Coupled-wire construction and quantum phase transition of two-dimensional fermionic crystalline higher-order topological phases

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Coupled-wire constructions have widely applied to quantum Hall system and symmetry protected topological (SPT) phases. In this Letter, we use the coupled one-dimensional nonchiral Luttinger liquids with domain-wall structured mass terms as quantum wires to construct the crystalline higher-order topological superconductors (HOTSC) in two-dimensional interacting fermionic systems by two representative examples: \(D_4\)-symmetric class-D HOTSC and \(C_4\)-symmetric class-BDI HOTSC, with Majorana corner modes on the edge. Furthermore, based on the coupled-wire constructions, the quantum phase transition between different phases of 2D HOTSC by tuning the inter-wire coupling are investigated in a straightforward way.

**Introduction** – Topological phases of quantum matter have become one of the greatest triumph of condensed matter physics since the discovery of fractional quantum Hall effect [1, 2]. Topological order defined by patterns of long-range entanglement provides a systematic way of understanding topological phases of quantum matter [3]. Furthermore, the interplay between symmetry and topology plays a central role in the topological phases of quantum matter. In particular, symmetry protected topological (SPT) phases have been systematically constructed and classified in short-range entangled systems [4–25]. A well-understood elegant example of SPT phases is topological insulator, which is protected by time-reversal and charge-conservation symmetry [26, 27]. Recently, crystalline SPT phases have been intensively studied [28–59], with great opportunities for experimental realization [60–63]. In particular, different from SPT phases with on-site symmetry, the boundaries of 2D crystalline SPT phases are usually almost gapped but with protected 0D corner zero modes, which are reflected by topological crystals composed by lower-dimensional block-states as crystalline symmetry domain walls. This type of topological phases are called higher-order topological phases [64–71].

The study of higher-order topological phases mainly focus on free-fermion systems, because interactions and crystalline symmetry restrictions limit the analytical study of lattice model, only numerics on finite size lattice can give some insights. On the other hand, a clear and powerful tool of studying topological phases of quantum matter is coupled-wire construction [72–80]. One decomposes a higher-dimensional system into an assembly of 1D quantum wires, and topological properties then arise from the suitable couplings of them. A unique advantage of coupled-wire construction is that different from higher-dimensional quantum field theory, the powerful bosonization technique of one-dimensional subsystems can be used to challenge the strong interaction effects. Different phases are manifested by patterns of coupled wires and the quantum phase transition of different phases is controlled by tuning inter-wire couplings directly. Therefore, an important open question arises: if the higher-order topological phases in strongly correlated systems can be constructed by coupled-wire perspective?

In this Letter, we systematically construct the crystalline HOTSC in two-dimensional interacting fermionic systems by coupling the 1D nonchiral Luttinger liquids with domain-wall structured mass terms as quantum wires, via two typical intriguing interacting examples: \(D_4\)-symmetric class-D HOTSC and \(C_4\)-symmetric class-BDI HOTSC, whose higher-order edge modes are Majorana zero modes [44, 45]. By suitable inter-wire tunneling/interaction, several 1D quantum wires are assembled and fully gapped, leaving few dangling quantum wires at the edge or near the center of the systems. Near the center, the dangling quantum wires are fully gapped by intra-wire interactions; on the edge, the dangling quantum wires explicitly manifest the higher-order topological edge modes of 2D HOTSC by their domain-wall structure. Different 2D higher-order topological phases are characterized by different patterns of coupled-wire. Furthermore, with concrete coupled-wire construction of 2D HOTSC, we directly investigate the quantum phase transitions by tuning different coupling constants of inter-wire interactions.

\(D_4\)-symmetric class-D HOTSC – For 2D \(D_4\)-symmetric systems with spinless fermions, there is an intriguing interacting 2D HOTSC with protected Majorana corner modes \(\xi_{1,2,3,4}\) that can be reformulated to complex fermions \(c^\dagger_{1,2,3,4} = (\xi_{1,2,3,4} + \xi^\dagger_{1,2,3,4})/\sqrt{2}\) (see Fig. 2). In this section we construct this phase by an “almost free” coupled-wires, with necessary interaction only defined near the \(D_4\)-center. These Majorana corner modes are also reformulated in terms of domain walls of 1D nonchiral Luttinger liquids [81]. Consider 2\(n\) decoupled 1D quantum wires with circular geometry (see Fig. 1), firstly we define a nonchiral Luttinger liquid as a 1D quantum wire with the Lagrangian:

\[
\mathcal{L}_0 = \frac{K_{ij}}{4\pi} \left( \partial_x \phi_i^j \right) \left( \partial_x \phi_i^j \right) + \frac{V_{ij}}{8\pi} \left( \partial_x \phi_i^j \right) \left( \partial_x \phi_j^j \right)
\]  

