ON THE EULER CHARACTERISTIC OF GENERALIZED KUMMER VARIETIES

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The aim of this note is to apply the Yau-Zaslow-Beauville method ([YZ], [B1]) to compute the Euler characteristic of the generalized Kummer varieties attached to a complex abelian surface (a calculation also done in [GS] by different methods). The argument is very geometric: given an ample line bundle L with \( h^0(L) = n \) on an abelian surface A, such that each curve in \( |L| \) is integral, we construct a projective symplectic \( (2n-2) \)-dimensional variety \( J^d(A) \) with a Lagrangian fibration \( J^d(A) \to |L| \) whose fiber over a point corresponding to a smooth curve C is the kernel of the Albanese map \( J^dC \to A \). The Yau-Zaslow-Beauville’s method shows that the Euler characteristic of \( J^d(A) \) is \( n \) times the number of genus 2 curves in \( |L| \), to wit \( n^2\sigma(n) \) (where \( \sigma(n) = \sum_{m|n} m \)). The latter computation was also done in [G], where a general conjecture (proved in [BL] for K3 surfaces) expresses these numbers in terms of quasi-modular forms: if \( N_n^r \) is the number of genus \( r+2 \) curves in \( |L| \) passing through \( r \) general points of A, one should have

\[
\sum_{n \in \mathbb{N}} N_n^r q^n = \left( \sum_{n \in \mathbb{N}} n\sigma(n)q^n \right)^r \left( \sum_{n \in \mathbb{N}} n^2\sigma(n)q^n \right).
\]

Unlike the case of K3 surfaces, none of these varieties \( J^d(A) \) seem to be birationally isomorphic to the generalized Kummer \( K_{n-1}(A) \) introduced by Beauville in [B2] (a symplectic desingularization of a fiber of the sum morphism \( A(n) \to A \)). However, we check that this is the case when A is a product of elliptic curves and \( d = n-2 \). Using a result of Huybrechts, we conclude that \( K_{n-1}(A) \) and \( J^{n-2}(A) \) are diffeomorphic hence have the same Euler characteristic \( n^3\sigma(n) \). I would like to thank D. Huybrechts very much for his help with theorem 3.4.

1. The symplectic variety \( J^d(A) \)

Let A be a complex abelian surface with a polarization \( \ell \) of type \((1,n)\). Assume that each curve with class \( \ell \) is integral (this holds for generic \((A,\ell)\)). Let \( \hat{A} \) be the dual abelian surface. Let \( \phi_\ell : A \to \hat{A} \) be the morphism associated with the polarization \( \ell \); there exists a factorization \( n\text{Id}_A : A \xrightarrow{\phi_\ell} \hat{A} \xrightarrow{\phi_{\hat{\ell}}} A \), where \( \hat{\ell} \) is a polarization on \( \hat{A} \) of type \((1,n)\).

We denote by \( \text{Pic}^\ell(A) \) the component of the Picard group of A corresponding to line bundles with class \( \ell \), by \( \{\ell\} \) the component of the Hilbert scheme that parametrizes curves in A with class \( \ell \), by \( C \to \{\ell\} \) the universal family, and by \( \tilde{\mathcal{C}} \to \{\ell\} \) the compactified Picard scheme of this family ([AK]).

The variety \( \tilde{\mathcal{C}} \) splits as a disjoint union \( \coprod_{d \in \mathbb{Z}} \tilde{J}^d\mathcal{C} \), where \( \tilde{J}^d\mathcal{C} \) is a projective variety of dimension \( 2n+2 \), which parameterizes pairs \((C,\mathcal{L})\) where C is a curve on A with class \( \ell \) and \( \mathcal{L} \) is a torsion free, rank 1 coherent sheaf on C of degree \( d \) (i.e. with \( \chi(\mathcal{L}) = d + 1 - g(C) = d - n \)). According to Mukai ([M1], ex. 0.5), \( \tilde{J}^d\mathcal{C} \) can be viewed as a connected component of the moduli space of simple sheaves \( \mathcal{L} \) on A, and therefore is smooth, and admits a (holomorphic) symplectic structure. There is a natural morphism

\[
\alpha : \tilde{J}^d\mathcal{C} \to \{\ell\} \to \text{Pic}^\ell(A)
\]

\[
(C,\mathcal{L}) \mapsto C \mapsto [\mathcal{O}_A(C)]
\]

