Optimal Stationary State Estimation Over Multiple Markovian Packet Drop Channels

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Abstract

In this paper, we investigate the state estimation problem over multiple Markovian packet drop channels. In this problem setup, a remote estimator receives measurement data transmitted from multiple sensors over individual channels. By the method of Markovian jump linear systems, an optimal stationary estimator that minimizes the error variance in the steady state is obtained, based on the mean-square (MS) stabilizing solution to the coupled algebraic Riccati equations. An explicit necessary and sufficient condition is derived for the existence of the MS stabilizing solution, which coincides with that of the standard Kalman filter. More importantly, we provide a sufficient condition under which the MS detectability with multiple Markovian packet drop channels can be decoupled, and propose a locally optimal stationary estimator but computationally more tractable. Analytic sufficient and necessary MS detectability conditions are presented for the decoupled subsystems subsequently. Finally, numerical simulations are conducted to illustrate the results on the MS stabilizing solution, the MS detectability, and the performance of the optimal and locally optimal stationary estimators.

Key words: State estimation, stabilizing solution, Markovian packet drops, Markovian jump linear systems.

1 Introduction

Networked control systems (NCSs) attract a great deal of attention from the control community due to their numerous advantages over conventional control systems. Much effort has been devoted to the study of control and estimation for NCSs over various communication channels in recent years. In the case of wireless channels, one major issue is the occurrence of data packet drops that may destroy the feedback stability of estimators and controllers. Hence, a large number of existing works are focused on the stability and stabilization of dynamic systems over packet drop channels [1–8].

From the stochastic point of view, the packet drop channel is commonly modeled as either an independent and identically distributed (i.i.d.) random process or a two-state Markov chain by taking the temporal correlation into consideration. Under such two modeling methods, the stability of Kalman filtering with intermittent measurements has been well studied. In [1], the authors show that over an i.i.d. packet drop channel, there exists a critical packet arrival rate below which the Kalman filter is unstable. Multi-sensor and distributed scenarios are further studied in [9] and [5], respectively. Compared with the i.i.d. case, the stability problem of Kalman filtering with Markovian packet drops is more complicated, yet many interesting results are obtained. Authors of [10] are the first to introduce the notion of peak covariance to evaluate the estimation performance. Some improved results are obtained in [11], in comparison with [10]. By showing the equivalent stability property for the estimation error covariances at packet reception times and each time instant, the stability for Kalman filtering has been sufficiently studied in [12] through exploiting the system structure. In [13], a necessary and sufficient condition is provided for diagonalizable systems with multiple sensors. Similar to the existence of the critical packet arrival rate shown in the i.i.d. case, the existence of the critical curve in terms of the failure-recovery rate is proved.
in [14]. For the NCS without an acknowledgment signal sent by the actuator to the estimator, authors of [15] derive the optimal and an approximate optimal estimator, and show the same stability for both of them.

Since NCSs over Markovian packet drop channels can be considered as a class of Markov jump linear systems (MJLS) [16], an alternative approach for studying the state estimation with Markovian packet drops is design of optimal stationary jump estimators [17, 18], instead of the time-varying Kalman filter (TVKF) mentioned above. Although the TVKF is known to be the optimal linear estimator, it does not converge in the steady state and its estimation gain explicitly depends on the realization of packet drops such that it needs to be computed online. In this paper, we consider the stationary state estimation problem over multiple Markovian packet drop channels. Different from the TVKF studied in the existing literature, e.g. [1, 10–12, 14], we are interested in the optimal stationary linear state estimator that remained unknown for which the estimator gains can be computed off-line, leading to a reduction in computational burden for the estimator. Compared to [17], where a jump estimator has been designed based on the last finite measurement loss modes for a single Markovian packet drop channel, we consider a nontrivial case of multiple Markovian packet drop channels. More importantly, we investigate two fundamental stability issues, the existence of the MS stabilizing solution to the corresponding coupled algebraic Riccati equations (CAREs) and the MS detectability for the NCS with Markovian packet drops, which have not been studied by [17].

It is well known that estimation and control are dual problems. Their optimal solutions are associated with their respective Riccati equations. Therefore, feedback control problems over packet drop or fading channels are also related to the study in this paper. In [19], it is shown that for the optimal linear quadratic Gaussian (LQG) control with i.i.d. packet drops, the separation principle still holds under a TCP-like protocol, and there exists a critical arrival probability for the control data. The case of multiple lossy channels is further considered in [20], while the LQG control with Markovian packet drops is studied in [21]. In the presence of i.i.d. fading channels, the existence of the MS stabilizing solution to the modified algebraic Riccati equation is studied recently in [22]. Compared to [22], the existence of the MS stabilizing solution in this paper is more involved, due to the temporal correlation of packet drops.

The contributions of this paper are summarized as follows.

1. An optimal stationary state estimator is obtained for the NCS over multiple Markovian packet drop channels, by making use of the MJLS method. This is a nontrivial generalization from a single channel studied in [17], since our case brings a challenging issue that the complexity of the optimal estimator increases exponentially with respect to the number of channels.

2. A necessary and sufficient condition is derived for the existence of the MS stabilizing solution to the associated filtering CAREs. It is shown that, in addition to the MS detectability, only the controllability of the eigenvalues on the unit circle is required, which is weaker than the stabilizability assumption in e.g. [1, 5, 12, 17].

3. We provide some sufficient and necessary conditions for the MS detectability and propose a locally optimal stationary estimator, through exploring the system structure. Specifically, the MS detectability with multiple Markovian packet drop channels can be decoupled and a locally optimal estimator, lowering the complexity from $N = 2^m$ to $2^m$, is obtained, resulting in significant reduction on the computational complexity. Moreover, some analytic MS detectability conditions are derived for the decoupled subsystems.

The remainder of this paper is organized as follows. Section 2 describes the problem considered in this paper and gives the optimal stationary estimator. Section 3 is focused on the existence of the MS stabilizing solution, which is hinged on the MS detectability and the controllability of the eigenvalues on the unit circle. The MS detectability is studied, and a locally optimal stationary estimator is proposed in Section 4. Numerical examples are provided in Section 5 to illustrate the stability results and the performance of the proposed estimators, followed by some concluding remarks in Section 6.

The notations in this paper are standard. $\mathbb{R}^{m \times n}$ ($\mathbb{C}^{m \times n}$) respectively denotes the set of $m \times n$ real (complex) matrices, with $\mathbb{R}^n := \mathbb{R}^{n \times 1}$ ($\mathbb{C}^n := \mathbb{C}^{n \times 1}$). $S^n_+$ is the set of $n \times n$ real positive semidefinite matrices. For a matrix or vector $X$, denote by $X^*$, $X'$, and $\bar{X}$ the conjugate transpose, transpose and conjugate of $X$, respectively. $\rho(\cdot)$ denotes the spectral radius of a matrix or operator, $\text{diag}\{\cdot\}$ the (block) diagonalization operation, $\text{vec}\{\cdot\}$ the vectorization operation, and $\text{tr}\{\cdot\}$ the trace of a square matrix. $\otimes$ represents the Kronecker product. $I_n$ represents the identity matrix of dimension $n \times n$ and $\mathbf{1}$ denotes the indicator function. Finally, the expectation operator and the probability of a random event are denoted by $\mathbb{E}\{\cdot\}$ and $\text{Pr}\{\cdot\}$, respectively. Other notations will be made clear as we proceed.

2 Optimal Stationary State Estimator

Consider a discrete-time shift-invariant system described by
with \(x(k) \in \mathbb{R}^n\) the system state, and \(y_i(k) \in \mathbb{R}\) the output measurement obtained by the \(i\)th sensor. \(w(k)\) and \(v(k) = \text{vec}\{v_1(k), \ldots, v_m(k)\}\) are mutually independent white noises having mean zero and covariances \(Q \geq 0\) and \(R > 0\), respectively. The initial state \(x_0\) is independent of \(w(k)\) and \(v(k)\), with mean \(\pi_0\) and covariance \(\Pi_0\). Define \(y(k) := \text{vec}\{y_1(k), \ldots, y_m(k)\}\) and \(C := [C_1', \ldots, C_m']'\). We assume without loss of generality that \(R\) is a diagonal matrix.

