The analytical contribution of some eighth-order graphs containing vacuum polarization insertions to the muon (g-2) in QED.

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Abstract.

The contributions to the $g-2$ of the muon from some eighth-order (four-loop) graphs containing one-loop and two-loop vacuum polarization insertions have been evaluated analytically in QED perturbation theory, expanding the results in the ratio of the electron to muon mass ($m_e/m_\mu$). The results agree with the numerical evaluations and the asymptotic analytical results already existing in the literature.
Recently, the sixth-order (three-loop) coefficient $a^{(3)}_{\mu} - a^{(3)}_e$ of the QED perturbative expansion of the difference between muon and electron $(g-2)$

$$a^{QED}_\mu - a^{QED}_e = \left( a^{(2)}_\mu - a^{(2)}_e \right)^2 + \left( a^{(3)}_\mu - a^{(3)}_e \right)^3 + \left( a^{(4)}_\mu - a^{(4)}_e \right)^4 + \ldots \quad (1)$$

was calculated in closed analytical form [1][2]. The eighth-order (four-loop) coefficient $a^{(4)}_\mu - a^{(4)}_e$ is known only in numerical form [3]; its more recent value is [4]:

$$a^{(4)}_\mu - a^{(4)}_e = 127.55(41) \quad . \quad (2)$$

In this work we have calculated in analytical form the contributions to $a^{(4)}_\mu - a^{(4)}_e$ from some graphs containing insertions on a single photon line of one-loop and two-loop vacuum polarization subdiagrams (see fig.1); the considered graphs are shown in figs.(2) and (3).

We found convenient to express the results expanding them in the ratio of the electron and muon masses ($m_e/m_\mu$); for the sake of extensive numerical checks we have calculated the terms containing up to $(m_e/m_\mu)^{16}$. Unfortunately the coefficient of each term becomes more and more cumbersome as the power of $(m_e/m_\mu)$ increases, so that we will list here the terms of the expansions up to $(m_e/m_\mu)^2$ only.

The analytical expressions of the contributions to the muon anomaly of the graphs shown in figs.(2) and (3), accounting for the proper multiplicity factors, are ($r \equiv m_e/m_\mu$):

$$a^{(4)}_\mu \text{[fig.2(a)]} = -\frac{4}{27} \ln^3 r - \frac{25}{27} \ln^2 r - \left( \frac{2}{27} \pi^2 + \frac{317}{162} \right) \ln r - \frac{2}{9} \zeta(3) - \frac{25}{162} \pi^2 - \frac{8609}{5832} + r \left[ \frac{101}{1536} \pi^4 \right] + r^2 \left[ \frac{16}{9} \ln^3 r + \frac{52}{9} \ln^2 r + \left( \frac{304}{27} + \frac{8}{9} \pi^2 \right) \ln r + \frac{136}{35} \zeta(3) + \frac{26}{27} \pi^2 + \frac{967}{315} \right] + O(r^3) \quad , \quad (3)$$

$$a^{(4)}_\mu \text{[fig.2(b)]} = -\frac{4}{9} \pi^2 - \frac{119}{27} \ln^2 r + \left( -\frac{2}{27} \pi^2 + \frac{61}{81} \right) \ln r - \frac{4}{45} \pi^4 + \frac{13}{27} \pi^2 + \frac{7627}{1944} + r^2 \left[ \left( -\frac{8}{9} \pi^2 + \frac{230}{27} \right) \ln r - \frac{4}{3} \pi^2 + \frac{227}{18} \right] + O(r^4 \ln^3 r) \quad , \quad (4)$$
\( a^{(4)}_{\mu}[\text{fig.2(c)}] = \left( -\frac{16}{3} \zeta(3) + \frac{8}{135} \pi^2 + \frac{943}{162} \right) \ln r + \frac{2}{27} \pi^4 - \frac{2}{45} \zeta(3) - \frac{5383}{4050} \pi^2 + \frac{57899}{9720} + r^2 \left[ -\frac{8}{3} \zeta(3) - \frac{26}{105} \pi^2 + \frac{458}{81} \right] + O(r^4 \ln r) \) \hfill (5) \\

