A SPECTRAL IDENTITY FOR SECOND MOMENTS OF EISENSTEIN SERIES OF $O(n,1)$

JOÃO PEDRO BOAVIDA

Abstract. Let $H = O(n) \times O(1)$ be an anisotropic subgroup of $G = O(n,1)$ and let $\mathbb{A}$ be the adele ring of $k = \mathbb{Q}$. Consider the periods

$$(E_\varphi, F)_H = \int_{H_k \backslash H_\mathbb{A}} E_\varphi \cdot F,$$

of an Eisenstein series $E_\varphi$ on $G$ against a form $F$ on $H$. Relying on a variant of Levi–Sobolev spaces, we describe certain Poincaré series as fundamental solutions for the Laplacian, and use them to establish a spectral identity concerning the second moments (in $F$-aspect) of $E_\varphi$.

Introduction

Let $k = \mathbb{Q}$. Consider the form represented by

$$
\begin{pmatrix}
1 & \text{id} \\
\text{id} & -1
\end{pmatrix}
$$

(here and elsewhere, omitted entries are zero) with respect to the decomposition $k^{n+1} = k^n \oplus (k \cdot e_-) = (k \cdot e_+) \oplus k^{n-1} \oplus (k \cdot e_-)$. Let $G = O(n+1)$, $H = O(n) \times O(1)$, and $\Theta = O(n-1)$, and note that $H$ and $\Theta$ are $k$-anisotropic.

As the form is isotropic, we consider the hyperbolic pair $e' = \frac{1}{2}e_+ - \frac{1}{2}e_-$ and $e = e_+ + e_-$. Changing coordinates, we see the form is represented by

$$
\begin{pmatrix}
1 & \text{id} \\
1 & -1
\end{pmatrix}
$$

Received August 2, 2013; received in final form May 1, 2014.

2010 Mathematics Subject Classification. Primary 11F67. Secondary 11E45, 11F72, 43A85, 43A90, 46E35.
with respect to $(k \cdot e') \oplus k^{n-1} \oplus (k \cdot e)$. We use these new coordinates for the remainder of the Introduction, and observe that while $H$ has no simple description in these coordinates, $\Theta$ can still be identified with $O(n - 1)$.

Write
\[
m_\lambda = \begin{pmatrix} \lambda & \text{id} \\ \lambda^{-1} \end{pmatrix} \quad \text{and} \quad n_a = \begin{pmatrix} 1 & a & -\frac{1}{2}aa^t \\ & \text{id} & a^t \\ & & 1 \end{pmatrix}.
\]
The parabolic $P$ stabilizing the isotropic line $k \cdot e$ can be written as $P = NM$, with unipotent radical $N = \{n_a\}$ and Levi component $M = \{m_\lambda\} \cdot \Theta$. The modular function on $P$ is given by $\delta_P(m_\lambda) = |\lambda|^n$.

Let $A$ be the adele ring of $k = \mathbb{Q}$. At non-archimedean $v$, choose a maximal (open) compact $K_v$. At the archimedean place $v = \infty$, put $K_\infty = H_\infty$; it is a maximal compact in $G_\infty$. Write $K = \prod K_v$; it is a maximal compact in $G_A$. Let us recapitulate briefly the most salient points about the spectral decomposition of (right $K$-invariant) functions in $L^2(G_k \backslash G_A / K)$.

The constant term of $f \in L^2(G_k \backslash G_A / K)$ is
\[
cf(g) = \int_{N_k \backslash N_k} f(ng) \, dn.
\]
We say $f$ is a cuspform if $\cf = 0$; the space $L^2_0(G_k \backslash G_A / K)$ of (right $K$-invariant) cuspforms decomposes discretely [18] into joint eigenfunctions of the center $\mathcal{Z}(g_\infty)$ of the universal enveloping algebra.

The constant term $\cf$ is left $N_k M_k$-invariant. If $\varphi \in \mathcal{D}(N_k M_k \backslash G_A / K)$ is a test function, we have
\[
\int_{N_k M_k \backslash G_A / K} \cf(g) \varphi(g) \, dg = \int_{N_k M_k \backslash G_A / K} f(ng) \, dn \varphi(g) \, dg
= \int_{P_k \backslash G_A / K} f(g) \varphi(g) \, dg
= \int_{G_k \backslash G_A / K} f(g) E_\varphi(g) \, dg,
\]
where
\[
E_\varphi(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma g)
\]
(the sum has finitely many non-zero terms) is a pseudo-Eisenstein series. Observing that $N_k M_k = M_k N_k$ and taking the Iwasawa decomposition $G_A = N_k M_k K$ into account, we see the right $K$-invariant functions on $N_k M_k \backslash G_A$ are the right $K \cap M_k -$invariant functions on $M_k \backslash G_k$.

Recall that $M \cong \Theta \times GL(1)$ and that, because $\Theta$ is $k$-anisotropic, $\Theta_k \backslash \Theta_A$ is compact. Let $\Psi$ run over an orthonormal basis of $L^2(\Theta_k \backslash \Theta_A / (K \cap \Theta_A))$. 

Let also \( \lambda \mapsto \delta_P(\lambda)^s \) be a character of \( \text{GL}(1) \) (with \( k = \mathbb{Q} \), there are no other characters to account for). Extend

\[
\varphi_s, \Psi(m \lambda \theta) = \delta_P(\lambda)^s \cdot \Psi(\theta) = |\lambda|^{ns} \cdot \Psi(\theta)
\]

by left \( N_\mathbb{A} \)- and right \( K \)-invariance, and define the Eisenstein series as the meromorphic continuation of

\[
E_{s, \Psi}(g) = \sum_{\gamma \in \mathcal{P} \setminus G} \varphi_{s, \Psi}(\gamma g)
\]

to all \( \mathbb{C} \). (We will not go into the details, but the sum converges if \( \Re s > 1 \) and does have a meromorphic extension [31].) If \( \Psi = 1 \), we write simply \( \varphi_s = \varphi_{s, \Psi} \) and \( E_s = E_{s, \Psi} \).

Given a function \( f \) in \( L^2(G_k \setminus G_\mathbb{A}/K) \), we have [1], [31], [37]

\[
f = \sum_{\Phi} \langle f, \Phi \rangle \cdot \Phi + \frac{1}{4\pi i} \sum_{\Psi} \int_{\Re s = \frac{1}{2}} \langle f, E_{s, \Psi} \rangle \cdot E_{s, \Psi} \, ds + \sum_R \langle f, R \rangle \cdot R
\]

with \( \Phi \) running over an orthonormal basis of \( L_0^2(G_k \setminus G_\mathbb{A}/K) \), \( \Psi \) over an orthonormal basis of \( L^2(\Theta_k \setminus \Theta_\mathbb{A}/(K \cap \Theta_\mathbb{A})) \), and \( R \) over an orthonormal basis of residues of Eisenstein series to the right of \( \Re s = \frac{1}{2} \). (The inner products are integrals over \( G_k \setminus G_\mathbb{A} \).)

We choose each component to be a joint eigenvector of \( Z(g_\infty) \). The corresponding Plancherel identity is

\[
\|f\|_{L^2}^2 = \sum_{\Phi} |\langle f, \Phi \rangle|^2 + \frac{1}{4\pi i} \sum_{\Psi} \int_{\Re s = \frac{1}{2}} |\langle f, E_{s, \Psi} \rangle|^2 \, ds + \sum_R |\langle f, R \rangle|^2.
\]

(We note that the Eisenstein series themselves are not in \( L^2 \), therefore the inner product and integral are obtained by isometric extension.)

