Deconstructing the $E_0$ SCFT to Solve the Orbifold Paradox of the Heterotic M Theory

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Abstract: Many heterotic orbifold models have massless twisted-sector particles with simultaneous $E_8$ and $E_8'$ charges. In the strong-coupling M-theory dual of the heterotic string, this poses a paradox: Since the $E_8$ and $E_8'$ live at opposite ends of the $x^{10}$ dimension, where could a massless particle with both types of charges possibly live? To key to this question are the 5D SCFTs living at the orbifold fixed planes going through the bulk of the M theory. We use dimensional deconstruction to understand how such a 5D SCFT (specifically, the $E_0$ SCFT at the $Z_3$ fixed point) works at the superconformal point (rather that at the Coulomb branch) and how it interacts with the boundaries of the $x^{10}$. We find that the massless twisted states are not localized in the $x^{10}$. Instead, they are non-local meson-like composite particles comprised of a quark living at one boundary of the $x^{10}$, an antiquark living at the other boundary, and the string of strongly-interacting 5D gluons connecting the quark to the antiquark.
1 Introduction and Summary

Heterotic orbifolds [1, 2] are among the oldest and best-known types of string models. Unlike the smooth Calabi–Yau compactifications of the $E_8 \times E_8$ heterotic string, many $T^6/(\text{discrete symmetry } D)$ compactifications have massless particles charged under (unbroken subgroups of) both $E_8$ and $E_8$. Such particles are perfectly normal from the perturbative heterotic string theory point of view, but in the strong-coupling dual of the heterotic string — the Hořava–Witten theory [3, 4] — they pose a paradox. Indeed, the Hořava–Witten theory is M theory whose eleventh dimension is a finite interval with boundaries; the supergravity lives in the 11D bulk, while the $E_8$ and $E_8$ SYM theories live on the 10D boundaries at the opposite ends of the $x^{10}$. In this setup, massless particles with both $E_8$ and $E_8$ charges raise a paradox: where in the $x^{10}$ can they possibly live?
In the heterotic string theory, the particles in question belong to the \textit{twisted sectors} of the world-sheet Hilbert space, where the string does not close on the $\mathbb{R}^{1,3} \times T^6$ but closes modulo the discrete symmetry $D$:

$$X^I(\sigma = 2\pi) = D_I^J X^J(\sigma = 0), \quad D \in D, \quad D \neq 1. \quad (1.1)$$

The massless states in a twisted sector obtain from a string which begins and ends on a \textit{fixed point} of the discrete symmetry $D$. Classically, this allows for an arbitrarily short length of the string loop and hence arbitrarily low energy; this does not guarantee massless particle states in the quantum theory, but many orbifold models do have massless twisted states. Also, if action of the symmetry $D$ breaks both $E_{81}$ and $E_{82}$ gauge symmetry groups, then all the twisted states — massless and massive — are charged under surviving subgroups of both $E_{81}$ and $E_{82}$.

The strong-coupling dual of a heterotic orbifold is M theory on

$$\mathbb{R}^{1,3}_{\text{Minkowski}} \times (T^6/D)_{\text{orbifold}} \times \text{Interval}.$$ 

The massless twisted states are localized in the $T^6/D$ dimensions at the fixed points of $D$, but their locations in the $x^{10}$ dimension is problematic, especially in light of their $E_{81} \times E_{82}$ charges. In the present paper we solve this problem, and the solution turns out to be very different from the 6D $\mathbb{R}^{1,5} \times T^4/D$ orbifolds explored in [5, 6]:

\begin{itemize}
  \item In the 6D orbifolds, the massless twisted states are localized at one end of the $x^{10}$ dimension.
  \item But in the 4D orbifolds, the massless twisted states span the whole $x^{10}$, from one end to the other end.
\end{itemize}

In both cases, the key to the twisted stated is the non-trivial physics on the fixed 7D or 5D planes of the discrete symmetry $D$ in the 11D bulk of the M theory. For the 6D orbifolds, a fixed plane locally looks like $\mathbb{R}^{1,6} \times (\mathbb{C}^2/D)$, which in M theory gives rise to the 7D SYM theory on the $\mathbb{R}^{1,6}$, or rather on the $R^{1,5} \times \text{Interval}$. At one end of the Interval, this 7D gauge theory locks onto an unbroken subgroup of the $E_{81}$ in a kind of inter-dimensional Higgs mechanism,

$$(G_{10D} \subset E_{81}) \times G_{7D} \rightarrow G_{\text{common}}. \quad (1.2)$$

In the perturbative heterotic string, this $G_{\text{common}}$ gauge symmetry appears to be a subgroup of the $E_{81}$, but in the M theory it lives on both the 10D boundary at $x^{10} = 0$ and the 7D fixed planes, and along those fixed planes it reaches all the way to the other end of the $x^{10}$ where it meets the $E_{82}$ gauge group (or rather its surviving subgroups).
Thanks to this meeting of gauge symmetries, the 6D hypermultiplets localized at the intersections of the fixed planes with the second end of the $x^{10}$ may carry both the $G_{\text{common}}$ and the $E_8$ charges — which in the heterotic limit looks like the simultaneous $G_{10D} \subset E_8$ and $E_8$ charges.

For an example, consider the $T^4/\mathbb{Z}_2$ orbifold in which the $E_8$ is broken down to $E_7 \times SU_2$ while the $E_8$ is broken down to $SO_{16}$. This orbifold has 16 fixed points on the $T^4$, each fixed point giving rise to massless half-hypermultiplets in the $(1, 2; 16)$ multiplet of the $E_7 \times SU_2 \times SO_{16}$ gauge group. The simultaneous $SU_2 \subset E_8$ and $SO_{16} \subset E_8$ quantum numbers are best explained pictorially, see figure 1.1:

The massless 6D twisted states live at the intersections of the 7D fixed planes of the orbifold with the 10D right boundary of the 11D bulk of M theory. Their gauge quantum numbers are 16 of the $SO_{16} \subset E_8$ living on the 10D right boundary of the $x^{10}$ and 2 of the $SU_2$ living on the 7D fixed plane; both gauge symmetries are present at the intersection, so the simultaneous $(2, 16)$ quantum numbers are OK. Note that locally — where the massless twisted states live — the $SU_2$ is the 7D gauge theory on the fixed plane rather than the $SU_{10D} \subset E_8$. However, at the other (left) end of the $x^{10}$, the 7D $SU_2$ locks onto the unbroken $SU_2$ subgroup of the 10D $E_8$, so globally we have a common $SU_2$ which lives both on the 10D left boundary of the $x^{10}$ and on all the 7D fixed planes; the massless twisted states end up being doublets of this common $SU_2$. From the heterotic string point of view, this common $SU_2$ appears to be a subgroup of the $E_8$, but in M theory it is not, and that’s what resolves the paradox of the simultaneous $(2, 16)$ charges of the massless twisted states!
For the 4D orbifolds, the situation is more complicated because the fixed planes — which locally look like $\mathbb{R}^{1,4} \times (C^3/D)$ in the bulk of M theory — carry 5D superconformal theories rather than super–Yang–Mills. For example, the $C^3/Z_3$ fixed planes of the $T^6/Z_3$ orbifold give rise to the $E_0$ superconformal theories.\footnote{The $E_0$ is one of the Morrison–Seiberg $E_n$ superconformal 5D theories \cite{Morrison:1996xf}. In M theory, the $E_n$ SCFTs obtain from Calabi–Yau singularities where a del-Pezzo surface collapses to a point. The $E_n$'s other than the $E_0$ also obtain in the infinite gauge coupling limit of the 5D $SU2$ gauge theories with $n − 1$ massless flavors. However, the $E_0$ theory is isolated and does not appear in the $g \to \infty$ limit of any gauge theory.} The $E_0$ theory is poorly understood: All we know is its moduli space — which is limited to a 1D Coulomb branch where the $E_0$ reduces to a $U(1)$ SYM with Chern–Simons level $k = 9$ — and the connection to the moduli spaces of other 5D theories. In this paper, we use the flop transition between the Coulomb branches of the $E_0$ and the $SU2$ SYM to deconstruct \cite{Benvenuti:2006qr} the fifth dimension of the $E_0$ theory. In other words, we latticize the $x^{10}$ dimension of the 5D fixed plane, and realize the $E_0$ theory as a long quiver

\begin{equation}
\begin{array}{cccccccc}
\cdots & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \cdots \\
\end{array}
\end{equation}

of strongly-coupled four-dimensional $SU(2)$ gauge theories. In the Hořava–Witten orbifold context, the quiver (1.3) should have large but finite length (corresponding to the finite length of the $x^{10}$ dimension), and there probably should be some extra fields at the two ends of the quiver corresponding to the 4D fields at the intersections of the 5D fixed plane with the 10D boundaries at the ends of the $x^{10}$.

In this paper, we focus on a particular $T^6/Z_3$ orbifold model where the $E8_1$ is broken down to the $E6_1 \times SU3_1$ while the $E8_2$ is also broken down to the $E6_2 \times SU3_2$. The $T^6/Z_3$ has 27 fixed points (but no fixed tori), and in this model each fixed point gives rise to 9 massless chiral multiplet in the $(1, 3; 1, \bar{3})$ multiplet of the unbroken gauge symmetry, that is, the bi-fundamental multiplet of the two $SU3$ subgroups, one from each $E8$. The chiral anomaly of these twisted-sector states cancels against the chiral states in the un-twisted sector, and this cancellation assures that the massless twisted states stay exactly massless despite any quantum corrections when the heterotic string becomes strongly coupled. Therefore, the bi-fundamental $(3, \bar{3})$ massless particles must exist in the Hořava–Witten regime of the orbifold, so where in the $x^{10}$ do these particles live?

