SPECIAL LAGRANGIAN WEBBING

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Abstract. We construct families of imaginary special Lagrangian cylinders near transverse Maslov index 0 or \( n \) intersection points of positive Lagrangian submanifolds in a general Calabi-Yau manifold. Hence, we obtain geodesics of open positive Lagrangian submanifolds near such intersection points. Moreover, we introduce a method for proving \( C^{1,1} \) regularity of geodesics of positive Lagrangians at the non-smooth locus. Along the way, we study geodesics of positive Lagrangian linear subspaces in a complex vector space, and prove an a priori existence result in the case of Maslov index 0 or \( n \).

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1. Introduction

1.1. Setting. Let \((X, \omega, J, \Omega)\) be a Calabi-Yau manifold. Namely, \(X\) is a Kähler manifold with symplectic form \(\omega\) and complex structure \(J\), and \(\Omega\) is a non-vanishing

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holomorphic volume form on $X$. We denote by $g$ the Kähler metric and by $n$ the complex dimension.

An oriented Lagrangian submanifold $\Lambda \subset X$, possibly immersed, is said to be positive if $\text{Re } \Omega|_{\Lambda}$ is a positive volume form. A positive Lagrangian submanifold is special if $\text{Im } \Omega|_{\Lambda} = 0$. An oriented Lagrangian submanifold is called imaginary special if $\text{Re } \Omega|_{\Lambda} = 0$ and $\text{Im } \Omega|_{\Lambda}$ is a positive volume form.

Let $\mathcal{O}$ be a Hamiltonian isotopy class of closed smoothly embedded positive Lagrangians diffeomorphic to a given manifold $L$. Then $\mathcal{O}$ is naturally a smooth Fréchet manifold. A Riemannian metric $G$ on $\mathcal{O}$ is defined in [24]. It is shown in [30] that the metric $G$ has a Levi-Civita connection and the associated sectional curvature is non-positive. The Levi-Civita connection, which we describe in detail in Section 2.2, gives rise to the notion of geodesics. The geodesic equation is a fully non-linear degenerate elliptic PDE [27]. A satisfactory existence theory for these geodesics would have far-reaching consequences for the uniqueness and existence of special Lagrangian submanifolds in $\mathcal{O}$ [29] as well as rigidity of Lagrangian intersections [27].

In [33], the authors introduce the cylindrical transform for geodesics of positive Lagrangians. A cylinder is a manifold of the form $N \times [0,1]$. The cylindrical transform of a geodesic of positive Lagrangians $(\Lambda_t)_{t \in [0,1]}$ is a one-parameter family of imaginary special Lagrangian cylinders. These cylinders satisfy the elliptic boundary condition that the boundary component corresponding to $N \times \{i\}$ is contained in $\Lambda_t$ for $i = 0,1$. Thus, the degenerate elliptic geodesic equation is transformed to a family of elliptic equations. When the endpoints of the geodesic $\Lambda_0, \Lambda_1$, are Lagrangian spheres intersecting transversally in two points, necessary and sufficient conditions are given for a family of imaginary special Lagrangian cylinders to arise as the cylindrical transform of a geodesic. As a consequence, it is shown that geodesics of such positive Lagrangian spheres persist under small perturbations of the endpoints. The cylindrical transform is naturally defined for geodesics of immersed Lagrangians that are smooth away from a finite number of conical singularities. Conversely, geodesics constructed from the inverse cylindrical transform a priori have a finite number of conical singularities and may not be embedded. We call such geodesics cone-smooth. The precise definition of cone-smooth is recalled in Section 2.4. Unless otherwise mentioned, all geodesics are cone-smooth.

1.2. Statement of results. The present paper proves the a priori existence of families of imaginary special Lagrangian cylinders near intersection points of positive Lagrangian submanifolds of Maslov index 0 or $n$. As explained in Section 1.3 this result is a first step toward the non-perturbative construction of geodesics of positive Lagrangian submanifolds in a general Calabi-Yau manifold. Moreover, we introduce a method for proving $C^{1,1}$ regularity of geodesics at the non-smooth locus. A geodesic is called $C^{1,1}$ if it admits a parameterization by a $C^{1,1}$ family of positive Lagrangian immersions.

Let $\Lambda_0, \Lambda_1 \subset X$ be positive Lagrangian submanifolds and let $N$ be a manifold of dimension $n-1$. We denote by $\mathcal{SLC}(N; \Lambda_0, \Lambda_1)$ the space of imaginary special Lagrangian submanifolds of $X$, perhaps immersed, diffeomorphic to $N \times [0,1]$, such that the boundary corresponding to $N \times \{i\}$ is embedded in $\Lambda_t$ for $i = 0,1$. Positive Lagrangian submanifolds are naturally graded in the sense of [20] [28], so the Maslov index of intersection points is defined absolutely. See Definition 3.12. In Definition 2.36 we recall the notion of regular convergence of a family

$$(Z_s)_{s \in (0,\varepsilon)} \subset \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_1)$$

to an intersection point $q \in \Lambda_0 \cap \Lambda_1$. Roughly speaking, regular convergence is a necessary and sufficient condition for $(Z_s)_s$ to arise from the cylindrical transform.
of a geodesic of open positive Lagrangians in a neighborhood of \( q \). Here, open means not compact and without boundary. Although the metric \( G \) is defined only on a Hamiltonian isotopy class of closed positive Lagrangians, the associated Levi-Civita connection and geodesic equation continue to be well-defined in the open setting as explained in Definition 2.20.

**Theorem 1.1.** Let \( \Lambda_0, \Lambda_1 \subset X \) be smoothly embedded positive Lagrangians intersecting transversally at a point \( q \) with Maslov index 0.

(a) There exists a one-parameter family \( (Z_s)_{s \in (0, \epsilon)} \subset SLC (S^{n-1}; \Lambda_0, \Lambda_1) \) converging regularly to \( q \).

