ARITHMETIC THEORY OF \(q\)-DIFFERENCE EQUATIONS

(\(G_q\)-FUNCTIONS AND \(q\)-DIFFERENCE MODULES OF TYPE \(G\),
GLOBAL \(q\)-GEVREY SERIES)

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ABSTRACT. In the first part of the paper we give a definition of \(G_q\)-function and we establish a regularity result, obtained as a combination of a \(q\)-analogue of the André-Chudnovsky Theorem [And89 VI] and Katz Theorem [Kat70 §13]. In the second part of the paper, we combine it with some formal \(q\)-analogous Fourier transformations, obtaining a statement on the irrationality of special values of the formal \(q\)-Borel transformation of a \(G_q\)-function.

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1. INTRODUCTION

A \(G\)-function, notion introduced by C.L. Siegel in 1929, is a formal power series \(y = \sum_{n \geq 0} y_n x^n\) with coefficients in the field of algebraic numbers \(\mathbb{Q}\), such that:

1. the series \(y\) is solution of a linear differential equation with coefficients in \(\mathbb{Q}(x)\) (condition that actually ensures that the coefficients of \(y\) are contained in a number field \(K\));
2. there exist a sequence of positive numbers \(N_n \in \mathbb{N}\) and a positive constant \(C\) such that \(N_n y_s\) is an integer of \(K\) for any \(0 \leq s \leq n\) and \(N_n \leq C^n\);
3. for any immersion \(K \hookrightarrow \mathbb{C}\), the image of \(y\) in \(\mathbb{C}[[x]]\) is a convergent power series for the usual norm.

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Roughly speaking, a $G$-module is a, a posteriori fuchsian, $K(x)/K$-differential module whose (uniform part of) solutions are $G$-functions (cf. [Bom81, CC85, And89, DGS94]). More formally, if $Y'(x) = G(x)Y(x)$ is the differential system associate with such a connection in a given basis, one can iterate it obtaining a family of the higher order differential systems $\frac{d^sY}{dx^s}(x) = G_{[n]}(x)Y(x)$. Our differential module is of type $G$ if there exist a constant $C > 0$ and a sequence of polynomials $P_n(x) \in \mathbb{Z}[x]$, such that

1. $P_n(x)G_{[s]}(x)$ is a matrix whose entries are polynomials with coefficients in the ring of integers of $K$, for any $s = 1, \ldots, n$;
2. the absolute value of the coefficients of $P_n(x)$ is smaller that $C^n$.

The unsolved Bombieri-Dwork’s conjecture says that $G$-modules come from geometry, in the sense that they are extensions of direct summands of Gauss-Manin connections: the precise conjecture is stated in [And89 II]. Y. André proves that a differential module coming from geometry is of type $G$ (cf. [And89 V, App.]). More recently, the theory of $G$-functions has been the starting point for the papers [And00a] and [And00b], where the author develops an arithmetic theory of Gevrey series, allowing for a new approach to some diophantine results, such as the Schidlovskii’s theorem.

The question of the existence of an arithmetic theory of $q$-difference equations was first asked in [And00b]. A naive analogue over a number field of the notion above clearly does not work. In fact, let $K$ be a number field and let $q \in K$, $q \neq 0$, not be a root of unity. We consider formal power series $y \in K[[x]]$ that satisfies conditions 2 and 3 of the definition of $G$-function given above and that is solution of a nontrivial $q$-difference equation with coefficients in $K(x)$, i.e.:

$$a_\nu(x)y(q^\nu x) + a_{\nu-1}(q^{nu-1}x) + \cdots + a_0(x)y(x) = 0,$$

with $a_\nu(x), \ldots, a_0(x) \in K(x)$, not all zero. Then the following result by Y. André is the key point of [DV02].

**Proposition 1.1 ([DV02 8.4.1]).** A series $y$ as above is the Taylor expansion at 0 of a rational function in $K(x)$.

Other unsuccessful suggestions for a $q$-analogue of a $G$-function are made in [DV00 App.]. These considerations may induce to conclude that $q$-difference equations do not come from geometry over $\mathbb{Q}$.

Here we propose another approach: we consider a finite extension $K$ of the field of rational function $k(q)$ in $q$ with coefficients in a field $k$. This is a very natural approach since in the literature, $q$ very often considered as a parameter. Since $K$ is a global field, we can define a $G_q$-function to be a series in $K[[x]]$, solution of a $q$-difference equation with coefficients in $K(x)$, satisfying a straightforward analogue of conditions 2 and 3 of the definition above. As far as the definition of $q$-difference modules of type $G$ is concerned only the places of $K$ modulo whom $q$ is a root of unity - that we will briefly call cyclotomic places - comes into the picture (cf. Proposition 3.1 below). In fact, consider a $q$-difference system

\[(1.1.1)\]

$$Y(qx) = A(x)Y(x),$$

with $A(x) \in GL_\nu(K(x))$: its solutions can be interpreted as the horizontal vectors of a $K(x)$-free module $M$ of rank $\nu$ with respect to a semilinear bijective operator $\Sigma_q$ verifying $\Sigma_q(f(x)m) = f(qx)\Sigma_q(m)$ for any $f(x) \in K(x)$ and any $m \in M$. We consider the $q$-derivation:

$$d_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x}$$

and its iterations:

$$\frac{d_q^n}{[n]_q}, \text{ with } [0]_q! = [1]_q! = 1 \text{ and } [n]_q! = \frac{q^n - 1}{q-1}[n - 1]_q!.$$  

We can obtain from (1.1.1) a whole family of systems:

$$\frac{d_q^n}{[n]_q}Y(x) = G_{[n]}(x)Y(x),$$

where $G_1(x) = \frac{A(x)}{(q-1)x}$ and $\frac{q^n-1}{q-1}G_{[n]}(x) = G_{[1]}(x)G_{[n-1]}(x) + d_qG_{[n-1]}(x)$. The fact that the denominators $[n]_q!$ of the iterated derivations $\frac{d^n}{[n]_q}$ have positive valuation only at the cyclotomic places has the consequences that “there is no arithmetic growth” at the noncyclotomic places (cf. §3 below for a
precise formulation). Moreover, an important role in the proofs is played by the reduction of \( q \)-difference systems modulo a cyclotomic place: this means that we specialize \( q \) to a root of unity and we study the nilpotence properties of the obtained system. In characteristic zero, one automatically obtain an iterative \( q \)-difference module, in the sense of C. Hardouin [Har07].

The role played by the cyclotomic valuations, and therefore by roots of unity, points out some analogies with other topics:

- The Volume Conjecture predicts a link between the hyperbolic volume of the complement of an hyperbolic knot and the asymptotic of the sequence \( J_n(\exp(2i\pi/n)) \), where \( J_n(q) \) is an invariant of the knot called \( n \)-th Jones polynomial. The Jones polynomials are Laurent polynomials in \( q \) such that the generating series \( \sum_{n \geq 0} J_n(q) x^n \) is solution of a \( q \)-difference equations with coefficients in \( \mathbb{Q}(q) \) (cf. [GL05]): the situation is quite similar to the one considered in the present paper. The \( q \)-difference equations appearing in this topological setting have, in general, irregular singularities, differently from the \( q \)-difference operators of type \( G \), that are regular singular. To involve some irregular singular operators in the present framework, one should consider some formal \( q \)-Fourier transformations and develop a global theory of \( q \)-Gevrey series, in the wake of [And00a]: this is the topic of the second part of the paper.

- As already point out, an important role is played by the reduction of \( q \)-difference systems modulo the cyclotomic valuations. Conjecturally, the growth at cyclotomic places should be enough to describe the whole theory (cf. §3). It is natural to ask whether \( q \)-difference equations, that seem not to “come from geometry over \( \mathbb{Q} \)”, may have some geometric origin, in the sense of the geometry over \( \mathbb{F}_1 \) (cf. [Sou04], [CC08]).

Notice that in [Man08], Y. Manin establish a link between the Habiro ring, which is a topological algebra constructed to deal with quantum invariants of knots, and geometry over \( \mathbb{F}_1 \), so that the two remarks above are not orthogonal.

* * *

In the present paper we give a definition of \( G_q \)-functions and \( q \)-difference modules of type \( G \). We test those definitions proving that a \( q \)-difference module having an injective solution whose entries are \( G \)-functions is of type \( G \): that is to say that “the minimal \( q \)-difference module generated by a \( G \)-function” is of type \( G \) (cf. Theorem 4.2 below). We also prove that \( q \)-difference module of type \( G \) are regular singular (cf. Theorem 4.1). These two results are the base for the development of a global theory of \( q \)-Gevrey series.

In part two, we define global \( q \)-Gevrey series. Via the study of two \( q \)-analogues the formal Fourier transformation, we establish some structure theorems for the minimal \( q \)-difference equations killing global \( q \)-Gevrey series (cf. Theorems 12.3, 12.4 and 12.6). We conclude with an irrationality theorem for special values of global \( q \)-Gevrey series of negative orders (cf. Theorem 13.6).

This paper won’t be submitted for publication since the results below can be obtained in a more direct way. Namely, one can prove that \( G_q \)-functions are all rational (cf. [DVH09]). Nevertheless, the construction of the coefficients of the \( q \)-difference module from an injective solution in the proof of Theorem 13.2 has an interest in itself, since it may be applied to other difference operators.

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Part 1. \( G_q \)-functions and \( q \)-difference modules of type \( G \)

2. Definition and first properties

Let us consider the field of rational function \( k(q) \) with coefficients in a fixed field \( k \). We fix \( d \in (0,1) \) and for any irreducible polynomial \( v = v(q) \in k[q] \) we set:

\[
|f(q)|_v = d^{\deg_q v(q) \cdot \ord_{v(q)} f(q)}, \quad \forall f(q) \in k[q].
\]

The definition of \( | \cdot |_v \) extends to \( k(q) \) by multiplicativity. To this set of norms one has to add the \( q^{-1}\)-adic one, defined on \( k[q] \) by:

\[
|f(q)|_{q^{-1}} = d^{-\deg_q f(q)};
\]
once again this definition extends by multiplicativity to \( k(q) \). Then the Product Formula holds:

\[
\prod_v \frac{f(q)}{g(q)} \mid_v = d^\sum_v \deg q v(q) \left( \text{ord}_{v(q)} f(q) - \text{ord}_{v(q)} g(q) \right) = d^\deg_g f(q) - \deg_g g(q) = \left| \frac{f(q)}{g(q)} \right|_{q^{-1}}^{-1}.
\]

For any finite extension \( K \) of \( k(q) \), we consider the family \( P \) of ultrametric norms, that extends the norms defined above, up to equivalence. We suppose that the norms in \( P \) are normalized so that the Product Formula still holds. We consider the following partition of \( P \):

- the set \( P_\infty \) of places of \( K \) such that the associated norms extend, up to equivalence, either \( \| \cdot \|_q \) or \( \| \cdot \|_{q^{-1}} \);
- the set \( P_f \) of places of \( K \) such that the associated norms extend, up to equivalence, one of the norms \( \| \cdot \|_v \) for an irreducible \( v = v(q) \in k[q] \), \( v(q) \neq q \).

Moreover we consider the set \( C \) of places \( v \in P_f \) such that \( v \) divides a valuation of \( k(q) \) having as uniformizer a factor of a cyclotomic polynomial. We will briefly call \( v \in C \) a cyclotomic place.

**Definition 2.1.** A series \( y = \sum_{n \geq 0} y_n x^n \in K[[x]] \) is a \( G_q \)-function if:

1. It is solution of a \( q \)-difference equations with solutions in \( K(x) \), i.e. there exists \( a_0(x), \ldots, a_\nu(x) \in K(x) \) not all zero such that

\[
\sigma(y) := \limsup_{n \to \infty} n \sum_{v \in P_n} \log^+ \left( \sup_{s \leq n} \left| y_s \right|_v \right) < \infty,
\]

where \( \log^+ x = \sup(0, \log x) \).

We will refer to the invariant \( \sigma \) as the size, using the same terminology as in the classical case of series over a number field.

**Remark 2.2.**

1. One can show that this definition of \( G_q \)-function is equivalent to the one given in the introduction (cf. [And89, I, 1.3]).
2. Let \( \overline{k(q)} \) be the algebraic closure of \( k(q) \). A formal power series with coefficients in \( \overline{k(q)} \) solution of a \( q \)-difference equations with coefficients in \( \overline{k(q)}(x) \) is necessarily defined over a finite extension \( K/\overline{k(q)} \).

**Proposition 2.3.** The set of \( G_q \)-functions is stable with respect to the sum and the Cauchy product\(^1\). Moreover, it is independent of the choice of \( K \), in the sense that we can replace \( K \) by any finite extension of \( K \).

**Proof.** The proof is the same as in the case of classical \( G \)-functions (cf. [And89, I, 1.4, Lemma 2]). \( \square \)

The field \( K(x) \) is naturally a \( q \)-difference algebra, i.e. is equipped with the operator

\[
\sigma_q: \quad K(x) \rightarrow K(x), \quad f(x) \mapsto f(qx).
\]

The field \( K(x) \) is also equipped with the \( q \)-derivation

\[
d_q(f)(x) = \frac{f(qx) - f(x)}{(q - 1)x},
\]

satisfying a \( q \)-Leibniz formula:

\[
d_q(fg)(x) = f(qx)d_q(g)(x) + d_q(f)(x)g(x),
\]

for any \( f, g \in K(x) \). A \( q \)-difference module over \( K(x) \) (of rank \( \nu \)) is a finite dimensional \( K(x) \)-vector space \( M \) (of dimension \( \nu \)) equipped with an invertible \( \sigma_q \)-semilinear operator, i.e.

\[
\Sigma_q(f)(x) = f(qx)\Sigma_q(m),
\]

---

\(^1\)It may be interesting to remark, although we won’t need it in the sequel, that the estimate of the size of a product of \( G \)-functions proved in [And89, I, 1.4, Lemma 2] holds also in the case of \( G_q \)-functions.
for any \( f \in K(x) \) and \( m \in M \). A morphism of \( q \)-difference modules over \( K(x) \) is a morphisms of \( K(x) \)-vector spaces, commuting to the \( q \)-difference structure (for more generalities on the topic, cf. [vdPS97], [DV02] Part I or [DvRSZ03]).

Let \( \mathcal{M} = (M, \Sigma_q) \) be a \( q \)-difference module over \( K(x) \) of rank \( \nu \). We fix a basis \( \mathcal{C} \) of \( M \) over \( K(x) \) and we set:

\[
\Sigma_q \mathcal{C} = \mathcal{C}(A(x)),
\]

with \( A(x) \in \text{GL}_n(K(x)) \). An horizontal vector \( \vec{y} \in K(x)^n \) with respect to \( \Sigma_q \) is a vector that verifies \( \vec{y}(x) = A(x)\vec{y}(qx) \). Therefore we call \( Y(qx) = A_1(x)Y(x) \), with \( A_1(x) = A(x)^{-1} \), the system associated to \( \mathcal{M} \) with respect to the basis \( \mathcal{C} \). Recursively we obtain the families of \( q \)-difference systems:

\[
Y(q^n x) = A_n(x)Y(x) \text{ and } d_n^q Y(x) = G_n(x)Y(x),
\]

with \( A_n(x) \in \text{GL}_n(K(x)) \) and \( G_n(x) \in M_n(K(x)) \). Notice that:

\[
A_{n+1}(x) = A_n(qx)A_1(x) \text{, } G_1(x) = \frac{A_1(x) - 1}{(q - 1)x} \text{ and } G_{n+1}(x) = G_n(qx)G_1(x) + d_q G_n(x).
\]

It is convenient to set \( A_0 = G_0 = 1 \). Moreover we set \( [n]_q = \frac{q^n - 1}{q - 1} \) for any \( n \geq 1 \), \( [n]_q! = [n]_q[n-1]_q \cdots [1]_q \), \( [0]_q! = 1 \) and \( G_{[n]}(x) = \frac{G_n(x)}{[n]_q!} \).

**Definition 2.4.** A \( q \)-difference module over \( K(x) \) is said to be of type \( G \) (or a \( G \)-\( q \)-difference module) if the following global \( q \)-Galoškin condition is verified:

\[
\sigma^G_q(\mathcal{M}) = \limsup_{n \to \infty} \frac{1}{n} \sum_{v \in \mathcal{C}} \log^+ \left( \sup_{s \leq n} |G[s]|_{v,Gauss} \right) < \infty,
\]

where

\[
\frac{\left| \sum a_i x^i \right|_{v,Gauss}}{\left| \sum b_j x^j \right|_{v,Gauss}} = \frac{\sup |a_i|_v}{\sup |b_j|_v},
\]

for all \( \sum a_i x^i, \sum b_j x^j \in K(x) \).

**Remark 2.5.** Notice that the definition of \( G \)-\( q \)-difference module involves only the cyclotomic places.

**Proposition 2.6.** The definition of \( G_q \)-module is independent on the choice of the basis and is stable by extension of scalars to \( K'(x) \), for a finite extension \( K' \) of \( K \).

