WEAK PULLBACK ATTRACTORS FOR STOCHASTIC GINZBURG-LANDAU EQUATIONS IN BOCHNER SPACES

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Abstract. In this paper we discuss the weak pullback mean random attractors for stochastic Ginzburg-Landau equations defined in Bochner spaces. We prove the existence and uniqueness of weak pullback mean random attractors for the stochastic Ginzburg-Landau equations with nonlinear diffusion terms. We also establish the existence and uniqueness of such attractors for the deterministic Ginzburg-Landau equations with random initial data. In this case, the periodicity of the weak pullback mean random attractors is also proved whenever the external forcing terms are periodic in time.

1. Introduction. In this paper, we consider the following stochastic Ginzburg-Landau equations defined in a bounded domain $O \subseteq \mathbb{R}^n (n \leq 2)$:

$$du = ((1 + i\nu)\Delta u + f(u) + g(x,t))dt + \varepsilon\sigma(t,u)dW, \quad x \in O, \quad t > \tau$$

with initial condition

$$u(x,\tau) = u_0(x), \quad x \in O,$$

and the homogeneous Dirichlet boundary condition, where $u = u(x,t)$ is a complex-valued function, $i$ is the imaginary unit, $\nu \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\varepsilon \in (0,1]$, $g$ is given in $L^2_{\text{loc}}(\mathbb{R},L^2(O))$, $f$ and $\sigma$ are given complex-valued nonlinear functions, and $W$ is a two-sided real-valued Wiener process defined on a complete filtered probability space. The stochastic Eq.(1) is understood in the sense of Itô's integration.

The generalized complex Ginzburg-Landau equation is one of the most important equations in mathematical physics, which can describe turbulent dynamics and has a long history in physics as a generic amplitude equation near the onset of instabilities in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity[2, 6]. The global existence and long time behavior of Ginzburg-Landau equation were studied in [7, 11, 16]. The random attractors and ergodicity of stochastic Ginzburg-Landau equations were got in [14, 15, 17, 25, 28, 29].

Stochastic partial differential equations arise naturally in a wide variety of applications with uncertainties or random influences, called noises. The study of global random attractors dates back to Ruelle [26]. The concept of pathwise pullback random attractor for random dynamical systems was first introduced in [5, 8, 27]. Since

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then, the existence and uniqueness of such pathwise random attractors have been
extensively studied in the literature, see, e.g., [1, 3, 9, 10] for the autonomous stochas-
tic equations; and [4, 13, 18, 19, 22, 23, 31, 32, 33, 34, 38] for the non-autonomous
stochastic equations. The theory of pathwise pullback random attractors has been
successfully applied to a variety of stochastic differential equations with nonlinear
drift terms, but this theory is very restrictive on the diffusion terms of the equa-
tions. For instance, as far as the author is aware, there is few results reported in the
literature on the existence of pathwise pullback random attractors for the stochas-
tic equations with a nonlinear diffusion term $\sigma$. In [12], the authors introduced the
concept of weak mean-square attractor to replace the concept of pathwise random
attractor. But the existence of such weak mean-square attractors is very restrictive
too, and as a result, the authors only proved the existence of weak attractors for the
stochastic equations when nonlinear drift term $f \equiv 0$. Recently, Wang [35, 36, 37]
introduce a new type of weak mean-square random attractor (more precisely, weak
pullback mean random attractor in spaces of Bochner integrable functions), and
prove the existence and uniqueness of such attractors for the stochastic equations
and lattice equations with a nonlinear drift term $f$ and a nonlinear diffusion term $\sigma$. Motivated by [35], In this paper we are interested in the long term dynamics
of problem (1)–(2), especially the existence and uniqueness of weak pullback mean
random attractors.

When we prove the existence and uniqueness of weak mean random attractors
in $L^2(\mathcal{O})$, the nonlinearity $f(u)$ is special form, i.e., $f(u) = -(1 + i\mu)|u|^2u, \mu \in \mathbb{R}$,
which is consistent with the general physical background for the Ginzburg-Landau
equation.

This paper is organized as follows. In the next section, we introduce the concept
of weak pullback mean random attractor for mean random dynamical systems in
$L_p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$, and provide a sufficient criterion to guarantee
the existence and uniqueness of such attractors. In Sect. 3, we discuss the determin-
istic Ginzburg-Landau equations with random initial data and prove the existence
of weak pullback mean random attractors for the corresponding random Ginzburg-
Landau Equations in $L^2(\Omega, \mathcal{F}; L^2(\mathcal{O}))$. When the function $g$ in (1) is periodic in
time, we also show the periodicity of the weak pullback random attractors. In the
last section, we study the stochastic Eq. (1) with random initial data and prove the
existence of weak pullback mean random attractors under some conditions on the
nonlinear function $\sigma$ in $L^2(\Omega, \mathcal{F}; L^2(\mathcal{O}))$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$.

Throughout this paper, we denote the inner product and the norm of $L^2(\mathcal{O})$ by
$(\cdot, \cdot)$ and $\|\cdot\|$, respectively. The symbol $c$ and $c_i$ are generic positive numbers which
may change their values from line to line.

2. Preliminaries. In this section, from [35], we first recall the concept of weak
pullback mean random attractor for a mean random dynamical system over a given
probability space, and then for a mean random dynamical system over a filtered
probability space. The former system can be generated by the solution operators of
deterministic differential equations with random initial data; while the latter system
can be generated by the solution operators of stochastic differential equations. We
start with random systems over a fixed probability space.

2.1. Weak pullback mean random attractors over probability spaces.

In this subsection, we introduce the concept of weak $\mathcal{D}$-pullback mean random
attractor for a mean random dynamical system over a given probability space.
Let $X$ be a Banach space with norm $\| \cdot \|_X$. Then a function $\psi : \Omega \to X$ is strongly measurable if there exists a sequence of simple functions $\psi_n : \Omega \to X$, such that $\lim_{n \to \infty} \| \psi_n - \psi \|_X = 0$ $P$-almost everywhere. Such a function $\psi$ is called Bochner integrable if there exists a sequence of simple functions $\psi_n : \Omega \to X$, such that

$$\lim_{n \to \infty} \int_{\Omega} \| \psi_n - \psi \|_X dP = 0.$$ 

The Bochner integral of $\psi$ on $\Omega$ is defined by

$$\int_{\Omega} \psi dP = \lim_{n \to \infty} \int_{\Omega} \psi_n dP.$$ 

For every $p \in (1, \infty)$, denote by $L^p(\Omega, F; X)$ the Banach space consisting of all (equivalence classes of) Bochner integrable functions $\psi : \Omega \to X$ such that

$$\| \psi \|_{L^p(\Omega, F; X)} = \left( \int_{\Omega} \| \psi \|_X^p dP \right)^{\frac{1}{p}} < \infty.$$ 

Let $D$ be a collection of some families of nonempty bounded subsets of $L^p(\Omega, X)$ parametrized by $\tau \in \mathbb{R}$, that is,

$$D = \{ D = \{ D(\tau) \subseteq L^p(\Omega, X) : D(\tau) \neq \emptyset \text{ bounded}, \tau \in \mathbb{R} \} : D \text{ satisfies some conditions} \}.$$ 

Such a collection $D$ is called inclusion-closed if $D = \{ D(\tau) : \tau \in \mathbb{R} \} \in D$ implies that every family $\tilde{D} = \{ \tilde{D}(\tau) : \emptyset \neq \tilde{D}(\tau) \subseteq D(\tau), \forall \tau \in \mathbb{R} \}$ also belongs to $D$.

**Definition 2.1.** A family $\Phi = \{ \Phi(t, \tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R} \}$ of mappings from $L^p(\Omega, X)$ to $L^p(\Omega, X)$ is called a mean random dynamical system on $L^p(\Omega, X)$ if for all $\tau \in \mathbb{R}$ and $t, s \in \mathbb{R}^+$, the following conditions (i) and (ii) are fulfilled:

(i) $\Phi(0, \tau)$ is the identity operator on $L^p(\Omega, X)$;

(ii) $\Phi(t + s, \tau) = \Phi(t, \tau + s) \circ \Phi(s, \tau)$.

If, in addition, $\Phi(t, \tau) : L^p(\Omega, X) \to L^p(\Omega, X)$ is continuous for all $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, then $\Phi$ is said to be a continuous mean random dynamical system on $L^p(\Omega, X)$.

If $\Phi(t, \tau) : L^p(\Omega, X) \to L^p(\Omega, X)$ is weakly continuous for all $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, then $\Phi$ is said to be a weakly continuous mean random dynamical system on $L^p(\Omega, X)$.

If there exists a positive number $T$ such that $\Phi(t, \tau + T) = \Phi(t, \tau)$ for every $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, then $\Phi$ is said to be periodic with period $T$.

