A comparative assessment of reliability indicators and parameters of the bearings using different estimation methods

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Abstract. The paper depicts the method of statistical processing of experimental results, obtained by testing bearings on the test stands, using the maximum likelihood estimators and/or Best Linear Invariant Estimator. To verify statistical assumptions regarding parameters and reliability indicators of bearings, as well as, to determine confidence intervals, the article contains two original algorithms that allow obtaining values of some specific random variables by Monte Carlo numerical simulation. The statistical inferences are made through these values. The paper also, contains two case studies using data from failures and time truncated tests of bearings.

1. Introduction

Bearings manufacturing is an industrial field where product testing is performed planned in specialized laboratories, using types of equipment to ensure identical testing conditions for all products analyzed. After the purpose of the test, tests shall be carried out, in the bearings manufacture, to check the reliability indicators of the bearings. This test is also called a typing test and shall be repeated periodically at well-specified intervals.

- Complete tests (type \(n\) out of \(n\)). Complete tests are considered to be those tests carried out until all the products under test have failed. This type of test was practiced only at the beginning of the manufacture of bearings, later giving them up, taking into account the very long duration of the test;
- Censored tests or failures-truncated tests type \(r\) out of \(n\), are commonly used. A test is considered to be censored at the level \(r\), if it is completed at the time of the occurrence of the \(r\) failure (the number of damages being established before the beginning of the tests), figure 1;
- Time - truncated tests (with a fixed testing time, \(t_c\)). In this reliability test, the test is completed after a previously set duration \(t_c\), figure 2.
- After the working mode, accelerated tests are used in the case of bearings reliability tests. To shorten the duration of the tests, the constraints shall be increased to the operating conditions [1, 4].
- The acceleration factor shall be determined with the relationship of the basic rating life for dynamically loaded ball bearings:

\[
L_{10h} = \frac{1000000}{60 \cdot n} \cdot \left(\frac{C}{P}\right)^q
\]  

(1)
where:
\[ C = \text{basic dynamic load rating} [N]; \]
\[ P = \text{equivalent load} [N]; \]
\[ n = \text{operating speed} [\text{rpm}]; \]
\[ q = \text{exponent, having the values 3.0 for ball bearings and } 10/3 \text{ for roller bearings.} \]

By volume of products, tests specific to the manufacture of bearings are tests carried out on small volume samples.

Depending on the nature of the products, the bearings are considered as irreparable elements.

![Figure 1. Censored tests.](image)

![Figure 2. Time-truncated tests.](image)

2. The reliability testing of bearings

The methodology of testing is a complex process (figure 3), which is carried out in several stages [5]:

- Sampling of the bearings lot;
- Visual appearance and measurement of external geometry (macro geometry);
- Measurement of the level of vibration;
- Measurement of internal geometry (microgeometry);
- Chemical-metallographic and physical-mechanical control;
- Reliability tests and statistical processing of experimental results;
- Post-mortem chemical-metallographic analysis of the damaged bearings;
- Witness bearings;
- Elaboration of the Experimental Report.

Of these steps, the most important is the reliability tests on the test stands, since the values of the reliability indicators obtained by statistical processing of the experimental results are extrapolated to the entire population.

For this reason, the parameters that characterize the test must be chosen very carefully, namely:

1. Sample volume \((n)\) and number of items damaged during a test \((r)\);
2. The operating condition of the bearings on the stands. There are two ways of accelerating bearings testing [68]:
   a. By increasing the test speed. It is recommended [1, 4]:
   \[ n \leq (0.4 \div 0.6) n_{lim} \]  \hspace{1cm} (2)
   where, \(n_{lim}\), represents the catalog value of the limit speed, [\text{rpm}].
The value of the speed of the test is limited by the dynamic effects that occur (changing the angle of the contact, a significant increase in centrifugal forces acting on rolling bodies, change in lubricant film sizes and load distribution in the case of contacts, uneven loading of rolling bodies, increasing the speed of the sliding of the rolling element and raceways, etc.), and which have not been considered in the calculation of the basic rating life.

b. By increasing the stresses applied to the tested bearings. Equivalent dynamic load values that require bearings during the test are limited by the behavior of the material to contact fatigue [4]. The maximum recommended value [4], for Hertzian stress in the case of point contact is $\sigma_{max} = 3300 \text{ MPa}$. Thus [4], are recommended:
   - For testing ball bearings: $C/P < 3.0$;
   - For oscillating ball bearings: $C/P < 8.0$;
   - For roller bearings: $C/P < 4.5$.

