A Randomized Rounding Algorithm for the
Asymmetric Traveling Salesman Problem

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Abstract

We present an algorithm for the asymmetric traveling salesman problem on instances which
satisfy the triangle inequality. Like several existing algorithms, it achieves approximation ratio
$O(\log n)$. Unlike previous algorithms, it uses randomized rounding.

1 Introduction

Let $V$ be a set of $n$ vertices and let $c : V \times V \rightarrow \mathbb{R}_+$ be a cost function. We assume the triangle in-
equality: $c_{i,j} \leq c_{i,k} + c_{k,j}$ for all vertices $i, j, k$. The asymmetric traveling salesman problem (ATSP)
is to solve $\min_{\pi} \sum_{v \in V} c_{v,\pi(v)}$ over all cyclic permutations $\pi$ on $V$. A subgraph $\{(v,\pi(v)) : v \in V\}$
is called a tour; we seek a minimum cost tour.

We will use the following standard notation. For $U \subseteq V$, define

$$
\delta^+(U) = \{(v, w) : v \in U, w \notin U\},
\delta^-(U) = \{(w, v) : v \in U, w \notin U\}.
$$

For any vector $x \in \mathbb{R}_+^{V \times V}$ and $F \subseteq V \times V$, we use the notation $x(F) = \sum_{e \in F} x_e$.

The Held-Karp linear programming relaxation of ATSP is as follows.

$$
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad x(\delta^-(\{v\})) = x(\delta^+(\{v\})) \quad \forall v \in V \\
& \quad x(\delta^+(U)) \geq 1 \quad \forall \emptyset \neq U \subseteq V \\
& \quad x \geq 0
\end{align*}
$$

By standard shortcutting arguments, we may assume that $x_{(v,w)} \leq 1$ for all $v, w$, and that

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Several polynomial-time algorithms \cite{3} \cite[pp. 125]{4} \cite{2} are known for computing a tour
whose cost is at most a factor $O(\log n)$ larger than the optimum. In addition, several proofs \cite{10,8}
are known showing that the integrality gap of the Held-Karp relaxation is $O(\log n)$. This note
provides another such algorithm and another such proof of the integrality gap.

2 The Algorithm

The algorithm proceeds in two steps. In the first step, we round the fractional solution using a
simple randomized rounding schema to obtain \textit{nearly-balanced} graph. In the second step, we solve
the \textit{patch up} problem to make the graph Eulerian. The algorithm succeeds in returning a connected
Eulerian subgraph of small cost with high probability.
2.1 Constructing a nearly balanced graph

Let \( x \) be any feasible solution to this linear program. Since \( x \) is balanced at every vertex, this implies that \( x \) is Eulerian, i.e., \( x(\delta^+(U)) = x(\delta^-(U)) \) for all \( U \subseteq V \). So \( x \) is a fractional solution for which all cuts are perfectly balanced. We now use \( x \) to construct an integral solution \( z \) for which all cuts are nearly-balanced, i.e., for each cut \( U \subseteq V \), \( z(\delta^+(U))/z(\delta^-(U)) \leq 2 \). Moreover, the cost of \( z \) is at most \( O(\log n) \cdot c^T x \).

The first observation is that \( x \) has equivalent cut values to an undirected graph. Formally, for \( U \subseteq V \), define \( \delta(U) = \{ \{ v, w \} : v \in U, w \notin U \} \). For \( y \in \mathbb{R}_{+}^{\binom{V}{2}} \) and \( F \subseteq \binom{V}{2} \), let \( y(F) = \sum_{e \in F} y_e \).

Claim 1. Since \( x \) is Eulerian, there exists \( y \in \mathbb{R}_{+}^{\binom{V}{2}} \) such that \( y(\delta(U)) = x(\delta^+(U)) \) for all \( U \subseteq V \).

Proof. Define \( y_{\{v,w\}} = (x_{v,w} + x_{w,v})/2 \) for all \( v, w \). Then
\[
y(\delta(U)) = \sum_{v \in U, w \notin U} \frac{x_{v,w} + x_{w,v}}{2} = \frac{1}{2} \left( x(\delta^+(U)) + x(\delta^-(U)) \right) = x(\delta(U)),
\]
as required. \( \square \)

We now apply a random sampling result of Karger \[5\]. For convenience, we reprove it here in our notation. For any undirected graph with minimum cut value \( c \), Karger \[5\] shows that the number of cuts of value at most \( \alpha c \) is less than \( \binom{n}{2} \alpha \). This result applies to the graph induced by \( y \) and hence, by Claim 1 also to \( x \):
\[
| \{ U : \emptyset \neq U \subseteq V, x(\delta^+(U)) \leq \alpha \} | \leq n^{2\alpha}. \tag{1}
\]

To round \( x \), we must first scale it so that its minimum cut value is large. Let \( G \) be the directed, weighted, multigraph obtained from \( x \) by taking \( K := 100 \ln n \) parallel copies of each edge, each of the same weight as in \( x \). Let \( c_i \) be the value of the \( i \)th cut, ordered such that \( K \leq c_1 \leq c_2 \leq \cdots \). We will construct a directed multigraph \( H \) by taking each edge of \( G \) with probability proportional to its weight. The expected number of edges chosen by \( H \) in the \( i \)th cut is \( c_i \). Let \( p_i \) be the probability that the actual number of edges chosen in the \( i \)th cut diverges from its expectation by more than an \( \epsilon \) fraction. By a Chernoff bound, \( p_i \leq 2e^{-c_i/3} \).