**Definition 2.2.** A family $K = \{ K(\tau) : \tau \in \mathbb{R} \} \in D$ is called a $D$-pullback absorbing set for $\Phi$ if for every $\tau \in \mathbb{R}$ and $D \in D$, there exists $T = T(\tau, D) > 0$ such that

$$\Phi(t, \tau - t)(D(\tau - t)) \subseteq K(\tau) \text{ for all } t \geq T.$$ 

If, in addition, $K(\tau)$ is a closed nonempty subset of $L^p(\Omega, X)$ for every $\tau \in \mathbb{R}$, then $K = \{ K(\tau) : \tau \in \mathbb{R} \}$ is called closed $D$-pullback absorbing set for $\Phi$.

If $K(\tau)$ is a weakly compact nonempty subset of $L^p(\Omega, X)$ for every $\tau \in \mathbb{R}$, then $K = \{ K(\tau) : \tau \in \mathbb{R} \}$ is called a weakly compact $D$-pullback absorbing set for $\Phi$.

If there exists a positive number $T$ such that $K(\tau + T) = K(\tau)$ for every $\tau \in \mathbb{R}$, then $K$ is said to be periodic with period $T$. 
Definition 2.3. A family \( K = \{ K(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D} \) is called a \( \mathcal{D} \)-pullback weakly attracting set of \( \Phi \) in \( L^p(\Omega, X) \) if for every \( \tau \in \mathbb{R} \), \( D \in \mathcal{D} \) and every weak neighborhood \( N^\omega(K(\tau)) \) of \( K(\tau) \), there exists \( T = T(\tau, D, N^\omega(K(\tau))) > 0 \) such that for all \( t \geq T \),

\[
\Phi(t, \tau - t)(D(\tau - t)) \subseteq N^\omega(K(\tau)).
\]

If, in addition, \( K(\tau) \) is a weakly compact subset of \( L^p(\Omega, X) \) for every \( \tau \in \mathbb{R} \), then \( K = \{ K(\tau) : \tau \in \mathbb{R} \} \) is called a \( \mathcal{D} \)-pullback weakly compact weakly attracting set for \( \Phi \) in \( L^p(\Omega, X) \).

Definition 2.4. A family \( \mathcal{A} = \{ \mathcal{A}(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D} \) is called a weak \( \mathcal{D} \)-pullback mean random attractor for \( \Phi \) in \( L^p(\Omega, X) \) if the following conditions (i)-(iii) are fulfilled:

(i) \( \mathcal{A}(\tau) \) is a weakly compact subset of \( L^p(\Omega, X) \) for every \( \tau \in \mathbb{R} \).

(ii) \( \mathcal{A} \) is a \( \mathcal{D} \)-pullback weakly attracting set of \( \Phi \) in \( L^p(\Omega, X) \) in the sense of Definition 2.3.

(iii) \( \mathcal{A} \) is the minimal element of \( \mathcal{D} \) with properties (i) and (ii): that is, if \( B = \{ B(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D} \) is a \( \mathcal{D} \)-pullback weakly compact weakly attracting set of \( \Phi \) in \( L^p(\Omega, X) \), then \( \mathcal{A}(\tau) \subseteq B(\tau) \) for all \( \tau \in \mathbb{R} \).

If, in addition, there exists a positive number \( T \) such that \( \mathcal{A}(\tau + T) = \mathcal{A}(\tau) \) for every \( \tau \in \mathbb{R} \), then \( \mathcal{A} \) is said to be periodic with period \( T \).

To construct such a random attractor for \( \Phi \), we need the concept of weak \( \Omega \)-limit set as defined below.

Definition 2.5. Let \( D = \{ D(\tau) \subseteq L^p(\Omega, X) : D(\tau) \neq \emptyset, \forall \tau \in \mathbb{R} \} \) be a family of nonempty subsets of \( L^p(\Omega, X) \). For every \( \tau \in \mathbb{R} \), let

\[
\Omega^\omega(D, \tau) = \bigcap_{\tau \geq 0 \geq t} \overline{\bigcup_{t \geq T} \Phi(t, \tau - t)(D(\tau - t))},
\]

where the closure is taken with respect to the weak topology of \( L^p(\Omega, X) \). Then the family \( \{ \Omega^\omega(D, \tau) : \tau \in \mathbb{R} \} \) is called the weak \( \Omega \)-limit set of \( D \) and is denoted by \( \Omega^\omega(D) \).

The weak \( \Omega \)-limit sets have the following properties.

Lemma 2.6. Let \( D = \{ D(\tau) \subseteq L^p(\Omega, X) : D(\tau) \neq \emptyset, \forall \tau \in \mathbb{R} \} \) be a family of nonempty subsets of \( L^p(\Omega, X) \) and \( \Omega^\omega(D) = \{ \Omega^\omega(D, \tau) : \tau \in \mathbb{R} \} \) be the weak \( \Omega \)-limit set of \( D \) given by Definition 2.5. Let \( \tau \in \mathbb{R} \) be given. Then a point \( \psi_0 \in L^p(\Omega, X) \) belongs to \( \Omega^\omega(D, \tau) \) if and only if for every \( \varepsilon > 0 \) and \( \phi_1, \cdots, \phi_m \in (L^p(\Omega, X))^* \), there exist two sequences \( t_n \to \infty \) and \( \psi_n \in D(\tau - t_n) \) such that

\[
\Phi(t_n, \tau - t_n)(\psi_n) \in N_{\varepsilon, \phi_1, \cdots, \phi_m}(\psi_0), \quad \forall n \in \mathbb{N}.
\]

We now present a sufficient criterion for the existence of weak \( \mathcal{D} \)-pullback mean random attractors.

Theorem 2.7. Suppose \( X \) is a reflexive Banach space and \( p \in (1, \infty) \). Let \( \mathcal{D} \) be an inclusion-closed collection of some families of nonempty bounded subsets of \( L^p(\Omega, X) \) as given by (3) and \( \Phi \) be a mean random dynamical system in \( L^p(\Omega, X) \). If \( \Phi \) has a weakly compact \( \mathcal{D} \)-pullback absorbing set \( K \in \mathcal{D} \), then \( \Phi \) has a unique weak \( \mathcal{D} \)-pullback mean random attractor \( \mathcal{A} \) in \( \mathcal{D} \), which is given by, for each \( \tau \in \mathbb{R} \),

\[
\mathcal{A}(\tau) = \Omega^\omega(K, \tau) = \bigcap_{r \geq 0 \geq t} \overline{\bigcup_{t \geq T} \Phi(t, \tau - t)(K(\tau - t))},
\]
where the closure is taken with respect to the weak topology of $L^p(\Omega, X)$. If, in addition, there exists a positive number $T$ such that both $\Phi$ and $K$ are periodic with period $T$, then $A$ is also $T$-periodic.

2.2. Weak pullback mean random attractors over filtered probability spaces.

In this subsection, we introduce the concept of weak $D$-pullback mean random attractor for mean random dynamical systems over filtered probability spaces, and extend Theorem 2.7 to such random systems.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ be a complete filtered probability space satisfying the usual condition, that is, $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ is an increasing right continuous family of sub-$\sigma$-algebras of $\mathcal{F}$ that contains all $P$-null sets. Given $p \in (1, \infty)$ and $t \in \mathbb{R}$, denote by $L^p(\Omega, \mathcal{F}; X)$ the subspace of $L^p(\Omega, \mathcal{F}; X)$, which consists of all functions $\psi \in L^p(\Omega, \mathcal{F}; X)$ such that $\psi$ is strongly $F_t$-measurable.

Let $D_0$ be a collection of some families of nonempty bounded subsets of $L^p(\Omega, \mathcal{F}_\tau; X)$ parametrized by $\tau \in \mathbb{R}$, that is,

$$D_0 = \{D = \{D(\tau) \subseteq L^p(\Omega, \mathcal{F}_\tau; X) : D(\tau) \neq \emptyset, \tau \in \mathbb{R}\}$$

$$: D satisfies some conditions\}. \quad (4)$$

We now introduce the concept of random dynamical systems over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$.

**Definition 2.8.** A family $\Phi = \{\Phi(t, \tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ of mappings is called a mean random dynamical system on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ if for all $\tau \in \mathbb{R}$ and $t, s \in \mathbb{R}^+$,

(i) $\Phi(t, \tau)$ maps $L^p(\Omega, \mathcal{F}_\tau, X)$ to $L^p(\Omega, \mathcal{F}_{t+\tau}, X)$;

(ii) $\Phi(0, \tau)$ is the identity operator on $L^p(\Omega, \mathcal{F}_\tau, X)$;

(iii) $\Phi(t + s, \tau) = \Phi(t, \tau + s) \circ \Phi(s, \tau)$.

