SYMMETRY AND MONOTONICITY OF SOLUTIONS FOR THE FULLY NONLINEAR NONLOCAL EQUATION

MENG QU
School of mathematics and statistics
Anhui normal university, Wuhu, 241002, China

PING LI
School of Information and Mathematics
Yangtze University, Jingzhou 434023, China

LIU YANG
School of mathematics and statistics
Anhui normal university, Wuhu, 241002, China

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Abstract. In this paper, we consider equations involving the fully nonlinear fractional order operator with homogeneous Dirichlet condition:

\[
\begin{align*}
F_\alpha(u)(x) &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
u &> 0, \quad \text{in } \Omega; \\
u &\equiv 0, \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{align*}
\]

where \( \Omega \) is a domain (bounded or unbounded) in \( \mathbb{R}^n \) which is convex in \( x_1 \)-direction.

By using some ideas of maximum principle, we prove that the solution is strictly increasing in \( x_1 \)-direction in the left half of \( \Omega \). Symmetry of solution is also proved. Meanwhile we obtain a Liouville type theorem on the half space \( \mathbb{R}^n_+ \).

1. Introduction. This paper is mainly devoted to investigate the symmetry and monotonicity properties for the solution of nonlinear equation involving fully nonlinear non-local operator \( F_\alpha \), which is defined as

\[
F_\alpha(u)(x) = C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy,
\]

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where P.V. stands for the Cauchy principle value, \( C_{n, \alpha} > 0 \) and \( 0 < \alpha < 2 \). This kind of operator was introduced by Caffarelli and Silvestre in [3]. In order to make sense for (1) we require that \( G \) being at least local Lipschitz continuous and \( G(0) = 0 \), while \( u \) be a Schwartz function initially. One can extend this operator to more wider spaces of functions. In this paper, we consider \( u \in C^{1,1}_{\text{loc}} \cap L_\alpha(\mathbb{R}^n) \), where

\[
L_\alpha(\mathbb{R}^n) = \{ u \in L^1_{\text{loc}} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \}.
\]

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* Corresponding author.
We note that in this article, we always assume that
\[ G \in C^1(\mathbb{R}), \ G(0) = 0, \ \text{and} \ G'(t) \geq c_0 > 0 \ \forall t \in \mathbb{R}. \]  

When \( G \) is an identity map, \( F_\alpha \) becomes the fractional Laplacian \( (-\Delta)^{\frac{\alpha}{2}} \). Compared with the fruitful achievement about solutions involving \( \Delta \), the nonlocal nature of \( (-\Delta)^{\frac{\alpha}{2}} \) make the equations involving \( (-\Delta)^{\frac{\alpha}{2}} \) difficult to study. The first method was introduced by Caffarelli and Silvestre [2], which turns the nonlocal problem involving the fractional Laplacian into a local one in higher dimensions. This method, often called “extension method”, has been applied successfully to study equations involving fractional Laplacian (see [1, 8] and the references therein). One can also use the \textit{integral equations method}, like the \textit{method of moving planes in integral forms} and \textit{regularity lifting} to investigate equations which involves fractional Laplacian ([4, 6, 7, 20]). For more articles concerning the method of moving planes for nonlocal equations, mainly for integral equations, please see [10, 11, 12, 13, 14, 15, 16, 17, 18] and the references therein. Recently, Chen-Li-Li [5] systematically developed the maximum principles for anti-symmetric functions, like decay at infinity, narrow region principle and then established symmetry and monotonicity of positive solutions for nonlinear equation associated with \( F_\alpha \). More precisely, they proved that

**Theorem 1.1** ([5]).

(a) If \( u \in C^{1,1}_0(B) \cap L_\alpha \) is a positive solution of
\[
\begin{cases}
F_\alpha(u(x)) = f(u(x)), \ x \in B, \\
u(x) = 0, \ x \notin B,
\end{cases}
\]
with \( f(\cdot) \) being Lipschitz continuous. Then \( u \) must be radially symmetric and monotone decreasing about the origin.

(b) Suppose that \( u \in C^{1,1}_0 \cap L_\alpha \) is a positive of
\[ F_\alpha(u(x)) = g(u(x)), \ x \in \mathbb{R}^n. \]
If for some \( \gamma > 0, \ u(x) = o(|x|^{-\gamma}) \) as \( x \to \infty \) and \( g'(s) \leq s^q \) with \( q_0 \geq \alpha \). Then \( u \) must be radially symmetric about some point in \( \mathbb{R}^n \).

(c) Suppose that \( u \in C^{1,1}_0 \cap L_\alpha \) is a nonnegative solution of
\[
\begin{cases}
F_\alpha(u(x)) = h(u(x)), \ x \in \mathbb{R}^n_+, \\
u(x) \equiv 0, \ x \notin \mathbb{R}^n_+.
\end{cases}
\]
If \( h(s) \) is Lipschitz continuous in the range of \( u \), and \( h(0) = 0 \). Then \( u \equiv 0 \).

On the other hand, Cheng-Huang-Li [9] considered the zero-Dirichlet problem with more generalized nonlinear term,
\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}}u = f(x, u, \nabla u) \text{ in } \Omega, \\
u > 0, \text{ in } \Omega; \ u \equiv 0, \text{ in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]  

In (3), \( \Omega \) has been considered as a convex domain in \( x_1 \)-direction (bounded or unbounded) in \( \mathbb{R}^n \) (we say a convex domain \( \Omega \) is convex in \( x_1 \)-direction if and only if \( (x_1, x'), (x_2, x') \in \Omega \) imply that \( ((1 - t)x_1 + tx_2, x') \in \Omega \) for any \( t \in (0, 1) \)). Moreover, the nonlinear term \( f \) belongs to the function space \( F \), where \( F \) is defined as the collections of functions \( f(x, u, p) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) such that for any \( M > 0, \forall u_1, u_2 \in [-M, M] \) and \( \forall x, p \in \mathbb{R}^n \),
\[ |f(x, u_1, p) - f(x, u_2, p)| \leq C_M|u_1 - u_2| \text{ for some } C_M > 0. \]
Under some other adaptable condition on \( u \) and \( f \), Cheng-Huang-Li [9] obtained the solution \( u \) for (3) is monotone and symmetric when \( \Omega \) bounded or unbounded respectively.

