WELL-POSEDNESS FOR THE THREE-DIMENSIONAL COMPRESSIBLE LIQUID CRYSTAL FLOWS

XIAOLI LI∗

College of Science, Beijing University of Posts and Telecommunications
Beijing 100876, China

BOLING GUO

Institute of Applied Physics and Computational Mathematics
Beijing 100088, China

Abstract. This paper is concerned with the initial-boundary value problem for the three-dimensional compressible liquid crystal flows. The system consists of the Navier-Stokes equations describing the evolution of a compressible viscous fluid coupled with various kinematic transport equations for the heat flow of harmonic maps into $S^2$. Assuming the initial density has vacuum and the initial data satisfies a natural compatibility condition, the existence and uniqueness is established for the local strong solution with large initial data and also for the global strong solution with initial data being close to an equilibrium state. The existence result is proved via the local well-posedness and uniform estimates for a proper linearized system with convective terms.

1. Introduction. In this paper, we establish the well-posedness of a simplified hydrodynamic equation, proposed by Ericksen and Leslie, modeling the flow of nematic liquid crystals formulated in [7]-[9] and [15] in the 1960’s. A simplified version of the Ericksen-Leslie model was introduced by Lin [17] and analyzed by Lin and Liu [18]-[20] who used a modified Galerkin approach, and by Shkoller [25] who relied on a contraction mapping argument coupled with appropriate energy estimates. When the Ossen-Frank energy configuration functional reduces to the Dirichlet energy functional, the hydrodynamic flow equation of liquid crystals in $\mathbb{R}^3$ can be written as follows (see [17]):

\begin{align}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) &= \mu \Delta u - \lambda \nabla \cdot \left( \nabla d \otimes \nabla d - \frac{\nabla d^2}{2} I_3 \right), \\
d_t + u \cdot \nabla d &= \theta \left( \Delta d + [\nabla d]^2 d \right),
\end{align}

where $u \in \mathbb{R}^3$ denotes the velocity, $d \in S^2$ (the unit sphere in $\mathbb{R}^3$) is the unit-vector field that represents the macroscopic molecular orientations, $p(\cdot)$ is the pressure with $p = p(\cdot) \in C^1[0, \infty)$, $p(0) = 0$; and they all depend on the spatial variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the time variable $t > 0$. $\sigma = \mu \nabla u - p I_3$ (the 3×3 identity matrix) is the Cauchy stress tensor given by Stokes’ law and $\mu \nabla u$ stands for the

2010 Mathematics Subject Classification. Primary: 35A05, 76D10, 76D03.

Key words and phrases. Liquid crystals, compressible, vacuum, strong solution, existence and uniqueness.

∗ Corresponding author: Xiaoli Li, xlli@bupt.edu.cn.
fluid viscosity part of the stress tensor. The term $\lambda \nabla \cdot (\nabla d \odot \nabla d)$ in the stress tensor represents the anisotropic feature of the system. The parameters $\mu, \lambda, \theta$ are positive constants standing for viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Deborah number for the molecular orientation field, respectively. The symbol $\nabla d \odot \nabla d$ denotes a matrix whose $(i,j)$-th entry is $\partial_i d \cdot \partial_j d$ for $1 \leq i, j \leq 3$, and $\nabla d \odot \nabla d = (\nabla d) \top \nabla d$, where $(\nabla d) \top$ denotes the transpose of the matrix $\nabla d$.

It is well known liquid crystals are states of matter which are capable of flow and in which the molecular arrangement gives rise to a preferred direction. Roughly speaking, the system (1.1) is a coupling between the compressible Navier-Stokes equations and a transport heat flow of harmonic maps into $\mathbb{S}^2$. It is a macroscopic continuum description of the evolution for the liquid crystals of nematic type under the influence of both the flow field $u$, and the macroscopic description of the microscopic orientation configurations $d$ of rod-like liquid crystals. As for the nonlinear constraint $d \in \mathbb{S}^2$, one of the methods used to relax it is to consider a form of penalization, that is, not $|\nabla d|^2$ in (1.1c), but the Ginzburg-Landau approximation $\frac{1}{\varepsilon^2}(1 - |d|^2)$ for small $\varepsilon$. There were some similar fundamental results starting from the work in [19], where the density is constant, Lin-Liu proved local existence of the classical solutions the global existence of weak solutions in the two-dimensional and three-dimensional spaces. For the density-dependent case, Liu [22] proved the global existence of weak solutions and classical solutions to the system of incompressible Smectic-A liquid crystals under the general condition of the initial density $\rho_0$ satisfying $0 < \alpha \leq \rho_0 \leq \beta$. The global existence of weak solutions in dimension three was established by Liu-Zhang [24] if $\rho_0 \in L^2(\Omega)$. Later Jiang-Tan [14] pointed out that the condition on initial density can be weaken to belong to $L^\gamma(\Omega)$ for any $\gamma \geq \frac{3}{2}$. As for the compressible case, Liu-Liu-Hao [23] established the global existence of strong solutions under the smallness conditions on the initial data in Sobolev spaces in dimension three.

Compared with the Ginzburg-Landau approximation problem, $|\nabla d|^2$ in (1.1c) brings us some new difficulties. Since the strong solutions of a harmonic map must be blowing up at finite time (see Chang-Ding-Ye [1] for the heat flow of harmonic maps), one cannot expect that there exists a global strong solution to system (1.1) with general initial data. In fact, the global existence of weak solutions to (1.1) with large initial data is an outstanding open problem for high dimensions. By so far, only results in one space dimension have been obtained, for instance, we refer to [5, 6]. For the homogeneous case of system (1.1), both the regularity and existence of global weak solutions in dimension two were established by Lin-Lin-Wang [21]. More explicitly, they obtained both interior and boundary regularity theorem for such a flow under smallness conditions, and the existence of global weak solutions that are smooth away from at most finitely many singular times in any bounded smooth domain of $\mathbb{R}^2$. Meanwhile, Hong [10] also showed the global existence of weak solution to this system in two dimensional space. Wang [26] established a global well-posedness theory for the incompressible liquid crystals for rough initial data, provided that $\|u_0\|_{\text{BMO}^{-1}} + |d_0|_{\text{BMO}} \leq \varepsilon_0$ for some $\varepsilon_0 > 0$. Assuming that the initial density $\rho_0$ has a positive bound from below and under smallness conditions on the initial data, Wen-Ding [27] got the global existence and uniqueness of the strong solution to the incompressible density-dependent case in Sobolev spaces in two dimensions, and Li-Wang obtained the result in Sobolev-Besov spaces for three dimensional case in [16]. Concerning the compressible case, Hu-Wu considered
the Cauchy problem for the three-dimensional compressible flow of nematic liquid crystals and obtained the existence and uniqueness of the global strong solution in critical Besov spaces provided that the initial data is close to an equilibrium state in a recent work [11]. Local existence of unique strong solutions were proved provided that the initial data \( \rho_0, u_0, d_0 \) are sufficiently regular and satisfy a natural compatibility condition in [12]. A criterion for possible breakdown of such a local strong solution at finite time was given in terms of blow up of the \( L^\infty \)-norms of \( \rho \) and \( \nabla d \). In [13], an alternative blow-up criteria was derived in terms of the \( L^\infty \)-norms of \( \nabla u \) and \( \nabla d \) in dimension three.

In this paper we consider the initial-boundary value problem of system (1.1) in a bounded smooth domain \( \Omega \subset \mathbb{R}^3 \), with the following initial-boundary conditions:

\[
\begin{align*}
(r, u, d) \big|_{t=0} &= (\rho_0, u_0, d_0), \quad x \in \Omega, \\
(u, d) \big|_{\partial \Omega} &= (0, d_0 |_{\partial \Omega}), \quad t > 0,
\end{align*}
\]

where \( \rho_0 \geq 0, \ d_0 : \Omega \to S^2 \) is given with compatibility. The boundary condition on the velocity implies non-slip on the boundary.

We are interested in the strong solutions to the initial-boundary problem (1.1)-(1.3). In order to obtain strong solutions, we make the following assumptions on the initial data:

\[
\rho_0 
\in
W^{1,q}, \quad u_0 \in (H^1_0) \cap (H^2)^3, \quad \nabla d_0 \in (H^2)^9, \quad (1.4)
\]

where \( 3 < q \leq 6 \). And, \( (\rho_0, u_0, d_0) \) satisfies a natural compatibility condition

\[
-\mu \Delta u + \nabla p + \lambda \nabla \cdot \nabla d_0 + \nabla \cdot (\nabla d_0 \otimes \nabla d_0) = \rho_0^\frac{4}{q+1} g, \quad \forall x \in \Omega \quad (1.5)
\]

with some \( (\rho_0, g) \in H^1 \times (L^2)^2 \). We note here that (1.5) is a compensation to the lack of a positive lower bound of the initial density (see [2]).

Our main result establishes the local well-posedness of the Ericksen-Leslie problem for any regular enough initial data:

**Theorem 1.1.** Let the initial data satisfies the regularity condition (1.4) and also the compatibility condition (1.5). There exists a time \( T^* \) such that the initial-boundary problem (1.1)-(1.3) has a unique local strong solution \( (\rho, u, d) \) satisfying

\[
\begin{align*}
\rho &\in C([0, T^*]; W^{1,q}), \quad \rho_t \in C([0, T^*]; L^q); \\
u &\in C([0, T^*]; H^1_0 \cap H^2) \cap L^2(0, T^*; W^{2,q}), \quad d \in C([0, T^*]; H^3); \\
u_t &\in L^2(0, T^*; H^1_0), \quad d_t \in C([0, T^*]; H^1_0) \cap L^2(0, T; H^2); \\
\sqrt{\rho} u_t &\in C([0, T^*]; L^2), \quad d_{tt} \in L^2(0, T^*; L^2).
\end{align*}
\]

Furthermore, if we suppose that \( (\tilde{\rho}, \tilde{u}, \tilde{d}) \) is another solution with initial-boundary conditions

\[
\begin{align*}
(\tilde{\rho}, \tilde{u}, \tilde{d}) \big|_{t=0} &= (\tilde{\rho}_0, \tilde{u}_0, \tilde{d}_0), \quad \tilde{x} \in \Omega, \\
(\tilde{u}, \tilde{d}) \big|_{\partial \Omega} &= (0, \tilde{d}_0 |_{\partial \Omega}), \quad \tilde{t} > 0,
\end{align*}
\]

then for any \( t \in (0, T^*], \) the quantities

\[
\begin{align*}
\|\rho - \tilde{\rho}\|_{L^q(t)}, \quad \|u - \tilde{u}\|_{L^q(H^1)}, \quad \|u - \tilde{u}\|_{L^q_2(H^2)}, \quad \|\sqrt{\rho} (u_t - \tilde{u}_t)\|_{L^q_2(L^2)}; \\
\|d - \tilde{d}\|_{L^2(H^2)}, \quad \|d - \tilde{d}\|_{L^q_2(H^3)}, \quad \|d_t - \tilde{d}_t\|_{L^q_2(H^3)}; \quad \|d_{tt} - \tilde{d}_{tt}\|_{L^q_2(H^3)}
\end{align*}
\]

tend to zero as \( (\tilde{\rho}_0, \tilde{u}_0, \tilde{d}_0) \to (\rho_0, u_0, d_0). \)
Moreover, we shall prove the existence of global strong solution for initial data that is close to an equilibrium state \((0, 0, n)\) with a constant vector \(n \in S^2\). More precisely,

**Theorem 1.2.** Let \(\rho\) be a nonnegative constant and \(n\) be a constant unit vector in \(\mathbb{R}^3\). Then there exists a suitable positive constant \(\xi_0\) such that if the initial data satisfies further 
\[
\max\{\|\rho_0 - \rho\|_{W^{1,q}}, \|u_0\|_{H^2}, \|d_0 - n\|_{H^s}, \|g\|_{L^2}\} \leq \xi
\]
for all \(\xi \in (0, \xi_0]\), the problem (1.1)-(1.3) has a unique global strong solution enjoying the regularities as in Theorem 1.1.

**Remark 1.1.** We only consider the case \(\rho = 0\) in this paper, because the other case \(\rho > 0\) can be induced to the problem with positive initial density.

From the viewpoint of partial differential equations, system (1.1) is a highly nonlinear system coupling hyperbolic equations and parabolic equations. It is very challenging to understand and analyze such a system, especially when the density function \(\rho\) may vanish or the fluid takes vacuum states (equation (1.1b) becomes a degenerate parabolic-elliptic couples system). Our approach is quite classical. Successive approximation method [3, 4] is employed for two variables, which was used in [23]. It consists in deriving energy estimates without loss of derivatives in sufficiently high order Sobolev spaces for a linearized version of (1.1), and then solve the nonlinear problem through an iterative scheme. There are some difficulties in both steps that will be pointed out along the detailed proofs. Moreover, in order to overcome the difficulty that the initial density has vacuum brings, as usual, the technique is to approximate the nonnegative initial density by a positive initial data.

We organize the rest of this paper as follows. In Sect. 2, we introduce a special linear problem of the original system (1.1)-(1.3) and prove local existence of a strong solution to the linear problem with positive initial density. We also derive a series of uniform a priori estimates, which ensure the local strong solution exists when the initial density allows vacuum. In Sect. 3, after constructing a sequence of approximate solutions, a strong solution of (1.1)-(1.3) is obtained. The uniqueness and continuity on the initial data are also proved. Finally, in Sect. 4, our main result on the global existence of a strong solution of (1.1)-(1.3) will be established via iteration and the convergence of the iteration.