**Proof.** Once again the proof is similar to the classical theory of \( G \)-functions and differential modules of type \( G \). \( \square \)

### 3. Role of the “noncyclotomic” places

**Proposition 3.1.** In the notation introduced above, for any \( q \)-difference module \( \mathcal{M} = (M, \Sigma_q) \) over \( K(x) \) we have:

\[
\sigma^G_{P_f \setminus \mathcal{C}}(\mathcal{M}) := \limsup_{n \to \infty} \frac{1}{n} \sum_{v \in P_f \setminus \mathcal{C}} \log^+ \left( \sup_{s \leq n} |G[s]|_v \right) < \infty.
\]

**Proof.** We recall that the sequence of matrices \( G_{[n]} \) satisfies the recurrence relation:

\[
G_{[n+1]}(x) = \frac{G_{[n]}(qx)G_1(x) + d_q G_{[n]}(x)}{[n+1]_q^2}.
\]

Since \( [n+1]_q! = 1 \) for any \( v \in P_f \setminus \mathcal{C} \), we conclude recursively that

\[
|G_{[n]}|_{v,Gauss} \leq 1,
\]

for almost all places \( v \in P_f \setminus \mathcal{C} \). For the remaining finitely many places \( v \in P_f \), one can deduce from the recursive relation there exists a constant \( C > 0 \) such that \( |G_{[n]}|_{v,Gauss} \leq C^n \). \( \square \)

We immediately obtain the equivalence of our definition of \( q \)-difference module of type \( G \) with the naive analogue of the classical definition of \( G \)-module (cf. [And89] IV, 4.1):
Corollary 3.2. A q-difference module is of type G if and only if
\[ \sigma^{(q)}_{P_f}(M) := \limsup_{n \to \infty} \frac{1}{n} \sum_{v \in P_f} \log^+ \left( \sup_{s \leq n} |G[s]_v| \right) < \infty. \]

We expect the same kind of result to be true for \( G_q \)-functions, namely:

Conjecture 3.3. Suppose that \( y = \sum_{n \geq 0} y_n x^n \in K[[x]] \) is solution of a q-difference equations with coefficients in \( K \) (cf. (2.1.1)). Then:
\[ \sigma_{P_f \setminus C}(y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{v \in P_f \setminus C} \log^+ \left( \sup_{s \leq n} |y_s|_v \right) < \infty. \]

The last statement would immediately imply that one can define \( G_q \)-functions in the following way:

Conjectural definition 3.4. We say that the series \( y = \sum_{n \geq 0} y_n x^n \in K[[x]] \) is a \( G_q \)-function if \( y \) is solution of a q-difference equations with coefficients in \( K \) and moreover
\[ \sigma_{C \cup P_{\infty}}(y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{v \in C \cup P_{\infty}} \log^+ \left( \sup_{s \leq n} |y_s|_v \right) < \infty. \]

Remark 3.5. The fact that for almost all \( v \in P_f \setminus C \) we have \( |G[n]|_v \leq 1 \) for any \( n \geq 1 \) implies that for almost all \( v \in P_f \setminus C \) a “solution” \( y(x) = \sup_n y_n x^n \in K[[x]] \) of a q-difference system with coefficient in \( K(x) \) is bounded, in the sense that \( \sup_n |y_n|_v < \infty \). Unfortunately, one would need some uniformity with respect to \( v \) and \( n \) to conclude something about \( \sigma_{P_f \setminus C}(y) \).

Notice that if \( 0 \) is an ordinary point, the conjecture is trivial since
\[ \sum_{n \geq 0} G[n](0)x^n \]
is a fundamental solution of the linear system \( Y(qx) = A_1(x)Y(x) \). A q-analogue of the techniques developed in [And89] V (cf. also [DGS94], Chap. VII) would probably allow to establish the conjecture under the assumption that \( 0 \) is a regular point. This is not satisfactory because one of the purposes of the whole theory is the possibility of reading the regularity of a q-difference equation on one single solution (cf. Theorem 4.3 below), so one does not want to assume regularity a priori.

4. Main results

A q-difference module \( (M, \Sigma_q) \) is said to be regular singular at \( 0 \) if there exists a basis \( e \) such that the Taylor expansion of the matrix \( A_1(x) \) is in \( GL_v(K[[x]]) \). It is said to be regular singular tout court if it is regular singular both at \( 0 \) and at \( \infty \). We have the following analogue of a well-known differential result (cf. [Kat70] §13; cf. also [DV02] §6.2.2] for q-difference modules over a number field):

Theorem 4.1. A G-q-difference module \( M \) over \( K(x) \) is regular singular.

Let \( \bar{y}(x) = ^t(y_0(x), \ldots, y_{\nu - 1}(x)) \in K[[x]]^\nu \) be a solution of the q-difference system associated to \( M = (M, \Sigma_q) \) with respect to the basis \( e \):
\[ \bar{y}(qx) = A_1(x)\bar{y}(x). \]

We say that \( \bar{y}(x) \) is an injective solution if \( y_1(x), \ldots, y_{\nu}(x) \) are linearly independent over \( K(x) \).

We have the following q-analogue of the André-Chudnovsky Theorem [And89] VI:

Theorem 4.2. Let \( \bar{y}(x) = ^t(y_0(x), \ldots, y_{\nu - 1}(x)) \in K[[x]]^\nu \) be an injective solution of the q-difference system associated to \( M = (M, \Sigma_q) \) with respect to the basis \( e \).
If \( y_0(x), \ldots, y_{\nu - 1}(x) \) are \( G_q \)-functions, then \( M \) is a G-q-difference module.

We can immediately state a corollary:

Corollary 4.3. Let \( \bar{y}(x) = ^t(y_0(x), \ldots, y_{\nu - 1}(x)) \in K[[x]]^\nu \) be an injective solution of the q-difference system associated to \( M = (M, \Sigma_q) \) with respect to the basis \( e \).
If \( y_1(x), \ldots, y_{\nu}(x) \) are \( G_q \)-functions, then \( M \) is regular singular.

Thanks to the cyclic vector lemma we can state the following (cf. [Sau00] Annexe B):
Corollary 4.4. Let \( y(x) \) a \( G_q \)-function and let
\[
(4.4.1) \quad a_0(x)y(x) + a_1(x)y(qx) + \cdots + a_{\nu}(x)y(q^\nu x) = 0.
\]
a \( q \)-difference equation of minimal order \( \nu \), having \( y(x) \) as a solution.

Then (4.4.1) is fuchsian, i.e. we have \( \text{ord}_v a_i \geq \text{ord}_v a_0 = \text{ord}_v a_\nu \) and \( \deg_v a_i \leq \deg_v a_0 = \deg_v a_\nu \), for any \( i = 0, \ldots, \nu \).

The proofs of Theorem 4.1 and Theorem 4.2 are the object of §6 and §7, respectively.

5. Nilpotent reduction at cyclotomic places

We denote by \( \mathcal{O}_K \) the ring of integers of \( K \), \( k_v \) the residue field of \( K \) with respect to the pace \( v \), \( \varpi_v \), the uniformizer of \( v \) and \( q_v \), the image of \( q \) in \( k_v \), which is defined for all places \( v \in \mathcal{P} \). Notice that \( q_v \) is a root of unity for all \( v \in \mathcal{C} \). Let \( \kappa_v \in \mathbb{N} \) be the order of \( q_v \), for \( v \in \mathcal{C} \).

Let \( M = (M, \Sigma_q) \) be a \( q \)-difference module over \( K(x) \). We can always choose a lattice \( \tilde{M} \) of \( M \) over an algebra of the form
\[
(5.0.2) \quad \mathcal{A} = \mathcal{O}_K \left[ x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \frac{1}{P(q^2x)}, \ldots \right],
\]
for some \( P(x) \in \mathcal{O}_K[x] \), such that for almost all \( v \in \mathcal{C} \) we can consider the \( q_v \)-difference module \( M_v = \tilde{M} \otimes_{\mathcal{A}} k_v(x) \), with the structure induced by \( \Sigma_q \). In this way, for almost all \( v \in \mathcal{C} \), we obtain a \( q_v \)-difference module \( M_v = (M_v, \Sigma_{q_v}) \) over \( k_v(x) \), having the particularity that \( q_v \) is a root of unity. This means that \( \sigma_{q_v}^{\kappa_v} = 1 \) and that \( \Sigma_{q_v}^{\kappa_v} \) is a \( k_v(x) \)-linear operator.

The results in [DV02 §2] apply to this situation: we recall some of them. Since we have:
\[
\sigma_{q_v}^{\kappa_v} = 1 + (q - 1)^{\kappa_v} x^{\kappa_v} d_{q_v}^{\kappa_v}
\]
and
\[
\Sigma_{q_v}^{\kappa_v} = 1 + (q - 1)^{\kappa_v} x^{\kappa_v} \Delta_{q_v}^{\kappa_v},
\]
where \( \Delta_{q_v} = \frac{\Sigma_{q_v} - 1}{(q_v - 1)x} \), the following facts are equivalent:

1. \( \Sigma_{q_v}^{\kappa_v} \) is unipotent;
2. \( \Delta_{q_v}^{\kappa_v} \) is a linear nilpotent operator;
3. the reduction of \( A_{\kappa_v}(x) - 1 \) modulo \( \varpi_v \) is a nilpotent matrix;
4. the reduction of \( G_{\kappa_v}(x) \) modulo \( \varpi_v \) is nilpotent;
5. there exists \( n \in \mathbb{N} \) such that \( |G_{n\kappa_v}(x)|_{v, \text{Gauss}} \leq |\varpi_v|_v \).

Definition 5.1. If the conditions above are satisfied we say that \( M \) has nilpotent reduction (of order \( n \)) modulo \( v \in \mathcal{C} \).

Remark 5.2. If the characteristic of \( k \) is 0 and if \( |G_{\kappa_v}(x)|_{v, \text{Gauss}} \leq |\kappa_v|_{q_v} \), the module \( M_v \) has a structure of iterated \( q \)-difference module, in the sense of [Har07 §3]. In particular, if \( v \) is a non ramified place of \( K/k(q) \), then \( |\kappa_v|_{q_v} = 1 \).

The following result is a \( q \)-analogue of a well-known differential \( p \)-adic estimate (cf. for instance [DGS94 page 96]). It has already been proved in the case of \( q \)-difference equations over a \( p \)-adic field in [DV02 §5.1]. We are only sketching the argument: only the estimate of the \( q \)-factorials are slightly different from the case of mixed characteristic.

Proposition 5.3. If \( M = (M, \Sigma_q) \) has nilpotent reduction(of order \( n \)) modulo \( v \in \mathcal{C} \) then
\[
\limsup_{m \to \infty} \left( \sup_{m \geq 1} \left| G_{[m]} \right|_{v, \text{Gauss}} \right)^{1/m} \leq |\varpi_v|^{1/n\kappa_v} \left| \kappa_v \right|_{q_v}^{-1/\kappa_v}.
\]

Proof. The Leibniz formula (cf. [DV02 Lemma 5.1.2] for a detailed proof in a quite similar situation) implies that for any \( s \in \mathbb{N} \) we have:
\[
|G_{sn\kappa_v}(x)|_{v, \text{Gauss}} \leq |\varpi_v|^s.
\]
Since $|G_1(x)|_{v,\text{Gauss}} \leq 1$, for any $m \in \mathbb{N}$ we have:

$$|G_m(x)|_{v,\text{Gauss}} \leq \frac{|G_{\lfloor m/n\kappa_v \rfloor} x|_{v,\text{Gauss}}}{|m|_q^v} \leq \frac{|\varpi_v|_{v,\text{Gauss}}}{|m|_q^v},$$

where $\lfloor m/n\kappa_v \rfloor = \max\{a \in \mathbb{Z} : a \leq \frac{m}{n\kappa_v} \}$. The following lemma on the estimate of $|m|_q$ allows to conclude. \(\square\)

**Lemma 5.4.** For $v \in \mathcal{C}$ we have $|m|_q|_v = |\kappa_v|_v$ if $\kappa_v|m$ and $|m|_q|_v = 1$ otherwise. Therefore:

$$\lim_{m \to \infty} |m|_q^{1/m} = |\kappa_v|^{1/\kappa_v}.\]

**Proof.** Let $m \geq 2$ and $m = s\kappa_v + r$, with $r, s \in \mathbb{Z}$ and $0 \leq r < \kappa_v$. If $\kappa_v$ does not divide $m$, i.e. if $r > 0$, we have

$$|m|_q = 1 + q + \cdots + q^{m-1} = |\kappa_v|_q + q^{\kappa_v} |\kappa_v|_q + \cdots + q^{s\kappa_v} (1 + q + \cdots + q^{r-1}).$$

Therefore $|m|_q|_v = 1$. On the other hand, if $r = 0$:

$$|m|_q = \left(1 + q^{\kappa_v} + \cdots + q^{\kappa_v(s-1)}\right) |\kappa_v|_q.$$

Since $q^{\kappa_v} \equiv 1$ modulo $\varpi_v$, we deduce that $1 + q^{\kappa_v} + \cdots + q^{\kappa_v(s-1)} \equiv s$ modulo $\varpi_v$. Therefore

$$|m|_q|_v = |s|_v |\kappa_v|_q = |\kappa_v|_v.$$

This implies that

$$|m|_q^{1/m} = |\kappa_v|^{1/\kappa_v},$$

which allows to calculate the limit. \(\square\)

We obtain the following characterization:

**Corollary 5.5.** The $q$-difference module $\mathcal{M} = (M, \Sigma_q)$ has nilpotent reduction modulo $v \in \mathcal{C}$ if and only if

$$(5.5.1) \quad \lim_{m \to \infty} \sup_{m \to \infty} \left(1, |G_m|_{v,\text{Gauss}}^{1/m} \right) \leq |\kappa_v|_v^{-1/\kappa_v}.\]

**Proof.** One side of the implication is an immediate consequence of the proposition above. On the other hand, the assumption $\text{(5.5.1)}$ implies that

$$\lim_{m \to \infty} \sup_{m \to \infty} \left(1, |G_m|_{v,\text{Gauss}}^{1/m} \right) < 1,$$

which clearly implies that there exists $n$ such that $|G_{\kappa_v n}|_{v,\text{Gauss}} \leq |\varpi_v|_v$. \(\square\)

We finally obtain the following proposition, that will be useful in the proof of Theorem 4.1

**Proposition 5.6.** Let $\mathcal{M}$ be $q$-difference module over $K(x)$ of type $G$. Let $\mathcal{C}_0$ be the set of $v \in \mathcal{C}$ such that $\mathcal{M}$ does not have nilpotent reduction modulo $v$. Then

$$\sum_{v \in \mathcal{C}_0} \frac{1}{\kappa_v} < +\infty.$$

In particular, $\mathcal{M}$ has nilpotent reduction modulo $v$ for infinitely many $v \in \mathcal{C}$.

The proof relies on the following lemma:

**Lemma 5.7.** The following limit exists:

$$\lim_{n \to \infty} \frac{1}{n} \log^+ \left(\sup_{s \leq n} |G_s(x)|_{v,\text{Gauss}}\right).$$
Proof. The proof is essentially the same as the proof of [DV02, 4.2.7], a part from the estimate of the $q$-factorials (cf. Lemma 6.1 above). The key point is the following formula:

$$G_{|n+s|}(x) = \sum_{i+j=n} \frac{[n]_q^i [s]_q^j}{[s+n]_q^j} \binom{d_i}{q} (G_{|s|}(q^i x)) G_{|i|}(x), \forall s,n \in \mathbb{N},$$

obtained iterating the Leibniz rule. \hfill \Box

Proof of Proposition 5.6 The Fatou lemma, together with Lemma 5.7, implies:

$$\sum_{v \in \mathfrak{C}} \lim_{n \to \infty} \frac{1}{n} \log^+ \left( \sup_{x \leq n} |G_{|s|}(x)|_{v, \text{Gauss}} \right) \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{v \in \mathfrak{C}} \log^+ \left( \sup_{x \leq n} |G_{|s|}(x)|_{v, \text{Gauss}} \right) \leq \sigma_c^*(\mathcal{M}) < \infty.$$

It follows from Corollary 5.5 that:

$$\sum_{v \in \mathfrak{C}_q} \log^+ \left( |\kappa_v| q^{-1} \right) < \infty$$

and hence that

$$\sum_{v \in \mathfrak{C}_q} \log d^{-1} \kappa_v^{-1} < \infty,$$

since only a finite number of places of $K/k(q)$ are ramified. \hfill \Box

6. Proof of Theorem 4.1

It is enough to prove that 0 is a regular singular point for $\mathcal{M}$, the proof at $\infty$ being completely analogous.

Let $r \in \mathbb{N}$ be a divisor of $\nu!$ and let $L$ be a finite extension of $K$ containing an element $\tilde{q}$ such that $\tilde{q}^r = q$. We consider the field extension $K(x) \to L(t), x \to t^r$. The field $L(t)$ has a natural structure of $\tilde{q}$-difference algebra extending the $q$-difference structure of $K(x)$. Remark that:

**Lemma 6.1.** The $q$-difference module $\mathcal{M}$ is regular singular at $x = 0$ if and only if the $\tilde{q}$-difference module $\mathcal{M}_{L(t)} := (M \otimes_{K(t)} L(t), \Sigma_{\tilde{q}} := \Sigma_q \otimes \sigma_{\tilde{q}})$ is regular singular at $t = 0$.

**Proof.** It is enough to notice that if $\varepsilon$ is a cyclic basis for $\mathcal{M}$, then $\varepsilon \otimes 1$ is a cyclic basis for $\mathcal{M}_{L(t)}$ and $\Sigma_{\tilde{q}}(\varepsilon \otimes 1) = \Sigma_q(\varepsilon) \otimes 1$. \hfill \Box

The next lemma can be deduced from the formal classification of $q$-difference modules (cf. [Pra83, Cor. 9 and §9, 3], [Sau04, Thm. 3.1.7]):

**Lemma 6.2.** There exist an extension $L(t)/K(x)$ as above, a basis $f$ of the $\tilde{q}$-difference module $\mathcal{M}_{L(t)}$, such that $\Sigma_{\tilde{q}} f = f B(t)$, with $B(t) \in \text{Gl}_r(L(t))$, and an integer $\ell$ such that

$$B(t) = B_\ell \frac{t^\ell}{\ell!} + B_{\ell-1} \frac{t^{\ell-1}}{(\ell-1)!} + \ldots,$$  (6.2.1)

as an element of $\text{Gl}_r(L(t))$;

$B_\ell$ is a constant non nilpotent matrix.

**Proof of Theorem 4.1.** Let $\mathcal{B} \subset L(t)$ be a $\tilde{q}$-difference algebra over the ring of integers $\mathcal{O}_L$ of $L$, of the same form as $[5.0.2]$, containing the entries of $B(t)$. Then there exists a $B$-lattice $\mathcal{N}$ of $\mathcal{M}_{L(t)}$ inheriting the $\tilde{q}$-difference module structure from $\mathcal{M}_{L(t)}$ and having the following properties:

1. $\mathcal{N}$ has nilpotent reduction modulo infinitely many cyclotomic places of $L$;
2. there exists a basis $f$ of $\mathcal{N}$ over $\mathcal{B}$ such that $\Sigma_{\tilde{q}} f = f B(t)$ and $B(t)$ verifies (6.2.1).

