Atomic polyadic algebras of infinite dimension are completely representable

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Abstract . Answering a question posed by Hodkinson, we show that for infinite ordinals \( \alpha \), every atomic polyadic algebra of dimension \( \alpha \) is completely representable. We infer that certain infinitary extensions of first order logic without equality enjoy a Vaught’s theorem; atomic theories have atomic models. Our proof uses a neat embedding theorem together with a simple topological argument.\(^1\)

Polyadic algebras were introduced by Halmos \([12]\) to provide an algebraic reflection of the study of first order logic without equality. Later the algebras were enriched by diagonal elements to permit the discussion of equality. That the notion is indeed an adequate reflection of first order logic was demonstrated by Halmos’ representation theorem for locally finite polyadic algebras (with and without equality). Daigneault and Monk proved a strong extension of Halmos’ theorem, namely that, every polyadic algebra of infinite dimension (without equality) is representable \([11]\).

There are several types of representations in algebraic logic. Ordinary representations are just isomorphisms from boolean algebras with operators to a more concrete structure (having the same similarity type) whose elements are sets endowed with set-theoretic operations like intersection and complementation. Complete representations, on the other hand, are representations that preserve arbitrary conjunctions whenever defined. The notion of complete representations has turned out to be very interesting for cylindric algebras, where it is proved in \([15]\) that the class of completely representable algebras is not elementary.

The correlation of atomicity to complete representations has caused a lot of confusion in the past. It was mistakenly thought for a long time, among algebraic logicians, that atomic representable relation and cylindric algebras

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are completely representable, an error attributed to Lyndon and now referred to as Lyndon’s error.

For boolean algebras, however this is true. The class of completely representable algebras is simply the class of atomic ones. An analogous result holds for certain countable reducts of polyadic algebras [10]. The notion of complete representations has been linked to Martin’s axiom, omitting types theorems and existence of atomic models in model theory [4], [2], [9]. Such connections will be further elaborated upon below in a new setting.

In this paper we show that an atomic polyadic algebra of infinite dimension is also completely representable, in sharp contrast to the cylindric and quasipolyadic equality cases [15], [6]. This result answers a question raised by Ian Hodkinson, see p. 260 in [14] and Remark 6.4, p. 283 in op.cit. Furthermore, we show this result has non-trivial model-theoretic consequences for certain infinitary logics.

The paper is organized as follows. In section 1 we prove our main result as in the title. In section 2, we provide a non-trivial logical counterpart of our algebraic result. In the final section we comment on related results and pose some questions.

1 Main result

Our notation is in conformity with [13] with only one deviation. We write \( f \upharpoonright A \) instead of \( A \upharpoonright f \), for the restriction of a function \( f \) to a set \( A \), which is the more usual standard notation. On the other hand, following [13], for given sets \( A, B \) we write \( A \sim B \) for the set \( \{ x \in A : x \notin B \} \). Gothic letters are used for algebras, and the corresponding Roman letter will denote their universes. Also for an algebra \( \mathfrak{A} \) and \( X \subseteq A \), \( \mathfrak{S} \mathfrak{g} X \), or simply \( \mathfrak{S} \mathfrak{g} X \) when \( \mathfrak{A} \) is clear from context, denotes the subalgebra of \( \mathfrak{A} \) generated by \( X \). \( \text{Id} \) denotes the identity function and when we write \( \text{Id} \) we will be tacitly assuming that its domain is clear from context. We now recall the definition of polyadic algebras as formulated in [13]. Unlike Halmos’ formulation, the dimension of algebras is specified by ordinals as opposed to arbitrary sets.

**Definition 1.1.** Let \( \alpha \) be an ordinal. By a polyadic algebra of dimension \( \alpha \), or a \( \text{PA}_\alpha \) for short, we understand an algebra of the form

\[
\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_{(\Gamma)}, s_\tau \rangle_{\Gamma \subseteq \alpha, \tau \in \sigma^\alpha}
\]

where \( c_{(\Gamma)} \) (\( \Gamma \subseteq \alpha \)) and \( s_\tau \) (\( \tau \in \sigma^\alpha \)) are unary operations on \( A \), such that postulates below hold for \( x, y \in A, \tau, \sigma \in \sigma^\alpha \) and \( \Gamma, \Delta \subseteq \alpha \)

1. \( \langle A, +, \cdot, -, 0, 1 \rangle \) is a boolean algebra
2. \( c(\Gamma)0 = 0 \)

3. \( x \leq c(\Gamma)x \)

4. \( c(\Gamma)(x \cdot c(\Gamma)y) = c(\Gamma)x \cdot c(\Gamma)y \)

5. \( c(\Gamma)c(\Delta)x = c(\Gamma \cup \Delta)x \)

6. \( s_\tau \) is a boolean endomorphism

7. \( s_{Id}x = x \)

8. \( s_{\sigma \circ \tau} = s_{\sigma} \circ s_\tau \)

9. if \( \sigma \upharpoonright (\alpha \sim \Gamma) = \tau \upharpoonright (\alpha \sim \Gamma) \), then \( s_\sigma c(\Gamma)x = s_\tau c(\Gamma)x \)

10. If \( \tau^{-1}\Gamma = \Delta \) and \( \tau \upharpoonright \Delta \) is one to one, then \( c(\Gamma)s_\tau x = s_\tau c(\Delta)x \).

