ADIABATIC EVOLUTION GENERATED BY A ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH DECREASING NUMBER OF EIGENVALUES

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Abstract. We study a one-dimensional non-stationary Schrödinger equation with a potential slowly depending on time. The corresponding stationary operator depends on time as on a parameter. It has a finite number of negative eigenvalues and absolutely continuous spectrum filling \([0, +\infty)\). The eigenvalues move with time to the edge of the continuous spectrum and, having reached it, disappear one after another. We describe the asymptotic behavior of a solution close at some moment to an eigenfunction of the stationary operator, and, in particular, the phenomena occurring when the corresponding eigenvalue approaches the absolutely continuous spectrum and disappears.

1. Introduction

As \(\varepsilon \to 0\), we study solutions to the Schrödinger equation

\[
\frac{i\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + v(x, \varepsilon t)\psi, \quad x > 0, \quad \psi|_{x=0} = 0.
\]  

When \(\varepsilon\) is small, one says that (1) describes the adiabatic evolution in \(L^2(\mathbb{R}_+)\) generated by the stationary operator \(H(\varepsilon t) = -\frac{\partial^2}{\partial x^2} + v(x, \varepsilon t)\) with the Dirichlet boundary condition at zero. Note that this operator depends on time as on a parameter.

The adiabatic evolution generated by differential operators is a classical object of study in mathematical physics and, in particular, in quantum mechanics, see the review in [1]. The formulation of one of the standard problems assumes that the spectrum of the corresponding stationary operator is discrete. For this problem, an important role is played by the solutions that, at some moment in time, are close to the eigenfunctions of the stationary operator. In the case of (1), instead of such solutions, physicists often use formal asymptotic series of the form

\[
e^{-\frac{\tau}{\varepsilon}} \int_{\tau_0}^{\tau} E_n(\tau) d\tau \sum_{m=0}^{\infty} \varepsilon^m \psi_{n,m}(x, \varepsilon t), \quad \varepsilon \to 0,
\]  

where \(E_n(\tau)\) is an eigenvalue of \(H(\tau)\), \(\psi_{n,0}(\cdot, \tau)\) is the corresponding eigenfunction, and \(\tau_0\) is a fixed number.

We consider a model problem with an operator \(H(\varepsilon t)\) having a finite number of negative eigenvalues and absolutely continuous spectrum filling \(\mathbb{R}_+\). The eigenvalues move with time toward the continuous spectrum and, having reached it,
disappear one after another. We study a solution having the asymptotics of the form (2) as long as the \( n \)-th eigenvalue exists. There is a whole bunch of effects occurring as a result of “absorption” of the eigenvalue by the continuous spectrum. In this paper, we begin to analyze them. Some of our results were announced in the note [5].

2. MAIN RESULTS

Here, we describe the potential \( v \) that we consider, the solution to (1) that we study, and the asymptotics of this solutions that we get in this paper.

2.1. The model we consider. We assume that \( 0 < \varepsilon < 1 \) and study the case where

\[
v(x, \tau) = \begin{cases} 
-1 & \text{if } 0 \leq x \leq 1 - \tau, \\
0 & \text{otherwise}.
\end{cases}
\]

This allows to construct solutions to (1) using ideas similar to ideas of the Sommerfeld-Malyuzhinets method developed to study scattering of waves on wedge-shaped domains [2]: by means of a suitable integral transform, one expresses solutions of (1) in terms of solutions of a difference equation on the complex plane. In our case, the translation parameter in this equation equals \( \varepsilon \). The smallness of \( \varepsilon \) enables to carry out an effective analysis.

In this paper, to keep its size reasonable, we mostly restrict ourselves to the analysis of solutions inside the potential well. We also assume that \( \varepsilon t \leq 1 \).

2.2. The solution we study. In section 3, we construct a solution \( \Psi \) having the following simple wave interpretation. It depends on a parameter \( p \in \mathbb{R} \), and inside the wedge \( W = \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq 1 - \varepsilon t\} \), \( \Psi \) is a linear combination of the plane wave \( e^{-i(p^2-1)t + ipx} \) and all the plane waves that can be obtained from it by reflections from the boundaries of \( W \). Outside the wedge, \( \Psi \) can be considered as a linear combination of all the refracted waves. Describe briefly the construction of \( \Psi \). We choose the Ansatz

\[
\Psi(x, t, p) = \frac{1}{\sqrt{\pi}} \left\{ \begin{array}{ll}

\sum_{k=p+\varepsilon t, t \in \mathbb{Z}} e^{-it} e^{-ik^2(1-1/\varepsilon)} \sin(kx) R(k), & 0 \leq x < 1 - \varepsilon t, \\

\sum_{k=p+\varepsilon t, t \in \mathbb{Z}} e^{-ip_1^2(k,t-1/\varepsilon)+ip_1(k)x} T(k) R(k), & 1 - \varepsilon t \leq x.
\end{array} \right.
\]

If the series in (4) converge sufficiently well, \( \Psi \) satisfies (1) and the Dirichlet boundary condition. The series appears to be convergent and \( \Psi \) appears to be continuously differentiable in \( x \) at \( x = 1 - \varepsilon t \) if \( R \) satisfies the equation

\[
R(p + \varepsilon/2) = \rho(p) R(p - \varepsilon/2), \quad \rho(p) = \frac{Q(p) - p}{Q(p) + p}, \quad Q(p) = \sqrt{p^2 - 1},
\]

(5)

\( T \) and \( p_1 \) are defined in terms of \( Q \) by formulas (30) and (29), and the branch of the function \( Q \) in the definition of \( \rho \) in (5) is chosen in a suitable way.

We call \( \Psi \) a generating solution. It is \( \varepsilon \)-periodic in \( p \). Its Fourier coefficients \( \Psi_n \) with \( n \geq 1 \) are the solutions that we study. They satisfy the Dirichlet condition at \( x = 0 \) and share with \( \Psi \) all the regularity properties in \( x \). In the potential well, \( x = \varepsilon t \),

\[
\Psi_n(x, t) = \frac{e^{it}}{\sqrt{\varepsilon \pi}} \int_{-\infty}^{\infty} e^{i(p^2(1-\tau)-2\pi np)/\varepsilon} \sin(px) R(p) dp, \quad \tau = \varepsilon t.
\]

(6)
Before analyzing $\Psi_n$, we study solutions to (5) in Section 4.

2.3. Properties of the operator $H(\tau)$. When describing the asymptotics of $\Psi_n$, we use the following elementary facts. Consider the operator $H(\tau)$ with the potential (3). Let

$$\tau_n = 1 - \pi(n - 1/2), \quad n \in \mathbb{N}. \quad (7)$$

The number of negative eigenvalues of $H(\tau)$ equals $N \in \mathbb{N}$ if $\tau_{N+1} < \tau < \tau_N$. If $\tau > \tau_1$, there are no eigenvalues. We use the notation

$$c_n = e^{\pi(2\tau_n-3)/4}. \quad (8)$$

Let $E_n(\tau)$ be the $n$th eigenvalue of $H(\tau)$. It can be represented in the form $E_n(\tau) = p_n^2(\tau) - 1$ with $p_n \in (0, 1)$ satisfying the equation

$$(1 - \tau) p_n(\tau) + \arcsin(p_n(\tau)) = \pi n. \quad (9)$$

The eigenfunction corresponding to $E_n(\tau)$ is given by the formulas

$$\psi_n(x, \tau) = \begin{cases} 
\sin(p_n(\tau)x) & \text{if } 0 \leq x \leq 1 - \tau, \\
-(1+n) p_n(\tau)e^{-(x-1+\tau)\sqrt{1-p_n^2(\tau)}} & \text{if } x \geq 1 - \tau. 
\end{cases} \quad (10)$$

2.4. Standard asymptotic behavior. Fix $n \in \mathbb{N}$. In section Section 5, we study the solution $\Psi_n(x, t)$ in the case when $E_n(\epsilon t)$ exists and is bounded away from zero. In the potential well, $\Psi_n$ has two-scale asymptotic expansion. Describe it.

Outside the potential well, $\Psi$ has two-scale asymptotic expansion. Describe it. The equation

$$(1 - \tau) \dot{p}_n + \arcsin \dot{p}_n - i\dot{p}_n \xi/(2 \sqrt{1 - \dot{p}_n^2}) = \pi n, \quad (12)$$

allows to define a continuous function $\dot{p}_n : \{\tau < 1, \xi > 0\} \rightarrow \mathbb{Q}_1 = \{p \in \mathbb{C} : \Re p, \Im p > 0\}$ such that $\dot{p}_n(\tau, 0) = p_n(\tau)$ when $\tau < \tau_n$. One has

**Theorem 2.2.** Fix $K \in \mathbb{N}$, $T_1 < T_2 < \tau_n$ and $X > 0$. Let $T_1 \leq \epsilon t \leq T_2$ and $1 - \epsilon t \leq x \leq X/\epsilon$. Let $\xi = \epsilon(x - (1 - \tau))$, $\tau = \epsilon t$. As $\epsilon \rightarrow 0$,

$$\Psi(x, t) = c_n \sqrt{\frac{\partial \ln \dot{p}_n}{\partial \tau}} e^{-\frac{1}{4} \int_{\tau_n}^{\tau} E_n(\tau) d\tau} \left( \sum_{k=0}^{K-1} \epsilon^k \alpha_{n,k}(\xi, \tau) + O(\epsilon^K) \right), \quad (13)$$

$$\phi_n(\xi, \tau) = (-1)^{n+1} \dot{p}_n e^{-\frac{1}{4} \int_{\tau_n}^{\tau} E_n(\tau) d\tau}, \quad (14)$$

the branch of the square root is such that $\Re \sqrt{1 - \dot{p}_n^2} > 0$ for $\dot{p} \in \mathbb{Q}_1$, $\alpha_{n,0} \equiv 1$, and $\alpha_{n,k}(\xi, \tau)$ are bounded. This representation is uniform in $x$ and $t$.

In this paper, we only outline the proof this theorem. Note that (1) as $\Re \sqrt{1 - \dot{p}_n^2} > 0$, $\phi_n(\xi, \tau)$ decays as $\xi$ increases; (2) if $\epsilon^{1/2}x \rightarrow 0$, $\phi_n(\epsilon x, \tau) = \psi_n(x, \tau)(1 + o(1))$. For large $\xi$, $\Psi_n$ is described by

**Lemma 2.1.** Fix $0 < c < 1$. There is a $C > 0$ such that, for sufficiently large $\xi = \epsilon(x - (1 - \epsilon t))$, one has $|\Psi_n(x, t)| \leq Ce^{-\epsilon x}/\epsilon$. 


2.5. **Destruction of the standard asymptotic behavior, \( \varepsilon t \leq \tau_n \).** As \( \varepsilon t \) grows and approaches \( \tau_n \), the eigenvalue \( E_n(\tau) \) approaches the edge of the absolutely continuous spectrum, and the asymptotic behavior of \( \Psi_n \) changes. Set

\[
F(z) = \sqrt{\pi} e^{-\frac{2z^3}{3} + \frac{i\pi}{4}} \left( z \text{Ai}(z^2) - \text{Ai}'(z^2) \right),
\]

where \( \text{Ai} \) is the Airy function. One has \( F(e^{\frac{i\tau}{\varepsilon}} z) = z^{1/2} e^{-4iz^3/3} (1 + o(1)) \) as \( z \to +\infty \), and \( F(e^{\frac{i\tau}{\varepsilon}} z) = (-i/8 + o(1)) (-z)^{-5/2} \) as \( z \to -\infty \). In section 6, we prove

**Theorem 2.3.** Fix a sufficiently small \( \delta > 0 \). Let \( \tau_n - \delta \leq \varepsilon t \leq \tau_n \) and \( 0 \leq x \leq 1 - \varepsilon t \). Then, as \( \varepsilon \to 0 \),

\[
\Psi_n(x, t) = c_n \sqrt{\frac{1}{Z_n}} \frac{\partial}{\partial \tau} \ln p_n \psi_n F(e^{\frac{i\tau}{\varepsilon}} Z_n) + O \left( \varepsilon^{\frac{3}{2}} (1 + |Z_n|^{\frac{1}{2}}) \right),
\]

where \( Z_n = \left( \frac{3}{4} \right)^{\frac{1}{4}} \int \tau_n E_n d\tau \), \( p_n = p_n(x, \varepsilon t) \), \( \psi_n = \psi_n(x, \varepsilon t) \), and \( E_n = E_n(\tau) \). The asymptotic representation is uniform in \( x \) and \( t \).

If \( \varepsilon \to 0 \) and \( \tau_n - \varepsilon t \) stays of the order of 1, then \( Z_n \to +\infty \), and the leading term in (16) turns into the leading term from (11).

In this paper, we do not study \( \Psi \) outside the potential well for \( \varepsilon t \sim \tau_n \). We mention only that, in the domain where the expression \( \varepsilon^{-2/3}(\varepsilon t - \tau_n)^2 + \varepsilon^{1/3} x \) is of order of one, \( \Psi_n \) is described in terms of Airy functions, and, for larger \( x \), it exponentially decays as \( x \) increases.