**Definition 2.9.** A family $K = \{K(\tau) : \tau \in \mathbb{R}\} \in D_0$ is called a $D_0$-pullback absorbing set for $\Phi$ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ if for every $\tau \in \mathbb{R}$ and $D \in D_0$, there exists $T = T(\tau, D) > 0$ such that

$$\Phi(t, \tau - t)(D(\tau - t)) \subseteq K(\tau) \text{ for all } t \geq T.$$  

If, in addition, $K(\tau)$ is a weakly compact nonempty subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for every $\tau \in \mathbb{R}$, then $K = \{K(\tau) : \tau \in \mathbb{R}\}$ is called a weakly compact $D_0$-pullback absorbing set for $\Phi$.

**Definition 2.10.** A family $K = \{K(\tau) : \tau \in \mathbb{R}\} \in D_0$ is called a $D_0$-pullback weakly attracting set of $\Phi$ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ if for every $\tau \in \mathbb{R}$, $D \in D_0$ and every weak neighborhood $N^\omega(K(\tau))$ of $K(\tau)$ in $L^p(\Omega, \mathcal{F}_\tau; X)$, there exists $T = T(\tau, D, N^\omega(K(\tau))) > 0$ such that for all $t \geq T$,

$$\Phi(t, \tau - t)(D(\tau - t)) \subseteq N^\omega(K(\tau)).$$

If, in addition, $K(\tau)$ is a weakly compact subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for every $\tau \in \mathbb{R}$, then $K = \{K(\tau) : \tau \in \mathbb{R}\}$ is called a $D_0$-pullback weakly compact weakly attracting set for $\Phi$.

**Definition 2.11.** A family $A = \{A(\tau) : \tau \in \mathbb{R}\} \in D_0$ is called a weak $D_0$-pullback mean random attractor for $\Phi$ on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ if the following conditions (i)-(iii) are fulfilled:

(i) $A(\tau)$ is a weakly compact subset of $L^p(\Omega, \mathcal{F}_\tau; X)$ for every $\tau \in \mathbb{R}$.

(ii) $A$ is a $D_0$-pullback weakly attracting set of $\Phi$. 
problem: Landau equations with random initial data. Mean random dynamical systems generated 3.1. compact system based on the solution operators of the equation, and then construct a weakly equations with random initial data. We first define a mean random dynamical D Mean random attractors for deterministic Ginzburg-Landau equations 3. L where the closure is taken with respect to the weak topology of L D King set D {Suppose Theorem 2.12. Similar to Theorem 2.7, we have the following result on the existence and uniqueness of weak D-pullback mean random attractors for Φ on Lp(Ω, F; X) over (Ω, F, {Fi}i∈R, P)

Theorem 2.12. Suppose X is a reflexive Banach space and p ∈ (1, ∞). Let D0 be an inclusion-closed collection of some families of nonempty bounded subsets of Lp(Ω, F; X) as given by (4) and Φ be a mean random dynamical system on Lp(Ω, F; X) over (Ω, F, {Fi}i∈R, P). If Φ has a weakly compact D0-pullback absorbing set K ∈ D0 on Lp(Ω, F; X) over (Ω, F, {Fi}i∈R, P), then Φ has a unique weak D0-pullback mean random attractor A ∈ D0 on Lp(Ω, F; X) over (Ω, F, {Fi}i∈R, P), which is given by, for each τ ∈ R,

\[ A(\tau) = \Omega^\omega(K, \tau) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \Phi(t, \tau - t)(K(\tau - t)) \]

where the closure is taken with respect to the weak topology of Lp(Ω, F, τ; X).

3. Mean random attractors for deterministic Ginzburg-Landau equations with random initial data. In this section, we prove the existence and uniqueness of weak D-pullback mean random attractors for the deterministic Ginzburg-Landau equations with random initial data. We first define a mean random dynamical system based on the solution operators of the equation, and then construct a weakly compact D-pullback random absorbing set.

3.1. Mean random dynamical systems generated by deterministic Ginzburg-Landau equations with random initial data.

Let O be a bounded domain in \( \mathbb{R}^n \). For every τ ∈ R and t > τ, consider the problem:

\[ \frac{\partial u}{\partial t} - (1 + i\nu)\Delta u = f(u) + g(x, t), \quad x \in O \] (5)

with boundary condition

\[ u(x, t) = 0, \quad x \in \partial O \] (6)

and initial condition

\[ u(x, \tau) = u_0(x), \quad x \in O, \] (7)

where g is given in \( L^2_{loc}(\mathbb{R}, L^2(O)) \).

It is well-known that if the initial condition \( u_0 \) is deterministic, then problem (5)–(7) is well-posed in \( L^2(O) \), see e.g., [30]. This result will be frequently used in this paper and for convenience, is stated below.

Proposition 1. Let τ ∈ R and \( u_0 \in L^2(O) \). Then problem (5)–(7) has a unique solution

\[ u \in C([\tau, \infty), L^2(O)) \cap L^2_{loc}((\tau, \infty), H^1_0(O)) \cap L^4_{loc}((\tau, \infty), L^4(O)). \]

Furthermore, the solution is continuous with respect to \( u_0 \) in \( L^2(O) \).

Next, we consider the well-posedness of problem (5)–(7) with random initial data in a probability space \( (\Omega, F, P) \). The definition of solution for system (5)-(7) in this case is given below.
Definition 3.1. Let $\tau \in \mathbb{R}$ and $u_0 \in L^2(\Omega, L^2(\mathcal{O}))$. Then a continuous mapping $u(\cdot, \tau, u_0) : [\tau, \infty) \to L^2(\Omega, L^2(\mathcal{O}))$ is called a solution of problem (5)–(7) if

\[
u e^{(1+i\nu)\Delta} u + \int_\tau^t (1 + i\nu) \nabla u, \nabla \xi) ds = (u_0, \xi) + \int_\tau^t \int_\mathcal{O} f(u) \xi dx ds + \int_\tau^t \int_\mathcal{O} g(x, s) \xi dx ds
\]

for every $t > \tau$ and $\xi \in H^1(\mathcal{O}) \cap L^4(\mathcal{O})$.

We now prove the existence and uniqueness of solutions to problem (5)–(7) in the sense of Definition 3.1.

Theorem 3.2. Let $|\mu| \leq \sqrt{3}$. For every $\tau \in \mathbb{R}$ and $u_0 \in L^2(\Omega, L^2(\mathcal{O}))$, problem (5)–(7) has a unique solution $u(\cdot, \tau, u_0)$ in the sense of Definition 3.1. This solution is $\mathcal{F}$-measurable with respect to $\omega \in \Omega$ and is continuous with respect to initial data $u_0$ in $L^2(\Omega, L^2(\mathcal{O}))$. Moreover, the solution $u$ satisfies the energy equation:

\[
\frac{d}{dt} E \left(\|u(t, \tau, u_0)\|^2\right) + 2E \left(\|u(t, \tau, u_0)\|_{H^1(\mathcal{O})}^2\right)
= -2E \left(\|u(t, \tau, u_0)\|_{L^4(\mathcal{O})}^4\right) + 2E \left(\Re \int_\mathcal{O} g(x, t) \xi dx\right)
\]

for almost all $t \geq \tau$.

Proof. The proof consists of four steps. We first construct a sequence of approximate solutions by the Galerkin method, and then consider the limit of these approximate solutions.

Step (i): Approximate solutions
Let $A = -\Delta$ with domain $D(A) = H^2(\mathcal{O}) \cap H^1(\mathcal{O})$. Then we know that $A$ has a family of eigenfunctions $\{e_j\}_{j=1}^\infty$ such that

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty \quad \text{as} \quad j \to \infty.
\]

Given $n \in \mathbb{N}$, let $X_n$ be the space spanned by $\{e_j : j = 1, \cdots, n\}$ and $P_n : L^2(\mathcal{O}) \to X_n$ be the projection given by

\[
P_n u = \sum_{j=1}^n (u, e_j)e_j, \quad \forall u \in L^2(\mathcal{O}).
\]

Extend $P_n$ to $H^{-1}(\mathcal{O})$ and $L^4(\mathcal{O})$ by

\[
P_n \phi = \sum_{j=1}^n (\phi(e_j))e_j, \quad \text{for } \phi \in H^{-1}(\mathcal{O}) \quad \text{or } \phi \in L^4(\mathcal{O}).
\]

Let $u_0 : \Omega \to L^2(\mathcal{O})$ be a $\mathcal{F}$-measurable mapping such that $E(\|u_0\|^2) < \infty$. Then for every fixed $\omega \in \Omega$, consider the following deterministic system for $u_n(t, \tau, \omega) \in X_n$ parametrized by $\omega$:

\[
\frac{du_n}{dt} - (1 + \nu) \Delta u_n = -(1 + i\mu)|u_n|^2 u_n + g(t, \omega), \quad t > \tau
\]

(11)
with initial condition
\[ u_n(\tau, \tau, \omega) = P_n u_0(\omega). \] (12)
We find that for every fixed \( \omega \in \Omega \) and \( \tau \in \mathbb{R} \), problem (11)–(12) has a maximal solution \( u_n(\cdot, \tau, \omega) \in C^1([\tau, \tau+T), X_n) \) for some \( T > 0 \). In addition, for each \( t \geq \tau, u_n(t, \tau, \omega) \) is \( F \)-measurable with respect to \( \omega \in \Omega \). Next, we show \( T = \infty \) by deriving the uniform estimates on \( u_n \).

