ON COMBINATORIAL PROPERTIES AND THE ZERO DISTRIBUTION OF CERTAIN SHEFFER SEQUENCES

GI-SANG CHEON\textsuperscript{1}, TAMÁS FORGÁCS\textsuperscript{2}, HANA KIM\textsuperscript{1}, AND KHANG TRAN\textsuperscript{2}

Abstract. We present combinatorial and analytical results concerning a Sheffer sequence with a generating function of the form \( G(x, z) = Q(z)^2 Q(-z)^{1-z} \), where \( Q \) is a quadratic polynomial with real zeros. By using the properties of Riordan matrices we address combinatorial properties and interpretations of our Sheffer sequence of polynomials and their coefficients. We also show that apart from two exceptional zeros, the zeros of polynomials with large enough degree in such a Sheffer sequence lie on the line \( x = 1/2 + it \).

MSC: 05A15, 05A40, 30C15, 30E15

Contents

1. Introduction
1.1. Organization
Acknowledgements
2. Part I - Combinatorial results concerning Sheffer sequences
Part II - the zeros of the Sheffer sequence
2.1. Deforming the path of integration
2.2. Approximating \( \int_{\Gamma_2} f(z, t) \, dz \) - the saddle point method
2.2.1. Properties of \( \zeta_1(t) \) and \( \zeta_2(t) \n 2.2.2. The main term of the approximation
2.2.3. The tails of the approximation
2.2.4. The asymptotics when \( 1/n^{2/3} \ll T - t \ll \ln^2 n/n^{2/3} \)
2.2.5. The asymptotics when \( T = T_2 \) and \( \frac{1}{n} \ll |T_1 - t| \ll \ln^2 n/n^{2/3} \)
2.2.6. The asymptotics when \( t \ll \ln^4 n/n \n 2.3. The zeros of the polynomials \( H_n \).
References

1. Introduction

A systematic study of the Sheffer sequence of polynomials was carried out by G.-C. Rota and his collaborators as part of their development of Umbral Calculus (\cite{14,15}). Recall that a Sheffer
sequence \((s_n(x))_{n \geq 0}\) is uniquely associated to a pair \((g, f)\) of formal power series in \(z\) generating the polynomials \(s_n(x)\) of degree \(n\) via the relation
\[
g(z) e^{zf(z)} = \sum_{n=0}^{\infty} s_n(x) \frac{z^n}{n!},
\]
where \(g\) and \(f\) are invertible with respect to the product and the composition of series, respectively. This notion led to the development of Riordan group theory by means of infinite lower triangular matrices called Riordan matrices or Riordan arrays generated by a pair of formal power series. It is shown in [16] that (exponential) Riordan matrices constitute a natural way of describing Sheffer sequences and various combinatorial situations such as ordered trees, generating trees, and lattice paths, etc. ([1] [3]).

As is well-known, a number of classical polynomial sequences are Sheffer sequences, e.g., Laguerre polynomials, Bernoulli polynomials, Hermite polynomials, Poisson-Charlier polynomials and Stirling polynomials. In addition to their combinatorial importance, many of these sequences have also been extensively studied from an analytical perspective, and in particular, from the perspective of their zero distribution. While there is a wealth of knowledge assembled about the zeros of the classical orthogonal polynomials and other special functions (see [13] for example), there are many Sheffer sequences whose zero distribution is not known. On one hand, the requirement that a polynomial sequence be a Sheffer sequence restricts the type of generating function the sequence may have. On the other hand, the sequences under consideration in the current paper have generating functions that are quite different from those that have been studied in some recent works (see [5], [6], [7], [10], [18], and [19]). Broadly speaking, these works study the zero distribution of sequences \((s_n)_{n \geq 0}\) generated by certain ‘rational’-type bivariate generating functions. Studying the zero distribution of Sheffer sequences \((H_n(x))_{n \geq 0}\) with generating functions of the form \(G(x, z) = Q(z)^x Q(-z)^{1-x}\), where \(Q\) is a quadratic polynomial is a new contribution to the growing body of work in this area.

While the basic ideas involved are standard, their implementation requires some careful asymptotic analysis and is at times tedious. The reader will be rewarded with an appealing result, namely, that Sheffer polynomials \(H_n(x)\) with generating functions like \(G(x, z)\) and large enough degree have all their zeros (apart from the ones at \(x = 0\) and \(x = 1\)) on the ‘critical line’ \(x = 1/2 + it, t \in \mathbb{R}\).

1.1. Organization. The paper consists of two main sections. Part I addresses the combinatorial properties and interpretations of our sequence of polynomials and their coefficients. After introducing the notion of a Riordan matrix, we proceed to translate the coefficient matrix of a Sheffer sequence \((H_n(x))_{n \geq 0}\) into an exponential Riordan matrix. This allows us to obtain two different possible combinatorial interpretations for the Sheffer sequence of polynomials. We arrive at the first interpretation by introducing generating trees associated to the production matrix of the exponential Riordan matrix (Theorem 1 Lemma 2 Theorem 3). Since the best known applications of Sheffer sequences occur in enumeration problems of lattice paths, we offer a second interpretation for our Sheffer sequence in relation to weighted lattice paths (Theorem 4). We obtain the results of this approach by employing the Stirling transform of the sequence \((x^n)_{n \geq 0}\). Part II provides the analysis of the zeros of the generated Sheffer sequence. It begins with the standard representation of the polynomial \(H_n(x)\) as a line integral on a small circle around the origin using the Cauchy integral formula. In subsection 2.1, we show that this line integral is essentially the imaginary part (or \(i\) times the real part) of the integral \(\int_{\Gamma_2} f(z, t)dz\) for an appropriately defined function \(f(z, t)\), and \(\Gamma_2\) the boundary of a small, tubular neighborhood of the ray \([z_1, \infty)\).
In order to estimate \( \int_{\Gamma_2} f(z,t)dz \), we use the saddle point method, which requires that after we write

\[
f(z,t) = e^{-n\phi(z,t)}\psi(z,t),
\]
we identify the critical points of \( \phi(z,t) \) (in \( z \)), as a function of \( t \). These critical points will trace two curves as \( 0 < t < T \), of which we select the one more suitable (Lemmas 7, 8, 9, Proposition 11, Lemmas 13, 14 and 15) for the saddle point method, which we call \( \Gamma(t) \). Sections 2.2.2 and 2.2.3 (Lemmas 16, 21, 23 and Proposition 24) are dedicated to proving that through each point of \( \Gamma(t) \), there is a deformation \( \Gamma \) (see Figure 2.3) of \( \Gamma_2 \) with certain desirable properties vis-à-vis the saddle point method. In these sections we also establish that for each \( t \), \( \Gamma \) can be divided three segments – two tails, and a segment centered on \( \zeta(t) \) – so that the integral on the central segment dominates those over the tails. Sections 2.2.4, 2.2.5 and 2.2.6 address ranges of \( t \) for which the estimates in the preceding sections are not good enough to ascertain that that the integral over the central segment dominates, and provide asymptotic expressions for \( \int_{\Gamma_2} f(z,t)dz \) for values of \( t \) in these ranges. With these expression in hand, in Section 2.3 we proceed to compute the change in the argument of the integral representing \( H_n(1/2 + int) \) along a slightly deformed version of the closed loop \( \zeta \cup -\zeta \cup -\zeta \cup \zeta \) which avoids the singularities of \( \zeta(t) \) (Lemmas 29-35) in order to get a lower bound on the number of zeros of \( H_n \) on the critical line. After accounting for a few exceptional zeros (Lemma 43), a quick computation of the degree of \( H_n \) (Lemma 44) and the Fundamental Theorem of Algebra complete the proof of the main result.

Aknowledgements. G.-S. Cheon was partially supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (2016R1A5A1008055 and 2019R1A2C1007518). T. Forgács and K. Tran would like to acknowledge the research support of the California State University, Fresno.

2. Part I - Combinatorial results concerning Sheffer sequences

We begin by briefly describing the notion of a Riordan array by focusing on exponential Riordan matrices, as these are ones we mostly use in this paper.

An infinite lower triangular matrix \( A = [a_{n,k}]_{n,k\geq 0} \) is called a Riordan matrix if its \( k \)th column has generating function \( gf^k \) for some \( g, f \in \mathbb{C}[[z]] \), where \( f(0) = 0 \). If, in addition, \( g(0) \neq 0 \) and \( f'(0) \neq 0 \) then \( A \) is invertible and it is said to be a proper Riordan matrix. We may write \( A = (g, f) \). If the \( k \)th column of \( A \) has an exponential generating function \( g(z)f(z)^k/k! \), then \( A \) is called an exponential Riordan matrix and we denote it by \( A = [g, f] \). By definition, if \( A = (g, f) = [a_{n,k}]_{n,k\geq 0} \) and \( B = [g, f] = [b_{n,k}]_{n,k\geq 0} \), then it is obvious that

\[
b_{n,k} = \frac{n!}{k!}a_{n,k} \quad \text{or} \quad B = EAE^{-1}
\]

where \( E = \text{diag}(0!,1!,2!,\ldots) \) is the diagonal matrix. A well-known fundamental property of a Riordan matrix is that if \( A = [g, f] \) and \( b^T = (b_n)_{n\geq 0} \) is generated by exponential function \( b(z) \), then the sequence \( Ab \) has exponential generating function given by \( gb(f) \); we simply write this property as \( [g, f]b = gb(f) \). In particular, if \( b = e^{xz} \) then

\[
[g, f]e^{xz} = ge^{xf}.
\]

From the definition it follows at once that if \( s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k \), then the Sheffer sequence \( (s_n(x))_{n\geq 0} \) can be rewritten as a matrix product:

\[
(s_0(x), s_1(x), \ldots)^T = [s_{n,k}](1, x, x^2, \ldots)^T,
\]
where \([s_{n,k}]_{n,k \geq 0}\) is a lower triangular matrix as the coefficient matrix of \((s_n(x))_{n \geq 0}\). Since the sequence \((1, x, x^2, \ldots)\) has exponential generating function \(e^{xz}\), by the fundamental property it follows from (1.1) and (2.2) that \((s_n(x))_{n \geq 0}\) is a Sheffer sequence for \((g, f)\) if and only if its coefficient matrix \([s_{n,k}]_{n,k \geq 0}\) is an exponential Riordan matrix given by \([g, f]\). It may be also shown that if \(A = [g, f] \text{ and } B = [h, \ell]\) then the product is given by \(AB = [gh(f), \ell(f)]\). Moreover, \(I = [1, z]\) is the usual identity matrix and \([g, f]^{-1} = [1/g(T), T]\) where \(T\) is the compositional inverse of \(f\), i.e., \(T(f) = f(T) = z\).

We now turn to the polynomial sequence \((H_n(x))_{n \geq 0}\) generated by a bivariate function

\[
G(x, z) := Q(z)^x Q(-z)^{1-x},
\]

where \(Q(z)\) is a quadratic polynomial whose zeros are real, \(Q(0) \neq 0\) and \(Q'(0) \neq 0\). Since

\[
Q(z)^x Q(-z)^{1-x} = -Q(-z) \exp \left( x \ln \frac{Q(z)}{Q(-z)} \right),
\]

the sequence \((H_n(x))_{n \geq 0}\) can be considered as a Sheffer sequence with the coefficient matrix:

\[
A := [a_{i,j}]_{i,j \geq 0} = \left[ Q(-z), \ln \frac{Q(z)}{Q(-z)} \right].
\]

Using the fundamental property, we thus have

\[
\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = Q(z)^x Q(-z)^{1-x} = \left[ Q(-z), \ln \frac{Q(z)}{Q(-z)} \right] e^{xz}. \tag{2.3}
\]

In this section, we are interested in finding combinatorial interpretations for the Sheffer polynomials \(H_n(x) = \sum_{k=0}^{n} a_{n,k} x^k\) of degree \(n\) where \(A = [a_{i,j}]_{i,j \geq 0}\). For combinatorial counting purposes, our interest is the polynomials with non-negative integer coefficients \(a_{n,k}\). Throughout this section, we assume that \(Q(z) = (1 + az)(1 + bz)\) and \(a, b \geq 0\) are integers. We first note that the coefficient matrix \(A\) might have negative entries, but \(\hat{A} := \left[ Q(z), \ln \frac{Q(z)}{Q(-z)} \right]\) has no negative entries whenever \(a, b > 0\). Since

\[
Q(z) \exp \left( x \ln \frac{Q(z)}{Q(-z)} \right) = Q(z)^{1+x} Q(-z)^{-x},
\]

the Sheffer sequence associated to \(\hat{A}\), say \((\hat{H}_n(x))_{n \geq 0}\) has the bivariate generating function given by \(G(-x, -z)\). In addition, using Riordan multiplication we obtain \(LD\)-decomposition \(\hat{A} = L_Q D\), where

\[
L_Q = [q_{n,k}]_{n,k \geq 0} = \left[ Q(z), \frac{1}{2(a+b)} \ln \frac{Q(z)}{Q(-z)} \right] \tag{2.4}
\]

is a unit lower triangular matrix with ones on the main diagonal, and \(D = [1, 2(a+b)z]\) is a diagonal matrix of the form \(\text{diag}(1, (2a+2b), (2a+2b)^2, \ldots)\). If we use the notation \([x^k]\) for the coefficient extraction operator, we have for \(k = 0, 1, \ldots, n,

\[
[x^k]H_n(x) = (-1)^{n+k}[x^k] \hat{H}_n(x) = (-1)^{n+k}(2a+2b)^k q_{n,k}, \quad q_{0,0} = 1. \tag{2.5}
\]
A few rows of the matrix $L_Q$ are displayed by

\[
L_Q = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
a + b & 1 & 0 & 0 & 0 \\
2ab & 2(a + b) & 1 & 0 & 0 \\
0 & 2(a + b)^2 & 3(a + b) & 1 & 0 \\
0 & 8(a^3 + b^3) & 4(2a^2 + 2b^2 + ab) & 4(a + b) & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]  

(2.6)

Thus from (2.5) and (2.6) we obtain:

\[
H_0(x) = 1,
\]
\[
H_1(x) = -(a + b) + 2(a + b)x,
\]
\[
H_2(x) = 2ab - 4(a + b)^2x + 4(a + b)^2x^2,
\]
\[
H_3(x) = 4(a + b)^3(x - 3x^2 + 2x^3),
\]

\[
\vdots
\]

The following theorem is useful for finding the combinatorial interpretation for $L_Q$.

**Theorem 1.** (3, 4) Let $A = [a_{n,k}]$ be an infinite lower triangular matrix with $a_{0,0} = 1$. Then $A$ is an exponential Riordan matrix given by $A = [g, f]$ if and only if there exists a horizontal pair \(\{c_n, r_n\}_{n \geq 0}\) of the sequences with $c = \sum_{n \geq 0} c_n z^n$ and $r = \sum_{n \geq 0} r_n z^n$ such that

\[
c(f(z)) = \frac{g'(z)}{g(z)} \quad \text{and} \quad r(f(z)) = f'(z),
\]

or for all $n \geq k \geq 0$,

\[
a_{n+1,k} = \frac{1}{k!} \sum_{i=k}^{n} i!(c_{i-k} + kr_{i-k+1})a_{n,i}, \quad (c_{-1} := 0).
\]

(2.8)

At times the exponential generating function approach provides explicit forms for various (increasing) tree counting problems. Under various guises, such trees have surfaced as tree representations of permutations, as data structures in computer science, and as probabilistic models in diverse applications. There is a unified generating function approach to the enumeration of parameters on such trees (2, 8). Indeed, it was shown in 2 that the counting generating functions for several basic parameters, e.g. root degree, number of leaves, path length, and level of nodes, are related to a simple ordinary differential equation:

\[
Y'(z) = \varphi(Y(z)), \quad Y(0) = 0.
\]

(2.9)

Comparing this differential equation with the second equation in (2.7), we see that an exponential Riordan matrix with non-negative integer entries is closely related to tree counting problems. For instance, if we consider $\varphi(z)$ to be the generating function for the degree-weight sequence under a ‘nonnegativity’ condition, then $Y(z) = \ln \frac{Q(z)}{Q(0)}$ satisfying (2.9) can be regarded as the generating function for total weights of certain simple family of increasing trees.

The formula in equation (2.8) can be rewritten as a matrix equality, $AP_A = UA$ where $U$ is the upper shift matrix with ones only on the superdiagonal and zeros elsewhere, and $P_A := [p_{i,j}]_{i,j \geq 0}$ where

\[
p_{i,j} = \frac{i!}{j!} (c_{i-j} + jr_{i-j+1}), \quad c_i, r_i = 0 \text{ for } i < 0.
\]

(2.10)
Thus it follows from (2.7) that
\[
Solving the above quadratic equation we obtain
\[
We call
\[
Proof. Note that \( f(z) = \frac{1}{2(a+b)} \ln \frac{1+(a+b)z+abz^2}{1+(a+b)z+abz^2} \). Since \( f(\overline{f}) = z = \overline{f}(f) \), we have
\[
Solving the above quadratic equation we obtain
\[
Thus it follows from (2.7) that
\[
In particular, if \( a = b \) then we obtain:
\[
\overline{f}(z) = \frac{a+b}{2ab} \left( \coth (a+b)z - \sqrt{\coth^2(a+b)z - \frac{4ab}{(a+b)^2}} \right).
\]
\[
Proof. Note that \( f(z) = \frac{1}{2(a+b)} \ln \frac{1+(a+b)z+abz^2}{1+(a+b)z+abz^2} \). Since \( f(\overline{f}) = z = \overline{f}(f) \), we have
\[
1 + (a+b)\overline{f} + ab\overline{f}^2 = e^{2(a+b)z} \left( 1 - (a+b)\overline{f} + ab\overline{f}^2 \right).
\]
\[
Lemma 2. Let \( L_Q = [g,f] \) denote the exponential Riordan matrix given by (2.4). Then the horizontal pair \( \{c_n,r_n\}_{n \geq 0} \) of \( L_Q \) is given by
\[
(2.11) \quad c(z) = \frac{a}{1+a\overline{f}} + \frac{b}{1+b\overline{f}},
\]
\[
(2.12) \quad r(z) = \frac{1}{a+b} \left( \frac{a}{1-(a\overline{f})^2} + \frac{b}{1-(b\overline{f})^2} \right),
\]
where
\[
where
\[
\overline{f}(z) = \frac{a+b}{2ab} \left( \coth (a+b)z - \sqrt{\coth^2(a+b)z - \frac{4ab}{(a+b)^2}} \right).
\]
\[
Thus it follows from (2.7) that
\[
\frac{g'(\overline{f})}{g(\overline{f})} = \frac{(a+b) + 2abz}{1+(a+b)z+abz^2} \bigg|_{z=\overline{f}} = \frac{a}{1+a\overline{f}} + \frac{b}{1+b\overline{f}} = (a+b) - (a^2+b^2)z + 2(a^3+b^3) \frac{z^2}{2!} - 2(a+b)^2(ab+2(a-b)^2) \frac{z^3}{3!} + \cdots,
\]
and
\[
r(z) = f'(\overline{f}) = \frac{1}{a+b} \left( \frac{a}{1-(a\overline{f})^2} + \frac{b}{1-(b\overline{f})^2} \right)
\]
In particular, if \( a = b \) then we obtain:
\[
\overline{f}(z) = \frac{1}{a} \left( e^{2az} - 1 \right) = \frac{1}{a} \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)B_{2n}(a\overline{z})^{2n-1}}{(2n)!},
\]
\[
\overline{f}(z) = \frac{1}{a} \left( e^{2az} - 1 \right) = \frac{1}{a} \sum_{n=0}^{\infty} \frac{(-2az)^n}{n!},
\]
\[
\overline{f}(z) = \frac{1}{a} \left( e^{2az} - 1 \right) = \frac{1}{a} \sum_{n=1}^{\infty} \frac{2^{2n-1}(a\overline{z})^{2n}}{(2n)!}.
\]
where \( B_n \) is the \( n \)th Bernoulli number.

Let \( P_{LQ} = [p_{i,j}]_{i,j \geq 0} \) denote the production matrix of \( L_Q \) in (2.4). Then

\[
(2.13) \quad P_{LQ} = \begin{pmatrix}
    a + b & 1 & 0 & 0 \\
    - (a^2 + b^2) & a + b & 1 & 0 \\
    2(a^3 + b^3) & -2ab & a + b & 1 \\
    - 2(a + b)^2(ab + 2(a - b)^2) & 6(a^3 + b^3) & 3(a - b)^2 & a + b \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

It is well-known [1, 4] that if a production matrix \( P_A \) is an integer matrix then every element of the Riordan matrix \( A \) has a combinatorial interpretation of counting marked or non-marked nodes in the associated generating tree. A marked generating tree [1] is a rooted labeled tree with the property that if \( v_1 \) and \( v_2 \) are any of two nodes with the same label \( k \) then for each label \( k = 0, 1, 2, \ldots, v_1 \) and \( v_2 \) have the same number of children. The nodes \( (k) \) with label \( k \) may or may not be marked depending on whether an element in \( P_A \) is negative or positive. To specify a generating tree it therefore suffices to specify:

(a) the label of the root;
(b) a set of production rules explaining how to derive the quantity of children and their labels, from the label of a parent.

Noticing that the \( k \)th row of the production matrix of a Riordan matrix defines a production rule for the node \( (k) \), we can similarly associate an exponential Riordan matrix with its production matrix to a marked generating tree specification using the notation

\[
(k)^p = (k) \cdots (k) \quad (p > 0) \quad \text{and} \quad (\overline{k})^p = (\overline{k}) \cdots (\overline{k}) \quad (p < 0)
\]

where \((k)^0\) is the empty sequence and \((\overline{k}) = (k)\). We are now ready to give a combinatorial interpretation for coefficients of the polynomials \( H_n(x) \) by means of a marked generating tree. Note that it follows from (2.5) that \([(-1)^n x^n]\ H_n(x) = q_{n,k} \) where \( L_Q = [q_{i,j}]_{i,j \geq 0} \).

**Theorem 3.** For \( k = 0, 1, \ldots, n \), let \( \mu_n(k) \) denote the number of nodes \( (k) \) with label \( k \) at level \( n \) in the marked generating tree specification where the root is at level \( 0 \):

\[
(2.14) \quad \left\{ \begin{array}{l}
\text{root :} \quad (0) \\
\text{rule :} \quad (k) \rightarrow ((0)^p (1)^{k+1} \cdots (k + 1)^{p_{k-1}}) \\
& \quad ((\overline{0})^{p_k} (\overline{1})^{p_{k+1}} \cdots (\overline{k + 1})^{p_{k+1}})
\end{array} \right.
\]

where \( p_{k,j} = \frac{\ell_j}{\mathcal{J}_j} (c_{k-j} + jr_k) \) for the horizontal pair \( (c_n, r_n)_{n \geq 0} \) in Lemma 2

Then

\[
(2.15) \quad [x^k] H_n(x) = (-1)^{n+k} (2a + 2b)^k \left( \mu_n(k) - \mu_n(\overline{k}) \right).
\]

**Proof.** Let \( P_{LQ} = [p_{i,j}]_{i,j \geq 0} \) be the production matrix of \( L_Q \). Since \( L_Q \) is a unit lower triangular matrix of nonnegative integers for \( a, b > 0 \), it is obvious from \( P_{LQ} = L_Q^{-1} U L_Q \) that all entries of \( P_{LQ} \) are integers. Thus it follows from (2.10) that \( p_{k,j} = \frac{\ell_j}{\mathcal{J}_j} (c_{k-j} + jr_k) \in \mathbb{Z} \), where \( (c_n)_{n \geq 0} \) and \( (r_n)_{n \geq 0} \) are the sequences obtained from Theorem 1. In addition, for \( n, k \geq 0 \) we have

\[
q_{0,0} = 1, \quad q_{n+1,k} = \sum_{i=k-1}^{n} \frac{\ell_i}{\mathcal{J}_i} (c_{i-k} + kr_{i-k+1}) q_{n,i} = \sum_{i=k-1}^{n} q_{n,i} p_{i,k}, \quad \text{for} \quad (c_{-1} := 0).
\]
By using the succession rule (2.14), we show that \( q_{n,k} = \mu_n(k) - \mu_n(k) \). In the marked generating tree at level zero we have only one node \((0)\) with label 0. This is represented by the row vector \( R_0 = (1, 0, \ldots) \). At the next levels of the generating tree, the distribution of the nodes \((1), (2), \ldots\) is given by the row vectors \( R_i, i \geq 1 \), defined by the recurrence relation \( R_i = R_{i-1}P_{LQ} \). Stacking these row vectors we obtain the matrix \( L_Q = [R_0, R_1, \ldots]^T \) satisfying \( L_Q P_{LQ} = U L_Q \). Since the marked nodes kill or annihilate the non-marked nodes with the same number in this process, it follows that \( q_{n,k} \) counts the difference between the number of non-marked nodes \((k)\) and the number of marked nodes \((\tilde{k})\) at level \( n \). Hence (2.15) immediately follows from (2.5), as required.

