Formation of closed timelike curves in a composite vacuum/dust asymptotically-flat spacetime

Amos Ori
Department of Physics,
Technion—Israel Institute of Technology, Haifa, 32000, Israel

We present a new asymptotically-flat time-machine model made solely of vacuum and dust. The spacetime evolves from a regular spacelike initial hypersurface $S$ and subsequently develops closed timelike curves. The initial hypersurface $S$ is asymptotically flat and topologically trivial. The chronology violation occurs in a compact manner; namely the first closed causal curves form at the boundary of the future domain of dependence of a compact region in $S$ (the core). This central core is empty, and so is the external asymptotically flat region. The intermediate region surrounding the core (the envelope) is made of dust with positive energy density. This model trivially satisfies the weak, dominant, and strong energy conditions. Furthermore it is governed by a well-defined system of field equations which possesses a well-posed initial-value problem.

I. INTRODUCTION

This paper deals with the possibility of formation of closed causal curves (CCCs) in spacetime, within the framework of General Relativity. By causal curves we mean either timelike or null curves. The main question at the background is the following: Is it possible that CCCs will spontaneously evolve from rather "normal", non-pathological, initial conditions? In most of the classic solutions of the Einstein equations (e.g. Minkowski, Schwarzschild, Robertson-Walker) no such CCCs occur, although several examples of spacetimes which do admit closed timelike curves (CTCs) are known [1, 2, 3, 4, 5]. We dub such spacetimes, which admit CTCs, as "time-machine (TM) spacetimes". However, most of these examples suffer from pathologies or problematic ingredients which question their relevance to physical reality. Following is a list of three basic requirements that eliminate most of the TM models proposed so far: (i) The spacetime should admit a regular spacelike initial hypersurface (a partial Cauchy surface) $S$; (ii) asymptotic flatness; (iii) the weak energy condition [9]. Thus, Gödel's rotating-dust cosmological model [1] violates conditions (i,ii), as does Tipler's rotating-string solution [2]; the wormhole model by Morris, Thorne and Yurtsever [3] violates condition (iii). Gott's solution [4] of two infinitely long cosmic strings violates condition (ii) [10], and Mallett's solution [5] violates condition (i) (see [11]).

A few of the previous models [2, 3, 4] do satisfy the above requirements (i-iii). These are all asymptotically flat, topologically trivial models, in which the CCCs are born inside a certain torus. These models, however, fail to satisfy some other basic requirements discussed below. In particular, the models [2, 3] violate the strong energy condition [9]. Here we present a new model which better satisfies all these requirements.

In considering the physical relevance of a TM model (like any other kind of spacetime model), one of the most important aspects is the physical suitability of the spacetime's matter content. One certainly would like to impose the weak energy condition, and preferably also the strong and dominant energy conditions. But compliance with the energy conditions is not enough. One would also like the spacetime's energy-momentum tensor to coincide with some known matter field, governed by a well-known, well-posed field equation, for at least two obvious reasons. First, if we cannot associate the energy-momentum tensor with some known matter field, then we cannot tell whether this energy-momentum distribution can be obtained in reality. Second, the following question concerning spacetime dynamics is at issue: Can one design a "normal" initial configuration such that the laws of evolution will subsequently force the spacetime to develop CCCs and to violate chronology? A system (spacetime +matter) which does not admit a well-defined set of evolution equations will be inadequate for addressing such a question.

Motivated primarily by the last argument, in Ref. [7] we presented a TM model in which the initial hypersurface is composed of three parts: An external asymptotically flat region, the "envelope" (an intermediate region), and a compact toroidal region at the center, to which we refer as the "compact core". In this model CCCs evolve inside the compact core, in a manner which is causally independent of the surrounding regions. The central compact core, and also the external asymptotically-flat region, are vacuum, but the envelope is made of matter. This matter satisfies all three energy conditions mentioned above, yet the envelope’s energy-momentum has not been recognized as any known form of matter field. Nevertheless, since the internal core is made of pure vacuum, and the formations of CCCs inside the core is guaranteed independently of the envelope’s evolution, this model fulfills its main goal at least to some extent—it successfully demonstrates how the laws of spacetime dynamics inevitably lead, in a certain situation, to the violation of chronology—provided that the initial configuration of energy-momentum at the envelope could be realized by some real matter field. But still the question remains whether such a matter field exists or not. One of the main goals of this paper is to address this difficulty.
The spacetime model constructed here is similarly composed of three parts: An internal vacuum core, an external asymptotically flat vacuum region (the Schwarzschild geometry), and a non-empty intermediate region (the "envelope"). Here, however, the envelope's matter will be simply dust (namely, a perfect fluid with zero pressure), with non-negative energy density. This kind of matter trivially satisfies the weak, dominant, and strong energy conditions. It also yields a well-posed initial-value problem [9]. Dust is probably not the most realistic or fundamental description of matter, yet it has been proven useful in addressing various issues of principle in General Relativity, e.g. the dust Robertson-Walker cosmology, the Oppenheimer-Snyder model of homogeneous dust collapse (the first model to demonstrate the formation of a black hole in gravitational collapse), and the formation of naked singularities in spherical dust collapse [13]. In the last two problems, significant progress was first made by exploring a dust model, but qualitatively similar results were subsequently observed in models with vanishing pressure (see e.g. [14]). In fact, it appears that in our problem as well it will not be difficult to generalize the present dust model to a perfect fluid with pressure, but this is beyond the scope of this paper.

The present model also differs from that of Ref. [7] in the type of vacuum metric employed for the compact core. In Ref. [7] we used a vacuum solution locally isometric to a pp-wave spacetime. Here we use a vacuum solution locally isometric to a "pseudo-Schwarzschild" geometry (namely, one obtained from the Schwarzschild geometry by a Wick rotation), which we describe in Sec. III. One of the differences is that the pseudo-Schwarzschild core metric can be easily represented in a diagonal form, which globally covers the core metric from the initial hypersurface and up to the Cauchy Horizon (CH). We are not aware of such a global diagonal representation of the core metric of Ref. [7]. Another difference is that the present core metric admits an initial hypersurface with an especially simple extrinsic curvature, as described in section III again, we are not aware of such a possible choice of initial hypersurface in the core metric of Ref. [7]. Both factors, the diagonal core metric and the simple form of the extrinsic curvature, greatly simplify the construction of the initial data for the TM model with dust envelope.

There is another, more significant, difference between the two core metrics: The locally pseudo-Schwarzschild metric is similar to the (four-dimensional version of the) Misner space [10], as its CH is entirely generated by closed null geodesics (CNGs). In the locally-pp metric, on the other hand, the CNGs are generically isolated. The pseudo-Schwarzschild core metric is also similar to the Misner space in that two non-equivalent analytic extensions beyond the CH exist [13]. On the contrary, only one possible extension beyond the CH appears to exist in the locally-pp core metric of Ref. [7].

The Misner-like form of our present core metric has both advantages and disadvantages. A CH generated by CNGs is claimed [18] to be unstable against "fragmentation" into a set of isolated null geodesics. This is a disadvantage which might motivate us to try construct a dust envelope for the locally-pp core of Ref. [7] as well, but this is beyond the scope of the present paper. At any rate we do not attempt to address issues of stability in this paper.

But the Misner-like core used here also has advantages. In our previous model [7], the question arises whether the closed null geodesic N at the CH is adequately protected against a singularity which might form at the future boundary of D+ (S) and approach arbitrarily close to N [19]. Originally we thought that the CNG N is causally protected against this scenario, for the following reason: The initial hypersurface S has a compact core S0, such that the CNG N is located at the boundary of D+ (S0). The vacuum solution in the entire region D+ (S0) is known explicitly, and is regular throughout. This ensures that no singularity can form at (the boundary of) D+ (S0) and endanger the regularity of the neighborhood of N. However, the domain D+ (S) is larger than D+ (S0). Although the boundaries of these two domains coincide at N, it may well be the case that there is a separation between these two boundaries, and the set D+ (S)–D+ (S0) gets arbitrarily close to N. This structure may allow the possibility of a singularity which evolves at the future boundary of D+ (S) (but outside D+ (S0)) and extends arbitrarily close to N. In that case, N would be a regular CNG (with a regular neighborhood) as viewed from D+ (S0), but would still lack a regular neighborhood in D+ (S). This might be harmful for an extended (test) observer attempting to cross the CH of S through one of the points on N, in order to penetrate into the region of CTCs. [20]

It is not clear if this potential problem is realized in the model [7]. It is hard to say, because the exact solution for the time-evolving metric is only known in the internal vacuum core and in the external region, not in the envelope. Therefore, we do not know the full structure of D+ (S), and, in particular, what kinds of singularities it develops, if any. Fortunately this potential problem does not apply to the present model. As will be shown in Sec. III due to the Misner-like form of the CH, the boundaries of D+ (S) and D+ (S0) do overlap in a set denoted H1 below. This set, which is a portion of the CH, includes a continuum of CNGs. A sufficiently small neighborhood of any such CNG, restricted to D+ (S), is entirely contained in D+ (S0). Since the metric throughout D+ (S0) is known explicitly and is perfectly smooth, no singularity which might evolve at (the boundary of) D+ (S) can get close to any of these CNGs.

