is contained in Section 3, which is divided into three parts, according to the properties of Lie derivative along the projective vector field, see the cases given in (7) below. Each part is subsequently divided further into three or four subsections, which follow the cases of Proposition 1, and whose exact structure will be explained below (see Proposition 3).

2 Methods

2.1 Main results used in the proof of Theorem 1

A metric $g$ given in an explicit system of coordinates $(x, y)$ gives rise, via its Levi-Civita connection, to a second order ordinary differential equation (ODE)

$$y'' = -\Gamma^1_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12}) y' - (\Gamma^2_{22} - 2\Gamma^1_{12}) y^2 + \Gamma^1_{22} y^3,$$

where $y = y(x)$ and $\Gamma^j_{ik}$ are the Christoffel symbols of $g$. The ODE (1) is called the projective connection associated to $g$. The name is justified by the fact that, for a solution $y(x)$ to (1), the curve $(x, y(x))$ is a geodesic of $g$ up to reparametrization. Thus, (local) diffeomorphisms $(x, y) \rightarrow (u(x, y), v(x, y))$ preserving (1) (finite point symmetries) are projective transformations of $g$, i.e. they send geodesics into geodesics. Infinitesimal point symmetries of (1) are projective vector fields of $g$ and generate a 1-parametric family of projective transformations. For a more detailed study of such transformations, from a general point of view, see e.g. [9, 15]. Let us consider a general (2-dimensional) projective connection,

$$y'' = f_0 + f_1 y' + f_2 y'^2 + f_3 y^3, \quad f_i = f_i(x, y).$$

(2)

A natural question is whether or not the projective connection (2) is metrizable, i.e. if there exists a 2-dimensional metric $g$ such that (2) is equal to (1). The problem of metrizability is examined from multiple perspectives in [5]. The metrizability condition can be expressed in terms of a system of 4 partial differential equations (PDE) on the components of $g$. Its solution is discussed in [13] and [6] for metrics with, respectively, one or more projective vector fields. These references introduce a major simplification to the PDE problem: The system turns into a linear system of PDE when replacing the unknowns, i.e. the components of the metric $g$, by the components of a weighted tensor section, denoted by $a = \psi^{-1}(g)$ and called the Liouville tensor associated to $g$ in the following.3 Liouville tensors take values in $S^2M \otimes (\text{vol } M)^{-4/3}$ ($S^2M$ is the symmetric tensor product of the module of 1-forms on $M$, whereas vol $M$ is the one-dimensional bundle of volume forms on $M$):

$$a = \psi^{-1}(g) = \frac{g}{|\det(g)|^{2/3}}, \quad g = \psi(a) = \frac{a}{|\det(a)|^{2/3}}.$$  

(3)

More precisely we have the following proposition.

Proposition 2 ([6, 12]). The projective connection associated to the Levi-Civita connection of a metric $g$ is (2) if and only if the entries of the matrix $a_{ij}$ describing $a = \psi^{-1}(g) = \frac{g}{|\det(g)|^{2/3}}$ satisfy the linear system of PDEs

$$\begin{cases}
 a_{11x} - \frac{2}{3} f_1 a_{11} + 2 f_0 a_{12} = 0 \\
 a_{11y} + 2 a_{12x} - \frac{4}{3} f_2 a_{11} + \frac{2}{3} f_1 a_{12} + 2 f_0 a_{22} = 0 \\
 2 a_{12y} + a_{22x} - 2 f_3 a_{11} - \frac{4}{3} f_2 a_{12} + \frac{2}{3} f_1 a_{22} = 0 \\
 a_{22y} - 2 f_3 a_{12} + \frac{4}{3} f_2 a_{22} = 0
\end{cases}$$

(4)

where the subscripts $x, y$ denote derivatives.

Of course, solutions to the linear system of PDE (4) span a linear space. Solutions $a$ of the linear system (4) correspond to solutions of the initial metrizability problem if $\det(a) \neq 0$. In Lemma 2 of [13] it is proven that, if $a \neq 0$, then the set of the points where $a$ is degenerate is nowhere dense (in the topological sense).

Definition 4. The space of non-zero solutions to the system (4) for a metric $g$ is denoted by $\mathfrak{A}(g)$ and its dimension is called the degree of mobility of $g$. We usually abbreviate $\mathfrak{A} = \mathfrak{A}(g)$ when there is no risk of confusion.

Remark 2. Let $\mathfrak{A}$ be the space of solutions to the linear system of PDE (4), and let $a \in \mathfrak{A}$ be a particular solution. The mapping $\psi$ defined in (3) is a bijection that identifies the solution $a$ of (4) with the metric $g = \psi(a)$ that admits one projective vector field. In view of this correspondence, also the spaces $\mathfrak{A}$ and $\mathfrak{P}(g)$ are identified, and in particular we have $\mathfrak{P}(g) = \psi(\mathfrak{A})$.

3The linearized system of PDE (4) has, to the best knowledge of the authors, first been discussed by Liouville in [12].
It is proven in [13] that, if we restrict our attention to metrics with an essential projective vector field, \(\dim(\mathfrak{A})\) is either 2 or 3. Therefore, if we denote by \(\{a_i\}\) a basis of \(\mathfrak{A}\), we have that \(a = \sum_{i=1}^{m} K_i a_i\), where \(m = \dim(\mathfrak{A}) \in \{2, 3\}\). The corresponding metric, via Formula (3), is given by

\[
g = g[K_1, \ldots, K_m] = \frac{\sum_{i=1}^{m} K_i |g(y)|^{2/3}}{\det \left( \sum_{i=1}^{m} K_i |g(y)|^{2/3} \right)^2}, \quad m = \dim(\mathfrak{A}) \in \{2, 3\}.
\]  

(5)

It can be easily seen that \(\mathcal{P}(g_1) = \ldots = \mathcal{P}(g_m) = \mathcal{P}(g[K_1, \ldots, K_m])\), with \((K_1, \ldots, K_m) \in \mathbb{R}^m \setminus \{0\}\), \(m = \dim(\mathfrak{A}) \in \{2, 3\}\).

**Notation.** In what follows we sometimes alternatively denote the projective class \(\mathcal{P}(g_1) = \ldots = \mathcal{P}(g_m)\), for projectively equivalent metrics \(g_1, \ldots, g_m\), by \(\mathcal{P}(g_1, \ldots, g_m)\), i.e.

\[
\mathcal{P}(g_1, \ldots, g_m) := \mathcal{P}(g_1) = \ldots = \mathcal{P}(g_m).
\]

In this way we underline that an arbitrary metric in the projective class can be obtained by \(g_1, \ldots, g_m\) via Formula (5). In the above notation we assume that the metrics \(g_1, \ldots, g_m\) form a basis of their projective class, in the sense that the corresponding Liouville tensors \(a_i = \psi^{-1}(g_i)\) form a basis of the linear space \(\mathfrak{A}\).

The space \(\mathfrak{A}\) is invariant under the action of the Lie derivative \(L_w\), where \(w\) is a projective vector field, the action of \(L_w\) on the space of Liouville tensor being induced by the action of \(L_w\) on the metrics. More precisely we have that

\[
L_w a = \det(g)^{-2/3} \cdot L_w g - \frac{2}{3} \det(g)^{-2/3} \operatorname{tr}(L_w g) \cdot g
\]

(6)

Consider the Lie derivative \(L_w|\mathfrak{A} : \mathfrak{A} \to \mathfrak{A}\). There exists a basis of \(\mathfrak{A}\) such that \(L_w|\mathfrak{A}\) is represented by a matrix in Jordan normal form. Since \(\dim(\mathfrak{A}) \in \{2, 3\}\), there exists a two-dimensional, \(L_w\)-invariant subspace \(\mathfrak{A} \subset \mathfrak{A}\) (for \(m = 2\), \(\mathfrak{A} = \mathfrak{A}\)). In a suitable basis, the (restricted) Lie derivative \(L_w|\mathfrak{A}\) is represented by one of the following three matrices (note that we can rescale the matrices by a constant factor since the projective vector field is defined up to a constant factor only [13]):

\[
(\text{I}) \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad |\lambda| \geq 1; \quad (\text{II}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad (\text{III}) \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \quad \text{with} \quad \lambda \in \mathbb{R}.
\]

(7)

By using the above normal forms for each type of metrics described in Proposition 1, the author of [13] was able to divide metrics that admit exactly one essential projective vector field into several non-equivalent projective classes. The following proposition, which is one of the main tools we shall use, summarizes this result.

**Proposition 3** ([13]). Let \(g\) be a 2-dimensional metric on \(M\) such that \(\dim(\mathfrak{p}(g)) = 1\). Moreover, let us assume that \(g|_U\) admits no homothetic vector field in any neighborhood \(U \subset M\). Then, in a neighborhood of almost every point there exists a coordinate system \((x, y)\) such that \(g\) is of the form (5), where \(g_i, i \in \{1, \ldots, m\}, m \in \{2, 3\}\), are described below.

A) Liouville-type metrics

\[
g_1 = (X(x) - Y(y))(X_1(x) \, dx^2 + Y_1(y) \, dy^2), \quad g_2 = \left( \frac{1}{X} - \frac{1}{Y} \right) \left( \frac{X_1}{X} \, dx^2 + \frac{Y_1}{Y} \, dy^2 \right)
\]

(8)

For \(h \in \mathbb{R} \setminus \{0\}\), \(\nu \in (0, 1) \cup (1, 4]\), \(\varepsilon \in \{\pm 1\}\), \(\lambda \in \mathbb{R}:

(I) \(X(x) = e^{\nu x}, Y(y) = h e^{\nu y}, X_1(x) = e^{2\nu x}, Y_1(y) = \varepsilon e^{2\nu y}\) (if \(\nu = 2\): \(h \neq -\varepsilon\))

(II) \(X(x) = \frac{\lambda x}{\varepsilon}, Y(y) = \frac{1}{\varepsilon}, X_1(x) = e^{\varepsilon^2 x}, Y_1(y) = h e^{\varepsilon^2 y}\)

(III) \(X(x) = \tan(x), Y(y) = \tan(y), X_1(x) = \frac{\varepsilon^2 y}{\cos(x)}, Y_1(y) = h \frac{\varepsilon^2 y}{\cos(y)}\) (if \(\lambda = 0\): \(\varepsilon \neq 0\))

B) complex Liouville metrics

\[
g_1 = (h(z) - \bar{h}(\bar{z})) (h_1(z) \, dz^2 - h_1(\bar{z}) \, d\bar{z}^2), \quad g_2 = \left( \frac{1}{h(z)} - \frac{1}{\bar{h}(\bar{z})} \right) \left( \frac{h_1(z)}{h(z)} \, dz^2 - \frac{h_1(\bar{z})}{\bar{h}(\bar{z})} \, d\bar{z}^2 \right)
\]

(9)

For \(C \in \mathbb{C}, |C| = 1\), \(\nu \in (0, 1) \cup (1, 4]\), \(\lambda \in \mathbb{R}:

(I) \(h(z) = C e^{\nu z}, h_1(z) = e^{2\nu z}\) (if \(\nu = 2\): \(C \neq \pm 1\))

(II) \(h(z) = \frac{\lambda}{\nu}, h_1(z) = C e^{\varepsilon^2 x}\)

(III) \(h(z) = \tan(z), h_1(z) = C \frac{\varepsilon^2 y}{\cos(x)}\) (if \(\lambda = 0\): \(\varepsilon \neq 0\))
C) Jordan-block metrics

\[ g_1 = (Y(y) + x) \, dx \, dy, \quad g_2 = -\frac{2(Y(y) + x)}{y^3} \, dx \, dy + \frac{(Y(y) + x)^2}{y^4} \, dy^2 \]  

For \( \lambda \in \mathbb{R} \) and \( \eta \in (0, \frac{1}{3}) \cup \left( \frac{1}{3}, 1 \right) \cup (1, 4) \):

(Ia) \( Y(y) = y^2 \), and there is a third metric:

\[ g_3 = \frac{y^2 + x}{(3x - y^2)y} \left( 9(y^2 + x) \, dx^2 - 4y(9x + y^2) \, dx \, dy + 12x(y^2 + x) \, dy^2 \right) \]  

(Ib) \( Y(y) = y^{1/n} \)

(III) \( Y(y) = e^{\alpha/2y} \sqrt{\frac{y}{y-1}} + y^{1/2} e^{\alpha/2y} \sqrt{\frac{y}{y-1}} \, ds \)

(III) \( Y(y) = e^{-\frac{3}{2} \lambda \arctan(y)} \sqrt{\frac{y^2 - 1}{y - 3\lambda}} + y^{3/2} e^{-\frac{3}{2} \lambda \arctan(y)} \sqrt{\frac{y^2 - 1}{y - 3\lambda}} \, ds \)

Remark 3. Note that two metrics belonging to different classes \( A(I) - C(III) \) of Proposition 3 are non-isometric, because isometries are projective transformations, and thus preserve the projective class.

2.2 General procedure of the proof of Theorem 1

We shall obtain the normal forms of Theorem 1 by thoroughly investigating possible isometries within the cases \( A(I) - C(III) \) of the list of Proposition 3, as metrics belonging to different types among \( A(I) - C(III) \) cannot be isometric since they are not even projectively equivalent. Of course, a (local) diffeomorphism that links two metrics in a fixed projective class amongst \( A(I) - C(III) \) is a projective transformation, so the original problem of characterizing metrics up to isometries (where the metrics admit exactly one, essential projective vector field) can be rephrased as follows: to characterize metrics with no homothetic vector field within the classes \( A(I) - C(III) \) of Proposition 3, and then to determine their normal forms (up to isometries).

The current section is organized as follows: As a preliminary step, by Proposition 4, we characterize the metrics in Proposition 3 that do not admit a homothetic vector field. The normal forms are then obtained using two basic steps (see below) and a number of specific techniques that are explained when needed. Let us begin with the characterization of metrics that admit a homothetic vector field.

Proposition 4. The projective vector field \( w \) of a metric \( g \) in a projective class of Proposition 3 is homothetic if and only if the Liouville tensor \( a = \psi^{-1}(g) \) is an eigenvector of \( L_w |_\mathfrak{a} \).

In other words, the projective vector field \( w \) satisfies \( L_w g = \eta g \) for some \( \eta \in \mathbb{R} \) if and only if there is an eigenvalue \( \mu \) such that \( L_w a = \mu a \), where \( a = \psi^{-1}(g) \).

Note that here the restricted Lie derivative \( L_w |_\mathfrak{a} \) is denoted simply by \( L_w \), as we are going to do from now on unless otherwise specified.

Proof. First, assume there is a homothetic vector field \( w \). Then, since \( \dim(\mathfrak{p}(g)) = 1 \), we have \( L_w g = \eta g \). Furthermore, according to Equation (6),

\[ L_w a = \det(g)^{-2/3} \cdot L_w g = \frac{2}{3} \det(g)^{-2/3} \text{tr}_g (L_w g) \cdot g = \eta a - \frac{4}{3} \eta \cdot a = \frac{2}{3} \mu a. \]

On the other hand, if \( L_w a = \mu a \), then compute the components of \( L_w g \), i.e.

\[ L_w g_{ij} = w^k \partial_k \left( \frac{a_{ij}}{\det(a)^2} \right) = L_w a_{ij} - \frac{\omega \partial_k w^k}{\det(a)^2} - 2 \det(a)^{-3} w^k \partial_k (\det(a)) a_{ij} \]

\[ = \frac{L_w a_{ij} - \omega \partial_k w^k a_{ij}}{\det(a)^2} - \frac{2 w^k \partial_k (\det(a)) a_{ij}}{\det(a)^2} \]

\[ \Rightarrow 3 \mu g_{ij} + (4 + 3 \omega) \partial_k w^k g_{ij} = -3 \mu g_{ij} \]

where \( \omega = -\frac{4}{3} \) is the weight of the Liouville tensor \( a \) and where we use \( L_w a = \mu a \) at (*). This confirms \( L_w g = \eta g \) with \( \eta \in \mathbb{R} \), implying \( w \) is a homothetic vector field.

Proposition 4 permits us to easily identify the metrics with one essential projective vector field. As announced in the beginning of this section, we shall, largely speaking, proceed following two steps, in each of which a suitable invariant is introduced. Thus we shall obtain, in Section 3, the list of mutually non-isometric metrics of Theorem 1.
Let us fix a system of coordinates \((x, y)\) as in Proposition 3. Metrics of type \(A\) (resp. \(B, C\) not of type \((Ia)\)) belong to \(\Psi(g_1, g_2)\) with \(g_1, g_2\) given by (8) (resp. (9), (10)). Metrics of type \(C(Ia)\) belong to \(\Psi(g_1, g_2, g_3)\) where \(g_i, i \in \{1, 2, 3\}\) are given by (10) and (11). We recall that a metric \(g\) belonging to one of the previous projective classes \(\Psi(g_1, \ldots, g_m), m \in \{2, 3\}\), is of the form (5). Moreover, since we are interested in metrics admitting exactly one essential projective vector field, in view of Proposition 4 we can infer that, for the purposes of Theorem 1, we have to consider exactly those combinations (5) of the metrics of Proposition 3, for which at least two parameters \(K_i, i = 1, \ldots, m\) are non-zero \((m = 2, 3)\). Recall that, whereas Formula (5) is of “geometric” character (it looks formally alike for any choice of coordinates), the coordinate expressions of the metrics \(g_i\) in Proposition 3 are not.

**Step 1 (projective orbits).** Let us consider a projective class among \(A(I) – C(III)\) of Proposition 3. Metrics belonging to this class are of the form (5) and admit, up to a non-vanishing constant factor, one and only one projective vector field: let us fix this vector field and denote it by \(w\), and its local flow by \(\phi\).

**Definition 5** (Projective orbits). Let \(w\) be a projective vector field and \(\phi\) be its local flow. An orbit of \(\phi\) in \(\mathfrak{A}\) that passes through a point (i.e. a Liouville tensor) \(a\) is called the projective orbit through \(a\). In view of Remark 2, the term of projective orbit will refer either to the space of Liouville tensors or the space of metrics, depending on the context.

The transformation \(\phi\) sends a metric of the form (5) into a metric of the same form, so it induces a map \(\mathbb{R}^m \setminus \text{eigenspaces} \to \mathbb{R}^m \setminus \text{eigenspaces}, m = \dim \mathfrak{A} \in \{2, 3\}\). Here, we removed eigenspaces of \(L_w\) as we are considering metrics admitting no homothetic vector fields (see discussions above). Below these eigenspaces are axes of \(\mathbb{R}^m\). The first step therefore is to describe \(\phi\)-orbits in terms of the parameters \(K_i\).

