Braiding and exponentiating noncommutative vector fields

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Abstract The purpose of this paper is to put into a noncommutative context basic notions related to vector fields from classical differential geometry. The manner of exposition is an attempt to make the material as accessible as possible to classical geometers. The definition of vector field used is a specialisation of the Cartan pair definition, and the paper relies on the idea of generalised braidings of 1-forms. The paper considers Kroneker deltas, interior products, Lie derivatives, Lie brackets, exponentiation of vector fields and parallel transport.

1 Introduction

Classical differential geometry is heavily reliant on the use of vector fields, and they also provide an intuitive way to think about the geometry, linking with ideas of flow or motion from physics. However noncommutative differential geometry has been largely concerned with forms. In this paper I have used a specialisation of the Cartan pair definition of a noncommutative vector field. My intention was to try to formulate noncommutative analogues of certain classical constructions requiring vector fields, especially the interior product. Note that the Cartan pair definition of vector fields on Hopf algebras was considered in [7].

The feature which allows us to make any sense of many classical constructions in the noncommutative world is the generalised ‘braiding’ (in some cases this word is interpreted rather loosely) between bimodules and 1-forms, which is described in [8]. If we consider the braiding in the commutative case, it is just order reversal of forms or vector fields. The fact that this is an honest braiding (i.e. satisfies the braid relation), and that it precisely determines all the differential forms given just the 1-forms by antisymmetry, becomes the dominant feature of the commutative case. Also of great geometrical importance is the interior product, a pairing between the vector fields and forms which reduces the degree of the form by one. A noncommutative differential calculus with these features (such as the calculus for the noncommutative torus given in [4]) behaves more or less the same as a commutative differential calculus.
One seemingly strange feature is the number of conditions needed on the differential calculus for some of the results to hold. A little while spent constructing differential calculi on algebras given in terms of generators and relations will reveal a reason for this. There are often a very large number of possible differential calculi once the constraints of commutativity are removed, and some calculi, and some covariant derivatives, are nicer than others. In particular we arrive at an idea of a compatibility between the differential calculus and covariant derivatives and their associated braidings.

The paper begins with standard material \[6\] on differential calculi and connections. Then it considers paired connections and braidings on vector fields. The central idea introduced in the paper is a noncommutative analogue of interior product of a vector field with an \(n\)-form. From here it is not difficult to introduce the Lie derivative of an \(n\)-form. Antisymmetric tensor products of fields are introduced, and are used to define vector field versions of curvature and torsion, as well as an idea of Lie bracket. From a noncommutative Kroneker delta we can define a differential dimension of the algebra, which depends on the differential calculus and the associated braiding. The example of the noncommutative torus \[4\] is considered, and proves to be very like the classical case. The noncommutative sphere \[5\] illustrates some rather less classical behaviour.

The paper ends by considering noncommutative analogues of exponentiation of vector fields, parallel transport and geodesics. The problem here is that the result of an exponentiation is not in general an algebra map. However it retains the structure of a cochain map, and is shown to be well behaved under the coaction of a Hopf algebra on the algebra. In the process of doing this we must consider exponentials of ‘Lie algebra’ elements for the Hopf algebra. An example of exponentiation is given on the noncommutative torus.

In the notation, I have made use of overloading certain symbols, with distinction being made by considering the domains, rather than have a multiplicity of symbols or indices. I use \(id^n\) to be \(id \otimes id \otimes \ldots \otimes id\) \(n\) times. All algebras are assumed to be unital and associative.

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2 Noncommutative differential calculi

Definition 2.1 A differential structure on an algebra $B$ is a graded algebra $\Omega^n B$ for $n \geq 0$ (i.e. there is a multiplication $\cdot : \Omega^n B \otimes \Omega^m B \to \Omega^{n+m} B$) with $\Omega^0 B = B$. In particular the graded algebra structure makes every $\Omega^n B$ into an $\Omega^0 B = B$-bimodule, and we use a dot for this operation, rather than $\cdot$. To every differential form $\omega \in \Omega^n B$ we assign a grade $|\omega| = n$. There is a differential $d : \Omega^n B \to \Omega^{n+1} B$ with $d^2 = 0$ and $d(\tau \wedge \omega) = d\tau \wedge \omega + (-1)^{|\tau|} \tau \wedge d\omega$. In addition we assume that $B.d\Omega^n B$ and $d\Omega^n B.B$ are dense in $\Omega^{n+1} B$.

Definition 2.2 Given differential structures on algebras $B$ and $C$, an algebra map $f : B \to C$ is called differentiable if there is a well defined map $f_* : \Omega^1 B \to \Omega^1 C$ defined by $f_*(b db') = f(b) df(b')$. Here ‘well defined’ means that if a sum of elements of the form $b db'$ vanishes in $\Omega^1 B$, then the corresponding sum of $f(b) df(b')$ vanishes in $\Omega^1 C$.

Then $f_*$ is a $B$-$B$ bimodule map, where the left and right action by $b \in B$ on $\Omega^1 C$ is respectively left and right multiplication by $f(b)$. If $g : C \to E$ is also differentiable, then $g \circ f : B \to E$ is differentiable, and $(g \circ f)_* = g_* \circ f_* : \Omega^1 B \to \Omega^1 E$.

Definition 2.3 Given differentiable structures on algebras $B$ and $C$, the tensor product differential structure on $B \otimes C$ is defined by $\Omega^1 (B \otimes C) = (\Omega^1 B \otimes \Omega^1 C) \oplus (B \otimes \Omega^1 C)$ and $d(b \otimes c) = db \otimes c + b \otimes dc$. We use this splitting to define projections $\Pi_1 : \Omega^1 (B \otimes C) \to \Omega^1 B \otimes C$ and $\Pi_2 : \Omega^1 (B \otimes C) \to B \otimes \Omega^1 C$. These splittings obey the functorial conditions that if $f : B \to M$ and $g : C \to A$ are differentiable algebra maps, then $\Pi_1 \circ (f \otimes g)_* = (f_* \otimes g) \circ \Pi_1 : \Omega^1 (B \otimes C) \to \Omega^1 M \otimes A$ and $\Pi_2 \circ (f \otimes g)_* = (f \otimes g_*) \circ \Pi_2 : \Omega^1 (B \otimes C) \to M \otimes \Omega^1 A$.

3 Covariant derivatives and braidings with 1-forms

We shall take $M$ to be an algebra with a specified differential calculus. If $M$ were the algebra of functions on a topological space, then given a bundle over the space, the sections of the bundle form a module. In the noncommutative setting, we consider modules in place of bundles.

Definition 3.1 Given a left $M$-module $E$, a left $M$-covariant derivative is a map $\nabla : E \to \Omega^1 M \otimes_M E$ which obeys the condition $\nabla(m.e) = dm \otimes e + m.\nabla e$ for all $e \in E$ and $m \in M$. 
Remark 3.2 The tensor product over \( M, F \otimes_M E \) for a left \( M \)-module \( E \) and a right \( M \)-module \( F \), is like the usual tensor product of vector spaces, but in addition we make the identification \( f.m \otimes e = f \otimes m.e \) for all \( f \in F, e \in E \) and \( m \in M \).

Definition 3.3 (See [8].) A bimodule covariant derivative on an \( M \)-bimodule \( E \) is a pair \((\nabla, \sigma)\), where \( \nabla : E \to \Omega^1 M \otimes_M E \) is a left \( M \)-covariant derivative, and \( \sigma : E \otimes_M \Omega^1 M \to \Omega^1 M \otimes_M E \) is a bimodule map called the ‘braiding’ (even when it isn’t) obeying

\[
\nabla(e.m) = \nabla(e).m + \sigma(e \otimes dm).
\]

Remark 3.4 Of course, given a left \( M \)-covariant derivative \( \nabla \) on a bimodule \( E \), we can try to define a compatible left braiding by \( \sigma(e \otimes a db) = \nabla(e.ab) - \nabla(e.a).b \). However this might not give a well defined result, but we do see that there is at most one braiding compatible with a given connection. From this formula we also see (using the fact that \( d \) is a derivation) that the resulting \( \sigma \) (if well defined) is an \( M \)-bimodule map. This means, as we are only concerned with braidings compatible with connections, that there is no point in weakening the definition of a braiding to a left or right module map.

Proposition 3.5 Given \((\nabla', \sigma')\) a bimodule covariant derivative on the bimodule \( E \), any other left covariant derivative on the bimodule \( E \) is of the form \( \nabla = \nabla' + \Gamma \), where \( \Gamma : E \to \Omega^1 M \otimes_M E \) is a left \( M \)-module map. We get \((\nabla, \sigma)\) a bimodule covariant derivative if and only if the braiding \( \sigma(e \otimes a db) = \sigma'(e \otimes a db) + \Gamma(e.ab) - \Gamma(e.a).b \) is well defined. In particular \( \sigma = \sigma' \) if and only if \( \Gamma \) is an \( M \)-bimodule map.

Proof Straightforward. \( \square \)

Proposition 3.6 (See [8].) Given \((\nabla, \sigma_E)\) a bimodule covariant derivative on the bimodule \( E \) and \( \nabla \) a left covariant derivative on the left module \( F \), there is a left \( M \)-covariant derivative on \( E \otimes_M F \) given by \( \nabla \otimes \text{id}_F + (\sigma_E \otimes \text{id}_F)(\text{id}_E \otimes \nabla) \). Further if \( F \) is also an \( M \)-bimodule with a bimodule covariant derivative \((\nabla, \sigma_F)\), then there is a compatible braiding on \( E \otimes_M F \) given by \( \sigma_E \otimes_F = (\sigma_E \otimes \text{id})(\text{id} \otimes \sigma_F) \).
Proof Applying the formula to \(e \otimes m.f\) we get \(\nabla e \otimes m.f + (\sigma_E \otimes \text{id}_F)(e \otimes dm \otimes f + e \otimes m.\nabla f)\).

Applying the formula to \(e.m \otimes f\) we get \(\nabla (e.m) \otimes f + (\sigma_E \otimes \text{id}_F)(e.m \otimes \nabla f)\), and these are the same by definition of \(\sigma_E\). This shows that the given formula is well defined on \(E \otimes_M F\). The left multiplication property is true because \(\sigma_E\) is a left \(M\)-module map.

