The digits of $n + t$

Lukas Spiegelhofer∗ and Michael Wallner†
Vienna University of Technology, Austria

Abstract
The binary sum-of-digits function $s$ counts the number of 1s in the base-2 expansion of a nonnegative integer. T. W. Cusick defined the asymptotic density
\[ c_t = \lim_{N \to \infty} \frac{1}{N} \# \{ 0 \leq n < N : s(n + t) \geq s(n) \} \]
for integers $t \geq 0$ and conjectured that $c_t > 1/2$ for all $t$. We prove that indeed $c_t > 1/2$ if the binary expansion of $t$ contains at least $K$ blocks of contiguous 1s, where $K$ is an absolute, effective constant.

1 Introduction and main result
The behaviour of the binary expansion of an integer under addition of a constant is not fully understood. This elementary problem is concerned with the occurrence of carries in an addition $n + t$; by carry propagation, these occurrences can interact, turning an explicit description of the situation into a complicated case distinction.

Studying the sum-of-digits function of $n$, $t$, and $n + t$ synchronously captures the difficulty of studying carry propagation. A complete characterization of the occurring cases is out of sight, and the present paper provides an approximation to the problem by making progress on Cusick’s conjecture on the binary sum-of-digits function.

We are interested in the asymptotic densities
\[ \delta(j, t) = \text{dens} \{ n \in \mathbb{N} : s(n + t) - s(n) = j \}, \]
where $j \in \mathbb{Z}$ and dens $A$ is the asymptotic density of a set $A \subseteq \mathbb{N}$, which exists in our case (Béssineau [1]). Cusick’s conjecture (private communication, 2011) states that for all $t \geq 0$,
\[ c_t > 1/2, \quad (1) \]
where
\[ c_t = \text{dens} \{ n \in \mathbb{N} : s(n + t) \geq s(n) \} = \delta(0, t) + \delta(1, t) + \cdots. \]

∗The author was supported by the FWF project F5502-N26, which is a part of the Special Research Program “Quasi Monte Carlo methods: Theory and Applications”.
†The author was supported by the Erwin Schrödinger Fellowship of the Austrian Science Fund (FWF): J 4162-N35.

2010 Mathematics Subject Classification. Primary: 11A63, 05A20; Secondary: 05A16,11T71
Key words and phrases. Cusick conjecture, Hamming weight, sum of digits
1 INTRODUCTION AND MAIN RESULT

We note the partial results [4, 6–8, 12, 13] on Cusick’s conjecture, among which we find an almost-all result by Drmota, Kauers, and the first author [4] and a central limit-type result by Emme and Hubert [6], moreover a lower bound due to the first author [13].

This conjecture arose while Cusick was working on the related Tu–Deng conjecture [15, 16] relevant in cryptography: assume that $k$ is a positive integer and $t \in \{1, \ldots, 2^k - 2\}$. Then this conjecture states that

$$\left| \left\{ (a, b) \in \{0, \ldots, 2^k - 2\}^2 : a + b \equiv t \mod 2^k - 1, s(a) + s(b) < k \right\} \right| \leq 2^{k-1}.$$  

While the full conjecture is open, partial results are known [2, 3, 9, 10, 14, 15]. The authors [14] proved an almost-all result on this conjecture, using the method of proof set forward in the paper [4] by Drmota, Kauers, and the first author. Moreover, we proved in that paper [14] that Tu and Deng’s conjecture implies Cusick’s conjecture.

We return to Cusick’s conjecture. The values $\delta(k, t)$ satisfy the following recurrence [4, 12, 13]: we have

$$\delta(j, 1) = \begin{cases} 0, & j > 1; \\ 2^{j-2}, & j \leq 1, \end{cases}$$

and for $t \geq 0$,

$$\delta(j, 2t) = \delta(j, t),$$

$$\delta(j, 2t + 1) = \frac{1}{2} \delta(j - 1, t) + \frac{1}{2} \delta(j + 1, t + 1). \quad (2)$$

It can be shown by induction that the sets defining $\delta(j, t)$ are finite unions of arithmetic progressions.

Using (2), we verified (1) by numerical computation for all $t \leq 2^{30}$. By considering the asymptotic analysis of a diagonal of a trivariate generating function and Chebyshev’s inequality, Drmota, Kauers, and the first named author [4] obtained the following result, giving an almost all-solution of Cusick’s conjecture: for all $\varepsilon > 0$, we have

$$\left| \{ t < T : 1/2 < c_t < 1/2 + \varepsilon \} \right| = T - O \left( \frac{T}{\log T} \right).$$

In the present paper, we prove the following near-solution to Cusick’s conjecture.

**Theorem 1.1.** There exists an absolute, effective constant $K$ with the following property. If the natural number $t$ has at least $K$ blocks of $1$s in its binary expansion, then $c_t > 1/2$.

**Notation.** In this paper, $0 \in \mathbb{N}$. We will use Big O notation, employing the symbol $O$. We let $e(x)$ denote $\lfloor 2^\pi x \rfloor$ for real $x$, and $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$ is the distance to the nearest integer. In our calculations, the number $\pi$ will often appear with a factor 2 in front of it. Therefore we use the abbreviation $\tau = 2\pi$.

We consider blocks of 0s or 1s in the binary expansion of an integer $t \in \mathbb{N}$. Writing “block of 1s of length $\nu$ in $t$”, we always mean a maximal subsequence $\varepsilon_\mu = \varepsilon_{\mu+1} = \cdots = \varepsilon_{\mu+\nu-1} = 1$ (where maximal means that $\varepsilon_{\mu+\nu} = 0$ and either $\mu = 0$ or $\varepsilon_{\mu-1} = 0$). “Blocks of 0s” are defined analogously; the number of blocks in $t$ is the sum of these two numbers.

All constants in this paper are absolute and effective. The letter $C$ is often used for constants; occurrences of $C$ at different positions need not designate the same value.

The remainder of this paper is dedicated to the proof of Theorem 1.1.
2 Proof of the main theorem

We begin with the definition of the characteristic function of the probability distribution given by the densities $\delta(j, t)$: let
\[ \gamma_t(\vartheta) = \sum_{j \in \mathbb{Z}} \delta(j, t) e^{j \vartheta}. \]  
(3)

Since $\delta(\cdot, t)$ is summable, orthogonality relations imply
\[ \delta(j, t) = \int_{-1/2}^{1/2} \gamma_t(\vartheta) e^{-j \vartheta} d\vartheta. \]  
(4)

The recurrence (2) carries over to characteristic functions: for all $t \geq 0$, we have
\[ \gamma_{2t}(\vartheta) = \gamma_t(\vartheta), \]
\[ \gamma_{2t+1}(\vartheta) = \frac{e(\vartheta)}{2} \gamma_t(\vartheta) + \frac{e(-\vartheta)}{2} \gamma_{t+1}(\vartheta), \]  
(5)
and in particular
\[ \gamma_1(\vartheta) = \frac{e(\vartheta)}{2 - e(-\vartheta)}. \]  
(6)

For all $t \geq 1$, we have
\[ \gamma_t(\vartheta) = \omega_t(\vartheta) \gamma_1(\vartheta), \]
where $\omega_t$ is a trigonometric polynomial such that $\omega_t(0) = 1$. These polynomials satisfy the same recurrence relation as $\gamma_t$. In particular, noting also that the denominator $2 - e(-\vartheta)$ is nonzero near $\vartheta = 0$, we have $\text{Re} \gamma_t(\vartheta) > 0$ for $\vartheta$ in a certain disk
\[ D_t = \{ \vartheta \in \mathbb{C} : |\vartheta| < \kappa \}, \]
where $\kappa = \kappa(t) > 0$. It follows that
\[ d_t = -\log \circ (\gamma_t|_{D_t}) \]  
(7)

is analytic in $D_t$ and therefore there exist complex numbers $A_j(t)$ for $j \in \mathbb{N}$ such that
\[ \gamma_t(\vartheta) = \exp \left( -\sum_{j \geq 0} A_j(t)(\tau \vartheta)^j \right) \]
for all $\vartheta \in D_t$. For $t = 0$, we have $A_j(t) = 0$ for all $j \geq 0$, as $\delta(k, 0) = 1$ if $k = 0$ and $\delta(k, 0) = 0$ otherwise. The recurrence (5) shows that
\[ \gamma_t(\vartheta) = 1 + O(\vartheta^2) \]
at 0, which implies $A_0(t) = A_1(t) = 0$. Let us write
\[ a_j = A_j(t), \quad b_j = A_j(t+1), \quad \text{and} \quad c_j = A_j(2t+1). \]  
(8)

