A note on the IBVP for wave equations with dynamic boundary conditions

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Abstract

In this paper, we investigate the controllability on the IBVP for a class of wave equations with dynamic boundary conditions by the HUM method as well as the wellposedness for the related back-ward problems. After proving a new observability inequality, we establish new wellposedness and controllability theorems for the IBVP.

Keywords: Wentzell boundary condition; wave equation; wellposedness; controllability

1 Introduction

In this paper, we consider the exact boundary controllability on the IBVP for wave equation with dynamic boundary condition as follows:

\begin{equation}
\begin{aligned}
\phi'' - \Delta \phi + f(\phi) &= 0, & (x,t) \in Q = \Omega \times (0, T), \\
-\Delta_T \phi + \frac{\partial \phi}{\partial n} &= v_1, & \text{on } \Gamma_1 \times (0, T), \\
\phi &= 0, & \text{on } \Gamma_0 \times (0, T), \\
\phi(0,x) &= \phi_0, & \phi_t(0,x) = \phi_1, & x \in \Omega,
\end{aligned}
\end{equation}

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\Delta_T$ is tangential Laplace operator. The boundary condition on $\Gamma_1$ is called the static Wentzell boundary condition and the dynamic Wentzell boundary condition is

\begin{equation}
\phi'' - \Delta_T \phi + \frac{\partial \phi}{\partial v} = v_1, \quad \text{on } \Gamma_1 \times (0, T).
\end{equation}

The system models an elastic body’s transverse vibration. For details, please see the paper of Lemrabet [1]. In [1–7] and the references therein, one can find more details as regards dynamic boundary conditions. Moreover, Heminna [3] gives the controllability for elasticity system with two controls: both tangential and normal, under the assumption of the wellposedness for the backward system, which is a key assumption for getting controllability. In this paper, we establish first of all the wellposedness theorem for back-ward systems based on the transposition method (cf. [8]) and then obtain the controllability on the IBVP for the wave equation above by using the method of HUM.
2 Boundary controllability for Wentzell systems

For simplicity, we write

\[ V = H^1_{\Gamma_0}(\Omega) := \{ v \in H^1(\Omega) : v|_{\Gamma_1} \in H^1(\Gamma_1), v|_{\Gamma_0} = 0 \}, \quad \mathcal{H} = V \times L^2(\Omega), \]

with the norm

\[
\| u \|_V^2 = \| \nabla u \|_{L^2(\Omega)}^2 + \| \nabla \tau u \|_{L^2(\Gamma_1)}^2,
\]

\[
\| (u, v) \|_\mathcal{H}^2 = \| u \|_V^2 + \| v \|_{L^2(\Omega)}^2.
\]

We study the controllability under the geometric condition:

\[ \exists x_0 \in \mathbb{R}^n, (x-x_0) \cdot \nu \leq 0, \quad \text{on } \Gamma_0. \]

Take a look at the linear homogeneous system first,

\[
\begin{cases}
  u'' - \Delta u = 0, & (x,t) \in Q = \Omega \times (0,T), \\
  -\Delta \tau u + \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1 \times (0,T), \\
  u = 0, & \text{on } \Gamma_0 \times (0,T), \\
  u(0,x) = u_0, & u_t(0,x) = u_1, \quad x \in \Omega.
\end{cases}
\tag{2.1}
\]

The wellposedness for the problem (2.1) is not hard to see. Define an operator \( A : D(A) \to \mathcal{H} \) by

\[ A \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \end{pmatrix}, \]

with

\[ D(A) := \{ (u, v) \in \mathcal{H} : \Delta u \in L^2(\Omega), v \in V, \partial \nu u - \Delta \tau u = 0 \}, \]

\[ D(A^2) = \{ (u, v)^T \in D(A) : A(u, v)^T \in \mathcal{H} \}. \]

Write

\[ E(t) := \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |u'|^2) \, dx + \frac{1}{2} \int_{\Gamma_1} |\nabla \tau u|^2 \, ds. \]

Then it is clear that \( E(t) = E(0) \).

