SPECTRAL ANALYSIS OF 1D NEAREST–NEIGHBOR RANDOM WALKS AND APPLICATIONS TO SUBDIFFUSIVE TRAP AND BARRIER MODELS

A. FAGGIONATO

Abstract. We consider a family $X^{(n)}$, $n \in \mathbb{N}_+$, of continuous–time nearest–neighbor random walks on the one dimensional lattice $\mathbb{Z}$. We reduce the spectral analysis of the Markov generator of $X^{(n)}$ with Dirichlet conditions outside $(0, n)$ to the analogous problem for a suitable generalized second order differential operator $-D_{m_n}D_x$, with Dirichlet conditions outside a given interval. If the measures $dm_n$ weakly converge to some measure $dm_\infty$, we prove a limit theorem for the eigenvalues and eigenfunctions of $-D_{m_\infty}D_x$ to the corresponding spectral quantities of $-D_{m_n}D_x$. As second result, we prove the Dirichlet–Neumann bracketing for the operators $-D_mD_x$ and, as a consequence, we establish lower and upper bounds for the asymptotic annealed eigenvalue counting functions in the case that $m$ is a self–similar stochastic process.

Finally, we apply the above results to investigate the spectral structure of some classes of subdiffusive random trap and barrier models coming from one–dimensional physics.

Key words: random walk, generalized differential operator, Sturm–Liouville theory, random trap model, random barrier model, self–similarity, Dirichlet–Neumann bracketing.

MSC-class: 60K37, 82C44, 34B24.

1. Introduction

Continuous–time nearest–neighbor random walks on $\mathbb{Z}$ are a basic object in probability theory with numerous applications, including the modeling of one–dimensional physical systems. A fundamental example is given by the simple symmetric random walk (SSRW) on $\mathbb{Z}$, of which we recall some standard results. It is well known that the SSRW converges to the standard Brownian motion under diffusive space–time rescaling. Moreover, the sign–inverted Markov generator with Dirichlet conditions outside $(0, n)$ has exactly $n - 1$ eigenvalues, which are all positive and simple. Labeling the eigenvalues in increasing order $(\lambda_k^{(n)} : 1 \leq k < n)$, the $k$–th one is given by $\lambda_k^{(n)} = 1 - \cos(\pi k/n)$ with associated eigenfunction $f_k^{(n)}(j) = \sin(k\pi j/n)$, $j \in \mathbb{Z} \cap [0, n]$. Extending $f_k^{(n)}$ to all $[0, n]$ by linear interpolation, one observes that

$$\lim_{n \uparrow \infty} n^2 \lambda_k^{(n)} = \frac{\pi^2 k^2}{2} =: \lambda_k$$

and

$$\lim_{n \uparrow \infty} f_k^{(n)}(nx) = \sin(k\pi x) =: f_k(x),$$

where the last limit is in the space $C([0, 1])$ endowed of the uniform norm. On the other hand, the standard Laplacian $-(1/2)\Delta$ on $[0, 1]$ with Dirichlet boundary conditions has $(\lambda_k : k \geq 1)$ as family of eigenvalues and $f_k$ as eigenfunction associated to the simple eigenvalue $\lambda_k$.

Work supported by the European Research Council through the “Advanced Grant” PTRELSS 228032.
Considering this simple example it is natural to ask how general the above considerations can be. In particular, given a family of continuous-time nearest-neighbor random walks $X^{(n)}$ defined on the rescaled interval $[0,1] \cap \mathbb{Z}_n$, $\mathbb{Z}_n := \{k/n : k \in \mathbb{Z}\}$, killed when reaching the boundary, one would like very general criteria to establish (i) the convergence of $X^{(n)}$ to some stochastic process $X^{(\infty)}$, (ii) the convergence of the eigenvalues and eigenfunctions of the Dirichlet Markov generator of $X^{(n)}$ to the corresponding spectral quantities of the Dirichlet Markov generator of some stochastic process $Y^{(\infty)}$. Note that we have not imposed $X^{(\infty)} = Y^{(\infty)}$ and the reason will be clarified soon.

Criteria to establish (i) also in a more general context have been developed by C. Stone in [S], while in the first part of this paper we develop a general criterion to establish (ii). In order to allow a better understanding of the connection between the two solutions of (i) and (ii), we briefly recall the approach of [S]. The starting observation is that $X^{(n)}$ can be expressed as $(S_n, dM_n)$–space–time change of the (suitably killed) standard Brownian motion $B$, for some scale function $S_n$ and some speed measure $dM_n$ (cf. [IM], [D], [L2]). If $S_n$ is the identity function and $dM_n$ converges to some measure $dM$ (as for the SSRW), one can apply Stone’s result and conclude, under suitable weak technical assumptions, that $X^{(n)}$ converges to the process $X^{(\infty)}$ obtained as $(1, dM)$–space–time change of $B$, suitably killed. If $S_n$ is not the identity function, one first introduces a new random walk $Y^{(n)}$ as follows. Observing that $dM_n$ must be of the form $dM_n = \sum_i w_i \delta_{y_i}$ for a countable set $\{y_i\}$, while $S_n$ is an increasing function on $\{y_i\}$, one sets $dm_n = \sum_i w_i \delta_{S_n(y_i)}$. Then $Y^{(n)}$ is defined as the nearest-neighbor random walk on $\{S_n(y_i)\}$ obtained as $(1, dm_n)$–space–time change of $B$, suitably killed. If $dm_n$ converges to some measure $dm$, then one can try to apply Stone’s result to get the convergence $Y^{(n)} \to Y^{(\infty)}$, $Y^{(\infty)}$ being the $(1, dm)$–space–time change of $B$, suitably killed. Afterwards, one can try to derive from this limit the convergence of $X^{(n)}$ to some process $X^{(\infty)}$ using the fact that $X^{(n)} = S_n^{-1}(Y^{(n)})$.

These methods have been successfully applied in order to study rigorously asymptotic behavior of nearest-neighbor random walks on $\mathbb{Z}$ with random environment, as the random barrier model [KK], [FJL] and the random trap model [FIN], [BC1], [BC2] (see below).

We briefly describe our spectral continuity theorem concerning problem (ii). As remarked above, one can always transform $X^{(n)}$ into the random walk $Y^{(n)}$ having identity scale function. This transformation reveals crucial, since the Markov generator of $Y^{(n)}$ can be defined on continuous and piecewise-linear functions and the convergence of eigenfunctions is simply in the uniform topology (otherwise one is forced to deal with rather complex function spaces as in [FJL]). We show that the sign-inverted Markov generator of $Y^{(n)}$ can be written as a generalized differential operator $-D_{m_n} D_x$ on $(0, S_n(1))$ with Dirichlet b.c. (boundary conditions), having $n - 1$ eigenvalues $\{\lambda^{(n)}_k : 1 \leq k < n\}$ which are all positive and simple. Suppose now that $S_n(1) \to \ell$ and that $dm_n$ vaguely converges to some measure $dm$, which is not given by a finite set of atoms and whose support has 0, $\ell$ as extremes. Then the eigenvalues and the associated eigenfunctions of $-D_{m_n} D_x$ converge to the corresponding quantities of the generalized differential operator $-D_m D_x$ on $(0, \ell)$ with Dirichlet b.c. It is well known (cf. [L1], [L2]) that this operator is the Markov generator of the above limit process $Y^{(\infty)}$, and we show that it has only positive and simple eigenvalues. We point out that a similar convergence result is proven by T.Uno and I. Hong in [UH] for a family of differential operators on $\Gamma_n$, where $\Gamma_n$ is a suitable sequence of subsets in $\mathbb{R}$ converging to the Cantor set. Some ideas in their proof have been applied to our context, while others are very model-dependent. The route followed here is more inspired by modern Sturm–Liouville theory [KZ], [Z2], where the continuity
of the spectral structure is related to the continuity properties of a suitable family of entire functions. Our continuity result is also near to Theorem 1 in [K]. There, the author considers generalized second order differential operators without boundary conditions.

As second step in our investigation we have proved the Dirichlet–Neumann bracketing for the generalized operator $-D_m D_x$ (Theorem 8.8). This is a key result in order to get estimates on the asymptotics of eigenvalues. We recall that the limit distribution of the eigenvalues has been studied for several operators, we mention the Weyl’s classical theorem for the Laplacian on bounded Euclidean domains (see [W1], [W2], [CH1], [RS4] Chapter XIII.15). A key ingredient in this analysis is given by the Dirichlet–Neumann bracketing. The form of the bracketing used in our investigation goes back to G. Métivier and M.L. Lapidus (cf. [Me], [L]) and has been successfully applied in [KL] to establish an analogue of Weyl’s classical theorem for the Laplacian on finitely ramified self–similar fractals. In order to apply the Dirichlet–Neumann bracketing to our context we have first analyzed the generalized differential operators $-D_m D_x$ with Dirichlet and Neumann b.c. as self–adjoint operators on suitable Hilbert spaces and we have studied the associated quadratic forms. Finally, from the Dirichlet–Neumann bracketing we have derived the behavior at $\infty$ of the averaged eigenvalue counting function of the operator $-D_m D_x$ on a finite interval with Dirichlet b.c. under the assumption that $m$ is a self–similar stochastic process (see Proposition 2.2). We point out that in [Fr], [H], [KL], [UH] the authors study the asymptotics of the eigenvalues for the Laplacian defined on self–similar geometric objects. In our case, the self–similarity structure enters into the problem through the self–similarity of $m$.

As application of the above analysis (Theorem 2.1, Theorem 8.8 and Proposition 2.2) we have investigated the small eigenvalues of some classes of subdiffusive random trap and barrier models (Theorems 2.3 and 2.5). Let $T = \{\tau_x : x \in \mathbb{Z}\}$ be a family of positive i.i.d. random variables belonging to the domain of attraction of an $\alpha$–stable law, $0 < \alpha < 1$. Given $T$, in the random trap model the particle waits at site $x$ an exponential time with mean $\tau_x$ and after that it jumps to $x - 1$, $x + 1$ with equal probability. In the random barrier model, the probability rate for a jump from $x - 1$ to $x$ equals the probability rate for a jump from $x$ to $x - 1$ and is given by $1/\tau(x)$. We consider also generalized random trap models, called asymmetric random trap models in [BC1]. Let us call $X^{(n)}$ the rescaled random walk on $\mathbb{Z}_n$ obtained by accelerating the dynamics of a factor of order $n^{1 + \frac{\alpha}{\alpha - 1}}$ (apart a slowing varying function) and rescaling the lattice by a factor $1/n$. As investigated in [KK], [FIN] and [BC1], the law of $X^{(n)}$ averaged over the environment $T$ equals the law of a suitable $V$–dependent random walk $\tilde{X}^{(n)}$ averaged over $V$, $V$ being an $\alpha$–stable subordinator. To this last random walk $\tilde{X}^{(n)}$ one can apply our general results, getting at the end some annealed spectral information about $X^{(n)}$.

Random trap and random barrier walks on $\mathbb{Z}$ have been introduced in Physics in order to model 1d particle or excitation dynamics, random 1d Heisenberg ferromagnets, 1d tight–binding fermion systems, electrical lines of conductances or capacitances [ABSO]. More recently (cf. [BCKM], [BDe] and references therein) subdiffusive random walks on $\mathbb{Z}$ have been used as toy models for slowly relaxing systems as glasses and spin glasses exhibiting aging, i.e. such that the time–time correlation functions keep memory of the preparation time of the system even asymptotically. Our results contribute to the investigation of the spectral properties of aging stochastic models. This analysis and the study of the relation between aging and the spectral structure of the Markov generator has been done in [BF1] for the REM–like trap model on the complete graph. Estimates on the first Dirichlet
such that arrived at site $c$ sive random walks improving some previous results (cf. [BD]) as described in Propositions applied here and we have followed a different route.

Finally, we mention that we have applied our spectral continuity theorem also to diffusive random walks improving some previous results (cf. [BD]) as described in Propositions 2.3 and 2.6.

2. Model and results

We consider a generic continuous–time nearest–neighbor random walk $(X_t : t \geq 0)$ on $\mathbb{Z}$. We denote by $c(x, y)$ the probability rate for a jump from $x$ to $y$: $c(x, y) > 0$ if and only if $|x - y| = 1$, while the Markov generator $L$ of $X_t$ can be written as

$$
L f(x) = c(x, x - 1)\left[ f(x - 1) - f(x) \right] + c(x, x + 1)\left[ f(x + 1) - f(x) \right]
$$

for any bounded function $f : \mathbb{Z} \to \mathbb{R}$. The random walk $X_t$ can be described as follows: arrived at site $x \in \mathbb{Z}$, the particle waits an exponential time of mean $1/(c(x, x - 1) + c(x, x + 1))$, after that it jumps to $x - 1$ and $x + 1$ with probability

$$
\frac{c(x, x - 1)}{c(x, x - 1) + c(x, x + 1)} \quad \text{and} \quad \frac{c(x, x + 1)}{c(x, x - 1) + c(x, x + 1)},
$$

respectively.

By a recursive procedure, one can always determine two positive functions $U$ and $H$ on $\mathbb{Z}$ such that

$$
c(x, y) = 1/\left[ H(x)U(x \lor y) \right], \quad \forall x, y \in \mathbb{Z} : |x - y| = 1.
$$

Moreover, the above functions $U$ and $H$ are univocally determined apart a positive factor $c$ multiplying $U$ and dividing $H$. Indeed, the system of equation (2.2) is equivalent to the system

$$
\begin{cases}
U(x + 1) = U(x)c(x, x - 1)/(c(x, x + 1)), \\
H(x) = 1/c(x, x - 1)U(x),
\end{cases}
\forall x \in \mathbb{Z}.
$$

We observe that $U$ is a constant function if and only if the jump rates $c(x, y)$ depend only on the starting point $x$. Taking without loss of generality $U \equiv 2$, we get that after arriving at site $x$ the random walk $X_t$ waits an exponential time of mean $H(x)$ and then jumps with equal probability to $x - 1$ and to $x + 1$. This special case is known in the physics literature as trap model [ABSO]. Similarly, we observe that $H$ is a constant function if and only if the jump rates $c(x, y)$ are symmetric, that is $c(x, y) = c(y, x)$ for all $x, y \in \mathbb{Z}$. Taking without loss of generality $H \equiv 1$, we get that $c(x, x - 1) = c(x - 1, x) = U(x)$.

This special case is known in the physics literature both as barrier model [ABSO] and as random walk among conductances, since $X_t$ corresponds to the random walk associated in a natural way to the linear resistor network with nodes given by the sites of $\mathbb{Z}$ and electrical filaments between nearest–neighbor nodes $x - 1, x$ having conductance $c(x - 1, x) = U(x)$ [DS]. If the rates $\{c(x, x \pm 1)\}_{x \in \mathbb{Z}}$ are random one speaks of random trap model, random barrier model and random walk among random conductances.

In order to describe some asymptotic spectral behavior as $n \uparrow \infty$, we consider a family $X(n)^{(x)}(t)$ of continuous–time nearest–neighbor random walks on $\mathbb{Z}_n := \{k/n : k \in \mathbb{Z}\}$ parameterized by $n \in \mathbb{N}_+ = \{1, 2, \ldots \}$. We call $c_n(x, y)$ the corresponding jump rates and
we fix positive functions $U_n, H_n$ satisfying the analogous of equation (2.3) (all is referred to $\mathbb{Z}_n$ instead of $\mathbb{Z}$). Below we denote by $L_n$ the pointwise operator

$$L_n f(x) = c_u(x, x - 1/n)[f(x - 1/n) - f(x)] + c_u(x, x + 1/n)[f(x + 1/n) - f(x)]$$  

(2.4)

defined at $x \in \mathbb{Z}_n$ for all functions $f$ whose domain contains $x - \frac{1}{n}, x, x + \frac{1}{n}$. The Markov generator of $X_t(\ell)$ with Dirichlet conditions outside $(0, 1)$ will be denoted by $L_n$. We recall that it is defined as the operator $L_n : \mathcal{V}_n \to \mathcal{V}_n$, where

$$\mathcal{V}_n := \{ f : [0, 1] \cap \mathbb{Z}_n \to \mathbb{C}, f(0) = f(1) = 0 \},$$

such that

$$L_n f(x) = \begin{cases} L_n f(x) & \text{if } x \in (0, 1) \cap \mathbb{Z}_n, \\ 0 & \text{if } x = 0, 1. \end{cases}$$

As discussed in Section 4 the operator $-L_n$ has $n - 1$ eigenvalues which are all simple and positive, while the related eigenvectors can be taken as real vectors. Below we write the eigenvalues as $\lambda_1^{(n)} < \lambda_2^{(n)} < \cdots < \lambda_{n-1}^{(n)}$.

In order to determine the suitable frame for the analysis of the eigenvalues and eigenvectors of $-L_n$, we recall some definitions from the theory of generalized second order differential operators $-D_m D_x$ (cf. [KK0], [DM], [K1][Appendix]), initially developed to analyze the behavior of a vibrating string. Let $m : \mathbb{R} \to [0, \infty)$ be a nondecreasing function with $m(x) = 0$ for all $x < 0$. Without loss of generality we can suppose that $m$ is càdlàg. We denote by $dm$ the Lebesgue–Stieltjes measure associated to $m$, i.e. the Radon measure on $\mathbb{R}$ such that

$$dm((a, b]) = m(b) - m(a),$$

$\forall a < b$.

We define $E_m$ as the support of $dm$, i.e. the set of points where $m$ increases:

$$E_m := \{ x \in [0, \infty) : m(x - \varepsilon) < m(x + \varepsilon) \forall \varepsilon > 0 \}. $$

(2.6)

We suppose that $E_m \neq \emptyset$, $0 = \inf E_m$ and $\ell_m := \sup E_m < \infty$. Then, $F \in C([0, \ell_m], \mathbb{C})$ is an eigenfunction with eigenvalue $\lambda$ of the generalized differential operator $-D_m D_x$ with Dirichlet boundary conditions if $F(0) = F(\ell_m) = 0$ and if it holds

$$F(x) = b x - \lambda \int_0^x dy \int_{(0, y]} dm(z) F(z), \quad \forall x \in [0, \ell_m],$$

(2.7)

for some constant $b$. We point out that (2.7) together with the boundary condition $F(0) = 0$ implies that $b = \lim_{x \to 0} (F(x + \varepsilon) - F(x))/\varepsilon$ and that $F$ must be linear on the intervals of $\mathbb{R} \setminus E_m$. The number $b$ is called derivative number and is denoted $F_x^\prime(0)$ (see Section 4 for further details).

As discussed in [L1], [L2], the operator $-D_m D_x$ with Dirichlet conditions outside $(0, \ell_m)$ is the generator of the quasidiffusion on $(0, \ell_m)$ with scale function $s(x) = x$ and speed measure $dm$, killed when reaching the boundary points $0, \ell_m$. This quasidiffusion can be suitably defined as time change of the standard one–dimensional Brownian motion [L2], [S].

The spectral analysis of $-L_n$ can be reduced to the spectral analysis of a suitable generalized differential operator $-D_m D_x$ as follows. We define the function $S_n : [0, 1] \cap \mathbb{Z}_n \to \mathbb{R}$ as

$$S_n(k/n) = \begin{cases} 0 & \text{if } k = 0, \\ \sum_{j=1}^k U_n(j/n) & \text{if } 1 \leq k \leq n. \end{cases}$$

(2.8)
To simplify the notation, we set
\[ x^{(n)}_k := S_n(k/n), \quad \text{for } k : 0 \leq k \leq n. \] (2.9)

Finally, we define the nondecreasing càdlàg function \( m_n : \mathbb{R} \to [0, \infty) \) as
\[ m_n(x) = \sum_{k:0 \leq k \leq n} H_n(k/n). \] (2.10)

Then
\[ dm_n = \sum_{k=0}^{n} H_n(k/n) \delta_{x^{(n)}_k}, \quad E_n := E_{m_n} = \{ x^{(n)}_k : 1 \leq k \leq n \}, \quad \ell_n := \ell_{m_n} = x^{(n)}_n. \]

We denote by \( C_n[0, \ell_n] \) the set of complex continuous functions on \([0, \ell_n]\) that are linear on \([0, \ell_n] \setminus E_n\). Then, the map
\[ T_n : C^{[0,1]} \cap \mathbb{Z} \ni f \to T_n f \in C_n[0, \ell_n], \] (2.11)
associating to \( f \) the unique function \( T_n f \in C_n[0, \ell_n] \) such that
\[ T_n f(x^{(n)}_k) = f(k/n), \quad 0 \leq k \leq n, \]
is trivially bijective. As discussed in Section 4, the map \( T_n \) defines also a bijection between the eigenvectors of \(-L_n\) with eigenvalue \( \lambda \) and the eigenfunctions of the differential operator \(-D_{m_n}D_x\) with Dirichlet conditions outside \((0, \ell_n)\) associated to the eigenvalue \( \lambda \).

We can finally state the asymptotic behavior of the small eigenvalues:

**Theorem 2.1.** Suppose that \( \ell_n \) converges to some \( \ell \in (0, \infty) \) and that \( dm_n \) weakly converges to a measure \( dm \), where \( m : \mathbb{R} \to [0, \infty) \) is a càdlàg function such that \( m(x) = 0 \) for all \( x \in (-\infty, 0) \). Assume that \( 0 = \inf E_m, \ell = \sup E_m \) and that \( dm \) is not a linear combination of a finite family of delta measures.

Then the generalized differential operator \(-D_{m}D_x\) with Dirichlet conditions outside \((0, \ell)\) has an infinite number of eigenvalues, which are all positive and simple. List these eigenvalues in increasing order as \( \{ \lambda_k : k \geq 1 \} \), and list the \( n-1 \) eigenvalues of the operator \(-\mathbb{I}_{m_n}\), which are all positive and simple, as \( \lambda^{(n)}_1 < \cdots < \lambda^{(n)}_{n-1} \). Then for each \( k \geq 1 \) it holds
\[ \lim_{n \to \infty} \lambda^{(n)}_k = \lambda_k. \] (2.12)

For each \( k \geq 1 \), fix an eigenfunction \( F_k \) with eigenvalue \( \lambda_k \) for the operator \(-D_{m}D_x\) with Dirichlet conditions. Then, by suitably choosing the eigenfunction \( F^{(n)}_k \in C([0, \ell_n]) \) of eigenvalue \( \lambda^{(n)}_k \) for the operator \(-D_{m_n}D_x\) with Dirichlet conditions, it holds
\[ \lim_{n \to \infty} F^{(n)}_k = F_k \quad \text{in } C([0, \ell + 1]) \quad \text{w.r.t. } \| \cdot \|_{\infty}, \] (2.13)
where \( F_k \) and \( F^{(n)}_k \) are set equal to zero on \((\ell, \ell + 1]\) and \((\ell_n, \ell_n + 1]\), respectively.

Since by hypothesis the supports of \( dm_n \) and \( dm \) are all included in a common compact subset, the above weak convergence of \( dm_n \) towards \( dm \) is equivalent to the vague convergence: \( \int_{\mathbb{R}} f(s)dm_n(s) \to \int_{\mathbb{R}} f(s)dm(s) \) for any function \( f \in C_c(\mathbb{R}) \) (i.e. continuous with compact support).

The proof of the above theorem is given in Section 7.
We describe now another general result relating self-similarity to the spectrum edge, whose application will be relevant below when studying subdiffusive random walks. Recall the definition (2.6) of $E_m$.

**Proposition 2.2.** Suppose that $m : [0, \infty) \to [0, \infty)$ is a random process such that

(i) $m(0) = 0$,

(ii) $m$ is càdlàg and increasing a.s.,

(iii) $m$ has stationary and independent increments,

(iv) extending $m$ to all $\mathbb{R}$ by setting $m \equiv 0$ on $(-\infty, 0)$, for any $x \in \mathbb{R}$ with probability one $x$ is not a jump point of $m$.

