Two subclasses of generalized Kenmotsu manifolds

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Abstract. The paper studies the geometry of the Riemann curvature tensor of generalized Kenmotsu manifolds. In this paper, several identities satisfied by the curvature tensor of generalized Kenmotsu manifolds are obtained. Two identities are distinguished from the obtained identities, called the first and second additional identities of curvature of the GK-manifold. Based on additional identities, two subclasses of GK-manifolds are distinguished, and a local characterization of the distinguished classes is also obtained. It is proved that the distinguished two subclasses of GK-manifolds coincide and have dimension 5. In addition, it is proved that the class of distinguished manifolds coincides with the class of almost contact metric manifolds obtained from a cosymplectic manifold by a canonical concircular transformation of a cosymplectic structure of dimension 5.

1. Introduction
Let $M$ be a connected smooth manifold of dimension $(2n+1)$, $C^\infty(M)$ - the algebra of smooth functions on $M$, $\text{X}(M) - C^\infty$ - module of smooth vector fields on the manifold $M$, $d$ - the operator of external differentiation. If a Riemannian metric $\langle \cdot , \cdot \rangle$ is given on $M$, then the corresponding Riemannian connection is denoted by $\nabla$. All manifolds, tensor fields (tensors), and other similar objects will be assumed smooth of class $C^\infty$.

Differential 1-forms of maximal rank on an odd-dimensional Riemannian manifold generate a special differential geometric structure called a contact metric structure, which naturally generalizes to the so-called almost contact metric structure.

Recall [1] that a contact form or contact structure on an odd-dimensional manifold $M$, $\dim M=2n+1$, is a 1-form $\eta$ on $M$, such that at every point of the manifold $\eta(\text{d}\eta)^{\wedge}\ldots\wedge(\text{d}\eta)^{\wedge}\neq 0$, i.e. $\text{rg}\eta=\dim M$, in each point from $M$. A manifold with a contact form fixed on it is called a contact manifold [2]. On such a manifold, distributions $L=\text{ker}\ \eta$, $\dim L=2n$, $\mu=\text{ker}(\text{d}\eta)$, $\dim \mu=1$ are internally defined. It is easy to deduce from Darboux’s theorem [1] that $L\cap \mu=\{0\}$, and it means that $\text{X}(M)=L\oplus \mu$. Let’s choose $\xi\in M$ so that $\eta(\xi)=1$. Then on $M$ one can define mutually complementary projectors $m=\xi\otimes\eta$ and $l=\text{id}-\xi\otimes\eta$ on distributions $M$ and $L$, respectively.

Let Riemannian metric $h$ be a fixed on $M$. Based on the metric $h|L$, it is easy to construct a metric $(\cdot , \cdot )$ on $L$, such that the operator $I:L\rightarrow L$ defined by identity $(X,IY)=\text{d}\eta(X,Y)\colon X,Y\in L$ will be involutive, i.e. $I^2=\text{id}$, and $(IX,IY)=(X,Y)$. Then vector $\xi$, covector $\eta$, operator $\Phi=I\circ I$ and Riemannian metric $(X,Y)=(IX,IY)+[h(\xi,\eta)]^{-1}h(mX,mY)$ on $M$ obviously have properties:
A contact manifold $M$ equipped with a Riemannian metric $(\cdot,\cdot)$ for which relations (1) are valid is called a contact metric manifold.

The above construction makes the following definition natural.

Definition 1.1 [3]. An almost contact metric (or almost Gray) structure on a manifold $M$ is a collection $\{g, \Phi, \xi, \eta\}$ of tensor fields on $M$, where $g = (\cdot,\cdot)$ is a (pseudo) Riemannian metric, $\Phi$ is a tensor of type $(1,1)$, called structural endomorphism, $\xi$ is a vector field called characteristic, $\eta$ is a differential 1-form called the contact form of the structure. Wherein:

1. $\Phi(\xi) = 0$;
2. $\eta \circ \Phi = 0$;
3. $\eta(\xi) = 1$;
4. $\Phi^2 = \text{id} + \eta \otimes \xi$;
5. $(\Phi X, \Phi Y) = (X, Y) - \eta(X)\eta(Y), X, Y \in \mathcal{X}(M)$ (2)

Note that these relations are not independent; for example, (1.2:1) and (1.2:2) follow from (1.2:3) and (1.2:4) [4]. Further, from (1.2:1), (1.2:3) and (1.2:5) it follows that $\eta(X) = (\xi, X)$, $X \in \mathcal{X}(M)$, and from (1.2:2), (1.2:4) and (1.2:5) - that the tensor $\Omega(X, Y) = (X, \Phi Y)$ is skew-symmetric; it is called the fundamental form of structure. The triple $\{\Phi, \xi, \eta\}$ satisfying conditions (1.2:3) and (1.2:4) is called an almost contact structure; in this form, it was introduced by Sasaki in [3].

The most important example of almost contact metric structures, which largely determines their role in differential geometry, is the structure induced on the hypersurface $N$ of a manifold $M$ equipped with an almost Hermitian structure $\{J, (\cdot,\cdot)\}$. Recall this construction. Let $n^0$ be the field of the unit normal to $N$. Then the vector $\xi = J(n^0) \in \mathcal{X}(N)$, and its orthogonal complement $\mathcal{L}$ on $N$ is invariant under $J$. Let’s define the linear operator $\Phi = J|\mathcal{L} \otimes 0|\mathcal{M}$ in $\mathcal{X}(N)$, where $\mathcal{M}$ is the linear span of the vector $\xi$, and the 1-form $\eta(X) = (\xi, X)$. Then $\{\cdot, \cdot, \Phi, \xi, \eta\}$ is an almost contact metric structure on $N$. In particular, such a structure is induced on the odd-dimensional sphere $S^{2n-1}$ considered as a hypersurface in the reification of $C^n$. This is the most important and, apparently, historically first concrete example of such a structure.

