Manifestly covariant canonical quantization III: Gravity, locality, and diffeomorphism anomalies in four dimensions

T. A. Larsson
Vanadisvägen 29, S-113 23 Stockholm, Sweden
email: thomas.larsson@hdd.se

March 1, 2022

Abstract

The recently introduced manifestly covariant canonical quantization scheme is applied to gravity. New diffeomorphism anomalies generating a multi-dimensional generalization of the Virasoro algebra arise. This does not contradict theorems about the non-existence of gravitational anomalies in four dimensions, because the relevant cocycles depend on the observer’s spacetime trajectory, which is ignored in conventional field theory. Rather than being inconsistent, these anomalies are necessary to obtain a local theory of quantum gravity.

PACS (2003): 02.20.Tw, 03.70.+k, 04.60.Ds

Keywords: Covariant canonical quantization, History phase space, Multi-dimensional Virasoro algebra, Diffeomorphism anomalies.
1 Introduction

The construction of a quantum theory of gravity is the outstanding problem in physics today [2, 25]. To make some progress, it is useful to consider the key concepts underlying its constituent subtheories.

Quantum field theory (QFT) rests on three conceptual pillars:

1. Quantum mechanics.
2. Lorentz invariance, i.e. special relativity.
3. Cluster decomposition.

General relativity rests on two pillars:

1. General covariance, i.e. diffeomorphism invariance (without background fields).
2. Locality.

Since Lorentz invariance may be viewed as a special case of general covariance, and cluster decomposition is a statement about locality, this suggests that quantum gravity should be based on three principles:

1. Quantum mechanics.
2. Diffeomorphism invariance.
3. Locality.

Alas, it is well known that diffeomorphism invariance and locality are incompatible in quantum theory [3]: “there are no local observables in quantum gravity”. Hence the quest for a theory satisfying all three principles above appears futile.

However, there is a loophole. Proper diffeomorphism invariance is a gauge symmetry, i.e. a mere redundancy of the description, but the situation might be changed in the presence of anomalies. In the treatment of quantum black holes in three dimensions [3], diffeomorphism anomalies (or “would-be diffeomorphisms”) on the boundary are responsible for entropy. More to the point, it is well known in the context of conformal field theory (CFT) [6], that it is possible to combine infinite spacetime symmetries with locality, in the sense of correlation functions depending on separation, but only in the presence of an anomaly. Since the infinite conformal group in two
dimensions is isomorphic to (twice) the diffeomorphism group in one dimension, all results in CFT apply to one-dimensional diffeomorphism-invariant QFT as well. This is a kinematical statement on the level of fields, correlation functions and Ward identities, which does not involve dynamics and hence it is independent of whether diffeomorphisms are gauge transformations or not.

Although it has been suggested before that anomalies could be responsible for gauge symmetry breaking \[10\], it would seem that this could not be relevant to quantum gravity, since no pure diffeomorphism anomalies exist in four dimensions \[1\]; see also \[26\], chapter 22. Or do they? Anomalies manifest themselves as extensions of the constraint algebra, and the diffeomorphism algebra in any number of dimensions certainly admits extensions; in fact, all extensions by modules of tensor fields were classified by Dzhumadildaev \[5\]. In particular, there is a higher-dimensional analogue of the Virasoro algebra, whose representation theory was developed in \[12, 13, 14, 22\]; see \[15\] for a recent review.

The key idea is the introduction of the observer’s trajectory in spacetime. In non-relativistic quantum mechanics, observation is a complicated, non-local process which assigns numbers to experiments. However, in a relativistic theory, a process must be localized; it happens somewhere. In order to maintain locality, we must assign the process of observation to some definite event in spacetime. As time proceeds, the observer (or detector or test particle) traces out a curve in spacetime. Like the quantum fields, the observer’s trajectory should be treated as a material, quantized object; it has a conjugate momentum, it is represented on the Hilbert space, etc.

To consider the observer’s trajectory as a material object is certainly a very small modification, which does not introduce any new physical ideas, and one may wonder why it should be important. The reason is that it makes it possible for new types of anomalies to arise. If the field theory has a gauge symmetry of Yang-Mills type, there is a gauge anomaly proportional to the quadratic Casimir, and the gauge (or current) algebra becomes a higher-dimensional generalization of affine Kac-Moody algebras. Similarly, a general-covariant theory, in any number of dimensions, acquires a diffeomorphism anomaly, which is described by the above-mentioned multidimensional Virasoro algebra. The reason why these anomalies can not be seen in conventional QFT, without explicit reference to the observer, is that the relevant cocycles are functionals of the observer’s trajectory. If this trajectory has not been introduced, it is of course impossible to write down the relevant anomalies.

The canonical formalism is not very well suited for quantization of rel-
ativistic theories, because the foliation of spacetime into fixed time slices breaks manifest covariance. In fact, canonical quantization proceeds in two steps, both of which violate covariance:

1. Replace Poisson brackets by commutators. Phase space is defined non-covariantly on a space-like surface.

2. Represent the resulting Heisenberg algebra on a Hilbert space with energy bounded from below. The Hamiltonian refers to a privileged time direction.

To remedy these problems, a novel quantization scheme has recently been proposed, called manifest covariant canonical quantization (MCCQ) [16, 18]. In the present paper, which is the last in this series, we finally apply MCCQ to general relativity. The two sources of non-covariance are addressed as follows:

1. Phase space points are identified with histories which solve the dynamics. We quantize in the phase space of arbitrary histories first, and impose dynamics as a first-class constraint afterwards.

2. The observer’s trajectory is introduced as a base point for a Taylor expansion of all fields. The Hamiltonian is identified with the generator of rigid translations parallel to the observer’s trajectory. The dynamical variables, i.e. the Taylor coefficients, live on this trajectory and are hence causally related. The notion of spacelike separation only arises implicitly by taking the (problematic) limit of infinite Taylor series.

History methods have recently been advocated by Savvidou and Isham [9, 23, 24]; in particular, the last reference contains a very good summary of the conceptual problems involved in non-covariant canonical quantization. Their formalism differs in details from MCCQ, e.g. because they do not use cohomological methods. There is also a substantial difference, namely that the observer’s trajectory is not introduced, and hence they can not see any diffeomorphism anomalies.

2 Anomalies, consistency, locality, and unitarity

At this point it is necessary to discuss the issue of gauge anomalies and consistency, in particular unitarity. Gauge and diffeomorphism anomalies are usually considered as a sign of inconsistency and should therefore be cancelled [20]. There is ample evidence that this is the correct prescription
for conventional gauge anomalies, arising from chiral fermions coupled to
gauge fields. However, the anomalies discussed in this paper are of a com-
pletely different type, depending on the observer’s trajectory, and intuition
derived from conventional anomalies needs not apply. In fact, one can give
a very simple algebraic argument why conventional anomalies must vanish
[17], and this argument does not apply to the observer-dependent anomalies
considered here.

A quantum theory is defined by a Hilbert space and a Hamiltonian which
generates time evolution. The main conditions for consistency are unitarity
and lack of infinities. If the theory has some symmetries, these must be real-
ized as unitary operators acting on the Hilbert space as well. In particular,
if time translation is included among the symmetries, which is the case for
the Poincaré and diffeomorphism algebras, a unitary representation of the
symmetry algebra is usually enough for consistency. From this viewpoint,
there is a 1-1 correspondence between general-covariant QFTs and unitary
representations of the diffeomorphism group on a conventional Hilbert space.
Namely, given the QFT, its Hilbert space carries a unitary representa-
tion of the diffeomorphism group. Conversely, if we have a unitary repre-
sentation of the diffeomorphism group, the Hilbert space on which it acts can be
interpreted as the Hilbert space of some general-covariant QFT.

Unfortunately, this observation is not so powerful, because no non-trivial,
unitary, lowest-energy irreps of the diffeomorphism algebra are known ex-
cept in one dimension. Nevertheless, we are able to make some very general
observations. Assume that some algebra \( g \) has a unitary representation \( R \)
and a subalgebra \( h \). Then the restriction of \( R \) to \( h \) is still unitary, and this
must hold for every subalgebra \( h \) of \( g \). In particular, let \( g = \text{vect}(N) \) be the
diffeomorphism algebra in \( N \) dimensions and \( h = \text{vect}(1) \) the diffeomorphism
algebra in one dimension. There are infinitely many such subalgebras, and
the restriction of \( R \) to each and every one of them must be unitary. Fortu-
nately, the unitary irreps of the diffeomorphism algebra in one dimension are
known. The result is that the only proper unitary irrep is the trivial one, but
there are many unitary irreps with a diffeomorphism anomaly. From this
it follows that the trivial representation is the only unitary representation
also in \( N \) dimensions. “There are no local observables in quantum gravity”.

However, there is one well-known case where we know how to combine
locality and infinite spacetime symmetry with quantum theory: conformal
field theory (CFT). Locality means that the correlation functions depend on
separation. For two points \( z \) and \( w \) in \( \mathbb{R} \) or \( \mathbb{C} \), the correlator is

\[
\langle \phi(z) \phi(w) \rangle \sim \frac{1}{(z-w)^{2h}} + \text{more},
\]

where \text{more} stands for less singular terms when \( z \to w \). That the correlation function has this form is a diffeomorphism-invariant statement. The \text{more} terms will change under an arbitrary diffeomorphism, but the leading singularity will always have the same form, and in particular the anomalous dimension \( h \) is well defined.

We can phrase this slightly differently. The short-distance singularity only depends on two points being infinitesimally close. This is good, because we cannot determine the finite distance between two points without knowing about the metric. General relativity does not have a background metric structure, but it does have a background differentiable structure (locally at least), and that is enough for defining anomalous dimensions. Diffeomorphisms move points around, but they do not separate two points which are infinitesimally close.

The relevant algebra in CFT is not really the one-dimensional diffeomorphism algebra (or the two-dimensional conformal algebra), but rather its central extension known as the Virasoro algebra:

\[
[L_m, L_n] = (n-m)L_{m+n} - \frac{c}{12}(m^3-m)\delta_{m+n}.
\]

(2.2)

A lowest-energy representation is characterized by a vacuum satisfying

\[
L_0 |0\rangle = h |0\rangle, \quad L_m |0\rangle = 0 \text{ for all } m < 0.
\]

(2.3)

In particular, the lowest \( L_0 \) eigenvalue can be identified with the anomalous dimension \( h \) in the correlation function (2.1). This means that locality, in the sense of correlation functions depending on separation, requires that \( h > 0 \). It is well known \cite{6} that unitarity either implies that

\[
c = 1 - \frac{6}{m(m+1)}, \quad h = h_{rs}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)},
\]

(2.4)

where \( m \geq 2 \) and \( 1 \leq r < m, 1 \leq s < r \) are positive integers, or that \( c \geq 1, \ h \geq 0 \). In particular, the central extension \( c \) is non-zero for any non-trivial, unitary irrep with \( h \neq 0 \). This leads to the important observation:

\[
\text{Locality and unitarity are compatible with diffeomorphism (and local conformal) symmetry only in the presence of an anomaly.}
\]
This is true in higher dimensions as well. Consider the correlator \( \langle \phi(x) \phi(y) \rangle \), where \( x \) and \( y \) are points in \( \mathbb{R}^N \). We could take some one-dimensional curve \( q(t) \) passing through \( x \) and \( y \), such that \( x = q(t) \) and \( y = q(t') \). Then the short-distance behaviour is of the form

\[
\langle \phi(x) \phi(y) \rangle \sim \frac{1}{(t - t')^{2h}} + \text{more},
\]

and \( h \) is independent of the choice of curve, provided that it is sufficiently regular. The subalgebra of \( \text{vect}(N) \) which preserves \( q(t) \) is a Virasoro algebra, so \( h > 0 \) implies that \( c > 0 \).

This observation is completely standard in the application of CFT to statistical physics in two dimensions. The simplest example of a unitary model is the Ising model, which consists of three irreps, with \( c = 1/2 \) and \( h = 0 \), \( h = 1/16 \), and \( h = 1/2 \). The Ising model is perfectly consistent despite the anomaly, both mathematically (unitarity), and more importantly physically (it is realized in nature, in soft condensed matter systems). The standard counter-argument is that infinite conformal symmetry in condensed matter is not a gauge symmetry, but rather an anomalous global symmetry. However, if we could take the classical limit of such a system, the conformal symmetry would seem to be a gauge symmetry. Namely, the anomaly vanishes in the classical limit, and we can write down a classical BRST operator which is nilpotent, and the symmetry is gauge on the classical level. There is no classical way to distinguish between such a “fake” gauge symmetry and a genuine gauge symmetry which extends to the quantum level.

