Unitary representations of centrally extended mapping class groups $\tilde{M}_{g,1}, g \geq 1$ are given in terms of a rational Hopf algebra $H$, and a related generalization of the Verlinde formula is presented. Formulae expressing the traces of mapping class group elements in terms of the fusion rules, quantum dimensions and statistics phases are proposed.
1. Introduction

Mapping class groups $M_{g,n}$ of $n$-holed genus $g$ oriented surfaces $\Sigma_{g,n}$ [1] occur as covariance groups of (anti)holomorphic blocks of $n$-point genus $g$ correlation functions in a RCFT [2]. From the $M_{g,n}$-representations given in [2] on a peripheral basis of vertex operators it is clear that only the superselection structure [3] of the corresponding chiral RCFT is involved in the game. However the superselection structure of a chiral unitary RCFT can also be extracted from the DHR-endomorphisms [4] of the underlying chiral observable algebra [5] and can be encoded in the global symmetry algebra $H$: the unitary representations of the global symmetry algebra and the DHR-endomorphisms of the observable algebra form equivalent braided monoidal rigid $C^*$-categories by definition [6] and the common monodromy matrix, which is connected to a twofold braiding, satisfies a non-degeneracy condition [7, 8]. Therefore one should be able to derive the same $M_{n,g}$ representations from the corresponding global symmetry algebra $H$ itself.

In [8] rational Hopf algebras (RHA) were proposed as global symmetry algebras of RCFT-s and the categorical equivalence of RHA representations and DHR-endomorphisms of a rational QFT was shown in [9]. In special cases of RHAs, namely, when $H$ is the (possibly deformed) double $\mathcal{D}(G)$ of a finite group $G$, the $M_{g,n}$ representations\footnote{For recent results in the case of a general quantum double see [10], and for a more categorical and knot theoretical settings see [11] and [12], respectively.} were already given in [13] and meeting our expectations one can extend this result to a general RHA. Here we give a short summary of our results [14] on unitary representation of the centrally extended mapping class groups $\tilde{M}_{g,1}$ in terms of RHAs. It was inspired by the geometrical presentation of $M_{g,1}$ on admissible tangles given in [12].

As a byproduct of the genus one case $M_{1,1}$, generalized fusion matrices $\{N_p(t), t \in \hat{H}\}$ labelled by the irrep $t$ of the hole are found, that satisfy the usual fusion algebra and are diagonalized by the $S(t)$-matrices, hence can be expressed by a generalized Verlinde formula.

The sequence of mapping class group representations associated in this way to the RHA should obey very specific consistency conditions. We propose to investigate these conditions through the study of the traces of the mapping class group transformations, and give a set of (conjectural) formulae expressing these traces in terms of the usual data characterizing the RHA. We also derive a curious result about the asymptotic (large genus) behaviour of the traces.

2. Unitary representations of $\tilde{M}_{g,1}$

Let $U$ and $A$ denote the universal and the left adjoint representations of the RHA $H$, i.e.

$$
U = \bigoplus_{p \in \hat{H}} D_p, \quad A = \bigoplus_{p \in \hat{H}} D_p \times \bar{D}_p, \tag{1}
$$

where $\hat{H}$ is the finite set of unitary equivalence classes of irreducible $H$-representations, $D_p$ is a representative of the class $p \in \hat{H}$, the bar indicates contragredient representations and
\( \times \) denotes the product of representations. Let \( I(g, n) \equiv (D(g, n)|D(g, n)) \) denote the *-algebra of self-intertwiners of the product representation \( D(g, n) := \mathcal{A} \times \ldots \times \mathcal{A} \times U \times \ldots \times U \) containing \( g \mathcal{A} \) and \( n \mathcal{U} \) factors respectively.\(^1\) \( I(g, n) \) is a direct sum of full matrix algebras

\[
I(g, n) = \bigoplus_{r \in H} I_r(g, n) \equiv \bigoplus_{r \in H} (D(g, n)|D_r)(D_r|D(g, n)),
\]

labelled by the intermediate irreducible representation of \( H \). The elements of \( M_{g, n} \) will be represented by unitary elements from \( I_0(g, n) \), where 0 refers to the trivial representation. \( I_0(g, n) \) acts on the Hilbert space \( \mathcal{H}_0(g, n) \equiv (D(g, n)|D_0) \) of intertwiners by left multiplication. If \( C_{pq}^{r\alpha} \in (D_p \times D_q|D_r); p, q, r \in H, \alpha = 1, \ldots, N_r^r \) are the basic intertwiners, i.e.