Then, a.s. all eigenvalues of the operator $-D_m D_x$ with Dirichlet conditions outside $(0, 1)$ are simple and positive, and form a diverging sequence $(\lambda_k(m) : k \geq 1)$ if labeled in increasing order. The same holds for the eigenvalues $(\lambda_k(m^{-1}) : k \geq 1)$ of the operator $-D_{m^{-1}} D_x$ with Dirichlet conditions outside $(0, m(1))$, where $m^{-1}$ denotes the càdlàg generalized inverse of $m$, i.e.

$$m^{-1}(t) = \inf\{s \geq 0 : m(s) > t\}, \quad t \geq 0.$$  (2.14)

Moreover, if there exists $x_0 > 0$ such that

$$\mathbb{E}[\mathbb{1}\{k \geq 1 : \lambda_k(m) \leq x_0\}] < \infty,$$  (2.15)

then there exist positive constants $c_1, c_2$ such that

$$c_1 x^{\frac{m}{m-1}} \leq \mathbb{E}[\mathbb{1}\{k \geq 1 : \lambda_k(m) \leq x\}] \leq c_2 x^{\frac{m}{m-1}}, \quad \forall x \geq 1.$$  (2.16)

Similarly, if there exists $x_0 > 0$ such that

$$\mathbb{E}[\mathbb{1}\{k \geq 1 : \lambda_k(m^{-1}) \leq x_0\}] < \infty,$$  (2.17)

then there exist positive constants $c_1, c_2$ such that

$$c_1 x^{\frac{m}{m-1}} \leq \mathbb{E}[\mathbb{1}\{k \geq 1 : \lambda_k(m^{-1}) \leq x\}] \leq c_2 x^{\frac{m}{m-1}}, \quad \forall x \geq 1.$$  (2.18)

Strictly speaking, in the above Proposition we had to write $-D_m D_x$ and $-D_{m^{-1}} D_x$ instead of $-D_m D_x$ and $-D_{m^{-1}} D_x$, respectively, where

$$m_+(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ m(x) & \text{if } 0 \leq x < 1, \\ m(1) & \text{if } x \geq 1, \end{cases}, \quad (m^{-1})_+(x) = \begin{cases} m^{-1}(x) & \text{if } 0 \leq x \leq m(1), \\ m^{-1}(m(1)) & \text{if } x > m(1). \end{cases}$$  (2.19)

This will be understood also below, in Theorems 2.3 and 2.5. Since $m$ is càdlàg, it has a countable (finite or infinite) number of jumps $\{z_i\}$. For $x > 0$ it holds

$$m^{-1}(x) = \begin{cases} y & \text{if } y = m(x), \ x \in [0, \infty) \setminus \{z_i\}, \\ z_i & \text{if } x \in [m(z_i-), m(z_i)] \text{ for some } i. \end{cases}$$  (2.20)

Since we have assumed $E_m = [0, \infty)$ a.s., $m^{-1}$ must be continuous a.s. (observe that the jumps of $m^{-1}$ correspond to the flat regions of $m$).

The proof of the above Proposition is given in Section 9 and is based on the Dirichlet–Neumann bracketing developed in Section 8 (cf. Theorem 8.8). When applying Proposition 2.2 we will present a simple argument to check (2.15) and (2.17).
As application of Theorem 2.1 and Proposition 2.2 we consider special families of subdiffusive random trap and barrier models (cf. ABSO, KK, FIN, BC1, BC2, FJL and references therein). To this aim we fix a family $\mathcal{T} := \{\tau(x) : x \in \mathbb{Z}\}$ of positive i.i.d. random variables in the domain of attraction of a one–sided $\alpha$–stable law, $0 < \alpha < 1$. This is equivalent to the fact that there exists some function $L_1(t)$ slowly varying as $t \to \infty$ such that

$$F(t) = \mathbb{P}(\tau(x) > t) = L_1(t)t^{-\alpha}, \quad t > 0.$$  

Let us define the function $h$ as

$$h(t) = \inf\{s \geq 0 : 1/F(s) \geq t\},$$  

Then, by Proposition 0.8 (v) in [R] we know that

$$h(t) = L_2(t)t^{1/\alpha} \quad t > 0,$$

for some function $L_2$ slowly varying as $t \to \infty$.

Finally, we denote by $V$ the double–sided $\alpha$–stable subordinator defined on some probability space $(\Xi, \mathcal{F}, \mathbb{P})$ (cf. B Section III.2). Namely, $V$ has a.s. càdlàg paths, $V(0) = 0$ and $V$ has non-negative independent increments such that for all $s < t$

$$E\left[\exp\{-\lambda[\tau(t) - \tau(s)]\}\right] = \exp\{-\lambda^\alpha(t-s)\}.$$  

(Strictly speaking, inside the exponential in the r.h.s. there should be an extra positive factor $c_0$ that we have fixed equal to 1). The sample paths of $V$ are strictly increasing and of pure jump type, in the sense that $V(u) = \sum_{0 < v \leq u} (V(v) - V(v-))$. Moreover, the random set $\{(u, V(u) - V(u-)) : u \in \mathbb{R}, V(u) > V(u-))\}$ is a Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity $cw^{-1-\alpha}du dw$, for some $c > 0$. Finally, we denote by $V^{-1}$ the generalized inverse function $V^{-1}(t) = \inf\{s \in \mathbb{R} : V(s) > t\}$. Since $V$ is strictly increasing $\mathcal{F}$–a.s., $V^{-1}$ has continuous paths $\mathcal{F}$–a.s.

For random trap models we obtain:

**Theorem 2.3.** Fix $a \geq 0$ and let $\mathcal{T} = \{\tau(x)\}_{x \in \mathbb{Z}}$ be a family of positive i.i.d. random variables in the domain of attraction of an $\alpha$–stable law, $0 < \alpha < 1$. If $a > 0$, assume also that $\tau(x)$ is bounded from below by a non–random positive constant a.s.

Given a realization of $\mathcal{T}$, consider the $\mathcal{T}$–dependent trap model $\{X(t)\}_{t \geq 0}$ on $\mathbb{Z}$ with transition rates

$$c(x, y) = \begin{cases} \tau(x)^{-1+\alpha} \tau(y)^{\alpha} & \text{if } |x - y| = 1 \\ 0 & \text{otherwise}. \end{cases}$$  

(2.24)

Call $\lambda_1^{(n)}(\mathcal{T}) < \lambda_2^{(n)}(\mathcal{T}) < \cdots < \lambda_{n-1}^{(n)}(\mathcal{T})$ the (simple and positive) eigenvalues of the Markov generator of $X(t)$ with Dirichlet conditions outside $(0, n)$. Then

i) For each $k \geq 1$, the $\mathcal{T}$–dependent random vector

$$\gamma^2 L_2(n)^{1+\frac{1}{n}} (\lambda_1^{(n)}(\mathcal{T}), \cdots, \lambda_k^{(n)}(\mathcal{T}))$$  

(2.25)

weakly converges to the $\mathcal{T}$–dependent random vector

$$(\lambda_1(V), \ldots, \lambda_k(V)),$$

where $\gamma = \mathbb{E}(\tau(0)^{-a})$, the slowly varying function $L_2$ has been defined in [2.22] and $\{\lambda_k(V) : k \geq 1\}$ denotes the family of the (simple and positive) eigenvalues of the generalized differential operator $-D_VD_x$ with Dirichlet conditions outside $(0, 1)$.  

ii) If \( a = 0 \) and \( \mathbb{E}(\exp\{-\lambda \tau(x)\}) = \exp\{-\lambda^a\} \), then in (2.25) the quantity \( L_2(n) \) can be replaced by the constant 1.

iii) There exist positive constants \( c_1, c_2 \) such that
\[
  c_1 x^{\frac{1}{1+\alpha}} \leq \mathbb{E}[\mathbb{E}(k \geq 1 : \lambda_k(V) \leq x)] \leq c_2 x^{\frac{1}{1+\alpha}}, \quad \forall x \geq 1. \tag{2.26}
\]

The above random walk \( X(t) \) can be described as follows: after arriving at site \( x \in \mathbb{Z} \) the particle waits an exponential time of mean
\[
  \tau(x)^{1-a} \over \tau(x-1)^a + \tau(x+1)^a
\]
after that it jumps to \( x-1 \) and \( x+1 \) with probability given respectively by
\[
  \frac{\tau(x-1)^a}{\tau(x-1)^a + \tau(x+1)^a} \quad \text{and} \quad \frac{\tau(x+1)^a}{\tau(x-1)^a + \tau(x+1)^a}.
\]

The random walk \( X(t) \) is called random trap model following [BC1], although according to our initial terminology the name would be correct only when \( a = 0 \). Sometimes we will also refer to the case \( a \in (0,1] \) as generalized random trap model. The additional assumption concerning the bound from below of \( \tau(x) \) when \( a > 0 \) can be weakened. Indeed, as pointed out in the proof, we only need the validity of strong LLN for a suitable triangular arrays of random variables.

Of course, one can consider also the diffusive case. Extending the results of [BD] we get

**Proposition 2.4.** Fix \( a \geq 0 \) and let \( \mathcal{T} = \{\tau(x)\}_{x \in \mathbb{Z}} \) be a family of positive random variables, ergodic w.r.t. spatial translations and such that \( \mathbb{E}(\tau(x)) < \infty, \mathbb{E}(\tau(x)^{-a}) < \infty \).

Given a realization of \( \mathcal{T} \), consider the \( \mathcal{T} \)-dependent trap model \( \{X(t)\}_{t \geq 0} \) on \( \mathbb{Z} \) with transition rates (2.24) and call \( \lambda_1^{(n)}(\mathcal{T}) < \lambda_2^{(n)}(\mathcal{T}) < \ldots < \lambda_{n-1}^{(n)}(\mathcal{T}) \) the (simple and positive) eigenvalues of the Markov generator of \( X(t) \) with Dirichlet conditions outside \((0,n)\). Then for each \( k \geq 1 \) and for a.a. \( \mathcal{T} \),
\[
  n^2 \mathbb{E}(\tau(x)^{-a}) \mathbb{E}(\tau(x)) \lambda_k^{(n)}(\mathcal{T}) \rightarrow \pi^2 k^2. \tag{2.27}
\]

Let us state our results concerning random barrier models:

**Theorem 2.5.** Let \( \mathcal{T} = \{\tau(x)\}_{x \in \mathbb{Z}} \) be a family of positive i.i.d. random variables in the domain of attraction of an \( \alpha \)-stable law, \( 0 < \alpha < 1 \).

Given a realization of \( \mathcal{T} \), consider the \( \mathcal{T} \)-dependent barrier model \( \{X(t)\}_{t \geq 0} \) on \( \mathbb{Z} \) with jump rates
\[
  c(x,y) = \begin{cases} 
    \tau(x \lor y)^{-1} & \text{if } |x-y| = 1 \\
    0 & \text{otherwise}. 
  \end{cases} \tag{2.28}
\]

Call \( \lambda_1^{(n)}(\mathcal{T}) < \lambda_2^{(n)}(\mathcal{T}) < \ldots < \lambda_{n-1}^{(n)}(\mathcal{T}) \) the eigenvalues of the Markov generator of \( X(t) \) with Dirichlet conditions outside \((0,1)\). Recall the definition (2.29) of the positive slowly varying function \( L_2 \). Then:

i) For each \( k \geq 1 \), the \( \mathcal{T} \)-dependent random vector
\[
  L_2(n)^{1+\frac{1}{\alpha}}(\lambda_1^{(n)}(\mathcal{T}),\ldots,\lambda_k^{(n)}(\mathcal{T}))
\]
weakly converges to the \( V \)-dependent random vector
\[
  (\lambda_1(V^{-1}),\ldots,\lambda_k(V^{-1}))
\]

(2.29)
where \( \{ \lambda_k(V^{-1}) : k \geq 1 \} \) denotes the family of the (simple and positive) eigenvalues of the generalized differential operator \(-D_{V^{-1}}D_x\) with Dirichlet conditions outside \((0, V(1))\).

i) If \( \mathbb{E}(e^{-\lambda(x)}V) = e^{-\lambda^o} \) then in (2.29) the quantity \( L_2(n) \) can be replaced by the constant 1.

iii) There exist positive constants \( c_1, c_2 \) such that

\[
\frac{c_1}{n^{1+\alpha}} \leq \mathbb{E} \left[ \sharp\{ k \geq 1 : \lambda_k(V^{-1}) \leq x \} \right] \leq \frac{c_2}{n^{1+\alpha}}, \quad \forall x \geq 1. \tag{2.30}
\]

Again, one can consider also the diffusive case. Extending the results of [BD] we get

**Proposition 2.6.** Let \( T = \{ \tau(x) \}_{x \in \mathbb{Z}} \) be a family of positive random variables, ergodic w.r.t. spatial translations and such that \( \mathbb{E}(\tau(x)) < \infty \). Given a realization of \( T \), consider the \( T \)-dependent barrier model \( \{ X(t) \}_{t \geq 0} \) on \( \mathbb{Z} \) with transition rates (2.28) and call \( \lambda_1^{(n)}(T) < \lambda_2^{(n)}(T) < \cdots < \lambda_{n-1}^{(n)}(T) \) the (simple and positive) eigenvalues of the Markov generator of \( X(t) \) with Dirichlet conditions outside \((0, n)\). Then for each \( k \geq 1 \) and for a.a. \( T \),

\[
n^2 \mathbb{E}(\tau(x)) \lambda_k^{(n)}(T) \to \pi^2 k^2. \tag{2.31}
\]

Theorem 2.3 and 2.5 cannot be derived by a direct application of Theorem 2.1. Indeed, for any choice of the sequence \( c(n) > 0 \), fixed a realization of \( T \) the measures \( dm_n \) associated to the space–time rescaled random walks \( X^{(n)}(t) = n^{-1}X(c(n)t) \) do not converge to \( dV \) or \( dV^{-1} \) restricted to \((0, 1), (0, V(1))\) respectively. On the other hand, for each \( n \) one can define a random field \( T_n \) in terms of the \( \alpha \)-stable process \( V \), i.e. \( T_n = F_n(V) \), having the same law of \( T \). Calling \( \tilde{X}^{(n)} \) the analogous of \( X^{(n)} \) with jump rates defined in terms of \( T_n \), one has that the associated measures \( d\tilde{m}_n \) satisfy the hypothesis of Theorem 2.1. This explains why Theorems 2.3 and 2.5 give an annealed and not quenched result. On the other hand, for the random walks \( X^{(n)} \) the result is quenched, i.e. the convergence of the eigenvalues holds for almost all realizations of the subordinator \( V \). We refer to Sections 10 and 11 for a more detailed discussion of the above coupling and for the proof of Theorems 2.3 and 2.5.

**2.1. Outline of the paper.** The paper is structured as follows. In Section 3 we explain how the spectral analysis of \(-L_n\) reduces to the spectral analysis of the operator \(-D_{m_n}D_x\). In Section 4 we recall some basic facts of generalized second order operators. In particular, we characterize the eigenvalues of \(-L_n\) as zeros of a suitable entire function. In Section 5 we apply some general theorem about the dependence on the parameter of the zeros of a continuously parameterized family of entire functions. In Section 6 we investigate the eigenvalues of \(-D_{m_n}D_x\) using the minimum–maximum characterization. This completes the preparation to the proof of Theorem 2.1 which is given in Section 7.

In Section 8 we prove the Dirichlet–Neumann bracketing. This result, interesting by itself, allows us to prove Proposition 2.2 in Section 9. Finally, we move to applications: in Section 10 we prove Theorem 2.3, in Section 11 we prove Theorem 2.5, while in Section 12 we prove Propositions 2.4 and 2.6.

3. FROM \(-L_n\) TO \(-D_{m_n}D_x\)

Recall the definition of the local operator \( L_n \) given in (2.4) and of the bijection \( T_n \) given in (2.11).
Lemma 3.1. Given functions \(f, g : [0, 1] \cap Z_n \to \mathbb{R}\), the system of identities
\[
L_n f(x) = g(x), \quad \forall x \in (0, 1) \cap Z_n ,
\]
is equivalent to the system
\[
f(x) = f(0) + \sum_{j=1}^{n_x} U_n(j/n) \left( \frac{f(1/n) - f(0)}{U_n(1/n)} + \sum_{k=1}^{j-1} H_n(k/n)g(k/n) \right) , \quad \forall x \in (0, 1) \cap Z_n ,
\]
where we convey to set \(\sum_{k=1}^{0} H_n(k/n)g(k/n) = 0\). Setting \(F = T_n f\), \(G = T_n g\) and
\[
b = \frac{F(x_1^{(n)}) - F(0)}{U_n(1/n)} - H_n(0)G(0) ,
\]
is equivalent to
\[
F(x) = F(0) + bx + \int_{0}^{x} dy \int_{(0,y)} G(z) dm_n(z) , \quad \forall x \in [0, \ell_n] .
\]
In particular, \(f : [0, 1] \cap Z_n \to \mathbb{R}\) is an eigenvector with eigenvalue \(\lambda\) of the operator \(-D_n\)
if and only if \(T_n f\) is an eigenfunction with eigenvalue \(\lambda\) of the generalized differential
operator \(-D_n D_x\) with Dirichlet conditions outside \((0, \ell_n)\).

Proof. For simplicity of notation we write \(U, H\) instead of \(U_n, H_n\). Moreover, we use the
natural bijection \(Z \ni k \to k/n \in Z_n\), denoting the point \(k/n\) of \(Z_n\) simply as \(k\). Setting \(\Delta f(j) = f(j) - f(j - 1)\), we can rewrite (3.1) by means of the recursive identities
\[
\frac{\Delta f(j + 1)}{U(j + 1)} = H(j)g(j) + \frac{\Delta f(j)}{U(j)} , \quad \forall j \in (0, n) \cap Z .
\]
This system of identities is equivalent to
\[
\Delta f(j + 1) = U(j + 1) \left( \frac{\Delta f(1)}{U(1)} + \sum_{k=1}^{j-1} H(k)g(k) \right) , \quad \forall j \in (0, n) \cap Z .
\]
Writing \(f(x) = f(1) + \sum_{j=2}^{x} \Delta f(j)\) for all \(x \in (0, n] \cap Z\), (3.4) becomes equivalent to
\[
f(x) = f(1) + \sum_{j=2}^{x} U(j) \left( \frac{\Delta f(1)}{U(1)} + \sum_{k=1}^{j-1} H(k)g(k) \right) = \]
\[
f(0) + \sum_{j=1}^{x} U(j) \left( \frac{\Delta f(1)}{U(1)} + \sum_{k=1}^{j-1} H(k)g(k) \right) , \quad \forall x \in (0, n] \cap Z .
\]
This proves that (3.1) is equivalent to (3.2). Using \(T_n, F, G, m_n\) we can rewrite (3.2) as
\[
F(x) = F(0) + \int_{0}^{x} dy \left( \frac{F(x_1^{(n)}) - F(0)}{U(1)} + \int_{(0,y)} G(z) dm_n(z) \right) , \quad \forall x \in (0, \ell_n] .
\]
Indeed, for \( x = x_k^{(n)} \), \( k \geq 1 \), one can write

\[
\int_0^x dy \int_{(0,y)} G(z) dm_n(z) = \sum_{j=1}^{k-1} [x_{j+1}^{(n)} - x_j^{(n)}] \sum_{i=1}^j G(x_i^{(n)}) dm_n(\{x_i^{(n)}\}) = \\
\sum_{j=1}^{k-1} U(j+1) \sum_{i=1}^j g(i) H(i) = \sum_{j=2}^k U(j) \sum_{i=1}^{j-1} g(i) H(i) + \sum_{j=1}^k g(i) H(i),
\]

where in the last identity we have used the convention that \( \sum_{i=1}^0 g(i) H(i) = 0 \). From the above identity it is simple to prove that \((3.2)\) is equivalent to \((3.5)\) for \( x = x_k^{(n)} \), \( k \geq 1 \).

The validity of \((3.5)\) for \( x \in \{x_k^{(n)} : k \geq 1\} \) automatically extends to all \( x \in (0, \ell_m) \).

This concludes the proof of the equivalence between \((3.2)\) and \((3.5)\). Trivially, equation \((3.5)\) is equivalent to \((3.3)\). Finally, the conclusion of the lemma follows from the previous observations and the discussion about the generalized differential operator \(-D_mD_x\) given in the Introduction. \( \square \)

4. Generalized second order differential operators

For the reader’s convenience and for next applications, we recall the definition of generalized differential operator. We mainly follow \([\text{KK}0]\), with some slight modifications that we will point out. We refer to \([\text{KK}0], \text{DM}\) and \([\text{Ma}]\) for a detailed discussion.

Let \( m : \mathbb{R} \to [0, \infty) \) be a càdlàg nondecreasing function with \( m(x) = 0 \) for all \( x < 0 \).

We define \( m_x \) as the magnitude of the jump of the function \( m \) at the point \( x \):

\[
m_x = m(x) - m(x-), \quad x \in \mathbb{R}.
\]

We define \( E_m \) as the support of \( dm \), i.e. the set of points where \( m \) increases (see \((2.6)\)). We suppose that \( E_m \neq \emptyset \), \( 0 = \inf E_m \) and \( \ell_m := \sup E_m < \infty \).

Given a continuous function \( F(x) \in C([0, \ell_m]) \) and a \( dm \)-integrable function \( G \) on \([0, \ell_m] \) we write \(-D_mD_x F = G\) if there exist complex constants \( a, b \) such that

\[
F(x) = a + bx - \int_0^x dy \int_{(0,y)} dm(z) G(z), \quad \forall x \in [0, \ell_m].
\]

We remark that the integral term in equation \((4.2)\) can be written also as

\[
\int_0^x dy \int_{(0,y)} dm(z) G(z) = \int_{[0,x]} (x - z) G(z) dm(z) = \int_0^x dy \int_{[0,y]} dm(z) G(z).
\]

We point out that equation \((4.2)\) implies that \( F \) is linear on \([x_1, x_2] \) if \( m \) is constant on \((x_1, x_2) \subset [0, \ell_m] \).

As discussed in \([\text{KK}0]\), the function \( G \) is not univocally determined from \( F \). To get uniqueness, one can for example fix the value of \( b \) and \( b - \int_{[0, \ell_m]} G(s) dm(s) \). These values are called derivative numbers and denoted by \( F'_x(0) \) and \( F'_+ (\ell_m) \), respectively. Indeed, in \([\text{KK}0]\) the domain \( D_m \) of the differential operator \(-D_mD_x\) is defined as the family of complex–valued extended functions \( F[x] \), given by the triple \((F(x), F'_x(0), F'_+ (\ell_m))\), while the authors set \(-D_mD_x F[x] = G(x)\). We prefer to avoid the notion of extended functions here, since not necessarily.

It is simple to check that the function \( F \) satisfying \((4.2)\) fulfills the following properties:

for each \( x \in [0, \ell_m] \) the function \( F(x) \) has right derivative \( F'_+(x) \), for each \( x \in (0, \ell_m] \) the
function $F(x)$ has left derivative $F'_+(x)$ and
\[ F'_+(x) = b - \int_{[0,x]} G(s)dm(s), \quad x \in [0, \ell_m), \quad (4.3) \]
\[ F'_-(x) = b - \int_{[0,x]} G(s)dm(s), \quad x \in (0, \ell_m]. \quad (4.4) \]

In view of the definition of $F'_+(0)$ and $F'_-(\ell_m)$, the above identities extend to any $x \in [0, \ell_m]$. In addition, if $m_0 = 0$ then $F'_+(0) = \lim_{\varepsilon \downarrow 0} F'_+(\varepsilon)$, while if $m_{\ell_m} = 0$ then $F'_-(\ell_m) = \lim_{\varepsilon \uparrow 0} F'_-(\ell_m - \varepsilon)$.

As discussed in [KK0], fixed $\lambda \in \mathbb{C}$ there exists a unique function $F \in C([0, \ell_m])$ solving equation (4.2) with $G = \lambda F$ for fixed $a, b$. In other words, fixed $F(0)$ and $F'_-(0)$ there exists a unique solution of the homogeneous differential equation
\[ -D_m D_x F = \lambda F. \quad (4.5) \]

Given $\lambda \in \mathbb{C}$, we define $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ as the solutions (4.5) satisfying respectively the initial conditions
\[ \varphi(0, \lambda) = 1, \quad \varphi'_-(0, \lambda) = 0, \quad (4.6) \]
\[ \psi(0, \lambda) = 0, \quad \psi'_-(0, \lambda) = 1. \quad (4.7) \]

It is known that each function $F \in C([0, \ell_m])$ satisfying (4.5) is a linear combination of the independent solutions $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$. Finally, $F \neq 0$ is called an eigenfunction of the operator $-D_m D_x$ with Dirichlet [Neumann] b.c. if $F$ solves (4.5) for some $\lambda \in \mathbb{C}$, and moreover $F(0) = F(\ell_m) = 0 \ [F'_+(0) = F'_-(\ell_m) = 0]$. By the above observations, we get that $F$ is a Dirichlet eigenfunction if and only if $F$ is a nonzero multiple of $\psi(x, \lambda)$ for $\lambda \in \mathbb{C}$ satisfying $\psi(\ell_m, \lambda) = 0$, while $F$ is a Neumann eigenfunction if and only if $F(x)$ is a nonzero multiple of $\varphi(x, \lambda)$ with $\lambda \in \mathbb{C}$ satisfying
\[ \int_0^\ell \varphi(s, \lambda)dm(s) = 0. \quad (4.8) \]

In particular, the Dirichlet and the Neumann eigenvalues are all simple.