**Remark 1** It is worth mentioning that dimension one of the output measurement from each sensor does not pose constraints on the results in this paper from being generalized to the case of arbitrary dimensions. Since our focus is on the multiple Markovian packet drop processes, we consider the collective measurement \(y(k) \in \mathbb{R}^m\) for convenience.

We assume that the collective measurement \(y(k)\) is sent through unreliable channels suffering from packet drops; see Fig. 1. The signal received at the remote estimator is given by

\[
y_i(k) = \Gamma(k)y(k),
\]

where \(\Gamma(k) \in \mathbb{R}^{m \times m}\) represents the presence of the \(m\) packet drop channels in the diagonal form:

\[
\Gamma(k) = \text{diag}\{\gamma_1(k), \gamma_2(k), \ldots, \gamma_m(k)\}.
\]

Here, \(\gamma_i(k) \in \{0, 1\}\) for \(1 \leq i \leq m\). If \(\gamma_i(k) = 1\), \(y_i(k)\) arrives at the estimator; otherwise \(y_i(k)\) is dropped. Moreover, \(\gamma_1(k), \ldots, \gamma_m(k)\) are independent of each other, and each \(\gamma_i(k)\) is modeled as a time-homogeneous two-state Markov chain with the transition probability matrix (TPM)

\[
P_i = \begin{bmatrix} 1 - q_i & q_i \\ p_i & 1 - p_i \end{bmatrix}, \quad i = 1, \ldots, m,
\]

where \(q_i = \text{Pr}\{\gamma_i(k + 1) = 1|\gamma_i(k) = 0\}\) is the failure rate and \(p_i = \text{Pr}\{\gamma_i(k + 1) = 0|\gamma_i(k) = 1\}\) is the recovery rate. Denote \(\pi_{i,l}(k) := \text{Pr}\{\gamma_i(k) = l - 1\}\), \(l \in \{1, 2\}\). Assume that \(0 < p_i, q_i < 1\). Then, there exist the limit probability distribution \(\pi_i = [\pi_{i,1}, \pi_{i,2}]\) with

\[
\pi_{i,1} = \frac{p_i}{p_i + q_i}, \quad \pi_{i,2} = \frac{q_i}{p_i + q_i}.
\]

Different from the TVKF with Markovian packet drops studied in the existing literature [10–15], we are interested in the optimal stationary linear state estimator for which the estimator gains can be computed off-line. We will apply the filtering theory from MJLSs [16] to derive such an estimator. It is noted that the NCS described in (1) and (2) can be rewritten into the following jump form:

\[
x(k + 1) = Ax(k) + w(k), \quad x(0) = x_0 ,
\]

\[
y_i(k) = H_{\theta(k)}(x(k)) + D_{\theta(k)}v(k),
\]

where \(\theta(k) \in \mathcal{N} := \{1, \ldots, 2^m\}\), specified by

\[
\theta(k) = 1 + \sum_{i=1}^{m} 2^{i-1} \gamma_i(k),
\]

is the Markov jump variable. It follows that \(H_{\theta(k)} = \Gamma(k)C\) and \(D_{\theta(k)} = \Gamma(k)\). For simplicity, denote \(\mathcal{N} = 2^m\) in the remainder of the paper.

**Lemma 1** Given the way of computing \(\theta(k)\) in (6), the TPM \(P\) for the Markov process \(\{\theta(k)\}\) is given by

\[
P := P_m \otimes P_{m-1} \otimes \cdots \otimes P_1, \quad 1 \leq i, j \leq 2^m,
\]

where \(P = [p_{ij}]\), \(p_{ij} = \text{Pr}\{\theta(k + 1) = j|\theta(k) = i\}\). Moreover, by defining \(\mu_i(k) := \text{Pr}\{\theta(k) = i\}\) for each \(i \in \mathcal{N}\), and \(\mu(k) := [\mu_1(k) \cdots \mu_{2^m}(k)]\), \(\theta(k)\) has a unique stationary distribution \(\mu = [\mu_1 \cdots \mu_{2^m}]\), i.e.,

\[
\lim_{k \to \infty} \mu_{i,k}(k) = \mu_i \quad \text{for \(i \in \mathcal{N}\)}
\]

\[
\mu = \pi_m \otimes \pi_{m-1} \otimes \cdots \otimes \pi_1.
\]

**Proof.** We will prove that (7) holds for any \(m \geq 1\) by induction. Obviously, (7) holds when \(m = 1\). Assume that for \(m = l\) satisfying \(l \geq 1\), we have

\[
P^{(l)} := \left[ p^{(l)}_{ij} \right] = P_1 \otimes \cdots \otimes P_l, \quad 1 \leq i, j \leq 2^l.
\]

In this case, denote \(\vartheta(k)\) as the Markovian jump variable such that \(p^{(l)}_{ij} = \text{Pr}\{\vartheta(k + 1) = j|\vartheta(k) = i\}\). For \(m = l + 1\), suppose that sensor \(l + 1\) is the newly added sensor compared with the case \(m = l\). Define

\[
P^{(l+1)} := \left[ p^{(l+1)}_{rs} \right], \quad 1 \leq r, s \leq 2^{l+1}
\]
as the corresponding TPM. Then by (6) and the assumption that \( \{\gamma_i(k)\}_{i=1}^m \) are independent,

\[
\Pr\{\gamma_{i+1}(k+1), \vartheta(k+1) = j|\gamma_{i+1}(k), \vartheta(k) = i\} = p_{i,j} = \Pr\{\gamma_{i+1}(k+1)\mid \gamma_{i+1}(k)\} \Pr\{\vartheta(k+1) = j\mid \vartheta(k) = i\},
\]

\[\gamma_{i+1}(k+1), \gamma_{i+1}(k) \in \{0, 1\}, 1 \leq i, j \leq 2^l. \tag{11}\]

Therefore, we conclude that

\[P^{(l+1)} = P_{l+1} \otimes P^{(l)} = P_{l+1} \otimes P_{l} \otimes \cdots \otimes P_{1}. \tag{12}\]

Similarly, there holds

\[
\mu(k) = \pi_m(k) \otimes \pi_{m-1}(k) \otimes \cdots \otimes \pi_1(k). \tag{13}\]

Thus, \( \mu = \lim_{k \to \infty} \mu(k) = \pi_m \otimes \pi_{m-1} \otimes \cdots \otimes \pi_1 \). The uniqueness follows from the limit probability distributions \( \{\pi_i\}_{i=1}^m \). The proof is thus complete.


Let \( k(k) \) be the state estimation for \( x(k) \) in (5a). Now, consider a dynamic Markovian jump linear estimator described by

\[
\dot{x}(k+1) = A\hat{x}(k) + L_{\theta(k)}[y_k(k) - H_{\theta(k)}\hat{x}(k)]. \tag{14}\]

At each time instant, the estimator gain \( L_{\theta(k)} \) is chosen from a finite set of pre-computed values, i.e., \( L_{\theta(k)} \in \{L_i \in \mathbb{R}^{n \times m}\}_{i \in \mathcal{N}} \). This implies that the estimator gain \( L_{\theta(k)} \) depends only on \( \theta(k) \) (rather than on all the past modes \( \{\theta(0), \ldots, \theta(k)\} \), corresponding to the TVKF), which is an important feature of the jump estimator. We aim to find a set of optimal gains, denoted by \( \{K_i \in \mathbb{R}^{n \times m}\}_{i \in \mathcal{N}} \) such that with \( L_i = K_i \) for each \( i \in \mathcal{N} \), the stationary estimation cost

\[
J(\infty) = \lim_{k \to \infty} E[\|x(k) - \hat{x}(k)\|^2] \tag{15}\]

is bounded and minimized.

