The first passage time problem over a moving boundary for asymptotically stable Lévy processes

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Abstract

We study the asymptotic tail behaviour of the first-passage time over a moving boundary for asymptotically stable Lévy processes.

Our main result states that if the left tail of the Lévy measure is regularly varying with index $-\alpha$ and the moving boundary is equal to $1 - t^\gamma$ for some $\gamma < 1/\alpha$, then the probability that the process stays below the moving boundary has the same asymptotic polynomial order as in the case of a constant boundary. The same is true for the increasing boundary $1 + t^\gamma$ with $\gamma < 1/\alpha$ under the assumption of a regularly varying right tail with index $-\alpha$.

Key words and phrases: Lévy process; moving boundary; one-sided exit problem; one-sided boundary problem; first passage time; survival exponent; boundary crossing probability; boundary crossing problem; one-sided small deviations; lower tail probability

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1 Introduction and main results

This paper is concerned with the asymptotic tail behaviour of the first-passage time over a moving boundary. For a stochastic process $(X(t))_{t \geq 0}$ and a function $f : \mathbb{R}_+ \to \mathbb{R}$, the so-called moving boundary, the question is to determine the asymptotic rate of the probability

$$
P(X(t) \leq f(t), \ 0 \leq t \leq T), \quad \text{as } T \to \infty. \quad (1)$$

If this probability is asymptotically polynomial of order $-\delta$ (e.g. if it is regularly varying with index $-\delta$, in which case we will write $\mathcal{RV}(-\delta)$), the number $\delta$ is called the survival exponent or persistence exponent. If the function $f$ is constant then we are in the classical framework of first passage times over a constant boundary.

This problem is a classical question which is relevant in a number of different applications, a recent overview of results is presented in [3] and [12].

For Lévy processes, the study of the first passage time distribution over a constant boundary is a classical area of research. The results follow from fluctuation theory; e.g. [34] shows that $P(X(t) \leq c, \ 0 \leq t \leq T)$ varies regularly with index $-\rho \in (-1,0)$ if and only if $X$ satisfies Spitzer’s condition with $\rho \in (0,1)$, that is (cf. [7]), $P(X(t) > 0) \to \rho$ as $t \to \infty$.

Similar arguments as for Lévy processes were already used for random walks with zero mean (see e.g. [21]). If the process does not necessarily satisfy Spitzer’s condition, various results were obtained for a constant boundary by [4, 6, 10, 11, 15, 17, 27].

However, even for Brownian motion, the question involving moving boundaries in (1) is quite non-trivial. It is studied by [40, 22, 26, 36, 30, 32, 1] in different ways. Independently of each other [22] and [40] state an integral test for the boundary $f$, for which the survival exponent remains $1/2$. More precisely, they prove under some additional regularity assumptions that

$$\int_{1}^{\infty} |f(t)|t^{-3/2}dt < \infty \iff P(X(t) \leq f(t), \ 0 \leq t \leq T) \approx T^{-1/2}, \text{ as } T \to \infty. \quad (2)$$

Here and below we use the following notation for strong and weak asymptotics. We write $f \lesssim g$ if $\limsup_{x \to \infty} f(x)/g(x) < \infty$ and $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. Furthermore, $f \sim g$ if $f(x)/g(x) \to 1$ as $x \to \infty$.

In this paper we look at Lévy processes $(X(t))_{t \geq 0}$, in particular at Lévy processes which belong to the domain of attraction of a strictly stable Lévy process and do not require a centering function, that is, for $\alpha \in (0,2)$ there exists a deterministic function $c$ such that

$$\frac{X(t)}{c(t)} \overset{d}{\to} Z(1), \quad \text{as } t \to \infty,$$

where $Z$ is a strictly stable Lévy process with index $\alpha \in (0,2)$ and positivity parameter $\rho \in [0,1]$. It is well known that if such a function $c$ exists, then $c \in \mathcal{RV}(1/\alpha)$ (cf. [21], [13]). We will write $X \in D(\alpha, \rho)$.

Let us state our results. We distinguish decreasing and increasing moving boundaries.
For decreasing boundaries so far it was proved by [2] under the assumption that $X$ possesses negative jumps that

$$\gamma < \frac{1}{2} \implies \mathbb{P}(X(t) \leq 1 - t^\gamma, t \leq T) = T^{-\rho + o(1)}, \text{ as } T \to \infty.$$  

Note that [2] does not assume that $X \in \mathcal{D}(\alpha, \rho)$. Negative results (i.e. such that the survival exponent does change) are given in [23, 28]. Results similar to those for Brownian motion are only available under such heavy assumptions as bounded jumps or satisfying the Cramér condition, [29] or [31].

Our main result here is the following:

**Theorem 1.** Let $\alpha \in (0, 1) \cup (1, 2)$, $\rho \in (0, 1)$, and $\gamma > 0$. Suppose that $X \in \mathcal{D}(\alpha, \rho)$. If $1 - 1/\alpha < \rho$ and $\limsup_{t \to 0} \mathbb{P}(X(t) \geq 0) < 1$ then we have

$$\gamma < \frac{1}{\alpha} \implies \mathbb{P}(X(t) \leq 1 - t^\gamma, t \leq T) = T^{-\rho + o(1)}, \text{ as } T \to \infty. \quad (3)$$

For increasing boundaries until now it was known from combining [23] and [2] that

$$\gamma < \max\{\frac{1}{2}, \rho\} \implies \mathbb{P}(X(t) \leq 1 + t^\gamma, t \leq T) = T^{-\rho + o(1)}, \text{ as } T \to \infty.$$  

In this case, we have a similar statement as for decreasing boundaries.

**Theorem 2.** Let $\alpha \in (0, 1) \cup (1, 2)$, $\rho \in (0, 1)$, and $\gamma > 0$. Suppose that $X \in \mathcal{D}(\alpha, \rho)$. If $\rho < 1/\alpha$, then we have

$$\gamma < \frac{1}{\alpha} \implies \mathbb{P}(X(t) \leq 1 + t^\gamma, t \leq T) = T^{-\rho + o(1)}, \text{ as } T \to \infty. \quad (4)$$

Let us give a few comments on these results, in particular on the conditions on the Lévy process.

First, note that $\frac{1}{\alpha} \geq \max\{\frac{1}{2}, \rho\}$ for $X \in \mathcal{D}(\alpha, \rho)$ (cf. [12]) and thus, Theorem 1 improves the results of [2] and Theorem 2 improves the results of [2] and [23]. Note that [23] determines exact asymptotics so that [23] gives a more precise result for $\gamma < \rho$. Nevertheless, our approach provides a larger class of functions where $\rho$ remains to be the value of the survival exponent; and this was the main motivation of this work. Furthermore, our results indicate that the class of boundaries where the survival exponent remains the same as for the constant boundary case also depends on the tail of the Lévy measure and not only on $\rho$ in contrast to what the results of [23] seem to suggest.

Intuitively, a Lévy process with a higher fluctuation (i.e. a smaller index $\alpha$) follows a boundary easier and thus allows a larger class of boundaries where the survival exponent does not change compared to the constant boundary case. This intuition gives the main idea of the proofs.

Second, note that the assumption $1 - 1/\alpha < \rho$ (resp. $\alpha \rho < 1$) excludes the case where the stable process $Z$ is spectrally positive (resp. negative) with index $\alpha$. That means we assume a regularly varying left tail for the decreasing boundary and a regularly varying right tail for the increasing boundary. The regularly varying left (resp. right) tail with index $-\alpha$ of the Lévy measure of $X$ is an important property to show Theorem 1 (resp.
Theorem 2). Without these assumptions our approach does not work. Note that in the spectrally negative case \( \alpha \rho = 1 \) holds; and the relation (4), the increasing situation, was shown by [23] for \( \gamma < 1/\alpha \), even providing the exact strong asymptotics.

Essentially, our proof is based on transforming the moving boundary problem to the constant boundary case. For this purpose, the regularly varying left (resp. right) tail is used. Hence, our proof gives some hope to be generalized to other Lévy processes such as processes indicated in [15].

Remark 3. Since the process \( X \) in Theorem 1 (resp. Theorem 2) satisfies Spitzer’s condition with parameter \( \rho \in (0,1) \), [24] (or [3]) yields

\[
P(X(t) \leq 1, \ t \leq T) = T^{-\rho+o(1)}, \quad \text{as } T \to \infty.
\]

(5)

This property provides immediately the upper (resp. lower) bound for (3) (resp. (4)). Furthermore, note that for \( \rho = 1 \) the equation (5) is not necessarily true (cf. [15]).

