STATIONARY PHASE METHODS
AND THE SPLITTING OF SEPARATRICES

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Abstract. Using stationary phase methods, we provide an explicit formula for the Melnikov function of the one and a half degrees of freedom system given by a Hamiltonian system subject to a rapidly oscillating perturbation. Remarkably, the Melnikov function turns out to be computable without an explicit knowledge of the separatrix and in the case of non-analytic systems. This is related to a priori stable systems coupled with low regularity perturbations. It also applies to perturbations controlled by wave-type equations, so in particular we also illustrate this result with the motion of charged particles in a rapidly oscillating electromagnetic field. Quasi-periodic perturbations are discussed too.

1. Introduction

One of the most fundamental problems in celestial and Hamiltonian mechanics is to ascertain whether a given system is chaotic. Even though this question has been thoroughly studied for over a century, the only existing method to address this question, which dates back to Poincaré, is to consider perturbations of an explicit separatrix of the system (more precisely, a homoclinic or heteroclinic connection associated with a hyperbolic equilibrium) to produce transverse intersections of a stable manifold and an unstable manifold. This is well-known to imply the existence of a chaotic invariant set with positive topological entropy [25]. Under suitable technical hypotheses, the converse implication also holds in low dimension, so the existence of transverse homoclinic connections and positive entropy are in fact equivalent [18].

Again since Poincaré, the way to analyze the intersections of stable and unstable manifolds is through the computation of the so called displacement function, which measures the distance between these manifolds. The definition of the displacement function will be recalled in Section 2. For our present purposes it suffices to keep in mind that when the perturbation can be thought of in terms of a perturbation parameter, the leading order term of the displacement function is usually called the Melnikov function, which is given by an explicit integral that we will write down shortly. A serious difficulty in mechanics is that there are systems, called a priori stable, whose displacement function is exponentially small in the perturbation parameter, so one cannot define a nontrivial Melnikov function and the problem is not amenable to a perturbative analysis. It is well known that the appearance of a priori stable systems is associated with hyperbolic equilibria whose eigenvalues are small in the perturbation parameter or, equivalently modulo a rescaling of the time variable, to perturbations that oscillate rapidly in time.
To introduce the explicit expression of the Melnikov integral, let us for concreteness consider the classical perturbative setting of a one and a half degrees of freedom system of the form

\[ \ddot{x} = f(x) + \varepsilon^r g(x, \dot{x}, t; \varepsilon), \]

where \( r > 0 \) and \( x \) takes values in the real line. This is the kind of systems that one typically gets in the study of perturbations of Hamiltonian systems with one degree of freedom and in many reductions of a three-dimensional system. We can assume that the unperturbed system (\( \varepsilon = 0 \)) has a homoclinic trajectory given by a stable manifold and an unstable manifold of a hyperbolic equilibrium, which one can take to be the origin \((x, \dot{x}) = (0, 0)\). The homoclinic trajectory itself will be denoted by \( x_0(t) \), and the Melnikov integral \( \mathcal{M}(t_0) \) is then the one-variable function defined as [16, Section 4.5]

\[ \mathcal{M}(t_0) := \varepsilon^r \int_{-\infty}^{\infty} \dot{x}_0(t) g(x_0(t), \dot{x}_0(t), t + t_0; \varepsilon) \, dt. \]

When the function \( g \) and all its derivatives are locally bounded uniformly in \( \varepsilon \), it is standard that the Melnikov integral captures the leading order of the displacement function \( D(t_0) \) in the sense that \( \mathcal{M}(t_0) \) is of order \( \varepsilon^r \) and

\[ D(t_0) = \mathcal{M}(t_0) + O(\varepsilon^{r+1}). \]

When the perturbation \( g(x, \dot{x}, t; \varepsilon) \) and all its derivatives are well behaved at \( \varepsilon = 0 \), the evaluation of the Melnikov integral does not present any conceptual difficulties (although, of course, explicit formulas are often impossible to obtain). However, when the perturbation oscillates wildly as \( \varepsilon \to 0 \), the Melnikov function, whose connection with the displacement is also subtler, can become exponentially small and the analysis becomes much more involved [7, 3, 12, 23, 1, 14, 11]. This is not surprising as this highly oscillatory situation corresponds to the case of a priori stable systems. In order to see this, consider the model a priori stable problem of a small amplitude pendulum with a time-periodic perturbation,

\[ \frac{d^2x}{dt^2} = \varepsilon^r \sin x + \varepsilon^{r+2} G \left( x, \frac{dx}{dt} \right) \cos \tau, \]

where \( G(x, v) \) is a smooth function. After rescaling the time variable as \( t := \varepsilon \tau \), the system can be written in the form (1.1) with

\[ f(x) = \sin x, \quad g(x, \dot{x}, t; \varepsilon) := G(x, \varepsilon \dot{x}) \cos \frac{t}{\varepsilon}, \]

so in this particular example the \( C^j \) norm of \( g \) is

\[ \| \varepsilon^r g(\cdot, \cdot; \varepsilon) \|_{C^j} = O(\varepsilon^{r-j}) \]

for \( \varepsilon \) close to 0. More precisely, for all \( R \) one has

\[ C_j \varepsilon^{r-j+l} \leq \sup_{t \in \mathbb{R}, |x|+|\dot{x}| < R} \varepsilon^r |\partial_1^j \partial_2^k \partial_3^l g(x, \dot{x}, t; \varepsilon)| \leq C_2 \varepsilon^{r-j+l} \]

with a positive constant \( C_j \) that depends on \( R \) but not on \( \varepsilon \). For this kind of perturbations, it is standard (see e.g [20]) that the Melnikov function is exponentially small in \( \varepsilon \).

The present paper is centered around the observation that, however, it is possible to analyze the splitting of separatrices in an a priori stable system in the perturbative regime provided that the perturbation features fast oscillations in the
space variables. This is achieved by writing the Melnikov function as an oscillatory integral that can be evaluated using a robust method based on stationary phase arguments. Before explaining this mechanism in detail, let us discuss briefly the meaning of these perturbations. Basically, they correspond again to perturbations that are small in a low norm (say, the uniform norm) but large in higher norms, but with the additional twist that the norm gets larger not only through time derivatives as in (1.4) but also through space derivatives. Therefore, these systems model small perturbations of low regularity, in a quantitative way. These perturbations appear naturally in systems controlled through a PDE (prime examples would be non-smooth geometries generated via Nash–Moser iterations [21] or problems in fluid mechanics [9]), in systems with random Gaussian perturbations [22], or in equations on manifolds defined through diffeomorphisms that are close to the identity in the uniform norm but not in higher norms. The latter arise e.g. in Eliashberg and Mishachev’s approach to Gromov’s h-principle [5], which is central to many questions in contact geometry, and even in the study of magnetic fields generated by knotted wires [10]. For the benefit of the reader, a concrete physical example of a charged particle moving in a rapidly oscillating electromagnetic field is studied in detail in Section 3.

To explain how this kind of perturbation can split the separatrices of an a priori stable system in the perturbative regime, for concreteness let us consider perturbations of the form

\[ g(x, \dot{x}, t; \varepsilon) = q\left(\frac{x}{\varepsilon}, \dot{x}; \varepsilon\right) \cos\frac{t}{\varepsilon}, \]

where the function \( q(\xi, v; \varepsilon) \) is a \( C^1 \) function and its derivatives are locally bounded by \( \varepsilon \)-independent constants. To keep things simple, we will assume that the function \( q(\xi, v; \varepsilon) \) is \( 2\pi \)-periodic in \( \xi \). These perturbations, of course, do not satisfy the bound (1.4), as otherwise the Melnikov function would be exponentially small. It is worth mentioning that a perturbation that does not satisfy that bound either was studied in [15], where the authors took the analytic perturbation of the form

\[ g(x, t; \varepsilon) := \varepsilon^r \sin\left(\frac{x}{1 + (1 - \varepsilon^2) \sin x}\right) \cos\frac{t}{\varepsilon}, \]

which satisfies the uniform bounds

\[ C_1 \varepsilon^{r-2-4l} \leq \|g(\cdot, t; \varepsilon)\|_{C^l} \leq C_2 \varepsilon^{r-2-4l}. \]

Using residues, the authors manage to compute the leading order contribution to the displacement function, which is not exponentially small in \( \varepsilon \). Using harmonic analysis one can see that a significant part of the \( L^2 \) norm of the function is concentrated on frequencies of order \( \varepsilon^{-4} \), and a (straightforward but rather messy) extension of the ideas discussed in this paper could be used to analyze this kind of perturbation without any analyticity assumptions.

