INSCRIBED AND CIRCUMSCRIBED POLYGONS THAT CHARACTERIZE INNER PRODUCT SPACES

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Abstract. Let $X$ be a real normed space with unit sphere $S$. We prove that $X$ is an inner product space if and only if there exists a real number $\rho = \sqrt{(1 + \cos \frac{2k\pi}{2m+1})/2}$, $(k = 1, 2, \ldots; m = 1, 2, \ldots)$, such that every chord of $S$ that supports $\rho S$ touches $\rho S$ at its middle point. If this condition holds, then every point $u \in S$ is a vertex of a regular polygon that is inscribed in $S$ and circumscribed about $\rho S$.

1. Introduction and notation

Let $X$ be a real normed space with unit ball $B$ and unit sphere $S$. $X$ is an inner product space (i.p.s.) if and only if every chord of $S$ supports a sphere homothetic to $S$ at its middle point, namely, if it fulfils the “non-bias” condition

$$u, v \in S \Rightarrow \inf_{t \in [0, 1]} \| (1 - t)u + tv \| = \| \frac{1}{2}u + \frac{1}{2}v \|.$$  

([5]; see [1], p. 29, where this result is used to establish many characterizations of i.p.s.). But in order to characterize an i.p.s. we can only consider the chords of $S$ that supports $\rho S$ at its middle point for some $\rho \in (0, 1)$. Namely, given the following property (P-$\rho S$ from now on)

$$u, v \in S, \inf_{t \in [0, 1]} \| (1 - t)u + tv \| = \rho \Rightarrow \frac{1}{2}u + \frac{1}{2}v \in \rho S,$$  

(P-$\rho S$)

$X$ is an i.p.s. if and only if (P-$\rho S$) holds for $\rho = \frac{1}{2}$ (see [2]) or for any real number $\rho$ such that (see [3])

$$0 < \rho < 1, \quad \rho \neq \sqrt{(1 + \cos \frac{2k\pi}{n})/2}, \quad (2k < n; n = 3, 4, \ldots).$$

The aim of this paper is to prove (Theorem [1]) that $X$ is an i.p.s. if and only if (P-$\rho S$) holds for a real number on the set

$$M = \left\{ \rho \in (0, 1) / \rho = \sqrt{(1 + \cos \frac{2k\pi}{2m+1})/2} : k = 1, 2, \ldots; m = 1, 2, \ldots \right\}.$$  

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It is known that $X$ is an i.p.s. if and only if so are its 2-dimensional subspaces. This fact and the nature of the property $(P-$ ρ$S)$ allow us to consider that $X$ is a real 2-dimensional space from now on.

Given $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $X$, $[u, v]$ denotes the segment meeting $u$ and $v$, and $u \prec v$ means that $u$ precedes $v$ in the positive orientation (counterclockwise) of $X$, i.e., the following expression is positive:

$$u \wedge v := u_1 v_2 - u_2 v_1.$$  

We say that $u$ is orthogonal to $v$ in the sense of Birkhoff ([4], [6]), denoted by $u \perp v$, if

$$\|u\| \leq \|u + \lambda v\| \quad \forall \lambda \in \mathbb{R}.$$  

In other words, $u \perp v$ if and only if the homothetic copy of $S$ with scale factor $\|u\|$ is supported by the line $\{u + \lambda v : \lambda \in \mathbb{R}\}$ at $u$. If $(P-$ ρ$S)$ holds, and $[u, v]$ supports $\rho S$ for some $u, v \in S$, then $u + v \perp v - u$.

If $u, v \in S$, $u \prec v$, $B_u^v$ denotes the sector of $B$ bounded by $u$, $v$, and the arc of $S$ from $u$ to $v$ (counterclockwise); and $T_u^v$ denotes the subset of $B_u^v$ bounded by $[u, v]$ and the arc. If $\mathcal{A}(C)$ denotes the area of a set $C$, since $u \wedge v$ is the area of the parallelogram defined by $u$ and $v$, then obviously

$$\mathcal{A}(B_u^v) = \frac{1}{2}(u \wedge v) + \mathcal{A}(T_u^v).$$

This paper is organized as follows. Some preliminary lemmas related to useful maps from $[0, 2\pi]$ to $S$ are studied in Section 2. The concepts of $\rho$-ellipses and $\rho$-polygons, already used in [3], are reintroduced in Section 3 and some special results for $\rho \in M$ are proved in Section 4. The main Theorem 11 is obtained in Section 5 and an open problem is proposed in Section 6.

2. PRELIMINARY LEMMAS

Along this section, some essential maps from $[0, 2\pi]$ to $S$ and their properties are introduced. The first lemma and its proof appear in [3].

**Lemma 1.** Let $0 < \rho < 1$. For any $u \in S$, there is a unique $u^* \in S$, $u \prec u^*$, such that $[u, u^*]$ supports $\rho S$. The map $u \in S \to u^* \in S$ is a homeomorphism, and $u \prec v$ implies that $u^* \prec v^*$.

The next lemma summarizes a set of properties that are consequences of $(P-$ ρ$S)$.