Remark 4. An important idea for our proofs is to use a modified version of Theorem 3.1 in [27] to show the lower (resp. upper) bound of (3) (resp. (4)). For this purpose, the assumption \( \lim \sup_{t \to 0^+} P(X(t) \geq 0) < 1 \) in Theorem 1 is required (see Corollary 3.4 in [27] for more details). However, we believe that this may be of technical matter.

Remark 5. Theorems 1 and 2 are also true for \( \alpha \geq 2 \). However, in view of [2] these results become redundant.

Remark 6. An asymptotically stable Lévy process with index \( \alpha = 1 \) satisfies Spitzer’s condition with parameter \( \rho \in (0,1) \) if and only if the Lévy measure is symmetric (cf. Property 1.2.8 in [38]). In this case our approach does not work since in our proof a slight change is made to the skewness of the Lévy measure and thus, it is not symmetric anymore.

Let us mention that related topics have been discussed like the moments ([18, 25, 35]), the finiteness ([19]), and the stability ([24]) of the first passage time. Random boundaries were studied in [11, 33].

We briefly recall the basic facts of Lévy processes. A Lévy process \( (X(t))_{t \geq 0} \) possesses stationary and independent increments, càdlàg paths, and \( X(0) = 0 \) (see [5, 39]). By the Lévy-Khintchine formula, the characteristic function of a marginal of a Lévy process \( (X(t))_{t \geq 0} \) is given by

\[
\ln \mathbb{E}(e^{iuX(t)}) = t \Psi(u), \quad \text{for every } u \in \mathbb{R},
\]

where

\[
\Psi(u) = iu - \sigma^2/2u^2 + \int_{\mathbb{R}} (e^{ix} - 1 - |x| - 1_{\{|x| \leq 1\}}ix) \nu(dx),
\]

(6)

for parameters \( \sigma^2 \geq 0, \ b \in \mathbb{R}, \) and a positive measure \( \nu \) concentrated on \( \mathbb{R}\setminus\{0\} \), called Lévy measure, satisfying

\[
\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty.
\]

For a given triplet \( (b, \sigma^2, \nu) \) there exists a Lévy process \( (X(t))_{t \geq 0} \) such that [0] holds, and its distribution is uniquely determined by its triplet. For the tails of the Lévy measure we will write

\[
\nu_+(x) = \nu((x, \infty)) \quad \text{and} \quad \nu_-(x) = \nu((\infty, -x)).
\]
The proof of Theorem 1, the case of negative boundaries, is given in Section 2 whereas Section 3 contains the proof for positive boundaries, Theorem 2. For reasons of clarity and readability some parts of the proof are separated in Section 4 and may be of independent interest.

2 Proof of Theorem 1

Note that the upper bound is trivial due to Remark 3. Hence, the following proof is devoted to the lower bound.

Let $X$ be a Lévy process with Lévy triplet $(b, \sigma^2, \nu)$. By assumption $\nu_+ \in \mathcal{R}V(-\alpha)$. That means there exists a function $\ell$ slowly varying at zero such that

$$\nu(dx) = |x|^{-\alpha-1}f(1/|x|)dx, \quad \text{for } x < 0.$$ 

The main idea of the proof is to consider instead of the process $X$ the following two independent Lévy processes: Let $0 \leq \delta(T) := (\ln \ln T)^{-1} \wedge \frac{1}{2} \searrow 0$, for $T \to \infty$. Then, let $Z_T$ and $Y_T$ be Lévy processes with characteristic exponents

$$\Psi_{Z_T}(u) := \int_{-\infty}^{-1} (e^{iux} - 1)\nu_{Z_T}(dx), \quad u \in \mathbb{R}, \quad (7)$$

with

$$\nu_{Z_T}(dx) := \begin{cases} 
    0, & x \geq -1, \\
    \delta(T) \frac{\ell(\delta(T)x^{-1})}{\ell(1/|x|)} \nu(dx), & x < -1, 
  \end{cases}$$

and

$$\Psi_{Y_T}(u) := ibu - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - 1_{\{|x| \leq 1\}} iux)\nu_T(dx), \quad u \in \mathbb{R}, \quad (8)$$

with

$$\nu_T(dx) := \begin{cases} 
    \nu(dx), & x \geq -1, \\
    \left(1 - \delta(T) \frac{\ell(\delta(T)x^{-1})}{\ell(1/|x|)} \right) \nu(dx), & x < -1. 
  \end{cases}$$

Note that both $\nu_T$ and $\nu_{Z_T}$ are Lévy measures for every fixed $T > 1$. We also denote $S_T := -Z_T$.

Then, $X = Y_T - S_T$ for every fixed $T > 1$ and $S_T$ is a subordinator for fixed $T > 1$ by construction.

For $\lambda > 0$ sufficiently small Karamata’s Theorem (see Theorem 1.5.11 in [9]) implies
the following estimate for the Laplace exponent of $S_T$
\[
\mathbb{E} \left( e^{-\lambda S_T(1)} \right) = \exp \left( -\delta(T) \int_{-\infty}^{1} \left(1 - e^{-\lambda|x|}\right) \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)} \nu(dx) \right)
\leq \exp \left( -\frac{1}{2} \delta(T) \int_{-\infty}^{1/\lambda} \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)} \nu(dx) \right)
\leq \exp \left( -\frac{1}{4\alpha} \delta(T) \cdot \lambda^{\alpha} \ell(\lambda \delta(T)^{1/\alpha}) \right).
\] (9)

We record this formula here since it will be need very often. The decisive idea of our proof is the following observation: Due to the independence of $S_T$ and $Y_T$ we obtain that
\[
P \left( X(t) \leq 1 - t^{\gamma}, t \leq T \right) = P \left( Y_T(t) - S_T(t) \leq 1 - t^{\gamma}, t \leq T \right)
\geq P \left( Y_T(t) \leq \frac{1}{2}, t \leq T \right) \cdot P \left( -S_T(t) \leq \frac{1}{2} - t^{\gamma}, t \leq T \right).
\]

The theorem is proved by applying the following two lemmas.

**Lemma 7.** Let $T > 1$ and $\alpha \in (0, 1) \cup (1, 2)$. Furthermore, let $Y_T$ be the Lévy process defined in (8). Then, it holds, for fixed $x > 0$, that
\[
P \left( Y_T(t) \leq x, t \leq T \right) = T^{-\rho + o(1)}, \quad \text{as } T \to \infty.
\]

**Lemma 8.** Let $N > 1$ and $\alpha \in (0, 1) \cup (1, 2)$. Furthermore, let $S_N$ be a subordinator whose Laplace transform satisfies (9) for $\lambda > 0$ sufficiently small. Then, it holds, for $0 < \gamma < 1/\alpha$, that
\[
P \left( S_N(t) \geq \frac{1}{2} + t^{\gamma}, 0 \leq t \leq N \right) = N^{\alpha(1)}, \quad \text{as } N \to \infty.
\]

Lemmas 7 and 8 are proved in Section 4 using, among others, inequality (9).

### 3 Proof of Theorem 2

Contrary to the proof of Theorem 1 the lower bound is trivial due to Remark 3. Hence, the following proof is devoted to the upper bound, where the idea of the proof is essentially the same as for the lower bound of Theorem 2.

Let $X$ be a Lévy process with Lévy triplet $(b, \sigma^2, \nu)$. By assumption we have
\[
\nu(dx) = x^{-\alpha-1}f(1/x)dx, \quad x > 0
\]
where $f$ is a slowly varying at zero function. Again, the main idea of the proof is to consider instead of the process $X$ the following two independent Lévy processes: Let
0 \leq \delta(T) := (\ln \ln T)^{-1} \wedge \frac{1}{2} \searrow 0, \text{ for } T \to \infty. \text{ Then, let } S_T \text{ and } Y_T \text{ be Lévy processes with characteristic exponents}

\Psi_{S_T}(u) := \int_1^\infty (e^{iu x} - 1) \nu_{S_T}(dx), \quad u \in \mathbb{R},

(10)

with

\nu_{S_T}(dx) := \begin{cases} \delta(T) \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)} \nu(dx), & x > 1, \\ 0, & x \leq 1, \end{cases}

and

\Psi_{Y_T}(u) := i b u - \frac{\sigma^2}{2} u^2 + \int_\mathbb{R} (e^{iu x} - 1 - 1_{\{|x|\leq 1\}} i u x) \nu_T(dx), \quad u \in \mathbb{R},

(11)

with

\nu_T(dx) := \begin{cases} \nu(dx), & x \leq 1, \\ (1 - \delta(T) \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)}) \nu(dx), & x > 1. \end{cases}

Note that both \nu_T and \nu_{S_T} are Lévy measures for every fixed \( T > 1 \). Thus, \( X = Y_T + S_T \) for every \( T > 0 \). The process \( S_T \) is a subordinator by construction.

