FROM DISCRETE TO CONTINUOUS PAINLEVÉ EQUATIONS:
A BILINEAR APPROACH

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Abstract
We present the bilinear forms of the (continuous) Painlevé equations obtained from the continuous limit of the analogous expressions for the discrete ones. The advantage of this method is that it leads to very symmetrical results. A new and interesting result is the bilinearization of the $P_{VI}$ equation, something that was missing till now.
1. Introduction

The study of discrete Painlevé equations (d-P’s) has greatly profited from the parallel that exists between these discrete systems and their continuous counterparts. Once it was established that most properties of the continuous Painlevé equations could be transposed \textit{mutatis mutandis} to the discrete setting, the problem was greatly simplified. (It is much simpler to find something when you are convinced that it exists). Thus the vast majority of the existing results consists in establishing the discrete analog of something that is well known for the continuous Painlevé equations. The reverse process, starting from some discrete property and obtaining a new result for the continuous case, had not, to our knowledge, materialized yet. In this work we shall present such an approach for the bilinearization of the continuous Painlevé equations. In two recent work of ours we have, in fact, presented the discrete bilinear form for all d-P’s [1,2]. (Let us point out here that in some cases the bilinear form was not sufficient and we had to resort to trilinear forms, while higher multilinear ones are also mandatory in some cases).

Our approach was based on the observation that the bilinear structure is related in a certain way to the singularity structure of the mapping. More specifically, the number of different singularity patterns was an indication as to the number of $\tau$-functions necessary for the bilinearization. Moreover, the precise structure of the singularities was dictating the expression of the nonlinear variable in terms of the $\tau$-functions. As a result of this approach, we were able to obtain simple, highly symmetrical bilinear expressions for the discrete Painlevé equations. In what follows, we shall use these expressions as a starting point and, by implementing the appropriate limit, obtain the bilinear forms for the continuous Painlevé equations [3].

In their work [4], Hietarinta and Kruskal have presented such a study, which is close in spirit to ours since it was based on the consideration of the singularities of the continuous Painlevé equations. However, since the process of splitting a multilinear equation so as to reduce it to bilinear ones is non-systematic, in many instances, the final expressions of [4] were not very symmetric. This is avoided if one proceeds through the discrete case. As a matter of fact, the discrete setting introduces so many constraints that one is left with very little freedom as to the possible form of the bilinear equation. Another important point is that in [4] no bilinear form for $P_{VI}$ could be obtained. This is remedied here. Starting from the discrete bilinear expression of $q$-$P_{VI}$ [2], we were able to obtain its continuous counterpart.

A nice feature of this approach is that one does not have to worry about the alternate forms of d-P’s, i.e. the existence of many different expressions for d-P$_1$, d-P$_{II}$ and so on. In the continuous case they go over to the same continuous Painlevé equation. Thus one can start from the version of the d-P with the most convenient bilinear form and work out its continuous limit.

2. The case of the discrete P$_1$ equation

The d-P$_1$ equation has a particular status in the sense that it is the only equation involving only one $\tau$-function. This is related to the fact that d-P$_1$ has only one singularity pattern. However, when several forms of d-P$_1$ were studied, it turned out that these equations could not all be bilinearized. In some cases the resulting form was a trilinear one. In particular for the d-P$_1$:

$$\tau + z = \frac{z}{x} + \frac{a}{x^2}$$ (2.1)
where $\overline{x} = x(n+1)$, $\underline{x} = x(n-1)$ and $z = \alpha n + \beta$, we put:

$$x = \frac{\overline{x}F}{\overline{F}^2}$$

(2.2)

The latter suggested by the singularity structure $\{0, \infty^2, 0\}$ where $\infty^2$ is a shorthand notation for a singularity that behaves as $\epsilon^{-2}$ when the ‘0’ corresponds just to $\epsilon$ as $\epsilon \to 0$. This results to the form:

$$\overline{F}F^2 + \overline{F}F^2 = zFF + aF^3$$

(2.3)