As previously stated, the main underlying question is the possibility of triggering the onset of chronology violation. In other words, is it possible to design initial conditions for which the laws of dynamics will inevitably lead to violation of chronology? But there is a build-in logical difficulty in the formulation of this question: If
indeed CTCs form, then the portion of spacetime containing the CTCs is by definition outside the future domain of dependence of any initial hypersurface S. In what sense can one then state that the chronology violation has "emerged from the initial conditions on S"? This is indeed a difficulty, but nevertheless we propose a set of conditions which, when satisfied, provide meaning to the statement that the violation of chronology was triggered by the initial conditions on S. These conditions are: (i) $H_+(S)$ contains CNGs (therefore the Cauchy evolution of the initial data on S unambiguously leads to some sort of chronology violation); (ii) The analytic extension of the metric beyond $H_+(S)$ contains CTCs (or some portion of $H_+(S)$) in the immediate neighborhood of $H_+(S)$; (iii) Any smooth extension of the metric beyond $H_+(S)$ (or some portion of $H_+(S)$) will include CTCs in the immediate neighborhood of $H_+(S)$.

As a simple application of these criteria, consider the analytically-extended geometry of a Kerr black hole. This spacetime is known to admit CTCs deep inside the black hole. The CTCs are located beyond the inner horizon—a null hypersurface which serves as the CH for any initial hypersurface in the external universe. We shall not regard this spacetime as a "time-machine model" as it fails to satisfy any of the above criteria (i-iii). In particular, the inner horizon does not contain CNGs. On the other hand, the model presented in this paper does satisfy all three conditions.

When addressing the possibility of constructing a time machine, one would primarily be interested in the situation where the construction process takes place in a finite region of space. A simple criterion which captures this idea is the following: We shall say that the time machine is **compactly constructed** if the initial hypersurface S includes a compact set $S_0$ such that the Cauchy evolution of the initial data on $S_0$ leads to chronology violations; That is, the closure of $D_+(S_0)$ includes CCCs (specifically this means that $H_+(S_0)$ includes CNGs).

Hawking [18] earlier introduced a different notion of compactness called **compact generation**. A CH is said to be compactly generated if all its null generators, when past-propagated, enter a compact region of spacetime and never get out of it. This criterion differs from the notion of compact construction formulated above. The time-machine model presented in this paper, as well as our previous models [5][6][7], are all compactly constructed but might not be compactly generated.

The above discussion led to several criteria which one might apply to any candidate model attempting to describe the process of "constructing a time-machine" in our physical spacetime. In the next subsection we collect these criteria and list them in a more systematic manner. Then section [IV] outlines the structure of our model spacetime and its various parts (central core, envelope, and external region). In Sec. [V] we present our core metric and discuss its main properties. Section [VI] outlines the initial-value setup, and the constraint equations which must be satisfied by the initial data. Then Sections [VII] and [VIII] describe the construction of the desired initial data (3-metric and extrinsic curvature) on the envelope and external parts of the initial hypersurface S, respectively. In Sec. [VIII] we summarize and discuss some of the problems and open questions remaining for future research.

### A. Criteria for a physical time-machine model

Here we collect the various criteria which emerged in the discussion above (plus one more criterion related to the space topology). It should be emphasized that we do **not** attempt here to postulate a strict, formal definition of a "TM model". Rather, our goal here is to list the various criteria which we find relevant. This list may serve as a useful basis for discussing the physical relevance of various models which attempt to describe "TM construction":

1. The spacetime should admit a spacelike initial hypersurface (a partial Cauchy surface) S;
2. The initial data on S should be sufficiently regular; Namely, both the spatial 3-metric and the extrinsic curvature should be $C^{(k)}$ for a sufficiently large $k$. ($k$ should be, say, 4 or larger in order to guarantee a well-defined time evolution. The construction below yields $C^{(\infty)}$ initial data.)
3. Asymptotic flatness,
4. The spacetime’s matter content should satisfy the Energy conditions. This may be divided into two categories: (4a) The weak energy condition, and (4b) The dominant and strong energy conditions.
5. The causal connection between S and the chronology violation: (5a) $H_+(S)$ should contain CNGs; (5b) The analytic extension of the metric beyond (some portion of) $H_+(S)$ should include CTCs in the immediate neighborhood of $H_+(S)$; (5c) Any smooth ("hole-free" [22][23]) extension of the metric beyond (some portion of) $H_+(S)$ should include CTCs in the immediate neighborhood of $H_+(S)$.
6. "Causal protection" of the CNG: $H_+(S)$ includes a CNG N admitting a neighborhood in $D_+(S)$ which is perfectly regular.
7. Compact construction: S should include a compact set $S_0$ such that $H_+(S_0)$ contains CNGs. [Furthermore, the criteria 5,6 above should apply to a portion of $H_+(S)$ which is also contained in $H_+(S_0)$].
8. The topology of S should be trivial (R^3).
9. The energy-momentum tensor will correspond to a known matter field, which yields a well-posed initial-value problem. This is especially crucial for the core metric inside $D_+(S_0)$ (which itself develops...
chronology violation), but is also desired (though perhaps to a lesser extent) for the outer parts of the time-machine model.

We may also add the following, wider (and loosely formulated) requirement concerning the spacetime matter content:

10. The matter field will be as elegant and/or realistic as possible.

From the point of view of classical General Relativity, the most elementary and elegant type of energy-momentum tensor is obviously the vacuum, $T_{\alpha\beta} = 0$.

The present model satisfies all these requirements. Our previous model [2] does not fully satisfy criterion 9, because the envelope is made of an unrecognized matter field (though the core is vacuum). Also it is not clear if criterion 6 is satisfied by it. Our earlier models [2] also fail to satisfy criterion 4b. In fact, none of the previous models satisfy the combination of criteria 1, 2, 3, 9.

### II. AN OVERVIEW OF THE SPACETIME’S STRUCTURE

Our model spacetime is composed of three parts: the central active vacuum core, the dust envelope, and the external asymptotically-flat vacuum region. Correspondingly, the initial hypersurface $S$ (a partial Cauchy surface) will be composed of three parts:

1. The internal vacuum core, located inside a certain torus $T_0$.
2. The envelope, an intermediate region located between the torus $T_0$ and a two-sphere $R_s$ surrounding it. ($R_s$ is the two-sphere which is later denoted $R = R_s$.)
3. The external vacuum region, located outside the two-sphere $R_s$. (This external region actually corresponds to a certain spacelike hypersurface in the Schwarzschild geometry.)

These three parts of $S$ will be denoted $S_0$, $S_1$, and $S_2$, respectively.

In a similar manner we divide the evolving 4-dimensional spacetime into three regions. To be more precise, it is the predictable portion of spacetime, namely $D^+(S)$ and its closure, which we divide and associate with the various parts of $S$. The internal region $M_0$ is $D^+(S_0)$; the external region $M_2$ is $D^+(S_2)$, and the intermediate region $M_1$ is the intersection of $D^+(S)$ with $J^+(S_1)$. Both $M_0$ and $M_2$ are pure vacuum regions: $M_0$ is a compact region which includes CCCs and hence constitutes the TM core, and $M_2$ is a portion of the Schwarzschild geometry (the external part, which extends to spacelike infinity). The intermediate region $M_1$ is made of dust (though it also includes vacuum parts).

In both vacuum regions $M_0$ and $M_2$ the 4-geometry is known to us explicitly: It is the metric (12) in region $M_0$, and the Schwarzschild geometry in region $M_2$. In the dust region $M_1$, on the other hand, the evolving 4-geometry is not known explicitly. Instead, it is described in terms of the corresponding initial data on $S$ (section V below).