Let $\{g = (\cdot,\cdot), \Phi, \xi, \eta\}$ be an almost contact metric structure on the manifold $M$. It is well known [1] that in this case, the almost Hermitian structure $\{J, h\}$ is induced on the manifold $M \times \mathbb{R}$, where $J = \Phi|L \otimes J_1, h = g|\mathcal{L} \otimes g|_1$, $J_1$ is the canonical almost complex structure on the two-dimensional distribution $M \times \mathbb{R}$, $g_1$ is the metric on this distribution, which is the direct sum of the metric $g|\mathcal{M}$ and the canonical metric on $\mathbb{R}$. An almost contact structure $\{\Phi, \xi, \eta\}$ is called normal if the structure $\{J, h\}$ is integrable [4]. The necessary and sufficient condition for the normality of the structure has the form $N^+ \xi \otimes d\eta = 0$, where $N$ is the Nijenhuis tensor of the operator $\Phi$ [4].
Today, active research is being carried out on the geometry of almost contact metric structures on manifolds. One of the most pressing issues in this section of geometry is the question of studying individual classes of almost contact metric manifolds. In 1972, Kenmotsu [7] introduced a class of almost contact metric structures characterized by the identity \( \nabla_X (\Phi) Y = (\Phi X, Y) \xi - \eta (Y) \Phi X, X, Y \in \mathfrak{X} (M) \). Kenmotsu structures, for example, naturally arise in the Tanno classification of connected almost contact metric manifolds, whose automorphism group has maximum dimension [8]. They have a number of remarkable properties. For example, Kenmotsu structures are normal and integrable. They are not contact structures, and therefore, Sasakian. There are some known examples of Kenmotsu structures on odd-dimensional spaces of Lobachevsky curvature (-1). Such structures are obtained using the warped construction of the product \( \mathbb{R} \times \mathbb{C}^m \) in the sense of Bishop and O'Neill [9] of the complex Euclidean space and the real line, where \( f(t) = ce^t \) (see [7]).

Polarizing the identity characterizing Kenmotsu manifolds, S. Umnova singled out in her dissertation [10] a class of almost contact metric manifolds, which is a generalization of Kenmotsu manifolds and called a class of generalized (in short, GK-) Kenmotsu manifolds. It was proved in [10] that generalized Kenmotsu manifolds of constant curvature are Kenmotsu manifolds of constant curvature -1.

In [11], this class of manifolds is called the approximately Kenmotsu class of manifolds. The authors prove that a second-order symmetric closed recurrence tensor whose recurrence covector annihilates the characteristic vector \( \xi \) is a multiple of the metric tensor \( g \). In addition, the authors consider \( \Phi \)-recurrent manifolds near Kenmotsu. It is proved that \( \Phi \)-recurrent manifolds of approximately Kenmotsu are Einstein manifolds, and locally \( \Phi \)-recurrent manifolds of approximately Kenmotsu are manifolds of constant curvature -1.

In the work [12], Banaru M.B. investigated hypersurfaces of almost Hermitian manifolds of class \( W_3 \) endowed with a Kenmotsu structure, and obtained interesting properties of Kenmotsu manifolds.

The authors of [13-14] studied the Einstein generalized Kenmotsu manifolds, obtained contact analogues of Gray identities, distinguished three classes of this type of manifolds and obtained a local characterization of the distinguished classes of manifolds. In [15], identities of curvature on the Riemann curvature tensor of a special case of generalized Kenmotsu manifolds called special generalized Kenmotsu manifolds of the second kind were considered [10].

The article [16] is devoted to the study of integrability properties of generalized Kenmotsu manifolds. In this paper, we study GK-manifolds whose first fundamental distribution is completely integrable. It is shown that an almost Hermitian structure induced on integral manifolds of the maximum dimension of the first distribution of a GK-manifold is approximately Kähler. The local structure of a GK-manifold with a closed contact form is obtained, the expressions of the first and second structural tensors are given. Also, the components of the Nijenhuis tensor of a GK-manifold are calculated. Since the definition of the Nijenhuis tensor is equivalent to the definition of the four tensors \( N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)} \), we study the geometric meaning of the vanishing of these tensors. The local structure of an integrable and normal GK-structure is obtained. It is proved that the characteristic vector of the GK-structure is not a Killing vector.

From the above incomplete review of works devoted to generalized Kenmotsu manifolds, it can be seen that interest in the study of this class of manifolds does not fade, but rather grows.

In this paper, we continue the study of generalized Kenmotsu manifolds and investigate the geometry of the Riemann curvature tensor of this class of manifolds.

This paper is organized as follows. In Section 2, we give the preliminary information needed in the subsequent presentation, construct the space of the adjoint G-structure. In Section 3, we give a definition of generalized Kenmotsu manifolds, give a complete group of structural equations, and give the components of the Riemann-Christoffel tensor on the space of an adjoint G-structure. In Section 4, using the procedure for recovering identities [17-18], we obtain some identities that satisfy the tensor of the Riemann curvature of generalized Kenmotsu manifolds and on their basis distinguish two classes of generalized Kenmotsu manifolds. We prove that the distinguished classes coincide. The
The main result is a theorem that we called the main. The main theorem gives a local characterization of the distinguished classes.