We shall proceed as follows. Let \(g \in \Psi(g_1, \ldots, g_m), m \in \{2, 3\}\), and let \(a = \psi^{-1}(g)\) be its associated Liouville tensor, see (3). Furthermore, let \(a_i = \psi^{-1}(g_i)\) for each value of \(i\). It can easily be verified that \(a = \sum K_i a_i\) for some \(K_i \in \mathbb{R}\). We have already seen that the Lie derivative \(L_w\) acts on the space of Liouville tensors (see Section 2.1) in such a way that

\[ \phi_t (\sum_{i=1}^{m} K_i a_i) = \sum_{i=1}^{m} K_i \exp(tL_w) a_i = \sum_{i=1}^{m} K_i' a_i. \]  

(12)

From the last equality of (12), by eliminating the parameter \(t\), one can derive relations on \((K_i')_{i=1, \ldots, m}\) and \((K_i)_{i=1, \ldots, m}\) only, describing the \(\phi\)-orbit of (isometric) metrics in \(\Psi(g_1, \ldots, g_m)\) (such orbits do not change if the projective vector field is multiplied by a non-zero constant). This permits us to find a function \(f(K_1, \ldots, K_m)\) whose level sets are orbits of the projective flow in \(\mathfrak{A}\). In some cases it is helpful to replace the parameters \(K_i\) by another choice of parameters, using \(f\) itself as a parameter. Note that this amounts to a change of coordinates in the space \(\mathfrak{A}\).

Metrics on the same projective orbit are isometric, as they are linked by \(\phi\). A complication that can occur is that not all projective transformations are of type \(\phi_t\) for some \(t\), so that metrics belonging to different \(\phi_t\)-orbits could be isometric. In order to see if this happens, we need additional techniques. The most important one to be used in the proof is the thorough investigation of the function describing the (square of the) length of the projective vector field (Step 2 below). Other, more specific techniques, shall be introduced inside the proof when needed.

Another complication appears if \(\dim(\mathfrak{A}) = 3\), because the resulting metrics become more cumbersome to handle. In this case we use special elements of \(\mathfrak{A}\), taken from eigenspaces of \(L_w\), to obtain restrictions on possible isometries.

**Step 2 (isometries).** Let \(\tau\) be an isometry between \(g'\) and \(g = \tau^*(g')\). Let us fix the projective vector field \(w\) (we recall that it is defined up to a non-vanishing constant factor) by the condition (recall the action of \(L_w\) on the space of Liouville tensors, see formula (6))

\[ L_w \begin{pmatrix} a \\ \hat{a} \end{pmatrix} = M \begin{pmatrix} a \\ \hat{a} \end{pmatrix}, \]

(13)

where \(M\) is a \(2 \times 2\)-matrix as in (7). This condition depends only on the directions of \(a\) and \(\hat{a}\) in the 2-dimensional invariant subspace \(\mathfrak{A} \subset \mathfrak{A}\). The isometry \(\tau\) preserves eigenspaces and thus condition (13) fixes \(w\), in the sense that \(\tau^*(w) = w\). Now, let \(\phi\) be the local flow of \(w\). We have that \(t \to \tau(\phi_t(p_0))\), with \(p_0\) being a point on the manifold, is an integral curve of \(w\) starting at \(p_0 = \tau(p_0)\). Moreover, we have that

\[ g_{\phi_t(p_0)}(w_{\phi_t(p_0)}, w_{\phi_t(p_0)}) = g'_{\tau(\phi_t(p_0))}(w_{\tau(\phi_t(p_0))}, w_{\tau(\phi_t(p_0))}) = g'_{\phi_t(p_0)}(w_{\phi_t(p_0)}, w_{\phi_t(p_0)}). \]

(14)

We are going to study the poles and zeros of each side of Equation (14), which coincide for isometric metrics. Thus we obtain extra conditions on \(K_i\) and \(K_i'\) established in Step 1. We also find conditions on \(p'_0\) and \(p_0\), which provide information on the possible isometries between \(g'\) and \(g\).
3 Proof of Theorem 1

In Sections 3.1, 3.2 and 3.3 we shall consider, respectively, the cases in which the Lie derivative $L_w|\mathfrak{A}$ restricted to the 2-dimensional invariant subspace $\mathfrak{A} \subset \mathfrak{A}$ is of type (I), (II) or (III) of (7); we recall that $w$ is a projective vector field as in Step 1 of Section 2.2 and $\mathfrak{A}$ is the linear space of solutions of (4). For each of these three cases, we proceed as follows: Using mainly the methods described in Section 2, we study the three types of metrics from Proposition 1, thus obtaining the normal forms of Theorem 1. Specifically, the (real) Liouville-type metrics, complex Liouville-type metrics and Jordan block-type metrics from Proposition 1 yield, respectively, normal forms under (A), (B) and (C) in Theorem 1.

3.1 Case (I): $L_w$ has 2 real eigenvalues

Here we are in the case when $L_w$ is of the form (I) of (7). So, let $(a_1,a_2)$ be a basis of $\mathfrak{A} \subset \mathfrak{A}$ in which $L_w = L_w|\mathfrak{A}$ assumes such a form, i.e.

$$L_w \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

(15)

where $|\lambda| > 1$ and $w$ is the projective vector field fixed by condition (15) (see Step 2 of Section 2.2 for more details). Let us denote the flow of $w$ by $\phi_t$. Any metric $g$ from the projective class $\mathfrak{P}(g_1,g_2)$, where $g_i = \psi(a_i)$ with $\psi$ as in (3), is the $\psi$-image of $a = \sum_1^m K_i a_i$, $m \in \{2,3\}$. By Equation (15), taking into account (12), we have

$$\phi_t^*(K_1 a_1 + K_2 a_2) = K_1 e^{\lambda t} a_1 + K_2 e^{\lambda t} a_2 =: K_1' a_1 + K_2' a_2.$$  

(16)

The following lemma, that can be proved by a direct computation, characterizes metrics on the same $\phi_t$-orbit.

Lemma 1. The pairs $(K_1, K_2)$ and $(K'_1, K'_2)$ are related by (16) if and only if they satisfy one of the following cases:

(a) $K_1/K'_1 > 0$ and $K_2 = K'_2 = 0$  
(b) $K_2/K'_2 > 0$ and $K_1/K'_1 = K_1/K'_2 \lambda$.

Proof. Let us assume $K'_1, K_1, K'_2, K_2 \neq 0$ (otherwise the statement is trivial). Now, let us assume that (16) holds, i.e. $K'_1 = e^{\lambda t} K_1$ and $K'_2 = e^{\lambda t} K_2$, that in particular implies $K_1/K_2 > 0$. Exponentiating the absolute values in the second equation by $\lambda \neq 0$ and then eliminating $t$ we obtain

$$K_1/K_2 = K_1/K'_2 \lambda,$$

i.e. the case (b) above.

Conversely, if we assume either (a) or (b), we arrive, by a straightforward computation, to the condition (16). □

The graphs in Figure 1 illustrate the orbits of the projective flow inside $\mathfrak{A}$.

3.1.1 Normal forms (A.1) of Theorem 1

In the case of Liouville metrics, $A(I)$, we can construct any metric $g$ in the projective class, according to Proposition 3 and via Formula (5), from the following two metrics:

$$g_1 = (X - Y)(X_1 dx^2 + Y_1 dy^2) \quad \text{and} \quad g_2 = \left(\frac{1}{X} - \frac{1}{Y}\right)\left(\frac{X_1}{X} dx^2 + \frac{Y_1}{Y} dy^2\right),$$

(17)

where, after an obvious change of coordinates, $X = x^\xi$, $Y = hy^\xi$, $X_1 = 1$, $Y_1 = \varepsilon$, and where

$$\xi = 2 \frac{\lambda - 1}{2\lambda + 1}, \quad |\lambda| > 1 \text{ and } \lambda \neq 1.$$  

(18)

The projective vector field $w$ is given by

$$w = -2\frac{\lambda + 1}{2}(x \partial_x + y \partial_y)$$

and its flow reads

$$\phi_t(x_0, y_0) = (x_0 e^{-\frac{2\lambda + 1}{2}}, y_0 e^{-\frac{2\lambda + 1}{2}}) \quad \text{where} \quad x_0 > 0, \quad y_0 > 0.$$  

(19)

The projective connection (see (1)) of any metrics in $\mathfrak{P}(g_1,g_2)$, in the same coordinates $(x,y)$, reads

$$y'' = \frac{h \xi y^{\xi - 1}}{2 \varepsilon (hy^{\xi} - x^\xi)} + \frac{\xi x^{\xi - 1}}{2 (hy^{\xi} - x^\xi)} y' + \frac{h \xi y^{\xi - 1}}{2 \varepsilon (hy^{\xi} - x^\xi)} y'^2 + \frac{\xi x^{\xi - 1}}{2 (hy^{\xi} - x^\xi)} y'^3.$$
Figure 1: Graph of the orbits of the projective action of $\phi^*_t$ in the subspace $\mathfrak{A}$, for a metric $g = \psi(K_1a_1 + K_2a_2)$ of type $A(I)$ in Proposition 3, c.f. (5). The pictures show $a = K_1a_1 + K_2a_2$ as a point $(K_1, K_2)$, i.e. $a_1$ lies on the axis of abscissae, while $a_2$ is on the axis of ordinates.

The graphs in the upper half represent the case $\lambda > 0$, whereas the graphs in the lower half are for $\lambda < 0$. The plots to the left illustrate, for the generic situation, several orbits with different $K_1$ and/or $K_2$ (the thick curve is one example of such an orbit). The two graphs in the middle and those to the right represent the orbits for special cases, i.e. for projective classes that have discrete projective symmetries. In these plots, the thick solid line is the orbit of the projective action of $\phi^*_t$, and the dashed line represents isometric metrics that are reached due to a discrete projective symmetry.

It is different for different choices of $(h, \varepsilon, \xi)$, i.e. these parameters characterize the projective class. Obviously, if $h = 0$, the metric admits a Killing vector field. Thus, in the following, we require in particular that $h \neq 0$. Any metric $g$ belonging to the projective class $\mathfrak{P}(g_1, g_2)$ given by $g_1$ and $g_2$ from (17) reads, via Formula (5),

$$g = g[K_1, K_2] = \frac{x^\xi - hy^\xi}{(K_1 + K_2hy^\xi)(K_1 + K_2x^\xi)} dx^2 + \varepsilon \frac{x^\xi - hy^\xi}{(K_1 + K_2hy^\xi)(K_1 + K_2x^\xi)} dy^2. \quad (20)$$

In view of Step 2 of the general procedure outlined in Section 2.2, we from now on restrict to $K_1, K_2 \neq 0$, since we are only interested in metrics with an essential projective vector field. In this case, writing Equation (14) explicitly, taking into account (19), we obtain the requirement

$$\frac{x^\xi_0 - hy^\xi_0}{(K_1 + K_2hy^\xi_0 t)(K_1 + K_2x^\xi_0 t)^2} x^2 + \varepsilon \frac{x^\xi_0 - hy^\xi_0}{(K_1 + K_2hy^\xi_0 t)(K_1 + K_2x^\xi_0 t)} y^2 = \frac{x^\xi_0 - hy^\xi_0}{(K_1 + K_2hy^\xi_0 t)(K_1 + K_2x^\xi_0 t)^2} x^2_0 + \varepsilon \frac{x^\xi_0 - hy^\xi_0}{(K_1 + K_2hy^\xi_0 t)(K_1 + K_2x^\xi_0 t)} y^2_0. \quad (21)$$

Let us study poles of the functions in $t$ on the left and the right hand side of Equation (21). We find below that this is enough for our purposes. Possible poles are the zeros of the denominators, provided $x^\xi_0 \neq hy^\xi_0$ and $x^\xi_0 \neq hy^\xi_0$, respectively. Determining these possible poles and equating them pairwisely, we find the following cases:

(a) $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{x^\xi_0}{x^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$.

(b) $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{x^\xi_0}{x^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$.

(c) $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{x^\xi_0}{x^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$.

(d) $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{x^\xi_0}{x^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$; $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$, $\frac{y^\xi_0}{y^\xi_0} = \frac{K_1'K_2}{K_1K_2}$.

The first two cases (a) and (b) lead to $x^\xi_0 = hy^\xi_0$, i.e. to zeros of the left hand side of (21), contrary to our previous assumptions on $x_0, y_0$ (for the metric under consideration such points lie on light lines). Since we want to study only poles of (21), cases (a) and (b) can therefore be left out of further consideration.

In case (c), $x_0' = Kx_0$, $y_0' = Ky_0$ where $K = \left(\frac{K_1'K_2}{K_1K_2}\right)^{1/\xi}$. Isometries of the form $(x, y) \rightarrow (Kx, Ky)$, $K > 0$, are embedded in the flow of the projective vector field. Substitute these expressions into (21) and obtain

$$K \frac{K_1'}{K_1} = 1,$$

and thus $K_1' > 0$ and $K_2' > 0$, which suffices to prove $\frac{|K_1|}{|K_2|^\lambda} = \frac{|K_1'|}{|K_2'|^{1/\xi}}$. 

9
where the latter expression is identical to \( \frac{K_1^i}{|K_2|^i} \), in view of \( \frac{K_1^i}{K_1} > 0 \), so the condition of Lemma 1 is reobtained.

In case (d), supposing \( x_0, x_0', y_0, y_0' > 0 \), we similarly find

\[
x_0' = \left( \frac{K_1' K_2 h}{K_1 K_2} \right)^{1/\xi} y_0, \quad y_0' = \left( \frac{K_1' K_2 h}{K_1 K_2} \right)^{1/\xi} x_0.
\]

(22)

Such transformations are not embedded into the projective flow, cf. (19), since they include swapping \( x \leftrightarrow y \). One should therefore not expect the same results as in case (c).

Combining (22) and (21) we can deduce \(|h| = 1\) as follows: First substitute (22) into (21), multiply by the common denominator, and then collect terms w.r.t. \( t \). The terms of degree zero and three of this polynomial in \( t \) are enough to obtain \(|h| = 1\). Finally, using again (21), we obtain

\[
|K|^{\xi_2} \frac{K_3^2}{K_1^2} = -\varepsilon \text{sgn}(K) \in \{\pm 1\}, \quad \text{where} \quad K = \frac{K_1' K_2}{K_1 K_2},
\]

implying

\[
\frac{|K_1|}{|K_2|^\lambda} = \frac{|K_1'|}{|K_2'|^\lambda},
\]

so the constant \( \frac{|K_1|}{|K_2|^\lambda} \) is an isometric invariant for the metrics \( g[K_1, K_2] \). Moreover, from (23) we can infer

\[
\text{sgn}(K_1) \text{sgn}(K_1') = -\varepsilon \text{sgn}(K)
\]

and, using the definition of \( K \),

\[
\text{sgn}(K_2') = -\varepsilon \text{sgn}(K_2).
\]

Now consider a metric (20) with given \( k = \frac{|K_1|}{|K_2|^\lambda} \). After rescaling \( x \) and \( y \) by the same constant factor \( a^k = k |K_2|^\lambda - 1 \), we have the metric in the form

\[
g = \varepsilon_1 k^{\xi_2 - 2} k^3 \left( \frac{x^\xi - hy^\xi}{(1 + \varepsilon_2 hy^\xi)(1 + \varepsilon_2 x^\xi)^2} \right) dx^2 + \varepsilon \left( \frac{x^\xi - hy^\xi}{(1 + \varepsilon_2 hy^\xi)(1 + \varepsilon_2 x^\xi)^2} \right) dy^2
\]

with \( \varepsilon, \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \) where \( \varepsilon_1 = \text{sgn}(K_1) \) and \( \varepsilon_2 = \text{sgn}(K_2) \) \( \varepsilon_1 \). By renaming the multiplicative constant \( \kappa = \frac{|K_1|}{|K_2|^\lambda} \), we arrive at the following metric

\[
g = g[\kappa, \varepsilon_2; h, \varepsilon] = \kappa \left( \frac{x^\xi - hy^\xi}{(1 + \varepsilon_2 hy^\xi)(1 + \varepsilon_2 x^\xi)^2} \right) dx^2 + \varepsilon \left( \frac{x^\xi - hy^\xi}{(1 + \varepsilon_2 hy^\xi)(1 + \varepsilon_2 x^\xi)^2} \right) dy^2
\]

(24)

with \( \kappa \in \mathbb{R} \setminus \{0\}, h \in \mathbb{R} \setminus \{0\}, \varepsilon, \varepsilon_2 \in \{\pm 1\} \). Note that we redefined \( \varepsilon_2 \rightarrow \varepsilon_2 \text{sgn}(C_1) \).

**Notation.** The notation \( g[\kappa, \varepsilon_2; h, \varepsilon] \) in Equation (24) has the following meaning: the parameters after the semicolon specify the projective class. Once the projective class is specified, the parameters before the semicolon are those of the particular metric withing this class. This notation will be used several times in the remainder of the paper.

**Remark 4.** In case \( h = \pm 1 \), we may apply another change of coordinates and swap the coordinates \( x \leftrightarrow y \). Thus, we achieve additional restrictions on the parameters. In case \( h = 1 \), we can achieve \( \varepsilon_2 > 0 \). In case \( h = -1 \), we need to distinguish between the case \( \varepsilon = 1 \) and the case \( \varepsilon = -1 \). Let \( (\varepsilon, h) = (1, 1) \). Then we can achieve \( \kappa > 0 \). If, on the other hand, \( (\varepsilon, h) = (-1, -1) \), swapping \( x \) and \( y \) transforms the metric into itself.