We will sometimes add superscripts to cylindrifications and substitutions indicating the algebra they are evaluated in. The class of representable algebras is defined via set-theoretic operations on sets of \( \alpha \)-ary sequences. Let \( U \) be a set. For \( \Gamma \subseteq \alpha \) and \( \tau \in \alpha^\alpha \), we set

\[
c(\Gamma)X = \{s \in \alpha^U : \exists t \in X, \forall j /\in \Gamma, t(j) = s(j)\}
\]

and

\[
s_\tau X = \{s \in \alpha^U : s \circ \tau \in X\}.
\]

For a set \( X \), let \( \mathcal{B}(X) \) be the boolean set algebra \((\varnothing(X), \cup, \cap, \sim)\). The class of representable polyadic algebras, or \( \text{RPA}_\alpha \) for short, is defined by

\[
SP\{(\mathcal{B}(\alpha^\alpha U), c(\Gamma), s_\tau)_{\Gamma \subseteq \alpha, \tau \in \alpha^\alpha} : U \text{ a set } \}.
\]

Here \( SP \) denotes the operation of forming subdirect products. It is straightforward to show that \( \text{RPA}_\alpha \subseteq \text{PA}_\alpha \). Daigneault and Monk [11] proved that for \( \alpha \geq \omega \) the converse inclusion also holds, that is \( \text{RPA}_\alpha = \text{PA}_\alpha \). This is a completeness theorem for certain infinitary extensions of first order logic without equality [16].

In this paper we are concerned with the following question: If \( \mathfrak{A} \) is a polyadic algebra, is there a representation of \( \mathfrak{A} \) that preserves infinite meets and joins, whenever they exist? (A representation of a given abstract algebra is basically a non-trivial homomorphism from this algebra into a set algebra). To make the problem more tangible we need a few preparations. In what follows \( \prod \) and \( \sum \) denote infimum and supremum, respectively. We will encounter situations where we need to evaluate a supremum of a given set in more than one algebra, in which case we will add a superscript to the supremum indicating the algebra we want. For set algebras, we identify notationally the algebra
with its universe, since the operations are uniquely defined given the unit of the algebra.

Let \( A \) be a polyadic algebra and \( f : A \to \wp(\alpha U) \) be a representation of \( A \). If \( s \in X \), we let
\[
f^{-1}(s) = \{ a \in A : s \in f(a) \}.
\]
An atomic representation \( f : A \to \wp(\alpha U) \) is a representation such that for each \( s \in V \), the ultrafilter \( f^{-1}(s) \) is principal. A complete representation of \( A \) is a representation \( f \) satisfying
\[
f(\prod X) = \bigcap f[X]
\]
whenever \( X \subseteq A \) and \( \prod X \) is defined.

The following is proved by Hirsch and Hodkinson for cylindric algebras. The proof works for \( PA \)'s without any modifications.

**Lemma 1.2.** Let \( A \in PA_\alpha \). A representation \( f \) of \( A \) is atomic if and only if it is complete. If \( A \) has a complete representation, then it is atomic.

**Proof.**

By Lemma 1.2, a necessary condition for the existence of complete representations is the condition of atomicity. We now prove a converse to this result, namely, that when \( A \) is atomic, then \( A \) is completely representable. We need to recall from [13, definition 5.4.16], the notion of neat reducts of polyadic algebras, which will play a key role in our proof of the main theorem.

**Definition 1.3.** Let \( J \subseteq \beta \) and \( A = \langle A, +, \cdot, -, 0, 1, c_j, s_\tau \rangle_{\tau \in \beta, j \in \beta} \) be a \( PA_\beta \). Let \( Nr_J \mathcal{B} = \{ a \in A : c_{(\beta \sim J)} a = a \} \). Then
\[
\mathfrak{N}_r J \mathcal{B} = \langle Nr_J \mathcal{B}, +, \cdot, -, c_j, s_\tau' \rangle_{\tau \subseteq J, \tau \in ^\alpha \beta}
\]
where \( s_\tau' = s_{\bar{\tau}} \). Here \( \bar{\tau} = \tau \cup \text{Id}_{\beta \sim \alpha} \). The structure \( \mathfrak{N}_r J \mathcal{B} \) is an algebra, called the \( J \) compression of \( \mathcal{B} \). When \( J = \alpha \), \( \alpha \) an ordinal, then \( \mathfrak{N}_r \alpha \mathcal{B} \in PA_\alpha \) and is called the neat \( \alpha \)-reduct of \( \mathcal{B} \) and its elements are called \( \alpha \)-dimensional.

The notion of neat reducts is also extensively studied for cylindric algebras [7]. We also need, [11, theorem 2.1] and top of p.161 in op.cit:

**Definition 1.4.** Let \( A \in PA_\alpha \).

(i) If \( J \subseteq \alpha \), an element \( a \in A \) is independent of \( J \) if \( c_{(J)} a = a \); \( J \) supports \( a \) if \( a \) is independent of \( \alpha \sim J \).

(ii) The effective degree of \( A \) is the smallest cardinal \( \epsilon \) such that each element of \( A \) admits a support whose cardinality does not exceed \( \epsilon \).
(iii) The local degree of $\mathfrak{A}$ is the smallest cardinal $m$ such that each element of $\mathfrak{A}$ has support of cardinality $< m$.

(iv) The effective cardinality of $\mathfrak{A}$ is $c = |N_{r,J}A|$ where $|J| = c$. (This is independent of $J$).

We shall use the following elementary known facts about boolean algebras and topological spaces.

(1) Let $\mathcal{B}$ be a boolean algebra, and let $S$ be its Stone space whose underlying set consists of all ultrafilters of $\mathcal{B}$. The topological space $S$ has a clopen base of sets of the form $N_b = \{ F \in S : b \in F \}$ for $b \in B$. Assume that $X \subseteq B$ and $c \in B$ are such that $\sum X = c$. Then the set $N_c \sim \bigcup_{x \in X} N_x$ is nowhere dense in the Stone topology. In particular, if $c$ is the top element, then it follows that $S \sim \bigcup_{x \in X} N_x$ is nowhere dense. (A nowhere dense set is one whose closure has empty interior).