For the second part, we use the formula from 3.4

\[
\sigma_E \otimes_F (e \otimes f \otimes a.db) = \nabla (e \otimes f.ab) - \nabla (e \otimes f.a).b
= \nabla (e \otimes f.ab) + (\sigma_E \otimes \text{id})(e \otimes \nabla (f.ab))
- \nabla (e \otimes f.a).b - (\sigma_E \otimes \text{id})(e \otimes \nabla (f.a)b).
\]

\[
\Box
\]

Definition 3.7 A left module map \(\theta : E \to F\) is said to be preserved by the covariant derivatives \(\nabla\) on \(E\) and \(F\) if \(\nabla \circ \theta = (\text{id} \otimes \theta)\nabla : E \to \Omega^1_M \otimes_M F\).

Proposition 3.8 Given bimodules \(E\) and \(F\) with left covariant derivatives \((\nabla, \sigma)\), the bimodule map \(\theta : E \to F\) obeys the condition \((\text{id} \otimes \theta)\sigma = \sigma(\theta \otimes \text{id})\) if and only if the map \(\nabla \circ \theta - (\text{id} \otimes \theta)\nabla : E \to \Omega^1_M \otimes_M F\) is an \(M\)-bimodule map.

Proof The left module map property of \(\nabla \theta - (\text{id} \otimes \theta)\nabla\) is fairly simple. The right module map property is given by subtracting the following equations:

\[
(\text{id} \otimes \theta)(e \otimes db) = (\text{id} \otimes \theta)(\nabla (e.b) - \nabla (e).b),
\sigma(\theta(e) \otimes db) = \nabla (\theta(e).b) - \nabla (\theta(e)).b. \quad \Box
\]

Example 3.9 The simplest \(M\)-bimodule is \(M\) itself. Unless otherwise stated, we take the covariant derivative \(\nabla = d : M \to \Omega^1_M \otimes_M M = \Omega^1_M\). The corresponding \(\sigma\) is the identity.

Definition 3.10 Given a left covariant derivative \(\nabla\) on a left \(M\)-module \(F\) and a left submodule \(G \subset F\), \(\nabla\) is said to restrict to \(G\) if \(\nabla G \subset \Omega^1_M \otimes_M G\). If a covariant derivative preserves a sub-bimodule, then its associated braiding also preserves the sub-bimodule, i.e. \(\sigma(G \otimes_M \Omega^1_M) \subset \Omega^1_M \otimes_M G\).

4 Finitely generated projective modules

General modules over algebras can be quite badly behaved, so here we offer a definition and some results about a well known nice class of modules, the finitely generated projective modules. See [II] for more details.
Definition 4.1 The dual $E^*$ of a right $M$-module $E$ is defined to be $\text{Hom}_M(E, M)$, the right module maps from $E$ to $M$. Then $E^*$ has a left module structure given by $(m.\alpha)(e) = m.\alpha(e)$ for all $\alpha \in E^*$ and $e \in E$. If $E$ is a bimodule, then $E^*$ has a right module structure given by $(\alpha.m)(e) = \alpha(m.e)$, and there is a bimodule map evaluation $\text{ev} : E^* \otimes_M E \to M$.

Definition 4.2 A right $M$-module $E$ is said to be finitely generated projective if there are $e_i \in E$ and $\alpha_i \in E^*$ (for integer $1 \leq i \leq n$) (the ‘dual basis’) so that for all $e \in E$, $e = \sum e_i.\alpha_i(e)$. From this it follows directly that $\alpha = \sum \alpha(e_i).\alpha_i$ for all $\alpha \in E^*$.

Example 4.3 This condition may seem rather esoteric, but it has a simple example. In $C^\infty(\mathbb{R}^n)$ with coordinates $\{x^1, \ldots, x^n\}$ the sections $\Omega^1(\mathbb{R}^n)$ of the cotangent $T^*\mathbb{R}^n$ bundle has a module basis $dx^1 \ldots dx^n$ (i.e. every section of $T^*\mathbb{R}^n$ can be written as a sum of functions times the basis elements). The dual basis is $\partial_j \in \Omega^1(\mathbb{R}^n)^*$ $(1 \leq j \leq n)$ where $\partial_j(dx^i) = \delta^i_j$. The reader should note that the dual basis $(dx^i, \partial_i)$ is definitely not unique, though we will see shortly that a unique object can be made by combining them. Classically the complication comes when considering a manifold made by patching together coordinate charts. Then we have to apply partitions of unity to the previous construction on each coordinate chart. Of course, the dual of the 1-forms is the vector fields, but we should save that fact for later.

Proposition 4.4 If an $M$-bimodule $E$ is finitely generated projective, and $F$ is a right $M$-module, there is an isomorphism $\vartheta : F \otimes_M E^* \to \text{Hom}_M(E, F)$ defined by $\vartheta(f \otimes \alpha)(e) = f.\alpha(e)$.

Proof The inverse map is $\vartheta^{-1}(T) = \sum T(e_i) \otimes \alpha_i$. □

Proposition 4.5 If an $M$-bimodule $E$ is finitely generated projective, and $F$ is a left $M$-module, there is an isomorphism $\varphi : E \otimes_M F \to M\text{Hom}(E^*, F)$ (the left module maps from $E^*$ to $F$) defined by $\varphi(e \otimes f)(\alpha) = \alpha(e).f$.

Proof The inverse map is $\varphi^{-1}(T) = \sum e_i \otimes T(\alpha_i)$. □

Corollary 4.6 Suppose that we have a map $T : F \to H$ between left $M$-modules, with kernel $K \subset F$. Then for a finitely generated projective $M$-bimodule $E$, the map $\text{id} \otimes T : E \otimes_M F \to E \otimes_M H$ has kernel $E \otimes_M K$. 

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Proof If we use the isomorphism in 4.5, we get the map $T \circ M \text{Hom}(E^*, F) \to M \text{Hom}(E^*, H)$, and this has kernel $M \text{Hom}(E^*, K)$. □

5 Evaluations and coevaluations

From now on, we take all right modules considered in the paper to be finitely generated projective. Also suppose that the bimodules have a bimodule covariant derivative $(\nabla, \sigma)$, and that $\sigma$ is invertible.

Proposition 5.1 Given a bimodule covariant derivative $(\nabla, \sigma_E)$ on the $M$-bimodule $E$ for which the braiding is invertible, there is a unique bimodule covariant derivative $(\nabla, \sigma_{E^*})$ on $E^*$ so that the map $\text{ev} : E^* \otimes_M E \to M$ is preserved by the covariant derivatives (see 3.7). It is defined in terms of the dual basis $(e_i, \alpha_i)$ of $E$ given in 4.2 by

$$
\sigma_{E^*}(\alpha \otimes \xi) = \sum (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma^{-1}_E)(\alpha \otimes \xi \otimes e_i) \otimes \alpha_i,
$$

$$
\nabla \alpha = \sum d(\alpha(e_i)) \otimes \alpha_i - \sum (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma^{-1}_E)(\text{id} \otimes \nabla)(\alpha \otimes e_i) \otimes \alpha_i.
$$

Proof First we check that the formulae give a left covariant derivative:

$$
\nabla(m.\alpha) = \sum d(m.\alpha(e_i)) \otimes \alpha_i - \sum (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma^{-1}_E)(\text{id} \otimes \nabla)(m.\alpha \otimes e_i) \otimes \alpha_i
$$

$$
= dm \otimes \sum \alpha(e_i).\alpha_i + m.\nabla(\alpha) = dm \otimes \alpha + m.\nabla(\alpha).
$$

Given that the braiding on $M$ is trivial, the condition that $\text{ev} : E^* \otimes_M E \to M$ preserves the braiding is

$$
\text{ev} \otimes \text{id} = (\text{id} \otimes \text{ev})\sigma_{E^*} \otimes E = (\text{id} \otimes \text{ev})(\sigma_{E^*} \otimes \text{id})(\text{id} \otimes \sigma_E) : E^* \otimes E \otimes M^1 \to M^1 M.
$$

We check this by

$$
(\text{id} \otimes \text{ev})(\sigma_{E^*} \otimes \text{id})(\alpha \otimes \xi \otimes e) = \sum (\text{id} \otimes \text{ev})((\text{ev} \otimes \text{id})(\text{id} \otimes \sigma^{-1}_E)(\alpha \otimes \xi \otimes e_i) \otimes \alpha_i \otimes e)
$$

$$
= \sum (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma^{-1}_E)(\alpha \otimes \xi \otimes e_i.\alpha_i(e))
$$

$$
= (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma^{-1}_E)(\alpha \otimes \xi \otimes e).
$$

(2)

The $\sigma_{E^*}$ with this property (2) is unique by 4.3. To see that $\nabla$ preserves the evaluation:

$$
(\text{id} \otimes \text{ev})(\nabla(\alpha) \otimes e) = \sum d(\alpha(e_i)).\alpha_i(e) - \sum (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma^{-1}_E)(\text{id} \otimes \nabla)(\alpha \otimes e_i).\alpha_i(e)
$$
\[
= d(\alpha(e)) - (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma_E^{-1})(\text{id} \otimes \nabla)(\alpha \otimes e)
- \sum \alpha(e_i) d(\alpha_i(e)) + \sum (\text{ev} \otimes \text{id})(\alpha \otimes e_i \otimes d(\alpha_i(e)))
= d(\alpha(e)) - (\text{id} \otimes \text{ev})(\sigma_{E^*} \otimes \text{id})(\text{id} \otimes \nabla)(\alpha \otimes e) .
\] (3)

The \(\nabla\) with this property (3) is unique by \(4.4\). Finally we check the compatibility condition in \(3.3\), using (3):

\[
(\text{id} \otimes \text{ev})(\nabla(\alpha.m \otimes e))
= d((\alpha.m)(e)) - (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma_E^{-1})(\text{id} \otimes \nabla)(\alpha.m \otimes e)
= d(\alpha(m.e)) - (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma_E^{-1})(\text{id} \otimes \nabla)(\alpha \otimes m.e)
+ (\text{ev} \otimes \text{id})(\text{id} \otimes \sigma_E^{-1})(\alpha \otimes dm \otimes e)
= (\text{id} \otimes \text{ev})(\nabla(\alpha.m \otimes e)) + (\text{id} \otimes \text{ev})(\sigma_{E^*} \otimes \text{id})(\alpha \otimes dm \otimes e) .
\] □

Definition 5.2 Given an \(M\)-bimodule \(E\), the Kroneker delta \(\delta_E \in E \otimes_M E^*\) is defined so that
\[
(\text{id}_E \otimes \text{ev})(\delta_E \otimes e) = e \text{ for all } e \in E. \text{ In terms of tensor categories, } \delta_E \text{ is a coevaluation. For } E \text{ finitely generated projective (see }4.2\text{), we have } \delta_E = \sum e_i \otimes \alpha_i.
\]

Proposition 5.3 For a \(\delta_E\) given in 5.2:

a) \(\delta_E\) is unique.
b) \(m.\delta_E = \delta_E.m\) for all \(m \in M\).
c) \(\sigma_{E \otimes M} E^*(\delta_E \otimes \xi) = \xi \otimes \delta_E\) for all \(\xi \in \Omega^1 M\).
d) \(\nabla(\delta_E) = 0\).