(1)

where \(\phi^j = (\phi_1^j, \phi_2^j)^T\) is the 2-component bosonic field of \(j^{th}\) quantum wire and \(K^j = \sigma^z\) as the K-matrix of the
FIG. 1. Coupled-wire construction of 2D fermionic crystalline higher-order topological phases. The 1\textsuperscript{st} wire is on the edge, and 2\textsuperscript{n}th wire is near the center.

topological term [82]. The total Lagrangian of decoupled wires is: \( \mathcal{L}_0 = \sum_{j=1}^{2n} \mathcal{L}_j^0 \). The \( D_4 \) symmetry properties of these bosonic fields are \((R \in C_4/M_1) \in \mathbb{Z}_2^M \) is rotation/reflection generator of \( D_4 = C_4 \times \mathbb{Z}_2^M \) symmetry):

\[
R: \begin{cases} 
\phi_1^1(\theta) \mapsto -\phi_1^1(\theta + \pi/2) \\
\phi_2^1(\theta) \mapsto -\phi_2^1(\theta + \pi/2) + \pi 
\end{cases}
M_1: \begin{cases} 
\phi_1^1(\theta) \mapsto -\phi_2^1(2\pi - \theta) + \pi/2 \\
\phi_2^1(\theta) \mapsto -\phi_1^1(2\pi - \theta) + \pi/2 
\end{cases}
\tag{2}
\]

To figure out the Majorana corner modes of \( D_4 \)-symmetric HOTSC, we should further introduce the mass term with domain wall structure of each quantum wire:

\[
\mathcal{L}_\text{wall}^j = m sin(2\theta) \cdot \cos \left[ \phi_1^1(\theta) + \phi_2^1(\theta) \right] 
\tag{3}
\]

where \( \mathcal{L}_\text{wall}^j \) is symmetric under (2), and \( \mathcal{L}_\text{wall} = \sum_{j=1}^{2n} \mathcal{L}_\text{wall}^j \). For each quantum wire with a domain-wall structured mass term, there are four complex fermion zero modes at poles of \( c_{1,2,3,4} \) of the circle, with \( \theta = 0, \pi/2, \pi, 3\pi/2 \), which are equivalent to eight Majorana corner modes, as illustrated in Fig. 2. These dangling 0D gapless modes cannot be gapped in a \( D_4 \)-symmetric way.

Subsequently we define two types of \( D_4 \)-symmetric (2) inter-wire tunneling that couple the \((2j + k)\)th and \((2j + 1 + k)\)th quantum wires \((m_1, m_2 < m, k = 1, 2)\):

\[
\mathcal{L}_\text{ck}^j = m_1 \sum_{\alpha=1}^{2} \cos \left[ \phi_\alpha^{2j+2+k}(\theta) - \phi_\alpha^{2j+1+k}(\theta) \right] 
\tag{4}
\]

and \( \mathcal{L}_{ck} = \sum_{j=1}^{n} \mathcal{L}_{ck}^j \). There are two extreme cases: \( m_1 \neq 0, m_2 = 0/m_1 = 0, m_2 \neq 0 \) corresponds to the phase that \( \mathcal{L}_{c1}/\mathcal{L}_{c2} \) dominates the inter-wire physics. For \( \mathcal{L}_{c1} \)-dominant phase, the \((2j - 1)\)th and \(2j\)th wires are paired up and gapped, hence the corresponding system is fully gapped on a open circle, and the corresponding phase is topological trivial.

