On generalized universal irrational rotation algebras and the operator $u + v$

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Abstract

We introduce a class of generalized universal irrational rotation $C^*$-algebras $A_{\theta, \gamma} = C^*(x, w)$ which is characterized by the relations $w^*w = ww^* = 1$, $x^*x = \gamma(w)$, $xx^* = \gamma(e^{-2\pi i \theta} w)$, and $xw = e^{-2\pi i \theta} wx$, where $\theta$ is an irrational number and $\gamma(z) \in C(T)$ is a positive function. We characterize tracial linear functionals, simplicity, and $K$-groups of $A_{\theta, \gamma}$ in terms of zero points of $\gamma(z)$. We show that if $A_{\theta, \gamma}$ is simple then $A_{\theta, \gamma}$ is an $AT$-algebra of real rank zero. We classify $A_{\theta, \gamma}$ in terms of $\theta$ and zero points of $\gamma(z)$. Let $A_\theta = C^*(u, v)$ be the universal irrational rotation $C^*$-algebra with $vu = e^{2\pi i \theta} uv$. Then $C^*(u + v) \cong A_{\theta, |1+\gamma|^2}$. As an application, we show that $C^*(u + v)$ is a proper simple $C^*$-subalgebra of $A_\theta$ which has a unique trace, $K_1(C^*(u + v)) \cong \mathbb{Z}$, and there is an order isomorphism of $K_0(C^*(u + v))$ onto $\mathbb{Z} + \mathbb{Z} \theta$. Moreover, $C^*(u + v)$ is a unital simple $AT$-algebra of real rank zero. We also calculate the spectrum and the Brown measure of $u + v$.

1 Introduction

The irrational rotation $C^*$-algebra $A_\theta$ has been one of most studied $C^*$-algebras. It is known now that $A_\theta$ is a unital simple $C^*$-algebra with a unique tracial state. There is an order isomorphism of $K_0(A_\theta)$ onto $\mathbb{Z} + \mathbb{Z} \theta$ and $K_1(A_\theta) \cong \mathbb{Z}^2$ [32, 34]. Moreover, $A_\theta$ is a unital simple $AT$-algebra of real rank zero [11].

Let $u$ and $v$ be two unitary generators of the universal irrational rotation $C^*$-algebra $A_\theta$ such that $vu = e^{2\pi i \theta} uv$. Then $u + v$ is an abnormal operator of $A_\theta$ and $C^*(u + v)$ is a proper $C^*$-subalgebra of $A_\theta$. In this paper, we study the algebraic structure of $C^*(u + v)$ and the spectral theory of $u + v$. Our motivation comes from our attempt to relate the theory of strongly irreducible operators relative to $\text{II}_1$ factors with irreducible subfactors (cf. Prop. [10.7] and the question that follows).

In fact, we study a class of generalized universal irrational rotation $C^*$-algebras $A_{\theta, \gamma} = C^*(x, w)$,
which is the universal $C^*$-algebra satisfying the following properties:

\begin{align}
  w^*w = ww^* &= 1, \\
  x^*x &= \gamma(w), \\
  xx^* &= \gamma(e^{-2\pi i\theta}w), \\
  xw &= e^{-2\pi i\theta}wx,
\end{align}

where $\theta \in (0, 1)$ and $\gamma(z) \in C(T)$ is a positive continuous function of the unit circle $T$. As we will see that $C^*(u + v) \cong A_{\theta,|1+iz|^2}$. If $\theta$ is an irrational number and $\gamma(z) \equiv 1$, then $A_{\theta,\gamma}$ is the irrational rotation $C^*$-algebra $A_\theta$. In fact, if $\gamma$ is invertible, then $A_{\theta,\gamma} = A_\theta$. However, the main interest of this paper is to study $A_{\theta,\gamma}$ when the set of the zero points of $\gamma(z)$ is nonempty.

It turns out that, when $\theta$ is fixed, the $C^*$-algebra $A_{\theta,\gamma}$ only depends on the set of zero points and therefore the algebraic property of $A_{\theta,\gamma}$ is completely determined by the zero points of $\gamma(z)$. For example, we characterize simplicity and uniqueness of trace of $A_{\theta,\gamma}$ as follows. Let $Y$ be the set of zero points of $\gamma(z)$ and let $\phi : \mathbb{T} \to \mathbb{T}$ be the rotation of the unit circle determined by $\theta$, i.e., $\phi(z) = e^{2\pi i\theta}z$. Denote by $\text{Orb}(\xi) = \{\phi^n(\xi) : n \in \mathbb{Z}\}$ for $\xi \in \mathbb{T}$. Then the following properties are equivalent:

1. $A_{\theta,\gamma}$ is simple;
2. $A_{\theta,\gamma}$ has a unique tracial state;
3. $\phi^n(Y) \cap Y = \emptyset$ for all integer $n \neq 0$;
4. For each $\xi \in \mathbb{T}$, $\text{Orb}(\xi) \cap Y$ contains at most one point.

If $Y$ is not empty, then $K_1(A_{\theta,\gamma}) \cong \mathbb{Z}$ and $K_0(A_{\theta,\gamma})$ is determined by the following splitting exact sequence

$$0 \to \mathbb{Z} \to K_0(A_{\theta,\gamma}) \to C(Y,\mathbb{Z}) \to 0.$$ 

We also show that if $A_{\theta,\gamma}$ is simple, then $A_{\theta,\gamma}$ has tracial rank zero and is an inductive limit of recursive subhomogenous $C^*$-algebras. As a result, the classification of $A_{\theta,\gamma}$ falls into Elliott’s classification program. Indeed, we obtain the following result. Let $\theta_1$ and $\theta_2$ be two irrational numbers, $\gamma_1$ and $\gamma_2 \in C(T)$ be non-negative functions and let $Y_i$ be the set of zeros of $\gamma_i$, $i = 1, 2$. Let $\phi_1, \phi_2 : \mathbb{T} \to \mathbb{T}$ be rotations of the unit circle determined by $\theta_1$ and $\theta_2$ respectively. Suppose that $\phi^n(Y_i) \cap Y_i = \emptyset$ for all integers $n \neq 0$, $i = 1, 2$. Then $A_{\theta_1,\gamma_1} \cong A_{\theta_2,\gamma_2}$ if and only if the following hold:

$$\theta_1 = \pm \theta_2 \text{mod}(\mathbb{Z}) \quad \text{and} \quad C(Y_1,\mathbb{Z})/\mathbb{Z} \cong C(Y_2,\mathbb{Z})/\mathbb{Z}.$$ 

In particular, when $\gamma_1$ has only finitely many zeros, then $A_{\theta_1,\gamma_1} \cong A_{\theta_2,\gamma_2}$ if and only if $\theta_1 = \pm \theta_2 \text{mod}(\mathbb{Z})$ and $\gamma_2$ has the same number of zeros as those of $\gamma_1$. 

A special case of interest is
\[ C^*(u + v) = C^*(u + v, u^*v) = C^*(u(1 + u^*v), u^*v) \cong A_{\theta, \gamma}, \]
where \(\gamma(z) = |1 + z|^2\). As an application of the above results of generalized universal irrational rotation \(C^*\)-algebras, we show that \(C^*(u + v)\) is a proper simple \(C^*\)-subalgebra of \(A_{\theta}\) which has a unique trace, \(K_1(C^*(u + v)) \cong \mathbb{Z}\), and there is an order isomorphism of \(K_0(C^*(u + v))\) onto \(\mathbb{Z} + \mathbb{Z}\theta\). Moreover, \(C^*(u + v)\) is a unital simple \(\mathcal{A}\)-algebra with real rank zero. Therefore, \(C^*(u + v)\) has tracial rank zero.

The second part of the paper is to study the spectrum of \(u + v\), which is motivated by the “the Ten Martini Problem” on the almost Mathieu operator. In mathematical physics, the almost Mathieu operator is given by

\[ (H_{\lambda, \theta, \beta} u)(n) = u(n + 1) + u(n - 1) + 2\lambda \cos(2\pi(n\theta + \beta))u(n), \]

acting as a self-adjoint operator on the Hilbert space \(\ell^2(\mathbb{Z})\). Here \(\theta, \beta, \lambda \in \mathbb{R}\) are parameters. Almost Mathieu operator was firstly introduced by R. Peierls [27] and has been extensively studied (see [22] for a recent historical account and for the physics background). In pure mathematics, its importance comes from the fact of being one of the best-understood examples of an ergodic Schrödinger operator. For example, three problems (now all solved) of Barry Simon’s fifteen problems [36] about Schrödinger operators “for the twenty-first century” featured the almost Mathieu operator. The fourth problem in [36] (known as the “the Ten Martini Problem” after Kac and Simon) conjectures that the spectrum of the almost Mathieu operator is a Cantor set for all \(\lambda \neq 0\) and irrational numbers \(\theta\). Recently, Avila and Jitomirskaya confirmed this conjecture in [1]. For a history of this problem and earlier partial results see [22, 7, 36, 16, 8, 2, 31].

Recall that the irrational rotation \(C^*\)-algebra \(A_{\theta}\) can be represented on \(\ell^2(\mathbb{Z})\), by mapping \(u\) into the bilateral shift (taking \(\phi\) into \((\phi(n - 1))_{n\in\mathbb{Z}}\)), and \(v\) into the operation of multiplication by \(e^{2\pi i\theta}\) (taking \(\phi\) into \(e^{2\pi in\theta}(\phi(n))_{n\in\mathbb{Z}}\)), and then the polynomial \((u + \lambda e^{2\pi i\beta}v) + (u + \lambda e^{2\pi i\beta}v)^*\) is mapped into the bounded self-adjoint operator \(H_{\lambda, \theta, \beta}\). Since \(A_{\theta}\) is simple (when \(\theta\) is irrational), the spectrum of \(H_{\lambda, \theta, \beta}\) is the same as the spectrum of the element \((u + \lambda e^{2\pi i\beta}v)^*\). A natural question is that what is the spectrum of \(u + \lambda e^{2\pi i\beta}v\)? If \(\theta\) is an irrational number, then by the uniqueness of \(A_{\theta}\) the spectrum of \(u + \lambda e^{2\pi i\beta}v\) is the same as \(u + |\lambda|v\). So from now on, we always assume that \(\lambda > 0\) and \(\beta = 0\).

Let \(\tau\) be the unique tracial state on \(A_{\theta}\). By the GNS-construction, we obtain a faithful representation \(\pi\) of \(A_{\theta}\) on \(L^2(A_{\theta}, \tau)\). The weak operator closure of \(\pi(A_{\theta})\) is the hyperfinite II_1 factor \(R\). Since the spectrum of \(u + \lambda v\) is same as the spectrum of \(\pi(u + \lambda v)\) in \(R\), we need only to calculate the
spectrum of $\pi(u + \lambda v)$ in $R$. In the following we identify $A_\theta$ with $\pi(A_\theta)$ and thus identify $u + \lambda v$ with $\pi(u + \lambda v)$.

One of the main results of the present paper is that the spectrum of $u + \lambda v$ is given by

$$\sigma(u + \lambda v) = \begin{cases} T & 0 < \lambda < 1, \\ B(0, 1) & \lambda = 1, \\ \lambda T & \lambda > 1. \end{cases}$$

Another result of spectral theory is related to the Brown measure. L. G. Brown introduced in the paper [3] a spectral distribution measure $\mu_T$ for not necessarily normal operators $T$ in a von Neumann algebra $M$ with a faithful normal tracial state $\tau$, which is called the Brown measure of $T$. Recently, U. Haagerup and H. Schultz [17] proved a remarkable result about the existence of nontrivial hyperinvariant subspaces of operators in type $\text{II}_1$ factors. They proved that if the support of $\mu_T$ contains more than two points, then $T$ has a nontrivial hyperinvariant space. However, the calculation of Brown measures of nonnormal operators is difficult in general (see [15, 6, 13]). In particular, Haagerup and Larsen in [15] showed that the Brown measure of the sum of two free Haar unitary operator $T = u_1 + u_2$ is rotation invariant, has support equal to $B(0, \sqrt{2}) (= \sigma(T))$, and has radial density

$$f_T(r) = \begin{cases} \frac{4}{4\pi(4-r^2)^2}, & 0 < r < \sqrt{2} \\ 0, & \text{otherwise}. \end{cases}$$

In section 12, we will show that the Brown measure of $u + v$ (in $R$) is the Haar measure on the unit circle.

This paper is organized as follows. In section 2 we introduce the class of generalized universal irrational rotation $C^*$-algebras $A_{\theta,\gamma} = C^*(x, w)$. We prove that, in fact, $A_{\theta,\gamma}$ can be viewed as a $C^*$-subalgebra of $A_\theta$. We also fix some notation that will be used in the later sections. In section 3, we give some descriptions of the tracial state space of $A_{\theta,\gamma}$ in terms of zero points of $\gamma(z)$. In particular, we show that $A$ has a unique tracial state if and only if each rotation orbit contains at most one zero point of $\gamma$. In section 4, we characterize simplicity of $A_{\theta,\gamma}$ in terms of zero points of $\gamma(z)$. We show that $A_{\theta,\gamma}$ is simple if and only if it has a unique tracial state which is also equivalent to the condition that each rotation orbit contains at most one zero point of $\gamma$. In section 5, we construct Rieffel’s projections in every simple generalized universal irrational rotation algebra $A_{\theta,\gamma}$. In section 6, we calculate $K$-groups of $A_{\theta,\gamma}$. In section 7, using results of section 3-6 and recent development in the Elliott’s classification program, we show that when $A_{\theta,\gamma}$ is simple, then $A_{\theta,\gamma}$ is an $\mathcal{AT}$-algebra of real rank zero. We obtain a classification result of simple $C^*$-algebras of $A_{\theta,\gamma}$ in terms $\theta$ and zero points of $\gamma(z)$. In section 8 we prove that the von Neumann subalgebra generated by $u + \lambda v$ is $R$ for
all $0 < \lambda < \infty$, and the $C^*$-algebra generated by $u + \lambda v$ is $C^*(u, v)$ if $\lambda \neq 1$. However, for $\lambda = 1$, $C^*(u + v)$ is isomorphic to $A_{\theta, |1+1|^2}$. Therefore $C^*(u + v)$ is a unital simple $A\mathbb{T}$-algebra of real rank zero which has $K_1(C^*(u + v)) \cong \mathbb{Z}$ and $K_0(u + v)$ is order isomorphic to $\mathbb{Z} + \mathbb{Z}\theta$. In particular, $C^*(u + v)$ is not $\ast$-isomorphic to $C^*(u, v)$.

In section 9 we show that the spectral radius of $u + \lambda v$ is 1 if $0 < \lambda \leq 1$. A key idea in the calculation is using Birkhoff’s Ergodic theorem. Then in section 10 we show that the relative commutant of $u + v$ in $R$ does not contain any nontrivial idempotent. By the Riesz spectral decomposition theorem, the spectrum of $u + v$ is connected. Combining the fact that the spectrum of $u + v$ is rotation symmetric, in section 11 we obtain that $\sigma(u + v) = B(0, 1)$. We show that the spectral radius of $(u + \lambda v)^{-1}$ is less or equal than 1 for $0 < \lambda < 1$, which implies that $\sigma(u + \lambda v)$ is contained in the unit circle $\mathbb{T}$. Since the spectrum of $u + \lambda v$ is rotation symmetric, $\sigma(u + \lambda v) = \mathbb{T}$. By the symmetry of $u$ and $v$, $\sigma(u + \lambda v) = \lambda \sigma(\lambda^{-1}u + v) = \lambda \mathbb{T}$ for $\lambda > 1$. In section 12, we calculate Brown measure of $u + \lambda v$.

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2 Generalized universal irrational rotation $C^*$-algebras

Let $u$ and $v$ be two unitary generators of the universal irrational rotation $C^*$-algebra $A_\theta$ such that $vu = e^{2\pi i \theta} uv$. To study the properties of $C^*$-algebras generated by $u + v$, we will consider the universal $C^*$-algebra satisfying the following properties:

\begin{align}
    w^* w &= ww^* = 1, \\
    x^* x &= \gamma(w), \\
    xx^* &= \gamma(e^{-2\pi i \theta} w), \\
    xw &= e^{-2\pi i \theta} wx,
\end{align}

where $\gamma(z) \in C(\mathbb{T})$ is a positive function.

A $C^*$-algebra $A_{\theta, \gamma}$ is universal for the above relations provided that it is generated by operators $x, w$ satisfying (2.1)-(2.4) and whenever $\mathfrak{A} = C^*(x', w')$ is another $C^*$-algebra satisfying (2.1)-(2.4), there is a homomorphism of $A_{\theta, \gamma}$ onto $\mathfrak{A}$ which carries $x$ to $x'$ and $w$ to $w'$. By (2.1), $w$ is a unitary operator. So (2.2) implies that $\|x\| \leq \|\gamma\|^{1/2}$. We may consider the collection of all operators $x_\alpha, w_\alpha$ in $B(H_\alpha)$ satisfying (2.1)-(2.4). Then form the operator

\[ x = \sum \oplus x_\alpha \quad \text{and} \quad w = \sum \oplus w_\alpha. \]
Let $A_{\theta,\gamma} = C^*(x, w)$. Then $A_{\theta,\gamma}$ is the desired universal algebra. Note that if $\gamma(z) \equiv 1$, then $A_{\theta,\gamma}$ is precisely the universal irrational rotation algebra. So we call $A_{\theta,\gamma}$ a **generalized universal irrational rotation algebra**.

Let $A_\theta$ be the universal irrational rotation $C^*$-algebra with two unitary generators $u, v$ with $vu = e^{2\pi i \theta}uv$. Then $u\gamma(v)^{1/2}$ and $v$ satisfy (2.1)-(2.4). So there is a $*$-homomorphism from $A_{\theta,\gamma}$ onto the $C^*$-subalgebra of $A_\theta$ generated by $u\gamma(v)^{1/2}$ and $v$. We will show that we may view $A_{\theta,\gamma}$ as the $C^*$-subalgebra of $A_\theta$ generated by $u\gamma(v)^{1/2}$ and $v$ and $C^*(u + v) \cong A_{\theta,\gamma}$. 

By (2.1)-(2.4) and simple calculations, we have the following equations.

\begin{align}
x^*w &= e^{2\pi i \theta} wx^*, \quad (2.5) \\
x f(w) &= f(e^{-2\pi i \theta} w)x, \quad \forall f(z) \in C(\mathbb{T}), \quad (2.6) \\
x^* f(w) &= f(e^{2\pi i \theta} w)x^*, \quad \forall f(z) \in C(\mathbb{T}), \quad (2.7)
\end{align}

\begin{align}
(x^*)^r x^r &= \gamma(e^{2\pi i (r-1) \theta} w) \gamma(e^{2\pi i (r-2) \theta} w) \cdots \gamma(w), \quad (2.8) \\
x^r(x^*)^r &= \gamma(e^{-2\pi i \gamma \theta} w) \gamma(e^{-2\pi i (r-1) \theta} w) \cdots \gamma(e^{-2\pi i \theta} w). \quad (2.9)
\end{align}

We apply the universal property to obtain certain special automorphisms of $A_{\theta,\gamma}$. For any constant $\lambda = e^{2\pi i \theta}$ on the unit circle, the pair $(\lambda x, w)$ also satisfy (2.1)-(2.4). Thus there is an endomorphism of $A_{\theta,\gamma}$ such that $\rho_t(x) = \lambda x$ and $\rho_t(w) = w$. By symmetry, $\rho^{-t}(x) = \lambda x$ and $\rho^{-t}(w) = w$. Hence, $\rho^{-t}(\rho_t(x)) = \rho_t(\rho^{-t}(x)) = x$ and $\rho^{-t}(\rho_t(w)) = \rho_t(\rho^{-t}(w)) = w$. This implies that $\rho_t$ is an automorphism of $A_{\theta,\gamma}$.

For each fixed $a$ in $A_{\theta,\gamma}$, the map from $[0, 1]$ to $A_{\theta,\gamma}$ given by $f(t) = \rho_t(a)$ is norm continuous. To verify this, notice that it is true for all noncommutative polynomials in $x, x^*, w, w^*$. These are dense and automorphisms are contractive; so the rest follows from a simple approximation argument.

Define a map of $A_{\theta,\gamma}$ into itself by

$$
\Phi(a) = \int_0^1 \rho_t(a) dt.
$$

Then the integral makes sense as Riemann sum because the integrand is a norm continuous function. By (2.1)-(2.9) and simple calculations, we can see that the following set

$$
\left\{ \sum_{n=1}^N x^n f_n(w) + f_0(w) + \sum_{n=1}^N f_{-n}(w)(x^*)^n \mid N \in \mathbb{N}, f_n(z), f_{-n}(z) \in C(\mathbb{T}) \right\}
$$
We conclude that $a$ for any scalar $0 < a < 1$. Since, $\phi(a) = 0$, $C(a)$ $\phi(w) = \phi(f(w)(w^*)^k) = 0$ for all $f(z) \in C(T)$ and $k \in \mathbb{N}$. In addition, for every $a \in A_{\theta, \gamma}$, $\phi(a) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} w^j a(w^*)^j$.

**Proof.** Since, $\|\rho_t(a)\| = \|a\|$, $\left|\sum_{j=1}^{n} \rho_{t_j}(a)\beta_j\right| \leq \|a\|$ for any scalar $0 \leq \beta_j \leq 1$ such that $\sum_{j=1}^{n} \beta_j = 1$. It follows that $\|\phi(a)\| = \left|\int_{0}^{1} \rho_{t}(a)dt\right| \leq \|a\|$.