2. Preliminary information

Let $M$ be a smooth manifold, dimensions $2n+1$, $X(M)$ — $C^\infty$-module of smooth vector fields on the manifold $M$. Further, all manifolds, tensor fields, and other objects are assumed to be smooth of $C^\infty$ class.

Definition 2.1 [17,18]. An almost contact structure on a manifold $M$ is a triple $(\eta,\xi,\Phi)$ of tensor fields on this manifold, where $\eta$ is a differential 1-form called a contact form of a structure, $\xi$ is a vector field called a characteristic field, $\Phi$ is an endomorphism of the module $X(M)$ called structural endomorphism. Wherein

1) $\eta(\xi)=1$; 2) $\Phi\Phi=0$; 3) $\Phi(\xi)=0$; 4) $\Phi^2=\text{id}+\eta\otimes\xi$. (3)

If, in addition, such a Riemannian structure $g=\langle \cdot,\cdot \rangle$ is fixed on $M$ that

$$\langle \Phi X,\Phi Y \rangle=\langle X,Y \rangle-\eta(X)\eta(Y), \ X,Y\in X(M)$$ (4)

tetrad $(\eta,\xi,\Phi,g=\langle \cdot,\cdot \rangle)$ is called an almost contact metric (in short, AC-) structure. A manifold on which an almost contact (metric) structure is fixed is called an almost contact (metric (in short, AC-) manifold.

The skew-symmetric tensor $\Omega(X,Y)=\langle X,\Phi Y \rangle, X,Y\in X(M)$ is called the fundamental form of the AC-structure [17,18].

Let $(\eta,\xi,\Phi,g)$ be an almost contact metric structure on the manifold $M^{2n+1}$. In the module $X(M)$, two mutually complementary projectors $m=\eta\otimes\xi$ and $l=\text{id}-m=\Phi^2$ are internally defined [18]; thus, $X(M)=L\oplus M$, where $L=\text{Im}(\Phi)=\ker\eta$ is the so-called contact (or first fundamental) distribution, $\dim L=2n$, $M=\text{Im}(\Phi)=L(\xi)$ - the linear span of the structural vector or the so-called second fundamental distribution (moreover, $l$ and $m$ are projectors on the submodules $L$, $M$, respectively) [17, 18]. Obviously, the distributions of $L$ and $M$ are invariant with respect to $\Phi$ and are mutually orthogonal. It is also obvious that $\Phi^2=\text{id}$, $\langle \Phi X,\Phi Y \rangle=\langle X,Y \rangle, X,Y\in X(M)$, where $\Phi=\Phi|L$. Therefore, $\{\Phi|L,g|L\}$ is a Hermitian structure on the space $L_p$.

The complexification $(X(M))^C$ of the module $X(M)$ splits into the direct sum

$$X(M)^C=D_\Phi^{-1} \oplus D_\Phi^{-1} \oplus D_\Phi^0$$

of proper subspaces of the structural endomorphism $\Phi$ corresponding to the eigenvalues $-1$, $-1$ and 0, respectively. Moreover, the projectors on the terms of this direct sum are, respectively, endomorphisms [17, 18] $\sigma=\sigma_0l=\frac{1}{2}\left(\Phi^2+\sqrt{-1}\Phi\right), \bar{\sigma}=\bar{\sigma}_0l=\frac{1}{2}\left(\Phi^2+\sqrt{-1}\Phi\right), m=\text{id}+\Phi^2$,

where $\sigma=\frac{1}{2}\left(\text{id}+\sqrt{-1}\Phi\right), \bar{\sigma}=\frac{1}{2}\left(\text{id}+\sqrt{-1}\Phi\right)$.

The expressions $\sigma_p: L_p\to D_\Phi^{-1}$ and $\bar{\sigma}_p: L_p\to D_\Phi^{-1}$ are an isomorphism and anti-isomorphism, respectively, of Hermitian spaces. Therefore, to each point $p\in M^{2n+1}$, we can add a family of frames of the space $T_p(M)^C$ of the form $(p,e_0,e_1,...,e_m,e_1,...,e_0)$, where $e_a=\sqrt{2}\sigma(p)e_a, e_b=\sqrt{2}\bar{\sigma}(p)e_a, e_0=\xi p$; where $\{e_i\}$ is the orthonormal basis of the Hermitian space $L_p$. Such a frame is called an A-frame [18]. It is easy to see that the matrices of the components of the tensors $\Phi|L$ and $g|L$ in the A-frame have the form, respectively:
\[
(\Phi_i) = \begin{pmatrix}
0 & 0 & 0 \\
0 & -1I_n & 0 \\
0 & 0 & -1I_n
\end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1_n \\
0 & 1_n & 0
\end{pmatrix}
\]

where \(I_n\) – an identity matrix of order \(n\). It is well known [17, 18] that the set of such frames defines a G-structure on \(M\) with the structural group \(\{1\} \times U(n)\) represented by matrices of the form 
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & A
\end{pmatrix}
\]
where \(A \in U(n)\). This G-structure is called adjoint [17, 18].