For \( \mathcal{L}_{c2} \)-dominant phase, the \(2j\)th and \((2j + 1)\)th wires are paired up and gapped, hence all 1D quantum wires except \(1^{st}\) and \(2n^{th}\) are gapped. The \(1^{st}\) quantum wire on the edge of the system presents 4 complex fermion zero modes/8 Majorana zero modes at poles of circle, which are exactly the Majorana corner modes of 2D \( D_4 \)-symmetric HOTSC with spinless fermions. Near the \( D_4 \)-center, there are also gapless modes on the \(2n^{th}\) quantum wire which should be fully gapped in order to obtain a nontrivial HOTSC. Distinct from quantum wires away from \( D_4 \)-center, bosonic field \( \phi^{2n} \) of the \(2n^{th}\) quantum wires with different polar angles can tunnel to/interact with the field at other places. Consider two interacting terms of \(2n^{th}\) quantum wire near the \( D_4 \)-center:

\[
\mathcal{L}_{\text{int}} = m' \sum_{\beta=1}^{4} \cos \left( \sum_{\alpha=1}^{2} \left[ \phi_\alpha^{2n}(\theta) - \phi_\alpha^{2n}(\beta \pi - \theta) \right] \right) 
\tag{5}
\]

i.e., the intra-wire couplings of the \(2n^{th}\) quantum wire lead to a fully gapped bulk and an edge with dangling gapless modes. Equivalently, a nontrivial 2D \( D_4 \)-symmetric HOTSC with spinless fermions are described by 1D coupled quantum wires with Lagrangian \( \mathcal{L}_{\text{D4}} = \mathcal{L}_0 + \mathcal{L}_{\text{wall}} + \mathcal{L}_{c2} + \mathcal{L}_{\text{int}} \). The intriguing interacting nature of this HOTSC is reflected by \( \mathcal{L}_{\text{int}} \) near the \( D_4 \)-center. On the other hand, the physics away from the \( D_4 \)-center is well-understood on the noninteracting level. The classification of 2D class-D HOTSC is \( \mathbb{Z}_2 \), composed by phases dominated by inter-wire coupling \( \mathcal{L}_{c1} \) and \( \mathcal{L}_{c2} \) (see Fig. 5).

\( C_4 \)-symmetric class-BDI HOTSC – For 2D BDI-class systems with \( C_4 \)-symmetry, there is another type of intriguing interacting 2D HOTSC with protected Majorana corner modes \( \xi_{1,2,3,4} \) and \( \xi'_{1,2,3,4} \) (similar to above case, see Fig. 2) [44]. In this section we construct this phase by an “interacting” coupled-wires. Consider 4\( n \) decou-
decoupled 1D quantum wires is a K\textsuperscript{j} bosonic field of φ\textsuperscript{n}\textsuperscript{ch} Luttinger liquids described by Lagrangian like

Each four quantum wires with narrower intervals are coupled by inter-wire interactions and fully gapped.

\begin{align}
\mathcal{L}_\text{wall} = m \sum_{\alpha=1}^{2} \cos \left( \theta - \frac{\alpha \pi}{2} \right) \cdot \cos \left[ \phi_\alpha^j(\theta) - \phi_{5-\alpha}^j(\theta) \right] 
\end{align}

and \( \mathcal{L}_\text{wall} = \sum_{j=1}^{4n} \mathcal{L}_\text{wall} \). For each quantum wire, there are 4 gapless complex fermions \( c_{1,2,3,4} \) (8 Majorana zero modes \( \xi_{1,2,3,4} \) and \( \xi'_{1,2,3,4} \)) at poles of the circle, two of them at north and south poles are from the first term of (8) and other two at east and west poles are from the second term of (8). These dangling gapless modes cannot be gapped in a \((C_4 \times Z_2^T)\)-symmetric way.

Subsequently we consider \((C_4 \times Z_2^T)\)-symmetric [cf. Eqs. (6) and (7)] inter-wire interactions including four 1D quantum wires \( (k = 1, 2, 3, 4) \) [82]:
order to obtain a HOTSC, we should further add some intra-wire interactions to fully gap the $4n$th quantum wire in order to get a fully-gapped bulk state. Consider the 4-body interacting terms of $4n$th quantum wire, composed by the backscatterings of bosonic fields $\phi_{A,B,C,D}^{4n}$ with different polar angles:

$$\mathcal{L}_{\text{int}} = m^2 \sum_{\alpha, \beta = 1}^{2} \cos \left[ \phi_{A}^{4n}(\theta) - \phi_{B}^{4n}(\theta) + \phi_{C}^{4n}(\theta + \beta \pi/2) - \phi_{D}^{4n}(\theta + \beta \pi/2) \right]$$

