GRADIENT ESTIMATES FOR A NONLINEAR PARABOLIC EQUATION WITH DIRICHLET BOUNDARY CONDITION

XUENAN FU AND JIA-YONG WU

Abstract. In this paper, we prove Souplet-Zhang type gradient estimates for a nonlinear parabolic equation on smooth metric measure spaces with the compact boundary under the Dirichlet boundary condition when the Bakry-Emery Ricci tensor and the weighted mean curvature are both bounded below. As an application, we obtain a new Liouville type result for some space-time functions on such smooth metric measure spaces. These results generalize previous linear equations to a nonlinear case.

1. Introduction

In [33], Yau proved gradient estimates for harmonic functions on complete manifolds with the Ricci curvature bounded below. In [24], Souplet and Zhang generalized Yau’s gradient estimate to the heat equation by adding a necessary logarithmic correction term. Recently, Kunikawa and Sakurai [9] extended Yau and Souplet-Zhang type gradient estimates to the case of manifolds with the boundary under some Dirichlet boundary condition. Shortly later, H. Dung, N. Dung and Wu [6] generalized Kunikawa-Sakurai results to the $f$-Laplacian equation and the $f$-heat equation on smooth metric measure spaces with the compact boundary under some Dirichlet boundary condition; see also N. Dung and Wu [7] for further generalizations in this direction.

From above, we see that previous gradient estimates on manifolds with the boundary only focused on linear equations. In this paper we will investigate Souplet-Zhang type gradient estimates for a nonlinear parabolic equation on smooth metric measure spaces with the compact boundary under some Dirichlet boundary condition.

Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold, and let $f$ be a smooth potential function on $M$. The triple $(M, g, e^{-f}dv_g)$ is called a smooth metric measure space, where $dv_g$ is the Riemannian volume element of metric $g$ and $e^{-f}dv_g$ is the weighted volume element. On $(M, g, e^{-f}dv_g)$, Bakry and Emery [11] introduced the Bakry-Emery Ricci tensor

$$\text{Ric}_f := \text{Ric} + \text{Hess} f,$$

where Ric is the Ricci tensor of $(M, g)$ and Hess is the Hessian with respect to metric $g$. The Bakry-Emery Ricci tensor is a natural generalization of Ricci curvature on Riemannian manifolds, which plays an important role in smooth metric measure spaces, see for example [11], [27], [28], [30], [31], [32] and references therein. In particular, a smooth metric measure space satisfying

$$\text{Ric}_f = \lambda g$$

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for some $\lambda \in \mathbb{R}$ is called a gradient Ricci soliton, which can be considered as a natural generalization of an Einstein manifold. The gradient Ricci soliton is called shrinking, steady, or expanding, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Gradient Ricci solitons play important roles in the Ricci flow and Perelman’s resolution of the Poincaré conjecture, see for example [2], [8], [16], [17], [18] and references therein. Our interest of this paper in the Bakry-Emery Ricci tensor is largely due to a nonlinear parabolic equation related to gradient Ricci solitons.

On smooth metric measure space $(M, g, e^{-f} dv_g)$, the $f$-Laplacian (also called the weighted Laplacian or the Witten Laplacian) is defined by

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

which is usually linked with the Bakry-Emery Ricci tensor by the following generalized Bochner-Weitzenböck formula

$$\Delta_f |\nabla u|^2 = 2|\text{Hess } u|^2 + 2\langle \nabla \Delta_f u, \nabla u \rangle + 2\text{Ric}_f (\nabla u, \nabla u)$$

for any $u \in C^\infty(M)$. When $\text{Ric}_f$ is bounded below and $f$ is assumed to some restriction, there have been a lot of work in this direction; see for example [14], [15], [27], [28], [31] and [32], etc. On $(M, g, e^{-f} dv_g)$ with the compact boundary $\partial M$, the associated $f$-mean curvature (also called weighted mean curvature) is defined by

$$H_f := H - \nabla f \cdot \nu,$$

where $\nu$ is the unit outer normal vector to $\partial M$ and $H$ is the mean curvature of $\partial M$ with respect to $\nu$. When $f$ is constant, the above notations all return to the manifold case.

We now study Souplet-Zhang gradient estimates for positive solutions to the nonlinear parabolic equation

(1.1) \[ u_t = \Delta_f u + au \ln u, \]

where $a \in \mathbb{R}$, on $(M, g, e^{-f} dv)$ with compact boundary $\partial M$ under some Dirichlet boundary condition. It is known that all solutions to its Cauchy problem exist for all time. Equation (1.1) is closely related to gradient Ricci solitons and weighted log-Sobolev constants; see for example [12] and [29] for detailed explanations. By improving the argument of [6] especially for the boundary case, we prove Souplet-Zhang type gradient estimates for positive bounded solutions to the equation (1.1) without any assumption on the potential function $f$.

