Uniform families of minimal rational curves on Fano manifolds

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Abstract. It is a well-known fact that families of minimal rational curves on rational homogeneous manifolds of Picard number one are uniform, in the sense that the tangent bundle to the manifold has the same splitting type on each curve of the family. In this note we prove that certain stronger uniformity conditions on a family of minimal rational curves on a Fano manifold of Picard number one allow to prove that the manifold is homogeneous.

1. Introduction

In the framework of higher dimensional complex algebraic geometry, rational homogeneous manifolds constitute one of the most important classes of examples of Fano manifolds. Their geometric properties may be written in the language of the representation theory of complex reductive Lie groups, which makes this class one of the best understood. One of the most important questions in this context was inspired by Mori and posed by Campana and Peternell: Can rational homogeneity be described, within the class of Fano varieties, in terms of positivity properties of its tangent bundle? Namely, does the nefness of $T_X$ imply that a Fano manifold $X$ is homogeneous?

The strategy towards a solution of the Campana-Peternell problem that we have been considering recently (see [21, 22, 23]) is based on reconstructing the rational homogeneous structure upon families of minimal rational curves contained in the manifold $X$. Philosophically speaking, rational curves on Fano manifolds play a role as central as the one of $\mathfrak{s}(2)$ in the representation theory of reductive groups, and this analogy should become obvious in our candidates to be rational homogeneous manifolds.

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One of the fundamental problems in this approach is to understand how nefness is reflected on properties of certain families of rational curves in $X$. For instance, one may pose the following question, which is still unanswered, to our best knowledge:

**Question 1.** Let $\mathcal{M}$ be a locally unsplit dominating family of rational curves on a Fano manifold $X$ with nef tangent bundle. Is $\mathcal{M}$ uniform or, at least, locally uniform? (see Definition 3.3).

Associated to $\mathcal{M}$ one may consider the family of minimal sections of $\mathbb{P}(T_X)$ over curves of the family $\mathcal{M}$ (see Definition 4.3), that we denote by $\overline{\mathcal{M}}$, and study the different splitting types of the tangent bundle of $\mathbb{P}(T_X)$ with respect to curves of $\overline{\mathcal{M}}$. Looking at the rational homogeneous examples, one may check that the family $\overline{\mathcal{M}}$ is, in general, not uniform, and the different splitting types are related to the singular locus stratification of the crepant contraction of $\mathbb{P}(T_X)$ associated with the line bundle $\mathcal{O}(1)$ –note that it is not yet clear whether the nefness assumption on the tangent bundle of a Fano manifold implies its semiampleness. This singular locus stratification is related, for a rational homogeneous manifold, to the orbit decomposition of the corresponding Lie algebra $\mathfrak{g} = \mathcal{O}(1)$. In fact, the image of every stratum into $\mathbb{P}(\mathfrak{g}^\vee)$ is the closure of (the set of classes modulo homotheties of) a nilpotent orbit (see [3] for an account on these orbits).

In this paper we analyze the simplest case, in which the families $\mathcal{M}$ and $\overline{\mathcal{M}}$ are both uniform (we simply say that $\mathcal{M}$ is 2-uniform). The main result shows that, up to a technical assumption, this property is only fulfilled by rational homogenous manifolds:

**Theorem 1.1.** Let $X$ be a Fano manifold of Picard number one, not isomorphic to a projective space, supporting an unsplit 2-uniform family $\mathcal{M}$ of rational curves. Assume moreover that the anticanonical degree $-K_X \cdot \mathcal{M}$ of the family is smaller than or equal to $2(\dim(X) + 2)/3$. Then $X = G/P$, where $G$ is a semisimple Lie group with Dynkin diagram $\mathcal{D}$, $P$ is the parabolic subgroup associated to the $i$-th node of the diagram, and the pair $(\mathcal{D}, i)$ is one of the following:

$$(A_{k+1}, 2), \ k \geq 2, \ (B_k, 1), \ k = 2, 3, \ (D_5, 5), \ (E_6, 1).$$

In the above statement the cited rational homogeneous manifolds are described in terms of their marked Dynkin diagrams (where the nodes have been numbered as in [8] p. 58).

Note that the number $-K_X \cdot \mathcal{M} - 2$ equals the dimension of the variety of minimal rational tangents of $\mathcal{M}$ at $x$, denoted by $\mathcal{C}_x \subseteq \mathbb{P}(\Omega_{X,x})$. In the range $-K_X \cdot \mathcal{M} > 2(\dim(X) + 2)/3$ one could expect, via a positive answer to Hartshorne’s conjecture in our particular situation, that $\mathcal{C}_x$ is a complete intersection in $\mathbb{P}(\Omega_{X,x})$. If this is case, we may conclude the following:

**Corollary 1.2.** Let $X$ be a Fano manifold of Picard number one, not isomorphic to a projective space, supporting an unsplit 2-uniform family $\mathcal{M}$ of rational curves, and assume that, for the general point $x$, $\mathcal{C}_x$ is a complete intersection. Then $X$ is isomorphic to a smooth quadric.

Putting this back into the context of the Campana–Peternell Conjecture and Question 1, let us assume $X$ is a manifold of Picard number one such that $T_X$ is nef and big, and let $\epsilon$ be the birational crepant contraction of $\mathbb{P}(T_X)$ associated
to $\mathcal{O}_{\mathbb{P}(T_X)}(1)$. It was proved in [24] that if the restriction of $\epsilon$ to $\text{Exc}(\epsilon)$ has 1-dimensional fibers, then necessarily $X$ is a smooth hyperquadric. Our main result in this paper implies the following extension of [24]:

**Corollary 1.3.** Let $X$ be a Fano manifold of Picard number one, different from the projective space, with nef and big tangent bundle, and assume that, with the same notation as above, the restriction of $\epsilon$ to $\text{Exc}(\epsilon)$ is a smooth morphism. Assume moreover that $X$ supports an unsplit uniform family of rational curves of anticanonical degree smaller than or equal to $2(\dim(X) + 2)/3$. Then $X$ is rational homogeneous.

