NOVEL CONDITIONS OF EUCLIDEAN SPACE CONTROLLABILITY FOR SINGULARLY PERTURBED SYSTEMS WITH INPUT DELAY

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Abstract. A singularly perturbed linear time-dependent controlled system with a point-wise nonsmall (of order of 1) delay in the input (the control variable) is considered. Sufficient conditions of the complete Euclidean space controllability for this system, robust with respect to the parameter of singular perturbation, are derived. This derivation is based on an asymptotic analysis of the controllability matrix for the considered system and on such an analysis of the determinant of this matrix. However, this derivation does not use a slow-fast decomposition of the considered system. The theoretical result is illustrated by an example.

1. Introduction. Differential systems with a small multiplier \( \epsilon > 0 \) for a part of the highest order derivatives, called singularly perturbed systems, serve as adequate mathematical models for various real-life processes with two-time-scale dynamics, \cite{16}. The small multiplier \( \epsilon > 0 \) is called the parameter of singular perturbation. An important class of singularly perturbed differential systems represents the systems with time delays. Such systems also arise in various applications (see e.g. \cite{3, 21, 22, 23, 25, 28, 29}).

Different issues in theory and applications of singularly perturbed controlled systems without/with delays are extensively investigated in the literature (see e.g. \cite{2, 16, 19, 31} and references therein).

Controllability of a controlled system is one of its basic properties, used in many issues of control theory and applications (see e.g. \cite{12, 15, 24}). This property means the ability to transfer a system from any admissible initial position to any admissible terminal position in a finite time by a proper choice of the control. Controllability conditions for various controlled systems without/with delays were extensively studied in the literature (see e.g. \cite{1, 4, 13, 14, 15} and references therein). To check whether a singularly perturbed system is controllable in a proper sense,
the corresponding controllability conditions can be directly applied for any specified value of $\varepsilon$. However, the stiffness and a possible high Euclidean dimension of this system, can considerably complicate this application. Moreover, such an application depends on the value of $\varepsilon$, i.e., it is not robust with respect to this parameter, while in many real-life problems this value is unknown.

Controllability of singularly perturbed systems, robust with respect to $\varepsilon$, was studied in the literature in a number of works. Mainly, this property was studied either for undelayed systems [16, 26, 27], or for systems with only state delays (see e.g. [5, 8, 11, 17, 18, 20, 30] and references therein). To the best of our knowledge, there are only few works in the literature [7, 9, 10] where some kinds of controllability for singularly perturbed systems with input (control) delays are analyzed. In these works, based on the slow-fast decomposition of the considered systems, the case of the small delays of order of $\varepsilon$ was treated.

In the present paper, we consider a singularly perturbed linear time-dependent system with a nonsmall (of order of 1) delay in the input (control variable). For this system, the complete Euclidean space controllability, robust with respect to $\varepsilon$, is studied. It is shown that the considered system cannot be decomposed into the slow and fast subsystems, because its fast subsystem does not exists. Therefore, the slow-fast decomposition of the considered system is not applicable to the above mentioned study. The present paper is rather theoretical. The main motivation to consider the above mentioned controllability problem is twofold. Namely, the one part of this motivation is to extend the investigation of the controllability property for singularly perturbed systems with control delays to the case where such a delay is nonsmall (of order of 1). The other part of the motivation (not less important than the previous one) is to analyse a singularly perturbed problem for which the slow-fast decomposition method is not applicable.

In the previous works [7, 9, 10], the case of a small control delay of order of the parameter of singular perturbation $\varepsilon > 0$ was treated. In this case, the study of the complete Euclidean space controllability is based on a transformation of this type of controllability of the original system with delays in the state and the control to an equivalent output controllability of a new singularly perturbed system with only state delays. In the new system, the original control variable becomes an additional fast state variable with small delay, while a new control variable is undelayed. The Euclidean dimension of the slow mode equation in the new system is the same as in the original system, while the Euclidean dimension of the fast mode equation is larger than such a dimension in the original system. Further analysis is carried out based on the slow-fast decomposition of the original and transformed systems. This analysis yields the following result derived in [7, 9, 10]. If both, slow and fast, subsystems of the original system are completely Euclidean space controllable, then the original system is completely Euclidean space controllable for all sufficiently small values of $\varepsilon$. However, this is not a case in the present paper. Namely, if the above mentioned transformation would be applied to the system of the present paper, we would obtain the new system with a nonsmall delay in the fast state variable (the former control variable) appearing in a fast mode equation. Such a feature of the new system means that the slow-fast decomposition approach is not applicable to its analysis, like this approach is not applicable to the analysis of the original system of the present paper. This circumstance requires an essentially new approach to the controllability analysis of the original system and a
2. Problem statement and preliminary discussion.

2.1. Original singularly perturbed system. Consider the following controlled system:

\[
\frac{dx(t)}{dt} = A_{11}(t,\varepsilon)x(t) + A_{12}(t,\varepsilon)y(t) + B_{01}(t,\varepsilon)u(t) + B_{11}(t,\varepsilon)u(t - h(\varepsilon)), \quad t \geq 0,
\]

\[
\varepsilon \frac{dy(t)}{dt} = A_{21}(t,\varepsilon)x(t) + A_{22}(t,\varepsilon)y(t) + B_{02}(t,\varepsilon)u(t) + B_{12}(t,\varepsilon)u(t - h(\varepsilon)), \quad t \geq 0,
\]

where \(x(t) \in \mathbb{R}^n\), \(y(t) \in \mathbb{R}^m\) are state variables; \(u(t) \in \mathbb{R}^r\) is a control; \(\varepsilon > 0\) is a small parameter; \(h(\varepsilon) \geq 0\) is a time delay given for \(\varepsilon \in [0, \varepsilon_0]\), \((\varepsilon_0 > 0)\); \((u(t), u(t + \eta))\), \(\eta \in [-h(\varepsilon), 0)\) is a control variable; \(A_{ij}(t,\varepsilon)\) and \(B_{kl}(t,\varepsilon)\), \((i = 1, 2; j = 1, 2; k = 0, 1; l = 1, 2)\), are matrix-valued functions given for \(t \geq 0\) and \(\varepsilon \in [0, \varepsilon_0]\); \(A_{ij}(t,\varepsilon)\) and \(B_{kl}(t,\varepsilon)\), \((i = 1, 2; j = 1, 2; k = 0, 1; l = 1, 2)\), are continuous in \((t,\varepsilon) \in [0, +\infty) \times [0, \varepsilon_0]\); \(h(\varepsilon)\) is continuous in \(\varepsilon \in [0, \varepsilon_0]\).

