**Fuzzy 0-semi-generalized closed sets.**

N A Abdul Wahab$^1$ and Z Salleh$^2$

$^1,2$Department of Mathematics, Faculty of Science and Technology, Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu, MALAYSIA.

Email: $^1$adilla_aw@yahoo.com, $^2$zabidin@umt.edu.my

**Abstract.** In this paper, a new notion of the fuzzy generalized closed sets called fuzzy 0-semi-generalized closed sets in fuzzy topological spaces is introduced and studied their properties. Furthermore, we study these new sets in relation to some other types of already known fuzzy sets.

1. Introduction.

The concept of fuzzy sets due to Zadeh [16] naturally plays important role in the study of fuzzy topological spaces which has been introduced by Chang [4]. In the year of 1970, Levine [7] initiated the study of generalized closed set in general topology. Generalized closed (open) sets play an important role in general topology. After that, Balasubramanian and Sundaram [3] defined fuzzy generalization of closed set in fuzzy topological spaces. Later, El-Shafei [5] introduced fuzzy semi-generalized closed and fuzzy generalized semi-closed set with some of their properties in year 2007.

In this paper, we introduced another new notion of fuzzy generalized closed set called fuzzy 0-semi-generalized closed set which is an alternative generalization of fuzzy semi-closed set by utilizing semi-0-closure operator in fuzzy topological spaces. The relationships among some fuzzy generalized closed sets and this new notion are obtained. Some examples and their properties were discussed in details.

2. Preliminaries.

Throughout this paper, let $X$ be a set and $I$ the unit interval. A fuzzy set in $X$ is an element of the set of all functions from $X$ to $I$. The family of all fuzzy sets in $X$ is denoted by $I^X$. A fuzzy singleton $x_\alpha$ is a fuzzy set in $X$ define by $x_\alpha(x) = \alpha$, $x_\alpha(y) = 0$ for all $y \neq x, \alpha (0 < \alpha \leq 1)$. The set of all fuzzy singletons in $X$ is denoted by $S(X)$. For every $x_\alpha \in S(X)$ and $\mu \in I^X$, we define $x_\alpha \in \mu$ if and only if $\alpha \leq \mu(x)$. Space $(X, \tau)$ (simply $X$) always means fuzzy topological spaces. By $1_X$ and $0_X$, we mean the fuzzy sets with constant function 1 (unit function) and 0 (zero function) respectively. A fuzzy set $\mu$ is quasi-coincident with a fuzzy set $\eta$ denoted by $\mu \eta \eta$ if and only if there exist $x \in X$ such that $\mu(x) + \eta(x) > 1$. If $\mu$ and $\eta$ are not quasi-coincident, it is denoted as $\mu \eta \eta$. A fuzzy subset $\mu$ of $X$ is called fuzzy semi-open [2] if $\text{int}(\eta) \leq \mu \leq \eta$ where $\eta$ is fuzzy closed while $\mu$ is fuzzy semi-open if $\eta \leq \mu \leq cl(\eta)$ where $\eta$ is fuzzy open set. The family of all fuzzy semi-open and fuzzy semi-closed sets in $(X, \tau)$ will be denoted by $FSO(X, \tau)$ and $FSC(X, \tau)$, respectively. A fuzzy set $\mu$ of $X$ is called fuzzy semi-0-closed if $\mu = scl_0(\mu)$ and fuzzy semi-0-open if $\mu = sint_0(\mu)$. The operators fuzzy semi-closure and fuzzy semi-interior of $\mu$ are defined by $scl(\mu) = \{ \gamma : \mu \leq \gamma, \gamma \in FSC(X, \tau) \}$ and $sint(\mu) = \{ \gamma : \mu \geq \gamma, \gamma \in FSO(X, \tau) \}$ respectively. Saeid Jafari and Miguel Caldas [12] introduced $\theta$-semi-generalized closed set in topological spaces and now we proceed $\theta$-semi-generalized closed set in fuzzy topological spaces.
Now we introduce some basic notions that were used in the sequel.

**Definition 2.1.** A fuzzy set \( \mu \) in \( (X, \tau) \) is called:

(i) Fuzzy generalized closed set [3] (briefly, fg-closed set) if \( \text{cl}(\mu) \leq \eta \) whenever \( \mu \leq \eta \) and \( \eta \) is fuzzy open;

(ii) Fuzzy semi-generalized closed set [5] (briefly, fsg-closed set) if \( \text{scl}(\mu) \leq \eta \) whenever \( \mu \leq \eta \) and \( \eta \) is fuzzy semi open;

(iii) Fuzzy \( \theta \)-generalized closed set [6] (briefly, f-\( \theta \)g-closed set) if \( \text{cl}_\theta(\mu) \leq \eta \) whenever \( \mu \leq \eta \) and \( \eta \) is fuzzy open.

