Simple interval observers for linear impulsive systems with applications to sampled-data and switched systems

Corentin Briat and Mustafa Khammash*

Abstract

Sufficient conditions for the design of a simple class of interval observers for linear impulsive systems subject to minimum and range dwell-time constraints are obtained and formulated in terms of infinite-dimensional linear programs. The proposed approach is fully constructive in the sense that suitable observer gains can be extracted from the solution of the optimization problems and is flexible enough to be extended to include performance constraints and parametric uncertainties. In order to be solvable, the infinite-dimensional linear programs are relaxed using a method based on sum of squares which is known to be asymptotically exact in the present case. Three examples are given for illustration: the first one pertains on the interval observation of an impulsive system under a minimum dwell-time constraint, the second one is about the interval observation of an aperiodic sampled-data system and the last one is about the interval observation of a linear switched system.

1 Introduction

Impulsive linear systems are an important class of hybrid systems that can be used to model a wide variety of real world processes [1] and to represent switched and sampled-data systems [1–5]. Among this class of systems, the subclass of linear positive impulsive systems have been recently studied in [6] and several stability and stabilization conditions under various dwell-time constraints have been obtained and formulated in terms of, finite- or infinite-dimensional, linear programs which can then be solved using recent optimization techniques. Although restrictive, the class of linear positive impulsive systems can be used as comparison systems for more general impulsive systems or may be helpful in the design of interval observers. Interval observers have been first introduced in [7] in order to account for the presence of disturbances acting on the observed system. The key idea is to build two, possibly coupled, observers whose goal will be the real-time estimation of a lower bound and an upper bound for the state of the system. The problem of designing such observers have

*Corentin Briat and Mustafa Khammash are with the Department of Biosystems Science and Engineering, ETH-Zürich, Switzerland; email: mustafa.khammash@bsse.ethz.ch, corentin.briat@bsse.ethz.ch, corentin@briat.info; url: https://www.bsse.ethz.ch/ctsb/, [http://www.briat.info](http://www.briat.info)
then been considered for many different classes of systems including systems with inputs, linear systems, uncertain systems, delay systems, impulsive systems, LPV systems, discrete-time systems, systems with samplings, etc.

Some classes of interval observers for linear impulsive systems have been proposed in [13]. Unfortunately, these reported conditions, which are based on the discrete-time quadratic conditions obtained in [3], only characterize the asymptotic stability of the error dynamics and cannot be used for design purposes because of their strong non-convex structure. Moreover, this approach does not exploit the positive nature of the error dynamics which may help in the design of suitable observer gains by, for instance, considering diagonal or linear Lyapunov functions. Finally, it is also difficult to guarantee the stability under a desired dwell-time constraint or to ensure a desired performance level as these properties can only be checked a posteriori; i.e. after manually selecting the gains of the interval observer. On the other hand, very few results have been devoted to the interval observation of sampled-data systems while, to the author’s best knowledge, no results have been reported so far in the context of switched systems.

The main objective of the current paper will be derivation of constructive sufficient conditions for the design of a simple class of interval observers for linear impulsive systems that can guarantee prescribed range and minimum dwell-time constraints. As sampled-data systems and switched systems both admit an impulsive system representation, the developed approach also readily applies to those classes of systems. The overall method relies on the linear programming stability conditions recently obtained in [6] for linear positive impulsive/switched systems referred to as clock-dependent conditions because of their dependence of the (relative) time elapsed since the last discrete-event (i.e. the last impulse time or the last switching time). These conditions are the linear programming analogues of the semidefinite programming conditions obtained in [3,4,18] in the context of general linear impulsive, sampled-data and switched systems. These conditions have the benefits of being linear in the system matrices and to readily allow for the derivation of design conditions – in the current context, for the design of interval observers. The conditions can be checked using discretization methods [19], using linear programming via the use of Handelman’s theorem [20,21] or using semidefinite programming via the use of Putinar’s Positivstellensatz [22] combined with computational sum of square methods [23,24]. The proposed approach is flexible enough to incorporate performance constraints (e.g. in the $L_1$ or the $L_\infty$ sense [9,21]) and uncertain parameters. These extensions, however, are left for future research due to space restrictions.

Outline. The structure of the paper is as follows: in Section 2 preliminary definitions and results are given. Section 3 is devoted to the derivation of design conditions for the considered class of interval observers. Computational issues are discussed in Section 4 and examples, finally, are treated in Section 5.

Notations. The set of nonnegative integers is denoted by $\mathbb{N}_0$. The cones of positive and nonnegative vectors of dimension $n$ are denoted by $\mathbb{R}_{>0}^n$ and $\mathbb{R}_{\geq 0}^n$, respectively. The set of diagonal matrices of dimension $n$ is denoted by $\mathbb{D}^n$ and the subset of those being positive definite is denoted by $\mathbb{D}_{>0}^n$. The $n$-dimensional vector of ones is denoted by $\mathbf{1}_n$. For some
elements, \( \{x_1, \ldots, x_n\} \), the operator \( \text{diag}_{i=1}^{n}(x_i) \) builds a matrix with diagonal entries given by \( x_1, \ldots, x_n \) whereas \( \text{col}_{i=1}^{n}(x_i) \) creates a vector by vertically stacking them with \( x_1 \) on the top.