Due to the presence of the small multiplier \(\varepsilon > 0\) for the derivative \(dy(t)/dt\), the system (1)-(2) is singularly perturbed (see e.g. [16]). The equation (1) and the state variable \(x(t)\) are the slow mode and the slow state variable, while the equation (2) and the state variable \(y(t)\) are the fast mode and the fast state variable. An additional feature of (1)-(2) is the control delay.

Let \(t_c > 0\) be a given time instant.

Definition 2.1. For a given \(\varepsilon \in (0, \varepsilon_0]\), the system (1)-(2) is said to be completely Euclidean space controllable at the time instant \(t_c\) if for any \(x_0 \in \mathbb{R}^n\), \(y_0 \in \mathbb{R}^m\), \(\varphi_u(\cdot) \in L^2[-h(\varepsilon), 0; \mathbb{R}^r]\), \(x_c \in \mathbb{R}^n\) and \(y_c \in \mathbb{R}^m\) there exists a control function \(u(\cdot) \in L^2[0, t_c; \mathbb{R}^r]\), for which the system (1)-(2) has a solution \(\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\), \(t \in [0, t_c]\) satisfying the initial and terminal conditions

\[
u(\eta) = \varphi_u(\eta), \quad \eta \in [-h(\varepsilon), 0); \quad x(0) = x_0, \quad y(0) = y_0,\]

\[
x(t_c) = x_c, \quad y(t_c) = y_c.
\]
2.2. Discussion on the slow-fast decomposition of the original system. One of the approaches, widely used in the literature for study of a singularly perturbed system, is its slow-fast decomposition. This approach is applied in qualitative analysis and control design of various systems (see e.g. [16] and references therein). In particular, this approach is used in analysis of controllability properties for different singularly perturbed systems without/with delays in state and control (see e.g. [7, 8, 16, 26, 27] and references therein). Let us try to decompose asymptotically the system (1)-(2) into the ε-free slow and fast subsystems. Due to the above mentioned works, the slow subsystem is obtained from (1)-(2) by setting there formally ε = 0 which yields

\[
\frac{dx_s(t)}{dt} = A_{11}(t,0)x_s(t) + A_{12}(t,0)y_s(t) + B_{01}(t,0)u_s(t) + B_{11}(t,0)u_s(t - h(0)), \quad t \geq 0,
\]

where \( x_s(t) \in \mathbb{R}^n \) and \( y_s(t) \in \mathbb{R}^m \) are state variables; \( u_s(t) \in \mathbb{R}^r \) is a control; \( (u_s(t), u_s(t + \tau)) \), \( \tau \in [-h(0), 0) \) is a control variable.

If \( \det A_{22}(t,0) \neq 0, t \geq 0 \), the slow subsystem can be reduced to the differential equation

\[
\frac{dx_s(t)}{dt} = A_s(t)x_s(t) + B_{0s}(t)u_s(t) + B_{1s}(t)u_s(t - h(0)), \quad t \geq 0,
\]

where

\[
A_s(t) = A_{11}(t,0) - A_{12}(t,0)A_{22}^{-1}(t,0)A_{21}(t,0),
B_{0s}(t) = B_{01}(t,0) - A_{12}(t,0)A_{22}^{-1}(t,0)B_{02}(t,0),
B_{1s}(t) = B_{11}(t,0) - A_{12}(t,0)A_{22}^{-1}(t,0)B_{12}(t,0).
\]

The differential equation (3) also is called the slow subsystem associated with the original system (1)-(2). This equation has a delay in the control, it is of a lower Euclidean dimension than (1)-(2) and it is ε-free.

The fast subsystem, associated with the original system, is obtained from the fast mode equation (2) of the latter in the following way. First, we remove the slow state variable \( x(\cdot) \) from (2), which yields the auxiliary equation

\[
\varepsilon \frac{dy_{aux}(t)}{dt} = A_{22}(t,\varepsilon)y_{aux}(t) + B_{02}(t,\varepsilon)u_{aux}(t) + B_{12}(t,\varepsilon)u_{aux}(t - h(\varepsilon)), \quad t \geq 0.
\]

Then, in the auxiliary equation (5), we make the following transformation of the independent variable (time):

\[
\xi = (t - t_1)/\varepsilon,
\]

where \( \xi \) is a new independent variable, \( t_1 \geq 0 \) is any specified number.

Based on the transformation (6), we transform the state variable \( y_{aux}(\cdot) \) and the control variable \( u_{aux}(\cdot) \) in (5) as:

\[
y_f(\xi) = y_{aux}(\varepsilon \xi + t_1), \quad u_f(\xi) = u_{aux}(\varepsilon \xi + t_1),
\]

where \( y_f(\cdot) \) and \( u_f(\cdot) \) are new state and control variables, respectively.
Due to the transformations (6)-(7), the auxiliary equation (5) becomes:

\[
\frac{dy_f(\xi)}{d\xi} = A_{22}(\varepsilon \xi + t_1, \varepsilon) y_f(\xi) + B_{02}(\varepsilon \xi + t_1, \varepsilon) u_f(\xi) + B_{12}(\varepsilon \xi + t_1, \varepsilon) u_f(\xi - h(\varepsilon)/\varepsilon).
\]

Note that the transformation (6), along with the transformations in (7), allowed us to remove the small multiplier \(\varepsilon > 0\) for the derivative in the auxiliary equation (5).