Figure 3. The methodology of reliability testing.

3. The estimation method used. For the statistical processing of the experimental results obtained from the reliability tests of the bearings and the estimation of the values of the reliability indicators are used:
   a. Graphical estimation methods [2, 3]. These are only recommended to provide consistent information on experimental data quickly;
   b. The least squares method [1, 2];
   c. Maximum likelihood method (MLE);
   d. Linear estimators, applied to censored type tests [6, 7].

3. Estimation methods used
   The statistical model used for modelling reliability of bearings is the Weibull distribution. It is recommended to use the general case of the tri-parametric Weibull distribution, $W(x, \beta, \eta, \gamma)$, for an accurate estimation, having the probability density function:
   \[
   f(T) = \frac{\beta}{\eta} \left(\frac{T-\gamma}{\eta}\right)^{\beta-1} \cdot e^{-\left(\frac{T-\gamma}{\eta}\right)^{\beta}}, \quad (T > \gamma; \ \beta, \eta > 0)
   \]  (3)
Manufacturing and operating conditions: durability associated with a reliability of 90%, at a conventional material quality basic rating life.

The main indicator of the reliability of bearings, besides the Weibull distribution parameters, is the life, for a bearing or for a group of identical bearings which operate under the same conditions.

Solution of the system of equations:

$$f(T) = \frac{\beta}{\eta} \cdot \left(\frac{t}{\eta}\right)^{\beta - 1} \cdot e^{-\left(\frac{t}{\eta}\right)^{\beta}},$$

where: $\beta = \text{shape parameter of Weibull distribution}$; $\eta = \text{scale parameter of Weibull distribution}$; $\gamma = \text{location parameter of Weibull distribution}$.

If we estimate the value of the scale parameter, $\eta$, by a transformation of a random variable,

$$t = T - \gamma,$$

is obtained the model of the bi-parametric Weibull distribution, $W(x, \beta, \eta)$, having the probability density function:

$$f(T) = \frac{\beta}{\eta} \cdot \left(\frac{t}{\eta}\right)^{\beta - 1} \cdot e^{-\left(\frac{t}{\eta}\right)^{\beta}},$$

where $t > 0; \beta, \eta > 0$

The location parameter, $\gamma$, is estimated using the correlation coefficient ($\rho$) under conditions of a maximum value of this coefficient [5]:

$$\rho(T, C) = \frac{\sum_{i=1}^{n} \ln(T_i - \gamma) \cdot c_i - \sum_{i=1}^{n} \ln(T_i - \gamma) \cdot \sum_{i=1}^{n} c_i \cdot \sum_{i=1}^{n} \ln(T_i - \gamma)}{\left\{\sum_{i=1}^{n} \ln^{2}(T_i - \gamma) - \left[\frac{\sum_{i=1}^{n} \ln(T_i - \gamma)}{n}\right]^{2}\right\} \left[\sum_{i=1}^{n} c_i^{2} - \left(\frac{\sum_{i=1}^{n} c_i \cdot \sum_{i=1}^{n} \ln(T_i - \gamma)}{n}\right)^{2}\right]}^{1/2}$$

In the previous equation, the values of the deterioration probabilities values, $c_i$, are calculated using the approximate computing relationship [2, 3]:

$$c_i = \frac{1}{1 + \frac{n - i + 1}{n} \cdot F_{0.5; 0.2}(n - i + 1; 2; i)}, i = 1, n$$

where: $F_{p; v1; v2}$ is the $p$ quantile of Fisher-Snedecor’s $F$-distribution, with $v1$ and $v2$ degrees of freedom.

Among the estimation methods, listed above, the maximum likelihood method, MLE, is usually used, due to the remarkable properties of these point estimators.

MLE is based on the likelihood function, $L(t_i, \beta, \eta)$. For failures-truncated tests type $r$ out of $n$, the likelihood function is:

$$L(t_i, \beta, \eta) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(t_i, \beta, \eta) \cdot [1 - F(t_r, \beta, \eta)]^{n-r}$$

For the convenience of the calculation, it is usual to work with the logarithm of the probability function and the punctual estimates are obtained from the maximum condition of this function, as a solution of the system of equations:

$$\left\{ \begin{array}{l}
\frac{1}{\beta} + \frac{1}{r} \cdot \sum_{i=1}^{r} \ln t_i \cdot \ln t_i + (n-r) \cdot t_r \cdot \ln t_r = 0 \\
\eta \beta = \frac{1}{r} \cdot \left[ \sum_{i=1}^{r} t_i^{\beta} + (n-r) \cdot t_r \cdot \ln t_r^{\beta} \right]
\end{array} \right.$$  \hspace{1cm} (9)