### III. INTERNAL CORE METRIC

We start our construction from the vacuum metric

$$ds^2 = -(1-2\mu/r)^{-1}dr^2 + (1-2\mu/r)dt^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

(1) where $\mu$ is an arbitrary positive constant. Here $\theta$ takes all positive values, whereas $\varphi$ admits the usual periodicity $0 \leq \varphi < 2\pi$. This metric is obtained from the standard Schwarzschild metric by a Wick rotation $\theta \rightarrow i\theta$, and we shall refer to it as the pseudo-Schwarzschild metric. The coordinate $t$ is assumed here to be periodic,

$$0 \leq t < l,$$

where $l$ is a free parameter which we take to be greater than some minimal value $t_{\text{min}}$ [specified in Eq. (23) below].

The above metric has a coordinate singularity at $r = 2\mu$, analogous to the Schwarzschild’s horizon (later we shall remove this singularity by transforming to Eddington-like coordinates). At this stage, and throughout most of this paper, we shall primarily be interested in the range $r > 2\mu$. Note that in this range $t$ is a spatial coordinate (i.e., $g_{tt} > 0$) and $r$ is a time coordinate, which we take to be past-directed (namely, $r$ decreases on moving from the past to the future). Thus, all hypersurfaces $r = \text{const} > 2\mu$ are spacelike.

To overcome the coordinate singularity at $r = 2\mu$ we now transform to “Eddington-like” coordinates in the usual manner: We define

$$v = -(t + r^*),$$

where

$$r^* \equiv r + 2\mu \ln(r/2\mu - 1).$$

In the $(r, v)$ coordinates the metric becomes

$$ds^2 = (1-2\mu/r)dv^2 + 2dv dr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

(2) The coordinate $v$ has the same periodicity as that of $t$, namely, for given $(r, \theta, \varphi)$ the point $v = l$ is identified with $v = 0$. [24]

#### A. Formation of CTCs

The closed orbits of constant $r, \theta, \varphi$ admit the one-dimensional line element

$$ds^2 = (1-2\mu/r)dv^2.$$
These orbits are spacelike throughout $r > 2\mu$, but they become timelike—namely CTCs—at $r < 2\mu$. Note that the metric \( (2) \) passes $r = 2\mu$ in a perfectly regular manner, as all components are regular and

$$\det(g) = -r^4 \sin^2 \theta$$

is nonvanishing there.

The "hypersurface" $r = 0$ is a true, timelike, curvature singularity. Our analysis throughout this paper is restricted to the range $r > 0$.

The hypersurface $r = 2\mu$ is null, and its generators are the curves of constant $\theta, \varphi$, which are all CNGs. This hypersurface is in fact the CH for any partial Cauchy surface $r = \text{const.} > 2\mu$. It also serves as the chronology horizon for the metric \( (2) \). Namely, all points at $0 < r < 2\mu$ sit on CTCs (e.g. the curves of constant $r, \theta, \varphi$), but none of the points at $r > 2\mu$ do (because the region $r > 2\mu$ is foliated by the spacelike hypersurfaces $r = \text{const}$).

### B. Initial hypersurface for the internal core

When discussing the initial-value problem for the above spacetime, we shall consider an initial hypersurface located at $r > 2\mu$. It will be convenient to express the metric in the diagonal form \( (1) \) (the coordinate singularity at $r = 2\mu$ will not pose any difficulty, as it takes place away from the initial hypersurface).

For any spacelike hypersurface $r = \text{const.} > 2\mu$, the spatial 3-metric is

$$ds^2 = (1 - 2\mu/r) dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \equiv h_{ab} dx^a dx^b .$$

(3)

Hereafter the indices $a, b$ run over the three spatial coordinates. The extrinsic curvature $K^a_b$ of such a hypersurface is

$$K^a_b = \frac{\mu}{r^2} (1 - 2\mu/r)^{-1/2} , \quad K^a_\theta = K^\varphi_b = r^{-1} \sqrt{(1 - 2\mu/r)} ,$$

and all other components vanish. The two distinct eigenvalues coincide at $r = 3\mu$ in which $K^a_a = k_0 \delta^a_a$, or

$$K_{ab} = k_0 h_{ab} ,$$

where

$$k_0 = \frac{1}{\sqrt{2} r \mu} .$$

(4)

(5)

The form \( (4) \) greatly simplifies the constraint equations, therefore we shall take our initial hypersurface (for the internal core) to be at

$$r = 3\mu \equiv r_0 .$$

We denote this hypersurface by $\Sigma$.

For later convenience we transform the periodic coordinate $t$ of the internal vacuum core into a new coordinate $\phi$ with a standard periodicity

$$0 \leq \phi < 2\pi ,$$

namely $\phi = (2\pi/l)t$. Then the inner-core 3-metric becomes

$$ds^2 = r_0^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + L_0^2 d\phi^2 ,$$

(6)

where

$$L_0 = (l/2\pi) \sqrt{1 - 2\mu/r_0} = \frac{l}{2\pi \sqrt{3}} .$$

Note that the 3-metric \( (6) \) is cylindrically-symmetric, with $\theta$ serving as the "radial" coordinate and $\varphi$ as the azimuthal coordinate. [Later we embed this 3-metric as the core of a global asymptotically-flat hypersurface (sections \[\Box\] below). This global hypersurface is axially-symmetric. It should be clarified that it is $\phi$, not $\varphi$, which becomes the global azimuthal coordinate.]

### C. Truncating the internal core metric

The hypersurface $r = 3\mu$ with the spatial 3-metric \( (6) \) is not asymptotically-flat. In order to match it to an asymptotically-flat exterior, we have to truncate the internal 3-metric at a certain two-surface. Any two-surface $\theta = \text{const.} > 0$ on the three-surface $r = r_0$ is a torus (parametrized by the two periodic coordinates $\varphi, \phi$). We shall truncate the initial 3-metric \( (6) \) on such a 2-surface $\theta = \text{const.} \equiv \theta_0$. We denote the portion $\theta \leq \theta_0$ of the three-surface $r = r_0$ by $S_0$.

Consider now the set $D^+(S_0)$, namely the future domain of dependence of $S_0$. This set has a limited extent in the time $r$ (because none of the points at $r < 2\mu$ belong to this set). At $2\mu < r < r_0$, $D^+(S_0)$ will be bounded by the null geodesics of constant $\varphi, \phi$ which emanate at $r = r_0$ from $\theta = \theta_0$ and propagate towards smaller $\theta$ values. [Note that geodesics with different $\varphi, \phi$ will have the same orbit $\theta(r)$, due to the cylindrical symmetry.] For our time-machine construction it is crucial that $D^+(S_0)$ will include a portion of the chronology horizon at $r = 2\mu$. This demand will impose a minimal value for $\theta_0$, as we now discuss.

The above null orbits of constant $\varphi, \phi$ which bound $D^+(S_0)$ satisfy the differential equation

$$\frac{d\theta}{dr} = \frac{\sqrt{-g_{rr}}}{\sqrt{g_{\theta\theta}}} = \frac{1}{r \sqrt{1 - 2\mu/r}} .$$

The general solution of this equation is

$$\theta(r) = 2 \ln \left[ \sqrt{r/(2\mu)} + \sqrt{r/(2\mu) - 1} \right] + C_\theta ,$$

(7)

where $C_\theta$ is an integration constant. This constant is determined from the initial value $\theta = \theta_0$ at $r = 3\mu$, namely

$$C_\theta = \theta_0 - \theta_c ,$$

where

$$\theta_c = \ln \left[ 2 + \sqrt{3} \right] \approx 1.317 .$$
At \( r = 2\mu \) the term in squared brackets in Eq. (7) vanishes. We therefore demand \( C_0 > 0 \). Thus we shall take the cutoff value \( \theta_0 \) to be \( > \theta_c \). This ensures that the portion

\[ 0 \leq \theta < \theta_0 - \theta_c \]

of the chronology horizon at \( r = 2\mu \) will be included in the boundary of \( D^+(S_0) \). In particular this portion includes an open set of CNGs.

D. The relation between \( H_+ (\Sigma) \), \( H_+ (S_0) \), and \( H_+(S) \)

In the non-truncated core metric (2) the CH associated with the (complete) initial hypersurface \( \Sigma \), \( H_+ (\Sigma) \), is the hypersurface \( r = 2\mu \). Once the core metric is truncated (as described above), the structure of the CH changes. The truncated part \( S_0 \) has its own CH, denoted \( H_+ (S_0) \). In addition the global asymptotically-flat initial hypersurface \( S \) (which contains \( S_0 \) as its core) has its own CH, denoted \( H_+(S) \). Here we shall briefly discuss the relation between these various CHs.