Now, if there exists a finite limit

\[
\lim_{\varepsilon \to +0} h(\varepsilon)/\varepsilon \overset{\triangle}{=} \bar{h} > 0,
\]

then, setting formally \(\varepsilon = 0\), replacing \(h(\varepsilon)/\varepsilon\) with \(\bar{h}\) and replacing \(t_1\) with \(t\), we obtain the fast subsystem

\[
\frac{dy_f(\xi)}{d\xi} = A_{22}(t, 0) y_f(\xi) + B_{02}(t, 0) u_f(\xi) + B_{12}(t, 0) u_f(\xi - \bar{h}), \quad \xi \geq 0.
\]

In this equation, \(t \geq 0\) is a parameter, while \(\xi\) (called the stretched time) is an independent variable. Due to (6), for any given \(t_1 \geq 0\) and any \(t > t_1\), we have that \(\xi \to +\infty\) as \(\varepsilon \to +0\). Therefore, in (9), it is supposed that the independent variable \(\xi\) varies from 0 to \(+\infty\). The equation (9) is with a control delay, it is of a lower Euclidean dimension than (1)-(2) and it is \(\varepsilon\)-free. In \([7, 9, 10]\), the case \(h(\varepsilon) = \varepsilon \bar{h}\) was studied, where \(\bar{h} > 0\) is a given number. In this case it was shown that, subject to some smoothness assumptions on the coefficients of the original system and the assumption on the asymptotic stability of the unforced \((u_f(\cdot) \equiv 0)\) fast subsystem for all \(t \in [0, t_c]\), the complete Euclidean space controllability of both, slow and fast, subsystems yields such a kind of controllability of the original system at \(t = t_c\) for all sufficiently small values of \(\varepsilon\). However, if the limit value in (8) does not exists or this value is infinite, then the fast subsystem of the original system (1)-(2) does not exist. In this case, the slow-fast decomposition of the original system fails to be used for the controllability analysis of this system.

2.3. Objective of the paper. In this paper, we derive novel \(\varepsilon\)-free controllability conditions for (1)-(2) subject to the following assumption. There exists a positive number \(\bar{\varepsilon}\), \((\bar{\varepsilon} \leq \varepsilon^0)\), such that

\[
\min_{\varepsilon \in [0, \bar{\varepsilon}]} h(\varepsilon) \overset{\triangle}{=} h_{\min} > 0.
\]

The latter means that the control delay \(h(\varepsilon)\) does not vanish when the small parameter \(\varepsilon\) vanishes. Thus, the delay in the control \(u(\cdot)\) of the system (1)-(2) remains to be essential \((h(\varepsilon) \geq h_{\min})\) for all sufficiently small values of \(\varepsilon > 0\), i.e., this delay is of order of 1 for \(\varepsilon \to +0\). Also, due to the assumption (10), the limit in (8) equals positive infinity. Therefore, the derivation of the controllability conditions for (1)-(2) cannot be based on the slow-fast decomposition of this system.
3. **Auxiliary results.** For a given \( \varepsilon \in (0, \varepsilon^0] \), let us consider the block matrices

\[
A(t, \varepsilon) = \begin{pmatrix} A_{11}(t, \varepsilon) & A_{12}(t, \varepsilon) \\ \frac{1}{2}A_{21}(t, \varepsilon) & \frac{1}{2}A_{22}(t, \varepsilon) \end{pmatrix},
\]

\[
B_0(t, \varepsilon) = \begin{pmatrix} B_{01}(t, \varepsilon) \\ \frac{1}{2}B_{02}(t, \varepsilon) \end{pmatrix}, \quad B_1(t, \varepsilon) = \begin{pmatrix} B_{11}(t, \varepsilon) \\ \frac{1}{2}B_{12}(t, \varepsilon) \end{pmatrix}.
\]

(11)

Let, for a given \( \varepsilon \in (0, \varepsilon^0] \), the \((n+m) \times (n+m)\)-matrix-valued function \( \Psi(\sigma, \varepsilon) \), \( \sigma \in [0, +\infty) \) be the solution of the terminal-value problem

\[
\frac{d\Psi(\sigma, \varepsilon)}{d\sigma} = -A^T(\sigma, \varepsilon)\Psi(\sigma, \varepsilon), \quad \sigma \in [0, t_c),
\]

\[
\Psi(t_c, \varepsilon) = I_{n+m}, \quad \Psi(\sigma, \varepsilon) = 0, \quad \sigma > t_c.
\]

(12)

Consider the following matrix-valued functions

\[
B_0(\sigma, \varepsilon) = \Psi^T(\sigma, \varepsilon)B_0(\sigma, \varepsilon),
\]

\[
B_1(\sigma + h(\varepsilon), \varepsilon) = \Psi^T(\sigma + h(\varepsilon), \varepsilon)B_1(\sigma + h(\varepsilon), \varepsilon),
\]

\[
W(\sigma, \varepsilon) = B_0(\sigma, \varepsilon) + B_1(\sigma + h(\varepsilon), \varepsilon).
\]

(13)

Based on \( W(\sigma, \varepsilon) \), we construct the matrix

\[
W(t_c, \varepsilon) = \int_0^{t_c} W(\sigma, \varepsilon)W^T(\sigma, \varepsilon)d\sigma.
\]

(14)

The matrix \( W(t_c, \varepsilon) \) is called the controllability matrix of the system (1)-(2). As a direct consequence of the results of [6] (Theorem 3), we obtain the following assertion.