(2) Let $X = (X, \tau)$ be a topological space. Let $x \in X$ be an isolated point in the sense that there is an open set $G \in \tau$ containing $x$, such that $G \cap X = \{ x \}$. Then $x$ cannot belong to any nowhere dense subset of $X$.

The proofs of these facts are entirely straightforward. They follow from the basic definitions.

Now we formulate and prove the main result. The proof is basically a Henkin construction together with a simple topological argument. The proof also has affinity with the proofs of the main theorems in [11] and [5].

**Theorem 1.5.** Let $\alpha$ be an infinite ordinal. Let $\mathfrak{A} \in \mathcal{PA}_\alpha$ be atomic. Then $\mathfrak{A}$ has a complete representation.

**Proof.** Let $c \in A$ be non-zero. We will find a set $U$ and a homomorphism from $\mathfrak{A}$ into the set algebra with universe $\wp(\alpha U)$ that preserves arbitrary suprema whenever they exist and also satisfies that $f(c) \neq 0$. $U$ is called the base of the set algebra. Let $m$ be the local degree of $\mathfrak{A}$, $c$ its effective cardinality and $n$ be any cardinal such that $n \geq c$ and $\sum_{s<m} n^s = n$. The cardinal $n$ will be the base of our desired representation.

Now there exists $\mathcal{B} \in \mathcal{PA}_\alpha$ such that $\mathfrak{A} \subseteq \mathfrak{N}_\alpha \mathcal{B}$ and $A$ generates $\mathcal{B}$. The local degree of $\mathcal{B}$ is the same as that of $\mathfrak{A}$, in particular each $x \in \mathcal{B}$ admits a support of cardinality $< m$. Furthermore, $|n \sim \alpha| = |n|$ and for all $Y \subseteq A$, we have $\mathfrak{Sg}_A^Y = \mathfrak{N}_\alpha \mathfrak{Sg}_B^Y$. All this can be found in [11], see the proof of theorem 1.6.1 therein; in such a proof, $\mathcal{B}$ is called a minimal dilation of $\mathfrak{A}$. Without loss of generality, we assume that $\mathfrak{A} = \mathfrak{N}_\alpha \mathcal{B}$, since $\mathfrak{Sg}_A^Y = \mathfrak{N}_\alpha \mathfrak{Sg}_B^Y$. (In the last equality we are using that $A$ generates $\mathcal{B}$). Hence $\mathfrak{A}$ is first order interpretable in $\mathcal{B}$. In particular, any first order sentence (e.g. the one expressing that $\mathfrak{A}$ is atomic) of the language of $\mathcal{PA}_\alpha$ translates effectively to
a sentence $\hat{\sigma}$ of the language of $\text{PA}_\beta$ such that for all $\mathcal{C} \in \text{PA}_\beta$, we have $\mathcal{N}_\alpha \mathcal{C} \models \sigma \leftrightarrow \mathcal{C} \models \hat{\sigma}$. Here the languages are uncountable, even if $\mathfrak{A}$ is countable and has countable dimension, so we use effective in a loose sense, but it roughly means that there is an effective procedure or algorithm that does this translation. Since $\mathfrak{A} = \mathcal{N}_\alpha \mathcal{B}$ and $\mathfrak{A}$ is atomic, it follows that $\mathcal{B}$ is also atomic. Let $\Gamma \subseteq \alpha$ and $p \in \mathfrak{A}$. Then in $\mathcal{B}$ we have, see [11] the proof of theorem 1.6.1,

$$c_{(\Gamma)p} = \sum \{s_\tau p : \tau \in {}^\alpha n, \ \tau \upharpoonright \alpha \sim \Gamma = \text{Id}\}. \tag{1}$$

Here, and elsewhere throughout the paper, for a transformation $\tau$ with domain $\alpha$ and range included in $n$, $\bar{\tau} = \tau \cup \text{Id}_{n \setminus \alpha}$. Let $X$ be the set of atoms of $\mathfrak{A}$. Since $\mathfrak{A}$ is atomic, then $\sum_{\mathfrak{A}} X = 1$. By $\mathfrak{A} = \mathcal{N}_\alpha \mathcal{B}$, we also have $\sum_{\mathfrak{B}} X = 1$. We will further show that for all $\tau \in {}^\alpha n$ we have,

$$\sum s^{\mathfrak{B}}_\tau X = 1. \tag{2}$$

It suffices to show that for every $\tau \in {}^\alpha n$ and $a \neq 0 \in \mathfrak{A}$, there exists $x \in X$, such that $s_\tau x \leq a$. This will show that for any $\tau \in {}^\alpha n$, the sum $\sum s^{\mathfrak{B}}_\tau X$ is equal to the top element in $\mathfrak{A}$, which is the same as that of $\mathcal{B}$. The required will then follow since for $Y \subseteq A$, we have $\sum_{\mathfrak{A}} Y = \sum_{\mathfrak{B}} Y$ by $\mathfrak{A} = \mathcal{N}_\alpha \mathcal{B}$, and for all $x \in A$ and $\tau \in {}^\alpha n$, we have $s^{\mathfrak{B}}_\tau x = s^{\mathfrak{B}}_\tau x$, since $\mathcal{B}$ is a dilation of $\mathfrak{A}$.