The structure of the paper is the following: Sections 2 and 3 contain some preliminary material on dual varieties and rational curves on projective varieties, respectively. Section 4 deals with the family of minimal sections of $\mathbb{P}(T_X)$ over curves of a family $\mathcal{M}$, and the duality relation of its locus with the variety of minimal rational tangents of $\mathcal{M}$. Finally Section 5 contains the proofs of the main results of this paper.

2. Preliminaries on duals of varieties with nodal singularities

We will recall here some standard facts about dual varieties; for the reader’s convenience, we will consider here explicitly the concrete case we are interested in, which is the one in which $C$ has only **nodal singularities**; though these results are probably well known, we weren’t able to find a proper reference. For the smooth case see [25] or [4].

**Definition 2.1.** Let $C \subset \mathbb{P}^r = \mathbb{P}(V)$ be an irreducible projective variety. Denoting by $C_0 \subset C$ its subset of smooth points, the Euler sequence provides a surjection $\mathcal{O}_{C_0} \otimes V^\vee \to N_{C_0, \mathbb{P}^r}(-1)$, so that we have a morphism: $p_2 : \mathbb{P}(N_{C_0, \mathbb{P}^r}(-1)) \to \mathbb{P}(V^\vee)$. Then the closure of the image $C^\vee$ is called the **dual variety** of $C$.

In other words, $C^\vee$ may be described as the closure of the set of tangent hyperplanes of $C$. That is, we may consider $\mathbb{P}(N_{C_0, \mathbb{P}^r}(-1))$ as a subset of $F(0, r-1) := \mathbb{P}(T_{F(r)}) \subset \mathbb{P}^r \times \mathbb{P}^r$ and denote by $\mathcal{P}$ its closure (this is the so-called **conormal variety** of $C \subset \mathbb{P}^r$). Then the restrictions ($p_1$ and $p_2$) to $\mathcal{P}$ of the canonical projections have images $C$ and $C^\vee$, respectively:

$$
\begin{array}{ccc}
\mathbb{P}^r & \overset{\pi_1}{\leftarrow} & F(0, r-1) & \overset{\pi_2}{\rightarrow} & \mathbb{P}^r \\
\downarrow & & \downarrow & & \uparrow \\
C & & \mathcal{P} & & C^\vee
\end{array}
$$

Finally, let us recall that the biduality theorem states that $C^{\vee\vee} = C$, so that the diagram above is reversible, and we may assert that the general fiber of $p_2$ (the so-called **tangency locus** of a hyperplane) is a linear space. In particular one expects $p_2$ to be, indeed, birational for most projective varieties.

**Definition 2.2.** With the same notation as above, the number $e(C) := r - 1 - \dim(C^\vee)$ is called the **dual defect** of $C$. If $e(C) > 0$, we say that $C$ is **dual defective**.

It is well known that the dual defect of a smooth variety $C$ is smaller than or equal to $\text{codim}(C) - 1$. We will need an extension of this result to mildly singular varieties, a topic that may be understood in the context of the theory of discriminant varieties of linear systems (see, for instance, [19]).
whose image is precisely the dual variety \( C \), its inverse image in \( \mathbb{P} \) equal to one, whose general element satisfies that \( N_{\mathbb{P}(V)} \) to one to a linear subspace in \( \mathbb{C} \). Assume that, with the same notation as above, the defect \( N_{\mathbb{P}(V)} \) is swept out by a family of projective spaces \( \mathbb{P}^e \) of rank equal to \( \text{codim}(C, \mathbb{P}(V)) = r - c \). Furthermore, denoting \( O_M(1) := t^*(\mathcal{O}_{\mathbb{P}(V)}(1)) \) and considering the pull-back to \( M \) of the Euler sequence on the projective space \( \mathbb{P}(V) \), we see that \( N(-1) \) is globally generated by \( V := H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))^\vee \). In particular we then have a well-defined morphism:

\[
\mathbb{P}(N(-1)) \to \mathbb{P}(V^\vee),
\]

whose image is precisely the dual variety \( C^\vee \); furthermore, we have a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{P}(N(-1)) & \to & \mathbb{P}(V^\vee) \\
\downarrow & & \downarrow \\
M & \to & C
\end{array}
\]

where, again, \( \mathcal{P} \) is the conormal variety of \( C \). Note that \( \mathcal{P} \) is a \( \mathbb{P}^{r-c-1} \)-bundle over a certain open set of \( C \), isomorphic to its inverse image in \( \mathbb{P}(N(-1)) \).

**Proposition 2.3.** Let \( C \subset \mathbb{P}(V) \) be a variety with only nodal singularities and assume that, with the same notation as above, the defect \( c \) is bigger than zero. Then its normalization \( M \) is swept out by a family of projective spaces \( \mathbb{P}^e \) of degree equal to one, whose general element satisfies that \( N_{\mathbb{P}(V)} \) is uniform with respect to lines in \( \mathbb{P}^e \), with splitting type \((0, 1, \ldots, 1)\) (see Definition 3.3 below).

