q-vertex operator from 5D Nekrasov function

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Abstract
The five-dimensional AGT correspondence implies the connection between the \(q\)-deformed Virasoro block and the 5d Nekrasov partition function. In this paper, we determine a \(q\)-deformation of the four-point block in the Coulomb gas representation from the 5d Nekrasov function, and obtain an expression of the \(q\)-deformed vertex operator. If we use only one kind of the \(q\)-vertex operators, one of the insertion points of them must be modified in order to hold the 2d/5d correspondence.

Keywords: \(q\)-vertex operator, 5d Nekrasov function, 2d/5d correspondence

1. Introduction

Twenty years ago, a \(q\)-deformation of the Virasoro algebra (\(q\)-Virasoro algebra) was introduced in\textsuperscript{3} \cite{1–3}. It is closely related to the one-dimensional XYZ Heisenberg chain model or to a two-dimensional solvable lattice model (the Andrews–Baxter–Forrester model). One motivation for the \(q\)-deformation is to study thermodynamic limit of these models using the representation theory of this deformed algebra.

In ordinary minimal conformal field theories (CFT), the singular vectors have connection with the Jack symmetric functions indexed by a rectangular partition \cite{6}. There is a generalization of the Jack functions, called Macdonald symmetric functions \cite{7}. The guiding principle for deformation in \cite{3} is such that the singular vectors of the deformed algebra are
expressed in terms of the Macdonald functions. The defining relation of the $q$-Virasoro algebra and the screening currents are well established in [3].

On the other hand, the representations of the $q$-Virasoro algebra is still not well understood. In particular, we have no proper definition of $q$-deformed primary fields (vertex operators). The vertex operators or intertwining operators of the $q$-Virasoro algebra for the minimal cases are considered in [8–10]. These $q$-vertex operators have ‘good’ commutation relations with the $q$-Virasoro generators. But general criterion for goodness is not known.

Recently, the $q$-Virasoro algebra has collected renewed interests due to the $q$-deformed/lifted version (or K-theoretic five-dimensional version) of the (W)AGT conjecture. The (W)AGT relation [11, 12] implies the two-dimensional correlation functions (conformal blocks) of the Virasoro/W algebra are identical to the four-dimensional Nekrasov partition functions. For this 2d/4d correspondence, see for example [13–32]. The $q$-deformed/lifted (W)AGT blocks in two-dimensional theories are identified with the Nekrasov partition function of the five-dimensional gauge theories [33, 34]. See also [35–43] for the 5d AGT. More general 2d/6d correspondence is also discussed in [37, 44–46].

Assuming the $q$-deformed/lifted version of (W)AGT conjecture, one can fix the form of the $q$-vertex operator. The conformal blocks have a Coulomb gas representation (the Dotsenko–Fateev integral representation). A simple recipe for the $q$-deformation of the conformal block in the Coulomb gas representation is proposed in [35, 47] without constructing the $q$-operators. We consider the problem of operator realization of the deformed block in order to determine the $q$-vertex operator.

In [19], it is shown that the Dotsenko–Fateev representation of the four-point conformal block is related to multiple integrals with the Selberg measure. The Kadell’s formula gives the average of the Jack polynomials with respect to this measure. Using the formula, we compared the conformal block with the 4d Nekrasov function and found agreement.

In this paper, we consider a straightforward $q$-deformation of [19]. Kaneko obtained a $q$-deformed version of the Kadell’s formula [48]. This formula gives the average of the Macdonald polynomials with respect to the $q$-deformed Selberg measure. By adopting it as a calculational tool, and starting from the 5d SU(2) Nekrasov function, we obtained the $q$-deformed (Jackson) integral representation of the four-point block. Using a free field representation of the $q$-screening charges, we have determined an explicit form of the $q$-vertex operator. This is one of our main results.

If we use only one kind of the $q$-vertex operators, it turns out that one of the insertion points of them must be modified in order to match the $q$-block with the Nekrasov function. We have no simple explanation on this modification. Similar modification of an insertion point of $q$-vertex operator can be found in [47] where one of $q$-primary operators is degenerate at level 2.

This paper is organized as follows. In the next section, after reviewing the $q$-Virasoro algebra and its screening charges, we give explicitly the vertex operator for the $q$-Virasoro algebra such that the 2d/5d correspondence is established. The $q$-deformed version of the Coulomb gas representation of the four-point block is constructed. In section 3, a brief review of the 5d Nekrasov partition function is given. In section 4, we perform the $\Lambda_0$-expansion of the $q$-Virasoro block. The parameter dictionary of 2d/5d correspondence is presented. We display the explicit form of the first order term in $\Lambda_0$ expansion. In section 5, we carry out the comparison with the 5d Nekrasov partition function. In this process, we obtain non-trivial relations among some quantities. In appendix A, the two-point correlation function for the $q$-Virasoro vertex operator and the $q$-screening current are given. In appendix B, the $q$-Selberg integral and Kaneko’s formula are presented.
2. **q-deformed conformal block**

In this section, we first explain our convention for the $q$-deformed Virasoro algebra and its screening charges. Then, we propose a $q$-deformed vertex operator. We also introduce a ‘modified’ four-point block $B(\Lambda_0)$.

### 2.1. $q$-Virasoro algebra

Let us consider a $q$-Heisenberg algebra with generators

$$\alpha_n \ (n \in \mathbb{Z}), \quad Q$$

satisfying the following defining relations:

$$[\alpha_n, \alpha_m] = -\frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})}{(1 + p^n)} \delta_{n+m,0}, \quad (n \neq 0),$$

(2.2)

$$[\alpha_n, Q] = \delta_{n,0}.$$

(2.3)

Here $p = q/t$. The $q$-deformed Virasoro algebra can be realized by this algebra [3]

$$T(z) := \exp \left( \sum_{n=0} \alpha_n z^{-n} \right) :e^{z/2}Q^ {1/2}e^{z/2}: + \exp \left( -\sum_{n \neq 0} \alpha_n (pz)^{-n} \right) :p^{-1/2}q^{-v_{z0}}.$$

(2.4)

Here $\beta$ is defined by the relation $t = q^\beta$.

This operator satisfies the defining relation of the $q$-Virasoro algebra:

$$f(z')T(z)T(z') - f(z/z')T(z')T(z) = \frac{(1 - q)(1 - t^{-1})}{(1 - p)} [\delta(pz/z') - \delta(p^{-1}z/z')]$$

(2.5)

where

$$f(z) = \exp \left( \sum_{n=1}^{\infty} \frac{1 - q^n - t^{-n}}{(1 + p^n)z^n} \right), \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n.$$

(2.6)

In the following part, we assume that $|q| < 1$.