To our surprise, we found that these particles are not localized at any particular place in $x^{10}$ but spread out over the entire $x^{10}$ dimension, from one boundary to another.
Specifically, the twisted states are meson-like composite particles comprising a quark at one end of the $x^{10}$, an antiquark at the other end, and a whole string of 5D gluons connecting the quark to the antiquark across the whole length of the $x^{10}$, as shown on figure 1.2 below:

| Left end of the $x^{10}$ | Right end of the $x^{10}$ |
|--------------------------|--------------------------|
| $E_8 \rightarrow E_6 \times SU_3$ | $E_8 \rightarrow E_6 \times SU_3$ |
| 4D quark in $(\mathbf{q}, \mathbf{3})$ of colour $\times SU_3$ | 4D antiquark in $(\bar{\mathbf{q}}, \bar{\mathbf{3}})$ of colour $\times SU_3$ |
| $E_0$ on a 5D fixed plane | $E_0$ on a 5D fixed plane |
| 5D gluons | 5D gluons |

**Figure 1.2**: Twisted states of a 4D $T^6/Z_3$ orbifold

To be precise, in the deconstructed $E_0$ theory on the fixed plane, the twisted $(\mathbf{3}, \bar{\mathbf{3}})$ are composites

$$
\mathcal{M}_{ij} = Q_i \Omega_1 \Omega_2 \cdots \Omega_{N-1} \tilde{Q}_j
$$

of the quark $Q_i$ at one end of the quiver, the antiquark $\tilde{Q}_j$ at the other end, as well as every bifundamental field $\Omega_\ell$ of the quiver. In the continuum limit, the bifundamental fields becomes components of the 5D gluons and their superpartners. or at least they become 5D gluons in a 5D gauge theory. We are not quite sure what exactly do they become in a superconformal theory like $E_0$, but we call them ‘gluons’ simply because we do not have a better name.

The rest of this paper is the explanation and the justification of the above picture of the massless twisted states of the $T^6/Z_3$ orbifold. Sections 2 and 3 are introductory: In section 2 we describe our orbifold model from the heterotic string point of view. We also consider what happens when a fixed point is ‘blown up’ from both geometrical and field-theoretical points of view. In section 3 we discuss the $E_0$ SCFT at the 5D fixed plains in the M-theory dual of the orbifold. In §3.1 we focus on the moduli space of $E_0$ theory in the infinite $4 + 1$ dimensions and the flop transitions between the moduli spaces of the $E_0$ and the $SU_2$ SYM, while in §3.2 we focus on the boundaries of the $x^{10}$ (which acts as the fifth dimension of the $E_0$) Basically, we summarize the results of Ganor & Sonnenschein results [8] about the Coulomb branch of the $E_0$ living on the blown-up fixed plane.
In section 4 we dimensionally deconstruct the $E_0$ theory and explore the deconstructed theory from the 4D point of view; the §4 is the core of this paper. In §4.1 we deconstruct the $E_0$ in infinite $4 + 1$ dimensions: We start by deconstructing the 5D $SU2$ SYM, then move along the Coulomb branch of the moduli space across the flop transition to the Coulomb branch of the $E_0$ theory, and eventually reach the superconformal point of the $E_0$. We note that in the long quiver limit, there is an abrupt transition between the semiclassical Higgs regime and the strongly-coupled confinement regime of the infrared dynamics; these regimes correspond respectively to the Coulomb branch and the SCFT point of the 5D $E_0$ theory. In §4.2 we deconstruct the ‘quarks’ and the ‘antiquarks’ at the boundaries of the $E_0$ theory in the $\mathbb{Z}_3$ orbifold context, and we check that the Higgs regime of the resulting quiver theory agrees with the Ganor–Sonnenschein description of the $E_0$ Coulomb branch at the blown-up fixed point of the orbifold. In §4.3 we focus on the confinement regime of the quiver theory. We show that the $SU(3)_1 \times SU(3)_2$ flavor symmetry of the quiver is not broken in this regime; instead, there are nine massless meson-like composite particles (1.4) which deconstruct the massless twisted states at the un-blown fixed point. And in §4.4 we focus on the transition between the confinement and the Higgs regimes of the deconstructed theory; in orbifold terms, this transition corresponds to beginning to blow up the fixed point. We show how the slow increase of the scalar VEVs in the quiver theory leads to the abrupt change of masses of the 4 out of 9 twisted states, from very light to very heavy. This explain how these 4 states — which are exactly massless at the on-blown fixed point and should be light at the early stages of the blow-up — fail to appear in the light spectrum of the Ganor–Sonnenschein description of the blown-up fixed point.

Finally, section 5 lists the open questions we would like to address in the future.

2 The Heterotic Model

The compactification of $E_8 \times E_8$ heterotic string theory on $T^6 / \mathbb{Z}_N$ is defined by two vectors. The first of these, the twist vector $\vec{r}$, defines how the compactified directions are identified under the modding group. It is three complex-dimensional and depends on the particular $\mathbb{Z}_N$ group that the six-torus is modded by. In this work, we are considering the case $N = 3$ with $\vec{r} = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$. Specifically, we begin with the complex three-plane, $\mathbb{C}^3$, and mod by the $SU(3)^3$ root lattice, $\Lambda_{SU(3)^3}$, by making the following identifications for the three complex coordinates $z_i$:

$$z_i \sim z_i + 1, \quad z_i \sim z_i + e^{\frac{2\pi i}{3}}.$$ (2.1)
The result is a specific six-torus, \( T^6 = \frac{\mathbb{C}^4}{\Lambda_{SU(3)^3}} \). This \( T^6 \) is then operated on with a twist parametrized by the twist vector, identifying points as

\[
z_i \rightarrow e^{(2\pi i)r_i} z_i.
\]

In all, there are \( 3^3 = 27 \) fixed points of this twist on the \( T^6 \) corresponding to each of the \( z_i \)'s being either 0, \( \frac{1}{\sqrt{3}}e^{\frac{2\pi}{6}} \), or \( \frac{2}{\sqrt{3}}e^{\frac{2\pi}{6}} \).

The other vector, known as the shift vector \( \vec{s} \), defines the boundary conditions of the gauge fields under the twist above. This vector acts like a Wilson line, breaking the gauge group \( E_8 \times E_8 \) to some appropriate subgroup invariant under the twist. Modular invariance highly constrains the allowed values of \( \vec{s} \) so that there are only five distinct consistent 4D \( \mathcal{N} = 1 \) SQFT’s differentiated by their resulting gauge groups. These SQFT’s are

\[
(0^8; 0^8) : E_8 \times E_8,
\]

\[
\frac{1}{3}(1, 1, -2, 0^5; 0^8) : (E_6 \times SU(3)) \times E_8,
\]

\[
\frac{1}{3}(1, 1, -2, 0^5; 1, 1, -2, 0^5) : (E_6 \times SU(3)) \times (E_6 \times SU(3)),
\]

\[
\frac{1}{3}(1, 1, 0^6; -2, 0^7) : (E_7 \times U(1)) \times (SO(14) \times U(1)),
\]

\[
\frac{1}{3}(1, 1, 1, -2, 0^3; -2, 0^7) : SU(9) \times (SO(14) \times U(1)).
\]

Each of these models has their own unique massless spectrum. For each spectrum, there are untwisted states present everywhere on the orbifold and twisted states that are located at the fixed points. For our purposes, we are interested in the theory where \( E_8 \times E_8 \rightarrow (E_6 \times SU(3)) \times (E_6 \times SU(3)) \), which has untwisted states

\[
3 (3, 27; 1, 1) \oplus 3 (1, 1; 3, 27) \oplus 9 \text{ moduli},
\]

and twisted states,

\[
27 (\overline{3}, 1; \overline{3}, 1),
\]

with one localized at each of the 27 fixed points.

### 2.1 The Moduli Space

Let us consider the twisted states \( T^A_{ij}, A = 1, ..., 27, \ i, j = 1, 2, 3 \) as \( 27 \times 3 \times 3 \) matrices \( T^A \). The moduli space for these fields must satisfy the F-flatness constraints stemming from the superpotential

\[
W(T) = \sum_A \det(T^A).
\]
Specifically, the constraints are
\[
\frac{\partial W}{\partial T^A_{ij}} = \text{minor}(T^A)_{ij} = 0.
\] (2.7)

Note that these constraints do not relate different fixed points to each other. On the other hand, each $3 \times 3$ matrix $T^A_{ij}$ must have zero minors, which means that its rank should be at most 1. Consequently, each $T^A_{ij}$ must be a tensor product of a row vector and a column vector,
\[ T^A_{ij} = u^A_i \otimes (v^A_j)^\top, \quad \text{i.e.,} \quad T^A_{ij} = u^A_i \times v^A_j. \] (2.8)

From the $SU(3)_1 \times SU(3)_2$ point of view, the $u^A \in (3, 1)$ while $v^A \in (1, 3)$. Hence, a non-zero VEV of a $u^A$ triplet breaks $SU(3)_1 \rightarrow SU(2)_1$ while a non-zero VEV of a $v^A$ triplet breaks $SU(3)_2 \rightarrow SU(2)_2$. However, since the $u^A$ and the $v^A$ are not physical fields but only their product (2.8) is physical, the $\langle T^A \rangle$ VEV leaves an extra hypercharge $Y = Y_1 + Y_2$ unbroken. Thus, the overall Higgs effect of a single blown-up fixed point is
\[ SU(3)_1 \times SU(3)_2 \rightarrow SU(2)_1 \times SU(2)_2 \times U(1)_{1+2}. \] (2.9)

Thus far, we have looked at the F-flat directions while ignoring the D-terms due to Higgsed down $SU(3)_1 \times SU(3)_2$. Unlike the F-terms, the D-terms do relate different fixed points to each other and require
\[ \sum_A T^A (T^A)^\dagger = \sum_A (T^A)^\dagger T^A = 1_{3 \times 3} \times (\text{a number}). \] (2.10)

For the rank = 1 matrices $T^A$, this requires simultaneous blowing up of at least 3 fixed points with different gauge-group directions of the $u^A$ and $v^A$ triplets, which together break the $SU(3)_1 \times U(3)_2$ down to nothing.