2. **A linear problem.** In this section, let us consider the following auxiliary linear problem:
\begin{align}
\rho_t + \nabla \cdot (\rho v) &= 0, \quad (2.1a) \\
d_t + v \cdot \nabla d &= \theta (\Delta d + (\nabla f : \nabla d) f), \quad (2.1b) \\
\rho u_t + \rho v \cdot \nabla u + \nabla p(\rho) &= \mu \Delta u - \lambda \nabla \cdot \left(\nabla d \odot \nabla d - \frac{[\nabla d]^2}{2} I_3\right), \quad (2.1c)
\end{align}
with \(v \in \mathbb{R}^3\) and \(f \in \mathbb{R}^3\) being given vector functions and enjoying the regularities such that
\begin{align}
v &\in C([0, T]; H^1_0 \cap H^2) \cap L^2(0, T; W^{2,q}), \quad v_t \in L^2(0, T; H^1_0), \\
f &\in C([0, T]; H^3), \quad f_t \in C([0, T]; H^1_0) \cap L^2(0, T; H^2), \quad (2.2)
\end{align}
for all \(T > 0\).
Conformally to the initial-boundary conditions of the original problem, we suppose
\[
(v, f) |_{t=0} = (u_0, d_0), \quad x \in \Omega, \\
(v, f) |_{\partial \Omega} = (0, d_0 |_{\partial \Omega}), \quad t \in (0, T).
\] (2.3)

2.1. **Existence of approximate solutions.** For each \( \delta > 0 \), for example, \( \delta \in (0,1) \), let \( u_0^\delta \) solve the elliptic boundary value problem:
\[
\begin{aligned}
\mu &\Delta u_0^\delta - \lambda \nabla \cdot (\nabla d_0 \otimes \nabla d_0) - \frac{[\nabla d_0]^2}{2} \|_3) - \nabla p(\rho_0^\delta) = (\rho_0^\delta)^{\frac{3}{2}} g, \\
\rho_0^\delta = \rho_0 + \delta, \quad u_0^\delta \rightarrow u_0 \text{ in } H_0^1 \cap H^2 \text{ as } \delta \rightarrow 0.
\end{aligned}
\]

Moreover, if \( \rho_0, u_0 \) satisfy (1.2) and (2.1), respectively. Then the system (2.1) has a global unique strong solution such that
\[
\rho \in C([0, T]; W^{1,q}), \quad \rho_t \in C([0, T]; L^q),
\]
\[
u \in C([0, T]; H_0^1 \cap H^2) \cap L^q(0, T; W^{2,q}), \quad u_t \in C([0, T]; L^2) \cap L^q(0, T; H_0^1),
\]
\[
d \in C([0, T]; H^1), \quad d_t \in C([0, T]; H_0^1 \cap L^q(0, T; H^2),
\]
\[
u_{tt} \in L^q(0, T; H^{-1}), \quad d_{tt} \in L^q(0, T; L^2).
\]

We will prove Theorem 2.1 through a series of lemmas. To begin with, assume there are three constants \( c_0, c_1 \) and \( c_2 \) such that
\[
c_0 > 1 + \|\rho_0\|_{W^{1,q}} + \|u_0\|_{H^2} + \|\nabla d_0\|_{H^2} + \|\nabla u_0\|_{W^{1,q}},
\] (2.4)
\[
c_1 > \text{sup}_{0 \leq t \leq T} (\|\nabla v\|_{H_0^1} + \|f\|_{H^2} + \|f_t\|_{H_0^1}) + \int_0^T \|\nabla v_t\|_{L^2} + \|v\|_{W^{2,q}} + \|\nabla^2 f\|_{L^2} dt,
\] (2.5)
\[
c_2 > \text{sup}_{0 \leq t \leq T} (\|\nabla^2 v\|_{L^2} + \|\nabla^3 f\|_{L^1}),
\] (2.6)
\[
c_2 > c_1 > c_0.
\] (2.7)

Throughout of the whole paper, sometimes, we make use of \( A \lesssim B \) in place of \( A \leq C_0 B \), where \( C_0 \) stands for a “harmless” constant whose exact meaning depends on the context, and \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \).

**Lemma 2.1.** Let \( \rho_0 \) and \( v \) satisfy (1.4) and (2.2), respectively. Then the system (2.1a) (1.2) has a global unique strong solution such that
\[
\rho \in C([0, T]; W^{1,q}), \quad \rho_t \in C([0, T]; L^q).
\]
Moreover, if \( \rho_0, v \) satisfy (2.4)-(2.7), then there exists a time \( T_1 = \min\{c_1^{-1}, T\} \) such that for all \( t \in [0, T_1] \),
\[
\|\rho\|_{W^{1,q}}(t) \lesssim c_0, \quad \|\rho_t\|_{L^q}(t) \lesssim c_0 c_2.
\] (2.8)

In particular,
\[
\|p\|_{L^q}(t) \lesssim M(c_0), \quad \|\nabla p\|_{L^q}(t) \lesssim M(c_0) c_0, \quad \|p_t\|_{L^q}(t) \lesssim M(c_0) c_0 c_2,
\] (2.9)
where \( M(c_0) := \text{sup}_{0 \leq s \leq c_0} (1 + p(s) + p'(s)) \).
Proof. The existence of the solution follows from the method of characteristics. If we define \( V(t, \tau, x) \) to be the solution of
\[
\begin{aligned}
\frac{d}{d\tau} V(\tau, t, x) &= v(\tau, V(\tau, t, x)), \quad \tau \in [0, T], \; x \in \mathbb{R}, \\
V(t, t, x) &= x,
\end{aligned}
\]
(2.1a) can be rewritten as
\[
\frac{d}{d\tau} \rho(\tau, V(\tau, t, x)) = -\rho(\tau, V(\tau, t, x)) \nabla \cdot v(\tau, V(\tau, t, x)),
\]
and the explicit formula for \( \rho \) is
\[
\rho(t, x) = \rho_0(V(0, t, x)) \exp \left( -\int_0^t \nabla \cdot v(\tau, V(\tau, t, x)) \, d\tau \right).
\] (2.10)

Applying the gradient operator \( \nabla \) to (2.1a), we have
\[
\nabla \rho_t + \nabla \cdot (\nabla \rho) + \nabla v \cdot \nabla \rho + \rho \nabla (\nabla \cdot v) + \nabla \cdot (\nabla \rho) = 0,
\]
(2.11)

where \( \nabla v \cdot \nabla \rho = (\nabla v)^T \nabla \rho \).

Multiplying (2.11) by \( ||\nabla \rho||^q \) and integrating over \( \Omega \), we obtain
\[
\frac{1}{q} \frac{d}{dt} ||\nabla \rho||^q_{L^q} + \int_{\Omega} \left( \frac{1}{q} \nabla \cdot (||\nabla \rho||^q) + ||\nabla \rho||^q \nabla \cdot (\nabla v \cdot \nabla \rho) \right)
\]
\[
+ \rho ||\nabla \rho||^{q-2} \nabla \rho \cdot (\nabla \cdot v) + ||\nabla \rho||^q (\nabla \cdot v) \, dx = 0.
\]

Bearing in mind that
\[
\int (v \cdot \nabla (||\nabla \rho||^q) + ||\nabla \rho||^q (\nabla \cdot v)) \, dx = \int \nabla \cdot (||\nabla \rho||^q v) \, dx = 0,
\]
we find, by Hölder’s inequality and the imbedding
\[
W^{1, q} \hookrightarrow L^\infty, \text{ as } 3 < q \leq 6,
\]
(2.12)

\[
\frac{1}{q} \frac{d}{dt} ||\nabla \rho||^q_{L^q}
\]
\[
\leq ||\nabla \rho||^{q-1}_{L^q} (||\rho||_{L^\infty} ||\nabla (\nabla \rho)||_{L^q} + ||\nabla \rho||_{L^q} ||\nabla v||_{L^\infty}) + ||\nabla \rho||^q_{L^q} ||\nabla \cdot v||_{L^\infty}
\]
\[
\lesssim ||\nabla \rho||^{q-1}_{L^q} (||\nabla \rho||_{L^q} + ||\rho||_{L^\infty}) ||v||_{W^{2, q}},
\]
and, by Gronwall’s inequality,
\[
||\nabla \rho||_{L^q}(t) \leq \exp(\int_0^t ||v||_{W^{2, q}} ds) (||\nabla \rho_0||_{L^q} + \int_0^t ||\rho||_{L^\infty} ||v||_{W^{2, q}} ds)
\] (2.13)

for all \( t \leq T \).

Using (2.10), (2.13) and choosing \( T_1 = \min\{c_1^{-1}, T\} \), we get, for all \( t \in [0, T_1] \),
\[
||\rho||_{W^{1, q}(t)}
\]
\[
\lesssim ||\rho_0||_{W^{1, q}} \left( \exp(-\int_0^t \nabla \cdot v \, d\tau) \right) ||L^q
\]
\[
+ \exp\left( (\int_0^t ||v||_{W^{2, q}} ds)^{\frac{1}{2}} (1 + \int_0^t \exp(-\int_0^s \nabla \cdot v \, d\tau) ||v||_{W^{2, q}} ds) \right)
\]
\[
\lesssim c_0 \left( \exp(\int_0^t ||v||_{L^\infty} \, ds) + \exp(\sqrt{c_1 t})(1 + \int_0^t \exp(\int_0^s ||v||_{W^{2, q}} \, ds) \, ds) \right)
\]
\[
\begin{aligned}
&\lesssim c_0 \left(\exp\left(\int_0^t \|v\|_W^2 \, ds\right)^{\frac{2}{3}} \sqrt{t} + \exp\left(c_1 t\right) \left(1 + \int_0^t \exp\left(c_1 s\right) \|v\|_W^2 \, ds\right)\right) \\
&\lesssim c_0 \exp\left(c_1 T\right) \left(1 + \exp\left(c_1 T\right) \|v\|_W^2\right) \lesssim c_0,
\end{aligned}
\]

\[
\|p_t\|_{L^q}(t) \leq \|v\|_{L^\infty} \|\nabla \rho\|_{L^q} + \|\rho\|_{L^\infty} \|\nabla v\|_{L^q}
\]

\[
\lesssim \|v\|_{H^2} \|\rho\|_{W^{1,q}} \lesssim c_0 c_2,
\]

and, by using (2.12), it follows obviously that for all \(t \in [0, T_1]\),
\[
\|p\|_{L^q}(t) \lesssim M(c_0), \quad \|\nabla p\|_{L^q}(t) \lesssim M(c_0) c_0, \quad \|p_t\|_{L^q}(t) \lesssim M(c_0) c_0 c_2,
\]

where
\[
M(c_0) := \sup_{0 \leq t \leq c_0} \left(1 + p(t) + p'(t)\right).
\]

\[
\text{Lemma 2.2. Let } v, f \text{ satisfy (2.2) and (2.3). Then the system (2.1b) and (1.3) has a global unique strong solution such that}
\]
\[d \in C([0, T]; H^3), \quad d_t \in C([0, T]; H^1_0) \cap L^2(0, T; H^2), \quad d_{tt} \in L^2(0, T; L^2).
\]

Moreover, if \(v, f\) satisfy (2.4)-(2.7), then there exists a time \(T_3 = \min\{c_0^{-1} c_1^{-1} c_2^{-2}, T\}\) such that
\[
\sup_{0 \leq t \leq T_3} \left(\|d\|_{H^1} + c_1^{-4} \|\nabla^2 d\|_{L^2} + c_1^3 \|d_t\|_{H^3} + c_1^{-7} c_2^{-1} \|\nabla d\|_{H^2}\right)
\]

\[
+ \int_0^{T_3} \left(\|d_t\|_{H^2}^2 + c_0^3 \|d\|_{H^3}^2\right) \, dt \lesssim c_0^3.
\]

\[
\text{Proof. Since (2.1b) is a linear parabolic-type system in } d, \text{ the existence and uniqueness of } d \text{ to the problem (2.1b), (1.2) and (1.3) can be obtained by the standard Faedo-Galerkin method, and also the regularity of } d \text{ described in the lemma.}
\]

Differentiating (2.1b) with respect to \(t\), multiplying the result by \(d_t\) and then integrating over \(\Omega\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |d_t|^2 \, dx + \theta \int_\Omega |\nabla d_t|^2 \, dx 
\]

\[
\lesssim \|v\|_{L^\infty} \|\nabla d\|_{L^2} \|d_t\|_{L^3} + \|v\|_{L^\infty} \|\nabla d_t\|_{L^2} \|d_t\|_{L^3} + \|\nabla f\|_{L^\infty} \|\nabla d_t\|_{L^2} \|f\|_{L^\infty} \|d_t\|_{L^3}
\]

\[
+ \|\nabla f\|_{L^\infty} \|\nabla d_t\|_{L^2} \|f\|_{L^\infty} \|d_t\|_{L^3} + \|\nabla f\|_{L^\infty} \|\nabla d_t\|_{L^2} \|f\|_{L^\infty} \|d_t\|_{L^3}
\]

\[
:= \sum_{l=1}^5 I_l.
\]

Now we estimate the right-hand side of (2.16) term by term.
\[
I_1 \lesssim \|v\|_{H^1} \|\nabla d\|_{L^2} \|d_t\|_{L^2} \|d_t\|_{L^6}^2
\]

\[
\lesssim \|\nabla v\|_{L^2} \|\nabla d\|_{L^2} \|d_t\|_{L^2} \|\nabla d_t\|_{L^2}
\]

\[
\leq \eta \|\nabla v\|_{L^2}^2 \|\nabla d_t\|_{L^2}^2 + \eta^{-1} \|d_t\|_{L^2}^2 + \frac{\theta}{8} \|\nabla d_t\|_{L^2}^2,
\]

where we have used the fact that \(d_t \mid_{\partial \Omega} = 0\).
\[
I_2 \leq C \|v\|_{H^2}^2 \|d_t\|_{L^2}^2 + \frac{\theta}{8} \|\nabla d_t\|_{L^2}^2,
\]
From (2.1b), using the elliptic estimates, we get

\[ I_3 + I_5 \leq \eta^2 \left( \| \nabla f_i \|_{H^2}^2 \| f_i \|_{L^2}^2 + \| \nabla f_i \|_{L^2}^2 \| f_i \|_{H^2}^2 \right) \| \nabla d_i \|_{L^2}^2 + \eta^{-3} \| \partial_t d_i \|_{L^2}^2 + \frac{\theta}{4} \| \nabla d_i \|_{L^2}^2. \]

\[ I_4 \leq C \| \nabla f_i \|_{L^2}^2 \| f_i \|_{L^2}^2 + \| \partial_t d_i \|_{L^2}^2 \leq C \| \nabla f_i \|_{H^1}^2 \| f_i \|_{L^2}^2 + \frac{\theta}{4} \| \nabla d_i \|_{L^2}^2. \]

Note here that the positive constant \( \eta \) will be determined later.