The complement of fuzzy generalized closed (resp. fuzzy semi-generalized closed, fuzzy \( \theta \)-generalized closed) set is fuzzy generalized open (resp. fuzzy semi-generalized open, fuzzy \( \theta \)-generalized open) set.

**Lemma 2.1.** [13] Let \( \mu \) be a fuzzy set in \( (X, \tau) \). Then,

\[
\mu \leq \text{scl}(\mu) \leq \text{scl}_\theta(\mu)
\]

and hence fuzzy semi-\( \theta \)-closed set is a fuzzy semi-closed set.

3. **Fuzzy \( \theta \)-Semi-Generalized Closed Sets.**

In this section, we introduce fuzzy \( \theta \)-semi-generalized closed sets in fuzzy topological space and study some of their characteristics and their relationships with other notions.

**Definition 3.1.** Let \( (X, \tau) \) be a fuzzy topological space and \( \mu \) be a fuzzy set of \( X \). The operators of fuzzy semi-\( \theta \)-closure and fuzzy semi-\( \theta \)-interior of \( \mu \) are defined as follows,

\[
\text{scl}_\theta(\mu) = \bigwedge \{ \text{scl}(\gamma) : \gamma \leq \mu, \gamma \in \text{FSO}(X, \tau) \},
\]

\[
\text{sint}_\theta(\mu) = \bigvee \{ \text{sint}(\eta) : \eta \geq \mu, \eta \in \text{FSC}(X, \tau) \}.
\]

**Definition 3.2.** A fuzzy subset \( \mu \) of \( (X, \tau) \) is said to be fuzzy \( \theta \)-semi generalized closed set (briefly f-\( \theta \)sg-closed set) if \( \text{scl}_\theta(\mu) \leq \eta \) whenever \( \mu \leq \eta \) and \( \eta \in \text{FSO}(X, \tau) \).

The complement of fuzzy \( \theta \)-semi generalized closed set is fuzzy \( \theta \)-semi generalized open set (briefly f-\( \theta \)sg-open set).

**Lemma 3.1.** Every fuzzy semi-\( \theta \)-closed set in \( X \) is fuzzy \( \theta \)-semi generalized closed.

**Proof.** Let \( \mu \) be a fuzzy semi-\( \theta \)-closed set in \( X \), then \( \mu = \text{scl}_\theta(\mu) \). Suppose that \( \mu \leq \eta \) and \( \eta \in \text{FSO}(X, \tau) \). It follows that \( \text{scl}_\theta(\mu) \leq \eta \) and hence \( \mu \) is fuzzy \( \theta \)-semi generalized closed set in \( X \). \( \square \)

Example below shows that the converse of Lemma 3.1 does not true.
Example 3.1. Let \( X = \{a, b, c\} \) and \( \tau = \{0_X, \mu_1, \mu_2, \mu_3, 1_X\} \) where \( \mu_1 = a_0 \vee b_0 \vee c_{0.4}, \mu_2 = a_{0.9} \vee b_{0.6} \vee c_0, \) and \( \mu_3 = a_{0.9} \vee b_{0.6} \vee c_{0.4}. \) The family of all fuzzy semi-open sets is

\[
FSO(X, \tau) = \left\{\begin{array}{ll}
0 \leq x \leq 0.1 & \text{or} & 0.6 \leq y \leq 1 \quad \text{or} & 0.6 \leq z \leq 0.6
\end{array}\right. \quad \text{either}
\]

Hence the family of all fuzzy semi-closed sets is

\[
FSC(X, \tau) = \left\{\begin{array}{ll}
0 \leq x \leq 0.1 & 0.9 \leq x \leq 1
\end{array}\right. \quad \text{either}
\]

If \( \mu = a_{0.7} \vee b_{0.7} \vee c_{0.6}, \) then \( \mu \) is fuzzy \( \theta \)-semi-generalized closed set. But \( \mu \) is not fuzzy semi-\( \theta \)-closed set since \( scl_\theta(\mu) \neq \mu. \)

**Lemma 3.2.** Every fuzzy \( \theta \)-semi-generalized closed set is fuzzy \( \theta \)-generalized closed.

**Proof.** Let \( \mu \) be a fuzzy \( \theta \)-semi-generalized closed set of \( (X, \tau). \) Let \( \mu \leq \eta \) and \( \eta \in FSO(X, \tau). \) Since \( \mu \) be a fuzzy \( \theta \)-semi-generalized closed, \( scl_\theta(\mu) \leq \eta. \) It follows that \( scl(\mu) \leq \eta \) since \( scl(\mu) \leq scl_\theta(\mu) \) by Lemma 2.1. Hence \( \mu \) is fuzzy semi-generalized closed. \( \square \)

The following example shows that the converse of Lemma 3.2 is not true.