**Proposition 1.** For a given \( \varepsilon \in (0, \varepsilon^0] \), the original system (1)-(2) is completely Euclidean space controllable at the time instant \( t_c \) if and only if the matrix \( W(t_c, \varepsilon) \) is invertible, i.e., \( \det W(t_c, \varepsilon) \neq 0 \).

Let us partition the matrix \( \Psi(\sigma, \varepsilon) \) into blocks as:

\[
\Psi(\sigma, \varepsilon) = \begin{pmatrix} \Psi_{11}(\sigma, \varepsilon) & \Psi_{12}(\sigma, \varepsilon) \\ \Psi_{21}(\sigma, \varepsilon) & \Psi_{22}(\sigma, \varepsilon) \end{pmatrix},
\]

(15)

where the blocks \( \Psi_{11}(\sigma, \varepsilon), \Psi_{12}(\sigma, \varepsilon), \Psi_{21}(\sigma, \varepsilon) \) and \( \Psi_{22}(\sigma, \varepsilon) \) are of the dimensions \( n \times n, n \times m, m \times n \) and \( m \times m \), respectively.

Along with the terminal-value problem (12), let us consider the following terminal-value problem for the \( n \times n \)-matrix-valued function \( \Psi_s(\sigma) \), \( \sigma \in [0, +\infty) \):

\[
\frac{d\Psi_s(\sigma)}{d\sigma} = -A_s^T(\sigma)\Psi_s(\sigma), \quad \sigma \in [0, t_c),
\]

\[
\Psi_s(t_c) = I_n, \quad \Psi_s(\sigma) = 0, \quad \sigma > t_c.
\]

(16)

Also, we consider the following \( m \times m \)-matrix-valued function:

\[
\Psi_f(\chi) = \begin{cases} 0, & \chi < 0, \\ \exp\left(A_{22}^T(t_c, 0)\right)\chi, & \chi \geq 0. \end{cases}
\]

(17)

In what follows, we assume:

**A1** The matrix-valued functions \( A_{ij}(t, \varepsilon), B_{kl}(t, \varepsilon), (i = 1, 2; j = 1, 2; k = 0, 1; l = 1, 2) \) are continuously differentiable for \( (t, \varepsilon) \in [0, t_c] \times [0, \varepsilon^0] \).
(AII) The scalar function $h(\varepsilon)$ is continuously differentiable for $\varepsilon \in [0, \varepsilon^0]$.

(AIII) For all $t \in [0, t_c]$, all the eigenvalues $\lambda_q(t), \,(q = 1, \ldots, m)$ of the matrix $A_{22}(t, 0)$ satisfy the inequality $\Re\lambda_q(t) \leq -2\beta$, where $\beta > 0$ is some constant.

The assumption (AIII) directly yields the inequality
\[
\left\|\Psi_f(\chi)\right\| \leq a\exp(-\beta\chi), \quad \chi \geq 0,
\]
where $\|\cdot\|$ denotes the Euclidean norm of a matrix; $a > 0$ is some constant.

By virtue of the results of [7] (Lemma 11), we have the following assertion.

**Proposition 2.** Let the assumptions (A1)-(AIII) be valid. Then, there exists a number $0 < \varepsilon_0 \leq \varepsilon^0$, such that for all $\varepsilon \in (0, \varepsilon_0]$ the matrix-valued functions $\Psi_{11}(\sigma, \varepsilon), \Psi_{12}(\sigma, \varepsilon), \Psi_{21}(\sigma, \varepsilon), \Psi_{22}(\sigma, \varepsilon)$ satisfy the inequalities:
\[
\left\|\Psi_{11}(\sigma, \varepsilon) - \Psi_s(\sigma)\right\| \leq a\varepsilon, \quad \left\|\Psi_{12}(\sigma, \varepsilon)\right\| \leq a,
\]
\[
\left\|\Psi_{21}(\sigma, \varepsilon) - \Psi_s(\sigma)\right\| \leq a\varepsilon[\varepsilon + \exp(-\beta\chi)],
\]
\[
\left\|\Psi_{22}(\sigma, \varepsilon) - \Psi_f(\chi)\right\| \leq a\varepsilon,
\]
where $\sigma \in [0, t_c]$; $a > 0$ is some constant independent of $\varepsilon$;
\[
\tilde{\Psi}_s(\sigma) = -(A_{22}^T(\sigma, 0))^{-1}A_{12}^T(\sigma, 0)\Psi_s(\sigma),
\]
\[
\chi = (t_c - \sigma)/\varepsilon.
\]

**Remark 1.** Since $\Psi(\sigma, \varepsilon) = 0, \Psi_s(\sigma) = 0$ for all $\sigma > 0$ (see the equations (12) and (16)) and $\Psi_f(\chi) = 0$ for all $\chi < 0$ (see the equation (17)), then the inequalities in (19) also are valid for all $\sigma > t_c$.

4. **Main results.** In what follows, we assume that the time instant $t_c$, introduced in Section 2, satisfies the inequality
\[
t_c > \max_{\varepsilon \in [0, \varepsilon^0]} h(\varepsilon).
\]

4.1. **Asymptotic analysis of the controllability matrix** $W(t_c, \varepsilon)$. Let us partition the symmetric matrix $W(t_c, \varepsilon)$ into blocks as:
\[
W(t_c, \varepsilon) = \begin{pmatrix}
W_{11}(t_c, \varepsilon) & W_{12}(t_c, \varepsilon) \\
W_{21}(t_c, \varepsilon) & W_{22}(t_c, \varepsilon)
\end{pmatrix},
\]
where the blocks $W_{11}(t_c, \varepsilon)$, $W_{12}(t_c, \varepsilon)$ and $W_{22}(t_c, \varepsilon)$ are of the dimensions $n \times n$, $n \times m$ and $m \times m$, respectively.

