A derivation of the soft modes effective action for the SYK spectral form factor

Matteo A. Cardella

Dipartimento di Fisica, Università degli Studi di Milano,
via Celoria 16, 20133 Milan, Italy.
and

Dipartimento di Scienza e Alta Tecnologia, Università dell’Insubria,
via Valleggio 11, 22100 Como, Italy.

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Abstract

The SYK model spectral form factor exhibits absence of information loss in the form of a ramp and a plateau, that are typical of random matrix theory. In a large $N$ collective fields description, the ramp was reproduced by Saad, Shenker and Stanford [1], by a connected component of the analytically continued thermal partition function two point correlator. The existence of an off shell connected component for the real time partition function is a consequence of disorder averaging, over a statistical ensemble of two replica systems. In particular, the ramp arises from a family of vanishing action replica-non-diagonal conformal saddles. We provide a derivation of the soft modes two Schwarzians effective action for fluctuations around the ramp critical saddles, by adapting to the two replica system a method used by Kitaev and Suh [2] in regular SYK.
1 Introduction

The Bekenstein-Hawking [3],[4] black hole entropy formula in conjunction with the holographic principle [5],[6] predict that any region of space, surrounded by a boundary surface of finite area, should be described by a finite dimensional Hilbert space. It is not known in general how such an holographic description works, except in few notable cases. In the well studied case over the years AdS/CFT [7], despite a large amounts of results, several relevant problems related to black holes cannot be formulated in quantitative and explicit terms, due to the complexity of the boundary theory at finite temperature.

One of the forms of the black hole information problem, formulated by Maldacena sometime ago [8], involves the long time behavior of correlation functions. In any finite entropy system, correlation functions cannot decay to zero at large times, since this would violate quantum mechanics in the form of quantum information loss. However, this requirement is hard to be realized from the bulk point of view, where typical correlation functions decay to zero at large times, because of black hole quasi normal modes relaxation [9].

This problem can be stated more explicitly in the AdS/CFT context, in terms of correlation functions of boundary operators. Let us consider for example a thermal two point function for a boundary operator $O$

$$G_{\beta}(t) = \frac{1}{Z(\beta)} Tr \left( e^{-\frac{\beta}{2}H} O(t) e^{-\frac{\beta}{2}H} O(0) \right) = \frac{1}{Z(\beta)} \sum_{m,n} e^{-(\frac{\beta}{2}+it)E_n} e^{-(\frac{\beta}{2}+it)E_m} |\langle n|O|m \rangle|^2,$$  \hspace{1cm} (1)

where $Z(\beta)$ is the thermal partition function, the dimension of the Hilbert space is finite and exponentially large in the entropy and we omit possible dependence on spacial coordinates. At early times $G_{\beta}(t)$ decays exponentially as the effect of thermalization, which in the bulk corresponds the black hole quasi-normal modes relaxation. However, the finite sum in (1) does not go to zero at large times, instead it keeps oscillating with an amplitude exponentially small in the entropy [10],[11],[12]. As we remarked already, this effect is not visible from the bulk point of view in terms of perturbative corrections to classical gravity, which is a manifestation of the black hole information loss problem. More generally, it is a manifestation of the lack of understanding of how to fully reconcile quantum mechanics with a smooth geometry.
A somehow simpler quantity than (1), that does not contain the matrix elements $|\langle n | O | m \rangle|^2$ but still exhibits the same phenomenon [13], is the so called spectral form factor

$$|Z(\beta,t)|^2 = Z(\beta-it)Z(\beta+it) = \sum_{m,n} e^{-\beta(E_n+E_m)} e^{it(E_n-E_m)}.$$  (2)

In the context of quantum chaos its $\beta = 0$ version has been studied extensively. While at short times $|Z(\beta,t)|^2$ is of the order of the square of the thermal partition function $Z(\beta)^2$, at very long times, of the order of the inverse of the mean energy level spacing, the spectral form factor reaches a limiting value $Z(2\beta)$ usually called the plateau, due to a cancellation between the off diagonal contributions in the sum. A direct computation at large times of the spectral form factor in theories with a well defined gravity dual such as super Yang Mills is currently impossible. However, the Sachdev-Ye-Kitaev model (SYK) [14, 15, 16, 17, 18, 19, 20] offers both a numerical and an analytic handle for studying the spectral form factor. An accurate numerical analysis of the SYK spectral form factor was done in [21], (see also [22] for earlier related work), while a subsequent work [1] explains part of the observed behavior in a large $N$ collective fields description.

SYK is a statistical mechanics model over an ensemble of quantum mechanical many body systems of Majorana fermions, with random couplings of even order $q \geq 4$ all to all interactions.

A statistical average over an ensemble of quantum systems is not equivalent to a single quantum mechanical model, a fact that may cast doubts upon using SYK for discussing delicate issues like unitarity in a black hole holographic description. Yet, there are qualitative and quantitative features that survive the ensemble disorder average that make SYK an interesting playground for discussing certain quantum gravity issues related to black holes\footnote{There are also the so called colored tensor models, quantum mechanical models without disorder that exhibit the same diagrammatic as SYK [23].}

SYK has a collective fields description that in the large $N$ limit at low temperatures/strong coupling exhibits an interesting quasi conformal behavior, with aspects of a gravity dual. In particular, as an holographic model for nearly extremal black holes it provides an interesting arena for testing various ideas and proposals and sharpen open problems. One of its attractive features is a dominant soft mode dynamics at low temperatures [14],[15],[24],[25] that from a bulk perspective encodes the full gravitational backreaction [26],[2],[27],[28]; and, under certain circumstances, allows for a quantum mechanical description of a black hole interior [28],[29],[30],[31],[32],[33]. One of the exciting recent results, that is understood also from the SYK perspective, is a gravitational description of the quantum teleportation protocol [28],[34],[35]. This is based on recent observations on how certain double trace deformations that violate the average null energy condition make a wormhole temporarily traversable [36]. In the teleportation protocol those double trace deformations implement both the transmission of classical instruction for the quantum protocol between the two boundaries through external space-time and make the wormhole connecting them temporarily traversable.

Another interesting feature is that SYK exhibits quantum chaotic behavior both at short and long times scales. At short times scales, out of time ordered four point thermal correlators involving a perturbation on a typical operator saturate [14],[15],[37] a universal chaos bound [38] for their Liapunov exponent. In AdS/CFT, saturation of the chaos bound in the boundary is
understood by a corresponding bulk near horizon scattering dynamics that involves Dray and 't Hooft shock waves. The requirement for the boundary theory to be a fast scrambler avoids black hole quantum cloning in certain gedanken experiments. The fast scrambler property is satisfied by boundary theories that are $q$-local quantum many body systems with all to all interactions. In the SYK case there are no shock waves in the two dimensional bulk and the only bulk gravitational degrees of freedom are boundary modes that undergo classical chaotic dynamics by moving in hyperbolic space. On the other hand, SYK exhibits also a large time scales chaotic behavior, related to the fine details of correlations among its energy eigenvalues. The connected part of the two point correlation function for the SYK spectral density controls the large time behavior of the spectral form factor, a quantity of interest to diagnose the existence of quantum information loss. Numerics on the SYK spectral form factor exhibit the emergence at long time scales of typical behaviour of quantum chaotic systems, in particular random matrix theory (RMT) universality. We refer to for relevant numerical plots and typical values of the SYK spectral form factor, here we would like to mention the main qualitative features of the time behavior of the spectral form factor. The plot in figure 1, starts at early times with a decaying phase from its initial value $Z(\beta)^2$ with a characteristic decreasing power law up to a minimum value. This decaying part of the plot is called the slope and the minimum value is called the dip, which occurs at $t_{\text{dip}} \sim e^{N/2}$. After the dip, the SFF starts a linear rising behavior called the ramp. The ramp occurs up to a saturation time $t_{\text{plateau}} \sim e^N$ where $|Z(\beta, T)|^2$ reaches the limiting value $Z(2\beta)$. The ramp and the plateau are typical of quantum chaotic systems in particular in random matrix theory (RMT). The emergence of RMT universality in itself is not surprising from a Hilbert space perspective, the interest, motivated by the black hole information problem, is to understand the behavior of the spectral form factor by using a large $N$ collective fields description. This problem was discussed in [1] for the SYK model and by the same authors in [45] for Jackiw Teitelboim (JT) gravity, where a non perturbative completion of JT gravity in terms of RMT is proposed. In [48] the relations between JT gravity and RMT are extended to the case where the boundary theory has time-reversal symmetry and have fermions with or without supersymmetry.