### 2.2. Screening charges

A screening current for the $q$-Virasoro algebra is defined by

$$S_z(z) := e^{\varphi^{(+)}}(z),$$

(2.7)

where

$$\varphi^{(+)}(z) = \sqrt{\beta} Q + 2\sqrt{\beta} \alpha_0 \log z + \sum_{n \neq 0} \frac{1 + p^{-n}}{1 - q^n} \alpha_n z^{-n}.$$

(2.8)

This screening current $S_z$ commutes with the $q$-Virasoro generators up to a total $q$-derivative:

$$[T(z), S_z(z')] = -(1 - q)(1 - t^{-1}) \frac{d}{dqz'} [\delta(z'/z)p^{-1/2}z'A_+(z')],$$

(2.9)
where

\[ A_+ (z) := \exp \left( \sum_{n=0}^{\infty} \frac{(1 + r^n)}{n!} \alpha_n z^{-n} \right) \exp \left( \sum_{n=0}^{\infty} \frac{(1 - q^n)}{n!} \beta_n z^{-n} \right). \tag{2.10} \]

The \( q \)-derivative is given by

\[ \frac{d_q f (z)}{dz} = \frac{f (z) - f (qz)}{z - qz}. \tag{2.11} \]

There is another screening current \( S_+ (z) \) which commutes with the \( q \)-Virasoro generators up to a total \( t \)-derivative. But we will not use \( S_+ (z) \) in this paper, hence we do not introduce it.

Using two different ‘integration ranges’, two screening charges \( Q_+ \), \( Q'_+ \) are defined by

\[ Q_+ := \int_0^{\Lambda_0} d_q z S_+ (z), \quad Q'_+ := \int_1^\infty d_q z S_+ (z) \equiv \int_0^1 \frac{d_q y}{y^2} S_+ (1/y). \tag{2.12} \]

Here we use the Jackson integral

\[ \int_0^a d_q z f (z) = (1 - q) \sum_{n=0}^{\infty} f (aq^n) aq^n. \tag{2.13} \]

2.3. Vertex operators for the \( q \)-Virasoro algebra

Vertex operators for the \( q \)-Virasoro algebra are considered in [8–10, 34]. In [8–10], a \( q \)-deformation of primary operators \( V_\mu (z) = e^{(1/2) \alpha_u \phi (z)} e^{\alpha_0 \log z} \) in the minimal CFT model is introduced. Here \( \alpha_{u,s} = (1 - r) \sqrt{\beta} - (1 - s) \sqrt{\beta} \) with rational \( \beta \). The normalization of chiral boson is chosen as \( \langle \phi (z) \phi (z') \rangle = 2 \log (z - z') \). But we are not interested in these types of \( q \)-vertex operators.

The \( q \)-vertex operator determined from the 5d Nekrasov function is the following. For a complex parameter \( u \), let us define a vertex operator of the \( q \)-deformed Virasoro algebra by

\[ V_\mu (z) := : e^{\Phi_u (z)} : \tag{2.14} \]

where

\[ \Phi_u (z) := \frac{u}{\sqrt{\beta}} \left( \frac{1}{2} \alpha_0 \log z \right) + \sum_{n=0}^{\infty} \frac{(q^{mu} - 1)}{(1 - q^n) (1 - r^n)} \alpha_n z^{-n}. \tag{2.15} \]

This is essentially equivalent to the vertex operator \(^5\) [34, 47].

In the \( q \rightarrow 1 \) limit with keeping \( \beta = \log t / \log q \) fixed, this vertex operator becomes a free field representation of the Virasoro primary operator with scaling dimension \( \Delta = u (u - 2 \beta + 2) / (4 \beta) \).

2.4. \( q \)-deformed Coulomb gas representation

A \( q \)-deformed version of the Coulomb gas representation of the four-point block is defined by

\[ \langle V_{a_0} (0) V_{a_1} (\Lambda_0) V_{a_2} (1) V_{a_3} (\infty) (Q_+^N (Q'_+)^N) \rangle. \tag{2.16} \]

\(^5\) By identification of the fundamental bosons \( h_1^i \) in [34] with our \( \alpha_u \), the vertex operator \( V_\mu^i (z) \) in [34] are related to our operator as \( V_\mu^i (z) = V_i (q^{1/2}) q^{2 \alpha_u (1/2 \beta)} \). Here the parameter \( U \) is identified with \( q^{-n} \).
But we will consider the following ‘modified’ four-point block:

$$B(\Lambda_0) = \langle V_{a_0}(0) V_{a_1}(\Lambda_0) V_{a_2}(q^{\nu+1}) V_{a_3}(\infty) \rangle (Q_{+})^N (Q_{-})^{N_{-}}.$$  \hfill (2.17)

The four vertex operators are inserted at $z_1 = 0, z_2 = \Lambda_0, z_3 = q^{\nu+1}$ and $z_4 = \infty$. Note that $z_3 = q^{\nu+1}$, instead of $z_3 = 1$. We have no simple explanation on the position of the third vertex operator. With this choice of $z_3$, the modified four-point conformal block (2.17) coincides with the five-dimensional Nekrasov partition function of the $SU(2)$ gauge theory with $N_l = 4$.

In (2.17), the parameters $u_i$ should obey the ‘momentum conservation condition’:

$$u_1 + u_2 + u_3 + u_4 + 2\beta (N_+ + N_-) + 2(1 - \beta) = 0. \hfill (2.18)$$

The parameters $u_i$ used here is related to the parameters $\alpha_i$ in [19] as $u_i = \sqrt{\beta} \alpha_i$. The four point block (2.17) is a $q$-deformation of the conformal block with the intermediate scaling dimension $\Delta_i = u_i (u_i - 2\beta + 2)/(4\beta)$. Here $u_i$ is the ‘internal momentum’:

$$u_i = u_1 + u_2 + 2\beta N_+ = -u_3 - u_4 - 2\beta N_- - 2(1 - \beta). \hfill (2.19)$$

In the Jackson integral representation, this internal momentum is discretized because $N_i$ and $N_-$ are number of integration variables. But after performing the Jackson integrals, these parameters can be analytically continued to any values.

For simplicity, we assume that

$$0 < |\Lambda_0| < |q^{\nu+1}| < 1. \hfill (2.20)$$

Using the two-point correlators in appendix A, we have

$$B(\Lambda_0) = V_0(\Lambda_0) (N_i) ! (N_i') ! \int_{C_{\infty}([0, \Lambda_0])} d^{N_{-}} \zeta' \int_{C_{\infty}([1, \infty])} d^{N_{+}} \zeta \times \prod_{i=1}^{N_+} \frac{z_{i}^{u_i}}{(z_{i}^{2}; q_{\infty}) (q_{\infty}^{u_{i+1}} z_{i}^{2}; q_{\infty})} \times \prod_{j=1}^{N_-} \frac{z_{j}^{u_j}}{(z_{j}^{2}; q_{\infty}) (q_{\infty}^{u_{j+1}} z_{j}^{2}; q_{\infty})} \times \prod_{1 \leq i < j \leq N_+} \zeta_{i}^{2\beta} \left(1 - \frac{z_{i}^{2}}{z_{j}^{2}}\right) \frac{q^{1-\beta} z_{i}^{2} / z_{j}^{2}}{q^{1-\beta} z_{i}^{2} / z_{j}^{2}} \frac{q_{\infty}}{q_{\infty}} \times \prod_{1 \leq i < j \leq N_-} \zeta_{j}^{2\beta} \left(1 - \frac{z_{i}^{2}}{z_{j}^{2}}\right) \frac{q^{1-\beta} z_{i}^{2} / z_{j}^{2}}{q^{1-\beta} z_{i}^{2} / z_{j}^{2}} \frac{q_{\infty}}{q_{\infty}} \times \prod_{i=1}^{N_+} \prod_{j=1}^{N_-} \left(\zeta_{i}^{2\beta} \left(1 - \frac{z_{i}^{2}}{z_{j}^{2}}\right) \frac{q^{1-\beta} z_{i}^{2} / z_{j}^{2}}{q^{1-\beta} z_{i}^{2} / z_{j}^{2}} \frac{q_{\infty}}{q_{\infty}} \right). \hfill (2.21)$$