i.e., the intra-wire interactions of the $4n$th quantum wire lead to a fully gapped bulk, and a nontrivial 2D ($C_4 \times Z_4^2$)-symmetric HOTSC with spinless fermions are described by 1D coupled quantum wires with Lagrangian $\mathcal{L}_{\text{BDI}}^{C_4} = \mathcal{L}_0 + \mathcal{L}_{\text{wall}} + \mathcal{L}_{c4} + \mathcal{L}_{\text{int}}$. Similar for $\mathcal{L}_{c2}$ and $\mathcal{L}_{c3}$ dominant phases, and there are 4 topological distinct phases for 2D BDI-class ($C_4 \times Z_4^2$)-symmetric system, see Fig. 3. The interacting nature of these topological phases are reflected by inter-wire interactions $\mathcal{L}_{\text{int}}$ and intra-wire interactions near the $C_4$-center, $\mathcal{L}_{\text{int}}$. The classification of 2D class-BDI HOTSC is $\mathbb{Z}_4$, composed by phases dominated by inter-wire couplings $\mathcal{L}_{\text{int}} (k = 1, 2, 3, 4)$.

Quantum phase transition of HOTSC – Coupled-wire picture serves a unique platform for investigating the quantum phase transition (QPT) of 2D HOTSC because of its clear formulations. In this section, we elucidate the QPT of 2D intriguing interacting $D_4$-symmetric HOTSC as a representative example. Consider the $D_4$-symmetric Lagrangian $\mathcal{L}_0 + \mathcal{L}_{c1} + \mathcal{L}_{c2} + \mathcal{L}_{\text{int}}$, above we have discussed two extreme cases with $m_1 = 0/m_2 = 0$, derive two distinct phases characterized by appearance of Majorana corner modes on the edge (1st quantum wire). Now we suppose $m = 10m_1$ and set both $m_1$ and $m_2$ finite and study the possible QPT by tuning their ratio $m_2/m_1$. As summarized in Fig. 4, turn on $m_2$ in $m_2 < m_1$ regime, the system remains fully gapped with narrower gap; at $m_1 = m_2$, the gap closes and the system becomes critical; keep increasing $m_2$ toward $m_2 > m_1$ regime, the system reopens a bulk gap but leaving several gapless modes on the edge, which are exactly the Majorana domain walls of 1D quantum wire on the edge. Therefore, we conclude that there is a clear quantum phase transition from trivial state to 2D $D_4$-symmetric HOTSC at $m_1 = m_2$ point. Equivalently, this quantum phase transition is characterized by different inter-wire entanglement patterns of 1D quantum wires, as illustrated in Fig. 5.

For 2D $C_4$-symmetric class-BDI system, the quantum phase transitions can be described in a similar way with little complications. For this case, there are four distinct phases controlled by four different parameters $m_{1,2,3,4}$ (see Fig. 3). As an example, for the quantum phase transition between phase-2 and phase-3, we set $m_1 = m_4 = 0$ and investigate the bulk gap by tuning the ratio $m_2/m_3$. Heuristically, we see that the system will be critical for $m_2 = m_3$, hence there will be a quantum phase transition at this point [82]. As a matter of fact, distinct phases of 2D HOTSC are controlled by different patterns of inter-wire entanglements, and their quantum phase transitions can be manipulated by tuning the intensities of different types of inter-wire couplings. In other words, coupled-wire construction provides a straightforward way of comprehending the quantum phase transitions of 2D HOTSCs, by tuning the inter-wire couplings to control the patterns of inter-wire entanglement directly.

Conclusion and Discussion – Coupled-wire construction is a celebrated aspect in topological phases of quantum matter, for both long-range and short-range entangled systems. In this Letter, we establish the coupled-wire construction of 2D intriguing interacting fermionic crystalline HOTSC, with two representative examples: 2D $D_4$-symmetric class-D and $C_4$-symmetric class-BDI HOTSC phases. An indispensable advantage of coupled-wire construction is that the powerful bosonization technique can be utilized, and the inter-wire couplings can be straightforwardly involved by many-body backscattering terms in the Lagrangian. With this advantage, we use the 1D nonchiral Luttinger liquid with a domain-wall structured mass term as a “almost gapped” 1D quan-
quantum wire (only several dangling 0D modes are gapless). Based on these quantum wires, we introduce some suitable inter-wire couplings in order to gap out the bulk by assemblies of quantum wires. The remaining ungapped quantum wires on the edge are treated as the edge theory of 2D HOTSC, and near the center of point group, the ungapped quantum wires are gapped by interactions of bosonic fields at different places. Distinct HOTSC phases are manifested by different patterns of inter-wire entanglement. Furthermore, the concrete coupled-wire constructions serve a straightforward way to comprehend the quantum phase transitions of 2D HOTSCs, by directly tuning the inter-wire couplings to control the inter-wire entanglement patterns. The coupled-wire construction can also be generated to the systems with arbitrary crystalline symmetry $SG$ and internal symmetry $G_0$ in arbitrary dimensions, with more complicated inter-wire entanglement patterns, and the quantum phase transitions of crystalline higher-order topological phases in arbitrary dimensions should also be controlled by inter-wire entanglement patterns of quantum wires.