An interesting point that emerges from the numerics on the SYK spectral form factor is that for a single realization of the disorder, the time plot exhibits erratic oscillations in its ramp and plateau regions, (figure 1). Oscillations are washed out either by averaging SYK over a large enough ensemble or by taking a suitable time average of the spectral form factor. Although part of the information that corresponds to the erratic wild oscillations is washed out by the disorder average, the main trend that does not exhibit information loss is still present after ensemble average. This provides an interest for a quantitative study of the SYK spectral form factor in terms of large $N$ collective fields, in relation to the black hole information paradox.

It is easy to check that the connected part of the disorder averaged two point function for the analytically continued thermal partition function

\[
\langle Z(\beta + iT)Z(\beta - iT) \rangle_c \equiv \langle Z(\beta + iT)Z(\beta - iT) \rangle - \langle Z(\beta + iT) \rangle \langle Z(\beta - iT) \rangle = \int dE dE' \langle \rho(E)\rho(E') \rangle_c e^{-\beta(E+E')} e^{iT(E-E')} \]

(3)
Figure 1: A log-log plot [21] of the SYK spectral form factor for $q = 4$, $N = 34$, $\beta J = 5$. In red the time plot for a single realization of the disorder, wild erratic oscillations are quite visible in the ramp and the plateau parts of the plot. Oscillations are washed out by disorder average over a large enough number of samples without tough throwing away the main trend. Statistical average (in black) still exhibits an interested lack of information loss in the form of a random matrix universality. The black plot refers to numerics from an ensemble of 90 samples [21].

...exhibits a linear ramp for a contribution of the form $\langle \rho(E)\rho(E') \rangle_c \sim \frac{1}{(E-E')^2}$ in the connected component of the two point correlator for the spectral density of states $\rho(E)$. In the above expression brackets in the l.h.s. denote disorder average, while brackets in the r.h.s. denotes statistical correlation. It is indeed the disorder average over a statistical ensemble that provides a non zero connected component $\langle Z(\beta + iT)Z(\beta - iT) \rangle_c$ to the analytically continued partition function. In a single quantum mechanical model the partition function is just a number and there would be no two points connected contributions whatsoever. The above relation indicates that in order to study the connected component of the spectral form factor, in terms of a functional integral in collective fields, one should look to a two replica system. In fact, disorder average creates interactions between distinct replica, which leads to a connected component for the spectral form factor. Another indication that the non decaying contributions for the SYK spectral form factor may be understood by a two replica systems may come from the ER = EPR conjecture [49],[8],[50]. Indeed, it turns out [1] that the family of off replica diagonal saddle points responsible for the SYK ramp are obtained with good approximation from a sum over images of SYK correlators on the thermofield double state. On the other hand, from the JT gravity side, the ramp is reproduced by a JT double trumpet instanton [45], a connected Euclidean baby universe that connects two identical Euclidean black holes at the same temperature.

In this paper we provide a derivation of the soft modes effective action that corrects the SYK connected contribution of the disorder average spectral form factor (3), evaluated in the family of replica off diagonal conformal saddles. We follow a method inspired by the one used in [2] to derive the Schwarzian effective action in regular SYK. Our result agrees with the two Schwarzians
soft modes action employed in [1]\textsuperscript{4}.

The organization of the paper is the following, in section 2 we recap the construction [1] of the two replica functional integral for the connected part of the SYK spectral form factor (3). In section 3 we analyze enhanced effects of a regularized version of the two replica conformal symmetry breaking kinetic operator, on soft modes fluctuations around the SYK spectral form factor conformal saddles and deduce the two replica soft modes two Schwarzians effective action. In appendix, we review the method developed in [2] in regular SYK, for studying the effects of the non conformal SYK kinetic operator on soft modes fluctuations around the conformal point. This includes a derivation of the Schwarzian effective action.

2 Two replica functional integral representation for the connected part of the spectral form factor

The Sachdev-Ye-Kitaev model (SYK) [16, 17, 18, 19, 20, 14, 15] is a statistical mechanics model over an ensemble of many body systems made by \( N \) (even) Majorana fermions \( \Psi_i = \Psi_i^\dagger \)

\[
\{ \Psi_i, \Psi_j \} = \delta_{ij},
\]

(4)
on a complete hypergraph of even order \( q \geq 4 \). Each system of the ensemble has Hamiltonian

\[
H = \left( \frac{i}{q!} \right)^{\frac{q}{2}} \sum_{i_1...i_q} J_{i_1...i_q} \Psi_{i_1} \ldots \Psi_{i_q},
\]

(5)

where the couplings \( J_{i_1...i_q} \) are random variables, independently taken from a Gaussian probability distribution with zero mean \( \langle J_{i_1...i_q} \rangle = 0 \) and variance \( \langle J_{i_1...i_q}^2 \rangle = J^2 / N^{q-1} \). The parameter \( J \) has the dimension of an energy and fixes the characteristic energy scales of the model. In the large \( N \) limit, at high temperatures/weak coupling \( \beta J \ll 1 \), the model is asymptotically free, while at low temperatures/strong coupling \( \beta J \gg 1 \), SYK develops an interesting quasi-conformal dynamics. Conformal symmetry is slightly explicitly broken by \( \beta J N \) corrections, the leading contribution coming from a local Schwarzian effective action. The same pattern of symmetry breaking and a Schwarzian effective action occur in a certain limit of two dimensional Jackiw-Teitelboim (JT) gravity. Moreover, JT gravity describes the classical dynamics of the dimensionally reduced near horizon region of a low temperature nearly extremal black hole. In the following we mainly focus on the \( q = 4 \) case, extensions of the results for generic even \( q \) are usually straightforward\textsuperscript{5}.

We consider the following connected contribution to the SYK disorder averaged analytically continued thermal partition function two point correlator

\textsuperscript{4}After the first version of this paper was posted on the archive, a new version of [1] appeared, that includes a computation of the one loop determinant from fluctuations around the spectral form factor families of connected conformal saddles. Contributions to the one loop determinant include, besides the left and right SYK soft modes considered here, also two hydrodynamic modes that correspond to fluctuations of the two phase space moduli \( \beta_a \) and \( \Delta_c \) (described here at the end of section 2), that parametrize the families of conformal saddles. In particular [1] have shown that the one loop determinant reproduces consistently the linear rising ramp in the spectral form factor.

\textsuperscript{5}For a nice review of the SYK model and its relations to JT gravity see [51].
\[ \langle Z(\beta-iT)Z(\beta+iT) \rangle_c \equiv \langle Z(\beta-iT)Z(\beta+iT) \rangle - \langle Z(\beta-iT) \rangle \langle Z(\beta+iT) \rangle, \]  
(6)

where brackets denote disorder average. As already remarked in the introduction, in order to have a non vanishing connected component we shall look for a system of two replicas on which to perform disorder average. In fact, disorder average creates an interaction among non interacting replica. By following [1] we construct a functional integral representation for \( \langle Z(\beta-iT)Z(\beta+iT) \rangle_c \) in terms of two copies or replica of SYK, \( SYK_L \) and \( SYK_R \). We consider the following two replica representation for the spectral form factor for one particular realization of the disorder

\[ Z(\beta-iT)Z(\beta+iT) = Tr \left( e^{-(\beta-iT)H_L}e^{-(\beta+iT)H_R} \right) = Tr \left( e^{-T(J_LH_L+J_RH_R)} \right) \]

(7)

where

\[ H_I = -\frac{1}{4!} \sum_{i,j,k,l} J_{ijkl} \Psi^I_i \Psi^I_j \Psi^I_k \Psi^I_l, \quad I = L, R \]

(8)

and

\[ J_L = \frac{\beta}{T} - i, \quad J_R = \frac{\beta}{T} + i = J^*_L. \]

(9)

By disorder averaging (7) one finds the following path integral representation

\[
\langle Z(\beta+iT)Z(\beta-iT) \rangle_c = \int D\Psi^I_\tau \exp \left( -\int_0^T d\tau \left( \Psi^I_\tau(\tau) \partial_\tau \Psi^I_\tau(\tau) + \frac{NJ^2J_IJ_J}{8} \int_0^T d\tau d\tau' \left( \frac{1}{N} \Psi^I_\tau(\tau) \Psi^I_\tau(\tau') \right)^4 \right) \right),
\]

(10)

where replica indexes \( I, J = L, R \) are summed up.

