Zero-sum stopping games with asymmetric information.

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Abstract

We study a model of two-player, zero-sum, stopping games with asymmetric information. We assume that the payoff depends on two continuous-time Markov chains \((X_t), (Y_t)\), where \((X_t)\) is only observed by player 1 and \((Y_t)\) only by player 2, implying that the players have access to stopping times with respect to different filtrations. We show the existence of a value in mixed stopping times and provide a variational characterization for the value as a function of the initial distribution of the Markov chains. We also prove a verification theorem for optimal stopping rules in the case where only one player has information.

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1 Introduction

In this paper we consider a two player zero-sum stopping game with asymmetric information. The payoff depends on two independent continuous time Markov chains \((X_t, Y_t)_{t \geq 0}\) with finite state space \(K \times L\), commonly known initial law \(p \otimes q\) and infinitesimal generators \(R = (R_{k,k'})_{k,k' \in K}\) and \(Q = (Q_{\ell,\ell'})_{\ell,\ell' \in L}\). We assume that \(X\) is only observed by player 1 while \(Y\) is only observed by player 2. The fact that the game was not stopped up to some time gives to each player some additional information about the unknown state. This is a crucial point as it implies that players have to take into account which information they generate about their private state when searching for optimal strategies. In consequence, our analysis is significantly different to that of classical stopping games.

We prove the existence of the value \(V(p, q)\) in mixed stopping times. We work under the common assumptions on the payoffs used by Lepeltier and Maingueneau [26], in order to provide a variational characterization for \(V\). In the particular case of information on one side, meaning that there is only one Markov chain \(X\) observed by player 1 only, this characterization is used to establish a verification theorem allowing to certify that a mixed stopping time of player 1 is optimal. Moreover we provide in this case a probabilistic characterization for the value function \(V\) which can be thought as a characterization of optimal processes of revelation (martingales of posteriors induced by optimal strategies).

Mixed stopping times have already been studied by Baxter and Chacon, Bismut and Meyer (see e.g. [27] and the references therein) in a continuous time setting and applied to stopping games by Vieille and Touzi [34] and Laraki and Solan [25]. We also refer to the recent work of Shmaya and Solan [33] for a concise study of this type of stopping times.

The variational characterization for \(V\) that we provide can be seen as an extension (in a simple case) of the classical semi-harmonic characterization for stopping games of Markov processes with symmetric information (see e.g. Friedman [15], Eckström and Peskir [13]) to models with asymmetric information. It is reminiscent of the variational representation for the value of repeated games with asymmetric information given by Rosenberg and Sorin [31] and Laraki [24]. It is also equivalent to a first-order PDE with convexity constraints as introduced by Cardaliaguet in [4]. The probabilistic characterization for the value function \(V\) is reminiscent of the formula introduced by De Meyer in [10] for a class of discrete-time financial games. Using appropriate PDE with convexity constraints, the mentioned results of [4, 10] were later extended to particular classes of continuous-time games by Cardaliaguet and Rainer [5, 6, 7], Grün [22], to continuous time stopping games by Grün [21] and to continuous time limit of repeated games by Gensbittel [16, 17].

Most of the literature on dynamic games with asymmetric information deals with models where the payoff-relevant parameters of the game that are partially unknown (say information parameters) do not evolve over time. Some recent works focus on models of dynamic games with asymmetric information and evolving information parameters (see e.g. Renault [30], Neyman [29], Gensbittel-Renault [19], Cardaliaguet et al. [8], Gensbittel [18]). All these works consider dynamic games with current or terminal payoffs while in the present work we study the case of continuous-time stopping games with time-evolving information parameters.

In a forthcoming companion paper [20], our results are applied to study a class of examples where explicit expressions for the value as well as optimal strategies for both players are provided. These examples show in particular that the value may not exist if we restrict the players to use only classical (non-mixed) stopping times, and therefore that optimal stopping
times are not anymore hitting times in stopping games with asymmetric information.

The paper is structured as follows. First we give a description of the model and the main definitions. In the third section we assemble the main theorems and show how they can be applied to characterize optimal strategies for the informed player. Section 4 is devoted to the proof of the existence of the value and section 5 to the proof of the alternative probabilistic characterization. The appendix collects auxiliary results and some technical proofs.

2 Model

2.1 Notations

For any topological space $E$, $\mathcal{B}(E)$ denotes its Borel $\sigma$-algebra, $\Delta(E)$ denotes the set of Borel probability distributions on $E$ and $\delta_x$ denotes the Dirac measure at $x \in E$. Finite sets are endowed with the discrete topology and Cartesian products with the product topology. If $E$ is finite, then $|E|$ denotes its cardinal and $\Delta(E)$ is identified with the canonical simplex of $\mathbb{R}^E$. $(\cdot, \cdot)$ and $|\cdot|$ applied to vectors stand the usual scalar product and Euclidean norm while $|\cdot|$ applied to matrices denotes the spectral norm.

For a continuous-time process $Z = (Z_t)_{t \geq 0}$ (or $t \in [0, +\infty)$) defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let us denote $\mathcal{F}_t^Z = (\mathcal{F}_t^Z)_{t \geq 0}$ the raw filtration generated by $Z$, i.e. for all $t \geq 0$, $\mathcal{F}_t^Z := \sigma(Z_s, s \leq t)$, and the right-continuous filtration $\mathcal{F}_t^{Z,+}$ by $\mathcal{F}_t^{Z,+} := \cap_{s \geq t} \mathcal{F}_s^Z$ for all $t \geq 0$.

Definition 2.1. Let us consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\mathcal{F}_t)_{t \geq 0}$ a filtration on $\Omega$. 

- A mixed stopping time of the filtration $\mathcal{F}$ on $\Omega$ is an $\mathcal{A} \otimes \mathcal{B}([0,1])$-measurable map $\mu$ defined on $\Omega \times [0,1]$ such that for all $u \in [0,1]$, $\mu(., u)$ is an $\mathcal{F}$ stopping time.
- A random time is a $\mathcal{B}([0,1])$-measurable map $\mu : [0,1] \to [0, +\infty]$. The set of random times is denoted $\mathcal{T}_p^\emptyset$.

2.2 Model

Let $K$, $L$ be two non-empty finite sets. We consider two independent continuous-time, homogeneous Markov chains $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ with state space $K$ and $L$, initial laws $p \in \Delta(K)$, $q \in \Delta(L)$ and infinitesimal generators $R = (R_{k,k'})_{k,k' \in K}$ and $Q = (Q_{k,k'})_{k,k' \in L}$ respectively. $R_{k,k'}$ represents as usual the jump intensity of the process $X$ from state $k$ to state $k'$ when $k' \neq k$ and $R_{k,k} = -\sum_{k' \neq k} R_{k,k'}$. Let us denote $\mathbb{P}_p^X$ the law of the process $X$ defined on the canonical space of $K$-valued càdlàg trajectories $\Omega_X = \mathbb{D}([0,\infty), K)$ and $\mathbb{P}_q^Y$ the law of the process $Y$ defined on the space $\Omega_Y = \mathbb{D}([0,\infty), L)$. Furthermore, let us define

$$(\Omega, \mathcal{A}; \mathbb{P}_{p,q}) := (\Omega_X \times \Omega_Y, \mathcal{F}_\infty^X \otimes \mathcal{F}_\infty^Y, \mathbb{P}_p^X \otimes \mathbb{P}_q^Y).$$

We will identify $\mathcal{F}_\infty^X$ and $\mathcal{F}_\infty^Y$ as filtrations defined on $\Omega$ as well as $\mathcal{F}_\infty^X$-measurable random variables on $\Omega$ as $\mathcal{F}_\infty^X$-measurable variables defined on $\Omega_X$ (and similarly for $Y$).

We consider a zero-sum stopping game, where player 1 observes the trajectory of $X$, while player 2 observes the trajectory of $Y$, these two processes being defined on the canonical space $(\Omega, \mathcal{A}, \mathbb{P}_{p,q})$. Recall that the transition semigroup of the process $X$ is represented by the matrix-valued process $(e^{tY})_{t \geq 0}$ acting on probabilities seen as column vectors (resp $(e^{tQ})_{t \geq 0}$...
for $Y$), where $\tilde{R}$ denotes the transpose of $R$. In particular, we have for all $k \in K$, $\ell \in L$, $s \leq t$

$$(\mathbb{P}_{p,q}[X_t = k | F^X_s])_{k \in K} = e^{(t-s)\tilde{R}\delta X_s}, \quad (\mathbb{P}_{p,q}[Y_t = \ell | F^Y_s])_{\ell \in L} = e^{(t-s)\tilde{Q}\delta Y_s},$$

while, using the independence of $X$ and $Y$, we have

$$(\mathbb{P}_{p,q}[Y_t = \ell | F^\infty_s])_{\ell \in L} = e^{t\tilde{Q}q}, \quad (\mathbb{P}_{p,q}[X_t = k | F^\infty_s])_{k \in K} = e^{t\tilde{R}p}.$$

Both players are allowed to use mixed stopping times adapted to the filtration generated by the process they observe.

**Notation 2.2.**

- $\mathcal{T}^X$ denotes the set of $F^X$ stopping times and $\mathcal{T}^X_r$ the set of mixed $F^X$-stopping times.
- $\mathcal{T}^Y$ denotes the set of $F^Y$ stopping times and $\mathcal{T}^Y_r$ the set of mixed $F^Y$-stopping times.

**Remark 2.3.** The filtrations $F^X$ and $F^Y$ are right-continuous (see Theorem 26 p.304 in [3]).

Let $r > 0$ denote a fix discount rate and $f \geq h$ two real-valued functions defined on $K \times L$. The players choose mixed stopping times $\mu(\omega, u) \in \mathcal{T}^X_r$ and $\nu(\omega, v) \in \mathcal{T}^Y_r$ respectively, in order to maximize (resp. minimize) the expected payoff:

$$E_{p,q}\left[ \int_0^1 \int_0^1 J(\mu, \nu)(\omega, u, v)dudv \right],$$

where

$$J(\mu, \nu)(\omega, u, v) := e^{-rv}f(X_\nu, Y_\nu)\mathbb{1}_{\nu < \mu} + e^{-r\mu}h(X_\mu, Y_\mu)\mathbb{1}_{\mu \leq \nu}. \quad (2.2)$$

Furthermore we set

$$\bar{J}(\mu, \nu) := \int_0^1 \int_0^1 J(\mu, \nu)(\omega, u, v)dudv.$$

The upper value of the game is defined by

$$V^+(p, q) := \inf_{\nu \in \mathcal{T}^Y_r} \sup_{\mu \in \mathcal{T}^X_r} E_{p,q}\left[ \bar{J}(\mu, \nu) \right],$$

the lower value by

$$V^-(p, q) := \sup_{\mu \in \mathcal{T}^X_r} \inf_{\nu \in \mathcal{T}^Y_r} E_{p,q}\left[ \bar{J}(\mu, \nu) \right],$$

where by definition $V^-(p, q) \leq V^+(p, q)$. When there is equality, we say that the game has a value $V := V^- = V^+$. 

4
3 Results

3.1 Existence and characterization of the value

Our first result is the existence of the value together with a variational characterization, which is a first-order PDE with convexity constraints. These constraints are expressed using the notion of extreme points as in [7, 24, 31].

**Definition 3.1.** Let $g : \Delta(K) \times \Delta(L) \rightarrow \mathbb{R}$. For any $q \in \Delta(L)$ the set of extreme points $\text{Ext}(g(.,q))$ is defined as the set of all $p \in \Delta(K)$ such that

$$(p, g(p, q)) = \lambda(p_1, g(p_1, q)) + (1 - \lambda)(p_2, g(p_2, q))$$

with $\lambda \in (0, 1)$ and $p_1, p_2 \in \Delta(K)$ implies $p_1 = p_2 = p$. The set of extreme points $\text{Ext}(g(.,.))$ for any $p \in \Delta(K)$ is defined in a similar way.

Let $f, h$ be extended linearly on the set $\Delta(K)$, i.e.

$$\forall (p, q) \in \Delta(K) \times \Delta(L), f(p, q) := \sum_{(k, \ell) \in K \times L} p_{k\ell} f(k, \ell), \quad h(p, q) := \sum_{(k, \ell) \in K \times L} p_{k\ell} h(k, \ell).$$

**Theorem 3.2.** For all $(p, q) \in \Delta(K) \times \Delta(L)$, the game has a value

$$V(p, q) := V^+(p, q) = V^-(p, q),$$

and $V$ is the unique concave-convex Lipschitz function on $\Delta(K) \times \Delta(L)$ such that:

(Subsolution) $\forall p \in \text{Ext}(V(.,q)), \forall q \in \Delta(L)$,

$$\max\{\min\{rV(p, q) - \bar{D}_1V(p, q; \uparrow R p) - \bar{D}_2V(p, q; \uparrow Q q); V(p, q) - h(p, q)\}; V(p, q) - f(p, q)\} \leq 0,$$  \hspace{1cm} (3.1)

(Supersolution) $\forall p \in \Delta(K), \forall q \in \text{Ext}(V(.,.))$,

$$\max\{\min\{rV(p, q) - \bar{D}_1V(p, q; \uparrow R p) - \bar{D}_2V(p, q; \uparrow Q q); V(p, q) - h(p, q)\}; V(p, q) - f(p, q)\} \geq 0,$$  \hspace{1cm} (3.2)

where $\bar{D}_1 V(p, q; \xi), \bar{D}_2 V(p, q; \zeta)$ denote respectively the directional derivatives of $V$ at $(p, q)$ with respect to the first and second variables in the directions $\xi$ and $\zeta$.

Let us comment (3.1). If we have that $p$ is an extremal point, i.e. $p \in \text{Ext}(V(.,q))$, (3.1) is a standard subsolution property for an obstacle PDE. This PDE can be associated to a stopping game with a deterministic dynamic on $\Delta(K) \times \Delta(L)$ given by the marginal distribution of $(X_t, Y_t)$, i.e. $(e^{t R p}, e^{t Q q})$ for some initial values $(p, q) \in \Delta(K) \times \Delta(L)$. This corresponds exactly to the characterization of the value of the game where both players do not observe $(X_t, Y_t)$. A similar explanation holds for (3.2).

3.2 The case of information on one side

Next, we consider the particular case where the set $L$ is reduced to a single point. Equivalently, we may assume that player 2 has no private information while player 1 still observes a Markov chain $X$. In this context, admissible stopping times of player 2 are simply random times\(^1\).

\(^1\)Precisely, we identify mixed stopping times of the trivial filtration with random times, see Definition 2.1.
3.2.1 A characterization of optimal strategies for the informed player

In this context, the value function $V$ depends only on $p \in \Delta(K)$ and Theorem 3.2 reduces to the following corollary.

Corollary 3.3. The value function $V = V^+ = V^-$ is the unique concave Lipschitz function on $\Delta(K)$ such that:

$$\forall p \in \text{Ext}(V(.)), \max\{\min\{rV(p) - \tilde{D}V(p; \bar{R}p); V(p) - h(p)\}; V(p) - f(p)\} \leq 0,$$

$$\forall p \in \Delta(K), \max\{\min\{rV(p) - \tilde{D}V(p; \bar{R}p); V(p) - h(p)\}; V(p) - f(p)\} \geq 0,$$

where $\tilde{D}V(p; \xi)$ denotes the directional derivative of $V$ at $p$ in the direction $\xi$.

In the case of information on one side, it is possible to give a verification theorem (or certificate of optimality) for a mixed stopping time $\mu$. To this end, we introduce the belief process of the uninformed player over $X$. Let us fix a mixed stopping time $\mu \in T_X$. Despite the fact that player 2 has no information, he can compute for any $t \geq 0$ the conditional distribution $\psi_\mu$ (defined below) of $X_t$ given the event that player 1 did not stop before time $t$. Consider the product probability space

$$(\Omega', \mathcal{F}', \mathbb{P}_p') := (\Omega_X \times [0,1], \mathcal{F}^X \otimes \mathcal{B}([0,1]), \mathbb{P}_p \otimes \text{Leb}),$$

where Leb stands for the Lebesgue measure. The stopping time $\mu(\omega, u)$ is thus seen as a random variable defined on $\Omega'$. Define the belief process $\pi$ taking values in $\Delta(K)$ as a c\'adl\'ag version of:

$$(\mathbb{P}'_p[X_t = k|\mathcal{H}_t^\mu])_{k \in K}, \ t \geq 0,$$

where $\mathcal{H}^\mu$ is the usual right-continuous augmentation of $\sigma(\mathbb{1}_{t \leq s}, 0 \leq s \leq t)$. By construction, the process $\pi$ has the following property:

$$\forall 0 \leq s \leq t, \mathbb{E}_{\mathbb{P}_p'}[\pi_t|\mathcal{F}_s^\pi] = e^{(t-s)\bar{R}\pi_s}. \quad (3.5)$$

Let $T_\mu \in \mathbb{R}_+$ be the infimum of $t$ such that $\mathbb{P}_p'(\mu > t) = 0$, and define the c\'adl\'ag function $\psi_\mu$ on $[0, T_\mu]$ by:

$$\psi_\mu(t)(k) := \mathbb{P}_p'(X_t = k|\mu > t).$$

Note that with probability 1, we have $\pi_t = \psi_\mu(t)$ on $\{t < \mu\}$. The law of the process $\pi$ is therefore completely characterized by $\psi_\mu$ and the joint law of $(\mu, \pi_\mu)$.