### 2 Preliminaries on linear positive impulsive systems

The objective of this section is to recall few results about linear positive impulsive systems \([6]\). To this aim, let us consider the following class of linear impulsive system:

\[
\begin{align*}
\dot{x}(t_k + \tau) &= A(\tau)x(t_k + \tau) + E_c(\tau)w_c(t_k + \tau), \quad \tau \in (0, T_k] \\
x(t_k^+) &= Jx(t_k) + E_d w_d(k)
\end{align*}
\]

where \( k \in \mathbb{N}_0 \), \( x(t_k^+) := \lim_{s \downarrow t_k} x(s) \) and the matrix-valued functions \( A(\tau) \in \mathbb{R}^{n \times n} \) and \( E(\tau) \in \mathbb{R}^p \) are continuous and bounded. The sequence of impulse times \( \{t_k\}_{k \in \mathbb{N}_0} \), \( t_0 = 0 \), is assumed to verify the properties: (a) \( T_k := t_{k+1} - t_k > 0 \) for all \( k \in \mathbb{N}_0 \) and (b) \( t_k \to \infty \) as \( k \to \infty \). When all the above properties hold, the solution of the system (1) exists for all times.

We have the following result regarding the state positivity of the impulsive system (1).

#### Proposition 1

The following statements are equivalent:

(a) The system (1) is state positive, i.e. for any \( x_0 \geq 0, w_c(t) \geq 0 \) and \( w_d(k) \geq 0 \), we have that \( x(t) \geq 0 \) for all \( t \geq 0 \).

(b) The matrix-valued function \( A(\tau) \) is Metzler for all \( \tau \geq 0 \), the matrix-valued function \( E_c(\tau) \) is nonnegative for all \( \tau \geq 0 \) and the matrices \( J, E_d \) are nonnegative.

#### 2.1 Stability under range dwell-time

The following result provides a sufficient condition for the stability of the system (1) under a range dwell-time constraint; i.e. \( T_k \in [T_{\min}, T_{\max}], k \geq 0 \), for some given \( 0 < T_{\min} \leq T_{\max} < \infty \). It is an extension of the range dwell-time result derived in \([6]\).

#### Theorem 2

Let us consider the system (1) with \( w_c \equiv 0, w_d \equiv 0 \) and assume that it is state positive; i.e. \( A(\tau) \) is Metzler for all \( \tau \in [0, T_{\max}] \) where \( 0 < T_{\min} \leq T_{\max} < \infty \) are given real numbers. Then, the following statements are equivalent:

(a) There exists a vector \( \lambda \in \mathbb{R}_n^{>0} \) such that

\[
\lambda^T (J\Phi(\theta) - I_n) < 0 
\]

holds for all \( \theta \in [T_{\min}, T_{\max}] \) where

\[
\Phi(s) = A(s)\Phi(s), \quad \Phi(0) = I_n, s \in [0, T_{\max}].
\]
(b) There exist a differentiable vector-valued $\zeta : [0,T] \mapsto \mathbb{R}^n$, $\zeta(0) > 0$, and a scalar $\varepsilon > 0$ such that the conditions

$$
\dot{\zeta}(\tau) + \zeta(\tau)^T A(\tau) \leq 0
$$

and

$$
\zeta(0)^T J - \zeta(\theta)^T + \varepsilon 1^T \leq 0
$$

hold for all $\tau \in [0,T]$ and all $\theta \in [T_{\text{min}}, T_{\text{max}}]$. Moreover, when one of the above statements holds, then the positive impulsive system (1) is asymptotically stable under range dwell-time $[T_{\text{min}}, T_{\text{max}}]$. △

Proof: The proof of this result is similar to the one of the corresponding result for LTI positive system in [6]. However, since the system is not time-invariant, we have to use state-transition matrices instead of matrix exponentials to prove the equivalence as in [18]. Due to space limitations, this result is not proved here.

The condition stated in the statement (b) in the above result forms an infinite-dimensional linear program that cannot be solved per se. However, relaxed conditions taking the form of finite-dimensional semidefinite/linear programs can be obtained by restricting $\zeta(\tau)$ to be a vector-valued polynomial. Note that in this case, the derivative $\dot{\zeta}(\tau)$, which is also polynomial, is immediate to get by differentiation. This procedure will be explained in more details in Section 4.

### 2.2 Stability under minimum dwell-time

The following result provides a sufficient condition for the stability of the system (1) under a minimum dwell-time constraint; i.e. $T_k \geq \bar{T}$, $k \geq 0$, for some given $\bar{T} > 0$. It is an extension of the minimum dwell-time result derived in [6].