Iterating the operator $\Sigma_{\tilde{q}}$ we obtain:

$$\Sigma_{\tilde{q}}^m(f) = f B(t) B(\tilde{q}t) \cdots B(\tilde{q}^{m-1} t) = \sum_{s \neq 0} B_{\ell}^m \left( \frac{B_{\ell}^m}{q^{m(t^{m-1}) x^m M}} + h.o.t. \right)$$

We know that for infinitely many cyclotomic places $w$ of $L$, the matrix $B(t)$ verifies

$$(B(t) B(\tilde{q}t) \cdots B(\tilde{q}^{w-1} t) - 1)^{n(w)} \equiv 0 \mod \varpi_w,$$  (6.2.2)
where \( \varpi_w \) is an uniformizer of the place \( w \), \( \kappa_w \) is the order \( \bar{q} \) modulo \( \varpi_w \) and \( n(w) \) is a convenient positive integer. Suppose that \( \ell \neq 0 \). Then \( B_{\ell}^{n(w)} = 0 \) modulo \( \varpi_w \), for infinitely many \( w \), and hence \( B_{\ell} \) is a nilpotent matrix, in contradiction with lemma 6.2. So necessarily \( \ell = 0 \).

Finally we have \( \Sigma_q(f) = f(B_0 + h.o.t) \). It follows from (6.2.1) that \( B_0 \) is actually invertible, which implies that \( M_{L(t)} \) is regular singular at 0. Lemma 6.1 allows to conclude. \( \square \)

### 7. Proof of Theorem 4.2

#### 7.1. Idea of the proof

The hypothesis states that there exists a vector \( \vec{y} = (y_0, \ldots, y_{\nu-1}) \in K[[x]]^\nu \), which is solution of the \( q \)-difference system:

\[
(7.0.3) \quad \vec{y}(qx) = A_1(x)\vec{y}(x),
\]

and therefore of the systems \( d_n^q \vec{y} = G_n(x)\vec{y} \) and \( \sigma_q^n \vec{y} = A_n(x)\vec{y} \) for any \( n \geq 1 \), having the property that \( y_0, \ldots, y_{\nu-1} \) are linearly independent over \( K(x) \). We recall that

\[
G_{n+1}(x) = G_n(qx)G_1(x) + d_qG_n(x)
\]

and that

\[
A_{n+1}(x) = A_n(qx)A_1(x).
\]

Let us consider the operator:

\[
\Lambda = A_1(x)^{-1} \circ (d_q - G_1(x)).
\]

We know that there exists an extension \( \mathcal{U} \) of \( K(x) \) (for instance the universal Picard-Vessiot ring constructed in [vdPS97, §12.1]) such that we can find an invertible matrix \( \mathcal{Y} \) with coefficient in \( \mathcal{U} \) solution of our system \( d_q^\mathcal{Y} = G_1^\mathcal{Y} \). An explicit calculation shows that:

\[
d_q \circ \mathcal{Y}^{-1} = (\sigma_q \mathcal{Y})^{-1} (d_q - G_1(x)) = \mathcal{Y}^{-1} A_1(x)^{-1} (d_q - G_1(x))
\]

and therefore that:

\[
(7.0.4) \quad \Lambda^n = \mathcal{Y} \circ d_q^n \circ \mathcal{Y}^{-1}, \text{ for all integers } n \geq 0.
\]

We set \( \binom{n}{i}_q = \frac{[n]!}{[i]![n-i]!_q} \), for any pair of integers \( n \geq i \geq 0 \). The twisted \( q \)-binomial formula shows that \( \left| \binom{n}{i}_q \right|_v \leq 1 \) for any \( v \in \mathcal{P}_f \).

The proof of Theorem 4.2 is based on the following \( q \)-analogue of [And89, VI, §1]:

**Proposition 7.1.** There exist \( \alpha_0^{(n)}, \ldots, \alpha_n^{(n)} \in K \) such that for all \( \vec{P} \in K[[x]]^\nu \) and all \( n \geq 0 \) we have:

\[
(7.1.1) \quad G_{[n]}^{\vec{P}} = \sum_{i=0}^{n} \frac{(-1)^i}{[n]!_q} \binom{n}{i}_q \alpha_i^{(n)} d_q^{n-i} \circ A_i(x) \Lambda^i(\vec{P}),
\]

with \( |\alpha_i(n)|_v \leq 1 \), for any \( v \in \mathcal{P}_f \) and \( n \geq i \geq 0 \).

**Proof.** The iterated twisted Leibniz Formula (cf. for instance [DV02, 1.1.8.1])

\[
d_q^n(fg) = \sum_{j=0}^{n} \binom{n}{j}_q \sigma_q^j (\sum_{i=0}^{n-j} \binom{n-j}{i}_q A_i(x) \Lambda^i(\vec{P})), \forall f, g \in \mathcal{U}
\]
\begin{align*}
\sum_{i=0}^{n} \frac{(-1)^i}{[n]_q} \binom{n}{i} q^i \alpha_i^{(n)} d_q^{n-i} \circ A_i(x) \circ \Lambda^i(\vec{P}) \\
= \sum_{i=0}^{n} \frac{(-1)^i}{[n]_q} \binom{n}{i} q^i \alpha_i^{(n)} d_q^{n-i} \circ \sigma_q^i(Y) \circ d_q \circ Y^{-1}(\vec{P}) \\
= \sum_{i=0}^{n} \frac{(-1)^i}{[n]_q} \binom{n}{i} \alpha_i^{(n)} \sum_{j=0}^{n-i} \binom{n-i}{j} q^{ij} \sigma_q^{n-j} (d_q^j(Y)) \circ d_q^{n-j} \circ Y^{-1}(\vec{P}) \\
= \sum_{j=0}^{n-j} \frac{1}{[n-j]_q q^j} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} q^{ij} \alpha_i^{(n)} \sigma_q^{n-j} (d_q^j(Y)) \circ d_q^{n-j} \circ Y^{-1}(\vec{P}).
\end{align*}

We have to solve the linear system:
\[
\sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} q^{ij} \alpha_i^{(n)} = \begin{cases} 1 & \text{if } n = j, \\ 0 & \text{otherwise}. \end{cases}
\]

For \( n = j \) we obtain \( \alpha_0^{(n)} = 1 \). We suppose that we have already determined \( \alpha_0^{(n)}, \ldots, \alpha_{k-1}^{(n)} \). For \( n-j = k \) we get:
\[
\sum_{i=0}^{k-1} (-1)^i \binom{k}{i} q^{i(n-k)} \alpha_i^{(n)} = (-1)^{k+1} \alpha_k^{(n)} q^{k(n-k)}.
\]
This proves also that \( |\alpha_k^{(n)}|_v \leq 1 \) for ant \( v \in \mathcal{P}_f \).

For all \( \vec{P} = t(P_0, \ldots, P_v - 1) \in K[x]^v \) and \( n \geq 0 \) we set:
\[
\vec{R}_n = \frac{\Lambda^n}{[n]_q}(\vec{P})
\]
and:
\[
R^{<n>} = \left( \binom{n}{n}_q \vec{R}_n \ (n+1)_q \vec{R}_{n+1} \ldots \ (n+v-1)_q \vec{R}_{n+v-1} \right).
\]
Therefore we obtain the identity:

**Corollary 7.2.**

\[
G_{[n]} R^{<n>} = \sum_{i=0}^{n-j} (-1)^i \alpha_i^{(n)} \frac{d_q^{n-i}}{[n-i]_q} \circ A_i(x) R^{<i>}
\]

**Remark 7.3.** In order to obtain an estimate of \( \sigma_{\mathcal{P}_f}^{[q]}(\mathcal{M}) \) we want to estimate the matrices \( G_{[n]}(x) \). The main point of the proof is the construction of a vector \( \vec{P} \), linked to the solution vector \( \vec{y} \) of \((7.0.3)\), such that \( R^{<0>} \) is an invertible matrix.

The proof is divided in step: in step 1 we construct \( \vec{P} \); in step 2 we prove that \( R^{<0>} \) is invertible; step 3 and 4 are devoted to the estimate of \( G_{[n]}(x) \) and of \( \sigma_{\mathcal{P}_f}^{[q]}(\mathcal{M}) \).

**7.2. Step 1. Hermite-Padé approximations of \( \vec{y} \).** We denote by \( \deg \) the usual degree in \( x \) and by \( \ord \) the order at \( x = 0 \). We extend their definitions to vectors as follows:
\[
\deg \vec{P}(x) = \sup_{i=0,\ldots,v-1} \deg P_i(x), \text{ for all } \vec{P} = \tau(P_0(x), \ldots, P_v(x)) \in K[x]^v.
\]
\[
\ord \vec{P}(x) = \inf_{i=0,\ldots,v-1} \ord P_i(x), \text{ for all } \vec{P} = \tau(P_0(x), \ldots, P_v(x)) \in K((x))^v.
\]
Moreover we set:
\[
\begin{aligned}
\left(\sum_{n \geq 0} \bar{a}_n x^n\right)_{\leq N} &= \sum_{n \leq N} \bar{a}_n x^n, \\
\left(\sum_{n \geq 0} \bar{a}_n x^n\right)_{> N} &= \sum_{n > N} \bar{a}_n x^n,
\end{aligned}
\]
for all \(\sum_{n \geq 0} \bar{a}_n x^n \in K[[x]]^\nu\).

Finally, for \(g(x) = \sum_{n \geq 0} g_n x^n \in K[x]\) and for \(\vec{y} = \sum_{n \geq 0} \bar{y}_n x^n \in K[[x]]^\nu\) we set:
\[
h(g, v) = \sup_n |g_n|_v, \quad \forall \, v \in \mathcal{P},
\]
\[
h(g) = \sum_{v \in \mathcal{P}} h(g, v)
\]
and
\[
\tilde{h}(n, v) = \sup_{\bar{y}_s \leq n} |\bar{y}_s|_v, \quad \forall \, v \in \mathcal{P},
\]
where \(|\bar{y}_s|_v\) is the maximum of the \(v\)-adic absolute value of the entries of \(\bar{y}_s\).

The following lemma is proved in [And89, VI, §3] or [DGS94, Chap. VIII, §3] in the case of a number field. The proof in the present case is exactly the same, apart from the fact that there are no archimedean places in \(\mathcal{P}\):

**Proposition 7.4.** Let \(\tau \in (0, 1)\) be a constant and \(\vec{y} = \sum_{n \geq 0} \bar{y}_n x^n \in K[[x]]^\nu\). For all integers \(N > 0\) there exists \(\vec{g}(x) \in K[x]^\nu\) having the following properties:

\[(7.4.1) \quad \deg g(x) \leq N;\]
\[(7.4.2) \quad \ord(g\vec{y})_{\leq N} \geq 1 + N + \left[N \frac{1 - \tau}{\nu}\right];\]
\[(7.4.3) \quad h(g) \leq \text{const} + \frac{1 - \tau}{\tau} \sum_{v \in \mathcal{P}} \tilde{h} \left(N + \left[N \frac{1 - \tau}{\nu}\right], v\right).
\]

From now on we will assume that \(\vec{P}(x) = (g\vec{y})_{\leq N}\).

**Proposition 7.5.** Let \(Q_1(x) \in \mathcal{V}_K[x]\) be a polynomial such that \(Q_1(x) A_1^{-1}(x) \in M_{\nu \times \nu}(K[x])\). We set:
\[
Q_0 = 1 \quad \text{and} \quad Q_n(x) = Q_1(x) Q_{n-1}(qx), \quad \text{for all} \quad n \geq 1,
\]
and
\[
t = \sup \left\{ \deg(Q_1(x) A_1^{-1}(x)), \deg Q_1(x) \right\}.
\]
If \(n \leq \frac{N}{1 - \tau} \), then
\[
\left( x^n Q_n(x) \frac{d^n_q}{[n]_q^\nu}(x) \bar{y}(x) \right)_{\leq N + nt} = x^n Q_n(x) \bar{R}_n.
\]

The proposition above is a consequence of the following lemmas:

**Lemma 7.6.** For each \(n \geq 0\) we have:

\[(7.6.1) \quad x^n Q_n(x) \bar{R}_n(x) \in K[x]^\nu;\]
\[(7.6.2) \quad \deg x^n Q_n(x) \bar{R}_n(x) \leq N + nt.
\]

**Proof.** Clearly \(\bar{R}_0 = (g\vec{y})_{\leq N} \in K[x]^\nu\). We recall that there exist \(c_{i,n} \in K\) such that (cf. [DV02, 1.1.10]):
\[
d_q^n = \frac{(-1)^n}{(q - 1)^n x^n (\sigma_q - q - 1) \cdots (\sigma_q - q^{-n} - 1)} = \frac{(-1)^n}{(q - 1)^n x^n} \sum_{i=1}^n c_{i,n} \sigma_q^i,
\]
for each \( n \geq 1 \). Therefore we obtain:

\[
x^nQ_n(x)\tilde{R}_n = x^nQ_n(x)Y \frac{d^n_q}{[n]_q!} \left( Y^{-1}P_\bar{q} \right) = \frac{Q_n(x)Y}{[n]_q! (q-1)^n} \sum_{i=0}^{n} c_{i,n} \sigma_q^i \left( Y^{-1}P_\bar{q} \right) = \frac{1}{[n]_q! (q-1)^n} \sum_{i=0}^{n} c_{i,n} Q_n(x) A_i^{-1}(x) \sigma_q^i (\bar{P}).
\]

Since \( A_i(x) = A_1(q^{i-1}) \cdots A_1(x) \), we conclude that \( x^nQ_n(x)\tilde{R}_n \in K[x]^\nu \) and:

\[
\deg x^nQ_n(x)\tilde{R}_n \leq \sup_{i=0}^{n} \deg \left( Q_n(x) A_i^{-1}(x) \sigma_q^i (\bar{P}) \right) \leq \sup_{i=0}^{n} \left( \deg(\bar{Q}_i A_i^{-1}(x)) + \deg Q_n^{-1}(q^i) + \deg \sigma_q^i (\bar{P}) \right) \leq N + nt.
\]

\[\square\]

**Lemma 7.7.**

\[
\text{ord} \left( x^nQ_n(x) \frac{d^n_q(g)}{[n]_q!} (g)(x)\bar{y}(x) - x^nQ_n(x)\tilde{R}_n \right) \geq 1 + N + \left[ N \frac{1 - \tau}{\nu} \right].
\]

**Proof.** We have:

\[
x^nQ_n(x) \frac{d^n_q(g)}{[n]_q!} (g)(x)\bar{y}(x) - x^nQ_n(x)\tilde{R}_n = \frac{1}{[n]_q! (q-1)^n} \sum_{i=0}^{n} c_{i,n} Q_n(x) \left( \sigma_q^i (g)(x)\bar{y}(x) - Y \sigma_q^i \left( Y^{-1}P_\bar{q} \right) \right) = \frac{1}{[n]_q! (q-1)^n} \sum_{i=0}^{n} c_{i,n} Q_n(x) \left( \sigma_q^i (g)(x)\bar{y}(x) - A_i^{-1}(x) \sigma_q^i (\bar{P}) \right).
\]

Let \( \bar{H}_i = Q_i(x) \sigma_q^i (g)(x)\bar{y}(x) - Q_i(x) A_i^{-1}(x) \sigma_q^i (\bar{P}) \). Since:

\[
A_i^{-1}(x)Q_1(x) \sigma_q \left( \bar{H}_i \right) = \bar{H}_{i+1}.
\]

by induction on \( l \) we obtain:

\[
\text{ord} \bar{H}_i \geq \text{ord} \bar{H}_{i-1} \geq \text{ord} \left( g(x)\bar{y}(x) - \bar{P}(x) \right) \geq 1 + N + \left[ N \frac{1 - \tau}{\nu} \right].
\]

\[\square\]

### 7.3. Step 2. The matrix \( R^{<0>} \).

**Theorem 7.8.** Let \( \bar{y}(x) = \begin{pmatrix} y_0(x), \ldots, y_{\nu-1}(x) \end{pmatrix} \in K[[x]]^\nu \) a solution vector of \( \Lambda Y = 0 \), such that \( y_0(x), \ldots, y_{\nu-1}(x) \) are linearly independent over \( K(x) \). Then there exists a constant \( C(\Lambda) \), depending only on \( \Lambda \), such that if

\[
\bar{P} = \begin{pmatrix} P_0, \ldots, P_{\nu-1} \end{pmatrix} \in K[x]^\nu \setminus \{0\}
\]

has the following property:

\[
(7.8.1) \quad \text{ord det} \begin{pmatrix} P_i & P_j \\ y_i & y_j \end{pmatrix} \geq \deg \bar{P}(x) + C(\Lambda), \forall i, j = 0, \ldots, \nu - 1,
\]

then the matrix \( R^{<0>} \) is invertible.

**Remark 7.9.** We remark that if we choose \( g \) as in Propositions 7.4 and 7.5 and \( \bar{P} = (g\bar{y})_{\leq N} \), for \( N >> 0 \) we have:

\[
N \frac{1 - \tau}{\nu} \geq C(\Lambda).
\]
Therefore the condition (7.8.1) is satisfied since:
\[ \text{ord det } \begin{pmatrix} P_i & P_j \\ y_i & y_j \end{pmatrix} = \text{ord det } \begin{pmatrix} (gy_i)_{> N} & (gy_j)_{> N} \\ y_i & y_j \end{pmatrix} \geq 1 + N + \frac{1 - \tau}{\nu}. \]

We recall the Shidlovsky’s Lemma that we will need on the proof of Theorem 7.8.

**Definition 7.10.** We define total degree of \( \frac{f(x)}{g(x)} \in K(x) \) as:
\[ \deg \operatorname{tot} \frac{f(x)}{g(x)} = \deg f(x) + \deg g(x). \]

**Lemma 7.11** (Shidlovsky’s Lemma; cf. for instance [DGS94] Chap. VIII, 2.2). Let \( \mathcal{G}/K(x) \) be a field extension and let \( V \subset \mathcal{G} \) a \( K \)-vector space of finite dimension. Then the total degree of the elements of \( K(x) \) that can be written as quotient of two element of \( V \) is bounded.

