Abstract

We consider (effective) Quantum General Relativity coupled to the Standard Model and study its transversality. To this end, we provide all propagator and three-valent vertex Feynman rules. Then we examine the longitudinal, identical and transversal projection tensors for the de Donder gauge fixing and the Lorenz gauge fixing. In particular, we recall several identities from Quantum Yang–Mills theory and introduce their counterparts in (effective) Quantum General Relativity: This includes decompositions of the longitudinal projection tensors as well as expressions of the corresponding propagators in terms of their transversal structure, together with longitudinal contraction identities for all three-valent vertex Feynman rules. In addition, we introduce the notion of an optimal gauge fixing as the natural choice for a given gauge theory: In particular, we find that this is the de Donder gauge fixing in General Relativity and the Lorenz gauge fixing in Yang–Mills theory.

1 Introduction

The quantization of gauge theories introduces several new challenges: Most notably the necessity to choose a gauge fixing in order to calculate the propagator. This is due to the fact that the equations of motion of the gauge field determine only the evolution of its horizontal (i.e. physical) degrees of freedom. The vertical (i.e. gauge) degrees of freedom are unconstrained due to the gauge invariance of the theory. This is an obstruction to calculate the propagator of the gauge field, as it is given as the inverse of the differential operator of the corresponding quadratic monomial. Thus, to be able to invert this differential operator, we need to add a gauge fixing term such that said differential operator obtains full rank and becomes invertible. This choice has important consequences for the corresponding Quantum Field Theory, as it introduces the notion of longitudinal and transversal degrees of freedom. These degrees of freedom, which characterize propagating gauge fields, become especially important when Feynman integrals are considered: This is due to the fact that only the transversal degrees of freedom are physical. Thus, physically consistent theories should be such that unphysical (longitudinal) degrees of freedom are suppressed in scattering processes. It turns out, however, that for Feynman integrals this requires the introduction of ghost fields together with their corresponding Feynman integrals. Ghost fields (at least in the Faddeev–Popov construction [1]) are constructed to satisfy residual gauge transformations as equations of motion. With this setup it has been shown that Quantum Yang–Mills theory is indeed transversal [2, 3, 4, 5, 6]. In this article, we want to discuss the situation of (effective) Quantum General Relativity, possibly coupled to matter from the Standard Model. To this end, we consider (effective) Quantum General Relativity with a de Donder gauge fixing (QGR) and Quantum Yang–Mills theory with a Lorenz gauge fixing (QYM)
together with a vector of complex scalar fields and a vector of spinor fields, both subjected to the action of the gauge group.

More precisely, we consider (effective) Quantum General Relativity coupled to the Standard Model, given via the following Lagrange density:

$$\mathcal{L}_{QGR-SM} := \mathcal{L}_{QGR} + \mathcal{L}_{QYM} + \mathcal{L}_{Matter}$$  \hspace{1cm} (1)

Here, $\mathcal{L}_{QGR}$ is the Lagrange density for (effective) Quantum General Relativity, $\mathcal{L}_{QYM}$ is the Lagrange density for Quantum Yang–Mills theory and finally $\mathcal{L}_{Matter}$ is the Lagrange density for the matter fields in the Standard Model.

Specifically, the setup for (effective) Quantum General Relativity is given as follows:

$$\mathcal{L}_{QGR} := \mathcal{L}_{GR} + \mathcal{L}_{GR-GF-Ghost}$$  \hspace{1cm} (2a)

with the Einstein–Hilbert Lagrange density

$$\mathcal{L}_{GR} := -\frac{1}{2 \kappa^2} R dV_g$$  \hspace{1cm} (2b)

and the symmetric gauge fixing and ghost Lagrange density, as derived in \[7\],

$$\mathcal{L}_{GR-GF-Ghost} := \left( -\frac{1}{4 \kappa^2 \zeta} \eta^{\mu \nu} dD_{\mu}^{(1)} dD_{\nu}^{(1)} + \frac{1}{2 \zeta} \eta^{\mu \nu} \left( \partial_\mu C^{\rho} - \partial_\rho C^{\mu} \right) dV_\eta ight)$$

\begin{align*}
+ & \frac{1}{2} \eta^{\mu \nu} C^{\sigma} \left( \frac{1}{2} \partial_\mu \left( \Gamma^{\sigma}_{\mu \lambda} C^{\lambda} \right) - \partial_\mu \left( \Gamma^{\sigma}_{\lambda \mu} C^{\lambda} \right) \right) dV_\eta \\
- & \frac{1}{2} \eta^{\mu \nu} \left( \frac{1}{2} \partial_\rho \left( \Gamma^{\sigma}_{\mu \lambda} C^{\lambda} \right) - \partial_\mu \left( \Gamma^{\sigma}_{\lambda \mu} C^{\lambda} \right) \right) C^{\rho} dV_\eta \\
& \frac{\kappa^2 \zeta}{8} \eta^{\mu \nu} \left( C^{\sigma} \left( \partial_\sigma C^{\rho} \right) \right) dV_\eta 
\end{align*}  \hspace{1cm} (2c)

We remark that we consider its linearization with respect to the metric decomposition $g_{\mu \nu} \equiv \eta_{\mu \nu} + \kappa h_{\mu \nu}$, where $h_{\mu \nu}$ is the graviton field and $\kappa := \sqrt{\kappa}$ the graviton coupling constant (with $\kappa := 8 \pi G$ the Einstein gravitational constant). In addition, $R := g^{\rho \sigma} R_{\rho \mu \sigma}$ is the Ricci scalar (with $R^\rho_{\sigma \mu \nu} := \partial_\mu \Gamma^\rho_{\nu \sigma} - \partial_\nu \Gamma^\rho_{\mu \sigma} + \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\nu \sigma} - \Gamma^\rho_{\nu \lambda} \Gamma^\lambda_{\mu \sigma}$ the Riemann tensor). Furthermore, $dV_g := \sqrt{- \text{Det} (g)} dt \wedge dx \wedge dy \wedge dz$ denotes the Riemannian volume form and $dV_\eta := dt \wedge dx \wedge dy \wedge dz$ the Minkowskian volume form. In addition, $dD_{\mu}^{(1)} := \eta^{\rho \sigma} \Gamma_{\rho \mu \sigma}$ denotes the linearized de Donder gauge fixing functional and $\zeta$ is the gauge fixing parameter. Finally, $C^{\mu}$ and $C^{\mu \nu}$ are the graviton-ghost and graviton-antighost, respectively.

In addition, the setup for Quantum Yang–Mills theory is given as follows:

$$\mathcal{L}_{QYM} := \mathcal{L}_{YM} + \mathcal{L}_{YM-GF-Ghost}$$  \hspace{1cm} (3a)

with the Yang–Mills Lagrange density

$$\mathcal{L}_{YM} := -\frac{1}{4 g^2} \delta_{ab} g^{\mu \nu} g^{\rho \sigma} A^a_{\mu b} A^b_{\nu \rho} dV_g$$  \hspace{1cm} (3b)

and the symmetric gauge fixing and ghost Lagrange density, as derived in \[7\] cf. \[8\],

$$\mathcal{L}_{YM-GF-Ghost} := \frac{1}{\zeta} \left( -\frac{1}{2 g^2} \delta_{ab} L^a L^b + g^{\mu \nu} \left( \partial_\mu \tau_a \right) \left( \partial_\nu \bar{c}^a \right) \right) dV_g$$

\begin{align*}
+ & \frac{g}{2} \left( \partial_\mu \tau_a \right) \left( g c^b A^a_{\mu b} - \left( \partial_\mu \bar{c}^b \right) A^a_{\nu b} \right) dV_g \\
& \frac{g^2 \kappa}{16} f^a_{bc} f^{a}_{d e} \bar{c}^b c^e d^e dV_g .
\end{align*}  \hspace{1cm} (3c)
Then, we introduce the corresponding counterpart of QGR in Definition 3.14, which is given by
and

Here, \( \Phi \) and \( \Psi \) denote the respective vectors of complex scalar fields and spinor fields, with
they are given as follows:
Finally, the setup for matter in the Standard Model is represented via a vector of complex scalar
boson, respectively.