However, in this paper we are interested in the individual fixed points rather than interconnections between them. Therefore, assume the $T^6$ torus to be very large — formally, we take the radius $\rightarrow \infty$ limit, — so that each fixed point may be treated as an independent $\mathbb{C}^3/\mathbb{Z}_3$ singularity. In 4D field theory terms, the couplings of the $SU(3)_1 \times SU(3)_2$ gauge theories go to zero in the radius $\rightarrow \infty$ limit, so the D-terms in the scalar potential drop to zero, and the constraints (2.10) go away. Also, in the zero gauge coupling limit, the $SU(3)_1 \times SU(3)_2$ can be thought as a flavor symmetry of each $\mathbb{C}^3/\mathbb{Z}_3$ fixed point rather than a gauge symmetry.
2.2 Geometry

Instead of compactifying the heterotic string theory and looking at the massless spectrum, we can attempt to go to the low energy 10D heterotic supergravity theory first, compactify it on the orbifold, and then look at the resulting effective 4D theory. This, however, is difficult because the orbifold is not a manifold and the geometric description at the fixed points is not well-defined. The procedure, then, would be to blow these fixed points up, compactify the heterotic supergravity theory on the resulting Calabi-Yau threefold and then study the blow-down limit.

In general, the fixed point of $\mathbb{C}^n/\mathbb{Z}_n$ can be smoothed out by removing the fixed point and replacing it with a $\mathbb{CP}^{n-1}$. In the limit that this $\mathbb{CP}^{n-1}$ shrinks to zero size, the fixed point is restored. From the field theory perspective, the resolution of this fixed point is controlled by the moduli of the theory; giving a VEV to a scalar field along a flat direction in the moduli space will result in a deformation of the Kähler structure and a smoothing of the corresponding singularities.

3 Hořava–Witten Theory

Similar to type IIA superstring, the strong string coupling limit of the $E_8 \times E_8$ heterotic string is dual to the M-theory with a compact eleventh dimension, but in the heterotic case the $x^{10}$ is compactified on a finite interval with boundaries [3, 4]. The local anomaly cancellation at each boundary requires a 10D $E8$ gauge theory localized on that boundary, hence the two boundaries of the $x^{10}$ dimension give rise to the $E8_1 \times E8_2$ gauge symmetry of the heterotic string theory. In the bulk between the two boundaries, the M-theory included the 11D supergravity; its zero modes on the finite $x^{10}$ interval give rise to the $d=10, \mathcal{N}=1$ supergravity of the heterotic string.

Equipped with this, we see that another possible description of the orbifold theory would be to first compactify M-theory on the resolved orbifold. The resulting 5D theory could then be compactified on the interval, where the singular orbifold limit theory should match that of heterotic supergravity in the low energy limit. While this is generally not an issue to consider, there are specific examples of theories that seem to pose a quandary. For instance, since heterotic M-theory assigns each $E_8$ gauge group to opposite ends the interval, it is clear to see that states charged under the first $E_8$ reside on one end, while states charged under the other $E_8$ will be on the opposite end. However, there are unique cases, such as the one we are studying, where a state is somehow charged under both $E_8$’s. The bulk theory that separates these two boundaries is simply 11D supergravity, so it is highly nontrivial to describe how a state could be charged across this bulk.
In fact, such a description was found for the somewhat simpler case of heterotic M-theory compactified on a Calabi-Yau twofold to a 6D field theory [5, 6]. It was shown that the theories at these fixed points could be described as a 7D SYM theory compactified on the interval with particular boundary conditions. It was also shown that the fixed points could be described locally as multi–Taub–NUT spaces. This allowed for the alternative description in terms of type I' string theory, and brane-engineering was employed to naturally describe the theory and boundary conditions.

Unfortunately, the generalization of this procedure from 6D to 4D becomes complicated. First, there is no 6D equivalent to the multi-Taub-NUT space at the fixed points of the orbifold, so the procedure using brane engineering is not valid. Second, the 5D theories present at the fixed points when M-theory is compactified on an orbifold are not SYM theories as we would expect from the case above, but 5D interacting superconformal field theories [9]. Compactifying these theories on an interval is extremely nontrivial, as the theories themselves can be quite difficult to study.

### 3.1 The Bulk Theory: $E_0$ SCFT

Let’s now discuss M-theory compactified on $T^6/Z_3$ in more detail. As stated above, the orbifold can be smoothed out to a Calabi-Yau threefold by replacing the 27 singular fixed points with $\mathbb{CP}^2$’s. Blowing these $\mathbb{CP}^2$’s down to points will, in turn, reproduce the orbifold. More generally, $\mathbb{CP}^2$ is a del Pezzo surface, and it is has been shown [9] that M-theory compactified on a Calabi-Yau threefold with a collapsing del Pezzo surface results in a 5D $\mathcal{N} = 1$ SCFT at that point with a global $E_N$ symmetry. In the case of $\mathbb{CP}^2$, the resulting SCFT has $E_0$ global symmetry, which is no global symmetry at all.

The $E_0$ SCFT can best be illustrated in terms of brane webs [10, 11]. The webs are formed by $(p, q)$ 5-branes in Type IIB string theory. They share $(x^0, x^1, x^2, x^3, x^4)$, and form $(p, q)$-lines in the $(x^5, x^6)$-plane (for an appropriate choice of complexified string coupling $\tau = \chi + i e^{-\phi}$, namely $\tau = i$). Different $(p, q)$-lines are allowed to meet as long as $(p, q)$-charge is conserved:

$$\sum p_i = \sum q_i = 0.$$  \hspace{1cm} (3.1)

The resulting webs formed by these $(p, q)$ 5-branes in the $(x^5, x^6)$-plane describe 5D $\mathcal{N} = 1\ SU(N)$ gauge theories in the common $(x^0, x^1, x^2, x^3, x^4)$ volume.

For our interests, let us consider $SU(2)$ supersymmetric Yang-Mills theory. Similar to 4D where $\pi_3(SU(2)) = \mathbb{Z}$ leads to a vacuum $\theta$-angle that can take values in $\{2\pi \mathbb{Z}\}$, in 5D we have $\pi_4(SU(2)) = \mathbb{Z}_2$ which leads to a $\theta$-angle that can take values $\{0, \pi\}$. We thus have two $SU(2)$ SYM theories, differing by the value of a $\theta$-angle. The corresponding brane webs take the forms in Fig. 3.1. In both cases, the VEV of
the real scalar field $\phi$ in the vector multiplet is associated to the “breathing” mode, or contraction and expansion, of the quadrilateral. As can be seen, varying $\phi$ does not affect the asymptotic configuration of the external legs in the $(x^5, x^6)$-plane, and hence parametrizes a local symmetry, i.e., the gauge symmetry. When $\phi = 0$, the boxes collapse, as is illustrated by the dotted lines. In this limit, the length of the horizontal dotted line in each corresponds to its bare coupling $h = 4\pi^2/g_0^2$. Varying $h$ changes the asymptotic configuration, signaling that it parametrizes a global symmetry. Specifically, it corresponds to a global $U(1)$ symmetry associated to the conserved instanton current $j = *(F \wedge F)$ that can be defined in 5D.

\[ \theta = 0 \]  \hspace{1cm}  \[ \theta = \pi \]

**Figure 3.1:** $SU(2)$ SYM with $\theta = 0$ and $\theta = \pi$.

To investigate the $E_0$ SCFT, we will need to start with $SU(2)$ SYM with $\theta = \pi$. We can vary the parameters $\phi$ and $h$, and explore the various limits that result. Starting at $h > 0$, $\phi > 0$ as in fig. 3.2.(a), we can let $h \to 0$. At $h = 0$, the quantum corrected coupling is still positive, so the theory is still $SU(2)$ SYM with $\theta = \pi$, as in fig. 3.2.(b). As we continue into negative $h$, we eventually reach a point $h = h_{\text{flop}}$ where the coupling diverges and a quark becomes massless (the mass depends on the length of the bottom brane, which goes to zero at $h_{\text{flop}}$, fig. 3.2.(c)). Continuing past this point, there is a flop transition as seen in fig. 3.2.(d). This new phase has a massive quark with a mass proportional to $h$. In the low energy effective theory, we only care about the massless spectrum, so we discard this massive quark. The resulting theory — represented by the brane web in fig. 3.2.(e) — is the $E_0$ theory along its Coulomb branch. Its massless spectrum consists of a single $U(1)$ vector multiplet whose scalar field $\hat{\phi}$, a linear combination of $\phi$ and $h$, characterizes the breathing mode of the resulting triangle. There is no possible global deformation, so there is no global symmetry in the $E_0$ theory, as stated earlier. When $\hat{\phi} = 0$ as in Fig. 3.2.(f), the coupling diverges and we reach the $E_0$ SCFT point in the moduli space.
Figure 3.2: Brane webs for the $SU(2)_{\theta=\pi}$ SYM and the $E_0$ SCFT. $\phi > 0$ for all 6 webs, while $h$ keeps decreasing: (a) $h > 0$; (b) $h = 0$; (c) $h = h_{\text{flop}} = -\phi < 0$; (d) $-3\phi < h < -\phi$; (e) $h \to -\infty$, $\phi \to +\infty$ but finite $\hat{\phi} = \phi + \frac{1}{3}h > 0$ — the $E_0$ Coulomb branch; (f) $h \to -\infty$, $\phi \to +\infty$, $\hat{\phi} = 0$ — the superconformal $E_0$.