**Theorem 3** Let us consider the system (1) with $w_c \equiv 0$, $w_d \equiv 0$, $A(\tau) = A(\bar{T})$ for all $\tau \geq \bar{T} > 0$, where $\bar{T} > 0$ is given, and assume that it is state positive. Then, the following statements are equivalent:

(a) There exists a vector $\lambda \in \mathbb{R}_{>0}^n$ such that

$$
\lambda^T A(\bar{T}) < 0
$$

and

$$
\lambda^T (J\Phi(\bar{T}) - I_n) < 0
$$

hold where

$$
\dot{\Phi}(s) = A(s)\Phi(s), \quad \Phi(0) = I_n, s \in [0,\bar{T}].
$$

(b) There exist a differentiable vector-valued $\zeta : [0,\bar{T}] \mapsto \mathbb{R}^n$, $\zeta(\bar{T}) > 0$, and a scalar $\varepsilon > 0$ such that the conditions

$$
\zeta(\bar{T})^T A(\bar{T}) < 0
$$

and

$$
-\dot{\zeta}(\bar{T}) + \zeta(\bar{T})^T A(\bar{T}) \leq 0
$$

and

$$
\zeta(\bar{T})^T J - \zeta(0)^T + \varepsilon 1^T \leq 0
$$

4
hold for all \( \tau \in [0, \bar{T}] \).

Moreover, when one of the above statements holds, then the positive impulsive system (1) is asymptotically stable under minimum dwell-time \( \bar{T} \).

**Proof:** For the same reasons as for Theorem 6, the proof is omitted. \( \triangle \)

### 2.3 Extensions to systems with inputs

It seems important to clarify the fact that under the assumption of the asymptotic stability of the system without input, we have that the state of the system remains bounded provided that the inputs are bounded. Such a result is stated below where only the range dwell-time case is considered for brevity:

**Proposition 4** Assume that the linear impulsive system (1) is state positive and that the conditions of Theorem 6 hold for some \( 0 < T_{\text{min}} \leq T_{\text{max}} < \infty \). Then, the system (1) is input-to-state stable under range dwell-time \([T_{\text{min}}, T_{\text{max}}]\).

**Proof:** Assume that the conditions of statement (a) of the result on range dwell-time (Theorem 6) are met. Then, there exists an \( \epsilon \in (0, 1) \) such that \( \lambda^T (J e^{A T_k} - I_n) \leq (\epsilon - 1) \lambda^T \) for all \( T_k \in [T_{\text{min}}, T_{\text{max}}] \). Letting then \( V(x) = \lambda^T x \) and \( \Psi(t, s) = \Phi(t) \Phi(s)^{-1} \), we get from (13) that

\[
V(t_{k+1}^+) - V(t_k^+) = \lambda^T (J \Phi(T_k) - I_n) x(t_k^+) + \lambda^T E_d w_d(k + 1) + \lambda^T \int_{T_k}^{T_k} J \Psi(T_k, s) E_c(s) w(t_k + s) ds 
\leq (\epsilon - 1)V(t_k^+) + \mu
\]

where \( \mu := \lambda^T (E_d 1 \|w_d\|_{\ell_{\infty}} + JM 1 \|w_c\|_{L_{\infty}}) \) and \( M \) is such that

\[
\int_{0}^{T_k} \Psi(T_k, s) E_c(s) ds \leq M
\]

for all \( T_k \in [T_{\text{min}}, T_{\text{max}}] \). This then leads to

\[
V(t_{k}^+) \leq (1 - \epsilon)^k V(0) + \frac{1 - \epsilon^k}{1 - \epsilon} \mu
\]

and hence that

\[
\limsup_{k \to \infty} V(t_k^+) \leq \frac{\mu}{1 - \epsilon} < \infty.
\]

The boundedness of \( V(x(t)) \) for all \( t \geq 0 \), then simply follows from an application of the Bellman-Grönwall lemma. The proof is completed. \( \diamond \)
3 Main results

Let us consider the following class of linear impulsive system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + E_c w_c(t), \quad t \notin \{t_k\}_{k \in \mathbb{N}} \\
x(t_k^+) &= Jx(t_k) + E_d w_d(k), \quad k \in \mathbb{N} \\
y_c(t) &= C_c x(t) + F_c w_c(t), \quad t \notin \{t_k\}_{k \in \mathbb{N}} \\
y_d(k) &= C_d x(t_k) + F_d w_d(k), \quad k \in \mathbb{N} \\
x(t_0) &= x_0, \quad t_0 = 0
\end{align*}
\]

(13)

where \(x, x_0 \in \mathbb{R}^n_{\geq 0}, w_c \in \mathbb{R}^{p_c}_{\geq 0}, w_d \in \mathbb{R}^{p_d}_{\geq 0}, y_c \in \mathbb{R}^r_{\geq 0} \) and \(y_d \in \mathbb{R}^{q_d}_{\geq 0} \) are the state of the system, the initial condition, the continuous-time exogenous input, the discrete-time exogenous input, the continuous-time measured output and the discrete-time measured output, respectively. The sequence of impulse instants \(\{t_k\}_{k \in \mathbb{Z}_{\geq 0}} \) is assumed to satisfy the same properties as for the system (1). The input signals are all assumed to be bounded functions and that some bounds are known; i.e. we have \(w_c^-(t) \leq w_c(t) \leq w_c^+(t) \) and \(w_d^-(k) \leq w_d(k) \leq w_d^+(k) \) for all \(t \geq 0 \) and \(k \geq 0 \) and for some known \(w_c^-(t), w_c^+(t), w_d^-(k), w_d^+(k) \).