\( L^\alpha := g_\mu^\nu (\nabla^\mu \Phi)^\dagger (\nabla^\nu \Phi) + \sum_{i \in I_\Phi} \frac{\alpha_i}{2} (\Phi^\dagger \Phi)^i + \bar{\Psi} \left( \sqrt{\Sigma^M} - m_\Psi \right) \Psi \) \quad dV_g \quad (4)

Here, \( \Phi \) and \( \Psi \) denote the respective vectors of complex scalar fields and spinor fields, with

\( \bar{\Phi} := (\gamma_0 \Psi)^\dagger \), where \( \gamma_0 \) denotes the corresponding diagonal
denotes the diagonal matrix with all fermion masses as entries. We

\( \Sigma^M \) and \( \bar{\Sigma}_M \) are the gauge ghost and gauge
transverse structures, that is the sets of longitudinal,
and \( \bar{\gamma}_m \) the Minkowski space Dirac matrix. Moreover, \( I_{\Phi} \) denotes

\( \alpha_2 := -m_\Psi \). Finally, \( m_\Psi \) denotes the diagonal matrix with all fermion masses as entries. We

This article is organized as follows: We start by displaying the QGR-SM Feynman rules for
all propagators and all three-valent vertices in Section 2. Then we proceed by studying
their complicated and nontrivial counterparts in QGR. Specifically, our results are as follows: We

and recalling known and obvious identities in QYM. Then we proceed in Subsection 3.2 by studying

They are optimal in Corollaries 3.7 and 3.18, which highlights their special
roles. Then we present the respective transversal structures, that is the sets of longitudinal,
and transversal projection tensors introduced in Definition 3.14 together with their
corresponding metrics: First we recall the situation of QYM in Definition 3.3, which is given by

\( T_{\mu} := I_{\mu} - L_{\mu} \).

Then, we introduce the corresponding counterpart of QGR in Definition 3.14 which is given by

\( \mathcal{T}_{QGR} := \{ \mathcal{L}, \mathcal{I}, \mathcal{T} \} \) \quad (7)

We remark that \( F^a_\mu := g (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g^2 f^a_{bc} A^b_\mu A^c_\nu \) is the local curvature form of the gauge
boson \( A^a_\mu \). Furthermore, \( dV_g := \sqrt{-\det(g)} \, dt \wedge dz \wedge dy \wedge dz \) denotes again the Riemannian
volume form. In addition, \( L^a := g g^\mu_\nu (\nabla^T_M A^a_\mu) \equiv 0 \) denotes the covariant Lorenz gauge fixing
functional and \( \xi \) is the gauge fixing parameter. Finally, \( \alpha^a \) and \( \tau_a \) are the gauge ghost and gauge
antighost, respectively.

Finally, the setup for matter in the Standard Model is represented via a vector of complex scalar
fields and a vector of spinor fields, both subjected to the action of the gauge group. Specifically,
they are given as follows:

\( \mathcal{L}_{\text{Matter}} := \left( g^\mu_\nu \left( \nabla^H_\mu \Phi \right)^\dagger \left( \nabla^H_\nu \Phi \right) + \sum_{i \in I_\Phi} \frac{\alpha_i}{2} (\Phi^\dagger \Phi)^i + \bar{\Psi} \left( \sqrt{\Sigma^M} - m_\Psi \right) \Psi \right) \, dV_g \quad (4) \)

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all propagators and all three-valent vertices in Section 2. Then we proceed by studying their
local longitudinal and transversal properties in Section 3. More precisely, we start in Subsection 3.1 by
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\( \bar{\gamma}_m \) and the Minkowski space Dirac matrix. Moreover, \( I_\Phi \) denotes

\( \Sigma^M \) and \( \bar{\Sigma}_M \) denote the respective covariant derivatives, where \( \bar{\Sigma}_a \) and \( \Sigma_a \) denote

\( \epsilon^{\mu \nu \alpha} \) is the inverse vielbein and \( \gamma_m \) the Minkowski space Dirac matrix. Moreover, \( I_\Phi \) denotes

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corresponding metrics: First we recall the situation of QYM in Definition 3.3, which is given by

\( \mathcal{T}_{QYM} := \{ L, I, T \} \) \quad (5)

with

\( L^\nu_\mu := \frac{1}{p^\nu} p^\nu p_\mu \), \quad (6a)

\( I^\nu_\mu := \delta^\nu_\mu \) \quad (6b)

and

\( T^\nu_\mu := I^\nu_\mu - L^\nu_\mu \). \quad (6c)

Then, we introduce the corresponding counterpart of QGR in Definition 3.14 which is given by

\( \mathcal{T}_{QGR} := \{ \mathcal{L}, \mathcal{I}, \mathcal{T} \} \) \quad (7)

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with
\[
\mathcal{L}_{\mu\nu}^{\rho\sigma} := \frac{1}{2\mu^2} \left( \delta_{\rho}^{\mu} p^\sigma p_\nu + \delta_{\sigma}^{\mu} p^\rho p_\nu + \delta_{\sigma}^{\nu} p^\rho p_\mu + \delta_{\rho}^{\nu} p^\sigma p_\mu - 2\eta^{\rho\sigma} p_\mu p_\nu \right),
\]
(8a)
\[
\mathcal{I}_{\mu\nu}^{\rho\sigma} := \frac{1}{2} \left( \delta_{\rho}^{\mu} \delta_{\sigma}^{\nu} + \delta_{\sigma}^{\mu} \delta_{\rho}^{\nu} \right)
\]
(8b)
and
\[
\mathcal{T}_{\mu\nu}^{\rho\sigma} := \mathcal{I}_{\mu\nu}^{\rho\sigma} - \mathcal{L}_{\mu\nu}^{\rho\sigma}.
\]
(8c)
Thus, the transversal structure of QGR-SM is given via the union
\[
\mathcal{T}_{\text{QGR-SM}} := \mathcal{T}_{\text{QGR}} \cup \mathcal{T}_{\text{QYM}}
\]
and is acting on gravitons, gluons, photons, Z-bosons and W\(^\pm\)-bosons. Next we study the decomposition of the longitudinal projection tensor into the product of a gauge transformation together with the gauge fixing projection in Lemmata 3.5 and 3.16. In particular, we show that the provided longitudinal, identical and transversal projection tensors are indeed projectors in Propositions 3.6 and 3.17. Furthermore, we show that gauge transformations and the gauge fixing projections are eigentensors of the respective transversal structures in Corollaries 3.7 and 3.18. Thereafter, we study the action of the corresponding metrics on said tensors in Lemmata 3.8 and 3.19 and Corollaries 3.9 and 3.20. This allows us ultimately to simplify the gluon and graviton propagators as follows:

\[
\Phi(g) = -\frac{i p^2}{p^2 + i\varepsilon} \delta^{ab} \left( T_{\mu\nu} + \xi L_{\mu\nu} \right)
\]
(10)
and

\[
\Phi(g) = -\frac{2i p^2}{p^2 + i\varepsilon} \left( T_{\mu\nu\rho\sigma} + \zeta L_{\mu\nu\rho\sigma} \right).
\]
(11)
Furthermore, their corresponding ghost propagators then relate to them via a gauge fixing projection (denoted via \(l\) and \(L\), respectively, cf. Definitions 3.3 and 3.14):

\[
\Phi(\cdots) = \Phi(l \cdots)
\]
(12)
and

\[
\Phi(\cdots) = \Phi(L \cdots).
\]
(13)
These results can be found in Theorema 3.10 and 3.21 respectively. Additionally, we provide cancellation identities for the gluon and graviton vertex Feynman rules in Theorema 3.11 and 3.23 as well as for their corresponding couplings to matter from the Standard Model (SM) in Theorema 3.13 and 3.25. These identities have important consequences for the renormalization of QGR-SM. We have extensively discussed in [10, Subsection 3.3] possible solutions to a fundamental problem that frequently occurs in the renormalization of gauge theories: It may be that there exists divergent amplitudes without corresponding monomial in the Lagrange density to absorb the appearing divergences. This occurs famously in Quantum Electrodynamics with the three- and four-valent photon-interactions. Here, luckily, it turns out that the sum over all three-valent photon-interaction Feynman integrals adds up to zero due to the virtue of Furry’s theorem.\(^1\) Additionally, with even more luck, it turns out that the sum over all