The main result of the paper is the following, which provides an explicit formula for the leading order of the displacement function for rapidly oscillating perturbations as above. We will state it in terms of the Fourier coefficients \( q_k(v) \) of \( q(\xi, v; 0) \), defined as

\[ q_k(v) := \frac{1}{2\pi} \int_0^{2\pi} q(\xi, v; 0) e^{-ik\xi} \, d\xi. \]
We will state the theorem in the case of homoclinic connections, but obviously an analogous result holds for heteroclinic connections:

**Theorem 1.1.** Let us consider the system (1.1) with \( r > \frac{5}{6} \) and a perturbation of the form (1.3), where the functions \( f \) and \( g \) are of class \( C^3 \). Suppose that the unperturbed system (\( \varepsilon = 0 \)) has a hyperbolic equilibrium with a homoclinic connection corresponding to a stable manifold and an unstable manifold, which we describe through an integral curve \( x_0(t) \). Setting

\[
C_k := \{ t^* \in \mathbb{R} : \dot{x}_0(t^*) = 1/k \},
\]

assume moreover that \( |f(x_0(t^*))| \neq 0 \) for all \( t^* \in C_k \) and any nonzero integer \( k \). Then the displacement function is

\[
D(t_0) = \varepsilon^{r+\frac{3}{2}} \left( A \cos \frac{t_0}{\varepsilon} + B \sin \frac{t_0}{\varepsilon} \right) + O(\varepsilon^{r+1} + \varepsilon^{2r-2}),
\]

where \( A \) and \( B \) are the real constants

\[
A := \sqrt{2\pi} \Re \left[ \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{t^* \in C_k} \frac{q_k(k^{-1}) e^{ikx_0(t^*)}}{k|k|^{1/2}|f(x_0(t^*))|^{1/2}} \right],
\]

\[
B := \sqrt{2\pi} \Im \left[ \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{t^* \in C_k} \frac{q_k(k^{-1}) e^{ikx_0(t^*)}}{k|k|^{1/2}|f(x_0(t^*))|^{1/2}} \right],
\]

where \( \sigma^* := k f(x_0(t^*))/|k f(x_0(t^*))| \) is plus or minus one, and the cardinality of \( C_k \) is uniformly bounded in \( k \).

This theorem presents three unusual features that should be carefully noticed. Firstly, unlike essentially all the other results in the literature, analyticity is not required: both the unperturbed system and the perturbation can be of class \( C^3 \). Secondly, to compute this formula one does not need to know the separatrix explicitly, which makes it quite versatile. Thirdly, to effectively apply this formula, one only needs to know that \( A \) and \( B \) are not both zero. This condition is satisfied generically, and also computable, so in a concrete example one could even verify this condition by means of a computer assisted proof. The proof of this theorem is presented in Section 2.

Since the main application of the splitting of separatrices is to establish that a certain dynamics is chaotic, it is worth stating the following immediate corollary, which provides a computable generic condition for the existence of positive topological entropy:

**Corollary 1.2.** If the generic condition \( A^2 + B^2 \neq 0 \) is satisfied, the perturbed system exhibits transverse homoclinic intersections for all small enough \( \varepsilon > 0 \). In particular, it has positive topological entropy.

It is worth mentioning that the ideas of the proof of Theorem 1.1 apply to much more general perturbations. As a rule of thumb, one can handle perturbations that oscillate roughly like (1.5), meaning that a significant part of the perturbation (say, as measured with the \( L^2 \) norm) is concentrated over frequencies of order \( 1/\varepsilon \). In particular, in Section 4 we will consider perturbations that are quasiperiodic in \( t \), say of the form

\[
g(x, \dot{x}, t; \varepsilon) = q \left( \frac{x}{\varepsilon}, \frac{\dot{x}}{\varepsilon}; \varepsilon \right) F \left( \frac{\omega_1 t}{\varepsilon}, \ldots, \frac{\omega_n t}{\varepsilon} \right)
\]
with a $C^3$ function $F : \mathbb{T}^n \to \mathbb{R}$, and derive an analogous expression for the displacement function (Theorem 1.1). To keep the technicalities to a minimum, we have chosen not to state the result in the Introduction.

2. Proof of Theorem 1.1

Without any loss of generality we can assume that the hyperbolic equilibrium of the unperturbed system
\[ \dot{x} = f(x) \]
is located at the origin $z = 0$, with
\[ z := (x, \dot{x}) . \]
Likewise, it is convenient to introduce the notation
\[ z_0(t) := (x_0(t), \dot{x}_0(t)) \]
for the integral curve parametrizing the homoclinic connection. Since the function $f$ is of class $C^3$, $z_0$ is of class $C^3$ function of time.

Step 1: The Melnikov function and estimate for the error term. Let us fix a point in the separatrix (understood as a curve in the two-dimensional phase space of coordinates $z = (x, \dot{x})$), for example $Z := z_0(0)$, and consider a normal section to the separatrix at that point, which is given by a segment $\Sigma$ passing through the point $Z$ and parallel to the vector $(-f(x_0(0)), \dot{x}_0(0))$. In the rest of the proof, $K$ will denote a compact subset of the plane containing the homoclinic connection. Since the perturbation is $C^1$-bounded as
\[ \|\varepsilon g(\cdot, \cdot, \cdot, \cdot \varepsilon)\|_{C^1(K \times \mathbb{R})} < C\varepsilon^{-1} \]
because $g$ is of the form (1.2), the invariant manifold theorem [17] ensures that for small enough $\varepsilon$ the perturbed system also has a hyperbolic point $p_{\varepsilon, t_0}$ close to the origin, and that it has a stable manifold and an unstable manifold locally close to the separatrix. Here, with some abuse of notation, we are saying that $p_{\varepsilon, t_0}$ is a hyperbolic point in the sense that it is a fixed point of the return map corresponding to the section $\{t = t_0\}$ for the extended system $\ddot{x} = f(x) + \varepsilon g(x, \dot{x}, t, \varepsilon)$, $t = 1$.

More precisely, the first intersection points $Z^s_{\varepsilon, t_0}$, $Z^u_{\varepsilon, t_0}$ of the stable and unstable manifolds of the perturbed system with the segment $\Sigma$ are at a distance at most $C\varepsilon^{-1}$ of $Z$. Furthermore, the integral curves
\[ z^s_{\varepsilon}(t; t_0) = (x^s_{\varepsilon}(t; t_0), \dot{x}^s_{\varepsilon}(t; t_0)), \quad z^u_{\varepsilon}(t; t_0) = (x^u_{\varepsilon}(t; t_0), \dot{x}^u_{\varepsilon}(t; t_0)) \]
of the perturbed system, defined through the initial conditions
\[ z^s_{\varepsilon}(t_0; t_0) = Z^s_{\varepsilon, t_0}, \quad z^u_{\varepsilon}(t_0; t_0) = Z^u_{\varepsilon, t_0}, \]
satisfy
\[ \sup_{t > t_0} |z^s_{\varepsilon}(t; t_0) - z_0(t - t_0)| + \sup_{t < t_0} |z^u_{\varepsilon}(t; t_0) - z_0(t - t_0)| < C\varepsilon^{-1} . \]
In the case of the unperturbed system, notice that the eigenvalues of the hyperbolic equilibrium are of the form $\pm \lambda$ with $\lambda > 0$ a constant independent of $\varepsilon$, so it is well known that the unperturbed integral curve satisfies
\[ |x_0(t)| + |\dot{x}_0(t)| + |\ddot{x}_0(t)| < C e^{-\lambda|t|} \]
for all $t$. 


It is standard [15] Section 4.5] that the displacement function is defined as
\begin{equation}
\mathcal{D}(t_0) := \alpha [z^u_{\varepsilon}(t_0; t_0) - z^s_{\varepsilon}(t_0; t_0)] ,
\end{equation}
where for later convenience we have introduced an inessential nonzero constant independent of the perturbation parameter,
\[ \alpha := \left[ \dot{x}_0(0)^2 + f(x_0(0))^2 \right]^{\frac{1}{2}} . \]
Since the perturbations that we are considering are rapidly oscillating, the connection between the Melnikov integral and the displacement function is rather different from the usual one:

**Proposition 2.1.** The Melnikov function determines the leading order contribution to the displacement function in the sense that
\[ |\mathcal{D}(t_0) - \mathcal{M}(t_0)| \leq C\varepsilon^{2r-2} \]
uniformly on compact time intervals.