**Proof.** The biduality theorem tells us that, given a general point \( h \in C^\vee \) (smooth point is enough), its inverse image in \( \mathcal{P} \) is isomorphic to \( \mathbb{P}^e \), mapping one-to-one to a linear subspace in \( C \). Its inverse image \( F_h^e \) into \( \mathbb{P}(N(-1)) \) is smooth and maps birationally to \( \mathbb{P}^e \subset \mathcal{P} \) (by the birationality of \( \mathbb{P}(N(-1)) \to \mathcal{P} \)). Consequently, it maps birationally also to \( \mathbb{P}^e \subset \mathcal{P} \). Moreover, it maps finite-to-one onto its image \( F_h \subset M \) (because \( \mathbb{P}(N(-1)) \to \mathbb{P}(V^\vee) \) is determined by its tautological \( \mathcal{O}_{\mathbb{P}(N(-1))}(1) \)). Finally, since \( t \) is finite, it follows that the composition

\[
F_h^e \to F_h \to \mathbb{P}^e \subset C
\]

is finite and birational from a smooth variety, hence it is an isomorphism. Thus \( F_h \to \mathbb{P}^e \) is an isomorphism, too.

For the second part of the statement, note that we have short exact sequences on \( F_h \cong F_h^e \):

\[
\begin{align*}
0 & \longrightarrow T_M \xrightarrow{dt} t^*T_{\mathbb{P}(V)} \longrightarrow N \longrightarrow 0 \\
0 & \longrightarrow \mathcal{O}_{F_h} \longrightarrow N^\vee(1)_{|F_h} \longrightarrow T_{\mathbb{P}(N)|M|F_h} \\
0 & \longrightarrow \mathcal{O}_{F_h} \longrightarrow N_{F_h,M} = \mathcal{O}_{F_h}^{r-c-1} \longrightarrow N_{F_h,M} \\
0 & \longrightarrow \mathcal{O}_{F_h}^{r-c} \longrightarrow N(-1)_{|F_h} 
\end{align*}
\]
The first is the restriction of the relative Euler sequence of $\mathbb{P}(\mathcal{N})$, the second comes from the differential of the natural map $\mathbb{P}(\mathcal{N}_{|F_h}) \rightarrow F_h$, and the third from the differential of the immersion $\tau_x : \mathcal{M}_x \rightarrow \mathbb{P}(\Omega_{X,x})$. The three sequences fit in the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}_{F_h} & \rightarrow & \mathcal{N}^\vee(1)_{|F_h} \\
| & | & | \\
| & | & | \\
\mathcal{O}_{F_h} & \rightarrow & \mathcal{O}_{F_h}^{r-e} \\
| & | & | \\
| & | & | \\
N_{F_h,M}^\vee(1) & \rightarrow & N_{F_h,M}
\end{array}
\]

We then have an isomorphism

\[
N_{F_h,M}^\vee(1) \cong N_{F_h,M}.
\]

We consider now any line $\ell \subseteq F_h \subset M$. Being $(a_1, \ldots, a_{c-e})$ the splitting type of $N_{F_h,M}$ on $\ell$, $a_1 \leq \cdots \leq a_{c-e}$, we note first that, since $N_{F_h,M}$ is nef, $a_1 \geq 0$. Then the isomorphism (2) tells us that $a_{c-e} \leq 1$ and $\sum_i a_i = \frac{c-e}{2}$, so the splitting type of $N_{F_h,M}$ on $\ell$ is necessarily $(0, \frac{c-e}{2}, \ldots, 1)$. This concludes the proof.

**Corollary 2.4.** With the same notation as in 2.3, assume moreover that $C \subset \mathbb{P}(V)$ is not a linear subspace. Then the defect $e$ is smaller than or equal to $r - c - 1$. Moreover, if equality holds, then the tangent map $t$ is an embedding.

**Proof.** As an immediate consequence of Proposition 2.3 the defect $e$ is smaller than or equal to $c$, and $c \equiv e$ modulo 2. If $e = c$, then $C \subset \mathbb{P}(V)$ would be a linear subspace, a contradiction. Hence we may assert that $e \leq c - 2$.

Denoting $h := r - c - 1 - e$, this inequality reads as $c - 2 \geq e = r - c - 1 - h$, that is $2e \geq r + 1 - h > r - h$. Now, by a corollary of Fulton-Hansen Theorem (cf. [20], Cor. 3.4.1), if $h \leq 0$ it follows that $t$ is an embedding, which in turn implies, by Zak’s Theorem on Tangencies, that $\dim(C^\vee) \geq \dim(C)$, that is $h \geq 0$.

3. Preliminaries on rational curves on Fano manifolds

Let $X$ be a Fano manifold of Picard number one and dimension $m$, defined over the field of the complex numbers. For the reader’s convenience we will briefly review, in our setting, some well-known results on deformations of rational curves on $X$. Most of the listed results work on broader settings: we refer to [9, 17, 18] for more details.

A family of rational curves on $X$ is, by definition, the normalization $\mathcal{M}$ of an irreducible component of the scheme $\text{RatCurves}^a(X)$. Each of these families comes equipped with a smooth $\mathbb{P}^1$ fibration $p : \mathcal{U} \rightarrow \mathcal{M}$ and an evaluation morphism $q : \mathcal{U} \rightarrow X$. Given a point $x \in q(\mathcal{U})$, we denote by $\mathcal{M}_x$ the normalization of the set $pq^{-1}(x)$, and by $\mathcal{U}_x$ the normalization of its fiber product with $\mathcal{U}$ over $\mathcal{M}$. We say that the family $\mathcal{M}$ is:

- **dominating** if $q$ is dominant,
- **locally unsplit** if $\mathcal{M}_x$ is proper for general $x \in X$,
• unsplit if \( \mathcal{M} \) is proper.

Note that RatCurves\(^n\)(\(X\)) is quasi-projective, hence the properness of \( \mathcal{M}_x \) or \( \mathcal{M} \)
implies their projectivity. The anticanonical degree \(-K_X \cdot \Gamma\) is the same for every
curve \( \Gamma \) of the family \( \mathcal{M} \); thus we will denote by \(-K_X \cdot \mathcal{M}\); moreover, we will denote
by \( c(\mathcal{M}) \), or simply \( c \) if there is no possible confusion, the number \( c := -K_X \cdot \mathcal{M} - 2 \).