The above reasoning ensures that, if \( g = g[\kappa, \varepsilon_2; h, \varepsilon] \) and \( g' = g[\kappa', \varepsilon_2'; h, \varepsilon] \) of the form (24) are isometric, then \( |\kappa'| = |\kappa| \). Thus, it remains to prove that two metrics of the above type are isometric, (if and) only if \( \varepsilon_2' = \varepsilon_2 \) and \( \text{sgn}(\kappa') = \text{sgn}(\kappa) \). Examine the poles of either side of (21). We have the four cases

\[
\begin{array}{cccc}
\varepsilon_1' & \varepsilon_1 & \varepsilon_2' & \varepsilon_2 \\
\varepsilon_1' & \varepsilon_1 & 1 & 1 \\
\varepsilon_2' & \varepsilon_2 & 1 & 1 \\
\varepsilon_1' & 1 & \varepsilon_1 & \varepsilon_2 \\
\varepsilon_2' & 1 & \varepsilon_2 & \varepsilon_1 \\
\end{array}
\]

In view of Remark 4, two different copies of (24) might be isometric. However, we now prove that Remark 4 provides all projective transformations that do not embed into the flow (19). This permits us to obtain the following list of mutually non-isometric normal forms.
Proposition 5. A metric of the type \( A(I) \) in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

(i) For \( \kappa \in \mathbb{R} \setminus \{0\} \), \( \varepsilon, \varepsilon_2 \in \{\pm 1\} \) and \( h \in \mathbb{R} \setminus \{0, -1\} \), \( (h, \varepsilon) \neq (1, 1) \):

\[
g = \kappa \left( \frac{x^\xi - hy^\xi}{(1 + \varepsilon hy^\xi)(1 + \varepsilon_2 x^\xi)} \, dx^2 + \frac{\varepsilon(x^\xi - hy^\xi)}{(1 + \varepsilon hy^\xi)^2(1 + \varepsilon_2 x^\xi)} \, dy^2 \right),
\]

(ii) Special case \( h = -1 \) (we choose \( \varepsilon = 1 \), cf. Remark 4). For \( \varepsilon \in \{\pm 1\} \) and \( \kappa \in \mathbb{R} \setminus \{0\} \), the normal forms are:

\[
g = \kappa \left( \frac{x^\xi + y^\xi}{(1 - y^\xi)(1 + x^\xi)} \, dx^2 + \frac{\varepsilon(x^\xi + y^\xi)}{(1 - y^\xi)^2(1 + x^\xi)} \, dy^2 \right),
\]

(iii) Special case \( h = +1, \varepsilon = +1 \) (we choose \( \kappa > 0 \)). For \( \varepsilon_2 \in \{\pm 1\} \), the normal forms are:

\[
g = \kappa \left( \frac{x^\xi - y^\xi}{(1 + \varepsilon_2 y^\xi)(1 + \varepsilon_2 x^\xi)} \, dx^2 + \frac{(x^\xi - y^\xi)}{(1 + \varepsilon_2 y^\xi)^2(1 + \varepsilon_2 x^\xi)} \, dy^2 \right),
\]

The parameter \( \xi \) can assume values \( \xi \in (0, 1) \) or \( \xi \in (1, 4) \). In case \( \xi = 2 \), we require \( h \neq -\varepsilon \).

Proof. We need to show that two different copies of the above metrics are non-isometric. This can be achieved by a thorough examination of all the cases in the table before Proposition 5. We need to confirm that in order to have isometric metrics we need \( \varepsilon_2 = \varepsilon_2 \) and \( \kappa' = \kappa \).

Analogously to Equation (21), we obtain the following requirement (we divide by \( k \) on both sides)

\[
\text{sgn}(\kappa) \left( \frac{x^\xi_0 - hy^\xi_0}{(1 + \varepsilon hy^\xi_0)(1 + \varepsilon_2 x^\xi_0)} x^2_0 + \varepsilon \frac{x^\xi_0 - hy^\xi_0}{(1 + \varepsilon hy^\xi_0)(1 + \varepsilon_2 x^\xi_0)} y^2_0 \right) = \text{sgn}(\kappa') \left( \frac{x^\xi_0 - hy^\xi_0}{(1 + \varepsilon hy^\xi_0)(1 + \varepsilon_2 x^\xi_0)} x^2_0 + \varepsilon \frac{x^\xi_0 - hy^\xi_0}{(1 + \varepsilon hy^\xi_0)(1 + \varepsilon_2 x^\xi_0)} y^2_0 \right) \tag{25}
\]

(1) Start with case (a). Recalling \( x_0 > 0, y_0 > 0 \) and \( x'_0 > 0, y'_0 > 0 \), we conclude from the second equation that \( \varepsilon'_2/\varepsilon_2 > 0 \) and thus \( \varepsilon_2 = \varepsilon'_2 \). Thus, \( x_0 = x'_0 \), again from the second equation. Now, infer \( hy_0^\xi = hy'^\xi_0 \) and thus \( y_0 = y'_0 \) from the first equation.

From (25) we then find \( \kappa = \kappa' \). To see this, replace unprimed objects by the relations we have just derived. Comparing the left and the right hand side of (25), we have to equate \( x^\xi_0 \text{sgn}(\kappa \kappa') = x^\xi_0 = x^\xi_0 \), proving \( \text{sgn}(\kappa) = \text{sgn}(\kappa') \) and therefore \( \kappa' = \kappa \).

(2) In case (b), we have \( \varepsilon_2 x^\xi_0 = \varepsilon_2 y^\xi_0 \) and thus \( \varepsilon'_2/\varepsilon_2 > 0 \) from the second equation. This implies in particular that \( \varepsilon'_2 = \text{sgn}(h)\varepsilon_2 \) (and thus case (b) can only occur in case \( \text{sgn}(h) = 1 \)).

Using the relations obtained so far, we can replace unprimed objects in (25). From the resulting expression it follows that \( |h| = 1 \).

From the above we conclude \( h = 1 \) so that \( \varepsilon'_2 = \varepsilon_2 \). Assuming \( \varepsilon = -1 \) and resubstituting this into (25) confirms \( \kappa' = \kappa \). If, however, \( \varepsilon = 1 \), we see that case (b) cannot occur.

(3) Cases (c) and (d) can be discarded, as in these cases one can derive from the first and second equation that \( x^\xi_0 = hy^\xi_0 \). We may assume \( (x_0, y_0) \) to be chosen anywhere except for light lines of the metric \( g \). But, \( x^\xi_0 = hy^\xi_0 \) would obviously put both sides of (25) to zero, contrary to the assumption.

\( \square \)

3.1.2 Normal forms (B.4) of Theorem 1

In the complex Liouville case, \( B(I) \), we can construct any metric \( g \) in the projective class, via Formula (5), from the following two metrics:

\[
g_1 = \frac{1}{4} \left( C z^\xi - C z^\xi \right) \left( dz^2 - d\bar{z}^2 \right)
\]

\[
g_2 = \frac{1}{4} \left( C^{-1} z^{-\xi} - C^{-1} z^{-\xi} \right) \left( C^{-1} z^{-\xi} dz^2 - C^{-1} z^{-\xi} d\bar{z}^2 \right),
\]

with \( C \in \mathbb{C}, |C| = 1 \) and \( \xi = \frac{2}{2\lambda + 1} \) as in (18). Furthermore, we have

\[
L_{w\left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) = \left( \begin{array}{cc} \lambda & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \quad \text{and} \quad w = -\frac{2\lambda + 1}{2} \left( z \frac{\partial_z}{z} + \bar{z} \frac{\partial_{\bar{z}}}{\bar{z}} \right)}
\]

11
For convenience, we use real polar coordinates \((\rho, \theta)\) so that \(z = \rho \cos(\theta) + i \rho \sin(\theta)\). In these new coordinates, the projective vector field \(w\) takes the concise form

\[
w = \frac{3}{2(\xi - 1)} \rho \partial_\rho.
\]

The flow of this projective vector field is

\[
(\rho, \theta) = (\rho_0 e^{-\frac{\xi}{\rho_0}}, \theta_0), \quad \rho_0 > 0, \theta_0 \in [0, 2\pi).
\]

For the projective connection (1) we find, with \(\theta = \theta(\rho)\),

\[
\theta'' = \frac{\xi}{\rho} \theta' - \frac{\xi}{\tan(\xi \theta + \varphi)} \theta^2,
\]

where \(\varphi\) is defined by \(C = e^{i\varphi}\). Equation (26) shows that the projective connection is determined by the data \((\xi, \tan \varphi)\), i.e. by \((\xi, \varphi \mod \pi)\). Let us now consider a general metric from the projective class \(\Psi(g_1, g_2)\),

\[
g = \frac{C\zeta - \zeta}{(1 + K_2 C\zeta)(1 + K_2 C\zeta)} dz^2 - \frac{C\zeta - \zeta}{(1 + K_2 C\zeta)(1 + K_2 C\zeta)} dz^2
\]

(27)

From Lemma 1, we know that all metrics of the form (27) with the same value of \(\frac{K_1}{|K_2|^2} = k\) are isometric if also the signs \(\text{sgn}(K_1)\) and \(\text{sgn}(K_2)\) are the same. Given \(k\) and \(\varepsilon_i = \text{sgn}(K_i), i \in \{1, 2\}\), we can therefore choose a representative on each orbit, cf. Figure 1. For convenience let us choose the representative \(K_1 = \pm 1\), i.e. in (27) we substitute \(K_1 \to \text{sgn}(K_1) = 1\). Thus, Metric (27) may in the following w.l.o.g. be assumed to be

\[
g = \frac{C\zeta - \zeta}{(1 + K_2 C\zeta)(1 + K_2 C\zeta)} dz^2 - \frac{C\zeta - \zeta}{(1 + K_2 C\zeta)(1 + K_2 C\zeta)} dz^2
\]

(28)

Now perform a change of coordinates \(z \to \eta z\) with \(\eta = |K_2|^{-1/\xi}\), so (28) is put into the form

\[
g = K \left( \frac{e^{i\varphi} \zeta - e^{-i\varphi} \zeta}{(1 + e^{i\varphi} \zeta)(1 + e^{-i\varphi} \zeta)} dz^2 - \frac{e^{i\varphi} \zeta - e^{-i\varphi} \zeta}{(1 + e^{i\varphi} \zeta)(1 + e^{-i\varphi} \zeta)} dz^2 \right).
\]

(29)

Here, we introduced \(K = |K_2|^{-1/\xi}\). Rewriting (29), we find

\[
g = \varepsilon_1 K \left( \frac{e^{i\varphi} \zeta - e^{-i\varphi} \zeta}{(1 + e^{i\varphi} \zeta)(1 + e^{-i\varphi} \zeta)} dz^2 - \frac{e^{i\varphi} \zeta - e^{-i\varphi} \zeta}{(1 + e^{i\varphi} \zeta)(1 + e^{-i\varphi} \zeta)} dz^2 \right),
\]

where we introduced \(\varepsilon = \varepsilon_1 \varepsilon_2\).

**Observation 1.** We observe that under conjugation of the coordinates, \(z \leftrightarrow \bar{z}\), the metric \(g\) is transformed into a metric of the same form, with the parameters being transformed according to \((\kappa, \varphi) \to (\kappa, -\varphi)\).

The parameter \(\varphi\) specifies the projective class (together with \(\xi\)), and w.l.o.g. we may assume \(\varphi \in (0, \pi]\) in view of (26). Now suppose \(\varepsilon = -1 = e^{i\pi}\) and introduce \(\tilde{\varphi}\) such that

\[
e^{i\varphi} = e^{i\tilde{\varphi}}.
\]

where \(\tilde{\varphi} \in (-\pi, 0]\). By a conjugation \(z \leftrightarrow \bar{z}\) we can subsequently achieve again \(\tilde{\varphi} \in (0, \pi]\).

In view of Observation 1 we arrive at the normal forms

\[
g = g[\kappa; h, \xi] = \kappa \left( \frac{h \zeta - \bar{h} \bar{\zeta}}{(1 + h \zeta)(1 + h \bar{\zeta})} dz^2 - \frac{h \zeta - \bar{h} \bar{\zeta}}{(1 + h \zeta)(1 + h \bar{\zeta})} dz^2 \right)
\]

(30)

where \(\kappa = \varepsilon_2 K \in \mathbb{R} \setminus \{0\}\) and \(h \in \mathbb{P}^1\).

**Lemma 2.** Two different copies \(g = g[\kappa; h, \xi]\) and \(g' = g[\kappa'; h, \xi]\) of (30) are non-isometric.

**Proof.** Consider Equation (14) for the metrics \(g\) and \(g'\), i.e. metrics of the form (30). The flow of the projective vector fields integrates (with \(s = e^{(1-\lambda)\theta}\)) to \(z = s z_0\) with integration constant \(z_0\) (and analogously \(\bar{z} = s \bar{z}_0\) where \(z_0\) and \(\bar{z}_0\) are complex conjugates). Thus we need to investigate the equation

\[
\kappa \left( \frac{h \zeta - \bar{h} \bar{\zeta}}{(1 + h \zeta)(1 + h \bar{\zeta})} dz^2 - \frac{h \zeta - \bar{h} \bar{\zeta}}{(1 + h \zeta)(1 + h \bar{\zeta})} dz^2 \right) = \kappa' \left( \frac{h \zeta - \bar{h} \bar{\zeta}}{(1 + h \zeta)(1 + h \bar{\zeta})} dz^2 - \frac{h \zeta - \bar{h} \bar{\zeta}}{(1 + h \zeta)(1 + h \bar{\zeta})} dz^2 \right).
\]

(31)
Comparing the left and right hand side of (31), we infer from an investigation of possible poles the two possibilities
\[ (a) \quad z'_0 = z_0 \quad \text{and} \quad (b) \quad z'_0 = h^{1/ξ} z_0. \]
In case (a), we thus obtain \( κ' = κ \), which concludes the proof. In case (b), we consider (31) for \( s = 0 \), which implies
\[ -κ' |h|^2 (h^{1/ξ} z'_0 + h^{-1/ξ} z_0) = κ (z'_0 + z_0). \]
Since \((z_0, z_0)\) is a generic pair of complex conjugates, this implies \( h = \overline{h} \), thus \( h = 1 \), and finally \( κ' = κ \). \( \square \)

The reasoning made above assures that if the metrics \( g \) and \( g' \) are isometric then \( κ' = κ \) (\( h \) and \( ξ \) define the projective class).

**Proposition 6.** A metric of the type \( B(I) \) of Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

\[ g = κ \left( \frac{h_z^ξ - h_\overline{z}^ξ}{(1 + h_\overline{z}^ξ)(1 + h_z^ξ)^2} \, dz^2 - \frac{h_z^ξ - h_\overline{z}^ξ}{(1 + h_\overline{z}^ξ)(1 + h_z^ξ)^2} \, d\overline{z}^2 \right), \]
where \( κ \in \mathbb{R} \setminus \{0\} \) and \( h \in \mathbb{P}^1, ξ \in (0, 1) \cup (1, 4) \). If \( ξ = 2, h \neq ±1 \).

### 3.1.3 Normal forms (C.7) of Theorem 1

Consider metrics of case \( C(Ib) \) in Proposition 3. Such metrics are of Jordan block type and can be constructed, via Formula (5), from the following two metrics:

\begin{align*}
g_1 &= 2(Y(y) + x) dx dy \\
g_2 &= -2Y(y) + x \frac{dx}{y^3} + \frac{Y(y) + x^2}{y^4} dy^2
\end{align*}

where \( y > 0 \) and \( Y(y) = Cy^\frac{1}{ξ} \) with \( C \in \mathbb{R}, ξ \) as in (18). The projective connection (1) is found to be, with \( y = y(x) \),

\[ y'' = \frac{y'}{Cy^\frac{1}{ξ} + x} - \frac{2λ + 1}{2(λ - 1)} \frac{Cy^\frac{1}{ξ} + x}{Cy^\frac{1}{ξ} + 1 + x} \frac{y'}{y}, \]

i.e. the projective class is determined by \( λ \) such that \( |λ| \geq 1, \lambda \neq 1 \).

**Remark 5.** If \( C = 0 \), the metric has constant curvature, then we can ignore this case. For \( C \neq 0 \), we can w.l.o.g. restrict to \( C = 1 \), by grace of a coordinate transformation \( x \to Cx \) and subsequent rescaling of the metrics (32). In the following, however, we therefore continue only with \( C = 1 \).

We recall the definition (18) of \( ξ \) in terms of the eigenvalue \( λ \): \( ξ = \frac{λ - 1}{2λ + 1} \). The normal forms (C.7) of Theorem 1 have degree of mobility 2 (see Proposition 3), i.e. \( \dim(\mathcal{P}(g_1, g_2)) = 2 \) according to Definition 4. This requires

\[ λ \neq \frac{5}{2}, \quad \text{i.e.} \quad ξ \neq \frac{1}{2}, \]

for the present section, since for \( ξ = \frac{1}{2} \) the degree of mobility is 3 (this case is discussed in Section 3.1.4). The projective vector field, fixed by (13) and case (I) of (7), reads, in the coordinates of (32),

\[ w = -\left( λ + \frac{1}{2} \right) x∂_x - (λ - 1)y∂_y. \]

From (33) we infer the projective flow

\[ ϕ_t(x_0, y_0) = (x_0 e^{-(λ + \frac{1}{2})t}, y_0 e^{-(λ - 1)t}) \]

A metric from the projective class \( \mathcal{P}(g_1, g_2) \) is of the form

\[ g[K_1, K_2] = 2\frac{y^\frac{1}{ξ} + x}{(K_1 - K_2y)^3} \, dx dy + K_2\left( y^\frac{1}{ξ} + x \right)^2 dy^2 \]

Using this expression, we can write the square of the length of the projective vector field as

\[ ||w||_0^2(x, y) = 2\frac{y^\frac{1}{ξ} + x}{(K_1 - K_2y)^3} \left( λ + \frac{1}{2} \right) (λ - 1)xy + K_2\left( y^\frac{1}{ξ} + x \right)^2 (λ - 1)^2 y^2. \]
Restricting (36) to the flow of \( w \), we obtain

\[
\|w\|^2_\theta(x_0, y_0; t) = 2 \left( y_0 e^{-\frac{(\lambda-1) t}{2}} + x_0 e^{-\frac{(\lambda+\frac{1}{2}) t}{2}} \right) \left( \lambda - 1 \right) x_0 y_0 e^{-2\lambda \frac{1}{2} t} + \left( y_0 e^{-\frac{(\lambda-1) t}{2}} + x_0 e^{-\frac{(\lambda+\frac{1}{2}) t}{2}} \right)^2
\]

\[
+ K_2 \left( \frac{y_0 e^{-\frac{(\lambda-1) t}{2}} + x_0 e^{-\frac{(\lambda+\frac{1}{2}) t}{2}}}{(K_1 - K_2 y_0 e^{-2(\lambda-1) t})^2} \right) (\lambda - 1)^2 y_0^2 e^{-2(\lambda-1) t}
\]

\[
= e^{-3t} \left( 2 \left( y_0 + x_0 \right) \left( \lambda - 1 \right) x_0 y_0 + \frac{2}{(K_1 s - K_2 y_0)^3} (\lambda - 1)^2 y_0^2 \right)
\]