\[
C_{pq}^{r\alpha*}C_{pq}^{r'\alpha'} = \delta_{rr'}\delta_{\alpha\alpha'}D_r(1), \quad \sum_{r \in H} \sum_{\alpha=1}^{N_p^r} C_{pq}^{r\alpha}C_{pq}^{r\alpha*} = (D_p \times D_q)(1),
\]

then an orthonormal basis of \( \mathcal{H}_0(g, n) \), the so-called path basis, can be given by chains of basic intertwiners

\[
|\mathbf{x}\rangle = C_{0p_1}^{c_1}C_{c_1p_2}^{s_1\sigma_1}C_{c_2p_2}^{s_2\sigma_2} \ldots C_{c_n-1p_g}^{s_n\sigma_n}C_{c_gp_g}^{s_g\sigma_g}C_{g, n}^{r_1r_2r_2r_3} \ldots C_{N_{p_1}^r}^{0}\]

labelled by the multiindex \( \mathbf{x} = \{p_i, c_i, s_i, \gamma_i, \sigma_i; t_j, r_j, \tau_j|i = 1, \ldots, g; k = 1, \ldots, n\} \). A basis element can be visualized as the labelled skeleton of a \( \Sigma_{g,n} \) surface cutted along the \( d_i \) \((i = 1, \ldots, g)\) curves of Fig. 1. The irreps \( p_i, \tilde{p}_i, t_k \) flow in the holes, the irreps \( c_i, s_i, r_k \) label the intermediate channels.

In every RHA there is a central unitary balance element \( b = S(R_2^*\varphi_1\lambda^*)l^*R_1^*\varphi_2lS(\rho\varphi_3) \) expressed by universal elements of \( H \), which obeys the properties \([9]\]

\[
R_{21}R_{12} = (b^* \otimes b^*)\Delta(b), \quad b = S(b),
\]

where \( R \equiv R_{12} \) is the universal \( R \)-matrix of \( H \) and \( S \) denotes the antipode. The pure phases \( \omega_p = \omega_{\tilde{p}}, p \in \tilde{H} \) that appear in the central decomposition of \( b \) are called statistics phases. The universal elements \( r, l \in H \) lead to the statistics (or quantum) dimensions \( d_p \geq 1 \) of the irreps: \( d_p = \text{Tr}D_p(rr^*) = \text{Tr}D_p(l*l) \), which satisfy the fusion rule algebra \( d_p d_q = \sum_r N_{pq}^r d_r \) and the equality \( d_p = d_{\tilde{p}} \). They can have non-integer values only in case of unit-nonpreserving coproducts. In case of RHAs the quantity \( \sigma := \sum_{p \in \tilde{H}} \omega_p^2 d_p^2 \) satisfies \(|\sigma|^2 = \sum_{p \in \tilde{H}} d_p^4 \) and defines the ‘central charge’ of the RHA: \( c = (8/2\pi i) \log |\sigma|^2/|\sigma| \in [0, 8) \).

Generalized 6j-symbols, \( F_{\alpha\beta\gamma\nu\delta}^{(pqr)} \), of a RHA are given \([15]\) in terms of the basic intertwiners and the associator \( \varphi \) of \( H \) by

\[
F_{\alpha\beta\gamma\nu\delta}^{(pqr)} \cdot D_{t}(1)\delta_{tt'} = C_{ur}^{t\beta\alpha*}C_{pq}^{u\alpha*}(D_p \otimes D_q \otimes D_r)(\varphi)C_{qr}^{u\gamma*}C_{pv}^{t\delta*}.
\]

\(^1\) Although for a precise definition of \( D(g, n) \) we have to prescribe a bracketing in the product since \( H \) is not necessarily coassociative, all of the bracketings lead to canonically equivalent representations, hence canonically equivalent self-intertwiner spaces.
They are unitary matrices in the lower multiindices describing the change of the orthonormal product bases of basic intertwiners. An appropriate choice of basic intertwiners [15] leads to a simple form of them together with $S_4$ symmetry for the correctly normalized symbols.