**Proof.** To derive the formula, let us write the perturbed equation as a first order differential equation
\[ \dot{z} = X_f(z) + \varepsilon^r X_g(z, t; \varepsilon) , \]
where we are using the notation
\[ X_f(z) := (\dot{x}, f(x)) , \quad X_g(z, t; \varepsilon) := (0, g(x, \dot{x}, t, \varepsilon)) . \]
It is clear that
\[ \mathcal{D}(t_0) = \mathcal{D}^u(t_0; t_0) - \mathcal{D}^s(t_0; t_0) , \]
where the time-dependent displacement functions are defined as
\[ \mathcal{D}^u(t; t_0) := X_f(z_0(t - t_0)) \times (z^u_{\varepsilon}(t; t_0) - z_0(t - t_0)) , \]
\[ \mathcal{D}^s(t; t_0) := X_f(z_0(t - t_0)) \times (z^s_{\varepsilon}(t; t_0) - z_0(t - t_0)) \]
and we have defined the cross product of two vectors in \( \mathbb{R}^2 \) as the scalar quantity
\[ X \times Y := X_1Y_2 - X_2Y_1 . \]

Taking the case of the unstable manifold for concreteness, let us compute the derivative of \( \mathcal{D}^u \) with respect to \( t \). To keep the expressions simple, we will omit the arguments of the functions when no confusion may arise:
\[ \mathcal{D}^u = DX_f(z_0) \dot{z}_0 \times (z^u_{\varepsilon} - z_0) + X_f(z_0) \times (\dot{z}^u_{\varepsilon} - \dot{z}_0) \]
\[ = DX_f(z_0) \dot{z}_0 \times (z^u_{\varepsilon} - z_0) + X_f(z_0) \times [X_f(z^u_{\varepsilon}) + \varepsilon^r X_g(z^u_{\varepsilon}) - X_f(z_0)] \]
\[ = DX_f(z_0) \dot{z}_0 \times (z^u_{\varepsilon} - z_0) + X_f(z_0) \times [DX_f(z_0)(z^u_{\varepsilon} - z_0) + \varepsilon^r X_g(z_0)] + \mathcal{R} \]
\[ = (\text{tr} DX_f) \dot{z}_0 \times (z^u_{\varepsilon} - z_0) + \varepsilon^r X_f(z_0) \times X_g(z_0) + \mathcal{R} \]
\[ = \varepsilon^r \dot{x}_0(t - t_0) g(x_0(t - t_0), \dot{x}_0(t - t_0), t, \varepsilon) + \mathcal{R} , \]
where the trace of \( DX_f \) (which is zero) appears because
\[ (\text{tr} DX_f) \dot{z}_0 \times (z^u_{\varepsilon} - z_0) = DX_f(z_0) \dot{z}_0 \times (z^u_{\varepsilon} - z_0) + X_f(z_0) \times [DX_f(z_0)(z^u_{\varepsilon} - z_0)] \]
and the error term,
\[ \mathcal{R} := X_f(z_0) \times [X_f(z^u_{\varepsilon}) - X_f(z_0) - DX_f(z_0)(z^u_{\varepsilon} - z_0)] \]
\[ + \varepsilon^r X_f \times [X_g(z^u_{\varepsilon}) - X_g(z_0)] , \]
is bounded for \( t \leq t_0 \) as
\[
|R| \leq C\|X_f\|_{C^2(K \times R)}|X_f||z_u - z_0|^2 + C\varepsilon^r\|X_g\|_{C^1(K \times R)}|X_f||z_u - z_0| \\
\leq C\varepsilon^{2r-2}|X_f|,
\]
where we have used the bounds (2.2). The stable displacement function can be estimated for \( t \geq t_0 \) using a completely analogous argument. Since \( f(0) = 0 \), using the bound (2.3) together with the obvious fact that
\[
\lim_{t \to \infty} D_s(t; t_0) = \lim_{t \to -\infty} D_u(t; t_0) = 0,
\]
we can integrate the above expression for the derivative of the stable and unstable parts of the displacement function to obtain
\[
|D(t_0) - M(t_0)| \leq C\varepsilon^{2r-2},
\]
as claimed. \( \square \)

Remark 2.2. In what follows we will only need a uniform bound for the difference \( D(t_0) - M(t_0) \), which is what we prove. The method of proof, however, permits to estimate higher order derivatives too if \( f \) and \( g \) are regular enough, and one should notice that they are not of order \( \varepsilon^{2r-2} \), even if \( f \) and \( g \) are analytic, due to the rapid oscillations of the perturbation.

Step 2: Estimates for the critical points of the phase functions. Let us now write the Melnikov integral as
\[
M(t_0) = \varepsilon^r \int_{-\infty}^{\infty} \dot{x}_0(t) q \left( \frac{x_0(t)}{\varepsilon}, \dot{x}_0(t); \varepsilon \right) \cos \frac{t + t_0}{\varepsilon} \, dt \\
= \widetilde{M}(t_0) + R_1(t_0),
\]
where
\[
\widetilde{M}(t_0) := \varepsilon^r \int_{-\infty}^{\infty} \dot{x}_0(t) q \left( \frac{x_0(t)}{\varepsilon}, \dot{x}_0(t); 0 \right) \cos \frac{t + t_0}{\varepsilon} \, dt
\]
and the error
\[
R_1(t_0) := \varepsilon^r \int_{-\infty}^{\infty} \dot{x}_0(t) \left[ q \left( \frac{x_0(t)}{\varepsilon}, \dot{x}_0(t); \varepsilon \right) - q \left( \frac{x_0(t)}{\varepsilon}, \dot{x}_0(t); 0 \right) \right] \cos \frac{t + t_0}{\varepsilon} \, dt
\]
is uniformly bounded as
\[
|R_1(t_0)| \leq \varepsilon^r|q(\cdot, \cdot; \varepsilon) - q(\cdot, \cdot; 0)|_{C^0(K)} \int_{-\infty}^{\infty} |\dot{x}_0(t)| \, dt \leq C\varepsilon^{r+1}
\]
because \( q(\xi, v; \varepsilon) \) depends differentiably on \( \varepsilon \) and \( |\dot{x}_0(t)| \) falls off exponentially by (2.3).

Writing \( \widetilde{M}(t_0) \) in terms of the Fourier coefficients \( q_k(v) \) of \( q(\xi, v; 0) \), we have
\[
\widetilde{M}(t_0) := \varepsilon^r \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \dot{x}_0(t) q_k(\dot{x}_0(t)) e^{i k x_0(t)/\varepsilon} \cos \frac{t + t_0}{\varepsilon} \, dt.
\]
Setting
\[
\varphi_k(t) := k x_0(t) - t,
\]
this yields
\[
\tilde{M}(t_0) = \varepsilon^r \sum_{k = -\infty}^{\infty} \text{Re} \left[ e^{-it_0/\varepsilon} \int_{-\infty}^{\infty} \dot{x}_0(t) q_k(t) e^{i\phi_k(t)/\varepsilon} dt \right] = \varepsilon^r \sum_{k = -\infty}^{\infty} \text{Re}[e^{-it_0/\varepsilon} m_k].
\]
(2.7)

Our goal is to compute the constants \( m_k \) to leading order in \( \varepsilon \). Notice that the set \( C_k \) defined in the statement of Theorem 1.1 is the locus of the critical points of the phase function \( \phi_k \). We will need the following estimates for the behavior of the phase function \( \phi_k \) around points in the set \( C_k \):

**Lemma 2.3.** There is a number \( N \), independent of \( k \) and \( \varepsilon \), such that \( C_k \) consists of at most \( N \) points \( t_{k,j}^* \). Moreover, there are positive constants \( \eta, c, C \) and \( T_0 \), independent of \( k \) and \( \varepsilon \), such that the function

\[
\phi_k(t) := kx_0(t) - t
\]
(2.8)

satisfies

\[
|\dot{\phi}_k(t)| > c
\]
(2.9)

for any \( k \) provided that \( t \) does not lie on any of the pairwise disjoint intervals \( I_{k,j} := (t_{k,j}^* - 2\eta, t_{k,j}^* + 2\eta) \). Furthermore, for all \( t \in I_{k,j} \) one has:

(i) If \( |t_{k,j}^*| \leq T_0 \),

\[
|\dot{\phi}_k(t)| \geq c|k||t - t_{k,j}^*|, \quad |\ddot{\phi}_k(t_{k,j}^*)| > c|k|, \quad |\dot{\phi}_k(t)| + |\ddot{\phi}_k(t)| < C|k|.
\]

(ii) If \( |t_{k,j}^*| > T_0 \),

\[
|\dot{\phi}_k(t)| \geq c|t - t_{k,j}^*|, \quad |\ddot{\phi}_k(t_{k,j}^*)| > c, \quad |\dot{\phi}_k(t)| + |\ddot{\phi}_k(t)| < C.
\]

**Proof.** Let us start by noticing that \( C_0 \) is the empty set, so in what follows we will assume that \( k \neq 0 \). A first observation is that \( C_k \) is a discrete set. This follows from the fact that the zeros of the function

\[
\dot{x}_0(t^*) - \frac{1}{k}
\]
(2.10)
(which are the points in \( C_k \)) are non-degenerate, as its derivative

\[
\ddot{x}_0(t^*) = f(x_0(t^*))
\]
(2.11)
is nonzero whenever \( t^* \in C_k \) by hypothesis. Hence, there is a finite number of zeros of Equation (2.10) on any compact interval, for any given \( k \). We will show next that the bound is uniform in \( k \) by considering the case \( |k| \to \infty \).