3.2 The Boundaries of $E_0$

Now that we have established M-theory compactified on $T^6/\mathbb{Z}_3$, let’s compactify it further on $S^1/\mathbb{Z}_2$ to relate it to $E_8 \times E_8$ heterotic string theory. At the fixed point, this amounts to compactifying the $E_0$ SCFT on $S^1/\mathbb{Z}_2$. This introduces boundaries and, as a result, anomalies. Specifically, the 11D supergravity Chern-Simons term introduces anomalies. Ganor and Sonnenschein investigated this in [8] and found that, when compactified to 5D, the resulting Chern-Simons term introduces three times the usual anomaly to each boundary. Looking at the low-energy effective theory of $E_0$ along the Coulomb branch with $\langle \phi \rangle \gg 1/R$, they canceled the anomalies at each boundary with the addition of 3 4D chiral multiplets $X$ with $U(1)$ charge $+1$ at $x^4 = 0$ and 3 4D chiral multiplets $Y$ with $U(1)$ charge $-1$ at $x^4 = \pi$. The $X$’s transform under the fundamental of an $SU(3)_L$ global symmetry, and the $Y$’s transform under the fundamental of another $SU(3)_R$ global symmetry. The introduction of these chiral multiplets modifies the D-term, which imposes the boundary conditions

$$\frac{1}{2}\dot{\phi}^2|_{x^4=0} = |X_0|^2, \quad \frac{1}{2}\dot{\phi}^2|_{x^4=\pi} = |Y_0|^2. \quad (3.2)$$
where $X_0, Y_0$ are the scalars in the corresponding multiplets. We can thus see that the VEVs of these boundary fields are related to the $E_0$ scalar field VEV.

To make contact with the theory at the $T^6/Z_3$ fixed points, it is necessary to extrapolate from these results down to $\langle \hat{\phi} \rangle \ll 1/R$, specifically $\langle \hat{\phi} \rangle = 0$. Ganor and Sonnenschein established a conjecture in [8] that the global $SU(3)_L \times SU(3)_R$ is generically broken to $SU(2)_L \times SU(2)_R \times U(1)_V$ for $\langle \hat{\phi} \rangle \neq 0$. In this case, the boundary fields introduced for anomaly cancellation have charges

$$X : (2, 1)_{(1,1)} + (1, 1)_{(-2,1)}, \quad Y : (1, 2)_{(-1,-1)} + (1, 1)_{(2,-1)},$$

in notation $(SU(2)_L, SU(2)_R)_{U(1)_C}$, where $U(1)_B$ is the gauge symmetry. The $X$ and $Y$ VEVs are not invariant under $U(1)_V$, but are invariant under a combination of $U(1)_V$ and $U(1)_B$, with charge $Q_C \equiv \frac{1}{2} Q_V + Q_B$. Rewriting the charges now as $(SU(2)_L, SU(2)_R)_{U(1)_C}$, we have

$$X : (2, 1)^{\frac{1}{2}} + (1, 1)_0, \quad Y : (1, 2)^{-\frac{1}{2}} + (1, 1)_0.$$

Let us examine the anomalies due to these states at one of the boundaries, say $x^4 = 0$. If there is to be $SU(3)_L \times SU(3)_R$ restoration at $\langle \hat{\phi} \rangle = 0$, then 't Hooft anomaly matching requires that the $SU(3)_L^3$ triangle anomaly must match the $SU(2)_L^2 - U(1)_C$ triangle anomaly. In order for this to occur in the presence of $X$, it is necessary for there to be three triplets under $SU(3)_L$. Similar analysis at the $x^4 = \pi$ boundary indicates that we need three triplets under $SU(3)_R$. The simplest collection of states that satisfies this requirement is a single state with charge $(3, 3)$ under $SU(3)_L \times SU(3)_R$. This is at least suggestive that a state of this form could be present at the restoration point $\langle \hat{\phi} \rangle = 0$.

## 4 Dimensional Deconstruction

In this section, we deconstruct the fifth dimension $x^4$ (néé $x^{10}$) of the $E_0$ SCFT living on a fixed plane of the $T^6/Z_3$ orbifold. But let us start with a brief review of the dimensional deconstruction [7] in general. Basically, it amounts to latticizing one or more dimensions of a theory followed by reinterpreting the lattice as a quiver. For example, consider a 5D Yang–Mills theory with an $SU(N)$ gauge group. We begin by discretizing the fifth dimension with some lattice spacing $a$ while keeping the other dimensions $x^{0,1,2,3}$ continuous. On the resulting lattice, the $A^{0,1,2,3}$ components of the vector field $A^M(x)$ live on the nodes $x^4 = \ell \times a$ while the $A^4$ lives on the lattice links, encoded in unitary matrices

$$U_\ell = \text{Pexp} \left( \int_{\ell a}^{(\ell+1)a} A_4 \, dx^4 \right).$$

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Next, we reinterpret the $A^{0,1,2,3}(x^{0,1,2,3}, \ell)$ as 4D gauge fields of the $\ell^{th}$ factor of

$$G_{4D} = \prod_{\ell} SU(N_\ell),$$

and the $U_\ell(x^{0,1,2,3})$ as 4D scalar fields in bifundamental multiplets ($\square_\ell, \bar{\square}_{\ell+1}$) of the gauge group (4.2), hence the whole theory can be described by the quiver

\begin{center}
\begin{tikzpicture}
  \node (n0) at (0,1) {$N$};
  \node (n1) at (1,1) {$N$};
  \node (n2) at (2,1) {$N$};
  \node (n3) at (3,1) {$N$};
  \node (n4) at (4,1) {$N$};
  \node (n5) at (5,1) {$N$};
  \node (n6) at (6,1) {$N$};
  \node (n7) at (7,1) {$N$};
  \draw[->] (n0) -- (n1);
  \draw[->] (n1) -- (n2);
  \draw[->] (n2) -- (n3);
  \draw[->] (n3) -- (n4);
  \draw[->] (n4) -- (n5);
  \draw[->] (n5) -- (n6);
  \draw[->] (n6) -- (n7);
\end{tikzpicture}
\end{center}

(4.3)

To make the 4D quiver theory renormalizable, we may replace the non-linear scale fields $U_\ell$ with the linear bifundamental fields subject to a scalar potential with degenerate minima spanning the $SU(N)$ group manifold. Alternatively, we may realize the $U_\ell$ as technipions of confining gauge theories $SU(M)_\ell$ with $N$ massless flavors.

For another example, consider the SQCD with $n_c$ colors and $n_f$ flavors. As explained in [12, 13], this theory deconstructs to the 4D quiver

\begin{center}
\begin{tikzpicture}
  \node (nc1) at (0,1) {$n_c$};
  \node (nc2) at (1,1) {$n_c$};
  \node (nc3) at (2,1) {$n_c$};
  \node (nc4) at (3,1) {$n_c$};
  \node (nc5) at (4,1) {$n_c$};
  \node (nc6) at (5,1) {$n_c$};
  \node (nc7) at (6,1) {$n_c$};
  \draw[->] (nc1) -- (nc2);
  \draw[->] (nc2) -- (nc3);
  \draw[->] (nc3) -- (nc4);
  \draw[->] (nc4) -- (nc5);
  \draw[->] (nc5) -- (nc6);
  \draw[->] (nc6) -- (nc7);
\end{tikzpicture}
\end{center}

(4.4)

The deconstruction preserves 4 out of 8 supercharges of the 5D SUSY, so the quiver has $N = 1$ SUSY in 4D. Together, the 4D vector multiplets and the bifundamental chiral multiplets of the quiver deconstruct the 5D vector multiplets. Some of the 4D anti/fundamental fields deconstruct the 5D quarks and antiquarks, while additional 4D anti-fundamentals may be used to adjust the Chern–Simons level of the 5D theory (or the $\theta$ angle for $n_c = 2$).

### 4.1 Deconstructing $E_0$ Without Boundaries

Now let’s turn our attention to the $E_0$ theory in infinite $4 + 1$ dimensions; we shall deal with the finite $x^4$ (née $x^{10}$) dimension in the next subsection. There is no standard procedure for deconstructing superconformal theories, so we are going to exploit the connection between the moduli/parameter spaces of the $E_0$ SCFT and the $SU(2)$ SYM with $\theta = \pi$, see figure 4.1 for the phase diagram and figures 3.2.(a–f) for the brane webs for all the phases. Thus, we shall start by deconstructing the $SU(2)_{\theta=\pi}$ theory, go to the Coulomb branch, identify the flop transition to the $E_0$ Coulomb branch, and eventually reach the SCFT point in the moduli space.
**Figure 4.1:** Moduli/parameter space of the 5D $SU(2)_{\theta=\pi}$ SYM, the $E_0$ SCFT, and their Coulomb branches. The horizontal axis is the Coulomb modulus $\phi$ of the $SU(2)$ while the vertical axis is the inverse gauge coupling $h$ according to
\[ h = \frac{4\pi^2}{g_{5d}^2[SU2]}, \quad \langle \Phi_{SU2} \rangle = \begin{pmatrix} +\phi & 0 \\ 0 & -\phi \end{pmatrix}. \] (4.6)

The $E_0$ Coulomb modulus is the $\hat{\phi} = \phi + \frac{1}{3}h$.