Since \( 0 < \alpha \gamma < 1 \) there exists a constant \( \varepsilon > 0 \) such that \( \gamma \alpha + \varepsilon < 1 \) and \( \gamma \alpha - \varepsilon > 0 \). Moreover, define \( T_0 := \lfloor (\ln T)^{\frac{1}{1-\gamma}} \rfloor \). Then, we obtain the following estimate

\[ \mathbb{P}(X(t) \leq 1 + t^\gamma, \ t \leq T) = \mathbb{P}(Y_T(t) + S_T(t) \leq 1 + t^\gamma, \ t \leq T) \]

\[ \leq \mathbb{P}(Y_T(n) + S_T(n) \leq 1 + n^\gamma, \ \forall n = T_0, ..., [T]) \]

\[ \leq \mathbb{P}(\{Y_T(n) \leq 1 + n^\gamma - S_T(n), \ \forall n = T_0, ..., [T]\}) \]

\[ \cap \{n^\gamma - S_T(n) \leq 0, \ \forall n = T_0, ..., [T]\} \]

\[ + \mathbb{P}(\exists n \in \{T_0, ..., [T]\} : S_T(n) < n^\gamma) \]

\[ \leq \mathbb{P}(Y_T(n) \leq 1, \ \forall n = T_0, ..., [T]) \]

\[ + \mathbb{P}(\exists n \in \{T_0, ..., [T]\} : S_T(n) < n^\gamma). \]

(12)

On the one hand, for \( \lambda > 0 \) sufficiently small Karamata’s Theorem (see Theorem 1.5.11 in [9]) implies the following estimate for the Laplace exponent of \( S_T \)

\[ \mathbb{E}

\left( e^{-\lambda S_T(1)} \right) = \exp \left( -\delta(T) \int_1^\infty \left( 1 - e^{-\lambda x} \right) \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)} \nu(dx) \right) \]

\[ \leq \exp \left( -\delta(T) \int_1^\infty \left( 1 - e^{-\lambda x} \right) \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)} \nu(dx) \right) \]

\[ \leq \exp \left( \frac{1}{2} \delta(T) \int_{1/\lambda}^{\infty} \frac{\ell(\delta(T)^{1/\alpha}/x)}{\ell(1/x)} \nu(dx) \right) \]

\[ \leq \exp \left( -\frac{1}{4\alpha} \delta(T) \cdot \lambda^\alpha \ell(\delta(T)^{1/\alpha}) \right). \]

(13)

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Then, Chebyshev’s inequality gives, for $T > 1$ sufficiently large,

\[
\mathbb{P} (\exists n \in \{ T_0, \ldots, [T] \} : S_T(n) < n^\gamma) \leq \sum_{n=T_0}^{[T]} \mathbb{P} (S_T(n) < n^\gamma) = \sum_{n=T_0}^{[T]} \mathbb{P} (e^{-n^{-\gamma} S_T(n)} \geq e^{-1}) \leq \sum_{n=T_0}^{[T]} e^{-1 - \frac{4\alpha}{1-\alpha} n^{1-\alpha - \gamma} (\delta(T))^{1/\alpha}}(T) \tag{14}
\]

Proposition 1.3.6 in [9] implies $\ell(\lambda) \geq \lambda^{\varepsilon /\gamma}$, for $\lambda > 0$ sufficiently small, and thus we get for $T > 1$ sufficiently large

\[
\mathbb{P} (\exists n \in \{ T_0, \ldots, [T] \} : S_T(n) < n^\gamma) \leq \sum_{n=T_0}^{[T]} e^{1 - \frac{4\alpha}{1-\alpha} n^{1-\alpha - \gamma - \delta(T)} T^{1+\varepsilon/(\alpha \gamma)}} \leq e^{1 + \ln |T| - \frac{4\alpha}{1-\alpha} (\ln |T|)^3 T^{1+\varepsilon/(\alpha \gamma)}} \leq e^{- (\ln |T|)^2} \leq T^{-\rho + o(1)}, \tag{15}
\]

where we used that $\alpha \gamma - \varepsilon > 0$ in the second last step.

On the other hand, using the fact that the process $Y_T$ is associated (cf. [20] or [2], Lemma 14) implies

\[
\mathbb{P} (Y_T(n) \leq 1, \forall n = T_0, \ldots, [T]) \leq \frac{\mathbb{P} (Y_T(n) \leq 1, \forall n = 0, \ldots, [T])}{\mathbb{P} (Y_T(n) \leq 1, \forall n = 0, \ldots, T_0)}. \tag{16}
\]

Note that $S_T \geq n$ a.s. since $S_T$ is a subordinator. Hence, due to the definition of $Y_T$ and $S_T$ and Remark [9] we obtain that

\[
\mathbb{P} (Y_T(n) \leq 1, \forall n = 0, \ldots, T_0) = \mathbb{P} (X(n) - S_T(n) \leq 1, \forall n = 0, \ldots, T_0) = \mathbb{P} (X(n) \leq 1, \forall n = 0, \ldots, T_0) = \mathbb{P} (X(n) \leq 1, \forall n = 0, \ldots, T_0) = T_0^{-\rho + o(1)}. \tag{16}
\]

Now, setting (16) and (15) in (1) gives

\[
\mathbb{P} (Y_T(n) \leq 1, \forall n = 0, \ldots, [T]) \leq T^{-\rho + o(1)} + \mathbb{P} (Y_T(n) \leq 1, \forall n = 0, \ldots, [T]) \cdot T^{o(1)}
\]

The theorem is proved by applying the following lemma:

**Lemma 9.** Let $T > 1$ and $\alpha \in (0,1) \cup (1,2)$. Furthermore, let $Y_T$ be a Lévy process defined in (17). Then, for $x > 0$

\[
\mathbb{P} (Y_T(n) \leq x, \forall n = 1, \ldots, [T]) \leq T^{-\rho + o(1)}, \quad \text{as} \; T \to \infty.
\]
4 First passage times of a time-dependent Lévy process

In the first section, we briefly recall the basic notations of fluctuation theory for Lévy processes. Furthermore, we give some properties of $Y_T$ and $S_T$ defined in (8) and (7) (resp. (11) and (10)). In the subsequent section, we prove Lemma 7 and 9. The Section 4.3 provides the proof of Lemma 8.

4.1 Preliminaries and Notations

Let $L$ be the local time of a general Lévy process $X$ reflected at its supremum $M$ and denote by $L^{-1}$ the right-continuous inverse of $L$, the inverse local time. This is a (possibly killed) subordinator, and $H(s) := X(L^{-1}(s))$ is another (possibly killed) subordinator called ascending ladder height process. The Laplace exponent of the (possibly killed) bivariate subordinator $(L^{-1}(s), H(s)) (s \leq L(\infty))$ is denoted by $\kappa(a, b)$,

$$\kappa(a, b) = c \exp \left( \int_0^\infty \int_{[0, \infty)} (e^{-t} - e^{-at-bx}) t^{-1} \mathbb{P}(X(t) \in dx) dt \right), \quad (17)$$

where $c$ is a normalization constant of the local time. Since our results are not effected by the choice of $c$ we assume $c = 1$.

Following [5], we define the renewal function of the process $H$ by

$$V(x) = \int_0^\infty \mathbb{P}(H(s) < x) ds \quad (18)$$

and for $z \geq 0$

$$V^z(x) = \mathbb{E} \left( \int_0^\infty e^{-z t_1_{[0, x]}(M(t))} dL(t) \right).$$

Until further notice, we denote by $\kappa_T$ the Laplace exponent of the inverse local time $L_T^{-1}$ of $Y_T$ defined in (8) (resp. (11)) and $V_T$ the renewal function of the ladder height process $H_T$ of $Y_T$. Furthermore, denote by $L^{-1}$ the inverse local time of $X$ defined in Theorem 1 (resp. Theorem 2) and $H$ be the corresponding ladder height process. Let $\kappa$ be the Laplace exponent of $(L^{-1}, H)$.