Here the constant $V_0(\Lambda_0)$ is given by

$$V_0(\Lambda_0) := \Lambda_0^{\frac{1}{(2\beta)} (u_1 u_2 + u_1 N) - (u_1 + u_2 + 2\beta (N_+ + N_-) + 1) / (2\beta)} \times \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(q^{-n+1} - 1)(q^{-n+1} - 1)}{(1 - q^n) (1 - q^n) (1 + p^2)} \Lambda_0^n \right). \hfill (2.22)$$

The ‘range’ $C_{\infty}([0, \Lambda_0])$ of the Jackson integral for $\{z_{i}\}_{i=1, 2, \ldots, N_i}$ are the interval $[0, \Lambda_0]$ with the additional condition
Recall that we have assumed \( |q| < 1 \). Hence, this condition means that, for a parameterization \( z_i = \Lambda_0 q^k \) in the Jackson integral, the non-negative integers \( k_i \) are summed over with
\[
0 \leq k_1 < k_2 < \cdots < k_N.
\] (2.24)

Similarly, the ‘range’ \( C'_N \) ([1, \infty]) is chosen such that
\[
1 \leq |z'_1| < |z'_2| < \cdots < |z'_N| < \infty.
\] (2.25)

It means that \( z'_j = q^{-N_j} \) with \( 0 \leq k'_1 < k'_2 < \cdots < k'_N \).

By the following coordinate transformation:
\[
z_i = \Lambda_0 x_i, \quad (i = 1, 2, \ldots, N), \quad z'_j = 1/q_j, \quad (j = 1, 2, \ldots, N'),
\] (2.26)

we have
\[
B(\Lambda_0) = \mathcal{V}^N_0(\Lambda_0) (N_0)! (N_0')! \int_{C_{N_0}(1,1)} d_N^x \int_{C_{N_0'}(1,1)} d_N^y
\times \prod_{i=1}^{N_0} x_i^{N_0} \left( \frac{q(x_i^N; q \infty)}{(q^{u_i+1} x_i; q \infty)_{1 \leq i \leq N}} \right) \prod_{1 \leq i < j \leq N} x_i^{2j} \left( 1 - \frac{x_j}{x_i} \right) \frac{(q^{1-\beta} x_j/x_i; q \infty)}{(q^{3 \beta} x_j/x_i; q \infty)}
\times \prod_{j=1}^{N_0'} y_j^{N_0'} \left( \frac{q(y_j^N; q \infty)}{(q^{u_j+1} y_j; q \infty)_{1 \leq i \leq N}} \right) \prod_{1 \leq i < j \leq N_0} y_j^{2j} \left( 1 - \frac{y_j}{y_i} \right) \frac{(q^{1-\beta} y_j/y_i; q \infty)}{(q^{3 \beta} y_j/y_i; q \infty)}
\times \prod_{i=1}^{N_0} \prod_{j=1}^{N_0'} \left( \frac{q(1-\beta)^{\Lambda_0 x_i y_j^N; q \infty}}{(q^{1-\beta \Lambda_0 x_i y_j^N}; q \infty)} \right),
\] (2.27)

where
\[
\mathcal{V}^N_0(\Lambda_0) = \mathcal{V}_0(\Lambda_0) \Lambda_0^{N_0(u_0+1)+3N_0(N_0-1)}
\times \exp \left( -\frac{\sum_{n=1}^{\infty} (q^{n-1}-1)(q^{-n-1}-1)}{(1-q^n)(1-q^n)(1+q^n)} \Lambda_0^N \right).
\] (2.28)

The Jackson integrals in (2.27) are taken for the range [0, 1] under the following conditions
\[
0 \leq |x_N| < |x_{N-1}| < \cdots < |x_2| < |x_1| \leq 1,
\] (2.29)
\[
0 \leq |y_N| < |y_{N-1}| < \cdots < |y_2| < |y_1| \leq 1.
\] (2.30)

Let
\[
B_0(\Lambda_0) = B(\Lambda_0)/\mathcal{V}^N_0(\Lambda_0).
\] (2.31)

At \( \Lambda_0 = 0 \), \( B_0(\Lambda_0) \) factorizes into a product of two \( q \)-deformed Selberg integrals:
\[
B_0(0) = S^{(q)}_{N_0}(u_1, u_2, \beta) S^{(q)}_{N_0'}(u_3, u_3, \beta),
\] (2.32)
where

\[
S_N^{(q)}(u_1, u_2, \beta) = N! \int_{C(0,1)} d^{N-2} \mathbf{z} \prod_{i=1}^{N} z_i^{n_i} (q^{z_i}; q)_\infty \prod_{1 \leq i < j \leq N} z_i^{2\beta} \left(1 - \frac{z_j}{z_i}\right) \frac{(q^{1-\beta z_j/z_i}; q)_\infty}{(q^{z_j/z_i}; q)_\infty},
\]  

\text{(2.33)}

When \( \beta \) is a positive integer, (2.33) becomes

\[
S_N^{(q)}(u_1, u_2, \beta) = \int_{(0,1)^N} d^{N-2} \mathbf{z} \prod_{i=1}^{N} z_i^{n_i} (q^{z_i}; q)_\infty \prod_{1 \leq i < j \leq N} (z_i - z_j) \left(1 - q^{k_j - k_i - \beta}ight) \frac{(q^{k_j - k_i + 1 - \beta}; q)_\infty}{(q^{k_j - k_i}; q)_\infty}.
\]  

This integral (2.34) is evaluated as

\[
S_N^{(q)}(u_1, u_2, \beta) = N! q^{A_N(u_1, u_2)} \prod_{j=1}^{N} \Gamma_q(u_1 + 1 + (N - j)\beta) \Gamma_q(u_2 + 1 + (N - j)\beta) \Gamma_q(j\beta),
\]  

\text{(2.35)}

where \( \Gamma_q(x) \) is the \( q \)-Gamma function and

\[
A_N(u_1, \beta) = \frac{1}{2} N(N - 1)(u_1 + 1)\beta + \frac{1}{3} N(N - 1)(N - 2)\beta^2,
\]  

\text{(2.36)}

We expect that (2.35) also holds when \( \beta \) is not a positive integer with a modification of \( A_N(u_1, \beta) \) from (2.36).

For example, for positive \( u_1 \) and \( u_2 \), the small \( q \) behavior of \( S_N^{(q)} \) is given by \( N! q^{A_N} \cdots \). When \( 0 < \beta < 2 \), the leading contribution in the sum (2.33) comes from the term with \( k_i = (i-1) \). Hence, it seems that in this case

\[
A_N(u_1, \beta) = \frac{1}{2} N(N - 1)(u_1 + 1) + \frac{1}{3} N(N - 1)(N - 2)\beta,
\]  

\text{(2.37)}

2.5. \textit{Remark:} \( q \rightarrow 1 \) limit

In this subsection, we comment on the \( q \rightarrow 1 \) limit of various objects in previous subsections.