Our proof proceeds by certain non-trivial manipulations of substitutions. Assume that $\tau \in {}^\alpha n$ and non-zero $a \in A$ are given. Suppose for the time being that $\tau$ is onto. We define a right inverse $\sigma$ of $\tau$ the usual way. That is for $i \in \alpha$, choose $j \in \tau^{-1}(i)$ and set $\sigma(i) = j$. Then $\sigma \in {}^\alpha n$ and in fact $\sigma$ is one to one. Let $a' = s_\sigma a$. Then $a' \in A$ and $a' \neq 0$, for if it did, then by polyadic axioms (7) and (8), we would get $0 = s_\tau a' = s_{\tau \circ \sigma} a = a$ which is not the case, since $a \neq 0$. Since $\mathfrak{A}$ is atomic, then there exists an atom $x \in X$ such that $x \leq a'$. Hence using that substitutions preserve the natural order on the boolean algebras in question, since they are boolean endomorphisms, and axioms (7) and (8) in the polyadic axioms, we obtain

$$s_\tau x \leq s_\tau a' = s_\tau s_\sigma a = s_{\tau \circ \sigma} a = s_{\text{Id}} a = a.$$ 

Here all substitutions are evaluated in the algebra $\mathfrak{A}$, and we are done in this case. Now assume that $\tau$ is not onto. Here we use the spare dimensions of $\mathcal{B}$. By a simple cardinality argument, baring in mind that $|n \sim \alpha| = |n|$, we can find $\bar{\tau} \in {}^n n$, such that $\bar{\tau} \upharpoonright \alpha = \tau$, and $\bar{\tau}$ is onto. We can also assume that $\bar{\tau} \upharpoonright (n \sim \alpha)$ is one to one since $|n \sim \alpha| = |n \sim \text{Range}(\tau)| = n$. Notice also that we have $\bar{\tau}^{-1}(n \sim \alpha) \cap \alpha = \emptyset$. Indeed if $x \in \bar{\tau}^{-1}(n \sim \alpha) \cap \alpha$, then $\bar{\tau}(x) \notin \alpha$, while $x \in \alpha$, which is impossible, since $\bar{\tau} \upharpoonright \alpha = \tau$, so that $\bar{\tau}(x) = \tau(x) \in \alpha$. Now let $\sigma \in {}^n n$ be a right inverse of $\tau$ (on $n$), so that for chosen $j \in \bar{\tau}^{-1}(i)$, we have $\sigma(i) = j$. We distinguish between two cases:
(i) \( \sigma(\alpha) \subseteq \alpha \). Let \( a' = s^\alpha_\tau a = s^\alpha_{\sigma(\alpha)} a \). The last equation holds because \( B \) is a dilation of \( A \). Then \( a' \in A \), since \( \sigma(\alpha) \subseteq \alpha \). Also \( a' \neq 0 \), by same reasoning as above. Now there exists an atom \( x \in X \), such that \( x \leq a' \). A fairly straightforward computation, using that \( \text{polyadic axioms applied to } B \) with the fact that substitutions preserve order, and axioms (7) and (8) in the polyadic axioms applied to \( B \), gives
\[
s^\alpha_\tau x \leq s^\alpha_\tau a' = s^\alpha_\tau s^\alpha \sigma a = s^\alpha_{\sigma \sigma} a = s^\alpha_{\sigma \sigma} a = a,
\]
which finishes the proof in the first case.

(ii) \( \sigma(\alpha) \) is not contained in \( \alpha \).

We proceed as follows. Let \( a' = s^\alpha_\tau a \in B \), then by the same reasoning as above, \( a' \neq 0 \). But \( a' \in B \), hence, there exists \( \Gamma \subseteq n \sim \alpha \), such that \( a'' = c_\Gamma a' \in A \), since \( A = \mathfrak{N}_\tau B \). But \( a' \leq a'' \), so \( a'' \) is also a non-zero element in \( A \). Let \( x \in X \) be an atom below \( a'' \). Then we have
\[
s^\alpha_\tau x \leq s^\alpha_\tau a'' = s^\alpha_\tau c_\Gamma a' = s^\alpha_\tau c_\Gamma s^\alpha \sigma a = c_\Gamma s^\alpha s^\alpha \sigma a = a.
\]
Here \( \Delta = \vec{\tau}^{-1} \Gamma \), and the last equality follows from axiom (10) in the polyadic axioms noting that \( \vec{\tau} \upharpoonright \Delta \) is one to one since \( \vec{\tau} \upharpoonright (n \sim \alpha) \) is one to one and \( \Delta \subseteq n \sim \alpha \). The last inclusion holds by noting that \( \vec{\tau}^{-1}(n \sim \alpha) \cap \alpha = \emptyset \), \( \Gamma \subseteq n \sim \alpha \), and \( \Delta = \vec{\tau}^{-1} \Gamma \). But then going on, we have also by axioms (7) and (8) of the polyadic axioms
\[
c_\Delta s^\alpha \sigma a = c_\Delta s^\alpha s^\alpha \sigma a = c_\Delta s^\alpha s^\alpha \sigma a = c_\Delta a = a.
\]
The last equality follows from the fact that \( \Delta \subseteq n \sim \alpha \), and \( a \) is \( \alpha \)-dimensional, that is \( a \in A = \mathfrak{N}_\tau B \). We have proved that \( s^\alpha_\tau x \leq a \), and so we are done with the second slightly more difficult case, as well.