**Notation 3.1.** Along the paper we will often consider vector bundles on the
projective line \( \mathbb{P}^1 \). For simplicity, we will denote by \( E(a_1^{k_1}, \ldots, a_r^{k_r}) \) the vector
bundle \( \bigoplus_{j=1}^r \mathcal{O}(a_j)^{\otimes k_j} \) on \( \mathbb{P}^1 \).

**Proposition 3.2.** Let \( X \) be a Fano manifold of Picard number one, \( \mathcal{M} \) be a
locally unsplit dominating family of rational curves, and set \( c = -K_X \cdot \mathcal{M} - 2 \). With
the same notation as above, we have the following:

1. The variety \( \mathcal{M} \) has dimension \( m + c - 1 \) and, for a general point \( x \in X \),
\( \mathcal{M}_x \) is a smooth projective variety of dimension \( c \).
2. There exists a nonempty open set \( X_0 \) of \( X \) satisfying that, for every \( x \in X_0 \),
the normalization \( f : \mathbb{P}^1 \to \Gamma \) of any element \( \Gamma \) of \( \mathcal{M} \) passing by \( x \) is free,
that \( f^* \mathcal{T}_X \) is a nef vector bundle.
3. \( \pi|_{\mathcal{U}_x} : \mathcal{U}_x \to \mathcal{M}_x \) is a \( \mathbb{P}^1 \)-bundle, for a general \( x \in X \), admitting a section
\( \sigma_x \) whose image lies in \( q^{-1}(x) \).
4. If a rational curve \( \Gamma \) is a general member of \( \mathcal{M} \), then \( \Gamma \) is standard, i.e.
denoting by \( f : \mathbb{P}^1 \to X \) its normalization, \( f^* \mathcal{T}_X \cong E(2, 1^c, 0^{m-c-1}) \).

**Proof.** For the first and second statement, see [18 II. 1.7 and 3.11]. (3)
follows by [18 II. Theorem 2.12], and by the fact that the point \( x \) determines a
section of \( \pi_{p^{-1}(\mathcal{M}_x)} : p^{-1}(\mathcal{M}_x) \to \mathcal{M}_x \); in fact, [15 Thm. 3.3] implies that \( q^{-1}(x) \)
consists of a finite set of points and a unique component mapping one to one onto
\( \mathcal{M}_x \) via \( p \). Finally (4) follows from [18 IV. Corollary 2.9].

**Definition 3.3.** With the same notation as above, given the normalization
\( f : \mathbb{P}^1 \to X \) of an element \( \Gamma \) of \( \mathcal{M} \), we say that \( \Gamma \) has splitting type \( (a_1^{k_1}, \ldots, a_r^{k_r}) \)
on \( X \), with \( k_1 + \cdots + k_r = \dim(\mathcal{M}) \), \( a_1, \ldots, a_r \), if \( f^* \mathcal{T}_X \cong E(a_1^{k_1}, \ldots, a_r^{k_r}) \). We then say that \( \mathcal{M} \) is
uniform (resp. locally uniform) if every curve \( \Gamma \) of \( \mathcal{M} \) (resp.
every curve of \( \mathcal{M}_x \), \( x \in X \) general) has the same splitting type.

**3.1. Variety of minimal rational tangents of a family \( \mathcal{M} \).** With the same
notation as above, if \( \mathcal{M} \) is a locally unsplit dominating family, then \( \mathcal{M}_x \) is smooth
for general \( x \in X \). Moreover, (2) from Proposition 3.2 allows us to claim that there
exists a rational map \( \tau_x \) from \( \mathcal{M}_x \) to the Grothendieck projectivization of \( \Omega_{X,x} \),
\( \mathbb{P}(\Omega_{X,x}) \), called the tangent map of \( \mathcal{M} \) at \( x \), sending the general element of \( \mathcal{M}_x \)
to its tangent direction at \( x \). It is known (cf. [15 Theorem 3.4], [12 Theorem 1]) that
\( \tau_x \) is a finite morphism and birational (hence it is the normalization of its image)
onto a variety \( C_x \) usually called the variety of minimal rational tangents (VMRT,
for short) of \( \mathcal{M} \) at \( x \). The closure of the union of all the \( C_x \), \( x \in X \) general, into
the Grothendieck projectivization of \( \Omega_X \), \( \mathbb{P}(\Omega_X) \), is denoted by \( C \).

**Remark 3.4.** Given a general point \( x \in X \), the map \( \tau_x \) may be understood
as follows. Let us consider the section \( \sigma_x : \mathcal{M}_x \to \mathcal{U}_x \) defined in 3.2 (3), and let
\( K_{\mathcal{U}_x/\mathcal{M}_x} \) denote the relative canonical divisor of \( \mathcal{U}_x \to \mathcal{M}_x \). Via the composition
of the natural morphisms \( T_{\mathcal{U}_x/\mathcal{M}_x}|_{q^{-1}(x)} \subset T_{\mathcal{U}_x}|_{q^{-1}(x)} \) and \( T_{\mathcal{U}_x}|_{q^{-1}(x)} \to T_{X,x} \otimes
\mathcal{O}_{q^{-1}(x)} \), it follows from [15 Theorem 3.3] that \( T_{\mathcal{U}_x/\mathcal{M}_x}|_{q^{-1}(x)} \) is a subbundle of
$T_{X,x} \otimes O_{q^{-1}(x)}$. Restricting this inclusion of vector bundles to $\mathcal{M}_x$ via $\sigma_x$ yields a morphism $\mathcal{M}_x \to \mathbb{P}(T^\vee_{X,x})$, which is nothing but the tangent map $\tau_x$. Hence the tangent map $\tau_x$ satisfies

$$\tau_x^*O_{\mathbb{P}(\Omega_{X,x})}(1) = O_{\mathcal{M}_x}(\sigma_x^* K_{U/M_x}).$$

In particular, this line bundle (that we will denote it by $O_{\mathcal{M}_x}(1)$, or simply $O(1)$ where there is no possible confusion) is ample and globally generated.