**Example 3.2.** Let \( X = \{y\} \) with fuzzy topology \( \tau = \{0_X, y_{2/3}, y_{3/4}, 1_X\}. \) The family of all fuzzy semi-open sets is

\[
FSO(X, \tau) = \left\{0_X, 1_X, y_p \quad \text{where} \quad \frac{2}{3} \leq p < 1\right\}
\]

and the complement is

\[
FSC(X, \tau) = \left\{0_X, 1_X, y_q \quad \text{where} \quad 0 < q \leq \frac{1}{3}\right\}
\]

Let \( \mu = y_{1/3} \) then \( \mu \) is fuzzy semi-generalized closed set. But \( scl(\mu) = 1_X \not\leq y_{2/3} \) where \( y_{2/3} \in FSO(X, \tau) \) hence \( \mu \) is not fuzzy \( \theta \)-semi-generalized closed.

**Lemma 3.3.** Every fuzzy \( \theta \)-generalized closed set is fuzzy \( \theta \)-semi-generalized closed but the converse may not be true in general.

**Example 3.3.** Let \( X = \{a\} \) and \( \tau = \{0_X, a_{0.3}, 1_X\}. \) The family of all fuzzy semi-open sets is;

\[
FSO(X, \tau) = \{0_X, 1_X, a_x \quad \text{where} \quad 0.3 \leq x \leq 0.7\}
\]

Hence the family of all fuzzy semi-closed sets is
If $\mu = a_{0.2}$, then $\mu$ is fuzzy $\theta$-semi-generalized closed set but $\mu$ is not fuzzy $\theta$-generalized closed since $cl_\theta(\mu) \nleq a_{0.3}$ where $a_{0.3} \in \tau$.

Base on the discussion above, every fuzzy semi-closed set is fuzzy semi-generalized closed set but the converse is not true (see [5]). Moreover, fuzzy semi-$\theta$-closed implies fuzzy $\theta$-semi-generalized closed but the converse may not be true as in Example 3.1. Lemma 3.1 shows that fuzzy $\theta$-semi-generalized closed imply fuzzy semi-generalized closed but the reverse implication is not true in general as shown in Example 3.2. Example 3.3 above shows that fuzzy $\theta$-semi-generalized closed set does not implies fuzzy $\theta$-generalized closed. Furthermore, fuzzy $\theta$-closed implies fuzzy $\theta$-generalized closed but the converse is not true (see [6]). The Figure 1 below summarizes the relationships among some fuzzy generalized closed sets discussed above where none of these implications are reversible.

![Figure 1. Relationship among some fuzzy generalized closed sets.](image)

4. Some Properties of Fuzzy $\theta$-Semi-Generalized Closed Sets.
In this section, we study some properties of fuzzy $\theta$-semi-generalized closed set.

**Theorem 4.1.** If $\mu$ is fuzzy semi-open and fuzzy $\theta$-semi-generalized closed set in $X$, then $\mu$ is fuzzy semi-\(\theta\)-closed in $X$.

**Proof.** Let $\mu$ be fuzzy semi-open and f-$\theta$sg closed set in $X$. Obviously that $\mu \leq scl_\theta(\mu)$. Since $\mu \leq \mu$, by definition $scl_\theta(\mu) \leq \mu$. This implies $\mu = scl_\theta(\mu)$. Hence $\mu$ is fuzzy semi-\(\theta\)-closed in $X$. 

**Lemma 4.1.** [11] $A \leq B$ if and only if $A$ and $B^c$ are not quasi-coincident; particularly, $x_\alpha \in A$ if and only if $x_\alpha$ is not quasi-coincident with $A^c$.

**Theorem 4.2.** A fuzzy set $\mu$ is a fuzzy $\theta$-semi-generalized closed set if and only if $\mu \bar{\eta}$ implies $scl_\theta(\mu) \bar{\eta}$ for every fuzzy semi-closed set $\eta$ in $X$. 