The best known applications of Sheffer sequences occur in enumeration problem of lattice paths (see [11]). We propose now a second combinatorial interpretation for coefficients of the polynomials \( H_n(x) \) by weighted lattice paths. This approach can be obtained from the Stirling transform of the sequence \((x^k)_{k \geq 0}\) given by

\[
(2.16) \quad x^n = \sum_{k=0}^{n} S(n, k)(x)_k,
\]

where \( S(n, k) \) is the Stirling number of the second kind and \((x)_k := \prod_{i=0}^{k-1}(x - i)\) is the \( k \)-th falling factorial with \((x)_0 = 1\). Using the exponential generating functions \( e^{xz} \) for \((x^k)_{k \geq 0}\), and \((1 + z)^x\) for \((x)_k\) we obtain from (2.16) that \( e^{xz} = [1, e^x - 1](1 + z)^x \) where \([1, e^x - 1]\) is the Stirling matrix of the second kind whose \((i,j)\)-entry is \( S(i,j) \). Hence we obtain

\[
(2.17) \quad \left[ Q(-z), \ln \frac{Q(z)}{Q(-z)} \right] e^{xz} = \left[ Q(-z), \ln \frac{Q(z)}{Q(-z)} \right] [1, e^x - 1](1 + z)^x
\]

It follows from (2.3), (2.16) and (2.17) that

\[
(2.18) \quad H_n(x) = \sum_{k=0}^{n} a_{n,k} x^k = \sum_{k=0}^{n} \left( \sum_{i=0}^{n} a_{n,i} S(i, k) \right) (x)_k = \sum_{k=0}^{n} c_{n,k}(x)_k,
\]

where \( A = \left[ a_{n,k} \right]_{n,k \geq 0} = \left[ Q(-z), \ln \frac{Q(z)}{Q(-z)} \right] \) and \( C := \left[ c_{n,k} \right]_{n,k \geq 0} = \left[ Q(-z), \frac{Q(z)}{Q(-z)} - 1 \right] \). A few rows of \( C \) shown below:

\[
(2.19) \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-(a + b) & 2(a + b) & 0 & 0 & 0 \\
2ab & 0 & 4(a + b)^2 & 0 & 0 \\
0 & 0 & 12(a + b)^3 & 8(a + b)^3 & 0 \\
0 & 0 & 48(a + b)^2(a^2 + ab + b^2) & 64(a + b)^4 & 16(a + b)^4 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

We show that every element of the matrix \( C \) can be represented in terms of the weights of some lattice path. For this purpose, consider a weighted lattice path in the plane from \((0,0)\) to \((n,k)\) with up-steps \( U = (1,1) \), and level-steps \( H = (1,0) \) or double level-steps \( H^2 = (2,0) \). We denote by \( \omega(U) \) the weight of \( U \), and by \( \omega_{ij}(H) \) and \( \omega_{ij}(H^2) \) the weights of \( H \) for \((i-1,j) \rightarrow (i,j) \) and \( H^2 \) for \((i-2,j) \rightarrow (i,j) \), respectively. As usual, the weight of a lattice path is defined by the product of all weights assigned to the steps along the path.
Theorem 4. For \( k = 0, 1, \ldots, n \), let \( \sigma(n,k) \) denote the sum of the weights of lattice paths in the plane from \((0,0)\) to \((n,k)\) with the steps in \(\{U,H,H^2\}\) where \(w(D) = 2(a+b)\), \(w_j(H) = (i+j-2)(a+b)\), and \(w_j(H^2) = -(i-1)(i+2j-4)ab\). Then
\[
\sigma(n,k) = c_{n,k} = [z^n]Q(-z) \left( \frac{Q(z)}{Q(-z)} - 1 \right)^k.
\]

In particular,
\[
[x^k]H_n(x) = \sum_{i=0}^{n} (-1)^{i+k} \sigma(n,i) s(i,k)
\]
where \( s(i,k) \) is the Stirling number of the first kind.

Proof. Let \( H_n(x) = \sum_{k=0}^{n} c_{n,k}(x) \). Consider a weighted lattice path from \((0,0)\) to \((n,k)\) with the step set \(\{U,H,H^2\}\). Since a lattice point \((n,k)\) may be approached from any of the lattice points \((n-1,k-1)\), \((n-1,k)\), or \((n-2,k)\), it is immediate that \( \sigma(n,k) = c_{n,k} \) if and only if \( c_{n,k} \) satisfies the following recurrence relation for \( n \geq 2 \) and \( k \geq 1 \):
\[
\begin{align*}
\sigma(n,k) &= (2(a+b)c_{n-1,k-1} + (n+k-2)(a+b)c_{n-1,k} - (n-1)(n+2k-4)abc_n-2,k, \\
\end{align*}
\]
with the initial conditions \( c_{0,0} = 1, c_{1,0} = -(a+b), c_{1,1} = 2(a+b), c_{2,0} = 2ab \), and \( c_{n,0} = 0 \) for \( n \geq 3 \). It is clear that (2.21) together with the initial conditions determines \( c_{n,k} \).

For simplicity, we substitute by \( d_{n,k} = \frac{k!}{n!} c_{n,k} \). Then the recurrence (2.21) is equivalent to (2.22)
\[
nd_{n,k} = (2(a+b)kd_{n-1,k-1} + (n+k-2)(a+b)d_{n-1,k} - (n+2k-4)abcd_{n-2,k,}
\]
where \( d_{0,0} = 1, d_{1,0} = -(a+b), d_{1,1} = 2(a+b), d_{2,0} = 2ab \), and \( d_{n,0} = 0 \) for \( n \geq 3 \). If we put \( \varphi_k(z) = \sum_{n \geq 0} d_{n,k} z^n \), it follows from (2.22) that
\[
\begin{align*}
\varphi_k' &= (2(a+b)k\varphi_{k-1} + (a+b)(z\varphi_k') + (k-2)(a+b)\varphi_k - ab(z^2\varphi_k') - ab(2k-4)z\varphi_k \\
&= (2(a+b)k\varphi_{k-1} + ((a+b)z - abz^2)\varphi'_k + (k-1)(a+b - 2abz)\varphi_k.
\end{align*}
\]
Using \( Q(z) = 1 + (a+b)z + abz^2 \) we obtain the following differential equation for \( \varphi_k \):
\[
(2.23) \quad Q(-z)\varphi_k' - (k-1)Q'(-z)\varphi_k = 2(a+b)k\varphi_{k-1}, \quad \varphi_0 = Q(-z).
\]
If \( k = 1 \) then \( \varphi_1' = 2(a+b). \) Since \( \varphi_1(0) = 0 \), we get that
\[
\varphi_1 = 2(a+b)z = Q(z) - Q(-z) = Q(-z) \left( \frac{Q(z)}{Q(-z)} - 1 \right).
\]
If \( k = 2 \) then it follows from (2.23) that
\[
Q(-z)\varphi_2' - Q'(-z)\varphi_2 = (Q(-z)\varphi_2)' = 4(a+b)\varphi_1 = 8(a+b)^2 z.
\]
Since \( \varphi_2(0) = 0 \), we find in this case that \( Q(-z)\varphi_2 = 4(a+b)^2 z^2 = (Q(z) - Q(-z))^2 \) Thus
\[
\varphi_2 = \frac{(Q(z) - Q(-z))^2}{Q(-z)} = Q(-z) \left( \frac{Q(z)}{Q(-z)} - 1 \right)^2.
\]
Repeated application of recurrence in the differential equation (2.23) gives
\[
(2.24) \quad \varphi_k = Q(-z) \left( \frac{Q(z)}{Q(-z)} - 1 \right)^k.
\]
Since \( \varphi_k(z) = \sum_{n \geq 0} d_{n,k} z^n \) and \( d_{n,k} = 0 \) for \( n < k \), the array \( D := [d_{n,k}]_{n,k \geq 0} \) is a lower triangular matrix with \( \varphi_k \) in (2.24) as the \( k \)th column generating function for \( k \geq 0 \). By definition, \( D \) is a
Theorem 5. For $Q$ of trees or lattices paths, etc. In terms of enumeration, one may consider a Sheffer sequence as allowing several different kinds of generating functions counting increasing trees, generating trees, or weighted lattice paths, with negative integer coefficients, combinatorially the sequence may be interpreted as corresponding to a critical line generating function $Q^2$ which completes the proof.

Deforming the path of integration.

2.1. The integrand (as a function of $x$) gives

\[ H_n(x) = \sum_{k=0}^{n} c_{n,k}(x)k = \sum_{k=0}^{n} \left( \sum_{i=0}^{n} (-1)^{i+k} c_{n,i}s(i,k) \right) x^k, \]

which completes the proof.

In concluding this section, we remark that if the polynomials in a Sheffer sequence have non-negative integer coefficients, combinatorially the sequence may be interpreted as corresponding to a (pair of) generating functions counting increasing trees, generating trees, or weighted lattice paths, etc. In terms of enumeration, one may consider a Sheffer sequence as allowing several different kinds of trees or lattices paths.

Part II - the zeros of the Sheffer sequence

We now turn our attention to the zero distribution of the Sheffer sequence $(H_n(x))_{n \geq 0}$ with generating function $Q(z)zQ(-z)^{1-x}$, where $Q(z)$ is a quadratic polynomial whose zeros are real, $Q(0) \neq 0$ and $Q'(0) \neq 0$.

The main result of this part of the paper is the following theorem.

Theorem 5. For $z_2 > z_2 > 0$, let $Q(z) = (z_1 - z)(z_2 - z)$ and $(H_n(x))_{n \geq 0}$ be the sequence of polynomials generated by

\[ \sum_{n=0}^{\infty} H_n(x)z^n n! = Q(z)zQ(-z)^{1-x} \quad (z, x \in \mathbb{C}). \]

Then for all large $n$, other than the two trivial zeros at $x = 0, 1$, all the zeros of $H_n(x)$ lie on the critical line $\Re x = 1/2$.

2.1. Deforming the path of integration. The substitution $x = 1/2 + \text{int}$ and the Cauchy integral formula give

\[ H_n \left( \frac{1}{2} + \text{int} \right) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{Q(z)^{1/2 + \text{int}}Q(-z)^{1/2 - \text{int}}}{z^{n+1}} dz, \quad (n \in \mathbb{N} \cup \{0\}). \]

The integrand (as a function of $z$) has an analytic continuation to the complement of $\{0\} \cup [z_1, \infty) \cup (-\infty, -z_1]$ defined by

\[
  f(z, t) := \left( \frac{1}{2} + \text{int} \right) \exp \left( \left( \frac{1}{2} + \text{int} \right) \log(z_1 - z) + \left( \frac{1}{2} + \text{int} \right) \log(z_2 - z) + \left( \frac{1}{2} - \text{int} \right) \log(z_1 + z) + \left( \frac{1}{2} - \text{int} \right) \log(z_2 + z) \right),
\]
where \( \log z \) denotes the principal logarithm\(^1\). On any circular arc \( C_R \) in the complement of \{0\} \cup \{z_1, \infty\} \cup (\infty, -z_1] \) centered at the origin with large radius \( R \), the expression

\[
\left| \int_{C_R} f(z, t) \, dz \right|
\]

is at most

\[
\int_{|z|=R} |Q(z)Q(-z)|^{1/2} \exp^{-nt} \left( \frac{\arg(z_1 - z) + \arg(z_2 - z) + \arg(z_1 + z) + \arg(z_2 + z)}{R^{n+1}} \right) |dz|.
\]

Using the estimates

\[
|\arg(z_1 - z) + \arg(z_2 - z) + \arg(z_1 + z) + \arg(z_2 + z)| \leq 2\pi,
\]

and

\[
|Q(z)Q(-z)|^{1/2} = O(R^2)
\]

for \( |z| = R \) we conclude that

\[
\lim_{R \to \infty} \int_{C_R} f(z, t) \, dz = 0.
\]

As a consequence,

\[
H_n \left( \frac{1}{2} + i n t \right) = \frac{n!}{2\pi i} \int_{\Gamma_1 \cup \Gamma_2} f(z, t) \, dz,
\]

where \( \Gamma_1 \) and \( \Gamma_2 \) are two loops around two cuts \((\infty, -z_1] \) and \([z_1, \infty) \) with counter clockwise orientation. Using the substitution \( z \mapsto -z \) we see that

\[
\frac{1}{2\pi i} \int_{\Gamma_1} f(z, t) \, dz = \frac{(-1)^{n+1}}{2\pi i} \int_{\Gamma_2} f(z, -t) \, dz.
\]

On the other hand, the substitution \( z \mapsto \bar{z} \) leads to the identity

\[
\frac{(-1)^{n+1}}{2\pi i} \int_{\Gamma_2} f(z, -t) \, dz = \frac{(-1)^{n+1}}{2\pi i} \int_{\Gamma_2} f(z, t) \, dz.
\]

We deduce that \( \pi H_n(1/2 + i nt) \) is either the imaginary part, or \(-i\) times the real part of the integral

\[
\int_{\Gamma_2} f(z, t) \, dz,
\]

depending on the parity of \( n \).

2.2. **Approximating** \( \int_{\Gamma_2} f(z, t) \, dz \) - the saddle point method. In order to approximate \( (2.26) \) using the saddle point method (see for example [17, Ch. 4]), we write

\[
f(z, t) = e^{-\phi(z, t)} \psi(z),
\]

where

\[
\phi(z, t) = \log z - it (\log(z_1 - z) + \log(z_2 - z) - \log(z_1 + z) - \log(z_2 + z)),
\]

and

\[
\psi(z) = \frac{1}{z} \exp^{1/2} (\log(z_1 - z) + \log(z_2 - z) + \log(z_1 + z) + \log(z_2 + z)).
\]

\(^1\)In the remaining of the paper, we always use the principle cut for complex power functions without explicitly stating so. If we need a different cut, it will be made clear and explicit in the text.
As a function in $z$, the critical points of $\phi(z, t)$ are the solutions of the equation

$$0 = \frac{1}{z} - it \left( \frac{1}{z - z_1} + \frac{1}{z - z_2} - \frac{1}{z + z_1} - \frac{1}{z + z_2} \right),$$

which (after clearing denominators) is equivalent to

$$(2.30) \quad z^4 - 2it(z_1 + z_2)z^3 - (z_1^2 + z_2^2)z^2 + 2itz_1z_2(z_1 + z_2)z + z_1^2z_2^2 = 0.$$  

The discriminant in $z$ of $(2.30)$ is a polynomial in $t$, whose positive zeros (in $t$) are

$$T_1 := \frac{z_2 - z_1}{z_1 + z_2} \quad \text{and} \quad T_2 := \frac{z_1 + z_2}{4\sqrt{z_1z_2}}.$$  

We note that $T_2 \geq T_1$ since

$$T_2^2 - T_1^2 = \frac{(z_1^2 - 6z_1z_2 + z_2^2)^2}{16z_1z_2(z_1 + z_2)^2} \geq 0.$$  

Set

$$(2.31) \quad T = \begin{cases} 
T_1 & \text{if } z_1^2 - 6z_1z_2 + z_2^2 \geq 0 \\
T_2 & \text{if } z_1^2 - 6z_1z_2 + z_2^2 < 0.
\end{cases}$$  

For each $t \in (0, T)$, the four solutions of $(2.30)$ are $\zeta_1(t)$, $\zeta_2(t)$, $-\zeta_1(t)$, $-\zeta_2(t)$ where

$$(2.32) \quad \zeta_1(t) = \frac{z_1 + z_2}{2} \left( it - \sqrt{T_1^2 - t^2} + \sqrt{1 - 2t^2 - 2it\sqrt{T_1^2 - t^2}} \right) \quad \text{for } 0 \leq t < T_1,$$

$$(2.33) \quad \zeta_1(t) = \frac{z_1 + z_2}{2} \left( it + i\sqrt{t^2 - T_1^2} + \sqrt{1 - 2t^2 - 2it\sqrt{t^2 - T_1^2}} \right) \quad \text{for } T_1 \leq t \leq T_2,$$

and

$$(2.34) \quad \zeta_2(t) = \frac{z_1 + z_2}{2} \left( it + \sqrt{T_1^2 - t^2} + \sqrt{1 - 2t^2 + 2it\sqrt{T_1^2 - t^2}} \right) \quad \text{for } 0 \leq t < T_1,$$

$$(2.35) \quad \zeta_2(t) = \frac{z_1 + z_2}{2} \left( it - i\sqrt{t^2 - T_1^2} + \sqrt{1 - 2t^2 + 2it\sqrt{t^2 - T_1^2}} \right) \quad \text{for } T_1 \leq t \leq T_2.$$  

We next establish some properties of the two curves $\zeta_1(t)$ and $\zeta_2(t)$ and their images under the map $\phi(z, t)$. These properties provide the justification for the choice of the loops we use in applying the argument principle in Section 2.3.

2.2.1. Properties of $\zeta_1(t)$ and $\zeta_2(t)$. Using the definitions of $\zeta_1(t)$ and $\zeta_2(t)$ it is straightforward to verify that in the case $T = T_2$ and $t \in [T_1, T_2)$, the two curves $\zeta_1(t)$ and $\zeta_2(t)$ are parts of the circle centered at the origin with radius $\sqrt{z_1z_2}$.

**Lemma 6.** If $\zeta_k(t)$, $1 \leq k \leq 2$, is defined as in equations $(2.32) - (2.35)$, then for $0 < t < T$ these curves lie in the first open quadrant.

**Proof.** We need to study the solutions to equation $(2.30)$. For the purposes of this proof, we set

$$f(z) = z^4 - 2it(z_1 + z_2)z^3 - (z_1^2 + z_2^2)z^2 + 2itz_1z_2(z_1 + z_2)z + z_1^2z_2^2,$$

$$h(z) = 4t^2z_1z_2z^2.$$
Note first that $\Gamma$ on for any $\epsilon \in \Gamma$. It is clear that the roots $0 \leq r_1, r_2$. By assumption, $0 < t < T$, and the reader will recall that $0 < z_1 < z_2$. If $T = T_1$, then trivially $0 < t < 1$, since $T_1 < 1$. On the other hand, if $T = T_2$, then we are in the case $z_1^2 - 6z_1z_2 + z_2^2 < 0$. This implies that $z_1^2 + 2z_1z_2 + z_2^2 < 8z_1z_2 < 16z_1z_2$, and consequently, $z_1 + z_2 < 4\sqrt{z_1z_2}$. We conclude that $T_2 < 1$ and hence $t < 1$. This means that we may rewrite the roots of $g$ as $r_{1,2,3,4} = \pm z_k \sqrt{1 - t^2} + iz_k t, \quad k = 1, 2, 3, 4$.

It is clear that the roots $r_{1,2,3,4}$ lie on the two circular arcs $|z| = z_1$ and $|z| = z_2$. We wish to employ Rouché’s theorem to show that $g$, and $g + h = f$ have the same number of zeros inside the curve $\Gamma$, which consists of the arcs $z = \epsilon e^{i\theta}$ and $z = Re^{i\theta}$ for $0 < \theta < \pi/2$ and the two line segments $\epsilon \leq z \leq R$, and $iz$, with $\epsilon \leq z \leq R$. Here $0 < \epsilon \ll 1$, and $0 < R - z_2 < 1$. Since $g$ has two zeros there for any $0 < t < T$, so does $f$, thereby establishing the claim. We now demonstrate that $|h| < |g|$ on $\Gamma$, which in turn will imply that $g$ and $f$ have the same number of zeros inside $\Gamma$.

Note first that

$$
|h(z)|^2 = 16t^4 z_1^2 z_2^2 |z|^4, \quad \text{and}
$$

$$
|g(z)|^2 = 16t^4 z_1^2 z_2^2 |z|^4 + 4t^2 |z|^2 (z_1^2 (|z|^2 - z_2^2)^2 - \Re(4itz_2z(z_2^2 - z^2))) + z_2^2 (|z|^2 - z_1^2)^2 - \Re(4itz_1z(z_1^2 - z^2)))
$$

\begin{equation}
(2.36) \quad + (|z|^2 - z_1^2)^2 - \Re(4itz_1z(z_1^2 - z^2)) (|z|^2 - z_2^2)^2 - \Re(4itz_2z(z_2^2 - z^2)))
\end{equation}
Consider first the piece \( \epsilon \leq z \leq R \) of \( \Gamma \), which lies on the real axis. In this case every expression in equation (2.36) that involves taking the real parts is zero. Consequently,

\[
|g(z)|^2 = 16t^4 z_1^2 z_2^2 |z|^4 + 4t^2 |z|^2 (z_1^2 (|z_2|^2 - |z|^2) + z_2^2 (|z_1|^2 - |z|^2)) + |z|^2 - |z|^2 |z|^2 - z_2^2
\]

Next we consider the segment of \( \Gamma \) lying on the imaginary axis: \( z = is \), with \( \epsilon \leq s \leq R \). On this segment,

\[
\text{Re}(4it z_k (z_k^2 - z^2)) = 4t z_k (z_k^2 + s^2), \quad k = 1, 2.
\]

It follows that

\[
|g(z)|^2 = 16t^4 z_1^2 z_2^2 |z|^4 + 4t^2 |z|^2 (z_1^2 (|s|^2 + z_2^2)^2 + 4t z_2 s (z_2^2 + s^2)) + z_2^2 (|s|^2 + z_1^2)^2 + 4t z_1 s (z_1^2 + s^2)) + (|s|^2 + z_2^2)^2 (|z|^2 + z_2^2)^2
\]

\[
> 16t^4 z_1^2 z_2^2 |z|^4 = |h(z)|^2.
\]

We now turn our attention to the arc \( z = \epsilon e^{i\theta}, 0 \leq \theta \leq \pi/2 \). On this segment of \( \Gamma \), we have

\[
|z|^2 - z_k^2 |z|^2 > (z_k^2 - \epsilon^2)^2, \quad k = 1, 2
\]

\[
\text{Re}(4it z_k (z_k^2 - z^2)) = -4t z_k \epsilon (\epsilon^2 \cos \theta \sin(2\theta) - \sin \theta \cos(2\theta)) + \sin \theta z_2^2
\]

\[
= -4t z_k \epsilon \sin \theta (\epsilon^2 + z_2^2)
\]

\[
\leq 0, \quad k = 1, 2.
\]

Therefore,

\[
|g(z)|^2 > 16t^4 z_1^2 z_2^2 |z|^4 + 4t^2 \epsilon^2 (z_1^2 ((z_2^2 - \epsilon^2)^2) + z_2^2 ((z_1^2 - \epsilon^2)^2)) + (z_1^2 - \epsilon^2)^2 (z_2^2 - \epsilon^2)^2
\]

\[
> 16t^4 z_1^2 z_2^2 |z|^4 = |h(z)|^2.
\]

Finally, on the arc \( z = Re^{i\theta}, 0 \leq \theta \leq \pi/2 \), we have

\[
|z|^2 - z_k^2 |z|^2 > (R - z_k)^2 (R^2 + z_k^2), \quad k = 1, 2
\]

\[
\text{Re}(4it z_k (z_k^2 - z^2)) = -4t z_k R (R^2 \cos \theta \sin(2\theta) - \sin \theta \cos(2\theta)) + \sin \theta z_2^2
\]

\[
= -4t z_k R \sin \theta (R^2 + z_2^2)
\]

\[
\leq 0, \quad k = 1, 2.
\]

We conclude that

\[
|g(z)|^2 > 16t^4 z_1^2 z_2^2 |z|^4 + 4t^2 R^2 (z_1^2 ((R - z_2)^2 (R^2 + z_2^2) + z_2^2 (R - z_1)^2 (R^2 + z_1^2)) + (R - z_1)^2 (R^2 + z_2^2) (R^2 + z_2^2)
\]

\[
> 16t^4 z_1^2 z_2^2 |z|^4 = |h(z)|^2
\]

on this piece as well. In summary, \( |h(z)| < |g(z)| \) on \( \Gamma \), and by Rouché’s Theorem, \( f \) has two zeros in the first open quadrant for all \( 0 < t < T \) as desired. \( \square \)
Lemma 7. Let $T$ be as defined in (2.31). Then for any $t \in [0, T]$, 
$$|\zeta_1(t)| \leq \sqrt{z_1 z_2} \leq |\zeta_2(t)|$$
with equality only when $T = T_2$ and $t \in [T_1, T_2]$.