**Theorem 1** For the system dynamics described in (5), assume that there exists the MS stabilizing solution (see Definition 2 in Section 3) \( Y = (Y_1, \ldots, Y_N) \) to the following CAREs

\[
Y_j = \sum_{i=1}^{N} p_{ij} [(A - L_i(k)H_i)Y_i(k)(A - L_i(k)H_i)^T + \mu_i(k)D_iRD_i^T] + \mu_i(k)Q, \quad j \in \mathcal{N}. \tag{16}\]

Then the optimal stationary gains for the jump estimator in (14) are given by

\[
K_j = K_j(Y) := AY_jH_j^T(H_jY_jH_j^T + \mu_jR)^{-1} \tag{17}\]

for \( j \in \mathcal{N} \), and the stationary optimal cost is given by

\[
\lim_{k \to \infty} J(k) = \sum_{i=1}^{N} \text{tr}(Y_i). \tag{18}\]

**Proof.** The result can be obtained by using the filtering theory of MJLSs (Chapter 5 in [16]). To be more clear, define the estimation error by \( e(k) := x(k) - \hat{x}(k) \) and set \( Y_j(k) := E[e(k)e(k)^T1_{\theta(k)=j}], j \in \mathcal{N} \) such that

\[
E[e(k)e(k)^T] = \sum_{j=1}^{N} Y_j(k). \tag{19}\]

From (5) and (14) we have

\[
e(k+1) = (A - L_{\theta(k)}(k)H_{\theta(k)}(k)e(k) + w(k) - L_{\theta(k)}(k)D_{\theta(k)}v(k) \tag{20}\]

with \( L_{\theta(k)}(k) \) replaced by \( L_{\theta(k)}(k) \). From Proposition 3.35 2) in [16], we have

\[
Y_j(k+1) = \sum_{i=1}^{N} p_{ij} [(A - L_i(k)H_i)Y_i(k)(A - L_i(k)H_i)^T + \mu_i(k)D_iRD_i^T] + Y_j(k), \quad j \in \mathcal{N}. \tag{21}\]

By solving \( \partial \text{tr}(Y_j(k+1))/\partial L_i(k) = 0 \) for all \( i \in \mathcal{N} \), it gives \( L_i(k) = K_i(k) \) with

\[
K_i(k) = AY_i(k)H_i^T(H_iY_i(k)H_i^T + \mu_i(k)D_iRD_i^T) + \mu_i(k)Q, \tag{22}\]

which minimizes \( \text{tr}(Y_j(k+1)) \). The Moore-Penrose inverse is used, since matrix \( H_iY_iH_i^T + \mu_i(k)D_iRD_i^T \) in general is positive semi-definite and the relation \( R\{H_iY_i(k)A_i^T \} \subseteq R\{H_iY_iH_i^T + \mu_i(k)D_iRD_i^T \} \) holds for any \( i \in \mathcal{N} \) (\( R\{\cdot\} \) denotes the range space). With \( L_i(k) = K_i(k) \), (19) becomes

\[
Y_j(k+1) = \sum_{i=1}^{N} p_{ij} [AY_i(k)A_i^T - AL_i(k)H_i^T(H_iY_i(k)H_i^T + \mu_i(k)D_iRD_i^T) + \mu_i(k)Q]. \tag{21}\]

As \( k \to \infty \), (21) converges to the CAREs (16), \( Y_j(k+1) \to Y_j \), and \( K_i(k) \to K_i \), where \( (H_iY_iH_i^T + \mu_iR)^{-1} \) takes place of \( (H_iY_iH_i^T + \mu_iD_iRD_i^T)^{-1} \) without changing the values of \( Y_j \) and \( K_i \), following from the special form of \( \Gamma(k) \). Also, the stationary optimal cost is \( \sum_{i=1}^{N} \text{tr}(Y_i) \).

**Remark 2** It is seen from Theorem 1 that there are \( N = 2^m \) CAREs and gains for the optimal stationary estimator, which may cause a difficulty in implementation when the number of sensors is large. We will deal
where this issue in Section 4 by proposing a locally optimal stationary estimator.

3 MS Stabilizing Solution

From Theorem 1, it is known that the optimal stationary estimator is based on the existence of the MS stabilizing solution to the CAREs (16). Therefore, we focus on the necessary and sufficient condition for this existence problem in this section.

3.1 Preliminaries of MJLSs

Let $\mathbb{H}^{m,n}$ represent the linear space composed of all $N$-sequences of real matrices $V = (V_1, \ldots, V_N)$ with $V_i \in \mathbb{R}^{m,n}$. In the case of $m = n$, we denote $\mathbb{H}^{n} = \mathbb{H}^{n,n}$, and define

$$\mathbb{H}^{n,*} := \{ V = (V_1, \ldots, V_N) \in \mathbb{H}^{n}; V_i = V_i' \forall i \in \mathcal{N} \},$$

$$\mathbb{H}^{n,+} := \{ V = (V_1, \ldots, V_N) \in \mathbb{H}^{n,*}; V_i \geq 0 \forall i \in \mathcal{N} \}.$$

For $V = (V_1, \ldots, V_N) \in \mathbb{H}^{n,*}$ and $S = (S_1, \ldots, S_N) \in \mathbb{H}^{n,*}$, write that $V \geq S$ (or $V > S$) if $V - S = (V_1 - S_1, \ldots, V_N - S_N) \in \mathbb{H}^{n,+}$ (or $V_i - S_i > 0$). It is known that $\mathbb{H}^{m,n}$ can be equipped with the inner product:

$$\langle V, S \rangle = \sum_{i=1}^{N} \text{tr}(V_i' S_i), \quad (22)$$

where $V = (V_1, \ldots, V_N)$ and $S = (S_1, \ldots, S_N)$ are in $\mathbb{H}^{m,n}$. For $V = (V_1, \ldots, V_N) \in \mathbb{H}^{n}$ and $L = (L_1, \ldots, L_N) \in \mathbb{H}^{m,m}$, define the operators $\mathcal{L}(\cdot) = (\mathcal{L}_1(\cdot), \ldots, \mathcal{L}_N(\cdot))$ and $\mathcal{L}^*(\cdot) = (\mathcal{L}_1^*(\cdot), \ldots, \mathcal{L}_N^*(\cdot))$ as

$$\mathcal{L}_j(V) := \sum_{i=1}^{N} p_{ij} (A + L_i H_i) V_i (A + L_i H_i)', \quad (23)$$

$$\mathcal{L}_j(V) := \sum_{i=1}^{N} p_{ij} A V_i A', \quad j \in \mathcal{N}. \quad (24)$$

Their respective adjoint operators $\mathcal{L}^*$ and $\mathcal{L}^*$ are given by

$$\mathcal{L}_j^*(V) := \sum_{i=1}^{N} p_{ij} (A + L_i H_i)' V_i (A + L_i H_i), \quad (25)$$

$$\mathcal{L}_i^*(V) := \sum_{j=1}^{N} p_{ij} A' V_j A, \quad i \in \mathcal{N}, \quad (26)$$

satisfying the following equalities

$$\langle \mathcal{L}(V), S \rangle = \langle V, \mathcal{L}^*(S) \rangle, \quad (27a)$$

$$\langle \mathcal{L}(V), S \rangle = \langle V, \mathcal{L}^*(S) \rangle. \quad (27b)$$

Recall the inner product defined in (22). Denote

$$A := (A_1, \ldots, A_N), \quad Q := (Q_1, \ldots, Q_N),$$

$$H := (H_1, \ldots, H_N), \quad p := \{ p_{ij} \}, \quad i,j \in \mathcal{N}.$$

Next, we introduce several notions regarding MJLSs.

**Definition 1 (Def. 1, [23])** The system described in (5), or simply $(H, A, p)$ is said to be MS detectable, if there exists $L = (L_1, \ldots, L_N) \in \mathbb{H}^{n,m}$ such that $\rho(\mathcal{L}) < 1$.

**Definition 2 (Def. 5.7, [16])** $Y = (Y_1, \ldots, Y_N) \in \mathbb{H}^{n,*}$ is said to be the MS stabilizing solution for the CAREs (16), if $\rho(\mathcal{L}) < 1$ holds with $L_i = -K_i(Y)$ for $i \in \mathcal{N}$.