**Step (ii): Uniform estimates** By (11) we get
\[ \frac{d}{dt} \|u_n\|^2 + 2\|\nabla u_n\|^2 = -2\|u_n\|^4_{L^4(\mathcal{O})} + 2Re \int_{\mathcal{O}} g(x, t) \pi_n dx. \] (13)
By the Young inequality, we have
\[ Re \int_{\mathcal{O}} g(x, t) \pi_n dx \leq \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|u_n\|^2. \] (14)
By (13)–(14) we get
\[ \frac{d}{dt} \|u_n\|^2 + 2\|\nabla u_n\|^2 + 2\|u_n\|^4_{L^4(\mathcal{O})} \leq \|u_n\|^2 + \|g(t)\|^2. \] (15)
Multiplying (15) by \( e^{-t} \) and then integrating on \((\tau, t)\) with \( t > \tau \), we obtain
\[ \|u_n(t, \tau, \omega)\|^2 + 2 \int_{\tau}^{t} e^{t-s} \|\nabla u_n(s, \tau, \omega)\|^2 ds + 2 \int_{\tau}^{t} e^{t-s} \|u_n(s, \tau, \omega)\|^4_{L^4(\mathcal{O})} ds \leq e^{t-\tau} \|u_0(\omega)\|^2 + 2 \int_{\tau}^{t} e^{t-s} \|g(s)\|^2 ds, \]
and hence for every fixed \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \),
\[ \|u_n(t, \tau, \omega)\|^2 \leq e^T \left( \|u_0(\omega)\|^2 + \int_{\tau}^{\tau+T} \|g(s)\|^2 ds \right), \quad \forall t \in [\tau, \tau+T], \] (16)
\[ \int_{\tau}^{\tau+T} \|\nabla u_n(s, \tau, \omega)\|^2 ds \leq \frac{1}{2} e^T \left( \|u_0(\omega)\|^2 + \int_{\tau}^{\tau+T} \|g(s)\|^2 ds \right), \] (17)
and
\[ \int_{\tau}^{\tau+T} \|u_n(s, \tau, \omega)\|^4_{L^4(\mathcal{O})} ds \leq \frac{1}{2} e^T \left( \|u_0(\omega)\|^2 + \int_{\tau}^{\tau+T} \|g(s)\|^2 ds \right). \] (18)
It follows from (16)–(18) that for every fixed \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \),
\[ \{u_n(\cdot, \tau, \omega)\}_{n=1}^{\infty} \text{ is bounded in } L^\infty([\tau, \tau+T), L^2(\mathcal{O})) \cap L^2((\tau, \tau+T), H_{0}^1(\mathcal{O})) \cap L^4((\tau, \tau+T), L^4(\mathcal{O})). \] (19)
Then we get
\[ \int_{\tau}^{\tau+T} \int_{\mathcal{O}} \left| (1 + i\mu)|u_n(s, \tau, \omega)|^2 u_n(s, \tau, \omega) \right|^2 ds \leq c \int_{\tau}^{\tau+T} \int_{\mathcal{O}} |u_n(s, \tau, \omega)|^4 ds \]
which together with (18) implies that
\[ \{f(u_n(\cdot, \tau, \omega))\}_{n=1}^{\infty} \text{ is bounded in } L^\frac{4}{3}((\tau, \tau+T), L^\frac{4}{3}(\mathcal{O})). \] (20)
By (11), (17) and (20) we find that
\[ \left\{ \frac{d u_n}{dt} \right\}_{n=1}^{\infty} \text{ is bounded in } L^2((\tau, \tau+T), H^{-1}(\mathcal{O})) + L^\frac{4}{3}((\tau, \tau+T), L^\frac{4}{3}(\mathcal{O})). \] (21)
Step (iii): Existence of solutions Let \( t_0 \in (\tau, \tau + T) \) be fixed. Then by (19)–(21) we infer that there exist \( u(\cdot, \tau, \omega) \in L^\infty((\tau, \tau + T), L^2(\mathcal{O})) \) and \( v \in L^2(\mathcal{O}) \), and a subsequence \( \{u_{n_k}\}_{k=1}^\infty \) of \( \{u_n\}_{n=1}^\infty \) such that

\[
\begin{align*}
    u_{n_k}(\cdot, \tau, \omega) &\rightarrow u(\cdot, \tau, \omega) \text{ weak–star in } L^\infty((\tau, \tau + T), L^2(\mathcal{O})), \\
    u_{n_k}(\cdot, \tau, \omega) &\rightarrow u(\cdot, \tau, \omega) \text{ weakly in } L^2((\tau, \tau + T), H^1_0(\mathcal{O})), \\
    u_{n_k}(\cdot, \tau, \omega) &\rightarrow u(\cdot, \tau, \omega) \text{ weakly in } L^4((\tau, \tau + T), L^4(\mathcal{O})),
\end{align*}
\]

and

\[
u_{n_k}(t_0, \tau, \omega) \rightarrow v \text{ weakly in } L^2(\mathcal{O}).
\]

Note that \( 1 < \frac{4}{3} \leq 2 \) for \( 4 \geq 2 \). Then by (21) we find that \( \left\{ \frac{du_{n_k}}{dt} \right\}_{n=1}^\infty \) is bounded in \( L^4((\tau, \tau + T), (H^1_0(\mathcal{O}) \cap L^4(\mathcal{O}))^*) \). Since the embedding \( H^1_0(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \) is compact and \( L^2(\mathcal{O}) \hookrightarrow (H^1_0(\mathcal{O}) \cap L^4(\mathcal{O}))^* \) is continuous, by (17) we see that there exists a subsequence \( \{u_{n_k}\}_{k=1}^\infty \) (not relabeled) such that

\[
\begin{align*}
    u_{n_k}(\cdot, \tau, \omega) &\rightarrow u(\cdot, \tau, \omega) \text{ strongly in } L^2((\tau, \tau + T), L^2(\mathcal{O})).
\end{align*}
\]

By (26), there exists a further subsequence of \( \{u_{n_k}\}_{k=1}^\infty \) (not relabeled) such that

\[
u_{n_k}(t, \tau, \omega)(x) \rightarrow u(t, \tau, \omega)(x) \text{ for almost all } (t, x) \in (\tau, \tau + T) \times \mathcal{O}.
\]

By (27) and the continuity of \( f \), we get

\[
f(u_{n_k}(t, \tau, \omega)(x)) \rightarrow f(u(t, \tau, \omega)(x)) \text{ for almost all } (t, x) \in (\tau, \tau + T) \times \mathcal{O}.
\]

By (20), (28) and the arguments of [21], we obtain

\[
f(u_{n_k}(\cdot, \tau, \omega)) \rightarrow f(u(\cdot, \tau, \omega)) \text{ weakly in } L^\frac{4}{3}(\tau, \tau + T; L^\frac{4}{3}(\mathcal{O})).
\]

Using (22)–(24) and (29), by taking the limit of (11) as \( n \rightarrow \infty \) in a standard way, one can verify that for all \( \xi \in H^1_0(\mathcal{O}) \cap L^4(\mathcal{O}) \),

\[
\frac{d}{dt}(u, \xi) + ((1 + i\nu)\nabla u, \nabla \xi) = \left(-(1 + i\mu)|u|^2 u, \xi\right)_{L^\frac{4}{3}(\mathcal{O})} + (g(t), \xi)
\]

in the sense of distribution on \( (\tau, \tau + T) \). In addition, \( u(\cdot, \tau, \omega) \in C([\tau, \tau + T], L^2(\mathcal{O})) \) and

\[
u(\tau, \tau, \omega) = u_0(\omega), \quad u(t_0, \tau, \omega) = v,
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|u(t, \tau, \omega)\|^2 + \|\nabla u(t, \tau, \omega)\|^2 = -\|u(t, \tau, \omega)\|_{L^4(\mathcal{O})}^4 + Re \int_{\mathcal{O}} g(x, t) u(t, \tau, \omega) dx
\]

for almost all \( t \in (\tau, \tau + T) \), which implies (10).