The next lemma shows the convergence of the renewal function $V_T$ to $V$.

**Lemma 10.** Let $T > 1$. Then, for every $x > 0$ we have

$$\lim_{T \to \infty} V_T(x) = V(x).$$

**Proof.** The Continuity Theorem (cf. [21], Theorem XIII.1.2) gives $Y_T(s) \xrightarrow{d} X(s)$, as $T \to \infty$, for all $s \geq 0$. Since $e^{-\lambda x}$ is bounded for all $x, \lambda \geq 0$, Theorem VIII.1 in [21] implies that

$$\mathbb{E} \left( e^{-\lambda H_T(s)} \right) \longrightarrow \mathbb{E} \left( e^{-\lambda H(s)} \right) = e^{-s \kappa(0, \lambda)}$$

$$= \exp \left( -s \exp \left( \int_0^\infty \int_0^\infty t^{-1} (e^{-t} - e^{-\lambda x}) \mathbb{P}(X(t) \in dx) dt \right) \right), \quad \text{as } T \to \infty.$$
Hence, again due to the Continuity Theorem (cf. [21, Theorem XIII.1.2]) \( H_T(s) \xrightarrow{d} H(s) \), as \( T \to \infty \), for all \( s \geq 0 \).

Since \( X \in \mathcal{D}(\alpha, \rho) \) with \( \rho \in (0,1) \) there exists a constant \( c > 0 \) such that \( \kappa(0,1) > c \). Since \( \kappa_T(0,1) \to \kappa(0,1) > c \), as \( T \to \infty \), there exists a \( T_0 > 1 \) such that \( \kappa_T(0,1) \geq \frac{1}{T} \kappa(0,1) \geq \frac{1}{2} c \) for all \( T > T_0 \). Furthermore, by construction of \( Y_T \) there exists for every fixed \( T > 1 \) a constant \( c(T) > 0 \) such that \( \kappa_T(0,1) > c(T) \). Combining this two facts gives that there exists a constant \( \tilde{c} > 0 \) indpending of \( T \) such that \( \kappa_T(0,1) \geq \tilde{c} \) for all \( T > 1 \).

Hence, we have for all \( s \geq 0 \) and \( T > 1 \),

\[
E \left( e^{-H_T(s)} \right) \leq e^{-s\tilde{c}}.
\] (19)

Then, Chebyshev's inequality and (19) lead to

\[
P(H_T(s) < x) = P \left( e^{-H_T(s)} > e^{-x} \right) \leq e^{x-s\tilde{c}}, \quad \text{for all } s \geq 0.
\]

The dominated convergence theorem with \( P(H_T(t) < x) \leq e^{x-t\tilde{c}} \), for every \( T > 1 \), implies that

\[
\lim_{T \to \infty} V_T(x) = \lim_{T \to \infty} \int_0^\infty P(H_T(t) < x)dt = \int_0^\infty \lim_{T \to \infty} P(H_T(t) < x)dt = \int_0^\infty P(H(t) < x)dt = V(x),
\]

as required.

The next lemma characterises the tail behaviour of \( S_T \) defined in (10).

**Lemma 11.** Let \( T > 1 \) and \( S_T \) be the subordinator defined in (10). Let \( c \) be the norming sequence of \( X \). Then, there exists a constant \( C > 0 \) such that for all \( t > \delta(T)^{-1/2} \) and \( T \) sufficiently large

\[
P(S_T(t) > c(t)\delta(T)^{1/\alpha}) \leq C\delta(T)^{1/\alpha}. \quad (20)
\]

**Proof.** The idea of the proof is to apply a large deviation principle. For this purpose, define the following Lévy processes

\[
\Psi_X(u) := \int_1^\infty (e^{iu} - 1) \nu(dx), \quad u \in \mathbb{R},
\]

\[
\Psi_{S_T}(u) := \int_{\delta(T)^{1/\alpha}}^\infty (e^{iu} - 1) \nu_{S_T}(dx), \quad u \in \mathbb{R},
\]

with

\[
\nu_{S_T}(dx) := \begin{cases} \delta(T)x^{-\alpha-1} \ell(\delta(T)^{1/\alpha}/x)dx, & x \geq \delta(T)^{1/\alpha}, \\ 0, & x < \delta(T)^{1/\alpha}. \end{cases}
\]
1st. Step: This step shows that
\[ P(S_T(t) > \lambda) \leq P(\tilde{S}_T(t) > \lambda), \quad \text{for every } \lambda, t > 0, \] (21)

By construction we have for every \( \lambda > 0 \)
\[ \nu_{S_T}(x \in \mathbb{R} : x > \lambda) \leq \nu_{\tilde{S}_T}(x \in \mathbb{R} : x > \lambda). \]
Then, it follows from [37] that (21) holds. Let us mention that (21) is an extension of Slepian’s inequality for Lévy processes.

2nd. Step: Now, we will prove that for all \( T > 1 \)
\[ \frac{\tilde{S}_T(t)}{\delta(T)^{1/\alpha}} \frac{d}{d t} \tilde{X}(t), \quad \text{for every } t \geq 0. \] (22)

Integration by substitution gives for every \( \lambda > 0 \)
\[
\begin{align*}
t \Psi_{\tilde{S}_T} \left( \frac{\lambda}{\delta(T)^{1/\alpha} \delta(t)} \right) &= t \exp \left( \int_{\delta(T)^{-1/\alpha}}^{\infty} \left( e^{i \left( \delta(T)^{1/\alpha} c(t) \right) x} - 1 \right) \nu_{S_T}(dx) \right) \\
&= t \exp \left( \int_{1}^{\infty} \left( e^{i \left( \delta(T)^{1/\alpha} \delta(t) \right) x} - 1 \right) \delta(t)^{1/\alpha} \nu_{S_T}(dx) \delta(t)^{1/\alpha} \right) \\
&= t \exp \left( \int_{1}^{\infty} \left( e^{i \left( \delta(T)^{1/\alpha} \delta(t) \right) x} - 1 \right) \nu(dx) \right) \\
&= t \Psi_{\tilde{X}}(\lambda), \quad \text{for all } t \geq 0,
\end{align*}
\]
and this proves (22). Recall that \( X \) belongs to the domain of attraction of a strictly stable Lévy process with norming sequence \( \delta \) and \( \nu_+ \in \mathcal{RV}(-\alpha) \). Hence, by construction it implies that
\[ P(\tilde{X}(1) > \cdot) \in \mathcal{RV}(-\alpha). \] (23)

3rd. Step: Here, we finally show (20).

The large deviation principle (see Theorem 2.1 in [14]) gives that there exists a constant \( C > 1 \) such that for all \( t > \delta(T)^{-1/2} \) and \( T \) sufficiently large
\[
P \left( \frac{\tilde{X}(t)}{c(t)} > \delta(T)^{-\frac{1}{2}} \right) \leq 2P \left( \frac{\tilde{X}(|t|)}{c(|t|)} > \delta(T)^{-\frac{1}{2}} \right) \leq C \frac{1}{2} |t| P \left( \tilde{X}(1) > c(|t|) \delta(T)^{-\frac{1}{2}} \right) \leq C \delta(T)^{\frac{1}{3}},
\]
where we used (23) in the second step. Note that the constant \( C \) does not depend on \( t \). Hence, combining this with (21) and (22) leads to for all \( t > \delta(T)^{-1/2} \) and \( T \) sufficiently large
\[
P \left( S_T(t) > c(t) \delta(T)^{\frac{1}{3}} \right) \leq P \left( \frac{\tilde{S}_T(t)}{c(t) \delta(T)^{1/\alpha}} > \delta(T)^{-\frac{1}{3}} \right) = P \left( \frac{\tilde{X}(t)}{c(t)} > \delta(T)^{-\frac{1}{3}} \right) \leq C \delta(T)^{1/3}.
\]
Remark 12. Inequality (20) holds as well for the subordinator \( S_T \) defined in (7). The proof is essentially the same and is omitted.

4.2 A time-dependent Lévy process over a constant boundary

Now, we show Lemma 7 and Lemma 9. We calculate the asymptotic tail behaviour as \( T \) tends to infinity of the first-passage time over a constant boundary for a Lévy process which depends on the end time point \( T \). Lemma 7 and Lemma 9 differ only in the considered process as well as the time scale.

