Modelling of deformation process for the layer of elastoviscoelastic media under surface action of periodic force of arbitrary type

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Abstract. Description of deflected mode for different types of materials under action of external force plays special role for wide variety of applications – from construction mechanics to circuits engineering. This article considers the problem of plastic deformation of the layer of elastoviscoplastic soil under surface periodic force. The problem was solved with use of the modified lumped parameters approach which takes into account close to real distribution of normal stress in the depth of the layer along with changes in local mechanical properties of the material taking place during plastic deformation. Special numeric algorithm was worked out for computer modeling of the process. As an example of application suggested algorithm was realized for the deformation of the layer of elastoviscoplastic material by the source of external lateral force with the parameters of real technological process of soil compaction.

1. Introduction

Modelling of deformational processes in materials has important applications in theoretical researches as well as for the solution of practical problems in numerous fields of science and engineering. Analysis of evolution of deflected mode of material under dynamic action is necessary for design of complex mechanical systems and vibration protection systems, is in use for building construction and optimization of working regimes of different machines and mechanisms.

In case of dynamic deformation of elastoviscoelastic materials numeric modelling can describe accumulation of plastic strain which can cause the change of mechanical properties of the material and in consequence can lead to strengthening of the material. New properties may sufficiently widen the field of technical applications for modified material.

Classic results for the problems in the given field in case of isotropic elastic medium were obtained by Boussinesq and later generalized by Westergaard [1]. Approaches and methods developed by by Boussinesq and Westergaard are quite useful even now for computations of bases and foundations of constructions as well as in other problems of construction mechanics.

Dynamic deformation with accumulation of plastic strain and following modification of elastoviscoelastic media is a case of special interest of the investigators. The cause of the constant interest is wide spectrum of properties of materials and sources of surface loading which occur in technical applications [2], [3], [4], [5].

2. Formulation of the problem

The object of investigation chosen in this paper is the process of deformation of the layer of elastoviscoelastic material under the action of external surface force of given characteristics (amplitude, time of loading, dependence on time) acting on the contact spot of given shape and dependence on deformation. Patterns of accumulation of plastic strains will sufficiently differ for the material of the layer at different depths. Since that we suggest to considerate different zones of layer as
massive elements with different elastic and viscous characteristics which determine their replies on stress.

Choice of the shape of zones is to be done according to the shape of equal normal stress surfaces in the depth of the soil. Surfaces we choose to restrict elements correspond to the values of yield point, plastic limit and elastic limit of the material. Their exact values may be obtained from the stress-strain diagram. Calculation of lumped mechanical characteristics are to be done taking into account the dependences of characteristics of the material on the density depending on plastic strain accumulated by the layer.

Numeric solution of equation of motion for the active zone of the material under action of the surface loading will give the values of plastic strain and density of the material achieved by the time when dynamic loading finishes

3. Theory

The main point of suggested approach to theoretical description of deformation processes in layers of elastoviscoplastic media is that different parts of active zone of the layer are in different states in respect to their response on deforming stress. Boundaries of the parts may be chosen as equal stress surfaces corresponding to limit values from the stress-strain diagram. Boussinesq theory for point vertical loading \( Q \) gives following result for stress distribution

\[
\sigma_z = \frac{3Q}{2\pi z^2} \frac{1}{(1 + (r/z)^2)^{5/2}},
\]

Westergaard’s theory for the same case of point vertical load \( Q \) states

\[
\sigma_z = \frac{Q}{2\pi z^2} \frac{\gamma}{\gamma^2 + (r/z)^2}^{3/2},
\]

Where coefficient \( \gamma \) accounts redistribution of stress in the depth of the soil

\[
\gamma = \left(1 - 2\mu\right) \left(\frac{2}{2 - \mu}\right)^{1/2},
\]

\( \mu \) - Poisson coefficient.

In all of the expressions above \( r \) - stands for the distance from the point of observation to the vertical line going deep from the point of loading, variable \( z \) - stands for depth of observation point in respect to the surface of the layer.