The continuous limit of this equation is obtained through $z = 6 + \epsilon^4 \zeta/2$, $a = -4$ while $x = 1 + \epsilon^2 w/2$ leads to $w = 2(\log F) \zeta \zeta$. We point out here that the continuous limit of $F$ is just $F$ considered now as a function of $\zeta$ (rather than $n$). Thus while (2.1) reduces to

$$w'' + 3w^2 = \zeta$$

(2.4)

the bilinear equation for $F$ becomes:

$$F(D^4_\zeta F \cdot F) = \zeta F^3$$

(2.5)

or, after a trivial division by $F$, to a bilinear equation. This is in fact the bilinear equation of $P_1$ obtained by Hietarinta and Kruskal. Another form of d-$P_1$, the so-called ‘standard’ form, is:

$$\overline{x} + x + \underline{x} = \frac{z}{x} + a$$

(2.6)

and has a slightly different trilinearization. Indeed the substitution

$$x = \frac{\overline{x}F}{\overline{F}F}$$

(2.7)

suggested by the singularity structure $\{0, \infty, \infty, 0\}$ is not applied to (2.6) but rather to its discrete derivative leading to:

$$\overline{F}F^2 - \overline{F}F^2 = zF^2 F - z\overline{F}^2$$

(2.8)

The continuous limit of the latter, obtained through $z = -3 + \epsilon^4 \zeta$ is, expectedly:

$$\frac{d}{d\zeta} \left( \frac{D^4_\zeta F \cdot F}{F^2} \right) = 1$$

(2.9)

With $a = 6$ and $x = 1 + \epsilon^2 w$, so that again $w = 2(\log F) \zeta \zeta$, equation(2.9) gives upon integration in $\zeta$ the same bilinear form of $P_1$ as (2.5).

Finally there are also multiplicative forms of d-$P_1$ [5], the simplest of which is:

$$\overline{x} = \frac{z}{x} + \frac{a}{x^2}$$

(2.10)

where in this case $z = \mu \lambda^n$. The singularity structure $\{0, \infty^2, 0\}$ suggests again $x = \overline{F}F/\overline{F}^2$, whereupon (2.10) becomes directly the bilinear equation

$$\overline{F}F = zFF + aF^2$$

(2.11)
Its continuous limit corresponds to $\lambda = 1 + \epsilon^5/8$, $\mu = 4$, $a = -3$, and reads again:

$$D_+^2 F \cdot F = \zeta F^2$$

(2.12)

with $\zeta = n\epsilon$. Again, we recover (2.4) with $x = 1 + \epsilon^2 w/2$. In fact, all the forms of $P_1$ analyzed in [1] lead to the same continuous bilinear form, which coincides with the one obtained in [4]. While no new result is obtained for this simplest $d$-$P$, this will no more be the case for the ‘higher’ ones.

3. Bilinear forms for the Painlevé equations II to V

In [1] we have examined the discrete Painlevé equations II to V and obtained their bilinear forms. We have shown that the number of $\tau$-functions needed for the bilinearization was as follows. For $d$-$P_{II}$, 2 $\tau$-functions. For $d$-$P_{III}$, 2 or 4 $\tau$-functions, the latter choice resulting in a more symmetrical form. For $d$-$P_{IV}$, 4 or 6 $\tau$-functions, where again the latter choice is more symmetrical, while for $d$-$P_{V}$ 6 $\tau$-functions are needed. Based already on this counting argument, we expect our results to be different from those of Hietarinta and Kruskal who in [4] have presented the bilinearization of the Painlevé equations with a number of $\tau$-functions not exceeding three.

Here are our results. The standard form of $d$-$P_{II}$ is:

$$\mathcal{P} + x = \frac{zx + a}{1 - x^2}$$

(3.1)

where $z$ is linear in $n$. Singularity structure suggests:

$$x = -1 + \frac{FG}{FG} = 1 - \frac{FG}{FG}$$

(3.2)

hence the bilinear equations:

$$\overline{FG} + \overline{FG} = 2FG$$

$$\overline{FG} - \overline{FG} = z(\overline{FG} - \overline{FG}) + 2aFG$$

(3.3)

(3.4)

The continuous limit is obtained through $z = 2 + \epsilon^2 \zeta$, $a = \epsilon^3 \alpha$:

$$D_+^2 F \cdot G = 0$$

$$D_+^3 \zeta F \cdot G = 0$$

(3.5)

(3.6)

This is compatible with the continuous limit to $P_{II}$

$$w_{\zeta \zeta} = 2w^3 + \zeta w + \alpha$$

(3.7)

through $x = \epsilon w$, $w = (\log F/G)_\zeta$.