Let \( S \) be the Stone space of \( B \), whose underlying set consists of all boolean ultrafilters of \( B \). Let \( X^* \) be the set of principal ultrafilters of \( B \) (those generated by the atoms). These are isolated points in the Stone topology, and they form a dense set in the Stone topology since \( B \) is atomic. So we have \( X^* \cap T = \emptyset \) for every nowhere dense set \( T \) (since principal ultrafilters, which are isolated points in the Stone topology, lie outside nowhere dense sets). For \( a \in B \), let \( N_a \) denote the set of all boolean ultrafilters containing \( a \). Now for all \( \Gamma \subseteq \alpha \), \( p \in B \) and \( \tau \in a^n \), we have, by the suprema, evaluated in (1) and (2):
\[
G_{\Gamma,p} = N_{c_\Gamma(p)} \sim \bigcup_{\tau \in a^n} N_{s\tau p} \quad (3)
\]
and
\[
G_{X,\tau} = S \sim \bigcup_{x \in X} N_{s\tau x}. \quad (4)
\]
are nowhere dense. Let $F$ be a principal ultrafilter of $S$ containing $c$. This is possible since $B$ is atomic, so there is an atom $x$ below $c$; just take the ultrafilter generated by $x$. Then $F \in X^*$, so $F \notin G_{\Gamma,p}$, for every $\Gamma \subseteq \alpha, p \in A$ and $\tau \in \alpha^n$. Now define for $a \in A$

$$f(a) = \{ \tau \in \alpha^n : s^{B_\tau} a \in F \}.$$

Then $f$ is a homomorphism from $\mathfrak{A}$ to the full set algebra with unit $\alpha^n$, such that $f(c) \neq 0$. We have $f(c) \neq 0$ because $c \in F$, so $Id \in f(c)$. The rest can be proved exactly as in [3]: the preservation of the boolean operations and substitutions is fairly straightforward. Preservation of cylindrifications is guaranteed by the condition that $F \notin G_{\Gamma,p}$ for all $\Gamma \subseteq \alpha$ and all $p \in A$. (Basically an elimination of cylindrifications, this condition is also used in [11] to prove the main representation result for polyadic algebras.) Moreover $f$ is an atomic representation since $F \notin G_{\chi,\tau}$ for every $\tau \in \alpha^n$, which means that for every $\tau \in \alpha^n$, there exists $x \in X$, such that $s^{B_\tau} x \in F$, and so $\bigcup_{x \in X} f(x) = \alpha^n$. We conclude that $f$ is a complete representation by Lemma 1.2.

Contrary to cylindric algebras, we have:

**Corollary 1.6.** The class of completely representable polyadic algebras of infinite dimension is elementary.

**Proof.** Atomicity can be expressed by a first order sentence.

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### 2 A metalogical reading; an extension of a theorem of Vaught

Polyadic algebras of infinite dimension correspond to a certain infinitary logic studied by Keisler, and referred to in the literature as Keisler’s logic. Keisler’s logic allows formulas of infinite length and quantification on infinitely many variables, with semantics defined as expected. While Keisler [16], and independently Monk and Daigneault [11], proved a completeness theorem for such logics, our result implies a ‘Vaught’s theorem’ for such logics, namely, that atomic theories have atomic models, in a sense to be made precise.

Let $\mathcal{L}$ denote Keisler’s logic with $\alpha$ many variables ($\alpha$ an infinite ordinal). For a structure $\mathfrak{M}$, a formula $\phi$, and an assignment $s \in \mathfrak{M}$, we write $\mathfrak{M} \models [s]$ if $s$ satisfies $\phi$ in $\mathfrak{M}$. We write $\phi^{\mathfrak{M}}$ for the set of all assignments satisfying $\phi$.

**Definition 2.1.** Let $T$ be a given $\mathcal{L}$ theory.

1. A formula $\phi$ is said to be complete in $T$ iff for every formula $\psi$ exactly one of

   $$T \models \phi \rightarrow \psi, T \models \phi \rightarrow \neg \psi$$

   holds.
(2) A formula \( \theta \) is completable in \( T \) iff there is a complete formula \( \phi \) with \( T \models \phi \rightarrow \theta \).

(3) \( T \) is atomic iff every formula consistent with \( T \) is completable in \( T \).

(4) A model \( \mathcal{M} \) of \( T \) is atomic if for every \( s \in {}^aM \), there is a complete formula \( \phi \) such that \( \mathcal{M} \models \phi[s] \).

We denote the set of formulas in a given language by \( \mathfrak{Fm} \) and for a set of formula \( \Sigma \) we write \( \mathfrak{Fm}_\Sigma \) for the Tarski-Lindenbaum quotient (polyadic) algebra.

**Theorem 2.2.** Let \( T \) be an atomic theory in \( \mathcal{L} \) and assume that \( \phi \) is consistent with \( T \). Then \( T \) has an atomic model in which \( \phi \) is satisfiable.

**Proof.** Assume that \( T \) and \( \phi \) are given. Form the Lindenbaum Tarski algebra \( \mathfrak{A} = \mathfrak{Fm}_T \) and let \( a = \phi/T \). Then \( \mathfrak{A} \) is an atomic polyadic algebra, since \( T \) is atomic, and \( a \) is non-zero, because \( \phi \) is consistent with \( T \). Let \( \mathfrak{B} \) be a set algebra with base \( M \), and \( f : \mathfrak{A} \rightarrow \mathfrak{B} \) be a complete representation such that \( f(a) \neq 0 \). We extract a model \( \mathfrak{M} \) of \( T \), with base \( M \), from \( \mathfrak{B} \) as follows. For a relation symbol \( R \) and \( s \in {}^aM \), \( s \) satisfies \( R \) if \( s \in h(R(x_0, x_1, \ldots))/T \). Here the variables occur in their natural order. Then one can prove by a straightforward induction that \( \phi^{\mathfrak{M}} = h(\phi/T) \). Clearly \( \phi \) is satisfiable in \( \mathfrak{M} \). Moreover, since the representation is complete it readily follows that \( \bigcup \{ \phi^{\mathfrak{M}} : \phi \text{ is complete} \} = {}^aM \), and we are done.

For ordinary first order logic atomic theories in countable languages have atomic models, as indeed Vaught proved, but in the first order context countability is essentially needed.