Let \((M^{2n+1}, \Phi, \xi, \eta, g=g(\cdot, \cdot))\) be the almost contact metric manifold. We agree that throughout the work, unless otherwise indicated, the indices \(i,j,k,l,…\) run through the values from 1 to 2\(n\), the indices \(a,b,c,d,…\) - the values from 1 to \(n\), and assume that \(\bar{a}=a+\eta, \bar{a}=a, \bar{0}=0\). Let \((U, \varphi)\) be a local map on the manifold \(M\). According to the Main tensor analysis theorem [17], defining the structural endomorphism \(\Phi\) and the Riemannian structure \(g=g(\cdot, \cdot)\) on the manifold \(M\) induces the setting on the total space \(BM\) of the bundle of frames over \(M\) of a system of functions \(\{\Phi_i\}, \{g_{ij}\}\) satisfying in the coordinate neighborhood \(W=\pi^{-1}(U)\subset BM\) a system of differential equations of the form

\[
d\Phi_i^j+\Phi_k^i\theta_j^k-\Phi_k^j\theta_i^k, \quad dg_{ij}-\omega_i^j=g_{ik}\theta_i^j-g_{ij,k}\omega^k
\]

where \(\{\omega_i^j\}, \{\theta_i^j\}\) – the components of the displacement forms and the Riemannian connection \(\nabla\), respectively. \(\Phi_{ij,k}, \ g_{ij,k}\) are the components of the covariant differential of the tensors \(\Phi\) and \(g\) in this connection, respectively. Moreover, by the definition of the Riemannian connection, \(\nabla g=0\) and, therefore,

\[
g_{ij,k}=0
\]

Relations (6) on the space of the adjoint G-structure are written in the form [17, 18]

\[
\Phi_{h,k}=0, \ \Phi_{b,k}=0, \ \Phi_{0,k}=0
\]

\[
\theta_{b}^\pm = \frac{\sqrt{-1}}{2} \Phi_{b,k} \omega^k, \ \theta_{b}^\pm = -\frac{\sqrt{-1}}{2} \Phi_{b,k} \omega^k
\]

\[
\theta_{0}^a = -\sqrt{-1} \Phi_{0,k} \omega^k, \ \theta_{0}^a = \sqrt{-1} \Phi_{0,k} \omega^k
\]

\[
\theta_{a}^0 = -\sqrt{-1} \Phi_{a,k} \omega^k, \ \theta_{a}^0 = \sqrt{-1} \Phi_{a,k} \omega^k
\]

\[
\theta_{j}^i + \theta_{j}^i = 0, \ \theta_{0}^0 = 0
\]

In addition, we note that since the corresponding forms and tensors are real \(\bar{\omega}=\omega, \bar{\theta}_j=\theta_j, \bar{\nabla} \Phi_{ij,k}^\pm = \nabla \Phi_{ij,k}^\pm\), where \(t=\bar{t}\) is the complex conjugation operator.

The first group of structural equations of the Riemannian connection \(d\omega^j=\theta_i^j \Lambda \omega^j\) on the space of the adjoint G-structure of an almost contact metric manifold can be written in the following form called the first group of structural equations of an almost contact metric manifold [17, 18]:

\[
d\omega = C_{ab} \omega^a \Lambda \omega^b + C_{ab} \omega^a \Lambda \omega^b + C_{ab} \omega^a \Lambda \omega^b + C_{ab} \omega^a \Lambda \omega^b + C_{ab} \omega^a \Lambda \omega^b
\]

\[
d\omega^a = -\theta_i^j \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b
\]

\[
d\omega_i = \theta_i^j \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b
\]

\[
d\omega^a = -\theta_i^j \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b + B_{abc} \omega^a \Lambda \omega^b
\]
where \( \omega = \omega^0 = \pi^* (\eta) \); \( \pi \) – natural projection of the space of the adjoint G-structure onto the manifold \( M \), \( \omega^i = g_{ij} \omega^j \),

\[
B^{ab}=\frac{\sqrt{-1}}{2} \Phi_{b,\alpha}^a, \quad B^{ab}=\frac{\sqrt{-1}}{2} \Phi_{b,\beta}^a, \quad B^{abc}=\frac{\sqrt{-1}}{2} \Phi_{b,c}^a,
\]
\[
B_{abc}=\frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad B_{ab}=\sqrt{-1} \Phi_{a,b}^0, \quad B_{ab}^b=\sqrt{-1} \Phi_{a,b}^0
\]
\[
B^{ab}=\sqrt{-1} \Phi_{[a,b]}^0, \quad \mathbf{C}_{ab}=\sqrt{-1} \Phi_{[a,b]}^0, \quad \mathbf{C}_{ab}=\sqrt{-1} \Phi_{[a,b]}^0 (10)
\]
\[
\mathbf{C}_{ab}=\sqrt{-1} \Phi_{[a,b]}^0, \quad \mathbf{C}_{ab}=\sqrt{-1} \Phi_{[a,b]}^0 (11)
\]

We introduce the notation \([17]\)

\[
\mathbf{C}_{abc}=\frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad \mathbf{C}_{abc}=\frac{\sqrt{-1}}{2} \Phi_{b,c}^a, \quad \mathbf{F}_{ab}=\sqrt{-1} \Phi_{a,b}^0, \quad \mathbf{F}_{ab}=\sqrt{-1} \Phi_{a,b}^0 (12)
\]

### 3. Generalized Kenmotsu manifolds

Let \( (\tilde{M}^{2n+1}, \Phi, \tilde{\xi}, \tilde{\eta}, g^{\tilde{\cdot}, \tilde{\cdot}}) \) be an almost contact metric manifold.

**Definition 3.1** \([10,11]\). The class of almost contact metric manifolds characterized by the identity

\[
\nabla_X (\Phi) Y + \nabla_Y (\Phi) X = - \eta(Y) \Phi X + \eta(X) \Phi Y; X, Y \in \mathfrak{X}(M)
\]

is called generalized Kenmotsu manifolds (in short, GK-manifolds).

Note that in literature, this class of manifolds is called the nearly Kenmotsu class of manifolds \([11]\) and others. We will call these manifolds, as in \([10]\), generalized Kenmotsu manifolds, and shortly - GK-manifolds.

The following theorem holds.