Proof. One can easily verify from equations (2.33) and (2.35) that 
$$|\zeta_1(t)| = \sqrt{z_1 z_2} = |\zeta_2(t)|$$
in case $T = T_2$ and $t \in [T_1, T_2]$. Suppose now that $t \in (0, T_1)$. Then
$$\text{Im} \left( 1 - 2t^2 - 2it \sqrt{T_1^2 - t^2} \right) < 0,$$
and hence
$$\text{Im} \left( \sqrt{1 - 2t^2 - 2it \sqrt{T_1^2 - t^2}} \right) < 0$$
as well. It follows from equations (2.32) and (2.34) that $|\text{Re}(\zeta_1(t))| < |\text{Re}(\zeta_2(t))|$ and $|\text{Im}(\zeta_1(t))| < |\text{Im}(\zeta_2(t))|$, and consequently,
$$|\zeta_1(t)| < |\zeta_2(t)|.$$ 
Since the product of the four roots of the polynomial in equation (2.30) is equal to its constant term, we see that
$$|\zeta_1(t)|^2 |\zeta_2(t)|^2 = z_1^2 z_2^2.$$ 
Taking square roots and applying the preceding inequality finishes the proof. \hfill \Box

Lemma 8. Let $T$ be as defined in (2.31). Then as $t \to T$,
$$\zeta_1(t) - \zeta_1(T) = \begin{cases} 
\frac{z_1 + z_2}{2} \left( -\sqrt{T_1^2 - t^2} + \frac{T_1 \sqrt{T_1^2 - t^2}}{\sqrt{2T_1^2 - 1}} + O(T_1 - t) \right) & \text{if } T = T_1, \\
\sqrt{2(z_1 z_2)^{3/4}} \left( -\sqrt{T_1^2 - t^2} + \frac{T_1 \sqrt{T_1^2 - t^2}}{\sqrt{2T_1^2 - 1}} + O(T_1 - t) \right) & \text{if } T = T_2.
\end{cases}$$

Proof. Suppose first that $T = T_1$. We note that
$$1 - 2T_1^2 = -z_1^2 - 6z_1 z_2 + z_2^2 < 0,$$
and conclude that as $t \to T_1$,
$$\sqrt{1 - 2t^2 - 2it \sqrt{T_1^2 - t^2}} = \sqrt{\left(1 - 2T_1^2\right) \left( 1 - 2iT_1 \frac{\sqrt{T_1^2 - t^2}}{1 - 2T_1^2} + O(T_1 - t) \right)}$$
$$= -i \sqrt{2T_1^2 - 1} \left( 1 - iT_1 \frac{\sqrt{T_1^2 - t^2}}{1 - 2T_1^2} + O(T_1 - t) \right).$$
Thus equation (2.32) implies that as $t \to T_1$$$
$$\zeta_1(t) - \zeta_1(T_1) = \frac{z_1 + z_2}{2} \left( -\sqrt{T_1^2 - t^2} + \frac{T_1 \sqrt{T_1^2 - t^2}}{\sqrt{2T_1^2 - 1}} + O(T_1 - t) \right).$$
In the case $T = T_2$, using equation (2.33) and a CAS we obtain that as $t \to T_2$,
\[
\zeta_1(t) - \zeta_1(T_2) = \frac{\sqrt{32}(z_1 z_2)3/4}{\sqrt{(z_1 + z_2)\left(-z_1^2 + 6z_1 z_2 - z_2^2\right)}} \sqrt{T_2 - t} + O(T_2 - t).
\]

**Lemma 9.** Suppose $T = T_2$, and let
\[
d = \frac{(z_1 + z_2)\sqrt{T_1}}{\sqrt{2}} \left(1 + \frac{iT_1}{\sqrt{1 - 2T_1^2}}\right).
\]
Then for $k = 1, 2$ the following hold:
\[
\zeta_k(t) - \zeta_k(T_1) = \begin{cases} (-1)^{k+1} d \sqrt{T_1 - t} + O(T_1 - t) & \text{if } t \to T_1^- \\ (-1)^k d \sqrt{T - T_1} + O(t - T_1) & \text{if } t \to T_1^+ \end{cases}.
\]

**Proof.** We treat the case $t \to T_1^-$ and $k = 2$. The remaining cases follow from similar computations. Using
\[
1 - 2T_1^2 = -\frac{z_1^2 - 6z_1 z_2 + z_2^2}{(z_1 + z_2)^2} > 0
\]
we conclude as $t \to T_1$,
\[
\sqrt{1 - 2t^2 + 2t} \sqrt{T_1^2 - t^2} = \sqrt{(1 - 2T_1^2) \left(1 + 2iT_1 \frac{\sqrt{T_1^2 - t^2}}{1 - 2T_1^2} + O(T_1 - t)\right)} = \sqrt{1 - 2T_1^2} \left(1 + iT_1 \frac{\sqrt{T_1^2 - t^2}}{1 - 2T_1^2} + O(T_1 - t)\right).
\]
Thus (2.34) implies that as $t \to T_1^-$
\[
\zeta_2(t) - \zeta_2(T_1) = \frac{z_1 + z_2}{2} \left(\sqrt{T_1^2 - t^2} + \frac{iT_1 \sqrt{T_1^2 - t^2}}{\sqrt{1 - 2T_1^2}} + O(T_1 - t)\right)
\]
\[
= -d \sqrt{T_1 - t} + O(T_1 - t).
\]

**Remark 10.** Suppose that $\phi$ is as defined in equation (2.28) and $d$ as in equation (2.37). Using a CAS one can verify that
\[
d^3 \cdot \left(\frac{\partial^3 \phi(z, t)}{\partial z^3}\right)_{\zeta_2(T_1), T_1} = d^3 \cdot \left(\frac{\partial^3 \phi(z, t)}{\partial z^3}\right)_{\zeta_1(T_1), T_1} = 2\sqrt{2}(z_1 + z_2) \sqrt{\frac{z_1^2 - z_2^2}{z_2 - z_1} \sqrt{z_1^2 - 6z_1 z_2 + z_2^2}} \in \mathbb{R}^+
\]
and
\[
d \cdot \phi_{z, t}(\zeta_2(T_1), T_1) = \sqrt{2}(z_1 + z_2) \sqrt{\frac{z_1^2 - z_2^2}{z_2 - z_1} \sqrt{z_1^2 - 6z_1 z_2 + z_2^2}} \in \mathbb{R}^+.
\]
The fact that these quantities are positive and real will play important roles as we develop asymptotic expressions for the integral in (2.26) in the coming sections.

Our next result provides a key guiding component of the implementation of the saddle point method in our asymptotic approximation of the integral in (2.26).
Proposition 11. Let $T$ be as defined in (2.31). Then for any $t \in (0, T)$,
\[
\text{Re } \phi(\zeta_1(t), t) \leq \text{Re } \phi(\zeta_2(t), t),
\]
with equality only when $T = T_2$ and $T_1 \leq t < T_2$.

Proof. We first consider the case $t \in (0, T_1)$ regardless of whether $T = T_1$ or $T = T_2$. Since $\zeta_1(t)$ and $\zeta_2(t)$ approach $z_1$ and $z_2$ respectively as $t \to 0$, the definition of $\phi(z, t)$ (c.f. equation (2.28)) implies that
\[
\log(z_1) = \lim_{t \to 0} \text{Re } \phi(\zeta_1(t), t) < \lim_{t \to 0} \text{Re } \phi(\zeta_2(t), t) = \log(z_2).
\]
Suppose by way of contradiction that $\text{Re } \phi(\zeta_1(t), t) \geq \text{Re } \phi(\zeta_2(t), t)$ for some $t \in (0, T_1)$. Then the equality
\[
\text{Re } \phi(\zeta_1(T_1), T_1) = \text{Re } \phi(\zeta_2(T_1), T_1)
\]
along with the Mean Value Theorem implies the existence of a $t^* \in (0, T_1)$ for which
\[
(2.38) \quad \frac{d}{dt} \left( \text{Re } \phi(\zeta_1(t), t) - \text{Re } \phi(\zeta_2(t), t) \right) \bigg|_{t=t^*} = 0.
\]
We use the chain rule to compute
\[
\frac{d\phi(\zeta_k(t), t)}{dt} = \frac{\partial\phi}{\partial z} \bigg|_{\zeta_k(t), t} \frac{d\zeta_k(t)}{dt} + \frac{\partial\phi}{\partial t} = \frac{\partial \phi}{\partial t}
\]
since $\frac{\partial \phi}{\partial t} = 0$ for $k = 1, 2$ by virtue of $\zeta_k(t)$ being a critical point of $\phi$. Using this relation, we rewrite equation (2.38) to obtain
\[
0 = \text{Re } \left( \frac{\partial \phi(z, t)}{\partial t} \bigg|_{\zeta_1(t^*), t^*} - \frac{\partial \phi(z, t)}{\partial t} \bigg|_{\zeta_2(t^*), t^*} \right)
\]
\[= \text{Im } \left( F(\zeta_1(t^*)) - F(\zeta_2(t^*)) \right),
\]
where
\[
(2.39) \quad F(z) = \log(z_1 - z) + \log(z_2 - z) - \log(z_1 + z) - \log(z_2 + z).
\]
It is straightforward to check that
\[
\text{Im } (F(\zeta_1(T_1)) - F(\zeta_2(T_1))) = 0,
\]
hence applying the Mean Value theorem to the function $\text{Im } (F(\zeta_1(t)) - F(\zeta_2(t)))$ provides a $t^{**} \in (t^*, T_1)$ so that
\[
(2.40) \quad \text{Im } \left( \frac{d}{dt} F(\zeta_1(t^{**})) - \frac{d}{dt} F(\zeta_2(t^{**})) \right) = \text{Im } \left( \frac{dF}{dz} \bigg|_{\zeta_1(t^{**}), t^{**}} \zeta_1'(t^{**}) - \frac{dF}{dz} \bigg|_{\zeta_2(t^{**}), t^{**}} \zeta_2'(t^{**}) \right) = 0.
\]
Since $\phi(z, t) = \log z - itF(z)$, and
\[
\left. \frac{\partial \phi(z, t)}{\partial z} \right|_{\zeta_k(t), t} = 0 \text{ for all } t \in (0, T),
\]
we see that
\[
\frac{1}{\zeta_k(t)} - it \frac{dF}{dz} \bigg|_{\zeta_k(t)} = 0,
\]
or equivalently,
\[
\frac{dF}{dz} \bigg|_{\zeta_k(t)} = \frac{1}{it\zeta_k(t)}.
\]
This, together with the second equation in (2.40), implies that
\[
(2.41) \quad 0 = \text{Re } \left( \frac{\zeta_1'(t^{**})}{\zeta_1(t^{**})} - \frac{\zeta_2'(t^{**})}{\zeta_2(t^{**})} \right).
\]
Using the definition of $\zeta_1(t)$ and $\zeta_2(t)$ in equations (2.32) and (2.34) we find the explicit expressions
\[
\frac{\zeta_1'(t)}{\zeta_1(t)} = \frac{t + i\sqrt{T_1^2 - t^2}}{\sqrt{T_1^2 - t^2}\sqrt{-2it\sqrt{T_1^2 - t^2} - 2t^2 + 1}}, \quad \text{and}
\]
\[
\frac{\zeta_2'(t)}{\zeta_2(t)} = \frac{i\left(\sqrt{T_1^2 - t^2} + it\right)}{\sqrt{T_1^2 - t^2}\sqrt{2it\sqrt{T_1^2 - t^2} - 2t^2 + 1}},
\]
from which we readily deduce that
\[
\frac{\zeta_1'(t)}{\zeta_1(t)} = -\frac{\zeta_2'(t)}{\zeta_2(t)}.
\]
It follows that
\[
\frac{\zeta_1'(t)}{\zeta_1(t)} - \frac{\zeta_2'(t)}{\zeta_2(t)} = \frac{\zeta_1'(t)}{\zeta_1(t)} + \frac{\zeta_1'(t)}{\zeta_1(t)} = 2\Re\left(\frac{\zeta_1'(t)}{\zeta_1(t)}\right),
\]
and hence equation (2.41) can be reformulated as
\[
\Re\left(\frac{\zeta_1'(t^{**})}{\zeta_1(t^{**})}\right) = 0.
\]
But then $\Im\left(\frac{\zeta_1'(t^{**})}{\zeta_1(t^{**})}\right)^2 = 0$, which implies that $1 - T_1^2 = 0$, contradicting the fact that $0 < T_1 < 1$.

We point out here that in light of the arguments above, $\lim_{t \to T_1} \zeta_1'(t)/\zeta_1(t) > 0$ implies that
\[
\Re\left(\frac{\zeta_1'(t)}{\zeta_1(t)}\right) > 0, \quad \forall t \in (0, T_1),
\]
a fact we shall use shortly (see the proof of Lemma 14).

We now turn our attention to the second statement in the proposition. If $T = T_2$, then $\forall t \in [T_1, T_2)$
\[
(2.42) \quad \Re\left(\frac{\zeta_1'(t)}{\zeta_1(t)}\right) = \Re\left(\frac{\zeta_2'(t)}{\zeta_2(t)}\right) = 0,
\]

since on this range of the parameter $t$ we have the explicit formulas
\[
\frac{\zeta_1'(t)}{\zeta_1(t)} = \frac{i\left(\sqrt{T_2^2 - T_1^2} + t\right)}{\sqrt{T_1^2 - T_2^2}\sqrt{1 - 2t^2 - 2t\sqrt{T_1^2 - T_2^2}}}, \quad \text{and}
\]
\[
\frac{\zeta_2'(t)}{\zeta_2(t)} = \frac{i\left(\sqrt{T_2^2 - T_1^2} - t\right)}{\sqrt{T_1^2 - T_2^2}\sqrt{1 - 2t^2 + 2t\sqrt{T_1^2 - T_2^2}}},
\]

Using equation (2.42) and reversing the direction in the argument leading to equation (2.41) we conclude that $d/dt(\Im F(\zeta_1(t)) - F(\zeta_2(t)) = 0$. That is, $\Im F(\zeta_1(t)) = \Im F(\zeta_2(t)) = C$ for some constant $C$, and all $t \in [T_1, T_2)$. Recalling (c.f. equations (2.33) and (2.35)) that
\[
|\zeta_1(t)| = |\zeta_2(t)| = \sqrt{1 + 2t}, \quad t \in [T_1, T_2),
\]
and that $\Re\phi(\zeta_k(t), t) = \ln|\zeta_k(t)| + t\Im F(\zeta_k(t))$ (c.f. equation (2.28)), the conclusion $\Re\phi(\zeta_1(t), t) = \Re\phi(\zeta_2(t), t)$ readily follows. □
Lemma 13. Let φ be as defined in (2.28), T be as defined in (2.31), and ζ_k be as defined in (2.32)-(2.35). Then for \( k = 1,2, \)

(i) the function \( \text{Im} \, \phi(ζ_k(\cdot), \cdot) : (0,T) \to (0, \pi/2) \) is strictly monotone increasing and onto, and

(ii) the function \( \text{Re} \, \phi(ζ_k(\cdot), \cdot) \) is strictly decreasing on \( (0,T) \).

Proof. (i) Since \( ζ_k(t) \) is a critical point of \( φ \), we have

\[
\frac{dφ}{dt} = \frac{∂φ}{∂z}(ζ_k(t), t) \frac{dζ_k(t)}{dt} + \frac{∂φ}{∂t} \frac{∂ζ_k(t)}{∂t}.
\]

We take the imaginary part of both sides to obtain

\[
\frac{d}{dt} (\text{Im} \, φ(ζ_k(t), t)) = -\ln \left| \frac{z_1 - ζ_k(t)}{z_1 + ζ_k(t)} \right|.
\]

which is positive for \( t \in (0, T) \) since \( ζ_k(t) \) lies in the first open quadrant by Lemma 6. Finally, since \( ζ_k(t) \) approaches \( z_k \) (resp. a purely imaginary number) as \( t \to 0 \) (resp. \( t \to T \)), the definition of \( φ \) implies that

\[
\lim_{t \to 0} \text{Im} \, φ(ζ_k(t), t) = 0,
\]

\[
\lim_{t \to T} \text{Im} \, φ(ζ_k(t), t) = \pi/2.
\]

For part (ii), we compute

\[
\frac{d}{dt} (\text{Re} \, φ(ζ_k(t), t)) = \text{Arg}(z_1 - ζ_k(t)) + \text{Arg}(z_2 - ζ_k(t)) - \text{Arg}(z_1 + ζ_k(t)) - \text{Arg}(z_1 + ζ_k(t)) < 0,
\]

since \( ζ_k(t) \) lies in the first open quadrant. The proof is complete. \( \square \)

Lemma 14. Let φ be as defined in (2.28), T be as defined in (2.31), and ζ_k be as defined in (2.32)-(2.35). Then for any \( t, t_0 \in (0,T), \)

\[
\text{Re} \, φ(ζ_1(t_0), t_0) \leq \text{Re} \, φ(ζ_1(t), t_0) \quad \text{and} \quad \text{Re} \, φ(ζ_2(t_0), t_0) \geq \text{Re} \, φ(ζ_2(t), t_0),
\]

with equality only when (i) \( t = t_0 \) or (ii) \( T = T_2 \) and \( t, t_0 \in [T_1, T_2) \).

Proof. We begin with the first inequality. Let \( F \) be as defined in equation (2.39). Then

\[
\frac{d}{dt} \text{Re} \, φ(ζ_1(t), t_0) = \text{Re} \, \frac{d}{dt} φ(ζ_1(t), t_0)
\]

\[
= \frac{ζ_1'(t)}{ζ_1(t)} + t_0 \text{Im} \, \frac{d}{dt} F(ζ_1(t))
\]

\[
= \frac{ζ_1'(t)}{ζ_1(t)} + \frac{t_0}{t} \text{Re} \, \frac{ζ_1'(t)}{ζ_1(t)}
\]

\[
= \text{Re} \, \frac{ζ_1'(t)}{ζ_1(t)} \left( 1 - \frac{t_0}{t} \right).
\]

Remark 12. Using the definition of \( F \) in equation (2.39) and a CAS we find that for any \( (z,t) \) in the domain of \( φ(z,t) \) with \( z = \sqrt{z_1 z_2} e^{iθ} \), \( 0 \leq θ \leq π/2, \)

\[
\frac{d}{dθ} (\text{Im} \, F(z)) = 0.
\]

Consequently, \( \text{Re} \, φ(z,t) = \lim_{θ \to 0} \text{Re} \, φ(\sqrt{z_1 z_2} e^{iθ}, t) = \ln(\sqrt{z_1 z_2}) - 2πt. \)
If \( t \in (0, T_1) \setminus \{t_0\} \), the claim \( \Re(\phi(\zeta_1(t_0), t_0)) < \Re(\phi(\zeta_1(t), t_0)) \) follows from the fact that \( \Re(\zeta_1(t)/\zeta_1(0)) > 0 \). If \( T = T_2 \) and \( t \in [T_1, T_2) \), then by equation (2.42) we have \( \Re(\zeta_1(t)/\zeta_1(0)) = 0 \), which implies

\[
\Re(\phi(\zeta_1(t), t_0)) = \Re(\phi(\zeta_1(t_0), t_0))
\]

if \( t_0 \in [T_1, T_2) \). If on the other hand \( t_0 \in (0, T_1) \), then

\[
\Re(\phi(\zeta_1(t_0), t_0)) > \Re(\phi(\zeta_1(t_1), t_1)) = \Re(\phi(\zeta_1(t), t_0)).
\]

The second inequality in the lemma follows from analogous arguments, utilizing now that \( \Re(\zeta_2(t)/\zeta_2(0)) < 0 \) for \( t \in (0, T_1) \).

**Lemma 15.** Let \( \phi \) be as defined in (2.28). \( T \) be as defined in (2.31). For any \( t \in (0, T) \), the function \( \Re(\phi(\cdot, t)) \) is increasing on the positive imaginary axis, and decreasing on the negative imaginary axis.

**Proof.** We first show that \( \frac{\partial}{\partial y} \Re(\phi(iy, t)) > 0 \) for all \( y > 0 \). By the chain rule, this is equivalent to showing that for all \( y > 0 \),

\[
\Re(i\phi_z(iy, t)) = \frac{y^4 - 2t(z_1 + z_2)y^2 + (z_1^2 + z_2^2)y^2 - 2tz_1z_2(z_1 + z_2)y + z_1^2z_2^2}{y(y^2 + z_1^2)(y^2 + z_2^2)} > 0.
\]

Since equation (2.30) has no purely imaginary solutions when \( t \in (0, T), \Re(i\phi_z(iy, t)) \neq 0 \) for any \( y \in \mathbb{R} \). By evaluating the numerator of the fraction above at \( y = 0 \), we conclude \( \Re(\phi(iy, t)) \) is increasing at every \( y > 0 \). Applying the same reasoning mutatis mutandis, we obtain that \( \Re(\phi(iy, t)) \) decreasing at every \( y < 0 \).

Heuristically, for each \( t \in (0, T) \setminus T_1 \), the saddle point method gives the (non-uniform in \( t \)) approximation

\[
\int_{\Gamma_2} e^{-n\phi(z, t)}\psi(z, t)dz \sim \pm \frac{\sqrt{2\pi}e^{-n\phi(\zeta_1, t)}}{\sqrt{n\phi_z(\zeta_1, t)}}\quad (k = 1, 2, n \to \infty).
\]

When estimating the integral, we must therefore consider the quantity

\[
\left| e^{-n\phi(\zeta_1(0), t)} \right| = e^{-n\Re(\phi(\zeta_1(0), t))},
\]

which, together with Proposition [11] suggests that \( \zeta_1(t) \) plays a more important role than \( \zeta_2(t) \). Thus for the remainder of the paper we denote \( \zeta := \zeta_1(t) := \zeta(t) \). The goal of the ensuing sections is to provide rigorous arguments for this approximation and to provide a condition under which the approximation is uniform in \( t \).

### 2.2.2. The main term of the approximation.

The aim of this section, given \( \zeta \), is to find a curve \( \Gamma : I \to \mathbb{C} \) on some real interval containing the origin so that \( z(0) = \zeta(t) \), and so that the integral over this curve becomes the dominant term in estimating the integral \( \int_{\Gamma_2} e^{-n\phi(z, t)}\psi(z, t)dz \). We begin our quest by studying the behavior of \( \phi \) near the curve \( \zeta(t) \).