The above connected component has to be contrasted with the disconnected part of the two point function \( \langle Z_{SYK}(\beta-iT) \rangle \langle Z_{SYK}(\beta+iT) \rangle \), obtained by the standard SYK disorder averaged thermal partition function \( \langle Z_{SYK}(\beta) \rangle \) by analytic continuation \( \beta \to \beta \pm iT \). \( \langle Z_{SYK}(\beta) \rangle \) can be computed at full quantum level in the Schwarzian approximation, since the Schwarzian path integral over soft modes is one loop exact [52]. Alternatively, the same result for \( \langle Z_{SYK}(\beta) \rangle \) can be obtained by solving a quantum mechanical problem for a particle scattered by a Liouville potential [53],[54]. Accurate methods for computing Schwarzian amplitudes at full quantum level are developed in [55], [56]. The full quantum answer is given by

\[ \langle Z_{SYK}(\beta) \rangle \sim \frac{e^{\alpha_S 2\pi^2}}{(\beta^3)^{3/2}}, \]

(11)

which gives the following disconnected contribution to the spectral form factor

\[ \langle Z_{SYK}(\beta+iT) \rangle \langle Z_{SYK}(\beta-iT) \rangle \sim \frac{e^{\alpha_S 4\pi^2}}{(\beta^3+(\beta^2+T^2)^3/2)}. \]

(12)

This disconnected contribution reproduces accurately the decaying slope of the spectral form factor in figure 1. It is not surprising that this contribution manifests information loss, since
it is given by a product of analytic continuations of the SYK thermal partition function. From
the two replica system perspective, it is the result of a replica diagonal saddle plus the one loop
determinant from fluctuations, that together give the full quantum answer. In contrast, the
connected contribution from off diagonal replica saddle is somehow related to a purification of the
thermal density matrix, obtained by doubling the system. Indeed it turns out that the connected
saddles can be written in terms of an antisymmetrized version of the thermofield double correlators [1], (see eq. (32) and related discussion).

Concerning the functional integral in (10), before switching from the representation in terms of
Majorana fermions to a more convenient equivalent description in terms of $O(N)$ singlets collective
fields, let us notice that in any regime where the effects of the bilocal kinetic operator in (10)
\begin{equation}
\hat{\sigma}_{IJ}(\tau,\tau') \equiv \delta_{IJ} \delta(\tau,\tau') \partial_\tau \tag{13}
\end{equation}
can be neglected, the eight fermions interaction vertex in (10) is invariant under the following
time reparametrization transformations
\begin{equation}
\Psi_I^I(\tau_I) \rightarrow (f_I^I(\tau_I))^\Delta \Psi_I^I(f_I^I(\tau_I)), \quad \Delta = \frac{1}{4}, \quad I = L, R, \tag{14}
\end{equation}
for two independent time reparametrization diffeomorphisms, (aka left and right soft modes), $f_I^I(\tau_I), I = L, R$. In fact, it can be checked easily that this transformation corresponds to a change of integration variables in the double integral interaction term in (10). Therefore in any regime where $\hat{\sigma}_{IJ}(\tau,\tau')$ (13) can be neglected, the system develops the time reparametrization symmetry (14), where fermions are primary fields of weight $\Delta = \frac{1}{4}$. Let us notice also that fermions appears in the interaction term in (10) as the following $O(N)$ singlet collective field
\begin{equation}
G_{IJ}(\tau,\tau') = \frac{1}{N} \sum_{i=1}^{N} \Psi_i^I(\tau) \Psi_i^J(\tau'). \tag{15}
\end{equation}
As a consequence of (14), in any regime where the kinetic operator $\hat{\sigma}_{IJ}(\tau,\tau')$ (13) can be ignored, the action is invariant under the following reparametrization of the bilocal field
\begin{equation}
G_{IJ}(\tau_I,\tau_J) \rightarrow (f_I^I(\tau_I)f_J^J(\tau_J))^\Delta G_{IJ}(f_I^I(\tau_I)f_J^J(\tau_J)), \quad \Delta = \frac{1}{4}, \quad I = L, R. \tag{16}
\end{equation}
In order to study $\langle Z(\beta + iT)Z(\beta - iT) \rangle_c$ in the large $N$ limit, it is convenient to recast the functional integral (10) in terms of the collective field $G_{IJ}(\tau,\tau')$ (15) and a corresponding Lagrangian multiplier $\Sigma_{IJ}(\tau,\tau')$, by integrating out the Majorana fermion fields. This is achieved by inserting in the path integral (10) the identity
\begin{equation}
1 = \int \mathcal{D}G_{IJ} \delta \left( G_{IJ}(\tau,\tau') - \frac{1}{N} \sum_{i=1}^{N} \Psi_i^I(\tau) \Psi_i^J(\tau') \right) \nonumber
\end{equation}
\begin{equation}
= \int \mathcal{D}G_{IJ} \int \mathcal{D}\Sigma_{IJ} e^{\Sigma_{IJ}(\tau,\tau')(G_{IJ}(\tau,\tau') - \frac{1}{N} \sum_{i=1}^{N} \Psi_i^I(\tau) \Psi_i^J(\tau'))}, \tag{17}
\end{equation}
where integration on $\Sigma_{IJ}$ is performed along an imaginary direction. By then integrating out fermions one finds
\[
\langle Z(\beta - iT)Z(\beta + iT) \rangle_c = \int \mathcal{D}G_{ij}(\tau,\tau')\mathcal{D}\Sigma_{ij}(\tau,\tau')e^{-I(G_{ij},\Sigma_{ij})},
\]
where
\[
\frac{I(G_{ij},\Sigma_{ij})}{N} = -\frac{1}{2} \log \det (\delta(\tau - \tau')\delta_{ij}\partial_{\tau} - \Sigma_{ij}(\tau,\tau')) + \frac{1}{2} \sum_{i,j} \int_0^T d\tau d\tau' \left( \Sigma_{ij}(\tau,\tau')G_{ij}(\tau,\tau') - \frac{J^2J_{ij}}{4}G_{ij}(\tau,\tau')^4 \right).
\]

Notice that for notational convenience from now on we use lowercase indexes to denote left and right replicas entries, by switching our previous notation \( I = L, R \) to \( i = L, R \). This should not be source of confusion, since fermions have been integrated out. We also use the notation \( J_{ij} = J_iJ_j \) where
\[
J_{ij} = \begin{pmatrix} J_LJ_L & J_LJ_R \\ J_RJ_L & J_RJ_R \end{pmatrix} = \begin{pmatrix} \left( \frac{\beta}{T} + i \right)^2 & \frac{\beta^2}{T^2} + 1 \\ \frac{\beta^2}{T^2} + 1 & \left( \frac{\beta}{T} - i \right)^2 \end{pmatrix},
\]
which follow from the definitions (9).

The two replica action (19) gives the following saddle point Schwinger Dyson equations
\[
\frac{\delta I}{\delta \Sigma_{ij}} = 0,
\]
gives
\[
(\delta(\tau - \tau'')\delta_{ij}\partial_{\tau} - \Sigma_{ij}(\tau,\tau'')) * G_{ij}(\tau'',\tau') = -\delta(\tau - \tau'),
\]
where * is the convolution product
\[
f * g(x) \equiv \int dy f(x - y)g(y),
\]
and
\[
\frac{\delta I}{\delta G_{ij}} = 0,
\]
gives
\[
\Sigma_{ij}(\tau,\tau') = J^2J_{ij}G_{ij}^2(\tau,\tau').
\]

The strong coupling/low temperature condition \( \beta J >> 1 \), where an almost conformal regime holds, it can be extracted by looking at the Fourier transform of eq. (22)
\[
(-i\omega\delta_{ij} - \Sigma_{ij}(\omega))G_{ij}(\omega) = -1.
\]
By taking the self consistent ansatz for the self energy \( \Sigma_{ij}(\omega) = \Sigma_{ij}\sqrt{J\omega} \), where \( \Sigma_{ij} \) is a constant invertible matrix, in the low energy regime \( \omega << J \), the first term in the above equation can be neglected w.r.t. second term, thus giving the following conformal Swinger Dyson equation
\[
\Sigma_{ij}^c(\omega)G_{ij}^c(\omega) = 1.
\]
The above equation in position space reads to
\[ \Sigma^c_{ij}(\tau, \tau'') \ast G^c_{ij}(\tau''', \tau') = \delta(\tau, \tau'). \] (28)

In the conformal limit, the family of replica off diagonal saddles for the SYK spectral form factor are given [1] by the following antiperiodic in $T$ version of the following correlators

\[ G^\beta_{\alpha \alpha}(t) = \frac{1}{(4\pi)^{1/4}\sqrt{J}} \frac{1}{\beta + iT} \left( \frac{\beta - iT}{\beta + iT} \right)^{1/4} \frac{\text{sgn}(t)}{\left[ \frac{\beta_a}{\pi} \sinh \left( \frac{\pi t}{\beta_a} \sqrt{1 + \frac{\beta^2}{T^2}} \right) \right]^{1/2}}, \]
\[ G^\beta_{LR}(t) = \frac{1}{(4\pi)^{1/4}\sqrt{J}} \frac{i}{\beta + iT} \left( \frac{\beta - iT}{\beta + iT} \right)^{1/4} \left[ \frac{\beta_a}{\pi} \cosh \left( \frac{\pi (t+\Delta)}{\beta_a} \sqrt{1 + \frac{\beta^2}{T^2}} \right) \right]^{1/2}, \] (29)

where the missing diagonal entry is given by complex conjugation $G^\beta_{RR} = (G^\beta_{LL})^*$ and $\Delta \in [-T, T]$.