Our theorem can also be regarded as an omitting types theorem, for possibly uncountable languages, for the representation constructed in our theorem omits the set (or infinitary type) \( X^- = \{ -x : x \in X \} \), in the sense that the representation \( f \), defined in our main theorem, satisfies \( \bigcap_{x \in X^-} f(x) = \emptyset \). A standard omitting types theorem for Keisler’s logic, addressing the omission of a family of types, not just one, is highly problematic since, even in the countable case, i.e when the universe of algebras is countable, since we have uncountably many operations.

Nevertheless, a natural omitting types theorem can be formulated as follows. Let \( \mathcal{L} \) denote Keisler’s logic, and let \( T \) be an \( \mathcal{L} \) theory. A set \( \Gamma \subseteq \mathfrak{Fm} \) is principal, if there exist a formula \( \phi \) consistent with \( T \), such that \( T \models \phi \rightarrow \psi \) for all \( \psi \in \Gamma \). Otherwise \( \Gamma \) is non-principal. A model \( \mathfrak{M} \) of \( T \) omits \( \Gamma \), if \( \bigcap_{\phi \in \Gamma} \phi^{\mathfrak{M}} = \emptyset \). Then the omitting types theorem in this context says: If \( \Gamma \) is non-principal, then there is a model \( \mathfrak{M} \) of \( T \) that omits \( \Gamma \). Algebraically:

**OTT.** Let \( \mathfrak{A} \in \mathfrak{PA}_a \), and \( a \in A \) be non-zero. Assume that \( X \subseteq A \), is such that \( \prod X = 0 \). Then there exists a set algebra \( \mathfrak{B} \) and a homomorphism \( f : \mathfrak{A} \rightarrow \mathfrak{B} \) such that \( f(a) \neq 0 \) and \( \bigcap_{x \in X} f(x) = \emptyset \).
Note that if $X$ satisfies $\prod_s X = 0$, for every $\tau \in \alpha$, then \OTT holds (by the proof of our main theorem). We do not know whether \OTT holds for arbitrary $X$ with $\prod X = 0$.

Unlike omitting types theorems for countable languages, our proof does not resort to the Baire Category theorem [10], [4]. We believe that our result is an important addition to the model theory of Keisler’s logics, a highly unexplored area. Beth definability and Craig interpolation for Keisler’s logic are proved in [3].

3 Concluding Remarks

(1) The technique used above is a central technique in the representation theory of both polyadic algebras [5] and cylindric algebras [7], [10] together with a simple topological argument, namely, that isolated points in the Stone topology lie outside nowhere dense sets. The former technique is called the neat embedding technique, which is an algebraic version of Henkin’s completeness results, where the spare dimensions of $\mathfrak{B}$ are no more than added witnesses that eliminate quantifiers in a sense.

(2) The class of completely representable quasipolyadic equality algebras of infinite dimension, where only substitutions corresponding to finite transformations, is not elementary. This can be proved exactly like the cylindric case [15] using a simple cardinality argument. However, the case of quasipolyadic algebras of infinite dimensions without equality remains an open problem. The finite dimensional case for such algebras is completely settled in [6].

(3) It is commonly accepted that polyadic algebras and cylindric algebras belong to different paradigms. For example, any universal axiomatization of the class of representable cylindric algebras of dimension $> 2$ has an inevitable degree of complexity [1], but the class of representable polyadic algebras is axiomatized by a fairly simple schema. While the class of polyadic algebras has the superamalgamation property (which is a strong form of amalgamation) [5], the class of (representable) cylindric algebras of dimension $> 1$ fails to have even the amalgamation property [8]. This paper manifests another dichotomy between those two paradigms, because the class of completely representable cylindric algebras of dimension $> 2$ is not elementary [15], while the class of completely representable polyadic algebras is.

(4) The case of polyadic algebras of infinite dimension with equality is much more involved. In this case the class of representable algebras is not a variety; it is not closed under ultraproducts, although every algebra has
the neat embedding property (can be embedded into the neat reduct of algebras in every higher dimension). In particular, we do not guarantee that atomic algebras are even representable, let alone admit a complete representation. Still we can ask whether atomic representable algebras are completely representable. The question seems to be a hard one, because we cannot resort to a neat embedding theorem as we did here for the equality free case.

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The class of quasipolyadic equality algebras is not axiomatizable by a finite schema over the class of representable cylindric algebras.

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Abstract . Using a construction of Andréka, Németi, and Sayed Ahmed, we show that the class of representable quasi-polyadic equality algebras of dimension \( \omega \) is not axiomatized by a finite schema over the class of representable cylindric algebras of dimension \( \omega \). The proof consists of defining a sequence of non representable quasipolyadic equality algebras with representable cylindric reduct, whose ultraproduct is representable.

The main result in this paper, is a contribution to the representation theory of quasi-polyadic algebras, that has recently been investigated in the literature, coming to the forefront of algebraic logic again \[7, 2, 13\]. Let \( U \) be a set and \( \alpha \) be a ordinal. Then \( B(\alpha U) \) is the boolean set algebra with unit \( \alpha U \). Let \( \tau : \alpha \rightarrow \alpha, i, j < \alpha \) and \( X \subseteq \alpha U \). Then

\[
\begin{align*}
&s_\tau^U X = \{ s \in \alpha U : s \circ \tau \in X \}, \\
c_i^U X = \{ t \in \alpha U : \exists s \in X \text{ and } t(j) = s(j) \text{ for all } j \neq i \}, \\
\text{and} \\
d_{ij}^U = \{ s \in \alpha U : s_i = s_j \}.
\end{align*}
\]

