ON GENERALIZED $\phi$-RECURRENT GENERALIZED $(k, \mu)$-CONTACT METRIC MANIFOLDS

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ABSTRACT. The present paper deals with the study of generalized $\phi$-recurrent generalized $(k, \mu)$-contact metric manifolds with the existence of such notion by a proper example.

1. INTRODUCTION

In 1995 Blair, Koufogiorgos and Papantoniou [5] introduced the notion of $(k, \mu)$-contact metric manifolds, where $k$ and $\mu$ are real constants and a full classification of such manifolds was given by Boeckx [6]. Assuming $k, \mu$ be smooth functions, Koufogiorgos and Tschichias [7] introduced the notion of generalized $(k, \mu)$-contact metric manifolds with the existence of such notions.

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [15] introduced the notion of local $\phi$-symmetry on a Sasakian manifold. Recently Shaikh [14] studied the locally $\phi$-symmetry generalized $(k, \mu)$-contact metric manifolds. Also Baishya, Eysasim and Shaikh [2] introduced and studied the locally $\phi$-recurrent $(k, \mu)$-contact metric manifolds and locally $\phi$-recurrent generalized $(k, \mu)$-contact metric manifolds. Generalizing all these notions of local $\phi$-symmetry, in the present paper we introduce generalized $\phi$-recurrent generalized $(k, \mu)$-contact metric manifolds.

In 1979, Dubey [10] introduced generalized recurrent manifolds. We note that generalized recurrent manifolds are also studied in ([1], [8]). A Riemannian manifold $(M, g)$ is called generalized recurrent [8] if its curvature tensor $R$ satisfies the condition

(1.1) \[ \nabla R = A \otimes R + B \otimes G \]

where $A$ and $B$ are two non-vanishing 1-forms defined by $A(.) = g(., \rho_1)$, $B(.) = g(., \rho_2)$ and the tensor $G$ is defined by

(1.2) \[ G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \]

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for all $X, Y, Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of the smooth vector fields and $\nabla$ denotes covariant differentiation with respect to the metric $g$. Here $\rho_1$ and $\rho_2$ are vector fields associated with 1-forms $A$ and $B$ respectively. Especially, if the 1-form $B$ vanishes, then (1.1) turns into the notion of recurrent manifold introduced by Walker [17].

A Riemannian manifold $(M, g)$ is called a generalized Ricci-recurrent [9] if its Ricci tensor $S$ of type $(0, 2)$ satisfies the condition

\begin{equation}
\nabla S = A \otimes S + B \otimes g
\end{equation}

where $A$ and $B$ are defined in (1.1).

In particular, if $B = 0$, then (2.3) reduces to the notion of Ricci-recurrent manifolds introduced by Patterson [11].

Recently Shaikh and Ahmad [12] introduced the notion of generalized $\phi$-recurrent Sasakian manifolds. The present paper deals with the study of generalized $\phi$-recurrent generalized $(k, \mu)$-contact metric manifolds. The paper is organized as follows. Section 2 is concerned with some preliminaries. In section 3, we study generalized $\phi$-recurrent generalized $(k, \mu)$-contact metric manifolds. Finally, we construct an example of a generalized $\phi$-recurrent generalized $(k, \mu)$-contact metric manifold which is neither $\phi$-symmetric nor $\phi$-recurrent in the last section.

### 2. Preliminaries

A contact manifold is a $C^\infty$ manifold $M^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M^{2n+1}$. Given a contact form $\eta$ it is well known that there exists a unique vector field $\xi$, called the characteristic vector field of $\eta$, such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for every vector field $X$ on $M^{2n+1}$. A Riemannian metric is said to be associated metric if there exists a tensor field $\phi$ of type $(1, 1)$ such that

\begin{align}
&d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \\
&\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \\
&g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\end{align}

for all vector fields $X, Y$ on $M^{2n+1}$. Then the structure $(\phi, \xi, \eta, g)$ on $M^{2n+1}$ is called a contact metric structure and the manifold $M^{2n+1}$ equipped with such structure is called a contact metric manifold [3].

