Locating-Total Dominating Sets in Twin-Free Graphs

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Abstract

A total dominating set of a graph \( G \) is a set \( D \) of vertices of \( G \) such that every vertex of \( G \) has a neighbor in \( D \). A locating-total dominating set of \( G \) is a total dominating set \( D \) of \( G \) with the additional property that every two distinct vertices outside \( D \) have distinct neighbors in \( D \); that is, for distinct vertices \( u \) and \( v \) outside \( D \), \( N(u) \cap D \neq N(v) \cap D \) where \( N(u) \) denotes the open neighborhood of \( u \). A graph is twin-free if every two distinct vertices have distinct open and closed neighborhoods. The location-total domination number of \( G \), denoted \( \gamma_{L}^{t}(G) \), is the minimum cardinality of a locating-total dominating set in \( G \). It is well-known that every connected graph of order \( n \geq 3 \) has a total dominating set of size at most \( \frac{2}{3}n \). We conjecture that if \( G \) is a twin-free graph of order \( n \) with no isolated vertex, then \( \gamma_{L}^{t}(G) \leq \frac{3}{4}n \). We prove the conjecture for graphs without 4-cycles. We prove that if \( G \) is a twin-free graph of order \( n \), then \( \gamma_{L}^{t}(G) \leq \frac{3}{4}n \).

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1 Introduction

A dominating set in a graph \( G \) is a set \( D \) of vertices of \( G \) such that every vertex outside \( D \) is adjacent to a vertex in \( D \). The domination number, \( \gamma(G) \), of \( G \) is the minimum cardinality of a dominating set in \( G \). A total dominating set, abbreviated TD-set, of \( G \) is a set \( D \) of vertices of \( G \) such that every vertex of \( G \) is adjacent to a vertex in \( D \). The total domination number of \( G \),
denoted by \( \gamma_L(G) \), is the minimum cardinality of a TD-set in \( G \). The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books [15, 16], and a recent book on total dominating sets is also available [22].

Among the existing variations of (total) domination, the one of location-domination and location-total domination are widely studied. A set \( D \) of vertices locates a vertex \( v \) if the neighborhood of \( v \) within \( D \) is unique among all vertices in \( V(G) \setminus D \). A locating-dominating set is a dominating set \( D \) that locates all the vertices, and the location-domination number of \( G \), denoted \( \gamma_L(G) \), is the minimum cardinality of a locating-dominating set in \( G \). A locating-total dominating set, abbreviated LTD-set, is a TD-set \( D \) that locates all the vertices, and the location-total domination number of \( G \), denoted \( \gamma_T(G) \), is the minimum cardinality of a LTD-set in \( G \). The concept of a locating-dominating set was introduced and first studied by Slater [25, 26] (see also [9, 10, 13, 24, 27]), and the additional condition that the locating-dominating set be a total dominating set was first considered in [17] (see also [1, 2, 3, 5, 6, 7, 19, 20]).

We remark that there are (twin-free) graphs with total domination number two and arbitrarily large location-total domination number. For \( k \geq 3 \), let \( G_k \) be the graph obtained from \( K_{2,k} \) by selecting one of the two vertices of degree \( k \) and subdividing every edge incident with it, and then adding an edge joining the two vertices of degree \( k \). The resulting graph, \( G_k \), has order \( 2k + 2 \), total domination number 2, and location-total domination number exactly one-half the order (namely, \( k + 1 \)). The graph \( G_4 \), for example, is illustrated in Figure 1 where the darkened vertices form a minimum LTD-set in \( G_4 \).

![Figure 1: The twin-free graph \( G_4 \).](image)

A classic result due to Cockayne et al. [8] states that every connected graph of order at least 3 has a TD-set of cardinality at most two-thirds its order. While there are many graphs (without isolated vertices) which have location-total domination number much larger than two-thirds their order, the only such graphs that are known contain many twins, that is, pairs of vertices with the same closed or open neighborhood. We conjecture that in fact, twin-free graphs have location-total domination number at most two-thirds their order. In this paper we initiate the study of this conjecture.

**Definitions and notations.** For notation and graph theory terminology, we in general follow [15]. Specifically, let \( G \) be a graph with vertex set \( V(G) \), edge set \( E(G) \) and with no isolated vertex. The open neighborhood of a vertex \( v \in V(G) \) is \( N_G(v) = \{u \in V \mid uv \in E(G)\} \) and its closed neighborhood is the set \( N_G[v] = N_G(v) \cup \{v\} \). The degree of \( v \) is \( d_G(v) = |N_G(v)| \). Given a set \( S \subset V(G) \) and a vertex \( v \in S \), an \( S \)-external private neighbor of \( v \) is a vertex outside \( S \) that is adjacent to \( v \) but to no other vertex of \( S \) in \( G \). The set of all \( S \)-external private neighbors of \( v \), abbreviated epn\(_{G}(v,S) \), is the \( S \)-external private neighborhood. The subgraph induced by a set \( S \) of vertices in \( G \) is denoted by \( G[S] \). If the graph \( G \) is clear from the context, we simply write \( V, E, N(v), N[v], d(v) \) and epn\(_{G}(v,S) \) rather than \( V(G), E(G), N_G(v), N_G[v], d_G(v) \) and

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epn\(_G(v, S)\), respectively.

Given a set \(S\) of edges in \(G\), we will denote by \(G - S\) the subgraph obtained from \(G\) by deleting all edges of \(S\). For a set \(S\) of vertices, \(G - S\) is the graph obtained from \(G\) by removing all vertices of \(S\) and removing all edges incident to vertices of \(S\). A cycle on \(n\) vertices is denoted by \(C_n\) and a path on \(n\) vertices by \(P_n\). The girth of \(G\) is the length of a shortest cycle in \(G\).

A set \(D\) is a dominating set of \(G\) if \(N[v] \cap D \neq \emptyset\) for every vertex \(v\) in \(G\), or, equivalently, \(N[D] = V(G)\). A set \(D\) is a TD-set of \(G\) if \(N(v) \cap D \neq \emptyset\) for every vertex \(v\) in \(G\), or, equivalently, \(N(D) = V(G)\). Two distinct vertices \(u\) and \(v\) in \(V(G) \setminus D\) are located by \(D\) if they have distinct neighbors in \(D\); that is, \(N(u) \cap D \neq N(v) \cap D\). If a vertex \(u \in V(G) \setminus D\) is located from every other vertex in \(V(G) \setminus D\), we simply say that \(u\) is located by \(D\).

A set \(S\) is a locating set of \(G\) if every two distinct vertices outside \(S\) are located by \(S\). In particular, if \(S\) is both a dominating set and a locating set, then \(S\) is a locating-dominating set. Further, if \(S\) is both a TD-set and a locating set, then \(S\) is a locating-total dominating set. We remark that the only difference between a locating set and a locating-dominating set in \(G\) is that a locating set might have a unique non-dominated vertex.

Two distinct vertices \(u\) and \(v\) of a graph \(G\) are open twins if \(N(u) = N(v)\) and closed twins if \(N[u] = N[v]\). Further, \(u\) and \(v\) are twins in \(G\) if they are open twins or closed twins in \(G\). A graph is twin-free if it has no twins.

For two vertices \(u\) and \(v\) in a connected graph \(G\), the distance \(d_G(u, v)\) between \(u\) and \(v\) is the length of a shortest \((u, v)\)-path in \(G\). The maximum distance among all pairs of vertices of \(G\) is the diameter of \(G\), which is denoted by diam\((G)\). A nontrivial connected graph is a connected graph of order at least 2. A leaf of graph \(G\) is a vertex of degree 1, while a support vertex of \(G\) is a vertex adjacent to a leaf.

A rooted tree \(T\) distinguishes one vertex \(r\) called the root. For each vertex \(v \neq r\) of \(T\), the parent of \(v\) is the neighbor of \(v\) on the unique \((r, v)\)-path, while a child of \(v\) is any other neighbor of \(v\). A descendant of \(v\) is a vertex \(u \neq v\) such that the unique \((r, u)\)-path contains \(v\). Thus, every child of \(v\) is a descendant of \(v\). We let \(D(v)\) denote the set of descendants of \(v\), and we define \(D[v] = D(v) \cup \{v\}\). The maximal subtree at \(v\) is the subtree of \(T\) induced by \(D\{v\}\), and is denoted by \(T_v\).