The following fact should be more or less standard. Since we were unable to find a self-contained reference, for the reader's convenience we sketch its (very short) proof in Appendix [A].

Lemma 4.1. Let $m : \mathbb{R} \to [0, \infty)$ be a nondecreasing càdlàg function such that $m(x) = 0$ for $x < 0$, $0 = \inf E_m$, $\ell_m := \sup E_m < \infty$. Then the differential operator $-D_m D_x$ with Dirichlet conditions outside $(0, \ell_m)$ has a countable (finite or infinite) family of eigenvalues, which are all positive and simple. The set of eigenvalues has no accumulation points. In particular, if there is an infinite number of eigenvalues $\{\lambda_n\}_{n \geq 1}$, listed in increasing order, it must be $\lim_{n \to \infty} \lambda_n = \infty$.

The above eigenvalues coincide with the zeros of the entire function $\mathbb{C} \ni \lambda \to \psi(\ell_m, \lambda) \in \mathbb{C}$. The eigenspace associated to the eigenvalue $\lambda$ is spanned by the real function $\psi(\cdot, \lambda)$. Moreover, $F$ is an eigenfunction of $-D_m D_x$ with Dirichlet conditions outside $(0, \ell_m)$ and associated eigenvalue $\lambda$ if and only if
\[ F(x) = \lambda \int_{[0,\ell_m]} G_{0,\ell_m}(x, y) F(y)dm(y), \quad \forall x \in [0, \ell_m], \quad (4.9) \]
where, given an interval \([a, b]\), the Dirichlet Green function \(G_{a,b} : [a, b]^2 \to \mathbb{R}\) is defined as
\[
G_{a,b}(x, y) = \begin{cases} \frac{(y-a)(b-x)}{(x-a)(b-y)} & \text{if } y \leq x, \\ \frac{(x-a)(b-y)}{(x-a)(b-y)} & \text{if } x \leq y. \end{cases}
\] (4.10)

In particular, for any Dirichlet eigenvalue \(\lambda\) it holds
\[
\lambda \geq [\ell_m m(\ell_m)]^{-1}.
\] (4.11)

As discussed in [KK0], page 30, the function \(\varphi\) can be written as \(\lambda\)-power series
\[
\varphi(s, \lambda) = \sum_{j=0}^{\infty} (-\lambda)^j \varphi_j(s)
\]
for suitable functions \(\varphi_j\). Therefore the l.h.s. of (4.8) equals
\[
\sum_{j=0}^{\infty} (-\lambda)^j \int_{(0,1)} \varphi_j(s) dm(s).
\]
From the bounds on \(\varphi_j\) one derives that the l.h.s. of (4.8) is an entire function in \(\lambda\), thus implying that its zeros (or equivalently the eigenvalues of the operator \(-D_m D_x\) with Neumann b.c.) form a discrete subset of \([0, \infty)\). Moreover (cf. [KK0]) the eigenvalues are nonnegative and 0 itself is an eigenvalue.

5. Characterization of the eigenvalues as zeros of entire functions

At this point, we have reduced the analysis of the spectrum of the differential operator \(-D_m D_x\) with Dirichlet conditions outside \((0, \ell_m)\) to the analysis of the zeros of the entire function \(\psi(\ell, \cdot)\). As in [KZ] and [Za] a key tool is the following result:

Lemma 5.1. Let \(\Xi\) be a metric space, \(f : \Xi \times \mathbb{C} \to \mathbb{C}\) be a continuous function such that for each \(\alpha \in \Xi\) the map \(f(\alpha, \cdot)\) is an entire function. Let \(V \subset \mathbb{C}\) be an open subset whose closure \(\overline{V}\) is compact, and let \(\alpha_0 \in \Xi\) be such that no zero of the function \(f(\alpha_0, \cdot)\) is on the boundary of \(V\). Then there exists a neighborhood \(W\) of \(\alpha_0\) in \(\Xi\) such that:

(1) for any \(\alpha \in W\), \(f(\alpha, \cdot)\) has no zero on the boundary of \(V\),
(2) the sum of the orders of the zeros of \(f(\alpha, \cdot)\) contained in \(V\) is independent of \(\alpha\) as \(\alpha\) varies in \(W\).

Proof. See page 248 in [Di]. □

From now on, let \(m_n\) and \(m\) be as in Theorem 2.1. Given \(\lambda \in \mathbb{C}\), define \(\varphi(x, \lambda)\) and \(\psi(x, \lambda)\) as the solutions on the homogeneous differential equation (4.3) satisfying the initial conditions (4.6) and (4.7) respectively. Define similarly \(\varphi^{(n)}(x, \lambda)\) and \(\psi^{(n)}(x, \lambda)\) by replacing \(m\) with \(m_n\).

By applying Lemma 4.1 and Lemma 5.1 we obtain:

Lemma 5.2. Let \(m_n\) and \(m\) be as in Theorem 2.1. Fix a constant \(L > 0\) different from the Dirichlet eigenvalues of \(-D_m D_x\), and let \(\{\lambda_i : 1 \leq i \leq k_0\}\) be the Dirichlet eigenvalues of \(-D_m D_x\) smaller than \(L\). Let \(\varepsilon > 0\) be such (i) \(\lambda_{k_0} + \varepsilon < L\) and (ii) each interval \(J_i := [\lambda_i - \varepsilon, \lambda_i + \varepsilon]\) intersects \(\{\lambda_i : 1 \leq i \leq k_0\}\) only at \(\lambda_i\), for any \(i : 1 \leq i \leq k_0\). Then there exists an integer \(n_0\) such that:

i) for all \(n \geq n_0\), the spectrum of \(-L_n\) has only one eigenvalue in \(J_i\),
ii) for all \(n \geq n_0\), \(-L_n\) has no eigenvalue inside \((0, L) \setminus \bigcup_{i=1}^{k_0} J_i\).

Proof. As discussed in [KK0], page 30, one can write explicitly the power expansion of the entire functions \(\mathbb{C} \ni \lambda \to \psi^{(n)}(x, \lambda), \psi(x, \lambda) \in \mathbb{C}\). In particular, it holds
\[
\psi(x, \lambda) = \sum_{j=0}^{\infty} (-\lambda)^j \psi_j(x), \quad \psi^{(n)}(x, \lambda) = \sum_{j=0}^{\infty} (-\lambda)^j \psi^{(n)}_j(x),
\]
(5.1)
where
\[
\psi_0(x) \equiv x, \quad \psi_{j+1}(x) = \int_0^x (x-s)\psi_j(s)\,dm(s), \quad \forall j \geq 0, \ x \in \[0, \ell]\,
\]
\[
\psi_0^{(n)}(x) \equiv x, \quad \psi_{j+1}^{(n)}(x) = \int_0^x (x-s)\psi_j^{(n)}(s)\,dm_n(s), \quad \forall j \geq 0, \ x \in \[0, \ell_n]\.
\]
In the above integrals we do not need to specify the border of the integration domain since the integrand functions vanish both at 0 and at \(x\).

We already know that the Dirichlet eigenvalues of the operator \(-D_{mn}D_x \, [-D_mD_x]\) are given by the zeros of the entire function \(\psi^{(n)}(\ell_n, \cdot) [\psi(\ell, \cdot)]\). Hence, it is natural to derive the thesis by applying Lemma 5.1 with different choices of \(V\). More precisely, we take \(\alpha_0 = \infty\) and \(\Xi = \mathbb{N}_+ \cup \{\infty\}\) endowed of any metric \(d\) such that all points \(n \in \mathbb{N}_+\) are isolated w.r.t. \(d\) and \(\lim_{n \to \infty} d(n, \infty) = 0\). We define \(f : \Xi \times \mathbb{C} \to \mathbb{C}\) as
\[
f(\alpha, \lambda) = \begin{cases} 
\psi^{(n)}(\ell_n, \lambda) & \text{if } \alpha = n, \\
\psi(\ell, \lambda) & \text{if } \alpha = \infty.
\end{cases}
\]
Finally, we choose \(V = (\lambda_i - \varepsilon, \lambda_i + \varepsilon)\) as \(i\) varies in \(\{1, \ldots, k_0\}\) and after that we take \(V = (0, L) \setminus \left( \bigcup_{r=1}^{k_0} J_r \right)\).

By construction, \(f(\alpha, \cdot)\) is an entire function for any \(\alpha \in \Xi\). Moreover, \(f(\alpha_0, \cdot)\) has no zero at the border of \(V\) for any of the above choices of \(V\), \(f(\alpha_0, \cdot)\) has only one zero (which is of order 1) in \((\lambda_i - \varepsilon, \lambda_i + \varepsilon)\) for any \(i : 1 \leq i \leq k_0\), while it has no zero in the set \((0, L) \setminus \left( \bigcup_{r=1}^{k_0} J_r \right)\). Hence, the thesis follows from Lemma 5.1 if we prove that \(f : \Xi \times \mathbb{C} \to \mathbb{C}\) is continuous, i.e.
\[
\lim_{n \to \infty} \psi^{(n)}(\ell_n, \lambda_n) = \psi(\ell, \lambda)
\]
for any sequence of complex numbers \(\{\lambda_n\}_{n \geq 1}\), converging to some \(\lambda \in \mathbb{C}\).

In order to prove the above statement, we observe that \(\psi_j(x) \geq 0\), \(\psi_0(\ell) = \ell\) and that for \(j \geq 1\) it holds
\[
\psi_j(\ell) = \int_{(0,\ell)} dm(s_1) \int_{(0,s_1)} dm(s_2) \int_{(0,s_2)} dm(s_3) \cdots \\
\int_{(0,s_{j-1})} dm(s_j) (\ell - s_1)(s_1 - s_2)(s_2 - s_3) \cdots (s_{j-1} - s_j)s_j \leq \\
\ell \int_{(0,\ell)} dm(s_1)dm(s_2) \cdots dm(s_j) \left[ \prod_{u=1}^{j} (\ell - s_u) \right] \mathbb{I}(s_j < s_{j-1} < \cdots < s_2 < s_1).
\]
Above, \(\mathbb{I}(\cdot)\) denotes the characteristic function. By symmetry we can remove the characteristic function and earn a factor \(1/j!\). Therefore we get
\[
\psi_j(\ell) \leq \frac{\ell}{j!} \left[ \int_{(0,\ell)} dm(s)(\ell - s) \right]^j. \tag{5.4}
\]
Similarly we get
\[
\psi_j^{(n)}(\ell_n) \leq \frac{\ell_n}{j!} \left[ \int_{(0,\ell_n)} dm_n(s)(\ell_n - s) \right]^j. \tag{5.5}
\]
Since \(\ell_n \to \ell\) and \(\sup_n m_n(\ell_n) < \infty\), we can find positive constants \(c\) and \(A\) such that the r.h.s. of \((5.4)\) and the r.h.s. of \((5.5)\) are bounded by \(Ac^j/j!\).
Let us come back to (5.2) considering first the case $\lambda = 0$. Then $\psi(\ell, \lambda) = \ell$ while by the above bound $\psi^{(n)}(\ell_n, \lambda_n) = \ell_n + \mathcal{E}$, where

$$|\mathcal{E}| \leq \sum_{j=1}^{\infty} A(c|\lambda_n|)^j/j! = A\exp(c|\lambda_n|) - A.$$ 

Since $\lambda_n \to \lambda = 0$ as $n \to \infty$, the above bound implies (5.2).

Let us consider now the case $\lambda \neq 0$. Since $\lambda_n \to \lambda$ we restrict to $n$ large enough that $|\lambda_n/\lambda| \leq 2$. We introduce a complex-valued measure $\nu$ on $\mathbb{N}$ setting $\nu(j) = (-\lambda)^j/j!$. Moreover we write $|\nu|$ for the positive measure on $\mathbb{N}$ such that $|\nu|(j) = |\nu(j)|$. Finally, we set

$$a(j) = j!\psi_j(\ell), \quad a^{(n)}(j) = j!(\lambda_n/\lambda)^j\psi_j^{(n)}(\ell_n), \quad c(j) = A(2c)^j.$$ 

Then we can write

$$\psi^{(n)}(\ell_n, \lambda_n) = \sum_{j \in \mathbb{N}} \nu(j)a^{(n)}(j), \quad \psi(\ell, \lambda) = \sum_{j \in \mathbb{N}} \nu(j)a(j).$$ 

Since $|a^{(n)}(j)|, |a(j)| \leq c(j)$ and $c(\cdot) \in L^1(\mathbb{N}, |\nu|)$, by the Lebesgue Theorem in order to conclude we only need to show that $\lim_{n \to \infty} a^{(n)}(j) = a(j)$ for all $j \geq 0$, i.e. $\lim_{n \to \infty} \psi_j^{(n)}(\ell_n) = \psi_j(\ell)$ for all $j \geq 0$. The case $j = 0$ follows from our assumption $\ell_n \to \ell$. In order to avoid heavy notation, we discuss only the case $j = 2$ (the general case is completely similar). Let us set

$$C_n := \int_0^\infty \, dm_n(s_1) \int_{(0,s_1)} \, dm_n(s_2)(\ell - s_1)(s_1 - s_2)s_2.$$ 

Recalling (5.3), we can bound

$$|\psi_2^{(n)}(\ell_n) - C_n| \leq |\ell - \ell_n|m_n(\ell_n)^2\ell_n^2 \to 0, \quad \text{as } n \to \infty. \quad (5.6)$$ 

Let us fix $\gamma > \ell$, thus implying that $\gamma > \ell_n$ for $n$ large enough as we assume. Moreover, we fix a continuous function $\rho : [0, \infty) \to [0, 1]$ such that $\rho \equiv 1$ on $[0, \gamma]$ and $\rho \equiv 0$ on $[\gamma + 1, \infty)$. Then, the function

$$F(s_1, s_2) := (\ell - s_1)(s_1 - s_2)s_2[0 < s_2 < s_1]\rho(s_1)$$ 

is continuous on $[0, \infty)^2$ with support in the triangle $\{(s_1, s_2) : 0 \leq s_2 \leq s_1 \leq \gamma + 1\}$. Writing $dm_n \otimes dm(F)$ for the integral of the function $F$ w.r.t. the product measure $dm \otimes dm$ and similarly for $dm_n \otimes dm_n(F)$, we get $\psi_2(\ell) = dm \otimes dm(F)$ and $C_n = dm_n \otimes dm_n(F)$. Since $dm_n$ weakly converges to $dm$, the same property holds for $dm_n \otimes dm_n$ and $dm \otimes dm$. Using that $F \in C_c([0, \infty)^2)$ we conclude that $C_n = \psi_2(\ell) + o(1)$. Together with the above result $C_n = \psi_2^{(n)}(\ell_n) + o(1)$ (see (5.6)), we get the thesis. \hfill \Box

The above lemma is still not enough in order to prove that $-D_mD_x$ has infinite eigenvalues $\lambda_k$ and that $\lambda_k^{(n)} \to \lambda_k$. As explained in Section [7] we only need to prove that the sequence $\{\lambda_k^{(n)}\}_{n \geq k}$ is bounded. This will be done in the next section, using a different characterization of the eigenvalues $\lambda_k^{(n)}$.
6. Minimum–maximum characterization of the eigenvalues

For the reader’s convenience, we list some vector spaces that will be repeatedly used in what follows. We introduce the vector spaces \( \mathcal{A}(n) \) and \( \mathcal{B}(n) \) as

\[
\mathcal{A}(n) := \{ f : [0, 1] \cap \mathbb{Z}_n \to \mathbb{R} : f(0) = f(1) = 0 \}, \quad \mathcal{B}(n) = T_n \mathcal{A}(n),
\]

where the map \( T_n \) has been defined in (2.11). Hence \( F \in \mathcal{B}(n) \) if and only if (i) \( F(0) = F(1) = 0 \), (ii) \( F \) is continuous and (iii) \( F \) is linear on all subintervals \([x_{j-1}^{(n)}, x_j^{(n)}], 1 \leq j \leq n\).

Since we already know that the eigenvalues and suitable associated eigenfunctions of \(-\mathbb{L}_n\) are real, we can think of \(-\mathbb{L}_n\) as operator defined on \( \mathcal{A}(n) \). Finally, given \( a < b \) we write \( C_0[a, b] \) for the family of continuous functions \( f : [a, b] \to \mathbb{R} \) such that \( f(a) = f(b) = 0 \).

Let us recall the min–max formula characterizing the \( k \)-th eigenvalue \( \lambda_k^{(n)} \) of \(-\mathbb{L}_n\), or equivalently of the differential operator \(-\mathcal{D}_{m_n} \mathcal{D}_x\) with Dirichlet conditions outside \((0, \ell_n)\). We refer to [CH], [RS] for more details. First we observe the validity of the detailed balance equation:

\[
H_n(x) c_n(x, x + \frac{1}{n}) = \frac{1}{U_n(x + 1/n)} = H_n(x + \frac{1}{n}) c_n(x + \frac{1}{n}, x) \quad \forall x \in \mathbb{Z}_n.
\]

Identifying \( \mathcal{A}(n) \) with \( \{ f : (0, 1) \cap \mathbb{Z}_n \to \mathbb{R} \} \), this implies that \(-\mathbb{L}_n\) is a symmetric operator in \( L^2((0, 1) \cap \mathbb{Z}_n, \mu_n) \), where \( \mu_n := \sum_{x \in (0, 1) \cap \mathbb{Z}_n} H_n(x) \delta_x \). Indeed, writing \( \tilde{\mu}_n = \sum_{x \in \mathbb{Z}_n} H_n(x) \delta_x \), \(-\mathbb{L}_n\) for the generator on the random walk on \( \mathbb{Z}_n \) with jump rates \( c_n(x, y) \) and defining \( \tilde{f} : \mathbb{Z}_n \to \mathbb{R} \) as \( \tilde{f}(x) = f(x) \mathbb{I}(x \in (0, 1)) \) for any \( f \in \mathcal{A}(n) \), it holds

\[
\mu_n(f, -\mathbb{L}_n g) = \tilde{\mu}_n(\tilde{f}, -\mathbb{L}_n \tilde{g}) = \tilde{\mu}_n(\tilde{g}, -\mathbb{L}_n \tilde{f}) = \mu_n(g, -\mathbb{L}_n f), \quad f, g \in \mathcal{A}(n).
\]

Note that the second identity follows from (6.2).

Given \( f \in \mathcal{A}(n) \) we write \( D_n(f) \) for the Dirichlet form \( D_n(f) := \mu_n(f, -\mathbb{L}_n f) \). By simple computations, we obtain

\[
D_n(f) = \sum_{j=1}^n U_n(j/n)^{-1} \left[ f(j/n) - f((j - 1)/n) \right]^2.
\]

Note that \( D_n(f) = 0 \) with \( f \in \mathcal{A}(n) \) if and only if \( f \equiv 0 \). The min–max characterization of \( \lambda_k^{(n)} \) is given by the formula

\[
\lambda_k^{(n)} = \min_{V_k} \max_{f \in V_k : f \neq 0} \frac{D_n(f)}{\mu_n(f)^2}, \quad (6.3)
\]

where \( V_k \) varies among the \( k \)-dimensional subspaces of \( \mathcal{A}(n) \). Moreover, the minimum is attained at \( V_k = V_k^{(n)} \), defined as the subspace spanned by the eigenvectors \( f_j^{(n)} \) associated to the first \( k \) eigenvalues \( \{ \lambda_j^{(n)} : 1 \leq j \leq k \} \).

We can rewrite the above min–max principle in terms of \( F = T_n f \) and \( dm_n \). Indeed, given \( f \in \mathcal{A}(n) \), the function \( F = T_n f \) is linear between \( x_{j-1}^{(n)} \) and \( x_j^{(n)} \), thus implying that

\[
U_n(j/n)^{-1} \left[ f(j/n) - f((j - 1)/n) \right]^2 = \left[ x_j^{(n)} - x_{j-1}^{(n)} \right]^{-1} \left[ F(x_j^{(n)}) - F(x_{j-1}^{(n)}) \right]^2 = \int_{x_{j-1}^{(n)}}^{x_j^{(n)}} D_s F(s)^2 ds.
\]
Hence,
\[ D_n(f) = \int_0^{\ell_n} D_s F(s)^2 ds. \] (6.4)

Since trivially, \( \mu_n(f^2) = \int_0^{\ell_n} F(s)^2 dm_n(s) \) for \( f \in \mathcal{A}(n) \) and \( F = T_n f \), from (6.3) and (6.4) we get that
\[ \lambda_k^{(n)} = \min_{S_k} \max_{F \in S_k : F \neq 0} \Phi_n(F), \] (6.5)
where \( S_k \) varies among all \( k \)-dimensional subsets of \( \mathcal{B}(n) \) (recall (6.1)), while for a generic function \( F \in C_0[0, \ell_n] \) we define
\[ \Phi_n(F) := \frac{\int_0^{\ell_n} D_s F(s)^2 ds}{\int_0^{\ell_n} F(s)^2 dm_n(s)} \] (6.6)
whenever the denominator is nonzero. Here and in what follows, we write \( \ell_n \) instead of \( \int_{[0, \ell_n]} \).

The following observation will reveal very useful:

**Lemma 6.1.** Let \( F \in \mathcal{B}(n) \) and let \( G \in C_0[0, \ell_n] \) be any function satisfying \( F(x_j^{(n)}) = G(x_j^{(n)}) \) for all \( 0 \leq j \leq n \). Then
\[ \int_0^{\ell_n} D_s F(s)^2 ds \leq \int_0^{\ell_n} D_s G(s)^2 ds. \] (6.7)

In particular, if \( F \neq 0 \) then \( \Phi_n(F) \) and \( \Phi_n(G) \) are both well defined and \( \Phi_n(F) \leq \Phi_n(G) \).

**Proof.** In order to get (6.7) it is enough to observe that by Schwarz’ inequality it holds
\[ \int_{x_j^{(n)} \downarrow x_j}^{x_j^{(n)}} D_s F(s)^2 ds = \left[ \frac{F(x_j^{(n)}) - F(x_{j-1}^{(n)})}{x_j^{(n)} - x_{j-1}^{(n)}} \right]^2 \]
\[ = \frac{[G(x_j^{(n)}) - G(x_{j-1}^{(n)})]^2}{x_j^{(n)} - x_{j-1}^{(n)}} \]
\[ \leq \frac{\int_{x_j^{(n)} \downarrow x_j}^{x_j^{(n)}} D_s G(s) ds}{x_j^{(n)} - x_{j-1}^{(n)}} \]
From (6.7) one derives the last issue by observing that \( dm_n(F^2) = dm_n(G^2) \) (\( dm_n(\cdot) \) denoting the average w.r.t. \( dm_n \)). \( \square \)

We have now all the tools in order to prove that the eigenvalues \( \lambda_k^{(n)} \) are bounded uniformly in \( n \):

**Lemma 6.2.** For each \( k \geq 1 \), it holds
\[ \sup_{n \geq k} \lambda_k^{(n)} =: a(k) < \infty. \] (6.8)

**Proof.** Given a function \( f \in C_0[0, \ell_n] \) and \( n \geq 1 \), we define \( K_n f \) as the unique function in \( \mathcal{B}(n) \) such that \( f(x_j^{(n)}) = K_n f(x_j^{(n)}) \) for all \( 0 \leq j \leq n \). Note that \( K_n \) commutes with linear combinations: \( K_n(a_1 f_1 + \cdots + a_k f_k) = a_1 K_n f_1 + \cdots + a_k K_n f_k \).