By (25) and (31) we get

\[
u_{n_k}(t_0, \tau, \omega) \rightarrow u(t_0, \tau, \omega) \text{ weakly in } L^2(\mathcal{O}.
\]

Note that (30) and (31) imply that \( u(\cdot, \tau, \omega) \) is a solution of the deterministic system (5)–(7) with initial condition \( u_0(\omega) \) for a fixed \( \omega \). This along with (33) and the uniqueness of solutions given by Proposition 1 shows that the whole sequence \( u_n(t_0, \tau, \omega) \rightarrow u(t_0, \tau, \omega) \) weakly in \( L^2(\mathcal{O}) \). Since \( t_0 \in (\tau, \tau + T) \) is arbitrary, we find that \( u_n(t, \tau, \omega) \rightarrow u(t, \tau, \omega) \) weakly in \( L^2(\mathcal{O}) \) for any \( t \geq \tau \) and \( \omega \in \Omega \). Since \( u_n(t, \tau, \omega) \) is measurable in \( \omega \in \Omega \), so is the weak limit \( u(t, \tau, \omega) \).
By Hölder inequality we get
\[ \|u(t, \tau, \omega)\|^2 \leq e^T \left( \|u_0(\omega)\|^2 + \int_{\tau}^{\tau+T} \|g(s)\|^2 ds \right), \quad \forall \ t \in [\tau, \tau + T], \quad (34) \]
\[ \int_{\tau}^{\tau+T} \|\nabla u(s, \tau, \omega)\|^2 ds \leq \frac{1}{2} e^T \left( \|u_0(\omega)\|^2 + \int_{\tau}^{\tau+T} \|g(s)\|^2 ds \right), \quad (35) \]
and
\[ \int_{\tau}^{\tau+T} \|u(s, \tau, \omega)\|^{4}_{L^4(\Omega)} ds \leq \frac{1}{2} e^T \left( \|u_0(\omega)\|^2 + \int_{\tau}^{\tau+T} \|g(s)\|^2 ds \right). \quad (36) \]
Since \( u_0 \in L^2(\Omega, L^2(\mathcal{O})) \), we see from (34)–(36) that
\[ u \in L^\infty((\tau, \infty), L^2(\Omega, L^2(\mathcal{O}))) \bigcap \bigcap_{\tau} \bigcap_{\tau} L^4_{\text{loc}}((\tau, \infty), L^4(\Omega, L^4(\mathcal{O}))). \quad (37) \]
Since \( u(\tau, \tau, \omega) \in C((\tau, \infty), L^2(\mathcal{O})) \) for every fixed \( \omega \), by (34) and the Lebesgue dominated convergence theorem we obtain
\[ u \in C([\tau, \infty), L^2(\Omega, L^2(\mathcal{O}))). \quad (38) \]

By (30)–(31) and (37)–(38) we see that \( u \) is a solution to problem (5)–(7) in the sense of Definition 3.1.

**Step (iv): Uniqueness of solutions** Let \( u_1 \) and \( u_2 \) be two solutions of (5)–(7) and \( v = u_1 - u_2 \). Then we have
\[ \frac{dv}{dt} - (1 + i\nu) \Delta v = -(1 + i\mu)(|u_1|^2 u_1 - |u_2|^2 u_2). \quad (39) \]
From (39), we have
\[ \frac{1}{2} \frac{d}{dt} \|v\|^2 + \|\nabla v\|^2 = -Re[(1 + i\mu) \int_{\mathcal{O}} (|u_1|^2 u_1 - |u_2|^2 u_2) \overline{v} dx]. \quad (40) \]
From [20], we get
\[ -Re[(1 + i\mu) \int_{\mathcal{O}} (|u_1|^2 u_1 - |u_2|^2 u_2) \overline{v} dx] \leq c \int_{\mathcal{O}} (|u_1|^2 + |u_2|^2) |\nabla v|^2 dx. \quad (41) \]
By Hölder inequality we get
\[ \int_{\mathcal{O}} (|u_1|^2 + |u_2|^2) |\nabla v|^2 dx \leq c(\|u_1\|_{L^4}^4 + \|u_2\|_{L^4}^4) \|\nabla v\|^2. \quad (42) \]
By the interpolation inequality, we get
\[ \|v\|_{L^4}^4 \leq c\|v\|_{L^2}^2 \|\nabla v\|^{2(1-\delta)}, \quad \delta = \frac{4 - n}{4}. \quad (43) \]
By (43) and Young’s inequality, we get
\[ c((\|u_1\|_{L^4}^4 + \|u_2\|_{L^4}^4)) \|v\|_{L^2}^2 \|\nabla v\|^{2(1-\delta)} \leq \|\nabla v\|^2 + c((\|u_1\|_{L^4}^4 + \|u_2\|_{L^4}^4)) \theta \|v\|^2. \quad (44) \]
By (41)–(44), we have
\[ -Re[(1 + i\mu) \int_{\mathcal{O}} (|u_1|^2 u_1 - |u_2|^2 u_2) \overline{v} dx] \leq \|\nabla v\|^2 + c((\|u_1\|_{L^4}^4 + \|u_2\|_{L^4}^4)) \theta \|v\|^2, \quad (45) \]
\[ \theta = \frac{2}{4 - n}. \]
By (40) and (45), we have
\[
\frac{d}{dt} \|v\|^2 \leq c(\|u_1\|_4^4 + \|u_2\|_4^4)\|v\|^2.
\] (46)

By the Growall lemma, we obtain for \( t \geq \tau \),
\[
\|v(t, \tau, \omega)\|^2 \leq e^{\int_0^t c(\|u_1(s)\|_4^4 + \|u_2(s)\|_4^4)ds}\|v(\tau, \tau, \omega)\|^2,
\] (47)
which implies the uniqueness as well as the continuity of solutions in initial data in \( L^2(\Omega, L^2(O)) \). This concludes the proof. \( \square \)

We will investigate the long term dynamics of the solutions of problem (5)–(7). To that end, we need to define a random dynamical system based on the solution operators. Let \( \Phi \) be a mapping from \( \mathbb{R}^+ \times \mathbb{R} \times L^2(\Omega, L^2(O)) \) to \( L^2(\Omega, L^2(O)) \) given by
\[
\Phi(t, \tau, u_0) = u(t + \tau, \tau, u_0),
\]
where \( t \geq 0, \tau \in \mathbb{R}, u_0 \in L^2(\Omega, L^2(O)), \) and \( u \) is the solution of problem (5)–(7) with initial data \( u_0 \). By the uniqueness of solutions, we find that for every \( t, s \geq 0, \tau \in \mathbb{R} \) and \( u_0 \in L^2(\Omega, L^2(O)) \),
\[
\Phi(t + s, \tau, u_0) = \Phi(t, s + \tau, (\Phi(s, \tau, u_0))).
\]
In addition, \( \Phi(t, \tau, u_0) \) is continuous with respect to \( u_0 \) in \( L^2(\Omega, L^2(O)) \) for every \( t \in \mathbb{R}^+ \) and \( \tau \in \mathbb{R} \). We abuse the notation a little bit, and sometimes write \( \Phi(t, \tau, u_0) \) as \( \Phi(t, \tau)(u_0) \). Then we find that \( \Phi \) is a continuous mean random dynamical system on \( L^2(\Omega, L^2(O)) \) in the sense of Definition 2.1.

Recall the Poincare inequality in \( H^1_0(O) \): there exists \( \lambda > 0 \) such that
\[
\|\nabla v\|^2 \geq \lambda \|v\|^2, \quad \forall v \in H^1_0(O).
\] (48)
Let \( B \) be a bounded subset of \( L^2(\Omega, L^2(O)) \) and denote by
\[
\|B\|_{L^2(\Omega, L^2(O))} = \sup_{v \in B} \|v\|_{L^2(\Omega, L^2(O))}.
\]
We will consider a family of nonempty bounded subsets of \( L^2(\Omega, L^2(O)) \), \( D = \{D(\tau) \subseteq L^2(\Omega, L^2(O)) : D(\tau) \neq \emptyset \) and \( D(\tau) \) is bounded for each \( \tau \in \mathbb{R} \}, \) which satisfies
\[
\lim_{\tau \to -\infty} e^{\lambda \tau} \|D(\tau)\|_{L^2(\Omega, L^2(O))}^2 = 0,
\] (49)
where \( \lambda \) is the same positive constant as in (48). From now on, we will use \( \mathcal{D} \) to denote the collection of all families of nonempty bounded subsets of \( L^2(\Omega, L^2(O)) \) with property (49):
\[
\mathcal{D} = \{D = \{D(\tau) \subseteq L^2(\Omega, L^2(O)) : D(\tau) \neq \emptyset \) bounded, \( \tau \in \mathbb{R} \} : D \) satisfies (49)\}.
\] (50)
For the non-autonomous external \( g \), we assume:
\[
\int_{-\infty}^\tau e^{\lambda s} \|g(s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}.
\] (51)
3.2. Existence of weak mean random attractors for deterministic ginzburg-landau equations with random initial data.