Consider first the structure of \( H_+ (S_0) \). From the discussion in the previous subsection it follows that this hypersurface is composed of two parts: (i) the portion \( 0 \leq \theta \leq \theta_0 - \theta_c \) of the hypersurface \( r = 2\mu \), which we denote \( H_1 \), and (ii) a null hypersurface denoted \( H_2 \), associated with the truncation of the core metric, whose generators emanate from the truncation 2-surface \( r = 3\mu \), \( \theta = \theta_0 \). These generators follow the orbit (7) (for each \( \varphi \) and \( \phi \)). The part \( H_1 \) is a portion of \( H_+ (\Sigma) \). This part (unlike \( H_2 \)) is entirely generated by CNGs.

The structure of \( H_+ (S) \) is more complicated and still needs be explored. It is easy to show, however, that \( H_+ (S) \) contains \( H_1 \) as a subset: Since \( H_1 \subset H_+ (S_0) \) belongs to the closure of \( D_+ (S_0) \) (and since \( S_0 \subset S \)), it must also be included in the closure of \( D_+ (S) \). But since all points of \( H_1 \) sit on CNGs, none of them belong to \( D_+ (S) \). Therefore all points of \( H_1 \) must be located at the boundary of \( D_+ (S) \) (but not on \( S \) itself), namely on \( H_+ (S) \).

The fact that \( H_1 \) is contained in \( H_+ (S) \) guarantees the "causal protection" discussed in the previous sections. Consider a point \( P \) located in \( H_1 \) but away from its intersection with \( H_2 \) (i.e. at some \( \theta < \theta_0 - \theta_c \)). The boundaries of \( D_+(S) \) and \( D_+(S_0) \) overlap in the neighborhood of \( N \). Therefore, any sufficiently-small neighborhood of \( N \) in the closure of \( D_+(S) \) is contained in the closure of \( D_+(S_0) \), and is hence guaranteed to be regular. This ensures that no singularity which evolves at the boundary of \( D_+ (S) \) may get close to \( P \). Obviously this argument also applies to the entire CNGs located at \( \theta < \theta_0 - \theta_c \).

IV. THE INITIAL-VALUE SET-UP

A. Basic strategy

Our construction of the TM spacetime is formulated in terms of the corresponding initial data on the initial hypersurface \( S \). These initial data, which include the three-metric \( h_{ab} \) and the extrinsic curvature \( K_{ab} \), are constructed so as to satisfy the constraint equations (discussed below). The evolution of geometry will in turn be determined by the evolution equations. The set of equations relevant to our model is the Einstein-dust system, namely

\[ G^{\alpha\beta} = 8\pi T^{\alpha\beta} = 8\pi \epsilon u^\alpha u^\beta, \quad (8) \]

where \( \epsilon \) is a scalar field and \( u^\alpha \) is a normalized vector field. These quantities correspond to the dust density and four-velocity, respectively. The Energy conditions are all satisfied if \( \epsilon \geq 0 \) and \( u^\alpha \) is a timelike vector. The system (8) along with the initial data on \( S \), uniquely determine the evolution of geometry (and matter) throughout \( D_+ (S) \) (22). As was described in the previous section, the initial hypersurface \( S \) is composed of three parts: The inner core \( S_0 \), the envelope \( S_1 \), and the external region \( S_2 \). In \( S_0 \) and \( S_2 \) the initial data correspond to vacuum (i.e. \( \epsilon = 0 \)). This [along with the mathematical properties of Eq. (8)] guarantees that the evolving geometry will be vacuum throughout \( D_+ (S_0) \) and \( D_+ (S_2) \). In the rest of \( D_+ (S) \), \( \epsilon \) will in general be positive [though it may also vanish in certain portions of \( J_+(S_1) \)].

The evolving vacuum metric is known explicitly throughout \( D_+ (S_0) \), it is given in Eq. (2) [or Eq. (11)]. The vacuum solution in \( D_+(S_2) \), the external part of \( D_+ (S) \), is also known analytically: It is just the Schwarzschild solution. Our construction of \( S \) thus guarantees that the evolving spacetime in \( D_+ (S) \) will be asymptotically-flat and will admit future null infinity. In addition, the way we construct \( S_0 \) (in particular the requirement \( \theta_0 > \theta_c \)) guarantees that the conditions relevant to the central core region [e.g. features (5-7) in subsection (A) above] are satisfied.

The "envelope" part of the evolving spacetime is the region between \( D_+ (S_0) \) and \( D_+ (S_2) \). It may be expressed as \( D_+(S) \cap J^+(S_1) \). The evolving metric in this part is not known to us. In particular, we do not know which kinds of singularities (if any) develop there, and where. This limits our present ability to analyze the full causal structure of our spacetime, e.g. whether an event horizon form, and where exactly is the CH (outside of \( D_+ (S_0) \)). It seems that a numerical solution of the evolution equation will be required in order to fill this gap. Nevertheless, the known analytic vacuum solutions throughout \( D_+(S_0) \) and \( D_+(S_2) \), along with the properties of \( S \) and the initial data on it, guarantee that all the conditions (1-10) in subsection (A) above are satisfied by our spacetime.
B. The constraint equations

The initial data $h_{ab}$ and $K_{ab}$, to be specified on $S$, are subject to four constraint equations, which correspond to four combinations of the Einstein tensor that are completely determined by $h_{ab}$ and $K_{ab}$. Let $N^a$ be a normalized timelike vector (defined on $S$) orthogonal to $S$, and let $x^a$ be a set of three spacelike coordinates parametrizing $S$. Then the four constrained components of the Einstein tensor are

$\hat{G}_a = G_{\alpha\beta}N^\alpha K^b_{ab} - K^K_{b\alpha}$

and

$\hat{G} = G_{\alpha\beta}N^\alpha N^\beta = \frac{1}{2} \left[ R^{(3)} + (K_a^a)^2 - K_{ab}K^{ab} \right]$, 

where $R^{(3)}$ is the Ricci scalar associated with the 3-metric $h_{ab}$, and a colon denotes covariant differentiation with respect to $h_{ab}$. (Indices of $K$ are rased and lowered with the three-metric $h_{ab}$.)

The Einstein equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ then imposes the three Momentum equations

$K^b_{ab} - K^b_{b\alpha} = 8\pi T_{\alpha\beta} N^\alpha \equiv 8\pi \hat{T}_a$

and the Energy equation

$R^{(3)} + (K_a^a)^2 - K_{ab}K^{ab} = 16\pi T_{\alpha\beta} N^\alpha N^\beta \equiv 16\pi \hat{T}$.

In our dust model $T_{\alpha\beta} = \epsilon u^a u^\beta$. To simplify the analysis we now choose the initial dust velocity $u^a$ to coincide with $N^a$. Then $\hat{T}_a$ vanishes (because $N_a = 0$), and $\hat{T} = \epsilon$. We obtain the momentum equation

$K^b_{ab} - K^b_{b\alpha} = 0 \quad (9)$

and the Energy equation

$R^{(3)} + (K_a^a)^2 - K_{ab}K^{ab} = 16\pi \epsilon \quad (10)$.

In the internal part $S_0$ the extrinsic curvature takes the simple form (10). We shall now adopt this same form of $K_{ab}$ for the envelope $S_1$ as well. Then in both $S_0$ and $S_1$ the Momentum equation (9) is trivially satisfied, and the only non-trivial constraint equation is the Energy equation, which now reads

$R^{(3)} + \frac{2}{9\mu^2} = 16\pi \epsilon \quad (11)$

In the internal vacuum region $S_0$ this equation reduces to

$R^{(3)} = -\frac{2}{9\mu^2} \quad (12)$

and one can easily verify that the 3-metric (3) indeed satisfies this relation. In the envelope region $S_1$, we only require $\epsilon$ to be non-negative, therefore the Energy equation becomes an inequality

$R^{(3)} \geq -\frac{2}{9\mu^2} \quad (13)$.

In the external vacuum region the extrinsic curvature will no longer take the form (10). In this region the initial data will need to satisfy the Momentum equation (9) as well as the vacuum Energy equation, namely

$R^{(3)} + (K_a^a)^2 - K_{ab}K^{ab} = 0 \quad (14)$.

The initial data for the core $S_0$ are given in Eqs. (9) and (10). We shall now proceed to construct the initial data for $S_1$ and $S_2$ as well.

V. INITIAL DATA FOR THE ENVELOPE

The envelope $S_1$ interpolates between the internal vacuum core solution and the external Schwarzschild geometry. Throughout $S_1$ the extrinsic curvature is given by Eqs. (10), and the 3-metric only needs to satisfy the dust inequality (13).