**Proof of the Theorem 7.8.** Let \( \mathcal{Y} \) be an invertible matrix with coefficients in an extension \( \mathcal{U} \) of \( K(x) \) such that \( \Lambda \mathcal{Y} = 0 \) and let \( C \) be the field of constant of \( \mathcal{U} \) with respect to \( d_q \). The matrix
\[ \mathcal{Y}^{-1} \mathcal{P}, \mathcal{Y}^{-1} \mathcal{P}, \ldots, \frac{\nu - 1}{[\nu - 1]_q} \left( \mathcal{Y}^{-1} \mathcal{P} \right) \]
is invertible if and only if
\[ \text{rank } (\mathcal{Y}^{-1} \mathcal{P}) = \text{rank } \left( \mathcal{Y}^{-1} \mathcal{P}, \sigma_q \left( \mathcal{Y}^{-1} \mathcal{P} \right), \ldots, \sigma_q^{\nu - 1} \left( \mathcal{Y}^{-1} \mathcal{P} \right) \right) \]
is maximal. Let us suppose that
\[ \text{rank } (\mathcal{Y}^{-1} \mathcal{P}) = r < \nu. \]

Then the \( q \)-analogue of the wronskian lemma (cf. for instance [DV02] §1.2) implies that there exists an invertible matrix \( \mathcal{M} \) with coefficients in \( K(x) \) such that the first column of \( \mathcal{M} \mathcal{Y}^{-1} \mathcal{P}^{<0>} \) is equal to:
\[ \mathcal{M} \mathcal{Y}^{-1} \mathcal{P} = t \left( \bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{\nu - 1}, 0, \ldots, 0 \right). \]

The matrix \( \mathcal{M}^{-1} \) still verifies the \( q \)-difference equation \( \Lambda \mathcal{Y} = 0 \), so we will write \( \mathcal{Y} \) instead of \( \mathcal{Y} \mathcal{M}^{-1} \), to simplify notation. We set:
\[ \mathcal{S} = \mathcal{Y} \circ \sigma_q^n \circ \mathcal{Y}^{-1} \mathcal{P}, \forall n \geq 0, \]
\[ \mathcal{S}^{<0>} = (\mathcal{S}_0, \ldots, \mathcal{S}_{\nu - 1}) = \left( \begin{array}{cc} \mathcal{S}_I & \mathcal{S}_I' \\ \mathcal{S}_{I'} & \mathcal{S}_{I'}' \end{array} \right) \]
and
\[ \mathcal{Y}^{-1} = \left( \begin{array}{cc} \mathcal{Y}_{JL} & \mathcal{Y}_{JL'} \\ \mathcal{Y}_{J'L} & \mathcal{Y}_{J'L'} \end{array} \right), \]
where \( I = J = L = \{0, 1, \ldots, r - 1\} \) and \( I' = J' = L' = \{r, \ldots, \nu - 1\} \). We have:
\[ \left( \begin{array}{cc} \mathcal{Y}_{JL} & \mathcal{Y}_{JL'} \\ \mathcal{Y}_{J'L} & \mathcal{Y}_{J'L'} \end{array} \right) \left( \begin{array}{cc} \mathcal{S}_{IJ} & \mathcal{S}_{IJ'} \\ \mathcal{S}_{IJ'} & \mathcal{S}_{IJ'}' \end{array} \right) = \left( \begin{array}{cc} \sigma_q^{i} \left( \mathcal{Y}^{-1} \mathcal{P} \right) \end{array} \right)_{i=0, \ldots, \nu - 1} = \left( \begin{array}{cc} A \end{array} \right), \]
with \( A \in M_{r \times \nu}(K(x)) \), and therefore:
\[ \mathcal{Y}_{J,L} \mathcal{S}_{I,J} + \mathcal{Y}_{J,L'} \mathcal{S}_{I',J} = 0. \]

Because of our choice of \( \mathcal{Y} \), the vectors \( \mathcal{S}_0, \ldots, \mathcal{S}_{\nu - 1} \) are linearly independent over \( K(x) \), so by permutation of the entries of the vector \( \mathcal{P} \) we can suppose that the matrix \( \mathcal{S}_{I,J} \) is invertible.

Let \( B = \mathcal{S}_{I,J}^{-1} \mathcal{S}_{I,J}^{-1} \). Since \( \mathcal{S}^{<0>} \in M_{\nu \times \nu}(K(x)) \) is independent of the choice of the matrix \( \mathcal{Y} \), the same is true for \( B \). The matrix \( \mathcal{Y} \) is invertible and
\[ \left( \begin{array}{cc} \mathcal{Y}_{J,L} & \mathcal{Y}_{J,L'} \end{array} \right) = \mathcal{Y}_{J,L'} \left( \begin{array}{cc} -B & I_{\nu - r} \end{array} \right), \]
therefore the matrix \( \mathcal{Y}_{J,L'} \) is also invertible and we have:
\[ B = -\mathcal{Y}^{-1}_{J,L} \mathcal{Y}_{J,L'} \]
The coefficients of the matrix \( B \) can be written in the form \( \xi/\eta \), where \( \xi \) and \( \eta \) are elements of the \( K \)-vector space of polynomials of degree less or equal to \( \nu - r \) with coefficients in \( K \) in the entries of the matrix \( \mathcal{Y} \). By Shidlovsky’s lemma the total degree of the entries of the matrix \( B \) is bounded by a constant depending only on the \( q \)-difference system \( \Lambda \).
Let us consider the matrices:
\[ Q_1 = \begin{pmatrix} y_{v-1} & 0 & 0 & \cdots & 0 \\ y_1 & -y_0 & 0 & \cdots & 0 \\ y_2 & 0 & -y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{r-1} & 0 & 0 & \cdots & -y_0 \end{pmatrix} \in M_{r \times r}(K[[x]]) \]
and
\[ Q_2 = \begin{pmatrix} 0 & \cdots & 0 & -y_0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{r \times r}(K[[x]]); \]
we set:
\[ T = (Q_1 \ Q_2) \begin{pmatrix} S_{IJ} \\ S_{IJ} \end{pmatrix} = (Q_1 \ Q_2) \begin{pmatrix} I_r \\ B \end{pmatrix} S_{IJ}. \]
Let \((b_0, \ldots, b_{r-1})\) be the last row of \(B\). We have:
\[ \det (TS_{IJ}^{-1}) = \det (Q_1 + Q_2 B) \]
\[ = \det \begin{pmatrix} y_{v-1} - y_0 b_0 & -y_0 b_1 & -y_0 b_2 & \cdots & -y_0 b_{r-1} \\ y_1 & -y_0 & 0 & \cdots & 0 \\ y_2 & 0 & -y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{r-1} & 0 & 0 & \cdots & -y_0 \end{pmatrix} \]
\[ = \det \begin{pmatrix} y_{v-1} - y_0 b_0 - y_1 b_1 - \cdots - y_{r-1} b_{r-1} & 0 & 0 & \cdots & 0 \\ y_1 & -y_0 & 0 & \cdots & 0 \\ y_2 & 0 & -y_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{r-1} & 0 & 0 & \cdots & -y_0 \end{pmatrix} \]
\[ = (-y_0)^{r-1} (y_{v-1} - y_0 b_0 - y_1 b_1 - \cdots - y_{r-1} b_{r-1}). \]
We notice that \(\det (TS_{IJ}^{-1}) \neq 0\), since by hypothesis \(y_0, \ldots, y_{v-1}\) are linearly independent over \(K(x)\). Our purpose is to find a lower and an upper bound for \(\text{ord det}(TS_{IJ}^{-1})\).
Since the total degree of the entries of \(B\) is bounded by a constant depending only on \(\Lambda\), there exists a constant \(C_1\), depending on \(\Lambda\) and not on \(\tilde{P}\), such that:
\[ \text{ord det}(TS_{IJ}^{-1}) \leq C_1. \]
Now we are going to determine a lower bound. Let:
\[ \tilde{S}_n = \left( S_{n,0}, S_{n,2}, \ldots, S_{n,v-1} \right), \quad \text{pour tout } n \geq 0; \]
then we have:
\[ S^{<0>} = (S_{i,j})_{i,j \in \{0,1,\ldots,v-1\}}; \]
moreover we set:
\[ A_{\tilde{1}}^{-1} = (A_{i,j})_{i,j \in \{0,1,\ldots,v-1\}}. \]
The elements of the first row of \(T\) are of the form:
\[ \det \begin{pmatrix} y_{v-1} & S_{s,v-1} \\ y_0 & S_{s,0} \end{pmatrix}, \quad \text{pour } s = 0, \ldots, r - 1, \]
and the ones of the \(i\)-th row, for \(i = 1, \ldots, r - 1:\)
\[ \det \begin{pmatrix} y_j & S_{s,i} \\ y_0 & S_{s,0} \end{pmatrix}, \quad \text{pour } s = 0, \ldots, r - 1. \]
Since $\tilde{S}_{n+1} = A_1(x)^{-1}\sigma_q(\tilde{S}_n)$ we have:

$$\det \begin{pmatrix} y_i & S_{x+1,i} \\ y_j & S_{x+1,j} \end{pmatrix} = \det \begin{pmatrix} y_i & \sum_l A_{ij}(x) \sigma_q(S_{ij}) \\ y_j & \sum_l A_{ij}(x) \sigma_q(S_{ij}) \end{pmatrix},$$

d therefore:

$$\inf_{i,j=0,\ldots,\nu-1} \ord \det \begin{pmatrix} y_i & S_{x+1,i} \\ y_j & S_{x+1,j} \end{pmatrix} \geq (s + 1) \ord A_1(x)^{-1} + \inf_{i,j=0,\ldots,\nu-1} \ord \det \begin{pmatrix} y_i & P_i \\ y_j & P_j \end{pmatrix}. $$

Finally,

$$\ord \det T \geq r(\nu - 1) \ord A_1(x)^{-1} + r \inf_{i,j=0,\ldots,\nu-1} \ord \det \begin{pmatrix} y_i & P_i \\ y_j & P_j \end{pmatrix}. $$

By Lemma \[\ref{lemma:ord},\] we obtain:

$$\ord \det S_{x,i,j} \leq \deg \text{(numerator of } \det S_{x,i,j}) \leq \sum_{i=0}^{n-1} \deg \text{(numerator of } S_i) \leq r \deg \tilde{P} + \frac{r(r - 1)}{2}.$$ 

We deduce that:

$$\ord \det \left( T S_{x,i,j}^{-1} \right) \geq \ord \det (T) - \ord \det (S_{x,i,j}) \geq r \left( (\nu - 1) \ord A_1(x)^{-1} + \inf_{i,j=0,\ldots,\nu-1} \ord \det \begin{pmatrix} y_i & P_i \\ y_j & P_j \end{pmatrix} - \deg \tilde{P} - \frac{r(r - 1)}{2} \right) \geq r \inf_{i,j} \ord \det \begin{pmatrix} y_i & P_i \\ y_j & P_j \end{pmatrix} - \deg \tilde{P} + C_2,$$

where $C_2$ is a constant depending only on $\Lambda$. To conclude it is enough to choose a constant $C(\Lambda) > \tilde{C}_1 - C_2$. \qed

### 7.4. Step 3. First part of estimates.

We set:

$$y = \sum_{n \in K'} x^n, \text{ with } y_n \in K',$$

$$\sigma_f(y) = \limsup_{n \to +\infty} \frac{1}{n} \left( \sum_{v \in P, s \leq n} \log^+ |y_v| \right),$$

$$\sigma_\infty(y) = \limsup_{n \to +\infty} \frac{1}{n} \left( \sum_{v \in P, s \leq n} \log^+ |y_v| \right).$$

We recall that we are working under the assumption:

$$\sigma(y) = \limsup_{n \to +\infty} \frac{1}{n} \left( \sum_{v \in P} \tilde{h}(n, v) \right) = \sigma_f(y) + \sigma_\infty(y) < +\infty$$

and that we want to show that $\sigma_c^{(q)}(M) \leq \infty$. Since $\sigma_c^{(q)}(M) \leq \sigma_p^{(q)}(M)$, we will rather show that:

$$\sigma_p^{(q)}(M) = \limsup_{n \to +\infty} \frac{1}{n} \left( \sum_{v \in P} h(M, n, v) \right) < \infty,$$

where:

$$h(M, n, v) = \sup_{s \leq n} \log^+ \frac{G_v}{n! v}. $$

In the sequel $g$ will be a polynomial constructed as in Proposition \[\ref{proposition:polynomial},\] For such a choice of $g$ and for $\tilde{P} = (g\tilde{y})_{\leq N}$, the hypothesis of Corollary \[\ref{corollary:hypothesis},\] Proposition \[\ref{proposition:polynomial},\] and Theorem \[\ref{theorem:main},\] are satisfied.
Proposition 7.12. We have:
\[ \sigma^{(q)}_{\mathcal{P}_f}(\mathcal{M}) \leq \sigma_f(\hat{y}) \left( \frac{\nu^2 t}{1 - \tau} + t \right) + \Omega + \sum_{v \in \mathcal{P}_f} \log^+ |A_1(x)|_{v, \text{Gauss}}, \]
where:
\[ \Omega = \limsup_{n \to +\infty} \frac{1}{n} \left( \nu \sum_{v \in \mathcal{P}_f} h(g, v) + \sum_{v \in \mathcal{P}_f} \log^+ \left| \prod_{i=1}^{\nu-1} Q_i(x) \Delta(x) \right|_{v, \text{Gauss}}^{-1} \right). \]

Proof. We fix \( N, n \gg 0 \) such that:
\[ n + \nu - 1 \leq \frac{N}{t} \frac{1 - \tau}{\nu}. \]

Proposition 7.5 and Corollary 7.2 implies that for all integers \( s \leq n + \nu - 1 \), we have:
\[ (7.12.2) \quad x^s Q_s(x) \frac{d^s_q g(x)}{[s]_q} \hat{g}(x) \leq x^s Q_s(x) \hat{R}_s \]
and:
\[ G[s] = \sum_{i \leq s} (-1)^i \alpha_{i}^{(n)} \frac{d^{s-i}_q (A_i(x) R^{<i>})}{[s-i]_q} (R^{<0>})^{-1}. \]

For all \( v \in \mathcal{P}_f \) we deduce:
\[ |G[s]|_{v, \text{Gauss}} \leq \left( \sup_{i \leq s} \left| \frac{d^{s-i}_q (A_i(x) R^{<i>})}{[s-i]_q} \right|_{v, \text{Gauss}} \right) \left| \text{adj } R^{<0>}_{v, \text{Gauss}} \right| \left| \det R^{<0>}_{v, \text{Gauss}} \right|^{-1}_{v, \text{Gauss}} \]
\[ \leq \left( \sup_{i \leq s} |A_i(x) R^{<i>}_{v, \text{Gauss}}| \right) \left| \text{adj } R^{<0>}_{v, \text{Gauss}} \right| \left| \det R^{<0>}_{v, \text{Gauss}} \right|^{-1}_{v, \text{Gauss}} \]
\[ \leq C_{1,v} \left( \sup_{i \leq s} |\hat{R}|_{v, \text{Gauss}} \right)^{\nu-1} \left( \sup_{i \leq s} |\hat{R}|_{v, \text{Gauss}} \right)^{\nu-1} |\Delta(x)|^{-1}_{v, \text{Gauss}}, \]
where we have set:
\[ C_{1,v} = \sup(1, |A_1(x)|_{v, \text{Gauss}}) \]
and
\[ \Delta(x) = \det R^{<0>}(x). \]

Taking into account our choice of \( N \) and \( n \) and (7.12.2), for all \( i \leq n + \nu - 1 \) we have:
\[ |\hat{R}|_{v, \text{Gauss}} \leq \left| Q_i(x) \right|_{v, \text{Gauss}}^{-1} |\hat{g}|_{v, \text{Gauss}}^{\leq N+it} \]
\[ \leq |\hat{g}|_{v, \text{Gauss}}^{\leq N+it}, \]
therefore:
\[ \sup_{s \leq n} \log^+ |G[s]|_{v, \text{Gauss}} \leq n \log C_{1,v} + \tilde{h} (N + (n + \nu - 1) t, v) + \nu h(g, v) + \log^+ |\Delta|^{-1}_{v, \text{Gauss}}. \]

We set:
\[ \overline{\Delta}(x) = \hat{R}_0 \wedge xQ_1(x) \hat{R}_1 \wedge \cdots \wedge x^{\nu-1} Q_{\nu-1}(x) \hat{R}_{\nu-1} \]
\[ = x^s \left( \prod_{i=1}^{\nu-1} Q_i(x) \right) \Delta(x). \]

The fact that \( |Q_i(x)|_{v, \text{Gauss}} \leq 1 \) and \( x^n Q^n(x) \hat{R}_n \in K[x]^n \), for all integers \( n \geq 1 \), implies that \( |\overline{\Delta}(x)|_{v, \text{Gauss}} \leq |\Delta(x)|_{v, \text{Gauss}} \), with \( \overline{\Delta}(x) \in K[x] \), and:
\[ \sup_{s \leq n} \log^+ |G[s]|_{v, \text{Gauss}} \leq n \log C_{1,v} + \tilde{h} (N + (n + \nu - 1) t, v) + \nu h(g, v) + \log^+ |\overline{\Delta}|^{-1}_{v, \text{Gauss}}. \]
Taking into account condition (7.12.1), we fix a positive integer \( k \) such that:

\[
\begin{aligned}
&k > \frac{\nu(\nu - 1)t}{1 - \theta} \\
&N = \frac{\nu t}{1 - \theta} + \frac{k - \varepsilon_n}{n}, \text{ for some } \varepsilon_n \in (0, 1) \text{ fixed.}
\end{aligned}
\]

Let us set:

\[ C_1 = \sum_{v \in \mathcal{P}^f} \log^+ |A_1(x)|_v \]

and

\[ \Omega = \limsup_{n \to +\infty} \frac{1}{n} \left( \nu \sum_{v \in \mathcal{P}^f} h(g, v) + \sum_{v \in \mathcal{P}^f} \log^+ \left| \overline{\Delta}(x) \right|^{-1}_{v, \text{Gauss}} \right). \]

We obtain:

\[
\sigma_{\mathcal{P}_f}(\mathcal{M}) = \limsup_{n \to +\infty} \frac{1}{n} \left( \sum_{v \in \mathcal{P}^f} \sup_{s \leq n} \log^+ \frac{|G_s|}{|\mathcal{N}|^t_{\mathcal{Q}^f_{v, \text{Gauss}}}} \right) \\
\leq \sigma_{\mathcal{P}_f}(\bar{\mathcal{Y}}) \limsup_{n \to +\infty} \left( \frac{N + (n + \nu - 1)t}{n} + (\nu - 1) \frac{N + (\nu - 1)t}{n} \right) + C_1 + \Omega \\
\leq \sigma_{\mathcal{P}_f}(\bar{\mathcal{Y}}) \left( \frac{\nu^2 t}{1 - \theta} + t + (\nu - 1) \frac{\nu t}{1 - \theta} \right) + C_1 + \Omega \\
\leq \sigma_{\mathcal{P}_f}(\bar{\mathcal{Y}}) \left( \frac{\nu^2 t}{1 - \theta} + t \right) + C_1 + \Omega.
\]

\[ \square \]

7.5. Step 4. Conclusion of the proof of Theorem 4.2

Lemma 7.13. Let \( \Omega \) be as in the previous proposition. Then:

\[ \Omega \leq \frac{\nu^2 t}{1 - \theta} \sigma_{\infty}(\bar{\mathcal{Y}}) + \frac{\nu^2 t(\nu - 1)}{1 - \theta} C_2 + \limsup_{n \to +\infty} \frac{\nu}{n} h(q), \]

where

\[ C_2 = \sum_{v \in \mathcal{P}_\infty} \log(1 + |q|_v) \]

is a constant depending on the \( v \)-adic absolute value of \( q \), for all \( v \in \mathcal{P}_\infty \).