**Lemma 2.1** (Observability inequality) For \( T > 2R \),

\[ E(0) \leq C \int_{\Sigma_1} \left( u^2 + u^2 + |\nabla \tau u|^2 + |\Delta \tau u|^2 \right) ds \, dt, \tag{2.2} \]

where \( R = \max_{x \in \Omega} |x - x_0|, \Sigma_1 = (0, T) \times \Gamma_1. \)
Proof Multiply the equation with the radial multiplier \((x - x_0) \cdot \nabla u + \frac{n-1}{2} u\) and integrate by parts in \(Q\). Then we obtain

\[
\frac{1}{2} \int_Q \left( |u'|^2 + |
abla u|^2 \right) \, dx \, dt + \frac{1}{2} \int_{\Sigma_1} |\nabla_T u|^2 \, ds \, dt + \left| \left( u', (x-x_0) \cdot \nabla u + \frac{n-1}{2} u \right) \right|^T_0 \\
= \frac{1}{2} \int_{\Sigma_1} (x-x_0) \cdot v |u'|^2 \, ds \, dt + \int_{\Sigma_1} \frac{\partial u}{\partial v} (x-x_0) \cdot \nabla u \, ds \, dt \\
+ \frac{n-1}{2} \int_{\Sigma_1} u \frac{\partial u}{\partial v} \, ds \, dt + \frac{1}{2} \int_0^T (x-x_0) \cdot \left| \frac{\partial u}{\partial v} \right|^2 \, ds \, dt \\
+ \frac{1}{2} \int_{\Sigma_1} (|\nabla_T u|^2 - (x-x_0) \cdot v |\nabla u|^2) \, ds \, dt. \tag{2.3}
\]

It is easy to see that

\[
\left| \left( u', (x-x_0) \cdot \nabla u + \frac{n-1}{2} u \right) \right|^T_0 \leq 2RE(0) + c(T) \int_{\Sigma_1} (u^2 + u'^2) \, ds \, dt.
\]

Combining with the geometric condition \((x-x_0) \cdot v \leq 0\) on \(\Gamma_0\), we deduce from (2.3) and (2.1) that

\[
(T - 2R)E_0 \leq c_1 \int_{\Sigma_1} |u'|^2 \, ds \, dt + \int_{\Sigma_1} \frac{\partial u}{\partial v} (x-x_0) \cdot \nabla u \, ds \, dt \\
+ c(T) \int_{\Sigma_1} u^2 \, ds \, dt + \frac{n-1}{2} \int_{\Sigma_1} u \frac{\partial u}{\partial v} \, ds \, dt + \frac{1}{2} \int_{\Sigma_1} |\nabla_T u|^2 \, ds \, dt \\
\leq c \int_{\Sigma_1} \left( |u'|^2 + |\Delta_T u|^2 + u^2 + |\nabla_T u|^2 \right) \, ds \, dt.
\]

So, the observability inequality (2.2) holds. \(\square\)

The observability inequality (2.2) enables us to define the following norm:

\[
\| (u_0, u_1) \|_F^2 := \int_{\Sigma_1} \left( |u'|^2 + |\Delta_T u|^2 + u^2 + |\nabla_T u|^2 \right) \, ds \, dt,
\]

and the corresponding inner product

\[
\langle (u_0, u_1), (v_0, v_1) \rangle_F := \int_{\Sigma_1} \left( u'v' + \Delta_T u \Delta_T v + uv + \nabla_T u \nabla_T v \right) \, ds \, dt,
\]

where \(u\) (or \(v\)) is the solution of (2.1) with initial data \((u_0, u_1)\) (or \((v_0, v_1)\)). Let

\[
F := \left\{ (u_0, u_1) \in C^\infty(\bar{\Omega}) \times C^\infty(\bar{\Omega}) : \partial_{\nu} u_0 - \Delta_T u_0 = 0 \right\}^\perp_F. \tag{2.4}
\]

Then \((F, \langle \cdot, \cdot \rangle_F)\) is a Hilbert space.