**Proposition 3.5.** [1] Proposition 2.7 | With the same notation as above, given a locally unsplit dominating family of rational curves $\mathcal{M}$ on $X$, $\tau_x$ is immersive at $[C] \in \mathcal{M}_x$ if and only if $C$ is standard.

One of the key results we will use is the following theorem by Hong and Hwang, which tells us that within the class of Fano manifolds of Picard number one, a large class of rational homogeneous manifolds is determined by $C_x$ and its embedding in $\mathbb{P}(T^\vee_{X,x})$.

**Theorem 3.6.** [7] Let $X$ be a Fano manifold of Picard number one, $S = G/P$ a rational homogeneous manifold corresponding to a long simple root and $C_o \subset \mathbb{P}(T^\vee_{S,o})$ the VMRT at a reference point $o \in S$. Assume $C_o \subset \mathbb{P}(T^\vee_{S,o})$ and $C_x \subset \mathbb{P}(T^\vee_{X,x})$ are isomorphic as projective subvarieties. Then $X$ is isomorphic to $S$.

### 4. Minimal sections of $\mathbb{P}(T_X)$ over rational curves

Along the rest of the paper, unless otherwise stated, we will always consider the following setup:

**Setup 4.1.** $X$ is a Fano manifold of Picard number one and dimension $m$, not isomorphic to a projective space. $\mathcal{M}$ is a locally unsplit dominating family of rational curves in $X$, of anticanonical degree $c + 2$. We will use for them the notations introduced in Section 3. Furthermore, we will denote by $X$ the Grothendieck projectivization of $T_X$ and by $\phi : X \to X$ the corresponding natural projection.

**Definition 4.2.** Given a free element $\Gamma$ of $\mathcal{M}$, with normalization $f : \mathbb{P}^1 \to X$, a section of $X$ over $\Gamma$ of $\mathcal{M}$ corresponding to a surjective map $f^*T_X \to O_{\mathbb{P}^1}$ will be called minimal. The corresponding rational curve will be denoted by $\Gamma$.

We may now consider a nonempty open subset $\mathcal{M}_0 \subset \mathcal{M}$ parametrizing standard curves, and denote by $p_0 : U_0 \to \mathcal{M}_0$ and $q_0 : U_0 \to X$ the corresponding family and an evaluation morphism respectively. Then $p_0^*q_0^*\Omega_X$ and $p_0^*p_0^*q_0^*\Omega_X$ are rank $m - 1 - c$ vector bundles over $\mathcal{M}_0$ and $U_0$, respectively, and the natural morphism $\overline{\rho} : U_0 := \mathbb{P}(p_0^*q_0^*\Omega_X) \to \overline{\mathcal{M}} := \mathbb{P}((p_0^*q_0^*\Omega_X)^\vee)$ is a smooth $\mathbb{P}^1$-fibration. Together with the natural morphism $\overline{\pi} : U_0 \to \mathbb{P}(T_X) = X$, this data provides a family of rational curves in $X$, and hence a morphism from $\overline{\mathcal{M}}_0$ to $\text{RatCurves}^n(X)$, which is injective. Moreover, by construction, the image of every element of $\overline{\mathcal{M}}_0$ in $\text{RatCurves}^n(X)$ corresponds to a minimal section of $X$ over a curve of $\mathcal{M}_0$. Let us denote by $\overline{\mathcal{M}}$ the normalization of the closure of its image, and by $\overline{\rho} : U \to \overline{\mathcal{M}}, \overline{\pi} : U \to \overline{\mathcal{M}}$ the corresponding universal family and evaluation morphism. Furthermore, as we will see later, $\overline{\mathcal{M}}$ is, in fact, the normalization of an irreducible component of $\text{RatCurves}^n(X)$.

**Definition 4.3.** With the same notation as above, given a locally unsplit dominating family $\mathcal{M}$, the family of rational curves parametrized by $\overline{\mathcal{M}}$ is called the family of minimal sections of $X$ over curves of $\mathcal{M}$. 
Moreover, we have natural morphisms \( \phi_M : \overline{\mathcal{M}} \to \mathcal{M} \), \( \phi_U : \overline{\mathcal{U}} \to \mathcal{U} \), fitting in the following commutative diagram:

\[
\begin{array}{ccc}
\overline{\mathcal{M}} & \xrightarrow{\overline{p}} & \overline{\mathcal{U}} \\
\phi_M & & \phi_U \\
\downarrow & & \downarrow \phi \\
\mathcal{M} & \xrightarrow{p} & \mathcal{U} \\
\end{array}
\]

Note that (4) implies that the fibers of \( \phi_M \) over every standard curve of \( \mathcal{M}_0 \subset \mathcal{M} \) are isomorphic to \( \mathbb{P}^{m-c-2} \), so \( \overline{\mathcal{M}} \) has dimension \( 2m - 3 \). However, the image of \( \overline{q} \), that we will denote by \( D \), may have dimension smaller than \( \dim(\overline{\mathcal{U}}) = 2m - 2 \).

### 4.1. Projective duality for VMRT's.

We will study now the relation between the minimal sections of \( X \) over rational curves of a locally unsplit dominating family \( \mathcal{M} \), and its VMRT's. We first recall that \( X \) supports a contact structure \( F \), defined as the kernel of the composition of the differential of the natural projection \( \phi : X \to X \) with the co-unit map \( \theta : T_X \to \phi^* T_X = \phi^* \phi^* O(1) \to O(1) \).

