Robustness of non-Gaussian entanglement against noisy amplifier and attenuator environments

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The recently developed Kraus representation for bosonic Gaussian channels is employed to study analytically the robustness of non-Gaussian entanglement against evolution under noisy attenuator and amplifier environments, and compare it with the robustness of Gaussian entanglement. Our results show that some non-Gaussian states with one ebit of entanglement are more robust than all Gaussian states, even the ones with arbitrarily large entanglement, a conclusion of direct consequence to the recent conjecture by Allegra et al. [PRL, 105, 100503 (2010)].

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Early developments in quantum information technology of continuous variable (CV) systems largely concentrated on Gaussian states and Gaussian operations [1]. The symplectic group of linear canonical transformations [2] is available as a handy and powerful tool in this Gaussian scenario, leading to an elegant classification of permissible Gaussian processes or channels [3]. The fact that states in the non-Gaussian sector could offer advantages for several quantum information tasks has resulted in a proliferation of non-Gaussian states in noiseless or quantum-limited cases as a handy and powerful tool in this Gaussian scenario, leading to an elegant classification of permissible Gaussian processes or channels [3]. The fact that states in the non-Gaussian sector could offer advantages for several quantum information tasks has resulted in a proliferation of non-Gaussian states in noiseless or quantum-limited cases.

Since noise is unavoidable in any actual realization of these information processes, robustness of entanglement and other nonclassical effects against noise becomes an important consideration. Allegra et al. [7] have thus studied the evolution of what they call 'photon number entangled states' (PNES) (i.e., two-mode states of the form $|\psi\rangle = \sum_n c_n |n, n\rangle$) in a noisy attenuator environment. They conjectured based on numerical evidence that, for a given energy, Gaussian entanglement is more robust than the non-Gaussian ones. Earlier Agarwal et al. [8] had shown that entanglement of the NOON state is more robust than Gaussian entanglement in the quantum limited amplifier environment. More recently, Nha et al. [9] have shown that nonclassical features, including entanglement, of several non-Gaussian states survive the quantum limited amplifier environment much longer than Gaussian entanglement. Since the conjecture of Ref. [7] refers to the noisy environment and the analysis in Ref. [8] to the noiseless or quantum-limited case, the conclusions of the latter do not necessarily amount to refutation of the conjecture of Ref. [7]. Indeed, Adesso has argued very recently [10] that the well known extremality [11] of Gaussian states implies proof and rigorous validation of the conjecture of Ref. [7].

In the present work we employ the recently developed Kraus representation of bosonic Gaussian channels to study analytically the behaviour of non-Gaussian states in noisy attenuator and amplifier environments. Both NOON states and a simple form of PNES are considered. Our results show conclusively that the conjecture of Ref. [7] is too strong to be maintainable.

**Noisy attenuator environment:** Under evolution through a noisy attenuator channel $C_1(\kappa, a)$, $\kappa \leq 1$, an input state $\rho_1$ with characteristic function (CF) $\chi_{W,1}^{in}(\xi)$ goes to state $\rho_{out}$ with CF

$$\chi_{W,1}^{out}(\xi) = \chi_{W,1}^{in}(\kappa \xi) e^{-\frac{1}{4}(1-\kappa^2+a)\xi^2},$$

where $\kappa$ is the attenuation parameter [3]. In this notation, quantum limited channels [3] correspond to $a = 0$, and so the parameter $a$ stands for the additional Gaussian noise. Thus, $\rho_1$ can stationary under the two-sided symmetric action of $C_1(\kappa, a)$ to $\rho_{out} = C_1(\kappa, a) \otimes C_1(\kappa, a) (\rho_1)$ with CF

$$\chi_{W,1}^{out}(\xi_1, \xi_2) = \chi_{W,1}^{in}(\kappa \xi_1, \kappa \xi_2) e^{-\frac{1}{4}(1-\kappa^2+a)(|\xi_1|^2 + |\xi_2|^2)}.$$ (2)

To test for separability of $\rho_{out}$ we may implement the partial transpose test on $\rho_{out}$ in the Fock basis or on $\chi_{W,1}^{out}(\xi_1, \xi_2)$. The choice could depend on the state.