---

2012 iCAST: Contemporary Mathematics, Mathematical Physics and their Applications  
Journal of Physics: Conference Series 435 (2013) 012008  
doi:10.1088/1742-6596/435/1/012008

---

\[
FSC(X, \tau) = \{0_X, 1_X, a_x \mid 0.3 \leq x \leq 0.7\}.
\]
Proof. Suppose that $\mu$ is a $f$-0sg closed subset of $X$. Let $\eta$ be a fuzzy semi-closed set in $X$ such that $\mu q\eta$. Then by Lemma 4.1, $\mu \leq 1 - \eta$ and $1 - \eta$ is fuzzy semi-open set in $X$. Therefore, $\text{scl}_\theta(\mu) \leq 1 - \eta$ as $\mu$ is $f$-0sg closed in $X$. Hence, $\text{scl}_\theta(\mu) q\eta$.

Conversely, let $\eta$ be a fuzzy semi-open set in $X$ such that $\mu \leq \eta$ where $\mu$ is a fuzzy subset of $X$. By definition, $\mu q(1 - \eta)$ and $(1 - \eta)$ is fuzzy semi-closed set in $X$. Then we have $\text{scl}_\theta(\mu) q(1 - \eta)$ which implies $\text{scl}_\theta(\mu) \leq \eta$. Hence $\mu$ is $f$-0sg closed in $X$. □

Lemma 4.2.[9] If $\mu$ is a fuzzy semi-open set in $X$ then $\text{scl}(\mu) = \text{scl}_\theta(\mu)$.

Theorem 4.3. Let $\mu$ be a fuzzy semi-open set in a fuzzy topological space $(X, \tau)$. The fuzzy set $\mu$ is a fuzzy $\theta$-semi-generalized closed if and only if $\mu$ is fuzzy semi-generalized closed.

Proof. (Necessity). It is obvious by Lemma 3.2.

(Sufficiency). Let $\mu$ be a fuzzy semi-generalized closed set and $\mu \leq \eta$ where $\eta \in \text{FSO}(X, \tau)$. Hence $\text{scl}(\mu) \leq \eta$ and since $\mu$ is fuzzy semi-open in $X$, $\text{scl}_\theta(\mu) \leq \eta$ by Lemma 4.2. Thus $\mu$ is fuzzy $\theta$-semi-generalized closed set.

□

Theorem 4.4. A fuzzy set $\mu$ is fuzzy $\theta$-semi generalized open set if and only if $\eta \leq \text{sint}_\theta(\mu)$ whenever $\eta$ is fuzzy semi-closed in $X$ and $\eta \leq \mu$.

Proof. (Necessity) Let $\mu$ be $f$-0sg-open set in $X$ and $\eta \leq \mu$ where $\eta$ is fuzzy semi-closed. Obvious that $\mu^c$ is contained in $\eta^c$. Since $\mu^c$ is $f$-0sg-closed set then $\text{scl}_\theta(\mu^c) \leq \eta^c$ and hence $\text{scl}_\theta(\mu) = (\text{sint}_\theta(\mu))^c \leq \eta^c$ such that $\eta \leq \text{sint}_\theta(\mu)$.

(Sufficiency) If $\mu$ is a fuzzy semi-closed set with $\eta \leq \text{sint}_\theta(\mu)$ whenever $\eta \leq \mu$, then it follows that $\mu^c \leq \eta^c$ and $(\text{sint}_\theta(\mu))^c \leq \eta^c$ such that $\text{scl}_\theta(\mu^c) \leq \eta^c$. Hence $\mu^c$ is $f$-0sg-closed and therefore $\mu$ is $f$-0sg-open set. □

Lemma 4.3. If $\alpha$ and $\beta$ are fuzzy subsets of a fuzzy topological space $(X, \tau)$, then:

(i) $\text{scl}_\theta(\alpha \vee \beta) = \text{scl}_\theta(\alpha) \vee \text{scl}_\theta(\beta)$;
(ii) $\text{scl}_\theta(\alpha \wedge \beta) \leq \text{scl}_\theta(\alpha) \wedge \text{scl}_\theta(\beta)$;
(iii) $\text{sint}_\theta(\alpha \vee \beta) = \text{sint}_\theta(\alpha) \wedge \text{sint}_\theta(\beta)$;
(iv) $\text{sint}_\theta(\alpha \wedge \beta) \geq \text{sint}_\theta(\alpha) \vee \text{sint}_\theta(\beta)$;
(v) $(\text{scl}_\theta(\alpha))^c = \text{sint}_\theta(\alpha^c)$