Superscripts are omitted if no confusion is likely to ensue. \( \mathbf{SP} \) stands for the operation of forming subdirect products.

\[
\mathbf{RCA}_\alpha = \mathbf{SP}\{ (B(\alpha U), c_i^U, d_{ij}^U)_{i,j<\alpha} : U \text{ is a set} \}
\]

Let

\[
\mathbf{RQEA}_\alpha = \mathbf{SP}\{ (B(\alpha U), c_i^U, d_{ij}^U, s_{[i,j]}^U)_{i,j<\alpha} : U \text{ is a set} \}.
\]

\([i,j] \) is the transposition that interchanges \( i \) and \( j \) that is \([i,j](i) = j \) and \([i,j](j) = i \) and \([i,j]x = x \) for \( x \notin \{i,j\} \). \([i\downarrow] \) is the replacement that sends \( i \) to
and is the identity otherwise. In $\text{RCA}_\alpha$ and $\text{RQEA}_\alpha$, $s_{[i|j]}$ is term definable by $c_i(x \cdot d_{ij})$ for $i \neq j$; according to a widespread custom, we denote $s_{[i|j]}$ by $s_i$. $s_i x$ is just $x$. Now we turn to proving that $\text{RQEA}_\omega$ is not finitely axiomatizable over $\text{RCA}_\omega$. The proof uses the idea in [2]. We shall construct a sequence of algebras $(B_n : n \in \omega)$, such that each is a non-representable quasipolyadic equality algebra, that has a representable cylindric algebra. The ultraproduct relative to a cofinite ultrafilter will be representable as a quasipolyadic equality algebra.

The construction

Let $U = \mathbb{N}$. Let $Z \in \omega^\varphi(\mathbb{N})$ be defined by $Z_0 = Z_1 = n = \{0, 1, 2, \ldots, n-1\}$ and $Z_i = \{(n-1)i - 1, (n-1)i\}$ for $i > 1$. Let $p : \omega \to \omega$ be defined by $p(i) = (n-1)i$. Let $V = \omega^U(p) = \{s \in \omega^U : |\{i \in \omega : s_i \neq (n-1)i\}| < \omega\}$. We will work inside the weak set algebra with universe $\varphi(V)$ and cylindrifications and diagonal elements for $i, j < \omega$ defined for $X \subseteq V$ by:

$c_i X = \{s \in V : \exists t \in X, t(j) = s(j) \ \forall j \neq i\}$

and

$d_{ij} = \{s \in V : s_i = s_j\}$.

Let

$PZ = \{s \in V : (\forall i \in \omega)s_i \in Z_i\}$.

Let

$t = \{s \in \omega^{\sim 2}U : |\{i \in \omega \sim 2 : s_i \neq (n-1)i\}| < \omega, (\forall i > 2)s_i \in Z_i\}$.

Let

$X = \{s \in t : |\{i \in \omega \sim 2 : s(i) \neq (n-1)i\}| \text{ even}\}$,

$Y = \{s \in t : |\{i \in \omega \sim 2 : s(i) \neq (n-1)i\}| \text{ odd}\}$,

$R = \{(u, v) : u \in n, v = u + 1(modn)\}$,

$B = \{(u, v) : u \in n, v = u + 2(modn)\}$,

and

$a = \{s \in PZ : (s \upharpoonright 2 \in R \text{ and } s \upharpoonright \omega \sim 2 \in X) \text{ or } (s \upharpoonright 2 \in B \text{ and } s \upharpoonright \omega \sim 2 \in Y)\}$.

Let $Eq(\omega)$ be the set of all equivalence relations on $\omega$. For $E \in Eq(\omega)$, let $e(E) = \{s \in V : kers = E\}$. Note that $e(E)$ may be empty. Let

$d = PZ \cap d_{01}$. 


\( \pi(\omega) = \{ \tau \in FT_\omega : \tau \text{ is a bijection} \} \). For \( \tau \in FT_\omega \) and \( X \subseteq V \), recall that the substitution (unary) operation \( S_\tau \) is defined by

\[
S_\tau X = \{ s \in V : s \circ \tau \in X \}.
\]

Let

\[
P' = \{ S_\tau a : \tau \in \pi(\omega) \}, \quad P = P' \cup \{ S_\delta d : \delta \in \pi(\omega) \}.
\]

More concisely,

\[
P = \{ S_\tau x : \tau \in \pi(\omega), \ x \in \{ a, d \} \}.
\]

For \( W \in \omega RgZ^{(Z)} \), let

\[
PW = \{ s \in V : (\forall i \in \omega) s_i \in W_i \}.
\]

Let

\[
T = \{ PW \cdot e(E) : W \in \omega RgZ^{(Z)}, (\forall \delta \in \pi(\omega)) W \neq Z \circ \delta, E \in Eq(\omega) \},
\]

\[
At = P \cup T,
\]

and

\[
A_n = \bigcup X : X \subseteq At.
\]

**Theorem 8.**

(1) \( A_n \) is a subuniverse of the full cylindric weak set algebra

\[
\langle \wp(V), +, \cdot, -, c_i, d_{ij} \rangle_{i,j \in \omega}.
\]

Furthermore \( A_n \) is atomic and \( AtA_n = At \sim \{ 0 \} \).

(2) \( A_n \) can be expanded to a quasi-polyadic equality algebra \( B_n \) such that \( B_n \) is not representable.

(3) Let \( F \) be the cofinite ultrafilter over \( \omega \), then \( \prod B_n/F \) is representable.

**Proof.**

(1) Like [2] Claims 1 undergoing the obvious changes

(2) Like [2] Claim 2, also undergoing the obvious changes.