Expression for stress caused by surface point load allow to find the stress from distributed surface loads. One of the most interesting cases here is infinite stripe \( 2b \) wide vertically loaded with surface stress \( \sigma_0 \). Distribution of stress in the depth of the material looks like (Boussinesq) looks as follows

\[
\sigma_z = \frac{3\sigma_0}{\pi} \left(\arctan\left(\frac{x+b}{z}\right) - \arctan\left(\frac{x-b}{z}\right)\right) - \frac{2b}{(x^2+b^2+z^2)^{3/2}}\left(x^2+z^2-b^2\right).
\]

Where after transition to the dimensionless variables \( \zeta = z/b, \xi = x/b \), one gets

\[
\sigma_z = \frac{\sigma_0}{\pi} \left(\arctan\left(\frac{\xi+1}{\zeta}\right) - \arctan\left(\frac{\xi-1}{\zeta}\right)\right) - \frac{2\zeta}{(\zeta^2+1)^2}\left(\zeta^2+4\zeta^2\right).
\]

Graphs which represent sections of equal stress surfaces \( f(\xi, \zeta) = \frac{\sigma_z}{\sigma_0} = C \), for different values of \( C \) are given on Figure1. Function \( f(\xi, \zeta) = C \) connects variables \( \xi, \zeta \) implicitly and it’s impossible to find analytic dependence \( \xi = \xi(\zeta) \) because of transcendental equations to be solves for that purpose.

The result for Westergaard case can be written down as

\[
\sigma_z = \frac{\sigma_0}{\pi} \int_{-b}^{b} \frac{zd\sigma_z(t)}{\gamma\zeta} = \frac{\sigma_0}{\pi} \int_{-b}^{b} \frac{zd\left(\gamma\zeta\arctan\left(\frac{x+b}{\gamma\zeta}\right) - \arctan\left(\frac{x-b}{\gamma\zeta}\right)\right)}{(\gamma\zeta)^2 + (x-t)^2},
\]
which in dimensionless variables $\zeta = z/b$, $\xi = x/b$ looks like

$$
\sigma_\zeta (\xi, \zeta) = \frac{\sigma_0}{\pi} \left( \arctan\left(\frac{\xi + 1}{\gamma}\right) - \arctan\left(\frac{\xi - 1}{\gamma}\right) \right).
$$

Graphs which represent sections of equal stress surfaces for Westergaard distribution is given on Figure 2 in coordinates $\zeta = z/b$, $\xi = x/b$.

![Figure 1](image1.png)

**Figure 1.** Equal stress surfaces distribution according to Boussinesque (uniformly distributed load on infinitely long stripe) for stress equal 0.1, 0.2, 0.4 of surface value (in variables $\zeta$, $\xi$)

![Figure 2](image2.png)

**Figure 2.** Equal stress surfaces distribution according to Westergaard (uniformly distributed load on infinitely long stripe) for stress equal 0.1, 0.2, 0.4 of surface value (in variables $\zeta$, $\xi$)

The sections of Westergaard distribution may have analytic representation and appear to be ellipses going through points (-1,0) and (1,0). Since that surfaces of equal stress represent family of intersecting elliptic cylinders. Let’s write down equation of sections of equal stress surfaces as

$$
\frac{\pi \sigma}{\sigma_0} = \arctan\left(\frac{\xi + 1}{\gamma}\right) - \arctan\left(\frac{\xi - 1}{\gamma}\right)
$$

Let’s transform it to canonical view, assuming $C = \frac{\pi \sigma}{\sigma_0} = const$
\[
\frac{\left(\xi - \frac{1}{c^2}\right)^2}{\frac{1}{c^2} \left(1 + \frac{x^2}{c^2}\right)} + \frac{\xi^2}{\left(1 + \frac{x^2}{c^2}\right)^2} = 1.
\]

Distribution of equal stress surfaces as a family of intersecting cylinders is a good approximation for different parts of active zone in interesting case of loading when force is applied to the rectangular contact spot with length much greater than width.