In what follows, we shorten these formulas to read

\[
(1) \quad f = \int \langle f, \Phi \rangle \cdot \Phi \, d\Phi \quad \text{and} \quad \|f\|_{L^2}^2 = \int |\langle f, \Phi \rangle|^2 \, d\Phi
\]

(when writing thusly, \( \Phi \) runs over all relevant spectral components).

We may consider the periods

\[
(\Phi, F)_H = \int_{H_k \setminus H_\mathbb{A}} \Phi \cdot F
\]

of spectral components \( \Phi \) on \( G \) against cuspforms \( F \) on \( H \), or even

\[
(\Phi)_H = (\Phi, 1)_H = \int_{H_k \setminus H_\mathbb{A}} \Phi.
\]

Such periods contain information about the underlying representations. These same periods (called there global Shintani functions) were used by Katu, Murase, and Sugano [29], [38] to obtain and study integral expressions for
standard $L$-functions of the orthogonal group. And the Gross–Prasad conjecture [19], [20], [21] predicts that a representation of $O(n)$ occurs in a representation of $O(n+1)$ if and only if the corresponding tensor product $L$-function is non-zero on $\text{Re } s = \frac{1}{2}$. Ichino and Ikeda [24] discuss further details and broader context is provided in papers by Gross, Reeder [22], Jacquet, Lapid, Offen, and/or Rogawski [27], [33], [32], Jiang [28] and Sakellaridis and Venkatesh [39], [40].

The periods also help study the asymptotics of moments of automorphic $L$-functions. Often, the Phragmén–Lindelöf principle yields (so-called) convex bounds for such asymptotics [4], [26]. Diaconu and Garrett [9], [10] used a specific spectral identity to first break convexity for the asymptotics of second moments of automorphic forms in $GL(2)$, over any number field $k$. In fact, their strategy produces families of spectral identities, explored in other papers by them and/or Goldfeld [10], [11], [12] and used by Letang [34]. In the present paper, we carry out that strategy to obtain a spectral identity for second moments of Eisenstein series of $O(n,1)$.

Given a function $f \in L^2(G_k \backslash G_A/K)$, the spectral decomposition (1) above invites us to consider the effect of an operator $X \in \mathcal{Z}(g_\infty)$:

\begin{equation}
Xf = \int \bigoplus \langle f, \Phi \rangle \cdot \lambda_{X,\Phi} \cdot \Phi \, d\Phi \quad \text{and} \quad \|Xf\|^2_{L^2} = \int \bigoplus |\langle f, \Phi \rangle|^2 |\lambda_{X,\Phi}|^2 \, d\Phi,
\end{equation}

where $\lambda_{X,\Phi}$ is the $X$-eigenvalue of $\Phi$ (if $X = \Omega$, we write simply $\lambda_\Phi = \lambda_{\Omega,\Phi}$). The conditions for these decompositions to converge (even in the sense of isometric extensions) are most naturally discussed in the context of automorphic Sobolev spaces. The literature on automorphic Sobolev spaces is scarce; it includes papers by Bernstein and Reznikov [2], [3], Krötz and Stanton [30] and Michel and Venkatesh [36], as well as Garrett’s [17] notes and DeCelles’s [8] very detailed discussion. We discuss them (and their zonal counterparts) in Sections 1 and 2, following the approach in the author’s dissertation [5].

The automorphic Sobolev spaces we discuss in Section 1 are closures (with respect to the relevant norms) of the space $\mathcal{D}(G_k \backslash G_A)$ of global test functions. Even though we only take into account the eigenvalues of $\Omega$ in their definition, we rely on a global spectral decomposition, and the norms are defined from integrals over $G_k \backslash G_A$. So we should see these spaces as spaces of global functions.

A crucial point is that, using a pre-trace kernel, we can obtain an estimate

\[ \int_{|\lambda_\Phi| < T^2} |\Phi(g)|^2 \ll T^m \]

similar to Weyl’s Law, from which we can characterize an automorphic delta $\delta_A$. Then, it is just a matter of using the techniques one habitually uses with classical Sobolev spaces to obtain fundamental solutions of PDEs.
By contrast, the zonal Sobolev spaces we discuss in Section 2 are closures of test functions on $K_\infty \backslash G_\infty / K_\infty$; these are local (archimedean) functions. From them, we shall obtain a different construction of the (global) fundamental solutions just mentioned, which will help us extract some archimedean information.

In Section 3, we use those techniques to obtain fundamental solutions (following Diaconu and Garrett [9], we call them Poincaré series) for certain polynomials in $\Omega$. The spectral decomposition of these Poincaré series $P\ell$ involves the periods $(\Phi)_H$ discussed above. Given an automorphic function $f \otimes f'$ on $G \times G$, we expand $\langle f \cdot f', P\ell \rangle_G$ in two distinct ways, yielding an identity between a spectral expansion (along $G$) and a moment expansion (in $F$-aspect, with $F$ running over an orthonormal basis of cuspforms on $H$).

In Section 4, we apply those ideas to Eisenstein series. In particular, we see how the moment expansion involves the second moments of the Eisenstein series in $F$-aspect, as well as the periods of Eisenstein series. (Elsewhere [5], [6], [7], this author has computed these periods at non-archimedean primes. As discussed there, for the cases used in the present paper, the local factor at the archimedean place is 1.)

In the Appendix, we explain the regularization used in Section 4.

1. Automorphic Sobolev spaces

In the continuation, we will rely heavily on some $L^2$ Sobolev spaces, adapted to the automorphic case. Classically, the Sobolev space of order $\ell$ is defined as the space of functions whose weak derivatives up to order $\ell$ are square-integrable. The topology induced by that family of seminorms (one for each derivative up to order $\ell$) can also be described by a norm obtained from Plancherel formula. For example, in $\mathbb{R}^n$, we set

$$\|f\|_{H^\ell}^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|)^\ell \, d\xi.$$ 

Under Fourier transform, the Laplacian $\Delta$ acts (up to a constant) by multiplication by $|\xi|^2$. In the Plancherel identity, the effect of $\Delta$ is as described in (2).

In our case, the effect of the Casimir element $\Omega$ of $G_\infty$ on the Plancherel identity is also as in (2). Thus, with inner products obtained from integrals over $G_k \backslash G_\mathbb{A}$ (or by isometric extension), we define the automorphic Sobolev norm by

$$\|f\|_{H^\ell}^2 = \int_{\mathcal{H}} |\langle f, \Phi \rangle|^2 (1 + |\lambda\Phi|)^\ell \, d\Phi$$

and the automorphic Sobolev space as

(3) \[ H^\ell_{\text{auto}} = \text{closure of } \mathcal{D}(G_k \backslash G_\mathbb{A}) \text{ with respect to } \| \|_{\ell}. \]
We are specifically interested in the effect of the center \( Z(\mathfrak{g}_\infty) \) of the universal enveloping algebra (and the corresponding differential operators), so the only modification to the usual \( L^2 \) norm involves only archimedean information. However, the norm itself depends on the global automorphic spectral decomposition.