Proof. (i) Let $\text{scl}_\theta(\alpha)$ and $\text{scl}_\theta(\beta)$ be both fuzzy semi-$\theta$-closed sets and

$$\text{scl}_\theta(\alpha) \leq \text{scl}_\theta(\alpha \vee \beta), \text{scl}_\theta(\beta) \leq \text{scl}_\theta(\alpha \vee \beta).$$

By combining both of them,
\[ \text{scl}_\theta(\alpha) \lor \text{scl}_\theta(\beta) \leq \text{scl}_\theta(\alpha \lor \beta). \]

Conversely,
\[ \alpha \leq \text{scl}_\theta(\alpha) \quad \text{and} \quad \beta \leq \text{scl}_\theta(\beta) \]
implies,
\[ \alpha \lor \beta \leq \text{scl}_\theta(\alpha) \lor \text{scl}_\theta(\beta) \]

And \( \text{scl}_\theta(\alpha) \lor \text{scl}_\theta(\beta) \) is a fuzzy semi-\( \theta \)-closed. But since \( \text{scl}_\theta(\alpha \lor \beta) \) is the smallest fuzzy semi-\( \theta \)-closed set containing \( \alpha \lor \beta \), hence
\[ \text{scl}_\theta(\alpha \lor \beta) = \text{scl}_\theta(\alpha) \lor \text{scl}_\theta(\beta). \]
This gives the equality.

(ii) Let \( \alpha \land \beta \leq \alpha \) and \( \alpha \land \beta \leq \beta \). Therefore
\[ \text{scl}_\theta(\alpha \land \beta) \leq \text{scl}_\theta(\alpha) \quad \text{and} \quad \text{scl}_\theta(\alpha \land \beta) \leq \text{scl}_\theta(\beta). \]

Combining,
\[ \text{scl}_\theta(\alpha \land \beta) \leq \text{scl}_\theta(\alpha) \land \text{scl}_\theta(\beta) \]
or,
\[ \alpha \land \beta \leq \text{scl}_\theta(\alpha) \land \text{scl}_\theta(\beta) \]
implies that \( \text{scl}_\theta(\alpha \land \beta) \leq \text{scl}_\theta(\text{scl}_\theta(\alpha) \land \text{scl}_\theta(\beta)) = \text{scl}_\theta(\alpha) \land \text{scl}_\theta(\beta). \)

(iii) is the complement of (ii).
(iv) is the complement of (i).

(v) Observe that
\[
\left( \text{scl}_\theta(\alpha) \right)^c = 1 - \text{scl}_\theta(\alpha)
= 1 - \land \left\{ \text{scl}(\varepsilon) : \alpha \leq \varepsilon, \varepsilon \in \text{FSO}(X, \tau) \right\}
= \lor \left\{ \text{sint}(1 - \varepsilon) : 1 - \alpha \geq 1 - \varepsilon, \varepsilon \in \text{FSO}(X, \tau) \right\}
\]
By letting \( \eta = 1 - \varepsilon \), we have
\[ \lor \left\{ \text{sint}(\eta) : 1 - \alpha \geq \eta, 1 - \eta \in \text{FSO}(X, \tau) \right\} = \text{sint}_\theta(1 - \alpha). \]
\[ \square \]

**Theorem 4.5.** In a fuzzy topological space \( (X, \tau) \), the collection of all fuzzy semi-\( \theta \)-open sets in \( (X, \tau) \) is a fuzzy topological space.

**Proof.** Note that \( 0_X, 1_X \in \tau \) and obviously that \( 0_X \) and \( 1_X \) also fuzzy semi-\( \theta \)-open set since \( \text{sint}_\theta(0_X) = 0_X \) and \( \text{sint}_\theta(1_X) = 1_X \).
Suppose that \( \{ \mu_i, i \in I \} \) be a collection of arbitrary fuzzy semi-\( \theta \)-open sets in \( X \). Then \( \mu_i = \text{sint}_\theta(\mu_i) \) for each \( i \in I \). Let \( \mu = \bigvee \{ \mu_i, i \in I \} \). It is obvious that \( \text{sint}_\theta(\mu) \leq \mu \). On the other hand, since \( \mu_i \leq \mu \) then \( \text{sint}_\theta(\mu_i) \leq \text{sint}_\theta(\mu) \) for each \( i \in I \). So, \( \bigvee \{ \text{sint}_\theta(\mu) : i \in I \} \leq \text{sint}_\theta(\mu) \). Thus we have \( \mu = \bigvee \{ \mu_i, i \in I \} \leq \text{sint}_\theta(\mu) \). Hence we have \( \mu = \text{sint}_\theta(\mu) \) and this shows that union of arbitrary fuzzy semi-\( \theta \)-open sets is fuzzy semi-\( \theta \)-open set.