Notice that

\[
\frac{d}{dt} \int_\Omega |\nabla d|^2 \, dx = 2 \int_\Omega \nabla d : \nabla d_t \, dx \leq C \| \nabla d \|_{L^2}^2 + \frac{\theta}{8} \| \nabla d_t \|_{L^2}^2,
\]

and

\[
\frac{d}{dt} \int_\Omega |d - n|^2 \, dx \leq \| d - n \|_{L^2}^2 + \| d_t \|_{L^2}^2.
\]

Therefore, combining with (2.16), we get

\[
\frac{d}{dt} \int \left( |d_i|^2 + |d - n|^2 + |\nabla d|^2 \right) \, dx + \int_\Omega |\nabla d_i|^2 \, dx
\]

\[
\lesssim \left( \| d_i \|_{L^2}^2 + \| d - n \|_{L^2}^2 + \| \nabla d_i \|_{L^2}^2 \right) \left( 1 + \eta^{-1} + \eta^{-3} + \eta \| \nabla v_i \|_{L^2}^2 + \| v \|_{H^2}^2 \right) (2.17)
\]

\[
+ \| \nabla f_i \|_{H^1}^2 \| f_i \|_{L^2}^2 + \eta^3 \left( \| \nabla f_i \|_{H^1}^2 \| f_i \|_{L^2}^2 + \| \nabla f_i \|_{L^2}^2 \| f_i \|_{H^2}^2 \right).
\]

It follows from (2.1b) that

\[
\| d_i \|_{L^2}(0) \lesssim \| \Delta d_i \|_{L^2} + \| u_0 \|_{H^2} \| \nabla d \|_{L^2} + \| \nabla d_0 \|_{H^1}^2 d_0 \|_{H^1}
\]

\[
\leq \| \Delta d_0 \|_{L^2} + \| u_0 \|_{H^2} \| \nabla d_0 \|_{L^2} + \| \nabla d_0 \|_{H^1}^2 (\| d_0 - n \|_{H^1} + \| n \|_{H^1})
\]

\[
\leq \| \Delta d_0 \|_{L^2} + \| u_0 \|_{H^2} \| \nabla d_0 \|_{L^2} + \| \nabla d_0 \|_{H^1}^2 (\| \nabla d_0 \|_{L^2} + \| n \|_{H^1})
\]

\[
\lesssim c_0^2.
\]

Hence, by virtue of Gronwall’s inequality, we obtain from (2.17) that

\[
\int_\Omega \left( |d_i|^2 + |d - n|^2 + |\nabla d|^2 \right) \, dx + \int_0^t \int_\Omega |\nabla d_i|^2 \, dx \, ds
\]

\[
\lesssim c_0^6 \exp \left( \int_0^t (1 + \eta^{-1} + \eta^{-3} + \eta \| \nabla v_i \|_{L^2}^2 + \| v \|_{H^2}^2 + \eta^3 \| \nabla^2 f_i \|_{L^2}^2 \| f_i \|_{H^2}^2)ight.
\]

\[
+ \left. \eta^3 \| \nabla f_i \|_{L^2}^2 \| f \|_{H^3}^2 + \| \nabla f_i \|_{H^1}^2 \| f \|_{H^2}^2 \right) \, ds.
\]

(2.18)

By taking \( \eta = c_1^{-1} \) and \( T_2 = \min \{ c_1^{-2} c_2^{-2}, T \} \), it follows from (2.18) that

\[
\sup_{0 \leq t \leq T_2} \int_\Omega \left( |d_i|^2 + |d - n|^2 + |\nabla d|^2 \right) \, dx + \int_0^{T_2} \int_\Omega |\nabla d_i|^2 \, dx \, dt \lesssim c_0^6.
\]

From (2.1b), using the elliptic estimates, we get

\[
\| d - n \|_{H^2} \lesssim \| d_0 - n \|_{H^2} + \| d_t \|_{L^2} + \| v \cdot \nabla d \|_{L^2} + \| (\nabla f : \nabla d) f \|_{L^2}
\]

\[
\lesssim c_0^3 + \| v \|_{H^1} \| \nabla d \|_{L^2} + \| \nabla f \|_{L^2} \| \nabla d \|_{H^1}^2 \| f \|_{L^2}
\]

\[
\lesssim c_0^3 + (\| v \|_{H^1}^2 + \| \nabla f \|_{H^2}^2 \| f \|_{H^2}^2) \| \nabla d \|_{L^2}^2 \| \nabla d \|_{H^1}^2
\]

\[
\leq C \left( c_0^3 + (\| v \|_{H^1}^2 + \| f \|_{H^2}^2) \| \nabla d \|_{L^2}^2 + \frac{1}{2} \| \nabla d \|_{H^2}^2 \right),
\]

which implies \( \| \nabla^2 d \|_{L^2} \lesssim c_0^4 c_4^2 \).

Similarly, let us estimate \( \| \nabla d \|_{H^2} \). Since

\[
\theta \Delta (\nabla d) = \nabla d_t + (\nabla (v \cdot \nabla d) - \theta \nabla ((\nabla f : \nabla d) f),
\]

(2.19)
then
\[ \|\nabla d\|_{L^2} \lesssim \|\nabla d_0\|_{L^2} + \|\nabla d\|_{L^2} + \|\nabla v\|_{L^6}\|\nabla d\|_{L^3} + \|\nabla v\|_{L^2}\|\nabla^2 d\|_{L^2} \]
\[ + \|\nabla^2 f\|_{L^6}\|\nabla d\|_{L^3}\|f\|_{L^\infty} + \|\nabla f\|_{L^6}\|\nabla^2 d\|_{L^3}\|f\|_{L^\infty} + \|\nabla f\|_{L^6}^2\|\nabla d\|_{L^6} \]
\[ \lesssim c_0 + \|\nabla d\|_{L^2} + (\|\nabla v\|_{H^2} + \|f\|_{H^2}\|f\|_{H^3})\|\nabla d\|_{L^2}^2 \|\nabla d\|_{L^2}^{1/2} + \|\nabla v\|_{H^2}\|\nabla^2 d\|_{L^2} \]
\[ + \|f\|_{H^2}^2\|\nabla^2 d\|_{L^2}^{1/2} + \|f\|_{H^2}^2\|\nabla d\|_{L^6} \]
\[ \leq C\left(c_0 + \|\nabla d\|_{L^2} + (\|\nabla v\|_{H^2} + \|f\|_{H^2}\|f\|_{H^3})\|\nabla d\|_{L^2}^2 \|\nabla d\|_{L^2}^{1/2} + \|\nabla v\|_{H^2}\|\nabla^2 d\|_{L^2} \right) \]
\[ + \|f\|_{H^2}^2\|\nabla^2 d\|_{L^2} \]}
and therefore,
\[ \|\nabla d\|_{H^2} \lesssim \|\nabla d\|_{L^2} + c_0^3c_1^7c_2. \]  

Differentiating (2.1b) with respect to time and taking inner product with \( \Delta d \), we have
\[ \int_\Omega \frac{d}{dt} |\nabla d|^2 \, dx + \theta \int_\Omega |\Delta d|^2 \, dx \]
\[ = \int_\Omega (\nabla \cdot \nabla d) \cdot (\nabla d) \, dx + \int_\Omega (\nabla \cdot \nabla d) \cdot (\nabla d) \, dx - \int_\Omega (\nabla f : \nabla) \cdot (\nabla d) \, dx \]
\[ - \int_\Omega (\nabla f : \nabla d) \cdot (\nabla d) \, dx - \int_\Omega (\nabla f : \nabla d) \cdot (\nabla d) \, dx \]
\[ := \sum_{l=1}^{5} J_l. \]  

Here
\[ J_1 \lesssim \|\nabla v\|_{L^2}\|\nabla d\|_{L^6}\|\nabla d\|_{L^3} + \|\nabla v\|_{L^2}\|\nabla^2 d\|_{L^6}\|\nabla d\|_{L^2} \]
\[ \lesssim \eta\|\nabla v\|_{L^2}^2 + \eta^{-1}\|\nabla d\|_{L^2}^{2/3}\|\nabla d\|_{L^2}^{1/3} + \eta\|\nabla v\|_{L^2}^2 + \eta^{-1}\|\nabla^2 d\|_{L^6}^2 \]
\[ \lesssim \eta\|\nabla v\|_{L^2}^2 + \eta^{-1}c_1^6c_2^6\|\nabla d\|_{L^2}^2 + \eta\|\nabla v\|_{L^2}^2 + \eta\|\nabla v\|_{L^2}^2 + \eta^{-1}\|\nabla^2 d\|_{L^6}^2 \]
\[ + \eta^{-1}(\|\nabla d\|_{L^2}^2 + c_0^6c_1^{14}c_2^2) \]
\[ J_2 \lesssim \|\nabla v\|_{H^2}\|\nabla d\|_{L^2}\|\Delta d\|_{L^2} \]
\[ \leq \epsilon^{-1}c_2^2\|\nabla d\|_{L^2}^2 + \epsilon\|\Delta d\|_{L^2}^2, \]
\[ J_3 + J_5 \lesssim (\|\nabla v\|_{L^2}\|\nabla v\|_{L^6})\|\nabla d\|_{H^2}\|\Delta d\|_{L^2} \]
\[ \leq \epsilon^{-1}c_1^4(\|\nabla d\|_{L^2}^2 + c_0^6c_1^{14}c_2^2) + \epsilon\|\Delta d\|_{L^2}^2, \]
\[ J_4 \lesssim \|\nabla f\|_{L^\infty}\|\nabla d\|_{L^2}\|\Delta d\|_{L^2} \]

Bearing in mind that the elliptic estimate
\[ \|\nabla^2 d\|_{L^2} \lesssim |\Delta d|_{L^2} + 1 \]  

while substituting the above estimates into (2.21) and taking \( \epsilon \) small enough \((< \frac{\theta}{4})\), we have
\[ \frac{d}{dt} \int_\Omega |\nabla d| dx \leq X_\eta(t)|\nabla d|_{L^2}^2 + Y_\eta(t), \]  

where
\[ X_\eta(t) = \eta^{-2}c_0^2c_1^{14}c_2^2 + \eta \|\nabla v\|_{L^2}^2 + \eta^{-1}c_1^2 + c_2^2 + c_1^2c_2^2, \]
\[ Y_\eta(t) = \eta \|\nabla v\|_{L^2}^2 + \eta^{-1}c_0^6c_1^{14}c_2^2 + c_0^6c_1^{18}c_2^2. \]
Lemma 2.3. Under the hypotheses of Theorem 2.1, there exists a global unique solution $u$ of (2.1c) with the initial data $u_0^3$ and the boundary condition (1.3) such that

$$u \in C([0, T]; H_0^1 \cap H^2) \cap L^2(0, T; W^{2, q}), \quad u_t \in C([0, T]; L^2) \cap L^2(0, T; H_0^1),$$

$$u_{tt} \in L^2(0, T; H^{-1}).$$

In addition, if $v, f$ satisfy (2.4)-(2.7), then

$$\sup_{0 \leq t \leq T_5} \left( M(c_0)c_0^2 \|u\|_{H_0^1} + M(c_0)c_1^{-10}\|\nabla u\|_{H^1} + M(c_0)c_0^2\|\nabla u\|_{L^2} \right)$$

$$+ \int_0^{T_5} (c_0^2\|\nabla u\|_{L^2} + \|u\|_{W^{2, q}}) \, dt \lesssim M^2(c_0)c_0^2 + c_0^{12}.$$

(2.24)

for all $\delta > 0$ small enough, where $T_5 = \min\{T, c_1^{-28}c_2^{-4}\}$.

Proof. First, it comes easily from (2.10) that

$$\rho(x, t) \geq \delta \exp(-\int_0^t \|\nabla v\|_{W^{1, q}} \, ds) \geq \delta,$$

for all $(x, t) \in \bar{\Omega} \times [0, T]$, where $\delta > 0$ is a constant.

Now we can rewrite (2.1c) into

$$u_t + v \cdot \nabla u + \rho^{-1} \nabla p(\rho) = \mu \rho^{-1} \Delta u - \lambda \rho^{-1} \nabla \cdot \left( \nabla d \odot \nabla d - \frac{\|\nabla d\|^2}{2} I_3 \right).$$

(2.25)
Applying the Galerkin method again to (2.25) with the initial data \( \mathbf{u}_0 \) and the non-slip boundary condition, we can deduce the existence and regularity of \( \mathbf{u} \) described in the lemma.