Given a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ we define a $(1, 1)$ tensor field $h$ by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then $h$ is symmetric and satisfies $h\phi = -\phi h$. Thus, if $\lambda$ is an eigenvalue of $h$ with eigenvector $X$, $-\lambda$ is also an eigenvalue with eigenvector $\phi X$. Also we have $Tr. h = Tr. \phi h = 0$ and $h\xi = 0$. 

Moreover, if \( \nabla \) denotes the Riemannian connection of \( g \), then the following relation holds:

\[
(2.4) \quad \nabla_X \xi = -\phi X - \phi h X.
\]

The vector field \( \xi \) is Killing vector with respect to \( g \) if and only if \( h = 0 \). A contact metric manifold \( M^{2n+1}(\phi, \xi, \eta, g) \) for which \( \xi \) is a Killing vector is said to be a \( K \)-contact manifold. A contact structure on \( M^{2n+1} \) gives rise to an almost complex structure on the product \( M^{2n+1} \times R \). If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is Sasakian if and only if the relation

\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y
\]

holds for all \( X, Y \), where \( R \) denotes the curvature tensor of the manifold.

**Lemma 2.1.** \cite{4} Let \( M^{2n+1}(\phi, \xi, \eta, g) \) be a contact metric manifold with \( R(X, Y)\xi = 0 \) for all vector fields \( X, Y \) tangent to \( M \). Then \( M \) is locally isometric to the Riemannian product \( E^{n+1}(0) \times S^n(4) \).

For a contact metric manifold \( M^{2n+1}(\phi, \xi, \eta, g) \), the \((k, \mu)\)-nullity distribution is

\[
p \to N_p(k, \mu) = [Z \in T_pM : R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\} + \mu\{g(Y, Z)hX - g(X, Z)hY\}]
\]

for any \( X, Y \in T_pM \), \( k, \mu \) are real numbers. Hence, if the characteristic vector field \( \xi \) belongs to the \((k, \mu)\)-nullity distribution, then we have

\[
(2.5) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].
\]

Thus a contact metric manifold satisfying relation \((2.5)\) is called a \((k, \mu)\)-contact metric manifold \cite{5}. In particular, if \( \mu = 0 \), then the notion of \((k, \mu)\)-nullity distribution reduces to the notion of \( k \)-nullity distribution reduces to the notion of \( k \)-nullity distribution, introduced by Tanno \cite{16}. A \((k, \mu)\)-contact metric manifold is Sasakian if and only if \( k = 1 \). In a \((k, \mu)\)-contact metric manifold the following relations hold (\cite{5}, \cite{13}):

\[
(2.6) \quad h^2 = (k - 1)\phi^2, \quad k \leq 1,
\]

\[
(2.7) \quad (\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX),
\]

\[
(2.8) \quad (\nabla_X h)(Y) = \{((1 - k)g(X, \phi Y) + g(X, h\phi Y))\xi + \eta(Y)[h(\phi X + \phi hX)] - \mu\eta(X)\phi hY,
\]

\[
(2.9) \quad R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX],
\]

\[
(2.10) \quad \eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)],
\]
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(2.11) \[ S(X, \xi) = 2nk\eta(X), \]

(2.12) \[ Q\phi - \phi Q = 2[2(n - 1) + \mu]h\phi, \]

(2.13) \[ S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) + [2(1 - n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1, \]

(2.14) \[ r = 2n(2n - 2 + k - n\mu), \]

(2.15) \[ S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \]

where \( S \) is the Ricci tensor of type \( (0, 2) \), \( Q \) is the Ricci-operator, i.e., \( g(QX, Y) = S(X, Y) \) and \( r \) is the scalar curvature of the manifold. From (2.4), it follows that

(2.16) \[ (\nabla_X \eta)(Y) = g(X + hX, \phi). \]

