Quantization of Superstrings in Ramond-Ramond Backgrounds

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Abstract: We present a perturbative study of Ramond-Ramond backgrounds in the NSR formalism. We show how to perform $\sigma$-model computations and discuss in detail the structure of the BRST charge and picture changing operators. Contact terms play a vital role in the analysis. We also give evidence for a two loop non-renormalization theorem for the background beta functions.

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1. Introduction

The quantization of superstrings in Ramond-Ramond (RR) backgrounds is an important theoretical problem, as it has direct significance to the large $N$ behaviour of strongly coupled gauge theories [1]. For a review see [2]. Several attacks at this problem have been made in the literature. The Green-Schwarz formalism for the interesting case of $AdS_5 \times S_5$ has been studied in the classical theory by a number of authors, [3, 4] and aspects of the quantization of this action as an expansion around flat spacetime has been considered in Ref. [5]. Light cone gauge has been considered further in Ref. [6]. The RR background with $AdS_3 \times S_3$ geometry has been considered in Ref. [7, 8] and from a Green-Schwarz perspective in Refs. [9, 10].

Either way one looks at the quantization problem, there are technical questions which need to be answered in order to have a string quantization. In a Green-Schwarz formulation of the superstring, we only know how to perform a quantization in the lightcone gauge. For a general background, one can not expect to have a well-defined
The notion of lightcone. Moreover, if there is a choice of lightcone, it is not clear that in a curved background one can decouple the lightcone degrees of freedom from the rest of the worldsheet CFT.

If one attempts to work in a covariant formulation of the superstring, one must confront technical questions about the worldsheet CFT. In particular we are used to dealing with NS backgrounds, which respect worldsheet supersymmetry. The BRST structure and the picture changing operations follow directly from worldsheet supersymmetry, so one can calculate the constraints on physical states without difficulty. As we discuss below, an RR background breaks worldsheet supersymmetry. Moreover, the usual picture-changing operator is defined in terms of the worldsheet supersymmetry generator; in RR backgrounds, one must find an alternate way to remove the ambiguities of picture. It is important to stress here that although we are used to thinking of the superstring in flat spacetime as a simple theory, it is not a free theory. Rather, it is usually described via a free field realization; the price we pay for that is the complications of screening and picture. It is then not surprising that we must carefully consider the modifications to these structures in RR backgrounds.

It is the purpose of this paper to attack these problems and suggest an answer as to how the BRST structure and the picture changing operations are modified by the RR background within conformal perturbation theory.

In an earlier paper, we began a perturbative analysis of superstring theory in Ramond-Ramond backgrounds using the familiar Ramond-Neveu-Schwarz (RNS) formalism. We consider (Type IIB) string theory on an arbitrary smooth target space with non-zero Ramond-Ramond fieldstrength background. For simplicity, we assume the dilaton is constant, and all other moduli are set to zero. We also assume that the background is such that the \(\alpha'\) expansion and the string loop expansion is well-behaved. The closed string \(\sigma\)-model was derived explicitly through a process of summing over open string worldsheets in a D-brane background. The \(\sigma\)-model action is deduced by requiring conformal invariance order by order in the expansion in \(1/r\), the distance from the branes, an application of the Fischler-Susskind mechanism. This \(\sigma\)-model has the form

\[
S = \int d^2z d^2\theta \, g_{\mu\nu}(X) DX^\mu DX^\nu + \int d^2z \, P_{1/2} P_{1/2} h_{\alpha\beta}(X)e^{-\phi/2} S^\alpha e^{-\bar{\phi}/2} \bar{S}^\beta \quad (1.1)
\]

In this expression, the fields \(\phi\) and \(\bar{\phi}\) are those found in the bosonization of the \(\beta\gamma\) superghost system of the flat spacetime theory, and \(\alpha, \beta\) are spacetime chiral spinor indices. Further notational details may be found in the Appendix. The formal dimension zero operators \(P_{1/2}\) and \(\bar{P}_{1/2}\) are thought of as the (anti-)holomorphic square root of the picture-changing operator. These remove the ambiguity of ghost-picture, so that to each order in perturbation theory the RR-background vertex does...
not affect the pictures of external states. For the low order calculations presented in Ref. [11], the lack of knowledge of the detailed form of $P_{1/2}$ was not important, as for any (non-zero) S-matrix element, these operators always appear in pairs. In the present paper, we confront this issue more directly and discuss how to consistently deal with these operators, order by order in perturbation theory.

An important aspect of these backgrounds is that the CFT in no sense splits holomorphically. Aspects of this were pointed out in [11], such as the realization of spacetime symmetry currents and their algebras. Regardless, we will show in this paper that conformal invariance may be consistently maintained, and a holomorphic stress tensor constructed order by order. Furthermore, the BRST operator may be constructed; the resulting restrictions on physical states are consistent with spacetime supersymmetry.

Much of the discussions presented in this paper involve a careful analysis of contact terms. Indeed, the $\sigma$-model action should be supplemented with a consistent choice of contact terms in order to define the theory. (In fact, as we will discuss, the presentation of the NSNS part of the action as an integral over superspace corresponds to a choice of contact terms, valid at lowest order.) The guiding principles for choosing contact terms will be worldsheet conformal invariance and spacetime supersymmetry.

In the presence of the RR background, we expect operator mixing between operators of $NS$ and $R$ type. Thus it is a possibility that the BRST charge and picture-changing operator itself contains RR contributions. In the latter case, such effects would seriously complicate the perturbative expansion.

One of the principal results of this paper is that the corrections to the picture-changing operator are of a very simple form at low orders—it is simply covariantized with respect to the background metric. In particular, there appears to be no RR contribution at tree level, and the ghost-picture structure of this theory is then very much as it is in flat spacetime. We will comment more fully on this subject in later sections.

In general, because of the conformal invariance of the background, we expect that there exists a nilpotent BRST current which gets contributions from both $NS$ and $R$ sectors. The results of this paper are consistent with the assumption that only the $NS$ part of $J_{BRST}$ contributes to $P_{+1}$ at tree level. In particular, the $\eta$-dependence of $Q_{BRST}$ is severely restricted.

$$J_{BRST} = \gamma g_{\mu\nu}(X)\psi^\mu \partial X^\nu + \text{ghosts} + \ldots = e^{\phi} \eta \ G_{\text{matter}} + \ldots$$

where the pure ghost pieces are independent of the background.

Since the $\sigma$-model is interacting, all statements are perturbative in $\alpha'$; to verify these equations we use an expansion around a well known background (we choose flat backgrounds for our example), which is devoid of the RR field strength and which satisfies the string equations of motion.
The paper is organized as follows. In the next section, we discuss the issue of (the absence of) worldsheet supersymmetry in $RR$ backgrounds. In Section 3, we discuss the importance of contact terms in string perturbation theory. In Section 4, we present the details of one-loop $\sigma$-model computations. In Section 4, we discuss the form of the $BRST$ charge. After a short review of $BRST$ symmetry within the standard superstring, we show explicitly that the $RR$ background does not modify the picture-changing operator at tree level, thus validating the one-loop computations performed in Section 4. We argue however that at one loop there are non-trivial contact terms which imply that the $BRST$ charge is modified by the $RR$ background. This follows from the $BRST$ invariance of the background. We find a description in which the the $BRST$ charge and the stress tensor are modified holomorphically. Using this description, we are able to show that the resulting $BRST$ constraints on spacetime boson and fermion states are precisely equivalent to the respective equations of motion. Finally, in Section 4, we consider briefly higher loops in $\sigma$-model perturbation theory. We give worldsheet evidence that two and three loop contributions to the $\beta$-functions vanish. This result is of course consistent with expectations from spacetime arguments. We conclude the paper with additional comments.