\( a^{(4)}_{\mu}[\text{fig.2(d)}] = \frac{1}{3} \ln^2 r + \left( -\frac{2}{3} \zeta(3) + \frac{5}{4} \right) \ln r - \frac{25}{18} \zeta(3) + \frac{1}{18} \pi^2 + \frac{509}{432} + r \left[ \frac{101}{72} \pi^3 \beta_2 - \frac{101}{72} \pi^4 \ln 2 + \frac{707}{144} \pi^2 \zeta(3) + \frac{9035}{6912} \pi^4 - \frac{821}{135} \pi^3 - \frac{5081}{648} \pi^2 \right] \hfill (6) \\

\( + r^2 \left[ -\frac{16}{3} \ln^3 r - \frac{38}{3} \ln^2 r + \left( 8 \zeta(3) - \frac{8}{3} \pi^2 - 22 \right) \ln r \\
+ \frac{766}{2025} \pi^4 + \frac{176}{135} \pi^2 \ln^2 2 - \frac{1408}{45} a_4 - \frac{176}{135} \ln^2 2 \\
- \frac{3706}{225} \zeta(3) - \frac{19}{9} \pi^2 - \frac{31571}{2700} \right] + O(r^3) \)

\( a^{(4)}_{\mu}[\text{fig.2(e)}] = \left( \frac{1}{3} \pi^2 - \frac{119}{36} \right) \ln r - \frac{2}{3} \pi^2 \zeta(3) + \frac{119}{18} \zeta(3) - \frac{1}{9} \pi^2 + \frac{473}{432} + r^2 \left[ \left( \frac{4}{3} \pi^2 - \frac{115}{9} \right) \ln r + \frac{8}{9} \pi^2 - \frac{893}{108} \right] + O(r^3) \) \hfill (7) \\

\( a^{(4)}_{\mu}[\text{fig.2(f)}] = \ln r \left( \frac{14}{405} \pi^4 - \frac{128}{9} a_4 + \frac{16}{27} \pi^2 \ln^2 2 - \frac{16}{27} \ln^4 2 - \frac{26}{27} \zeta(3) + \frac{16}{27} \pi^2 \ln 2 \\
+ \frac{164}{243} \pi^2 - \frac{673}{81} \right) - \frac{128}{9} a_5 + \frac{73}{9} a_4 - \frac{98}{405} \pi^4 \ln 2 + \frac{38}{27} \pi^2 \zeta(3) \\
- \frac{16}{81} \pi^2 \ln^3 2 + \frac{16}{135} \ln^5 2 + \frac{22}{405} \pi^4 - \frac{32}{3} a_4 + \frac{4}{27} \pi^2 \ln^2 2 \\
- \frac{4}{9} \ln^4 2 + \frac{1213}{162} \zeta(3) - \frac{8}{3} \pi^2 \ln 2 + \frac{4873}{2916} \pi^2 - \frac{33335}{3888} \\
+ r^2 \left[ \frac{7}{135} \pi^4 - \frac{64}{3} a_4 + \frac{8}{9} \pi^2 \ln^2 2 - \frac{8}{9} \ln^4 2 - \frac{521}{30} \zeta(3) - \frac{12}{5} \pi^2 \ln 2 \\
+ \frac{821}{225} \pi^2 + \frac{56}{15} \right] + O(r^3) \) \hfill (8)
\[ a_{\mu}^{(4)}[\text{fig.3(a)}] = \left( \frac{2}{3} \zeta(3) - \frac{4}{9} \pi^2 \ln 2 + \frac{10}{27} \pi^2 - \frac{31}{18} \right) \ln^2 r + \ln r \left( -\frac{11}{162} \pi^4 + \frac{32}{9} a_4 + \frac{4}{27} \ln^4 2 + \frac{8}{27} \pi^2 \ln^2 2 + \frac{14}{3} \zeta(3) - \frac{20}{9} \pi^2 \ln 2 + \frac{158}{81} \pi^2 - \frac{115}{18} \right) \\
+ \frac{32}{9} a_5 - \frac{143}{36} \zeta(5) - \frac{1}{9} \pi^2 \zeta(3) - \frac{41}{810} \pi^4 \ln 2 \right. \\
- \frac{8}{27} \pi^2 \ln^2 2 + \frac{10}{27} \ln^4 2 + \frac{133}{18} \zeta(3) - \frac{221}{81} \pi^2 \ln 2 \\
+ \frac{1133}{486} \pi^2 - \frac{8719}{1296} \\
- r \left[ \frac{2}{45} \pi^2 \right] \\
+ r^2 \left[ -\frac{8}{3} \ln^2 r + \left( -\frac{28}{3} \zeta(3) + \frac{56}{9} \pi^2 \ln 2 - \frac{37}{9} \pi^2 + 6 \right) \ln r \\
+ \frac{77}{162} \pi^4 - \frac{224}{9} a_4 - \frac{56}{27} \pi^2 \ln^2 2 - \frac{28}{27} \ln^4 2 - \frac{397}{18} \zeta(3) \\
+ \frac{257}{27} \pi^2 \ln 2 - \frac{178}{27} \pi^2 + \frac{157}{27} \right] + O(r^3) , \] 