For the second identity, we have made use of Equation (18). Let \( g = g[K_1, K_2] \) and \( g' = g[K'_1, K'_2] \) be two isometric copies of (35). Let us consider Equation (14) and introduce the new parameter \( s := e^{(\lambda-1) t} \), which yields

\[
2 \left( y_0 + x_0 \right) \left( \lambda - 1 \right) x_0 y_0 + \frac{2}{(K_1 s - K_2 y_0)^3} (\lambda - 1)^2 y_0^2
\]

\[
= 2 \left( y_0 + x_0' \right) \left( \lambda - 1 \right) x_0' y_0 + \frac{2}{(K_1 s - K_2 y'_0)^3} (\lambda - 1)^2 y'_0^2
\]

(37)

where \( \prime \) has the obvious meaning. If \( K_1 = 0 \) then \( K'_1 = 0 \), and analogously \( K'_2 = 0 \) if \( K_2 = 0 \). Therefore, for metrics with an essential projective vector field, we have \( K_1 \neq 0 \), \( K_2 \neq 0 \), so that also \( K'_1 \neq 0 \), \( K'_2 \neq 0 \). Consider Equation (37). The left hand side has the pole \( s = \frac{K_2}{K_1} y_0 \), and therefore also the right hand side has to have a pole at \( s = \frac{K'_2}{K'_1} y_0 \). From this, we infer the condition \( \frac{K'_2}{K'_1} y_0 - K_2 y_0 = 0 \), which implies

\[
H := \frac{y'_0}{y_0} = \frac{K'_1 K_2}{K_1 K'_2} > 0.
\]

(38)

The constant \( H \) is greater than zero since, in view of Lemma 1, \( K_1 \) and \( K'_1 \) have the same sign, as well as \( K_2 \) and \( K'_2 \). The same result follows from (34), from which we can deduce that the signs of \( y_0 \) and \( y'_0 \) (\( x_0 \) and \( x'_0 \)) are the same. If we substitute \( y'_0 = H y_0 \) into Equation (37), we obtain

\[
2 \left( y_0 + x_0 \right) \left( \lambda - 1 \right) x_0 y_0 + \frac{2}{(K_1 s - K_2 y_0)^3} (\lambda - 1)^2 y_0^2
\]

\[
= 2 \frac{K_2 K'_2}{K_1 K'_1} \left( \lambda - 1 \right) x_0 y_0 + \frac{2}{(K_1 s - K_2 y_0)^3} (\lambda - 1)^2 y_0^2
\]

Multiplying the previous equation by \( (K_1 s - K_2 y_0)^3 \), we obtain a polynomial of first degree in \( s \). If we put equal to zero the coefficients of first degree and zero degree we obtain the following system

\[
\begin{cases}
(y_0 + x_0) x_0 = H (y_0 + x'_0) x'_0 \\
(y_0 + x_0)^2 = H (y_0 + x'_0)^2
\end{cases}
\]

(39)

that implies (by considering the square of the first equation and then considering the quotient of the two equations)

\[
x_0^2 = \tilde{H} x'_0^2, \quad \tilde{H} > 0
\]

(40)

(note that \( \tilde{H} > 0 \) also follows from the second equation of system (39)). \( \tilde{H} > 0 \) implies

\[
\frac{K_2}{K'_2} > 0 \quad \text{which in view of (38) implies} \quad \frac{K'_1}{K_1} > 0
\]

(41)

Substituting (40) into the first equation of (39), we obtain

\[
x_0 = \tilde{H} H^\frac{1}{2} x'_0
\]

(42)
By considering its square and taking into account (40), we find 1 = \( \tilde{H} \tilde{H} \tilde{\tau} \). Now, substitute the explicit expressions of \( H \) and \( \tilde{H} \) (see (38) and (39)) into this equation and obtain

\[
1 = \frac{K_2 K_1^2}{K_2^2 K_1} \left( K_1 K_2 \right) \tilde{\tau} = (41) \left( \frac{K_2}{K_1} \left( K_1 \right) \right) \tilde{\tau} + 1 \quad \text{and} \quad \frac{K_2}{K_1} \left( K_1 \right) \tilde{\tau} + 2 = (18) \left( \frac{K_2}{K_1} \left( K_1 \right) \right) \tilde{\tau} + 2,
\]

so

\[
1 = \left| \frac{K_2}{K_1} \right|^{\lambda} \left| \frac{K_1}{K_2} \right|,
\]

which together with (41) implies the second case of Lemma 1. So far we have essentially proven that different values of \( K := \frac{K_1}{K_2} \) give non-isometric metrics of the type (35). Now, we substitute \( K_1 = \varepsilon_1 K|K_2|^{\lambda} \) into (35) and obtain

\[
g = 2 \frac{y + x}{(\varepsilon_1 K|K_2|^{\lambda} - K|x)^2} \left\{ x \right\} dy^2 + K_2 \left( \frac{y + x}{(\varepsilon_1 K|K_2|^{\lambda} - K|x)^2} \right)^2 dy^2
\]

By performing the change of coordinates

\[
x \rightarrow (K|K_2|^{\lambda-1})^\varepsilon x \quad \text{and} \quad y \rightarrow K|K_2|^{\lambda-1} y,
\]

and taking into account (18), metrics (43) become

\[
g = k \left( 2 \frac{y + x}{(\varepsilon_1 - K|x)^3} \right) dy^2 + K_2 \left( \frac{y + x}{(\varepsilon_1 - K|x)^3} \right)^2 dy^2, \quad \varepsilon_1 \in \{\pm 1\}, \quad k > 0
\]

With \( k := K^{\varepsilon-2}, \varepsilon_1 := \text{sgn}(K_1), \varepsilon_2 := \text{sgn}(K_2) \). Let us rename \( k \rightarrow k/\varepsilon_2 \) and define \( \varepsilon := \varepsilon_1 \varepsilon_2 \), then we obtain

\[
g[k, \varepsilon; \xi] = k \left( -2 \frac{y + x}{(y - \varepsilon)^3} \right) dy^2 + \left( \frac{y + x}{(y - \varepsilon)^3} \right)^2 dy^2, \quad \varepsilon \in \{\pm 1\}, \quad k \in \mathbb{R} \setminus \{0\}
\]

(note that \( k \) can now also be negative).

**Lemma 3.** Two different copies \( g = g[k, \varepsilon; \xi] \) and \( \tilde{g} = g[k, \varepsilon; \xi] \) of (45) are non-isometric.

**Proof.** We need to prove \( \tilde{k} = k \) and \( \tilde{\varepsilon} = \varepsilon \). By construction, we already know that \( |k| = |k| \), so it suffices to prove that the sign of \( k \) and the sign \( \varepsilon \) remain unchanged under isometries. From (38), where we are only interested in the signs, we have \( \varepsilon \tilde{\varepsilon} = +1 \), and thus \( \tilde{\varepsilon} = \varepsilon \). Moreover, from (42), we similarly conclude \( \tilde{\varepsilon} = +1 \) (using the definition of \( \varepsilon_2 \) from (44)), proving \( \tilde{k} = k \) since \( \text{sgn} k = \text{sgn}(K^{2/\varepsilon-2} \varepsilon_2) = \varepsilon \).

Altogether, we have found:

**Proposition 7.** A metric of the type \( C(Ib) \) in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

\[
g = k \left( -2 \frac{y + x}{(y - \varepsilon)^3} \right) dy^2 + \left( \frac{y + x}{(y - \varepsilon)^3} \right)^2 dy^2
\]

where \( \varepsilon \in \{\pm 1\}, k \in \mathbb{R} \setminus \{0\} \) and \( \xi \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \cup (1, 4] \).

### 3.1.4 Normal forms (C.10) of Theorem 1; degree of mobility 3

Amongst the projective classes of Proposition 3, almost all cases have a 2-dimensional space of solutions of the linearized metrization equations (4), cf. Section 2.1. However, there is one case of dimension 3. This case corresponds to a superintegrable system\(^4\), see the Remark 6 below. In the superintegrable case, i.e. \( C(Ia) \) of

---

\(^4\)Actually, there are several superintegrable systems but the solution spaces \( \mathcal{B} \) are isomorphic.
Proposition 3, we can construct any metric $g$ in the projective class, via Formula (5), from the following three metrics:

\[
g_1 = (x + y^2) \, dx \, dy
\]

\[
g_2 = -2 \frac{x + y^2}{y^2} \, dx \, dy + \frac{(x + y^2)^2}{y^4} \, dy^2
\]

\[
g_3 = \frac{y^2 + x}{(3x - y^2)^2} (9(y^2 + x) \, dx^2 - 4y(9x + y^2) \, dx \, dy + 12x(y^2 + x) \, dy^2),
\]

and the projective vector field, fixed by (13) and case (I) of (7), has the form

\[
w = 2x \, \partial_x + y \, \partial_y.
\] (47)

**Remark 6** (Projective equivalence and superintegrable metrics). Given a metric $g \in \mathcal{P}(g_1, g_2, g_3)$, the existence of a metric $\bar{g}$ projectively equivalent to $g$ implies the existence of quadratic Killing tensors $\iota(g)$ for the geodesic motion (i.e. integrals of motion that are homogeneous quadratic polynomials in the fiber coordinates of $T^*M$).

The correspondence is given by the mapping $[6, 14]$:

\[
\iota(g) = \left| \frac{\det(g)}{\det(\bar{g})} \right|^{2/3} \bar{g}.
\]

For the purposes of the present paper, a superintegrable system is a triple $(g, L_1, L_2)$ on a 2-dimensional manifold with (pseudo-)Riemannian metric $g$ and two quadratic Killing tensors $L_1$ and $L_2$ such that $(H, I_1, I_2)$ are functionally independent integrals of motion, where we define:

\[
H = \sum_{ij} g^{ij} p_i p_j, \quad \text{and} \quad I_k = \sum_{ij} L_k^{ij} p_i p_j \quad \text{for} \ k \in \{1, 2\}.
\]

Note that this definition of a superintegrable system is different from the definition used in [6]. In [6], a (Darboux-)superintegrable system is defined by the requirement $\dim \mathfrak{X} = 4$, i.e. the existence of four linearly independent integrals of motion. The superintegrable metrics discussed in the present paper, in contrast, have $\dim \mathfrak{X} = 3$.

Before we proceed further, let us make the following observation:

**Remark 7.** For the metrics $g_1$, $g_2$ and $g_3$, the projective vector field is homothetic. This follows from Proposition 4 and is easily verified by computing the Lie derivatives w.r.t. (47):

\[
L_w g_1 = 5g_1, \quad L_w g_2 = 2g_2, \quad L_w g_3 = -4g_3.
\]

Any isometry is a projective transformation, and moreover any isometry preserves eigenspaces of $L_w$, as eigenspaces are geometric objects. Therefore, for an isometry $\tau$ we have the identities

\[
\tau^* g_1 = \mu_1 g_1, \quad \tau^* g_2 = \mu_2 g_2, \quad \tau^* g_3 = \mu_3 g_3, \quad \text{where} \ \mu_i \in \mathbb{R} \setminus \{0\}, \ i \in \{1, 2, 3\}.
\]

On the other hand, (46) suffices, via Formula (5), to describe any metric of the projective class $\mathcal{P}(g_1, g_2, g_3)$, using the same system of coordinates $(x, y)$. In order to identify which metrics in $\mathcal{P}(g_1, g_2, g_3)$ could be isometric, we therefore need to understand those coordinate transformations that preserve the form of the metric.

**Lemma 4.** Any isometry of a metric in a projective class of the type $C(1a)$ that preserves the form of a metric $g \in \mathcal{P}(g_1, g_2, g_3)$ in the coordinates of (46), is given by

\[
(x_{new}, y_{new}) = (k^2 x_{old}, k y_{old})
\]

where $k \in \mathbb{R} \setminus \{0\}$.

**Proof.** Consider the metrics $g_1$ and $g_2$ as in (46), and a generic coordinate transformation $(x, y) = (x(u, v), y(u, v))$. For $g_1$ obtain the equation

\[
g_1 = (x + y^2) (x_u y_v \, du \, dv + x_v y_u \, dv \, du + (x_u y_v + x_v y_u) \, du \, dv) = \mu_1 (u + v^2) \, du \, dv
\] (48)

(by ! we mark the condition that eigenspaces are preserved). Thus, around almost every point, we have one of the following possibilities:

(i) $x = x(u), \ y = y(v)$ \quad or \quad (ii) $x = x(v), \ y = y(u)$,
so that (48) gives
\[ \mu_1 (u + v^2) = (x + y^2) (x u_y + x_v y_u). \]  
(49)

Now, do an analogous computation for \( g_2 \). Case (ii) turns out to be incompatible since it would replace the \( dy^2 \)-term by a term in \( du^2 \) instead of \( dv^2 \). For alternative (i), we obtain
\[
g_2 = -2 \frac{x + y^2}{y^3} x_u y_v \, du \, dv + \frac{(x + y^2)^2}{y^4} x_v^2 \, dv^2
\]
\[= -2 \mu_1 \frac{u + v^2}{y^3} \, du \, dv + \frac{\mu_2^2}{x_u^2} (u + v^2)^2 \, dv^2 \quad \text{for } (49) \]
\[
2 \mu_2 \frac{u + v^2}{y^3} \, du \, dv + \frac{\mu_2^2}{y^4 x_u^2} (u + v^2)^2 \, dv^2. \]
(50)
The coefficients in (50) of, respectively, \( du \, dv \) and \( dv^2 \), yield the equations
\[
\frac{\mu_1}{y^3} = \frac{\mu_2}{y^3} \quad \text{and} \quad \frac{\mu_1^2}{y^4 x_u^2} = \mu_2^2,
\]
from which we can deduce
\[
y = \sqrt{\frac{\mu_1}{\mu_2}} v =: \mu v \quad \text{and} \quad x_u^2 = \frac{\mu_2^4}{\mu_1^2} \sqrt{\frac{\mu_2}{\mu_1}} = \mu_1^{2/3} \mu_2 =: \eta v,
\]
(51)
where the root takes account of the sign and where \( \mu, \eta \in \mathbb{R} \setminus \{0\} \) are constants. In view of (50), we can integrate the second equation of (51) and obtain \( x = \eta u \). Note that \( \mu \) and \( \eta \) are not independent, but, since their definition is rather complicated, we shall continue with the investigation of the transformation \((x, y) = (\eta u, \mu v)\) without prior study of the relation of \( \mu \) and \( \eta \). Apply this coordinate transformation to \( g_3 \) of (46). The result again has to be in the same eigenspace. Thus,
\[
g_3 = \frac{\mu^2 v^2 + \eta u}{(3 \mu u - \mu^2 v^2)^2} \left( 9 \left( \mu^2 v^2 + \eta u \right) \eta^2 \, du^2 - 4 \mu y \left( 9 \eta u + \mu^2 v^2 \right) \eta \mu \, du \, dv + 12 \eta u (\mu^2 v^2 + \eta u) \mu^2 \, dv^2 \right)
\]
\[= \eta^2 \left( \frac{\mu^2 v^2 + \eta u}{(3 \mu u - \mu^2 v^2)^2} \right)^2
\]
\[
= \frac{\mu^2 (v^2 + u)^2}{(3u - v^2)^6} = \eta \mu^2 \left( \frac{\mu^2 v^2 + \eta u}{(3 \mu u - \mu^2 v^2)^2} \right)^2.
\]
The coefficients w.r.t. \( du^2 \) and \( dv^2 \) of this equation give two equations from which we may infer
\[
\eta^2 \left( \frac{\mu^2 v^2 + \eta u}{(3 \mu u - \mu^2 v^2)^2} \right)^2 = \mu_3 \left( \frac{v^2 + u}{3u - v^2} \right)^2 = \eta \mu^2 \left( \frac{\mu^2 v^2 + \eta u}{(3 \mu u - \mu^2 v^2)^2} \right)^2.
\]
Therefore, we have verified \( \eta = \mu^2 \) and thus, in order to preserve the form of the metric, it is necessary that the isometry takes the form \((x, y) = (\mu^2 u, \mu v)\).

**Observation 2.** Consider an arbitrary metric \( g \in \mathcal{P}(g_1, g_2, g_3) \). By Formula (5), \( g \) is of the form
\[
g = \frac{\sum_{i=1}^{3} K_i \frac{g_i}{\det(g)}}{\det \left( \sum_{i=1}^{3} K_i \frac{g_i}{\det(g)} \right)^2},
\]
Note that if we apply a change of coordinates \((x, y) \rightarrow (k^2 x, ky)\) then
\[
g = \frac{\sum_{i=1}^{3} K_i k^{- \omega_i/3} \frac{g_i}{\det(g)}}{\det \left( \sum_{i=1}^{3} K_i k^{- \omega_i/3} \frac{g_i}{\det(g)} \right)^2},
\]
with \((\alpha_1, \alpha_2, \alpha_3) = (5, 2, -12)\). Thus, by a suitable choice of \( k \), we can normalize one of the coefficients \( K_1 \), \( K_2 \) or \( K_3 \) (up to its sign).

In view of Observation 2, we need to distinguish cases when one of the parameters \( K_i \), \( i \in \{1, 2, 3\} \), is zero, since in such cases the respective parameters cannot be normalized. On the other hand, if one of the \( K_i \) is zero, then the other two parameters definitely need to be non-zero since otherwise the metric is proportional to one of the metrics (46) and thus admits a homothety.

We therefore proceed in three steps: first, if \( K_1 = 0 \), we normalize \( K_2 \) (or \( K_1 \)). Second, if \( K_2 = 0 \), we normalize \( K_1 \). Finally, if both \( K_2 \) and \( K_3 \) are non-zero, we normalize \( K_2 \).
Vanishing coefficient $K_3 = 0$. In this case we have to assume $K_1, K_2 \neq 0$ because otherwise the projective vector field would not be essential. In view of Observation 2 we can put the metric into the form

$$g[\varepsilon, c] = \varepsilon \left( -2\frac{y^2 + x}{(y - c)^3} \, dx \, dy + \frac{(y^2 + x)^2}{(y - c)^4} \, dy^2 \right),$$

(52)

where $c \in \mathbb{R} \setminus \{0\}$ and $\varepsilon \in \{\pm 1\}$. Alternatively, we could normalize the other coefficient and obtain

$$g[\kappa, \varepsilon] = \kappa \left( -2\frac{y^2 + x}{(y - \varepsilon)^3} \, dx \, dy + \frac{(y^2 + x)^2}{(y - \varepsilon)^4} \, dy^2 \right),$$

(53)

where $\kappa \in \mathbb{R} \setminus \{0\}$ and $\varepsilon \in \{\pm 1\}$.