For each \( t \in (0, T) \), the function \( \phi(z, t) \) is analytic as a function in \( z \) on the open ball with center \( \zeta \) and radius \( \min(\Re(\zeta), \Im(\zeta)) \), with the power series representation

\[
\phi(z, t) = \phi(\zeta, t) + \frac{\phi_z(\zeta, t)}{2}(z - \zeta)^2 + \frac{\phi_z(\zeta, t)}{k!}(z - \zeta)^k. \tag{2.45}
\]
Letting $t \to 0$ in (2.32) we obtain
\[ z_1 - \zeta = z_1 - \frac{z_1 + z_2}{2} \left( 1 - T_1 + i t - i t T_1 + \mathcal{O}(t^2) \right) \]
\[ = -i z_1 t + \mathcal{O}(t^2). \]  
(2.46)
This means that the curve $\zeta(t)$ approaches the real axis at an angle of $\pi/2$. Consequently,
\[ z_1 - \zeta = \text{Im}(\zeta - z_1) = \text{Im} \zeta. \]
In addition, as $t \to 0$, we also have the relation
\[ \phi_{z_2}(\zeta, t) = \frac{t}{(\zeta - z_1)^2} = \frac{1}{t}, \]
and hence $t^2 \phi_{z_2}(\zeta, t) = \text{Im} \zeta$. Since $\phi_{z_2}(\zeta(T), T) \neq 0$, expanding $\phi_{z_2}$ in a Taylor series and using Lemma 8 we see that as $t \to T$,
\[ \phi_{z_2}(\zeta, t) \approx \zeta - \zeta(T) = \text{Re} \zeta. \]
Putting all this together we conclude that there exists small $\xi$ (independent of $t$) such that given any $t \in (0, T)$ the expansion in (2.45) is valid for all $z$ satisfying
\[ |z - \zeta| < \xi|\phi_{z_2}(\zeta, t)|t^2, \]
as the right side the above inequality is less than $\min(\text{Re} \zeta, \text{Im} \zeta)$. For $k \geq 3$, the definition of $\phi$ and our preceding discussion also yield the estimate
\[ \phi_{z_k}(\zeta, t) \ll \frac{(k-1)!}{|\zeta|^k} + \frac{t(k-1)!}{|\zeta - z_1|^k} + \frac{t(k-1)!}{|\zeta + z_1|^k} + \frac{t(k-1)!}{|\zeta - z_2|^k} + \frac{t(k-1)!}{|\zeta + z_2|^k} \]
\[ \ll \frac{tk!}{|\zeta - z_1|^k} + \frac{k!}{\eta^k} \ll \frac{tk!A^k}{|\zeta - z_1|^k} \]
for some small $\eta > 0$ and large $A > 0$ independent of $t$, $\zeta$ and $k$. Combining (2.47) and (2.48) we conclude that for sufficiently small $\xi$,
\[ \sum_{k=3}^{\infty} \frac{\phi_{z_k}(\zeta, t)}{k!} (z - \zeta)^k = (z - \zeta)^3 \sum_{k=3}^{\infty} \frac{\phi_{z_k}(\zeta, t)}{k!} (z - \zeta)^{k-3} \]
\[ \ll t |z - \zeta|^3 \sum_{k=0}^{\infty} A^3 |\zeta - z_1|^3 |A\zeta \phi_{z_2}(\zeta, t)|t^{2k} \]
\[ \ll \frac{|z - \zeta|^3}{t^2} \sum_{k=0}^{\infty} |A\zeta \phi_{z_2}(\zeta, t)|t^k \]
\[ \ll \frac{|z - \zeta|^3}{t^2}. \]
(2.49)
Using the above estimate for the tail of the series we write
\[ \phi(z, t) = \phi(\zeta, t) + \frac{\phi_{z_2}(\zeta, t)}{2} (z - \zeta)^2 (1 + h(z, t)), \]
where
\[ h(z, t) = \mathcal{O} \left( \frac{z - \zeta}{t^2 \phi_{z_2}(\zeta, t)} \right) \quad (t \neq T_1). \]
(2.50)
Consequently, if $\xi \ll 1$, $t \in (0, T) \setminus \{ T_1 \}$ and $z$ satisfies (2.47), then $|h(z, t)| < 1/2$, and in turn
\[
|\phi(z, t) - \phi(\zeta, t)| < \frac{3\xi^2}{4} |\phi_{z^2}(\zeta, t)|^3 t^4.
\]
We now establish the existence of the desired curve. For $\xi \ll 1$ and for any fixed $y \in \mathbb{C}$ satisfying (2.52)
\[
|y| < \frac{\xi}{2} |\phi_{z^2}(\zeta, t)|^{3/2} t^2,
\]
we apply Rouché’s theorem to the functions
\[
\frac{\sqrt{\phi_{z^2}(\zeta, t)}}{\sqrt{2}} (z - \zeta) \sqrt{1 + h(z, t)} \quad \text{and} \quad \frac{\sqrt{\phi_{z^2}(\zeta, t)}}{\sqrt{2}} (z - \zeta) \sqrt{1 + h(z, t) - y}
\]
to demonstrate that the equation
\[
y = \frac{\sqrt{\phi_{z^2}(\zeta, t)}}{\sqrt{2}} (z - \zeta) \sqrt{1 + h(z, t)}
\]
has exactly one solution in $z$ satisfying (2.47). Indeed, the equation
\[
0 = \frac{\sqrt{\phi_{z^2}(\zeta, t)}}{\sqrt{2}} (z - \zeta) \sqrt{1 + h(z, t)}
\]
has exactly one solution $z = \zeta$ satisfying (2.47). In addition, if
\[
|z - \zeta| = \xi |\phi_{z^2}(\zeta, t)| t^2,
\]
then the inequality $\sqrt{1 + h(z, t)} > 1/\sqrt{2}$ implies that
\[
\frac{\sqrt{|\phi_{z^2}(\zeta, t)|}}{\sqrt{2}} |z - \zeta| \sqrt{1 + h(z, t)} \geq \frac{\xi}{2} |\phi_{z^2}(\zeta, t)|^{3/2} t^2 > |y|.
\]
It follows then that the equation
\[
y = \frac{\sqrt{\phi_{z^2}(\zeta, t)}}{\sqrt{2}} (z - \zeta) \sqrt{1 + h(z, t)}
\]
also has exactly one solution in $z$ satisfying (2.47). Thus, for any $y$ satisfying (2.52), we may invert the relation
\[
y = \frac{\sqrt{\phi_{z^2}(\zeta, t)}}{\sqrt{2}} (z - \zeta) \sqrt{1 + h(z, t)}
\]
and - after using an analytic continuation argument - obtain a function $z = z(y)$ which is analytic in the entire ball defined by (2.52).
We continue by developing an asymptotic expression for the integral on a section of the curve parametrized by $z(y)$ under suitable conditions. In essence, for a given $\zeta$, we wish to understand how the integrand $e^{-n\phi(z, t)} \psi(z)$ compares to the 'central' value $e^{-n\phi(\zeta, t)} \psi(\zeta)$ for $z$ in the ball defined by (2.47). Consider the smooth curve $\Gamma_{\epsilon}(y)$ parameterized by $z(y)$, $-\epsilon \leq y \leq \epsilon$, for any small $\epsilon = \epsilon(n)$ and $t \in K \subset (0, T)$ for which
\[
\frac{\epsilon}{\sqrt{|\phi_{z^2}(\zeta, t)|^{3/2} t^2}} = o(1)
\]
\[\footnote{By virtue of $\xi$ being small enough, the square roots are all defined using the principle cut of the logarithm, and are analytic on the domain under consideration. Although it is possible that $\sqrt{\phi_{z^2}(\zeta, t)}$ is not continuous in $t$, we will later remove potential discontinuities by squaring this quantity.}
uniformly on $K$ as $n \to \infty$. If $z \in \Gamma_{\epsilon}(y)$, then
\[
\phi(z, t) - \phi(\zeta, t) = y^2,
\]
and using (2.51) and (2.53) we get
\[
z = \zeta + \frac{\sqrt{2}y}{\sqrt{\phi_{z^2}(\zeta, t)}} \left( 1 + O \left( \frac{\epsilon}{t^2 \sqrt{\phi_{z^2}(\zeta, t)}} \right) \right) \tag{2.55}
\]
Rearranging (2.55) and invoking condition (2.54) shows that given any $\epsilon > 0$, if $n > 1$, then
\[
|z - \zeta| = \frac{\sqrt{2}|y|}{\sqrt{\phi_{z^2}(\zeta, t)}} \left( 1 + O \left( \frac{\epsilon}{t^2 \sqrt{\phi_{z^2}(\zeta, t)}} \right) \right)
\]
and hence the curve $\Gamma_{\epsilon}(y)$ lies in the first open quadrant. On the ball defined by (2.47), we write
\[
\psi(z) = e^{\psi(z)},
\]
where $\psi(z) = e^{\Psi(z)}$, with
\[
\Psi(z) = - \log z + \frac{1}{2} \log(z_1 - z) + \frac{1}{2} \log(z_2 - z) + \frac{1}{2} \log(z_1 + z) + \frac{1}{2} \log(z_2 + z).
\]
We apply arguments similar to those leading to (2.49) along with equation (2.55) and the fact that $\sqrt{\Psi_z(\zeta)} |\zeta - z_1| = 1$ for small $t$, to obtain
\[
\frac{|z - \zeta|}{|\zeta - z_1|} = o(1),
\]
where the last equality relies on the facts $\phi_{z^2}(\zeta, t) = 1/t$ and $\zeta - z_1 = t$ for small $t$, as well as condition (2.54). In summary, for $z \in \Gamma_{\epsilon}$ and $t \in K \subset (0, T)$ for which condition (2.54) holds,
\[
e^{-n(\phi(z, t) + y^2)} = e^{-n(\phi(\zeta, t) + y^2)}
\]
and
\[
dz = \frac{dz}{dy} = \frac{\sqrt{2}}{\sqrt{\phi_{z^2}(\zeta, t)}} (1 + o(1)) dy.
\]
Consequently,
\[
\int_{\Gamma_{\epsilon}} e^{-n(\phi(z, t))} \psi(z) dz = \frac{\sqrt{2} \psi(\zeta) e^{-\phi(\zeta, t)}}{\sqrt{\phi_{z^2}(\zeta, t)}} \int_{-\epsilon}^{\epsilon} e^{-ny^2} (1 + o(1)) dy
\]
If $z \in \Gamma_{\epsilon}$, then
\[
\phi(z, t) - \phi(\zeta, t) = y^2,
\]
and using (2.51) and (2.53) we get
\[
z = \zeta + \frac{\sqrt{2}y}{\sqrt{\phi_{z^2}(\zeta, t)}} \left( 1 + O \left( \frac{\epsilon}{t^2 \sqrt{\phi_{z^2}(\zeta, t)}} \right) \right) \tag{2.55}
\]
Rearranging (2.55) and invoking condition (2.54) shows that given any $\xi < 1$, if $n > 1$, then
\[
|z - \zeta| = \frac{\sqrt{2}|y|}{\sqrt{\phi_{z^2}(\zeta, t)}} \left( 1 + O \left( \frac{\epsilon}{t^2 \sqrt{\phi_{z^2}(\zeta, t)}} \right) \right)
\]
and hence the curve $\Gamma_{\epsilon}(y)$ lies in the first open quadrant. On the ball defined by (2.47), we write
\[
\psi(z) = e^{\psi(z)},
\]
where $\psi(z) = e^{\Psi(z)}$, with
\[
\Psi(z) = - \log z + \frac{1}{2} \log(z_1 - z) + \frac{1}{2} \log(z_2 - z) + \frac{1}{2} \log(z_1 + z) + \frac{1}{2} \log(z_2 + z).
\]
We apply arguments similar to those leading to (2.49) along with equation (2.55) and the fact that $\sqrt{\Psi_z(\zeta)} |\zeta - z_1| = 1$ for small $t$, to obtain
\[
\frac{|z - \zeta|}{|\zeta - z_1|} = o(1),
\]
where the last equality relies on the facts $\phi_{z^2}(\zeta, t) = 1/t$ and $\zeta - z_1 = t$ for small $t$, as well as condition (2.54). In summary, for $z \in \Gamma_{\epsilon}$ and $t \in K \subset (0, T)$ for which condition (2.54) holds,
\[
e^{-n(\phi(z, t) + y^2)} = e^{-n(\phi(\zeta, t) + y^2)}
\]
and
\[
dz = \frac{dz}{dy} = \frac{\sqrt{2}}{\sqrt{\phi_{z^2}(\zeta, t)}} (1 + o(1)) dy.
\]
Consequently,
\[
\int_{\Gamma_{\epsilon}} e^{-n(\phi(z, t))} \psi(z) dz = \frac{\sqrt{2} \psi(\zeta) e^{-\phi(\zeta, t)}}{\sqrt{\phi_{z^2}(\zeta, t)}} \int_{-\epsilon}^{\epsilon} e^{-ny^2} (1 + o(1)) dy
\]
This requirement implies for example that $t \neq T_1$. In addition, the condition may fail to hold for certain ranges of $t$ near $0$, $T_1$, and $T_2$, which is the principal reason for us having to handle these cases separately in Sections 2.2.4, 2.2.5 and 2.2.6.
The reader will note that for any $\epsilon, n > 0$,
\[
\sqrt{\pi} = \int_{\mathbb{R}} e^{-y^2} dy = 2 \int_{\epsilon \sqrt{\pi}}^{\infty} e^{-y^2} dy + \int_{-\epsilon \sqrt{\pi}}^{\epsilon \sqrt{\pi}} e^{-y^2} dy,
\]
and that
\[
\int_{\epsilon \sqrt{\pi}}^{\infty} e^{-y^2} dy \leq \int_{\epsilon \sqrt{\pi}}^{\infty} \frac{y}{\epsilon \sqrt{\pi}} e^{-y^2} dy = \frac{e^{-\epsilon^2 n}}{2\epsilon \sqrt{n}}.
\]
As a result, we obtain the following asymptotic expression for the integral over $\Gamma_e$:
\[
(2.56) \quad \int_{\Gamma_e} e^{-n\phi(z,t)} \psi(z) dz = \frac{\sqrt{2\pi} \psi(z) e^{-n\phi(z,t)}}{\sqrt{n\phi z^2(z,t)}} \left( 1 + O \left( \frac{e^{-\epsilon^2 n}}{\epsilon \sqrt{n}} \right) + o(1) \right)
\]
for $\epsilon$ and $t \in K \subset (0, T)$ satisfying (2.54).

In the next section we extend append the curve $\Gamma_e$ with two tails going to $\infty$ in the upper and lower half planes respectively, for $t \neq T_1$ (i.e. $\phi_z(z,t) \neq 0$). We will also demonstrate that the integrals over these two tails are dominated by the integral in (2.56). We shall employ the convention that a complex number approaches $\infty$ in the upper half (resp. lower half) plane if its modulus approaches $\infty$ and its argument lies in $(0, \pi)$ (resp. $(-\pi, 0)$).

2.2.3. The tails of the approximation. To ensure the integrals over the two tails are dominated by the integral in (2.56), we will choose these tails so that $\text{Re } \phi(z,t) > \text{Re } \phi(z)(\Gamma \pm \epsilon, t) = \text{Re } \phi(z(t), t) + \epsilon^2$ for all $z$ in the tails. This section is dedicated to showing that such tails exist, and to demonstrating the claimed dominance.

**Lemma 16.** Let $\psi$ be as defined in (2.25); $T$ be as defined in (2.31). Let $r \in \mathbb{R}$ and $t \in (0, T)$. If $R_r$ denotes the intersection of a connected component of $\{z : \text{Re } \phi(z,t) > r\}$ with the first open quadrant, then $\partial R_r \cap (0, \infty)$ contains a non-empty open interval.

**Proof.** The claim is trivial if $R_r$ is unbounded since $\{z : \text{Re } \phi(z,t) > r\}$ contains all $z$ with large modulus. Assume next that $R_r$ is bounded, and note that $\text{Re } \phi(z,t)$ is continuous at every $z \in \overline{R_r}$, except perhaps at $z_1$ and $z_2$, should they lie on $\partial R_r$. Thus, it suffices to show that there exists a real $z_0 \in \partial R_r \setminus \{z_1, z_2\}$ such that $\text{Re } \phi(z_0,t) > r$. Suppose, by way of contradiction, that there is no such $z_0$. By Lemma 15, the intersection of $\partial R_r$ and the $y$-axis is either a point where $\text{Re } \phi(z,t) = r$ or it is empty. We deduce that $\text{Re } \phi(z,t) = r$ for all $z \in \partial R_r$, except at $z_1$ or $z_2$ were they lie on $\partial R_r$. For $\delta > 0$ set

\[ R_{r,\delta} = \{z \in R_r : |z - z_1| > \delta \text{ and } |z - z_2| > \delta\}, \]

and select $z^* \in R_{r,\delta}$. Then $z^* \in R_{r,\delta}$ for sufficiently small $\delta$, and since $\text{Re } \phi(z^*, t) > r$ and $\text{Im } \phi(z,t) \to -\infty$ as $z \to z_1$ or $z \to z_2$, we see that the map $\phi(\cdot, t) - \phi(z^*, t) : \partial R_{r,\delta} \to \mathbb{C}$ maps $\partial R_{r,\delta}$ into the complement of the closed first quadrant. Since $\phi(z,t)$ is analytic on a region containing $\overline{R_{r,\delta}}$, we may apply the argument principle theorem to conclude that

\[ 0 = i\Delta \arg_{z \in \partial R_{r,\delta}} \phi(z, t) - \phi(z^*, t) \int_{\partial R_{r,\delta}} \frac{\phi'(z,t)}{\phi(z,t) - \phi(z^*, t)} dz. \]

On the other hand,
\[
\frac{1}{2\pi i} \int_{\partial R_{r,\delta}} \frac{\phi'(z,t)}{\phi(z,t) - \phi(z^*, t)} dz \geq 1,
\]
since $z^* \in R_{r,\delta}$ is a zero of $\phi(z,t) - \phi(z^*, t)$ inside the curve $\partial R_{r,\delta}$, and we have reached a contradiction. The result follows. \qed
Remark 17. For each fixed \( t \in (0, T) \), as a function in \( x \), \( \Re \phi(x + i\delta, t) \) converges uniformly to the function
\[
u(x) = \begin{cases} 
\ln x & \text{if } 0 < x < z_1 \\
\ln x - t\pi & \text{if } z_1 < x < z_2 \\
\ln x - 2t\pi & \text{if } z_2 < x 
\end{cases}
\]
on any real compact subset of \((0, \infty) \setminus \{z_1, z_2\}\) as \( \delta \to 0^+ \). Since \( u(x) \) is increasing, if \((a, b) \subseteq \partial \mathcal{R}_t \cap (0, z_1)\), then so is \((a, z_1)\) and the similar conclusions hold for the intervals \((z_1, z_2)\) and \((z_2, \infty)\).

Definition 18. Let \( T \) be as defined in (2.31). For each \( t \in (0, T) \), we set \( C_t := C_1(t) \) (resp. \( C_2 \)) to be the intersection of the first open quadrant with the connected component of \( \{z : \Re \phi(z, t) - \Re \phi(\zeta, t) > 0\} \) containing \( \Gamma_c(y) \) for \( -\epsilon \leq y < 0 \) (resp. \( 0 < y \leq \epsilon \)).

Before we present our next result, we recall two standard theorems from complex analysis.

Lemma 19. [12] Lemma 1.2, p.511 . Let \( \gamma : [a, b] \to \mathbb{C} \) be a simple, closed path, and let \( c \) be a point of \((a, b)\) at which \( \gamma \) is differentiable with \( \gamma(c) \neq 0 \). There exists an \( \epsilon > 0 \) for which it is true that the sets \( I_s^+ = \{\gamma(c) + si\gamma(c) : 0 < s \leq \epsilon\} \) and \( I_s^- = \{\gamma(c) + si\gamma(c) : -\epsilon \leq s < 0\} \) lie in different components of \( \mathbb{C} \setminus |\gamma|\).

In the following theorem \( n(\gamma, z) \) denotes the winding number of a simple closed curve \( \gamma \) about the point \( z \in \mathbb{C} \).

Theorem 20. [12] Theorem 1.3, p.553 Let \( \gamma : [a, b] \to \mathbb{C} \) be a simple, closed, piecewise smooth path and let \( D \) be the bounded component of \( \mathbb{C} \setminus |\gamma|\). Then either \( n(\gamma, z) = 1 \) for every \( z \in D \) or \( n(\gamma, z) = -1 \) for all such \( z \).

Lemma 21. \( T \) be as defined in (2.31), and let \( C_1 \) and \( C_2 \) be as in Definition 18 . Then \( C_1 \neq C_2 \)

Proof. If \( C_1 = C_2 \), then in this component we can extend \( \Gamma_c(y) \) to a simple, closed, piecewise smooth curve \( \gamma \) on which \( \Re \phi(z, t) \geq \Re \phi(\zeta, t) \) with equality only when \( z = \zeta \). This implies that if \( z_0 \in \mathbb{C} \) is any point for which \( \Re \phi(\zeta, t) > \Re \phi(z_0, t) \), then the winding number of \( \phi(\gamma, t) - \phi(z_0, t) \) about the origin is zero, and consequently, \( n(\gamma, z_0) = 0 \). For small \( y > 0 \), consider the points
\[
z_0^\pm = \zeta \pm i \frac{\sqrt{2y}}{\phi_{z_0}(\zeta, t)}.
\]
Using the expression in (2.50) we find that
\[
\phi(z_0^+, t) - \phi(\zeta, t) = \frac{\phi_{z_0}(\zeta, t)}{2} \left( \pm i \frac{\sqrt{2y}}{\phi_{z_0}(\zeta, t)} \right)^2 (1 + \mathcal{O}(y^3)) = -y^2 + \mathcal{O}(y^3).
\]
It follows that \( \Re \phi(z_0^+, t) - \Re \phi(\zeta, t) < 0 \), and hence \( z_0^\pm \notin \gamma \), and \( n(\gamma, z_0^\pm) = 0 \). By Lemma 19 the points \( z_0^\pm \) lie in different components of the complement of the trace of \( \gamma \) in the first open quadrant, and by [12] Theorem 1.3, page 553 these components are both unbounded. We have reached contradiction since the complement of the trace of \( \gamma \) has only one unbounded component. \( \square \)

Remark 22. Using similar arguments we also establish that there are two distinct connected components of the set \( \{z : \Re \phi(z, t) > \Re \phi(\zeta_2, t)\} \) whose boundaries contain \( \zeta_2 \).

The next result shows that the endpoints of the curve \( \Gamma_c \) lie in regions of the plane from which it is possible to continue the curve to the point at infinity both in the upper and in the lower half planes, while maintaining the desired relation \( \Re \phi(z, t) - \Re \phi(\zeta, t) > 0 \) for all \( z \) in the tails.
Lemma 23. Suppose \( T = T_2 \) and (2.54). Let \( S \) be the region enclosed by \( \zeta_2(\tau) \), \( 0 \leq \tau \leq T_1 \), the circular arc radius \( \sqrt{z_1 z_2} \), and the segment from \( \sqrt{z_1 z_2} \) to \( z_2 \) (see Figure 2.2). If \( \epsilon > 0 \) is such that condition (2.54) holds, then \( \Gamma_\epsilon(\pm \epsilon) \) lies outside \( S \).

Proof. Suppose first that \( t \in [T_1, T_2] \). Then for all \( y \in [-\epsilon, \epsilon] \) and for all \( \tau \in (0, T) \),

\[
\text{Re} \phi(z(y), t) > \text{Re} \phi(\zeta_1(t), t) \quad \text{(Def. of } \Gamma_\epsilon) \\
= \text{Re} \phi(\zeta_2(t), t) \quad \text{(Remark 12)} \\
\geq \text{Re} \phi(\zeta_2(\tau), t). \quad \text{(Lemma 14)}
\]

Were \( \Gamma_\epsilon(\pm \epsilon) \in S \), the continuity of \( \Gamma_\epsilon \) would necessitate its crossing of either the \( \zeta_2 \) curve, or the circular arc. Either of these cases would furnish a \( y \in [-\epsilon, \epsilon] \) for which the above inequality would fail.