2. Worldsheet Supersymmetry

In eq. (1.1), we have written the $NSNS$ term in terms of worldsheet superspace for brevity. It is important to consider here the significance, if any, of worldsheet supersymmetry. To this end, recall the sum over spin structures of the flat spacetime superstring.

$$Z = \sum_{\sigma} Z_\sigma$$

(2.1)

Superconformal symmetry acts simply within each spin structure $Z_\sigma$, but not of course in any sense on the whole. Note that the superconformal transformations have branch cuts at the positions of $RR$ insertions (equivalently, the OPE is non-local).

In summing over worldsheets in a D-brane background, we are effectively summing over insertions of tadpoles produced by the D-brane. As is well-known,[13] there are tadpoles for both $NSNS$ fields as well as $RR$ fields. Thus summation over worldsheets exponentiates these fields into the $\sigma$-model action (1.1). Note however the effect of the sum over spin structures–because of the insertions of arbitrary numbers of $RR$ operators, it is no longer the case that the partition function can (necessarily\(^2\)) be represented in the form (2.1). This fact is consistent with the absence of worldsheet supersymmetry in the $\sigma$-model. The superconformal symmetry of the flat spacetime $\sigma$-model is broken to conformal symmetry by the $RR$ background. There

\(^2\)There may exist special cases where this indeed does occur.
are several puzzles associated with this fact that we would like to consider here. First, the flat spacetime theory has a superghost conformal field theory, and one may ask the significance of this when worldsheet supersymmetry is broken. In the present formulation, this ghost sector becomes irrevocably tied up with the matter CFT. A related issue is the appearance of picture. Recall first the source of picture in the flat spacetime theory. The $\beta\gamma$ system is bosonized in terms of fields $\eta, \xi, \phi$, and the field $\xi$ has a zero mode on the sphere. One must then include a factor of $\xi$ in the string path integral; BRST invariance is guaranteed if the operator $P_{+1} = \{Q_{BRST}, \xi\}$ is independent of $\xi$. We will argue in what follows that these properties persist in Ramond-Ramond backgrounds: although the detailed forms of $Q_{BRST}$ and $P_{+1}$ are modified, there is still a $\xi$ zero mode, and hence the picture structure is maintained.

3. Contact terms

Contact terms in string theory are required for various reasons. As Green and Seiberg [14] pointed out, contact terms arise in the OPE of vertex operators. Contact terms are required in order to cancel contributions to S-matrix elements due to unphysical states, and therefore they serve to ensure the analyticity of the S-matrix and worldsheet supersymmetry. In some other cases contact terms are required to preserve spacetime supersymmetry. Alternatively, [16] contact terms may be thought of in terms of our inability to enforce the superconformal gauge globally when operators collide.

Contact terms also arise [17] when one studies the moduli space of a family of conformal field theories. In this case they represent the connection coefficients of the metric in moduli space. As the connection depends on a choice of coordinates, contact terms are not uniquely determined at a given point in moduli space.

Regardless, it is clear that to properly define the theory, we have to give, in addition to the field content and $\sigma$-model action, a prescription for contact terms. In the backgrounds considered in this paper, this is non-trivial, since worldsheet supersymmetry is not present in the sense discussed above. Instead, the appropriate prescription is to enforce spacetime symmetries.

As an example, let us consider deforming a string theory compactified on a torus by the $(1,1)$ operator $\epsilon_{ij} \int d^2 z \, \partial X^i \bar{\partial} X^j$, which corresponds to a graviton at zero momentum. The propagator $X^i(z)X^j(0) \sim \delta^{ij} \ln |z - z'|^2$ implies that there results a quadratic divergence proportional to $\epsilon_{ij} \delta^{ij} \delta^{(2)}(0)$. This may be cancelled by extending the vertex operator to an integrated superfield

$$V = \epsilon_{ij} \int d^2 z \left( \partial X^i \bar{\partial} X^j + \psi^i \bar{\partial} \psi^j + \bar{\psi}^i \partial \bar{\psi}^j + F^i F^j \right).$$

The additional terms are contact terms and they vanish on-shell, $F^i = 0 = \bar{\partial} \psi^i$. However, in a given correlation function, they potentially contribute when the insertion collides with other operators. It is at these points that one cannot use the
equations of motion, and the terms which are zero on-shell produce a contact term contribution to amplitudes.

For general deformations $\phi^i(x)$ produced by $(1,1)$ operators on the worldsheet we expect the following structure for the OPE

$$\phi^i(x,z)\phi^j(x',z') = \frac{\Delta^{ij}(x)}{|z-z'|^4} + \frac{1}{|z-z'|^2} O^{ij}_k \phi^k(x,z)$$

$$+ \frac{b^{ij}}{|z-z'|^2} \delta^{(2)}(z-z') + c^{ij} \delta^{(2)}(z-z')$$

$$+ \delta^{(2)}(z-z') \Gamma^{ij}_k \phi^k(x,z) + \text{higher order} \quad (3.2)$$

The higher order terms correspond to $\alpha'$ corrections to the OPE. The power singularities will contribute a quadratic divergence on the worldsheet (proportional to $\Delta^{ij}$) and a one loop $\beta$-function of $\phi^k$, whereas the $\delta$-function contact terms serve to cancel the quadratic divergences by making them into total worldsheet derivatives. If one keeps the calculation to one loop order the contact terms also serve to modify propagators to give the correct background field renormalization. The contact term OPE corresponds to

$$\phi^i(x,z)\phi^j(x',z')|_{\alpha'} = \frac{b^{ij}}{|z-z'|^2} \delta(z-z') + c^{ij} \delta^2(z-z') + \delta^{(2)}(z-z') \Gamma^{ij}_k \phi^k(x,z)$$

and $b^{ij}, c^{ij}$ are related to $\Delta^{ij}$, so that when they are combined they form a total derivative on the worldsheet. This is how the tachyon decouples from the CFT.

One has to keep in mind that the coefficient $\Gamma^{ij}_k$ depends on the background, and gets corrected order by order; so when one does a set of calculations to a given order, one has to give a prescription for these corrections. This is the coefficient which is interpreted as the connection on the moduli space of CFT [17].

In the superspace formulation, cancellation of quadratic divergences is automatic, and thus it is easier to enforce conformal invariance on the worldsheet. Indeed, these are the contact terms which are required to cancel contributions to the S-matrix elements by unphysical particles, and they always show up as total derivatives on the worldsheet. One can, on the other hand, integrate out the auxiliary fields, and one gets explicit contact terms in the Lagrangian. In the non-linear $\sigma$-model, these contact terms serve to cancel the quadratic divergences on the worldsheet, and they also serve to restore the worldsheet supersymmetry, which is non-linearly realized. In the absence of worldsheet supersymmetry, this is the only way to proceed, so at each order in the perturbation one has to calculate the appropriate contact terms.

In this paper we wish to understand $RR$ backgrounds, so one has two types of problems to address. First, one still needs contact terms to cancel the divergences on the worldsheet field theory, but these are more complicated due to the fact that the $RR$ backgrounds break worldsheet supersymmetry, and thus the superspace formulation is ill-defined. One could hope that the contact terms introduced by the
RR background are all zero, and that one could use the superspace formulation for
the NS operators and thus obtain a result which is free of quadratic divergences.
Indeed, as we will see later, this is precisely what happens in the \( \alpha' \) expansion, but
only to one loop order.

