ON A TRANSVERSAL THEOREM OF MONTEJANO AND KARASEV

ANDREAS F. HOLMSEN

Abstract. We give a new proof of a theorem of Montejano and Karasev [5] regarding $k$-dimensional transversals to small families of convex sets. While the result in [5] uses technical algebraic and topological tools, our proof is a simple application of the Borsuk–Ulam theorem. Additionally, in certain cases we obtain stronger results than those in [5].

1. Introduction

Let $F_1, \ldots, F_n$ be finite families of convex sets in $\mathbb{R}^d$. We say that the families $F_1, \ldots, F_n$ satisfy the colorful intersection property if $C_1 \cap \cdots \cap C_n \neq \emptyset$ for any choice of $C_1 \in F_1, \ldots, C_n \in F_n$.

Given $n = d + 1$ finite families of convex sets in $\mathbb{R}^d$ which satisfy the colorful intersection property, the colorful Helly theorem [1] asserts that there is a point that intersects every member of one of the families $F_i$. (Even more can be said; If $n = d + k$ there is a point that intersects every member of $k$ of the families.)

On the other hand, what can be said when $n \leq d$? In this case, the conclusion of the colorful Helly theorem clearly fails; Simply take each $F_i$ to be a family of hyperplanes in general position. Nevertheless, the colorful intersection property imposes non-trivial geometric constraints on the families. For instance, if $n = d$ it is easy to see that for any direction there exists a family $F_i$ whose members can be intersected by a line in that given direction. (Just project to a hyperplane orthogonal to the given direction and apply the colorful Helly theorem.)

These observations have led to a number of interesting results and conjectures concerning families satisfying the colorful intersection property [3, 4, 5]. Here we focus on the remarkable results by Montejano and Karasev [5], which state that under certain conditions on $n$, $|F_i|$ and $d$, the colorful intersection property implies that that one of the families $F_i$ admits a $k$-transversal, that is, a $k$-dimensional affine flat that intersects every member of $F_i$. In its simplest non-trivial form, their theorem can be formulated as follows (See also [4, Theorem 3.2]):

Consider 3 red convex sets and 3 blue convex sets in $\mathbb{R}^3$, and suppose every red set intersects every blue set. Then all the red sets can be intersected by a line or all the blue sets can be intersected by a line.

The general theorems of Montejano and Karasev rely on a number of technical tools (multiplication formulas for Schubert cocycles, Steifel–Whitney characteristic classes, Lusternik–Schnirelmann category of the Grassmannian), and simpler
proofs are of considerable interest. In this respect, Strausz [6] recently gave an elementary proof of the statement above based on the non-planarity of the complete bipartite graph $K_{3,3}$.

The purpose of this note is to give an elementary proof of the Montejano–Karasev theorem. (Our proof is based on the Borsuk–Ulam theorem, but it seems unlikely that this type of results can be proven without using any topology.) In particular, we establish the following.

**Theorem 1.** Let $k_1, \ldots, k_n$ be non-negative integers and set $m = k_1 + \cdots + k_n$. For each $1 \leq i \leq n$, let $F_i$ be a family of $k_i + 2$ convex sets in $\mathbb{R}^{n+m-1}$, and suppose the families $F_1, \ldots, F_n$ satisfy the colorful intersection property. Then one of the families $F_i$ admits a $k_i$-transversal.

**Remark.** Let us point out some particular cases of Theorem 1.

(a) The case $n = 1$ is tautological, as this corresponds to a single family $F_1$ of $k+2$ convex sets in $\mathbb{R}^k$.

(b) The case $n = 2$ with $k_1 = d - 1$ and $k_2 = 0$ follows from the separation theorem for convex sets in $\mathbb{R}^d$. If the two members of $F_2$ are disjoint (i.e. they do not have a 0-transversal), then there is a hyperplane that strictly separates them. By the colorful intersection property every member of $F_1$ must cross this hyperplane.

(c) The case $n \geq 2$ and $k_1 = \cdots = k_n = 0$ corresponds to $n$ families, each consisting of two convex sets in $\mathbb{R}^{n-1}$. The colorful Helly theorem implies that the members of one of the $F_i$ must intersect.

(d) In the case $n = 2$ we have $k_1 + 2$ red convex sets and $k_2 + 2$ blue convex sets in $\mathbb{R}^{k_1+k_2+1}$ such that every red set intersects every blue set. The conclusion is that the red sets admit a $k_1$-transversal or the blue sets admit a $k_2$-transversal. This is a special case of a result of Montejano and Karasev [5, Corollary 7 with $k = 1$]. (See also [4, Theorem 3.2].)

(e) For $n > 2$ Theorem 1 strengthens a result of Montejano and Karasev [5, Theorem 8]. Their result states that if $F_1, \ldots, F_n$ are families of $k+2$ convex sets in $\mathbb{R}^{n+k}$ which satisfy the colorful intersection property, then one of the families has a $k$-transversal. (See also Theorems 3.3 and 3.4 in [4].) Our Theorem 1 shows that dimension of the ambient space can be increased to $n(k+1) - 1$ without affecting the conclusion.