For \( \ell > 0 \), as usual, \( H_{\text{auto}}^{-\ell} \) is the dual of \( H_{\text{auto}}^{\ell} \). Let \( f \in \mathcal{D}(G_k \setminus G_\Lambda) \cap H_{\text{auto}}^{-\ell} \) and \( \varphi \in \mathcal{D}(G_k \setminus G_\Lambda) \cap H_{\text{auto}}^{\ell} \). In the expanded notation, we define \( \langle f, \varphi \rangle \) by

\[
\sum_\Phi \langle f, \Phi \rangle \langle \varphi, \Phi \rangle + \frac{1}{4\pi i} \sum_\psi \int_{\Re s = \frac{1}{2}} \langle f, E_s, \psi \rangle \langle \varphi, E_s, \psi \rangle \, \mathrm{d}s + \sum_R \langle f, R \rangle \langle \varphi, R \rangle.
\]

From Cauchy–Schwarz–Bunyakowsky, we obtain (now in the compressed notation)

\[
\langle f, \varphi \rangle = \int \oplus |\lambda_\Phi| < T^2 |\Phi(g)|^2 \, \mathrm{d}\Phi \ll \int \oplus |\langle f, \Phi \rangle| (1 + |\lambda_\Phi|)^{-\ell/2} \cdot |\langle \varphi, \Phi \rangle| (1 + |\lambda_\Phi|)^{\ell/2} \, \mathrm{d}\Phi
\]

\[
\ll \sqrt{\int \oplus |\langle f, \Phi \rangle|^2 (1 + |\lambda_\Phi|)^{-\ell} \, \mathrm{d}\Phi} \cdot \sqrt{\int \oplus |\langle \varphi, \Phi \rangle|^2 (1 + |\lambda_\Phi|)^{\ell} \, \mathrm{d}\Phi} = \| f \|_{-\ell} \cdot \| \varphi \|_{\ell}.
\]

**Proposition 1.** With \( X = G_\infty / K_\infty \) and \( n = \dim_\mathbb{R} X \), we have

\[
\int_{|\lambda_\Phi| < T^2} |\Phi(g)|^2 \ll T^n.
\]

(This is unsurprising, in light of Weyl’s Law [13], [25], [35].)

**Proof of Proposition 1.** We follow Garrett [15], with adjustments to a different group and attempting to avoid tedious computations.

We use the “ball”

\[
B = \left\{ n_a m_\lambda : \max |a_i| < \frac{1}{T} \text{ and } |\log \lambda| < \frac{1}{T} \right\}
\]

of radius \( 1/T \) in \( P_\infty \). Then \( BK_\infty \) is a tubular neighborhood of \( K_\infty \) in \( G_\infty \). Considering the action by \( \eta = \text{ch}_{BK_\infty} \otimes \otimes_{v<\infty} \text{ch}_{K_v} \), we have

\[
\langle \eta \cdot f \rangle(g) = \int_{G_k} \eta(h) f(gh) \, \mathrm{d}h = \int_{G_\Lambda} \eta(g^{-1}h) f(h) \, \mathrm{d}h = \int_{G_k \setminus G_\Lambda} \sum_{\gamma \in G_k} \eta(g^{-1}gh) f(h) \, \mathrm{d}h = \langle \eta_g, f \rangle,
\]

where

\[
\eta_g(h) = \sum_{\gamma \in G_k} \eta(g^{-1}\gamma h).
\]
If the radius $1/T$ is sufficiently small, this sum has one single term. (Note that $\eta$ is not smooth; such a choice avoids cut-off functions.) Still following Garrett,
\[
\|\eta_g\|^2 = \int_{G_k \backslash G} \eta_g(h) \sum_{\gamma \in G_k} \bar{\eta}(g^{-1}\gamma h) \, dh = \int_{G_A} \eta_g(h) \bar{\eta}(g^{-1}h) \, dh
\]
\[
= \int_{G_A} \sum_{\gamma \in G_k} \eta(g^{-1}\gamma gh) \bar{\eta}(h) \, dh.
\]
We are led to
\[
\|\eta_g\|^2 \ll \text{radius}^n = \frac{1}{T^n}.
\]

On the other hand, because $\Phi$ is right $K$-invariant and generates an irreducible representation, it must be that
\[
\langle \eta_g, \Phi \rangle = (\eta \cdot \bar{\Phi})(g) = C\Phi(g),
\]
where the constant $C$ depends only on $\eta$ and the archimedean parameters of $\Phi$. Let $s$ be the archimedean parameter, seen as the parameter of a principal series representation $\varphi_s$ at $\infty$. Then, if $n_a m_\lambda \in B$, $k \in K_\infty$, and $s \ll T$ (which is the case if $|\lambda_\Phi| < T^2$, as $\lambda_\Phi \asymp s^2$), we have
\[
\varphi_s(n_a m_\lambda k) = \delta(m_\lambda) = |\lambda|^{ns} < e^{ns/T} \ll 1.
\]
In particular, assume that $g$ lies in a fixed compact and the radius $1/T$ is sufficiently small. Then
\[
(\eta \cdot \bar{\Phi})(g) = \int_{G_A} \eta(h) \bar{\Phi}(gh) \, dh \gg \int_{G_A} \eta(h) \, dh \cdot \bar{\Phi}(g) \asymp \text{radius}^n \cdot \bar{\Phi}(g)
\]
and we see that
\[
C \gg \text{radius}^n = \frac{1}{T^n}.
\]

Combining all this information, we conclude
\[
\frac{1}{T^n} \gg \|\eta_g\|^2 = \int_{|\lambda_\Phi| < T^2} \|\langle \eta_g, \Phi \rangle \|^2 \, d\Phi \geq \int_{|\lambda_\Phi| < T^2} \|\langle \eta_g, \Phi \rangle \|^2 \gg \int_{|\lambda_\Phi| < T^2} \frac{|\Phi(g)|^2}{T^{2n}}.
\]

**Lemma 2.** Let $\delta_A$ be the distribution defined, for right $K$-invariant $f$, by
\[
f \cdot \delta_A = f(1).
\]
We have $\delta_A \in H_{\text{auto}}^{-n/2-\varepsilon}$, for any $\varepsilon > 0$.

(As the definition of $H_{\text{auto}}^\ell$ depends on the global spectral decomposition, it is not possible to reduce this to classical lemmas, of which it is a direct analogue.)
Proof of Lemma 2. Indeed, let
\[ a_N = \int_{|\lambda \Phi| < N^2}^\oplus |F(1)|^2. \]

Then
\[ \int_{|\lambda \Phi| < N^2}^\oplus \frac{|\langle \delta, \Phi \rangle|^2}{(1 + |\lambda \Phi|)^{n/2+\varepsilon}} \, d\Phi \ll \sum_{N \geq 0} a_{N+1} - a_N \frac{1}{(1 + N)^{n/2+\varepsilon}} \]
\[ = \sum_{N \geq 0} a_N \left( \frac{1}{N^{n+2\varepsilon}} - \frac{1}{(N + 1)^{n+2\varepsilon}} \right) \]
\[ \ll \sum_{N \geq 0} \frac{a_N}{N^{n+1+\varepsilon}} \ll \sum_{N \geq 0} \frac{N^n}{N^{n+1+\varepsilon}} < \infty. \]

Therefore, in $H_{\text{auto}}^{-n/2-\varepsilon}$, we have
\[ \delta = \int_{\lambda}^\oplus \Phi(1) \cdot \Phi \, d\Phi. \]

Proposition 3. If $|\lambda \Phi - \lambda| \geq r > 0$ for all $\Phi$, then $(\Omega - \lambda) : H_{\text{auto}}^\ell \rightarrow H_{\text{auto}}^{\ell-2}$ is an isomorphism.