\[ a_{\mu}^{(4)}[\text{fig.3(b)}] = \left( -\frac{2}{27} \pi^4 + \frac{35}{6} \zeta(3) + \frac{16}{9} \pi^2 \ln 2 - \frac{62}{81} \pi^2 - \frac{227}{54} \right) \ln r \\
+ \frac{20}{27} \pi^2 \zeta(3) - \frac{409}{2160} \pi^4 - \frac{52}{9} a_4 - \frac{35}{54} \pi^2 \ln^2 2 - \frac{13}{54} \ln^4 2 - \frac{1475}{324} \zeta(3) \\
+ \frac{308}{81} \pi^2 \ln 2 + \frac{187}{1458} \pi^2 - \frac{11891}{1944} \\
+ r^2 \left[ -\frac{14}{135} \pi^4 - \frac{1199}{1080} \zeta(3) - \frac{64}{27} \pi^2 \ln 2 + \frac{9959}{6075} \pi^2 + \frac{18367}{1620} \right] + O(r^3) , \] 

\[ a_{\mu}^{(4)}[\text{fig.3(c)}] = \left( -\frac{3}{4} \zeta(3) + \frac{1}{2} \pi^2 \ln 2 - \frac{5}{12} \pi^2 + \frac{31}{16} \right) \ln r + \frac{3}{2} \zeta^2(3) - \pi^2 \zeta(3) \ln 2 \\
+ \frac{5}{6} \pi^2 \zeta(3) + \frac{11}{288} \pi^4 - 2 a_4 - \frac{1}{6} \pi^2 \ln^2 2 - \frac{1}{12} \ln^4 2 - \frac{99}{16} \zeta(3) \\
+ \frac{25}{24} \pi^2 \ln 2 - \frac{133}{144} \pi^2 + \frac{535}{192} \\
+ r \left[ -\frac{8}{9} \pi^2 \ln 2 - \frac{13}{36} \pi^3 + \frac{79}{54} \pi^2 \right] \\
+ r^2 \left[ 6 \ln^2 r + \ln r \left( -14 \pi^2 \ln 2 + 21 \zeta(3) + \frac{37}{4} \pi^2 - \frac{47}{2} \right) - \frac{77}{72} \pi^4 \\
+ 56 a_4 + \frac{14}{3} \pi^2 \ln^2 2 + \frac{7}{3} \ln^4 2 + \frac{185}{8} \zeta(3) - \frac{39}{4} \pi^2 \ln 2 + \frac{57}{8} \pi^2 \\
+ \frac{35}{3} \right] + O(r^3 \ln r) . \]
Here $\zeta(p)$ is the Riemann $\zeta$-function of argument $p$, $\zeta(p) \equiv \sum_{n=1}^{\infty} \frac{1}{n^p}$, (whose first values are $\zeta(2) = \pi^2/6$, $\zeta(3) = 1.202056903...$, $\zeta(4) = \pi^4/90$, $\zeta(5) = 1.036927755...$), $a_4 \equiv \sum_{n=1}^{\infty} \frac{1}{2^n n^4} = 0.517479061...$, $a_5 \equiv \sum_{n=1}^{\infty} \frac{1}{2^n n^5} = 0.508400579...$, and $\beta_2$ is the Catalan constant $\beta_2 \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594...$