Proof of Lemma 7. Recall that

\[ Y_T = X + S_T, \]

where \( X \) is defined in Theorem 1 and \( S_T \) is a subordinator defined in (7). Since \( X \in D(\alpha, \rho) \) with \( 1 - 1/\alpha < \rho \) there exists a deterministic function \( c : (0, \infty) \to (0, \infty) \) such that

\[ \frac{X(t)}{c(t)} \xrightarrow{d} Z(1), \quad \text{as } t \to \infty, \]

where \( Z \) is strictly stable Lévy process with index \( \alpha \in (0, 2) \) and positivity parameter \( \rho \in (0, 1) \).

The upper bound is trivial since \( S_T \geq 0 \) a.s. and thus

\[ \mathbb{P}(Y_T(t) \leq x, t \leq T) = \mathbb{P}(X(t) + S_T(t) \leq x, t \leq T) \leq \mathbb{P}(X(t) \leq x, t \leq T) = T^{-\rho + o(1)}, \quad \text{as } T \to \infty, \quad (24) \]

see Remark 3.

The idea of the proof of the lower bound is to apply some parts of Theorem 3.1 and Corollary 3.2 in [27]. The main difference between our result and [27] is that the process \( Y_T \) depends on \( T \). For this purpose, we define

\[ M_T(t) := \sup_{s \leq t} Y_T(s), \]

and thus,

\[ \mathbb{P}(Y_T(t) < x, t \leq T) = \mathbb{P}(M_T(T) < x). \]

1st. Step:

Let \( z = z(T) \) with \( T^{-1 - |o(1)|} < z < T^{-1} \). By the estimate (3.7) in the proof of Theorem 3.1 in [27] we have for \( M_T \):

\[ \mathbb{P}(M_T(T) < x) \geq \frac{\kappa_T(z, 0)\mathbb{P}(M_T(1/z) \geq x)V_T(x)}{e} - z \int_0^T e^{-zt} \mathbb{P}(M_T(t) < x) dt. \]

Then, since \( \mathbb{P}(M_T(t) < x) \leq \mathbb{P}(M(t) < x) \), for all \( t \geq 0 \), (cf. [27]) the upper bound of (3.5) in [27] applied to \( \mathbb{P}(M(t) < x) \) gives, for every \( T > 1 \) and \( x > 0 \),

\[ \mathbb{P}(M_T(T) < x) \geq \frac{\kappa_T(z, 0)\mathbb{P}(M_T(1/z) \geq x)V_T(x)}{e} - \frac{e}{e - 1}V(x)z \int_0^T \kappa(1/t, 0) dt \]

\[ = \frac{\kappa_T(z, 0)\mathbb{P}(M_T(1/z) \geq x)V_T(x)}{e} - \frac{e}{e - 1}V(x)zK(1/T), \]

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where
\[ K(s) = \frac{\kappa(z,0)}{z} \int_{s}^{\infty} \frac{\kappa(z,0)}{z} \, dz. \]

The assumption \( \limsup_{t \to 0^+} \mathbb{P}(X(t) \geq 0) < 1 \) implies that \( K(s) \) is well-defined (see Corollary 3.4 in [27] for more details). Since \( \kappa(z,0) \) is regularly varying at zero, by Karamata’s theorem ([8], Theorem 1.5.11) we have \( K(1/T) \leq c_1(\kappa) T \kappa(1/T,0) \), for \( T \geq 1 \). Hence,
\[
\mathbb{P}(M_T(T) < x) \geq \frac{\kappa_T(z,0) \mathbb{P}(M_T(1/z) \geq x) V_T(x)}{e} - \frac{e}{e-1} V(x) z c_1(\kappa) T \kappa(1/T,0).
\]

Lemma 10 implies, for \( T > 1 \) sufficiently large, that
\[
V_T(x) \geq \frac{1}{2} V(x).
\]
Furthermore, the inequality (24) gives, for \( T \) sufficiently large,
\[
\mathbb{P}(M_T(1/z) \geq x) = 1 - \mathbb{P}(M_T(1/z) < x) \geq 1 - \mathbb{P}(M(1/z) < x) \geq \frac{1}{2}.
\]
Hence, we obtain that
\[
\mathbb{P}(M_T(T) < x) \geq \frac{\kappa_T(z,0) V_T(x)}{4e} - \frac{e}{e-1} V(x) z c_1(\kappa) \kappa(1/T,0).
\]

2nd. Step:
In this step we estimate \( \kappa_T \) from below by \( \kappa \). Recall that the bivariate Laplace exponent \( \kappa \) corresponds to \( X \). We get, for every \( a > 0 \), that
\[
\kappa_T(a,0) = \kappa(a,0) \cdot \exp \left( \int_{0}^{\infty} (e^{-t} - e^{-at}) t^{-1} (\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0)) \, dt \right).
\]
First, we will prove
\[
\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0) \leq o(1), \quad \text{as } T \to \infty,
\]
for all \( t \geq 0 \) to obtain finally an estimate for \( \kappa \). For this purpose, we distinguish between \( 0 < t \leq \delta(T)^{-1/2} \) and \( t > \delta(T)^{-1/2} \).
For \( 0 < t \leq \delta(T)^{-1/2} \) we have
\[
\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0) \
\leq \mathbb{P}(X(t) \geq -S_T(t), S_T(t) = 0) + \mathbb{P}(S_T(t) > 0) - \mathbb{P}(X(t) \geq 0) \
= \mathbb{P}(X(t) \geq 0) + 1 - \mathbb{P}(S_T(t) = 0) - \mathbb{P}(X(t) \geq 0) \
= 1 - e^{-\delta(T)\nu_{+}(1)} \
\leq 1 - e^{-\delta(T)^{-1/2}\delta(T)\nu_{+}(1)} \
\leq C_1 \cdot \delta(T)^{1/2},
\]
(26)
where \( C_1 = \nu_+(1) \). Now, let \( t > \delta(T)^{-1/2} \). It holds that
\[
\mathbb{P}(Y_T(t) \geq 0) - \mathbb{P}(X(t) \geq 0) = \mathbb{P}(-S_T(t) \leq X(t) < 0) \\
\leq \mathbb{P} \left( -S_T(t) \leq X(t) < 0, S_T(t) < c(t)\delta(T)^{1/2} \right) + \mathbb{P} \left( S_T(t) \geq c(t)\delta(T)^{1/2} \right) \\
\leq \mathbb{P} \left( -c(t)\delta(T)^{1/2} \leq X(t) < 0 \right) + \mathbb{P} \left( S_T(t) \geq c(t)\delta(T)^{1/2} \right).
\]

Due to Stone’s local limit theorem (see Theorem 8.4.2 in [9] for non-lattice random walks resp. Proposition 13 in [16] for \( \text{\textit{\textsc{S}}} \text{\textsc{L}} \) processes) and the fact that the density of any \( \alpha \)-stable law is bounded there exists a constant \( C_2 > 0 \) such that for all \( t > \delta(T)^{-1/2} \)
\[
\mathbb{P} \left( -c(t)\delta(T)^{1/2} \leq X(t) < 0 \right) \leq C_2\delta(T)^{1/(3\alpha)}.
\]
Combining this with Remark 12 and (26) gives uniformly in \( t \)
\[
\mathbb{P}(-S_T(t) \leq X(t) < 0) \leq C_2\delta(T)^{1/6} = o(1), \text{ as } T \to \infty.
\]

Hence, for \( T > 1 \) sufficiently large we obtain by Frullani’s integral for \( a \in (0,1] \) that
\[
\kappa_T(a,0) \geq \kappa(a,0) \cdot \exp \left( (\ln a) \cdot C_2 \cdot \delta(T)^{1/6} \right).
\]

3rd. Step:

Inserting this upper bound of \( \kappa_T \) in (26) leads to
\[
\mathbb{P} \left( M_T(T) < x \right) \geq \frac{\kappa_T(z,0)V(x)}{4e} - \frac{e}{e-1}V(x)zTc_1(\kappa)\kappa(1/T, 0) \\
\geq \exp \left( (\ln z) \cdot C_2\delta(T)^{1/6} \right) \frac{\kappa(z,0)V(x)}{4e} - \frac{e}{e-1}V(x)zTc_1(\kappa)\kappa(1/T, 0) \\
\geq z^{C_2\delta(T)^{1/6}} \cdot \frac{\kappa(z,0)V(x)}{4e} \cdot \left( 1 - \frac{e^2}{e-1} \frac{zTc_1(\kappa)\kappa(1/T, 0)}{\kappa(z,0)} \right)^{-z^{C_2\delta(T)^{1/6}}}.
\]

