Schubert calculus and the Hopf algebra structures of exceptional Lie groups

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Abstract. Let $G$ be an exceptional Lie group with a maximal torus $T$. Based on the Schubert presentation of the cohomology ring $H^*(G/T; \mathbb{F}_p)$ of the flag manifold $G/T$ obtained by the authors in a previous paper, we present a unified approach to the structure of $H^*(G; \mathbb{F}_p)$ as a Hopf algebra over the Steenrod algebra $\mathbb{A}_p$.

Keywords. Lie groups, Hopf algebra, Steenrod algebra, Schubert calculus.

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1 Introduction

Let $G$ be a compact, connected Lie group with a maximal torus $T \subset G$. According to A. Weil [38, p. 331] the classical Schubert calculus amounts to the determination of the integral cohomology ring $H^*(G/T)$ of the complete flag manifold $G/T$ (with respect to the canonical Schubert classes on $G/T$). The purpose of this paper is to extend this calculus to construct the cohomology $H^*(G; \mathbb{F}_p)$ for $G$ over $\mathbb{F}_p$, and to determine its structure as a Hopf algebra over the Steenrod algebra.

More precisely, let $G$ be an exceptional Lie group and let

$$\{\omega_1, \ldots, \omega_n\} \in H^2(G/T)$$

be a set of fundamental dominant weights ([7]), $n = \dim T$. Based on the Schubert presentation of the integral cohomology ring $H^*(G/T)$ obtained in [13], we have derived for each prime $p = 2, 3, 5$ an explicitly set $\{\theta_{s_1}, \ldots, \theta_{s_n}\}$, $\deg \theta_s = 2s$, of basic generalized Weyl invariants for $G$ over $\mathbb{F}_p$ (see Section 2 and Section 5.2). We demonstrate how this set $\{\theta_{s_1}, \ldots, \theta_{s_n}\}$ of polynomials in $\omega_1, \ldots, \omega_n$ gives rise naturally to a set $\{a_{2s_1-1}, \ldots, a_{2s_n-1}\}$ of $p$-transgressive generators on the ring $H^*(G; \mathbb{F}_p)$ (see Definition 3.3), and the structure of $H^*(G; \mathbb{F}_p)$ as a Hopf

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algebra over the Steenrod algebra can be effectively calculated by computing with these polynomials. We restrict ourselves to the cases where the integral cohomology $H^*(G)$ contains non-trivial $p$-torsion subgroup, for exactly in these cases $H^*(G;\mathbb{F}_p)$ fails to be a primitive generated exterior algebra.

Let $\mathcal{A}_p$ be the mod $p$ Steenrod algebra with $\mathcal{P}^k \in \mathcal{A}_p$, $k \geq 1$, the $k$th reduced power ([35]) and $\delta_p \in \mathcal{A}_p$ the Bockstein operator. If $p = 2$, it is also customary to write $Sq^{2k}$ instead of $\mathcal{P}^k$, $Sq^1$ in place of $\delta_2$. Theorem 1.1 below, together with Corollaries 4.2 and 4.4, gives a complete characterization of $H^*(G;\mathbb{F}_p)$ as an algebra over the Steenrod algebra $\mathcal{A}_p$, with respect to the $p$-transgressive generators $\alpha_{2s-1}$ and the generators $x_{2r}$ explicitly constructed from the presentation of $H^*(G/T;\mathbb{F}_p)$ in Lemma 2.1.

**Theorem 1.1.** Let $(G, p)$ be a pair with $G$ an exceptional Lie group and $H^*(G)$ containing non-trivial $p$-torsion subgroup.

(1.1) The algebra $H^*(G;\mathbb{F}_2)$ has the presentation

$$H^*(G_2;\mathbb{F}_2) = \mathbb{F}_2[x_6]/\langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\alpha_3, \alpha_5),$$

$$H^*(F_4;\mathbb{F}_2) = \mathbb{F}_2[x_6]/\langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\alpha_3, \alpha_5, \alpha_{15}, \alpha_{23}),$$

$$H^*(E_6;\mathbb{F}_2) = \mathbb{F}_2[x_6]/\langle x_6^2 \rangle \otimes \Delta_{\mathbb{F}_2}(\alpha_3, \alpha_5, \alpha_9, \alpha_{15}, \alpha_{17}, \alpha_{23}),$$

$$H^*(E_7;\mathbb{F}_2) = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}]}{\langle x_6^2, x_{10}^2, x_{18}^2 \rangle} \otimes \Delta_{\mathbb{F}_2}(\alpha_3, \alpha_5, \alpha_9, \alpha_{15}, \alpha_{17}, \alpha_{23}, \alpha_{27}),$$

$$H^*(E_8;\mathbb{F}_2) = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, x_{30}]}{\langle x_6^2, x_{10}^2, x_{18}^2, x_{30}^2 \rangle} \otimes \Delta_{\mathbb{F}_2}(\alpha_3, \alpha_5, \alpha_9, \alpha_{15}, \alpha_{17}, \alpha_{23}, \alpha_{27}, \alpha_{29}),$$

on which all nontrivial $\mathcal{A}_2$-actions are given by

$$\delta_2(\alpha_5) = x_6 \text{ in } G_2, F_4, E_6, E_7, E_8,$$

$$\delta_2(\alpha_{2r-1}) = x_{2r}, \quad r = 5, 9,$$

$$\delta_2(\alpha_{15}) = x_6 x_{10}, \quad \delta_2(\alpha_{27}) = x_{10} x_{18} \text{ in } E_7, E_8,$$

$$\delta_2(\alpha_{23}) = x_6 x_{18} \text{ in } E_7,$$

$$\delta_2(\alpha_{23}) = x_6 x_{18} + x_6^4, \quad \delta_2(\alpha_{29}) = x_{30} + x_6^2 x_{18} \text{ in } E_8,$$

$$\mathcal{P}^1 \alpha_3 = \alpha_5 \text{ in } G_2, F_4, E_6, E_7, E_8,$$

$$\mathcal{P}^4 \alpha_{15} = \alpha_{23} \text{ in } F_4, E_6, E_7, E_8,$$

$$\mathcal{P}^2 \alpha_5 = \alpha_9, \quad \mathcal{P}^4 \alpha_9 = \mathcal{P}^1 \alpha_{15} = \alpha_{17} \text{ in } E_6, E_7, E_8,$$

$$\mathcal{P}^2 \alpha_{23} = \alpha_{27} \text{ in } E_7, E_8, \quad \mathcal{P}^3 \alpha_{23} = \mathcal{P}^1 \alpha_{27} = \alpha_{29} \text{ in } E_8.$$
(1.2) The algebra $H^*(G; \mathbb{F}_3)$ has the presentation
\[ H^*(F_4; \mathbb{F}_3) = \mathbb{F}_3[x_8]/ \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\alpha_3, \alpha_7, \alpha_{11}, \alpha_{15}), \]
\[ H^*(E_6; \mathbb{F}_3) = \mathbb{F}_3[x_8]/ \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\alpha_3, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{15}, \alpha_{17}), \]
\[ H^*(E_7; \mathbb{F}_3) = \mathbb{F}_3[x_8]/ \langle x_8^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\alpha_3, \alpha_7, \alpha_{11}, \alpha_{15}, \alpha_{19}, \alpha_{27}, \alpha_{35}), \]
\[ H^*(E_8; \mathbb{F}_3) = \mathbb{F}_3[x_8, x_{20}]/ \langle x_8^3, x_{20}^3 \rangle \]
\[ \otimes \Lambda_{\mathbb{F}_3}(\alpha_3, \alpha_7, \alpha_{15}, \alpha_{19}, \alpha_{27}, \alpha_{35}, \alpha_{39}, \alpha_{47}) \]
on which all nontrivial $A_3$-actions are given by
\[ \delta_3(\alpha_7) = -x_8, \quad \delta_3(\alpha_{15}) = -x_8^2 \text{ in } F_4, E_6, E_7, E_8, \]
\[ \delta_3(\alpha_{19}) = x_{20}, \quad \delta_3(\alpha_{27}) = -x_8 x_{20}, \quad \delta_3(\alpha_{35}) = x_8^2 x_{20}, \]
\[ \delta_3(\alpha_{39}) = x_{20}^2, \quad \delta_3(\alpha_{47}) = x_8 x_{20}^2 \text{ in } E_8, \]
\[ \mathcal{P}^1 \alpha_3 = \alpha_7 \text{ in } F_4, E_6, E_7, E_8, \]
\[ \mathcal{P}^1 \alpha_{11} = \alpha_{15} \text{ in } F_4, E_6, E_7, \]
\[ \mathcal{P}^1 \alpha_{15} = \mathcal{P}^3 \alpha_7 = \alpha_{19}, \quad \mathcal{P}^3 \alpha_{15} = \alpha_{27} \text{ in } E_7, E_8, \]
\[ \mathcal{P}^2 \alpha_{11} = -\alpha_{19} \text{ in } E_7, \]
\[ \mathcal{P}^1 \alpha_{35} = \mathcal{P}^3 \alpha_{27} = \alpha_{39}, \quad \mathcal{P}^3 \alpha_{35} = \alpha_{47} \text{ in } E_8. \]

(1.3) The algebra $H^*(E_8; \mathbb{F}_5)$ has the presentation
\[ \mathbb{F}_5[x_{12}]/ \langle x_{12}^5 \rangle \otimes \Lambda(\alpha_3, \alpha_{11}, \alpha_{15}, \alpha_{23}, \alpha_{27}, \alpha_{35}, \alpha_{39}, \alpha_{47}) \]
on which all nontrivial $A_5$-actions are given by
\[ \delta_5(\alpha_{11}) = -x_{12}, \quad \delta_5(\alpha_{23}) = -x_{12}^2, \quad \delta_5(\alpha_{35}) = x_{12}^3, \quad \delta_5(\alpha_{47}) = 2x_{12}^4, \]
\[ \mathcal{P}^1 \alpha_i = \alpha_{i+8}, \quad i = 3, 15, 27, 39. \]