(b) There exist open neighborhoods, \( q \in U_i \subset \Lambda_i, \ i = 0, 1 \), which are connected by a \( C^{1,1} \) geodesic \( (U_t)_{t \in [0,1]} \) of open positive Lagrangians.

**Remark 1.2.** For \( \Lambda_0, \Lambda_1 \subset X \) positive Lagrangians and \( q \in \Lambda_0 \cap \Lambda_1 \), the Maslov index \( m(q; \Lambda_0, \Lambda_1) \) satisfies

\[
m(q; \Lambda_0, \Lambda_1) = n - m(q; \Lambda_1, \Lambda_0).
\]

Also, the time parameter of a geodesic from \( \Lambda_0 \) to \( \Lambda_1 \) can be reversed to obtain a geodesic from \( \Lambda_1 \) to \( \Lambda_0 \). So, Theorem 1.1 holds for \( q \) of Maslov index \( n \) as well as 0.

The family of imaginary special Lagrangian cylinders \( (Z_s)_s \) of Theorem 1.1 is depicted in Figure 1. We call such a family special Lagrangian webbing.

![Figure 1](image-url)
Theorem 1.3. Let $\Lambda_0, \Lambda_1 \in \mathcal{O}$ intersect transversally at exactly two points. Suppose there exists a $C^1$ geodesic $(\Lambda_t)_{t \in [0, 1]}$ between $\Lambda_0$ and $\Lambda_1$. Let $\alpha \in (0, 1)$. Then, there exists a $C^{2,\alpha}$-open neighborhood $\mathcal{Y}$ of $\Lambda_1$ in $\mathcal{O}$ and a weak $C^{1,\alpha}$-open neighborhood $X$ of $(\Lambda_t)_{t \in [0, 1]}$ in $\mathcal{O}$ such that for every $\Lambda \in \mathcal{Y}$ there exists a unique geodesic between $\Lambda_0$ and $\Lambda$ in $X$. This geodesic has regularity $C^{1,1}$ and depends continuously on $\Lambda$ with respect to the $C^{2,\alpha}$ topology on $\mathcal{Y}$ and the strong $C^{1,\alpha}$ topology on $X$.

In [32], there are examples of geodesics of positive Lagrangians of arbitrary dimension, many of which satisfy the hypothesis of Theorem 1.3. However, they are all preserved by an isometric action of $O(n)$ on the ambient manifold $X$. From Theorem 1.3 we obtain the following.

Corollary 1.4. There exist $C^{1,1}$ geodesics of positive Lagrangians in arbitrary dimension that are not invariant under any isometries of the ambient manifold.

To prove Theorems 1.1 and 1.3 we use a blow-up argument, which can be summarized as follows. The critical locus of a geodesic of positive Lagrangians $(\Lambda_t)_t$ is defined by

$$\text{Crit}((\Lambda_t)_t) = \bigcap_t \Lambda_t.$$ 

To each point $p$ of each Lagrangian $\Lambda_i$, we associate a unique tangent cone $T_C p \Lambda_i$. If $\Lambda_i$ is $C^1$ regular at $p$, then $T_C p \Lambda_i = T_p \Lambda_i$. For $q \in \text{Crit}((\Lambda_t)_t)$, Lemma 5.1 shows that the family of tangent cones $T_C q \Lambda_i \subset T_q X$ is a geodesic from $T_q \Lambda_0$ to $T_q \Lambda_1$. Conversely, given a geodesic of positive Lagrangian cones in $T_q X$ from $T_q \Lambda_0$ to $T_q \Lambda_1$, Lemma 6.1 shows how to construct open neighborhoods, $q \in U_i \subset \Lambda_i$, $i = 0, 1$, and a geodesic $(U_t)_{t \in [0, 1]}$ of open positive Lagrangians connecting them. Non-smooth points of a cone-smooth geodesic are always critical points. Lemma 5.3 shows that if the tangent cones at a critical point $q$ of a geodesic are all linear, then the geodesic is of regularity $C^{1,1}$ at $q$. The final ingredient in Theorem 1.1 is the following a priori existence result for geodesics of positive Lagrangian linear subspaces. Let $L^G^+(n)$ denote the Grassmannian of positive Lagrangian linear subspaces in $\mathbb{C}^n$ with the standard Calabi-Yau structure. For $\Lambda \in L^G^+(n)$, the tangent space $T_{\Lambda} L^G^+(n)$ is canonically isomorphic to the space of quadratic forms on $\Lambda$. For $\Lambda_0, \Lambda_1 \in L^G^+(n)$, we abbreviate $m(\Lambda_0, \Lambda_1)$ for the Maslov index of $0 \in \Lambda_0 \cap \Lambda_1$.

Theorem 1.5. Let $\Lambda_0, \Lambda_1 \in L^G^+(n)$ with $m(\Lambda_0, \Lambda_1) = 0$. Then there exists a geodesic in $L^G^+(n)$ between $\Lambda_0$ and $\Lambda_1$ with negative semi-definite derivative. If, in addition, $\Lambda_0$ and $\Lambda_1$ intersect transversally, the geodesic has negative definite derivative.

By Remark 1.2, Theorem 1.5 applies also when $m(\Lambda_0, \Lambda_1) = n$, only the derivative of the geodesic is positive instead of negative. The final ingredient in the proof of Theorem 1.5 is Proposition 4.9 which asserts that geodesics of positive Lagrangian linear subspaces are stable under deformations in the space of geodesics of positive Lagrangian cones.

1.3. Directions for future research.

1.3.1. From local to global. Theorem 1.1 can be viewed as a special Lagrangian analog of Bishop’s result [1] on the existence of a family of holomorphic disks near a point of an $n$-dimensional real submanifold of $\mathbb{C}^n$ with tangent space containing a complex line. The extension of Bishop’s local families of holomorphic disks to global families is the basis for foundational results in symplectic and contact geometry [12] [19]. A fundamental question for future research is the following.