Theorem 3.4. Define

$$\mathcal{H} := \{p \in \Delta(K) | rV(p) - \tilde{D}V(p; \bar{R}p) \leq 0\}.$$ 

Let $\mu \in T_X$ such that

(i) $\psi_\mu(t)$ has bounded variation on $[0, T_\mu],$

(ii) $\psi_\mu(t) \in \mathcal{H}$ on $[0, T_\mu],$

(iii) $V(\pi_s) - V(\pi_{s-}) = \tilde{D}V(\pi_{s-}; \pi_s - \pi_{s-}) \mathbb{P}_p'$-a.s. on $\{s \leq \mu\},$

$$\forall p \in \Delta(K), \max\{\min\{rV(p) - \tilde{D}V(p; \bar{R}p); V(p) - h(p)\}; V(p) - f(p)\} \leq 0,$$

$$\forall p \in \Delta(K), \max\{\min\{rV(p) - \tilde{D}V(p; \bar{R}p); V(p) - h(p)\}; V(p) - f(p)\} \geq 0,$$
(iii) \( V(\pi_\mu) = h(\pi_\mu) \mathbb{P}_p'-a.s. \),
where \( \pi \) and \( \psi_\mu \) are defined as above. Then \( \mu \) is optimal, i.e.
\[
\inf_{\nu \in \mathcal{T}_\theta^0} \mathbb{E}_{\mathbb{P}_p}[J(\mu, \nu)] \geq V(p).
\]

**Proof.** Note at first that (3.5) implies that we can decompose \( \pi \) as
\[
\pi_t = p + \int_0^t R\pi_s ds + M_t, \tag{3.6}
\]
where \( M \) is a càdlàg martingale under \( \mathbb{P}_p' \) with respect to the augmentation of \( \mathcal{F}^{\pi,+} \). The process \( M \) stopped at \( \mu \) has bounded variation. As \( V \) is concave and Lipschitz, for any \( t \in \mathbb{R}_+ \), we can apply the following pathwise chain rule formula (actually a simple particular case of the general chain rule given by Theorem 3.101 in [2])
\[
V(p) = e^{-r(\mu \wedge t)}V(\pi_{\mu \wedge t}) + \int_0^{\mu \wedge t} e^{-rs}V(\pi_s)ds - \int_0^{\mu \wedge t} e^{-rs} \bar{\partial}V(\pi_{s-}; \pi_s - \pi_{s-}) ds
\]
\[
- \sum_{0 < s \leq \mu \wedge t} e^{-rs} \left( V(\pi_s) - V(\pi_{s-}) - \tilde{\partial}V(\pi_{s-}; \pi_s - \pi_{s-}) \right),
\tag{3.7}
\]
where \( |\pi|_s \) denote the total variation of \( \pi \) on \([0, s] \) and \( \frac{d\pi_s}{d|\pi|_s} \) the Radon-Nikodym derivative of \( d\pi \) with respect to \( d|\pi|_s \), taking values in the tangent cone of \( \Delta(K) \) at \( \pi_{s-} \).

The last term in (3.7) cancels out by (iii).

Given \( p \in \Delta(K) \) and \( z \) in the tangent cone of \( \Delta(K) \) at \( p \), let \( S(p, z) \) be a measurable selection of \( \arg\min_{y \in \theta^+} V(p)(z, x) \). Define the process \( \rho_s(\omega) := S(\pi_{s-}, \frac{d\pi_s}{d|\pi|_s}) \), and note that (see Lemma A.1):
\[
\int_0^{\mu \wedge t} e^{-rs} \tilde{\partial}V(\pi_{s-}; \frac{d\pi_s}{d|\pi|_s}) d|\pi|_s = \int_0^{\mu \wedge t} e^{-rs} \langle \rho_s, d\pi_s \rangle.
\]

Since \( \pi \) has bounded variation, the martingale \( M \) given in (3.6) also has bounded variation. We have therefore:
\[
d\pi_s = \tilde{\partial}R\pi_s ds + dM_s.
\]

It follows that:
\[
\int_0^{\mu \wedge t} e^{-rs} \langle \rho_s, d\pi_s \rangle = \int_0^{\mu \wedge t} e^{-rs} \langle \rho_s, \tilde{\partial}R\pi_s \rangle ds + \int_0^{\mu \wedge t} e^{-rs} \langle \rho_s, dM_s \rangle.
\]

The second term is a martingale under \( \mathbb{P}_p' \) (measurability of the Radon-Nikodym densities follows from Proposition I.3.13 in [23]). On the other hand, we have
\[
\int_0^{\mu \wedge t} e^{-rs} \langle \rho_s, \tilde{\partial}R\pi_s \rangle ds \geq \int_0^{\mu \wedge t} e^{-rs} \tilde{\partial}V(\pi_{s-}; \tilde{\partial}R\pi_s) ds.
\]

Taking expectation with respect to \( \mathbb{P}_p' \), we obtain
\[
V(p) \leq \mathbb{E}_{\mathbb{P}_p'} \left[ e^{-r(\mu \wedge t)}V(\pi_{\mu \wedge t}) \right] + \mathbb{E}_{\mathbb{P}_p'} \left[ \int_0^{\mu \wedge t} e^{-rs} \left( rV(\pi_s) - \tilde{\partial}V(\pi_s; \tilde{\partial}R\pi_s) \right) ds \right]. \tag{3.8}
\]
By (ii) we have that \( rV(\pi_s) - \tilde{D}V(\pi_s; TR\pi_s) \leq 0 \) on \( \{ s < \mu \} \). Together with (iv), this implies that for all \( t \):

\[
V(p) \leq \mathbb{E}_{p}[e^{-r(\mu \wedge t)} V(\pi_{\mu \wedge t})] = \mathbb{E}_{p}[\mathbb{I}_{t < \mu} e^{-rt} V(\pi_t) + \mathbb{I}_{\mu \leq t} e^{-r\mu} V(\pi_{\mu})]
\]

\[
\leq \mathbb{E}_{p}[e^{-rt} f(\pi_t) \mathbb{I}_{t < \mu} + e^{-r\mu} h(\pi_{\mu}) \mathbb{I}_{\mu \leq t}]
\]

\[
= \mathbb{E}_{p}[e^{-rt} f(X_t) \mathbb{I}_{t < \mu} + e^{-r\mu} h(X_{\mu}) \mathbb{I}_{\mu \leq t}],
\]

where the last equality follows by conditioning with respect to \( H_t^\mu \) and \( H_{\mu}^\mu \). Finally, we obtain

\[
V(p) \leq \inf_{t \in \mathbb{R}^+} \mathbb{E}_{p}[e^{-rt} f(X_t) \mathbb{I}_{t < \mu} + e^{-r\mu} h(X_{\mu}) \mathbb{I}_{\mu \leq t}]
\] (3.9)

which yields the optimality of \( \mu \).

### 3.2.2 Alternative characterization of the value

We provide now an alternative probabilistic representation of the value. In this representation, the belief process \( \pi \) of the uninformed player over \( X \) introduced in the preceding subsection is interpreted as an external random source chosen by player 1 before playing a game with symmetric information. Let us define the set of “beliefs” in a very general way.

**Definition 3.5.** Let \( K(p) \) denote the set of distributions \( P \) on \( \mathbb{D}([0, +\infty), \Delta(K)) \) of càdlàg processes \( (\pi_t)_{t \geq 0} \) taking values in \( \Delta(K) \) such that

\[
\forall 0 \leq s \leq t, \mathbb{E}_{P}[\pi_t | F_{\pi s}] = e^{(t-s) \top R_{\pi s}}.
\]

**Remark 3.6.** Equivalently, \( K(p) \) is the set of distributions \( P \) of càdlàg processes \( (\pi_t)_{t \geq 0} \) such that \( (e^{-t \top R_{\pi t}})_{t \geq 0} \) is a martingale.

To motivate this definition, note that the law of any belief process as constructed in the previous subsection belongs to this family. However, \( K(p) \) is a very large set compared to the set of laws of achievable processes in this model since its definition does not take into account that the filtration \( H_{\mu}^\mu \) is generated by a single jump occurring at the stopping decision of player 1. Let \( T^\pi \) denote the set of stopping times of the filtration \( F_{\pi \cdot}^\pi \). For \( \tau, \sigma \in T^\pi \) we set

\[
J'(\tau, \sigma) := e^{-r\sigma} f(\pi_{\sigma}) \mathbb{I}_{\sigma < \tau} + e^{-r\tau} h(\pi_{\tau}) \mathbb{I}_{\tau \leq \sigma}
\]

i.e. the payoff of a stopping game with two equally informed players having belief \( \pi \) over \( X \). The following result provides a probabilistic characterization for the value function \( V \).

**Theorem 3.7.** For all \( p \in \Delta(K) \)

\[
V(p) = \sup_{P \in K(p)} \left( \sup_{\tau \in T_{\pi}^+} \inf_{\sigma \in T_{\pi}^-} \mathbb{E}_{P}[J'(\tau, \sigma)] \right),
\] (3.10)

**Remark 3.8.** Note that the sup and inf in between the parentheses in (3.10) commute (see e.g. Lepeltier and Maingueneau [26]).
3.3 Open questions

Several open problems arise. A natural question is whether a verification theorem to characterize optimal stopping times of player 1 can be also established in the case of asymmetric information with two Markov chains. A straightforward generalization of the proof of Theorem 3.4 is impossible. The set $\mathcal{H}$ will then depend on both variables $p$ and $q$. Thus in order to mimic the proof player 1 would have to compute both belief processes. Such a computation, however, is not possible for player 1 because it requires to know beforehand which stopping time player 2 is using. The same difficulty arises in the study of differential games with asymmetric information where the construction of optimal strategies is still an open problem.

Another important point would be to investigate the case where the Markov process $X, Y$ have infinite state space, e.g. diffusion processes. The main difficulty is that our approach leads formally to study partial differential equations in infinite dimensional spaces of probabilities. The only results in this direction consider differential games where the information parameters of the game do not evolve over time (see [7]).

In view of possible applications to stopping games arising e.g. in financial mathematics, one may also consider models with publicly observed diffusive dynamics. The particular case where the information parameters were not evolving was considered in [21]. A generalization of these results is an interesting point for further research.

4 Existence and characterization of the value

In this section we give the proof for the existence of the value and its PDE characterization in Theorem 3.2.

4.1 Properties of $V^+, V^-$ and their concave, convex conjugates

First we note the following fact.

**Remark 4.1.** We can replace the infimum over $\mathcal{T}_r^X$ by an infimum over $\mathcal{T}^X$ in the definition of $V^-$ (and the symmetric property for $V^+$) i.e.

$$
V^+(p, q) = \inf_{\nu \in \mathcal{T}_r^Y} \sup_{\mu \in \mathcal{T}^X} \bar{J}(\mu, \nu), \\
V^-(p, q) = \sup_{\mu \in \mathcal{T}_r^X} \inf_{\nu \in \mathcal{T}_r^Y} \bar{J}(\mu, \nu).
$$

(4.1)

Indeed, using Fubini theorem, for any $\nu \in \mathcal{T}_r^Y$, we have

$$
\mathbb{E}_{p, q}[\bar{J}(\mu, \nu)] = \int_0^1 \mathbb{E}_{p, q}[\int_0^1 J(\mu, \nu)du]dv.
$$

For any $\varepsilon > 0$, one can choose $v_\varepsilon \in [0, 1]$ such that

$$
\mathbb{E}_{p, q}[\int_0^1 J(\mu, \nu(\cdot, v_\varepsilon))du] \leq \int_0^1 \mathbb{E}_{p, q}[\int_0^1 J(\mu, \nu)du]dv + \varepsilon,
$$

and $\nu(\cdot, v_\varepsilon)$ is a stopping time in $\mathcal{T}^Y$. The conclusion follows by sending $\varepsilon$ to zero.

We summarize the properties of $V^+, V^-$ in the following lemma.
Lemma 4.2. \( V^+ \) and \( V^- \) are concave-convex, Lipschitz functions and
\[
\forall p, q \in \Delta(K) \times \Delta(L), \ h(p, q) \leq V^-(p, q) \leq V^+(p, q) \leq f(p, q). \tag{4.2}
\]

Proof. We prove the claim only for \( V^+ \) since the proof \( V^- \) is similar. It is easily seen that inequality (4.2) follows immediately from the definitions as both players can stop at time zero. Furthermore we note that for any \( \mu \in \mathcal{T}^X_r, \nu \in \mathcal{T}^Y_r, \) it holds by conditioning
\[
\mathbb{E}_{p,q} \left[ J(\mu, \nu) \right] = \sum_{k \in K, \ell \in L} \mathbb{P}\left[X_0 = k\right] \mathbb{P}\left[Y_0 = \ell\right] \mathbb{E}_{p,q} \left[ J(\mu, \nu) | X_0 = k, Y_0 = \ell \right] = \sum_{k \in K, \ell \in L} p_k q_{\ell} \mathbb{E}_{\delta_k, \delta_{\ell}} \left[ J(\mu, \nu) \right],
\]
\( \delta_k, \delta_{\ell} \) denoting the Dirac masses at \( k, \ell \) identified with the \( k\)-th, \( \ell\)-th vectors in the canonical bases of \( \mathbb{R}^K \) and \( \mathbb{R}^L \) respectively.

For the Lipschitz continuity let \( p, p' \in \Delta(K), q, q' \in \Delta(L) \) such that \( 0 < V^+(p, q) - V^+(p', q') \). Choosing \( \nu^* \in \mathcal{T}^Y_r \) \( \varepsilon \)-optimal for \( V^+(p', q') \) and \( \mu^* \in \mathcal{T}^Y_r \) \( \varepsilon \)-optimal for \( \sup_{\mu \in \mathcal{T}^X_r} \mathbb{E}_{p,q} \left[ J(\mu, \nu^*) \right] \) we have
\[
0 < V^+(p, q) - V^+(p', q') \leq \mathbb{E}_{p,q} \left[ J(\mu^*, \nu^*) \right] - \mathbb{E}_{p',q'} \left[ J(\mu^*, \nu^*) \right] + 2\varepsilon \tag{4.4}
\]
for \( \varepsilon \) arbitrarily small. The claim follows then immediately by (4.3). Furthermore we claim that
\[
V^+(p, q) = \inf_{\nu \in \mathcal{T}^Y_r} \sup_{\mu \in \mathcal{T}^X_r} \mathbb{E}_{p,q} \left[ J(\mu, \nu) \right] = \inf_{\nu \in \mathcal{T}^Y_r} \sum_{k \in K} p_k \left( \sup_{\mu \in \mathcal{T}^X_r} \mathbb{E}_{\delta_k,q} \left[ J(\mu, \nu) \right] \right). \tag{4.5}
\]
Indeed, \( V^+(p, q) \) is clearly less or equal than the second line in the above equation. To prove the reverse inequality, for any \( \nu \in \mathcal{T}^Y_r \), and any \( k \in K \), let \( \mu^k \) be some \( \varepsilon \)-optimal stopping time for the problem \( \sup_{\mu \in \mathcal{T}^X_r} \mathbb{E}_{\delta_k,q} \left[ J(\mu, \nu) \right] \). Construct the stopping time \( \mu := \sum_{k \in K} \mathbb{1}_{X_0 = k} \mu^k \), and note that
\[
\sum_{k \in K} p_k \mathbb{E}_{\delta_k,q} \left[ J(\mu^k, \nu) \right] = \mathbb{E}_{p,q} \left[ J(\mu, \nu) \right].
\]
Equation (4.5) follows by sending \( \varepsilon \) to zero.

We deduce from (4.5) that \( p \rightarrow V^+(p, q) \) is concave as an infimum of affine functions.