Consider first the Gaussian case, and in particular the two-mode squeezed state $|\psi(\mu)\rangle = \text{sech} \mu \sum_n \tanh^\mu |n, n\rangle$ with variance matrix $V_{sq}(\mu)$. Under the two-sided action of noisy attenuator channels $C_1(\kappa, a)$, the output two-mode Gaussian state $\rho_{out}(\mu) = C_1(\kappa, a) \otimes C_1(\kappa, a) (|\psi(\mu)\rangle \langle \psi(\mu)|)$ has variance matrix

$$V_{out}(\mu) = \kappa^2 V_{sq}(\mu) + (1 - \kappa^2 + a) \mathbb{I}_4, \quad V_{sq}(\mu) = \begin{pmatrix} c_{2\mu} \mathbb{I}_2 & s_{2\mu} \sigma_3 \cr s_{2\mu} \sigma_3 & c_{2\mu} \mathbb{I}_2 \end{pmatrix},$$ (3)

where $c_{2\mu} = \cosh 2\mu$, $s_{2\mu} = \sinh 2\mu$. Note that our variance matrix differs from that of some authors by a factor 2; in particular, the variance matrix of vacuum is the unit matrix in our notation. Partial transpose test [12] shows that $\rho_{out}(\mu)$ is separable if $a \geq \kappa^2(1 - e^{-2\mu})$. The additional noise $a$ required to render $\rho_{out}(\mu)$ separable is an increasing function of the squeeze (entanglement) parameter $\mu$ and saturates at $\kappa^2$. In particular, $|\psi(\mu_1)\rangle$, $\mu_1 \approx 0.5185$ corresponding to one ebit of entanglement is...
rendered separable when $a \geq \kappa^2 (1 - e^{-2\mu_1})$. For $a \geq \kappa^2$, $\rho^\text{out}(\mu)$ is separable, independent of the initial squeeze parameter $\mu$. Thus $a = \kappa^2$ is the additional noise that renders separable all Gaussian states.

Behaviour of non-Gaussian entanglement may be handled directly in the Fock basis using the recently developed Kraus representation of Gaussian channels \[12\]. In this basis quantum-limited attenuator $C_1(\kappa;0)$, $\kappa \leq 1$ and quantum-limited amplifier $C_2(\kappa;0)$, $\kappa \geq 1$ are described, respectively, by Kraus operators $\{2\}$. The matrix elements of interest are:

$$B_\ell(\kappa) = \sum_{m=0}^{\infty} \sqrt{m+\ell}C_\ell ((1-\kappa^2)^{\ell} \kappa^m |m\rangle \langle m+\ell|,$$

$$A_\ell(\kappa) = \frac{1}{\kappa} \sum_{m=0}^{\infty} \sqrt{m+\ell}C_\ell ((1+\kappa^2)^{-\ell} \frac{1}{\kappa^m} |m\rangle \langle m+\ell|,$$

$\ell = 0, 1, 2, \ldots$. In either case, the noisy channel $C_j(\kappa; a)$, $j = 1, 2$ can be realized in the form $C_2(\kappa;2) \circ C_1(\kappa;1)$, so that the Kraus operators for the noisy case is simply the product set $\{A_\ell(\kappa)B_\ell(\kappa)\}$. Indeed, the composition rule $C_2(\kappa;2) \circ C_1(\kappa;1) = C_1(\kappa;2\kappa^2) \circ C_2(\kappa;2\kappa^2)$ or $C_2(\kappa;2\kappa^2) \circ C_1(\kappa;1) = C_1(\kappa;2\kappa^2)$ according as $\kappa^2 \leq 1$ or $\kappa^2 \geq 1$ implies that the noisy attenuator $C_1(\kappa;\kappa)$, $\kappa \leq 1$ is realised by the choice $\kappa_2 = \sqrt{1+\kappa^2}$, $\kappa_1 = \kappa/\kappa_2 \leq \kappa \leq 1$, and the noisy amplifier $C_2(\kappa;\kappa)$, $\kappa \geq 1$ by $\kappa_2 = \sqrt{\kappa^2 + a^2}$, $\kappa_1 = \kappa/\kappa_2 \leq \kappa \leq 1$. Note that one goes from realization of $C_1(\kappa;\kappa)$, $\kappa \leq 1$ to that of $C_2(\kappa;\kappa)$, $\kappa \geq 1$ simply by replacing $(1+a/2)$ by $(\kappa^2 + a^2)$; this fact will be exploited later.