**Theorem 1.1.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional smooth metric measure space with the compact boundary. Assume that

$$\text{Ric}_f \geq -(n-1)K \quad \text{and} \quad H_f \geq -L$$

for some non-negative constants $K$ and $L$. Let $0 < u \leq B$ for some positive constant $B$ be a solution to (1.1) in

$$Q_{R,T}(\partial M) := B_R(\partial M) \times [-T,0] \subset M \times (-\infty, \infty),$$

where $B_R(\partial M) := \{x | d(x, \partial M) < R \}$ and $T > 0$. If $u$ satisfies the Dirichlet boundary condition (that is, $u(x,t)|_{\partial M}$ is constant for each time slice $t \in [-T,0]$),

$$u_{\nu} \geq 0 \quad \text{and} \quad u_t \leq au \ln u$$
over $\partial M \times [-T, 0]$, then there exists a constant $c(n)$ depending on $n$ such that
\[\sup_{Q_{R/2,T/2}(\partial M)} \frac{|\nabla u|}{u} \leq c(n) \left( \frac{\sqrt{D} + 1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} + L + \sqrt{E} \right) \sqrt{1 + \ln \frac{B}{u}},\]
where $D := 1 + \ln B - \ln(\inf_{Q_{R,T}(\partial M)} u)$ and $E := \max\{(n-1)K + \frac{4}{3}, 0\} + \max\{\frac{4}{3}(1 + \ln B), 0\}$.

**Remark 1.2.** When $a = 0$ and $L = 0$, the theorem returns to linear cases [9] and [6]. Notice that our weighted mean curvature assumption here could be bounded below by a non-negative constant (not just non-negative) and hence it indicates that our result is suitable to a little more general setting.

In Theorem 1.1, $\text{Ric}_f \geq -(n-1)K$ means that the infimum of $\text{Ric}_f$ on the unit tangent bundle in the interior of $M$ is more than $-(n-1)K$, while $H_f \geq -L$ means that boundary $\partial M$ has some weak convex property. In previous works, Kunikawa and Sakurai [9] proved Souplet-Zhang type gradient estimates for the heat equations under some Dirichlet boundary condition; H. Dung, N. Dung and Wu [6] generalized their result to the $f$-heat equation; N. Dung and Wu [7] further studied the case of the $f$-heat equation. Now we further generalize these linear cases to a nonlinear parabolic equation.

The second author [29] ever proved that Souplet-Zhang gradient estimates for positive bounded solutions to equation (1.1) only hold for radius $R \geq 2$ of a geodesic ball on non-compact manifolds without boundary, but the gradient estimates in Theorem 1.1 could hold for any radius $R > 0$. Because in the proof of our case we may apply a new weighted Laplacian comparison on neighborhoods of the boundary (see Theorem 2.1 in Section 2) instead of the Wei-Wylie comparison [27].

The proof of Theorem 1.1 employs the arguments of [9], [6] and [29]. The outline of the proof is as follows. In the interior of the manifold, as in [29], we essentially use the standard Souplet-Zhang argument for the nonlinear parabolic equation. On the boundary of the manifold, adapting arguments of [9] and [6], we apply a derivative equality (see Proposition 2.3 in Section 2) to prove the desired estimate. Compared with the previous proof, here we need to carefully deal with an extra nonlinear term.

Finally we would like to mention that there exist some geometric results on manifolds with the Neumann boundary condition, e.g. [3], [25], [10] and [19].

As a consequence of Theorem 1.1, we get the following Liouville type result.

**Corollary 1.3.** Let $(M, g, e^{-f}dv)$ be a complete smooth metric measure space with compact boundary satisfying
\[\text{Ric}_f \geq 0 \quad \text{and} \quad H_f \geq 0.\]

Let $1 \leq u \leq B$ for some constant $B$ be an ancient solution to (1.1) satisfying $a \leq 0$ with the Dirichlet boundary condition. If
\[u_\nu \geq 0 \quad \text{and} \quad u_t \leq au \ln u\]
over $\partial M \times (-\infty, 0]$, then $u$ is constant if $a = 0$; $u(t) = \exp\{ce^{at}\}$ for some constant $c \leq 0$ if $a < 0$.