The key is to show that the solutions to the limit equation

\[
\dot{x}_0(t) = 0
\]
(2.12)
are finite. Notice that the derivative of this function, also given by (2.11), is nonzero on the solutions of (2.12) because there cannot be any equilibria on the homoclinic connection (recall that this simply means that

\[
(\dot{x}_0(t), f(x_0(t))) \neq (0, 0)
\]
for all \( t \in \mathbb{R} \)). This implies that the set of solutions of (2.12), which we call \( \{T^*_m\} \) is finite on any compact subset of the real line. Restricting our attention to a
compact interval $|t^*| < T$, it is then a straightforward consequence of the implicit
function theorem and the analysis of the limit equation (2.12) that the number
of zeros of (2.10) with $|t^*| < T$ remains bounded as $|k| \to \infty$. Notice that it
follows from this argument that the intersection of the set $C_k$ with any large finite
interval $[-T, T]$ (with $T$ independent of $k$ and $\varepsilon$) converges as $|k| \to \infty$ to the set of
solutions of the limit equation (2.12), so in particular the cardinality of $C_k \cap [-T, T]$
is bounded uniformly in $k$. Notice moreover that, since
\[
\dot{\varphi}_k(t) = k \dot{x}_0(t) - 1, \quad \ddot{\varphi}_k(t) = k f'(x_0(t)) \dot{x}_0(t),
\]
one immediately infers the bounds presented in item (i).

Hence we can concentrate on controlling what happens for large $|t|$. Specifically,
we will show that the number of zeros of (2.10) with $t^* > T$ or $t^* < -T$
is at most one, for large enough $T$. To this end, let us recall that, since $x_0(t)$ is a
homoclinic connection of a hyperbolic equilibrium, it is standard that there are
nonzero constants $c_+, c_-$ such that
\[
\dot{x}_0(t) = \begin{cases} 
\lambda c_+ e^{-\lambda t} (1 + \varepsilon_+(t)) & \text{for } t > 0, \\
\lambda c_- e^{\lambda t} (1 + \varepsilon_-(t)) & \text{for } t < 0,
\end{cases}
\]
where the errors $\varepsilon_{\pm}(t)$ are bounded as
\[
|\varepsilon_{\pm}(t)| + |\dot{\varepsilon}_{\pm}(t)| + |\ddot{\varepsilon}_{\pm}(t)| < C e^{-\lambda |t|}
\]
as $t \to \pm \infty$. In order to see this, observe that, by the stable manifold theorem and
the fact that $f$ is of class $C^3$, the unstable manifold contained in the homoclinic
connection of the unperturbed system can be locally written as a $C^3$ graph in the
$(x, \dot{x})$-plane, that is,
\[
\dot{x} = h(x)
\]
with $h$ a $C^3$ function defined in a neighborhood of 0 with the asymptotic behavior
\[
h(x) = \lambda x + O(x^2).
\]
As a hyperbolic one-dimensional system of class $C^3$ can be conjugated to its linear
part through a $C^3$ diffeomorphism, there is a local variable $y := \Phi(x)$
in a neighborhood of 0, with
\[
\Phi \in C^3, \quad \Phi(0) = 0, \quad \Phi'(0) = 1,
\]
such that the integral curve describing the unstable manifold can be written in this
variable as
\[
y(t) = c_- e^{\lambda t}.
\]
Inverting the diffeomorphism to write the unstable manifold in the original vari-
able $x$, one readily obtains that for $t \to -\infty$ it is parametrized as
\[
x(t) = c_- e^{\lambda t}[1 + O(e^{\lambda t})],
\]
where the small term can be differentiated three times with the same bound. This
proves the bound (2.14) for $\varepsilon_-(t)$. The argument for the stable manifold is analo-
gous.

An immediate consequence of the estimate (2.13) is that $\dot{x}_0(t) \neq 0$ for all large
enough $|t|$. Therefore, we infer that the number of solutions of Equation (2.12) is
finite. To control the solutions with large $t^*$, let us rewrite the equation for the zeros of (2.10) as

$$c_+ \lambda e^{-\lambda t^*} (1 + E_+(t^*)) = \frac{1}{k}.$$  

Here we are supposing that $k$ is a large enough number of the same sign as $c_+$, since otherwise there cannot be any solutions with $t^*$ larger than some fixed constant $T$ (independent of $\varepsilon$ and $k$) by the bounds on $E_+(t^*)$.

If one takes out the error $E_+(t^*)$, it is clear that the equation

$$c_+ \lambda e^{-\lambda t^*} = \frac{1}{k}$$  

then has a unique solution with $t^* > T$, given in terms of a logarithm. A judicious application of the inverse function theorem then shows that (2.15) has exactly one solution with $t^* > T$ too. To see this, define a new variable $	au := e^{-\lambda t^*}$ and let $E_+(\tau)$ be the expression in this variable of the function $e^{-\lambda t^*} E_+(t^*)$. The bounds (2.14) ensure that $|E_+(\tau)| \leq C\tau^2$, $|E'_+(\tau)| \leq C\tau$ for small positive $\tau$. As Equation (2.15) can be written as

$$c_+ [\lambda \tau + E_+(\tau)] = \frac{1}{k},$$  

the existence of exactly one solution in the interval follows from our knowledge of Equation (2.16). The analysis for $t^* < -T$ is completely analogous. Notice that, as a byproduct of the above argument, if $t^*_{k,j} > 0$ one infers that

$$t^*_{k,j} = \frac{1 + e^*_{k,j}}{\lambda} \log(c_+ \lambda k), \quad |e^*_{k,j}| < \frac{C}{\log |k|},$$

which tends to infinity for large $|k|$, and

$$(2.17) \quad x_0(t^*_{k,j}) = -\frac{1 + e^*_{k,j}}{\lambda k}, \quad |e_{k,j}| < \frac{C}{|k|},$$

which tends to the equilibrium, located at $0$. Of course, the preceding argument also shows that, if $|t - t^*_{k,j}| < c_0$ with $c_0$ a small uniform constant, then

$$(2.18) \quad \frac{1}{C|k|} < |x_0(t)| + |\dot{x}_0(t)| + |\ddot{x}_0(t)| + |\dddot{x}_0(t)| < \frac{C}{|k|}.$$  

We record this fact here for later use.

The previous analysis shows that the cardinality of $C_k$ is uniformly bounded and that the distance between the points $t^*_{k,j} \in C_k$ is larger than some positive uniform constant. Let us now prove the uniform bound (2.14). Since $f(x_0(t^*_{k,j})) \neq 0$ by hypothesis, it suffices to obtain uniform bounds when $|k| \to \infty$ or $|t^*| \to \infty$. Of course, if $|k|$ is large but $t^*$ remains in a compact interval, it follows from the above comparison argument with the limit equation (2.12) that the uniform bound (2.9) holds, so we can focus on the case of large $|t^*|$. Assuming for concreteness that $t > T$, with $T$ a large positive constant, one can now use the asymptotic formula (2.13) to write

$$k\dot{x}_0(t) - 1 = k\lambda c_+ e^{-\lambda t} (1 + E_+(t)) - 1.$$  

If $t = t^*_{k,j} + s$, using that

$$k\lambda c_+ e^{-\lambda t^*_{k,j}} (1 + E_+(t^*_{k,j})) = 1$$  

(2.19)
it is possibly to simplify the above expression to obtain
\[ k \dot{x}_0(t) - 1 = e^{-\lambda_s (1 + \mathcal{E}_+(t_{k,j}^* + s))} - 1. \]

In view of the bound (2.13), it is now clear that
\[ |\dot{\varphi}_k(t)| = |k \dot{x}_0(t) - 1| > c \]
if \(|s|\) is larger than some uniform constant, say \(2\eta\). This proves the estimate (2.9).

Finally, we differentiate the asymptotic expression (2.13) and employ the formula (2.19) to obtain
\[ \ddot{\varphi}_k(t_{k,j}^*) = k \ddot{x}_0(t_{k,j}^*) = -\lambda + O(e^{-\lambda t_{k,j}^*}), \]
and thus the desired lower bounds
\[ |\ddot{\varphi}_k(t_{k,j}^*)| > c \quad \text{and} \quad |\dot{\varphi}_k(t)| > c|t - t_{k,j}^*| \]
for all \(t \in I_{k,j}\). Furthermore, the above asymptotic expression also ensures that
\[ |\dot{\varphi}_k(t)| \leq C \]
for all \(t \in I_{k,j}\), which implies the estimates presented in item (ii). The lemma then follows.

\[ \square \]

Step 3: The oscillatory integrals of the Melnikov function. Let us consider the intervals \(I_{k,j}\) introduced in Lemma 2.3 and take a smooth compactly supported function \(\chi_{k,j}(t)\) that is equal to 1 on the interval \([t_{k,j}^* - \eta, t_{k,j}^* + \eta]\) and is supported in the larger interval \(I_{k,j}\). Since \(\eta\) does not depend on \(\varepsilon\) and \(k\), the functions \(\chi_{k,j}\) and their derivatives can be assumed to be bounded by constants independent of \(k\) and \(\varepsilon\). We will also set
\[ \chi_{k,0}(t) := 1 - \sum_{j=1}^{N_k} \chi_{k,j}(t), \]
where \(N_k\) is the cardinality of \(\mathcal{C}_k\).