In [32], there are examples of geodesics of positive Lagrangians of arbitrary dimension, many of which satisfy the hypothesis of Theorem 1.3. However, they are all preserved by an isometric action of $O(n)$ on the ambient manifold $X$. From Theorem 1.3 we obtain the following.

Corollary 1.4. There exist $C^{1,1}$ geodesics of positive Lagrangians in arbitrary dimension that are not invariant under any isometries of the ambient manifold.

To prove Theorems 1.1 and 1.3 we use a blow-up argument, which can be summarized as follows. The critical locus of a geodesic of positive Lagrangians $(\Lambda_t)_t$ is defined by

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Theorem 1.5. Let $\Lambda_0, \Lambda_1 \in L^G^+(n)$ with $m(\Lambda_0, \Lambda_1) = 0$. Then there exists a geodesic in $L^G^+(n)$ between $\Lambda_0$ and $\Lambda_1$ with negative semi-definite derivative. If, in addition, $\Lambda_0$ and $\Lambda_1$ intersect transversally, the geodesic has negative definite derivative.

By Remark 1.2, Theorem 1.5 applies also when $m(\Lambda_0, \Lambda_1) = n$, only the derivative of the geodesic is positive instead of negative. The final ingredient in the proof of Theorem 1.5 is Proposition 4.9 which asserts that geodesics of positive Lagrangian linear subspaces are stable under deformations in the space of geodesics of positive Lagrangian cones.
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Question 1.6. Under what conditions can one extend the family of imaginary special Lagrangian cylinders $Z_s$ of Theorem 1.1 to obtain the cylindrical transform of a geodesic of positive Lagrangians between $\Lambda_0$ and $\Lambda_1$?

Beyond the existence problem for geodesics of special Lagrangians, progress on Question 1.6 could potentially lead to progress on the nearby Lagrangian conjecture of Arnol’d as we now explain. Let $M$ be a smooth closed manifold and let $\alpha$ denote the Liouville 1-form on the cotangent bundle $T^* M$. A Lagrangian submanifold $\Lambda \subset T^* M$ is exact if $\alpha|_\Lambda$ is exact. The nearby Lagrangian conjecture is the following.

Conjecture 1.7 (Arnol’d). An exact closed Lagrangian submanifold $\Lambda \subset T^* M$ is Hamiltonian isotopic to the zero section.

Recently, there has been dramatic progress on this conjecture [1, 2, 13, 21, 25, 26], culminating in the result that an exact closed Lagrangian $\Lambda \subset T^* M$ is simply homotopy equivalent to $M$. The full conjecture has been proved for $M = S^2$ in [18]. Nonetheless, in general, we are still far from being able to construct a Hamiltonian isotopy from an exact closed Lagrangian $\Lambda \subset T^* M$ to the zero section.

Suppose we are given a compatible complex structure and a holomorphic volume form on a neighborhood $U$ of the zero section of $T^* M$ such that the zero section is positive Lagrangian. These always exist by the Grauert tube construction [15, 16, 22]. Let $\Lambda_0$ be the zero section of $T^* M$ and let $\Lambda_1$ be an exact closed positive Lagrangian in $U \subset T^* M$ intersecting $\Lambda_0$ transversally. It follows from the above cited results on the nearby Lagrangian conjecture that there exists at least one intersection point of $\Lambda_0$ and $\Lambda_1$ of Maslov index zero. So, Theorem 1.1 applies. A geodesic of positive Lagrangians is in particular a Hamiltonian isotopy, so progress on Question 1.6 gives progress on Conjecture 1.7.

1.3.2. Local questions. In our proof of Theorem 1.5, we use in an essential way the assumption that $m(\Lambda_0, \Lambda_1) = 0$. Thus, the following seems fundamental.

Question 1.8. For general $\Lambda_0, \Lambda_1 \in LG^+(n)$, does there exist a geodesic in $LG^+(n)$ between $\Lambda_0$ and $\Lambda_1$?

A positive answer to Question 1.8 would provide a model at the level of tangent cones for geodesics of positive Lagrangians near critical points of index strictly between 0 and $n$, completing the picture given by Theorem 1.5. However, even when $m(\Lambda_0, \Lambda_1) = 0$, as in Theorem 1.5, it is not clear whether there are more complicated models for tangent cones at critical points.

Question 1.9. For $\Lambda_0, \Lambda_1 \in LG^+(n)$, is it possible that there exists a geodesic from $\Lambda_0$ to $\Lambda_1$ of positive Lagrangian cones that are not linear subspaces?

If the answer to Question 1.9 is the affirmative for $m(\Lambda_0, \Lambda_1) = 0$, Lemma 6.1 gives geodesics of open positive Lagrangians in an arbitrary Calabi-Yau manifold that are not $C^1$.

1.3.3. Regularity limitations. The limited regularity of geodesics of positive Lagrangian submanifolds is reminiscent of the limited regularity of geodesics in the space of Kähler metrics [6, 7, 9, 11] and in the space of positive $(1, 1)$-forms [8, 10]. However, the above results on $C^{1,1}$ regularity are of a different nature than $C^{1,1}$ regularity in the Kähler metric or positive $(1, 1)$-form context. On the one hand, as explained in [27], a geodesic of positive Lagrangian submanifolds can be viewed locally as the graph of the gradient of a potential function analogous to the potential functions for Kähler metrics and positive $(1, 1)$-forms. The $C^{1,1}$ regularity of Theorems 1.1 and 1.3 translates to $C^{2,1}$ regularity for the local potential function unlike the $C^{1,1}$ optimal result for potential functions of geodesics of Kähler
metrics and positive (1,1)-forms. In fact, it may well be possible to obtain even higher regularity in the settings of Theorem 1.1 and 1.3 On the other hand, we consider only the case that all critical points of geodesics are non-degenerate with Maslov index either 0 or n. In greater generality, it is likely that geodesics of positive Lagrangian submanifolds exhibit regularity limits similar to geodesics of Kähler metrics or positive (1,1)-forms.