Following the definition of the observability for MJLSs in Theorem 3 of [24], we introduce the following definition for the uncontrollable eigenvalue of the MJLS in (5).

**Definition 3** A real number $\lambda \geq 0$ is said to be an uncontrollable eigenvalue for the pair $(A, Q)$ if there exists an eigenvector $V = (V_1, \ldots, V_N) \in \mathbb{H}^{n}\setminus\{0\}$ of $\mathcal{L}^*$ such that

$$\text{(a) } \mathcal{L}^*(V) = \lambda V, \quad \text{ (b) } Q V_i = 0 \forall i \in \mathcal{N}. \quad (28)$$

3.2 Existence of MS Stabilizing Solution

We begin with the notion of the maximal solution. For $X = (X_1, \ldots, X_N) \in \mathbb{H}^{n,*}$, denote $P_j(X) = \sum_{i=1}^{N} p_{ij} X_i$ and define the operators $\mathcal{X}(\cdot) = (\mathcal{X}_1(\cdot), \ldots, \mathcal{X}_N(\cdot))$ and $\mathcal{R}(\cdot) = (\mathcal{R}_1(\cdot), \ldots, \mathcal{R}_N(\cdot))$ as

$$\mathcal{X}_j(X) := \begin{bmatrix} AP_j(X) A' + \mu_j Q - X_j & AP_j(X) H_j' \\ H_j P_j(X) A' & \mu_j R + H_j P_j(X) H_j' \end{bmatrix},$$

$$\mathcal{R}_j(X) := \mu_j R + H_j P_j(X) H_j', \quad j \in \mathcal{N}. \quad (29)$$

Then define the following set

$$\Omega := \{ X \in \mathbb{H}^{n,*}; \mathcal{X}(X) \geq 0, \mathcal{R}(X) > 0 \}. \quad (30)$$

**Definition 4** A solution $Y^+ = (Y_1^+, \ldots, Y_N^+)$ to the CAREs (16) is said to be the maximal solution if $Y^+ \geq Y$ for any $Y = (Y_1, \ldots, Y_N)$ with $Y_j = P_j(X), j \in \mathcal{N}, X \in \Omega$.

The maximal solution can be numerically computed by solving the following convex programming problem [25]:

$$\max \left\{ \text{tr} \left( \sum_{j=1}^{N} X_j \right) : \quad X = (X_1, \ldots, X_N) \in \Omega \right\}. \quad (32)$$
Lemma 2 ([25]) Recall Definitions 1 and 2.

(i) Suppose that \((H, A, p)\) is MS detectable. Then there exists the maximal solution \(Y^+(1,\ldots,N)\) to the CAREs (16). Moreover, with \(L_i = -K_i(Y^+)\) for \(1 \leq i \leq N\), there holds \(\rho(L^*) \leq 1\).

(ii) There exists at most one MS stabilizing solution to the CAREs (16), which coincides with the maximal solution.

Having the solution \(X^+ = (X_1^+,\ldots,X_N^+)\) for the convex programming problem (32), the maximal solution is \(Y^+ = (Y_1^+,\ldots,Y_N^+)\) with \(Y_j^+ = P_j(X^+)\). By Lemma 2 (ii), \(Y^+\) is also the MS stabilizing solution to the CAREs (16), if the latter exists. The following result is important, which can be applied directly to the main result in this section.

Theorem 2 The CAREs in (16) admit the MS stabilizing solution, if and only if

1) \((H, A, p)\) is MS detectable.
2) \(\lambda = 1\) is not an uncontrollable eigenvalue for \((A, Q)\).

Proof. See Appendix.

We remark that a similar result to Theorem 2 for the control CAREs is given in Corollary 14 of [26]. Nonetheless, a self-contained and independent proof for the CAREs (16) is provided in Appendix, in which the equalities (27) in terms of inner product play an important role. Before presenting the main result in this section, we need the following technical lemma.

Lemma 3 Let \(\lambda > 0\). If \(A'X = \lambda X\) and \(QX = 0\) admit a solution \(X \in S_+^n \setminus \{0\}\), then there exist a nonzero vector \(x_0 \in \mathbb{C}^n\) and \(\omega_0 \in R\) such that

\[A'x_0 = \sqrt{\lambda}e^{j\omega_0}x_0, \quad Qx_0 = 0.\] (33)

That is, \(\sqrt{\lambda}e^{j\omega_0}\) is an uncontrollable eigenvalue for \((A, Q)\) according to the well-known Popov–Belevitch–Hautus (PBH) test.

Proof. The hypothesis on \(X \in S_+^n \setminus \{0\}\) implies that \(X = GG^*\) with \(G \in \mathbb{R}^{n \times r}\) and \(r = \text{rank}\{X\} > 0\). It follows that \(\lambda GG^* = A'GG'A\) and \(QGG^* = 0\). Hence, there exists an orthogonal matrix \(U \in \mathbb{R}^{n \times r}\) such that

\[\sqrt{\lambda}GU = A'G, \quad QGU = 0.\] (34)

Since all the eigenvalues of an orthogonal matrix are on the unit circle, there exists a nonzero vector \(u_0 \in \mathbb{C}^r\) such that \(Uu_0 = e^{j\omega_0}u_0\) for some \(\omega_0 \in \mathbb{R}\). Multiplying the two equalities in (34) by \(u_0\) from right verifies the eigenvalue–eigenvector equation and \(Qx_0 = 0\) in (33) by taking \(x_0 = Gu_0\).

Theorem 3 The CAREs in (16) admit the MS stabilizing solution, if and only if

1) \((H, A, p)\) is MS detectable;
2) \((A, Q)\) does not have uncontrollable eigenvalues on the unit circle, i.e.,

\[\text{rank}\{\lambda I_n - A Q\} = n, \quad \forall |\lambda| = 1,\] (35)

where \(\lambda\) is an eigenvalue of \(A\).

Proof. By Theorem 2, it suffices to show that 1 is an uncontrollable eigenvalue for \((A, Q)\), if and only if there exists some eigenvalue on the unit circle that is uncontrollable for \((A, Q)\). We first show the sufficiency. By the PBH test, there exist some \(|\lambda| = 1\) and \(0 \neq v \in \mathbb{C}^n\) such that

\[A'v = \lambda v, \quad Qv = 0.\] (36)

Let \(V_i := vv^* + \bar{v}\bar{v}^* \in S_+^n\) for all \(i \in \mathcal{N}\) such that \(V = (V_1,\ldots,V_N) \in \mathbb{H}_+^n\). Then by (26),

\[L_i^*(V) = \sum_{j=1}^N p_{ij}A'V_iA = A'(vv^* + \bar{v}\bar{v}^*)A = A'(A'v)^* + A'\bar{v}(A'\bar{v})^* = \lambda vv^* + \lambda \bar{v}\bar{v}^* = V_i \quad \forall i \in \mathcal{N}.\] (37)

In addition, it is easy to see that \(QV_i = 0 \forall i \in \mathcal{N}\). Therefore, 1 is an uncontrollable eigenvalue for the pair \((A, Q)\).