We conclude that $\|\phi\| \leq 1$. Since $\phi(1) = 1$, $\|\phi\| = 1$. Since $\rho_t(w) = w$ for all $t$, $\rho_t(a) = a$ for all $a \in C^*(w)$. Hence $\phi(a) = a$ for all $a \in C^*(w)$. By the definition of $\phi$, $\phi(a_1a_2) = \int_{0}^{1} \rho_{t}(a_1a_2)dt = \int_{0}^{1} \rho_{t}(a_1)\rho_{t}(a_2)dt = \int_{0}^{1} a_1\rho_{t}(a_2)dt = a_1\phi(a)$ for all $a_1, a_2 \in C^*(w)$ and $a \in A_{\theta, \gamma}$.

Suppose $a = x^k f(w)$ for $f(z) \in C(T)$ and $k \in \mathbb{N}$. Then $\phi(a) = \int_{0}^{1} \rho_{t}(x^k f(w))dt = \int_{0}^{1} \rho_{t}(x^k)\rho_{t}(f(w))dt = \int_{0}^{1} e^{2\pi ikt} x^k f(w)dt = \left(\int_{0}^{1} e^{2\pi ikt} dt\right) a = 0$.

Suppose $a = f(w)(x^*)^k$ for $f(z) \in C(T)$ and $k \in \mathbb{N}$. Then $\phi(a) = \int_{0}^{1} \rho_{t}(f(w)(x^*)^k)dt = \int_{0}^{1} \rho_{t}(f(w))\rho_{t}((x^*)^k)dt = \int_{0}^{1} e^{-2\pi ikt} f(w)(x^*)^k dt = \left(\int_{0}^{1} e^{-2\pi ikt} dt\right) a = 0$.

Since $\|\phi\| = 1$, $\phi(A_{\theta, \gamma}) \subseteq C^*(w)$. By Tomiyama’s Theorem [37], $\phi$ is a conditional expectation of $A_{\theta, \gamma}$ onto $C^*(w)$. If $a$ is positive and nonzero, then $\rho_t(a)$ is positive and nonzero for all $t$. Thus the integral $\phi(a)$ is positive and nonzero. Hence $\phi$ is faithful.

Suppose $a = x^k f(w)$ for $f(z) \in C(T)$ and $k \in \mathbb{N}$. By equations (2.4) and (2.5), $\lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} w^j a(w^*)^j = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} e^{2\pi i j \theta} a = \lim_{n \to \infty} \frac{1}{2n+1} \left(\frac{\sin(2n+1)\pi k \theta}{\sin \pi k \theta}\right) a = 0$. 

is dense in $A_{\theta, \gamma}$.

The proof of the following proposition is similar to the proof of Theorem VI.1.1 of [9]. For the sake of completeness, we include a detailed proof.

**Proposition 2.1.** The map $\phi$ is a faithful conditional expectation of $A_{\theta, \gamma}$ onto $C^*(w)$ such that $\phi(x^k f(w)) = \phi(f(w)(x^*)^k) = 0$ for all $f(z) \in C(T)$ and $k \in \mathbb{N}$. In addition, for every $a \in A_{\theta, \gamma}$,
Hence
\[ \Phi(a) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} w^j a(w^*)^j = 0. \]

Similarly, we can show that if \( a = f(w)(x^*)^k \) for \( f(z) \in C(\mathbb{T}) \) and \( k \in \mathbb{N} \) then
\[ \Phi(a) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} w^j a(w^*)^j = 0. \]

If \( a = f(w) \) for some \( f(z) \in C(\mathbb{T}) \), then
\[ \Phi(a) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{j=-n}^{n} w^j a(w^*)^j = a. \]

By linearity and continuity, this formula is valid for all \( a \) in \( A_{\theta, \gamma} \). \( \Box \)

**Corollary 2.2.** \( \forall a \in A_{\theta, \gamma}, \rho_t(a) = a \) for all \( 0 \leq t \leq 1 \) if and only if \( a \in C^*(w) \).

**Proof.** If \( a \in A_{\theta, \gamma} \) and \( \rho_t(a) = a \) for all \( 0 \leq t \leq 1 \), then \( a = \Phi(a) \in C^*(w) \) by Proposition 2.1.

Conversely, since \( \rho_t(w) = w \) for all \( t \), \( \rho_t(a) = a \) for all \( t \) and \( a \in C^*(w) \). \( \Box \)

Let \( m = dz/2\pi \) be the unique Haar measure on \( \mathbb{T} \).

**Remark 2.3.** If \( \gamma(z) \in C(\mathbb{T}) \) is a positive function with \( m(\{z|\gamma(z) = 0\}) = 0 \), then \( 2.3 \) can be replaced by a weaker condition
\[ xx^* \in C^*(w). \] (2.10)

To see this, let \( xx^* = h(w) \) for some \( h(z) \in C(\mathbb{T}) \). Then by \( 2.5 \)
\[ \gamma(w)^2 = x^* xx^* x = x^* h(w) x = h(e^{2\pi i \theta} w) x^* x = h(e^{2\pi i \theta} w) \gamma(w). \]

Hence \( \gamma(z)^2 = h(e^{2\pi i \theta} z) \gamma(z) \). Let \( E = \{z|\gamma(z) = 0\} \). Then for \( z \in \mathbb{T} \setminus E \), \( \gamma(z) = h(e^{2\pi i \theta} z) \). Since \( m(\mathbb{T} \setminus E) = 1 \), \( \gamma(z) = h(e^{2\pi i \theta} z) \) for all \( z \in \mathbb{T} \). Thus \( h(z) = \gamma(e^{-2\pi i \theta} z) \), which is \( 2.3 \).

Note that in the irrational rotation \( C^*-\)algebra \( C^*(u, v) \) with \( vu = e^{2\pi i \theta} uv, u \gamma(v)^{1/2} \) and \( v \) satisfy \( 2.1 \)-\( 2.4 \). So there exists a homomorphism \( \varphi \) from \( A_{\theta, \gamma} \) onto \( C^*(u \gamma(v)^{1/2}, v) \) such that \( \varphi(x) = u \gamma(v)^{1/2} \) and \( \varphi(w) = v \). Since the spectrum \( \sigma(v) \) is \( \mathbb{T} \), \( \sigma(w) = \mathbb{T} \). Hence \( C^*(w) \cong C(\mathbb{T}) \). In the following, we identify \( C^*(w) \) with \( C(\mathbb{T}) \). Let \( \rho \) be the state on \( C(\mathbb{T}) \) induced by the Haar measure \( m \) on \( \mathbb{T} \). Then \( \rho \) is faithful on \( C^*(w) \).

**Lemma 2.4.** For \( a \in A_{\theta, \gamma} \), let \( \tau(a) = \rho \cdot \Phi(a) \). Then \( \tau \) is a faithful trace on \( A_{\theta, \gamma} \).
Proof. Since \( \rho \) is a faithful state on \( C^*(w) \) and \( \Phi \) is a faithful conditional expectation of \( A_{\theta,\gamma} \) onto \( C^*(w) \), \( \tau \) is a faithful state on \( A_{\theta,\gamma} \). We only need to verify \( \tau \) is a trace. Note that the following set
\[
\left\{ \sum_{n=1}^{N} x^n f_n(w) + f_0(w) + \sum_{n=1}^{N} f_{-n}(w)(x^n)^N \mid N \in \mathbb{N}, f_n(z), f_{-n}(z) \in C(\mathbb{T}) \right\}
\]
is dense in \( A_{\theta,\gamma} \). By boundedness, linearity and positivity of \( \tau \), we need only to verify \( \tau(ab) = \tau(ba) \) for the following two cases.

Case 1. \( a = x^r f(w), b = x^s g(w), r, s \geq 0 \). If \( r + s = 0 \), i.e., \( r = s = 0 \), then \( \tau(ab) = \tau(ba) \) is trivial. Suppose \( r + s > 0 \). Then
\[
\tau(ab) = \tau(x^r f(w) x^s g(w)) = \tau(x^{r+s} f(e^{2\pi i s \theta} w) g(w)) = \rho(\Phi(x^{r+s} f(e^{2\pi i s \theta} w) g(w)) = 0,
\]
and
\[
\tau(ba) = \tau(x^s g(w) x^r f(w)) = \tau(x^{r+s} g(e^{2\pi i r \theta} w) f(w)) = \rho(\Phi(x^{r+s} g(e^{2\pi i r \theta} w) f(w)) = 0.
\]
So \( \tau(ab) = \tau(ba) \).

Case 2. \( a = x^r f(w), b = g(w)(x^*)^s, r, s \geq 0 \). If \( r > s \), then
\[
\tau(ab) = \tau(x^r f(w) g(w)(x^*)^s) = \tau(x^{r-s} f(e^{-2\pi i s \theta} w) g(e^{-2\pi i s \theta} w) x^s (x^*)^s)
\]
\[
= \rho(\Phi(x^{r-s}) f(e^{-2\pi i s \theta} w) g(e^{-2\pi i s \theta} w) x^s (x^*)^s) = 0,
\]
and
\[
\tau(ba) = \tau(g(w)(x^*)^s x^r f(w)) = \tau(g(w)(x^*)^s x^s f(e^{-2\pi i (r-s) \theta} w) x^{r-s})
\]
\[
= \rho(g(w)(x^*)^s x^s f(e^{-2\pi i (r-s) \theta} w) \Phi(x^{r-s})) = 0.
\]
So \( \tau(ab) = \tau(ba) \). Similarly, we can show that if \( r < s \) then \( \tau(ab) = \tau(ba) \). If \( r = s \), then we have
\[
\tau(ab) = \rho(ba) = \rho(g(w)(x^*)^s x^r f(w)) = \rho(g(w) f(w) \gamma(e^{2\pi i (r-1) \theta} w) \gamma(e^{2\pi i (r-2) \theta} w) \cdots \gamma(w)),
\]
\[
\tau(ba) = \rho(ab) = \rho(x^r f(w) g(w)(x^*)^r) = \rho(f(e^{-2\pi i r \theta} w) g(e^{-2\pi i r \theta} w) x^r (x^*)^r)
\]
\[
= \rho(f(e^{-2\pi i r \theta} w) g(e^{-2\pi i r \theta} w) \gamma(e^{-2\pi i r \theta} w) \gamma(e^{-2\pi i r \theta} w) \cdots \gamma(e^{-2\pi i r \theta} w))
\]
\[
= \int_{\mathbb{T}} f(e^{-2\pi i r \theta} z) g(e^{-2\pi i r \theta} z) \gamma(e^{-2\pi i r \theta} \cdot e^{2\pi i (r-1) \theta} z) \gamma(e^{-2\pi i r \theta} \cdot e^{2\pi i (r-2) \theta} z) \cdots \gamma(e^{-2\pi i r \theta} z) dm(z).
\]
Since \( m \) is the Haar measure on \( \mathbb{T} \), \( \tau(ab) = \tau(ba) \).

\[\square\]
Theorem 2.5. The homomorphism \( \varphi \) from \( A_{\theta,\gamma} \) onto \( C^*(u\gamma(v)^{1/2}, v) \) such that \( \varphi(x) = u\gamma(v)^{1/2} \) and \( \varphi(w) = v \) is an isomorphism.

Proof. Consider the GNS-construction of \( A_{\theta,\gamma} \) with respect to the faithful trace \( \tau \). Then we may assume that \( A_{\theta,\gamma} \) faithfully acts on the Hilbert space \( L^2(A_{\theta,\gamma}, \tau) \). Let \( \tau' \) be the unique trace on \( C^*(u, v) \), and let \( x' = u\gamma(v), \ w' = v \). For a noncommutative polynomial \( p \) in four variables, we have \( \tau(p(x, x^*, w, w^*)) = \tau'(p(x', (x')^*, w', (w')^*)) \). Hence the operator \( U : p(x, x^*, w, w^*) \rightarrow p(x', (x')^*, w', (w')^*) \) extends to a unitary operator from \( L^2(A_{\theta,\gamma}, \tau) \) onto \( L^2(C^*(u\gamma(v)^{1/2}, v), \tau') \). So \( \varphi(x) = U^*xU \) is an isomorphism.

In what follows, we will identify \( A_{\theta,\gamma} \) with the \( C^* \)-subalgebra of \( A_\theta \) generated by \( u\gamma^{1/2}(v) \) and \( v \). We will take advantage of the knowledge of \( A_\theta \) to study \( A_{\theta,\gamma} \). We will use the following conventions:

Definition 2.6. We may view \( A_\theta = C(\mathbb{T}) \rtimes_\phi \mathbb{Z} \), where \( \phi : \mathbb{T} \rightarrow \mathbb{T} \) is defined by \( \phi(z) = e^{2\pi i \theta}z \) for all \( z \in \mathbb{T} \). Define \( \alpha_\theta : C(\mathbb{T}) \rightarrow C(\mathbb{T}) \) by \( \alpha_\theta(f) = f \circ \phi \) for all \( f \in C(\mathbb{T}) \). Denote by \( u \) the unitary in \( A_\theta \) implementing \( \alpha_\theta \), i.e., \( u^*fu = \alpha_\theta(f) = f \circ \phi \) for all \( f \in C(\mathbb{T}) \).

Let \( \gamma : \mathbb{T} \rightarrow \mathbb{R}_+ \) be a nonnegative continuous function and let

\[
Y = \{ z \in \mathbb{T} : \gamma(z) = 0 \}.
\]

Viewing \( A_{\theta,\gamma} \) as a \( C^* \)-subalgebra of \( A_\theta \), it is easy to check that

\[
A_{\theta,\gamma} = C^*(C(\mathbb{T}), uC_0(\mathbb{T} \setminus Y)),
\]

the \( C^* \)-subalgebra of \( A_\theta \) generated by \( C(\mathbb{T}) \) and \( \{ uf : f \in C_0(\mathbb{T} \setminus Y) \} \).

Let \( \xi \in \mathbb{T} \) denote by

\[
\text{Orb}(\xi) = \{ \phi^n(\xi) : n \in \mathbb{Z} \}
\]

the orbit of \( \xi \) under the rotation \( \phi \).

The following is an easy fact:

Proposition 2.7. Let \( \theta \in (0, 1) \) be an irrational number and let \( Y \subset \mathbb{T} \) be a subset. Then the following are equivalent:

(1). \( \phi^n(Y) \cap Y = \emptyset \) for any integer \( n \neq 0 \);

(2). For each \( \xi \in \mathbb{T} \), \( \text{Orb}(\xi) \cap Y \) contains at most one point;

(3). \( Y_1 \cap Y_2 = \emptyset \), where \( Y_1 = \cup_{n \geq 0} \phi^n(Y) \) and \( Y_2 = \cup_{k \geq 1} \phi^{-k}(Y) \).
Proof. (1) ⇒ (2): Suppose that \( \phi^{k_1}(\xi), \phi^{k_2}(\xi) \in Y \) for integers \( k_1 \neq k_2 \). Then \( \phi^{k_1} \xi \in \phi^{k_1-k_2}(Y) \cap Y \). This is contradiction. So (2) holds.

(2) ⇒ (3): If \( \xi \in Y_1 \cap Y_2 \), then there are \( \xi_1, \xi_2 \in Y \) such that \( \xi = e^{2\pi i n \theta} \xi_1 = e^{-2\pi i k \theta} \xi_2 \) for some \( n \geq 0 \) and \( k \geq 1 \). It follows that \( \xi_1 \in Y \) and \( e^{2\pi i (n+k) \theta} \xi_1 \in Y \). By (2), this is impossible. So (3) holds.

(3) ⇒ (1): Suppose that \( \xi \in \phi^{-n}(Y) \cap Y \) for some integer \( n \neq 0 \). If \( n \leq -1 \), then \( \xi \in Y_1 \cap Y_2 \). If \( n \geq 1 \), then \( \phi^{-n}(\xi) \in Y \cap \phi^{-n}(Y) \subset Y_1 \cap Y_2 \).

\( \square \)

3 Traces on generalized universal irrational rotation \( C^* \)-algebras

We will continue to study the traces on \( A_{\theta, \gamma} \). Here, again, \( \gamma \in C(\mathbb{T}) \) is a positive function and \( Y \) is the set of zeros of \( \gamma \). The proof of Lemma 2.4 indeed implies the following result.

Proposition 3.1. If \( \mu \) is a complex regular Borel measure on \( \mathbb{T} \) which satisfies that

\[
\int_{\mathbb{T}} f(e^{-2\pi i \theta}z) d\mu(z) = \int_{\mathbb{T}} f(z) d\mu(z) \quad \text{for all } f(z) \text{ in } C_0(\mathbb{T} \setminus Y), \text{ the unitization of } C_0(\mathbb{T} \setminus Y), \text{ and let } \sigma(f) = \int_{\mathbb{T}} f(z) d\mu(z) \text{ for } f(z) \in C(\mathbb{T}),
\]

then \( \sigma \cdot \Phi \) is a bounded tracial linear functional on \( A_{\theta, \gamma} \). Conversely, every bounded tracial linear functional on \( A_{\theta, \gamma} \) is given in this way.

Proof. If \( \mu \) satisfies (3.1) for all \( f(z) \) in \( C_0(\mathbb{T} \setminus Y) \), then by a similar argument of the proof of Lemma 2.4, \( \sigma \cdot \Phi \) is a bounded tracial linear functional on \( A_{\theta, \gamma} \). Conversely, suppose \( \sigma \) is a bounded tracial linear functional on \( A_{\theta, \gamma} \). By Proposition 2.1, \( \sigma(a) = \sigma(\Phi(a)) \). By the Riesz representation theorem, \( \sigma \) induces a complex regular Borel measure \( \mu \) on \( \mathbb{T} \).

Since for all \( f(z) \in C(\mathbb{T}) \),

\[
\sigma(xf(w)\overline{f(w)x^*}) = \sigma(|f|^2(e^{-2\pi i \theta}w)\gamma(e^{-2\pi i \theta}w)) = \int_{\mathbb{T}} |f|^2(e^{-2\pi i \theta}z)\gamma(e^{-2\pi i \theta}z) d\mu(z) = \int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(e^{2\pi i \theta}z)
\]

and

\[
\sigma(\overline{f(w)x^*xf(w)}) = \sigma(|f|^2(w)\gamma(w)) = \int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(z),
\]

we have

\[
\int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(e^{2\pi i \theta}z) = \int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(z), \quad \forall f(z) \in C(\mathbb{T}).
\]

Since every continuous function is a linear combinations of positive functions,

\[
\int_{\mathbb{T}} f(z)\gamma(z) d\mu(e^{2\pi i \theta}z) = \int_{\mathbb{T}} f(z)\gamma(z) d\mu(z), \quad \forall f(z) \in C(\mathbb{T}).
\]
This implies that \((3.1)\) is true for all \(f(z) \in \overline{\gamma(z)C(T)} = C_0(T \setminus Y)\). Since \((3.1)\) is true for \(f(z) \equiv 1\), \(\mu\) is a regular Borel measure on \(T\) which satisfies
\[
\int_T f(e^{-2\pi i \theta}z) d\mu(z) = \int_T f(z) d\mu(z)
\]
for all \(f(z)\) in \(\widetilde{C_0(T \setminus Y)}\).

\[\square\]

Recall that \(\phi : T \to T\) is the rotation of circle by \(\theta\), i.e., \(\phi(z) = e^{2\pi i \theta}z\) for \(z \in T\).

**Theorem 3.2.** Let \(Y\) be the set of zero points of \(\gamma(z)\). Then the following conditions are equivalent:

1. There exists a unique trace on \(A_{\theta, \gamma}\);
2. \(\phi^n(Y) \cap Y = \emptyset\) for all integers \(n \neq 0\);
3. For each \(\xi \in T\), \(\text{Orb}(\xi) \cap Y\) contains at most one point.

**Proof.** The equivalence of 2 and 3 follows from Proposition \(2.7\).

“1 \(\Rightarrow\) 2.” Suppose that \(\phi^k(Y) \cap Y \neq \emptyset\) for some integers \(k \neq 0\). Assume that \(z_1 \in Y\) and \(z_2 = \phi(z_1) = e^{2\pi i k \theta}z_1 \in Y\). By symmetry, we may assume that \(k > 0\). Let
\[
\mu = \frac{\delta_{e^{2\pi i \theta}z_1} + \delta_{e^{2\pi i \theta}z_1} + \cdots + \delta_{z_2}}{k},
\]
where \(\delta_t\) is the point-mass at \(t\). Then \(C_0(T \setminus Y) \subseteq \{f \in C(T) : f(z_1) = f(z_2)\}\). Note that for \(f(z) \in C(T)\) with \(f(z_1) = f(z_2)\) we have
\[
\int_T f(e^{-2\pi i \theta}z) d\mu(z) = \frac{f(z_1) + f(e^{2\pi i \theta}z_1) + \cdots + f(e^{2\pi i (k-1)\theta}z_1)}{k} = \frac{f(e^{2\pi i \theta}z_1) + \cdots + f(e^{2\pi i (k-1)\theta}) + f(z_2)}{k} = \int_T f(z) d\mu(z).
\]

By Proposition \(3.1\) \(\mu\) induces a trace different from the trace given in Lemma \(2.4\).