Proof. Let \( \xi \) a root of unity such that:

\[ \overline{\Delta}(\xi) \neq 0 \neq Q_i(\xi) \forall i = 0, \ldots, \nu - 1. \]

Since \( |\overline{\Delta}(\xi)|_v \leq |\overline{\Delta}(x)|_{v, \text{Gauss}} \) for all \( v \in \mathcal{P}_f \), the Product Formula implies that:

\[ \sum_{v \in \mathcal{P}_f} \log^+ |\overline{\Delta}(x)|^{-1}_{v, \text{Gauss}} \leq \sum_{v \in \mathcal{P}_f} \log^+ |\overline{\Delta}(\xi)|^{-1}_v \leq \sum_{v \in \mathcal{P}_\infty} \log^+ |\overline{\Delta}(\xi)|_v. \]

We recall that:

\[ \overline{\Delta}(x) = \det (\hat{R}_0, Q_1(x)\hat{R}_1, \ldots, Q_{\nu - 1}(x)\hat{R}_{\nu - 1}) \]

and that for all \( s \leq \nu - 1 \), (7.12.2) is verified. Moreover we have:

\[
\begin{aligned}
Q_s(x) \cdot \frac{d^n_s}{[s]_q!} (g)(x) &\cdot \bar{g}(x) \\
&= \sum_{n \geq 0} \left( \sum_{i+j+n = n} \left( Q_s \right)_i \left( \frac{d^n_s}{[s]_q!} \right)_j \bar{g}_h \right) x^n \\
&= \sum_{n \geq 0} \left( \sum_{i+j+n = n} \left( Q_s \right)_i \left( \frac{s+j}{[s]_q} \right)_j \bar{g}_{s+j} \right) x^n,
\end{aligned}
\]

where we have used the notation:

\[ \text{for all } P \in K[[x]] \text{ and for all } n \in \mathbb{N}, P_n \text{ is the coefficient of } x^n \text{ in } P. \]
We deduce that $Q_s(\xi)\tilde{R}_s(\xi)$ is a sum of terms of the type:

$$(Q_s)_i\left(\begin{array}{c}s+j \\ j \end{array}\right)_q g_{s+j}\tilde{y}_h\xi^n$$

with:

$$0 \leq s \leq \nu - 1, \quad 0 \leq i \leq \deg Q_s(x), \quad 0 \leq j \leq N, \quad s + j \leq N, \quad 0 \leq h \leq N + (\nu - 1)t.$$ 

For all $v \in \mathcal{P}_\infty$ we obtain:

$$|Q_s(\xi)\tilde{R}_s(\xi)|_v \leq c_v \left(\sup_{s \leq j \leq N} \left|\left(\begin{array}{c}s+j \\ j \end{array}\right)_q\right| \left(\sup_{h \leq N + (\nu - 1)t} |\tilde{y}_h|_v\right) \left(\sup_{j \leq N} |g_j|_v\right)\right),$$

with:

$$c_v = \sup_{s=0, \ldots, \nu-1} \left(\sup_{v=0, \ldots, \deg Q_s} |(Q_s)_i|_v\right).$$

Since $|q|_v \neq 1$, for all $v \in \mathcal{P}_\infty$, we have:

$$\left|\left(\begin{array}{c}s+j \\ j \end{array}\right)_q\right|_v \leq \frac{(1 - q^j) \cdots (1 - q^{j-s+1})}{(1 - q^s) \cdots (1 - q)} |\frac{1 - q^s}{1 - q^j}|^{-1}$$

if $|q|_v < 1$;

$$\left(\frac{1 + |q|_v^N}{|q|_v^{\nu - 1} - 1}\right)^{\nu - 1} \leq \frac{1 + |q|_v^N}{|q|_v^{\nu - 1} - 1}$$

if $|q|_v > 1$;

hence:

$$\sup_{j=s, \ldots, N} \left|\left(\begin{array}{c}s+j \\ j \end{array}\right)_q\right|_v \leq \frac{(\sup(1 + |q|_v, 1 + |q|_v^N)\nu - 1}{\inf((1 - |q|_v, |1 - |q|_v^\nu - 1))^{\nu - 1}} \leq \frac{(1 + |q|_v^N)^{\nu - 1}}{(1 - |q|_v, |1 - |q|_v^\nu - 1))^{\nu - 1}}$$

We obtain the following estimate:

$$|Q_s(\xi)\tilde{R}_s(\xi)|_v \leq c_v \left(\sup_{h \leq N + (\nu - 1)t} |\tilde{y}_h|_v\right) \left(\sup_{j \leq N} |g_j|_v\right).$$

Finally we get:

$$|\Delta(\xi)|_v \leq c_v'^{\nu} \left(\frac{(1 + |q|_v)^{N(\nu - 1)}}{\inf((1 - |q|_v, |1 - |q|_v^\nu - 1))^{\nu - 1}} \right) \left(\sup_{h \leq N + (\nu - 1)t} |\tilde{y}_h|_v\right) \left(\sup_{j \leq N} |g_j|_v\right)^\nu$$

and therefore:

$$\sum_{v \in \mathcal{P}_\infty} \log^+ |\Delta(\xi)|_v \leq \text{const} + N\nu(\nu - 1)C_2$$

$$-\nu(\nu - 1) \sum_{v \in \mathcal{P}_\infty} \log \inf((1 - |q|_v, |1 - |q|_v^\nu - 1))^{\nu - 1}$$

$$+ \nu \sum_{v \in \mathcal{P}_\infty} \bar{h}(g, v) + \nu \sum_{v \in \mathcal{P}_\infty} \bar{h} \left(N + (\nu - 1)t, v\right).$$

where:

$$C_2 = \sum_{v \in \mathcal{P}_\infty} \log (1 + |q|_v).$$

We recall that by (7.12.3), we have:

$$\lim_{n \to +\infty} \frac{N}{n} = \frac{\nu}{1 - \tau}. $$
and:
\[
\lim_{n \to +\infty} \frac{\log N}{n} = 0.
\]
So we can conclude since:
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{v \in P_f} \log^+ |\mathfrak{M}(x)|_{v, \text{Gauss}}^{-1} \leq \limsup_{n \to +\infty} \frac{1}{n} \sum_{v \in P_{\infty}} \log^+ |\mathfrak{M}(\xi)|_{v}
\]
\[
\leq \limsup_{n \to +\infty} \frac{N\nu(\nu - 1)C_2}{n} + \nu n \sum_{v \in P_{\infty}} h(g, v) + \nu n \sum_{v \in P_{\infty}} \bar{h}(N + (\nu - 1)t, v)
\]
\[
\leq \frac{t\nu^2(\nu - 1)}{1 - \tau} C_2 + \limsup_{n \to +\infty} \nu n \sum_{v \in P_{\infty}} h(g, v)
\]
\[
\leq \frac{t\nu^2}{1 - \tau} \sigma_\infty(\bar{g}) + \frac{t\nu^2(\nu - 1)}{1 - \tau} C_2 + \limsup_{n \to +\infty} \frac{\nu}{n} \sum_{v \in P_{\infty}} h(g, v)
\]

**Conclusion of the proof of Theorem 4.2.** Proposition 7.4 implies that:
\[
\limsup_{n \to +\infty} \frac{\nu}{n} h(g) \leq \limsup_{n \to +\infty} \frac{\nu}{n} \left( \text{const} + \frac{1 - \tau}{\tau} \sum_{v \in P} \bar{h} \left( N + \frac{1 - \tau}{\nu}, v \right) \right)
\]
\[
\leq \limsup_{n \to +\infty} \frac{1 - \tau}{\tau} \sum_{v \in P} \bar{h} \left( N + \frac{1 - \tau}{\nu}, v \right)
\]
\[
\leq \frac{1 - \tau}{\tau} \nu \sigma(\bar{g}) \limsup_{n \to +\infty} \frac{1}{n} \left( N + \frac{1 - \tau}{\nu} \right)
\]
\[
\leq \frac{1 - \tau}{\tau} \nu t \left( 1 + \frac{\nu}{1 - \tau} \right) \sigma(\bar{g})
\]

which, combined with Propositions 7.12 and 7.13 implies that:
\[
\sigma^\nu_{P_f}(\mathcal{M}) \leq \sigma_f(\bar{g}) \left( \frac{\nu^2 t}{1 - \tau} + t \right) + \sigma_{\infty}(\bar{g}) \frac{\nu^2 t}{1 - \tau} + \sigma(\bar{g}) \frac{1 - \tau}{\tau} vt \left( 1 + \frac{\nu}{1 - \tau} \right)
\]
\[
+ \log C_1 + \frac{\nu^2(\nu - 1)t}{1 - \tau} C_2
\]
\[
\leq \sigma(\bar{g}) \left( \frac{\nu^2 t}{1 - \tau} + \nu^2 t \left( \frac{1}{\nu} + \frac{1 - \tau}{\nu^2 t} \right) + t \right) + \log C_1 + \frac{\nu^2(\nu - 1)t}{1 - \tau} C_2
\]
\[
\leq \sigma(\bar{g}) \left( \nu^2 t \left( \frac{\nu + 1}{\nu} + \frac{1}{1 - \tau} \right) - \nu t + t \right) + \log C_1 + \frac{\nu^2(\nu - 1)t}{1 - \tau} C_2.
\]
The function \(\frac{\nu + 1}{\nu} + \frac{1}{1 - \tau}\) has a minimum for
\[
\tau = \left( 1 + \sqrt{\frac{\nu}{\nu + 1}} \right)^{-1};
\]
for this value of \(\tau\) we get:
\[
\frac{\nu + 1}{\nu} + \frac{1}{1 - \tau} = \left( 1 + \sqrt{\frac{\nu + 1}{\nu}} \right) \leq \begin{cases} 4.95 & \text{for } \nu \geq 2 \\ 5.9 & \text{for } \nu = 1 \end{cases}.
\]
Finally we have:
\[
\sigma^\nu_{P_f}(\mathcal{M}) \leq \log C_1 + \frac{\nu^2(\nu - 1)t}{1 - \tau} C_2 + \begin{cases} \sigma(\bar{g}) \left( 4.95\nu^2 t - \nu t + (t - 1) \right) & \text{for } \nu \geq 2 \\ \sigma(\bar{g})5.9t & \text{for } \nu = 1 \end{cases},
\]
where
\[
C_1 = \sum_{v \in P_f} \log^+ |A_1(x)|_{v, \text{Gauss}}
\]
and
\[ C_2 = \sum_{v \in \mathcal{P}_\infty} \log(1 + |q|_v). \]

\[
\text{Part 2. Global } q\text{-Gevrey series}
\]

8. Definition and first properties

The notation is the same as in Part 1. We recall that \( K \) is a finite extension of \( k(q) \), equipped with its family of ultrametric norms, normalized so that the Product Formula holds. The field \( K(x) \) is naturally a \( q \)-difference algebra with respect to the operator \( \sigma_q : f(x) \mapsto f(qx) \).

**Definition 8.1.** We say that the series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in K[[x]] \) is a global \( q \)-Gevrey series of orders \((s_1, s_2) \in \mathbb{Q}^2\) if it is solution of a \( q \)-difference equation with coefficients in \( K(x) \) and

\[
\sum_{n=0}^{\infty} \left( q^{\frac{n(n-1)}{2}} \right)^{\frac{a_n}{(n!)_q^{s_1}(n!)_q^{s_2}}} x^n
\]

is a \( G_q \)-function.

**Remark 8.2.** We point out that:

1. The definition above forces \( s_2 \) to be an integer, in fact the \( q \)-holonomy condition implies that the coefficients \( |n|_q^{s_2} \), for \( n \geq 1 \), are all contained in a finite extension of \( k(q) \).

2. Being a global \( q \)-Gevrey series of orders \((s_1, s_2) \) implies being a \( q \)-Gevrey series of order \( s_1 + s_2 \) in the sense of [BB92] for all \( v \in \mathcal{P}_\infty \) extending the \( q^{-1} \)-adic norm, i.e. for the norms that verify \( |q|_v > 1 \): this simply means that \( |q|^{-\frac{n(n-1)}{2}} |n|_q^{s_2} \), as the same growth as \( |q|_v^{(s_1+s_2)\frac{1}{|n|_q}} \).

If \( v \in \mathcal{P}_\infty \) and \( |q|_v < 1 \), then \( |n|_q \geq 1 \). Therefore a global \( q \)-Gevrey series of orders \((s_1, s_2) \) is a \( q \)-Gevrey series of order \( s_1 \) in the sense of [BB92]. This remark actually justifies the choice of considering two orders, instead of one as in the analytic theory.

In the local case, both complex (cf. [Béz92], [MZ00], [Zha99]) and \( p \)-adic (cf. [BB92]), the \( q \)-Gevrey order is not uniquely determined. The global situation considered here is much more rigid: the same happens in the differential case.

**Proposition 8.3.** The orders of a given global \( q \)-Gevrey series \( \sum_{n=0}^{\infty} a_n x^n \in K[[x]] \setminus K[x] \) are uniquely determined.

*Proof.* Suppose that \( \sum_{n=0}^{\infty} a_n x^n \) is a global \( q \)-Gevrey series of orders \((s_1, s_2) \) and \((t_1, t_2) \). By definition

\[
\sum_{n=0}^{\infty} \left( q^{\frac{n(n-1)}{2}} \right)^{\frac{a_n}{(n!)_q^{s_1}(n!)_q^{s_2}}} x^n \quad \text{and} \quad \sum_{n=0}^{\infty} \left( q^{\frac{n(n-1)}{2}} \right)^{\frac{a_n}{(n!)_q^{t_1}(n!)_q^{t_2}}} x^n
\]

have finite size. We have:

\[
\sum_{n=0}^{\infty} \left( q^{\frac{n(n-1)}{2}} \right)^{\frac{a_n}{(n!)_q^{t_1}(n!)_q^{t_2}}} x^n = \sum_{n=0}^{\infty} \left( q^{\frac{n(n-1)}{2}} \right)^{\frac{a_n}{(n!)_q^{t_1-t_2}}} \left( \frac{a_n}{(n!)_q^{s_2-t_1}} \right) x^n.
\]

One observes that having finite size implies having finite radius of convergence for all \( v \in \mathcal{P} \), therefore for all \( v \) such that \( |q|_v \neq 1 \) we must have:

\[
\limsup_{n \to \infty} \left( q^{\frac{n(n-1)}{2}} \right)^{\frac{a_n}{(n!)_q^{s_2-t_1}}} \left( \frac{a_n}{(n!)_q^{s_2-t_1}} \right) x^n < \infty.
\]

If \( |q|_v > 1 \) this implies:

\[
\limsup_{n \to \infty} |q|_v^{\frac{n(n-1)}{2}} (s_1 + s_2 - (t_1 + t_2)) < \infty.
\]

Since for all \( v \in \mathcal{P} \) such that \( |q|_v < 1 \) the limit \( \limsup_{n \to \infty} |n|_q^{1/n} \) is bounded we get:

\[
\limsup_{n \to \infty} |q|_v^{\frac{n(n-1)}{2}} (s_1 - t_1) < \infty.
\]
We deduce that necessarily \( s_1 + s_2 \leq t_1 + t_2 \) and \( t_1 \leq s_1 \) and \( s_2 \leq t_2 \). Since the role of \((t_1, t_2)\) and \((s_1, s_2)\) is symmetric, one obviously obtain the opposite inequalities in the same way. \(\square\)

8.1. Changing \(q\) in \(q^{-1}\). One can transform a \(q\)-difference equations in a \(q^{-1}\)-difference equations, obtaining:

**Proposition 8.4.** Let \( f(x) \in K[[x]] \) be a global \(q\)-Gevrey series of orders \((-s_1, -s_2) \in \mathbb{Q}^2\), then \( f(x) \) is a global \(q^{-1}\)-Gevrey series of orders \((s_1 + s_2, -s_2)\).

In particular, if \( f(x) \) is a global \(q\)-Gevrey series of orders \((t_1, -t_2)\), with \( t_1 \geq t_2 \geq 0 \), then \( f(x) \) is a global \(q^{-1}\)-Gevrey series of negative orders \((-t_1 - t_2, -t_2)\).

**Proof.** It is enough to write \( f(x) \) in the form:

\[
f(x) = \sum_{n=0}^{\infty} \frac{a_n}{q^{n(n+1)/2}} \left( \frac{[n]_q}{[n]_q!} \right)^{s_2} x^n = \sum_{n=0}^{\infty} \frac{a_n}{q^{-n(n-1)/2} \left( [n]_q! \right)^{s_1} \left( [n]_q^{-1} \right)^{s_2} x^n},
\]

where \( \sum_n a_n x^n \) is a convenient \(G_q\)-function. \(\square\)

8.2. Rescaling of the orders. Clearly we can always look at a global \(q\)-Gevrey series of orders \((s,0)\) as a global \(q^{-1}\)-Gevrey series of orders \((s/t,0)\), for any \( t \in \mathbb{Q}, t \neq 0 \), the holonomy condition being always satisfied:

**Lemma 8.5.** Let \( t \in \mathbb{Q}, t \neq 0 \). If \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) is solution of a \(q\)-difference equation then it is solution of a \(q^t\)-difference equation.