3.1 Proposed interval observer

We are interested in finding an interval-observer of the form

\[
\begin{align*}
x^\bullet(t) &= Ax^\bullet(t) + E_c w^\bullet_c(t) \\
&\quad + L_c(t)(y_c(t) - C_c x^\bullet(t) - F_c w^\bullet_c(t)) \\
x^\bullet(t_k^+) &= Jx^\bullet(t_k) + E_d w^\bullet_d(t) \\
&\quad + L_d(y_d(k) - C_d x^\bullet(t_k) - F_d w^\bullet_d(t)) \\
x^\bullet(0) &= x^\bullet_0
\end{align*}
\]

(14)

where \(\bullet \in \{-, +\} \). Above, the observer with the superscript “+” is meant to estimate an upper-bound on the state value whereas the observer with the superscript “−” is meant to estimate a lower-bound, i.e. \(x^- (t) \leq x(t) \leq x^+(t) \) for all \(t \geq 0 \) provided that \(x^-_0 \leq x_0 \leq x^+_0 \), \(w_c^-(t) \leq w_c(t) \leq w_c^+(t) \) and \(w_d^-(k) \leq w_d(k) \leq w_d^+(k) \).

The errors dynamics \(e^+(t) := x^+(t) - x(t) \) and \(e^-(t) := x(t) - x^-(t) \) are then described by

\[
\begin{align*}
\dot{e}^\bullet(t) &= (A - L_c(t)C_c)e^\bullet(t) + (E_c - L_c(t)F_c)\delta^\bullet_c(t) \\
\dot{e}(t_k^+) &= (J - L_dC_d)e^\bullet(t_k) + (E_d - L_dF_d)\delta^\bullet_d(k)
\end{align*}
\]

(15)

where \(\bullet \in \{-, +\}, \delta^\bullet_c(t) := w_c^+(t) - w_c(t) \in \mathbb{R}^{p_c}_{\geq 0}, \delta^\bullet_c(t) := w_c(t) - w_c^-(t) \in \mathbb{R}^{p_c}_{\geq 0}, \delta^\bullet_d(k) := w_d^+(k) - w_d(k) \in \mathbb{R}^{p_d}_{\geq 0} \) and \(\delta^\bullet_d(k) := w_d(k) - w_d^-(k) \in \mathbb{R}^{q_d}_{\geq 0} \). Note that both errors have exactly the same dynamics and, consequently, it is unnecessary here to consider different observer gains. Note that this would not be the case if the observers were coupled in a non-symmetric way.
3.2 Range dwell-time result

In the range-dwell-time case, the time-varying gain \( L_c(t) \) in (14) is defined as follows

\[
L_c(t) = \tilde{L}_c(t - t_k), \quad t \in (t_k, t_{k+1}]
\]  

where \( \tilde{L}_c : [0, T_{max}] \mapsto \mathbb{R}^{n \times q_c} \) is a matrix-valued function to be determined. The rationale for considering such structure is to allow for the derivation of convex design conditions. The observation problem is defined, in this case, as follows:

**Problem 5** Find an interval observer of the form (14) (i.e. a matrix-valued function \( L_c(\cdot) \) of the form (16) and a matrix \( L_d \in \mathbb{R}^{n \times q_d} \)) such that the error dynamics (15) is

(a) state-positive, that is

\[
A - L_c(\tau)C_c \text{ is Metzler for all } \tau \in [0, T_{max}],
\]

\[
E_c - L_c(\tau)F_c \text{ is nonnegative for all } \tau \in [0, T_{max}],
\]

\[
J - L_dC_d \text{ and } E_d - L_dF_d \text{ are nonnegative}; \text{ and}
\]

(b) asymptotically stable under range dwell-time \([T_{min}, T_{max}]\) when \( w_c \equiv 0 \) and \( w_d \equiv 0 \).

Note that by virtue of Proposition 4, the property (b) implies that the error dynamics (15) will be stable and positive for any bounded \( w_c \) and \( w_d \) satisfying the conditions below (15).

The following result provides a sufficient condition for the solvability of Problem 5:

**Theorem 6** Assume that there exist a differentiable matrix-valued function \( X : [0, T_{max}] \mapsto \mathbb{D}^n, X(0) > 0 \), a matrix-valued function \( U_c : [0, T_{max}] \mapsto \mathbb{R}^{n \times q_c} \), a matrix \( U_d \in \mathbb{R}^{n \times q_d} \) and scalars \( \varepsilon, \alpha > 0 \) such that the conditions

\[
X(\tau)A - U_c(\tau)C_c + \alpha I_n \geq 0 \]  

(17a)

\[
X(0)J - U_dC_d \geq 0 \]  

(17b)

\[
X(\tau)E_c - U_c(\tau)F_c \geq 0 \]  

(17c)

\[
X(0)E_d - U_dF_d \geq 0 \]  

(17d)

and

\[
1_T n \left[ \dot{X}(\tau) + X(\tau)A - U_c(\tau)C_c \right] \leq 0 \]  

(18a)

\[
1_T n \left[ X(0)J - U_dC_d - X(\theta) + \varepsilon I \right] \leq 0 \]  

(18b)

hold for all \( \tau \in [0, T_{max}] \) and all \( \theta \in [T_{min}, T_{max}] \). Then, there exists an interval observer of the form (14)-(16) that solves Problem 5 and suitable observer gains are given by

\[
\tilde{L}_c(\tau) = X(\tau)^{-1}U_c(\tau) \quad \text{and} \quad L_d = X(0)^{-1}U_d.
\]  