“2 \(\Rightarrow\) 1” Let \(C = \widetilde{C_0(T \setminus Y)}\) be the unitization of \(C_0(T \setminus Y)\), and let \(\rho\) be a tracial state on \(A_{\theta, \gamma}\). It follows from \(3.1\) that \(\rho = \mu \circ \Phi\), where \(\mu\) is a Borel probability measure on \(T\) such that
\[
\int_T f(\phi^{-1}(z)) d\mu(z) = \int_T f(z) d\mu(z) \quad (3.2)
\]
for all \(f \in C\). Define \(X_0 = Y\) and \(X_n = \phi^n(Y), n = \pm 1, \pm 2, \ldots\). By the assumption, \(\{X_n : n \in \mathbb{Z}\}\) are mutually disjoint closed subsets of \(T\). We claim that
\[
\mu(X_n) = 0, \quad n \in \mathbb{Z}.
\]
Let \( k \geq 1 \) be an integer. One can find an open subset \( G \subset \mathbb{T} \) such that
\[
X_0 \subset G \quad \text{and} \quad \phi^j(G) \cap \phi^i(G) = \emptyset
\]  
(3.4)
if \( i \neq j \) and \(-k \leq i, j \leq k\). Define \( 0 \leq g \leq 1 \) in \( C(\mathbb{T}) \) such that \( g(z) = 0 \) if \( z \in X_0 \) and \( g(z) = 1 \) if \( z \in \mathbb{T} \setminus G \). Then \( h = 1 - g \). Then \( h(z) = 1 \) if \( z \in X_0 \) and \( h(z) = 0 \) if \( z \in \mathbb{T} \setminus G \). Moreover, \( h \in C \). Let \( h_j = h \circ \phi^{-j} \), \(-k \leq j \leq k\). Note that \( h_j(z) = 1 \) if \( z \in \phi^j(X_0) \) and \( h_j(z) = 0 \) if \( z \in \mathbb{T} \setminus \phi^j(G) \) for \(-k \leq j \leq k\). In particular, if \(-k \leq j \leq k \) and \( j \neq 0 \), then \( h_j(z) = 0 \) for \( z \in X_0 \). Therefore \( h_j \in C_0(\mathbb{T} \setminus Y) \subset C \) for \(-k \leq j \leq k \) and \( j \neq 0 \). It follows from (3.2) that
\[
\int_{\mathbb{T}} h_j d\mu = \int_{\mathbb{T}} hd\mu, \quad -k \leq j \leq k.
\]  
(3.5)
Since \( h_j \) has disjoint support, (3.5) implies that
\[
0 \leq \int_{\mathbb{T}} h_j d\mu = \int_{\mathbb{T}} hd\mu < \frac{1}{2k + 1}.
\]  
(3.6)
Therefore,
\[
\mu(X_j) < \frac{1}{2k + 1}, \quad -k \leq j \leq k.
\]  
(3.7)
Since (3.7) holds for any integer \( k \geq 1 \), we conclude that the claim (3.3) holds.

Let \( f \in C(\mathbb{T}) \) and let \( \epsilon > 0 \). Since \( \mu(X_0) = \mu(X_{-1}) = 0 \), we can choose an open subset \( O \subset \mathbb{T} \) such that
\[
Y \subset O, \mu(O) < \epsilon/(2\|f\| + 1) \quad \text{and} \quad \mu(\phi^{-1}(O)) < \epsilon/(2\|f\| + 1).
\]  
(3.8)
Define a continuous function \( g_1 \in C \) such that \( 0 \leq g_1 \leq 1 \),
\[
g_1(z) = 0, \quad \text{when} \quad z \in Y \quad \text{and} \quad g_1(z) = 1, \quad \text{when} \quad z \in \mathbb{T} \setminus O.
\]  
(3.9)
Note that \( fg_1 \in C \). In particular,
\[
\int_{\mathbb{T}} fg_1 \circ \phi^{-1} d\mu = \int_{\mathbb{T}} fg_1 d\mu.
\]  
(3.10)
Then
\[
\left| \int_{\mathbb{T}} f(e^{-2i\pi \theta}z)d\mu(z) - \int_{\mathbb{T}} f(z)d\mu(z) \right| \leq \left| \int_{\mathbb{T}} (f - g_1 f) \circ \phi^{-1} d\mu \right| + \left| \int_{\mathbb{T}} (fg_1 - f g_1 \circ \phi^{-1})d\mu \right| \quad \text{(3.11)}
\]
\[
+ \left| \int_{\mathbb{T}} (f - g_1 f)d\mu \right| \quad \text{(3.12)}
\]
\[
\leq \left| \int_{\mathbb{T}} (f - g_1 f) \circ \phi^{-1} d\mu \right| + \left| \int_{\mathbb{T}} (fg_1 - f g_1 \circ \phi^{-1})d\mu \right| + \left| \int_{\mathbb{T}} (f - g_1 f)d\mu \right| \quad \text{(3.13)}
\]
Remark 3.3. If \( \gamma(z) \) has a single zero point, then there exists a unique tracial state on \( A_{\theta,\gamma} \).

Remark 3.4. If \( \gamma(z) \) has two zero points \( z_1, z_2 \), then there exists a unique tracial state on \( A_{\theta,\gamma} \) if and only if there does not exist \( k \in \mathbb{N} \) such that \( z_2 = e^{2\pi ik \theta} z_1 \).

For a \( C^* \)-algebra \( \mathfrak{A} \), we denote by \( \text{Tr}(\mathfrak{A}) \) the space of bounded tracial linear functionals on \( \mathfrak{A} \). Denote by \( T(\mathfrak{A}) \) the tracial state space of \( \mathfrak{A} \).

Let \( \Delta \) be a subset of \( \mathbb{T} \) which contains exactly one point of each orbit \( \text{Orb}(\xi) \) and let \( Y \) be the set of zeros of \( \gamma(z) \).

Lemma 3.5. Let \( \xi_1, \xi_2, \ldots, \xi_r \in \Delta \) and \( Y_j = Y \cap \text{Orb}(\xi_j) \), \( j = 1, 2, \ldots, r \). Let \( Y'_j \subset Y_j \) be a finite subset of \( Y_j \) and let \( |Y'_j| \) be the cardinality of \( Y'_j \). Then \( \dim(\text{Tr}(A_{\theta,\gamma})) \geq 1 + \sum_{j=1}^{r} (|Y'_j| - 1) \).

Proof. Suppose that \( Y'_j = \left\{ z_{j,1}, (e^{2\pi i m_{j,1} \theta})z_{j,1}, \ldots, (e^{2\pi i m_{j,n_j} \theta})z_{j,1} \right\} \) with \( 1 < m_{j,1} < \cdots < m_{j,n_j} \), where \( |Y'_j| = n_j + 1 \), \( j = 1, 2, \ldots, r \). As in the proof of Proposition 3.2, the Haar measure \( m \) together with

\[
\begin{align*}
\mu_{j,1} &= \frac{\delta(e^{2\pi i \theta})z_{j,1} + \cdots + \delta(e^{2\pi i m_{j,1} \theta})z_{j,1}}{m_{j,1} - 1} \\
\mu_{j,2} &= \frac{\delta(e^{2\pi i (m_{j,1}+1) \theta})z_{j,1} + \cdots + \delta(e^{2\pi i m_{j,2} \theta})z_{j,1}}{m_{j,2} - m_{j,1}} \\
&\quad \vdots \\
\mu_{j,n_j} &= \frac{\delta(e^{2\pi i (m_{j,n_j-1}+1) \theta})z_{j,1} + \cdots + \delta(e^{2\pi i m_{j,n_j} \theta})z_{j,1}}{m_{n_j} - m_{n_j-1}}
\end{align*}
\]

induce \( 1 + \sum_{j=1}^{r} (|Y'_j| - 1) \) linearly independent tracial states on \( A_{\theta,\gamma} \). This proves that \( \dim(\text{Tr}(A_{\theta,\gamma})) \geq 1 + \sum_{j=1}^{r} (|Y'_j| - 1) \). \( \square \)

Corollary 3.6. Let \( \xi \in \mathbb{T} \) and let \( N(\xi) \) be the number of points in \( Y \cap \text{Orb}(\xi) \). If \( \sum_{\xi \in \Delta} N(\xi) = \infty \), then \( A_{\theta,\gamma} \) has infinitely many extreme points in its tracial state space \( T(A_{\theta,\gamma}) \) and \( \dim(\text{Tr}(A_{\theta,\gamma})) = \infty \).
Proof. For any integer $N \geq 1$, since $\sum_{\xi \in \Delta} N(\xi) = \infty$, one can find $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{T}$ and finite subsets $Y_j \subset Y \cap \text{Orb}(\xi_j)$, $j = 1, 2, \ldots, n$ such that

$$\sum_{j=1}^{n} (|Y_j| - 1) > N.$$  

It follows from Lemma 3.5 that $\dim(\text{Tr}(A_{\theta, \gamma})) \geq N$. It follows that $\dim(\text{Tr}(A_{\theta, \gamma})) = \infty$. The corollary follows.  

Proposition 3.7. Let $\xi_1, \xi_2, \ldots, \xi_r \in \Delta$ and $Y_j = Y \cap \text{Orb}(\xi_j)$, $j = 1, 2, \ldots, r$, such that $Y = \bigcup_{j=1}^{r} Y_j$. Suppose that $Y$ is a finite set. Then $\dim(\text{Tr}(A_{\theta, \gamma})) = 1 + \sum_{j=1}^{r} (|Y_j| - 1)$, where $|Y_j|$ is the number of elements in $Y_j$.

Proof. By Lemma 3.5, $\dim(\text{Tr}(A_{\theta, \gamma})) \geq 1 + \sum_{j=1}^{r} (|Y_j| - 1)$. We need to show $\dim(\text{Tr}(A_{\theta, \gamma})) \leq 1 + \sum_{j=1}^{r} (|Y_j| - 1)$. By Proposition 3.1, a regular Borel probability measure $\mu$ on $\mathbb{T}$ induces a trace on $A_{\theta, \gamma}$ if and only if

$$\int f(z) d\mu(e^{2\pi i \theta}z) = \int f(z) d\mu(z)$$  

for all $f(z) \in C_0(\mathbb{T} \setminus Y)$. Suppose that the zero points of $\gamma(z)$ are $z_1, \ldots, z_n$. Then the norm closure of $\gamma(z)C(\mathbb{T})$ in $C(\mathbb{T})$ is

$$J = \{ f(z) \in C(\mathbb{T}) : f(z_1) = \cdots = f(z_n) = 0 \}$$  

and so

$$C = C_0(\mathbb{T} \setminus Y) = \{ f(z) \in C(\mathbb{T}) : f(z_1) = \cdots = f(z_n) \} \subseteq C(\mathbb{T}).$$  

Therefore, $\mu$ induces a trace on $A_{\theta, \gamma}$ if and only if

$$\int f(z) d\mu(e^{2\pi i \theta}z) = \int f(z) d\mu(z)$$  

for all $f(z) \in C$.

Let $C^\perp = \{ \rho : \rho \in C(\mathbb{T})^* \text{ and } \rho(a) = 0 \text{ for all } a \in C \}$. Note that $C(\mathbb{T})/C \cong \mathbb{C}^{n-1}$. So $\dim C^\perp = n - 1$. Suppose that $Y_j = \{ z_{j,1}, (e^{2\pi i m_{j,1}} \theta)z_{j,1}, \ldots, (e^{2\pi i m_{j,n_j}} \theta)z_{j,1} \}$ with $1 < m_{j,1} < \cdots < m_{j,n_j}$. Define $\mu_j'$s as in the proof Lemma 3.5 and let $\nu_j = \delta_{z_{j,1}} - \delta_{z_{j+1,1}}$ for $1 \leq j \leq r - 1$. Then

$$\{ \mu_{j,k} - \mu_{j,k}(e^{2\pi i \theta} \cdot) : 1 \leq j \leq r, 1 \leq k \leq n_j \} \cup \{ \nu_j : 1 \leq j \leq r - 1 \}$$  

are $n - 1$ linearly independent elements in $C^\perp$. Therefore, there are real numbers $s_{j,k}$ and $t_j$ such that

$$\mu(e^{2\pi i \theta} E) - \mu(E) - \sum_{j=1}^{r-1} s_{j,k}(\mu_{j,k}(e^{2\pi i \theta} E) - \mu_{j,k}(E)) = \sum_{j=1}^{r-1} t_j \nu_j(E)$$  

and so $\mu$ induces a trace on $A_{\theta, \gamma}$. The corollary follows.  

for all Borel measurable subset $E$ of $\mathbb{T}$. Let $\bar{\mu}(E) = \mu(E) - \sum s_{j,k} \mu_{j,k}(E)$. Then $\bar{\mu}(\{z_{1,1}\}) = \mu(\{z_{1,1}\}) \geq 0$ and
\[
\bar{\mu}(e^{2\pi i \theta} E) - \bar{\mu}(E) = \sum_{j=1}^{r-1} t_j (\delta_{z_{j,1}}(E) - \delta_{z_{j+1,1}}(E)).
\] (3.17)

Claim $t_1 = \cdots = t_{r-1} = 0$. Otherwise, we may assume that $t_1 > 0$. In (3.17) let $E = \{z_{1,1}\}$, then we have $\delta_{z_{j,1}}(E) = 0$ for all $2 \leq j \leq r$. Hence, $\bar{\mu}(\{e^{2\pi i \theta} z_{1,1}\}) \geq t_1 + \bar{\mu}(\{z_{1,1}\}) \geq t_1$. In (3.17) let $E = \{e^{2\pi i \theta} z_{1,1}\}$, then we have $\delta_{z_{j,1}}(E) = 0$ for all $1 \leq j \leq r$. Hence, $\bar{\mu}(\{e^{2\pi i \theta} z_{1,1}\}) \geq \bar{\mu}(\{e^{2\pi i \theta} z_{1,1}\}) \geq t_1 > 0$. By induction, we have $\bar{\mu}(\{e^{2\pi i \theta} z_{1,1}\}) \geq t_1 > 0$ for all $n \in \mathbb{N}$. This contradicts to the fact that $\bar{\mu}$ is a bounded real measure.

Therefore,
\[
\mu(e^{2\pi i \theta} E) - \mu(E) = \sum s_{j,k} (\mu_{j,k}(e^{2\pi i \theta} E) - \mu_{j,k}(E))
\]
for all Borel measurable subset $E$ of $\mathbb{T}$, i.e.,
\[
\mu(e^{2\pi i \theta} E) - \sum s_{j,k} (\mu_{j,k}(e^{2\pi i \theta} E)) = \mu(E) - \sum s_{j,k} (\mu_{j,k}(E))
\]
for all Borel subset $E$ of $\mathbb{T}$. Let
\[
\nu = \mu - \sum s_{j,k} \mu_{j,k}.
\]
Then
\[
\nu(e^{2\pi i \theta} E) = \nu(E)
\]
for all Borel subsets of $\mathbb{T}$. Therefore, for every $n \in \mathbb{N}$, $\nu(e^{2\pi i n \theta} E) = \nu(E)$ for all Borel subsets $E$ of $\mathbb{T}$. Since $\theta$ is an irrational number, $\{e^{2\pi i n \theta} : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$. By the Lebesgue dominated theorem, $\nu(z E) = \nu(E)$ for all Borel subsets $E$ of $\mathbb{T}$ and $z \in \mathbb{T}$. By the uniqueness of the Haar measure on $\mathbb{T}$, there exists $t \in \mathbb{R}$ such that $\nu = tm$, i.e., $\mu = \sum s_{j,k} \mu_{j,k} + tm$. This implies that $\dim(\text{Tr}(A_{\theta,\gamma})) \leq 1 + \sum_{j=1}^{r} |Y_j| - 1$. So $\dim(\text{Tr}(A_{\theta,\gamma})) = 1 + \sum_{j=1}^{r} |Y_j| - 1$.

**Proposition 3.8.** Suppose $\gamma(z)$ has finitely many points in its zero set $Y$ and there are $\xi_1, \xi_2, \ldots, \xi_r \in \triangle$ such that $Y = \cup_{j=1}^{r} Y_j$, where $Y_j = Y \cap \text{Orb}(\xi_j)$. Then $\tau$ and the traces induced by $\mu_{j,k}'s$ constructed in Lemma 3.7 are precisely the extreme points of $T(A_{\theta,\gamma})$.

**Proof.** Let $\sigma$ be a tracial state on $A_{\theta,\gamma}$ induced by a regular Borel probability measure $\mu$ on $\mathbb{T}$. Then by the proof of Proposition 3.7, there are real numbers $t, s_{j,k}$ such that
\[
\mu(E) = tm(E) + \sum s_{j,k} \mu_{j,k}(E)
\]
for all Borel subsets $E$ of $\mathbb{T}$. Since $m$ and $\mu_{j,k}'s$ are mutually disjoint measures, $t, s_{j,k} \geq 0$ and $t + \sum s_{j,k} = 1$. This shows that $\tau$ and the traces induced by $\mu_{j,k}'s$ constructed in Lemma 3.7 are precisely the extreme points of $T(A_{\theta,\gamma})$.

**Corollary 3.9.** Suppose $\gamma(z)$ has finite zero points. Then $\tau$ is the unique extreme point in $T(A_{\theta,\gamma})$ which is faithful on $A_{\theta,\gamma}$. 

\[\square\]
4 Simplicity of generalized universal $C^*$-algebras

In this section, we provide a characterization of simplicity of a generalized universal algebra $A_{\theta,\gamma}$ in terms of the zero points of $\gamma(z)$. We begin with the following lemma.

Lemma 4.1. Let $f_n(z) \in C(\mathbb{T})$ for $-M \leq n \leq N$. Then

$$\|x^k f_k(w)\| \leq \left\| \sum_{n=1}^{N} x^n f_n(w) + f_0(w) + \sum_{m=1}^{M} f_m(w) (x^*)^m \right\|,$$

and

$$\|f_k(w)(x^*)^k\| \leq \left\| \sum_{n=1}^{N} x^n f_n(w) + f_0(w) + \sum_{m=1}^{M} f_m(w) (x^*)^m \right\|.$$

Proof. There is a function $\gamma_k \in C(\mathbb{T})_+$ such that $x^k = u^k \gamma_k(w)$, $k = 1, 2, \ldots$, Therefore $u^{-k} x^k f_k(w) = \gamma_k(w) f_k(w)$.

Put $a = \sum_{i=0}^{N} x^i f_i(w) + \sum_{j=1}^{M} f_{-j}(w)(x^*)^k$. Let $\Phi$ be the conditional expectation. Then

$$\|x^k f_k(w)\| = |u^{-k} x^k f_k(w)| = |\Phi(u^{-k} a)| \leq |u^{-k} a| = \|a\|.$$

So the first part of the lemma follows. The second part follows similarly. □

Lemma 4.2. Let $Y_1$ be the set of zero points of functions $\gamma(e^{2\pi i \theta} z)$ for $n \geq 0$, and let $Y_2$ be the set of zero points of functions $\gamma(e^{-2\pi i \theta} z)$ for $n \geq 1$. Then $A_{\theta,\gamma}$ is a simple algebra if and only if $Y_1 \cap Y_2 = \emptyset$.

Proof. Suppose $Y_1 \cap Y_2 = \emptyset$ and $J$ is a non-zero ideal of $A_{\theta,\gamma}$. Then there is a positive nonzero element $x$ in $J$. Since $w^j x (w^*)^j \in J$, the limit formula for $\Phi(x)$ in Proposition 2.1 shows that $\Phi(x) \in J \cap C^*(w)$. Since $\Phi$ is faithful, $\Phi(x) > 0$. So $J \cap C^*(w)$ is a nontrivial ideal in $C^*(w)$, which is contained in a maximal nontrivial ideal

$$I = \{ f(w) | f(z) \in C(\mathbb{T}) \text{ and } f(z_0) = 0 \text{ for some } z_0 \in \mathbb{T} \}$$

of $C^*(w)$.

Let $f(z) \in C(\mathbb{T})$ such that $f(w) \in J \cap C^*(w) \subset I$. Then $f(z_0) = 0$. By (2.7) and (2.8), we have

$$x^* f(w) x = f(e^{2\pi i \theta} w) \gamma(w) \in J \cap C^*(w) \subset I. \quad (4.1)$$

By (2.6) and (2.9), we have

$$x f(w) x^* = f(e^{-2\pi i \theta} w) \gamma(-e^{2\pi i \theta} w) \in J \cap C^*(w) \subset I. \quad (4.2)$$

Case 1. Suppose $z_0 \in Y_2$. Then the assumption of the theorem implies that $z_0 \notin Y_1$. So (4.1) implies that $f(e^{2\pi i \theta} z_0) = 0$. Repeat using (4.1), we have for all $n \in \mathbb{N}$,

$$(x^*)^n f(w) x^n = f(e^{2\pi i \theta} w) \gamma(e^{2\pi i (n-1) \theta} w) \gamma(e^{2\pi i (n-2) \theta} w) \cdots \gamma(w) \in J \cap C^*(w) \subset I.$$
Thus \( f(e^{2\pi in\theta}z_0) = 0 \) for all \( n \in \mathbb{N} \). Since \( \{e^{2\pi in\theta} : n \in \mathbb{N}\} \) is dense in \( \mathbb{T} \), \( f(z) = 0 \) for all \( z \in \mathbb{T} \). This implies that \( J \cap C^*(w) \) is trivial and we obtain a contradiction.

Case 2. Suppose \( z_0 \notin Y_2 \). Then (1.2) implies that \( f(e^{-2\pi i\theta}z_0) = 0 \). Repeat using (1.2), we have for all \( n \in \mathbb{N} \),

\[
x^n f(w)(x^*)^n = f(e^{-2\pi in\theta}w)\gamma(e^{-2\pi in\theta}w)\gamma(e^{-2\pi i(n-1)\theta}w)\cdots\gamma(e^{-2\pi i\theta}w) \in J \cap C^*(w) \subset I.
\]

Thus \( f(e^{-2\pi in\theta}z_0) = 0 \) for all \( n \in \mathbb{N} \). Since \( \{e^{-2\pi in\theta}z_0 : n \in \mathbb{N}\} \) is dense in \( \mathbb{T} \), \( f(z) = 0 \) for all \( z \in \mathbb{T} \). This implies that \( J \cap C^*(w) \) is trivial and we obtain a contradiction.