Let \( \tau, \delta \in FT_\omega \). We say that “\( \tau, \delta \) transpose” iff \( (\delta 0 - \delta 1).(\tau 0 - \tau 1) \) is negative.
Now we first define \( p_\sigma : At \to A \) for every \( \sigma \in FT_\omega \).

\[
p_\sigma(s_\delta a) = \begin{cases} 
  s_{\sigma \circ \delta \circ [0,1]}a & \text{if } "\sigma, \delta \text{ transpose"} \\
  s_{\sigma \circ \delta}a & \text{otherwise}
\end{cases}
\]

\[
p_\sigma(x) = s_\sigma x \text{ if } x \in At \sim P'.
\]

Then we set:

\[
p_\sigma(\sum X) = \sum \{ p_\sigma(x) : x \in X \} \text{ for } X \subseteq At.
\]

The defined operations satisfy the polyadic axioms, so that the expanded algebra \( \mathfrak{B}_n \) with the polyadic operations is a quasipolyadic equality algebra. To show that the resulting quasipolyadic equality algebra is non representable, we proceed like [2]. Assume \( \mathfrak{B}_n \in \text{RQEA}_\omega \). Then \( \mathfrak{B}_n \) is isomorphic to some weak set algebra \( \mathfrak{C} \) since \( \mathfrak{R}_\omega \mathfrak{B}_n \) is weakly subdirectly indecomposable. Let \( U' \) be the base of \( \mathfrak{C} \). The unit of \( \mathfrak{C} \) is of the form \( aU' \) for some sequence \( p \). Let \( h : \mathfrak{B} \to \mathfrak{C} \) be an isomorphism. Let \( x = Z_0 \times U \times U \times U \times \ldots \times Z_{n+2} \times Z_{n+3} \ldots \). That is \( x = \{ s \in V : s_0 \in Z_0 : (\forall i > n+1)(s_i \in Z_i) \} \). Then \( x \in \mathfrak{A}_n \) and \( c_i x = x \) for \( i \in n \). So \( c_i h(x) = h(x) \) for \( i \in n \), thus \( h(x) = Z' \times U' \times U' \times U' \times U' \times U' \times U' \times U' \ldots \). For a relation \( R \), recall that \( d(R) = \prod_{(i,j) \in R} -d_{ij} \). Then we have \( x \cdot d(n \times n) \neq 0 \) and \( x \cdot d(n + 1 \times n + 1) = 0 \) imply the same for \( h(x) \), therefore \( |Z'| = n \). Let \( b' = h(b), a' = h(a), g = s_{[0,1]}a, g' = h(g) \). Then \( b \leq x \cdot s_{[0,1]}x = d_{01} \) hence \( b' \subseteq h(x) \cdot s_{[0,1]}h(x) - d_{01}, \) thus

\[ \forall s \in b' \quad (s_0, s_1) \in Z' \sim d_{01} \quad \text{and} \quad |Z'| = n. \quad (*) \]

In \( \mathfrak{A}_n \) we have \( a + g = b \neq 0, a \cdot g = 0, s_{[0,1]}a = a, s_{[0,1]}g = g \) and \( c_i a = c_i g = c_i b \quad \forall i \in 2 \). Therefore

\[
(*) \quad a' + g' = b' \neq 0, a' \cdot g' = 0
\]

\[
(**) \quad s_{[0,1]}a' = a', s_{[0,1]}g' = g' \quad \text{and}
\]

\[
(***) \quad c_i a' = c_i g' = c_i b' \quad \forall i \in 2
\]

Let \( q \in b' \) be arbitrary. \( q_{uv}^{01} \) is the function \( q' \) that agrees with \( q \) everywhere except that \( q'(0) = u \) and \( q'(1) = v \). Define

\[
\tilde{a} = \{ (u, v) : q_{uv}^{01} \in a' \}
\]

and

\[
\tilde{g} = \{ (u, v) : q_{uv}^{01} \in g' \}.
\]
Then by (\(\ast\)) \(-\) (\(\ast\)) we have

\[(\ast)' \quad \bar{a} + \bar{g} = 2Z' \sim d_{01}, \bar{a} \cdot \bar{g} = 0,\]

\[(**)' \quad S_{[0,1]}\bar{a} = \bar{a}, S_{[0,1]}\bar{g} = \bar{g} \quad \text{and}\]

\[(***)' \quad c_0\bar{a} = c_0\bar{g} = c_0^2Z'.\]

We show that (\(\ast\)') \(-\) (\(\ast\ast\ast\)') together with \(|Z'| = n\) is impossible. By (\(\ast\ast\ast\)') we have \(Rg\bar{a} = Rg\bar{g} = Z'\), hence \(|\bar{a}| \geq n\) and \(|\bar{g}| \geq n\). By (\(\ast\)) we have then \(|\bar{a}| = |\bar{g}| = n\) by \(\bar{a} \cdot \bar{g} = 0\) and \(|2Z_1' \sim d_{01}| = 2n\). But by (\(\ast\ast\))' and \(\bar{a} \leq -d_{01}\) we have \(|\bar{a}| \geq n + 1\), contradiction.

(3) To be filled in (Could not do it actually though I am almost sure its right.)

We note that a recursive axiomatization of \(\mathbf{RQEA}_\alpha\alpha\) an infinite ordinal is given in [13], by using the methods of Hirsch and Hodkinson of synthesising axioms by games. The interaction between the theory of cylindric algebras of Tarski and that of (quasi)polyadic algebras of Halmos, which constitute the most two important algebrizations of first order logic and certain infinitary expansions of it, has been extensively studied in the literature, with differences and similarities illuminating both theories. In fact the definability of substitutions in cylindric algebras have initiated a lot of research, we refer to [2] for more references dealing with such a topic.

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