(19)
Proof: From the diagonal structure of the matrix-valued function $X(\cdot)$ and the changes of variables (19), the inequalities (17a) to (17d) are readily equivalent to saying that the statement (a) of Problem 5 holds. Using now the changes of variables $\lambda(\tau) = X(\tau)I_n$ and (19), we get that the feasibility of (18a)-(18b) is equivalent to saying that the error dynamics (15) with (16) verifies the range dwell-time conditions of Theorem 2 with the same $\lambda(\tau)$. ♦

3.3 Minimum dwell-time result

In the minimum dwell-time case, the time-varying gain $L_c$ is defined as follows

$$L_c(t) = \begin{cases} \tilde{L}_c(t - t_k) & \text{if } t \in (t_k, t_k + \tau] \\ L_c(T) & \text{if } t \in (t_k + T, t_{k+1}] \end{cases}$$

where $\tilde{L}_c : \mathbb{R}_{>0} \mapsto \mathbb{R}^{n \times q_c}$ is a function to be determined. As in the range dwell-time case, the structure is chosen to facilitate the derivation of convex design conditions. The observation problem is defined, in this case, as follows:

Problem 7 Find an interval observer of the form (14) (i.e., a matrix-valued function $L_c(\cdot)$ of the form (20) and a matrix $L_d \in \mathbb{R}^{n \times q_d}$) such that the error dynamics (15) is

(a) state-positive, that is

- $A - L_c(\tau)C_c$ is Metzler for all $\tau \in [0, T]$,  
- $E_c - L_c(\tau)F_c$ is nonnegative for all $\tau \in [0, T]$,  
- $J - L_dC_d$ and $E_d - L_dF_d$ are nonnegative; and

(b) asymptotically stable under minimum dwell-time $T$ when $w_c \equiv 0$ and $w_d \equiv 0$.

The following result provides a sufficient condition for the solvability of Problem 7:

Theorem 8 There exists a differentiable matrix-valued function $X : [0, T] \mapsto \mathbb{D}^n$, $X(T) \succ 0$, a matrix-valued function $U_c : [0, T] \mapsto \mathbb{R}^{n \times q_c}$, a matrix $U_d \in \mathbb{R}^{n \times q_d}$ and scalars $\varepsilon, \alpha > 0$ such that the conditions

$$X(\tau)A - U_c(\tau)C_c + \alpha I_n \geq 0$$
$$X(T)J - U_dC_d \geq 0$$
$$X(\tau)E_c - U_c(\tau)F_c \geq 0$$
$$X(T)E_d - U_dF_d \geq 0$$

and

$$1_T^T \left[ X(T)A - U_c(T)C_c + \varepsilon I_n \right] \leq 0$$
$$1_T^T \left[ -X(\tau) + X(\tau)A - U_c(\tau)C_c \right] \leq 0$$
$$1_T^T \left[ X(T)J - U_dC_d - X(0) + \varepsilon I \right] \leq 0$$

8
hold for all $\tau \in [0, T]$. Then, there exists an interval observer of the form (14)-(20) that solves Problem 7 and suitable observer gains are given by

$$
\tilde{L}_c(\tau) = X(\tau)^{-1}U_c(\tau) \quad \text{and} \quad L_d = X(T)^{-1}U_d.
$$

Proof: The proof is similar to the one of Theorem 6 and is omitted. \(\Diamond\)

4 Computational considerations

Several methods can be used to check the conditions stated in Theorem 6 and Theorem 8. We opt here for an approach based on Putinar’s Positivstellensatz [22] and semidefinite programming [23]. Before stating the main result of the section, we need to define first some terminology. A multivariate polynomial $p(x)$ is said to be a sum-of-squares (SOS) polynomial if it can be written as $p(x) = \sum_i q_i(x)^2$ for some polynomials $q_i(x)$. A polynomial matrix $p(x) \in \mathbb{R}^{n \times m}$ is said to be componentwise sum-of-squares (CSOS) if each of its entries is an SOS polynomial. Checking whether a polynomial is SOS can be exactly cast as a semidefinite program [23] that can be easily solved using semidefinite programming solvers such as SeDuMi [25]. The package SOSTOOLS [24] can be used to formulate and solve SOS programs in a convenient way.

Below is the SOS implementation of the conditions of statement (b) of Theorem 6:

**Proposition 9** Let $d \in \mathbb{N}$, $\varepsilon > 0$ and $\epsilon > 0$ be given and assume that there exist polynomials $\chi_i : \mathbb{R} \mapsto \mathbb{R}$, $i = 1, \ldots, n$, $U_c : \mathbb{R} \mapsto \mathbb{R}^{n \times q_c}$, $\Gamma_1 : \mathbb{R} \mapsto \mathbb{R}^{n \times n}$, $\Gamma_2 : \mathbb{R} \mapsto \mathbb{R}^{n \times q_c}$ and $\gamma_1, \gamma_2 : \mathbb{R} \mapsto \mathbb{R}^n$ of degree $2d$, a matrix $U_d \in \mathbb{R}^{n \times q_d}$ and a scalar $\alpha \geq 0$ such that