**Proof.** If \( f(x) \) is solution of a \(q\)-difference equation, then it is also solution of a \(q^{-1}\)-difference equation. Therefore we can suppose \( t > 0 \). Let \( t = \frac{p}{r} \), with \( p, r \in \mathbb{Z}_{>0} \). Since \( f(x) \) is solution of a \(q\)-difference operator, we have:

\[
\dim_{K(x)} \sum_{i \geq 0} K(x) \sigma_q^i(f(x)) < +\infty.
\]

Then:

\[
\dim_{K(x)} \sum_{i \geq 0} K(x) \sigma_q^{it}(f(x)) = \dim_{K(x)} \sum_{i \geq 0} K(x) \sigma_q^{it}(f(x)) \leq \dim_{K(x)} \sum_{i \geq 0} K(x) \sigma_q^i(f(x)) < +\infty,
\]

so \( f(x) \) is solution of a \(q^t\)-difference operator. Finally we can conclude since \( \sum_{i=0}^{\infty} a_i(x)f(q^{itr}x) = 0 \) implies that \( \sum_{i=0}^{\infty} a_i(x)f(\bar{q}^{itr}x) = 0 \).

Unfortunately, the same is not true for global \(q\)-Gevrey series of orders \((0,s)\). To prove it, one can calculate size of the series

\[
\Phi(x) = \sum_{n \geq 0} \frac{(\bar{q}; q)_n}{(q; q)_n} x^n,
\]

where \( \bar{q} \) is a \( r \)-th root of \( q \), for some positive integer \( r \), \( K = \mathbb{Q}(\bar{q}) \) and \( t \) is an integer. The Pochhammer symbols \((\bar{q}; q)_n \) and \((q; q)_n\) are both polynomials in \( \bar{q}^{1/2} \) of degree \( tn(n+1) \) and \( rn(n+1) \), respectively. If we want \( \Phi(x) \) to have finite size, we are forced to take \( t \leq r \), so that it has positive radius of convergence at any place \( v \) such that \( |q|_v > 1 \). Notice that \( \Phi(x) \) is convergent for any place \( v \) such that \( |q|_v < 1 \) and that the noncyclotomic places give a zero contribution to the size. As far as the cyclotomic places of \( K \) is concerned, we obtain

\[
\sigma_C(y) = \lim_{n \to \infty} \sup_n \frac{1}{n} \sum_{k=1}^{n} \left( \frac{n}{K}(k, r) - t \left[ \frac{n}{K} \right] \right) \log^+ d^{-1} \sim \lim_{n \to \infty} \sup_n \frac{1}{n} \sum_{k=1}^{n} \left( (k, r) - t \right) \log d^{-1}.
\]

The limit above is infinite.
9. Formal Fourier transformations

The following natural two $q$-analogues of the usual formal Borel transformation

\[ (\cdot)^+ : \quad K[[x]] \longrightarrow K[[z^{-1}]] \]

\[ F = \sum_{n=0}^{\infty} a_n x^n \quad \longrightarrow \quad F^+ = \sum_{n=0}^{\infty} [n]_q a_n z^{-n-1} \]

and

\[ (\cdot)^\# : \quad K[[x]] \longrightarrow K[[z^{-1}]] \]

\[ F = \sum_{n=0}^{\infty} a_n x^n \quad \longrightarrow \quad F^\# = \sum_{n=0}^{\infty} q^{n(n-1)} z^{-n-1} \]

are equally considered in the literature on $q$-difference equations. From an archimedean analytical point of view, they are equivalent as soon as one works under the hypothesis that $|q| \neq 1$ (cf. [MZ00 §8] and [DVZ07 Part II]). As already noticed in [And00b], from a global point of view, $(\cdot)^+$ and $(\cdot)^\#$ have a completely different behavior: for the same reason the definition of global $q$-Gevrey series involves two orders.

Let $p = q^{-1}$ and let $\sigma_p : z \mapsto pz$, $d_p = \frac{\sigma_q-1}{(p-1)x}$. The Borel transformations that we have introduced above have the following properties:

**Lemma 9.1.** For all $F = \sum_{n=0}^{\infty} a_n x^n \in K[[x]]$ we have:

\[ (xF)^+ = -pd_p F^+ , \quad (d_q F)^+ = z F^+ - F(0) , \]

\[ (xF)^\# = \frac{p}{z} \sigma_p F^\# , \quad (\sigma_q F)^\# = p \sigma_p F^\# . \]

**Proof.** We deduce the first equality using the relation:

\[ -pd_p \frac{1}{z^n} = [n]_q \frac{1}{z^{n+1}} . \]

All the other formulas easily follow from the definitions. \hfill \Box

For any $q$-difference operator $\sum_{i=0}^{N} a_i(x) \sigma_q^i \in K(x)[\sigma_q]$ (resp. $\sum_{i=0}^{N} b_i(x) d_q^i \in K(x)[d_q]$) we set:

\[ \deg_{\sigma_q} \sum_{i=0}^{\nu} a_i(x) \sigma_q^i = \sup \{ i \in \mathbb{Z} : 0 < i < \nu , \quad a_i(x) \neq 0 \} \]

(resp. $\deg_{d_q} \sum_{i=0}^{\nu} b_i(x) d_q^i = \sup \{ i \in \mathbb{Z} : 0 < i < \nu , \quad b_i(x) \neq 0 \}$)

Obviously we have $K(x)[d_q] = K(x)[\sigma_q]$ and $\deg_{d_q} = \deg_{\sigma_q}$ (for explicit formulas cf. [DV02 1.1.10] and [10.0.3] below). The previous lemma justifies the definition of the formal Fourier transformations below, acting on the skew rings $K[x,d_q]$ and $K[x,\sigma_q]$:

**Definition 9.2.** We call the maps:

\[ F_{q^+} : K[x,d_q] \longrightarrow K[z,d_p] \quad \text{and} \quad F_{q^\#} : K[x,\sigma_q] \longrightarrow K[\frac{1}{z},\sigma_p] \]

\[ d_q \longrightarrow z \quad \sigma_q \longrightarrow p \sigma_p \]

\[ x \longrightarrow -pd_p \quad x \longrightarrow \frac{1}{qz} \sigma_p \]

the $q^+$-Fourier transformation and the $q^\#$-Fourier transformation respectively.
Remark 9.3. Let $\mathcal{F}_p : K[z, d_p] \to K[x, d_q]$ and let $\lambda : K [x, d_q] \to K [x, d_q], d_q \mapsto -\frac{1}{q} d_q$, $x \mapsto -qx$ Then $\mathcal{F}_q^{-1} = \lambda \circ \mathcal{F}_p$. 

As far as $\mathcal{F}_{q^+}$ is concerned, if $\mathcal{L} = \sum_{i=0}^{\nu} a_i(\frac{1}{x}) \sigma_q^i \in K \left[ \frac{1}{x}, \sigma_q \right]$ is such that $\deg_{x} a_i \left( \frac{1}{x} \right) \leq i$, there exists a unique $\mathcal{N} \in K [x, \sigma_q]$ such that $\mathcal{F}_{q^+} (\mathcal{N}) = \mathcal{L}$ and we note $\mathcal{F}_{q^+}^{-1} (\mathcal{L}) = \mathcal{N}$.

In the following lemma we verify that the formal Fourier transformations we have just defined are compatible with the Borel transformations $(\cdot)^+$ and $(\cdot)^\#$.

Lemma 9.4. Let $F \in K[[x]]$ be a series solution of a $q$-difference linear operator $\mathcal{N} \in K [x, d_q]$, such that $\nu = \deg_{d_q} \mathcal{N}$ (resp. $\mathcal{N} \in K [x, \sigma_q]$). Then $d_{q^{-1}}^\nu \circ \mathcal{F}_{q^+} (\mathcal{N}) F^+ = 0$ (resp. $\mathcal{F}_{q^+}(\mathcal{N}) F^\# = 0$).

Inversely:

1. If $F^+$ is a solution of $\mathcal{L}_1 \in K [z, d_p]$, then $\mathcal{F}_{q^+}^{-1} (\mathcal{L}_1) F = 0$.

2. If $\mathcal{L}_2 \in K \left[ \frac{1}{x}, \sigma_p \right]$ is such that $\mathcal{L}_2 F^\# = 0$, for all $n \in \mathbb{N}$, $n >> 0$, we have: $\mathcal{F}_{q^+}^{-1} (\sigma_p^n \circ \mathcal{L}_2) F = 0$.

Proof. We prove the statements for $(\cdot)^+$. The proof for $(\cdot)^\#$ is quite similar. We write $\mathcal{N}$ in the form:

$$\mathcal{N} = \sum_{j=0}^{\nu} \sum_{i=0}^{N} a_{i,j} x^i d^j_q \in K [x, d_q].$$

Lemma 9.1 implies that $\mathcal{F}_{q^+} (\mathcal{N}) F^+$ is a polynomial of degree less or equal to $\nu$, therefore $d_{q^{-1}}^\nu \circ \mathcal{F}_{q^+} (\mathcal{N}) F^+ = 0$. Let us now write $\mathcal{L}_1$ as:

$$\mathcal{L}_1 = \sum_{j=0}^{\nu} \sum_{i=0}^{N} a_{i,j} z^i d^j_p \in K [z, d_p].$$

Then $\left( \mathcal{F}_{q^+}^{-1} (\mathcal{L}_1) F \right)^+$ is a polynomial of degree less or equal to $\nu$. Hence we obtain:

$$d_{q^{-1}}^\nu \left( \mathcal{F}_{q^+}^{-1} (\mathcal{L}_1) F \right)^+ = \left( (-qx)^\nu \mathcal{F}_{q^+}^{-1} (\mathcal{L}_1) F \right)^+ = 0$$

and finally $(-qx)^\nu \mathcal{F}_{q^+}^{-1} (\mathcal{L}_1) F = 0$. \hfill $\Box$

Remark 9.5. In the following we will use the formal Fourier transformations above composed with the symmetry $S: z \mapsto 1/x$:

$$S \circ \mathcal{F}_q^+ : K [x, d_q] \to K \left[ \frac{1}{x}, x, d_q \right] \quad \text{and} \quad S \circ \mathcal{F}_q^\# : K [x, \sigma_q] \to K [x, \sigma_q]$$

\begin{align*}
\begin{array}{c c c c c c c c c}
\text{Lemma 9.1} & & & & & & & & & \\
\end{array}
\end{align*}

(9.5.1) $$
\begin{array}{c c c c c c c c c}
\begin{array}{cccc}
\nu & \nu & \nu & \nu & \nu & \nu & \nu & \nu & \nu \\
\end{array}
\end{array}
\begin{array}{cccc}
\text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} \\
\text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} \\
\end{array}
\end{align*}

(9.5.1) $d_q \mapsto \frac{1}{x}, \quad \sigma_q \mapsto \frac{1}{q} \sigma_q, \quad x \mapsto x^2 d_q, \quad x \mapsto \frac{x}{q} \sigma_q$

Notice that $S \circ \mathcal{F}_q^+ (d_q \circ x) = xd_q$.

10. Action of the formal Fourier transformations on the Newton polygon

Let as consider a linear $q$-difference operator:

$$\mathcal{N} = \sum_{i=0}^{\nu} a_i (x) x^i d_q^i = \sum_{i=0}^{\nu} b_i (x) \sigma_q^i,$$

such that $b_j(x), a_j(x) \in K[x]$. Applying formulas [DV02, 1.1.10], we obtain:

$$\mathcal{N} = \sum_{j=0}^{\nu} b_j (x) \sum_{i=0}^{j} \binom{j}{i} \frac{1}{q} (1-q)^i q^{(i-1)/2} x^i d_q^i$$

(10.0.3) $\begin{array}{c c c c c c c c c}
\begin{array}{cccc}
\nu & \nu & \nu & \nu & \nu & \nu & \nu & \nu & \nu \\
\end{array}
\end{array}
\begin{array}{cccc}
\text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} \\
\text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} \\
\end{array}
\end{align*}

(10.0.3) $\begin{array}{c c c c c c c c c}
\begin{array}{cccc}
\nu & \nu & \nu & \nu & \nu & \nu & \nu & \nu & \nu \\
\end{array}
\end{array}
\begin{array}{cccc}
\text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} \\
\text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} & \text{and} \\
\end{array}
\end{align*}$

Therefore $a_j (x) = (1-q)^i q^{(i-1)/2} \sum_{j=i}^{\nu} \binom{j}{i} \frac{1}{q^i} b_j (x)$.

We recall the definition of the Newton-Ramis Polygon:
Definition 10.1. Let $N = \sum_{i=0}^{\nu} a_i(x) x^i d_q^i = \sum_{i=0}^{\nu} b_i(x) \sigma_q^i$ be such that $b_j(x), a_j(x) \in K[x]$. Then we define the Newton-Ramis Polygon of $N$ with respect to $\sigma_q$ (and we write $\text{NRP}_{\sigma_q}(N)$) (resp. with respect to $d_q$ (and we write $\text{NRP}_{d_q}(N)$)) to be the convex hull of the following set:

$$\bigcup_{b_i(x) \neq 0} \{(u, v) \in \mathbb{R}^2 : u = i, \deg_x b_i(x) \geq v \geq \text{ord}_x b_i(x)\} \subset \mathbb{R}^2.$$  

For an operator with rational coefficient $N$, we set $\text{NRP}_{\sigma_q}(N) = \text{NRP}_{\sigma_q}(f(x)N)$ and $\text{NRP}_{d_q}(N) = \text{NRP}_{d_q}(f(x)N)$, where $f(x)$ is a polynomial in $K[x]$ such that $f(x)N$ can be written as above. In this way the Newton-Ramis polygon is defined up to a vertical shift, so that its slopes are actually well-defined.

Lemma 10.2. We have:

$$\text{NRP}_{d_q}(N) = \bigcup_{(u_0, v_0) \in \text{NRP}_{\sigma_q}(N)} \{(u, v) \in \mathbb{R}^2 : u \leq u_0\}.$$  

Proof. The statement follows from (10.0.3). \qed

The following proposition describes the behavior of the Newton-Ramis Polygon with respect to $\mathcal{F}_q^+$ and $\mathcal{F}_q^\#$.

Proposition 10.3. The map\footnote{To make the notation clear, we underline that we denote $\text{NRP}_{\sigma_q} \left( \mathcal{F}_q^\# \right)$ the Newton-Ramis Polygon of $\mathcal{F}_q^\#$ defined with respect to $z$ and $\sigma_q$ and $\text{NRP}_{d_q} \left( \mathcal{F}_q^+ \right)$ the Newton-Ramis Polygon of $\mathcal{F}_q^+$ defined with respect to $z$ and $d_q$.}:

$$\begin{align*}
\text{NRP}_{\sigma_q}(N) &\rightarrow \text{NRP}_{\sigma_q} \left( \mathcal{F}_q^\# \right) \\
(u, v) &\mapsto (u + v, -v) \\
\text{NRP}_{d_q}(N) &\rightarrow \text{NRP}_{d_q} \left( \mathcal{F}_q^+ \right) \\
(u, v) &\mapsto (u + v, -v)
\end{align*}$$

is a bijection between $\text{NRP}_{\sigma_q}(N)$ and $\text{NRP}_{\sigma_q} \left( \mathcal{F}_q^\# \right)$ (resp. $\text{NRP}_{d_q}(N)$ and $\text{NRP}_{d_q} \left( \mathcal{F}_q^+ \right)$).

Proof. As far as $\mathcal{F}_q^\#$ is concerned, it is enough to notice that:

$$\mathcal{F}_q^\# \left( \sum_{i=0}^N \sum_{j=0}^N b_{i,j} x^i z^j \right) = \sum_{i=0}^\nu \sum_{j=0}^N \frac{b_{i,j}}{q^{j(j+1)/2} z^j} \sigma_q^{i+j}.$$  

Let

$$N = \sum_{i=0}^\nu \sum_{j=0}^N a_{i,j} d_q^i.$$  

We have:

$$\begin{align*}
\mathcal{F}_q^+ \left( N \right) &= \sum_{i=0}^\nu \sum_{j=0}^N \frac{(-1)^j a_{i,j}}{q^j} d_q^j \circ z^i \\
&= \sum_{i=0}^\nu \sum_{j=0}^N \sum_{h=0}^j \frac{(-1)^j a_{i,j}}{q^j} \left( \begin{array}{c} j \\ h \end{array} \right) \\
&\quad \frac{[i]_q!}{[h-i]_q!} \left( \begin{array}{c} j-h \\ h \end{array} \right) q^{j-h} d_p^{j-h} \circ h^i \circ z^{i-h} \circ d_p^{j-h}.
\end{align*}$$

Then if $(i, j - i) \in \text{NRP}_{d_q}(N)$ we have:

$$(j - h, i - j) \in \text{NRP}_{d_p} \left( \mathcal{F}_q^+ \left( N \right) \right) \text{ for all } h = 0, \ldots, j.$$  

The statement follows from this remark. \qed
Corollary 10.4. In the notation of the previous proposition, $F_{q^*}$ (resp. $F_{q^+}$) acts in the following way on the slopes of the Newton-Ramis Polygon:

\[
\{ \text{slopes of } NRP_{\sigma_p}(\mathcal{N}) \} \quad \rightarrow \quad \{ \text{slopes of } NRP_{\sigma_p}(F_{q^*}(\mathcal{N})) \}
\]

(resp. \[
\{ \text{slopes of } NRP_{d_q}(\mathcal{N}) \} \quad \rightarrow \quad \{ \text{slopes of } NRP_{d_q}(F_{q^*}(\mathcal{N})) \}
\].

\[
\begin{align*}
\lambda & \quad \mapsto \quad -\frac{\lambda}{1+\lambda} \\
\infty & \quad \mapsto \quad -1
\end{align*}
\]

11. Solutions at points of $K^*$

We have described what happens at zero and at $\infty$ when the Fourier transformations act. Now we want to describe what happens at a point $\xi \in K^* = \mathbb{P}^1(K) \smallsetminus \{ 0, \infty \}$.