The main indicator of the reliability of bearings, besides the Weibull distribution parameters, is the basic rating life, $L_{10}$, for a bearing or for a group of identical bearings which operate under the same conditions is durability associated with a reliability of 90%, at a conventional material quality manufacturing and operating conditions:

$$L_{10} = \gamma + \eta \cdot \left( \ln \left( \frac{1}{1 - 0.10} \right) \right)^{\frac{1}{\beta}}$$
In other words, nominal durability is the 10th quantile of the functioning period, $t_{0.10}$, depending on the test procedure used in the manufacturing process of bearings.

Because the sample size is very small, the asymptotically properties of maximum likelihood estimators cannot be applied in this case, the estimations with confidence intervals were calculated based on three random variables, independent of the sample size $(n)$ and the type of test $[3, 7, 8]$: \[
\begin{align*}
v(r, n) &= \frac{\hat{\beta}}{\hat{\beta}} \\
k(r, n) &= \hat{\beta} \cdot \ln \left( \frac{\hat{\eta}}{\eta} \right) \\
u(r, n, p) &= \hat{\beta} \cdot \ln \left( \frac{\hat{t}_p}{t_p} \right)
\end{align*}
\] (11)

The determination of these distributions is performed by the numerical simulation Monte Carlo. The original algorithm, developed in this sense, involves following steps presented in the figure 4.

### Figure 4. The algorithm used for calculating the values for $v(r, n), k(r, n)$ and $u(r, n, p)$.

1. **Entry data:**  
   - $n$ – sample size;  
   - $r$ – censoring level

2. **Generating random uniform numbers ($n_{ai}, i = 1, n$) in the interval [0,1]**

3. **Calculating the damaging times ($t_i$) by using the inverse distribution function:**  
   \[ t_i = \ln \left( \frac{i}{1-n_{ai}} \right) \]

4. **Increasing order of values $t_i$, obtained in step 3**

5. **Truncating these values at the $r$ level, and retention of values $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(j)} \leq \cdots \leq t_{(r)}$**

6. **Estimating the Weibull distribution parameters by taking into account the $r$ values previously determined.**

7. **Computing the specific random variables**  
   $\hat{\beta}_1 = v(r, n), \hat{\beta}_1 \cdot \ln \hat{\eta}_1 = k(r, n)$ and $\hat{\beta}_1 \cdot \ln \hat{t}_{1.0} = u(r, n, 0.10)$

8. **Repeat the steps 1 ÷ 7 for $N_{sim}$ times**

9. **Determining the random variables quantiles by taking into consideration the $N_{sim}$ made for each case.**

10. **Output data:**  
    ASCII file (*.prn) which contains the quantiles values.
Based on the original algorithm was carried out a Mathcad program for:

\[ N_{\text{sim}} = 10000 \text{ simulations.} \]

For the shape parameter, the confidence interval (CI) corresponding to a confidence level \( 1 - \alpha \), it is obtained as the solution of the equation of the probability:

\[ Pr \left[ v_{\alpha/2}(r, n) \leq \frac{\hat{\beta}}{\beta} \leq v_{1-\alpha/2}(r, n) \right] = 1 - \alpha \quad (12) \]

and it has the form, [9]:

\[ \frac{\hat{\beta}}{v_{1-\alpha/2}(r, n)} \leq \beta \leq \frac{\hat{\beta}}{v_{\alpha/2}(r, n)} \quad (13) \]

For the basic rating life, \( L_{10} \), the corresponding CI is:

\[ \hat{L}_{10} \cdot \left( \frac{-u_{\alpha/2}(r, n, 0.10)}{\hat{\beta}} \right) \leq L_{10} \leq \hat{L}_{10} \cdot \left( \frac{-u_{\alpha/2}(r, n, 0.10)}{\hat{\beta}} \right) \quad (14) \]

For Time - truncated tests with a fixed testing time, \( t_c \), the likelihood function is:

\[ L(t_i, \beta, \eta) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(t_i, \beta, \eta) \cdot [1 - F(t_c, \beta, \eta)]^{n-r} \quad (15) \]

The maximum likelihood estimates are, therefore, the solutions of the following system of equations:

\[
\begin{align*}
\frac{1}{\beta} + \frac{1}{r} \sum_{i=1}^{r} \ln t_i - \frac{\sum_{i=1}^{r} t_i^\beta \cdot \ln t_i + (n-r) \cdot t_c^\beta \cdot \ln t_c^\beta}{\sum_{i=1}^{r} t_i^\beta + (n-r) \cdot t_c^\beta \cdot \ln t_c^\beta} &= 0 \\
\hat{\eta}^\beta &= \frac{1}{r} \left[ \sum_{i=1}^{r} t_i^\beta + (n-r) \cdot t_c^\beta \cdot \ln t_c^\beta \right]
\end{align*}
\]

(16)

The estimations with confidence intervals were calculated based on two random variables, \( v(r, n) \) and \( k(r, n) \).