**Lemma 2.** Let $0 < \rho < 1$. If $X$ fulfils $(P-$ ρ$S)$, then:

1. $X$ is regular (strictly convex and smooth).
2. For any $u \in S$, there is an unique $u^\perp \in S$, $u \prec u^\perp$, such that $u \perp u^\perp$ and the map $u \in S \to u^\perp \in S$ is a homeomorphism. If $v \in S$ and $u \prec v$, then $u^\perp \prec v^\perp$.
3. For any $u \in S$, there exists an unique $\mu > 0$, such that $\rho(u - \mu v^\perp)$ and $\rho(u + \mu v^\perp)$ belong to $S$. 

Proof. Property (P-ρS) implies that \([u, u^*]\) supports \(ρS\) at \(\frac{1}{2}u + \frac{1}{2}u^*\). Due to this fact and using Lemma 1, the proof of (1) and (2) presented in [2] for the case \(\rho = \frac{1}{2}\) can be applied for every \(0 < \rho < 1\).

Having in mind (2) and fixed \(u ∈ S\), the convexity of the function \(F : λ ∈ \mathbb{R} → F(λ) = \|ρu + λρu^⊥\|\) implies that there exist only two real numbers \(μ, ν ∈ \mathbb{R}_+\) such that \(\|ρu - μρu^⊥\| = \|ρu + νρu^⊥\| = 1\). Applying Lemma 1 to \(ρu - μρu^⊥\) and (P-ρS), it is concluded that \(μ = ν\).

Since \(S\) is a convex curve, the following natural parametrization \(s\) is continuous and of bounded variation

\[
\begin{align*}
  s : [0, 2π] & \rightarrow S \\
  θ & \mapsto s(θ) = (s_1(θ), s_2(θ)),
\end{align*}
\]

where \((s_1(θ), s_2(θ)) = \|((cos θ, sin θ))^{-1}(cos θ, sin θ)\).

And as a consequence of Lemma 1 the following parametrization (non-natural, in general) is also continuous and of bounded variation

\[
\begin{align*}
  s^∗ : θ ∈ [0, 2π] & \rightarrow s^∗(θ) := s(θ)^* ∈ S.
\end{align*}
\]

Moreover, if \(X\) fulfils (P-ρS), the continuity and bounded variation hold for the parametrizations \(s^⊥, ρ(s + μs^⊥)\), and \(ρ(s - μs^⊥)\), and for the application \(μs^⊥\) (by (2) and (3) of Lemma 2) defined as follows:

\[
\begin{align*}
  s^⊥ : θ ∈ [0, 2π] & → s^⊥(θ) := s(θ)^⊥ ∈ S, \\
  ρ(s + μs^⊥) : θ ∈ [0, 2π] & → ρ(s(θ) - ρ(θ)μs^⊥(θ)) ∈ S, \\
  ρ(s - μs^⊥) : θ ∈ [0, 2π] & → ρ(s(θ) + ρ(θ)μs^⊥(θ)) ∈ S, \\
  μs^⊥ : θ ∈ [0, 2π] & → μ(θ)s^⊥(θ) ∈ X,
\end{align*}
\]

where \(μ(θ)\) is the real number considered for \(u = s(θ)\) in (3) of Lemma 2. Therefore all the Riemann-Stieltjes integrals that we shall write from now on make sense. For example, if \(t\) is any of the parametrizations of \(S\) introduced above, and \(u = t(α)\) and \(v = t(β)\) \((0 ≤ α < β ≤ 2π)\), then

\[
\mathcal{A}(B_u^V) = \frac{1}{2} \int_α^β t(θ) \wedge dt(θ) = \frac{1}{2} \int_α^β [t_1(θ)dt_2(θ) - t_2(θ)dt_1(θ)].
\]

Lemma 3. Let \(0 < ρ < 1\) and \(X\) fulfill (P-ρS). Let \(s : θ ∈ [0, 2π] → s(θ) ∈ S\) be a natural parametrization for \(S\), and \(s^⊥(θ)\) and \(μs^⊥(θ)\) as they are defined above. Then, for any \(0 ≤ α < β ≤ 2π\):

1. \(f_α^β μ(θ)s^⊥(θ) \wedge ds(θ) = 0\).
2. \(f_α^β s(θ) \wedge ds^⊥(θ) = s(β) \wedge s^⊥(β) - s(α) \wedge s^⊥(α)\).
3. \(f_α^β s(θ) \wedge d[μ(θ)s^⊥(θ)] = s(β) \wedge μ(β)s^⊥(β) - s(α) \wedge μ(α)s^⊥(α)\).

Proof. Let \(α = θ_0 < θ_1 < \ldots < θ_n = β\) be a partition of \([α, β]\). By the Mean Value Theorem, there exist \(η_1, η_2, ..., η_n ∈ \mathbb{R}\) such that \(θ_0 ≤ η_1 ≤ θ_1 ≤ ... ≤ η_{n-1} ≤ η_n ≤ θ_n\) and \(s^⊥(η_k) \wedge [s(θ_k) - s(θ_{k-1})] = 0\). Thus, the Riemann-Stieltjes sum related to this partition is equal to 0, and (1) holds. (2) and (3) result from the integration by parts of \(f_α^β d[s(θ) \wedge s^⊥(θ)]\) and of \(f_α^β d[s(θ) \wedge μs^⊥(θ)]\), respectively, and (1).
Lemma 4. Let $0 < \rho < 1$. If $X$ fulfills ($P - \rho S$), the function $u \in S \rightarrow A(T^u)$ is constant.