Proof. For $\lambda \in \mathbb{C}$ and $f \in H_{\text{auto}}^\ell$, we have
\[ \| (\Omega - \lambda) f \|_{\ell^{-2}}^2 = \int_{\lambda}^\oplus \left| \langle f, \Phi \rangle (\lambda \Phi - \lambda) \right|^2 (1 + |\lambda \Phi|)^{-2} \, d\Phi \]
\[ \ll \int_{\lambda}^\oplus \left| \langle f, \Phi \rangle \right|^2 (1 + |\lambda \Phi|)^{\ell} \, d\Phi = \| f \|_{\ell}^2, \]
showing that $(\Omega - \lambda) : H_{\text{auto}}^\ell \rightarrow H_{\text{auto}}^{\ell-2}$ is continuous and injective. On the other hand, if $|\lambda \Phi - \lambda| \geq r > 0$ for all $\Phi$, then
\[ 1 + |\lambda \Phi| \leq 1 + |\lambda \Phi - \lambda| + |\lambda| \ll |\lambda \Phi - \lambda|. \]

For example, there is a unique solution $u_\delta$ of
\[ (\Omega - \lambda)^N u_\delta = \delta, \]
for $\delta$ as defined above. In $H_{\text{auto}}^{2N-n/2-\varepsilon}$, it can be expressed as
\[ u_\delta = \int_{\lambda}^\oplus \frac{\Phi(1)}{(\lambda \Phi - \lambda)^N} \cdot \Phi \, d\Phi. \]
2. Zonal spherical functions

In this section, we work at the single archimedean place (suppressed). The facts we need on spherical functions were taken from the monographs by Helgason [23] and Gangolli and Varadarajan [14]. In this summary, we follow mostly Helgason, as well as Garrett [15], with adaptations for the rank one case.

Let \( X = G_\infty/K_\infty \), \( n = \dim_\mathbb{R} X \), and \( \Delta \) be the image of the Casimir element \( \Omega \) on \( X \). A smooth function \( f \) on \( K \setminus G/K \) is a zonal spherical function if it is an eigenfunction of \( \Delta \) normalized by \( f(1) = 1 \). For any given eigenvalue \( \lambda \), there is only one such \( f \).

Recall that we defined 
\[
\varphi_s(n_a m_\lambda k) = \delta_P(m_\lambda)^s = |\lambda|^s.
\]
By a theorem of Harish-Chandra, all zonal spherical functions are of the form 
\[
\psi_s(g) = \int_K \varphi_s(kg) \, dk,
\]
for some \( s \in \mathbb{C} \).

The spherical transform is defined for \( f \in L^2(K \setminus G/K) \) by 
\[
\tilde{f}(s) = \int_G f \cdot \psi_{1-s}.
\]
The inversion formula (up to a constant) is 
\[
f(g) = \int_{\text{Re } s = \frac{1}{2}} \tilde{f}(s) \cdot \frac{\psi_s(g)}{|c(s)|^2} \, ds,
\]
with corresponding Plancherel identity 
\[
\|f\|_{L^2}^2 = \int_{\text{Re } s = \frac{1}{2}} \frac{|\tilde{f}(s)|^2}{|c(s)|^2} \, ds.
\]
(We need to use the Plancherel identity to establish an isometric extension.)

The Harish-Chandra function \( c(s) \) is given [23] by the Gindikin–Karpelevič formula, which, in our case, is 
\[
c(s) = \frac{\Gamma((s - \frac{1}{2}) \frac{n-1}{2}) \cdot \Gamma(\frac{3(n-1)}{4})}{\Gamma((s + \frac{1}{2}) \frac{n-1}{2}) \cdot \Gamma(\frac{n-1}{4})}.
\]
(Helgason’s \( i \lambda \) relates to our \( s \) by \( \rho + i\lambda = 2\rho s \). The positive simple root \( \alpha \) has \( \langle \alpha, \alpha \rangle = n - 1 \) and multiplicity \((n - 1)\). Therefore, \( 2\rho = (n - 1)\alpha \).) The main fact we need is that 
\[
c(s) \asymp |s|^{\frac{n-1}{2}}.
\]

We define the zonal Sobolev norm by 
\[
\|f\|_{L}^2 = \int_{\text{Re } s = \frac{1}{2}} \frac{|\tilde{f}(s)|^2}{|c(s)|^2} \left(1 + |\lambda_s|\right)^\ell \, ds,
\]
where \( \lambda_s = \lambda \psi_s \gg |s|^2 \). We define the zonal spherical Sobolev space as

\[
H^\ell_{\text{zonal}} = \text{closure of } \mathcal{D}(K \setminus G/K) \text{ with respect to } \| \cdot \|_\ell.
\]

**Lemma 4.** If \( \delta_\infty \) is the delta distribution centered at \( 1 \cdot K \), we have \( \delta_\infty \in H^{-n/2-\ell}_{\text{zonal}} \), for any \( \varepsilon > 0 \).

(This is compatible with the outcome for \( \delta_A \), in Lemma 2.)

**Proof of Lemma 4.** We have

\[
\tilde{\delta}_\infty(s) = \psi_{1-s}(1) = \int_K \varphi_{1-s}(k) \, dk = 1.
\]

On the other hand,

\[
\frac{(1 + |\lambda_s|)^{-\ell}}{|c(s)|^2} \ll \frac{|s|^{-2\ell}}{|s|^{-(n-1)}}
\]

and the requirement for

\[
\int_{\text{Re } s = \frac{1}{2}} \frac{(1 + |\lambda_s|)^{-\ell}}{|c(s)|^2} \, ds < \infty
\]

is \( 2\ell - (n - 1) > 1 \), or \( \ell > n/2 \).

The spherical expansion of \( \delta_\infty \), valid in \( H^{-n/2-\varepsilon}_{\text{zonal}} \), is

\[
\delta_\infty = \int_{\text{Re } s = \frac{1}{2}} \frac{\psi_s}{|c(s)|^2} \, ds.
\]

Exactly as in Proposition 3, \((\Delta - \lambda) : H^\ell_{\text{zonal}} \to H^{\ell-2}_{\text{zonal}}\) is an isomorphism provided \( \lambda \) is away from all eigenvalues of \( \Delta \) (with \( s \) lying on \( \text{Re } s = \frac{1}{2} \) this is not at all an issue). In that case, also as before, there is a solution \( u_\infty \) of \((\Delta - \lambda)^N u_\infty = \delta_\infty \). In \( H^{2N-n/2-\varepsilon}_{\text{zonal}} \),

\[
u_\infty = \int_{\text{Re } s = \frac{1}{2}} \frac{\psi_s}{(\lambda_s - \lambda)^N \cdot |c(s)|^2} \, ds.
\]

### 3. Poincaré series and spectral identities

We return to the global picture and follow Diaconu and Garrett [9], [10], [15], [16].

At non-archimedean \( v \), let \( u_v \) be the characteristic function of \( H_v \cdot K_v \) and \( u = u_\infty \otimes \bigotimes_{v < \infty} u_v \). Noting that \( u_\infty \) inherits the left \( H_\infty \)-invariance of \( \delta_\infty \), we define the Poincaré series

\[
\text{Pé}(g) = \sum_{\gamma \in H_k \setminus G_k} u(\gamma g).
\]

(This is a function on \( G_k \setminus G_A \), while \( u_\infty \) is a function on \( G_\infty \).)
For brevity, write $P(\Omega) = (\Omega - \lambda)^N$. Require $P(\lambda_\phi) \gg 0$. Note that Pe is left $G_k$-invariant and that $\Omega$ acts only on the archimedean information. It is clear that the Poincaré series is a solution of

$$P(\Omega) \text{Pe} = \delta_A.$$  

This same solution was shown in Section 1 to be unique in the automorphic Sobolev space. Therefore, it must be that

$$\sum_{\gamma \in H_k \backslash G_k} u(\gamma g) = \text{Pe} = \int \Phi(1) \cdot \Phi d\Phi.$$  

(Unremarkably, Pe has a larger support than $\delta_A$. The same phenomenon occurs already with fundamental solutions of $\Delta$ in $\mathbb{R}^n$, whose support is all of $\mathbb{R}^n$, while the support of $\delta$ is only $\{0\}$.)