This region will be further divided into three sub-regions:

The inner part: A region which extends between the torus $\theta = \theta_0$ (the inner boundary of $S_1$) and a certain larger torus $\theta = \theta_3 > \theta_0$;

The outer part: An inhomogeneous spherically-symmetric dust solution near the outer boundary of the envelope. This sub-region extends between the 2-sphere $R = R_1$ bounding $S_1$ and a certain 2-sphere $R = R_2$ (which contains the torus $\theta = \theta_3$);

The intermediate part: A homogeneous dust solution which extends between the inner and outer parts (i.e. between the torus $\theta = \theta_3$ and the 2-sphere $R = R_1$).

Our construction of the initial data for $S_1$ will start from the inner part, proceed with the intermediate part, and conclude with the outer part.

A. The inner part

This region interpolates between the internal vacuum solution and a homogeneous dust solution. At its inner boundary the 3-metric is

$ds^2 = r_0^2(d\theta^2 + \sinh^2 \theta d\varphi^2) + L_0^2 d\varphi^2 \quad (15)$

and on approaching the outer boundary it will become the flat 3-metric (21) below. A general form which covers both metrics (as well as the entire region in between) is
where \( \Delta \) denotes the 2-dimensional flat-space Laplacian in polar coordinates, namely

\[
\Delta L \equiv L_{,\theta\theta} + L_{,\theta} + L_{,\varphi\varphi}/\theta^2.
\]

The dust inequality (13) then implies that \( L(\theta, \varphi) \) must satisfy

\[
\Delta L \leq L.
\]

(along with \( L > 0 \)). We now take \( L(\theta, \varphi) \) in the form

\[
L(\theta, \varphi) = L_0 + f(\hat{\theta}) r_0 \theta \cos \varphi,
\]

where

\[
\hat{\theta} = (\theta - \theta_2)/\delta \theta,
\]

and \( \delta \theta \equiv \theta_3 - \theta_2 \). The function \( f(\hat{\theta}) \) may be any smooth function satisfying the following:

(i) it joins smoothly on \( f = 0 \) at \( \hat{\theta} \leq 0 \) (corresponding to \( L(\theta, \varphi) = L_0 \) at \( \theta \leq \theta_2 \)),

(ii) it joins smoothly on \( f = 1 \) at \( \hat{\theta} \geq 1 \) (corresponding to \( L(\theta, \varphi) = L_0 + r_0 \theta \cos \varphi \) at \( \theta \geq \theta_3 \)),

(iii) it is monotonous in between.

For this choice of \( L(\theta, \varphi) \) one finds

\[
\Delta L = r_0 \cos \varphi \left( \frac{3 f' (\hat{\theta})}{\delta \theta} + \frac{\theta f'' (\hat{\theta})}{\delta \theta^2} \right),
\]

where a prime denotes here a differentiation with respect to \( \hat{\theta} \). Using \( \theta \leq \theta_3 \) we obtain

\[
\Delta L \leq \frac{r_0}{\delta \theta} \left( 3 f_{,\max} + \frac{\theta_3}{\delta \theta} f''_{,\max} \right),
\]

where \( f_{,\max} \) and \( f''_{,\max} \) denote the maximal absolute values of \( df/d\theta \) and \( d^2 f/d\theta^2 \), respectively, throughout the range \( 0 < \hat{\theta} \leq 1 \). Noting that

\[
L(\theta, \varphi) \geq L_0 - r_0 \theta_3,
\]

we conclude that the dust inequality \( \Delta L \leq L \) is satisfied if \( L_0 \) is greater than some minimal value

\[
L_0_{,\min} = r_0 \theta_3 + \frac{r_0}{\delta \theta} \left( 3 f_{,\max} + \frac{\theta_3}{\delta \theta} f''_{,\max} \right).
\]

We shall thus take \( L_0 \) to be \( > L_0_{,\min} \) as required, hence the dust inequality is satisfied. This corresponds to

\[
l > l_{,\min} \equiv 2\pi \sqrt{\Delta L_0_{,\min}}
\]

in terms of the parameters of the original core metric (11). Note that Eq. (21) and the above expression for \( L_0_{,\min} \) also guarantees that \( L(\theta, \varphi) \) is strictly positive throughout \( \theta < \theta_3 \).
B. The intermediate homogeneous part

Next, in the range θ > θ_3 the 3-metric takes the form

\[ ds^2 = r_0^2 [d\theta^2 + \phi^2 d\varphi^2] + (L_0 + r_0 \theta \cos \varphi)^2 d\phi^2 . \]  

(24)

(Recall that both ϕ and φ admit a 2π periodicity.) This is a flat metric in somewhat unusual coordinates. To bring it to a standard form we perform the following coordinate transformation:

\[ \rho = r_0 \theta \cos \varphi + L_0 , \ z = r_0 \theta \sin \varphi , \]

and the metric becomes

\[ ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 . \]

(25)

Note that

\[ \rho \geq L_0 - r_0 \theta > L_0^\text{min} - r_0 \theta , \]

hence 𝜌 is positive on the torus θ = θ_3 and in its neighborhood.

In summary, outside the torus θ = θ_3 the 3-metric is flat and the dust has a constant positive density,

\[ \epsilon = \frac{1}{72\pi \mu^2} \equiv \epsilon_0 . \]

The flat 3-metric and the uniform extrinsic curvature indicate that the initial data at θ > θ_3 are just those of the (contracting) spatially-flat dust Robertson-Walker geometry.

C. The outer part: inhomogeneous spherical dust geometry

The outermost layer of the envelope S_3 will be constructed to be a spherically-symmetric, inhomogeneous, dust region, which interpolates between the constant density 𝜀_0 > 0 at θ > θ_3 and the vanishing density at the Schwarzschild exterior. To this end we first transform the flat metric (25) into standard spherical coordinates (R, Θ, φ) by

\[ z = R \cos \Theta , \ \rho = R \sin \Theta . \]

The 3-metric becomes

\[ ds^2 = dR^2 + R^2 d\Omega^2 , \]

(26)

where \( d\Omega^2 \) is the unit 2-sphere,

\[ d\Omega^2 = d\Theta^2 + \sin^2 \Theta d\phi^2 . \]

R may be expressed directly in terms of the original toroidal coordinates θ, ϕ:

\[ R^2 = \rho^2 + z^2 = L_0^2 + (r_0 \theta)^2 + 2L_0r_0 \theta \cos \varphi . \]

This implies an inequality

\[ L_0 - r_0 \theta \leq R \leq L_0 + r_0 \theta . \]

(27)

We now truncate the flat metric (26) at a two-sphere \( R = R_1 \). We take

\[ R_1 > L_0 + \theta_3 r_0 . \]

(28)

This ensures, by virtue of the second inequality in Eq. (27), that the torus θ = θ_3 is entirely contained at \( R < R_1 \). The flat metric (26) thus holds throughout the region between the torus θ = θ_3 and the sphere \( R = R_1 \) surrounding it.

The 3-metric at \( R > R_1 \) is assumed to be spherically symmetric, and we write it in the general form

\[ ds^2 = g_{RR}(R)dR^2 + R^2 d\Omega^2 . \]

(29)

It is convenient to substitute

\[ g_{RR} = [1 - 2M(R)/R + R^2/27\mu^2]^{-1} . \]

(30)

The Ricci scalar is then

\[ R^{(3)} = -\frac{2}{9\mu^2} + \frac{4}{R^2} \frac{dM}{dR} . \]

(31)

and Eq. (11) yields

\[ \epsilon = \frac{1}{4\pi R^2} \frac{dM}{dR} . \]

(32)

Therefore, to ensure non-negative dust density, the function \( M(R) \) must be a monotonously-increasing (or at least non-decreasing) one.

Consider next the boundary conditions on \( M(R) \). In the homogeneous dust region at \( R < R_1 \) the 3-metric is flat, \( g_{RR} = 1 \), which corresponds to

\[ M(R) = R^3/54\mu^2 . \]

(33)

The external boundary of the spherically-symmetric dust region is at the two-sphere \( R = R_2 \) for some \( R_2 > R_1 \), where the dust solution is to be matched to a spherically-symmetric vacuum solution at \( R > R_2 \) (see next section). From Eq. (32) this vacuum solution is characterized by

\[ M = \text{const} \equiv m . \]

Thus, \( M(R) \) is required to be a monotonously-increasing function which smoothly joins \( M = R^3/54\mu^2 \) at \( R \leq R_1 \) and \( M = m \) at \( R \geq R_2 \). For later convenience we also demand

\[ M(R) \leq R^3/54\mu^2 . \]

(34)

It is straightforward to construct a function \( M(R) \) satisfying all these requirements, for any given \( R_2 > R_1 \) and \( m > R_1^3/54\mu^2 \). We shall take

\[ R_2 > 2m . \]

(35)
From Eq. (31) it follows that

\[ 1 - 2M(R)/R + R^2/2\mu^2 \geq 1. \]

This guarantees that the 3-metric (29, 30) is regular throughout \( R_1 \leq R \leq R_2 \).