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also defined by $\alpha(L) = \det L$. For each smooth curve $C$ in $\{\ell\}$, the inclusion $C \subset A$ induces an Abel-Jacobi map $J^dC \to A$; this defines a rational map

$$\beta : J^dC \to A \times \{\ell\} \to A$$

which is regular since $A$ is an abelian variety and $J^dC$ is normal. Let $J^d(A)$ be a fiber of the map $(\alpha, \beta) : J^dC \to \text{Pic}^\ell(A) \times A$ (they are all isomorphic). Note that $J^d(A)$, $J^{d+2n}(A)$ and $J^{-d}(A)$ are isomorphic.

**Proposition 1.1.**– The symplectic structure on $\tilde{J}^dC$ induces a symplectic structure on the $(2n-2)$-dimensional variety $J^d(A)$.

**Proof.** Recall that there is a canonical isomorphism $T_L\tilde{J}^dC \cong \text{Ext}^1(L, L)$, and that the symplectic form $\omega$ is the pairing

$$\text{Ext}^1(L, L) \otimes \text{Ext}^1(L, L) \to \text{Ext}^2(L, L) \xrightarrow{\text{Tr}} H^2(A, O_A) \cong C$$

The map $T_L\alpha$ is the trace map $T : \text{Ext}^1(L, L) \to H^1(A, O_A)$, whereas the tangent map at the origin to the map $\iota : \text{Pic}^0(A) \to \tilde{J}^dC$ defined by $\iota(P) = P \otimes L$ is the dual $T^* : H^1(A, O_A) \to \text{Ext}^1(L, L)$. Since $\alpha\iota$ is constant, $T \circ T^* = 0$; in particular

$$\text{Ker} T \supset \text{Im} T^* = (\text{Ker} T)^\perp .$$

Note also that $\beta \iota \phi_\ell = n\text{Id}_A$ (use the Morikawa-Matsusaka endomorphism), hence $T_L\beta \circ T^* = T\phi_\ell$ and $\text{Ker} T_L\beta \cap \text{Im} T^* = \{0\}$. Since both $\text{Ker} T$ and $\text{Ker} T_L\beta$ have codimension 2, this implies

$$\text{Ker} T = (\text{Ker} T \cap \text{Ker} T_L\beta) \oplus (\text{Ker} T)^\perp ,$$

and the restriction of $\omega$ to $\text{Ker} T \cap \text{Ker} T_L\beta = T_LJ^d(A)$ is non-degenerate. $\blacksquare$

The map $\alpha$ restricts to a morphism $\alpha : J^d(A) \to |L|$ whose fiber $K^d(C)$ over the point corresponding to a smooth curve $C$ is the (connected) kernel of the Abel-Jacobi map $\beta : J^dC \to A$; it is a Lagrangian fibration.

**2. The Euler characteristic of $J^d(A)$**

We calculate the Euler characteristic of $J^d(A)$ by using the Lagrangian fibration $\alpha : J^d(A) \to |L|$, as in [B1].

**Proposition 2.1.**– Let $C$ be an integral element of $|L|$. The Euler characteristic of $K^d(C)$ is $n$ if the normalization of $C$ has genus 2, and 0 otherwise.