We note the appearance of transcendental constants $\beta_2$ in eq.(6) and $a_5$ in eq.(8) and eq.(9); these constants appear for the first time in a contribution to the lepton anomaly.

Of the above expressions, eqs.(3)-(11), only the leading terms of eq.(3) and eq.(6) were already worked out in analytical form using renormalization group techniques [3][5]. (1). An examination of eqs.(3)-(11) shows that the expressions of the contributions of the graphs containing only electron loops, eqs.(3),(6),(9) and (11), have a linear $(m_e/m_\mu)$ term; the numerical values of the corresponding coefficients are respectively 6.41, -5.61, -0.44, -2.84. The $(m_e/m_\mu)$ expansions of the contributions of the other graphs (containing also muon loops) begin with the $(m_e/m_\mu)^2$ term.

We list now the numerical values of eqs.(3)-(11) obtained using the experimental value [7] $(m_\mu/m_e) = 206.768262(30)$ and taking into account all the calculated terms of the expansions in $(m_e/m_\mu)$, compared with the corresponding numerical values given in

(1) We want point out that, contrarily to the assertion of Appendix A of ref.[3], even the leading terms of the sum of the contributions from the graphs of fig.2(e) and fig.3(c) can be calculated in analytical form using renormalization group techniques. In fact, the integral $I_1$ of eq.(A.25) of ref.[3], evaluated numerically in that work, can be worked out in analytical form using the contributions of sixth-order graphs containing vacuum polarization insertions which are known in analytical form by long time (see eq.(2.22) of ref.[6]); the analytical value of this integral is found to be $I_1 = \frac{11}{72} \pi^4 - \frac{2}{3} \pi^2 \ln^2 2 - \frac{1}{3} \ln^4 2 - 8a_4 - 8\zeta(3) + \frac{10}{3} \pi^2 \ln 2 - 4\pi^2 + \frac{641}{36}$ which, inserted in eq. (A.26) of ref.[3], gives the leading terms of the sum of our eq.(7) and eq.(11).
ref.[3]:

\[ a_\mu^{(4)}[\text{fig.2(a)}] = 7.2230764(8) \quad \text{(our)}; \quad 7.2237(13) \quad \text{(ref.[3])}; \]

\[ a_\mu^{(4)}[\text{fig.2(b)}] = 0.49407203(3) \quad \text{(our)}; \quad 0.4942(2) \quad \text{(ref.[3])}; \]

\[ a_\mu^{(4)}[\text{fig.2(c)}] = 0.027988322(7) \quad \text{(our)}; \quad 0.0280(1) \quad \text{(ref.[3])}; \]

\[ a_\mu^{(4)}[\text{fig.2(d)}] = 7.1280084(2) \quad \text{(our)}; \quad 7.1289(23) \quad \text{(ref.[3])}; \]

\[ a_\mu^{(4)}[\text{fig.2(e)}] = 0.119602460(2) \quad \text{(our)}; \quad 0.1195(1) \quad \text{(ref.[3])}; \]

\[ a_\mu^{(4)}[\text{fig.2(f)}] = 0.33366468(1) \quad \text{(our)}; \quad 0.3337(1) \quad \text{(ref.[3])}; \]

\[ a_\mu^{(4)}[\text{fig.3(a+b)}] = -9.3427221(5) \quad \text{(our)}; \quad -9.3571(40) \quad \text{(ref.[3])}; \]

\[ a_\mu^{(4)}[\text{fig.3(c)}] = -2.77885233(5) \quad \text{(our)}; \quad -2.7864(45) \quad \text{(ref.[3])}. \]