Proof \(\delta_E\) corresponds to the identity map under the isomorphism in \(4.4\) proving (a). By \(4.4\) again, to prove (b) we only have to show that
\[
m.e = (\text{id} \otimes \text{ev})(\delta_E.m \otimes e) .
\] (4)

But the right hand side of (4) is
\[
\sum e_i.(\alpha_i.m)(e) = \sum e_i.(\alpha_i)(m.e) = m.e
\]
for all \(e \in E\) as required. By \(4.4\) again and using the fact that \(\sigma_E\) is invertible, to prove (c) we only have to show that, for all \(\xi \in \Omega^1 M\) and \(e \in E\),
\[
\sigma_E(e \otimes \xi) = (\text{id}^2 \otimes \text{ev})(\sigma_{E \otimes E^*} \otimes \text{id})(\delta_E \otimes \sigma_E(e \otimes \xi)) .
\] (5)


From (1), the right hand side of (5) is

\[(\text{id}^2 \otimes \text{ev})(\sigma_E \otimes \text{id}^2)(\text{id} \otimes \sigma_{E^*} \otimes \text{id})(\delta_E \otimes \sigma_E(e \otimes \xi)) = \sigma_E(\text{id} \otimes \text{ev} \otimes \text{id})(\delta_E \otimes e \otimes \xi) = \sigma_E(e \otimes \xi),\]

as required. By 4.4 again, to prove (d) we only have to show that \((\text{id}^2 \otimes \text{ev})/(\nabla \delta_E \otimes e) = 0\) for all \(e \in E\). Then, using (3),

\[(\text{id}^2 \otimes \text{ev})(\nabla \delta_E \otimes e) = (\text{id}^2 \otimes \text{ev}) \sum \left(\nabla e_i \otimes \alpha_i \otimes e + (\sigma_E \otimes \text{id}^2)(e_i \otimes \nabla \alpha_i \otimes e)\right) = \sum \left(\nabla(e_i \alpha_i(e)) - \sigma_E(e_i \otimes \text{ev})(\sigma_{E^*} \otimes \text{id})(\alpha_i \otimes \nabla e)\right) = \nabla(e) - \sum \sigma_E(e_i \otimes (\text{ev} \otimes \text{id})(e_i \otimes \nabla e)).\]

Now substitute \(\sigma^{-1}_E \nabla e = \sum f_j \otimes \eta_j \in E \otimes \Omega^1 M\), giving

\[(\text{id}^2 \otimes \text{ev})(\nabla \delta_E \otimes e) = \nabla(e) - \sum \sigma_E(e_i \otimes \alpha_i(f_j) \otimes \eta_j) = \nabla(e) - \sum \sigma_E(f_j \otimes \eta_j) = 0. \square\]

### 6 Vector fields

In this section we assume that the \(M\)-bimodule \(\Omega^1 M\) is finitely generated projective as a right module. We use the dual basis \(\xi_i \in \Omega^1 M\) and \(X_i \in (\Omega^1 M)^*\) so that \(\sum \xi_i.X_i(\eta) = \eta\) for all \(\eta \in \Omega^1 M\).

**Definition 6.1** Define the vector fields on an algebra \(M\) by \(\text{Vec} M = (\Omega^1 M)^*\), the right \(M\)-module maps from \(\Omega^1 M\) to \(M\). Then evaluation gives a bimodule map \(\text{ev} : \text{Vec} M \otimes M \Omega^1 M \rightarrow M\). If \(f : B \rightarrow M\) is a differentiable algebra map, we define \(f^* : \text{Vec} M \rightarrow \text{Vec} B\) by \((f^*X)(\xi) = X(f \cdot \xi)\) for \(\xi \in \Omega^1 B\).

**Definition 6.2** An \(X \in \text{Vec} M\) gives a ‘directional derivative’ map \(D_X : M \rightarrow M\) defined by

\[\text{Vec} M \otimes M \xrightarrow{\text{id} \otimes d} \text{Vec} M \otimes \Omega^1 M \xrightarrow{\text{eval}} M .\]
This map is a derivation on $M$ if and only if $X : \Omega^1 M \to M$ is also a left $M$-module map. In general $D_X(a b) = D_X(a) b + D_{X.a}(b)$.

If the left $M$-module $E$ has a left $M$-covariant derivative $\nabla$, then given $X \in \text{Vec}_M$ we define the covariant directional derivative by $\nabla_X e = (ev \otimes \text{id})(X \otimes \nabla e)$. The reason for defining vector fields as right $M$-module maps was so that this would be well defined.

**Definition 6.3** The vector fields are braided by $\sigma^{-1} : \text{Vec}_M \otimes \text{Vec}_M \to \text{Vec}_M \otimes \text{Vec}_M$ given by the formula $\sigma^{-1}(X \otimes Y) = \sum (ev \otimes \text{id})(X \otimes \sigma_{\text{Vec}_M}(Y \otimes \xi_i)) \otimes X_i$. Note that we use the notation $\sigma^{-1}$ to fit with the crossings in a braided category - we do not claim that $(\sigma^{-1})^{-1}$ exists!

**Proposition 6.4** The braiding in 6.3 is the unique braiding for which

$$(\text{id} \otimes ev)(\sigma^{-1} \otimes \text{id}) = (ev \otimes \text{id})(\text{id} \otimes \sigma_{\text{Vec}_M}) : \text{Vec}_M \otimes \text{Vec}_M \otimes \Omega^1 M \to \text{Vec}_M .$$

**Proof** Uniqueness follows from 4.4 again. For all $\eta \in \Omega^1 M$ and $X, Y \in \text{Vec}_M$,

$$(\text{id} \otimes ev)(\sigma^{-1} \otimes \text{id})(X \otimes Y \otimes \eta) = \sum (ev \otimes \text{id})(X \otimes \sigma_{\text{Vec}_M}(Y \otimes \xi_i)) \cdot X_i(\eta) = (ev \otimes \text{id})(X \otimes \sigma_{\text{Vec}_M}(Y \otimes \eta)) .$$

**Definition 6.5** Using the fact that $\Omega^1 M$ is finitely generated projective, we have a unique Kroneker delta (see 5.4) $\delta \in \Omega^1 M \otimes_M \text{Vec}_M$. In addition we define $\hat{\delta} = \sigma^{-1} \delta \in \text{Vec}_M \otimes_M \Omega^1 M$, and $\dim M = ev(\hat{\delta}) \in M$. Note that $\dim M$ is a central element in $M$ by 5.3(b).

**Remark 6.6** Following from 4.3, note that $\delta$ and $\hat{\delta}$ in classical differential geometry are just the usual Kroneker deltas, $\delta^i_j$ and $\delta_i^j$. It is then immediate that $\dim M$ is a constant function with value the dimension of the manifold.

### 7 Interior products

In this section we would like to define the interior product of a vector field with an $n$-form. However we must remember that the $n$-forms are not realised as a subspace of the $n$-fold tensor product of the 1-forms, but rather as a quotient of them by $\Theta^n M = \ker : \bigotimes^n_M \Omega^1 M \to \Omega^n M$. This leads us to a compatibility condition between the braiding and the differential calculus which is necessary to define interior products with $\Omega^n M$. 

10
Definition 7.1 Recursively define the $M$-bimodule map $\sigma_n : \bigotimes_M^n \Omega^1 M \to \bigotimes_M^n \Omega^1 M$, beginning with $\sigma_1 = \text{id}$ and $\sigma_2 = \sigma$, and continuing with $\sigma_{n+1} = (\sigma \otimes \text{id}^{n-1})(\text{id} \otimes \sigma_n)$. It is easy to see that for all $r, s \geq 0$, $(\sigma_{s+1} \otimes \text{id}^r)(\text{id}^s \otimes \sigma_{r+1}) = \sigma_{r+s+1}$.

Definition 7.2 Define the interior product $X \llcorner z \in \bigotimes_M^{n-1} \Omega^1 M$ for $X \in \text{Vec}_M$ and $z \in \bigotimes_M^n \Omega^1 M$ as $(\text{ev} \otimes \text{id}^{n-1})(X \otimes T_n(z))$, where

$$T_n = - \sum_{r=1}^{n} (-1)^r \sigma_r \otimes \text{id}^{n-r} : \bigotimes_M^n \Omega^1 M \to \bigotimes_M^n \Omega^1 M.$$ 

For $\omega \in \Omega^1 M$ we have $X \llcorner \omega = X(\omega)$.

Proposition 7.3 The map $\llcorner : \text{Vec}_M \otimes_M (\bigotimes_M^{n+1} \Omega^1 M) \to \bigotimes_M^n \Omega^1 M$ is an $M$-bimodule map.

Proof All its component maps are bimodule maps. □

Definition 7.4 The interior product operation is said to be compatible with the differential calculus if $T_{n+1} (\Theta^{n+1} M) \subset \Omega^1 M \otimes_M \Theta^n M$ for all $n \geq 1$. In this case, we get an interior product $\llcorner : \text{Vec}_M \otimes_M \Omega^{n+1} M \to \Omega^n M$. We conventionally add $X \llcorner m = 0$ for $m \in \Omega^0 M$.