For a prime $p$ the multiplication $\mu : G \times G \to G$ in $G$ induces the reduced co-product $\phi_p : H^*(G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p)$ by
\[ \phi_p(x) = \mu^*(x) - (x \otimes 1 + 1 \otimes x) \]
by virtue of the Kunneth formula. It furnishes $H^*(G; \mathbb{F}_p)$ with the structure of a Hopf algebra. The proof of the next result reveals that, with respect to our presentation of $H^*(G; \mathbb{F}_p)$ in Theorem 1.1, the structure of $H^*(G; \mathbb{F}_p)$ as a Hopf algebra is entirely determined by its structure as an algebra over $A_p$. 
Theorem 1.2. With respect to the presentation of $H^*(G; \mathbb{F}_p)$ in (1.1)–(1.3), the reduced co-product $\phi_p$ on $H^*(G; \mathbb{F}_p)$ is determined, respectively, by

(1.4) for an exceptional Lie group $G$ and $p = 2$:

$$\phi_2(\alpha_i) = \begin{cases} 
0 & \text{if } i = 3 \text{ for all } G \text{ and } i = 15 \text{ for } G = F_4, \\
x_6 \otimes \alpha_9 & \text{if } i = 15 \text{ and } G = E_6, \\
x_{10} \otimes \alpha_5 + x_6 \otimes \alpha_9 & \text{if } i = 15 \text{ and } G = E_7, \\
x_{10} \otimes \alpha_5 + x_6 \otimes \alpha_9 + x_6^2 \otimes \alpha_3 & \text{if } i = 15 \text{ and } G = E_8.
\end{cases}$$

(1.5) for an exceptional Lie group $G \neq G_2$ and $p = 3$:

$$\phi_3(\alpha_i) = \begin{cases} 
0 & \text{if } i = 3, 7, 9, 17, 19, \\
-x_8 \otimes \alpha_3 & \text{if } i = 11 \text{ and } G = F_4, E_6, E_7, \\
-x_8 \otimes \alpha_7 & \text{if } (G, i) = (E_8, 15), \\
x_8 \otimes \alpha_{27} + x_8^2 \otimes \alpha_{19} & \text{if } (G, i) = (E_7, 35), \\
x_8 \otimes \alpha_{27} + x_8^2 \otimes \alpha_{19} & \text{if } (G, i) = (E_7, 35), \\
+ x_8x_{20} \otimes \alpha_7 - x_{20} \otimes \alpha_{15} & \text{if } (G, i) = (E_8, 35).
\end{cases}$$

(1.6) for $(G, p) = (E_8, 5)$:

$$\phi_5(\alpha_i) = \begin{cases} 
0 & \text{if } i = 3, \\
2x_{12} \otimes \alpha_3 & \text{if } i = 15, \\
2x_{12} \otimes \alpha_{15} + 2x_{12}^2 \otimes \alpha_3 & \text{if } i = 27, \\
3x_{12} \otimes \alpha_{27} + 3x_{12}^2 \otimes \alpha_{15} + 2x_{12}^3 \otimes \alpha_3 & \text{if } i = 39.
\end{cases}$$

The multiplication $\mu : G \times G \to G$ induces also the ring map

$$\mu^* : H^*(G) \to H^*(G \times G)$$

(1.7)

on the integral cohomology $H^*(G)$ of $G$, called the near-Hopf ring structure on $H^*(G)$ in [10], where Theorem 1.2 is applied to determine the near-Hopf ring structure of all exceptional Lie groups.

This paper is arranged as follows. In Section 2 we recall from [13] common properties in the Schubert presentation of the ring $H^*(G/T; \mathbb{F}_p)$. In particular, the role of the set $\{\theta_{s_1}, \ldots, \theta_{s_n}\}$ of generalized Weyl invariants is emphasized. In Section 3 we construct $H^*(G; \mathbb{F}_p)$ from the presentation of $H^*(G/T; \mathbb{F}_p)$ in Section 2, and reduce the $R^k$ action on $H^*(G; \mathbb{F}_p)$ to certain relations among the polynomials $\theta_{s_1}, \ldots, \theta_{s_n}$. Section 5 is created to record expressions of these polynomials and to handle computational aspects in this paper. With these preliminaries, Theorems 1.1 and 1.2 are established in Section 4.
To determine the cohomology ring of a space it is often necessary to describe
the generators rigorously at the beginning, so that the subsequent calculation could
be performed without any uncertainty. In the classical descriptions of the rings
$H^*(E_7; \mathbb{F}_2)$ and $H^*(E_8; \mathbb{F}_2)$ in [3, 4, 20, 22, 27, 29, 36] the generators were spec-
ified mainly up to their degrees. As a result the action of $Sq^1 = \delta_2$ on the gen-
erators in degrees 15, 23, 27 was absent. In comparison, results in (1.1) provide a
complete characterization of $H^*(G; \mathbb{F}_2)$ as an algebra over $A_2$, see Remark 4.3.

In [25] Kono and Mimura largely determined the $A_3$ action on $H^*(E_7; \mathbb{F}_3)$ and
$H^*(E_8; \mathbb{F}_3)$ with respect also to a set of transgressive generators, except an inde-
determinacy $\epsilon = \pm 1$ occurred in their expressions of $P^2\alpha_{11}, P^1\alpha_{15}$ in $E_7$, and of
$P^1\alpha_{15}, P^1\alpha_{35}$ in $E_8$. Again, with respect to our explicit construction these ambi-
guities are clarified in (1.2).

Historically, the Hopf algebras structure on $H^*(G; \mathbb{F}_p)$ were studied by quite
different methods, presented by generators with various origins and using case by
case computations depending on $G$ and $p$. As examples, see

- Borel [5] for $(G, p) = (G_2; 2), (F_4; 2),$
- Araki [2] for $(F_4; 3),$
- Toda, Kono, Mimura, Shimada [23, 26, 37] for $(E_i; 2), i = 6, 7, 8,$
- Kono, Mimura and Toda [24, 37] for $(E_6; 3),$
- Kono-Mimura [25] for $(E_7; 3)$ and $(E_8; 3),$
- Kono [21] for $(E_8; 5).$

In [19] Kač initiated a unified approach to $H^*(G; \mathbb{F}_p)$ using techniques from Schu-
bert calculus. He succeeded in showing that the algebra structure for $p \neq 2$ and
additive structure for $p = 2$ are entirely determined by the degrees of basic Weyl-
invariants over $\mathbb{Q}$, and the degrees of basic generalized Weyl-invariant over $\mathbb{F}_p$.

With our explicit construction of $H^*(G; \mathbb{F}_p)$ out of $H^*(G/T; \mathbb{F}_p)$ we have ar-
rive at the structure of $H^*(G; \mathbb{F}_p)$ as a Hopf algebra over $A_p$.

2 Schubert presentation of $H^*(G/T; \mathbb{F}_p)$

For a compact Lie group $G$ with a maximal torus $T$ consider the fibration

$$G/T \xrightarrow{\psi} BT \xrightarrow{\pi} BG \quad (2.1)$$

induced by the inclusion $T \subset G$, where $BT$ (resp. $BG$) is the classifying space
of $T$ (resp. $G$). Since $H^{\text{odd}}(BT) = H^{\text{odd}}(G/T) = 0$, the cohomology exact se-
quence of the pair $(BT, G/T)$ with $\mathbb{F}_p$-coefficients contains the section

$$0 \longrightarrow H^{\text{even}}(BT, G/T; \mathbb{F}_p) \xrightarrow{j} H^{\text{even}}(BT; \mathbb{F}_p) \xrightarrow{\psi^p} H^{\text{even}}(G/T; \mathbb{F}_p) \quad (2.2)$$
where, as is classical, $H^*(BT; \mathbb{F}_p)$ can be identified with the free polynomial ring $\mathbb{F}_p[\omega_1, \ldots, \omega_n]$ in a set of fundamental dominant weights $\omega_1, \ldots, \omega_n \in H^2(BT; \mathbb{F}_p)$ of $G$, and where the ring map $\psi_p^*$ induced by the fiber inclusion $\psi$ is well known as the Borel’s characteristic map in characteristic $p$ ([6, 7]).

According to Borel [6] if the integral cohomology $H^*(G)$ is free of $p$-torsion, then $\psi_p^*$ is surjective and induces an isomorphism

$$H^*(G/T; \mathbb{F}_p) = H^*(BT; \mathbb{F}_p)/(H^+(BT; \mathbb{F}_p)^W(G))$$

where $\langle H^+(BT; \mathbb{F}_p)^W(G) \rangle$ is the ideal in $H^*(BT; \mathbb{F}_p)$ generated by Weyl invariants in positive degrees (see Demazure [8] for another proof of this result). Without any restriction on the torsion subgroup of $H^*(G)$ we extend this classical result in Lemma 2.1 below.

For simplicity we shall make no difference in notation between a polynomial $2H^*(BT; \mathbb{F}_p)$ and its $p$-image in $H^*(G/T; \mathbb{F}_p)$. Given a subset $\{f_1, \ldots, f_m\}$ in a ring, write $\langle f_1, \ldots, f_m \rangle$ for the ideal generated by $f_1, \ldots, f_m$.

**Lemma 2.1** ([13, Proposition 3]). For each 1-connected Lie group $G$ with rank $n$ and for a prime $p$, there exist a set $\{s_1, \ldots, s_n\} \subset H^*(BT; \mathbb{F}_p)$ of $n$ polynomials, and a set $\{y_{t_1}, \ldots, y_{t_k}\} \subset H^*(G/T; \mathbb{F}_p)$ of Schubert classes on $G/T$ with $\deg \theta_s = 2s$, $\deg y_t = 2t > 2$, such that

$$H^*(G/T; \mathbb{F}_p) = \mathbb{F}_p[\omega_1, \ldots, \omega_n, y_t]/\langle \theta_s, y_t^{k_t} + \beta_t \rangle_{s \in r(G, p), t \in e(G, p)}$$

where $r(G, p) = \{s_1, \ldots, s_n\}$, $e(G, p) = \{t_1, \ldots, t_k\}$, and where

(i) $\ker \psi_p^* = \langle \theta_1, \ldots, \theta_n \rangle$,

(ii) $\beta_t \in \langle \omega_1, \ldots, \omega_n \rangle$,

(iii) the three sets of integers $r(G, p)$, $e(G, p)$ and $\{k_t\}_{t \in e(G, p)}$ are subject to the constraints

$$e(G, p) \subset r(G, p), \quad \dim G = \sum_{s \in r(G, p)} (2s - 1) + \sum_{t \in e(G, p)} 2(k_t - 1)t.$$

Since the set $\{\omega_1, \ldots, \omega_n\}$ of weights consists of all Schubert classes on $G/T$ with cohomology degree 2, Lemma 2.1 describes the ring $H^*(G/T; \mathbb{F}_p)$ by certain Schubert classes on $G/T$ and, therefore, is called a Schubert presentation of the ring $H^*(G/T; \mathbb{F}_p)$ (see [13]). In addition to $\{\omega_1, \ldots, \omega_n\}$, generators in the set $\{y_t\}_{t \in e(G, p)}$ will be called the $p$-special Schubert classes on $G/T$. For each exceptional $G$ and prime $p$, a set of $p$-special Schubert classes on $G/T$ has been determined in [13], and is specified by their Weyl coordinates in Table 1.
Hopf algebra structures of exceptional Lie groups

In view of (i) of Lemma 2.1 we shall call \( \{\theta_s\}_{s \in r(G,p)} \) a set of generating polynomials for \( \ker \psi_p^* \). These polynomials have been emphasized by Kač [19] as a regular sequence of homogeneous generators for \( \ker \psi_p^* \), or a set of basic generalized Weyl invariants for the group \( G \) over the field \( \mathbb{F}_p \) (because of the relation \( \langle \theta_{s_1}, \ldots, \theta_{s_n} \rangle = (H^+(BT; \mathbb{F}_p)^{W(G)}) \) in \( H^*(BT; \mathbb{F}_p) \).