**Lemma 5.** Two different copies $g[\varepsilon, c]$ and $g[\varepsilon', c']$ of (52) are non-isometric, as are two different copies $g = g[\kappa, c]$ and $g' = g[\kappa', c']$ of (53).

**Proof.** We refer to use Metric (53) for the proof. Recall that the flow of the projective vector field (47) is $\phi_x(x_0, y_0) = (s^2 x_0, s y_0)$ with $s = e^t > 0$. Considering Equation (14), i.e.

$$\kappa \left( -2\frac{y_0^2 + x_0}{(s y_0 - x_0)^3} \, s^5 \, x_0 y_0 + \frac{s^6 (y_0^2 + x_0)^2}{(s y_0 - x_0)^4} \right) = \kappa' \left( -2\frac{y_0^2 + x_0}{(s y_0' - x_0')^3} \, s^5 \, x_0' y_0' + \frac{s^6 (y_0'^2 + x_0')^2}{(s y_0' - x_0')^4} \right),$$

(54)

and comparing poles of either side, we find

$$\frac{y_0}{\varepsilon} = \frac{y_0'}{\varepsilon'}$$

Substituting this back into (54), an equation polynomial in the parameter $s$ is obtained,

$$[(\kappa - \kappa')y_0^2 + (\kappa'x_0'^2 - \kappa x_0^2)] y_0^6 + 2\varepsilon y_0 \left[ (\kappa x_0 - \kappa' x_0') y_0^3 + (\kappa x_0^2 - \kappa' x_0'^2) \right] s^5 = 0$$

(55)

Consider the system of polynomial equations obtained from the coefficients of (55) w.r.t. $s$. For generic values of $y_0, x_0, x_0'$, it implies $\kappa' = \kappa$ and $x_0' = x_0$. Resubstituting these relations into (54), we have

$$-2\frac{y_0^2 + x_0}{(s y_0 - x_0)^3} \, x_0 y_0 + \frac{s^6 (y_0^2 + x_0)^2}{(s y_0 - x_0)^4} y_0^6 = -2\frac{\varepsilon' y_0^2 + x_0}{(s \varepsilon' y_0 - \varepsilon')^3} \, \varepsilon' x_0 y_0 + \frac{s^6 (\varepsilon' y_0^2 + x_0')^2}{(s \varepsilon' y_0 - \varepsilon')^4} \varepsilon' y_0^6,$$

from which we finally infer $\varepsilon' = \varepsilon$. \qed

Vanishing coefficient $K_2 = 0$. To exclude metrics with homotheties we have to assume $K_1, K_3 \neq 0$. Via Equation (5), we obtain

$$g = \frac{K_3^{-3}}{F(K_1; x, y)} \left( 9(y^2 + x)^2 \, dx^2 - 2 \left( y^3 + 9xy - \frac{2K_1}{K_3} \right) (y^2 + x) \, dx \, dy + 12x(y^2 + x) \, dy^2 \right),$$

with

$$F(c; x, y) = y^6 - 9xy^4 + 27x^2y^2 - 27x^3 + 4c^2 - (36xy + 4y^3) \, c.$$  

In view of Observation 2, we may perform a change of coordinates $(x, y) \rightarrow (k^2 x, ky)$ so that the metric becomes

$$g[\kappa, \varepsilon] = \frac{\kappa}{F(\varepsilon; x, y)} \left( 9(y^2 + x)^2 \, dx^2 - 2(y^3 + 9xy - 2\varepsilon) (y^2 + x) \, dx \, dy + 12x(y^2 + x) \, dy^2 \right),$$

(56)

where $\varepsilon \in \{\pm 1\}$ and $\kappa \in \mathbb{R} \setminus \{0\}$.

**Lemma 6.** Two different copies $g = g[\kappa, \varepsilon]$ and $g' = g[\kappa', \varepsilon']$ of metrics (56) are non-isometric.

**Proof.** If two metrics are isometric, they must, by virtue of Lemma 4, be related by a change of coordinates of the form

$$(x_{new}, y_{new}) = (k^2 x_{old}, ky_{old})$$

with $k \in \mathbb{R} \setminus \{0\}$. Plugging this relation into one of the metrics, we obtain an expression polynomial in $dx, dy, x, y$. Its coefficients form a system of algebraic equations on $\kappa, \varepsilon$ and $\kappa', \varepsilon'$ as well as on $k$ (the primed objects refer to the parameters of another copy of the metric (56)). Solving this system of equations, we find

$$\kappa' = \kappa, \quad \varepsilon' = \varepsilon, \quad k = 1$$

as the only real solution with $k \neq 0$. This concludes the proof. \qed
Both coefficients $K_2, K_3 \neq 0$. The constant $K_1$ may or may not be zero. As before, let us first factor out $K_3$ in (5), which gives a conformal factor $K_3^{-3/2}$. Then, recalling Observation 2, we perform a change of coordinates with $(x, y) \rightarrow (k^{2}x, ky)$ with $k = \left|\frac{K_3}{R}\right|^{3/14} \neq 0$. Thus we arrive at the metric

$$g = \frac{\kappa}{F(\epsilon, c, x, y)^{2}} \left(9(y^{2} + x)^{2} dx^{2} - 2(y^{3} + 2\epsilon y + 9xy - 2c)(y^{2} + x) dx dy + 4(3x + \epsilon)(y^{2} + x)^{2} dy^{2}\right),$$

where $\epsilon = \text{sgn}(K_2) \in \{\pm 1\}$, $\kappa \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}$, and where the function $F$ is defined by

$$F(\zeta, c, x, y) = y^{6} - 9xy^{4} + 27x^{2}y^{2} - 27x^{3} + 4c^{2} - (36xy + 4y^{3})c + (18xy^{2} - 5y^{4} - 9x^{2} - 8cy)\zeta + 4y^{2}\zeta^{2}.$$  

**Lemma 7.** Two different copies $g = g[\kappa, c, \epsilon]$ and $g' = g[\kappa', c', \epsilon']$ of (57) are non-isometric.

**Proof.** If two metrics are isometric, they must, by virtue of Lemma 4, be related by a change of coordinates of the form

$$(x_{\text{new}}, y_{\text{new}}) = (k^{2}x_{\text{old}}, ky_{\text{old}})$$

with $k \in \mathbb{R} \setminus \{0\}$. Plugging this relation into one of the metrics, we obtain an expression polynomial in $dx, dy, x, y$. Its coefficients form a system of algebraic equations on $\kappa, c, \epsilon$ and $\kappa', c', \epsilon'$ as well as $k$ (the primed objects refer to the parameters of another copy of the metric (57)). Solving this system of equations, we find

$$\kappa' = \kappa, \quad c' = c, \quad \epsilon' = \epsilon, \quad k = 1$$

as the only real solution with $k \neq 0$. This concludes the proof. \qed

Finally, we have to make sure that two metrics of different form (53), (56) or (57) cannot be mutually isometric.

**Lemma 8.** Metrics of the form (53), (56) or (57) are pairwise non-isometric.

**Proof.** Let $a = \sum K_{i}a_{i}$ and $\bar{a} = \sum K'_{i}a_{i}$ be the Liouville tensors of a pair of metrics $g, \bar{g}$, as in the statement of Lemma 8. In view of Lemma 4 and Observation 2, isometries between $g$ and $\bar{g}$ rescale $K_{1}, K_{2}, K_{3}$,

$$K'_{i} = K_{i} k^{-a_{i}/3}, \quad i = 1, 2, 3,$$

with $k \in \mathbb{R} \setminus \{0\}$, $(\alpha_1, \alpha_2, \alpha_3) = (5, 2, -12)$. Therefore, if $K_2 = 0$, then $K_3' = 0$, and if $K_3 = 0$, then $K_3' = 0$. If $K_2, K_3 \neq 0$, then $K_2', K_3' \neq 0$. So, $g$ and $\bar{g}$ cannot be isometric. \qed

Summarizing the statements of Lemmas 5, 6, 7 and 8, we have the following:

**Proposition 8.** A metric of the type $C(Ia)$ in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

(i) $g = \kappa \left(-2\frac{y^{2} + x^{2}}{(y^{2} + x^{2})} dx dy + \frac{y^{2} + x^{2}}{(y^{2} + x^{2})} dy^{2}\right),$

(ii) $g = \frac{\kappa}{\gamma_{(0, c, x, y)}} \left(9(y^{2} + x)^{2} dx^{2} - 2(y^{3} + 9xy - 2c)(y^{2} + x) dx dy + 12x(y^{2} + x^{2}) dy^{2}\right),$

(iii) $g = \frac{\kappa}{\gamma_{(\epsilon, c, x, y)}} \left(9(y^{2} + x)^{2} dx^{2} - 2(y^{3} + 2\epsilon y + 9xy - 2c)(y^{2} + x) dx dy + 4(\epsilon + 3x)(y^{2} + x^{2}) dy^{2}\right).$

Here, $\kappa \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}$, $\epsilon \in \{\pm 1\}$, and

$$F(\zeta, c, x, y) = y^{6} - 9xy^{4} + 27x^{2}y^{2} - 27x^{3} + 4c^{2} - (36xy + 4y^{3})c + (18xy^{2} - 5y^{4} - 9x^{2} - 8cy)\zeta + 4y^{2}\zeta^{2}.$$  

### 3.2 Case (II): $L_{w}$ has 1 real eigenvalue

Here we are in the case when $L_{w}$ has the form (II) of (7). Let $(a_{1}, a_{2})$ be a suitable basis of $\mathfrak{X}$, in which $L_{w}$ assumes such form. We have that

$$\exp(tL_{w}) = \begin{pmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{pmatrix}. \quad (58)$$

Let $g_{1} = \psi(a_{1})$, $g_{2} = \psi(a_{2})$ according to Equation (3). Furthermore let $g = \psi(K_{1}a_{1} + K_{2}a_{2})$ be an arbitrarily chosen metric in the projective class $\mathcal{P}(g_{1}, g_{2})$. By virtue of Equation (58), we have

$$\phi^{*}_{a_{1}}(K_{1}a_{1} + K_{2}a_{2}) = K_{1}e^{\epsilon}a_{1} + K_{1}te^{\epsilon}a_{2} + K_{2}e^{\epsilon}a_{2} =: K_{1}'a_{1} + K_{2}'a_{2}. \quad (59)$$

**Lemma 9.** The pairs $(K_1, K_2)$ and $(K'_1, K'_2)$ are related by (59) if and only if they satisfy one of the following cases

(a) $K_1' = K_1 = 0$ and $\frac{K_2'}{K_2} > 0$ \hspace{1cm} (b) $\frac{K_2'}{K_2} > 0$ and $\frac{1}{K_1'} e^{\frac{K_2}{K_2}} = \frac{1}{K_1} e^{\frac{K_2}{K_1}}$. 

19
The metric $g$ has the form
\[
g = \frac{(y - x)e^{-3x}}{(K_1x + K_2)^2(K_1y + K_2)}dx^2 + \frac{h(y - x)e^{-3y}}{(K_1x + K_2)(K_1y + K_2)^2}dy^2.
\]

Since we are only interested in metrics with an essential projective vector field, we assume $K_1 \neq 0$ (for $K_1 = 0$ the metric lies in an eigenspace of $L_w$). The projective vector field, using the same coordinates $(x, y)$, reads $w = \partial_x + \partial_y$. Therefore, the projective flow is given by
\[
\phi_t(x_0, y_0) = (x_0 + t, y_0 + t).
\]

The projective connection (1) is
\[
y'' = \frac{e^{3(y-x)}}{2h(y-x)} + \frac{1 + 3(x-y)}{2(y-x)} y' + \frac{1 - 3(y-x)}{2(y-x)} y'^2 + \frac{h e^{3(x-y)}}{2(y-x)} y'^3,
\]
meaning that $h \in \mathbb{R} \setminus \{0\}$ determines the projective class. We find the following equation from (14).
\[
\frac{(y_0 - x_0)e^{-3x_0}}{(K_1(t + x_0) + K_2)^2(K_1(t + y_0) + K_2)} + \frac{h(y_0 - x_0)e^{-3y_0}}{(K_1(t + x_0) + K_2)(K_1(t + y_0) + K_2)^2}
\]
\[
= \frac{(y_0' - x_0')e^{-3x_0'}}{(K_1'(t + x_0') + K_2')^2(K_1'(t + y_0') + K_2')} + \frac{h(y_0' - x_0')e^{-3y_0'}}{(K_1'(t + x_0') + K_2')(K_1'(t + y_0') + K_2')^2}
\]

Looking at the poles of this expression (respectively, those of the left and right side of this equation, which need to be the same) we obtain the following cases:

| Case | Equation |
|------|----------|
| (a)  | $x'_0 - x_0 + \frac{K'_2}{K'_1} - \frac{K_2}{K_1} = 0$ |
| (b)  | $y'_0 - x_0 + \frac{K'_2}{K'_1} - \frac{K_2}{K_1} = 0$ |
| (c)  | $x'_0 - x_0 + \frac{K'_2}{K'_1} - \frac{K_2}{K_1} = 0$ |
| (d)  | $y'_0 - y_0 + \frac{K'_2}{K'_1} - \frac{K_2}{K_1} = 0$ |

Figure 2 illustrates the orbits of a metric in the projective class $\Psi(g_1, g_2)$ under the flow of $\phi_t$.

### 3.2.1 Normal forms (A.2) of Theorem 1

In the case of Liouville metrics $A(II)$, we can construct any metric $g$ in the projective class, via Formula (5), from the following two metrics, where $h \in \mathbb{R} \setminus \{0\}$:
\[
g_1 = \frac{1}{x} \left( \frac{1}{x} - \frac{1}{y} \right) e^{-3x} dx^2 + \frac{h}{y} \left( \frac{1}{x} - \frac{1}{y} \right) e^{-3y} dy^2
\]
\[
g_2 = (y - x)e^{-3x} dx^2 + (y - x)h e^{-3y} dy^2.
\]

The converse direction is straightforward. □
The first two cases (a) and (b) imply \( x_0 = y_0 \) and are thus irrelevant.

For the case (c) we find \( y_0 - x_0 = y_0' - x_0' \) and thus the above equation reduces to

\[
\left( K_1(t + x_0) + K_2 \right)^2 \left( K_1(t + y_0) + K_2 \right)^2 - e^{-3x_0} + h e^{-3y_0} = \left( K_1(t + x_0') + K_2 \right)^2 \left( K_1(t + y_0') + K_2 \right)^2 - e^{-3x_0'} + h e^{-3y_0'}.
\]

By denoting \( K = \frac{K_1'}{K_1} - \frac{K_2'}{K_2} \), we have that \( x_0' = x_0 + K, \ y_0' = y_0 + K \), which is in the flow of \( \phi_t \). Thus, \( (62) \) simplifies to

\[
\frac{K_1'}{K_1} e^{-3x_0} (K_1(t + y_0) + K_2) + h \frac{K_1'}{K_1} e^{-3y_0} (K_1(t + x_0) + K_2) = e^{-3x_0} e^{-3K} (K_1(t + y_0) + K_2) + h e^{-3y_0} e^{-3K} (K_1(t + x_0) + K_2),
\]

which is a polynomial in \( t \). By considering its first-degree component, we obtain

\[
e^{-K} = \frac{K_1'}{K_1} \left( \text{thus } K_1' \frac{K_1}{K_1} > 0 \right),
\]

which, in view of the definition of \( K \), is the relation from Lemma 9.

Finally, in case (d) we have

\[
x_0' = y_0 + K, \quad y_0' = x_0 + K, \quad \text{with } K = \frac{K_2}{K_1} - \frac{K_2}{K_1}
\]

(note that this transformation is not inside the flow of \( \phi_t \)), which implies \( y_0 - x_0 = x_0' - y_0' \). Analogously to case (c) we find

\[
e^{-K} = -\frac{K_1'}{K_1} \quad \Rightarrow \quad K_1' \frac{K_1}{K_1} < 0.
\]

Thus, in view of Lemma 9, the pairs \((K_1, K_2)\) and \((K_1', K_2')\) are not related by \( (59) \). However, this case is structurally similar to case (c). Consider the exchange of coordinates \( x \leftrightarrow y \), so Metric \( (61) \) becomes

\[
g = \frac{h (x - y) e^{-3x}}{(K_1 y + K_2) (K_1 x + K_2)^2} dx^2 + \frac{(x - y) e^{-3y}}{(K_1 y + K_2) (K_1 x + K_2)^2} dy^2.
\]

Note that if we set \((K_1', K_2') = (K_1, -K_2)\), we obtain

\[
g = \frac{h (y - x) e^{-3x}}{(K_1 y + K_2) (K_1 x + K_2)^2} dx^2 + \frac{(y - x) e^{-3y}}{(K_1 y + K_2) (K_1 x + K_2)^2} dy^2
\]

and this is identical to \( (61) \) if \( h = 1 \).

Let us now look for the normal forms for metrics \( (61) \). Our discussions imply that different values of \( K = \frac{K_2}{K_1} \) give non-isometric metrics. By substituting \( K_2 = K_1 \ln(K K_1) \) into \( (61) \), we hence obtain

\[
g = \frac{(y - x) e^{-3x}}{K_1^2 (x + \ln(K K_1))^2 (y + \ln(K K_1))} dx^2 + \frac{h (y - x) e^{-3y}}{K_1^2 ((y + \ln(K K_1)) (y + K_1 \ln(K))^2} dy^2.
\]

We have the following proposition:

**Proposition 9.** A metric of the type \( A(II) \) in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

\[
g = k \left( \frac{(y - x) e^{-3x}}{x^2 y} dx^2 + \frac{h (y - x) e^{-3y}}{x y^2} dy^2 \right)
\]

with \( k \in \mathbb{R} \setminus \{0\} \). If \( h = 1 \), we require \( k > 0 \).

**Proof.** In \( (63) \), let us perform the change of coordinates

\[
x \to x - \ln(K K_1), \quad y \to y - \ln(K K_1),
\]

and let us introduce \( k = K^3 \).

For \( h \neq 1 \), the statement is clear by construction of the metric. For the case \( h = 1 \), recall our freedom to swap coordinates, \( x \leftrightarrow y \), which produces the conformal factor \(-1\). Therefore, we obtain the additional requirement \( k > 0 \).