Conversely, suppose \( Y_1 \cap Y_2 \neq \emptyset \). We may assume that \( \lambda \in \mathbb{T} \) is a zero point of \( \gamma(e^{2\pi in\theta}z) \) and \( \gamma(e^{-2\pi im\theta}z) \). Consider the subset

\[
J = \{f(w)|f(z) \in C(\mathbb{T}) \text{ and } f(e^{2\pi in\theta}w) = \cdots = f(\lambda) = \cdots = f(e^{-2\pi im\theta}w) = 0\}
\]

of \( C^*(w) \). Claim that \( I = A_{\theta,\gamma}J A_{\theta,\gamma} \) is a two-sided ideal of \( A_{\theta,\gamma} \). Otherwise, there exists \( f_i(w) \in J \),

\[
a_i = \sum_{n=1}^{K}(x^*)^ng_{-n}^i(w) + g_i^1(w) + \sum_{n=1}^{K}g_n^i(w)x^n,
\]

and

\[
b_i = \sum_{n=1}^{K}(x^*)^nh_{-n}^i(w) + h_i^1(w) + \sum_{n=1}^{K}h_n^i(w)x^n,
\]

for sufficiently large \( K \in \mathbb{N} \) such that

\[
\left\| \sum_{i=1}^{N}a_if_i(w)b_i - 1 \right\| < 1,
\]

where \( g_n^i, g^i, h_n^i, h^i \in C(\mathbb{T}) \). By Lemma 1.1 and simple computations, we have

\[
\left\| \sum_{i=1}^{N}g_{-K}^i(e^{2\pi iK\theta}w)f_i(e^{2\pi iK\theta}w)h_i^1(e^{2\pi iK\theta}w)\gamma(e^{2\pi i(K-1)\theta}w)\cdots\gamma(w) + \right.
\]

\[
g_{-(K-1)}^i(e^{2\pi i(K-1)\theta}w)f_i(e^{2\pi i(K-1)\theta}w)h_i^1(e^{2\pi i(K-1)\theta}w)\gamma(e^{2\pi i(K-2)\theta}w)\cdots\gamma(w) + \cdots +
\]

\[
g_1^i(e^{2\pi i\theta}w)f_i(e^{2\pi i\theta}w)h_1^i(e^{2\pi i\theta}w)\gamma(w) + g_i^1(w)f_i(w)h_i^1(w) + g_i^1(w)f_i(e^{-2\pi i\theta}w)h_i^{-1}(w)\gamma(e^{-2\pi i\theta}w) + \cdots +
\]

\[
g_{K-1}^i(w)f_i(e^{-2\pi i(K-1)\theta}w)h_{-(K-1)}^i(w)\gamma(e^{-2\pi i(K-1)\theta}w)\cdots\gamma(e^{-2\pi i\theta}w) +
\]

\[
g_K^i(w)f_i(e^{-2\pi iK\theta}w)h_{-K}^i(w)\gamma(e^{-2\pi iK\theta}w)\cdots\gamma(e^{-2\pi i\theta}w) - 1\right\| < 1.
\]

Let

\[
\bar{f}(z) = \sum_{i=1}^{N}g_{-K}^i(e^{2\pi iK\theta}z)f_i(e^{2\pi iK\theta}z)h_i^1(e^{2\pi iK\theta}z)\gamma(e^{2\pi i(K-1)\theta}z)\cdots\gamma(z) + \cdots +
\]
Suppose Corollary 4.5. If \( \gamma(z) \) has a unique tracial state; \( \theta, \gamma \) is simple, then by Theorem 4.3, \( \gamma(e^{2\pi i \theta z}) \) for \( n \geq 0 \), and let \( Y_2 \) be the set of zero points of functions \( \gamma(e^{-2\pi i m \theta z}) \) for \( n \geq 1 \). By Proposition \( 2.7 \) condition (3) is equivalent to \( Y_1 \cap Y_2 = \emptyset \). By Lemma \( 4.2 \) (1) is equivalent to (3).

**Corollary 4.4.** Suppose \( \gamma(z) \in C(\mathbb{T}) \) is a positive function with a single zero point. Then \( A_{\theta, \gamma} \) is a simple \( C^* \)-algebra with a unique tracial state.

**Corollary 4.5.** Suppose \( \gamma(z) \in C(\mathbb{T}) \) is a positive function with two zero points \( z_1, z_2 \). Then \( A_{\theta, \gamma} \) is a simple \( C^* \)-algebra with a unique tracial state if and only if there does not exist integer \( k \) such that \( z_2 = e^{2\pi ik \theta} z_1 \).

**Corollary 4.6.** If \( m(\{ z | \gamma(z) = 0 \} ) > 0 \), then \( A_{\theta, \gamma} \) is not simple.

**Proof.** Let \( Y = \{ z | \gamma(z) = 0 \} \). If \( A_{\theta, \gamma} \) is simple, then by Theorem 4.3 \( \phi^n(Y) \cap Y = \emptyset \) for all integers \( n \neq 0 \). Then \( \{ \phi^n(Y) : n \in \mathbb{Z} \} \) is a sequence of mutually disjoint subsets. Therefore \( m(Y) = 0 \).
5 Rieffel’s projections in generalized universal algebras

Lemma 5.1. If \( \lambda \in \mathbb{T} \), then \( A_{\theta, \gamma(z)} \cong A_{\theta, \gamma(\lambda z)} \).

Proof. Let \( A_{\theta, \gamma(z)} = C^*(x, w) \) and \( A_{\theta, \gamma(\lambda z)} = C^*(x', w') \). Then \( x', \lambda w' \) satisfy (2.1)-(2.4) for \( \gamma(z) \). So there is a homomorphism \( \varphi : A_{\theta, \gamma(z)} \to A_{\theta, \gamma(\lambda z)} \) such that \( \varphi(x) = x', \varphi(w) = \lambda w' \). By symmetry, there is a homomorphism \( \psi : A_{\theta, \gamma(\lambda z)} \to A_{\theta, \gamma(z)} \) such that \( \psi(x') = x, \psi(w') = \lambda w \). Hence \( \psi \cdot \varphi(x) = x \) and \( \psi \cdot \varphi(w) = w; \varphi \cdot \psi(x') = x' \) and \( \varphi \cdot \psi(w') = w' \). So \( \varphi \) is an isomorphism from \( A_{\theta, \gamma(z)} \) onto \( A_{\theta, \gamma(\lambda z)} \).

Lemma 5.2. \( A_{\theta, \gamma} \cong A_{1-\theta, \gamma} \).

Proof. Let \( A_{\theta, \gamma} = C^*(x, w) \) and \( A_{1-\theta, \gamma} = C^*(x', w') \). Then \( x', (w')^* \) satisfy (2.1)-(2.4) for \( \theta \) and \( \gamma \). So there is a homomorphism \( \varphi : A_{\theta, \gamma} \to A_{1-\theta, \gamma} \) such that \( \varphi(x) = x', \varphi(w) = (w')^* \). By symmetry, there is a homomorphism \( \psi : A_{1-\theta, \gamma} \to A_{\theta, \gamma} \) such that \( \varphi(x') = x, \varphi(w') = w^* \). Hence \( \psi \cdot \varphi(x) = x \) and \( \psi \cdot \varphi(w) = w; \varphi \cdot \psi(x') = x' \) and \( \varphi \cdot \psi(w') = w' \). So \( \varphi \) is an isomorphism from \( A_{\theta, \gamma} \) onto \( A_{1-\theta, \gamma} \).

The proof the following theorem is similar to the proof of Theorem 1.1 of [31]. However, some details should be treated carefully.

Theorem 5.3. Suppose \( \gamma \) is a positive function in \( C(\mathbb{T}) \) and there exists \( \lambda \in \mathbb{T} \) such that \( \gamma(\lambda e^{2\pi i n}) \neq 0 \) for all nonnegative integers \( n \). Then for every \( \alpha \in (\mathbb{Z} + \mathbb{Z} \theta) \cap [0, 1] \), there is a projection \( p \) in \( A_{\theta, \gamma} \) such that \( \tau(p) = \alpha \).

Proof. By Lemma 5.1 we may assume that \( \lambda = 1 \). Firstly we prove if \( \alpha = \theta \in (0, 1) \) then there exists a projection \( p \) in \( A_{\theta, \gamma} \) such that \( \tau(p) = \theta \). By Lemma 5.2 we may assume that \( 0 < \theta < 1/2 \).

A dense set of elements of \( A_{\theta, \gamma} \) can be represented by a finite sum of the form \( \sum_{i=1}^n f_i(w)x^i + f(w) + \sum_{j=1}^m f_{-j}(w)(x^*)^j \), where \( f_k(z), f(z) \in C(\mathbb{T}) \). Note that the set \( C(\mathbb{T})x^i, C(\mathbb{T}), C(\mathbb{T})(x^*)^j \) are mutually orthogonal to each other in \( L^2(A_{\theta, \gamma}, \tau) \). In the following we identify \( C^*(w) \) with \( C(\mathbb{R}/\mathbb{Z}) \). For \( f(t) \in C(\mathbb{R}/\mathbb{Z}) \), define \( f_\theta(t) = f(t - \theta) \). Let \( \beta(t) = (\gamma(e^{2\pi i t}))^{1/2} \). Then \( \beta(n\theta) \neq 0 \) for all nonnegative integers \( n \).

We look for a projection \( p = g(t)x + f(t) + h(t)x^* \) such that \( \tau(p) = \theta \). Since \( p = p^* \), by (2.4) and (2.5),

\[
g(t)x + f(t) + h(t)x^* = x^*g(t) + \tilde{f}(t) + x\tilde{h}(t) = \tilde{g}_{-\theta}(t)x^* + \tilde{f}(t) + \tilde{h}_{\theta}(t)x.
\]

By comparing coefficients, we see that \( f = \tilde{f} \) is a real valued function; and that \( h(t) = \frac{g(t + \theta)}{g(t)} \) or equivalently \( h(t - \theta) = g(t) \). Since \( p = p^2 \), (2.6)-(2.9) imply

\[
g(t)x + f(t) + h(t)x^* = g(t)g_\theta(t)x^2 + (g(t)(f(t) + f_\theta(t)))x + [g(t)h_\theta(t)\beta^2(t - \theta) + f^2(t) + h(t)g_\theta(t)\beta^2(t)]
\]

\[+h(t)(f(t) + f_\theta(t))x^* + h(t)h_\theta(t)(x^*)^2.
\]
By comparing coefficients and replacing \( h \)'s with \( g \)'s using the relation between them, we arrive at the necessary and sufficient conditions:

\[
g(t)g(t - \theta) = 0, \tag{5.1}
\]

\[
g(t)(1 - f(t) - f(t - \theta)) = 0, \tag{5.2}
\]

\[
f(t) - f(t)^2 = |g(t)\beta(t - \theta)|^2 + |g(t + \theta)\beta(t)|^2. \tag{5.3}
\]

Pick any positive \( \epsilon > 0 \) such that \( \theta + \epsilon < 1/2 \). Define \( f \) to be the piece-wise linear function

\[
f(t) = \begin{cases} 
\epsilon^{-1}t & \text{for } 0 \leq t \leq \epsilon \\
1 & \text{for } \epsilon \leq t \leq \theta \\
\epsilon^{-1}(\theta + \epsilon - t) & \text{for } \theta \leq t \leq \theta + \epsilon \\
0 & \text{for } \theta + \epsilon \leq t \leq 1 
\end{cases}
\]

and define

\[
g(t) = \begin{cases} 
\sqrt{f(t) - f(t)^2/\beta(t - \theta)} & \text{for } \theta \leq t \leq \theta + \epsilon \\
0 & \text{otherwise}
\end{cases}
\]

Since \( \beta(0) \neq 0 \), \( g(t) \in C(\mathbb{T}) \) for sufficiently small \( \epsilon > 0 \). Then \( f(t) \) and \( g(t) \) satisfy equations (5.1), (5.2), and (5.3). So \( \tau(p) = \int_0^1 f(t)dt = \theta \). We also get the projection \( 1 - p \) with trace \( \tau(1 - p) = 1 - \theta \).

In the following we show that for \( k \geq 2 \) there is a projection \( q \) such that \( \tau(q) \) is the fractional part \( k\theta \) of \( k\theta \). Let \( \alpha = \{k\theta\} \). We may assume that \( \alpha < 1/2 \). The idea is similar. Let \( q = g_1(t)(u + v)^k + f_1(t) + h_1(t)((u + v)^*)^k \). Then we will have the following equations

\[
g_1(t)g_1(t - \alpha) = 0, \tag{5.4}
\]

\[
g_1(t)(1 - f_1(t) - f_1(t - \alpha)) = 0, \tag{5.5}
\]

\[
f_1(t) - f_1(t)^2 = |g_1(t)\beta(t - k\theta)\cdots\beta(t - \theta)|^2 + |g_1(t + \alpha)\beta(t + (k - 1)\theta)\cdots\beta(t)|^2
\]

\[
= |g_1(t)\beta(t - k\theta)\beta(t - (k - 1)\theta)\cdots\beta(t - \theta)|^2 \\
+ |g_1(t + \alpha)\beta((t + \alpha) - k\theta)\beta((t + \alpha) - (k - 1)\theta)\cdots\beta((t + \alpha) - \theta)|^2. \tag{5.6}
\]

Pick any positive \( \epsilon > 0 \) such that \( \theta + \epsilon < 1/2 \). Define \( f_1 \) to be the piece-wise linear function

\[
f_1(t) = \begin{cases} 
\epsilon^{-1}t & \text{for } 0 \leq t \leq \epsilon \\
1 & \text{for } \epsilon \leq t \leq \alpha \\
\epsilon^{-1}(\alpha + \epsilon - t) & \text{for } \alpha \leq t \leq \alpha + \epsilon \\
0 & \text{for } \alpha + \epsilon \leq t \leq 1
\end{cases}
\]
and define
\[ g_1(t) = \begin{cases} \sqrt{f_1(t)} - f_1(t)/\beta(t) \cdots \beta(t) & \text{for } \alpha \leq t \leq \alpha + \epsilon \\ 0 & \text{otherwise} \end{cases} \]

Since \( \beta(0) \neq 0, \beta(t) \neq 0, \cdots, \beta((k - 1) \theta) \neq 0, \) \( g_1(t) \in C(\mathbb{T}) \) for sufficiently small \( \epsilon > 0. \) Then \( f_1(t) \) and \( g_1(t) \) satisfy equations (5.4), (5.5), and (5.6). So \( \tau(q) = \int_0^1 f_1(t) \, dt = \alpha. \)

\[ \square \]

**Corollary 5.4.** If \( m(\{z|\gamma(z) = 0\}) = 0, \) e.g., the zero points of \( f(z) \) is countable, then for every \( \alpha \) in \( (\mathbb{Z} + \mathbb{Z} \theta) \cap [0, 1], \) there is a projection \( p \) in \( A_{\theta, \gamma} \) such that \( \tau(p) = \alpha. \)

**Proof.** We divide \( \mathbb{T} \) into equivalent classes \( F_\alpha, \) where \( x, y \in F_\alpha \) if and only if \( x = e^{2\pi ik\theta}y \) for some \( k \in \mathbb{Z}. \) Suppose \( \forall \alpha, F_\alpha \cap \{z|\gamma(z) = 0\} \neq \emptyset. \) By axiom of choice we can choose a representative set \( \{x_\alpha\}_{\alpha \in Y} \) of \( \{F_\alpha\}_{\alpha \in Y} \) such that \( x_\alpha \in F_\alpha \cap \{z|\gamma(z) = 0\} \) for each \( \alpha \in Y. \) Then \( m(\{x_\alpha\}_{\alpha \in Y}) = 0. \) On the other hand it is well-known that \( \{x_\alpha\}_{\alpha \in Y} \) is not Lebesgue measurable. This is a contradiction. Therefore, there exists \( \alpha \in Y \) such that the intersection of \( F_\alpha \) and the set of zero points of \( \gamma \) is empty. Now the corollary follows from Theorem 5.3.

Combining Corollary 4.6 and Corollary 5.4, we obtain the following result.

**Corollary 5.5.** If a generalized universal \( C^* \)-algebra \( A_{\theta, \gamma} \) is simple, then for every \( \alpha \) in \( (\mathbb{Z} + \mathbb{Z} \theta) \cap [0, 1], \) there is a projection \( p \) in \( A_{\theta, \gamma} \) such that \( \tau(p) = \alpha. \)

This corollary also follows from 7.2.

### 6 K-groups of generalized universal irrational rotation algebras

Let \( A_\theta \) be the universal irrational rotation \( C^* \)-algebra with two unitary generators \( u, v \) satisfying \( vu = e^{2\pi i \theta}uv. \) Then there exists an action \( \alpha_z \) of \( \mathbb{T} \) on \( A_\theta \) defined by \( \alpha_z(u) = zu \) and \( \alpha_z(v) = v. \) By Theorem 2.5 we may identify \( A_{\theta, \gamma} \) with the unital \( C^* \)-subalgebra \( B \) of \( A_\theta \) generated by \( u\gamma^{1/2}(v) \) and \( v. \) Then \( x = u\gamma^{1/2}(v) \) and \( w = v. \) Let \( A \) be the unital \( C^* \)-algebra generated by \( v. \) The following definition is introduced by Ruy Excel in 12.

**Definition 6.1.** For each \( n \in \mathbb{Z} \) the \( n^{th} \) spectral subspace for \( \alpha \) is defined by

\[ B_n = \{b \in A_{\theta, \gamma} : \alpha_z(b) = z^n b \text{ for } z \in \mathbb{T}\}. \]

**Lemma 6.2.** \( B_0 = A \) and \( B_1 = \{uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y\}. \)
Proof. By Corollary 2.2, \( B_0 = A \). We need to show \( B_1 = \{ uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y \} \). Note that 
\[ \alpha_z(xg(v)) = zxg(v) \]
Since the norm closure of \( \{ xg(v) : g \in C(\mathbb{T}) \} \) is \( \{ uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y \} \subseteq B_1 \). On the other hand, if \( b \in B_1 \), then \( \alpha_z(u^*b) = u^*b \) for all \( z \in \mathbb{T} \). This implies that \( b = uf(v) \) for some \( f(z) \in C(\mathbb{T}) \). Suppose \( f(\lambda_0) \neq 0 \) for some \( \lambda_0 \in Y \). Then for any \( y = \sum_{n=1}^{N} x^n f_n(z) + f_0(z) + \sum_{n=1}^{N} f_{-n}(z)(x^*)^n \), by Lemma 6.1 we have
\[
\|y - z\| = \|xf_1(v) - uf(v)\| = \|uh(v)f_1(v) - uf(v)\| = \|h(v)f_1(v) - f(v)\| \geq |f(\lambda_0)| > 0.
\]
Thus for any \( y \in A_{\theta, \gamma} \) we have \( \|y - uf(v)\| \geq |f(\lambda_0)| > 0 \). This is a contradiction. So \( B_1 = \{ uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y \} \).

\[
\text{Definition 6.3. If } X \text{ and } Y \text{ are subsets of a } C^*\text{-algebra, then } XY \text{ denotes the closed linear span of the set of products } xy \text{ with } x \in X \text{ and } y \in Y.
\]

\[
\text{Corollary 6.4. } B_1^*B_1 = \{ f(v) : f(\lambda) = 0 \text{ for } \lambda \in Y \} \subseteq A \text{ and } B_1B_1^* = uB_1B_1^*u^* \subseteq A.
\]

\[
\text{Lemma 6.5. The action of } \mathbb{T} \text{ on } A_{\theta, \gamma} \text{ is regular in the sense of [12] (see Definition 4.4), i.e., there exist an isomorphism } \theta : B_1^*B_1 \rightarrow B_1B_1^* \text{ and a linear isometry } \phi \text{ from } B_1^* \text{ onto } B_1B_1^* \text{ such that for any } y_1, y_2 \in B_1, a \in B_1^*B_1 \text{ and } b \in B_1B_1^*,
\]

1. \( \phi(y_1^*b) = \phi(y_1^*)b \);
2. \( \phi(ay_1^*) = \theta(a)\phi(y_1^*) \);
3. \( \phi(y_1^*)^*\phi(y_2^*) = y_1y_2^* \);
4. \( \phi(y_1^*)\phi(y_2^*) = \theta(y_1^*y_2) \).

\[
\text{Proof. By Corollary 6.4, } B_1^*B_1 = uB_1B_1^*u^*. \text{ Let } \theta(f(v)) = uf(v)u^*. \text{ Then } \theta \text{ is an isomorphism of } B_1B_1^* \text{ onto } B_1^*B_1. \text{ Define } \phi(f(v)u^*) = uf(v)u^*. \text{ By Lemma 6.2 and Corollary 6.4, } \phi \text{ is a linear isometry from } B_1^* \text{ onto } B_1B_1^*. \text{ Let } y_1 = uf_1(v) \text{ and } y_2 = uf_2(v) \text{ such that } y_1, y_2 \in B_1, a = g_1(v) \in B_1^*B_1 \subseteq A \text{ and } b = g_2(v) \in B_1B_1^* \subseteq A. \text{ Then}
\]

\[
\phi(y_1^*b) = \phi(\bar{f}_1(v)u^*g_2(v)) = \phi(g_1(v)^{-1}(g_2(v))u^*) = uf_1(v)\theta^{-1}(g_2(v))u^* = u\bar{f}_1(v)u^*g_2(v) = \phi(y_1^*)b,
\]

\[
\phi(ay_1^*) = \phi(g_1(v)\bar{f}_1(v)u^*) = u\bar{f}_1(v)u^* = \theta(g_1(v))uf_1(v)u^* = \theta(a)\phi(y_1^*),
\]

\[
\phi(y_1^*)^*\phi(y_2^*) = (uy_1^*)^*(uy_2^*) = y_1y_2^*,
\]

\[
\phi(y_1^*)\phi(y_2^*) = (uy_1^*)(uy_2^*) = uy_1^*y_2u^* = \theta(y_1^*y_2).
\]
Lemma 6.6. Let $\Theta = (\theta, B_1^*B_1, B_1B_1^*)$ be the partial automorphism of the fixed point algebra $A$ as in [12]. Then there exists an isomorphism

$$\varphi : C^*(A, \Theta) \to A_{\theta, \gamma}.$$  

Proof. Clearly $A_{\theta, \gamma}$ is generated by the fixed point algebra $A$ and the first spectral subspace $B_1$. So the action $\alpha$ of $T$ on $A_{\theta, \gamma}$ is semi-saturated (see Definition 4.1 of [12]). By Lemma 6.5 $\alpha$ is also regular. By Theorem 4.21 of [12], there exists an isomorphism

$$\varphi : C^*(A, \Theta) \to A_{\theta, \gamma}.$$  

\[\square\]

Theorem 6.7. Let $Y$ be the set of zeros of $\gamma$. If $T \neq Y \neq \emptyset$, then

$$K_1(A_{\theta, \gamma}) = \mathbb{Z}$$  

(6.1)

and there exists a splitting short exact sequence:

$$0 \to \mathbb{Z} \to K_0(A_{\theta, \gamma}) \to C(Y, \mathbb{Z}) \to 0.$$  

(6.2)

In particular, if $Y$ has $n$ points, then

$$K_0(A_{\theta, \gamma}) = \mathbb{Z}^{n+1}.$$  

(6.3)

Proof. Let $J = B_1B_1^*$. By Lemma 6.6 and Theorem 7.1 of [12], we have the following exact sequence of $K$-groups

$$K_0(J) \xrightarrow{i_* - \theta_{-1}^*} K_0(A) \xrightarrow{i_*} K_0(A_{\theta, \gamma}) \xrightarrow{j_*} K_1(A_{\theta, \gamma}) \xrightarrow{i_* - \theta_{-1}^*} K_1(J).$$

It is easy to see that $K_0(J) = 0$, $K_1(J) \cong C(Y, \mathbb{Z})$, $K_0(A) \cong K_1(A) \cong \mathbb{Z}$. Note that

$$K_1(J) \xrightarrow{i_* - \theta_{-1}^*} K_1(A)$$

is the composition of maps

$$K_1(J) \xrightarrow{id} K_1(A)$$

and

$$K_1(A) \xrightarrow{id - \theta_{-1}} K_1(A).$$

Since

$$K_1(A) \xrightarrow{id - \theta_{-1}} K_1(A).$$
is zero map,
\[ K_1(J) \xrightarrow{i_* - \theta_1} K_1(A) \]
is zero map. This gives the short exact sequence (6.2).