More generally, let us assume that we have some phase space, and a Lie algebra \( \mathfrak{g} \) with generators \( J_a \), satisfying

\[
[J_a, J_b] = f^{c}_{ab} J_c,
\]

acts on this phase space. The Einstein convention is used; repeated indices, one up and one down, are implicitly summed over. If the bracket with the Hamiltonian gives us a new element in \( \mathfrak{g} \),

\[
[J_a, H] = C^b_a J_b,
\]

we say that \( \mathfrak{g} \) is a symmetry of the Hamiltonian system. If \( \mathfrak{g} \) in addition contains arbitrary functions of time, the symmetry is a gauge symmetry. In this case, a solution to Hamilton’s equations depends on arbitrary functions of time and is thus not fully specified by the positions and momenta at time \( t = 0 \). The standard example is electromagnetism, where the zeroth
component $A_0$ of the vector potential is arbitrary, because its canonical momentum $F^{00} = 0$. An arbitrary time evolution is of course not acceptable. The reason why this seems to happen is that a gauge symmetry is a redundancy of the description; the true dynamical degrees of freedom are fewer than what one naively expects. In electromagnetism, the gauge potential has four components but the photon has only two polarizations.

There are various ways to handle quantization of gauge systems. One is to eliminate the gauge degrees of freedom first and then quantize. This is cumbersome and it is usually preferable to quantize first and eliminate the gauge symmetries afterwards. The simplest way is to require that the gauge generators annihilate physical states,

$$J_a |\text{phys}\rangle = 0,$$

(2.8)

and also that two physical states are equivalent if the differ by some gauge state, $J_a |\rangle$. This procedure produces a Hilbert space of physical states.

However, one thing may go wrong. Upon quantization, a symmetry may acquire some quantum corrections, so that $g$ is replaced by

$$[J_a, J_b] = f_{abc} J_c + \hbar D_{ab} + O(\hbar^2).$$

(2.9)

The operator $D_{ab}$ is called an anomaly. We can also have anomalies of the type

$$[J_a, H] = C_{ab} J_b + \hbar E_a + O(\hbar^2).$$

(2.10)

If we now try to keep the definition of a physical state, we see that we must also demand that

$$D_{ab} |\text{phys}\rangle = 0.$$

(2.11)

This implies further reduction of the Hilbert space. In the case that $D_{ab}$ is invertible, there are no physical states at all, so the Hilbert space is empty. However, this does not necessarily mean that the anomaly by itself is inconsistent, only that our definition of physical states is. In the presence of an anomaly, additional states become physical. So our Hilbert space becomes larger, containing some, or even all, of the previous gauge degrees of freedom. A gauge anomaly implies that the gauge symmetry is broken on the quantum level.

It is important to realize that such a “fake” gauge symmetry may well be consistent. The Virasoro algebra is obviously anomalous, with the central charge playing the role of the $D_{ab}$, and still it has unitary representations.
with non-zero $c$. Of course, a gauge anomaly may be inconsistent, if the anomalous algebra does not possess any unitary representations. This is apparently what happens for the chiral-fermion type anomaly which is relevant e.g. in the standard model.

3 Multi-dimensional Virasoro algebra

All non-trivial, unitary, lowest-energy irreps of the diffeomorphism algebra are anomalous, in any number of dimensions. This is well-known in one dimension, where the diffeomorphism algebra acquires an extension known as the Virasoro algebra. It is also true in several dimensions, which one proves by considering the restriction to the many Virasoro subalgebras living on lines in spacetime. This is perhaps rather surprising, in view of the following two no-go theorems:

- The diffeomorphism algebra has no central extension except in one dimension.

- In field theory, there are no pure gravitational anomalies in four dimension.

However, the assumptions in these no-go theorems are too strong; the Virasoro extension is not central except in one dimension, and one needs to go slightly beyond field theory by explicitly specifying where observation takes place.

To make contact with the Virasoro algebra in its most familiar form, we describe its multi-dimensional sibling in a Fourier basis on the $N$-dimensional torus. Recall first that the algebra of diffeomorphisms on the circle, $\text{vect}(1)$, has generators

$$L_m = -i \exp(\text{i}mx) \frac{d}{dx},$$

where $x \in S^1$. $\text{vect}(1)$ has a central extension, the Virasoro algebra:

$$[L_m, L_n] = (n-m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n},$$

where $c$ is a $c$-number known as the central charge or conformal anomaly. This means that the Virasoro algebra is a Lie algebra; anti-symmetry and the Jacobi identities still hold. The term linear in $m$ is unimportant, because it can be removed by a redefinition of $L_0$. The cubic term $m^3$ is a non-trivial extension which cannot be removed by any redefinition.
The generators (3.1) immediately generalize to vector fields on the \(N\)-dimensional torus:

\[
L_\mu(m) = -i \exp(im_\rho x^\rho) \partial_\mu,
\]

where \(x = (x^\mu)\), \(\mu = 1, 2, \ldots, N\) is a point in \(N\)-dimensional space and \(m = (m_\mu) \in \mathbb{Z}^N\). These operators generate the algebra \(\mathfrak{vect}(N)\):

\[
[L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n).
\]

The question is now whether the Virasoro extension, i.e. the \(m^3\) term in (3.2), also generalizes to higher dimensions.

Rewrite the ordinary Virasoro algebra (3.2) as

\[
\begin{align*}
[L_m, L_n] &= (n - m)L_{m+n} + cm^2 nS_{m+n}, \\
[L_m, S_n] &= (n + m)S_{m+n}, \\
[S_m, S_n] &= 0, \\
mS_m &\equiv 0.
\end{align*}
\]

It is easy to see that the two formulations of the Virasoro algebra are equivalent (the linear cocycle has been absorbed into a redefinition of \(L_0\)). The second formulation immediately generalizes to \(N\) dimensions. The defining relations are

\[
\begin{align*}
[L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m + n) - m_\nu L_\mu(m + n) \\
&\quad + (c_1 m_\rho n_\mu + c_2 m_\mu n_\nu) m_\rho S^\rho(m + n), \\
[L_\mu(m), S^\nu(n)] &= n_\mu S^\nu(m + n) + \delta^\nu_\rho m_\rho S^\rho(m + n), \\
[S^\mu(m), S^\nu(n)] &= 0, \\
m_\mu S^\mu(m) &\equiv 0.
\end{align*}
\]

This is an extension of \(\mathfrak{vect}(N)\) by the abelian ideal with basis \(S^\mu(m)\). Geometrically, we can think of \(L_\mu(m)\) as a vector field and \(S^\mu(m)\) as a dual one-form (and \(S_{\nu_2..\nu_N}(m)\) as an \((N-1)\)-form); the last condition expresses closedness. The cocycle proportional to \(c_1\) was discovered by Rao and Moody [22], and the one proportional to \(c_2\) by this author [11].

There is also a similar multi-dimensional generalization of affine Kac-Moody algebras, sometimes called the central extension. This term is somewhat misleading because the extension does not commute with diffeomorphisms, although it does commute with all gauge transformations. Let \(\mathfrak{g}\)
be a finite-dimensional Lie algebra with structure constants $f_{ab}^c$ and Killing metric $\delta_{ab}$. The central extension of the current algebra map$(N, g)$ is defined by the brackets

$$[J_a(m), J_b(n)] = f_{abc} J_c(m + n) + k\delta_{ab} m_\rho S^\rho(m + n),$$
$$[J_a(m), S^\mu(n)] = [S^\mu(m), S^\nu(n)] = 0,$$
$$m_\mu S^\mu(m) \equiv 0.$$  

(3.7)

This algebra admits an intertwining action of the $N$-dimensional Virasoro algebra (3.8):

$$[L_\mu(m), J_a(n)] = n_\mu J_a(m + n).$$

(3.8)

The current algebra map$(N, g)$ also admits another type of extension in some dimensions. The best known example is the Mickelsson-Faddeev algebra, relevant for the conventional anomalies in field theory, which arise when chiral fermions are coupled to gauge fields in three spatial dimensions. Let $d_{abc} = \text{tr}\{J_a, J_b\} J_c$ be the totally symmetric third Casimir operator, and let $\epsilon^{\mu\nu\rho}$ be the totally anti-symmetric epsilon tensor in three dimensions. The Mickelsson-Faddeev algebra [19] reads in a Fourier basis:

$$[J_a(m), J_b(n)] = f_{abc} J_c(m + n) + d_{abc}^{\mu\nu\rho} m_\mu n_\nu A^\rho(m + n),$$
$$[J_a(m), A_\nu^\rho(n)] = -f_{ac}^\mu A_\nu^\rho(m + n) + \delta^\rho_\mu m_\nu \delta(m + n),$$
$$[A_\mu^\rho(m), A_\nu^\rho(n)] = 0.$$  

(3.9)

$A_\mu^\rho(m)$ are the Fourier components of the gauge connection.

Note that $Q_a \equiv J_a(0)$ generates a Lie algebra isomorphic to $g$, whose Cartan subalgebra is identified with the charges. Moreover, the subalgebra of (3.7) spanned by $J_a(m_0) \equiv J_a(m)$, where $m = (m_0, 0, ..., 0) \in \mathbb{Z}$, reads

$$[J_a(m_0), J_b(n_0)] = f_{ab}^c J_c(m_0 + n_0) + k\delta_{ab} m_0 n_0 \delta(m_0 + n_0),$$

(3.10)

where $k = S^0(0)$, which we recognize as the affine algebra $\hat{g}$. Since all non-trivial unitary irreps of $\hat{g}$ has $k > 0$ [4], it is impossible to combine unitary and non-zero $g$ charges also for the higher-dimensional algebra (3.7). This follows immediately from the fact that the restriction of a unitary irrep to a subalgebra is also unitary (albeit in general reducible).

In contrast, the Mickelsson-Faddeev algebra (3.9) has apparently no faithful unitary representations on a separable Hilbert space [21]. A simple way to understand this is to note that the restriction to every loop subalgebra is proper and hence lacks unitary representations of lowest-weight type [17]. This presumably means that this kind of extension should be avoided. Indeed, Nature appears to abhor this kind of anomaly, which is proportional to the third Casimir.
4 DGRO algebra

The Fourier formalism in the previous section makes the analogy with the usual Virasoro algebra manifest, but it is neither illuminating nor a useful starting point for representation theory. To bring out the geometrical content, we introduce the DGRO (Diffeomorphism, Gauge, Reparametrization, Observer) algebra \( \text{DGRO}(N, g) \), whose ingredients are spacetime diffeomorphisms which generate \( \text{vect}(N) \), reparametrizations of the observer’s trajectory which form an additional \( \text{vect}(1) \) algebra, and gauge transformations which generate a current algebra. Classically, the algebra is \( \text{vect}(N) \ltimes \text{map}(N, g) \oplus \text{vect}(1) \).

Let \( \xi = \xi^\mu(x)\partial_\mu, \, x \in \mathbb{R}^N, \, \partial_\mu = \partial/\partial x^\mu \), be a vector field, with commutator \([\xi, \eta] \equiv \xi^\mu \partial_\mu \eta^\nu \partial_\nu - \eta^\nu \partial_\nu \xi^\mu \partial_\mu \), and greek indices \( \mu, \nu = 1, 2, ..., N \) label the spacetime coordinates. The Lie derivatives \( L_\xi \) are the generators of \( \text{vect}(N) \).

Let \( f = f(t)d/dt, \, t \in S^1 \), be a vector field in one dimension. The commutator reads \([f, g] = (f \dot{g} - g \dot{f})d/dt \), where the dot denotes the \( t \) derivative: \( \dot{f} \equiv df/dt \). We will also use \( \partial_t = \partial/\partial t \) for the partial \( t \) derivative. The choice that \( t \) lies on the circle is physically unnatural and is made for technical simplicity only (quantities can be expanded in Fourier series). However, this seems to be a minor problem at the present level of understanding. Denote the reparametrization generators \( \mathcal{L}_\xi \).