The convexity in \( q \) follows by the classical splitting method. Let \( q_1, q_2, q \in \Delta(L), \lambda \in (0, 1) \) such that
\[
q = \lambda q_1 + (1 - \lambda)q_2.
\]
We choose \( \nu_1 \in \mathcal{T}^Y_r, \nu_2 \in \mathcal{T}^Y_r \) that are \( \varepsilon \)-optimal for \( V^+(p, q_1) \) and \( V^+(p, q_2) \) respectively. Then we will construct \( \nu \in \mathcal{T}^Y_r \) such that
\[
\mathbb{E}_{p,q} \left[ J(\mu, \nu) \right] = \lambda \mathbb{E}_{p,q_1} \left[ J(\mu, \nu_1) \right] + (1 - \lambda)\mathbb{E}_{p,q_2} \left[ J(\mu, \nu_2) \right]. \tag{4.6}
\]

The intuition of the construction is the following: At time \( t = 0 \), player 2, knowing \( Y_0 \), can choose at random a decision \( d \in \{1, 2\} \) such that the conditional law of \( Y_0 \) given that \( d = 1 \) is \( q_1 \) and the conditional law of \( Y_0 \) given that \( d = 2 \) is \( q_2 \). He will then play \( \nu_1 \) if \( d = 1 \) and \( \nu_2 \) when \( d = 2 \). This strategy can be easily described by the mixed stopping time \( \nu \) below.

\( \nu(\omega, u) \) is constructed in such a way that the probability to choose \( \nu_1 \) given that \( Y_0 = \ell \) is \( \frac{\lambda(q_1)}{q_\ell} \) whenever \( q_\ell > 0 \) and the probability to choose \( \nu_2 \) is \( \frac{(1 - \lambda)(q_2)}{q_\ell} \). Precisely:
\[
\nu(\omega, u) := \sum_{\ell=1}^L \mathbb{1}_{Y_0 = \ell} \left( \mathbb{1}_{u \in [0, \frac{\lambda(q_1)}{q_\ell}]} \nu_1(\omega, \frac{q_\ell}{\lambda(q_1)}) \right) + \mathbb{1}_{u \in [\frac{\lambda(q_1)}{q_\ell}, 1]} \nu_2(\omega, \frac{q_\ell u - \lambda(q_1)}{(1 - \lambda)(q_2)}) \tag{4.7}
\]
It follows that:

\[ E_{p,q} \left[ J(\mu, \nu) \right] = E_{p,q} \left[ \int_0^1 \bar{J}(\mu, \nu(\cdot, u))du \right] \]

\[ = E_{p,q} \left[ \sum_{\ell \in L} I_{Y_0 = \ell} \left( \int_0^{\lambda (q_1) \ell_{\nu_1}} \bar{J}(\mu, \nu_1(\cdot, \frac{q_\ell u - \lambda (q_1) \ell_{\nu_1}}{1 - \lambda (q_2) \ell_{\nu_2}}))du \right. \right. \]

\[ + \left. \left. \int_1^{\lambda (q_1) \ell_{\nu_2}} \bar{J}(\mu, \nu_2(\cdot, \frac{q_\ell u - \lambda (q_1) \ell_{\nu_1}}{1 - \lambda (q_2) \ell_{\nu_2}}))du \right) \right] \]

\[ = \lambda E_{p,q_1} \left[ J(\mu, \nu_1) \right] + (1 - \lambda) E_{p,q_2} \left[ J(\mu, \nu_2) \right]. \]

Maximizing (4.6) over \( \mu \in T_r^X \) yields then, using the \( \varepsilon \) optimality of \( \nu_1 \) and \( \nu_2 \),

\[ V^+(p, q) \leq \lambda V^+(p, q^1) + (1 - \lambda) V^+(p, q^2) + \varepsilon \]

and the convexity in \( q \) follows since \( \varepsilon \) can be chosen arbitrarily small.

Next we define the concave conjugate in \( p \) of \( V^{+, \ast} \) as

\[ \forall x \in \mathbb{R}^K, q \in \Delta(L), V^{+, \ast}(x, q) := \inf_{p \in \Delta(K)} \{ \langle x, p \rangle - V^+(p, q) \} \]

and the convex conjugate in \( q \) of \( V_{\ast}^- \) as

\[ \forall p \in \Delta(K), y \in \mathbb{R}^L, V_{\ast}^-(p, y) := \sup_{q \in \Delta(L)} \{ \langle q, y \rangle - V^-(p, q) \}. \]

Immediately from the previous lemma it follows that:

\[ f^\ast(x, q) \leq V^{+, \ast}(x, q) \leq h^\ast(x, q) \]

\[ f\ast(p, y) \leq V_{\ast}^-(p, y) \leq h\ast(p, y), \]

where the functions \( h^\ast, f^\ast \) and \( h\ast, f\ast \) are defined as

\[ h^\ast(x, q) := \inf_{p \in \Delta(K)} \{ \langle x, p \rangle - h(p, q) \}, f^\ast(x, q) := \inf_{p \in \Delta(K)} \{ \langle x, p \rangle - f(p, q) \}, \]

\[ h\ast(p, y) := \sup_{q \in \Delta(L)} \{ \langle q, y \rangle - h(p, q) \}, f\ast(p, y) := \sup_{q \in \Delta(L)} \{ \langle q, y \rangle - f(p, q) \}. \]

The next lemma provides an alternative formulation.

**Lemma 4.3.** We have the following, alternative representations:

\[ \forall x \in \mathbb{R}^K, q \in \Delta(L), V^{+, \ast}(x, q) = \sup_{\nu \in T_r^Y} \inf_{\mu \in T_r^X} \inf_{p \in \Delta(K)} \{ \langle x, p \rangle - E_{p,q}[\bar{J}(\mu, \nu)] \} \]  

\[ \forall p \in \Delta(K), y \in \mathbb{R}^L, V_{\ast}^-(p, y) = \inf_{\mu \in T_r^X} \sup_{\nu \in T_r^Y} \sup_{q \in \Delta(L)} \{ \langle y, q \rangle - E_{p,q}[\bar{J}(\mu, \nu)] \}. \]
Proof. Using remark 4.1, we have

\[ V^{+,*}(x, q) = \inf_{p \in \Delta(K)} \sup_{\nu \in \mathcal{T}_r^Y} \inf_{\mu \in \mathcal{T}^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q}[\bar{J}(\mu, \nu)] \right). \]

Then, we will apply Fan’s minmax theorem (see [14]) to deduce that:

\[ V^{+,*}(x, q) = \inf_{p \in \Delta(K)} \sup_{\nu \in \mathcal{T}_r^Y} \inf_{\mu \in \mathcal{T}^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q}[\bar{J}(\mu, \nu)] \right). \]

In order to apply Fan’s minmax theorem, we have to check that the function

\[ (p, \nu) \in \Delta(K) \times \mathcal{T}_r^Y \to \inf_{\mu \in \mathcal{T}^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q}[\bar{J}(\mu, \nu)] \right) \]

is concave-like with respect to \( \nu \in \mathcal{T}_r^Y \) and affine (hence continuous) with respect to \( p \) in the compact convex set \( \Delta(K) \).

To prove the concave-like property, given \( \nu_1, \nu_2 \in \mathcal{T}_r^Y \), and \( \lambda \in (0, 1) \), we define the mixed stopping time \( \nu(\omega, u) = \nu_1(\omega, \frac{u}{\lambda})1_{u \in [0, \lambda]} + \nu_2(\omega, \frac{u - \lambda}{1 - \lambda})1_{u \in [\lambda, 1]} \).

A simple change of variables gives:

\[ \inf_{\mu \in \mathcal{T}^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q}[\bar{J}(\mu, \nu)] \right) \geq \lambda \inf_{\mu \in \mathcal{T}^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q}[\bar{J}(\mu, \nu_1)] \right) + (1 - \lambda) \inf_{\mu \in \mathcal{T}^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q}[\bar{J}(\mu, \nu_2)] \right), \]

which is exactly the the concave-like property we need to apply Fan’s theorem.

The second property follows from the relation:

\[ \inf_{\mu \in \mathcal{T}^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q}[\bar{J}(\mu, \nu)] \right) = \langle x, p \rangle - \sum_{k \in K} p^k \sup_{\mu \in \mathcal{T}^X} \mathbb{E}_{\delta_k, q}[\bar{J}(\mu, \nu)]. \]

This last equation is proved in the same way as equation (4.5) in Lemma 4.2.

\[ \square \]

### 4.2 Dynamic Programming for \( V^{+,*}, V_*^- \)

In order to prove the dynamic programming inequalities, we need to recall the definition of the shift operator \( \theta^X \) on \( \Omega_X \).

**Definition 4.4.** For all \( t \geq 0 \), the map \( \theta_t^X : \Omega_X \to \Omega_X \) is defined by

\[ \forall s \geq 0, \quad \theta_t^X(\omega_X)(s) = \omega_X(s + t). \]

The shift operator \( \theta^Y \) on \( \Omega_Y \) is defined similarly.

We will now prove a dynamic programming inequality for \( V^{+,*} \) and \( V_*^- \).
Proposition 4.5. For all $\varepsilon > 0$
\[ V^{+,*}(x, q) \geq \inf_{t \in [0, \varepsilon]} \inf_{\mu \in T^X} \left( e^{-rt}h^*(x_t, q_t)\mathbb{1}_{t < \varepsilon} + e^{-r\varepsilon}V^{+,*}(x_\varepsilon, q_\varepsilon)\mathbb{1}_{t = \varepsilon} \right), \tag{4.12} \]
where the dynamic $(x_t, q_t)$ is given by
\[ \forall t \geq 0, \; x_t = x + \int_0^t (rI - R)x_s ds \quad \text{and} \quad q_t = q + \int_0^t Qq_s ds. \]
Similarly, for all $\varepsilon' > 0$
\[ V_*(x, q) \leq \sup_{t \in [0, \varepsilon']} \left( e^{-rt}f_*(p_t, y_t)\mathbb{1}_{s < \varepsilon'} + e^{-r\varepsilon'}V^*_*(p_{\varepsilon'}, y_{\varepsilon'})\mathbb{1}_{t = \varepsilon'} \right), \tag{4.13} \]
with $(p_t, y_t)$ given by
\[ \forall t \geq 0, \; p_t = p + \int_0^t R_p ds \quad \text{and} \quad y_t = x + \int_0^t (rI - Q)y_s ds. \]

Proof. We only give the proof of (4.12), the proof of (4.13) being similar. Given $\varepsilon > 0$, we consider the family $T^Y_{\rho, \varepsilon}$ of mixed stopping times $\nu \in T^Y_{\rho, \varepsilon}$ such that there exists a mixed stopping time $\nu' \in T^Y_\rho$ and $\nu(\omega, v) = \varepsilon + \nu'(\theta^Y_\varepsilon(\omega_Y), v)$.

As $T^Y_{\rho, \varepsilon} \subset T^Y_\rho$, we have:
\[ V^{+,*}(x, q) \geq \sup_{\nu \in T^Y_{\rho, \varepsilon}} \inf_{\mu \in T^X} \left( \langle x, p \rangle - \mathbb{E}_{p, q} \left[ e^{-r\mu}h(X_\mu, Y_\mu)\mathbb{1}_{\mu < \varepsilon} \right. \right. \]
\[ + e^{-r\varepsilon} \int_0^1 \left( e^{-r(\nu - \varepsilon)}f(X_\nu, Y_\nu)\mathbb{1}_{\nu < \mu} + e^{-r(\mu - \varepsilon)}h(X_\mu, Y_\mu)\mathbb{1}_{\mu < \nu}dv\mathbb{1}_{\mu \geq \varepsilon} \right) \right]. \tag{4.14} \]

Let us fix $\nu \in T^Y_{\rho, \varepsilon}$ (or equivalently $\nu' \in T^Y_{\rho, \varepsilon}$), $\mu \in T^X$ and $p \in \Delta(K)$. By conditioning, we obtain:
\[ \mathbb{E}_{p, q} \left[ e^{-r\mu}h(X_\mu, Y_\mu)\mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon} \int_0^1 \left( e^{-r(\nu - \varepsilon)}f(X_\nu, Y_\nu)\mathbb{1}_{\nu < \mu} + e^{-r(\mu - \varepsilon)}h(X_\mu, Y_\mu)\mathbb{1}_{\mu < \nu}dv\mathbb{1}_{\mu \geq \varepsilon} \right) \right] \]
\[ = \mathbb{E}_{p, q} \left[ e^{-r\mu}\mathbb{E}_{p, q} \left[ h(X_\mu, Y_\mu)\mathcal{F}^X_\mu \right] \mathbb{1}_{\mu \leq \varepsilon} \right. \]
\[ + e^{-r\varepsilon} \mathbb{E}_{p, q} \left[ \int_0^1 \left( e^{-r(\nu - \varepsilon)}f(X_\nu, Y_\nu)\mathbb{1}_{\nu < \mu} + e^{-r(\mu - \varepsilon)}h(X_\mu, Y_\mu)\mathbb{1}_{\mu < \nu}dv\mathcal{F}^X_\nu \right) \mathbb{1}_{\mu \geq \varepsilon} \right]. \]

Recall that the stopping time $\mu$ of the filtration $\mathcal{F}^X$ can be identified with a stopping time defined on $\Omega_X$. It is well-known that on the event $\mu \geq \varepsilon$, we have $\mu = \mu'(\omega_X, \theta^X_\varepsilon(\omega_X)) + \varepsilon$, where $\mu'$ is $\mathcal{F}^X_\varepsilon \otimes \mathcal{F}^X_\infty$ measurable and for all $\omega, \mu'(\omega, .)$ is an $\mathcal{F}^X$ stopping time (see theorem 103 p. 151 in [9]). Then, using the Markov property, we deduce that,
\[ \mathbb{1}_{\mu \geq \varepsilon} \mathbb{E}_{p, q} \int_0^1 \left( e^{-r(\nu - \varepsilon)}f(X_\nu, Y_\nu)\mathbb{1}_{\nu < \mu} + e^{-r(\mu - \varepsilon)}h(X_\mu, Y_\mu)\mathbb{1}_{\mu < \nu} \right) dv\mathcal{F}^X_\varepsilon \]
\[ = \mathbb{1}_{\mu \geq \varepsilon} \mathbb{E}_{\delta X_\varepsilon, \delta Y_\varepsilon} [\mathcal{J}(\mu'(\omega, .), \nu')]. \]
On the other hand, since \(X\) and \(Y\) are independent we have:
\[
\mathbb{E}_{p,q}[\delta_{X_t,Y_t}|\mathcal{F}^X_t] = \delta_{X_t} \otimes q_t \in \Delta(K \times L).
\]

Using the usual properties of the optional projection (see e.g. [9]) the previous equality implies
\[
\mathbb{E}_{p,q}[h(X_{\mu},Y_{\mu})|\mathcal{F}^X_{\mu}] = h(\delta_{X_\mu},q_\mu),
\]
and inequality (4.14) may be rewritten as:
\[
V^{+,*}(x,q) \geq \sup_{\nu' \in \mathcal{T}^Y} \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \left(\langle x,p \rangle - \mathbb{E}_{p,q}[e^{-r\mu}h(\delta_{X_\mu},q_\mu)1_{\mu_\nu} + e^{-r\varepsilon}E_{X_\nu,Y_\nu}[\tilde{J}(\mu'(\cdot),\nu')1_{\mu_\nu}]]\right).
\]

Defining
\[
\forall (\nu',p,q) \in \mathcal{T}^Y \times \Delta(K) \times \Delta(L), F(\nu',p,q) = \sup_{\hat{\mu} \in \mathcal{T}^X} \mathbb{E}_{p,q}[\tilde{J}(\hat{\mu},\nu')]
\]
we have, using the same arguments as for (4.3):
\[
F(\nu',p,q) = \sum_{\ell \in L} \sum_{k \in K} q^\ell p^k F(\nu',\delta_k,\delta_\ell).
\]

The previous inequality implies therefore:
\[
V^{+,*}(x,q) \geq \sup_{\nu' \in \mathcal{T}^Y} \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \left(\langle x,p \rangle - \mathbb{E}_{p,q}[e^{-r\mu}h(\delta_{X_\mu},q_\mu)1_{\mu_\nu} + e^{-r\varepsilon}F(\nu',\delta_{X_\nu},\delta_{Y_\nu})1_{\mu_\nu}]]\right).
\]

Taking conditional expectation with respect to \(\mathcal{F}^X_\varepsilon\), we obtain that
\[
\mathbb{E}_{p,q}[F(\nu',\delta_{X_\varepsilon},\delta_{Y_\varepsilon})1_{\mu_\varepsilon}] = F(\nu',\delta_{X_\varepsilon},q_\varepsilon)1_{\mu_\varepsilon}.
\]

Applying the optional sampling theorem with the \(\mathcal{F}^X\)-stopping time \(\mu \wedge \varepsilon\), we obtain
\[
\langle x,p \rangle = \mathbb{E}_{p,q}[\langle x,e^{-r\mu R}\delta_{X_\mu},q_\mu1_{\mu_\varepsilon} + \langle x,e^{-\varepsilon R}\delta_{X_\varepsilon},q_\varepsilon1_{\mu_\varepsilon} \rangle1_{\mu_\varepsilon}]
\]
\[
= \mathbb{E}_{p,q}[\langle e^{-\mu R}x,\delta_{X_\mu},1_{\mu_\varepsilon} + e^{-\varepsilon R}x,\delta_{X_\varepsilon},1_{\mu_\varepsilon} \rangle1_{\mu_\varepsilon}].
\]