Next, we prove the estimate (2.24). Differentiating (2.1c) with respect to \( t \), multiplying the result by \( \mathbf{u}_t \) and then integrating over \( \Omega \), one has

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho \| \mathbf{u}_t \|^2 \, dx + \mu \int_{\Omega} \| \nabla \mathbf{u}_t \|^2 \, dx
\]

\[
= \int_{\Omega} (\mathbf{u}_t \rho_t \cdot \mathbf{v} - \mathbf{u}_t \cdot (2 \mathbf{v} \cdot \nabla \mathbf{u} - \rho \mathbf{v}_t \cdot \nabla \mathbf{u})) \cdot \mathbf{u}_t \, dx
\]

\[- \lambda \int_{\Omega} ( (\nabla \mathbf{d})^T \Delta \mathbf{d})_t \cdot \mathbf{u}_t \, dx \tag{2.26}
\]

\[
:= \sum_{i=1}^{5} K_i.
\]

Here

\[
K_1 = -\int_{\Omega} \nabla p_t \cdot \mathbf{u}_t \, dx \leq C \| p_t \|_{L^2}^2 + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2} \leq CM^2 (\epsilon_0) c_0^2 c_2^2 + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2},
\]

\[
K_2 = -\int_{\Omega} \rho \mathbf{v} \cdot \nabla \mathbf{u} \cdot \mathbf{u}_t \, dx \leq C \| \rho \|_{L^\infty} \| \mathbf{v} \|^2_{L^2} \| \nabla \mathbf{u} \|^2_{L^2} + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2}
\]

\[
\leq C \| \rho \|_{L^{2/1}} \| \nabla \mathbf{u} \|^2_{L^2} + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2}
\]

\[
K_3 = -2 \int_{\Omega} \mathbf{v} \cdot \nabla \mathbf{u}_t \cdot \mathbf{u}_t \, dx \leq C \| \mathbf{v} \|_{L^\infty} \| \mathbf{u}_t \|^2_{L^2} + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2}
\]

\[
\leq C c_0^2 \| \sqrt{\rho} \mathbf{u}_t \|^2_{L^2} + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2},
\]

\[
K_4 = -\int_{\Omega} \rho \mathbf{v}_t \cdot \nabla \mathbf{u}_t \, dx \leq \| \sqrt{\rho} \mathbf{u}_t \|_{L^2} \| \mathbf{v}_t \|_{L^2} \sqrt{\rho} \| \mathbf{u}_t \|^2_{L^2} \sqrt{\rho} \| \mathbf{u}_t \|^2_{L^2}
\]

\[
\leq \| \sqrt{\rho} \mathbf{u}_t \|_{L^2} \| \mathbf{v}_t \|_{L^2} \sqrt{\rho} \| \mathbf{u}_t \|^2_{L^2} \sqrt{\rho} \| \mathbf{u}_t \|^2_{L^2}
\]

\[
\leq \| \sqrt{\rho} \mathbf{u}_t \|_{L^2} \| \mathbf{v}_t \|_{L^2} + C \eta^{-2} \| \rho \|^2_{H^{1/2}} \| \mathbf{u}_t \|^2_{L^2} + \| \nabla \mathbf{u}_t \|^2_{H^1},
\]

\[
K_5 = -\lambda \int_{\Omega} ( (\nabla \mathbf{d})^T \Delta \mathbf{d})_t \cdot \mathbf{u}_t \, dx
\]

\[
\leq \| \nabla \mathbf{d} \|_{L^2} \| \nabla^2 \mathbf{d} \|_{L^2} \| \mathbf{u}_t \|_{L^6} + \| \nabla \mathbf{d} \|_{L^\infty} \| \nabla \mathbf{d} \|_{L^2} \| \mathbf{u}_t \|_{L^2}
\]

\[
\leq C ( \| \nabla \mathbf{d} \|^2_{L^2} \| \nabla^2 \mathbf{d} \|_{H^1} + \| \nabla \mathbf{d} \|^2_{H^1} \| \nabla \mathbf{d} \|^2_{L^2}) + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2}
\]

\[
\leq C c_0^2 c_1 c_2^2 + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2}.
\]

Meanwhile, since

\[
\frac{1}{2} \frac{d}{dt} \| \mathbf{u}_t \|^2_{L^2} = \int_{\Omega} \mathbf{u}_t \cdot \nabla \mathbf{u}_t \, dx \leq C \| \mathbf{u}_t \|^2_{L^2} + \frac{\mu}{6} \| \nabla \mathbf{u}_t \|^2_{L^2},
\]

then, combining with the above estimates and (2.26), we have

\[
\frac{d}{dt} \int_{\Omega} (\rho \| \mathbf{u}_t \|^2 + | \nabla \mathbf{u}_t |^2) \, dx + \int_{\Omega} | \nabla \mathbf{u}_t |^2 \, dx
\]

\[
\lesssim \lambda \eta (\| \sqrt{\rho} \mathbf{u}_t \|^2_{L^2} + \| \mathbf{u}_t \|^2_{L^2}) + \mathcal{Y} + \| \nabla \mathbf{u}_t \|^2_{H^1}, \tag{2.27}
\]
Choosing $\eta = \epsilon_1^{-1}$, we have
\[
\int_0^t \mathcal{X}_\eta(s) \, ds \lesssim 1 + (c_0^2 \epsilon_2 + c_0^2 \epsilon_1^2) \, t, \quad \forall \, t \in [0, T_3].
\]
Multiplying (2.1c) by $u$, integrating the result over $\Omega$ and using the Cauchy-Schwarz inequality, we obtain
\[
\int_\Omega \rho |u|^2 \, dx = \int_\Omega \rho^2 u \cdot (-\rho^2 v \cdot \nabla u + \rho - \frac{\lambda}{2} (\mu \Delta u - \lambda (\nabla d)^T \Delta d - \nabla p(\rho))) \, dx \\
\leq \frac{1}{2} \int_\Omega \rho |u|^2 \, dx + \frac{1}{2} \int_\Omega \rho |v|^2 |\nabla u|^2 + \rho^{-1} |\mu \Delta u - \lambda (\nabla d)^T \Delta d - \nabla p(\rho)|^2 \, dx,
\]
i.e.,
\[
\int_\Omega \rho |u|^2 \, dx \leq \int_\Omega \rho |v|^2 |\nabla u|^2 + \rho^{-1} |\mu \Delta u - \lambda (\nabla d)^T \Delta d - \nabla p(\rho)|^2 \, dx,
\]
and hence
\[
\limsup_{t \to 0^+} \int_\Omega \rho |u|^2 \, dx \leq c_0^5 + ||g||_{L_2}^2 \leq c_0^5.
\]
Here we have used the compatibility condition (1.5).

By virtue of the Gronwall inequality, from (2.27), we have
\[
\int_\Omega (\rho |u|^2 + |\nabla u|^2) \, dx + \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds \\
\lesssim c_0^5 + M^2(\epsilon_0) + \int_0^t ||\nabla u||_{H^1}^2 \, ds, \tag{2.28}
\]
for all $t \in [0, T_3]$, where $T_3 = \min\{T_3, c_0^{-2} \epsilon_2^{-4}\}$.

Next, we need to estimate $||\nabla u||_{H^1}^2$. The classical elliptic estimates applied to (2.1c), using equation (2.1b) and the assumption (2.4)-(2.7), it gives rise to
\[
||\nabla u||_{H^1} \lesssim ||\rho u||_{L^2} + ||\rho v \cdot \nabla u||_{L^2} + ||\nabla p||_{L^2} + ||\nabla u||_{L^2} + ||(\nabla d)^T \Delta d||_{L^2} \\
\lesssim \sqrt{\rho} ||\nabla u||_{L^2} + ||\nabla \rho||_{L^2} \sqrt{\rho} ||\nabla u||_{L^2} + ||\nabla u||_{L^2} + ||\nabla u||_{L^2} \\
+ ||\nabla d||_{L^2} ||d||_{L^6} + ||\nabla d||_{L^2} ||v||_{H_0^1} + ||\nabla f||_{H^1} ||\nabla u||_{L^2} \\
\leq C (M(\epsilon_0) c_0 + c_0^2 \epsilon_1^2 ||\nabla u||_{L^2} + ||\nabla u||_{L^2} + c_0^6 \epsilon_1^10) + \frac{1}{2} ||\nabla u||_{H^1}.
\]
Thus, we deduce
\[
||\nabla u||_{H^1} \lesssim M(\epsilon_0) c_0 + c_0^2 \epsilon_1^2 ||\nabla u||_{L^2} + ||\nabla u||_{L^2} + c_0^6 \epsilon_1^10. \tag{2.29}
\]
Taking (2.29) into (2.28) and using Gronwall’s inequality, we have for all $t \in [0, T_3]$,
\[
\int_\Omega (\rho |u|^2 + |\nabla u|^2) \, dx + \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds \lesssim c_0^5 + M^2(\epsilon_0), \tag{2.30}
\]
and
\[
||\nabla u||_{H^1} \lesssim M(\epsilon_0) c_0^2 \epsilon_1^2 + c_0^6 \epsilon_1^10.
\]
The term $\|\nabla^2 u\|_{L^q}$ can be estimated by the same way as $\|\nabla u\|_{H^1}$ above. In fact,
\[
\|\nabla^2 u\|_{L^q} \lesssim \|\rho u\|_{L^q} + \|\rho u \cdot \nabla u\|_{L^q} + \|\nabla p\|_{L^q} + \|\nabla u\|_{L^q} + \|\Delta d\|_{L^q} \\
\lesssim \|\rho\|_{L^\infty} \|u\|_{L^2} + (\|\rho\|_{L^\infty} \|v\|_{L^\infty} + 1) \|\nabla u\|_{L^q} + \|\nabla p\|_{L^q} + \|\nabla d\|_{H^2}^2 \\
\lesssim c_0 \|\nabla u\|_{L^2} + M(c_0)c_1^2c_2 + c_0^{\delta}c_1^{14}c_2.
\]
(2.31)

Integrating (2.31) over time and using the estimate (2.30), one has for all $t \in [0,T_5]$,
\[
\int_0^t \|\nabla^2 u\|_{L^q}^2 \, ds \lesssim M^2(c_0)c_0^2 + c_0^2,
\]
where $T_5 = \min\{T, c_5^{-28}c_2^{-4}\}$. \hfill \Box

In conclusion, it is obvious that Lemmas 2.1-2.3 imply Theorem 2.1.

2.2. Local existence of a strong solution to the linear problem (2.1). Notice that all the estimates obtained in Lemmas 2.1-2.3 are uniform for all small $\delta > 0$, we have the following theorem:

**Theorem 2.2.** If the initial data satisfies the regularity assumption (1.4) and the compatibility condition, then there exists a unique strong solution $(\rho,u,d)$ of (2.1) with the initial-boundary conditions (1.2) (1.3) such that

\[
\rho \in C([0,T_3];W^{1,q}), \quad \rho_t \in C([0,T_3];L^q);
\]
\[
u \in C([0,T_5];H^1 \cap H^2) \cap L^3(0,T_5;W^{2,q}), \quad \nu_t \in L^2(0,T_5;H^1_0),
\]
\[
\sqrt{\rho}u_t \in C([0,T_5];L^2) \cap C([0,T_5];H^3), \quad \rho \in C([0,T_5];H^1_0) \cap L^2(0,T_5;H^2).
\]

Moreover, $(\rho,u,d)$ also satisfies the inequalities (2.8),(2.9),(2.15) and (2.24).

**Proof.** From Theorem 2.1 and the estimates (2.8),(2.9),(2.15) and (2.24), by virtue of the compactness theorem, there exists a trio $(\rho^\delta, u^\delta, d^\delta)$ such that
\[
(\rho^\delta, u^\delta, d^\delta) \to (\rho,u,d) \text{ in } L^2(0,T_5;L^r \times W^{1,\alpha} \times H^3), \quad \forall \ r \in (1, +\infty), \quad \forall \ \alpha \in [2, +\infty),
\]
\[
p^\delta \to \bar{p} \text{ in } L^\infty(0,T_5;W^{1,q}) \text{ as } \delta \to 0.
\]

Hence $p = \bar{p}$, a.e..

According to the lower semi-continuity of various norms, the estimates (2.8), (2.15) and (2.24) hold also for $(\rho,u,d)$. Therefore, $(\rho,u,d)$ satisfies the system (2.1) almost everywhere on $[0,T_5] \times \Omega$, which means, $(\rho,u,d)$ is a strong solution to (2.1) with the initial-boundary condition (1.2) (1.3).

We claim the solution $(\rho,u,d)$ is unique. From Lemma 2.1, $\rho$ is the unique solution to the linear equation (2.1a). Once $\rho$ is computed uniquely from (2.1a), and then (2.1b) and (2.1c) are linear parabolic equations in terms of $d$ and $u$, respectively. Uniqueness is obvious.

Next, we prove the time continuity of the solution $(\rho,u,d)$. Since the solution to the linear equation (2.1a) is unique, then the solution from Lemma 2.1 is the same as from the approximation above, i.e.,
\[
\rho \in C([0,T_5];W^{1,q}).
\]

And we can easily obtain
\[
\rho_t \in C([0,T_5];L^q).
\]

We know from Lemma 2.2, $d_t \in L^2(0,T_5;H^2), \ d \in L^2(0,T_5;H^3)$, then
\[
\rho \in C([0,T_5];H^3).
\]
Differentiating (2.1b) with respect to both time and space, we have
\[ \nabla \dot{d}_t + \nabla (v \cdot \nabla d)_t = \theta (\nabla \Delta d_t + \nabla ((\nabla f : \nabla f)_t). \]
Combining with the estimate (2.15), we deduce
\[ \nabla d_t \in L^2(0, T_5; H^{-1}). \]
Because \( \nabla d_t \in L^2(0, T_5; H^1) \), we have \( \nabla d_t \in C([0, T_5]; L^2) \), and then
\[ d_t \in C([0, T_5]; H^1_0). \]
Coming back to the linear equation (2.1b) again, we have
\[ \Delta d \in C([0, T_5]; H^1). \]
Similarly, from Lemma 2.3, \( u \in L^2(0, T_5; H^1_0) \), \( u \in L^2(0, T_5; W^{2,q}) \), then
\[ u \in C([0, T_5]; H^1_0). \]
From the linear equation (2.1c) and the estimates (2.8), (2.15) and (2.24), we obtain
\[ \rho u_t \in L^2(0, T_5; H^1) \] and \( (\rho u_t)_t \in L^2(0, T_5; H^{-1}) \), then
\[ \rho u_t \in C([0, T_5]; L^2). \]
Coming back to the linear equation (2.1c) and taking advantage of the elliptic regularity estimate, we have
\[ u \in C([0, T_5]; H^2). \]
The proof of Theorem 2.2 is now completed. \( \Box \)

3. **Local existence in Theorem 1.1.** In this section, we prove the local existence and uniqueness of strong solution in Theorem 1.1. The proof will be divided into several steps, including constructing the approximate solutions by iteration, obtaining the uniform estimate, showing the convergence, consistency and uniqueness.