Assume in the remainder of this section that \( (G, p) \) is a pair with \( G \) exceptional and \( H^*(G) \) containing non-trivial \( p \)-torsion. Explicitly, we shall have

- \( p = 2: G = G_2, F_4, E_6, E_7, E_8 \),
- \( p = 3: G = F_4, E_6, E_7, E_8 \),
- \( p = 5: G = E_8 \).

In these cases a set of generating polynomials for \( \ker \psi_p^* \) is presented in Propositions 5.5–5.7, and the three sets \( r(G, p), e(G, p) \) and \( \{k_t\}_{t \in e(G, p)} \) of integers appearing in Lemma 2.1 are tabulated in Table 2 where \( e(G, p) \) is given as the subset of \( r(G, p) \) whose elements are underlined.

Combining (2.2) with the presentation of \( H^*(G/T; \mathbb{F}_p) \) in Lemma 2.1, we get the short exact sequence

\[
0 \rightarrow H^\text{even}(BT, G/T; \mathbb{F}_p) \xrightarrow{j} H^*(BT; \mathbb{F}_p) \xrightarrow{\psi_p^*} H^*(BT; \mathbb{F}_p) \xrightarrow{\langle \theta_s \rangle_{s \in r(G,p)}} 0. \tag{2.3}
\]

It implies that the map \( j \) identifies \( H^\text{even}(BT, G/T; \mathbb{F}_p) \) with

\[
\ker \psi_p^* = \langle \theta_s \rangle_{s \in r(G,p)}.
\]

In particular, we have \( \{\theta_s\}_{s \in r(G,p)} \subset H^*(BT, G/T; \mathbb{F}_p) \). It follows that, for any pair \( \{s, t\} \subset r(G, p) \) with \( t = s + k(p - 1) \), there exists a unique \( b_{s,t} \in \mathbb{F}_p \) so

| \( y_i \) | \( G_2/T \) | \( F_4/T \) | \( E_n/T, n = 6, 7, 8 \) | \( p \) |
|---|---|---|---|---|
| \( y_3 \) | \( \sigma_{[1,2,1]} \) | \( \sigma_{[5,4,2]} \), \( n = 6, 7, 8 \) | 2 |
| \( y_4 \) | \( \sigma_{[3,2,1]} \) | \( \sigma_{[6,5,4,2]} \), \( n = 6, 7, 8 \) | 3 |
| \( y_5 \) | \( \sigma_{[4,3,2,1]} \) | \( \sigma_{[7,6,5,4,2]} \), \( n = 7, 8 \) | 2 |
| \( y_6 \) | \( \sigma_{[1,3,6,5,4,2]} \), \( n = 8 \) | 5 |
| \( y_9 \) | \( \sigma_{[1,5,4,3,7,6,5,4,2]} \), \( n = 7, 8 \) | 2 |
| \( y_{10} \) | \( \sigma_{[1,6,5,4,3,7,6,5,4,2]} \), \( n = 8 \) | 3 |
| \( y_{15} \) | \( \sigma_{[5,4,2,3,1,6,5,4,3,8,7,6,5,4,2]} \), \( n = 8 \) | 2 |

Table 1. The \( p \)-special Schubert classes on \( G/T \) and their abbreviations.
Table 2. The three sets $r(G,p)$, $e(G,p)$ and $\{k_t\}_{t \in e(G,p)}$ of integers appearing in Lemma 2.1, where $e(G,p)$ is given as the subset of $r(G,p)$ whose elements are underlined.

| $(G, p)$ | $e(G, p) \subset r(G, p)$ | $\{k_t\}_{t \in e(G,p)}$ |
|----------|-----------------------------|-------------------------|
| $(G_2, 2)$ | $\{2, 3\}$ | $\{2\}$ |
| $(F_4, 2)$ | $\{2, 3, 8, 12\}$ | $\{2\}$ |
| $(E_6, 2)$ | $\{2, 3, 5, 8, 9, 12\}$ | $\{2\}$ |
| $(E_7, 2)$ | $\{2, 3, 5, 8, 9, 12, 14\}$ | $\{2, 2, 2\}$ |
| $(E_8, 2)$ | $\{2, 3, 5, 8, 9, 12, 14, 15\}$ | $\{8, 4, 2, 2\}$ |
| $(F_4, 3)$ | $\{2, 4, 6, 8\}$ | $\{3\}$ |
| $(E_6, 3)$ | $\{2, 4, 5, 6, 8, 9\}$ | $\{3\}$ |
| $(E_7, 3)$ | $\{2, 4, 6, 8, 10, 14, 18\}$ | $\{3\}$ |
| $(E_8, 3)$ | $\{2, 4, 8, 10, 14, 18, 20, 24\}$ | $\{3, 3\}$ |
| $(E_8, 5)$ | $\{2, 6, 8, 12, 14, 18, 20, 24\}$ | $\{5\}$ |

that a relation of the form

$$\mathcal{P}^k(\theta_s) = b_{s,t}\theta_t + \tau_t \quad \text{with} \quad \tau_t \in \langle \theta_s \rangle_{s \in r(G,p)}, s < t$$

(2.4)

holds in $H^*(BT, G/T; \mathbb{F}_p)$ (and in $H^*(BT; \mathbb{F}_p)$). Based on concrete expression of a set $\{\theta_s\}_{s \in r(G,p)}$ of generating polynomials for $\ker \psi^*_p$ in Section 5.2, Lemma 2.2 below is proved in Section 5.3 by direct computation in the simpler ring $H^*(BT; \mathbb{F}_p)$:

**Lemma 2.2.** With respect to the degree set $r(G, p)$ of a generating polynomials for $\ker \psi^*_p$ given in Table 2, all non-zero $b_{s,t} \in \mathbb{F}_p$ in (2.4) are

- $p = 2$: $b_{2, 3} = 1$ for $G_2, F_4, E_6, E_7, E_8$,
  $b_{8, 12} = 1$ for $F_4, E_6, E_7, E_8$,
  $b_{3, 5} = b_{5, 9} = b_{8, 9} = 1$ for $E_6, E_7, E_8$,
  $b_{12, 14} = 1$ for $E_7, E_8$,
  $b_{12, 15} = b_{14, 15} = 1$ for $E_8$.

- $p = 3$: $b_{2, 4} = 1$ for $F_4, E_6, E_7, E_8$,
  $b_{6, 8} = 1$ for $F_4, E_6, E_7$,
  $b_{4, 10} = b_{8, 14} = b_{8, 10} = 1$ for $E_7, E_8$,
  $b_{6, 10} = -1$ for $E_7$,
  $b_{18, 20} = b_{14, 20} = b_{18, 24} = 1$ for $E_8$.

- $p = 5$: $b_{k, k+4} = 1$ for $G = E_8$ and $k = 2, 8, 14, 20$. 


Remark 2.3. In [30, 31] M. Nakagawa obtained also presentations of the integral cohomology rings $H^*(G/T)$ for $G = E_7, E_8$ using the method due to Borel and Toda. From these one may deduce the presentations of the ring $H^*(G/T; \mathbb{F}_p)$ analogous to that in Lemma 2.1. However, in Nakagawa’s presentations the generators of $H^*(G/T)$ in the degrees exceeding 2 are given by their degrees, while ours are certain Schubert classes on $G/T$ whose geometric constructions are transparent in view of their Weyl coordinates ([12]).

3  $H^*(G; \mathbb{F}_p)$ as a module over $A_p$

In this section we construct the $\mathbb{F}_p$-module $H^*(G; \mathbb{F}_p)$ from the presentation of $H^*(G/T; \mathbb{F}_p)$ in Lemma 2.1, and relate the $P^k$ action on $H^*(G; \mathbb{F}_p)$ to the values of $b_{s,t} \in \mathbb{F}_p$ in (2.4).