In this subsection, we prove the existence of weak \( \mathcal{D} \)-pullback mean random attractors for problem (5)–(7). We first derive uniform estimates on the solutions and then construct a \( \mathcal{D} \)-pullback absorbing set in \( L^2(\Omega, L^2(\Omega)) \).

**Lemma 3.3.** Suppose (51) holds. Then for every \( \tau \in \mathbb{R} \) and \( D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D} \), there exists \( T = T(\tau, D) > 0 \) such that for all \( t \geq T \),

\[
E \left( \|u(\tau, \tau - t, u_0)\|^2 \right) \leq M + Me^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} g(s)^2 ds,
\]

\[
\int_{\tau-t}^{\tau} e^{\lambda s} E \left( \|u(s, \tau - t, u_0)\|^4_{L^4(\Omega)} \right) ds \leq M e^{\lambda \tau} + M \int_{-\infty}^{\tau} e^{\lambda s} \|g(s)\|^2 ds,
\]

and

\[
\int_{\tau-t}^{\tau} e^{\lambda s} E \left( \|u(s, \tau - t, u_0)\|^2 \right) ds \leq M e^{\lambda \tau} + M \int_{-\infty}^{\tau} e^{\lambda s} \|g(s)\|^2 ds,
\]

where \( u_0 \in D(\tau - t) \), and \( M \) is a positive constant independent of \( \tau \) and \( D \).

**Proof.** By (10) and (48) we get

\[
\frac{d}{ds} E(\|u(s, \tau - t, u_0\|^2) + \frac{5}{3} \lambda E(\|u(s, \tau - t, u_0\|^2) + \frac{1}{3} E\|\nabla u(s, \tau - t, u_0\|^2)
\leq -2E(\|u(s, \tau - t, u_0\|^4_{L^4(\Omega)}) + 2E \left( Re \int_{\Omega} g(x, s) \overline{u(s, \tau - t, u_0)} dx \right).
\]

By the Young inequality, we have

\[
2E \left( Re \int_{\Omega} g(x, s) \overline{u(s, \tau - t, u_0)} dx \right) \leq \frac{2}{3} \lambda E(\|u(s, \tau - t, u_0\|^2) + \frac{3}{2\lambda} \|g\|^2.
\]

By (52)–(53) we get

\[
\frac{d}{ds} E(\|u(s, \tau - t, u_0\|^2) + \lambda E(\|u(s, \tau - t, u_0\|^2) + \frac{1}{3} E\|\nabla u(s, \tau - t, u_0\|^2)
\leq -2E(\|u(s, \tau - t, u_0\|^4_{L^4(\Omega)}) + \frac{3}{2\lambda} \|g(s)\|^2.
\]

Multiplying (54) by \( e^{\lambda s} \) and then integrating on \( (\tau - t, \tau) \) with \( t \geq 0 \), we get

\[
E(\|u(\tau, \tau - t, u_0\|^2) + \frac{1}{3} e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda s} E(\|\nabla u(s, \tau - t, u_0\|^2) ds + 2e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda s} E(\|u(s, \tau - t, u_0\|^4_{L^4(\Omega)}) ds
\leq e^{-\lambda \tau} E(\|u_0\|^2) + \frac{3}{2\lambda} e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda s} \|g(s)\|^2 ds = e^{-\lambda \tau} e^{\lambda(\tau-t)} E(\|u_0\|^2) + \frac{3}{2\lambda} e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda s} \|g(s)\|^2 ds.
\]

Since \( u_0 \in D(\tau - t) \) and \( D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D} \) we get

\[
e^{-\lambda \tau} e^{\lambda(\tau-t)} E(\|u_0\|^2) \leq e^{-\lambda \tau} e^{\lambda(\tau-t)} \|D(\tau - t)\|^2_{L^2(\Omega, L^2(\Omega))} \to 0 \text{ as } t \to \infty.
\]

Therefore, there exists \( T_1 = T_1(\tau, D) > 0 \) such that for all \( t \geq T_1 \),

\[
e^{-\lambda \tau} e^{\lambda(\tau-t)} E(\|u_0\|^2) \leq 1.
\]
By (55)–(56) we obtain, for all $t \geq T_1$,
\[
E(\|u(\tau, \tau - t, u_0)\|^2) + \frac{1}{3} e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda s} E(\|\nabla u(s, \tau - t, u_0)\|^2) \, ds \\
+ 2 e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda s} \left(\|u(s, \tau - t, u_0)\|_{L^4(O)}^4\right) \, ds \\
\leq 1 + 3 \left(\frac{2}{\lambda} e^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} \|g(s)\|^2 \, ds\right).
\]
This completes the proof. \[\Box\]

Next, we present the existence of weakly compact $D$-pullback absorbing sets for $\Phi$ in $L^2(\Omega, L^2(O))$.

**Lemma 3.4.** Suppose (51) hold. For each $\tau \in \mathbb{R}$, let
\[
K(\tau) = \{u \in L^2(\Omega, L^2(O)) : E(\|u\|^2) \leq R(\tau)\},
\]
where $R(\tau)$ is given by
\[
R(\tau) = M + M e^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda s} \|g(s)\|^2 \, ds
\]
with $M$ being the same constant as in Lemma 3.3. Then the family $K = \{K(\tau) : \tau \in \mathbb{R}\}$ belongs to $D$ and is a weakly compact $D$-pullback absorbing set for $\Phi$.

**Proof.** Note that for each $\tau \in \mathbb{R}$, $K(\tau)$ is a bounded closed convex subset of $L^2(\Omega, L^2(O))$, and hence it is weakly compact in $L^2(\Omega, L^2(O))$. Further, by Lemma 3.3, we find that for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in D$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,
\[
\Phi(t, \tau - t, D(\tau - t)) \subseteq K(\tau).
\]
It remains to show $K \in D$, i.e., $K$ satisfies (49). By (57) we get
\[
\lim_{\tau \to -\infty} e^{\lambda \tau} \|K(\tau)\|_{L^2(\Omega, L^2(O))}^2 = \lim_{\tau \to -\infty} e^{\lambda \tau} R(\tau) = \lim_{\tau \to -\infty} e^{\lambda \tau} M + \lim_{\tau \to -\infty} M \int_{-\infty}^{\tau} e^{\lambda s} \|g(s)\|^2 \, ds.
\]
By (51) we have
\[
\int_{-\infty}^{0} e^{\lambda s} \|g(s)\|^2 \, ds < \infty.
\]
Then it follows from (59)–(60) that
\[
\lim_{\tau \to -\infty} e^{\lambda \tau} \|K(\tau)\|_{L^2(\Omega, L^2(O))}^2 = 0.
\]
By (58) and (61), we conclude the proof. \[\Box\]

Finally, we prove the existence of weak $D$-pullback mean random attractors of $\Phi$.

**Theorem 3.5.** Let (51) holds. Then problem (5)–(7) has a unique weak $D$-pullback mean random attractor $A = \{A(\tau) : \tau \in \mathbb{R}\} \in D$ in $L^2(\Omega, L^2(O))$. Furthermore, if there exists a positive number $T$ such that $g : \mathbb{R} \to L^2(O)$ is T-periodic, then the attractor $A$ is also T-periodic; that is, $A(\tau + T) = A(\tau)$ for all $\tau \in \mathbb{R}$. 
Proof. Note that $\Phi$ has a weakly compact $D$-pullback absorbing set $K = \{K(\tau) : \tau \in \mathbb{R}\}$ as given by (57). Then the existence and uniqueness of weak $D$-pullback mean random attractor $A \in D$ of $\Phi$ follows from Theorem 2.7 immediately. In addition, if $g$ is $T$-periodic, then so are $\Phi$ and the absorbing set $K$. As a result, the $T$-periodicity of $A$ follows also from Theorem 2.7.

4. Weak mean random attractors for stochastic ginzburg-landau equations. In this section, we prove the existence and uniqueness of weak mean random attractors for stochastic Ginzburg-Landau equations. We first define a mean random dynamical system associated with the stochastic Ginzburg-Landau equations over a filtered probability space, and then establish the existence of weakly compact pullback random absorbing sets.