Let’s apply to the mechanical system of active zone of material modified Lagrange formalism which makes it possible to take into account dissipative forces of viscous friction and dry friction as described in [5].

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} = F(x, \dot{x}, t) + F_{mp}, \tag{8}
\]

where \( T(x, \dot{x}) = \frac{m(x) \dot{x}^2}{2} \) - kinetic energy of active zone,

\( F(x, \dot{x}, t) \) - internal forces of elasticity and viscous friction and external force of surface loading,

\( F_{mp} \) - separately taken force of dry friction to model plastic element when contact stress exceeds yield point.

At the start of loading process material of the layer while stress does not exceed value 1 on Figure 3 undergo only elastic deformation (it’s known that if the material is not dense enough then point 1 coincides with origin of coordinates); between points 1 and 2 deformations recover partially after deloading. If stress takes values greater than at 2 deformations are becoming plastic but stress must grow in order to obtain greater strain. Point 3 corresponds to the yield limit and after it material is gaining plastic deformations without change in external stress.

![Figure 3. “Strain-stress” diagram for elastoviscoplastic material for normal axial loading with consequent reload (thin lines stand for reload)](image)

The difference of suggested method from usual is consideration of dynamics of the mass of active zone under loading. In this case right side of (8) looks like

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} = \frac{d}{dt} \left( m(x) \dot{x} - \frac{\partial m(x)}{\partial x} \frac{\dot{x}^2}{2} \right) = m(x) \ddot{x} + \frac{\partial m(x)}{\partial x} \dot{x}, \tag{9}
\]

and modifies equation of motion of active zone, obtained in [6].

Elastic or quasi-elastic force for the layer of elastoviscoplastic media is also assumed to be nonlinear in respect to the value of deformation (strain). Nonlinear behaviour is to be modelled by introduction of non-constant stiffness of active zone

\[
F_{mp} = c(x) x. \tag{10}
\]
Unequal deformational properties for loading and deloading are to be taken into account by following model description:

$$\sigma_{def} = E \varepsilon, \quad E = (E_{pl} \theta(\dot{\sigma}) + E_{el}) \theta(\sigma_{pl})$$,  

where $E_{pl}$ - plastic deformation modulus of the material, $E_{el}$ - Young modulus, $\dot{\sigma}$ - velocity of change in contact stress, $\theta(z)$ - Heaviside function, $\sigma_{pl}$ - limit of plasticity of the material. Transition to the lumped elastic element gives:

$$F_{def} = c(x)\chi, \quad c(x) = (c_{pl}(x)\theta(\dot{\sigma}) + c_{el}(x))\theta(\sigma_{pl})$$  

It must be mentioned that force and elasticity must depend on coordinate of the centre of mass of active zone not on surface coordinate and for transition to coordinate of the surface must be recalculated.

Viscous friction force of the active zone is modelled by lumped viscosity element

$$F_{visc} = b(x)\chi$$,

And it must be pointed that dependence of viscous force on the coordinate is quite slow and mostly is influenced by the change in viscosity with accumulation of plastic strain.

In the framework of given approach behaviour of the material after yield point is exceeded must be modelled by the element of dry friction which “turns on” when the following condition is satisfied

$$\sigma(t) \geq \sigma_{pl}$$, 

where $\sigma(t)$ - value of contact stress. Corresponding constant resistance force does not depend on deformation

$$F_{fr}(t) = S_{cont}(t)\sigma_{pl}$$

where $S_{cont}(t)$ - contact spot area between the source of loading and material surface in the moment $t$.

At the same moment of time elastic force (12) “turns off”.