Let \( \text{map}(N, g) \) be the current algebra corresponding to the finite-dimensional semisimple Lie algebra \( g \) with basis \( J_a \), structure constants \( f^{abc} \), and Killing metric \( \delta_{ab} \). The brackets in \( g \) are given by \( (2.6) \). A basis for \( \text{map}(N, g) \) is given by \( g \)-valued functions \( X = X^a(x)J_a \) with commutator \([X, Y] = f_{abc}X^aY^bJ_c \). The intertwining \( \text{vect}(N) \) action is given by \( \xi X = \xi^\mu \partial_\mu X^aJ_a \).

Denote the \( \mathcal{L}_\xi \) generators by \( \mathcal{J}_X \).

Finally, let \( \text{Obs}(N) \) be the space of local functionals of the observer’s trajectory \( q^\mu(t) \), i.e. polynomial functions of \( q^\mu(t), \dot{q}^\mu(t), \ldots \partial^k q^\mu(t)/dt^k, \, k \) finite, regarded as a commutative algebra. \( \text{Obs}(N) \) is a \( \text{vect}(N) \) module in a natural manner.

\( \text{DGRO}(N, g) \) is an abelian but non-central Lie algebra extension of \( \text{vect}(N) \ltimes \text{map}(N, g) \oplus \text{vect}(1) \) by \( \text{Obs}(N) \):

\[ 0 \rightarrow \text{Obs}(N) \rightarrow \text{DGRO}(N, g) \rightarrow \text{vect}(N) \ltimes \text{map}(N, g) \oplus \text{vect}(1) \rightarrow 0. \]
The brackets are given by

\[
[L_\xi, L_\eta] = L_{[\xi, \eta]} + \frac{1}{2\pi i} \int dt \, \dot{\eta}^\mu(t) \left\{ c_1 \partial_\rho \partial_\nu \xi^\mu(q(t)) \partial_\mu \eta^\nu(q(t)) + 
+ c_2 \partial_\rho \partial_\mu \xi^\mu(q(t)) \partial_\nu \eta^\nu(q(t)) \right\},
\]

\[
[L_\xi, J_X] = J_{\xi X},
\]

\[
[J_X, J_Y] = J_{[X,Y]} - \frac{c_5}{2\pi i} \delta_{ab} \int dt \, \dot{\eta}^\rho(t) \partial_\rho X^a(q(t)) Y^b(q(t)),
\]

\[
[L_f, L_\xi] = \frac{c_3}{4\pi i} \int dt \, (\dddot{f}(t) - i \dot{f}(t)) \partial_\mu \xi^\mu(q(t)),
\]

\[
[L_f, J_X] = 0,
\]

\[
[L_f, L_g] = L_{[f,g]} + \frac{c_4}{24\pi i} \int dt (\dddot{f}(t) \dot{g}(t) - \dddot{g}(t) \dot{f}(t)),
\]

\[
[L_\xi, q^\mu(t)] = \xi^\mu(q(t)),
\]

\[
[L_f, q^\mu(t)] = -\dot{f}(t) \dot{q}^\mu(t),
\]

\[
[J_X, q^\mu(t)] = [q^\mu(s), q^\nu(t)] = 0,
\]

extended to all of \(\text{Obs}(N)\) by Leibniz’ rule and linearity. The numbers \(c_1 - c_5\) are called \textit{abelian charges}, in analogy with the central charge of the Virasoro algebra. In \cite{12, 13} slightly more complicated extensions were considered, which depend on three additional abelian charges \(c_6 - c_8\). However, these vanish automatically when \(\mathfrak{g}\) is semisimple.

## 5 Representations of the DGRO algebra

To construct Fock representations of the ordinary Virasoro algebra is straightforward:

- Start from classical modules, i.e. primary fields = scalar densities.
- Introduce canonical momenta.
- Normal order.

The first two steps of this procedure generalize nicely to higher dimensions. The classical representations of the DGRO algebra are tensor fields over \(\mathbb{R}^N \times S^1\) valued in \(\mathfrak{g}\) modules. The basis of a classical DGRO module \(\mathcal{Q}\) is thus a field \(\phi^\alpha(x, t)\), \(x \in \mathbb{R}^N, t \in S^1\), where \(\alpha\) is a collection of all kinds of
indices. The $DGRO(N, g)$ action on $Q$ can be succinctly summarized as

$$
\begin{align*}
[\mathcal{L}_\xi, \phi^\alpha(x, t)] &= -\xi^\mu(x)\partial_\mu \phi^\alpha(x, t) - \partial_\nu \xi^\mu(x)T^{\alpha\nu}_{\beta\mu}\phi^\beta(x, t), \\
[J_X, \phi^\alpha(x, t)] &= -X^\alpha(x)J^\alpha_{\beta\delta}\phi^\delta(x, t), \\
[L_f, \phi^\alpha(x, t)] &= -f(t)\partial_t \phi^\alpha(x, t) - \lambda(f(t) - \dot{f}(t))\phi^\alpha(x, t).
\end{align*}
$$

Here $J_a = (J^a_{\alpha\delta})$ and $T^\mu_\nu = (T^\alpha_{\beta\mu \nu})$ are matrices satisfying $g(N)$ and $gl(N)$, respectively:

$$
[T^\mu_\nu, T^\sigma_\tau] = \delta^\sigma_\nu T^\mu_\tau - \delta^\mu_\tau T^\sigma_\nu. 
$$

The tensor field representations of the DGRO algebra can thus be expressed in matrix form as

$$
\begin{align*}
\mathcal{L}_\xi &= -\int d^N x \int dt \left( \xi^\mu(x)\partial_\mu \phi^\alpha(x, t) + \partial_\nu \xi^\mu(x)T^{\alpha\nu}_{\beta\mu}\phi^\beta(x, t) \right)\pi_\alpha(x, t), \\
J_X &= -\int d^N x \int dt \left( X^\alpha(x)J^\alpha_{\beta\delta}\phi^\delta(x, t) \right)\pi_\alpha(x, t), \\
L_f &= -\int d^N x \int dt \left( f(t)\partial_t \phi^\alpha(x, t) - \lambda(f(t) - \dot{f}(t))\phi^\alpha(x, t) \right)\pi_\alpha(x, t),
\end{align*}
$$

where the conjugate momentum $\pi_\alpha(x, t) = \delta/\delta\phi^\alpha(x, t)$ satisfies

$$
[\pi_\alpha(x, t), \phi^\beta(x', t')]=\delta^\beta_\alpha \delta(x - x')\delta(t - t').
$$

However, the normal-ordering step simply does not work in several dimensions, because

- It requires that a foliation of spacetime into space and time has been introduced, which runs against the idea of diffeomorphism invariance.
- Normal ordering of bilinear expressions always results in a central extension, but the Virasoro cocycle is non-central when $N \geq 2$.
- It is ill defined. Formally, attempts to normal order result in an infinite central extension, which of course makes no sense.

To avoid this problem, the crucial idea in [12] was to expand all fields in a Taylor series around the observer’s trajectory and truncate at order $p$, before introducing canonical momenta. Hence we expand e.g.,

$$
\phi^\alpha(x, t) = \sum_{|\mathbf{m}| \leq p} \frac{1}{\mathbf{m}!} \phi^\alpha_{\mathbf{m}}(t)(x - q(t))^\mathbf{m},
$$

14
where \( \mathbf{m} = (m_1, m_2, ..., m_N) \), all \( m_\mu \geq 0 \), is a multi-index of length \( |\mathbf{m}| = \sum_{\mu=1}^{N} m_\mu \), \( \mathbf{m}! = m_1! m_2! ... m_N! \), and

\[
(x - q(t))^\mathbf{m} = (x^1 - q^1(t))^{m_1} (x^2 - q^2(t))^{m_2} ...(x^N - q^N(t))^{m_N}.
\] (5.6)

Denote by \( \mu \) a unit vector in the \( \mu \):th direction, so that \( \mathbf{m} + \mu = (m_1, ..., m_\mu + 1, ..., m_N) \), and let

\[
\phi^{\alpha}_{,\mathbf{m}}(t) = \partial_{\mathbf{m}} \phi^\alpha(q(t), t) = \partial_{m_1} ... \partial_{m_N} \phi^\alpha(q(t), t)
\] (5.7)

be an \( |\mathbf{m}| \):th order mixed partial derivative of \( \phi^\alpha(x, t) \) evaluated on the observer’s trajectory \( q^\mu(t) \).

Given two jets \( \phi_{,\mathbf{m}}(t) \) and \( \psi_{,\mathbf{m}}'(t) \), we define their product

\[
(\phi(t)\psi(t'))_{,\mathbf{m}} = \sum_n \left( \begin{array}{c} \mathbf{m} \\ \mathbf{n} \end{array} \right) \phi_{,\mathbf{n}}(t)\psi_{,\mathbf{n}}(t'),
\] (5.8)

where

\[
\left( \begin{array}{c} \mathbf{m} \\ \mathbf{n} \end{array} \right) = \frac{\mathbf{m}!}{\mathbf{n}!(\mathbf{m} - \mathbf{n})!} = \left( \begin{array}{c} m_1 \\ n_1 \end{array} \right) \left( \begin{array}{c} m_2 \\ n_2 \end{array} \right) ... \left( \begin{array}{c} m_N \\ n_N \end{array} \right).
\] (5.9)

It is clear that \( (\phi(t)\psi(t'))_{,\mathbf{m}} \) is the jet corresponding to the field \( \phi(x, t)\psi(x, t') \).

For brevity, we also denote \( (\phi\psi)_{,\mathbf{m}}(t) = (\phi(t)\psi(t))_{,\mathbf{m}} \).

\( p \)-jets transform under \( DGRO(N, g) \) as

\[
[\mathcal{L}_\xi, \phi^{\alpha}_{,\mathbf{m}}(t)] = \partial_{\mathbf{m}} ([\mathcal{L}_\xi, \phi^\alpha(q(t), t)]) + [\mathcal{L}_\xi, q^\mu(t)] \partial_\mu \partial_{\mathbf{m}} \phi^\alpha(q(t), t)
\]

\[
= - \sum_{|\mathbf{m}| \leq |\mathbf{n}| \leq \rho} T_{\beta\mathbf{m}}^{\alpha\mathbf{n}}(\xi(q(t)))\phi^\beta_{,\mathbf{n}}(t),
\]

\[
[J_X, \phi^{\alpha}_{,\mathbf{m}}(t)] = \partial_{\mathbf{m}} ([J_X, \phi^\alpha(q(t), t)])
\]

\[
= - \sum_{|\mathbf{m}| \leq |\mathbf{n}| \leq \rho} J_{\beta\mathbf{m}}^{\alpha\mathbf{n}}(X(q(t)))\phi^\beta_{,\mathbf{n}}(t),
\]

\[
[L_f, \phi^{\alpha}_{,\mathbf{m}}(t)] = -f(t)\phi^\alpha_{,\mathbf{m}}(t) - \lambda(\dot{f}(t) - if(t))\phi^\alpha_{,\mathbf{m}}(t),
\]

where

\[
T_{\mathbf{n}}(\xi) \equiv (T_{\beta\mathbf{m}}^{\alpha\mathbf{n}}(\xi)) = \sum_{\mu\nu} \left( \begin{array}{c} \mathbf{n} \\ \mathbf{m} \end{array} \right) \partial_{\mathbf{n} - \mathbf{m} + \nu} \xi^\mu T_{\mu}^\nu
\]

\[
+ \sum_{\mu} \left( \begin{array}{c} \mathbf{n} \\ \mathbf{m} - \mu \end{array} \right) \partial_{\mathbf{n} - \mathbf{m} + \mu} \xi^\mu - \sum_{\mu} \delta_{\mathbf{n} - \mathbf{m} - \mu}^\mu \xi^\mu,
\] (5.11)

\[
J_{\mathbf{n}}(X) \equiv (J_{\beta\mathbf{m}}^{\alpha\mathbf{n}}(X)) = \left( \begin{array}{c} \mathbf{n} \\ \mathbf{m} \end{array} \right) \partial_{\mathbf{n} - \mathbf{m}} X^a J_a.
\]

15
We thus obtain a non-linear realization of $\mathfrak{vect}(N)$ on the space of trajectories in the space of tensor-valued $p$-jets\footnote{$p$-jets are usually defined as an equivalence class of functions: two functions are equivalent if all derivatives up to order $p$, evaluated at $q^\mu$, agree. However, each class has a unique representative which is a polynomial of order at most $p$, namely the Taylor expansion around $q^\mu$, so we may canonically identify jets with truncated Taylor series. Since $q^\mu(t)$ depends on a parameter $t$, we deal in fact with trajectories in jet space, but these will also be called jets for brevity.}; denote this space by $J^pQ$. Note that $J^pQ$ is spanned by $q^\mu(t)$ and $\{\phi^\alpha_m(t)\}_{|m|\leq p}$ and thus it is not a $DGRO(N, g)$ module by itself, because diffeomorphisms act non-linearly on $q^\mu(t)$, as can be seen in \ref{DGRO}. However, the space $C(J^pQ)$ of functionals on $J^pQ$ (local in $t$) is a module, because the action on a $p$-jet can never produce a jet of order higher than $p$. The space $C(q) \otimes_q J^pQ$, where only the trajectory itself appears non-linearly, is a submodule.