As explained in [12], the 5D $SU(2)$ SYM with $\theta = \pi$ deconstructs to the following 4D quiver

\[
\begin{array}{cccccccc}
\vdots & \Phi & q & \Omega & q & \Omega & q & \Omega & q & \Omega & \vdots \\
\downarrow & 2 & \downarrow & \Omega & \downarrow & \Omega & \downarrow & \Omega & \downarrow & \Omega & \downarrow \\
\ldots & \Phi & \ldots & \Omega & \ldots & \Omega & \ldots & \Omega & \ldots & \Omega & \ldots \\
\downarrow & \tilde{q} & \downarrow & \tilde{q} & \downarrow & \tilde{q} & \downarrow & \tilde{q} & \downarrow & \tilde{q} & \downarrow \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\] (4.7)

with the superpotential
\[ W = \sum_{\ell} s_{\ell} (\det(\Phi_{\ell}) - v^2) + \sum_{\ell} \tilde{q}_{\ell} \Omega_{\ell} q_{\ell+1} \] (4.8)

where the $s_{\ell}$ are gauge singlets (not shown on the quiver (4.7)). Each $SU(2)_{\ell}$ factor of the quiver has the same 4D gauge coupling $g$, or in quantum terms the same $\Lambda_{\ell} = \Lambda$; the 5D gauge coupling of the deconstructed $SU(2)_{5D}$ obtains as

\[ h = \frac{4\pi^2}{g_5^2} = \frac{1}{a} \log \left| \frac{v_1^3}{\Lambda} \right| \] (4.9)
where $a \approx 1/(g v)$ is the lattice spacing. Finally, the moduli space of the quiver is the complexified Coulomb moduli space of the $SU(2)_{5D}$; it’s parametrized by equal (modulo gauge symmetries) VEVs of all the bifundamental scalars,

$$\langle \Omega_{\ell} \rangle = \begin{pmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{pmatrix}, \quad \omega_\pm = v \times \exp(\pm a \phi)$$

(4.10)

where $\text{Re} \phi$ is the 5D modulus from eq. (4.6) while $\text{Im} \phi$ is irrelevant for an infinitely long quiver; without loss of generality we shall henceforth assume real $\phi > 0$. The $\phi = 0$ point — and hence all $\langle \Omega_{\ell} \rangle = (v \ 0 \ 0 \ v)$ — corresponds to the unbroken $SU(2)$ in 5D while $\phi > 0$ spans the Coulomb branch.

When $|\Lambda| \geq |v|$, the quiver theory becomes strongly coupled in the IR. For the deconstructed theory, this means $h < 0$, which puts us in the bottom half of the moduli/parameter space (4.1). For $h = -\phi$ in that bottom half, we should have a flop transition to the $E_0$ Coulomb branch. To see how this works in the quiver, consider the mass terms for the fundamental $Q_{\ell}$ and $\tilde{Q}_{\ell}$ fields as functions of the $\omega_\pm$:

$$W \supset \omega_+ \times \sum_{\ell} \tilde{q}_{\ell,(1)} q_{\ell+1}^{(1)} + \omega_- \times \sum_{\ell} \tilde{q}_{\ell,(2)} q_{\ell+1}^{(2)} + \frac{\Lambda^3}{\omega_+^2} \times \sum_{\ell} \tilde{q}_{\ell-1,(2)} q_{\ell+1}^{(2)},$$

(4.11)

where the first two terms on the RHS are tree-level while the third term stems from the one-instanton effects in the $SU(2)_\ell$ Higgsed down by the $\langle \tilde{q}_{\ell-1,(2)} \rangle$ and $\langle q_{\ell+1}^{(2)} \rangle$.

To see the origin of these instanton terms, let’s focus on a single $SU(2)_\ell$ gauge group factor and ignore all the others. Let’s temporarily turn on VEVs of the $\tilde{q}_{\ell-1,(2)}^{(1)}$ and $q_{\ell+1}^{(2)}$ scalars (which are both neutral WRT $SU(2)_\ell$) while turning off the $\omega_-$ eigenvalue of the bifundamental VEVs $\langle \Omega_{\ell-1} \rangle$ and $\langle \Omega_{\ell} \rangle$. The $\langle \tilde{q}_{\ell-1,(2)} \rangle$ VEV gives mass to the $SU(2)_\ell$ doublets $\Omega_{\ell-1}^{(1)}$ and $\Omega_{\ell-1}^{(2)}$ ($\alpha = 1, 2$), while the $\langle q_{\ell+1}^{(2)} \rangle$ VEV gives masses to the $\tilde{q}_{\ell,(2)}$ and $\Omega_{\ell+1}^{(2)}$ doublets. Integrating out these doublets from the $SU(2)_\ell$ gauge theory leaves us with two massless doublets $\Omega_{\ell-1}^{(1)}$ and $\Omega_{\ell}^{(2)}$ and effective strong-interaction scale $\Lambda_{\text{eff}}^5 = \Lambda^3 \langle \tilde{q}_{\ell-1,(2)} \rangle \langle q_{\ell+1}^{(2)} \rangle$. The remaining doublets have $\omega_+$ VEVs which Higgs the $SU(2)_\ell$ down to nothing exactly as in the Affleck–Dine–Seiberg setup [14], and just like in that setup, the instantons of the broken gauge theory generate the superpotential

$$W_{\text{inst}} = \frac{\Lambda_{\text{eff}}^5}{\omega_+^2} = \frac{\Lambda^3 \langle \tilde{q}_{\ell-1,(2)} \rangle \langle q_{\ell+1}^{(2)} \rangle}{\omega_2}$$

(4.12)

for the $\Omega_{\ell-1}^{(1)}$ and $\Omega_{\ell}^{(2)}$ doublets. Now let’s analytically continue this one-instanton superpotential to zero $\langle \tilde{q}_{\ell-1} \rangle$ and $\langle q_{\ell+1} \rangle$ (but non-zero $\omega_+$). In this regime, the superpotential (4.12) becomes the $\omega_+$
eigenvalues of the doublets’ mass matrix follow from the Fourier transform from $\ell$ to
the momentum $p_4$ in the $x^4$ direction, thus for the $SU(2)$ color $\alpha = 1$
\begin{equation}
m_1(p_4) = \omega_+ \times e^{iap_4},
\end{equation}
while for the $\alpha = 2$ color
\begin{equation}
m_2(p_4) = \omega_+ \times e^{iap_4} + \frac{\Lambda^3}{\omega_+^2} \times e^{2iap_4}.
\end{equation}
For generic values of the eigenvalues $\omega_{\pm}$ these masses never come close to zero, so
the effective low-energy theory is simply the deconstructed $U(1)_{5D} \subset SU(2)_{5D}$ of the
Coulomb branch. However, when
\begin{equation}
|\omega_-| = \left|\frac{\Lambda^3}{\omega_+^2}\right|,
\end{equation}
the mass $m_2(p_4)$ crosses zero for some momentum $p_4$. Without loss of generality we
may assume this happens for $p_4 = 0$ (otherwise, we can shift the $p_4$ by a constant by
a suitable redefinition of the quark field phases), thus $m_2 \sim p_4$ + lattice corrections,
which means the deconstructed 5D quark with color $\alpha = 2$ has zero 5D mass. And
the massless charged particle is exactly what should happen at the flop transition!
Moreover, the flop condition (4.16) corresponds in 5D terms to
\begin{equation}
v \times e^{-a\phi} = v \times e^{-2a\phi} \times e^{-ah} \iff h = -\phi,
\end{equation}
which is precisely where the 5D $SU2_{\theta=\pi}$ SYM should have the flop transition to the $E_0$
Coulomb branch.

Going further below the flop transition line of the diagram (4.1) we eventually
reach the line of the superconformal $E_0$ at $h = -3\phi$. On this line, the 5D moduli space
ends — the $E_0$ Coulomb modulus
\begin{equation}
\hat{\phi} = \phi + \frac{1}{a}h
\end{equation}
ever becomes negative in infinite 4+1 dimensions. Although the $E_0$ compactified on a
circle allows for $\hat{\phi} < 0$, the negative–$\hat{\phi}$ chamber of the moduli space shrinks to nothing
in the decompactification limit due to $g_{\hat{\phi}\hat{\phi}} \to 0$.

dependent mass term for the $\tilde{q}_{\ell-1,2}$ and $q_{\ell+1}^{(2)}$ fields,
\begin{equation}
W_{\text{inst}} = \frac{\Lambda^3}{\omega_+} \times \tilde{q}_{\ell-1,2} q_{\ell+1}^{(2)}.
\end{equation}
Adding such mass terms for all the $SU(2)_\ell$ gauge groups of the quiver gives us the double-hopping
third term in the superpotential (4.11).
In the quiver terms, the $E_0$ modulus is

$$\hat{\phi} = \frac{1}{a} \log \left| \frac{\omega_+}{\Lambda} \right|$$

while the superconformal line $\hat{\phi} = 0$ (for $\phi > 0$ and $h < 0$) corresponds to

$$|\omega_+| = |\Lambda| \gg |v| \gg |\omega_-|$$

From the 4D point of view, this line is the transitions between the semiclassical Higgs regime of the quiver theory and the confinement regime. Indeed, for $\omega_+ \gg \Lambda$, we have semiclassical Higgsing of the $\prod_\ell SU(2)_\ell$ theory down to a single $U(1)$, with a Kaluza-Klein tower of light 4D photons corresponding to the deconstructed $U(1)_{5D}$. On the other hand, for $|\Lambda| \gg |\omega_+| \geq |\omega_-|$, the Higgs effects of the scalar VEVs become negligible compared to the non-perturbative 4D effects such as confinement.

For a single $SU(2)$ gauge theory, there is a smooth crossover between the Higgs and the confinement regimes of the theory rather than a phase transition. But for for the $[SU(2)]^N$ quiver theory with $N \to \infty$, the transition seems to become abrupt. Thus, any scalar VEV $> \Lambda$ — even if its just a little bit larger than $\Lambda$ — puts the quiver in the semiclassical Higgs regime. On the other hand, a scalar VEV $< \Lambda$ is as good as zero. So when all the VEVs are smaller than $\Lambda$ — even if they are just a hair smaller — the quiver is in the confinement regime, and the Higgs effects of the scalar VEVs are of no importance.