**Remark 8.** Note that the normal forms \( (64) \) are constant multiples of metric \( g_1 \) of \( (60) \). In Figure 2, these metrics lie on the axis of abscissa. In the special case \( h = 1 \), the normal forms \( (64) \) lie in the right half of the horizontal axis only.
3.2.2 Normal forms (B.5) of Theorem 1

In the case of complex Liouville metrics $B(II)$, we can construct any metric $g$ in the projective class, via Formula (5), from the following two metrics:

\[
g_1 = \left(\frac{1}{z} - \frac{1}{\overline{z}}\right) \left( C e^{-3z} \frac{dz^2}{z} - \overline{C e^{-3z}} \frac{d\overline{z}^2}{\overline{z}} \right),
\]
\[
g_2 = (\overline{z} - z) \left( C e^{-3\overline{z}} d\overline{z}^2 - C e^{-3z} dz^2 \right).
\]

A metric from the projective class $\mathfrak{P}(g_1, g_2)$ is given by

\[
g = (\overline{z} - z) \left( \frac{C e^{-3z}}{(K_2 - K_1z)(K_2 - K_1\overline{z})^2} dz^2 - \frac{\overline{C e^{-3\overline{z}}}}{(K_2 - K_1\overline{z})(K_2 - K_1z)^2} d\overline{z}^2 \right), \tag{65}
\]

where we require $K_1 \neq 0$ because we only consider metrics with one essential vector field (metrics with $K_1 = 0$ admit a homothetic vector field). The projective vector field is $w = \partial_z + \partial_{\overline{z}}$, and thus the projective flow reads

\[
\phi_t(z_0, \overline{z}_0) = (z_0 + t, \overline{z}_0 + t).
\]

**Remark 9.** Let $C = e^{i\varphi}$. The projective connection (1) is specified completely by the value of $\varphi$ mod $\pi$. Instead of writing down (1) explicitly, which can be done but is rather lengthy in the present case, we give the following qualitative argument: Since multiplication by a constant conformal factor does not affect the Christoffel symbols of a metric, we may divide $g_2$ by $\cos(\varphi)$. In view of Euler’s formula, $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$, one therefore finds that

\[
g_{\text{new}} = \frac{g_2}{\cos(\varphi)} = (z - \overline{z}) \left( (1 + i \tan(\varphi)) e^{-3z} dz^2 - (1 - i \tan(\varphi)) e^{-3\overline{z}} d\overline{z}^2 \right)
\]

lies in the projective class $\mathfrak{P}(g_1, g_2)$. Thus the projective connection (1) depends on $\tan(\varphi)$ only, and therefore it depends on $\varphi$ mod $\pi$ only, which is exactly the result of the full computation.

The square of the length of the projective vector field $w = \partial_z + \partial_{\overline{z}}$ w.r.t. the metric (65) is

\[
\|w\|^2 = (\overline{z}_0 - z_0) \left( \frac{C e^{-3z_0} - 3t}{(K_2 - K_1z_0 - K_1\overline{z}_0)(K_2 - K_1\overline{z}_0 - K_1t)} - \frac{\overline{C e^{-3\overline{z}_0}}}{(K_2 - K_1\overline{z}_0 - K_1t)(K_2 - K_1z_0 - K_1\overline{z}_0)} \right).
\]

We therefore need to study the following equation (see (14)):

\[
(\overline{z}_0 - z_0) \left( \frac{C e^{-3z_0} - 3t}{(K_2 - K_1z_0 - K_1\overline{z}_0)(K_2 - K_1\overline{z}_0 - K_1t)} - \frac{\overline{C e^{-3\overline{z}_0}}}{(K_2 - K_1\overline{z}_0 - K_1t)(K_2 - K_1z_0 - K_1\overline{z}_0)} \right) = (\overline{z}_0' - z_0') \left( \frac{C e^{-3z_0'} - 3t}{(K_2 - K_1z_0' - K_1\overline{z}_0')(K_2 - K_1\overline{z}_0' - K_1t')} - \frac{\overline{C e^{-3\overline{z}_0'}}}{(K_2 - K_1\overline{z}_0' - K_1t')(K_2 - K_1z_0' - K_1\overline{z}_0')} \right). \tag{66}
\]

In particular, poles of the left hand side must also be poles of the right hand side. Thus, proceeding in the same way as earlier and introducing $K = \frac{K_1'}{K_1} = \frac{K_2}{K_2}$, we can deduce the following possibilities

\[
\begin{align*}
\frac{z_0' - z_0}{\overline{z}_0' - \overline{z}_0} = K & \quad \frac{z_0 - z_0'}{\overline{z}_0 - \overline{z}_0'} = K & \quad \frac{z_0 - z_0'}{\overline{z}_0' - \overline{z}_0} = K & \quad \frac{z_0' - z_0}{\overline{z}_0 - \overline{z}_0'} = K.
\end{align*}
\]

In the case (a), we have $z_0' = z_0 + K, \overline{z}_0' = \overline{z}_0 + K$. From (66) we can then infer the equation

\[
\left( \frac{K_1'}{K_1} \right)^3 C e^{-3z_0} (K_2 - K_1\overline{z}_0 - K_1t) - \left( \frac{K_1'}{K_1} \right)^3 \overline{C e^{-3\overline{z}_0}} (K_2 - K_1\overline{z}_0 - K_1t) = e^{-3K} C e^{-3z_0} (K_2 - K_1\overline{z}_0 - K_1t) - e^{-3K} \overline{C e^{-3\overline{z}_0}} (K_2 - K_1\overline{z}_0 - K_1t).
\]

From this follows $e^{-K} = \frac{K_1'}{K_1}$, which is covered by Lemma 9.

Continuing with case (b), we have $\overline{z}_0' = z_0 + K, z_0' = \overline{z}_0 + K$. By the same reasoning as before, we can again derive $e^{-K} = \frac{K_1'}{K_1}$ from it using Equation (66).

Cases (c) and (d) imply $\overline{z}_0 = z_0$, so they do not have to be considered.
Let us now look for the normal forms of metrics (27). Using the definition $K = \frac{e^{K_2/K_1}}{K_1}$, we have from (65) that
\[
g = \frac{\overline{\sigma} - z}{K_1^2} \left( \frac{Ce^{-3z}}{(z + \ln(KK_1))^2(z + \ln(KK_1))^2} dz^2 - \frac{C e^{-3\varpi}}{(z + \ln(KK_1))^2(z + \ln(KK_1))^2} d\varpi^2 \right).
\]
Performing a transformation $z \rightarrow z - \eta$ with $\eta = \ln(KK_1)$, we obtain (note that $KK_1$ is a positive number)
\[
g = k(\overline{\sigma} - z) \left( \frac{e^{i\varpi} e^{-3z}}{z^{2\overline{\varpi}}} dz^2 - \frac{e^{-i\varpi} e^{-3\varpi}}{z^{2\varpi}} d\varpi^2 \right),
\]
where $k = K^3$ and $C = e^{i\varphi}$ as in Remark 9.
Therefore, metrics (67) are mutually non-isometric for different values of $k \neq 0$ and $\varphi \in \mathbb{R} \mod \pi$. Summing up, we have thus proven the following result.

**Proposition 10.** A metric of the type $B(II)$ in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:
\[
g = k(\overline{\sigma} - z) \left( \frac{h e^{-3z}}{z^{2\overline{\varpi}}} dz^2 - \frac{\overline{h} e^{-3\varpi}}{z^{2\varpi}} d\varpi^2 \right),
\]
with $h \in \mathbb{P}^1$ and $k \in \mathbb{R} \setminus \{0\}$.

### 3.2.3 Normal forms (C.8) of Theorem 1

We now discuss the case of Jordan block metrics, $C(II)$. We can construct any metric $g$ in the projective class, via Formula (5), from the following two metrics:
\[
g_1 = 2(Y + x) dxdy \\
g_2 = -2 \frac{Y + x}{y^3} dxdy + (Y + x)^2 \frac{1}{y^4} dy^2
\]
where $Y = Y(y)$ is a function of one variable,
\[
Y = \left( C_2 e^{3z/2} \sqrt{|y|} \frac{\sqrt{|y|}}{y - 3} \right) + \left( C_1 + C_2 \int_0^y e^{3z} \sqrt{|\xi|} \frac{d\xi}{\xi^{3/2}} \right).
\]
By grace of a change of coordinates, we may w.l.o.g. assume $C_1 = 0$ and $C_2 = 1$ in the following. Furthermore, using integration by parts, one can achieve the simplified form
\[
Y(y) = \int_0^y e^{3z} \sqrt{|\xi|} \frac{d\xi}{\xi^{3/2}}.
\]

**Remark 10.** In Section 3.1.3. of [13], the function $Y$ is obtained as the derivative of a function $Y_1$, where $Y = Y_1'$ satisfies the ODE
\[
y'^2 Y_1'' - \frac{1}{2} (y - 3) Y_1' + \frac{1}{2} Y_1 = 0
\]
This equation is similar to Kummer’s equation. Indeed we can write the solution in terms of the confluent hypergeometric function $1F_{1}$,
\[
Y_1(y) = C_1 \left( 2\sqrt{6} (y - 3) 1F_1 \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2y} \right) + 6\sqrt{6} e^{3z/2} \right) + C_2 (y - 3)
\]
and w.l.o.g.
\[
Y(y) = Y_1 \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2y} \right).
\]
For $y \neq 0$, the special function $1F_1 \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2y} \right)$ can be represented by the (imaginary) error function erf (erfi),
\[
Y_1(y) = C_1 \left( \sqrt{6\pi} (y - 3) \text{erfi} \left( \sqrt{\frac{6}{2\sqrt{y}}} \right) + 6\sqrt{6} e^{3z/2} \right) + C_2 (y - 3), \quad \text{for } y < 0
\]
\[
Y_1(y) = C_1 \left( \sqrt{6\pi} (y - 3) \text{erf} \left( \sqrt{\frac{6}{2\sqrt{y}}} \right) + 6\sqrt{6} e^{3z/2} \right) + C_2 (y - 3), \quad \text{for } y > 0.
\]
Therefore, w.l.o.g. one may assume
\[
Y = \text{erfi} \left( \sqrt{\frac{1}{\pi}} \right) \quad \text{for } y < 0 \quad \text{and} \quad Y = \text{erf} \left( \sqrt{\frac{1}{\pi}} \right) \quad \text{for } y > 0.
\]
By formula (5), a metric from the projective class \( \mathcal{P}(g_1, g_2) \) can be written as

\[
g = 2 \frac{Y(y) + x}{(K_1 - K_2 y)^2} dxdy + \frac{K_2 (Y(y) + x)^2}{(K_1 - K_2 y)^4} dy^2,
\]

using the same coordinates as in (68). Metrics with \( K_1 = 0 \) are eigenvectors of \( L_w \) and admit a homothetic vector field. We therefore require \( K_1 \neq 0 \) in the remainder of this section. In view of Lemma 9, we may now reparametrize the metrics (70), for \( K_1 \neq 0 \), by replacing \((K_1, K_2) \to (K, \alpha) \) according to the definitions

\[
K = \frac{e^{\kappa_2 / \kappa_1}}{K_1}, \quad \alpha = \frac{K_2}{K_1}.
\]

We thus obtain the metric (70), in the same local coordinates, as

\[
g = K^3 e^{-3\alpha} \left( 2 \frac{Y(y) + x}{(1 - \alpha y)^3} dxdy + \frac{\alpha (Y(y) + x)^2}{(1 - \alpha y)^4} dy^2 \right),
\]

with \( \alpha \in \mathbb{R} \) and \( K \in \mathbb{R} \setminus \{0\} \). In Lemma 9, it is proven that \( K \) characterizes orbits of the projective flow. Below we shall see that the condition is also sufficient, i.e. metrics belonging to different orbits of the projective flow are non-isometric.

**Lemma 10.** A metric (71) is locally isometric to a constant multiple of metric \( g_1 \) of (68), i.e.

\[
K^3 e^{-3\alpha} \left( 2 \frac{Y(y) + x}{(1 - \alpha y)^3} dxdy + \frac{\alpha (Y(y) + x)^2}{(1 - \alpha y)^4} dy^2 \right) = 2K^3 (Y(v) + u) dudv.
\]

in terms of new coordinates \( u = u(x, y), v = v(y) \).

**Proof.** Metrics on the same projective orbit are isometric, with the isometries linking them being given by the flow of the projective vector field, i.e. solutions to

\[
\dot{x} = \frac{1}{2} (y - 3) x + \frac{1}{2} Y_1(y),
\]

\[
\dot{y} = y^2.
\]

The new coordinates are thus given by \( u = x(t_0; x, y), v = y(t_0; y) \). Now consider the metric on the left hand side of (72),

\[
g = K^3 e^{-3\alpha} \left( 2 \frac{Y(y) + x}{(1 - \alpha y)^3} dxdy + \frac{\alpha (Y(y) + x)^2}{(1 - \alpha y)^4} dy^2 \right).
\]

Applying a coordinate transformation as just described, the metric \( g \) is transformed into another metric of the same form, as determined by (59). This is because for the metrics \( g \) and another metric \( \tilde{g} \) isometric to it, \( \tilde{g} = \phi_t^*(g) \) (for a fixed \( t \) where \( \phi_t \) is the projective flow), we have with \( \alpha = \psi^{-1}(g) \) and \( \tilde{a} = \psi^{-1}(\tilde{g}) \) that

\[
\tilde{a} = \psi^{-1}(\tilde{g}) = \psi^{-1}(\phi_t^*(g)) = \phi_t^*(\psi^{-1}(g)) = \phi_t^*(a).
\]

The metric \( g \) is determined by two parameters,

\[
\alpha = -\frac{K_2}{K_1}, \quad K = \frac{e^{\kappa_2 / \kappa_1}}{K_1},
\]

which are here represented in terms of the initially introduced parameters \( K_1, K_2 \). Looking at Figure 2, the canonical choice for a normal form is the intersection of the (thick=projective orbit) curve with the axis of abscissa. The value \( t_0 \) hence is determined by

\[
K'_2 = (K_1 t_0 + K_2) e^{t_0} = 0 \quad \Rightarrow \quad t_0 = -\frac{K_2}{K_1} = -\alpha.
\]

The resulting metric, isometric to \( g \), is thus a multiple of \( g_1 \), and the multiplicative factor is determined by

\[
K' = K_1 e^{\alpha} = e^{\alpha} K e^{-\alpha} = \frac{1}{K}
\]

and thus, in new coordinates \((u, v)\), the metric (73) assumes the form

\[
\tilde{g} = g[K] = 2K^3 (Y(v) + u) dudv.
\]
Figure 3: The normal forms for metrics of type $C(II)$ of Proposition 3 are chosen to lie on the intersection of the orbits (lines in the plot) with the axis of abscissa determined by $g_1$ (marked by points). Such metrics are multiples of the metric $g_1$.

Figure 3 illustrates geometrically how the normal forms are chosen on the orbits of the projective vector field.

Lemma 11. Two different copies $g = g[k]$ and $\tilde{g} = g[\tilde{k}]$ of (74) are non-isometric.

Proof. One only needs to prove that the condition $\tilde{k} = k$ is necessary. Thus assume $g$ and $\tilde{g}$ are isometric. This means that there exists a diffeomorphism $(x, y) \mapsto (u, v)$, $u = u(x, y), v = v(x, y)$ mapping one metric into the other. There are only two possibilities: $u = u(x), v = v(y)$ or $u = u(y), v = v(x)$. Assume first $u = u(x), v = v(y)$, so that the following equation holds:

$$k(Y(y) + x) \, dx \, d Little mark and comma

The above equation implies

$$k(Y(y) + x) = \tilde{k}(Y(v(y)) + u(x)) \, u_x \, v_y \, dx \, dy. \tag{75}$$

By differentiating Equation (75) twice w.r.t. $x$, we obtain

$$0 = \tilde{k} u_y (Y(v(y)) u_{xxx} + 3 u_x u_{xx} + uu_{xxx}). \tag{76}$$

Since $v(y)$, being a coordinate transformation, is invertible, we can infer that $Y'(v(y)) u_{xxx} = 0$ (by differentiating the right hand side of (76)). Thus, $u_{xxx} = 0$, since $Y'(v(y))$ is not identically zero, so that

$$u(x) = \frac{e_1 x^2}{2} + c_1 x + d_1.$$ 

Substituting this back into (75) yields

$$k(Y(y) + x) = \tilde{k} \left( Y'(v(y)) + \frac{e_1 x^2}{2} + c_1 x + d_1 \right) (e_1 x + c_1) v_y,$$

which is polynomial in $x$. Analyzing each coefficient w.r.t. $x$ separately, one finds from the cubic term $e_1 = 0$, and from the linear term that $v_y$ is a non-zero constant, i.e. $v = c_2 y + d_2, c_2 \in \mathbb{R} \setminus \{0\}, d_2 \in \mathbb{R}$. Taking this into account, by differentiating Equation (75) w.r.t. $y$, we obtain

$$kY' = \tilde{k} c_2^n Y_v \ u_x \tag{77}$$

implying that $u_x$ is a non-zero constant, i.e. $u = c_1 x + d_1, c_1 \in \mathbb{R} \setminus \{0\}, d_1 \in \mathbb{R}$, so that equation (76) becomes

$$k = \tilde{k} c_1 c_2 \tag{78}$$

whereas Equation (77) assumes the form

$$k \frac{e_1^3}{2 |y|^{3/2}} = \tilde{k} c_1^2 c_2^2 |e_2 y + d_2|^{2/3}. \tag{79}$$

which, on account of (78), reduces to

$$c_1 \frac{e_1^3}{|y|^{3/2}} = c_2 \frac{e_2^3 (e_2 y + d_2)}{|e_2 y + d_2|^{2/3}}. \tag{80}$$

Since Equations (79) and (80) hold for any $y$, we have that $d_2 = 0, c_2 = 1$ and finally $c_1 = 1$, and therefore $k = \tilde{k}$ in view of (78).
Now assume $u = u(y)$ and $v = v(x)$. The analogue of Equation (75) becomes

$$k(Y(y) + x) = \tilde{k} \ (Y(v(x)) + u(y)) \ u_y v_x.$$  

(81)

Taking two derivatives w.r.t. $x$, we arrive at

$$\tilde{k} u_y \left( Y''(v(x)) v_x^3 + 2v_x v_{xx} Y'(v(x)) + Y'(v(x)) v_{xx} + Y(v(x)) v_{xxx} + u(y) v_{xxx} \right) = 0.$$  

(82)

Dividing by $\tilde{k} u_y$ (recall that $u_y$ is non-zero) and taking a derivative w.r.t. $y$ of this equation, we thus find $v_{xxx} = 0$, which implies that $v(x)$ is of the form $v(x) = \frac{ax^2}{2} + c_1 x + d_1$, $c_1, c_1, d_1 \in \mathbb{R}$. Since $v_{xxx} = 0$, from (82) we obtain

$$Y''(v(x)) v_x^3 + (2v_x + 1) v_{xx} Y'(v(x)) = 0.$$  

(83)

Differentiating (69) w.r.t. $y$, we obtain

$$y^2 Y''(y) + \frac{3}{2} (y + 1) Y'(y) = 0.$$  

(84)

Eliminating $Y''$ from (83) by grace of (84), and replacing $v(x)$, a polynomial equation in $x$ is obtained. The system of equations on $c_1, c_1$ and $d_1$ obtained from its coefficients w.r.t. $x$ can straightforwardly be solved and yields the solution $c_1 = c_1 = 0$, $d_1 = -1$. This, however, is a contradiction since it would imply $v(x) = -1$, but $v(x)$, being part of a coordinate transformation, cannot be constant. \hfill $\square$

**Remark 11.** Note that in the present case there are no parameters for the projective class. The projective connection (1) for the projective class defined by a metric $g_1 = (Y(y) + x) \ dx \ dy$, with $y = y(x)$,

$$y'' = \frac{y'}{Y(y) + x} \cdot \frac{Y'(y) y^2}{Y(y) + x}.$$  

Summing up, we arrive at the following Proposition.