Potter’s theorem (cf [8], Theorem 1.5.6) implies
\[
\frac{\kappa(1/T, 0)}{\kappa(z,0)} \leq \frac{c_2(\kappa)}{(zT)^{(1+\rho)/2}}, \text{ for } z \leq \frac{1}{T}.
\]

Now, set
\[
z(T) := \left( \frac{1}{T} \right)^{\left( 1 + \frac{2c_2(\delta(T)^{1/6})}{1+\rho} \right)^{-1}} \left( \frac{e-1}{8e^2c_1(\kappa)c_2(\kappa)} \right)^{-\left( \frac{1}{1+\rho} + C_2\delta(T)^{1/6} \right)^{-1}} \\
= T^{-1-\omega(1)} \leq T^{-1}.
\]
Thus,

\[
4\frac{e^2}{e-1} z c_1(\kappa(1/T, 0) \kappa(z, 0)) \leq 4\frac{e^2}{e-1} (zT)^{1-(1+\rho)/2} c_1(\kappa(1/T, 0) \kappa(z, 0)) z^{-C_2 \delta(T)^{1/6}} \\
\leq \frac{1}{2} zT.
\]

(27)

Since \( zT \leq 1 \), Frullani’s integral gives

\[
\kappa(z, 0) = \exp \left( -\int_0^{\infty} (e^{-t} - e^{-t/T}) t^{-1} \mathbb{P}(Y_T(t) \geq 0) dt \right) \\
\cdot \exp \left( \int_0^{\infty} (e^{-t/T} - e^{-tz}) t^{-1} \mathbb{P}(Y_T(t) \geq 0) dt \right) \\
\geq zT \kappa(1/T, 0).
\]

Hence, we obtain finally with \( zT \geq T - |o(1)| \) that

\[
\mathbb{P}(M_T(T) < x) \geq zT \cdot z^{-C_2 \delta(T)^{1/6}} \cdot \kappa(1/T, 0) V(x) \\
= T^{-\rho+o(1)},
\]

where we used that \( \kappa(1/T, 0) = T^{-\rho+o(1)} \) by [34].

Next, we show Lemma 9. Here, we look at the tail behaviour of the first passage time of \( Y_T \) defined in (11). Note that this lemma deals with Lévy processes in discrete time.

Proof of Lemma 9. Recall that \( Y_T = X - S_T \), where \( X \) is defined in Theorem 2 and \( S_T \) is a subordinator defined in (10). Furthermore, note that \( (Y_T(n))_{n \in \mathbb{N}} \) with \( Y_T \) defined in (11) is a time discrete Lévy process. The same holds for \( (X(n))_{n \in \mathbb{N}} \). By construction, it is clear that \( (X(n))_{n \in \mathbb{N}} \) satisfies Spitzer’s condition with parameter \( \rho \in (0, 1) \).

The basics of fluctuation theory for the time discrete case are essentially the same as for the time continuous case. In the following we keep the notation for the inverse local time \( L^{-1} \) and the ascending ladder process \( H \). The bivariate Laplace exponent of \((L^{-1}, H)\) is denoted by

\[
\kappa(a, b) = \exp \left( \sum_{n=0}^{\infty} \int_{(0, \infty)} (e^{-n} - e^{-an-bx}) n^{-1} \mathbb{P}(X(n) \in dx) \right).
\]

1st. Step:

By Proposition 2.4 in [27] we obtain that

\[
\mathbb{P}(Y_T(n) \leq x, \forall n = 1, \ldots, \lfloor T \rfloor) \leq \frac{1}{T(1-e^{-1})} \sum_{m=0}^{\infty} e^{-m} \mathbb{P}(Y_T(n) \leq x, \forall n = 1, \ldots, m).
\]

By repeating the argument used for the continuous-time case in [5], Formula (VI.8), we obtain, for fixed \( T > 1 \), that

\[
\sum_{m=0}^{\infty} e^{-m} \mathbb{P}(Y_T(n) \leq x, \forall n = 1, \ldots, m) dt \leq T \kappa_T(1/T, 0) V_T^{1/T}(x), \quad \text{for } x \geq 0.
\]

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By definition we get $V_T^{1/T}(x) \leq V_T(x)$, for all $T > 1$ and $x \geq 0$. Hence,

$$
P(Y_T(n) \leq x, \forall n = 1, \ldots, [T]) \leq \min \left( 1, \frac{e}{e - 1} \kappa_T(1/T, 0)V_T(x) \right).$$

(28)

Note that [27] proves this statement for the time-continuous case by using the same arguments.

The proof is complete as soon as we know that $\kappa_T(1/T, 0) \leq T^{-\rho + o(1)}$ and $V_T(x) \leq 2V(x)$, for $T > 1$ sufficiently large.

2nd. Step:

In this step, we show the upper bound of $\kappa_T$. Due the definition of $Y_T$ and $\kappa_T$ we have for every $T > 1$

$$
\kappa_T(1/T, 0) = \kappa(1/T, 0) \cdot \frac{\kappa_T(1/T, 0)}{\kappa(1/T, 0)}
= \kappa(1/T, 0) \cdot \exp \left( \sum_{n=0}^{\infty} (e^{-n} - e^{-n/T})n^{-1}(\mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(X(n) \geq 0)) \right).
$$

Now, we will prove

$$
\mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(X(n) \geq 0) \geq -o(1), \quad \text{as } T \to \infty,
$$

for all $n \geq 0$ to obtain finally an estimate for $\kappa$. For this purpose, we distinguish $n \leq \lceil \delta(N)^{-1/2} \rceil$ and $n > \lceil \delta(N)^{-1/2} \rceil$. Due to the independence of $S_T$ and $Y_T$ we get, for $T > 1$ and $n \leq \lceil \delta(N)^{-1/2} \rceil$,

$$
\mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(X(n) \geq 0)
\geq \mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(Y_T(n) \geq -S_T(1), S(1) = 0) - \mathbb{P}(S_T(n) > 0)
\geq \mathbb{P}(Y_T(n) \geq 0) - \mathbb{P}(Y_T(n) \geq 0) \cdot \mathbb{P}(S(n) = 0) - 1 + \mathbb{P}(S_T(n) = 0)
\geq 1 + e^{-\lceil \delta(N)^{-1/2} \rceil} \mathbb{P}(S(n) = 0)
\geq 1 + \nu(n) \cdot \delta(T)^{1/2}.
$$

(29)

Now, let $n > \lceil \delta(N)^{-1/2} \rceil$. It holds that

$$
\mathbb{P}(X(n) > 0) - \mathbb{P}(Y_T(n) \geq 0)
\leq \mathbb{P}(0 < X(n) < S_T(n), S_T(n) < c(n)\delta(T)^{1/2}) + \mathbb{P}(S_T(n) > c(n)\delta(T)^{1/2})
\leq \mathbb{P} \left( 0 < X(n) < c(n)\delta(T)^{1/2} \right) + \mathbb{P}(S_T(n) > c(n)\delta(T)^{1/2}).
$$

Due to Stone’s local limit theorem (see Theorem 8.4.2 in [9] for non-lattice random walks) and the fact that the density of any $\alpha$-stable law is bounded there exists a constant $C_1 > 0$ such that for $n > \lceil \delta(N)^{-1/2} \rceil$

$$
\mathbb{P} \left( 0 < X(n) < c(n)\delta(T)^{1/2} \right) \leq C_1 \delta(T)^{1/(3\alpha)}.
$$

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Combining this with Lemma 11 and (29) gives uniformly in $t$

$$\mathbb{P}(Y_T(1) \geq 0) - \mathbb{P}(X(1) \geq 0) \geq -C_2\delta(T)^{1/6} = -o(1), \text{ as } T \to \infty,$$

with a constant $C_2 > 0$. Hence, Frullani’s integral implies that

$$\kappa_T(1/T, 0) \leq \kappa(1/T, 0)T^{-C_2\delta(T)^{1/6}}. \quad (30)$$

3rd. Step: Lemma 10 gives for $T > 1$ sufficiently large,

$$V_T(x) \leq 2V(x). \quad (31)$$

Since $\mathbb{P}(X(n) > 0) \to \rho$, as $n \to \infty$, it follows from \[34\] that $\kappa(1/T, 0) = T^{-\rho+o(1)}$. Thus, inserting (31) and (30) in (28) leads, for $T > 1$ sufficiently large, to

$$\mathbb{P}(Y_T(n) \leq 1, \forall n = 1, \ldots, \lfloor T \rfloor) \leq 2\frac{e^{e^{-1}\kappa(1/T, 0)T^{-C_2(\ln\ln T)^{-1/6}}}}{\kappa(1/T, 0)}V(x) = T^{-\rho+o(1)},$$

as desired. \[\square\]

4.3 First passage time of a time-dependent subordinator

This section deals with the asymptotic behaviour of the first passage time for a subordinator depending on $T$ over an increasing boundary as $T$ converges to infinity. Lemma 13 serves as an auxiliary tool to prove the main result of this section, Lemma 8.