Thus, for the shape parameter the CI corresponding to a confidence level \( 1 - \alpha \), has the form:

\[ \frac{\hat{\beta}}{v_{1-\alpha/2}(r+1, n)} \leq \beta \leq \frac{\hat{\beta}}{v_{\alpha/2}(r, n)} \quad (17) \]

and for the basic rating life, \( L_{10} \):

\[ \hat{L}_{10} \cdot \left( \frac{-u_{\alpha/2}(r+1, n, 0.10)}{\hat{\beta}} \right) \leq L_{10} \leq \hat{L}_{10} \cdot \left( \frac{-u_{\alpha/2}(r+1, n, 0.10)}{\hat{\beta}} \right) \quad (18) \]

Another type of estimator, developed after the 1970’s and which can be used to estimate the parameters and reliability indicators of bearings, are Best Linear Invariant Estimator (BLIE).

The point estimated values, based on the sampling data obtained, are determined by solving the system of equations [3, 6, 7]:

\[
\begin{align*}
\hat{\gamma}_{\text{BLIE}} &= A(n, r, 1) \cdot \ln t_1 + A(n, r, 2) \cdot \ln t_2 + \cdots + A(n, r, r) \cdot \ln t_r \\
\hat{\delta}_{\text{BLIE}} &= C(n, r, 1) \cdot \ln t_1 + C(n, r, 2) \cdot \ln t_2 + \cdots + C(n, r, r) \cdot \ln t_r
\end{align*}
\]

(19)

where, \( A(n, r, i) \) and \( C(n, r, i) \) are numerical coefficients.
The punctual estimations of Weibull parameters are:

\[
\begin{aligned}
\hat{\beta}_{BLIE} &= \frac{1}{\delta_{BLIE}} \\
\hat{\eta}_{BLIE} &= e^{\gamma_{BLIE}}
\end{aligned}
\]  

(20)

In [7] it is shown that in the case of BLIE estimators, can be established the following random variables, independent of the population parameters:

\[
\begin{aligned}
W(r, n) &= \beta \\
z(r, n) &= \beta \cdot \ln\left(\frac{\hat{\eta}}{\eta}\right) \\
U(r, n, p) &= \beta \cdot \ln\left(\frac{\hat{t}_p}{t_p}\right)
\end{aligned}
\]  

(21)

which allow the determination of confidence intervals, [9], for:

- **Shape parameter**:
  \[
  \hat{\beta} \cdot W_{\alpha/2}(r, n) \leq \beta \leq \hat{\beta} \cdot W_{1-\alpha/2}(r, n)
  \]  

(22)

- **Basic rating life,** \(L_{10}\):
  \[
  \hat{L}_{10} \cdot \left[ e^{-\frac{U_{1-\alpha/2}(r, n, 0.10)}{\hat{\beta}}} \right] \leq L_{10} \leq \hat{L}_{10} \cdot \left[ e^{-\frac{U_{\alpha/2}(r, n, 0.10)}{\hat{\beta}}} \right]
  \]  

(23)

The original algorithm, developed in this sense, involves following steps presented in the figure 5.

4. **Case studies**

Currently, it has been found that there is a trend in the procedure of testing radial ball bearings, to use complex devices that allow simultaneous testing, under characteristic conditions and on a large number of radial ball bearings. Through reliability tests, it can be obtained a series of information on reliability, durability, quality of materials, processing methods and technologies used, and manufacturing accuracy.

After testing on the test machines a lot of ball bearing \((n = 20)\), type 6307, requested with radial load \(P_r = 1160 [daN]\) at \(n = 4000 [rpm]\), using failures-truncated tests [5], at level \(r = 8\), we get the following failures times, \([h]\):

59, 98, 154, 172, 191, 232, 248, 300.