Let us now write the integral \(m_k\) in (2.7) as
\[ m_k = \sum_{j=0}^{N_k} m_{k,j}, \]
where
\[ m_{k,j} := \int_{-\infty}^{\infty} \dot{x}_0(t) q_k(\dot{x}_0(t)) e^{i\varphi_k(t)/\varepsilon} \chi_{k,j}(t) \, dt \]
Our first goal is to estimate the terms \(m_0\) and \(m_{k,0}\), where the phase function does not have any critical points in the region where the integrand is nonzero. In the proof of these estimates we will use that [13, Theorem 3.3.9], as the function \(q(\xi, v; 0)\) is of class \(C^3(K)\), its Fourier coefficients decay as
\[ (2.20) \quad \|q_k\|_{C^l(K_2)} \leq CM (1 + |k|)^{l-3} \]
for all \(0 \leq l \leq 3\), where
\[ M := \|q\|_{C^3(K)} \]
and the constant \(C\) does not depend on \(q\). Here and in what follows, \(K_l \subset \mathbb{R}\) denotes the projection of the set \(K \subset \mathbb{R}^2\) on the \(l\)th coordinate, with \(l = 1, 2\).
Lemma 2.4. The terms \( m_0 \) and \( m_{k,0} \) with \( k \neq 0 \) are bounded as
\[
|m_0| < CM\varepsilon, \quad |m_{k,0}| < CM\varepsilon|k|^{-2}.
\]

Proof. Let us begin with \( m_0 \). As \( \varphi_0(t) = -t \) and \( \dot{x}_0 = f(x_0) \), one can integrate by parts to show
\[
|\dot{x}_0(t) q_0(\dot{x}_0(t)) e^{-it/\varepsilon} dt | = \varepsilon \int_{-\infty}^{\infty} e^{-it/\varepsilon} \frac{d}{dt} \left( \dot{x}_0(t) q_0(\dot{x}_0(t)) \right) dt \leq \varepsilon \int_{-\infty}^{\infty} |f(x_0(t))| \left( |q_0(\dot{x}_0(t))| + |\dot{x}_0(t) q_0'(\dot{x}_0(t))| \right) dt \leq CM\varepsilon.
\]

Let us pass now to \( m_{k,0} \) with \( k \neq 0 \). Since \( |\dot{\varphi}_k(t)| > c \) on the support of \( \chi_{k,0} \) by Lemma 2.3 one can use the identity
\[
e^{-i\dot{\varphi}_k(t)/\varepsilon} = -i\varepsilon \frac{d}{dt} e^{-i\dot{\varphi}_k(t)/\varepsilon}
\]
and the exponential fall off (2.3) to integrate by parts, which permits to write the integral for \( m_{k,0} \) as
\[
m_{k,0} = -i\varepsilon \int_{-\infty}^{\infty} \frac{d}{dt} \left( \frac{\dot{x}_0(t) q_k(\dot{x}_0(t)) \chi_{k,0}(t)}{\dot{\varphi}_k(t)} \right) e^{i\dot{\varphi}_k(t)/\varepsilon} dt.
\]
The derivative under the integral sign is
\[
\frac{\ddot{x}_0 q_k \chi_{k,0} + \dot{x}_0 \ddot{q}_k \chi_{k,0} + \dot{x}_0 q_k \dot{\chi}_{k,0} - \frac{\ddot{q}_k \dot{x}_0 q_k \chi_{k,0}}{\dot{\varphi}_k^2}}{\dot{\varphi}_k^2},
\]
so we get the obvious bound
\[
|m_{k,0}| \leq C \varepsilon \frac{\|q_k\|_{C^1(K_2)} \|\chi_{k,0}\|_{C^1(\mathbb{R})}}{c} \int_{-\infty}^{\infty} \left( |\ddot{x}_0| + |\dot{x}_0 \ddot{x}_0| + |\dot{x}_0| \right) dt
\]
\[
+ C\varepsilon \frac{|k||q_k|_{C^0(K_2)} \|\chi_{k,0}\|_{C^0(\mathbb{R})}}{c^2} \int_{-\infty}^{\infty} |\dot{x}_0| dt
\]
where we have employed that
\[
|\dot{\varphi}_k| = |k\ddot{x}_0| = |kf(x_0)| < C|k|
\]
and \( |\dot{\varphi}_k| > c \) by (2.9). Using now the bound (2.20) for the decay of the Fourier coefficients, the lemma follows. \( \square \)

Let us now pass to estimate the numbers \( m_{k,j} \):

Lemma 2.5. For all \( j \in C_k \),
\[
m_{k,j} = \left( \frac{2\pi \varepsilon}{|f(x_0(t_{k,j}^*))|} \right)^j \frac{q_k(k^{-1})}{k|k|^{1/2}} e^{ikx_0(t_{k,j}^*) - \alpha_k(t_{k,j}^*) + i\sigma_k(t_{k,j}^*)} + R_{k,j},
\]
where
\[
\sigma_{k,j} := \frac{kf(x_0(t_{k,j}^*))}{|kf(x_0(t_{k,j}^*))|}
\]
is plus or minus one and the error is bounded as
\[ |R_{k,j}| < C M \varepsilon |k|^{-5/2}. \]

**Proof.** The proof is a little different depending on whether \(|t_{k,j}^*|\) is larger than the uniform constant \(T_0\) introduced in Lemma 2.3 or not:

**First case:** \(|t_{k,j}^*| \leq T_0\). Let us consider a function \(\chi_{k,j}(t)\) that is equal to 1 if \(|t - t_{k,j}^*| < \frac{1}{2}c_0|k|^{-1/2}\), vanishes identically for \(|t - t_{k,j}^*| > c_0|k|^{-1/2}\), and is bounded as
\[ \sup_{t \in \mathbb{R}} \left( |\chi_{k,j}^1(t)| + |k|^{-\frac{1}{2}} |\chi_{k,j}^1(t)| + |k|^{-1} |\chi_{k,j}^1(t)| \right) < C, \]
where \(c_0 < \eta\) and \(C\) are uniform constants. We will also set
\[ \chi_{k,j}^2(t) := \chi_{k,j}(t) - \chi_{k,j}^1(t). \]

This allows us to write
\[ m_{k,j} = m_{k,j}^1 + m_{k,j}^2, \]
where we have set
\[ m_{k,j}^1 := \int_{-\infty}^{\infty} \bar{x}_0(t) q_k(\bar{x}_0(t)) e^{i\varphi_k(t)/\varepsilon} \chi_{k,j}^1(t) \, dt \]
with \(l = 1, 2\). Using an integration by parts argument similar to that of Lemma 2.4, one can readily write
\[ m_{k,j}^2 = -i \varepsilon \int_{-\infty}^{\infty} \bar{x}_0 \chi_{k,j} \frac{q_k(\bar{x}_0(t))}{\varphi_k} \frac{d}{dt} e^{i\varphi_k/\varepsilon} \, dt = \int_{-\infty}^{\infty} \frac{d}{dt} \left( \frac{\bar{x}_0 \chi_{k,j}^2 q_k(\bar{x}_0(t))}{\varphi_k} \right) e^{i\varphi_k/\varepsilon} \, dt. \]

Hence one can utilize the estimates for \(\varphi_k(t)\) presented in item (i) of Lemma 2.3 to bound this term as
\[ |m_{k,j}^2| \leq C \varepsilon \left[ \|q_k\|_{C^0(K_2)} |k|^{-\frac{1}{2}} + \|q_k\|_{C^1(K_2)} |k|^{-1} \right] \int_{\frac{2\varepsilon}{|k|} < |s| < 2\eta} \frac{ds}{s} \]
\[ + C \varepsilon \|q_k\|_{C^0(K_2)} |k|^{-\frac{1}{2}} \int_{\frac{2\varepsilon}{|k|} < |s| < 2\eta} \frac{ds}{s^2} \]
\[ \leq C \varepsilon \left[ \|q_k\|_{C^0(K_2)} |k|^{-\frac{1}{2}} + \|q_k\|_{C^1(K_2)} \right] \frac{\log(1 + |k|)}{|k|}. \]

To see why these integrals appear in the estimate, notice that the way one bounds each of the terms in the above integral using Lemma 2.3 (together with the bounds (2.21) and (2.23)) is as follows:
\[ \left| \int_{-\infty}^{\infty} \frac{\bar{x}_0 \chi_{k,j}^2 q_k}{\varphi_k} e^{i\varphi_k/\varepsilon} \, dt \right| \leq C \|q_k\|_{C^0(K_2)} \left| \chi_{k,j}^2 \right|_{C^0} |k|^{-\frac{1}{2}} \int_{\frac{2\varepsilon}{|k|} < |t - t_{k,j}^*| < 2\eta} \frac{dt}{|t - t_{k,j}^*|} \]
\[ \leq C \|q_k\|_{C^0(K_2)} |k|^{-\frac{1}{2}} \int_{\frac{2\varepsilon}{|k|} < |s| < 2\eta} \frac{ds}{s}. \]