4.1. Mean random dynamical systems generated by stochastic ginzburg-landau equations.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$ be a complete filtered probability space satisfying the usual hypothesis as before. Throughout this section, we assume that $W$ is a two-sided real-valued $Q$-Wiener process with respect to $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$, where $Q$ is the identity operator on a separable Hilbert space $U$. Let $L_2(U, L^2(\mathcal{O}))$ be the Hilbert space consisting of all Hilbert Schmidt operators from $U$ to $L^2(\mathcal{O})$.

Given $\tau \in \mathbb{R}$, consider the non-autonomous stochastic Ginzburg-Landau equations defined in a bounded domain $\mathcal{O} \subseteq \mathbb{R}^n (n \leq 2)$ for $t > \tau$:

$$
\begin{align*}
\frac{du}{dt} & = ((1 + i\nu)\Delta u + f(u) + g(x,t))dt + \varepsilon \sigma(t, u)dW, \quad x \in \mathcal{O} \\
\text{with boundary condition} & \quad u(x, t) = 0, \quad x \in \partial \mathcal{O}
\end{align*}
$$

with initial condition

$$
\begin{align*}
u(x, \tau) & = u_0(x), \quad x \in \mathcal{O},
\end{align*}
$$

where $\varepsilon \in (0, 1]$ is a constant, $g$ is given in $L^2_{loc}(\mathbb{R}, L^2(\mathcal{O}))$, and the stochastic term in (62) is understood in the sense of Itô's integration.

Assume $\sigma : \mathbb{R} \times H^1_0(\mathcal{O}) \to L_2(U, L^2(\mathcal{O}))$ satisfies the conditions:

$$
\begin{align*}
\gamma_1 \|u - v\|^2 + \|\sigma(t, u) - \sigma(t, v)\|^2_{L_2(U, L^2(\mathcal{O}))} \leq 2\|\nabla u - \nabla v\|^2 + \gamma_2 \|u - v\|^2
\end{align*}
$$

for some $\gamma_1 \geq 0, \gamma_2 \in \mathbb{R}$, for all $t \in \mathbb{R}$, and for all $u, v \in H^1_0(\mathcal{O})$; and

$$
\begin{align*}
\gamma_3 \|\nabla u\|^2 + \|\sigma(t, u)\|^2_{L^2(U, L^2(\mathcal{O}))} \leq 2\|\nabla u\|^2 + \gamma_4 \|u\|^2 + h(t)
\end{align*}
$$

for some $\gamma_3 > 0, \gamma_4 \in \mathbb{R}$ and for all $t \in \mathbb{R}$, where $h \in L^1_{loc}(\mathbb{R})$.

We first discuss the existence of solutions for problem (62)–(64).

**Definition 4.1.** Let $\tau \in \mathbb{R}$ and $u_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$. A $L^2(\mathcal{O})$-valued $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$-adapted stochastic process $\{u(t)\}_{t \in [\tau, \infty)}$ is called a solution of (62)–(64) with initial data $u_0$ if $u \in C([\tau, \infty), L^2(\mathcal{O})) \cap L^2_{loc}([\tau, \infty), H^1_0(\mathcal{O}))$ $P$-a.s., and satisfies for every $t > \tau$ and $\xi \in H^1_0(\mathcal{O}),$

$$
\begin{align*}
(u(t), \xi) + \int_\tau^t ((1 + i\nu)\nabla u, \nabla \xi)ds = (u_0, \xi) + \int_\tau^t \int_O f(u)\xi dxds \\
+ \int_\tau^t (g(s), \xi)ds + \varepsilon \int_\tau^t (\xi, \sigma(s, u)dW(s))
\end{align*}
$$

$P$-almost everywhere.
It follows from Theorem 4.2.4 in [24] (also see [25]) that if conditions (65)–(66) are fulfilled, then for every \( \tau \in \mathbb{R} \) and \( u_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) \), problem (62)–(64) has a unique solution \( u \) in the sense of Definition 4.1. Furthermore, for every \( T > 0 \),

\[
E \left( \sup_{t \in [\tau, \tau+T]} \|u(t)\|_{L^2(\mathcal{O})}^2 \right) < \infty. \tag{67}
\]

Since \( u \in C([\tau, \infty), L^2(\mathcal{O})) \) P-a.s., by (67) and the Lebesgue dominated convergence theorem we find that \( u \in C([\tau, \infty), L^2(\mathcal{O})) \). This enables us to define a mean random dynamical system for (62)–(64). More precisely, let \( \Phi \) be a mapping from \( \mathbb{R}^+ \times \mathbb{R} \times L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) \) to \( L^2(\Omega, L^2(\mathcal{O})) \) given by

\[
\Phi(t, \tau, u_0) = u(t + \tau, \tau, u_0),
\]

where \( t \geq 0, \tau \in \mathbb{R}, u_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) \), and \( u \) is the solution of system (62)–(64) with initial data \( u_0 \). By the uniqueness of solutions, we get that for every \( t, s \geq 0, \tau \in \mathbb{R} \) and \( u_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) \),

\[
\Phi(t + s, \tau, u_0) = \Phi(t, s + \tau, (\Phi(s, \tau, u_0))).
\]

Therefore, \( \Phi \) is a mean random dynamical system on \( L^2(\Omega, \mathcal{F}; L^2(\mathcal{O})) \) over \( (\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau \in \mathbb{R}}, P) \) in the sense of Definition 2.8.

Let \( D = \{D(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : \tau \in \mathbb{R}\} \) be a family of nonempty bounded sets such that

\[
\lim_{\tau \to -\infty} e^{\lambda \tau} \|D(\tau)\|_{L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))}^2 = 0. \tag{68}
\]

Let \( D_0 \) be the collection of all families of nonempty bounded sets with property (68):

\[
D_0 = \{D = \{D(\tau) \subseteq L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : D(\tau) \neq \emptyset \text{ bounded}, \tau \in \mathbb{R}\} : D \text{ satisfies } (68)\}. \tag{69}
\]

From now on, we assume:

\[
\int_{-\infty}^\tau e^{\lambda \tau}(\|g(s)\|^2 + |h(s)|)ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{70}
\]

Next, we investigate the existence and uniqueness of weak \( D_0 \)-pullback random attractors for system (62)–(64).

### 4.2. Existence of weak mean random attractors for stochastic ginzburg-landau equations.

In this subsection, we prove the existence of weak \( D_0 \)-pullback mean random attractors for problem (62)–(64). We first derive uniform estimates on the solutions and then construct a \( D_0 \)-pullback absorbing set for the system.

**Lemma 4.2.** Suppose (65)–(66) and (70) hold. Then there exists \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon \leq \varepsilon_0 \) and for every \( \tau \in \mathbb{R} \) and \( D = \{D(t)\}_{t \in \mathbb{R}} \in D_0 \), there exists \( T = T(\tau, D) > 0 \) such that for all \( t \geq T \), the solution \( u \) of problem (62)–(64) satisfies

\[
E(\|u(\tau, \tau - t, u_0)\|^2) \leq M_1 + M_1 e^{-\lambda \tau} \int_{-\infty}^\tau e^{\lambda s}(\|g(s)\|^2 + |h(s)|)ds,
\]

where \( u_0 \in D(\tau - t) \), and \( M_1 \) is a positive constant independent of \( \tau \) and \( D \).
Proof. By Itô’s formula, P-a.s, we get, for $r \geq \tau - t$,

$$
\|u(r, \tau - t, u_0)\|^2 + 2 \int_{\tau - t}^{r} \|\nabla u(s, \tau - t, u_0)\|^2 ds \\
= \|u_0\|^2 - 2 \int_{\tau - t}^{r} \|u(s, \tau - t, u_0)\|_{L^4(O)}^4 ds \\
+ 2 \int_{\tau - t}^{r} Re \int_{O} g(x, s)u(s, \tau - t, u_0)dx ds + \varepsilon^2 \int_{\tau - t}^{r} \|\sigma(s, u(s, \tau - t, u_0))\|_{L^2(U,L^2(O))}^2 ds \\
+ 2\varepsilon \int_{\tau - t}^{r} Re(u(s, \tau - t, u_0), \sigma(s, u(s, \tau - t, u_0))dW(s)
$$

which implies that for $r \geq \tau - t$,

$$
E(\|u(r, \tau - t, u_0)\|^2) + 2 \int_{\tau - t}^{r} E(\|\nabla u(s, \tau - t, u_0)\|^2) ds \\
= E(\|u_0\|^2) - 2 \int_{\tau - t}^{r} E\left(\|u(s, \tau - t, u_0)\|_{L^4(O)}^4\right) ds \\
+ 2 \int_{\tau - t}^{r} E\left(Re \int_{O} g(x, s)u(s, \tau - t, u_0)dx\right) ds \\
+ \varepsilon^2 \int_{\tau - t}^{r} E\left(\|\sigma(s, u(s, \tau - t, u_0))\|_{L^2(U,L^2(O))}^2\right) ds.
$$