The rest of this paper is organized as follows. In Section 2 we will list some basic results about smooth metric measure spaces with the compact boundary, which will be used in our proof of Theorem 1.1 In particular, we introduce the weighted Laplacian comparison,
the derivative equality, the Bochner type formula and the space-time cut-off function. In Section 3, we will adopt arguments of [29] and [9] to prove Theorem 1.1. In the end we will apply Theorem 1.1 to prove Corollary 1.3.

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2. Background

In this section, we mainly recall some basic results about smooth metric measure spaces with the compact boundary, which will be used in the proof of our result. We refer the interested reader to [21, 22, 23] for more results. On an $n$-dimensional smooth metric measure space $(M^n, g, e^{-f} dv)$ with the boundary $\partial M$, for any point $x \in M$, we let

$$\rho(x) = \rho_{\partial M}(x) = d(x, \partial M),$$

denote the distance function from the boundary $\partial M$. From the argument of [21], we may assume that distance function $\rho$ is smooth outside of the cut locus for the boundary $\text{Cut}(\partial M)$. In [26], Wang, Zhang and Zhou proved weighted Laplacian comparisons for the distance function on smooth metric measure spaces with the boundary under some assumptions (see also [23]). Later, Sakurai [22] proved a general comparison result, which plays an important role in the proof of Theorem 1.1.

Theorem 2.1. Let $(M^n, g, e^{-f} dv)$ be an $n$-dimensional complete smooth metric measure space with the compact boundary $\partial M$. Assume that

$$\text{Ric}_f \geq -(n - 1)K \quad \text{and} \quad \text{H}_f \geq -L$$

for some constants $K \geq 0$ and $L \in \mathbb{R}$. Then

$$\Delta_f \rho(x) \leq (n - 1)KR + L$$

for all $x \in B_R(\partial M)$ outside of $\text{Cut}(\partial M)$, where $B_R(\partial M) := \{x|d(x, \partial M) < R\}$.

Remark 2.2. This comparison result was used in the proof of gradient estimate for linear equations in [6] and [7]. We would like to point out that this comparison theorem holds for all $R > 0$, which is quite different from the Wei-Wylie comparison on smooth metric measure spaces without boundary. Indeed, the Wei-Wylie comparison theorem requires the radius of geodesic ball $R \geq R_0$ for some constant $R_0 > 0$; see Theorem 3.1 of [27].

Next, we recall a useful derivative equality, which can be derived from the proof of the weighted Reilly formula in [13]; see also (24) in Appendix of [4]. The present derivative equality can be regarded as a weighted version of the classical setting [20] and will be used in the proof of gradient estimate for the boundary case.

Proposition 2.3. Let $(M^n, g, e^{-f} dv)$ be an $n$-dimensional complete smooth metric measure space with the compact boundary $\partial M$. For any $u \in C^\infty(M)$,

$$\frac{1}{2} (|\nabla u|^2)_\nu = u_\nu [\Delta_f u - \Delta_{\partial M, f}(u|_{\partial M}) - \text{H}_f u_\nu] + g_{\partial M}(\nabla_{\partial M}(u|_{\partial M}), \nabla_{\partial M} u_\nu) - \Pi(\nabla_{\partial M}(u|_{\partial M}), \nabla_{\partial M}(u|_{\partial M})).$$
where \( \nu \) is the outer unit normal vector to \( \partial M \), and \( \Pi \) is the second fundamental form of \( \partial M \) with respect to \( \nu \).

Meanwhile, we need an important Bochner type formula for equation (1.1) in the proof of our result, which was similarly discussed in \([5]\). Let \( 0 < u \leq B \) for some positive constant \( B \) be a solution to (1.1) in \( Q_{R,T}(\partial M) \). Consider the function

\[
h(x,t) := \sqrt{1 + \ln u(x,t)} = \sqrt{\ln A} \frac{u(x,t)}{u(x,t)}
\]

in \( Q_{R,T}(\partial M) \). Obviously,

\[
A = Be \quad \text{and} \quad h(x,t) \geq 1.
\]

As in \([5]\), we have the following Bochner type formula, which holds on smooth metric measure spaces without any assumption on \( f \).

**Lemma 2.4.** Under the same assumptions of Theorem 1.1, for any \( (x,t) \in Q_{R,T}(\partial M) \), the function \( w = |\nabla h|^2 \) satisfies

\[
\Delta_fw - w_t \geq 2(2h - h^{-1})\langle \nabla w, \nabla h \rangle + 2(2 + h^{-2})w^2
\]

\[
- \left[ \max \{ 2(n-1)K + a, 0 \} \right] w,
\]

where \( A := Be \).