We express the coefficients $c_j$ as functions of the coefficients $a_j$ and $b_j$. By the recurrence (5) for $\gamma_t(\vartheta)$, these quantities are related via the fundamental identity
\[ \exp(-c_2(\tau \vartheta)^2 - c_3(\tau \vartheta)^3 - \cdots) = -\frac{1}{2} \exp \left( i \tau \vartheta - a_2(\tau \vartheta)^2 - a_3(\tau \vartheta)^3 - \cdots \right) \]
\[ + \frac{1}{2} \exp(-i \tau \vartheta - b_2(\tau \vartheta)^2 - b_3(\tau \vartheta)^3 - \cdots), \]  
(9)
valid for $\vartheta \in D = D_t \cap D_{t+1} \cap D_{2t+1}$. From this equation, we derive the following lemma by comparing coefficients of the appearing analytic functions.
Lemma 2.1. Assume that \( t \geq 0 \) and let \( a_j, b_j, \) and \( c_j \) be defined by (8). We have

\[
\begin{align*}
c_2 &= \frac{a_2 + b_2}{2} + \frac{1}{2}, \\
c_3 &= \frac{a_3 + b_3}{2} + i \frac{a_2 - b_2}{2}; \\
c_4 &= \frac{a_4 + b_4}{2} + i \frac{a_3 - b_3}{2} - \frac{(a_2 - b_2)^2}{8} + \frac{1}{12}, \\
c_5 &= \frac{a_5 + b_5}{2} + i \frac{a_4 - b_4}{2} - \frac{(a_2 - b_2)(a_3 - b_3)}{4} + i \frac{a_2 - b_2}{6}.
\end{align*}
\]

In particular,

\[
A_2(1) = 1, \quad A_3(1) = -i, \quad A_4(1) = -\frac{13}{12}, \quad A_5(1) = \frac{5i}{4}.
\]

Proof. Extracting the coefficient of \( \vartheta^2 \) in (9), we obtain

\[
c_2 = -\frac{1}{2\tau^2} \left[ \vartheta^2 \right] \left( 1 + i \tau \vartheta - a_2(\tau \vartheta)^2 + \frac{1}{2} (i \tau \vartheta - a_2(\tau \vartheta)^2)^2 \right) + 1 - i \tau \vartheta - b_2(\tau \vartheta)^2 + \frac{1}{2} (-i \tau \vartheta - b_2(\tau \vartheta)^2)^2 \right) = \frac{a_2 + b_2}{2} + \frac{1}{2},
\]

which is (10). The sequence \( \tau^{-2}[\vartheta^2]d_4(\vartheta) \) appears in another context too: it is the discrepancy of the Van der Corput sequence [5, 11]. Our calculation gives another proof of the fact [6, 13] that this sequence describes the second moment of the probability distribution \( j \mapsto \delta(j,t) \).

Similarly, we handle the higher coefficients. We proceed to \( \left[ \vartheta^3 \right]d_4(\vartheta) \). From (9) we obtain by collecting the cubic terms

\[
c_3 = -\frac{1}{2\tau^3} \left( -a_3 \tau^3 - \frac{1}{2} i a_2 \tau^3 + \frac{1}{6} (i \tau)^3 - b_3 \tau^3 + \frac{1}{2} i a_2 \tau^3 - \frac{1}{6} (i \tau)^3 \right) = \frac{a_3 + b_3}{2} + i \frac{a_2 - b_2}{2},
\]

which is (11). For the next coefficient \( \left[ \vartheta^4 \right]d_4(\vartheta) \), we have to take the quadratic term of the exponential on the left hand side of (9) into account. This yields, inserting the recurrence for \( c_2 \) obtained before,

\[
\left[ \vartheta^4 \right] \exp(-c_2 \tau^2 \vartheta^2 - c_3 \tau^3 \vartheta^3 - c_4 \tau^4 \vartheta^4) = \tau^4 \left( -c_4 + \frac{c_2^2}{2} \right) = \tau^4 \left( -c_4 + \frac{1}{8} + \frac{a_2 + b_2}{4} + \frac{(a_2 + b_2)^2}{8} \right).
\]

The right hand side of (9) yields by collecting the quartic terms

\[
\frac{\tau^4}{2} \left( -a_4 + \frac{1}{2} (-2i a_3 + a_2^2) + \frac{1}{6} (-3a_2 i^2) + \frac{1}{24} - b_4 + \frac{1}{2} (2i b_3 + b_2^2) + \frac{1}{6} (-3b_2 i^2) + \frac{1}{24} \right) = -\tau^4 \left( \frac{a_4 + b_4}{2} + i \frac{a_3 - b_3}{2} - \frac{a_2 + b_2}{4} - \frac{a_2 + b_2}{4} - \frac{1}{24} \right).
\]

Equation (12) follows. Finally, we need the quintic terms. The left hand side of (9) yields

\[
\left[ \vartheta^5 \right] \exp(-c_2 \tau^2 \vartheta^2 - c_3 \tau^3 \vartheta^3 - c_4 \tau^4 \vartheta^4 - c_5 \tau^5 \vartheta^5) = \tau^5 \left( -c_5 + c_2 c_3 \right) = \tau^5 \left( -c_5 + \frac{a_2 + b_2}{2} + \frac{1}{2} \left( \frac{a_3 + b_3}{2} + i \frac{a_2 - b_2}{2} \right) \right),
\]
while the right hand side of (9) yields
\[
\frac{\tau^5}{2} \left( -a_5 + \frac{1}{2} (2a_2a_3 - 2i a_4) + \frac{1}{6} (-3a_3i^2 + 3i a_2^2) + \frac{1}{24} (-4a_2 i^3) + \frac{i^5}{120} \right)
\]
\[
- b_5 + \frac{1}{2} (2b_2b_3 + 2i b_4) + \frac{1}{6} (-3b_3i^2 - 3i b_2^2) + \frac{1}{24} (4b_2 i^3) - \frac{i^5}{120}
\]
\[
= -\tau^5 \left( \frac{a_5 + b_5}{2} + \frac{a_4 - b_4}{2} - \frac{a_3 + b_3}{2} - \frac{a_2 a_3 + b_2 b_3}{2} - \frac{i a_2^2 - b_2^2}{4} - \frac{i}{6} (a_2 - b_2)^2 \right),
\]
which implies (13) after a short calculation. Finally, we compute the values \(A_2(1), \ldots, A_5(1)\) by substituting \(t = 0\) in (10)–(13).
\[\square\]

By the same method of proof (or alternatively, by concatenating the power series for \(\log\) and \(\gamma_t(\vartheta)\)) this list can clearly be prolonged indefinitely. For the proof of our main theorem however, we only need the terms up to \(A_5\). Note the important property that lower coefficients always appear as differences; we believe that this behaviour continues for higher coefficients.

In the following, we are not concerned with the original definition of \(A_j\), involving a disk \(D_t\) with potentially small radius. Instead, we only work with the recurrences

\[
A_2(2t) = A_2(t), \quad A_2(2t + 1) = \frac{A_2(t) + A_2(t + 1)}{2} + \frac{1}{2};
\]

\[
A_3(2t) = A_3(t), \quad A_3(2t + 1) = \frac{A_3(t) + A_3(t + 1)}{2} + \frac{1}{2} \left( A_2(t) - A_2(t + 1) \right);
\]

\[
A_4(2t) = A_4(t), \quad A_4(2t + 1) = \frac{A_4(t) + A_4(t + 1)}{2} + \frac{1}{2} \left( A_3(t) - A_3(t + 1) \right)
\]
\[
- \frac{(A_2(t) - A_2(t + 1))^2}{8} + \frac{1}{12};
\]

\[
A_5(2t) = A_5(t), \quad A_5(2t + 1) = \frac{A_5(t) + A_5(t + 1)}{2} + \frac{1}{2} \left( A_4(t) - A_4(t + 1) \right)
\]
\[
- \frac{(A_2(t) - A_2(t + 1))(A_3(t) - A_3(t + 1))}{4} + \frac{A_2(t) - A_2(t + 1)}{6},
\]
valid for all integers \(t \geq 0\).