Now we consider the wellposedness for the linear backward problem

\[
\begin{align*}
\phi'' - \Delta \phi &= 0, & \text{in } Q, \\
\frac{\partial \phi}{\partial \nu} - \Delta_T \phi &= \nu, & \text{on } \Gamma_1 \times (0, T), \\
\phi &= 0, & \text{on } \Gamma_0 \times (0, T),
\end{align*} \tag{2.5}
\]
with terminal data

\[ \phi(T) = \phi_0, \quad \phi'(T) = \phi_1, \quad \text{in } \Omega, \]  

(2.6)

where

\[ \nu(x,t) = -\partial_t u' + \Delta T (\Delta T u) - \Delta T u + u \]

and \( \partial_t \) is taken in the following sense:

\[ \langle -\partial_t u', \psi \rangle = \langle u', \psi' \rangle, \quad \forall \psi \in H^1(0,T;L^2(\Omega)). \]

For every

\[ (\theta, \theta') \in C([0, T + \varepsilon); D(A^2)) \cap C^1([0, T + \varepsilon); D(A)) \cap C^2([0, T + \varepsilon); H) \]

with \( \theta(0) = \theta'(0) = 0 \), we say \( \phi \in L^\infty(0,T;V') \) is the solution of (2.5)-(2.6) if it satisfies the following equality:

\[
\begin{align*}
\int_Q \phi f \, dQ &+ \left( \langle \phi'(T), \theta(T) \rangle_{F,F} - \langle \phi(T), \theta'(T) \rangle_{F,F} \right) \\
&= -\int_{\Sigma_1} (\nabla_T u \nabla_T \theta + \Delta T \theta u \Delta T \theta + u \theta' + u \theta) \, ds \, dt,
\end{align*}
\]

(2.7)

where

\[ f = \theta'' - \Delta \theta \in L^1(0,T;V). \]

It is clear that \( \theta \) satisfies

\[
\begin{cases}
\theta'' - \Delta \theta = f, & \text{in } Q, \\
\frac{\partial \theta}{\partial n} - \Delta_T \theta = 0, & \text{on } \Gamma_1, \\
\theta = 0, & \text{on } \Gamma_0, \\
\theta(0) = 0, \quad \theta'(0) = 0, & \text{in } \Omega.
\end{cases}
\]

(2.8)

**Theorem 2.2** *In the sense of (2.7), the problem (2.5)-(2.6) has a unique solution \( \phi \) satisfying*

\[ \phi \in L^\infty(0,T;V'). \]

**Proof** First of all, we give the energy estimate for the nonhomogeneous system (2.8).

For the general energy (the low-order energy), since

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} \theta^2 + |\nabla \theta|^2 \, dx + \int_{\Gamma_1} |\nabla_T \theta|^2 \, ds \right) = \int_{\Omega} f \theta_1 \, dx
\]

and

\[ E(T) = E(t) + \int_t^T \int_{\Omega} f \theta' \, dx \, dt, \]


we have

\[ E(t) \leq C_T \left( E(T) + \|f\|_{L^2(\Omega)}^2 \right), \quad \forall t \in (0, T). \]

For the high-order energy, we have

\[ E_1(t) = \frac{1}{2} \int_{\Omega} |\nabla \theta'|^2 + |\Delta \theta|^2 \, dx + \frac{1}{2} \int_{\Gamma_1} |\nabla_{\Gamma} \theta'|^2 \, ds \]

and

\[ E_1(T) = E_1(t) + \int_t^T \int_{\Omega} f \Delta \theta' \, dx \, dt. \]

Hence,

\[
E_1(t) = E_1(T) + \int_t^T \int_{\Omega} \frac{\partial \theta'}{\partial v} \, ds \, dt - \int_t^T \int_{\Omega} \nabla f \nabla \theta' \, dx \, dt \\
= E_1(T) + \int_t^T \int_{\Gamma_1} f \Delta_{\Gamma} \theta' \, ds \, dt - \int_t^T \int_{\Omega} \nabla f \nabla \theta' \, dx \, dt \\
\leq E_1(T) + \int_t^T \left( \int_{\Gamma_1} \|\nabla f\|^2 \, ds \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla_{\Gamma} \theta'|^2 \, ds \right)^{\frac{1}{2}} \, dt \\
+ \int_t^T \left( \int_{\Omega} |\nabla f|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \theta'|^2 \, dx \right)^{\frac{1}{2}} \, dt \\
\leq E_1(T) + \|E_1(t)\|_{L^2(0,T;V)} \|f\|_{L^1(0,T;V)},
\]

which implies that

\[ E_1(t) \leq C \left( E_1(T) + \|f\|_{L^2(0,T;V)}^2 \right), \quad 0 \leq t \leq T. \]