To see it splits, note that \( Y \) may be identified with a compact subset of the unit line segment which in turn is viewed as a compact subset of the plane. Note also that \( K_1(C(Y)) = \{0\} \). It follows from the BDF-theory [3] that \( Ext(C(Y)) = \{0\} \). Let \( E \) be a unital essential extension of the form:
\[ 0 \to \mathcal{K} \to E \to C(Y) \to 0. \]
The fact that \( Ext(C(Y)) = \{0\} \) implies, in particular, the short exact sequence
\[ 0 \to K_0(\mathcal{K}) \to K_0(E) \to K_0(C(Y)) \to 0 \]
splits for any such \( E \), or,
\[ 0 \to \mathbb{Z} \to K_0(E) \to C(Y, \mathbb{Z}) \to 0 \]
splits for any such group \( K_0(E) \). It follows (from Brown’s UCT) that \( Ext_{\mathbb{Z}}(\mathbb{Z}, C(Y, \mathbb{Z})) = \{0\} \). Therefore the short exact sequence (6.2) splits.

**Corollary 6.8.** \( K_i(A_{\theta, \gamma}) \) is torsion free, \( i = 0, 1 \). If \( \gamma \) has finitely many zeros, then \( K_i(A_{\theta, \gamma}) \) is free, \( i = 0, 1 \).

## 7 Classification of simple \( C^* \)-algebras of \( A_{\theta, \gamma} \)

In this section, we will discuss the structure of \( A_{\theta, \gamma} \) when it is simple. For recursive subhomogeneous algebras see [28], Section 1. Recall also that the Jiang-Su algebra \( \mathcal{Z} \) is a unital simple \( C^* \)-algebra of recursive subhomogeneous \( C^* \)-algebra with one dimensional base spaces with a unique tracial state and with \( K_0(\mathcal{Z}) = \mathbb{Z} \) and \( K_1(\mathcal{Z}) = \{0\} \) (see [18]).

**Lemma 7.1.** Let \( \theta \) be an irrational number. Suppose that \( \gamma \) has at least one zero. Then \( A_{\theta, \gamma} \) is an inductive limit of recursive subhomogeneous \( C^* \)-algebras with one dimensional base spaces. In particular, if \( A_{\theta, \gamma} \) is simple, then \( A_{\theta, \gamma} \) is \( \mathcal{Z} \)-stable, where \( \mathcal{Z} \) is the Jiang-Su algebra [18].

**Proof.** Let \( Y \) be the set of zeros of \( \gamma \). It is a closed subset of \( \mathbb{T} \). Let \( 1/4 > \epsilon_n > 0 \) be such that \( \lim_{n \to \infty} \epsilon_n = 0 \). Define
\[ Y_n = \{ x \in \mathbb{T} : \text{dist}(x, Y) \leq \epsilon_n \}, \quad n = 1, 2, \ldots. \]
Define
\[ A_Y = C^*(C(\mathbb{T}), uC_0(\mathbb{T} \setminus Y)) \quad \text{and} \quad (7.1) \]
By Theorem 2.3 of \cite{Z} (or Example 1.6 of \cite{Y}), $A_{Y_n}$ is a recursive subhomogeneous $C^*$-algebra with one dimensional base spaces. Since $A_Y = \lim_{n \to \infty} A_{Y_n}$ (with inclusion maps), the first part of the lemma follows.

To see the second part, it follows from Theorem 1.6 of \cite{Z} that each $A_{Y_n}$ has decomposition rank at most one. Therefore $A_Y$ has decomposition rank one. Since we assume that $A_Y$ is simple, by Theorem 5.1 of \cite{Y}, $A_Y$ is $\mathcal{Z}$-stable. Note that $A_Y = A_{\theta, \gamma}$.

**Lemma 7.2.** Suppose that $A_{\theta, \gamma}$ is simple and $Y$ is the set of zeros of $\gamma$. Let $i$ be the embedding of $A_{\theta, \gamma} = C^*(C(\mathbb{T}), uC_0(\mathbb{T} \setminus Y)) \subset A_{\theta}$, and let $\rho_{A_{\theta, \gamma}}$ be the induced map of $K_0(A_{\theta, \gamma})$ into $K_0(A_{\theta})$. Then

$$
\rho_{A_{\theta, \gamma}}(K_0(A_{\theta, \gamma})) = \mathbb{Z} + \mathbb{Z} \theta \text{ and } \ker \rho_{A_{\theta, \gamma}} \cong C(Y, \mathbb{Z})/\mathbb{Z},
$$

(7.3)

where $\mathbb{Z}$ is identified with constant functions in $C(Y, \mathbb{Z})$. Thus one has the following splitting short exact sequence:

$$
0 \to C(Y, \mathbb{Z})/\mathbb{Z} \to K_0(A_{\theta, \gamma}) \xrightarrow{\rho_{A_{\theta, \gamma}}} \mathbb{Z} + \mathbb{Z} \theta \to 0.
$$

(7.4)

Moreover, in this case,

$$
K_0(A_{\theta, \gamma})_+ = \{0\} \cup \{x \in K_0(A_{\theta, \gamma}) : \rho_{A_{\theta, \gamma}}(x) > 0\}
$$

(7.5)

and $K_0(A_{\theta, \gamma})$ is weakly unperforated and has the Riesz interpolation property.

**Proof.** Denote by $\phi : \mathbb{T} \to \mathbb{T}$ the rotation of the unit circle by $\theta$, i.e., $\phi(z) = e^{2\pi i \theta} z$ for $z \in \mathbb{T}$. By the assumption of the lemma and Theorem 1.3 of \cite{Z}, $\phi^n(Y) \cap Y = \emptyset$ for all integers $n \neq 0$. By Theorem 2.4 and Example 2.6 of \cite{Z}, one obtains the following six term exact sequence:

$$
\xymatrix{ K_0(C(Y)) \ar[r] & K_0(A_{\theta, \gamma}) \ar[r]^{\iota_{\theta}} & K_0(A_{\theta}) \ar[d] & K_1(A_{\theta}) \ar[l]_{i_{\theta, \gamma}} \ar[r] & K_1(A_{\theta, \gamma}) \ar[r] & K_1(C(Y)) }
$$

Note that $Y$ is a proper closed subset of the circle. It follows that $K_1(C(Y)) = \{0\}$ and $\iota_{\theta, \gamma} = \rho_{A_{\theta, \gamma}}$ is surjective. Since $K_0(A_{\theta}) = \mathbb{Z} + \mathbb{Z} \theta$ as an ordered subgroup of $\mathbb{R}$, $\operatorname{Ran} \rho_{A_{\theta, \gamma}} = \mathbb{Z} + \mathbb{Z} \theta$. By Theorem 6.7 of \cite{Z}, $K_1(A_{\theta, \gamma}) = \mathbb{Z}$. One then computes that

$$
\ker \rho_{A_{\theta, \gamma}} \cong C(Y, \mathbb{Z})/\mathbb{Z}.
$$

It is proved in Lemma 7.1 that $A_{\theta, \gamma}$ is $\mathcal{Z}$-stable. In particular, $K_0(A_{\theta, \gamma})$ has the strict comparison. Therefore

$$
K_0(A_{\theta, \gamma})_+ = \{0\} \cup \{x \in K_0(A_{\theta, \gamma}) : \rho_{A_{\theta, \gamma}}(x) > 0\}.
$$

(7.6)

It follows that $K_0(A_{\theta, \gamma})$ is weakly unperforated and has the Riesz interpolation property.
For the convenience of the reader, we recall the meaning of tracial rank zero (or tracial topological
rank zero) for simple $C^*$-algebras.

**Definition 7.3.** Let $A$ be a simple unital $C^*$-algebra. Then $A$ has tracial rank zero if for every
subset $F \subset A$, every $\epsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$
and a unital finite dimensional subalgebra $E \subset pAp$ such that:

1. $\| [a, p] \| < \epsilon$ for all $a \in F$.
2. $\text{dist}(pap, E) < \epsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $cAc$.

This definition is equivalent to the original one following from [23], Proposition 3.8.

**Theorem 7.4.** Let $A_{\theta, \gamma}$ be a unital simple $C^*$-algebra. Then $A_{\theta, \gamma}$ is a unital simple $AT$-algebra of
real rank zero. In particular, $A_{\theta, \gamma}$ has tracial rank zero.

**Proof.** By Lemma 7.1 and Theorem 7.2, $A_{\theta, \gamma}$ is $Z$-stable, $K_0(A_{\theta, \gamma})$ is weakly unperforated and has
the Riesz interpolation property. Since $A_{\theta, \gamma}$ is an inductive limit of type I $C^*$-algebras, it satisfies the
universal coefficient theorem. By Corollary 6.7, $K_1(A_{\theta, \gamma})$ is torsion free. Therefore, by [10], there is a
unital simple $AT$-algebra $C$ of real rank zero such that

\[
(K_0(C), K_0(C)_+, [1_C], K_1(C)) \cong (K_0(A_{\theta, \gamma}), K_0(A_{\theta, \gamma}), [1_{A_{\theta, \gamma}}], K_1(A_{\theta, \gamma})).
\]  

(7.7)

Let $U$ be a UHF-algebra of infinite type. Consider $B = A_{\theta, \gamma} \otimes U$. $B$ has a unique tracial state and
is approximately divisible. Therefore its projections separate the tracial state space. It follows from [4] that $B$
has real rank zero. Since $B$ is $Z$-stable, $B$ has strict comparison for projections. Therefore
$K_0(B)$ is weakly unperforated. It follows from Lemma [7.1] that $B$ is a locally type I $C^*$-algebra. Then,
by applying 5.16 of [24], $B$ has tracial rank zero. We also note that since $A_{\theta, \gamma}$ satisfies the universal
coefficient theorem, so does $B$.

It follows from the classification theorem of [25] (Theorem 5.4) that $C \otimes Z \cong A_{\theta, \gamma} \otimes Z$. However,
$C$ is $Z$-stable and, by Lemma [7.1], $A_{\theta, \gamma}$ is also $Z$-stable, one actually has

\[
C \cong A_{\theta, \gamma}.
\]  

(7.8)

$\square$
Corollary 7.5. Let \( \theta \) be an irrational number, \( \gamma \in C(\mathbb{T}) \) be a non-negative function, let \( Y \) be the set of zeros of \( \gamma \) and let \( \phi: \mathbb{T} \to \mathbb{T} \) be the homeomorphism by rotation of \( \theta \). Then the following are equivalent:

1. \( A_{\theta, \gamma} \) is simple;
2. \( A_{\theta, \gamma} \) has a unique tracial state;
3. \( \phi^n(Y) \cap Y = \emptyset \) for all integers \( n \neq 0 \);
4. For each \( \xi \in \mathbb{T} \), \( \text{Orb}(\xi) \cap Y \) contains at most one point;
5. \( A_{\theta, \gamma} \) is a unital simple \( \mathbb{A} \mathbb{T} \)-algebra of real rank zero.

Theorem 7.6. Let \( \theta_1 \) and \( \theta_2 \) be two irrational numbers, \( \gamma_1 \) and \( \gamma_2 \in C(\mathbb{T}) \) be non-negative functions and let \( Y_i \) be the set of zeros of \( \gamma_i \), \( i = 1, 2 \). Suppose that \( A_{\theta_i, \gamma_i} \) is simple, or one of the equivalent conditions in Corollary 7.5 satisfies. Then \( A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2} \) if and only if the following hold:

\[
\theta_1 = \pm \theta_2 \mod(\mathbb{Z}) \quad \text{and} \quad C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}.
\] (7.9)

In particular, when \( \gamma_1 \) has only finitely many zeros, then \( A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2} \) if and only if \( \theta_1 \equiv \pm \theta_2 \mod \mathbb{Z} \) and \( \gamma_2 \) has the same number of zeros.

Proof. We will prove the “if” part only. Note that we have \( K_1(A_{\theta_1, \gamma_1}) \cong K_1(A_{\theta_2, \gamma_2}) \). We may write, by Lemma 7.2, that

\[
K_0(A_{\theta_i, \gamma_i}) = C(Y_i, \mathbb{Z})/\mathbb{Z} \oplus (\mathbb{Z} + \mathbb{Z} \theta_i).
\] (7.10)

It follows that \( K_0(A_{\theta_1, \gamma_1}) \cong K_0(A_{\theta_2, \gamma_2}) \). In fact they are order isomorphic. By Theorem 7.4 both \( C^* \)-algebras are unital simple \( \mathbb{A} \mathbb{T} \)-algebras of real rank zero. By the classification theorem they are isomorphic.

\[\square\]

Corollary 7.7. With the same assumption as in 7.6, if \( Y_1 \) and \( Y_2 \) are homeomorphic and \( \theta_1 = \pm \theta_2 \mod \mathbb{Z} \), then \( A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2} \).

Theorem 7.8. Let \( \theta_1, \theta_2 \in (0, 1) \) be two irrational numbers, \( \gamma_1, \gamma_2 \in C(\mathbb{T}) \) be non-negative functions and let \( Y_i \) be the set of zeros of \( \gamma_i \), \( i = 0, 1 \). Suppose that \( A_{\theta_i, \gamma_i} \) is simple, or one of the equivalent conditions in Corollary 7.5 satisfies. Then \( A_{\theta_1, \gamma_1} \) and \( A_{\theta_2, \gamma_2} \) are Morita equivalent if and only if \( \mathbb{Z} + \mathbb{Z} \theta_1 \) and \( \mathbb{Z} + \mathbb{Z} \theta_2 \) are order isomorphic and

\[
C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}.
\] (7.11)

In particular, assuming, in addition, \( Y_1 \) and \( Y_2 \) are both finite subsets, then \( A_{\theta_1, \gamma_1} \) and \( A_{\theta_2, \gamma_2} \) are Morita equivalent if and only if \( \mathbb{Z} + \mathbb{Z} \theta_1 \) and \( \mathbb{Z} + \mathbb{Z} \theta_2 \) are order isomorphic and \( Y_1 \) and \( Y_2 \) have the same number of points.
Proof. Suppose that \( h_1 : \mathbb{Z} + \mathbb{Z}\theta_1 \to \mathbb{Z} + \mathbb{Z}\theta_2 \) is an order isomorphism and \( h_2 : C(Y_1, \mathbb{Z})/\mathbb{Z} \to C(Y_2, \mathbb{Z})/\mathbb{Z} \) is an isomorphism as groups. There is an injective homomorphism \( \iota_i : \mathbb{Z} + \mathbb{Z}\theta_i \to K_0(A_{\theta_i, \gamma_i}) \) such that

\[
\rho_{A_{\theta_i, \gamma_i}} \circ \iota_i = \text{id}_{\mathbb{Z} + \mathbb{Z}\theta_i}, \quad i = 1, 2.
\]

We write

\[
K_0(A_{\theta_i, \gamma_i}) = C(Y_i, \mathbb{Z})/\mathbb{Z} \oplus \iota_i(\mathbb{Z} + \mathbb{Z}\theta_i),
\]

\( i = 1, 2. \)

Define \( h_3 : K_0(A_{\theta_1, \gamma_1}) \to K_0(A_{\theta_2, \gamma_2}) \) by

\[
h_3|_{\text{ker} \rho_{A_{\theta_1, \gamma_1}}} = h_2
\]

(7.12)

and

\[
h_3(x) = \iota_2 \circ h_1 \circ \rho_{A_{\theta_1, \gamma_1}}(x).
\]

(7.13)

for \( x \in \iota_1(\mathbb{Z} + \mathbb{Z}\theta_1) \). It is easy to verify that \( h_3 \) is an order isomorphism from \( K_0(A_{\theta_1, \gamma_1}) \) onto \( K_0(A_{\theta_2, \gamma_2}) \). We also have \( K_1(A_{\theta_1, \gamma_1}) = \mathbb{Z} = K_1(A_{\theta_2, \gamma_2}) \). Since both \( A_{\theta_1, \gamma_1} \) and \( A_{\theta_2, \gamma_2} \) are unital simple \( AT \)-algebras of real rank zero, by the classification results mentioned earlier, \( A_{\theta_1, \gamma_1} \) and \( A_{\theta_2, \gamma_2} \) are stably isomorphic. In other words, \( A_{\theta_1, \gamma_1} \) and \( A_{\theta_2, \gamma_2} \) are Morita equivalent.

Conversely, if \( A_{\theta_1, \gamma_1} \otimes \mathcal{K} \cong A_{\theta_2, \gamma_2} \otimes \mathcal{K} \), then \( K_0(A_{\theta_1, \gamma_1}) \) and \( K_0(A_{\theta_2, \gamma_2}) \) are order isomorphic. Denote by \( h_0 \) the order isomorphism. This implies, in particular, \( h_0 \) maps \( \text{ker} \rho_{A_{\theta_1, \gamma_1}} \) isomorphically onto \( \text{ker} \rho_{A_{\theta_2, \gamma_2}} \) which implies that

\[
C(Y_1, \mathbb{Z})/\mathbb{Z} = \text{ker} \rho_{A_{\theta_1, \gamma_1}} \cong \text{ker} \rho_{A_{\theta_2, \gamma_2}} = C(Y_2, \mathbb{Z})/\mathbb{Z}.
\]

Therefore \( h_0 \) induces an order isomorphism from \( \rho_{A_{\theta_1, \gamma_1}}(K_0(A_{\theta_1, \gamma_1})) \) onto \( \rho_{A_{\theta_2, \gamma_2}}(K_0(A_{\theta_2, \gamma_2})) \) which implies that \( \mathbb{Z} + \mathbb{Z}\theta_1 \) and \( \mathbb{Z} + \mathbb{Z}\theta_2 \) are order isomorphic. \( \square \)

Let \( \text{GL}(2, \mathbb{Z}) \) denote the group of \( 2 \times 2 \) matrices with entries in \( \mathbb{Z} \) and with determinant \( \pm 1 \), and let \( \text{GL}(2, \mathbb{Z}) \) act on the set of irrational numbers by

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \alpha = \frac{a\alpha + b}{c\alpha + d}.
\]

By Corollary 2.6 of [34] (or Lemma 4.7 of [35]), \( \mathbb{Z} + \mathbb{Z}\theta_1 \) and \( \mathbb{Z} + \mathbb{Z}\theta_2 \) are ordered isomorphic if and only if \( \theta_1 \) and \( \theta_2 \) are in the same orbit of \( \text{GL}(2, \mathbb{Z}) \). Thus we obtain the following corollary.

**Corollary 7.9.** Let \( \theta_1, \theta_2 \in (0, 1) \) be two irrational numbers, \( \gamma \in C(\mathbb{T}) \) be non-negative functions and let \( Y \) be the set of zeros of \( \gamma \). Suppose that \( A_{\theta_i, \gamma} \) is simple, or one of the equivalent conditions in Corollary 7.3 satisfies. Then \( A_{\theta_1, \gamma} \) and \( A_{\theta_2, \gamma} \) are Morita equivalent if and only if \( \theta_1 \) and \( \theta_2 \) are in the same orbit under the action of \( \text{GL}(2, \mathbb{Z}) \) on irrational numbers.
8 The $C^*$-algebra generated by $u + \lambda v$

**Proposition 8.1.** Let $R$ be the hyperfinite type II$_1$ factor with two unitary generators $u, v$ such that $vu = e^{2\pi i \theta} uv$. If $f(z) \in C(T)$ and $m(\{z|f(z) = 0\}) = 0$, then the von Neumann subalgebra generated by $uf(v)$ and $v$ is $R$. Furthermore, $C^*(uf(v), v) = C^*(u, v)$ if and only if $f(z) \neq 0$ for all $z \in T$.