\(^1\)We remark the generalization of Furry’s theorem to the coupling of (effective) Quantum General Relativity to Quantum Electrodynamics in [10, Theorem 3.55].
four-valent photon-interaction Feynman integrals is finite, despite the superficial degree of
divergence suggesting otherwise [11]. Similar problems do also occur at numerous places when
the coupling of (effective) Quantum General Relativity to the Standard Model is considered: In
particular, Feynman integrals with external matter particles attached together with gravitons
are superficially divergent to arbitrary valences. To this end, we suggest the application of [10,
Solution 3.39]: Here, we have suggested to absorb the divergences of Feynman integrals into the
symmetrized sum over trees, with gravitons as virtual particles. In particular, with reference to
Theorema [3.13] and [3.25], we argue that this a priori non-local operation can become effectively
local, if corresponding Slavnov–Taylor-like identities hold (with the corresponding four-valent
residue being zero) [1]. Finally, we conclude our investigations with a comment on the differences
of the two most prominent definitions of the graviton field: The metric decomposition and the
metric density decomposition of Goldberg [12] in Remark [3.22].

The present article integrates as follows into the present literature: Ultimately, we aim to prove
the renormalizability of (effective) Quantum General Relativity via generalized Slavnov–Taylor
identities in the sense of [13, 14]. More precisely, we aim to show them graphically via so-called
cancellation identities, cf. [15, 16, 17, 18, 19, 20]. These identities will then be implemented
on the algebra of Feynman graphs via a modified version of the Feynman graph cohomology
introduced in [21, 22]. In particular, we argue that the compatibility of cancellation identities
with renormalization is reflected in the well-definedness of the corresponding differential-graded
renormalization Hopf algebra, which will be introduced in [23]. Additionally, we argue in [23]
that this differential-graded renormalization Hopf algebra does also constitute the perturbative
version of BRST cohomology. BRST cohomology is a powerful tool to study the gauge fixing
and ghost Lagrange densities of gauge theories and has been studied for QGR-SM in [24].
Furthermore, we highlight the interesting connection between Feynman graph cohomology and
the Corolla polynomial, which constructs a relation between $\phi^3$ scalar field theory amplitudes
and Yang–Mills gauge theory amplitudes [21, 25, 26, 27].

\section{Explicit Feynman rules}

After these general results we additionally provide the concrete gravity-matter Feynman rules for
all propagators and three-valent vertices of (effective) Quantum General Relativity coupled to
the Standard Model. In this section and the section thereafter we use the symmetric (hermitian)
ghost Lagrange densities associated to the Lorenz and de Donder gauge fixing conditions, which
are given in Equations (2c) and (3c) and have been derived in [7, cf. 8]. In particular, we
highlight the benefit of significantly simpler cancellation identities for residues involving ghost
fields, which is the reason to choose them here, cf. Theorema [3.13] and [3.25].

\subsection{Gravity-matter propagators}

\begin{align}
\Phi ( \text{............} ) &= \frac{i}{p^2 - m^2 + i\epsilon} \tag{14} \\
\Phi ( \text{------} ) &= \frac{i(p + m)}{p^2 - m^2 + i\epsilon} \tag{15} \\
\Phi ( \text{----------} ) &= -\delta^{a_1 a_2} \frac{i}{p^2 - m^2 + i\epsilon} \left( \eta_{\rho_1 \rho_2} - \frac{1 - \xi}{p^2} p_{\rho_1} p_{\rho_2} \right) \tag{16} \\
\Phi ( \text{.........} ) &= \delta^{a_1 a_2} - \frac{\xi}{p^2 + i\epsilon} \tag{17}
\end{align}

\footnote{We remark that a similar situation also appears for gauge ghosts with the Faddeev–Popov ghost construction.}
\[ \Phi = -\frac{2i}{p^2 + i\varepsilon} \left( \eta_{\mu_1 \nu_2} \eta_{\mu_1 \nu_2} + \eta_{\mu_1 \nu_2} \eta_{\mu_1 \nu_2} - \eta_{\mu_1 \nu_1} \eta_{\mu_2 \nu_2} \right) \]

\[ \Phi = -\frac{2i \zeta}{p^2 + i\varepsilon} \eta_{\rho_1 \rho_2} \]

2.2 Gravity-matter vertices

\[ \Phi = -\frac{ig}{2} (q_1 - q_2)^\rho \delta_{\rho \alpha \beta \lambda} \]

\[ \Phi = -ig \gamma^\rho \mathcal{G}_{\alpha \beta \lambda} \]

\[ \Phi = -gf_{\alpha \beta \lambda} \sum_{\mathcal{S}_3} \left( \eta_{\alpha (1) \beta (2)} (p_{s(1)} - p_{s(2)})^{\rho (3)} \right) \]

\[ \Phi = -\frac{ig}{2} f_{\alpha \beta \lambda} (q_1 - q_2)^\rho \]

\[ \Phi = \frac{i\zeta}{2} \left( q_1^\mu q_2^\nu + q_1^\nu q_2^\mu - \eta^{\mu \nu} (q_1 \cdot q_2 + m^2) \right) \]

\[ \Phi = \frac{i\zeta}{8} \left( 2\eta^{\mu \nu} \left( q_1 \cdot q_2 - 2m \right) - (q_1 - q_2)_\mu \gamma_\nu - (q_1 - q_2)_\nu \gamma_\mu \right) \]

\[ \Phi = \frac{i\zeta}{2} \delta_{\alpha \beta} \left( (q_1 \cdot q_2) (\eta^{\mu \nu} \eta_{\rho_1 \rho_2} - \eta^{\mu \rho_1} \eta_{\rho_2 \nu} - \eta^{\rho_1 \rho_2} \eta^{\nu \mu}) \right) \]

\[ \Phi = \frac{i\zeta}{2} \left( (q_1 \cdot q_2) \eta^{\mu \nu} - q_1^\mu q_2^\nu - q_2^\mu q_1^\nu \right) \]
\[ \Phi \left( \Phi \right) = \frac{i \pi}{32} \sum_{\mu_1, \nu_1, s \in S_3} \left( \frac{1}{2} p^{\mu_1(1)} p^{\nu_1(1)} \eta^{\mu_1(2) \nu_1(3)} \eta^{\mu_1(1) \nu_1(2)} - p^{\mu_1(1)} p^{\nu_1(1)} \eta^{\mu_1(2) \nu_1(3)} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \right) \]

\[ + \left( p_{s(1)} \cdot p_{s(2)} \right) \left( - \frac{1}{2} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} + \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \right) \]

\[ + \left( p_{s(1)} \cdot p_{s(2)} \right) \left( - \frac{1}{4} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} + \frac{1}{8} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \eta^{\mu_1(1) \nu_1(2)} \eta^{\mu_1(2) \nu_1(3)} \right) \]