Proof. Let $u, v \in S$, $u < v$, and $s : \theta \in [0, 2\pi] \rightarrow s(\theta) \in S$ be a parametrization of $S$. By (3) of Lemma 2, there exist $0 \leq \alpha < \beta \leq 2\pi$ such that
\[ u = \rho[s(\alpha) - \mu(\alpha)s^+(\alpha)], \quad u^* = \rho[s(\alpha) + \mu(\alpha)s^+(\alpha)] \]
\[ v = \rho[s(\beta) - \mu(\beta)s^+(\beta)], \quad v^* = \rho[s(\beta) + \mu(\beta)s^+(\beta)]. \]

By (A),
\[ A(B^*_u) = \frac{\rho^2}{2} \int_\alpha^\beta [s(\theta) - \mu(\theta)s^+(\theta)] \land [s(\theta) - \mu(\theta)s^+(\theta)], \]
\[ A(B^*_v) = \frac{\rho^2}{2} \int_\alpha^\beta [s(\theta) + \mu(\theta)s^+(\theta)] \land [s(\theta) + \mu(\theta)s^+(\theta)]. \]

Therefore, (1) and (3) of Lemma 3 imply that
\[ A(B^*_u) - A(B^*_v) = \rho^2 \left( \int_\alpha^\beta s(\theta) \land [\mu(\theta)s^+(\theta)] + \rho^2 \int_\alpha^\beta [s(\theta) \land [\mu(\theta)s^+(\theta)] \land ds(\theta) = \rho^2 [s(\beta) \land [\mu(\beta)s^+(\beta)] - s(\alpha) \land [\mu(\alpha)s^+(\alpha)]] = \frac{1}{2} (v \land v^*) - \frac{1}{2} (u \land u^*). \]

If $v < u^*$, it is easy to check that $A(B^*_u) = A(B^*_v) + A(B^*_w)$ and $A(B^*_v) = A(B^*_w) + A(B^*_v)$. Similarly, $A(B^*_u) = A(B^*_v) + A(B^*_w)$ and $A(B^*_w) = A(B^*_v) + A(B^*_w)$. In both cases, it is verified that
\[ A(B^*_u) - A(B^*_v) = \frac{1}{2} (v \land v^*) - \frac{1}{2} (u \land u^*) + A(T^*_v) - A(T^*_w), \]
and it is concluded that $A(T^*_v) - A(T^*_w) = 0$. \hfill \Box

3. $\rho$-Ellipses and $\rho$-Polygons

Fixed $0 < \rho < 1$ and $u \in S$, the $\rho$-polygon associated to $u$ is the set of ordered points $P_u = \{u_1, u_2, u_3, \ldots\}$ of $S$ and the segments $[u_i, u_{i+1}]$, such that $u = u_1 < u_2 < u_3, \ldots$, and every segment $[u_i, u_{i+1}]$ supports $\rho S$, i.e., $u_{i+1} = u_i^*$. Each $u_i \in P_u$ is a vertex of $P_u$ and each $[u_i, u_{i+1}]$ is a side of $P_u$. The $\rho$-ellipses $C_u$ associated to $u \in S$ is the unique ellipse centered at $(0,0)$ that contains the points $u, \frac{1}{2\rho}(u + u^*)$, and $u^*$ (see Figure 3).

Some properties and examples of $\rho$-polygons and $\rho$-ellipses are presented in [3]. For instance, if $S$ is an ellipse and $\rho = (1 + \cos \frac{2k\pi}{n})/2$ with $k$ irreducible, it is proved that $P_u$ is convex with $n$ vertices for $k = 1$ and $n = 3, 4, \ldots$; $P_u$ is star-shaped with $n$ vertices for either $k = 2, \ldots, \frac{n}{2} - 1$ when $n$ is even, or for $k = 2, \ldots, \frac{n-1}{2}$ when $n$ is odd; $P_u$ is dense if $\rho$ is not in any of the previous cases. On the other hand, if $S$ is the unit sphere of $\ell_\infty^2$.
(with vertices \{((\pm 1, \pm 1))\}) and \(\rho = \frac{1}{2}\), then \(P_{(1,0)} = \{(0, \pm 1), (\pm 1, 0)\}\); and if \(u \notin \{(0, \pm 1), (\pm 1, 0)\}\), then \(P_u\) has infinite vertices, but it is not dense in \(S\) (the points \{(0, \pm 1), (\pm 1, 0)\} are the unique points of accumulation of \(P_u\)).

And the same happens for \(I^2_p\) (\(p > 2\)) and \(\rho = \frac{1}{2}\frac{p-1}{p}\).

The next lemma presents some properties of \(\rho\)-polygons.

**Lemma 5.** Let \(0 < \rho < 1\). Then:

1. If there exists \(u \in S\) such that \(P_u\) is dense in \(S\), then \(P_v\) is dense in \(S\) for any \(v \in S\).
2. If there exist \(u, v \in S\) such that \(P_u\) and \(P_v\) have a finite number of points, then both have the same number of vertices, and either both are convex or both are star-shaped. Besides, in this last case the number of vertices that are (geometrically situated on) \(P_u\) and \(P_{v+1}\) is equal to the number of vertices between \(v_i\) and \(v_{i+1}\).
3. If \(P_u = \{u_1, u_2, u_3, \ldots\}\), then \(-P_u = \{-u_1, -u_2, -u_3, \ldots\} = P_{-u}\).
4. If \(P_u\) has an odd number of vertices, then \(P_u \cap P_{-u} = \emptyset\).