Consider the following matrix-valued functions
\[
B_{0s}(\sigma) = \Psi_s^T(\sigma)B_{0s}(\sigma),
\]
\[
B_{1s}(\sigma + h(0)) = \Psi_s^T(\sigma + h(0))B_{1s}(\sigma + h(0)),
\]
\[
W_s(\sigma) = B_{0s}(\sigma) + B_{1s}(\sigma + h(0)),
\]
where $B_{0s}(t)$ and $B_{1s}(t)$ are given in (4).

Using $W_s(\sigma)$, we construct the matrix
\[
W_s(t_c) = \int_0^{t_c} W_s(\sigma)W_s^T(\sigma)d\sigma.
\]

Also, we introduce into the consideration the matrix
\[
W_f(t_c) = \int_0^{+\infty} \Psi_f(\chi)S_f(t_c)\Psi_f(\chi)d\chi,
\]
It is clear that

\[ S_f(t_c) = B_{02}(t_c)B_{02}^T(t_c) + B_{12}(t_c)B_{12}^T(t_c). \]  

Due to (18), the integral in (26) converges.

**Lemma 4.1. (Main Lemma)** Let the assumptions (AI)-(AIII) be valid. Then, there exists a number \( 0 < \varepsilon_1 \leq \varepsilon_0 \), such that for all \( \varepsilon \in (0, \varepsilon_1] \) the matrices \( W_{11}(t_c, \varepsilon), W_{12}(t_c, \varepsilon), W_{22}(t_c, \varepsilon) \) satisfy the inequalities

\[
\begin{align*}
|W_{11}(t_c, \varepsilon) - W_s(t_c)| &\leq a\varepsilon, \\
|W_{12}(t_c, \varepsilon)| &\leq a, \\
|\varepsilon W_{22}(t_c, \varepsilon) - W_f(t_c)| &\leq a\varepsilon,
\end{align*}
\]

where \( a > 0 \) is some constant independent of \( \varepsilon \).

Proof of the lemma is presented in Appendix (Section 7).

4.2. \( \varepsilon \)-free controllability conditions for the system (1)-(2).

**Theorem 4.2.** Let the assumptions (AI)-(AIII) be valid. Let

\[ \det W_s(t_c) \neq 0, \quad \det W_f(t_c) \neq 0. \]  

Then, there exists a number \( 0 < \varepsilon^* \leq \varepsilon_1 \), such that for all \( \varepsilon \in (0, \varepsilon^*] \) the system (1)-(2) is completely Euclidean space controllable at the time instant \( t_c \).

**Proof.** For a given \( \varepsilon > 0 \), consider the \((n + m) \times (n + m)\)-matrix

\[ L(\varepsilon) = \begin{pmatrix} I_n & 0 \\ 0 & \sqrt{\varepsilon}I_m \end{pmatrix}. \]  

Using the equations (23) and (30) yields

\[ L(\varepsilon)W(t_c, \varepsilon)L(\varepsilon) = \begin{pmatrix} W_{11}(t_c, \varepsilon) & \sqrt{\varepsilon}W_{12}(t_c, \varepsilon) \\ \sqrt{\varepsilon}W_{12}^T(t_c, \varepsilon) & \varepsilon W_{22}(t_c, \varepsilon) \end{pmatrix}. \]

Let us calculate the limit of the determinant of this matrix as \( \varepsilon \to +0 \). Taking into account that the determinant of a quadratic matrix is a continuous functions of its entries, and using the inequalities (28), we obtain

\[
\lim_{\varepsilon \to +0} \det \left( L(\varepsilon)W(t_c, \varepsilon)L(\varepsilon) \right) = \det \left( \lim_{\varepsilon \to +0} \left( L(\varepsilon)W(t_c, \varepsilon)L(\varepsilon) \right) \right) = \det \begin{pmatrix} W_s(t_c) & 0 \\ 0 & W_f(t_c) \end{pmatrix} = \det W_s(t_c) \det W_f(t_c).
\]

The equation (32), along with the inequalities (29) and (31), directly implies the existence of a number \( 0 < \varepsilon^* \leq \varepsilon_1 \), such that \( \det W(t_c, \varepsilon) \neq 0 \) for all \( \varepsilon \in (0, \varepsilon^*] \). The latter, along with Proposition 1, yields the statement of the theorem. \( \square \)

**Remark 2.** Note that the first inequality in (29) is equivalent to the complete Euclidean space controllability at the time instant \( t_c \) of the slow subsystem (3). The second inequality in (29) cannot, in general, be interpreted as a controllability condition of a system with \( m \)-dimensional state and \( r \)-dimensional control variables. However, if there exists an \( m \times r \)-matrix \( \tilde{B}(t_c) \) such that \( \tilde{B}(t_c)\tilde{B}^T(t_c) = S_f(t_c) \),
then the second inequality in (29) is equivalent to the complete controllability of the system
\[
\frac{d\tilde{u}(\chi)}{d\chi} = A_{22}(t_c, 0)\tilde{y}(\chi) + \tilde{B}(t_c)\tilde{u}(\chi), \quad \chi \geq 0,
\]
where \(\tilde{y}(\chi) \in \mathbb{R}^m\) is a state variable; \(\tilde{u}(\chi) \in \mathbb{R}^r\) is a control.