(28)

\[ \Phi \left( \Phi \Phi \right) = \frac{i \pi}{8} \left( 2 (q_1 \cdot q_2) (\eta^{\mu \nu_1} \eta^{\nu_2} + \eta^{\mu \nu_2} \eta^{\nu_1}) \right) \]

\[ - q_1^\mu \left( \eta^{\mu \nu_1} + \eta^{\nu_1 \mu} \right) \]

\[ - q_2^\nu \left( \eta^{\mu \nu_1} + \eta^{\nu_1 \mu} \right) \]

\[ + p^\mu \left( q_1^\nu \eta^{\mu \nu_1} + q_2^\nu \eta^{\mu \nu_2} - q_1^\nu \eta^{\mu \nu_2} - q_2^\nu \eta^{\mu \nu_1} \right) \]

\[ + p^\nu \left( - q_1^\nu \eta^{\mu_1 \nu} + q_2^\nu \eta^{\mu_2 \nu} \right) \]

\[ + (q_1 \cdot p) \left( \eta^{\mu \nu \nu_1} + \eta^{\nu \nu_1 \mu} \right) \]

\[ + (q_2 \cdot p) \left( \eta^{\nu \nu_1 \mu} + \eta^{\nu_1 \mu \nu} \right) \]

(29)

3 Longitudinal and transversal projections

In this section, we introduce the transversal structure \( \mathcal{T}_Q \) of a given quantum gauge theory \( Q \) as the set of all its longitudinal, identical and transversal projection tensors \( \{L, I, T\} \) in Definition 3.1. This then directly leads us to introduce the notion of an optimal gauge fixing as the most natural gauge fixing in Definition 3.2. In particular, we will later show that this is given via the Lorenz gauge fixing for Yang–Mills theory and the de Donder gauge fixing for General Relativity in Corollaries 3.7 and 3.18. Then, we will study the cases of Quantum Yang–Mills theory and (effective) Quantum General Relativity in detail in Subsections 3.1 and 3.2.

**Definition 3.1** (Transversal structure). Let \( Q \) be a quantum gauge theory. Then each independent gauge fixing term induces a longitudinal projection operator \( L \) for the propagator of the corresponding gauge field. Together with the respective identity operator \( I \) we define the associated transversal projection operator \( T \) via

\[ T := I - L. \]  

(30)

We refer to the set \( \{L, I, T\} \) as transversal structure. Additionally, let \( |Q| \) denote the number
of independent gauge fixing terms of $Q$. Then we consider the union

$$\mathcal{T}_Q := \bigcup_{k=1}^{i_Q} \{ L, I, T \}_k$$

and refer to it as the transversal structure of $Q$.

**Definition 3.2 (Optimal gauge fixing).** Let $Q$ be a quantum gauge theory with Lagrange density

$$\mathcal{L}_Q = \mathcal{L}_{\text{Classical}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}},$$

We call a gauge fixing functional *optimal*, if the following three conditions are satisfied:

- The tensor $T$ is proportional to the Feynman rule of the quadratic term in $\mathcal{L}_{\text{Classical}}$
- The tensor $L$ is proportional to the Feynman rule of the quadratic term in $\mathcal{L}_{\text{GF}}$
- The tensors satisfy $T + L = I$, where $I$ denotes the corresponding identity tensor

### 3.1 Quantum Yang–Mills theory with matter

We recall known and rather trivial identities for the transversal structure of Quantum Yang–Mills theory with a Lorenz gauge fixing.

**Definition 3.3 (Longitudinal and transversal structure in QYM).** Consider Quantum Yang–Mills theory with a Lorenz gauge fixing. Then we set its longitudinal and transversal structure $\mathcal{T}_{\text{QYM}} := \{ L, I, T \}$ as follows:\(^3\)

$$L^\nu_{\mu} := \frac{1}{p^2} p^\nu p_{\mu},$$

$$I^\nu_{\mu} := \delta^\nu_{\mu},$$

and

$$T^\nu_{\mu} := I^\nu_{\mu} - L^\nu_{\mu},$$

where we have set $p^2 := \eta_{\mu\nu} p^\mu p^\nu$. Lorentz indices on $L$, $I$ and $T$ are raised and lowered with the metric $G$, defined via

$$G_{\mu\nu} := \frac{1}{p^2} \eta_{\mu\nu}$$

and its inverse

$$G^{\mu\nu} := p^2 \eta^{\mu\nu}.$$  

Finally, we define the following two tensors

$$g_{\mu} := \frac{1}{p^2} p_{\mu}$$

and

$$l^\nu := p^\nu.$$  

\(^3\)This includes in particular the coupling of gravity to gauge theories, which requires independent gauge fixing terms for the diffeomorphism invariance and the gauge invariance, cf. e.g. [10, 9]. With that we also obtain two separate transversal structures: $\{ L, I, T \}$ for the Quantum Yang–Mills theory part and $\{ L, I, T \}$ for the (effective) Quantum General Relativity part, cf. [14, Examples 5.2 and 5.3].

\(^4\)We remark that the color indices are implicitly included in the tensors $L, I$ and $T$ by considering their tensor product with the identity matrix $\delta$. We suppress this to simplify the notation.
Remark 3.4. The tensor $g$ corresponds to a gauge transformation and the tensor $l$ describes the gauge fixing projection. Furthermore, their degree in $p^2$ is chosen such that the contraction with $g$ corresponds to the contraction with half of a longitudinal gauge boson propagator.

Lemma 3.5. The following identities hold, i.e. $g$ and $l$ are inverse to each other and $L$ decomposes into the product of $g$ and $l$:

\[
\begin{align*}
g_{\mu}^{\nu}l^{\mu} &= 1 \quad (36a) \\
l_{\nu}^{\mu}g_{\mu} &= L_{\mu}^{\nu} \quad (36b)
\end{align*}
\]

Proof. This follows immediately from basic tensor calculations. ■

Proposition 3.6. The following identities hold, i.e. the tensors $L$, $I$ and $T$ are projectors:

\[
\begin{align*}
L_{\mu}^{\tau}L_{\tau}^{\nu} &= L_{\mu}^{\nu} \quad (37a) \\
I_{\mu}^{\tau}I_{\tau}^{\nu} &= I_{\mu}^{\nu} \quad (37b) \\
T_{\mu}^{\tau}T_{\tau}^{\nu} &= T_{\mu}^{\nu} \quad (37c)
\end{align*}
\]

Additionally, the tensor $I$ is the identity with respect to the metric $G$ and its inverse $G^{-1}$:

\[
G_{\mu\nu}G^{\mu\nu} = I_{\nu}^{\mu} \quad (38)
\]

Proof. This follows immediately from Lemma 3.5 and basic tensor calculations. ■

Corollary 3.7. The two tensors $g$ and $l$ are eigenvectors of the tensors $L$, $I$ and $T$ with respective eigenvalues 1 and 0. In particular, the Lorenz gauge fixing is the optimal gauge fixing condition for Quantum Yang–Mills theory:

\[
\begin{align*}
L_{\mu}^{\tau}g_{\nu} &= g_{\nu} \quad (39a) \\
L_{\mu}^{\tau}l_{\nu} &= l_{\nu} \quad (39b) \\
I_{\mu}^{\tau}g_{\nu} &= g_{\nu} \quad (39c) \\
I_{\mu}^{\tau}l_{\nu} &= l_{\nu} \quad (39d) \\
T_{\mu}^{\tau}g_{\nu} &= 0 \quad (39e) \\
T_{\mu}^{\tau}l_{\nu} &= 0 \quad (39f)
\end{align*}
\]

Proof. This follows immediately from Lemma 3.5 and basic tensor calculations. ■

Lemma 3.8. The following identities hold, i.e. $g$ and $l$ are related via $G$:

\[
\begin{align*}
G_{\mu\nu}l^{\nu} &= g_{\mu} \quad (40a) \\
G^{\mu\nu}g_{\mu} &= l^{\nu} \quad (40b)
\end{align*}
\]

Proof. This follows immediately from basic tensor calculations. ■
Corollary 3.9. The following identities hold, i.e. \( L \) with raised and lowered indices decomposes into products of two \( g \) or \( l \) tensors, respectively:

\[
L_{\mu\nu} = g_\mu g_\nu \quad (41a) \\
L^{\mu\nu} = l^{\mu} l^{\nu} \quad (41b)
\]

Proof. This follows immediately from Lemma 3.5 and Lemma 3.8.