Let’s turn to the mapping class group representations. It is known [1] that $M_{g,1}$ is generated by Dehn twists $a_i, b_i, d_i, e_i, i = 1, \ldots, g$ around the corresponding curves in Fig. 1. and a presentation can be given in terms of $a_i, b_i, i = 1, \ldots, g$ and $e_2$ [16, 12]. Knowing the geometrical realization of $M_{g,1}$ by the isomorphic group, $T_{2g}$ of certain equivalence classes of admissible tangles [12], a central extension $\tilde{M}_{g,1} := \tilde{T}_{2g}$ of $M_{g,1}$ can be easily obtained by splitting up equivalence classes in $T_{2g}$; the central generator of $z$, which has been the $K_1$-equivalence in [12], will correspond to an insertion (elimination) of a $\pm 1$ framed separated unknot into (from) a tangle diagram. On the level of tangles it clearly defines a central generator, which in terms of a RHA is represented by multiplication by the phase $\exp(\pm 2\pi i c/8)$. Now $\tilde{M}_{g,1}$ has a presentation in terms of the generators $a_i, b_i, i = 1, \ldots, g; e_2, z$ together with the relations

$$b_i a_i b_i = a_i b_i a_i, \quad b_i a_{i+1} b_i = a_{i+1} b_i a_{i+1}, \quad b_2 e_2 b_2 = e_2 b_2 e_2, \quad (7a)$$

and every other pair of generators commute,

$$(a_2 b_1 a_1)^4 = z^4 k e_2 k^{-1} e_2, \quad k = b_2 a_2 b_1 a_1^2 b_1 a_2 b_2,$$

$$(a_3 a_2 a_1)^{-1} g_1 g_2 e_2 = w e_2 w^{-1}, \quad g_2 = (b_2 a_3 a_2 b_2) e_2 (b_2 a_3 a_2 b_2)^{-1},$$

$$g_1 = (b_1 a_2 a_1 b_1) g_2 (b_1 a_2 a_1 b_1)^{-1}, \quad w = b_3 a_3 b_2 a_2 b_1 a_3^{-1} b_3^{-1} g_2 b_3 a_3 a_1^{-1} b_1^{-1} a_2^{-1} b_1^{-1}.$$

Having constructed a homomorphism [14] — suggested by the tangle diagrams — from $\tilde{T}_{2g}$ into unitary $I_0(g, 1)$ intertwiners of a RHA we obtain unitary representations of $\tilde{M}_{g,1}$. Using the multiindex notation of (4) the explicit form of this representation is as follows:

$$Z_{xx'} = \delta_{xx'} \exp(2\pi i c/8), \quad D(i)_{xx'} = \delta_{xx'} \omega_{p_i}, \quad E(i)_{xx'} = \delta_{xx'} \omega_{c_i},$$

$$A(i)_{xx'} = \delta_{xx'} (\sigma_{i-1} s_{i-1} \gamma_{i-1}) \sum_{q, \alpha, \beta} F_{\sigma_{i-1} s_{i-1} \gamma_{i-1}, \omega}^{(c_{i-1} - 1 p_i - 1) c_i} \omega_{q \alpha} \cdot \omega_{p_i} \cdot F_{\sigma_{i-1} s_{i-1} \gamma_{i-1}}^{(c_{i-1} - 1 p_i - 1) c_i} \omega_{q \alpha},$$

$$B(i)_{xx'} = \delta_{xx'} (p_i \gamma_i e_i) \exp(2\pi i c/8) \omega_{p_i} \cdot \omega_{p_i}^* \sum_{q, \alpha, \beta} F_{\sigma_{i-1} s_{i-1} \gamma_{i-1}}^{(c_{i-1} - 1 p_i - 1) c_i} S(q) \cdot \alpha_{p_i} \cdot \alpha_{p_i}^* F_{\sigma_{i-1} s_{i-1} \gamma_{i-1}}^{(c_{i-1} - 1 p_i - 1) c_i} \omega_{q \alpha},$$

where $\delta_{xx'}(\cdot)$ means the Kronecker delta in the multiindex except the subindices in its argument.