1.4. Outline. In Section 2 we recall relevant definitions and results from [33]. In Section 3 we study geodesics of linear positive Lagrangians and prove Theorem 1.5. Section 4 shows that geodesics of positive Lagrangian linear subspaces are stable under deformations in the space of geodesics of positive Lagrangian cones. The precise statement is given in Proposition 4.9. In Section 5 we develop a blowup procedure that relates the behavior of a geodesic of positive Lagrangians near a critical point and the associated geodesic of tangent cones. In particular, if the tangent cones are linear, the geodesic is $\mathcal{C}^{1,1}$ regular. The section concludes with the proof of Theorem 1.3. In Section 6 we show how to construct a special Lagrangian webbing from a geodesic of special Lagrangian cones and we prove Theorem 1.1.

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2. Background

This article relies on the notation and results of [33] summarized below.

2.1. Immersed Lagrangians. Let $N,M$ be smooth manifolds, $M$ perhaps with boundary. Denote by $\text{Diff}(M)$ the diffeomorphisms of $M$ preserving each boundary component. That is, if $\varphi \in \text{Diff}(M)$ and $B \subset \partial M$ is a component, then $\varphi(B) = B$.

Definition 2.1. An immersed (resp. embedded) submanifold of $N$ of type $M$ is an equivalence class of immersions (resp. embeddings),

$$K = [f : M \to N],$$

where the equivalence is with respect to the $\text{Diff}(M)$-action: The immersions $f$ and $f'$ are equivalent if there exists $\varphi \in \text{Diff}(M)$ such that $f' = f \circ \varphi$.

We say that $K = [f]$ is free if $f$ has trivial isotropy subgroup. We say that $K$ has boundary if $M$ does. In this case, to each boundary component of $M$ we associate a boundary component of $K$, which is itself an immersed submanifold. A differential form on $K$ is an equivalence class of pairs $\eta = [(f, \tau)]$ where $f$ is a representative of $K$ and $\tau \in \Omega^r(M)$. The pairs $(f, \tau)$ and $(f', \tau')$ are equivalent if there exists $\varphi \in \text{Diff}(M)$ such that $f' = f \circ \varphi$ and $\tau' = f^* \tau$.

Let $(X, \omega)$ be a symplectic manifold of dimension $2n$ and let $L$ be a smooth manifold of dimension $n$. An immersion $f : L \to X$ is Lagrangian if it satisfies $f^* \omega = 0$. The diffeomorphism group $\text{Diff}(L)$ acts on Lagrangian immersions $L \to X$ by composition. An immersed Lagrangian submanifold in $X$ of type $L$ is a submanifold of type $L$ that can be represented by a Lagrangian immersion and thus all representatives are Lagrangian.

Suppose now that $L$ is closed. We let $\mathcal{L}(X, L)$ denote the space of free immersed Lagrangian submanifolds in $X$ of type $L$. The space $\mathcal{L}(X, L)$ is a smooth Fréchet
Choose a horizontal lifting (Ψ : L → X) of (Λ, Ω). Let (Λ, Ω) be an exact isotopy class of positive Lagrangians in X, and let (Λ, Ω) be an oriented smooth Lagrangian submanifold perhaps immersed. Then Ω|Λ is non-vanishing \([17]\). In fact, we have

\[ Ω|Λ = ρ e^{−iθ_Λ} vol_Λ, \]

where \( vol_Λ \) denotes the Riemannian volume form with respect to the Kähler metric, the positive function \( ρ : X \to \mathbb{R} \) is determined by

\[ ρ^2Ω^n = (−1)^{\frac{n(n−1)}{2}} \left( \frac{\sqrt{−1}}{2} \right)^n Ω ∧ \bar{Ω}, \]

and \( θ_Λ : Λ \to S^1 \) is called the phase function of Λ. We say Λ is positive if \( Re Ω|Λ \) is positive. In this case, the phase may be regarded as a real-valued function admitting values in the interval \( (−\frac{π}{2}, \frac{π}{2}) \). We say Λ is special-Lagrangian if \( θ_Λ ≡ 0 \). If we have \( θ_Λ ≡ ±\frac{π}{2} \), we say Λ is imaginary special-Lagrangian.

Let \( L \) be a closed manifold, let \( O \) be an exact isotopy class of positive Lagrangians in \( X \) of type \( L \), and let \( Λ ∈ O \). Positivity gives rise to the isomorphism

\[ T_Λ O ≅ \text{C}^∞(Λ) := \left\{ h ∈ C^∞(Λ) \middle| ∫_Λ h Re Ω = 0 \right\}. \]

Hence, we think of vectors tangent to \( O \) as functions.

**Definition 2.3.** Let \( O \) be as above and let \( (Λ_t)_{t ∈ [0,1]} \) be a smooth path in \( O \). A lifting of \( (Λ_t) \) is a smooth family of Lagrangian immersions, \( (Ψ_t : L → X)_{t ∈ [0,1]} \), such that \( Ψ_t \) represents \( Λ_t \) for \( t ∈ [0,1] \). A lifting \( (Ψ_t) \) is horizontal if it satisfies

\[ i_{\frac{d}{dt}} Ψ_t Re Ω = 0, \quad t ∈ [0,1]. \]