**Proof.** The statement (1) is proved in Lemma 2.2 of [B].

In order to prove (2), let us consider \(P_u = \{u_1, u_2, \ldots, u_n\}\) and \(P_v = \{v_1, v_2, \ldots, v_m\}\) with \(m > n\). Let us assume without loss of generality that \(u_1 < v_1 < u_p\), and there is not any other vertex of \(P_u\) between \(u_1\) and \(u_p\) (\(p < n\)). By Lemma 1

\[ u_2 < v_2 < u_{p+1}, \ldots, u_n < v_n < u_{p+n-1}, \quad u_1 < v_{n+1} < u_p, \quad \ldots, \]

and it is concluded that \(m = kn, k \in \mathbb{N}\). Thus, \(u_1 < v_1 < u_p\) and \(u_1 < v_{n+1} < u_p\). But if \(v_1 < v_{n+1}\), then \(v_1 < v_{kn+1}\) (\(\forall k \in \mathbb{N}\)) by Lemma 1 and this is a contradiction because \(v_1 = v_{m+1} = v_{kn+1}\). The analysis is similar if \(v_{n+1} < v_1\).

Let us assume now that \(P_u\) is convex, \(P_v\) is star-shaped, and each set has \(n\) vertices. Then, \(u_1 < v_1 < u_2 < v_2 < \ldots < u_n < v_n < u_1\), leads to a contradiction.

For the last assertion of (2), it is enough to consider that \(u_i < u_n < u_{i+1}\) implies (Lemma 1) \(u_{i+1} < u_{m+1} < u_{i+2}\).

The equality \(-P_u = P_{-u}\) of (3) is a consequence of the symmetry of \(S\).

Let us see (4). Let us assume that \(P_u\) has an odd number of vertices and that there exist \(-u_p \in -P_u\) and \(u_q \in P_u\) such that \(-u_p = u_q\). Then, \((-u_p)^* = -u_{p+1}\) (by symmetry of \(S\)) and \((-u_p)^* = u_{q+1}\) (by the construction of \(P_u\)). I.e., \(-u_{p+1} = u_{q+1}\), and, in general, \(-u_{p+k} = u_{q+k}\), \(k \in \mathbb{N}\). Consequently, \(P_u\) is symmetric, and this is not possible because it has an odd number of vertices. 

**Lemma 6.** Let \(u, v \in S\) such that \(P_u = \{u_1, u_2, \ldots, u_n\}\) and \(P_v = \{v_1, v_2, \ldots, v_n\}\) for some \(0 < \rho < 1\). If \(X\) fulfills \((\rho\)-\(S\)), then

\[ u_1 \land u_2 \land \ldots \land u_{n-1} \land u_n \land u_1 = v_1 \land v_2 \land \ldots \land v_{n-1} \land v_n \land v_1. \]

**Proof.** Let \(r\) be the constant number of vertices of \(P_u\) that are (geometrically situated) between \(u_i\) and \(u_{i+1}\) (or, equivalently, between \(v_i\) and \(v_{i+1}\) by (2) of
The statement is a consequence of Lemma 4 and these equalities
\[(r + 1)A(B) = A(B^{u_1}_{v_1}) + \ldots + A(B^{u_n}_{v_n}) = \frac{1}{2}(u_1 \wedge u_2) + A(T^{u_2}_{u_1}) + \ldots + \frac{1}{2}(u_n \wedge u_1) + A(T^{u_1}_{u_n}),\]
\[(r + 1)A(B) = A(B^{v_2}_{v_1}) + \ldots + A(B^{v_n}_{v_1}) = \frac{1}{2}(v_1 \wedge v_2) + A(T^{v_2}_{v_1}) + \ldots + \frac{1}{2}(v_n \wedge v_1) + A(T^{v_1}_{v_n}).\]

\[\square\]

The following result presents some properties about \(\rho\)-ellipses and spheres that are tangent. It is said that \(C_u\) and \(S\) are tangent at \(v \in S \cap C_u\) if both curves have the same supporting line at \(v\). If \(X\) fulfills (P-\(\rho\)S), then \(C_u\) and \(S\) are tangent at \(u\) if and only if the common supporting line at \(u\) is
\[\{u + \lambda[(1 - 2\rho^2)u + u^*] : \lambda \in \mathbb{R}\},\]
that is, if and only if \(u\) has the following property (see Figure 1)
\[u \perp (1 - 2\rho^2)u + u^*.\]

Likewise, \(C_u\) and \(S\) are tangent at \(u^* \in S \cap C_u\) if and only if \(u^*\) verifies
\[u^* \perp -u - (1 - 2\rho^2)u^*.\]

\[\text{Figure 1. } \rho\text{-ellipses } C_u \text{ associated to } u \in S. \text{ On the right, } S \text{ and } C_u \text{ are tangent at } u, \frac{u + u^*}{2\rho}, \text{ and } u^*.\]