The crucial observation is that the jet space $J^pQ$ consists of finitely many functions of a single variable $t$, which is precisely the situation where the normal ordering prescription works. After normal ordering, denoted by double dots $\vdots$, we obtain a Fock representation of the DGRO algebra:

\begin{align*}
\mathcal{L}_\xi &= \int dt \left\{ :\xi^\mu(q(t))p_\mu(t): - \sum_{|n| \leq |m| \leq p} T^n_{3m}(\xi(q(t))) :\phi^\beta_n(t)\pi^m_\alpha(t): \right\}, \\
\mathcal{J}_X &= -\int dt \left\{ \sum_{|n| \leq |m| \leq p} J^n_{3m}(\xi(q(t))):\phi^\beta_n(t)\pi^m_\alpha(t): \right\}, \quad (5.12) \\
L_f &= \int dt \left\{ - f(t) :\phi^\alpha_0(t)\pi^m_\alpha(t) : - \lambda(\dot{f}(t) - if(t)) :\phi^\alpha_m(t)\pi^m_\alpha(t) : \right\},
\end{align*}

where we have introduced canonical momenta $p_\mu(t) = \delta/\delta q^\mu(t)$ and $\pi^m_\alpha(t) = \delta/\delta \phi^\alpha_m(t)$. The field $\phi^\alpha(x, t)$ can be either bosonic or fermionic but the trajectory $q^\mu(t)$ is of course always bosonic.

Normal ordering is defined with respect to frequency; any function of $t \in S^1$ can be expanded in a Fourier series, e.g.

\begin{equation}
p_\mu(t) = \sum_{m=-\infty}^{\infty} \hat{p}_\mu(m)e^{-int} \equiv p^<_\mu(t) + \hat{p}_\mu(0) + p^>_\mu(t), \quad (5.13)
\end{equation}

where $p^<_\mu(t)$ ($p^>_\mu(t)$) is the sum over negative (positive) frequency modes only. Then

\begin{equation}
:\xi^\mu(q(t))p_\mu(t): \equiv \xi^\mu(q(t))p^<_\mu(t) + p^>_\mu(t)\xi^\mu(q(t)), \quad (5.14)
\end{equation}

where the zero mode has been included in $p^>_\mu(t)$.\footnote{\textsuperscript{1}$p$-jets are usually defined as an equivalence class of functions: two functions are equivalent if all derivatives up to order $p$, evaluated at $q^\mu$, agree. However, each class has a unique representative which is a polynomial of order at most $p$, namely the Taylor expansion around $q^\mu$, so we may canonically identify jets with truncated Taylor series. Since $q^\mu(t)$ depends on a parameter $t$, we deal in fact with trajectories in jet space, but these will also be called jets for brevity.}
It is clear that (5.12) defines a Fock representation for every $gl(N)$ irrep $\rho$ and every $g$ irrep $M$; denote this Fock space by $J^pF$, which indicates that it also depends on the truncation order $p$. Namely, introduce a Fock vacuum $|0\rangle$ which is annihilated by half of the oscillators, i.e.

$$\phi^\alpha_m(t)|0\rangle = \pi^m_\alpha(t)|0\rangle = q^\mu_m(t)|0\rangle = p^\mu_m(t)|0\rangle = 0.$$ (5.15)

Then $DGRO(N, g)$ acts on the space of functionals $C(q^\mu_<, p^\mu_>, \phi^\alpha_<, m^\mu, \pi^m_\alpha, \mu^\mu_<)(t)$ of the remaining oscillators; this is the Fock module. Define numbers $k_0(g)$, $k_1(g)$, $k_2(g)$ and $y_M$ by

$$\begin{align*}
\text{tr}_g T^\mu_{\nu'} &= k_0(g)\delta^\mu_{\nu'}, \\
\text{tr}_g T^\mu_{\nu'} T^\sigma_{\tau'} &= k_1(g)\delta^\mu_{\nu'}\delta^\sigma_{\tau'} + k_2(g)\delta^\mu_{\nu'}\delta^\sigma_{\tau'}, \\
\text{tr}_M J_a J_b &= y_M\delta_{ab}.
\end{align*}$$ (5.16)

For an unconstrained tensor with $p$ upper and $q$ lower indices and weight $\kappa$, we have

$$\begin{align*}
dim(g) &= N^{p+q}, & k_0(g) &= -(p - q - \kappa N)N^{p+q-1}, \\
k_1(g) &= (p + q)N^{p+q-1}, & k_2(g) &= ((p - q - \kappa N)^2 - p - q)N^{p+q-2}.
\end{align*}$$ (5.17)

Note that if $\kappa = (p - q)/N$, $g$ is an $sl(N)$ representation. For the symmetric representations on $\ell$ lower indices, $S_{\ell}$, and on $\ell$ upper indices, $S^\ell$, we have

$$\begin{align*}
dim(S_{\ell}) &= \dim(S^\ell) = \binom{N - 1 + \ell}{\ell}, \\
k_0(S_{\ell}) &= -k_0(S^\ell) = \binom{N - 1 + \ell}{\ell - 1}, \\
k_1(S_{\ell}) &= k_1(S^\ell) = \binom{N + \ell}{\ell - 1}, \\
k_2(S_{\ell}) &= k_2(S^\ell) = \binom{N - 1 + \ell}{\ell - 2}.
\end{align*}$$ (5.18)
The values of the abelian charges $c_1 - c_5$ \[4.1\] were calculated in \[12\], Theorems 1 and 3, and in \[13\], Theorem 1:

\begin{align*}
c_1 &= 1 - u \binom{N+p}{N} - x \binom{N+p+1}{N+2}, \\
c_2 &= -v \binom{N+p}{N} - 2w \binom{N+p}{N+1} - x \binom{N+p}{N+2}, \\
c_3 &= 1 + (1 - 2\lambda)(w \binom{N+p}{N} + x \binom{N+p}{N+1}), \\
c_4 &= 2N - x(1 - 6\lambda + 6\lambda^2) \binom{N+p}{N}, \\
c_5 &= y \binom{N+p}{N},
\end{align*} \[5.19\]

where

\begin{align*}
u &= \mp k_1(q) \dim M, \\
v &= \mp k_2(q) \dim M, \\
w &= \mp k_0(q) \dim M,
\end{align*} \[5.20\]

and the sign factor depends on the Grassmann parity of $\phi^\alpha$; the upper sign holds for bosons and the lower for fermions, respectively. The $p$-independent contributions to $c_1$, $c_3$ and $c_4$ come from the trajectory $q^\mu(t)$ itself.

6 MCCQ: Manifestly Covariant Canonical Quantization

Consider a classical dynamical system with action $S$ and degrees of freedom $\phi^\alpha$. As is customary in the antifield literature \[8\], we use an abbreviated notation where the index $\alpha$ stands for both discrete indices and spacetime coordinates. Dynamics is governed by the Euler-Lagrange (EL) equations,

$$\mathcal{E}_\alpha = \partial_\alpha S = \frac{\delta S}{\delta \phi^\alpha} = 0. \tag{6.1}$$

We do not assume that the EL equations are all independent; rather, let there be identities of the form

$$r^\alpha_\alpha \mathcal{E}_\alpha \equiv 0, \tag{6.2}$$

where the $r^\alpha_\alpha$ are some functionals of $\phi^\alpha$. 

Introduce an antifield $\phi^*_\alpha$ for each EL equation (6.1), and a second-order antifield $\zeta_a$ for each identity (6.2). Replace the space of $\phi$-histories $Q$ by the extended history space $Q^*$, spanned by both $\phi^\alpha$, $\phi^*_\alpha$, and $\zeta_a$. In $Q^*$ we define the Koszul-Tate (KT) differential $\delta$ by

$$\begin{align*}
\delta \phi^\alpha &= 0, \\
\delta \phi^*_\alpha &= E\alpha, \\
\delta \zeta_a &= r^a_\alpha \phi^*_\alpha.
\end{align*}$$

(6.3)

One checks that $\delta$ is nilpotent, $\delta^2 = 0$. Define the antifield number $\text{afn} \phi^\alpha = 0$, $\text{afn} \phi^*_\alpha = 1$, $\text{afn} \zeta_a = 2$. The KT differential clearly has antifield number $\text{afn} \delta = -1$.

The space $C(Q^*)$ decomposes into subspaces $C^k(Q^*)$ of fixed antifield number

$$C(Q^*) = \sum_{k=0}^\infty C^k(Q^*)$$

(6.4)

The KT complex is

$$0 \xleftarrow{\delta} C^0 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^2 \xleftarrow{\delta} \ldots$$

(6.5)

The cohomology spaces are defined as usual by $H^\bullet_{cl}(\delta) = \ker \delta / \text{im} \delta$, i.e. $H^k_{cl}(\delta) = (\ker \delta)_k / (\text{im} \delta)_k$, where the subscript $\text{cl}$ indicates that we deal with a classical phase space. It is easy to see that

$$(\ker \delta)_0 = C(Q),$$

$$(\text{im} \delta)_0 = C(Q)E\alpha \equiv N.$$ 

(6.6)

Thus $H^0_{cl}(\delta) = C(Q)/N = C(\Sigma)$. The higher cohomology groups vanish, because although $r^a_\alpha \phi^*_\alpha$ is KT closed ($\delta(r^a_\alpha \phi^*_\alpha) = r^a_\alpha E\alpha \equiv 0$), it is also KT exact ($r^a_\alpha \phi^*_\alpha = \delta \zeta_a$). The complex (6.5) thus gives us a resolution of the covariant phase space $C(\Sigma)$, which by definition means that $H^0_{cl}(\delta) = C(\Sigma)$, $H^k_{cl}(\delta) = 0$, for all $k > 0$.

Alas, the antifield formalism is not suited for canonical quantization. We can define an antibracket in $Q^*$, but in order to do canonical quantization we need an honest Poisson bracket. To this end, we introduce canonical momenta conjugate to the history and its antifields, and obtain an even larger space $P^*$, which may be thought of as the phase space corresponding to the extended history space $Q^*$. Introduce canonical momenta $\pi_\alpha =$
\[ \frac{\delta}{\delta \phi^\alpha}, \ \pi^\alpha = \frac{\delta}{\delta \phi^*_\alpha} \] and \( \chi^a = \frac{\delta}{\delta \zeta_a} \), which satisfy the graded canonical commutation relations (\( \phi^\alpha \) is assumed bosonic),

\[ [\pi_\beta, \phi^\alpha] = \delta^\alpha_\beta, \quad \{\pi_\beta^\ast, \phi^*_\alpha\} = \delta^\beta_\alpha, \quad [\chi^a, \zeta_b] = \delta^a_b, \] (6.7)

where \{\cdot, \cdot\} is the symmetric bracket. Let \( \mathcal{P} \) be the phase space of histories with basis \( (\phi^\alpha, \pi_\beta) \), and let \( \mathcal{P}^* \) be the extended phase space with basis \( (\phi^\alpha, \pi_\beta, \phi^*_\alpha, \pi^*_\beta, \zeta_a, \chi^b) \). The definition of the KT differential extends to \( \mathcal{P}^* \) by requiring that \( \delta F = [Q_{KT}, F] \) for every \( F \in C(\mathcal{P}^*) \), where the nilpotent KT operator is

\[ Q_{KT} = \mathcal{E}_a \pi^\alpha_a + r^\alpha_a \phi^*_\alpha \chi^a. \] (6.8)

From this explicit expression we can read off the action of \( \delta \) on the canonical momenta. As explained in [16], the momenta are killed in cohomology, provided that the Hessian \( \delta^2 S / \delta \phi^\alpha \delta \phi^\beta \) is non-singular.