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[82] see Supplementary Materials for more details.
Supplemental Materials

I. K-matrix formalism of fSPT

In the main text, we define the 1D quantum wire based on the nonchiral Luttinger liquid with topological K-matrix. In this section we review the K-matrix formalism of fermionic symmetry-protected topological (fSPT) phases. A U(1) Chern-Simons theory has the form:

$$\mathcal{L} = \frac{K_{I J}}{4\pi} e^{\mu\nu\lambda} \partial_{\mu} a_{I}^{\dagger} \partial_{\nu} a_{J}^{\dagger} + a_{I}^{\dagger} a_{I} + \cdots$$  \hspace{1cm} (S1)

where $K$ is a symmetric integral matrix, $\{a^{I}\}$ is a set of one-form gauge fields, and $\{j_{I}\}$ are the corresponding currents that couple to the gauge fields $a^{I}$. The symmetry is defined as: two theories $\mathcal{L}[a^{I}]$ and $\mathcal{L}[\tilde{a}^{I}]$ correspond to the same phase if there is an $n \times n$ integral unimodular matrix $W$ satisfying $\tilde{a}^{I} = W_{I J} a^{J}$.

The topological order described by Abelian Chern-Simons theory hosts Abelian anyon excitations. An anyon is labeled by an integer vector $l = (l_{1}, l_{2}, \ldots, l_{n})$. The self and mutual statistics of anyons are:

$$\theta_{I} = \pi l^{T} K^{-1} l$$
$$\theta_{I, I'} = 2\pi l^{T} K^{-1} l'$$  \hspace{1cm} (S2)

The total number of anyons and the ground-state degeneracy (GSD) on a torus are both given by $|\text{det}K|$. For SPT phase, there is no GSD or anyon, hence we require $|\text{det}K| = 1$ for SPT phases.

The K-matrix Chern-Simons theory has a well-known bulk-boundary correspondence [1, 2]. In a system with open boundary, the edge theory of (S1) has the form:

$$\mathcal{L}_{\text{edge}} = \frac{K_{I J}}{4\pi} (\partial_{\nu} \phi^{I}) (\partial_{\nu} \phi^{J}) + \frac{V_{I J}}{8\pi} (\partial_{\nu} \phi^{I}) (\partial_{\nu} \phi^{J})$$  \hspace{1cm} (S3)

where $\phi = \{\phi^{I}\}$ are chiral bosonic fields on the edge related to dynamical gauge field $a_{I}^{\dagger}$ in the bulk by $a_{I}^{\dagger} = \partial_{\mu} \phi^{I}$, and an anyon on the edge can be created by the operator $e^{il^{T} \phi}$.

II. Assembly of quantum wires of $C_{4}$-symmetric class-BDI HOTSC

In the main text, the inter-wire couplings of 2D $C_{4}$-symmetric class-BDI HOTSC are defined by backscatterings of four 1D quantum wires as an assembly. In this section we demonstrate that the minimal number of quantum wires of an assembly should be four, equivalently, two quantum wires cannot be gapped in a symmetric way.