Secondly, contact terms for theories with RR backgrounds are subtle because of
picture. Given a prescription for contact terms in a fixed picture, there could arise
additional contact terms from picture-changing operators. This would imply then
that contact terms are picture dependent. Fixing the RR operators to be in picture
zero (by introducing the square root of the picture-changing operator) should remove
this ambiguity

### 4. One-Loop \( \sigma \)-model Calculations

Let us now demonstrate some features of \( \sigma \)-model perturbation theory. We do this
for several reasons, mainly to demonstrate that the technique is straightforward and
systematic. We begin with one-loop computations, and comment on higher order
calculations later in the paper. As mentioned previously, we assume for simplicity
throughout that the dilaton is constant, and the NS \( B \) field is unexcited. The NSNS
part of the action is

\[
I_{\text{NSNS}} = \frac{1}{2} \int d^2 x \ \left\{ g_{ij}(\phi) \partial^\mu X^i \partial_\mu X^j + i g_{ij}(\phi) \overline{\psi}^i \slashed{D} \psi^j + \frac{1}{6} R_{ijk\ell} \overline{\psi}^i \psi^k \overline{\psi}^j \psi^\ell \right\} \tag{4.1}
\]

A choice of contact terms must be made at lowest order. As we have discussed, it
is consistent (at least at leading orders) to define these as if the NSNS part of the
theory possessed (off-shell) worldsheet supersymmetry.

\[
I_{\text{NSNS}} = \frac{1}{4i} \int d^2 x \ d^2 \theta \ g_{ij}(X) \overline{\psi}^i \slashed{D} \psi^j \tag{4.2}
\]

As is well-known, this choice cancels all quadratic divergences at one loop. Alternate
(on-shell) contact term prescriptions also exist, but we find the present formulation
most convenient computationally.

The RR part of the \( \sigma \)-model is written

\[
I_{\text{RR}} = \int d^2 z \ S^\alpha \tilde{S}^\beta h_{\alpha\beta}(X) \tag{4.3}
\]

where \( S^\alpha \) are spin fields

\[
S^\alpha = P_{1/2} S^\alpha e^{-\phi/2} \tag{4.4}
\]

To define \( S^\alpha \), we introduce an auxiliary orthonormal frame \( e^\mu_a \), such that \( \psi^\mu = e^\mu_a \chi^a \). The \( \chi^a \) fermions are conventionally normalized and may be bosonized in the
standard fashion. The spin fields \( S^\alpha \) are then defined in the usual way, mutual locality
restricting them to one chirality. These are spinors of the spacetime manifold, and
the monodromies introduced by $S^\alpha$ on the worldsheet fermions are also carried by the superghost system. $P_{1/2}$ and its right moving counterpart will be defined implicitly by their square, which is also formally $P_{\pm 1} = \{Q_{BRST}, \xi\}$. The ghost system is the standard one,[18] with $\gamma = e^{\phi} \eta$, $\beta = e^{-\phi} \xi$, where $\eta, \xi$ are fermions of conformal dimensions one and zero respectively. The point here is simply that in the local frame, the ghost and picture-changing structure is identical to the flat spacetime quantities. We will argue that in fact the picture-changing operator receives no further corrections in $\sigma$-model perturbation theory, at least to low orders.

Relevant OPE’s may be written

$$S^\alpha(z) \ S^\beta(0) \sim \frac{(\Gamma C^{-1} \cdot \partial X)^{\alpha\beta}}{z} \quad (4.5)$$

$$\chi^\alpha(z) \ S^\alpha(z') \sim \frac{(\Gamma^a S)^\alpha}{(z - z')^{1/2}} \quad (4.6)$$

$$: \chi^a \chi^b : (z) \ S^\alpha(z') \sim \frac{([\Gamma^b, \Gamma^a] S)^\alpha}{z - z'} \quad (4.7)$$

For the calculations which follow, we will employ a normal coordinate expansion (see Ref. [22] for details).

Let us consider the graviton $\beta$-function. At one loop, there are two contributions linear in curvature.\(^3\) From the NSNS graph, we obtain

$$\frac{1}{2} \int \frac{d^2z}{2\pi} \int d^2z' \ R_{ikl}k_{lj} \partial X^i \bar{\partial} X^j \ e^{k_1}_a(X) e^{k_2}_b(X) \langle \xi^a(z) \xi^b(z') \rangle$$

$$= \frac{1}{4} \Delta(\epsilon) \int d^2z \ R_{ij} \partial X^i \bar{\partial} X^j$$

where $\xi^a$ are canonically normalized bosonic fluctuations, with $\langle \xi^a(z) \xi^b(z') \rangle = \eta^{ab} \Delta(z - z')$.

From the RR graph we find

$$\frac{1}{2} \int \frac{d^2z}{2\pi} \ S^\alpha \bar{S}^\beta \ h_{\alpha\beta}(z) \cdot S^\gamma \bar{S}^\delta h_{\gamma\delta}(z')$$

$$= \frac{1}{2!} \int d^2z \int d^2z' \frac{d^2z''}{|z - z''|^2} \partial X^i \bar{\partial} X^j e^k_i(X) e^l_j(X) \text{tr} \left( h \Gamma_{a} C^{-1}(\Gamma_{b} C^{-1}h)^T \right) \quad (4.8)$$

To evaluate this further, let us specialize to the case of a single $p$-form field strength

$$h_{\alpha\beta} = \frac{1}{p!} H_{\mu_1...\mu_p} (C T^{\mu_1...\mu_p})_{\alpha\beta} \quad (4.9)$$

We need then evaluate

$$\text{tr} (\Gamma^{\mu_1...\mu_p} \Gamma^a_{\alpha} \Gamma^{\nu_1...\nu_p} \Gamma_{\beta}) \ H_{\mu_1...\mu_p} H_{\nu_1...\nu_p} = 32 \ p! \left( H^2_{ab} - \frac{1}{2p} \eta_{ab} H^2 \right) \quad (4.10)$$

\(^3\)Two powers of the field strength $h$ count consistently as a single power of curvature.
Thus, we may write the $\beta$-function at this order as

$$\beta_{ij} = R_{ij} - \frac{1}{2(p-1)!} \left( H_{ij}^2 - \frac{1}{2p} g_{ij} H^2 \right)$$

(4.11)

Note that this is the correct equation of motion for constant dilaton (for which $R = 0$; in the case of $AdS_p \times S_p$, this is achieved by cancellation between the two factors.)

We have computed the $\beta$-function to the operator of order $\xi^2$; there are of course also $\beta$-functions for terms of higher order. These however may be interpreted geometrically as the normal coordinate expansion of the curvature tensors appearing in the above $\beta$-function. There are tree-level corrections to the picture-changing operator itself, which come only from the NSNS background (we discuss this fact more fully later in the paper). The modification simply involves a covariantizaton

$$\eta_{\mu
u} \psi^\mu \partial X^\nu \rightarrow g_{\mu
u}(X) \psi^\mu \partial X^\nu.$$  

This is simply related to the normalization of the field propagators; the spin field itself however is given directly in an orthonormal frame (see Appendix), and so does not undergo such renormalizations in the NSNS background.

Let us make a few comments concerning the contraction of the two $RR$ operators. At lowest order, there are no contact terms, and the divergence is as given. In particular, the operators $P_{1/2}$ and $\bar{P}_{1/2}$ are harmless; there are two of each, and at this order, they simply combine to $P_{+1}\bar{P}_{+1}$. At higher orders, there is a possible contact term between $P_{1/2}$ and $\bar{P}_{1/2}$ (as there would be between $P_{+1}$ and $\bar{P}_{+1}$), and this needs to be taken into account consistently. We will return to this discussion in the next section.

