Relating Quarkonium Wave Functions at the Origin

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Within the context of nonrelativistic potential models, we obtain several formulas (with varying degrees of rigor) relating the wave functions at the origin of the $c\bar{c}$, $b\bar{c}$ and $b\bar{b}$ S-wave quarkonium systems. One of our main results is a model-independent relation which seems to hold to within 3\% for any reasonable choice of interquark potential and any choice of radial quantum number — namely, $|\Psi_{b\bar{c}}(0)|^2 \simeq |\Psi_{c\bar{c}}(0)|^{1.3} |\Psi_{b\bar{b}}(0)|^{0.7}$ (the exponents are motivated in the text). One of the physical consequences of this result is the following relationship between heavy meson masses which we expect to hold at about the 10\% level: $M_{B^*_c} - M_{B_c} \simeq (0.7)(M_{J/\psi} - M_{\eta_c})^{0.65}(M_{\Upsilon} - M_{\eta_b})^{0.35}$.

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Recent major advances in our understanding of the nonrelativistic limit of Quantum Chromodynamics (QCD) have generated renewed interest in the calculation of the production and decay rates of heavy quark bound states. However, these computations still contain numerous nonperturbative “parameters” which cannot as yet be accurately determined analytically from first principles. They must either be fit to experiment, determined numerically from lattice simulations of QCD, or extracted by more phenomenological considerations (for example, from potential models). One important class of such parameters is the wave function at the origin (WFO), $\Psi(0)$, for an S-wave bound state of a heavy quark and anti-quark. (More generally, for a bound state with angular momentum $\ell$, we should consider the quantity $d^\ell \Psi / dr^\ell$ evaluated at the origin. However, in what follows we will concentrate on the case $\ell = 0$.) The WFO enters not only into the production and decay amplitudes for heavy quarkonium systems, but also into the determination of the hyperfine splitting in their mass spectra. Large QCD and relativistic corrections to the simple first order formulas which relate the WFO’s to the above observables (especially for charmonium) make it difficult to extract precise information about the WFO’s from the experimental data. Moreover, while there has been much recent progress (see, for example, ), lattice simulations of QCD are not yet accurate enough to be very useful. This leaves us at the mercy of a more model-dependent approach.

In nonrelativistic potential model descriptions of heavy quarkonia, it is a simple numerical exercise to extract highly accurate values of the WFO’s for any given choice of the static potential between a heavy quark and anti-quark. The problem lies in which potential to choose. Two potentials which yield very similar spectra for the heavy mesons can give very different WFO’s for any given state. The WFO’s seem to be quite sensitive to the global details of the potential, while the energy levels are only sensitive to the shape of the potential in the vicinity of the RMS radii of the states being studied. In other words, when this sensitivity of the WFO’s is coupled with our ignorance regarding the details of the heavy quark potential, we again arrive at the sobering conclusion that an accurate determination of the WFO’s seems beyond our reach.

So what’s a theorist to do? Well, the situation isn’t quite so dire as we have made it sound. There are certain rigorous, qualitative statements that one can make about WFO’s within the context of potential models which hold for very large classes of potentials — including those believed to be relevant for heavy quark systems. For example, in a concave downward potential $V(r)$ (defined by $V'(r) > 0$ and $V''(r) < 0$ for all values of the interquark distance $r$), it can be shown that the square of the WFO for the 1S state is
larger than that of the 2S state \[3\]. (Most likely it is more generally true that the square of the WFO decreases monotonically with increasing radial quantum number, but this has not, to our knowledge, been shown yet.) Moreover, it can be proven from lattice QCD that the static potential between two heavy color sources is concave downward \[4\]. An accurate enough determination of the WFO’s for low-lying heavy mesons can be made from the experimental data in order to test this qualitative result, despite the size of QCD and relativistic corrections. And indeed it holds true in both the $c\bar{c}$ and $b\bar{b}$ systems. Another such result is that the square of the WFO for the 1S state increases with the two-body reduced mass $\mu$ faster than linearly for a concave downward potential \[5\]. (This result is also conjectured, but not proven, for higher states.) A comparison of the WFO’s extracted from the $c\bar{c}$ and $b\bar{b}$ data is consistent with this theorem as well.