In this paper we are interested in the following problem, which can be regarded as the problem (3) while \((-\Delta)^{\gamma} u\) be replaced by \( F_\alpha \). More precisely, we consider

\[
\begin{align*}
F_\alpha(u)(x) &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
\alpha > 0, & \text{ in } \Omega, \\
\alpha \equiv 0, & \text{ in } \Omega^c.
\end{align*}
\]  

Unlike the operator \((-\Delta)^{\gamma}\) is linear, while the operator \( F_\alpha \) is nonlinear, some technique in [9], for example the key lemma (Lemma 1 in page 4 [9]) can not be used here. The idea to overcome the difficulty of nonlinearity of \( F_\alpha \) was coming from [5] and [19]. Our results are listed in the following three theorems, where the domain \( \Omega \) be considered as the bounded domain, unbounded domain and the half space of \( \mathbb{R}^n \), respectively.

**Theorem 1.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) which is convex in \( x_1 \)-direction and symmetrical about \( x_1 = 0 \). We suppose that \( u \in C(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\Omega) \) solves (4) for \( \alpha \in (0, 2) \) and \( f(x, v, p) \in \mathcal{F} \) satisfies

\[
\begin{align*}
&\begin{cases}
    f(x_1, x', v, p_1, p_2, \ldots, p_n) \leq f(x_1, x', v, -p_1, p_2, \ldots, p_n), \\
    \forall -x_1, p_1 \geq 0, \ x_1 \leq x_1' \leq -x_1
    \end{cases}
\end{align*}
\]

Then \( u(x_1, x') \) is strictly increasing in the left half of \( \Omega \) in \( x_1 \)-direction and

\[
\begin{align*}
u(x_1, x') &\leq u(-x_1, x'), \ \forall x_1 < 0, \ (x_1, x') \in \Omega. \\
\text{Moreover, if } f(x_1, x', v, p_1, p_2, \ldots, p_n) &= f(-x_1, x', v, -p_1, p_2, \ldots, p_n), \text{ then }
\end{align*}
\]

\[
u(x_1, x') = u(-x_1, x').
\]

To state the second theorem in unbounded domain \( \Omega \), we must impose a growth condition on \( f(x, v, p) \):

\[
\begin{align*}
\frac{|f(x_1, u_1, p) - f(x_2, u_2, p)|}{|u_1 - u_2|} \leq C(|u_1|^s + |u_2|^s) \text{ as } u_1, u_2 \to 0, \text{ for some } s > 0 \quad (6)
\end{align*}
\]

**Theorem 1.3.** Let \( \Omega \) be an unbounded domain in \( \mathbb{R}^n \), which is convex in \( x_1 \)-direction and symmetric about \( x_1 = 0 \). Suppose that \( u \in C(\mathbb{R}^n) \cap C^{1,1}_{\text{loc}}(\Omega) \cap L_\alpha \) solves (4) for \( \alpha \in (0, 2) \) and \( f(x, v, p) \in \mathcal{F} \). If \( f(x, v, p) \) satisfies (5), (6) and \( u(x) \) has the following asymptotic:

\[
\begin{align*}
u(x) &= o(|x|^{-\alpha/s}) \text{ as } |x| \to +\infty, \ x_1 < 0, \quad (7)
\end{align*}
\]

then there exists \( \mu_0 \leq 0 \) such that \( u(x_1, x') \) is strictly increasing in \( \Omega \cap \{x_1 < \mu_0\} \) in \( x_1 \)-direction and

\[
u(x_1, x') \leq u(2\mu_0 - x_1, x'), \ (x_1, x') \in \Omega \cap \{x_1 < \mu_0\}, \quad (8)
\]

where

\[
\mu_0 = \sup_{\mu \leq 0} \{ \mu \mid u(x_1, x') \leq u(2\lambda - x_1, x'), \ \forall (x_1, x') \in \Omega \cap \{x_1 < \lambda\}, \ \forall \lambda \leq \mu \}.
\]

Furthermore, if \( \mu_0 < 0 \), then \( u(x_1, x') = u(2\mu_0 - x_1, x'), \ (x_1, x') \in \Omega \cap \{x_1 < \mu_0\} \).
Now we consider the following problem,
\[
\begin{cases}
F_\alpha(u(x)) = f(x, u, \nabla u), & x \in \mathbb{R}^n, \\
u \equiv 0, & x \in \mathbb{R}^n \setminus \mathbb{R}_+^n.
\end{cases}
\] (9)

We have the following theorem.

**Theorem 1.4.** We suppose that \( u \in C(\mathbb{R}_+^n) \cap C^{1,1}_{\text{loc}}(\mathbb{R}_+) \) is a nonnegative solution of (9) and \( f(x, v, p) \in F \) satisfies:
\[
\begin{cases}
f(x', x_n, v, p_1, \cdots, p_{n-1}, p_n) \leq f(x', x_n, v, p_1, \cdots, p_{n-1}, -p_n), \\
\forall x_n, p_n \geq 0, \ x_n \leq x_n.
\end{cases}
\]

Suppose that \( \lim_{|x| \to \infty} u(x) = 0 \), (10)
and
\( f(x, v, p) = 0, \) if \( v = 0 \). (11)

Then \( u \equiv 0 \).

This note is organized as follows. In Section 2 and 3, we will prove Theorem 1.2 and Theorem 1.3 respectively. While in section 4, we will give the proof of Theorem 1.4.