In the homogeneous dust region Eq. (33) implies

\[ 2M/R = R^2/2\mu^2. \]

Applying this to \( R = R_1 \), using \( R_1 > 2\theta_3 r_0 \) (obtained from Eqs. (22, 28) and \( \theta_3 > \theta_c > 1 \), one finds that \( 2M(R_1) > R_1 \). Namely, the dust solution includes spherical trapped surfaces at \( R = R_1 \) and its neighborhood.

**Non-spherical modification**

The above construction of the inhomogeneous dust region was spherically-symmetric. However it is easy to generalize it to obtain nonspherical configurations. This increases the space of solutions, and also allows for new kinds of causal structures. Since this modification goes beyond the main course of this paper, we shall only sketch it briefly here.

In the first stage, one chooses the function \( M(R) \) such that at a certain region \( R_a < R < R_b \), for some \( R_1 < R_a < R_b < R_2 \), it takes the form

\[ M(R) = \alpha R^3/54\mu^2, \]

where \( 0 < \alpha < 1 \) is a fixed number. The 3-metric then becomes

\[ ds^2 = [1 + \frac{1 - \alpha}{2\mu^2} R^2]^{-1} dR^2 + R^2 d\Omega^2. \]

This is a maximally-symmetric 3-metric of negative-curvature. Recalling the uniform extrinsic curvature [1], one realizes that the initial data at \( R_a < R < R_b \) correspond to a contracting, unbounded (i.e. “\( k = -1 \)”), Robertson-Walker solution. Indeed from Eq. (32) the dust density is constant, \( \epsilon = \alpha\epsilon_0 \). One can easily arrange the parameters \( \alpha, R_a, R_b \) (and the mass function) such that the region \( R < R_a \) is free of trapped surfaces (namely \( M(R) < R/2 \)).

Next, one picks a point \( P \) (corresponding to certain \( R, \Theta, \phi \)) on \( S \) somewhere at \( R_a < R < R_b \), and re-express the 3-metric in spherical coordinates centered on the point \( P \) (this is possible because the metric is maximally-symmetric). We denote these new spherical coordinates \((\hat{R}, \hat{\Theta}, \hat{\phi})\). The 3-metric in these new coordinates still takes the form (37), but with \( R, \Theta, \phi \) replaced by \( \hat{R}, \hat{\Theta}, \hat{\phi} \), respectively. This 3-metric may be expressed by Eqs. (29, 30, 31), with \( R \) and \( \Omega \) replaced by \( \hat{R} \) and \( \hat{\Omega} \), respectively.

Finally one picks a two-sphere \( \hat{R} = \hat{R}_0 \) around \( P \) which is entirely contained in \( R_a < R < R_b \). At \( \hat{R}_0 \) one modifies the mass function and picks a (monotonously increasing) smooth function \( M(\hat{R}) \) at will. (Optionally one may also modify the extrinsic curvature, namely replace Eq. (1) by a more generally spherically-symmetric form, in a manner described in the next section.) One then obtains an inhomogeneous dust solution at \( \hat{R} < \hat{R}_0 \).

This modified solution at \( \hat{R} < \hat{R}_0 \) is on itself spherically-symmetric, but is not concentric with the spherical shells at e.g. \( R_1 < R < R_a \). However it is easy to arrange that this singularity will be globally naked. This spacetime will fail to be future asymptotically predictable.

**VI. INITIAL DATA FOR THE EXTERNAL VACUUM REGION**

In the region \( R_2 \leq R \leq R_3 \), for some \( R_3 > R_2 \), we set the 3-metric

\[ ds^2 = (1 - 2m/R + R^2/2\mu^2)^{-1} dR^2 + R^2 d\Omega^2, \]

and the uniform extrinsic curvature [4]. The Ricci scalar is \( R^{(3)} = -2/9\mu^2 \) and \( \epsilon \) vanishes. Note that the term \( 1 - 2m/R + R^2/2\mu^2 \) was shown to be positive (if fact \( > 1 \)) at \( R = R_2 \) and it is also monotonously increasing in \( R \), hence it is positive throughout \( R_2 \leq R \leq R_3 \).

Since the initial geometry in this range is both vacuum and spherically symmetric, it must correspond to that of a certain spherically-symmetric initial hypersurface in the Schwarzschild geometry. Although the Schwarzschild spacetime is asymptotically-flat, the initial 3-metric (38) is obviously not. (In fact this three-metric is the same as that of a time-symmetric hypersurface in the de Sitter spacetime.) As it turns out, the initial data we have constructed in the range \( R_2 \leq R \leq R_3 \) correspond to a "hyperbolic" (rather than time-symmetric) initial hypersurface in the Schwarzschild geometry. Since we do want S to be asymptotically-flat, essentially what we need is to deform this initial hypersurface at \( R > R_3 \) (say), so that at large \( R \) it approaches a time-symmetric hypersurface in Schwarzschild, namely one with vanishing extrinsic curvature and asymptotically-flat 3-metric.

Here we describe the construction of S in terms of the initial data for \( h \) and \( K \) (rather than through its embedding in a given spacetime). The discussion above makes it obvious, though, that in order to make S asymptotically flat we must relax the condition (4) on \( K_{ab} \) at \( R > R_3 \). (This would amount to ”changing the embedding of S in spacetime”.)

In the next two subsections we shall construct \( K_{ab} \) and \( h_{ab} \), respectively, in the range \( R > R_3 \). Our only presumption is that both tensors are spherically-symmetric,
with \( h_{ab} \) given by Eq. (29). The extrinsic curvature will be obtained from the momentum equation (3), and the 3-metric [namely the function \( g_{RR}(R) \)] will in turn be derived from the vacuum Energy equation (14).

**A. The extrinsic curvature**

Being a spherically-symmetric tensor, we write \( K_{ab} \) as

\[
K_{ab} = K_{0}(R)h_{ab} + \delta K(R)n_{a}n_{b},
\]

(39)

where \( n_{a} \) is the unit radial vector field. This expression must satisfy the momentum equation

\[
K_{ab}^{b} - K_{b}^{b}a = 0.
\]

(39)

Owing to the linearity of this equation, we may consider the contribution of each term in Eq. (39) separately. Since \( K_{0}h_{ab}^{b} = 3K_{0} \), the first term contributes \(-2K_{0,a} \). The contribution of the second term is

\[
[\delta K(R)n_{a}n_{b}]_{;b} \cdot [\delta K(R)n_{a}n_{b}]_{;a} \cdot R.
\]

(40)

Since the gradient of \( \delta K \) is tangent to \( n \), the contribution coming from the derivative of \( \delta K \) cancels out between the two terms in Eq. (40). Also \((n_{b}h_{ab})_{;a}\) vanishes due to normalization. In addition the term \( n_{a}n_{ab} \) vanishes by the geodesic equation, because \( n_{a} \) is the tangent vector to a congruence of geodesics (the radial rays). The expression (40) therefore reduces to \( \delta K n_{a}n_{b}^{b} \), and the momentum equation becomes

\[
2K_{0,a} = \delta K n_{a}n_{b}^{b}.
\]

The angular components trivially satisfy this equation. In evaluating the radial component, a straightforward calculation yields

\[
n_{b}^{b} = (2/R)(g_{RR})^{-1/2}.
\]

Since \( n_{R} = (g_{RR})^{1/2} \), the momentum equation reduces to the simple relation

\[
\delta K = R \frac{dK_{0}}{dR}.
\]

(41)

Note that the choice \( 4 \) at \( R \leq R_{3} \) corresponds to \( K_{0}(R) = \text{const} = k_{0} \). Then \( K_{0}(R) \) varies in the range \( R_{3} \leq R \leq R_{4} \), for some \( R_{4} > R_{3} \). At \( R \geq R_{4} \), the outermost layer of \( S_{2} \), we choose \( K_{0}(R) = 0 \), which yields \( K_{ab} = 0 \) (this would correspond to a time-symmetric hypersurface in Schwarzschild). Thus, in the transition region \( R_{3} \leq R \leq R_{4} \) we take \( K_{0}(R) \) to be any smooth function which smoothly joins on \( K_{0}(R) = k_{0} \) at \( R \leq R_{3} \) and on \( K_{0}(R) = 0 \) at \( R \geq R_{4} \). The function \( \delta K(R) \) is then defined by Eq. (41), hence the momentum equation is satisfied.