We note that the given representation does not involve intertwiners that change the representation $t$ of the hole therefore the representation space decomposes as $\mathcal{H}_0(g, 1) = \oplus_t \mathcal{H}_0(g; t)$.

As an illustration of the $\tilde{M}(g, 1)$ representations (8) let us consider the case of the Lee-Yang fusion rules:

$$N_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$
There are two inequivalent RHAs having these fusion rules [15], they correspond to the
unitary level one $G_2$ and $F_4$ KM-algebra representations: the common statistical
dimensions are $d_0 = 1, d_1 \equiv d = (1 + \sqrt{5})/2$ and the statistics phases are $\omega_0 = 1, \omega_1 = \omega^4$, with
$\omega = \exp(\pm 2\pi i/10)$. The corresponding central charges are $14/5$ and $26/5$.

For a simple form of the nondiagonal $M_{g,1}$ generators in (8) we have decomposed
the unit into orthogonal projections: $P_{[s_i,1]}^{p_i-1, p_i}$ for the generators $A(i)$ and $P_{[p_i, c_i]}^{p_i-1, p_i}$
for the generators $B(i)$. The upper indices (with fixed admissible irrep values) show
the rank of these projections by listing the admissible irrep values, i.e. the basis for
this subspace. The order of the list shows how the connected matrix is understood.

$$Z_{x x'} = \delta_{x x'} \exp 2\pi i \frac{\pm 7}{20}, \quad D(i)_{x x'} = \delta_{x x'} \omega_{p_i}, \quad E(i)_{x x'} = \delta_{x x'} \omega_{c_i},$$

$$A(i)_{x x'} = \delta_{x x'} (s_i-1) \left\{ P_{[0]}^{0000} + P_{[1]}^{0110} + P_{[1]}^{1001} + \frac{1}{d^2} \left( \begin{array}{c} 1 + \omega d \\ \sqrt{d(1 - \omega)} \omega + d \end{array} \right) P_{[0,1]}^{1111} \\
+ \omega \left( P_{[0]}^{0111} + P_{[1]}^{1010} + P_{[1]}^{1010} + P_{[1]}^{1011} + P_{[1]}^{1100} + P_{[0]}^{1100} + P_{[1]}^{1110} + P_{[1]}^{1111} \right) \right\}$$

$$B(i)_{x x'} = \delta_{x x'} (p_i, c_i) \left\{ P_{[1]}^{0110} + P_{[1]}^{1010} + \frac{\exp 2\pi i c/8}{|\sigma|} \left( \begin{array}{c} 1 \\ \omega^{-1} d \omega^3 \end{array} \right) P_{[0,1,11]}^{00} \\
+ \frac{\omega^{-1} \exp 2\pi i c/8}{|\sigma|} \left( \begin{array}{ccc} 1 \\
\sqrt{d} & d^{-2}(\omega^4 \mp i|\sigma|) & d^{-3/2}(\omega^4 \pm i|\sigma|) \\
\sqrt{d} & d^{-3/2}(\omega^4 \pm i|\sigma|) & d^{-2}(\omega^4 d \mp i|\sigma|) \end{array} \right) P_{[01,10,11]}^{11} \right\}$$

In case of mapping class group generators $A(1), B(1)$ irreps with index $i = 0$ are always
understood as the trivial ones: $p_0 = s_0 = c_0 = 0$. Moreover, $c_1 = p_1$ and $s_g = \ell$ are always
valid. An explicit check of (7) can be easily performed by using that $d = \omega + \omega^{-1}$ and
$d^2 - d - 1 = 0$.