### 2.1 An approximation to the characteristic function

Let us define the following approximation to \(\gamma_t\). Set
\[
\gamma'_t(\vartheta) = \exp \left( - \sum_{2 \leq j \leq 5} A_j(t)(\tau \vartheta)^j \right).
\]

We are going to replace \(\gamma_t\) by \(\gamma'_t\), and for this purpose we have to bound the difference
\[
\tilde{\gamma}_t(\vartheta) = \gamma_t(\vartheta) - \gamma'_t(\vartheta).
\]

Clearly, we have \(\tilde{\gamma}_{2t}(\vartheta) = \tilde{\gamma}_t(\vartheta)\). Moreover,
\[
\tilde{\gamma}_{2t+1} = \frac{e(\vartheta)}{2} (\tilde{\gamma}_t(\vartheta) + \gamma'_t(\vartheta)) + \frac{e(-\vartheta)}{2} (\tilde{\gamma}_{t+1} + \gamma'_{t+1}(\vartheta)) - \gamma'_{2t+1}(\vartheta)
\]
\[
= \frac{e(\vartheta)}{2} \gamma_t(\vartheta) + \frac{e(-\vartheta)}{2} \tilde{\gamma}_{t+1}(\vartheta) + \xi_t(\vartheta),
\]
\[
\xi_t(\vartheta) = \gamma'_t(\vartheta) - \gamma'_{2t+1}(\vartheta),
\]

where \(\xi_t(\vartheta)\) is a bounded expression.
where
\[ \xi_t(\vartheta) = \frac{e(\vartheta)}{2} \gamma_t'(\vartheta) + \frac{e(-\vartheta)}{2} \gamma_{t+1}'(\vartheta) - \gamma_{2t+1}'(\vartheta). \] (17)

We prove the following rough bounds on the numbers \( A_j \) and their differences.

**Lemma 2.2.** We have

\[
\begin{align*}
|A_2(t + 1) - A_2(t)| &\leq 1; \tag{18} \\
|A_3(t + 1) - A_3(t)| &\leq 1; \tag{19} \\
|A_4(t + 1) - A_4(t)| &\leq 17/12; \tag{20} \\
|A_5(t + 1) - A_5(t)| &\leq 9/4. \tag{21}
\end{align*}
\]

**Proof.** We prove these statements by induction, inserting the recurrence (14). We have

\[
A_2(2t + 1) - A_2(2t) = \frac{A_2(t) + A_2(t + 1) + 1}{2} - A_2(t) = \frac{A_2(t + 1) - A_2(t) + 1}{2}
\]
and

\[
A_2(2t + 2) - A_2(2t + 1) = A_2(t + 1) - \frac{A_2(t) + A_2(t + 1) + 1}{2} = \frac{A_2(t + 1) - A_2(t) - 1}{2}
\]
and by induction, the first statement is an easy consequence. We prove the second inequality.

\[
A_3(2t + 1) - A_3(2t) = \frac{A_3(t + 1) - A_3(t)}{2} + \frac{i A_2(t) - A_2(t + 1)}{2},
\]
and similarly

\[
A_3(2t + 2) - A_3(2t + 1) = \frac{A_3(t + 1) - A_3(t)}{2} + \frac{i A_2(t + 1) - A_2(t)}{2}
\]
and using the first part and induction, the claim follows. Concerning (20),

\[
A_4(2t + 1) - A_4(2t) = \frac{A_4(t + 1) - A_4(t) + i A_3(t) - A_3(t + 1) + (A_2(t) - A_2(t + 1))^2}{2} - \frac{1}{12}, \tag{22}
\]
and the last three terms sum up to a value bounded by 17/24 in absolute value, using the first two estimates. An analogous statement for \( A_4(2t + 2) - A_4(2t + 1) \) holds. This implies the third line. The proof of the last line is completely analogous. \( \square \)

**Corollary 2.3.** There exists a constant \( C \) such that for all \( t \) having \( M \) blocks of 1s we have

\[ |A_2(t)| \leq CM, \quad |A_3(t)| \leq CM, \quad |A_4(t)| \leq CM, \quad |A_5(t)| \leq CM. \]

**Proof.** We proceed by induction on the number of blocks of 1s in \( t \). Appending \( 0^r \) to the binary expansion, there is nothing to show by the identity \( A_j(2^r) = A_j(t) \). We append a block of 1s of length \( r \): we have the trivial identity

\[
A_j(2^r t + 2^r - 1) = A_j(t) + (A_j(2^r t + 2^r - 1) - A_j(t + 1)) - (A_j(t) - A_j(t + 1))
\]
and since \( A_j(2t) = A_2(t) \) we have \( A_j((2^r t + 2^r - 1) + 1) = A_j(t + 1) \) and we can conclude by Lemma 2.2. \( \square \)
The following lower bound is [11, Lemma 3.1], and essentially contained in [5]; see also [6].

**Lemma 2.4.** Let $M$ be the number of blocks of $1s$ in $t$. Then $A_2(t) \geq M/2$.

We prove the following upper bound for $\xi_t(\vartheta)$, using the recurrence (14) as an essential input.

**Proposition 2.5.** There is an absolute constant $C$ such that for $|\vartheta| \leq \min \left( M^{-1/6}, 1/(2\tau) \right)$ we have
\[
|\tilde{\gamma}_t(\vartheta)| \leq CM\vartheta^6,
\]
\[
|\xi_t(\vartheta)| \leq C\vartheta^6,
\]

where $M$ is the number of blocks in $t$.

**Proof.** We proceed by induction on the length $L$ of the binary expansion of $t$. In order to start the process, we note that $\tilde{\gamma}_0(\vartheta) = 0$; moreover, we need to treat $\tilde{\gamma}_1(t)$ separately. Since $d_1 = -\log(\gamma_1 D_1)$ is analytic, Lemma 2.1 and the Taylor expansion of $\exp$ of degree 2 yield
\[
\gamma'_1(\vartheta) = \exp(- \tau \vartheta^2) + i \tau \vartheta^3 + \frac{12}{4!}(\tau \vartheta)^4 - \frac{i}{2}(\tau \vartheta)^5 + O(\vartheta^6)
\]
\[
= 1 - \tau \vartheta^2 + i \tau \vartheta^3 + \frac{12}{4!}(\tau \vartheta)^4 - \frac{i}{2}(\tau \vartheta)^5 + O(\vartheta^6)
\]
for $|\vartheta| \leq \rho$, with some absolute $\rho$ and absolute implied constants. This implies that $\tilde{\gamma}_1(\vartheta) \ll \vartheta^6$ for all $\vartheta$, with an absolute implied constant.

Let $L \geq 1$; assume that $t$ contains $M$ blocks (of $0s$ or $1s$) and that $\tilde{\gamma}_1(\vartheta) \leq CM\vartheta^6$ has already been established. In order to estimate $\xi_t(\vartheta)$, we factor out $\gamma'_1(\vartheta)$ in the equation (17).

Inserting the definition, we obtain
\[
\xi_t(\vartheta) = \gamma'_1(\vartheta) \left( \frac{e(\vartheta)}{2} + \frac{e(-\vartheta)}{2} \exp \left( \sum_{2 \leq j \leq 5} (A_j(t + 1) - A_j(t)) (\tau \vartheta)^j \right) ight.
\]
\[
- \exp \left( \sum_{2 \leq j \leq 5} (A_j(2t + 1) - A_j(t)) (\tau \vartheta)^j \right)
\]

By the triangle inequality and our induction hypothesis, we have
\[
|\gamma'_1(\vartheta)| \leq |\gamma_1(\vartheta)| + |\tilde{\gamma}_t(\vartheta)| \leq 1 + CM\vartheta^6
\]
and since $\vartheta \leq M^{-1/6}$, we obtain
\[
|\gamma'_1(\vartheta)| = O(1).
\]

We have the bounds for $A_j(t + 1) - A_j(t)$ stated in Lemma 2.2, moreover the same bounds hold for $A_j(2t + 1) - A_j(t) = A_j(2t + 1) - A_j(2t)$.

