Complex analysis/Dynamical systems

Brody curves in complicated sets

Courbes de Brody dans quelques ensembles compliqués

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A B S T R A C T

For a hyperbolic generalized Hénon mapping (in the sense of [3]), $J^+$, the boundary of the set of points with bounded orbit is known as a complicated set and also known to admit a lamination by biholomorphic images of $\mathbb{C}$ (see [3,6]). We prove that there exists a leaf, which is an injective Brody curve in $\mathbb{P}^2$, in the lamination of $J^+$ for certain generalized Hénón mappings (for Brody curves and injective Brody curves, see Subsection 2.2).

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R É S U M É

L’ensemble $J^+$ des points d’orbite bornée est connu, pour une application de Hénon généralisée hyperbolique (dans le sens de [3]), comme étant un ensemble compliqué admettant une lamination par images biholomorphes de $\mathbb{C}$ (voir [3,6]). Nous montrons que, pour certaines applications de Hénon généralisées hyperboliques, une feuille de cette lamination $J^+$ est une courbe de Brody injective dans $\mathbb{P}^2$ (voir la sous-section 2.2 pour les notions de courbes de Brody et courbes de Brody injectives).

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1. Introduction

A generalized Hénon mapping $f$ is defined simply by a polynomial diffeomorphism $f(z, w) = (p(z) - aw, z)$ of $\mathbb{C}^2$, where $p$ is a monic polynomial of one complex variable and $a$ is a non-zero constant. Then, $f^{-1}(z, w) = (w, (p(w) - z)/a)$. Define

$$K^\pm = \{ p \in \mathbb{C}^2 : |f^{\pm n}(p)| \text{ is a bounded sequence of } n \}.$$ 

and $J^\pm = \partial K^\pm$, $K = K^+ \cap K^-$, $J = J^+ \cap J^-$ and $U^\pm = \mathbb{C}^2 \setminus K^\pm$. Let $g^+ : \mathbb{C}^2 \to \mathbb{R}$ denote the Green function associated with $f$. Then $U^+ = \{ g^+ > 0 \}$ and $K^+ = \{ g^+ = 0 \}$, and $U^+$ is open and $K^+$ is closed.

In [7], it was proved that the level set $\{ g^+ = c \}$ for $c > 0$ is foliated by biholomorphic images of $\mathbb{C}$ and that each leaf is dense in $\{ g^+ = c \}$. In [1] and [2], it was proved that every leaf is actually an injective Brody curve.

In this note, we study the same or a similar property for $J^+$. In [3,4] and [6], the lamination structure of $J^+$ was studied. In particular, in [3], Bedford and Smillie proved that $J^+$ admits a lamination $\mathcal{F}^+$ by biholomorphic images of $\mathbb{C}$ for
a hyperbolic generalized Hénon mapping \( f \). Then, one might ask "Is \( J^+ \) foliated by injective Brody curves as \( U^+ \) is?" Since \( U^+ \) and \( J^+ \) have completely different dynamical natures, we cannot apply the method of [1] and [2]. So, we rather consider a preceding question in this note: "Is there any leaf of \( J^+ \) which is injective Brody?"

The purpose of this note is to prove the following:

**Theorem 1.1.** Let \( f(z, w) = (p(z) - aw, z) \) where \( p \) is a monic polynomial of one complex variable and \( a \) is a non-zero constant. Assume that \( f \) is hyperbolic (in the sense of [3]) and \( |a| \leq 1 \). Then, in the natural lamination \( \mathcal{F}^+ \) of \( J^+ \), there exists a leaf that is an injective Brody curve of \( \mathbb{P}^2 \).

**Remark 1.** If we restrict Theorem 1.1 to the Hénon mappings in [6], then \( J^+ \) has fractional Hausdorff dimension and, as in [1] and [2], the injective Brody curve leaf is dense in \( J^+ \).

**Remark 2.** Since \( J^+ \) is closed in \( \mathbb{C}^2 \), due to Theorem 2.1, it is not difficult to find a Brody curve in \( J^+ \) for an arbitrary generalized Hénon mapping by applying the Brody reparametrization lemma with Theorem 2.1. However, it is not clear whether this Brody curve stays inside a single leaf, whether the Brody curve is injective, and whether Remark 1 is true for this Brody curve. The main point of Theorem 1.1 is that we have these properties.