The 1D quantum wire building block for coupled-wire construction of 2D $C_{4}$-symmetric class-BDI HOTSC is described by 1D nonchiral Luttinger liquid on a circle, with the Lagrangian:

$$\mathcal{L}_{0} = \frac{K_{I J}}{4\pi} (\partial_{\nu} \phi^{I}) (\partial_{\nu} \phi^{J}) + \frac{V_{I J}}{8\pi} (\partial_{\nu} \phi^{I}) (\partial_{\nu} \phi^{J})$$  \hspace{1cm} (S4)

where $\theta$ is the polar angle of the circle, $\phi^{I}(\theta) = (\phi_{1}^{I}(\theta), \phi_{2}^{I}(\theta), \phi_{3}^{I}(\theta), \phi_{4}^{I}(\theta))^{T}$ as 4-component bosonic fields of $j$th quantum wire, and $K^{I J} = \sigma^{+} \otimes \sigma^{z}$ as the topological K-matrix. In the main text, the (C4 $\times Z_{2}$) symmetry has been defined on the bosonic fields as Eqs. (6) and (7), which can be reformulated to:

$$R : \phi^{I} \mapsto W_{R}^{I J} \phi^{J} + \delta \phi_{R}^{I}$$
$$T : \phi^{I} \mapsto W_{T}^{I J} \phi^{J} + \delta \phi_{T}^{I}$$  \hspace{1cm} (S5)

where $(\Theta)$ is the operator that transforms the polar angle $\theta \mapsto \theta + \pi/2$.

$$W_{R}^{I J} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \Theta, \quad \delta \phi_{R}^{I} = \frac{\pi}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$  \hspace{1cm} (S6)

and

$$W_{T}^{I J} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \quad \delta \phi_{T}^{I} = \begin{pmatrix} 0 \\ \pi \\ \pi \\ 0 \end{pmatrix}$$  \hspace{1cm} (S7)

Now we consider two copies of such quantum wire and investigate whether they can be fully gapped out. The corresponding Lagrangian has the similar form to Eq. (S4), with $\phi(\theta) = (\phi_{1}(\theta), \phi_{2}(\theta))^{T}$ as 4-component bosonic fields, and $K = \sigma^{+} \otimes \sigma^{z}$ as the topological K-matrix. For this case, the C4 $\times Z_{2}$ symmetry is defined on $\phi$ as $(1_{2 \times 2} \otimes \Theta)$:

$$R : \phi \mapsto W_{R}^{I J} \phi + \delta \phi_{R}^{I}$$
$$T : \phi \mapsto W_{T}^{I J} \phi + \delta \phi_{T}^{I}$$  \hspace{1cm} (S8)

where

$$W_{R} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix} \otimes 1_{2 \times 2} \cdot \Theta$$  \hspace{1cm} (S9)

$$\delta \phi_{R}^{I} = \delta \phi_{R}^{1} \oplus \delta \phi_{R}^{2}, \quad \delta \phi_{T}^{I} = \delta \phi_{T}^{1} \oplus \delta \phi_{T}^{2},$$

and

$$W_{T} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \otimes 1_{2 \times 2}$$  \hspace{1cm} (S10)

We now try to construct interaction terms that gap out the edge without breaking $R$ and $T$ symmetries, neither
explicitly nor spontaneously. Consider the backscattering term of the form:

$$ U = \sum_k U(\Lambda_k) = \sum_k U(\theta) \cos [\Lambda_k^T K \phi - \alpha(\theta)]$$  \hspace{1cm} (S11)

The backscattering term (S11) can gap out the edge as long as the vectors \( \{\Lambda_k\} \) satisfy the “null-vector” conditions [3] for \( \forall i, j \):

$$ \Lambda_i^T K \Lambda_j = 0 $$  \hspace{1cm} (S12)

For the present case, there are only two linear independent solutions to this problem:

$$ \Lambda_1^T = (1, 0, 0, 1, 0, 1, 0, 1)^T $$
$$ \Lambda_2^T = (0, 1, 1, 0, 0, 1, 1, 0)^T $$  \hspace{1cm} (S13)

Nevertheless, there are eight bosonic fields \( \phi^{1,2} \), we should introduce at least 4 independent backscattering terms to fully gap them out, hence we cannot fully gap out the two copies of 1D quantum wires for the cases of 2D \( C_4 \)-symmetric class-BDI HOTSCs.

For the case with 4 copies of 1D quantum wires, we have introduced eight linear independent 4-component backscattering terms in the main text that can fully gap all four quantum wires. As the consequence, in the coupled-wire construction of 2D \( C_4 \)-symmetric class-BDI HOTSCs, we should assemble the 1D quantum wires by four, hence the root phase is modulo-4, and the corresponding classification from the coupled-wire constructions is \( \mathbb{Z}_4 \).

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