The 2-corona of a graph \(H\) is the graph of order \(3|V(H)|\) obtained from \(H\) by adding a vertex-disjoint copy of a path \(P_2\) for each vertex \(v\) of \(H\) and adding an edge joining \(v\) to one end of the added path.

We use the standard notation \([k] = \{1, 2, \ldots, k\}\). If \(A\) and \(B\) are sets, then \(A \times B = \{(a, b) \mid a \in A, b \in B\}\).

**Conjectures and known results.** As a motivation for our study, we pose and state the following conjecture:

**Conjecture 1.** Every twin-free graph \(G\) of order \(n\) without isolated vertices satisfies \(\gamma_t^1(G) \leq \frac{2}{3}n\).

In an earlier paper, Henning and Löwenstein [19] proved that every connected cubic claw-free
graph (not necessarily twin-free) has a LTD-set of size at most one-half its order, which implies that Conjecture 1 is true for such graphs. Moreover they conjectured this to be true for every connected cubic graph, with two exceptions — which, if true, would imply Conjecture 1 for all cubic graphs.

A similar conjecture for locating-dominating sets, that motivated the present study, was posed in [14].

Conjecture 2 (Garijo et al. [14]). Every twin-free graph $G$ of order $n$ without isolated vertices satisfies $\gamma_L(G) \leq \frac{n}{2}$.

Conjecture 2 remains open, although it was proved for a number of graph classes such as bipartite graphs and graphs with no 4-cycles [14], split and co-bipartite graphs [13], cubic graphs [11] and line graphs [12]. Some of these results were obtained using selected vertex covers and matchings, but none of these techniques seems to be useful in the study of Conjecture 1.

Our results. We prove the bound $\gamma_t(G) \leq \frac{3}{4}n$ in Section 3. We then give support to Conjecture 1 by proving it for graphs without 4-cycles in Section 4 where we also characterize all extremal examples without 4-cycles.

2 Preliminary Results

This section contains a number of preliminary results that will be useful in the next sections.

Theorem 3 (Cockayne et al. [8]; Brigham et al. [4]). If $G$ is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq \frac{2}{3}n$. Further, $\gamma_t(G) = \frac{2}{3}n$ if and only if $G$ is isomorphic to a 3-cycle, a 6-cycle, or the 2-corona of some connected graph $H$.

We will need the following property of minimum TD-sets in a graph established in [18].

Theorem 4 ([18]). If $G$ is a connected graph of order $n \geq 3$, and $G \not\cong K_n$, then $G$ has a minimum TD-set $S$ such that every vertex $v \in S$ satisfies $|\text{epn}(v, S)| \geq 1$ or has a neighbor $x$ in $S$ of degree 1 in $G[S]$ satisfying $|\text{epn}(x, S)| \geq 1$.

Given a graph $G$, the set $L \cup T$, where $L$ is a locating-dominating set of $G$, and $T$ is a TD-set of $G$ is both a TD-set and a locating set, implying the following observation.

Observation 5. For every graph $G$ without isolated vertices, we have $\gamma_t(G) \leq \gamma_L(G) + \gamma_t(G)$.

3 A general upper bound of three-quarters the order

In this section we prove a general upper bound on the location-total domination number of a graph in terms of its order. The proof is similar to the bound $\gamma_L(G) \leq \frac{2}{3}n$ proved for locating-dominating sets in [13].
Theorem 6. If $G$ is a twin-free graph of order $n$ without isolated vertices, then $\gamma_t^L(G) \leq \frac{2}{3}n$.

Proof. By linearity, we may assume that $G$ is connected. By the twin-freeness of $G$, we note that $n \geq 4$ and that $G \not\cong K_n$. For an arbitrary subset $S$ of vertices in $G$, let $\mathcal{P}_S$ be a partition of $\overline{S} = V(G) \setminus S$ with the property that all vertices in the same part of the partition have the same open neighborhood in $S$ and vertices from different parts of the partition have different open neighborhood in $S$. Let $|\mathcal{P}_S| = k(S)$. Let $X_S$ be the set of vertices in $\overline{S}$ that belong to a partition set in $\mathcal{P}_S$ of size 1 and let $Y_S = \overline{S} \setminus X_S$. Hence every vertex in $Y_S$ belongs to a partition set of size at least 2. Let $n_1(S) = |X_S|$ and let $n_2(S) = k(S) - n_1(S)$. Let $S$ be a minimum TD-set in $G$ with the property that every vertex $v \in S$ satisfies $\gamma(v, S) \geq 1$ or has a neighbor $x$ in $S$ of degree 1 in $G[S]$ satisfying $\gamma(v', S) \geq 1$. Such a set exists by Theorem 3. We note that at least half the vertices in $S$ have an $S$-external private neighbor, implying that $n_1(S) + n_2(S) \geq \frac{1}{2}|S|$. Among all supersets $S'$ of $S$ with the property that $n_1(S') + n_2(S') \geq \frac{1}{2}|S'|$, let $D$ be chosen to be inclusion-wise maximal. (Possibly, $D = S$.)

Claim 6.A. The vertices in each partition set of size at least 2 in $\mathcal{P}_D$ have distinct neighborhoods in $X_D$, and $D \cup X_D$ is a LTD-set of $G$.

Proof of claim. Let $u$ and $v$ be two vertices that belong to a partition set $T$, of size at least 2 in $\mathcal{P}_D$. Since $G$ is twin-free, there exists a vertex $w \notin \{u, v\}$ that is adjacent to exactly one of $u$ and $v$. Since $u$ and $v$ have the same neighbors in $D$, we note that $w \notin D$. Hence, $w \in V(G) \setminus D$. Suppose that $w \in Y_D$ and consider the set $D' = D \cup \{w\}$. Let $R$ be an arbitrary partition set in $\mathcal{P}_D$ that might or might not contain $w$. If $w$ is either adjacent to every vertex of $R \setminus \{w\}$ or adjacent to no vertex in $R \setminus \{w\}$, then $R \setminus \{w\}$ is a partition set in $\mathcal{P}_{D'}$. If $w$ is adjacent to some, but not all, vertices of $R \setminus \{w\}$, then there is a partition $R \setminus \{w\} = (R_1, R_2)$ of $R \setminus \{w\}$ where $R_1$ are the vertices in $R \setminus \{w\}$ adjacent to $w$ and $R_2$ are the remaining vertices in $R \setminus \{w\}$. In this case, both sets $R_1$ and $R_2$ form a partition set in $\mathcal{P}_{D'}$. In particular, we note that there is a partition $T \setminus \{w\} = (T_1, T_2)$ of $T \setminus \{w\}$ where both sets $T_1$ and $T_2$ form a partition set in $\mathcal{P}_{D'}$. Therefore, $n_1(D') + n_2(D') \geq n_1(D) + n_2(D) + 1 \geq \frac{1}{2}|D| + 1 > \frac{1}{2}(|D| + 1) = \frac{1}{2}|D'|$, contradicting the maximality of $D$. Hence, $w \notin Y_D$. Therefore, $w \in X_D$. Hence, $u$ and $v$ are located by the set $X_D$ in $G$. Moreover, $D \cup X_D$ is a TD-set since every vertex of $X_D$ is dominated by a vertex of $D$, and $D$ itself is a TD-set. \((\Box)\)

Let $Y'_D$ be obtained from $Y_D$ by deleting one vertex from each partition set of size at least 2 in $\mathcal{P}_D$, and let $D' = D \cup Y'_D$. Then, $|D'| = n - n_1(D) - n_2(D)$. By definition of the partition $\mathcal{P}_D$, every vertex in $V(G) \setminus D'$ has a distinct nonempty neighborhood in $D$ and therefore in $D'$. Moreover, $D'$ is a TD-set since every vertex of $Y_D$ is dominated by a vertex of $D$, and $D$ itself is a TD-set. Hence we have the following claim.