3. Representation of $M_{1,1}$ and the generalized Verlinde formula

An equivalent (redundant) presentation of $M_{1,1}$ to (7) can be given by

$$\langle T, S, R | (ST)^3 = S^2, S^4 = R^{-1} \rangle,$$ \hspace{1cm} (9)

where $R$ has the geometrical meaning of a twist around the hole of the surface $\Sigma_{1,1}$. The
correspondence between the $M_{1,1}$ generators in (7) and (9) is as follows

$$S = (ABA)^{-1} \exp(2\pi i c/8), \quad T = A \exp(-2\pi i c/24).$$ \hspace{1cm} (10)

As we have already mentioned the representation space decomposes according to the irrep
$\ell$ flowing through the hole, $\mathcal{H}_0(1,1) = \oplus_\ell \mathcal{H}_0(1; t)$, and the mapping class group transformations decompose into a direct sum accordingly. In these subspaces the path basis is
characterized by the pair $p, \alpha$ with $\alpha = 1, \ldots, N_{\bar{p} \bar{p}}$ and the generators are represented by the unitary matrices

$$T(t)_p^\alpha = \delta_{pp'} \delta_{\alpha \alpha'} \omega_p e^{-2\pi i \frac{t}{p}}, \quad S(t)_p^\alpha = \frac{1}{|\sigma|} \sum_r \frac{\omega_r}{\omega_p \omega_p'} d_r N_r(t)_p^\alpha',$$

where the generalized fusion matrices $N_r(t)$ read as

$$N_r(t)_p^\alpha = \sum_{\beta=1}^{N_{\bar{p} \bar{p}}'} F_{\beta p \alpha, \bar{\beta} \bar{p} \alpha'}.$$

They satisfy the usual fusion algebra

$$N_p(t)N_q(t) = \sum_r N_{pq}^r N_r(t) \quad t \in \hat{H},$$

and are diagonalized by the corresponding $S(t)$ transformations:

$$N_q(t) s^p(\alpha) = \frac{S(0)q_p}{S(0)_0} s^p(\alpha),$$

where $s^p(\alpha)(t)$ are the column vectors of $S(t)$. Therefore a generalized Verlinde formula holds:

$$N_r(t)_p^\alpha' = \sum_{q}^{N_{\bar{p} \bar{p}}'} S(0)_q^p S(t)_{pq}^{\beta} S(t)^*_{p'q'}.$$ 

Due to the $S_4$-symmetry of the normalized 6j-symbols the generalized fusion coefficients obey also the properties

$$N_r(t) = N_r(t)^* = \overline{N_r(t)} = N_{\bar{r}}(\bar{t})^T,$$

where star, overline and $^T$ indicate adjoint, complex conjugate and transposed matrices.

As an illustration of the generalized fusion matrices and the Verlinde formula let us consider the $(7, 2)$ fusion rules:

$$N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

There are two inequivalent RHAs corresponding these fusion rules [15]: the common statistical dimensions are $d_0 = 1, d_1 = d, d_2 = d^2 - 1$ with $d = 2 \cos(\pi/7)$ and the statistics phases are $\omega_0 = 1, \omega_1 = \omega^2, \omega_2 = \omega^{10}$ with $\omega = \exp(\pm i\pi/7)$. The corresponding central charges are $48/7$ and $8/7$ and $|\sigma| = d^2 + d - 2$. The generalized fusion matrices read as

$$N_0(1) = 1, \quad N_1(1) = \frac{1}{d^2 - 1}, \quad N_2(1) = -\frac{d}{d^2 - 1}.$$
\[ N_0(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_1(2) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{d}} \\ -\frac{1}{\sqrt{d}} & \frac{d}{d^2-1} \end{pmatrix}, \quad N_2(2) = \begin{pmatrix} -\frac{1}{d^2-1} & \frac{\sqrt{d}}{d^2-1} \\ \frac{\sqrt{d}}{d^2-1} & \frac{1}{d^2+1} \end{pmatrix}. \]

The corresponding \( S(t) \)-matrices are

\[ S(0) = \frac{1}{|\sigma|} \begin{pmatrix} 1 & d & d_2 \\ d & -d_2 & 1 \\ d_2 & 1 & -d \end{pmatrix}, \quad S(1) = \omega^{10}, \quad S(2) = \frac{\omega^{-3}}{|\sigma|} \begin{pmatrix} \omega^{-1} + \omega^2 & \sqrt{d}(\omega + 1) \\ \sqrt{d}(\omega + 1) & -\omega^{-1} - \omega^2 \end{pmatrix}. \]

That the \( S(t) \)-matrices diagonalize the corresponding fusion rules and \( S(t)^4 = \omega_{t}^{-1}N_0(t) \) can be easily checked by noting that \( d = \omega + \omega^* \) and \( d^3 - d^2 - 2d + 1 = 0 \).