On the other side, let \( \mu_1 \) and \( \mu_2 \) be two fuzzy semi-\( \theta \)-open sets in \( X \). Then \( \mu_1 = \text{sint}_\theta(\mu_1) \) and \( \mu_2 = \text{sint}_\theta(\mu_2) \). Let \( \mu = \mu_1 \land \mu_2 \). Obviously that \( \text{sint}_\theta(\mu) \leq \mu \). Since \( \text{sint}_\theta(\mu_1) \land \text{sint}_\theta(\mu_2) \leq \mu_1 \land \mu_2 \) then by Lemma 4.3(iii),

\[
\begin{align*}
\text{sint}_\theta\left(\text{sint}_\theta(\mu_1) \land \text{sint}_\theta(\mu_2)\right) & \leq \text{sint}_\theta(\mu_1 \land \mu_2) \\
\text{sint}_\theta\left(\text{sint}_\theta(\mu_1) \lor \text{sint}_\theta(\mu_2)\right) & \leq \text{sint}_\theta(\mu) \\
\mu_1 \land \mu_2 & \leq \text{sint}_\theta(\mu) \\
\mu & \leq \text{sint}_\theta(\mu)
\end{align*}
\]

Hence we have \( \mu = \text{sint}_\theta(\mu) \) and this shows that the intersection of two fuzzy semi-\( \theta \)-open sets is also fuzzy semi-\( \theta \)-open set. It’s completes the proof.

**Remark.** The collection of all fuzzy semi-\( \theta \)-open sets in \( (X, \tau) \) is a fuzzy topological spaces denoted by \( \tau_{\text{semi}} \).

By Theorem 4.5, the collection of all fuzzy semi-\( \theta \)-open sets in \( (X, \tau) \) is a fuzzy topological space. Similar argument as previous discussion, we will have

**Proposition 4.1.** The union of two fuzzy \( \theta \)-semi-generalized closed sets is always fuzzy \( \theta \)-semi-generalized closed set.

**Proof.** Suppose that \( \alpha \) and \( \beta \) are fuzzy \( \theta \)-semi-generalized closed sets in \( X \) and let \( \eta \in \text{FSO}(X, \tau) \) such that \( \alpha \lor \beta \leq \eta \). Since \( \alpha \) and \( \beta \) are fuzzy \( \theta \)-semi-generalized closed, then we have \( \text{scl}_\theta(\alpha) \lor \text{scl}_\theta(\beta) \leq \eta \) and by Lemma 4.3(i), \( \text{scl}_\theta(\alpha \lor \beta) \leq \eta \). Hence, \( \alpha \lor \beta \) is fuzzy \( \theta \)-semi-generalized closed.

Intersection of two fuzzy \( \theta \)-semi-generalized closed sets is not necessarily a fuzzy \( \theta \)-semi-generalized closed set as the following example.

**Example 4.1.** Let \( X = \{a, b\} \) and \( \tau = \{0_X, a_{0.4}, b_{0.5}, a_{0.4} \lor b_{0.5}, 1_X\} \). The family of all fuzzy semi-open sets is:

\[
\text{FSO}(X, \tau) = \left\{ 0_X, 1_X, a_x \lor b_y, \text{ either } 0.4 \leq x \leq 0.6 \text{ or } 0 \leq y \leq 0.5 \right\}
\]

Hence the family of all fuzzy semi-closed sets is
If \(\mu = a_{0.8} \lor b_{0.3}\) and \(\eta = a_{0.5} \lor b_{0.7}\), then \(\mu\) and \(\eta\) are fuzzy \(\theta\)-semi-generalized closed sets since the only semi-open superset of \(\mu\) and \(\eta\) is \(1_X\). But \(\mu \land \eta = a_{0.5} \lor b_{0.3}\) is not fuzzy \(\theta\)-semi-generalized closed set since \(\text{scl}_\theta(\mu \land \eta) = a_{0.5} \lor b_{0.5} \neq a_{0.1} \lor b_{0.5}\) where \(a_{0.1} \lor b_{0.5} \in FSO(X, \tau)\).

**Theorem 4.6.** If \(\mu\) is a fuzzy \(\theta\)-semi-generalized closed set in \(X\) and \(\mu \leq \beta \leq \text{scl}_\theta(\mu)\) then \(\beta\) is also a fuzzy \(\theta\)-semi-generalized closed set in \(X\).