Proposition 7.5 If $\llcorner$ is compatible with the differential calculus, then

a) $\sigma_{n+1} (\Theta^n M \otimes_M \Omega^1 M) \subset \Omega^1 M \otimes_M \Theta^n M$ for all $n \geq 1$.

b) $\Theta^2 M$ is contained in the $+1$ eigenspace of $\sigma : \Omega^1 M \otimes_M \Omega^1 M \to \Omega^1 M \otimes_M \Omega^1 M$.

Proof To prove (a), given $z \in \Theta^n M$ and $\xi \in \Omega^1 M$, we know that $z \otimes \xi \in \Theta^{n+1} M$, so $T_{n+1} (z \otimes \xi) \in \Omega^1 M \otimes_M \Theta^n M$ by our assumption. But

$$T_{n+1} (z \otimes \xi) = T_n(z) \otimes \xi + (-1)^n \sigma_{n+1}(z \otimes \xi),$$

and, also by our assumption, $T_n(z) \otimes \xi \in \Omega^1 M \otimes \Theta^{n-1} M \otimes \Omega^1 M \subset \Omega^1 M \otimes_M \Theta^n M$. We deduce that $\sigma_{n+1}(z \otimes \xi) \in \Omega^1 M \otimes_M \Theta^n M$.

To prove (b), note that $\Theta^1 M = 0$, so we have $T_2 \Theta^2 M = 0$ by our assumption. □

8 Lie derivatives of forms

Having defined interior products of vector fields with $\Omega^n M$ in section 7, we are in the happy position of being able to define the Lie derivative of an $n$-form with respect to a vector field. We assume that we have covariant derivatives and braidings satisfying 7.4.
Definition 8.1 We define the Lie derivative of $\omega \in \Omega^n M$ with respect to $X \in \text{Vec} M$ to be $\mathcal{L}_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega)$. Note that $\mathcal{L}_X (m) = D_X m$ for $m \in \Omega^0 M = M$.

Proposition 8.2 The Lie derivative $\mathcal{L} : \text{Vec} M \otimes \Omega^n M \to \Omega^n M$ obeys the following rules:

a) $\mathcal{L}_X (m \lrcorner \omega) = \mathcal{L}_X m \lrcorner \omega + X \lrcorner (dm \wedge \omega)$.

b) $\mathcal{L}_m.X (\omega) = dm \wedge (X \lrcorner \omega) + m.\mathcal{L}_X \omega$.

c) $\mathcal{L}_X (\omega \lrcorner m) = \mathcal{L}_X \omega \lrcorner m + (-1)^{|\omega|} (X \lrcorner (\omega \wedge dm) - (X \lrcorner \omega) \wedge dm)$.

d) $d\mathcal{L}_X = \mathcal{L}_X d : \Omega^n M \to \Omega^{n+1} M$.

Proof More or less immediate from the definition. \[\Box\]

9 Covariant derivatives of higher forms

Remark 9.1 Given a bimodule covariant derivative $(\nabla, \sigma)$ on $\Omega^1 M$, the discussion in Remark 9.1 gives a covariant derivative $(\nabla, \sigma_{n+1})$ on $\bigotimes^n_M \Omega^1 M$, given by

$$\nabla = \sum_{i=1}^n (\sigma_i \otimes \text{id}^{n+1-i}) (\text{id}^{i-1} \otimes \nabla \otimes \text{id}^{n-i}) : \bigotimes^n_M \Omega^1 M \to \Omega^1 M \otimes_M \left( \bigotimes^n_M \Omega^1 M \right).$$

Proposition 9.2 If $\nabla$ on $\bigotimes^n_M \Omega^1 M$ preserves (in the sense of Proposition 9.2) the submodule $\Theta^n M = \ker \wedge : \bigotimes^n_M \Omega^1 M \to \Omega^n M$, then we get a covariant derivative on $\Omega^n M$ by quotienting.

Proof Reasonably direct from the previous statements. \[\Box\]

10 Antisymmetry and Lie brackets of vector fields

Definition 10.1 An $x \in \text{Vec} M \otimes \text{Vec} M$ is called antisymmetric if $\text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(\pi x \otimes k) = 0$ for all $k \in \Theta^2 M$, where $\pi$ is the quotient map from $\text{Vec} M \otimes \text{Vec} M$ to $\text{Vec} M \otimes_M \text{Vec} M$. We call $A^2 M$ the set of antisymmetric elements in $\text{Vec} M \otimes \text{Vec} M$.

Remark 10.2 The map $(\text{ev} \otimes \text{ev})(\text{id} \otimes \sigma \otimes \text{id}) : \text{Vec} M \otimes_M \text{Vec} M \otimes_M \Omega^1 M \otimes_M \Omega^1 M \to M$ can also be written as $\text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(\sigma^{-1} \otimes \text{id} \otimes \text{id})$ and as $\text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(\text{id} \otimes \text{id} \otimes \sigma^{-1})$. If $\text{7.4}$ holds, it follows that all eigenspaces of $\sigma^{-1}$ except the $+1$ eigenspace are contained in $\pi A^2 M$.

Definition 10.3 Define a map $\phi : A^2 M \to \text{Vec} M$ by the following formula, where $\xi \in \Omega^1 M$:

$$\phi(X \otimes Y)(\xi) = D_X (Y(\xi)) + \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(X \otimes Y \otimes z),$$

where $z \in \Omega^1 M \otimes_M \Omega^1 M$ is chosen so that $z = \delta \xi$. By the previous discussion the choice does not matter. To check that its image is in $\text{Vec} M$ we use the following proposition.

**Proposition 10.4** The image of the map $\phi$ in $\Omega^1 M \otimes_M \Omega^1 M$ is in $\text{Vec} M$. Further $\phi$ is a left $M$-module map, but not in general a right module map, as $\phi(X \otimes Y).m = \phi(X \otimes Y.m) + X.D_Y(m)$. Also $\phi(X \otimes m.Y) = \phi(X.m \otimes Y) + D_X(m).Y$.

**Proof** To see that $\phi(X \otimes Y)$ is a right module map use the following, where $z = \delta \xi$,

$$
\phi(X \otimes Y)(\xi.m) = D_X(Y(\xi).m) + \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(X \otimes Y(\delta \xi.m - \xi \otimes \delta m)) = \phi(X \otimes Y)(\xi.m) + X(Y(\xi).dm) - X(Y(\xi).dm).
$$

It is quite easy to see that $\phi((m.X \otimes Y)(\xi)) = m.\phi(X \otimes Y)(\xi)$. For the right action,

$$
\phi(X \otimes Y)(m.\xi) = D_X(Y(m.\xi)) + \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(X \otimes Y(m.\xi + \delta m \otimes \xi)) = \phi(X \otimes m.Y)(\xi) + \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(X \otimes Y \delta m \otimes \xi).
$$

Finally we calculate

$$
\phi(X \otimes m.Y)(\xi) = D_X(m.X(\xi)) + \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(X \otimes m.Y \otimes \xi) = D_X(m).Y(\xi) + D_X(m.Y(\xi)) + \text{ev}(\text{id} \otimes \text{ev} \otimes \text{id})(X \otimes m.Y \otimes \xi).
$$

\[\Box\]

**Remark 10.5** Now $\phi$ is the non-commutative version of the Lie bracket, but it needs to be applied to elements of $\text{Vec} M \otimes \text{Vec} M$ which are already antisymmetric. In the commutative case we would have $\phi(X \otimes Y - Y \otimes X) = [X,Y]$. In the non-commutative case the problem is the analogue of the $X \otimes Y \mapsto X \otimes Y - Y \otimes X$ operation, i.e. how to antisymmetrise elements.

If $\sigma^2 = \text{id}$, then we can antisymmetrise on $\text{Vec} M \otimes \text{Vec} M$ by $X \otimes Y \mapsto X \otimes Y - \sigma(X \otimes Y)$, but $\phi$ does not descend to a well defined map on a subspace of $\text{Vec} M \otimes \text{Vec} M$. Any definition of Lie bracket will depend on being given an antisymmetrisation operation on $\text{Vec} M \otimes \text{Vec} M$.

## 11 Commutative differential geometry

In this case $\sigma : \Omega^1 M \otimes_M \Omega^1 M \to \Omega^1 M \otimes_M \Omega^1 M$ is just transposition $\sigma(\xi \otimes \eta) = \eta \otimes \xi$, and satisfies the braid relation. We have $\sigma^2 = 1$, and $\Omega^1 M \otimes_M \Omega^1 M$ splits into a direct sum of the $+1$ and $-1$ eigenspaces of $\sigma$. The $+1$ eigenspace is $\Theta^2 M = \ker \wedge : \Omega^1 M \otimes_M \Omega^1 M \to \Omega^2 M$, and we can use this to identify $\Omega^2 M$ with the $-1$ eigenspace (antisymmetric tensors) in $\Omega^1 M \otimes_M \Omega^1 M$. 


12 The non-commutative torus

Take the algebra $T^2_q$ generated by invertible elements $u$ and $v$, subject to $uv = qvu$, where $q$ is a unit norm complex number. The simplest differential calculus (there are many to choose from) on $T^2_q$ is generated by \{u, v, du, dv\}, subject to the relations

\[
du \wedge dv = -q dv \wedge du, \quad u dv = q dv u, \quad v du = q^{-1} du v, \\
[u, du] = [v, dv] = 0, \quad du \wedge du = dv \wedge dv = 0.
\] (6)

Then Vec $T^2_q$ is generated as a left $T^2_q$ module by the elements $\partial_u$ and $\partial_v$, where $\partial_u (du) = \partial_v (dv) = 1$ and $\partial_u (dv) = \partial_v (du) = 0$. In fact for any $X \in \text{Vec} T^2_q$, $X = X(du) \partial_u + X(dv) \partial_v$.