3.1. **Construction of approximate solutions and uniform estimates.** We initialize the construction of approximate solutions by choosing an initial data \( \rho \in L^2(0, T_5; H^1_0) \), \( u \in L^2(0, T_5; W^{2,q}) \), \( d \in C([0, T_5]; H^1_0). \)
Coming back to the linear equation (2.1c) and the parabolic equation (2.1b) enable us to define \( \rho^{k+1}(t, x), u^{k+1}(t, x), d^{k+1}(t, x) \) inductively (replacing \( (v, f) \) successively by \( (u^k, d^k) \), \( k = 0, 1, 2, \ldots \), and using Theorem 2.2) as the (global, i.e., \( [0, T_*] \times \bar{\Omega} \) solution of
\[
\begin{cases}
\rho^{k+1}_t + \nabla \cdot (\rho^{k+1} u^k) = 0, \\
\rho^{k+1} u^{k+1}_t + u^k \cdot \nabla d^{k+1} = \theta (\nabla d^{k+1} + (\nabla d^k : \nabla d^k + \nabla f)_t)(d^k), \\
\rho^{k+1} u^{k+1}_t + \rho^{k+1} u^k \cdot \nabla u^{k+1} + \nabla p(\rho^{k+1}) = \mu \Delta u^{k+1} - \lambda \nabla \cdot (\nabla d^{k+1} \otimes \nabla d^{k+1} - \frac{\nabla d^{k+1} \otimes \nabla d^{k+1}}{2} \bar{d}_3),
\end{cases}
\]
with the initial-boundary conditions:
\[ \rho^{k+1} \big|_{t=0} = \rho_0, \quad u^{k+1} \big|_{t=0} = u_0, \quad d^{k+1} \big|_{t=0} = d_0, \]
\[ u^{k+1} |_{\partial \Omega} = 0, \quad d^{k+1} |_{\partial \Omega} = d_0, \quad |d_0| = 1. \]

Then we get a solution sequence \( \{ (\rho^k, u^k, d^k) \} \) and every \( (\rho^k, u^k, d^k) \) satisfies the following estimates with the same \( \{ c_0, c_1, c_2, T_s \} \):

\[
\sup_{0 \leq t \leq T_s} \left( \| u^k \|_{H^1}^2 + \| d^k \|_{H^1}^2 + \| d_x^k \|_{H^1}^2 + c_1^{-10}(\| \nabla u^k \|_{H^1} + \| \nabla^2 d^k \|_{L^2}) \right) \\
+ \int_0^{T_s} (\| \nabla u^k \|_{L^2}^2 + \| u^k \|_{W^{2,\infty}}^2 + \| d_x^k \|_{H^2}^2 + \| d^k \|_{H^3}^2) \, dt \\
\lesssim M^2(c_0)^2 + c_0^2,
\]

and

\[
\sup_{0 \leq t \leq T_s} (\| \rho^k \|_{W^{1,\infty}} + \| \rho_x^k \|_{L^2}) \lesssim c_0 c_2, \\
\sup_{0 \leq t \leq T_s} \| \sqrt{\rho^k} u^k \|_{L^2} \lesssim c_0^2 + M(c_0), \\
\sup_{0 \leq t \leq T_s} (\| p^k \|_{W^{1,\infty}} + \| p_x^k \|_{L^2}) \lesssim M(c_0) c_0 c_2, \\
\sup_{0 \leq t \leq T_s} \| \nabla d^k \|_{H^2} \lesssim c_0^2 c_0 c_2.
\]

### 3.2. Convergence of the approximate sequence.

We claim that \( \{ (\rho^k, u^k, d^k) \} \) is a Cauchy sequence and thus converges. In fact, define

\[
\tilde{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \tilde{u}^{k+1} = u^{k+1} - u^k, \quad \tilde{d}^{k+1} = d^{k+1} - d^k,
\]

and

\[
\tilde{\phi}^{k+1}(t) = \| \tilde{\rho}^{k+1} \|_{L^2}^2 + \| \tilde{d}^{k+1} \|_{L^2}^2 + \| \tilde{d}^{k+1} \|_{L^2}^2 + \| \sqrt{\rho^{k+1}} u^{k+1} \|_{L^2}^2.
\]

A straightforward calculation shows that \( (\tilde{\rho}^{k+1}, \tilde{u}^{k+1}, \tilde{d}^{k+1}) \) satisfies

\[
\begin{cases}
\tilde{\rho}_t^{k+1} + \nabla \cdot (\tilde{\rho}^{k+1} u^k) + \nabla \cdot (\rho^k \tilde{u}^k) = 0, \\
\tilde{d}_t^{k+1} - \theta \Delta \tilde{d}^{k+1} = -\tilde{u}^k \cdot \nabla \tilde{d}^{k+1} - u^{k-1} \cdot \nabla \tilde{d}^{k+1} + \theta(\nabla d^k : \nabla d^k) \tilde{d}^{k+1} + \theta(\nabla d^k : \nabla d^k) \tilde{d}^{k+1}, \\
\rho^{k+1} \tilde{u}_t^{k+1} + \rho^{k+1} u^k \cdot \nabla \tilde{d}^{k+1} - \mu \Delta \tilde{u}^{k+1} + \nabla (p(\rho^{k+1}) - p(\rho^k)) \\
- \lambda \nabla \cdot (\nabla \tilde{d}^{k+1} + \nabla \tilde{d}^{k+1} \cdot \nabla d^k - \nabla \tilde{d}^{k+1} \cdot (\nabla d^k + \nabla d^k)) \|_3.
\end{cases}
\]

From (3.5)\(_1\), we have

\[
\frac{d}{dt} \| \tilde{\rho}^{k+1} \|_{L^2}^2 \lesssim \| \nabla u^k \|_{W^{1,\infty}} \| \tilde{\rho}^{k+1} \|_{L^2}^2 + (\| \nabla \rho^k \|_{L^3} + \| \rho^k \|_{L^\infty}) \| \nabla \tilde{u}^k \|_{L^2} \| \tilde{\rho}^{k+1} \|_{L^2}^2 \\
\leq (\| \nabla u^k \|_{W^{1,\infty}} + \nu^{-1}(\| \nabla \rho^k \|_{L^2}^2 + \| \rho^k \|_{L^\infty}^2)) \| \tilde{\rho}^{k+1} \|_{L^2}^2 + \nu \| \nabla u^k \|_{L^2}^2.
\]

For (3.5)\(_2\), on the one hand, we multiply (3.5)\(_2\) by \( \tilde{d}^{k+1} \) and integrate over \( \Omega \) to obtain

\[
\frac{d}{dt} \| \tilde{d}^{k+1} \|_{L^2}^2 + \| \nabla \tilde{d}^{k+1} \|_{L^2}^2 \\
\leq C \| \tilde{d}^{k+1} \|_{L^2}^2 + \| \nabla \tilde{d}^{k+1} \|_{L^2}^2 + \| \nabla \tilde{d}^{k+1} \|_{L^2}^2 + \| \nabla \tilde{d}^{k+1} \|_{L^2}^2 + \| \tilde{d}^{k+1} \|_{L^2}^2 \\
+ \| \nabla \tilde{d}^{k+1} \|_{L^2}^2 + \| \nabla \tilde{d}^{k+1} \|_{L^2}^2 + \nu(\| \nabla \tilde{u}^k \|_{L^2}^2 + \| \nabla \tilde{d}^{k+1} \|_{L^2}^2),
\]

where we have used

\[
\frac{1}{2} \int_\Omega u^{k-1} \cdot \nabla d^{k+1} \cdot d^{k+1} \, dx = \frac{1}{2} \int_\Omega u^{k-1} \cdot \nabla |d^{k+1}|^2 \, dx = -\frac{1}{2} \int_\Omega |d^{k+1}|^2 \nabla \cdot u^{k-1} \, dx,
\]
and
\[
\int_{\Omega} (\nabla d^k : \nabla \bar{d}^{k+1}) \, \bar{d}^{k-1} \cdot \bar{d}^{k+1} \, dx
\leq \|\nabla d^k\|_{L^\infty} \|\bar{d}^{k-1}\|_{L^\infty} \|\nabla \bar{d}^{k+1}\|_{L^2} \|\bar{d}^{k+1}\|_{L^2}
\leq C_0 \|\nabla d^k\|_{H^2}^2 \|\bar{d}^{k-1}\|_{H^2}^2 \|\bar{d}^{k+1}\|_{L^2}^2 + \frac{\theta}{2} \|\nabla \bar{d}^{k+1}\|_{L^2}^2.
\]

On the other hand, we multiply (3.5) by \(\Delta \bar{d}^{k+1}\) and integrate over \(\Omega\) to get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \bar{d}^{k+1}\|_{L^2}^2 + \theta \|\Delta \bar{d}^{k+1}\|_{L^2}^2 := \sum_{i=1}^{5} N_i. \tag{3.8}
\]

Bearing in mind that \(\|\nabla^2 \bar{d}^{k+1}\|_{L^2} \approx \|\Delta \bar{d}^{k+1}\|_{L^2}\), we estimate the terms \(N_i\) \((i = 1, 2, \cdots, 5)\) as follows:

\[N_1 = \int_{\Omega} \bar{u}^k \cdot \nabla \bar{d}^{k+1} \cdot \Delta \bar{d}^{k+1} \, dx \leq \|\nabla \bar{u}^k\|_{L^2} \|\nabla \bar{d}^{k+1}\|_{H^2} \|\bar{d}^{k+1}\|_{L^2} + \|\bar{u}^k\|_{H^2} \|\nabla^2 \bar{d}^{k+1}\|_{L^2} \|\nabla \bar{d}^{k+1}\|_{L^2} \tag{3.9}
\]

\[\leq C_0 \|\nabla \bar{d}^{k+1}\|_{H^2}^2 \|\nabla \bar{d}^{k+1}\|_{L^2}^2 + \frac{\theta}{4} \|\Delta \bar{d}^{k+1}\|_{L^2}^2, \tag{3.10}
\]

\[N_3 = -\theta \int_{\Omega} (\nabla d^k : \nabla \bar{d}^{k+1}) \, \bar{d}^k \cdot \Delta \bar{d}^{k+1} \, dx \leq \|\nabla d^k\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{L^2} + \frac{1}{2\nu} \|\nabla d^k\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{L^2}^2 + \frac{1}{2\nu} \|\nabla \bar{d}^{k+1}\|_{L^2}^2 \tag{3.11}
\]

\[\leq \|\bar{d}^k\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{L^2} + \frac{1}{2\nu} \|\nabla d^k\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{L^2}^2 + \frac{\theta}{4} \|\Delta \bar{d}^{k+1}\|_{L^2}^2.
\]

\[N_4 = -\theta \int_{\Omega} (\nabla d^k : \nabla \bar{d}^{k+1}) \, \bar{d}^{k-1} \cdot \Delta \bar{d}^{k+1} \, dx \leq \|\nabla d^k\|_{L^\infty} \|\bar{d}^{k-1}\|_{L^\infty} \|\nabla \bar{d}^{k+1}\|_{L^2} \|\Delta \bar{d}^{k+1}\|_{L^2} \tag{3.12}
\]

\[\leq C_0 \|\nabla d^k\|_{H^2}^2 \|\bar{d}^{k-1}\|_{H^2}^2 \|\bar{d}^{k+1}\|_{L^2}^2 + \frac{\theta}{4} \|\Delta \bar{d}^{k+1}\|_{L^2}^2,
\]

\[N_5 = -\theta \int_{\Omega} (\nabla d^k : \nabla \bar{d}^k) \, d^{k-1} \cdot \Delta \bar{d}^{k+1} \, dx \leq \|\nabla d^k\|_{H^1} \|\nabla \bar{d}^{k-1}\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{L^2} \|\nabla \bar{d}^{k+1}\|_{L^2} \tag{3.13}
\]

\[\leq \|\nabla d^k\|_{H^1} \|\nabla \bar{d}^{k-1}\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{L^2} \|\nabla \bar{d}^{k+1}\|_{L^2} + \|\nabla d^k\|_{H^1} \|\nabla \bar{d}^{k+1}\|_{L^2} \|\Delta \bar{d}^{k+1}\|_{L^2} \|\nabla \bar{d}^{k+1}\|_{L^2}.
\]
\[
\frac{d}{dt}\|\nabla \tilde{d}^{k+1}\|_{L^2}^2 + \|\Delta \tilde{d}^{k+1}\|_{L^2}^2 \\
\lesssim \left( C_\nu(\|\nabla \tilde{d}^{k+1}\|_{H^2}^2 + \|\nabla^2 \tilde{d}^{k+1}\|_{H^1}^2) + \|\nabla \tilde{u}^{k-1}\|_{W^{1,q}} + \|\tilde{u}^{k-1}\|_{W^{1,q}} \\
+ \|\nabla \tilde{d}^{k}\|_{H^2}^2 \|\tilde{d}^{k-1}\|_{H^2}^2 + C_\nu(\|\nabla^2 \tilde{d}^{k}\|_{H^1}^2 \|\nabla \tilde{d}^{k+1}\|_{H^1}^2 + \|\nabla \tilde{d}^{k}\|_{H^1}^1 \|\nabla^2 \tilde{d}^{k+1}\|_{H^1}^2 \\
+ \|\nabla \tilde{d}^{k}\|_{H^2}^4 \|\nabla \tilde{d}^{k+1}\|_{H^1}^4) + C_\nu(\|\tilde{d}^{k}\|_{L^2}^2 \|\tilde{d}^{k-1}\|_{H^2}^2 + \|\nabla^2 \tilde{d}^{k}\|_{H^1}^2 \|\tilde{d}^{k-1}\|_{H^2}^2 \\
+ \|\nabla \tilde{d}^{k}\|_{H^2}^2 \|\nabla \tilde{d}^{k+1}\|_{H^1}^2)\right) \|\nabla \tilde{d}^{k+1}\|_{L^2}^2 + \nu(\|\nabla \tilde{u}^{k}\|_{L^2}^2 + \|\nabla \tilde{d}^{k}\|_{L^2}^2 + \|\Delta \tilde{d}^{k}\|_{L^2}^2).
\]