The standard form of $d$-$P_{III}$ is:

$$\mathcal{P} x = \frac{cd(x - az)(x - bz)}{(x - c)(x - d)}$$

(3.8)

with $z = \mu \lambda^n$. We present first the more symmetrical bilinear form involving 4 $\tau$-functions. The singularity structures suggest:

$$x = c \left(1 - \frac{FG}{FG}\right) = d \left(1 - \frac{FG}{FG}\right) = \frac{HK}{FG}$$
\[
\frac{1}{x} = \frac{1}{az} \left( 1 - \frac{HK}{HK} \right) = \frac{1}{bz} \left( 1 - \frac{HK}{HK} \right) = \frac{FG}{HK} \quad (3.9)
\]
leading to the bilinear equations:
\[
(c - d)FG - cFG + dG = 0 \quad (3.10a)
\]
\[
(a^{-1} - b^{-1})HK - a^{-1}HK - b^{-1}HK = 0 \quad (3.10b)
\]
\[
(c + d)FG - cFG - dG = 2HK \quad (3.10c)
\]
\[
(a^{-1} + b^{-1})HK - a^{-1}HK - b^{-1}HK = 2zFG \quad (3.10d)
\]
The continuous limit is obtained through \( \lambda = 1 + \epsilon, a = \epsilon - a_0 \epsilon^2, b = -\epsilon - b_0 \epsilon^2, c = 1/\epsilon + c_0, d = -1/\epsilon + d_0 \) leading to:
\[
D_\varsigma F \cdot G = HK \\
D_\varsigma H \cdot K = -zFG \\
D_\varsigma^2 F \cdot G = (c_0 + d_0)HK \\
D_\varsigma^2 H \cdot K = (a_0 + b_0)zFG 
\]
where \( z = e^\varsigma \). In fact, since \( x = HK/FG \), it goes, at the continuous limit, to \( x = -z \log (F/G)_z \) (and also \(-\log (H/K)_z)^{-1}\)) and satisfies the continuous P_{III} (though not exactly under its canonical form):
\[
x_{zz} = \frac{x^2}{x} - \frac{x}{z} + \frac{x^3}{z^2} - \frac{c_0 + d_0}{z^2}x^2 + \frac{a_0 + b_0}{z} - \frac{1}{x}. \quad (3.12)
\]
Instead of (3.10) there also exists a bilinear form for d-P_{III} involving only the two \( \tau \)-functions \( F, G \). It consists of (3.10a) and
\[
\frac{cd}{c - d} (cFG - dG) + (a - c)(b - d)FG + c(b - d)FG + d(a - c)GF = 0 \quad (3.13)
\]
Its continuous limit is:
\[
D_\varsigma^2 F \cdot G + (c_0 + d_0)D_\varsigma F \cdot G = 0 \\
D_\varsigma^3 F \cdot G + (c_0 + d_0)D_\varsigma^2 F \cdot G + 2(a_0 + b_0)zD_\varsigma^2 F \cdot G + 2z^2FG = 0 \quad (3.14)
\]
We now proceed to d-P_{IV} the standard form of which is:
\[
(\overline{F} + x)(x + z) = \frac{(x + a)(x - a)(x + b)(x - b)}{(x + z + c)(x + z - c)} \quad (3.15)
\]
We introduce the transformation
\[
x = a - \frac{HK}{FG} = -a - \frac{HK}{FG} = b \left( 1 - \frac{MN}{FG} \right) = -b \left( 1 - \frac{MN}{FG} \right) \\
= -z - c \left( 1 - \frac{FG}{FG} \right) = -z + c \left( 1 - \frac{FG}{FG} \right) \quad (3.16)
\]
Note that although \( a, b \) play a symmetrical role in (3.15), equation (3.16) treats them in an asymmetrical way. This is done with some hindsight, in view of the continuous limit below. This leads to:
\[
2aFG - HK + HK = 0 \quad (3.17a)
\]
\[ 2FG - MN - MN = 0 \]  
\[ 2FG - \overline{FG} - \overline{FG} = 0 \]  
\[ HK + HK = 2zFG - c(\overline{FG} - \overline{FG}) \]  
\[ b(MN - MN) = 2zFG - c(\overline{FG} - \overline{FG}) \]  
\[ \overline{HK} - \overline{HK} + a(\overline{MN} + \overline{MN}) + \frac{2a}{c^2}(a^2 - b^2)FG = 0 \]  