For notational simplicity, let us introduce the following functions:

\[
D_N^{(q)}(u_1, u_2, \beta; z) = \prod_{j=1}^{N} z_i^{n_j} (q^{z_j}; q)_\infty \prod_{1 \leq i < j \leq N} z_i^{2\beta - 1} \left(1 - \frac{z_j}{z_i}\right) \frac{(q^{1-\beta z_j/z_i}; q)_\infty}{(q^{z_j/z_i}; q)_\infty}(z_i - z_j),
\]  

\text{(2.38)}
\[ F_{N,\Lambda}^{(q)}(u_2, u_3, \beta, \Lambda_0; x, y) = \prod_{i=1}^{N-1} \left( \Lambda_0 q^{-u_i x_i} q^{-u_i y_i} ; q \right)_{\infty} \prod_{j=1}^{N-1} \left( \Lambda_0 q^{-u_j y_j} q^{-u_j x_j} ; q \right)_{\infty} \prod_{i=1}^{N-1} \prod_{j=1}^{N-1} \left( q^{1-\beta} \Lambda_0 x_i y_j q^{-u_i x_j} ; q \right)_{\infty} \left( 1 - \Lambda_0 x_i y_j \right). \] (2.39)

In the \( q \to 1 \) limit, these objects behave as follows:

\[ \lim_{q \to 1} D_N^{(q)}(u_1, u_2, \beta; z) = \prod_{j=1}^{N} z_j^{n_j} (1 - z_j)^{y_j} \prod_{1 \leq i < j \leq N} (z_i - z_j)^{2\beta}. \] (2.40)

\[ \lim_{q \to 1} F_{N,\Lambda}^{(q)}(u_2, u_3, \beta, \Lambda_0; x, y) = \prod_{i=1}^{N} (1 - \Lambda_0 x_i)^{n_i} \prod_{j=1}^{N} (1 - \Lambda_0 y_j)^{n_j} \prod_{i=1}^{N} \prod_{j=1}^{N} (1 - \Lambda_0 x_i y_j)^{2\beta}. \] (2.41)

Hence, (2.27) goes to equation (2.8) of [19].

Using (2.38), the \( q \)-deformed Selberg integral (2.33) can be written as

\[ S_N^{(q)}(u_1, u_2, \beta) = N! \int_{C_0(0,1)} d^N z D_N^{(q)}(u_1, u_2, \beta; z). \] (2.42)

In the \( q \to 1 \) limit, the \( q \)-Selberg integral (2.33) with (2.35) goes to the ordinary Selberg integral:

\[ \lim_{q \to 1} S_N^{(q)}(u_1, u_2, \beta) = \int_0^1 d^N z \prod_{j=1}^{N} z_j^{n_j} (1 - z_j)^{y_j} \prod_{1 \leq i < j \leq N} (z_i - z_j)^{2\beta} \prod_{j=1}^{N} (1 - x_j u_2 + 2 + (2N - j - 1) \beta \Gamma(1 + \beta)). \] (2.43)

Here we have used \( \lim_{q \to 1} \Gamma_q(x) = \Gamma(x) \) and

\[ N! \prod_{j=1}^{N} \frac{\Gamma(j \beta)}{\Gamma(\beta)} = \prod_{j=1}^{N} \frac{\Gamma(1 + j \beta)}{\Gamma(1 + \beta)}. \] (2.44)

The \( q \to 1 \) limit is related to the 2d CFT and the 4d gauge theory on the flat space. While root of unity limits of \( q \) are related to 2d supersymmetric/parafermionic theories and the 4d gauge theories on ALE spaces [49–61].

### 3. 5d Nekrasov partition function

In this section, we briefly review the Nekrasov partition function on \( \mathbb{R}^{1,1} \times S^1 \). We denote the radius of \( S^1 \) by \( R \). The five-dimensional \( SU(N) \) Nekrasov partition function with \( N_f = 2N \) fundamental matters can be found in [62, 63] (see also [64]). We follow the notation of [34] and consider the \( N = 2 \) case.

The instanton part of the five-dimensional \( SU(2) \) Nekrasov partition function with \( N_f = 4 \) fundamental matters is given by
\[ Z_2^{\text{inst}}(Q; \Lambda) = \sum_{\lambda, \mu} Z_{\lambda, \mu}^{(+)} \left( \frac{\Lambda^2}{\nu^2} \right)^{|\lambda|+|\mu|}, \]  

(3.1)

where

\[ Z_{\lambda, \mu}^{(+)} = \prod_{j=1}^{2} N_{\lambda, \mu}(vQ_j/Q_j^\pm)N_{\mu, \lambda}(vQ_j/Q_j^\mp)N_{0, \lambda}(vQ_j/Q_j^\mp)N_{0, \mu}(vQ_j/Q_j^\pm). \]  

(3.2)

Here \( v = (q/t)^{1/2} = p^{1/2} \). The summation in (3.1) is over a pair of partitions \((\lambda, \mu)\). The function \( N_{\nu}(Q) \) is defined by

\[ N_{\nu}(Q) = \prod_{(i,j) \in \lambda} (1 - Qq^{\nu-j}p^{i-j+1}) \prod_{(i,j) \in \mu} (1 - Qq^{\nu-i}p^{j-i+1}) = \prod_{(i,j) \in \lambda} (1 - Qq^{\nu-j}p^{i-j+1}) \prod_{(i,j) \in \mu} (1 - Qq^{\nu-i}p^{j-i+1}). \]  

(3.3)

Here \( \lambda' \) is the conjugate partition of \( \lambda \).

The parameters in (3.1) are \( q, t, \lambda, Q_1, Q_2, Q_1^\mp \) and \( Q_2^\pm \). \( q \) and \( t \) are related to the \( \Omega \)-background parameters \( \epsilon_1 \) and \( \epsilon_2 \):

\[ q = e^{R_1}, \quad t = e^{-R_1}. \]  

(3.4)

The parameters \( Q_1 \) and \( Q_2 \) are related to the vev of the adjoint scalar in the 4d theory:

\[ vQ_1 = e^{R_1}, \quad vQ_2 = e^{-R_1}, \]  

(3.5)

while \( Q_i^\pm \) are related to the mass of fundamental matters:

\[ Q_1^+ = e^{-R_m}, \quad Q_2^+ = e^{R_m}, \quad Q_1^- = e^{-R_m}, \quad Q_2^- = e^{R_m}. \]  

(3.6)

The expansion parameters \( \Lambda_{\alpha}^\pm \) are defined by

\[ \Lambda_{\alpha}^\pm = \frac{q_{\alpha}^\pm}{Q_{\alpha}^\pm} \]  

(3.7)

Hence the relation between \( Z_{\lambda, \mu}^{(+)} \) is given by

\[ Z_{\lambda, \mu}^{(+)} = e^{(\lambda|+|\mu|R(m_2+m_4-m_1-m_2))}Z_{\lambda, \mu}^{(-)}. \]  

(3.8)

### 3.1. First order: \( Z_{(3,0)}^{(\pm)} \) and \( Z_{(0,1)}^{(\pm)} \)

Using (3.4)–(3.6), \( Z_{\lambda, \mu}^{(+)} \) with \(|\lambda| + |\mu| = 1\) can be written as

\[ Z_{(3,0)}^{(+)}(Q_{1,0}, Q_{0,1}) = \frac{1 - e^{R(a+m_4)}(1 - e^{-R(a+m_3)})(1 - e^{-R(a+m_1)})}{(1 - e^{-R_1})(1 - e^{-R_2})(1 - e^{-R_4})}, \]  