The main ingredients for Theorem 1.1 are the hyperbolicity of \( f \) and flow-boxes of the lamination of \( J^+ \) and they are quite different from those for [1] and [2].

**Notation.** We use \( \Delta(a, r) \) for the disc in \( \mathbb{C} \) centered at \( a \in \mathbb{C} \) and of radius \( r > 0 \) and \( \Delta \) for the standard unit disc in \( \mathbb{C} \). We denote by \( \|\cdot\| \) the standard Euclidean metric of \( \mathbb{C}^2 \) and by \( ds(P, V) \) the standard Fubini-Study metric on \( \mathbb{P}^2 \) of \( V \in T_P \mathbb{P}^2 \) at \( P \in \mathbb{P}^2 \). In this note, we are interested in the Fubini-Study metric on \( \mathbb{C}^2 \subset \mathbb{P}^2 \). With respect to the affine coordinate chart \( (z, w) \in \mathbb{C}^2 \subset \mathbb{P}^2 \), the standard Fubini-Study metric is defined by \( ds((z, w), (z', w')) = (|z|^2 + |w|^2 + |zw' - z'w|^2)/(1 + |z|^2 + |w|^2) \) for \( (z', w') \in T_{(z, w)} \mathbb{P}^2 \). For a holomorphic curve \( \gamma : U \to \mathbb{P}^2 \) and for \( \theta' \in U \), \( \|\gamma\|_{\mathcal{F}, \theta'} \) denotes \( ds(\gamma(\theta')), d\gamma_{|\theta=\theta'}(\frac{d\theta}{d\theta'}) \), where \( U \) is an open set in \( \mathbb{C} \).

2. Preliminaries

2.1. Generalized Hénon mappings

Let \( \mathbb{P}^2 \) be the 2-dimensional complex projective space. We denote by \( I_+ := \{0 : 1 : 0\} \) in the homogeneous coordinate system of \( \mathbb{P}^2 \). Then, \( f \) has the natural extension to \( \tilde{f} : \mathbb{P}^2 \setminus \{I_+\} \to \mathbb{P}^2 \setminus \{I_+\} \) by 
\[
\tilde{f}(\{z : w : t\}) = \left[t^d p\left(\frac{z}{t}\right) - awt^{d-1} : zt^{d-1} : r^d\right].
\]

The following proposition and theorem describe the behavior of \( J^+ \).

**Proposition 2.1.** (See [9].) \( \overline{K^+} = K^+ \cup I_+ \) in \( \mathbb{P}^2 \).

**Theorem 2.1.** (See Theorem 1.3 in [2].) There is no non-trivial holomorphic curve, which passes through \( I_+ \), and is supported in \( \overline{K^+} \subset \mathbb{P}^2 \).

We recall hyperbolicity for generalized Hénon mappings in [3] (see [8] and also [6]). Recall that \( J \) is an invariant set for \( f \). If a generalized Hénon mapping \( f \) is hyperbolic, there are continuous subbundles \( E_u \) and \( E_s \) such that \( T \mathbb{C}^2 = E_s \oplus E_u \), and \( Df(E^u) = E^u \) and \( Df(E^s) = E^s \), and there exist constants \( c > 0 \) and \( 0 < \lambda < 1 \) such that 
\[
\|Df^n|_{E^s}\| < c\lambda^n, \ n \geq 0 \quad \text{and} \quad \|Df^{-n}|_{E^u}\| < c\lambda^n, \ n \geq 0.
\]

The Stable Manifold Theorem and Theorem 5.4 in [3] imply that, for every \( x \in J \), there exists a leaf \( \mathcal{L}_x \) in \( \mathcal{F}^+ \) such that \( x \in \mathcal{L}_x \) and \( T_x \mathcal{L}_x = E^s_x \) where \( \mathcal{F}^+ \) is the natural lamination of \( J^+ \).

2.2. Brody curves

In this subsection, we briefly introduce **Brody curves** and **injective Brody curves**.

**Definition 2.2 (Brody curve).** Let \( M \) be a compact complex manifold with a smooth metric \( ds_M \). Let \( \psi : \mathbb{C} \to M \) be a non-constant holomorphic map.