2.6. **Aftermath.** If

\[
0 \leq x \leq 1 - \tau, \quad \tau_n \leq \tau \leq 1, \quad \tau = \varepsilon t,
\]

then, up to some error terms, the solution \( \Psi_n \) appears to be the sum of three terms “responsible” for three different phenomena. First, we describe these terms, and then, formulate a theorem.

**“Transition” term.** The first term is described by the formula

\[
\mathcal{T}_0(x, \varepsilon t) = \left( \frac{4\varepsilon^3}{\tau_n} \right)^\frac{1}{2} c_n \sin x F \left( e^{\frac{i\tau}{\varepsilon} Z_n(\varepsilon t)} \right), \quad Z_n(\tau) = \frac{\tau_n - \tau}{(4\varepsilon^3)^{1/3}}.
\]

One can see that, when \( \varepsilon t - \tau_n \) is of the order of \( \varepsilon^{1/3} \), the leading term in (16) turns into \( \mathcal{T}_0 \). On the other hand, as \( (\varepsilon t - \tau_n)/\varepsilon^{1/3} \to +\infty \), the function \( F \) in (18) can be “replaced” by its asymptotics. This leads to an asymptotics of \( \mathcal{T}_0 \) with the leading term \( \frac{1}{2} \left( \tau_n - \tau \right)^{\frac{1}{2}} \sin x \), and \( \mathcal{T}_0 \) becomes of the order of \( \varepsilon \).

**“Resonance” term.** Put

\[
a(z) = \int_0^{\infty} e^{\frac{-u^2}{4} + izu^2} u \, du.
\]

The function \( a \) is a close relative of the Airy function. By means of the method of steepest descents, one checks that

\[
a(z) = \left( \frac{i}{2} \right) z + O \left( z^{-\frac{3}{2}} \right), \quad z \to \pm \infty.
\]

This representation can be differentiated infinitely many times. The second term is given by the formula

\[
\mathcal{R}_0(x, \varepsilon t) = \frac{c_n \sin x}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{16} f_k \left( \frac{(\varepsilon t)^3}{2} a(z_{n-k}(\tau)) - \frac{i(1-\tau)\varepsilon}{16} a''(z_{n-k}(\tau)) \right),
\]

where
where \( \tau = \varepsilon t \), and
\[
f_k = \frac{(-1)^k}{k^{3/2}}, \quad k > 0, \quad f_0 = -\sum_{k=1}^{\infty} f_k.
\] (22)

Let us discuss the term \( \mathcal{R}_0 \). If \( \varepsilon t \) is outside a fixed neighborhood of \( \tau_1, \tau_2 \ldots \tau_n \), then, in view of (20),
\[
\mathcal{R}_0(x, \varepsilon t) = \frac{i\varepsilon c_n \sin x}{2\pi} \sum_{k=0}^{\infty} \frac{f_k}{\tau_{n-k} - \varepsilon t} + O(\varepsilon^2) = O(\varepsilon).
\]

On the other hand, assume that, for an integer \( 1 \leq N \leq n-1 \), one has \( \tau_N - \tau \approx \varepsilon^{1/3} \). Then, (21) turns into
\[
\mathcal{R}_0(x, \varepsilon t) = \frac{c_n \sin x}{\pi} \left( \frac{\varepsilon}{N} \right)^{3/2} F_n \left( \varepsilon t - \tau \right) + O(\varepsilon).
\]

So, \( \mathcal{R}_0 \) becomes relatively large near the moments \( t = \frac{\tau_l}{\varepsilon}, l = 1, 2 \ldots n - 1 \), i.e., the moments of “death” of the eigenvalues of the stationary operator \( H_\varepsilon \). Note that these can be interpreted as the moments of birth of its resonances.

**Between the moments \( \tau_n, \tau_{n-1} \ldots \)** If \( \varepsilon t - \tau_n \leq \varepsilon^{1/3} \), we set \( \mathcal{G}_0(x, \varepsilon t) = 0 \), and otherwise we define \( \mathcal{G}_0 \) by the formula
\[
\mathcal{G}_0(x, \varepsilon t) = ic_n \sqrt{\frac{2}{\pi}} \frac{\varepsilon \sin x}{\varepsilon t - \tau_n} \Re \int_0^{\infty} e^{-2s(\varepsilon t - \tau_n)} \left( e^{i\pi/4} \zeta(is) + 2\sqrt{s} \right) ds,
\] (23)
where \( \zeta \) is analytic in \( \mathbb{C} \setminus [1/2, \infty) \) and given there by (51). Thanks to (52), the integral in (23) converges uniformly in \( \varepsilon t \geq \tau_n \).

The \( \mathcal{G}_0 \) is the third term that contributes to the leading term of the asymptotics of \( \Psi_n \). If the distance from \( \varepsilon t \) to \( \tau_1, \tau_2 \ldots \tau_n \) is bounded away from zero by a fixed constant, then, as \( \varepsilon \to 0 \), all the three terms \( \mathcal{T}_0, \mathcal{R}_0 \) and \( \mathcal{G}_0 \) are of the order of \( \varepsilon \).

In Section 7 we prove

**Theorem 2.4.** Let \( x \) and \( t \) satisfy (17). As \( \varepsilon \to 0 \),
\[
\Psi_n(x, t) = \mathcal{T}_0(x, \varepsilon t) \left( 1 + O(\varepsilon^{1/2}) \right) + \mathcal{R}_0(x, \varepsilon t) + \mathcal{G}_0(x, \varepsilon t) + O(\varepsilon^{7/6}) + O(\varepsilon^{2/3}/(1 + |z_n(\varepsilon t)|)^{3/2}).
\]

2.7. **Final remarks.** In the course of investigation of the sound propagation in a narrow water wedge near a sea shore, problems similar to ours were studied (non-rigorously). For example, in [6], the author got an elegant partial differential equation for the sound field in the case similar to the case where \( \varepsilon t \sim \tau_n \). This enabled him to describe the the leading order approximation of the field in terms of Airy functions. It looks like that the physicists have not discovered the effects described by the term \( \mathcal{G}_0 \), neither have they found the function \( \zeta \) determining the term \( \mathcal{G}_0 \).

We hope that our results can be generalized to the case where the potential \( v \) in equation (1) is non-positive, quickly vanishes as \( x \to \infty \), and the eigenvalues \( E(\tau) \) behave for \( \tau \sim \tau_n \) as in the model problem. In particular, we expect that the leading terms of the asymptotics of \( \Psi_n \) for \( \varepsilon t \geq \tau_n \), can be obtained from \( \mathcal{T}_0 \), \( \mathcal{R}_0 \) and \( \mathcal{G}_0 \) by replacing \( \sin x \) with a solution of the equation \( -\psi''_{xx} + v(x, \tau)\psi = 0 \) satisfying the Dirichlet condition at \( x = 0 \).

The results we got admit the following physical interpretation. For \( t < \tau_n/\varepsilon \), the quantum particle described by the wave function \( \Psi_n \) is in the state with the
energy $E_n(\varepsilon t)$ and is localized in the potential well $0 \leq x \leq \varepsilon t$. When $t$ approaches $\tau_n/\varepsilon$, the moment of "death" of the energy level $E_n(\varepsilon t)$, the probability to find the quantum particle in the potential well decreases and becomes small when $t > \tau_n/\varepsilon$. The energy of the particle moves into the continuous spectrum. After that, when $t$ is close to $\tau_{n-1}/\varepsilon$, $\tau_{n-2}/\varepsilon$, ..., i.e., to the moments of birth of the resonances, thanks to tunneling effects, the probability to find the particle in the potential well again becomes noticeable.

3. Generating solution

Here, we construct the generating solution as described in Section 2.2.

3.1. Relations for the functions $R$, $T$ and $p_1$. Assume that the series in formulas (4) converge sufficiently well. The function $\Psi$ defined by these formulas is continuous in $x$ at $x = 1 - \varepsilon t$ for all $t < 1/\varepsilon$, if, for all $p \in \mathbb{R}$ and for all $x \in \mathbb{R}$,

$$e^{i(1-x)/\varepsilon} \left( e^{ip^2 x/\varepsilon + i p x} R(p) - e^{i(p+\varepsilon)^2 x/\varepsilon - i(p+\varepsilon)x} R(p+\varepsilon) \right) = 2i T(p) R(p) e^{i p_1(p)^2 x/\varepsilon + i p_1(p)x}. \quad (24)$$

As $p^2 + \varepsilon p = (p + \varepsilon)^2 - \varepsilon(p + \varepsilon)$, (24) implies that

$$-1 + p^2 + \varepsilon p = p_1(p)^2 + \varepsilon p_1(p), \quad (25)$$

$$R(p) - R(p + \varepsilon) = 2ie^{-i/\varepsilon} T(p) R(p). \quad (26)$$

Furthermore, $\partial \Psi/\partial x$ becomes continuous in $x$ at $x = 1 - \varepsilon t$ for all $t < 1/\varepsilon$, if, for all $p \in \mathbb{R}$ and for all $x \in \mathbb{R}$,

$$e^{i(1-x)/\varepsilon} \left( e^{ip^2 x/\varepsilon + i p x} p R(p) + e^{i(p+\varepsilon)^2 x/\varepsilon - i(p+\varepsilon)x} (p + \varepsilon) R(p + \varepsilon) \right) = 2ie^{ip_1(p)^2 x/\varepsilon + i p_1(p)x} p_1(p) T(p) R(p). \quad (27)$$

which leads to the relation

$$p R(p) + (p + \varepsilon) R(p + \varepsilon) = 2ie^{-i/\varepsilon} p_1(p) T(p) R(p). \quad (28)$$

Let us discuss (25), (26) and (28). Equation (25) implies that

$$p_1(p) = -\frac{\varepsilon}{2} + Q \left( p + \frac{\varepsilon}{2} \right), \quad Q(p) = \sqrt{p^2 - 1}. \quad (29)$$

From (26) and (28), we deduce the equation for $R$ from (5) and the formula

$$T(p) = -i \left. \frac{pe^{i/\varepsilon}}{Q(p) + p} \right|_{p = p + \varepsilon/2}. \quad (30)$$

3.2. Function $R$. Here, we construct a solution to equation (5) for complex $p$.

3.2.1. A branch of the $Q$. The branch points of $Q$ are the points $\pm 1$. Let $C_0 = \mathbb{C} \setminus \{p \in \mathbb{R} : |p| \geq 1\}$. In $C_0$, we fix the single-valued analytic branch $Q_0$ of $Q$ so that $Q_0(0) = i$. Note that

$$Q_0(p) \in i\mathbb{R}_+ \quad \text{if} \quad -1 < p < 1, \quad \text{and} \quad Q_0(p + i0) \in \mathbb{R}_+ \quad \text{if} \quad \pm p > 1. \quad (31)$$

The function $Q_0$ is even and $\rho_0$, the coefficient $\rho$ defined by (5) for $Q = Q_0$, satisfies the relation $\rho_0(-p) = 1/\rho_0(p)$. One has

$$\rho_0(p + i0) \asymp 1/p^2 \quad \text{as} \quad |p| \to \infty, \quad p \in \mathbb{R}. \quad (32)$$
3.2.2. Constructing a solution to (5). To describe a solution to (5), we need some notations. In $\mathbb{C}_0$, we define an analytic function by the formulas

$$l_0(p) = -i \ln \rho_0(p) = -i \ln \frac{Q_0(p) - p}{Q_0(p) + p}, \quad l_0(0) = 0. \quad (33)$$

The definition of $l_0$ implies that

$$l_0 = 2 \text{arcsin}(p), \quad -1 \leq p \leq 1,$$

and that

$$\frac{\partial l_0}{\partial \theta} = \frac{\partial l_0}{\partial \theta} \rho_0(\gamma), \quad 0 \leq \theta \leq 2\pi.$$

We call a curve $\gamma \subset \mathbb{C}$ vertical if, along $\gamma$, $p$ is a piecewise $C^1$-function of $\text{Im} \ p \in \mathbb{R}$, and $dp/d\text{Im} \ p$ is uniformly bounded. For $p_0 \in \mathbb{C}$, we denote by $\gamma(p_0) \subset \mathbb{C}$ a vertical curve containing $p_0$. One has

**Proposition 3.1.** For $p \in \mathbb{C}_0$, let

$$R_0(p) = \exp \left( \frac{i}{\varepsilon} \int_0^p L_0(p) \, dp \right), \quad L_0(p) = \frac{\pi}{2i\varepsilon} \int_{\gamma(p)} \frac{l_0(\zeta) \, d\zeta}{\cos^2 \left( \frac{\pi(p - \zeta)}{\varepsilon} \right)}, \quad (36)$$

and $\gamma(p) \subset \mathbb{C}_0$. The functions $L_0$ and $R_0$ are analytic in $\mathbb{C}_0$, and $R_0$ is continuous up to its boundary. The $R_0$ is a solution to the equation in (5), and $L_0$ satisfies the relations

$$L_0(p + \varepsilon/2) - L_0(p - \varepsilon/2) = \varepsilon l_0'(p), \quad p \pm \varepsilon/2 \in \mathbb{C}_0, \quad (37)$$

$$L_0(\bar{p}) = \overline{L_0(p)}, \quad L_0(-p) = -L_0(p), \quad p \in \mathbb{C}_0. \quad (38)$$

**Proof.** The analyticity of $L_0$ (and $R_0$) follows from the choice of $\gamma(p)$ and the estimate

$$|l_0(p)| \leq C \ln(2 + |p|), \quad p \in \mathbb{C}_0,$$

$C > 0$ being a constant. Using the residue theorem, one checks that $L_0$ solves (37). To prove (38), one checks by means of (35) that, for $p \in i\mathbb{R}$, $L_0(p) \in i\mathbb{R}$ and $L_0(-p) = -L_0(p)$.