Now we find the right actions

\[
(X.u)(du) = X(du) u \quad \text{,} \quad (X.u)(dv) = q X(dv) u, \\
(X.v)(du) = q^{-1} X(du) v \quad \text{,} \quad (X.v)(dv) = X(dv) v.
\] (7)

A covariant derivative $\nabla$ on $\Omega^1 T^2_q$ is specified by $\nabla(du), \nabla(dv) \in \Omega^1 T^2_q \otimes_{T^2_q} \Omega^1 T^2_q$. We calculate the corresponding braiding from (3.3) by

\[
\sigma(du \otimes du) = \nabla(du) - \nabla(du).u = \nabla(u du) - \nabla(du).u \\
= du \otimes du + u.\nabla(du) - \nabla(du).u, \\
\sigma(dv \otimes du) = \nabla(dv) - \nabla(dv).u = q^{-1} \nabla(u dv) - \nabla(dv).u \\
= q^{-1} du \otimes dv + q^{-1} u.\nabla(dv) - \nabla(dv).u, \\
\sigma(dv \otimes dv) = dv \otimes dv + v.\nabla(dv) - \nabla(dv).v, \\
\sigma(du \otimes dv) = q dv \otimes du + q v.\nabla(dv) - \nabla(dv).v.
\] (8)

To have a compatible interior product (see (3.3), from (3.3) we must have $\Theta^2 T^2_q = \ker \wedge : \Omega^1 T^2_q \otimes_{T^2_q} \Omega^1 T^2_q \to \Omega^2 T^2_q$ contained in the +1 eigenspace of $\sigma$. As $du \otimes du, dv \otimes dv$ and $du \otimes dv + q dv \otimes du$ are in $\Theta^2 T^2_q$, we deduce that

\[
\nabla(du).u = u.\nabla(du), \quad \nabla(dv).v = v.\nabla(dv), \\
q \nabla(dv).u - u.\nabla(dv) = q v.\nabla(du) - \nabla(du).v.
\] (9)

The general solution to this is, where the coefficients $r_{**}$ and $s_{**}$ are numbers,

\[
\nabla(du) = r_{uu} du \otimes du.u^{-1} + r_{vu} dv \otimes du.v^{-1} + r_{uv} du \otimes dv.v^{-1} + r_{vv} dv \otimes dv.u v^{-2},
\]

14
\[ \nabla(dv) = s_{vv} dv \otimes dv.v^{-1} + s_{vu} dv \otimes du.u^{-1} + s_{uv} du \otimes dv.u^{-1} + s_{uu} du \otimes du.v u^{-2}. \] (10)

Putting this back into the equations for the braiding, we find

\[ \sigma(du \otimes du) = du \otimes du, \quad \sigma(dv \otimes du) = q^{-1} du \otimes dv, \]
\[ \sigma(dv \otimes dv) = dv \otimes dv, \quad \sigma(du \otimes dv) = q dv \otimes du. \] (11)

From (2) and 6.4 we have

\[ \sigma(\partial u \otimes du) = du \otimes \partial u, \quad \sigma(\partial v \otimes du) = q^{-1} du \otimes \partial v, \]
\[ \sigma(\partial v \otimes dv) = dv \otimes \partial v, \quad \sigma(\partial u \otimes dv) = q dv \otimes \partial u. \] (12)

The paired left covariant derivative on Vec $T^2_q$ can be calculated using 5.1 as

\[ \nabla(\partial v) = -r_{uv} du.u^{-1} \otimes \partial u - q^{-1} r_{vu} dv.v^{-1} \otimes \partial v - q^{-1} s_{uv} du.u^{-1} \otimes \partial v - s_{uu} du.v u^{-2} \otimes \partial u, \]
\[ \nabla(\partial u) = -s_{vu} dv.v^{-1} \otimes \partial u - q s_{uv} du.u^{-1} \otimes \partial v - q r_{uv} dv.v^{-1} \otimes \partial v - r_{vu} dv.v u^{-2} \otimes \partial u. \] (13)

From 7.2, the interior product on $\Omega^2 T^2_q$ is

\[ \partial_v \lrcorner (du \wedge dv) = dv, \quad \partial_u \lrcorner (du \wedge dv) = -q du. \]

From 10.1 we see that $\partial_v \otimes \partial_u - q^{-1} \partial_u \otimes \partial_v \in A^2 T^2_q$, and from 10.3 $\phi(\partial_v \otimes \partial_u - q^{-1} \partial_u \otimes \partial_v) = 0$. The given covariant derivatives restrict to $\Theta^2 T^2_q$, and the covariant derivative on $\Omega^2 T^2_q$ is

\[ \nabla(du \wedge dv) = du \otimes (r_{uu}q^{-1} + s_{uv})du \wedge dv.u^{-1} + dv \otimes (r_{vu} + q s_{vv})du \wedge dv.v^{-1}. \] (14)

The Kroneker delta (see 5.2) is $\delta = du \otimes \partial_u + dv \otimes \partial_v$, and from 12 dim $T^2_q = 2$.

Remark 12.1 In summary; from the given differential calculus (6) we have derived a unique compatible braiding (11) and a small number of compatible covariant derivatives (10). At this point, the reader may express some alarm: This is far too restrictive! Should we not be able to have a wide variety of covariant derivatives on a given algebra?

Of course, the reader would be correct, we cannot restrict ourselves to such a small class of possible covariant derivatives in the geometry of the noncommutative torus. However the problem is that we fixed the differential calculus (6). We can indeed have many more covariant derivatives, but only if we allow different differential calculi on our algebra.
Remark 12.2 In [5] the classical symplectic connection associated to the semiclassical \((q \approx 1)\) noncommutative torus and the given differential calculus is computed. The braiding calculated from the connection is shown to agree with the exact result here to lowest order in \(q - 1\).

13 A noncommutative sphere

Following [5], we describe a differential calculus on a deformed sphere \(S_q^2\) using a stereographic projection. The algebra on the coordinate chart of the projection is generated by \(z\) and \(\bar{z}\) with commutation relation \(z\bar{z} = q^{-2}\bar{z}z + q^{-2} - 1\). This can be made into a \(C^*\) algebra with the involution \(z^* = \bar{z}\). There is a left covariant (with respect to the action of \(q\)-deformed \(SU_2\)) differential calculus given by

\[
dz \wedge d\bar{z} = -q^{-2} d\bar{z} \wedge dz \quad , \quad z.dz = q^{-2} dz.z \quad , \quad z.d\bar{z} = q^{-2} d\bar{z}.z \quad ,
\]

\[
dz \wedge dz = d\bar{z} \wedge d\bar{z} = 0 \quad , \quad \bar{z}.dz = q^2 dz.\bar{z} \quad , \quad \bar{z}.d\bar{z} = q^2 d\bar{z}.\bar{z} \quad . \tag{15}
\]

It will be convenient to set \(R = \bar{z}z\), and as \(R\) is a positive element in \(S_q^2\), \(R + 1\) is invertible. The commutation relation can then be written \(z(R + 1) = q^{-2}(R + 1)z\). For a function \(f(R + 1)\) we have \(z f(R + 1) = f(q^{-2}(R + 1)) z\). Also we write \(z^1 = z\) and \(z^2 = \bar{z}\).

Proposition 13.1 The kernel of \(\wedge : \Omega^1 S_q^2 \otimes S_q^2 \Omega^1 S_q^2 \rightarrow \Omega^2 S_q^2\) is contained in the \(+1\) eigenspace of \(\sigma\) if the covariant derivative is of the form

\[
\nabla(dz) = q^{-2} (R + 1)^{-1} \bar{z}.dz \otimes dz + (R + 1)^{-1} \sum (g_{ij0} + q^6 h_{ij1} z) dz^i \otimes dz^j ,
\]

\[
\nabla(d\bar{z}) = q^{4} (R + 1)^{-1} z.d\bar{z} \otimes d\bar{z} + (R + 1)^{-1} \sum (h_{ij0} + h_{ij1} \bar{z}) d\bar{z}^i \otimes d\bar{z}^j ,
\]

where \(h_{ij0}, h_{ij1}\) and \(g_{ij0}\) (for \(1 \leq i, j \leq 2\)) are constants. The braiding is

\[
\sigma(dz \otimes dz) = dz \otimes dz \quad , \quad \sigma(d\bar{z} \otimes d\bar{z}) = d\bar{z} \otimes d\bar{z} \quad ,
\]

\[
\sigma(dz \otimes d\bar{z}) = q^{-2} d\bar{z} \otimes dz + (q^2 - 1) \sum h_{ij1} d\bar{z}^i \otimes dz^j ,
\]

\[
\sigma(q^{-2} d\bar{z} \otimes dz) = dz \otimes d\bar{z} - (q^2 - 1) \sum h_{ij1} dz^i \otimes dz^j . \tag{16}
\]

Proof Begin with

\[
dz \otimes dz = \sigma(dz \otimes dz) = \nabla(dz.z) - \nabla(dz)z
\]

\[
= q^2 \nabla(z.dz) - \nabla(dz).z = q^2 dz \otimes dz + q^2 z.\nabla(dz) - \nabla(dz).z . \tag{17}
\]
If we set $\nabla(dz) = \sum c_{ij} dz^i \wedge dz^j$, we have the equations $z c_{ij} = q^2 c_{ij} z$ for $(i, j) \neq (1, 1)$, and $1 - q^2 = q^2 (z c_{11} - q^2 c_{11} z)$. The equation $z c_{ij} = q^2 c_{ij} z$ has solution $c_{ij} = (R + 1)^{-1} g_{ij}(z)$ for any non-singular function $g_{ij}$. Now the equation for $c_{11}$ has solution

$$c_{11} = q^{-2} z^{-1} + (R + 1)^{-1} f_{11}(z).$$

Unfortunately $z$ is not invertible, but we can write $(1 - (R + 1)^{-1}) z^{-1} = (R + 1)^{-1} \bar{z}$, and then

$$c_{11} = q^{-2} (R + 1)^{-1} \bar{z} + (R + 1)^{-1} g_{11}(z), \quad (18)$$

where $g_{11}(z)$ is non-singular. Next,

$$d\bar{z} \otimes d\bar{z} = \sigma(d\bar{z} \otimes d\bar{z}) = \nabla(d\bar{z}, \bar{z}) - \nabla(d\bar{z}, \bar{z}) = q^{-2} \nabla(d\bar{z}, \bar{z}) = q^{-2} d\bar{z} \otimes d\bar{z} + q^{-2} \bar{z} \nabla(d\bar{z}) - \nabla(d\bar{z}, \bar{z}). \quad (19)$$