**Proof of Lemma 2.4** The proof is essentially the same as Lemma 2.1 in \([5]\) and we include it for the sake of completeness. Since \( u = Ae^{-h^2} \), we compute that

\[
u_t = -2Ahe^{-h^2}h_t, \quad \nabla u = -2Ahe^{-h^2}\nabla h
\]

and

\[
\Delta_fu = -2Ahe^{-h^2}\Delta_fh - 2A(1 - 2h^2)|\nabla h|^2e^{-h^2}.
\]

By (1.1), we get that

\[
-2Ahe^{-h^2}h_t = u_t = \Delta_fu + au\ln u
\]

\[
= -2Ahe^{-h^2}\Delta_fh - 2A(1 - 2h^2)|\nabla h|^2e^{-h^2} - aA(h^2 - \ln A)e^{-h^2},
\]

which is equivalent to

\[
h_t = \Delta_fh + (h^{-1} - 2h)|\nabla h|^2 + \frac{a}{2} (h - \ln A \cdot h^{-1}).
\]

On the other hand, by the generalized Bochner-Weitzenböck formula and the curvature assumption \( \text{Ric}_f \geq -(n-1)K \), we have

\[
\Delta_f|\nabla h|^2 = 2|\text{Hess} h|^2 + 2\langle \nabla \Delta_fh, \nabla h \rangle + 2\text{Ric}_f(\nabla h, \nabla h)
\]

\[
\geq 2\langle \nabla \Delta_fh, \nabla h \rangle - 2(n-1)K|\nabla h|^2
\]

and hence

\[
\Delta_fw - w_t \geq 2\langle \nabla \Delta_fh, \nabla h \rangle - 2(n-1)Kw - w_t.
\]
Combining this with (2.1) and using the definition of $w$, we get

$$
\Delta_f w - w_t \geq 2 \langle \nabla h_t, \nabla h \rangle + 2 \langle \nabla [(2h - h^{-1})w], \nabla h \rangle
$$

(2.2)

$$
+ a \langle \nabla (\ln A \cdot h^{-1} - h), \nabla h \rangle - 2(n - 1) Kw - w_t
$$

$$
= 2 \langle \nabla h_t, \nabla h \rangle + 2 \langle \nabla (2h - h^{-1}), \nabla h \rangle w + 2(2h - h^{-1}) \langle \nabla w, \nabla h \rangle
$$

$$
- a \ln A \cdot h^{-2}w - aw - 2(n - 1) Kw - w_t.
$$

Notice that

$$
2 \langle \nabla h_t, \nabla h \rangle = w_t \quad \text{and} \quad \nabla (2h - h^{-1}) = (2 + h^{-2}) \nabla h.
$$

Therefore (2.2) becomes

$$
\Delta_f w - w_t \geq 2(2 + h^{-2})w^2 + 2(2h - h^{-1}) \langle \nabla w, \nabla h \rangle - [2(n - 1)K + a] w - a \ln A \cdot h^{-2}w
$$

and the result follows.

To prove Theorem 1.1, we also need a space-time cut-off function, which was ever used in [10], [24], [9] and [6].

**Lemma 2.5.** Let $(M^n, g, e^{-f} dv)$ be an $n$-dimensional complete smooth metric measure space with the compact boundary $\partial M$. There exists a smooth cut-off function $\psi = \psi(\rho, t) \equiv \psi(\rho_{\partial M}(x), t)$ supported in $Q_{R,T}(\partial M)$ and a constant $C_\varepsilon > 0$ depending only on $0 < \varepsilon < 1$ such that

(i) $0 \leq \psi(\rho, t) \leq 1$ in $Q_{R,T}(\partial M)$ and $\psi(\rho, t) = 1$ in $Q_{R/2,T/2}(\partial M)$.

(ii) $\psi$ is decreasing as a radial function of parameter $r$.

(iii)

$$
\frac{|\psi_t|}{\psi^{1/2}} \leq \frac{C}{T}, \quad |\psi_\rho| \leq \frac{C_\varepsilon \psi^\varepsilon}{R} \quad \text{and} \quad |\psi_{\rho\rho}| \leq \frac{C_\varepsilon \psi^\varepsilon}{R^2},
$$

where $C > 0$ is a universal constant.

### 3. Souplet-Zhang gradient estimate

In this section, we will apply some results of Section 2 to prove Theorem 1.1 by adapting arguments of [9], [6] and [29]. It is worth to point out that we need carefully to discuss the boundary case.