Let us prove that $R_0$ satisfies (5). To simplify the notations, we write below $l$ and $L$ instead of $l_0$ and $L_0$. For $p \pm \varepsilon/2 \in \mathbb{C}_0$, one has

$$\int_0^{p+\varepsilon/2} L(p) \, dp - \int_0^{p-\varepsilon/2} L(p) \, dp = \int_{-\varepsilon/2}^{p} L\left(p + \frac{\varepsilon}{2}\right) \, dp - \int_{\varepsilon/2}^{p} L\left(p - \frac{\varepsilon}{2}\right) \, dp = \int_0^{\varepsilon/2} \left( L\left(p + \frac{\varepsilon}{2}\right) \, dp - \varepsilon \int_0^p l'(p) \, dp \right) = \int_{-\varepsilon/2}^{\varepsilon/2} L(p) \, dp + \varepsilon(l(p) - l(0)).$$

As $L_0$ is odd and $l_0(0) = 0$, this implies the needed.

Finally, let us check that $L_0$ is continuous up to the boundary of $\mathbb{C}_0$. As $(Q_0(p) + p)(Q_0(p) - p) = -1$, the factor $\rho_0$ is continuous in $\mathbb{C}_0$ and along its boundary. This and (5) imply that, being analytic in $\mathbb{C}_0$, $R_0$ is continuous up to its boundary. □
3.3. Completing the construction of the solution to the Schrödinger equation. Let \( R_0 \) be the solution to (5) described in Proposition 3.1. For \( p \in \mathbb{R} \), we define
\[
R(p) = \begin{cases} 
R_0(p-i0), & p \leq 0, \\
R_0(p+i0), & p > 0. 
\end{cases}
\] (39)
The function \( R \) is continuous and satisfies (5) (we assumed that \( 0 < \varepsilon < 1 \)). Let \( p_1 \) and \( T \) be defined by (29) and (30) with \( Q = Q_0 \). One has

**Theorem 3.1.** The constructed function \( \Psi \) is continuous in \((t,x,p) \in (-\infty,1/\varepsilon] \times [0,\infty) \times \mathbb{R}\). Both for \( 0 < x < 1 - \varepsilon t \) and for \( x > 1 - \varepsilon t \), it is infinitely differentiable in \( x \) and \( t \) and satisfies the Schrödinger equation (1). At \( x = 0 \), \( \Psi \) satisfies the Dirichlet boundary condition. At \( x = 1 - \varepsilon t \), it is continuously differentiable in \( x \).

As a function of \( p \), \( \Psi \) is \( \varepsilon \)-periodic.

**Proof.** We need only to check that both the series in (4) converge sufficiently well. By (5), for \( N \in \mathbb{N} \),
\[
R(p + N\varepsilon) = \prod_{l=1}^{N} \rho(p + \varepsilon(l - 1/2)) R(p). \] (40)
In view of Proposition 3.1 and (39), \( R \) is bounded on the interval \([-1,1]\). This, (40) and (32) imply that \( R(p+i0) = O(p^{-\infty}) \) as \( p \to +\infty \). As \( R_0 \) is even, \( R_0(p-i0) = O(|p|^{-\infty}) \) as \( p \to -\infty \). These estimates imply the needed. \(\square\)

4. Asymptotics of the function \( L_0 \) as \( \varepsilon \to 0 \)

The results of this section are used to get the asymptotics of the solutions \( \Psi_n \).

Below, \( C \) denotes positive constants independent of \( \varepsilon \). For any given function \( z \to f(z) \), \( O(f(z)) \) denotes a function such that \( |O(f(z))| \leq C|f(z)| \).

4.1. The asymptotics “between” the branching points of \( Q \). For \( a > 0 \), we define \( K_0(a) = \{ p \in \mathbb{C} : \text{Im} p > a(|\text{Re} p| - 1) \} \). One has

**Theorem 4.1.** Fix \( a > 0 \). For \( p \in K_0(a) \),
\[
L_0(p) = l_0(p) + O\left( \frac{\varepsilon^2(|p|+1)}{|p^2-1|^{3/2}} \right). \] (41)

This theorem describes the asymptotic behavior of \( L_0 \) both for \( \varepsilon \to 0 \) and for \( p \to \infty \).

**Proof.** Below, we consider only \( p \in K_0(a) \). Remind that \( \gamma(p) \), the integration contour in (36), is vertical. Assume that \( \gamma(p) \subset K_0(a) \).

Note that \( l_0''(p) = O \left( \frac{|p|+1}{|p^2-1|^{3/2}} \right) \). This implies that, for \( t \in \gamma(p) \),
\[
l_0(t) = l_0(p) + l_0'(p)(t - p) + O(|t - p|^2 M(t,p)), \quad M(t,p) = \max_{\zeta \in \gamma_{p,t}(p)} \left| \frac{|\zeta|+1}{|\zeta|^2-1|^{3/2}} \right|, \] (42)
where \( \gamma_{p,t}(p) \) is the segment of \( \gamma(p) \) between \( p \) and \( t \). Substituting this representation in the formula for \( L_0 \) in (36), we get
\[
L_0(p) = l_0(p) + I(p), \quad I(p) = O\left( \frac{1}{\varepsilon} \int_{\gamma(p)} \frac{M(t,p)|t - p|^2 d\text{Im} t}{|\cos^2 \frac{\pi}{2}(t-p)|} \right). \] (43)
First, we consider the case where $\text{Im} \, p \geq 1$. Then, we can assume that the distance between $\gamma(p)$ and the points $\pm 1$ is bounded away from zero by a constant independent of $p$. We obtain

$$|I(p)| \leq \frac{C}{\varepsilon} \int_{\gamma(p), \text{Im} \, t \leq \frac{1}{2}} \frac{|t - p|^2 d \text{Im} \, t}{|\cos^2 \frac{\pi}{2} (t - p)|} + \frac{C}{\varepsilon |p|^2} \int_{\gamma(p), \text{Im} \, t \geq \frac{1}{2}} \frac{|t - p|^2 d \text{Im} \, t}{|\cos^2 \frac{\pi}{2} (t - p)|}$$

$$\leq \frac{C}{\varepsilon} \int_{\gamma(p), \text{Im} \, t \leq \frac{1}{2}} e^{-2 \pi |\text{Im} \, (p-t)|} |t - p|^2 d \text{Im} \, t + \frac{C \varepsilon^2}{|p|^2} \leq \frac{C \varepsilon^2}{|p|^2}.$$ 

Now, consider the case of $0 \leq \text{Im} \, p \leq 1$. We can choose the contour $\gamma(p)$ so that, for $t \in \gamma(p), \ M(t, p) \leq C |p^2 - 1|^{-3/2}$. Then,

$$|I(p)| \leq \frac{C}{\varepsilon |p^2 - 1|^{3/2}} \int_{\gamma(p)} |t - p|^2 d \text{Im} \, p = \frac{C \varepsilon^2}{|p^2 - 1|^{3/2}}.$$ 

The estimates for $I(p)$ imply (41) in the case where $\text{Im} \, p \geq 0$. The complementary case is treated similarly. □

Replacing (42) with a similar formula containing more terms of the Taylor series for $l_0$, one proves

**Lemma 4.1.** Fix $a, \delta > 0$ and $J \in \mathbb{N}$. In $K_0(a)$ without the $\delta$-neighborhood of $p = \pm 1$,

$$L_0(p) = \sum_{j=0}^{J-1} \varepsilon^{2j} l_j(p) + O\left(\varepsilon^{2J}(1 + |p|)^{-2J}\right), \quad (44)$$

where $l_j$ being analytic and satisfying the estimate $l_j(p) = O((1 + |p|)^{-2j})$.

### 4.2. Analytic properties of $l_0$

In the sequel, we use some properties of $l_0$, the leading term in the asymptotics of $L_0$. Let us discuss them here.

Formula (33) implies that

- for $p \in \mathbb{C}_0$,

$$l_0(p) = \pi - 2^{3/2} z + O(\varepsilon^3), \quad z \to 0, \quad (45)$$

where $z = \sqrt{1 - p}$, the function $p \mapsto z$ is analytic, and $z > 0$ if $p < 1$;

- $l_0$ is analytic in $z = \sqrt{1 - p}$ in a neighborhood of $z = 0$;

- in $\mathbb{C}_+ \cup (\mathbb{R} + i0)$, one has

$$l_0(p) = 2i \ln (2p) + \pi + O(1/|p|^2), \quad |p| \to \infty, \quad (46)$$

where the function $p \mapsto \ln p$ is analytic and defined so that $\ln i = i\pi/2$.

One also has

**Lemma 4.2.** The function $l_0$ conformally maps the first quadrant $Q_1 = \{ p \in \mathbb{C} : \text{Im} \, p, \text{Re} \, p > 0 \}$ onto the half-strip $\Pi = \{ z \in \mathbb{C} : 0 < \text{Re} \, z < \pi, \text{Im} \, z > 0 \}$. The boundary of $Q_1$ is bijectively mapped onto the boundary of $\Pi$ in the following way: $i\mathbb{R}_+$ is mapped onto itself, the interval $[0, 1]$ is mapped onto the interval $[0, \pi]$, and the half-line $[1, +\infty)$ is mapped onto $\pi + i\mathbb{R}_+$.

The statements of the lemma easily following from (33), we omit the details. Finally, we discuss the integral $\int_0^p l_0(p) \, dp$. Integrating by parts, we check that

$$\int_0^p l_0(p) \, dp = pl_0(p) - 2iQ_0(p) - 2, \quad p \in \mathbb{C}_0, \quad (47)$$
where the integral is taken along a smooth curve in $C_0$. This formula implies, in particular, that
\[
\int_0^1 l_o(p) \, dp = \pi - 2, \tag{48}
\]
and that
\[
\int_1^p (l_o(p) - \pi) \, dp = 2i p (\ln(2p) - 1) + O(1/|p|), \quad p \in \mathbb{C}_+ \cup (\mathbb{R} + i0), \quad |p| \to \infty. \tag{49}
\]

4.3. **Asymptotic behavior of $L_0$ near the point $p = 1$.** The function $L_0$ is analytic in $C_0$ and, in particular, between the points $p = \pm 1$. We have assumed that $0 < \varepsilon < 1$. This allows to use (37) to study $L_0$ to the right of $p = 1$. The function $l'_0$ (staying in the right hand side in (37)) has a square root singularity at $p = 1$: representation (45) shows that, in a neighborhood of $p = 1$, $l'_0(p) = \left(2/(1 - p)\right)^{1/2} + O((1 - p)^{1/2})$. Therefore, $L_0$ has square root singularities at $p = 1 + \varepsilon(1/2 + l)$, $l = 0, 1, 2, 3, \ldots$. So, the asymptotic behavior of $L_0$ near the point $p = 1$ is quite non-trivial. Here, we discuss $L_0$ to left of $p = 1$.

**Theorem 4.2.** Fix $a > 0$ and sufficiently small $\delta > 0$. In $K_0(a)$, in the $\delta$-neighborhood of $p = 1$,
\[
L_0(p) = \pi + \sqrt{2\varepsilon} \zeta((p - 1)/\varepsilon) + O(\varepsilon^{3/2} + |p - 1|^{3/2}), \tag{50}
\]
where $\zeta$ is analytic in $C \setminus [1/2, \infty)$ and defined there by the formula
\[
\zeta(t) = \lim_{L \to +\infty} \left( \sum_{l=0}^{L-1} \frac{1}{\sqrt{l + 1/2 - t}} - 2\sqrt{L} \right), \tag{51}
\]
the branch of the square root being fixed so that $\sqrt{\mathbb{R}_+} = \mathbb{R}_+$. In $C_a = \{ t \in \mathbb{C} : |\text{Im}t| \geq a \text{Re}t \}$, the function $\zeta$ admits the asymptotic representation
\[
\zeta(t) = -2\sqrt{-t} + O(t^{-3/2}), \quad |t| \to \infty. \tag{52}
\]

**Remark 4.1.** In the case of Theorem 4.2, the second term in (50) satisfies the estimate
\[
\sqrt{2\varepsilon} \zeta((p - 1)/\varepsilon) = O(\varepsilon^{1/2} + |p - 1|^{1/2}). \tag{53}
\]

**Proof.** Assume that $\delta$ is sufficiently small, and that $p \in K_0(a)$ is in the $\delta$-neighborhood of $1$. In view of (46), one can use (45) as a rough approximation for $l_0$ on the whole curve $\gamma(p)$. Substituting (45) into the second formula in (36), we get
\[
\zeta(t) = i \pi \int_{\gamma(t)} \frac{\sqrt{-s} \, ds}{\cos^2(\pi(t - s))}, \quad \Delta = \frac{1}{\varepsilon} \int_{\gamma(p)} \frac{O((1 - q)^{3/2}) \, dq}{\cos^2(\pi(p - q)/\varepsilon)}, \tag{54}
\]
where $\gamma(t) \subset C_a$, $\gamma(p) \subset K_0(a)$, and $\sqrt{-s} > 0$ when $s < 0$. Using the Residue theorem, we obtain
\[
\zeta(t) = \lim_{L \to \infty} \left( \sum_{l=0}^{L-1} \frac{1}{\sqrt{l + 1/2 - t}} - I_L \right), \quad I_L = -i \pi \int_{\gamma(t-L)} \frac{\sqrt{-s} \, ds}{\cos^2(\pi(t - s))}.
\]
In this formula, for \( L \) sufficiently large, we can choose \( \gamma(t - L) = i\mathbb{R} + t - L \). So, as \( L \to \infty \),
\[
I_L = -i\pi \int_{i\mathbb{R}} \frac{\sqrt{L - t - \tau}}{\cos^2(\pi \tau)} \, d\tau = -i\pi \int_{-iL}^{iL} \left( \frac{\sqrt{L} + O(\tau/\sqrt{L})}{\cos^2(\pi \tau)} \right) d\tau + O \left( \int_{|\text{Im} \tau| \geq L} \frac{\sqrt{\tau} \, d\text{Im} \tau}{\chi^2(\pi \text{Im} \tau)} \right) = 2\sqrt{L} + O(1/\sqrt{L}).
\]
This implies (51).