Note that \( \theta \) fits in the following commutative diagram, with exact rows and columns:

\[
\begin{array}{ccc}
T_{X/X} & \xrightarrow{T_X} & F \\
\downarrow & & \downarrow \Omega_{X/X}(1) \\
T_{X/X} & \xrightarrow{T_X} & \phi^* T_X \\
\downarrow \rho & & \downarrow \\
O(1) & \xrightarrow{} & O(1)
\end{array}
\]

The distribution \( F \) being contact means precisely that it is maximally non integrable, i.e. that the morphism \( d\theta : F \otimes F \to T_X/F \cong O(1) \) induced by the Lie bracket is everywhere non-degenerate. This fact can be shown locally analytically, by considering, around every point, local coordinates \((x_1, \ldots, x_m)\) and vector fields \((\zeta_1, \ldots, \zeta_m)\), satisfying \( \zeta_i(x_j) = \delta_{ij} \). Then the contact structure is determined, around that point, by the 1-form \( \sum_{i=1}^m \zeta_i dx_i \) (see [16] for details).

The next proposition describes the infinitesimal deformations of a general minimal section \( \Gamma \).

**Proposition 4.4.** With the same notation as above, let \( \overline{\mathcal{F}} : \mathbb{P}^1 \to X \) denote the normalization of a minimal section \( \Gamma \) of \( X \) over a standard rational curve in the class \( \Gamma \). Then \( \overline{\mathcal{M}} \) is smooth at \( \Gamma \), of dimension \( 2m - 3 \), and

\[
\overline{\mathcal{F}} T_X \cong E(-2, 2, (-1)^e, 1^c, 0^{2m-3-2e}), \text{ for some } e \leq c.
\]

**Proof.** Writing \( f^* T_X \cong E(2, 1^c, 0^{m-c-2}) \) and taking into account that \( \overline{\mathcal{F}} O(1) = O \), the relative Euler sequence of \( X = \mathbb{P}(T_X) \) over \( X \), pulled-back via \( \overline{\mathcal{F}} \) provides \( \overline{\mathcal{F}} T_{X/X} = E(-2, (-1)^c, 0^{m-c-2}) \). The upper exact row of diagram (5) provides:

\[
0 \to E(-2, (-1)^c, 0^{m-c-2}) \to \overline{\mathcal{F}} F \to E(2, 1^c, 0^{m-c-2}) \to 0.
\]
On the other hand, $\mathcal{F}$ $\mathcal{O}(1) = \mathcal{O}$ also implies that $d\mathcal{F} : T_{\mathcal{O}} \to \mathcal{F} T_{\mathcal{X}}$ factors via $\mathcal{F}$. Hence this bundle has a direct summand of the form $\mathcal{O}(2)$. Being $\mathcal{F}$ a contact structure, it follows that $\mathcal{F} \mathcal{F} \cong \mathcal{F} \mathcal{F}^\vee$, so this bundle has a direct summand $\mathcal{O}(-2)$, as well.

From this we may already conclude that

$$\mathcal{F} \equiv E(-2, 2, (e^2, 0^{2m-2e-4}) \text{ for } e \leq c,$$

hence the bundle $\mathcal{T}^{T_X}$ is isomorphic either to $E(-2, 2, (1^2, 0^{2m-2e-3})$ or to $E(2, (1)^{-2}, 1^0, 0^{2m-2e-4})$. On the other hand, the fact that $\dim \mathcal{M} = 2m - 3$ implies that $h^0(\mathcal{T}^{T_X}) \geq 2m$, which allows us to discard the second option. Finally, in the first case $h^0(\mathcal{T}^{T_X})$ is precisely equal to $\dim \mathcal{M}$ $\text{Hom}(\mathcal{P}^1, \mathcal{X}) = 2m,$ hence this scheme is smooth at $[f]$ and $\mathcal{M}$ is smooth at $[f]$.

**Definition 4.5.** Given a minimal section $[\Gamma]$ over a general $\Gamma$ $\in \mathcal{M}$, the number $e$ provided by the proposition above will be called the *defect of $\mathcal{M}$ at $[\Gamma]$.*

The next result establishes the relation between $D = \text{Im} [\mathcal{T}] \subset \mathcal{X}$ and the VMRT $\mathcal{C} \subset \mathcal{X} = \mathcal{P}(\Omega_X)$, at the general point $x$. This was first obtained in [13 Corollary 2.2] in their study of the moduli space of stable vector bundles on a curve. Our line of argumentation here is based on the proof of [9 Proposition 1.4].

**Proposition 4.6.** With the same notation as above, being $x \in \mathcal{X}$ general, $D_x := D \cap \mathcal{P}(T_{\mathcal{X},x}) \subset \mathcal{P}(T_{\mathcal{X},x})$ is the dual variety of $\mathcal{C}_x \subset \mathcal{P}(\Omega_{\mathcal{X},x})$.

**Proof.** Let $x \in \mathcal{X}$ be a general point and $f : \mathcal{P}^1 \to \mathcal{X}$ be the normalization of a general $\Gamma$, satisfying $f(O) = x$, for some $O \in \mathcal{P}^1$. By Proposition 3.5, the tangent map $\tau_x : \mathcal{M}_x \to \mathcal{C}_x \subset \mathcal{P}(\Omega_{\mathcal{X},x})$ is immersive at $[\Gamma]$ and we may use it to identify the tangent space of $\mathcal{C}_x$ at $P := \tau_x(\Gamma)$.