(3.9)

\[ Z_{(0,1)}^{(+)}(Q_{1,0}, Q_{0,1}) = \frac{1 - e^{-R(a-m_1)}(1 - e^{-R(a-m_2)})(1 - e^{-R(a-m_3)})}{(1 - e^{-R_1})(1 - e^{-R_2})(1 - e^{-R_4})}, \]  

(3.10)

\[ Z_{(3,0)}^{(-)}(Q_{1,0}, Q_{0,1}) = \frac{1 - e^{-R(a+m_4)}(1 - e^{R(a+m_3)})(1 - e^{R(a+m_1)})}{(1 - e^{-R_1})(1 - e^{-R_2})(1 - e^{-R_4})}, \]  

(3.11)

\[ Z_{(0,1)}^{(-)}(Q_{1,0}, Q_{0,1}) = \frac{1 - e^{R(a-m_1)}(1 - e^{R(a-m_2)})(1 - e^{R(a-m_3)})}{(1 - e^{-R_1})(1 - e^{-R_2})(1 - e^{-R_4})}. \]  

(3.12)

Here \( \epsilon = \epsilon_1 + \epsilon_2 \).
In the $R \to 0$ limit, these terms reproduce the 4d results:
\[
\lim_{R \to 0} Z_{(1),0}^{(4)} = \frac{(a + m_1)(a + m_2)(a + m_3)(a + m_4)}{2a(2a + \epsilon)g^2_f}.
\] (3.13)
\[
\lim_{R \to 0} Z_{(0),1}^{(4)} = \frac{(a - m_1)(a - m_2)(a - m_3)(a - m_4)}{2a(2a - \epsilon)g^2_f}.
\] (3.14)

4. $\Lambda_0$ expansion of $q$-block

In this section, we study the expansion of the modified blocks in the $\Lambda_0$ parameter.

4.1. Method of calculation in $q$-block

In order to compare the $q$-block (2.31) with the 5d Nekrasov function (3.1), let us introduce the following function
\[
\mathcal{A}(\Lambda_0) = \frac{B_0(\Lambda_0)}{B_0(0)} = \langle \langle F_{N,N}^{(q)}(u_2, u_3, \beta, \Lambda_0; x, y) \rangle \rangle = 1 + \sum_{n=1}^{\infty} \Lambda_0^n A_n.
\] (4.1)

Here
\[
\langle \langle f(x, y) \rangle \rangle = \langle \langle f(x, y) \rangle \rangle_+.
\] (4.2)

and $\langle \rangle_+$ is the average with respect to the $q$-deformed Selberg measure:
\[
\langle f(x) \rangle_+ = \frac{1}{S_N^{(q)}(u_1, u_2, \beta)} \int_{C_N([0,1])} d^N x \, D_N^{(q)}(u_1, u_2, \beta; x) f(x),
\] (4.3)
\[
\langle f(y) \rangle_+ = \frac{1}{S_N^{(q)}(u_4, u_3, \beta)} \int_{C_N([0,1])} d^N y \, D_N^{(q)}(u_4, u_3, \beta; y) f(y).
\] (4.4)

Note that
\[
F_{N,N}^{(q)} = \exp \left[ -\sum_{k=1}^{\infty} \frac{\Lambda_0^k}{k} \frac{(1 - t^k)}{(1 - q^k)} \left\{ \left( p_k(x) - \frac{1 - q^{-nk}}{1 - t^k} \right) p_k(y) 
\right.
\right.
\]
\[
+ p_k(x) \left( \frac{q/t^k}{1 - t^k} \right) \left( p_k(y) - \frac{1 - q^{-nk}}{1 - t^k} \right) \right]\] (4.5)

where $p_k$ denotes the power sum:
\[
p_k(x) = \sum_{i=1}^{N} x_i^k, \quad p_k(y) = \sum_{j=1}^{N} y_j^k.
\] (4.6)

We conjecture that the Kaneko’s formula (B.8) also holds for the contour $C_N([0, 1])$. We have checked it for small values of $N$ and $\beta$. 

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Then, we can calculate $\mathcal{A}_n$ by using the average of the Macdonald polynomial $P_l$:

$$
\langle P_l(x'; q, t) \rangle_\lambda = \frac{(t^{N_l} x^{(q,l)})_\lambda (q^{m_0+l(N_0-1)} x^{(q,l)})_\lambda}{h_\lambda(q, t)(q^{m_0+l_2} + 2t^{2N_0-1} q^{(q,l)})_\lambda}
= \prod_{(i,j) \in \lambda} \frac{(t^{i-1} - q^{-(i-j)N}) (t^{i-1} - q^{2u_i+j_0 t N + l_2 2N_0 - 2})}{(1 - q^{j_0 t N + l_2 2N_0 - 1} (t^{i-1} - q^{m_0+l_i + l_j} N_i - 1))},
$$

(4.7)

$$
\langle P_l(y'; q, t) \rangle_\lambda = \frac{(t^{N_l} y^{(q,l)})_\lambda (q^{m_0+l(N_0-1)} y^{(q,l)})_\lambda}{h_\lambda(q, t)(q^{m_0+l_2} + 2t^{2N_0-1} q^{(q,l)})_\lambda}
= \prod_{(i,j) \in \lambda} \frac{(t^{i-1} - q^{-(i-j)N}) (t^{i-1} - q^{2u_i+j_0 t N + l_2 2N_0 - 2})}{(1 - q^{j_0 t N + l_2 2N_0 - 1} (t^{i-1} - q^{m_0+l_i + l_j} N_i - 1))},
$$

(4.8)

4.2. Parameter dictionary

In [19], we have used the following identification of $\beta$ with the $\Omega$-background parameters $\epsilon_1$ and $\epsilon_2$:

$$
\epsilon_1 = \sqrt{3} g_s, \quad \epsilon_2 = - \frac{1}{\sqrt{3}} g_s, \quad \epsilon = \epsilon_1 + \epsilon_2 = \left( \sqrt{3} - \frac{1}{\sqrt{3}} \right) g_s.
$$

(4.9)

In this choice, it holds that $\epsilon_1 \epsilon_2 = - g_s^2$.

With this identification, (3.4) yields the following relations:

$$
q = e^{R \epsilon_2} = e^{-R g_s / \sqrt{3}}, \quad t = q^{\beta} = e^{-R \epsilon_1} = e^{-R g_s / \sqrt{3}}, \quad v = (q/t)^{1/2} = e^{(1/2) R \epsilon}. 
$$

(4.10)

The momentum conservation condition (2.18) can be rewritten as

$$
q^{u_1+u_2+u_3+2} t^{2(N_1+N_2)-2} = 1.
$$

(4.11)

The 2d/4d dictionary used in [19] now converted to the following 2d/5d dictionary:

$$
\begin{align*}
T^N_0 &= e^{-R (a-m_2)}, & T^{-N}_0 &= e^{R (a+m_2)}, \\
q^{u_1} &= e^{-R (m_2-m_1+1)}, & q^{u_2} &= e^{-R (m_3-m_2)}, \\
q^{u_3} &= e^{-R (m_1+m_3)}, & q^{u_4} &= e^{-R (m_3-m_4+1)}.
\end{align*}
$$

(4.12)

(4.13)

(4.14)