Let us estimate \( \Delta \). Set \( t = p - 1 \). First, we check that
\[
|\Delta| \leq \frac{C}{\varepsilon} \int_{-\infty}^{\infty} e^{-2\pi|\text{Im}(t-\tau)|/\varepsilon} (|t| + |\text{Im} \tau|)^{3/2} \, d\text{Im} \tau.
\]
One has \( \Delta = \frac{1}{\varepsilon} \int_{\gamma(t)}^\infty \frac{O(\sigma^{3/2}) \, d\sigma}{\cos^2(\pi(t-\sigma/\varepsilon))} \). If \( \text{Re} \, t < 0 \), we choose \( \gamma(t) = \{ \tau \in \mathbb{C} : \text{Re} \, \tau = \text{Re} \, t \} \), then, \( |\tau| \leq |\text{Re} \, t| + |\text{Im} \, \tau| \leq |t| + |\text{Im} \, \tau| \). This leads to (55). If \( \text{Re} \, t \geq 0 \), we pick \( \gamma(t) = \{ \tau \in \mathbb{C} : \text{Im} \, \tau = k(t) \text{Re} \, \tau \} \) with \( k(t) = \text{Im} /\text{Re} \, t \). Clearly, \( |k(t)| \geq a \).

Now, we have \( |\tau| \leq |\text{Re} \, \tau| + |\text{Im} \, \tau| \leq (1 + 1/a)|\text{Im} \, \tau| \). This again implies (55).

If \( |t| \leq \varepsilon \), then, after the change of the variable \( \tau := \tau/\varepsilon \) in (55), we see that \( \Delta = O(\varepsilon^{3/2}) \). Otherwise, we use the estimates
\[
|\Delta| \leq \frac{C}{\varepsilon} \left( \int_{-2|t|}^{-2|t|} + \int_{-\infty}^{-2|t|} + \int_{-2|t|}^{\infty} \right) e^{-2\pi|\text{Im} \, y|/\varepsilon} (|t| + |y|)^{3/2} \, dy
\]
\[
\leq \frac{C}{\varepsilon} \int_{-\infty}^{\infty} e^{-2\pi|\text{Im} \, y|/\varepsilon} |t|^{3/2} \, dy + \frac{C}{\varepsilon} \left( \int_{-\infty}^{-2|t|} + \int_{2|t|}^{\infty} \right) e^{-2\pi|\text{Im} \, y|/\varepsilon} |y|^{3/2} \, dy.
\]
They imply that \( |\Delta| \leq C|t|^{3/2} \). The obtained estimates for \( \Delta \) justify the error term estimate in (50).

Finally, let us study the function \( \zeta \). Its analyticity follows from (54). For \( t \in C_a \), sufficiently large \( |t| \) and \( l \geq 0 \),
\[
\frac{1}{2} (l + 1/2 - t)^{-1/2} = (l + 1 - t)^{1/2} (l - t)^{1/2} + O((l - t)^{-5/2})
\]
uniformly in \( t \) and \( l \). Substituting this representation into (51), we easily arrive at (52). This completes the proof. \( \square \)

4.4. **The asymptotics of \( L_0 \) to the right of \( p = 1 \)**. Fix \( \alpha > 0 \). Here, we describe the asymptotics for \( L_0 \) in the domain \( K_1(\alpha) = \{ p \in \mathbb{C} : \text{Im} \, p \geq \alpha \, \text{Re} \, (1 - p) \} \).

Let \( C_1 = \mathbb{C} \setminus \{ z = x \in \mathbb{R} : x \leq 1 \} \). We continue analytically the function \( l_0 \) from \( C_0 \) to \( C_1 \) across \( \mathbb{C}_+ \) and denote the obtained function by \( l_1 \).

Let \( L_1 \) be a solution to the equation
\[
L_1(p + \varepsilon/2) - L_1(p - \varepsilon/2) = \varepsilon l_1'(p), \quad p \pm \varepsilon/2 \in \mathbb{C}_1.
\]
Assume that \( L_1 \) is analytic in \( \mathbb{C}_1 \). Then, in \( \mathbb{C}_+ \), both the functions \( L_0 \) and \( L_1 \) are analytic and satisfy one and the same difference equation. This implies that, in \( \mathbb{C}_+ \), the function \( P = L_0 - L_1 \) is an analytic and \( \varepsilon \)-periodic. To get the asymptotics of \( L_0 \), we construct \( L_1 \) and to analyze \( L_1 \) and \( P \).
The function $L_1$ is constructed similarly to $L_0$. It equals the right hand side of the second formula from (36) with $\gamma(p) \subset \mathbb{C}_1$. One has
\[
L_1(p) = l_1(p) + O\left(\frac{\varepsilon^2 p}{(p^2 - 1)^{3/2}}\right), \quad p \in K_1(a),
\]
(57)
This representation is proved in the same way as (41).

Fix $\delta > 0$. Reasoning as in the proof of Theorem 4.2, one checks that, in the $\delta$-neighborhood of $p = 1$ in $K_1(a)$,
\[
L_1(p) = \pi - i\sqrt{2\varepsilon} \zeta((1 - p)/\varepsilon) + O(\varepsilon^{3/2} + |p - 1|^{3/2}).
\]
(58)

Finally, we discuss the function $P$. Its Fourier series is described by

**Lemma 4.3.** For $p \in \mathbb{C}_+$,
\[
P(p) = \sum_{k=1}^{\infty} e^{2\pi ik(p - 1 - \varepsilon/2)/\varepsilon} P_k,
\]
(59)
\[
P_k = 2e^{i\pi/4} \left(\frac{\varepsilon}{k}\right)^{1/2} + O\left(\frac{\varepsilon}{k}\right)^{3/2}.
\]
(60)

**Proof.** Note that
\[
l_1(1) = l_0(1) = \pi, \quad \text{and} \quad l_1(1 - it + 0) - l_1(1) = -(l_0(1 - it - 0) - l_0(1)), \quad t > 0.
\]

This and the integral representations for $L_0$ and $L_1$ imply that
\[
P(p) = \frac{\pi}{i\varepsilon} \int_{1-i\infty}^{1+i\infty} \frac{(l_0(\zeta) - l_0(1)) d\zeta}{\cos^2 \frac{\pi(p - \zeta)}{\varepsilon}} , \quad \text{Im} \, p > 0.
\]

Integrating by parts, we get
\[
P(p) = -i \int_{1-i\infty}^{1+i\infty} \left(\tan \frac{\pi(p - \zeta)}{\varepsilon} - i\right) l_0'/(\zeta) d\zeta = \int_{1-i\infty}^{1+i\infty} \frac{-4i d\zeta}{(e^{2\pi i(p - \zeta)/\varepsilon} + 1) Q_0(\zeta)},
\]
where $Q_0$ is the branch of the function $\zeta \mapsto \sqrt{\zeta^2 - 1}$ from the definition of $l_0$. As $p \in \mathbb{C}_+$, this implies (59) with $P_k = 4i \int_{-i\infty}^{0} \frac{e^{-2\pi i k t/\varepsilon} dt}{Q_0(t + 1)}$. Clearly,
\[
P_k = 4i \int_{-i\infty}^{0} e^{-2\pi i k t/\varepsilon} (1 + O(t)) \frac{dt}{\sqrt{2t}} + \int_{-i\infty}^{-i} e^{-2\pi i k t/\varepsilon} O(1/t) dt
\]
with $\sqrt{t} = e^{3\pi/4|t|}$ for $t \in i\mathbb{R}_-$. This implies (60).

We finish this section with

**Corollary 4.1.** For $p$ in the first quadrant $Q_1$,
\[
i \int_{p_0}^{p} L_0(p) \, dp = -2p \ln |p| + O(|p|), \quad |p| \to \infty.
\]

**Proof.** In $\mathbb{C}_+$, one has $\int_{p_0}^{p} L_0 \, dp = \int_{p_0}^{p} P \, dp + \int_{p_0}^{p} L_1 \, dp$. In view of Lemma 4.3, $\int_{p_0}^{p} P \, dp$ is bounded in $\mathbb{C}_+$. So, representations (57) and (49) imply the needed. \hfill \Box
5. Standard asymptotic behavior of $\Psi_n$

Here, we prove Theorem 2.1, Lemma 2.1 and outline the proof of Theorem 2.2. We fix $n > 0$ and use the notations introduced in Section 2.3.

5.1. A convenient integral representation for $\Psi$. First, we check

**Lemma 5.1.** Pick $0 < \theta < \pi/2$. One has

$$\Psi_n(x, t) = \frac{e^{xt}}{\sqrt{2\pi}} \int_{e^{i\theta}\mathbb{R}} A(p) \sin(px) e^{iS(p, \tau)} dp, \quad \tau = \varepsilon t,$$

where

$$S(p, \tau) = p^2 (1 - \tau) - 2\pi n p + \int_0^p l_0(p) dp,$$

$$A(p) = e^{\frac{i}{2} \int_0^p (l_0(p) - l_0(p)) dp}.$$

**Proof.** In view of formulas (39) and (36), the integral in (6) equals the integral of $A(p) \sin(px) e^{iS(p, \tau)}$ along the path going in $\mathbb{C}_0$, first, along $\mathbb{R} - i0$ from $-\infty$ to $-1$, next, along $\mathbb{R}$ to $1$, and then, along $\mathbb{R} + i0$ to $+\infty$. Thanks to Corollary 4.1, one can deform the part of the integration path going along $\mathbb{R}_+$ to $e^{i\theta}\mathbb{R}_+$. As $p \to \int_0^p L_0(p) dp$ is even, see (38), one can deform the whole contour to $e^{i\theta}\mathbb{R}$. □

**Remark 5.1.** Fix sufficiently small $a, b > 0$. Let $V(b)$ be the $b$-neighborhood of the points $\pm 1$. Fix $M \in \mathbb{N}$. In view of Lemma 4.1, the factor $A$ admits the uniform asymptotic representation

$$A(p, \varepsilon) = \sum_{m=0}^{M-1} \varepsilon^m A_m(p) + O(\varepsilon^M), \quad p \in K_0(a) \setminus V(b), \quad \varepsilon \to 0,$$

with $A_0 \equiv 1$ and $A_k$ independent of $\varepsilon$, analytic and bounded uniformly in $p$.

In the next sections, we study the asymptotic behaviour of the integral in (61) as $\varepsilon \to 0$ by means of the method of steepest descents, see, e.g. [7].

5.2. Saddle point and lines of steepest descent. Discuss the saddle points, i.e., the zeros of the function $p \mapsto S_p(p, \tau)$. One has

**Lemma 5.2.** Fix $\tau < \tau_n$, $\tau_n$ being defined in (7). Then, in $\mathbb{C}_0$, there is only one zero of $S_p$. Denote it by $p_n(\tau)$. The $p_n$ is simple, is located on $[0, 1]$ and satisfies (9). One has $S_{pp}(p, \tau) > 0$ if $-1 < p < 1$.