Lemma 13. Let $\alpha \in (0, 1) \cup (1, 2)$ and $\gamma > 0$ with $0 < \gamma\alpha < 1$. Then, there exists a constant $\varepsilon > 0$ such that

$$\gamma\alpha - \varepsilon > 0 \quad \text{and} \quad \gamma\alpha + \varepsilon < 1.$$

For $N > 1$ define $\delta(N) = (\ln \ln N)^{-1} \wedge \frac{1}{2}$ and $N_1(N) = \lfloor (\ln \ln N)^{4/(1-\gamma\alpha-\varepsilon)} \rfloor$. Furthermore, let $S_N$ be a subordinator with Laplace transform

$$\mathbb{E}(\exp(-\lambda S_N(1))) \leq \exp \left( -\delta(N)\lambda^\alpha \ell(\lambda\delta(N)^{1/\alpha}) \right),$$

for $\lambda > 0$ sufficiently small and $\ell$ a slowly varying function at zero.

Then, it holds

$$\mathbb{P} \left( S_N(n) \geq (n + 1)^\gamma, \forall n = N_1(N), \ldots, N \right) \sim 1, \text{ as } N \to \infty.$$

Proof. Denote $N_1 := N_1(N)$ and define $N_0 := N_0(N) = \lfloor (\ln N)^{4/(1-\gamma\alpha-\varepsilon)} \rfloor$. Observe that

$$\mathbb{P} \left( S_N(n) \geq \frac{1}{2} - (n + 1)^\gamma, \forall n = N_0, \ldots, N \right) = 1 - \mathbb{P}(\exists n \in \{N_0, \ldots, N\} : S_N(n) < (n + 1)^\gamma).$$
By Chebyshev’s inequality we obtain, for \( N \) sufficiently large, that
\[
\mathbb{P}(\exists n \in \{N_1, \ldots, N\} : S_N(n) < (n + 1)^\gamma) \\
\leq \sum_{n=N_1}^{N_0} \mathbb{P}(S_N(n) < (n + 1)^\gamma) + \sum_{n=N_1}^{N_0} \mathbb{P}(S_N(n) < (n + 1)^\gamma) \\
= \sum_{n=N_1}^{N_0} \mathbb{P}((n + 1)^{-\gamma} S_N(n) < 1) + \sum_{n=N_1}^{N_0} \mathbb{P}((n + 1)^{-\gamma} S_N(n) < 1) \\
= \sum_{n=N_1}^{N_0} \mathbb{P}(e^{-(n+1)^{-\gamma}} S_N(n) > e^{-1}) + \sum_{n=N_1}^{N_0} \mathbb{P}(e^{-(n+1)^{-\gamma}} S_N(n) > e^{-1}) \\
\leq \sum_{n=N_1}^{N_0} e^{1\mathbb{E}\left(e^{-(n+1)^{-\gamma}} S_N(n)\right)} + \sum_{n=N_1}^{N_0} e^{1\mathbb{E}\left(e^{-(n+1)^{-\gamma}} S_N(n)\right)} \\
\leq \sum_{n=N_1}^{N_0} \exp\left(1 - n(n + 1)^{-\gamma\alpha} \ell((n + 1)^{-\gamma} \delta(N)^{1/\alpha}) \delta(N)\right) \\
+ \sum_{n=N_0}^{N_0} \exp\left(1 - n(n + 1)^{-\gamma\alpha} \ell((n + 1)^{-\gamma} \delta(N)^{1/\alpha}) \delta(N)\right).
\]

Then, Proposition 1.3.6 in [9] gives \( \ell(\lambda) \geq \lambda^{\ell/\gamma} \), for \( \lambda > 0 \) sufficiently small and thus, for \( N \) sufficiently large,
\[
\mathbb{P}(\exists n \in \{N_1, \ldots, N\} : S_N(n) < (n + 1)^\gamma) \\
\leq \sum_{n=N_1}^{N_0} \exp\left(1 - n(n + 1)^{-\gamma\alpha - \varepsilon} \delta(N)^{1+\varepsilon/(\gamma\alpha)}\right) \\
+ \sum_{n=N_1}^{N_0} \exp\left(1 - n(n + 1)^{-\gamma\alpha - \varepsilon} \delta(N)^{1+\varepsilon/(\gamma\alpha)}\right) \\
\leq \sum_{n=N_1}^{N_0} \exp\left(1 - \frac{1}{2} n^{-\gamma\alpha - \varepsilon} \delta(N)^{1+\varepsilon/(\gamma\alpha)}\right) + \sum_{n=N_0}^{N_0} \exp\left(1 - \frac{1}{2} n^{-\gamma\alpha - \varepsilon} \delta(N)^{1+\varepsilon/(\gamma\alpha)}\right),
\]
where we used in the last last step the fact that \( \gamma\alpha + \varepsilon < 1 \) and thus,
\[
(n + 1)^{\gamma\alpha + \varepsilon} \leq n^{\gamma\alpha + \varepsilon} + 1 \leq 2n^{\gamma\alpha + \varepsilon}, \quad n \geq 1.
\]

Since \( \gamma\alpha - \varepsilon > 0 \) and \( \gamma\alpha + \varepsilon < 1 \) we get
\[
\mathbb{P}(\exists n \in \{N_1, \ldots, N\} : S_N(n) < (n + 1)^\gamma) \\
\leq \exp\left(1 + \ln(N_0) - \frac{1}{2} N_1^{-\gamma\alpha - \varepsilon} \delta(N)^{1+\varepsilon/(\gamma\alpha)}\right) \\
+ \exp\left(1 + \ln N - \frac{1}{2} N_0^{-\gamma\alpha - \varepsilon} \delta(N)^{1+\varepsilon/(\gamma\alpha)}\right) \\
\leq \exp\left(1 + (3/(1 - \gamma\alpha)) \ln \ln N - \frac{1}{2}(\ln \ln N)^2\right) + \exp\left(1 + \ln N - (\ln N)^2\right) \\
\leq \exp\left(-\ln \ln N\right).
\]

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Hence, we obtain finally
\[
P(S_N(n) \geq \frac{1}{2} - (n+1)^\gamma, \; \forall n = N_0, ..., N) \\
= 1 - P(\exists n \in \{N_0, ..., N\} : S_N(n) < (n+1)^\gamma) \\
\geq 1 - e^{-\ln N} \rightarrow 1, \; \text{as } N \rightarrow \infty.
\]

Now, we proceed with the proof of Lemma 8.

Proof of Lemma 8. We start by transforming this problem to the discrete time as follows
\[
P(S_N(t) \geq \frac{1}{2} + t^\gamma, \; 0 \leq t \leq N) \geq P(S_N(n) \geq \frac{1}{2} + (n+1)^\gamma, \; \forall n = 1, ..., N),
\]
where we used that $S_N$ is nondecreasing.