**Lemma 7.** Let \(0 < \rho < 1\). If \(X\) fulfills (P-\(\rho\)S), then:

1. \(C_u\) and \(S\) are tangent at \(\frac{1}{2\rho}(u + u^*)\) for every \(u \in S\).
2. If \(C_u\) and \(S\) are tangent at \(u \in S\) (equivalently, if \(u\) verifies \([\parallel]\)), then \(C_u\) and \(S\) are tangent at every point of \(P_u \cup P_w\), where \(w = \frac{1}{2\rho}(u + u^*)\).
3. There exists \(v \in S\) such that \(C_v\) and \(S\) are tangent at \(v\).
Proof. Lemma 3.2 in [3] proves (1), (3), and that if $C_u$ and $S$ are tangent at $u$, then $C_u$ and $S$ are tangent at $u^*$. The proof of Lemma 3.3 in [3] can be applied for every $u \in S$ such that $C_u$ and $S$ are tangent at $u$, and (2) holds.

\[ \square \]

4. $\rho$-ELLIPSES AND $\rho$-POLYTOPES: THE SPECIAL CASE $\rho \in M$

We remind the reader the definition of $M$

\[ M = \left\{ \rho \in (0, 1) / \rho = \sqrt{(1 + \cos \frac{2k\pi}{2m+1})/2} : k = 1, 2, \ldots, m; m = 1, 2, \ldots \right\}. \]

Lemma 8. Let $\rho \in M$. If $X$ fulfills (P-$\rho$S), the following properties hold for every $v \in S$ such that $C_v$ and $S$ are tangent at $v$ (equivalently, for every $v$ that verifies (4)):

1. $P_v = \{v_1, v_2, \ldots, v_n\}$ has $n = 2m + 1$ vertices, and $C_v$ and $S$ are tangent at every $v_i \in P_v$.
2. If $w = \frac{1}{2\rho}(v + v^*)$, $P_w$ has $n = 2m + 1$ vertices. Such as vertices are the points $w_i = \frac{1}{\rho}(v_i + v_{i+1})$, and $C_v$ and $S$ are tangent at every $w_i \in P_w$.
3. If $k$ is an odd number, $P_w = P_{-v}$. If $k$ is an even number, $P_w = P_v$.
4. $v_1 \land v_2 = \ldots = v_{n-1} \land v_n = v_n \land v_1$.
5. $A(B_{\rho\sigma}^{v_1}) = \ldots = A(B_{\rho\sigma}^{v_n}) = A(B_{\rho\sigma}^{v_1})$.
6. The vertices of $P_v$ and $P_{-v}$ split $B$ into $2n$ disjoint sectors of equal area.

Proof. Lemma 7 ensures the existence of $v \in S$ such that $C_v$ and $S$ are tangent at every vertex of $P_v \cup P_w$. Since $C_v$ is an ellipse, $P_v$ has $n = 2m + 1$ vertices for $\rho \in M$ (see Example 1 in [3] or the comments at the beginning of Section 3) and (1) holds, as well as (2).

It is easy to see that (3) and (4) are true when $S$ is an ellipse (see Figure 2). But in the general case, the vertices of $P_v$ and $P_w$ are always the vertices of $\rho$-polytopes inscribed in the $\rho$-ellipse $C_v$ and circumscribed about its homothetic ellipse of ratio $\rho$ (Lemma 7). Therefore, (3) and (4) hold for every $S$.

From (4) and Lemma 4 (5) is obtained.

Let us see (6). Since $n = 2m + 1$ is an odd number, then (see (4) of Lemma 5), the vertices of $P_v \cup P_{-v}$ determine $2n$ different vectors in $S \cap C_v$: \{v_1, \ldots, v_n, -v_1, \ldots, -v_n\}. For every $v_i \in P_v$, let $-v_{\sigma(i)} \in P_{-v}$ be such that $v_i < -v_{\sigma(i)}$ and there is not any other vertex of $P_v \cup P_{-v}$ between (counterclockwise) $v_i$ and $-v_{\sigma(i)}$ (see Figure 2).

Due to the symmetry of $S$ and (5), $A(B_{\rho\sigma}^{v_i+1}) = A(B_{-\rho\sigma(i)}^{v_i+1})$. Using arguments similar to those in Lemma 4 it holds that:

\[
A(B_{\rho\sigma}^{v_i+1}) = A(B_{-\rho\sigma(i)}^{v_i+1}) + A(B_{\rho\sigma}^{v_i+1}), \\
A(B_{-\rho\sigma(i)}^{v_i+1}) = A(B_{-\rho\sigma(i)}^{v_i+1}) + A(B_{\rho\sigma}^{v_i+1}) .
\]
Hence, the \( n \) disjoint sectors \( B_{v_i}^{-v_{\sigma(i)}} \) have the same area. Since \( n \) is an odd number (and again the symmetry of \( S \)), (6) holds.

\[
\text{Figure 2. } \rho\text{-polygon } P_v \text{ with } n = 7 \text{ and } k = 1 \text{ (left), } k = 2 \text{ (center), and } k = 3 \text{ (right).}
\]

**Lemma 9.** Let \( \rho \in M \). If \( X \) fulfills (P-\( \rho S \)), then for every \( u \in S \) it is verified that \( P_u = \{v_1, v_2, v_3, \ldots \} \) has \( n = 2m + 1 \) vertices and \( C_u \) supports \( S \) at every vertex of \( P_u \). Besides, the function \( u \in S \rightarrow u \wedge u^* \) is constant, and \( A(B_u^+) = A(B_u^{-}) \) for any \( v \in S \) such that \( u \prec v \).