**Proof.** Differentiating (62), we get

$$S_p(p, \tau) = 2p(1 - \tau) + l_0(p) - 2\pi n.$$

Lemma 4.2 and (35) imply that, in $\mathbb{C}_0$, $\text{Im} S_p(p, \tau)$ vanishes only on $[-1, 1]$. As $\tau < \tau_n < 1$, Lemma 4.2 and formula (65) imply that $S_p(\cdot, \tau)$ is monotonously increasing on $[-1, 1]$. As $S_p(0, \tau) = l_0(0) - 2\pi n = -2\pi n < 0$ and as $S_p(1, \tau) = 2(1 - \tau) + \pi - 2\pi n = 2(\tau_n - \tau) > 0$, we conclude that, in $\mathbb{C}_0$, $S_p$ has a unique zero $p_n$, that it is simple and that $0 < p_n < 1$. Formulas (65) and (34) imply (9). Finally, using the inequality $\tau < 1$ and (34), one checks that $S_{pp}(p, \tau) > 0$ for $-1 < p < 1$. □

Now, we discuss the paths of steepest descents “beginning” at $p_n(\tau)$. Remind that they are described by the equation $\text{Re} S(p, \tau) = \text{Re} S(p_n(\tau), \tau)$. As $S_{pp}(p_n, \tau) > 0$, there are four of them, and, at $p = p_n$, the angles between $\mathbb{R}_+$ and these curves are...
equal to \( \pi/4 + \pi l/2, \ l = 0, 1, 2, 3 \). We denote the paths of steepest decent by \( \gamma_l, \ l = 0, 1, 2, 3 \), respectively. Along \( \gamma_0 \) and \( \gamma_2 \), \( \text{Im} S(p, \tau) \) is monotonously increasing as \( p \) moves away from \( p_n \). One has

**Lemma 5.3.** If \( \tau < \tau_n \), then \( \gamma_0, \gamma_2 \subset \mathbb{C}_0 \), \( \gamma_0 \) goes to infinity inside \( \mathbb{C}_+ \), and \( \gamma_2 \) goes to infinity inside \( \mathbb{C}_- \). Along these curves,

\[
\text{Im} \ p = \text{Re} \ p + O(\ln \text{Re} \ p), \quad |p| \to \infty. \tag{66}
\]

**Proof.** We prove the statement concerning \( \gamma_0 \). The analysis of \( \gamma_2 \) is similar.

1. Let us study \( \gamma_0 \) for large \( |p| \). Formula (49) implies that

\[
S(p, \tau) = p^2 (1 - \tau) + O(p \ln (2p)), \quad p \in \mathbb{C}_+ \cap (\mathbb{R} + i 0), \quad |p| \to \infty. \tag{67}
\]

As, along \( \gamma_0 \), \( \text{Re} S(p, \tau) = \text{Re} S(p_n, \tau) \) and \( \text{Im} S(p, \tau) \) increases as \( p \to \infty \), (67) leads to (66). \( \square \)

5.3. **Proof of Theorem 2.1.** Let, as before, \( V(b) \) be the \( b \)-neighborhood of the points \( \pm 1 \). Lemma 5.3 implies that there exist sufficiently small positive numbers \( a \) and \( b \) such that \( \gamma_0, \gamma_2 \subset K_0(a) \setminus V(b) \). The paths of steepest descent continuously depend on the parameter \( \tau \). So, there are \( a, b > 0 \) such that \( \gamma_0, \gamma_2 \subset K_0(a) \setminus V(b) \) for all \( T_1 < \tau < T_2 \), i.e., for all \( \tau = \varepsilon t \) considered in Theorem 2.1.

We deform the integration path in integral in (61) to \( \gamma = \gamma_0 \cup \gamma_2 \) and replace the factor \( A \) in the integrand by the expression in the right hand side in (64). As a result, the integral in the right hand side in (61) becomes the sum of integrals along \( \gamma \). Applying the method of steepest descent to these integrals, we arrive to the asymptotic representation

\[
\Psi_n(x, \tau) = \sqrt{\frac{2}{S_{pp}(p_n, \tau)}} e^{i\tau + \frac{1}{2} S_{pp}(p_n, \tau)} \left[ \sin(p_n x) + \sum_{k=1}^{L-1} \varepsilon^k \psi_{n,k}(x, \tau) + O(\varepsilon^L) \right], \quad \tau = \varepsilon t, \quad \varepsilon \to 0,
\]

where the coefficients \( \psi_{n,k} \) are bounded uniformly in \( 0 \leq x \leq 1 - \tau \) and \( T_1 < \tau < T_2 \), and the error estimate is uniform in \( 0 \leq x \leq 1 - \varepsilon t \) and \( T_1 < \varepsilon t < T_2 \). One competes the proof of Theorem 2.1 using

**Lemma 5.4.** For \( \tau < \tau_n \),

\[
\tau + S(p_n(\tau), \tau) = \int_{\tau}^{\tau_n} E_n(\tau) \ d\tau + 2\tau_n - 3, \quad \frac{1}{S_{pp}(p_n(\tau), \tau)} = -\frac{1}{2} d \ln p_n^2(\tau), \tag{68}
\]

where \( E_n(\tau) = p_n^2(\tau) - 1 \).

**Proof.** As \( S_p(p_n, \tau) = 0 \), we get

\[
\frac{d S(p_n(\tau), \tau)}{d\tau} = S_p(p_n(\tau), \tau) p_n'(\tau) + S_{pp}(p_n(\tau), \tau) = S_{pp}(p_n(\tau), \tau) = -p_n^2(\tau).
\]

Therefore,

\[
\tau + S(p_n(\tau), \tau) = \tau_n + S(p_n(\tau_n), \tau_n) + \int_{\tau}^{\tau_n} E_n(\tau) \ d\tau. \tag{69}
\]
Note that \( p_n(\tau_n) = 1 \). This and (48) imply that
\[
\tau_n + S(p_n(\tau_n), \tau_n) = 1 - 2\pi n + \int_0^1 l_0(p) \, dp = -1 - 2\pi(n - 1/2) = 2\tau_n - 3. \tag{70}
\]
Formulas (69) and (70) imply the first relation in (68). Using the definition of \( p_n \), we get
\[
0 = \frac{d}{d\tau} S_p(p_n(\tau), \tau) = S_{pp}(p_n(\tau), \tau) p'_n(\tau) + S_{\tau p}(p_n(\tau), \tau).
\]
This leads to the second relation in (68). \( \square \)

5.4. **Outside the potential well.**

5.4.1. *The proof of Theorem 2.2* is parallel to the proof of Theorem 2.1. We only outline it. Below, \( \xi = \varepsilon(x - (1 - \tau)) \), \( \tau = \varepsilon t \), \( \xi \geq 0 \) and \( \tau < 1 \).

Instead of (61), we obtain
\[
\Psi_n(x, t) = \frac{(-1)^{n+1} e^{i t - i \xi/2 - i(1 - \tau)/4}}{\sqrt{\pi}} \int_{i\mathbb{R}} \hat{A}(p) p e^{i\hat{S}(p, \tau, \xi)} \, dp, \tag{71}
\]
\[
\hat{S}(p, \tau, \xi) = S(p, \tau) + Q_0(p)\xi, \quad \hat{A}(p) = A(p) e^{-\varepsilon \int_{p_{\varepsilon_0}}^{p_{\varepsilon}} (l_0(q) - l_0(p)) \, dq}.
\tag{72}
\]
We apply the method of steepest desents to the integral in (71). One has

**Lemma 5.5.** If \( \xi > 0 \), then in \( \mathbb{C}_0 \) there is only one saddle point \( \tilde{p}_n(\tau, \xi) \) of the function \( p \to \hat{S}(p, \tau, \xi) \). It is simple, satisfies (12) and is located in the first quadrant \( Q_1 \). If \( \tau < \tau_n \) and \( \xi = 0 \), then \( \tilde{p}_n(\tau, \xi) = p_n(\tau) \).

The paths of steepest decent continuously depend on \( \xi \). This allows to use for them the notations introduced in the case when \( \xi = 0 \) and \( \tau < \tau_n \). With these notations, the statement of Lemma 5.3 remains true.

Applying the method of steepest decent and using the formulas
\[
\tau + \hat{S}(\tilde{p}_n(\tau, \xi), \tau, \xi) = \int_\tau^{\tau_n} E_n(\tau) \, d\tau + \int_0^\xi Q_0(\tilde{p}_n(\tau, \xi')) \, d\xi' + (2\tau_n - 3),
\]
\[
\frac{1}{\hat{S}_{pp}(\tilde{p}_n(\tau, \xi), \tau, \xi)} = \frac{1}{2} \frac{\partial \ln \hat{p}_n(\tau, \xi)}{\partial \tau},
\]
alongous to formulas (68), one obtains representation (13).

5.4.2. *Proof of Lemma 2.1.* If \( \xi \) is sufficiently large, the integration path in (71) can be deformed to \( i\mathbb{R} \). Indeed, Theorem 4.1 implies that, for any fixed \( \alpha > 0 \), in \( K_0(a) \), the factor \( \hat{A} \) stays bounded as \( p \to \infty \). On the other hand, in view of (49), there is a constant \( C > 0 \) such that \( \text{Im} S(p, \tau) \geq -C|p| \) as \( p \) tends to infinity inside the sector \( 0 \leq \arg p \leq \pi/2 \). As \( l_0 \) is odd, one has an estimate of the same form in the sector \( -\pi \leq \arg p \leq -\pi/2 \). These observations imply the needed.

As along the imaginary axis \( l_0(p) \in i\mathbb{R} \), see (35), for sufficiently large \( \xi \), we get
\[
|\Psi_n(x, t)| \leq \frac{1}{\sqrt{\pi}} \sup_{p \in i\mathbb{R}} |\hat{A}| \int_{-\infty}^\infty |t| e^{i(2\pi nt - \sqrt{1 + t^2}\xi)} \, dt.
\]
To complete the proof of the lemma, we estimate the last integral by means of the Laplace method, see [7]. We omit elementary details.
6. Destruction of the standard adiabatic behavior

Here, we prove Theorem 2.3. When \( \tau = \varepsilon t \) increases, \( \tau < \tau_n \), the saddle point \( p_n \) in (61) moves to the point \( p = 1 \), a branch point of the action \( S \) (a branch point of \( l_0 \) in (62)). After a natural change of variables, \( S \) becomes an analytic function having two saddle points approaching one to another as \( \tau \to \tau_n \). This effect determines the asymptotic behavior of \( \Psi_n \) for \( \tau \sim \tau_n \).

6.1. The factor \( A \). To control the factor \( A \) in (61), we often use quite a rough

**Lemma 6.1.** For sufficiently small \( \varepsilon \),
\[
\frac{1}{\varepsilon} \int_0^p (L_0(p) - l_0(p)) \, dp = O(\varepsilon^{1/2}), \quad p \in \mathbb{C}_0.
\]  
(73)

**Proof.** In view of (35) and (38), it suffices to prove (73) only for \( p \in \mathbb{Q}_1 \). Assume that \( p \in \mathbb{Q}_1 \) and fix \( a > 0 \). If \( p \in K_0(a) \) and \( |p - 1| \geq \varepsilon \), the statement follows from Theorem 4.1. If \( p \in K_0(a) \) and \( |p - 1| \leq \varepsilon \), we prove (73) using also (50) and (45).

Assume that \( p \in K_1(a) \). Then, the analysis is based on the formula \( L_0 = L_1 + P \) and representations (59) – (60) describing the function \( P \). If \( p \in K_1(a) \) and \( |p - 1| \geq \varepsilon \), we come to (73) using also (57) to control \( L_1 \), and if \( p \in K_1(a) \) and \( |p - 1| \leq \varepsilon \), then, in addition, we use (58).

6.2. Reducing the problem to a “local one”. Pick \( 0 < b < 1 \). Denote by \( V(b) \) the \( b \)-neighborhood of \( p = 1 \). Let \( \tau_n - \delta \leq \tau \leq \tau_n \) for \( \delta > 0 \). Bellow, we assume that \( \delta \) is sufficiently small. Then, in particular, the saddle point \( p_n(\tau) \) is in \( V(b) \).

Consider \( \gamma_0 \) and \( \gamma_2 \), two paths of the steepest descents beginning at \( p_n \). Let \( \gamma = \gamma_0 \cup \gamma_2 \), and let \( \gamma(\tau) \) be the connected component of \( \gamma \cap V(b) \) containing \( p_n \). There is \( C > 0 \) such that, as \( \varepsilon \to 0 \),
\[
\Psi_n(x, t) = \frac{e^{i \tau}}{\sqrt{2 \pi}} \int_{\gamma(\tau)} A(p) \sin(px) e^{xS(p, \tau)} \, dp + O(e^{-C/\varepsilon}).
\]  
(74)

One proves this representation using an argument standard for the method of steepest descents. Let us outline it. First, fix \( \tau \). Note that the derivative of \( \text{Im} S(p, \tau) \) along \( \gamma \) equals \( |S_p(p, \tau)| \), and that, along \( \gamma \), \( S_p(p, \tau) \to \infty \) as \( p \to \infty \), and \( S_p(p, \tau) \neq 0 \) if \( p \neq p_n(\tau) \). Using these observations, one checks that \( \int_{\gamma(\tau)} A \sin(px) e^{xS} \, dp = O(e^{-C/\varepsilon}) \). As \( S \), \( p_n \) and the paths of steepest decent continuously depend on \( \tau \), this estimate is uniform both in \( x \) and \( \tau \).

6.3. Local analysis of the action \( S \). Let us study \( S \) near the point \( p = 1 \). We begin with

**Lemma 6.2.** For \( p \in \mathbb{C}_0 \) and \( |p - 1| < 1 \), consider the action \( S \) as a function of the variable \( z = z(p) = \sqrt{1 - p} \) fixed by the condition \( z(p) > 0 \) for \( p < 1 \). One has
\[
S(p(z), \tau) = S(1, \tau) - 2(\tau_n - \tau) z^2 + \frac{4\sqrt{2}}{3} z^3 + (1 - \tau) z^4 + z^5 f(z),
\]  
(75)
\[
S(1, \tau) = -3 + 2\tau_n - \tau,
\]  
(76)
where \( f \) is independent of \( \tau \) and analytic in \( z \) in the 1-neighborhood of zero.