The identity (6.2) implies that \( J_a = r^\alpha_a \pi^\alpha_a \) generate a Lie algebra under the Poisson bracket. Namely, all \( J_a \)'s preserve the action, because

\[ [J_a, S] = r^\alpha_a [\pi^\alpha_a, S] = r^\alpha_a \mathcal{E}_a \equiv 0, \] (6.9)

and the bracket of two operators which preserve some structure also preserves the same structure. We will only consider the case that the \( J_a \)'s generate a proper Lie algebra \( \mathfrak{g} \) as in (2.6). The formalism extends without too much extra work to the more general case of structure functions \( f^{abc}(\phi) \), but we will not need this complication here. It follows that the functions \( r^\alpha_a \) satisfy the identity

\[ \partial_\beta r^\alpha_b r^\beta_a - \partial_\beta r^\beta_a r^\alpha_b = f_{abc} r^\alpha_c. \] (6.10)

The Lie algebra \( \mathfrak{g} \) also acts on the antifields:

\[ [J_a, \phi^\alpha] = r^\alpha_a, \quad [J_a, \phi^*_\alpha] = -\partial_\alpha r^\beta_a \phi^*_\beta, \quad [J_a, \zeta_b] = f_{abc} \zeta_c. \] (6.11)

In particular, it follows that \( \phi^*_\alpha \) carries a \( \mathfrak{g} \) representation because it transforms in the same way as \( \pi^\alpha_a \) does.

Classically, it is always possible to reduce the phase space further, by identifying points on \( \mathfrak{g} \) orbits. To implement this additional reduction, we introduce ghosts \( c^a \) with anti-field number \( \text{afn} c^a = -1 \), and ghost momenta \( b_a \) satisfying \( \{b_a, c^b\} = \delta^b_a \). The Lie algebra \( \mathfrak{g} \) acts on the ghosts as \( [J_a, c^b] = \).
\(-f_{ac}^{\ bc}\). The full extended phase space, still denoted by \(P^*\), is spanned by 
\((\phi^a, \pi^\beta, \phi_\alpha^*, \pi_\alpha^*, \zeta_a, \chi^b, e^a, b_b)\). The generators of \(g\) are thus identified with the following vector fields in \(P^*\):

\[
J_a = r_a^\alpha \pi_{\alpha} - \partial_\alpha r_b^\beta \phi_\beta^{*\alpha} + f_{ab}^{\ c} \zeta_{\alpha}^{b} = J_a^{\text{field}} + J_a^{\text{ghost}}, \tag{6.12}
\]

where \(J_a^{\text{ghost}} = -f_{ab}^{\ c} \zeta_{\alpha}^{b}\) and \(J_a^{\text{field}}\) is the rest.

Now define the longitudinal derivative \(d\) by

\[
d c^a = -\frac{1}{2} f_{bc} c^b c^c, \quad d \phi^\alpha = r_a^{\alpha} c^a, \quad d \phi_\alpha^{*} = \partial_\alpha r_b^\beta \phi_\beta^{*\alpha}, \quad d \zeta_a = -f_{ab}^{\ c} \zeta_{\alpha}^{b}. \tag{6.13}
\]

The longitudinal derivative can be written as \(dF = [Q_{\text{Long}}, F]\) for every \(F \in C(P^*)\), where

\[
Q_{\text{Long}} = J_a^{\text{field}} c^a - \frac{1}{2} f_{ab}^{\ c} c^b c^c = J_a^{\text{field}} c^a + \frac{1}{2} J_a^{\text{ghost}} c^a. \tag{6.14}
\]

We note that \(Q_{\text{Long}}\) can be considered as smeared gauge generators, \(J_X = X^a J_a\), where the smearing function \(X^a\) is the fermionic ghost \(c^a\):

\[
Q_{\text{Long}} = J_c^{\text{field}} + \frac{1}{2} J_c^{\text{ghost}}. \tag{6.15}
\]

One verifies that \(d^2 = 0\) when acting on the fields and antifields by means of the identify (6.10) and the Jacobi identities for \(g\). Moreover, it is straightforward to show that \(d\) anticommutes with the KT differential, \(d\delta = -\delta d\); the proof is again done by checking the action on the fields. Hence we may define the nilpotent \(BRST\) derivative \(s = \delta + d\),

\[
s c^a = -\frac{1}{2} f_{bc} c^b c^c, \quad s \phi^\alpha = r_a^{\alpha} c^a, \quad s \phi_\alpha^{*} = \delta_\alpha + \partial_\alpha r_b^\beta \phi_\beta^{*\alpha}, \quad s \zeta_a = r_a^\alpha \phi_\alpha^{*} - f_{ab}^{\ c} \zeta_{\alpha}^{b}. \tag{6.16}
\]

21
Nilpotency immediately follows because $s^2 = \delta^2 + \delta d + d\delta + d^2 = 0$. The BRST operator can be written in the form $sF = [Q_{BRST},F]$ with

$$Q_{BRST} = Q_{KT} + Q_{Long}$$

$$= \mathcal{E}_\alpha \pi^\alpha + r^\alpha_a \phi^* \chi^a + J^{field}_a c^a + \frac{1}{2} J^{ghost}_a c^a$$

$$= -\frac{1}{2} f_{ab} c^a c^b b_c + r^\alpha_a c^a \pi^\alpha + (\mathcal{E}_\alpha + \partial^\beta r^\alpha_\beta \phi^* c^a) \pi^\alpha$$

$$+ (r^\alpha_a \phi^* - f_{ab} \zeta c^b) \chi^a.$$  

(6.17)

7  MCCQ: Jets and covariant quantization

Like $C(Q^*)$, the space $C(P^*)$ of functions over the extended history phase space decomposes into subspaces $C^k(P^*)$ of fixed antifield number,

$$C(P^*) = \sum_{k=-\infty}^{\infty} C^k(P^*)$$  

(7.1)

The complexes associated with $Q_{BRST}$, $Q_{KT}$, and $Q_{Long}$ now extend to infinity in both directions:

$$\ldots \triangleleft Q \triangleleft C^{-2} \triangleleft Q \triangleleft C^{-1} \triangleleft 0 \triangleleft Q \triangleleft C^0 \triangleleft Q \triangleleft C^1 \triangleleft Q \triangleleft C^2 \triangleleft Q \triangleleft \ldots$$  

(7.2)

It is important that the spaces $C^k(P^*)$ are phase spaces, equipped with the Poisson bracket (6.7). Unlike the resolution (6.5), the new resolution (7.2) therefore allows us to do canonical quantization: replace Poisson brackets by commutators and represent the graded Heisenberg algebra (6.7) on a Hilbert space. However, the Heisenberg algebra can be represented on different Hilbert spaces; there is no Stone-von Neumann theorem in infinite dimension. To pick the correct one, we must impose the physical condition that there is an energy which is bounded from below.

To define the Hamiltonian, we must single out a privileged variable $t$ among the $\alpha$’s, and declare it to be time. However, this amounts to an a-priori foliation of spacetime into space and time, which clashes with the philosophy of diffeomorphism invariance. Therefore, we now introduce the observer’s trajectory $q^\mu(t)$, which locally defines a time direction (namely $\dot{q}^\mu(t)$). Thus, we reformulate the classical theory in jet coordinates. To the fields $c^\alpha(x)$, $\phi^\alpha(x)$, $\phi^*^\alpha(x)$ and $\zeta_\alpha(x)$ we associate $p$-jets $c^\alpha_m(t)$, $\phi^\alpha_m(t)$, $\phi^*_\alpha_m(t)$ and $\zeta_{\alpha,m}(t)$, as in (5.7). The jets have canonical momenta
\[ b_\alpha^m(t) = \delta/\delta c_\alpha^m(t), \pi_\alpha^m(t) = \delta/\delta \phi_\alpha^m(t), \pi_\alpha^m(t) = \delta/\delta \phi_\alpha^m(t) \text{ and } \chi_\alpha^m(t) = \delta/\delta \zeta_\alpha^m(t), \]

defined as usual by commutation relations

\[ [\pi_\beta^n(t), \phi_\alpha^m(t)] = \delta_\alpha^\beta \delta_\alpha^m \delta(t-t'), \]

\[ \{ \pi_\beta^n(t), \phi_\alpha^m(t) \} = \delta_\alpha^\beta \delta_\alpha^m \delta(t-t'), \]

(7.3)

etc. Note that the jet momenta \( \pi_\alpha^m(t) \) are not the Taylor coefficients of the momentum \( \pi_\alpha(x) \); a jet, defined by (5.5), always has a multi-index downstairs.

The jets depend on an additional parameter \( t \), and a Taylor series like (5.5) defines a field \( \phi_\alpha(x,t) \) rather than \( \phi_\alpha(x) \). But the fields in the physical phase space do not depend on the parameter \( t \), wherefore we must impose extra conditions, e.g.

\[ \partial_t \phi_\alpha(x,t) \equiv \frac{\partial \phi_\alpha(x,t)}{\partial t} = 0. \]

(7.4)

We can implement this condition by introducing new antifields \( \tilde{\phi}_\alpha(x,t) \). However, the identities \( \partial_t \mathcal{E}_\alpha(x,t) \equiv 0 \) give rise to unwanted cohomology. To kill this condition, we must introduce yet another antifield \( \tilde{\phi}_\alpha^*(x,t) \), etc., as described in detail in [16]. This means that our extended jet space \( Q^* \) is spanned by the jets

\[ c_\alpha^m(t), \phi_\alpha^m(t), \phi_\alpha^m(t), \zeta_\alpha^m(t), \tilde{c}_\alpha^m(t), \tilde{\phi}_\alpha^m(t), \phi_\alpha^m(t), \phi_\alpha^m(t), \zeta_\alpha^m(t), \]

and that the extended jet phase space \( P^* \) in addition has basis vectors

\[ b_\alpha^m(t), \pi_\alpha^m(t), \pi_\alpha^m(t), \chi_\alpha^m(t), \tilde{b}_\alpha^m(t), \tilde{\pi}_\alpha^m(t), \tilde{\pi}_\alpha^m(t), \tilde{\chi}_\alpha^m(t). \]

The cohomology is not changed by the new, barred, variables, since all we have done is to introduce an extra variable \( t \) and then eliminate it in cohomology.

The Taylor expansion requires that we introduce the observer’s trajectory as a physical field, but what equation of motion does it obey? The obvious answer is the geodesic equation, which we compactly write as \( G_\mu(t) = 0 \). The geodesic operator \( G_\mu(t) \) is a function of the metric \( g_{\mu\nu}(q(t),t) \) and its derivatives on the curve \( q^\mu(t) \). To eliminate this ideal in cohomology we introduce the trajectory antifield \( q^*_\mu(t) \), and extend the KT differential to it:

\[ \delta q^\mu(t) = 0, \]

\[ \delta q^*_\mu(t) = G_\mu(t). \]

(7.5)
For models defined over Minkowski spacetime, the geodesic equation simply becomes $\ddot{q}^\mu(t) = 0$, and the KT differential reads

$$\delta q^\mu(t) = \eta_{\mu\nu} q^\nu(t).$$

(7.6)

$H_{cl}^0(\delta)$ only contains trajectories which are straight lines,

$$q^\mu(t) = u^\mu t + a^\mu,$$

(7.7)

where $u^\mu$ and $a^\mu$ are constant vectors. We may also require that $u^\mu$ has unit length, $u_\mu u^\mu = 1$. This condition fixes the scale of the parameter $t$ in terms of the Minkowski metric, so we may regard it as proper time rather than as an arbitrary parameter. We define momenta $p_\mu(t) = \delta/\delta q^\mu(t)$ and $p^*_\mu(t) = \delta/\delta q^*_\mu(t)$ for the observer’s trajectory and its antifield.