**B. The 3-metric**

The 3-metric at \( R > R_{3} \) will be determined from the vacuum Energy equation

\[
R^{(3)} + (K_{a}^{a})^{2} - K_{a}^{a}K_{b}^{b} = 0.
\]

With the substitutions (39, 41) this equation becomes

\[
R^{(3)} = -2K_{0}(3K_{0} + 2R) \frac{dK_{0}}{dR}.
\]

(42)

In the 3-metric (29) we now set

\[
g_{RR}(R) = [1 - 2\dot{M}(R)/R]^{-1}.
\]

(43)

The Ricci scalar is then found to be

\[
R^{(3)} = \frac{d\dot{M}}{dR}.
\]

(44)

This, combined with Eq. (42), yields a closed expression for \( d\dot{M}/dR \) in terms of \( K_{0}(R) \). After integration one obtains

\[
\dot{M}(R) = m - R^{3}(K_{0})^{2}/2.
\]

The integration constant \( m \) is determined from the boundary condition at \( R = R_{3} \). Thus, in the range \( R_{3} \leq R \leq R_{4} \) (and, in fact, throughout \( R > R_{2} \) the 3-metric is \( 29, 14 \) with

\[
g_{RR} = [1 - 2m/R + R^{2}K_{0}(R)2]^{-1}.
\]

(44)

Note that \( R_{3} > R_{2} > 2m \), hence the 3-metric (29, 14) is regular throughout \( R \geq R_{3} \) (its regularity at \( R < R_{3} \) was already established above).

Finally, at \( R \geq R_{4} \) we have \( K_{0} = 0 \), hence the 3-metric and extrinsic curvature are

\[
ds^{2} = (1 - 2m/R)^{-1}dR^{2} + R^{2}d\Omega^{2}
\]

(45)

and

\[
K_{ab} = 0.
\]

(46)

Obviously this corresponds to a time-symmetric initial hypersurface in a Schwarzschild spacetime with mass \( m \).

**VII. DISCUSSION**

With the substitutions (39, 41) this equation becomes

\[
R^{(3)} = -2K_{0}(3K_{0} + 2R) \frac{dK_{0}}{dR}.
\]

(42)

In the 3-metric (29) we now set

\[
g_{RR}(R) = [1 - 2\dot{M}(R)/R]^{-1}.
\]

(43)

The Ricci scalar is then found to be

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ds^{2} = (1 - 2m/R)^{-1}dR^{2} + R^{2}d\Omega^{2}
\]

(45)

and

\[
K_{ab} = 0.
\]

(46)

Obviously this corresponds to a time-symmetric initial hypersurface in a Schwarzschild spacetime with mass \( m \).
realistic description of matter, it nevertheless provides a rather simple paradigm, which proved in the past to be useful in addressing various issues of principle in General Relativity—e.g., gravitational collapse, formation of naked singularities, and cosmological models. Furthermore, the system of dust+gravity is known to yield a well-posed initial-value problem.

The spacetime constructed in this way satisfies all the requirements (1-10) listed in subsection 1A. In particular it is smooth, asymptotically flat, and topologically trivial (to be precise, the initial hypersurface $S$ is of topology $\mathbb{R}^3$). It trivially satisfies the energy conditions (weak, strong, and dominant).

The vacuum core metric was taken here to be the pseudo-Schwarzschild metric. We point out that we could also use the standard Schwarzschild metric (with the coordinate $t$ identified on a circle), or even the 4-dimensional Misner space, for the core metric, and obtain a TM model with similar properties. However, it is only the pseudo-Schwarzschild metric which admits a homogeneous initial hypersurface (namely $r = 3\mu$) with a uniform extrinsic curvature. This simplifies the construction of the initial data for the envelope, because the momentum equation is automatically satisfied. With the alternative core metrics previously mentioned, the construction of initial data will be slightly more complicated.

Several problems and important questions are still left open. Perhaps the most important one is the issue of stability. A stability analysis is beyond the scope of this paper, but there is comment worth noting. Although there are indications for classical and semiclassical instabilities in various TM models (see e.g. [27] [18] [28]), the robustness and effectiveness of these instabilities are still unclear [27] [29] [30] [31] [32]. Further research is required in order to assess the robustness and effectiveness of the various instability phenomena. The model constructed here may provide a more solid basis for a systematic stability analysis. A few of its features that could be important for a genuine stability analysis include: (i) having a regular initial hypersurface, (ii) asymptotic flatness, and (iii) admitting a well-posed system of evolution equations. None of the previous TM models demonstrated all these properties.

Two other important open questions should be mentioned here:

1. It may turn out that the evolving spacetime includes a black hole, and all CCCs are imprisoned inside the event horizon. In such a case the formation of CCCs might still have crucial implications to various aspects of the internal black-hole physics and geometry (e.g. singularity formation), but nevertheless the external universe will not be influenced.

2. In our present construction, the initial data on $S$ involve strong (though finite) gravitational fields. Is it possible to create a TM spacetime of this kind starting from weak-field initial data on some earlier initial hypersurface? Rephrasing this question: is it possible to create such a TM spacetime by sending weak gravitational waves (from past null infinity) and diluted dust shells (from past timelike infinity) in the inward direction?

As a first step towards addressing these questions one may numerically evolve the initial data on $S$ in both the future and past directions. Future time evolution will tell us whether a black hole forms, and if it does, whether it engulfs the CCCs. Past time evolution will probably indicate one of the two possibilities: (a) the backpropagated fields disperse and weakens—the dust expands and dilutes, and the gravitational field spreads to past null infinity as weak gravitational waves, or (b) the fields (back-) focus to form a white hole, a naked singularity, or pathologies of some other kind. Such numerical simulations will thus answer the questions (1,2) above at least with regards of the specific TM model constructed here. Note, however, that even if this specific model is indeed found to form a black hole in the future evolution, and/or a white hole in the past evolution, it leaves open the possibility that a modified TM model will be free of these undesired properties. I am not aware of any theorem or argument which establishes a firm link between the formation of CCCs and the subsequent formation of a black hole, or the presence of a white-hole etc. to the past of $S$.

One might hope to gain insight into these two questions by exploring the initial data on $S$ for trapped and/or anti-trapped (i.e. “past-trapped”) surfaces. Consider first the issue of trapped surfaces and their relation to black-hole formation in future time evolution. The external spherically-symmetric vacuum region is free of trapped surfaces, because $S_2$ is restricted to $R > R_2$ and $R_2 > 2m$. But as previously mentioned, the dust region in the neighborhood of $R = R_1$ does include spherical trapped surfaces. However, the role of trapped surfaces as indicators for black hole formation is not so clear in our case. The theorems establishing the connection between trapped surfaces and black hole formation assume either global hyperbolicity, or lack of CTCs, or asymptotic predictability, or similar properties. Here none of these properties can be assumed apriori, especially because the spacetime in consideration is guaranteed to develop CCCs and a CH. For example, on the basis of proposition 9.2.1 in Ref. [9], the occurrence of trapped surfaces on $S$ basically tells us that one of the following two scenarios will be realized: (i) a black hole will form in $D_+(S)$ and engulf the CCCs, or (ii) the CH will extend to future null infinity, thus invalidating the condition of future asymptotic predictability.

The situation regarding anti-trapped surfaces seems to be different. If anti-trapped surfaces were found to be present on $S$, this would provide firm evidence that in past evolution, the field cannot just back-spread to infinity. It would indicate that $D_-(S)$ fails to be past asymptotically simple, meaning that some pathology must have taken place in the past (prior to $S$): for example, a white
hole, or a naked singularity. Thus, the presence of anti-trapped surfaces on S would severely reduce the relevance of the spacetime in consideration as a physical model describing the construction of a TM. (Presumably, the future “spacetime engineers” will not have white holes or naked singularities at their disposal.)

Fortunately, in the specific model constructed here it appears that no anti-trapped surfaces exist on S. In regions $S_0$ and $S_1$, and also in the part $R \leq R_3$ of $S_2$, the simple form of the extrinsic curvature means that $K^a_b$ only has positive eigenvalues ($\text{triple } k_0 > 0$), which does not allow for anti-trapped surfaces. In the part $R \geq R_3$ of $S_2$ the geometry is Schwarzschild with $R > 2m$, hence again there are no anti-trapped surfaces.

Although the lack of anti-trapped surfaces is encouraging, recall that it is a necessary but not a sufficient condition for the non-pathological asymptotic structure of $D_-(S)$. A numerical simulation of the initial data on S towards both the past and future directions could therefore provide valuable insight into this issue of weak-field initial data prior to S, as well as into the problem of black hole formation in the future of S.