Claim 6.B. The set $D'$ is a LTD-set of $G$.

Let $n_1 = n_1(D)$ and $n_2 = n_2(D)$. By Claim 6.A, the set $D \cup X_D$ is a LTD-set of $G$ of cardinality $|D| + n_1$. By Claim 6.B, the set $D'$ is a LDT-set of $G$ of cardinality $n - n_1 - n_2$. Hence,

$$\gamma_t^L(G) \leq \min\{|D| + n_1, n - n_1 - n_2\}. \quad (1)$$
Inequality (1) implies that if \( n - n_1 - n_2 \leq \frac{3}{4}n \), then \( \gamma_L(G) \leq \frac{3}{4}n \). Hence we may assume that \( n - n_1 - n_2 > \frac{3}{4}n \), for otherwise the desired upper bound on \( \gamma_L(G) \) follows. With this assumption, \( n_1 + n_2 < \frac{1}{4}n \). By our choice of the set \( D \), we recall that \( |D| \leq 2(n_1 + n_2) \). Therefore,

\[
|D| + n_1 \leq 3n_1 + 2n_2 \leq 3(n_1 + n_2) < \frac{3}{4}n.
\]

Hence, by Inequality (1), \( \gamma_L(G) < \frac{3}{4}n \). This completes the proof of Theorem \( \Box \)

4 Graphs without 4-cycles

In this section, we prove Conjecture 1 for graphs with no 4-cycles. We also characterize all graphs with no 4-cycles that achieve the bound of Conjecture 1. Surprisingly, these are precisely those graphs that have no 4-cycles and no twins and that are extremal for the bound on the total domination number from Theorem 3. This is in stark contrast with Conjecture 2 for the location-domination number, where many graphs (without 4-cycles) are known that are extremal for the conjecture but have much smaller domination number than one-half the order, see [13].

**Theorem 7.** Let \( G \) be a twin-free graph of order \( n \) without isolated vertices and 4-cycles. Then, \( \gamma_L(G) \leq \frac{2}{3}n \). Further, \( \gamma_L(G) = \frac{2}{3}n \) if and only if \( G \) is isomorphic to a 6-cycle or is the 2-corona of some connected nontrivial graph that contains no 4-cycles.

**Proof.** We prove the claim by induction on \( n \). By linearity, we may assume that \( G \) is connected, for otherwise we apply induction to each component of \( G \) and we are done. By the twin-freeness of \( G \), we note that \( n \geq 4 \). Further if \( n = 4 \), then since \( G \) is \( C_4 \)-free, the graph \( G \) is the path \( P_4 \) and \( \gamma_L(P_4) = 2 < \frac{2}{3}n \). This establishes the base case. Let \( n \geq 5 \) and assume that every twin-free graph \( G' \) without isolated vertices and with no 4-cycles of order \( n' \), where \( n' < n \), satisfies \( \gamma_L(G') \leq \frac{2}{3}n' \), and that the only graphs achieving the bound are the extremal graphs described in Theorem 3 that are twin-free and have no 4-cycles. Let \( G \) be a twin-free graph without isolated vertices and with no 4-cycles of order \( n \). The general idea will be to partition \( V(G) \) into two sets \( V_1 \) and \( V_2 \). If \( G[V_1] \) and/or \( G[V_2] \) are twin-free, we apply induction, and use the obtained LTD-sets of \( G[V_1] \) and/or \( G[V_2] \) to build one of \( G \). We proceed further with the following series of claims.

**Claim 7.A.** If \( G \) is a tree, then \( \gamma_L(G) \leq \frac{2}{3}n \). Further, \( \gamma_L(G) = \frac{2}{3}n \) if and only if \( G \) is the 2-corona of a nontrivial tree.

**Proof of Claim 7.A** Suppose that \( G \) is a tree. Since \( n \geq 5 \) and \( G \) is twin-free, we note that \( \text{diam}(G) \geq 4 \). If \( \text{diam}(G) = 4 \), then either \( G = P_5 \) or \( G \) is obtained from a star \( K_{1,k+1} \), where \( k \geq 2 \), by subdividing at least \( k \) edges of the star exactly once. In this case, the set of vertices of degree at least 2 in \( G \) forms a LTD-set of size strictly less than two-thirds the order. Hence, we may assume that \( \text{diam}(G) \geq 5 \), for otherwise the desired result follows.

Let \( P \) be a longest path in \( T \) and let \( P \) be an \((r,u)\)-path. Necessarily, both \( r \) and \( u \) are leaves. Since \( \text{diam}(G) \geq 5 \), we note that \( P \) has length at least 5. We now root the tree at the vertex.
Let \( v \) be the parent of \( u \), and let \( w \) be the parent of \( v \), \( x \) the parent of \( w \), and \( y \) the parent of \( x \) in the rooted tree. Since \( |V(P)| \geq 5 \), we note that \( y \neq r \). Since \( G \) is twin-free, the vertex \( w \) has at most one leaf-neighbor and every child of \( w \) that is not a leaf has degree 2 in \( G \). In particular, \( d(v) = 2 \). We now consider the subtree \( G_w \) of \( G \) rooted at the vertex \( w \). If \( d_G(w) = 2 \), then \( G_w = P_3 \), while if \( d_G(w) \geq 3 \), then \( G_w \) is obtained from a star \( K_{1,k+1} \), where \( k \geq 1 \), by subdividing at least \( k \) edges of the star exactly once. Let \( G' = G - V(G_w) \).

We now define a twin-free subtree \( G_2 \) of \( G \) as follows. If the tree \( G' \) is twin-free, then we let \( V_1 = V(G_w) \) and \( V_2 = V(G) \setminus V_1 \), and we let \( G_1 = G[V_1] \) and \( G_2 = G[V_2] \). We note that in this case, \( G_2 = G' \). Suppose that the tree \( G' \) is not twin-free. Then, the parent \( x \) of \( w \) has a twin \( x' \) in \( G' \), and \( N_G(x') = N_G(x) \setminus \{w\} = \{y\} \). Thus, \( d_G(x) = 2 \) and the vertex \( x' \) is a leaf-neighbor of \( y \) in \( G \). If \( x' = r \), then our choice of \( P \) as a longest path in \( G \) implies that \( G' \) is the path \( rxy \). If now \( G_w \neq P_3 \), then the set of vertices of degree at least 2 in \( G \) form a LTD-set of \( G \) of size strictly less than two-thirds the order, while if \( G_w = P_3 \), then \( G \) is a path \( P_6 \), which is the 2-corona of a tree \( K_2 \), and \( \gamma^L_t(G) = \frac{7}{3}n \). Hence, we may assume that \( x' \neq r \), for otherwise the desired result follows. We now let \( V_1 = V(G_w) \cup \{x\} \), \( V_2 = V(G) \setminus V_1 \), and we let \( G_1 = G[V_1] \) and \( G_2 = G[V_2] \). We note that in this case, \( G_2 = G' - x \). Our assumption that \( x' \neq r \) implies that in this case, \( G_2 \) is a twin-free tree.