2. **Proof of Theorem 1.2.** Let \( T_\lambda \) be a hyperplane in \( \mathbb{R}^n \), without loss of generality, we assume that:
\[
T_\lambda = \{x = (x_1, x') \in \mathbb{R}^n | x_1 = \lambda, \lambda \in \mathbb{R}\},
\]
where \( x' = (x_2, x_3, \cdots, x_n) \). Let
\[
x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)
\]
be the reflection of \( x \) about \( T_\lambda \). Set
\[
\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 < \lambda\},
\]
\[
u_\lambda(x) = u(x^\lambda), \ \omega_\lambda(x) = u_\lambda(x) - u(x), \ \forall x \in \mathbb{R}^n.
\]

We carry out the proof of Theorem 1.2 in two steps. Firstly, we show that for \( \lambda > -1 \) and sufficiently close to \(-1\), we have
\[
\omega_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda.\] (12)

Next we move the plane \( T_\lambda \) along the \( x_1 \)-axis to the right in the left half of \( \Omega \) as long as inequality (12) holds. The plane will just stop at the limiting position \( \lambda = 0 \).

**Step 1.** Since \( \Omega \) is bounded and convex in \( x_1 \)-direction, without loss of generality, we may assume
\[
\Omega \subset \{|x_1| \leq 1\} \text{ and } \partial \Omega \cap \{x_1 = -1\} \neq \emptyset.
\]

We claim that there exists \( \delta > 0 \) small enough such that
\[
\omega_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda, \ \forall \lambda \in [-1, -1 + \delta).\] (13)

Suppose not, we set
\[
A = \inf_{\lambda \in (-1, -1 + \delta)} \omega_\lambda(x) < 0.
\]
Since \( 0 \leq u(x) \in C(\mathbb{R}^n) \) and \( u(x) \equiv 0, x \in \Omega^c \), \( A \) can be obtained for some \((\lambda_0, x_0) \in \{(\lambda, x) | (\lambda, x) \in [-1, -1 + \delta] \times \Sigma_\lambda \cap \Omega\}\) for any \( \delta \in (0, 1) \).
We have \( \omega_{\lambda_0} \geq 0 \) on \( \partial (\Omega \cap \Sigma_{\lambda_0}) \), then \( x_0 \in \Sigma_{\lambda_0} \cap \Omega \). By the fact that if \( \lambda = -1 \), then we have \( \Sigma_{-1} \cap \Omega = \partial \Sigma_{-1} \cap \Omega \), one gets \( \omega_{-1}(x) \geq 0 \) in \( \Sigma_{-1} \cap \Omega \). This implies \( \lambda_0 > -1 \).

Now we set \( u_i(x) = \frac{\partial \omega_{\lambda}}{\partial \lambda}(x) \) for \( i = 1, 2, \ldots, n \). Since \( (\lambda_0, x_0) \) is a minimizing point, we have when \( \lambda_0 \) in \( (0, \delta) \), \( \frac{\partial \omega_{\lambda}(x)}{\partial \lambda} \bigg|_{(\lambda_0, x_0)} = 0 \), while \( \frac{\partial \omega_{\lambda}(x)}{\partial \lambda} \bigg|_{(\delta, x_0)} \leq 0 \). Moreover, we have

1. \( u_{x_1}(x_0^{\lambda_0}) \leq 0 \). This is because that
   \[
   0 \geq \frac{\partial \omega_{\lambda}(x)}{\partial \lambda} \bigg|_{(\lambda_0, x_0)} = \frac{\partial}{\partial \lambda} (u(x) - u(\lambda)) \bigg|_{(\lambda_0, x_0)} = 2u_{x_1}(x_0)(\lambda_0, x_0),
   \]
   and
2. \( \nabla_x \omega(x_0^{\lambda_0}) = 0 \), where \( \nabla_x \) denotes the gradient with respective to \( x \).
   By (b), we have that
   \[
   \nabla_x u(x_0^{\lambda_0}) = \nabla_x u(x_0),
   \]
   which gives that
   \[
   u_{x_i}(x_0^{\lambda_0}) = u_{x_i}(x_0) \text{ for } i = 2, \ldots, n \quad \text{(14)}
   \]
   and
   \[
   u_{x_1}(x_0^{\lambda_0}) = -u_{x_1}(x_0). \quad \text{(15)}
   \]

Moreover with (a), we get
\[
\lambda_0(x_0) \geq 0. \quad \text{(16)}
\]

Then using the property (5) of \( f(x, u, p) \) and the condition (14)-(16), we have that
\[
\begin{align*}
F_{n}(u_{\lambda_0}(x_0)) - F_{n}(u(x_0)) &= f(x_0^{\lambda_0}, u_{\lambda_0}(x_0), \nabla_x u(x_0)) - f(x_0, u(x_0), \nabla_x u(x_0)) \\
&\geq f(x_0, u_{\lambda_0}(x_0), \nabla_x u(x_0)) - f(x_0, u(x_0), \nabla_x u(x_0)) \\
&= c(x_0) \omega_{\lambda_0}(x_0).
\end{align*}
\]

Here we set
\[
c(x_0) = f(x_0, u_{\lambda_0}(x_0), \nabla_x u(x_0)) - f(x_0, u(x_0), \nabla_x u(x_0)).
\]