2. Optimality of dimension in Theorem 1

Before getting to the proof of Theorem 1 we give an example which shows that the dimension $n + m - 1$ can not be increased to $n + m$. Let $X$ be a set of $2n + m$ points in $\mathbb{R}^{n+m}$ and consider a partition $X = X_1 \cup \cdots \cup X_n$ where $|X_i| = k_i + 2$. We assume that $X$ is in general position in the sense that any subset of size $n + m$ spans a unique affine hyperplane and the affine spans of the $X_i$ form a generic collection of flats. Let $\pi_i$ denote the orthogonal projection map from $\mathbb{R}^{n+m}$ to the affine span of $X_i$, and let

$$F_i = \{ \pi_i^{-1}(x) : x \in X_i \}.$$
Since each member of $F_i$ is a flat of dimension $n + m - (k_i + 1)$, if we pick one member from each $F_i$, then they will generically intersect in a unique point. In other words, the families $F_1, \ldots, F_n$ satisfy the colorful intersection property. Moreover, $F_i$ does not admit a $k_i$-transversal. To see this, suppose $\gamma$ is a $k_i$-dimensional flat that intersects every member of $F_i$. Then $X_i$ must be contained in $\pi_i(\gamma)$, contradicting the affine independence of $X_i$.

3. Two geometric lemmas

The proof of Theorem 1 requires two simple geometric lemmas. They are both quite standard (and most likely known in some form or another), but for completeness we include proofs.

The first lemma can be thought of as an extension of Radon’s lemma. It was previously observed by Goodman and Pollack [2], and also plays a role in the argument by Strausz [6].

**Lemma 2.** Let $F = \{C_1, \ldots, C_{k+2}\}$ be a family of convex sets in $\mathbb{R}^d$. Then $F$ has a $k$-dimensional transversal if and only if there exists a partition $[k+2] = A \cup B$ such that

$$\text{conv}(\bigcup_{i \in A} C_i) \cap \text{conv}(\bigcup_{j \in B} C_j) \neq \emptyset$$

**Proof.** Suppose there is a $k$-dimensional transversal $\gamma$ to $F$. For each member $C_i \in F$ choose a point $p_i \in \gamma \cap C_i$. Applying Radon’s lemma to the set $\{p_1, \ldots, p_{k+2}\}$ contained in the $k$-dimensional affine subspace $\gamma$ gives us the desired partition.

For the opposite direction suppose there is a partition $[k+2] = A \cup B$ and a point $p \in \text{conv}(\bigcup_{i \in A} C_i) \cap \text{conv}(\bigcup_{j \in B} C_j)$. For each $i \in A$ we can choose a point $p_i \in C_i$ such that $p \in \text{conv}(\{p_i\}_{i \in A})$. Similarly, we can choose points $p_j \in C_j$ such that $p \in \text{conv}(\{p_j\}_{j \in B})$. Clearly the affine span of $\{p_i\}_{i \in A \cup B}$ intersects every member of $F$ and has dimension at most $|A| + |B| - 2 = k$. \hfill \Box

The second lemma is a simple application of the separation theorem for convex sets.

**Lemma 3.** Let $v_1, \ldots, v_n \in \mathbb{R}^d$ and $(b_1, \ldots, b_n) \in \mathbb{R}^n$. Suppose there exists vectors $u, w \in \mathbb{R}^d$ such that

$$w \cdot v_i < b_i < u \cdot v_i$$

for every $1 \leq i \leq n$. Then there exists a vector $v \in \mathbb{R}^d$ such that $v \cdot v_i > 0$ for every $1 \leq i \leq n$.

**Proof.** If the conclusion of the Lemma does not hold then the origin is contained in the convex hull of the $v_i$, so there exists a linear dependency $\sum_{i \in I} t_i v_i = 0$ with $t_i > 0$ for some $\emptyset \neq I \subset [n]$. This is a contradiction since

$$0 = w \cdot (\sum_{i \in I} t_i v_i) = \sum_{i \in I} t_i (w \cdot v_i) < \sum_{i \in I} t_i b_i < \sum_{i \in I} t_i (u \cdot v_i) = u \cdot (\sum_{i \in I} t_i v_i) = 0.$$

\hfill \Box
4. Proof of Theorem 1

Set \( d = n + m - 1 \), and suppose that none of the subfamilies \( F_i \) admits a \( k_i \)-transversal. Define \( K_i \) to be the abstract simplicial complex whose vertex set \( V(K_i) \) consists of the nonempty proper subfamilies of \( F_i \) and whose faces are the chains ordered by inclusion. In other words,

\[
K_i := \{ \sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_r: \emptyset \neq \sigma_j \subseteq F_i \}.
\]