Let \( D_2 \) be a minimum LTD-set of \( G_2 \). Applying the induction hypothesis to the twin-free tree \( G_2 \), the set \( D_2 \) satisfies \( |D_2| \leq \frac{2}{3}|V_2| \). Further, if \( |D_2| = \frac{2}{3}|V_2| \), then \( G_2 \) is the 2-corona of a nontrivial tree. Let \( D_1 \) consist of \( w \) and every child of \( w \) of degree 2. Then, \(|D_1| \leq \frac{2}{3}|V_1| \) with strict inequality if \( G_1 \) is not the path \( www \). We claim that \( D = D_1 \cup D_2 \) is a LTD-set of \( G \). Since \( D_1 \) and \( D_2 \) are TD-sets of \( G_1 \) and \( G_2 \), respectively, the set \( D \) is a TD-set of \( G \). Every vertex of \( G \) is located by \( D \) except possibly for the vertex \( x \) and a leaf-neighbor of \( w \) in \( G \), if such a leaf-neighbor exists. If \( x \in V(G_2) \), then it is located in \( G_2 \) and hence in \( G \). If \( x \in V(G_1) \), then its twin \( x' \) in \( G' \) is a leaf-neighbor of \( y \), implying that in \( G_2 \) the support vertex \( y \in D_2 \). Thus, \( x \) is located by \( w \) and \( y \). If \( w \) has a leaf-neighbor in \( G \), then such a leaf-neighbor is located by \( w \) only. Therefore, \( D \) is a LTD-set of \( G \), and so

\[
\gamma^L_t(G) \leq |D| = |D_1| + |D_2| \leq \frac{2}{3}|V_1| + \frac{2}{3}|V_2| = \frac{2}{3}n.
\]

This establishes the desired upper bound. Suppose next that \( \gamma^L_t(G) = \frac{2}{3}n \). Then we must have equality throughout the Inequality Chain (2). In particular, \( |D_1| = \frac{2}{3}|V_1| \) and \( |D_2| = \frac{2}{3}|V_2| \), implying that \( G_1 = P_3 \) (and \( G_1 \) consists of the path \( www \)) and \( G_2 \) is the 2-corona of a nontrivial tree, say \( T_2 \). Let \( A \) and \( B \) be the set of leaves and support vertices, respectively, in \( G_2 \), and let \( C \) be the remaining vertices of \( G_2 \). We note that \( C = V(T_2) = V_2 \setminus (A \cup B) \) and \( |C| \geq 2 \) (since \( T_2 \) is a nontrivial tree). If \( x \in A \), then \( x \) is a leaf in \( G_2 \) and its neighbor \( y \) is a support vertex in \( G_2 \) and belongs to the set \( B \). If \( x \in B \), then \( x \) is a support vertex in \( G_2 \) and its parent \( y \) belongs to \( C \). In both cases, the set \( (B \cup (C \cup \{v, w\})) \setminus \{y\} \) is a LTD-set of \( G \) of size \( |D_1| + |D_2| - 1 = \frac{2}{3}n - 1 \), a contradiction to our supposition that \( \gamma^L_t(G) = \frac{2}{3}n \). Hence, \( x \in C \), implying that \( G \) is the 2-corona of a nontrivial tree, namely the tree \( G[C \cup \{w\}] \) obtained from \( T_2 \) by adding to it the vertex \( w \) and the edge \( wx \). This completes the proof of Claim 7.4.3.

By Claim 7.4.3 we may assume that \( G \) is not a tree, for otherwise the desired result follows. Hence, \( G \) contains a cycle. We consider next the case when \( G \) contains a triangle.
Claim 7.B. If $G$ contains a triangle, then $\gamma_t^L(G) \leq \frac{2}{3}n$. Further, $\gamma_t^L(G) = \frac{2}{3}n$ if and only if $G$ is isomorphic to a 6-cycle or is the 2-corona of some connected nontrivial graph that contains no 4-cycles but contains a triangle.

Proof of Claim 7.B. Suppose that $G$ contains a triangle $C$. Let $G' = G - V(C)$. We build a subset $V_1$ of vertices of $G$ as follows. Let $V_0$ consist of $V(C)$ together with all vertices that belong to a component of $G'$ isomorphic to $P_1$, $P_2$ or $P_3$. We remark that if $C'$ is a $P_1$- or $P_2$-component of $G'$, then at most one edge joins it to $C$, for otherwise there would be a 4-cycle or a pair of twins in $G$. Suppose that $S$ is a set of mutual twins of $G - V_0$. Since $G$ is twin-free, all but possibly one vertex in $S$ must be adjacent to a vertex of $C$. For each such set $S$ of mutual twins of $G - V_0$, we select $|S| - 1$ vertices from $S$ that have a neighbor in $C$, and add these vertices to the set $V_0$ to form the set $V_1$ (possibly, $V_1 = V_0$). Let $V_2 = V(G) \setminus V_1$. Let $G_1 = G[V_1]$ and if $V_2 \neq \emptyset$, let $G_2 = G[V_2]$. We note that $G_1$ is connected, while $G_2$ may possibly be disconnected.

Subclaim 7.B.1 $G_2$ is twin-free and has no isolated vertices.

Proof of Subclaim 7.B.1. We first prove that $G_2$ is twin-free. Suppose, to the contrary, that there is a pair $\{t, t'\}$ of twins in $G_2$. By construction of $V_2$, the vertices $t$ and $t'$ are not twins in $G - V_0$, implying that there exists a vertex $v$ in $V_1 \setminus V_0$ such that $v$ is adjacent to exactly one of $t$ and $t'$, say to $t$. Let $v'$ be the twin of $v$ in $G - V_0$ that was not added to the set $V_1$ (recall that by construction, all but one vertex from a set of mutual twins in $G - V_0$ is added to the set $V_1$). But then, $v'$ is a vertex in $G_2$ that is adjacent to $t$ but not to $t'$, contradicting our supposition that $t$ and $t'$ are twins in $G_2$. Therefore, $G_2$ is twin-free. The proof that $G_2$ has no isolated vertices, again by the construction, an isolated vertex $x$ would have been a neighbor of a set of twins of $G - V_0$. But at least one twin still belongs to $G_2$, and $x$ is not isolated. (\(\square\))

By Subclaim 7.B.1, $G_2$ is twin-free. Let $D_2$ be a minimum LTD-set of $G_2$. Applying the induction hypothesis to each component of $G_2$, the set $D_2$ satisfies $|D_2| \leq \frac{2}{3}|V_2|$. Further, if $|D_2| = \frac{2}{3}|V_2|$, then each component of $G_2$ is isomorphic to a 6-cycle or is the 2-corona of some connected nontrivial graph that contains no 4-cycles.

We note that the graph $G_1$ could have twins. For example, this would occur if $V_1 = V(C)$, in which case $G_1$ is the 3-cycle $C$. A more complicated possibility is if there were twins $t$ and $t'$ in $G - V_0$; then at least one of them belongs to $G_1$ and could be, in $G_1$, a twin with the vertex of some $P_1$-component of $G'$. Let us build a set $D_1 \subset V_1$. As observed earlier, if $C'$ is a $P_1$- or $P_2$-component of $G'$, then at most one edge joins it to $C$. For every $P_3$-component $C'$ of $G'$, select the central vertex of $C'$ and one of its neighbors in $C'$ that is not a leaf in $G$ and add these two vertices of $C'$ to $D_1$. For every $P_3$-component $C'$ of $G'$, add to $D_1$ the unique vertex of $C'$ adjacent to a vertex of $C$, as well as its neighbor in $C$. For every $P_1$-component of $G'$ consisting of a vertex $v'$, add to $D_1$ the unique neighbor of $v'$ in $C$. For every vertex in $V_1 \setminus V_0$ that has a twin in $G - V_0$, add its neighbor in $C$ to $D_1$. We note that if $t$ and $t'$ are two vertices in $V_1 \setminus V_0$ that are twins in $G - V_0$, then $t$ and $t'$ have no common neighbor on $C$, otherwise $G$ would contain a 4-cycle. Now, if there is at most one vertex of $C$ in the resulting set $D_1$, then we augment $D_1$ so that exactly two vertices of $C$ belong to $D_1$. By construction the resulting set $D_1$ is a TD-set of $G_1$ and $|D_1| \leq \frac{2}{3}|V_1|$. 

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Subclaim 7.B.2 $D = D_1 \cup D_2$ is a LTD-set of $G$.