Due to the assumption that \( dm \) is not a linear combination of a finite number of delta measures, for some \( \varepsilon > 0 \) we can divide the interval \([0, \ell - \varepsilon]\) in \( k \) subintervals \( I_j = [a_j, b_j] \) such that \( dm(\text{int}(I_j)) > 0 \), \( \text{int}(I_j) = (a_j, b_j) \).
Since $dm_n$ converges to $dm$ weakly, it must be $dm_n(\int(I_j)) > 0$ for all $j : 1 \leq j \leq k$, and for $n$ large enough. For each $j$ we fix a piecewise–linear function $f_j : \mathbb{R} \to \mathbb{R}$, with support in $I_j$ and strictly positive on $\operatorname{int}(I_j)$. Since $\ell_n \to \ell > \varepsilon$, taking $n$ large enough, all functions $f_j$ are zero outside $(0, \ell_n)$, hence we can think of $f_j$ as function in $C_0[0, \ell_n]$. Having disjoint supports, the functions $f_1, f_2, \ldots, f_k$ are independent in $C_0[0, \ell_n]$.

We claim that $K_n f_1, K_n f_2, \ldots, K_n f_k$ are independent functions in $B(n)$ for $n$ large enough. Indeed, we know that $dm_n(\int(I_j)) > 0$ for all $j : 1 \leq j \leq k$, if $n$ is large enough. Hence, for $n$ large, each set $\int(I_j)$ contains at least one point $x_r^{(n)}$ with $1 \leq r \leq n$. Since $f_j(x_r^{(n)}) > 0$ while $f_u(x_r^{(n)}) = 0$ for all $u \neq j$ such that $1 \leq u \leq k$, $K_n f_j$ cannot be written as linear combination of the functions $K_n f_u, u \neq j, 1 \leq u \leq k$.

Due to the above independence, we can apply the min–max principle \eqref{eq:min-max-principle}. Let us write $S_k$ for the real vector space spanned by $K_n f_1, K_n f_2, \ldots, K_n f_k$ and $S_k$ for the real vector space spanned by $f_1, f_2, \ldots, f_k$. As already observed, $S_k = K_n(S_k)$. Using also Lemma \ref{lem:lambda-n(k)} we conclude that for $n$ large enough

$$\lambda_k^{(n)} \leq \max\{\Phi_n(f) : f \in S_k, \quad dm_n(f^2) > 0\} \leq \max\{\Phi_n(f) : f \in S_k, \quad dm_n(f^2) > 0\}. \quad (6.9)$$

Take $f = a_1 f_1 + a_2 f_2 + \cdots + a_k f_k$ such that $\int_0^{\ell_n} f(s)^2 dm_n(s) > 0$. Since $\Phi_n(f) = \Phi_n(cf)$, without loss of generality we can assume that $\sum_{i=1}^k a_i^2 = 1$. Since the functions $f_j$ have disjoint supports, it holds $(D_s f)^2 = \sum_{j=1}^k a_j^2 (D_s f_j)^2$ a.e., while $f^2 = \sum_{j=1}^k a_j^2 f_j^2$. In particular, we can write

$$\Phi_n(f) = \frac{\sum_{j=1}^k a_j^2 \int_0^{\ell_n} D_s f_j(s)^2 ds}{\sum_{j=1}^k a_j^2 \int_0^{\ell_n} f_j(s)^2 dm_n(s)}.$$

Taking $n$ large enough that $\ell - \varepsilon \leq \ell_n$, equations \eqref{eq:lambda-n(k)} and \eqref{eq:phi-n(f)} together imply that

$$\lambda_k^{(n)} \leq \max\{\int_0^{\ell} D_s f_j(s)^2 ds : 1 \leq j \leq k\} \min\{\int_0^{\ell} f_j(s)^2 dm_n(s) : 1 \leq j \leq k\}. \quad (6.11)$$

Since $dm_n$ weakly converges to $dm$, the $k$ terms appearing in the denominator converge to positive numbers as $n \uparrow \infty$. Hence, the r.h.s. converges to a positive number, thus implying \eqref{eq:bound-on-lambda-n}.

\section*{7. Proof of Theorem \ref{thm:main-result}}

Most of the work necessary for the convergence of the eigenvalues has been done for proving Lemma \ref{lem:existence-of-eigenvalues} and Lemma \ref{lem:bound-on-lambda-n}. Due to Lemma \ref{lem:properties-of-L-m-n}, we know that the eigenvalues of $-L_n$ and the eigenvalues of the differential operator $-D_m D_x$ with Dirichlet conditions outside $(0, \ell)$ are simple, positive and form a set without accumulation points. Since $-L_n$ is a symmetric operator on the $(n - 1)$–dimensional space $L^2((0, 1) \cap \mathbb{Z}, \mu_n)$, where $\mu_n$ has been introduced in Section \ref{sec:definitions}, we conclude that $-L_n$ has $n - 1$ eigenvalues.

Given $k \geq 1$ we take $a(k)$ as in Lemma \ref{lem:bound-on-lambda-n} and we fix $L \geq a(k)$ such that $L$ is not an eigenvalue of $-D_m D_x$ with Dirichlet conditions. Let $k_0$, $\varepsilon$, and $n_0$ be as in Lemma \ref{lem:existence-of-eigenvalues}. Then for $n > n_0$ the following holds: in each interval $J_i = [\lambda_i - \varepsilon, \lambda_i + \varepsilon]$ there is exactly one eigenvalue of $-L_n$ and in $[0, L] \setminus \bigcup_{i=1}^{k_0} J_i$ there is no eigenvalue of $-L_n$. Since we know by Lemma \ref{lem:bound-on-lambda-n} that $-L_n$ has at least $k$ eigenvalues in $[0, L]$ it must be $k \leq k_0$ and $\lambda_i^{(n)} \in J_i$.
for all $i : 1 \leq i \leq k$. In particular, it holds

$$\limsup_{n \to \infty} |\lambda^{(n)}_i - \lambda_i| \leq \varepsilon, \quad \forall i : 1 \leq i \leq k. \quad (7.1)$$

Using the arbitrariness of $\varepsilon$ and $k$ we conclude that the operator $-D_nD_x$ with Dirichlet conditions outside $(0, \ell)$ has infinite eigenvalues satisfying (2.12).

### 7.1. Convergence of the eigenfunctions

Having proved (2.12), the convergence of the eigenfunctions can be derived by arguments close to the ones of [UH]. Alternatively, one could try to estimate $\psi^{(n)}(x, \lambda^{(n)}_k) - \psi(x, \lambda_k)$ with $\psi^{(n)}$ and $\psi$ defined as before Lemma 5.2. Below, we follow the first route.

Let us define $L = \ell + 1$. By restricting to $n$ large enough, we can assume that $\ell_n \leq L$. Using (4.10), we define the function $G_n$ on $[0, L] \times [0, L]$ as

$$G_n(x, y) := \begin{cases} G_{0, \ell_n}(x, y) & \text{if } x, y \in [0, \ell_n], \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$|G_n(x, y)| \leq L, \quad (7.2)$$

$$|G_n(x, y) - G_n(x', y)| \leq |x - x'|, \quad \forall x, x' \in [0, \ell_n]. \quad (7.3)$$

Since we want to work with the space $C([0, L])$ endowed of the uniform norm, in what follows we think of functions $F \in C_0([0, \ell_n))$ as elements of $C_0([0, L]) \subset C([0, L])$ by setting $F(x) = 0$ if $\ell_n \leq x \leq L$. This identification will be often understood.

We fix $k \geq 1$ and take $n > k$. Then we denote by $F^{(n)}_k$ any nonzero real solution of the integral equation (4.9) where $\lambda$ and $m$ are replaced by $\lambda^{(n)}_k$ and $m_n$, respectively. Moreover, we require that

$$\int_{[0, L]} F^{(n)}_k(x)^2 dm_n(x) = 1. \quad (7.4)$$

Note that considering $F^{(n)}_k$ as element of $C([0, L])$ equation (4.9) can be rewritten as

$$F^{(n)}_k(x) = \lambda^{(n)}_k \int_{[0, L]} G_n(x, y) F^{(n)}_k(y) dm_n(y), \quad \forall x \in [0, L]. \quad (7.5)$$

**Lemma 7.1.** For each fixed $k \geq 1$, the sequence $\{F^{(n)}_k\}_{n > k}$ has compact closure in $C([0, L])$.

**Proof.** Since we know that $\lim_{n \to \infty} \lambda^{(n)}_k = \lambda_k \in (0, \infty)$, it is enough to prove that the sequence $\{f_{n,k}\}_{n > k}$, where $f_{n,k} := (1/\lambda^{(n)}_k)F^{(n)}_k$, has compact closure in $C([0, L])$. To this aim we only need to apply Ascoli–Arzelà Theorem, showing that the sequence is uniformly bounded and uniformly continuous. Indeed, from (7.2), (7.4) and (7.5), we get

$$|f_{n,k}(x)| \leq \left[ \int_{[0, L]} G_n(x, y)^2 dm_n(y) \right]^{1/2} \left[ \int_{[0, L]} (F^{(n)}_k(y))^2 dm_n(y) \right]^{1/2} \leq Lm_n(L)^{1/2}. \quad (7.6)$$

Moreover, from (7.4) and (7.5), we get

$$|f_{n,k}(x) - f_{n,k}(x')| \leq \left[ \int_{[0, L]} (G_n(x, y) - G_n(x', y))^2 dm_n(y) \right]^{1/2} \quad (7.7)$$
which by (7.3) is bounded by \(|x - x'|m_n(L)^{1/2}\) if \(x, x' \leq \ell_n\), by 0 if \(x, x' > \ell_n\) and by
\[|\ell_n - x|m_n(L)^{1/2} \leq |x - x'|m_n(L)^{1/2}\]
if \(x \leq \ell_n \leq x'\). The thesis now follows from the above bounds and from the limit \(\lim_{n \to \infty} m_n(L) = m(L)\), consequence of the weak convergence of \(dm_n\) to \(dm\). □

It remains now to characterize the limit points of \(\{F_k^{(n)}\}_{n>k}\). We fix a point \(s_0 \in (0, \ell)\) such that \(\psi(s_0, \lambda_k) \neq 0\). Then for \(n\) large, at cost to multiply \(F_k^{(n)}\) by \(\pm 1\), we can assume that \(F_k^{(n)}(s_0)\) is not zero and has the same sign of \(\psi(s_0, \lambda_k)\).

We come back to (7.5). Since \(\ell_n \to \ell\) we know that \(G_n \to G\) in \(C([0, L] \times [0, L])\), where
\[G(x, y) = \begin{cases} G_{0, \ell}(x, y) & \text{if } x, y \in [0, \ell], \\ 0 & \text{otherwise}. \end{cases}\]

Hence, from (7.5) and from the convergence \(\lambda_k^{(n)} \to \lambda_k\), we derive that any limit point \(F_k \in C([0, L])\) of \(\{F_k^{(n)}\}_{n>k}\) must satisfy
\[F_k(x) = \lambda_k \int_0^L G(x, y)F_k(y)dm(y). \tag{7.8}\]

By the weak convergence \(dm_n \to dm\) it holds
\[
\lim_{n \to \infty} \int_{[0, L]} F_k(s)^2dm_n(s) = \int_{[0, L]} F_k(s)^2dm(s). \tag{7.9}
\]

On the other hand,
\[
\int_{[0, L]} F_k(s)^2dm_n(s) = \int_{[0, L]} F_k^{(n)}(s)^2dm_n(s) + \mathcal{E} = 1 + \mathcal{E}, \tag{7.10}
\]

where \(|\mathcal{E}| \leq \|F_k^{(n)} - F_k\|_2^2m_n(L)\). The above bound, together with (7.9) and (7.10), implies the normalization \(\int_{[0, L]} F_k(s)^2dm(s) = 1\). Finally, we observe that \(F_k\) is a real function and \(F_k(s) = 0\) if \(s \in (\ell, L]\). Lemma 4.1 together with (7.8) and the normalization of \(F_k\), implies that \(F_k(s) = \pm C\psi(s, \lambda_k)\) for all \(s \in [0, \ell]\), where \(1/C = \int_{[0, \ell]} \psi(s, \lambda_k)^2dm(s)\). Since by construction \(F_k^{(n)}(s_0)\) is not zero and has the same sign of \(\psi(s_0, \lambda_k)\), we conclude that \(F_k = C\psi(\cdot, \lambda_k)\). In particular, the exists a unique limit point of the sequence \(\{F_k^{(n)}\}_{n>k}\). That concludes the proof of Theorem 2.1.

8. Dirichlet–Neumann Bracketing

Let \(m : \mathbb{R} \to [0, \infty)\) be a càdlàg nondecreasing function with \(m(x) = 0\) for all \(x < 0\). We recall that \(E_m\) denotes the support of \(dm\), i.e. the set of points where \(m\) increases (see (2.6)) and that \(m_x\) denotes the magnitude of the jump of the function \(m\) at the point \(x\), i.e. \(m_x := m(x^+) - m(x^-) = m(x) - m(x^-)\). We suppose that \(E_m \neq \emptyset\), \(0 = \inf E_m\) and \(\ell_m := \sup E_m < \infty\). We want to compare the eigenvalue counting function for the generalized operator \(-D_xD_m\) with Dirichlet boundary conditions to the same function when taking Neumann boundary conditions. In order to apply the Dirichlet–Neumann bracketing as stated in Section XIII.15 of [RS4] and as developed by Métivier and Lapidus (cf. [Me] and [L]), we need to study generalized differential operators as self–adjoint operators on suitable Hilbert spaces.
In the rest of the Section we assume that
\[ m_0 = m_{\ell_m} = 0. \] (8.1)
The reason will become clear soon. We consider the real Hilbert space \( \mathcal{H} := L^2([0, \ell_m], dm) \) and denote its scalar product as \( \langle \cdot, \cdot \rangle \). When writing \( \int dm(y)g(y) \) we mean \( \int_{[0,\ell_m]} dm(y)g(y) \).

8.1. The operator \(-\mathcal{L}_D\). We define the operator \(-\mathcal{L}_D : \mathcal{D}(-\mathcal{L}_D) \subset \mathcal{H} \rightarrow \mathcal{H}\) as follows.

First, we set that \( f \in \mathcal{D}(-\mathcal{L}_D) \) if there exists a function \( g \in \mathcal{H} \) such that
\[
f(x) = bx - \int_0^x dy \int_{[0,y]} dm(z)g(z), \quad b := \frac{1}{\ell_m} \int_0^{\ell_m} dy \int_{[0,y]} dm(z)g(z). \quad (8.2)
\]

We note that the above identity implies that \( f \) has a representative given by a continuous function in \( C[0,\ell_m] \) such that \( f(0) = f(\ell_m) = 0 \). Moreover, by the discussion following (4.2) (cf. (4.3) and (4.4)) and the assumption \( m_0 = m_{\ell_m} = 0 \), we derive from identity (8.2) that the function \( g \in \mathcal{H} \) satisfying (8.2) is unique. Hence, we define \(-\mathcal{L}_D f = g\).

Always due to (4.3) and (4.4), we know that if \( f \in \mathcal{D}(-\mathcal{L}_D) \), then \( f \) has derivative \( D_x^+ f \) on \([0,\ell_m)\), \( f \) has left derivative \( D_x^- f \) on \((0,\ell_m] \) and has derivative \( D_x f \) on \((0,\ell_m)\) apart a countable set of points. In particular, \( f \) has derivative Lebesgue a.e. on \((0,\ell_m)\). The operator \(-\mathcal{L}_D\) is simply the operator \(-D_x D_m\) with Dirichlet boundary conditions thought on the space \( \mathcal{H}\).

**Proposition 8.1.** The following holds:

(i) the operator \(-\mathcal{L}_D : \mathcal{D}(-\mathcal{L}_D) \subset \mathcal{H} \rightarrow \mathcal{H}\) is self-adjoint;
(ii) consider the symmetric compact operator \(\mathcal{K} : \mathcal{H} \rightarrow \mathcal{H}\) defined as
\[
\mathcal{K}g(x) = \int K(x,y)g(y)dm(y), \quad g \in \mathcal{H},
\]
where the function \(K(x,y) := G_{0,\ell_m}(x,y)\) is given by (4.10). Then, \(\text{Ran}(\mathcal{K}) = \mathcal{D}(-\mathcal{L}_D)\) and \(-\mathcal{L}_D \circ \mathcal{K} = \mathcal{I}\) on \(\mathcal{H}\). In particular, the operator \(-\mathcal{L}_D\) admits a complete orthonormal set of eigenfunctions and therefore \(-\mathcal{L}_D\) has pure point spectrum. Moreover, the above eigenvalues and eigenfunctions coincide with the ones in Lemma 4.7.
(iii) for all \(f, \hat{f} \in \mathcal{D}(-\mathcal{L}_D)\) it holds
\[
(\hat{f}, -\mathcal{L}_D \hat{f}) = \int_0^{\ell_m} D_x f(x)D_x \hat{f}(x)dx.
\]

**Proof.** It is trivial to check that (8.2) can be rewritten as
\[
f(x) = \int K(x,y)g(y)dm(y).
\]
Hence, by definition \(\mathcal{D}(-\mathcal{L}_D) = \text{Ran}(\mathcal{K})\) and \(-\mathcal{L}_D(\mathcal{K}(g)) = g\) for all \(g \in \mathcal{H}\) and \(\mathcal{K}\) is injective (see the discussion on the well definition of \(-\mathcal{L}_D\)). Since \(K(x,y) = K(y,x)\), the operator \(\mathcal{K}\) is symmetric. Since \(K \in L^2(dm \otimes dm) \) (\(K\) is bounded and \(dm\) has finite mass), by [RSI][Theorem VI.23] \(\mathcal{K}\) is an Hilbert–Schmidt operator and therefore is compact (cf. [RSI][Theorem VI.22]). In particular, \(\mathcal{H}\) has an orthonormal basis \(\{\psi_n\}\) such that \(\mathcal{K}\psi_n = \gamma_n \psi_n\) for suitable eigenvalues \(\gamma_n\) (cf. Theorems VI.16 in [RSI]). Since \(\mathcal{K}\) is injective, we conclude that \(\gamma_n \neq 0\), \(\psi_n = \mathcal{K}(1/\gamma_n)\psi_n \in \text{Ran}(\mathcal{K}) = \mathcal{D}(-\mathcal{L}_D)\) and
\[
-\mathcal{L}_D \psi_n = -\frac{1}{\gamma_n} \mathcal{L}_D(\mathcal{K}\psi_n) = \frac{1}{\gamma_n} \psi_n.
\]
It follows that \( \{ \psi_n \} \) is an orthonormal basis of eigenvectors of \(-\mathcal{L}_D\). By (8.2), the function \( \psi_n \in L^2(dm) \) must have a representative in \( C[0,\ell_m] \). Taking this representative, the identity \( \psi_n = -(1/\gamma_n)\mathcal{L}_D\psi_n \) simply means that \( \psi_n \) is an eigenfunction with eigenvalue \( 1/\gamma_n \) of the generalized differential operator \( -D_xD_m \) with Dirichlet boundary conditions as defined in Section 4. Finally, since \(-\mathcal{L}_D\) admits an orthonormal basis of eigenvectors, its spectrum is pure point and is given by the family of eigenvalues. This concludes the proof of point (ii).

In order to prove (i), we observe that \( \mathcal{D}(-\mathcal{L}_D) \) contains the finite linear combinations of the orthonormal basis \( \{ \psi_n \} \) and therefore it is a dense subspace in \( \mathcal{H} \). Given \( f, \hat{f} \in \mathcal{D}(-\mathcal{L}_D) \), let \( g, \hat{g} \in \mathcal{H} \) such that \( f = \mathcal{K}g \), \( \hat{f} = \mathcal{K}\hat{g} \). Then, using the symmetry of \( \mathcal{K} \) and point (ii), we obtain

\[
(-\mathcal{L}_D f, \hat{f}) = (\mathcal{K}g, \hat{g}) = (f, -\mathcal{L}_D \hat{f}) .
\]

This proves that \(-\mathcal{L}_D\) is symmetric. In order to prove that it is self-adjoint we need to show that, given \( v, w \in \mathcal{H} \) such that \((\mathcal{L}_D f, v) = (f, w)\) for all \( f \in \mathcal{D}(-\mathcal{L}_D) \), it must be \( v \in \mathcal{D}(-\mathcal{L}_D) \) and \( -\mathcal{L}_D v = w \). To this aim, we write \( g = -\mathcal{L}_D f \). Then, by the symmetry of \( \mathcal{K} \), it holds

\[
(g, v) = (-\mathcal{L}_D f, v) = (f, w) = (\mathcal{K}g, w) = (g, \mathcal{K}w) .
\]

Since this holds for any \( f \in \mathcal{D}(-\mathcal{L}_D) \) and therefore for any \( g \in \mathcal{H} \), it must be \( v = \mathcal{K}w \). By point (ii), this is equivalent to the fact that \( w \in \mathcal{D}(-\mathcal{L}_D) \) and \( w = -\mathcal{L}_D v \). This concludes the proof of (i).

In order to prove (iii), we set \( g = -\mathcal{L}_D f \) and \( \hat{g} = -\mathcal{L}_D \hat{f} \). Then

\[
(f, -\mathcal{L}_D \hat{f}) = \int dm(x)\hat{g}(x) \int dm(y)g(y)K(x, y) .
\]

By (4.3), \( d(D^+_xf)(x) = -dm(x)g(x) \) as Stieltjes measure (similarly for \( \hat{f} \) and \( \hat{g} \)). Therefore, the above integral can be rewritten as

\[
(f, -\mathcal{L}_D \hat{f}) = \int_{[0,\ell_m]} d(D^+_x\hat{f})(x) \int_{[0,\ell_m]} d(D^+_xf)(y)K(x, y) . \tag{8.6}
\]

Since \( K(x, y) \) is zero if \( y \in \{ 0, \ell_m \} \), by integration by parts we get

\[
-\int_{[0,\ell_m]} d(D^+_xf)(y)K(x, y) = \frac{\ell_m - x}{\ell_m} \int_{[0,\ell_m]} D^+_x f(y)dy - \frac{x}{\ell_m} \int_{[0,\ell_m]} D^+_x f(y)dy = \int_{[0,\ell_m]} D^+_x f(y)dy - \frac{x}{\ell_m} \int_{[0,\ell_m]} D^+_x f(y)dy . \tag{8.7}
\]

We observe that \( \int_0^{\ell_m} D^+_x f(y)dy = 0 \), since by (4.3) and (8.2),

\[
\int_0^{\ell_m} D^+_x f(y)dy = \int_0^{\ell_m} dy \int_{[0,y]} m(ds)g(s) = 0 .
\]

The above remark allows us to rewrite (8.7) as

\[
\int_{[0,\ell_m]} d(D^+_xf)(y)K(x, y) = -\int_0^x D^+_x f(y)dy .
\]

Substituting this expression in (8.6) and making another integration by parts, we get that

\[
(f, -\mathcal{L}_D \hat{f}) = \int_0^{\ell_m} D^+_x f(x)D^+_x \hat{f}(x)dx .
\]
The thesis then follows recalling that $D_x f$, $D_x \hat{f}$ are well defined Lebesgue a.e. and that on the definition points it holds $D_x f = D_x^+ f$, $D_x \hat{f} = D_x^\hat{f}$.

8.2. **The operator $-\mathcal{L}_N$.** We define the operator $-\mathcal{L}_N : \mathcal{D}(-\mathcal{L}_N) \subset \mathcal{H} \to \mathcal{H}$ as follows. First, we say that $f \in \mathcal{D}(-\mathcal{L}_N)$ if there exist a function $g \in \mathcal{H}$ and a constant $a \in \mathbb{R}$ such that

$$f(x) = a - \int_0^x dy \int_{[0,y]} dm(z)g(z)$$

(8.8) and

$$\int_{[0,\ell_m]} dm(z)g(z) = 0.$$

(8.9)

We note that the above identity implies that $f$ has a representative given by a continuous function in $C[0,\ell_m]$. Moreover, by the discussion following (1.2) (cf. 1.3 and 1.4) and the assumption $m_0 = m_{\ell_m} = 0$, we derive from identity (8.8) that the function $g \in \mathcal{H}$ satisfying (8.8) is unique. Hence, we define $-\mathcal{L}_N f = g$. Always due to (1.3) and (1.4), we know that if $f \in \mathcal{D}(-\mathcal{L}_N)$, then $f$ has right derivative $D_x^+ f$ on $[0,\ell_m]$, $f$ has left derivative $D_x^- f$ on $(0,\ell_m]$ and has derivative $D_x f$ on $(0,\ell_m)$ apart a countable set of points. In addition, $D_x^+ f(0)$ and $D_x^- f(\ell_m)$ are zero due to (8.8) and (8.9). The operator $-\mathcal{L}_D$ is simply the operator $-D_x D_m$ with Neumann boundary conditions thought of on the space $\mathcal{H}$.