5. BRST

Let us turn our attention now to the structure of BRST in theories with RR backgrounds. In order to emphasize the various features, we begin with a short review of standard material for the ordinary superstring. This discussion will also serve to set notation. For a more complete discussion we refer the reader to [19, 20].

Traditional string compactifications are described in terms of a worldsheet supersymmetric $\sigma$-model coupled to worldsheet supergravity. The full theory is superconformally invariant. For simplicity of the discussion we will assume that the dilaton is constant (in this case the theory is also classically superconformally invariant).

Locally on the worldsheet one can choose the superconformal gauge and in this gauge one has eliminated the local worldsheet gravity degrees of freedom. The supergravity ghost system in this gauge is conformally invariant, and is described by a free field theory which is decoupled from the matter superconformal field theory.

The metric and gravitino enter in the action as Lagrange multipliers, and their equations of motion imply that the (super) stress tensor of the system vanishes. Classically one has $T_{zz} = 0$ (because the theory is conformally invariant).
holomorphic and anti-holomorphic pieces of the matter stress tensor do not vanish as operators, and they can only be made to vanish on states. Because there is a conformal anomaly for the matter system, the stress tensor can not be made to vanish identically, independent of the ghost system. Solving the constraints is done via the BRST quantization of the string theory.

The string theory is well posed (the target space $\sigma$-model is a solution of string theory) if the total central charge of the conformal field theory of the combined matter and superghost system vanishes. The ghost system is described by two conjugate superfields $B = \beta + \theta b$ and $C = c + \theta \gamma$, of dimensions $3/2$ and $-1$. In this case, one has an (anti)holomorphic BRST operator $Q$ which squares to zero, and is described by the superfield

$$Q = \oint d\sigma d\theta (CT_{Gm} + \frac{1}{2} CT_{Gg})$$

where $T_{Gm} = G_m + \theta T_m$ is the super-stress tensor of the matter system and $T_g = G_g + \theta T_g$ is the stress tensor of the decoupled ghost system.

Because the ghost system is free, one has two separately conserved ghost numbers (one for the $b,c$ system, and another for the $\beta,\gamma$ system), and the BRST operator splits into pieces of different fermionic ghost charge, namely

$$Q = Q_0 + Q_1 + Q_2$$

where

$$Q_0 = \oint c(T_m + T_{\beta\gamma} + \frac{1}{2} T_{bc})$$

$$Q_1 = \oint \gamma(G_m)$$

$$Q_2 = \oint \frac{1}{2} b\gamma^2$$

$Q^2 = 0$ gives independent equations for each ghost number, so we get $Q_0Q_0 = 0$, $Q_1^2 = -\{Q_0,Q_2\}$ and $Q_2^2 = 0$. The first of these equations is the nilpotency of the standard conformal BRST charge, which says that the superconformal field theory is conformally invariant.

Physical states are associated with the cohomology of $Q$. To each state one associates a vertex operator $\mathcal{O}_i$ which is superconformally invariant, $Q\mathcal{O}_i = 0$. This solves the constraints produced by the equations of motion of the worldsheet supergraviton. The constraints put the physical state on-shell in the spacetime theory.

The operators are given with fixed ghost charges. For the $b,c$ system, unintegrated vertex operators are built as $\mathcal{O}_i = c\bar{c}V_{1,1}^{(i)}$, where the $V_{1,1}^{(i)}$ is a vertex operator of conformal dimension $(1,1)$ that is independent of $b,c$ (but which might depend

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4Notice that for conformal field theories with more than one worldsheet supersymmetry we actually have a choice as to which supersymmetry we use to define $G_m$. All of these choices should be unitarily equivalent.
on \(\beta,\gamma\). The integrated operators are obtained by removing the \(c\bar{c}\) and replacing it by \(\int d^2 z\). \(Q\mathcal{O}_i = 0\) implies that \(Q_0\mathcal{O}_i = 0\), so the operator is also conformally invariant. Correlation functions are constructed by integrating the vertex operators \(\mathcal{O}_i\) over a Riemann surface with fixed complex structure; and then integrating over the moduli space of complex structures (these are the non-local degrees of freedom that the superconformal gauge does not remove globally). Because of the worldsheet fermions one also needs to pick a spin structure on the Riemann surface.

The different parts of the BRST charge play different roles. Indeed, vertex operators are conformally invariant if \(Q_0V = 0\). The extra pieces of the BRST operator give additional constraints on the system. It is particularly important to notice that \(Q_0\) acts as a second order differential operator in spacetime, so it is an appropriate tool to give equations of motion to spacetime bosons.

For spacetime fermions, \(Q_0\) gives a Laplace equation, with double the appropriate number of degrees of freedom. The extra pieces of the BRST operator give an additional constraint, namely the Dirac equation.

Notice that one can interpret this fact in a slightly different way. First, ignoring the superconformal invariance, one could just have required conformal invariance and say that supersymmetry was accidental. This is how one builds the bosonic string in the absence of supersymmetry after all. In the special case where the theory is supersymmetric, there is an extra operator \(Q_1\) which squares to zero in the \(Q_0\) BRST cohomology classes, \((Q_1)^2 = -Q_0Q_2 = 0\). Thus, in the cohomology of the \(Q_0\) BRST operator, \(Q_1\) acts as the \(d\) operator of an exact sequence (graded by the ghost charge), so one can use it to give further restrictions on cohomology (the idea behind spectral sequences). From a spacetime point of view, we are getting a first order differential operator acting on states.

5.1 BRST for \(RR\) backgrounds

In the calculations presented so far, the picture-changing operator for RR backgrounds was defined without any reference to a BRST structure on the worldsheet, indeed we used the standard picture-changing operator. The purpose of this section is to give a (somewhat heuristic) description of BRST that justifies our comments; this should serve as a basis for a full formal perturbation expansion to all orders.

It is instructive to analyze how the BRST charge gets modified as we change the background in which the string propagates. We have a family of conformal field theories (parametrized by the perturbation parameters), and the BRST operator is a section of an operator bundle over the moduli space of conformal field theories.

Let us begin with a supersymmetric \(\sigma\)-model parametrized by a metric on target space \(g^{(a)}_{\mu\nu}(x)\); the stress tensor is given by

\[
T = \frac{1}{2} g^{(a)}_{\mu\nu}(x) \partial X^\mu \partial X^\nu + \text{fermions}
\]  

(5.6)
with a similar expression for the supersymmetry generator. As we change $g$, the stress tensor changes appropriately, at tree level.

As the BRST charge is intimately tied to the stress tensor, we expect that it is modified just as the stress tensor is. Indeed

$$\delta Q_{\text{BRST}} = \delta C T_{m}$$

(5.7)

We want to justify this answer directly from the perturbative approach to the family of conformal field theories. Indeed, the tangent space in the moduli space is generated locally by the massless $(1,1)$ vertex operators, which are BRST invariant, so it does not seem necessary to modify the BRST operator locally on the moduli space.

However, there is a caveat in the argument, which stems from the fact that the perturbations are done by integrated vertex operators. When one is checking the invariance of the background one needs to make sure to take into account the contact terms between the integrated vertex operator and the BRST contour. These contact terms need to be cancelled, as they would spoil the BRST invariance of the background, and they provide the necessary corrections to the BRST charge.

The fact that to first order one can ignore contact terms is justified by how one parametrizes moduli space. Indeed, the contact terms act as the connection on the operator bundle on the moduli space of conformal field theories [17]. One can choose a local parameterization of the moduli space in which the connection vanishes, which is also the reason why contact terms are not fully determined when one considers string amplitudes. In spacetime this choice is characterized by the different gauge choices in the supergravity low energy effective field theory.