Let \( \theta = \theta_1 + \theta_2 \), where \( \theta_1 \) satisfies

\[
\begin{cases}
\theta''_1 - \Delta \theta_1 = 0, & \text{in } Q, \\
\frac{\partial \theta_1}{\partial n} - \Delta_{\Gamma} \theta_1 = 0, & \text{on } \Sigma_1, \\
\theta_1 = 0, & \text{on } \Sigma_0, \\
\theta_1(T) = \theta(T), & \theta'_1(T) = \theta'(T), \quad \text{in } \Omega,
\end{cases}
\]

and \( \theta_2 \) satisfies

\[
\begin{cases}
\theta''_2 - \Delta \theta_2 = f, & \text{in } Q, \\
\frac{\partial \theta_2}{\partial n} - \Delta_{\Gamma} \theta_2 = 0, & \text{on } \Sigma_1, \\
\theta_2 = 0, & \text{on } \Sigma_0, \\
\theta_2(T) = 0, & \theta'_2(T) = 0, \quad \text{in } \Omega.
\end{cases}
\]

Let

\[
L(\theta(T), \theta'(T), f) = \int_{\Sigma_1} (\nabla_{\Gamma} u \nabla_{\Gamma} \theta + \Delta_{\Gamma} u \Delta_{\Gamma} \theta + u \theta_d + u \theta) \, ds \, dt.
\]
Then we obtain

\[
L(\theta(T), \theta'(T), f) = \int_{\Sigma_1} \left( \nabla_T u \nabla_T \theta + \Delta_T u \Delta_T \theta_1 + u_1 \theta_1 + u \theta \right) ds dt \\
\leq \int_{\Sigma_1} \left( \nabla_T u \nabla_T \theta_1 + \Delta_T u \Delta_T \theta_1 + u_1 \theta_1 + u \theta \right) ds dt \\
+ \int_{\Sigma_1} \left( \nabla_T u \nabla_T \theta_2 + \Delta_T u \Delta_T \theta_2 + u_1 \theta_2 + u \theta \right) ds dt \\
\leq C \left( \|\theta(T)\|_{L^2}^2 + \|f\|_{L^2(0, T; V')}^2 \right)^{1/2}. 
\]

Therefore, \( L : F \times L^1(0, T; V) \rightarrow L^\infty(0, T; V') \) is a bounded operator. So \( \exists \phi \in L^\infty(0, T; V') \), \((\rho_1, -\rho_0) \in F'\) such that

\[
\int_Q \phi f dx dt - \langle \rho_1, \theta(T) \rangle - \langle \rho_0, \theta'(T) \rangle \\
= \int_{\Sigma_1} \left( \nabla_T u \nabla_T \theta + \Delta_T u \Delta_T \theta + u_1 \theta + u \theta \right) ds dt, 
\]

where \( \int_Q \phi f dx dt \) means \( \langle \cdot, \cdot \rangle_{L^\infty(0, T; V'), L^1(0, T; H^1(\Omega))} \).

Next, we prove that

\[
\phi(T) = \rho_0, \quad \phi'(T) = \rho_1. 
\]

Let \( \lambda \) be the eigenvalue for the \( \Delta \) operator with mixed Wentzell, Dirichlet boundary conditions and \( m \) be the corresponding eigenvector. The existence of eigenvalue for the \( \Delta \) operator with mixed Wentzell, Dirichlet boundary condition is based on the fact that \( \Delta^{-1} : L^2(\Omega) \rightarrow V \) is a compact operator. That is,

\[
\begin{cases}
-\Delta m = \lambda m, & \text{in } \Omega, \\
\frac{\partial m}{\partial \nu} - \Delta_T m = 0, & \text{on } \Gamma_1, \\
m = 0, & \text{on } \Gamma_0.
\end{cases}
\]