We can now define a genuine Hamiltonian $H$, which translates the fields relative to the observer or vice versa. Since the formulas are shortest when $H$ acts on the trajectory but not on the jets, we make that choice, and define

$$H = i \int dt (\dot{q}^\mu(t)p_\mu(t) + \dot{q}^*_\mu(t)p^*_\mu(t)).$$

(7.8)

Note the sign; moving the fields forward in $t$ is equivalent to moving the observer backwards. From (5.5) we get the energy of the fields:

$$[H, \phi^\alpha(x,t)] = -i \dot{q}^\mu(t) \partial_\mu \phi^\alpha(x,t).$$

(7.9)

This a crucial result, because it allows us to define a genuine energy operator in a covariant way. In Minkowski space, the trajectory is a straight line (7.7), and $\dot{q}^\mu(t) = u^\mu$. If we take $u^\mu$ to be the constant four-vector $u^\mu = (1, 0, 0, 0)$, then (7.9) reduces to

$$[H, \phi^\alpha(x,t)] = -i \partial_\mu \phi^\alpha(x,t).$$

(7.10)

Equation (7.8) is thus a genuine covariant generalization of the energy operator.

Now we quantize the theory. Since all operators depend on the parameter $t$, we can define the Fourier components as in (5.13). The the Fock vacuum $|0\rangle$ is defined to be annihilated by all negative frequency modes, $\phi^\alpha_m(\omega)$, $q^\mu_m(\omega)$, etc. with $\omega \sim m < 0$. The normal-ordered form of the Hamiltonian (7.8) reads, in Fourier space,

$$H = -\sum_{m=-\infty}^{\infty} m (\pm q^\mu(m)p_\mu(-m) + q^*_\mu(m)p^*_\mu(-m)\).$$

(7.11)
where double dots indicate normal ordering with respect to frequency. This ensures that $H|0\rangle = 0$. The classical phase space $H^0_{\text{cl}}(\delta)$ is thus the space of fields $\phi^\alpha(x)$ which solve $\mathcal{E}_\alpha(x) = 0$, and trajectories $q^\mu(t) = u^\mu t + a^\mu$, where $u^2 = 1$. After quantization, the fields and trajectories become operators which act on the physical Hilbert space $\mathcal{H} = H^0_{\text{qm}}(Q_{KT})$, which is the space of functions of the positive-energy modes of the classical phase space variables.

This construction differs technically from conventional canonical quantization, but there is also a physical difference. Consider the state $|\phi^\alpha(x)\rangle = \phi^\alpha(x)|0\rangle$ which excites one $\phi$ quantum from the vacuum. The Hamiltonian yields

$$H|\phi^\alpha(x)\rangle = -i\hat{q}^\mu(t)\partial_\mu\phi^\alpha(x)|0\rangle = -i\hat{q}^\mu(t)\partial_\mu\phi^\alpha(x)$$  \hspace{1cm} (7.12)

If $u^\mu$ were a classical variable, the state $|\phi^\alpha(x)\rangle$ would be a superposition of energy eigenstates:

$$H|\phi^\alpha(x)\rangle = -iu^\mu\partial_\mu|\phi^\alpha(x)\rangle.$$  \hspace{1cm} (7.13)

In particular, let $u^\mu = (1, 0, 0, 0)$ be a unit vector in the $x^0$ direction and $\phi^\alpha(x) = \exp(ik \cdot x)$ be a plane wave. We then define the state $|0; u, a\rangle$ by

$$q^\mu(t)|0; u, a\rangle = (u^\mu t + a^\mu)|0; u, a\rangle.$$  \hspace{1cm} (7.14)

Now write $|k; u, a\rangle = \exp(ik \cdot x)|0; u, a\rangle$ for the single-quantum energy eigenstate.

$$H|k; u, a\rangle = k^\mu u_\mu|k; u, a\rangle,$$  \hspace{1cm} (7.15)

so the eigenvalue of the Hamiltonian is $k^\mu u_\mu = k_0$, as expected. Moreover, the lowest-energy condition ensures that only quanta with positive energy will be excited; if $k^\mu u_\mu < 0$ then $|k; u, a\rangle = 0$.

8 MCCQ: Gauge anomalies

To be concrete, consider the case that the symmetry is the DGRO algebra $\mathfrak{h}$. To each symmetry, we assign ghosts as in the following table:

| Gen Smear | Ghost | Momentum | $Q_{\text{Long}}$ |
|-----------|-------|----------|------------------|
| Diffeomorphisms | $\xi^\mu(x)$ | $\partial^\mu_{\text{diff}}(x,t)$ | $b^\mu_{\text{diff}}(x,t)$ | $Q_{\text{diff}}$ |
| $\mathcal{J}_X$ | $X^a(x)$ | $\partial^a_{\text{gauge}}(x,t)$ | $b^a_{\text{gauge}}(x,t)$ | $Q_{\text{gauge}}$ |
| Reparametrizations | $f(t)$ | $c_{\text{rep}}(t)$ | $b_{\text{rep}}(t)$ | $Q_{\text{rep}}$ | $Q_{\text{Long}}$ |
The BRST operator is $Q_{BRST} = Q_{Long} + Q_{KT}$, where the longitudinal operator is given by the prescription (6.15). For brevity, we only write down the formulas for the fields $\phi^\alpha(x,t)$ and the ghosts; the antifields do of course give rise to additional terms.

\[
Q_{Long}^{diff} = - \int d^N x \int dt \left\{ (c^{\mu}_{\text{diff}}(x,t) \partial^\mu \phi^\alpha(x,t) \\
+ \partial^\nu c^{\mu}_{\text{diff}}(x,t) T_{\beta\mu}^{\nu} \phi^\beta(x,t) ) \pi_\alpha(x,t) \right. \\
+ c^{\mu}_{\text{diff}}(x,t) \partial^\mu c^{\nu}_{\text{diff}}(x,t) b^{\text{diff}}(x,t) \right\},
\]

\[
Q_{Long}^{gauge} = - \int d^N x \int dt \left\{ c^a_{\text{gauge}}(x,t) J_{\beta a}^\alpha \phi^\beta(x,t) \pi_\alpha(x,t) \\
+ \frac{1}{2} f_{ab} c^a_{\text{gauge}}(x,t) c^b_{\text{gauge}}(x,t) b^{\text{gauge}}_c(x,t) \right\}
\]

\[
Q_{Long}^{rep} = - \int d^N x \int dt \left\{ (c_{\text{rep}}(t) \partial_t \phi^\alpha(x,t) \\
+ \lambda(c_{\text{rep}}(t) - i c_{\text{rep}}(t)) \phi^\alpha(x,t) ) \pi_\alpha(x,t) \right. \\
- \int dt c_{\text{rep}}(t) c_{\text{rep}}(t) b^{\text{rep}}(t) \right\}.
\]

These formulas assume that the field $\phi^\alpha(x)$ transforms as a tensor field. There is an additional term if the field is a connection, but this terms does not lead to any complications. After passage to jet space and normal ordering, we use the prescription (6.15) to find the longitudinal derivative, i.e.
\[ \partial_m \xi^\mu \to e^\mu_{diff,m}, \quad \partial_m X^a \to e^a_{gauge,m}, \quad \text{and } f \to c_{rep} : \]

\[
Q^\text{diff}_{Long} = \int dt \left\{ c^\mu_{diff,0} p_\mu(t) - \sum_{|n| \leq |m| \leq p} \pi^m_n (c^\mu_{diff}(t)) \phi^\beta_n(t) \phi^\mu_n(t) : \right\},
\]

\[
Q^\text{gauge}_{Long} = - \int dt \left\{ \sum_{|n| \leq |m| \leq p} J^\mu_n (c_{gauge}(t)) \phi^\mu_n(t) \pi^m_n(t) : \right\},
\]

\[
Q^\text{rep}_{Long} = - \int dt \left\{ \sum_{|m| \leq p} c_{rep}(t) \phi^\mu_n(t) \pi^m_n(t) : \right\} + \lambda \sum_{|m| \leq p} (c_{rep}(t) - ic_{rep}(t)) \phi^\mu_n(t) \pi^m_n(t) : \right\} + c_{rep}(t) c_{rep}(t) b^{rep}(t) : \right\}.
\]

The matrices are given by (cf. (5.211))

\[
T^\mu_n(c_{diff}(t)) \equiv (T^\mu_{\beta n}(c_{diff}(t))) = \sum_{\mu'} \left( \begin{array}{c} n \\ m \end{array} \right) e^\mu_{\mu', n-m+\mu}(t) T^\mu_{\mu'} + \sum_{\mu'} \left( \begin{array}{c} n \\ m - \mu \end{array} \right) e^\mu_{\mu', n-m+\mu}(t) - \sum_{\mu'} \delta^m_{\mu'} e^\mu_{\mu', 0}(t),
\]

\[
J^\mu_n(c_{gauge}(t)) \equiv (J^\mu_{\beta n}(c_{gauge}(t))) = \left( \begin{array}{c} n \\ m \end{array} \right) e^a_{gauge, n-m}(t) J_a. \] (8.3)

\[ T^\mu_{\beta n} \] and \[ J^\mu_{\beta n} \] denote the specializations of \( T^\mu_{\beta n} \) and \( J^\mu_{\beta n} \) to the adjoint representations; \( \sum_m T^\mu_{\beta n}(c_{\mu'}) e^\nu_{\mu', m} = (c_{\mu'} e^\nu_{\mu', n}) \) and \( \sum_m J^\mu_{\beta n}(c_{\mu'}) e^b_{\mu', m} = (c_{\mu'} e^b_{\mu', n}) \).

The condition for \( Q^2_{Long} = 0 \), and thus \( Q^2_{BRST} = 0 \), is that the algebra generated by the normal-ordered gauge generators is anomaly free. However, even if this condition fails, which is the typical situation, everything is not lost. The KT operator is still nilpotent, and we can implement dynamics as the KT cohomology in the extended phase space without ghosts. The physical phase space now grows, because some gauge degrees of freedom become physical upon quantization.
9 Gravity

Finally we are ready to apply the MCCQ formalism to general relativity. For simplicity we consider only pure gravity. The only field is the symmetric metric $g_{\mu\nu}(x)$. The inverse $g^{\mu\nu}$, the determinant $g = \det(g_{\mu\nu})$, the Levi-Civita connection $\Gamma_{\nu\rho}^{\mu}$, Riemann’s curvature tensor $R^\rho_{\sigma\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$ and the Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ are defined as usual. The covariant derivative is

$$\nabla_\mu = \partial_\mu + \Gamma_{\nu\rho}^{\mu}T_\rho^{\nu},$$

(9.1)

where $T_\nu^\mu$ are finite-dimensional matrices satisfying $gl(N)$. The Einstein action

$$S_E = \frac{1}{16\pi} \int d^4x \sqrt{g(x)}R(x).$$

(9.2)

leads to Einstein’s equation of motion

$$G^{\mu\nu}(x) = 0,$$

(9.3)

which is subject to the identity

$$\nabla_\nu G^{\mu\nu}(x) \equiv 0$$

(9.4)

We introduce a fermionic antifield $g^{\mu\nu}_* (x)$ for (9.3), a bosonic second-order antifield $\zeta_\mu(x)$ for (9.4), and a ghost $\chi_\mu(x)$ to eliminate diffeomorphisms. The total field content in the extended history phase space is thus

| afn | Field | Momentum | Parity |
|-----|-------|----------|--------|
| $-1$ | $c^{\mu}_{diff}(x)$ | $b^{\mu}_{diff}(x)$ | $F$ |
| $0$  | $g_{\mu\nu}(x)$ | $\pi^{\mu\nu}(x)$ | $B$ |
| $1$  | $g_{\mu\nu}^*(x)$ | $\pi^{\mu\nu}_*(x)$ | $F$ |
| $2$  | $\zeta_\mu(x)$ | $\chi_\mu(x)$ | $B$ |

The KT differential $\delta$ is defined by

$$\delta c^{\mu}_{diff}(x) = 0,$$

$$\delta g_{\mu\nu}(x) = 0,$$

$$\delta g^{\mu\nu}_*(x) = G^{\mu\nu}(x),$$

$$\delta \zeta_\mu(x) = \nabla_\nu G^{\mu\nu}(x),$$

(9.5)
i.e. the KT operator is

\[ Q_{KT} = \int d^4x \ (G^{\mu\nu}(x)\pi^*_\mu(x) + \nabla_\nu G^{\mu\nu}(x)\chi_\mu(x)). \]  

(9.6)

The longitudinal operator was written down in (8.1), \( Q_{\text{Long}} = Q_{\text{diff}}^{\text{Long}} \), and the BRST operator is the sum of the KT and the longitudinal operators, as usual.