The pull back of the universal $T$-bundle $E_T \to BT$ via the fiber inclusion $\psi$ in (2.1) gives rise to the principle $T$-bundle on $G/T$

$$T \to G \to G/T.$$  (3.1)

Since $G/T$ is 1-connected, the Borel transgression

$$\tau : H^1(T; \mathbb{F}_p) \to H^2(G/T; \mathbb{F}_p)$$

defines a basis $\{t_i\}_{1 \leq i \leq n}$ of $H^1(T; \mathbb{F}_p)$ by $\tau(t_i) = \omega_i$. Consequently,

$$H^*(T; \mathbb{F}_p) = \Lambda^*_{\mathbb{F}_p}(t_1, \ldots, t_n).$$

In the Leray–Serre spectral sequence $\{E_r^{s,t}(G; \mathbb{F}_p), d_r\}$ of (3.1) one has ([19])

$$E_2^{s,t}(G; \mathbb{F}_p) = H^s(G/T; \mathbb{F}_p) \otimes \Lambda^t_{\mathbb{F}_p}(t_1, \ldots, t_n),$$  (3.2)

the differential $d_2 : E_2^{s,t}(G; \mathbb{F}_p) \to E_2^{s+2,t-1}(G; \mathbb{F}_p)$ is given by

$$d_2(x \otimes t_k) = x \omega_k \otimes 1, \quad x \in H^s(G/T; \mathbb{F}_p), \; 1 \leq k \leq n.$$ (3.3)

Over $\mathbb{F}_p$ the subring $H^+(BT; \mathbb{F}_p)$ has the canonical additive basis

$$\{\omega_{b_1}^{b_1} \cdots \omega_{b_n}^{b_n} \mid b_i \geq 0, \; \sum b_i \geq 1\}.$$  

Consider the $\mathbb{F}_p$-linear map

$$\mathcal{D} : H^+(BT; \mathbb{F}_p) \to E_2^{s,1}(G; \mathbb{F}_p) = H^*(G/T; \mathbb{F}_p) \otimes \Lambda^1_{\mathbb{F}_p}$$  (3.4)

by

$$\mathcal{D}(\omega_{b_1}^{b_1} \cdots \omega_{b_n}^{b_n}) = \omega_{b_1}^{b_1} \cdots \omega_{b_s}^{b_s-1} \cdots \omega_{b_n}^{b_n} \otimes t_s,$$

where $s \in \{1, \ldots, n\}$ is the smallest one such that $b_s \geq 1$. Immediate but useful properties of the map $\mathcal{D}$ are as follows.
Lemma 3.1. Let $\beta_1, \beta_2 \in H^+(BT; \mathbb{F}_p)$ and write $[\theta] \in E_3^{s,t}(G; \mathbb{F}_p)$ for the cohomology class of a $d_2$-cocycle $\theta \in E^{s,t}_2(G; \mathbb{F}_p)$. Then

(i) $D(\ker \psi_p^*) \subset \ker d_2$,
(ii) $D(\beta_1 \beta_2) - \beta_1 D(\beta_2) \in \text{Im} \ d_2$.

In particular, $[D(\beta_1 \beta_2)] = 0$ if either $\beta_1$ or $\beta_2 \in \ker \psi_p^*$.

Proof. Statement (i) is shown by $d_2(D(\theta)) = 0$ in $H^*(G/T; \mathbb{F}_p)$ for all $\theta \in \ker \psi_p^*$. For (ii) it suffices to consider the cases where $\beta_1, \beta_2$ are monomials in $w_1, \ldots, w_n$, and the result comes directly from the definition of $D$. \hfill \Box

According to (i) of Lemma 3.1, the map $D$ assigns each generating polynomial $\theta_s$ an element

$$\alpha_{2s-1} := [D(\theta_s)] \in E_3^{2s-2,1}(G; \mathbb{F}_p).$$

Since $E_2^{s,t}(G; \mathbb{F}) = 0$ for $s$ odd, one has the canonical monomorphism

$$E_3^{2k,1}(G; \mathbb{F}_p) = E_\infty^{2k,1}(G; \mathbb{F}_p) = \mathcal{F}^{2k} H^{2k+1}(G; \mathbb{F}_p) \subset H^{2k+1}(G; \mathbb{F}_p)$$

which interprets directly the element $\alpha_{2s-1}$ as a cohomology class of $G$, where $\mathcal{F}$ is the filtration on $H^*(G; \mathbb{F}_p)$ induced from the bundle projection $\chi$. Furthermore, by Lemma 3.1 if we write $\mathcal{T}$ for the subspace of $H^*(G; \mathbb{F}_p)$ spanned by the set $\{\alpha_{2s-1}\}_{s \in r(G,p)}$, the map $D$ in (3.4) restricts to a surjection

$$[D] : \ker \psi_p^* \subset H^+(BT, G/T; \mathbb{F}_p) \to \mathcal{T} \subset H^{0\text{dd}}(G; \mathbb{F}_p).$$

Let $\{y_t\}_{t \in \mathcal{E}(G,p)}$ be the set of $p$-special Schubert classes on $G/T$ and put

$$x_{2t} := \chi^* y_t \in H^{2t}(G; \mathbb{F}_p).$$

Denote by $\Delta(\alpha_{2s-1})_{s \in r(G,p)}$ the $\mathbb{F}_p$-module in the simple system $\{\alpha_{2s-1}\}_{s \in r(G,p)}$ of generators. We formulate $H^*(G; \mathbb{F}_p)$ from the presentation of $H^*(G/T; \mathbb{F}_p)$ in Lemma 2.1, and relate the $\mathcal{P}^k$-action on $H^*(G; \mathbb{F}_p)$ to the coefficients $b_{s,t} \in \mathbb{F}_p$ in (2.4).

Lemma 3.2. The inclusion $\{\alpha_{2s-1}\}_{s \in r(G,p)}, \{x_{2t}\}_{t \in \mathcal{E}(G,p)} \subset H^*(G; \mathbb{F}_p)$ induces an isomorphism of $\mathbb{F}_p$-modules

(i) $H^*(G; \mathbb{F}_p) = \mathbb{F}_p[x_{2t}] / \langle x_{2t}^k \rangle_{t \in \mathcal{E}(G,p)} \otimes \Delta(\alpha_{2s-1})_{s \in r(G,p)}$.

Moreover, $\mathcal{T} \subset H^*(G; \mathbb{F}_p)$ is an invariant subspace of $P^k$ and

(ii) equation (2.4) implies that $P^k \alpha_{2s-1} = b_{s,t} \alpha_{2t-1}, \ t = s + k(p-1)$. 

Proof. From Lemma 2.1 and (3.3) one finds that
\[ E^*_3,0 = \text{Im} \chi^* = F_p[x_{2i}] / (x_{2i}^{k_i}) \subset H^*(G; F_p). \] (3.7)

The same method as that used in establishing [15, Lemma 3.4] is applicable to show that as a module over \( E^*_3,0 \), the term \( E^*_3,1 \) is spanned by \( \{ \alpha_{2s-1} \}_{s \in \sigma(G, p)} \). Further, since \( E^*_3, \ast \) is generated multiplicatively by \( E^*_3,0 \) and \( E^*_3,1 \) (see [19, 33]), and since
\[ E^*_3,3 = \text{spanned by} \Gamma \sum_{r(G, p)} \]

Further, since \( E^*_3,3 \) is generated multiplicatively by \( E^*_3,0 \) and \( E^*_3,1 \) (see [19, 33]), and since
\[ E^*_3,3 = \text{generated by} \Gamma \sum_{r(G, p)} \]

The proof for (i) is completed by
\[ E^*_3,3 = \text{generated by} \Gamma \sum_{r(G, p)} \]

Turning to (ii) the short exact sequence (2.3) induces the exact sequence of complexes
\[ 0 \to H^*(BT, G/T; F_p) \to \Lambda^* \to H^*(BT; F_p) \]
in which \( \Lambda^* = \Lambda^*_p(t_1, \ldots, t_n) \), \( \mathcal{B}_2^*, \ast = H^*(BT; F_p) \to \Lambda^* \) and
\[ H^*(BT, G/T; F_p) \otimes \Lambda^* = E^*_2,*(E_T, G; F_p), \]
\[ H^*(BT; F_p) \otimes \Lambda^* = E^*_2,*(E_T; F_p), \]

where \( E_T \) is the total space of the universal \( T \)-bundle on \( BT \). It is clear that \( \mathcal{B}_2^*, \ast \) is a subcomplex of \( E^*_2,*(G; F_p) \) with
\[ \mathcal{B}_3^*,1 = T \quad \text{and} \quad \mathcal{B}_3^*, \ast = \Delta(\alpha_{2i-1})_{i \in \sigma(G, p)} \subset H^*(G; F_p). \]

Since \( E^*_3,(E_T; F_p) = 0 \), the connecting homomorphisms in cohomologies give rise to the isomorphisms
\[ \beta : \mathcal{B}_3^*,1 = T \to E_3^*,0(E_T, G; F_p), \quad \beta' : H^{odd}(G; F_p) \to H^{even}(E_T, G; F_p) \]

that fit into the commutative diagrams
\[ 0 \to H^{odd}(G; F_p) \overset{\beta'}{\to} H^{even}(E_T, G; F_p) \to 0 \]
\[ \bigcup \kappa \]
\[ 0 \to T \overset{\beta}{\to} E_3^{even,0}(E_T, G; F_p) \to 0 \] (3.8)

\[ \bigcup [\mathcal{D}] \quad \kappa^* \]

\[ H^{even}(BT, G/T; F_p) \]
where the inclusion $\kappa$ identifies $E_3^{\text{even},0}(E_T, G; \mathbb{F}_p)$ with the subring
\[
\text{Im} \chi^*[H^{\text{even}}(BT, G/T; \mathbb{F}_p) \to H^{\text{even}}(E_T, G; \mathbb{F}_p)]
\]
(by a standard property of Leray–Serre spectral sequence). As $[\mathcal{D}] = (\beta')^{-1} \circ \chi^*$ by (3.8) and as both $\beta'$ and $\chi^*$ commute with $\mathcal{P}_k$, we obtain (ii). \hfill $\square$

In view of (3.5) and (i) of Lemma 3.2, we can introduce the next

**Definition 3.3.** Elements in the set $\{\alpha_{2s-1}\}_{s \in r(G, p)}$ are called $p$-transgressive generators on $H^*(G; \mathbb{F}_p)$.

## 4 Proofs of Theorems 1.1 and 1.2

Assume in this section that $G$ is exceptional with $H^*(G)$ containing non-trivial $p$-torsion. Let $\{\alpha_{2s-1} =: [\mathcal{D}(\theta_s)]\}_{s \in r(G, p)}$ be the set of $p$-transgressive generators on $H^*(G; \mathbb{F}_p)$ with $\theta_s$ being given as those in Proposition 5.5–5.7.