Also we have from (2.5) that

(2.17) \[ (\nabla_W R)(X, Y)\xi = k[g(W + hW, \phi Y)X - g(W + hW, \phi X)Y] + \mu[g(W + hW, \phi Y)hX - g(W + hW, \phi Y)hY] + \{(1 - k)g(W, \phi X) + g(W, h\phi X)\}\eta(Y)\xi - \{(1 - k)g(W, \phi Y) + g(W, h\phi Y)\}\eta(X)\xi + \mu\eta(W)\{\eta(X)\phi hY - \eta(Y)\phi hX\} + R(X, Y)\phi W + R(X, Y)\phi hW. \]

3. GENERALIZED \( \phi \)-RECURRENT \((k, \mu)\)-CONTACT METRIC MANIFOLDS

**Definition 3.1.** A generalized \((k, \mu)\)-contact metric manifold \((M^n, g)\) is said to be a generalized \( \phi \)-recurrent generalized \((k, \mu)\)-contact metric manifold if the relation

(3.1) \[ \phi^2((\nabla_W R)(X, Y)Z) = A(W)\phi^2(R(X, Y)Z) + B(W)\phi^2(G(X, Y)Z) \]

holds for all \( X, Y, Z, W \in \chi(M) \) and \( A \) and \( B \) are two non vanishing 1-forms such that \( A(X) = g(X, \rho_1) \), \( B(X) = g(X, \rho_2) \). Here \( \rho_1 \) and \( \rho_2 \) are vector fields associated with 1-forms \( A \) and \( B \) respectively.

Let us consider a generalized \( \phi \)-recurrent generalized \((k, \mu)\)-contact metric manifold. Then by virtue of (2.2), we have from (3.1) that

(3.2) \[ - (\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)[-R(X, Y)Z + \eta(R(X, Y)Z)\xi] + B(W)[-G(X, Y)Z + \eta(G(X, Y)Z)\xi] \]

from which it follows that
\[
\begin{align*}
(3.3) \quad -g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) \\
= A(W)[-g(R(X, Y)Z, U) + \eta(R(X, Y)Z)\eta(U)] \\
+ B(W)[-g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)].
\end{align*}
\]

Taking an orthonormal frame field and then contracting (3.3) over \(X\) and \(U\) and then using \((1.2)\) and \((2.9)\), we get
\[
\begin{align*}
&-g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) \\
= A(W)[-g(R(X, Y)Z, U) + \eta(R(X, Y)Z)\eta(U)] \\
+ B(W)[-g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)].
\end{align*}
\]

Plugging \(Z = \xi\) in (3.4), we obtain
\[
(3.5) \quad (\nabla_W S)(Y, \xi) = A(W)S(Y, \xi) + 2nB(W)\eta(Y).
\]

By virtue of \((2.4)\), \((2.11)\) and \((2.16)\) it follows from (3.3) that
\[
(3.6) \quad 2nkg(W + hW, \phi Y) + S(Y, \phi W + \phi hW) = 2n[kA(W) + B(W)]\eta(Y).
\]

Setting \(Y = \xi\) in (3.6) and using \((2.2)\) and \((2.11)\), we get
\[
(3.7) \quad kA(W) + B(W) = 0.
\]

In view of \((3.7)\), \((3.6)\) yields
\[
(3.8) \quad S(Y, \phi W + \phi hW) = 2nkg(Y, \phi W + \phi hW).
\]

Replacing \(Y\) by \(\phi Y\) in (3.8) and using \((2.3)\) and \((2.15)\), we get
\[
(3.9) \quad S(Y, W + hW) = 2nkg(Y, W + hW) + 2(2n - 2 + \mu)g(hY, W + hW).
\]

Again replacing \(Y\) by \(hY\) in (3.9) and using \((2.2)\) and \((2.6)\), we get
\[
(3.10) \quad S(Y, hW) - (k - 1)S(Y, W) = -2nk(k - 1)g(Y, W) \\
- 2(k - 1)(2n - 2 + \mu)g(hY, W) \\
+ 2(k - 1)(2n - 2 + \mu)\eta(W)\eta(hY).
\]

Subtracting (3.10) from (3.9), we get
\[
(3.11) \quad kS(Y, Z) = 2nk^2g(Y, W) + 2k(2n - 2 + \mu)g(hY, W) \\
- 2(k - 1)(2n - 2 + \mu)\eta(W)\eta(hY).
\]