**Remark 3.** Theorem 4.2 represents several important features in the controllability analysis of the original singularly perturbed system (1)-(2). First, this theorem allows to avoid a necessity to solve the terminal-value problem for the stiff (ill posed) matrix differential equation (12). Namely, this theorem replaces such a solution with the solution of the well posed terminal-value problem (16) and the calculation of the matrix (17). Second, the problem (16) and the matrix (17) are not only well posed, but they also are of lower dimensions than the problem (12). The latter feature is mainly considerable in the simplification of the controllability analysis of the original singularly perturbed system (1)-(2) if the dimensions of both, the problem (16) and the matrix (17), are essentially smaller than the dimension of the problem (12). Third, Theorem 4.2 allows to analyze the controllability of the original singularly perturbed system (1)-(2) in the case where the parameter \(\varepsilon\) of the singular perturbation is unknown, i.e., the original system is uncertain. Theorem 4.2 reduces the controllability analysis of the singularly perturbed system (1)-(2) to analysis of the nonsingularity of the \(\varepsilon\)-free matrices \(W_s(t_c)\) and \(W_f(t_c)\). Such a reduction is the main result of the paper. The analysis of the nonsingularity of the matrices \(W_s(t_c)\) and \(W_f(t_c)\), especially in the case of their large dimensions, is out of the scope of this paper.

5. **Example.** Consider a particular case of (1)-(2), where \(n = 2, m = 2, r = 1, h(\varepsilon) = 1 - \varepsilon\), and

\[
\begin{align*}
A_{11}(t, \varepsilon) = & \begin{pmatrix} t^2 + 3 & 1 - \varepsilon \\ \varepsilon^2 - 2 & -(t + 1)^2 \end{pmatrix}, \\
A_{12}(t, \varepsilon) = & \begin{pmatrix} t + \varepsilon & -1 \\ 2 - \varepsilon & t + 1 \end{pmatrix}, \\
A_{21}(t, \varepsilon) = & \begin{pmatrix} -t & 1 + \varepsilon \\ 2 & t + 1 \end{pmatrix}, \\
A_{22}(t, \varepsilon) = & \begin{pmatrix} \varepsilon t - 1 & 0 \\ 0 & \varepsilon t - 1 \end{pmatrix}, \\
B_{01}(t, \varepsilon) = & \begin{pmatrix} \varepsilon - 2t \\ t^2 + t - 4 \end{pmatrix}, \\
B_{11}(t, \varepsilon) = & \begin{pmatrix} (t + 1)^2 \\ 3 \exp(\varepsilon) \end{pmatrix}, \\
B_{02}(t, \varepsilon) = & \begin{pmatrix} 2 - \varepsilon \\ -t \end{pmatrix}, \\
B_{12}(t, \varepsilon) = & \begin{pmatrix} -(t + 2) \\ \exp(-\varepsilon) \end{pmatrix}.
\end{align*}
\]

(33)

We study the complete Euclidean space controllability of the system (1)-(2) with the data (33) at the time instant \(t_c = 2\) for all sufficiently small \(\varepsilon > 0\).

It is clear that for any given \(\varepsilon^0 \in (0, 1)\), the inequality (22) is valid and the assumptions (AI)-(AIII) are fulfilled. Due to the equation (4), we obtain

\[
A_s(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_{0s}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad B_{1s}(t) = \begin{pmatrix} 0 \\ -t \end{pmatrix}.
\]
Using these matrices, as well as the terminal-value problem (16) and the equations (24)-(25), we have

\[
W_s(\sigma) = \begin{cases}
\begin{pmatrix}
\sigma \exp(2 - \sigma) \\
-(\sigma + 1) \exp(2(1 - \sigma))
\end{pmatrix}, & \sigma \in [0, 1], \\
\begin{pmatrix}
\sigma \exp(2 - \sigma) \\
0
\end{pmatrix}, & \sigma \in (1, 2],
\end{cases}
\]

\[
W_s(2) = \begin{pmatrix}
10.3995 & -7.1911 \\
-7.1911 & 34.5488
\end{pmatrix}.
\]

Hence, \( \det W_s(2) \neq 0 \), i.e., the first inequality in (29) is fulfilled. Proceed to the second inequality in (29). From the equations (17), (26)-(27) and (33), we obtain

\[
\Psi_f(\chi) = \text{diag}\left(\exp(-\chi), \exp(-\chi)\right), \quad \chi \geq 0,
\]

\[
S_f(2) = \begin{pmatrix}
20 & -8 \\
-8 & 4
\end{pmatrix}, \quad W_f(2) = \begin{pmatrix}
10 & -4 \\
-4 & 2
\end{pmatrix}.
\]

Thus, \( \det W_f(2) \neq 0 \), i.e., the second inequality in (29) also is fulfilled. Therefore, by virtue of Theorem 4.2, there exists a positive number \( \varepsilon^* \), such that for all \( \varepsilon \in (0, \varepsilon^*) \) the system (1)-(2),(33) is completely Euclidean space controllable at the time instant \( t_c = 2 \).

6. **Conclusions.** In this paper, the singularly perturbed linear time-dependent controlled system with a point-wise control delay was considered. The delay is of order of 1. The complete Euclidean space controllability of this system, robust with respect to the parameter of singular perturbation, was studied. Sufficient conditions of this controllability, which are not based on the slow-fast decomposition of the considered system, were derived. Although these conditions are independent of the parameter of singular perturbation, they guarantee the complete Euclidean space controllability of the considered system for all sufficiently small values of this parameter.