**Proof.** We remind that if \( X \) fulfills (P-\( \rho S \)) and \( u \in S \), then \( C_u \) and \( S \) are tangent at \( u \) if and only \( u \) has property \([\#]\). The proof is organized in four steps.

1. **Step 1:** there exists \( v \in S \) such that the condition \([\#]\) is verified for every \( z \in P_v \cup P_{-v} \).

   Since \( S \) and the \( \rho \)-ellipses are symmetric, it is deduced (by Lemma 7, Lemma 8, and (3) and (4) of Lemma 5) that there exists \( v \in S \) such that the condition \([\#]\) is verified for every \( z \in P_v \cup P_{-v} \).

Let \( P_v = \{v_1, v_2, \ldots, v_n\} \) be the polygon generated by \( v \). As in Lemma 8, let us denote \( -v_{\sigma(i)} \) to the unique point in \( P_{-v} \) such that \( v_i \prec -v_{\sigma(i)} \) and there is not any other vertex of \( P_v \cup P_{-v} \) between (counterclockwise) \( v_i \) and \( -v_{\sigma(i)} \).

2. **Step 2:** for every \( i \in \{1, 2, \ldots, n\} \), there exists \( \bar{v} \in S \) such that \( \bar{v} \) verifies \([\#]\) and \( v_i \prec \bar{v} \prec -v_{\sigma(i)} \).

   Without loss of generality, let us assume that \( i = 1 \). Let us consider a parametrization \( s : \theta \in [0, 2\pi] \rightarrow s(\theta) \in S \) of \( S \). By (3) of Lemma 2 there exist \( 0 \leq \theta_1 < \theta'_1 \leq 2\pi \) such that

\[
\begin{align*}
v_1 &= \rho[s(\theta_1) - \mu(\theta_1)s^+(\theta_1)], \quad -v_{\sigma(1)} = \rho[s(\theta'_1) - \mu(\theta'_1)s^+(\theta'_1)], \\
v_2 &= \rho[s(\theta_1) + \mu(\theta_1)s^+(\theta_1)], \quad -v_{\sigma(2)} = \rho[s(\theta'_1) + \mu(\theta'_1)s^+(\theta'_1)].
\end{align*}
\]
It is proved just some lines below that
\[
\int_{\theta_1}^{\theta_1'} [(1 - \rho^2)s(\theta) + \rho^2 \mu(\theta)s^\perp(\theta)] \land d[s(\theta) - \mu(\theta)s^\perp(\theta)] = 0,
\] (1)
and as a consequence, there exists \( \theta_1 < \bar{\theta} < \theta_1' \) such that
\[
s(\bar{\theta}) - \mu(\bar{\theta})s^\perp(\bar{\theta}) \perp (1 - \rho^2)s(\bar{\theta}) + \rho^2 \mu(\bar{\theta})s^\perp(\bar{\theta}).
\]
Thus the points
\[
\bar{v} := \rho[s(\bar{\theta}) - \mu(\bar{\theta})s^\perp(\bar{\theta})], \quad \bar{v}^* := \rho[s(\bar{\theta}) + \mu(\bar{\theta})s^\perp(\bar{\theta})],
\]
verify \( \bar{v} \perp (1 - 2\rho^2)\bar{v} + \bar{v}^* \), with \( v_1 < \bar{v} < -v_{\sigma(1)} \), as Step 2 claims.

In order to see (1), the integral is separated into four parts as follows:
\[
\int_{\theta_1}^{\theta_1'} [(1 - \rho^2)s(\theta) + \rho^2 \mu(\theta)s^\perp(\theta)] \land d[s(\theta) - \mu(\theta)s^\perp(\theta)] =
\]
\[
(1 - \rho^2) \int_{\theta_1}^{\theta_1'} s(\theta) \land ds(\theta) - (1 - \rho^2) \int_{\theta_1}^{\theta_1'} s(\theta) \land d[\mu(\theta)s^\perp(\theta)]
\]
\[
+ \rho^2 \int_{\theta_1}^{\theta_1'} \mu(\theta)s^\perp(\theta) \land ds(\theta) - \rho^2 \int_{\theta_1}^{\theta_1'} \mu(\theta)s^\perp(\theta) \land d[\mu(\theta)s^\perp(\theta)].
\]
Let us denote \( w_1 \) and \( -w_{\sigma(1)} \), respectively, to \( \frac{1}{2\rho}(v_1 + v_2) \) and \( -\frac{1}{2\rho}(v_{\sigma(1)} + v_{\sigma(2)}) \). Then \( w_1 = s(\theta_1) \) and \( -w_{\sigma(1)} = s(\theta_1') \). By the calculus of area (see (A)), the first part is
\[
(1 - \rho^2) \int_{\theta_1}^{\theta_1'} s(\theta) \land ds(\theta) = 2(1 - \rho^2) A(B_{w_1}^{-w_{\sigma(1)}}).
\]
By (3) of Lemma 3 and (4) of Lemma 8 the second part is
\[
\int_{\theta_1}^{\theta_1'} s(\theta) \land d[\mu(\theta)s^\perp(\theta)] =
\]
\[
s(\theta_1') \land \mu(\theta_1')s^\perp(\theta_1') - s(\theta_1) \land \mu(\theta_1)s^\perp(\theta_1) =
\]
\[
\frac{1}{2\rho^2} [(-v_{\sigma(1)} \land -v_{\sigma(2)}) - (v_1 \land v_2)] = 0.
\]
And by (1) of Lemma 3 the third part is
\[
\rho^2 \int_{\theta_1}^{\theta_1'} \mu(\theta)s^\perp(\theta) \land ds(\theta) = 0.
\]
Regarding the last part of the decomposition, using again the calculus of the area (A), and the same statements of Lemma 3 and Lemma 8 it is
deduced that