This leads to the following:

**Theorem 3.1.** In a generalized \(\phi\)-recurrent generalized \((k, \mu)\)-contact metric manifold, the 1-forms \(A\) and \(B\) are related by the relation \((3.7)\) and the Ricci tensor \(S\) is of the form \((3.11)\).
Changing $W, X, Y$ cyclically in (3.3) and adding them we get by virtue of Bianchi identity and using (3.7), we get

\begin{equation}
A(W)[-g(R(X,Y)Z, U) + kg(G(X,Y)Z, U) + \{\eta(R(X,Y)Z)
- k\eta(G(X,Y)Z)\}\eta(U)] + A(X)[-g((R(Y,W)Z, U) + kg(G(Y,W)Z, U)
+ \{\eta(R(Y,W)Z) - k\eta(G(Y,W)Z)\}\eta(U)]) + A(Y)[-g(R(W,X)Z, U)
+ kg(G(W,X)Z, U) + \{\eta(R(W,X)Z) - k\eta(G(W,X)Z)\}\eta(U)] = 0.
\end{equation}

Contracting (3.12) over $Y$ and $Z$, we get

\begin{equation}
A(W)[-S(X, U) + 2nkg(X, U)] - A(X)[-S(W, U) + 2nkg(W, U)]
+ A(R(W, X)U) + k\{A(X)g(W, U) - A(W)g(X, U)\} - A(R(W, X)\xi)\eta(U)
- k\{A(X)\eta(W) - A(W)\eta(X)\}\eta(U) = 0.
\end{equation}

Again contracting (3.13) over $X$ and $U$ and using (2.5), we get

\begin{equation}
2A(QW) - [r - 2n(2n - 1)]A(W) - \mu A(hW) = 0.
\end{equation}

This leads to the following:

**Theorem 3.2.** In a generalized $\phi$-recurrent generalized $(k, \mu)$-contact metric manifold, the relation (3.14) holds for all $W$.

Using (2.9), (2.17) and the relation $g((\nabla_W R)(X,Y)Z, U) = -g((\nabla_W R)(X,Y)U, Z)$, we have

\begin{equation}
(3.15) \quad g((\nabla_W R)(\xi, Y)Z, \xi) = \mu\{(1 - k)g(W, \phi Y) + g(W, h\phi Y)
- g(hY, \phi(W + hW))\}\eta(Z) - \mu\eta(W)g(\phi hY, Z).
\end{equation}

By virtue of (3.7) and (3.15) it follows from (3.1) that

\begin{equation}
(3.16) \quad (\nabla_W S)(Y, Z) = A(W)S(Y, Z) - 2nka(W)g(Y, Z)
+ \mu\{A(W)\eta(hY) - (1 - k)g(W, \phi Y)
- g(W, h\phi Y) + g(hY, \phi(W + hW))\}\eta(Z)
- A(W)g(hY, Z) + \mu\eta(W)g(\phi hY, Z).
\end{equation}

This leads the following:

**Theorem 3.3.** A generalized $\phi$-recurrent generalized $(k, \mu)$-contact metric manifold is generalized Ricci recurrent if and only if the following relation holds:

\[\{A(W)\eta(hY) - (1 - k)g(W, \phi Y) - g(W, h\phi Y) + g(hY, \phi(W + hW))\}\eta(Z)
- A(W)g(hY, Z) + \mu\eta(W)g(\phi hY, Z) = 0.\]

Again from (3.2), we get

\begin{equation}
(3.17) (\nabla_W R)(X,Y)\xi = A(W)[k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}]
+ B(W)[\eta(Y)X - \eta(X)Y].
\end{equation}
From (2.17) and (3.17) we obtain

\[
(3.18) \quad k \left[ g(W + hW, \phi Y)X - g(W + hW, \phi X)Y \right] + \mu \left[ g(W + hW, \phi Y)hX - g(W + hW, \phi X)hY + \{(1 - k)g(W, \phi X) + g(W, h\phi X)\} \eta(X)\xi \right. \\
+ g(W, h\phi X)\eta(Y)\xi - \{(1 - k)g(W, \phi Y) + g(W, h\phi Y)\} \eta(X)\xi \\
+ \mu \eta(W) \{ \eta(X)\phi hY - \eta(Y)\phi hX \} + R(X, Y)\phi W \\
+ R(X, Y)\phi hW - B(W)\eta(Y)X - \eta(X)Y \\
- A(W) [k\{ \eta(Y)X - \eta(X)Y \} + \mu \{ \eta(Y)hX - \eta(X)hY \} ] = 0.
\]