Let us notice that the above expressions correspond to SYK correlators in the conformal limit on the double field thermal state $|TFD\rangle$

\[ |TFD\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_{n=1}^{2N} e^{-\frac{\beta^2}{4}E_n} |E_n\rangle_L \otimes |E_n\rangle_R. \] (30)

at the following effective inverse temperature

\[ \tilde{\beta}(\beta_a, T) = \frac{\beta_a}{\sqrt{1 + \frac{\beta^2}{T^2}}}. \] (31)

In fact the saddle point correlators (29) can be written in the following equivalent form, where the dependence on the effective inverse temperature $\tilde{\beta}$ (31) is emphasized

\[ G^\beta_{LL}(t) = \frac{1}{(J_{LL})^{1/4}} G^c_{\tilde{\beta}}(t), \]
\[ G^\beta_{LR}(t) = \frac{i}{(J_{LR})^{1/4}} \left[ G^c_{\tilde{\beta}} \left( t + \Delta - \frac{i\tilde{\beta}}{2} \right) \right]^*, \] (32)

where

\[ G^\beta_c(t) = \frac{1}{\sqrt{J(4\pi)^{1/4}}} \frac{\text{sgn}(t)}{\left[ \frac{\beta}{\pi} \sinh \left( \frac{\pi t}{\beta} \right) \right]^{1/2}}, \] (33)

is the analytic continuation to real time $\tau = it$ of the regular SYK conformal thermal Green function at inverse temperature $\beta$. Let us notice that the left-right correlator $G^\beta_{LR}(t)$ in (32) is obtained from the diagonal one by the shift in the real time argument $t \to t - i\frac{\beta}{2}$. This is the
standard prescription that allows to express a correlator on the double field thermal state (30) as a thermal correlator on one copy of the system, the latter given by the trace over the thermal density matrix. For a generic operator moved from one copy of the system to the other one, besides the shift on the time argument, one has also to take a CPT conjugation on the operator itself. In the SYK case, CPT conjugation reduces to the identity on Majorana fermions. From the bulk point of view the imaginary time shift by $\pm i\beta$ allows to map points of one side to corresponding points in the other side in a wormhole geometry.

The meaning of the parameter $\beta_a$ is explained in [1], by considering the spectral form factor in the infinite temperature limit and showing that it can be approximated by a family of thermal partition functions labeled by the auxiliary inverse temperature $\beta_a$. From a bulk gravity perspective $\beta_a$ can be understood in the following way [1]. In the $AdS_2$ eternal wormhole, described holographically by the double field thermal state (30), on each of the two thermal Rindler pathces, the metric and the dilaton have the following form

$$ds^2 = -\sinh(\rho)^2d\tilde{t}^2 + d\rho^2, \quad \Phi = \Phi_h \cosh(\rho). \tag{34}$$

In the bulk solution relevant for the spectral form factor there is a Rindler time periodic identification $\tilde{t}_R \sim \tilde{t}_R + \bar{T}$ and $\tilde{t}_L \sim \tilde{t}_L - \bar{T}$. This is compatible with the asymmetric analytic continuation in the spectral form factor $\langle Z(\beta - iT)Z(\beta + iT)\rangle_c$. This Rindler time identifications give rise to a Lorentzian manifold with the topology of a double cone with closed time curves [1]. $\beta_a$ arises in relating Rindler time $\bar{T}$ to boundary time $T$ in the following way. By using an holographic renormalization parameter $\epsilon$ one can relate the boundary proper time $t$ to the bulk Rindler time $\tilde{t}$ from the bulk metric (34) at a large fixed $\rho = \pm \rho_c$

$$dt^2 = \epsilon ds^2|_{\rho=\pm \rho_c}. \tag{35}$$

This gives the following relation between SYK boundary time $t$ and bulk Rindler time $\tilde{t}$

$$t = \epsilon \frac{\rho_c}{2} \frac{\beta_a}{2\pi} \tilde{t}. \tag{36}$$

This relation was also checked numerically [1] by computing geodesics distances from boundaries points in the $AdS_2$ wormhole geometry and getting agreement with (29). The idea is that in the large $N$ limit the bulk theory classicizes and boundary CFT two points correlators reduce to simple functions of the length of the geodesic connecting the pair of points on which the two point function is evaluated. Besides $\beta_a$, there is at least a second phase space parameter, since phase space is always even dimensional. In the case of pure JT gravity the dimension of phase space is two, (see for example [57] for a detailed account). This missing phase space parameter is the compact parameter $\Delta \in [-T,T]$ that appears in the off diagonals saddle correlator $G_{LR}^{\beta_a} = G_{RL}^{\beta_a}$ in (29) and (32) and it is responsible for the linear ramp behavior in the SYK spectral form factor [1]. $\Delta$ corresponds to a relative shift between the time coordinate origins on the right and left boundary. It is not surprise that it appears only on the left-right diagonal correlator in (29), since only there a relative shift on the origins of times coordinates is relevant. The fact that $\Delta$ is responsible for the ramp goes as follows [1]. It turns out that the conformal saddles (29) have zero action. In order to compute the functional integral in the large $N$ limit, one has to still integrate over the phase space parameters $\beta_a$ and $\Delta$. Integration over $\beta_a$ gives just a constant overall constant to the spectral form factor. On the other hand, $\Delta$ integration on $[-T,T]$ gives a linear $T$ factor which reproduces
3 Derivation of the soft modes two Schwarzians effective action

In this section we deduce the leading contribution to the effective action for fluctuations controlled by $f_L(\tau)$ and $f_R(\tau)$ around the conformal saddles (29), (32). Our analysis is inspired by a method developed in [2] for regular SYK. For studying the effects of non conformal perturbations on soft modes, we perform a translation $\Sigma_{ij}(\tau, \tau') \rightarrow \Sigma_{ij}(\tau, \tau') + \hat{\sigma}_{ij}(\tau, \tau')$ in the functional integral (18). This recasts the action (19) in the following equivalent form

$$\frac{I(G_{ij}, \Sigma_{ij})}{N} = -\frac{1}{2} \log \det (-\Sigma_{ij}(\tau, \tau')) + \frac{1}{2} \sum_{i,j} \int_0^T d\tau d\tau' \left( \Sigma_{ij}(\tau, \tau') G_{ij}(\tau, \tau') - \frac{J^2}{4} \hat{j}_{ij} G_{ij}(\tau, \tau')^4 \right) + \frac{1}{2} \sum_{i,j} \int_0^T d\tau d\tau' \hat{\sigma}_{ij}(\tau, \tau') G_{ij}(\tau, \tau'),$$

(37)

where $\hat{\sigma}_{ij}(\tau, \tau') = \delta_{ij} \delta(\tau - \tau') \partial_\tau$ is the replica diagonal kinetic operator in (19). As it was remarked in the previous section, (see the discussion that leads to eq. (16)), the first three terms of the above action form the critical action, which is invariant under the simultaneous time reparametrization $\tau \rightarrow f_1(\tau), \tau' \rightarrow f_2(\tau')$, where $G_{ij}(\tau_1, \tau_2)$ transforms as a primary bilocal field with weights $\Delta = \frac{1}{4}$ and $\Sigma_{ij}(\tau_1, \tau_2)$ transforms with weights $1 - \Delta$.