Should we have panicked if either of these two results were violated by the WFO’s obtained from the data? No, not necessarily. We could have attributed the discrepancy to (at least) two possible sources. First, the uncertainty in any WFO extracted from the data can be estimated from an educated guess at the size of the higher order QCD and relativistic corrections not included in the determination (as well as the experimental uncertainty in the measurement). It is possible that these corrections are larger than we expect, and therefore a WFO obtained with the truncated series is not as accurate as we thought. Second, it is possible that our naive picture of, say, the $J/\psi$ as a simple bound state of a $c$ quark and a $\bar{c}$ quark in a relative S-wave interacting via a static potential is incorrect. For example, one may have to consider a dynamical treatment of excited glue inside the meson, or allow mixing with other angular momentum states and/or continuum states. There is a nice example of this latter possibility \[6\]. It is strongly believed that in a concave downward potential, the energy splitting between the $(n + 2)S$ and $(n + 1)S$ states is always less than the splitting between the $(n + 1)S$ and $nS$ states, for any $n \geq 1$. However, in the charmonium system, though the measured 3S-2S splitting is less than the 2S-1S splitting, the 4S-3S splitting shows an increase over the 3S-2S difference. Accepting the truth of the above conjecture concerning energy splittings in a concave downward potential, how do we explain the experimental numbers? The answer is that the threshold for open charm production occurs between the 2S and 3S levels and induces substantial mixing of the 3S and 4S $c\bar{c}$ states with continuum states. Thus, what we experimentally identify as the 3S and 4S levels of charmonium actually have substantial $D\bar{D}$ and $D^*\bar{D}^*$ components, among others. A similar state of affairs occurs in the bottomonium system. Here, the measured 5S-4S splitting is greater than the 4S-3S splitting. The open bottom
threshold occurs between the 3S and 4S levels and causes substantial mixing of the 4S and 5S $b\bar{b}$ states with continuum states. It is this mixing in both the $c\bar{c}$ and $b\bar{b}$ systems that seems to be responsible for the apparent violations of the above energy splitting conjecture. This interpretation is supported by a coupled-channel analysis in the charmonium system \cite{7}. The only purpose in showing this example is to remind the reader that no matter how generally a certain result may apply within the context of nonrelativistic potential models, there are still assumptions that must be made in order to relate these potential model results to real observations. And these assumptions may not hold for all states in all systems.

With this disclaimer behind us, we can now describe the results of this note. The starting point for our investigation is a recent paper by Eichten and Quigg \cite{8} which tabulates the WFO's for various quarkonium states in an assortment of “successful” potential models. We list them below (in natural units).

(1) The Martin potential \cite{9}: \[ V(r) = Ar^{0.1} + C, \] where $A = 6.898$ GeV$^{1.1}$, $m_c = 1.8$ GeV and $m_b = 5.174$ GeV.

(2) The log potential \cite{10}: \[ V(r) = A\ln(r/r_0), \] where $A = 0.733$ GeV, $m_c = 1.5$ GeV and $m_b = 4.906$ GeV.

(3) The Cornell potential \cite{11}: \[ V(r) = -A/r + Br + C, \] where $A = 0.52$, $B = (1/2.34)^2$ GeV$^2$, $m_c = 1.84$ GeV and $m_b = 5.17$ GeV.

(4) The Buchmüller-Tye potential \cite{12}: This potential has a rather complicated position space form. It is linear at large distances and quasi-Coulombic at short distances. The deviations from pure Coulombic behavior reproduce the running of the strong coupling constant to next-to-leading order in QCD. The global shape of the potential is essentially determined by two parameters — namely, the QCD scale (in the modified minimal subtraction scheme) $\Lambda_{\overline{MS}}$ which the authors of \cite{12} fit to be 509 MeV, and the QCD string tension which they take to be 0.153 GeV$^2$ (motivated by the light meson data). The potential also depends on the number of “light” flavors $n_f$. The authors take $n_f = 3$ for $r \geq 0.01$ fm, and $n_f = 4$ for $r < 0.01$ fm. The quark masses used are $m_c = 1.48$ GeV and $m_b = 4.88$ GeV.