### 4.1 The Bockstein $\delta_p : H^*(G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p)$

Instead of the set $\{\alpha_{2s-1}\}_{s \in r(G, p)}$ of $p$-transgressive generators on $H^*(G; \mathbb{F}_p)$, in [15] the ring $H^*(G; \mathbb{F}_p)$ was also described by the set
\[
\mathcal{O}_{G, \mathbb{F}_p} = \left\{ \zeta_{2s-1} \in E_3^{2s-2,1}(G; \mathbb{F}_p) \mid s \in r(G, p) \right\}
\]
of $p$-primary generators on $H^*(G; \mathbb{F}_p)$ (see [15, Definition 2.9; Theorems 4–5]). These classes $\zeta_{2s-1}$ behave well with respect to the Bockstein $\delta_p$ in the sense that (see [15, Lemma 3.5])
\[
\delta_p(\zeta_{2s-1}) = \begin{cases} 
-x_{2s} & \text{if } s \in e(G, p), \\
0 & \text{if } s \notin e(G, p). 
\end{cases} \tag{4.1}
\]

On the other hand, from the proof of Lemma 3.2 we find that

(i) the subring $E_3^{*,0} = \mathbb{F}_p[x_{2t}]/\langle x_{2t}^{k_t} \rangle_{t \in e(G, p)}$ has the additive basis
\[
\mathcal{C} = \left\{ \prod_{t \in e(G, p)} x_{2t}^{r(t)} \mid r : e(G, p) \to \mathbb{Z} \text{ is a map with } 0 \leq r(t) < k_t \right\},
\]

(ii) $E_3^{*,1}(G, \mathbb{F}_p)$ is an $E_3^{*,0}$-module with basis $\{\alpha_{2s-1}\}_{s \in r(G, p)}$.

As a result each $\zeta_{2s-1} \in E_3^{2s-2,1}(G; \mathbb{F}_p)$ can be expressed uniquely by
\[
\zeta_{2s-1} = \sum_{g_i \in \mathcal{C}} g_i \alpha_{2t-1} \quad \text{with } \deg g_i = 2(s - t). \tag{4.2}
\]
The main idea in obtaining the formulae for \( \delta_p(\alpha_{2s-1}) \) in Theorem 1.1 is to clarify the expression (4.2), and to apply formula (4.1).

**Theorem 4.1.** We have \( \zeta_{2s-1} = \alpha_{2s-1} \) with the following exceptions:

(i) for \( p = 2 \) and in \( E_7, E_8 \):

\[
\begin{align*}
\zeta_{15} &= \alpha_{15} + x_6 \alpha_9, \\
\zeta_{27} &= \alpha_{27} + x_{10} \alpha_{17} \text{ in } E_7, E_8, \\
\zeta_{23} &= \alpha_{23} + x_6 \alpha_{17} \text{ in } E_7, \\
\zeta_{23} &= \alpha_{23} + x_6 \alpha_{17} + x_6^3 \alpha_5 \\
\zeta_{29} &= \alpha_{29} + x_6^2 \alpha_{17} \text{ in } E_8.
\end{align*}
\]

(ii) for \( p = 3 \):

\[
\begin{align*}
\zeta_{15} &= \alpha_{15} - x_8 \alpha_7 \text{ in } F_4, E_6, E_7, E_8, \\
\zeta_{35} &= \alpha_{35} + x_8 \alpha_{27} \text{ in } E_7, E_8, \\
\zeta_{19} &= -\alpha_{19}, \quad \zeta_{27} = \alpha_{27} + x_8 \alpha_{19}, \\
\zeta_{39} &= \alpha_{39} - x_{20} \alpha_{19}, \\
\zeta_{47} &= \alpha_{47} - x_8 \alpha_{39} \text{ in } E_8.
\end{align*}
\]

(iii) for \( p = 5 \) and in \( E_8 \):

\[
\zeta_s = \begin{cases} 
3\alpha_{15} & \text{for } s = 15, \\
3\alpha_{23} + 2x_{12} \alpha_{11} & \text{for } s = 23, \\
-\alpha_{35} - x_{12}^2 \alpha_{11} & \text{for } s = 35, \\
3\alpha_{47} + x_{12}^3 \alpha_{11} & \text{for } s = 47.
\end{cases}
\]

**Proof.** Let \( \Phi_{G,F_p} = \{ \gamma_s \}_{s \in e(G,p)} \), deg \( \gamma_s = 2s \), be the set of \( p \)-primary polynomials on \( G \) ([15, Definition 2.7]). In the context of [13, Section 6] each \( \gamma_s \in \Phi_{G,F_p} \) has been presented as

\[
\gamma_s = \beta_s + \sum_r \beta_r y^r \quad \text{with } \beta_s, \beta_r \in \ker \psi^*_p, \tag{4.3}
\]

where

(i) the sum is over all functions \( r : e(G,p) \to \mathbb{Z} \) that satisfy \( 0 \leq r(t) < k_t \) and \( \sum r(t) > 0 \),

(ii) \( y^r = \prod_{t \in e(G,p)} y_t^{r(t)} \) with \( \{ y_t \mid t \in e(G,p) \} \) the set of \( p \)-special Schubert classes on \( G/T \) (see Section 2).
Applying the operator $\varphi$ in [15, (2.7)] to both sides of equation (4.3) yields in $E^2_{3s-2,1}(G; \mathbb{F}_p)$ the relation
\[
\zeta_{2s-1} = [\varphi(\gamma_s)] = \mathcal{D}(\beta_s) + \sum x^r \mathcal{D} (\beta_r) \quad \text{with} \quad x^r = \prod_{t \in e(G, p)} x^{r(t)}_{2t},
\]  
(4.4)
where the first equality comes from the definition of the class $\zeta_{2s-1}$ (see [15, Definition 2.9]), the second is obtained by comparing the definitions of $\mathcal{D}$ in [15, (2.7)] with that of $\mathcal{D}$ in (2.4), and where $\mathcal{D}(\beta_s), \mathcal{D}(\beta_r) \in \mathcal{T}$ by (3.6).

Assume that $\deg \beta_r = c$. By (i) of Lemma 2.1, $\beta_s, \beta_r \in \ker \psi^*$ implies that
\[
\beta_s = b_s \theta_s + \tau_s, \quad \beta_r = \begin{cases} 
\tau_c & \text{if } c \notin r(G, p), \\
 b_r \theta_c + \tau_c & \text{if } c \in r(G, p), 
\end{cases}
\]  
(4.5)
where $b_s, b_r \in \mathbb{F}_p, \tau_h \in \langle \theta_t \rangle_{t \in r(G, p), t < h}$. Consequently
\[
\mathcal{D}(\beta_s) = b_s \alpha_{2s-1}, \quad \mathcal{D}(\beta_r) = \begin{cases} 
0 & \text{if } c \notin r(G, p), \\
 b_r \alpha_{2c-1} & \text{if } c \in r(G, p), 
\end{cases}
\]  
(4.6)
by Lemma 3.1. Substituting the expressions (4.6) in (4.4), we get the desired expression (4.2) of $\zeta_{2s-1}$ in terms of the $\alpha_{2s-1}$. This explains the algorithm to obtain the relations in Theorem 4.1.

Finally, we remark that, in the context of [13], the polynomials $\gamma_s$ have been explicitly presented in the form of (4.3) (as examples, see in [13, (6.2), (6.3)] for the cases $G = E_7$ and $p = 2, 3$) and the method computing $b_s, b_r \in \mathbb{F}_p$ will be illustrated by our latter computation for the $b_s, t \in \mathbb{F}_p$ in (2.4) (see Section 5.3 and Remark 5.9).

\section{4.2 Proof of Theorem 1.1}

The presentations of $H^*(G; \mathbb{F}_p)$ in Theorem 1.1 come from (i) of Lemma 3.2, together the degree set $r(G, p)$ given in Table 2 in Section 2. We note that, in a characteristic $p \neq 2$, the factor $\Delta(\alpha_{2s-1})_{s \in r(G, p)}$ in Lemma 3.2 can be replaced by the exterior algebra $\Lambda(\alpha_{2s-1})_{s \in r(G, p)}$ because odd-dimensional cohomology classes are all square free.

By (ii) of Lemma 3.2, results on $\mathcal{P}^k(\alpha_{2s-1})$ are verified by Lemma 2.2.

Combining formula (4.1) with the expressions of $\zeta_{2s-1}$ in Theorem 4.1 yields the formulae of $\delta_p(\alpha_{2s-1})$ in Theorem 1.1. As examples consider the case
\[
(G, p) = (E_8, 2).
\]
From Theorem 4.1 we have

\[ \zeta_{2s-1} = \alpha_{2s-1}, \quad s = 2, 3, 5, 9, \]
\[ \zeta_{15} = \alpha_{15} + x_6\alpha_9, \]
\[ \zeta_{27} = \alpha_{27} + x_{10}\alpha_{17}, \]
\[ \zeta_{23} = \alpha_{23} + x_6\alpha_{17} + x_6^3\alpha_5, \]
\[ \zeta_{29} = \alpha_{29} + x_6^2\alpha_{17}. \]

With \( e(E_8, 2) = \{3, 5, 9, 15\} \subset r(E_8, 2) = \{2, 3, 5, 8, 9, 12, 14, 15\} \) (see Table 2 in Section 2) we get from (4.1) that

\[ \delta_{2s} = 0, \]
\[ \delta_{2s-1} = x_{2s}, \quad s = 3, 5, 9, \]
\[ \delta_{2s} + x_6\delta_{2s} = 0, \]
\[ \delta_{2s} + x_{10}\delta_{2s} = 0, \]
\[ \delta_{2s} + x_6^3\delta_{2s} = 0, \]
\[ \delta_{2s} + x_6^2\delta_{2s} = x_{30}. \]

These justify the formulae for \( \delta_{2s-1} \) in (1.1) for the case \( G = E_8 \).

**Proof of Theorem 1.2.** Let \( \phi_p : H^*(G; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p) \otimes H^*(G; \mathbb{F}_p) \) be the reduced co-product. In the notation developed in Section 3, Lemma 2.1 in Ishitoya, Kono, and Toda [18] can be rephrased as

\[ \phi_p(\alpha_{2s-1}) \in E_3^{*, 0} \otimes T, \quad (4.7) \]

where

\[ E_3^{*, 0} = \mathbb{F}_p[x_{2t}] / \langle x_{2t} \rangle_{t \in e(G, p)}. \]

Granted with Theorem 1.1, coefficients comparison based on (4.7) suffices to establish Theorem 1.2. As examples we show the formulae for \( \phi_p(\alpha_{2s-1}) \) in Theorem 1.2 for the cases of \( G = E_8 \) and \( p = 2, 3 \).