**Proof of Theorem 1.1.** Let $w$ be the function in Lemma 2.4 and $\psi$ denote the cut-off function in Lemma 2.5. Our aim is to estimate $(\Delta_f - \partial_t)(\psi w)$ and carefully analyze the result at a space-time point where the function $\psi w$ attains its maximum. Assume that the space-time maximum of $\psi w$ is reached at some point $(x_1, t_1)$ in $Q_{R,T}(\partial M)$. We will prove the Souplet-Zhang gradient estimate according to two cases: $x_1 \notin \partial M$ and $x_1 \in \partial M$.

**Case 1:** If $x_1 \notin \partial M$, we may assume without loss of generality that $x_1 \notin \text{Cut}(\partial M)$ by the Calabi’s argument. Since at $(x_1, t_1)$, we have

$$
\Delta_f (\psi w) \leq 0, \quad (\psi w)_t \geq 0 \quad \text{and} \quad \nabla (\psi w) = 0.
$$
Using the above properties and Lemma 2.4 at \((x_1, t_1)\), we have

\[
0 \geq \Delta f(\psi w) - (\psi w)_t
= \Delta f \psi \cdot w + 2(\nabla w, \nabla \psi) + \psi(\Delta f w - w_t) - \psi_t w
\geq \Delta f \psi \cdot w - 2\left(\frac{\nabla \psi}{\psi}\right)^2 w - \psi_t \cdot w + 2(2h - h^{-1})\psi(\nabla w, \nabla h) + 2(2 + h^{-2})\psi w^2
- \left[ \max \{2(n-1)K + a, 0\} + h^{-2} \max \{a \ln A, 0\} \right] \psi w
= \Delta f \psi \cdot w - 2\left(\frac{\nabla \psi}{\psi}\right)^2 w - \psi_t \cdot w - 2(2h - h^{-1})(\nabla \psi, \nabla h) w + 2(2 + h^{-2})\psi w^2
- \left[ \max \{2(n-1)K + a, 0\} + h^{-2} \max \{a \ln A, 0\} \right] \psi w.
\]

This inequality can be written as

\[
2\psi w^2 \leq \frac{2h^2}{1 + 2h^2} \left(\frac{\nabla \psi}{\psi}\right)^2 w - \frac{2h(1 - 2h^2)}{1 + 2h^2}(\nabla \psi, \nabla h) w + \frac{h^2}{1 + 2h^2} \psi_t \cdot w - \frac{h^2}{1 + 2h^2} \Delta f \psi \cdot w
+ \left[ \frac{h^2}{1 + 2h^2} \max \{2(n-1)K + a, 0\} + \frac{1}{1 + 2h^2} \max \{a \ln A, 0\} \right] \psi w
\]

at \((x_1, t_1)\). Since \(h \geq 1\), and then

\[
0 < \frac{h^2}{1 + 2h^2} \leq \frac{1}{2} \quad \text{and} \quad 0 < \frac{1}{1 + 2h^2} \leq \frac{1}{3},
\]

so the above inequality can be further simplified as

\[
(3.1) \quad 2\psi w^2 \leq \frac{\left(\frac{\nabla \psi}{\psi}\right)^2}{\psi} w - \frac{2h(1 - 2h^2)}{1 + 2h^2}(\nabla \psi, \nabla h) w + \frac{1}{2} |\psi_t| \cdot w - \frac{h^2}{1 + 2h^2} \Delta f \psi \cdot w + E \psi w
\]

at \((x_1, t_1)\), where \(E := \max \{(n-1)K + \frac{a}{2}, 0\} + \max \{\frac{a}{3} \ln A, 0\}\).

Below we will apply Lemma 2.5 to estimate upper bounds for each term of the right-hand side of (3.1). We remark that for any real numbers \(M\) and \(N\), the Young’s inequality

\[
C_1C_2 \leq \frac{|C_1|^p}{p} + \frac{|C_2|^q}{q}, \quad \forall \ p, q > 0 \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

where \(C_1, C_2 \in \mathbb{R}\), will be repeatedly used in the following estimates. We let \(c\) denote a constant may depending on \(n\) whose value may change from line to line. First,

\[
(3.2) \quad \frac{\left(\frac{\nabla \psi}{\psi}\right)^2}{\psi} w = \psi^{1/2} w \cdot \frac{\left(\frac{\nabla \psi}{\psi^{3/2}}\right)^2}{\psi^{3/2}}
\leq \frac{1}{5} \psi w^2 + c \left(\frac{\left(\frac{\nabla \psi}{\psi^{3/2}}\right)^2}{\psi^{3/2}}\right)^2
\leq \frac{1}{5} \psi w^2 + \frac{c}{R^4}.
\]
Second,

\[-\frac{2h(1-2h^2)}{1+2h^2}(\nabla \psi, \nabla h)w \leq 2h|\nabla \psi||\nabla h|w\]