**Proof.** Let $M$ be the von Neumann algebra generated by $uf(v)$ and $v$. Since $m(\{z|f(z) = 0\}) = 0$, $f(v)^{-1}$ is affiliated with $M$, i.e., the spectral projections of the unbounded operator $f(v)^{-1}$ are in $M$. Hence $u = uf(v) \cdot f(v)^{-1}$ is affiliated with $M$. Since $u$ is a bounded operator, $u \in M$ and therefore $R \subseteq M$ and $M = R$.

If $f(z) \neq 0$ for all $z \in T$, then $f(v)$ is an invertible operator in $C^*(v)$. Hence $u = uf(v) \cdot f(v)^{-1}$ is in the $C^*$-subalgebra generated by $uf(v)$ and $v$. Therefore, $C^*(uf(v), v) = C^*(u, v)$. Conversely, suppose $f(z_0) = 0$ for some $z_0 \in T$. By Theorem 6.7, $K_1(C^*(uf(v), v)) \cong \mathbb{Z}$. Therefore, $C^*(uf(v), v) \neq C^*(u, v)$.

**Theorem 8.2.** Let $R$ be the hyperfinite type II$_1$ factor with two unitary generators $u, v$ such that $vu = e^{2\pi i \theta} uv$. Then the von Neumann subalgebra generated by $u + \lambda v$ is $R$ for $\lambda > 0$. Furthermore, $C^*(u + \lambda v) = C^*(u, v)$ if $\lambda \neq 1$ while $C^*(u + v)$ is a proper simple $C^*$-subalgebra of $C^*(u, v)$ which has a unique trace, $K_1(C^*(u + v)) \cong \mathbb{Z}$, and there is an order isomorphism of $K_0(C^*(u + v))$ onto $\mathbb{Z} + \mathbb{Z}\theta$. Moreover, $C^*(u + v)$ is a unital simple $\mathcal{AT}$-algebra of tracial rank zero.

**Proof.** Note that

$$(u + \lambda v)(u + \lambda v)^* = (u + \lambda v)(u^* + \lambda v^*) = \lambda e^{-2\pi i \theta} u^* v + \lambda uv^* + 1 + \lambda^2$$

and

$$(u + \lambda v)^*(u + \lambda v) = (u^* + \lambda v^*)(u + \lambda v) = u^* v + e^{-2\pi i \theta} \lambda uv^* + 1 + \lambda^2.$$ 

Hence $u^* v, uv^* \in C^*(u + \lambda v)$. Let $w = u^* v$. Thus $C^*(u + \lambda v) = C^*(u + \lambda v, w) = C^*(u(1 + \lambda w), w)$. By Proposition 8.1, the von Neumann subalgebra generated by $u + \lambda v$ is $R$ for $\lambda > 0$, and $C^*(u + \lambda v) = C^*(u, v)$ if $\lambda \neq 1$ while $C^*(u + v)$ is a proper $C^*$-subalgebra of $C^*(u, v)$. Note that $u + v$ and $w$ satisfy (2.1)-(2.4) for $\theta$ and $\gamma(z) = |1 + z|^2$. By Proposition 3.2 Theorem 4.2, Theorem 5.3 and Theorem 6.7 $C^*(u + v)$ is a simple algebra with a unique trace, $K_1(C^*(u + v)) \cong \mathbb{Z}$, and there is an order isomorphism of $K_0(C^*(u + v))$ onto $\mathbb{Z} + \mathbb{Z}\theta$. By Theorem 7.4 $C^*(u + v)$ is a unital simple $\mathcal{AT}$-algebra of tracial rank zero.

**Corollary 8.3.** $C^*(u + v)$ is not *-isomorphic to $C^*(u, v)$.
9 Spectral radius of $u + \lambda v$

In this section, we assume that $0 \leq \lambda \leq 1$. Let $\alpha = e^{2\pi i \theta}$ and $w = u^*v$. Then $w$ is a Haar unitary operator in $R$, i.e., $\tau(w^n) = \tau((w^*)^n) = 0$ for all $n \in \mathbb{N}$. Note that

$$u + \lambda v = u(1 + \lambda u^*v) = u(1 + \lambda w),$$

$$(u + \lambda v)^2 = (u + \lambda v)(u + \lambda w) = (u^2 + \alpha \lambda u^2v)(1 + \lambda w) = u^2(1 + \alpha \lambda w)(1 + \lambda w),$$

$$(u + \lambda v)^3 = (u + \lambda v)(u^2 + \alpha \lambda u^2v)(1 + \lambda w) = (u^3 + \alpha \lambda u^2v)(1 + \alpha \lambda w)(1 + \lambda w) = u^3(1 + \alpha^2 \lambda w)(1 + \alpha \lambda w)(1 + \lambda w).$$

By induction, we have

$$(u + \lambda v)^n = u^n(1 + \lambda w)(1 + \alpha \lambda w) \cdots (1 + \alpha^{(n-1)} \lambda w), \quad \forall n \in \mathbb{N} \quad (9.1)$$

Let $r(u + \lambda v)$ be the spectral radius of $u + \lambda v$. Then

$$r(u + \lambda v) = \lim_{n \to +\infty} \|(u + \lambda v)^n\|^{1/n} = \lim_{n \to +\infty} \|(1 + \lambda w)(1 + \alpha \lambda w) \cdots (1 + \alpha^{(n-1)} \lambda w)\|^{1/n}.$$ 

Since $w = u^*v$ is a Haar unitary operator, we may identify $w$ with the multiplication operator $M_z$ on $L^2(\mathbb{T}, m)$, where $m$ is the Haar measure on $\mathbb{T}$. Hence,

$$\|(u + \lambda v)^n\|^{1/n} = \|(1 + \lambda w)(1 + \alpha \lambda w) \cdots (1 + \alpha^{(n-1)} \lambda w)\|^{1/n} = \left(\max_{z \in \mathbb{T}}\left|(1 + \lambda z)(1 + \alpha \lambda z) \cdots (1 + \alpha^{(n-1)} \lambda z)\right|\right)^{1/n}.$$ 

Let $z = e^{i2\pi x}$, $x \in [0, 1]$. Then simple calculation shows that

$$\left|(1 + \lambda z)(1 + \alpha \lambda z) \cdots (1 + \alpha^{(n-1)} \lambda z)\right| = \left(\prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)))\right)^{1/n}. $$

So

$$\|(u + \lambda v)^n\|^{1/n} = \max_{x \in [0, 1]} \left(\prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)))\right)^{1/n}. \quad (9.2)$$

**Lemma 9.1.** For $0 \leq \lambda \leq 1$,

$$\int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x)dx = 0.$$

**Proof.** For $0 \leq \lambda \leq 1$, let

$$f(\lambda) = \int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x)dx.$$
Then \( f(\lambda) \) is continuous on \([0, 1]\), differentiable in \((0, 1)\), and \(f(0) = 0\). Note that for \(0 < \lambda < 1\),

\[
f'(\lambda) = \int_{0}^{1} \frac{2\lambda + 2 \cos 2\pi x}{1 + \lambda^2 + 2\lambda \cos 2\pi x} \, dx
\]

\[
= \frac{1}{2\pi i} \int_{\mathcal{T}} \frac{2\lambda + z + \frac{1}{z}}{1 + \lambda^2 + \lambda \left( z + \frac{1}{z} \right)} \, \frac{dz}{z}
\]

\[
= \frac{1}{2\pi i} \int_{\mathcal{T}} \frac{2\lambda z + z^2 + 1}{(1 + \lambda^2)z + \lambda z^2 + \lambda} \, \frac{dz}{z}
\]

\[
= \frac{1}{2\pi i} \int_{\mathcal{T}} \frac{2\lambda z + z^2 + 1}{(\lambda z + 1)(z + \lambda)} \, dz
\]

\[
= \text{Res} \left( \frac{2\lambda z + z^2 + 1}{(\lambda z + 1)(z + \lambda)} ; 0 \right) + \text{Res} \left( \frac{2\lambda z + z^2 + 1}{(\lambda z + 1)(z + \lambda)} ; -\lambda \right)
\]

\[
= \frac{1}{\lambda} - \frac{1}{\lambda} = 0.
\]

So for \(0 \leq \lambda \leq 1\), \(f(\lambda) = 0\).

\[
\square
\]

**Lemma 9.2.** Let \(0 < \lambda \leq 1\). Then for almost all \(x \in [0, 1]\),

\[
\lim_{n \to +\infty} \left( \prod_{k=0}^{n-1} \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right) \right)^{\frac{1}{2n}} = 1.
\]

**Proof.** We only need to show that for almost all \(x \in [0, 1]\),

\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right) = 0.
\]

Let \(f(x) = \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x)\). If \(0 < \lambda < 1\), then

\[
2 \ln(1 - \lambda) \leq f(x) \leq 2 \ln(1 + \lambda), \quad \forall x \in [0, 1].
\]

So \(f(x) \in L^1[0, 1]\). If \(\lambda = 1\), then

\[
f(x) = \ln(2 + 2 \cos 2\pi x) = 2 \ln 2 + 2 \ln |\cos \pi x|
\]

and so

\[
|f(x)| \leq 2 \ln 2 - 2 \ln |\cos \pi x|, \quad \forall x \in [0, 1].
\]

By Lemma 9.1, \(\int_{0}^{1} f(x) \, dx = 0\), which implies that \(\int_{0}^{1} 2 \ln |\cos \pi x| \, dx = -2 \ln 2\). Therefore, \(\int_{0}^{1} |f(x)| \, dx \leq 4 \ln 2\) and \(f(x) \in L^1[0, 1]\).

Let \(T : x \to x + \theta \pmod{1}\). Then \(T\) is a measure preserving ergodic transformation of \([0, 1]\). By Birkhoff’s Ergodic theorem and Lemma 9.1 for almost all \(x \in [0, 1]\)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right) = \int_{0}^{1} \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) \, dx = 0.
\]

\[
\square
\]
Corollary 9.3. For $0 < \lambda \leq 1$, $r(u + \lambda v) \geq 1$.

Proof. Let $\epsilon > 0$. By Lemma 9.2, there is an $x \in [0, 1]$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$
\left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^{\frac{1}{2\pi}} \geq 1 - \epsilon.
$$

By equation (9.2), for $n \geq N$,

$$
\|(u + \lambda v)^n\|^{1/n} \geq 1 - \epsilon.
$$

This implies that

$$
r(u + \lambda v) = \lim_{n \to +\infty} \|(u + \lambda v)^n\|^{1/n} \geq 1 - \epsilon.
$$

Since $\epsilon > 0$ is arbitrary, $r(u + \lambda v) \geq 1$. \qed

Let $\theta \in (0, 1)$ be an irrational number and let $\alpha = e^{2\pi i \theta}$.

Lemma 9.4. Given $\epsilon > 0$ and $N \in \mathbb{N}$. Then there exists $N' \in \mathbb{N}$ such that for $n \geq N'$ and every arc $\Gamma$ of the unit circle $\mathbb{T}$ with length $\frac{2\pi}{N}$, there exists $\frac{n}{N} + r$ points of $1, \alpha, \ldots, \alpha^{n-1}$ in $\Gamma$ with $|\frac{r}{n}| < \epsilon$.

Proof. Since $\theta \in (0, 1)$ is irrational, $\{\alpha^k : k \in \mathbb{N}\}$ is dense in the unit circle $\mathbb{T}$. Therefore, there exists $m \in \mathbb{N}$ such that for every $0 \leq \varphi \leq 2\pi$, there exists $1 \leq k \leq m$ such that $|(\varphi - 2k\pi \theta) \bmod 2\pi| < \frac{\epsilon}{8}$. By Birkhoff’s ergodic theorem, there exists an arc $\Gamma_1$ of the unit circle with length $l(\Gamma_1) = 2\pi \left(\frac{1}{N} - \frac{\epsilon}{4}\right)$ and

$$
\lim_{n \to +\infty} \frac{\chi_{\Gamma_1}(1) + \chi_{\Gamma_1}(\alpha) + \cdots + \chi_{\Gamma_1}(\alpha^{n-1})}{n} = \frac{l(\Gamma_1)}{2\pi} = \frac{1}{N} - \frac{\epsilon}{4}.
$$

Let $N_1$ be sufficiently large such that $\frac{m}{N_1} < \frac{\epsilon}{2}$ and if $n \geq N_1$ then

$$
\frac{\chi_{\Gamma_1}(1) + \chi_{\Gamma_1}(\alpha) + \cdots + \chi_{\Gamma_1}(\alpha^{n-1})}{n} \geq \frac{1}{N} - \frac{\epsilon}{4} - \frac{\epsilon}{4} = \frac{1}{N} - \frac{\epsilon}{2}.
$$

Let $e^{2\pi i \theta}$ and $e^{2\pi i (\theta + 2\pi/N)}$ be the ending points of the arc $\Gamma$. Let $\Gamma_1' \subset \Gamma$ be the arc of $\mathbb{T}$ with ending points $e^{2\pi i \theta + (\pi/4)i}$ and $e^{2\pi i (\theta + 2\pi/N) - (\pi/4)i}$. Then there exists an $\varphi$ with $0 \leq \varphi \leq 2\pi$ such that we can rotate $\Gamma_1$ by angle $\varphi$ to obtain $\Gamma_1'$. So if $\{\alpha^{k_1}, \ldots, \alpha^{k_s}\} \subseteq \Gamma_1$ with $0 \leq k_1 < k_2 < \cdots < k_s \leq n - 1$, then $\{\alpha^{k_1} e^{i\varphi}, \ldots, \alpha^{k_s} e^{i\varphi}\} \subseteq \Gamma_1' \subseteq \Gamma$. Since $|(\varphi - 2k\pi \theta) \bmod 2\pi| < \frac{\epsilon}{8}$ for some $1 \leq k \leq m$,

$$
\{\alpha^{k_1} e^{i2k\pi \theta}, \ldots, \alpha^{k_s-m} e^{i2k\pi \theta}\} \subset \Gamma.
$$

Since $k_{s-m} \leq n - m$, $\{\alpha^{k_1+k}, \ldots, \alpha^{k_{s-m}+k}\} \subset \Gamma$. So $\Gamma$ contains at least $n \left(\frac{1}{N} - \frac{\epsilon}{4}\right) - m = n \left(\frac{1}{N} - \epsilon\right)$ points of $1, \alpha, \ldots, \alpha^{n-1}$.

By Birkhoff’s ergodic theorem, there exists an arc $\Gamma_2$ of the unit circle with length $l(\Gamma_2) = 2\pi \left(\frac{1}{N} + \frac{\epsilon}{4}\right)$ and

$$
\lim_{n \to +\infty} \frac{\chi_{\Gamma_2}(1) + \chi_{\Gamma_2}(\alpha) + \cdots + \chi_{\Gamma_2}(\alpha^{n-1})}{n} = \frac{l(\Gamma_2)}{2\pi} = \frac{1}{N} + \frac{\epsilon}{4}.
$$
Let \( N_2 \) be sufficiently large such that \( \frac{m}{N_2} < \frac{\epsilon}{2} \) and if \( n \geq N_2 \) then
\[
\frac{\chi r_2(1) + \chi r_2(\alpha) + \cdots + \chi r_2(\alpha^{n-1})}{n} \leq \frac{1}{N} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{1}{N} + \frac{\epsilon}{2}.
\]
Let \( e^{2\pi i \varphi'} \) and \( e^{2\pi i \varphi' + 2\pi (1/N + \epsilon/4)i} \) be the ending points of the arc \( \Gamma \). Let \( \Gamma_2 \subset \Gamma_2 \) be the arc of \( \mathbb{T} \) with ending points \( e^{2\pi i \varphi' + (\pi/4)i} \) and \( e^{2\pi i (\varphi' + 1/N) + (\pi/4)i} \). Then there exists an \( \varphi' \) with \( 0 \leq \varphi' \leq 2\pi \) such that we can rotate \( \Gamma \) by angle \( \varphi' \) to obtain \( \Gamma' \). So if \( \{\alpha^j, \cdots, \alpha^j\} \subseteq \Gamma \) with \( 0 \leq j_1 < j_2 < \cdots < j_s \leq n-1 \), then \( \{\alpha^j e^{\varphi'}, \cdots, \alpha^j e^{\varphi'}\} \subseteq \Gamma' \subseteq \Gamma \). Since \( |(\varphi' - 2k'\pi \theta) \mod 2\pi| < \frac{\epsilon}{8} \) for some \( 1 \leq k' \leq m \),
\[
\{\alpha^j e^{2k' \pi \theta i}, \cdots, \alpha^j e^{2k' \pi \theta i}\} \subset \Gamma_2.
\]
So \( \Gamma \) contains at most \( n \left( \frac{1}{N} + \frac{\epsilon}{2} \right) + m = n \left( \frac{1}{N} + \epsilon \right) \) points of \( 1, \alpha, \cdots, \alpha^{n-1} \). Let \( N' = \max\{N_1, N_2\} \). Then we obtain the lemma. \( \square \)

Now we prove the main result of this section.

**Lemma 9.5.** For \( 0 < \lambda \leq 1 \), \( r(u + \lambda v) = 1 \).

**Proof.** By Corollary 9.3, we need to prove that \( r(u + \lambda v) \leq 1 \). Let \( \epsilon > 0 \). Note that
\[
\lim_{n \to \infty} \frac{1}{2n} \left( \sum_{k=0}^{n-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) + \sum_{k=n+1}^{2n} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) \right) = \int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) \, dx = 0.
\]
There is \( N \in \mathbb{N} \) such that
\[
\frac{1}{2N} \left( \sum_{k=0}^{N-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \sum_{k=N+1}^{2N} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) < \epsilon.
\]
Let
\[
M(\lambda) = \max_{0 \leq k \leq 2N, k \neq N} \left| \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right|.
\]
Then for \( 0 < \lambda \leq 1 \), \( M(\lambda) < \infty \). Divide the unit circle \( \mathbb{T} \) into \( 2N \) equal sections \( A_1, \cdots, A_{2N} \). By Lemma 9.4, there exists \( N' \) such that for all \( n \geq N' \) and all \( x \in [0, 1] \), if \( A_k \) contains \( n/(2N) + r_k(x) \) points of \( e^{2\pi i x}, \alpha e^{2\pi i x}, \cdots, \alpha^{n-1} e^{2\pi i x} \), then \( \frac{\sum_{k=1}^{2N} |r_k(x)|}{n} < \frac{\epsilon}{M(\lambda)} \). Note that \( \cos 2\pi x \) is decreasing for \( x \in [0, 1/2] \) and increasing for \( x \in [1/2, 1] \). Therefore, for all \( x \in [0, 1] \),
\[
\frac{1}{n} \sum_{k=0}^{n-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right) \leq \frac{1}{n} \sum_{k=0}^{N-1} \left( \frac{n}{2N} + r_{k+1}(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right)
\]
\[ + \frac{1}{n} \sum_{k=N+1}^{2N} \left( \frac{n}{2N} + r_k(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \]
\[ = \frac{1}{2N} \left( \sum_{k=0}^{N-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \sum_{k=N+1}^{2N} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) \]
\[ + \frac{1}{n} \sum_{k=0}^{N-1} r_{k+1}(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \frac{1}{n} \sum_{k=N+1}^{2N} r_k(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \]
\[ < \epsilon + \frac{1}{n} \sum_{k=1}^{2N} |r_k(x)| M(\lambda) < 2\epsilon. \]

This implies that for all \( n \geq N' \) and \( x \in [0, 1] \),
\[ \left( \prod_{k=0}^{n-1} \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right) \right)^{\frac{1}{n}} \leq e^{2\epsilon}. \]

By equation(9.2), \( \|(u + \lambda v)^n\|^{1/n} \leq e^{2\epsilon} \) for all \( n \geq N' \). So \( r(u + \lambda v) \leq e^{2\epsilon} \). Since \( \epsilon > 0 \) is arbitrary, \( r(u + \lambda v) \leq 1 \).

\section{Strongly irreducible operators relative to type II\(_1\) factors}

An operator \( T \) in a type II\(_1\) factor \( M \) is called \emph{irreducible} if \( \{T, T^*\}' \cap M = \mathbb{C}1 \), i.e., the von Neumann subalgebra generated by \( T \) is an irreducible subfactor of \( M \).

**Proposition 10.1.** Every separable type II\(_1\) factor \( M \) contains an irreducible operator.

**Proof.** By [29], every separable type II\(_1\) factor \( M \) contains an irreducible hyperfinite factor. Since hyperfinite factor is generated by an operator \( T \), it follows that \( T \) is an irreducible operator in \( M \). \( \square \)

Recall that an operator \( T \) in \( B(H) \) is a \emph{strongly irreducible operator} if there is no nontrivial idempotents in \( \{T\}' \). Strongly irreducible operators are generalizations of Jordan blocks in matrix algebras. A rich theory has been set up on this class of operators in the past twenty years (see [19, 20]). Let \( M \) be a type II\(_1\) factor. An operator \( T \in M \) is called a strongly irreducible operator relative to \( M \) if \( \{T\}' \cap M = \mathbb{C}1 \). In this section we will give explicit examples of strongly irreducible operators in hyperfinite II\(_1\) factors.