**Proposition 8.2.** The following holds:

(i) the operator $-\mathcal{L}_N : \mathcal{D}(-\mathcal{L}_N) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint;

(ii) the operator $-\mathcal{L}_N$ admits a complete orthonormal set of eigenfunctions and therefore $-\mathcal{L}_N$ has only pure point spectrum. The eigenvalues and eigenfunctions are the same as the ones associated to the operator $-D_x D_m$ with Neumann boundary conditions as defined in Section 4;

(iii) for all $f, \hat{f} \in \mathcal{D}(-\mathcal{L}_N)$ it holds

$$(f, -\mathcal{L}_N \hat{f}) = \int_0^{\ell_m} D_x f(x) D_x \hat{f}(x) dx. $$

(8.10)

*Proof.* We start with point (i). First we prove that $-\mathcal{L}_N$ is symmetric. Take $f, g, a$ as in (8.8) and (8.9), and take $\hat{f}, \hat{g}, \hat{a}$ similarly. Then,

$$(f, -\mathcal{L}_N \hat{f}) = \int dm(x) f(x) \hat{g}(x) = a \int dm(dx) \hat{g}(x) - \int dm(x) \hat{g}(x) \int_0^x dy \int_{[0,y]} dm(z)g(z).$$

Using that $\int dm(x) \hat{g}(x) = 0$ by (8.9), we conclude that

$$(f, -\mathcal{L}_N \hat{f}) = \int dm(x) \int dm(z) \hat{g}(x) g(z) I_z < x(z-x).$$

Since, by (8.9) and its analogous version for $\hat{g}$, it holds $\int dm(x) \int dm(z) g(x) \hat{g}(z)(z-x) = 0$, we can rewrite the above expression in the symmetric form

$$(f, -\mathcal{L}_N \hat{f}) = -\frac{1}{2} \int dm(x) \int dm(z) \hat{g}(x) g(z) I_{x-z}.$$

(8.11)

which immediately implies that $-\mathcal{L}_N$ is symmetric.
Let us consider the Hilbert subspace \( W = \{ f \in \mathcal{H} : (1, f) = 0 \} \), namely \( W \) is the family of functions in \( \mathcal{H} \) having zero mean w.r.t. \( dm \). Then we define the operator \( T : \mathcal{H} \to \mathcal{H} \) as

\[
Tg(x) = -\int_0^x dy \int_{[0,y]} dm(z)g(z) = \int dm(z)g(z)(z-x)v_0 \leq z \leq x .
\]

Finally, we write \( P : \mathcal{H} \to W \) for the orthogonal projection of \( \mathcal{H} \) onto \( W \): \( Pf = f - (1,f)/(1,1) \). Note that \( [P \circ T]g(x) = \int dm(z)g(z)H(x,z) \), where

\[
H(x,z) = (z-x)v_0 \leq z \leq x - \int_{(z,\ell_m)} dm(u)(z-u)/\int dm(u)
\]

Since \( H \in L^2(dm \otimes dm) \), due to [RS][Theorem VI.23] \( P \circ T \) is an Hilbert–Schmidt operator on \( \mathcal{H} \), and therefore a compact operator. In particular, the operator \( W : W \to W \) defined as the restriction of \( P \circ T \) to \( W \) is again a compact operator. We claim that \( W \) is symmetric. Indeed, setting \( f = Wg \) and \( f' = Wg' \); due to the first identity in (8.12) we get that \( f, f' \in D(-\mathcal{L}_N) \) and \( -\mathcal{L}_N f = g, -\mathcal{L}_N f' = g' \). Then, using that \( \mathcal{L}_N \) is symmetric as proven above, we conclude

\[
(Wg, g') = (f, -\mathcal{L}_N f') = (-\mathcal{L}_N f, f') = (g, Wg') .
\]

Having proved that \( W \) is a symmetric compact operator, from [RS][Theorem VI.16] we derive that \( W \) has an orthonormal basis \( \{ \psi_n \}_n \) of eigenvectors of \( W \), i.e. \( W\psi_n = \gamma_n\psi_n \) for suitable numbers \( \gamma_n \). Since \( W \) is injective (recall the discussion on the well definition of \( -\mathcal{L}_N \)), it must be \( \gamma_n \neq 0 \). From the identity \( W\psi_n = \gamma_n\psi_n \) we conclude that

\[
\psi_n(x) = a_n - \frac{1}{\gamma_n} \int_0^x dy \int_{[0,y]} dm(z)\psi_n(z)
\]

for some constant \( a_n \in \mathbb{R} \). The above identity implies that \( \psi_n \in D(-\mathcal{L}_N) \) and \( -\mathcal{L}_N \psi_n = (1/\gamma_n)\psi_n \). On the other hand \( 1 \in D(-\mathcal{L}_N) \) and \( -\mathcal{L}_N 1 = 0 \). Since \( \mathcal{H} = \{ c : c \in \mathbb{R} \} \oplus W \), we obtain that \( \mathcal{H} \) admits an orthonormal basis of eigenvectors of \( -\mathcal{L}_N \). This also implies that \( -\mathcal{L}_N \) has only pure point spectrum. Trivially, all eigenvectors (as all elements in \( D(-\mathcal{L}_N) \)) are continuous and are eigenvectors of \( -D_x D_m \) with Neumann b.c. in the sense of Section 4. This concludes the proof of (i) and (ii).

Let us now prove (iii). From (8.8) we derive that for Lebesgue a.e. \( x \) it holds \( D_x f(x) = -\int_{[0,x]} dm(z)g(z) \) and \( D_x f(x) = -\int_{[0,x]} dm(z)\hat{g}(z) \). This implies that

\[
\int_0^{\ell_m} D_x f(x)D_x \hat{f}(x) dx = \int_{[0,\ell_m]} dm(z)g(z) \int_{[0,\ell_m]} dm(u)\hat{g}(u) \int_0^{\ell_m} dx\mathbb{I}(x > z, x > u)
\]

\[
= \int_{[0,\ell_m]} dm(z)g(z) \int_{[0,\ell_m]} dm(u)\hat{g}(u)\left[\ell_m - \max(z, u)\right] .
\]

(8.13)

We now observe that in the last term \( \ell_m \) can be erased due to (8.9). Using again (8.9), we can write \( D_x f(x) = \int_{[x,\ell_m]} dm(z)g(z) \) and \( D_x \hat{f}(x) = \int_{[x,\ell_m]} dm(z)\hat{g}(z) \). This implies that

\[
\int_0^{\ell_m} D_x f(x)D_x \hat{f}(x) dx = \int dm(z)g(z) \int dm(u)\hat{g}(u) \int_0^{\ell_m} dx\mathbb{I}(x \leq z, x \leq u)
\]

\[
= \int dm(z)g(z) \int dm(u)\hat{g}(u) \min(z, u) .
\]

(8.14)
Taking the symmetric average between (8.13) (after removing ℓm) and (8.14), observing that \( \min(z,u) - \max(z,u) = -|z-u| \), we conclude that
\[
\int_0^{\ell m} D_x f(x) D_x f(x) dx = -\frac{1}{2} \int dm(z) g(z) \int dm(u) g(u) |z-u|.
\]
(8.15)
Comparing the above identity with (8.11), we get point (iii).

\[\square\]

8.3. The quadratic forms \( q_D \) and \( q_N \). We call \( q_D, q_N \) the quadratic forms associated to \( -\mathcal{L}_D, -\mathcal{L}_N \), respectively, and write \( Q(q_D), Q(q_N) \) for the associated form domains (see [RS1] [Section VIII.6] for their definitions). Due to Exercises 15(b) and 16(b) in [RS1] [Chapter VIII], \( q_D, q_N \) can be defined also as follows: the domain \( Q(q_D) \) of \( q_D \) is given by the elements \( f \in \mathcal{H} \) such that there exists a sequence \( f_n \in \mathcal{D}(\mathcal{L}_D) \) with \( f_n \to f \in \mathcal{H} \), \( (f_n - f_m, -\mathcal{L}_D(f_n - f_m)) \to 0 \) as \( n, m \to \infty \). For \( f, f' \in Q(q_D) \), defining \( f'_n \) similarly to \( f_n \), it holds
\[
q_D(f, f') = \lim_{n \to \infty} (f_n - \mathcal{L}_D f'_n),
\]
while the limit does not depend on the sequences \( \{f_n\}, \{f'_n\} \).

**Lemma 8.3.** It holds \( Q(q_N) \supset Q(q_D) \) and \( q_N(f) = q_D(f) \) for all \( f \in Q(q_D) \). In particular, \( 0 \leq -\mathcal{L}_N \leq -\mathcal{L}_D \) according to the definition on [RS1] [page 269].

For the reader’s convenience and for later use, we recall the definition given in [RS1] [page 269]: given nonnegative self–adjoint operators \( A, B \), where \( A \) is defined on a dense subset of a Hilbert space \( \mathcal{H}' \) and \( B \) is defined on a dense subset of a Hilbert subspace \( \mathcal{H}'_1 \subset \mathcal{H}' \), one says that \( 0 \leq A \leq B \) if (i) \( Q(A) \subset Q(B) \), and (ii) \( 0 \leq q_A(\psi) \leq q_B(\psi) \) for all \( \psi \in Q(B) \), where \( Q(A) \) and \( Q(B) \) denote the domains of the quadratic forms \( q_A \) and \( q_B \) associated to the operators \( A \) and \( B \), respectively.

**Proof of Lemma** [8.3] Due to Lemma [8.4] below, for any \( f \in \mathcal{D}(\mathcal{L}_D) \) there exists a sequence \( f_n \in \mathcal{D}(\mathcal{L}_D) \) such that \( f_n \to f \) in \( \mathcal{H} \) and \( f'_n - (D_x f(x) - D_x f_n(x))^2 dx \to 0 \), as \( n \to \infty \). This implies that \( f \in Q(q_N) \). By (8.3) and (8.10), we also deduce that \( q_N(f) = q_D(f) \).

Given now a generic \( f \in \mathcal{D}(q_D) \) we fix a sequence \( f_n \in \mathcal{D}(\mathcal{L}_D) \) such that (i) \( f_n \to f \) in \( \mathcal{H} \), and (ii) \( \lim_{n \to \infty} q_D(f_n - f_m, -\mathcal{L}_D(f_n - f_m)) \to 0 \). By definition, it holds (iii) \( q_D(f) = \lim_{n \to \infty} q_D(f_n) \).

On the other hand, for what proven at the beginning, we know that \( f_n \in Q(q_N) \), while (ii) and (iii) remain valid with \( q_D \) replaced with \( q_N \). Since \( Q(q_N) \) is an Hilbert space with respect to the scalar product \( \langle \cdot, \cdot \rangle_1 := \langle \cdot, \cdot \rangle + q_N(\cdot, \cdot) \) (cf. Exercise 16 in [RS1] [Chapter VIII]), we conclude that \( f \in Q(q_N) \) and \( q_N(f) = q_D(f) \).

\[\square\]

**Lemma 8.4.** Let \( f \in \mathcal{H} \) be a function such that
\[
f(x) = a + bx - \int_0^x dy \int_{[0,y]} dm(z) g(z),
\]
for some function \( g \in \mathcal{H} \) and some constants \( a, b \in \mathbb{R} \). Then there exists a family of functions \( f_\varepsilon \) parameterized by \( \varepsilon \in (0, \ell_m/2) \) such that (i) \( f_\varepsilon \in \mathcal{D}(\mathcal{L}_N) \), (ii) \( f_\varepsilon \to f \) in \( \mathcal{H} \) as \( \varepsilon \to 0 \) and (iii) \( \int_0^{\ell m} (D_x f_\varepsilon(x) - D_x f(x))^2 dx \to 0 \) as \( \varepsilon \to 0 \).

We point out that if \( f \) is of the form (8.17), then \( D_x f \) is well defined for (Lebesgue) almost every \( x \in (0, \ell_m) \).

**Proof.** In order to simplify the notation we set \( \ell := \ell_m \). Given \( \varepsilon \in (0, \ell/2) \) we define \( g_\varepsilon \in \mathcal{H} \) as
\[
g_\varepsilon(z) := g(z) + A(\varepsilon) \mathbb{I}(z \in [0, \varepsilon)) + B(\varepsilon) \mathbb{I}(z \in [\ell - \varepsilon, \ell]),
\]
where \( A(\varepsilon) = -b/dm([0, \varepsilon]) \), \( B(\varepsilon) = C/dm([\ell - \varepsilon, \ell]) \) and \( C = b - \int_{[0, \ell]} dm(z)g(z) \). Recall that \( m_x = m(x) - m(x-) \). Since by assumption \( m_0 = m_\ell = 0 \) while \( 0 = \inf E_m \) and \( \ell = \sup E_m \), the quantities \( A(\varepsilon) \) and \( B(\varepsilon) \) are well defined. Finally, we set

\[
f_\varepsilon(x) = a - \int_0^x dy \int_{[0,y]} dm(z)g_\varepsilon(z), \quad x \in [0, \ell_m].
\]

By the definition of \( A(\varepsilon) \) and \( B(\varepsilon) \), it holds \( \int_{[0, \ell]} dm(z)g_\varepsilon(z) = 0 \). In particular, \( f_\varepsilon \) belongs to \( \mathcal{D}(-\mathcal{L}_N) \) and \( D_x f_\varepsilon(x) = -\int_{[0,x]} dm(z)g_\varepsilon(z) \) for almost every \( x \in (0, \ell) \). Using again the definition of \( A(\varepsilon) \) and \( B(\varepsilon) \) and since \( D_x f(x) = b - \int_{[0,x]} dm(z)g(z) \) a.e., we get

\[
D_x f(x) - D_x f_\varepsilon(x) = \begin{cases} 
b - b \frac{dm((0,x))}{dm([0,\varepsilon])} & \text{if } x \in [0, \varepsilon), \\
0 & \text{if } x \in [\varepsilon, \ell - \varepsilon), \\
c \frac{dm([\ell-\varepsilon, \ell])}{dm([\ell-\varepsilon, \ell])} & \text{if } x \in [\ell - \varepsilon, \ell].
\end{cases}
\]

In particular, \(|D_x f(x) - D_x f_\varepsilon(x)| \) is bounded uniformly in \( x \) by some constant \( c_0 \) and is zero on \([\varepsilon, \ell - \varepsilon]\). Therefore,

\[
\int_0^\ell (D_x f(x) - D_x f_\varepsilon(x))^2 \leq 2c_0^2 \varepsilon. \tag{8.18}
\]

Since \( f(0) = f_\varepsilon(0) = a \), the above bound and Schwarz’ inequality imply that

\[
(f(y) - f_\varepsilon(y))^2 \leq \int_0^y (D_x f(x) - D_x f_\varepsilon(x))^2 dx \leq 2c_0^2 \varepsilon. \tag{8.19}
\]

From the above estimate we derive that

\[
\int dm(y) (f(y) - f_\varepsilon(y))^2 \leq dm([0, \ell])2c_0^2 \varepsilon. \tag{8.20}
\]

At this point, the thesis follows from (8.15) and (8.16). \( \square \)

Due to Lemma 8.3 and the Lemma preceding Proposition 4 in [BS4] [Section XIII.15], keeping in consideration that all eigenvalues are simple (cf. Section 3), we conclude that, given \( x \geq 0 \),

\[
\sharp \{ \lambda \in \mathbb{R} : \lambda \leq x, \lambda \text{ is eigenvalue of } -\mathcal{L}_D \} \leq \sharp \{ \lambda \in \mathbb{R} : \lambda \leq x, \lambda \text{ is eigenvalue of } -\mathcal{L}_N \}. \tag{8.20}
\]

We will recover the above result in Subsection 8.3 following the approach of [Mc].

Up to now we have defined \( -\mathcal{L}_D \) and \( -\mathcal{L}_N \) referring to the interval \((0, \ell_m)\), where \( 0 = \inf E_m \), \( \ell_m = \sup E_m \), \( m_0 = 0 \) and \( m_{\ell_m} = 0 \). In general, given an open interval \( I = (u, v) \subset (0, \ell_m) \), such that

\[
m_u = m_v = 0, \quad dm((u, u + \varepsilon)) > 0 \text{ and } dm((v - \varepsilon, v)) > 0 \forall \varepsilon > 0, \tag{8.21}
\]

we define \( -\mathcal{L}_D^I, -\mathcal{L}_N^I \) as the operators \( \mathcal{L}_D \) and \( -\mathcal{L}_N \) but with the measure \( dm \) replaced by its restriction to \( I \). For simplicity, we write \( L^2(I, \tilde{dm}) \) for the space \( L^2(I, \tilde{dm}) \) where \( \tilde{dm} \) denotes the restriction of \( dm \) to the interval \( I \). Then, \( f \in \mathcal{D}(-\mathcal{L}_D^I) \subset L^2(I, \tilde{dm}) \) if and only if there exists \( g \in L^2(I, \tilde{dm}) \) such that, writing \( I = (u, v) \),

\[
f(x) = b(x - u) - \int_u^x dy \int_{[u,y]} dm(z)g(z), \quad \forall x \in I,
\]
where \( b = (v - u)^{-1} \int_u^v dy \int_{[u,y]} dm(z)g(z) \). Then, the above \( g \in L^2(I, dm) \) is unique and one sets \(-L_D^f f = g \). The definition is similar for \(-L_N^f \). Propositions 8.1 and 8.2 extend trivially to \(-L_D^f \) and \(-L_N^f \). Indeed, the restriction of \( dm \) to \( I \) equals \( dm \), where the function \( \bar{m} \) is defined as \( \bar{m}(x) = m(u)\|x \leq u\| + m(x)\|x \in I\| + m(v)\|x \geq v\| \). We write \( q_D^f, q_N^f \) for the corresponding quadratic forms. Finally, for \( x \geq 0 \) we define

\[
N^f_{m,D}(x) := \{ \lambda \in \mathbb{R} : \lambda \leq x, \lambda \text{ is eigenvalue of } -L_D^f \}, \quad (8.22)
\]

\[
N^f_{m,N}(x) := \{ \lambda \in \mathbb{R} : \lambda \leq x, \lambda \text{ is eigenvalue of } -L_N^f \}. \quad (8.23)
\]

Note that \( 8.20 \) can be rewritten as \( N^{(0,\ell,m)}_{m,D}(x) \leq N^{(0,\ell,m)}_{m,N}(x) \).

**Lemma 8.5.** Let \( I_1 = (a_1, b_1), \ldots, I_k = (a_k, b_k) \) be a finite family of disjoint open intervals, where \( a_1 < b_1 \leq a_2 < b_2 \leq \cdots < a_k < b_k \) and

\[
m_{a_r} = 0, \quad m_{b_r} = 0 \quad \forall r = 1, \ldots, k, \quad \text{and} \quad \rho_m((a_r, a_r + \varepsilon)) > 0, \quad \rho_m((b_r - \varepsilon, b_r)) > 0 \quad \forall \varepsilon > 0, \forall r = 1, \ldots, k. \]

Then for any \( x \geq 0 \) it holds

\[
N^{(a_1, b_1)}_{m,D}(x) \geq \sum_{r=1}^k N^{(a_r, b_r)}_{m,D}(x). \quad (8.24)
\]

If in addition the intervals \( I_r \) are neighboring, i.e. \( b_r = a_{r+1} \) for all \( r = 1, \ldots, k - 1 \), then for any \( x \geq 0 \) it holds

\[
N^{(a_1, b_1)}_{m,N}(x) \leq \sum_{r=1}^k N^{(a_r, b_r)}_{m,N}(x). \quad (8.25)
\]

The above result is the analogous to Point c) in Proposition 4 in [RS4] [Section XIII.15].

**Proof.** We begin with \( 8.21 \). We consider the direct sum \( \oplus_{r=1}^k L^2(I_r, dm) \). We define \( A = \oplus_{r=1}^k (-L_D^f) \) as the operator with domain

\[
D(A) = \oplus_{r=1}^k D(-L_D^f) \subset \oplus_{r=1}^k L^2(I_r, dm)
\]

such that \( A[(f_r)_{r=1}^k] = (-L_D^f f_r)_{r=1}^k \). Due to the properties listed in [RS4] [page 268] and due to Proposition 8.1 the operator \( A \) is a nonnegative self–adjoint operator.

Trivially, the map

\[
\psi : \oplus_{r=1}^k L^2(I_r, dm) \rightarrow L^2([a_1, b_k], dm), \quad (8.26)
\]

where

\[
\psi[(f_r)_{r=1}^k](x) = \begin{cases} f_r(x) & \text{if } x \in I_r \text{ for some } r, \\ 0 & \text{otherwise} \end{cases}
\]

is injective and conserves the norm. In particular, the image of \( \psi \) is a closed (and therefore Hilbert) subspace of \( L^2([a_1, b_k], dm) \). Consider, the operator

\[
A' : \psi(D(A)) \subset \psi \left[ \oplus_{r=1}^k L^2(I_r, dm) \right] \rightarrow \psi \left[ \oplus_{r=1}^k L^2(I_r, dm) \right],
\]

defined as \( A'(\psi(f)) = \psi(Af) \) for all \( f \in D(A) \). Then, \( A' \) is a nonnegative self–adjoint operator.

**Claim:** It holds \( L_D^{(a_1, b_k)} \leq A' \), where the inequality has to be thought in the sense specified after Lemma 8.3.
Assuming the above claim, the conclusion (8.24) then follows from the Lemma stated in [RS4][page 270] and property (5) on page 268 of [RS4]. It remains then to prove our claim.

Proof of the claim. For simplicity of notation we restrict to the case $k = 2$ (the arguments are completely general). We take $(f_1, f_2) \in \mathcal{D}(A)$. Then there exist constants $\kappa_1, \kappa_2$ and functions $g_1 \in L^2(I_1, dm), g_2 \in L^2(I_2, dm)$ such that

\[ f_i(x) = \kappa_i(x - a_i) - \int_{a_i}^{x} dy \int_{[a_i, y]} dm(z) g_i(z), \quad x \in [a_i, b_i], \quad i = 1, 2. \tag{8.27} \]

We recall that $f_1(a_1) = f_1(b_1) = f_2(a_2) = f_2(b_2) = 0$. Let $f = \psi(f_1, f_2) \in \mathcal{D}(A)$. We need to exhibit a family of functions $f_\varepsilon \in L^2((a_1, b_2), dm), \varepsilon > 0$, such that (i) $f_\varepsilon \in \mathcal{D}(\mathcal{L}_D^{(a_1,b_2)})$, (ii) $f_\varepsilon \to f$ in $L^2((a_1, b_2), dm)$ as $\varepsilon \to 0$ and (iii) $D_x f_\varepsilon \to D_x f$ in $L^2((a_1, b_2), dx)$ as $\varepsilon \to 0$. This would assure that $f$ belongs to the form domain associated to $\mathcal{L}_D^{(a_1,b_2)}$. Note that, due to (S.4), at this point the conclusion of the claim becomes trivial.

In order to prove the claim, we set

\[ f_\varepsilon(x) := (\kappa_1 + \delta_\varepsilon)(x - a_1) - \int_{a_1}^{x} dy \int_{[a_1, y]} dm(z) g_\varepsilon(z), \tag{8.28} \]

where

\[ g_\varepsilon(x) := g_1(x)\mathbb{1}(x \in [a_1, b_1]) + g_2(x)\mathbb{1}(x \in [a_2, b_2]) + D_\varepsilon \frac{R_\varepsilon}{dm([b_1 - \varepsilon, b_1])}(x \in [b_1 - \varepsilon, b_1]) - R_\varepsilon \frac{R_\varepsilon}{dm([a_2, a_2 + \varepsilon])}(x \in [a_2, a_2 + \varepsilon]), \]

and the constants $\delta_\varepsilon, D_\varepsilon$ and $R_\varepsilon$ are defined imposing $f_\varepsilon(b_1) = 0, f_\varepsilon(b_2) = 0$ and $D_x f_\varepsilon(b_1) = 0$ (i.e. $\kappa_1 + \delta_\varepsilon = \int_{[a_1, b_1]} dm(z) g_\varepsilon(z)$). It is simple to check that $D_\varepsilon, R_\varepsilon = O(1)$, while $\delta_\varepsilon = o(1)$.