(a) $\Gamma_i(\tau), \gamma_i(\tau), i = 1, 2$ are CSOS,

(b) $X(0) - \epsilon I_n \geq 0$ (or is CSOS),

(c) $X(\tau) A - U_c(\tau) C_c + \alpha I_n - \Gamma_1(\tau) f(\tau)$ is CSOS,

(d) $X(0) J - U_d C_d \geq 0$ (or is CSOS),

(e) $X(\tau) E_c - U_c(\tau) F_c - \Gamma_2(\tau) f(\tau)$ is CSOS,

(f) $X(0) E_d - U_d F_d \geq 0$ (or is CSOS),

(g) $-1_n^T \left[ \dot{X}(\tau) + X(\tau) A - U_c(\tau) C_c \right] - f(\tau) \gamma_1(\tau)^T$

is CSOS,

(h) $-1_n^T [X(0) J - U_d C_d - X(\theta) + \varepsilon I] - g(\theta) \gamma_2(\theta)^T$

is CSOS.

\(^1\)See [6] for a comparison of all these methods.
where \( X(\tau) := \text{diag}_{i=1}^{n}(\chi_{i}(\tau)) \), \( f(\tau) := \tau(T_{\text{max}} - \tau) \) and \( g(\theta) := (\theta - T_{\text{min}})(T_{\text{max}} - \theta) \).

Then, the conditions of statement (b) of Theorem 6 hold with the same \( X(\tau), U_{c}(\tau), U_{d}, \alpha \) and \( \varepsilon \).

Proof: The proof follows from the same arguments as the proof of Proposition 3.15 in [6]. ♦

The conditions stated in the above results can be readily implemented using SOSTOOLS [24]. However, it seems important to explain the meaning of those conditions. Clearly, the condition (a) implies that \( \Gamma_{i}(\tau) \geq 0, \gamma_{i}(\tau) \geq 0 \), for all \( i = 1, 2 \) and all \( \tau \in \mathbb{R} \). The conditions (b), (d) and (f) are here to indicate that the corresponding expressions are nonnegative and are equivalent to the conditions \( X(0) \succ 0, (17b) \) and (17d), respectively. The condition (c) implies that \( X(\tau)A - U_{c}(\tau)C_{c} + \alpha I_{n} - \Gamma_{1}(\tau)f(\tau) \geq 0 \) for all \( \tau \in \mathbb{R} \). This equivalent to saying that \( X(\tau)A - U_{c}(\tau)C_{c} + \alpha I_{n} \geq \Gamma_{1}(\tau)f(\tau) \) for all \( \tau \in \mathbb{R} \) and, hence, that \( X(\tau)A - U_{c}(\tau)C_{c} + \alpha I_{n} \geq 0 \) for all \( \tau \in [0, T_{\text{max}}] \), which coincides with the condition (17a). The other conditions can analyzed in the same way.

Remark 10 (Asymptotic exactness) The above relaxation is asymptotically exact under very mild conditions [22] in the sense that if the original conditions of Theorem 6 hold then we can find a degree \( d \) for which the conditions in Proposition 9 are feasible. See [6] for more details.

5 Examples

5.1 An impulsive system

Let us consider here the example from [3] to which we add disturbances as also done in [13]. The matrices of the system are given by

\[
A = \begin{bmatrix}
-1 & 0 \\
1 & -2
\end{bmatrix},
E_{c} = \begin{bmatrix}
0.1 \\
0.1
\end{bmatrix},
J = \begin{bmatrix}
2 & 1 \\
1 & 3
\end{bmatrix},
E_{d} = \begin{bmatrix}
0.3 \\
0.3
\end{bmatrix},
C_{c} = C_{d} = \begin{bmatrix}
0 & 1
\end{bmatrix},
F_{c} = F_{d} = 0.03.
\]

Define also \( w_{c}(t) = \sin(t), w^{-}(t) = -1, w^{+}(t) = 1, w_{d}(k) \) is a stationary random process that follows the uniform distribution \( U(-0.5, 0.5), w_{d}^{-} = -0.5 \) and \( w_{d}^{+} = 0.5 \). Letting a desired minimum dwell-time of \( T = 0.7 \) and solving the conditions of Theorem 8 with polynomials of degree 4 yield the observer gains

\[
L_{d} = \begin{bmatrix}
0.9977 \\
1.6460
\end{bmatrix}
\text{and } L_{c}(\tau) = \begin{bmatrix}
n_{1}(\tau)d_{1}(\tau)^{-1} \\
n_{2}(\tau)d_{2}(\tau)^{-1}
\end{bmatrix}
\]

(25)
Figure 1: Trajectories of the system (13)-(24) and the interval observer (14)-(20)-(25) for some randomly chosen impulse times satisfying the minimum dwell-time $T = 0.7$.

where

\[
\begin{align*}
n_1(\tau) &= 0.3064\tau^4 - 0.4410\tau^3 + 0.2132\tau^2 \\
&\quad -0.0409\tau + 0.0043 \\
d_1(\tau) &= 0.1999\tau^4 - 0.0447\tau^3 - 1.0739\tau^2 \\
&\quad +2.6471\tau - 2.7157 \\
n_2(\tau) &= -0.5400\tau^4 + 0.6047\tau^3 - 0.0939\tau^2 \\
&\quad -0.0771\tau + 0.0251 \\
d_2(\tau) &= 0.0633\tau^4 - 0.1169\tau^3 + 0.0868\tau^2 \\
&\quad -0.0359\tau^1 + 0.0101.
\end{align*}
\]  

(26)

For information, the semidefinite program has 242 primal variables, 76 dual variables and it takes 2.18 seconds to solve on an i7-2620M with 8GB of RAM. To illustrate this result, we generate random impulse times satisfying the minimum dwell-time condition and we obtain the trajectories depicted in Fig. 1 where we can observe the ability of the interval observer to properly frame the trajectory of the system.