Conformal saddles (29) or (32) break spontaneously the twofold conformal symmetry down to the diagonal $SL_{diag}(2, \mathbb{R})$ subgroup of $SL(2\mathbb{R})_L \times SL(2\mathbb{R})_R$, a fact that can be checked directly by acting with Moebius transformations on the correlators in (29) or (32). On the other hand, the
source term in (37) breaks explicitly conformal invariance. By treating \( \hat{\sigma}_{ij}(\tau, \tau') \) as a perturbation on the soft modes fluctuations around the conformal saddles, we will derive an effective action in the \( f_R(t) \) and \( f_L(t) \) soft modes. In order to do that we need to regularize \( \hat{\sigma}_{ij}(\tau, \tau') \), in particular by replacing the singular kernel \( \delta(\tau - \tau') \) with a smooth version that accounts for the fact that in the SYK model times shorter then \( 1/J \) cannot be resolved. The regularized operator \( \sigma_{ij}(\tau, \tau') \) contains a smooth kernel with compact support that vanishes for \( |\tau - \tau'| < 1/J \) and for \( |\tau - \tau'| > 2\beta \) in the finite temperature case. We consider the following expansion

\[
\sigma_{ij}(\tau, \tau') = \sum_J a_J \text{sgn}(\tau - \tau') |\tau - \tau'| h_J U^{ij}(\log (J|\tau - \tau'|)),
\]

where the \( h_J = 1 \) contribution gives the effect of the time derivative operator \( \partial_\tau \). However, a priori various scaling weights \( h_J \) might be relevant, in order to account for non linear responses of the perturbation on soft modes fluctuations [2]. In (38) the functions \( U' \)'s are smooth and do vanish for \( \xi = \log (J|\tau - \tau'|) \leq 0 \), which implements the uv cutoff for the impossibility to resolve times intervals that elapse less then \( 1/J \). Moreover, the \( U' \)'s have support on a subregion of the positive \( \xi \) axis. When the infrared cutoff is given by the inverse temperature \( \beta \), they are demanded to vanish for \( \xi \geq \log (2\beta J) \). Finally all the \( U' \)'s are requested to be normalized as follows

\[
\int_{-\infty}^{\infty} d\xi \ U^{ij}_J(\xi) = 1.
\]

Next, we expand the action (37) around the two parameter family of critical saddle points (32)

\[
G_{ij}^c = G_{ij}^{\beta_0} \quad \text{and} \quad \Sigma_{ij}^c = \Sigma_{ij}^{\beta_0}
\]

and keep terms up to quadratic order in the fluctuations. Let us notice that the chosen normalization in front of the fluctuations in (40) preserve the functional integral measure in going from fields to fluctuations. By expanding up to quadratic order in the fluctuations one finds

\[
I = I_c - \frac{1}{12J^2} \langle \delta \Sigma_{ij} | K^{ij}_c | \delta \Sigma_{ij} \rangle + \frac{1}{2} \langle \delta \Sigma_{ij} | \delta G_{ij} \rangle - \frac{3}{4} J^2 \langle \delta G_{ij} | \delta G_{ij} \rangle
\]

where \( I_c \) is the action evaluated in the conformal saddles and replica indexes \( i,j = R,L \) are summed over. In (41), \( \langle f | g \rangle \) denotes standard scalar product on the space of bilocal functions

\[
\langle f | g \rangle \equiv \int d\tau d\tau' f^*(\tau, \tau') g(\tau, \tau'),
\]

while the kernel \( K^{ij}_c \) is given by
\[ K_{ij}(\tau_1, \tau_2; \tau_3, \tau_4) = 3J^2G_{ij}^c(\tau_{12})G_{ij}^c(\tau_{13})G_{ij}^c(\tau_{24})G_{ij}^c(\tau_{34}), \]  

(43)

with \( \tau_{pq} = \tau_p - \tau_q \). This is the analogous of the four point function symmetrized kernel in regular SYK [24]

\[ \tilde{K}(\tau_1, \tau_2; \tau_3, \tau_4) = J^2(q - 1)G_{ij}^{q-2}(\tau_{12})G(\tau_{13})G(\tau_{24})G_{ij}^{q-2}(\tau_{34}). \]  

(44)

One can integrate out either \( \delta G_{ij} \) or \( \delta \Sigma_{ij} \) or both from (41). Integrating out \( \delta \Sigma_{ij} \) yields

\[ \tilde{I} = I - I_c \\
= \frac{3}{4} J^2 \left\langle \delta G_{ij} | (\tilde{K}_{ij}^c)^{-1} - J_{ij} | \delta G_{ij} \right\rangle \\
+ \frac{1}{2} \left\langle \sigma_{ij} | \delta G_{ij}^c \right\rangle + \frac{1}{2} \left\langle \sigma_{ij} | G_{ij}^c \right\rangle. \]  

(45)

By integrating out also the \( \delta G_{ij} \) fluctuations in (45) one is left with the following expression that depends only on the source

\[ \tilde{I} = \frac{1}{12 J^2} \left\langle s_{ij} | \tilde{K}_{ij}^c \left( 1 - J_{ij} \tilde{K}_{ij}^c \right)^{-1} | s_{ij} \right\rangle + \frac{1}{2} \left\langle s_{ij} | G_{ij}^c \right\rangle, \]  

(46)

where \( s_{ij}(\tau, \tau') \) is the following rescaled version of the source \( \sigma_{ij}(\tau, \tau') \) (38)

\[ s_{ij}(\tau, \tau') \equiv \frac{\sigma_{ij}(\tau, \tau')}{|G_{ij}^c(\tau, \tau')|}. \]  

(47)

A soft mode transformation for the conformal saddle correlators (32) is given by

\[ G_{ij}(\tau_1, \tau_2) = (f_1'(\tau_1))^\Delta (f_2'(\tau_2))^\Delta G_{ij}^c(f_1(\tau_1), f_2(\tau_2)), \quad \Delta = \frac{1}{4}, \quad i = L, R. \]  

(48)

It corresponds to a twofold diffeomorphism given by the two independent functions \( f_1(\tau) \) and \( f_2(\tau) \). This is an invariance for the action (37) without the source term \( \sigma_{ij}(\tau, \tau') \). Let us denote as \( \delta_c G_{ij} \) the infinitesimal version of the twofold conformal transformation (48), given by the choice \( f_1(\tau) = \tau + \epsilon_1(\tau) \) and \( f_2(\tau) = \tau + \epsilon_2(\tau) \).

We now show that fluctuations \( \delta_c G_{ij}^c \) along soft modes directions (48) do satisfy the following zero mode equation

\[ (1 - J_{ij} \tilde{K}_{ij}^c) \delta_c G_{ij}^c = 0. \]  

(49)

Let us consider the saddle point equations for the collective fields action (37) without the source term. This action is invariant under (48) and therefore its saddle point equations are by construction invariant under the infinitesimal conformal transformation operator \( \delta_c \). The two saddle point conformal Schwinger-Dyson equations for the connected part of the spectral form factor were discussed in the previous section, eq. (28) and eq. (25). We write them again for convenience.
\[
\Sigma^c_{ij}(\tau, \tau'') * G^c_{ij}(\tau'', \tau') = \delta(\tau, \tau'),
\]
\[
\Sigma_{ij}(\tau, \tau') = J^2 J_{ij} G^3_{ij}(\tau, \tau').
\]

(50)

While the first equation holds only in the conformal limit \( \beta J >> 1 \), the second equation in (50) holds in any regime. Conformal invariance of the first Schwinger Dyson equation in (50) gives

\[
\delta_c \left( \Sigma^c_{ij}(\tau, \tau'') * G^c_{ij}(\tau'', \tau') \right) = \delta(\tau, \tau')
\]

(51)

By taking now the convolution product of this equation by \( G^c_{ij} = (\Sigma^c_{ij})^{-1} \) one finds

\[
\delta_c G^c_{ij} + G^c_{ij} * \delta_c \Sigma^c_{ij} * G^c_{ij} = 0.
\]

(52)

On the other hand, a conformal variation of the second Schwinger Dyson equation in (50) gives

\[
\delta_c \Sigma^c_{ij} = 3 J^2 J_{ij} (G^c_{ij})^2 \delta_c G^c_{ij}.
\]

(53)

If one inserts the above expression in (52) one finds

\[
\delta_c G^c_{ij} + 3 J^2 J_{ij} G^c_{ij} * (G^c_{ij})^2 \delta_c G^c_{ij} * G^c_{ij} = (1 - J_{ij} \tilde{K}^c_{ij}) \delta_c G^c_{ij} = 0.
\]

(54)

which proves the claim (49).