Some useful relations are given by

$$
\begin{align*}
q^{m_0+l_m-1} &= e^{-R (m_2-m_1)}, & q^{m_1} &= e^{-R (m_1+m_2)}, \\
q^{m_3} &= e^{-R (m_3-m_2)}, & q^{u_4+1} &= e^{-R (m_2-m_3)}, \\
q^{u_3+u_4+1} 2N_0 - 1 &= e^{-2 R a}, & q^{m_3+u_4+1} 2N_0 - 1 &= e^{2 R a}.
\end{align*}
$$

(4.15)

(4.16)

(4.17)
4.3. First order: $A_1$

The first order term in the $\Lambda_0$-expansion (4.1) is given by

$$A_1 = -\frac{(1-t)}{(1-q)}(1 + v^2)\langle p_1(x)\rangle_+\langle p_1(y)\rangle_- + \frac{1 - q^{-u_1}}{1 - q}\langle p_1(x)\rangle_+ + \frac{1 - q^{-u_2}}{1 - q}\langle p_1(y)\rangle_-.$$  \hspace{1cm} (4.18)

Note that the following identity holds for any $A$ and $v$:

$$\frac{1 - v^2A}{1 - A} + \frac{1 - v^2A^{-1}}{1 - A^{-1}} = 1 + v^2.$$  \hspace{1cm} (4.19)

Specializing this relation by setting $A$ to $q^m + u_1 + 1, 2N_1 - 1$, $A^{-1} = q^m + u_1 + 1, 2N_1 - 1$, we have

$$1 + v^2 = -\frac{1 - q^{m+u_1+2}2N_1-2}{1 - q^{m+u_1+1}2N_1-1} + \frac{1 - q^{m+u_1+2}2N_1-2}{1 - q^{m+u_1+1}2N_1-1}.$$  \hspace{1cm} (4.20)

With help of (4.21), a non-trivial decomposition of $A_1$ is obtained:

$$A_1 = A_{(1),(0)} + A_{(0),(1)}.$$  \hspace{1cm} (4.22)

where

$$A_{(1),(0)} = \left\{ \frac{(1-t)(1 - q^{m+u_1+2}2N_1-2)}{(1-q)(1 - q^{m+u_1+1}2N_1-1)}\langle p_1(x)\rangle_+ + \frac{(1 - q^{-u_2})}{(1 - q)} \right\}\langle p_1(y)\rangle_-,$$  \hspace{1cm} (4.23)

$$A_{(0),(1)} = \langle p_1(x)\rangle_+\left\{ \frac{(1-t)(1 - q^{m+u_1+2}2N_1-2)}{(1-q)(1 - q^{m+u_1+1}2N_1-1)}\langle p_1(y)\rangle_- + \frac{(1 - q^{-u_2})}{(1 - q)} \right\}.$$  \hspace{1cm} (4.24)

The averages for the Macdonald polynomial with $\lambda = (1)$ are given by

$$\langle P_1(x; q, t)\rangle_+ = \langle p_1(x)\rangle_+ = \frac{(1 - t^{N_1})(1 - q^{m+1}t^{N_1-1})}{(1-t)(1 - q^{m+u_1+2}t^{2N_1-2})},$$  \hspace{1cm} (4.25)

$$\langle P_1(y; q, t)\rangle_- = \langle p_1(y)\rangle_- = \frac{(1 - t^{N_1})(1 - q^{m+1}t^{N_1-1})}{(1-t)(1 - q^{m+u_1+2}t^{2N_1-2})}.$$  \hspace{1cm} (4.26)

By substituting these expressions into factors in curly bracket of (4.23) and (4.24), we can see that

$$\begin{align*}
A_{(1),(0)} &= -\frac{(1-t)(1 - q^{m+u_1+2}2N_1-2)}{(1-q)(1 - q^{m+u_1+1}2N_1-1)}\langle p_1(x)\rangle_+ + \frac{(1 - q^{-u_2})}{(1 - q)} \\
&= -\frac{(1 - t^{N_1})(1 - q^{m+1}t^{N_1-1})}{(1-t)(1 - q^{m+u_1+2}t^{2N_1-2})} + \frac{(1 - q^{-u_2})}{(1 - q)} \\
&= q^{m+u_1+1}t^{2N_1-1}(1 - q^{-u_1}t^{N_1})(1 - q^{-u_1-u_2-1}t^{1-N_1}) \\
&= \frac{(1 - q)^2(q^{m+u_1+1}t^{2N_1-1})(1 - q^{m+u_1+2}t^{2N_1-1})}{(1-q)(1 - q^{m+u_1+1}t^{2N_1-1})}.
\end{align*}$$  \hspace{1cm} (4.27)
\[ \begin{align*}
\left\{ \frac{(1 - t)\left(1 - q^{m_1+m_2+2}t^{2N, -2}\right)}{(1 - q)\left(1 - q^{m_1+m_2+1}t^{2N, -1}\right)} p_1(y) \right\}_+ + \frac{(1 - q^{-m_2})}{(1 - q)} \\
= - \frac{(1 - t)\left(1 - q^{m_1+m_2+1}t^{N, -1}\right)}{(1 - q)\left(1 - q^{m_1+m_2}t^{2N, -1}\right)} + \frac{(1 - q^{-m_2})}{(1 - q)} \\
= - q^{m_1+m_2+1}t^{2N, -1}\left(1 - q^{m_1+m_2+1}t^{N, -1}\right) - q^{m_1+m_2}t^{2N, -1}\left(1 - q^{m_1+m_2}t^{N, -1}\right). 
\end{align*} \]

Consequently, we have explicit form of \( A_{(1,0)} \) and \( A_{(0,1)} \):
\[ \begin{align*}
A_{(1,0)} &= - q^{m_1+m_2+1}t^{2N, -1} \\
&\times \frac{(1 - q^{m_1+m_2}t^{N, -1})(1 - q^{m_1+m_2-1}t^{1-N, -1})(1 - q^{m_1+m_2-1}t^{1-N, -1})(1 - q^{m_1+m_2-1}t^{N, -1})}{(1 - q)(1 - t)(1 - q^{m_1+m_2+1}t^{2N, -1})(1 - q^{m_1+m_2+2}t^{2N, -2})}, \\
A_{(0,1)} &= - q^{m_1+m_2+1}t^{2N, -1} \\
&\times \frac{(1 - q^{m_1+m_2}t^{N, -1})(1 - q^{m_1+m_2-1}t^{1-N, -1})(1 - q^{m_1+m_2-1}t^{1-N, -1})(1 - q^{m_1+m_2-1}t^{N, -1})}{(1 - q)(1 - t)(1 - q^{m_1+m_2+1}t^{2N, -1})(1 - q^{m_1+m_2+2}t^{2N, -2})}. 
\end{align*} \]

4.3.1. In terms of parameters of gauge theory. Let us rewrite the parameters in (4.29) and (4.30) by the gauge theory parameters.