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APPENDIX A

We present here the construction of the function $F(\theta)$ of Eq. (18) in the range $\theta_0 < \theta < \theta_1$, where $\theta_0$ and $\theta_1$ are given numbers satisfying $\theta_0 > \theta_c$ and $\theta_1 > \theta_{\text{min}} \equiv 2\theta_0 + 1$. Throughout this appendix a prime will denote differentiation with respect to $\theta$.

Recall the required properties of the function $F(\theta)$: (i) It is strictly positive, (ii) $F'' \leq F$, (iii) It is smooth in the range $\theta_0 < \theta < \theta_1$, (iv) It joins smoothly on $F(\theta) = \sinh \theta$ at $\theta = \theta_0$, (v) It joins smoothly on $F(\theta) = \theta$ at $\theta_1$.

This construction is naturally divided into two stages: In the first one (stage A below) we construct a function $\tilde{F}(\theta)$ which satisfies all the above requirements except that it is non-smooth ($\tilde{F}'$ is discontinuous) at two points, denoted $\tilde{\theta}_0$ and $\tilde{\theta}_1$, which satisfy $\theta_0 < \tilde{\theta}_0 < \tilde{\theta}_1 < \theta_1$. Nevertheless $\tilde{F}(\theta)$ is continuous at $\theta = \tilde{\theta}_0$ and $\theta = \tilde{\theta}_1$. Furthermore, we make sure that the "jump" in $\tilde{F}'(\theta)$ is negative at both points $\tilde{\theta}_0$ and $\tilde{\theta}_1$ (namely, the one-sided derivative in the direction $\theta > \tilde{\theta}_{0.1}$ is smaller than the corresponding one in the direction $\theta < \tilde{\theta}_{0.1}$). Then in the next stage (referred to as stage B) the function $\tilde{F}(\theta)$ is modified—smoothened—in narrow neighborhoods of both $\theta = \tilde{\theta}_0$ and $\theta = \tilde{\theta}_1$, to obtain a smooth function $\hat{F}(\theta)$ in the entire range $\theta_0 < \theta < \theta_1$ which satisfies all the above requirements (i-v).

Here we shall describe in some detail the procedure comprising stage A. The main statement underlying stage B, namely, that a function with a "kink" (of the correct sign) can be smoothened without violating properties (i,ii) above, is quite obvious, but the full presentation of the detailed smoothening procedure is lengthy. We shall therefore skip the detailed description of stage B.

**Stage A: Constructing the rough function $\hat{F}(\theta)$**

Let us define

$$\hat{F}(\theta) \equiv \cosh(\theta - 2\theta_0) .$$  \hspace{1cm} (47)

We take the function $\hat{F}(\theta)$ to be $\tilde{F}(\theta) = \hat{F}(\theta) = \tilde{\theta}_0 \leq \theta \leq \tilde{\theta}_1$, along with $\tilde{F}(\theta) = \sinh \theta$ at $\theta \leq \tilde{\theta}_0$ and $\tilde{F}(\theta) = \theta$ at $\theta \geq \tilde{\theta}_1$. The points $\tilde{\theta}_0$ and $\tilde{\theta}_1$ will thus be the intersection points of $\tilde{F}(\theta)$ with $\sinh \theta$ and $\theta$, respectively. (The smooth matching at points $\tilde{\theta}_0$ and $\tilde{\theta}_1$ is thus trivially satisfied, but as we mentioned above the challenge remains to arrange the smooth matching at $\tilde{\theta}_0$ and $\tilde{\theta}_1$—the central task in stage B.)

We take

$$\hat{\theta}_1 = 2\theta_0 + 1 ,$$

hence $\theta_0 < \hat{\theta}_1 < \theta_1$ as desired. The parameter $a$ will then be derived from the requirement of continuity at $\theta = \hat{\theta}_1$, and subsequently continuity at $\theta = \tilde{\theta}_0$ will determine the value of $\tilde{\theta}_0$. Note that Eq. (47) satisfies $\tilde{F}'' = \tilde{F}$ (hence $\epsilon$ vanishes—and condition (ii) above holds—throughout the range where $\tilde{F} = \tilde{F}$).

Continuity at $\theta = \hat{\theta}_1$, namely $\tilde{F}(\hat{\theta}_1) = \hat{F}(\hat{\theta}_1)$, yields

$$a = (2\theta_0 + 1)/\cosh 1 .$$ \hspace{1cm} (48)

Note that

$$a > (2\theta_c + 1)/\cosh 1 \equiv 2.35 ,$$

hence in particular

$$a > 1 .$$

Also note that

$$a < 2\theta_0 + 1 < e^{2\theta_0} .$$

The parameter $\tilde{\theta}_0$ is taken to be the point where $\tilde{F}(\theta)$ intersects $\sinh \theta$, namely it satisfies

$$a \cosh(\tilde{\theta}_0 - 2\theta_0) = \sinh \tilde{\theta}_0 .$$

This equation has a single real root:

$$\tilde{\theta}_0 = \theta_0 + (1/2) \ln \left[ \frac{ae^{2\theta_0} + 1}{e^{2\theta_0} - a} \right] .$$
Note that $\hat{\theta}_0 > \theta_0$ as desired. Later we shall also need the inequality $\theta_0 < 2\theta_0$. To see this, one compares $F'(\theta)$ to sinh $\theta$ at the two points $\theta = \theta_0$ and $\theta = 2\theta_0$. In the former

$$F'(\theta_0) = a \cosh \theta_0 > \cosh \theta_0 > \sinh \theta_0.$$ 

On the other hand, at $\theta = 2\theta_0$ one obtains

$$F'(2\theta_0) = a = (2\theta_0 + 1)/\cosh 1,$$

which is to be compared to sinh$(2\theta_0)$. One finds (e.g., numerically) that

$$(2\theta_0 + 1)/\cosh 1 < \sinh(2\theta_0)$$

for any $\theta_0 > \theta_c$ (the above inequality in fact holds for any $\theta_0$ greater than 0.57, whereas $\theta_c \approx 1.317$). Since $F'(\theta)$ is smaller than sinh $\theta$ at $\theta = 2\theta_0$ but greater than sinh $\theta$ at $\theta = \theta_0$, the (single) intersection point $\hat{\theta}_0$ must be located in between, namely

$$\theta_0 < \hat{\theta}_0 < 2\theta_0.$$ 

Finally we compare the two one-sided values of $F'$ at each of the matching points $\theta_0$ and $\hat{\theta}_1$. Starting at $\theta_0$, the directional derivative corresponding to $\theta < \theta_0$ is

$$F' = \cosh \theta_0 > 0,$$

whereas the one corresponding to $\theta > \hat{\theta}_0$ is

$$\hat{F}' = \hat{F}' = a \sinh(\hat{\theta}_0 - 2\theta_0) < 0,$$

due to the jump in $\hat{F}'$ is negative. Consider next the two one-sided values of $F'$ at $\hat{\theta}_1$. For the direction $\theta > \hat{\theta}_1$ we have $\hat{F}' = 1$, and for $\theta < \hat{\theta}_1$ we have

$$\hat{F}' = \hat{F}' = a \sinh 1.$$ 

Substituting the value of $a$, Eq. 15, we get (at $\theta = \hat{\theta}_1$)

$$\hat{F}' = (2\theta_0 + 1) \tanh 1 > (2\theta_0 + 1) \tanh 1 \approx 2.77 > 1.$$ 

We conclude that at both $\theta = \theta_0$ and $\theta = \hat{\theta}_1$ the jump in $\hat{F}'$ is negative (namely, the directional derivative corresponding to $\theta > \hat{\theta}_0,1$ is smaller than the corresponding one corresponding to $\theta < \hat{\theta}_{0,1}$, as desired.

### Stage B: Smoothening the rough function $\hat{F}(\theta)$

In the next stage one constructs the function $F(\theta)$ in the range $\theta_0 < \theta < \theta_1$ to be the same as $\hat{F}(\theta)$ except at two narrow ranges, one in the neighborhood of $\theta = \theta_0$ and one in the neighborhood of $\theta = \hat{\theta}_1$, in which one replaces the non-smooth function $\hat{F}(\theta)$ by a smooth one. This must be done without violating the two equalities $F > 0$ and $F'' \leq 0$. This procedure is quite straightforward though a bit tedious, and we shall skip the details here.

It should be emphasized that this smoothening is only possible if the “jump” in $\hat{F}'$ is negative at both $\theta_0$ and $\hat{\theta}_1$ (which was indeed shown above to be the case). For, only in this case $\hat{F}(\theta)$ satisfies the condition (ii) above in a distributional sense.
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