Consequently, in view of (3.9)-(3.13) and (3.8), we obtain

\[
\frac{d}{dt}\|\nabla \tilde{d}^{k+1}\|_{L^2}^2 + \frac{\mu}{2} \int_\Omega |\nabla \tilde{u}^{k+1}|^2 \, dx + \mu \int_\Omega |\nabla \tilde{u}^{k+1}|^2 \, dx \\
= -\int_\Omega \rho^{k+1}(\nabla \tilde{u}^k + \tilde{u}^{k+1}) \cdot \tilde{u}^{k+1} \, dx - \int_\Omega \rho^{k+1} \tilde{u}^k \cdot \nabla \tilde{u}^k \cdot \tilde{u}^{k+1} \, dx \\
+ \mu \int_\Omega (p(\rho^{k+1}) - p(\rho^k)) \nabla \cdot \tilde{u}^{k+1} \, dx \\
+ \lambda \int_\Omega \left( \nabla \tilde{d}^{k+1} \circ \nabla \tilde{u}^{k+1} + \nabla \tilde{d}^{k+1} \circ \nabla \tilde{d}^k - \frac{\nabla \tilde{d}^{k+1} \cdot (\nabla \tilde{u}^{k+1} + \nabla \tilde{d}^k)_{\parallel}}{2} \right) : \nabla \tilde{u}^{k+1} \, dx \\
:= \sum_{i=6}^{9} N_i,
\]
where

\[
N_6 \leq \|\tilde{\rho}^{k+1}\|_{L^2} \|\tilde{u}^{k-1}\|_{L^2} \|\nabla \tilde{u}^k\|_{L^6} \|\tilde{u}^{k+1}\|_{L^6} + \|\tilde{\rho}^{k+1}\|_{L^2} \|\nabla \tilde{u}^k\|_{L^6} \|\tilde{u}^{k+1}\|_{L^6} \|\tilde{u}^k\|_{L^6} \|\tilde{u}^{k+1}\|_{L^6} \\
\lesssim \|\rho^{k+1}\|_{L^2} \|\nabla \tilde{u}^{k-1}\|_{L^2} \|\nabla \tilde{u}^k\|_{H^1} + \|\tilde{u}^k\|_{L^2} \|\nabla \tilde{u}^k\|_{L^6}^2 \|\nabla \tilde{u}^{k+1}\|_{L^2} \\
\leq C_\nu \|\rho^{k+1}\|_{L^2}^2 \|\nabla \tilde{u}^{k-1}\|_{L^2}^2 \|\nabla \tilde{u}^k\|_{H^1} + \|\tilde{u}^k\|_{L^2} \|\nabla \tilde{u}^k\|_{L^2} \|\tilde{u}^{k+1}\|_{L^2} \|\tilde{u}^k\|_{L^2} \|\tilde{u}^{k+1}\|_{L^2}^2, \\
N_7 \leq \|\sqrt{\rho^{k+1}}\|_{L^6} \|\nabla \tilde{u}^k\|_{L^6} \|\nabla \tilde{u}^k\|_{L^6} \|\sqrt{\rho^{k+1}} \tilde{u}^{k+1}\|_{L^2} \\
\lesssim \|\sqrt{\rho^{k+1}}\|_{L^6} \|\nabla \tilde{u}^k\|_{L^6} \|\nabla \tilde{u}^k\|_{H^1} \|\sqrt{\rho^{k+1}} \tilde{u}^{k+1}\|_{L^2} \\
\leq C_\nu \|\rho^{k+1}\|_{L^2} \|\nabla \tilde{u}^k\|_{H^1} \|\sqrt{\rho^{k+1}} \tilde{u}^{k+1}\|_{L^2}^2 + \nu \|\nabla \tilde{u}^k\|_{L^2}^2, \\
N_8 \lesssim \|p(\rho^{k+1}) - p(\rho^k)\|_{L^2} \|\nabla \tilde{u}^{k+1}\|_{L^2} \\
\leq C_\mu M^2(c_0) \|\tilde{\rho}^{k+1}\|_{L^2}^2 + \frac{\mu}{4} \|\nabla \tilde{u}^{k+1}\|_{L^2}^2, \\
N_9 \lesssim (\|\nabla \tilde{d}^k\|_{H^2} + \|\nabla \tilde{d}^{k+1}\|_{H^2}) \|\nabla \tilde{d}^{k+1}\|_{L^2} \|\nabla \tilde{u}^{k+1}\|_{L^2} \\
\leq C_\mu (\|\nabla \tilde{d}^k\|_{H^2} + \|\nabla \tilde{d}^{k+1}\|_{H^2}) \|\nabla \tilde{d}^{k+1}\|_{L^2}^2 + \frac{\mu}{4} \|\nabla \tilde{u}^{k+1}\|_{L^2}^2.
\]
Accordingly, relation (3.15) reduces to
\[
\frac{d}{dt} \left( \sqrt{\rho^{k+1}} \nu^{k+1} \right)_{L^2} + \frac{d}{dt} \left( \nabla \nu^{k+1} \right)_{L^2} \\
\lesssim (\|\nabla u^{k-1}\|_{L^2}^2 + \| \nabla u^k \|_{H^1} + \| \nabla u^k \|_{L^2} + M^2(\epsilon_o)) \left( \|\rho^{k+1}\|_{L^2}^2 + \nu \|\nabla \nu^{k+1}\|_{L^2}^2 \right) \\
+ C \nu \|\rho^{k+1}\|_{L^2} \left( \|\nabla \nu^k \|_{H^1}^2 + \|\nabla u^{k+1}\|_{L^2}^2 + (\|\nabla d^k\|_{H^1}^2) \right)
\]
Finally, by summing (3.6), (3.7), (3.14) and (3.16), it yields to
\[
\frac{d}{dt} \Phi^{k+1}(t) + \| \nabla d^{k+1} \|_{L^2}^2 + \| \Delta d^{k+1} \|_{L^2}^2 + \| \nabla u^{k+1} \|_{L^2}^2 \leq A^k(t) \Phi^{k+1}(t) + \nu \left( \| \nabla d^k \|_{L^2}^2 + \| \Delta d^k \|_{L^2}^2 + \| \nabla u^k \|_{L^2}^2 \right)
\]
where
\[
A^k(t) = \| \nabla u^k \|_{H^1} + \| \nabla u^{k-1} \|_{L^2}  + \| \nabla u^k \|_{H^1} + \| \nabla u^k \|_{L^2} + \| \nabla d^k \|_{H^2} + \| \nabla d^{k+1} \|_{L^2} \left( \|\rho^k\|_{L^2}^2 + \|\rho^{k+1}\|_{L^2} \right) \\
+ \| \nabla d^{k+1} \|_{H^1} + \| \Delta d^{k+1} \|_{L^2} + \| \nabla u^{k+1} \|_{L^2} \right)
\]
Making use of the uniform estimates (3.2)-(3.4), we have
\[
\int_0^t A^k(s) \, ds \leq 1 + (1 + \nu^{-1}) \, t
\]
for all \( t \in [0, T_*] \).
Now we can apply Gronwall’s inequality to (3.17) in order to get the following estimate:
\[
\Phi^{k+1}(t) + \int_0^t (\| \nabla d^{k+1} \|_{L^2}^2 + \| \Delta d^{k+1} \|_{L^2}^2 + \| \nabla u^{k+1} \|_{L^2}^2 ) \, ds
\]
\[
\leq C \nu \exp \left( C + C(1 + \nu^{-1}) \, t \right) \int_0^t (\| \nabla d^k \|_{L^2}^2 + \| \Delta d^k \|_{L^2}^2 + \| \nabla u^k \|_{L^2}^2 ) \, ds.
\]
Since
\[
C \nu \exp \left( C + C(1 + \nu^{-1}) \, t \right) \leq \frac{1}{2}
\]
for suitable (small) \( \nu, T_*(\leq T_*) \) and all \( t \in [0, T_*] \), then it is easy to derive that
\[
\sum_{k=1}^\infty \sup_{0 \leq t \leq T_*} \Phi^{k+1}(t) + \sum_{k=1}^\infty \int_0^{T_*(k)} (\| \nabla d^{k+1} \|_{L^2}^2 + \| \Delta d^{k+1} \|_{L^2}^2 + \| \nabla u^{k+1} \|_{L^2}^2 ) \, dt
\]
\[
\leq C < \infty,
\]
and which implies that \( \{ \rho^k, u^k, d^k \}_{k=1}^\infty \) is a Cauchy sequence and yields strong convergence:
\[
\rho^k \to \rho \quad \text{in} \quad L^\infty(0, T_*; L^2), \quad u^k \to u \quad \text{in} \quad L^2(0, T_*; H^1_0), \quad \text{and} \quad d^k \to d \quad \text{in} \quad L^\infty(0, T_*; H^1) \cap L^2(0, T_*; H^2).
\]
We remark here that the time of existence $T_*$ depends (continuously) on the norms of the data, on the bound for the density, on the domain and on the regularity parameters.

3.3. The limit is a solution. From the argument above, we know the initial-boundary problem (1.1)-(1.3) has a weak solution $(\rho, u, d)$. Passing to a subsequence of $\{(\rho^k, u^k, d^k)\}_{k=1}^{\infty}$ if necessary the limit for $k \to \infty$, it follows from the estimates (3.2)-(3.4) that $(\rho^k, u^k, d^k)$ converges to $(\rho, u, d)$ in an obvious weak or weak$^*$ sense. By using the lower semi-continuity of various norms, we also have $(\rho, u, d)$ satisfies the regularity estimate:

$$
\sup_{0 \leq t \leq T_*} \left( \|\rho\|_{W^{1, q}} + \|\rho_t\|_{L^q} + \|u\|_{H_0^3} + \|p\|_{W^{1, q}} + \|p_t\|_{L^q} + \|\dot{d}\|_{H^1} + \|d_t\|_{H_0^1} 
+ \|\nabla u\|_{H^1} + \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{H^2} \right) 
+ \int_0^{T_*} \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|u\|_{W^{2, q}}^2 + \|d_t\|_{H^2}^2 + \|d\|_{H^3}^2 \right) \, dt 
\leq C.
$$

(3.18)

We claim all those nonlinear terms in (3.1) converge to their corresponding terms in (1.1) almost everywhere in $\Omega \times (0, T_*)$. Indeed,

$$
\begin{align*}
&\|\nabla \cdot (\rho^{k+1}u^k) - \nabla \cdot (\rho u)\|_{L_{T_*}^2(L^{\frac{3q}{2q-3}})} \\
&\leq \|u^k - u\|_{L_{T_*}^2(L^2)} \|\nabla \rho^{k+1}\|_{L_{T_*}^q(L^q)} + \|u\|_{L_{T_*}^2(L^2)} \|\nabla \rho^{k+1} - \nabla \rho\|_{L_{T_*}^q(L^q)} \\
&\quad + \|\nabla u - \nabla u\|_{L_{T_*}^2(L^2)} \|\rho^{k+1}\|_{L_{T_*}^q(L^q)} + \|\nabla u\|_{L_{T_*}^2(L^2)} \|\rho^{k+1} - \rho\|_{L_{T_*}^q(L^q)} \\
&\leq \|u^k - u\|_{L_{T_*}^2(H^1)} \|\rho^{k+1}\|_{L_{T_*}^q(W^{1,q})} + \|u\|_{L_{T_*}^2(L^2)} \|\rho^{k+1} - \rho\|_{L_{T_*}^q(W^{1,q})} \\
&\quad + \|u^k - u\|_{L_{T_*}^2(H^1)} \|\rho^{k+1}\|_{L_{T_*}^q(L^q)} + \|u\|_{L_{T_*}^2(W^{1,q})} \|\rho^{k+1} - \rho\|_{L_{T_*}^q(L^q)} \\
&\to 0 \quad \text{as } k \to \infty,
\end{align*}
$$

and

$$
\begin{align*}
&\|\rho^{k+1}u^k \cdot \nabla u^{k+1} - \rho u \cdot \nabla u\|_{L_{T_*}^3(L^2)} \\
&\leq \|\rho^{k+1} - \rho\|_{L_{T_*}^q(L^q)} \|u^k\|_{L_{T_*}^2(L^2)} \|\nabla u^{k+1}\|_{L_{T_*}^q(L^q)} \\
&\quad + \|\rho\|_{L_{T_*}^q(L^q)} \|u^k\|_{L_{T_*}^2(L^2)} \|\nabla u^{k+1} - \nabla u\|_{L_{T_*}^2(L^2)} \\
&\quad + \|\rho\|_{L_{T_*}^q(L^q)} \|u^k - u\|_{L_{T_*}^2(L^2)} \|\nabla u\|_{L_{T_*}^2(L^2)} \\
&\to 0 \quad \text{as } k \to \infty.
\end{align*}
$$