Since $\alpha, \gamma \alpha \in (0, 1)$ there exist constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$ such that $\alpha - \varepsilon_1 > 0,$ $\alpha + \varepsilon_2 < 1,$ $\gamma - \varepsilon_3 > 0$ and $\alpha \gamma + \varepsilon_4 < 1.$ Define
\[
\varepsilon := \min \left\{ \varepsilon_1, \varepsilon_2, \varepsilon_3, 1 + \varepsilon_1, \varepsilon_4 \right\}.
\]
Furthermore, define $N_1 := \lfloor (\ln \ln N)^{\frac{1}{4}}/(1-\gamma \alpha - \varepsilon) \rfloor.$

Note that since $\gamma \alpha + \varepsilon < 1$ we have
\[
(n+1)^{\gamma \alpha + \varepsilon} \leq n^{\gamma \alpha + \varepsilon} + 1 \leq 2n^{\gamma \alpha + \varepsilon}, \; n \geq 1. \quad (32)
\]

Since $S_N$ is associated (cf. [20] or [2], Lemma 14) we obtain that
\[
P(S_N(n) \geq \frac{1}{2} + (n+1)^\gamma, \; \forall n = 1, ..., N) \\
\geq P(S_N(n) \geq \frac{1}{2} + (n+1)^\gamma, \; \forall n = 1, ..., N_1 - 1) \\
\cdot P(S_N(n) \geq \frac{1}{2} + (n+1)^\gamma, \; \forall n = N_1, ..., N). \quad (33)
\]

Due to Lemma 13 it is left to show that
\[
P(S_N(n) \geq \frac{1}{2} + (n+1)^\gamma, \; \forall n = 1, ..., N_1 - 1) = N^{o(1)}. \quad (34)
\]

In order to show (34) we distinguish $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$.

1st. Case: Let $\alpha \in (0, 1).$ Then, again the fact that $S_N$ is associated leads to
\[
P(S_N(n) \geq \frac{1}{2} + (n+1)^\gamma, \; \forall n = 1, ..., N) \geq P(S_N(n) \geq (n+1)^\gamma, \; \forall n = 1, ..., N_1) \\
\geq \prod_{n=1}^{N_1} P(S_N(n) \geq (n+1)^\gamma).
\]
Since \( \frac{2\alpha}{\alpha - 1} < 0 \) we obtain \((n + 1)^{-\gamma}(\ln N)^{\frac{2\alpha}{\alpha - 1}} \to 0\), as \( N \to \infty \), for all \( n \geq 1 \). Then, applying Chebyshev’s inequality, for \( N \) sufficiently large, implies

\[
\mathbb{P} \left( S_N(n) \geq -\frac{1}{2} + (n + 1)^{\gamma}, \ \forall n = 1, ..., N_1 \right)
\]

\[
= \prod_{n=1}^{N_1} 1 - \mathbb{P} \left( S_N(n) < (n + 1)^{\gamma} \right)
\]

\[
= \prod_{n=1}^{N_1} 1 - \mathbb{P} \left( \exp \left( -\left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} (n + 1)^{-\gamma} S_N(n) \right) > \exp \left( -\left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} \right) \right)
\]

\[
\geq \prod_{n=1}^{N_1} 1 - \exp \left( \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} \right) - \frac{1}{4\alpha} \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} (n + 1)^{-\gamma} \ell \left(\left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} \right).
\]

By Proposition 1.3.6 in [9] we get \( \ell(\lambda) \geq \lambda^{\varepsilon/\gamma} \) for \( \lambda > 0 \) sufficiently small and thus, combining this with (32) gives, for \( N \) sufficiently large,

\[
\mathbb{P} \left( S_N(n) \geq -\frac{1}{2} + (n + 1)^{\gamma}, \ \forall n = 1, ..., N_1 \right)
\]

\[
\geq \prod_{n=1}^{N_1} 1 - \exp \left( \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} - \frac{1}{8\alpha} \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} \right)
\]

\[
\geq \prod_{n=1}^{N_1} 1 - \exp \left( \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} - \frac{1}{8\alpha} \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} \right)
\]

Recall that \( \varepsilon \leq \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3}{1 + \varepsilon_1} \). Thus,

\[
2\alpha + 2(\varepsilon/\gamma) - \alpha + 1 + \varepsilon/(\alpha \gamma) - \varepsilon/\gamma = \alpha + 1 + \varepsilon \left( \frac{1}{\gamma} + \frac{1}{\gamma \alpha} \right)
\]

\[
\leq \alpha + 1 + \varepsilon \frac{1 + \varepsilon_1}{\varepsilon_1 \varepsilon_2}
\]

\[
\leq \alpha + 1 + \varepsilon_2
\]

\[
< 2.
\]

Thus, we have, for \( N \) sufficiently large,

\[
\left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} - \frac{1}{8\alpha} \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} \leq -\frac{1}{10\alpha} \left(\ln \ln N\right)^{\frac{2\alpha}{\alpha - 1}} < 0.
\]
Thus, we obtain, for $N$ sufficiently large, that
\[
\begin{aligned}
\mathbb{P} \left( S_N(n) \geq -\frac{1}{2} + (n + 1)^\gamma, \ n = 1, ..., N_1 \right) &\geq \frac{1}{2} \prod_{n=1}^{N_1} 1 - \exp \left( -\frac{1}{10\alpha} \left( \ln \ln N \right)^{\frac{\alpha + 1 + \varepsilon}{\alpha}} \right) \\
&\geq \frac{1}{2} \left( \frac{1}{10\alpha} \right)^{N_1} \left( \ln \ln N \right)^{-\frac{\alpha + 1 + \varepsilon}{\alpha}} \\
&= N_0^{(1)},
\end{aligned}
\]
and this proves (34) for $\alpha \in (0, 1)$.

2nd. Case: Now, let $\alpha \in (1, 2)$. Note that in this case $\gamma < 1$. Hence,
\[
(n + 1)^\gamma \leq n + 1.
\]
Then, due to the independent and stationary increments we obtain that
\[
\begin{aligned}
\mathbb{P} \left( S_N(n) \geq -\frac{1}{2} + (n + 1)^\gamma, \ \forall n = 1, ..., N_1 - 1 \right) &\geq \mathbb{P} \left( \bigcap_{n=0}^{N_1} (S_N(n) - S_N(n - 1)) > 2 \right) \\
&= \mathbb{P} (S_N(1) > 2)^{N_1} \\
&= \mathbb{P} \left( \frac{S_N(1)}{\delta(N)^{1/\alpha}} > 2(\ln \ln N)^{1/\alpha} \right)^{N_1}.
\end{aligned}
\]
Define now the following Lévy process
\[
\Psi_S(u) := \int_{\delta(T)^{1/\alpha}}^{\infty} (e^{iux} - 1) \nu_S(dx), \ u \in \mathbb{R},
\]
with
\[
\nu_S(dx) := \begin{cases} 
\nu(dx), & x \geq \delta(T)^{-1/\alpha}, \\
0, & x < \delta(T)^{-1/\alpha}.
\end{cases}
\]
Integration by substitution gives for every $\lambda > 0$
\[
\begin{aligned}
t \Psi_S \left( \frac{\lambda}{\delta(N)^{1/\alpha} c(t)} \right) &= t \exp \left( \int_{1}^{\infty} \left( e^{i\left( \frac{\lambda}{\delta(N)^{1/\alpha} c(t)} \right)^x} - 1 \right) \nu_S(dx) \right) \\
&= t \exp \left( \int_{\delta(N)^{-1/\alpha}}^{\infty} \left( e^{i\left( \frac{\lambda}{\delta(N)^{1/\alpha} c(t)} \right)^x} - 1 \right) \delta(N)^{1/\alpha} \nu_S(dx) \right) \\
&= t \exp \left( \int_{\delta(N)^{-1/\alpha}}^{\infty} \left( e^{i\left( \frac{\lambda}{\delta(N)^{1/\alpha} c(t)} \right)^x} - 1 \right) \nu(dx) \right) \\
&= t \Psi_S(\lambda), \text{ for all } t \geq 0.
\end{aligned}
\]
Hence, for all $N > 1$,
\[
\mathbb{P} \left( \frac{S_N(1)}{\delta(N)^{1/\alpha}} > 2(\ln \ln N)^{1/\alpha} \right) = \mathbb{P} \left( \tilde{S}_N(1) > 2(\ln \ln N)^{1/\alpha} \right).
\]
Since $\tilde{S}_N$ does possess jumps larger that $\delta(N)^{-1/\alpha}$ it follows, for $N$ sufficiently large, that

$$\mathbb{P}\left(\tilde{S}_N(1) > 1\right) \geq (\ln \ln N)^{-1/2}.$$  

Hence, we obtain finally

$$\mathbb{P}\left(S_N(n) \geq -\frac{1}{2} + (n + 1)^\gamma, \ \forall n = 1, ..., N_1 - 1\right) \geq \left((\ln \ln N)^{-1/2}\right)^{N_1} = N^{o(1)},$$

and this proves (34) for $\alpha \in (1, 2)$.

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