Due to the integral representation (8.28) and since $f_\varepsilon(a_1) = f_\varepsilon(b_2) = 0$, we conclude that $f_\varepsilon \in \mathcal{D}(\mathcal{L}_D^{(a_1,b_2)})$ (property (i) above). Moreover, we point out that $f_\varepsilon(x) = 0$ for all $x \in [b_1, a_2]$ and that

\[ f_\varepsilon(x) = R_\varepsilon(x - a_2 - \varepsilon)\mathbb{1}(x \in [a_2 + \varepsilon, b_2]) + R_\varepsilon \int_{a_2}^{x} dy \frac{dm([a_2, y])}{dm([a_2, a_2 + \varepsilon])}(x \in [a_2, a_2 + \varepsilon]) - \int_{a_2}^{x} dy \int_{[a_2, y]} dm(z) g_\varepsilon(z) \]

for all $x \in [a_2, b_2]$. From the above observations one can easily check that the functions $f_\varepsilon$ satisfy also properties (ii) and (iii).

In the general case, i.e. $k \ge 2$, the idea is the following: by a small perturbation near $b_1, b_2, \ldots, b_{k-1}$ one modifies $f_i$ into a function $f_i(\varepsilon)$ such that $f_i(\varepsilon)(a_i) = f_i(\varepsilon)(b_i) = D_x f_i(\varepsilon)(b_1) = 0$, while by a small perturbation near $a_k$ one modifies $f_k$ into a function $f_k(\varepsilon)$ such that $f_k(\varepsilon)(a_k) = f_k(\varepsilon)(b_k) = D_x f_k(\varepsilon)(a_k) = 0$. Then the good approximating function is $f_\varepsilon = \psi((f_i(k))_{r=1}^k)$.

In order to prove (8.29) under the hypothesis $b_r = a_{r+1}$ for all $r = 1, \ldots, k - 1$, we first observe that the map (8.26) is indeed an isomorphism of Hilbert spaces (recall that
Given Lemma 8.6, let $(f_r)_{r=1}^k = \psi^{-1}(f)$. Then, we denote by $a$ and $g$ the unique constant $a \in \mathbb{R}$ and the unique function $g \in L^2([a_1, b_k], dm)$ satisfying
\[ f(x) = a - \int_{a_1}^x dy \int_{[a_1, y]} dm(z)g(z), \quad x \in [a_1, b_k], \tag{8.29} \]
derives that, given $r = 1, \ldots, k$, there exist suitable constants $A_r, B_r \in \mathbb{R}$ such that
\[ f_r(x) = A_r + B_r(x - a_r) - \int_{a_r}^x dy \int_{[a_r, y]} dm(z)g(z), \quad \forall x \in I_r. \]

Applying Lemma 8.24, we get that $f_r \in Q(q^{k}_N)$, i.e. $f_r$ belongs to the domain of the quadratic form $q^k_N$ associated to the operator $-\mathcal{L}^k_N$ and moreover $q^k_N(f_r) = \int_{I_r} D_x f_r(x)^2 dx$.

Since $f_r$ is simply the restriction of $f$ to the interval $I_r$, we get that $D_x f_r(x)$ exists and equals $D_x f(x)$ for almost all $x \in I_r$. In particular, since $dm$ gives zero mass to the complement of $\bigcup_{r=1}^k I_r$, invoking (8.10) we get $\sum_{r=1}^k q^k_N(f_r) = q^{(a_1, b_k)}(f)$. The above considerations imply that
\[ 0 \leq \bigoplus_{r=1}^k (-\mathcal{L}^k_N f^{a_1, b_k}) \leq \psi^{-1}(\mathcal{L}^k_N f^{a_1, b_k}) \circ \psi, \]
where the operator on the right is simply the self-adjoint operator on $\bigoplus_{r=1}^k L^2(I_r, dm)$ with domain $\{ \psi^{-1}(f) : f \in D(-\mathcal{L}^k_N f^{a_1, b_k}) \}$, mapping $\psi^{-1}(f)$ into $\psi^{-1}(-\mathcal{L}^k_N f^{a_1, b_k})$. At this point, (8.24) follows from the Lemma on page 270 of [RS4] and property (5) on page 268 of [RS4]. \hfill \square

### 8.4 Variational triple

In order to go beyond the estimates (8.24) and (8.25) (obtained mainly by adapting the arguments presented in [RS4] Chapter XIII) we need the abstract approach to the eigenvalue counting functions developed in [Me]. To this aim we consider the space $Q(q_N)$ endowed of the scalar product
\[(f, g) = q_N(f, g) + (f, g), \quad f, g \in Q(q_N),\]
where $(\cdot, \cdot)$ denotes the scalar product in $\mathcal{H}$. We write $\| \cdot \|_1$ for the associated norm. Due to Lemma 8.3 we know that $Q(q_D) \subset Q(q_N)$ and that on $Q(q_D)$ the scalar product $(\cdot, \cdot)_1$ coincides with $q_D(\cdot, \cdot) + (\cdot, \cdot)$.

In order to investigate better the spaces $Q(q_N)$ and $Q(q_D)$ endowed of the scalar product $(\cdot, \cdot)_1$ we need the following technical fact:

**Lemma 8.6.** Given $f \in Q(q_N)$, there exists a function $F \in C([0, \ell_m])$ such that (i) $f = F dm$–almost everywhere and (ii)
\[ |F(x) - F(y)| \leq \sqrt{q_N(f)(y - x)}, \quad \forall x < y \in [0, \ell_m]. \tag{8.30} \]
Moreover, $\lim_{x \downarrow 0} F(x)$ and $\lim_{x \uparrow \ell_m} F(x)$ are the same for all functions $F \in C([0, \ell_m]$ satisfying the above properties (i) and (ii).

**Proof.** Since $f \in Q(q_N)$ there exists a sequence of functions $f_n \in D(-\mathcal{L}_N)$ such that $f_n \rightarrow f$ in $\mathcal{H}$ and $(f_n - f_m, -\mathcal{L}_N(f_n - f_m)) \rightarrow 0$ as $n, m \rightarrow \infty$. At cost to take a subsequence, we can assume that $f_n$ converges to $f$ $dm$–almost everywhere, namely there exists a Borel subset $A \subset [0, \ell_m]$ such that $dm(A^c) = 0$ and $f_n(x) \rightarrow f(x)$ for all $x \in A$. Due to (8.3) it holds
\[ f_n(y) - f_n(x) = \int_x^y D_z f_n(z)dz, \quad \forall x < y \in [0, \ell_m]. \tag{8.31} \]
We point out that the limit \( \lim_{n,m \to \infty} (f_n - f_m, -\mathcal{L}_N (f_n - f_m)) \) is equivalent to the fact that \( (D_x f_n)_{n \geq 0} \) is a Cauchy sequence in \( L^2([0, \ell_m], dx) \), hence converging to some function \( g \in L^2([0, \ell_m], dx) \). Since \( (f_n, -\mathcal{L}_N f_n) \to q_N(f) \), it must be
\[
\int_0^{\ell_m} g(z)^2 dz = q_N(f).
\] (8.32)

In particular, passing to the limit \( \lim \) for \( x < y \) in \( \mathcal{A} \) we get
\[
f(y) - f(x) = \int_x^y g(z) dz, \quad \forall x < y \text{ in } \mathcal{A}.
\] (8.33)

At this point, we fix \( x_0 \in \mathcal{A} \) and set \( F(x) = f(x_0) + \int_{x_0}^x g(z) dz \) for all \( x \in [0, x_0] \). Then \( F(y) - F(x) = \int_x^y g(z) dz \) for all \( x < y \in [0, \ell_m] \). This identity, \( \text{(8.32)} \) and Schwarz' inequality trivially imply \( \text{(8.31)} \). Moreover, by \( \text{(8.33)} \) we conclude that \( f(y) = F(y) \) for all \( y \in \mathcal{A} \), and therefore \( f = F \) \( dm \)-almost everywhere.

Let us now take generic functions \( F, F' \in C([0, \ell_m]) \), satisfying (i) and (ii). We know that \( F = F' \) \( dm \)-almost everywhere. Since \( 0 = \inf E_m \) and \( m_0 = 0 \), it must be \( dm(\{(0, \varepsilon)\}) > 0 \) for all \( \varepsilon > 0 \). In particular, \( F = F' \) on a set having 0 as accumulation point, thus implying that \( \lim_{x \to 0} F(x) = \lim_{x \to 0} F'(x) \). A similar argument holds for \( \ell_m \) instead of 0. \( \square \)

Motivated by the above result, given \( f \in Q(q_N) \) we write \( f(0) \) and \( f(\ell_m) \) for the limits \( \lim_{x \to 0} F(x) \) and \( \lim_{x \to \ell_m} F(x) \), respectively, where \( F \) is any continuous function satisfying properties (i) and (ii) of Lemma 8.6

We can now prove the following fact:

**Lemma 8.7.** The following holds:

(i) The subset \( Q(q_N) \) is dense in \( \mathcal{H} \).

(ii) The space \( Q(q_N) \) endowed of the scalar product \( (\cdot, \cdot)_1 \) is an Hilbert space.

(iii) The inclusion map
\[
\iota : (Q(q_N), \| \cdot \|_1) \ni f \to f \in (\mathcal{H}, \| \cdot \|)
\]
is a continuous compact operator.

(iv) \( Q(q_D) \) is a closed subspace of the Hilbert space \( (Q(q_N), (\cdot, \cdot)_1) \). Moreover,
\[
Q(q_D) = \{ f \in Q(q_N) : f(0) = f(\ell_m) = 0 \}
\] (8.34)

and \( Q(q_D) \) has codimension 2 in \( Q(q_N) \).

**Proof.** (i) The set \( Q(q_N) \) includes the domain \( \mathcal{D}(\mathcal{L}_N) \), which we know to be dense in \( \mathcal{H} \).

(ii) This is a general fact, stated in Exercise 16 of [RS1] Chapter VIII.

(iii) Since \( \| f \| \leq \| f \|_1 \) for each \( f \in Q(q_N) \), the inclusion map \( \iota \) is trivially continuous. In order to prove compactness, we need to show that each sequence \( f_n \in Q(q_N) \) with \( \| f_n \|_1 \leq 1 \) admits a subsequence \( f_{n_k} \), which converges in \( \mathcal{H} \). Using Lemma 8.6 we can assume that \( f_n \in C([0, \ell_m]) \) and that \( |f_n(x) - f_n(y)| \leq \sqrt{y - x} \) for all \( x, y \in [0, \ell_m] \). Applying Ascoli–Arzelà Theorem, we then conclude that \( f_n \) admits a subsequence \( f_{n_k} \), which converges in the space \( C([0, \ell_m]) \) endowed of the uniform norm. Trivially, this implies the convergence in \( \mathcal{H} \).

(iv) We first prove the following:

**Claim:** If \( h \in Q(q_N) \) satisfies \( h(0) = h(\ell_m) = 0 \), then \( h \in Q(q_D) \).
Proof of the claim. To simplify the notation, we think \( h \) as the continuous representative described in Lemma \[8.34\]. We take \( h_n \in \mathcal{D}(-\mathcal{L}_N) \) such that \( h_n \to h \) in \( \mathcal{H} \) and \((h_n - h_m, -\mathcal{L}_N(h_n - h_m)) \to 0 \) as \( n, m \to \infty \). By definition of \( \mathcal{D}(-\mathcal{L}_N) \), we can write

\[
  h_n(x) = a_n - \int_0^x dy \int_{[0,y)} dm(z) g_n(z),
\]

where \( g_n \in \mathcal{H} \) satisfies \( \int_{[0,\ell_m]} dm(z) g_n(z) = 0 \). Due to \[8.35\], \( h_n \) can be thought of as a continuous function on \([0,\ell_m]\).

We claim that \( \lim_{n \to \infty} h_n(0) = \lim_{n \to \infty} h_n(\ell_m) = 0 \), at cost to take a subsequence. Indeed, the convergence in \( \mathcal{H} \) implies that, at cost to take a subsequence, there exists a subset \( \mathcal{A} \subset [0,\ell_m] \) with \( dm(\mathcal{A}^c) = 0 \) and \( h_n(x) \to h(x) \) for all \( x \in \mathcal{A} \). Since by assumption \( dm((0,\varepsilon)), dm((\ell - \varepsilon, \ell)) > 0 \) for all \( \varepsilon > 0 \), \( 0 \) and \( \ell_m \) are accumulation points of \( \mathcal{A} \). Using that \( h(0) = 0 \) and applying Lemma \[8.6\] we can write for \( x \in [0,\ell_m] \)

\[
  |h_n(0)| < |h_n(0) - h_n(x)| + |h_n(x) - h(x)| + |h(x) - h(0)| \leq \sqrt{q_N(h_n)} x + |h_n(x) - h(x)| + \sqrt{q_N(h)} x. \tag{8.36}
\]

Taking \( x \in \mathcal{A} \), the middle term in the r.h.s. disappears as \( n \to \infty \). Using now that \( q_N(h_n) \to q_N(h) < \infty \) and that \( 0 \) is an accumulation point for \( \mathcal{A} \) we conclude that \( h_n(0) \to 0 \). Similarly, we can prove that \( h_n(\ell_m) \to 0 \).

Now we define \( h_n(x) = h_n(x) - h_n(0) + c_n x \), where \( c_n \) is defined by the identity \( \tilde{h}_n(\ell_m) = h_n(\ell_m) - h_n(0) + c_n \ell_m = 0 \). Comparing with \[8.35\] and the definition of \( \mathcal{D}(-\mathcal{L}_D) \) we get that \((1) \tilde{h}_n \in \mathcal{D}(-\mathcal{L}_D) \). Since \( h_n(0) \to 0 \) and \( h_n(\ell_m) \to 0 \), we get that \( \|\tilde{h}_n - h_n\|_{\infty} \to 0 \) and therefore \( \|h_n - h\| \to 0 \). It follows that \((2) \tilde{h}_n \to h \) in \( \mathcal{H} \). Moreover, by definition \( D_x h_n(x) = D_x h(x) + c_n x \). This implies that \( D_x h_n(x) \) converges to zero in \( L^2([0,\ell_m], dx) \). Comparing with \[8.4\] we conclude that \((3) (h_n - h_m, -\mathcal{L}_D(h_n - h_m)) \to 0 \) as \( n, m \to \infty \). The above properties \((1),(2),(3)\) implies that \( h \in Q(q_D) \), thus concluding our proof.

Due to Exercise 16 in \[RS1\][Chapter VIII], the space \( Q(q_D) \) endowed of the scalar product \((\cdot, \cdot) + q_D(\cdot, \cdot)\) is an Hilbert space, hence complete. Since, as already observed, the above scalar product coincides with \((\cdot, \cdot)_1\) we conclude that \( Q(q_D) \) is a complete, and therefore close, subspace of \((Q(q_N), (\cdot, \cdot)_1)\).

Let us now prove \[8.34\]. To this aim we call \( W \) the set appearing in r.h.s. of \[8.34\]. Due to the above claim, we know that \( W \subset Q(q_D) \). By definition, the domain \( \mathcal{D}(-\mathcal{L}_D) \) is included in \( W \). Since, by Exercise 16 in \[RS1\][Chapter VIII], \( \mathcal{D}(-\mathcal{L}_D) \) is a dense subset of \((Q(q_D), (\cdot, \cdot)_1)\), in order to prove \[8.34\] we only need to show that \( W \) is closed. To this aim, take \( f_n \in W \) with \( f_n \to f \in Q(q_N) \) w.r.t. \( \|\cdot\|_1 \). Again, we suppose \( f_n \) and \( f \) to be continuous functions in \([0,\ell_m]\) as in Lemma \[8.6\]. At cost to take a subsequence, we can assume that \( f_n(x) \to f(x) \) for all \( x \in \mathcal{A} \subset [0,\ell_m] \), where \( dm(\mathcal{A}^c) = 0 \). By Lemma \[8.6\] for such an \( x \) we can bound \( |f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq |f(x) - f_n(x)| + c \sqrt{x} \) for a positive constant \( c \) independent from \( n \) and \( x \). Taking the limit we get \( |f(x)| \leq c \sqrt{x} \) for all \( x \in \mathcal{A} \), thus implying that \( f(0) = 0 \). Similarly, one get that \( f(\ell_m) = 0 \). This concludes the proof of \[8.34\].

The fact that \( Q(q_D) \) has codimension 2 in \( Q(q_N) \) follows immediately from Lemma \[8.4\] and the characterization \[8.34\].

Considering the space \( Q(q_N) \) endowed of the scalar product \((\cdot, \cdot)_1\), the above Lemma \[8.7\] implies that \((Q(q_N), \mathcal{H}, q_N(\cdot, \cdot))\) is a variational triple (cf. \[Mc\][Section II-2]). Indeed,
the following holds: (i) $Q(q_N)$ and $H$ are Hilbert spaces, (ii) the inclusion map gives a continuous injection of $Q(q_N)$ into $H$, (iii) $q_N(\cdot,\cdot)$ is a continuous scalar product on $Q(q_N)$ since $|q_N(f,g)| \leq \|f\|_1\|g\|_1$ for all $f,g \in Q(q_N)$, (iv) the scalar product $q_N(\cdot,\cdot)$ is coercive with respect to $H$: $\|f\|_1^2 - \|f\|_2^2 \leq q_N(f,f)$ for all $f \in Q(q_N)$ (the inequality is indeed a strict inequality).

Finally, by Lemma 8.7 the inclusion map $\iota : Q(q_N) \hookrightarrow H$ is compact and $Q(q_D)$ is a closed subspace in $Q(q_N)$. Applying Proposition 2.9 in [Me] we get the equality $\mathcal{N}_{m_N}^{0,\ell_m}(x) = N(x;Q(q_N),H,q_N)$ and $\mathcal{N}_{m,D}^{0,\ell_m}(x) = N(x;Q(q_D),H,q_D)$, where the functions $N(x;Q(q_N),H,q_N)$ and $N(x;Q(q_D),H,q_D)$ are defined in [Me] [Page 131]. As byproduct of Proposition 8.7(iv), Proposition 2.7 in [Me] and the arguments used in Corollary 4.7 in [KL], we obtain that

$$\mathcal{N}_{D,m}^{0,\ell_m}(x) \leq \mathcal{N}_{m,m}^{0,\ell_m}(x) \leq \mathcal{N}_{D,m}^{0,\ell_m}(x) + 2, \quad \forall x \geq 0. \quad (8.37)$$

8.5. Conclusion. We can now conclude stating the Dirichlet–Neumann bracketing in our context:

**Theorem 8.8.** (Dirichlet–Neumann bracketing). Let $I = [a,b]$, let

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

be a partition of the interval $I$ and set $I_r := [a_r,a_{r+1}]$ for $r = 0,\ldots,n-1$. Suppose that $m : I \to \mathbb{R}$ is a nondecreasing function such that

(i) $m_{a_r} = 0$ for all $r = 0,\ldots,n$,

(ii) $dm([a_r,a_r+\varepsilon]) > 0$ for all $r = 0,\ldots,n-1$ and $\varepsilon > 0$,

(iii) $dm([a_r-\varepsilon,a_r]) > 0$ for all $r = 1,\ldots,n$ and $\varepsilon > 0$.

Then, for all $x \geq 0$ it holds

$$\mathcal{N}_{m,D}^I(x) \leq \mathcal{N}_{m,m}^I(x) \leq \mathcal{N}_{m,D}^I(x) + 2, \quad (8.38)$$

$$\mathcal{N}_{m,D}^I(x) \geq \sum_{i=0}^{n-1} \mathcal{N}_{m,D}^I(x) \quad (8.39)$$

$$\mathcal{N}_{m,m}^I(x) \leq \sum_{i=0}^{n-1} \mathcal{N}_{m,m}^I(x). \quad (8.40)$$

**Proof.** The bounds in (8.38) have been obtained in (8.37) (note that the first bound follows also from (8.20)). The inequalities (8.39) and (8.40) follow from Lemma 8.8. 

As immediate consequence of (8.38) and (8.40) we get a bound which will reveal very useful to derive (2.15) and (2.17):

**Corollary 8.9.** In the same setting of Theorem 8.8 it holds $\mathcal{N}_{m,D}^I(x) \leq 2n + \sum_{i=0}^{n-1} \mathcal{N}_{m,D}^I(x)$. 

9. Proof of Proposition 2.22

We first consider how the eigenvalue counting functions change under affine transformations (recall the notation introduced after (8.20)):

**Lemma 9.1.** Let $m : \mathbb{R} \to \mathbb{R}$ be a nondecreasing càdlàg function. Given the interval $I = [a,b]$, suppose that $m_a = m_b = 0$ and $dm((a,a+\varepsilon)) > 0$, $dm((b-\varepsilon,b)) > 0$ for
all $\varepsilon > 0$. Given $\gamma, \beta > 0$, set $J = [\gamma a, \gamma b]$ and define the function $M : \mathbb{R} \to \mathbb{R}$ as $M(x) = \gamma^{1/\beta} m(x/\gamma)$. Then

$$N_{m,D/N}(x) = N_{M,D/N}(x/\gamma^{1+1/\beta}).$$

(9.1)

Trivially, $M_{\gamma a} = M_{\gamma b} = 0$ and $dM((\gamma a, \gamma a + \varepsilon)) > 0$, $dM((\gamma b - \varepsilon, \gamma b)) > 0$ for all $\varepsilon > 0$

Proof. For simplicity of notation we take $a = 0$. Suppose that $\lambda$ is an eigenvalue of the operator $-D_M D_x$ on $[0,b]$ with Dirichlet b.c. at 0 and $b$. This means that for a nonzero function $F \in C(I)$ with $F(0) = 0$ and a constant $c$ it holds

$$F(x) = cx - \lambda \int_0^x dy \int_{(0,y)} dm(z) F(z), \quad \forall x \in I.$$ (9.2)

Taking $X \in J$, the above identity implies that

$$F(X/\gamma) = \frac{cX}{\gamma} - \lambda \int_0^{X/\gamma} dy \int_{(0,y)} dm(z) F(z) = \frac{cX}{\gamma} - \lambda \int_0^X dY \int_{(0,Y)} dm(z) F(z) =$$

$$\frac{cX}{\gamma} - \frac{\lambda}{\gamma^{1+1/\beta}} \int_0^X dY \int_{(0,Y)} dm(Z) F(Z/\gamma).$$ (9.3)

Since trivially $F(X/\gamma) = 0$ for $X = b\gamma$, the above identity implies that $\lambda/\gamma^{1+1/\beta}$ is an eigenvalue of the operator $-D_M D_x$ on $J$ with Dirichlet b.c. and eigenfunction $F(-\cdot/\gamma)$. This implies (9.1) in the case of Dirichlet b.c. The Neumann case is similar.$\Box$

We have now all the tools in order to prove Proposition 2.2.