The BRST operator, being essential to the quantization of the string, should be a canonical object in this moduli space of backgrounds. One can construct the BRST operator by following a path from our starting point (flat space in our case) to another SCFT in the same moduli space. The BRST operator of the perturbed theory is constructed by parallel transport along the path. As there might be monodromies in the bundle over moduli space, the best one can hope for is that given two different paths between the same points in moduli space, the two BRST operators give string theories which are unitarily equivalent. It is in this sense that the BRST operator associated to a point in moduli space is canonical.

For RR backgrounds, we will take the approach that the BRST operator is defined by parallel transport of the BRST operator in the NS conformal field theories on the path that connects it to the point we are studying.

From this construction, and the fact that the physical states can always be taken to be independent of the $\xi$ zero modes coming from the $\beta\gamma$ ghost system, we learn that these zero modes survive even in the presence of the RR perturbation of the CFT. The BRST operator thus obtained will give us a picture-changing operator $P_{+1} = \{Q_{\text{BRST}}, \xi\}$. 
Now, we will argue that there is no contact term that makes the picture-changing operator depend on the $RR$ field strength at tree level on the worldsheet, as we have assumed in Section 4. To analyze this question we will concentrate on the $\eta$-dependent part of the BRST operator, $Q_1$. Since we are considering a perturbation expansion, we indicate the expansion order in the BRST current by an upper index; the lower index continues to refer to the ghost charge. Thus, if we write the background vertex corresponding to the full deformation as

$$V_B = \sum_{i=1}^{\infty} \epsilon^i V_{(i)}$$

(5.8)

where $\epsilon$ is the formal perturbation expansion parameter (in normal coordinates the expansion parameter is the coordinate itself), we also write

$$Q_j = \sum_i \epsilon^i Q_j^{(i)}$$

(5.9)

for the BRST charge. $Q_j^{(0)}$ are the pieces of the unperturbed BRST charge. For our purposes, we will only analyze the first order corrections to $Q$, namely $Q_0^{(1)}$ and $Q_1^{(1)}$ which are the corrections coming from the $RR$ vertex operator. $Q_0^{(2)}$ is determined by the contact term between a $NS$ operator and the $Q_0$ BRST current (a standard computation).

Contact terms usually arise because a left moving field and a right moving field are located at the same point, e.g.

$$\partial X^\mu(z) \bar{\partial} X^\nu(z') \sim \delta^{(2)}(z - z') \eta^{\mu\nu}$$

(5.10)

in a free field theory. It is easy to see that these type of terms are precisely the ones that make modifications to the BRST operator in the $NS$ backgrounds. So let us consider the left moving BRST operator (just the piece with ghost charge 1), and the insertion of a $RR$ background vertex. We want to analyze a possible contact term between $Q_0^{(1)} \sim \gamma G_m$ and the vertex.

Now, we would like to be able to directly calculate this effect with $V_{RR}$ in the $(0, 0)$-picture (i.e., with the factors of $P_{1/2}$ present). However, one must be careful, because there could be contact terms with the $P_{1/2}$’s. Indeed, we would like to motivate that there is definitely such a contact term present. To do so, we will consider the $RR$ background insertion in various pictures, and show that the resulting contact terms are picture-dependent.\textsuperscript{5} This ambiguity is removed if we agree to take the background insertions always in picture $(0, 0)$. In a later section, we will work out the contact terms in that picture, using as a guiding principle, worldsheet conformal invariance.

\textsuperscript{5}Here we specifically mean that results are different if we move picture-changing operators to be coincident with the background insertion.
As a first step, which justifies the treatment at one-loop that we gave in Section 4, we argue that this ambiguity is not present at tree-level – it is a one-loop effect.

If one uses a \((-1/2,-1/2)\)-picture operator, there seems to be no apparent contact term, because the left and right moving fermion field theories are decoupled. On the other hand, suppose we consider a picture \((-1/2,1/2)\) vertex operator, 

\[
\bar{I}^\alpha \equiv S^\alpha e^{\phi/2}
\]

Here, we have defined

\[
I^\alpha \equiv S^\alpha e^{\phi/2}
\]

There is a contact term coming from the \(\partial X^\mu\) in \(G_m\) and the \(\bar{\partial} X^\mu\) in the vertex. Also, the \(\gamma \psi^\mu\) of \(Q_1^{(0)}\) has an OPE with the spin fields. The resulting contact term counts as a 1-loop effect, so we learn that the tree level correction to the picture-changing operator vanishes. Thus, we have justified our claims regarding the effects of picture-changing in the one-loop computations of Section 4. In particular, since the \(RR\) correction to picture changing vanishes at tree level, the picture-changing operator acts, in one-loop computations, as if there were a supersymmetric \(\sigma\)-model. Indeed, the one loop result is globally well-defined.

If one works out the details in picture \((-1/2,1/2)\), one finds that the the contact term is equivalent to a one-loop correction to the BRST operator takes the form

\[
Q_1^{(1)} \sim \eta \mathcal{T}^{\dot{\alpha}} \tilde{\mathcal{T}}^{\dot{\beta}} \bar{I}^\alpha \tilde{I}^\beta
\]

which is seen to have ghost charge \((1/2,1/2)\). The fact that the contact terms are picture dependent, shows that there are ambiguities in the definition of the contact terms. Note that the operator \(\mathcal{T}^\alpha \equiv S^{\dot{\alpha}} e^{+\phi/2}\) is of conformal dimension \((0,0)\). However, since the CFT is not unitary, it is not the case that \(\mathcal{T}^\alpha\) is trivial. Indeed it has non-trivial OPE’s with itself and other operators.

Note that \(Q_1^{(1)} \sim \eta \ h_{\dot{\alpha} \dot{\beta}} \mathcal{T}^{\dot{\alpha}} \tilde{\mathcal{T}}^{\dot{\beta}}\) is not holomorphic. This is certainly a problematic feature. However, we remind the reader that this explicit form was derived starting with a picture \((-1/2,1/2)\) background insertion. It’s presence is meant to merely indicate the necessity of having a contact term. The fact that it is non-holomorphic, we believe, may be traced to the fact that the superstring is not really a free field theory (rather, it only appears so if one ignores picture). In the next section, we will give a holomorphic prescription for the contact term (for picture \((0,0)\) backgrounds) which has the right structure.

That this switch to a holomorphic form is necessary is clearly indicated by looking at the contact term related to \(Q_0^{(1)}\), i.e., the term proportional to \(c\). Again, it is picture-dependent and if we choose the \(RR\) background in \((-1/2,1/2)\) picture, we find

\[
\sim \int dz \ c \ h_{\alpha \dot{\beta}} (\Gamma^\mu)^\alpha_{\dot{\beta}} S^\alpha \partial X^\mu \ \tilde{\mathcal{T}}^{\dot{\beta}}
\]

\[
 h_{\alpha \dot{\beta}} \sim h_{\alpha \dot{\beta}} (\Gamma^\mu)^\alpha_{\dot{\beta}} (\Gamma^\nu)^\beta_{\dot{\beta}}. \tag{5.13}
\]

\[\text{Here,} \ h_{\alpha \dot{\beta}} \sim h_{\alpha \dot{\beta}} (\Gamma^\mu)^\alpha_{\dot{\beta}} (\Gamma^\nu)^\beta_{\dot{\beta}}.\]
This correction may be interpreted as a correction to $T_{zz}$, and so must be holomorphic by conformal invariance. Again, we will find the proper resolution in the next section.