Replacing \( W \) by \( \phi W \) in (3.18) and using (2.2) we get

\[
(3.19) \quad R(X, Y)W + R(X, Y)hW \\
= k \left[ g(W + hW, Y)hX - g(W + hW, X)hY \right] + \mu \left[ g(W + hW, Y)hX - g(W + hW, X)hY \right. \\
+ \{(1 - k)g(W, X) - g(W, hX) + \eta(W)\eta(hX)\} \eta(Y)\xi \\
- \{(1 - k)g(W, X) - g(W, hY) + \eta(W)\eta(hY)\} \eta(X)\xi \\
- B(\phi W) [\eta(Y)X - \eta(X)Y ] - A(\phi W) [k\{ \eta(Y)X - \eta(X)Y \} \\
+ \mu \{ \eta(Y)hX - \eta(X)hY \} ].
\]

This leads to the following:

**Theorem 3.4.** In a generalized \( \phi \)-recurrent generalized \( (k, \mu) \)-contact metric manifold, the curvature tensor \( R \) satisfies the relation (3.19).

### 4. Example of Generalized \( \phi \)-Recurrent Generalized \( (k, \mu) \)-contact Metric Manifold

**Example 4.1.** We consider a 3-dimensional manifold \( M = \{x, y, z\} \in \mathbb{R}^3 : x \neq 0, y \neq 0\}, where \((x, y, z)\) are the standard coordinates in \( \mathbb{R}^3 \). Let \( \{E_1, E_2, E_3\} \) be a linearly independent global frame on \( M \) given by

\[
E_1 = \frac{\partial}{\partial y}, \quad E_2 = 2xy \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.
\]

Let \( g \) be the Riemannian metric defined by \( g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1 \). Let \( \eta \) be the 1-form defined by \( \eta(U) = g(U, E_3) \) for any \( U \in \chi(M) \). Let \( \phi \) be the \((1, 1)\) tensor field defined by \( \phi E_1 = -E_2, \phi E_2 = E_1 \) and \( \phi E_3 = 0 \). Then using the linearity of \( \phi \) and \( g \) we have \( \eta(E_3) = 1, \phi^2 U = -U + \eta(U)E_3 \) and \( g(\phi U, \phi W) = g(U, W) - \phi(U)\phi(W) \) for any \( U, W \in \chi(M) \). Moreover \( hE_1 = -E_1, hE_2 = E_2 \) and \( hE_3 = 0 \). Thus for \( E_3 = \xi, (\phi, \xi, \eta, g) \) defines a
contact metric structure on $M$. Let $\nabla$ be the Riemannian connection of $g$. Then we have

$$[E_1, E_2] = \frac{1}{y} E_2, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$ 

Using Koszul formula for the Riemannian metric $g$, we can easily calculate

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = 0,$$

$$\nabla_{E_2} E_1 = -\frac{1}{y} E_2, \quad \nabla_{E_2} E_2 = \frac{1}{y} E_1, \quad \nabla_{E_2} E_3 = 0,$$