Under the action of $C_j(\kappa; a)$, $C_j(\kappa;2) \circ C_1(\kappa;1)$, $j = 1, 2$ the elementary operators $|m\rangle \langle n|$ go to

$$C_2(\kappa;2) \circ C_1(\kappa;1) (|m\rangle \langle n|) = \kappa^2 \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \left[ m-\ell+j C_j n-\ell+j C_j m C_\ell n C_\ell \right]^{1/2} \times (\kappa_2^{-1} \kappa_1^{m+n-2\ell}) (1-\kappa_2^{-2})^{\ell} (1-\kappa_2^{-2})^{\ell} \times |m-\ell+j\rangle \langle n-\ell+j|.$$

Substitution of $\kappa_2 = \sqrt{1+\kappa^2}$, $\kappa_1 = \kappa/\kappa_2$ gives realization of $C_1(\kappa;\kappa)$, $\kappa \leq 1$ while $\kappa_2 = \sqrt{\kappa^2 + a^2}$, $\kappa_1 = \kappa/\kappa_2$ gives that of $C_2(\kappa;\kappa)$, $\kappa \geq 1$.

As our first non-Gaussian example we study the NOON state $|\psi\rangle = (|n0\rangle + |0n\rangle)/\sqrt{2}$ with density matrix

$$\rho = \frac{1}{2} (|n\rangle \langle n| \otimes |0\rangle \langle 0| + |0\rangle \langle 0| \otimes |n\rangle \langle n| + |0\rangle \langle 0| \otimes |n\rangle \langle n| + |n\rangle \langle n| \otimes |0\rangle \langle 0|).$$

The output state $\rho^\text{out} = C_1(\kappa; \kappa) \otimes C_1(\kappa; \kappa) (\rho)$ can be detailed in the Fock basis through use of Eq. \[3\].

To test for inseparability, we project $\rho^\text{out}$ onto the $2 \times 2$ subspace spanned by the four bipartite vectors $\{|00\rangle, |0n\rangle, |n0\rangle, |nn\rangle\}$, and test for entanglement in this subspace; this simple test proves sufficient for our purpose! The matrix elements of interest are: $\rho^\text{out}_{00,00}$, $\rho^\text{out}_{00,nn}$, and $\rho^\text{out}_{nn,00} = \rho^\text{out}_{0n,0n}$. Negativity of $\delta_1(\kappa, a) \equiv \rho^\text{out}_{00,00} \rho^\text{out}_{nn,nn} - \rho^\text{out}_{0n,0n}^2$ will prove for $\rho^\text{out}$ not only NPT entanglement, but also one-copy distillability \[14\].

To evaluate $\rho^\text{out}_{00,00}$, $\rho^\text{out}_{0n,0n}$, and $\rho^\text{out}_{nn,nn}$, it suffices to evolve the four single-mode operators $|0\rangle \langle 0|$, $|0\rangle \langle n|$, $|n\rangle \langle 0|$, and $|n\rangle \langle n|$ through the noisy attenuator $C_1(\kappa; a)$ using Eq. \[4\], and then project the output to one of these operators. For our purpose we need only the following single mode matrix elements:

$$x_1 \equiv \langle n| C_1(\kappa; a) |n\rangle \langle n| = (1+a/2)^{-1} \sum_{\ell=0}^{n} [C_{\ell}^2 (1+a/2)^{-2\ell}] \times (1-\kappa^2) (1+a/2)^{-1} (1-1+a/2)^{-1} \langle n-\ell|,$$