*Proof.* Let \(\eta\) be a fuzzy semi-open subset of \(X\) such that \(\beta \leq \eta\). Then, \(\mu \leq \eta\). Since \(\mu\) is fuzzy \(\theta\)-semi-generalized closed, its follows that \(\text{scl}_\theta(\mu) \leq \eta\). Now, \(\beta \leq \text{scl}_\theta(\mu)\) implies \(\text{scl}_\theta(\beta) \leq \text{scl}_\theta(\text{scl}_\theta(\mu)) = \text{scl}_\theta(\mu)\). Thus, \(\text{scl}_\theta(\beta) \leq \eta\). This proves that \(\beta\) is also fuzzy \(\theta\)-semi-generalized closed subset of \(X\).

**Theorem 4.7.** Let \(\mu\) be a f-\(\theta\)-sg-closed subset of \((X, \tau)\). Then

(i) \(\text{scl}_\theta(\mu) - \mu\) does not contain a nonzero fuzzy semi-closed set.

(ii) \(\text{scl}_\theta(\mu) - \mu\) is f-\(\theta\)-sg-open set

*Proof.* (i) Let \(\mu\) be a fuzzy set of \((X, \tau)\) and suppose that there exists a nonzero fuzzy semi-closed subset \(\rho\) of \(X\) such that \(\rho \leq \text{scl}_\theta(\mu) - \mu\) and \(\rho \neq 0_X\). Now, \(\rho \leq \text{scl}_\theta(\mu) - \mu\), i.e. \(\rho \leq \mu^c\) which implies \(\mu \leq \rho^c\). Since \(\rho^c\) is fuzzy semi-open and \(\mu\) is f-\(\theta\)-sg-closed set, \(\text{scl}_\theta(\mu) \leq \rho^c\), i.e. \(\rho \leq (\text{scl}_\theta(\mu))^c\). Then \(\rho \leq \text{scl}_\theta(\mu) \land (\text{scl}_\theta(\mu))^c = 0_X\) and hence \(\rho = 0_X\) which is a contradiction.

(ii) Suppose that \(\mu\) be f-\(\theta\)-sg-closed and \(h\) be a fuzzy semi-closed set such that \(h \leq \text{scl}_\theta(\mu) - \mu\), then by (i), \(h\) is zero and therefore \(h \leq \text{sint}_\theta(\text{scl}_\theta(\mu) - \mu)\). Hence, \(\text{scl}_\theta(\mu) - \mu\) is f-\(\theta\)-sg-open by Theorem 4.4.

Recall that a fuzzy topological space \((X, \tau)\) is said to be fuzzy semi-\(T_{1/2}\) [8] if and only if every fuzzy semi-generalized closed set is fuzzy semi-closed in \(X\).

**Theorem 4.8.** A fuzzy topological space \((X, \tau)\) is said to be fuzzy semi-\(T_{1/2}\)-space if and only if:

(i) every fuzzy singleton is fuzzy semi-open or fuzzy semi-closed.

(ii) every fuzzy \(\theta\)-semi-generalized closed set is fuzzy semi-closed.

*Proof.* (i) Let \((X, \tau)\) be fuzzy semi-\(T_{1/2}\)-space and for some \(x_\alpha \in S(X)\), \(x_\alpha\) is not fuzzy semi-closed. Then \(1-x_\alpha\) is not fuzzy semi-open and hence \(1_X\) is the only fuzzy semi-open set containing \(1-x_\alpha\). Therefore, \(1-x_\alpha\) is fsg-closed in \((X, \tau)\). Since \(X\) is a fuzzy semi-\(T_{1/2}\)-space, then \(1-x_\alpha\) is fuzzy semi-closed set or equivalently \(x_\alpha\) is fuzzy semi-open set.
Conversely, assume that every singleton set of \((X, \tau)\) is either fuzzy semi-closed or fuzzy semi-open set. Let \(\mu\) be a fsg-closed set of \((X, \tau)\). Let \(x_\alpha \in \mathcal{S}(X)\) and we consider the following two cases:

**Case I:** Suppose that \(x_\alpha\) is fuzzy semi-closed. If \(x_\alpha \notin \mu\), then \(x_\alpha \in \text{scl}(\mu) - \mu\). Now \(\text{scl}(\mu) - \mu\) contains a nonzero fuzzy semi-closed set. Since \(\mu\) is fsg-closed set, by Theorem 4.7(i), which is contradiction. Hence \(x_\alpha \in \mu\).