For the quivers with large but finite $N$, we expect to have a continuous crossover between the Higgs and the confinement regimes, but the crossover should become sharper and sharper with larger $N$. As a heuristic explanation of this behavior, note that the holomorphic gauge-invariant order parameters of the quiver behave line $(\text{VEV}/\Lambda)^N$, so the crossover between the large-VEV and small-VEV regimes should become sharper and sharper with larger $N$. Ultimately, in the $N \to \infty$ limit, the crossover becomes infinitely sharp and turns into an abrupt phase transition.

For the quiver (4.7) at hand, this means that

- For $\Lambda \geq \omega_+$, every $SU(2)_\ell$ of the quiver confines and the effect of the scalar eigenvalues $\omega_\pm$ is negligible. This regime deconstructs the $E_0$ SCFT.

- For $\Lambda < \omega_+$, every $SU(2)_\ell$ is Higgsed down and only the diagonal $U(1)_{\text{diag}} \subset SU(2)_{\text{diag}} \subset [SU(2)]^N$ survives. This regime deconstructs the Coulomb branch of the $E_0$ (or perhaps the Coulomb branch of the $SU(2)_{\theta=\pi}$).

This completes our survey of the deconstructed $SU(2) + E_0$ moduli space.
Since in this paper we are interested in the $E_0$ theory rather than the $SU(2)$ SYM, we are going to take the limit of $\phi \to +\infty$, $h \to -\infty$ while $\hat{\phi}$ stays finite, see figures 3.2.(d–e) for the brane web illustration. In quiver terms, this corresponds to setting $v = 0$ and hence enforcing $\omega_- = 0$ while $\omega_+$ remains unconstrained, thus

$$\langle \Omega_\ell \rangle = \begin{pmatrix} \omega_+ & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\phi} = \frac{1}{a} \log \left| \frac{\omega_+}{\Lambda} \right| . \quad (4.21)$$

Thanks to the instanton term in the mass matrix (4.11), all the fundamental fields remain massive despite $\omega_- = 0$, so the low-energy degree of freedom are comprised of the gauge and bifundamental fields. Or rather, these are the dominant degrees of freedom in an infinite quiver. In a finite quiver with boundaries, the extra ‘quark’ and ‘antiquark’ fields at the boundaries are also very important.

### 4.2 Deconstructing the Boundaries of the $E_0$

In the Hořava–Witten orbifold context, the $x^{10}$ dimension has boundaries where the $SU(3)_1 \subset E_8_1$ (at the left boundary) and the $SU(3)_2 \subset E_8_2$ (at the right boundary) act as flavor symmetries of the $E_0$ theory. Although these $SU(3)_1 \times SU(3)_2$ symmetries are gauged, their couplings become weak (from the 4D point of view) when the volume of the $T^6/\mathbb{Z}_3$ orbifold becomes large. Since we want to focus on the single fixed point rather than on the entire orbifold, we take the infinitely large 6–volume limit, and in that limit we may treat the $SU(3)_1 \times SU(3)_2$ flavor symmetry of the $E_0$ boundaries as global rather than gauged. Thus, in the finite quiver which deconstructs the $E_0$ with boundaries, each boundary should have a global $SU(3)$ symmetry.

Moreover, according to Ganor and Sonnenschein [8], on the Coulomb branch of the $E_0$, the $SU(3) \times SU(3)$ global symmetry should be spontaneously broken down to the $SU(2) \times SU(2) \times U(1)$. Or taking into account the $U(1)$ gauge symmetry in the bulk which is Higgsed down by the squark VEVs at the boundaries,

$$SU(3)_{\text{left end}} \times SU(3)_{\text{right end}} \times U(1)_{\text{bulk}} \to SU(2)_{\text{left}} \times SU(2)_{\text{right}} \times U(1)_{\text{left+right+bulk}} . \quad (4.22)$$

The Higgs regime of the quiver which deconstructs the $E_0$ with boundaries should faithfully reproduce this symmetry breaking pattern. It should also not yield any massless 4D particles besides the 5 twisted states of the orbifold which remain massless when the fixed-point singularity is blown up.
The simplest solution to these requirements is the following quiver:

\[ W = \sum_{\ell=1}^{N-1} \left( s_\ell \Omega_\ell^2 + \tilde{q}_\ell \Omega_\ell q_{\ell+1} \right) + \epsilon^{ijk} s_i Q_j Q_k + \epsilon^{ijk} \tilde{S}_i \tilde{Q}_j \tilde{Q}_k \]  

(4.24)

Note the $SU(3)_1 \times SU(3)_2$ flavor symmetry of this superpotential: the $SU(3)_1$ symmetry acts on the $Q_i$ at the left end of the quiver, while the $SU(3)_2$ acts on the $\tilde{Q}_i$ and the $\tilde{S}_i$ at the right end.

The simplest way to obtain this theory is to start with the infinite $SU(2)$ quiver (4.7) and the superpotential (4.8), turn off the gauge fields of the $\ell = 0$ and $\ell = N + 1$ nodes, and throw away all chiral and gauge fields which decouple from the $\ell = 1, \ldots, N$ nodes. The surviving fields left of the $\ell = 1$ node comprise the $\Omega_0$ (which becomes two $SU(2)$ doublets), the $\tilde{q}_{-1}$ (which becomes two singlets), and the singlet $s_0$. Together with the $q_0$ doublet, they become three doublets $Q_i$ plus three singlets $S_i$ with the Yukawa couplings between them amounting to $W \supset \epsilon^{ijk} S_i Q_j Q_k$. Likewise, the surviving fields to the right of the $\ell = N$ node comprise the $\Omega_N$, the $q_{N+1}$ and the $s_0$, which together with the $\tilde{q}_N$ become three $SU(2)_N$ doublets $\tilde{Q}_i$ and three singlets $\tilde{S}_i$, with the Yukawa couplings $W \supset \epsilon^{ijk} \tilde{S}_i \tilde{Q}_j \tilde{Q}_k$ to each other.

Now consider the Higgs regime of the quiver (4.23). The Yukawa couplings to the singlets restrict the scalar VEV matrices $\langle \Omega_\ell \rangle$, $\langle Q \rangle$ and $\langle \tilde{Q} \rangle$ to rank $\leq 1$, while the $[SU(2)]^N$ D-terms require all these VEVs to have similar magnitudes. Thus, modulo gauge and flavor symmetries of the theory, the only flat direction of the scalar potential of the quiver is

\[
\begin{align*}
\langle Q_\ell \rangle &= \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}, &
\langle Q \rangle &= \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}, &
\langle \tilde{Q} \rangle &= \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix},
\end{align*}
\]  

(4.25)

with the same modulus $\omega$ governing all these VEVs. For $|\omega| > |\Lambda|$ we expect the quiver to be in the Higgs regime, so the semiclassical analysis should adequately describe the low-energy physics. Here are the highlights:
• The entire $[SU(2)]^N$ gauge theory of the quiver is Higgsed down to nothing.\footnote{For a long quiver, the $[U(1)]^N$ photons have a Kaluza-Klein-like tower of light modes with $O(1/(Na))$ masses, but there is no zero mode due to Higgsing by $\langle Q \rangle$ and $\langle \tilde{Q} \rangle$ at the ends of the quiver. For the charged $W^\pm$ gauge fields, all the modes have heavy $O(\omega)$ masses.}

• Most of the chiral superfields of the theory that are not eaten by the Higgs mechanism get $O(\omega)$ masses from the the superpotential (4.24) or $O(\Lambda^3/\omega^2)$ masses from the one-instanton effects.

• The $SU(3)_1 \times SU(3)_2$ global symmetry of the theory is spontaneously broken down to $SU(2)_1 \times SU(2)_2 \times U(1)_{\text{combined}}$, in perfect agreement with Ganor and Sonnenschein.

• The only massless particles are the 9 Goldstone bosons of the global symmetry breakdown and their superpartners, packaged into 5 chiral multiplets, namely:
  
  \begin{itemize}
    \item The modulus $\omega$ or rather its variation $\delta \omega$, which affects the $Q_{1,(1)}$, the $\tilde{Q}_{1}^{(1)}$, and all the $\Omega_{\ell,(1)}^{(1)}$ fields.
    \item The quarks $Q_{2,(1)}$ and $Q_{3,(1)}$.
    \item The antiquarks $\tilde{Q}_{2}^{(1)}$ and $\tilde{Q}_{3}^{(1)}$.
  \end{itemize}

• The $x^{10}$ locations of these particles are precisely as in Ganor and Sonnenschein:

\begin{equation}
\begin{aligned}
\text{bulk+boundaries} & \rightarrow \begin{pmatrix}
\delta \omega \\
Q_{2}^{(1)} \\
Q_{3}^{(1)}
\end{pmatrix} \\
\text{left boundary} & \rightarrow \begin{pmatrix}
\tilde{Q}_{2}^{(1)} & * & *
\end{pmatrix} \\
& \quad \quad \leftarrow \begin{pmatrix}
Q_{3}^{(1)} & * & *
\end{pmatrix} \\
& \quad \quad \text{right boundary}
\end{aligned}
\end{equation}

\begin{equation}
4.26
\end{equation}

* To summarize, the low-energy physics of the Higgs regime of the quiver (4.23) is in good agreement with the Coulomb branch of the $E_0$ theory with boundaries.

### 4.3 The Confinement Regime

Now consider the confinement regime of the quiver (4.23), which corresponds to the superconformal point of the $E_0$. For the infinite quiver, this regime obtains for any $|\omega| < |\Lambda|$, so for simplicity’s sake, let’s assume $\omega = 0$, i.e., no scalar VEVs whatsoever.