On the other hand, let $f : G_k \backslash G_h \to \mathbb{C}$ be an eigenfunction of $\Omega$ with eigenvalue $\lambda_f \neq \lambda$. Then

$$\langle f, \text{Pe} \rangle_G = \int_{G_k \backslash G_h} f \cdot \text{Pe} = \int_{H_k \backslash G_h} \int_{H_k \backslash H_h} f \cdot \bar{u} = \int_{H_k \backslash H_h} \int_{H_k \backslash H_h} f \cdot \frac{P(\Omega)}{P(\lambda_f)} \cdot \bar{u} = \frac{1}{P(\lambda_f)} \int_{H_k \backslash H_h} f \cdot (\delta_\infty \otimes \bigotimes_{v < \infty} u_v) = \frac{1}{P(\lambda_f)} \int_{H_k \backslash H_h} f$$

That is, the Poincaré series can be used to extract information about periods.

In $H_{auto}^{2N-n/2-\varepsilon}$, we decompose

$$\text{Pe} = \int \Phi(1) \cdot \Phi d\Phi = \int \text{Pe} \cdot \Phi d\Phi = \int \Phi(1) \cdot \Phi d\Phi.$$  

Diaconu and Garrett [9] discuss a similar decomposition for $G = \text{GL}_2$.

**Application to spectral identities.** Still following Diaconu and Garrett [9], consider two chains of inclusions: $H^\Delta \subseteq G^\Delta \subseteq G \times G$ and $H^\Delta \subseteq H \times H \subseteq G \times G$, where $G^\Delta$ denotes the image of $G \to G \times G : g \mapsto (g,g)$, and similarly for $H^\Delta$.

Let $f \otimes f'$ be an automorphic function on $G \times G$. The two inclusions suggest two different evaluations of

$$\langle f \cdot f', \text{Pe} \rangle_G = \int_{H_k \backslash G_k} f \cdot f' \cdot u;$$

a spectral decomposition along $G^\Delta$ (we will call it the *spectral expansion*) or along $H \times H$ (we will call it the *moment expansion*).
Decomposing along $G$ (the spectral expansion), we have

$$\langle f \cdot f', \text{Pe} \rangle_G = \int_{\Phi \text{ on } G} \langle f \cdot f', \Phi \rangle \langle \Phi, \text{Pe} \rangle$$

$$= \int_{\Phi} (\Phi)_H \int_{G \backslash G_h} f \cdot f' \cdot \Phi d\Phi,$$

involving triple products as well as the periods $(\Phi)_H$ of each component $\Phi$.

Note that $f$ has a discrete decomposition along $H$. Writing $(g \cdot f)(h) = f(hg)$:

$$g \cdot f = \sum_F (g \cdot f, F)_H \cdot F,$$

with $F$ running over an orthonormal basis of eigenfunctions of $\mathcal{Z}(\mathfrak{h}_\infty)$. We obtain the moment expansion:

$$\langle f \cdot f', \text{Pe} \rangle_G = \int_{H \backslash G_h} f \cdot f' \cdot u = \int_{H \backslash G_h} \int_{H \backslash H_h} f(hg)f'(hg)u(g) \, dh \, dg$$

$$= \int_{H \backslash G_h} (g \cdot f, \overline{g \cdot f'})_H u(g) \, dg$$

$$= \int_{H \backslash G_h} \sum_F (g \cdot f, F)_H (g \cdot f', \overline{F})_H u(g) \, dg.$$

Often, it can be rewritten in the form

$$\sum_F (f, F)_H (f', \overline{F})_H \cdot \text{weight}\left(f_\infty, f'_\infty, F_\infty\right),$$

with the weight depending only on the archimedean parameters. In the next section, we show the details of such a rewriting when $f$ and $f'$ are spherical Eisenstein series, unramified at non-archimedean places. For applications, one would need to study its asymptotics.

In sum, we establish the following theorem.

**Theorem 5.** Let $f \otimes f'$ be an automorphic function on $G \times G$, where $f$ and $f'$ are spherical Eisenstein series, unramified at non-archimedean places. Then $\langle f \cdot f', \text{Pe} \rangle_G$ has two expansions: the spectral expansion

$$\int_{\Phi} \langle \Phi \rangle_H \int_{G \backslash G_h} f \cdot f' \cdot \overline{\Phi} d\Phi$$

is a decomposition along $G$, while the moment expansion

$$\sum_F \int_{H \backslash G_h} (g \cdot f, F)_H (g \cdot f', \overline{F})_H u(g) \, dg$$

$$= \sum_F (f, F)_H (f', \overline{F})_H \cdot \text{weight}\left(f_\infty, f'_\infty, F_\infty\right)$$

is a decomposition along $H$. 
4. Eisenstein series and their second moments

We want to specialize to spherical, unramified, Eisenstein series \( f = E_a \) and \( f' = E_b \). Here, \( a, b \in \mathbb{C} \), and \( E_a \) and \( E_b \) are parametrized as discussed in the Introduction.

One first obstacle is that \( f \cdot f' \) is not in \( L^2(G_k \backslash G_A) \) and it is unclear whether we can integrate

\[
\langle f \cdot f', \text{P} \rangle_G = \int_{G_k \backslash G_A} f \cdot f' \cdot \text{P}
\]
directly. It is possible to subtract finitely many singular terms from \( f \cdot f' \) so that the difference is square-integrable; we discuss that in the Appendix.

The exact choice of singular terms will depend on where \( a \) or \( b \) lie. For definiteness, say

\[
\mathcal{F} = E_a E_b + \sum_s c_s E_s
\]
(with finitely many \( s \) occurring) is the regularized expression.

For the spectral expansion, we have, as in (5),

\[
\langle \mathcal{F}, \text{P} \rangle_G = \int_{\Phi \text{ on } G} \langle \mathcal{F}, \Phi \rangle \langle \Phi, \text{P} \rangle = \int \langle \mathcal{F}, \Phi \rangle \frac{(\Phi)_H}{P(\lambda_\Phi)} d\Phi.
\]

The moment expansion starts as (6),

\[
\langle \mathcal{F}, \text{P} \rangle_G = \int_{G_k \backslash G_A} \mathcal{F} \cdot \text{P} = \int_{H_k \backslash G_A} \mathcal{F} \cdot u = \int_{H_k \backslash G_k} \int_{H_k \backslash H_A} \mathcal{F}(hg) \, dh \cdot u(g) \, dg,
\]
where the convergence of the inner integral is justified by the compactness of \( H_k \backslash H_A \). Recall that at non-archimedean \( v \) we chose \( u_v = ch_{H_v \cdot K_v} \), so we assume \( g_v \in H_v \cdot K_v \). Therefore, we can simplify further:

\[
\langle \mathcal{F}, \text{P} \rangle_G = \int_{H_\infty \backslash G_\infty} \int_{H_k \backslash H_A} \mathcal{F}(hg) \, dh \cdot u_\infty(g) \, dg.
\]

The inner integral is

\[
\int_{H_k \backslash H_A} \mathcal{F}(hg) \, dh = \int_{H_k \backslash H_A} E_a(hg) E_b(hg) \, dh + \sum_s c_s \int_{H_k \backslash H_A} E_s(hg) \, dh.
\]

The “main” part. For the \( E_a E_b \) summand, we have, as before,

\[
\int_{H_k \backslash H_A} E_a(hg) E_b(hg) \, dh = \sum_F (g \cdot E_a, F)_H (g \cdot E_b, F)_H.
\]