Since \( u(x) \in C(\mathbb{R}^n) \) with compact support and \( f(x, u, p) \in F \), then \( c(x) \) is uniformly bounded. On the other hand, by the definition of \( F_{n} \),
\[
\begin{align*}
F_{n}(u_{\lambda_0}(x_0)) - F_{n}(u(x_0)) &= C_{n, \alpha}P.V. \int_{\mathbb{R}^n} \frac{G(u_{\lambda_0}(x_0) - u_{\lambda_0}(y)) - G(u(x_0) - u(y))}{|x_0 - y|^{n+\alpha}} dy \\
&= C_{n, \alpha}P.V. \int_{\Sigma_{\lambda_0}} \frac{G(u_{\lambda_0}(x_0) - u_{\lambda_0}(y)) - G(u(x_0) - u(y))}{|x_0 - y|^{n+\alpha}} dy \\
&\quad + C_{n, \alpha} \int_{\Sigma_{\lambda_0}} G(u_{\lambda_0}(x_0) - u_{\lambda_0}(y)) - G(u(x_0) - u(y)) |x_0 - y|^{n+\alpha} dy \\
&= C_{n, \alpha}P.V. \int_{\Sigma_{\lambda_0}} \frac{G(u_{\lambda_0}(x_0) - u_{\lambda_0}(y)) - G(u(x_0) - u(y))}{|x_0 - y|^{n+\alpha}} dy \\
&\quad + C_{n, \alpha} \int_{\Sigma_{\lambda_0}} G(u_{\lambda_0}(x_0) - u(y)) - G(u(x) - u_{\lambda_0}(y)) |x_0 - y|^{n+\alpha} dy.
\end{align*}
\]
Now we set \( A_{x_0, \lambda_0}(y) = G(u_{\lambda_0}(x_0) - u_{\lambda_0}(y)) - G(u(x_0) - u(y)) \) and \( B_{x_0, \lambda_0}(y) = G(u_{\lambda_0}(x_0) - u(y)) - G(u(x) - u_{\lambda_0}(y)) \), respectively, then
\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) = C_{n, \alpha} P.V. \int_{\Sigma_{\lambda_0}} \left[ \frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y^{\lambda_0}|^{n+\alpha}} \right] A_{x_0, \lambda_0}(y) dy \]
\[+ C_{n, \alpha} \int_{\Sigma_{\lambda_0}} A_{x_0, \lambda_0}(y) + B_{x_0, \lambda_0}(y) \frac{dy}{|x_0 - y^{\lambda_0}|^{n+\alpha}} = I_1 + I_2. \tag{18} \]

To estimate \( I_1 \), we have that
\[
\frac{1}{|x_0 - y|^{n+\alpha}} > \frac{1}{|x_0 - y^{\lambda_0}|^{n+\alpha}}, \quad \forall x, y \in \Sigma_{\lambda_0}.
\]

While for second part of \( I_1 \), we have for \( y \in \Sigma_{\lambda_0} \),
\[
u_{\lambda_0}(x_0) - u_{\lambda_0}(y) - u(x_0) - u(y) = \omega_{\lambda_0}(x_0) - \omega_{\lambda_0}(y) \leq 0, \quad \text{but} \neq 0,
\]
due to \( u > 0 \) in \( \Omega \) with \( u = 0 \) in \( \Omega^c \). Then with (2) , we have
\[
A_{x_0, \lambda_0}(y) = G(u_{\lambda_0}(x_0) - u_{\lambda_0}(y)) - G(u(x_0) - u(y)) = G'(-\cdot)(\omega_{\lambda_0}(x_0) - \omega_{\lambda_0}(y)) \leq 0, \quad \text{but} \neq 0
\]
for \( y \in \Sigma_{\lambda_0} \), and then
\[
I_1 < 0. \tag{19}\]

To estimate \( I_2 \), we use the fact
\[
A_{x_0, \lambda_0}(y) + B_{x_0, \lambda_0}(y) = G(u_{\lambda_0}(x_0) - u_{\lambda_0}(y)) - G(u(x_0) - u(y)) + G(u_{\lambda_0}(x_0) - u(y)) - G(u(x_0) - u(y))
\]
\[
= G'(-\cdot)\omega_{\lambda_0}(x_0) + G'(-\cdot)\omega_{\lambda_0}(x_0),
\]
with
\[
u_{\lambda_0}(x_0) - u_{\lambda_0}(y) < \xi(x) < u(x_0) - u_{\lambda_0}(y),
\]
and
\[
u_{\lambda_0}(x_0) - u(y) < \eta(x) < u(x_0) - u(y).
\]

Then using (2) and the fact \( \omega_{\lambda_0}(x_0) < 0 \), we have
\[
I_2 = C_{n, \alpha} \int_{\Sigma_{\lambda_0}} \frac{G'(-\cdot)\omega_{\lambda_0}(x_0) + G'(-\cdot)\omega_{\lambda_0}(x_0)}{|x_0 - y^{\lambda_0}|^{n+\alpha}} dy
\]
\[= C_{n, \alpha} \omega_{\lambda_0}(x_0) \int_{\Sigma_{\lambda_0}} \frac{G'(-\cdot) + G'(-\cdot)}{|x_0 - y^{\lambda_0}|^{n+\alpha}} dy \tag{20}\]
\[\leq 2c_0 \omega_{\lambda_0}(x_0) C_{n, \alpha} \int_{\Sigma_{\lambda_0}} \frac{1}{|x_0 - y^{\lambda_0}|^{n+\alpha}} dy.
\]

Combining (18)-(20), we deduce
\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) \leq I_2 \leq 2c_0 \omega_{\lambda_0}(x_0) C_{n, \alpha} \int_{\Sigma_{\lambda_0}} \frac{1}{|x_0 - y^{\lambda_0}|^{n+\alpha}} dy.
\]
Let \( \Sigma_{\lambda_0}^c = \mathbb{R}^n \setminus \Sigma_{\lambda_0} \), we choose a point in \( \Sigma_{\lambda_0}^c : x_0^* = (x_0', 3\delta + (x_0)_n) \), where we write \( x_0 = ((x_0)', (x_0)_n) \) and \((x_0)_n\) denotes for the last coordinate of \( x_0 \). It follows that \( B_{\delta}(x_0^*) \subset \Sigma_{\lambda_0}^c \) and for any \( y \in B_{\delta}(x_0^*) \), \( |y - x_0| \leq 4\delta \). Then

\[
\int_{\Sigma_{\lambda_0}^c} \frac{1}{|x_0 - y|^{n+\alpha}} dy = \int_{\Sigma_{\lambda_0}^c} \frac{1}{|x_0 - y|^{n+\alpha}} dy 
\geq \int_{B_{\delta}(x_0^*)} \frac{1}{|x_0 - y|^{n+\alpha}} dy 
\geq \int_{B_{\delta}(x_0^*)} \frac{1}{4^{n+\alpha}|\delta|^{n+\alpha}} dy = \frac{C}{\delta^\alpha},
\]

where \( C \) is a positive constant only relies on \( n \) and \( \alpha \).