Meanwhile,

$$
\begin{align*}
&\|u^k \cdot \nabla d^{k+1} - u \cdot \nabla d\|_{L_{T_*}^2(L^3)} \\
&\leq \|u^k - u\|_{L_{T_*}^2(L^3)} \|\nabla d^{k+1}\|_{L_{T_*}^q(L^q)} + \|u\|_{L_{T_*}^2(L^2)} \|\nabla d^{k+1} - \nabla d\|_{L_{T_*}^q(L^q)} \\
&\leq \|u^k - u\|_{L_{T_*}^2(H^1)} \|d^{k+1}\|_{L_{T_*}^q(H^1)} + \|u\|_{L_{T_*}^2(L^2)} \|d^{k+1} - d\|_{L_{T_*}^2(H^2)} \\
&\to 0 \quad \text{as } k \to \infty,
\end{align*}
$$
Thus, passing to the limit in (3.1) as $k \to \infty$, we have used the fact that $\nabla \cdot (\nabla d \circ \nabla d) = \nabla (\nabla d^2) + (\nabla d)^T \Delta d$, and that the body force term $\nabla \cdot (\nabla d^2 E_3) = \nabla (\nabla d^2)$. Therefore, we obtain

$$
\frac{1}{2} \langle |d|^2 \rangle_t + \frac{1}{2} u \cdot \nabla (|d|^2) = \Delta d \cdot d + |\nabla d|^2 |d|^2.
$$

Since

$$
\Delta (|d|^2) = 2 |\nabla d|^2 + 2d \cdot (\Delta d),
$$

then it follows that

$$
\frac{1}{2} \langle |d|^2 \rangle_t + \frac{1}{2} u \cdot \nabla (|d|^2) = \frac{1}{2} \Delta (|d|^2) - |\nabla d|^2 + |\nabla d|^2 |d|^2.
$$

Therefore, it is easy to deduce that

$$
(|d|^2 - 1)_t - \Delta (|d|^2 - 1) + u \cdot \nabla (|d|^2 - 1) - 2|\nabla d|^2 (|d|^2 - 1) = 0. \tag{3.19}
$$

Multiplying (3.19) by $|d|^2 - 1$ and then integrating over $\Omega$, using (1.3), we get

$$
\frac{d}{dt} \int_{\Omega} (|d|^2 - 1)^2 \, dx \leq \int_{\Omega} \nabla \cdot u \, (|d|^2 - 1)^2 \, dx + 4 \int_{\Omega} |\nabla d|^2 (|d|^2 - 1)^2 \, dx \tag{3.20}
$$

$$
\leq \left( \|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2 \right) \int_{\Omega} (|d|^2 - 1)^2 \, dx.
$$

Recalling (3.2), we know that $\|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}^2 \in L^1(0, T_*)$. Notice that

$$
\int_{\Omega} (|d|^2 - 1)^2 \, dx = 0, \quad \text{at time } t = 0.
$$
Thus, using estimate (3.20) together with Grönwall’s inequality, it yields $|d| = 1$ in $\Omega \times (0, T_*)$.

3.4. **Uniqueness and continuity.** Let $(\rho_1, u_1, d_1)$ and $(\rho_2, u_2, d_2)$ be two solutions to (1.1) with the initial-boundary conditions (1.2) (1.3). Denote

$$
\bar{\rho} = \rho_1 - \rho_2, \quad \bar{u} = u_1 - u_2, \quad \bar{d} = d_1 - d_2.
$$

Note that $(\bar{\rho}, \bar{u}, \bar{d})$ satisfies the following system:

$$
\begin{aligned}
\dot{\bar{\rho}} + \nabla \cdot (\bar{\rho} \bar{u}) + \nabla \cdot (\rho_1 \bar{u}) &= 0, \\
\rho_1 \dot{\bar{u}} + \rho_1 \bar{u} \cdot \nabla u_2 + \rho_1 \Upsilon u_1 \cdot \nabla \bar{u} - \mu \Delta \bar{u} + \nabla (\rho_1 - p(\rho_1)) \\
&= -\bar{\rho}(u_{2t} + u_2 \cdot \nabla u_2) - \lambda(\nabla \bar{d})^T \Delta \bar{d}_1 - \lambda(\nabla \bar{d}_2)^T \Delta \bar{d}_1, \\
\dot{\bar{d}}_t - \theta \Delta \bar{d} &= -u_1 \cdot \nabla \bar{d} - \bar{u} \cdot \nabla d_2 + \theta (|\nabla d_1|^2 \bar{d} + (\nabla d_1 + \nabla d_2) : \nabla d_2)
\end{aligned}
$$

with the initial-boundary conditions:

$$(\bar{\rho}, \bar{u}, \bar{d})|_{t=0} = (0, 0, 0), \quad (\bar{u}, \bar{d})|_{\partial \Omega} = (0, 0).$$

Define a function

$$
\Psi(t) = \|\bar{\rho}\|_{L^2}^2 + \|\bar{d}\|_{L^2}^2 + \|\nabla \bar{d}\|_{L^2}^2 + \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2.
$$

Repeating the similar argument in subsection 3.2, we get the following estimates:

$$
\begin{aligned}
\frac{d}{dt} \|\bar{\rho}\|_{L^2}^2 &\lesssim (\|\nabla u_2\|_{W^{1,4}} + \|\nabla \rho_1\|_{L^\infty} + \|\nabla \rho_1\|_{L^\infty}) \|\bar{\rho}\|_{L^2}^2 + \frac{1}{4} \|\nabla \bar{u}\|_{L^2}^2, \\
\frac{d}{dt} \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 &\lesssim (\|\nabla u_1\|_{L^3}^2 + \sqrt{\rho_1} \|\nabla \bar{u}\|_{L^2}^2) \|\sqrt{\rho_1} \bar{u}\|_{L^2}^2 \\
&+ (\|\nabla u_2\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2) \|\nabla \bar{u}\|_{L^2}^2, \\
\frac{d}{dt} \|\bar{d}\|_{L^2}^2 + \|\nabla \bar{d}\|_{L^2}^2 &\lesssim (\|\nabla u_1\|_{W^{1,4}} + \|\nabla \bar{d}\|_{L^2}^2) \|\nabla \bar{d}\|_{L^2}^2 \\
&+ \frac{1}{4} \|\nabla \bar{u}\|_{L^2}^2, \\
\frac{d}{dt} \|\nabla \bar{d}\|_{L^2}^2 + \|\Delta \bar{d}\|_{L^2}^2 &\lesssim (\|\nabla d_1\|_{L^2}^2 + \|\nabla d_1\|_{L^2}^2) \|\nabla \bar{d}\|_{L^2}^2 \\
&+ \frac{1}{4} \|\nabla \bar{u}\|_{L^2}^2,
\end{aligned}
$$

and

$$
\frac{d}{dt} \Psi(t) + \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla \bar{d}\|_{L^2}^2 + \|\Delta \bar{d}\|_{L^2}^2 \lesssim A(t) \Psi(t),
$$

where

$$
A(t) = \|\nabla u_1\|_{W^{1,4}} + \|\nabla u_2\|_{W^{1,4}} + \|u_1\|_{W^{1,4}} + \|\nabla \rho_1\|_{L^\infty} + \|\rho_1\|_{L^\infty} + \|\nabla u_2\|_{L^2}^2 \|\nabla u_2\|_{L^4}^2 \\
+ \|u_2\|_{L^2} \|\nabla u_2\|_{L^2} + \|\rho_1\|_{L^\infty} \|\nabla u_1\|_{L^4}^2 + \sqrt{\rho_1} \|\nabla \bar{u}\|_{L^2}^2 \\
+ \|\nabla d_1\|_{L^2}^2 + \|\nabla d_1\|_{L^2}^2 + \|\nabla u_2\|_{L^2}^2 \|\nabla u_2\|_{L^4}^2 + \|\nabla d_1\|_{L^2}^2 + \|\nabla d_2\|_{L^2}^2)(1 + \|d_2\|_{L^2}^2),
$$

and obviously, it follows from the estimate (3.18) that $A(t) \in L^1(0, T_*)$. 

Now a straightforward result follows from the Gronwall’s inequality, that
\[ \Psi(t) + \int_0^t (\|\Delta \bar{d}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2) \, ds \leq 0, \quad t \in [0, T_*], \]
and it in turn yields:
\[ \bar{\rho} = 0, \quad \bar{\bar{u}} = 0, \quad \bar{d} = 0, \]
which implies the property of uniqueness.

Following the argument of uniqueness, we can also easily prove that if \((\rho, u, d)\) and \((\bar{\rho}, \bar{u}, \bar{d})\) are solutions to (1.1) and (1.3) with different initial data \((\rho_0, u_0, d_0)\) and \((\bar{\rho}_0, \bar{u}_0, \bar{d}_0)\), then for all \(t \in [0, T_*]\), we have
\[ (\|\bar{\rho}\|_{L^2}^2 + \|\bar{d}\|_{L^2}^2 + \|\sqrt{\rho} \bar{u}\|_{L^2}^2)(t) + \int_0^t (\|\Delta \bar{d}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2) \, ds \to 0 \quad (3.23) \]
as \((\bar{\rho}_0, \bar{u}_0, \nabla \bar{d}_0) \to (\rho_0, u_0, \nabla d_0)\) in \(W^{1,q} \times (H^2)^3 \times (H^2)^3\), where
\[ \bar{\rho} = \rho - \bar{\bar{\rho}}, \quad \bar{\bar{u}} = u - \bar{u}, \quad \bar{\bar{d}} = d - \bar{d}. \]

Since \(d\) and \(\bar{d}\) satisfy (1.1c), then
\[ \bar{d}_t + \bar{\bar{u}} \cdot \nabla d + \bar{u} \cdot \nabla \bar{d} = \theta (\Delta \bar{d} + |\nabla d|^2 \bar{\bar{d}} + (\nabla d + \nabla \bar{d}) : \nabla \bar{d} \bar{d}). \quad (3.24) \]
Now we multiply (3.24) by \(\Delta \bar{d}\), and integrate over \(\Omega\) while bearing in mind the fact that \(\sup_{t} \|\nabla d\|_{H^2} \leq C\) and \(\sup_{t} \|\nabla \bar{u}\|_{H^1} \leq C\) (see (3.18)), that the Sobolev imbedding \(H^2(\Omega) \subset L^\infty(\Omega)\), and that the boundary conditions, that
\[ \frac{d}{dt} \int_{\Omega} |\Delta \bar{u}|^2 \, dx + \int_{\Omega} |\nabla \bar{d}|^2 \, dx \]
\[ \lesssim \|\nabla \bar{u}\|_{L^2}^2 + \left(\|\nabla \bar{u}\|_{H^1}^2 + \|\nabla d\|_{H^2}^4 + \|\nabla d\|_{L^\infty}^2 \|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^6}^2 \right) \]
\[ + \|\nabla \bar{d}\|_{L^6}^2 \right) \|\Delta \bar{d}\|_{L^2}^2. \quad (3.25) \]
Recalling the estimate (3.18), applying Gronwall’s inequality to (3.25), making use of the elliptic estimate \(\|\bar{d}\|_{H^2} \lesssim \|\Delta \bar{d}\|_{L^2}\) and the convergence (3.23), we get, for all \(t \in [0, T_*]\),
\[ \int_{\Omega} |\Delta \bar{d}|^2 \, dx + \int_0^t \|\nabla \bar{d}_t\|_{L^2}^2 \, ds \to 0 \quad (3.26) \]
as \((\rho_0, u_0, \nabla d_0) \to (\rho_0, u_0, \nabla d_0)\) in \(W^{1,q} \times (H^2)^3 \times (H^2)^3\).