We remark that

\[
(g \cdot E_s, F)_H = \int_{H_k \backslash H_A} E_s(hg) \overline{F}(h) \, dh = \int_{\Theta_h \backslash H_k} \varphi_s(hg) \overline{F}(h) \, dh
\]

\[
= \int_{\Theta_h \backslash H_k} \varphi(hg) \int_{\Theta_k \backslash \Theta_h} \overline{F}(\theta h) \, d\theta \, dh.
\]
The function
\[ F_\Theta(h) = \int_{\Theta_k \setminus \Theta_H} F(\theta h) \, d\theta \]
is a spherical vector in \( \text{Ind}^H_{\Theta} 1 \), normalized by \( F_\Theta(1) = (F)_\Theta \). Therefore, with \( \eta \) a spherical vector normalized by \( \eta(1) = 1 \), we obtain
\[
(g \cdot E_s, F)_H = \int_{\Theta_k \setminus \Theta_H} \varphi_s(hg) \overline{F_\Theta(h)} \, dh = (F)_\Theta \cdot \int_{\Theta_k \setminus \Theta_H} \varphi_s(hg) \overline{\eta(h)} \, dh.
\]
Recalling that \( g_v \in H_v \cdot K_v \) for non-archimedean \( v \), we see that all but the archimedean factor are independent of \( g \) and
\[
(g \cdot E_s, F)_H = \left( \int_{\Theta_k \setminus \Theta_H} \varphi_{s,v}(h) \overline{F_\Theta(h)} \, dh \right) \cdot \prod_{v < \infty} \left( \int_{\Theta_k \setminus \Theta_H} \varphi_{s,v}(h) \overline{H_\Theta(h)} \, dh \right).
\]
We abbreviate this as follows:
\[
\psi_{s,F}(g_\infty) = \int_{\Theta_k \setminus \Theta_H} \varphi_{s,\infty}(hg_\infty) \overline{F_\Theta(h)} \, dh;
\]
\[
(E_s, F)'_H = \prod_{v < \infty} \int_{\Theta_k \setminus \Theta_H} \varphi_{s,v}(h) \overline{F_\Theta(h)} \, dh;
\]
\[
(g \cdot E_s, F)_H = \psi_{s,F}(g_\infty)(E_s, F)'_H.
\]
Combining this with (9), we see that the “main” part of the moment expansion (7) is
\[
\sum_{F} (E_a, F)'_H (E_b, F)'_H \int_{H_\infty \setminus G_\infty} \psi_{s,F}(g) \psi_{b,F}(g) u_\infty(g) \, dg.
\]
Suppress for a moment the \( \infty \) indices, and use \( G = HP \), with measure \( d(hp) = dh \, dp \) and \( dp \) being a right Haar measure. We have
\[
\int_{H \setminus G} \psi_{a,F} \cdot \psi_{b,F} \cdot u = \int_{H} \int_{H} \int_{P} \varphi_a(hp) \overline{F_\Theta(h)} \cdot \varphi_b(h'p) F_\Theta(h') \cdot u(p) \, dp \, dh \, dh'.
\]
With a nod to Diaconu and Garrett [9], set
\[
X_{a,b}(h, h') = \int_{P} \varphi_a(hp) \varphi_b(h'p) u_\infty(p) \, dp
\]
and conclude
\[
\int_{H \setminus G} \psi_{a,F} \cdot \psi_{b,F} \cdot u = \int_{\Theta \setminus H} \int_{\Theta \setminus H} \overline{F_\Theta(h)} F_\Theta(h') X_{a,b}(h, h') \, dh \, dh'.
\]
Resuming (10), we see that the “main” part of the moment expansion is
\[
\sum_{F} (E_a, F)'_H (E_b, F)'_H \int_{\Theta_k \setminus \Theta_H} \int_{\Theta_k \setminus \Theta_H} \overline{F_\Theta(h)} F_\Theta(h') X_{a,b}(h, h') \, dh \, dh'.
\]
Recalling that $F_{\Theta}(h) = (F)_{\Theta} \eta_F(h)$, where $\eta_F$ is a spherical vector in $\text{Ind}_H^H 1$ normalized by $\eta_F(1) = 1$, we can make the periods even more apparent.

**The "singular" part.** For the other summands in (8), we observe that, by Witt’s lemma, $P_k \backslash G_k$ is the space of isotropic lines in $k^{n+1}$, on which $H_k$ acts transitively. As $\Theta = H \cap P$, we have $P_k \backslash G_k = \Theta_k \backslash H_k$ and

$$
\int_{H_k \backslash H_k} E_s(hg) \, dh = \int_{H_k \backslash H_k} \sum_{\varphi_s(\gamma h) \in \Theta_k \backslash H_k} \varphi_s(\gamma h) \, dh = \int_{\Theta_k \backslash H_k} \varphi_s(hg) \, dh
$$

$$
= \text{vol}(\Theta_k \backslash \Theta_h) \int_{\Theta_k \backslash H_k} \varphi_s(hg) \, dh.
$$

Normalizing $\text{vol}(\Theta_k \backslash \Theta_h) = 1$ and recalling that $g_v \in H_v \cdot K_v$ for non-archimedean $v$, we obtain

$$
\int_{H_k \backslash H_k} E_s(hg) \, dh = \left( \int_{\Theta_h \backslash H_h} \varphi_s(hg) \, dh \right) \cdot \prod_{v<\infty} \left( \int_{\Theta_v \backslash H_v} \varphi_s(h) \, dh \right)
$$

$$
= \psi_s(g)(E_s)_{H}.
$$

Additionally, because $u_\infty$ is a solution of $P(\Delta)u_\infty = (\Delta - \lambda)^N u_\infty = \delta_\infty$, we have

$$
\int_{H_\infty \backslash G_\infty} \psi_s \cdot u_\infty = \int_{H_\infty \backslash G_\infty} \frac{P(\Delta) \psi_s}{P(\lambda_s)} \cdot u_\infty
$$

$$
= \int_{H_\infty \backslash G_\infty} \psi_s \cdot \frac{P(\Delta) u_\infty}{P(\lambda_s)} = \int_{H_\infty \backslash G_\infty} \psi_s \cdot \frac{\delta_\infty}{P(\lambda_s)} = \frac{1}{P(\lambda_s)}.
$$

Therefore, the "singular" part of the moment expansion (7) becomes

$$
(12) \quad \int_{H_\infty \backslash G_\infty} \sum_s c_s(g \cdot E_s, 1)_H = \sum_s c_s(E_s)_H \int_{H_\infty \backslash G_\infty} \psi_s \cdot u_\infty
$$

$$
= \sum_s c_s \frac{(E_s)_H}{P(\lambda_s)}.
$$

Combining this with the "main" part (11), we obtain the complete moment expansion:

**Proposition 6.** Let

$$
F = E_a E_b + \sum_s c_s E_s
$$

be the regularized expression (with finitely many $s$ occurring) for $E_a E_b$, and

$$
X_{a,b}(h, h') = \int_{P_\infty} \varphi_{a,\infty}(hp) \varphi_{b,\infty}(h'p) u_\infty(p) \, dp,
$$

$$
\text{weight}_{a,b,F} = \| (F)_{\Theta} \|^2 \int_{H_\infty} \int_{H_\infty} \overline{\eta_F}(h) \eta_F(h') X_{a,b}(h, h') \, dh \, dh',
$$
where $\eta_F$ is a spherical vector in $\text{Ind}_{\mathfrak{H}}^{\mathfrak{H}} 1$ normalized by $\eta_F(1) = 1$. Then the moment expansion of $\langle \mathcal{F}, \mathcal{P} \mathcal{E} \rangle_G$ is

$$
\langle \mathcal{F}, \mathcal{P} \mathcal{E} \rangle_G = \sum_F (E_a, F)'_H (E_b, F)'_H \cdot \text{weight}_{a,b,F} + \sum_s c_s \frac{(E_s)_H}{P(\lambda_s)}.
$$

The actual computation of $X_{a,b}$ and weight $a,b,F$ can get quite involved, as illustrated, for example, in the GL($r$) $\times$ GL($r-1$) case discussed by Diaconu, Garrett and Goldfeld [12].