To show the necessity, assume that there exists \(V = (V_1,\ldots,V_N) \in \mathbb{H}_+^n \setminus \{0\}\) such that \(L_i^*(V) = V_i\) and \(QV_i = 0\). Recall the limit probability distribution \(\mu\) in (8) of Lemma 1 where \(\mu = [\mu_1, \mu_2, \ldots, \mu_N] \neq 0\) is a positive row vector. Since \(\mu P = \mu\), there holds

\[\mu_j = \sum_{i=1}^N \mu_i p_{ij} \quad \forall j \in \mathcal{N}.\]

By the above equality, \(V_i = L_i^*(V)\) with \(L_i^*(V)\) defined
in (26), and \( QV_i = 0 \) \( \forall i \in \mathcal{N} \), we have
\[
X := \sum_{i=1}^{N} \mu_i V_i = \sum_{i=1}^{N} \mu_i L_i^*(V) = \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i p_{ij} A' V_j A \\
= \sum_{j=1}^{N} A' V_j A \left( \sum_{i=1}^{N} \mu_i p_{ij} \right) = \sum_{j=1}^{N} \mu_j A' V_j A \\
= A' \left( \sum_{j=1}^{N} \mu_j V_j \right) A = A' X A,
\]
and
\[
QX = \sum_{i=1}^{N} \mu_i QV_i = 0.
\]

There thus hold \( A' X A = X \) and \( QX = 0 \). An application of Lemma 3 with \( \lambda = 1 \) concludes that \((A, Q)\) has at least an uncontrollable eigenvalue on the unit circle. The proof is complete. \( \blacksquare \)

Condition 2) in Theorem 3 shows that the controllability requirement is the same as that arising in the study of the standard ARE [27], which is an interesting result. In Theorem 3, the matrix rank condition (35) is simple to check, while the MS detectability for \((H, A, p)\) is not straightforward, which needs to be numerically verified by solving a feasibility problem generally, in terms of \(2^m\) linear matrix inequalities (LMIs); see (42) in the next section. The other issue is the exponential complexity of the optimal stationary estimator in Theorem 1. In the next section, we will derive some sufficient and necessary conditions for the MS detectability by exploring the system structure, which show directly how system parameters influence the MS detectability, and propose a locally optimal stationary estimator that has a linear complexity.

4 MS Detectability and Locally Optimal Stationary Estimator

4.1 MS detectability for \((H, A, p)\)

Theorem 3.9 of [16] describes the MS stability of MJLSs, based on which we also have the following definition for the MS detectability for system (5), coinciding exactly with Definition 1.

**Definition 5** We say that \((H, A, p)\) is **MS detectable**, if there exist \( \{X_i > 0\}_{i=1}^{N} \) and \( L = (L_1, \ldots, L_N) \in \mathbb{H}^{n \times m} \) such that either of the following two inequalities holds:

\[
X_i > \sum_{j=1}^{N} p_{ij} (A + L_j H_j)' X_j (A + L_j H_j) \quad \forall i \in \mathcal{N}, \quad (38)
\]

\[
X_j > \sum_{i=1}^{N} p_{ij} (A + L_i H_i) X_i (A + L_i H_i)' \quad \forall j \in \mathcal{N}. \quad (39)
\]

The pair \((C, A)\) is assumed to be detectable throughout the section, which is clearly weaker than the MS detectability of \((H, A, p)\) that relates to channel parameters of packet drops. We first provide an analytic necessary condition for the MS detectability.

**Theorem 4** The triple \((H, A, p)\) is **MS detectable**, only if
\[
\prod_{i=1}^{N} (1 - q_i) p^2(A) < 1. \quad (40)
\]

**Proof.** By (39), the MS detectability for \((H, A, p)\) implies that there exist \( \{X_i > 0\}_{i=1}^{N} \) such that
\[
X_1 > \sum_{i=2}^{N} p_{i1} (A + L_i H_i)' X_i (A + L_i H_i) \quad (41)
\]
\[
+ \prod_{i=1}^{N} (1 - q_i) AX_1 A \geq \prod_{i=1}^{N} (1 - q_i) AX_1 A'.
\]

This implies that \( A (\prod_{i=1}^{N} (1 - q_i)) \) must be a Schur stability matrix, leading to the inequality in (40). \( \blacksquare \)

The MS detectability for \((H, A, p)\) can be numerically verified by solving the following feasibility problem of LMIs that can be obtained by using the MS detectability condition in (38) and the Schur complement repeatedly.

**Proposition 1** Denote \( X_d := \text{diag}\{X_1, \ldots, X_N\} \) and \( \Psi_i(X, \Omega) = [ \Psi_{i1} \Psi_{i2} \cdots \Psi_{iN} ] \) where \( \Psi_{ij} = \sqrt{p_{ij}} (A' X_j + H_j' \Omega_j) \) for \( 1 \leq j \leq N \). Then, \((H, A, p)\) is **MS detectable**, if and only if there exist \( \{X_i > 0\}_{i=1}^{N} \) and \( \{\Omega_i \in \mathbb{R}^{n \times m}\}_{i=1}^{N} \) such that LMI
\[
\begin{bmatrix}
X_1 & \Psi_1(X, \Omega) \\
\Psi_1(X, \Omega)' & X_d
\end{bmatrix} > 0 \quad (42)
\]
holds for each \( i \in \mathcal{N} \).

A significant challenge to checking the MS detectability from Proposition 1 lies in the complexity, due to \( N = 2^m \) LMIs in (42). We will focus on lowering the complexity from \( N = 2^m \) to \( 2^m \). This is indeed possible through decoupling the MS detectability for \((H, A, p)\) into that for \( m \) subsystems, respectively.

Denote the state estimation error by
\[
e(k) := x(k) - \hat{x}(k). \quad (43)
\]

Taking the difference between (1a) and (14) yields
\[
e(k + 1) = (A - L_\theta(k) H_{\theta(k)}) e(k) \quad (44)
\]
after removing the terms about noises. Then the existence of \( \{L_\theta(k)\} \) that achieve the MS stability for the error dynamics described in (44) is equivalent to the MS detectability for \((H, A, p)\). Without loss of generality, the pair \((C, A)\) for system (1) is assumed to be of the following Wonham decomposition form [28]:

\[
A = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
A_{21} & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{m1} & \cdots & A_{m(m-1)} & A_m
\end{bmatrix}, \quad C = \begin{bmatrix}
c_1 & 0 & \cdots & 0 \\
0 & c_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_m
\end{bmatrix},
\]

(45)

where \( A_i \in \mathbb{R}^{n_i \times n_i}, c_i \in \mathbb{R}^{1 \times n_i}, \sum_{i=1}^{m} n_i = n \), and each pair \((c_i, A_i)\) is detectable under the detectability of \((C, A)\). Define

\[
\theta_i(k) := \gamma_i(k) + 1, \quad h_{i,\theta_i(k)} := \gamma_i(k)c_i,
\]

\[
A_i := (A_i, A_i), \quad h_i := (h_{i,1}, h_{i,2}), \quad p_i := \{p_{ij}^{(i)}\},
\]

for \(1 \leq i \leq m\), where \(p_{ij}^{(i)}\) is the \((l, j)\)th element of the TPM \(P_i\) in (4). Let us introduce a particular \(L_\theta(k)\) that is in the block diagonal form, conformal to that of \(C\):

\[
\bar{L}_\theta(k) := \text{diag}\{\ell_{\theta_1(k)}, \ldots, \ell_{\theta_m(k)}\}, \quad \ell_{\theta_i(k)} \in \mathbb{R}^{n_i}.
\]

(46)

**Theorem 5** If \((h_i, A_i, p_i)\) is MS detectable for all \(1 \leq i \leq m\), then \((H, A, p)\) is MS detectable.

**Proof.** For system (5), consider a similarity transform \(x(k) = S\hat{x}(k)\) with

\[
S = \text{diag}\{I_{n_1}, \epsilon^{-1}I_{n_2}, \ldots, \epsilon^{1-m}I_{n_m}\}, \quad \epsilon > 0.
\]

(47)

Then the MS detectability for \((H, A, p)\), i.e., system (5), is equivalent to that for

\[
\hat{x}(k + 1) = \hat{A}\hat{x}(k) + S^{-1}w(k),
\]

(48a)

\[
y_i(k) = \hat{H}_{\theta_i(k)}\hat{x}(k) + D_{\theta_i(k)}v(k),
\]

(48b)

where \(\hat{A} = S^{-1}AS\) and \(\hat{H}_{\theta_i(k)} = H_{\theta_i(k)}S = \Gamma_{\theta_i(k)}CS\) that is in the block diagonal form. Define \(\bar{L}_{\theta_i(k)} := S^{-1}\bar{L}_{\theta_i(k)}\), where \(\bar{L}_{\theta_i(k)}\) is given by (46). It follows from (44) that

\[
\bar{e}(k + 1) = (\hat{A} - \bar{L}_{\theta_i(k)}\hat{H}_{\theta_i(k)})\bar{e}(k),
\]

(49)

where \(\bar{e}(k) = S^{-1}e(k)\). Specifically, \(\hat{A} - \bar{L}_{\theta_i(k)}\hat{H}_{\theta_i(k)}\) has the following lower triangular form

\[
\hat{A} - \bar{L}_{\theta_i(k)}\hat{H}_{\theta_i(k)} = S^{-1}(A - L_{\theta_i(k)}H_{\theta_i(k)})S
\]

\[
= \begin{bmatrix}
A_{\theta_1(k)} & 0 & \cdots & 0 \\
\epsilon A_{21} & A_{\theta_2(k)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon^{m-1}A_{m1} & \cdots & \epsilon A_{m(m-1)} & A_{\theta_m(k)}
\end{bmatrix},
\]

where \(A_{\theta_i(k)} = A_i - \ell_{\theta_i(k)}h_{i,\theta_i(k)}\). Hence as \(\epsilon \to 0\), estimation error dynamics (49) approaches to a diagonal form, implying that with the block diagonal gain \(\bar{L}_{\theta_i(k)}\) in (46), the MS detectability for \(\{h_i, A_i, p_i\}_{i=1}^{m}\) is equivalent to the MS stability for the error dynamics in (49). The proof is thus complete.