\[= 2h|\nabla \psi|^{3/4}(\psi^2)^{3/4}\]

\[\leq \frac{1}{5}\psi w^2 + h^4|\nabla \psi|^3
\leq \frac{1}{5}\psi w^2 + \frac{cD^2}{R^4},\]

where \(D := \ln A - \ln(\inf_{\Omega \cup \partial M} u)\). Third,

\[
\frac{1}{2}|\psi_t| \cdot w = \frac{1}{2} \psi^{1/2} \cdot \psi^{1/2} w\]

\[\leq \frac{1}{5}\psi w^2 + c\frac{|\psi_t|^2}{\psi}\]

\[\leq \frac{1}{5}\psi w^2 + \frac{c}{T^2}.
\]

Fourth, we will apply Theorem 2.1 to the following term

\[-\frac{h^2}{1+2h^2}\Delta f \psi \cdot w = - \frac{h^2}{1+2h^2}(\psi_\rho \Delta f \rho + \psi_{\rho\rho} |\nabla \rho|^2)w\]

\[\leq \frac{h^2}{1+2h^2}(|\psi_\rho|(n-1)KR + |\psi_\rho|L + |\psi_{\rho\rho}|)w\]

\[\leq \frac{|\psi_{\rho\rho}|}{2\psi^{1/2}} \psi^{1/2} w + \frac{(n-1)KR + L}{2} \psi^{1/2} \frac{|\psi_\rho|}{\psi^{1/2}} w\]

\[\leq \frac{1}{5}\psi w^2 + c\left(\frac{|\psi_{\rho\rho}|}{\psi^{1/2}}\right)^2 + (K^2 R^2 + L^2) \left(\frac{|\psi_\rho|}{\psi^{1/2}}\right)^2\]

\[\leq \frac{1}{5}\psi w^2 + \frac{c}{R^4} + cK^2 + c\frac{L^2}{R^2}\]

\[\leq \frac{1}{5}\psi w^2 + \frac{c}{R^4} + cK^2 + cL^4.\]

Fifth,

\[E \psi w \leq \frac{1}{5}\psi w^2 + cE^2.\]

Substituting (3.2)-(3.6) into the right hand side of (3.1), we have

\[\psi w^2 \leq c \left(\frac{D^2 + 1}{R^4} + L^4 + \frac{1}{T^2} + K^2 + E^2\right)\]
at \((x_1, t_1)\). Then for all \((x, t) \in Q_{R/2,T/2}(\partial M)\), we have \(\psi(x, t) \equiv 1\) and therefore
\[
\begin{align*}
w^2(x, t) &= \psi(x, t)w^2(x, t) \\
&\leq \psi(x_1, t_1)w^2(x_1, t_1) \\
&\leq c \left( \frac{D^2 + 1}{R^4} + L^4 + \frac{1}{T^2} + K^2 + E^2 \right).
\end{align*}
\]
Since \(w = |\nabla h|^2\), then
\[
|\nabla h|(x, t) \leq c \left( \frac{\sqrt{D} + 1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} + L + \sqrt{E} \right)
\]
for all \((x, t) \in Q_{R/2,T/2}(\partial M)\). The desired gradient estimate follows by using the equality
\[
|\nabla h| = \frac{|\nabla u|}{2u \log(A/u)}.
\]

**Case 2:** If \(x_1 \in \partial M\), the gradient estimate of Theorem 1.1 still holds. Indeed at the maximum point \((x_1, t_1)\) of \(\psi w\), we have \((\psi w)_\nu \geq 0\) and hence
\[
\psi_\nu w + \psi w_\nu = \psi w_\nu \geq 0,
\]
which implies \(w_\nu \geq 0\).