We set $\nabla(d\bar{z}) = \sum e_{ij} dz^i \wedge dz^j$, and get the equations $\bar{z} e_{ij} = q^{-2} e_{ij} \bar{z}$ for $(i, j) \neq (2, 2)$ and

$$1 - q^{-2} = q^{-2} \bar{z} e_{22} - q^{-4} e_{22} \bar{z}. \quad (20)$$

Now we use the result $\bar{z} (R + 1)^{-1} = q^{-2} (R + 1)^{-1} \bar{z}$ to find that $e_{ij} = (R + 1)^{-1} h_{ij}(\bar{z})$ for $(i, j) \neq (2, 2)$ and

$$e_{22} = q^4 (R + 1)^{-1} z + (R + 1)^{-1} h_{11}(\bar{z}).$$

Now we use the fact that $dz \otimes d\bar{z} + q^{-2} d\bar{z} \otimes dz$ is in the kernel of $\wedge$:

$$\sigma(dz \otimes d\bar{z}) = \nabla(dz, \bar{z}) - \nabla(dz) = q^{-2} \nabla(dz, \bar{z}) - \nabla(dz, \bar{z}) = q^{-2} \bar{z} \nabla(dz) - \nabla(dz, \bar{z}),$$

$$q^{-2} \sigma(dz \otimes dz) = q^{-2} \nabla(dz, z) = q^{-2} \nabla(dz) - \nabla(dz, \bar{z}) = \nabla(z, d\bar{z}) = q^{-2} \nabla(d\bar{z}, z) = dz \otimes d\bar{z} + z. \nabla(d\bar{z}) - q^{-2} \nabla(d\bar{z}, z). \quad (20)$$

Then, using $z (R + 1)^{-1} = q^2 (R + 1)^{-1} z$,

$$q^{-2} \bar{z} \nabla(dz) - \nabla(dz, \bar{z}) = \sum (q^{-2} \bar{z}, c_{ij} - q^{-4} c_{ij} \bar{z}) dz^i \otimes dz^j = q^{-4} (R + 1)^{-1} \sum (\bar{z} g_{ij}(z) - g_{ij}(z) \bar{z}) dz^i \otimes dz^j, \quad (21)$$

$$q^{-2} \nabla(d\bar{z}, z) = \sum (q^2 e_{ij} z - z e_{ij}) dz^i \otimes dz^j = q^2 (R + 1)^{-1} \sum (h_{ij}(\bar{z}) z - z h_{ij}(\bar{z})) dz^i \otimes dz^j, \quad (21)$$

17
and we find that
\[ q^b(h_{ij}(\tilde{z}) z - z h_{ij}(\tilde{z})) = \tilde{z} g_{ij}(z) - g_{ij}(\tilde{z})\tilde{z} . \]

A solution to this is given by constants \( h_{ij0}, h_{ij1} \) and \( g_{ij0} \) (for \( 1 \leq i, j \leq 2 \)), when
\[ h_{ij}(\tilde{z}) = h_{ij0} + h_{ij1} \tilde{z}, \quad g_{ij}(z) = g_{ij0} + q^b h_{ij1} z . \]

The braiding is calculated from \( [20] \). \( \square \)

**Proposition 13.2** Assume that \( q \) is nonzero and \( q^4 \neq 1 \). The cases for which the braiding \( [16] \) gives an interior product which is compatible with the differential calculus \([7, 4]\) are

\[ a) \quad h_{221} = h_{211} = 0, \quad h_{121} = 1/(q^2 - 1). \]
\[ b) \quad h_{111} = h_{121} = 0, \quad h_{211} = 1/(q^2 - q^4). \]
\[ c) \quad h_{111} = h_{221} = 0, \quad h_{211} h_{121} = 0. \]
\[ d) \quad h_{111} = h_{221} = 0, \quad h_{121} = 1/(q^2 - 1), \quad h_{211} = 1/(q^2 - q^4). \]

**Proof** The only case left to check is that \( T_3\Theta^3 S_q^2 \subset \Omega^1 S_q^2 \otimes S_q^2 \Theta^2 S_q^2 \), where \( \Theta^3 S_q^2 \) is all of \( \Omega^1 S_q^2 \otimes S_q^2 \Omega^1 S_q^2 \otimes S_q^2 \), and explicit calculation gives the answer. \( \square \)

**Proposition 13.3** Assume that \( q \) is nonzero and \( q^4 \neq 1 \). The cases for which the generalised braiding \( [16] \) actually satisfies the braid relation are

\[ a) \quad h_{111} = h_{221} = 0, \quad h_{211} h_{121} = 0. \]
\[ b) \quad h_{111} = h_{221} = 0, \quad h_{121} = 1/(q^2 - 1), \quad h_{211} = 1/(q^2 - q^4). \]

**Proposition 13.4** Assume that \( q \) is nonzero and \( q^4 \neq 1 \). The condition for the braiding \( [16] \) to be invertible is that \( (q^2 - 1)(h_{121} - h_{211} q^2) \neq 1 \), and the cases for which \( \sigma^2 \) is the identity are

\[ a) \quad h_{121} - h_{211} q^2 = 0. \]
\[ b) \quad h_{111} = h_{221} = 0, \quad h_{121} = 1/(q^2 - 1), \quad h_{211} = 1/(q^2 - q^4). \]

**Proposition 13.5** The vector fields are generated by \( \partial_z \) and \( \partial_{\bar{z}} \), where \( \partial_z(dz) = \partial_{\bar{z}}(d\bar{z}) = 1 \) and \( \partial_{\bar{z}}(d\bar{z}) = \partial_z(dz) = 0 \). Then the ‘Lie bracket’ is \( \phi(\partial_z \otimes \partial_z - q^2 \partial_{\bar{z}} \otimes \partial_{\bar{z}}) = 0 \), and the braiding \( \sigma : \text{Vec} \otimes \Omega^1 \rightarrow \Omega^1 \otimes \text{Vec} \) is given by

\[
\sigma(\partial_z \otimes dz) = dz \otimes \partial_z + \frac{h_{111}(1 - q^3)}{x} dz \otimes \partial_z + \frac{h_{211}(q^2 - q^4)}{x} d\bar{z} \otimes \partial_{\bar{z}},
\]
\[
\sigma(\partial_{\bar{z}} \otimes d\bar{z}) = \frac{h_{111}(q^4 - q^2)}{x} dz \otimes \partial_z + \frac{h_{121}(q^4 - q^2)}{x} d\bar{z} \otimes \partial_{\bar{z}},
\]

 avoided.
\[
\sigma(\partial z \otimes dz) = \frac{h_{211}}{x} (1 - q^2) - 1/q^2 dz \otimes \partial z + \frac{h_{221}}{x} (1 - q^2) d\bar{z} \otimes \partial \bar{z},
\]
\[
\sigma(\partial \bar{z} \otimes d\bar{z}) = \frac{x}{x^2 + q^2 (q^2 - 1)^2 (h_{121} h_{211} - h_{111} h_{221})}.
\]

where \(x = (q^2 - 1)(h_{121} - q^2 h_{211}) - 1\). The differential dimension is

\[
\dim S^2_q = \frac{x(x - 1)}{x^2 + q^2 (q^2 - 1)^2 (h_{121} h_{211} - h_{111} h_{221})}.
\]

**Remark 13.6** In summary; we now have a wider variety of possibilities. The differential calculus does not uniquely specify the braiding. There are generalised braidings (13.2 (a) with \(h_{111} \neq 0\) and (b) with \(h_{221} \neq 0\)) which are compatible with the differential calculus but do not satisfy the braid relation. There are generalised braidings (13.2 (c) with exactly one of \(h_{121}\) or \(h_{211}\) vanishing) which are compatible with the differential calculus and satisfy the braid relation, but do not square to the identity and give (in general) fractional differential dimension.

## 14 Curvature and Torsion

Using vector fields, we can define the curvature and torsion by some remarkably familiar classical formulae, rather than the usual noncommutative formulae using forms. Remember that \(\phi\) defined in 10.3 is the analogue of the Lie bracket.

**Definition 14.1** Given a left \(M\)-covariant derivative \(\nabla\) on a left \(M\)-module \(E\), define the curvature \(R : A^2 M \otimes E \rightarrow E\) as

\[
R(X \otimes Y)(e) = \nabla_X \nabla_Y (e) - \nabla_{\phi(X \otimes Y)} (e).
\]

**Proposition 14.2** The curvature descends to a well defined left \(M\)-module map \(R : \pi A^2 M \otimes_M E \rightarrow E\), where \(\pi : \text{Vec} M \otimes \text{Vec} M \rightarrow \text{Vec} M \otimes_M \text{Vec} M\) is the quotient map.

**Proof** The left module property is quite simple. Next, for \(X \otimes Y \in A^2 M\):

\[
R(X \otimes m.Y)(e) = \nabla_X \nabla_{m.Y} (e) - \nabla_{\phi(X \otimes m.Y)} (e)
\]

\[
= \nabla_X (m.\nabla_Y (e)) - \nabla_{\phi(X,m \otimes Y)} (e) - \nabla_{D_X(m).Y} (e)
\]

\[
= R(X.m \otimes Y)(e).
\]

Next, using 10.34,

\[
R(X \otimes Y)(m.e) = \nabla_X \nabla_Y (m.e) - \nabla_{\phi(X \otimes Y)} (m.e)
\]

19
\[
\begin{align*}
= \nabla_X(D_Y(m).e + \nabla_{Y,m}(e)) - \phi(X \otimes Y)(dm).e - \nabla_{\phi(X \otimes Y),m}(e) \\
= R(X \otimes Y)(e) + \nabla_X(D_Y(m).e) - \phi(X \otimes Y)(dm).e - \nabla_{X,D_Y(m)}(e) \\
= R(X \otimes Y)(e).
\end{align*}
\]

**Definition 14.3** Given a bimodule connection on \(\Omega^1 M\), we define the torsion \(T : A^2 M \to \text{Vec} M\) as \(T(X \otimes Y) = \nabla_X(Y) - \phi(X \otimes Y)\).