**Remark 3** The Wonham decomposition (45), which gives \(m\) detectable subsystems \((c_i, A_i)\) from the original detectable system \((C, A)\), plays a critical role in lowering the complexity from \(N = 2^m\) to \(2m\), without which the complexity reduction is not possible. This decomposition is also a powerful tool when dealing with networked stabilization for multi-input systems over logarithmic quantization and i.i.d. fading channels [3, 4, 29].

Now we obtain a sufficient condition on decoupling the MS detectability, but the complexity of the stationary optimal estimator remains exponential. In the next subsection, we propose a locally optimal stationary estimator, which has the complexity \(2m\) as well.

### 4.2 A locally optimal stationary estimator

In this subsection, instead of the general estimator gain \(L_\theta(k)\) in estimator (14), we consider the block diagonal gain in (46) that reduces estimator (14) to

\[
\hat{x}(k + 1) = A\hat{x}(k) + \bar{L}_{\theta(k)}[y_k(k) - H_{\theta(k)}\hat{x}(k)].
\]

(50)

Write system state \(x(k) = [x_1(k)’, \ldots, x_m(k)’]^t\) with \(x_i(k) \in \mathbb{R}^{n_i}\) and measurement \(y_k(k) = [y_{i,1}(k), \ldots, y_{i,n_i}(k)]’\), and let \(\hat{x}_i(k)\) be the state estimation of \(x_i(k)\). It is observed that estimator (50) can be written as the following \(m\) sub-estimators:

\[
\begin{align*}
(\hat{x}_i(k + 1) &= A_1\hat{x}_i(k) + \ell_{\theta_i(k)}[y_{i,1}(k) - h_{i,\theta_i(k)}\hat{x}_i(k)], \\
\hat{x}_i(k + 1) &= A_i\hat{x}_i(k) + \ell_{\theta_i(k)}[y_{i,1}(k) - h_{i,\theta_i(k)}\hat{x}_i(k)] + \sum_{j=1}^{i-1}A_{ij}\hat{x}_j, \quad 2 \leq i \leq m.
\end{align*}
\]

(51)

Let \(\{Q_i\}_{i=1}^{m}\) and \(\{R_i\}_{i=1}^{m}\) be the sub-matrices on the diagonal of \(Q\) and \(R\), respectively. We have the following estimation result that is similar to that in Theorem 1. Hence, its proof is omitted.
Lemma 4 Assume that there exists the MS stabilizing solutions \( \{Z_i = (Z_{i,1}, Z_{i,2})\} \) to the following CAREs

\[
Z_{i,r} = \sum_{j=1}^{2} p_{ij} \left\{ A_j Z_{i,j} \left[ I_{n_i} + h_{i,j}(\pi_{i,j} R_i)^{-1} h_{i,j} Z_{i,j} \right]^{-1} A_j^T + \pi_{i,j} Q_j \right\}, \quad 1 \leq i \leq m, \ 1 \leq r \leq 2. \quad (52)
\]

Then for each sub-estimator in (51), the optimal stationary gains that minimize

\[
J_i(\infty) = \lim_{k \to \infty} E[\|x_i(k) - \hat{x}_i(k)\|^2] \quad (53)
\]

for \( i = 1, \ldots, m \) are given by

\[
\ell_{\theta_i(k)} = K_i(Z_i) := A_i Z_{i,2} c_{i}^T (c_i Z_{i,2} c_i + \pi_{i,2} R_i)^{-1} \quad (54)
\]

for both \( \theta_i(k) = 1, 2 \).

Note that we have intentionally allowed the value of \( l_{\theta_i(k)} \) for the case \( \theta_i(k) = 1 \) to be the same as that for \( \theta_i(k) = 2 \), without any effect, such that only a gain is required for each sub-estimator. Define \( \Sigma(k) := E[e(k)e(k)^T] \) and \( \Sigma_i(k) := E[e_i(k) e_i(k)^T 1_{\theta_i(k)=i}] \). Notice that

\[
\Sigma(k) = \sum_{i=1}^{N} \Sigma_i(k). \quad (55)
\]

In light of Lemma 4, we present the following locally optimal estimator.

Proposition 2 For the system dynamics described in (5) with \( (C, A) \) in the Wonham decomposition form (45), a locally optimal stationary estimator of which sub-estimator \( i \) in (51) minimizes cost \( J_i(\infty) \) in (53) is given by

\[
\hat{x}(k+1) = A \hat{x}(k) + \bar{K} [y_i(k) - H_{\theta_i(k)} \hat{x}(k)], \quad (56)
\]

where \( \bar{K} = \text{diag}\{K_1(Z_1), \ldots, K_m(Z_m)\} \}. Moreover, the corresponding stationary error covariance \( \Sigma(\infty) \) is given by the solution of following coupled Lyapunov equations

\[
\Sigma_j(\infty) = \sum_{i=1}^{N} p_{ij} \left\{ [A - \bar{K} H_i] \Sigma_i(\infty) [A - \bar{K} H_i]^T + \mu_i [\bar{K} D_i, R_i \bar{K}^T + Q_i] \right\}, \quad j \in \mathcal{N}. \quad (57)
\]

Remark 4 Compared with the complexity of solving \( 2^m \) CAREs for the optimal stationary estimator, the estimator in (56) only needs to solve \( 2m \) CAREs and one estimator gain \( \bar{K} \). This is achieved by restricting the estimator gain to the block diagonal form (46) and the estimation cost to local costs (53). As it is expected, the estimator in (56) has a performance loss compared with the optimal one, which will be illustrated in the simulation. Nonetheless, it follows from Theorems 3 and 5 that if \( (h_i, A_i, p_i) \) is MS detectable for \( 1 \leq i \leq m \) and the rank condition (35) for each \( (A_i, Q_i) \) holds, the error dynamics described in (44) with \( L_{\theta(k)} = \bar{K} \) is MS stable, implying that the covariance of the locally optimal stationary estimator (56) converges to a finite value \( \Sigma(\infty) \).

\[
\square
\]

Remark 5 If \( m > n \), it is not possible to obtain the Wonham decomposition form in (45), but in the following form:

\[
A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_{n+1} & A_2 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ A_n & \cdots & A_{n(n-1)} & A_n \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_n \end{bmatrix}_0
\]

where \( C_0 = [c_{n+1}', \ldots, c_m]'T \) with \( T \) the Wonham decomposition transform matrix. In this case, we may not obtain the ideal locally optimal estimator (56) that admits the low complexity \( 2m \) and uses measurement information from all \( m \) sensors. One simple way is to only use the measurements from the \( n \) sensors and abandon the left \( m - n \) sensors. One can also expect a better method to fuse the measurements from both the \( n \) and \( m - n \) sensors, which deserves a further study in the future.

\[
\square
\]

Since the MS detectivity for each subsystem \( (h_i, A_i, p_i) \) is the sufficient MS detectability condition for \( (H, A, p) \), and is also required by the locally optimal stationary estimator, we will investigate the MS detectability for \( (h_i, A_i, p_i) \) by providing some analytic MS detectability conditions in the next subsection.