Since \(w = |\nabla h|^2\), and \(h = \sqrt{\log(A/u)}\) satisfies the Dirichlet boundary condition, by Proposition 2.3 we have
\[
0 \leq w_\nu = (|\nabla h|^2)_\nu = 2h_\nu(\Delta fh - H_fh_\nu).
\]
Notice that since \(u\) satisfies the Dirichlet boundary condition, then \(|\nabla u| = u_\nu\) and hence
\[
(3.8) \quad h_\nu = \frac{-u_\nu}{2u \log(A/u)} = \frac{-|\nabla u|}{2u \log(A/u)} = -w^{1/2}.
\]
We also have
\[
\Delta fh = \text{div} \left( \frac{-\nabla u}{2u \log(A/u)} \right) - \langle \nabla f, \nabla h \rangle
\]
\[
= \frac{-\Delta u}{2u \log(A/u)} - \frac{1}{2} \left\langle \nabla u, \nabla \left( \frac{1}{u \log(A/u)} \right) \right\rangle + \left\langle \nabla f, \frac{\nabla u}{2u \log(A/u)} \right\rangle
\]
\[
= \frac{-\Delta fh}{2u \log(A/u)} + \frac{1}{2} \left( \frac{|\nabla u|^2}{u^2 \log(A/u)} - \frac{|\nabla u|^2}{2u^2(\log(A/u))^{3/2}} \right)
\]
\[
= \frac{-u_t}{2uh} + \frac{a \ln u}{2h} + \left( 2h - \frac{1}{h} \right) w.
\]
Substituting (3.8) and (3.9) into (3.7) yields
\[
-\frac{u_t}{2uh} + \frac{a \ln u}{2h} + \left( 2h - \frac{1}{h} \right) w + Hfw^{1/2} \leq 0
\]
at \((x_1, t_1)\). Since \(\partial_t u \leq au \ln u\) over \(\partial M \times [-T, 0]\) by our theorem assumption, then

\[-\frac{u_t}{2uh} + \frac{a \ln u}{2h} \geq 0\]

and hence

\[(2h - \frac{1}{h})w + Hfw^{1/2} \leq 0\]

at \((x_1, t_1)\). Since \(h \geq 1\), then \((2h - h^{-1}) \geq 1\) and therefore we get

\[w + Hfw^{1/2} \leq 0\]

at \((x_1, t_1)\). This implies

\[w(x_1, t_1) = 0 \quad \text{or} \quad w^{1/2}(x_1, t_1) \leq L,\]

where we used the theorem assumption \(H_f \geq -L\). It means that

\[\psi w \equiv 0 \quad \text{or} \quad (\psi w)(x_1, t_1) \leq L^2\]

on \(Q_{R,T}(\partial M)\). The former indicates that \(u\) is constant and the conclusion follows; the latter gives that for all \((x, t) \in Q_{R/2,T/2}(\partial M), \psi(x, t) \equiv 1\) and

\[|\nabla h|^2(x, t) = w(x, t) = \psi(x, t)w(x, t) \leq \psi(x_1, t_1)w(x_1, t_1) \leq L^2,\]

which also implies the conclusion by using

\[|\nabla h| = \frac{|\nabla u|}{2u \sqrt{\ln(A/u)}}.\]

□

In the end, we apply Theorem 1.1 to prove Corollary 1.3.

**Proof of Corollary 1.3.** We assume that \(\text{Ric}_f \geq 0\) and \(H_f \geq 0\) on smooth metric measure space \((M, g, e^{-f}dv)\) with the compact boundary. Let \(1 \leq u \leq B\) be an ancient solution to (1.1) satisfying \(a \leq 0\) in \(Q_{R,T}(\partial M)\) with the Dirichlet boundary condition. If \(u_t \geq 0\) and \(u_t \leq au \ln u\) over \(\partial M \times (-\infty, 0]\), then by Theorem 1.1, there exists a constant \(c(n)\) depending on \(n\) such that

\[
\sup_{Q_{R/2,T/2}(\partial M)} \frac{|\nabla u|}{u} \leq c(n) \left( \frac{\sqrt{1 + \ln B} + 1}{R} + \frac{1}{\sqrt{T}} \right) \sqrt{1 + \ln B}.
\]

Letting \(R \to \infty\) and \(T \to \infty\), we obtain \(\nabla u = 0\) and hence \(u(x, t) = u(t)\) only depends on time parameter \(t\). Substituting \(u(t)\) into the equation (1.1) we get the following ordinary differential equation

\[u_t = au \ln u.\]

If \(a = 0\), we can get \(u_t = 0\) and \(u\) is constant. If \(a < 0\), we directly solve the above equation and obtain

\[u(t) = \exp\{ce^{at}\} \]
for some $c \in \mathbb{R}$. Moreover since $u$ is an ancient solution in $(-\infty, 0]$, we claim that $c \leq 0$. Indeed if $c > 0$ then $u = \exp\{ce^{at}\}$ is unbounded in $(-\infty, 0]$, which contradicts with our theorem assumption.

**References**

[1] D. Bakry and M. Émery, Diffusion hypercontractivitives, in Séminaire de Probabilités XIX, 1983/1984, in: Lecture Notes in Math. **1123**, Springer-Verlag, Berlin, 1985, 177-206.

[2] H.-D. Cao, Recent progress on Ricci solitons, in: Recent advances in geometric analysis, Adv. Lect. Math. (ALM), **11**, International Press, Somerville, 2010, 1-38.