$$x_2 \equiv \langle 0| C_1(\kappa; a) |n\rangle \langle n| = (1+a/2)^{-1} (1-\kappa^2) (1+a/2)^{-1} \langle n\rangle,$$

$$x_3 \equiv \langle 0| C_1(\kappa; a) |0\rangle \langle 0| = (1+a/2)^{-1} \langle 0\rangle,$$

$$x_4 \equiv \langle n| C_1(\kappa; a) |0\rangle \langle 0| = (1+a/2)^{-1} (1-1+a/2)^{-1} \langle n\rangle,$$

$$x_5 \equiv \langle n| C_1(\kappa; a) |0\rangle \langle 0| = \kappa^2 (1+a/2)^{-1} \langle n\rangle,$$

$$x^*_6 \equiv \langle 0| C_1(\kappa; a) |0\rangle \langle 0| = 0.$$
the matrix elements $\rho_{0m,0n}^{\text{out}}$, $\rho_{0n,0n}^{\text{out}}$, and $\rho_{0n,nn}^{\text{out}}$, for if $\delta_2(\kappa, a) \equiv \rho_{0m,0n}^{\text{out}} - |\rho_{00,nn}^{\text{out}}|^2$ is negative then $\rho_{0n,nn}^{\text{out}}$ is NPT entangled, and one-copy distillable.

Once again, the matrix elements listed in (6) prove sufficient to determine $\delta_2(\kappa, a)$: $\rho_{0m,0n}^{\text{out}} = \rho_{00,nn}^{\text{out}} = (x_1 x_2 + x_3 x_4)/2$, and $\rho_{0n,nn}^{\text{out}} = |x_5|^2/2$, and so

$$\delta_2(\kappa, a) = (|x_5|^2/2)^2 - (|x_5|^2/2)^2. \quad (9)$$

Let $a_2(\kappa)$ denote the solution to $\delta_2(\kappa, a) = 0$. That is, entanglement of our PNES survives all $a \leq a_2(\kappa)$. This $a_2(\kappa)$ is shown as the curve labelled $P_5$ in Fig. 2 for the PNES $\langle|00\rangle + |55\rangle\rangle/\sqrt{2}$. The lines $g_1$ and $g_\infty$ have the same meaning as in Fig. 1. The region $R$ above $g_\infty$ but below $P_5$ corresponds to channels $(\kappa, a)$ under whose action all two-mode Gaussian states are rendered separable, while entanglement of the non-Gaussian PNES $\langle|00\rangle + |55\rangle\rangle/\sqrt{2}$ definitely survives.

**Noisy amplifier environment:** We turn our attention now to the amplifier noise. Under the symmetric two-sided action of a noisy amplifier channel $C_2(\kappa; a)$, $\kappa \geq 1$, the two-mode CF $\chi_W^{\text{out}}(\xi_1, \xi_2)$ is taken to

$$\chi_W^{\text{out}}(\xi_1, \xi_2) = \chi_W^{\text{in}}(\kappa \xi_1, \kappa \xi_2) e^{-\frac{1}{2}(\kappa^2 - 1 + a)|\xi_1|^2 + |\xi_2|^2}. \quad (10)$$

In particular, the two-mode squeezed vacuum state $|\psi(\mu)\rangle$ with variance matrix $V_{\text{sq}}(\mu)$ is taken to a Gaussian state with variance matrix

$$V^{\text{out}}(\mu) = \kappa^2 V_{\text{sq}}(\mu) + (\kappa^2 - 1 + a) I_4. \quad (10)$$

The partial transpose test [13] readily shows that the output state is separable when $a \geq 2 - \kappa^2(1 + e^{-2\mu})$: the additional noise $a$ required to render the output Gaussian state separable increases with the squeeze or entanglement parameter $\mu$ and saturates at $a = 2 - \kappa^2$: for $a \geq 2 - \kappa^2$ the output state is separable for every Gaussian input. The noise required to render the two-mode squeezed vacuum state $|\psi(\mu)\rangle$ with 1 ebit of entanglement ($\mu_1 \approx 0.5185$) separable is $a = 2 - \kappa^2(1 + e^{-2\mu})$.