The bounds (2.20) for the decay of the Fourier coefficients then ensure that
\[ |m_{k,j}^2| < C M \varepsilon |k|^{-3} \log(1 + |k|), \]
Let us now estimate $m_{k,j}^1$. Using Taylor’s formula at $t_{k,j}^*$ and the fact that

$$f_{k,j} := f(x_0(t_{k,j}^*))$$

satisfies $|f_{k,j}| > c$ by Lemma 2.3 (item (i)), one can write

$$\varphi_k(t) = \alpha_{k,j} + h(t) (t - t_{k,j}^*)^2,$$

where $\alpha_{k,j} := \varphi_k(t_{k,j}^*), h(t_{k,j}^*) = \frac{1}{2}k f_{k,j}$ and

$$\frac{|h(t) - \frac{1}{2}k f_{k,j}|}{|t - t_{k,j}^*|} \leq \sup_{|t - t_{k,j}^*| < c_0 |k|^{-1/2}} |\varphi_k(t)|$$

$$\leq \sup_{|t - t_{k,j}^*| < c_0 |k|^{-1/2}} |k f'(x_0(t)); x_0(t)|$$

$$\leq C|f'|^C_{0(K_j)} |k|$$

for $|t - t_{k,j}^*| < c_0 |k|^{-1/2}$. Likewise,

$$|\dot{h}(t)| + |\ddot{h}(t)| + |\dddot{h}(t)| \leq C|k|$$

for all $|t - t_{k,j}^*| < c_0 |k|^{-1/2}$, with $C$ a constant that depends on $\|f\|_{C^3(K_j)}$. In particular, if the constant $c_0$ is small enough (but independent of $k$ and $\varepsilon$), one can define a real function $\tau = \tau(t)$ as

$$\tau := |h(t)|^{1/2} (t - t_{k,j}^*),$$

with

$$|t - t_{k,j}^*| < c_0 |k|^{-1/2}.$$

In terms of this variable, one can write

$$\varphi_k = \alpha_{k,j} + \sigma_{k,j} \tau^2,$$

where the multiplicative factor $\sigma_{k,j} := k f_{k,j} / |k f_{k,j}|$ is plus or minus one. Moreover, the interval (2.23) can be described in terms of the new variable as $\tau \in I$, where the new interval satisfies

$$I \subset \{ |\tau| < c_1 \}$$

with $c_1$ a constant independent of $\varepsilon$ and $k$.

It is easy to see that both $\dot{\tau}$ and

$$V := \frac{1}{|\tau|} = \frac{2|h|^3/2}{2|h|^2 + h h(t - t_{k,j}^*)}$$

do not vanish in the interval $|t - t_{k,j}^*| < c_0 |k|^{-1/2}$ if $c_0$ is small enough (independently of $k$ and $\varepsilon$), and that

$$V|_{t=t_{k,j}^*} = 2^{1/2} |k f_{k,j}|^{-1/2}.$$

In fact, using that

$$\frac{dV}{d\tau} = \frac{\dot{V}}{\dot{\tau}}$$

and an analogous formula for the second derivative, it is easy to see that in the region $|t - t_{k,j}^*| < c_0 |k|^{-1/2}$ the function $V$ is bounded as

$$|V| < C|k|^{-1/2}, \quad \left| \frac{dV}{d\tau} \right| < C|k|^{-1}, \quad \left| \frac{d^2V}{d\tau^2} \right| < C|k|^{-1/2}.$$
Again, $C$ does not depend on $k$ or $\varepsilon$.

Still denoting by $\dot{x}_0$, $\chi^1_{k,j}$ and $q_k(\dot{x}_0)$ the expression of these functions in terms of the new variable $\tau$, with some abuse of notation, notice that

\begin{equation}
\left| \frac{d\chi^1_{k,j}}{d\tau} \right| + \frac{|k|^{5/2}}{M} \left| \frac{dq_k(\dot{x}_0)}{d\tau} \right| + |k|^{1/2} \left| \frac{d\dot{x}_0}{d\tau} \right| + \frac{|k|^2}{M} \left| \frac{d^2q_k(\dot{x}_0)}{d\tau^2} \right| + |k| \left| \frac{d^2\dot{x}_0}{d\tau^2} \right| < C
\end{equation}

with a constant independent of $\varepsilon$ or $k$. This follows from the bounds \textcolor{red}{(2.20)-(2.22)} after writing the derivatives with respect to $\tau$ in terms of derivatives with respect to $t$. The integral for $m^1_{k,j}$ can therefore be written as

\[ m^1_{k,j} = e^{i\alpha_{k,j}/\varepsilon} \int_{c_1}^{-c_1} V\dot{x}_0 \chi^1_{k,j} q_k(\dot{x}_0) e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau. \]

Notice that $V$ appears here as the Jacobian of the change of variables.

To evaluate this integral, let us decompose it as

\[ m^1_{k,j} = e^{i\alpha_{k,j}/\varepsilon} (m^3_{k,j} + m^4_{k,j}), \]

where

\[ m^3_{k,j} = [V\dot{x}_0 \chi^1_{k,j} q_k(\dot{x}_0)]_{\tau=0} \int_{c_1}^{-c_1} e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau = \frac{2^{1/2} q_k(\dot{x}_0(\tau_{k,j}^c))}{|k|/k |f_{k,j}|^{1/2}} \int_{c_1}^{-c_1} e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau, \]

\[ m^4_{k,j} = \int_{c_1}^{-c_1} [V\dot{x}_0 \chi^1_{k,j} q_k(\dot{x}_0) - [V\dot{x}_0 \chi^1_{k,j} q_k(\dot{x}_0)]_{\tau=0}] e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau. \]

It is clear that the first integral can be evaluated explicitly modulo higher order corrections using Fresnel integrals:

\[ \int_{c_1}^{-c_1} e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau = \varepsilon^{1/2} \int_0^{\infty} e^{i\sigma_{k,j} \tau^2} \, d\tau - \varepsilon^{1/2} \int_{|\tau| > c_1/\sqrt{\varepsilon}} e^{i\sigma_{k,j} \tau^2} \, d\tau \]

\[ = (\pi \varepsilon)^{1/2} e^{i\sigma_{k,j} \pi/4} + R_2, \]

with $|R_2| < C \varepsilon$. To estimate $m^4_{k,j}$, we use Taylor's formula to write

\[ V\dot{x}_0 \chi^1_{k,j} q_k(\dot{x}_0) - [V\dot{x}_0 \chi^1_{k,j} q_k(\dot{x}_0)]_{\tau=0} =: \tau F(\tau), \]

where the bounds \textcolor{red}{(2.25)-(2.26)} ensure that

\[ |k|^3 |F(\tau)| + |k|^{5/2} |F'(\tau)| < CM \]

in the domain of integration. It then follows that

\[ |m^4_{k,j}| = \left| \int_{-c_1}^{c_1} \tau F(\tau) e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau \right| \]

\[ = \frac{\varepsilon}{2} \left| \int_{-c_1}^{c_1} F(\tau) \frac{d}{d\tau} e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau \right| \]

\[ = \frac{\varepsilon}{2} \left| \int_{-c_1}^{c_1} F'(\tau) e^{i\sigma_{k,j} \tau^2/\varepsilon} \, d\tau \right| \]

\[ \leq \frac{\varepsilon}{2} \int_{-c_1}^{c_1} |F'(\tau)| \, d\tau \]

\[ \leq CM \varepsilon |k|^{-5/2}. \]
The expression for \(m_{k,j}\) in the lemma then follows upon recalling that \(\dot{x}_0(t^*_k) = 0\).

**Second case:** \(|t^*_k| > T_0\). In this case we take a small constant \(c_0 < \eta\) independent of \(\varepsilon\) and \(k\) and define a function \(\chi_{k,j}^1(t)\) that is equal to 1 if \(|t-t^*_k| < c_0/2\), vanishes identically for \(|t-t^*_k| > c_0\), and is bounded as

\[
\sup_{t \in \mathbb{R}} \left( |\chi_{k,j}^1(t)| + |\dot{\chi}_{k,j}^1(t)| + |\ddot{\chi}_{k,j}^1(t)| \right) < C,
\]

where \(C\) is a uniform constant. We set \(\chi_{k,j}^2(t) := \chi_{k,j}(t) - \chi_{k,j}^1(t)\), which allows us to write

\[
m_{k,j} = m_{k,j}^1 + m_{k,j}^2
\]

with

\[
m_{k,j}^l := \int_{-\infty}^{\infty} \dot{x}_0(t) q_k(\dot{x}_0(t)) e^{i\varphi_k(t)/\varepsilon} \chi_{k,j}^l(t) \, dt
\]

and \(l = 1, 2\).