Based on the experimental results, the estimated value of the location parameter is:

\[
\hat{\gamma} = -1.6868 [h]
\]  

(24)

This value was determined under a correlation coefficient equal to:

\[
\rho(\hat{\gamma}) = 0.9950
\]  

(25)

which indicates a very good correlation with the Weibull two-parameter model.

The estimated values of the shape and scale parameters of the Weibull two-parameter using maximum likelihood method are:

\[
\begin{aligned}
\hat{\beta} &= 1.878 \\
\hat{\eta} &= 430.006 [h] \\
\hat{L}_{10} &= 128.093 [h]
\end{aligned}
\]  

(26)
The 90% confidence intervals for Weibull distribution parameters estimated based on experimental results are:

\[
0.802 < \hat{\beta} < 2.730
\]  \hspace{1cm} (27)

\[
44.454 < \hat{L}_{10} < 188.025
\]  \hspace{1cm} (28)

**Figure 5.** The algorithm used for calculating the values for \( W(r, n) \), \( z(r, n) \) and \( U(r, n, p) \).

The estimated values of the shape and scale parameters of the Weibull two-parameter using BLIE estimators are:

\[
\begin{cases}
\hat{\beta} = 1.863 \\
\hat{\eta} = 449.505 \ [h] \\
\hat{L}_{10} = 128.093 \ [h]
\end{cases}
\]  \hspace{1cm} (29)
The 90% confidence intervals for Weibull distribution parameters estimated based on experimental results are:

\[
0.797 < \hat{\beta} < 2.720
\]  

\[
44.126 < \hat{L}_{10} < 188.338
\]

Analyzing the results obtained, it is found that they are similar. BLIE type estimators representing a convenient alternative to MLE.

After testing on the stands a lot of ball bearing (\(n = 20\), type 6007, requested with radial load \(Pr = 550 \text{ daN}[\text{daN}]\) at \(n = 4000 \text{ [rpm]}\), using time-truncated tests \[5\], at \(t_c = 250 \text{ [h]}\), we get the following failures times, [h]: 70, 92, 116, 120, 137, 152, 170, 172.

Based on the experimental results, the estimated value of the location parameter is:

\[
\hat{\gamma} = 31.329 \text{ [h]}
\]

This value was determined under a correlation coefficient equal to:

\[
\rho(\hat{\gamma}) = 0.9957,
\]

which indicates a very good correlation with the Weibull two-parameter model.

The estimated values of the shape and scale parameters of the Weibull two-parameter using maximum likelihood method are:

\[
\begin{align*}
\hat{\beta} & = 1.330 \\
\hat{\eta} & = 347.170 \text{ [h]} \\
\hat{L}_{10} & = 65.325 \text{ [h]}
\end{align*}
\]

The 90% confidence intervals for Weibull distribution parameters estimated based on experimental results are:

\[
0.622 < \hat{\beta} < 1.934
\]  

\[
23.775 < \hat{L}_{10} < 163.871
\]

The estimated values of the shape and scale parameters of the Weibull two-parameter using BLIE estimators are:

\[
\begin{align*}
\hat{\beta} & = 2.484 \\
\hat{\eta} & = 191.162 \text{ [h]} \\
\hat{L}_{10} & = 108.589 \text{ [h]}
\end{align*}
\]

These values were obtained when the time interval (172; 250] was ignored, thus obtaining a censored test. It is also found that the values obtained differ significantly from those obtained by the MLE.

5. Conclusion

From the analysis of the properties of the two types of estimators presented, as well as from the results obtained based on experimental data, it results:

- The maximum likelihood estimation is the only method that uses all the information obtained by tests of the failures and time truncated type;
- The Best Linear Invariant Estimator can be used only in the case of failures truncated tests;
- The punctual and/or confidence interval values of the parameters and reliability indicators for truncated tests are very similar;
• BLIE estimates can also be used for truncated attempts but in this case, a large amount of information is lost and results are very different;
• The maximum likelihood estimators have the great disadvantage that for certain combinations of \( t_i \) values cannot solve equations using numerical methods;
• BLIE have the great disadvantage that for the estimation of parameters and reliability indicators of it is necessary to know the values of the coefficients \( A(n, r, i) \) and \( C(n, r, i) \).

From the analysis of the experimental data, presented in subchapter 4, no conclusions can be drawn regarding the superiority of an estimation method. To demonstrate this aspect, another study is needed, by Monte Carlo numerical simulation. The accuracy of an estimation method can be assessed by analyzing the bias, consistency, mean square error and efficiency of the estimates obtained. These results will be the subject of future research.

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