Now we examine the behaviour of the NOON state $\langle|n0\rangle + |0n\rangle\rangle/\sqrt{2}$ under the symmetric action of noisy amplifiers $C_2(\kappa; a)$, $\kappa \geq 1$. Proceeding exactly as in the attenuator case, we know that $\rho_{0n,nn}^{\text{out}}$ is definitively entangled if $\delta_3(\kappa, a) \equiv \rho_{00,nn}^{\text{out}} - |\rho_{0n,nn}^{\text{out}}|^2$ is negative. As remarked earlier the expressions for $C_1(\kappa; a)$, $\kappa \leq 1$ in Eq. (10) are valid for $C_2(\kappa; a)$, $\kappa \geq 1$ provided $1 + a/2$ is replaced by $\kappa^2 + a/2$. For clarity we denote by $x_j$ the expressions resulting from $x_j$ when $C_1(\kappa; a)$, $\kappa \leq 1$ replaced by $C_2(\kappa; a)$, $\kappa \geq 1$ and $1 + a/2$ by $\kappa^2 + a/2$. For instance, $x_5' = \langle n|C_2(\kappa; a)(|n\rangle\langle 0|)|0\rangle = \kappa^n(\kappa^2 + a/2)^{-(n+1)}$ and $\delta_3(\kappa; a) = x_1 x_2 x_3 x_4 - (|x_5'|^2/2)^2$.

Let $a_3(\kappa)$ be the solution to $\delta_3(\kappa, a) = 0$. This is represented in Fig. 3 by the curve marked $N_5$, for the case of NOON state $\langle|05\rangle + |50\rangle\rangle/\sqrt{2}$. This curve is to be compared with the line $a = 2 - \kappa^2$, denoted $g_\infty$, above which no Gaussian entanglement survives, and with the line $a = 2 - \kappa^2(1 + e^{-2\mu})$, $\mu_1 = 0.5185$, denoted $g_1$, above which no Gaussian entanglement $\leq 1$ ebit survives. In particular, the region $R$ between $g_\infty$ and $N_5$ corresponds to noisy amplifier channels against which entanglement of the NOON state $\langle|05\rangle + |50\rangle\rangle/\sqrt{2}$ is robust, whereas no Gaussian entanglement survives.

Finally, we consider the behaviour of the PNES $\langle|00\rangle + |55\rangle\rangle/\sqrt{2}$ in this noisy amplifier environment. The output, denoted $\rho_{0n,nn}^{\text{out}}$, is certainly entangled if $\delta_3(\kappa, a) \equiv$
endure more noise than Gaussian states with arbitrarily large entanglement.

We conclude with a pair of remarks inspired by insightful comments by the Referee. First, our conclusion following Eq. (3) and Eq. (10) that entanglement of two-mode squeezed (pure) state $|\psi(\mu)\rangle$ does not survive, for any value of $\mu$, channels $(\kappa, a)$ which satisfy the inequality $|1 - \kappa^2| + a \geq 1$ applies to all Gaussian states. Indeed, for an arbitrary (pure or mixed) two-mode Gaussian state with variance matrix $V_G$ it is clear from Eqs. (3), (10) that the output Gaussian state has variance matrix $V^\text{out} = \kappa^2 V_G + ((1 - \kappa^2) + a)I_4$. Thus $|1 - \kappa^2| + a \geq 1$ immediately implies, in view of nonnegativity of $V_G$, that $V^\text{out} \geq I_4$, demonstrating separability of the output state for arbitrary Gaussian input [13].

Secondly, Gaussian entanglement resides entirely ‘in’ the variance matrix, and hence disappears when environmental noise raises the variance matrix above the vacuum or quantum noise limit. That our chosen states survive these environments shows that their entanglement resides in the higher moments, in turn demonstrating that their entanglement is genuine non-Gaussian. Indeed, the variance matrix of our PNES and NOON states for $N = 5$ is six times that of the vacuum state.

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