7. **Appendix: Proof of Lemma 4.1.** We start with the proof of the first inequality in (28). Using the equations (11),(13)-(15) and (23), we obtain after a routine algebra the expression for \( W_{11}(t_c, \varepsilon) \):

\[
W_{11}(t_c, \varepsilon) = \Phi_1(\varepsilon) + \Phi_2(\varepsilon) + \Phi_3(\varepsilon),
\]

where

\[
\Phi_1(\varepsilon) = \int_0^{t_c} \left[ \Psi_{11}^T(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{01}^T(\sigma, \varepsilon)\Psi_{11}(\sigma, \varepsilon) + \frac{1}{\varepsilon} \Psi_{21}^T(\sigma, \varepsilon)B_{02}(\sigma, \varepsilon)B_{01}^T(\sigma, \varepsilon)\Psi_{11}(\sigma, \varepsilon) + \frac{1}{\varepsilon} \Psi_{11}^T(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{02}^T(\sigma, \varepsilon)\Psi_{21}(\sigma, \varepsilon) + \frac{1}{\varepsilon^2} \Psi_{21}^T(\sigma, \varepsilon)B_{02}(\sigma, \varepsilon)B_{02}^T(\sigma, \varepsilon)\Psi_{21}(\sigma, \varepsilon) \right] d\sigma,
\]

(35)
\[ \Phi_2(\varepsilon) = \int_0^{\tau_d} \left[ \Psi_{11}^T(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{11}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{11}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon} \Psi_{21}^T(\sigma, \varepsilon)B_{02}(\sigma, \varepsilon)B_{11}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{11}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon} \Psi_{11}^T(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{12}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{21}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon^2} \Psi_{21}^T(\sigma, \varepsilon)B_{02}(\sigma, \varepsilon)B_{12}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{21}(\sigma + h(\varepsilon), \varepsilon) \right] d\sigma, \]

\[ (36) \]

\[ \Phi_3(\varepsilon) = \int_0^{\tau_d} \left[ \Psi_{11}^T(\sigma + h(\varepsilon), \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)B_{11}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{11}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon} \Psi_{21}^T(\sigma + h(\varepsilon), \varepsilon)B_{12}(\sigma + h(\varepsilon), \varepsilon)B_{11}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{11}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon} \Psi_{11}^T(\sigma + h(\varepsilon), \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)B_{12}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{21}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon^2} \Psi_{21}^T(\sigma + h(\varepsilon), \varepsilon)B_{12}(\sigma + h(\varepsilon), \varepsilon)B_{12}^T(\sigma + h(\varepsilon), \varepsilon)\Psi_{21}(\sigma + h(\varepsilon), \varepsilon) \right] d\sigma. \]

\[ (37) \]

Using the equations (35)-(37), as well as the inequalities for \( \Psi_{11}(\sigma, \varepsilon) \) and \( \Psi_{21}(\sigma, \varepsilon) \) in (19), Remark 1, and the assumptions (A1) and (AII), we obtain the existence of a number \( 0 < \varepsilon_1 \leq \varepsilon_0 \), such that for all \( \varepsilon \in (0, \varepsilon_1] \) the following inequalities are satisfied:

\[ \| \Phi_p(\varepsilon) - \Phi_{p,s} \| \leq a\varepsilon, \quad p = 1, 2, 3, \]

\[ (38) \]

where \( a > 0 \) is some constant independent of \( \varepsilon \), and

\[ \Phi_{1,s} = \int_0^{\tau_d} \left[ \Psi_{11}^T(\sigma)B_{01}(\sigma, 0)B_{01}^T(\sigma, 0)\Psi_{11}(\sigma) + \Psi_{21}^T(\sigma)B_{02}(\sigma, 0)B_{01}^T(\sigma, 0)\Psi_{21}(\sigma) \\
+ \Psi_{11}^T(\sigma)B_{01}(\sigma, 0)B_{12}^T(\sigma, 0)\Psi_{21}(\sigma) + \Psi_{21}^T(\sigma)B_{02}(\sigma, 0)B_{12}^T(\sigma, 0)\Psi_{21}(\sigma) \right] d\sigma, \]

\[ (39) \]

\[ \Phi_{2,s} = \int_0^{\tau_d} \left[ \Psi_{11}^T(\sigma)B_{01}(\sigma, 0)B_{11}^T(\sigma + h(0), 0)\Psi_{11}(\sigma + h(0)) \\
+ \Psi_{21}^T(\sigma)B_{02}(\sigma, 0)B_{11}^T(\sigma + h(0), 0)\Psi_{21}(\sigma + h(0)) + \Psi_{11}^T(\sigma)B_{01}(\sigma, 0)B_{12}^T(\sigma + h(0), 0)\Psi_{21}(\sigma + h(0)) \\
+ \Psi_{21}^T(\sigma)B_{02}(\sigma, 0)B_{12}^T(\sigma + h(0), 0)\Psi_{21}(\sigma + h(0)) \right] d\sigma, \]

\[ (40) \]

\[ \Phi_{3,s} = \int_0^{\tau_d} \left[ \Psi_{11}^T(\sigma + h(0))B_{11}(\sigma + h(0), 0)B_{11}^T(\sigma + h(0), 0)\Psi_{11}(\sigma + h(0)) \\
+ \Psi_{21}^T(\sigma + h(0))B_{12}(\sigma + h(0), 0)B_{11}^T(\sigma + h(0), 0)\Psi_{21}(\sigma + h(0)) + \Psi_{11}^T(\sigma + h(0))B_{11}(\sigma + h(0), 0)B_{12}^T(\sigma + h(0), 0)\Psi_{21}(\sigma + h(0)) \\
+ \Psi_{21}^T(\sigma + h(0))B_{12}(\sigma + h(0), 0)B_{12}^T(\sigma + h(0), 0)\Psi_{21}(\sigma + h(0)) \right] d\sigma. \]

\[ (41) \]
Substitution of the equation (20) into these expressions for $\Phi_{1,s}$, $(p = 1, 2, 3)$, and use of the expressions for $B_{0s}(\sigma), B_{1s}(\sigma), B_{0s}(\sigma), B_{1s}(\sigma + h(0))$, $W_s(\sigma), W_s(t_c)$ (see the equations (4), (24), (25)) yield after a routine algebra:

$$
\Phi_{1,s} = \int_0^{t_c} B_{0s}(\sigma)B_{0s}(\sigma)d\sigma,
$$

$$
\Phi_{2,s} = \int_0^{t_c} B_{0s}(\sigma)B_{0s}(\sigma + h(0))d\sigma,
$$

$$
\Phi_{3,s} = \int_0^{t_c} B_{1s}(\sigma + h(0))B_{1s}(\sigma + h(0))d\sigma,
$$

$$
\Phi_{1,s} + \Phi_{2,s} + \Phi_{2,s} + \Phi_{3,s} = W_s(t_c).
$$

The latter equality, along with the equation (34) and the inequalities in (38), immediately yields the first inequality in (28) for all $\varepsilon \in (0, \bar{\varepsilon})$.