We shall return to the effects of $\omega \neq 0$ on a finite-length quiver in the next subsection §4.4.

Note that each $SU(2)_\ell$ factor of the quiver (4.23) couples to 6 doublets, so it acts as SQCD with $n_c = 2$ colors and $n_f = 3 = n_c + 1$ flavors. The IR behavior of such theories is confinement without chiral symmetry breaking. Instead, there are massless composite
particles — the mesons and the baryons. However, tree-level Yukawa couplings of the quarks and antiquarks to singlets (or more general, to fields not charged under the $SU(2)_\ell$ in question) would render some of the mesons and the baryons massive, and the singlets would also become massive. We shall see momentarily that for the quiver (4.23), all the fundamental and the bifundamental fields become confined, while most of the composite particles and all the singlets become massive due to Yukawa couplings. The only particles which remain exactly massless are the 9 meson-like states comprising a quark at one end of the quiver, an antiquark at the other end, and all of the bifundamental fields,

$$M_{ij} = \Lambda^{-N} (Q_i \Omega_1 \Omega_2 \cdots \Omega_{N-1} \tilde{Q}_j). \tag{4.27}$$

The $SU(3)_1 \times SU(3)_2$ flavor symmetry of the quiver remains unbroken in the confinement regime, and the mesons (4.27) form the $(3,3)$ multiplet of this symmetry. This is in perfect agreement with the heterotic twisted states $T_{ij}$ at the un-blow-up fixed point. Also, we shall see that the mesons (4.27) have Yukawa couplings to each other of the form $W \supset \det(M_{ij})$, exactly as the heterotic twisted states $T_{ij}$.

To see how it works, let’s start with a warm-up exercise of the quiver of length $N = 1$: A single $SU(2)$ gauge group, with 3 quarks $Q_i$, 3 antiquarks $\tilde{Q}_i$, 6 singlets $S_i$ and $\tilde{S}_i$, and the Yukawa couplings

$$W = \epsilon^{ijk} S_i Q_j Q_k + \epsilon^{ijk} S_i \tilde{Q}_j \tilde{Q}_k. \tag{4.28}$$

This theory confines without chiral symmetry breaking, and without the Yukawa couplings it would produce 15 massless supermultiplets: 9 mesons $M_{ij} = Q_i \tilde{Q}_j / \Lambda$, 3 baryons $B^i = \epsilon^{ijk} Q_j Q_k / \Lambda$, and 3 antibaryons $\tilde{B}^i = \epsilon^{ijk} \tilde{Q}_j \tilde{Q}_k / \Lambda$. But the tree-level Yukawa couplings (4.28) become mass terms for the baryons, antibaryons, and all the singlets, so only the nine mesons $M_{ij}$ remains massless.

These mesons have non-perturbative Yukawa couplings to each other,

$$W_{NP} \sim \det(M) \sim \epsilon^{ijk} \epsilon^{lmn} M_{il} M_{jm} M_{kn}. \tag{4.29}$$

They also have Yukawa couplings to the baryons and to other massive particles, but for the present purposes we shall focus on the couplings among the massless particles only.
Now consider a more involved example of the two-node quiver, $N = 2$:

\begin{align}
\begin{array}{ccc}
3 & \xrightarrow{Q} & 2 \\
\downarrow & & \Omega \\
2 & \xleftarrow{\bar{Q}} & 3 \\
\downarrow & & \bar{q}
\end{array} + \text{ singlets } S_i, \bar{S}_i, s
\end{align}

For the moment, let’s give unequal gauge couplings to the two $SU(2)$ gauge groups of the quiver so that $\Lambda_1 \gg \Lambda_2$. In this case, the non-perturbative effects of the $SU(2)_1$ group are felt at higher energies, so we may focus on the $SU(2)_1$ non-perturbative effects first, truncate the resulting particle spectrum to the massless particles only, and only then couple them to the $SU(2)_2$. Thus, from the $SU(2)_1$ point of view, the $\bar{q}^{(\alpha)}$ and the $\Omega^{(\alpha)}_{(\beta)}$ are 3 antiquarks doublets while the $s$ and the $q^{(\beta)}$ are 3 singlets coupled to those antiquarks, just like the $\bar{S}_i$ couple to the $\bar{Q}_i$ in the single-node example. Thus, when the $SU(2)_1$ confines, it makes massless mesons

\begin{align}
P_{i(\beta)} & = \frac{1}{\Lambda_1} Q_i \Omega_{(\beta)} \\
R_i & = \frac{1}{\Lambda_1} Q_i \bar{q}
\end{align}

but the baryons $Q_{[i} Q_{j]}$, the antibaryons $\Omega^2$ and $\bar{q} \Omega_{(\beta)}$ and the singlets $S_i$, $s$, and $q^{(\beta)}$ become massive.

Now from the $SU(2)_2$ point of view, the $P_{i(\beta)}$ mesons are 3 doublets, so they act as quarks, while the $R_i$ mesons act as 3 singlets. Combining these fields with the antiquarks $\bar{Q}_i^\beta$ and singlets $\bar{S}_i$, we end up with the SQCD with 2 colors, 3 flavors, 6 singlets, and the Yukawa couplings

\begin{align}
W & = \varepsilon^{ijk} R_i P_j P_k + \varepsilon^{ijk} \bar{S}_i \bar{Q}_j \bar{Q}_k,
\end{align}

where the first term in the non-perturbative 3-meson coupling of the first $SU(2)$ while the second term is tree-level. Altogether, we get a theory exactly similar to the single-node example, so it behaves in the same way: confines the quarks and the antiquarks without breaking the $SU(3) \times SU(3)$ chiral symmetry, makes massless mesons

\begin{align}
M_{ij} & = \frac{1}{\Lambda_2} P_i \bar{Q}_j = \frac{1}{\Lambda_1 \Lambda_2} Q_i \Omega \bar{Q}_j,
\end{align}

while the baryons, the antibaryons, and the singlets $R_i$ and $\bar{S}_i$ become massive.

Now suppose $\Lambda_2 \gg \Lambda_1$ instead of the other way around. In this case, the $SU(2)_2$ confines first, makes massless mesons which from the $SU(2)_1$ point of view look like
3 antiquarks plus 3 singlets, and then the confinement in the $SU(2)_1$ makes massless meson-like particles exactly as in eq. (4.34). So, the quivers with $\Lambda_1 \gg \Lambda_2$ and with $\Lambda_2 \gg \Lambda_1$ have exactly the same massless composite particles, made from exactly the same quark, bifundamental, and antiquark fields, and with the same Yukawa couplings $W \supset \det(M_{ij})$ to each other, while every other particle in the theory — elementary or composite — becomes massive.

Based on this complementarity, we believe that two-node quivers with all $\Lambda_1/\Lambda_2$ ratios have the same spectrum of massless particles, namely the 3 quark-bifundamental-antiquark composites (4.34). In particular, such mesons should be the only massless particles for the quiver with $\Lambda_1 = \Lambda_2$.

Generalizing the above analysis to quivers (4.23) with any numbers of nodes is completely straightforward. For simplicity, we shall proceed by dealing with one $SU(2)_l$ factor at a time as would be appropriate for $\Lambda_1 \gg \Lambda_2 \gg \cdots \gg \Lambda_N$, but the massless spectrum obtaining at the end of the process should be valid for all ratios of confinement scales, and in particular for the $\Lambda_1 = \Lambda_2 = \cdots = \Lambda_N$. Thus we start with the confining $SU(2)_1$ which has 3 quarks, three other doublets acting as antiquarks, and 6 singlets (or fields without $SU(2)_1$ charges), exactly as in the two-node example, so the composite massless particles are the $P_{i,(\beta)}$ (which act as 3 quarks of the $SU(2)_2$) and the singlets $R_i$, while the baryons, the antibaryons, and the elementary $S_i$, $\sigma$, and $q_{2,(\beta)}$ fields become massive. Consequently, we end up with the quiver

\[
\begin{array}{cccccccc}
3 & \rightarrow & 2 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & 2 & \rightarrow & 3 \\
\downarrow P & & \downarrow \Omega_2 & & \downarrow \Omega_3 & & \cdots & & \downarrow \Omega_{N-1} & & \downarrow \tilde{Q} \\
\downarrow \tilde{q}_2 & & \downarrow q_3 & & \downarrow \tilde{q}_5 & & \cdots & & \downarrow \tilde{q}_{N-1} & & \downarrow q_N \\
\end{array}
\]

which looks exactly like the original quiver (4.23) except that it is shorter by one node.

At this point we repeat the procedure focusing on the confining $SU(2)_2$, and in the same manner end up with a quiver of length $N - 2$, etc., etc. Eventually, we arrive at a single-node quiver, and after dealing with the confining $SU(2)_N$, we end up with nine massless meson-like states

\[
M_{ij} = \frac{1}{\Lambda_1 \cdots \Lambda_N} (Q_i \Omega_1 \Omega_2 \cdots \Omega_{N-1} \tilde{Q}_j)
\]

while every particle — elementary or composite — is massive.

Note that while it is much simpler to deal with one confining $SU(2)$ factor at a time, we can handle them in any order we like. In particular, we may start in the
middle of the quiver and work our way outwards, or even jump around the quiver to non-adjacent nodes in a random fashion. While the technical details of such random-order procedure are too boring to be presented here, let us simply state the bottom line: regardless of the order in which we handle the $N\ SU(2)_f$ factors, we always end up with the same 9 massless particles (4.36). Consequently, we believe that the same 9 massless particles emerge for any ratios of the confinement scales $\Lambda_f$, including the equal-scales case of $\Lambda_1 = \Lambda_2 = \cdots = \Lambda_N$.