Proof of Subclaim 7.B.2. Since $D_1$ and $D_2$ are TD-sets of $G_1$ and $G_2$, respectively, the set $D$ is a TD-set of $G$. Suppose, to the contrary, that $D$ is not locating. Then there is a pair of vertices, $u$ and $v$, that is not located by $D$. If $(u, v) \in V_1 \times V_2$ (that is, $u \in V_1$ and $v \in V_2$), then $u$ is dominated by a vertex of $D_1$ and $v$ is dominated by a vertex of $D_2$. Hence, $u$ and $v$ must both be dominated by these two vertices. But then we have a 4-cycle in $G$, a contradiction. Hence, $(u, v) \notin V_1 \times V_2$. Analogously, $(u, v) \notin V_2 \times V_1$. Since $D_2$ is locating in $G_2$, we note that $(u, v) \notin V_1 \times V_1$. Hence, $(u, v) \in V_1 \times V_1$; that is, both $u$ and $v$ belong to $G_1$. Every vertex of $V_1$ that belongs to the 3-cycle $C$ or to a $P_2$- or a $P_3$-component of $G'$ is located by $D_1$, and hence $D$. Therefore, at least one of $u$ and $v$, say $u$, belongs to $V_1 \setminus V_0$ and had a twin in $G - V_0$. Let $u'$ be the twin of $u$ in $G - V_0$ that was not added to the set $V_1$, and so $u' \in V_2$. If $u$ and $u'$ are open twins in $G - V_0$, then $u'$ is a vertex of degree 1 in $G$, for otherwise $u$ and $u'$ belong to a 4-cycle. For the same reason, if $u$ and $u'$ are closed twins, then $u'$ has degree 2 in $G$. In both cases, $u'$ has degree 1 in $G_2$. The unique common neighbor of $u$ and $u'$ therefore belongs to $D_2$ in order to totally dominate the vertex $u'$ in $G_2$. Thus, $u$ is dominated by a vertex of $D_1$ and a vertex of $D_2$. Since $u$ and $v$ are not located, $v$ is also dominated by these two vertices, which implies that $u$ and $v$ belong to a common 4-cycle of $G$, a contradiction. Therefore, $D$ is a LTD-set of $G$. \( \blacksquare \)

By Subclaim 7.B.2, the set $D = D_1 \cup D_2$ is a LTD-set of $G$, implying that the Inequality Chain \( \blacksquare \) presented in the proof of Claim 7.A holds. This establishes the desired upper bound.

Suppose next that $\gamma_l^T(G) = \frac{2}{3}m$. Then we must have equality throughout the Inequality Chain \( \blacksquare \). In particular, $|D_1| = \frac{2}{3}|V_1|$ and $|D_2| = \frac{2}{3}|V_2|$. Since $|D_1| = \frac{2}{3}|V_1|$, our construction of the set $D_1$ implies that no component of $G'$ is isomorphic to $P_1$ and that $V_1 = V_0$. Further, if $G'$ contains a $P_2$-component, then it has exactly three $P_2$-components each being joined via exactly one edge to a distinct vertex of $C$. In addition, there may be some, including the possibility of none, $P_3$-components in $G'$. Suppose that $P'$ is a $P_3$-component in $G'$ and $x$ is a vertex of $P'$ that is adjacent to a vertex of $C$. Then, $x$ is a leaf of $P'$ and is adjacent to exactly one vertex of $C$, since $G$ is twin-free and has no 4-cycles. Suppose, further, that both leaves of $P'$ are adjacent to (distinct) vertices of $C$. Let $u$ and $v$ be two (distinct) vertices of $C$ joined to $P'$. If exactly one of $u$ and $v$ belong to $D_1$, then by our earlier observations, $G'$ contains no $P_2$-component. But then by the way in which the set $D_1$ is constructed and recalling that $G'$ contains no $P_1$-component and that $V_1 = V_0$, we would have chosen two arbitrary vertices of $C$ to add to $D_1$. Hence, we can replace the two vertices of $C$ that currently belong to $D_1$ with the two vertices $u$ and $v$. We may therefore assume that $D_1$ is chosen to contain both $u$ and $v$. With this assumption, we can replace the two vertices of $P'$ that currently belong to $D_1$ with one of the leaves of $P'$ to produce a new LTD-set of $G$ of size $|D| - 1 = \gamma_l^T(G) - 1$, a contradiction. Therefore, $P'$ is joined via exactly one edge to a vertex of $C$. Thus, there are two possible structures of the graph $G_1$, described as follows.

Structure 1. The graph $G_1$ is obtained from the 3-cycle $C$ by adding any number of vertex-disjoint copies of $P_3$, including the possibility of zero, and joining an end from each such added path to exactly one vertex of $C$. 

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Structure 2. The graph $G_1$ is obtained from the 2-corona of the 3-cycle $C$ by adding any number of vertex-disjoint copies of $P_3$, including the possibility of zero, and joining an end from each such added path to exactly one vertex of $C$.

We note that if $G_1$ has the structure described in Structure 2, then $G_1$ is the 2-corona of some connected nontrivial graph, $H_1$ say, that contains the triangle $C$ and contains no 4-cycles. Further we note that if $x \in V(H_1)$, then either $x \in V(C)$ or $x$ is the vertex of a $P_3$-component in $G'$ that is adjacent to a vertex of $C$.

Subclaim 7.B.3 If $G = G_1$, then the graph $G$ is the 2-corona of some connected nontrivial graph that contains the triangle $C$ and contains no 4-cycles.

Proof of Subclaim 7.B.3. Suppose that $G = G_1$, i.e., $V_2 = \emptyset$. We first show that $G$ has the structure described in Structure 2. Suppose to the contrary that $G$ has the structure described in Structure 1. Then, since $G$ is twin-free, the graph $G$ is obtained from the 3-cycle $C$ by adding $k \geq 2$ vertex-disjoint copies of $P_3$ and joining an end from each such added path to exactly one vertex of $C$. Further, by the twin-freeness of $G$, at least two vertices of $C$ are joined to an end of an added path. Let $u$ and $v$ be two (distinct) vertices of $C$ are joined to ends of added paths $P_3$. The set of $2k$ vertices of degree 2 in $G$ that belong to added paths, together with the vertex $u$, forms a LTD-set of $G$ of size $\frac{2n}{3} - 1$, a contradiction. Therefore, $G$ has the structure described in Structure 2. Thus, the graph $G$ is the 2-corona of some connected nontrivial graph that contains the triangle $C$ and contains no 4-cycles. ($\diamond$)

By Subclaim 7.B.3, we may assume that $G \neq G_1$, for otherwise the desired result follows. Hence, $V_2 \neq \emptyset$. Since $|D_2| = \frac{2}{3} |V_2|$, applying the inductive hypothesis to each component of $G_2$, we deduce that each component of $G_2$ is isomorphic to a 6-cycle or is the 2-corona of some connected nontrivial graph that contains no 4-cycles.

Subclaim 7.B.4 No component of $G_2$ is isomorphic to a 6-cycle.

Proof of Subclaim 7.B.4. Suppose, to the contrary, that $G_2$ contains a component $C'$ that is isomorphic to a 6-cycle. Since $G$ is connected, there is an edge that joins a vertex $x \in V(C)$ and a vertex $y \in V(C')$. Let $C'$ be given by $y_1y_2 \ldots y_6y_1$, where $y = y_1$. If $G_1$ has the structure described in Structure 1, then we can choose $D_1$ to contain any two vertices of $C$. Hence we may assume that in this case, $D_1$ is chosen to contain the vertex $x$. If $G_1$ has the structure described in Structure 2, then $V(C) \subset D_1$. In particular, $x \in D_1$. Hence, in both cases, $x \in D_1$. Replacing the four vertices of $D$ that belong to the component $C'$ with the three vertices \{y_3, y_4, y_5\} produces a LTD-set of $G$ of size $|D| - 1 = \frac{2}{3} n - 1$, a contradiction. ($\diamond$)

By Subclaim 7.B.4, each component of $G_2$ is the 2-corona of some connected nontrivial graph that contains no 4-cycles, implying that the graph $G_2$ is the 2-corona of some graph, $H_2$ say, that contains no 4-cycles. Moreover, since $G_2$ is twin-free, each component of $H_2$ is nontrivial. Let $A_2$ and $B_2$ be the set of leaves and support vertices, respectively, in $G_2$, and let $C_2$ be the remaining vertices of $G_2$. We note that $C_2 = V(H_2) = V_2 \setminus (A_2 \cup B_2)$. 

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Subclaim 7.B.5 G₁ has the structure described in Structure 2.