Substituting the last two equalities in the right-hand side of (4.15) yields:
\[
\langle x,p \rangle - \mathbb{E}_{p,q}[e^{-r\mu}h(\delta_{X_\mu},q_\mu)1_{\mu_\varepsilon} + e^{-r\varepsilon}F(\nu',\delta_{X_\varepsilon},q_\varepsilon)1_{\mu_\varepsilon}] = \mathbb{E}_{p,q}[e^{-r\mu}(\langle e^{r(I-R)}x,\delta_{X_\mu},h(\delta_{X_\mu},q_\mu)1_{\mu_\varepsilon} + e^{-r\varepsilon}((\langle e^{r(I-R)}x,\delta_{X_\varepsilon}\rangle - F(\nu',\delta_{X_\varepsilon},q_\varepsilon)1_{\mu_\varepsilon})1_{\mu_\varepsilon})]
\]
\[
= \mathbb{E}_{p,q}[e^{-r\mu}(\langle x\mu,\delta_{X_\mu}\rangle - h(\delta_{X_\mu},q_\mu)1_{\mu_\varepsilon} + e^{-r\varepsilon}((\langle x\varepsilon,\delta_{X_\varepsilon}\rangle - F(\nu',\delta_{X_\varepsilon},q_\varepsilon)1_{\mu_\varepsilon})1_{\mu_\varepsilon})]
\]

Given any \(\eta > 0\), let us choose \(\nu'\) as an \(\eta\)-optimal minimizer in the problem \(V^{+,*}(x,q)\) (note that these dynamics do not depend on \(p\) or \(\mu\), so that:
\[
\langle x\varepsilon,\delta_{X_\varepsilon}\rangle - F(\nu',\delta_{X_\varepsilon},q_\varepsilon) \geq \inf_{\nu' \in \Delta(K)} \langle x\varepsilon,p'\rangle - F(\nu',p',q_\varepsilon)
\]
\[
= \inf_{\nu' \in \Delta(K)} \inf_{\hat{\mu} \in \mathcal{T}^X} \langle x\varepsilon,p'\rangle - \mathbb{E}_{p',q_\varepsilon}[\tilde{J}(\hat{\mu},\nu')]
\]
\[
\geq V^{+,*}(x,q) - \eta.
\]
Proof. For all \( \exists \eta > 0 \) such that for all \( 0 < \varepsilon < \eta \),

\[
V^{+, \ast}(x, q) \geq \inf_{\mu \in T^X} \inf_{\rho \in \Delta(K)} \mathbb{E}_p[ e^{-r\mu}(\langle x_{\mu}, \delta x_{\mu} \rangle - h(\delta x_{\mu}, q_{\mu}))\mathbb{I}_{\mu < \varepsilon} + e^{-r\varepsilon}(V^{+, \ast}(x_{\varepsilon}, q_{\varepsilon}) - \eta)\mathbb{I}_{\mu \geq \varepsilon}].
\]

Putting together the preceding results, we deduce that for all \( \eta > 0 \):

\[
V^{+, \ast}(x, q) \geq \inf_{\mu \in T^X} \inf_{\rho \in \Delta(K)} \mathbb{E}_p[ e^{-r\mu}(\langle x_{\mu}, \delta x_{\mu} \rangle - h(\delta x_{\mu}, q_{\mu}))\mathbb{I}_{\mu < \varepsilon} + e^{-r\varepsilon}(V^{+, \ast}(x_{\varepsilon}, q_{\varepsilon}) - \eta)\mathbb{I}_{\mu \geq \varepsilon}].
\]

Note that in the preceding expression, only \( \mu \) is random, and thus we may replace the expectation with an integral with respect to the law of \( \mu \) on \( \mathbb{R}_+ \) denoted \( P_{p, \mu} \).

\[
V^{+, \ast}(x, q) \geq \inf_{\mu \in T^X} \inf_{\rho \in \Delta(K)} \int_{\mathbb{R}_+} (e^{-rt}(x_{\mu}(t), q_{\mu}(t)))\mathbb{I}_{\mu \leq r} + e^{-r\varepsilon}(V^{+, \ast}(x_{\varepsilon}, q_{\varepsilon}) - \eta)\mathbb{I}_{\mu \geq \varepsilon})dP_{p, \mu}(t).
\]

Using the linearity of the integral and arguing as in Remark 4.1, the infimum over all admissible distributions for \( \mu \) is equal to the infimum over Dirac masses (constant stopping times), and the conclusion follows by sending \( \eta \) to zero. \( \square \)

### 4.3 Subsolution property for \( V^{+} \)

We will prove the subsolution property for \( V^{+} \) by establishing a super-solution property for \( V^{+, \ast} \). The results rely on classical tools of convex analysis.

**Notation 4.6.** For any \( g : \mathbb{R}^K \times \mathbb{R}^L \to [-\infty, +\infty] \) and \( (x, y) \in \mathbb{R}^K \times \mathbb{R}^L \), we denote for any \( y \in \mathbb{R}^L \) the sub-differential of \( g \) with respect to the first variable \( x \) by

\[
\partial_{+} g(x, y) = \{ x^* \in \mathbb{R}^K \mid \forall x' \in \mathbb{R}^K, g(x, y) + \langle x^*, x' - x \rangle \leq g(x', y) \}.
\]

The super-differential \( \partial_{-} g(x, y) \) is defined similarly.

We use the index \( \partial_{-} \) (resp. \( \partial_{+} \)), whenever we consider derivatives with respect second variable \( y \in \mathbb{R}^L \). And we use \( \partial^{+} \) for the full super- and sub-differential.

**Proposition 4.7.** For all \( (x, q) \in \mathbb{R}^K \times \Delta(L) \), we have

\[
\max\{ (V^{+, \ast} - h^*)(x, q) ; \partial V^{+, \ast}(x, q ; (rI - R)x, \overline{Q}q) - rV^{+, \ast}(x, q) \} \leq 0. \tag{4.16}
\]

For all \( (p, y) \in \Delta(K) \times \mathbb{R}^L \), we have

\[
\min\{ V_{-}^{+}(p, y) ; \partial V_{-}^{+}(p, y ; (rI - Q)y, \overline{R}p) - rV_{-}^{+}(p, y) \} \geq 0. \tag{4.17}
\]

**Proof.** We only show the first statement. Recall that by Proposition 4.5, for all \( \varepsilon > 0 \),

\[
V^{+, \ast}(x, q) \geq \inf_{t \in [0, \varepsilon]} (e^{-rt}(x_{\mu}(t), q_{\mu}(t)))\mathbb{I}_{\mu \leq r} + e^{-r\varepsilon}(V^{+, \ast}(x_{\varepsilon}, q_{\varepsilon})\mathbb{I}_{\mu \geq \varepsilon})), \tag{4.18}
\]

where the dynamic \( (x_{t}, q_{t}) \) is given by \( x_{t} = x + \int_{0}^{t}(rI - R)x_{s}ds \) and \( q_{t} = q + \int_{0}^{t}Q_{s}ds \). We know that \( h^*(x, q) - V^{+, \ast}(x, q) \geq 0 \) by construction. In case \( h^*(x, q) - V^{+, \ast}(x, q) > 0 \), there exists by continuity \( \bar{\varepsilon} > 0 \) such that for all \( 0 < \varepsilon \leq \bar{\varepsilon} \), choosing \( t < \varepsilon \) would not be optimal in (4.18). Thus

\[
V^{+, \ast}(x, q) \geq e^{-r\varepsilon}V^{+, \ast}(x_{\varepsilon}, q_{\varepsilon}). \tag{4.19}
\]
We deduce that (4.16) holds since:
\[ \tilde{D}V^{+,*}(x,q;(rI - R)x,\bar{\langle}Qq) - rV^{+,*}(x,q) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( e^{-\varepsilon r}V^{+,*}(x,\varepsilon) - V^{+,*}(x,q) \right) \leq 0. \]
The last equality follows from the fact that \( V^{+,*} \) is Lipschitz, implying that:
\[ V^{+,*}(x,\varepsilon) - V^{+,*}(x + \varepsilon(rI - R)x,q + \varepsilon \bar{\langle}Qq) = o(\varepsilon). \]

\[ \]

**Proposition 4.8.** \( V^{+} \) is a subsolution of (3.1). \( V^{-} \) is a supersolution of (3.2).

**Proof.** Again it is sufficient to prove only the subsolution property for \( V^{+} \), as the proof of the supersolution property for \( V^{-} \) is similar due to the symmetry of the problem. We have to prove that for \( \bar{q} \in \Delta(L) \) and \( \bar{p} \in Ext(V(\cdot,\bar{q})) \), then \( V^{+}(\bar{p},\bar{q}) \leq f(\bar{p},\bar{q}) \) and if \( V^{+}(\bar{p},\bar{q}) > h(\bar{p},\bar{q}) \), then:
\[ rV^{+}(\bar{p},\bar{q}) - \tilde{D}_1V^{+}(\bar{p},\bar{q};\bar{\langle}R\bar{p})(\bar{q}) - \tilde{D}_2V^{+}(\bar{p},\bar{q};\bar{\langle}Q\bar{q}) \leq 0. \]  
(4.20)

We first assume that \( \bar{p} \) is an exposed point of \( V(\cdot,\bar{q}) \) (see Definition A.3) and then extend the property to extreme points.

We know that \( V^{+}(\bar{p},\bar{q}) \leq f(\bar{p},\bar{q}) \) by construction. Let us assume that \( V^{+}(\bar{p},\bar{q}) > h(\bar{p},\bar{q}) \). We will reformulate (4.20) using the conjugate function \( V^{+,*} \). Recall the definition of \( V^{+,*} \):
\[ V^{+,*}(x,q) := \inf_{p \in \Delta(K)} \langle x,p \rangle - V^{+}(p,q). \]

Let us choose \( \bar{x} \in \partial_1^+V^{+}(\bar{p},\bar{q}) \) and \( \bar{y} \in \partial_2^+V^{+}(\bar{p},\bar{q}) \) (see Lemma A.1), such that
\[ \tilde{D}_1V^{+}(\bar{p},\bar{q};\bar{\langle}R\bar{p})(\bar{q}) = \bar{x}, \quad \tilde{D}_2V^{+}(\bar{p},\bar{q};\bar{\langle}Q\bar{q}) = \bar{y}. \]

By construction \( V^{+}(\bar{p},\bar{q}) = \langle \bar{x},\bar{p} \rangle - V^{+,*}(\bar{x},\bar{q}) \) and (4.20) can be written as
\[ \langle (rI - R)\bar{x},\bar{p} \rangle - \langle \bar{Q}\bar{q},\bar{y} \rangle - rV^{+,*}(\bar{x},\bar{q}) \leq 0. \]  
(4.21)

As \( \bar{p} \) is an exposed point, we know that there exists some \( \bar{x} \in \partial_1^+V^{+}(\bar{p},\bar{q}) \) such that in the expression:
\[ \inf_{p \in \Delta(K)} \langle \bar{x},p \rangle - V^{+}(p,\bar{q}), \]
the minimum is uniquely attained in \( \bar{p} \). It follows that denoting \( u := \bar{x} - \bar{x} \), for all \( \varepsilon > 0 \), the minimum in the expression
\[ \inf_{p \in \Delta(K)} \langle \bar{x} + \varepsilon u,p \rangle - V^{+}(p,\bar{q}), \]
is uniquely attained in \( \bar{p} \). Fenchel’s lemma implies that the function \( V^{+,*}(\cdot,\bar{q}) \) is differentiable at \( \bar{x} + \varepsilon u \) with a gradient equal to \( \bar{p} \) and we have:
\[ V^{+,*}(\bar{x} + \varepsilon u,\bar{q}) = \langle \bar{x} + \varepsilon u,\bar{p} \rangle - V^{+}(\bar{p},\bar{q}). \]

Instead of proving directly (4.21), we will prove that for all \( \varepsilon > 0 \):
\[ \langle (rI - R)(\bar{x} + \varepsilon u),\bar{p} \rangle - \langle \bar{Q}\bar{q},\bar{y} \rangle - rV^{+,*}(\bar{x} + \varepsilon u,\bar{q}) \leq 0, \]  
(4.22)
and the conclusion will follow by sending $\varepsilon > 0$ to zero.

In order to apply Proposition 4.7, let us prove that $V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) < h^*(\bar{x} + \varepsilon u, \bar{q})$. By construction, we have $V^{+,*} \leq h^*$ since $V^+ \geq h$. Assume by contradiction that $V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = h^*(\bar{x} + \varepsilon u, \bar{q})$. Both functions being concave with respect to their first argument and since $D_1V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \bar{p}$, we would have

$$D_1h^*(\bar{x} + \varepsilon u, \bar{q}) = D_1V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \bar{p},$$

and therefore

$$V^+(\bar{p}, \bar{q}) = (\bar{x} + \varepsilon u, \bar{p}) - V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = (\bar{x} + \varepsilon u, \bar{p}) - h^*(\bar{x} + \varepsilon u, \bar{q}) = h(\bar{p}, \bar{q}),$$

which contradicts the assumption $V^+(\bar{p}, \bar{q}) - h(\bar{p}, \bar{q}) > 0$.

Note that $q \to V^+(\bar{p}, q)$ is convex on $\Delta(L)$ and that $\bar{y}$ was chosen so that the directional derivative verifies (see Lemma A.1 for the first equality):

$$\bar{D}_2V^+(\bar{p}, \bar{q}) = \{q\} = \max_{v \in \partial^*_2V^+(\bar{p}, \bar{q})} \langle v, \bar{Q} \bar{q} \rangle = \langle \bar{y}, \bar{Q} \bar{q} \rangle.$$ 

Since $V^{+,*}$ is a concave Lipschitz function on $\mathbb{R}^K \times \Delta(L)$, the envelope theorem (see Lemma A.2) implies:

$$\partial^+V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \{\bar{p}\} \times (-\partial_2^*V^+(\bar{p}, \bar{q})).$$

Indeed, note that the right-hand side of (4.23) is the superdifferential of the concave function:

$$(x, q) \in \mathbb{R}^K \times \Delta(L) \rightarrow \langle x, \bar{p} \rangle - V^+(\bar{p}, q),$$

at $(\bar{x} + \varepsilon u, \bar{q})$. We deduce that the directional derivatives of $V^{+,*}$ at $(\bar{x} + \varepsilon u, \bar{q})$ verify

$$\bar{D}V^{+,*}(\bar{x} + \varepsilon u, \bar{q}; (rI - R)\bar{x}, \bar{Q} \bar{q}) = \min_{w, v \in \partial^+V^{+,*}(\bar{x} + \varepsilon u, \bar{q})} \langle w, (rI - R)\bar{x} \rangle + \langle v, \bar{Q} \bar{q} \rangle = \langle \bar{p}, (rI - R)\bar{x} \rangle - \langle \bar{y}, \bar{Q} \bar{q} \rangle.$$ 

The conclusion follows therefore from Proposition 4.7.

It remains to extend the result from exposed points to extreme points. We know that exposed points are a dense subset of extreme points (see Theorem 18.6 in [32]). We may therefore use an approximation argument by using Lemma A.4 applied to $p \to -V(\cdot, \bar{q})$ together with the fact that $p \to \bar{D}_2V^+(p, \bar{q})$ is upper semi-continuous (use Lemma A.1 together with the fact that the correspondence of superdifferentials has closed graph).

4.4 Comparison principle

In the previous section we showed that $V^+(p, q)$ is a sub-solution to (3.1) while $V^-(p, q)$ verifies the super-solution property (3.2). Since $V^+(p, q) \geq V^-(p, q)$ by construction the following comparison principle will imply Theorem 3.2.

**Theorem 4.9.** Let $w_1, w_2$ be two Lipschitz, concave-convex functions defined on $\Delta(K) \times \Delta(L)$ such that $w_1$ verifies the sub-solution property (3.1) and $w_2$ verifies the super-solution property (3.2). Then $w_1 \leq w_2$. 

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Proof. We proceed by contradiction. Assume that $M := \max_{\Delta(K) \times \Delta(L)} w_1 - w_2 > 0$ and let $C$ denote the compact set of $(p, q) \in \Delta(K) \times \Delta(L)$ where the maximum is reached. Let $(\bar{p}, \bar{q}) \in C$ denote an extreme point of (the convex hull of) $C$. It follows that $\bar{p}$ is an extreme point of $w_1(\cdot, \bar{q})$ and that $\bar{q}$ is an extreme point of $w_1(\bar{p}, \cdot)$.