Suppose now that \( t \in (0, T_1] \). We claim that for any \( z = \sqrt{z_1 z_2} e^{i\theta} \), \( 0 < \theta < \pi/2 \), \( \text{Re} \phi(z, t) > \text{Re} \phi(\zeta(t), t) + \epsilon^2 \) from which the result will follow, because \( \text{Re} \phi(\zeta, t) + \epsilon^2 > \text{Re} \phi(z(y), t) \) for all \( y \in [-\epsilon, \epsilon] \) and \( t \in (0, T_1] \). In establishing the claim, it suffices to consider \( T_1 - t = o(1) \) due to the fact that \( \epsilon = o(1) \). For any \( z \) on the circular arc (i.e. of the form \( z = \sqrt{z_1 z_2} e^{i\theta} \), \( 0 < \theta < \pi/2 \)),

\[
\text{Re} \phi(z, t) - \text{Re} \phi(\zeta(t), t) = \text{Re} (\phi(\zeta(T_1), t) - \phi(\zeta(t), t)).
\]

Using the Taylor series expansion of \( \phi(z, t) \) as a function of two complex variables centered at \( (\zeta(T_1), T_1) \), and Lemma 9 (along with the constant \( d \) defined therein (c.f. equation (2.37))) yield

\[
\phi(\zeta(t), t) - \phi(\zeta(T_1), t) \\
= d \cdot \phi_{z,t}(\zeta(T_1), T_1)(t - T_1) \sqrt{T_1 - t} + \frac{1}{3!} d^3 \cdot \phi_{z+t}(\zeta(T_1), T_1)(T_1 - t)^{3/2} + \mathcal{O}(T_1 - t)^2.
\]

Taking real parts and invoking Remark 10 gives

\[
\text{Re} (\phi(\zeta(T_1), t) - \phi(\zeta(t), t)) \approx (T_1 - t)^{3/2}.
\]

4Recall that \( \Gamma_\epsilon \) lies entirely in the first open quadrant, so it certainly doesn’t cross the real axis.
The inequality \( \Re (\phi(T_1), t) - \phi(T(t), t)) > \epsilon^2 \) follows from condition (2.54) and the asymptotic equivalence
\[
\phi_{\delta^2} (T(t), t) = (T_1 - t)^{1/2},
\]
and the proof is complete.

We now complete the argument demonstrating the existence of the two tails of \( \Gamma \) needed to complete the asymptotic estimate for \( \int_{\gamma_2} f(z, t) dz \), provided that condition (2.54) holds. To this end, let \( R_1 \subset C_1 \) (resp. \( R_2 \subset C_2 \)) be a connected component of
\[
\{ z : \Re \phi(z, t) > \Re \phi(\Gamma, t), t) = \Re \phi(\Gamma, t)) + \epsilon^2 \}
\]
whose boundary contains \( \Gamma \) (resp. \( \Gamma \)). We argue that one of the two sets \( R_1 \) and \( R_2 \) is unbounded, while the other contains an interval of the form \((z_1 - \delta, z_1)\) in its closure for some \( \delta > 0 \). Since for any \( x_0 \in (0, z_1) \), the function \( \Re \phi(x_0 - iy, t) \) is increasing in \( y \in (0, \infty) \), the existence of the tail will follow because (i) in the unbounded component we can find a path to \( -\infty \) from one endpoint of \( \Gamma \), and (ii) in the bounded component we can connect the other endpoint of \( \Gamma \) to \( x_0 \in (z_1, -\delta, z_0) \) and then append a ray \( x_0 - iy, y \geq 0 \).

Let \( t \in (0, T) \), and suppose that condition (2.54) holds. Assume first that \( \Re \phi(\Gamma, t)) + \epsilon^2 \geq \Re \phi(\Gamma, t) \). Since \( \epsilon = o(1) \), Proposition 11 implies that \( T = T_2, T_1 < t < T_2 \) and \( \Re \phi(\Gamma, t) \). By Lemma 14 we have \( \Re \phi(\Gamma, t) \), which means that if \( z = \Gamma(t) \) for any \( t \in (0, T) \), then \( z \notin \mathcal{R}_k \), \( k = 1, 2 \). In addition, if \( z \in \partial S \) with \( |z| = \sqrt{z_1 z_2} \), then by Remark 12 \( \Re \phi(z, t) = \Re \phi(\Gamma, t) \) and once more we conclude that \( z \notin \mathcal{R}_k \), \( k = 1, 2 \). Thus, either \( \mathcal{R}_k \subset S \cup \{z_1, z_2\} \), or \( \mathcal{R}_k \cap S = \emptyset \). Since Lemma 23 shows that \( \Gamma(\pm \epsilon) \notin S \), we conclude that \( \mathcal{R}_k \cap S = \emptyset \) for \( k = 1, 2 \). Lemma 16 now establishes the claim, since it implies that one of the \( \mathcal{R}_k \) intersects the positive real axis in an interval \((A, \infty) \) (and hence unbounded), while the other intersects the positive real axis in an interval of the form \((z_1 - \delta, z_1)\) for some \( \delta > 0 \).

Next, we consider the case when \( t \in (0, T) \) is such that \( \Re \phi(\Gamma, t)) + \epsilon^2 < \Re \phi(\Gamma, t) \). In this case we must have \( t \in (0, T_1) \) (regardless off whether \( T = T_1 \) or \( T = T_2 \)). Thus \( \Gamma(t) \) lies outside the closed ball centered at the origin with radius \( \sqrt{z_1 z_2} \), and for all \( z \) in the boundary of this ball in the first quadrant we have
\[
\Re \phi(\Gamma, t)) > \Re \phi(z, t).
\]
Consider now the two distinct, connected components of the set \{ \( z : \Re \phi(z, t) > \Re \phi(\Gamma, t) \) \}, whose boundaries contain \( \Gamma(t) \). By the argument above, we see that neither of these two components contain any points inside the ball with radius \( \sqrt{z_1 z_2} \). By Lemma 16 one will intersect the positive real axis in an interval of the form \((z_2 - \nu, z_2)\), and the other in an interval of the form \((B, \infty) \) for some \( B > z_2 \). Since the connected components \( \mathcal{R}_k \), \( k = 1, 2 \) are distinct, we see that one of these has to have a common point with either \( (z_2 - \nu, z_2) \) for some \( \nu > 0 \), or with \((B, \infty) \). The inequality \( \Re \phi(\Gamma, t)) + \epsilon^2 < \Re \phi(\Gamma, t) \) now implies that both components of \{ \( z : \Re \phi(z, t) > \Re \phi(\Gamma, t) \) \} with \( \Gamma(t) \) in their boundary are contained in \( \mathcal{R}_k \) for \( k = 1 \) or \( k = 2 \) (w.o.l.g. we may assume that \( k = 2 \)). It follows now that \( \partial \mathcal{R}_2 \) contains two real intervals whose right endpoints are \( z_2 \) and \( \infty \), and \( \partial \mathcal{R}_1 \) contains an interval whose right endpoint is \( z_1 \) (recall that \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are distinct).

We summarize these discussions in the following proposition.

**Proposition 24.** Let \( \phi \) be as defined in (2.28), \( T \) be as defined in (2.31), and \( f \) be as defined in Section 2. Suppose \( t \in (0, T) \) and \( \epsilon > 0 \) are such that condition (2.54) holds. Then there exists a piecewise smooth curve \( \Gamma \) parameterized by \( z(y), -\infty < y < \infty \) such that

\footnote{we remind the reader that by convention, condition (2.54) implies that \( t \neq T_1 \).}
(i) \( \int_{\Gamma} f(z,t)dz = \pm \int_{\Gamma} f(z,t)dz \),
(ii) \( \lim_{y \to \pm \infty} z(y) = \infty \),
(iii) \( \phi(z(y), t) - \phi(\zeta, t) = y^2 \) for all \(-\epsilon < y < \epsilon\),
(iv) \( \Re \phi(z(y), t) \geq \Re \phi(\zeta, t) + \epsilon^2 \) for all \(|y| \geq \epsilon\), and with equality if and only if \( y = \pm \epsilon \).

Remark 25. Since \( \Re \phi(z, t) \) is large for large \( z \), we may assume that for large \( z \in \Gamma \), the curve \( \Gamma \) is given by \( z = iy \) and still have (iv) in Proposition 24 hold. Figure 2.3 illustrates the proposed path of integration.

![Figure 2.3. The curve \( \Gamma \)](image)

We break the path \( \Gamma \setminus \Gamma_\epsilon \) into the pieces \( \gamma_+, \gamma_-, \gamma_+, \) and \( \gamma_- \) as indicated in Figure 2.3, and give bounds on the integral of \( e^{-n\phi(z,t)}\psi(z) \) over these segments under the assumptions that

| Conditions |
|---|
| (i) Condition (2.54) holds, and |
| (ii) \( ne^2 \to \infty \) as \( n \to \infty \). |

When combined, these estimates provide an asymptotic bound for the integral

\[
(2.57) \quad \int_{\Gamma \setminus \Gamma_\epsilon} e^{-n\phi(z,t)}\psi(z)dz.
\]

Recall the definition of \( \psi \) (c.f. equation (2.29)) which implies that for \( z \in \Gamma \setminus \Gamma_\epsilon, \psi(z) = \mathcal{O}(z) \). On the portions \( \gamma_+ \) and \( \gamma_- \) of \( \Gamma \setminus \Gamma_\epsilon \), starting at the points \( \Gamma_\epsilon(\pm \epsilon) \) with length \( l(\gamma_+) = l(\gamma_-) = e^{n\epsilon^2/4} \) (which is large if \( n \gg 1 \) by assumption (ii) above), we have \( |z| = \mathcal{O}(e^{n\epsilon^2/4}) \), and consequently

\[
\int_{\gamma_{+/-}} e^{-n\phi(z,t)}\psi(z)dz = \mathcal{O}\left(e^{-n\phi(\zeta,t)}-ne^2\int_{\gamma_{+/-}} |z||dz| \right) = \mathcal{O}\left(e^{-n\phi(\zeta,t)}-ne^2/2 \right).
\]
On the other hand, by Remark 29 on the segments $\gamma_{++}$ and $\gamma_{--}$ we have
\[
\int_{\gamma_{++/--}} e^{-n\phi(z,t)} \psi(z) dz = O \left( \int_Y^{\infty} \frac{dy}{y^{n-1}} \right) = O \left( \frac{1}{(n-2)Y^{n-2}} \right),
\]
where $Y \gg 1$. Since $\epsilon = o(1)$ and $\phi(\zeta,t)$ is bounded on $(0,T)$, we see that
\[
\frac{1}{Y} = e^{-\phi(\zeta,t) - r^2/2},
\]
and consequently,
\[
\int_{\gamma_{++/--}} e^{-n\phi(z,t)} \psi(z) dz = O \left( e^{-n\phi(\zeta,t) - nr^2/2} \right).
\]
We conclude that under the assumptions in (\dagger),
\[
\int_{\Gamma_2} f(z,t) dz = \pm \frac{\sqrt{2\pi}}{\sqrt{n\phi(z,t)}} \left( 1 + O \left( \frac{e^{-\epsilon^2 n}}{\epsilon \sqrt{n}} + \frac{e^{-\epsilon^2 n/2}}{\psi(\epsilon) \sqrt{n}} \right) \right),
\]
which implies that
\[
\left( \int_{\Gamma_2} f(z,t) dz \right)^2 = \frac{2 \pi \psi(\epsilon) e^{-2n\phi(\zeta,t)}}{n\phi(z,t)} \left( 1 + O \left( \frac{e^{-\epsilon^2 n}}{\epsilon \sqrt{n}} + \frac{e^{-\epsilon^2 n/2}}{\psi(\epsilon) \sqrt{n}} \right) \right).
\]
We close this section by noting that we may not be able to satisfy the assumptions in (\dagger) when $\sqrt{\phi_z(\zeta,t)^2} t^2$ approaches 0 too rapidly. Since $\sqrt{\phi_z(\zeta,t)^2} t^2$ is small when $t$ is close to 0, $T_1$, or $T_2$ (when $T = T_2$), we need separate arguments to develop asymptotic expressions for $\int_{\Gamma_2} f(z,t) dz$ for $t$ in these ranges.

2.2.4. The asymptotics when $1/n^{2/3} \ll T - t \ll \ln^2 n/n^{2/3}$. We continue our work by looking at the case when $t$ is close to $T$ (as defined in equation (2.31)). We begin with developing an asymptotic expression for $\phi(z,t) - \phi(\zeta,t)$ for $z$ close to $\zeta$. Using Lemma 8 we conclude that for large $n$,
\[
\zeta(t) - \zeta(T) = c\sqrt{T - t} + O(T - t) \leq 2|c| \ln n/n^{1/3},
\]
where
\[
c = \begin{cases} \frac{z_1 + z_2}{2} \left( \frac{T_1 + T_2}{\sqrt{2T_1 T_2}} \right) & \text{if } T = T_1 \\ \frac{\sqrt{3z(z_1 z_2)}}{\sqrt{(z_1 + z_2)^3}} \left( \frac{T_1 T_2}{\sqrt{2T_1 T_2}} \right) & \text{if } T = T_2 \end{cases}
\]
For $z$ in a small neighborhood of $\zeta$, we expand $\phi(z,t)$ (as a function of $z$) about $\zeta$:
\[
\phi(z,t) = \phi(\zeta,t) + \frac{\phi_{zz}(\zeta,t)}{2!} (z - \zeta)^2 + \frac{\phi_{zzz}(\zeta,t)}{3!} (z - \zeta)^3 + O \left( (z - \zeta)^4 \right).
\]
Since $\phi_{zz}(\zeta(T),T) = 0$, expanding $\phi_{zz}(\zeta,t)$ in a bi-variate series centered at $(\zeta(T),T)$ yields
\[
\phi_{zz}(\zeta,t) = \phi_{zz}(\zeta(T),T) (\zeta - \zeta(T)) + O(T - t) = c\phi_{zz}(\zeta(T),T) \sqrt{T - t} + O(T - t).
\]
Similiarly, expanding $\phi_{zz}(\zeta,t)$ in a series centered at $(\zeta(T),T)$ gives
\[
\phi_{zz}(\zeta,t) = \phi_{zz}(\zeta(T),T) (1 + O(|\zeta - \zeta(T)| + |T - t|)) = \phi_{zz}(\zeta(T),T) \left( 1 + O \left( \sqrt{T - t} \right) \right).
\]
Thus
\[ \phi(z, t) - \phi(\zeta, t) = \frac{c^3 \phi_{z^3}(\zeta(T), T)}{6} (z/c - \zeta/c)^2 \left( 3\sqrt{T} - t - (z/c - \zeta/c) \right) \]
(2.60)
+ \mathcal{O} \left( (z - \zeta)^2 (T - t) + (z - \zeta)^3 \sqrt{T} - t + (z - \zeta)^4 \right).

We also remark that using the definition of \( \phi(\zeta, t) \) and a CAS, one easily verifies that if \( T = T_2 \), then
\[ c^3 \phi_{z^3}(\zeta(T), T) = \frac{128i\sqrt{2}(z_1 z_2)^{3/4}}{(z_1 + z_2)^3 \sqrt{(z_1 + z_2)(-z_1^2 + 6z_1 z_2 - z_2^2)}} \in i\mathbb{R}^+, \]
and if \( T = T_1 \), then
\[ c^3 \phi_{z^3}(\zeta(T), T) = \frac{2i\sqrt{2}(z_1 + z_2)^2}{\sqrt{-z_1^4 + 6z_1^3 z_2 - 6z_1 z_2^3 + z_2^4}} \in i\mathbb{R}^+. \]

**Proposition 26.** Let \( T \) be as defined in equation (2.31), \( c \) be as defined in equation (2.59), and let \( t \) satisfy \( 1/n^{2/3} < T - t \leq \ln^2 n/n^{2/3} \). Then there exists a function \( z(y) \) analytic in a neighborhood of \( \mathbb{R} \) such that
\[
(i) \quad z(0) = \zeta,
(ii) \quad 3\sqrt{T} - t - (z(y)/c - \zeta/c) \notin (-\infty, 0] \forall y \in \mathbb{R}, \text{ and }
(iii) \quad y = \frac{\sqrt{c^3 \phi_{z^3}(\zeta(T), T)}}{\sqrt{6}} (z(y)/c - \zeta/c) \sqrt{3\sqrt{T} - t - (z(y)/c - \zeta/c)}.

**Proof.** Let \( c, t \) and \( T \) be as in the statement. Let \( y \in \mathbb{R} \) and set
\[ A = -\frac{\sqrt{6}}{\sqrt{c^3 \phi_{z^3}(\zeta(T), T) \sqrt{3\sqrt{T} - t}}}, \]
\[ B = \frac{1}{2} c^3 \phi_{z^3}(\zeta(T), T) \sqrt{3\sqrt{T} - t}, \quad \text{and} \]
\[ Z = \frac{1}{3A\sqrt{B}} \left( \frac{r(y)}{\sqrt{2}} + \frac{\sqrt{2}}{r(y)} - 1 \right), \]
(2.62)
where
\[ r(y)^3 = 27A^2 y^2 - 2 + 2\sqrt{27} y \sqrt{27A^4 y^2 - 4A^2}. \]
(2.63)
The last remarks immediately preceding the proposition imply that \( A^4 \in \mathbb{R} \), and that \( A^2 \in i\mathbb{R}^- \). Consequently, \( 27A^4 y^2 - 4A^2 \notin \mathbb{R} \), which implies that \( r(y)^3 \) is analytic in a neighborhood of \( \mathbb{R} \). The existence of an analytic cube root of \( r(y)^3 \) will follow once we show that
\[ 27A^2 y^2 - 2 + \sqrt{27} y \sqrt{27A^4 y^2 - 4A^2} \notin \{0, \infty\}, \]
since in this case we can choose \([0, \infty)\) as the cut to define \( r(y) \). Suppose to the contrary, that
\[ 27A^2 y^2 - 2 + \sqrt{27} y \sqrt{27A^4 y^2 - 4A^2} \in \{0, \infty\}. \]
The identity
\[ (\star) \quad (27A^2 y^2 - 2 + \sqrt{27} y \sqrt{27A^4 y^2 - 4A^2}) (27A^2 y^2 - 2 - \sqrt{27} y \sqrt{27A^4 y^2 - 4A^2}) = 4 \quad (y \in \mathbb{R}) \]
implies that \( r^3(y) \neq 0 \forall y \in \mathbb{R} \, \text{and} \, 27A^2 y^2 - 2 - \sqrt{27} y \sqrt{27A^4 y^2 - 4A^2} \notin [0, \infty) \) for all \( y \in \mathbb{R} \). Consequently, the sum of the two factors in \((\star)\) is real, and non-negative. But this sum is equal
to $27A^2y^2 - 2$, which belongs to $\mathbb{C}\setminus[0, \infty)$, as $A^2 \in i\mathbb{R}^-$. We have reached a contradiction, and conclude that $r^3(y)$ has an analytic cube root $r(y)$ in a neighborhood of $\mathbb{R}$.

Define $z(y)$ by the relation

$$Z(y) = \sqrt{B}(z(y)/c - \zeta/c).$$

Since $Z(y)$ is analytic on a neighborhood or $\mathbb{R}$, so is $z(y)$. We now verify the claims (i)-(iii) in the statement of the proposition. For (i), we compute

$$Z(0) = \frac{1}{3A\sqrt{B}} \left( \frac{r(0)}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} \right) = \frac{1}{3A\sqrt{B}} \left( \frac{\sqrt{2}e^{\pi i/3}}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}e^{\pi i/3}} - 1 \right) = 0,$$

from which we readily deduce that $z(0) = \zeta$. For (ii), note that the definitions of $A, B, Z$ and $r^3(y)$ imply that

$$y^2 = A Z^3 + Z^2,$$

or equivalently,

$$y^2 = \frac{c^3 \phi_2(\zeta(T), T)}{6} (z/c - \zeta/c)^2 \left( 3\sqrt{T} - t - (z/c - \zeta/c) \right).$$

It is immediate then that if $3\sqrt{T} - t - (z/c - \zeta/c) \in (-\infty, 0]$, then $(z/c - \zeta/c) \in \mathbb{R}$, and hence we must have $\frac{c^3 \phi_2(\zeta(T), T)}{6} \in \mathbb{R}$, which we know to be false. We conclude that $3\sqrt{T} - t - (z/c - \zeta/c) \notin (-\infty, 0]$.

To establish (iii), we note that given (ii), we may deduce from the equation above that either

$$y = \frac{\sqrt{c^3 \phi_2(\zeta(T), T)}}{\sqrt{6}} (z(y)/c - \zeta/c) \sqrt{3\sqrt{T} - t - (z(y)/c - \zeta/c)}$$

or

$$-y = \frac{\sqrt{c^3 \phi_2(\zeta(T), T)}}{\sqrt{6}} (z(y)/c - \zeta/c) \sqrt{3\sqrt{T} - t - (z(y)/c - \zeta/c)}.$$

In the first case the result follows. In the second case chose the analytic function $z(-y)$ in place of $z(y)$. The proof is complete.

We continue our discussion by developing a path of integration akin to $\Gamma$ from the previous section, with a central segment and two tails, and appropriate asymptotic expressions for the integral of $f(z, t)$ on each. We begin with the central segment. The function $z(y)$ as defined in Proposition 26 is unbounded, as are the sets \{ $z(y) | y \in (0, \infty)$ \} and \{ $z(y) | y \in (-\infty, 0)$ \}. Thus, given $n \in \mathbb{N}$ and $c$ as in equation (2.59), there exists $a < 0 < b$ (depending on $n$, c.f. Lemma 28) such that

1. $|z(y)/c - \zeta/c| < 7(\ln n)/n^{1/3}$, $\forall y \in (a, b)$,
2. $|z(a)/c - \zeta/c| = |z(b)/c - \zeta/c| = 7\ln n/n^{1/3}$.

This means that the curve $z(y)$ cuts the boundary of the ball centered at $\zeta(t)$ with radius $c \cdot 7(\ln n)/n^{1/3}$ at $z(a)$ and $z(b)$ and for no other $y \in (a, b)$. While this ball may not, in its entirety, be contained in the first open quadrant, the curve $z(y)$ for $a < y < b$ is contained in the first open quadrant.