Set \( f := g(t)m \), where \( g \) is a smooth function in \([0, T + \epsilon]\), and let \( \theta := h(t)m \). Then

\[
\begin{cases}
h'' + \lambda h = g, \\
h(0) = 0, \quad h'(0) = 0.
\end{cases} \tag{2.9}
\]

Claim \( \exists g = g_0 \) such that

\[
h(T) = h'(T) = 0, \quad h''(T) \neq 0.
\]

If this is true, then

\[
\begin{align*}
\int_Q \phi g(t)m dx dt &- \langle \rho_1, h(T)m \rangle - \langle \rho_0, h'(T)m \rangle \\
&= \int_{\Sigma_1} \left( \Delta_T u \Delta_T m - u''m + \nabla_T u \nabla_T m + mu \right) h(t) ds dt.
\end{align*}
\]
Since \( h'' + \lambda h = g_0 \), we have
\[
\int_0^T \langle \phi'' + \lambda \phi, m \rangle h(t) dt + \langle \phi(T), m \rangle h'(T) - \langle \phi'(T), m \rangle h(T) + \langle \rho_1, m \rangle h(T) - \langle \rho_0, m \rangle h'(T) \\
= \int_{\Sigma_1} \Delta_T u \Delta_T m(t) - \Delta_T u \nabla_T m(t) + \rho_1 m(t) ds dt.
\] (2.10)

Differentiate (2.10) with respect to \( T \), we get
\[
\langle \phi'' + \lambda \phi, m \rangle h(T) + \langle \phi(T), m \rangle h''(T) + \langle \phi'(T), m \rangle h'(T) - \langle \phi''(T), m \rangle h(T) \\
- \langle \phi'(T), m \rangle h'(T) + \langle \rho_1, m \rangle h'(T) - \langle \rho_0, m \rangle h''(T) \\
= \int_{\Gamma_1} \Delta_T u \Delta_T m - \Delta_T u \nabla_T m + \rho_1 m \ ds dt.
\]

Therefore
\[
\langle \phi(T), m \rangle h''(T) - \langle \rho_0, m \rangle h''(T) = 0,
\]
which implies that \( \phi(T) = \rho_0 \). Similarly, we obtain \( \phi'(T) = \rho_1 \).

Now we prove the claim above. Write
\[
A := \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then, by the Kalman condition [9], we know that (2.9) is controllable. Set \( X(t) := (h(t), h'(t))^T \). Then \( \exists g_1(s) \), \( s \in (0, \frac{T}{2}) \), such that \( X(\frac{T}{2}) = X_0 \neq 0 \). Write
\[
g_2 \left( s - \frac{T}{2} \right) := B^T e^{\lambda(T-s)} w^{-1}(e^{\lambda \frac{T}{2}} X_0),
\]
where \( w = \int_{\frac{T}{2}}^T e^{\lambda(T-s)} B B^T e^{\lambda(T-s)} ds \). Then
\[
X(t) = e^{\lambda \left( t - \frac{T}{2} \right)} X_0 + \int_{\frac{T}{2}}^t e^{\lambda(T-s)} B g_2 \left( s - \frac{T}{2} \right) ds.
\]
Clearly, \( X(T) = 0, X'(T) \neq 0 \). This proof is then complete. \( \square \)

The following is our exact controllability theorem.

**Theorem 2.3** Let \( T > 2R \) and \( F \) be the Hilbert space defined in (2.4). Then for every \( (\phi(0), -\phi'(0)) \in F' \), there are \( (u_0, u_1) \in F \) and a control function
\[
v(x, t) = -\partial_t u' + \Delta_T (\Delta_T u) - \Delta_T u + u,
\]
where \( u \) is the solution to (2.1), such that the solution \( \phi(t) \) of system (2.5) with initial data \( (\phi(0), \phi'(0)) \) satisfies
\[
\phi(T) = 0, \quad \phi'(T) = 0.
\]
For the nonlinear case, we assume that \( f \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \) satisfies \( f(0) = 0 \) and the superlinear condition (see [10]):

\[
\exists C > 0, p > 1: \quad |f'(s)| \leq C|s|^{p-1}, \quad \forall s \in \mathbb{R} \text{ with } p < \frac{n}{n - \frac{3}{2} + \varepsilon} \quad \text{if } n \geq 2. \tag{2.11}
\]