Proof of Proposition 2.2 Take $m$ as in the Proposition 2.2 and recall the notational convention stated after the Proposition. We first prove (2.19), assuming without loss of generality that (2.10) holds with $x_0 = 1$. By assumption, with probability one, for any $n \in \mathbb{N}_+$ and any $k \in \mathbb{N} : 0 \leq k \leq n$ it holds: (i) $dm\{k/n\} = 0$, (ii) $dm((k/n, k/n + \varepsilon)) > 0$ for all $\varepsilon > 0$ if $k < n$, (iii) $dm((k/n - \varepsilon, k/n)) > 0$ for all $\varepsilon > 0$ if $k > 0$. Below, we assume that the realization of $m$ satisfies (i), (ii) and (iii). This allows us to apply the Dirichlet–Neumann bracketing stated in Theorem 8.8 to the non–overlapping subintervals $I_k := [k/n, (k+1)/n]$, $k \in \{0, 1, \ldots, n-1\}$. Due to the superadditivity (resp. subadditivity) of the Dirichlet (resp. Neumann) eigenvalue counting functions (cf. (8.30) and (8.40) in Theorem 8.8), we get for any $x \geq 0$ that $N_{m,D}^{[0,1]}(x) \geq \sum_{k=0}^{n-1} N_{m,D}^{[k/n, (k+1)/n]}(x)$, while $N_{m,N}^{[0,1]}(x) \leq \sum_{k=0}^{n-1} N_{m,N}^{[k/n, (k+1)/n]}(x)$. By taking the average over $m$ and using that $m$ has independent stationary increments we get that $\mathbb{E}N_{m,D}^{[0,1]}(x) \geq n\mathbb{E}N_{m,D}^{[0,1/n]}(x)$ and $\mathbb{E}N_{m,N}^{[0,1]}(x) \leq n\mathbb{E}N_{m,N}^{[0,1/n]}(x)$. Using now the scaling property of Lemma 9.1 with $\gamma = n$, $\beta = \alpha$ and the self–similarity of $m$, we conclude that

$$\mathbb{E}N_{m,D}^{[0,1]}(x) \geq n\mathbb{E}N_{m,D}^{[0,1/n]}(x) = n\mathbb{E}N_{M,D}^{[0,1]}(x/n^{1+1/\alpha}) = n\mathbb{E}N_{m,B}^{[0,1]}(x/n^{1+1/\alpha}),$$

(9.4)

$$\mathbb{E}N_{m,N}^{[0,1]}(x) \leq n\mathbb{E}N_{m,N}^{[0,1/n]}(x) = n\mathbb{E}N_{M,N}^{[0,1]}(x/n^{1+1/\alpha}) = n\mathbb{E}N_{m,N}^{[0,1]}(x/n^{1+1/\alpha}),$$

(9.5)

where $M(x) := n^{1/\alpha} m(x/n)$. On the other hand, by (8.88) of Theorem 8.8

$$\mathbb{E}N_{m,B}^{[0,1]}(x) \leq \mathbb{E}N_{m,N}^{[0,1]}(x) \leq \mathbb{E}N_{m,B}^{[0,1]}(x) + 2.$$ (9.6)

From the above estimates (9.4), (9.5) and (9.6), we conclude that

$$\mathbb{E}N_{m,D}^{[0,1]}(1) \leq n^{-1}\mathbb{E}N_{m,D}^{[0,1]}(n^{1+1/\alpha}) \leq n^{-1}\mathbb{E}N_{m,N}^{[0,1]}(n^{1+1/\alpha}) \leq \mathbb{E}N_{m,N}^{[0,1]}(1) \leq \mathbb{E}N_{m,D}^{[0,1]}(1) + 2.$$ (9.7)
We remark that (2.15) with \( x_0 = 1 \) simply reads \( \mathbb{E}N_{m,D}^{[0,1]}(1) < \infty \). Since the eigenvalue counting functions are monotone, in the above estimate (9.7) we can think of \( n \) as any positive number larger than 1. Then, substituting \( n^{1+1/\alpha} \) with \( x \) we get (2.16).

In order to prove (2.18), we first prove the joint self–similarity of \( m, m^{-1} \): given \( \gamma > 0 \), it holds

\[
(m(x), m^{-1}(y) : x, y \geq 0) \sim (\gamma^{1/\alpha} m(x/\gamma), \gamma m^{-1}(\gamma^{-1/\alpha} y) : x, y \geq 0) \sim (\gamma m(x/\gamma), \gamma^{\alpha} m^{-1}(x/\gamma) : x, y \geq 0).
\]

To check the above claim, first we observe that for each \( x \geq 0 \) it holds

\[
\inf \left\{ t \geq 0 : \gamma^{1/\alpha} m(t/\gamma) > y \right\} = \gamma \inf \left\{ t \geq 0 : m(t) > \gamma^{-1/\alpha} y \right\} = \gamma m^{-1}(\gamma^{-1/\alpha} y).
\]

On the other hand, by the self–similarity of \( m \) and by the definition of the generalized inverse function, we get

\[
\left( \gamma^{1/\alpha} m(x/\gamma), \inf \left\{ t \geq 0 : \gamma^{1/\alpha} m(t/\gamma) > y \right\} : x, y \geq 0 \right) \sim (m(x), m^{-1}(y) : x, y \geq 0).
\]

The first identity in (9.8) follows from (9.7) and (9.10). The second identity follows by replacing \( \gamma^{1/\alpha} \) with \( \gamma \). This concludes the proof of (9.8).

Recall the convention established after (2.15). We already know that \( dm^{-1} \) is a continuous function a.s., hence a.s. it holds \( (P1) \ dm^{-1}\left(\{m(k/n)\}\right) = 0 \) for all \( n \in \mathbb{N} \) and \( k \in \mathbb{N} : 0 \leq k \leq n \). By identity (2.20) \( m^{-1}(x) = m^{-1}(y) \) if and only if \( x, y \in [m(z_i-), m(z_i)] \) for some jump point \( z_i \) of \( m \). Since by property (iv) in Proposition 2.2 \( m(k/n) \) is not a jump point for \( m \) a.s. (with \( k, n \) as above), the following properties hold a.s.: \( (P2) \ dm^{-1}\left(\{m(k/n), m(k/n) + \varepsilon\}\right) > 0 \) for all \( \varepsilon > 0 \) if \( 0 \leq k < n \) and \( (P3) \ dm^{-1}\left(\{m(k/n) - \varepsilon, m(k/n)\}\right) > 0 \) for all \( \varepsilon > 0 \) if \( 0 < k \leq n \). In what follows we assume that the realization of \( m \) satisfies the properties \( (P1), (P2) \) and \( (P3) \). This allows us to apply the Dirichlet–Neumann bracketing to the measure \( dm^{-1} \) and to the non–overlapping subintervals \( I_k = [m(k/n), m(k + 1/n)] \), \( k \in \{0, 1, \ldots, n - 1 \} \). We point out that the measure \( dm^{-1} \) restricted to each subinterval \( I_k \) is univocally determined by the values \( \{m(x) - m(k/n) : x \in [k/n, (k + 1)/n]\} \). The fact that \( m \) has independent and stationary increments, allows to conclude that the random functions \( N^{I_k}_{m^{-1},D/N}(\cdot) \) are i.i.d.

We observe now that (9.8) with \( \gamma = n^{1/\alpha} \) implies that

\[
(m(x), m^{-1}(y) : x, y \geq 0) \sim \left(n^{1/\alpha} m(x/n), nm^{-1}(x/n^{1/\alpha}) : x, y \geq 0\right).
\]

Then, using the Dirichlet–Neumann, Lemma (9.1) with \( \beta = 1/\alpha \) and \( \gamma = n^{1/\alpha} \) and the joint self–similarity (9.11), we conclude that

\[
\mathbb{E}N_{m^{-1}, D}^{(0, m(1))}(x) \geq n \mathbb{E}N_{m^{-1}, D}^{(0, m(1))}(x) = n \mathbb{E}N_{M, D}^{(0, n^{1/\alpha} m(1))}(x/n^{1+1/\alpha}) = n \mathbb{E}N_{m^{-1}, D}^{(0, m(1))}(x/n^{1+1/\alpha}),
\]

\[
\mathbb{E}N_{m^{-1}, D}^{(0, m(1))}(x) \leq n \mathbb{E}N_{m^{-1}, D}^{(0, m(1))}(x) = n \mathbb{E}N_{M, D}^{(0, n^{1/\alpha} m(1))}(x/n^{1+1/\alpha}) = n \mathbb{E}N_{m^{-1}, D}^{(0, m(1))}(x/n^{1+1/\alpha}),
\]

where now \( M(x) = nm^{-1}(x/n^{1/\alpha}) \). Note that (9.12) and (9.13) have the same structure of (9.14) and (9.15), respectively. The conclusion then follows the same arguments used for (2.16).
10. Proof of Theorem 2.3

As already mentioned in the Introduction, the proof of Theorem 2.3 is based on a special coupling introduced in [FIN] (and very similar to the coupling of [KK] for the random barrier model). If \( \tau(x) \) is itself the \( \alpha \)-stable law with Laplace transform \( E[e^{-\lambda \tau(x)}] = e^{-\lambda^\alpha} \), this coupling is very simple since it is enough to define, for each realization of \( V \) and for all \( n \geq 1 \), the random variables \( \tau_n(x) \)'s as

\[
\tau_n(x) = n^{1/\alpha} \left[ V\left( x + \frac{1}{n} \right) - V(x) \right], \quad \forall x \in \mathbb{Z}_n. \tag{10.1}
\]

Due to (2.23) and the fact that \( V \) has independent increments, one easily derives that the \( V \)-dependent random field \( \{ \tau_n(x) : x \in \mathbb{Z}_n \} \) has the same law of \( \{ \tau(nx) : x \in \mathbb{Z}_n \} \). In the general case one proceeds as follows. Define a function \( G : [0, \infty) \rightarrow [0, \infty) \) such that

\[ P(V(1) > G(x)) = P(\tau(0) > x), \quad \forall x \geq 0. \]

(Recall that \( V \) is defined on the probability space \((\Xi, \mathcal{F}, P)\).) The above function \( G \) is well defined since \( V(1) \) has continuous distribution, \( G \) is right continuous and nondecreasing. Then the generalized inverse function

\[ G^{-1}(t) = \inf\{ x \geq 0 : G(x) > t \} \]

is nondecreasing and right continuous. Finally, set

\[ \tau_n(x) = G^{-1}\left( n^{\frac{1}{\alpha}} \left[ V\left( x + \frac{1}{n} \right) - V(x) \right] \right), \quad x \in \mathbb{Z}_n. \tag{10.2} \]

It is trivial to check that the \( V \)-dependent random field \( \{ \tau_n(x) : x \in \mathbb{Z}_n \} \) has the same law of \( \{ \tau(nx) : x \in \mathbb{Z}_n \} \). Indeed, since \( V \) has independent and stationary increments one obtains that the \( \tau_n(x) \)'s are i.i.d., while since \( n^{\frac{1}{\alpha}} (V(x + \frac{1}{n}) - V(x)) \) and \( V(1) \) have the same law, one obtains that

\[ P(\tau_n(x) > t) = P(G^{-1}(V(1)) > t) = P(V(1) > G(t)) = P(\tau(nx) > t), \quad \forall t \geq 0. \]

We point out that the coupling obtained by this general method does not lead to (10.1) in the case that \( \tau(x) \) is itself the \( \alpha \)-stable law with Laplace transform \( E[e^{-\lambda \tau(x)}] = e^{-\lambda^\alpha} \).

10.1. Proof of Point (i). Let us keep definition (10.2). For any \( n \geq 1 \) we introduce the generalized trap model \( \{X^{(n)}(t)\}_{t \geq 0} \) on \( \mathbb{Z}_n \) with jump rates

\[ c_n(x, y) = \begin{cases} 
\gamma^2 L_2(n)n^{1+\frac{1}{\alpha}}\tau_n(x)^{-1+\alpha} \tau_n(y)^a & \text{if } |x - y| = 1/n \\
0 & \text{otherwise},
\end{cases} \]

where \( \gamma = E(\tau(x)^{-a}) \). The above jump rates can be written as \( c_n(x, y) = 1/H_n(x)U_n(x\lor y) \) for \( |x - y| = 1/n \) by taking

\[
\begin{align*}
U_n(x) &= \gamma^{-2}n^{-1}\tau_n(x - \frac{1}{n})^{-a}\tau_n(x)^{-a} \\
H_n(x) &= L_2(n)^{-1}n^{-\frac{3}{2}}\tau_n(x).
\end{align*}
\]

Note that in all cases both \( U_n \) and \( H_n \) are functions of the \( \alpha \)-stable subordinator \( V \).

Then the following holds

**Lemma 10.1.** Let \( m_n \) be defined as in (2.10) by means of the above functions \( U_n, H_n \). Then for almost any realization of the \( \alpha \)-stable subordinator \( V \), \( \ell_n \rightarrow 1 \) and the measures \( dm_n \) weakly converge to the measure \( dV \) (recall definition (2.19)).
Proof. Due to our definition (2.8) we have
\[ S_n\left(\frac{k}{n}\right) = \frac{1}{n} \sum_{j=1}^{k} \gamma^{-2} \tau_n\left(\frac{j-1}{n}\right)^{-a} \tau_n\left(\frac{j}{n}\right)^{-a}, \quad 0 \leq k \leq n, \]
with the convention that the sum in the r.h.s. is zero if \( k = 0 \). If \( a = 0 \) trivially \( \gamma = 1 \) and \( S(k/n) = k/n \). If \( a > 0 \) we can apply the strong law of large numbers for triangular arrays. Indeed, all addenda have the same law and they are independent if they are not consecutive, moreover they have bounded moments of all orders since \( \tau(x) \) is bounded from below by a positive constant a.s. (this assumption is used only here and could be weakened in order to assure the validity of the strong LLN). Due to the choice of \( \gamma \) we have that \( \gamma^{-2} \tau_n\left(\frac{j-1}{n}\right)^{-a} \tau_n\left(\frac{j}{n}\right)^{-a} \) has mean 1. By the strong law of large number we conclude that for a.a. \( V \) it holds \( \lim_{n \to \infty} S(|x_n|/n) = x \) for all \( x \geq 0 \). This proves in particular that \( \ell_n := S_n(1) \to 1 \). It remains to prove that for all \( f \in C_c(\mathbb{R}) \) it holds
\[ \lim_{n \to \infty} \sum_{k=0}^{n} f(S_n(k/n))H_n(k/n) = \int_{0}^{1} f(s)dV_s(s). \quad (10.3) \]
This limit can be obtained by reasoning as in the proof of Proposition 5.1 in [BC1], or can be derived by Proposition 5.1 in [BC1] itself together with the fact that \( P \) a.s. \( V \) has no jump at 0, 1. To this aim one has to observe that the constant \( c_\varepsilon \) (where \( \varepsilon = 1/n \)) in [FIN] and [BC1] (eq. (49)) equals our quantity \( 1/h(n) = 1/(n^{1/n}L_2(n)) \) (recall the definitions preceding Theorem 2.3). In particular, \( H_n(k/n) = c_{1/n}\tau_n(k/n) \).

Due to the above result, Point (i) in Theorem 2.3 follows easily from Theorem 2.1 and the fact that the random fields \( \{\tau_n(x) : x \in \mathbb{Z}_n\} \) and \( \{\tau(nx) : x \in \mathbb{Z}_n\} \) have the same law for all \( n \geq 1 \).

10.2. Proof of Point (ii). Point (i) can be proved in a similar and simpler way. In this case, we define \( \tau_n(x) \) as in (10.1) and we consider the generalized trap model \( \{X^{(n)}(t)\}_{t \geq 0} \) on \( \mathbb{Z}_n \) with jump rates
\[ c_n(x,y) = \begin{cases} \frac{1}{n} \tau_n(x)^{-1} & \text{if } |x-y| = 1/n \\ 0 & \text{otherwise}, \end{cases} \]
with associated functions
\[ U_n(x) = 1/n, \quad H_n(x) = n^{-\frac{3}{2}} \tau_n(x) = V(x + 1/n) - V(x) =: \Delta_nV(x). \]
By this choice, \( dm_n = \sum_{k=0}^{n} \delta_{k/n} \Delta_nV(k/n) \). Trivially, \( \ell_n = 1 \) and \( dm_n \to dV_s \) for all realizations of \( V \) giving zero mass to the extreme points 0 and 1. Since this event takes place \( P \)-almost surely, the proof of part (ii) is concluded.

10.3. Proof of Point (iii). Part (iii) of Theorem 2.3 (i.e. (2.26)) follows from Proposition 2.2 and Lemma 10.2 below. The self-similarity of \( V \) is the following: for each \( \gamma > 0 \) it holds
\[ (V(x), x \in \mathbb{R}) \sim (\gamma \frac{1}{n} V(x/\gamma), x \in \mathbb{R}). \quad (10.4) \]
Indeed, both processes are càdlàg, take value 0 at the origin and have independent increments with the same law due to (2.23).

Lemma 10.2. Taking \( m = V \), the bound (2.15) is satisfied.
Proof. Using the notation of Section 9 we denote by $\mathcal{N}^{[0,1]}_{V,D}(1)$ the number of eigenvalues not larger than 1 of the operator $-D_V D_x$ on $[0,1]$ with Dirichlet boundary conditions. We assume that $V$ has no jump at 0,1 (this happens $\mathcal{P}$-a.s.). We recall that $V$ can be obtained by means of the identity $dV = \sum_{j \in J} x_j \delta_{v_j}$, where the random set $\xi = \{(x_j, v_j) : j \in J\}$ is the realization of a inhomogeneous Poisson point process on $\mathbb{R} \times \mathbb{R}_+$ with intensity $cv^{-1-\alpha}dx dv$, for a suitable positive constant $c$. In order to distinguish between the contribution of big jumps and not big jumps it is convenient to work with two independent inhomogeneous Poisson point processes $\xi^{(1)}$ and $\xi^{(2)}$ on $\mathbb{R} \times \mathbb{R}_+$ with intensity $cv^{-1-\alpha}1(v \leq 1/2)dx dv$ and $cv^{-1-\alpha}1(v > 1/2)dx dv$. We write $\xi^{(1)} = \{(x_j, v_j) : j \in J_1\}$ and $\xi^{(2)} = \{(x_j, v_j) : j \in J_2\}$. The above point process $\xi$ can be defined as $\xi = \xi^{(1)} \cup \xi^{(2)}$. Moreover, a.s. it holds $\xi^{(1)} \cap \xi^{(2)} = \emptyset$ (this fact will be understood in what follows). By the Master Formula (cf. Proposition (1.10) in [RY]), it holds

$$E\left[ \sum_{j \in J_1 : x_j \in [0,1]} \nu_j \right] = c \int_0^1 dx \int_0^{1/2} dv v^{-\alpha} < \infty, \quad (10.5)$$

$$E\left[ \sum_{j \in J_2} \nu_j \right] = c \int_0^1 dx \int_{1/2}^{\infty} dv v^{-\alpha-1} < \infty. \quad (10.6)$$

We label in increasing order the points in $\{x_j : j \in J_2, x_j \in [0,1]\}$ as $y_1 < y_2 < \cdots < y_N$ (note that the set is finite due to (10.6)).

Given $\delta \in (0, 1/8)$, we take $\varepsilon \in (0,1)$ small enough that

(i) the intervals $(y_i - \varepsilon, y_i + \varepsilon)$ are included in $(0,1)$ and do not intersect as $i$ varies from 1 to $N$,

(ii) for all $i : 1 \leq i \leq N$, it holds $\sum_{j \in J_1 : x_j \in (y_i - \varepsilon, y_i + \varepsilon)} v_j < \delta$,

(iii) for all $i : 1 \leq i \leq N$, the points $y_i - \varepsilon$ and $y_i + \varepsilon$ do not belong to $\{x_j : j \in J_1\}$.

Defining $V^{(1)}(t) = \sum_{j \in J_1 : x_j \leq t} v_j$, the last condition (iii) can be stated as follows: for all $i : 1 \leq i \leq N$, the points $y_i - \varepsilon$ and $y_i + \varepsilon$ are not jump points for $V^{(1)}$.

By construction the function $V^{(1)}$ has jumps not larger than 1/2. In particular, all the intervals $A_0 = (0, y_1 - \varepsilon)$, $A_1 = (y_1 + \varepsilon, y_2 - \varepsilon)$, $A_2 = (y_2 + \varepsilon, y_3 - \varepsilon), \ldots$, $A_{N-1} = (y_{N-1} + \varepsilon, y_N - \varepsilon)$, $A_N = (y_N + \varepsilon, 1)$ can be partitioned in subintervals such that, on each subinterval, the function $V^{(1)}$ has increment in $[1/2,1)$ and has no jump at the border (recall property (iii) above). As a consequence, the total number $R$ of subintervals is bounded by $2V^{(1)}(1)$, which has finite expectation due to (10.5). By the bound (11.11) in Lemma 4.4 we get that the operator $-D_V D_x$ on any subinterval with Dirichlet boundary conditions has no eigenvalues smaller than 2. This observation and Corollary 8.9 imply that

$$\mathcal{N}^{[0,1]}_{D,V}(1) \leq 2R + \sum_{i=1}^N \mathcal{N}^{[y_i-\varepsilon,y_i+\varepsilon]}_{D,V}(1). \quad (10.7)$$

Claim: For each $i : 1 \leq i \leq N$ it holds $\mathcal{N}^{[y_i-\varepsilon,y_i+\varepsilon]}_{D,V}(1) \leq 1$.

Proof of the claim. We reason by contradiction supposing that $f_1$ and $f_2$ are eigenfunctions of the Dirichlet operator $-D_V D_x$ on $U = [y_i - \varepsilon, y_i + \varepsilon]$, whose corresponding eigenvalues $\lambda_1$ and $\lambda_2$ satisfy $0 < \lambda_1 < \lambda_2 \leq 1$. Due to this bound and Lemmata 8.3 and 8.6 we can take $f_1$ and $f_2$ continuous on $U$, satisfying $\int_U f_1^2(x) dV(x) = 1$ and

$$|f_j(x) - f_j(y_i)| \leq \sqrt{|x - y_i|}, \quad x \in U \quad (10.8)$$

Furthermore, since $f_1$ and $f_2$ are continuous on $U$ and $y_i - \varepsilon, y_i + \varepsilon$ are not jump points, we have $f_1(x) = f_2(x)$ for $x \in U$, hence $f_1 = f_2$ and $\|f_1 - f_2\|^2 = 0$. This contradicts the assumption that $f_1$ and $f_2$ are eigenfunctions of $-D_V D_x$ on $U$ with different eigenvalues, proving the claim.

38 A. FAGGIONATO
In order to prove the weak convergence of $d\nu^{(2.19))$. We point out that in [KK] a similar result is proved, but the definition given in $f$ as stated in Lemma 3.1 of [FIN] it holds $g$ with jump rates $H$

Proof of Point (i).

11.1. On each $n$ associate the measure $H$

where $H$

Combining (10.9) and (10.10), we conclude that

$$1/4 \leq |\Delta f_1(y_1)f_2(y_2)| \leq (1 + \sqrt{2})^2 \Delta f_1(y_1)f_2(y_2),$$

in contrast with the bound $\delta < 1/8$. □

Applying the above claim to (10.7) we conclude that $X_{D,V}^{[0,1]}(1) \leq 2R + N$. We have already observed that $R$ has finite expectation. The same trivially holds also for $N$ due to (10.6).

11. Proof of Theorem 2.5

Recall the definition of $T_n$ given in the previous section. Given a realization of $V$, for each $n \geq 1$ we consider the continuous-time nearest–neighbor random walk $X^{(n)}$ on $\mathbb{Z}_n$ with jump rates

$$c_n(x, y) = \begin{cases} L_2(n)n^{1+\frac{1}{\alpha}}\tau_n(x \lor y)^{-1} & \text{if } |x - y| = 1/n, \\ 0 & \text{otherwise}. \end{cases}$$

The rates $c_n(x, y)$ for $|x - y| = 1/n$ can be written as $c_n(x, y) = 1/[H_n(x, y)U_n(x \lor y)]$, where $H_n(x) = 1/n$ and $U_n(x) = L_2(n)^{-1}n^{-\frac{1}{\alpha}}\tau_n(x)$. To the above random walk we associate the measure $dm_n$ defined in (2.10).