At the continuous limit \((a = \epsilon a_1, b = 2/\epsilon + \epsilon b_1, c = 1/\epsilon + \epsilon c_1)\) we have:

\[ D_z H \cdot K = 2a_1 FG \]  
\[ MN = FG \]  
\[ D^2_z FG = 0 \]  
\[ HK = zFG - D_z F \cdot G \]  
\[ D_z M \cdot N = zFG - D_z F \cdot G \]  
\[ D^3_z H \cdot K + 2a_1 D^2_z M \cdot N = 8a_1(b_1 - 2c_1)FG \]

It is straightforward to check that the continuous variable \(x = -z + (\log F/G)_z = -HK/FG = -(\log M/N)_z\) satisfies the continuous \(P_{IV}\):

\[ x_{zz} = \frac{x_z^2}{2x} + \frac{3}{2} x^3 + 4z x^2 + 2z^2 x - \frac{2a_1^2}{x} + 2(b_1 - 2c_1)x \]  

It is also easy to obtain for \(P_{IV}\) a bilinear expression involving only four \(\tau\)-functions. Eliminating \(M, N\) between (3.17b), (3.17e) and (3.17f) we find:

\[ \overline{HK} - \overline{HK} + a(\overline{MN} + \overline{MN}) + \frac{2a}{c^2}(a^2 - b^2)FG = 0 \]  

and the continuous equivalent, which amounts to eliminate \(M\) and \(N\) between (3.18b), (3.18e) and (3.18f):

\[ D^3_z H \cdot K = 4a_1 zD_z F \cdot G - 2a_1(z^2 - 4b_1 + 8c_1)FG \]

For d-PV:

\[ (x - 1)(x - 1) = \frac{pq(x - u)(x - \frac{1}{p})(x - v)(x - \frac{1}{q})}{(x - p)(x - q)} \]  

we introduce:

\[ x = u + \frac{HK}{FG} = \frac{1}{u} + \frac{HK}{FG} = v + \frac{MN}{FG} = \frac{1}{v} + \frac{MN}{FG} \]

\[ = p \left( 1 - \frac{FG}{FG} \right) = q \left( 1 - \frac{FG}{FG} \right) \]  

and find

\[ \left( u - \frac{1}{u} \right) FG + HK - HK = 0 \]  
\[ \left( v - \frac{1}{v} \right) FG + MN - MN = 0 \]
\[(p-q)FG - pFG - qFG = 0\]  
\[(u + \frac{1}{u}) FG + HK + HK = (p+q)FG - pFG - qFG\]  
\[(v + \frac{1}{v}) FG + MN + MN = (p+q)FG - pFG - qFG\]  
\[\frac{1}{u-\frac{1}{u}} \left(\frac{u HK - uHK}{1} - \frac{1}{v-\frac{1}{v}} \left(\frac{v MN - vMN}{1}\right)\right) = -\left(\frac{u + \frac{1}{u} - v - \frac{1}{v}}{1}\right)FG\]  

For the continuous limit we take \(u = 1 + \epsilon u_1\), \(v = -1 - \epsilon v_1\), \(p = (1/\epsilon + p_0)/z\), \(q = (-1/\epsilon + p_0)/z\), and find:

\[2u_1FG - D_\zeta H\cdot K = 0\]  
\[2v_1FG + D_\zeta M\cdot N = 0\]  
\[D_\zeta^2 F\cdot G - 2p_0 D_\zeta F\cdot G = 0\]  
\[z(FG + HK) - D_\zeta F\cdot G = 0\]  
\[z(FG - MN) + D_\zeta F\cdot G = 0\]  
\[\frac{1}{u_1} D_\zeta^2 H\cdot K - \frac{1}{v_1} D_\zeta^2 M\cdot N + 2D_\zeta^2 H\cdot K - 2D_\zeta^2 M\cdot N = 0\]  

where \(z = e^\zeta\). At the continuous limit we have \(x = (\log F/G)_x = 1 + HK/FG = -1 + MN/FG\).