Let us prove this property for $\bar{p}$ (the case of $\bar{q}$ being symmetric). Assume that there exists $p_1, p_2 \in \Delta(K)$ and $\lambda \in (0, 1)$ such that $\bar{p} = \lambda p_1 + (1 - \lambda)p_2$ and $\lambda w_1(p_1, \bar{q}) + (1 - \lambda)w_1(p_2, \bar{q}) = w_1(\bar{p}, \bar{q})$. Using that $w_2(\cdot, \bar{q})$ is concave, we would have

$$\lambda(w_1(p_1, \bar{q}) - w_2(p_1, \bar{q})) + (1 - \lambda)(w_1(p_2, \bar{q}) - w_2(p_2, \bar{q})) \geq w_1(\bar{p}, \bar{q}) - w_2(\bar{p}, \bar{q}) = M.$$  

As $w_1 - w_2 \leq M$, we deduce that $(p_1, \bar{q})$ and $(p_2, \bar{q})$ belong to $C$ and therefore that $p_1 = p_2$.

At point $(\bar{p}, \bar{q})$, we have $w_1(\bar{p}, \bar{q}) > w_2(\bar{p}, \bar{q}) \geq h(\bar{p}, \bar{q})$ so that

$$rw_1(\bar{p}, \bar{q}) - \tilde{D}_1 w_1(\bar{p}, \bar{q}; \top R\bar{p}) - \tilde{D}_2 w_1(\bar{p}, \bar{q}; \top Q\bar{p}) \leq 0,$$

$$rw_2(\bar{p}, \bar{q}) - \tilde{D}_1 w_2(\bar{p}, \bar{q}; \top R\bar{p}) - \tilde{D}_2 w_2(\bar{p}, \bar{q}; \top Q\bar{p}) \geq 0.$$  

Note that $\top R\bar{p}$ (resp. $\top Q\bar{p}$) always belong to the tangent cone of $\Delta(K)$ at $p$ (resp. of $\Delta(L)$ at $q$) so that directional derivatives are well-defined and real-valued. We deduce that

$$\tilde{D}_1 w_1(\bar{p}, \bar{q}; \top R\bar{p}) + \tilde{D}_2 w_1(\bar{p}, \bar{q}; \top Q\bar{p}) \geq rw_1(\bar{p}, \bar{q}) \geq rw_2(\bar{p}, \bar{q}) + \frac{M}{r}$$

$$\geq \tilde{D}_1 w_2(\bar{p}, \bar{q}; \top R\bar{p}) + \tilde{D}_2 w_2(\bar{p}, \bar{q}; \top Q\bar{p}) + \frac{M}{r}.$$  

It follows that one of the following inequalities holds true:

$$\tilde{D}_1 w_1(\bar{p}, \bar{q}; \top R\bar{p}) > \tilde{D}_1 w_2(\bar{p}, \bar{q}; \top R\bar{p})$$

$$\tilde{D}_2 w_1(\bar{p}, \bar{q}; \top Q\bar{p}) > \tilde{D}_2 w_2(\bar{p}, \bar{q}; \top Q\bar{p}).$$  

In the first case, this would imply that for a sufficiently small $\varepsilon$,

$$w_1(\bar{p} + \varepsilon \top R\bar{p}, \bar{q}) - w_2(\bar{p} + \varepsilon \top R\bar{p}, \bar{q}) > w_1(\bar{p}, \bar{q}) - w_2(\bar{p}, \bar{q}) = M,$$

and thus a contradiction. The second case is similar and this concludes the proof.

\[\square\]

5 The probabilistic characterization

This section is devoted to the proof of Theorem 3.7. Let us define

$$W(p) := \sup_{\overrightarrow{\rho} \in \mathcal{K}(\rho)} \left( \sup_{\tau \in \mathcal{T}_\kappa} \inf_{\sigma \in \mathcal{T}_\kappa} \mathbb{E}_p[J'(\tau, \sigma)] \right),$$

where we have immediately

$$\forall p \in \Delta(K), \ h(p) \leq W(p) \leq f(p).$$  

The proof of Theorem 3.7 is divided in three parts: at first we establish that $W$ is concave. In the second part, we prove that $W$ is a supersolution of equation (3.2), and we deduce that $W \geq V$ using a slightly different version of the comparison principle Theorem 4.9. Finally, we prove that $W \leq V$ using a direct probabilistic approach which concludes the proof of Theorem 3.7 and also relates the maximizers of $W$ to the optimal strategies of player 1 in the original game.
5.1 Concavity of $W$

Despite the fact that the main idea to show the concavity of $W$ is simple and intuitive, its proof is surprisingly long and technical. The first element is the following lemma, which is itself very technical but relies deeply on existing results, so that its proof is postponed to the appendix.

**Lemma 5.1.** For any $p \in \Delta(K)$ and $\varepsilon > 0$, there exists a law $\mathbb{P}_\varepsilon \in \mathcal{K}(p)$ such that

$$\sup_{\mu \in T^\pi} \inf_{\tau \in T^\pi} \mathbb{E}_{\mathbb{P}_\varepsilon}[J'(\mu, \tau)] \geq W(p) - \varepsilon,$$

and for all $T > 0$, $(\pi_t)_{t \in [0,T)}$ takes only finitely many values under $\mathbb{P}_\varepsilon$.

**Proposition 5.2.** The function $W$ is concave on $\Delta(K)$.

**Proof.** Let $\varepsilon > 0$ and $C$ denote a bound on the uniform norm of $f, h$. For $i = 1,2$, let $p_i \in \Delta(K)$, $p_i \neq p_2$ and $\mathbb{P}_i \in \mathcal{K}(p_i)$ be an $\varepsilon$-optimal probability for the maximization problem $W(p_i)$ given by Lemma 5.1. In the following, in order to avoid confusions, $\pi^i$ will denote the canonical process under the probability $\mathbb{P}_i$.

At first, we need to slightly perturb the probabilities $\mathbb{P}_1, \mathbb{P}_2$. We claim that there exists $\delta \in [0, \frac{\varepsilon}{\mathbb{P}_1}]$ such that if we define the processes:

$$\forall i = 1,2, \forall x \in \mathbb{R}^K, \quad T_{i,t}(x) := (1 - \delta)x + \delta e^{tr}p_i,$$

then $\hat{\pi}^1_0$ and $\hat{\pi}^2_0$ have disjoint finite supports (under the respective probabilities $\mathbb{P}_1$ and $\mathbb{P}_2$).

Indeed, using that $\pi^1_0$ and $\pi^2_0$ have finite supports $S_1, S_2$, this follows directly from the fact that there exists $\delta \in [0, \frac{\varepsilon}{\mathbb{P}_1}]$ such that:

$$((1 - \delta)S_1 + \delta\{p_1\}) \cap ((1 - \delta)S_2 + \delta\{p_2\}) = \emptyset.$$

Let us denote $\hat{\mathbb{P}}_i$ the law of $\hat{\pi}^i$ and

$$\forall x \in \mathbb{R}^K, \quad T_{i,t}(x) := (1 - \delta)x + \delta e^{tr}p_i.$$

We claim that:

$$\sup_{\hat{\tau} \in T^\pi} \inf_{\hat{\sigma} \in T^\pi} \mathbb{E}_{\hat{\mathbb{P}}_i}[J'(\hat{\tau}, \hat{\sigma})] = \sup_{\tau \in T^\pi} \inf_{\sigma \in T^\pi} \mathbb{E}_{\mathbb{P}_i}[1_{\tau \leq \sigma}e^{-\tau r}h(\hat{\pi}^i) + 1_{\sigma < \tau}e^{-\tau r}f(\hat{\pi}^i)]. \quad (5.1)$$

Indeed, if $\omega_i \in \mathcal{D}([0, +\infty), \Delta(K))$ denotes a trajectory, to any stopping time $\hat{\sigma} \in T^\pi$ we can associate the stopping time $\sigma(\omega) := \hat{\sigma}(T_{i.}(\omega))$ and to any stopping time $\tau \in T^\pi$ we can associate the stopping time $\tau(\omega) := \tau(P_K(T_{i.}^{-1}(\omega)))$ where $P_K$ denotes the orthogonal projection on $\Delta(K)$ (say). With these definitions, we deduce that $\sigma(\hat{\pi}^i) = \hat{\sigma}(\hat{\pi}^i)$ (everywhere) and that $\tau(\hat{\pi}^i) = \tau(\pi^i)$ ($\hat{\mathbb{P}}_i$-almost-surely). The same construction also holds when inverting the role of $\sigma$ and $\tau$ and this proves (5.1).

Given two stopping times $\sigma, \tau \in T^\pi$, using that $h$, and $f$ are affine, we have

$$\mathbb{E}_{\mathbb{P}_i}[1_{\tau \leq \sigma}e^{-\tau r}h(\hat{\pi}^i_\tau) + 1_{\sigma < \tau}e^{-\tau r}f(\hat{\pi}^i_\sigma)] = (1 - \delta)\mathbb{E}_{\mathbb{P}_i}[1_{\tau \leq \sigma}e^{-\tau r}h(\pi^i_\tau) + 1_{\sigma < \tau}e^{-\tau r}f(\pi^i_\sigma)] + \delta \mathbb{E}_{\mathbb{P}_i}[1_{\tau \leq \sigma}e^{-\tau r}h(e^{\tau R}p_i) + 1_{\sigma < \tau}e^{-\tau r}f(e^{\tau R}p_i)].$$
We deduce that
\[
\sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}_p [J'(\tau, \sigma)] = \mathbb{E}_\bar{p} [J'(\tau, \sigma)].
\]
and therefore, combining this inequality with (5.1), that for \(i = 1, 2\), the probability \(\hat{p}_i\) is 2\(\varepsilon\)-optimal for \(W(p_i)\).

Given \(\lambda \in [0, 1]\), let us define a measure \(\tilde{p}\) on \([1, 2] \times \mathcal{D}(\mathbb{R}_+, \Delta(K))\) in the following way: If \((d, \pi)\) denotes the canonical element of \([1, 2] \times \mathcal{D}(\mathbb{R}_+, \Delta(K))\), then the random variable \(d\) has law \((\lambda, (1 - \lambda))\) and the conditional law of \(\pi\) given \(d\) equals \(\tilde{p}_d\).

We can check easily that under \(\tilde{p}\), the law of the canonical process \(\pi\) is \(p := \lambda \tilde{p}_1 + (1 - \lambda)\tilde{p}_2\) and belongs to \(K(p)\) for \(p := \lambda p_1 + (1 - \lambda)p_2\).

Let us denote \((\mathcal{F}_t)_{t \geq 0}\) the augmentation of the filtration \(\mathcal{F}^{\pi,+}\) by \(\tilde{p}\)-negligible sets on \([1, 2] \times \mathcal{D}(\mathbb{R}_+, \Delta(K))\). For \(i = 1, 2\), due to the construction of the probability \(\hat{p}_i\), we have:

\[
\hat{p}(d = i) = \tilde{p}(\pi_0 \in \{(1 - \delta)S_i + \delta p_i)\}).
\]

It follows that \(d\) is \(\mathcal{F}_0\)-measurable.

Let us denote \(\mathcal{T}^{\pi,d}\) the set of stopping times adapted to the filtration \(\sigma(d) \vee \mathcal{F}^{\pi,+}\). Since \(d\) is \(\mathcal{F}_0\)-measurable, any stopping time in \(\mathcal{T}^{\pi,d}\) is an \(\mathcal{F}\)-stopping time, and therefore is almost surely equal to an \(\mathcal{F}^{\pi,+}\)-stopping time (see Lemma I.1.19 in [23]). We deduce that

\[
\sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}[J' \tau, \sigma] = \sup_{\tau \in \mathcal{T}^{\pi,d}} \inf_{\sigma \in \mathcal{T}^{\pi,d}} \mathbb{E}[J' \tau, \sigma].
\]

On the other hand, the stopping times \(\sigma' \in \mathcal{T}^{\pi,d}\) are exactly the variables \(\sigma' = \sigma_1 \mathbb{1}_{d=1} + \sigma_2 \mathbb{1}_{d=2}\) where \(\sigma_i\) are stopping times in \(\mathcal{T}^{n}\). By conditioning with respect to \(d\), it follows that:

\[
W(p) \geq \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}[J' \tau, \sigma] = \sup_{\tau \in \mathcal{T}^{\pi,d}} \inf_{\sigma \in \mathcal{T}^{\pi,d}} \mathbb{E}[J' \tau, \sigma] = \lambda \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}[J' \tau, \sigma] + (1 - \lambda) \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbb{E}[J' \tau, \sigma]
\]

\[
\geq \lambda W(p_1) + (1 - \lambda) W(p_2) - 2\varepsilon.
\]

The concavity of \(W\) follows since \(\varepsilon\) can be chosen arbitrarily small. \(\square\)

**Remark 5.3.** *Recall that since \(W\) is concave on \(\Delta(K)\), it is lower-semi-continuous, hence measurable (see e.g. Theorem 10.2 in [32]).*

### 5.2 \(W \geq V\) using a variational method

We prove at first that \(W\) satisfies a super dynamic programming principle.

We recall the definition of the shift operator \(\theta\) on \(\mathcal{D}(\mathbb{R}_+, \Delta(K))\).

**Definition 5.4.** For all \(t \geq 0\), the map \(\theta_t : \mathcal{D}(\mathbb{R}_+, \Delta(K)) \to \mathcal{D}(\mathbb{R}_+, \Delta(K))\) is defined by

\[
\forall s \geq 0, \theta_t(\omega)(s) = \omega(s + t).
\]
Proposition 5.5. For all $p \in \Delta(K)$ and $\varepsilon > 0$:

$$W(p) \geq \inf_{s \in [0, \varepsilon]} \mathbb{1}_{s < \varepsilon} e^{-r s} f(p_s) + \mathbb{1}_{s = \varepsilon} e^{-r \varepsilon} W(p_\varepsilon),$$

where for all $t \geq 0$, $p_t := p + \int_{0}^{t} R p_s ds = e^{t R} p$.

Proof. Let us consider the subset $\mathcal{K}_\varepsilon(p)$ of probabilities $\mathbb{P}$ in $\mathcal{K}(p)$ such that the process $(\pi_t)_{t \geq 0}$ equals $(p_t)_{t \geq 0}$ on the interval $[0, \varepsilon)$.

Given any law $\mathbb{P}_\varepsilon$ in $\mathcal{K}(p_\varepsilon)$, we associate a probability $\mathbb{P} \in \mathcal{K}_\varepsilon(p)$ by defining $\pi_t = p_t$ for $t \in [0, \varepsilon)$ and then $\pi_t = \pi_{t-}$ for $t \geq \varepsilon$, where $\pi$ has law $\mathbb{P}_\varepsilon$. Reciprocally, for any law $\mathbb{P}$ in $\mathcal{K}_\varepsilon(p)$, the process $(\pi_t)_{t \geq 0}$ has a law $\mathbb{P}_\varepsilon \in \mathcal{K}(p_\varepsilon)$.

Let $\mathcal{T}_\varepsilon$ the set of stopping times $\tau \in \mathcal{T}_\varepsilon$ such that there exists a stopping time $\tau' \in \mathcal{T}_\varepsilon$ with $\tau = \varepsilon + \tau'((\theta_\varepsilon(\omega))$ where $\theta$ denotes the shift operator on the canonical space $\mathbb{D}([0, \infty), \Delta(K))$.

Recall that for any $\sigma \in \mathcal{T}_\varepsilon$, there exists a map $\sigma'(\omega, \omega')$ defined on $\mathbb{D}([0, \infty), \Delta(K))^2$ which is $\mathcal{F}_{\varepsilon-} \otimes \mathcal{F}_\infty$ measurable, such that $\sigma'(\omega, .)$ is an $\mathcal{F}_{\varepsilon}^{\sigma'}$ stopping time for all $\omega$ and $\sigma(\omega) = \varepsilon + \sigma'(\omega, \theta_\varepsilon(\omega))$ on the event $\sigma \geq \varepsilon$ (the proof is similar to theorem 103 p.151 in Dellacherie-Meyer [9]). It follows that:

$$W(p) \geq \sup_{\mathbb{P} \in \mathcal{K}_\varepsilon(p)} \sup_{\tau \in \mathcal{T}_\varepsilon} \sup_{\sigma \in \mathcal{T}_\varepsilon} \inf \mathbb{E}_\mathbb{P}[J'(\tau, \sigma)]$$

$$= \sup_{\mathbb{P} \in \mathcal{K}_\varepsilon(p)} \sup_{\tau \in \mathcal{T}_\varepsilon} \sup_{\sigma \in \mathcal{T}_\varepsilon} \inf \mathbb{E}_\mathbb{P}[\mathbb{1}_{\sigma < \varepsilon} e^{-r \sigma} f(p_\sigma) + \mathbb{1}_{\sigma \geq \varepsilon} e^{-r \varepsilon} \mathbb{E}_{\mathbb{P}_\varepsilon}[J'(\tau', \sigma'(\omega, .))]].$$

The second line of the above system was obtained by taking conditional expectation given $\mathcal{F}_{\varepsilon-}$ and using that $(\pi_{\varepsilon+t})_{t \geq 0}$ is independent of $\mathcal{F}_{\varepsilon-}$.