**Proof.** Let $\eta : \tilde{C} \to C$ be the normalization. There is a commutative diagram (as in §2 of [B1], we may restrict ourselves to the case $d = 0$)

$$\begin{array}{ccc}
K(C) & \longrightarrow & JC \\
\cup & & \uparrow \pi \\
JC & \xrightarrow{\eta^*} & \tilde{J}C \\
\cup & \uparrow & \\
(\eta^*)^{-1}(\text{Ker}\, \pi) & \longrightarrow & \text{Ker}\, \pi \\
\end{array}$$
By lemma 2.1 of loc.cit., the group $JC$ acts freely on $\tilde{J}C$. Note also that for $M$ in $JC$ and $L$ in $JC$,
\[ \beta(M \otimes L) = \beta(M) + \beta(L) \]
because this is true when $L$ is invertible, and $JC$ is dense in $\tilde{J}C$. It follows that $(\eta^*)^{-1}(\text{Ker} \pi)$ acts (freely) on $K(C)$. As in prop. 2.2 of loc.cit., it follows that $e(K(C)) = 0$ if $\text{Ker} \pi$ is infinite, that is if $g(C) > 2$.

Assume now that $\tilde{C}$ has genus 2. The situation here is much simpler than in loc.cit., because the normalization $\eta$ of $C$ is unramified: it is the restriction to $\tilde{C}$ of the isogeny $\pi : \tilde{J}C \to A$. If $\tilde{C} \to C$ is the minimal unibranch partial normalization (cf. loc.cit.), it follows that $\tilde{C} \to \tilde{C}$ is an unramified homeomorphism, hence an isomorphism (EGA IV, 18.12.6).

There is a commutative diagram
\[
\begin{array}{cccc}
0 & \to & \text{Ker} \pi & \to & \tilde{J}C & \xrightarrow{\pi} & A \\
\cap & & \cap \eta_* & & \| & & \\
0 & \to & K(C) & \to & \tilde{J}C & \xrightarrow{\beta} & A
\end{array}
\]
and an exact sequence
\[ 1 \to \mathcal{O}_C^*/\mathcal{O}_C^* \to JC \to \tilde{J}C \to 0. \]

If one chooses a line bundle $M$ on $C$ corresponding to a point of $\mathcal{O}_C^*/\mathcal{O}_C^*$ as in the proof of prop. 3.3 of loc.cit., it acts on $\tilde{J}C$, hence on $K(C)$. Beauville’s reasoning proves that $M$ acts freely on the complement of $\eta_* \tilde{J}C$ in $\tilde{J}C$, hence also on the complement of $\text{Ker} \pi$ in $K(C)$. It follows that $e(K(C)) = e(\text{Ker} \pi) = n$. \[\blacksquare\]

As a corollary, we get, assuming that each curve with class $\ell$ is integral,
\[ e(J^n(A)) = n \text{ Card}\{C \in |L| \mid g(C) = 2\}. \]

It remains to count the number of (integral) genus 2 curves $C$ in $|L|$. The normalization $\eta : \tilde{C} \to C$ induces an isogeny $\pi : \tilde{J}C \to A$ such that $\pi_* \tilde{C} \in |L|$. Let $r$ be the degree of $\pi$; then $\pi^*L$ is numerically equivalent to $r\tilde{C}$, hence $rL^2 = r^2\tilde{C}^2 = 2r^2$ and $r = n$, and $\pi^*\ell$ has type $(n,n)$.

The number of isomorphism classes of isogenies $\pi : \tilde{A} \to A$ such that $\pi^*\ell$ is of type $(n,n)$ is also the number of isomorphism classes of isogenies $\tilde{\pi} : \tilde{A} \to \tilde{A}$ where $\tilde{A}$ has a principal polarization $\theta$ such that $\tilde{\pi}^*\theta = \ell$, hence also the number of subgroups of $\ker \phi_{\ell}$ that are maximal totally isotropic for the Weil form. This kernel is isomorphic to $(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})^*$, and the Weil form is given by $e((x,x^*),(y,y^*)) = y^*(x) - x^*(y)$. Given a quotient group $H$ of $\mathbb{Z}/n\mathbb{Z}$ and any homomorphism $u : H^* \to H$, the set of pairs $(x,x^*)$ in $(\mathbb{Z}/n\mathbb{Z}) \times H^*$ such that the class of $x$ in $H$ is $u(x^*)$ is such a subgroup, and they are all of this form. Their number is
\[ \sum_{\mathbb{Z}/n\mathbb{Z} \to H} |H| = \sum_{m|n} m = \sigma(n). \]

To each isogeny $\pi : \tilde{J}C \to A$ correspond $n^2$ curves in of genus 2 in $|L|$, to wit the curves $\pi(C) + x$, for each $x \in \ker \phi_{\ell}$. So we get a total of $n^2\sigma(n)$ such curves, and they are all distinct.
Proposition 2.2.– Assume each curve with class $\ell$ is integral; then $e(J^d(A)) = n^3\sigma(n)$.