With respect to the presentation of \( H^*(E_8; \mathbb{F}_2) \) in (1.1) we can assume that

\[ \phi_2(\alpha_{15}) = ax_{10} \otimes \alpha_5 + bx_6 \otimes \alpha_9 + cx_6^2 \otimes \alpha_3, \quad a, b, c \in \mathbb{F}_2, \]

by (4.7). From \( \delta_2(\alpha_{15}) = x_6x_{10}, \delta_2(\alpha_5) = x_6, \delta_2(\alpha_3) = 0 \) by Theorem 1.1 and from the obvious relation \( \delta_2\phi_2 = \phi_2\delta_2 \), one finds that \( a = b = 1 \). Consequently

\[ \phi_2(\alpha_{15}) = x_{10} \otimes \alpha_5 + x_6 \otimes \alpha_9 + cx_6^2 \otimes \alpha_3. \]
Applying $P^1(= Sq^2)$ to both sides yields the equation
\[
\phi_2(P^1\alpha_{15}) = x_6^2 \otimes \alpha_5 + cx_6^2 \otimes \alpha_5.
\]
The relations $P^1\alpha_{15} = \alpha_{17} = P^4P^2P^1\alpha_3$ (by (1.1)) and $\phi_2(\alpha_3) = 0$ then imply $c = 1$. This verifies the formula for $\phi_2(\alpha_{15})$ in Theorem 1.2.

Similarly, with respect to the presentation of $H^*(E_8; F_3)$ in (1.2) we can assume by (4.7), for the degree reason, that
\[
\phi_3(\alpha_{15}) = ax_8 \otimes \alpha_7,
\]
\[
\phi_3(\alpha_{35}) = bx_8 \otimes \alpha_{27} + cx_8^2 \otimes \alpha_{19} + dx_8x_{20} \otimes \alpha_7 + ex_{20} \otimes \alpha_{15},
\]
where $a, b, c, d, e \in F_3$. From $\delta_3\phi_3 = \phi_3\delta_3$ and from the values of $\delta_3(\alpha_{2s-1})$ in (1.2) we get the equations in $H^*(E_8; F_3) \otimes H^*(E_8; F_3)$
\[
-ax_8 \otimes x_8 = x_8 \otimes x_8,
\]
and
\[
-bx_8 \otimes x_8x_{20} + cx_8^2 \otimes x_{20} - dx_8x_{20} \otimes x_8 - ex_{20} \otimes x_8^2
\]
\[
= -x_8 \otimes x_8x_{20} + x_8^2 \otimes x_{20} - x_8x_{20} \otimes x_8 + x_{20} \otimes x_8.
\]
Coefficients comparison yields that $a = e = -1$, $b = c = d = 1$. This verifies the formula for $\phi_3(\alpha_{15})$ and $\phi_3(\alpha_{35})$ in Theorem 1.2.

### 4.3 Applications: The algebra $H^*(G; F_2)$

It follows from the proof of Lemma 3.2 that the set of 2-transgressive generators on $H^*(G; F_2)$ is unique. Moreover, one can deduce from (1.1) the next result, that expresses $H^*(G; F_2)$ solely by these generators (i.e. without resorting to the classes $x_{2t}$ on $G/T$).

**Corollary 4.2.** With respect to the 2-transgressive generators on $H^*(G; F_2)$ one has the isomorphisms of algebras
\[
H^*(G_2; F_2) = F_2[\alpha_3]/(\alpha_3^4) \otimes \Lambda F_2(\alpha_5),
\]
\[
H^*(F_4; F_2) = F_2[\alpha_3]/(\alpha_3^4) \otimes \Lambda F_2(\alpha_5, \alpha_{15}, \alpha_{23}),
\]
\[
H^*(E_6; F_2) = F_2[\alpha_3]/(\alpha_3^4) \otimes \Lambda F_2(\alpha_5, \alpha_9, \alpha_{15}, \alpha_{17}, \alpha_{23}),
\]
\[
H^*(E_7; F_2) = \frac{F_2[\alpha_3, \alpha_5, \alpha_9]}{(\alpha_3^4, \alpha_5^4, \alpha_9^4)} \otimes \Lambda F_2(\alpha_{15}, \alpha_{17}, \alpha_{23}, \alpha_{27}),
\]
\[
H^*(E_8; F_2) = \frac{F_2[\alpha_3, \alpha_5, \alpha_9, \alpha_{15}]}{(\alpha_3^4, \alpha_5^8, \alpha_9^4, \alpha_{15}^4)} \otimes \Lambda F_2(\alpha_{17}, \alpha_{23}, \alpha_{27}, \alpha_{29}).
\]
Proof. In view of (1.1) it suffices to show that
\[
\alpha_{2s-1}^2 = \begin{cases} 
  x_6 & \text{for } s = 2 \text{ and in } G_2, F_4, E_6, E_7, E_8, \\
  x_{4s-2} & \text{for } s = 3, 5 \text{ and in } E_7, E_8, \\
  x_{30} + x_6^2 x_{18} & \text{for } s = 8 \text{ and in } E_8,
\end{cases}
\] (4.8)
and that
\[
(4.9) \quad \alpha_{2s-1}^2 = 0 \text{ for those } \alpha_{2s-1} \text{ belonging to the exterior part}.
\]
These can be deduced directly from (4.8) and (1.1), the Adem relation [1] and the fact \( P^{s-2} \alpha_{2s-1} \in T \) by Lemma 3.2.

Remark 4.3. Historically the algebras \( H^*(G; \mathbb{F}_2) \) (together with the \( P^k \) action) were first obtained by Borel, Araki, Shikata and Thomas [3–5, 36]. In terms of generator and relations, their results agree with those given in Corollary 4.2. However, in these classical results there is no indication on the effect of \( Sq^1 \) action on the generators in the exterior part. The formulae for \( \delta_2(\alpha_{2s-1}) \) in (1.1) implies that these actions are highly nontrivial:
\[
\begin{align*}
Sq^1(\alpha_{15}) &= \alpha_3^2 \alpha_5^2, \\
Sq^1(\alpha_{27}) &= \alpha_5^2 \alpha_5^2 \text{ in } E_7, E_8, \\
Sq^1(\alpha_{23}) &= \alpha_3^2 \alpha_5^2 \text{ in } E_7, \\
Sq^1(\alpha_{23}) &= \alpha_3^2 \alpha_5^2 + \alpha_3^8, \quad Sq^1(\alpha_{29}) = \alpha_3^{15} \text{ in } E_8.
\end{align*}
\]

In [15, Theorem 1] we have presented \( H^*(G; \mathbb{F}_2) \) additively by the 2-primary generators \( \{\xi_{2s-1}\} \) as
\[
H^*(G; \mathbb{F}_2) = \mathbb{F}_2[x_{2t}]/[x_{2t}]_{t \in e(G, 2)} \otimes \Delta(\xi_{2s-1})_{s \in e(G, 2)}.
\]
To specify its corresponding algebra structure, it suffices to find the expressions of all the squares \( \xi_{2s-1}^2 \). This can be done by (i) of Theorem 4.1, (4.8) and (4.9).

Corollary 4.4. With respect to the 2-primary generators on \( H^*(G; \mathbb{F}_2) \) one has the isomorphisms of algebras
\[
\begin{align*}
H^*(G_2; \mathbb{F}_2) &= \mathbb{F}_2[x_6]/[x_6^2] \otimes \Delta_{F_2}(\xi_3) \otimes \Lambda_{F_2}(\xi_5), \\
H^*(F_4; \mathbb{F}_2) &= \mathbb{F}_2[x_6]/[x_6^2] \otimes \Delta_{F_2}(\xi_3) \otimes \Lambda_{F_2}(\xi_5, \xi_{15}, \xi_{23}), \\
H^*(E_6; \mathbb{F}_2) &= \mathbb{F}_2[x_6]/[x_6^2] \otimes \Delta_{F_2}(\xi_3) \otimes \Lambda_{F_2}(\xi_5, \xi_9, \xi_{15}, \xi_{17}, \xi_{23}), \\
H^*(E_7; \mathbb{F}_2) &= \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}]}{[x_6^2, x_{10}^2, x_{18}^2]} \otimes \Delta_{F_2}(\xi_3, \xi_5, \xi_9) \otimes \Lambda_{F_2}(\xi_{15}, \xi_{17}, \xi_{23}, \xi_{27}), \\
H^*(E_8; \mathbb{F}_2) &= \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, x_{30}]}{[x_6^8, x_{10}^4, x_{18}^2, x_{30}^2]} \otimes \Delta_{F_2}(\xi_3, \xi_5, \xi_9, \xi_{15}, \xi_{23}) \otimes \Lambda_{F_2}(\xi_{17}, \xi_{27}, \xi_{29}).
\end{align*}
\]
where

\[ \zeta_3^2 = x_6 \quad \text{in} \quad G_2, F_4, E_6, E_7, E_8, \]
\[ \zeta_5^2 = x_{10}, \quad \zeta_9^2 = x_{18} \quad \text{in} \quad E_7, E_8, \]
\[ \zeta_{15}^2 = x_{30}, \quad \zeta_{23}^2 = x_6^6 x_{10} \quad \text{in} \quad E_8. \]

**Remark 4.5.** Corollary 4.4 is applied in [15, Section 6] to determine the integral cohomology ring \( H^*(G) \) with respect to the integral primary generators.

**Remark 4.6.** In [15, Theorems 3–5] the algebras \( H^*(G; \mathbb{F}_p) \) were presented by the \( p \)-primary generators. Theorems 1.1, 1.2 and 4.1 determine the structure of the algebra \( H^*(G; \mathbb{F}_p) \) as a Hopf algebra over \( A_p \) with respect to these generators, see [10, Lemma 3.3], where the result is applied to determine the near-Hopf ring structure on the integral cohomology \( H^*(G) \) for all exceptional Lie groups.

## 5 Proof of Lemma 2.2

In Section 5.1 we obtain formulae for the \( \mathcal{P}^k \) action on the universal Chern classes of complex vector bundles. In Section 5.2 we will present, for each exceptional \( G \) and prime \( p = 2, 3, 5 \), a set \( \{ \theta_s \}_{s \in r(G,p)} \) of generating polynomials for the ideal \( \ker \psi_p^* \) in terms of Chern classes of certain vector bundle on \( BT \). With these preliminaries Lemma 2.2 is established in Section 5.3.