The first exponential term in (24) has $(i \tau \vartheta)^k/k!$ as coefficients; the contribution of the coefficients for $k \geq 6$ is bounded by
\[
\sum_{k \geq 6} \frac{(\tau \vartheta)^k}{k!} \leq (\tau \vartheta)^6(e - 163/60) < \frac{1}{619}(\tau \vartheta)^6.
\]

We want to show that the contribution of the second exponential term (in fact, the product of two exponentials) is bounded by $C(\tau \vartheta)^6$, and the third term as well. An upper bound is given as follows. We consider the coefficients of
\[
f(\vartheta) = \exp \left( \frac{\vartheta}{4}(\tau \vartheta) + \cdots + (\tau \vartheta)^5 \right)
\]
with indices \( \geq 6 \). An exponent \( k \) in a \( j \)-fold product appears at most \( 5^j \) times (this is a very rough upper bound!), and therefore the coefficient \( a^k \) \( f(\vartheta) \) is bounded by
\[
\tau^k \sum_{1 \leq j \leq k} (9/4)^j \frac{1}{j!} 5^j \leq C \tau^k
\]
with an absolute constant \( C \). Consequently,
\[
\sum_{k \geq 0} a^k [a^k] f(\vartheta) \leq C' \vartheta^6
\]
for some absolute constant \( C \) (since \( \tau \vartheta \leq 1/2 \)). The same holds for the third exponential in (24).

By construction, the Taylor series of \( \xi_t \) starts with \( \vartheta^6 \), and consequently, multiplying by the power series \( (\gamma_1(\vartheta))^{-1} \), we see that the series of the expression in parentheses starts with \( \vartheta^6 \) too. It follows that
\[
\xi_t(\vartheta) = O(\vartheta^6),
\]
where the implied constant is absolute and effective. This is true as long as \( \vartheta \leq M^{-1/6} \) and \( |\tau \vartheta| \leq 1/2 \).

As a second step in our induction, we show the stronger statement that
\[
\tilde{\gamma}_t(\vartheta) \leq 2CM\vartheta^6; \\
\tilde{\gamma}_{t+1}(\vartheta) \leq 2CM\vartheta^6,
\]
(25)
under the hypotheses that
\[
|\tilde{\gamma}_{t'}(\vartheta)| \leq 2CM'\vartheta^6; \\
|\tilde{\gamma}_{t'+1}(\vartheta)| \leq 2CM'\vartheta^6,
\]
and
\[
|\xi_{t'}(\vartheta)| \leq C\vartheta^6
\]
for all \( t \) whose binary expansion is strictly shorter than the binary expansion of \( t \), and all \( \vartheta \) satisfying \( |\vartheta| \leq 1/(2\tau) \) and \( |\vartheta| \leq M'^{-1/6} \). Here \( M \) is the number of blocks in \( t \), and \( M' \) is the number of blocks in \( t' \). We can write (16) as a matrix recurrence:
\[
\begin{pmatrix}
\tilde{\gamma}_{2t}(\vartheta) \\
\tilde{\gamma}_{2t+1}(\vartheta)
\end{pmatrix} = A_0 \begin{pmatrix}
\tilde{\gamma}_t(\vartheta) \\
\tilde{\gamma}_{t+1}(\vartheta)
\end{pmatrix} + \begin{pmatrix} 0 \\
\xi_t(\vartheta)
\end{pmatrix};
\]
(26)
\[
\begin{pmatrix}
\tilde{\gamma}_{2t+1}(\vartheta) \\
\tilde{\gamma}_{2t+2}(\vartheta)
\end{pmatrix} = A_1 \begin{pmatrix}
\tilde{\gamma}_t(\vartheta) \\
\tilde{\gamma}_{t+1}(\vartheta)
\end{pmatrix} + \begin{pmatrix} \xi_t(\vartheta) \\
0
\end{pmatrix}
\]
for \( t \geq 1 \), where
\[
A_0 = \begin{pmatrix} 1 & 0 \\
\frac{e(\vartheta)}{2} & \frac{e(-\vartheta)}{2}
\end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{e(\vartheta)}{2} & \frac{e(-\vartheta)}{2} \\
0 & 1
\end{pmatrix}.
\]
We are interested in run of 0s or 1s at the very right of the binary expansion of \( t \). If we have a run of 0s, we have \( t = 2^k t' \), where \( t' \) is odd, which corresponds to powers of \( A_0 \):
\[
\begin{pmatrix}
\tilde{\gamma}_{2^k t'}(\vartheta) \\
\tilde{\gamma}_{2^k t'+1}(\vartheta)
\end{pmatrix} = A_0^k \begin{pmatrix}
\tilde{\gamma}_{t'}(\vartheta) \\
\tilde{\gamma}_{t'+1}(\vartheta)
\end{pmatrix} + A_0^{k-1} \begin{pmatrix} 0 \\
\xi_{t'}(\vartheta)
\end{pmatrix} + A_0^{k-2} \begin{pmatrix} 0 \\
\xi_{2^{t'}}(\vartheta)
\end{pmatrix} + \cdots + A_0^{0} \begin{pmatrix} 0 \\
\xi_{2^k t'-1}(\vartheta)
\end{pmatrix}
\]
\[
= A_0^k \begin{pmatrix}
\tilde{\gamma}_{t'}(\vartheta) \\
\tilde{\gamma}_{t'+1}(\vartheta)
\end{pmatrix} + \begin{pmatrix} 0 \\
E_0(\vartheta)
\end{pmatrix},
\]
(27)
where

\[ E_0(\vartheta) = \sum_{0 \leq j < k} e^{-\frac{(k-1-j)\vartheta}{2^{k-1-j}}} \xi_{2^k\gamma')(\vartheta) \]

satisfies

\[ |E_0(\vartheta)| \leq 2 \max_{0 \leq j < k} |\xi_{2^k\gamma'(\vartheta)}|. \]

The binary length of \(2^j t'\) is less than the binary length of \(t\), therefore we can conclude by our hypothesis that \(|E_0(\vartheta)| \leq 2C\vartheta^6\). Moreover, the number \(M'\) of blocks (of 0s or 1s) in \(t'\) is the number \(M\) of blocks in \(t\) decreased by one (since \(t'\) is odd). By hypothesis and the fact that \(A_0\) has row-sum norm equal to 1, we obtain (25).

Analogously, appending a block of 1s to an even integer \(t'\), we obtain

\[ \begin{pmatrix} \tilde{\gamma}_{2^k\gamma' + 2^{k-1}}(\vartheta) \\ \tilde{\gamma}_{2^k\gamma' + 1}(\vartheta) \end{pmatrix} = A_1^k \begin{pmatrix} \tilde{\gamma}_{\gamma'}(\vartheta) \\ \tilde{\gamma}_{\gamma'+1}(\vartheta) \end{pmatrix} + \begin{pmatrix} E_1(\vartheta) \\ 0 \end{pmatrix}, \tag{28} \]

where

\[ E_1(\vartheta) = \sum_{0 \leq j < k} \xi_{2^j\gamma' + 2j-1}(\vartheta) e^{-\frac{(k-1-j)\vartheta}{2^{k-1-j}}} \]

satisfies

\[ |E_1(\vartheta)| \leq 2 \max_{0 \leq j < k} |\xi_{2^j\gamma' + 2j-1}(\vartheta)|. \]

As above, we have by our induction hypothesis \(E_1(\vartheta) \leq 2C\vartheta^6\). The integer \(t'\) has one block less than \(t\); since \(A_1\) has row-sum norm equal to 1, and by the induction base (where we had to verify the cases \(t \in \{0, 1\}\)) we are done.

\[ \square \]

### 2.2 Evaluating the integral

We use the following representation of the values \(c_t\), which can be found in [13].