**Appendix: Regularizing functions not of rapid decay**

In the previous section, we needed the spectral expansion of $E_a E_b$, and observed that one difficulty was that the product is not in $L^2(G_k \backslash G_A / K)$. However, it is possible to subtract a linear combination of Eisenstein series (the singular part), so that the difference is an $L^2$ function.

The idea, which I learned from Garrett [15], [16] and he traces to Zagier [41], uses the constant terms of the Eisenstein series to guide the choice of singular terms, so as to assure cancellation of non-$L^2$ terms. We articulate the details in our specific case $G = O(n,1)$.

We saw in the Introduction that $M = \Theta A$, where $A = \{m_\lambda\} \cong \text{GL}(1)$. We will always write the elements of $P = N \Theta A$ in the form $p = n \theta m_\lambda$. Because $dn d\theta d(m_\lambda)$ is a right invariant measure, $dp = \delta_P(m_\lambda)^{-1} \cdot dn d\theta d(m_\lambda)$ is a left invariant measure on $P$. In the same manner, we always write the elements of $G = PK = N \Theta A K$ in the form $g = pk = n \theta m_\lambda k$, in which case $dg = dp dk = \delta_P(m_\lambda)^{-1} \cdot dn d\theta d(m_\lambda) dk$ is a Haar measure on $G$.

Recall now that we can choose a compact $C \subset N_A \Theta A$ and a real $t_0 > 0$ such that the Siegel set

$$
\mathcal{S} = \{ g = n \theta m_\lambda k : n \theta \in C \text{ and } \delta_P(m_\lambda) \geq t_0 \}
$$

satisfies $G_k \mathcal{S} = G_A$. We assume such a choice was made.

Supposing $f(g) \ll \delta_P(m_\lambda)^\sigma$ for some real $\sigma$, we have

$$
\int_{G_k \backslash G_A} f \leq \int_{\mathcal{S}} f \ll \int_K \int_{t_0}^\infty \int_C |f(n \theta m_\lambda k)| \cdot \delta_P(m_\lambda)^{-1} \cdot d(n \theta) \frac{d\lambda}{\lambda} dk
$$

$$
\ll \int_{t_0}^\infty \delta_P(m_\lambda)^{\sigma-1} \frac{d\lambda}{\lambda} = \int_{t_0}^\infty |\lambda|^{n(\sigma-1)-1} d\lambda
$$

(in the last step, we used $\delta_P(m_\lambda) = |\lambda|^n$). This last integral converges when $\sigma < 1$. We have thus shown that $f$ is integrable over $G_k \backslash G_A$ provided $\sigma < 1$.

For $L^2$ integrability, we need $\sigma < \frac{1}{2}$.

Recall next that a function $f$ on $P_k \backslash G_A$ is of moderate growth if $f(g) \ll \delta_P(m_\lambda)^\sigma$ for some $\sigma > 0$. 
and of rapid decay if
\[ f(g) \ll \delta_P(m_\lambda)^\sigma \quad \text{for all } \sigma < 0. \]

From the discussion above, it is apparent that if \( f \) is right \( G_k \)-invariant and of rapid decay, then it is integrable over \( G_k \setminus G_k \).

We also know [31], [37] that, choosing the normalization \( \text{vol}(N_k \setminus N_k) = 1 \), the constant term of the Eisenstein series is
\[ cE_s(g) = \delta_P(m_\lambda)^s + c_s \cdot \delta_P(m_\lambda)^{1-s}, \]
where \( c_s \) is the same constant as in the functional equation
\[ E_{1-s} = c_{1-s} \cdot E_s. \]

Moreover, it is a standard fact that \( f - cf \) is of rapid decay, so we can write
\[ E_s(g) = \delta_P(m_\lambda)^s + c_s \cdot \delta_P(m_\lambda)^{1-s} + \text{ fn rapid decay.} \]

We return to the case \( E_a \cdot E_b \) with \( a, b \in \mathbb{C} \). Clearly,
\[ E_a(g) \cdot E_b(g) = \delta_P(m_\lambda)^{a+b} + c_a \cdot \delta_P(m_\lambda)^{1-a+b} \]
\[ + c_b \cdot \delta_P(m_\lambda)^{a+1-b} + c_a \cdot c_b \cdot \delta_P(m_\lambda)^{1-a+1-b} \]
\[ + \text{ fn rapid decay.} \]

As we know that exponents less than \( \frac{1}{2} \) assure \( L^2 \) integrability, we usually can say more.

For example, if \( \text{Re} \ a > 1 \) and \( \text{Re} \ b = \frac{1}{2} \),
\[ E_a(g) \cdot E_b(g) = \delta_P(m_\lambda)^{a+b} + c_a \cdot \delta_P(m_\lambda)^{a+1-b} + L^2 \text{ function.} \]

Moreover,
\[ E_{a+b}(g) = \delta_P(m_\lambda)^{a+b} + c_{a+b} \cdot \delta_P(m_\lambda)^{1-a-b} + \text{ fn rapid decay} \]
\[ = \delta_P(m_\lambda)^{a+b} + L^2 \text{ function.} \]

In the same manner,
\[ E_{a+1-b}(g) = \delta_P(m_\lambda)^{a+1-b} + c_{a+1-b} \cdot \delta_P(m_\lambda)^{-a+b} + \text{ fn rapid decay} \]
\[ = \delta_P(m_\lambda)^{a+1-b} + L^2 \text{ function.} \]

Therefore,
\[ E_a \cdot E_b - E_{a+b} - c_b \cdot E_{a+1-b} = L^2 \text{ function.} \]

We may well have more than two singular terms. For example, if \( \text{Re} \ a = \text{Re} \ b = \frac{1}{2} \), we obtain:
\[ E_a(g) \cdot E_b(g) = \delta_P(m_\lambda)^{a+b} + c_a \cdot \delta_P(m_\lambda)^{1-a+b} \]
\[ + c_b \cdot \delta_P(m_\lambda)^{a+1-b} + c_a \cdot c_b \cdot \delta_P(m_\lambda)^{2-a-b} + L^2 \text{ function.} \]

Here all exponents have real part equal to 1. But the important point is that if one exponent in
\[ E_s(g) = \delta_P(m_\lambda)^s + c_s \cdot \delta_P(m_\lambda)^{1-s} + L^2 \text{ function} \]
has real part greater than \( \frac{1}{2} \), the other one will have it less than \( \frac{1}{2} \). In our case, we have

\[
E_{a+b}(g) = \delta_P(m_\lambda)^{a+b} + L^2 \text{ function;}
E_{1-a+b}(g) = \delta_P(m_\lambda)^{1-a+b} + L^2 \text{ function;}
E_{a+1-b}(g) = \delta_P(m_\lambda)^{a+1-b} + L^2 \text{ function;}
E_{2-a-b}(g) = \delta_P(m_\lambda)^{2-a-b} + L^2 \text{ function.}
\]

Therefore,

\[
E_a \cdot E_b - E_{a+b} - c_a \cdot E_{1-a+b} - c_b \cdot E_{a+1-b} - c_a \cdot c_b \cdot E_{2-a-b} = L^2 \text{ function.}
\]

Acknowledgments. This paper is based on part of the author’s doctoral dissertation [5], done under the supervision of Paul Garrett, and might well not exist without his continued encouragement since then. It is influenced by discussions with and talks by him [15], as well as multiple class discussions with Adrian Diaconu about their joint work [9], [10], [11], [16]. The author thanks also the referee for a few corrections and suggestions.