5.2 A sampled-data system

Let us consider now the sampled-data system

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{u}(t) \\
x(t_k^+) \\
u(t_k^+)
\end{bmatrix} =
\begin{bmatrix}
A & B \\
0 & 0 \\
I & 0 \\
K_1C_y & K_2
\end{bmatrix}
\begin{bmatrix}
x(t) \\
u(t) \\
x(t_k) \\
u(t_k)
\end{bmatrix} +
\begin{bmatrix}
E \\
0 \\
0 \\
K_1F_y
\end{bmatrix}
\begin{bmatrix}
w(t) \\
0
\end{bmatrix}
\]

(27)
which incorporates in its formulation the sampled-data static-output feedback control law
\[ u(t) = K_1(C_y x(t_k) + F_y w(t_k)) + K_2 u(t_k), \quad t \in (t_k, t_{k+1}] \]
The goal would be the design of an impulsive interval observer for this system and, to this aim, we consider the measured output
\[ y = \text{diag}(C_y x + F_y w_c, u) \]
Note that we have here \( y_c(t) = y(t), \ y_d(k) = y(t_k), \ w_c(t) = w(t) \) and \( w_d(k) = w(t_k) \). We then propose the observer (14) with the matrices
\[
L_c(t) = \begin{bmatrix} L^1_c(t) & B \\ 0 & L^2_c(t) \end{bmatrix} \quad \text{and} \quad L_d = \begin{bmatrix} L^1_d & 0 \\ K_1 & K_2 \end{bmatrix}. \tag{28}
\]
Note that this observer contains a continuous-time component which may be contradictory with the fact we are considering a sampled-data system. However, if the observer is sampled at a much higher frequency than the controller, this approximation is, in general, satisfying.

The dynamics of the observation error is given, in this case, by
\[
\begin{bmatrix}
\dot{e}^c_e(t) \\
\dot{e}^c_u(t)
\end{bmatrix} = \begin{bmatrix}
A - L^1_c(t) C_y & 0 \\
0 & -L^2_c(t)
\end{bmatrix}\begin{bmatrix}
e^c_e(t) \\
e^c_u(t)
\end{bmatrix} + \begin{bmatrix}
E - L^1_c(t) F_y \\
0
\end{bmatrix}\delta_c(t) \tag{29}
\]
\[
\begin{bmatrix}
e^c_e(t^+_k) \\
e^c_u(t^+_k)
\end{bmatrix} = \begin{bmatrix}
I - L^1_d C_y & 0 \\
0 & 0
\end{bmatrix}\begin{bmatrix}
e^c_e(t_k) \\
e^c_u(t_k)
\end{bmatrix}
\]
where we can see that with this observer gains, the dynamics of the errors are fully decoupled. Consequently, it is enough to choose \( L^2_c(t) \) to be constant, diagonal and large enough. The gains \( L^1_c(t) \) and \( L^1_d \) can then be designed exactly in the same way as in the previous example.

### 5.3 A switched system

Let us consider here the switched system
\[
\begin{align*}
\dot{x}(t) &= \tilde{A}_{\sigma(t)} x(t) + \tilde{E}_{\sigma(t)} w(t) \\
\tilde{y}(t) &= \tilde{C}_{\sigma(t)} x(t) + \tilde{F}_{\sigma(t)} w(t)
\end{align*} \tag{30}
\]
where \( \sigma : \mathbb{R}_{\geq 0} \mapsto \{1, \ldots, N\} \) is the switching signal, \( x \in \mathbb{R}^n \) is the state of the system, \( \tilde{w} \in \mathbb{R}^p \) is the exogenous input and \( \tilde{y} \in \mathbb{R}^p \) is the measured output.

This system can be rewritten into the following impulsive system with multiple jump maps \[6\]
\[
\begin{align*}
\dot{x}(t) &= \text{diag}_{i=1}^N(\tilde{A}_i) x(t) + \text{col}_{i=1}^N(\tilde{E}_i) w(t) \\
y(t) &= \text{diag}_{i=1}^N(\tilde{C}_i) x(t) + \text{col}_{i=1}^N(\tilde{F}_i) w(t) \\
x(t^+_k) &= J_{ij} x(t_k), \quad i, j = 1, \ldots, N, \ i \neq j
\end{align*} \tag{31}
\]
where \( J_{ij} := (b_i b_j^T) \otimes I_n \) and \( \{b_1, \ldots, b_N\} \) is the standard basis for \( \mathbb{R}^N \). Because of the particular structure of the system, we can define w.l.o.g. an interval observer of the form (14) for the system (31) with the gains \( L_c(t) = \text{diag}_{i=1}^N(L^1_c(t)) \) and \( L^2_d = (b_i b_j^T) \otimes \tilde{L}^2_d \). The
error dynamics is then given in this case by