It is shown in \([29]\) that, given a smooth path \( (Λ_t)_{t ∈ [0,1]} \) in \( O \), every representative \( Ψ_0 : L → X \) of \( Λ_0 \) extends uniquely to a horizontal lifting of \( (Λ_t) \). We thus define a connection on \( O \) as follows. Let \( (h_t)_{t ∈ [0,1]} \) be a vector field along a path \( (Λ_t) \). Choose a horizontal lifting \( (Ψ_t) \). Then the covariant derivative of \( (h_t) \) is given by

\[ \frac{D}{dt} h_t := (Ψ_t)_* \left( \frac{d}{dt} h_t \circ Ψ_t \right). \]

The definition is independent of the choice of \( (Ψ_t) \). This connection, to which we refer as the positive Lagrangian connection, is studied thoroughly in \([30]\), where it is shown to be the Levi-Civita connection of the Riemannian metric on \( O \) defined by

\[ (h, k) := ∫_Λ hk Re Ω, \quad h, k ∈ T_Λ O ≅ \text{C}^∞(Λ). \]
The main objects of the present work are geodesics with respect to the positive Lagrangian connection. Below, we extend the notion of geodesics to isotopy classes of Lagrangians that may be open and/or non-smooth. See Definition 2.20.

2.3. Cone smooth maps. The following notations and definitions are taken from Section 3 of [33].

Notation 2.4. Let $M$ be a smooth manifold and let $S \subset M$ be a finite subset. We denote by $\pi : \tilde{M}_S \to M$ the oriented blowup of $M$ at $S$. For $p \in S$, we denote by $E_p = \pi^{-1}(p)$ the exceptional sphere over $p$. For a detailed account of the definition of the oriented blowup, see [33] Definition 3.1.

Definition 2.5. Let $M$ and $N$ be smooth manifolds, let $p \in M$, and let $\Psi : M \to N$ be continuous. Let $\tilde{M}_p, E_p$ and $\pi : \tilde{M}_p \to M$ be as in Notation 2.4.

1. The map $\Psi$ is said to be cone-smooth at $p$ if there exists an open $E_p \subset \tilde{U} \subset \tilde{M}_p$ such that the composition $\Psi \circ \pi|_{\tilde{U}} : \tilde{U} \to N$ is smooth.
2. Suppose $\Psi$ is cone-smooth at $p$. The cone-derivative of $\Psi$ at $p$ is the unique map

$$d\Psi_p : T_p M \to T_{\Psi(p)} N$$

satisfying the equality

$$d(\Psi \circ \pi)_{\tilde{p}} = d\Psi_p \circ d\pi_{\tilde{p}}$$

for $\tilde{p} \in E_p$. One verifies that the cone-derivative is well-defined and homogeneous of degree 1. Also, the restricted map $d\Psi_p|_{T_p M \setminus \{0\}}$ is smooth. Nevertheless, the cone-derivative is not linear in general.

3. Suppose $\Psi$ is cone-smooth at $p$. We say $\Psi$ is cone-immersive at $p$ if the restricted map

$$d\Psi_p|_{T_p M \setminus \{0\}} : T_p M \setminus \{0\} \to T_{\Psi(p)} N$$

is a smooth immersion. In particular, in this case we have $d\Psi_p(v) \neq 0$ for $0 \neq v \in T_p M$.

Definition 2.6. Let $M$ and $N$ be smooth manifolds. Let $S \subset M$ be a finite subset and $\Psi : M \to N$ a continuous map.

(a) We say the pair $(\Psi, S)$ is cone-smooth if $\Psi$ is smooth away from $S$ and cone-smooth at every element of $S$. That is, the composition $\Psi \circ \pi : \tilde{M}_S \to N$ is cone smooth. For $\Theta$ a manifold with corners, a family of maps $((\Psi_t, S))_{t \in \Theta}$ is cone smooth if the composition $\Psi_t \circ \pi : \tilde{M}_S \to N$ gives a smooth map $\tilde{M}_S \times \Theta \to N$.

(b) We say the pair $(\Psi, S)$ is a cone-immersion from $(M, S)$ to $N$ if $\Psi$ is a smooth immersion away from $S$ and cone-immersive at every element of $S$.

(c) Suppose $N = M$ and $\Psi(p) = p$ for all $p \in S$. The pair $(\Psi, S)$ is a cone-diffeomorphism of $(M, S)$ if $\Psi$ is a smooth diffeomorphism away from $S$ and for $p \in S$, the cone derivative $d\Psi_p|_{T_p M \setminus \{0\}} : T_p M \setminus \{0\} \to T_p M$ is a diffeomorphism onto $T_p M \setminus \{0\}$. We let $\text{Diff}(M, S)$ denote the group of cone-diffeomorphisms of $(M, S)$ that act trivially on the set of connected components.

(d) Let the diffeomorphism group $\text{Diff}(M, S)$ act on cone-immersions from $(M, S)$ to $N$ by composition. A cone-immersed submanifold of $N$ of type $(M, S)$ is an orbit of the $\text{Diff}(M, S)$-action.

(e) Suppose $M$ is orientable and let $\text{Diff}^+(M, S) \circ \text{Diff}(M, S)$ denote the normal subgroup of orientation preserving cone-smooth diffeomorphisms. An orientation on a cone-immersed submanifold $K$ of $N$ of type $(M, S)$ is an equivalence class of pairs $(O, C)$ where $O$ is an orientation of $M$ and $C$
is a $\text{Diff}^+(M, S)$ orbit inside the $\text{Diff}(M, S)$ orbit $K$. There is a natural $\text{Diff}(M, S)/\text{Diff}^+(M, S)$ action on such pairs and this gives rise to the desired equivalence relation.