In order to see this, we denote by $\beta : X' \to X$ the blow-up of $X$ at $x$, with exceptional divisor $E := \mathcal{P}(\Omega_{\mathcal{X},x})$. Note that we have a filtration $T_{\mathcal{X},x} \supset V_1(f) \supset V_2(f)$, where $V_1(f)$ and $V_2(f)$ correspond, respectively, to the fibers over $O$ of the (unique) subbundles of $f^*T_X$ isomorphic to $E(2, 1^0)$ and $E(2)$. Moreover $T_{E,P}$ is naturally isomorphic to the quotient of $T_{\mathcal{X},x}$ by $V_2(f)$, hence our statement may be rewritten as $T_{E,P} = V_1(f)/V_2(f)$.

Let us then consider the irreducible component of $\text{Hom}(\mathcal{P}^1, \mathcal{X}; \mathcal{O}, x)$ (parametrizing morphisms from $\mathcal{P}^1$ to $\mathcal{X}$, sending $\mathcal{O}$ to $x$) containing $[f]$ and note that the evaluation morphism factors

$$\mathcal{P}^1 \times \text{Hom}(\mathcal{P}^1, \mathcal{X}; \mathcal{O}, x) \xrightarrow{\text{ev}} X' \xrightarrow{\beta} X$$

In this setting, we have $T_{E,P} = \text{ev}_f([\mathcal{O}, f]) (\{0\} \times H^0(\mathcal{P}^1, f^*T_X(-1))/V_2(f)$, and we may identify $H^0(\mathcal{P}^1, f^*T_X(-1))$ with the global sections of $f^*T_X$ vanishing at $O$. Choosing now a set of local coordinates $(t, t_2, \ldots, t_m)$ of $X$ around $x$ such that $f(\mathcal{P}^1)$ is given by $t_2 = \ldots, t_m = 0$ and $t$ is a local parameter of $f(\mathcal{P}^1)$, and writing the blow-up of $X$ at $x$ in terms of these coordinates, one may check that, modulo $V_2(f)$, $\text{ev}_f([\mathcal{O}, f])$ sends every section $s$ vanishing at $O$ to $\frac{d}{dt}(s) = \frac{d}{dt}(O)$, hence it follows that its image is $V_1(f)$.
Proposition 4.7. With the same notation as above, being \( x \in X \) general, let \( \Gamma \) be a minimal section of \( \mathcal{X} \) over a standard element of \( \mathcal{M}_x \). Then the dual defect of \( \mathcal{C}_x \) equals the defect of \( \mathcal{M} \) at \( \Gamma \).

Proof. For general \( x \), the dual defect of \( \mathcal{C}_x \) equals \( \text{codim}(D \subset \mathcal{X}) - 1 \) by Proposition 4.6. This number is the corank of \( \overline{\mathcal{T}} \) at the general point minus one, so we need to check that it is equal to \( e \).

Let us denote by \( \overline{f} : \mathbb{P}^1 \to \mathcal{X} \) the normalization of \( \overline{\mathcal{T}} \) satisfying \( f(O) = x \) for some \( O \in \mathbb{P}^1 \), and consider the evaluation morphism \( \text{ev} : \text{Hom} \overline{\mathcal{T}}(\mathbb{P}^1, \mathcal{X}) \times \mathbb{P}^1 \to \mathcal{X} \), where \( \text{Hom} \overline{\mathcal{T}}(\mathbb{P}^1, \mathcal{X}) \) stands for the irreducible component of \( \text{Hom}(\mathbb{P}^1, \mathcal{X}) \) containing \( \overline{f} \) (which is smooth at \( \overline{f} \)) and of dimension \( 2m \) by the proof of Proposition 4.4. The morphism \( \overline{T} \) is obtained from this morphism by quotienting by the action of the group of automorphisms of \( \mathbb{P}^1 \), hence in order to check that the corank of \( \overline{T} \) is equal to \( e + 1 \) it is enough to check that the corank of \( \text{ev} \) at \((\overline{f}, O)\) is \( e + 1 \). Using the description of this differential provided in [15], Proposition 3.4, the result follows then by noting that the cokernel of the evaluation of global sections \( H^0(\mathbb{P}^1, \overline{T}(T_x)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \overline{T}(T_x) \) is isomorphic to \( E(-2, (-1)^e) \), and hence it has constant rank equal to \( e + 1 \).

5. Proofs

We start by noting the following statement, that follows from the proof of Proposition 4.7 above:

Lemma 5.1. Under the assumptions of Theorem 1.1, the morphism \( \overline{T} \) is a submersion of corank \( e + 1 \). In particular \( D_x \subset \mathbb{P}(T_{X,x}) \) has only nodal singularities, at the general point \( x \).

Proof. The first part follows from Proposition 4.7. For the second we note that \( \overline{T} \) factors via the normalization of \( D \), and the corresponding Stein factorization \( \overline{q} : \mathcal{U} \to \overline{D} \) is now a smooth morphism. Hence the inverse image \( \overline{D}_x \) of a general point \( x \) in \( \overline{D} \) is smooth. Finally, since the map from \( \overline{D} \) to \( \mathcal{X} \) has injective differential at every point, it follows that \( D_x \), which is the image of \( \overline{D}_x \) via this morphism, has only nodal singularities.

Proof of Theorem 1.1. We claim first that, under the assumptions of Theorem 1.1, \( \mathcal{C}_x \) and \( D_x \) are smooth for the general \( x \in X \). In fact we know that they both have at worst nodal singularities by Proposition 5.3 and Lemma 5.1. On the other hand, we note that, neither \( \mathcal{C}_x \), nor \( D_x \) can be linear subspaces: in fact, this may only happen if \( X \) is itself a projective space (cf. [10], Proposition 5). Therefore we may apply Corollary 2.4 to both \( \mathcal{C}_x \) and \( D_x \) in order to obtain \( e = m - c - 2 \). Since we have assumed in 1.1 that \( c \leq 2(m - 1)/3 \), we may apply [4], Theorem 4.5] to get that \( \mathcal{C}_x \subset \mathbb{P}(\Omega_{X,x}) \) is one of the following:

- a hypersurface in \( \mathbb{P}^2 \) or \( \mathbb{P}^3 \);
- the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^{c-1} \subset \mathbb{P}^{2c-1} \);
- the Plücker embedding of \( G(1, 4) \);
- the Spinor variety \( S_4 \subset \mathbb{P}^{15} \).