**Proof.** Using formulas (62), (7), we obtain
\[
S(p, \tau) - S(1, \tau) = 2(\tau_n - \tau) (p - 1) + (1 - \tau) p(p - 1)^2 + \int_1^p (l_0(p) - \pi) \, dp.
\]  
(77)
This and (45) imply the representation

\[ S(p, \tau) = \frac{S(p(z), \tau) - S(1, \tau)}{2} = 2(\tau - \tau_n) (1 - p) + (1 - \tau) (1 - p)^2 + \frac{4\sqrt{2}}{3} (1 - p)^3 + O((1 - p)^4). \]  

This implies (75). The analyticity of \( \tilde{f} \) follows from the analyticity of \( l_0 \) in \( z \). Finally, by means of (48), we get

\[ S(1, \tau) = -1 + \pi - 2\pi n - \tau = -3 + 2\tau_n - \tau. \]  

Denote by \( S_z \) and \( S_{zz} \) the first and second derivatives of the function \( z \rightarrow S(p(z), \tau) \) with respect to \( z \). By means of Lemma 6.2, one checks

**Lemma 6.3.** If \( b > 0 \) is sufficiently small, then there exists \( \delta = \delta(b) \) such that for all \( |\tau - \tau_n| \leq \delta \), \( S_z \) has only two zeros in \( z \) in the \( \sqrt{b} \)-neighborhood \( z = 0 \). They are simple if \( \tau \neq \tau_n \), coincide if \( \tau = \tau_n \), and are located at the points

\[ z = 0 \quad \text{and} \quad z = z_n(\tau) := z(p_n(\tau)). \]  

The function \( \tau \rightarrow z_n(\tau) \) is analytic in \( \tau \), and

\[ z_n(\tau) = \frac{1}{\sqrt{2}}(\tau_n - \tau) + O((\tau - \tau_n)^2), \quad \tau \rightarrow \tau_n. \]  

Lemmas 6.2 and 6.3 show that after the change of variable \( p \rightarrow z(p) \) the asymptotic analysis of \( \Psi_n \) is reduced to the standard asymptotic analysis of an integral with two coalescing saddle points, a well understood classical problem see, e.g., [7].

**6.4. Change of variables.** The key to the asymptotic analysis of integrals with two nearby saddle points is a change of variables that transforms the action into a third order polynomial. This transformation is described by the Chester, Friedmann and Ursell theorem [3, 7]. We formulate it in a convenient for us form. For \( z \) close to \( 0 \), we set

\[ G(z, \tau) = \frac{S(p(z), \tau) - S(1, \tau)}{2} = -(\tau_n - \tau) z^2 + \frac{2\sqrt{2}}{3} z^3 + O(z^4), \]  

and, for \( \tau \) close to \( \tau_n \), using the formula

\[ \lambda(\tau) = (-6G(z_n(\tau), \tau))^{1/3}, \]  

we define a function \( \lambda \) analytic in \( \tau \) and such that \( \lambda(\tau) > 0 \) when \( \tau < \tau_n \) and

\[ \lambda(\tau) = \tau_n - \tau + o(\tau_n - \tau), \quad \tau \rightarrow \tau_n. \]  

Define a multivalued analytic function \( z \rightarrow u(z, \tau) \) by the equation

\[ G(z, \tau) = -\lambda(\tau) u^2 + \frac{2\sqrt{2}}{3} u^3. \]  

One has

**Theorem 6.1.** If \( c > 0 \) is sufficiently small, then, for sufficiently small \( \delta(c) > 0 \), there is just one branch of the function \( u \) that is analytic in \( (z, \tau) \in \{|z| < c\} \times \{|\tau - \tau_n| < \delta(c)\} \). For this branch,

\[ u(0, \tau) = 0 \quad \text{and} \quad u(z_n(\tau), \tau) = \lambda(\tau)/\sqrt{2}. \]  

The correspondence \( z \leftrightarrow u \) is \( (1, 1) \).

For \( |\tau - \tau_n| < \delta(c) \), we define \( u \rightarrow z(u, \tau) \), the function inverse to \( z \rightarrow u(z, \tau) \). Let \( u_n(\tau) = u(z_n(\tau), \tau) = \lambda(\tau)/\sqrt{2} \). We shall use
Corollary 6.1. For $|\tau - \tau_n| < \delta(c)$,

$$\frac{z}{u} \frac{\partial z}{\partial u} \bigg|_{u=0} = \frac{\lambda(\tau)}{\tau_n - \tau}, \quad \frac{z}{u} \frac{\partial z}{\partial u} \bigg|_{u=\eta_n(\tau)} = \sqrt{\frac{1}{\lambda(\tau)} \frac{d\ln p_n}{d\tau}(\tau)}, \quad (85)$$

where the square root is positive for $\tau < \tau_n$.

Proof. As $z(0, \tau) = 0$, we get $\frac{z}{u} \frac{\partial z}{\partial u} \bigg|_{u=0} = (\frac{\partial z}{\partial u} \bigg|_{u=0})^2$. Furthermore, relations (83) and (80) imply that $(\frac{\partial z}{\partial u} \bigg|_{u=0})^2 = \lambda(\tau)/(\tau_n - \tau)$. This proves the first formula in (85). The proof of the second one. Using (83), we obtain $\frac{1}{2} \left( (\frac{\partial z}{\partial u} \bigg|_{u=0})^2 S_{zz} + \frac{\partial^2 z}{\partial u^2} S_{zz} \right) = 4(-\lambda + \sqrt{2}u)$. Therefore, $(\frac{\partial z}{\partial u} \bigg|_{u=0})^2 S_{zz} \bigg|_{z=z_n} = 4(-\lambda + \sqrt{2}u)$. On the other hand, as $p = 1 - z^2$, $S_{zz} = 4z^2 S_{pp} - 2S_p$. This implies that $S_{zz} \bigg|_{z=z_n} = 4z_n^2 S_{pp} \bigg|_{p=p_n}$. Using these two observations and the second formula in (68), we obtain

$$\frac{z_n^2}{u_n^2} \left( \frac{\partial z}{\partial u} \right)^2 \bigg|_{u=\eta_n} = \frac{z_n^2}{u_n} S_{zz} \bigg|_{z=z_n} = \frac{\sqrt{2}}{u_n} S_{pp} \bigg|_{p=p_n} = \frac{2}{\lambda} S_{pp} \bigg|_{p=p_n} = \frac{1}{\lambda} \frac{d\ln p_n}{d\tau}(\tau).$$

This implies the second formula in (85) up to the sign of its right-hand side. The first formula in (85) and (82) imply that $\frac{z}{u} \frac{\partial z}{\partial u} \bigg|_{u=0, \tau=0} = 1$. This implies that the sign in the second formula in (85) is correct and completes the proof. \qed

6.5. Integration path. Let $\beta(b, \tau) = u(z(\gamma(b)), \tau)$, $\gamma(b)$ being the curve defined in section 6.2, and let

$$S(u, \lambda) = -\lambda u^2 + \frac{2\sqrt{2}}{3} u^3. \quad (86)$$

Relation (83) imply

Lemma 6.4. For the function $u \to S(u, \lambda(\tau))$,

- $u_n(\tau)$ is a saddle point, and $\beta(b, \tau)$ is a segment of $B(\tau)$, the path of steepest descent containing $u_n(\tau)$ and such that $\text{Im} S(u, \lambda(\tau))$ increases as $u$ moves away from $u_n(\tau)$ along $B(\tau)$;
- there is a positive constant $C(b)$ such that, for all $\tau$ we consider, $\text{Im} S(u, \lambda(\tau)) > C(b)$ at the ends of $\beta(b, \tau)$;
- the orientation of $\beta$ induced by the orientation of $\gamma(b)$ is such that, at $u = u_n$, the $\beta(b, \tau)$ is oriented downwards;
- $B(\tau)$ is a smooth curve having the asymptotes $-i \mathbb{R}_+$ and $e^{\frac{\pi}{2} i} \mathbb{R}_+$.

Proof. The first two properties follow from the analogous properties of $p_n$ and $\gamma(b)$. To prove the third one, remind that, at $p_n$, $\gamma(b)$ is oriented upwards. In view of the second formula in (85), $u_{z_n} > 0$. As $\frac{\partial u}{\partial p} \bigg|_{p_n} = -\frac{1}{2z_n} \frac{\partial u}{\partial z} \bigg|_{z_n} < 0$, at the point $u = u_n$, the path $\beta(b, \tau)$ is oriented downwards. The proof of the forth property is elementary and is omitted. \qed

6.6. The proof of Theorem 2.3. Define $F$ by (15). One has
Proposition 6.1. As $\varepsilon \to 0$,
\[
\Psi_n(x, t) = a_0(x, \tau) F(e^{i\pi/6} Z_n) + O(\varepsilon^{2/3}(1 + |Z_n|^{1/2})), \quad Z_n = \lambda(\tau)/(4\varepsilon)^{1/3},
\]  
\[
a_0(x, \tau) = (4\varepsilon)^{1/6} e^{i\phi(S(1, \tau)+\tau)/2} \sin(p_n(\tau)x) \frac{z}{u} \frac{\partial z}{\partial u} u^{-1}(\tau), \quad \tau = \varepsilon t.
\]  
Proof. Let $b > 0$ and $\delta > 0$ be sufficiently small, and let $\tau_n - \delta \leq \tau \leq \tau_n$. In the integral in (74) we change the variable $p \mapsto u = u(z(p), \tau)$. The leading term in (74) takes the form
\[
\Psi_n^{(0)}(x, \tau) = -\frac{2e^{i\phi(S(1, \tau)+\tau)}}{\sqrt{\pi \varepsilon}} \int_{\beta(b, \tau)} A(p) \varphi(u, \tau, x) e^{2iS(u, \lambda(\tau))} u \, du,
\]  
where
\[
\varphi(u, \tau, x) = \sin(px) \frac{z}{u} \frac{\partial z}{\partial u}, \quad p = 1 - z^2, \quad z = z(u, \tau).
\]  
Represent $\varphi$ in the form
\[
\varphi(u, \tau, x) = \varphi_0(\tau, x) + (u - u_n(\tau)) \varphi_1(u, \tau, x), \quad \varphi_0(\tau, x) = \varphi(u_n(\tau), \tau, x).
\]  
Note that $S(u, \tau) = 2\sqrt{2}(u - u_n(\tau))$. Using integration by parts, we get
\[
\Psi_n^{(0)}(x, \tau) = k (J_1 + J_2 + J_3 + J_4), \quad k = -\frac{2e^{i\phi(S(1, \tau)+\tau)}}{\sqrt{\pi \varepsilon}},
\]  
\[
J_1 = \int_{\beta(b, \tau)} (A(p) - 1) \varphi_0(\tau, x) e^{2iS} u \, du, \quad J_1 = \varphi_0(\tau, x) \int_{\beta} e^{2iS} u \, du,
\]  
\[
J_2 = \frac{\varepsilon}{4\sqrt{2}i} \varphi_1(u, \tau, x) e^{2iS} \bigg|_{\beta}, \quad J_3 = \frac{i\varepsilon}{4\sqrt{2}} \int_{\beta} (\varphi_1)'(u, \tau, x) e^{2iS} u \, du,
\]  
where $S = S(u, \lambda), \lambda = \lambda(\tau)$. In view of the second statement of Lemma 6.4, the term $J_2$ is exponentially small in $\varepsilon$, and, up to an exponentially small error, we can replace in the formula for $J_1$ the integration path by $B(\tau)$. We get
\[
J_1 + J_2 = \varphi_0(\tau, x) \int_{B(\tau)} e^{2iS(u, \lambda)} u \, du + O(e^{-C/\varepsilon}).
\]  
In the integral in (92), we change the variable $u \mapsto v = e^{-\varphi_0+(2+\frac{i}{2})} - (2-\frac{i}{2})$. This gives
\[
J_1 + J_2 = -2^{\frac{i}{2}} \pi e^{\varphi_0} e^{i\frac{2}{3} \varphi_0} e^{2iS} \varphi_0(\tau, x) e^{-\frac{i}{3}Z_n f(e^{i\pi/6} Z_n)} + O(e^{-C/\varepsilon}),
\]  
\[
f(s) = \frac{1}{2\pi i} \int_{\gamma} e^{-\frac{\nu^2}{\tau^2} + s^2 u} (s - v) \, dv,
\]  
where $Z_n$ is as in (87), and $\gamma$ is a smooth curve going from $e^{-2i\pi/3}$ to $e^{2i\pi/3}$. Recall that $\text{Ai}(s) = \frac{1}{2\pi i} \int_{\gamma} e^{-\frac{s^2}{\tau^2} + sv} \, dv$, see [4]. Therefore, $f(s) = s\text{Ai}(s^2) - \text{Ai}'(s^2)$. This implies that the leading term in (93) being multiplied by the factor $k$ is equal to the leading term in (87). To complete the proof, it suffices to check the estimates
\[
J_3 = O(e^{4/3}/(1 + |Z_n|^{1/2})), \quad J_4 = O(e^{7/6}/(1 + |Z_n|^{1/2})).
\]  
Begin with $J_3$. If $|\lambda/\varepsilon^{1/3}| \leq 1$, we change the variable $u$ to $w = u/\varepsilon^{1/3}$ and get
\[
J_3 = e^{4/3} \int_{\beta} O(1) e^{2iS(w, \lambda)/\varepsilon^{1/3}} \, dw = O(e^{4/3}). \quad \text{If } |\lambda/\varepsilon^{1/3}| \geq 1, \text{ we change the variable } u \text{ to } w = u/\lambda.
\]  
Then, $J_3 = \varepsilon \lambda \int_{\beta} O(1) e^{\frac{2i\lambda}{\varepsilon^{1/3}} S(w, 1)} \, dw$, the integral is taken along a segment of a path of the steepest descent of the function $w \mapsto S(w, 1)$,
Lemma 7.1. To prove the proposition, we use the following two lemmas. and set ρ



asymptotics of Ψ



viewing each of these terms and prove Theorem 7. Let us complete the proof of Theorem 2.3. First, check that Zn(εt) admits the representation from this theorem. Using (76) and the first relation in (68), we check that S(pn(τ), τ) − S(1, τ) = \int_τ^{∞} E_n(τ) dτ. This and (81) imply the representation λ(τ) = \left(3 \int_{τ_n}^{τ} E_n(τ) dτ\right)^{\frac{1}{3}}. Substituting it into the formula for Zn from (87), we get the needed representation for Zn. Next, we transform the expression for a0 from (87). Using (76), the second relation in (85), (10) and the expression for Zn from (87), we prove that a0 = c_n \sqrt{(\ln p_n)'(τ)/Z_n} \psi_n(x, τ). This formula and (87), imply (16).