Theorem 3.10. The Feynman rule for the gauge boson propagator can be written as follows:

\[
\Phi ( \begin{array}{c} \mu \\ \nu \end{array} ) = - \frac{i p^2}{p^2 + i\varepsilon} \delta^{a b} (T_{\mu\nu} + \xi L_{\mu\nu}) 
\]  

(42)

Furthermore, the Feynman rules for the gauge boson propagator and the gauge ghost propagator are related as follows:

\[
\Phi ( \begin{array}{c} \ldots \\ \ldots \end{array} ) = \Phi ( \begin{array}{c} l \\ l \end{array} )
\]

(43)

Proof. Equation (42) follows from the Feynman rule

\[
\Phi ( \begin{array}{c} \ldots \\ \ldots \end{array} ) = - \frac{i}{p^2 + i\varepsilon} \delta^{a b} \left( \eta_{\mu\nu} - \frac{(1 - \xi)}{p^2} p_\mu p_\nu \right)
\]

(44)

From this, Equation (43) follows from Equation (39f) together with the Feynman rule

\[
\Phi ( \begin{array}{c} \ldots \\ \ldots \end{array} ) = - \frac{i\xi}{p^2 + i\varepsilon} \delta^{a b}.
\]

(45)

Theorem 3.11. The Feynman rule for the three-valent gauge boson vertex satisfies the following identities:

\[
\Phi \left( \begin{array}{c} g \\ g \\ g \end{array} \right) = \Phi \left( \begin{array}{c} L \\ L \end{array} \right) = 0 \quad , \\
\Phi \left( \begin{array}{c} g \\ T \\ T \end{array} \right) \simeq \text{OS} 0
\]

(46a)

(46b)

and thus

\[
\Phi \left( \begin{array}{c} g \\ l \\ l \end{array} \right) \simeq \text{OS} \left\{ \Phi \left( \begin{array}{c} g \\ L \\ T \end{array} \right) + \Phi \left( \begin{array}{c} g \\ L \\ T \end{array} \right) \right\}
\]

(46c)

where \( \simeq \text{OS} \) indicates equality on-shell, i.e. modulo momentum conservation and equations of motion.

Proof. Starting with the first identity, we recall the decomposition \( L'_\mu = l'^\nu g_\mu \) of Equation (36b) and therefore only calculate the contraction with the \( g \) tensors:

\[
\Phi \left( \begin{array}{c} g \\ g \\ g \end{array} \right) = - gf_{a_1 a_2 a_3} \frac{1}{p_1 p_2 p_3} p_1^{\rho_1} p_2^{\rho_2} p_3^{\rho_3} \sum_{s \in S_3} \left\{ \eta^{\rho_1(1)\rho_2(2)} \left( p_{s(1)} - p_{s(2)} \right) \rho_3(3) \right\}
\]

(47)

\[
= 0
\]

10
For the two remaining identities we first recall the decomposition \( I^\mu_\nu = T^\mu_\nu + L^\mu_\nu \) due to Equation (33c) and calculate

\[
\Phi \left( g^{\mu\nu} \right) = -g f_{a_1 a_2 a_3} \frac{1}{p_1^2} \sum_{s \in S_3} \left\{ \eta^\rho_{s(1)} \left( p_s - p_{s(2)} \right) \rho^\lambda_{s(3)} \right\}
\]

\[
\simeq_{MC} g f_{a_1 a_2 a_3} \frac{1}{p_1^2} \left( I^{\mu \rho_{s(2)}} (p_2^s) - I^{\mu \rho_{s(2)}} (p_3^s) - L^{\rho_{s(2)} \mu} (p_2^s) + L^{\rho_{s(2)} \mu} (p_3^s) \right)
\]

\[
\simeq_{EoM} -g f_{a_1 a_2 a_3} \frac{1}{p_1^2} \left( L^{\rho_{s(2)} \mu} (p_2^s) - L^{\rho_{s(2)} \mu} (p_3^s) \right)
\]

by noting \( I^{\mu \nu} = p^2 \eta^{\mu \nu} \) and recalling the identity \( L^{\mu \nu} T^\rho_\mu = 0 \), where \( \simeq_{MC} \) indicates equality modulo momentum conservation and \( \simeq_{EoM} \) indicates equality modulo equations of motion. ■

Remark 3.12. Equations (46b) and (46c) imply that the longitudinal projection of a single gluon results in both, transversal on-shell cancellations and propagating longitudinal gluon modes. We will find out that this is equivalent in (effective) Quantum General Relativity, cf. Remark 3.24.

Theorem 3.13. The Feynman rules for three-valent interactions of gauge bosons with scalars, spinors and gauge ghosts satisfy the following on-shell contraction identities:

\[
\Phi \left( g^{\mu\nu} \right) \simeq_{OS} \Phi \left( L^{\mu\nu} \right) \simeq_{OS} 0
\]

(49)

\[
\Phi \left( g^{\mu\nu} \right) \simeq_{OS} \Phi \left( L^{\mu\nu} \right) \simeq_{OS} 0
\]

(50)

\[
\Phi \left( g^{\mu\nu} \right) \simeq_{OS} \Phi \left( L^{\mu\nu} \right) \simeq_{OS} 0,
\]

(51)

where \( \simeq_{OS} \) indicates equality on-shell, i.e. modulo momentum conservation and equations of motion.

Proof. Again, we only calculate the contraction with the \( g \) tensors due to the decomposition \( L^\nu_\mu = l^\nu g_\mu \) of Equation (36b). Furthermore, we consider all momenta incoming and denote the gluon momentum by \( p^\sigma \) and the matter momenta by \( q^\sigma_1 \) and \( q^\sigma_2 \). Furthermore, we denote the gauge boson Lorentz and color indices by \( \rho \) and \( a \), respectively. Moreover, \( \mathcal{H} \) and \( \mathcal{S} \) denote the infinitesimal gauge group actions on the Higgs bundle and spinor bundle, respectively. In Equations (53) and (54) number 1 denotes the particle and number 2 denotes the anti-particle. In particular, in Equation (53) this implies that the equations of motion differ in a relative sign. In addition, in Equation (54) we denote the gauge ghost color indices by \( c_1 \) and \( c_2 \), respectively. Additionally, in Equation (54) we use the symmetric (hermitian) gauge ghost Lagrange density.
of Equation (3c). With that, we perform the actual calculations:

\[
\Phi \left( g_{\mu\nu} \right) = -\frac{i g}{2} \left( \frac{1}{p^2} p_\rho \right) \left( (q_1 - q_2)^\rho \delta_{\alpha k l} \right)
\]

\[
\simeq_{\text{MC}} \frac{i g}{2p^2} \left( q_1^2 - q_2^2 \right) \delta_{\alpha k l}
\]

\[
= \frac{i g}{2p^2} \left( \left( q_1^2 - m^2 \right) - \left( q_2^2 - m^2 \right) \right) \delta_{\alpha k l}
\]

\[
\simeq_{\text{EoM}} 0
\]

\[
\Phi \left( g_{\mu\nu} \right) = -ig \left( \frac{1}{p^2} p_\rho \right) \left( \gamma^\rho \mathbb{S}_{\alpha k l} \right)
\]

\[
\simeq_{\text{MC}} \frac{i g}{p^2} \left( \left( \gamma_1 - m \right) \left( \gamma_2 + m \right) \right) \mathbb{S}_{\alpha k l}
\]

\[
\simeq_{\text{EoM}} 0,
\]

\[
\Phi \left( g_{\mu\nu} \right) = -\frac{i g}{2} \left( \frac{1}{p^2} p_\rho \right) \left( f_{ab_1 b_2} \left( q_1 - q_2 \right)^\rho \right)
\]

\[
\simeq_{\text{MC}} \frac{i g}{2p^2} f_{ab_1 b_2} \left( q_1^2 - q_2^2 \right)
\]

\[
\simeq_{\text{EoM}} 0,
\]

where \( \simeq_{\text{MC}} \) indicates equality modulo momentum conservation and \( \simeq_{\text{EoM}} \) indicates equality modulo equations of motion.