**Proposition 2.6.** Let \(t \geq 0\). We have

\[ c_t = \frac{1}{2} + \frac{\delta(0, t)}{2} + \frac{1}{2} \int_{-1/2}^{1/2} \text{Im} \gamma_1(\vartheta) \cot(\pi \vartheta) \, d\vartheta, \tag{29} \]

where the integrand is a bounded, continuous function.

We split the integral at the points \(\pm \vartheta_0\), where \(\vartheta_0 = M^{-1/2}R\). Here \(M = 4M' + 1\) is the number of blocks (of 0s or 1s) of \(t\) and \(R\) is a small parameter (of size \(\approx \log M\)) to be chosen later. For now, we assume that

\[ 4 \leq R \leq M^{1/6} \quad \text{and} \quad \vartheta_0 \leq \frac{1}{2R}, \tag{30} \]

for technical reasons, among others we need to apply Proposition 2.5. Note that under these hypotheses,

\[ \vartheta_0 \leq M^{-1/6}. \]

By our choice \(R \approx \log M\) these conditions will be satisfied for large \(M\). We have in [13] the following lemma, which we use for the estimation of the tails of the above integral.
Lemma 2.7. Assume that \( t \geq 1 \) has at least \( M = 4M' + 1 \) blocks. Then
\[
|\gamma_t(\vartheta)| \leq \left(1 - \frac{\vartheta^2}{2}\right)^{M'} \leq \exp\left(-\frac{M'\vartheta^2}{2}\right) \leq 2\exp\left(-\frac{M\vartheta^2}{8}\right)
\]
for \( |\vartheta| \leq 1/2 \).

We have \( \cot(x) = 1/x + O(1) \) for \( x \leq 1/2 \). The contribution of the tail can therefore be bounded by
\[
\int_{M^{-1/2}R}^{1/2} \exp\left(-\frac{M\vartheta^2}{8}\right) \cot(\pi\vartheta) \, d\vartheta \leq \frac{1}{\pi} I + O(J),
\]
where
\[
I = \int_{M^{-1/2}R}^{\infty} \exp\left(-\frac{M\vartheta^2}{8}\right) \, d\vartheta,
\]
and
\[
J = \int_{M^{-1/2}R}^{\infty} \exp\left(-\frac{M\vartheta^2}{8}\right) \, d\vartheta.
\]
The integral \( J \) is bounded by
\[
O\left(\exp\left(-M(M^{-1/2}R)^2/8\right)\right) = O\left(\exp\left(-R^2/8\right)\right).
\]

In order to estimate \( I \), we write
\[
I \leq \sum_{j \geq 0} \int_{2^j \vartheta_0}^{2^{j+1} \vartheta_0} \exp\left(-\frac{M\vartheta^2}{8}\right) \frac{d\vartheta}{2^j \vartheta_0} \leq \sum_{j \geq 0} \exp\left(-\frac{4^j R^2}{8}\right).
\]

Using the hypothesis \( R \geq 1 \), this is easily shown to be bounded by \( O\left(\exp\left(-R^2/8\right)\right) \) by a geometric series. For \( |\vartheta| \leq \vartheta_0 \), we replace \( \gamma_t(\vartheta) \) by \( \gamma'_t(\vartheta) \) in the integral in (29), using Proposition 2.5. Noting the hypotheses (30), we obtain \( |\gamma_t(\vartheta) - \gamma'_t(\vartheta)| \ll M\vartheta^6 \), where \( M \) is the number of blocks in \( t \). Therefore
\[
\int_{-1/2}^{1/2} \text{Im} \, \gamma_t(\vartheta) \cot(\pi\vartheta) \, d\vartheta = \int_{-\vartheta_0}^{\vartheta_0} \text{Im} \, \gamma_t(\vartheta) \cot(\pi\vartheta) \, d\vartheta + O(\exp(-R^2/8))
\]
\[
= \int_{-\vartheta_0}^{\vartheta_0} \text{Im} \, \gamma'_t(\vartheta) \cot(\pi\vartheta) \, d\vartheta + O\left(M \int_{-\vartheta_0}^{\vartheta_0} \vartheta^6 \, d\vartheta\right) + O(\exp(-R^2/8))
\]
\[
= \int_{-\vartheta_0}^{\vartheta_0} \text{Im} \, \gamma'_t(\vartheta) \cot(\pi\vartheta) \, d\vartheta + O(E),
\]
where
\[
E = M^{-2}R^6 + \exp(-R^2/8).
\]

Similarly,
\[
\delta(0, t) = \int_{-\vartheta_0}^{\vartheta_0} \text{Re} \, \gamma'_t(\vartheta) \, d\vartheta + O(E).
\]

By the Taylor expansion of \( \exp \), noting also that \( A_2 \) and \( A_4 \) are real, while \( A_3 \) and \( A_5 \) are imaginary, we have for \( |\vartheta| \leq \vartheta_0 \)
\[
\gamma'_t(\vartheta) = \exp(-A_2(t)(\tau\vartheta)^2) \times \left(1 - A_3(t)(\tau\vartheta)^3 - A_4(t)(\tau\vartheta)^4 - A_5(t)(\tau\vartheta)^5 + \frac{1}{2} A_3(t)^2(\tau\vartheta)^6 + A_3(t)A_4(t)(\tau\vartheta)^7 - \frac{1}{6} A_3(t)^3(\tau\vartheta)^9 \right) + O(M^2\vartheta^8 + M^3\vartheta^{10}) + iO(M^2\vartheta^9 + M^3\vartheta^{11}),
\]
where both error terms are real. We note that \( \cot(\pi \vartheta) = 2/(\pi \vartheta) - \tau \vartheta / 6 + \mathcal{O}(\vartheta^3) \) for \(|\vartheta| \leq 1/2\).

Splitting into real and imaginary summands, we obtain by (32) and (33)

\[
c_t = \frac{1}{2} + \frac{1}{2} \int_{-\vartheta_0}^{\vartheta_0} \exp \left( -A_2(\tau \vartheta)^2 \right) \left( 1 - A_4(t)(\tau \vartheta)^4 + \frac{1}{2} A_3(t)^2(\tau \vartheta)^6 + (i\, A_3(t)(\tau \vartheta)^3 + i\, A_5(t)(\tau \vartheta)^5 - i\, A_3(t)A_4(t)(\tau \vartheta)^7 + \frac{i}{6} A_3(t)(\tau \vartheta)^9 \right) \cot(\pi \vartheta) \right) d\vartheta + \mathcal{O}(E + E_2)
\]

\[
= \frac{1}{2} + \frac{1}{2} \int_{-\vartheta_0}^{\vartheta_0} \exp \left( -A_2(\tau \vartheta)^2 \right) \left( 1 - A_4(t)(\tau \vartheta)^4 + \frac{1}{2} A_3(t)^2(\tau \vartheta)^6 + 2i\, A_3(t)(\tau \vartheta)^2 + 2i\, A_5(t)(\tau \vartheta)^4 - 2i\, A_3(t)A_4(t)(\tau \vartheta)^6 + \frac{i}{3} A_3(t)(\tau \vartheta)^8 - \frac{i}{6} A_3(t)(\tau \vartheta)^4 \right) d\vartheta + \mathcal{O}(E + E_2),
\]

where

\[
E_2 = \int_{-\vartheta_0}^{\vartheta_0} \left( M \vartheta^6 + M^2 \vartheta^8 + M^3 \vartheta^{10} \right) d\vartheta \ll M^{-5/2} R^{11}.
\]

We extend the integration limits again, introducing an error

\[
E_3 \ll \int_{M^{-1/2} R}^{\infty} \exp \left( -A_2(t)(\vartheta^2) \right) (1 + M \vartheta^2 + M^2 \vartheta^4 + M^3 \vartheta^6 + M^3 \vartheta^8),
\]

using Corollary 2.3. In order to estimate this, we use the following lemma.