We now quantize by passing to jet space, introducing a Fock vacuum that is annihilated by the negative frequency modes of all fields and antifields, and normal ordering. The fields are symmetric tensor fields, i.e. they correspond to the symmetric \( gl(N) \) modules \( S_\ell \) and \( S^{(\ell)} \) in (5.18). The values of the parameters in (5.20) are

| \( \text{afn} \) | Field | \( q \) | \( u \) | \( v \) | \( w \) | \( x \) | \( p \) |
|---|---|---|---|---|---|---|---|
| -1 | \( c^\mu_{\text{diff}}(x) \) | \( S^1 \) | 1 | 0 | -1 | \( N \) | \( p - 2 \) |
| 0 | \( g_{\mu\nu}(x) \) | \( S_2 \) | -(\( N + 2 \)) | -1 | -(\( N + 1 \)) | \( -N(N + 1)/2 \) | \( p \) |
| 1 | \( g^\mu_{\nu}(x) \) | \( S^2 \) | \( (N + 2) \) | 1 | -(\( N + 1 \)) | \( N(N + 1)/2 \) | \( p - 2 \) |
| 2 | \( \zeta^\mu(x) \) | \( S^1 \) | -1 | 0 | 1 | \( -N \) | \( p - 3 \) |

The parameters were written down in arbitrary dimension \( N \) for generality, although we are primarily interested in the physical case \( N = 4 \). The last column is the truncation order for the corresponding jets. That the antifields with \( \text{afn} = 1 \) and 2 should be truncated at order \( p - 2 \) and \( p - 3 \), respectively, follow because Einstein’s equation is second order and the identity (9.4) third order. That the ghost contribution should be truncated at order \( p - 2 \) is less clear, and I have made a different assignment elsewhere [18]. The reason why I now favor \( p - 2 \) is that this leads to nice cancellation of infinities.

The diffeomorphism anomalies are now read off from (5.19); for definiteness, we only consider \( c_1 \). The abelian charge \( c_1 = 1 + c^\text{ghost}_1 + c^g_1 + c^{g*}_1 + c^\zeta_1 \), where

\[
\begin{align*}
    c^\text{ghost}_1 &= -\left(\frac{N + p - 2}{N}\right) - N\left(\frac{N + p - 1}{N + 2}\right), \\
    c^g_1 &= (N + 2)\left(\frac{N + p}{N}\right) + \frac{N(N + 1)}{2}\left(\frac{N + p + 1}{N + 2}\right), \\
    c^{g*}_1 &= -(N + 2)\left(\frac{N + p - 2}{N}\right) - \frac{N(N + 1)}{2}\left(\frac{N + p - 1}{N + 2}\right), \\
    c^\zeta_1 &= \left(\frac{N + p - 3}{N}\right) + N\left(\frac{N + p - 2}{N + 2}\right).
\end{align*}
\]

(9.7)
It is clear that $c_1$ does not vanish for generic $p$. The longitudinal operator \[ (8.2) \] thus acquires an anomaly, and we can only implement the KT cohomology. Hence the ghost plays no direct role, and I have previously argued that should be discarded. Nevertheless, it softens the divergence in the $p \to \infty$ limit, since

$$c_1^{\text{ghost}} + c_1^r = -\left(\frac{N + p - 3}{N - 1}\right) - N \left(\frac{N + p - 2}{N + 1}\right)$$  \hspace{1cm} (9.8)$$

only diverges like $p^{N+1}/(N+1)!$, while the leading term in (9.7) is proportional to $p^{N+2}/(N+2)!$.

The quantum KT operator becomes

$$Q_{KT} = \int dt \left\{ \sum_{|m| \leq p-2} :G_{\nu m}(t)\pi_{\nu \mu}(t): + \sum_{|m| \leq p-3} :\left(\nabla_{\nu} G^{\mu \nu}\right)_{m}(t)\chi_{\mu}(t): \right\},$$  \hspace{1cm} (9.9)$$

where $G_{\nu m}(t)$ and $(\nabla_{\nu} G^{\mu \nu})_{m}(t)$ are the corresponding jets.

10 Finiteness conditions

In the previous section we applied the MCCQ formalism to gravity, and found a well-defined but anomalous action of the DGRO algebra. However, the passage to the space of $p$-jets amounts to a regularization. The regularization is unique in that it preserves the full constraint algebra, but it must nevertheless be removed in the end. In order to reconstruct the original field by means of the Taylor series (5.5), we must take the limit $p \to \infty$. A necessary condition for taking this limit is that the abelian charges have a finite limit.

Taken at face value, the prospects for succeeding appear bleak. When $p$ is large, $\binom{m+p}{n} \approx p^n/n!$, so the abelian charges (5.19) diverge; the worst case is $c_1 \approx c_2 \approx p^{N+2}/(N+2)!$, which diverges in all dimensions $N > -2$. In [13] a way out of this problem was devised: consider a more general realization by taking the direct sum of operators corresponding to different values of the jet order $p$. Take the sum of $r+1$ terms like those in (5.12), with $p$ replaced by $p$, $p-1$, ..., $p-r$, respectively, and with $\varrho$ and $M$ replaced by $\varrho^{(i)}$ and $M^{(i)}$ in the $p - i$ term.

Such a sum of contributions arises naturally from the KT and BRST complexes, because the antifields are only defined up to an order smaller
than $p$ (e.g. $p - 1$ or $p - 2$). Denote the numbers $u, v, w, x, y$ in the modules $g^{(i)}$ and $M^{(i)}$, defined as in [5,20], by $u_i, v_i, w_i, x_i, y_i$, respectively. Of course, there is only one contribution from the observer’s trajectory. Then it was shown in [13], Theorem 3, that

$$
c_1 = -U \binom{N + p - r}{N - r}, \quad c_2 = -V \binom{N + p - r}{N - r},
$$

$$
c_3 = W \binom{N + p - r}{N - r}, \quad c_4 = -X \binom{N + p - r}{N - r},
$$

$$
c_5 = Y \binom{N + p - r}{N - r},
$$

where $u_0 = U, v_0 = V, w_0 = W, x_0 = X$ and $y_0 = Y$, provided that the following conditions hold:

$$
u_i + (-)^i \binom{r - 2}{i - 2} X = (-)^i \binom{r}{i} U,
$$

$$
v_i - 2(-)^i \binom{r - 1}{i - 1} W - (-)^i \binom{r - 2}{i - 2} X = (-)^i \binom{r}{i} V,
$$

$$
w_i - (-)^i \binom{r - 1}{i - 1} X = (-)^i \binom{r}{i} W,
$$

$$
x_i = (-)^i \binom{r}{i} X,
$$

$$
y_i = (-)^i \binom{r}{i} Y.
$$

The contributions from the observer’s trajectory have also been eliminated by antifields coming from the geodesic equation; this is not important in the sequel because these contributions are finite anyway.

Let us now consider the solutions to (10.2) for the numbers $x_i$, which can be interpreted as the number of fields and anti-fields. First assume that the field $\phi^\alpha_{\mathbf{m}}(t)$ is fermionic with $x_F$ components, which gives $x_0 = x_F$. We may assume, by the spin-statistics theorem, that the EL equations are first order, so the bosonic antifields $\phi^*_{\mathbf{m}}(t)$ contribute $-x_F$ to $x_1$. The barred antifields $\bar{\phi^\alpha_{\mathbf{m}}}(t)$ are also defined up to order $p - 1$, and so give $x_1 = -x_F$, and the barred second-order antifields $\bar{\phi^*_{\mathbf{m}}}(t)$ give $x_2 = x_F$.

For bosons the situation is analogous, with some exceptions: all signs are reversed, and the EL equations are assumed to be second order. Hence $\phi^\alpha_{\mathbf{m}}(t)$ yields $x_0 = -x_B$ and $\phi^*_{\mathbf{m}}(t)$ yields $x_2 = x_B$. Moreover, we have the second-order antifields $\zeta_{\mathbf{m}}(t)$ which contribute $x_3 = -x_G$, and the ghosts
We assume, only because this choice will lead to cancellation of unwanted terms, that the ghosts are truncated at order \( p-2 \), and thus they contribute \( x_2 = x_G \). Again, there are barred antifields of opposite parity at one order higher.

Previously I have argued \[14\] that the fermionic EL equations should also have \( x_S \) gauge symmetries, i.e. there are also fermionic second-order antifields \( \zeta_{a,m}(t) \) which give \( x_2 = x_S \). The motivation for this assumption was that a non-zero value for \( x_S \) is necessary to cancel some infinites in the \( p \to \infty \) limit. However, \( x_S = 0 \) in all established theories (a non-zero \( x_S \) may be thought of as some kind of supersymmetry), so this assumption was a major embarrassment. Because it turns out that infinites cancel precisely when \( x_S = x_G \), the role of these hypothetical fermionic second-order antifields can be played by the ghosts. This is the reason why I now favor that the ghosts be kept, despite \( Q_{BRST}^2 \neq 0 \), and that they should be truncated at one order higher than \( \zeta_{a,m}(t) \).

The situation is summarized in the following tables, where the upper half is valid if the original field is fermionic and the lower half if it is bosonic:

| afn | Jet | Order | \( x \) |
|-----|-----|-------|--------|
| 0   | \( \phi_{a,m}^\alpha(t) \) | \( p \) | \( x_F \) |
| 1   | \( \bar{\phi}_{a,m}^\alpha(t) \) | \( p-1 \) | \( -x_F \) |
| 1   | \( \phi_{a,m}^*(t) \) | \( p-1 \) | \( -x_F \) |
| 2   | \( \bar{\phi}_{a,m}^*(t) \) | \( p-2 \) | \( x_F \) |
| 0   | \( \phi_{a,m}^\alpha(t) \) | \( p \) | \( -x_B \) |
| 1   | \( \bar{\phi}_{a,m}^\alpha(t) \) | \( p-1 \) | \( x_B \) |
| 1   | \( \phi_{a,m}^*(t) \) | \( p-2 \) | \( x_B \) |
| 2   | \( \bar{\phi}_{a,m}^*(t) \) | \( p-3 \) | \( -x_B \) |
| -1  | \( c_{a,m}^\alpha(t) \) | \( p-2 \) | \( x_G \) |
| 0   | \( \bar{c}_{a,m}^\alpha(t) \) | \( p-3 \) | \( -x_G \) |
| 2   | \( \zeta_{a,m}(t) \) | \( p-3 \) | \( -x_G \) |
| 3   | \( \bar{\zeta}_{a,m}(t) \) | \( p-4 \) | \( x_G \) |

If we add all contributions of the same order, we see that the fourth relation
in (10.2) can only be satisfied provided that

\begin{align*}
  p & : \ x_F - x_B = X \\
  p - 1 & : \ -2x_F + x_B = -rX, \\
  p - 2 & : \ x_B + x_F + x_G = \left(\frac{r}{2}\right)X, \\
  p - 3 & : \ -x_B - 2x_G = -\left(\frac{r}{3}\right)X, \\
  p - 4 & : \ x_G = \left(\frac{r}{4}\right)X, \\
  p - 5 & : \ 0 = -\left(\frac{r}{5}\right)X, ...
\end{align*}

(10.4)

The last equation holds only if \( r \leq 4 \) (or trivially if \( X = 0 \)). On the other hand, if we demand that there is at least one bosonic gauge condition, the \( p - 4 \) equation yields \( r \geq 4 \). Such a demand is natural, because both the Maxwell/Yang-Mills and the Einstein equations have this property. Therefore, we are unambiguously guided to consider \( r = 4 \) (and thus \( N = 4 \)). The specialization of (10.4) to four dimensions reads

\begin{align*}
  p & : \ x_F - x_B = X \\
  p - 1 & : \ -2x_F + x_B = -4X, \\
  p - 2 & : \ x_B + x_F + x_G = 6X, \\
  p - 3 & : \ -x_B - 2x_G = -4X, \\
  p - 4 & : \ x_G = X.
\end{align*}

(10.5)

Clearly, the unique solution to these equations is

\begin{align*}
  x_F = 3X, \quad x_B = 2X, \quad x_G = X.
\end{align*}

(10.6)

The solutions to the remaining equations in (10.2) are found by analogous reasoning. The result is

\begin{align*}
  u_B &= 2U \\
  v_B &= 2V + 2W \\
  u_F &= 3U \\
  v_F &= 3V + 2W \\
  u_G &= U - X \\
  v_G &= V + 2W + X \\
  w_B &= 2W + X \\
  y_B &= 2Y \\
  w_F &= 3W + X \\
  y_F &= 3Y \\
  w_G &= W + X \\
  y_G &= Y
\end{align*}

(10.7)
This result expresses the fifteen parameters \( x_B - w_G \) in terms of the five parameters \( X, Y, U, V, W \). For this particular choice of parameters, the abelian charges in (10.8) are given by
\[
c_1 = -U, \quad c_2 = -V, \quad c_3 = W, \quad c_4 = -X, \quad c_5 = Y, \quad (10.8)
\]
independent of \( p \). Hence there is no manifest obstruction to the limit \( p \to \infty \).