Let \( A_\theta \) be the universal irrational rotation \( C^* \)-algebra with two unitary generators \( u, v \) such that \( vu = e^{2\pi i \theta} uv \). Then there exists a unique trace \( \tau \) on \( A_\theta \). Applying the GNS-construction to \( \tau \), we may assume that \( A_\theta \) acts on \( L^2(A_\theta, \tau) \). Let \( R \) be the strong operator closure of \( A_\theta \). Then \( R \) is the hyperfinite type II\(_1\) factor with a unique trace \( \tau \). Recall that \( u, v \) in \( R \) satisfy the following properties:
1. \( \tau(u^n) = \tau(v^n) = 0 \) for all integers \( n \neq 0 \);

2. \( vu = e^{2\pi i \theta} u v \);

3. \( \{ u^m v^n : m, n \in \mathbb{Z} \} \) is an orthonormal basis of \( L^2(R) = L^2(R, \tau) \), where \( u^m v^n \) is viewed as an element of \( L^2(R) \).

The following theorem is the main result of this section.

**Theorem 10.2.** For every irrational number \( \theta \in (0, 1) \), \( u + v \) is a strongly irreducible operator relative \( R \), i.e., there exists no nontrivial idempotents in \( \{ u + v \}' \cap R \).

**Proof.** Let \( x \in \{ u + v \}' \cap R \). By condition 3 above Theorem 10.2 \( x = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n \) and

\[
\sum_{m,n \in \mathbb{Z}} |\alpha_{m,n}|^2 = \tau(x^* x) < \infty.
\]

By condition 2 above Theorem 10.2

\[
(u + v) x = (u + v) \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i m \theta} u^m v^{n+1} \tag{10.1}
\]

and

\[
x(u + v) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n (u + v) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i n \theta} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^{n+1}. \tag{10.2}
\]

By condition 3 above Theorem 10.2 \( \{ u^m v^n : m, n \in \mathbb{Z} \} \) is an orthonormal basis of \( L^2(R) \). Comparing the coefficients of the term \( u^m v^n \) in (10.1) and (10.2), we have

\[
\alpha_{m-1,n} + \alpha_{m,n-1} e^{2\pi i m \theta} = \alpha_{m-1,n} e^{2\pi i n \theta} + \alpha_{m,n-1},
\]

which is equivalent to

\[
\alpha_{m-1,n} (1 - e^{2\pi i n \theta}) = \alpha_{m,n-1} (1 - e^{2\pi i m \theta}). \tag{10.3}
\]

Since \( \theta \) is an irrational number, \( 1 - e^{2\pi i k \theta} \neq 0 \) for \( k \neq 0 \). Let \( n = 0 \) in equation (10.3). We have \( \alpha_{m,-1} = 0 \) for \( m \neq 0 \). Let \( n = -1 \) in equation (10.3). We have \( \alpha_{m,-2} = 0 \) for \( m \neq 0, m \neq 1 \). In general, let \( n = -k \) in equation (10.3). We have \( \alpha_{m,-k-1} = 0 \) for \( m \neq 0, \cdots, m \neq k \). On the other hand, let \( m = 0 \) in equation (10.3). We have \( \alpha_{-1,n} = 0 \) for \( n \neq 0 \). Similarly, in general we have \( \alpha_{-k-1,n} = 0 \) for \( n \neq 0, \cdots, n \neq k \). So we have \( \alpha_{m,n} = 0 \) if either both \( m < 0 \) and \( n < 0 \) or \( m = -n \neq 0 \).

The motivation of the following part is to prove that \( \alpha_{m,n} = 0 \) if either \( m < 0 \) or \( n < 0 \). We only need to show that \( \alpha_{m,-k-m} = 0 \) and \( \alpha_{-k-m,m} = 0 \) for \( k \geq 1 \) and \( m \geq 0 \). Repeat using equation (10.3), we have

\[
\alpha_{m,-k-m} = \alpha_{0,-k} \frac{1 - e^{-2\pi i k \theta}}{1 - e^{2\pi i \theta}} \cdot \frac{1 - e^{-2\pi i (k+1) \theta}}{1 - e^{2\pi i 2\theta}} \cdots \frac{1 - e^{-2\pi i (k+m-1) \theta}}{1 - e^{2\pi i m \theta}}. \tag{10.4}
\]
and
\[ \alpha_{-k-m,m} = \alpha_{-k,0} \frac{1 - e^{-2\pi i k \theta}}{1 - e^{2\pi i \theta}} \frac{1 - e^{-2\pi i (k+1) \theta}}{1 - e^{2\pi i 2\theta}} \cdots \frac{1 - e^{-2\pi i (k+m-1) \theta}}{1 - e^{2\pi i m\theta}} \] (10.5)
for \( m \geq 0 \) and \( k \geq 1 \).

Let \( k = 1 \) in equation (10.4). We have
\[ |\alpha_{m,-1-m}| = |\alpha_{0,-1}| \frac{|1 - e^{-2\pi i (m+1) \theta}|}{|1 - e^{2\pi i \theta}|} \frac{|1 - e^{-2\pi i (m+2) \theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i (m+k-1) \theta}|}{|1 - e^{2\pi i (k-1) \theta}|} = |\alpha_{0,-1}|. \] (10.6)

In general for \( k \geq 2 \) and \( m \geq 0 \),
\[ |\alpha_{m,-k-m}| = |\alpha_{0,-k}| \frac{|1 - e^{-2\pi i (m+1) \theta}|}{|1 - e^{2\pi i \theta}|} \frac{|1 - e^{-2\pi i (m+2) \theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i (m+k-1) \theta}|}{|1 - e^{2\pi i (k-1) \theta}|}. \] (10.7)

To prove \( \alpha_{0,-k} = 0 \) and therefore \( \alpha_{m,-k-m} = 0 \) (by equation (10.4)), we use the following fact:
\[ \sum_{m,n} |\alpha_{m,n}|^2 < +\infty \Rightarrow \sum_{m \geq 0} |\alpha_{m,-k-m}|^2 < +\infty \Rightarrow \lim_{m \to +\infty} |\alpha_{m,-k-m}| = 0. \] (10.8)

If \( k = 1 \), then \( |\alpha_{m,-1-m}| = |\alpha_{0,-1}| \) by (10.6). By (10.8), we have \( |\alpha_{m,-1-m}| = |\alpha_{0,-1}| = 0 \) for all \( m \geq 0 \).

To prove the general case, we need to use a property of irrational rotation. Namely, there exists a sequence of increasing integers \( m_n \) such that
\[ \lim_{n \to +\infty} e^{2\pi i m_n \theta} = 1. \]

Now for each fixed \( k \geq 2 \), by (10.8) and equation (10.7),
\[
0 = \lim_{n \to +\infty} |\alpha_{m_n,-k-m_n}|
= \lim_{n \to +\infty} |\alpha_{0,-k}| \frac{|1 - e^{-2\pi i (m_n+1) \theta}|}{|1 - e^{2\pi i \theta}|} \frac{|1 - e^{-2\pi i (m_n+2) \theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i (m_n+k-1) \theta}|}{|1 - e^{2\pi i (k-1) \theta}|}
= |\alpha_{0,-k}| \frac{|1 - e^{-2\pi i \theta}|}{|1 - e^{2\pi i \theta}|} \frac{|1 - e^{-2\pi i 2\theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i (k-1) \theta}|}{|1 - e^{2\pi i (k-1) \theta}|}
= |\alpha_{0,-k}|.
\]

By equation (10.7), \( \alpha_{m,-k-m} = |\alpha_{0,-k}| = 0 \) for all \( m \geq 0 \) and \( k \geq 1 \). By equation (10.5) and similar arguments, \( |\alpha_{-k-m,m}| = |\alpha_{-k,0}| = 0 \) for all \( m \geq 0 \) and \( k \geq 1 \).

Above all, we have proved that \( \alpha_{m,n} = 0 \) if either \( m < 0 \) or \( n < 0 \). Hence
\[ x = \sum_{m \geq 0, n \geq 0} \alpha_{m,n} u^m v^n. \]

For \( k \geq 0 \), let \( x_k = \sum_{m \geq 0, n \geq 0, m+n=k} \alpha_{m,n} u^m v^n \). Then \( x = \sum_{k=0}^{\infty} x_k \) as a vector in \( L^2(R) \). Since \( x \in \{ u+v \}' \cap R \) and \( \{ u^m v^n : m, n \in \mathbb{Z} \} \) is an orthonormal basis of \( L^2(R) \), \( x_k \in \{ u+v \}' \cap R \). By
equation (10.3), \(\alpha_{m,k-m}\) is uniquely determined by \(\alpha_{0,k}\) for \(0 \leq k \leq m\). Since \((u + v)^k\) commutes with \(u + v\), \(x_k = \lambda_k (u + v)^k\) for some complex number \(\lambda_k\). This implies that \(x = \sum_{k=0}^{\infty} \lambda_k (u + v)^k\) and the decomposition is unique.

Suppose \(x \in \{u + v\}' \cap R\). Let \(x^2 = \sum_{k=0}^{\infty} \sigma_k (u + v)^k\). For \(a, b \in R\), let \(\langle a, b \rangle = \tau(b^*a)\). Then

\[
\sigma_k = \langle x^2, u^k \rangle = \langle x, x^* u^k \rangle = \left\langle \sum_{j=0}^{\infty} \lambda_j (u + v)^j, \sum_{j=0}^{\infty} \tilde{\lambda}_j ((u + v)^*)^j u^k \right\rangle = \sum_{j=0}^{k} \lambda_j \lambda_{k-j}, \quad \forall k \geq 0. \tag{10.9}
\]

If \(x^2 = x\), then \(\lambda_k = \sigma_k\) for all \(k\). Let \(k = 0\). Then (10.9) implies that \(\lambda_0 = \lambda_0^2\). So \(\lambda_0 = 0\) or \(\lambda_0 = 1\). By considering \(1 - x\), we may assume that \(\lambda_0 = 0\). Let \(k = 1\). Then (10.9) implies that \(\lambda_1 = \lambda_0 \cdot \lambda_1 + \lambda_1 \cdot \lambda_0 = 0\). By (10.9) and induction, we have \(\lambda_k = 0\) for all \(k \geq 0\). This implies that \(x = 0\), which completes the proof. \(\square\)

By the Riesz spectral decomposition theorem, we immediately have the following corollary.

**Corollary 10.3.** For every irrational number \(\theta \in (0,1)\), the spectrum of \(u + v\) is connected.

**Remark 10.4.** By the proof of Theorem 10.2, every operator in the commutant algebra of \(u + v\) can be written as a formal series \(\sum_{n=0}^{\infty} a_n (u + v)^n\). A similar argument can show that for \(0 < \lambda < 1\), every operator in the commutant algebra of \(u + \lambda v\) can be written as a formal series \(\sum_{n=\infty}^{\infty} a_n (u + \lambda v)^n\).

In the following, we will construct more examples of strongly irreducible operators relative to the hyperfinite type II\(_1\) factor. Precisely, we will prove the following result.

**Proposition 10.5.** For \(\theta\) in a second category subset of \([0,1]\), we have \(u + v^k\) is strongly irreducible relative to \(R\) for all \(k = 1, 2, \ldots\).

To prove Proposition 10.5 we need the following lemma.

**Lemma 10.6.** Let

\[
f_{s,r,k}(z) = \prod_{t=1}^{s} \frac{1 - z^{kt+r}}{1 - z^{kt}}, \quad E_{r,k} = \{ z \in \mathbb{T} : \lim_{s \to +\infty} f_{s,r,k}(z) = 0 \},
\]

where \(k\) and \(r\) are positive integers such that \(k \geq 2\) and \(r \not\equiv 0 \mod k\). Then \(E_{r,k}\) is a first category subset of \(\mathbb{T}\).

**Proof.** Let \(\epsilon > 0\). Note that \(f_{s,r,k}(z)\) is a meromorphic function with finite poles on \(\mathbb{T}\). So the set

\[
D_{s,r,k,\epsilon} \triangleq \{ z \in \mathbb{T} : |f_{s,r,k}(z)| \leq \epsilon \}
\]

is a closed subset of \(\mathbb{T}\). Let

\[
E_{s,r,k,\epsilon} \triangleq \{ z \in \mathbb{T} : |f_{s,r,k}(z)| \leq \epsilon, \forall a \geq s \}.
\]
Then \( E_{s,r,k,\epsilon} = \bigcap_{a \geq s} D_{a,r,k,\epsilon} \) is also a closed subset of \( \mathbb{T} \).

Let \( F_{s,r,k,\epsilon} = \mathbb{T} \setminus E_{s,r,k,\epsilon} \). Then \( F_{s,r,k,\epsilon} \) is an open subset of \( \mathbb{T} \), and

\[
F_{s,r,k,\epsilon} = \mathbb{T} \setminus \bigcap_{a \geq s} D_{a,r,k,\epsilon} \\
= \bigcup_{a \geq s} (\mathbb{T} \setminus D_{a,r,k,\epsilon}) \\
= \bigcup_{a \geq s} \{ z \in \mathbb{T} : |f_{a,r,k}(z)| > \epsilon \} \\
\supseteq \bigcup_{a \geq s} \{ \text{poles of } f_{a,r,k}(z) \} \\
= \bigcup_{a \geq s} \{ z : z^{ak} = 1 \}.
\]

So \( F_{s,r,k,\epsilon} \) is a dense open subset of \( \mathbb{T} \), which implies that \( E_{s,r,k,\epsilon} \) is a nowhere dense closed subset of \( \mathbb{T} \). Therefore, \( E_{r,k} \subseteq \bigcup_{s=1}^{\infty} E_{s,r,k,\epsilon} \) is a first category subset of \( \mathbb{T} \). □

**Proof of Proposition 10.5** Define \( f_{s,r,k}(z) \) and \( E_{r,k} \) as in Lemma 10.6. By Lemma 10.6, \( E_{r,k} \) is a first category subset of \( \mathbb{T} \). So \( \bigcup_{r \neq 0 \text{ (mod } k)} E_{r,k} \) is also a first category subset of \( \mathbb{T} \). Hence

\[
\mathbb{T} \setminus \{ e^{2\pi i \theta} : \theta \in [0,1] \text{ is a rational number} \} \setminus \bigcup_{r \neq 0 \text{ (mod } k)} E_{r,k}
\]

is a second category subset of \( \mathbb{T} \). Choose a \( \theta \in [0,1] \) such that \( e^{2\pi i \theta} \) is in the above set. Then for all \( r \) with \( r \neq 0 \mod k \),

\[
\lim_{s \to +\infty} f_{s,r,k}(z) = 0
\]
does not hold.

Let \( x \in \{ u+v^k \} \cap \mathbb{R} \) be an idempotent. By condition 3 above Theorem 10.2, \( x = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n \) and \( \sum_{m,n \in \mathbb{Z}} |\alpha_{m,n}|^2 = \tau(x^*x) < \infty \). By condition 2 above Theorem 10.2

\[
(u+v^k)x = (u+v^k) \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i km \theta} u^m v^{n+k} \quad (10.10)
\]

and

\[
x(u+v^k) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n (u+v) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i \theta} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^{n+k}. \quad (10.11)
\]

By condition 3 above Theorem 10.2, \( \{ u^m v^n : m, n \in \mathbb{Z} \} \) is an orthonormal of \( L^2(\mathbb{R}) \). Comparing the coefficients of the term \( u^m v^n \) in (10.10) and (10.11), we have

\[
\alpha_{m-1,n} + \alpha_{m,n-k} e^{2\pi i (km) \theta} = \alpha_{m-1,n} e^{2\pi i \theta} + \alpha_{m,n-k},
\]
which is equivalent to
\[
\alpha_{m-1,n}(1 - e^{2\pi i \theta}) = \alpha_{m,n-k}(1 - e^{2\pi i (km)\theta}).
\] (10.12)

Since \( \theta \) is an irrational number, \( 1 - e^{2\pi i k\theta} \neq 0 \) for \( k \neq 0 \). Let \( n = 0 \) in equation (10.12). We have \( \alpha_{m,-k} = 0 \) for \( m \neq 0 \). Let \( n = -k \) in equation (10.12). We have \( \alpha_{m,-2k} = 0 \) for \( m \neq 0, m \neq 1 \). In general, let \( n = -sk \) in equation (10.12). We have \( \alpha_{m,-(s+1)k} = 0 \) for \( m \neq 0, \cdots, m \neq s \). On the other hand, let \( m = 0 \) in equation (10.12). We have \( \alpha_{-1,n} = 0 \) for \( n \neq 0 \). Similarly, in general we have \( \alpha_{-s-1,n} = 0 \) for \( n \neq 0, k, \cdots, sk \).

Claim that \( \alpha_{m,n} = 0 \) if either \( m < 0 \) or \( n < 0 \). By the above arguments, we only need to show that \( \alpha_{0,-r} = 0 \) and \( \alpha_{-r,0} = 0 \) for \( r \geq 1 \). Firstly, we show that \( \alpha_{-r,0} = 0 \) for \( r \geq 1 \).

In equation (10.12), let \( m = -r \) and \( n = k \). We have
\[
\alpha_{-r-1,k}(1 - e^{2\pi i (k)\theta}) = \alpha_{-r,0}(1 - e^{-2\pi i (kr)\theta}).
\] (10.13)

In equation (10.12), let \( m = -r - 1 \) and \( n = 2k \). We have
\[
\alpha_{-r-2,2k}(1 - e^{2\pi i (2k)\theta}) = \alpha_{-r-1,k}(1 - e^{-2\pi i (k(r+1))\theta}).
\] (10.14)

In general, for a positive integer \( s \), let \( m = -r - s + 1 \) and \( n = (s-1)k \) in equation (10.12). We have
\[
\alpha_{-r-s,sk}(1 - e^{2\pi i (sk)\theta}) = \alpha_{-r-s+1,k}(1 - e^{-2\pi i (k(r+s-1))\theta}).
\] (10.15)

By equations (10.13), (10.14), (10.15),
\[
\alpha_{-r-s,sk} = \alpha_{-r,0} \prod_{t=1}^{s} \frac{1 - e^{-2\pi i (k(r+s-t))\theta}}{1 - e^{2\pi i (tk)\theta}}.
\]

So for \( s > r - 1 \), we have
\[
|\alpha_{-r-s,sk}| = |\alpha_{-r,0}| \prod_{t=1}^{r-1} \left| \frac{1 - e^{-2\pi i ((r+s)k\theta)} e^{2\pi i (tk)\theta}}{1 - e^{2\pi i (tk)\theta}} \right|.
\] (10.16)

Since \( \theta \) is an irrational number, there is a sequence positive integers \( s_n \) such that
\[
\lim_{n \to \infty} e^{-2\pi i (r+s_n)k\theta} = 1.
\]

By equation (10.16), \( \lim_{n \to \infty} |\alpha_{-r-s_n,s_n,k}| = |\alpha_{-r,0}| \). Since \( \sum_{n=1}^{\infty} |\alpha_{-r-s_n,s_n,k}|^2 < \infty \), \( |\alpha_{-r,0}| = \lim_{n \to \infty} |\alpha_{-r-s_n,s_n,k}| = 0 \).

Secondly, we show that \( \alpha_{0,-r} = 0 \) for \( r \geq 1 \). In equation (10.12), let \( m = 1 \) and \( n = -r \). We have
\[
\alpha_{0,-r}(1 - e^{2\pi i (-r)\theta}) = \alpha_{1,-r-k}(1 - e^{2\pi i (k)\theta}).
\] (10.17)
In equation (10.12), let \( m = 2 \) and \( n = -r - k \). We have
\[
\alpha_{1,-r-k}(1 - e^{2\pi i(-r-k)\theta}) = \alpha_{2,-r-2k}(1 - e^{2\pi i(2k)\theta}).
\] (10.18)

In general, for a positive integer \( s \), let \( m = s + 1 \) and \( n = -r - (s - 1)k \) in equation (10.12). We have
\[
\alpha_{0,-r-(s-1)k}(1 - e^{2\pi i(-r-(s-1)k)\theta}) = \alpha_{s,-r-sk}(1 - e^{2\pi i(sk)\theta}).
\] (10.19)

By equations (10.17), (10.18), (10.19),
\[
\alpha_{s,-r-sk} = \alpha_{0,-r} \prod_{t=1}^{s} \frac{1 - e^{-2\pi i(tk+r-k)\theta}}{1 - e^{2\pi i(tk)\theta}}.
\] (10.20)

We consider two cases. Case 1: \( r = 0 \pmod{k} \). By equation (10.20),
\[
\alpha_{s,-r-sk} = \alpha_{0,-r} \prod_{t=1}^{r-k} \frac{1 - e^{-2\pi i(sk+r)\theta}}{1 - e^{2\pi i(tk)\theta}}
\] (10.21)

for \( s > \frac{r-k}{k} \). Since \( \theta \) is an irrational number, there is a sequence positive integers \( s_n \) such that
\[
\lim_{n \to \infty} e^{-2\pi i(s_n k + r)\theta} = 1.
\]

By equation (10.21), \( \lim_{n \to \infty} \vert \alpha_{s_n,-r-s_nk} \vert = \vert \alpha_{0,-r} \vert \). Since \( \sum_{n=1}^{\infty} \vert \alpha_{s_n,-r-s_nk} \vert ^2 < \infty \),
\[
\vert \alpha_{0,-r} \vert = \lim_{n \to \infty} \vert \alpha_{s_n,-r-s_nk} \vert = 0.
\]

Case 2: \( r \neq 0 \pmod{k} \). Note that \( \sum_{s=1}^{\infty} \vert \alpha_{s,-r-sk} \vert ^2 < \infty \). So \( \lim_{s \to \infty} \alpha_{s,-r-sk} = 0 \). By the choice of \( \theta \),
\[
\lim_{s \to \infty} \prod_{t=1}^{s} \frac{1 - e^{-2\pi i(tk+r-k)\theta}}{1 - e^{2\pi i(tk)\theta}} = 0
\]
does not hold. So \( \alpha_{0,-r} \) has to be 0.