Given $\eta > 0$, for all $\mathbb{P} \in \mathcal{K}_\varepsilon(p)$, we can choose $\tau' \in \mathcal{T}_\varepsilon$ $\eta$-optimal in the problem

$$H(\mathbb{P}_\varepsilon) := \sup_{\tau' \in \mathcal{T}_\varepsilon} \inf_{\sigma' \in \mathcal{T}_\varepsilon} \mathbb{E}_{\mathbb{P}_\varepsilon}[J'(\tau', \sigma')].$$

It follows that

$$W(p) \geq \sup_{\mathbb{P} \in \mathcal{K}_\varepsilon(p)} \inf_{\sigma \in \mathcal{T}_\varepsilon} \mathbb{E}_{\mathbb{P}_\varepsilon}[J'(\tau, \sigma')] + \mathbb{P}(\sigma \geq \varepsilon) \left(e^{-r \varepsilon} H(\mathbb{P}_\varepsilon) - \eta\right).$$

One may clearly replace the infimum by an infimum over deterministic times $t \in \mathbb{R}_+$, using either linearity or the fact that the event $\tau < \varepsilon$ has probability 0 or 1 under $\mathbb{P}$ and that $\tau$ is almost surely constant on this event. We obtain:

$$W(p) \geq \sup_{\mathbb{P} \in \mathcal{K}_\varepsilon(p)} \inf_{t \in \mathbb{R}_+} \mathbb{1}_{t < \varepsilon} e^{-r t} f(p_t) + \mathbb{1}_{t \geq \varepsilon} \left(e^{-r \varepsilon} H(\mathbb{P}_\varepsilon) - \eta\right).$$

To conclude, note that we can choose $\mathbb{P} \in \mathcal{K}_\varepsilon(p)$ such that $\mathbb{P}_\varepsilon$ is $\eta$-optimal in the problem

$$\sup_{\mathbb{P}_\varepsilon \in \mathcal{K}(p_\varepsilon)} H(\mathbb{P}_\varepsilon),$$

and we obtain:

$$W(p) \geq \inf_{t \in \mathbb{R}_+} \mathbb{1}_{t \in \varepsilon} e^{-r t} f(p_t) + \mathbb{1}_{t \geq \varepsilon} \left(e^{-r \varepsilon} W(p_\varepsilon) - 2\eta\right).$$

The conclusion follows by sending $\eta$ to zero. \qed
In order to avoid proving regularity results for \( W \) (which would lead to very technical proofs as it was the case for the concavity), we prove now that \( W \) is supersolution of a modified version of (3.2).

**Proposition 5.6.** For all \( p \in \Delta(K) \), we have:

\[
\max\{\min\{rW(p) - G(W,p) ; W(p) - h(p)\} ; W(p) - f(p)\}\geq 0
\]

where

\[
G(W,p) := \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}(W(p_\varepsilon) - W(p))
\]

**Proof.** We know that \( W \) is lower semicontinuous on \( \Delta(K) \) and that \( W(p) - h(p) \geq 0 \). Let us assume that \( W(p) - f(p) < 0 \). It is then sufficient to prove that \( G(W,p) \geq 0 \). Using the dynamic programming inequality we know that

\[
W(p) \geq \inf_{t \in [0,\varepsilon]} \mathbbm{1}_{p_\varepsilon} e^{-rt}f(p_t) + \mathbbm{1}_{t=\varepsilon} e^{-r\varepsilon}W(p_\varepsilon).
\]

Since \( W(p) - f(p) < 0 \), it implies the existence of some \( \bar{\varepsilon} \) such that for all \( 0 < \varepsilon \leq \bar{\varepsilon} \):

\[
W(p) \geq e^{-r\varepsilon}W(e^{\varepsilon R}p),
\]

and therefore

\[
\frac{1}{\varepsilon}(W(p) - e^{-r\varepsilon}W(p_\varepsilon)) = \frac{1 - e^{-r\varepsilon}}{\varepsilon}W(p) - \frac{e^{-r\varepsilon}}{\varepsilon}(W(p_\varepsilon) - W(p)) \geq 0,
\]

and the conclusion follows by sending \( \varepsilon \) to zero. \( \square \)

Let us now adapt the proof of Theorem 4.9 to deduce that \( W \geq V \).

**Corollary 5.7.** For all \( p \in \Delta(K) \), \( W(p) \geq V(p) \).

**Proof.** Assume by contradiction that \( M := \max_{\Delta(K) \times \Delta(L)} V - W > 0 \) and let \( C \) denote the compact set of \( p \) which realize the maximum (\( V - W \) is upper semi-continuous). Let \( \bar{p} \in C \) denote an extreme point of (the convex hull of) \( C \). It follows as in Theorem 4.9 that \( \bar{p} \) is an extreme point of \( V \).

At point \( \bar{p} \), we have \( f(\bar{p}) \geq V(\bar{p}) > W(\bar{p}) \geq h(\bar{p}) \) so that

\[
rV(\bar{p}) - \hat{D}V(\bar{p}, \mathbb{T}R\bar{p}) \leq 0, \quad rW(\bar{p}) - G(W,\bar{p}) \geq 0.
\]

The second inequality implies that \( G(W,\bar{p}) < +\infty \). We deduce that

\[
\hat{D}V(\bar{p}, \mathbb{T}R\bar{p}) \geq rV(\bar{p}) \geq rW(\bar{p}) + \frac{M}{r} \geq G(W,\bar{p}) + \frac{M}{r}.
\]

It follows that \( \hat{D}V(\bar{p}, \mathbb{T}R\bar{p}) > G(W,\bar{p}) \) so that if we define \( p_t = e^{t \mathbb{T}R}\bar{p} \), then for a sufficiently small \( \varepsilon \),

\[
V(p_\varepsilon) - W(p_\varepsilon) > V(\bar{p}) - W(\bar{p}) = M,
\]

which leads a contradiction using the definition of \( G(W,\bar{p}) \) and the fact that

\[
\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}(V(p_\varepsilon) - V(\bar{p})) = \hat{D}V(\bar{p}, \mathbb{T}R\bar{p}).
\]

To prove the last assertion, note that \( V \) is Lipschitz so that \( V(p_\varepsilon) - V(\bar{p} + \varepsilon \mathbb{T}R\bar{p}) = o(\varepsilon) \). \( \square \)
5.3 \( W \leq V \) using a probabilistic method

In this subsection, we prove the inequality \( W \leq V \) which concludes the proof of Theorem 3.7. The next result allows to relate \( \varepsilon \)-optimizers in the problem \( W(p) \) to \( \varepsilon \)-optimal strategies of player 1 in the initial game.

**Theorem 5.8.** For any \((\mathbb{P}, \tau) \in \mathcal{K}(p) \times \mathcal{T}_\pi\) there exists \( \mu \in \mathcal{T}_r^X \) such that:

\[
\forall s \in \mathbb{R}_+, \quad \mathbb{E}_\mathbb{P}[J'(\tau, s)] = \mathbb{E}_\mathbb{P}[\tilde{J}(\mu, s)]
\]

We will need the following set of laws.

**Notation 5.9.** Let \( \mathcal{K}'(p) \subset \Delta(\mathbb{D}([0, \infty), K \times \Delta(K))) \) denote the set of laws of càdlàg processes \((X_t, \pi_t)_{t \geq 0}\) taking values in \( K \times \Delta(K) \) such that \( X \) is an \( \mathcal{F}^{(X, \pi)}\)-Markov chain of law \( \mathbb{P}_p \) and such that for all \( t \geq 0 \)

\[
\forall k \in K, \quad \mathbb{P}(X_t = k|\mathcal{F}_t^X) = \pi_t^k.
\]

**Remark 5.10.** Equation (5.2) expresses the fact that \( \pi \) is a belief process about \( X \) and it follows easily that the law of \((\pi_t)_{t \geq 0}\) belongs to \( \mathcal{K}(p) \) and that (5.2) extends to arbitrary stopping times \( \tau \) of \( \mathcal{F}^{X, +} \) on the set \( \tau < \infty \) (the property holds for stopping times taking finitely many values and then can then be extended using right-continuity of the processes and filtrations).

The following representation lemma is the continuous-time analog of Lemma 4 in [8]. The proof of Lemma 5.11 is postponed to the appendix.

**Lemma 5.11.** For any process law \( \mathbb{P} \in \mathcal{K}(p) \), there exists a process \((X_t, \pi_t)_{t \geq 0}\) with law in \( \mathbb{P}' \in \mathcal{K}'(p) \) such that \((\pi_t)_{t \geq 0}\) has law \( \mathbb{P} \).

**Proof of Theorem 5.8.** Let \( \mathbb{P} \in \mathcal{K}(p) \) and let \((X_t, \pi_t)_{t \geq 0}\) be the associated process given by Lemma 5.11 with law in \( \mathbb{P}' \in \mathcal{K}'(p) \). Let \( \tau \in \mathcal{T}_\pi \) and \( s \in \mathbb{R}_+ \), then using (5.2) we have:

\[
\mathbb{E}_{\mathbb{P}'}[e^{-rs}f(X_s)|\mathcal{F}_s^\pi] = \mathbb{E}_{\mathbb{P}'}[f(X_s)|\mathcal{F}_s^\pi] = \mathbb{1}_{s<\tau}e^{-rs}f(\pi_s).
\]

Using the same arguments with \( h \), we deduce that:

\[
\mathbb{E}_{\mathbb{P}'}[J'(\tau, s)] = \mathbb{E}_{\mathbb{P}'}[e^{-rs}f(X_s)\mathbb{1}_{s<\tau} + e^{-\tau r}h(X_\tau)\mathbb{1}_{\tau \geq s}].
\]

The construction of \( \mu \) relies on well-known ideas (see e.g. [27, 33, 34]). However, we provide a precise proof as it is quite short and the cited references do not cover the last step we need here. Define the process \((M_t)_{t \in [0, +\infty]}\) as a càdlàg non-decreasing version of:

\[
t \mapsto \mathbb{P}'(\tau \leq t|\bar{\mathcal{F}}_\infty^X),
\]

where \( \bar{\mathcal{F}}_\infty^X \) denotes the usual augmentation of \( \mathcal{F}^X \) with \( \mathbb{P}' \)-null sets. Using that \( X \) is an \( \mathcal{F}^{(X, \pi), +} \)-Markov process, we have that for all \( t \geq 0 \), \( M_t = \mathbb{P}'(\tau \leq t|\bar{\mathcal{F}}_t^X) \) almost surely, which proves that \( M \) is \( \mathcal{F}^X \)-adapted.

By construction, it holds for all \( t \in \mathbb{R}_+ \), \( \mathbb{P}'(\tau > t|\bar{\mathcal{F}}_\infty^X) = M_\infty - M_t \). Therefore, we have for any bounded \( \mathcal{F}^X_\infty \)-measurable process \((Z_t)_{t \in [0, +\infty]}\)

\[
\mathbb{E}_{\mathbb{P}'}[Z_\tau] = \mathbb{E}_{\mathbb{P}'} \left[ \int_{[0, \infty)} Z_s dM_s \right].
\]

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Indeed, the equality is true for processes \( Z = \mathbb{1}_{(T, +\infty) \times A} \) for \( A \in \mathcal{F}_X^\infty \) and then the conclusion follows from the monotone class theorem as both sides of the equation define a measure on \( \Omega \times [0, +\infty] \).

Then, consider the product space \((\Omega' \times [0, 1], \mathcal{F} \otimes \mathcal{B}([0, 1]), \mathbb{P}' \otimes \text{Leb})\) where \([0, 1]\) is endowed with the Lebesgue measure \(\text{Leb}\) and \(\Omega'\) denotes the canonical space on which the process \((X, \pi)\) is defined. Let \(\mathbb{P}' := \mathbb{P} \otimes \text{Leb}\) denote the product probability on this set and let us identify random variables defined on \(\Omega'\) with random variables defined on \([0, 1] \times \Omega'\) so that \(\bar{\mathcal{F}}_X^\infty\) can be seen as a filtration on \(\Omega' \times [0, 1]\) (we do not complete the filtration with \(\bar{\mathbb{P}}'\)-null sets).

Define \(\bar{\mu}(u, \omega) := \inf\{t \mid M_t(\omega) \geq u\}\). Note that \(\bar{\mu}(\cdot, \omega)\) is non-decreasing and left-continuous and that by construction \(t < \bar{\mu}(u, \omega) \iff M_t(\omega) < u\).

This implies that \(S\) is a mixed stopping time of \(\bar{\mathcal{F}}_X^\infty\) and we deduce easily that for any bounded \(\bar{\mathcal{F}}_X^\infty\)-measurable process \(Z\), we have

\[
\mathbb{E}_{\mathbb{P}'}[Z_{\bar{\mu}}] = \mathbb{E}_{\mathbb{P}'} \left[ \int_{[0, \infty]} Z_s dM_s \right].
\]

Indeed, the equality holds for processes \(Z = \mathbb{1}_{(T, +\infty) \times A}\) for \(A \in \mathcal{F}_X^\infty\) and the conclusion follows as above from the monotone class theorem.

We deduce that for any bounded \(\bar{\mathcal{F}}_X^\infty\)-measurable process \((Z_t)_{t \in [0, +\infty]}\),

\[
\mathbb{E}_{\mathbb{P}'}[Z_T] = \mathbb{E}_{\mathbb{P}'}[\int_{[0, +\infty]} Z_t dM_t] = \mathbb{E}_{\mathbb{P}'}[Z_{\bar{\mu}}].
\]

Given any \(s \in \mathbb{R}_+\), we may apply this result to the process

\[
t \to e^{-rs} f(X_s) \mathbb{1}_{s < t} + e^{-rt} h(X_t) \mathbb{1}_{t \geq s}.
\]

We conclude that

\[
\mathbb{E}_{\mathbb{P}'}[J'(\tau, s)] = \mathbb{E}_{\mathbb{P}'}[J(\bar{\mu}, s)].
\]

It remains to prove that \(\bar{\mu}\) is \(\mathbb{P}'\)-almost surely equal to a mixed stopping time \(\mu \in \mathcal{T}^\infty_X\). Indeed, the definition of \(\mathcal{T}^\infty_X\) requires \(\mu(u, .)\) to be a stopping time of the non-augmented filtration \(\mathcal{F}^X\) for all \(u \in [0, 1]\).

We will use that \(\bar{\mu}(\cdot, \omega)\) is non-decreasing and left-continuous for all \(\omega\) and that for all \(u \in [0, 1]\), \(\bar{\mu}(u, .)\) is a stopping time of \(\bar{\mathcal{F}}_X^\infty\). Recall that any stopping time of \(\bar{\mathcal{F}}_X^\infty\) is \(\mathbb{P}'\)-almost surely equal to a stopping time of \(\mathcal{F}^X\) (see e.g. Lemma I.1.19 in [23]). For all \(u \in [0, 1] \cap \mathbb{Q}\), let \(\bar{\mu}_u\) a stopping time of \(\mathcal{F}^X\) which is almost surely equal to \(\bar{\mu}(u, .)\).

There exists a \(\mathbb{P}'\)-negligible subset \(N\) such that for all \(\omega \in N^c\), we have

\[
\forall u \in [0, 1] \cap \mathbb{Q}, \bar{\mu}(u, \omega) = \bar{\mu}_u(\omega).
\]

Define \(\mu\) by

\[
\mu(u, \omega) := \sup_{u' \in [0, 1] \cap \mathbb{Q}, u' < u} \bar{\mu}_{u'}(\omega).
\]
\( \mu(u, \cdot) \) is an \( \mathcal{F}^X \)-stopping time as a supremum of \( \mathcal{F}^X \)-stopping times and \( S^* \) is jointly measurable as it is left-continuous with respect to \( u \) and \( \mathcal{F}^X_\infty \)-measurable with respect to \( \omega \). By construction, for all \( \omega \in \mathcal{N}^c \), we have

\[
\forall u \in [0, 1], \, \mu(u, \omega) = \bar{\mu}(u, \omega).
\]

We deduce that

\[
\mathbb{E}_p[J'(\tau, s)] = \mathbb{E}_{\bar{p}'}[J(\bar{\mu}, s)] = \mathbb{E}_p[J(\mu, s)]
\]

which concludes the proof. \( \square \)

As consequence of this result we have the following corollary which concludes the proof of Theorem 3.7.