The total sum is

\[ a_\mu^{(4)}[\text{fig.2 + 3}] = 3.2048378(8) \quad \text{(our)}; \quad 3.1845(66) \quad \text{(ref.[3])}. \quad (12) \]

The numerical error of our results is induced by the experimental uncertainty of \((m_\mu/m_e)\); in order to reach such a precision accounting of terms up to \((m_e/m_\mu)^4\) is needed. Note that the linear terms in \((m_e/m_\mu)\) are essential to check the results of ref.[3] within their precision: as an example, the contribution of the linear term of eq.(3) is 0.031 which is about 23 times the error of the correspondent numerical result of ref.[3].

We found that our results are in good agreement with numerical results of ref.[3]; only the contribution \(a_\mu^{(4)}[\text{fig.3(a+b)}]\) shows a slight disagreement, at the level of 3.6\(\sigma\). This is a remarkable cross check of results, due to the difference of the methods followed in the two derivations.

Let us now consider the contributions to the electron anomaly from the graphs shown in figs.(2) and (3) with \(\mu\) and \(e\) leptons exchanged \(^{(2)}\); we list only the leading terms of the expansions \((r \equiv m_e/m_\mu)\):

\[ a_e^{(4)}[\text{fig.2(a)}, e \leftrightarrow \mu] = r^4 \left[ -\frac{89}{15015} \zeta(3) + \frac{87709}{9729720} \right] + O(r^6 \ln r) , \quad (13) \]

\(^{(2)}\) The contributions to the electron anomaly of the graphs shown in figs.(2) and (3) with \(\mu\) leptons replaced by electrons can be found in ref.[8].

6
\[ a_e^{(4)}[\text{fig. 2(b), } e \leftrightarrow \mu] = r^4 \left[ \frac{2}{225} \ln^2 r + \frac{61}{27000} \ln r + \frac{5809}{1080000} \right] + O \left( r^6 \ln r^2 \right), \] (14)

\[ a_e^{(4)}[\text{fig. 2(c), } e \leftrightarrow \mu] = r^2 \left[ \frac{16}{45} \zeta(3) - \frac{203}{486} \right] + O \left( r^4 \ln^3 r \right), \] (15)

\[ a_e^{(4)}[\text{fig. 2(d), } e \leftrightarrow \mu] = r^4 \left[ -\frac{82}{1215} \ln r - \frac{3827}{374200} \pi^4 + \frac{712}{10395} a_4 - \frac{89}{31185} \pi^2 \ln^2 2 
+ \frac{89}{31185} \ln^4 2 - \frac{756121}{3201600} \zeta(3) + \frac{2268671641}{31120135200} \right] + O \left( r^6 \ln r \right), \] (16)

\[ a_e^{(4)}[\text{fig. 2(e), } e \leftrightarrow \mu] = r^2 \left[ -\frac{82}{729} \pi^2 + \frac{1681}{1458} \right] + O \left( r^4 \ln^2 r \right), \] (17)

\[ a_e^{(4)}[\text{fig. 2(f), } e \leftrightarrow \mu] = r^2 \left[ -\frac{14}{2025} \pi^4 + \frac{128}{45} a_4 - \frac{16}{135} \pi^2 \ln^2 2 + \frac{16}{135} \ln^4 2 + \frac{34}{27} \zeta(3) 
- \frac{344}{1215} \pi^2 + \frac{424}{405} \right] + O \left( r^4 \ln^2 r \right), \] (18)

\[ a_e^{(4)}[\text{fig. 3(a), } e \leftrightarrow \mu] = -\frac{529}{5832} r^2 + O \left( r^4 \ln^2 r \right), \] (19)