\[
\begin{align*}
\dot{e}^\star(t) &= \text{diag}_{i=1}^N(\bar{A}_i - L_i^s(t)\bar{C}_i)e(t) \\
&\quad + \text{col}_{i=1}^N(\bar{E}_i - L_i^s(t)\bar{F}_i)\delta(t) \\
\delta^\star(t_k) &= \left[(b_ib_j^T) \otimes I_n - \bar{L}_d^s ij\bar{C}_j\right]e(t_k) \\
&\quad - \left[(b_ib_j^T) \otimes (\bar{L}_d^s ij\bar{F}_j)\right]\delta(t_k).
\end{align*}
\]

(32)

Once again, the gains of the observer can be designed as in the previous examples. Note, however, that in this case the stability conditions will be slightly different due to the existence of multiple jump maps. See [6] for more details on how to straightforwardly adapt the conditions to this case.

References

[1] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid Dynamical Systems. Modeling, Stability, and Robustness. Princeton University Press, 2012.

[2] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel. Exponential stability of impulsive systems with application to uncertain sampled-data systems. Systems & Control Letters, 57:378–385, 2008.

[3] C. Briat. Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints. Automatica, 49(11):3449–3457, 2013.

[4] C. Briat. Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time. Systems & Control Letters, 78:63–72, 2015.

[5] J. C. Geromel and M. Souza. On an lmi approach to optimal sampled-data state feedback control design. International Journal of Control, 88(11):2369–2379, 2015.

[6] C. Briat. Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems. Nonlinear Analysis: Hybrid Systems, 24:198–226, 2017.

[7] J. L. Gouzé, A. Rapaport, and M. Z. Hadj-Sadok. Interval observers for uncertain biological systems. Ecological modelling, 133:45–56, 2000.

[8] F. Mazenc and O. Bernard. Interval observers for linear time-invariant systems with disturbances. Automatica, 47:140–147, 2011.

[9] C. Briat and M. Khammash. Interval peak-to-peak observers for continuous- and discrete-time systems with persistent inputs and delays. Automatica, 74:206–213, 2016.
[10] F. Mazenc, M. Kieffer, and E. Walter. Interval observers for continuous-time linear systems. In American Control Conference, pages 1–6, Montréal, Canada, 2012.

[11] M. Bolajraf and M. Ait Rami. A robust estimation approach for uncertain systems with perturbed measurements. International Journal of Robust and Nonlinear Control, 26(4):834–852, 2016.

[12] D. Efimov, W. Perruquet, and J.-P. Richard. On reduced-order interval observers for time-delay systems. In 12th European Control Conference, pages 2116–2121, Zürich, Switzerland, 2013.

[13] K. H. Degue, D. Efimov, and J.-P. Richard. Interval observers for linear impulsive systems. In 10th IFAC Symposium on Nonlinear Control Systems, 2016.

[14] D. Efimov, T. Raïssi, and A. Zolghadri. Control of nonlinear and LPV systems: Interval observer-based framework. IEEE Transactions on Automatic Control, 58(3):773–778, 2013.

[15] F. Mazenc, T. N. Dinh, and S.-I. Niculescu. Robust interval observers and stabilization design for discrete-time systems with input and output. Automatica, 49:3490–3497, 2013.

[16] F. Mazenc and T. N. Dinh. Construction of interval observers for continuous-time systems with discrete measurements. Automatica, 50:2555–2560, 2014.

[17] D. Efimov, E. Fridman, A. Polyakov, W. Perruquet, and J.-P. Richard. On design of interval observers with sampled measurement. Systems & Control Letters, 96:158–164, 2016.

[18] C. Briat. Theoretical and numerical comparisons of looped functionals and clock-dependent Lyapunov functions - The case of periodic and pseudo-periodic systems with impulses. International Journal of Robust and Nonlinear Control, 26:2232–2255, 2016.

[19] L. I. Allerhand and U. Shaked. Robust stability and stabilization of linear switched systems with dwell time. IEEE Transactions on Automatic Control, 56(2):381–386, 2011.

[20] D. Handelman. Representing polynomials by positive linear functions on compact convex polyhedra. Pacific Journal of Mathematics, 132(1):35–62, 1988.

[21] C. Briat. Robust stability and stabilization of uncertain linear positive systems via integral linear constraints - $L_1$- and $L_\infty$-gains characterizations. International Journal of Robust and Nonlinear Control, 23(17):1932–1954, 2013.

[22] M. Putinar. Positive polynomials on compact semi-algebraic sets. Indiana Univ. Math. J., 42(3):969–984, 1993.
[23] P. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, Pasadena, California, 2000.

[24] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. A. Parrilo. *SOSTOOLS: Sum of squares optimization toolbox for MATLAB v3.00*, 2013.

[25] J. F. Sturm. Using SEDUMI 1.02, a Matlab Toolbox for Optimization Over Symmetric Cones. *Optimization Methods and Software*, 11(12):625–653, 2001.