**Proposition 14.4** The torsion descends to a well defined left module map \(T : \pi A^2 M \to \text{Vec} M\).

**Proof** Using [10.4]

\[
T(X.m \otimes Y) = \nabla_{X,m}(Y) - \phi(X.m \otimes Y) = \nabla_{X,m}(Y) - \phi(X \otimes m.Y) + D_X(m).Y = T(X \otimes m.Y).
\]

For the left module map condition,

\[
T(m.X \otimes Y) = \nabla_{m.X}(Y) - \phi(m.X \otimes Y) = m.T(X \otimes Y). \qed
\]

15 Classical exponentiation and parallel transport

We consider the usual point based definition of differential geometry on a manifold \(M\), and translate it into a form more amenable to non-commutative geometry. The directional derivative notation is used, where \(f'(x; v)\) is the derivative of the function at \(x\) along the vector \(v\). We begin with a result about exponentiating a known time dependent vector field into a diffeomorphism:

**Proposition 15.1** Given a vector field \(w(t) \in \text{Vec} M\) for \(t \in \mathbb{R}\), define a function \(p : M \times \mathbb{R} \to M\) by \(p(x,0) = x\) and \(p(x,t) = w(t)(p(x,t))\). Now define functions \(J,K : \mathbb{R} \to L(C^\infty(M),C^\infty(M))\) (the set of linear maps from \(C^\infty(M)\) to itself) by

\[
J(t)(f)(x) = f'(x;w(x,t)) , \quad K(t)(f)(x) = f(p(x,t)) .
\]

Then \(K(t) \in L(C^\infty(M),C^\infty(M))\) is the solution to the differential equation

\[
\dot{K}(t) = K(t) \circ J(t) , \quad K(0) = \text{id} .
\]
Proof By differentiating the definition of $K(t)$ with respect to $t$,

$$\dot{K}(t)(f)(x) = f'(p(x,t);w(p(x,t),t)) = K(t)(J(t)(f))(x) . \quad \Box$$

Now we consider parallel transport in a bundle $E$ with connection $\nabla$, along a curve which is given by exponentiation of a time dependent vector field. The connection is specified in our coordinate system by Christoffel symbols $\Gamma$.

**Proposition 15.2** Take a curve $p(x,t)$ starting at $x$ given by exponentiating the time dependent vector field $w(t)$. Along the curve take $s(x,t) \in E_{p(x,t)}$ which is a solution to the parallel transport equation $\dot{s}(x,t) + \Gamma(p(x,t);\dot{p}(x,t),s(x,t)) = 0$. Define a time dependent section $c(t)$ of $E$ by $c(t)(p(x,t)) = s(x,t)$. Then $c(t)$ obeys the first order differential equation $\dot{c}(t) = -\nabla_{\dot{p}(x,t)}c(t)$.  

**Proof** Differentiating the definition of $c$ with respect to $t$ we find

$$\dot{s}(x,t) = \dot{c}(t)(p(x,t)) + c(t)'(p(x,t);\dot{p}(x,t)) = -\Gamma(p(x,t);\dot{p}(x,t),s(x,t)) .$$

We can rearrange this to give

$$\dot{c}(t)(p(x,t)) = -\nabla_{\dot{p}(x,t)}c(t)(p(x,t)) . \quad \Box$$

Now we can use a connection on the tangent bundle $TM$ to define geodesics on $M$.

**Corollary 15.3** Suppose that the vector field $c(t) \in \text{Vec} M$ is parallel transported along curves which are given by exponentiating $c(t)$ itself. Then $c(t)$ obeys the first order non-linear differential equation $\dot{c}(t) = -\nabla_{c(t)}c(t)$.

The reader will notice that the geodesic equation is, unlike the parallel transport equation, non-linear. First order linear equations tend (sweeping much under the carpet) to have solutions which can be extended for all time, whereas non-linear equations can easily have solutions which blow up at finite time. This phenomenon is well known in classical geometry, in fact a manifold is called complete just when its geodesics can be extended for all time.
16 Non-commutative vector fields and parallel transport

Now we translate the ideas of the last section into the non-commutative regime. There is a problem with exponentiation, if $X(t)$ is not a derivation then its exponentiation is not an algebra map. A partial answer is given in this section. We denote by $L(A, B)$ the linear maps from $A$ to $B$.

**Definition 16.1** A time dependent vector field $X(t) \in \text{Vec}(M)$ exponentiates to give $K^n_X(t) \in L(\Omega^n M, \Omega^n M)$ (for all $n \geq 0$) defined by

$$
\dot{K}^n_X(t) = K^n_X(t) \circ \mathcal{L}_{X(t)}, \quad K^n_X(0) = \text{id}.
$$

For $n = 0$ in the classical case, this gives the same result as 15.1.

**Proposition 16.2** The sequence of maps $K^n_X(t) : \Omega^* M \to \Omega^* M$ is a cochain complex map, i.e.

$$
d \circ K^n_X(t) = K^{n+1}_X(t) \circ d : \Omega^n M \to \Omega^{n+1} M.
$$

**Proof** This is given by the uniqueness of solutions to first order equations. Using 8.2,

$$
(d \circ \dot{K}^{n+1}_X(t))(\xi) = (d \circ K^{n+1}_X(t))(\mathcal{L}_{X(t)}(\xi)) = (d_X(\dot{K}^{n+1}_X(0))(\xi)) = (d_X(K^{n+1}_X(0)))(\mathcal{L}_{X(t)}(\xi)).
$$

Then $d \circ K^n_X(t)$ and $K^{n+1}_X(t) \circ d$ are both solutions to the differential equation $\dot{U}(t) = U(t) \circ \mathcal{L}_{X(t)}$ with initial condition $U(0) = d$ for $U(t) : \Omega^n M \to \Omega^{n+1} M$. □

**Proposition 16.3** The maps $\dot{K}^n_X(t)$ are cochain homotopic to 0 via the cochain homotopy $h^n_X(t) = K^n_X(t) \circ (X(t), \cdot) : \Omega^{n+1} M \to \Omega^n M$.

**Proof** We use the definition of the Lie derivative 8.1 and 16.2 to write

$$
\dot{K}^{n+1}_X(t) = K^{n+1}_X(t) \circ (d \circ (X(t), \cdot) + (X(t), \cdot) \circ d) = d \circ (K^n_X \circ (X(t), \cdot)) + (K^{n+1}_X \circ (X(t), \cdot)) \circ d.
$$

**Definition 16.4** Let $E$ be a left $M$-module with connection $\nabla$, and take a time dependent vector field $X(t) \in \text{Vec}(M)$. Then $c(t) \in E$ is parallel transported along the exponentiation of $X(t)$ if it obeys the first order differential equation $\dot{c}(t) = -\nabla_{X(t)} c(t)$.

**Definition 16.5** Given a connection $\nabla$ on $\text{Vec} M$, $c(t) \in \text{Vec} M$ is parallel transported along the exponentiation of $c(t)$ if it obeys the first order differential equation $\dot{c}(t) = -\nabla_{c(t)} c(t)$. 22
17 Exponentials of vector fields on the non-commutative torus

We will compare the exponentials of the time independent vector fields \( u \partial_u \) (which is a bimodule map) and \( \partial_u \) (which is only a right module map) on \( T_q^2 \).

**Lemma 17.1** We have \( d(v^ru^s) = r \, dv \cdot v^{r-1}u^s + s q^{-r} du \cdot v^{r-1}u^{s-1} \). This then gives

\[
\mathcal{L}_{\partial_u}(v^ru^s) = s q^{-r} v^r u^{s-1}, \quad \mathcal{L}_{u\partial_u}(v^ru^s) = s v^r u^s.
\]

**Proof** For the difficult case, first iterate the Lie derivative to get

\[
(L_{\partial_u})^n(v^ru^s) = s(s - 1) \ldots (s - n + 1) q^{-nr} v^r u^{s-n}
\]

\[
= s(s - 1) \ldots (s - n + 1) u^{-n} v^r u^s,
\]

and then use the binomial expansion

\[
\sum_{n \geq 0} \frac{s(s - 1) \ldots (s - n + 1)}{n!} (tu^{-1})^n = (1 + tu^{-1})^s . \quad \square
\]

**Lemma 17.3** On \( \Omega^1 T_q^2 \) we get

\[
\exp(t \mathcal{L}_{\partial_u})(du \cdot v^r u^s + dv \cdot v^n u^m) = (1 + tu^{-1})^s \cdot du \cdot v^r u^s + (1 + tu^{-1})^m \cdot dv \cdot v^n u^m ,
\]

\[
\exp(t \mathcal{L}_{u\partial_u})(du \cdot v^r u^s + dv \cdot v^n u^m) = e^{(s+1)t} \cdot du \cdot v^r u^s + e^{mt} \cdot dv \cdot v^n u^m .
\]

**Proof** From the following equations:

\[
\mathcal{L}_{\partial_u}(du \cdot v^r u^s + dv \cdot v^n u^m) = d(\partial_u \cdot (du \cdot v^r u^s + dv \cdot v^n u^m))
\]

\[
- \partial_u \cdot (du \wedge d(v^r u^s) + dv \wedge d(v^n u^m))
\]

\[
= d(v^r u^s) - \partial_u \cdot (du \wedge dv) \cdot r v^{r-1}u^s - \partial_u \cdot (dv \wedge du) \cdot m q^{-n} v^n u^{m-1}
\]

\[
= s q^{-r} du \cdot v^r u^{s-1} + dv \cdot m q^{-n-1} v^n u^{m-1} ,
\]

\[
\mathcal{L}_{u\partial_u}(du \cdot v^r u^s + dv \cdot v^n u^m) = d(u \partial_u \cdot (du \cdot v^r u^s + dv \cdot v^n u^m))
\]

\[
- u \partial_u \cdot (du \wedge d(v^r u^s) + dv \wedge d(v^n u^m))
\]

\[
= q^r d(v^r u^{s+1}) - u \partial_u \cdot (du \wedge dv) \cdot r v^{r-1}u^s
\]

\[
- u \partial_u \cdot (dv \wedge du) \cdot m q^{-n} v^n u^{m-1}
\]

\[
= (s + 1) du \cdot v^r u^s + m dv \cdot v^n u^m . \quad \square
\]
**Proposition 17.4** On $\Omega^2_T q$ we get

\[
\exp(t \mathcal{L}_\partial)(du \wedge dv^r u^s) = (1 + tu^{-1})^s . du \wedge dv^r u^s,
\]
\[
\exp(t \mathcal{L}_u\partial)(du \wedge dv^r u^s) = du \wedge dv^r u^s e^{(s+1)t}.
\]

**Proof** From the following equations:

\[
\mathcal{L}_\partial(du \wedge dv^r u^s) = d(\partial_u . (du \wedge dv).v^r u^s)
\]
\[
= d(dv^r u^s) = -dv \wedge d(v^r u^s)
\]
\[
= -dv \wedge du . s q^{-r} v^r u^{s-1} = du \wedge dv . s q^{-r-1} v^r u^{s-1},
\]
\[
\mathcal{L}_u\partial(du \wedge dv^r u^s) = d(u.\partial_u . (du \wedge dv).v^r u^s)
\]
\[
= d(u.dv^r u^s) = q^r+1 d(dv^r u^{s+1}) = -q^r+1 dv \wedge d(v^r u^{s+1})
\]
\[
= -q dv \wedge du . (s+1) v^r u^s = (s+1) du \wedge dv v^r u^s. \quad \Box
\]

### 18 Exponentiation and Hopf algebra coactions

It would be somewhat premature for me to claim that these exponentials of Lie derivatives really were significant in the non-commutative context, just because they reduce to the correct construction in the commutative case. Thus I would like to present some non-commutative supporting evidence.