Similarly, for the momentum conservation equation, we have
\[ \rho \bar{d}_t + \bar{\rho} \bar{u}_t + \rho \bar{u} \cdot \nabla \bar{u} + \rho \bar{\bar{u}} \cdot \nabla \bar{u} + \nabla (\rho(p) - p(\rho)) \]
\[ = \mu \Delta \bar{u} - \lambda (\nabla \bar{d})^\top \Delta \bar{d} - \lambda (\nabla \bar{d})^\top \Delta \bar{d}. \quad (3.27) \]
On the one hand, similarly from (3.21),(3.22), \(\bar{\rho}\) satisfies
\[ \frac{d}{dt} \|\bar{\rho}\|_{L^2}^2 \lesssim (\|\nabla \bar{u}\|_{W^{1,s}}^2 + \|\nabla \rho\|_{L^2}^2 + \|\rho\|_{L^\infty}^2) \|\bar{\rho}\|_{L^2}^2 + \eta \|\nabla \bar{u}\|_{L^2}^2 \]
for suitable (small) \(\eta\), and \(\bar{u}\) satisfies
\[ \frac{d}{dt} \|\sqrt{\rho} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 \]
\[
\begin{align*}
\lesssim (|\rho|_{L^3} \|\nabla u\|_{L^2}^2 & + \|\sqrt{\rho}\|_{L^\infty}^2 (\|\nabla u\|_{L^2}^2 + \|\nabla \tilde{u}\|_{L^2}^2)) \|\nabla \tilde{u}\|_{L^2}^2 \\
+ (\|\nabla \tilde{u}\|_{L^2}^2 \|\nabla \tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{L^2} \|\nabla \tilde{u}\|_{L^2} + M^2(c_0)) \|\tilde{p}\|_{L^2}^2 \\
+ (\|\nabla d\|_{L^2}^2 + \|\nabla \tilde{d}\|_{L^2}^2)) \|\nabla \tilde{d}\|_{L^2}^2.
\end{align*}
\]

On the other hand, multiplying (3.27) by $\tilde{u}$ and integrating over $\Omega$, we obtain
\[
\frac{d}{dt} \|\nabla \tilde{u}\|_{L^2}^2 + \|\sqrt{\tilde{\rho}} \tilde{u}\|_{L^2}^2
\lesssim (\|\rho\|_{L^3} \|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}\|_{L^\infty}^2 \|\nabla \tilde{u}\|_{L^2}^2) \|\nabla \tilde{u}\|_{L^2}^2
+ \|\tilde{u}\|_{L^2} (\|\nabla d\|_{H^2} + \|\nabla^2 \tilde{u}\|_{H^1}^2) \|\nabla \tilde{d}\|_{L^2}
+ (\|\tilde{u}\|_{L^3} + \|\tilde{u}\|_{L^6} \|\nabla \tilde{u}\|_{L^6} + M(c_0)) \|\nabla \tilde{u}\|_{L^2} \|\tilde{p}\|_{L^2} + \|\nabla \tilde{d}\|_{H^2} \|\nabla \tilde{u}\|_{L^2} \|\nabla \tilde{d}\|_{L^2}.
\]

With (3.23) and (3.26) at hand, evoking the Poincaré inequality and the estimate (3.18), we conclude that
\[
\|\tilde{u}\|_{H^1}(t) + \int_0^t \|\sqrt{\tilde{\rho}} \tilde{u}\|_{L^2}^2 \, ds \to 0 \tag{3.28}
\]

as $(\tilde{\rho}_0, \tilde{u}_0, \nabla \tilde{d}_0) \to (\rho_0, u_0, \nabla d_0)$ in $W^{1,q} \times (H^2)^3 \times (H^2)^9$.

Finally, it is easy to show that
\[
\tilde{\rho}_t + u \cdot \nabla \tilde{\rho} + \tilde{\rho} \nabla \cdot u + \nabla \tilde{\rho} \cdot \tilde{u} + \tilde{\rho} \nabla \cdot \tilde{u} = 0. \tag{3.29}
\]

Multiplying (3.29) by $q \tilde{\rho}^{q-1}$ and integrating over $\Omega$, using the Hölder inequality, we have
\[
\frac{d}{dt} \|\tilde{\rho}\|_{L^q}^q \lesssim \|\nabla u\|_{W^{1,q}} \|\tilde{p}\|_{L^2} + \|\nabla \tilde{u}\|_{L^q} \|\nabla \tilde{\rho}\|_{L^q} \|\tilde{p}\|_{L^q}^{-1}.
\]

Making use of the estimate (3.18) and the convergence (3.28), we can apply Gronwall's inequality to the above inequality to get, for all $t \in [0, T_\ast]$,
\[
\|\tilde{\rho}\|_{L^q}(t) \to 0 \tag{3.30}
\]

as $(\tilde{\rho}_0, \tilde{u}_0, \nabla \tilde{d}_0) \to (\rho_0, u_0, \nabla d_0)$ in $W^{1,q} \times (H^2)^3 \times (H^2)^9$.

Consequently, together with the equations (1.1b) and (1.1c), we can obtain, for all $t \in [0, T_\ast]$,
\[
\begin{cases}
\|\tilde{d}\|_{L^2}(t) \to 0, & \forall t \in [0, T_\ast], \\
\|\tilde{d}\|_{L^2_{\max}(H^2)} \to 0, & \|\tilde{u}\|_{L^2_{\max}(H^2)} \to 0
\end{cases} \tag{3.31}
\]

as $(\tilde{\rho}_0, \tilde{u}_0, \nabla \tilde{d}_0) \to (\rho_0, u_0, \nabla d_0)$ in $W^{1,q} \times (H^2)^3 \times (H^2)^9$.

To conclude, (3.23), (3.26), (3.28), (3.30) and (3.31) complete the proof of the continuity in Theorem 1.1.

4. Proof of Theorem 1.2. Suppose that there are two positive constants $\xi (\lt 1)$ and $C$ such that
\[
\max\{\|\rho\|_{W^{1,q}}, \|u_0\|_{H^2}, \|d_0 - u\|_{H^2}, \|g\|_{L^2}\} < \xi,
\]

\[
\sup_{0 \leq t \leq T} (\|v\|_{H^2} + \|f\|_{H^2} + \|\tilde{f}\|_{H^2}) + \int_0^T (\|\nabla v\|_{L^2}^2 + \|\nabla \tilde{f}\|_{L^2}^2 + \|\nabla^2 \tilde{f}\|_{L^2}^2) \, dt < C\xi^2.
\]

In this section, we assume the genuine constant $C$, maybe depending on the constant $M(1)$ which is defined by (2.14).

By Lemma 2.1, there exists a small $\xi_1 (\lt 1)$ such that for all $\xi \in (0, \xi_1]$,
\[
\|\rho\|_{W^{1,q}} \leq C\xi^{\frac{1}{2}}, \|\rho\|_{L^q} \leq C\xi^{\frac{1}{2}}, \|\tilde{p}\|_{W^{1,q}} \leq C\xi^{\frac{1}{2}}, \|p\|_{L^q} \leq C\xi^{\frac{1}{2}}, \forall t \in [0, T]. \tag{4.1}
\]
According to Lemma 2.2, we can find a small $\xi_2$ ($< 1$) such that for all $\xi \in (0, \xi_2]$,

$$\|d_t\|_{H^2_0}^2(t), \|d - n\|_{H^2}^2(t), \int_0^t \|d - n\|_{H^3}^2 \, ds \leq C \xi^2, \quad \forall \, t \in [0, T].$$

Similarly, from Lemma 2.3, a small $\xi_3$ ($\leq \min\{\xi_1, \xi_2\}$) can be found, satisfying that for all $\xi \in (0, \xi_3]$,

$$\|u\|_{H^2_0}^2(t), \|\sqrt{\rho}u\|_{L^2}^2(t), \int_0^t \|u\|_{H^1}^2 \, ds, \int_0^t \|u\|_{W^{2,q}}^2 \, ds \leq C \xi^2, \quad \forall \, t \in [0, T].$$

Making use of the estimates (4.1)-(4.3) and Theorem 2.1, we can obtain the global strong solution of the linear system (2.1), (2.2) with the initial-boundary conditions (1.2)(1.3) provided that

$$\max\{\|\rho_0\|_{W^{1,q}}, \|u_0\|_{H^2}, \|d_0 - n\|_{H^3}, \|g\|_{H^2}\} < \xi, \quad \forall \, \xi \in (0, \xi_3].$$

Now let us consider the possibility of the iteration. First, if we choose $\xi_3$ so small that $C \xi_3 \leq \bar{C}$, then the process of iteration can be continued for the same $\xi_3$. Next, we will focus on the convergence of the iteration.

Repeating the same procedure as in the Section 3, using the estimates (4.1)-(4.3), we can get

$$H^{k+1}(t) + \int_0^t (\|\nabla \tilde{d}^{k+1}\|_{L^2}^2 + \|\Delta \tilde{d}^{k+1}\|_{L^2}^2 + \|\nabla u^{k+1}\|_{L^2}^2) \, ds$$

$$\leq C \nu \exp \left( C(t + \xi^2 t + \xi^2 + \nu^{-1} \xi^2 t + \nu^{-1} \xi^2) \right)$$

$$\int_0^t (\|\nabla \tilde{d}^k\|_{L^2}^2 + \|\Delta \tilde{d}^k\|_{L^2}^2 + \|\nabla u^k\|_{L^2}^2) \, ds.$$

where

$$H^{k+1}(t) = \|\tilde{\rho}^{k+1}\|_{L^2}^2 + \|\tilde{d}^{k+1}\|_{L^2}^2 + \|\nabla \tilde{d}^{k+1}\|_{L^2}^2 + \|\sqrt{\tilde{\rho}^{k+1}u^{k+1}}\|_{L^2},$$

and

$$\tilde{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \tilde{u}^{k+1} = u^{k+1} - u^k, \quad \tilde{d}^{k+1} = d^{k+1} - d^k.$$

Now if we choose small constants $\nu, \xi_0$ such that for all $\xi \in (0, \xi_0]$,

$$C \nu \exp \left( C(t + \xi^2 t + \xi^2 + \nu^{-1} \xi^2 t + \nu^{-1} \xi^2) \right) \leq \frac{1}{2},$$

then it is easy to deduce from (4.4) that

$$\sum_{k=1}^{\infty} H^{k+1}(t) + \sum_{k=1}^{\infty} \int_0^T (\|\nabla \tilde{d}^{k+1}\|_{L^2}^2 + \|\Delta \tilde{d}^{k+1}\|_{L^2}^2 + \|\nabla u^{k+1}\|_{L^2}^2) \, ds$$

$$\leq C < \infty.$$
REFERENCES

[1] K. C. Chang, W. Y. Ding and R. Ye, Finite-time blow-up of the heat flow of harmonic maps from surfaces, *J. Diff. Geom.*, 36 (1992), 507–515.

[2] H. J. Choe and H. Kim, Strong solutions of the Navier-Stokes equations for isentropic compressible fluids, *J. Diff. Equations*, 190 (2003), 504–524.

[3] Y. Cho, H. J. Choe and H. Kim, Unique solvability of the initial boundary value problem for compressible viscous fluids, *J. Math. Pures Appl.*, 83 (2004), 243–275.

[4] Y. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum, *J. Diff. Equations*, 228 (2006), 377–411.

[5] S. Ding, C. Wang and H. Wen, Weak solution to compressible hydrodynamic flow of liquid crystals in dimension one, *Discrete Conti. Dyna. Sys. Ser. B*, 15 (2011), 357–371.

[6] S. Ding, J. Lin, C. Wang and H. Wen, Compressible hydrodynamic flow of liquid crystals in 1-D, *Discrete Conti. Dyna. Sys.*, 32 (2012), 539–563.

[7] J. Ericksen, Conservation laws for liquid crystals, *Trans. Soc. Rheol.*, 5 (1961), 22–34.

[8] J. Ericksen, Equilibrium theory for liquid crystals, in: G. Brown (Ed.), *Advances in Liquid Crystals*, Academic Press, New York, 2 (1976), 233–298.

[9] J. Ericksen, Continuum theory of nematic liquid crystals, *Molecular Crystals*, 7 (2007), 153–164.

[10] M. Hong, Global existence of solutions of the simplified Ericksen-Leslie system in dimension two, *Calc. Var. Partial Diff. Equations*, 40 (2011), 15–36.

[11] X. Hu and H. Wu, Global solution to the three-dimensional compressible flow of liquid crystals, *SIAM J. Math. Anal.*, 45 (2013), 2678–2699, arXiv:1206.2850.

[12] T. Huang, C. Wang and H. Wen, Strong solutions of the compressible nematic liquid crystal flow, *J. Diff. Equations*, 252 (2012), 2222–2265.

[13] T. Huang, C. Wang and H. Wen, Blow up criterion for compressible nematic liquid crystals flows in dimension three, *Arch. Rational Mech. Anal.*, 204 (2012), 285–311.

[14] F. Jiang and Z. Tan, Global weak solution to the flow of liquid crystals system, *Math. Meth. Appl. Sci.*, 32 (2009), 2243–2266.

[15] F. M. Leslie, Theory of flow phenomena in liquid crystals, in: G. Brown (Ed.), *Advances in Liquid Crystals*, Academic Press, New York, 4 (1979), 1–81.

[16] X. Li and D. Wang, Global strong solution to the density-dependent incompressible flow of liquid crystals, *Trans. Amer. Math. Soc.*, 367 (2015), 2301–2338.

[17] F. H. Lin, Nonlinear theory of defects in nematic liquid crystal: Phase transition and flow phenomena, *Comm. Pure Appl. Math.*, 42 (1989), 789–814.

[18] F. H. Lin, Existence of solutions for the Ericksen-Leslie system, *Arch. Rat. Mech. Anal.*, 154 (2000), 135–156.

[19] F. H. Lin and C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, *Comm. Pure Appl. Math.*, 48 (1995), 501–537.

[20] F. H. Lin and C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, *Discrete Conti. Dyna. Sys.*, 2 (1996), 1–22.

[21] F. H. Lin, J. Lin and C. Wang, Liquid crystal flows in two dimensions, *Arch. Ration. Mech. Anal.*, 197 (2010), 297–336.

[22] C. Liu, Dynamic theory for incompressible smectic-A liquid crystals, *Discrete Conti. Dyna. Sys.*, 6 (2000), 591–608.

[23] X. Liu, L. Liu and Y. Hao, Existence of strong solutions for the compressible Ericksen-Leslie model, arXiv:1106.6140.

[24] X. Liu and Z. Zhang, Existence of the flow of liquid crystals system, *Chinese Ann. Math.*, 30 (2009), 1–20.

[25] S. Shkoller, Well-posedness and global attractors for liquid crystals on Riemannian manifolds, *Comm. Partial Diff. Equations*, 27 (2001), 1103–1137.

[26] C. Wang, Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, *Arch. Ration. Mech. Anal.*, 200 (2011), 1–19.

[27] H. Wen and S. Ding, Solutions of incompressible hydrodynamic flow of liquid crystals, *Nonlinear Analysis: Real World Applications*, 12 (2011), 1510–1531.

Received July 2015; revised September 2016.

E-mail address: xlli@bupt.edu.cn
E-mail address: gbl@iapcm.ac.cn