**Theorem 3.1** \([13]\). The full group of structural equations of GK-manifolds on the space of the adjoint G-structure has the form:

\[
\begin{align*}
1) & \quad d \omega = F_{ab} \omega^a \wedge \omega^b + F_{ab} \omega_a \wedge \omega_b; \\
2) & \quad d \omega^a = - \Phi_{b}^{a \beta} \omega^b + \mathbf{C}_{abc} \omega^a \wedge \omega^b + \Phi_{b}^{a \beta} \omega_a \wedge \omega_b; \\
3) & \quad d \omega^a = - \Phi_{b}^{a \beta} \omega^b + \mathbf{C}_{abc} \omega^a \wedge \omega^b + \Phi_{b}^{a \beta} \omega_a \wedge \omega_b; \\
4) & \quad d \Phi_{b}^{a \beta} = \mathbf{C}_{abc} \Phi_{b}^{a \beta} + \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta}; \\
5) & \quad d \Phi_{b}^{a \beta} = \mathbf{C}_{abc} \Phi_{b}^{a \beta} + \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta}; \\
6) & \quad d \Phi_{b}^{a \beta} = \mathbf{C}_{abc} \Phi_{b}^{a \beta} + \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta}; \\
7) & \quad d \Phi_{b}^{a \beta} = \mathbf{C}_{abc} \Phi_{b}^{a \beta} + \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta}; \\
8) & \quad d \Phi_{b}^{a \beta} = \mathbf{C}_{abc} \Phi_{b}^{a \beta} + \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta}; \\
9) & \quad d \Phi_{b}^{a \beta} = \mathbf{C}_{abc} \Phi_{b}^{a \beta} + \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta} - \Phi_{b}^{a \beta} \mathbf{C}_{abc} \Phi_{b}^{a \beta}.
\end{align*}
\]

where

\[
\begin{align*}
& C_{[abc]} = C_{abc}, \quad C_{[abc]} = C_{abc}, \quad C_{[abc]} = C_{abc}, \quad F_{ab} + F_{ba} = 0, \quad F_{ab} + F_{ba} = 0, \quad F_{ab} = F_{ba}, \quad A_{[abc]}^{[ab]} = A_{[abc]}^{[ab]} = 0, \quad C_{[bcd]}^{[bc]} = 2 F_{[ab]} F_{[cd]}, \quad F_{ad} C_{abc} = 0
\end{align*}
\]
Corollary 1 [10]. If $C^{abc}=C_{abc}=0$ and $F^{ab}=F_{ab}=0$, then GK-manifold is a Kenmotsu manifold.

Definition 3.2 [10]. A GK-structure is called a special generalized Kenmotsu structure of the first kind (in short, an SGK-structure of the first kind) if $C^{abc}=C_{abc}=0$; a special generalized Kenmotsu structure of the second kind (in short, an SGK-structure of the second kind) if $F_{ad}=F^{ab}=0$.

Let $M$ be a GK-manifold. Recall the following theorems from [13,14].

Theorem 3.2. The nonzero essential components of the Riemann-Christoffel tensor on the space of the adjoint $G$-structure have the form:

1. $R^{ab}_{cd}=F^{ac}F_{cb}+\delta^{a}_{b},$ $R^{a}_{00b}=-F_{ac}F^{cb}+\delta^{a}_{c},$

2. $R^{a}_{bcd}=\frac{2}{3}\delta^{a}_{b}F_{cd}+\frac{1}{3}\delta^{a}_{d}F_{bc}+\frac{1}{3}\delta^{a}_{d}F_{be};$ $R^{a}_{bcd}=\frac{2}{3}\delta^{a}_{b}F_{cd}+\frac{1}{3}\delta^{a}_{d}F_{bc};$

3. $R^{a}_{bcd}=A^{ad}C^{bdc}+\frac{2}{3}\delta^{a}_{d}F_{be};$ $R^{a}_{bcd}=A^{ad}C^{bdc}+\frac{2}{3}\delta^{a}_{d}F_{be};$

4. $R^{a}_{bcd}=2C^{ab}C_{bcd}+\frac{2}{3}\delta^{a}_{d}F_{be};$ $R^{a}_{bcd}=2C^{ab}C_{bcd}+\frac{2}{3}\delta^{a}_{d}F_{be};$

5. $R^{a}_{bcd}=C^{acbd}+\frac{2}{3}(F^{ad}F_{be}+F^{ad}F_{be});$ $R^{a}_{bcd}=C^{acbd}+\frac{2}{3}(F^{ad}F_{be}+F^{ad}F_{be}).$ (15)

4. The curvature identities of GK-manifolds

In [17,18], a class of quasi-Sasakian manifolds was distinguished whose Riemann curvature tensor satisfies the identity $R(\xi,\Phi^{2}X)\Phi^{2}Y-R(\xi,\Phi^{2}X)\Phi^{2}Y=0; \forall X,Y\in\chi(M)$. Following the idea presented in these papers, in [10] we considered some identities that satisfy the Riemann curvature tensor of SGK-manifolds of the second kind. In this section, we consider similar identities that are satisfied by the Riemann curvature tensor of GK-manifolds.