We argue as in the first case to write

\[
m_{k,j}^2 = -i\varepsilon \int_{-\infty}^{\infty} \dot{x}_0 \chi_{k,j}^2 q_k(\dot{x}_0(t)) \frac{d}{dt} e^{i\varphi_k/\varepsilon} \, dt
= i\varepsilon \int_{-\infty}^{\infty} d \left( \frac{\dot{x}_0 \chi_{k,j}^2 q_k(\dot{x}_0(t))}{\varphi_k} \right) e^{i\varphi_k/\varepsilon} \, dt.
\]

The bound (2.18) together with the estimates for \(\varphi_k(t)\) presented in item (ii) of Lemma 2.3 then permit to estimate \(m_{k,j}^2\) using the same argument as in the first case:

\[
|m_{k,j}^2| \leq \frac{C\varepsilon}{|k|} \left[ \|q_k\|_{C^1(K_2)} \int_{\frac{\varepsilon}{k} < |s| < 2\eta} \frac{ds}{s} + \|q_k\|_{C^0(K_2)} \int_{\frac{\varepsilon}{k} < |s| < 2\eta} \frac{ds}{s^2} \right]
\]

\[
\leq \frac{C\varepsilon}{|k|} \|q_k\|_{C^1(K_2)}.
\]

Let us now compute the remaining integral, \(m_{k,j}^1\). Again using Taylor’s formula at \(t^*_k\) and the fact that the absolute value of

\[
f_{k,j} := f(x_0(t^*_k)) = \dot{x}_0(t^*_k)
\]

is bigger than \(c/|k|\) by Lemma 2.3 (item (ii)), one can write

\[
\varphi_k(t) = \alpha_{k,j} + h(t) (t - t^*_k)^2,
\]

where \(\alpha_{k,j} := \varphi_k(t^*_k), h(t^*_k) = \frac{1}{2} k f_{k,j}\) and

\[
\sup_{|t-t^*_k| < c_0} \left[ |\dot{h}(t)| + |\ddot{h}(t)| + |\dddot{h}(t)| \right] \leq C\|f\|_{C^3(K_1)}
\]

just as in Equation (2.15). Notice that, in contrast to the first case, no powers of \(k\) appear in the bounds. Hence, if the uniform constant \(c_0\) is small enough, one can define a real function \(\tau = \tau(t)\) as

\[
\tau := |h(t)|^\frac{1}{3} (t - t^*_k),
\]
with \(|t - t^*_{k,j}| < c_0\), and one can then write
\[
\varphi_k = \alpha_{k,j} + \sigma_{k,j}\tau^2,
\]
where the multiplicative factor \(\sigma_{k,j} := kf_{k,j}/|kf_{k,j}|\) is plus or minus one. Moreover, the interval \(|t - t^*_{k,j}| < c_0\) can be described in terms of the new variable as \(\tau \in \mathcal{I}\), where the new interval satisfies
\[
\mathcal{I} \subset \{|\tau| < c_1\}
\]
with \(c_1\) a constant independent of \(\varepsilon\) and \(k\). Analogously to the first case, one can show that the Jacobian of this change of variables is
\[
V := \frac{2|h|^{3/2}}{2h^2 + hh(t - t^*_{k,j})},
\]
which does not vanish in the interval \(|t - t^*_{k,j}| < c_0\) for a small enough uniform constant \(c_0\). Notice that \(V\) is well defined because the denominator does not vanish in the above interval. Its value at \(t^*_{k,j}\) is
\[
V|_{t=t^*_{k,j}} = 2^{1/2}|k|f_{k,j}|^{-1/2}.
\]
Notice that the fact that \(f_{k,j} = \tilde{x}_0(t^*_{k,j})\) and the asymptotics \(2.18\) show that, contrary to what happened in the first case, here \(V\) is uniformly bounded in the interval under consideration in the sense that
\[
\frac{1}{C} < |V| < C, \quad \left|\frac{dV}{d\tau}\right| + \left|\frac{d^2V}{d\tau^2}\right| < C.
\]

The estimates in \((2.26)\) are then replaced by
\[
\left|\frac{d\chi_{k,j}}{d\tau}\right| + \left|\frac{k^3}{M}\frac{dk_k(\hat{x}_0)}{d\tau}\right| + \left|\frac{k^3}{M}\frac{d\hat{x}_0}{d\tau}\right| + \left|k^3\frac{d^2\chi_{k,j}}{d\tau^2}\right| + \left|k^3\frac{d^2q_k(\hat{x}_0)}{d\tau^2}\right| + \left|k^3\frac{d^2\hat{x}_0}{d\tau^2}\right| < C
\]
with a constant independent of \(\varepsilon\) or \(k\).

To compute \(m^1_{k,j}\), we can use the same trick, obtaining completely analogous formulas. Specifically, just as in the first case one can decompose
\[
m^1_{k,j} = e^{i\sigma_{k,j}/\varepsilon}(m^3_{k,j} + m^4_{k,j}),
\]
where
\[
m^3_{k,j} = \int_{-c_1}^{c_1} e^{i\sigma_{k,j}\tau^2/\varepsilon} d\tau = \frac{1}{k|k|^{1/2}|f_{k,j}|^{1/2}} \int_{-c_1}^{c_1} e^{i\sigma_{k,j}\tau^2/\varepsilon} d\tau,
\]
\[
m^4_{k,j} = \int_{-c_1}^{c_1} \left[V\tilde{x}_0\chi_{k,j}q_k(\hat{x}_0) - V\tilde{x}_0\chi_{k,j}q_k(\hat{x}_0)\right] e^{i\sigma_{k,j}\tau^2/\varepsilon} d\tau.
\]
The first integral can be evaluated explicitly modulo higher order corrections using the formula \((2.27)\) and recalling that \(\hat{x}_0(t^*_{k,j}) = 1/k\), obtaining the same formula as in the first case. The second integral can be shown to be small upon using Taylor’s formula to write
\[
[V\tilde{x}_0\chi_{k,j}q_k(\hat{x}_0) - V\tilde{x}_0\chi_{k,j}q_k(\hat{x}_0)|_{\tau=0} =: \tau F(\tau)
\]
with
\[
k^4(|F(\tau)| + |F'(\tau)|) < CM
\]
in the domain of integration. Arguing as in the first case one then finds
\[
|m^1_{k,j}| \leq CM\varepsilon k^{-4},
\]
Step 4: Formula for the displacement function. To complete the proof of the theorem, we just need to combine the relation between the displacement and Melnikov functions, captured in Proposition 2.1, the formula for the Melnikov function in terms of the numbers $m_k$ and an error (Equations (2.5)–(2.7)), and the expressions for $m_{k,j}$ computed in Lemmas 2.4-2.5. Using the notation

$$\sigma^* := \frac{f(x_0(t^*))}{|f(x_0(t^*))|},$$

this immediately yields

$$D(t_0) = \varepsilon^r \sum_{k \in \mathbb{Z} \setminus \{0\}} \Re \left[ \sum_{t^* \in C_k} \left( \frac{2\pi\varepsilon}{|f(x_0(t^*))|} \right)^{\frac{1}{2}} g_k(k^{-1}) e^{i{kx_0(t^*)} - it^* + i\sigma^* \frac{\pi}{4} - i\frac{\pi}{4}} \right] + O(\varepsilon^{r+1} + \varepsilon^{2r-2}),$$

which amounts to the formula provided in the statement of Theorem 1.1.

3. An Example: Dynamics in a Rapidly Oscillating Electromagnetic Field

A context where rapid spacetime oscillations appear naturally is in dynamical systems subject to a perturbation controlled by a wave equation. Roughly speaking, the reason is that the wave equation

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

imply that the time frequencies and the space frequencies are equal, which corresponds to the situation modeled by the perturbation (1.5) that we have considered in this paper. A prime example of physical phenomena controlled by the wave equation are electromagnetic fields, which are described by the Maxwell equations in vacuum:

$$\frac{\partial E}{\partial t} = \text{curl } B, \quad \text{div } E = 0,$$
$$\frac{\partial B}{\partial t} = -\text{curl } E, \quad \text{div } B = 0.$$

The three-dimensional time-dependent vector fields $E(x, y, z, t)$ and $B(x, y, z, t)$ are the electric and magnetic field, respectively. For the benefit of the reader we will next present a simple physical model where one can encounter this kind of rapid spacetime oscillations.