We close this Section by mentioning that if we had allowed the presence of a nontrivial degenerate sector \( p \in \hat{H}, p \neq 0 \) — when, by definition, all monodromies of \( p \) are trivial, i.e. \( (D_p \otimes D_q)(R_{21}R_{12}) = (D_p \times D_q)(1) \) for all \( q \) — then the \( S(0) \) matrices would have been degenerate and the \( S(t), t \neq 0 \) matrices would have even contained identically zero rows corresponding to the degenerate sector. This means that a nontrivial sector should be braided at least with one of the others in order to obtain a modular structure. Sectors having completely permutation statistics are excluded.

### 4. Trace formulae

As we have seen in the previous sections, the knowledge of the RHA associated to a given theory enables us to compute explicitly the relevant mapping class group representations. But for some applications, e.g. the classification of RCFT-s or the study of the higher genus aspects of string theory, this is not really what is needed. In these problems the interesting question is to find out what kind of relations exist between the different higher genus characteristics of a given theory, for example between the (linear equivalence classes of) mapping class group representations. That there should be some relations is clear from the fact that all these representations are determined by a finite set of data, the 6j-symbols and statistics phases of the RHA. Our task is to make them explicit.

A natural strategy is to try to express the above mentioned relations in terms of the traces of the mapping class group transformations. We shall illustrate this idea first in the case of the one-holed torus, and we shall comment on the general case later.

A word of caution is in order here. All of the results to be presented in this Section have been derived in the context of orbifold models, i.e. the corresponding RHAs are (possibly deformed) doubles of a finite group, where the availability of group theory techniques made possible the computations. We have not been able to prove them for general RHAs, so the status of the following results is only conjectural. Nevertheless, we have checked them numerically in a variety of models, including those with \( \mathbb{Z}_2, \mathbb{Z}_3, \) Ising, Lee-Yang, and \((7,2)\) fusion rules, and in our opinion this strongly supports their validity in general.

Let’s turn to the results. It is more convenient to work with the class functions \( \chi_p \) instead of the partial traces \( \text{Tr}_t \) on \( \mathcal{H}_0(1; t) \). The former are defined for \( X \in M_{1,1} \) via the (invertible) rule

\[ \chi_p(X) = S_{0p} \sum_t \tilde{S}_{pt} \text{Tr}_t(X), \quad (20) \]
where we use $S_{pt} = S(0)^{t}_{p}$ for short. A simple observation is that the $\chi_{p}$-s are normalized, i.e. $\chi_{p}(1) = 1$. Moreover, $\chi_{p}(X^{-1}) = \bar{\chi}_{p}(X)$.

The first interesting result is that for many elements $X \in M_{1,1}$, the quantity $\chi_{p}(X)$ may be expressed in terms of the fusion rules, the quantum dimensions and the statistics phases of the sectors. Here is a sample (with the notation $\eta = \exp(-i\pi \frac{c_{I}^{2}}{4T})$):

| $T^{n}$ | $\eta^{n} \sum_{q} |S_{pq}|^{2} \omega_{q}^{n}$ |
|-------|----------------------------------|
| $S$   | $\frac{\eta^{6}}{\sigma^{3}} \sum_{q,r} N_{pq}^{r} d_{p} d_{q} d_{r} \frac{\omega_{r}^{4} \omega_{q}^{4}}{\omega_{q}^{2}}$ |
| $S^{2}T^{n}$ | $\frac{\eta^{8}}{\sigma^{4}} \sum_{q,r,s,t} N_{pq}^{t} N_{rs}^{t} d_{p} d_{q} d_{r} d_{s} \frac{\omega_{r}^{2} \omega_{q}^{2} \omega_{s}^{2}}{\omega_{q}^{2} \omega_{r}^{2} \omega_{s}^{2}}$ |
| $ST$  | $\frac{\eta^{7}}{\sigma^{3}} \sum_{q,r} N_{pq}^{r} d_{p} d_{q} d_{r} \frac{\omega_{p}^{6} \omega_{r}^{2}}{\omega_{q}^{2}}$ |
| $(ST)^{2}$ | $\frac{\eta^{9}}{\sigma^{4}} \sum_{q,r,s,t} N_{pq}^{t} N_{rs}^{t} d_{p} d_{q} d_{r} d_{s} \frac{\omega_{r}^{3} \omega_{q}^{3} \omega_{s}^{3}}{\omega_{s}^{3} \omega_{q}^{3} \omega_{r}^{3}}$ |