Using
\[ q^{-u_1} = e^{R(m_1+m_2)}, \quad q^{-u_2} = e^{R(m_1+m_3)}. \] (4.31)
\[ t^{N, -1} = e^{-R(a-m_2)}, \] (4.32)
\[ q^{m_1+m_2+2}t^{2N, -2} = e^{-R(2a-e)}, \] (4.33)
\[ t^{N, -1} = e^{R(a+m_3)}, \quad q^{m_1+m_2+1}t^{N, -1} = e^{R(a+m_4)}. \] (4.34)

e etc., we have
\[ \begin{align*}
\langle p_1(x) \rangle_+ &= \frac{(1 - e^{-R(a-m_1)})(1 - e^{-R(a-m_2)})}{(1 - e^{-R}(2a-e))}, \\
\langle p_1(y) \rangle_- &= \frac{(1 - e^{R(a+m_1)})(1 - e^{R(a+m_2)})}{(1 - e^{R}(2a-e))}, \\
- \frac{(1 - t)(1 - q^{m_1+m_2+2}t^{2N, -2})}{(1 - q)(1 - q^{m_1+m_2+1}t^{2N, -1})} \langle p_1(x) \rangle_+ + \frac{(1 - q^{-u_2})}{(1 - q)} \\
= \frac{(1 - e^{R(a+m_1)})(1 - e^{R(a+m_2)})}{(1 - e^{R}(2a-e))}, \\
- \frac{(1 - t)(1 - q^{m_1+m_2+2}t^{2N, -2})}{(1 - q)(1 - q^{m_1+m_2+1}t^{2N, -1})} \langle p_1(y) \rangle_- + \frac{(1 - q^{-u_2})}{(1 - q)} \\
= \frac{(1 - e^{-R(a-m_1)})(1 - e^{-R(a-m_2)})}{(1 - e^{-R}(2a-e))}. 
\end{align*} \]

etc., we have
Thus, we finally have the following expressions:

\[
A_{(1),(0)} = \frac{(1 - e^{R(a + m_i)}) (1 - e^{R(a + m_2)}) (1 - e^{R(a + m_3)}) (1 - e^{R(a + m_4)})}{(1 - e^{R(a)}) (1 - e^{-R(a)}) (1 - e^{2R(a)}) (1 - e^{R(2a + r)})}, \tag{4.38}
\]

\[
A_{(0),(1)} = \frac{(1 - e^{-R(a - m_i)}) (1 - e^{-R(a - m_2)}) (1 - e^{-R(a - m_3)}) (1 - e^{-R(a - m_4)})}{(1 - e^{R(a)}) (1 - e^{-R(a)}) (1 - e^{-2R(a)}) (1 - e^{-R(2a + r)})}. \tag{4.39}
\]

Remark: the following identity plays the crucial role in (4.36):

\[
e^{-2R(a)} (1 - e^{R(a - m_i)}) (1 - e^{R(a - m_2)}) (1 - e^{-2R(a)}) (1 - e^{-R(a + m_i)})
= (1 - e^{-R(a + m_i)}) (1 - e^{R(a + m_2)})
\tag{4.40}
\]

and similarly the following identity is used in (4.37):

\[
e^{-2R(a)} (1 - e^{R(a + m_i)}) (1 - e^{R(a + m_2)}) (1 - e^{-2R(a)}) (1 - e^{-R(a + m_i)})
= (1 - e^{-R(a + m_i)}) (1 - e^{-R(a + m_2)}),
\tag{4.41}
\]

5. Comparison with 5d Nekrasov partition function

In this section, we compare the modified $q$-block with the Nekrasov function.

We assume that the 2d/5d correspondence holds, i.e.

\[
A(A_0) = Z^{{\text{int}}}_2(Q, \Lambda).
\tag{5.1}
\]

By comparing (4.38) with (3.9) or (3.11), and (4.39) with (3.10) or (3.12), we can check that

\[
A_{(1),(0)} = \left( \frac{t}{q^2} \right) e^{R(m_2 + m_i)} Z^{(+)}_{(1),(0)} = \left( \frac{t}{q^2} \right) e^{R(m_1 + m_2)} Z^{(-)}_{(1),(0)}, \tag{5.2}
\]

\[
A_{(0),(1)} = \left( \frac{t}{q^2} \right) e^{R(m_2 + m_4)} Z^{(+)}_{(0),(1)} = \left( \frac{t}{q^2} \right) e^{R(m_1 + m_2)} Z^{(-)}_{(0),(1)}. \tag{5.3}
\]

Then we must have

\[
A_0 A_{(1),(0)} = \frac{\Lambda^+}{v^2} Z^{(+)}_{(1),(0)} = \frac{\Lambda^-}{v^2} Z^{(-)}_{(1),(0)}, \tag{5.4}
\]

\[
A_0 A_{(0),(1)} = \frac{\Lambda^+}{v^2} Z^{(+)}_{(0),(1)} = \frac{\Lambda^-}{v^2} Z^{(-)}_{(0),(1)}. \tag{5.5}
\]

The relations (5.2) and (5.3) lead to the following identification of the expansion parameters:

\[
A_0 = q e^{-R(m_1 + m_2)} \Lambda^+_i = q e^{-R(m_1 + m_2)} \Lambda^-_i = q e^{-(1/2) R(m_1 + m_2 + m_3 + m_4)} \Lambda^i.
\tag{5.6}
\]

This connection is also stated as follows:

\[
A_0 = q^{(1/2)(m_1 + m_4)} \Lambda_i. \tag{5.7}
\]

Using this identification, (5.1) decomposes into the following identities:

\[
A_k = \left( \frac{t}{q^2} \right)^k \sum_{|\lambda| + |\mu| = k} Z^{(+)}_{\lambda,\mu} = \left( \frac{t}{q^2} \right)^k \sum_{|\lambda| + |\mu| = k} Z^{(-)}_{\lambda,\mu}. \tag{5.8}
\]
We have checked these identities up to \( k = 4 \). These are quite non-trivial relations even for the cases of low order \( k \). We expect that these hold for all \( k \). Therefore, this gives strong evidence of the 2d/5d correspondence (5.1).

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**Appendix A. Two-point correlation functions**

In this section, we collect two-point functions utilized in section 2.4.

The (radial ordered) two-point correlation functions for the \( q \)-deformed vertex operators (2.14) and the screening current (2.7) are given by

\[
\langle V_{\nu}(z_1)V_{\nu}(z_2) \rangle = \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{n} (q_{\nu z_1} - 1)(q_{\nu z_2} - 1) \left( \frac{z_2}{z_1} \right)^n, & |z_1| > |z_2|, \\
\sum_{n=1}^{\infty} \frac{1}{n} (q_{\nu z_2} - 1)(q_{\nu z_1} - 1) \left( \frac{z_1}{z_2} \right)^n, & |z_2| > |z_1|, 
\end{cases}
\]

\[
\langle V_{\nu}(z_1)S_{\nu}(z_2) \rangle = \begin{cases} 
\frac{q_{\nu z_2 z_1}}{(q_{\nu z_2} - 1)q_{\nu z_1}}, & |z_1| > |z_2|, \\
\frac{q_{\nu z_1 z_2}}{(q_{\nu z_1} - 1)q_{\nu z_2}}, & |z_2| > |z_1|, 
\end{cases}
\]

\[
\langle S_{\nu}(z_1)S_{\nu}(z_2) \rangle = \begin{cases} 
\frac{1 - q_{\nu z_2 z_1}}{(q_{\nu z_2} - 1)q_{\nu z_1}}, & |z_1| > |z_2|, \\
\frac{1 - q_{\nu z_1 z_2}}{(q_{\nu z_1} - 1)q_{\nu z_2}}, & |z_2| > |z_1|, 
\end{cases}
\]

Here

\[ (x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n) \]

is the \( q \)-Pochhammer symbol.