**Proposition 2.4** Assume that \( f \) satisfies the super-linear condition (2.11). Then there exists \( T_0 > 0 \) such that for every \( T > T_0 \), there is a neighborhood \( \omega \) of \((0,0)\) in \( V \times L^2(\Omega) \) such that for each \((\phi_0, \phi_1) \in \omega\), there exists a control \( v_1 \in H^{-2}(\Gamma) \) such that the solution to (1.1) satisfies

\[
\phi(T) = 0, \quad \phi'(T) = 0.
\]

**Proof** From the results for the nonlinear system of Neumann problems (see [11]), we see that there exists a controllability \( v \in L^2(\Gamma_1) \) such that the solution \((\phi, \phi')\) of the following system:

\[
\begin{aligned}
\phi'' - \Delta \phi + f(\phi) &= 0, \quad \text{in } Q, \\
\frac{\partial \phi}{\partial n} &= v, \quad \text{on } \Sigma_1, \\
\phi &= 0, \quad \text{on } \Sigma_0, \\
\phi(0) &= \phi_0, \quad \phi'(0) = \phi_1, \quad \text{in } \Omega,
\end{aligned}
\]

satisfies \((\phi(T), \phi'(T)) = (0,0)\), and \( \phi \in H^\beta(\Omega) \) where \( \beta \leq \frac{3}{2} - \varepsilon \). The regularity of \( \phi \) for Neumann problems can be found in Theorem 1.4 of [11]. Let \( v_1 = v - \Delta \tau \phi \). Then

\[
\frac{\partial \phi}{\partial v} - \Delta \tau \phi = v_1,
\]

and \( v_1 \in H^{-2}(\Gamma_1) \) such that \( \phi(T) = 0, \phi'(T) = 0 \).

**Remark 2.1** For dynamic Wentzell systems with boundary condition (1.2), we can also prove the results as Theorem 2.3 and Proposition 2.4 by similar arguments.

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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**References**
1. Lemrabet, K: Le problème de Ventcel pour le systèmes de l’élaticité dans un domaine de \( \mathbb{R}^3 \). C. R. Acad. Sci. Paris, Sér. I Math. 304(1), 151-154 (1987)
2. Cavalcanti, M, Lasecka, I, Toundykov, D: Wave equation with damping affecting only a subset of static Wentzell boundary is uniformly stable. Trans. Am. Math. Soc. 364(11), 5693-5713 (2012)
3. Heminna, A: Contrôlabilité exacte d’un problème avec conditions de Ventcel évolutives pour le systèmes linéaire de l’élasticité. C. R. Acad. Sci. Paris, Sér. I Math. 324(2), 195-200 (1997)
4. Tong, C, Wang, Y: Existence of solutions for an initial control problem with dynamic boundary conditions. J. Shanghai Univ. Nat. Sci. 20(6), 741-748 (2014)
5. Xiao, T, Liang, J: Complete second order differential equations in Banach spaces with dynamic boundary conditions. J. Differ. Equ. 200, 105-136 (2004)
6. Xiao, T.J., Liang, J.: Second order differential operators with Feller-Wentzell type boundary conditions. J. Funct. Anal. 254, 1467-1486 (2008)
7. Xiao, T.J., Liang, J.: Nonautonomous semilinear second order evolution equations with generalized Wentzell boundary conditions. J. Differ. Equ. 252, 3953-3971 (2012)
8. Tucsnak, M., Weiss, G.: Observation and Control for Operator Semigroups. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Basel (2009)
9. Coron, J.M.: Control and Nonlinearity. Mathematical Surveys and Monographs, vol. 136. Am. Math. Soc., New York (2007)
10. Zuazua, E.: Exact controllability for the semilinear wave equation. J. Math. Pures Appl. 69(9), 1-31 (1990)
11. Lasiecka, I., Triggiani, R.: Sharp regularity theory for second order hyperbolic equations of Neumann type. Ann. Mat. Pura Appl. 157(1), 285-367 (1990)