The map \( \psi \) is said to be **Brody** if \( \sup_{\theta' \in \mathbb{C}} ds_M(\psi(\theta'), d\psi_{|\theta=\theta'}(\frac{d\theta}{d\theta'})) < C_\phi \) for some constant \( C_\phi > 0 \). We call the image \( \psi(\mathbb{C}) \) a **Brody curve** in \( M \). The curve \( \psi(\mathbb{C}) \) is said to be **injective Brody** if the parameterization \( \psi \) is injective.
Remark 3. Note that since $M$ is assumed to be compact, Brodyness is independent of the choice of the metric $d_{SM}$. For the purpose of simpler computations, in the remainder of the note, we will consider the Fubini–Study metric $ds$ on $\mathbb{P}^2$.

Below, we consider some trivial examples. The proofs are all straightforward and so, we omit them.

Proposition 2.3. Let $\alpha$ be a complex constant and $p, q$ polynomials of one complex variable $z$. Then, all curves of the form $[z : p(z) : 1]$ and of the form $[p(z) \exp(z) : q(z) \exp(az) : 1]$ are Brody.

However, not all holomorphic curves from $\mathbb{C}$ to $\mathbb{P}^2$ are Brody. The mapping $z \to [\exp(z) : \exp(iz^2) : 1]$ is not Brody. The following gives us some examples of injective but non-Brody curves.

Proposition 2.4. The map $f_n : z \to (z, \exp(z^2))$ is not Brody in $\mathbb{C}^2 \subset \mathbb{P}^2$ for $n \geq 3$. In particular, not all holomorphic images of $\mathbb{C}$ in $\mathbb{P}^2$ are Brody.

We close this section by pointing out a property of the injective Brody curves. Since the proof is straightforward, we omit it.

Proposition 2.5. For an injective Brody curve $C$ in $\mathbb{P}^2$, every parameterization of $C$ has uniformly bounded Fubini–Study metrics. In short, the injective Brodyncity property does not depend on the choice of the parameterization.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We first define a family of analytic discs. From Corollary 6.13 in [3], periodic points are dense in $J$. Pick a periodic point $P \in J$ and say $N$ its period. Let $L_p$ be a leaf in the laminar $\mathcal{J}^+$ of $J^+$ passing through $P$ as discussed in Section 2. Fix an analytic disc $\psi : \Delta \to L_p$ such that $\psi(0) = P$ and $\|\psi\|_{FS,0} > 0$. Then we consider a family of analytic discs as follows:

$$\varphi_n := f^{-Nn} \circ \psi : \Delta \to L_p.$$  

Then, $L_p$ is a stable manifold of $P$, from the hyperbolicity of $f$, $\|\varphi_n\|_{FS,0} \to \infty$ as $n \to \infty$.

Now we apply the Brody reparameterization lemma as in [5]. Note that $\varphi_n$‘s are holomorphic in a slightly larger disc. Define $H_n : \Delta \to \mathbb{R}^+$ by $H_n(\theta) := \|\varphi_n\|_{FS,0}(1 - |\theta|^2)$. Then, there exists $\theta_n \in \Delta$ with $H_n(\theta_n) = \max_{\theta \in \Delta} H_n(\theta)$. For each $n$, define a Möbius transformation $\mu_n(z) := (z + \theta_n)/(1 + \overline{\theta_n}z)$ mapping $0$ to $\theta_n$. Let $\eta_n := \varphi_n \circ \mu_n$. Then

$$\|\eta_n\|_{FS,\Delta}(1 - |\theta|^2) = \|\varphi_n\|_{FS,\Delta}(1 - |\mu_n(z)|^2) = \|\varphi_n\|_{FS,0}(1 - |\theta|^2).$$

So, $\|\eta_n\|_{FS,\Delta} \leq \|\varphi_n\|_{FS,0}(1 - |\theta|^2)$. Let $R_n = \|\varphi_n\|_{FS,0}$ and define $k_n(\theta) = \eta_n(\theta/R_n)$. Then,

$$\|k_n\|_{FS,0} = \|\eta_n\|_{FS,\Delta} \leq \frac{\|\varphi_n\|_{FS,0}}{R_n(1 - |\theta|/R_n^2)} \leq 2,$$

on $\Delta(0, R_n/2)$. Note that $\|k_n\|_{FS,0} = 1$ and that from the hyperbolicity of $f$, we see that $R_n \to \infty$ as $n \to \infty$. Hence, from a normal family argument applied to $\{k_n\}$ and the compactness of $\mathbb{P}^2$, there exists a holomorphic map $\Phi : \mathbb{C} \to \overline{\mathbb{C}^2} \subset \mathcal{J}^+ \subset \mathbb{P}^2$ with $\|\Phi\|_{FS,0} = 1$ and a subsequence $\{k_{n_j}\}$ locally uniformly converging toward $\Phi$. In particular, $\Phi$ is a Brody map. From Proposition 2.1, we have $\mathcal{J}^+ = J^+ \cup \{I_+\}$. However, Theorem 2.1 implies that $\Phi(\mathbb{C}) \subset J^+$.