**Lemma 27.** Let $n \gg 1$, and let $a < 0 < b$ be such that conditions (1) and (2) in the above discussion hold. Then for any $y \in [a, b]$, $z(y)$ lies in the first quadrant.
Similar computations establish the claim for $\zeta \to \zeta(T)$ as $n \to \infty$. In addition, the definition of $a$ and $b$ imply that $|z(y) - \zeta| \to 0$ as $n \to \infty$. Since $\text{Im}\,\zeta(T) > 0$, we deduce that $\text{Im}\,z(y) > 0$ for all large $n$. Note that

$$\frac{\pi}{2} - \text{Im}\,\phi(\zeta, t) = \text{Im}\,\phi(\zeta(T), T) - \text{Im}\,\phi(\zeta, t).$$

With $\zeta(T) - \zeta = c\sqrt{T-t} + \mathcal{O}(T-t)$, $\phi_z(\zeta(T), T) = 0$, $\phi_z(\zeta(T), T) = 0$, and $\text{Im}\,\phi_t(\zeta(T), T) = 0$, we apply the Taylor series expansion to the bivariate function $\phi(z, t)$ in a neighborhood of $(\zeta(T), T)$ to arrive at

$$\text{Im}\,\phi(\zeta(T), T) - \text{Im}\,\phi(\zeta, t) = \text{Im}\,(\langle \zeta(T) - \zeta \rangle(T-t)\phi_z(\zeta(T), T) + \mathcal{O}((T-t)^2)) \approx (T-t)^{3/2} \approx \frac{1}{n^2}.$$  

If there were a $y \in [a, b]$ such that $\text{Re}\,z(y) = 0$, then by the definition of $\phi(z, t)$, we would have $\text{Im}\,\phi(z(y), t) = \frac{\pi}{2}$, and consequently,

$$\text{Im}\,\phi(z(y), t) - \text{Im}\,\phi(\zeta, t) = \frac{\pi}{2} - \text{Im}\,\phi(\zeta, t) \approx \frac{1}{n^2}.$$  

This, however, contradicts the estimate (obtained from (2.60) and (2.61))

$$\text{Im}\,\phi(z(y), t) - \text{Im}\,\phi(\zeta, t) = \text{Im}\,(y^2 + \phi(z(y), t) - \phi(\zeta, t)) = \mathcal{O}\left(\frac{\ln^4 n}{n^{3/2}}\right).$$  

Since $\text{Re}\,z(0) = \text{Re}\,\zeta(t) > 0$, we conclude that $\text{Re}\,z(y) > 0$ for all $a < y < b$, and the result follows.

\begin{lemma}
Let $a$, $b$ be as in given in Lemma 27. Then $a \approx \ln^{3/2} n/n^{1/2}$ and $b \approx \ln^{3/2} n/n^{1/2}$.
\end{lemma}

\begin{proof}
We compute

$$\left|3\sqrt{T-t} - (z(b)/c - \zeta/c)\right|^2 = 9(T-t) + |z(b)/c - \zeta/c|^2 - 6\sqrt{T-t} \text{Re}(z(b)/c - \zeta/c)$$

$$\geq |z(b)/c - \zeta/c|(|z(b)/c - \zeta/c| - 6\sqrt{T-t})$$

$$\geq \frac{7\ln n}{n^{1/3}} \left(\frac{7\ln n}{n^{1/3}} - \frac{\ln n}{n^{1/3}}\right)$$

$$= \frac{7\ln^2 n}{n^{2/3}},$$

and

$$\left|3\sqrt{T-t} - (z(b)/c - \zeta/c)\right| \leq 3\sqrt{T-t} + |z(b)/c - \zeta/c| \approx \frac{10\ln n}{n^{1/3}}.$$  

Combining these estimates yields

$$b = \sqrt{\frac{|c^3\phi_z(\zeta(T), T)|}{6}} |z(b)/c - \zeta/c| \sqrt{\left|3\sqrt{T-t} - (z(b)/c - \zeta/c)\right|}$$

$$\approx \frac{\ln^{3/2} n}{n^{1/2}}.$$  

Similar computations establish the claim for $a$ as well.
\end{proof}
We now develop an estimate for the central integral (analogous to the estimate (2.56) of section 2.2.2). Let \( a, b \) and \( z(y) \) be as above, and let \( \Gamma(a, b) \) be the curve parameterized by \( z(y), a \leq y \leq b \). Since \( \psi \) is analytic near \( \zeta(T) \), we see that

\[
\int_{\Gamma(a,b)} e^{-n\phi(z,t)} \psi(z) dz = \psi(\zeta(T)) \int_{\Gamma(a,b)} e^{-n\phi(z,t)} dz (1 + o(1))
\]

(2.64)

\[
= \psi(\zeta(T)) \int_{\Gamma(a,b)} e^{-n(\phi(z,t)-\phi(\zeta,t)+\phi(\zeta,t))} dz (1 + o(1)).
\]

We use equation (2.60) to estimate \( \phi(z,t) - \phi(\zeta,t) \), along with the estimate

\[
\exp \left(n\mathcal{O}\left((z-\zeta)^2(T-t) + (z-\zeta)^3\sqrt{T-t} + (z-\zeta)^4\right)\right) = e^{\mathcal{O}(\ln^4 n/n^{1/3})} = 1 + \mathcal{O}(\ln^4 n/n^{1/3})
\]

to conclude that the expression in (2.64) is equal to

\[
\psi(\zeta(T)) e^{-n\phi(\zeta,t)} \int_{\Gamma(a,b)} \exp \left(-n\frac{\phi_3(\zeta(T),T)}{6} (z-\zeta)^2 \left(3\sqrt{T-t} - (z-\zeta)\right)\right) dz(1 + o(1))
\]

\[
= \psi(\zeta(T)) e^{-n\phi(\zeta,t)} \int_a^b e^{-ny^2} y' dy (1 + o(1)).
\]

In order to find an expression for \( y' \), we square both sides of equation (2.61) and compute the derivatives with respective to \( y \) to obtain

\[
2y = \frac{c^2 \phi_3(\zeta(T),T)}{2} (z/c - \zeta/c) \left(2\sqrt{T-t} - (z/c - \zeta/c)\right) y',
\]

which we solve for \( y' \):

\[
z'(y) = \frac{4y}{c^2 \phi_3(\zeta(T),T)(z/c - \zeta/c) \left(2\sqrt{T-t} - (z/c - \zeta/c)\right)}.
\]

Since

\[
(z/c - \zeta/c) \left(2\sqrt{T-t} - (z/c - \zeta/c)\right) = (z/c - \zeta/c) \left(3\sqrt{T-t} - (z/c - \zeta/c)\right) - \sqrt{T-t}(z/c - \zeta/c)
\]

\[
= \frac{6y^2}{c^3 \phi_3(\zeta(T),T)(z/c - \zeta/c)} - \sqrt{T-t}(z/c - \zeta/c),
\]

we conclude that

(2.65)

\[
z'(y) = \sqrt{\frac{4c}{\sqrt{c^3 \phi_3(\zeta(T),T)}}} \left(\frac{6y}{\sqrt{c^3 \phi_3(\zeta(T),T)(z/c - \zeta/c)} - \frac{\sqrt{T-t}(z/c - \zeta/c)}{y}}\right)^{-1}
\]

\[
= \frac{4c}{\sqrt{6 \sqrt{c^3 \phi_3(\zeta(T),T)}}} \left(\frac{\sqrt{3\sqrt{T-t} - (z/c - \zeta/c)} - \frac{\sqrt{T-t}}{\sqrt{3\sqrt{T-t} - (z/c - \zeta/c)}}}{\frac{\sqrt{T-t}}{\sqrt{3\sqrt{T-t} - (z/c - \zeta/c)}}}^{-1}ight),
\]
and consequently
\[
\int_{\Gamma(a,b)} e^{-n\phi(z,t)}\psi(z)dz
\]
(2.66)
\[
= \frac{4e\psi(\zeta(T))e^{-n\phi(\zeta,t)}}{\sqrt{6}c^3\psi(\zeta(T),T)} \int_a^b e^{-ny^2} \left( \frac{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} - \sqrt{T-t}}{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)}} \right)^{-1} dy(1+o(1)).
\]
In an effort to develop an asymptotic lower bound for \( \int_{\Gamma(a,b)} e^{-n\phi(z,t)}\psi(z)dz \), we start with the following result.

**Lemma 29.** Let \( a, b \) be as in Lemma 27. Then for any \( y \in (a,b) \),
\[
\text{Re} \left( \frac{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} - \sqrt{T-t}}{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)}} \right) \geq 0.
\]

**Remark 30.** The conclusion of Lemma 29 is equivalent to the non-negativity of the real part of the integrand of the right hand side of (2.66), but is easier to establish.

**Proof.** Since \( \sqrt{T-t} \in \mathbb{R} \),
\[
\text{Arg} \left( \frac{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} - \sqrt{T-t}}{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)}} \right) = -\text{Arg} \left( \frac{\sqrt{T-t}}{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)}} \right),
\]
and this common argument lies between \(-\pi/2\) and \(\pi/2\). Consequently, the sign of
\[
\text{Re} \left( \frac{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} - \sqrt{T-t}}{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)}} \right)
\]
is the same as the sign of
\[
\left| \frac{1}{\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)}} \right| E(y),
\]
where
\[
E(y) := \left| 3\sqrt{T-t}-(z/c-\zeta/c) \right| - \sqrt{T-t}.
\]
We now argue that \( E(y) \geq 0 \) for all \( y \in [a,b] \). We first check the endpoints. If \( y = a \) or \( y = b \), then \( |z(y)/c-\zeta/c| = \frac{7\ln n}{n^{1/3}} \), and hence
\[
\left| 3\sqrt{T-t}-(z(y)/c-\zeta/c) \right| - \sqrt{T-t} \geq |z(y)/c-\zeta/c| - 4\sqrt{T-t} \geq \frac{3\ln n}{n^{1/3}} > 0.
\]
Next we demonstrate that the condition \( E(y) = 0 \) holds at no more than one point in \([a,b]\). To this end, assume that \( E(y) = 0 \). Since
\[
\text{Re} \left( 3\sqrt{T-t}-(z/c-\zeta/c) \right) = 3\sqrt{T-t} - \text{Re}(z/c-\zeta/c),
\]
the assumption
\[
|3\sqrt{T - t} - (z/c - \zeta/c)| = \sqrt{T - t}
\]
implies that \(\text{Re}(z/c - \zeta/c) > 0\), or equivalently \(-\pi/2 < \text{Arg}(z/c - \zeta/c) < \pi/2\). In order to obtain more information on the quantity \(\text{Arg}(z/c - \zeta/c)\), consider the identity
\[
(2.68) \quad -(z/c - \zeta/c)e^{-i\text{Arg}(z/c - \zeta/c)} = (3\sqrt{T - t} - (z/c - \zeta/c))e^{-i\text{Arg}(z/c - \zeta/c)} - 3\sqrt{T - t}e^{-i\text{Arg}(z/c - \zeta/c)}.
\]
Taking the imaginary parts of both sides of (2.68) and using the assumption \(E(y) = 0\) we conclude that
\[
(2.69) \quad 0 = \sin \left( \text{Arg}(3\sqrt{T - t} - (z/c - \zeta/c)) - \text{Arg}(z/c - \zeta/c) \right) + 3\sin(\text{Arg}(z/c - \zeta/c)).
\]
On the other hand, taking the arguments of both sides of (2.61) yields
\[
(2.70) \quad \frac{\pi}{4} + \text{Arg}(z/c - \zeta/c) + \frac{1}{2} \text{Arg}(3\sqrt{T - t} - (z/c - \zeta/c)) = \begin{cases} 0 & \text{if } y > 0 \\ \pm \pi & \text{if } y < 0 \end{cases},
\]
which implies that
\[
\text{Arg}(3\sqrt{T - t} - (z/c - \zeta/c)) = -\frac{\pi}{2} - 2\text{Arg}(z/c - \zeta/c) \pmod{2\pi}.
\]
Substituting into equation (2.69) and rearranging gives the equation
\[
0 = \cos(3\text{Arg}(z/c - \zeta/c)) + 3\sin(\text{Arg}(z/c - \zeta/c)),
\]
which has exactly one solution \(\text{Arg}(z/c - \zeta/c) = -0.247872\ldots\) on \((-\pi/2, \pi/2)\). Now we take the real parts of both sides of (2.68) and employ the assumption \(E(y) = 0\) to solve for \(|z/c - \zeta/c|\):
\[
-\frac{|z/c - \zeta/c|}{\sqrt{T - t}} = \cos \left( \text{Arg}(3\sqrt{T - t} - (z/c - \zeta/c)) - \text{Arg}(z/c - \zeta/c) \right) - 3\cos(\text{Arg}(z/c - \zeta/c))
\]
\[
\quad = -\sin(3\text{Arg}(z/c - \zeta/c)) - 3\cos(\text{Arg}(z/c - \zeta/c))
\]
from which we conclude that \(E(y) = 0\) on \((a, b)\) can only occur at \(y\) corresponding to \(\text{Arg}(z/c - \zeta/c) = -0.247872\ldots\). The result now follows from the continuity of \(E(y)\). \(\square\)

Remark 31. Using (2.70) and the fact that a possible zero of \(E(y)\) on \([a, b]\) satisfies \(\text{Arg}(z/c - \zeta/c) = -0.247872\ldots\), we conclude that \(E(y)\) has no zero in \((a, 0)\).

The following lemma gives the range of \(\text{Arg}(z/c - \zeta/c)\) for \(y \in (a, 0)\).

**Lemma 32.** For any \(y \in (a, 0)\), \(\pi > \text{Arg}(z/c - \zeta/c) \geq \pi/4\) or \(-\pi < \text{Arg}(z/c - \zeta/c) \leq -3\pi/4\).

**Proof.** For ease of notation, set
\[
\left[ \begin{array}{c} \square \end{array} \right] = \sqrt{3\sqrt{T - t} - (z/c - \zeta/c)} - \frac{\sqrt{T - t}}{\sqrt{3\sqrt{T - t} - (z/c - \zeta/c)}}.
\]
Equation (2.65) implies that
\[
z'(y) \frac{\sqrt{c^4\phi_3(\zeta(T), T)}}{c} = \frac{4}{\sqrt{6}} \left[ \begin{array}{c} \square \end{array} \right]^{-1}.
\]
By Lemma 29 \(\text{Re} \left[ \begin{array}{c} \square \end{array} \right] \geq 0\), and hence
\[
\text{Re} \left( \frac{4}{\sqrt{6}} \left[ \begin{array}{c} \square \end{array} \right]^{-1} \right) = \text{Re} \left( \frac{\sqrt{c^4\phi_3(\zeta(T), T)}}{c} z'(y) \right) \geq 0.
\]
Combining this inequality with
\[ \Re \left( \sqrt{c^3 \phi_2(z(T), T)}(z(0)/c - \zeta/c) \right) = 0 \]
gives
\[ \Re \left( \sqrt{c^3 \phi_2(z(T), T)}(z(y)/c - \zeta/c) \right) \leq 0, \quad \forall y \in (a, 0). \]
The lemma now follows from the fact that
\[ \Arg \left( \sqrt{c^3 \phi_2(z(T), T)} \right) = \pi/4. \]

**Lemma 33.** Let \( z(y) \) be the function defined on \([a, b]\) by Proposition 27, where \( a, b \) are as in the discussion preceding Lemma 27. Then
\[ \left| \int_a^b \left( \frac{\sqrt{T - t} - (z/c - \zeta/c)}{3\sqrt{T - t} - (z/c - \zeta/c)} \right)^{-1} dy \right| \geq \frac{1}{n^{1/3} \sqrt{\ln n}}, \quad \text{as} \quad n \to \infty. \]

**Proof.** By Lemma 29, we have
\[ \left| \int_a^b \left( \frac{\sqrt{T - t} - (z/c - \zeta/c)}{3\sqrt{T - t} - (z/c - \zeta/c)} \right)^{-1} dy \right| \geq \int_a^b \Re \left( \frac{\sqrt{T - t} - (z/c - \zeta/c)}{3\sqrt{T - t} - (z/c - \zeta/c)} \right)^{-1} dy. \]
The integrand in the last integral is the quotient of
\[ \left( \frac{\sqrt{T - t} - (z/c - \zeta/c)}{|\sqrt{T - t} - (z/c - \zeta/c)|} \right) \cos \left( \Arg \sqrt{T - t} - (z/c - \zeta/c) \right) \]
and
\[ \left( \frac{\sqrt{T - t} - (z/c - \zeta/c)}{|\sqrt{T - t} - (z/c - \zeta/c)|} \right)^2. \]
Rearranging this quotient gives
\[ (2.71) \]
\[ \frac{\left( \sqrt{3\sqrt{T - t} - (z/c - \zeta/c)} - \sqrt{T - t} \right) \left( \sqrt{3\sqrt{T - t} - (z/c - \zeta/c)} \right) \cos \left( \Arg \sqrt{3\sqrt{T - t} - (z/c - \zeta/c)} \right)}{|2\sqrt{T - t} - (z/c - \zeta/c)|^2} \]
To bound the denominator, we compute
\[ |2\sqrt{T - t} - (z/c - \zeta/c)|^2 \leq \left( \frac{2 \ln n}{n^{1/3}} + \frac{7 \ln n}{n^{1/3}} \right)^2 = \frac{81 \ln^2 n}{n^{2/3}}. \]
Turning our attention to the numerator, we recall Lemma 32, which – since \( y \in (a, 0) \) – implies that
\[ -\frac{3\pi}{4} \leq \Arg \sqrt{3\sqrt{T - t} - (z/c - \zeta/c)} \leq \frac{\pi}{4}, \]
and consequently
\begin{equation}
\cos \left( \text{Arg} \sqrt{3T-t-(z/c-\zeta/c)} \right) \geq \cos \frac{3\pi}{8}.
\end{equation}

We now find a lower bound for the remaining part of the numerator. To this end, note that
\[
\left| 3\sqrt{T-t}-(z/c-\zeta/c) \right|^2 = 9(T-t) + |z/c-\zeta/c|^2 - 6\sqrt{T-t} \text{Re}(z/c-\zeta/c)
\]
\[
= 9(T-t) + |z/c-\zeta/c|^2 - 6\sqrt{T-t}|z/c-\zeta/c| \cos \text{Arg}(z/c-\zeta/c).
\]
By Lemma \[32\] \[32\]
\[
\cos (z/c-\zeta/c) \leq \sqrt{2}/2.
\]
Therefore
\[
9(T-t) + |z/c-\zeta/c|^2 - 6\sqrt{T-t}|z/c-\zeta/c| \cos \text{Arg}(z/c-\zeta/c)
\]
\begin{equation}
\geq 9(T-t) - 3\sqrt{2}\sqrt{T-t}|z/c-\zeta/c| + |z/c-\zeta/c|^2 \geq \frac{9}{2}(T-t).
\end{equation}

On the other hand,
\[
\left| \sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} \right|^3 - \sqrt{T-t} \left| \sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} \right|
\]
is an increasing function in terms of \[\sqrt{3\sqrt{T-t}-(z/c-\zeta/c)}\] when this quantity is at least \[\sqrt{T-t}/3\]. Since \[\sqrt{9/2(T-t)} > \sqrt{T-t}/3\], we conclude that
\[
\left| \sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} \right|^3 - \sqrt{T-t} \left| \sqrt{3\sqrt{T-t}-(z/c-\zeta/c)} \right|
\]
\[
\geq \frac{9}{2}(T-t)^{1/4} \left( \frac{3}{\sqrt{2}} \sqrt{T-t} - \sqrt{T-t} \right) \approx (T-t)^{3/4} \geq 1/\sqrt{n}.
\]

Putting all this together shows that the expression in \[2.71\] is at least a constant multiple of \[\frac{n^{1/6}}{\ln^2 n}\]. The result now follows from Lemma \[28\] as
\[
\int_0^a \frac{n^{1/6}}{\ln^2 n} dy = a \frac{n^{1/6}}{\ln^2 n} \approx \frac{1}{m^{1/3} \sqrt{\ln m}}.
\]

In order to extend \[\Gamma_{(a,b)}\] to the point at infinity we use arguments similar to those in Section 2.3 using \(a\) and \(b\) instead of \(\pm \epsilon\) and extend the curve \(z(y)\), \(a \leq y \leq b\), by adding two tails \(\Gamma_a\) and \(\Gamma_b\) starting from \(z(a)\) and \(z(b)\) to \(\infty\) in the lower and upper half planes respectively, such that \(\text{Re } \phi(z) \geq \text{Re } \phi(z(a)) + a^2\), \(\forall z \in \Gamma_a\) and \(\text{Re } \phi(z) \geq \text{Re } \phi(z(b)) + b^2\), \(\forall z \in \Gamma_b\) with equality only at \(z = z(a)\) or \(z = z(b)\) respectively. Repeating the arguments which provided the asymptotic bound for \[2.57\] we obtain
\[
\int_{\Gamma_a} e^{-n\phi(z,t)} \psi(z) dz = O \left( e^{-n\phi(\zeta,t)-na^2/2} \right)
\]
and
\[
\int_{\Gamma_b} e^{-n\phi(z,t)} \psi(z) dz = O \left( e^{-n\phi(\zeta,t)-nb^2/2} \right),
\]
Finally, from equation (2.66) and Lemmas 28 and 33 we obtain the estimate
\[
\int_{F_2} f(z,t)dz = K_c(n) \int_{a}^{b} e^{-ny^2} \left( \frac{\sqrt{3T - t - (z/c - \zeta/c)} - \sqrt{T - t}}{\sqrt{3T - t - (z/c - \zeta/c)}} \right)^{-1} dy(1+o(1)),
\]
where
\[
K_c(n) := \frac{4\psi(\zeta(T))e^{-n\phi(\zeta,t)}}{\sqrt{6}\sqrt{d^3\phi_3(\zeta(T),T)}}.
\]

2.2.5. The asymptotics when \(T = T_2\) and \(\frac{1}{n} \ll |T_1 - t| < \ln^2 n/n^{2/3}\). We focus on the case \(T_1 < t\). Recall from Lemma 9 that
\[
\zeta(t) - \zeta(T_1) = d\sqrt{T_1 - t} + \mathcal{O}(T_1 - t),
\]
where \(d^3 \cdot \phi_3(\zeta(T_1),T_1) \in \mathbb{R}^+\). Following arguments analogous to those in the previous section, replacing \(c\) with \(d\), and \(T\) with \(T_1\), we conclude that
\[
\int_{F_2} f(z,t)dz = K_d(n) \int_{a}^{b} e^{-ny^2} \left( \frac{\sqrt{3T_1 - t - (z/d - \zeta/d)} - \sqrt{T_1 - t}}{\sqrt{3T_1 - t - (z/d - \zeta/d)}} \right)^{-1} dy(1+o(1)),
\]
where
\[
K_d(n) = \frac{4d\psi(\zeta(T_1))e^{-n\phi(\zeta,t)}}{\sqrt{6}\sqrt{d^3\phi_3(\zeta(T_1),T_1)}}.
\]

For the sake of brevity, instead of reproducing the argument in its entirety, we content ourselves with highlighting the differences. The curve \(z(y)\) (c.f. Proposition 26) is now only piecewise smooth, and is defined by replacing the function in (2.63) by
\[
r(y)^3 = \begin{cases} 27A^2y^2 - 2 + \sqrt{27}y \sqrt{4A^2 - 27A^4y^2} & \text{if } 27A^4y^2 - 4A^2 < 0 \\ 27A^2y^2 - 2 + \sqrt{27}y \sqrt{27A^4y^2 - 4A^2} & \text{if } 27A^4y^2 - 4A^2 \geq 0 \end{cases}.
\]

When \(27A^4y^2 - 4A^2 < 0\) we use the cut \([0, \infty)\) to define \(r(y)\) and when \(27A^4y^2 - 4A^2 \geq 0\), we define \(r(y)\) by
\[
r(y) = \begin{cases} \sqrt{27A^2y^2 - 2 + \sqrt{27}y \sqrt{27A^4y^2 - 4A^2}} & \text{if } y > 0 \\ e^{-2\pi i/3} \sqrt{27A^2y^2 - 2 + \sqrt{27}y \sqrt{27A^4y^2 - 4A^2}} & \text{if } y < 0 \end{cases}.
\]

Lemma 27 holds trivially as \(\zeta(T_1)\) lies in the open first quadrant. In the proof of Lemma 29 equation (2.70) becomes
\[
\text{Arg}(z/d - \zeta/d) + \frac{1}{2} \text{Arg}(3\sqrt{T_1 - t} - (z/d - \zeta/d)) = \begin{cases} 0 & \text{if } y > 0 \\ \pm \pi & \text{if } y < 0 \end{cases},
\]
which implies that
\[
\text{Arg}(3\sqrt{T_1 - t} - (z/d - \zeta/d)) = -2 \text{Arg}(z/d - \zeta/d) \pmod{2\pi}.
\]
The analogue of equation (2.69) reads

$$0 = \sin(3 \text{Arg}(z/d - \zeta/d)) + 3 \sin(\text{Arg}(z/d - \zeta/d)),$$

which has exactly one solution, namely $\text{Arg}(z/d - \zeta/d) = 0$, on $(-\pi/2, \pi/2)$. Using this solution we also find $|z/d - \zeta/d|$. Since $d^3 \cdot \phi_{\zeta^3}(\zeta(T_1), T_1) \in \mathbb{R}^+$, the result corresponding to Lemma 32 is that for $y \in (a, 0)$

$$\text{Re}(z/d - \zeta/d) \leq 0,$$

while the inequality corresponding to (2.72) is

$$\cos \left( \text{Arg} \sqrt{3 \sqrt{T_1 - t} - (z/d - \zeta/d)} \right) \geq \frac{\sqrt{2}}{2}.$$

Using that $\cos \text{Arg}(z/d - \zeta/d) \leq 0$, inequality (2.73) is replaced by

$$9(T_1 - t) + |z/d - \zeta/d|^2 \geq 9(T_1 - t),$$

leading to the asymptotic lower bound

$$\left| \sqrt{3 \sqrt{T_1 - t} - (z/d - \zeta/d)} - \sqrt{T_1 - t} \right| \geq (9(T_1 - t))^{1/4} \left( 3 \sqrt{T_1 - t} - \sqrt{T_1 - t} \right) = (T_1 - t)^{3/4} \geq 1/n^{3/4}.$$

With these differences, the estimate in the statement of Lemma 33 becomes

$$\left| \int_a^b \left( \sqrt{3 \sqrt{T_1 - t} - (z/d - \zeta/d)} - \frac{\sqrt{T_1 - t}}{\sqrt{3 \sqrt{T_1 - t} - (z/d - \zeta/d)}} \right)^{-1} dy \right| \geq \frac{1}{n^{7/12} \ln n}.$$

For the case $t > T_1$, we note that as $t \to T_1$

$$\sqrt{1 - 2t^2 - 2\sqrt{t^2 - T_1^2}} = \sqrt{(1 - 2T_1^2) \left( 1 - 2T_1 \frac{\sqrt{t^2 - T_1^2}}{1 - 2T_1^2} + O(t - T_1) \right)} = \sqrt{1 - 2T_1^2 \left( 1 - T_1 \frac{\sqrt{t^2 - T_1^2}}{1 - 2T_1^2} + O(t - T_1) \right)},$$

and consequently equation (2.32) yields

$$\zeta(t) - \zeta(T_1) = \frac{z_1 + z_2}{2} \left( i \sqrt{t^2 - T_1^2} - \frac{T_1 \sqrt{t^2 - T_1^2}}{\sqrt{1 - 2T_1^2}} + O(t - T_1) \right)$$

$$= \hat{d} \cdot \sqrt{t - T_1} + O(t - T_1)$$

where

$$\hat{d} = \frac{(z_1 + z_2) \sqrt{T_1}}{\sqrt{2}} \left( i - \frac{T_1}{\sqrt{1 - 2T_1^2}} \right).$$

In order to be able to extend the curve $z(y)$, $a < y < b$ with two tails going to infinity we replace Lemma 23 with the following result.