**Corollary 5.12.** For all \( p \in \Delta(K) \), \( V(p) \geq W(p) \).

**Proof.** Let us choose \((\bar{p}^*, \tau^*)\) to be \( \varepsilon \)-optimal for \( W(p) \) and let \((X, \pi)\) be an associated process of law \( \bar{p}' \in K'(p) \) by Lemma 5.11. Using the notation of Proposition 5.8, the associated stopping time \( \mu^* \) is such that:

\[
V(p) \geq \inf_{s \in \mathbb{R}^+} \mathbb{E}_p[J(\mu^*, s)] = \inf_{s \in \mathbb{R}^+} \mathbb{E}_{\bar{p}'}[J(\bar{\mu}, s)] \geq \inf_{\sigma \in T^*} \mathbb{E}_{\bar{p}'}[J(\tau^*, \sigma)] \geq W(p) - \varepsilon.
\]

We deduce that \( V \geq W \) by sending \( \varepsilon \) to zero. \( \square \)

**Remark 5.13.** In particular, the above inequalities prove that the stopping time \( \mu^* \) associated to \((\bar{p}^*, \tau^*)\) is \( \varepsilon \)-optimal for player 1 in the stopping game with value \( V(p) \).

## A Technical proofs and auxiliary tools

### A.1 Auxiliary results of convex analysis

We prove here some elementary results in convex analysis that we are using in the proof of Proposition 4.7. The following lemmas are easy adaptations of classical and well-known results to Lipschitz convex functions with polyhedral domains. As references covering exactly what we need were difficult to find, we decided to add this appendix for the convenience of the reader. Note that we do not try to provide the most general version of these lemmas.

**Lemma A.1.** Let \( f \) be a Lipschitz convex function from \( \mathbb{R}^n \) to \( \mathbb{R} \cup \{+\infty\} \). Assume that \( C = \text{Dom}(f) := \{x \in \mathbb{R}^n | f(x) < +\infty\} \) is a polyhedron, then if \( T_C(x) \) denotes the tangent cone of \( C \) at \( x \):

\[
\forall x \in C, \forall v \in T_C(x), \partial f(x; v) = \max_{u \in \partial^{-} f(x)} \langle u, x \rangle.
\]

**Proof.** This relation holds for any point \( x \) in the relative interior of \( C \) (see Theorem 23.4 in [32]). For \( x \) in the relative boundary of \( C \), the left hand-side of the above equality is always greater than the right-hand side using the definition of subgradients.

For \( m \in \mathbb{N}^* \) with \( m \geq M \), where \( M \) is the Lipschitz constant of \( f \), define the Moreau-Yosida regularization

\[
\forall x \in \mathbb{R}^n, \, f_m(x) := \inf_{y \in C} f(y) + m|y - x|.
\]
The function \( f_n \) is convex, \( m \)-Lipschitz and coincides with \( f \) on \( C \). As \( f_m \leq f \), we have for all \( x \in C, \partial^- f_m(x) \subset \partial^- f(x) \). For any \( v \in T_C(x) \), we have that \( x + tv \in C \) for all sufficiently small \( t > 0 \) (here we use the polyhedron assumption) and therefore:

\[
\tilde{D}f(x; v) = \tilde{D}f_n(x; v) = \sup_{u \in \partial^- f_n(x)} \langle u, x \rangle \leq \sup_{u \in \partial^- f(x)} \langle u, x \rangle.
\]

\[\square\]

**Lemma A.2** (Danskin). Let \( P \) be a non-empty compact subset of \( \mathbb{R}^m \) and \( C \) a non-empty polyhedron in \( \mathbb{R}^n \). Let \( f \) be a real-valued Lipschitz function defined on \( P \times C \). Assume that for all \( p \in P \), the function \( f_p \) defined on \( \mathbb{R}^n \) by \( f_p(x) = f(p, x) \) is a convex function. Define \( g(x) = \sup_{p \in P} f(p, x) \). Then, if for \( \bar{x} \in C \), the maximum \( \max_{p \in P} f(p, \bar{x}) \) is uniquely attained in \( \bar{p} \), we have

\[
\partial^- g(\bar{x}) = \partial^- f_{\bar{p}}(\bar{x}).
\]

**Proof.** Note that both sets are non-empty due to the Lipschitz assumption, and that the inclusion \( \partial^- f_{\bar{p}}(\bar{x}) \subset \partial^- g(\bar{x}) \) is a direct consequence of the definitions. Let us prove the reverse inclusion. Assume by contradiction the there exists \( v \in \partial^- g(\bar{x}) \setminus \partial^- f_{\bar{p}}(\bar{x}) \). Then, using a separation argument, there exists \( \epsilon > 0 \) such that:

\[
\forall u \in \partial^- f_{\bar{p}}(\bar{x}), \langle z, v \rangle \geq \epsilon + \langle z, u \rangle. \tag{A.1}
\]

For all \( u \in \partial^- f_{\bar{p}}(\bar{x}), w \) in the normal cone of \( C \) at \( \bar{x} \) and \( j \in \mathbb{N}^* \), we have \( u + jw \in \partial^- f_{\bar{p}}(\bar{x}) \). Replacing \( u \) bu \( u + jw \) in (A.1), and taking the limit as \( j \to \infty \), we deduce that \( \langle z, w \rangle \leq 0 \), implying that \( z \) belongs to the tangent cone of \( C \) at \( \bar{x} \). Since \( C \) is a polyhedron, \( \bar{x} + tz \in C \) for all \( t \in (0, \alpha) \) for some \( \alpha > 0 \). Up to replace \( z \) by \( az \), we may assume that \( \bar{x} + z \in C \). Let \( p_n \) be a sequence maximizing \( f(p, \bar{x} + \frac{z}{n}) \). The sequence \( p_n \) converges to \( \bar{p} \) using the continuity of \( f \) and \( g \). For \( s \in (0, 1) \) and \( n \) such that \( \frac{1}{n} \leq s \) we have

\[
\frac{f(p_n, \bar{x} + sz) - f(p_n, \bar{x})}{s} \geq \frac{f(p_n, \bar{x} + \frac{z}{n}) - f(p_n, \bar{x})}{n^{-1}} \geq \frac{g(\bar{x} + \frac{z}{n}) - g(\bar{x})}{n^{-1}} \geq \langle z, v \rangle.
\]

Letting \( n \) go to \( +\infty \), we deduce that for all \( s \in (0, 1) \),

\[
\frac{f(\bar{p}, \bar{x} + sz) - f(\bar{p}, \bar{x})}{s} \geq \langle z, v \rangle.
\]

Taking the limit when \( s \to 0^+ \) and using the preceding lemma, we conclude that

\[
Df_{\bar{p}}(\bar{x}, z) = \sup_{u \in \partial^- f_{\bar{p}}(\bar{x})} \langle z, u \rangle \geq \langle z, v \rangle,
\]

which is a contradiction. \(\square\)

Before stating the next lemma, let us recall the definition of an exposed point.

**Definition A.3.** Let \( f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \) be a convex function. The set of exposed points of \( f \) is defined as the set of all \( x \in \mathbb{R}^n \) such that there exists \( x^* \in \partial^- f(x) \) such that

\[
\forall x' \neq x, f(x') > f(x) + \langle x^*, x' - x \rangle.
\]
Lemma A.4. Let $C \subset \mathbb{R}^n$ be a compact polyhedron and $f : C \to \mathbb{R}$ a convex Lipschitz function. If $x$ is an extreme point of $f$ and $z$ in the tangent cone of $C$ at $x$, then there exists a sequence $x_n$ of exposed points of $f$ with limit $x$ such that

$$\bar{D}f(x_n; z) \to \bar{D}f(x; z).$$

Proof. Let us choose $y \in \partial^- f(x)$ such that $\bar{D}f(x; z) = (z, y)$. We claim that $x$ is an extreme point of $\partial^- f^*(y)$. Note at first that Fenchel’s Lemma implies that $x \in \partial^- f^*(y)$. Assume then that $x$ is not an extreme point of $\partial^- f^*(y)$, so that there exists a segment $(x_1, x_2)$ containing $x$ and included in $\partial^- f^*(y)$. It follows that $y \in \partial^- f(x')$ for all $x' \in (x_1, x_2)$, and thus that $f$ restricted to this segment is affine, which contradicts the fact that $x$ is an extreme point of $f$.

Using Theorem 25.6 of [32], there exists therefore a sequence $y_n$ with limit $y$ such that $f^*$ is differentiable at $y_n$ and the sequence $x_n := \nabla f^*(y_n)$ has limit $x$. (In the proof of Theorem 25.6 of [32], this statement is proved only for exposed points of $\partial^- f^*(y)$, but this extends easily to extreme points by a diagonal extraction). The sequence of points $x_n$ is made of exposed points of $f$ by Corollary 25.1.2 in [32] and

$$\bar{D}f(x_n; z) = (y_n, z) \to (y, z) = \bar{D}f(x; z).$$

A.2 Proof of Lemma 5.1

Proof of Lemma 5.1. Let us start with an $\varepsilon$-optimal $\mathbb{P} \in \mathcal{K}(p)$ and the process $\pi$ of law $\mathbb{P}$ defined on the canonical space. Define the process $M_t = e^{-t \overline{J} \pi_t}$ which is a martingale under $\mathbb{P}$ using (3.5).

At first, we show that there exists an $\varepsilon$-optimal stopping time $\mu^*$ of the raw filtration $\mathcal{F}^\pi$ (which is not right-continuous) such that $\mu^*$ takes finitely many values in the set $E \cup \{+\infty\}$ where $E$ is the set of (almost sure) continuity points of $\pi$.

Let us briefly adapt Lemma 3.1 in Dolinsky [11]. Choose $\mu$ an $\varepsilon$-optimal stopping time. Let $E_n$ denote a finite subset of $E$ such that $[0, n] \subset \cup_{e \in E_n} (e - \frac{1}{n}, e]$ and define $\mu_n$ by

$$\mu_n = \inf \{ e \in E_n \mid e \geq \mu + \frac{1}{n} \}.$$

$\mu_n$ is clearly a stopping time of $\mathcal{F}^\pi$ and we have that

$$\sup_{\tau \in T^*} (J'(\mu, \tau) - J'(\mu_n, \tau))^+ \leq \sup_{\mu \leq \mu \leq \mu_n} (e^{-t \mu} \mathcal{H}(\pi) - e^{-t \mu} \mathcal{H}(\pi))^+ \rightarrow 0,$$

where convergence holds almost surely. Indeed, if $\mu < +\infty$, we have $\mu \leq \mu_n \leq \mu + \frac{2}{n}$ for $n$ sufficiently large (if $\mu \in [0, n - \frac{2}{n}]$) and otherwise the above quantity is zero. The conclusion follows by bounded convergence.

Let us fix $T > 0$. Note that since $\pi$ takes values in the compact set $\Delta(K)$, there exists a constant $C_T$ such that $\sup_{t \leq T} |M_t| \leq C_T$ (for all $\omega$).

For $n$ sufficiently large, let $\{t_i^n, i \geq 0\}$ be an increasing sequence such that $t_0^n = 0$, containing the support of $\mu^*$, having for limit $+\infty$ and such that $\sup_{i \geq 0} |t_{i+1}^n - t_i^n| \leq \frac{1}{n}$.
Define a first approximation of $M$ by $\tilde{M}_t^n := M_t^n$ for $t \in [t_0^n, t_{n+1}^0)$.

Let us consider a sequence of countable partitions $(U_i^n)_{i \geq 1}$ of $\mathbb{R}^R$ of mesh $\frac{1}{n}$ which is nested in the sense that for all $n$, each $U_i^n$ is the union of a family $(U_{j}^{n+1})_{j \in J}$ and locally finite in the sense that for all $n$, every compact set is covered by a finite family $(U_i^n)_{i \in I}$.

Let us define $\phi_n : \mathbb{R}^R \to \mathbb{N}$ by

$$\phi_n(x) := \sum_{i \geq 1} i 1_{x \in U_i^n}.$$  

For any $n \in \mathbb{N}^*$, define the piece-wise constant right-continuous filtration $\mathcal{F}_t^n$ by

$$\forall t \geq 0, \mathcal{F}_t^n := \sigma(\phi_n(\tilde{M}_t^n), 0 \leq s \leq t).$$

Note that for all $t$, $\mathcal{F}_t^n \subset \mathcal{F}_t^1$ and let $\mathcal{T}^n$ denote the set of $\mathcal{F}_t^n$ stopping times.

The important point of the construction is that any $\mathcal{F}_t^n$-measurable random variable takes at most finitely many values. Indeed, $M$ is uniformly bounded on $[0, T]$, so that $\phi_n(M_t^n)$ takes only finitely many values, and there are finitely many $t_i^n$ in $[0, T]$.

Define the process $M^n$ by

$$\forall t \geq 0, M_t^n := \mathbb{E}[\tilde{M}_t^n | \mathcal{F}_t^n].$$

Note that this process is an $\mathcal{F}_t^n$-martingale. Indeed it is sufficient to check the martingale property at times $t_i^n$ and

$$\mathbb{E}[M_{t_{i+1}^n}^n | \mathcal{F}_{t_i^n}] = \mathbb{E}[M_{t_{i+1}^n}^n | \mathcal{F}_{t_i^n}] = \mathbb{E}[\mathbb{E}[M_{t_{i+1}^n}^n | \mathcal{F}_{t_i^n}^n] | \mathcal{F}_{t_i^n}] = \mathbb{E}[M_{t_i^n}^n | \mathcal{F}_{t_i^n}^n] = M_{t_i^n}^n.$$  

Define then $\pi^n$ by

$$\forall t \geq 0, \pi_t^n := e^{t \mathbb{I}_n} M_t^n.$$  

Note that by construction $\pi^n$ has a law in $\mathcal{K}(p)$ and that for any $T > 0$, $(\pi_t^n)_{t \in [0, T)}$ take only finitely many values. More precisely, $t \in [t_i^n, t_{i+1}^n)$, we have the relation

$$\pi_t^n = e^{t \mathbb{I}_n} M_t^n = e^{t \mathbb{I}_n} \mathbb{E}[M_{t_i^n}^n | \mathcal{F}_{t_i^n}^n] = e^{(t - t_i^n) \mathbb{I}_n} \mathbb{E}[\pi_t^n | \mathcal{F}_{t_i^n}^n],$$

which proves that $\pi_t^n \in \Delta(K)$.

Let $\mathbb{P}_n$ denote the law of $\pi^n$ and $\mathcal{T}_0^n \subset \mathcal{T}^n$ denote the set of stopping times taking values in $\{t_i^n, i \geq 0, +\infty\}$. Given two stopping times $\mu, \tau$ in $\mathcal{T}^n$, define

$$J^n(\mu, \tau) = 1_{\tau < \mu} e^{-\tau \mu} f(\pi^n_t) + 1_{\mu \leq \tau} e^{-\tau \mu} h(\pi^n_\mu).$$  

Note that

$$\begin{equation}
\sup_{\mu \in \mathcal{T}^n} \inf_{\tau \in \mathcal{T}^n} \mathbb{E}_\mathbb{P}_n[J^n(\mu, \tau)] \geq \sup_{\mu \in \mathcal{T}_0^n} \inf_{\tau \in \mathcal{T}^n} \mathbb{E}_\mathbb{P}_n[J^n(\mu, \tau)].
\end{equation}$$

We now prove that for $n$ sufficiently large, we have:

$$\begin{equation}
\sup_{\mu \in \mathcal{T}_0^n} \inf_{\tau \in \mathcal{T}^n} \mathbb{E}_\mathbb{P}_n[J^n(\mu, \tau)] \geq \sup_{\mu \in \mathcal{T}_0^n} \inf_{\tau \in \mathcal{T}_0^n} \mathbb{E}_\mathbb{P}_n[J^n(\mu, \tau)] - \varepsilon. \quad (A.2)
\end{equation}$$

For any $\tau \in \mathcal{T}^n$, define $\tau'$ by $\tau' = \inf\{t_i^n | i : t_i^n \geq \tau\}$ which is a stopping time in $\mathcal{T}_0^n$. For all $\mu \in \mathcal{T}_0^n$, we have

$$J^n(\mu, \tau) - J^n(\mu, \tau') = 1_{\tau < \mu \leq \tau'}(e^{-\tau \mu} f(\pi^n_{\tau'}) - e^{-\tau \mu} h(\pi^n_{\mu})) \geq 1_{\tau < \mu \leq \tau'}(e^{-\tau \mu} h(\pi^n_{\tau'}) - e^{-\tau \mu} h(\pi^n_{\mu})).$$
To conclude note that if \( C \) denotes a bound for the supremum of \( h, f \) on \( \Delta(K) \) and the Lipschitz constants of \( h, f \), we have
\[
\mathbb{1}_{\tau < \mu \leq \tau'} |e^{-r\tau} h(\pi^n_\tau) - e^{-r\mu} h(\pi^n_\mu)| \leq C \left( \frac{\tau}{n} + \sup_{t \in [\tau, \tau']} |\pi^n_t - \pi^n_\tau| \right)
\leq C \left( \frac{\tau}{n} + \sup_{t \in [\tau, \tau']} |e^{-(t-\tau)} R \pi^n_{\tau} - \pi^n_\tau| \right)
\leq C \left( \frac{\tau}{n} + \sup_{s \in [0, \frac{1}{n}]} \|e^{-sR} - I\| \right) \to 0,
\]
which concludes the proof of A.2. Now, we claim that for \( n \) sufficiently large, we have
\[
\sup_{\mu \in T^n_0} \inf_{\tau \in T^n_0} \mathbb{E}[J^n(\mu, \tau)] \geq \sup_{\mu \in T^n_0} \inf_{\tau \in T^n_0} \mathbb{E}[J'(\mu, \tau)] - \varepsilon. \tag{A.3}
\]
Indeed, given \( \mu, \tau \in T^n \), we have
\[
J^n(\mu, \tau) - J'(\mu, \tau) \geq -C(e^{-r\tau} |\pi_\tau - \pi^n_\tau| + e^{-r\mu} |\pi_\mu - \pi^n_\mu|) \geq -C \sqrt{|K|} \frac{|K|}{n},
\]
where we used that \( \pi_t = e^{tR} M_t \), that \( \|e^{tR}\| \leq \sqrt{|K|} \) and \( |M^n_t - M^n_{\tau}| \leq \frac{1}{n} \) and also that contribution of the events \( \mu = +\infty \) and \( \tau = +\infty \) is equal to zero.