Since the only smooth nonlinear hypersurfaces with smooth dual have degree two (see, for instance, [25], 10.2]), in the first case \( \mathcal{C}_x \) is a conic or a two-dimensional quadric, hence in all cases the listed varieties are projectively equivalent to the
PROOF OF COROLLARY 1.2 Proceeding as above, we get that \( C_\epsilon \) and \( D_\epsilon \) are smooth for the general \( x \in X \). Being \( C_\epsilon \) a nonlinear complete intersection, it follows by \cite[Theorem 5.11]{25}] that \( D_\epsilon \) is a hypersurface, hence \( C_\epsilon \) is a hypersurface, too, and arguing as above it follows that \( C_\epsilon \) is a smooth hyperquadric, which allows to conclude by Theorem 3.6.

Remark 5.2. Since the determination of \( X \) is done upon the determination of its VMRT at the general point, the proofs of Theorem 1.1 and Corollary 1.2 still hold if we change the 2-uniformity assumption by the one in which \( \mathcal{M} \) is locally uniform and the defect \( e \) of every minimal section of \( \Gamma \) over every curve \( \Gamma \) passing a general point \( x \) is constant.

PROOF OF COROLLARY 1.3. Let \( p : \mathcal{U} \to \mathcal{M} \) denote the uniform unsplit family of rational curves on \( X \), and consider, with the same notation as above, the corresponding family \( \overline{p} : \overline{\mathcal{U}} \to \overline{\mathcal{M}} \) of minimal sections of \( X = \overline{\mathbb{F}(T_X)} \) over curves of \( \mathcal{M} \). It is enough to check that \( \overline{\mathcal{M}} \) is uniform. Let us then denote by \( e \) the defect of \( \overline{\mathcal{M}} \) at a general curve \( \Gamma \). Note that the contraction \( \epsilon : X \to Y \) determined by \( \mathcal{O}_{\overline{\mathbb{F}(T_X)}} \) can be obtained by as the quotient modulo homotheties of the contraction:

\[
\text{Spec}_X \left( \bigoplus_{r \geq 0} S^r T_X \right) \longrightarrow \text{Spec} \left( \bigoplus_{r \geq 0} H^0(X, S^r T_X) \right)
\]

It is known that this map is a symplectic contraction, hence it is in particular semismall (see \cite[24, 14]), and this property is then inherited by our contraction \( \epsilon \).

Now, let \( E \) be an irreducible component of \( \text{Exc}(\epsilon) \) containing \( D = \overline{\mathbb{F}(U)} \), \( L \) the locus of curves of the family \( \overline{\mathcal{M}} \) passing by a general point \( \overline{x} \in D \), and \( F \) the fiber of the restriction \( \epsilon : E \to \epsilon(E) \) passing through \( \overline{x} \). Then we have:

\[
\text{codim}(D \subset X) \geq \text{codim}(E \subset X) \geq \dim(F) \geq \dim(L).
\]

Proposition 1.4 tells us that \( e \) is equal to \( \text{codim}(D \subset X) - 1 \), and to the dimension of the general fiber of \( \overline{\mathbb{F}} : \overline{\mathcal{U}} \to D \), which implies that \( \dim(L) = e + 1 \). Summing up, we get that \( D = E \), and that \( F = L \).

Moreover \( F \) is an \((e + 1)\)-dimensional smooth variety swept out by curves of the family \( \overline{\mathcal{M}} \), which is unsplit, and the dimension of the subfamily of this curves passing by the general element of \( F \) is \( e \). It then follows by \cite[2] that \( F \cong \mathbb{P}^{e+1} \), and curves of \( \overline{\mathcal{M}} \) contained in \( F \) are lines. Furthermore, we claim that the smoothness of the map \( \epsilon_{|E} : E \to \epsilon(E) \) implies that every fiber \( F \) contained in \( E \) is necessarily isomorphic to \( \mathbb{P}^{e+1} \). In fact, this follows from a classical result of Hirzebruch-Kodaira \cite[6] (as noted in \cite[11, Theorem 1'])], but we may also infer it from a result of Fujita as follows: it is enough to check that the all the fibers of \( \epsilon_{|E} \) over any smooth curve \( C \) in \( \epsilon(E) \) passing through a general point are \( \mathbb{P}^{e+1} \)'s. Over an open Zariski subset \( U \) of such a curve, \( \epsilon_{|E} \) is a projective bundle, since the Brauer group of \( U \) is trivial. By taking the closure of an effective unisecant divisor on \( \epsilon^{-1}(U) \), we get a global relative hyperplane section divisor on \( \epsilon^{-1}(C) \). Since \( \epsilon_{|E} \) is equidimensional, then it follows by \cite[Lemma 2.12]{5} that any fiber over a point of \( C \) is a \( \mathbb{P}^{e+1} \).
In particular, this implies that every curve $\Gamma$ of $\overline{M}$ is a line on a fiber $F \cong \mathbb{P}^{e+1}$ contained in $E$, and so, denoting by $\overline{\Gamma}$ the normalization $\overline{\Gamma}$, $\overline{\Gamma}^* T_X$ contains a vector subbundle of the form $E(2,1^e)$. On the other hand, by Proposition 4.4 and semi-continuity, the splitting type of $T_X$ at $\Gamma$ is of the form $(-2,2,(-1)^e,1^{e'},0^{2m-3-2e'})$ for $e' \leq e$. We may then conclude that $e' = e$.

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