Thus,

\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) \leq \frac{C\omega_{\lambda_0}(x_0)}{\delta^\alpha},
\]

then combining (17) with (21), we have

\[
0 \leq F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) - c(x_0)\omega_{\lambda_0}(x_0) \leq \frac{C}{\delta^\alpha} - c(x_0)\omega_{\lambda_0}(x_0) < 0,
\]

if we choose \( \delta \) small enough. This yields a contradiction and proves claim (13).

**Step 2.** Set

\[
\lambda_0 = \sup_{-1 \leq \lambda \leq 0} \{ \lambda \mid \omega_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda \}.
\]

Then we must have

\[
\lambda_0 = 0.
\]

Otherwise, suppose that \( \lambda_0 < 0 \), since \( \omega_{\lambda_0} \neq 0 \), we will have

\[
\omega_{\lambda_0} > 0, \; x \in \Sigma_{\lambda_0} \cap \Omega.
\]

If not, there exists \( x_0 \in \Omega \cap \Sigma_{\lambda_0} \) such that \( \omega_{\lambda_0}(x_0) = 0 \). Using (17), we have

\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) - c(x_0)\omega_{\lambda_0}(x_0) \geq 0.
\]

On the other hand, the same as (18), we write

\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) = I_1 + I_2.
\]

The fact \( \omega_{\lambda_0}(x_0) = 0 \) gives \( I_2 = 0 \). Then

\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) - c(x_0)\omega_{\lambda_0}(x_0)
\leq C_{n, \alpha} P.V. \int_{\Sigma_{\lambda_0}} \left[ \frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y_0|^{n+\alpha}} \right] A_{x_0, \lambda_0}(y) dy
\leq C_{n, \alpha} P.V. \int_{\Sigma_{\lambda_0}} \left[ \frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y_0|^{n+\alpha}} \right] G'(\cdot)(-\omega_{\lambda_0}(y)) dy
\leq 0,
\]

since \( \omega_{\lambda_0}(y) \geq 0 \) and \( \omega_{\lambda_0}(y) \neq 0 \), which yields a contradiction and (23) holds.

We **claim** that there exists \( \epsilon > 0 \) small enough such that

\[
\omega_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda \cap \Omega, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon).
\]

(24)

Now we prove (24) is true. Since we have (23) and \( \omega_\lambda(x) \) is lower semi-continuous in \( \Omega \). Thus \( \forall \delta > 0 \),

\[
\omega_{\lambda_0}(x) \geq C_\delta > 0, \; \forall x \in \Sigma_{\lambda_0 - \delta} \cap \Omega.
\]
By the continuity of $\omega_\lambda$ with respect to $\lambda$, there exist $\epsilon > 0$, such that
\[
\omega_\lambda(x) \geq 0, \forall x \in \Sigma_{\lambda_0 - \delta} \cap \Omega, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon). \tag{25}
\]
Suppose (24) is not true, then we have
\[
\forall \epsilon > 0, A_\epsilon = \inf_{x \in \Sigma_\lambda} \omega_\lambda(x) < 0.
\]

Since $\Omega$ is a bounded domain, then by (25), $A_\epsilon$ can be obtained for some point
\[
(\mu, x_0) \in \{(\mu, x) | (\mu, x) \in [\lambda_0, \lambda_0 + \epsilon] \times (\Sigma_{\lambda_0 - \delta} \cap \Omega)\}, \text{ i.e.,}
\]
\[
\omega_\mu(x_0) = A_\epsilon < 0.
\]
Obviously we have $\omega_\mu \geq 0$ on $\partial(\Sigma_{\mu} \setminus \Sigma_{\lambda_0 - \delta} \cap \Omega)$, thus $x_0 \in \Sigma_{\mu} \setminus \Sigma_{\lambda_0 - \delta} \cap \Omega$, then going through the similar proof in Step 1 (17)-(22), we get
\[
0 \leq F_\alpha(u_\mu(x_0)) - F_\alpha(u(x_0)) - c(x_0) \omega_\mu(x_0) \leq (\frac{C}{(\delta + \epsilon)^\alpha} - c(x_0)) \omega_\mu(x_0) < 0,
\]
if we choose $\delta, \epsilon$ small enough. This yields a contradiction and proves the claim (24). This contradicts the definition of $\lambda_0$. Therefore, we must have $\lambda_0 = 0$. It follows that
\[
\omega_0(x) \geq 0, \ x \in \Sigma_0,
\]
which implies
\[
u(x_1, x') \leq u(-x_1, x'), \ \forall x_1 \leq 0, \ (x_1, x') \in \Omega.
\]

3. **Proof of Theorem 1.3.** We carry out the proof in two steps, first, to begin with, we show that for $\lambda$ sufficiently negative, we have
\[
\omega_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda. \tag{26}
\]

Then we move the plane $T_\lambda$ along the $x_1$-axis to the right in the left half of $\Omega$ as long as inequality (26) holds. The plane will eventually stop at some limiting position at $\lambda = \lambda_0 < 0$ or $\lambda = 0$.