**Lemma 34.** Let $S$ be defines as in Lemma 23, and let $a, b$ be defined as in the discussion preceding Lemma 27. If $\frac{1}{n} \ll T_1 - t \ll \ln^2 n/n^{2/3}$, then $z(a)$ and $z(b)$ lie outside the region $S$. 


Proof. We recall that
\[ y = \frac{\sqrt{d^3 \phi_3(\zeta(T_1), T_1)}}{\sqrt{6}} (z(y)/d - \zeta/d) \sqrt{3\sqrt{T_1} - t} - (z(y)/d - \zeta/d) \]
and
\[ A = \frac{\sqrt{6}}{\sqrt{d^3 \phi_3(\zeta(T_1), T_1)\sqrt{3\sqrt{T_1} - t}}} \in \mathbb{R}^+, \]
\[ B = \frac{1}{2} \frac{d^3 \phi_3(\zeta(T_1), T_1)\sqrt{3\sqrt{T_1} - t}}{\sqrt{3\sqrt{T_1} - t}} \in \mathbb{R}^+. \]
Using the conditions
\[ |z(a)/d - \zeta/d| = |z(b)/d - \zeta/d| = 7 \ln n/n^{1/3}, \]
we conclude that when \( y = a \),
\[ A^2 a^2 = |A^2 a^2| \geq \frac{7^2 \cdot 4}{3}, \]
which in turn implies that \( 27A^4a^2 - 4A^2 > 0 \). Combining this with the similar inequality \( 27A^4b^2 - 4A^2 > 0 \), we deduce that
\[ r(y) = \begin{cases} \sqrt[3]{27A^2b^2} - 2 + \sqrt[3]{27b^2A^2b^2 - 4A^2} & \text{if } y = b, \\ e^{-2ni/3} \sqrt[3]{27A^2a^2} - 2 + \sqrt[3]{27A^2a^2 - 4A^2} & \text{if } y = a. \end{cases} \]
Lemma 9 provides for any \( |\tau - T_1| = o(1) \) the estimate
\[ \zeta_2(\tau) - \zeta_1(t) \sim \begin{cases} -d\sqrt{T_1} - \tau - d\sqrt{T_1} - t & \text{if } \tau < T_1, \\ id\sqrt{\tau - T_1} - d\sqrt{T_1} - t & \text{if } \tau > T_1, \end{cases} \]
from which we deduce that if \( z \in \mathcal{S} \) and \( z - \zeta = o(1) \), then \( (z - \zeta)/d \) lies in the second quadrant. Consequently, the expression
\[(2.75) \quad r(y) = \begin{cases} \sqrt[3]{27A^2b^2} - 2 + \sqrt[3]{27b^2A^2b^2 - 4A^2} & \text{if } y = b, \\ e^{-2ni/3} \sqrt[3]{27A^2a^2} - 2 + \sqrt[3]{27A^2a^2 - 4A^2} & \text{if } y = a. \end{cases} \]
implies that \( z(b) \notin \mathcal{S} \). Using an analogous argument we also conclude \( (z(a) - \zeta)/d \) lies in the third quadrant (see discussion after equation (2.63) for why \( \sqrt[3]{2}/r(a) \) is greater than 1 in modulus), and hence \( z(a) \notin \mathcal{S} \). The proof is complete. \( \square \)

Repeating the arguments provided for the case \( t < T_1 \) \textit{mutatis mutandis}, we conclude that when \( t > T_1 \),
\[ \int_{T_2} f(z,t)dz = K_d(n) \int_a^b e^{-ny^2} \left( \frac{\sqrt{3\sqrt{t - T_1} - (z/d - \zeta/d)}}{\sqrt[3]{3\sqrt{t - T_1} - (z/d - \zeta/d)}} \right)^{-1} dy(1 + o(1)). \]
We hasten to note that Lemma 34 is not necessary in this case by the first paragraph in the proof of Lemma 23. This completes the asymptotic analysis of the key integral away from the point \( t = 0 \). We deal with this range in the next section.
2.2.6. The asymptotics when $t \ll \ln^4 n/n$. With a slight abuse of notation we let $\Gamma_2$ be the curve in Figure 2.4 where each circle around $z_1$ and $z_2$ has small radius $\xi$, and each horizontal line segment has distance $\delta$ from the $x$-axis.

On each arc $\gamma$ of the circles with radius $\xi$ centered at $z_1$ and $z_2$ we employ the basic estimates

$$\left| e^{-n\phi(z,t)} \right| \leq |z|^{-n} e^{4\pi t} \quad \text{and} \quad |\psi(z)| = \frac{|Q(z)Q(-z)|^{1/2}}{|z|}$$

to conclude that

$$\int_{\gamma} f(z,t) dz \to 0$$

as $\xi \to 0$. By letting $\delta \to 0$ and $\xi \to 0$ we may then rewrite the integral

$$\int_{\Gamma_2} \frac{Q(z)^{1/2+int}Q(-z)^{1/2-int}}{z^{n+1}} dz$$

as

$$\left( e^{-i\pi(1/2+int)} - e^{i\pi(1/2+int)} \right) \int_{z_1}^{z_2} (z - z_1)^{1/2+int}(z_2 - z)^{1/2+int}(z_1 + z)^{1/2-int}(z_2 + z)^{1/2-int} \frac{dz}{z^{n+1}}$$

$$+ \left( e^{-2i\pi(1/2+int)} - e^{2i\pi(1/2+int)} \right) \int_{z_2}^{\infty} (z - z_1)^{1/2+int}(z - z_2)^{1/2+int}(z_1 + z)^{1/2-int}(z_2 + z)^{1/2-int} \frac{dz}{z^{n+1}}.$$  

We make the substitutions $w = z/z_1$ and $w = z/z_2$ in the first and the second integral respectively to arrive at the expression

$$-i \left( e^{n\pi t} + e^{-n\pi t} \right) \int_{z_1^n}^{z_2/z_1^n} ((wz_1 - z_1)(z_2 - wz_1))^{1/2+int}((z_1 + wz_1)(z_2 + wz_1))^{1/2-int} \frac{dw}{w^{n+1}}$$

$$+ \left( e^{2n\pi t} + e^{-2n\pi t} \right) \int_{z_2^n}^{\infty} ((wz_2 - z_1)(wz_2 - z_2))^{1/2+int}((z_1 + wz_2)(z_2 + wz_2))^{1/2-int} \frac{dw}{w^{n+1}}.$$  

We claim that the first summand in (2.76) is asymptotically equivalent to

$$-i \left( e^{n\pi t} + e^{-n\pi t} \right) z_1^{1/2+int}(z_2 - z_1)^{1/2+int}(z_2 + z_1)^{1/2-int} \Gamma(3/2 + int)e^{-(3/2+int)\ln n}.$$  

We make the substitutions $w = z/z_1$ and $w = z/z_2$ in the first and the second integral respectively to arrive at the expression

$$-i \left( e^{n\pi t} + e^{-n\pi t} \right) \int_{z_1^n}^{z_2/z_1^n} ((wz_1 - z_1)(z_2 - wz_1))^{1/2+int}((z_1 + wz_1)(z_2 + wz_1))^{1/2-int} \frac{dw}{w^{n+1}}$$

$$+ \left( e^{2n\pi t} + e^{-2n\pi t} \right) \int_{z_2^n}^{\infty} ((wz_2 - z_1)(wz_2 - z_2))^{1/2+int}((z_1 + wz_2)(z_2 + wz_2))^{1/2-int} \frac{dw}{w^{n+1}}.$$  

We claim that the first summand in (2.76) is asymptotically equivalent to
In order to demonstrate the claim, we first apply the substitution $e^z = w$ to transform the first summand in (2.76) into
\[
-i \left( \frac{e^{\pi t} + e^{-\pi t}}{z_1^n} \right) \int_0^{\ln(z_2/z_1)} \left( z \left( \frac{z_1 e^z - z_1}{z} \right) (z_2 - e^z z_1) \right)^{1/2+int} ((z_1 + e^z z_1)(z_2 + e^z z_1))^{1/2-int} \frac{dz}{z^n e^{nz}}.
\]
Then we split the range of integration into the intervals $(0, \ln^5 n/n)$ and $(\ln^5 n/n, \ln(z_2/z_1))$. The contribution corresponding to the second interval is
\[
-i \left( \frac{e^{\pi t} + e^{-\pi t}}{z_1^n} \right) \mathcal{O} \left( e^{-\ln^5 n} \right),
\]
since
\[
\left| \left( z \left( \frac{z_1 e^z - z_1}{z} \right) (z_2 - e^z z_1) \right)^{1/2+int} ((z_1 + e^z z_1)(z_2 + e^z z_1))^{1/2-int} \right| = \mathcal{O}(1).
\]
Next we find the contribution corresponding to the first interval:
\[
-i \left( \frac{e^{\pi t} + e^{-\pi t}}{z_1^n} \right) \int_0^{\ln^5 n/n} z^{1/2+int} \left( z_2 - z_1 \right)^{1/2+int} (2z_1)^{1/2-int} (z_2 + z_1)^{1/2-int} \left( \int_0^{\ln^5 n/n} z^{1/2+int} e^{-nz} \frac{dz}{z^n} \right) (1 + \mathcal{O}(\ln^5 n/n)).
\]
We write the integral in the above expression as
\[
\int_0^{\ln^5 n/n} z^{1/2+int} e^{-nz} \frac{dz}{z^n} = \frac{1}{n^{3/2+int}} \int_0^{\ln^5 n} z^{1/2+int} e^{-z} \frac{dz}{z^n}
\]
\[
= \frac{1}{n^{3/2+int}} (\Gamma(3/2 + int) - \int_{\ln^5 n}^{\infty} z^{1/2+int} e^{-z} \frac{dz}{z^n})
\]
\[
= \frac{1}{n^{3/2+int}} (\Gamma(3/2 + int) + \mathcal{O}(e^{-\ln^5 n/2})),
\]
where $\Gamma(\cdot)$ denotes the (complete) gamma function. Using Sterling’s approximation to the gamma function
\[
\Gamma(z) = \exp \left( (z + 1/2) \log z - z + \frac{1}{2} \ln 2\pi + \mathcal{O}(z^{-1}) \right) \quad (|z| \gg 1)
\]
and the condition $t \ll \ln^4 n/n$ we find that
\[
|\Gamma(3/2 + int)| = \exp \left( 2 \ln |3/2 + int| - nt \text{ Arg}(3/2 + int) - \frac{3}{2} - \frac{1}{2} \ln 2\pi + \mathcal{O}(1/nt) \right)
\]
\[
\gg \exp(-\frac{\pi \ln^4 n}{4}).
\]
Consequently,
\[
\Gamma(3/2 + int) + \mathcal{O}(e^{-\ln^5 n/2}) = \Gamma(3/2 + int)(1 + o(1)),
\]
and hence
\[
\int_0^{\ln^5 n/n} z^{1/2+int} e^{-nz} \frac{dz}{z^n} = \frac{1}{n^{3/2+int}} \Gamma(3/2 + int)(1 + o(1)) = \Gamma(3/2 + int)e^{-(3/2+int)\ln n}(1 + o(1)).
\]
Assembling the estimates results in the claimed equivalence. Similar arguments show that the second summand in (2.76) is asymptotic to
\[
\left( -\frac{e^{2\pi t} + e^{-2\pi t}}{z_2^n} \right) \int_0^{\ln^5 n/n} z^{1/2-int} \left( z_2 - z_1 \right)^{1/2+int} (2z_1)^{1/2-int} \Gamma(3/2 + int)e^{-(3/2+int)\ln n}.
\]
Since \( t \ll \ln^4 n/n \) and \( z_2 > z_1 \),
\[
\frac{(-e^{2\pi n t} + e^{-2\pi n t})}{z_2^n} = \left( \frac{e^{\pi n t}(1 + e^{-2\pi n t})}{z_1 \left( \frac{z_2}{z_1} \right)^n} \right) \frac{(-e^{\pi n t} + e^{-\pi n t})}{z_1^{n-1}} = \left( \frac{-e^{\pi n t} + e^{-\pi n t}}{z_1^{n-1}} \right) \cdot o(1),
\]
and we conclude that the entire expression (2.76) is asymptotic to (2.77). We thus obtain the estimate
\[
\int_{\Gamma_2} f(z, t) dz \sim -i(\frac{e^{\pi n t} + e^{-\pi n t}}{z_1^{n-1}})2^{1/2-int}(z_2 - z_1)^{1/2+int}(z_2 + z_1)^{1/2-int} \Gamma(3/2 + int)e^{-(3/2+int) \ln n},
\]
completing the section. We are now in a position to account for the zeros of the polynomials in the Sheffer sequence \((H_n)_{n=1}\).

2.3. The zeros of the polynomials \(H_n\). Recall (c.f. the discussion preceding equation (2.26)) that given \( n \in \mathbb{N} \), the polynomial of interest, namely \( \pi H_n(1/2 + int) \), is the imaginary part or \( -i \) times the real part of
\[
\int_{\Gamma_2} f(z, t) dz
\]
when \( n \) is even or odd respectively. For large \( n \), we now find a lower bound for the number of real zeros of \( H_n(1/2 + int) \) on \( t \in (0, T) \) and compare this number to the degree of \( H_n \).

Recall that for each \( t \in (0, T) \), \( \zeta \) is a solution of (2.30). Using this relation we express \( t \) as a function of \( \zeta \):
\[
(2.78) \quad i t = \frac{(\zeta^2 - z_1^2)(\zeta^2 - z_2^2)}{2\zeta(z_1 + z_2)(\zeta^2 - z_1 z_2)},
\]
and rewrite \( \int_{\Gamma_2} f(z, t) dz \) as a function of \( \zeta \):
\[
h(\zeta) := \int_{\Gamma_2} f \left( z, \frac{(\zeta^2 - z_1^2)(\zeta^2 - z_2^2)}{2i\zeta(z_1 + z_2)(\zeta^2 - z_1 z_2)} \right) \ dz.
\]
Since zeros of \( H_n(1/2 + int) \) correspond to the intersections of \( h(\zeta) \) with the imaginary or the real axis depending on the parity of \( n \), we focus on the change of argument of \( h(\zeta) \), provided that it does not pass through the origin. Our next results describes ranges of \( t \) on which this condition on \( h(\zeta) \) is met. In order to ease the exposition, we introduce the following notation.

**Notation 35.** Given two functions \( f_1, f_2 : \mathbb{N} \to \mathbb{R} \), we will denote the set of \( t \in (0,T) \) satisfying \( f_1(n) < t < f_2(n) \) as \( n \to \infty \) by \( \langle f_1(n), f_2(n) \rangle \). Similarly, we will denote the set of \( t \in (0,T) \) satisfying \( f_1(n) < t < f_2(n) \) as \( n \to \infty \) by \( (f_1(n), f_2(n)) \).

**Lemma 36.** Let \( \phi, \psi \) and \( T \) be defined as in equations (2.28), (2.29) and (2.31). Let
\[
I_1 := \langle \ln^4 n/n, \ T - \ln^2 n/n^{2/3} \rangle
\]
\[
I_2 := \langle \ln^4 n/n, \ T_1 - \ln^2 n/n^{2/3} \rangle
\]
\[
I_3 := \langle T_1 + \ln^2 n/n^{2/3}, \ T_2 - \ln^2 n/n^{2/3} \rangle,
\]
and set
\[
(2.79) \quad g(\zeta) := \frac{2\pi \psi^2(\zeta)e^{-2n\phi(\zeta,t)}}{n\phi_{xx}(\zeta,t)}.
\]
(i) If \( T = T_1 \), then \( h(\zeta(t)) \neq 0 \) for \( t \in I_1 \).
The relevant estimate for $g$ is 

$$g(\zeta) = \frac{2\pi \psi^2(\zeta)e^{-2n\phi(\zeta)}}{n\phi_z(\zeta)} \sim \frac{2\pi \psi^2(\zeta(T))e^{-2n\phi(\zeta)}}{n\phi_z(\zeta(T))c\sqrt{T-t}},$$

as well as 

$$e^{-c^2n/2} \frac{\sqrt{n\phi_z(\zeta,t)}}{\psi(\zeta)} = o(1).$$

Consequently, (2.80)

$$h^2(\zeta) = \frac{2\pi \psi^2(\zeta)e^{-2n\phi(\zeta)}}{n\phi_z(\zeta)} (1 + o(1)) = g(\zeta)(1 + o(1))$$

from which the claims $h(\zeta) \neq 0$ and $2\Delta_t \arg h(\zeta) = \Delta_t \arg g(\zeta) + o(1)$ follow.

Lemma 37. Let $T$ be as defined in equation (2.31) and $g(\zeta)$ as defined in (2.79).

Set $I = T - \ln^2 n/n^{2/3}$, $1/n^{2/3}/\zeta$. Then $h(\zeta(t)) \neq 0$ on $I$, and $\Delta_t \arg h(\zeta) = \frac{1}{2} \Delta_t \arg g(\zeta) + C$, where $|C| < \pi/2 + o(1)$.

Proof. The relevant estimate for $h(\zeta)$ in this case (c.f. Section 2.2.4) is 

$$\frac{4c\psi(\zeta(T))e^{-n\phi(\zeta,t)}}{\sqrt{6e^n(\phi_z(\zeta(0),T))}} \int_a^b e^{-n\phi(\zeta)} \left( \sqrt{3\sqrt{T-t}} - (z/c - \zeta/c) - \frac{\sqrt{T-t}}{\sqrt{3\sqrt{T-t} - (z/c - \zeta/c)}} \right)^{-1} dy(1 + o(1)).$$

Since the real part of the integrand is positive, we immediately obtain that $h(\zeta) \neq 0$. To compute the change of argument of the above expression over $I$ we first note that 

$$\Delta_I \arg \frac{4c\psi(\zeta(T))e^{-n\phi(\zeta,t)}}{\sqrt{6e^n(\phi_z(\zeta(0),T))}} = \Delta_I \arg e^{-n\phi(\zeta,t)}.$$

Next we employ the estimate $\zeta(t) - \zeta(T) = c\sqrt{T-t} + O(T-t)$ to deduce that (2.81)
and hence
\[ \Delta_J \arg e^{-n\phi(z,t)} = \frac{1}{2} \Delta_J \arg g(\zeta) + o(1). \]

Since
\[ (2.82) \quad \int_a^b e^{-ny^2} \left( \frac{\sqrt{3T - t - (z/c - \zeta/c)} - \sqrt{t}}{\sqrt{3T - t - (z/c - \zeta/c)}} \right)^{-1} dy \]
is in the right half plane for \( t \in I \), the change in argument of this integral over \( I \) is no more than the difference in arguments corresponding to \( t = T - \ln^2 n/n^{2/3} \) and \( t = T - 1/n^{2/3} \). When \( t = T - \ln^2 n/n^{2/3} \), the square of the integral in (2.82) is
\[ h^2(\zeta) \left( \frac{16\psi^2(\zeta(T)) e^{-2n\phi(\zeta,t)}}{6c\phi_3(\zeta(T),T)} \right)^{-1} (1 + o(1)). \]
This expression by (2.80) is equal to
\[ g(\zeta) \left( \frac{16\psi^2(\zeta(T)) e^{-2n\phi(\zeta,t)}}{6c\phi_3(\zeta(T),T)} \right)^{-1} (1 + o(1)), \]
whose argument is \( o(1) \) by equation (2.81). Thus, regardless of what the argument is at \( t = T - 1/n^{2/3} \), the absolute value of the change in argument of (2.82) for \( t \in I \) is at most \( \pi/2 + o(1) \) and the result follows. \( \square \)

**Lemma 38.** Suppose that \( T = T_2 \) and \( h(\zeta(T_1)) \neq 0 \). Let
\[ I_1 = (T_1 - \ln^2 n/n^{2/3}, T_1 + \ln^2 n/n^{2/3}), \]
\[ I_2 = T_1 - (\ln^2 n/n^{2/3}, 1/n\zeta), \quad \text{and} \]
\[ I_3 = T_1 + (1/n, \ln^2 n/n^{2/3}). \]
Then the \( h(\zeta(t)) \neq 0 \) on \( I_1 \), and
\[ 2\Delta_{I_1} \arg h(\zeta) = \Delta_{I_2} \arg g(\zeta) + \Delta_{I_3} \arg g(\zeta) + C \]
where \( |C| \leq \pi + o(1) \).

**Proof.** On the interval \( I_1 \), we either have \( |T_1 - t| = o\left(\frac{1}{n}\right) \) or \( \frac{1}{n} \ll |T_1 - t| < \ln^2 n/n^{2/3} \). If \( |T_1 - t| = o\left(\frac{1}{n}\right) \), then the boundedness of
\[ \log(z_1 - z) + \log(z_2 - z) - \log(z_1 + z) - \log(z_2 + z) \]
on \( \Gamma_2 \) implies that
\[ \phi(z,t) = \phi(z,T_1) + i(t - T_1) (\log(z_1 - z) + \log(z_2 - z) - \log(z_1 + z) - \log(z_2 + z)) \]
\[ = \phi(z,T_1) + o(1/n) \quad (z \in \Gamma_2). \]
Thus
\[ h(\zeta(t)) = \int_{\Gamma_2} e^{-n\phi(z,t)} \psi(z) \, dz \]
\[ = h(\zeta(T_1))(1 + o(1)), \]
Recall from Section 2.2.6 that under the assumptions of the Lemma, we conclude that $h(\zeta(t)) \neq 0$ on $I_2 \cup I_3$, and that
\[
\Delta h_2 \arg h(\zeta) + \Delta h_3 \arg h(\zeta) \leq \frac{1}{2} \Delta h_2 \arg g(\zeta) + \frac{1}{2} \Delta h_3 \arg g(\zeta) + C,
\]
where $|C| \leq \pi + o(1)$. Combining these estimates establishes the claim.