[3] R. Chen, Neumann eigenvalue estimate on a compact Riemannian manifold, Proc. Amer. Math. Soc. **108** (1990), 961-970.

[4] X. Cheng, T. Mejia and D.-T. Zhou, Eigenvalue estimate and compactness for closed $f$-minimal surfaces, Pacific J. Math. **271** (2014), 347-367.

[5] H. T. Dung and N. T. Dung, Sharp gradient estimates for a heat equation in Riemannian manifolds, Proc. Amer. Math. Soc. **147** (2019), 5329-5338.

[6] H. T. Dung, N. T. Dung and J.-Y. Wu, Sharp gradient estimates on weighted manifolds with compact boundary, Commun. Pure Appl. Anal. (2021), doi:10.3934/cpaa.2021148.

[7] N. T. Dung and J.-Y. Wu, Gradient estimates for weighted harmonic function with Dirichlet boundary condition, Nonlinear Anal. **213**, (2021), 112498.

[8] R. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geometry, International Press, Boston, **2**, 1995, 7-136.

[9] K. Kunikawa and Y. Sakurai, Yau and Souplet-Zhang type gradient estimates on Riemannian manifolds with boundary under Dirichlet boundary condition, arXiv:2012.09374.

[10] P. Li and S.-T. Yau, On the parabolic kernel of the Schrodinger operator, Acta Math. **156** (1986), 153-201.

[11] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. **169** (2009), 903-991.

[12] L. Ma, Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds, J. Funct. Anal. **241** (2006), 374-382.

[13] L. Ma and S.-H. Du, Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians, C. R. Math. Acad. Sci. Paris **348** (2010), 1203-1206.

[14] O. Munteanu and J. Wang, Smooth metric measure spaces with nonnegative curvature, Comm. Anal. Geom. **19** (2011), 451-486.

[15] O. Munteanu and J. Wang, Analysis of weighted Laplacian and applications to Ricci solitons, Comm. Anal. Geom., **20** (2012), 55-94.

[16] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, (2002), arXiv:math.DG/0211159.

[17] G. Perelman, Ricci flow with surgery on three-manifolds, (2003), arXiv:math.DG/0303109.

[18] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, (2003), arXiv:math.DG/0307245.

[19] X. Ramos Olívè, Neumann Li-Yau gradient estimate under integral Ricci curvature bounds, Proc. Amer. Math. Soc. **147** (2019), 411-426.

[20] R.C. Reilly, Applications of the Hessian operator in a Riemannian manifold, Indiana Univ. Math. J. **26** (1977), 459-472.

[21] Y. Sakurai, Rigidity of manifolds with boundary under a lower Ricci curvature bound, Osaka J. Math. **54** (2017), 85-119.

[22] Y. Sakurai, Concentration of 1-Lipschitz functions on manifolds with boundary with Dirichlet boundary condition, preprint arXiv:1712.04212v4.

[23] Y. Sakurai, Rigidity of manifolds with boundary under a lower Bakry-Émery Ricci curvature bound, Tohoku Math. J. **71** (2019), 69-109.

[24] P. Souplet and Q S. Zhang, Sharp gradient estimate and Yau’s Liouville theorem for the heat equation on noncompact manifolds, Bull. London Math. Soc. **38** (2006), 1045-1053.
[25] J.-P. Wang, Global heat kernel estimates, Pacific J. Math. 178 (1997), 377-398.
[26] L.-F. Wang, Z.-Y. Zhang and Y.-J. Zhou, Comparison theorems on smooth metric measure spaces with boundary, Adv. Geom. 16 (2016), 401-411.
[27] G.-F. Wei and W. Wylie, Comparison geometry for the Bakry-Émery Ricci tensor, J. Diff. Geom. 83 (2009), 377-405.
[28] J.-Y. Wu, Upper bounds on the first eigenvalue for a diffusion operator via Bakry-Émery Ricci curvature II, Results Math. 63 (2013), 1079-1094.
[29] J.-Y. Wu, Elliptic gradient estimates for a nonlinear heat equation and applications, Nonlinear Analysis: TMA 151 (2017), 1-17.
[30] J.-Y. Wu, Gradient estimates for a nonlinear parabolic equation and Liouville theorems, Manuscr. Math. 159 (2019), 511-547.
[31] J.-Y. Wu and P. Wu, Heat kernel on smooth metric measure spaces with nonnegative curvature, Math. Ann. 362 (2015), 717-742.
[32] J.-Y. Wu and P. Wu, Heat kernel on smooth metric measure spaces and applications, Math. Ann. 365 (2016), 309-344.
[33] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975), 201-228.

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