7. Aftermath: the case of εt ≥ τn

Here, first, we single out the terms that become the leading terms of the asymptotics of Ψn for t and x satisfying (17). Then, we describe the asymptotic behavior of each of these terms and prove Theorem 2.4.

7.1. Singling out the leading terms. For τ ≥ τn, the main contribution to the integral in (5.1) comes from a small neighborhood of the point p = 1, the branch point of the action S. Let us formulate the precise statement. Pick 0 < α < 1/6 and set ρ(ε) = ε^{\frac{2}{3}−α}. For a, b ∈ C, denote by [a, b] the segment of straight line connecting a and b. We prove

**Proposition 7.1.** For t and x satisfying (17), as ε → 0,

\[ Ψ_n(x, εt) = e^{iφ(ε)}(T(x, εt) + R(x, εt) + G(x, εt)) + E(x, εt), \]

where

\[ T(x, τ) = \frac{e^{it}}{3} \int_{1-ρ(ε)}^{1} \sin(px) e^{iS(p, τ)} dp, \]

\[ R(x, τ) = \frac{i e^{it}}{3} \int_{1-ρ(ε)}^{1} \sin(px) e^{iS(p, τ)} \int_{p}^{1} P(q) dq dp, \]

\[ G(x, τ) = \frac{i e^{it}}{3} \int_{1-ρ(ε)}^{1} \sin(px) e^{iS(p, τ)} \int_{1}^{p} (L_0(q) - l_0(1)) dq dp + \int_{1-ρ(ε)}^{1} \sin(px) e^{iS(p, τ)} \int_{1}^{p} (L_1(q) - l_0(1)) dq dp, \]

E(x, τ) = O(ε^2), and φ(ε) = \frac{1}{ε} \int_{0}^{1} (L_0 - l_0) dp = O(ε^2).

To prove the proposition, we use the following two lemmas.

**Lemma 7.1.** For p ∈ C_0, we let p = 1 + e^{iφ}s, s ≥ 0, φ ∈ R. If 0 < φ < π, then

\[ \text{Im} S(p, τ) = (1 − τ)s^2 \sin(2φ) + 2s(\ln(2s) − 1) \cos φ + 2s \sin φ(τ_n − τ − φ) + O(1) \]

as s → ∞. If −π < φ < 0, then, as s → ∞,

\[ \text{Im} S(p, τ) = (1 − τ)s^2 \sin(2φ) − 2s(\ln(2s) − 1) \cos φ + 2s \sin φ(τ_n − τ + φ) + O(1). \]
These representations are uniform in φ.

Proof. The first statement follows from representations (77) and (49). The second follows from the first one and the relation $S(p, \tau) = S(p, \bar{\tau})$ valid for $p \in \mathcal{C}_0$. □

In the next lemma, we discuss $S(\cdot, \tau)$ on $\Gamma = \{1 - i\mathbb{R}_+\} \cup \{1\} \cup \{1 + \mathbb{R}_+ + i0\} \subset \mathcal{C}_0$.

Lemma 7.2. Let $\tau \geq \tau_n$.

- Along $\Gamma$, $\text{Im} \ S(p, \tau)$ monotonously increases as $p$ moves away from $p = 1$, and $\partial \text{Im} \ S(p, \tau) / \partial |p - 1| > 0$ when $p \neq 1$;
- Fix $b > 0$. There is $C > 0$ such that, along $\Gamma$, for $|p - 1| \leq b$, one has $\text{Im} \ S(p, \tau) > C|p - 1|^\frac{3}{2}$.

Proof. The first statement immediately follows from the following properties of $S$:

1) If $p > 1$, then $\text{Im} \ S(p + i0, \tau) = \int_1^p \text{Im} \ l_0(p + i0) \, dp$, and, according to Lemma 4.2, $\text{Im} \ l_0(p + i0) > 0$ for $p > 1$;
2) If $p \in 1 - i\mathbb{R}_+$, then $\text{Im} \ S = -2(\tau - \tau_n)\text{Im} \ (p - 1) - \int_{\text{Im} \ p}^0 \text{Re} \ (l_0(1 + is) - \pi) \, ds$, and, in view of Lemma 4.2, $\text{Re} \ l_0(p) < \pi$ inside $\mathcal{C}_0$.

The second statement follows the first one and representation (78). □

Now, we turn to the proof of Proposition 7.1.

Proof. Lemma 7.1 and estimate (73) imply that, for sufficiently small $\varepsilon$ and $\tau \geq \tau_n$, one can deform the integration path in (61) to $\Gamma$. Let $\Gamma_+ = \{1\} \cup (1 + \mathbb{R}_+ + i0)$, and let $\Gamma_- = \{1\} \cup (1 - i\mathbb{R}_+)$. Lemma 7.1 also shows that, as $p \to \infty$,

$$\text{Im} \ S(p, \tau) = 2p \ln p + O(p), \quad p \in \Gamma_+, \quad (100)$$

$$\text{Im} \ S(p, \tau) = 2(\tau - \tau_n + \pi/2) |\text{Im} p| + O(1), \quad p \in \Gamma_- \quad (101)$$

Using this and the first point of Lemma 7.2, one proves that, for any fixed $b > 0$, there is a $C > 0$, such that, for sufficiently small $\varepsilon$, modulo $O(e^{-C/\varepsilon})$, $\Psi_n$ equals the right hand side of (61) with the integration path replaced by $\Gamma \cap \{p - 1 \mid \leq b\}$. Finally, choosing $b$ sufficiently small and using the second point of Lemma 7.2, we prove that, modulo $O(e^{-C/\varepsilon^3})$, $\Psi_n$ equals the right hand side of (61) with the integration path replaced by $\Gamma \cap \{p - 1 \mid \leq \rho(\varepsilon)\}$.

Now, discuss the factor $A$ in the integrand in (61). In view of (73),

$$A(p) = e^{i\phi(\varepsilon)} \left(1 + \frac{i}{\varepsilon} \int_1^p (L_0 - l_0) \, dp + O(\varepsilon)\right), \quad p \in \Gamma. \quad (102)$$

We use this representation for $p \in \Gamma_-$. For $p \in \Gamma_+$, we represent $L_0$ in the form $L_0 = L_1 + P$, see Section 4.4, and get

$$A(p) = e^{i\phi(\varepsilon)} \left(1 + \frac{i}{\varepsilon} \int_1^p (L_1 - l_0) \, dp + \frac{i}{\varepsilon} \int_1^p P(p) \, dp + O(\varepsilon)\right), \quad p \in \Gamma_+. \quad (103)$$

Estimate (73) also implies that $\phi(\varepsilon) = O(\varepsilon^{1/2})$. Substituting (103) and (102) in (61) with the integration path replaced with $\Gamma \cap \{p - 1 \mid \leq \rho(\varepsilon)\}$ and with the corresponding error estimate, we arrive at (96) with

$$E = \int_{\Gamma \cap \{p - 1 \mid \leq \rho(\varepsilon)\}} e^\frac{1}{\varepsilon} S(p, \tau) \, dp + O(e^{-C/\varepsilon^3}). \quad (104)$$

Using the second point of Lemma 7.2, we get $|E| \leq C \varepsilon^\frac{3}{2} \int_0^\infty e^{-\frac{C}{\varepsilon^3}|p - 1|^\frac{3}{2}} \, dp \leq C \varepsilon^2$. This completes the proof of the proposition. □
7.2. The term $T$. Here, we prove that, in the case of (17), as $\varepsilon \to 0$,

$$
T(x, \tau) = T_0(x, \tau) + O\left(\varepsilon / (1 + |z_n(\tau)|^3)\right) \tag{105}
$$

with $z_n$ and $T_0$ given by (18).

Proof. 1. In the integral in (97), we pass to the variable $z = z(p)$ as in Lemma 6.2. The integration path turns into $c(\varepsilon^{1/3-\alpha/2})$, where, for $r > 0$,

$$
c(r) = [-ir, 0] \cup [0, e^{i\pi/4}] \subset -i\mathbb{R}_+ \cup \{0\} \cup e^{i\pi/4}\mathbb{R}_+.
$$

As a curve, $c(r)$ is oriented downwards. Since $0 < \alpha < 1/6$, on the integration path, the expression $z^4/\varepsilon = O(\varepsilon^{1/3-2\alpha})$ is small, and, in view of (75) and (86),

$$
\sin(p(x)e^{iS(p(x),\tau)}) = e^{i(S(1,\tau) + 2S(z,\tau_n-\tau))}(\sin x + O(z^2) + O(z^4/\varepsilon))
$$

Therefore,

$$
T(x, \tau) = -\frac{2c_0e^{-i\pi/4}}{(\pi \varepsilon)^{1/2}}(\sin x \cdot I_0 + I_1 + I_2/\varepsilon), \tag{106}
$$

$$
I_0 = \int_c e^{\frac{2iS}{z}} z \, dz,
I_1 = \int_c e^{\frac{2iS}{z}} O(z^3) \, dz,
I_2 = \int_c e^{\frac{2iS}{z}} O(z^5) \, dz,
$$

where $S = S(z, \tau_n - \tau)$ and $c = c(\varepsilon^{1/3-\alpha/2})$.

2. Along $c(\infty)$, one has $\left|e^{\frac{2iS}{z(z,\tau_n-\tau)}}\right| \leq e^{-C|z|^3/\varepsilon}$. Therefore, $I_0 = \int_c e^{\frac{2iS}{z}} z \, dz + O(e^{-C\varepsilon^{-3\alpha/2}})$. The last integral is computed as the analogous integral from the proof of Proposition 6.1. This gives

$$
I_0 = -2^{-2/3} \sqrt{\pi} e^{i\pi/4} \varepsilon^{2/3} F(e^{i\pi/6}z_n(\tau)) + O(e^{-C\varepsilon^{-3\alpha/2}}). \tag{107}
$$

3. To estimate the integral $I_1$, we change the variable $z$ to $w = z\varepsilon^{-1/3}$ and obtain $I_1 = \varepsilon^{4/3} \int_{c(-\varepsilon^2/2)} e^{2iS(w,4^{1/3}z_n(\tau))} O(w^3) \, dw$. Therefore, if $-1 \leq z_n(\tau) \leq 0$, then $I_1 = O(\varepsilon^{4/3})$. Let $z_n(\tau) \leq -1$. Changing the variable $w$ to $u = -w/z_n(\tau)$, we get $I_1 = \varepsilon^{4/3} z_n(\tau) \int_{c(\infty)} E^{(a)}(w^{-1/3}) O(w^3) \, dw$ with $b = -\varepsilon^{-\alpha/2}/z_n(\tau)$, and integrating twice by parts, we obtain $I_1 = O(\varepsilon^{4/3} z_n^2)$. 

4. Estimating the integral $I_2$ similarly, one checks that $I_2 = O(\varepsilon^2)$ if $|z_n| \leq 1$, and that $I_2 = O(\varepsilon^2/|z_n|^3)$ otherwise.

5. Representation (106), formula (107) for $I_0$ and our estimates for $I_1$ and $I_2$ lead to (105). \hfill \square

7.3. The term $R$. Here, we prove that, under conditions (17),

$$
R(x, \tau) = R_0(x, \tau) + O(\varepsilon^{4/3}), \quad \varepsilon \to 0, \tag{108}
$$

with $R_0$ given by (21).