Putting \(x = (1 + w)/(1 - w)\) we find that \(w\) obeys the continuous \(P_V\) equation in the form:

\[w_{zz} = \left(\frac{1}{2w} + \frac{1}{w-1}\right) w_z^2 - \frac{1}{z} w_z + \frac{(w-1)^2}{2z^2} \left(u_1^2 w - u_1^2\right) - \frac{4p_0 w}{z} - \frac{2w(w+1)}{w-1}\]  

An interesting novel feature is that some of the continuous bilinear equations are non-differential, e.g. (3.18b) or the difference of (3.25d) and (3.25e), relating 4 or 6 \(\tau\)-functions. This may be an explanation as to why our bilinear expressions were not obtained through the direct search of Hietarinta and Kruskal although these authors also used systematically the singularity structure argument (but directly on the continuous equations).

4. Bilinear form of \(P_{VI}\)

This is the most interesting result of this paper since it provides the bilinearization of \(P_{VI}\) that was unknown up to now. The main factor for this progress was the recent derivation of a discrete form of \(P_{VI}\) by Jimbo and Sakai [6]. The \(q\)-\(P_{VI}\) equation is written in form of a system:

\[\bar{x}x = \frac{(y - \alpha z)(y - \beta z)}{(y - \gamma)(y - \frac{1}{\gamma})}\]  
\[y\bar{y} = \frac{(x - az)(x - bz)}{(x - c)(x - \frac{1}{c})}\]

where \(z = \mu\lambda^n\), \(\bar{z} = z\sqrt{\lambda}\) and we have the constraint \(ab = \alpha\beta\). The \(\tau\)-functions are introduced through:

\[x = e \left(1 + (1-z)^{1/2} \frac{MN}{FG}\right) = \frac{1}{e} \left(1 + (1-z)^{1/2} \frac{MN}{FG}\right) = \frac{HK}{FG}\]
\[
\frac{1}{x} = \frac{1}{a z} \left( 1 - (1-z)^{1/2} \frac{P Q}{H K} \right) = \frac{1}{b z} \left( 1 - (1-z)^{1/2} \frac{P Q}{H K} \right) = \frac{F G}{H K}
\]
\[
y = \gamma \left( 1 + (1-\tilde{z})^{1/2} \frac{F G}{M N} \right) = \frac{1}{\gamma} \left( 1 + (1-\tilde{z})^{1/2} \frac{F G}{M N} \right) = \frac{P Q}{M N}
\]
\[
\frac{1}{y} = \frac{1}{\alpha \tilde{z}} \left( 1 - (1-\tilde{z})^{1/2} \frac{H K}{P Q} \right) = \frac{1}{\beta \tilde{z}} \left( 1 - (1-\tilde{z})^{1/2} \frac{H K}{P Q} \right) = \frac{M N}{P Q}
\]
leading to:
\[
2FG + (1-z)^{1/2}(MN + MN) = \left( c + \frac{1}{c} \right) HK
\]
\[
2HK - (1-z)^{1/2}(PQ + PQ) = (a + b)zFG
\]
\[
2MN + (1-\tilde{z})^{1/2}(F G + F G) = \left( \gamma + \frac{1}{\gamma} \right) PQ
\]
\[
2PQ - (1-\tilde{z})^{1/2}(\bar{H}K + H K) = (\alpha + \beta)\tilde{z}MN
\]
\[
\left( c - \frac{1}{c} \right) FG + (1-z)^{1/2} \left( cMN - \frac{1}{c^2}MN \right) = 0
\]
\[
\left( \frac{1}{a} - \frac{1}{b} \right) HK - (1-z)^{1/2} \left( \frac{1}{a} PQ - \frac{1}{b} PQ \right) = 0
\]
\[
\left( \gamma - \frac{1}{\gamma} \right) MN + (1-\tilde{z})^{1/2} \left( \gamma FG - \frac{1}{\gamma} F G \right) = 0
\]
\[
\left( \frac{1}{\alpha} - \frac{1}{\beta} \right) PQ - (1-\tilde{z})^{1/2} \left( \frac{1}{\alpha} HK - \frac{1}{\beta} HK \right) = 0
\]