**Definition 2.7.** Let $K = [(\Psi : M \to N, S)]$ be a cone-immersed submanifold of type $(M, S)$. We let $\text{im}(K)$ denote the image of $K$ in $M$. That is,

$$\text{im}(K) = \Psi(M).$$

A point $p$ in $K$ is an equivalence class of pairs $((\chi, S), q)$, where $(\chi : M \to N, S)$ is a representative of $K$ and $q \in M$. We let $\text{im}(p)$ denote the image of $p$ in $N$. That is,

$$\text{im}(p) = \chi(q)$$

for $((\chi, S), q)$ a representative of $p$. The cone-immersed submanifold $K$ thus has a well-defined cone locus

$$K^C := \{[((\Psi, S), c)] | c \in S\}.$$

A cone point is an element of the cone locus. We define the tangent cone of $K$ at a point $p = [((\Psi, S), q)]$ to be the cone-immersed submanifold

$$TC_p K := [(d\Psi_c : T_c M \to T_{\text{im}(p)} N, \{0\})]$$

of $T_{\text{im}(p)} N$. The tangent cone $TC_p K$ is indeed a cone, that is, invariant under scalar multiplication. Moreover, it is independent of the choice of $\Psi$. If $\Psi$ is smooth at $q$, then $TC_p K$ is smoothly embedded and recovers the usual notion of tangent space. For $h : V \to W$ a homogeneous map of real vector spaces that does not vanish on $V \setminus \{0\}$, we denote by $\mathbb{P}^+(h) : \mathbb{P}^+(V) \to \mathbb{P}^+(W)$ the oriented projectivization. We define the projective tangent cone of $K$ at $p$ by

$$\mathbb{P}^+(TC_p K) := [\mathbb{P}^+(d\Psi_c) : \mathbb{P}^+(T_c M) \to \mathbb{P}^+(T_{\text{im}(p)} N)].$$

This is a smooth immersed sphere in $\mathbb{P}^+(T_{\text{im}(p)} N)$.

A function on $K$ is an equivalence class of pairs $((\Psi, S), f)$, where $(\Psi, S)$ is a representative of $K$ and $f$ is a function on $M$. We say the function $h = [((\Psi, S), f)]$ is cone-smooth at the point $p = [((\Psi, S), q)]$ if $f$ is cone-smooth at $q$. In this case $h$ has a well-defined cone-derivative $dh_p : TC_p K \to \mathbb{R}$, which is a degree-1 homogeneous function.

Let $(\Psi : M \to N, S)$ be a cone-smooth map and let $p \in S$. Recall that the cone derivative $d\Psi_p : T_p M \to T_{\Psi(p)} N$ is homogeneous of degree 1 and $d\Psi_p |_{T_p M \setminus \{0\}}$ is smooth. It follows that for $0 \neq v \in T_p M$ and $\lambda > 0$, we have $d(d\Psi_p)_v = d(d\Psi_p)_0$ under the canonical identification $T_p \simeq T_{\Psi(p)} N \simeq T_{\Psi(p)} M$. Thus, for $\tilde{p} = [v] \in \mathbb{P}^+(T_p M)$, we define

$$d(d\Psi_p)_{\tilde{p}} := d(d\Psi_p)_v : T_p M \to T_{\Psi(p)} N.$$

The following is Lemma 3.16 from 

**Lemma 2.8.** Let $(\Psi : M \to N, S)$ be a cone-smooth map. Consider the map $d(\Psi|_{M \setminus S}) : TM|_{M \setminus S} \to \Psi^* TN|_{M \setminus S}$. Pulling back by $\pi$ gives a map

$$\pi^* d(\Psi|_{M \setminus S}) : \pi^* TM|_{\tilde{M}_S} \to \pi^* \Psi^* TN|_{\tilde{M}_S}.$$

This map extends uniquely to a map of bundles $\tilde{d}\psi : \pi^* TM \to \pi^* \Psi^* TN$. Moreover, for $p \in S$ and $\tilde{p} \in E_p$, we have

$$\tilde{d}\psi_{\tilde{p}} = d(d\Psi_p)_{\tilde{p}}.$$  

In particular, if $\Psi$ is a cone-immersion, then $\tilde{\psi}$ is an injective map of vector bundles.
Define 2.14. Let \( S \subset M \) be a finite subset. The blowup tangent bundle of \((M, S)\) is the bundle \( \widetilde{T}_M S := \pi^*T_M M \to \tilde{M}_S \). When clear from the context, we may omit the subscript \((\cdot)_S\). Let \( (\Psi : M \to N, S) \) be a cone-smooth map. The blowup differential of \( \Psi \) is the map
\[
\widetilde{d\Psi} : \widetilde{T}_M S \to \pi^*\Psi^*TN
\]
given by Lemma 2.8.

Let \( M \) be a smooth manifold and let \( S \subset M \) be a finite subset and let \( \pi : \tilde{M}_S \to M \) denote the blowup projection. Given a differential form \( \alpha \) on \( M \) we can pull-back \( \alpha \) as a section of \( \Lambda^*(T^*M) \) to obtain a section of \( \pi^*\Lambda^*(T^*M) \cong \Lambda^*(\tilde{T}M_S) \). We denote this pull-back by \( \pi^{-1}\alpha \). Observe that this pull-back is different from the pull-back of \( \alpha \) as a differential form, \( \pi^*\alpha \), which would be a section of \( \Lambda^*(\tilde{T}M_S) \). The following are taken from Definition 3.18, Remark 3.19, Remark 3.26, Definition 3.27 and Definition 3.28 of [33].

**Definition 2.13.** Let \( K = [([\Psi : M \to N, S])] \) be a cone-immersed submanifold of type \((M, S)\). A cone-smooth differential form on \( K \) is an equivalence class \( \tau = [([\chi, S, \alpha])] \) where \( (\chi, S) \) represents \( K \) and \( \alpha \) is a cone-smooth differential form on \((M, S)\). Two pairs are equivalent if they belong to the same orbit of the \( \text{Diff}(M, S) \) action given by Remark 2.12. We may write \( \alpha = \Psi^*\tau \). Given a smooth form \( \eta \) on \( N \), the restriction to \( K \) is the cone-smooth form given by
\[
\eta|_K := [([\Psi, S, \Psi^*\eta])].
\]
We say that \( \tau \) is closed if \( \alpha \) is. We say that \( \tau \) vanishes at the cone locus if \( \alpha \) vanishes on \( \partial M_S \).