3.2 (Effective) Quantum General Relativity with matter

We introduce novel and involved identities for the transversal structure of (effective) Quantum General Relativity with a de Donder gauge fixing.

**Definition 3.14** (Longitudinal and transversal structure in QGR). Consider (effective) Quantum General Relativity with a de Donder gauge fixing. Then we set its longitudinal and transversal structure \( \mathcal{T}_{\text{QGR}} := \{ \mathbb{L}, \mathbb{I}, \mathbb{T} \} \) as follows:

\[
\mathbb{L}_{\mu\nu}^{\alpha\sigma} := \frac{1}{2p^2} \left( \delta^\rho_\mu \rho^\sigma_\nu + \delta^\rho_\nu \rho^\sigma_\mu + \delta^\rho_\sigma \rho^\sigma_\mu + \delta^\rho_\mu \rho^\sigma_\nu - 2\eta^\rho_\sigma \rho_\mu \rho_\nu \right),
\]

\[
\mathbb{I}_{\mu\nu}^{\alpha\sigma} := \frac{1}{2} \left( \delta^\rho_\mu \delta^\sigma_\nu + \delta^\sigma_\mu \delta^\rho_\nu \right)
\]

\[
\mathbb{T}_{\mu\nu}^{\alpha\sigma} := \mathbb{I}_{\mu\nu}^{\alpha\sigma} - \mathbb{L}_{\mu\nu}^{\alpha\sigma},
\]

where we have set \( p^2 := \eta_{\mu\nu} p^\mu p^\nu \). Lorentz indices on \( \mathbb{L}, \mathbb{I} \) and \( \mathbb{T} \) are raised and lowered with the metric \( \mathcal{G} \), defined via

\[
\mathcal{G}_{\mu\nu\sigma} := \frac{1}{p^2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma} \right)
\]

---

5The reason for the asymmetric definition concerning the factor 1/4 is motivated by Equations [57].
and its inverse
\[ G^\mu\nu = \frac{p^2}{4} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}) . \] (56b)

Finally, we define the following two tensors
\[ G^\nu_\mu := \frac{1}{p^2} (p_\mu \delta^\nu_\nu + p_\nu \delta^\nu_\mu) \] (57a)
and
\[ L^{\rho\sigma} := \frac{1}{2} (p^\rho \delta^\sigma_\lambda + p^\sigma \delta^\rho_\lambda - p_\lambda \eta^{\rho\sigma}) . \] (57b)

Remark 3.15. The tensor \( G \) corresponds to a gauge transformation and the tensor \( L \) describes the gauge fixing projection. Furthermore, their degree in \( p^2 \) is chosen such that the contraction with \( G \) corresponds to the contraction with half of a longitudinal gauge boson propagator.

Lemma 3.16. The following identities hold, i.e. \( G \) and \( L \) are inverse to each other and \( L \) decomposes into the product of \( G \) and \( L \):
\[ G^\mu_\nu L^\nu_\lambda = \delta^\lambda_\mu \] (58a)
\[ L^{\rho\sigma} G^\rho_\mu = \delta^{\rho\sigma} \] (58b)

Proof. This follows immediately from basic tensor calculations. ■

Proposition 3.17. The following identities hold, i.e. the tensors \( L \), \( I \) and \( T \) are projectors:
\[ L^{\kappa\lambda} L^{\rho\sigma} = L^{\rho\sigma} \] (59a)
\[ I^{\kappa\lambda} I^{\rho\sigma} = I^{\rho\sigma} \] (59b)
\[ T^{\kappa\lambda} T^{\rho\sigma} = T^{\rho\sigma} \] (59c)

Additionally, the tensor \( I \) is the identity with respect to the metric \( G \) and its inverse \( G^{-1} \):
\[ G^{\mu\nu\kappa\lambda} G^\kappa\lambda\rho\sigma = I^{\rho\sigma} \] (60)

Proof. This follows immediately from Lemma 3.16 and basic tensor calculations. ■

Corollary 3.18. The two tensors \( G \) and \( L \) are eigentensors of the tensors \( L \), \( I \) and \( T \) with respective eigenvalues 1 and 0. In particular, the de Donder gauge fixing is the optimal gauge fixing condition for (effective) Quantum General Relativity:
\[ L^{\rho\sigma} G^\nu_\mu = G^\nu_\mu \] (61a)
\[ L^{\rho\sigma} L^\mu_\nu = L^{\rho\sigma} \] (61b)
\[ I^{\rho\sigma} G^\nu_\mu = G^\nu_\mu \] (61c)
\[ I^{\rho\sigma} L^\mu_\nu = L^{\rho\sigma} \] (61d)
\[ T^{\rho\sigma} G^\nu_\mu = 0 \] (61e)
\[ T^{\rho\sigma} L^\mu_\nu = 0 \] (61f)

(61g)
Proof. This follows immediately from Lemma 3.16 and basic tensor calculations. ■

Lemma 3.19. The following identities hold, i.e. $\mathcal{H}$ and $\mathcal{L}$ are related via $\mathcal{G} \otimes \eta$:

\begin{align}
\mathcal{G}_{\mu\nu\rho\sigma} & \eta^\kappa \mathcal{L}_\lambda^\rho = \mathcal{G}_\mu^\kappa \\
\mathcal{G}^{\mu\nu\rho\sigma} \eta_\kappa \lambda \mathcal{L}^\rho_\lambda &= \mathcal{L}^{\mu\nu}_\rho
\end{align}

(62a) (62b)

Proof. This follows immediately from basic tensor calculations. ■

Corollary 3.20. The following identities hold, i.e. $\mathcal{L}$ with raised and lowered indices decomposes into products of two $\mathcal{H}$ or $\mathcal{L}$ tensors, respectively:

\begin{align}
\mathcal{L}_{\mu\nu\rho\sigma} &= \eta^\kappa \mathcal{G}_{\mu\nu}^\kappa \mathcal{G}_\rho^\lambda \mathcal{L}^\lambda_\rho \\
\mathcal{L}^{\mu\nu\rho\sigma} &= \eta_\kappa \lambda \mathcal{L}^{\mu\nu}_\kappa \mathcal{L}^\rho_\lambda
\end{align}

(63a) (63b)

Proof. This follows immediately from Lemma 3.16 and Lemma 3.19. ■

Theorem 3.21. The Feynman rule for the graviton propagator can be written as follows:

$$
\Phi \left( \varepsilon \right) = -\frac{2ip^2}{p^2 + i\varepsilon} \left( \mathcal{H}_{\mu\nu\rho\sigma} + \zeta \mathcal{L}_{\mu\nu\rho\sigma} \right)
$$

(64)

Furthermore, the Feynman rules for the graviton propagator and the graviton-ghost propagator are related as follows:

$$
\Phi \left( \varepsilon \right) = \Phi \left( \varepsilon \mathcal{L}_{\mu\nu\rho\sigma} \varepsilon \right)
$$

(65)