**Lemma 39.** If $\tau$ is a large constant multiple of $\ln^4 n/n$, then $h(\zeta(t)) \neq 0$ for $0 < t \leq \tau$ and
\[
\Delta_{0 < t \leq \tau} \arg h(\zeta(t)) = \frac{1}{2} \lim_{\xi \to 0} \Delta_{\xi < t \leq \tau} \arg g(\zeta(t)) + o(1).
\]

**Proof.** Recall from Section 2.2.6 that under the assumptions of the Lemma,
\[
h(\zeta(t)) = \frac{-i(e^{\pi t} + e^{-\pi t})}{z_1 - e^{\pi t}} 2^{1/2 - int}(z_2 - z_1)^{1/2 + int}(z_2 + z_1)^{1/2 - int} \Gamma(3/2 + int) e^{-(3/2 + int) \ln n}
\]
for $0 < t \leq \tau$. It follows that $h(\zeta(t)) \neq 0$ on this interval. Using Stirling’s formula once more we find that the change of the argument $\Gamma(s)$ along the line $L: s = 3/2 + int$, $0 < t \leq \tau$ is
\[
\Delta_{0 < t \leq \tau} \arg \Gamma(3/2 + int) = \text{Im}((1/2 + int) \log(3/2 + int)) - n\tau + O\left(\frac{1}{\ln^4 n}\right)
\]
\[
= n\tau \ln(n\tau) + \pi + n\tau + O\left(\frac{1}{\ln^4 n}\right).
\]
The change of arguments of the factors $2^{1/2 - int}$, $(z_2 - z_1)^{1/2 + int}$, $(z_2 + z_1)^{1/2 - int}$, and $e^{-(3/2 + int) \ln n}$ are given by $-n\tau \ln 2$, $n\tau \ln(z_2 - z_1)$, $-n\tau \ln(z_2 + z_1)$, and $-n\tau \ln n$ respectively. Thus, the change of argument of $h(\zeta)$ over the range in question is
\[
\Delta_{0 < t \leq \tau} \arg h(\zeta(t)) = n\tau \ln \frac{\tau(z_2 - z_1)}{2(z_2 + z_1)} - n\tau + \frac{\pi}{4} + o(1).
\]
We next consider the change in argument of
\[
g(\zeta) = \frac{2\pi i^2(\zeta)e^{-2n\phi(\zeta)}n^2}{n^2 \phi(\zeta)} = \frac{2\pi i^2(\zeta)e^{-2n\phi(\zeta)}n^2}{n^2 \phi(\zeta)} \frac{(z_2 - z_1)^{1+2int}(z_2 - \zeta)^{1+2int}(z_2 + \zeta)^{-2int}(z_2 + \zeta)^{-2int}}{(z_2 + \zeta)^{2n^2+2}}.
\]
With $z_1 - \zeta = -iz_1 t + O(t^2)$ (c.f. equation (2.16)), the change of argument of the factor $(z_1 - \zeta(t))^{1+2int}$ is
\[
\lim_{\xi \to 0} \text{Im} \left((1 + 2int)(\ln |z_1 t| - \frac{\pi}{2} + O(t))\right) |\xi|
\]
\[
= 2n\tau \ln(z_1 t) + o(1),
\]
while the change of argument of $\zeta(t)^{2n^2+2}$ is
\[
\text{Im}((2n + 2)(\log \zeta(\tau)) = (2n + 2)(\tau + O(\tau^2)) = 2n\tau + o(1).
\]
Similarly, the change of argument of the factors $(z_2 - \zeta(t))^{1+2int}$, $(z_1 + \zeta(t))^{1+2int}$, and $(z_2 + \zeta(t))^{1-2int}$ are given by $2n\tau \ln(z_2 - z_1 + o(1)$, $-2n\tau \ln(2z_1) + o(1)$, and $-2n\tau \ln(z_1 + z_2)$ respectively. Since
\[
\phi(z_2, \zeta) = -\frac{1}{\zeta^2} - it \left(\frac{1}{(z_1 - \zeta)^2} + \frac{1}{(z_2 - \zeta)^2} + \frac{1}{(z_1 + \zeta)^2} + \frac{1}{(z_2 + \zeta)^2}\right),
\]
The curve $\gamma$ for $(z_1, z_2) = (1, 3)$ (left) and $(1, 7)$ (right)

the corresponding change in argument of $\phi_{z_2}(\zeta, t)$ for $0 < t \leq \tau$ is $\pi/2 + o(1)$. We conclude that the change in argument of $g(\zeta(t))$ is

$$2n\tau \ln \frac{\tau(z_2 - z_1)}{2(z_1 + z_2)} - 2n\tau + \pi \frac{\pi}{2} + o(1)$$

and the result follows. \hfill \qed

Lemmas 36, 37, 38, and 39 show that in order to understand the change in the argument of $h(\zeta(t))$, it suffices to study of the change of argument of $g(\zeta)$ defined in (2.79), where we (re)write $\phi_{z_2}$ as

$$\phi_{z_2}(\zeta, t) = \frac{(\zeta^2 + z_1 z_2)(\zeta^4 + (z_2^2 - 4z_1 z_2 + z_1^2)\zeta^2 + z_1^2 z_2^2)}{\zeta^2(\zeta^2 - z_1^2)(\zeta^2 - z_2^2)(\zeta^2 - z_1 z_2)}.$$

Let $\gamma$ be the simple closed curve with counter clockwise orientation formed by the traces of $\zeta(t), \zeta(t), -\zeta(t), -\zeta(t)$ for $0 \leq t \leq T$ and small deformations around

$$\pm i\sqrt{z_1 z_2, \pm \zeta(T_1), \pm \zeta(T_1)}$$

such that the region enclosed by $\gamma$ contains the points defined in (2.83). We also deform $\gamma$ around $\pm z_1$ so that the cuts $(-\infty, -z_1]$ and $[z_1, \infty)$ lie outside this region (see Figure 2.5).

By computing the logarithmic derivative of $g(\zeta)$, we find that

$$\text{Res} \left( \frac{g'(\zeta)}{g(\zeta)}, 0 \right) = \text{Res} \left( -2n \frac{d\phi(\zeta, t)}{d\zeta}, 0 \right),$$

since $\phi_{z_2}(\zeta, t)$ is analytic in a neighborhood of the origin. Furthermore,

$$\frac{d\phi(\zeta, t)}{d\zeta} = \frac{1}{\zeta} \frac{d}{d\zeta} \left( \frac{(\zeta^2 - z_1^2)(\zeta^2 - z_2^2)}{2\zeta(z_1 + z_2)(\zeta^2 - z_1 z_2)} \left( \frac{1}{\zeta - z_1} + \frac{1}{\zeta - z_2} - \frac{1}{\zeta + z_1} - \frac{1}{\zeta + z_2} \right) \right).$$

Figure 2.5. The curve $\gamma$ for $(z_1, z_2) = (1, 3)$ (left) and $(1, 7)$ (right)
and hence $d\phi(\zeta, t)/d\zeta - 1/\zeta$ is the derivative of a meromorphic function around the origin. Thus, its residue at 0 is 0 by its Laurent series expansion. We conclude that

$$\text{Res} \left( \frac{g'(\zeta)}{g(\zeta)}, 0 \right) = -2n \text{ Res} \left( \frac{d\phi(\zeta, t)}{d\zeta}, 0 \right) = -2n.$$  

Since the points in (2.83) are all simple poles of $g(\zeta)$ with residue 1, we conclude from the residue theorem that

$$(2.84) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{g'(\zeta)}{g(\zeta)} d\zeta = \begin{cases} -2n - 6 & \text{if } z_1^2 - 6z_1 z_2 + z_2^2 < 0 \\ -2n - 2 & \text{if } z_1^2 - 6z_1 z_2 + z_2^2 \geq 0. \end{cases}$$

Suppose that $\zeta$ not a singularity of $g(\zeta)$. Then $g(\zeta) = \overline{g(\zeta)}$. In addition, since

$$e^{-2n \log \zeta} = \frac{1}{\zeta^{2n}},$$

we also have

$$(2.85) \quad g(-\zeta) = g(\zeta).$$

If $\zeta \in i\mathbb{R}$, then $\zeta = -\overline{\zeta}$. In this case we apply (2.85) to conclude that

$$g(\zeta) = g(-\zeta) = g(\overline{\zeta}) = \overline{g(\overline{\zeta})},$$

and consequently $g(\zeta) \in \mathbb{R}$.

If we let $\gamma_1, \gamma_2, \gamma_3, \text{ and } \gamma_4$ be the portion of $\gamma$ on the first, second, third, and fourth quadrant respectively, then − using the above relations − we see that

$$\Delta_{\gamma_4} \arg g(\zeta) = -\Delta_{\gamma_4} \arg g(\overline{\zeta}) = -\Delta_{\gamma_4} \arg g(\overline{\overline{\zeta}}) = -\Delta_{\gamma_1} \arg g(\zeta),$$

and

$$\Delta_{\gamma_3} \arg g(\zeta) = \Delta_{\gamma_2} \arg g(\zeta).$$

Similarly

$$\Delta_{\gamma_3} \arg g(\zeta) = \Delta_{\gamma_3} \arg g(-\zeta) = \Delta_{\gamma_1} \arg g(\zeta).$$

We conclude that

$$(2.86) \quad \Delta_{\gamma_1} \arg g(\zeta) = \frac{\Delta_{\gamma_1} \arg g(\zeta)}{4} = \begin{cases} -(n + 3)\pi & \text{if } z_1^2 - 6z_1 z_2 + z_2^2 < 0 \\ -(n + 1)\pi & \text{if } z_1^2 - 6z_1 z_2 + z_2^2 \geq 0. \end{cases}$$

The following three lemmas demonstrate that the change in the argument of $g(\zeta(t))$ near the points $T_1$ and $T_2$ are small.

**Lemma 40.** Let $g(\zeta)$ be as defined in equation (2.79), and let $T$ be as defined in (2.31). If $\tau < T$ satisfies $T - \tau \ll 1/n^{2/3}$, then

$$\lim_{\zeta \to 0} \Delta_{\tau \leq t < T - \zeta} \arg g(\zeta(t)) \ll 1.$$  

**Proof.** Using the definition of $g(\zeta)$ and the estimates $\phi_{z_2}(\zeta, t) = \zeta(t) - \zeta(T) = \sqrt{T - t}$, it suffices to show that

$$\text{Im } \phi(\zeta(t), \tau) - \text{Im } \phi(\zeta(T), T) \ll \frac{1}{n}.$$
With \( \zeta(T) - \zeta(\tau) = c\sqrt{T - \tau} + \mathcal{O}(T - \tau) \), \( \phi_z(\zeta(T), T) = 0 \), \( \phi_{zz}(\zeta(T), T) = 0 \), and \( \text{Im} \phi_t(\zeta(T), T) = 0 \), we expand the bivariate function \( \phi(z, t) \) in a Taylor series in a neighborhood of \( (\zeta(T), T) \) to arrive at

\[
\text{Im} \phi(\zeta(\tau), \tau) - \text{Im} \phi(\zeta(T), T) = \text{Im} ((\zeta(T) - \zeta(\tau))(T - \tau)\phi_{zz}(\zeta(T), T) + \mathcal{O}((T - \tau))) = (T - \tau)^{3/2},
\]

and since \( (T - \tau)^{3/2} \ll \frac{1}{n} \), the result follows.

Since the proofs of the next two lemmas are essentially identical to the one we just gave, we omit them, and state only the results.

**Lemma 41.** Let \( T = T_2 \), and suppose that \( \tau > T_1 \) satisfies \( \tau - T_1 \ll 1/n \). Then

\[
\lim_{\xi \to 0} \Delta_{T_1 + \xi < \tau} \arg g(\zeta(t)) \ll 1.
\]

**Lemma 42.** Let \( T = T_2 \) and suppose that \( \tau < T_1 \) satisfies \( T_1 - \tau \ll 1/n \). Then

\[
\lim_{\xi \to 0} \Delta_{\tau < t < T_1 - \xi} \arg g(\zeta(t)) \ll 1.
\]

Next, we address the change in the argument of \( g(\zeta) \) on the small deformations near the points \( z_1, \zeta(T_1) \) and \( \zeta(T_2) \). We begin with the small arc of \( \gamma_1 \) around \( z_1 \). Note that for any fixed \( n \), as \( \zeta \to z_1 \),

\[
\psi^2(\zeta) \sim C_1(z_1 - \zeta), \quad \phi_{zz}(\zeta) \sim C_2/(z_1 - \zeta), \quad \text{and}
\]

\[
\exp(2n(\zeta - z_1)(z_1 - z_2) \log(z_1 - \zeta)/z_1) \to 1
\]

for some constants \( C_1, C_2 \). We conclude that as \( \zeta \to z_1 \),

\[
g(\zeta) = \frac{2\pi \psi^2(\zeta)e^{-2n\phi(\zeta)}}{n\phi_{zz}(\zeta)} \sim C_3(z_1 - \zeta)^2, \quad (C_3 \in \mathbb{C})
\]

Using the estimate \( z_1 - \zeta = -iz_1 t + \mathcal{O}(t^2) \) developed for small \( t \) in equation (2.46), we find the change of argument of \( g(\zeta) \) on the small arc of \( \gamma_1 \) around \( z_1 \) to be \( -\pi + o(1) \).

We continue by considering the small arc of \( \gamma_1 \) around \( \zeta(T) \). Using Lemma 8 we deduce that the change of argument of \( \zeta(t) \) on the arc of \( \gamma_1 \) around \( \zeta(T) \) approaches \( \pi/2 \) when the arc is small. Since \( g(\zeta) \) has a simple pole at \( \zeta(T) \), the change of argument of \( g(\zeta) \) on the piece of \( \gamma_1 \) around \( \zeta(T) \) approaches \( -\pi/2 \).

Finally, we look at the change of argument in \( g(\zeta) \) on the small arc near \( T_1 \) when \( T = T_2 \). If \( T = T_2 \), then equations (2.37) and (2.74) imply that \( d\hat{d} = -i \). Consequently, the change of argument of \( g(\zeta) \) on the arc of \( \gamma_1 \) around \( \zeta(T_1) \) approaches \( -3\pi/2 \).

We are now in position to count the number of zeros of our polynomials \( H_n \) thereby completing the proof of the main result – via bounding the change in the argument of \( h(\zeta(t)) \) from below on the interval \((0, T)\). There are two cases to consider, corresponding to whether \( T = T_1 \), or \( T = T_2 \).

If \( T = T_1 \), then the observations in the preceding paragraphs along with Lemmas 37 and 39 imply that for some \( |C| < \pi/2 + o(1) \),

\[
\Delta_{0 < t < T} \arg h(\zeta) = \frac{1}{2} \left( \Delta_{T_1} \arg g(\zeta) + \frac{3\pi}{2} \right) + C \tag{2.86}
\]

\[
\frac{n\pi}{2} + \frac{\pi}{4} + C.
\]
Proof. We apply the binomial expansion to each factor of degree of
Lemma 44. Since all the coefficients in the Taylor series of the right side is positive, we conclude and collect the zeros of H.
H is at least
\[ \left| \frac{\Delta_{n,\zeta < T} \arg h(\zeta)}{\pi} \right| \geq \left| \frac{n}{2} - \frac{3}{4} + o(1) \right|. \]
It follows that Hn(x) has at least
\[ 2 \left| \frac{n}{2} - \frac{3}{4} + o(1) \right| = \begin{cases} n - 3 & \text{if } 2 \nmid n \\ n - 2 & \text{if } 2 \mid n \end{cases} \]
nonreal zeros on the line Re x = 1/2. It remains to account for the missing 2 or 3 zeros depending on the parity of n. Once this is accomplished (see Lemma 44), and we establish a bound on the degree of Hn (see Lemma 44), Theorem 5 will follow from the fundamental theorem of algebra.

Lemma 43. If n > 2 is even, then
1. x = 0 and x = 1 are zeros of Hn(x)
2. Hn'(0) < 0 and Hn'(1) > 0.
If n > 1 is odd, then
1. x = 0, x = 1/2, and x = 1 are zeros of Hn(x)
2. Hn'(0) < 0 and Hn'(1) < 0.
Proof. With the substitution x by 1 - x and z by -z in (2.25) we conclude
\[ H_n(x) = (-1)^n H_n(1 - x). \]
Consequently Hn(1/2) = 0 when n is odd and it suffices to consider the case x = 0. Plugging x = 0 into the right side of equation (2.25) gives Hn(0) = 0 \( \forall n > 2 \). Similarly, evaluating the derivative of this expression as a function of x at x = 0 gives
\[ \sum_{n=0}^{\infty} H_n'(0) \frac{z^n}{n!} = (z_1 + z) (z_2 + z) \left( \log(z_1 - z) + \log(z_2 - z) - \log(z_1 + z) - \log(z_2 + z) \right) \]
\[- \left( \log \left( 1 - \frac{z}{z_1} \right) - \log \left( 1 + \frac{z}{z_1} \right) + \log \left( 1 - \frac{z}{z_2} \right) - \log \left( 1 + \frac{z}{z_2} \right) \right) \]
Since all the coefficients in the Taylor series of the right side is positive, we conclude Hn'(0) > 0. □

Lemma 44. Suppose that (Hn(x))n\geq0 is as in the statement of Theorem 5. Then for each n, the degree of Hn(x) is at most n. Furthermore, \( \lim_{x \to -\infty} H_n(x) = +\infty \) and
\[ \lim_{x \to +\infty} H_n(x) = \begin{cases} +\infty & \text{if } 2 \mid n \\ -\infty & \text{if } 2 \nmid n \end{cases} \]
Proof. We apply the binomial expansion to each factor of
\[ Q(z)^x Q(-z)^{1-x} = z_1 z_2 (1 - \frac{z}{z_1})^x (1 - \frac{z}{z_2})^x (1 + \frac{z}{z_1})^{1-x} (1 + \frac{z}{z_2})^{1-x} \]
and collect the \( z^n \)-coefficients to conclude that the degree of Hn(x) is at most n. Also from this binomial expansion, we see that all the coefficients in the power series in z of each factor are positive as \( x \to -\infty \). Thus \( \lim_{x \to -\infty} H_n(x) = +\infty \). We complete the proof by applying the identity Hn(x) = (-1)n Hn(1 - x). □
With these results, the proof of Theorem 5 is complete in the case when $T = T_1$.

If $T = T_2$ and $h(\zeta(T_1)) \neq 0$, Lemmas 37, 38, and 39 imply that for some $|C| < 3\pi/2 + o(1)$

\[
\Delta_{0<T} \arg h(\zeta) = \frac{1}{2} \left( \Delta_{T_1} \arg g(\zeta) + 3\pi \right) + C
\]

Consequently, the number of zeros of $H_n(1/2 + int)$ on $(0, T)$ is at least

\[
\left\lfloor \frac{\Delta_{0<T} \arg h(\zeta)}{\pi} \right\rfloor \geq \frac{n}{2} - \frac{3}{2} + o(1)
\]

and $H_n(x)$ has at least

\[
2 \left\lfloor \frac{n}{2} - \frac{3}{2} + o(1) \right\rfloor = \begin{cases} 
  n - 5 & \text{if } 2 \nmid n \\
  n - 4 & \text{if } 2 \mid n
\end{cases}
\]

non-real zeros on the line $\Re x = 1/2$. Since $H_n(x)$ has zeros at $x = 0$ and $x = 1$ and a zero at $x = 1/2$ when $n$ is odd, there are at most two possible zeros of $H_n(x)$ distinct from 0, 1 and not on the line $\Re x = 1/2$. By the symmetry of zeros of $H_n(x)$ along the line $\Re x = 1/2$ and the real axis, these two zeros must be real and symmetric about the line $\Re x = 1/2$.

If $n$ is even, then Lemmas 13 and 14 imply that $H_n(+\infty)H_n(0^-) > 0$, $H_n(0^+)H_n(1^-) > 0$, $H_n(1^+)H_n(+\infty) > 0$ and these two possible exceptional zeros do not exist. A similar argument applies to the case $n$ is odd.

If $T = T_2$ and $h(\zeta(T_1)) = 0$, then the number of real zeros of $H_n(1/2 + int)$ on $(0, T) \setminus \{T_1\}$ is at least

\[
\lim_{\xi \to 0} \left\lfloor \frac{\Delta_{0<T-\xi} \arg h(\zeta)}{\pi} \right\rfloor + \left\lfloor \frac{\Delta_{T_1-x} \arg h(\zeta)}{\pi} \right\rfloor \geq 1
\]

If we count $T_1$ as another zero of $H_n(1/2 + int)$ on $(0, T)$, we obtain the same number of zeros of this polynomial as in the case $h(\zeta(T_1)) \neq 0$. This completes the proof of Theorem 5.

**References**

[1] D. Baccherini, D. Merlini, and R. Sprugnoli, Level generating trees and proper Riordan arrays, Appl. Anal. Discrete Math. 2 (2008), 69–91.
[2] F. Bergeron, P. Flajolet, and B. Salvy, Varieties of increasing trees, In CAAP’92 (Rennes, 1992), volume 581 of Lecture Notes in Comput. Sci., pages 24–48. Springer, Berlin, 1992.
[3] G.-S. Cheon, J.-H. Jung, and P. Barry, Horizontal and vertical formulas for exponential Riordan matrices and their applications, Lin. Alg. Appl., 541 (2018), 266-284.
[4] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, Ann. Comb. 13 (2009), 65–85.
[5] T. Forgács and K. Tran, Polynomials with rational generating functions and real zeros. J. Math. Anal. Appl. 443(2) (2016), pp. 631-651. DOI 10.1016/j.jmaa.2016.05.041
[6] T. Forgács and K. Tran, Zeros of polynomials generated by a rational function with a hyperbolic-type denominator, Constr. Approx. 46 (2017), pp. 617-643. DOI 10.1007/s00365-017-9378-2
[7] T. Forgács, and K. Tran, Hyperbolic polynomials and linear-type generating functions, J. Math. Anal. Appl. 488 (2) (2020) DOI: 10.1016/j.jmaa.2020.124085
[8] A. Meir and J. W. Moon, On the altitude of nodes in random trees, Can. J. Math., Vol. XXX, 5 (1978), 997-1015.
[9] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, On some alternative characterizations of Riordan arrays, Can. J. Math. 49 (1997), 301–320.
[10] I. Ndikubwayo, Criterion of the reality of zeros in a polynomial sequence satisfying a three-term recurrence relation, *Czechoslovak Mathematical Journal* 70(3):1-12, 2020. DOI: 10.21136/CMJ.2020.0535-18

[11] H. Niederhausen, Lattice Path Enumeration and Umbral Calculus, Balakrishnan N. (eds) Advances in Combinatorial Methods and Applications to Probability and Statistics, Statistics for Industry and Technology. Birkhauser Boston, 1997.

[12] B. Palka, An introduction to complex function theory, UTM, Springer, 1991.

[13] E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.

[14] S. Roman and G.-C. Rota. The umbral calculus. Adv. Math. 27 (1978), 95-188.

[15] G.-C. Rota, D. Kahaner, and A. Odlyzko. On the foundations of combinatorial theory viii: finite operator calculus. J. Math. Anal. Appl. 42 (1973), 684-760.

[16] L. W. Shapiro, S. Getu, W.-J. Woan and L. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991), 229-239.

[17] N. M. Temme., Asymptotic Methods for Integrals. Series in Analysis, vol. 6. ISBN: 978-981-4612-15-9, World Scientific, 2014.

[18] K. Tran, The root distribution of polynomials with a three-term recurrence, *J. Math. Anal. Appl.* 421 (2015), 878-892.

[19] K. Tran, A. Zumba, Zeros of polynomials with four-term recurrence. *Involve, a Journal of Mathematics* Vol. 11 (2018), No. 3, 501-518.