(21)

Let’s observe that the above formulae lead to non-trivial relations even for the partial trace $\text{Tr}_{0}$, that is for the closed torus, e.g. for the transformation $S$ we get

$$\sum_{p} \chi_{p}(S) \equiv \text{Tr}_{0}(S) = \frac{\eta^{6}}{\sigma^{3}} \sum_{p,q,r} N_{pq}^{r} d_{p} d_{q} d_{r} \frac{\omega_{p}^{4} \omega_{q}^{4}}{\omega_{q}^{2}},$$

(22)

the last equality being far from trivial.

The fact that the transformation $S^{4}$ is nothing but a Dehn-twist around the hole implies

$$\chi_{p}(S^{4}X) = \eta^{3} \sum_{q} S_{pq} d_{p} \omega_{p} \omega_{q} \frac{d_{p} \omega_{p} \omega_{q} \chi_{q}(X)}{d_{q}}.$$

(23)

Two more important properties of the class functions $\chi_{p}$ will be of use later. The first is that if $d_{r} = 1$ and $N_{pq}^{q} = 1$, then $\chi_{p} = \chi_{q}$, in particular $\chi_{p} = \chi_{0}$ for all irreps $p$ whose quantum dimension is 1. The second one is the following interesting factorization property:

$$\chi_{0} \left( T^{k} S^{-1} T^{m} S \right) = \chi_{0} (T^{k}) \chi_{0} (T^{m}).$$

(24)

Up to now we have been concerned with the mapping class group $M_{1,1}$ of the one-holed torus, but clearly the above results may be applied to investigate the higher genus mapping class groups. This is so because sewing allows one to imbed naturally (in many
different ways) $M_{1,1}$ in $M_{g,0}$, and any such imbedding allows us to view a transformation $X \in M_{1,1}$ as an element $\hat{X}$ of $M_{g,0}$. A simple argument shows that

$$\text{Tr}(\hat{X}) = \sum_{p,q} S_{0p}^{1-2g} S_{qp} \text{Tr}_q(X) = \sum_p S_{0p}^{2-2g} \chi_p(X), \quad (25)$$

where the trace on the lhs. is over the space of genus $g$ characters. Note that (25) implies that asymptoticaly

$$\frac{\text{Tr}(\hat{X})}{\text{Tr}(1)} \rightarrow \chi_0(X) \quad \text{as} \quad g \rightarrow \infty, \quad (26)$$

which together with (24) leads to the following curious result:

If $\delta_1, \delta_2 \in M_{g,0}$ are Dehn-twists around simple closed curves whose linking number is one, then in the limit $g \rightarrow \infty$ the trace of their product factorizes up to a constant of proportionality equal to the trace of the identity transformation.

5. Concluding remarks

The morale of our work is that the modular geometry of a CFT is just a reflection of its superselection structure. One of the advantages of the algebraic approach developed in this paper is that it makes easier the deeper study of this connection. We feel that the trace formulae presented in the last section could be a first step in this direction. Of course, much more has to be done to achieve this goal.

We hope that our work would be useful in the study of field theories defined on a space-time of nontrivial topology as well as in a better understanding of the nonperturbative aspects of string theory.

As a final remark we note that although the emergence of mapping class group transformations seems to be natural in the context of conformal field theories, one should be able to give a meaning of these transformations when a RHA arises as a global symmetry of a QFT without conformal invariance (e.g. certain lattice field theories [17]). We think that Schroer’s vacuum polarization picture [18] is a natural answer: mapping class group transformations describe the possible unitary selfintertwiners in the presence of spectator charges and vacuum splittings that ‘sum up’ to the trivial sector. In our treatment the $M_{g,n}$ generators act exactly on this ‘spectator’ space, on $H_0(g,n)$, where the factors of universal and the left adjoint representations correspond to the spectator charges and the vacuum splittings, respectively.

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