3. A degeneration of $J^{n-2}(A)$

Our aim is to relate the symplectic variety $J^d(A)$ constructed above with the generalized Kummer variety $K_{n-1}(A)$. Contrary to the case of K3 surfaces, these varieties do not seem to be birational for general $A$ (except when $n = 2$). However, we will prove that it is the case when $A$ is a product of elliptic curves and $d = n - 2$. For this, we will use the Mukai-Fourier transform for sheaves on $A$.

For any sheaf $F$ on $A$, we denote by $F^\bullet$ the cohomology sheaves of the Mukai-Fourier transform of $F$ (see ([M2]). If only $F^jF$ is non-zero, we say that $F$ has weak index $j$, and we write $\hat{F} = F^\bullet F$; in that case, $\hat{F}$ has weak index $2 - j$, and $\hat{F}^\bullet \simeq (-1)^F$ ([loc.cit., cor. 2.4]). If $H^i(A,F \otimes P_2) = 0$ for all $\hat{x} \in \hat{A}$ and all $i \neq j$, we say that $F$ has index $j$; it implies that $F$ has weak index $j$.

For any $\hat{x}$ in $\hat{A}$, we denote by $P_{\hat{x}}$ the corresponding line bundle on $A$; we identify the dual of $\hat{A}$ with $A$, so that, for any $x$ in $A$, $P_x$ is a line bundle on $\hat{A}$.

Let $\mathcal{L}$ be a sheaf on $A$ corresponding to a point of $J^dC$ with smooth support $C$. For $\mathcal{L}$ generic in $J^dC$, the surface $\mathcal{L} \otimes \text{Pic}^0(A)$ does not meet the subvariety $W_d(C)$ of $J^dC$, as soon as $g(C) > 2 + d$, i.e. $d < n - 1$. In that case, one has $H^0(A,\mathcal{L} \otimes P_{\hat{x}}) = 0$ for all $\hat{x}$ in $\hat{A}$, so that $\mathcal{L}$ has index 1, and $\hat{\mathcal{L}}$ is a locally free simple sheaf on $\hat{A}$ of rank $n - d$, first Chern class $\hat{\ell}$ and Euler characteristic 0.

Proposition 3.1.– Assume that the Néron-Severi group of $A$ is generated by $\ell$. For $d < n - 1$ and $\mathcal{L}$ generic in $J^dC$, the vector bundle $\hat{\mathcal{L}}$ on $\hat{A}$ is $\hat{\ell}$-stable.

Proof. We follow [FL]: assume $\hat{\mathcal{L}}$ is not stable, and look at torsion-free non-zero quotients of $\hat{\mathcal{L}}$ of smallest degree, and among those, pick one, $\hat{Q}$, of smallest rank. Because $\text{NS}(\hat{A}) = \mathbb{Z}\hat{\ell}$, the degree of $Q$ is non-positive. The proofs of lemmes 2 and 3 of [FL] apply without change: $Q$ has index 1 and if $K$ be the kernel of $\hat{\mathcal{L}} \rightarrow Q$, the sheaf $\hat{F}^2K$ has finite support. Consider the exact sequence

$$0 \rightarrow F^1K \rightarrow (-1)^*\mathcal{L} \rightarrow \hat{Q} \rightarrow F^2K \rightarrow 0;$$

since $c_1(\hat{Q}) \cdot \ell = c_1(Q) \cdot \hat{\ell} \leq 0$ ([FL], lemme 1), the torsion sheaf $\hat{Q}$ has finite support, hence index 0. But this index is also $2 - \text{ind} Q = 1$; this contradiction proves the proposition. ■