Given a differentiable action of a Lie group on a manifold, an element of the Lie algebra gives a vector field on the manifold. Exponentiation of this vector field gives a diffeomorphism which is just action by the exponential of the Lie algebra element as an element of the Lie group. In this section I show an analogous result for Hopf algebra coactions on algebras. I shall use the Sweedler notation $\Delta(h) = h(1) \otimes h(2)$ for coproducts.

**Definition 18.1** Suppose that a Hopf algebra $H$ is given a differentiable structure so that the coproduct $\Delta : H \rightarrow H \otimes H$ is differentiable, where $H \otimes H$ is given the tensor product differential structure (see 2.3). The braided Lie algebra of $H$ is defined as

\[
\mathfrak{h} = \{ \alpha : \Omega^1 H \rightarrow k : \alpha(\xi.h) = \alpha(\xi).e(h) \quad \forall h \in H \ \forall \xi \in \Omega^1 M \}. \]

**Remark 18.2** This idea of differentiability is really the same as the more usual idea of bicovariance of the differential calculus. Given the existence of $\Delta_*$, we define right and left coactions...
of $H$ on $\Omega^1H$ by $\rho = \Pi_1\Delta_*$ and $\mu = \Pi_2\Delta_*$ respectively. The fact that these are coactions can be checked from the tensorial property in \ref{2.3}.

**Remark 18.3** See \cite{10} for more on the differential calculus on Hopf algebras. Note that the condition that $\Delta : H \to H \otimes H$ is differentiable is the same as requiring the bicovariance of the calculus, where the right and left coactions are $\Pi_1 \circ \Delta_* : \Omega^1H \to \Omega^1H \otimes H$ and $\Pi_2 \circ \Delta_* : \Omega^1H \to H \otimes \Omega^1H$.

**Proposition 18.4** There is a 1-1 correspondence between $\mathfrak{h}$ and left $H$-covariant vector fields on $H$ given by $\alpha \in \mathfrak{h}$ mapping to $L_\alpha = (\text{id} \otimes \alpha)\Pi_2\Delta_* : \Omega^1H \to H$, and a vector field $X$ mapping to $\epsilon \circ X \in \mathfrak{h}$.

**Proof** For a vector field $X$, $\epsilon(X(\xi,h)) = \epsilon(X(\xi).h) = \epsilon(X(\xi)) \epsilon(h)$, so $\epsilon \circ X \in \mathfrak{h}$. Also
\[
(id \otimes \alpha)\Pi_2\Delta_*(dh.a) = h(1) a(1) \alpha(dh(2).a(2)) = h(1) a(1) \epsilon(a(2)) \alpha(dh(2)) = h(1) \alpha(dh(2)) a ,
\]
so $L_\alpha$ is a right module map, i.e. a vector field on $H$. To check that $L_\alpha$ is left invariant,
\[
(id \otimes L_\alpha)\mu(dh.a) = h(1) a(1) \otimes L_\alpha(dh(2).a(2)) = h(1) a(1) \otimes h(2) a(2) \alpha(dh(2)) ,
\]
\[
\Delta \circ L_\alpha(dh.a) = \alpha(dh(2)) \Delta(h(1) a) = \alpha(dh(2)) h(1) a(1) \otimes h(1) a(2) ,
\]
and these are the same by coassociativity. To check the 1-1 correspondence,
\[
\epsilon(L_\alpha(dh)) = \epsilon(h(1)) \alpha(dh(2)) = \alpha(dh) ,
\]
and finally the more difficult bit. From the discussion above
\[
L_{\epsilon \circ X}(dh) = h(1) \epsilon(X(dh(2)) ,
\]
and if $X$ is left invariant then $\Delta(X(dh)) = h(1) \otimes X(dh(2))$, so
\[
L_{\epsilon \circ X}(dh) = (\text{id} \otimes \epsilon)\Delta(X(dh)) = X(dh) . \quad \square
\]

**Proposition 18.5** The exponentiation of the time independent vector field $L_\alpha$ on $H$ is
\[
\exp(tL_\alpha)(h) = \sum_{r \geq 0} \frac{t^r}{r!} h(1) \alpha(dh(2)) \ldots \alpha(dh(r+1)) .
\]
Proposition 18.7 Given a left tensor product algebra structure) and (if $M \to \alpha$ so $\Lambda(\alpha)$, as a brief check that this corresponds to the classical construction, the Lie algebra $g$ of a Lie group $G$ in our setting corresponds to the Hopf algebra $k(G)$ of functions on $G$. The exponential of an element of $g$ is in the dual algebra, the group algebra $kG$.

Now we turn to a differentiable left coaction $\lambda$ of a Hopf algebra $H$ on an algebra $M$. We suppose that $M$ is a left $H$-comodule algebra, i.e. $\lambda : M \to H \otimes M$ is an algebra map (with the tensor product algebra structure) and (if $M$ is unital) $\lambda(1_M) = 1_H \otimes 1_M$.

Proposition 18.7 Given a left $H$-comodule algebra $M$ with differentiable left $H$-coaction $\lambda : M \to H \otimes M$, there is a map $\Lambda : \mathfrak{h} \to \text{Vec} M$ given by $\Lambda(\alpha) = (\alpha \otimes \text{id})\Pi_1\lambda_\epsilon$.

Proof If we write the left coaction as $\lambda(m) = m_{[-1]} \otimes m_{[0]}$, then

\[
\lambda_\epsilon(dm.m) = d(\lambda(n)).\lambda(m) = d(n_{[-1]} \otimes n_{[0]}).m_{[-1]} \otimes m_{[0]},
\]

\[
\Pi_1\lambda_\epsilon(dm.m) = dn_{[-1]}m_{[-1]} \otimes n_{[0]}m_{[0]},
\]

\[
\Lambda(\alpha)(dm.m) = \alpha(dm_{[-1]}m_{[-1]})n_{[0]}m_{[0]} = \alpha(dm_{[-1]}m_{[-1]})m_{[0]} = \Lambda(\alpha)(dm).m,
\]

so $\Lambda(\alpha)$ is a right module map, i.e. a vector field on $M$. □

Proposition 18.8 The exponential of the time independent vector field $\Lambda(\alpha)$ on $M$ is

\[
\exp(t\mathcal{L}_{\Lambda(\alpha)})(m) = \sum_{r \geq 0} \frac{t^r}{r!} m_{[0]} \alpha(dm_{[-1](1)}) \ldots \alpha(dm_{[-1](r)}).
\]

Proof First calculate the Lie derivative $\mathcal{L}_{\Lambda(\alpha)}(m) = \Lambda(\alpha)(dm) = \alpha(dm_{[-1]})m_{[0]}$. Iterating this and using the coaction property, $(\mathcal{L}_{\Lambda(\alpha)})^2(m) = \alpha(dm_{[-1](1)}) \alpha(dm_{[-1](2)})m_{[0]}$ etc. □

Theorem 18.9 We have the following relation between the exponential of the vector field on $M$ generated by an element $\alpha \in \mathfrak{h}$ and the exponential of $\alpha$ as an element of $H^*$:

\[
\exp(\mathcal{L}_{\Lambda(\alpha)}) = (\exp(\alpha) \otimes \text{id}) \circ \lambda : M \to M.
\]
Proof  Directly from the preceeding results. □

References

[1] Anderson FW. & Fuller KR., Rings and categories of modules, Second edition, Graduate Texts in Mathematics, 13. Springer-Verlag, New York 1992.

[2] Beggs EJ. & Majid S., Semi-Classical differential structures.

[3] Borowiec A., Vector fields and differential operators: noncommutative case. Czechoslovak J. Phys. 47 (1997), no. 11, 1093–1100.

[4] T. Brzeziński, H. Dabrowski & J. Rembieliński, On the quantum differential calculus and the quantum holomorphicity, Jour. Math. Phys. 33 (1992), 19-24.

[5] Chu C-S., Ho P-M. & Zumino B., Some complex quantum manifolds and their geometry. Quantum fields and quantum space time (Cargse, 1996) , 281–322, NATO Adv. Sci. Inst. Ser. B Phys., 364, Plenum, New York, 1997.

[6] Connes A., Noncommutative Geometry, Academic Press 1994.

[7] Jara P. & Llena D., Lie bracket of vector fields in noncommutative geometry. arXiv:math.RA/0306044

[8] Madore J., An introduction to noncommutative differential geometry and its physical applications. London Mathematical Society Lecture Note Series, 257, CUP 1999.

[9] Majid S., Quantum and Braided Lie Algebras. J. Geom. Phys. 13 (1994) 307-356.

[10] Woronowicz SL., Differential calculus on compact matrix pseudogroups (quantum groups). Comm. Math. Phys. 122 (1989), no. 1, 125–170.