4.3 MS detectability for \( (h_i, A_i, p_i) \)

For convenience, we will omit the subscript \( i \) in this subsection, i.e., denote \( A := A_i, \ C := c_i, \ \gamma(k) := \gamma_i(k), \ \theta(k) := 1 + \gamma(k) \) for any \( i = 1, \ldots, m \) with a slight abuse of notation. Also, the TPM for the process of Markovian packet drops \{\gamma(k)\} is now given by

\[
P = [p_{ij}] = \begin{bmatrix} 1 - q & q \\ p & 1 - p \end{bmatrix}. \quad (58)
\]

We further denote \( C_\theta(k) := \theta(k)C \) and

\[
A := (A, A), \ \ C := (C_1, C_2), \ \ p := \{p_{ij}\}, \ i, j \in \{1, 2\}.
\]
For simplicity, define the following operators

\[
\begin{align*}
\phi_1(L, X_1, X_2) &:= (1 - q)A'X_1A + qA'X_2A_L, \\
\phi_2(L, X_1, X_2) &:= pA'X_1A + (1 - p)L_XX_2A_L, \\
\psi_1(L, X_1, X_2) &:= (1 - q)AX_1A' + pAX_2A_L', \\
\psi_2(L, X_1, X_2) &:= qAX_1A' + (1 - p)L_XX_2A_L', \\
g_1(X_1, X_2) &:= (1 - q)AX_1A' + pAX_2A_L' - pAX_2C'(CX_2C')^{-1}CX_2A', \\
g_2(X_1, X_2) &:= qAX_1A' + (1 - p)L_XX_2A' - (1 - p)AX_2C'(CX_2C')^{-1}CX_2A',
\end{align*}
\]

where \( A_L = A + LC \). The next lemma is useful.

**Lemma 5** The following statements are equivalent.

a) The triple \((p, C, A)\) is MS detectable.

b) There exist \( X_1 > 0, X_2 > 0, \) and \( L \in \mathbb{R}^{n_1 \times m} \) such that

\[
X_1 > \psi_i(L, X_1, X_2) \quad \text{for} \quad i = 1, 2.
\]

c) There exist \( X_1 > 0 \) and \( X_2 > 0 \) such that \( X_i > g_i(X_1, X_2) \) for \( i = 1, 2 \).

d) There exist \( X_1 > 0, X_2 > 0, \) and \( \Omega \in \mathbb{R}^{n_1 \times m} \) such that the following LMIs hold:

\[
\begin{bmatrix}
X_1 \sqrt{1 - qA'X_1} \sqrt{q}(A'X_2 + C'O') \\
* & X_1 \\
* & 0 & X_2 \\
X_2 \sqrt{pA'X_1} \sqrt{1 - p}(A'X_2 + C'O') \\
* & X_1 & 0 \\
* & 0 & X_2
\end{bmatrix} > 0,
\]

for any \( X_1 > 0, X_2 > 0, \) and \( L \in \mathbb{R}^{n_1 \times m} \). Therefore, if \( X_1 > g_1(X_1, X_2) \) and \( X_2 > g_2(X_1, X_2) \) hold for some \( X_1 > 0 \) and \( X_2 > 0 \), then \( X_1 > \psi_1(-L_XX_2, X_1, X_2) \) and \( X_2 > \psi_2(-L_XX_2, X_1, X_2) \), proving the statement of c) \( \Rightarrow \) b).

b) \( \Rightarrow \) c): This is clearly true by \( X_1 > \psi_1(L, X_1, X_2) \geq g_1(X_1, X_2) \) and \( X_2 > \psi_2(L, X_1, X_2) \geq g_2(X_1, X_2) \).

a) \( \Leftrightarrow \) d): This is straightforward from Proposition 1.

**Theorem 6** The triple \((p, C, A)\) is MS detectable, if

\[
\min\{q, 1 - p\} > \lambda_c = 1 - \frac{1}{\prod_i \max\{|\lambda_i(A)|^2, 1\}},
\]

where \( \lambda_i(A) \) is the \( i \)-th eigenvalue of \( A \).

**Proof.** Recall the hypothesis on the detectability of \((C, A)\), assumed throughout this section. Hence, \((c_i, A_i)\) is detectable with rank\(\{c_i\} = 1 \) for \( 1 \leq i \leq m \). Then by Lemma 5.4 of [19], there exists \( X > 0 \) such that

\[
X > AXA' - \lambda AXC'(CXC')^{-1}CXA',
\]

if and only if \( \lambda > \lambda_c \). Therefore, if (61) holds, there exists \( \bar{X} > 0 \) to (62) with \( \lambda = \min\{q, 1 - p\} \) such that

\[
\bar{X} > AXA' - qAXC'(CXC')^{-1}CXA',
\]

\[
\bar{X} > AXA' - (1 - p)AXC'(CXC')^{-1}CXA'.
\]

Let \( \bar{X} = X_1 = pX_2/q \). Then, the above inequalities reduce to

\[
X_1 > g_1(X_1, X_2), \quad X_2 > g_2(X_1, X_2).
\]

Therefore, the sufficient condition in (61) holds, following from the equivalence of a) and c) in Lemma 5.

When \( m = n \), i.e., the order of \( A_i \) is one for \( 1 \leq i \leq n \), we have the following analytic necessary and sufficient condition for the MS detectability of \((p, C, A)\).

**Theorem 7** If \( m = n \), then the triple \((p, C, A)\) is MS detectable, if and only if

\[
q > 1 - \frac{1}{\rho(A)^2}.
\]

**Proof.** The argument for necessity is the same as that in Theorem 4. So we only show the sufficiency. First, we observe that if there exists a matrix \( L \) such that \( A_L = A + LC = 0 \), then the MS detectability condition in (39) becomes the case that there exist \( X_1 > 0, X_2 > 0 \) such that

\[
X_1 > (1 - q)AX_1A', \quad X_2 > pAX_1A'.
\]

In this case, if \( q > 1 - \rho(A)^{-2} \), we can always find some \( X_1 > 0 \) and \( X_2 > 0 \), rendering inequalities in (64) true. Clearly, when \( A \) and \( C \) are both scalars, the choice of \( L = -A/C \) makes \( A_L = 0 \), which completes the proof.
Remark 6 It is worth mentioning that the results in Section 3 and this subsection can be applied to the dual optimal control problem studied in [21], of which focus is on the convergence issue under assumptions of stabilizability and detectability of the system. To be specific, consider the linear system described by

\[ x(k + 1) = Ax(k) + Bu_i(k), \quad u_i(k) = \gamma(k)u(k), \]  

where \( u(k) \in \mathbb{R}^m \) is the control signal sent from the remote controller via the Markovian packet drop channel. In accordance with the MJLS theory in [16], the optimal controller is given by \( u(k) = F(X)x(k) \), where \( F(X) \) is computed using the MS stabilizing solution to following control CAREs:

\[ X_i = A'\mathcal{E}_i(X)A + W + A'\mathcal{E}_i(X)B_iF_i(X), \quad F(X) = -(U + B'E_i(X)B)^{-1}B'E_i(X)A, \quad i = 1, 2. \]

Here, \( \mathcal{E}_i(X) = \sum_{j=1}^{2} p_{ij}X_j \), \( B_1 = 0, B_2 = B \) and \( (W \geq 0, U > 0) \) are the weighting matrices for the state and control signal, respectively. According to the notion of MS stabilizability in Definition 2 of [23], which is dual to the MS detectability, similar results can be obtained for such an optimal control problem.

\[ \blacksquare \]

5 Simulation Examples

5.1 Existence of the MS stabilizing solution

First, we illustrate the theoretical results about the MS stabilizing solution and MS detectability by a numerical example. Consider a third-order system of the form in 1 with \( C = R = I_3 \), and

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1.2 & 0 \\ 1 & 1.5 & 1.3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