References

[1] J. Arthur, The Selberg trace formula for groups of F-rank one, Ann. of Math. (2) 100 (1974), 326–385. MR 0360470
[2] J. Bernstein and A. Reznikov, Analytic continuation of representations and estimates of automorphic forms, Ann. of Math. (2) 150 (1999), no. 1, 329–352. MR 1715328
[3] J. Bernstein and A. Reznikov, Sobolev norms of automorphic functionals, Int. Math. Res. Not. IMRN 40 (2002), 2155–2174. MR 1930758
[4] J. Bernstein and A. Reznikov, Periods, subconvexity of L-functions and representation theory, J. Differential Geom. 70 (2005), no. 1, 129–141. MR 2192063
[5] J. P. P. Boavida, Compact periods of Eisenstein series of orthogonal groups of rank one, Ph.D. thesis, University of Minnesota, 2009. MR 2713897
[6] J. P. Boavida, Compact periods of Eisenstein series of orthogonal groups of rank one, Indiana Univ. Math. J. 62 (2013), no. 3, 869–890. MR 3164848
[7] J. P. Boavida, Compact periods of Eisenstein series of orthogonal groups of rank one at even primes, New York J. Math. 20 (2014), 153–181. MR 3177169
[8] A. DeCelles, An exact formula relating lattice points in symmetric spaces to the automorphic spectrum, Illinois J. Math. 56 (2012), no. 3, 805–823. MR 3161352
[9] A. Diaconu and P. Garrett, Integral moments of automorphic L-functions, J. Inst. Math. Jussieu 8 (2009), no. 2, 335–382. MR 2485795
[10] A. Diaconu and P. Garrett, Subconvexity bounds for automorphic L-functions, J. Inst. Math. Jussieu 9 (2010), no. 1, 95–124. MR 2576799
[11] A. Diaconu and P. Garrett, Averages of symmetric square L-functions, and applications, preprint, 2009; available at http://www.math.umn.edu/~garrett/m/v/sym_two.pdf.
[12] A. Diaconu, P. Garrett and D. Goldfeld, Moments for L-functions for \( GL_r \times GL_{r-1} \), Patterson 60++ International Conference on the Occasion of the 60th Birthday of Samuel J. Patterson (University of Göttingen, July 27–29, 2009), Contributions in analytic and algebraic number theory (V. Blomer and P. Mihăilescu, eds.), Springer Proc. Math., vol. 9, Springer, New York, 2012, pp. 197–227. MR 3060461
[13] H. Donnelly, *On the cuspidal spectrum for finite volume symmetric spaces*, J. Differential Geom. **17** (1982), no. 2, 239–253. MR 0664496

[14] R. Gangolli and V. S. Varadarajan, *Harmonic analysis of spherical functions on real reductive groups*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 101, Springer, Berlin, 1988. MR 0954385

[15] P. Garrett, *Talks, private communication*, unpublished notes, September 2006.

[16] P. Garrett, *Integral moments IIIa*, notes from a talk at ICMS, Edinburgh, August 2008; available at [http://math.umn.edu/~garrett/m/v/edinb_two.pdf](http://math.umn.edu/~garrett/m/v/edinb_two.pdf).

[17] P. Garrett, *Global automorphic Sobolev spaces*, preprint, 2011; available at [http://www.math.umn.edu/~garrett/m/v/auto_sob.pdf](http://www.math.umn.edu/~garrett/m/v/auto_sob.pdf).

[18] R. Godement, *The spectral decomposition of cusp-forms*, Symposium on algebraic groups (University of Colorado, Boulder, Colo., July 5–August 6, 1965), Algebraic groups and discontinuous subgroups (A. Borel and G. D. Mostow, eds.), Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, RI, 1966, pp. 225–234. MR 0210828

[19] B. H. Gross and D. Prasad, *Test vectors for linear forms*, Math. Ann. **291** (1991), no. 2, 343–355. MR 1129372

[20] B. H. Gross and D. Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}*, Canad. J. Math. **44** (1992), no. 5, 974–1002. MR 1186476

[21] B. H. Gross and D. Prasad, *On irreducible representations of SO_{2n+1} \times SO_{2m}*, Canad. J. Math. **46** (1994), no. 5, 930–950. MR 1295124

[22] B. H. Gross and M. Reeder, *From Laplace to Langlands via representations of orthogonal groups*, Bull. Amer. Math. Soc. (N.S.) **43** (2006), no. 2, 163–205. MR 2216109

[23] S. Helgason, *Groups and geometric analysis: Integral geometry, invariant differential operators, and spherical functions*, Mathematical Surveys and Monographs, vol. 83, Amer. Math. Soc., Providence, RI, 2000. Corrected reprint of the 1984 original. MR 1790156

[24] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross–Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425. MR 2585578

[25] H. Iwaniec, *Spectral methods of automorphic forms*, Graduate Studies in Mathematics, vol. 53, Amer. Math. Soc., Providence, RI, 2002. MR 1942691

[26] H. Iwaniec and P. Sarnak, *Perspectives on the analytic theory of L-functions*, Geom. Funct. Anal., special volume, part II, Visions in mathematics, towards 2000 (Tel Aviv University, August 25–September 3, 1999) GAFA 2000 (N. Alon, J. Bourgain, A. Connes, M. Gromov and V. Milman, eds.), Birkhäuser, Basel, 2000, pp. 705–741. MR 1826269

[27] H. Jacquet, E. Lapid and J. Rogawski, *Periods of automorphic forms*, J. Amer. Math. Soc. **12** (1999), no. 1, 173–240. MR 1625060

[28] D. Jiang, *Periods of automorphic forms*, Proceedings of the international conference on complex geometry and related fields (S. S.-T. Yau, Z. Chen, J. Wang and S.-L. Ten, eds.), AMS/IP Stud. Adv. Math., vol. 39, Amer. Math. Soc., Providence, RI, 2007. MR 2338623

[29] S.-i. Kato, A. Murase and T. Sugano, *Whittaker–Shintani functions for orthogonal groups*, Tohoku Math. J. (2) **55** (2003), no. 1, 1–64. MR 1956080

[30] B. Krätz and R. J. Stanton, *Holomorphic extensions of representations. I. Automorphic functions*, Ann. of Math. (2) **159** (2004), no. 2, 641–724. MR 2081437

[31] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544, Springer, Berlin, 1976. MR 0579181

[32] E. Lapid and O. Offen, *Compact unitary periods*, Compos. Math. **143** (2007), no. 2, 323–338. MR 2309989

[33] E. Lapid and J. Rogawski, *Periods of Eisenstein series*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 6, 513–516. MR 1860921
[34] D. Letang, *Automorphic spectral identities and applications to automorphic L-functions on GL_2*, J. Number Theory 133 (2013), no. 1, 278–317. MR 2981412

[35] E. Lindenstrauss and A. Venkatesh, *Existence and Weyl’s law for spherical cusp forms*, Geom. Funct. Anal. 17 (2007), no. 1, 220–251. MR 2306657

[36] P. Michel and A. Venkatesh, *The subconvexity problem for GL_2*, Publ. Math. Inst. Hautes Études Sci. 111 (2010), 171–271. MR 2653249

[37] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series: Une paraphrase de l’Écriture*, Cambridge Tracts in Mathematics, vol. 113, Cambridge University Press, Cambridge, 1995. MR 1361168

[38] A. Murase and T. Sugano, *Shintani function and its application to automorphic L-functions for classical groups. I. The case of orthogonal groups*, Math. Ann. 299 (1994), no. 1, 17–56. MR 1273075

[39] Y. Sakellaridis, *Spherical varieties and integral representations of L-functions*, Algebra Number Theory 6 (2012), no. 4, 611–667. MR 2966713

[40] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*, preprint, 2012; available at arXiv:1203.0039v1.

[41] D. Zagier, *The Rankin–Selberg method for automorphic functions which are not of rapid decay*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 415–437. MR 0656029

João Pedro Boavida, Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Prof. Dr. Cavaco Silva, 2744–016 Porto Salvo, Portugal

E-mail address: joao.boavida@tecnico.ulisboa.pt