To construct some formal solutions of our $q$-difference operators at $\xi \in K^*$, we are going to consider a ring defined as follows (cf. [DV04, §1.3]). For any $\xi \in K$ and any nonnegative integer $n$, we consider the polynomials

\[
T^n_q(x, \xi) = x^n \left( \frac{\xi}{x} q \right)_n = (x - \xi)(x - q\xi) \cdots (x - q^{n-1}\xi).
\]

One verifies directly that for any $n \geq 1$

\[
d_q T^n_q(x, \xi) = [n]_q T^n_q(x, \xi)
\]

and $d_q T^n_0(x, \xi) = 0$. The product $T^n_q(x, \xi)T^n_q(x, \xi)$ can be written as a linear combination with coefficients in $K$ of $T^n_q(x, \xi), T^n_1(x, \xi), \ldots, T^n_{n+m}(x, \xi)$ (cf. [DV04, §1.3]). It follows that we can define the ring:

\[
K[[x - \xi]]_q = \left\{ \sum_{n \geq 0} a_n T^n_q(x, \xi) : a_n \in K \right\},
\]

with the obvious sum and the Cauchy product described above, extended by linearity. The ring $K[[x - \xi]]_q$ is a $q$-difference algebra with the natural action of $d_q$. Notice that in general it makes no sense to look at the sum of those series. Nevertheless, they can be evaluated at the point of the set $\xi q^{Z_{\geq 0}}$, and they are actually in bijective correspondence with the sequences $\{ f(q^n) \}_{n \in \mathbb{Z}_{\geq 0}} \in \mathbb{C}^\mathbb{N}$.

Proposition 11.1. Let $\mathcal{N} \in K[x, d_q]$ be a linear $q$-difference operator such that $NRP_{d_q}(\mathcal{N})$ has only the zero slope at $\infty^4$; then the operator $F_{q^*}\mathcal{N}$ has a basis of solution in $K[[z - \xi]]_p$ for all $\xi \in K^*$.

Proof. The hypothesis on the Newton Polygon of $\mathcal{N}$ at $\infty$ implies that we can write $\mathcal{N}$ in the following form

\[
\mathcal{N} = \sum_{i=0}^{\nu} \sum_{j=0}^{N} a_{i,j} x^i d_q^j,
\]

with $a_{i,N} = 0$ for all $i = 0, \ldots, \nu - 1$ and $a_{\nu,N} \neq 0$. This implies that the coefficient of $d_q^N$ in

\[
F_{q^*}(\mathcal{N}) = \sum_{i=0}^{\nu} \sum_{j=0}^{N} a_{i,j} (-pd_q^j) \circ z^i
\]

\[
= \sum_{j=0}^{N-1} \sum_{i=0}^{\nu} c_{i,j} z^i d_q^j + a_{\nu,N} (-q)^{-N} z^\nu d_q^N
\]

does not have any zero in the set $\{ q^n \xi : n \in \mathbb{Z}_{\geq 0} \}$. Using the fact that $d_p T^n_p(z, \xi) = [n]_p T^n_{p-1}(z, \xi)$ and that $z T^n_p(z, \xi) = T^n_{p+1}(z, \xi) + p^\xi T^n_{p}(z, \xi)$, a basis of solutions of $F_{q^*}(\mathcal{L})$ in $K[[z - \xi]]_p$ can be constructed working with the recursive relation induced by $F_{q^*}(\mathcal{L})y = 0$ on the coefficients of a generic solution of the form $\sum_{n} a_n T^n(z, \xi)$. \hspace{1cm} \Box

\footnote{or equivalently, $NRP_{d_q}(\mathcal{N})$ has no negative slopes.}
Corollary 11.2. For any \( N \in K[z, d_p] \) (resp. \( N \in K[x, d_q] \), \( N \in K[z, d_p] \)) having only the zero slope at \( \infty \), the operator \( F_{q^+}^{-1}(N) \) (resp. \( S \circ F_{q^+}^{-1}(N) \), \( S \circ F_{q^+}^{-1}(N) \)) has a basis of solution in \( K[[x - \xi]]_q \) for any \( \xi \in K^* \).

Proof. The statement follows from the remark that \( F_{q^+}^{-1}(N) = \lambda \circ F_{q^+}^{-1}(N) \) and that the symmetry \( S : z \mapsto 1/z \) does not change the kind of singularity at the points of \( K^* \).

An analogous property holds for \( F_{q^+}^{-1} \).

Proposition 11.3. Let \( L = \sum_{i=0}^{\nu} a_i \left( \frac{1}{z} \right) \sigma_p \in K \left[ \frac{1}{z}, \sigma_p \right] \) such that \( \deg_\frac{1}{z} a_i(\frac{1}{z}) \leq i \). We suppose that

\[
N = \text{ord}_{\frac{1}{z}} a_i \left( \frac{1}{z} \right) \leq \text{ord}_{\frac{1}{z}} a_i \left( \frac{1}{z} \right),
\]

for all \( i = 0, \ldots, \nu - 1 \). Then \( F_{q^+}^{-1}(L) \) has a basis of solution in \( K[[x - \xi]]_q \) for all \( \xi \in K^* \).

Proof. We call \( a_{\nu, N} \in K \) the coefficients of \( \frac{1}{z}^N \) in \( a_{\nu} \left( \frac{1}{z} \right) \). Then we have:

\[
F_{q^+}^{-1}(L) = \sum_{i=0}^{\nu - N - 1} b_i(x) \sigma_q^i + a_{\nu, N} x^N \sigma_q^{\nu - N}.
\]

One ends the proof as above.

\[\square\]

12. Structure theorems

Inspired by [And00a], we want to characterize \( q \)-difference operators killing the global \( q \)-Gevrey series of orders \((-s_1, -s_2)\), with \((s_1, s_2) \in \mathbb{Z} := \mathbb{Q}_{\geq 0} \times \mathbb{Z}_{\geq 0} \setminus \{(0, 0)\}\).

The skew polynomial ring \( K(x)[d_q] \) is euclidean with respect to \( \text{deg}_{d_q} \). It follows that, if we have a formal power series \( y(x) \) solution of a \( q \)-difference operator, we can find a \( q \)-difference operator \( L \) killing \( y \) and such that \( \text{deg}_{d_q} L \) is minimal. All the other linear \( q \)-difference operators killing \( y \), minimal with respect to \( \text{deg}_{d_q} \), are of the form \( f(x)L \), with \( f(x) \in K(x) \). By abuse of language, we will call the minimal degree operator \( L \in K[x, d_q] \) (resp. \( K[x, \sigma_q] \)) with no common factors in the coefficients the minimal operator killing \( y \).

Remark 12.1. Let \( y(x) \in K[[x]] \) be a formal power series solution of the linear \( q \)-difference operator \( L_q = \sum_{i=0}^{\nu} a_i(x) \sigma_q^i \). We choose \( L_q \) such that \( \text{deg}_{\sigma_q} L_q \) is minimal. Then for all positive integers \( r \) the operator \( L_{q^{1/r}} = \sum_{i=0}^{\nu} a_i(x) \sigma_{q^{1/r}}^i \) is the minimal \( q^{1/r} \)-difference operator killing \( y(x) \). Moreover if \( \lambda \) is a slope of \( NRP_{\sigma_q}(L_q) \) (resp. \( NRP_{d_q}(L_q) \)) then \( \lambda/r \) is a slope of \( NRP_{\sigma_q}(L_{q^{1/r}}) \) (resp. \( NRP_{d_q}(L_{q^{1/r}}) \)).

In fact, let \( L \) be a \( q^{1/r} \)-difference operator killing \( y(x) \), minimal with respect to \( \text{deg}_{\sigma_q^{1/r}} \). Then \( L_{q^{1/r}} \) is a factor of \( L \) in \( K(x)[\sigma_q^{1/r}] \), hence \( \text{deg}_{\sigma_q^{1/r}} L \leq r \text{deg}_{\sigma_q} L_q \). On the other side we have:

\[
\dim_{K(x)} \sum_{i=0}^{\nu} K(x) \sigma_{q^{1/r}}^i(y) = \dim_{K(x)} \sum_{i=0}^{\nu} K(x) \sigma_q^i(y) = \dim_{K(x)} \sum_{i=0}^{\nu} K(x) \sigma_q^i(y),
\]

therefore \( \text{deg}_{\sigma_q^{1/r}} L \geq r \text{deg}_{\sigma_q} L_q \).

We recall the statement of Corollary 4.3, which is the starting point for this second part of the paper:

Proposition 12.2. Let \( F \in K[[x]] \) be a global \( q \)-Gevrey series of orders \((0, 0)\) and \( L \in K[x, d_q] \) the minimal \( q \)-difference operator such that \( LF = 0 \). Then \( L \) is regular singular.

Using the formal \( q \)-Fourier transformations introduced in the previous section, we will deduce the structure theorems below from Proposition 12.2.

Theorem 12.3. Let \( F \in K[[x]] \setminus K[x] \) be a global \( q \)-Gevrey series of orders \((-s_1, -s_2)\), with \((s_1, s_2) \in \mathbb{Z} \) and \( L \in K[x, d_q] \) be the minimal linear \( q \)-difference operator such that \( LF = 0 \). Then \( L \) has the following properties:

- the set of finite slopes of the Newton Polygon \( NRP_{d_q}(L) \) is \( \{-1/(s_1 + s_2), 0\} \);
- for all \( \xi \in K^* \), the \( q \)-difference operator \( L \) has a basis of solutions in \( K[[x - \xi]]_q \).

\[\text{or equivalently, that } NRP_{d_q}(S \circ L) \text{ does not have any positive slope.}\]
Proof. Let us write the formal power series $F$ in the form:

$$F = \sum_{n=0}^{\infty} a_n \frac{2!}{q^{n(n-1)}} [n]_q ! x^n,$$

where $\sum_{n=0}^{\infty} a_n x^n$ is a $G_q$-function. Let $\tilde{F}(x) = \sum_{n=0}^{\infty} a_n x^{n+s_2}$; then the series $\tilde{F}$ has finite size, therefore there exists a regular singular $q$-difference operator $L \in K[x, \sigma_q]$ such that $L\tilde{F} = 0$. The polygon $NRP_{s_2}(L)$ has only the zero slope (apart from the infinite slopes).

Let $S$ be the symmetry with respect to the origin:

$$S: \begin{aligned} x &\mapsto 1/x \\ \sigma_q &\mapsto \sigma_p \end{aligned}$$

Remark that the operator $F_{q^s}^{-1} \circ S(L)$ kill the formal power series $\sum_{n=0}^{\infty} a_n [n]_q ! x^{n+s_2}$. The polygon $NRP_s(S(L))$ is obtained by $NRP_{s_2}(L)$ applying a symmetry with respect to the line $v = 0$. It follows from Proposition [10.4] that the set of finite slopes of $NRP_{s_2}(F_{q^s}^{-1} \circ S(L))$ is $\{0, -1\}$. Iterating $s_2$ times this reasoning, we obtain a $q$-difference operator $\tilde{L} = F_{q^s}^{-1} \circ S \circ \cdots \circ F_{q^s}^{-1} \circ S(L)$, such that the set of finite slopes of $NRP_{s_2}(\tilde{L})$ is $\{0, -1/s_2\}$. We obtain:

$$\tilde{L} \left( \sum_{n=0}^{\infty} a_n \frac{2!}{q^{n(n-1)}} [n]_q ! x^n \right) = 0.$$

Because of [8.2] we can now suppose that $s_1$ is actually a positive integer. We conclude the proof applying the same argument to $\tilde{L} = (F_{q^s}^{-1} \circ S) \circ \cdots \circ (F_{q^s}^{-1} \circ S) (\sigma_q \circ \tilde{L} \circ x^s)$ for a suitable $n \in \mathbb{Z}_{\geq 0}$, and to the Newton-Ramis Polygon defined with respect to $\sigma_q$. We know that $L\tilde{F} = 0$.

The operator $L$ is a factor of $\tilde{L}$ in $K(x)[\sigma_q]$. We know (cf. for instance [Sau04]) that the slopes of the Newton Polygon of $L$ at zero (resp. $\infty$) are slopes of the Newton Polygon of $\tilde{L}$ at zero (resp. $\infty$). To obtain the desired result on the slopes of $NRP_{s_2}(L)$ one has to notice that $\tilde{L}$ must have a positive slope at $\infty$ because of [Ram92] Theorem 4.8. As far as $\xi \in K^*$ is concerned, the operator $\tilde{L}$ has a basis of solutions at $\xi$ in $K[[x-\xi]]_q$ (cf. Propositions [11.1] and [11.3], therefore the same is true for $L$. □

Proposition [10.9] implies that for a global $q$-Gevrey series of orders $(-s_1,0)$ we have actually proved a more precise result:

**Theorem 12.4.** Under the hypothesis of the previous theorem, we assume that $s_2 = 0$. Then $L$ has the following properties:

- the set of finite slopes of $NRP_{s_2}(L)$ is $\{0, -1/s_1\}$;
- for all $\xi \in K^*$, the $q$-difference operator $L$ has a basis of solutions in $K[[x-\xi]]_q$.

Changing $q$ in $q^{-1}$ we get the corollary:

**Corollary 12.5.** Let $F \in K[[x]] \setminus K[x]$ be a global $q$-Gevrey series of orders $(-s_1, -s_2)$, with $(s_1, s_2) \in \mathbb{Q} \times \mathbb{Z}$, such that $s_1 \geq s_2 \geq 0$ and either $s_1 \neq s_2$ or $s_2 \neq 0$. Let $L \in K[x, \sigma_q]$ be the minimal linear $q$-difference operator such that $L\tilde{F} = 0$. Then $L$ has the following properties:

- the set of finite slope of $NRP_{s_2}(L)$ is $\{0, 1/s_1\}$;
- for all $\xi \in K^*$, the $q$-difference operator $L$ has a basis of solutions in $K[[x-\xi]]_q$.

**Proof.** It follows by Proposition [8.3] taking into account that when one changes $q$ in $q^{-1}$, the slopes of the Newton Polygon change sign. □

Following [And00b] we can characterize the apparent singularities of such a $q$-difference equation:

**Theorem 12.6.** Let $F \in K[[x]] \setminus K[x]$ be a global $q$-Gevrey series of orders $(-s_1, -s_2)$, with $(s_1, s_2) \in \mathbb{Z}$. We fix a point $\xi \in K^*$. For all $v \in \mathcal{P}$ such that $|q|_v > 1$ we suppose that the $v$-adic function $F(x)$ has a zero at $\xi$. Let $L \in K[x, d_q]$ be the minimal linear $q$-difference operator such that $L\tilde{F} = 0$. Then $L$ has a basis of solution in

$$(x-\xi)K[[x-q\xi]]_q = \left\{ \sum_{n=1}^{\infty} a_n (x-\xi)_n : a_n \in K \right\}.$$
The proof is based on the following lemma, which is an analogue of \cite{And00b} Lemme 2.1.2 (cf. also \cite{And00b} Lemma 4.4.2).

**Lemma 12.7.** Let $F$ be a global $q$-Gevrey series of orders $(-s_1, -s_2)$, with $s_1, s_2 \in \mathbb{Q}_{\geq 0} \times \mathbb{Z}_{> 0}$. We fix a point $\xi \in K^*$. For all $v \in \mathcal{P}$ such that $|q|_v > 1$ we suppose that the $v$-adic entire function $F(x)$ has a zero at $\xi$. Then $G = (x - \xi)^{-1}F$ is a global $q$-Gevrey series of orders $(-s_1, -s_2)$.

**Proof of Theorem 12.7.** We fix some notation:

$$F = \sum_{n=0}^{\infty} \frac{a_n}{q^{n(n-1)/2} |n|_q^{s_1}} x^n, \quad G = \sum_{n=0}^{\infty} \frac{b_n}{q^{n(n-1)/2} |n|_q^{s_2}} x^n,$$

$$\tilde{h}(n, v, F) = \sup_{n \leq N} |a_n|_v \quad \text{and} \quad \tilde{h}(n, v, G) = \sup_{n \leq N} |b_n|_v.$$

Since $\frac{1}{\tilde{h}(n, v, F)} = -\sum_{n \geq N} \xi^n$, we obtain:

$$b_n = -\sum_{k=0}^{n} \left( q^{\frac{n(n-1)}{2} - \frac{k(k-1)}{2}} \right) \left( \frac{|n|_q^{s_1}}{|k|_q^{s_2}} \right)^{\xi^{k-n-1}} a_k$$

and therefore:

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{|q|_v \leq 1} \tilde{h}(n, v, G) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{|q|_v \leq 1} \tilde{h}(n, v, F) + \sum_{|q|_v \leq 1} |\xi|_v.$$

To conclude it is enough to prove that $G$ is a local $q$-Gevrey series of order $s_1 + s_2$ for all $v \in \mathcal{P}$ such that $|q|_v > 1$. This follows from \cite{Ram92} Prop. 2.1, since $F$ and $G$ have the same growth at $\infty$, because $F$ has a zero at $\xi$. \qed

**Proof.** Let $G = (x - \xi)^{-1}F$ and $\mathcal{L}$ be the minimal linear $q$-difference operator such that $\mathcal{L}F = 0$; then $\mathcal{L} \circ (x - \xi)$ is the minimal linear $q$-difference operator such that $\mathcal{L} \circ (x - \xi)(G) = 0$. By Lemma 12.7 and Theorem 12.3, $\mathcal{L} \circ (x - \xi)$ has a basis of solution in $K[[x - q\xi]]_q$, therefore the operator $\mathcal{L}$ has a basis of solution in $(x - \xi)K[[x - q\xi]]_q$. \qed

Once again, switching $q$ into $q^{-1}$ we obtain the corollary:

**Corollary 12.8.** Let $F \in K[[x]] \setminus K[x]$ be a global $q$-Gevrey series of orders $(s_1, -s_2)$, with $s_1, s_2 \in \mathbb{Q} \times \mathbb{Z}$, $s_1 \geq s_2 \geq 0$ and either $s_1 \neq s_2$ or $s_2 \neq 0$. We fix a point $\xi \in K^*$. For all $v \in \mathcal{P}$ such that $|q|_v < 1$ we suppose that the $v$-adic function $F(x)$ has a zero at $\xi$. Let $\mathcal{L} \in K[x]$, be the minimal linear $q$-difference operator such that $\mathcal{L}F = 0$. Then $\mathcal{L}$ has a basis of solution in $(x - \xi)K[[x - p\xi]]_p$. \qed

**Proof.** It follows from Proposition 8.4 and Theorem 12.6. \qed

We conclude the section with an example:

**Example 12.9.** Let us consider the $q$-exponential series $E_q(x) = \sum_{n \geq 0} \frac{x^n}{|n|_q}$, solution of the equation $d_q y = y$. A classical formula (cf. \cite{GR90} 1.3.16) says that for $|q|_v > 1$ the series $E_q(x)$ can be written as an infinite product:

$$E_q(x) = (-x(1 - q^{-1}); q^{-1})_\infty := \prod_{k=0}^{\infty} \left( 1 - x \frac{1 - q}{q^{k+1}} \right),$$

hence $E_q(\frac{a}{q}) = 0$ for all $v$ such that $|q|_v > 1$. Let us consider formal $q$-series:

$$G(x) = \frac{E_q(x)}{x - \frac{1}{1-q}} = \frac{q - 1}{q} E_q \left( \frac{x}{q} \right).$$

Obviously, $qd_q G(x) - G(x) = 0$ and actually:

$$(d_q - 1) \circ \left( x - \frac{q}{1-q} \right) G(x) = \left( x - \frac{1}{1-q} \right) (qd_q - 1) G(x) = 0.$$
Since $\sum_{n \geq 0} \frac{q^{-n}}{[n]_q} T_n^q \left( x, q^2 \right) \in K[[x - q^2/x]]_q$ is a formal solution of $qdy = y$, the series

$$
\left( x - \frac{q}{1 - q} \right) \sum_{n \geq 0} \frac{q^{-n}}{[n]_q} T_n^q \left( x, q^2 \right) \in \left( x - \frac{q}{1 - q} \right) K \left[ \left[ x - \frac{q^2}{1 - q} \right] \right]_q
$$

is a formal solution of $d_q y = y$.