By definition, we have:
\[
\sup_{\mu \in T^n_0} \inf_{\tau \in T^n_0} \mathbb{E}[J'(\mu, \tau)] \geq \sup_{\mu \in T^n_0} \inf_{\tau \in T^n} \mathbb{E}[J'(\mu, \tau)]. \tag{A.4}
\]
To conclude the proof, for \( n \) sufficiently large, we will construct a stopping time \( \mu \in T^n_0 \) such that \( \mathbb{P}(\mu^* \neq \mu) \leq \varepsilon \). This last construction already appears in Lemma 3.2 in Dolinsky [11] (see also Aldous [1]). It relies on the fact that \( \mu^* \) is a stopping time of the filtration \( \mathcal{F}^\pi \) and that the support of \( \mu^* \) is contained in the sequence \( \{t^n_i, i \geq 0\} \).

Note that \( \pi^n \) converges almost surely to \( \pi \) in the Skorokhod topology when \( n \) goes to \( +\infty \). Indeed, at first we have
\[
\sup_{t \geq 0} |\pi^n_t - e^{tR} M^n_t| \leq \frac{\sqrt{|K|}}{n},
\]
since \( \|e^{tR}\| \leq \sqrt{|K|} \) (\( R \) being a stochastic matrix). Secondly, \( M^n \) converges (in the sense of Skorokhod) to \( M \) by Lemma VI.6.37 in Jacod-Shiryaev [23], from which we deduce easily that \( (e^{tR} M^n_t)_{t \geq 0} \) converges (Skorokhod) almost surely to \( (\mu_t)_{t \geq 0} = (e^{tR} M_t)_{t \geq 0} \).

Let \( \{t_1 < \ldots < t_i\} \) denote the finite elements in the support of \( \mu^* \). For each \( t_i \), the event \( A_i = \{\mu^* = t_i\} \) belongs to \( \mathcal{F}^\pi_{t_i} \) and may therefore be approximated by some Skorokhod-continuous function \( \phi_i : \mathbb{D}([0, t_i], \Delta(K)) \to \mathbb{R} \) such that (apply Lusin’s theorem and then Urysohn’s lemma):
\[
\mathbb{E}[\mathbb{1}_{A_i} - \phi_i((\pi_s)_{s \in [0, t_i]}))] \leq \frac{\varepsilon}{2^i}.
\]
Since \( t_i \) is almost surely a continuity point of \( \pi \) to deduce that \( (\pi^n_s)_{s \in [0, t_i]} \) converges almost surely to \( (\pi_s)_{s \in [0, t_i]} \) for the Skorokhod topology on \( \mathbb{D}([0, t_i], \Delta(K)) \) (see e.g. Lemma VI.6.37 in [23]).
Define the stopping time $\mu_n := \inf\{t_i | i : \phi_i((\pi^n_s)_{s \in [0,t_i]}) > \frac{1}{2}\}$ and

$$D_i := (A_i \cap \{\phi_i((\pi_s)_{s \in [0,t_i]}) > \frac{1}{2}\}) \cup (A_i^c \cap \{\phi_i((\pi_s)_{s \in [0,t_i]}) < \frac{1}{2}\}), \ i = 1, \ldots, I$$

$$D := \cap_{i=1}^{I} D_i.$$ 

Using that $\phi_i$ is continuous, for any $\omega \in D$, there exists $N(\omega)$ such that for any $n \geq N(\omega)$, $\mu_n(\omega) = \mu(\omega)$. From Markov inequality, we have

$$\mathbb{P}(D) = 1 - \mathbb{P}(D^c) \geq 1 - \sum_{i=1}^{I} \mathbb{P}(|\phi_i((\pi_s)_{s \in [0,t_i]}) - 1_{A_i}| \geq \frac{1}{2}) > 1 - \varepsilon.$$ 

Define $B_n := \cap_{q=n}^{+\infty} \{\mu_n = \mu^*\}$. $B_n$ is a non-decreasing sequence of events and $D \subset \cup_{n \geq 1} B_n$. There exists therefore some $N$ such that for $n \geq N$, $\mathbb{P}(B_n) > 1 - \varepsilon$. Let us now choose $n$ sufficiently large so that $\mathbb{P}(\mu_n \neq \mu^*) < \varepsilon$. For all $\tau \in T^\pi$, we have:

$$J'(\mu_n, \tau) - J'(\mu^*, \tau) \geq C\mathbb{P}(\mu_n \neq \mu^*) \geq C\varepsilon.$$ 

Since $\mu_n \in T^\pi_0$, we deduce that

$$\sup_{\mu \in T^\pi_0} \inf_{\tau \in T^\pi} \mathbb{E}_\rho[J'(\mu, \tau)] \geq \sup_{\mu \in T^\pi} \inf_{\tau \in T^\pi} \mathbb{E}_\rho[J'(\mu, \tau)] - \varepsilon - C\varepsilon. \quad (A.5)$$

Putting together inequalities (A.2, A.3, A.4, A.5), we proved that the law $\mathbb{P}^n$ is $(3 + C)\varepsilon$-optimal for the problem $W(p)$ and fulfills the required properties for the Lemma. 

**A.3 Proof of lemma 5.11**

Let us consider a process $(\tilde{\pi}_i)_{i \geq 0}$ of law $\mathbb{P} \in K(p)$ defined on some probability space $\tilde{\Omega}$. Assume (up to enlarging the space) that a family of independent random variables $(U_i)_{i \in \mathbb{N}}$ uniformly distributed on $[0,1]$ and independent from $\tilde{\pi}$ are defined on $\tilde{\Omega}$.

We will also consider a second probability space $\Omega$ on which is defined a process $X$ of law $\mathbb{P}_p$ and a family of independent random variables $(V_i)_{i \in \mathbb{N}}$ uniformly distributed on $[0,1]$ and independent from $X$.

For all $n$, we will construct a process $\pi^n$ on $\Omega$ such that $(X, \pi^n)$ has a law in $K'(p)$. As $n$ goes to infinity, we will show that a subsequence of $(X, \pi^n)$ converges in law (in the sense of $\mathcal{L}(MZ)$ defined below) to some process $(X, \pi)$ having the required properties.

The first part of the proof is similar to Lemma 4 in [8]. Given $n \geq 1$, and the sequence $(t^n_i)_{i \in \mathbb{N}}$ with $t^n_i = i 2^{-n}$, we will construct a process $\pi^n$ on $\Omega$. Let us construct on $\Omega$ a variable $X_0$ as a function of $(\tilde{\pi}_0, U_0)$ such that:

$$\forall k \in K, \mathbb{P}(X_0 = k|\tilde{\pi}_0) = \tilde{\pi}_0(k).$$

Then construct $\pi^n_0$ on $\Omega$ as a function of $(X_0, V_0)$ such that the conditional law of $\pi_0$ given $X_0$ is the same as the conditional law of $\tilde{\pi}_0$ given $X_0$. It follows that for all $k \in K$, $\mathbb{P}(X_0 = k|\pi^n_0) = \pi^n_0(k)$.

Assume now, that the variables $\pi^n_i$ are constructed up to $i = m - 1$ and such that $\pi^n_i$ is measurable with respect to $(X^n_{t^n_0}, ..., X^n_{t^n_{m-1}}, V_0, ..., V_i)$ and for all $k \in K$, $\mathbb{P}(X^n_{t^n_i} = k|\pi^n_{0}, ..., p_i) = \pi^n_{t^n_i}(k)$. 

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Let us construct a variable $\tilde{X}_{t_n}$ on $\tilde{\Omega}$ as a function of $(\tilde{\pi}_{t_0}, ..., \tilde{\pi}_{t_{n-1}}, U_m)$ such that for all $k \in K$, $\mathbb{P}(\tilde{X}_{t_n} = k|\tilde{\pi}_{t_0}, ..., \tilde{\pi}_{t_{n-1}}) = \tilde{\pi}_{t_n}(k)$. Then construct $\pi^n_{t_n}$ as a function of $(X^n_{t_0}, \pi^n_{t_0}, ..., \pi^n_{t_{n-1}}, V_m)$ such that the conditional law of $\pi^n_{t_n}$ given $(X^n_{t_0}, \pi^n_{t_0}, ..., \pi^n_{t_{n-1}})$ is the same as the conditional law of $\tilde{\pi}_{t_n}$ given $(\tilde{X}_{t_n}, \tilde{\pi}_{t_0}, ..., \tilde{\pi}_{t_{n-1}})$. It follows that $\mathbb{P}(X^n_{t_m} = k|\pi^n_{t_0}, ..., \pi^n_{t_{m-1}}) = \pi^n_{t_m}(k)$.

By induction, we conclude that for all $m \geq 0$, the variable $\pi^n_{t_m}$ is measurable with respect to $(X^n_{t_0}, ..., X^n_{t_m}, V_0, ..., V_m)$ and for all $k \in K$, $\mathbb{P}(X^n_{t_m} = k|\pi^n_{t_0}, ..., \pi^n_{t_{m-1}}) = \pi^n_{t_m}(k)$.

Define now the process $\pi^n$ on the interval $[t^n_i, t^n_{i+1})$ by $\pi^n_t = e^{(t-t^n_i)^+}R\pi^n_{t^n_i}$ and it follows easily that the law of $(X, \pi^n)$ belongs to $K'(p)$.

It remains to let $n$ go to infinity. In order to avoid confusions in the sequel, let us denote $(X^n, \pi^n) := (X, \pi^n)$. We will consider the topology of convergence in measure of the trajectories in $\mathbb{D}(\mathbb{R}_+, K \times \Delta(K))$ when $\mathbb{R}_+$ is endowed with the measure $e^{-x}dx$. In the sequel, we will denote $MZ$ this topology, and $\mathcal{L}(MZ)$ the associated weak convergence of probabilities over $\mathbb{D}(\mathbb{R}_+, K \times \Delta(K))$.

We will need a slightly stronger convergence which corresponds to convergence in measure together with pointwise convergence for all $t$ in the dense countable set $D := \cup_n \{t^n_i, i \geq 0\}$.

Rather than defining another topology, we use the following idea: we enumerate the points in $D$ and denote them $(t_m)_{m \geq 0}$ and we associate the weight $2^{-m}$ to $t_m$. We define the deterministic change of time $\tau(t) := t + \sum_{t_m \in D: 0 \leq t_m \leq t} 2^{-m}$ so that $\tau$ is strictly increasing and right-continuous with limit $+\infty$ and its generalized inverse $\tau^{-1}$ is continuous and non-decreasing with limit $+\infty$. For any càdlàg trajectory $f(t)$ we consider the trajectory $\Phi(f)(t) := f(\tau^{-1}(t))$. We will apply the tightness and convergence result to the images of càdlàg trajectories by $\Phi$, and thus obtain the required convergence for the initial trajectories.

Using a diagonal argument and Theorem 4 in [28] for each coordinate, the set of laws of processes $\Phi(X^n, \pi^n)$ is relatively compact for the $\mathcal{L}(MZ)$-topology. Let us extract some convergent subsequence and assume that convergence holds almost surely using Skorokhod representation theorem for separable metric spaces (see e.g. Theorem 11.7.31 in [12]).

Let us denote the limit process $(X, \pi)$. We have to prove that $X$ has law $\mathbb{P}_p$, is an $\mathcal{F}(X, \pi)^{+,+}$ Markov process, and that for all $t \in \mathbb{R}_+$:

$$\forall k \in K, \mathbb{P}(X_t = k|\mathcal{F}^{\pi,+}) = \pi_t. \quad (A.6)$$

The fact that $X$ has law $\mathbb{P}_p$ is obvious since all the processes $X^n$ have law $\mathbb{P}_p$ and that the projection on the first coordinate is $MZ$-continuous on $\mathbb{D}(\mathbb{R}_+, K \times \Delta(K))$.

Let us prove the Markov property. At first we claim that for all $s \geq t$ in $D$, all finite family $t_1, ..., t_r \in D$ and bounded continuous function $\phi$ defined on $(K \times \Delta(K))^r$, we have:

$$\mathbb{E}[\delta_{X_s}\phi(X_{t_1}, \pi_{t_1}, ..., X_{t_r}, \pi_{t_r})] = \mathbb{E}[e^{(s-t)^+}R\delta_{X_t}\phi(X_{t_1}, \pi_{t_1}, ..., X_{t_r}, \pi_{t_r})].$$

Indeed, this property holds for $(X^n, \pi^n)$ (for $n$ sufficiently large), and we conclude by bounded convergence since we have almost sure convergence at all times in $D$. This extends to all $s \geq t$.

\[2\] In reference to the seminal paper of Meyer and Zheng [28]
and all finite families in \([0, t]\) using that \((X, \pi)\) are right-continuous. We deduce that for all \(t \in D\):

\[
E[\delta_{X_t} | \mathcal{F}_t^{X, \pi}] = e^{(s-t) \mathbf{1}R} \delta_{X_t}.
\]

To conclude with for an arbitrary \(t\), we simply use a decreasing sequence in \(D\) with limit \(t\), so that the filtration \(\mathcal{F}^{X, \pi}\) is replaced by \(\mathcal{F}^{(X, \pi), +}\) by using the backward martingale convergence theorem.

The proof that (A.6) is true for the limit process is similar. At first, we claim that or all \(t \in D\) and all finite families \(t_1 \leq t_2 \ldots \leq t_r \in D\) and bounded continuous function \(\phi\) on \(\Delta(K)^r\), we have

\[
E[(\delta_{X_{t_i}} \pi_t) \phi(\pi_{t_{1}}, ..., \pi_{t_r})] = 0.
\]

Indeed, this property holds true for \((X_n, \pi^n)\) and using almost sure convergence and Lebesgue bounded convergence theorem, it is true for the limit. Then, it can be extended to any family \(t_1, ..., t_r \in [0, t]\) using that \(\pi\) is right-continuous. It follows that for all \(t \in D\)

\[
\forall k \in K, \ P(X_t = k | \mathcal{F}_t^\pi) = \pi_t.
\]

To prove (A.6) for an arbitrary \(t\), we use an approximation by a decreasing sequence \(t_m \in D\) and the backward martingale convergence theorem.

To conclude the proof, one has to prove the law of \(\pi\) is \(\mathbb{P}\) (i.e. equal to the law of \(\hat{\pi}\)). To see this, note that the processes \(\hat{\pi}_t^n\) which are piecewise constant on the partitions \(\{[t^n_i, t^n_{i+1})\}_{i \geq 0}\) and equal to \(\pi^n_t\) at points \(t^n_i\) converge in law to \(\hat{\pi}\) since the laws of \((\pi^n_{t_i})_{i \geq 0}\) and \((\hat{\pi}_{t_i})_{i \geq 0}\) are the same and using Lemma VI.6.37 in [23]. Moreover, \(\pi^n\) and \(\hat{\pi}^n\) have the same limit points (in law) since:

\[
sup_t |\pi^n_t - \hat{\pi}_t^n| \leq \sup_{q \in \Delta(K), h \in [0, 2^{-n}]} |q - e^{h \mathbf{1}R} q| \longrightarrow 0.
\]

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