**Lemma 2.8.** For real numbers \( a > 0 \) and \( \delta \geq 0 \), and integers \( j \geq 0 \), we define

\[
I_j = \int_0^\infty \exp(-ax^2) x^j.
\]

Then

\[
I_2 \ll \frac{\delta}{a} \exp(-a\delta^2),
\]

\[
I_4 \ll \left( \frac{\delta^3}{a} + \frac{\delta}{a^2} \right) \exp(-a\delta^2),
\]

\[
I_6 \ll \left( \frac{\delta^5}{a} + \frac{\delta^3}{a^2} + \frac{\delta}{a^3} \right) \exp(-a\delta^2),
\]

\[
I_8 \ll \left( \frac{\delta^7}{a} + \frac{\delta^5}{a^2} + \frac{\delta^3}{a^3} + \frac{\delta}{a^4} \right) \exp(-a\delta^2).
\]

**Proof.** We have

\[
\frac{\partial}{\partial x} \exp(-ax^2) x^m = \left( mx^{m-1} - 2ax^{m+1} \right) \exp(-ax^2),
\]

therefore

\[
I_{m+1} = \frac{1}{2a} \exp(-ax^2) x^m \bigg|_{\delta} + \frac{m}{2a} I_{m-1}.
\]

Noting that \( I_0 \ll \exp(-a\delta^2) \), we obtain the above estimates by recurrence. \( \square \)

We insert \( a = A_2(t) \) and \( \delta = \vartheta_0 \). By Lemma 2.4 we have \( a \geq M/2 > 0 \), and by our hypothesis (30) we have \( R \leq M^{1/6} \), which implies in particular that \( \delta \leq 1 \). By these estimates and Lemma 2.8, using also \( \delta = M^{-1/2} R \), we obtain

\[
E_3 \ll \left( 1 + \frac{M^2 \delta^5}{a} + \frac{M^3 \delta^7}{a} + \frac{M^3 \delta^5}{a^2} \right) \exp \left( -A_2(t)(M^{-1/2} R)^2 \right)
\]

\[
\ll \exp \left( -R^2/2 \right) \ll E.
\]
Substituting \( \tau \vartheta \) by \( \vartheta \), we obtain
\[
c_t = \frac{1}{2} + \frac{1}{2\tau} \int_{-\infty}^{\infty} \exp(-A_2 \vartheta^2) \left( 1 + 2i A_3 \vartheta^2 + \left( 2i A_5 - A_4 - \frac{i A_3}{6} \right) \vartheta^4 \right) \, \vartheta^4
+ \left( \frac{A_3}{2} - 2i A_4 \right) A_3 \vartheta^6 + \frac{i A_3^3}{3} \vartheta^8 \, d\vartheta + \mathcal{O}(E + E_2).
\]
Inserting standard Gaussian integrals, it follows that
\[
c_t = \frac{1}{2} + \frac{1}{4\sqrt{\pi}} \left( A_2^{-1/2} + i A_2^{-3/2} A_3 + \frac{3}{4} A_2^{-5/2} \left( 2i A_5 - A_4 - \frac{i A_3}{6} \right) \right)
+ \frac{15}{8} A_2^{-7/2} \left( \frac{A_4}{2} - 2i A_4 \right) A_3 + \frac{35}{16} A_2^{-9/2} A_3^3 + \mathcal{O} \left( M^{-2} R^{11} + \exp(-R^2/8) \right),
\]
(34)
under the hypotheses that \( 4 \leq R \leq M^{1/6} \) and \( M^{-1/2} R \leq 1/(2\tau) \), where \( M \) is the number of blocks (of 0s or 1s) in \( t \). The implied constant is absolute, as usual in this paper.

We choose
\[ R = \log M \]
in order to simplify the error term. Using the hypothesis \( R \geq 4 \), we have \( \exp(-R^2/8) \leq M^{-2} \).

Hence, we see that for \( c_t > 1/2 \) it is sufficient to prove
\[
v(t) \geq 0,
\]
where
\[
v(t) = A_2(t)^4 + i A_2(t)^3 A_3(t) + \frac{3}{4} A_2(t)^2 \left( 2i A_5(t) - A_4(t) - \frac{i A_3(t)}{6} \right)
+ \frac{15}{8} \left( \frac{A_4(t)}{2} - 2i A_4(t) \right) A_2(t) A_3(t) + i \frac{35}{16} A_3(t)^3 - C A_2^{5/2} R^{11}.
\]
(36)
and \( C \) is large enough such that the error term in (34) is strictly dominated by \( C A_2^{5/2} R^{11} \).

Usually the first term is the dominant one; the critical cases occur when the first two terms in (36) almost cancel. We take the first two terms together and write
\[ D = A_2(t) + i A_3(t). \]

Let us rewrite the expression for \( v(t) \), eliminating \( A_3 \). We have \( A_2^3 = -A_2^2 + 2DA_2 - D^2 \) and \( A_3^3 = -i(A_2(t)^3 - 3DA_2(t)^2 - 3D^2 A_2 + D^3) \), therefore
\[
v(t) = DA_2^3 + \frac{3i}{2} A_2^2 A_5 - \frac{3}{4} A_2^2 A_4 + \frac{1}{8} A_2^3 - \frac{1}{8} DA_2^2 + \frac{15}{16} A_2 \left( -A_2^2 + 2DA_2 - D^2 \right)
+ \frac{15}{4} A_2 A_4 - \frac{15}{4} DA_2 A_4 + \frac{35}{16} \left( A_2(t)^3 - 3DA_2(t)^2 + 3D^2 A_2 - D^3 \right)
= \left( D + \frac{11}{8} \right) A_2^3 + 3A_2^2 A_4 + \frac{3i}{2} A_2^2 A_5 - \frac{77}{16} DA_2^2 - \frac{15}{4} DA_2 A_4 + \frac{45D^2}{8} A_2 + \frac{35D}{16} - CA_2^{5/2} R^{11}.
\]

We make use of the fact (expressed in Corollary 2.3) that \( A_1 \leq CM \) for some absolute constant \( C \). For all \( \varepsilon \) there is an \( M_0 \) such that for \( M \geq M_0 \) we obtain
\[
v(t) \geq \left( D + \frac{11}{8} - \varepsilon \right) A_2^3 + 3A_2^2 A_4 + \frac{3i}{2} A_2^2 A_5.
\]
(37)

We see that if \( D \) is larger than some absolute constant, then for a large number of blocks, the other terms cannot cancel out this large contribution (where we use Lemma 2.4 and Corollary 2.3), and we obtain \( c_t > 1/2 \). We have therefore proved the following result.
Lemma 2.9. Assume that $M$ is the number of blocks in $t$. If $M$ is larger than some constant $M_0$ and $A_2(t) + i A_3(t) \geq D_0$ for some constant $D_0$, then $c_t > 1/2$.

As usual, all of these constants are effective and absolute. We have

$$D(2t + 1) = \frac{A_2(t) + A_2(t + 1)}{2} + \frac{i A_3(t) + A_3(t + 1)}{2} + \frac{i^2 A_2(t) - A_2(t + 1)}{2} + \frac{1}{2},$$

therefore

$$D(2t) = D(t) \quad \text{and} \quad D(2t + 1) = \frac{D(t) + D(t + 1)}{2} + \frac{A_2(t + 1) - A_2(t)}{2} + \frac{1}{2}. \quad (38)$$

Obviously, we have $D(1) = D(2) = 2$, moreover the term $(A_2(t+1) - A_2(t+1))/2$ is nonnegative by Lemma 2.2. This implies

$$D(t) \geq 2. \quad (39)$$

Choosing $\varepsilon = 1/16$ in (37), we see that it remains to show that

$$53 A_2 + 48 A_4 + 24 i A_5 > 0 \quad (40)$$

if $t$ contains many blocks, and $D$ is bounded by some absolute constant.

This is done in two steps: first, we determine the structure of the exceptional set of integers $t$ such that $D(t)$ is bounded. We will see that such an integer has few blocks of $0$s of length $\geq 2$, and few blocks of $1$s of bounded length. As a second step, we prove lower bounds for the numbers $A_4(t)$ and $i A_5(t)$, if $t$ is contained in this exceptional set.

2.3 Determining the exceptional set

We begin with investigating the effect of appending a block of the form $01^k$ to an integer $t$.