Above all, we have proved that \( \alpha_{m,n} = 0 \) if either \( m < 0 \) or \( n < 0 \). Furthermore, we claim that \( \alpha_{m,n} = 0 \) for \( m, n \geq 0 \) and \( n \neq 0 \pmod{k} \). Let \( s \) be the least positive integer greater than \( n/k \). By equation (10.12), we have
\[
\alpha_{m,n}(1 - e^{2\pi i n\theta}) = \alpha_{m+1,n-k}(1 - e^{2\pi i (m+1)\theta}),
\]
\[
\alpha_{m+1,n-k}(1 - e^{2\pi i (n-k)\theta}) = \alpha_{m+2,n-2k}(1 - e^{2\pi i (m+2)\theta}),
\]
\[
\vdots
\]
\[
\alpha_{m+s-1,n-(s-1)k}(1 - e^{2\pi i (n-(s-1)k)\theta}) = \alpha_{m+s,n-sk}(1 - e^{2\pi i (m+s)\theta}).
\]
Since \( n - sk < 0 \), \( \alpha_{m+s,n-sk} = 0 \). The above equations imply that \( \alpha_{m,n} = 0 \) since \( 1 - e^{2\pi i (n-jk)\theta} \neq 0 \) for all \( j \).
Hence

\[ x = \sum_{m \geq 0, n \geq 0} \alpha_{m,n} u^m v^{kn}, \]

which implies that \( x \) is in the commutant algebra of \( u + v^k \) relative to the von Neumann subalgebra generated by \( u \) and \( v^k \). Since \( v^k u = e^{2\pi ik\theta} v^k u \) and \( k\theta \) is an irrational number, \( x = 0 \) or \( x = 1 \) by Theorem \( 10.2 \). So \( T \) is a strongly irreducible operator relative to \( R \). This completes the proof of Theorem \( 10.5 \).

**Proposition 10.7.** Let \( n \) be a positive integer. Then by Theorem \( 8.2 \), \( N = W^*(u + v^n) = W^*(u, v^n) \) is an irreducible subfactor of \( W^*(u + v) = R \) with Jones index \( [27] [R : N] = n. \)

**Proof.** Since \( R \) is generated by \( u, v \) and \( N \) is generated by \( u, v^n \), it is clear that every element of \( R \) can be written as finite linear combinations of elements in \( Nv^i, 0 \leq i \leq n - 1 \). Since \( Nv^i \) is orthogonal to \( Nv^j, 0 \leq i \neq j \leq n - 1 \), under the inner product defined by the trace on \( R \), it follows that \( R = N \oplus Nv \oplus Nv^2 \oplus \cdots \oplus Nv^{n-1} \), where \( Nv^i \) is orthogonal to \( Nv^j, 0 \leq i \neq j \leq n - 1 \). So by \( 30 \), \( v^i, 0 \leq i \leq n - 1 \) is a Pimsner-Popa basis of \( R \) over \( N \), and since \( v^i \) is unitary, it follows that \([R : N] = n\). \( \square \)

On the other hand, by Proposition \( 10.5 \) for \( \theta \) in a second category subset of \([0, 1]\), \( u + v^n \) is strongly irreducible relative to \( R \). So for every bounded invertible operator \( x \in R \), \( x(u + v^n)x^{-1} \) generates an irreducible subfactor \( W^*(x(u + v^n)x^{-1}) \) of \( R \). Is it true that \([R : W^*(x(u + v^n)x^{-1})] = n \) for all bounded invertible operators \( x \) in \( R \), at least when \( x \) is close to identity in norm?

By definitions if \( T \) is strongly irreducible relative to \( M \), then \( T \) is irreducible relative to \( M \). An operator \( T \) is strongly irreducible relative to a type \( \text{II}_1 \) factor if and only if \( XTX^{-1} \) is an irreducible operator relative to \( M \) for every bounded invertible operator \( X \in M \). However, if \( T \) is irreducible relative to \( M \), this is not true in general. The following result shows that an irreducible operator relative to \( M \) can be similar to a unitary operator.

**Proposition 10.8.** Let \( \theta \) be an irrational number in \([0, 1]\) and let \( n \) be any positive integer. Then in the hyperfinite type \( \text{II}_1 \) factor \( R \) there exists a bounded invertible operator \( x \) such that \( W^*(xux^{-1}) = W^*(u + v^n) = W^*(u, v^n) \).

**Proof.** Let \( \sigma \) be a nonempty open connected subset of \( \sigma(v^n) = \mathbb{T} \) such that \( \sigma \cap e^{2\pi in\theta} \sigma = \emptyset \). Let \( x = 1 - \frac{E_{\sigma}(\sigma)}{2} \in R \), where \( E_{\sigma}(\cdot) \) denotes the spectral measure of \( v^n \). Since \( v^n u = e^{2\pi in\theta} uv^n \), \( f(v^n)u = uf(e^{2\pi in\theta}v) \) for all \( f \in L^\infty(\mathbb{T}, m) \). Therefore,

\[ E_{v^n}(\sigma)u = uE_{v^n}(e^{2\pi in\theta} \sigma), \quad (10.22) \]
which implies that
\[ u^{-1} E_v^n(\sigma) = E_v^n(e^{2\pi i n \theta}) u^{-1}. \] (10.23)

Similarly, by \( v^n u^{-1} = e^{-2\pi i n \theta} u^{-1} v^n \), we have
\[ u E_v^n(\sigma) = E_v^n(e^{-2\pi i n \theta}) u. \] (10.24)

Combining the above equations, we have
\[ (xux^{-1})^*(xux^{-1}) = x^{-1} u^{-1} x x u x^{-1} = \left(1 - \frac{E_v^n(\sigma)}{2}\right)^{-1} \left(1 - \frac{E_v^n(e^{2\pi i n \theta})}{2}\right)^2 \left(1 - \frac{E_v^n(\sigma)}{2}\right)^{-1}. \]

We can write
\[ 1 - \frac{E_v^n(e^{2\pi i n \theta})}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \text{Ran } E_v^n(T \setminus (e^{2\pi i n \theta} \sigma \cup \sigma)) \\ \text{Ran } E_v^n(e^{2\pi i n \theta} \sigma) \\ \text{Ran } E_v^n(\sigma) \end{array} \]
and write
\[ 1 - \frac{E_v^n(\sigma)}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \text{Ran } E_v^n(T \setminus (e^{2\pi i n \theta} \sigma \cup \sigma)) \\ \text{Ran } E_v^n(e^{2\pi i n \theta} \sigma) \\ \text{Ran } E_v^n(\sigma) \end{array} \]

So we have
\[ (xux^{-1})^*(xux^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{c} \text{Ran } E_v^n(T \setminus (e^{2\pi i n \theta} \sigma \cup \sigma)) \\ \text{Ran } E_v^n(e^{2\pi i n \theta} \sigma) \\ \text{Ran } E_v^n(\sigma) \end{array} \in W^*(xux^{-1}). \]

Therefore,
\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{array}{c} \text{Ran } E_v^n(T \setminus (e^{2\pi i n \theta} \sigma \cup \sigma)) \\ \text{Ran } E_v^n(e^{2\pi i n \theta} \sigma) \\ \text{Ran } E_v^n(\sigma) \end{array} \in W^*(xux^{-1}). \]

This implies that
\[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \text{Ran } E_v^n(T \setminus (e^{2\pi i n \theta} \sigma \cup \sigma)) \\ \text{Ran } E_v^n(e^{2\pi i n \theta} \sigma) \\ \text{Ran } E_v^n(\sigma) \end{array} \in W^*(xux^{-1}). \]

Note that
\[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{SOT}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \text{Ran } E_v^n(T \setminus (e^{2\pi i n \theta} \sigma \cup \sigma)) \\ \text{Ran } E_v^n(e^{2\pi i n \theta} \sigma) \\ \text{Ran } E_v^n(\sigma) \end{array} \]
Hence, \( E_{v^n}(\sigma) \in W^*(xux^{-1}) \) and \( x = 1 - \frac{E_{v^n}(\sigma)}{2} \in W^*(xux^{-1}) \). Therefore, \( u \in W^*(xux^{-1}) \). Note that
\[
E_{v^n}(e^{2\pi i k n \theta} \sigma) = u^{-k} E_{v^n}(\sigma) u^k \in W^*(xux^{-1}), \ \forall k \in \mathbb{N}.
\]
Since \( \{e^{2\pi i k n \theta}\}_{k=1}^{\infty} \) is dense in \( \mathbb{T} \), \( E_{v^n}(\sigma_1) \in W^*(xux^{-1}) \) for every open connected subset \( \sigma_1 \) in \( \mathbb{T} \) which has same arc length as \( \sigma \). If \( \sigma_0 \) is an open connected subset of \( \mathbb{T} \) with arc length smaller than the arc length of \( \sigma \), then there are two open connected subsets \( \sigma_1, \sigma_2 \) of \( \mathbb{T} \) with arc length same as \( \sigma \) such that \( \sigma_1 \cap \sigma_2 = \sigma_0 \). Thus
\[
E_{v^n}(\sigma_0) = E_{v^n}(\sigma_1) \cap E_{v^n}(\sigma_2) \in W^*(xux^{-1}).
\]
This implies that for every measurable subset \( F \) of \( \mathbb{T} \), we have \( E_{v^n}(F) \in W^*(xux^{-1}) \). So \( v^n \in W^*(xux^{-1}) \) and we have proved that \( W^*(xux^{-1}) = W^*(u, v^n) \). \( \square \)

In general, we have the following observation.

**Proposition 10.9.** Let \( N \subseteq M \) be an inclusion of type II\(_1\) factors. Then there exists an operator \( S \in N \) which is similar to an irreducible operator \( T \) relative to \( M \).

**Proof.** We may identify \( N = M_3(\mathbb{C}) \otimes N_1 \) and \( M = M_3(\mathbb{C}) \otimes M_1 \). Choose complex numbers \( \alpha_1, \alpha_2, \alpha_3 \) such that \( \alpha_i \neq \alpha_j \) for \( i \neq j \). Let \( D \) be an irreducible operator in \( M_1 \),

\[
S = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & \alpha_2 & 0 \\
0 & 0 & \alpha_3
\end{pmatrix}, \quad T = \begin{pmatrix}
\alpha_1 & 1 & D \\
0 & \alpha_2 & 1 \\
0 & 0 & \alpha_3
\end{pmatrix}, \quad X = \begin{pmatrix}
1 & \frac{1}{\alpha_1 - \alpha_2} & \frac{1}{\alpha_1 - \alpha_3} + \frac{D}{\alpha_1 - \alpha_3} \\
0 & 1 & \frac{1}{\alpha_2 - \alpha_3} \\
0 & 0 & 1
\end{pmatrix}.
\]

Then direct calculations show that \( T \) is an irreducible operator in \( M \) and \( X S X^{-1} = T \). \( \square \)

## 11 Spectrum of \( u + \lambda v \)

**Theorem 11.1.** For every irrational number \( \theta \in (0, 1) \),

\[
\sigma(u + \lambda v) = \begin{cases}
\mathbb{T} & 0 < \lambda < 1, \\
\overline{B(0, 1)} & \lambda = 1, \\
\lambda \mathbb{T} & \lambda > 1,
\end{cases}
\]

where \( \mathbb{T} \) is the unit circle.

**Proof.** Note that \( u + v = u(1 + u^* v) \). Since \( u^* v \) is a Haar unitary operator, \( -1 \in \sigma(u^* v) \). This implies that \( u + v \) is not invertible and therefore \( 0 \in \sigma(u + v) \). For every \( \theta \in [0, 2\pi] \), \( e^{i\theta} u \) and \( e^{i\theta} v \) satisfy the
same irrational rotation relation as $u$ and $v$, so $\sigma(u + v)$ is rotation symmetric with respect to 0. By Corollary 10.3, $\sigma(u + v)$ is a closed disk with center 0. By Lemma 9.5, $\sigma(u + v) = B(0, 1)$.

For $0 < \lambda < 1$, $u + \lambda v = u(1 + \lambda u^*)v$ is invertible. In the following we prove that $r((u + \lambda v)^{-1}) \leq 1$. The proof is similar to the proof of Lemma 9.5. However, some details should be treated carefully, so we include the complete proof. By equation (9.1),

$$(u + \lambda v)^{-n} = (1 + \lambda w)^{-1}(1 + n\lambda w)^{-1} \cdots (1 + n^{n-1}\lambda w)^{-1}u^{-n}, \quad \forall n \in \mathbb{N}.$$ 

Hence,

$$\| (u + \lambda v)^{-n} \|^{1/n} = \| (1 + \lambda w)^{-1}(1 + n\lambda w)^{-1} \cdots (1 + n^{n-1}\lambda w)^{-1} \|^{1/n}$$

$$= \left( \max_{x \in \mathbb{T}} \left| (1 + \lambda z)^{-1}(1 + n\lambda z)^{-1} \cdots (1 + n^{n-1}\lambda z)^{-1} \right| \right)^{1/n}$$

$$= \max_{x \in [0, 1]} \left( \prod_{k=0}^{n-1} \left| 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right|^{-1} \right)^{1/n}.$$ 

Let $\varepsilon > 0$. Note that

$$\lim_{n \to \infty} \frac{1}{2n} \left( \sum_{k=1}^{n} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) + \sum_{k=N}^{2n-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) \right)$$

$$= - \int_{0}^{1} \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) dx = 0.$$ 

There is $N \in \mathbb{N}$ such that

$$\frac{1}{2N} \left( \sum_{k=1}^{N} - \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \frac{2N-1}{n} - \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) < \varepsilon/2.$$ 

Let

$$L(\lambda) = \max_{1 \leq k \leq 2N-1} \left| \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right|.$$ 

Then for $0 < \lambda < 1$, $L(\lambda) < \infty$ (Note that if $\lambda = 1$, then $L(\lambda) = \infty$). Divide the unit circle $\mathbb{T}$ into $2N$ equal sections $A_1, \cdots, A_{2N}$. By Lemma 9.4, there exists $N'$ such that for all $n \geq N'$ and all $x \in [0, 1]$, if $A_k$ contains $n/(2N) + r_k(x)$ points of $e^{2\pi i x}, e^{2\pi i x}, \cdots, e^{2\pi i x}$, then $\frac{\sum_{k=1}^{2N} |r_k(x)|}{n} < \frac{\varepsilon}{L(\lambda)}$. Note that $\cos 2\pi x$ is decreasing for $x \in [0, 1/2]$ and increasing for $x \in [1/2, 1]$. Therefore, for all $x \in [0, 1]$,

$$\frac{1}{n} \sum_{k=0}^{n-1} - \ln \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right) \leq \frac{1}{n} \sum_{k=1}^{N} - \left( \frac{n}{2N} + r_k(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right)$$

$$+ \frac{1}{n} \sum_{k=N}^{2N-1} - \left( \frac{n}{2N} + r_{k+1}(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right).$$
\[
\frac{1}{2N} \left( \sum_{k=1}^{N} - \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \sum_{k=N}^{2N-1} - \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) \\
+ \frac{1}{n} \sum_{k=1}^{N} -r_k(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \frac{1}{n} \sum_{k=N}^{2N-1} -r_{k+1}(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right)
\]

\[< \epsilon + \frac{1}{n} \sum_{k=1}^{2N} |r_k(x)| L(\lambda) < 2\epsilon.\]

This implies that for all \( n \geq N' \) and \( x \in [0,1] \),

\[
\left( \prod_{k=0}^{n-1} \left( 1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)) \right)^{-1} \right)^{\frac{1}{2\pi}} \leq e^{2\epsilon}.
\]

Therefore, \( \|(u + \lambda v)^{-n}\|^{1/n} \leq e^{2\epsilon} \) for all \( n \geq N' \). So \( r ((u + \lambda v)^{-1}) \leq e^{2\epsilon} \). Since \( \epsilon > 0 \) is arbitrary, \( r ((u + \lambda v)^{-1}) \leq 1 \). By Lemma 9.5, \( r(u + \lambda v) = 1 \) for \( 0 < \lambda < 1 \). This implies that \( \sigma(u + \lambda v) \subseteq \mathbb{T} \).

Since \( \sigma(u + \lambda v) \) is rotation invariant, \( \sigma(u + \lambda v) = \mathbb{T} \).

If \( \lambda > 1 \), then \( \sigma(u + \lambda v) = \lambda \sigma(\lambda^{-1}u + v) = \lambda \mathbb{T} \). This completes the proof.

\[\square\]

### 12 Brown’s spectral distribution of \( u + \lambda v \)

Let \( M \) be a finite von Neumann algebra with a faithful normal tracial state \( \tau \). The Fuglede-Kadison determinant \([14]\), \( \Delta : M \to [0, +\infty[ \), is given by

\[
\Delta(T) = \exp\{\tau(\ln|T|)\}, \quad T \in M,
\]

with \( \exp\{-\infty\} := 0 \). For an arbitrary element \( T \) in \( M \) the function \( \lambda \to \ln \Delta(T - \lambda 1) \) is subharmonic on \( \mathbb{C} \), and its Laplacian

\[
d\mu_T(\lambda) := \frac{1}{2\pi} \nabla^2 \ln \Delta(T - \lambda 1),
\]

in the distribution sense, defines a probability measure \( \mu_T \) on \( \mathbb{C} \), called the Brown’s spectral distribution or Brown measure of \( T \). From the definition, Brown measure \( \mu_T \) only depends on the joint distribution of \( T \) and \( T^* \), i.e., the (noncommutative) mixed moments of \( T \) and \( T^* \).

If \( T \) is normal, then \( \mu_T \) is the trace \( \tau \) composed with the spectral projections of \( T \). If \( M = M_n(\mathbb{C}) \), then \( \mu_T \) is the normalized counting measure \( \frac{1}{n} (\delta_{\lambda_1} + \delta_{\lambda_2} + \cdots + \delta_{\lambda_n}) \), where \( \lambda_1, \lambda_2, \cdots, \lambda_n \) are the eigenvalues of \( T \) repeated according to root multiplicity.

The following theorem is Theorem 2.2 of \([17]\).
**Theorem 12.1.** Let $T \in M$, and for $n \in \mathbb{N}$, let $\mu_n \in \text{Prob}([0,\infty))$ denote the distribution of $(T^n)^*T^n$ w.r.t $\tau$, and let $\nu_n$ denote the push-forward measure of $\mu_n$ under the map $t \to t^{\frac{1}{n}}$. Moreover, let $\nu$ denote the push-forward measure of $\mu_T$ under the map $z \to |z|^2$, i.e., $\nu$ is determined by

$$\nu([0,t^2]) = \mu_T(B(0,t)), \quad t > 0.$$  

Then $\nu_n \to \nu$ weakly in $\text{Prob}([0,\infty))$.

**Theorem 12.2.** The Brown measure of $u + \lambda v$ is the Haar measure on the unit circle $\mathbb{T}$ if $0 < \lambda \leq 1$ and the Haar measure on $\lambda\mathbb{T}$ if $\lambda > 1$.

**Proof.** By Theorem 11.1, $\sigma(u + \lambda v) = \mathbb{T}$ if $0 < \lambda < 1$ and $\sigma(u + \lambda v) = \lambda\mathbb{T}$ if $\lambda > 1$. Since $\mu_{(u+\lambda v)}$ is rotation invariant and the support of $\mu_{(u+\lambda v)}$ is contained in $\sigma(u + \lambda v)$, the Brown measure of $u + \lambda v$ is the Haar measure on the unit circle $\mathbb{T}$ if $0 < \lambda < 1$ and the Haar measure on $\lambda\mathbb{T}$ if $\lambda > 1$.

In the following, we consider the case $\lambda = 1$. Let $T = u + v$, and let $\nu$ and $\nu_n$ be the measures defined as in Theorem 12.1. Note that $((T^n)^*T^n)_{\frac{1}{n}} = |(1 + w) \cdots (1 + \alpha^{n-1}w)|^{\frac{2}{n}}$, where $w = u^*v$ is a Haar unitary operator. So we can view $((T^n)^*T^n)_{\frac{1}{n}}$ as the multiplication operator on $L^2[0,1]$ corresponding to the function

$$\prod_{k=0}^{n-1} \left| 2 + 2 \cos(2\pi(x + k\theta)) \right|^{\frac{1}{n}}.$$  

Let $m$ be the Lebesgue measure on $[0,1]$. For $0 < b < 1$, since $[0,b)$ is an open set relative to $[0,\infty)$ and $\nu_n \to \nu$ weakly in $\text{Prob}([0,\infty))$ (by Theorem 12.1),

$$\nu([0,b)) \leq \liminf_{n \to \infty} \nu_n([0,b)) = \liminf_{n \to \infty} m \left( \left\{ x : \prod_{k=0}^{n-1} \left| 2 + 2 \cos(2\pi(x + k\theta)) \right|^{\frac{1}{n}} \in [0,b) \right\} \right).$$  

By Lemma 9.2, for almost all $x \in [0,1]$,

$$\lim_{n \to \infty} \prod_{k=0}^{n-1} \left| 2 + 2 \cos(2\pi(x + k\theta)) \right|^{\frac{1}{n}} = 1.$$  

In particular, $\left| \prod_{k=0}^{n-1} \left| 2 + 2 \cos(2\pi(x + k\theta)) \right|^{\frac{1}{n}} \right|$ converges in measure to the constant function 1 on $[0,1]$. Since $b < 1$, $\nu([0,b)) = 0$. Let $r'(u + v)$ be the Brown spectral radius of $u + v$. Then $r'(u + v) \leq r(u + v) = 1$ (see 17, Corollary 2.6). So the support of $\nu$ is contained in $[0,1]$. Thus $\nu$ is the Dirac measure $\delta_1$ and the support of $\mu_T$ is contained in $\mathbb{T}$. Since $\mu_T$ is rotation invariant, $\mu_T$ is the Haar measure on $\mathbb{T}$.  \qed
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