For each $d < n - 1$, we have constructed a birational rational map between $J^dC$ and an irreducible component $\mathcal{M}^0_A(n - d, \hat{\ell}, 0)$ of the moduli space $\mathcal{M}_A(n - d, \hat{\ell}, 0)$ of $\hat{\ell}$-semi-stable sheaves on $\hat{A}$ of rank $n - d$, first Chern class $\hat{\ell}$ and Euler characteristic 0. This map is a morphism if $d < 0$. Let us interpret the maps $\alpha$ and $\beta$ in this context. Let $(C, \mathcal{L})$ be a pair corresponding to a point of $J^dC$; it follows from the exact sequence $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(x) \rightarrow C_x \rightarrow 0$ that $\text{det} \mathcal{O}_C(x) \simeq \text{det} \mathcal{O}_C \otimes P_{-x}$, hence

$$\text{det} \hat{\mathcal{L}} \simeq \text{det} \mathcal{O}_C \otimes P_{-\beta(\mathcal{L})} \simeq \text{det} \mathcal{O}_A(-C) \otimes P_{-\beta(\mathcal{L})}.$$

Hence, the fibers of $(\alpha, \beta)$ are also the fibers of the map $J^dC \rightarrow \text{Pic}^\ell(A) \times \text{Pic}^{-\ell}(\hat{A})$ which sends $\mathcal{L}$ to $(\text{det} \mathcal{L}, \text{det} F^\bullet \mathcal{L})$. Let $\gamma : \mathcal{M}^0_A(n - d, \hat{\ell}, 0) \rightarrow \text{Pic}^\ell(A) \times \text{Pic}^{-\ell}(\hat{A})$ be the map $E \mapsto (\text{det} F^\bullet E, \text{det} E)$, and let $M_{n-d}(\hat{A})$ be a fiber. We have proved the following.

Proposition 3.2.– Assume that the Néron-Severi group of $A$ is generated by $\ell$. For $d < n - 1$, the Fourier-Mukai transform induces a birational isomorphism between $J^dC$ and an irreducible component of $\mathcal{M}_A(n - d, \hat{\ell}, 0)$ which sends $J^d(A)$ onto $M_{n-d}(\hat{A})$. 

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We will now study the case where \( n - d = 2 \) and \( A \) is the product of two general elliptic curves \( F \) and \( G \), with \( \ell \) of bidegree \((1, n)\). One has \( \hat{A} = \hat{F} \times \hat{G} \), and \( \hat{\ell} \) has bidegree \((n, 1)\). To avoid non-stable semi-stable sheaves, we will study the moduli space \( \mathcal{M}_{\hat{A}}' \) of rank 2 sheaves on \( \hat{A} \) with first Chern class \( \hat{\ell} \) and Euler characteristic 0 which are semi-stable for the polarization \( \hat{\ell}' \) of bidegree \((n + 1, 1)\), and call \( M'(\hat{A}) \) a fiber of the map 
\[ \gamma : \mathcal{M}_{\hat{A}}' \rightarrow \text{Pic}^{-\ell}(A) \times \text{Pic}^\ell(\hat{A}) \] 
de fined above.

**Proposition 3.3.** The moduli space \( \mathcal{M}_{\hat{A}}' \) is smooth and birational to \( \hat{A}^{(n)} \times A \). The variety \( M'(\hat{A}) \) is smooth and birational to \( K_{n-1}(\hat{A}) \).

**Proof.** Let \( E \) be an \( \hat{\ell}' \)-semi-stable rank 2 torsion free sheaf on \( \hat{A} \) with first Chern class \( \hat{\ell} \) and Euler characteristic 0. Let \( x \in \hat{A} \) : by semi-stability of \( E^* \), one has \( H^2(\hat{A}, E \otimes P_x^{-1}) = 0 \), hence \( \ell^0(\hat{A}, E \otimes P_x^{-1}) = \ell^1(\hat{A}, E \otimes P_x^{-1}) \). Since \( \hat{F}E \) is non-zero, for at least one \( x \), these numbers are non-zero and there is an inclusion \( P_x \hookrightarrow E \); let \( K \) be the kernel of the exact sequence

\[ 0 \rightarrow K \rightarrow E \rightarrow I_{\hat{A}} \otimes K' \rightarrow 0 \]

where \( K' \) is a line bundle. The line bundle \( K \) has bidegree \((a, b)\), with \( a \) and \( b \) non-negative and \( b(n + 1) + a \leq (2n + 1)/2 \) (by \( \hat{\ell}' \)-semi-stability); hence \( b = 0 \) and \( Z \) is a subscheme of \( \hat{A} \) of length \( n - a \).