Proof. Equation (64) follows from the Feynman rule

$$
\Phi \left( \varepsilon \right) = -\frac{2ip^2}{p^2 + i\varepsilon} \left[ \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma} \right) \\
- \left( \frac{1 - \zeta}{p^2} \right) \left( \eta_{\mu\rho} \nu_{\nu\rho\sigma} + \eta_{\mu\sigma} \nu_{\nu\rho\rho} + \eta_{\nu\rho} \nu_{\mu\rho\sigma} + \eta_{\nu\sigma} \nu_{\rho\mu\rho} + \eta_{\rho\sigma} \nu_{\mu\nu\rho} \right) \right]
$$

(66)

From this, Equation (65) follows from Equation (61a) together with the Feynman rule

$$
\Phi \left( \varepsilon \right) = -\frac{2i\zeta}{p^2 + i\varepsilon} \eta_{\rho\sigma}
$$

(67)

Remark 3.22. Given the metric density decomposition of Goldberg and Capper et al. [12] [28] [29] [30], i.e.

$$
\phi^{\mu\nu} := \frac{1}{\sqrt{-\text{Det}(g)}} g^{\mu\nu} - \eta^{\mu\nu} \quad \iff \quad \sqrt{-\text{Det}(g)} g^{\mu\nu} \equiv \eta^{\mu\nu} + \phi^{\mu\nu},
$$

(68)

together with the gauge fixing functional

$$
C^\mu := \partial_\nu \phi^{\mu\nu} \equiv 0.
$$

(69)
Then, the corresponding graviton propagator is given via
\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) = -\frac{2i}{p^2 (p^2 + i\epsilon)} (T^{\mu\nu\rho\sigma} + \zeta L^{\mu\nu\rho\sigma}),
\]
Eq. (70), i.e. the roles of \( G \) and \( L \) are reversed. In particular, the gauge fixing functional \( C^\mu (\phi) \) is the optimal gauge fixing condition for the metric density decomposition. This is due to the fact that in this case the graviton field \( \phi^{\mu\nu} \) is a tensor density of weight 1, instead of the Feynman rules. This will be studied further in [23].

**Theorem 3.23.** The Feynman rule for the three-valent graviton vertex satisfies the following identities:

\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{MC} \Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{MC} 0, \quad (71a)
\]
\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} 0 \quad (71b)
\]

and thus

\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} \Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) + \Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right), \quad (71c)
\]

where \( \simeq_{MC} \) indicates equality modulo momentum conservation and \( \simeq_{OS} \) indicates equality on-shell, i.e. modulo momentum conservation and equations of motion.

**Proof.** All three identities are checked with a Python program written by the author, cf. [31]. In addition, we emphasize the relation between the three identities via the decompositions \( L^{\rho\sigma}_{\mu\nu} = L^{\rho\sigma}_{\tau\gamma} G^\tau_{\mu\nu} \) of Equation (58b) and \( T^{\rho\sigma}_{\mu\nu} = T^{\rho\sigma}_{\mu\nu} + L^{\rho\sigma}_{\mu\nu} \) due to Equation (55c).

**Remark 3.24.** Equations (71b) and (71c) imply that the longitudinal projection of a single graviton results in both, transversal on-shell cancellations and propagating longitudinal gluon modes. This is equivalent to Quantum Yang–Mills theory, cf. Remark 3.12.

**Theorem 3.25.** The Feynman rules for three-valent interactions of gravitons with scalars, spinors, gauge bosons, gauge ghosts and graviton-ghosts satisfy the following on-shell contraction identities:

\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} \Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} 0 \quad (72)
\]
\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} \Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} 0 \quad (73)
\]
\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} \Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} 0, \quad (74)
\]
\[
\Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} \Phi \left( \begin{array}{c}
\gamma_\mu \\
\gamma_\nu \\
\gamma_\rho \\
\gamma_\sigma
\end{array} \right) \simeq_{OS} 0, \quad (75)
\]
\[ \Phi \left( \begin{array}{c} G \\ \end{array} \right) \simeq_{\text{OS}} \Phi \left( \begin{array}{c} L \\ \end{array} \right) \simeq_{\text{OS}} 0, \]  

(76)

where \( \simeq_{\text{OS}} \) indicates equality on-shell, i.e. modulo momentum conservation and equations of motion.

**Proof.** Again, we only calculate the contraction with the \( G \) tensors due to the decomposition \( L_{\mu \nu} = L_{\tau}^{\alpha \beta} G_{\mu \nu}^{\alpha \beta} \) of Equation (58b). Furthermore, we consider all momenta incoming and denote the graviton momentum by \( p^\tau \) and the matter momenta by \( q_1^\tau \) and \( q_2^\tau \). Furthermore, we denote the graviton Lorentz indices by \( \mu \) and \( \nu \), respectively. In Equations (78), (80) and (81) number 1 denotes the particle and number 2 denotes the anti-particle. In particular, in Equation (78) this implies that the equations of motion differ in a relative sign. In addition, in Equation (80) we denote the gauge ghost color indices by \( c_1 \) and \( c_2 \), respectively, and in Equation (81) we denote the graviton-ghost Lorentz indices by \( \rho_1 \) and \( \rho_2 \), respectively. Additionally, in Equation (81) we use the symmetric (hermitian) graviton-ghost Lagrange density of Equation (2c). With that, we perform the actual calculations:

\[
\Phi \left( \begin{array}{c} G \\ \end{array} \right) = \frac{i\kappa}{2p^2} \left( p_\mu \delta_\nu^\tau + p_\nu \delta_\mu^\tau \right) \left( -\eta^{\mu \nu} \left( q_1 \cdot q_2 + m^2 \right) + q_1^\tau q_2^\nu + q_2^\tau q_1^\nu \right) \\
\simeq_{\text{MC}} \frac{i\kappa}{p^2} \left( \left( \frac{2}{q_1^2 - m^2} \right) q_2^\tau + \left( \frac{2}{q_2^2 - m^2} \right) q_1^\tau \right) \\
\simeq_{\text{EoM}} 0
\]

(77)

\[
\Phi \left( \begin{array}{c} G \\ \end{array} \right) = \frac{i\kappa}{8p^2} \left( p_\mu \delta_\nu^\tau + p_\nu \delta_\mu^\tau \right) \\
\times \left( 2\eta^{\mu \nu} \left( \slashed{q}_1 = \slashed{q}_2 - 2m \right) - \left( q_1 - q_2 \right)_\mu \gamma_\nu - \left( q_1 - q_2 \right)_\nu \gamma_\mu \right) \\
\simeq_{\text{MC}} \frac{i\kappa}{4p^2} \left( -2 \left( q_1 + q_2 \right)^\tau \left( \slashed{q}_1 - \slashed{q}_2 - 2m \right) \right) \\
+ \left( q_1 - q_2 \right)^\tau \left( \slashed{q}_1 + \slashed{q}_2 + m - m \right) + \left( q_1^2 - q_2^2 + m - m \right) \gamma^\tau \\
\simeq_{\text{EoM}} \frac{i\kappa}{4p^2} \left( \slashed{q}_1 - m \right) \left( -2 \left( q_1 + q_2 \right)^\tau + \left( q_1 - q_2 \right)^\tau + \left( \slashed{q}_1 + m \right) \gamma^\tau \right) \\
+ \left( \slashed{q}_2 + m \right) \left( 2 \left( q_1 + q_2 \right)^\tau + \left( q_1 - q_2 \right)^\tau - \left( \slashed{q}_2 - m \right) \gamma^\tau \right) \\
\simeq_{\text{EoM}} 0
\]