(The parameters $C$ in (1) and (3) and $r_0$ in (2) are irrelevant for $|\Psi(0)|^2$.) Eichten and Quigg treat the $c\bar{c}$, $b\bar{b}$ and $b\bar{c}$ systems.
The first thing that catches one’s eye in glancing at these tables is the apparent randomness of the entries. Of course one can spot the aforementioned general trends — namely, for a fixed quark content, the square of the S-wave WFO decreases with increasing radial excitation, and for fixed quantum numbers the square of the WFO gets bigger as one goes from the $c \bar{c}$ to the $b \bar{c}$ to the $b \bar{b}$ system (increasing reduced mass). However, besides these qualitative behaviors, no additional regularity is apparent. For example, the square of the WFO for the $\Upsilon$ changes by about a factor of 3 between the various potentials — potentials which yield basically the same low-lying spectrum! Things like the ratio of the $\psi(2S)$ and the $J/\psi$ WFO’s, or the ratio of the $\Upsilon$ and the $J/\psi$ WFO’s, also cover a large range of values. Can any additional statements about these numbers be made which possess some degree of model-independence?

Before we address this question, we would first like to present our own version of the S-wave portion of Tables I-III in [8], which corrects some small numerical errors made there. For instance, it is well known that for power-law potentials $V(r) = Ar^a + C$, the square of the S-wave WFO scales with reduced mass $\mu$ as $\mu^{3/(2+a)}$ [5]. This result can also be used for the log potential by putting $a = 0$. The results of [8] show a mild violation of this scaling (on the order of a few percent for all radial quantum numbers) which cannot be accounted for by rounding errors. Upon our redoing of the computations using the Runge-Kutta method for solving the nonrelativistic Schrödinger equation, we found results which satisfied the scaling laws (within rounding errors) for the log and Martin potentials, and typically disagreed with the results of [8] in the second significant figure. We also tested our program on potentials with analytically known WFO’s, such as the Coulomb, linear and harmonic oscillator potentials, and obtained agreement with the exact results to at least six significant figures. We then ran our program on the other potentials treated in [8], the Cornell and Bückmuller-Tye potentials, and found similar disagreements to those encountered in the log and Martin cases. It should be stressed however that these mild errors in no way affect the conclusions of [8]. We just want numerical results which are as accurate as possible in order to test some approximate formulas relating different WFO’s that we will derive later.

For all of these potentials we display results for the ground state as well as the next five radial excitations. This goes a little further than the results in [8]. Many of these states lie above the threshold for open flavor production, and hence in a region where the WFO’s have limited usefulness because of mixing with continuum states. However, these
numbers are still quite useful in checking the general validity of the analytic formulas which are to come.

We have also added one additional potential to the table.

(5) The Lichtenberg-Wills potential \[ V(r) = \frac{8\pi(1 - \Lambda r)^2}{[(33 - 2n_f)\pi n^2}(\Lambda r)], \] where we choose \( \Lambda = 0.7 \text{ GeV}, m_c = 1.84 \text{ GeV} \) and \( m_b = 5.17 \text{ GeV} \). At short distances, the running of the strong coupling constant to leading order in QCD is reproduced if one identifies \( \Lambda = e^\gamma \Lambda_{QCD}, \) where \( \gamma \) is Euler’s constant. We have also taken the number of light flavors \( n_f \) to be three in all systems studied with this potential.

Note that we have chosen the \( b \) and \( c \) constituent quark masses to be the same as for the Cornell potential. The parameter \( \Lambda \) was then chosen so as to obtain a low-lying meson spectrum reasonably close to that obtained from the Cornell potential. The differences in the WFO’s between the Cornell and Lichtenberg-Wills examples are then basically due to the different shapes of the potentials outside of the region between the RMS radii of the \( c\bar{c} \) and \( b\bar{b} \) systems. A substantial difference can still be seen between the two sets of WFO’s, again emphasising their sensitivity to global features of the interquark potential.