\[ a_e^{(4)}[\text{fig. 3(b), } e \leftrightarrow \mu] = r^2 \left[ \frac{46}{405} \ln^2 r + \frac{23}{225} \ln r + \frac{2}{225} \pi^4 + \frac{23}{27} \zeta(3) - \frac{2}{135} \pi^2 - \frac{709027}{364500} \right] + O \left( r^4 \ln^2 r \right), \] (20)

\[ a_e^{(4)}[\text{fig. 3(c), } e \leftrightarrow \mu] = r^2 \left[ \frac{943}{1458} \ln r + \frac{1771}{2592} \zeta(3) - \frac{41}{729} \pi^2 - \frac{1124}{2187} \right] + O \left( r^4 \ln r \right); \] (21)

due to the smallness of the ratio \((m_e/m_\mu)\), these contributions are almost negligible, the numerical value of the sum of eqs.(13)-(21) being

\[ a_e^{(4)}[\text{fig. 2 + 3}] = -1.796 \times 10^{-4}. \] (22)

Using eq.(2) and our results (12) and (22) we can work out a new slightly different value for \(a_\mu^{(4)} - a_e^{(4)}\):

\[ a_\mu^{(4)} - a_e^{(4)} = 127.57(41). \] (23)
We sketch the method used for obtaining the analytical expressions of eqs.(3)-(11) and eqs.(13)-(21). Quite in general the contribution to the anomaly of the muon from a vertex graph with vacuum polarization insertions with electron loops can be written as

\[ a_{ln} \left( \frac{m_e}{m_\mu} \right) = \frac{1}{\pi} \int_{4m_e^2}^{\infty} \frac{db}{b} K_l \left( \frac{b}{m_\mu^2} \right) \text{Im}\Pi_n \left( \frac{b}{m_e^2} \right), \tag{24} \]

where \( \text{Im}\Pi_n \left( \frac{b}{m_e^2} \right) \) is the imaginary part of the vacuum polarization insertion in \( n \)th order and \( K_l \left( \frac{b}{m_\mu^2} \right) \) is the anomaly contribution from some set of \( l \)th-order vertex graphs in which a photon line has been given a mass \( b \). Both quantities are analytically known up to fourth order \[6][9][10].

Once that the suitable expressions of \( K \) and \( \text{Im}\Pi \) are inserted in eq.(24), the contribution to the muon \( (g-2) \) becomes a sum of one-dimensional integrals containing square roots, logarithms, dilogarithms and at worst trilogarithms of the variable \( b \).

Unlike ref.[2], where analogous sixth-order integrals were calculated in closed analytical form, we found convenient to calculate the integrals of eq.(24) expanding in \((m_e/m_\mu)\). We split the integration region into two parts by introducing a cut \( \Lambda \), such that \( m_e^2 \ll \Lambda \ll m_\mu^2 \): in the region where \( b \leq \Lambda \) we find \( \frac{b}{m_\mu^2} \ll 1 \) so that we can expand \( K_l \left( \frac{b}{m_\mu^2} \right) \) for small values of the argument, while in the region where \( b > \Lambda \) we expand \( \text{Im}\Pi_n \left( \frac{b}{m_e^2} \right) \) for large \( \left( \frac{b}{m_e^2} \right) \). Integrals over these regions were calculated analytically using the method described in ref.[2] as functions of \( m_e, m_\mu \) and \( \Lambda \); summing up the analytical contributions of the two regions the dependence on \( \Lambda \) drops out, as expected. A similar method is used when the \( e \) and \( \mu \) leptons are exchanged.

Finally, we checked that the direct numerical evaluations of the integrals (24) are in perfect agreement with our analytical expressions for the same quantities.

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Figure captions

Fig.1: The irreducible fourth-order vacuum polarization subdiagram.

Fig.2: Eighth-order vertex graphs obtained with insertions of second- and fourth-order vacuum polarization subdiagrams on the second-order vertex graph.

Fig.3: Examples of eighth-order vertex graphs obtained with the insertion of second- and fourth-order vacuum polarization subdiagrams on fourth-order vertex graphs.