5.2 Background BRST invariance

When the background is taken in $(0,0)$ picture, the correction to the BRST operator may be determined unambiguously. We can determine this by requiring that the one-loop $\beta$-functions are recovered explicitly through BRST invariance of the background. In particular, a tree-level correction to the BRST operator, proportional to the $RR$ fieldstrength, is required.

As a warmup, let us recall how this works in a purely gravitational $NSNS$ background. The bosonic term in the $\sigma$-model is at second order proportional to

$$R_{\mu\nu\rho\sigma}X^\mu X^\rho \partial X^\nu \bar{\partial} X^\sigma$$  \hspace{1cm} (5.14)

and the correction to the worldsheet stress tensor results from exchanging $\bar{\partial} X^\sigma \rightarrow \partial X^\sigma$. Note that the resulting operator is holomorphic on-shell at this order in $R$.

The BRST current applied to the correction to the background gives

$$Q^{(0)} \cdot \int d^2 z \ V^{(1)} \sim \int d^2 z \ \partial c \cdot R_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu + \text{total derivative}$$  \hspace{1cm} (5.15)

Requiring that this vanishes is completely equivalent to the vanishing of the graviton $\beta$-function.

Now, we would like to analyze this with the $RR$ background. In order to see an effect, we need to expand to second order in the background. To this order then, we want to test

$$Q^{(0)}_0 \cdot \left( - \int d^2 z \ V^{(2)} + \frac{1}{2} \left( \int d^2 z \ V^{(1)} \right)^2 \right) - Q^{(1)}_0 \cdot V^{(1)} = 0$$  \hspace{1cm} (5.16)

Each vertex is meant to be inside the contour of $Q$. The factors of $V^{(1)}$ are $RR$ background insertions, while $V^{(2)}$ is the $NSNS$ quantity given in (5.14). $Q^{(1)}_0 = Q^{(0)}_0 V^{(1)}|_{c.t}$ is the contact term between the original BRST operator and the $RR$ background.

We work to one-loop. The term involving $(\int V^{(1)})^2$ may be dropped because its contribution either vanishes on-shell (from $Q^{(0)}_0 \cdot V^{(1)} = 0$) or is of higher loop order (a contact contribution from $V^{(1)} V^{(1)} \sim \delta$ is already of one-loop order).

The result of $Q^{(0)}_0 \cdot V^{(2)}$ is as calculated above for the NSNS background, and results in a contribution to the graviton $\beta$-function $\sim R_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu$.

Next, we need to calculate $Q^{(1)}_0 \cdot V^{(1)}$. Previously, we considered $V^{(1)}$ in picture $(-1/2, 1/2)$ and found the result

$$Q^{(1)}_0 \sim \int dz \ c \ h_{\alpha\beta} (\Gamma_{\mu})^\beta_{\beta} e^{-\phi/2} S^\alpha \partial X^\mu \bar{\partial} X^\nu$$  \hspace{1cm} (5.17)
We would now like to find a suitable $Q_0^{(1)}$ when $V_{(1)}$ is in picture $(0, 0)$, such that eq. (5.16) is satisfied. This will uniquely fix $Q_0^{(1)}$, including its coefficient.

Let us demonstrate that eq. (5.17) is problematic if we use it literally, including appropriately chosen picture-changing operators to bring the full result to $(0, 0)$ picture. Indeed there is a non-holomorphic singularity present.

\[ \oint dz \frac{1}{|z - z'|^2} c h_{\alpha\beta} h_{\gamma\delta} \partial X^\mu \partial X^\nu (\Gamma^\mu)^{\beta\delta} (\Gamma^\nu)^{\alpha\gamma} \]  
(5.18)

To avoid this, we choose a contact term, i.e., a new version of $Q_0^{(1)}$.

\[ Q_0^{(1)} \sim \oint dz c h_{\alpha\beta} S^\alpha \Sigma^\beta \]  
(5.19)

If we require the OPE

\[ \Sigma^\alpha(z) \tilde{S}^\beta(z') \sim \frac{1}{z - z'} \partial X^\mu (\Gamma^\mu)^{\alpha\beta} \]  
(5.20)

then we obtain

\[ Q_0^{(1)} \cdot V_{(1)} \sim (hh)_{\mu\nu} \partial X^\mu \tilde{X}^\nu \]  
(5.21)

The tensor form is precisely right such that eq. (5.15) reduces to the full graviton $\beta$-function, (4.11).

Next, note that the above form for $Q_0^{(1)}$ could be deduced via a contact term calculation if we have also have

\[ T(z) \cdot \tilde{S}^\beta(z') \bigg|_{c.t.} \sim \delta^{(2)}(z - z') \Sigma^\beta(z') \]  
(5.22)

\[ T(z) \cdot S^\beta(z') \bigg|_{c.t.} \sim 0 \]  
(5.23)

Furthermore, note that we may equivalently write a correction to the stress tensor from the RR background of the form

\[ T^{(1)} \sim h_{\alpha\beta} S^\alpha \Sigma^\beta \]  
(5.24)

The stress tensor is thus properly holomorphic, and has holomorphic OPE’s with other operators.

5.3 BRST and Physical States

In our previous paper, we demonstrated mixing between $NS$ and $R$ operators in the presence of $RR$ backgrounds. We can no show, with our prescription for $Q_0^{(1)}$, that those results can be consistently reproduced via BRST methods. Indeed, this is an important check of our methods.

As an example, consider the massless case. Physical vertex operators in unintegrated form are

\[ O_{NS} = P_{+1} \bar{P}_{+1} c \bar{c} f_{\mu\nu} e^{ikx} \psi^\mu e^{-\phi} \bar{\psi}^\nu e^{-\phi} \]  
(5.25)
and
\[
O_{RR} = \frac{P_{1/2} \bar{P}_{1/2}}{g} e^{ikx} e^{-\phi/2} S^\gamma e^{-\tilde{\phi}/2} \tilde{S}^\delta
\] (5.26)

Consider a state which, at zeroth order, is a RR state. We know that this mixes with an NSNS state, and so the condition, at first order, for the full state \(O_{RR} + \epsilon O_{NSNS}\) to be physical is
\[
Q^{(0)}_{NSNS} + Q^{(1)}_{O_{RR}} = 0
\] (5.27)

To check this, we need only \(Q^{(1)}_0\), which we have given above. Performing the OPE's, we find
\[
c\partial \bar{c} \psi^\mu e^{-\phi} \bar{\psi}^\nu e^{-\bar{\phi}} \left( k^2 f_{\mu\nu} + h_{\alpha\beta} g_{\gamma\delta} (\Gamma^\mu)_\alpha^\gamma (\Gamma^\nu)_\beta^\delta \right)
\] (5.28)

This is precisely of the correct form; its vanishing determines the appropriate mixing (see Ref. [11] or [21]).