Lemma 2.10. For $t \geq 0$ and $k \geq 2$ we have

$$A_2(2^k t + 2^k - 1) = \frac{(2^{k-1} + 1) A_2(t)}{2^k} + \frac{(2^{k-1} - 1) A_2(t + 1)}{2^k} + \frac{3 (2^{k-1} - 1)}{2^k}, \quad (41)$$

$$D(2^k t + 2^k - 1) = \frac{2^{k-1} + 1}{2^k} D(t) + \frac{2^{k-1} - 1}{2^k} D(t + 1)$$

$$+ \left( \frac{1}{2} + \frac{k - 2}{2^k} \right) (A_2(t + 1) - A_2(t)) + \frac{1}{2} + \frac{3k - 4}{2^k}. \quad (42)$$

Proof. The proof of the first part is easy, using induction and the recurrence (14).

For $k = 2$, the second statement is clear from (41). We use the abbreviations $\rho_k = 1/2 +
\[(k-2)/2^k\text{ and } \sigma_k = 1/2 + (3k-4)/2^k.\text{ For } k \geq 2 \text{ we have by induction, using the first part,}\]
\[
D(2^{k+1}t + 2^k - 1) = \frac{D(2^kt + 2^{k-1} - 1) + D(2t + 1) + A_2(2t + 1) - A_2(2^kt + 2^{k-1} - 1)}{2} + \frac{1}{2}
\]
\[
= \frac{2^{k-1} + 1}{2^{k+1}}D(t) + \frac{2^{k-1} - 1}{2^{k+1}}D(t + 1) + \frac{\rho_k}{2}(A_2(t + 1) - A_2(t)) + \frac{\sigma_k}{2}
\]
\[
+ \frac{1}{4} + \frac{1}{4} + \frac{3}{2^{k+1}}.
\]

which implies the statement. \(\Box\)

By the bound \(|A_2(t + 1) - A_2(t)| \leq 1\) from Lemma 2.2, we obtain the following corollary.

**Corollary 2.11.** For all \(t \geq 0\) and \(k \geq 2\) we have
\[
D(2^kt + 2^{k-1} - 1) \geq \min(D(t), D(t+1)) + \frac{k-1}{2^{k-1}}.
\]

We can now extract the contribution to the value of \(D\) of a block of the form \(011 \cdots 10\). For this, we use the notation
\[
m(t) = \min(D(t), D(t+1)).
\]

Note that \(m(t) \geq 2\) by (39).

**Corollary 2.12.** Assume that \(t \geq 0\). For an integer \(k \geq 2\) let
\[
t' = 2^{k+1}t + 2^k - 2.
\]

Then
\[
m(t') \geq m(t) + \frac{k-1}{2^{k-1}}.
\]

**Proof.** We have \(D(t') = D(2^kt + 2^{k-1} - 1)\), and by Corollary 2.11 this is bounded below by \(\min(D(t), D(t+1)) + \frac{k-1}{2^{k-1}}\). Also, \(D(t'+1) = D(2^{k+1}t + 2^k - 1) \geq \min(D(t), D(t+1)) + \frac{k}{2}\). \(\Box\)

Moreover, we want to find the contribution of a block of 0s of length \(\geq 2\). For this, we append 001 and look what happens: note that
\[
A(4t + 1) = \frac{3A_2(t)}{4} + \frac{A_2(t + 1)}{4} + \frac{3}{4},
\]
\[
D(4t + 1) = \frac{3D(t)}{4} + \frac{D(t + 1)}{4} + \frac{A_2(t + 1) - A_2(t)}{2} + 1
\]
by (41) and (42). Therefore, by the recurrence (38), we obtain
\[
D(8t + 1) = \frac{D(t) + D(4t + 1)}{2} + \frac{A_2(4t + 1) - A_2(t)}{2} + \frac{1}{2}
\]
\[
= \frac{7}{8}D(t) + \frac{1}{8}D(t + 1) + \frac{3}{8}(A_2(t + 1) - A_2(t)) + \frac{11}{8}.
\]
These formulas together with \( D(8t + 2) = D(4t + 1) \) and \(|A_2(t + 1) - A_2(t)| \leq 1\) show that
\[
m(8t + 1) \geq m(t) + 1/2. \tag{43}
\]

**Corollary 2.13.** Assume that \( k \geq 2 \) and \( t \geq 1 \) are integers. Let \( K \) be the number of blocks of 0s of length at least 2 in the binary expansion of \( t \), and \( L \) be the number of blocks of 1s of length \( \leq k \). Then
\[
m(t) \geq 2 + \frac{K - 1}{2} + \max \left( 0, \left\lfloor \frac{L - 1 - 2K}{2} \right\rfloor \right) \frac{k - 1}{2^k - 1}.
\]

In particular, for all integers \( D_0 \geq 2 \) and \( k \geq 2 \), there exist a bound \( B = B(D_0, k) \) with the following property: for all integers \( t \geq 1 \) such that \( D(t) \leq D_0 \), the number of blocks of 0s of length \( \geq 2 \) in \( t \) and the number of blocks of 1s of length \( \leq k \) in \( t \) are bounded by \( B \).

**Proof.** Each block of 0s of length \( \geq 2 \) with the possible exception of the rightmost one belongs to a factor 001 in the binary expansion, therefore (43) explains the contribution \((K - 1)/2\). Each such block (which cannot be guaranteed to have length \( > 2 \)) renders the adjacent blocks of 1s unusable for the application of Corollary 2.12. Moreover, we cannot use the first and the last blocks of 1s. If 1 or 2 blocks remain, we can apply Corollary 2.12 once, for 3 or 4 blocks twice, and so on. This explains the last summand.

The constant 2 is explained by \( m(1) = \min(D(1), D(2)) = 2 \). As a final ingredient, we use the monotonicity of \( m(t) \) following from (38): we have \( m(2t) \geq m(t) \) and \( m(2t + 1) \geq m(t) \). \( \square \)

In the following, we will only use the “in particular”-statement of Corollary 2.13.

**2.4 Bounds for \( A_4 \) and \( A_5 \)**

**Lemma 2.14.** Assume that \( t \) contains \( M \) blocks of 1s. Then
\[
A_4(t) \geq -\frac{13(M + 1)}{12}.
\]

**Proof.** We have \( A_4(1) = A_4(1) = -13/12 \). By (14) and the estimates from Lemma 2.2 we have
\[
A_4(2t + 1) \geq \frac{A_4(t) + A_4(t + 1)}{2} - 13/24.
\]

Using the geometric series, this implies
\[
A_4(2^k t + 2^k - 1) \geq \frac{A_4(t)}{2^k} + \frac{(2^k - 1)A_4(t + 1)}{2^k} - 13/12. \tag{44}
\]

The statement for \( M = 1 \) easily follows. We also study \( t' = 2^k t + 1 \): in this case, we have
\[
A_4(2^k t + 1) \geq \frac{(2^k - 1)A_4(t)}{2^k} + \frac{A_4(t + 1)}{2^k} - 13/12 \tag{45}
\]
by induction. We consider the values \( n(t) = \min(A_4(t), A_4(t + 1)) \) and prove the stronger statement that \( n(t) \geq -13(M + 1)/12 \) by induction. We append a block \( 1^k \) to \( t \) and obtain \( t' = 2^k t + 2^k - 1 \). Then
\[
A_4(t') \geq \frac{A_4(t)}{2^k} + \frac{(2^k - 1)A_4(t + 1)}{2^k} - \frac{13}{12} \geq \min(A_4(t), A_4(t + 1)) - \frac{13}{12} = n(t) - 13/12,
\]
and $A_4(t' + 1) = A_4(t + 1)$. Analogously, we append $1^k$ to $t$ and obtain $t' = 2^k t$. Clearly, $A_4(t') = A_4(t)$, and

$$A_4(t' + 1) \geq \frac{(2^k - 1) A_4(t)}{2^k} + \frac{A_4(t + 1)}{2^k} - \frac{13}{12} \geq n(t) - 13/12.$$  

This implies the statement.