11.1. Proof of Point (i). Let us show that $dm_n$ weakly converges to $d(V^{-1})_*$ (recall (2.19)). We point out that in [KK] a similar result is proved, but the definition given in [KK] of the analogous of $dm_n$ is different, hence that proof cannot be adapted to our case. In order to prove the weak convergence of $dm_n$ to $d(V^{-1})_*$, we use some results and ideas developed in Section 3 of [FIN]. Recall that the constant $c_\varepsilon$ of [FIN] equals our quantity $1/h(n) = 1/(n^{1/\alpha}L_2(n))$ if $\varepsilon = 1/n$. Given $n \geq 1$ and $x > 0$ we define

$$g_n(x) = (L_2(n)n^{\frac{1}{\alpha}})^{-1}G^{-1}(n^{\frac{1}{\alpha}}x).$$

We point out that $g_n$ coincides with the function $g_\varepsilon$ defined in [FIN]((3.12)) if $\varepsilon = 1/n$. As stated in Lemma 3.1 of [FIN] it holds $g_n(x) \to x$ as $n \to \infty$ for all $x > 0$. Since $g_n$ is nondecreasing, we conclude that

$$g_n(x_n) \to x \text{ as } n \to \infty, \quad \forall x > 0, \forall \{x_n\}_{n \geq 1} : x_n > 0, \ x_n \to x.$$
As stated in Lemma 3.2 of [FIN], for any \( \delta' > 0 \) there exist positive constants \( C' \) and \( C'' \) such that
\[
g_n(x) \leq C' x^{-\delta'} \text{ for } n^{-\frac{1}{T}} x \leq 1 \text{ and } n \geq C''. \quad (11.3)
\]

Since \( U_n(x) = g_n(V(x + 1/n) - V(x)) \), we can write
\[
S_n(k/n) = \sum_{j=0}^{k-1} g_n(V((k+1)/n) - V(k/n)) . \quad (11.4)
\]

**Lemma 11.1.** For \( \mathcal{P} \)-almost all \( V \) it holds
\[
\lim_{n \uparrow \infty} \max_{0 \leq k \leq n} |S_n(k/n) - V(k/n)| = 0 . \quad (11.5)
\]

**Proof.** We recall that \( V \) can be obtained by means of the identity \( dV = \sum_{j \in J} x_j \delta_{v_j} \), where the random set \( \xi = \{(x_j, v_j) : j \in J\} \) is the realization of a inhomogeneous Poisson point process on \( \mathbb{R} \times \mathbb{R}_+ \) with intensity \( c v^{-1-\alpha} \, dx \, dv \), for a suitable positive constant \( c \). Given \( y > 0 \), let us define
\[
J_{n,y} := \{ r \in \{0, 1, \ldots, n - 1\} : V((r + 1)/n) - V(r/n) \geq y \}, \\
J_y := \{ j \in J : v_j \geq y, \, x_j \in [0, 1]\} .
\]

Note that the set \( J_y \) is always finite. Reasoning as in the Proof of Proposition 3.1 in [FIN], and in particular using also (11.3), one obtains for \( \mathcal{P} \)-a.a. \( V \) that
\[
\lim_{n \uparrow \infty} \sum_{r:0 \leq r < n, r \notin J_{n,\delta}} g_n(V((r + 1)/n) - V(r/n)) = 0 , \quad \forall \delta > 0 . \quad (11.6)
\]

We claim that, given \( \delta > 0 \), for a.a. \( V \) it holds
\[
J_{n,\delta} = \{ r \in \{0, 1, \ldots, n - 1\} : \exists j \in J_\delta \text{ such that } x_j \in (r/n, (r+1)/n]\} \quad (11.7)
\]
eventually in \( n \). Let us suppose that (11.7) is not satisfied. Since the set in the r.h.s.
is trivially included in \( J_{n,\delta} \), there exists a sequence of integers \( r_n \) with \( 0 \leq r_n < n \) such that \( \phi_n := V((r_n + 1)/n) - V(r_n/n) \geq \delta \) while \( v_j < \delta \) for all \( x_j \in (r_n, (r_n + 1)/n] \). We introduce the càdlàg function \( \hat{V}(t) = \sum_{j \in J : x_j \leq t} v_j I(v_j < \delta) \) and we note that, if \( \forall j \in J \) with \( x_j \in (r_n/n, (r_n + 1)/n] \) it holds \( v_j < \delta \), then \( \phi_n = \hat{V}((r_n + 1)/n) - \hat{V}(r_n/n) \). At cost to take a subsequence, we can suppose that \( r_n/n \) converges to some point \( x \). It follows then that \( \hat{V}(x+) - \hat{V}(x-) \geq \delta \), in contradiction with the fact that \( \hat{V} \) has only jumps smaller than \( \delta \). This concludes the proof of our claim.

Due to the above claim and due to (11.2), we conclude that a.s., given \( \delta > 0 \), it holds
\[
\lim_{n \uparrow \infty} \sup_{1 \leq k \leq n} \left| \sum_{r \in J_{n,\delta}, r < k} g_n(V((r + 1)/n) - V(r/n)) - \sum_{j \in J_{\delta} : x_j \leq k/n} v_j \right| = 0 . \quad (11.8)
\]
Combining (11.8) and (11.6), we conclude that for any \( \varepsilon > 0 \) one can fix a.s. \( \delta > 0 \) small enough such that
\[
\max_{0 \leq k \leq n} \left| S(k/n) - \sum_{j \in J_{\delta} : x_j \leq k/n} v_j \right| \leq \varepsilon \quad (11.9)
\]
for \( n \) large enough. On the other hand, a.s. one can fix \( \delta \) small enough that \( \sum_{j \in J_{\delta} : x_j \in [0, 1]} v_j \) is bounded by \( \varepsilon \). This last bound and (11.9) imply (11.5). \( \square \)
Lemma 11.2. For $\mathcal{P}$–almost all $V$ and for any function $f \in C_c(\mathbb{R})$ it holds
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} f(S_{n}(k/n)) = \int_{[0,V(1)]} f(x) dV^{-1}(x). \tag{11.10}
\]

Proof. Since $f$ is uniformly continuous, by Lemma 11.1 it is enough to prove (11.10) with $S_{n}(k/n)$ replaced by $V(k/n)$. Approximating $f$ by stepwise functions with jumps on rational points, it is enough to prove that, fixed $t \in \mathbb{Q}$, for $\mathcal{P}$–a.a. $V$ the limit (11.10) holds with $S_{n}(k/n)$ replaced by $V(k/n)$ and with $f(x) = \mathbb{1}(x \leq t)$. This last check is immediate. \qed

We have now all the tools in order to prove Point (i) of Theorem 2.5. Indeed, by Lemma 11.1 $\ell_{n} = S_{n}(1) \to V(1) \mathcal{P}$–a.s. Moreover, by Lemma 11.2 the measure $dm_{n}$ defined in (2.10) weakly converges to the measure $d(V^{-1})$. In order to get Point (i) of Theorem 2.5 it is enough to apply Theorem 2.1.

11.2. Proof of Point (ii). If $\mathbb{E}(e^{-\lambda \tau(x)}) = e^{-\lambda^2}$ one can replace $L_{2}(n)$ with 1 in (11.1) and in the above definition of $U_{n}(x)$, and one can define $\tau_{n}(x)$ directly by means of (11.11). In this case, definition (2.8) gives $S_{n}(k/n) = V((k+1)/n)$ and therefore $dm_{n} = \frac{1}{n} \sum_{k=1}^{n+1} \delta_{V(k/n)}$. It is simple to prove that a.s. $dm_{n}$ weakly converges to $dm := d(V^{-1})$. Hence, one gets that the assumptions of Theorem 2.1 are fulfilled with $\ell_{n} = V((n+1)/n)$, $\ell = V(1)$ and $dm = (V^{-1})$, for almost all realization of $V$. As a consequence, one derives Point (ii) in Theorem 2.5.

11.3. Proof of Point (iii). The proof of point (iii) of Theorem 2.5 follows from Proposition 2.2 once we prove (2.17) with $m = V$. As in the proof of Lemma 11.2 we denote by $0 < y_{1} < y_{2} < \cdots < y_{N} < 1$ the points in $[0,1]$ where $V$ has a jump larger than 1/2 (note that $V$ is continuous in 0 and 1 a.s.). We set $a_{i} := V(y_{i})$, $b_{i} = V(y_{i})$ and remark that the function $V^{-1}$ is constant on $[a_{i}, b_{i}]$. Then we fix $\varepsilon > 0$ (which is a random number) such that the following properties holds:

(i) the intervals $U_{i} := [a_{i} - \varepsilon, b_{i} + \varepsilon]$, $i = 1, \ldots, N$, are disjoint and included in $[0, V(1)]$,

(ii) $V$ has no jump at $a_{i} - \varepsilon$ and $b_{i} + \varepsilon$, for all $i = 1, \ldots, N$,

(iii) for all $i = 1, \ldots, N$,
\[
(b_{i} - a_{i} + 2\varepsilon)(V^{-1}(b_{i} + \varepsilon) - V^{-1}(a_{i} - \varepsilon)) \leq 1/2. \tag{11.11}
\]

Note that, since $V^{-1}$ is continuous a.s. and flat on $U_{i}$, condition (iii) is satisfied for $\varepsilon$ small enough. Moreover, due to condition (ii) it holds $V^{-1}(x) < V^{-1}(y) < V^{-1}(z)$ if $y \in \{a_{i} - \varepsilon, b_{i} + \varepsilon\}$ and $x < y < z$.

Let now $f$ be an eigenfunction of the operator $-D_{V^{-1}}D_{x}$ on $U_{i}$ with Dirichlet boundary conditions. Writing $\lambda$ for the associated eigenvalue, by equation (4.9) in Lemma 11.1 it holds
\[
f(x) = \lambda \int_{U_{i}} G_{a_{i} - \varepsilon, b_{i} + \varepsilon}(x, y) f(y) dV^{-1}(y).
\]

Using that $\|G_{a_{i} - \varepsilon, b_{i} + \varepsilon}\|_{\infty} \leq b_{i} - a_{i} + 2\varepsilon$ we get
\[
|f(x)| \leq \lambda(b_{i} - a_{i} + 2\varepsilon) \|f\|_{\infty}(V^{-1}(b_{i} + \varepsilon) - V^{-1}(a_{i} - \varepsilon)). \tag{11.12}
\]

Combining (11.11) and (11.12) we conclude that $\lambda \geq 2$. Hence $N_{V^{-1}, D}^{U_{i}}(1) = 0$. We now observe that the set $W = [0, V(1)] \setminus \cup_{i=1}^{N} U_{i}$ is the union of $N + 1$ intervals and its total length is smaller than $V^{1}(1)$ (see the proof of Lemma 11.2 for the definition of $V^{1}$).
It follows that we can partition $W$ in at most $2V^{(1)}(1) + N$ subintervals $A_r$ of length bounded by $1/2$. Since the $dV^{-1}$-mass of any subinterval $A_r$ is bounded by the total $dV^{-1}$-mass of $[0, V(1)]$ (which is a.s. 1), by the estimate (1.11) in Lemma 2.1 we get that all eigenvalues of the operator $-D_{V^{-1}}D_x$ restricted to any subinterval $A_r$ (with Dirichlet b.c.) is at least 2, hence $\mathcal{N}_{V^{-1}, D}(1) = 0$. We now apply Corollary 8.9 observing that we are in the same setting on Theorem 8.8 (recall that $V^{-1}$ is continuous a.s. and recall our condition (ii), thus leading to (i)–(iii) in Theorem 8.8). By Corollary 8.9, we conclude that $\mathcal{N}_{V^{-1}, D}(1) \leq V^{(1)}(1) + 4N$ a.s. As already observed in the proof of Lemma 10.2 both $V^{(1)}(1)$ and $N$ have finite expectation, thus leading to (2.17).

12. The diffusive case: Proof of Propositions 2.4 and 2.6

12.1. Proof of Proposition 2.4

We consider the diffusively rescaled random walk $X^{(n)}$ on $\mathbb{Z}_n$ with jump rates

$$c_n(x, y) = \begin{cases} \mathbb{E}(\tau(0)^{-a})^2 \mathbb{E}(\tau(0)) n^2 \tau(nx) \frac{\tau(ny)}{\tau(y)} & \text{if } |x - y| = 1/n \\ 0 & \text{otherwise} \end{cases}.$$

The above jump rates can be written as $c_n(x, y) = 1/H_n(x)U_n(x \lor y)$ for $|x - y| = 1/n$ by taking

$$\begin{cases} U_n(x) = \mathbb{E}(\tau(0)^{-a})^{-2} \mathbb{E}(\tau(nx - 1)^{-a} \tau(nx)^{-a} \\ H_n(x) = \mathbb{E}(\tau(0))^{-1} \mathbb{E}(\tau(nx)) \end{cases}.$$

Due to our definition (2.8) we have

$$S_n(k/n) = \frac{1}{n \mathbb{E}(\tau(0)^{-a})^2} \sum_{j=1}^k \tau(j - 1)^{-a} \tau(j)^{-a}, \quad 0 \leq k \leq n.$$

By the ergodic theorem and the assumption $\mathbb{E}(\tau(0)^{-a}) < \infty$, it holds $\lim_{n \to \infty} S_n(|xn|/n) = x$ for all $x \geq 0$ (a.s.). In particular, it holds $\ell_n = S_n(1) \to 1$. Since $\pi^2 k^2$ is the $k$-th eigenvalue of $-\Delta$ with Dirichlet conditions outside $(0, 1)$, by Theorem 2.4 it remains to prove that, a.s., for all $f \in C_C([0, \infty))$ it holds

$$\lim_{n \to \infty} dm_n(f) = \lim_{n \to \infty} \sum_{k=0}^n f(S_n(k/n)) H_n(k/n) = \int_0^1 f(s) ds. \quad (12.1)$$

By the ergodic theorem and the assumption $\mathbb{E}(\tau(0)) < \infty$, the total mass of $dm_n$, i.e. $\sum_{k=0}^n H_n(k/n)$, converges to 1 a.s. Hence, by a standard approximation argument with stepwise functions, it is enough to prove (12.1) for functions $f$ of the form $f = 1([0, t])$. By the ergodic theorem a.s. it holds: for any $\varepsilon > 0$ there exists a random integer $n_0$ such that $S_n(k/n) < t$ for all $k \leq (t - \varepsilon)n$ and $S_n(k/n) > t$ for all $k \geq (t + \varepsilon)/n$. Therefore, for $f$ as above and $n \geq n_0$, we can bound

$$\frac{1}{n \mathbb{E}(\tau(0))} \sum_{k \in \mathbb{N}, k \leq (t-\varepsilon)n} \tau(k) \leq dm_n(f) \leq \frac{1}{n \mathbb{E}(\tau(0))} \sum_{k \in \mathbb{N}, k \leq (t+\varepsilon)n} \tau(k).$$

Applying again the ergodic theorem, it is immediate to conclude.
12.2. **Proof of Proposition 2.6.** We sketch the proof since the technical steps are very easy and similar to the ones discussed above. We consider the diffusively rescaled random walk $X^{(n)}$ on $\mathbb{Z}_n$ with jump rates

$$c_n(x, y) = \begin{cases} n^2 \mathbb{E}(\tau(0)) \tau(nx \vee ny)^{-1} & \text{if } |x - y| = 1/n, \\
0 & \text{otherwise.} \end{cases}$$

The rates $c_n(x, y)$ for $|x - y| = 1/n$ can be written as $c_n(x, y) = 1/[H_n(x, y)U_n(x \vee y)]$, where $H_n(x) = 1/n$ and $U_n(x) = \tau(nx)/n\mathbb{E}(\tau(0))$. By the ergodic theorem and the assumption $\mathbb{E}(\tau(0)) < \infty$, a.s. it holds $\lim_{n \uparrow \infty} S_n(\lfloor nx \rfloor) = x$ for all $x \geq 0$. In particular, a.s. $S_n(n) \to 1$ and

$$\lim_{n \uparrow \infty} dm_n(f) = \lim_{n \uparrow \infty} \frac{1}{n} \sum_{k=0}^{n} f(S_n(k/n)) = \int_{0}^{1} f(x) dx,$$

for all $f \in C_c([0, \infty))$. At this point it is enough to apply Theorem 2.1.

**Appendix A. Proof of Lemma 4.1**

For simplicity of notation we write $\ell = \ell_m$. As already observed, the Dirichlet eigenvalues are all simple and the $\lambda$-eigenspace is spanned by $\psi(\cdot, \lambda)$. The fact that $\psi(\cdot, \lambda)$ is a real function for any real $\lambda$ is a simple consequence of the expression of $\psi(\cdot, \lambda)$ as series given at page 30 in [KK0] and recalled in the proof of Lemma 5.2.

As discussed in [KK0] Section 2, the function $\mathbb{C} \ni \lambda \to \psi(\ell, \lambda) \in \mathbb{C}$ is an entire function, having only positive zeros, which are all simple. It is well known that the set of zeros of any entire functions on $\mathbb{C}$ is given by $\mathbb{C}$ or is a countable (finite or infinite) set without accumulation points. We can exclude the first alternative since we know that the zeros of $\psi(\ell, \cdot)$ must lie on the halfline $(0, \infty)$. In particular, if there are infinite eigenvalues they must diverge to $+\infty$.

It remains to prove the last statement concerning (4.9) and the estimate (4.11). By definition, $F$ is a Dirichlet eigenfunction of $-D_{m}D_x$ with eigenvalue $\lambda$ if and only if for some $b \in \mathbb{C}$ $F$ solves the integral equation

$$F(x) = bx - \lambda \int_{0}^{x} dy \int_{[0,y]} F(z) dm(z), \quad \forall x \in [0, \ell]. \quad (A.1)$$

We can rewrite (A.1) as:

$$F(x) = bx - \lambda \int_{[0,x]} dm(y)(x - y) F(y), \quad \forall x \in [0, \ell]. \quad (A.2)$$

Then the condition $F(\ell) = 0$ is equivalent to

$$b\ell = \lambda \int_{[0,\ell]} dm(y)(\ell - y) F(y). \quad (A.3)$$

Equations (A.2) and (A.3) imply that

$$F(x) = \lambda \int_{[0,\ell]} dm(y) \frac{x(\ell - y)}{\ell} F(y) - \lambda \int_{[0,x]} dm(y)(x - y) F(y), \quad \forall x \in [0, \ell]. \quad (A.4)$$

It is simple to check that the above identity (A.4) is equivalent to (4.9). On the other hand we know that (A.4) is equivalent to equation (A.1) together with (A.3), and the latter is equivalent to $F(\ell) = 0$. 
To conclude, we observe that (4.9) implies \( \| F \|_\infty \leq \lambda \| F \|_\infty \ell_m dm([0, \ell_m]) \), since trivially \( 0 \leq G_{0,\ell_m}(x,y) \leq \ell_m \). (4.11) then follows.

**Acknowledgements.** I thank Jean–Christophe Mourrat and Eugenio Montefusco for useful discussions, and acknowledge the financial support of the European Research Council through the “Advanced Grant” PTRELSS 228032.

At 3:32 a.m. on Monday April 6th, 2009 an earthquake has destroyed several buildings and killed hundreds of people in L’Aquila, the city where I lived with my family. The ending of a preliminary version of this work has been possible thanks to the help of several people. I kindly acknowledge the public library of Codroipo for the computer facilities, the “Protezione Civile” of Codroipo for the burocratic help, the colleagues who have sent me files and who have substituted me in teaching, the Department of Mathematics and the office “Affari Sociali” of the University “La Sapienza”.

**References**

[ABSO] S. Alexander, J. Bernasconi, W.R. Schneider, R. Orbach, *Excitation dynamics in random one-dimensional systems*. Rev. Mod. Phys. 53, 175–198 (1981).

[B] J. Bertoin, *Lévy Processes*. Cambridge Tracts in Mathematics 121. Cambridge University Press, Cambridge (1996).

[BC1] J. Ben Arous, J. Cerný, *Bouchaud’s model exhibits two different aging regimes in dimension one*. Ann. Appl. Prob. 15, 1161–1192 (2005).

[BC2] J. Ben Arous, J. Cerný, *Dynamics of Trap Models*. In ”Mathematical Statistical Mechanics”, Proceedings of the 83rd Les Houches Summer School, July 2005. A. Bovier, F. Dunlop, A.C.D. van Enter, F. den Hollander, and J. Dalibard (Eds.), Elsevier (2006).

[BD] D. Boivin, J. Depauw, *Spectral homogenization of reversible random walks on \( \mathbb{Z}^d \) in a random environment*. Stochastic Process. Appl. 104, no. 1, 29–56 (2003).

[BCKM] J.P. Bouchaud, L. Cugliandolo, J. Kurchan, M. Mézard, *Out–of–equilibrium dynamics in spin–glasses and other glassy systems*. In Spin–Glasses and Random Fields (A.P. Young, ed.), Singapore, Word Scientific (1998).

[BDe] J.-P. Bouchaud, D.S. Dean, *Aging on Parisi’s tree*. J. Phys. I France 5, 265286 (1995).

[BF1] A. Bovier, A. Faggionato, *Spectral characterization of aging: the REM–like trap model*. Ann. Appl. Prob. 15, 1997–2037 (2005).

[BF2] A. Bovier, A. Faggionato, *Spectral analysis of Sinai’s walk for small eigenvalues*. Ann. Probab. 36, 198–254 (2008).

[CH1] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Vol. I, Berlin, Springer Verlag (1980).

[DM] H. Dym, H.P. McKean, *Gaussian processes, function theory and the inverse spectral problem*, New York, Academic Press (1976).

[Di] J. Dieudonne, *Foundations of modern analysis*, Academic Press, New York (1969).

[D] E.B. Dynkin, *Markov processes*, Volume II, Grundlehren der mathematischen Wissenschaften 122, Berlin, Springer Verlag (1965).

[DS] P.G. Doyle, J.L. Snell, *Random walks and electric networks*, The Carus mathematical monographs 22, Mathematical Association of America, Washington, 1984.

[FJL] A. Faggionato, M. Jara, C. Landim, *Hydrodynamic behavior of one dimensional subdiffusive exclusion processes with random conductances*. Prob. Theory Relat. Fields 144, 633–667 (2009).

[FIN] L.R. Fontes, M. Isopi, C. Newmann, *Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension*. Ann. Probab. 30 (2), 579–604 (2002).

[Fr] U. Freiberg, *Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets*. Forum Math. 17, 87–104 (2005).

[H] B.M. Hambly, *On the asymptotics of the eigenvalue counting function for random recursive Sierpinski gaskets*. Probab. Theory. Relat. Fields 117, 221–247 (2000).

[IM] K. Ito, H. P. McKean, *Diffusion processes and their sample paths*, Grundlehren der mathematischen Wissenschaften 125, Berlin, Springer Verlag (1996).
[KK0] I.S. Kac, M.G. Krein, *On the spectral functions of the string.* Amer. Math. Soc. Transl. (2), Vol. 103, 19–102 (1974).

[K] Y. Kasahara, *Spectral theory of generalized second order differential operators and its applications to Markov processes.* Japan J. Math. 1, 67–84 (1975).

[KK] K. Kawazu, H. Kesten, *On birth and death processes in symmetric random environment.* Journal of Stat. Physics 37, Nos 5/6, 561–576, (1984).

[KL] J. Kigami, M.L. Lapidus, *Weyl’s problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals.* Comm. Math. Phys. 158, 93-125 (1993).

[KZ] Q. Kong, A. Zettl, *Eigenvalues of regular Sturm–Liouville problems.* J. Differential Equations 131, no. 1, 1–19 (1996).

[K] U. Kühler, *Some asymptotic properties of the transition densities of one–dimensional diffusions.* Publ. RIMS, Kyoto Univ. 16, 245–268 (1980).

[L] M.L. Lapidus, *Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture.* Trans. Am. Math. Soc. 325, 465–529 (1991).

[L1] J.-U. Löbus, *Generalized second order differential operators.* Math. Nachr. 152, 229-245 (1991).

[L2] J.-U. Löbus, *Construction and generators of one-dimensional quasi-diffusions with applications to selfaffine diffusions and brownian motion on the Cantor set.* Stoch. and Stoch. Rep. 42, 93–114, (1993).

[Ma] P. Mandl, *Analytical treatment of one-dimensional Markov processes.* Grundlehren der mathematischen Wissenschaften 151. Berlin, Springer Verlag (1968).

[Me] G. Métévier, *Valeurs propres de problèmes aux limites elliptiques irrégulier.* Bull. Soc. Math. France, Mém. 51–52, 125–219 (1977).

[Mo] J.-C. Mourrat, *Principal eigenvalue for random walk among random traps on Zd.* Preprint arXiv:0805.0706v2 (2009).

[RS1] M. Reed, B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis.* New York, Academic Press (1972).

[RS4] M. Reed, B. Simon. *Methods of Modern Mathematical Physics IV: Analysis of Operators.* New York, Academic Press (1978).

[R] S.I. Resnick, *Extreme values, regular variation, and point processes.* Berlin, Springer Verlag (1987).

[RY] D. Revuz, M. Yor. *Continuous martingales and Brownian motion.* Grundlehren der mathematischen Wissenschaften 293. Third edition. Berlin, Springer Verlag (1999).

[S] C. Stone. *Limit theorems for random walks, birth and death processes, and diffusion processes.* Ill. J. Math. 7, 638–660 (1963).

[UH] T. Uno, I. Hong, *Some consideration of asymptotic distribution of eigenvalues for the equation d^2u/dx^2 + \lambda_0 u = 0.* Japanese Journal of Math., 29, 152–164 (1959).

[W1] H. Weyl. *Über die asymptotische Verteilung der Eigenwerte.* Göttinger Nach., 110–117 (1911).

[W2] H. Weyl. *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen.* Math. Ann. 71, 441–479 (1912).

[Z] A. Zettl. *Sturm–Liouville theory.* Mathematical Surveys and Monographs, 121, AMS, Providence (2005).

**Alessandra Faggionato. Dipartimento di Matematica “G. Castelnuovo”, Università “La Sapienza”. P.le Aldo Moro 2, 00185 Roma, Italy. e-mail: faggionatamat.uniroma1.it**