Now, let us prove the third inequality in (28). Similarly to (34), we obtain

$$
\varepsilon W_{22}(t, \varepsilon) = \Theta_1(\varepsilon) + \Theta_2(\varepsilon) + \Theta_3(\varepsilon) + \Theta_4(\varepsilon), \quad (42)
$$

where

$$
\Theta_1(\varepsilon) = \int_0^{t_c} \left[ \varepsilon \Psi_{12}(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{11}(\sigma, \varepsilon)\Psi_{12}(\sigma, \varepsilon) \right. \\
+ \Psi_{12}(\sigma, \varepsilon)B_{02}(\sigma, \varepsilon)B_{11}(\sigma, \varepsilon)\Psi_{12}(\sigma, \varepsilon) \\
+ \Psi_{12}(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{12}(\sigma, \varepsilon)\Psi_{12}(\sigma, \varepsilon) \\
+ \frac{1}{\varepsilon} \varepsilon \Psi_{22}(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{11}(\sigma, \varepsilon)\Psi_{22}(\sigma, \varepsilon)d\sigma,
$$

$$
\Theta_2(\varepsilon) = \int_0^{t_c} \left[ \varepsilon \Psi_{12}(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)\Psi_{12}(\sigma + h(\varepsilon), \varepsilon) \right. \\
+ \Psi_{12}(\sigma, \varepsilon)B_{02}(\sigma, \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)\Psi_{12}(\sigma + h(\varepsilon), \varepsilon) \\
+ \Psi_{12}(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{12}(\sigma + h(\varepsilon), \varepsilon)\Psi_{12}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon} \varepsilon \Psi_{22}(\sigma, \varepsilon)B_{01}(\sigma, \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)\Psi_{22}(\sigma + h(\varepsilon), \varepsilon)d\sigma,
$$

$$
\Theta_3(\varepsilon) = \int_0^{t_c} \left[ \varepsilon \Psi_{12}(\sigma + h(\varepsilon), \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)\Psi_{12}(\sigma + h(\varepsilon), \varepsilon) \right. \\
+ \Psi_{12}(\sigma + h(\varepsilon), \varepsilon)B_{12}(\sigma + h(\varepsilon), \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)\Psi_{12}(\sigma + h(\varepsilon), \varepsilon) \\
+ \Psi_{12}(\sigma + h(\varepsilon), \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)B_{12}(\sigma + h(\varepsilon), \varepsilon)\Psi_{12}(\sigma + h(\varepsilon), \varepsilon) \\
+ \frac{1}{\varepsilon} \varepsilon \Psi_{22}(\sigma + h(\varepsilon), \varepsilon)B_{11}(\sigma + h(\varepsilon), \varepsilon)B_{12}(\sigma + h(\varepsilon), \varepsilon)\Psi_{22}(\sigma + h(\varepsilon), \varepsilon)d\sigma.
$$

In the integrals of the equations (43) and (44), we change the variable as $\chi = (t_c - \sigma)/\varepsilon$. In the integral of the equation (45), we change the variable as $\chi = (t_c - \sigma - h(\varepsilon))/\varepsilon$. These changes of the variables, along with the inequalities for $\Psi_{12}(\sigma, \varepsilon)$ and $\Psi_{22}(\sigma, \varepsilon)$ in (19), Remark 1, the properties (17)-(18) of $\Psi_f(\chi)$, the
assumptions (AI)-(AII) and the inequality (22), yield after a routine rearrangement the existence of a number $0 < \tilde{\varepsilon}_3 \leq \varepsilon_0$, such that for all $\varepsilon \in (0, \tilde{\varepsilon}_3]$ the following inequalities are satisfied:

\[
\left\| \Theta_1(\varepsilon) - \Theta_{1,f} \right\| \leq a \varepsilon, \quad \left\| \Theta_3(\varepsilon) - \Theta_{3,f} \right\| \leq a \varepsilon, \quad \left\| \Theta_2(\varepsilon) \right\| \leq a \varepsilon, \tag{46}
\]

where $a > 0$ is some constant independent of $\varepsilon$, and

\[
\Theta_{1,f} = \int_0^{+\infty} \Psi_f(\chi) B_{q2}(t_c, 0) B_{q2}^T(t_c, 0) \Psi_f(\chi) d\chi, \\
\Theta_{3,f} = \int_0^{+\infty} \Psi_f(\chi) B_{12}(t_c, 0) B_{12}^T(t_c, 0) \Psi_f(\chi) d\chi.
\]

These two equations, along with the equations (26)-(27), (42) and the inequalities in (46), yield the third inequality in (28) for all $\varepsilon \in (0, \tilde{\varepsilon}_3]$.

Similarly to the proofs of the first and third inequalities in (28), one can obtain the existence of a number $0 < \tilde{\varepsilon}_2 \leq \varepsilon_0$, such that for all $\varepsilon \in (0, \tilde{\varepsilon}_2]$ the second inequality in (28) is valid. Finally, the choice of $\varepsilon_1$ as $\varepsilon_1 = \min_{p=1,2,3} \tilde{\varepsilon}_p$ completes the proof of the lemma.

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