**Case II:** Assume that \(x_\alpha\) is fuzzy semi-open. Since \(x_\alpha \in \text{scl}(\mu)\), then \(x_\alpha \wedge \mu \neq 0_X\). So, \(x_\alpha \in \mu\).

Thus in both case \(x_\alpha \in \mu\). So \(\text{scl}(\mu) \subseteq \mu\). Therefore \(\mu = \text{scl}(\mu)\) or equivalently \(\mu\) is a fuzzy semi-closed set. Thus every fsg-closed is fuzzy semi-closed set. Hence, \((X, \tau)\) is fuzzy semi-\(T_{1/2}\)-spaces.

(ii) **Necessity.** Let \(\mu\) be a \(f\)-\(\theta\)sg-closed set in \((X, \tau)\). By Lemma 3.2, \(\mu\) is fsg-closed set. Since \((X, \tau)\) is a fuzzy semi-\(T_{1/2}\)-space, \(\mu\) is fuzzy semi-closed set.

**Sufficiency.** Let \(x_\alpha \in \mathcal{S}(X)\). If \(x_\alpha\) is not fuzzy semi-closed, then \(1 - x_\alpha\) is not fuzzy semi-open set and thus the only superset of \(1 - x_\alpha\) is \(1_X\). So, \(1 - x_\alpha\) is \(f\)-\(\theta\)sg-closed. By hypothesis, \(1 - x_\alpha\) is fuzzy semi-closed or equivalently \(x_\alpha\) is fuzzy semi-open. Hence \((X, \tau)\) is a fuzzy semi-\(T_{1/2}\)-space. \(\Box\)

**Conclusion.**

In this paper, we introduce \(\theta\)-semi-generalized closed set in fuzzy setting. We also investigate the relationship of some generalized closed sets that already known which related to fuzzy \(\theta\)-semi-generalized closed set. Those give some new relationships which has been found to be useful in the study of generalized closed sets in fuzzy topological spaces.

**Acknowledgement.**

This research has been partially supported by the Universiti Malaysia Terengganu under the Geran Galakan Penyelidikan (GGP) Ref. No.: 68007/2013/80.

**References**

[1] Athar M and Ahmad B 2008 Fuzzy boundary and fuzzy semiboundary Hindawi Publishing Corporation. (doi: 10.1155/2008/586893)

[2] Azad K K. 1981 On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 14-32.

[3] Balasubramanian G and Sundaram P 1997 On some Generalizations of fuzzy continuous functions Fuzzy sets and systems. 86 93-100.

[4] Chang C L. 1968 Fuzzy topological spaces J. Math. Anal. Appl. 24 182-190.

[5] El-Shafei M E & Zakari A 2007 Semi-generalized continuous mappings in Fuzzy Topological Spaces Journal of the Egyptian Mathematical Society. 15 No.1 57-67.

[6] El-Shafei M E & Zakari A 2005 \(\theta\) - generalized closed sets in Fuzzy Topological Spaces. The Arabian Journal for Science and Engineering. 31 No.2A 197-206.

[7] Levine N. 1970. Generalized closed sets in topology, Rend. Circ. Mat. Palermo. 19 89-96.

[8] Maki H et.al 1998 Generalized closed sets in fuzzy topological spaces I. Meetings on Topological Spaces Theory and its Applications. 23-36.

[9] Malakar S 1992 On fuzzy semi- irresolute and strongly irresolute functions Fuzzy sets and systems. 45 239-244.

[10] Murugesan S & Thangavelu P 2008 Fuzzy Pre-semi-closed sets Bulletin of the Malaysian Mathematical Sciences Society. 31 No.2 223-232.
[11] Palaniappan N 2007, Fuzzy Topology, Second Edition, Oxford, Alpha Science International Ltd., 20.

[12] Saeid Jafari & Miguel Caldas 2003 On θ-semi-generalized closed set in Topology Kyungpook Math. J. 43 135-148.

[13] Seong Hoon Cho and Jae Keun Park 2003 A Note on Fuzzy Irresolute and Strongly Irresolute Functions Commun. Korean Math. Soc. 18 No.2 355-366.

[14] Sinha S P. On S*-Closedness in fuzzy setting, preprint.

[15] Yalvac T H 1988 Semi-interior and semi-closure of a fuzzy set J. Math. Anal. Appl. 132 No.2 356-364.

[16] Zadeh L A 1965 Fuzzy sets Information and control. 8 338-352.