$$\mathcal{A}(B_{v_i}^{\nu_{\sigma}(1)}) = \frac{\rho^2}{2} \int_{\theta_1}^{\theta_i} [s(\theta) - \mu(\theta) s^\perp(\theta)] \wedge d[s(\theta) - \mu(\theta) s^\perp(\theta)] =$$

$$\frac{\rho^2}{2} \int_{\theta_1}^{\theta_i} s(\theta) \wedge ds(\theta) - \frac{\rho^2}{2} \int_{\theta_1}^{\theta_i} s(\theta) \wedge d[\mu(\theta) s^\perp(\theta)] - \frac{\rho^2}{2} \int_{\theta_1}^{\theta_i} \mu(\theta) s^\perp(\theta) \wedge ds(\theta) + \frac{\rho^2}{2} \int_{\theta_1}^{\theta_i} \mu(\theta) s^\perp(\theta) \wedge d[\mu(\theta) s^\perp(\theta)] =$$

$$\rho^2 \mathcal{A}(B_{w_i}^{\nu_{\sigma}(1)}) + \frac{\rho^2}{2} \int_{\theta_1}^{\theta_i} \mu(\theta) s^\perp(\theta) \wedge d[\mu(\theta) s^\perp(\theta)],$$

and concluded that

$$\rho^2 \int_{\theta_1}^{\theta_i} \mu(\theta) s^\perp(\theta) \wedge d[\mu(\theta) s^\perp(\theta)] = 2\mathcal{A}(B_{v_i}^{\nu_{\sigma}(1)}) - 2\rho^2 \mathcal{A}(B_{w_i}^{\nu_{\sigma}(1)}).$$

Since $\mathcal{A}(B_{v_i}^{\nu_{\sigma}(i)}) = \mathcal{A}(B_{w_i}^{\nu_{\sigma}(i)})$ by (3) and (6) of Lemma 8, then

$$\rho^2 \int_{\theta_1}^{\theta_i} \mu(\theta) s^\perp(\theta) \wedge d[\mu(\theta) s^\perp(\theta)] = 2(1 - \rho^2)\mathcal{A}(B_{w_i}^{\nu_{\sigma}(1)}),$$

and the equality (1) holds.

**Step 3:** If $\bar{v}$ verifies (1) and $v_i < \bar{v} < -v_{\sigma(i)}$, then there exist $\bar{v}', \bar{v}'' \in S$ such that both $\bar{v}'$ and $\bar{v}''$ verify (1), and $v_i < \bar{v}' < \bar{v} < \bar{v}'' < -v_{\sigma(i)}$.

Let us prove the existence of $\bar{v}'$ (the existence of $\bar{v}''$ can be proved similarly). Only for simplicity, let us assume that $i = 1$. Let us consider

$$\bar{v}_1 = \bar{v}, \quad v_{j+1} = \bar{v}' \quad P_\bar{v} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$$

$$w_j = \frac{1}{2\rho}(\bar{v}_j + \bar{v}_{j+1}) \quad w_{j+1} = \bar{w}_j \quad P_{\bar{w}} = \{w_1, \bar{w}_2, \ldots, \bar{w}_n\}$$

Since $\bar{v}$ verifies (1), then ((2) of Lemma 7) $C_\bar{v}$ and $S$ are tangent at every point of $P_\bar{v} \cup P_{\bar{w}}$. Moreover:

(a) $\bar{v}_j \wedge \bar{v}_{j+1} = v_i \wedge v_{i+1}$ (by (4) of Lemma 8 and Lemma 6).
(b) $\mathcal{A}(B_{v_i, \bar{v}_j})$ is constant (by (a) and Lemma 4).
(c) $\mathcal{A}(B_{v_i, \bar{v}_j}) = \mathcal{A}(B_{w_i, \bar{w}_j})$ (by (3) of Lemma 8).

Using the above properties, the existence of $\bar{v}'$ of this Step 3 (such that $v_i < \bar{v}' < \bar{v}$) can be proved applying the arguments of Step 2 to $v_i$ and $\bar{v}$ (instead of $v_i$ and $-v_{\sigma(i)}$).

**Step 4:** The set of points of $S$ that verify (1) is dense on $S$.

Let $x, y$ be a pair of points of $S$ that verify (1). Step 3 can be applied to $x$ and $y$ (instead of $v_i$ and $\bar{v}$) because $x$ and $y$ have the required properties: both verify property (1) and the conditions for $x$ and $y$ equivalent to (a), (b), and (c) remind true (by the same reasons). Therefore, there exists $z \in S$ such that $z$ verifies (1) and $x \prec z \prec y$. 
As a consequence of Step 4, the statements of Lemma 9 are true for every \( u \in S \). Particularly, because conditions similar to (a) and to (b) are verified for any pair of points of \( S \) (instead of \( v_i \) and \( \bar{v}_i \)), then \( u_v = v_v \) and \( \mathcal{A}(B_v) = \mathcal{A}(B_v^* \) for any \( u, v \in S \) such that \( u < v \).

\[ \square \]

5. Main Result

The last lemma describes some properties of a natural parametrization \( s \) and the related parametrization \( s^* \).