We first became interested in finding regularities in these numbers after a comment made to one of us by Ira Rothstein. He was able to prove that, in nonrelativistic QCD in the limit as \( m_b \to m_c \), the square of the WFO for any state in the \( b\bar{c} \) system is equal to the average of the squares of the WFO’s of the corresponding \( c\bar{c} \) and \( b\bar{b} \) states, plus a correction of order \( \delta^2 \) where \( \delta = (m_b - m_c)/(m_b + m_c) \) \[ 14 \]. That is, \[ |\Psi_{b\bar{c}}(0)|^2 = \left( |\Psi_{c\bar{c}}(0)|^2 + |\Psi_{b\bar{b}}(0)|^2 \right)/2 + O(\delta^2). \] (1)

We have shown that this is also true in an arbitrary nonrelativistic potential model. Indeed, one can prove a slightly stronger result:

\[ |\Psi_{b\bar{c}}(0)| = \left( |\Psi_{c\bar{c}}(0)| + |\Psi_{b\bar{b}}(0)| \right)/2 + O(\delta^2). \] (2)

A similar formula also holds for the geometric mean instead of the arithmetic mean:

\[ |\Psi_{b\bar{c}}(0)|^2 = |\Psi_{c\bar{c}}(0)||\Psi_{b\bar{b}}(0)| + O(\delta^2). \] (3)

These last two results can be easily demonstrated from perturbation theory in \( \delta \). However, even though these results are independent of the nature of the interquark forces, they are
unfortunately not very useful in real applications since the quantity $\delta$ is approximately $1/2$ for reasonable values of $m_b$ and $m_c$. The order $\delta^2$ corrections in the above equations are therefore large, which a simple check using the numbers in Table I will show.

What we want is a relation with the model-independence of Eqs.(1)-(3), but with much more quantitative accuracy. For the class of power-law potentials $V(r) = Ar^a + C$ discussed earlier, there is a very simple, exact relationship between the WFO’s of the $c\bar{c}$, $b\bar{c}$ and $b\bar{b}$ systems. In order to derive this relation, we first recall that simple scaling arguments for the above power-law potentials tell us that for any reduced mass $\mu$ we have $|\Psi_\mu(0)|^2 = f(n,a)(A\mu)^{3/(2+a)}$, where $f(n,a)$ is only a function of the radial quantum number $n$ and the power $a$. Using this fact alone, it is straightforward to obtain, for reduced masses $\mu_1 < \mu_2 < \mu_3$ and any fixed $n$,

$$|\Psi_{\mu_2}(0)|^2 = |\Psi_{\mu_1}(0)|^{2(1-q)}|\Psi_{\mu_3}(0)|^{2q}, \quad (4)$$

where $q = \ell n(\mu_2/\mu_1)/\ell n(\mu_3/\mu_1)$. Choosing $\mu_1 = m_c/2$, $\mu_2 = m_b m_c/(m_b + m_c)$ and $\mu_3 = m_b/2$, this becomes

$$|\Psi_{b\bar{c}}(0)|^2 = |\Psi_{c\bar{c}}(0)|^{2(1-q)}|\Psi_{b\bar{b}}(0)|^{2q}, \quad (5)$$

where $q = \ell n(2m_b/(m_b + m_c))/\ell n(m_b/m_c)$. This result is nice not only for its simplicity, but also because it does not depend on any of the parameters ($A$, $a$ and $C$) appearing in the potential. It depends only on the constituent quark masses $m_b$ and $m_c$. It is easy to check this result on the log ($a = 0$) and Martin ($a = 0.1$) potentials in Table I. Since this formula has no dependence on parameters in the potential, we can also check it on the other examples in Table I. Of course it will no longer be exact in these cases since the above scaling law for $|\Psi_\mu(0)|^2$ is true (for all $\mu$) only for power-law potentials. And these other potentials are far from being power-like. They each have a (quasi-)Coulombic nature at small $r$, motivated from one gluon exchange, and a (quasi-)linear behavior at large $r$, motivated by a stringy picture of confinement. In this sense, they are more “realistic” than the power-law potentials. In the intermediate $r$ range containing the RMS radii of the heavy quarkonium states, they are quasi-logarithmic, just like the log and Martin potentials. But, though Eq.(5) is not exact here, we can still ask if it is a reasonably accurate approximation.