Proof of Subclaim 7.B.5. Suppose, to the contrary, that G₁ has the structure described in Structure 1. Then, the graph G₁ is obtained from the 3-cycle C by adding k ≥ 0 vertex-disjoint copies of P₃ and joining an end from each such added path to exactly one vertex of C. Let V(C) = {u, v, w}. If at least two vertices of C are joined to an end of an added path, then analogously as in the proof of Subclaim 7.B.3, we produce a LTD-set of G of size 2n − 1, a contradiction. Hence, either G₁ = C₂ or G₁ is obtained from the 3-cycle C by adding k ≥ 1 vertex-disjoint copies of P₃ and joining an end from each such added path to the same vertex of C, say to u. In both cases, both v and w have degree 2 in G₁. Since G is twin-free, at least one of v and w, say v, is adjacent to a vertex of V₂. If v is adjacent to a vertex of A₂ ∪ B₂, then an analogous argument as in the last paragraph of the proof of Claim 7.A produces a LTD-set of G of size 2n − 1, a contradiction. Hence, the neighbors of v in V₂ all belong to C₂. Analogously, the neighbors of u and w in V₂, if any, all belong to C₂. The set of 2k vertices of degree 2 in G that belong to the added P₃-paths in G₁, together with the set B₂ ∪ C₂ ∪ {v}, is a LTD-set of G of size 2n − 1, a contradiction. (c)

By Subclaim 7.B.5, G₁ has the structure described in Structure 2, implying that G₁ is the 2-corona of some connected nontrivial graph, H₁ say, that contains the triangle C and contains no 4-cycles. Let A₁ and B₁ be the set of leaves and support vertices, respectively, in G₁, and let C₁ be the remaining vertices of G₁. We note that, C₁ = V(H₁) = V₁ \ (A₁ ∪ B₂).

Since G is connected, there is an edge in G joining a vertex x ∈ V₁ and a vertex y ∈ V₂. Let a₁b₁c₁ be the path in G₁ containing x, where a₁ ∈ A₁, b₁ ∈ B₁ and c₁ ∈ C₁. Similarly, let a₂b₂c₂ be the path in G₂ containing y, where a₂ ∈ A₂, b₂ ∈ B₂ and c₂ ∈ C₂. We show that x = c₁. Suppose, to the contrary, that x ∈ {a₁, b₁}. Let D* = C₁ ∪ C₂ ∪ B₁ ∪ B₂. If xy = a₁a₂, let X = (D* ∪ {a₁, a₂}) \ {b₁, b₂, c₁}. If xy ∈ {a₁b₁, a₁c₁}, let X = (D* \ {a₁}) \ {b₁, c₁}. If xy = b₁a₂, let X = (D* ∪ {a₂}) \ {b₂, c₂}. If xy = b₁b₂, let X = D* \ {c₂}. If xy = b₁c₂, let X = D* \ {c₁}. In all cases, the set X is a LTD-set of G of size |X| − 1 = 2n − 1, a contradiction. Therefore, x = c₁. Analogously, y = c₂. This is true for every edge xy joining a vertex x ∈ V₁ and a vertex y ∈ V₂, implying that G is the 2-corona of some connected nontrivial graph that contains no 4-cycles but contains a triangle. This completes the proof of Claim 7.B (c).

By Claim 7.B, the graph G contains no triangle, for otherwise the desired result follows. Hence, the girth of G is at least 5. Let C: u₀u₁ . . . uₖu₀ be a smallest cycle in G. Let G′ = G − V(C). We build a subset V₀ of vertices of G as follows (similarly to the proof of Claim 7.B). Let V₀ consist of V(C) together with all vertices that belong to a component of G′ isomorphic to P₁, P₂ or P₃. Since G is twin-free and has girth at least 5, we note that G[V₀] is twin-free. Suppose that S is a set of mutual twins of G − V₀. Since G is twin-free, all but possibly one vertex in S must be adjacent to a vertex of C. For each such set S of mutual twins of G − V₀, we select |S| − 1 vertices from S that have a neighbor in C, and add these vertices to the set V₀ to form the set V₁ (possibly, V₁ = V₀). Let T = V₁ − V₀. We note that since G has girth at least 5, the vertices in each set S of mutual twins of G − V₀ are open twins, and have degree 1 in G − V₀. Moreover they can have at most one neighbor in V₀, for otherwise they would have two or more neighbors in V(C), but this would create a shorter cycle than C, contradicting its minimality. Hence, every
vertex in $T$ has exactly one neighbor in $V_0$ (more precisely, in $V(C)$). Let $V_2 = V(G) \setminus V_1$. Let $G_1 = G[V_1]$ and if $V_2 \neq \emptyset$, let $G_2 = G[V_2]$.

**Claim 7.C.** If $G = G_1$, then $\gamma^L_t(G) \leq \frac{2}{3}n$. Further, $\gamma^L_t(G) = \frac{2}{3}n$ if and only if $G$ is isomorphic to a 6-cycle or is the 2-corona of the cycle $C$.

**Proof of Claim 7.C.** Suppose that $G = G_1$. If $T \neq \emptyset$, then this would imply that $V_2 \neq \emptyset$, contradicting our supposition that $V(G) = V_1$. Hence, $T = \emptyset$, and so $V_1 = V_0$. Thus, either $G$ is the $k$-cycle $C$ or $V(G) \neq V(C)$ and every component in $G' = G - V(C)$ is isomorphic to $P_1$, $P_2$ or $P_3$. Suppose that $G = C$. Then, $n = k$. If $k = 5$, then $G = C_5$ and $\gamma^L_t(G) = 3 < \frac{2}{3}n$. If $k = 6$, then $G = C_6$ and $\gamma^L_t(G) = \frac{2}{3}n$. If $G = C$ and $k > 6$, then, as observed in [17], $\gamma^L_t(G) = \gamma_t(G) = [n/2] + [n/4] - [n/4] \leq \frac{1}{2}n + 1 < \frac{2}{3}n$. Hence we may assume that $G \neq C$, for otherwise the desired result follows. As observed earlier, every component of $G'$ is isomorphic to $P_1$, $P_2$ or $P_3$. Among all component of $G'$, let $P'$ be chosen so that its order is maximum. We now consider the graph $F = G - V(P')$. Since the graph $G$ is twin-free, so too is the graph $F$. Applying the inductive hypothesis to the graph $F$, $\gamma^L_t(F) \leq \frac{2}{3}|V(F)|$. Further, $\gamma^L_t(F) = \frac{2}{3}|V(F)|$ if and only if $F$ is isomorphic to a 6-cycle, $C_6$, or is the 2-corona of some connected nontrivial graph that contains no 4-cycles.

**Subclaim 7.C.1** If $\gamma^L_t(F) < \frac{2}{3}|V(F)|$, then the desired result of Claim 7.C holds.

**Proof of Subclaim 7.C.1.** Suppose that $\gamma^L_t(F) < \frac{2}{3}|V(F)|$. If $P' = P_3$, consider a minimum LTD-set $D_F$ of $F$, and note that $D_F$ together with the two vertices of $P'$ that have degree at least 2 in $G$, forms a LTD-set of $G$ of size strictly less than $\frac{2}{3}n$. Hence, we may assume that $P'$ is isomorphic to $P_1$ or $P_2$. By our choice of $P'$, this implies that every component of $G'$ is isomorphic to $P_1$ or $P_2$. We now construct a set $Q$ with $V(P') \subset Q$. Renaming vertices of $C$, if necessary, we may assume that $u_1$ is the vertex of $C$ adjacent to a vertex of $P'$. We initially define $Q$ to contain both $u_1$ and $u_2$, as well as all vertices that belong to a $P_1$- or $P_2$-component of $G - \{u_1, u_2\}$. If $u_3$ has degree 2 in $G$ and $u_4$ has a leaf-neighbor in $G$, say $u_4'$, then $u_3$ and $u_4'$ are (open) twins in $G - Q$. In this case, we add the vertex $u_3$ to the set $Q$. Analogously, if $u_0$ has degree 2 in $G$ and $u_{k-1}$ has a leaf-neighbor in $G$, then we add the vertex $u_0$ to the set $Q$. By construction, the resulting graph $G - Q$ is twin-free, unless we have the special case when $k = 5$, both $u_0$ and $u_3$ have degree 2 in $G$, and $u_4$ has degree 3 in $G$ with a leaf-neighbor in $G$. In this case, graph $G$ is determined and the set $\{u_0, u_1, u_2, u_4\}$ together with the vertices of every $P_2$-component in $G'$ that have a neighbor in $V(C)$ forms a LTD-set of $G$ of size strictly less that $\frac{2}{3}n$. Hence, we may assume that the graph $F' = G - Q$ is twin-free.