We will ensure that no cut diverges significantly from its expectation by choosing \( \epsilon \) appropriately and applying a union bound. Define \( \epsilon = \sqrt{1/10} \). Since \( c_i \geq K = 100 \ln n \), we have \( p_i \leq n^{-3} \) for all \( i \). For the small cuts, we use the bound
\[
\sum_{i=1}^{2n^2} p_i = O(1/n). \tag{2}
\]
For the large cuts, we use a different bound. Eq. (1) implies that \( c_{n^{2\alpha}} \geq \alpha K \). Letting \( i = n^{2\alpha} \), we have \( c_i \geq K \ln i/(2 \ln n) \), and hence \( p_i \leq i^{-3/2} \). Thus
\[
\sum_{i>n^2} p_i \leq \sum_{i>n^2} i^{-3/2} < \int_{n^2}^{\infty} x^{-3/2} \, dx = O(1/n). \tag{3}
\]
Combining Eq. (2) and Eq. (3) shows that with probability $1 - O(1/n)$, no cut in $H$ diverges from its expectation by more than an $\epsilon$ fraction. The expected cost of $H$ is $K \cdot c^T x$, so a Chernoff bound again implies that the cost of $H$ is $O(\log n) \cdot c^T x$ with high probability.

Let $z \in \mathbb{Z}^{V \times V}_+$ be the vector giving the total weight of the edges in the multigraph $H$. Assuming that no cut in $H$ diverges significantly from its expectation, we have

$$\frac{z(\delta^+(U))}{z(\delta^-(U))} \leq \frac{1 + \epsilon}{1 - \epsilon} \leq 2 \quad \forall \emptyset \neq U \subseteq V. \quad (4)$$

The last inequality follows because $\epsilon < 1/3$. Thus $z$ is a nearly-balanced graph with high probability.

### 2.2 Patching Up

We now make $z$ Eulerian by “patching it up” with another graph $w$. That is, we seek another vector $w \in \mathbb{Z}^{V \times V}_+$ such that $z + w$ is connected and Eulerian — an integral, feasible solution to the Held-Karp relaxation of ATSP. Indeed, we show that a subgraph of $z$ can be used to patch up $z$.

Consider the transshipment problem on $V$ where each vertex $v$ has demand $b(v) := z(\delta^+(v)) - z(\delta^-(v))$. Hoffman’s circulation theorem \[7, Corollary 11.2f\] implies that there exists a subgraph of $z$ giving a feasible, integral transshipment for these demands if the capacity of each cut is at least its demand:

$$z(\delta^-(U)) \geq \sum_{v \in U} b(v) = z(\delta^+(U)) - z(\delta^-(U)).$$

This inequality is implied by Eq. (4), so the desired transshipment $w$ exists, and its cost is at most $c^T z$. Thus $z + w$ gives a connected, Eulerian graph of cost at most $2c^T x$, which is $O(\log n) \cdot c^T x$, as argued above. By shortcutting, we obtain a tour of no worse cost. If $x$ is an optimum solution of the linear program then the resulting tour is at most a factor $O(\log n)$ larger than the optimum tour. Consequently, the integrality gap of this linear program is at most $O(\log n)$.

### 3 Tight Example

We now show that the analysis of the algorithm given above is tight to within constant factors — we give an example where we must choose $K$ to be $\Omega(\log n)$ in the first step of the algorithm. This condition is necessary not only to ensure the nearly-balanced condition but also to ensure that $H$ is connected.

Consider any extreme point $x$ such that $x_a < \frac{2}{3}$ for every arc $a \in A$ and let $E$ be the support of $x$. Such extreme points exist of arbitrarily large size \[11\]. Using the fact that $|E| < 3n - 2$ where $n = |V|$, we obtain that there in an independent set of vertices $V_1$ of size at least $\frac{n}{6}$. Since $x_a < \frac{2}{3}$ for each $a \in A$ and $x(\delta^-(v)) + x(\delta^+(v)) = 2$, the probability that $v$ is a not an isolated vertex in $H$ is at most $1 - \frac{2}{27n}$ for each $v \in V_1$. Since $V_1$ is an independent set, these events are independent. Hence, the probability that none of the vertices in $V_1$ is an isolated vertex in $H$ is at most $(1 - \frac{1}{27n})^{\frac{n}{6}}$. In order for $H$ to be connected with constant probability, we must take $K = \Omega(\log n)$. 

3
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