### 5.1 The mod \( p \)-Wu formulae

Let \( U(n) \) be the unitary group of rank \( n \), and let \( BU(n) \) be its classifying space. It is well known that, for a prime \( p \),

\[ H^*(BU(n), \mathbb{F}_p) = \mathbb{F}_p[c_1, \ldots, c_n], \]

where \( 1 + c_1 + \cdots + c_n \in H^*(BU(n), \mathbb{F}_p) \) is the total Chern class of the universal complex bundle \( \xi_n \) on \( BU(n) \). This implies that each \( \mathcal{P}^k c_m \) can be written as a polynomial in the \( c_1, \ldots, c_n \), and such expression may be called the mod \( p \)-Wu formula for \( \mathcal{P}^k c_m \) ([32, 34]). In the next result we present such formulae for certain \( \mathcal{P}^k c_m \) that are barely sufficient for a proof of Lemma 2.2.

**Proposition 5.1.** The following relations hold in \( H^*(BU(n), \mathbb{F}_p) \):

(i) \( p = 2 \):

\[ \mathcal{P}^r c_m = \sum_{0 \leq t \leq r} \binom{r-m}{t} c_{r-t} c_{m+t}, \]

where \( \binom{n}{i} = n(n-1) \cdots (n-i+1)/i! \).
(ii) $p = 3$:

\[
\mathcal{P}_1^c m = (m + 2)c_{m+2} - c_1 c_{m+1} + (c_1^2 + c_2)c_m.
\]

\[
\mathcal{P}_2^c m = c_2^2 c_m + c_1 c_3 c_m - c_4 c_m - c_1 c_2 c_{1+m} + (m + 1)c_1^2 c_{2+m}
\]

\[
+ (m - 1)c_2 c_{2+m} - (m + 1)c_1 c_{3+m}
\]

\[
+ \frac{1}{2}(m^2 + 3m + 2)c_{4+m}.
\]

\[
\mathcal{P}_3^c m = c_3^2 c_m + c_2 c_4 c_m - c_1 c_5 c_m + c_6 c_m - c_2 c_3 c_{1+m} + c_5 c_{1+m}
\]

\[
+ mc_2^2 c_{2+m} + (1 + m)c_1 c_3 c_{2+m} - (1 + m)c_4 c_{2+m}
\]

\[
- mc_1 c_2 c_{3+m} - c_3 c_{3+m} + \frac{1}{2}(m^2 + m)c_1^2 c_{4+m}
\]

\[
- m^2 c_2 c_{4+m} - \frac{1}{2}(m^2 + m)c_1 c_{5+m}
\]

\[
+ \frac{1}{6}(m^3 + 3m^2 + 2m - 6)c_{6+m}.
\]

(iii) $p = 5$:

\[
\mathcal{P}_1^c m = (m + 4)c_{m+4} - c_1 c_{m+3} + (c_1^2 - 2c_2)c_{m+2}
\]

\[
+ (-c_1^3 - 2c_1 c_2 + 2c_3)c_{m+1}
\]

\[
+ (c_1^4 + c_1^2 c_2 + 2c_2^2 - c_1 c_3 + c_4)c_m.
\]

Proof. For $p = 2$ the expansion of $\mathcal{P}_k^c m$ comes from the classical Wu formula ([39]) as $c_r \mod 2$ is the $2r^{th}$ Stiefel–Whitney class of the real reduction of $\xi_n$.

For $p > 2$ we have the general expansion of $\mathcal{P}_k^c m$ in terms of the Schur symmetric functions $s_\lambda$ by the formula ([9, (1.2)])

\[
\mathcal{P}_k^c m = \sum K_{(1^{m-k},p^k)\lambda}^{-1} s_\lambda \mod p,
\]  

(5.1)

where $K_{(\mu,\lambda)}^{-1}$ is the inverse Kostka number associated to the pair $\{\mu; \lambda\}$ of partitions, and where the sum is over all partitions $\lambda$ of $m + 2k(p - 1)$. We note in (5.1) that

(5.2) for those $(p, k)$ concerned by Proposition 5.1, [16, Corollary 2] and [9, Corollary 5] can be applied to evaluate the coefficients $K_{(1^{m-k},p^k)\lambda}$,

(5.3) each Schur function $s_\lambda$ can be expanded as a polynomial in the $c_r$ by the classical Giambelli formula $s_\lambda = \det(c_{\lambda'_j + j - i})$ (see [28, p. 36]), where we denote by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ the partition conjugate to $\lambda$.

Combining (5.2) and (5.3) one obtains the relations in Proposition 5.1. \qed
**Remark 5.2.** For the cases relevant to Proposition 5.1, explicit expressions of (5.1) are recorded in Remark 5.2 of the more detailed version [14] of the present paper.

### 5.2 Generating polynomials for \( \ker \psi_p^* \)

We have seen from Lemma 3.2 the crucial role that the set \( \{s \}_s \in \mathbb{R}(G, p) \) of polynomials has played in a direct construction of \( H^*(G; \mathbb{F}_p) \) as a module over \( \mathcal{A}_p \). Although certain leading terms of them are sufficient for a proof of Lemma 2.2, we choose to present them in full for the sake of completeness.

For \( n \) indeterminacies \( t_1, \ldots, t_n \) of degree 2 we set

\[
1 + e_1 + \cdots + e_n = \prod_{1 \leq i \leq n} (1 + t_i), \tag{5.2}
\]

That is, \( e_i \) is the \( i \)th elementary symmetric functions in \( t_1, \ldots, t_n \) with degree \( 2i \).

For an exceptional \( G \) with rank \( n \), assume that the set

\[
\{\omega_i\}_{1 \leq i \leq n} \subset H^2(BT)
\]

of fundamental weights (see Lemma 2.1) is so ordered as the vertices in the Dynkin diagram of \( G \) in [17, p. 58]. We introduce a set of intermediate polynomials

\[
c_k(G) \in H^{2k}(BT)
\]

useful in simplifying the expression of \( \theta_s \).

**Definition 5.3.** If \( G = F_4 \), we let \( c_k(F_4), 1 \leq k \leq 6, \) be the polynomial obtained from \( e_k(t_1, \ldots, t_6) \) in (5.4) by letting

\[
t_1 = \omega_4, \quad t_2 = \omega_3 - \omega_4, \\
t_3 = \omega_2 - \omega_3, \quad t_4 = \omega_1 - \omega_2 + \omega_3, \\
t_5 = \omega_1 - \omega_3 + \omega_4, \quad t_6 = \omega_1 - \omega_4.
\]

If \( G = E_n, n = 6, 7, 8, \) we let \( c_k(E_n), 1 \leq k \leq n, \) be the polynomial obtained from \( e_k(t_1, \ldots, t_n) \) in (5.4) by letting

\[
t_1 = \omega_n, \quad t_2 = \omega_{n-1} - \omega_n, \\
\vdots \\
t_{n-3} = \omega_4 - \omega_5, \quad t_{n-2} = \omega_3 - \omega_4 + \omega_2, \\
t_{n-1} = \omega_1 - \omega_3 + \omega_2, \quad t_n = -\omega_1 + \omega_2.
\]
Lemma 5.4. The class \(1 + c_1(F_4) + \cdots + c_6(F_4)\) (resp. \(1 + c_1(E_n) + \cdots + c_n(E_n)\), \(n = 6, 7, 8\)) is the total Chern class of a 6-dimensional (resp. \(n\)-dimensional) complex bundle \(\xi_G\) on \(BT\).

Moreover, the class \(c_1(G)\) can be expressed in terms of the weights as

\[
c_1(G) = \begin{cases} 
3\omega_1 & \text{for } G = F_4, \\
3\omega_2 & \text{for } G = E_6, E_7, E_8.
\end{cases}
\]

Proof. For a 2-dimensional cohomology class \(t \in H^2(BT)\) let \(L_t\) be the complex line bundle on \(BT\) with Euler class \(t\). Then

\[
\xi_{F_4} = \bigoplus_{1 \leq i \leq 6} L_{t_i}, \quad \text{resp.} \quad \xi_{E_n} = \bigoplus_{1 \leq i \leq n} L_{t_i}, \quad n = 6, 7, 8,
\]

where \(t_i\) is the linear form in the weights given in Definition 5.3.

The expressions of all \(c_r(G)\) by the special Schubert classes on \(G/T\) were deduced in [13, Lemma 4], by which the formula for \(c_1(G)\) is a special case. \(\Box\)

Let \((G, p)\) be a pair with \(H^*(G)\) containing non-trivial \(p\)-torsion. In Propositions 5.5–5.7 below we present, in accordance to \(p = 2, 3, 5\), a set \(\{\theta_s\}_{s \in r(G, p)}\) of generating polynomial for \(\ker \psi_p^*\) (derived in [13]).

Proposition 5.5. For \(G = G_2, F_4\) and \(E_8\), a set \(\{\theta_i\}_{i \in r(G, 2)}\) of generating polynomials for \(\ker \psi_2^*\) is given by

| \(\{\theta_i\}_{i \in r(G, 2)}\) | \(G_2\) | \(F_4\) | \(E_8\) |
|-------------------------------|--------|--------|--------|
| \(\theta_2\) | \(\omega_1^2 + \omega_1\omega_2 + \omega_2^2\) | \(c_2\) | \(c_2\) |
| \(\theta_3\) | \(\omega_2^3\) | \(c_3\) | \(c_3\) |
| \(\theta_5\) | \(\omega_2^4 + \omega_1 c_6\) | \(c_8 + c_4^2 + \omega_2^2 c_6 + \omega_2^3 c_5 + \omega_2^8\) |
| \(\theta_8\) | \(\omega_2^2 c_7 + \omega_2 c_8 + \omega_2^3 c_6\) |
| \(\theta_9\) | \(c_4^2 + c_4^3\) | \(c_6^2 + c_4^3\) |
| \(\theta_{12}\) | \(c_7^2 + c_4^2 c_6 + \omega_2^2 c_6^2\) |
| \(\theta_{14}\) | \(c_7^2 + \omega_2^2 c_8 + \omega_2^3 c_4 c_8\) |
| \(\theta_{15}\) | \(c_7 c_8 + \omega_2^2 c_8 + \omega_2^3 c_4 c_8\) |

and for \(G = E_6, E_7\) by

\[
\{\theta_i\}_{i \in r(E_6, 2)} = \{\theta_i \mid c_7 = c_8 = 0\}_{i \in r(E_8, 2)} \setminus \{14, 15\},
\]

\[
\{\theta_i\}_{i \in r(E_7, 2)} = \{\theta_i \mid c_8 = 0\}_{i \in r(E_8, 2)} \setminus \{15\}.
\]
Proposition 5.6. For an exceptional $G$ with $G \neq G_2$, a set $\{\theta_i\}_{i \in \tau(G, 3)}$ of generating polynomials for $\ker \psi^*_3$ is given by

$$
\begin{array}{|c|c|c|c|}
\hline
\{\theta_i\} & F_4 & E_6 & E_7 \\
\hline
\theta_2 & \omega^2_2 - c_2 & \omega^2_2 - c_2 & \omega^2_2 - c_2 \\
\theta_4 & c_2^2 - c_4 & c_2^2 - c_4 & c_2^2 - c_4 \\
\theta_5 & & c_5 + c_2c_3 & \\
\theta_6 & c_2c_4 - c_6 & c_2c_4 + c_3^2 - c_6 & -\omega^2_2c_3 + c_2c_4 - \omega^2_2c_5 + c_3^2 - c_6 \\
\theta_8 & -c_2c_6 & -c_4^2 & -c_4^2 + c_2c_3^2 - \omega^2_2c_7 + c_3c_5 \\
\theta_9 & & c_6c_3 & \\
\theta_10 & & -c_4c_3^2 + c_2c_3c_5 + c_3c_7 - c_5^2 & \\
\theta_14 & c_4c_3^2 + c_2c_5c_7 + c_7^2 & & \\
\theta_18 & c_2c_3^2c_7 + c_3^6 + c_3^2c_5c_7 + c_3c_5^3 & & \\
\hline
\end{array}
$$