Let us focus now on the action \( \tilde{I} \) for the fluctuations around the spectral form factor conformal saddles, written in the form (46) that depends only on the rescaled source \( s_{ij}(\tau, \tau') \) (47). It appears that a component for the source \( s_{ij}(\tau, \tau') \) in field space along the zero mode direction

\[
(1 - J_{ij} \tilde{K}^c_{ij}) \psi_{ij}(\tau, \tau') = 0
\]

(55)

produces an enhanced effect that naively makes \( \tilde{I} \) in eq. (46) to diverge. Therefore, in order to find enhancing directions in field space for the source \( s_{ij}(\tau, \tau') \) one has to solve the following eigenvectors problem

\[
\tilde{K}^c_{ij} \psi_{ij}(\tau, \tau') = \frac{1}{J_{ij}} \psi_{ij}(\tau, \tau')
\]

(56)

on the space of the antisymmetric bilocal functions \( \psi_{ij}(\tau, \tau') \), and find in particular the corresponding eigenvectors. Actually, since \( \sigma_{ij}(\tau, \tau') \) and its rescaled form \( s_{ij}(\tau, \tau') \) (47) are a regularized version of the replica diagonal kinetic operator \( \delta_{ij}(\tau, \tau') = \delta_{ij} \delta(\tau - \tau') \partial_\tau \), it is therefore natural to assume the regularized source to be diagonal in replica indexes. This leads to restrict (56) to replica diagonal indexes

\[
\tilde{K}^c_{ii} \psi_i(\tau, \tau') = \frac{1}{J_{ii}} \psi_i(\tau, \tau'), \quad i = L, R.
\]

(57)

At this point it is worth to notice that by inserting the explicit form of the spectral form factor conformal correlators written in the convenient form (32) into the definition of \( \tilde{K}^c_{ii} \) (43) one finds the following relation
where $\tilde{K}^{\tilde{\beta}}_c$ is the analytic continuation to Lorentzian time $\tau = it$ of the regular SYK, $q = 4$ symmetrized four point function kernel (44) in the conformal point at inverse temperature $\tilde{\beta}$ (31).

Therefore the eighenvalues equation (57) simplifies to

$$\tilde{K}^{\tilde{\beta}}_c \psi(t, t') = \psi(t, t').$$

(59)

A similar condition occurs in regular SYK [2],[24] (see appendix). The solution of the eighenvalues problem

$$\tilde{K}^{\tilde{\beta}}_c \psi_k(t, t') = k_c \psi_k(t, t').$$

(60)

don the space of bilocal antisymmetric function and in particular, the enhancing eighenfunction $\psi_{k=1}(t, t')$ can be found by the following method. By exploiting the spectral form factor conformal saddle point residual symmetry, given by the diagonal $SL_{diag}(2, \mathbb{R})$ subgroup of $SL_L(2, \mathbb{R}) \times SL_R(2, \mathbb{R})$. The $SL_{diag}(2, \mathbb{R})$ Casimir $C$ commutes with $\tilde{K}^{\tilde{\beta}}_c$, and thus by diagonalizing $C$ one diagonalizes also $\tilde{K}^{\tilde{\beta}}_c$. The Casimir $C$ eighenvalue problem reads

$$C \psi_h(t, t') = h(h-1) \psi_h(t, t').$$

(61)

$C$ has a spectrum with both a continuum and a discrete component [24]. The continuum component is given by the points on the critical strip $h_s = \frac{1}{2} + is$, $s \in \mathbb{R}$, while the discrete component is given by $h_n = 2n$, $n \in \mathbb{N}_{>0}$. Notice that for both the discrete and continuous components of the spectrum, the Casimir $C$ eighenvalues $h(h-1)$ are real. Moreover, the Casimir $C$ eighenfunctions have the following form [24]

$$\Psi_h(\tau_1, \tau_2) = \int d\tau_0 g_h(\tau_0) \frac{\text{sgn}(\tau_{12})}{|\tau_{01}|^h |\tau_{02}|^h |\tau_{12}|^{1-h}},$$

(62)

which corresponds to a generic linear combination of conformal three points functions. Let us notice that the above integral gives an antisymmetric bilocal conformal function with scaling dimension $\Delta_h = \frac{h}{2}$.

By inserting (62) into the $\tilde{K}^{\tilde{\beta}}_c$ eighenvalues equation (60) one finds the explicit form for $k_c(h)$. For the four order coupling $q = 4$ the result is

$$k_c(h) = -\frac{3}{2} \tan\left(\frac{\pi}{2} \left( h - \frac{1}{2} \right) \right).$$

(63)

The discrete component of the spectrum $\text{Spec}(C)$ gives

$$k_c(2n) = \frac{3}{4n - 1} \quad n = 1, 2, \ldots,$$

(64)

and in particular $k_c(2) = 1$ is the eighenvalue with the enhanced effect for (46). On the other hand, the continuum component of the spectrum gives

$$k_c(h_s) = i\frac{3}{2} \tanh\left( \frac{\pi}{2} s \right), \quad s \in \mathbb{R}.$$
Therefore the enhancing component for the rescaled source has the following form

$$s_{ii}(t, t') = a_0 \frac{\text{sgn}(t - t')}{\left[ \frac{\beta}{\pi} \text{sinh}\left( \frac{\pi}{\beta} |\tau - \tau'| \right) \right]^2}.$$  \hspace{1cm} (66)

We are finally in the position to derive from the regularized source term (37) the soft modes effective action that corrects the connected part of the spectral form factor conformal saddle points

$$I_s = \int_0^T dt dt' (G_{LL}(t, t') \sigma_{LL}(t, t') + G_{RR}(t, t') \sigma_{RR}(t, t'))$$
$$= \langle \sigma_{LL}|G_{LL}\rangle + \langle \sigma_{RR}|G_{RR}\rangle$$
$$= \langle s_{LL}|G_{cLL}|G_{LL}\rangle + \langle s_{RR}|G_{cRR}|G_{RR}\rangle.$$  \hspace{1cm} (67)

By a soft modes transformation one finds

$$G_{ii}(t, t') = \frac{1}{(J_{ii})^{1/4}} G_{\beta=\infty}(f_i(t), f_i(t')) f'_i(t) f'_i(t')^\Delta, \quad \Delta = \frac{1}{4} \quad i = L, R$$  \hspace{1cm} (68)

where

$$G_{\beta=\infty}(t, t') = \frac{1}{(4\pi)^{1/4} \sqrt{J} |t - t'|^{1/2}}.$$  \hspace{1cm} (69)

By switching from $(t, t')$ coordinates to $(t_+, t_-)$, slow time and fast time coordinates, $t_+ = \frac{t + t'}{2}$, $t_- = t - t'$, one can Taylor expand (68) at lowest order in $t_-$ as follows

$$G_{ii}(t_+, t_-) = \frac{1}{(J_{ii})^{1/4}} G_{\beta=\infty}(i t_-) \left( 1 + \frac{\Delta}{6} \text{Sch}(f_i(t_+), t_+) t_-^2 + O(t_-^4) \right).$$  \hspace{1cm} (70)

On the other hand, the diagonal components (32) of the conformal spectral form factor Green functions, obtained from (68) for $f_L(t) = \tanh\left( \frac{\pi}{\beta} t \right)$ and $f_R(t) = -\frac{1}{\tanh\left( \frac{\pi}{\beta} t \right)}$ have the explicit form

$$G_{ci}^c(\tau, \tau') = \frac{1}{(J_{ii})^{1/4} (4\pi)^{1/4} \sqrt{J}} \text{sgn}(t - t') \left[ \frac{\beta}{\pi} \text{sinh}\left( \frac{\pi}{\beta} |t - t'| \right) \right]^{1/2}.$$  \hspace{1cm} (71)

By inserting in eq. (67) the short time $t_-$ expansions (70), the rescaled source along the enhanced direction (66) and the conformal Green function (71) at inverse temperature $\tilde{\beta}$ (31) one finds
\[
\frac{I_s}{N} = \frac{a_0 (4\pi)^{-1/4}}{24\sqrt{JJ_{LL}}} \int_0^T dt_+ \text{Sch}(f_L(t_+), t_+) \int_{-T}^T dt_- G_{\beta=\infty}(it_-) t_-^2 \frac{\text{Sgn}(t_-)}{\left| \frac{\beta}{\pi} \sinh \left( \frac{\pi t_-}{\beta} \right) \right|^{2+1/2}} U \left( \log \left( J |t_-| \right) \right) \\
+ L \leftrightarrow R \\
\approx \frac{\alpha_S}{\mathcal{J}} \left( \frac{1}{\sqrt{JJ_{LL}}} \int_0^T dt_+ \text{Sch}(f_L(t_+), t_+) + \frac{1}{\sqrt{JJ_{RR}}} \int_0^T dt_+ \text{Sch}(f_R(t_+), t_+) \right) \int_{-T}^T \frac{dt_-}{|t_-|} U \left( \log \left( J |t_-| \right) \right) \\
= \frac{\alpha_S}{\mathcal{J}} \left( \frac{1}{\sqrt{JJ_{LL}}} \int_0^T dt_+ \text{Sch}(f_L(t_+), t_+) + \frac{1}{\sqrt{JJ_{RR}}} \int_0^T dt_+ \text{Sch}(f_R(t_+), t_+) \right) \\
= \frac{\alpha_S}{\mathcal{J} \left( \frac{\beta}{T} + i \right)} \int_0^T dt \text{Sch}(f_L(t), t) + \frac{\alpha_S}{\mathcal{J} \left( \frac{\beta}{T} - i \right)} \int_0^T dt \text{Sch}(f_R(t), t). 
\]  

(72)

In the above expression, the coefficient \(a_0\) and thus \(\alpha_S\) have to be fitted numerically. The above result agrees with the effective action employed in [1].