**Proposition 11.** A metric of the type $C(II)$ in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric of the form

$$g = k (Y(y) + x) \ dx \ dy,$$

where $k \in \mathbb{R} \setminus \{0\}$.

### 3.3 Case (III): $L_w$ has 2 complex eigenvalues

Here we are in the case when $L_w$ has the form (III) of (7). So, let $(a_1, a_2)$ be a basis of $\mathfrak{A}$ in which $L_w$ assume such a form. We have that

$$\exp(t L_w) = e^{\lambda t} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$  

(85)

Consider a general metric $g = \psi(K_1 a_1 + K_2 a_2)$ from the projective class $\mathcal{P}(g_1, g_2)$. By virtue of Equation (85), we can compute

$$\phi_*(K_1 a_1 + K_2 a_2) = e^{\lambda t} \ (K_1 \cos(t) a_1 - K_1 \sin(t) a_2 + K_2 \sin(t) a_2 + K_2 \cos(t) a_2).$$  

(86)

Thus, the orbits of the projective flow describe logarithmically spiraling curves in $\mathfrak{A}$. For what follows, it will be convenient to reformulate the problem in terms of new parameters $(\theta, K)$, i.e. we chose new coordinates in $\mathfrak{A}$,

$$K_1 = Ke^{\lambda \theta} \sin(\theta) \quad \text{and} \quad K_2 = Ke^{\lambda \theta} \cos(\theta).$$  

(87)

This reparametrization is invertible,

$$\theta = \arctan \left( \frac{K_1}{K_2} \right) \quad \text{and} \quad K = \sqrt{K_1^2 + K_2^2} \ e^{-\lambda \arctan K_1/K_2}.$$  

**Remark 12.** Note that in case $\lambda = 0$, the new parametrization is by polar parameters, with $K > 0$ being the radius parameter and $\theta \in [0, 2\pi)$ being the angle parameter. On the other hand, if $\lambda \neq 0$, it is convenient to take $\theta \in \mathbb{R}$ as parameter along spiraling curves in $\mathfrak{A}$, whereas $K$ should be considered as an angle via

$$K = e^{\lambda \alpha}, \quad \alpha \in [0, 2\pi).$$

The parametrization (87) is adjusted to the problem in the sense that $K$ is an invariant of the orbits of the projective flow.
Figure 4: The flow of the projective vector field in $\mathfrak{A}$, sketched for different values of $(K_1,K_2)$ and different $\lambda$.

First line: The left panel shows orbits for $\lambda<0$, the middle and right panel show orbits for $\lambda=0$ and $\lambda>0$, respectively. Note that for $\lambda=0$, the orbits have constant radius $\zeta = \sqrt{K_1^2 + K_2^2}$, so $\zeta$ is invariant under the action of the projective flow.

Second line: The graphs show the situation when there are additional discrete projective symmetries. The left panel illustrates the situation for $\lambda<0$, the right panel for $\lambda>0$.

**Lemma 12.** The pairs $(K,\theta)$ and $(K',\theta')$ are related by the transformation (86) if and only if they satisfy $K' = K$.

**Proof.** Consider (86), i.e.

$$Ke^{\lambda(t+\theta)}\left((\sin(\theta)\cos(t) + \cos(\theta)\sin(t))a_1 + (-\sin(\theta)\sin(t) + \cos(\theta)\cos(t))a_2\right) = K'e^{\lambda}\theta'\sin(\theta')a_1 + K'e^{\lambda}\cos(\theta')a_2.$$  

Using standard trigonometrical identities, one finds $K' = K$ and $\theta' = \theta + t$ (in case $\lambda = 0$ we understand the second equality modulo $2\pi$).

### 3.3.1 Normal forms (A.3) of Theorem 1

In the Liouville case, $A(III)$, any metric $g$ in the projective class can be constructed, via Formula (5), from the following two metrics:

$$g_1 = (\tan(x) - \tan(y)) \left(\frac{e^{-3\lambda x}}{\cos(x)}dx^2 + \frac{h e^{-3\lambda y}}{\cos(y)}dy^2\right),$$

$$g_2 = (\cot(x) - \cot(y)) \left(\frac{e^{-3\lambda x}}{\sin(x)}dx^2 + \frac{h e^{-3\lambda y}}{\sin(y)}dy^2\right).$$

The projective connection is obtained according to (1) as

$$y'' = \frac{e^{-3\lambda(x-y)}}{2h \sin(y-x)} + \frac{1}{2} (\cot(y-x) - 3\lambda) y' + \frac{1}{2} (\cot(y-x) + 3\lambda) y'^2 + \frac{h e^{3\lambda(x-y)}}{2 \sin(y-x)} y'^3,$$

so the projective class is determined by the data $(h,\lambda)$ with $h \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$. In the same coordinates $(x,y)$, the projective vector field, fixed by (13) and case (III) of (7), assumes the form $w = \partial_x + \partial_y$. Therefore,

$$\phi_t(x_0,y_0) = (x_0 + t, y_0 + t).$$

Any metric $g \in \mathfrak{P}(g_1,g_2)$ is given via (5) by

$$g = (\tan(y) - \tan(x)) \left(\frac{c_1 e^{-3\lambda x}}{\cos(x)(K_2\tan(y) - K_1)(K_2\tan(x) - K_1)^2}dx^2 + \frac{c_2 e^{-3\lambda y}}{\cos(y)(K_2\tan(y) - K_1)^2(K_2\tan(x) - K_1)}dy^2\right)$$

27
Using the parametrization (87) and standard trigonometrical identities, this becomes
\[
 g = \frac{\sin(y - x)}{K^3} \left( \frac{e^{-3\lambda(x+\alpha)} \sin(y - x) d\alpha}{\sin^2(y - x)} + \frac{h e^{-3\lambda(y+\alpha)} \sin(y - x) d\alpha}{\sin^2(y - x)} \right). 
\]  
(88)

After a change of coordinates \((x, y) \to (x - \theta, y - \theta)\), and a redefinition of \(\theta\), we arrive at
\[
 g = \frac{\sin(y - x)}{K^3} \left( \frac{e^{-3\lambda x} dx^2}{\sin^2(y - \theta)} + \frac{h e^{-3\lambda y} dy^2}{\sin^2(y - \theta)} \right). 
\]  
(89)

**Assume** \(\lambda = 0\). In this case metric (89) becomes
\[
 g = K^{-3} \frac{\sin(y - x)}{\sin(y - \theta) \sin(x - \theta)} \left( \frac{dx^2}{\sin(x - \theta)} + \frac{h dy^2}{\sin(y - \theta)} \right), 
\]
and by a change of coordinates \((x, y) \to (x - \theta, y - \theta)\) we obtain
\[
 g[\kappa; h] = \kappa \frac{\sin(y - x)}{\sin(y)} \left( \frac{dx^2}{\sin(x)} + \frac{h dy^2}{\sin(y)} \right), 
\]  
(90)
where we introduced \(\kappa = K^{-3} > 0\).

**Lemma 13.** Two different copies \(g = g[\kappa; h], g' = g[\kappa'; h]\) of (90) are non-isometric.

**Proof.** This follows from an investigation of Equation (14), i.e.
\[
 \kappa \sin(y_0 - x_0) \left( \frac{1}{\sin(y_0 + t) \sin^2(x_0 + t)} + \frac{h}{\sin^2(y_0 + t) \sin(x_0 + t)} \right) = \kappa' \sin(y_0 - x_0') \left( \frac{1}{\sin(y_0' + t) \sin^2(x_0' + t)} + \frac{h}{\sin^2(y_0' + t) \sin(x_0' + t)} \right) 
\]  
(91)

As discussed in Step 2 on page 7, and done in the previous sections, poles on the left must correspond to poles on the right of (91). From this requirement, we infer the following cases:
(a) \(x_0' = x_0 + N\pi, y_0' = y_0 + M\pi\)
(b) \(x_0' = y_0 + N\pi, y_0' = x_0 + M\pi\)
(c) \(x_0' = y_0 + N\pi, x_0' = y_0 + N\pi\)
(d) \(y_0' = y_0 + M\pi, y_0' = x_0 + M\pi\)

with \(N, M \in \mathbb{Z}\) and \(H = \arctan \frac{K_1}{\kappa_1} - \arctan \frac{K_1}{\kappa_1} = \theta' - \theta\).

In the case (a), plugging the relations back into (91) we obtain the conditions
\[
(-1)^N = \left( \frac{\kappa'}{\kappa} \right)^3 \quad \text{and} \quad (-1)^M = \left( \frac{\kappa'}{\kappa} \right)^3, 
\]
from which we infer, since \(\kappa', \kappa > 0\), that \(N, M \in 2\mathbb{Z}\). However, this means \(\kappa' = \kappa\).

Cases (b) or (c) would imply \(x_0' - y_0' = 0\) which would send the right hand side of (91) to zero for all values of \(t\) (light line). Thus, we can discard these cases.

In case (d), we find the conditions
\[
(-1)^{N+1} = h \left( \frac{\kappa'}{\kappa} \right)^3 \quad \text{and} \quad (-1)^{M+1} h = \left( \frac{\kappa'}{\kappa} \right)^3, 
\]  
(92)
which imply \(M - N \in 2\mathbb{Z}\) and \(|h| = 1\). If \(M, N\) are even, this implies \(h = -1\), and if \(M, N\) is odd, \(h = 1\). Substituting this back into (92), we arrive, in both cases, to \(\kappa' = \kappa\). \(\square\)

**Assume** \(\lambda \neq 0\). In this case metric (89) can be rewritten as
\[
 g = \varepsilon \sin(x - y) \left( \frac{e^{-3\lambda(x+\alpha)}}{\sin(y - \theta) \sin^2(x - \theta)} dx^2 + \frac{h e^{-3\lambda(y+\alpha)}}{\sin^2(y - \theta) \sin(x - \theta)} dy^2 \right) 
\]  
(93)
where we defined \(e^{3\lambda \alpha} = K^{-3}\). By a change of coordinates \((x, y) \to (x - \theta + \alpha, y - \theta + \alpha)\) and by renaming \(\theta \to 2\theta - \alpha\), we thus arrive at the normal form
\[
 g[\theta; \lambda, h] = \varepsilon \sin(x - y) \left( \frac{e^{-3\lambda x}}{\sin(y - \theta) \sin^2(x - \theta)} dx^2 + \frac{h e^{-3\lambda y}}{\sin^2(y - \theta) \sin(x - \theta)} dy^2 \right), 
\]  
(94)
with parameters \(\theta \in [0, 2\pi)\) and \(h \in \mathbb{R} \setminus \{0\}\). If \(h = e^{-3\lambda n\pi}\) for some \(n \in \mathbb{Z}\), then we require \(\theta \in [0, \pi)\).
Lemma 14. Two different copies \( g = g[\theta; \lambda, h] \) and \( g' = g[\theta'; \lambda, h] \) of (94) are non-isometric.

Proof. Consider again the requirement (14).

\[
\sin(x_0 - y_0) \left( e^{-3\lambda(x_0 + t)} \frac{e^{-3\lambda(x_0 + t)}}{\sin^2(x_0 + t) - \sin^2(y_0 + t)} + \frac{h e^{-3\lambda(y_0 + t)}}{\sin^2(y_0 + t) - \sin^2(x_0 + t)} \right)
\]

\[
= \sin(x'_0 - y'_0) \left( e^{-3\lambda(x'_0 + t')} \frac{e^{-3\lambda(x'_0 + t')}}{\sin^2(x'_0 + t') - \sin^2(y'_0 + t')} + \frac{h e^{-3\lambda(y'_0 + t')}}{\sin^2(y'_0 + t') - \sin^2(x'_0 + t')} \right)
\]

In order that both sides can be equal, poles of the left hand side must correspond to poles on the right hand side. Thus, we infer the following possibilities:

- \( (a) \) \( x'_0 = x_0 + H + N\pi \) \( y'_0 = y_0 + H + M\pi \)
- \( (b) \) \( x'_0 = x_0 + H + N\pi \) \( y'_0 = y_0 + H + M\pi \)
- \( (c) \) \( x'_0 = y_0 + H + N\pi \) \( y'_0 = x_0 + H + M\pi \)
- \( (d) \) \( x'_0 = y_0 + H + N\pi \) \( y'_0 = x_0 + H + M\pi \)

with \( N, M \in \mathbb{Z} \) and \( H = \arctan\frac{K_1}{K_2} - \arctan\frac{K_1}{K_2} = \theta' - \theta \). Again, we can discard cases (b) and (c).

Let us consider case (a): Plugging the relations \( x'_0 = x_0 + H + N\pi \) and \( y'_0 = y_0 + H + M\pi \) into (95), we find

\[
(1 - M)^{-1} e^{-3\lambda(H + N\pi)} = 1 \quad \text{and} \quad (1 - N)^{-1} e^{-3\lambda(H + M\pi)} = 1,
\]

from which we infer \( N, M \in 2\mathbb{Z} \) and thus \( \theta' = \theta + 2n\pi \) with \( n \in \mathbb{Z} \).

In case (d), we follow the same procedure and obtain two equations

\[
(1 - N)^{-1} e^{-3\lambda(H + M\pi)} = h \quad \text{and} \quad (1 - M)^{-1} e^{-3\lambda(H + N\pi)} = h = 1.
\]

From these, we conclude

\[
(1 - N)^{-1} e^{-3\lambda(N - M)} = h^2
\]

and hence \( (N - M) \in 2\mathbb{Z} \). Resubstituting, we obtain \( \text{sgn}(h) = (N + 1)/2\mathbb{Z} = (M + 1)/2\mathbb{Z} \) and thus \( (N - M) \in 2\mathbb{Z} \). Using (96), this implies

\[
e^{-6\lambda H} e^{-3\lambda(M + N)n} = 1,
\]

which means \( H = n\pi \) with \( n \in \mathbb{Z} \), i.e. \( \theta' = \theta + n\pi \). Moreover, we obtain

\[
h = e^{3\lambda m\pi} \quad \text{with} \quad m \in \mathbb{Z}.
\]

In summary, putting together Lemmas 13 and 14, we conclude:

Proposition 12. A metric of the type A(III) in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

(i) for \( \lambda = 0 \):

\[
g = \kappa \sin(x - y) \left( \frac{dx^2}{\sin(y) \sin^2(x)} + \frac{h dy^2}{\sin^2(y) \sin(x)} \right),
\]

(ii) for \( \lambda \in \mathbb{R} \setminus \{0\} \):

\[
g = \sin(x - y) \left( \frac{e^{-3\lambda x}}{\sin(y - \theta) \sin^3(x - \theta)} dx^2 + \frac{h e^{-3\lambda y}}{\sin^2(y - \theta) \sin(x - \theta)} dy^2 \right).
\]

The parameters may assume values \( h \in \mathbb{R} \setminus \{0\} \), \( \kappa > 0 \) and \( \theta \in [0, 2\pi) \), but if \( h = e^{-3\lambda n\pi} \) for some \( n \in \mathbb{Z} \) then we require \( \theta \in [0, \pi) \). Moreover, if \( \lambda = 0 \), \( h \neq \pm 1 \).

3.3.2 Normal forms (B.6) of Theorem 1

In case of the complex Liouville metric, B(III), we can construct any metric \( g \) in the projective class, via Formula (5), from the following two metrics, where \( C \in \mathbb{C} \), \( |C| = 1 \):

\[
g_1 = (\tan(z) - \tan(\overline{z})) \left( C \frac{e^{-3\lambda}}{\cos(z)} dz^2 - \overline{C} \frac{e^{-3\lambda}}{\cos(z)} dz^2 \right),
\]

\[
g_2 = (\cot(z) - \cot(\overline{z})) \left( C \frac{e^{-3\lambda}}{\sin(z)} dz^2 - \overline{C} \frac{e^{-3\lambda}}{\sin(z)} dz^2 \right).
\]

The projective vector field, fixed by condition (13) and the case (III) of (7), assumes the form \( w = \partial_x = \partial_z + \partial_{\overline{z}} \), and thus its flow reads

\[
\phi_t(z_0, \overline{z}_0) = (z_0 + t, \overline{z}_0 + t).
\]
Let us now analyze the parameters of the projective class. Taking advantage of the fact that a conformal constant does not affect the Christoffel symbols, we can divide the metric by \( \cos(\varphi) \) to obtain

\[
g_{\text{new}} = \frac{\sin(z - \overline{z})}{\cos(z) \cos(\overline{z})} \left( (1 + i \tan \varphi) \frac{e^{-3\lambda z}}{\cos(z)} dz^2 - (1 - i \tan \varphi) \frac{e^{-3\lambda \overline{z}}}{\cos(\overline{z})} d\overline{z}^2 \right).
\]

Thus the projective connection should depend on \( \varphi \mod \pi \) and \( \lambda \in \mathbb{R} \) only, and this is indeed confirmed by a straightforward computation. An arbitrary metric in the projective class \( \mathfrak{P}(g_1, g_2) \) is

\[
g = (\tan(z) - \tan(z)) \left( \frac{C e^{-3\lambda z}}{\cos(z)(K_2 \tan(z) - K_1)(K_2 \tan(z) - K_1)} dz^2 - \frac{C e^{-3\lambda \overline{z}}}{\cos(\overline{z})(K_2 \tan(z) - K_1)^2(K_2 \tan(z) - K_1)} d\overline{z}^2 \right).
\]

We work again with the parametrization (87) and obtain

\[
g = \frac{\sin(z - \overline{z})}{K^3} \left( \frac{C e^{-3\lambda (z + \theta)}}{\sin(z - \theta) \sin^2(z - \theta)} dz^2 - \frac{C e^{-3\lambda (\overline{z} + \theta)}}{\sin(\overline{z} - \theta) \sin^2(\overline{z} - \theta)} d\overline{z}^2 \right). \tag{98}
\]

Assume \( \lambda = 0 \). Metric (98) becomes, after the change of coordinates \( (x, y) \rightarrow (x + \theta, y + \theta) \)

\[
g[\kappa; h] = \kappa \sin(z - \overline{z}) \left( \frac{h dz^2}{\sin(\overline{z}) \sin^2(z)} - \frac{\overline{h} d\overline{z}^2}{\sin(\overline{z}) \sin^2(z)} \right), \tag{99}
\]

where we introduce \( \kappa = K^{-3} > 0 \).