We prove that the Brody curve $\Phi(\mathbb{C})$ sits inside a single leaf of the lamination $\mathcal{J}^+$ of $J^+$. Suppose the contrary. Then, there exist two points $\alpha, \beta \in \mathbb{C}$ such that $\Phi(\alpha)$ and $\Phi(\beta)$ live in two different leaves and that $\alpha, \beta$ are sufficiently close so that some small piece of the complex curve $\Phi(\mathbb{C})$ connecting $\Phi(\alpha)$ and $\Phi(\beta)$ sits in a single flow-box of the lamination of $J^+$. Let $\gamma \subset \Phi(\mathbb{C})$ denote the piece of the complex curve $\Phi(\mathbb{C})$ connecting $\Phi(\alpha)$ and $\Phi(\beta)$. Then, there exists a constant $\epsilon > 0$ such that for any plaque $T$ in the flow-box, $sup_{(z,w) \in \gamma} dist((z,w), T) > \epsilon$ where dist $(\cdot, \cdot)$ is with respect to the standard Euclidean distance of $\mathbb{C}^2$. This is a contradiction to the local uniform convergence of $\{k_{n_j}\}$ to $\Phi$, since the image of each reparameterized analytic disc sits inside a single leaf of the lamination $\mathcal{J}^+$ of $J^+$. This proves that $\Phi(\mathbb{C})$ sits in a single leaf of the lamination $\mathcal{J}^+$ of $J^+$.

We show that $\Phi$ is one-to-one. Suppose on the contrary that $\Phi$ is not one-to-one. Then, there are $\alpha : \beta \in \mathbb{C}$ and $q \in \Phi(\mathbb{C})$ such that $\alpha = \beta$ and $\Phi(\alpha) = \Phi(\beta) = q$. Consider a sufficiently large $R_q > 1$ such that $\alpha, \beta \in \Delta(0, R_q)$. Let $F$ be a compact set of $\mathbb{C}^2$ such that its interior contains $J$. Consider a finite covering of $J^+ \cap F$ consisting of flow-boxes of $\mathcal{J}^+$. Note that since $|\alpha| \leq 1$, Theorem 5.9 in [3] says that for any leaf $\mathcal{L}$ in $J^+$, there exists a point $x \in J$ such that $\mathcal{L}$ is a stable manifold of the point $x$. Since $\Phi(\Delta(0, 2R_q))$ lies in a single leaf, there exists sufficiently large $N_q \in \mathbb{N}$ such that the analytic disc $f^{N_q}(\Phi(\Delta(0, 2R_q)))$ is entirely contained in a flow-box in the finite covering. By passing to a subsequence, we may assume that $k_n$ converges to $\Phi$ locally uniformly and $f^{N_q}(k_n(\Delta(0, 2R_q)))$ is entirely contained in the same flow-box. Let $\pi$ denote the projection onto the base in the flow-box. Then, since the image of each reparameterized analytic disc sits inside a single
leaf of the lamination $F^+$ of $J^+$, $\pi \circ f^N_q \circ k_\nu$’s are injective. Since the convergence of $\{k_\nu\}$ to $\Phi$ is locally uniform and $\Phi$ is not a constant map, the Hurwitz theorem of one complex variable implies that $\pi \circ f^N_q \circ \Phi$ is injective. So, we have $\pi \circ f^N_q \circ \Phi(\alpha) \neq \pi \circ f^N_q \circ \Phi(\beta)$, which contradicts $\Phi(\alpha) = \Phi(\beta) = q$. Hence, we just proved that $\Phi$ is injective.

Note that each leaf in the lamination $F^+$ of $J^+$ is biholomorphic to $\mathbb{C}$ (see [3]). Since there is no proper biholomorphic image of $\mathbb{C}$ inside $\mathbb{C}$, the leaf containing the injective Brody curve itself should be an injective Brody curve. This proves our theorem. □

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