Applying the inductive hypothesis to the graph $F'$ there exists a LTD-set, $D'_F$, of $F'$ of size at most $\frac{2}{3}|V(F')|$. Although $G[Q]$ is not necessarily twin-free, by similar arguments as before we can easily choose a set $D_Q$ of size at most $\frac{2}{3}|Q|$ such that $D'_F \cup D_Q$ is a LTD-set of $G$ of size at most $\frac{2}{3}n$. Moreover, if $|D'_F \cup D_Q| = \frac{2}{3}n$, then $F'$ must be either the 2-corona of the path $G[V(C) \setminus Q]$, or $F' = P_6$. Furthermore, $|Q| = 6$ and $G[Q]$ is either a $P_6$, a $P_4$ with an additional leaf attached to each central vertex, or a $P_3$ with an additional leaf forming a twin with another leaf. If $F' = P_6$ or $G[Q] \neq P_6$, we can readily find a LTD-set of $G$ strictly smaller than $\frac{2}{3}n$. 

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Otherwise, \( G \) is the 2-corona of \( C \), and we are done. This completes the proof of Subclaim 7.C.1.

By Subclaim 7.C.1, we may assume that \( \gamma_t^t(F) = \frac{2}{3}|V(F)| \), for otherwise the desired result follows. If \( F = C_6 \), then \( \gamma_t^t(G) < \frac{2}{3}n \), irrespective of whether \( P' \) is isomorphic to \( P_1 \), \( P_2 \) or \( P_3 \). Hence, we may assume that \( F \neq C_6 \), for otherwise the desired result follows. Thus, \( F \) is the 2-corona of some connected nontrivial graph, \( F' \) say, that contains no 4-cycles. Let \( A_F \) and \( B_F \) be the set of leaves and support vertices, respectively, in \( F \), and let \( C_F \) be the remaining vertices of \( F \). Thus, \( F' = F[C_F] \). If \( P' \) is not isomorphic to \( P_3 \), or if \( P' \) is isomorphic to \( P_3 \) and contains a vertex adjacent to \( A_F \) or \( B_F \), then it is a simple exercise to see that \( \gamma_t^t(G) < \frac{2}{3}n \). Further, if \( P' \) is isomorphic to \( P_3 \) and contains exactly one vertex adjacent to vertices of \( C_F \), then \( \gamma_t^t(G) = \frac{2}{3}n \) and \( G \) is the 2-corona of some connected nontrivial graph that contains no 4-cycles. This completes the proof of Claim 7.C. (c)

By Claim 7.C, we may assume that \( G \neq G_1 \), i.e., \( V_2 \neq \emptyset \) and \( G = G_2 \). An identical proof as in the proof of Subclaim 7.B.1 shows that \( G_2 \) is twin-free. Let \( D_2 \) be a minimum LTD-set of \( G_2 \). Applying the induction hypothesis to each component of \( G_2 \), the set \( D_2 \) satisfies \( |D_2| \leq \frac{2}{3}|V_2| \). Further, if \( |D_2| = \frac{2}{3}|V_2| \), then each component of \( G_2 \) is isomorphic to a 6-cycle or is the 2-corona of some connected nontrivial graph that contains no 4-cycles.

Recall that \( G[V_0] \) is twin-free. We now build sets \( V'_t \) and \( T' \) such that \( V_0 \subseteq V'_t \subseteq V_1 = V_0 \cup T \) and \( T' \subseteq T \), as follows. Initially, we let \( V'_1 = V_0 \) and \( T' = T \). We consider the vertices of \( T \) sequentially. Let \( t \) be a vertex in \( T \), and recall that \( t \) has exactly one neighbor, \( u_t \) say, in \( V_0 \), and such a neighbor belongs to \( V(C) \). If \( u_t \) has no leaf-neighbor in \( G[V'_1] \), we add \( t \) to \( V'_1 \) and remove \( t \) from \( T' \). We iterate this process until all vertices of \( T \) have been considered. Let \( G'_1 \) be the resulting graph \( G[V'_1] \). This process yields a new partition of \( V(G) \) into sets \( V_2 \), \( V'_1 \) and \( T' \). Since \( G[V_0] \) is twin-free, by construction of the set \( V'_1 \), the graph \( G'_1 \) is also twin-free. Since \( V_2 \neq \emptyset \), the order of \( G'_1 \) is less than \( n \) and we can therefore apply the induction hypothesis to the connected twin-free graph \( G'_1 \). Let \( D'_1 \) be a minimum LTD-set of \( G'_1 \). By the induction hypothesis, the set \( D'_1 \) satisfies \( |D'_1| \leq \frac{2}{3}|V'_1| \). Further, if \( |D'_1| = \frac{2}{3}|V'_1| \), then \( G'_1 \) is isomorphic to a 6-cycle or is the 2-corona of some connected nontrivial graph that contains no 4-cycles.

We claim that \( D = D'_1 \cup D_2 \) is a LTD-set of \( G \). By the construction of the set \( T' \), for each vertex \( t \) of \( T' \), there is a twin, \( t' \) say, of \( t \) in \( G - V_0 \) that belongs to \( V_2 \) and has degree 1 in \( G_2 \). The common neighbor of \( t \) and \( t' \) in \( V_2 \) must belong to \( D_2 \). Further, since \( t \) has not been removed from \( T' \) during the construction of \( T' \), the vertex \( t \) has a neighbor \( u_t \) in \( V(C) \) which has a leaf-neighbor in \( G'_1 \), implying that the vertex \( u_t \) belongs to \( D'_1 \). Hence, \( t \) is dominated by two vertices of \( D'_1 \cup D_2 \) and is therefore located by \( D \), for otherwise we would have a 4-cycle in \( G \). Thus, every vertex of \( T' \) is located by \( D \). Since \( D'_1 \) and \( D_2 \) are TD-sets of \( G'_1 \) and \( G_2 \), respectively, and since every vertex in \( T' \) is dominated by \( D \), the set \( D \) is a TD-set of \( G \). Suppose, to the contrary, that \( D \) is not locating. Then there is a pair of vertices, \( u \) and \( v \), that is not located by \( D \). As observed earlier, neither \( u \) nor \( v \) belong to \( T' \). Since \( D_2 \) is locating in \( G_2 \), we note that \( (u, v) \notin V_2 \times V_2 \). Analogously, since \( D'_1 \) is locating in \( G'_1 \), we note that \( (u, v) \notin V'_1 \times V'_1 \). If \( (u, v) \in V'_1 \times V_2 \), then \( u \) is dominated by a vertex of \( D'_1 \) and \( v \) is dominated by a vertex of \( D_2 \). Hence, \( u \) and \( v \) must both be dominated by these two vertices. But then these four vertices

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would form a 4-cycle, a contradiction. Hence, \((u, v) \notin V_1' \times V_2\). Analogously, \((u, v) \notin V_2 \times V_2'\). This contradicts our supposition that \(u\) and \(v\) are not located by \(D\). Therefore, \(D\) is a LTD-set of \(G\), and so

\[
\gamma^L_t(G) \leq |D| = |D'_1| + |D_2| \leq \frac{2}{3}|V_1'| + \frac{2}{3}|V_2| \leq \frac{2}{3}|V_1| + \frac{2}{3}|V_2| = \frac{2}{3}n. \tag{3}
\]

This establishes the desired upper bound. Suppose next that \(\gamma^L_t(G) = \frac{2}{3}n\). Then we must have equality throughout the Inequality Chain (3). In particular, \(|D'_1| = \frac{2}{3}|V_1'| = \frac{2}{3}|V_1|\) and \(|D_2| = \frac{2}{3}|V_2|\). This in turn implies that \(T' = \emptyset\). Using an analogous proof as in the proof when equality holds in the Inequality Chain (2) in the proof of Claim 7.B, the graph \(G\) can be shown to be the 2-corona of some connected nontrivial graph that contains no 4-cycles. Since the proof is very similar, we omit the details. This completes the proof of Theorem 7.

\[\square\]

5 Graphs with given minimum degree

We now discuss the special case of graphs of given minimum degree.