Proof. 1. Put

$$
Q(p) = \frac{1}{\sqrt{2\pi}} \int_1^p P(q) \, dq. \tag{109}
$$

Representation (59) implies that, for $p \in \mathbb{R}$, as $\varepsilon \to 0$,

$$
Q(p) = \frac{e^{-i\pi}}{\pi} f \left(\frac{p - 1}{\varepsilon}\right) + O(\varepsilon), \quad f(p) = \sum_{k=0}^{\infty} f_k e^{2\pi i kp}, \tag{110}
$$

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where $f_k$ are given in (22).

2. Using the second point of Lemma 7.2, (110) and the representation $\sin(px) = \sin x + O(p - 1)$, one proves that

$$\mathcal{R}(x, \tau) = \frac{e^{it + \frac{4\pi}{x}}}{\pi^2} \sin x \int_0^{1+p(x)} e^{it(p, \tau)} f \left( \frac{p-1}{x} \right) dp + O(\varepsilon^{\frac{5}{3}}). \quad (111)$$

3. Using (78), we rewrite (111) in the form

$$\mathcal{R}(x, \tau) = \frac{c_n \sin x}{\pi^2} \int_0^\rho e^{i \left( (2(\tau_n - \tau) + \frac{4\pi}{x} \right) t^2 + (1 - \tau) t^2 + O(\varepsilon^2) \right) f(t/\varepsilon) dt + O(\varepsilon^{\frac{5}{3}}).$$

4. As in the definition of $\rho$, $0 < \alpha < 1/6$, the last integral can be transformed to the form

$$\int_0^\rho e^{it(\tau_n - \tau) + \frac{4\pi}{x} t^2 + O(\varepsilon^2)} \left( 1 + i(1 - \tau) t^2/\varepsilon + O(t^2/\varepsilon^2) + O(t^4/\varepsilon^2) \right) f(t/\varepsilon) dt.$$

As, for $c > 0$ and $l > 0$, both the integrals $\int_0^\rho e^{-\frac{it^2}{\varepsilon} + \frac{4\pi}{x} t^2} dt$ and $\int_0^\rho e^{-\frac{it^2}{\varepsilon} + \frac{4\pi}{x} t^2} t^4 dt$ are $O(\varepsilon^{\frac{5}{3}})$, and $\int_0^\rho e^{-\frac{it^2}{\varepsilon} + \frac{4\pi}{x} t^2} dt = O(e^{-C\varepsilon^{-3b/2}})$, we get

$$\mathcal{R}(x, \tau) = \frac{c_n \sin x}{\pi^2} \int_0^\infty e^{it(\tau_n - \tau) + \frac{4\pi}{x} t^2} \left( 1 + i(1 - \tau) t^2/\varepsilon + O(t^2/\varepsilon^2) + O(t^4/\varepsilon^2) \right) f(t/\varepsilon) dt + O(\varepsilon^{\frac{5}{3}}).$$

5. Consider the integral $I_1 = \int_0^\infty e^{it(\tau_n - \tau) + \frac{4\pi}{x} t^2} f(t/\varepsilon) dt$. Representing the function $f$ by its Fourier series and using the definition of $\tau_N$, see (7), we get $I_1 = \sum_{k=0}^\infty f_k \int_0^\infty e^{it(\tau_n - \tau) + \frac{4\pi}{x} t^2} dt$. Changing the variable $t \mapsto u = 2\varepsilon^{-\frac{1}{3}} t^{\frac{3}{2}}$, and using the definition of the function $a$, see (19), we get finally

$$I_1 = \left( \frac{\varepsilon}{2} \right)^{\frac{5}{3}} \sum_{k=0}^\infty f_k a \left( \frac{\tau_n - \tau}{(4\varepsilon)^\frac{1}{3}} \right).$$

6. Similarly, one proves that

$$I_2 := \frac{1}{\varepsilon} \int_0^\infty e^{it(\tau_n - \tau) + \frac{4\pi}{x} t^2} f(t/\varepsilon) t^2 dt = -\frac{\varepsilon}{16} \sum_{k=0}^\infty f_k a'' \left( \frac{\tau_n - \tau}{(4\varepsilon)^\frac{1}{3}} \right).$$

7. The last three steps lead to the formula (108).

7.4 The term $\mathcal{G}$. Here, we assume that the condition (17) is satisfied, and we check that, as $\varepsilon \to 0$,

$$\mathcal{G}(x, \tau) = O(\varepsilon^{\frac{5}{3}}), \quad \text{and} \quad \mathcal{G}(x, \tau) = \mathcal{G}_0(x, \tau) + O(\varepsilon^{2/3} / \varepsilon^{5/2}) \quad \text{if} \quad \varepsilon = (\tau - \tau_n)/\varepsilon^x \to \infty,$$

$\mathcal{G}_0$ being described in (23).

Proof. According to (99), we can write $\mathcal{G} = \mathcal{G}_- + \mathcal{G}_+$, where

$$\mathcal{G}_- = \int_{1-i\rho(x)}^1 g_-(p, \tau) e^{it(p, \tau)} dp, \quad \mathcal{G}_+ = \int_1^{1+p(x)} g_+(p, \tau) e^{it(p, \tau)} dp, \quad (113)$$

g_- and $g_+$ are functions analytic in $\mathbb{C}_0 = \mathbb{C} \setminus \{ p \in \mathbb{C} : |p| \geq 1 \}$ and $\mathbb{C}_1 = \mathbb{C} \setminus \{ p \in \mathbb{C} : p \leq 1 \}$ respectively and satisfy the estimates

$$g_- = O(1), \quad g_+ = O(1), \quad 1 \leq \Re p \leq 2, \quad -1 \leq \Im p \leq 0. \quad (114)$$
that follow from estimate (73) for the integral of \( L_0 - l_0 \) and its analogue for the integral of \( L_1 - l_0 \).

The estimate \( G = O(\varepsilon^{\frac{3}{2}}) \) follows directly from (113), (114) and the second point of Lemma 7.2. The key to the representation in the second line of (112) is

**Lemma 7.3.** Pick \( 0 < \beta < 1 \). Set \( \rho_0 = \frac{\rho}{\tau - \tau_n} \varepsilon^\beta \). If \( \varepsilon \to 0 \) and \( z \to \infty \), then

\[
G_\pm = \mp e^{\frac{iS(1, \tau)}{\varepsilon}} \int_{1-i\rho_0}^1 g_\pm e^{-\frac{2i(\tau - \tau_n)(p-1)}{\varepsilon}} dp + O \left( \varepsilon^{\frac{3}{2}} z^{-\frac{3}{2}} \right). \tag{115}
\]

**Proof.** Let us explain the way to obtain (115) for \( G_+ \). For the analytic continuation of \( S \) from \( C_+ \) to \( C_1 \) across \( 1 + \mathbb{R}_+ \), we keep the “old” notation \( S \). It suffices to justify the equalities

\[
G_+(x, \tau) = \int_1^{1+e^{-\pi i/4} \rho_0} g_+ e^{\frac{iS(p, \tau)}{\varepsilon}} dp + O \left( e^{-C_\varepsilon^{3\alpha/2}} \right)
\]

\[
= \int_1^{1+e^{-\pi i/4} \rho_0} g_+ e^{\frac{iS(p, \tau)}{\varepsilon}} dp + O \left( \varepsilon^2 e^{-C_\varepsilon^{\beta}} + e^{-C_\varepsilon^{3\alpha/2}} \right)
\]

\[
= \int_1^{1+e^{-\pi i/4} \rho_0} g_+ e^{\frac{iS(1, \tau) - 2(\tau - \tau_n)(p-1)}{\varepsilon}} dp + O \left( \varepsilon^2 \varepsilon^{-\frac{3}{2}} \right)
\]

\[
= \int_1^{1-i\rho_0} g_+ e^{\frac{iS(1, \tau) - 2(\tau - \tau_n)(p-1)}{\varepsilon}} dp + O \left( \varepsilon^2 \varepsilon^{-\frac{3}{2}} \right).
\]

One proves them using (78), (114) and the definitions of \( \rho \) and \( \rho_0 \). Similarly one obtains the announced representation for \( G_- \); first, one “replaces” in (113) \( \rho \) with \( \rho_0 \), and then, one “replaces” \( S(p, \tau) \) with \( S(1, \tau) - 2i(\tau - \tau_n)(p-1) \). We omit further details. \( \square \)

Using (115) and the definition of \( g_- \), we obtain

\[
G_- = C_0 e^{i\pi/4} \pi^{1/2} \int_{1-i\rho_0}^1 e^{-\frac{2i(\tau - \tau_n)(p-1)}{\varepsilon}} \sin(px) \frac{1}{\varepsilon^{3/2}} \int_1^p (L_0(q) - l_0(q)) dq dp,
\]

\[
G_-^0 = \int_1^{1-i\rho_0} e^{-\frac{2i(\tau - \tau_n)(p-1)}{\varepsilon}} \frac{1}{\varepsilon^{3/2}} \int_1^p (L_0(q) - l_0(q)) dq dp + O \left( \varepsilon^{4/3} z^{3/2} \right).
\]

Let us study \( G_-^0 \) as \( \varepsilon \to 0 \) and \( z \to \infty \). As \( x \) is bounded, in view of (73), we have

\[
G_-^0 = \sin x \int_{1-i\rho_0}^1 e^{-\frac{2i(\tau - \tau_n)(p-1)}{\varepsilon}} \frac{1}{\varepsilon^{3/2}} \int_1^p (L_0(q) - l_0(q)) dq dp + O \left( \varepsilon^{3/3} z^{3/2} \right).
\]

In view of (45) and (50), we get

\[
L_0(p) - l_0(p) = \sqrt{2\varepsilon} f((p-1)/\varepsilon) + O(\varepsilon^{3/2} + |p-1|^{3/2}), \quad f(s) = \zeta(s) + 2\sqrt{-s},
\]

where the branch of \( s \mapsto \sqrt{-s} \) is analytic in \( \mathbb{C} \) cut along \( \mathbb{R}_+ \) and is positive when \( s < 0 \). This observation implies that

\[
G_-^0 = \varepsilon \sqrt{2} \sin x \int_{1-i\rho_0}^1 e^{-\frac{2i(\tau - \tau_n)t}{\varepsilon}} \int_0^t f(s) ds dt + O \left( \varepsilon^{4/3} z^{3/2} \right).
\]

\[
+ \varepsilon^2 \int_{1-i\rho_0}^1 e^{-\frac{2i(\tau - \tau_n)t}{\varepsilon}} O(|t| + |t|^{5/2}) dt + O \left( \varepsilon^{4/3} z^{3/2} \right).
\]
Integrating by parts in the first integral and changing the variable \( t \mapsto (\tau - \tau_n) t \) in the second one, we easily get

\[
G_0^- = \frac{\varepsilon \sqrt{2} \sin x}{2i(\tau - \tau_n)} \left( e^{-2i\rho_0/\varepsilon} \int_0^{-i\rho_0/\varepsilon} f(s) \, ds + \int_0^0 e^{-2i(\tau - \tau_n)t} f(t) \, dt \right) + \mathcal{O} \left( \varepsilon^{4/3} / z^2 + \varepsilon^{5/6} / z^{7/2} \right).
\]

In view of (52), \( \int_0^{-i\rho_0/\varepsilon} f(s) \, ds \) is bounded, and, as \( 2 \omega(\tau - \tau_n) = z^{3/2} \), the first term in the brackets is \( \mathcal{O}(e^{-2z^{3/2}}) \). Estimate (52) implies that \( \int_{-i\infty}^{0} e^{-2i(\tau - \tau_n)t} g(t) \, dt = \mathcal{O}(\sqrt{\tau - \tau_n} z^{-3/2} e^{-2z^{3/2}}) = \mathcal{O}(e^{-2z^{3/2}}) \). Therefore,

\[
G_0^- = \frac{\varepsilon \sqrt{2} \sin x}{2i(\tau - \tau_n)} \int_{-i\infty}^{0} e^{-2i(\tau - \tau_n)t} f(t) \, dt + \mathcal{O} \left( \varepsilon^{\frac{2}{3}} e^{-2z^{3/2}} + \varepsilon^{\frac{4}{3}} z^{-2} + \varepsilon^{\frac{5}{6}} z^{-\frac{7}{2}} \right).
\]

Substituting this representation into (116), we get finally

\[
G_- = \frac{c_n e^{i\pi/4} \varepsilon \sin x}{\sqrt{2}\pi(\tau - \tau_n)} \int_{0}^{\infty} e^{-2i(\tau - \tau_n)t} \left( \zeta(-it) + 2e^{i\pi/4} \sqrt{t} \right) \, dt + \mathcal{O} \left( \varepsilon^{2/3} / z^{5/2} \right),
\]

where \( \sqrt{t} \geq 0 \). The term \( G_+ \) is analyzed similarly. It is described by (118) where \( \zeta(-it) \) is replaced with \( i\zeta(it) \). In view of (51), on the integration path \( \zeta(-it) = \zeta(it) \), and, as \( G = G_- + G_+ \), we come to (112). \( \Box \)

7.5. Proof of Theorem 2.4. The theorem follows from Proposition 7.1 and representations (105), (108) and (112).

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