**Step 1.** Start moving the plane $T_\lambda$ along the $x_1$-axis from near $-\infty$ to the right. **We claim:** there exists $R_0 > 0$ large enough such that
\[
\omega_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda, \forall \lambda \leq -R_0. \tag{27}
\]
Suppose not, so there exist $\lambda_k \to -\infty$ such that
\[
A = \inf_{x \in \Sigma_{\lambda_k}} \omega_\lambda(x) < 0.
\]
Since $u(x)$ is decay at infinity, so we have $\omega_\lambda(x) = u_\lambda(x) - u(x) \geq -u(x) \geq \frac{A}{2}$ if $|x| \geq R_k$ for some $R_k$ large enough. Hence we can get
\[
\omega_{\mu_k}(x^k) = A \text{ for some } \mu_k \in [-|x^k|, \lambda_k], |x^k| \leq R_k.
\]
Now on the one hand, the same as (18), we write $F_\alpha(u_{\mu_k}(x^k)) - F_\alpha(u(x^k)) = I_1 + I_2$. The fact we have $I_1 \leq 0$, then going through the similar proof in (17)-(22), we get
\[
F_\alpha(u_{\mu_k}(x^k)) - F_\alpha(u(x^k)) \leq I_2 \leq 2c_0 C_n \alpha \int_{\Sigma_{\mu_k}} \frac{\omega_{\mu_k}(x^k)}{|x^k - y^\mu_k|^{n+\alpha}} dy \leq \frac{C \omega_{\mu_k}(x^k)}{|x^k|^{\alpha}}. \tag{28}
\]
To get the last inequality, we notice the fact that \( 0 < u_{\mu_k}(x^k) < u(x^k) \). For \( x^k \in \Sigma_{\mu_k} \) and \( |x^k| \) sufficiently large, let \( \Sigma_{\mu_k}^c = \mathbb{R}^n \setminus \Sigma_{\mu_k} \), choose a point in \( \Sigma_{\mu_k}^c \) : 
\[
x^m = (3|x^k| + x^k_1, (x^k_2'))
\]
then \( B_{|x^k|}(x^m) \subset \Sigma_{\mu_k}^c \). there exists a \( C > 0 \) such that 
\[
\int_{\Sigma_{\mu_k}} \frac{1}{|x^k - y|^n + \alpha} \, dy \geq \int_{\Sigma_{\mu_k}} \frac{1}{|x^k - y|^n + \alpha} dy
\]
\[
\geq \int_{B_{|x^k|}(x^m)} \frac{1}{|x^k - y|^n + \alpha} dy \sim \frac{C}{|x^k|^\alpha}.
\]
On the other hand, from (17), we have 
\[
c(x^k)\omega_{\mu_k}(x^k) \leq F_\alpha(u_{\mu_k}(x^k)) - F_\alpha(u(x^k)),
\]
where 
\[
c(x^k) = \frac{f(x^k, u_{\mu_k}(x^k), \nabla u(x^k)) - f(x_0, u(x^k), \nabla u(x^k))}{u_{\mu_k}(x^k) - u(x^k)}
\]
\[
\leq C(|u_{\mu_k}(x^k)|^s + |u(x^k)|^s) \leq C|u(x^k)|^s.
\]
Then from (7) and (28)-(29), we can get 
\[
C|u(x^k)|^s \omega_{\mu_k}(x^k) \leq F_\alpha(u_{\mu_k}(x^k)) - F_\alpha(u(x^k)) \leq \frac{C\omega_{\mu_k}(x^k)}{|x^k|^\alpha},
\]
that is 
\[
o(1) = C|x^k|^\alpha u^r(x^k) \geq \tilde{C}_{n, \alpha},
\]
since \( |x^k| \geq |\lambda_k| \to \infty \). This yields a contradiction and ends the proof of Step 1.

**Step 2**. Set 
\[
\lambda_0 = \sup_{\lambda \leq 0} \{ \lambda \mid \omega_{\mu}(x) \geq 0, \forall x \in \Sigma_{\mu}, \forall \mu \leq \lambda \}.
\]

By the definition of \( \lambda_0 \) and the continuity of \( u(x) \), we have \( \omega_{\lambda_0} \geq 0 \) for all \( x \in \Sigma_{\lambda_0} \). And then we must have 
\[
\lambda_0 = 0,
\]
or 
\[
\lambda_0 < 0 \text{ and } \omega_{\lambda_0}(x) \equiv 0.
\]
The second case will happen only when \( \sup_{x \in \Omega} |x_1| = +\infty \). Now suppose \( \lambda_0 < 0 \) and \( \omega_{\lambda_0} \neq 0 \), for \( x \in \Sigma_{\lambda_0} \). We claim that there exists \( \delta > 0 \) small enough such that 
\[
\omega_{\lambda}(x) \geq 0, \forall x \in \Sigma_{\lambda}, \forall \lambda \in [\lambda_0, \lambda_0 + \delta].
\]
For \( x \in \Sigma_{\lambda_0} \cap \Omega^c \), it is easy to see that \( \omega_{\lambda_0}(x) \geq 0 \). Then with the same proof as (23), we will have 
\[
\omega_{\lambda_0} > 0, x \in \Sigma_{\lambda_0} \cap B_{R_0}(0) \cap \Omega.
\]
It follow that for any positive number \( \sigma \), 
\[
\omega_{\lambda_0}(x) \geq C_\delta > 0, \forall x \in \Sigma_{\lambda_0 - \sigma} \cap B_{R_0}(0) \cap \Omega,
\]
where \( R_0 \) is defined in Step 1. Since \( \omega_{\lambda} \) depends on \( \lambda \) continuously, for all sufficiently small \( \delta > 0 \), we have 
\[
\omega_{\lambda}(x) \geq 0, \forall x \in \Sigma_{\lambda_0 - \sigma} \cap B_{R_0}(0) \cap \Omega, \forall \lambda \in [\lambda_0, \lambda_0 + \delta].
\]
Suppose (30) is false, then we have for any \( \delta > 0 \), 
\[
A = \inf_{x \in \Sigma_{\lambda_0}} \omega_{\lambda}(x) < 0.
\]
From Step 1, we know that minimizing point can not hold on \( B_{R_0}^c \), then using (32), \( A \) can be obtained for some point \( (\mu_k, x_0) \in \{(\lambda, x)| (\lambda, x) \in [\lambda_0, \lambda_0 + \delta] \times \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta} \cap B_{R_0}(0) \} \). Obviously we have \( \omega_{\lambda_0} \geq \frac{A}{2} \) on \( \partial(\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta} \cap B_{R_0}) \), thus \( x_0 \in \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta} \cap B_{R_0}(0) \). Then going through the similar proof in Theorem 2.1 (17)-(22), one can see that