5.1 Minimum degree two

If we forbid six graphs of small orders (at most 10), then it is known (see [18]) that if \(G\) is a connected graph of order \(n\) with \(\delta(G) \geq 2\), then \(\gamma_t(G) \leq 4n/7\). However, for graphs with minimum degree 2, the location-total domination number can be much larger than the total domination number. For example, let \(G\) be the graph obtained by taking the disjoint union of \(k \geq 2\) 5-cycles, adding a new vertex \(v\) and joining \(v\) with an edge to exactly one vertex from each 5-cycle. The resulting twin-free graph \(G\) has order \(n = 5k + 1\), minimum degree \(\delta(G) = 2\) and satisfies \(\gamma^L_t(G) = 3k = \frac{3}{5}(n - 1)\) and \(\gamma_t(G) = 2(k + 1) = \frac{2}{5}(n - 1) + 2\).

We believe that Conjecture [11] can be strengthened for graphs with minimum degree at least 2 and pose the following question.

**Question 8.** Is it true that every twin-free graph with order \(n\), no isolated vertices and minimum degree 2 satisfies \(\gamma^L_t(G) \leq \frac{3n}{5}\)?

If Question 8 is true, then the bound is asymptotically tight by the examples given earlier.

5.2 Large minimum degree

The following is an upper bound on \(\gamma_t(G)\) according to the minimum degree \(\delta\) of \(G\).

**Theorem 9** (Henning, Yeo [21]). If \(G\) is a graph with minimum degree \(\delta \geq 1\) and order \(n\), then

\[
\gamma_t(G) \leq \left(1 + \ln \frac{\delta}{\delta}\right)n.
\]
Using Observation 5, we obtain the following corollary of the results in \[12, 13, 14\] and Theorem 9.

**Corollary 10.** Let $G$ be a twin-free graph of minimum degree $\delta \geq 1$. We have

$$\gamma^L_t(G) \leq \left(\frac{2}{3} + \frac{1 + \ln \delta}{\delta}\right) n.$$  

Moreover, if $G$ is a bipartite, co-bipartite, split or line graph, then

$$\gamma^L_t(G) \leq \left(\frac{1}{2} + \frac{1 + \ln \delta}{\delta}\right) n.$$  

If Conjecture 2 holds, we always have $\gamma^L_t(G) \leq \left(\frac{1}{2} + \frac{1 + \ln \delta}{\delta}\right) n$.

It follows from Corollary 10 that Conjecture 1 asymptotically holds for large minimum degree, in the sense that $\lim_{\delta \to \infty} \left(\frac{2}{3} + \frac{1 + \ln \delta}{\delta}\right) = \frac{2}{3}$. Moreover, Conjecture 1 holds for bipartite, co-bipartite, split and line graphs with minimum degree $\delta \geq 26$. Finally, if Conjecture 2 holds, then Conjecture 1 holds whenever $\delta \geq 26$.

6 Conclusion

A classic result in total domination theory in graphs is that every connected graph of order $n \geq 3$ has a total dominating set of size at most $\frac{2}{3}n$. In this paper, we conjecture that every twin-free graph of order $n$ with no isolated vertex has a locating-total dominating set of size at most $\frac{2}{3}n$ and we prove our conjecture for graphs with no 4-cycles. We also prove that our conjecture, namely Conjecture 1 holds asymptotically for large minimum degree. Since Conjecture 2 was proved for bipartite graphs \[14\] and cubic graphs \[11\], can we prove Conjecture 1 for these classes as well?

References

[1] M. Blidia, M. Chellali, F. Maffray, J. Moncel, and A. Semri. Locating-domination and identifying codes in trees, Australas. J. Combin. 39 (2007), 219–232.

[2] M. Blidia and W. Dali. A characterization of locating-total domination edge critical graphs, Discuss. Math. Graph Theory 31(1) (2011), 197–202.

[3] M. Blidia, O. Favaron, and R. Lounes. Locating-domination, 2-domination and independence in trees, Australas. J. Combin. 42 (2008), 309–319.

[4] R. C. Brigham, J. R. Carrington, and R. P. Vitray. Connected graphs with maximum total domination number. J. Combin. Comput. Comb. Math. 34 (2000), 81–96.
[5] M. Chellali. On locating and differentiating-total domination in trees, *Discuss. Math. Graph Theory* **28**(3) (2008), 383–392.

[6] M. Chellali and N. Jafari Rad. Locating-total domination critical graphs, *Australas. J. Combin.* **45** (2009), 227–234.

[7] X.G. Chen and M.Y. Sohn. Bounds on the locating-total domination number of a tree, *Discrete Appl. Math.* **159** (2011), 769–773.

[8] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi. Total domination in graphs. *Networks* **10** (1980), 211–219.

[9] C. J. Colbourn, P. J. Slater, and L. K. Stewart. Locating-dominating sets in series-parallel networks. *Congr. Numer.* **56** (1987), 135–162.

[10] A. Finbow and B. L. Hartnell. On locating dominating sets and well-covered graphs. *Congr. Numer.* **65** (1988), 191–200.

[11] F. Foucaud and M. A. Henning. Location-domination and matching in cubic graphs, manuscript. [http://arxiv.org/abs/1412.2865](http://arxiv.org/abs/1412.2865)

[12] F. Foucaud and M. A. Henning. Locating-dominating sets in line graphs, manuscript.

[13] F. Foucaud, M. A. Henning, C. Löwenstein, and T. Sasse. Locating-dominating sets in twin-free graphs, manuscript. [http://arxiv.org/abs/1412.2376](http://arxiv.org/abs/1412.2376)

[14] D. Garijo, A. González and A. Márquez. The difference between the metric dimension and the determining number of a graph. *Applied Mathematics and Computation* **249** (2014), 487–501.

[15] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[16] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc. New York, 1998.

[17] T. W. Haynes, M. A. Henning, and J. Howard. Locating and total dominating sets in trees. *Discrete Appl. Math.* **154** (2006), 1293–1300.

[18] M. A. Henning. Graphs with large total domination number. *J. Graph Theory* **35**(1) (2000), 21–45.

[19] M. A. Henning and C. Löwenstein. Locating-total domination in claw-free cubic graphs. *Discrete Math.* **312**(21) (2012), 3107–3116.

[20] M. A. Henning and N. J. Rad. Locating-total domination in graphs, *Discrete Appl. Math.* **160** (2012), 1986–1993.

[21] M. A. Henning and A. Yeo. A transition from total domination in graphs to transversals in hypergraphs. *Quaestiones Math.* **30** (2007), 417–436.
[22] M. A. Henning and A. Yeo, *Total domination in graphs*, Springer-Verlag, 2013.

[23] O. Ore, *Theory of graphs*. *Amer. Math. Soc. Transl.* **38** (Amer. Math. Soc., Providence, RI, 1962), 206–212.

[24] D. F. Rall and P. J. Slater. On location-domination numbers for certain classes of graphs. *Congr. Numer.* **45** (1984), 97–106.

[25] P. J. Slater, Dominating and location in acyclic graphs. *Networks* **17** (1987), 55–64.

[26] P. J. Slater, Dominating and reference sets in graphs. *J. Math. Phys. Sci.* **22** (1988), 445–455.

[27] P. J. Slater. Locating dominating sets and locating-dominating sets. In Y. Alavi and A. Schwenk, editors, *Graph Theory, Combinatorics, and Applications, Proc. Seventh Quad. Internat. Conf. on the Theory and Applications of Graphs*, pages 1073–1079. John Wiley & Sons, Inc., 1995.