I) We apply the procedure for the restoration of identity [17,18] to the equalities: $R_{00b}=F^{0c}F_{cb}+\delta^{0}_{b}=0; R^{a}_{00b}=-F^{ac}F_{cb}+\delta^{a}_{b}; R^{a}_{00b}=F^{ac}F_{cb}+\delta^{a}_{b}$. At a fixed point $p \in \mathcal{M}$, the last equality is equivalent to the relation $R(\xi,\xi)\Sigma=F^{2}(\xi_{a})+\xi_{a}$. Since $\xi_{a}=\xi_{0},$ and the vectors $\{\xi_{a}\}$ form the basis of the subspace $p \in \mathcal{D}_{\Phi}^{1}$, given that the projectors of the module $X(M)^{C}$ onto the submodules $D_{\Phi}^{1}$ and $D_{\Phi}^{0}$ will be endomorphisms $\pi=se^{-1/2}(\Phi^{2}+\sqrt{-1}\Phi), m=id+\Phi^{2}$, the identity $R(\xi,\xi)\Sigma=F^{2}(\xi_{a})+\xi_{a}$ can be rewritten in the form $R(\xi,\Phi^{2}X+\sqrt{-1}\Phi)\xi=F^{2}\left(\Phi^{2}X+\sqrt{-1}\Phi\right)\xi+\Phi^{2}X+\sqrt{-1}\Phi \xi; \forall \xi \in \chi(M)$. By revealing this relation in linearity and highlighting the real and imaginary parts of the resulting equality, we get the equivalent identity:

$$R(\xi,\Phi^{2}X)\xi=F^{2}(\Phi^{2}X)+\Phi^{2}X; \forall X \in \chi(M)$$ (16)

Let’s call identity (15) the first additional identity of curvature of a GK-manifold.

Since $\Phi^{2}=id+\eta\otimes\xi$, then, in view of Theorem 4.3, identity (15) takes the form:

$$R(\xi,\xi)\Sigma=F^{2}(\xi)-\xi+\eta(\xi)\xi; \forall \xi \in \chi(M)$$ (17)

Definition 4.1. Let’s call an AC-manifold a manifold of class $R_{1}$ if its curvature tensor satisfies the equality

$$R(\xi,\xi)\xi=0; \forall \xi \in \chi(M)$$ (18)

Theorem 4.1. A GK-manifold is a manifold of class $R_{1}$ if and only if $R^{a}_{00b}=-R^{a}_{00b}=0$ on the space of an adjoint $G$-structure.
Proof. Let M be a GK-manifold being a manifold of class R₄. Then the tensor of the Riemann curvature of the manifold satisfies condition (17). On the space of the adjoint G-structure, relation (17) can be written in the form: \(R_{\mu
u}^{\lambda}X^{\lambda}=0\). If now M is a GK-manifold, then, taking into account (3.4), the last equality can be written in the form: \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\). The resulting equality is satisfied if and only if \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\).

Conversely, let M be a GK-manifold for which \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\). Since \(R_{\mu
u}^{\lambda}X^{\lambda}=0\) for a GK-manifold, then applying the procedure for restoring the identity \([17,18]\) to the equalities: \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\), we obtain the identity (15).

Theorem 4.2. A GK-manifold is a manifold of class R₄ if and only if the equality \(F=F^2=-I_n\) holds, \(I_n\) is the identity matrix of order n.

Proof. Since \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\), then, according to theorem 4.1, a GK-manifold is a manifold of class R₄ if and only if the equality \(F=F^2=-I_n\) holds. That is, the matrix \(F=F^2\) is scalar.

Conversely, suppose that the matrix \(F=F^2\) is scalar for a GK-manifold, i.e. \(F^2F_{ij}=-\delta_{ij}\). Then, in particular, \(F^{ab}F_{cb}=-\delta_{ab}\), and therefore \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\). Since \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\), the GK-manifold is a manifold of class R₄.

Theorem 4.3. A GK-manifold of class R₅ has dimension 5.

Proof. From identity \(2F_{\mu
u}^{ab}F_{\mu
u}^{cd}=F^{ac}F_{\mu
u}^{db}+F^{ad}F_{\mu
u}^{bc}\), we obtain: \(F_{\mu
u}^{ab}F_{\mu
u}^{bc}=\delta_{ab}\). Let us collapse this identity by indices b and c, then \(F_{\mu
u}^{ac}F_{\mu
u}^{db}=\delta_{ab}\). From the last equality we have that \(F_{\mu
u}^{ab}=0\), which means that the manifold is an SGK-manifold of the second kind. Then the proof of the theorem follows from the local structure of the SGK-manifold of the second kind [5].

II) Since \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\), \(R_{\mu
u}^{\lambda}X^{\lambda}+R_{\mu
u}^{\lambda}X^{\lambda}=0\), and the vectors \(\{e_n\}\) form the basis of the subspace \(D_{\Phi}^{\lambda}\), and the projections of the module \(X(M)^C\) onto the submodules \(D_{\Phi}^{\lambda}\) and \(D_{\Phi}^{\lambda}\) are endomorphisms \(\pi=\sigma=1/2(\Phi^2,-1\Phi)\), \(m=\text{id}+\Phi^2\), the identity \(R(\alpha_\mu,\alpha_\nu)\xi=0\) can be rewritten in the form \(R(\Phi^2X\mu,\Phi^2X\nu)\xi=0\); \(\forall X, Y \in X(M)\). By revealing this relation in linearity and highlighting the real and imaginary parts of the resulting equality, we get the equivalent identity:

\[
R(\Phi^2X\mu,\Phi^2Y\nu)\xi=0; \forall X, Y \in X(M) \tag{19}
\]

Taking into account the equality \(\Phi^2=-\text{id}+\eta\otimes\xi\), we write the last equality in the form...
\( R(X,Y) \xi - \eta(X) R(X,Y) \xi + \eta(Y) R(X,Y) \xi + \eta(X) R(X,Y) \xi = 0 \) which, taking into account (17), takes the form:

\[
R(\Phi X, \Phi Y) \xi - R(X,Y) \xi = -\eta(X) F^2(Y) + \eta(Y) F^2(X) + \eta(X) X + \eta(Y) Y; \quad \forall X,Y \in X(M)
\]