Proposition 5.7. For $G = E_8$, a set of generating polynomials for $\ker \psi^*_5$ is given by

$$
\begin{align*}
\theta_2 &= -\omega^2_2 - c_2, \\
\theta_6 &= 2\omega^6 - 2\omega^3_2c_3 - 2\omega^2_2c_5 - 2c_3^2 - c_6, \\
\theta_8 &= -\omega^8_2 - \omega^4_2c_4 - 2\omega^3_2c_5 - \omega^2_2c_3c_4 - \omega^2_2c_7 - c_3c_5 - c_4^2 - c_8.
\end{align*}
$$
\[ \theta_{12} = -2\omega_2^4c_4^2 - \omega_2^4c_8 + \omega_2^3c_3^3 + 2\omega_2^3c_4c_5 - 2\omega_2^2c_3^2c_4 - \omega_2^2c_3c_7 - 2\omega_2c_3c_4^2 + c_3^4 - c_3c_4c_5 - 2c_5c_7 + 2c_6^2, \]

\[ \theta_{14} = -2\omega_2^{10}c_4 + 2\omega_2^8c_3^2 - 2\omega_2^7c_7 + \omega_2^5c_3c_6 - 2\omega_2^4c_3c_7 + 2\omega_2^2c_5^2 + \omega_2^3c_3^2c_5 + \omega_2^3c_4c_7 + \omega_2c_3c_4c_6 - \omega_2c_4^2c_5 + \omega_2c_5c_8 - 2\omega_2c_6c_7 + c_5^2c_4^2 - c_3^2c_8 + 2c_3c_4c_7 + c_4^2c_6 + c_4c_5^2 + c_7^2, \]

\[ \theta_{18} = -2\omega_2^8c_5^2 + 2\omega_2^7c_3^2c_5 - 2\omega_2^6c_3^2c_6 + \omega_2^6c_3c_4c_5 + 2\omega_2^5c_3^2c_7 + 2\omega_2^4c_3^2c_8 + \omega_2^4c_4c_5^2 + 2\omega_2^3c_4c_6 + \omega_2^3c_3^2c_5 + \omega_2^3c_3c_4c_7 - 2\omega_2^2c_3^3c_4 - 2\omega_2^2c_3c_4c_5 + 2\omega_2^2c_2c_4^2 - \omega_2c_2c_3c_5 - 2\omega_2c_4c_5c_6 - \omega_2^2c_4^2c_8 - \omega_2^2c_4c_5c_7 - 2\omega_2c_6c_7 + 2c_3c_4c_5c_6 - c_3^2c_5^2 - 2c_3c_7c_8 + c_4c_7^2, \]

\[ \theta_{20} = -\omega_2^{17}c_3 - \omega_2^{13}c_7 + 2\omega_2^{12}c_4 + 2\omega_2^{12}c_8 + 2\omega_2^{11}c_3c_6 + \omega_2^{10}c_3^2c_4 - \omega_2^9c_4c_7 + 2\omega_2^8c_3^3 - \omega_2^7c_3c_5^2 - \omega_2^6c_3c_5 - \omega_2^6c_3c_8 + \omega_2^6c_4c_5^2 - \omega_2^5c_3^5 + \omega_2^5c_3c_4^2 + \omega_2^5c_2c_4c_7 + 2\omega_2^4c_3^3 - \omega_2^4c_4^2c_5 - 2\omega_2^3c_4c_5c_7 + \omega_2^3c_3c_4c_5 - \omega_2^3c_3c_4c_5 + \omega_2^2c_3c_4^2c_5 + \omega_2^2c_3^2c_5^2 - \omega_2^2c_3^2c_5c_7 - 2\omega_2^2c_3c_4c_5 + 2\omega_2^2c_3^2c_5 + \omega_2^2c_3^2c_5c_7 + 2\omega_2^2c_3c_4c_5 + 2\omega_2c_3^2c_5c_7 + 2c_3c_4c_5c_7 + c_3^2c_7c_8 + c_4c_7^2 + c_3^2c_5^2 + 2c_3c_4c_5 + 2c_3^2c_4c_8 + c_3^2c_4c_8 + c_3^2c_4c_8 + c_3c_4c_5 + 2c_3^2c_4c_8 - 2c_5^4, \]

\[ \theta_{24} = -\omega_2^{16}c_8 - \omega_2^{13}c_3c_8 - 2\omega_2^9c_3c_4c_8 + 2\omega_2^7c_4c_5c_8 + \omega_2^6c_4c_6c_8 - 2\omega_2^6c_5^2c_8 + 2\omega_2^5c_3^2c_8 + \omega_2^5c_4c_7c_8 - \omega_2^5c_5c_6c_8 + 2\omega_2^4c_4c_8 - \omega_2^4c_5c_7c_8 + \omega_2^3c_3c_4c_8 - 2\omega_2^3c_3c_7c_8 + \omega_2^3c_3c_4c_6c_8 - 2\omega_2^3c_3c_5^2c_8 + \omega_2^3c_6c_7c_8 + \omega_2^2c_4c_5^2c_8 - \omega_2^2c_6c_8^2 - 2\omega_2c_4c_5c_6c_8 - 2\omega_2c_7c_8^2 + c_4^2c_4c_8 + 2c_3c_5c_8^2 + c_3c_6c_7c_8 - 2c_5^2c_6c_8. \]

### 5.3 Proof of Lemma 2.2

Granted with the explicit expressions of the set \( \{ \theta_s \}_{s \in \mathcal{R}(G, p)} \) of generating polynomials for \( \ker \psi_p^* \) in Propositions 5.5, 5.6, 5.7 and the mod \( p \)-Wu formulae in Proposition 5.1, we prove Lemma 2.2.
If \((G, p) = (G_2, 2)\), Lemma 2.2 is directly shown by the computation (see in Proposition 5.5 for the expressions of \(\theta_2, \theta_3\) in \(G_2\))

\[
\mathcal{P}^1 \theta_2 = \mathcal{P}^1(\omega_1^2 + \omega_1 \omega_2 + \omega_2^2) = \omega_1^2 \omega_2 + \omega_1 \omega_2^2 = \theta_3 + \omega_1 \theta_2.
\]

So we can assume from now on that \(G \neq G_2\).

Let \(\mathbb{F}_p[G]\) be the subring of \(H^*(BT; \mathbb{F}_p)\) generated by \(c_i = c_i(G)\) and the weight \(\omega_r\) with \(r = 1\) for \(F_4\), and \(r = 2\) for \(E_6, E_7, E_8\). Then

\[
\{\theta_i\}_{i \in r(G, p)} \subset \mathbb{F}_p[G]
\]

by Propositions 5.5–5.7. Since the polynomials \(c_r(G)\) are the mod \(p\) reduction of the Chern classes of a vector bundle on \(BT\), the Wu formulae in Proposition 5.1, together with the Cartan formula ([35]), are applicable to express \(\mathcal{P}^k \theta_r\) as an element in \(\mathbb{F}_p[G]\). It remains for us to sort out the number \(b_{s,t} \in \mathbb{F}_p\) in equation (2.4).

Based on certain built-in functions of MATHEMATICA, an algorithm to evaluate the number \(b_{s,t}\) in (2.4) is given below.

Furnish the monomials in \(\mathbb{F}_p[G]\) with the term order given by the total degree lexicographical order with respect to the following ordering on the generators of the subring \(\mathbb{F}_p[G]\)

\[
c_8 > \cdots > c_1 > \omega_r,
\]

where \(r = 1\) for \(F_4\), and \(r = 2\) for \(E_6, E_7, E_8\). For an element \(i \in r(G, p)\) we denote by \(\mathcal{G}_i(G, p) \subset \mathbb{F}_p[G]\) a Gröbner basis of the ideal generated by the subset \(\{\theta_j\}_{j \in r(G, p), j < i}\). Let \(\{s, t\} \subset r(G, p)\) be a pair with \(t = s + k(p - 1)\).

Step 1 Call GroebnerBasis[ , ] to compute \(\mathcal{G}_t(G, p)\).

Step 2 Call PolynomialReduce[ , , ] to compute the residue \(h_a\) of \(P^k \theta_s - a \theta_t\) module \(\mathcal{G}_t(G, p)\) with \(a \in \mathbb{F}_p\) an indeterminacy.

Step 3 Take \(b_{s,t} = a_0\) with \(a_0\) the solution to the equation \(h_a = 0\).

We note that in Step 2, the residue \(h_a\) obtained is always linear in \(a\).

\[\square\]

**Remark 5.8.** As for a proof of Lemma 2.2 certain leading terms of the polynomials \(\theta_s\) are sufficient. This is illustrated by the second proof of Lemma 2.2 in the more detailed version [14] of the present paper.

**Remark 5.9.** The algorithm in the proof of Lemma 2.2 has also been applied in [11, Section 7.1] to justify three enumerative problems due to Schubert (in 1879). This indicates that the computation requested by the proofs of Lemma 2.2 and Theorem 4.1 is a routine task for the computer.
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