**Definition 2.14.** Let \( K = [([\Psi : M \to N, S])] \) be a cone-immersed submanifold, let \( p = [([\Psi, S], q)] \) be a cone point, and let \( \tilde{p} = [([\tilde{p}, d\Psi q], \tilde{q})] \in \mathbb{P}^*(T\tilde{C}_p K) \). The tangent space of \( K \) at \( \tilde{p} \) is defined by
\[
T_p K := d\Psi_{\tilde{q}}(\tilde{T}\tilde{M}_{\tilde{q}}) \subset T\tilde{p}_N,
\]
which is independent of the choice of representatives. At a smooth point \( p = [([\Psi, S], q)] \) of \( K \), we define the tangent space \( T_p K := d\Psi_q(T_q M) \subset \text{im}(p)N \).
The following are Definition 3.30, Remark 3.31 and Definition 3.32 from [33].

**Definition 2.15.**

(1) Let \((f: M \to \mathbb{R}, S)\) be a cone-smooth function, and let \(p \in S\). The point \(p\) is said to be a **critical point** of \(f\) if the cone-derivative \(df_p: T_pM \to \mathbb{R}\) vanishes identically.

(2) Let \(K = [((\Psi: M \to N, S))\) be a cone-immersed submanifold, let \(h = [((\Psi, S), f)]\) be a cone-smooth function on \(K\), and let \(p = [((\Psi, S), q)]\) be a cone point. The point \(p\) is a **critical point** of \(h\) if \(q\) is a critical point of \(f\).

**Remark 2.16.** Let \(S \subset M\) be a finite subset and let \((f: M \to \mathbb{R}, S)\) be a cone-smooth function. It follows from Lemma 2.8 that in the situation of part (1) of the preceding definition, if \(p\) is a critical point of \(f\), then \(df_\sim p = 0\) for all \(\sim p \in E_p\). The analogous statement holds in the situation of part (2).

**Definition 2.17.** Let \((f: M \to \mathbb{R}, S)\) be a cone-smooth function, and let \(p \in S\) be a critical point of \(f\).

(1) The **cone-Hessian** of \(f\) at \(p\) is the map \(\nabla df: T_pM \to T_pM^*\), smooth away from 0 and homogeneous of degree 1, defined as follows. By Remark 2.16, the blowup differential \(\sim df\) vanishes on the exceptional sphere \(E_p \subset \sim M_S\). So, the restriction of the second covariant derivative \(\nabla \sim df \in \text{Hom}(\sim M_S, \sim M_S^*)\) to \(E_p\) is independent of the choice of connection. Moreover, \(\nabla \sim df\) vanishes on \(TE_p \subset \sim M_S|_{E_p}\). Recall that a vector \(0 \neq v \in T_pM\) gives rise to a point \([v] \in \mathbb{P}(T_pM) \simeq E_p\). For \(v \in T_pM\), and \(\tilde{v} \in T_{[v]}\sim M_S\) such that \(d\pi_{[v]}(\tilde{v}) = v\), we define

\[
\nabla v df := \nabla \sim df \in \left(\sim M_S^*\right)_{[v]} = T_p^*M.
\]

(2) The critical point \(p\) is said to be **degenerate** if there exists a tangent vector \(0 \neq v \in T_pM\) with \(\nabla v df = 0\).

The following is Definition 3.3 from [33].

**Definition 2.18.** Let \(M\) be a smooth manifold, let \(p \in M\). Let \(\sigma\) be a smooth section of the quotient map \(T_pM \setminus \{0\} \to \mathbb{P}(T_pM) \simeq S^{n-1}\). A **polar coordinate map** centered at \(p\) associated with \(\sigma\) is a smooth map

\[
\kappa: S^{n-1} \times [0, \epsilon) \to M
\]
satisfying the following.

(1) \(\kappa|_{S^{n-1} \times \{0\}}\) is the constant map to \(p\).

(2) \(\kappa|_{S^{n-1} \times (0, \epsilon)}\) is an open embedding.

(3) For \(r \in S^{n-1}\),

\[
\frac{\partial \kappa}{\partial s}(r, 0) = \sigma(r).
\]

An elementary argument shows that for every \(\sigma\) there exist many polar coordinate maps. We may sometimes speak of a polar-coordinate map \(\kappa\) without mentioning \(\sigma\) explicitly. In this case, we refer to the section \(\sigma\) determined by condition (3) as the section associated with \(\kappa\).

The following is Lemma 3.36 from [33].
Lemma 2.19. Let $M$ be a smooth manifold of dimension $m + 1$ and let $p \in M$. Let $h : M \to \mathbb{R}$ be cone-smooth at $p$ such that $p$ is a non-degenerate critical point and an extremum point of $h$. Then there exist a positive $\epsilon$ and a polar coordinate map $\kappa : S^m \times [0, \epsilon) \to M$ centered at $p$ such that for each $s \in (0, \epsilon)$ the restricted map $\kappa|_{S^m \times \{s\}}$ parameterizes a level set of $h$.

2.4. Lagrangians with cone points. Let $(X, \omega)$ be a symplectic manifold and let $L$ be a smooth manifold. A cone-immersed submanifold $\Lambda = [(\Psi : L \to X, S)]$ is said to be Lagrangian if it is Lagrangian in the smooth locus. It follows in this case that, for a cone point $p \in \Lambda$ and $\tilde{p} \in \mathbb{P}^+ (TC_p \Lambda)$, the tangent space $T_p \Lambda$ is also Lagrangian.