Since that equation of motion for active zone (9) may be rewritten under model assumption made above

$$m(x)\ddot{x} + \frac{\partial m(x)}{\partial x} \dot{x}^2 = -(c_{pl}(x)\theta(\dot{\sigma}) + c_{el}(x)(1 - \theta(\sigma_{pl})))x - b(x)\dot{x} - \frac{\dot{x}}{k_0} \theta(\sigma_{pl})S_{cont}(t)\sigma_{pl} + F_{ext}(t)$$  

Here $F_{ext}(t)$ - is total external surface force generated by the compactor. Time dependence in it is determined by the technical properties of the machine and may be chosen arbitrary. Lumped parameter model of corresponding mechanical system is given on Figure 4.

More detailed consideration of the process is possible in the framework of multy-body approach which can show the evolution of the stressed state in the depth of the layer. The general form of the equation of motion (14) for interacting parts of the active zone must be corrected according to their properties of the reply on deforming stress. Lumped parameter model of multy-body system is given on Figure 5Suggested multy-body model generates system of differential equations which requires simultaneous solution in order to calculate plastic strain accumulated by the layer. Index 0 stands for the source of surface loading, indices 1,2,3 – for different parts of the active zone.

\[
\begin{align*}
& m_0(\dot{x}_0)\dot{x}_0 = -c_0(x_0)(x_{p0} - x_0) - b_0(x_0)\dot{x}_0 + F_{ext}(t)
\end{align*}
\]

\[
\begin{align*}
& m_1(\dot{x}_1)\dot{x}_1 + \frac{\partial m(x_1)}{\partial x_1} \dot{x}_1^2 = c_0(x_0)(x_{p0} - x_0) - b_0(x_0 - x_2)(\dot{x}_1 - \dot{x}_2) - \frac{\dot{x}_1 - \dot{x}_2}{k_1 - k_0} \theta(\sigma_{pl})S_{cont}(t)\sigma_{pl} \\
& m_2(\dot{x}_2)\dot{x}_2 + \frac{\partial m(x_2)}{\partial x_2} \dot{x}_2^2 = \frac{\dot{x}_1 - \dot{x}_2}{k_1 - k_0} \theta(\sigma_{pl})S_{cont}(t)\sigma_{pl} - (c_{pl}(x_2)\theta(\dot{\sigma}) + c_{el}(x_2)(1 - \theta(\sigma_{pl}))(x_2 - x_1) - b_2(x_2 - x_1)(\dot{x}_2 - \dot{x}_1)
\end{align*}
\]

\[
\begin{align*}
& m_3(\dot{x}_3)\dot{x}_3 + \frac{\partial m(x_3)}{\partial x_3} \dot{x}_3^2 = (c_{pl}(x_3)\theta(\dot{\sigma}) + c_{el}(x_3)(1 - \theta(\sigma_{pl}))(x_3 - x_1) - (c_{pl}(x)\theta(\dot{\sigma}) + c_{el}(x))(1 - \theta(\sigma_{pl}))(x_3 - x_1) - b_3(x_3)\dot{x}_3)
\end{align*}
\]
In the case under consideration index 0 corresponds to the work tool of the compactor, indices 1,2,3 consequently to the part of soil layer without elastic response to the stress, then to the part of soil with different properties for loading and deloading and, at last for usual viscoelastic part.

![Figure 4. One-body model for active zone of material layer under surface loading](image)

![Figure 5. Three-body model for active zone of material layer under surface loading](image)

System (15) is definitely nonlinear and requires numeric methods which must allow taking into account motion of the surface under deforming loading and redistribution of masses and mechanical characteristics between different parts of active zone with accumulation of plastic strain.

Very important issue for program realization of given numeric algorithm for the solution of system (15) is selection of partition for time interval. Transition to the finite elements is to be performed the usual way by substitution for first and second derivatives of strain in respect to time following expressions

\[
x^{(k+1)}_j = \frac{x^{(k+1)}_i - x^{(k)}_i}{\Delta t},
\]

\[
x^{(k+2)}_i = \frac{x^{(k+2)}_i - 2x^{(k+1)}_i + x^{(k)}_i}{\Delta t^2},
\]

Where upper index numerates step of partition of time interval and lower index numerates part of active zone and therefore corresponding model. Values of strain for each part allow to calculate full amount of work that was spent on the deformation and work that was spent to provide accumulated plastic strain and since that to determine net power of external source of dynamic action.