\[
0 \leq F_\alpha(u_{\mu_0}(x^0)) - F_\alpha(u(x^0)) - c(x^0)\omega_{\mu_0}(x^0) \leq \left( \frac{C}{(\delta + \sigma)^2} - c(x^0) \right)\omega_{\mu_0}(x^0) < 0,
\]

if we choose \( \delta, \sigma \) small enough. This yields a contradiction, and hence (30) holds, which contradicts the definition of \( \lambda_0 \). The strict monotonicity follows from \( \omega_\lambda(x) > 0 \) in \( \Sigma_\lambda \cap \Omega \) for \( \lambda < \lambda_0 \).

4. Proof of Theorem 1.4. First, we claim that

\[
u(x) > 0, \ x \in \mathbb{R}^n_+ \text{ or } u(x) \equiv 0, \ x \in \mathbb{R}^n_+.
\]

To proof (33), we assume that there exist \( x_0 \in \mathbb{R}^n_+ \) such that \( u(x_0) = 0 \), then by (11), we have that

\[
F_\alpha(u(x_0)) = f(x_0, u(x_0), \nabla u(x_0)) = 0.
\]

On the other hand, using the condition (2), we have

\[
F_\alpha(u(x_0)) = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{G(u(x_0) - u(y))}{|x - y|^{n+\alpha}} dy.
\]

\[
= -C_{n, \alpha} G'(\cdot) P.V. \int_{\mathbb{R}^n_+} \frac{u(y)}{|x - y|^{n+\alpha}} dy < 0,
\]

where the last inequality is obtained by \( u(y) \geq 0 \) and \( u(y) \neq 0 \) with \( u \in C(\mathbb{R}^n_+) \). This yields a contradiction, and so (33) holds. Now we assume that

\[
u(x) > 0, \ x \in \mathbb{R}^n_+.
\]

We carry on the method of moving planes on the solution \( u \) along \( x_n \)-direction. Let \( T_\lambda \) be a hyperplane in \( \mathbb{R}^n \) with

\[
T_\lambda = \{ x = (x', x_n) \in \mathbb{R}^n | x_n = \lambda, \lambda \in \mathbb{R} \}, \ \lambda > 0,
\]

where \( x' = (x_1, x_2, \ldots, x_{n-1}) \). Let

\[
x_\lambda = (x_1, \ldots, x_{n-1}, 2\lambda - x_n)
\]

be the reflection of \( x \) about the plane \( T_\lambda \). Set

\[
\Sigma_\lambda = \{ x \in \mathbb{R}^n : x_n < \lambda \},
\]

and

\[
u_\lambda(x) = u(x_\lambda), \ \omega_\lambda(x) = u_\lambda(x) - u(x), \ \forall x \in \mathbb{R}^n.
\]

Step 1. We prove the claim that there exists \( \delta > 0 \) small enough such that

\[
\omega_\lambda(x) \geq 0 \ \forall x \in \Sigma_\lambda, \ \forall \lambda \in [0, \delta].
\]

Suppose the claim is not true, then we have for any \( \delta > 0 \) (we can restrict \( \delta \in (0, 1) \) for simplicity),

\[
A = \inf_{\lambda \in [0, \delta]} \omega_\lambda(x) < 0.
\]
First we know that $\omega_\lambda(x) = u_\lambda(x) - u(x) \geq -u(x)$ uniformly with respect to $\lambda$, so by the condition (10), we have
\[
\lim_{|x| \to \infty} \omega_\lambda(x) \geq \lim_{|x| \to \infty} -u(x) = 0,
\]
uniformly with respect to $\lambda$. Then there must exist $R_0 > 0$ large enough such that $\omega_\lambda(x) \geq \frac{4}{2}, x \in B_{R_0}(0)$ uniformly with respect to $\lambda$. So we have $A$ in (36) can be obtained for some point $(\lambda_0, x_0) \in \{(\lambda, x)|\lambda, x \in [0, \delta] \times \Sigma_\lambda \setminus \Sigma_0 \cap B_{R_0}(0)\}$.

Noticing that $u \equiv 0$ in $\mathbb{R}^n$ and $u \geq 0$ in $\mathbb{R}^n$ and $\omega_{\lambda_0}(x) \geq \frac{A}{\lambda}, x \in B_{R_0}(0)$, we have $\omega_{\lambda_0} \geq \frac{4}{2}$ on $\partial(\Sigma_{\lambda_0} \setminus \Sigma_0 \cap B_{R_0}(0))$, so $x_0$ must in $B_{R_0}(0) \cap \Sigma_{\lambda_0} \setminus \Sigma_0$. On the other hand it is easy to see that $\lambda_0 \neq 0$ and $\lambda_0 \in (0, \delta]$. Since $(\lambda_0, x_0)$ is a minimizing point, we can get:

(a) $u_x(\lambda_0) \leq 0$;
(b) $\nabla_x \omega(\lambda_0) = 0$.