We want to find a lower bound for $i A_5(t)$. In the following, we consider the behaviour of the differences $A_j(t) - A_j(t + 1)$ when a block of 1s is appended to $t$. We do so step by step, starting with $A_2$. Assume that $k \geq 1$ is an integer and set $t^{(k)} = 2^k t + 2^k - 1$. By the recurrence (10) we obtain

$$A_2(t^{(k)}) - A_2(t^{(k)} + 1) = \frac{A_2(t^{(k-1)}) + A_2(t + 1) + 1}{2} - A_2(t + 1)$$

$$= \frac{A_2(t^{(k-1)}) - A_2(t + 1)}{2} + \frac{1}{2},$$

which gives by induction

$$A_2(t^{(k)}) - A_2(t^{(k)} + 1) = \frac{A_2(t) - A_2(t + 1)}{2^k} + \frac{2^k - 1}{2^k} = 1 + O(2^{-k}).$$  

We proceed to $A_3$. For $k \geq 1$, we have

$$A_3(t^{(k)}) - A_3(t^{(k)} + 1) = \frac{A_3(t^{(k-1)}) - A_3(t + 1)}{2} + i + O(2^{-k})$$

by (11) and (46), and by induction and the geometric series we obtain

$$A_3(t^{(k)}) - A_3(t^{(k)} + 1) = \frac{A_3(t) - A_3(t + 1)}{2^k} + i + O(k 2^{-k})$$

(47)

Concerning $A_4$, we have by (12), (46), and (47)

$$A_4(t^{(k)}) - A_4(t^{(k)} + 1) = \frac{A_4(t^{(k-1)}) - A_4(t + 1)}{2} + \frac{A_3(t^{(k-1)}) - A_3(t + 1)}{2}$$

$$= \frac{(A_2(t^{(k-1)}) - A_2(t + 1))^2}{8} + \frac{1}{12}$$

$$= \frac{A_4(t^{(k-1)}) + A_4(t + 1)}{2} - \frac{1}{2} + O(k 2^{-k}) - \frac{(1 + O(2^{-k}))^2}{8} + \frac{1}{12}$$

$$= \frac{A_4(t^{(k-1)}) - A_4(t + 1)}{2} - \frac{13}{24} + O(k 2^{-k})$$

and by induction we obtain

$$A_4(t^{(k)}) - A_4(t^{(k)} + 1) = -\frac{13}{12} + O(k^2 2^{-k}).$$  

(48)
Finally, we have by (13), (46), (47), and (48)

\[
A_5(t^{(k)}) - A_5(t^{(k)} + 1) = \frac{A_5(t^{(k-1)}) - A_5(t + 1)}{2} + i\frac{A_4(t^{(k-1)}) - A_4(t + 1)}{2}
\]

\[
- \frac{(A_2(t^{(k-1)}) - A_2(t + 1)) (A_3(t^{(k-1)}) - A_3(t + 1))}{4} + i\frac{A_2(t^{(k-1)}) - A_2(t + 1)}{6}
\]

\[
= \frac{A_5(t^{(k-1)}) - A_5(t + 1)}{2} - \frac{13i}{24} \left(1 + \mathcal{O}(2^{-k})\right) \left(i + \mathcal{O}(k2^{-k})\right) + \frac{i}{6} + \mathcal{O}(k2^{-k})
\]

and therefore

\[
A_5(t^{(k)}) - A_5(t^{(k)} + 1) = -\frac{5i}{4} + \mathcal{O}(k32^{-k}).
\]

(49)

**Proposition 2.15.** Let \( k \geq 1 \) be an integer. Assume that the integer \( t \geq 1 \) has \( N_1 \) blocks of zeros of length \( \geq 2 \), and \( N_2 \) blocks of 1s of length \( \leq k \). Define \( N = N_1 + N_2 \). If \( M \) is the number of blocks of 1s of length \( > k \), we have

\[
iA_5(t) \geq \frac{5M}{4} - C(N + Mk32^{-k})
\]

with an absolute constant \( C \).

**Proof.** We proceed by induction on the number of blocks of 1s in \( t \). The statement obviously holds for \( t = 0 \). Clearly, by the identity \( A_5(2t) = A_5(t) \) we may append 0s, preserving the truth of the statement (since \( N \) and \( M \) are unchanged). We therefore consider, for \( r \geq 1 \), appending a block of the form \( 01^r \) to \( t \), obtaining \( t' = 2r+1 t + 2 + r - 1 \). Define the integers \( N' \) and \( M' \) according to this new value \( t' \). If \( t \) is even, an additional block of zeros of length \( \geq 0 \) thus appears, therefore \( N' \geq N + 1 \), moreover \( M' \leq M + 1 \). By (22), the bound \( |A_5(m + 1) - A_5(m)| \leq 9/4 \) from (21), and the induction hypothesis we have

\[
iA_5(t') \geq iA_5(t) - 9/2 \geq \frac{5M}{4} - C(N + Mk32^{-k}) - 9/2 \geq \frac{5M'}{4} - C(N' + M'k32^{-k})
\]

(50)

if \( C \) is chosen large enough.

The case of odd \( t \) odd remains. The integer \( t \) ends with a block of 1s of length \( s \geq 1 \). We distinguish between three cases. First, let \( r \leq k \). In this case, \( N' = N + 1 \) and \( M' = M \), and a similar calculation as in (50) yields the claim. In the case \( r > k \), we have \( N' = N \) and \( M' = M + 1 \). This case splits into two subcases. Assume first that \( s \neq k \). We consider the integer \( 2t + 1 \) (corresponding to the quantity \( t^{(k)} + 1 \) in (50)). The quantities \( N'' \) and \( M'' \) corresponding to the integer \( t'' = 2t + 1 \) satisfy \( N'' = N \) and \( M'' = M \) due to the restriction \( s \neq k - 1 \), and by hypothesis we have

\[
iA_5(2t + 1) \geq \frac{5M}{4} - C(N + Mk32^{-k}).
\]

In this case, we need to extract the necessary gain \( 5/4 \) from (49); this formula yields

\[
iA_5(t) = iA_5(2t + 1) + \frac{5}{4} + \mathcal{O}(k32^{-k})
\]

\[
\geq \frac{5M'}{4} - C(N + M'k32^{-k})
\]
if $C$ is chosen appropriately. Finally, we consider the case $s = k$. In this case, we have $N'' = N - 1 = N' - 1$ and $M'' = M + 1 = M'$, and therefore by hypothesis

$$iA_5(2t + 1) \geq \frac{5(M + 1)}{4} - C((N - 1) + (M + 1)k^{32^{-k}}).$$

By the bound (21) we have

$$iA_5(t') \geq iA_5(2t + 1) - \frac{9}{4} \geq \frac{5M'}{4} - C(N' + M'k^{32^{-k}})$$

if we choose $C$ large enough. This finishes the proof of Proposition 2.15.

2.5 Finishing the proof of the main theorem

By Lemma 2.9 there are constants $D_0$ and $M_0$ such that $c_t > 1/2$ if $t$ contains at least $M_0$ blocks and $D(t) \geq D_0$.

Assume that $C$ is the constant from Proposition 2.15 and choose $k$ large enough such that $Ck^{32^{-k}} \leq 1/6$. Choose $B = B(D_0, k)$ as in Corollary 2.13 and assume that $D(t) \leq D_0$. The number $N_1$ of blocks of 0s of length $\geq 2$ in $t$ and the number $N_2$ of blocks of 1s of length $\leq k$ in $t$ are bounded by $f$ by this corollary. Therefore by Proposition 2.15,

$$iA_5(t) \geq \frac{13M}{12} - 2CB.$$ 

If $t$ contains sufficiently many blocks of 1s, we therefore have by Lemmas 2.4 and 2.14

$$53A_2(t) + 48A_4(t) + 24iA_5(t) \geq \frac{53M}{2} + 48 \left( -\frac{13M}{12} - \frac{13}{12} \right) + 26M - 48CB$$

$$\geq \frac{M}{2} - 48CB - 52 > 0.$$ 

By (40) it follows that $c_t > 1/2$ for sufficiently many (greater than some absolute bound) blocks of 1s. The proof is complete.

Acknowledgements.

The first named author is grateful to Johannes Morgenbesser, who introduced him to Cusick’s sum-of-digits conjecture in 2011, on one of the author’s first days as a PhD student. Moreover, we wish to thank Dmitry Badziahin, Michael Drmota, Jordan Emme, Wolfgang Steiner, and Thomas Stoll for fruitful discussions on the topic. Finally, we thank Thomas W. Cusick for constant encouragement and interest in our work.

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