Set \( M = K' \otimes K^{-1} \). By Serre duality, \( \text{Ext}^1(\hat{A}, K' \otimes M) = 0 \), one has

\[ h^1(\hat{A}, I_{\hat{A}} \otimes K') = \text{length}(z) - \chi(\hat{A}, M) = a \]

and \( a > 0 \) (otherwise \( I_{\hat{A}} \otimes K' \) would be a subsheaf of \( E \) with \( \hat{\ell}' \)-slope \( 2n + 1 \)), and \( E \) depends on at most \( 2n + 3 - a \) parameters. For \( E \) generic this forces \( a = 0 \), \( Z \) reduced and \( h^0(\hat{A}, I_{\hat{A}} \otimes M) = 1 \). This yields a component of \( \mathcal{M}_{\hat{A}}' \) parametrized in this fashion. Assume now \( H^0(\hat{A}, I_{\hat{A}} \otimes M) \neq 0 \); one checks (by projecting onto \( |M| \)), that the set of pairs \((\hat{Z}, D)\) with \( D \in |M| \) and \( \hat{Z} \subset D \), has dimension \( \leq n - 2a - 1 + n - a \). Hence \( E \) depends on at most \( 2n - 3a - 1 - \chi(\hat{A}, I_{\hat{A}} \otimes M) + 4 = 2n - 2a + 3 \) parameters. For \( E \) generic, this forces \( a = 0 \), \( Z \) reduced and \( h^0(\hat{A}, I_{\hat{A}} \otimes M) = 1 \). This yields a component of \( \mathcal{M}_{\hat{A}}' \) which can be parametrized as follows. Let \( Z = (\hat{f}_1, \hat{g}_1) + \ldots + (\hat{f}_n, \hat{g}_n) \) be generic in \( \hat{A}^{(n)} \), set \( L = O_{\hat{F}}(\hat{f}_1 + \ldots + \hat{f}_n) \), and let \( f \in F \) and \( \hat{g} \in \hat{G} \). The vector space \( \text{Ext}^1(\hat{A}, I_{\hat{A}} \otimes p^*_F L \otimes p^*_G O_G(\hat{g}), O_{\hat{A}}) \) has dimension 1, hence there is a unique extension

\[ 0 \rightarrow p^*_F P_f \rightarrow E \rightarrow I_{\hat{A}} \otimes p^*_F (L \otimes P_f) \otimes p^*_G O_G(\hat{g}) \rightarrow 0 \]

where \( E \) is locally free (it satisfies the Cayley-Bacharach condition; see for example th. 5.1.1 of [HL]) and stable (the only thing to check is \( H^0(\hat{A}, E \otimes P_x) \neq 0 \), and this is true because the extension is non-trivial). This yields a rational map

\[ \phi : \hat{A}^{(n)} \times F \times \hat{G} \rightarrow \mathcal{M}_{\hat{A}}' \]

which is birational onto its image : given a locally free \( E \) as above, one recovers \( f \) and the \((\hat{g}_i - \hat{g})\)'s by noting that the set \( C_E = \{ x \in A \mid H^0(\hat{A}, E \otimes P_x) \neq 0 \} \) is