So let us re-iterate the bottom line of this subsection: The massless twisted states $T_{ij}$ at an un-blown fixed point of the $T^6/\mathbb{Z}_3$ orbifold — or rather of the Hořava–Witten dual of the orbifold — deconstruct to the massless meson-like states (4.36) of the quiver (4.23) in its confinement regime.

4.4 Deconstructing the Blow Up

In the heterotic string theory, blowing up a fixed point is parametrized by the VEVs $\langle T_{ij} \rangle$ of twisted-sector scalars. Up to an $SU(3) \times SU(3)$ symmetry, the VEV matrix looks like

$$
\langle T_{ij} \rangle = \begin{pmatrix} t & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}
$$

(4.37)

and the $W \supset \det(T)$ Yukawa couplings give mass $= t$ to the 4 of the $T_{ij}$ states, while the remaining 5 states remain massless. For the Hořava–Witten dual of the blown-up orbifold, Ganor and Sonnenschein have identified the 5 massless states $T_{11}, T_{12}, T_{13}, T_{21}, T_{31}$ of the Coulomb-branch $E_0$ theory with boundaries, but they saw no sign of the 4 massive states $T_{22}, T_{23}, T_{32}, T_{33}$. However, for small $t$ — i.e., when the fixed point is just a little bit blown up — the $T_{22}, T_{23}, T_{32}, T_{33}$ states become very light so they should be easy to identify. In this subsection, we resolve this paradox for the deconstructed $E_0$.

Consider the Coulomb-branch moduli of the quiver (4.23). In terms of independent holomorphic gauge-invariant combinations of the chiral superfields, there are 9 such moduli, namely the

$$
\mathcal{M}_{ij} = Q_i \Omega_1 \Omega_2 \cdots \Omega_{N-1} \tilde{Q}_j, \quad i, j = 1, 2, 3
$$

(4.38)

subject to the $\text{rank}(\mathcal{M}) = 1$ constraint due to superpotential

$$
W(\mathcal{M}) \sim \frac{1}{\Lambda^{3N}} \det(\mathcal{M}).
$$

(4.39)

At the origin $\mathcal{M}_{ij} = 0$ of the moduli space, all 9 moduli (4.38) give rise to massless particles $T_{ij}$, so the moduli space metric $g_{\mathcal{M},\mathcal{M}}$ should be non-singular at the origin. In
terms of the Kähler function of the moduli space, this means
\[ K(\mathcal{M}, \mathcal{M}) = \frac{1}{|\Lambda|^{2N}} \text{tr}(\mathcal{M} \mathcal{M}) + \mathcal{O}\left(\frac{|\mathcal{M} \mathcal{M} \mathcal{M} \mathcal{M}|}{|\Lambda|^{4N+2}}\right) \]  
(4.40)

A small blow-up (4.37) of the fixed point corresponds to
\[ \langle \mathcal{M}_{11} \rangle = \Lambda^N \times t, \quad t \ll \Lambda, \]  
(4.41)

which gives the twisted-sector particles \( T_{22}, T_{23}, T_{32}, T_{33} \) physical masses
\[ m = |\langle \mathcal{M}_{11} \rangle| \times \frac{\text{Yukawa coupling from (4.39)}}{g_{\mathcal{M},\mathcal{M}} \text{ from (4.40)}} = |t|. \]  
(4.42)

As long as \( t \ll \Lambda \), this mass remains much lighter than the \( O(\Lambda) \) masses of all the other massive particles of the quiver theory.

In terms of the \( \omega \) parameter of the scalar VEVs (4.25),
\[ \langle \mathcal{M}_{11} \rangle = \omega^{N+1} \implies t = \frac{\omega^{N+1}}{\Lambda^N}. \]  
(4.43)

Hence, in the long quiver limit \( N \to \infty \), any \( \omega < \Lambda \) corresponds to a very small \( t \ll \Lambda \). Physically, this means negligible blow-up and hence negligible masses of the \( T_{22}, T_{23}, T_{32}, T_{33} \) twisted particles. This is why for a long quiver, any scalar VEV \( \omega < \Lambda \) is as good as \( \omega = 0 \): the confinement regime is un-affected and the fixed point of the deconstructed orbifold remains un-blown.

Now consider the Higgs regime of the quiver with \( \omega > \Lambda \), which corresponds to very large \( \langle \mathcal{M}_{11} \rangle \) and \( t \gg \Lambda \). In this regime, the Kähler function of the the quiver’s moduli space is quite different from eq. (4.40). From the semiclassical Higgs fields (4.25), we expect
\[ K \approx (N + 1)|\omega|, \]  
(4.44)

so re-expressing \( K \) in terms of \( \mathcal{M}_{ij} \) and \( \mathcal{M}_{ij} \) and requiring the \( SU(3) \times SU(3) \) symmetry gives us
\[ K \approx (N + 1)^{N+1}|\text{tr}(\mathcal{M} \mathcal{M})|. \]  
(4.45)

Consequently, the Kähler metric for the fields \( \mathcal{M}_{22}, \mathcal{M}_{23}, \mathcal{M}_{32}, \mathcal{M}_{33} \) is
\[ g_{\mathcal{M} \mathcal{M}} = \frac{1}{|\omega|^{2N}}, \]  
(4.46)

hence the physical mass of the \( T_{22}, T_{23}, T_{32}, T_{33} \) twisted particles is
\[ m = \left| \frac{\omega^{3N+1}}{\Lambda^{3N}} \right| \gg \Lambda. \]  
(4.47)
This extremely large mass explains why we do not see these particles in the Ganor–Sonnenschein construction of the blown-up fixed point of the Hořava–Witten orbifold.

We hope the above arguments explain the abrupt transition at \( \omega = \Lambda \) between the confinement and the Higgs regimes in the \( N \to \infty \) limit of the quiver (4.23). In \( E_0 \) terms, this is the transition between the superconformal theory on the un-blown fixed plane and the Coulomb-branch \( U(1)_{5D} \) theory on the blown-up plane. For the quivers of large but finite length \( N \), the transition takes a finite but narrow range of scalar VEVs \( \omega \),

\[
1 - O \left( \frac{1}{N} \right) < \frac{\omega}{\Lambda} < 1 + O \left( \frac{1}{N} \right).
\]

Thus, for the Hořava–Witten theory with a finite length of \( x^{10} \) — which corresponds to a large but finite heterotic string coupling, — we expect a sharp but continuous transition between the Calabi–Yau regime of a blown-up fixed point and some non-geometric regime hiding behind the un-blown fixed point.

The similar sharp crossover is well known for many 5D theories (including the \( E_0 \)) compactified on a large circle. For the 5D theories compactified on the interval with boundaries, the general behavior should be similar, but the details need to be worked out. We hope the present paper sheds some light on the \( E_0 \) theory on the interval.

5 Work in Progress and Open Questions

Through most of this paper we have focused on deconstructing the twisted states of a particular \( T^6/Z_3 \) orbifold. The obvious next step is to apply the same method to other heterotic orbifolds: Deconstruct the 5D SCFT at each fixed plane, work out the boundary ‘quarks’ and incorporate them into the quiver, and to see if the confining regime of the quiver indeed produces massless meson-like states with quantum numbers matching the twisted states of the heterotic string. This work is in progress: Thus far, we have worked out a few \( T^6/Z_4 \) and \( T^6/Z_6 \) models [15]; the boundary ‘quarks’ and hence the quiver’s ends in these models are more complicated than in the \( T^6/Z_3 \) model, but the non-local massless mesons of the quivers do match the massless twisted states of the corresponding orbifold. We hope to work out a few more models to see how the quiver’s ending depend on a particular model before we present all the deconstructed orbifolds in a separate paper.

But besides the technical issues of the quiver boundaries, the very fact that in the Hořava–Witten theory the twisted states become non-local ‘mesons’ spanning the entire length of the \( x^{11} \) dimensions raises a several deep questions:

* First of all, what is the physical meaning of the gluon string connecting the quark at one end of the \( x^{10} \) to the antiquark at the other end? In the deconstructed
theory, this string is the product of all the bifundamental scalar fields of the quiver, but what does it become in the continuum limit? A flux tube? A Wilson line? Something else?

- Second, what is the M-theory origin of this gluon string? It does not look like an M2 or M5 brane wrapped around some cycle of the $\mathbb{C}^3/\mathbb{Z}_3$ singularity, so what else can it be?

- Regardless of the gluon string’s origin, it is tensionless: It’s the only way to keep the twisted states including this string massless in the long $x^{10}$ regime dual to the strong heterotic coupling. So what happens when the tensionless string becomes long? Does it run straight from one end of the $x^{10}$ to the other end, or can it wiggle in the $x^{1,2,3}$ directions of the ordinary 3D space? Can the quark at one end of the string move away (in the $x^{1,2,3}$ directions) from the antiquark at the other end? If yes, does it mean that the twisted particles become ‘fat’ rather than nearly-pointlike in the strong heterotic coupling regime?

  - Tentatively, the answers to these questions depend on the higher-derivative terms in the world-sheet Lagrangian for the gluon string. If all such terms vanish with the tension, then the string can wiggle as much as it wants, the quarks and the antiquark can separate in 3D space, and the twisted-sector particles have effective size comparable to the length of the $x^{11}$ (since this is the only scale of the locally-conformal 5D theory on the fixed plane). But if the higher-derivative terms do not vanish, then they stiffen the string and might force it to run in a straight line in the $x^{10}$ direction, which in turn would keep the quark and the antiquark from separating in $x^{1,2,3}$ directions. Alas, without knowing the nature of the string we cannot say if it has any higher-derivative terms or not.

  - Without delving into the nature of the gluon string, can we probe for the quark-antiquark separation through some gedankenexperiment? What would be a good probe of such separation? The form-factor in some scattering process involving both the $SU(3)_1$ and the $SU(2)_2$ charges of the twisted states? Something else?

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