**Appendix B. \( q \)-Selberg integral and Kaneko’s formula**

In this section, we shortly summarize the \( q \)-Selberg integral and the Kaneko’s formula [48].
### B.1. q-Selberg integral

Using (2.38), let us consider the following q-deformation of the Selberg integral:

\[
S_N^{(q)}(u_1, u_2, \beta; \xi) := \int_{[0, \xi]^{N}} \mathbf{d}^N_q z \ D_N^{(q)}(u_1, u_2, \beta; z),
\]

with

\[
\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in (\mathbb{C}^*)^N.
\]

For certain value of the parameters \(u_1, u_2, \beta, \xi\) such that the Jackson integral (B.1) converges, the Aomoto’s formula \[65\] implies that (see also \[48\])

\[
S_N^{(q)}(u_1, u_2, \beta; \xi) = q^{(1/2)N(N-1)/2} \prod_{j=1}^{N} \xi_{j} \prod_{1 \leq i < j \leq N} q^{u_1 + u_2 + (j-1)(N-j)-N+2} \times \prod_{j=1}^{N} \frac{\vartheta(\xi_j q^{u_1 + u_2 + (j-1)(N-j)}) \vartheta(q^{u_1 + u_2 + (j-2)(N-j)+1})}{\vartheta(q^{u_1 + u_2 + (j-1)(N-j)}) \vartheta(q^{u_1 + u_2 + (j-2)(N-j)+1})} \prod_{1 \leq i < j \leq N} \frac{\vartheta(\xi_i/\xi_j)}{\vartheta(q^{2(2(N-j)-1)} \xi_i/\xi_j)} \times \prod_{j=1}^{N} \Gamma_q(u_1 + 1 + (j-1) \beta) \Gamma_q(u_2 + 1 + (j-1) \beta) \Gamma_q(j \beta)
\]

Here \(\vartheta(x)\) is the Jacobi elliptic function

\[
\vartheta(x) = (x; q)_\infty (q/x; q)_\infty (q; q)_\infty
\]

and \(\Gamma_q(x)\) is the \(q\)-Gamma function

\[
\Gamma_q(x) = (1 - q)^{1-x} (q^x; q)_\infty / (q^4; q)_\infty
\]

### B.2. Kaneko’s formula

Let \(P_\lambda(z; q, t)\) be the Macdonald polynomial for the variables \(z = (z_1, z_2, \ldots, z_N)\). The average over the Macdonald polynomials is defined by

\[
\langle P_\lambda(z; q, t) \rangle := \frac{1}{S_N^{(q)}(u_1, u_2, \beta; \xi)} \int_{[0, \xi]^{N}} \mathbf{d}^N_q z \ P_\lambda(z; q, t) \ D_N^{(q)}(u_1, u_2, \beta; z).
\]

It is given by [48]

\[
\langle P_\lambda(z; q, t) \rangle = \frac{(t^{N})^{(q,t)}}{h_t(q, t)} \prod_{(i,j) \in \lambda} \frac{(t^{i-1} - q^{i-1} j^{N-1} t^{j-1} - q^{i-1} j^{N-1})}{(1 - q^{i-1} j^{N-1} t^{j-1})},
\]

with

\[
h_t(q, t) := \prod_{(i,j) \in \lambda} \frac{(t^{i-1} - q^{i-1} j^{N-1} t^{j-1} - q^{i-1} j^{N-1})}{(1 - q^{i-1} j^{N-1} t^{j-1})}.
\]
Here
\[
(A)^{(k)}_{\lambda} = \prod_{s \in \lambda} (t^s - q^{-i} t^{s-1}) = \prod_{(i,j) \in \lambda} (t^{i+1} - q^{j+1} t^{i+1}), \tag{B.9}
\]
\[
h_{\lambda}(q, t) = \prod_{s \in \lambda} \left(1 - q^{s} t^{s+1}\right) = \prod_{(i,j) \in \lambda} \left(1 - q^{i-j} t^{i-j+1}\right). \tag{B.10}
\]

For a square \(s = (i, j)\) in a partition \(\lambda\), the arm-length, the leg-length, the arm-colength and the leg-colength are respectively denoted by \(a(s), \ell(s), a'(s)\) and \(\ell'(s)\).

Note that the average (B.8) does not depend on the choice of \(\xi\).

B.3. Special case

When \(\beta\) is a positive integer, \(\beta = k\), by choosing \(\xi = (1, q^{2/3}, \ldots, q^{(N-1)/3})\), the Aomoto’s formula (B.4) reduces to the Askey–Habsieger–Kadell’s formula [66–68]:

\[
\int_{[0,1]^N} d^N z^1 \wedge d^N z^2 \wedge \cdots \wedge d^N z^N = q^{kN} \prod_{i=1}^{N} \frac{\Gamma_q(u_1 + 1 + (N - i)k) \Gamma_q(u_2 + 1 + (N - i)k) \Gamma_q(1 + ik)}{\Gamma_q(u_1 + u_2 + 2 + (2N - i - 1)k) \Gamma_q(1 + k)}, \tag{B.11}
\]

where
\[
A_N = \frac{1}{2} (u_1 + 1) k N (N - 1) + \frac{1}{3} k^2 N (N - 1) (N - 2). \tag{B.12}
\]

Notice that there is a slight difference between (2.34) and (B.11). In (2.34), the integrand is symmetric under a permutation of \(z_i\) and \(z_j\), while in (B.11) it is not the case.

Also, Kadell’s formula [48, 68] is obtained as a special case of Kaneko’s formula (B.8):

\[
\int_{[0,1]^N} d^N z^1 \wedge d^N z^2 \wedge \cdots \wedge d^N z^N P_A(z; q, q^k)
\times \prod_{i=1}^{N} z^i_{\infty} \frac{(q z_i; q)_{\infty}}{(q^{2N+1} z_i; q)_1} \prod_{1 \leq i < j \leq N} z^i_{\infty} \frac{q^{1-k} z_j z_i; q_{\infty}}{(q^{1+k} z_j z_i; q_{\infty})}
= q^{kN} P_A(1, q^k, q^{2k}, \ldots, q^{(N-1)k}; q, q^k)
\times \prod_{i=1}^{N} \frac{\Gamma_q(u_1 + 1 + (N - i)k + \lambda) \Gamma_q(u_2 + 1 + (N - i)k) \Gamma_q(1 + ik)}{\Gamma_q(u_1 + u_2 + 2 + (2N - i - 1)k + \lambda) \Gamma_q(1 + k)}. \tag{B.13}
\]

Here
\[
P_A(1, q^k, q^{2k}, \ldots, q^{(N-1)k}; q, q^k) = \frac{(q^{Nk} t^k)}{h_{\lambda}(q, t^k)}. \tag{B.14}
\]

It can be rewritten as follows:

\[
\langle P_A(z; q, q^k) \rangle
= \frac{(q^{Nk} t^k)}{h_{\lambda}(q, t^k)} \prod_{i=1}^{N} \frac{\Gamma_q(u_1 + 1 + (N - j)k + \lambda) \Gamma_q(u_1 + u_2 + 2 + (2N - j - 1)k)}{\Gamma_q(u_1 + 1 + (N - j)k) \Gamma_q(u_1 + u_2 + 2 + (2N - j - 1)k + \lambda)}
= \frac{(q^{Nk} t^k)}{h_{\lambda}(q, t^k)} \frac{(q^{Nk+1+N(N-1)k}) t^k}{h_{\lambda}(q, t^k)} \tag{B.15},
\]
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