\[ \{ -f \} \times G \cup \bigcup_{i=1}^n (F \times \{ [O_G(\hat{g}_i - \hat{g})] \}) \]
the \( \hat{f}_i \)'s because \( \text{Ext}^1_{\hat{A}}(\mathcal{I}_Z \otimes p_F^*L \otimes p_G^*O_{\hat{G}}(\hat{g}), O_{\hat{A}}) \) must be non-zero, and \( \hat{g} \) by noting that
\[
\det E \simeq p_F^*(L \otimes P_{2f}) \otimes p_G^*O_{\hat{G}}(\hat{g}).
\]
Because \( H^0(\hat{A}, E \otimes p_F^*(P_{-f} \otimes O_{\hat{F}}(-f_1)) \otimes p_G^*O_{\hat{G}}(\hat{g}_1 - \hat{g})) \) is non-zero, there exists an exact sequence \((*)\) with \( K \) of bidegree \((1,0)\). This proves that the set \( \mathcal{M}' \) defined above is contained in the image of \( \phi \), which must therefore be \( \mathcal{M}'_{\hat{A}} \).

Finally, \( E \) has weak index 1, \( \mathcal{F}'E \) has support on \( C_E \), and fixing \( \det \mathcal{F}' \) amounts to fixing \( [O_A(C_E)] \). It follows that taking a fiber of \( \gamma \) amounts to fixing \( f, \sum(\hat{g}_i - \hat{g}), \sum \hat{f}_i \) and \( \hat{g} \); hence \( M'(\hat{A}) \) is birational to \( K_{n-1}(\hat{A}) \).

The following proof is due to D. Huybrechts, and uses ideas from prop. 2.2 of [GH].

**Theorem 3.4.** -- Let \((A, \ell)\) be a polarized abelian surface of type \((1,n)\) whose Néron-Severi group is generated by \( \ell \). The symplectic varieties \( J^{n-2}(A), M_2(\hat{A}) \) and \( K_{n-1}(\hat{A}) \) are deformation equivalent. In particular, they are all irreducible symplectic.

**Proof.** Let \( f : \hat{A} \rightarrow S \) be a family of polarized abelian surfaces, where \( S \) is smooth quasi-projective, with a relative polarization \( \hat{L} \) of type \((1,n)\), such that the fiber over a point \( 0 \in S \) is \( \hat{F} \times \hat{G} \) with a polarization of bidegree \((n,1)\); assume also that the Néron-Severi group of a very general fiber of \( f \) has rank 1. Let \( g : \mathcal{M} \rightarrow S \) be the (projective) relative moduli space of \( \hat{L} \)-semi-stable sheaves of rank 2 with first Chern class \( \ell \) and Euler characteristic 0 on the fibers of \( f \) (cf. [HL], th. 4.3.7, p. 92).

**Lemma 3.5.** -- Under the hypothesis of the proposition, any rank 2 torsion free sheaf on \( \hat{A} \) with first Chern class \( \ell \) which is either simple or semi-stable is stable.

**Proof.** Assume that a sheaf \( E \) with these numerical characters is not stable. There exists an exact sequence
\[
0 \rightarrow K \rightarrow E \rightarrow \mathcal{I}_Z \otimes K' \rightarrow 0,
\]
where \( K \) and \( K' \) are line bundles on \( \hat{A} \) with \( c_1(K) = k\hat{\ell}, \ c_1(K') = (1-k)\hat{\ell} \) and \( k > 0 \). This proves that \( E \) is not semi-stable; moreover, \( K \otimes K'^{-1} \) is ample, hence there exists a non-zero morphism \( u : K' \rightarrow K \), which induces an endomorphism \( E \rightarrow \mathcal{I}_Z \otimes K' \xrightarrow{u} K \rightarrow E \) which is not a homothety, and \( E \) is not simple.