(20)

III) Similarly, applying identity recovery to equalities \( R^0_{\alpha \beta} = 0 \), \( R^0_{\alpha \beta} = 0 \), i.e. \( R^1_{\alpha \beta} = 0 \), we get the identity:

\[
R(\Phi^2 X, \Phi^2 Y) \xi + R(\Phi X, \Phi Y) \xi = 0; \quad \forall X,Y \in X(M)
\]

(21)

From (19) and (21) we have:

\[
R(\Phi^2 X, \Phi^2 Y) \xi = R(\Phi X, \Phi Y) \xi = 0; \quad \forall X,Y \in X(M)
\]

(22)

Taking into account the equality \( \Phi^2 = -\text{id} + \eta \otimes \xi \) and identity (4.2), identity \( R(\Phi^2 X, \Phi^2 Y) \xi = 0 \) will take the form:

\[
R(X,Y) \xi = \eta(X) F^2(Y) - \eta(Y) F^2(X) + \eta(X) X - \eta(Y) Y; \quad \forall X,Y \in X(M)
\]

(23)

IV) Applying identity recovery to relations \( R^0_{\alpha \beta} = 0 \), \( R^0_{\alpha \beta} = 0 \), i.e. \( R^1_{\alpha \beta} = 0 \), we get the identity:

\[
R(\xi, \Phi^2) \Phi^2 Y - R(\xi, \Phi X) \Phi Y = 0; \quad \forall X,Y \in X(M)
\]

(24)

Taking into account (17) and the relation \( \Phi^2 = -\text{id} + \eta \otimes \xi \), the last identity can be rewritten as:

\[
R(\xi, X) Y - R(\xi, \Phi X) \Phi Y = \eta(Y) F^2(X) - \eta(Y) X + \eta(X) \eta(Y) \xi; \quad \forall X,Y \in X(M).
\]

(24)

Summing up the above, we can formulate the following theorem.

Theorem 4.5. The Riemann curvature tensor of a GK-manifold satisfies the following identities:

1) \( R(\Phi^2 X, \Phi^2 Y) \xi = R(\Phi X, \Phi Y) \xi = 0 \);
2) \( R(X,Y) \xi = \eta(X) F^2(Y) - \eta(Y) F^2(X) + \eta(X) X - \eta(Y) Y \);
3) \( R(\xi, \Phi^2) \Phi^2 Y - R(\xi, \Phi X) \Phi Y = 0 \);
4) \( R(\xi, X) Y - R(\xi, \Phi X) \Phi Y = \eta(Y) F^2(X) - \eta(Y) X + \eta(X) \eta(Y) \xi; \quad \forall X,Y \in X(M) \).

V) Now apply the procedure for restoring identity to the equalities \( R^0_{\alpha \beta} = (F^b c a + \delta^b c) \xi^0, R^0_{\alpha \beta} = (F^b c a + \delta^b c) \xi^0 = 0, R^0_{\alpha \beta} = (F^b c a + \delta^b c) \xi^0 = 0 \), i.e. \( R^i_{\alpha \beta} = (F^b c a + \delta^b c) \xi^0 \). At a fixed point \( p \in M \), the last equality is equivalent to relation \( R(\xi, e_6) e_a \equiv \{F(e_a), F(e_6)\} - \{e_a, e_6\} \xi \). Since \( \xi_p = e_0 \), vectors \( \{e_a\} \) form the basis of the subspace \( D^1 p \), vectors \( \{e_\beta\} \) form the basis of the subspace \( D_p^0 \), and projectors of module \( X(M)^C \) to submodules \( D^1 p, D^0 p \) will be endomorphisms \( \pi = \sigma = -\frac{1}{2} (\Phi^2 - \sqrt{-1} \Phi) \), \( \pi = \sigma = -\frac{1}{2} (\Phi^2 + \sqrt{-1} \Phi) \), \( m = \text{id} + \Phi^2 \), identity \( R(\xi, e_6) e_a \equiv \{F(e_a), F(e_6)\} - \{e_a, e_6\} \xi \) can be rewritten in the form

\[
R(\xi, \Phi^2 X + \sqrt{-1} \Phi X) \left(\Phi^2 Y + \sqrt{-1} \Phi Y\right) = \left\{\{F \left(\Phi^2 X + \sqrt{-1} \Phi X\right), F \left(\Phi^2 Y + \sqrt{-1} \Phi Y\right)\} - \{\Phi^2 X + \sqrt{-1} \Phi X, \Phi^2 Y + \sqrt{-1} \Phi Y\}\right\} \xi; \quad \forall X,Y \in X(M)
\]

By revealing this relation in linearity and highlighting the real and imaginary parts of the resulting equality, taking into account Theorem 4.3, we obtain the equivalent identity:
\[ R(\xi, \Phi^2 X)(\Phi^2 Y) + R(\xi, \Phi X)\Phi Y = 2\{(F(X, F(Y)) - (X, Y) + \eta(X)\eta(Y))\xi; \forall X, Y \in X(M) \] (25)

Let’s call identity (25) the second additional identity of curvature of a GK-manifold.