(78)
\[ \Phi \left( \begin{array}{c} \Phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\
 \end{array} \right) \begin{array}{c} T \\ T \\ T \\ T \\ T \\ T \\
 \end{array} \right) = \frac{i\kappa}{2p^2} \left( p_\mu \delta^\nu_\mu + p_\nu \delta^\mu_\mu \right) \\
\times \delta_{\alpha_1\alpha_2} \left( (q_1 \cdot q_2) (\eta^{\mu
u} \eta^{\sigma_1\sigma_2} - \eta^{\mu\sigma_1} \eta^{\nu\sigma_2} - \eta^{\mu\sigma_2} \eta^{\nu\sigma_1}) \right. \\
- \eta^{\mu
u} q_1^{\sigma_2} q_2^{\sigma_1} - \eta^{\sigma_1\sigma_2} \left( q_1^{\mu} q_2^{\nu} + q_2^{\mu} q_1^{\nu} \right) \\
+ q_1^{\sigma_2} \left( \eta^{\mu\sigma_1} q_2^{\nu} + \eta^{\nu\sigma_1} q_2^{\mu} \right) + q_2^{\sigma_1} \left( \eta^{\mu\sigma_2} q_1^{\nu} + \eta^{\nu\sigma_2} q_1^{\mu} \right) \\
- \frac{1}{\xi} \eta^{\mu\nu} (q_1^{\sigma_1} q_2^{\sigma_2} + p^{\sigma_1} q_2^{\sigma_2} + p^{\sigma_2} q_1^{\sigma_2}) \\
+ \frac{1}{\xi} q_1^{\sigma_1} \left( \eta^{\mu\sigma_2} q_2^{\nu} + \eta^{\nu\sigma_2} q_2^{\mu} + \eta^{\mu\sigma_2} p^{\nu} + \eta^{\nu\sigma_2} p^{\mu} \right) \\
+ \frac{1}{\xi} q_2^{\sigma_2} \left( \eta^{\mu\sigma_1} q_1^{\nu} + \eta^{\nu\sigma_1} q_1^{\mu} + \eta^{\mu\sigma_1} p^{\nu} + \eta^{\nu\sigma_1} p^{\mu} \right) \right) \\
\times T_{\sigma_1}^\rho\left( q_1 \right) \times T_{\sigma_2}^\rho\left( q_2 \right) \right) \\
\simeq_{\text{EoM}} 0 \quad (79) \\
\]

\[ \Phi \left( \begin{array}{c} \Phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\
 \end{array} \right) \begin{array}{c} T \\ T \\ T \\ T \\ T \\ T \\
 \end{array} \right) \right) = \frac{i\kappa}{2p^2} \left( p_\mu \delta^\nu_\mu + p_\nu \delta^\mu_\mu \right) \\
\times \delta_{\alpha_1\alpha_2} \left( (q_1 \cdot q_2) (\eta^{\mu
u} q_1^{\sigma_2} q_2^{\sigma_1} - q_2^{\sigma_2} q_1^{\sigma_1}) \right. \\
+ \eta^{\sigma_1\sigma_2} \left( q_1^{\mu} q_2^{\nu} + \eta^{\mu\sigma_1} q_2^{\nu} + \eta^{\nu\sigma_1} q_2^{\mu} \right) \\
+ \eta^{\sigma_2\sigma_1} \left( q_1^{\mu} q_2^{\nu} + \eta^{\mu\sigma_2} q_1^{\nu} + \eta^{\nu\sigma_2} q_1^{\mu} \right) \\
\times T_{\sigma_1}^\rho\left( q_1 \right) \times T_{\sigma_2}^\rho\left( q_2 \right) \right) \\
\simeq_{\text{EoM}} 0 \quad (80) \\
\]
\[
\Phi \left( \begin{bmatrix} q_1 & q_2 \end{bmatrix} \right) = \frac{i\kappa}{8p^2} \left( p_\mu \delta^\nu_\mu + p_\nu \delta^\mu_\nu \right) \\
\qquad \left( 2 (q_1 \cdot q_2) (\eta^{\mu_1 \rho_1} \eta^{\nu_2 \rho_2} + \eta^{\mu_2 \rho_2} \eta^{\nu_1 \rho_1}) \\
\qquad - q_1^{\rho_1} \left( p^{\mu_1} \eta^{\rho_2 \nu} + p^{\nu} \eta^{\rho_2 \mu} - p^{\rho_2} \eta^{\mu_1 \nu} \right) \\
\qquad - q_2^{\rho_2} \left( p^{\nu} \eta^{\rho_1 \mu} + p^{\rho_1 \nu} - p^{\rho_1} \eta^{\nu \mu} \right) \\
\qquad + p^{\rho_1} \left( q_1^{\mu} \eta^{\rho_2 \nu} \eta^{\nu} + q_1^{\nu} \eta^{\rho_2 \mu} - q_2^{\mu} \eta^{\rho_2 \nu} - q_2^{\nu} \eta^{\rho_2 \mu} \right) \\
\qquad + p^{\rho_2} \left( - q_1^{\mu} \eta^{\rho_1 \nu} - q_1^{\nu} \eta^{\rho_1 \mu} + q_2^{\mu} \eta^{\rho_1 \nu} + q_2^{\nu} \eta^{\rho_1 \mu} \right) \\
\qquad + (q_1 \cdot p) \left( \eta^{\rho_2 \nu} \eta^{\nu} + \eta^{\rho_2 \mu} \eta^{\mu} \right) \\
\qquad + (q_2 \cdot p) \left( \eta^{\rho_1 \nu} \eta^{\nu} + \eta^{\rho_1 \mu} \eta^{\mu} \right) \right) \\
\simeq_{\text{MC}} \frac{i\kappa}{2p^2} \left( q_2^{2} \sqrt{p_1} p^{p_2} + q_1^{2} \sqrt{p_2} p^{p_1} \right) \\
\simeq_{\text{EoM}} 0,
\]
where \( \simeq_{\text{MC}} \) indicates equality modulo momentum conservation and \( \simeq_{\text{EoM}} \) indicates equality modulo equations of motion.

Remark 3.26. We emphasize that the longitudinal projection of the graviton in Equation (74) also induces longitudinal gluon-modes, cf. Equation (79). We remove them via transversal gluon projectors, cf. Definition 3.3, as we are here interested in physical external particles (which are on-shell and transversal). In general, however, these longitudinal gluon legs are important as they lead to further cancellations, cf. Theorems 3.11 and 3.13. This is, as we have seen, equivalent to the three-valent gluon and graviton vertex Feynman rules, cf. Remarks 3.12 and 3.24. We will study this in detail in future work.

4 Conclusion

We have studied the transversal structure of (effective) Quantum General Relativity coupled to the Standard Model. To this end, we provided the corresponding propagator and three-valent Feynman rules in Section 2. Then we discussed several aspects of the corresponding longitudinal, identical and transversal projection tensors in Section 3. In particular, we recalled known and trivial identities of Quantum Yang–Mills theory in Subsection 3.1 and then proceeded by analogy to introduce their involved counterparts in (effective) Quantum General Relativity in Subsection 3.2. Our main results are the following: First we discussed the decomposition of the gluon and graviton propagators into their physical and unphysical degrees in Theorems 3.10 and 3.21. Next we studied the corresponding cancellation identities for the pure theories in Theorems 3.11 and 3.23 and then ultimately for their couplings to matter from the Standard Model in Theorems 3.13 and 3.25. We believe that these results provide further insight into the involved tensorial structure of gravitational Feynman integrals. This continues the aim to
prove the renormalizability of (effective) Quantum General Relativity (QGR), possibly coupled to matter from the Standard Model (SM), as was suggested in [13] and then worked out in [14]. In particular, we refer to [9, 10] for introductions to the corresponding perturbative expansions. Additionally, the present work also relates to the BRST double complex of diffeomorphisms and gauge transformations [24] and the respective symmetric (hermitian) ghost Lagrange densities [7]. Finally, we aim to combine these different angles on the renormalization problem of QGR-SM in [23] with the introduction of a differential-graded renormalization Hopf algebra.

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