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4 Appendix

4.1 Soft modes enhancement: review of the standard SYK case

We review a method to derive in regular SYK the soft mode Schwarzian effective action, according to Kitaev and Suh [2]. The SYK action in terms of collective fields, after a translation \(\Sigma(\tau, \tau') \rightarrow \Sigma(\tau, \tau') + \hat{\sigma}(\tau, \tau')\) can be casted in the following form

\[
I(G, \Sigma) = -\frac{1}{2} \log \det (-\Sigma(\tau, \tau')) \\
+ \frac{1}{2} \int d\tau d\tau' \left( \Sigma(\tau, \tau') G(\tau, \tau') - \frac{J^2}{4} G(\tau, \tau')^4 \right) + \frac{1}{2} \int d\tau d\tau' \hat{\sigma}(\tau, \tau') G(\tau, \tau'), 
\]  

(73)

where \(\hat{\sigma}\) is the SYK kinetic kernel

\[
\hat{\sigma}(\tau, \tau') = \delta(\tau - \tau') \partial_\tau. 
\]  

(74)

Notice that all the terms in (73) except the last source term in \(\hat{\sigma}(\tau, \tau')\), do form a conformal invariant. The last term breaks explicitly conformal invariance and it will be treated as a perturbation on fluctuations around the SYK conformal saddle point, in order to construct the leading contribution to an effective action. Since in the SYK model times shorter then \(1/J\) cannot
be resolved, we replace the kinetic operator $\hat{\sigma}(\tau, \tau')$ that contains the singular Dirac delta kernel $\delta'(\tau - \tau')$ by a regularized version $\sigma(\tau, \tau')$ that contains a smoothed kernel, that in particular vanishes for $|\tau - \tau'| < \frac{1}{J}$.

We assume the following expansion for the regularized kinetic operator $\sigma(\tau, \tau')$

$$\sigma(\tau, \tau') = \sum_J a_J \frac{\text{sgn}(\tau - \tau')}{|\tau - \tau'|} U_J (|J|\tau - \tau'|) ,$$  \hspace{1cm} (75)

where the $h_J = 1$ term has the effect as the time derivative operator $\partial_\tau$, but, a priori various scaling weights $h_J$ might be relevant, due to non linear effects in the responses to the perturbation. In (75) the function $U$’s are the smoothed kernel, they are smooth have compact support and in particular they do vanish for $\xi = \log (|J|\tau - \tau'|) \leq 0$. This implements the uv cutoff for time resolutions less then $1/J$. Moreover, these functions have support on a subregion of the positive $\xi$ axis. In the finite temperature case, the infrared cutoff is given by the inverse temperature $\beta$, and the $U$’s are demanded to vanish for $\xi \geq \log (2\beta J)$ in order to implement the temperature cutoff.

Moreover the following normalization condition of Dirac $\delta(\tau - \tau')$ type is imposed

$$\int_{-\infty}^\infty d\xi U_J(\xi) = 1. \hspace{1cm} (76)$$

In order to study the effects of fluctuations on the SYK conformal saddle $G_c(\tau_1, \tau_2), \Sigma_c(\tau_1, \tau_2)$, the action (73) is expanded up to quadratic order in the fluctuations

$$G = G_c + |G_c|^{-1}\delta G$$

$$\Sigma = \Sigma_c + |G_c|\delta \Sigma ,$$

(77)

Let us notice that (77) preserve the measure in the functional integral in going from fields to fluctuations. By keeping terms up to quadratic order in the fluctuations one gets from (73)

$$I = I_c - \frac{1}{12J^2} \langle \delta \Sigma | \tilde{K}_c | \delta \Sigma \rangle + \frac{1}{2} \langle \delta \Sigma | \delta G \rangle - \frac{3}{4} J^2 J_{ij} \langle \delta G | \delta G \rangle$$

$$+ \frac{1}{2} \langle \sigma | G_c + |G_c|^{-1}\delta G \rangle ,$$

where $I_c$ is the action evaluated in the conformal saddle, $\tilde{K}_c$ is the symmetrized four point function ladder kernel at the conformal point, which for generic $q$ has the following form

$$\tilde{K}_c(\tau_1, \tau_2; \tau_3, \tau_4) = J^2 (q - 1) G_c^{q-2} (\tau_{12}) G_c(\tau_{13}) G_c(\tau_{24}) G_c^{q-2}(\tau_{34}) ,$$

(79)

and $\langle f | g \rangle$ denotes the standard scalar product on the space of antisymmetric bilocal functions

$$\langle f | g \rangle \equiv \int_0^\beta d\tau d\tau' f^*(\tau, \tau') g(\tau, \tau') .$$

(80)
By integrating out $\delta \Sigma_{ij}$ from eq. (78) one then finds

$$\tilde{I} = I - I_c = \frac{3}{4} J^2 \langle \delta G \left( \tilde{K}_c^{-1} - 1 \right) | \delta G \rangle + \frac{1}{2} \left\langle \sigma | \delta G \right| \frac{\sigma}{|G_c|} \rangle + \frac{1}{2} \langle \sigma | G_c \rangle$$

(81)

One can also integrate out the $\delta G$ fluctuation and get

$$\tilde{I} = \frac{1}{12 J^2} \langle s | \tilde{K}_c \left( 1 - \tilde{K}_c \right)^{-1} | s \rangle + \frac{1}{2} \langle s | G_c |^2 \rangle. \tag{82}$$

In this last form for $\tilde{I}$ the perturbation source appears in the following rescaled version

$$s(\tau_1, \tau_2) = \frac{\sigma(\tau_1, \tau_2)}{|G_c(\tau_1, \tau_2)|}. \tag{83}$$

By inspection of (82) it follows that a component in functions space for the rescaled source $s(\tau, \tau')$ along the zero mode direction

$$\left( 1 - \tilde{K}_c \right) \psi_1(\tau, \tau') = 0 \tag{84}$$

gives an enhanced effect on $\tilde{I}$.

This consideration leads to study the eigenfunctions and the spectrum of $\tilde{K}_c$

$$\tilde{K}_c \psi_{k_c}(\tau, \tau') = k_c \psi_{k_c}(\tau, \tau'). \tag{85}$$

In the SYK model around the conformal point this task is made easier by exploiting the $SL(2, \mathbb{R})$ symmetry. In fact, the $SL(2, \mathbb{R})$ Casimir $C$ commutes with $\tilde{K}_c$, and by diagonalizing $C$ one diagonalizes also $\tilde{K}_c$. The Casimir eigenvalues equation

$$C \psi_h(\tau, \tau') = h(h - 1) \psi_h(\tau, \tau') \tag{86}$$

gives rise to a spectrum with both a discrete and a continuum components. The discrete component is given by $h_n = 2n$ with $n \in \mathbb{N}_{>0}$, while the continuum one is given by points on the critical strip $h_s = \frac{1}{2} + is, s \in \mathbb{R}$. The Casimir $C$ eigenfunctions have the following form [24]

$$\Psi_h(\tau_1, \tau_2) = \int d\tau_0 g_h(\tau_0) \frac{\text{sgn}(\tau_{12})}{|\tau_0|^h |\tau_{02}|^h |\tau_{12}|^{1-h}}. \tag{87}$$

This corresponds to a generic linear combination of conformal three points functions. Let us notice that the integral in (87) gives an antisymmetric two variables conformal function with scaling dimension $\Delta_h = \frac{h}{2}$.