From (23) and (25) we have:
\[ R(\xi, \Phi^2 X)(\Phi^2 Y) = R(\xi, \Phi X)\Phi Y = (F(X, F(Y)) - (X, Y) + \eta(X)\eta(Y))\xi; \forall X, Y \in X(M) \] (26)

Note 1. Identity \[ R(\xi, \Phi^2 X)(\Phi^2 Y) = (F(X, F(Y)) - (X, Y) + \eta(X)\eta(Y))\xi; \forall X, Y \in X(M) \]
and equality \[ \Phi^2 = -\text{id} + \eta \otimes \xi, \] can be written in the form:
\[ R(\xi, X)Y = \eta(Y)F^2(X) - \eta(X)\eta(Y)F(Y) + (X, Y)\xi + (F(X, F(Y)) - (X, Y)\xi + \eta(X)\eta(Y)\xi; \forall X, Y \in X(M) \] (27)

Definition 4.2. An AC-manifold is called a manifold of class \( R_2 \) if its curvature tensor satisfies the equality:
\[ R(\xi, \Phi^2 X)(\Phi^2 Y) + R(\xi, \Phi X)\Phi Y = 0; \forall X, Y \in X(M) \] (28)

Theorem 4.6. A GK-manifold is a manifold of class \( R_2 \) if and only if \( \hat{R}_{aob} = \hat{R}_{aob}' = 0 \) on the space of an adjoint G-structure.

The proof of this theorem is similar to the proof of Theorem 4.1.

Theorem 4.7. A GK-manifold is a manifold of the class \( R_2 \) if and only if it is a manifold of the class \( R_1 \).

The proof follows directly from Theorems 4.1, 4.6 and the properties of the Riemann curvature tensor.

Thus, Theorem 4.4 is also valid for GK-manifolds that are manifolds of the class \( R_2 \).

Note that in [19], Blair and Showers proved that every five-dimensional precisely cosymplectic manifold is symplectic. Kirichenko V.F. proved in [20] that the class of Kenmotsu manifolds coincides with the class of almost contact metric manifolds obtained from symplectic manifolds by a canonical concircular transformation of a symplectic structure. Taking into account the remark made, the main theorem can be formulated.

The main theorem. A GK-manifold of class \( R_2 \) is a five-dimensional Kenmotsu manifold, which means that it coincides with an almost contact metric manifold obtained from a symplectic manifold by a canonical concircular transformation of a symplectic structure of dimension 5.

References

[1] Kobayashi Sh, Nomizu K 1981 Fundamentals of differential geometry vol 2, 414 p

[2] Gray J W 1959 Some global properties of contact structures Ann. Math Vol 69, No2, pp 421 - 450

[3] Sasaki S 1960 On differentiable manifolds with certain structures which are closely related to almost contact structures. Tôhoku Math. J Vol 12, No3, pp 456 – 476

[4] Blair D E 1976 Contact manifolds in Riemannian geometry. Lect. Notes Math. Vol 509, pp 146

[5] Bouzon J 1964 Structures Presque cocomplexes. Univ. et Politechn. Torino. Rend. Sem. Nat., Vol 24, pp 53 – 123

[6] Chern S S 1953 Pseudo-groupes continus infinis. Colloq. Internat. Centre nat. rech. Scient. Vol 52 pp119 – 136

[7] Kenmotsu K A 1972 class of almost contact Riemannian manifolds Tôhoku Math. J., vol 24, pp 93 – 103

[8] Tanno S 1969 The automorphisms groups of almost contact Riemannian manifolds Tôhoku Math. J., vol 21 pp 21 – 38

[9] Bishop R L, O’Neil B 1969 Manifolds of negative curvature. Trans. Amer. Math. Soc., vol 145, pp 1 – 50

[10] Umnova S V 2002 The geometry of Kenmotsu manifolds and their generalizations: Dis. of cand. Phys.-Math. sciences. (Moscow State Pedagogical University) p 88
[11] Najafi B, Kashani N H 2013 On nearly Kenmotsu manifolds. *Turkish Journal of Mathematics*, Vol 37 pp 1040 – 1047
[12] Banaru M B 2004 On Kenmotsu hypersurfaces of special Hermitian manifolds *Siberian Mathematical Journal* Vol 45, No 1, pp 11-15
[13] Abu-Salem A, Rustanov A R 2018 Some aspects of the geometry of generalized Kenmotsu manifolds *Far East Journal of Mathematical Sciences* (FJMS), Vol 103, N 9 Pp 1407-1432, dx.doi.org/10.17654/MS103091407
[14] Abu-Salem A, Rustanov A R 2017 Analogs of Gray Identities for the Riemannian Curvature Tensor of Generalized Kenmotsu Manifolds *International Mathematical Forum*, Vol 12, №2, pp 87 – 95 https://doi.org/10.12988/imf.2017.611149
[15] Abu-Saleem A and Rustanov A R 2015 Curvature Identities Special Generalized Manifolds Kenmotsu Second Kind *Malaysian Journal of Mathematical Sciences* Vol 9(2) pp 187-207
[16] Abu-Saleem A, A.R. Rustanov S V 2018 Kharitonova. Integrability properties of generalized Kenmotsu manifolds *Vladikavkaz Mathematical Journal*, Vol 20, Issue 3, pp 4-20 DOI 10.23671/VNC.2018.3.13829.
[17] Kirichenko V F 2013 Differential and geometric structures on manifolds. Second edition, supplemented p 458
[18] Kirichenko V F, Rustanov A R 2002 Differential geometry of quasi-Sasakian manifolds *Mathematical collection*, vol 193, no 8 pp 71-100
[19] Blair D E, Showers D K 1974 Almost contact manifolds with killing structures tensors. II *J. Differential Geometry*, Vol 9 pp 577-582
[20] Kirichenko V F On the geometry of Kenmotsu manifolds. *Reports of the Russian Academy of Sciences*, vol 380, No 5, pp 585-587