Note that \( K_i \) is isomorphic to the barycentric subdivision of the \( k_i \)-skeleton of the \((k_i + 1)\)-dimensional simplex, and is therefore homeomorphic to \( S^{k_i} \). Furthermore, we observe that \( \upsilon_i(\sigma) := F_i \setminus \sigma \) defines a free simplicial involution on \( K_i \) since taking complements reverses inclusions

\[
\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_r \overset{\upsilon_i}{\longrightarrow} \upsilon_i(\sigma_1) \subset \cdots \subset \upsilon_i(\sigma_r) \subseteq \upsilon_i(\sigma_2) \subseteq \upsilon_i(\sigma_1).
\]

If we group the vertices of \( K_i \) into complementary pairs \( \{ \sigma, \upsilon_i(\sigma) \} \), then Lemma 2 implies that for each complementary pair there exists an affine hyperplane \( h_{(\sigma, \upsilon_i(\sigma))} \) in \( \mathbb{R}^d \) which strictly separates the members of \( \sigma \) from the members of \( \upsilon_i(\sigma) \). Let \( H_\sigma \) and \( H_{\upsilon_i(\sigma)} \) denote the opposite open halfspaces bounded by \( h_{(\sigma, \upsilon_i(\sigma))} \) which contain the members of \( \sigma \) and \( \upsilon_i(\sigma) \), respectively, and let \( f(\sigma) \) and \( f(\upsilon_i(\sigma)) \) be their outward unit normal vectors. In this way, we obtain a mapping

\[
f_i : V(K_i) \to S^{d-1},
\]

which satisfies \( f_i(\sigma) = -f_i(\upsilon_i(\sigma)) \).

Let \( K \) be the join \( K = K_1 * K_2 * \cdots * K_n \) which comes equipped with the free simplicial involution \( \upsilon = \upsilon_1 * \upsilon_2 * \cdots * \upsilon_n \). Note that \( K \) is homeomorphic to \( S^d \). We define a map \( f : V(K) \to S^{d-1} \) given by

\[
f(\sigma) = f_i(\sigma) \iff \sigma \in V(K_i),
\]

which obviously satisfies \( f(\sigma) = -f(\upsilon(\sigma)) \). By taking the affine extension of \( f \) we get a continuous equivariant map \( \hat{f} : K \to \mathbb{R}^d \) for which the following holds.

Claim. For any simplex \( S \in K \), the origin is not contained in \( \hat{f}(S) \)

Assuming this claim is true, the proof the theorem is complete since \( \hat{f} \) would contradict the Borsuk–Ulam theorem.

It remains to prove the claim. Consider a maximal simplex \( S \in K \). The vertices of \( S \) can be expressed as \( n \) chains

\[
\begin{align*}
\sigma_1^{(1)} & \subset \sigma_2^{(1)} \subset \cdots \subset \sigma_{k_1+1}^{(1)} \\
\sigma_1^{(2)} & \subset \sigma_2^{(2)} \subset \cdots \subset \sigma_{k_2+1}^{(2)} \\
& \vdots \\
\sigma_1^{(n)} & \subset \sigma_2^{(n)} \subset \cdots \subset \sigma_{k_n+1}^{(n)}
\end{align*}
\]
where $\sigma^{(i)}_j$ is a subfamily of $F_i$ with $|\sigma^{(i)}_j| = j$. For every $1 \leq i \leq n$ let us denote $\sigma^{(i)}_1 = \{C_i\}$ and $\nu_i(\sigma^{(i)}_{k+1}) = \{D_i\}$. We observe that
\[C_i \subset \bigcap_{j=1}^{k+1} H_{\sigma^{(i)}_j} \quad \text{and} \quad D_i \subset \bigcap_{j=1}^{k+1} H_{\nu_i(\sigma^{(i)}_j)}\]
so by the colorful intersection property, it follows that the intersections $\bigcap_{i=1}^n C_i$ and $\bigcap_{i=1}^n D_i$ are both nonempty. This implies
\[\bigcap_{\sigma \in S} H_{\sigma} \neq \emptyset \quad \text{and} \quad \bigcap_{\sigma \in S} H_{\nu(\sigma)} \neq \emptyset.
\]
Since $H_{\sigma}$ and $H_{\nu(\sigma)}$ are opposite open halfspaces bounded by $h_{\{\sigma, \nu(\sigma)\}}$, it follows from Lemma 3 that $0 \notin \text{conv} f(S) = \hat{f}(S)$. This completes the proof of the claim and of Theorem 1.

5. Final remarks

The paper by Montejano and Karasev [5] contains several other interesting results that are not covered by our Theorem 1. In particular, Corollary 7 in [5] gives conditions for when families of size greater than $k + 2$ have a $k$-transversal. For example they show the following:

Let $F_1, F_2, F_3$ be families of convex sets in $\mathbb{R}^5$ satisfying the colorful intersection property, and suppose $|F_1| = |F_2| = |F_3| = 5$. Then one of the families $F_i$ admits a $2$-transversal.

We have not been able to prove the statement above using the proof method of Theorem 1. The main issue seems to be that for families of size greater than $k + 2$ we do not have a good combinatorial characterization for the existence of a $k$-transversal as in Lemma 2.

Question. What is the optimal dimension in Corollary 7 in [5]?

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