(71)

Therefore, we obtain, for almost all $r \geq \tau - t$

$$
\frac{d}{dr}E(\|u(r, \tau - t, u_0)\|^2) + 2E(\|\nabla u(r, \tau - t, u_0)\|^2) \\
= -2E\left(\|u(r, \tau - t, u_0)\|_{L^4(O)}^4\right) \\
+ 2E\left(Re \int_{O} g(x, r)u(r, \tau - t, u_0)dx\right) + \varepsilon^2 E\left(\|\sigma(r, u(r, \tau - t, u_0))\|_{L^2(U,L^2(O))}^2\right).
$$

(72)

By Young’s inequality, we get

$$
Re \int_{O} g(x, r)u(r, \tau - t, u_0)dx \leq \frac{\lambda}{4} \|u(r, \tau - t, u_0)\|^2 + \frac{1}{\lambda} \|g(r)\|^2,
$$

Then

$$
2E\left(Re \int_{O} g(x, r)u(r, \tau - t, u_0)dx\right) \leq \frac{\lambda}{2} E(\|u(r, \tau - t, u_0)\|^2) + \frac{2}{\lambda} \|g(r)\|^2.
$$

(73)

Put

$$
\varepsilon_0 = \min \left\{ 1, \sqrt[4]{\frac{\lambda}{4\lambda + 2|\gamma_4|}} \right\}
$$

Then for all $0 < \varepsilon \leq \varepsilon_0$, by (66) we obtain

$$
\varepsilon^2 E\left(\|\sigma(r, u(r, \tau - t, u_0))\|_{L^2(U,L^2(O))}^2\right) \\
\leq 2\varepsilon^2 E(\|\nabla u(r, \tau - t, u_0)\|^2) + \varepsilon^2 |\gamma_4| E(\|u(r, \tau - t, u_0)\|^2) + \varepsilon^2 |h(r)| \\
\leq \frac{2\lambda}{4\lambda + 2|\gamma_4|} E(\|\nabla u(r, \tau - t, u_0)\|^2) \\
+ \frac{\lambda|\gamma_4|}{4\lambda + 2|\gamma_4|} E(\|u(r, \tau - t, u_0)\|^2) + |h(r)|.
$$

(74)
It follows from (72)–(74) that, for almost all \( r \geq \tau - t \),
\[
\frac{d}{dr} E(\|u(r, \tau - t, 0)\|^2) + \left( 2 - \frac{2\lambda}{4\lambda + 2|\gamma_4|} \right) E(\|\nabla u(r, \tau - t, 0)\|^2) \\
\leq -2E \left( \|u(r, \tau - t, 0)\|_{L^4(\Omega)}^4 \right) + \frac{1}{2} \left( \lambda + \frac{\lambda|\gamma_4|}{2\lambda + |\gamma_4|} \right) E(\|u(r, \tau - t, 0)\|^2) \\
+ \frac{2}{\lambda} g(r)^2 + |h(r)|
\]
which together with (48) implies that, for almost all \( r \geq \tau - t \),
\[
\frac{d}{dr} E(\|u(r, \tau - t, 0)\|^2) + \left( 2 - \frac{2\lambda}{4\lambda + 2|\gamma_4|} \right) \lambda E(\|u(r, \tau - t, 0)\|^2) \\
\leq -2E \left( \|u(r, \tau - t, 0)\|_{L^4(\Omega)}^4 \right) + \frac{1}{2} \left( \lambda + \frac{\lambda|\gamma_4|}{2\lambda + |\gamma_4|} \right) E(\|u(r, \tau - t, 0)\|^2) \\
+ \frac{2}{\lambda} g(r)^2 + |h(r)|. \tag{75}
\]
By simple computations, we find
\[
\left( 2 - \frac{2\lambda}{4\lambda + 2|\gamma_4|} \right) \lambda - \frac{1}{2} \left( \lambda + \frac{\lambda|\gamma_4|}{2\lambda + |\gamma_4|} \right) = \lambda.
\]
Therefore, by (75) we get, for almost all \( r \geq \tau - t \),
\[
\frac{d}{dr} E(\|u(r, \tau - t, 0)\|^2) + \lambda E(\|u(r, \tau - t, 0)\|^2) \\
\leq -2E \left( \|u(r, \tau - t, 0)\|_{L^4(\Omega)}^4 \right) + \frac{2}{\lambda} g(r)^2 + |h(r)|. \tag{76}
\]
Multiplying (76) by \( e^{\lambda t} \) and then integrating on \((\tau - \tau, \tau)\) with \( t \geq 0 \), we get
\[
E(\|u(\tau, \tau - t, 0)\|^2) + 2e^{\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda r} E \left( \|u(r, \tau - t, 0)\|_{L^4(\Omega)}^4 \right) \, dr \\
\leq e^{-\lambda \tau} E(\|u_0\|^2) + e^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda r} \left( \frac{2}{\lambda} g(r)^2 + |h(r)| \right) \, dr \\
\leq e^{-\lambda \tau} e^{\lambda |\tau-t|} E(\|u_0\|^2) + ce^{-\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda r} (g(r))^2 + |h(r)|) \, dr. \tag{77}
\]
Since \( u_0 \in D(\tau - t) \) and \( D = \{ D(t) \}_{t \in \mathbb{R}} \subset D_0 \) we get
\[
e^{-\lambda \tau} e^{\lambda |\tau-t|} E(\|u_0\|^2) \leq e^{-\lambda \tau} e^{\lambda |\tau-t|} \|D(\tau - t)\|_{L^2(\Omega, L^2(\Omega))}^2 \to 0 \text{ as } t \to \infty.
\]
Therefore, there exists \( T_1 = T_1(\tau, D) > 0 \) such that for all \( t \geq T_1 \),
\[
e^{-\lambda \tau} e^{\lambda |\tau-t|} E(\|u_0\|^2) \leq 1. \tag{78}
\]
By (77)–(78) we obtain, for all \( t \geq T_1 \),
\[
E(\|u(\tau, \tau - t, 0)\|^2) + 2e^{\lambda \tau} \int_{\tau-t}^{\tau} e^{\lambda r} E \left( \|u(r, \tau - t, 0)\|_{L^4(\Omega)}^4 \right) \, dr \\
\leq 1 + ce^{-\lambda \tau} \int_{-\infty}^{\tau} e^{\lambda r} (g(r))^2 + |h(r)| \, dr.
\]
The proof is completed. \( \square \)

Next, we present the existence of weakly compact \( D_0 \)-pullback absorbing sets for (62)–(64).
Lemma 4.3. Suppose (65)–(66) and (70) hold. Then for each $\tau \in \mathbb{R}$, there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$, the random dynamical system $\Phi$ for problem (62)–(64) has a weakly compact $\mathcal{D}_0$-pullback absorbing set $K_0 = \{K_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$, which is given by

$$K_0(\tau) = \{ u \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) : E(\|u\|^2) \leq R_0(\tau) \},$$

where $R_0(\tau)$ is given by

$$R_0(\tau) = M_1 + M_1 e^{-\lambda r} \int_{-\infty}^{\tau} e^{\lambda s}(\|g(s)\|^2 + |h(s)|)ds$$

with $M_1$ being the same constant as in Lemma 4.2.

Proof. Since for each $\tau \in \mathbb{R}$, $K_0(\tau)$ given by (79) is a bounded closed convex subset of $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$, and hence it is weakly compact in $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$. In addition, by Lemma 4.2, we find that for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t \in \mathbb{R}} \in \mathcal{D}_0$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$ and $0 < \epsilon \leq \epsilon_0$,

$$\Phi(t, \tau-t, D(\tau-t)) \subseteq K_0(\tau).$$

On the other hand, by the arguments of Lemma 3.4, one can verify $K_0 \in \mathcal{D}_0$. Therefore, $K_0$ is a weakly compact $\mathcal{D}_0$-pullback absorbing set for $\Phi$. \hfill \Box

We are now ready to present the existence of weak $\mathcal{D}_0$-pullback mean random attractors of problem (62)–(64).

Theorem 4.4. Suppose (65)–(66) and (70) hold. Then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$, the mean random dynamical system $\Phi$ for problem (62)–(64) has a unique weak $\mathcal{D}_0$-pullback mean random attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$ in $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, P)$.

Proof. The existence and uniqueness of the weak $\mathcal{D}_0$-pullback mean random attractor $\mathcal{A}_0 \in \mathcal{D}_0$ of $\Phi$ follows from Lemma 4.3 and Theorem 2.12 immediately. \hfill \Box

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