By the lemma, the (closed) locus of non-stable points in \( \mathcal{M} \) does not project onto \( S \). By replacing \( S \) with an open subset, we may assume that there are no such points. Let now \( S \rightarrow \mathcal{M} \) be the (smooth) relative moduli space of simple sheaves on the fibers of \( f \) (see [AK]). There are embeddings \( \mathcal{M} \subset S \) and \( \mathcal{M}'_{\hat{F} \times \hat{G}} \subset S_0 \) as closed and open subsets. Let \( S' = S \setminus (S_0 - \mathcal{M}'_{\hat{F} \times \hat{G}}) \); it is open in \( S \), hence smooth over \( S \). Let \( \mathcal{M}' \) be the closure of \( g^{-1}(S \setminus \{0\}) \) in \( S' \); the fibers of \( g' : \mathcal{M}' \rightarrow S \) are projective off 0, and contained in \( \mathcal{M}'_{\hat{F} \times \hat{G}} \) over 0. Norton’s criterion ([N]) shows that points in \( \mathcal{M}'_0 \) are separated in the moduli space of simple sheaves on \( \hat{A} \) (because they are stable), hence in \( S \); therefore, \( \mathcal{M}' \) is separated. By semi-continuity, \( \mathcal{M}'_0 \) is a closed subset of \( \mathcal{M}'_{\hat{F} \times \hat{G}} \) of the same dimension, hence they are equal. Using the lemma, we get, after shrinking \( S \) again, a proper family \( g' : \mathcal{M}' \rightarrow S \) with projective irreducible smooth fibers which coincide with \( g : \mathcal{M} \rightarrow S \) off 0.

By prop. 3.2, \( J^{n-2}(A) \) is birationally isomorphic to \( M_2(\hat{F} \times \hat{G}) \), and we just saw that the latter deforms to \( M'(\hat{A}) \), itself birationally isomorphic to \( K_{n-1}(\hat{F} \times \hat{G}) \) by prop. 3.3; in particular, these symplectic varieties are all irreducible symplectic. Since birationally isomorphic smooth projective irreducible symplectic varieties are deformation equivalent ([H], th. 10.12), the theorem is proved.
Corollary 3.6.– Let \((A, \ell)\) be a general polarized abelian surface of type \((1, n)\). The moduli space \(M_A(2, \ell, 0)\) is smooth irreducible.

Corollary ([GS]) 3.7.– Let \(A\) be an abelian surface. The Euler characteristic of \(K_{n-1}(A)\) is \(n^3\sigma(n)\).

REFERENCES

[AK] A. Altman, S. Kleiman : Compactifying the Picard Scheme, Adv. Math. 35 (1980), 50–112.
[B1] A. Beauville : Counting rational curves on K3 surfaces, e-print alg-geom 9701019.
[B2] A. Beauville : Variétés kähleriennes dont la première classe de Chern est nulle, J. Diff. Geom. 18 (1983), 755–782.
[BL] J. Bryan, N.C. Leung : The Enumerative Geometry of K3 Surfaces and Modular Forms, e-print alg-geom 9711031.
[FL] M. Fakhlaoui, Y. Laszlo : Transformée de Fourier et stabilité sur les surfaces abéliennes, Comp. Math. 79 (1991), 271–278.
[G] L. Göttsche : A conjectural generating function for numbers of curves on surfaces, e-print alg-geom 9711012.
[GS] L. Göttsche, W. Soergel : Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces, Math. Ann. 296 (1993), 235–245.
[GH] L. Göttsche, D. Huybrechts : Hodge Numbers of Moduli Spaces of Stable Bundles on K3 surfaces, Intern. J. Math. 7 (1996), 359–372.
[H] D. Huybrechts : Compact Hyperkähler Manifolds, Habilitationsschrift, Universität-GH Essen, 1997.
[HL] D. Huybrechts, M. Lehn : The Geometry of Moduli Spaces of Sheaves, Aspects of Mathematics, Vieweg, Braunschweig/Wiesbaden, 1997.
[M1] S. Mukai : Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, Invent. Math. 77 (1984), 101–116.
[M2] S. Mukai : Duality between \(D(X)\) and \(D(\hat{X})\) with its application to Picard sheaves, Nagoya Math. J. 81 (1981), 153–175.
[N] A. Norton : Analytic moduli of complex vector bundles, Indiana Univ. Math. J. 28 (1979), 365–387.
[YZ] S.-T. Yau, E. Zaslow : BPS states, string duality, and nodal curves on K3, Nuclear Physics B 471 (1996), 503–512; also preprint hep-th 9512121.

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