**Lemma 15.** Two different copies \( g = g[\kappa; h] \) and \( g' = g[\kappa'; h] \) of (99) are non-isometric.

**Proof.** This follows from inspection, in the current context, of Equation (14),

\[
\kappa \sin(z_0 - \overline{z}_0) \left( \frac{h}{\sin(z_0 + t) \sin^2(z_0 + t)} - \frac{\overline{h}}{\sin(\overline{z}_0 + t) \sin^2(\overline{z}_0 + t)} \right) = \kappa' \sin(z'_0 - \overline{z}_0) \left( \frac{h}{\sin(z'_0 + t) \sin^2(z'_0 + t)} - \frac{\overline{h}}{\sin(\overline{z}_0 + t) \sin^2(\overline{z}_0 + t)} \right). \tag{100}
\]

Examining the poles in \( t \), we obtain two cases: (a) \( z'_0 = z_0 + N\pi, \overline{z}_0 = \overline{z}_0 + N\pi \), and (b) \( z'_0 = \overline{z}_0 + N\pi, \overline{z}_0 = z_0 + N\pi \) (with \( N \in \mathbb{Z} \)).

Substituting back into (100), we conclude in both cases that \( \kappa' = (-1)^N \kappa \), but since \( \kappa, \kappa' > 0 \) this requires that \( N \) is even and thus \( \kappa' = \kappa \).

Assume \( \lambda \neq 0 \). We may rewrite metric (98),

\[
g = \sin(z - \overline{z}) \left( \frac{he^{-3\lambda (z + \alpha)}}{\sin(z - \theta) \sin^2(z - \theta)} dz^2 - \frac{\overline{h} e^{-3\lambda (\overline{z} + \theta)}}{\sin(\overline{z} - \theta) \sin^2(\overline{z} - \theta)} d\overline{z}^2 \right),
\]

where \( h \in \mathbb{P}^1 \) and \( e^{-3\lambda \alpha} = K^{-3} e^{-\lambda \theta} \). By a change of coordinates \( z \rightarrow z + \alpha \) and redefining \( \theta \), we arrive at the normal form

\[
g[\theta; h] = \sin(z - \overline{z}) \left( \frac{he^{-3\lambda z}}{\sin^2(z + \theta) \sin(z + \theta)} dz^2 - \frac{\overline{h} e^{-3\lambda \overline{z}}}{\sin^2(z + \theta) \sin^2(z + \theta)} d\overline{z}^2 \right) \tag{101}
\]

with \( h \in \mathbb{P}^1 \) and \( \theta \in [0, 2\pi) \).

**Lemma 16.** Two different copies \( g = g[\theta; h] \) and \( g' = g[\theta'; h] \) of (101) are non-isometric.

**Proof.** Consider again the requirement (14),

\[
\sin(z_0 - \overline{z}_0) \left( \frac{he^{-3\lambda (z_0 + t)}}{\sin(z_0 + t - \theta) \sin^2(z_0 + t - \theta)} - \frac{\overline{h} e^{-3\lambda (\overline{z}_0 + t)}}{\sin(\overline{z}_0 + t - \theta) \sin(\overline{z}_0 + t - \theta)} \right) = \sin(z'_0 - \overline{z}_0) \left( \frac{he^{-3\lambda (z'_0 + t)}}{\sin(z'_0 + t - \theta') \sin^2(z'_0 + t - \theta')} - \frac{\overline{h} e^{-3\lambda (\overline{z}_0 + t)}}{\sin(\overline{z}_0 + t - \theta') \sin(\overline{z}_0 + t - \theta')} \right). \tag{102}
\]

In order that both sides can be equal, poles of the left hand side must correspond to poles on the right hand side. Thus, we obtain the possibilities:
with \( N \in \mathbb{Z} \) and \( H = \arctan \frac{K^2}{N^2} - \arctan \frac{hN}{z_0} = \theta' - \theta \). Again, in the same way as we did in Section 3.3.1, we can discard cases (b) and (c). Let us consider case (a): Plugging the relations \( z_0' = z_0 + H + N \pi \) and \( \varpi_0 = z_0 + H + N \pi \), we find
\[
e^{-3\lambda (H+N\pi)} = (-1)^N \quad \Rightarrow \quad N \in 2\mathbb{Z} \quad \text{and} \quad \theta' = \theta,
\]
where we view \( \theta', \theta \) as angles. In case (d), we follow the same procedure and obtain
\[
h(-1)^N e^{-3\lambda (H+N\pi)} = \overline{h},
\]
and the real part of the requirement amounts to the requirement \( e^{-3\lambda (H+N\pi)} = (-1)^N \), which is only possible if \( N \) is an even integer. In such case we have again \( \theta' = \theta \) if \( \theta', \theta \) are viewed as angles. Furthermore, we obtain the restriction \( \overline{h} = h \), and thus \( h = 1 \) for case (d) to happen. This completes the proof. \( \square \)

We conclude, putting together Lemma 15 and Lemma 16:

**Proposition 13.** A metric of the type \( B(III) \) in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

(i) for \( \lambda = 0 \):
\[
g = \kappa \sin(z - \varpi) \left( \frac{h dz^2}{\sin^2(z) \sin \varpi} - \frac{\varpi d\varpi}{\sin(z) \sin \varpi} \right),
\]

(ii) for \( \lambda \in \mathbb{R} \setminus \{0\} \):
\[
g = \sin(z - \varpi) \left( \frac{y e^{-3\lambda z}}{\sin(z + \theta) \sin(z + \theta)} \right) dz^2 - \frac{\varpi e^{-3\lambda z}}{\sin(z + \theta) \sin(z + \theta)} d\varpi^2.
\]

Here, \( h \in \mathbb{P}^1 \), \( \kappa > 0 \) and \( \theta \in [0, 2\pi) \). If \( \lambda = 0 \), \( h \neq \pm 1 \).

### 3.3.3 Normal forms (C.9) of Theorem 1

This case, \( C(III) \), is similar to the case \( C(II) \) discussed in Section 3.2.3. Any metric \( g \) in the projective class can be constructed, via Formula (5), from the following two metrics:

\[
g_1 = 2(Y + x) \, dxdy,
\]
\[
g_2 = -2 \frac{Y + x}{y^3} \, dxdy + \frac{(Y + x)^2}{y^4} \, dy^2.
\]

The function \( Y = Y(y) = Y_1'(y) \) is given w.l.o.g. by
\[
Y_1(y) = (y - 3\lambda) \int y e^{-\frac{3\lambda \arctan(\xi)}{(\xi - 3\lambda)^2}} \frac{\sqrt{\xi^2 + 1}}{\xi^2 + 1} d\xi = (y - 3\lambda) \int y e^{-\frac{3\lambda \arctan(\xi)}{(\xi - 3\lambda)^2}} \frac{\sqrt{\xi^2 + 1}}{\xi^2 + 1} d\xi.
\]

The function \( Y = Y_1'(y) \) reads, w.l.o.g. [13],
\[
Y(y) = e^{-\frac{3\lambda \arctan(y)}{y - 3\lambda}} \frac{\sqrt{y^2 + 1}}{y - 3\lambda} + \int Y(y) e^{-\frac{3\lambda \arctan(\xi)}{(\xi - 3\lambda)^2}} \frac{\sqrt{\xi^2 + 1}}{\xi^2 + 1} d\xi = \int \frac{y e^{-\frac{3\lambda \arctan(\xi)}{(\xi - 3\lambda)^2}}}{\xi^2 + 1} d\xi.
\]

**Remark 13.** In [13], \( Y_1 \) is obtained from the ODE
\[
(y^2 + 1) Y_1'' - \frac{1}{2} (y - 3\lambda) Y_1' + \frac{1}{2} Y_1 = 0.
\]

The solution of this ODE can be obtained and, arguing analogously to Section 3.2.3, we can w.l.o.g. assume it to be identical to the integral in (104). Another representation, similar to that of Remark 10 in Section 3.2.3, could be achieved in terms of a confluent hypergeometric function with complex arguments. However, this representation does not appear to be convenient for our purposes in this paper, so we are not going to discuss it further.

By Formula (5), a metric from the projective class \( \mathfrak{P}(g_1, g_2) \) can be written
\[
g = \frac{Y(y) + x}{(K_1 - K_2 y)^2} \, dxdy + \frac{K_2 (Y(y) + x)^2}{(K_1 - K_2 y)^4} \, dy^2,
\]

31
illustrates the metrics chosen as representative for each orbit of the projective flow in the (\[103\]), using the same coordinates as in (\[103\]). Note that we do not have real eigenvalues in the present case, so no further restrictions on \(K_1, K_2\) apply. Let us again use the parametrization (\[87\]). With this parametrization, the metric (\[70\]), in the same local coordinates, reads

\[
g = K^{-3} e^{-3\lambda t} \left( 2 \frac{Y(y) + x}{(\sin \theta - y \cos \theta)^3} dxdy + \cos(\theta) \left( \frac{(Y(y) + x)^2}{(\sin \theta - y \cos \theta)^4} dy^2 \right) \right),
\]

with \(\theta \in [0, 2\pi)\) and \(K \in \mathbb{R} \setminus \{0\}\). We prove that the orbits of \(g\) under isometries and under the action of the projective flow coincide, i.e. the constant \(K\) completely characterizes the orbits under isometries.

**Lemma 17.** A metric (\[71\]) is locally isometric to a constant multiple of the metric \(g_1\) of (\[103\]), i.e.

\[
K^{-3} e^{-3\lambda t} \left( 2 \frac{Y(y) + x}{(\sin \theta - y \cos \theta)^3} dxdy + \cos(\theta) \left( \frac{(Y(y) + x)^2}{(\sin \theta - y \cos \theta)^4} dy^2 \right) \right) = 2 K^3 (Y(v) + u) dudv.
\]

**Proof.** The proof is analogous to that of Lemma 10. For the normal forms we make the following choices:

**Case \(\lambda = 0\):** We have \(K = \text{const}\) and thus the projective orbits are concentric circles around the origin in Figure 4. For the normal forms we can therefore choose freely (by grace of a rotation) the value of \(\theta\) in (\[71\]), and if we choose \(\theta = \pi/2\), then \(\cos(\theta) = 0\). The resulting metric is a constant multiple of \(g_1\), with the factor being a positive real number.

**Case \(\lambda \neq 0\):** The orbital invariant is still \(K\), but the orbits are spirals in \(\mathfrak{A}\), see Figure 4. Since the radial distance of any of these spirals is strictly growing with increasing \(\theta\), each orbit crosses the unit circle exactly once. The isometry is obtained from the solution of the projective flow, where the parameter value has to be chosen according to the following computation:

\[
\zeta' = K' e^{\lambda t'} = K e^{\lambda t} e^{\lambda t_1} = 1 \quad \Rightarrow \quad t = -\frac{\ln K}{\lambda} - \theta,
\]

where \(\zeta', \zeta\) denote the radial distance from the origin.

However, it is more convenient to choose the normal forms again on the axis given by multiples of \(g_1\). We use the parametrization (\[87\]), but recall Remark 12, i.e. we let \(K = e^{\lambda t} \) where \(\alpha \in [0, 2\pi)\). Choosing \(\theta = \pi/2 + 2\pi N\) (\(N \in \mathbb{Z}\) for the normal form, we obtain a metric of the form (\[105\]) with

\[
K_1 = K e^{\lambda(\pi/2 + 2\pi N)} \quad \text{and} \quad K_2 = 0,
\]

and choosing \(N\) appropriately permits us further to have

\[
K_1 = e^{\lambda \beta} \quad \text{with} \quad \beta \in [0, 2\pi).
\]

Figure 5 illustrates the metrics chosen as representative for each orbit of the projective flow in the \((K_1, K_2)\)-space.

**Lemma 18.** Two metrics \(g = k (Y(y) + x) dxdy\) and \(\tilde{g} = \tilde{k} (Y(v) + u) dudv\) with \(k, \tilde{k} \in \mathbb{R} \setminus \{0\}\) are isomorphic if and only if \(\tilde{k} = k\).
Proof. The proof is analogous to that of Lemma 11. Instead of (79) we find
\[ k \frac{e^{\frac{1}{2}\lambda \arctan(y)}}{|y^2 + 1|^{3/4}} = \tilde{k} c_1^2 \frac{e^{\frac{1}{2}\lambda \arctan(c_2 y + d_2)}}{|(c_2 y + d_2)^2 + 1|^{3/4}}, \]
and instead of (80) we find
\[ c_1 \frac{e^{\frac{1}{2}\lambda \arctan(y)}}{|y^2 + 1|^{3/4}} = c_2 \frac{e^{\frac{1}{2}\lambda \arctan(c_2 y + d_2)}}{|(c_2 y + d_2)^2 + 1|^{3/4}}. \]
Thus, it follows that \( d_2 = 0, c_2 = 1 \) and finally \( c_1 = 1 \). Therefore, \( \tilde{k} = k \) as claimed.

Putting together Lemma 17 and Lemma 18, we arrive at

**Proposition 14.** A metric of the type \( C(III) \) in Proposition 3 that admits exactly one essential projective vector field is isometric to one and only one metric below:

For \( \lambda = 0 \):
\[ g = k (Y(\lambda = 0, y) + x) \, dx \, dy \quad \text{with } k > 0 \]
For \( \lambda \in \mathbb{R} \setminus \{0\} \):
\[ g = e^{\lambda \alpha} (Y(\lambda, y) + x) \, dx \, dy \quad \text{with } \alpha \in [0, 2\pi). \]

The collection of Propositions 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 form the statement and the proof of Theorem 1.

**Acknowledgments**

The authors wish to thank Vladimir Matveev for numerous suggestions, helpful discussions and comments on the manuscript. Andreas Vollmer is a research fellow of Istituto Nazionale di Alta Matematica (INdAM). Both authors are members of GNSASA of INdAM and also gratefully acknowledge support from the project FIR-2013 Geometria delle equazioni differenziali. Gianni Manno was also partially supported by “Starting grant per giovani ricercatori” of Politecnico di Torino, code 53_RSG16MANGIO.

**References**

[1] A. V. Aminova. “Projective Transformations of Pseudo-Riemannian Manifolds”. In: *Journal of Mathematical Sciences* 113.3 (2003), pp. 367–470.

[2] A. V. Aminova and N. A. Aminov. “Projective geometry of systems of second-order differential equations”. In: *Sbornik: Mathematics* 197.7 (2006), p. 951.

[3] E. Beltrami. *Risoluzione del problema: riportar i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette*. Vol. 1. 7. 1865, pp. 185–204.

[4] A. V. Bolsinov, V. S. Matveev, and G. Pucacco. “Normal forms for pseudo-Riemannian 2-dimensional metrics whose geodesic flows admit integrals quadratic in momenta”. In: *Journal of Geometry and Physics* 59.7 (2009), pp. 1048 –1062.

[5] R. Bryant, M. Dunajski, and M. Eastwood. “Metrisability of two-dimensional projective structures”. In: *J. Differential Geom.* 83.3 (Nov. 2009), pp. 465–500.

[6] R. L. Bryant, G. Manno, and V. S. Matveev. “A solution of a problem of Sophus Lie: normal forms of two-dimensional metrics admitting two projective vector fields”. In: *Mathematische Annalen* 340.2 (2008), pp. 437–463.

[7] U. Dini. “Sopra un problema che si presenta nella teoria generale delle rappresentazioni geografiche di una superficie su di un’altra”. In: *Annali di Matematica Pura ed Applicata (1867-1897)* 3.1 (1869), pp. 269–293.

[8] G. Fubini. “Sui gruppi trasformazioni geodetiche”. In: *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Natur* 53.2 (1903), 261–313.

[9] B. Kruglikov. “Point Classification of Second Order ODEs: Tresse Classification Revisited and Beyond”. In: *Differential Equations - Geometry, Symmetries and Integrability: The Abel Symposium 2008*. Ed. by B. Kruglikov, V. Lychagin, and E. Straume. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 199–221.

[10] S. Lie. “Classification und Integration von gewöhnlichen Differentialgleichungen zwischen \( x, y \), die eine Gruppe von Transformationen gestatten”. In: *Archiv for Mathematik og Naturvidenskab. Christiana*. 9 (1883), pp. 371–393.
[11] S. Lie. “Untersuchungen über geodätische Curven”. In: Math. Ann. 20 (1882).

[12] R. Liouville. “Sur les invariants de certaines équations différentielles et sur leurs applications”. In: Journal de l’École Polytechnique 59 (1889), pp. 7–76.

[13] V. S. Matveev. “Two-dimensional metrics admitting precisely one projective vector field”. In: Mathematische Annalen 352.4 (2012), pp. 865–909.

[14] P. Topalov and V. S. Matveev. “Geodesic Equivalence via Integrability”. In: Geometriae Dedicata 96.1 (2003), pp. 91–115.

[15] A. Tresse. Détermination des invariants ponctuels de l’équation différentielle ordinaire du second ordre $y'' = \omega(x, y, y')$. Hirzel, Leipzig, 1896.