Then going through the similar proof in Theorem 2.1 Step 1 (17)-(21), we have that
\[
0 \leq F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) - c(x_0)\omega_{\lambda_0}(x_0) \leq (\frac{C}{\beta^\alpha} - c(x_0))\omega_{\lambda_0}(x_0) < 0,
\]
if we choose $\delta$ small enough. This yields a contradiction and proves claim (35).

**Step 2.** Set
\[
\lambda_0 = \sup\{\lambda > 0|\omega_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda\}.
\]

We claim that
\[
\lambda_0 = \infty.
\]
Otherwise, if $\lambda_0 < \infty$, then by the definition of $\lambda_0$, we have
\[
\omega_{\lambda_0} > 0, x \in \Sigma_{\lambda_0} \text{ or } \omega_{\lambda_0} = 0, x \in \Sigma_{\lambda_0}.
\]

To prove (39), we assume there exists $x_0 \in \Sigma_{\lambda_0}$, such that $\omega_{\lambda_0}(x_0) = 0$. Then use (17) we have
\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) - c(x_0)\omega_{\lambda_0}(x_0)
\geq f(x_0, u_{\lambda_0}(x_0), \nabla_x u(x_0)) - f(x_0, u(x_0), \nabla_x u(x_0)) = 0.
\]

On the other hand, using (18) and $\omega_{\lambda_0} \geq 0$. We write $F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) = I_1 + I_2$. The fact $\omega_{\lambda_0}(x_0) = 0$ gives $I_2 = 0$. we have
\[
F_\alpha(u_{\lambda_0}(x_0)) - F_\alpha(u(x_0)) - c(x_0)\omega_{\lambda_0}(x_0),
\]
\[
= C_{n,\alpha}P.V. \int_{\Sigma_{\lambda_0}} \left[ \frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y_{\lambda_0}|^{n+\alpha}} \right] A_{x_0,\lambda_0}(y)dy
\]
\[
= C_{n,\alpha}G'(-)P.V. \int_{\Sigma_{\lambda_0}} \left[ \frac{1}{|x_0 - y|^{n+\alpha}} - \frac{1}{|x_0 - y_{\lambda_0}|^{n+\alpha}} \right] [-\omega_{\lambda_0}(y}]dy
< 0.
\]
Since $\omega_{\lambda_0} \neq 0$, it will yield a contradiction. So (39) holds.

Now we suppose
\[
\omega_{\lambda_0}(x) > 0, x \in \Sigma_{\lambda_0}.
\]
Then we claim that there exists $\epsilon > 0$ small enough such that
\[
\omega_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon].
\]
Now we prove (41) is true. Since we have (40) and \( \omega_\lambda(x) \) is lower semi-continuous in \( \Omega \). Thus \( \forall \delta > 0 \forall R > 0 \),
\[
\omega_{\lambda_0}(x) \geq C_\delta > 0, \ \forall x \in \Omega_{\lambda_0} \cap B_R(0).
\]
By the continuity of \( \omega_\lambda \) with respect to \( \lambda \), there exist \( \epsilon > 0 \), such that
\[
\omega_\lambda(x) \geq 0, \ \forall x \in \Omega_{\lambda_0} \cap B_R(0), \ \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon).
\]  
(42)
Suppose (41) is not true, then we have \( \forall \epsilon > 0 \),
\[
A = \inf \lambda \epsilon \in \Omega_{\lambda_0}, \lambda_0 + \epsilon \omega_\lambda(x) < 0.
\]
First we know that \( \omega_\lambda(x) = u_\lambda(x) - u(x) \geq -u(x) \), so by the condition (10), we have
\[
\lim_{|x| \to \infty} \omega_\lambda(x) \geq \lim_{|x| \to \infty} -u(x) = 0.
\]
So there must exist \( R_0 > 0 \) large enough such that \( \lim_{|x| \to \infty} \omega_\lambda(x) \geq 0, x \in B_{R_0}(0) \cap \Omega_\lambda \). Then using (42), \( A \) can be obtained for some point \( (\mu_k, x_0) \in \{(\lambda, x)| (\lambda, x) \in [\lambda_0, \lambda_0 + \epsilon) \times \Omega_{\lambda_0} \cap B_{R_0}(0)\} \).
Then going through the similar proof in Theorem 2.1 (17)-(22), one can see that
\[
0 \leq (-\Delta)^\frac{2}{\alpha} \omega_{\mu_k}(x_0) - c(x_0)\omega_{\mu_k}(x_0) \leq \frac{C}{(\delta + \epsilon)^\alpha} - c(x_0)\omega_{\mu_k}(x_0) < 0,
\]
if we choose \( \delta, \epsilon \) small enough. This yields a contradiction and proves claim (41).
However this contradicts the definition of \( \lambda_0 \). So (40) is not true. Therefore, we must have
\[
\omega_{\lambda_0} \equiv 0, x \in \Omega_{\lambda_0}, \ \text{i.e.}
\]
\[
u_{\lambda_0}(x) = u(x), x \in \Omega_{\lambda_0},
\]
If we choose the point \( \bar{x} = (x_1, x_2, \ldots, x_n, 0) \) in the hyperplane \( \{x_n = 0\} \), then \( \bar{x}^\lambda \in \mathbb{R}^n_+ \) and
\[
u(\bar{x}^\lambda) = u(x_1, x_2, \ldots, x_n - 1, 0) = u(x_1, x_2, \ldots, x_n - 1, 2\lambda_0) = u(x_1, x_2, \ldots, x_n, 0) = 0.
\]
This contradict the assume (34).
Therefore we proved the claim (38): \( \lambda_0 = \infty \) and consequently the solution \( u(x) \) is monotone increasing with respect to \( x_n \). The condition (10): \( \lim_{|x| \to \infty} u(x) = 0 \) gives us the claim (38) is not true and then \( u(x) \equiv 0, x \in \mathbb{R}^n_+ \).

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