The answer is yes. For every choice of $n$ in Table I, the relation in Eq.(5) holds to within 4% (except for the 1S state of the Cornell potential where it is off by about 7% ).
The least accurate results are obtained for the ground state. As \( n \) increases, the results get better. This is a substantial improvement over the accuracy of Eqs.(1)-(3). The only price that we have had to pay is the introduction of the constituent quark masses into the relation. It is interesting to note that the left-hand side of Eq.(5) is less than or equal to the right-hand side for each potential considered and each choice of \( n \). Is it possible that this is always the case — at least for a wide class of potentials? A numerical study of numerous examples, as well as an analysis of the question within the context of various approximation schemes, has led us to the following conjecture:

**Conjecture:** Consider a potential \( V(r) \) such that \( p(r) \equiv rV''(r)/V'(r) \) is monotonically increasing with increasing \( r \). Then for each choice of radial quantum number, and for reduced masses \( \mu_1 < \mu_2 < \mu_3 \), we have

\[
|\Psi_{\mu_2}(0)|^2 < |\Psi_{\mu_1}(0)|^{2(1-q)}|\Psi_{\mu_3}(0)|^{2q},
\]

where \( q \) is as in Eq.(4). For \( p(r) \) monotonically decreasing with increasing \( r \), the inequality in Eq.(6) is reversed.

Of course when \( p(r) \) is independent of \( r \), \( V(r) \) is a power-law potential and the inequality in Eq.(6) is replaced by the equality of Eq.(4). One can think of \( 1 + p(r) \) as the “effective power” of \( V(r) \) at quark separation \( r \). We will call a potential power increasing, or PI, when \( p(r) \) is monotonically increasing, and power decreasing, or PD, when \( p(r) \) is monotonically decreasing. Each of the non-power-law quarkonium potentials in Table I is PI (we have checked this numerically for the Buchmüller-Tye potential), and satisfies the inequality in Eq.(6) with the appropriate choices of \( \mu_1, \mu_2 \) and \( \mu_3 \) — namely

\[
|\Psi_{b\bar{c}}(0)|^2 < |\Psi_{c\bar{c}}(0)|^{2(1-q)}|\Psi_{bb}(0)|^{2q},
\]

where \( q \) is as in Eq.(5). Indeed, all of the popular potentials used in quarkonium studies seem to be PI. But, unlike the concave downward property, we know of no QCD-motivated reason why this must be so. But we conjecture that Eq.(7) holds in the nonrelativistic limit of QCD, and in all realistic potential models.

Actually, there is an even better result which is completely parameter-independent. To obtain this, we first note that \( q = \ln(2m_b/(m_b + m_c))/\ln(m_b/m_c) \) lies between about 0.36 and 0.38 for any reasonable choices of \( m_b \) and \( m_c \). However, as noted above, when substituted into Eq.(5) this yields WFO’s for the \( b\bar{c} \) system which are too high. In the
context of the general form of Eq.(5), the potentials of interest seem to favor a slightly lower value of $q$. We have found that if $q$ is simply taken to be 0.35 independent of the interquark potential and quark masses being considered, very accurate results are obtained. That is, we have

$$|\Psi_{b\bar{c}}(0)|^2 \simeq |\Psi_{c\bar{c}}(0)|^{1.3} |\Psi_{b\bar{b}}(0)|^{0.7}. \quad (8)$$

Although no longer exact for power-law potentials, this simple formula holds to within 2.5% for all cases in Table I. This is quite remarkable given the range of radial quantum numbers covered and the global differences in the potentials treated. We fully expect it to have a similar accuracy for any reasonable quarkonium potential. Though not on the same rigorous footing as the two qualitative theorems discussed earlier, it is reasonably well motivated by Eqs.(5) and (7) above. Moreover, it gives us a better quantitative understanding of the jumble of numbers in Table I.

Can we extract any simple physical consequences of this result? Certainly it implies relationships between the production (and decay) amplitudes for the $J/\psi$, $B_c$ and $\Upsilon$ systems. However, it is perhaps simpler to discuss the implications for the hyperfine mass splittings in these systems. To leading order in $\alpha_s$ and $v^2/c^2$, the mass splitting $(\Delta M)_{ij}$ (for any fixed $n$) between the vector and pseudoscalar mesons composed of a quark of flavor $i$ and an antiquark of flavor $j$ (of mass $m_i$ and $m_j$, respectively) is given by

$$(\Delta M)_{ij} = 32\pi\alpha_s(2\mu_{ij})|\Psi_{ij}(0)|^2/9m_im_j, \quad (9)$$

where $\mu_{ij} = m_im_j/(m_i + m_j)$, and we have assumed the standard Breit-Fermi hyperfine interaction [13]. Putting this together with Eq.(8) gives

$$(\Delta M)_{b\bar{c}} = \alpha_s(2\mu_{b\bar{c}})(m_c/m_b)^{0.3}[(\Delta M)_{c\bar{c}}/\alpha_s(m_c)]^{0.65}[(\Delta M)_{b\bar{b}}/\alpha_s(m_b)]^{0.35}. \quad (10)$$