Consider as the unperturbed system a charged particle moving in the electric field generated by a harmonic time-independent potential, such as

$$E_0 := -\nabla V, \quad V(x, y, z) := \cos x \cosh z.$$

With $X := (x, y, z)$, the equations of motion are then $\ddot{X} = -\nabla V$, that is,

$$\ddot{x} = \sin x \cosh z, \quad \ddot{y} = 0, \quad \ddot{z} = -\cos x \sinh z.$$

The subset of the phase space

$$(3.2) \quad \{(X, \dot{X}) \in \mathbb{R}^6 : y = \dot{y} = z = \dot{z} = 0\}$$
is obviously invariant, and the equation of motion there is just that of a pendulum: $$\ddot{x} = \sin x.$$ 

Let us now perturb this system by adding a small but rapidly oscillating electromagnetic field of the form 

$$B_1 := \frac{\partial f(x,t)}{\partial t} (0,-z,y), \quad E_1 := \left(2 f(x,t), -\frac{\partial f(x,t)}{\partial x} y, -\frac{\partial f(x,t)}{\partial x} z\right).$$

A straightforward computation shows that the electromagnetic fields that we have considered satisfy the Maxwell equations if and only if $f(x,t)$ satisfies the wave equation (3.1). A simple choice for $f$ is to set 

$$f(x,t) := \varepsilon^r \sin \frac{x}{\varepsilon} \cos \frac{t}{\varepsilon}$$

with $r > \frac{5}{2}$. The perturbed equations of motion in the total electromagnetic field, 

$$\ddot{X} = E_0 + E_1 + \dot{X} \times B_1,$$

can then be written as 

$$\ddot{x} = \sin x \cosh z - (y\dot{y} + z\dot{z}) \frac{\partial f(x,t)}{\partial t} + 2f(x,t),$$

$$\ddot{y} = -y \left(\frac{f(x,t)}{\partial x} - \dot{x} \frac{\partial f(x,t)}{\partial t}\right),$$

$$\ddot{z} = -\cos x \sinh z - z \left(\dot{x} \frac{\partial f(x,t)}{\partial t} + \frac{\partial f(x,t)}{\partial x}\right).$$

Notice that the set (3.2) is invariant also for the perturbed system for any choice of the function $f(x,t)$. The equations of motion on this subset reduce to the one and a half degrees of freedom system 

$$\ddot{x} = \sin x + 2f(x,t),$$

which with $f(x,t)$ as in (3.3) (or with many other rapidly oscillating solutions of the wave equation (3.1)) is exactly of the form studied in Theorem 1.1. In this particular case, an easy computation shows that the displacement function reads as 

$$D(t_0) = \varepsilon^{r+\frac{1}{2}} \left(\frac{16\pi}{\sqrt{3}}\right) \frac{1}{2} \cos \left(\frac{12\pi}{\sqrt{3}} + \log(2 + \sqrt{3})\right) \frac{t_0}{\varepsilon} + O(\varepsilon^{r+1} + \varepsilon^{2r-2}).$$

Now, by Corollary 1.2, the perturbed system with $f$ given by (3.3) is chaotic in the sense that it has positive topological entropy for all small enough $\varepsilon > 0$.

4. QUASI-PERIODIC PERTURBATIONS

In this section we will study rapidly oscillating perturbations that are not time-periodic, but quasiperiodic. While the periodic case is a model for one and a half degrees of freedom systems, the quasiperiodic case is a model for higher dimensional systems. This model allows us to understand the splitting of invariant manifolds of whiskered tori with $n$ frequencies in a nearly integrable Hamiltonian system, and hence it is relevant to the study of the mechanisms of diffusion. Arguing as in
Equation (1.3), the corresponding a priori stable problems can be understood in terms of perturbations of the form
\[ g(x, \dot{x}, t; \varepsilon) := G(x, \varepsilon \dot{x})F \left( \frac{\omega t}{\varepsilon} \right), \]
where \( \omega \in \mathbb{R}^n \) is a non-resonant frequency vector and \( F : \mathbb{T}^n \to \mathbb{R} \) is a periodic function of all its arguments.

As is well known, the quasiperiodic case is not well understood and its analysis is usually rather involved. In fact, the literature on this matter is rather scarce. In general [5, 24, 19, 1, 6], one has only been able to prove the splitting of separatrices for very concrete systems (usually the quasiperiodically forced pendulum) and frequency vectors with very concrete arithmetic properties (always in very low dimensions, with \( n = 2 \) or 3, and frequencies usually given by the golden ratio). Needless to say, all the systems that have been considered are analytic.

What we will show in this section is that, when the quasiperiodic perturbation features fast oscillations in the space variable, the study of the Melnikov function can be addressed using the tools developed in Section 2. This permits to prove the splitting of separatrices in fairly general contexts (provided, of course, that these fast oscillations appear). For concreteness we will consider perturbations of the form
\[ g(x, \dot{x}, t; \varepsilon) := q \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon} \right) F \left( \frac{\omega t}{\varepsilon} \right), \]
where \( \omega \in \mathbb{R}^n \) is a non-resonant frequency vector and \( F : \mathbb{T}^n \to \mathbb{R} \) is a trigonometric polynomial of zero mean, which we write as
\[ F(\theta) =: \sum_{m \in \mathbb{Z}} F_m e^{im\theta} \]
with harmonics in a finite subset \( Z \subset \mathbb{Z}^n \setminus \{0\}. \)

**Theorem 4.1.** Let us consider the system (1.1) with \( r > \frac{5}{2} \) and a perturbation of the form (4.1), where the functions \( f \) and \( g \) are of class \( C^3 \). Suppose that the unperturbed system \( (\varepsilon = 0) \) has a hyperbolic equilibrium with a homoclinic connection corresponding to a stable manifold and an unstable manifold, which we describe through an integral curve \( x_0(t) \). Setting for each nonzero \( k \in \mathbb{Z} \) and each \( m \in \mathbb{Z} \)
\[ C_{k,m} := \left\{ t^* \in \mathbb{R} : x_0(t^*) = -\frac{m \cdot \omega}{k} \right\}, \]
assume moreover that \( |f(x_0(t^*))| \neq 0 \) for all \( t^* \in C_{k,m}, k \in \mathbb{Z} \setminus \{0\} \) and \( m \in \mathbb{Z} \). Then the displacement function is
\[ D(t_0) = \varepsilon^{r+\frac{1}{2}} \sum_{m \in \mathbb{Z}} \left( A_m \cos \frac{m \cdot \omega t_0}{\varepsilon} + B_m \sin \frac{m \cdot \omega t_0}{\varepsilon} \right) + O(\varepsilon^{r+1} + \varepsilon^{2r-2}), \]
where \( A_m \) and \( B_m \) are the real constants
\[ A_m := \sqrt{2\pi} \text{Re} \left[ F_m \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{t^* \in C_k} \frac{q_k(k^{-1}) e^{i(kx_0(t^*)-t^*+\sigma^* \pi/4)}}{|k|^{1/2}|f(x_0(t^*))|^{1/2}} \right], \]
\[ B_m := \sqrt{2\pi} \text{Im} \left[ F_m \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{t^* \in C_k} \frac{q_k(k^{-1}) e^{i(kx_0(t^*)-t^*+\sigma^* \pi/4)}}{|k|^{1/2}|f(x_0(t^*))|^{1/2}} \right]. \]
and the cardinality of $C_{k,m}$ is uniformly bounded in $k$.

The proof of Theorem 4.1 goes just as that of Theorem 1.1, mutatis mutandis. One starts by introducing a new phase function

$$\varphi_{k,m}(t) := kx_0(t) + m \cdot \omega t$$

that takes into account the quasiperiodic dependence (simply because it depends on $m \in \mathbb{Z}$). We then observe that Proposition 2.1 holds true when the perturbation is quasiperiodic. The Melnikov function now reads as

$$M(t_0) = \varepsilon^r \int_{-\infty}^{\infty} \dot{x}_0(t) q \left( \frac{x_0(t)}{\varepsilon}, \dot{x}_0(t); \varepsilon \right) F \left( \frac{\omega(t + t_0)}{\varepsilon} \right) dt,$$

which one can approximate as

$$\tilde{M}(t_0) = \varepsilon^r \int_{-\infty}^{\infty} \dot{x}_0(t) q \left( \frac{x_0(t)}{\varepsilon}, \dot{x}_0(t); 0 \right) F \left( \frac{\omega(t + t_0)}{\varepsilon} \right) dt,$$

and the error is bounded as

$$|R_1(t_0)| \leq \varepsilon^r \|q(\cdot, \cdot, \varepsilon) - q(\cdot, \cdot, 0)\|_{C^0(K)} \|F\|_{C^0(T^d)} \int_{-\infty}^{\infty} |\dot{x}_0(t)| dt \leq C \varepsilon^{r+1}.$$

The leading part of the Melnikov function is then written as

$$\tilde{M}(t_0) = \varepsilon^r \sum_{m \in \mathbb{Z}} \sum_{k = -\infty}^{\infty} F_m e^{i \frac{m \cdot \omega t_0}{\varepsilon}} M_{k,m},$$

where

$$M_{k,m} := \int_{-\infty}^{\infty} \dot{x}_0(t) q_k(\dot{x}_0(t)) e^{i \frac{x_{k,m}(t)}{\varepsilon}} dt.$$
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