**Lemma 10.** Let \( \rho \in M \) and \( s : [0, 2\pi] \to S \) be a natural parametrization for \( S \). If \( X \) fulfills \( \text{P-} \rho S \), then:

(i) \( s \) is continuously differentiable and there is a continuous function \( p : [0, 2\pi] \to \mathbb{R}_+ \) such that \( s'(\theta) = p(\theta)s^*(\theta) \).

(ii) \( s^* \) is continuously differentiable and there is a continuous function \( q : [0, 2\pi] \to \mathbb{R}_+ \) such that \( s'^*(\theta) = q(\theta)s^{**}(\theta) \).

**Proof.** The following conditions holds: \( X \) is smooth (by Lemma 2); the function \( u \in S \to u \wedge u^* \) is constant (by Lemma 9); and \( s, s^* \), and \( s^\perp \) are continuous functions (see Section 2). Besides, \( s'(\theta) \wedge s^*(\theta) \neq 0 \) as a consequence of \( s(\theta) \perp (1 - 2\rho^2)s(\theta) + s^*(\theta) \) for every \( \theta \in [0, 2\pi] \) (by Lemma 9 and because \( 1 - 2\rho^2 \neq 0 \)). Therefore, the proof of the statement for the case \( \rho = \frac{1}{2} \) (Lemma 2.8 in [2]) can be rewritten for \( \rho \in M \). \( \square \)

And finally, the main result is presented.

**Theorem 11.** Given the set

\[ M = \left\{ \rho \in (0, 1) / \rho = \sqrt{(1 + \cos \frac{2k\pi}{2m+1})/2} : k = 1, 2, \ldots, m; m = 1, 2, \ldots \right\}, \]

a real normed space \( X \) is an i.p.s. if and only if there exists \( \rho \in M \) such that \( X \) fulfills

\[ u, v \in S, \quad \inf_{t \in [0, 1]} \|tu + (1 - t)v\| = \rho \Rightarrow \frac{1}{2}u + \frac{1}{2}v \in \rho S. \quad \text{(P-} \rho S) \]

**Proof.** Let \( X \) be an i.p.s. such that the scalar product of \( u, v \in X \) is \( (u|v) \).

It is easy to see that for any \( u, v \in S \), \( u \prec v \), the convex function

\[ F(t) = \|(1 - t)u + tv\|^2 = 1 - 2t + 2t^2 + 2t(1 - t)(u|v) \]

attains its minimum at \( t = \frac{1}{2} \) when \( (u|v) < 1 \). Thus, \( X \) fulfills \( \text{P-} \rho S \) for every \( \rho \in (0, 1) \).

In order to prove the converse, let us fixed a natural parametrization \( s : [0, 2\pi] \to S \) for \( S \). The following conditions holds:

1. \( X \) is smooth (Lemma 2).
2. If \( u, v \in S \) such that \( u \prec v \), then \( u \wedge u^* = v \wedge v^* \); \( \mathcal{A}(B_u^*) = \mathcal{A}(B_v^*) \); \( u \perp (1 - 2\rho^2)u + u^* \); and \( u^* \perp -u - (1 - 2\rho^2)u^* \) (Lemma 8); property \( \text{**} \) for \( u \), and property \( \text{***} \) for \( u^* \).
(3) There exist some continuous functions $p, q : [0, 2\pi] \to \mathbb{R}_+$ such that $s'(\theta) = p(\theta)s^+(\theta)$ and $s''(\theta) = q(\theta)s^+\perp(\theta)$ (Lemma 10).

Using (1), (2), and (3), the proof of the statement for the case $\rho = \frac{1}{2}$ (Theorem 3.1 in [2]) can be rewritten for $\rho \in M$ with only very slight and not significant changes. For example, the (non restrictive) initial data $s(0) = (1, 0)$ and $s^*(0) = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ considered for $\rho = \frac{1}{2}$ would be replaced by $s(0) = (1, 0)$ and $s^*(0) = (\cos \frac{2k\pi}{2m+1}, \sin \frac{2k\pi}{2m+1})$. $\square$

6. CONCLUSION AND OPEN PROBLEM

We conjecture that a real normed space $X$ is an i.p.s. if and only if there exists $0 < \rho < 1$ such that $X$ fulfills property (P-$\rho S$). If $2k < n$ and $n = 3, 4, \ldots$, the case $\rho \neq \sqrt{(1 + \cos \frac{2k\pi}{n})/2}$ is proved in [3]. The case $\rho = \sqrt{(1 + \cos \frac{2k\pi}{n})/2}$ is proved in this paper when $n$ is odd (for $n = 3$, $\rho = \frac{1}{2}$, it was solved previously in [2]), but it is left open when $n$ is even.

For this unsolved situation, the results of Section 2 and Section 3 as well as some assertions of Lemma 8 (4), (5); also (1) and (2) considering $n = 2m$) remain true. Besides, regarding (3) of Lemma 8 it is easy to see that $P_v = P_{-v}$ and $P_v \cap P_w = \emptyset$ when $n$ is even. Nevertheless, the authors are not able to prove that the vertices of $P_v$ and $P_w$ split $B$ into $2n$ disjoint sectors of equal area, which would be the property equivalent to (6) of Lemma 8.

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