It is interesting to note that for any reasonable choices of $m_b$, $m_c$, and $\Lambda_{QCD}$, the quantity $\alpha_s(2\mu_{b\bar{c}})/\alpha_s(m_c)^{0.65}\alpha_s(m_b)^{0.35}$ has a numerical value which is within about 3% of 1. (In a similar fashion, both QCD and relativistic corrections to Eq.(9) approximately cancel when fed into Eq.(10).) Moreover, $(m_b/m_c)^{0.3}$ is always within a few percent of 0.7. Therefore, we can write

$$(\Delta M)_{b\bar{c}} \simeq (0.7)(\Delta M)_{c\bar{c}}^{0.65}(\Delta M)_{b\bar{b}}^{0.35}. \quad (11)$$

For the ground state, this reads

$$M_{B_c^*} - M_{B_c} \simeq (0.7)(M_{J/\psi} - M_{\eta_c})^{0.65}(M_{\Upsilon} - M_{\eta_b})^{0.35}. \quad (12)$$
Given the sources of uncertainty enumerated above, we expect this result to hold at about the 10% level. Only three of the six mesons appearing in Eq.(12) have been found experimentally thus far — namely, the $J/\psi$, $\eta_c$ and $\Upsilon$. Their masses are known quite accurately \cite{16}. There is some hope that the remaining three mesons will be detected in the near future at either the Fermilab Tevatron or LEP, allowing a test of the above result.

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| Potential       | System | 1S  | 2S  | 3S  | 4S  | 5S  | 6S  |
|-----------------|--------|-----|-----|-----|-----|-----|-----|
| Martin          | $cc$   | 0.979 | 0.545 | 0.390 | 0.309 | 0.257 | 0.222 |
|                 | $bc$   | 1.720 | 0.957 | 0.685 | 0.542 | 0.452 | 0.390 |
|                 | $bb$   | 4.423 | 2.461 | 1.763 | 1.394 | 1.164 | 1.004 |
| logarithmic     | $cc$   | 0.796 | 0.406 | 0.277 | 0.211 | 0.172 | 0.145 |
|                 | $bc$   | 1.508 | 0.770 | 0.524 | 0.401 | 0.325 | 0.275 |
|                 | $bb$   | 4.706 | 2.401 | 1.636 | 1.250 | 1.015 | 0.857 |
| Cornell         | $cc$   | 1.458 | 0.930 | 0.793 | 0.725 | 0.683 | 0.654 |
|                 | $bc$   | 3.191 | 1.769 | 1.449 | 1.297 | 1.205 | 1.141 |
|                 | $bb$   | 14.06 | 5.681 | 4.275 | 3.672 | 3.322 | 3.088 |
| Buchmüller-Tye  | $cc$   | 0.794 | 0.517 | 0.441 | 0.404 | 0.381 | 0.365 |
|                 | $bc$   | 1.603 | 0.953 | 0.785 | 0.705 | 0.658 | 0.625 |
|                 | $bb$   | 6.253 | 3.086 | 2.356 | 2.032 | 1.845 | 1.721 |
| Lichtenberg-Wills | $cc$   | 1.121 | 0.693 | 0.563 | 0.496 | 0.453 | 0.423 |
|                 | $bc$   | 2.128 | 1.231 | 0.975 | 0.846 | 0.766 | 0.711 |
|                 | $bb$   | 6.662 | 3.370 | 2.535 | 2.139 | 1.902 | 1.740 |

**Table 1:** Numerical values of the radial wave function at the origin squared, $|R(0)|^2 = |\Psi(0)|^2/4\pi$, for the first six S-wave states of heavy quarkonium systems in various potential models. The parameters used in the potentials are discussed in the text.