Numeric approach suggested in the article makes the motion equations linear because coefficients which may cause significant non-linear effects and lead to the problems are constant and are to be recalculated on each step of scheme in order to take into account modification of elastoviscoplastic material when density grows.

4. Results of numeric experiments

The algorithm developed above for numeric integration of non-linear equation of motion for multi-body model of active zone of the material was realized in the framework of computer algebra package Maple 2016.
It must be pointed out that numeric methods as well as experimental investigations are the most common and effective research tools for interaction of different types of media with source of surface force. Numeric methods can be used for revealing of the properties of interaction process [4], [7], as well as for solution of applied problems [8], [9], [10]. Case of periodic vibratory surface loading is one of the most interesting from both points. During numeric modelling media were represented by widely used in construction clay loam and sandy loam and work tool (drum) of the vibratory road roller represented source of surface loading. The case of soil media quite common for investigation of the vibratory loading [11], [12], [13], [14].

Numeric modelling was performed with use of parameters of external force normal for midsize vibratory road roller manufactured by Rybinsk road machine plant DM-13-VD acting on layer of sandy loam (required density 1700 kg/m$^3$) and clay loam (required density 1900 kg/m$^3$) 0.3 thick. Static loading on vibratory drum is 65 kN; vibratory force for sandy loam was assumed to be of amplitude 62 kN and frequency 40 Hz and for clay loam of amplitude 62 kN and frequency 40 Hz. Results of modelling in graphic form are presented on figures 6, 7, 8, 9. Initial parameters of the modelling along with properties of the roller included roller speed which varied from 0.2 to 1.6 m/c and initial density of the soil defined by the value of compaction degree – rate between initial density and required one.

![Figure 6](image1.png)

**Figure 6.** Accumulated plastic strain for one roller pass depending on initial compaction degree of a soil (a) and speed of the roller (b) (sandy loam)
Figure 7. Energy spent on plastic deformation of the soil layer depending on initial compaction degree of a soil a) and speed of the roller b) (sandy loam)
Figure 8. Accumulated plastic strain for one roller pass depending on initial compaction degree of a soil a) and speed of the roller b) (clay loam)

Figure 9. Energy spent on plastic deformation of the soil layer depending on initial compaction degree of a soil a) and speed of the roller b) (clay loam)
Accumulated plastic strain of the layer and amount of mechanical work for that on one roller pass may be considered parameters for the choice of energy and time efficient regime of soil compaction by the given type of the roller for given initial state of the soil layer.

5. Discussion
Results of computer modelling and numeric experiments well correspond with results of contemporary researches and show that new modified approach of lumped parameters for description of accumulation of plastic strains in elastoviscoplastic media. Mathematical model developed in the work for evolution of deformation processes in depth of material layer under surface force as well as corresponding numeric algorithm allow effective application for evaluation of strain accumulation dynamics in case of arbitrary force and evaluation of efficiency of power transmission to the material layer under surface loading.

6. Conclusion
Accumulation of plastic strain by elastoviscoplastic media is still a problem of constant attention for specialists in the field of construction mechanics, road and highway engineering because of its interest from the point of compaction processes in soils and other media. Effective theoretical solution of the problem can allow to perform so called Intelligent Soil Compaction technology in order to achieve required densities of soil media with least possible spent time and resources [15], [16], [17], [3]. Approach applied in the article for the first time for description of interaction between elastoviscoplastic material layer and work tool of compactor makes it possible to predict necessary parameters of surface force by given physical properties of the material in order to provide the most efficient accumulation of plastic strain. Functional dependences revealed by the numeric experiment presented in the article in graphic form may be used for design of energy efficient work regimes of road rollers and construction of new work tools.

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