By inserting (87) in the $\tilde{K}_c$ eigenvalues equation

$$\tilde{K}_c \psi_h(\tau_1, \tau_2) = k_c(h) \psi_h(\tau_1, \tau_2), \tag{88}$$

leads to the explicit form for the function $k_c(h)$. In the SYK model with $q = 4$ coupling one finds

$$k_c(h) = \frac{3 \tan \left( \frac{\pi}{2} \left( h - \frac{1}{2} \right) \right)}{h - \frac{1}{2}}, \tag{89}$$
In particular $k_c(2) = 1$. This means that there is an enhancement for $\tilde{I}$ (82) whenever the perturbation source $s(\tau_1, \tau_2)$ has a component along the $\Psi_{h=2}(\tau_1, \tau_2)$ direction in functional space. This direction is also the soft mode direction given by infinitesimal time reparametrizations of the conformal two point function $G_c(\tau_1, \tau_2)$, which we denote by $\delta_c G_c(\tau_1, \tau_2)$,

$$(1 - \tilde{K}_c) \delta_c G_c = 0.$$  \hfill (90)

In fact, a soft mode transformation on $G(\tau, \tau')$ is given by

$$G(\tau, \tau') = G_{\beta=\infty}(f(\tau), f(\tau')) f'(\tau)^\Delta f'(\tau')^\Delta, \quad \Delta = \frac{1}{4}$$  \hfill (91)

where

$$G_{\beta=\infty}(\tau, \tau') = \frac{1}{\sqrt{J}} \frac{\text{sgn}(\tau - \tau')}{|\tau - \tau'|^{1/2}}.$$  \hfill (92)

The conformal finite temperature Green function $G_c(\tau, \tau')$ is obtained by (91) for $f(\tau) = \tan\left(\frac{\pi}{\beta} \tau\right)$, its explicit form is given by

$$G_c(\tau, \tau') = \frac{1}{\sqrt{J}} \frac{\text{sgn}(\tau - \tau')}{\left[\frac{\beta}{\pi} \sin\left(\frac{\pi}{\beta} |\tau - \tau'|\right)\right]^{1/2}}.$$  \hfill (93)

An infinitesimal conformal transformation $\delta_c G_c$ is given by the transformation law (91) applied on $G_c(\tau, \tau')$ for a periodic soft mode $f : S^1_\beta \rightarrow S^1_\beta$ closed to the identity map $f(\tau) = \tau + \epsilon(\tau)$. In order to see that eq. (90) holds, let us consider the two conformal Schwinger-Dyson equations that are obtained as saddle point equations from the action (82) without the source term

$$G_c(\tau, \tau'') * \Sigma_c(\tau'', \tau') = \delta(\tau - \tau'),$$
$$\Sigma(\tau, \tau') = J^2 G(\tau, \tau')^3,$$  \hfill (94)

where * denotes the convolution product.

These equations are derived from a time reparametrization invariant action and are therefore themselves time reparametrization invariant. Therefore a variation by $\delta_c$ on both the Schwinger-Dyson equations (94) gives zero. In particular a variation of the first of the two Schwinger Dyson equations gives

$$\delta_c (G_c * \sigma_c) = \delta_c G_c * \Sigma_c + G_c * \delta_c \Sigma_c = 0.$$  \hfill (95)

We then convolute the above equation by $G_c$, which is the inverse of $\Sigma_c$ w.r.t. the convolution product *, $G_c = (\Sigma_c)^{-1}$ and get

$$\delta_c G_c + G_c * \delta_c \Sigma_c * G_c = 0.$$  \hfill (96)

By making an infinitesimal variation along time reparametrization of the second Schwinger Dyson equation in (94) one also finds

$$\delta_c \Sigma_c = 3J^2 J(G_c)^2 \delta_c G_c.$$  \hfill (97)
that inserted in (96) gives
\[ \delta_c G_c + 3J^2 G_c * (G_c)^2 \delta_c G_c * G_c = (1 - \tilde{K}_c) \delta^c G_c = 0. \quad (98) \]

We are ready to derive the Schwarzian soft mode effective action in the finite temperature case from the source term (73)
\[ I_s = \int_0^\beta d\tau d\tau' G(\tau, \tau') \sigma(\tau, \tau') = \langle \sigma | G \rangle = \langle s | G_c | G \rangle. \quad (99) \]

We switch from \((\tau, \tau')\) coordinates to \((\tau_+, \tau_-)\), slow time and fast time coordinates, \(\tau_+ = \frac{\tau + \tau'}{2}\), \(\tau_- = \tau - \tau'\) and compute in (99) the leading effect in \(\tau_-\) in the \(\tau_- \to 0\) limit. By Taylor expanding (91) in powers of \(\tau_-\) one finds
\[ G(\tau_+, \tau_-) = G_{\beta=\infty}(\tau_-) \left( 1 + \frac{\Delta}{6} \text{Sch}(f(\tau_+), \tau_+)(\tau - \tau')^2 + O((\tau - \tau')^4) \right). \quad (100) \]

The finite temperature version of the rescaled source \(s(\tau, \tau')\) (83) along the enhancement direction reads
\[ s(\tau_1, \tau_2) = a_0 \frac{\text{sgn}(\tau - \tau')}{\left[ \frac{\beta}{\pi} \sin \left( \frac{\pi}{\beta} |\tau - \tau'| \right) \right]^2}, \quad (101) \]
and the finite temperature conformal Green function has the form
\[ G_c(\tau, \tau') = \frac{1}{\sqrt{J}} \frac{\text{sgn}(\tau - \tau')}{\left[ \frac{\beta}{\pi} \sin \left( \frac{\pi}{\beta} |\tau - \tau'| \right) \right]^{1/2}}. \quad (102) \]

By inserting in \(I_s\), the rescaled source (99), the leading non trivial order of the \(\tau_-\) expansion (100), and the thermal conformal Green function (102) finally gives
\[ \frac{I_s}{N} = \frac{a_0}{24\sqrt{J}} \int_0^\beta d\tau_+ \text{Sch}(f(\tau_+), \tau_+) \int_{-\beta}^\beta d\tau_- G_{\beta=\infty}(\tau_-) \tau_-^2 \frac{\text{Sgn}(\tau_-)}{\left[ \frac{\beta}{\pi} \sin \left( \frac{\pi}{\beta} |\tau_-| \right) \right]^{2+1/2}} U(\log(J|\tau_-|)) \]
\[ \sim \frac{a_0}{24J} \int_0^\beta d\tau_+ \text{Sch}(f(\tau_+), \tau_+) \int_{-\beta}^\beta d\tau_- \left| \frac{\tau_-}{|\tau_-|} \right| U(\log(J|\tau_-|)) \]
\[ = \frac{\alpha s}{J} \int_0^\beta d\tau \text{ Sch}(f(\tau), \tau). \quad (103) \]

In the above expression, the coefficient \(a_0\) has to be fitted numerically. We omitted an overall normalization \(b = \frac{1}{(4\pi)^{1/4}}\) for the SYK Green functions, which should be included in the definition of the overall coefficients in the last line, (see [2] for full details).
4.2 Green function UV response to a enhancing source

Another interesting calculable effect is the uv response of the Green function to different weights components of a source [2], and in particular to the $h = 2$ enhancing component. Let us consider a generic component for the rescaled source

$$s_I(\tau, \tau') = a_I \frac{\text{sgn}(\tau - \tau')}{|\tau - \tau'| h_I} \int \frac{d\eta}{2\pi} \tilde{U}(\eta) e^{i\eta \xi},$$

(104)

where we used a Fourier integral representation for the smoothed kernel $U(\xi) = U(J|\tau - \tau'|)$. The Fourier transform $\tilde{U}(\eta)$ has a narrow support in $\eta$ since $U(\xi)$ has a broad support in $\xi$.

The above expression can be equivalently written as

$$s_I(\tau, \tau') = a_I \text{sgn}(\tau - \tau') \int \frac{d\eta}{2\pi} \frac{\tilde{U}(\eta)}{|\tau - \tau'| h_I + i\eta},$$

(105)

which shows that the conformal scaling $h_I$ has acquired a narrow imaginary part.

The response on the Green function to this component of the source is then given by

$$g_I(\tau, \tau') = a_I \text{sgn}(\tau - \tau') \int \frac{d\eta}{2\pi} \frac{k_c(h_I + i\eta)}{1 - k_c(h_I + i\eta) |\tau - \tau'| h_I + i\eta} \tilde{U}(\eta),$$

(106)

where $g_I$ is the rescaled Green function $g_I = |G_c| G_I$.

In particular, the enhanced uv response is obtained by expanding up to the first order in $\eta$,

$$K_c(2 + i\eta) \sim 1 + i\eta k_c'(2)$$

$$g_{UV}(\tau, \tau') = a_2 \frac{\text{sgn}(\tau - \tau')}{k_c'(2) |\tau - \tau'|^3/2} \int \frac{d\eta}{2\pi} \frac{\tilde{U}(\eta)}{i\eta} e^{i\eta \xi}$$

(107)

From that it follows that

$$G_{UV}(\tau, \tau') = \frac{g_{UV}}{|G_c|} \sim \alpha_G \frac{\text{sgn}(\tau - \tau')}{|\tau - \tau'|^{3/2}},$$

(108)

with

$$\alpha_G = \frac{a_2}{k_c'(2)}.$$

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