Next, if we want to demonstrate the mixing for a physical state of the form \(O_{NSNS} + \epsilon O_{RR}\), we need a prescription for \(Q^{(1)}_1\):
\[
Q^{(1)}_{NSNS} + Q^{(0)}_1 O_{RR} = 0
\] (5.29)

Recall that previously we had \(Q^{(1)}_1 \sim \eta h_{\bar{a}\bar{b}} \mathcal{I}^{\bar{a}} \mathcal{I}^{\bar{b}}\). We wish to replace this by a holomorphic expression. To do so, introduce a new field \(\Sigma^{\hat{a}}\) with OPE
\[
\Sigma^{\hat{a}} \partial X^\mu \sim \frac{1}{z - z'} (\Gamma^\mu)^{\hat{a}}_\alpha \tilde{S}^\alpha
\] (5.30)

Then, the choice
\[
Q^{(1)}_1 \sim a \oint P_{-1/2} e^{3\phi/2} S^\alpha \partial X^\mu (\Gamma^\mu)^{\hat{a}}_\alpha h_{\bar{a}\bar{b}} \Sigma^{\bar{b}}
\] (5.31)

is holomorphic and of the appropriate dimension and picture. \(a\) is a normalization constant. Eq. (5.29) gives
\[
P_{1/2} \bar{P}_{1/2} e^{\phi/2} S^\alpha \partial X^\mu (\Gamma^\mu)^{\hat{a}}_\alpha h_{\bar{a}\bar{b}} \Sigma^{\bar{b}} e^{ikx}
\] (5.32)

One can think of the extra term as arising from the variation of the covariant derivative of the background; there is a choice of the normalization constant \(a\) which makes this consistent with spacetime supergravity. This equation can be solved to find the required mixing for the physical state.

5.4 Comments

In the previous sections on BRST, we have noted that in order to obtain consistent results, we must modify the BRST charge. In order for this to retain a holomorphic structure, we are forced to introduce new fields, with somewhat ‘exotic’ OPE’s. Let us review the structure of these fields. We have fields \(\Sigma^\alpha\) and \(\Sigma^{\hat{a}}\) of dimension (1, 0).
(Fields of dimension $(0,1)$ would also be required for the antiholomorphic part of the BRST charge.) These fields intertwine the matter CFT’s via

\begin{align}
\Sigma^\alpha(z)\tilde{S}^\beta(z') & \sim \frac{1}{z - z'} \bar{\partial}X^\mu(z')(\Gamma^\mu)_{\alpha\beta} \\
\Sigma^{\dot{\alpha}}(z)\bar{\partial}X^\mu(z') & \sim \frac{1}{z - z'} \tilde{S}^\alpha(z')(\Gamma^\mu)^{\dot{\alpha}}_{\alpha}
\end{align}

(5.33) (5.34)

Note that such a structure, mixing holomorphic and anti-holomorphic quantities, is allowed here because the CFT is not unitary. The alternative would be to discard BRST entirely. Although this system has a simple structure, we caution that it is by no means clear that it is complete. We have not for example checked that the operator algebra closes.

Let us also comment further on the structure of the ghost CFT’s. Since we have enforced conformal invariance ($T_{zz} = 0$), a conformal gauge choice is possible, at least locally. Thus the conformal ghost CFT is unmodified. The superconformal ghosts are another matter. We have seen that although some of the structure, such as picture, is retained in the full theory, although the ghost and matter CFT’s are mixed. Since worldsheet supersymmetry is lost in the RR background, one should simply think of the presence of $\eta, \xi, \phi$ as a parameterization of the CFT which is convenient close to a trivial background.

Perhaps one puzzling aspect of the BRST analysis is the lack of worldsheet supersymmetry. When supersymmetry is present, it provides a canonical square root for the spacetime Laplacian, related to $Q_1$, which squares to zero in $Q_0$ cohomology. The existence of such an operator allows for the appropriate constraints on spacetime fields. Worldsheet supersymmetry is not a necessary condition however, and it is possible to find an appropriate nilpotent operator which leads to constraints consistent with spacetime supersymmetry. Note that we have a theory with constraints, which are not derived from any apparent gauge principle. It is possible of course that such a gauge principle exists.

6. Two loops and higher order

In Section 5, we have obtained one-loop $\beta$-functions in a normal coordinate expansion. In the last section, we have seen that the picture-changing operator receives corrections from the NSNS background only at tree level— it is covariantized with respect to the full background metric. This means that there are no subtleties for one-loop $\beta$-functions, and they are defined globally as a power series in the normal coordinate expansion. The inclusion of the RR background led to no quadratic divergences, and there appeared to be no need for contact terms of the RR insertions. The BRST analysis however led to the conclusion that contact terms may appear at higher orders, and we were able to accommodate such terms effectively.
In fact, contact terms for RR insertions are required by spacetime supersymmetry at higher loop order. If there were no such contact terms, then each pair of RR insertions corresponds to an extra loop in the $\alpha'$ expansion, and one obtains a logarithmic divergent factor for each pair—as a result they do not contribute to the graviton $\beta$-function. Thus, in the absence of RR contact terms, the only contribution to the graviton $\beta$-function would be of order $h^2$. On the other hand, it is expected that at four loops in the $\alpha'$ expansion, the graviton $\beta$-function receives a correction of order $R^4$. A spacetime supersymmetric completion would include terms with at least four factors of the RR fieldstrength $h$.

Contact terms for the RR insertions can reduce the divergence of terms of higher order in $h$ to a single logarithm, thus giving a contribution to the $\beta$-function. Thus, such contact terms are required by spacetime supersymmetry.

However, we do need to show that at low loop orders, there are nevertheless no contributions to the $\beta$-functions, which is a generalization of the results of [22]. Let us now discuss the case of two loops. Consider the presence of three RR background insertions—this would potentially give a contribution to the RR $\beta$-function. If we bring two of the insertions together, we obtain a logarithmic divergence proportional to $R_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu$ (here we have, consistently, used the one-loop equations of motion). To this, we should add $R_{\mu\nu}(\psi^\mu \bar{\partial} \psi^\nu + \bar{\psi}^\nu \partial \bar{\psi}^\nu)$ The resulting operator could have a contact term with the third RR insertion. A short calculation reveals that this is proportional to
\[
\delta^{(2)}(z - z') R_{\mu\nu} h_{\alpha\beta} (\Gamma^\mu \Gamma^\nu)_\gamma^\alpha S^\gamma \tilde{S}^\beta
\]
Since $R_{\mu\nu}$ is symmetric, this expression simplifies, and the $\beta$-function is proportional to $Rh_{\alpha\beta}$. However, since we have restricted the backgrounds to constant dilaton, $R = 0$. Thus, there is no two-loop contribution to the RR $\beta$-function from three RR insertions. One should also take into account a possible contribution from loop corrections to the picture-changing operator. However, one can again argue that this is proportional to $R$ and thus vanishes for this special class of backgrounds.

Another possible source for the RR $\beta$-function would be a single RR insertion plus terms of up to order $X^4$ in NSNS fields. However, again we find that all such contributions are proportional to the Ricci scalar, and so vanish.

We can also discuss the two-loop graviton $\beta$-function. From our previous discussion, we know that the contribution of four RR insertions would lead to a $(\log \epsilon)^2$ singularity in the absence of RR contact terms. However, such contact terms are of one-loop origin; a pair of such contact terms then counts as two-loops, and there is no room left, at two loop order, for a further logarithmic divergence. We conclude that a two-loop contribution of order $h^4$ to the graviton $\beta$-function vanishes.

Thus, all two-loop contributions to the $\beta$-functions vanish. These results imply that there is a choice of contact terms which is compatible with spacetime supersymmetry to this order. Indeed for products of homogeneous spaces (like $AdS_5 \times S^5$) it
is easy to see that the possible contact terms between the NS and RR field vanish at one loop, again because they are proportional to $R_{\mu\nu}g^{\mu\nu}$, and therefore there is no $\beta$-function to two loops at all for the RR fields.