11 Comparison with known physics

The analysis in the previous section yields highly non-trivial constraints on the possible field content, if we demand that it should be possible to remove the jet regularization. It is therefore interesting to compare the numbers in (10.6), with the field content of gravity coupled to the standard model in four dimensions. This is a slightly modified form of an analysis given in [14, 15].

The bosonic content of the theory is given by the following table. Standard notation for the fields is used, and one must remember that it is the naive number of components that enters the equation, not the gauge-invariant physical content. E.g., the photon is described by the four components \( A^\mu \) rather than the two physical transverse components. Also, the gauge algebra \( su(3) \oplus su(2) \oplus u(1) \) has 8 + 3 + 1 = 12 generators.

| Field | Name | EL equation | \( x_B \) |
|-------|------|-------------|--------|
| \( A^\mu_a \) | Gauge bosons | \( D^\mu_a F^{a\mu\nu} = j^{a\mu} \) | \( 12 \times 4 = 48 \) |
| \( g_{\mu\nu} \) | Metric | \( G^{\mu\nu} = \frac{1}{8\pi} T^{\mu\nu} \) | \( 10 \) |
| \( H \) | Higgs field | \( g^{\mu\nu} \partial_\mu \partial_\nu H = V(H) \) | \( 2 \) |

(11.1)

The total number of bosons in the theory is thus \( x_B = 48 + 10 + 2 = 60 \), which implies \( X = 30 \) by (10.6). The number of gauge conditions is \( x_G = 16 \), which implies \( X = 16 \). There is certainly a discrepancy here.

The fermionic content in the first generation is given by

| Field | Name | EL equation | \( x_F \) |
|-------|------|-------------|--------|
| \( u \) | Up quark | \( \not\!\!D u = ... \) | \( 2 \times 3 = 6 \) |
| \( d \) | Down quark | \( \not\!\!D d = ... \) | \( 2 \times 3 = 6 \) |
| \( e \) | Electron | \( \not\!\!D e = ... \) | \( 2 \) |
| \( \nu_L \) | Left-handed neutrino | \( \not\!\!D \nu_L = ... \) | \( 1 \) |

(11.2)
The number of fermions in the first generation is thus $x_F = 6 + 6 + 2 + 1 = 15$. Counting all three generations and anti-particles, we find that the total number of fermions is $x_F = 2 \times 3 \times 15 = 90$, which implies $X = 30$.

The prediction that spacetime has $N = 4$ dimensions is of course nice, but the values for $X$ (30, 16, 30) are not mutually consistent. One could think of various modifications. By identifying $x_i$ with the number of field components, we have assumed that the weight $\lambda = 0$ in (5.10). A non-zero $\lambda$ would modify the abelian charge. Unfortunately, this is only possible for the fermions which obey linear equations of motion, so the discrepancy $x_B \neq 2x_G$ remains. Another problem, found in [18], is that the gauge anomaly $c_5$ does not have a finite $p \to \infty$ limit for reasonable choices of field content.

Hence it is presently unclear how to remove the regulator and take the field limit, and this is of course a major unsolved problem. Nevertheless, it should be emphasized that already the regularized theories carry representations of the full gauge and diffeomorphism algebras. One may also hope that some overlooked idea may improve the situation, as happened with the inclusion of ghosts.

12 Conceptual issues

One of the most important tasks of any putative quantum theory of gravity is to shed light on the various conceptual difficulties which arise when the principles of quantum mechanics are combined with general covariance [2, 24]. These issues include:

1. In conventional canonical quantization, the canonical commutation relations are defined on a “spacelike” surface. However, a surface is spacelike w.r.t. some particular spacetime metric $g_{\mu\nu}$, which is itself a quantum operator.

2. Microcausality requires that the field variables defined in spacelike separated regions commute. Again, it is unclear what this means when the notion of spacelikeness is dynamical.

3. Different choices of foliation lead to a priori different quantum theories, and it by no means clear that these are unitarily equivalent.

4. The problem of time: The Hamiltonian of general relativity is a first class constraint, hence it vanishes on the reduced phase space. This
means that there is no notion of time evolution among diffeomorphism-invariant degrees of freedom.

5. The arrow of time: Dynamics is invariant under time reversal (combined with CP reversal), but we nevertheless experience a difference between past and future.

6. The notion of time as a causal order is lost. This is not really a problem in the classical theory, where one can solve the equations of motion first, but in quantum theory causality is needed from the outset.

7. QFT rests on two pillars: quantum mechanics and locality. However, locality is at odds with diffeomorphism invariance underlying gravity; “there are no local observables in quantum gravity”.

Let us see how MCCQ addresses these conceptual issues.

1. The canonical commutation relations are defined throughout the history phase space $\mathcal{P}$, and hence not restricted to variables living on a spacelike surface. Dynamics is implemented as a first class constraint in $\mathcal{P}$. Only if we solve this constraint prior to quantization need we restrict quantization to a spacelike surface.

2. By passing to $p$-jet space, we eliminate the notion of spacelikeness altogether. The $p$-jets live on the observer’s trajectory, and the observer moves along a timelike curve. It might seem strange to dismiss the notion of spacelike separation, but distant events can never be directly observed, and a physical theory only needs to describe directly observable events. What can be observed are indirect effects of distant events. E.g., a terrestrial detector does not directly observe the sun, but only photons emanating from the sun eight light-minutes ago. The detector signals are of course compatible with the existence of the sun, but a physical theory only needs to deal with directly observed events, i.e. the absorption of photons in the detector.

3. In MCCQ there is no foliation, but rather an explicit observer, or detector. The theory is unique since the observer’s trajectory is a quantum object; we do not deal with a family of theories parametrized by the choice of observer, but instead the observer’s trajectory is represented on the Hilbert space in the same way as the quantum fields.

4. By introducing an explicit observer, we can define a genuine energy operator which translates the fields relative to the observer, or
vice versa. In contrast, there is also a Hamiltonian constraint, which translates both the observer and the fields the same amount. This constraint is killed in KT cohomology and is thus identically zero on physical observables.

5. The Fock vacuum $|0\rangle$ \[5.15\] treats positive and negative energy modes differently, thus introducing an asymmetry between forward and backward time translations.

6. The $p$-jets live on the observer’s trajectory $q(t)$ and are thus causally related; causal order is defined by the parameter $t$. The relation between this order and the fields is encoded in the geodesic equation.

7. As we saw in Section 2, locality is compatible with infinite-dimensional spacetime symmetries, but only in the presence of an anomaly. This is the key lesson from CFT.

It is gratifying that the MCCQ formalism yields natural explanations of many of the conceptual problems that plague quantum gravity.

13 Conclusion

The key insight underlying the present work is that the process of observation must be localized in spacetime in order to be compatible with the philosophy of QFT. The innocent-looking introduction of the observer’s trajectory leads to dramatic consequences, because new gauge and diffeomorphism anomalies arise. On the mathematical side, this construction leads to well-defined realizations of the constraint algebra generators as operators on a linear space, as least for the regularized theory.

We have also further developed the manifestly covariant canonical quantization method, based on the form of the DGRO algebra modules, which was introduced in [16, 18]. This formalism is convenient due to its relation to representation theory, but it is presumably possible to repeat the analysis in any sensible quantization scheme, at the cost of additional work. In contrast, the introduction of the observer’s trajectory is absolutely crucial, because the new anomalies can not be formulated without it. Anomalies matter!

Four critical problems remain to be solved. As was discussed in Section 10, the original fields must be reconstructed from the $p$-jets, i.e. we must take the limit $p \to \infty$. This limit is problematic because the abelian charges diverge. Second, the issue of unitarity needs to be understood. So far we only
noted that an extension is necessary for unitarity by restriction to Virasoro subalgebras, and then we proceeded to construct anomalous representations. The main problem is to find an invariant inner product. Third, perturbation theory and renormalization must be transcribed to this formalism, to make contact with numerical predictions of ordinary QFT. Finally, we know from CFT that reducibility conditions analogous to Kac’ formula are needed in physically interesting situations. Unfortunately, none of these problems appears to be easy.

References

[1] L. Bonora, P. Pasti and M. Tonin, The anomaly structure of theories with external gravity, J. Math. Phys. 27 (1986) 2259–2270.

[2] S. Carlip, Quantum gravity: a progress report, gr-qc/0108040

[3] S. Carlip, Conformal Field Theory, (2+1)-Dimensional Gravity, and the BTZ Black Hole, gr-qc/0503022

[4] B.S. de Witt, Quantum theory of gravity II: the manifestly covariant theory, Phys. Rev. D. 162 (1967) 1195.

[5] A. Dzhumadildaev, Virasoro type Lie algebras and deformations, Z. Phys. C 72 (1996) 509–517.

[6] P. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal field theory, New York: Springer-Verlag, 1996

[7] P. Goddard and D. Olive, Kac-Moody and Virasoro algebras in relation to quantum physics. Int. J. Mod. Phys. 1 (1986) 303–414.

[8] M. Henneaux and C. Teitelboim, Quantization of gauge systems, Princeton Univ. Press (1992)

[9] C. J. Isham and N. Linden, Continuous histories and the history group in generalised quantum theory, J. Math. Phys. 36 (1995) 5392.

[10] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219.

[11] T.A. Larsson, Central and non-central extensions of multi-graded Lie algebras, J. Phys. A. 25 (1992) 1177–1184.

[12] T.A. Larsson, Extended diffeomorphism algebras and trajectories in jet space. Comm. Math. Phys. 214 (2000) 469–491. math-ph/9810003
[13] T.A. Larsson, *Multi-dimensional diffeomorphism and current algebras from Virasoro and Kac-Moody Currents*, [math-ph/0101007](http://arxiv.org/abs/math-ph/0101007) (2001)

[14] T.A. Larsson, *Koszul-Tate cohomology as lowest-energy modules of non-centrally extended diffeomorphism algebras*, [math-ph/0210023](http://arxiv.org/abs/math-ph/0210023) (2002)

[15] T.A. Larsson, *Multi-dimensional Virasoro algebra and quantum gravity*, in *Mathematical physics research on the leading edge*, ed. C. V. Benton, New York: Nova Science Publishers, 2003.

[16] T.A. Larsson, *Manifestly covariant canonical quantization I: the free scalar field*, [hep-th/0411028](http://arxiv.org/abs/hep-th/0411028) (2004)

[17] T.A. Larsson, *Why the Mickelsson-Faddeev algebra lacks unitary representations*, [hep-th/0501023](http://arxiv.org/abs/hep-th/0501023) (2005)

[18] T.A. Larsson, *Manifestly covariant canonical quantization II: Gauge theory and anomalies*, [hep-th/0501043](http://arxiv.org/abs/hep-th/0501043) (2005)

[19] J. Mickelsson, *Current algebras and groups*, Plenum Monographs in Nonlinear Physics, London: Plenum Press, 1989.

[20] P. Nelson and L. Alvarez-Gaumé, *Hamiltonian interpretation of anomalies*, Comm. Math. Phys. **99** (1985) 103–114.

[21] D. Pickrell, *On the Mickelsson-Faddeev extensions and unitary representations*, Comm. Math. Phys. **123** (1989) 617.

[22] S.E. Rao and R.V. Moody, *Vertex representations for N-toroidal Lie algebras and a generalization of the Virasoro algebra*, Comm. Math. Phys. **159** (1994) 239–264.

[23] K. Savvidou, *The action operator in continuous time histories*, J. Math. Phys. **40** (1999) 5657.

[24] N. Savvidou, *General relativity histories theory*, [gr-qc/0412059](http://arxiv.org/abs/gr-qc/0412059) (2004).

[25] L. Smolin, *How far are we from the quantum theory of gravity?*, [hep-th/0303185](http://arxiv.org/abs/hep-th/0303185) (2003).

[26] S. Weinberg, *The quantum theory of fields, volume II*, Cambridge University Press, 1996.