Note that \((C, A)\) is already in the form of Wonham decomposition. Clearly, \( \lambda = 1.3 \) is an uncontrollable eigenvalue for \((A, Q)\). Nonetheless, according to the condition 2) in Theorem 3, we only require that \(|\lambda| = 1\) be controllable. This condition is satisfied in this example. Let the parameters of the three channels be

\[ p_1 = 0.50, \quad p_2 = 0.60, \quad p_3 = 0.70, \]

\[ q_1 = 0.20, \quad q_2 = 0.32, \quad q_3 = 0.51. \]

So, \( q_1 > 0, q_2 > 1 - 1.2^2 = 0.3056, \) and \( q_3 > 1 - 1.3^2 = 0.4083. \) Then by Theorems 5 and 7, the system is MS detectable. Therefore, by Theorem 3, the MS stabilizing solution to the CAREs (16) exists. By solving the convex programming problem (32) using YALMIP [30], we can obtain the stabilizing solution \( Y \). Then with \( \{K_i(Y)\}_{i=1}^{8} \) computed by (17), we have \( \rho(C^*) = \rho(A) = 0.9297 < 1, \) according to Remark 3.5 of [16], where

\[ A = (P^i \otimes I_3)\text{diag}\{ (A - K_i(Y)H_i) \otimes (A - K_i(Y)H_i) \}_{i \in N}. \]

Thus, \( Y \) is indeed the MS stabilizing solution from Definition 2.

5.2 Performance of the optimal and locally optimal stationary estimators

We will use a target tracking example [31] to show the estimation performance of the optimal and locally optimal stationary estimators in Theorem 1 and Proposition 2, respectively. For brevity, the two estimators will be abbreviated as OS estimator and LOS estimator, respectively. The system dynamics is described by [31]

\[ x(k + 1) = \begin{bmatrix} 1 & 0 & 0 \\ T & 1 & 0 \\ T^2/2 & T & 1 \end{bmatrix} x(k) + w(k), \]

where \( T \) is the sampling period and \( w(k) \) is the Gaussian noise with covariance

\[ Q = 2\alpha\sigma_m^2 \begin{bmatrix} T & T^2/2 & T^3/6 \\ T^2/2 & T^3/3 & T^4/8 \\ T^3/6 & T^4/8 & T^5/20 \end{bmatrix}, \]

with \( \alpha \) the reciprocal of the maneuver time constant and \( \sigma_m^2 \) the variance of the target acceleration. The first, second and third entries of \( x(k) \) represent the acceleration, speed and position of the target, respectively. Suppose that there are three sensors measuring the target acceleration, speed and position, respectively. As a result, the measurement model is given by

\[ y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + v(k). \]

The covariance of the Gaussian noise \( v(k) \) is assumed to be \( R = 0.01I_3 \). The other system parameters are set to \( T = 1s, \alpha = 0.01, \) and \( \sigma_m^2 = 10. \) In this example, the original \((C, A)\) in [31] is already in the form of Wonham decomposition. By Theorems 5 and 7, the conditions \( q_1 > 0, q_2 > 0 \) and \( q_3 > 0 \) are sufficient to guarantee the MS detectability of the system.

Set the channel parameters as

\[ p_1 = 0.20, \quad p_2 = 0.30, \quad p_3 = 0.20, \]

\[ q_1 = 0.85, \quad q_2 = 0.75, \quad q_3 = 0.80. \]
Fig. 2. The target position error variance of the OS and LOS estimators. The theoretical value for OS estimator is the $(3,3)$th element in the matrix $\sum_{i=1}^{N} Y_i$, and the theoretical value for LOS one is the $(3,3)$th element in the matrix $\lim_{k \to \infty} \sum_{i=1}^{N} \Sigma_i(k)$.

For both OS and LOS estimators, we perform a Monte Carlo simulation with 50,000 trials over the time horizon of $k \in [1, 50]$ to show the estimation performance represented by the target position error variance. Fig. 2 shows the empirical variances of the OS and LOS estimators, of which both are close to their respective theoretical values. As expected, the performance of the LOS estimator is inferior to that of the globally optimal one that has much higher complexity. On the other hand, numerical results for the target position tracking shown in Fig. 3, illustrate fairly good estimation performance for both estimators. Fig. 4 shows the corresponding packet drop sequences of sensor data for the tracking in Fig. 3.

6 Conclusion

This paper studies the stationary state estimation over multiple Markovian packet drop channels. We have investigated the existence of the MS stabilizing solution, which is pivotal to the proposed estimator. It is shown that the known stabilizability condition in the existing literature is not necessary; only the controllability of the eigenvalues on the unit circle is required. In addition, a sufficient condition is derived showing that the MS detectability with multiple Markovian packet drop channels can be decoupled. Based on the decoupling method, a locally optimal stationary estimator with much lower complexity is proposed. In fact, the exponential complexity of the original optimal estimator is reduced to the linear complexity of the locally optimal estimator. Some analytic sufficient and necessary MS detectability conditions are also derived for the decoupled subsystems, each of which corresponds to the scenario of single Markovian packet drop channel. The results in this paper are potentially applicable to smart and optimal manufacturing in which a network of sensors collaboratively sense the interested process states [32].

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Appendix

The following technical Lemma will be useful in the proof of Theorem 2.

Lemma 6 The following two statements are true:

(i) For any $A, B$ in $\mathbb{S}_+^n$, $\text{tr}(AB) \geq 0$, and the equality holds if and only if $AB = 0$.

(ii) For any $A \in \mathbb{C}^{n \times n}$, $AA^* = 0$ if and only if $A = 0$. 

Fig. 3. A realization of target position estimates of the OS and LOS estimators.

Fig. 4. A sample path of packet drops.
Proof of Theorem 2. Necessity: It is obvious that the MS detectability for \((H, A, p)\) is necessary. In order to prove that condition 2) is also necessary, assume on the contrary that condition 2) does not hold but the CAREs in (16) has the MS stabilizing solution \(Y\), implying that \(\hat{L}^*(V)\) with \(L_i = -K_i(Y)\) is a stable operator. Since condition 2) is false, from Definition 3 there holds

\[
\mathcal{L}^*(V) = V, \quad Q^{1/2}V_i = 0, \quad \forall i \in \mathcal{N}, \tag{1.1}
\]

which means that \(\hat{L}^*(V)\) with \(L_i = -K_i(Y)\) is unstable, contradicting the assumption on the MS stabilizing solution. This concludes the necessity of condition 2).

Sufficiency: It suffices to show that \(\rho(\hat{L}^*) < 1\) under conditions 1) and 2). Assume on the contrary, \(\rho(\hat{L}^*) \geq 1\). Setting \(L_i = -K_i(Y^+)\) implies that \(\rho(\hat{L}^*) = 1\), by Lemma 2 (i). Let \((\lambda = 1, V)\) be an eigenvalue–eigenvector pair for \(\hat{L}^*\) such that \(\hat{L}^*(V) = V\). Rewrite CAREs (16) as

\[
Y_j^+ = \sum_{i=1}^{N} p_{ij} \left\{ [A - K_i(Y^+)H_i]Y_i^+ [A - K_i(Y^+)H_i]' + \mu_i Q \right\}. \tag{5.5}
\]

The same manipulations as in (.3) and the adjoint relation \(\sum_{j=1}^{N} \text{tr}(\mathcal{L}_j(Y^+)V_j) = \sum_{j=1}^{N} \text{tr}\{Y_j^+ \mathcal{L}_j^*(V)\}\) lead to

\[
\sum_{j=1}^{N} \text{tr}(Y_j^+V_j) = \sum_{j=1}^{N} \text{tr}(Y_j^+ \mathcal{L}_j^*(V)) + \sum_{j=1}^{N} \sum_{i=1}^{N} p_{ij} \mu_i \text{tr}(K_i(Y^+)RK_i(Y^+)')V_j + QV_j.
\]

Since \(\hat{L}^*(V) = V\), the above equation implies that \(QV_j = 0\), and \(K_i(Y^+)V_j = 0\), by again Lemma 6, further leading to \(\hat{L}^*(V) = V\). This concludes condition 2) and concludes the sufficiency proof. The proof is now complete. □

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