We go to the continuous limit through: \( a = 1 + \epsilon a_1 + \epsilon^2 a_2, b = 1 - \epsilon a_1 + \epsilon^2 b_2, c = 1 + \epsilon c_1, \alpha = 1 + \epsilon \alpha_1 + \epsilon^2 \alpha_2, \beta = 1 - \epsilon \alpha_1 + \epsilon^2 \beta_2, \gamma = 1 + \epsilon \gamma_1 \). The constraint \( ab = \alpha \beta \) translates into \( a_2 + b_2 - a_1^2 = \alpha_2 + \beta_2 - \alpha_1^2 \). We then find:
\[
FG + (1-z)^{1/2}MN = HK
\]
\[
HK - (1-z)^{1/2}PQ = zFG
\]
\[
c_1 FG + (1-z)^{1/2}(D_{\zeta} + c_1) M \cdot N = 0
\]
\[
a_1 HK + (1-z)^{1/2}(D_{\zeta} - a_1) P \cdot Q = 0
\]
\[
\gamma_1 MN + (1-z)^{1/2}(D_{\zeta} + \gamma_1) F \cdot G = 0
\]
\[
\alpha_1 PQ + (1-z)^{1/2}(D_{\zeta} - \alpha_1) H \cdot K = 0
\]
\[
(1-z)D_{\zeta}^2 F \cdot G - (1-z)^{1/2}D_{\zeta}^2 M \cdot N + (1-z)^{1/2}D_{\zeta}^2 P \cdot Q
\]
\[
= -(a_2 + b_2)zFG + c_1^2 HK + \gamma_1^2 (1-z)^{1/2}PQ
\]
\[
(1-z)D_{\zeta}^2 H \cdot K - z(1-z)^{1/2}D_{\zeta}^2 M \cdot N + (1-z)^{1/2}D_{\zeta}^2 P \cdot Q
\]
\[
= -(a_2 + b_2)zFG - c_1^2 zHK - (\alpha_2 + \beta_2)z(1-z)^{1/2}MN
\]

where \( z = e^{\zeta/2}, x = HK/FG \) and, at the continuous limit, we have in addition \( x = 1 + (1-z)^{1/2}MN/FG, 1/x = (1-(1-z)^{1/2}PQ/HK)/z \). We obtain thus the continuous PVI:
\[
x_{zz} = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-z} \right) x_z^2 + \left( \frac{1}{z} + \frac{1}{z-1} - \frac{1}{z-x} \right) x_z
- \frac{x(x-1)(x-z)}{z^2(z-1)^2} \left( \frac{\gamma_1^2}{2} - \frac{\alpha_1^2}{2} \frac{z}{x^2} + \frac{\alpha_2^2}{2} \frac{z-1}{(x-1)^2} + \frac{1 - \alpha_1^2}{2} \frac{z(z-1)}{(x-z)^2} \right)
\] (4.6)

We remark here also that there exist non-differential relations between the \(\tau\)-functions and they are unavoidable for the bilinearization of \(P_{VI}\).

5. Conclusion

In the previous sections, we have presented the bilinearization of the six Painlevé transcendental equations starting from the results for the discrete ones and implementing the appropriate continuous limit. This approach has made possible the derivation of the bilinear form for \(P_{VI}\), a result that was obtained here for the first time.

The analogy between discrete and continuous case is also useful in a broader scope. In our analysis of discrete equations, it became clear that the ‘right’ number of \(\tau\)-functions is identical to the number of different singularity patterns. The same appears to be true in the continuous case (with one possible exception for \(P_{IV}\) where the use of 6 \(\tau\)-functions leads to a more symmetrical result). Now that results on discrete Painlevé equations start accumulating, it would be interesting to translate them back to the continuous case. New results for continuous equations may thus make their appearance and old results may be transcribed in more symmetrical, easier to use, forms.

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