Notice that in principle one can set the contact terms to zero up to two loops for the RR fields, and then there is no contribution from the RR field to the three loop $\beta$ function, which is then consistent with supergravity. If one introduces contact terms at three loops, they would contribute to a four loop $\beta$ function for the graviton and the RR fields. In spacetimes where one is supposed to have spacetime supersymmetry, we have to worry that we can define the spacetime supersymmetry current on the worldsheet consistently order by order in the perturbation expansion. Indeed, contact terms should contribute to show that the worldsheet symmetry current associated to supersymmetry is conserved. This makes it hard to believe that the procedure of setting all these contact terms to zero will work up to that high an order. We do not wish however, to explicitly carry out the four loop computations.

7. Conclusions and Comments

In this paper we have further explored $\sigma$-model perturbation theory including RR backgrounds. The expansion appears to be consistent to all orders in $\alpha'$, with contact terms being determined by spacetime supersymmetry. Worldsheet supersymmetry is broken by these backgrounds; however, at least at low orders in perturbation theory, it is possible to mimic the appropriate contact terms by taking the NSNS part of the $\sigma$-model off-shell in a supersymmetric form. This stems from the fact that the contact terms between the RR operators don’t contribute at lowest orders for the special class of backgrounds where the dilaton is constant.

Certain aspects of the theory remain quite simple. One has a notion of BRST symmetry and the BRST current is of the form

$$J_{\text{BRST}} = J_{\text{conf}} + J_1 + J_2$$

(7.1)

where $J_{\text{conf}}$ is the contribution from the conformal field theory stress tensor, and it annihilates physical states. The stress tensor includes contributions from the RR background. The remaining piece, $J_1$, as usual, implements the Dirac equation on physical fermionic states, and it is also conformally invariant. We have constructed the leading corrections to $J$ and $J_1$ coming from the RR background. In principle one should expect corrections to all orders in the $\alpha'$ expansion to each of these quantities. As well, since there is no symmetry restricting the form of $Q$, there may be additional terms of other ghost charges present. We suspect that $Q$ in fact does truncate, as given, although we have not identified the mechanism.

We have not checked that $(Q_1)^2 = 0$ in the BRST cohomology of $Q_0$, which would make the set of constraints consistent. We have found that the conditions on
massless vertex operators seem to be compatible with supergravity results. This is an explicit check on the consistency of the perturbation expansion.

Calculations become complicated at higher orders because we don’t have worldsheet supersymmetry as a guiding principle. Contact terms must be introduced order by order in the perturbation expansion, and one would like a systematic approach in which to do calculations, that is a prescription for the contact terms to all orders. The guiding principle we have available is compatibility of the $\alpha'$ expansion with spacetime supersymmetry; however, spacetime supersymmetry is corrected to all orders in $\alpha'$ as well, so in practice it may be difficult to implement. Our $\sigma$-model approach has given us plausibility arguments that there are no two loop corrections to the $\beta$ function of the background in the $\alpha'$ expansion. Given such a prescription, if one shows that one has compatibility with spacetime supersymmetry, one should be able to give a worldsheet proof of the non-renormalization theorems which have heretofore been based on spacetime supersymmetry arguments[23, 24].

The techniques in this paper if developed further should make it possible to understand the spectrum of massive string states in $RR$ backgrounds. Particularly, one should be able to compute the perturbative spectrum of the string in the $AdS_5 \times S^5$ geometry in the infinite $N$ limit, as an expansion in the inverse of the t’Hooft coupling, and it should depend only on defining the full OPE of the fields appearing in the BRST current.

One should also be able to include the dilaton and the $NS$ $B$-field for the most general supergravity background. The results should also generalize without difficulty to any situation where we might have a good knowledge of the CFT, as in the case of orbifolds.

Going to finite $N$ is more difficult, as it involves calculating the one loop partition function of the string in these backgrounds. Perturbation theory in $\alpha'$ is probably not sufficient for this calculation, as one may need to go to all orders to understand issues such as modular invariance, or a stringy exclusion principle[25].

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A. Appendix: Conventions

The fields in the superstring are described by the spacetime coordinates $X^\mu$, their
superpartners $\psi^\mu$, the bosonic ghosts $b, c$ and the fermionic ghosts $\beta, \gamma$. One has $\bar{\psi}^\mu, \bar{b}, \bar{c}, \bar{\beta}, \bar{\gamma}$ for the antiholomorphic fields. One also includes the auxiliary field $F^\mu$ to close supersymmetry off shell. The $\beta\gamma$ system is bosonized by $\gamma = e^\phi \eta, \beta = e^{-\phi} \partial \xi$ with $\eta, \xi$ fermions of conformal dimension 1 and 0 respectively.

The OPE’s of these fields that are used in calculations around flat space are given as follows

$$X^\mu(z)X^\nu(z') = -\eta^{\mu\nu} \log |z - z'|^2 \quad (A.1)$$
$$\psi^\mu(z)\psi^\nu(z') = \eta^{\mu\nu} \frac{1}{z - z'} \quad (A.2)$$
$$F^\mu(z)F^\nu(z') = \eta^{\mu\nu} \partial \bar{\partial} \log |z - z'|^2 \quad (A.3)$$
$$S^\alpha e^{-\phi/2}(z)S^\beta e^{-\phi/2}(z') \sim \frac{1}{z - z'} \Gamma^\alpha\beta_j \psi^\mu e^{-\phi} \quad (A.4)$$
$$\psi^\mu(z)S^\alpha(z') \sim \frac{1}{(z - z')^{1/2}} \Gamma^\alpha\beta_j S^\beta \quad (A.5)$$
$$\psi^\mu(z)S^\alpha(z') \sim \frac{1}{(z - z')^{1/2}} \Gamma^\mu\alpha_j S^\beta \quad (A.6)$$

with similar OPE’s for the right moving fields.

Divergent quantities are defined through

$$\langle X(z)X(z) \rangle = \log \epsilon = \int \frac{d^2 p}{(2\pi)^2 |p|^2} \quad (A.7)$$

It is easy to show also that

$$\int \frac{d^2 z}{|z - z'|^2} = \log \epsilon \quad (A.8)$$

For curved spaces, fermions are bosonized by pairing in an orthonormal frame

$$\psi^\mu = e^\mu_a \chi^a \quad (A.9)$$

and it is $\chi^a$ that is bosonized in a standard fashion. With this bosonization it is possible to define spin fields in the curved manifold (with respect to the given orthonormal frame). The OPE of these spin fields is

$$S^\alpha e^{-\phi/2}(z)S^\beta e^{-\phi/2}(z') \sim \frac{\Gamma^\alpha\beta_j a}{z - z'} \chi^a e^{-\phi} \quad (A.10)$$

The picture changing operator is defined by

$$P_{+1} = Q_{BRST} \xi \quad (A.11)$$

In flat backgrounds, the relevant piece of the picture changing operator for integrated vertex operators is given by

$$P_{+1} \sim e^\phi \psi^\mu \partial X^\mu = e^\phi G_m \quad (A.12)$$
We also formally define $P_{1/2}$ to be the holomorphic square root of the picture changing operator
\[ P_{1/2}^2 = P_{+1} \]  
(A.13)
The relevant piece of the picture-changing operator in a curved geometry will be
\[ P_{+1} \sim e^\phi \chi^a \partial \xi^a \sim e^\phi \psi^\nu \partial X^\mu g_{\mu\nu}(X), \]  
(A.14)
where $\xi^a$ is the normalized tangent vector. This result which is independent of the $RR$ background to first order in the $\alpha'$ loop expansion.

Occasionally, we also use the shorthand notation
\[ S^\alpha = P_{1/2} e^{-\phi/2} S^\alpha \]  
(A.15)
with a similar expression for the right moving fields $\bar{S}^\beta$.

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