13. An irrationality result for global $q$-Gevrey series of negative orders

In this section we are going to give a simple criteria to determine the $q$-orbits where a global $q$-Gevrey series does not satisfy the hypothesis of Theorem 12.6. We will deduce an irrationality result for values of a global $q$-Gevrey series $F(x) \in K[[x]] \setminus K[x]$ of negative orders.

**Remark 13.1.** The arithmetic Gevrey series theory in the differential case has applications to transcendence theory (cf. \[\text{And00b}\]). In the global $q$-Gevrey series framework this cannot be true, since the set of global $q$-Gevrey series has only a structure of $k(q)$-vector space. We mean that the product of two global $q$-Gevrey series of nonzero orders doesn’t need to be a global $q$-Gevrey series, as the following example shows:

$$
e_q(x)^2 = \left( \sum_{n=0}^{\infty} \frac{x^n}{[n]_q} \right)^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \right) \frac{x^n}{[n]_q}.
$$

In fact, because of the estimate at the cyclotomic places $e_q(x)^2$ should be a global $q$-Gevrey series of order $(0,-1)$, while the local $q$-Gevrey order at places $v \in \mathcal{P}_\infty$ such that $|q|_v > 1$ is $2$. For this reason a global $q$-Gevrey series theory can only have applications to the irrationality theory.

Let

$$
\mathcal{L} = a_v(x) \sigma_q^v + \cdots + a_1(x) \sigma_q + a_0(x) \in K[x, \sigma_q],
$$

and let $u_0, \ldots, u_{\nu-1}$ a basis of solution of $\mathcal{L}$ is a convenient $q$-difference algebra extending $K(x)$. The Casorati matrix

$$
\mathcal{U} = \begin{pmatrix} u_0 & \cdots & u_{\nu-1} \\ \sigma_q u_0 & \cdots & \sigma_q u_{\nu-1} \\ \vdots & \ddots & \vdots \\ \sigma_q^{\nu-1} u_0 & \cdots & \sigma_q^{\nu-1} u_{\nu-1} \end{pmatrix},
$$

is a fundamental solution of the $q$-difference system

$$
\sigma_q \mathcal{U} = \begin{pmatrix} 0 & \cdots & 1_{\nu-1} \\ \vdots & \ddots & \vdots \\ -a_0(x) a_{\nu}(x) & \cdots & -a_{\nu-1}(x) a_{\nu}(x) \end{pmatrix} \mathcal{U},
$$

so that $\mathcal{L} = \det \mathcal{U}$ is solution of the equation:

$$
\sigma_q \mathcal{C} = (-1)^{\nu} \frac{a_0(x)}{a_{\nu}(x)} \mathcal{C}.
$$

Notice that the “$q$-Wronskian lemma” (cf. for instance \[\text{DV02} \ §1.2\]) implies that the determinant of the Casorati matrix of a basis of solutions of an operator $\mathcal{L}$ is nonzero.

**Proposition 13.2.** Let $F \in K[[x]] \setminus K[x]$ be a global $q$-Gevrey series of orders $(-s_1, -s_2)$, with $s_1, s_2 \in \mathbb{Z}$. We fix a point $\xi \in K^*$. Let

$$
\mathcal{L} = a_v(x) \sigma_q^v + \cdots + a_1(x) \sigma_q + a_0(x) \in K[x, \sigma_q]
$$

be the minimal $q$-difference operator such that $\mathcal{L} F = 0$. If $F(x)$ has a zero at $\xi$ for all $v$ such that $|q|_v > 1$, then there exists an integer $m \geq 0$ such that $q^m \xi$ is a zero of $a_0(x)$.

**Proof.** The determinant of the Casorati matrix of a basis of solutions of $\mathcal{L}$ satisfies the equation

$$
y(qx) = (-1)^{\nu} \frac{a_{\nu}(x)}{a_0(x)} y(x).
$$
On the other hand we know that $\mathcal{L}$ has a basis of solution $u_0, \ldots, u_{r-1} \in (x - \xi)K[[x - q\xi]]$. This means that the $u_i$’s are formal series of the form $\sum_{n \geq 1} a_n T_n^q(x, \xi)$, for some $a_n \in K$. Since $(qx - \xi) = q(x - q^{n-1}\xi) + (q^n - 1)\xi$, one obtain that

$$
\sigma_q \left( \sum_{n \geq 1} a_n T_n^q(x, \xi) \right) = qa_1 + \sum_{n \geq 1} (q^n a_n + q^{n+1} a_{n+1} \xi(q^n - 1)) T_n^q(x, \xi).
$$

This implies that the determinant $C$ of the Casorati matrix of $u_0, \ldots, u_{r-1}$ is an element of $(x - \xi)K[[x - q\xi]]_q$. Let $m \geq 1$ be the larger integer such that $C \in T_m^q(x, \xi)K[[x - q^n\xi]]_q$. The formula above implies that $\sigma_q \mathcal{L} \in T_m^q(x, \xi)K[[x - q^{m-1}\xi]]_q \subseteq T_m^q(x, \xi)K[[x - q^n\xi]]_q$, and therefore that $q^{m-1}\xi$ is a zero of $a_0(x)$.

In the same way we can prove the following result:

**Corollary 13.3.** Let $F \in K[[x]] \setminus K[x]$ be a global $q$-Gevrey series of orders $(s_1, -s_2)$, with $s_1, s_2 \in \mathbb{Q} \times \mathbb{Z}$, $s_1 \geq s_2 \geq 0$ and either $s_1 \neq s_2$ or $s_2 \neq 0$. We fix a point $\xi \in K^*$. Let $\mathcal{L} = a_0(x)\sigma_q + \cdots + a_1(x)\sigma_q + a_0(x) \in K[x, \sigma_q]$ be the minimal linear $q$-difference operator such that $\mathcal{L}F = 0$. If $F(x)$ has a zero at $\xi$ for all $v \in \mathcal{P}$ such that $|q|_v < 1$ then there exists an integer $m \leq -\nu$ such that $q^m \xi$ is a zero of $a_v(x)$.

**Proof.** It follows from Proposition [2.4] that $F(x)$ is a global $q^{-1}$-Gevrey series of negative orders $(-(s_1 - s_2), -s_2)$ and the minimal linear $q^{-1}$-difference operator killing $F(x)$ is $a_0(q^{-1}x) + \cdots + a_1(q^{-1}x)\sigma_q^{-1} + a_0(q^{-1}x)\sigma_q^{-1}$.

**Example 13.4.** Let us consider the field $K = k(q)$ and the Thakaloff series:

$$
T_q(x) = \sum_{n \geq 0} \frac{x^n}{q^{n(n-1)/2}}.
$$

Together with $E_q(x)$, $T_q(x)$ is a $q$-analogue of the exponential function. The minimal linear $q$-difference equation killing $T_q(x)$ is

$$
\mathcal{L} = (\sigma_q - 1) \circ (\sigma_q - qx) = (\sigma_q - q^2x) \circ (\sigma_q - 1) = \sigma_q^2 - (1 + q^2x)\sigma_q + q^2x.
$$

Notice that $1, T_q(x)$ is a basis of solutions of $\mathcal{L}$ at zero. We conclude that $T_q(\xi) \neq 0$ for all $\xi \in K^*$, as the value a $q^{-1}$-adic entire analytic function, i.e. the hypothesis of Theorem 12.6 are never satisfied.

In particular, let $K = k(\bar{q})$, where $\bar{q} = q$ for some positive integer $r$. For any $\xi \in k(\bar{q}), \xi \neq 0$, the $\bar{q}^{-1}$-adic value $T_q(\xi)$ of $T_q(x)$ at $\xi$ can be formally written as a Laurent series in $k((\bar{q}^{-1}))$, which is the completion of $k(\bar{q})$ at the $\bar{q}^{-1}$-adic place. The theorem above says that $T_q(\xi)$ cannot be the expansion of a rational function in $k(\bar{q})$. In fact, if it was, there would exists $c \in k(\bar{q})$ such that $T_q(x) + c$ has a zero at $\xi$ and is solution of $\mathcal{L}$, which would imply that $\mathcal{L}$ has a basis of solutions having a zero at $\xi$, against the fact that the constants are solution of $\mathcal{L}$.

As in [And00b], we can also deduce a Lindemann-Weierstrass type statement:

**Corollary 13.5.** Let $K = k(\bar{q})$, where $\bar{q}$ is a root of $q$. We consider the $q$-exponential function $e_q(x) = \sum_{n \geq 0} \frac{x^n}{[q^n]}$ and a set of element $a_1, \ldots, a_r \in K$, which are multiplicatively independent modulo $q^2$ (i.e. $a_1^2 \cdot \cdots \cdot a_r^2 \cap q^2 = \{1\}$). Then the Laurent series $e_q(a_1\xi), \ldots, e_q(a_r\xi) \in k((\bar{q}^{-1}))$ are linearly independent over $k(\bar{q})$ for any $\xi \in K^*$.

**Proof.** It is enough to notice that $e_q(a_1\xi), \ldots, e_q(a_r\xi)$ is a basis of solutions of the operator

$$(d_q - a_1) \circ \cdots \circ (d_q - a_r).$$

If there exist $\lambda_1, \ldots, \lambda_r \in K$ such that $\lambda_1 e_q(a_1\xi) + \cdots + \lambda_r e_q(a_r\xi) = 0$, then $e_q(a_i\xi) = 0$ for any $i = 1, \ldots, r$, because of Theorem 12.6. Since $e_q(x)$ satisfies the equation $y(qx) = (1 + (q - 1)x)e_q(x)$, we deduce that $\xi \in \frac{\bar{q}^2 - 1}{(1 - q^2)a_i}$, for any $i = 1, \ldots, r$. The last assertion would imply that $a_i a_j^{-1} \in \bar{q}^2$ for any pair of distinct $i, j$, against the assumption.

We can deduce by Theorem 12.6 an irrationality result for all global $q$-Gevrey series $F(x)$ such that zero is not a slope of the Newton Polygon at $\infty$ of the minimal $q$-difference operator that kills $F(x)$:
Theorem 13.6. Let \( \overline{k(q)} \) be a fixed algebraic closure of \( k(q) \) and \( \overline{K} \subset \overline{k(q)} \) the maximal extension of \( k(q) \) such that the \( q^{-1} \)-adic norm of \( k(q) \) extends uniquely to \( \overline{K} \).

Let \( F(x) \in \overline{K}[x] \backslash \overline{K}[x] \) be a global \( q \)-Gevrey series of orders \((-s_1, -s_2)\), with \((s_1, s_2) \in \mathbb{Z} \), and \( \mathcal{L} \) the minimal linear \( q \)-difference operator such that \( \mathcal{L} F(x) = 0 \). We suppose that zero is not a slope of \( \mathcal{L} \) at \( \infty \). Then for all \( \xi \in K^* \) the value \( F(\xi) \) of the \( q^{-1} \)-adic analytic entire function \( F(x) \) is not an element of \( \overline{K} \) (but of its \( \tilde{q}^{-1} \)-adic completion).

Before proving the theorem, we give an example, which illustrates the proof:

Example 13.7. Let us consider the \( q \)-analogue of a Bessel series

\[
B_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q!}.
\]

The series \( B_q(x) \) is solution of the linear \( q \)-difference operator \((xd_q)^2 - x\) that can be written also in the form:

\[
\mathcal{L} = \sigma_q^2 - 2\sigma_q + (1 - (q - 1)^2)x.
\]

There is a unique factorization of a linear \( q \)-difference operator linked to the slopes of its Newton Polygon (cf. [Sau04]): we deduce that \( \mathcal{L} \) is the minimal \( q \)-difference operator killing \( B_q(x) \) from the fact that the only slope of the Newton-Polygon of \( \mathcal{L} \) at \( \infty \) is \(-1/2\). We conclude that \( B_q(\xi) = 0 \) for all \( \nu \) such that \(|\nu|_\nu > 1\), with \( \xi \in \mathbb{P}^1(K) \), implies \( \xi = q^m/(q - 1)^2 \) for some integer \( m \geq 2 \).

Let \( K = k(\tilde{q}) \), with \( \tilde{q} = q \) for some positive integer \( r \). In this case the \( \tilde{q}^{-1} \)-adic norm is the only one such that \(|\nu|_\nu > 1\). For any \( c \in K \) we have:

\[
(q\sigma_q - 1) \circ \mathcal{L}(B_q(x) + c) = 0.
\]

One notices that the slopes of the Newton Polygon of \((q\sigma_q - 1) \circ \mathcal{L} \) at \( \infty \) are \( \{0, -1/2\} \), therefore we deduce from the uniqueness of the factorization that \((q\sigma_q - 1) \circ \mathcal{L} \) is the minimal \( q \)-difference operator killing \( B_q(x) + c \). Since constants are solutions of \((q\sigma_q - 1) \circ \mathcal{L} \), Theorem 13.6 implies that no solution of \((q\sigma_q - 1) \circ \mathcal{L} \) can have a zero at any point \( \xi \in K^* \) as \( \tilde{q}^{-1} \)-adic holomorphic functions. This means that the function \( B_q(x) + c \) cannot have a zero as a \( \tilde{q}^{-1} \)-adic analytic function at \( \xi \in K^* \), which means that \( B_q(x) \) takes values in \( k((\tilde{q}^{-1})) \backslash k(\tilde{q}) \) at each \( \xi \in K^* \).

Proof of Theorem 13.6. Let \( c \in \overline{K}, c \neq 0 \), \( G(x) = F(x) + c \), \( \mathcal{L} = \sum_{i=1}^\nu a_i(x) d_q^i \in \overline{K}[x, d_q] \) (resp. \( \mathcal{N} = \sum_{i=1}^\mu b_i(x) d_q^i \in \overline{K}[x, d_q] \)) be the minimal \( q \)-difference operator killing \( F(x) \) (resp. \( G(x) \)). Of course we may assume that \( a_i(x), b_j(x) \in \overline{K}(x) \) and \( a_\nu(x) = b_\mu(x) = 1 \), and that everything is defined over a finite extension \( K \subset \overline{K} \) of \( k(q) \).

Since:

\[
\left( d_q - \frac{d_q(a_0(x))}{a_0(x)} \right) \circ \mathcal{L}(G(x)) = 0 \text{ and } \left( d_q - \frac{d_q(b_0(x))}{b_0(x)} \right) \circ \mathcal{N}(F(x)) = 0,
\]

we must have \( \nu - 1 \leq \mu \leq \nu + 1 \). Let us suppose first \( \nu = \mu \). Then

\[
\left( d_q - \frac{d_q(a_0(x))}{a_0(x)} \right) \circ \mathcal{L} = \left( d_q - \frac{d_q(b_0(x))}{b_0(x)} \right) \circ \mathcal{N}
\]

since they have the same set of solutions and they are both monic operators. By hypothesis, zero is not a slope of the Newton Polynomial of \( \mathcal{L} \) at \( \infty \), while \( \left( d_q - \frac{d_q(a_0(x))}{a_0(x)} \right) \) has only the zero slope at \( \infty \): we conclude by the uniqueness of the factorization that \( \mathcal{L} = \mathcal{N} \). We remark that the equality \( \mathcal{L} = \mathcal{N} \) implies that constants are solutions of \( \mathcal{L} \) and that \( \mathcal{L} \) has a zero slope at \( \infty \), hence we obtain a contradiction. So either \( \mu = \nu - 1 \) or \( \mu = \nu + 1 \). If \( \mu = \nu - 1 \), then

\[
\mathcal{L} = \left( d_q - \frac{d_q(b_0(x))}{b_0(x)} \right) \circ \mathcal{N}
\]

since both \( \mathcal{L} \) and \( \mathcal{N} \) are monic. Once again, constants are solution of \( \mathcal{L} \) and this is a contradiction. Finally, we have necessarily \( \mu = \nu + 1 \) and

\[
\mathcal{N} = \left( d_q - \frac{d_q(b_0(x))}{b_0(x)} \right) \circ \mathcal{L}.
\]
Let us suppose that there exists $\xi \in K^*$, such that $F(x)$ takes a value in $K$ at $\xi$, as $q^{-1}$-adic analytic function. Then all the solutions of $\mathcal{N}$ would have a zero at $\xi$ against the fact that the constants are solutions of $\mathcal{N}$, hence $F(\xi) \neq 0$ is not in $K$.

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