$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$ 

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is a generalized $(k, \mu)$-contact metric structure on $M$. Consequently $M^3(\phi, \xi, \eta, g)$ is a generalized $(k, \mu)$-contact metric manifold with $k = -\frac{1}{y}$ and $\mu = -\frac{1}{y}$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_1, E_2)E_1 = \frac{2}{y^2} E_2, \quad R(E_1, E_2)E_2 = \frac{2}{y^2} E_1,$$

and the components which can be obtained from these by the symmetry properties. We shall now show that such a generalized $(k, \mu)$-contact metric manifold is generalized $\phi$-recurrent. Since $\{E_1, E_2, E_3\}$ forms a basis of $M^3$, any vector field $X, Y, Z \in \mathcal{X}(M)$ can be written as

$$X = a_1 E_1 + b_1 E_2 + c_1 E_3,$$

$$Y = a_2 E_1 + b_2 E_2 + c_2 E_3,$$

$$Z = a_3 E_1 + b_3 E_2 + c_3 E_3,$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1, 2, 3$. Then

$$R(X, Y)Z = \frac{2}{y^2} (a_1 b_2 - a_2 b_1) (a_3 E_2 - b_3 E_1)$$

and

$$G(X, Y)Z = (a_2 a_3 + b_2 b_3 + c_2 c_3) (a_1 E_1 + b_1 E_2 + c_1 E_3)$$

$$- (a_1 a_3 + b_1 b_3 + c_1 c_3) (a_2 E_1 + b_2 E_2 + c_2 E_3).$$

By virtue of (4.1) we have the following:

$$\nabla_{E_1} R(X, Y)Z = \frac{4}{y^3} (a_1 b_2 - a_2 b_1) (b_3 E_1 - a_3 E_2),$$

$$\nabla_{E_2} R(X, Y)Z = 0,$$

$$\nabla_{E_3} R(X, Y)Z = 0.$$
From (4.1) and (4.2), we get
\[ \phi^2(R(X, Y))Z = u_1 E_1 + u_2 E_2 \quad \text{and} \quad \phi^2(G(X, Y))Z = v_1 E_1 + v_2 E_2, \]
where
\[ u_1 = \frac{2b_3}{y^2}(a_1b_2 - a_2b_1), \quad u_2 = -\frac{2a_3}{y^2}(a_1b_2 - a_2b_1), \]
\[ v_1 = a_2(b_1b_3 + c_1c_3) - a_1(b_2b_3 + c_2c_3), \]
\[ v_2 = b_2(a_1a_3 + c_1c_3) - b_1(a_2a_3 + c_2c_3). \]

Also from (4.3)-(4.5), we obtain
\[ \phi^2((\nabla_{E_i} R)(X, Y))Z = p_i E_1 + q_i E_2 \quad i = 1, 2, 3, \]
where
\[ p_1 = -\frac{4b_3}{y^3}(a_1b_2 - a_2b_1), \quad q_1 = \frac{4a_3}{y^3}(a_1b_2 - a_2b_1), \]
\[ p_2 = 0, \quad q_2 = 0, \quad p_3 = 0, \quad q_3 = 0. \]

Let us consider the 1-forms as
\[ A(E_1) = \frac{v_2 p_1 - v_1 q_1}{u_1 v_2 - u_2 v_1}, \quad B(E_1) = \frac{u_1 q_1 - u_2 p_1}{u_1 v_2 - u_2 v_1}, \]
\[ A(E_2) = 0, \quad B(E_2) = 0, \]
\[ A(E_3) = 0, \quad B(E_3) = 0, \]
where \( a_1b_2 - a_2b_1 \neq 0, v_2 p_1 - v_1 q_1 \neq 0, u_1 q_1 - u_2 p_1 \neq 0, u_1 v_2 - u_2 v_1 \neq 0. \)

From (3.1), we have
\[ \phi^2((\nabla_{E_i} R)(X, Y))Z = A(E_i)\phi^2(R(X, Y))Z + B(E_i)\phi^2(G(X, Y))Z, \quad i = 1, 2, 3. \]

By virtue of (4.6)-(4.8), it can be easily shown that the manifold satisfies the relation (4.9). Hence the manifold under consideration is a 3-dimensional generalized \( \phi \)-recurrent generalized \((k, \mu)\)-contact metric manifold which is neither \( \phi \)-symmetric nor \( \phi \)-recurrent.

This leads to the following:

**Theorem 4.1.** There exists a 3-dimensional generalized \( \phi \)-recurrent generalized \((k, \mu)\)-contact metric manifold, which is neither \( \phi \)-symmetric nor \( \phi \)-recurrent.

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