We wish to study paths of cone-immersed Lagrangians with static cone locus. For a finite subset $S \subset L$ and an extremum point of $h$. Let $(\Lambda_t)_{t \in [0, 1]}$ be a family of cone-immersions, $((\Psi_t : L \to X, S))_{t \in \Theta}$, such that $(\Psi_t, S)$ represents $\Lambda_t$ for $t \in \Theta$. The family $(\Lambda_t)_{t \in \Theta}$ is smooth if it admits a smooth lifting, that is, if the family of maps $\tilde{\Psi}_t \circ \pi$ is smooth, where $\pi : \tilde{L}_S \to L$ is the blowup projection.

For a path $(\Lambda_t)_{t \in [0, 1]}$ in $L(X, L; S, C_0)$, the time derivative $\frac{d}{dt} \Lambda_t$ is a closed cone-smooth 1-form on $\Lambda_t$ that vanishes at the cone locus. See Section 3.2 of [33] for details of how the argument in the smooth case generalizes to the cone-smooth case. For a cone smooth immersed submanifold $\Lambda$ of $X$, let $C^\infty(\Lambda)$ denote the space of cone-smooth functions on $\Lambda$. A path $(\Lambda_t)_{t \in [0, 1]}$ in $L(X, L; S, C_0)$ is said to be exact if, for $t \in [0, 1]$, we have

$$\frac{d}{dt} \Lambda_t = dh_t$$

for some $h_t \in C^\infty(\Lambda_t)$.

Let $(X, \omega, J, \Omega)$ be Calabi-Yau and let $L$ be a smooth manifold. A cone-immersed Lagrangian $\Lambda = [(\Psi : L \to X, S)]$ is said to be positive Lagrangian if it is positive Lagrangian in the smooth locus and, in addition, the tangent space $T_p \Lambda \subset T_{\text{im}(p)} X$ is positive Lagrangian for $p \in \Lambda^C$ and $\tilde{p} \in \mathbb{P}^+ (TC_p \Lambda)$. Let $C_0 \subset X$ be finite, let $O \subset L(X, L; S, C_0)$ be an exact isotopy class of cone-immersed positive Lagrangians, and let $(\Lambda_t)_{t \in [0, 1]}$ be a path in $O$. A lifting $(\Psi_t : L \to X)$ of $(\Lambda_t)_{t}$ is said to be horizontal if it is horizontal in the smooth locus. Below is the definition of geodesics in its general form, which is taken from Definition 3.39 of [33].

Definition 2.20. Let $(X, \omega, J, \Omega)$ be a Calabi-Yau manifold, let $L$ be a connected smooth manifold, not necessarily closed, and let $S \subset L$ be a finite subset. Let $C_0 \subset X$ be finite, let $O \subset L(X, L; S, C_0)$ be an exact isotopy class of cone-immersed Lagrangians, and let $(\Lambda_t)_{t \in [0, 1]}$ be a path in $O$. The path $(\Lambda_t)_{t}$ is a geodesic if it admits a horizontal lifting $((\Psi_t, S))_{t \in [0, 1]}$ and a family of functions $h_t \in C^\infty(\Lambda_t)$ satisfying

$$\frac{d}{dt} \Lambda_t = dh_t, \quad \frac{d}{dt} (h_t \circ \Psi_t) = 0.$$

We call the family $(h_t)_{t \in [0, 1]}$ the Hamiltonian or the derivative of the geodesic. We also call the time independent function $h = h_t \circ \Psi_t : L \to \mathbb{R}$ the Hamiltonian with respect to the horizontal lifting $(\Psi_t)_{t}$. Observe that $h_t = [(\Psi_t, h)]$. If $L$ is not compact, the Hamiltonian is only well-defined up to a time-independent constant. If $C_0$ is empty, we say $(\Lambda_t)_{t}$ is a smooth geodesic or geodesic of smooth Lagrangians.

Recalling that $\text{im}(\Lambda_t)$ denotes the image of $\Lambda_t$ in $X$, we define the critical locus of the geodesic $(\Lambda_t)_{t}$ by

$$\text{Crit}((\Lambda_t)_{t}) = \bigcap_{t \in [0, 1]} \text{im}(\Lambda_t) \subset X.$$
Observe that $C_0 \subset \text{Crit}((\Lambda_t)_t)$ but the opposite inclusion need not hold. We say the cone-smooth geodesic $(\Lambda_t)_t$ is of class $C^{k,\alpha}$ if it admits a lifting $(\Psi_t : L \to X)_t$ that belongs to the space $C^{k,\alpha}(L \times [0,1], X)$.

**Notation 2.21.** Suppose $(\Lambda_t)_{t \in [0,1]}$ is a geodesic with Hamiltonian $(h_t)_t$ and $\Lambda_0, \Lambda_1$, are embedded. Let $(\Psi_t : L \to X, S)_t$ be a horizontal lifting of $(\Lambda_t)_t$. Then, for each $q \in \text{Crit}((\Lambda_t)_t)$ there exists a unique $p \in L$ such that $\Psi_t(p) = q$ for $t \in [0,1]$. Thus, we let $\hat{q}_t$ denote the point of $\Lambda_t$ given by

$$
\hat{q}_t = [(\Psi_t, p)].
$$

**Remark 2.22.** In the setting of Notation 2.21, it follows that $\hat{q}_t$ is a critical point of the Hamiltonian function $h_t$.

The following is Lemma 5.10 of [33].

**Lemma 2.23.** Suppose $\Lambda_0$ and $\Lambda_1$ intersect transversally at $q \in C_0$. Then, $\hat{q}_t$ is a non-degenerate critical point of $h_t$, $t \in [0,1]$.

### 2.5. Special Lagrangian cylinders and geodesics.

In this section we recall results from [33] concerning special Lagrangian cylinders, families thereof and their relation to geodesics of positive Lagrangians.

**Definition 2.24.** Let $(X, \omega, J, \Omega)$ be a Calabi-Yau manifold of complex dimension $n$, let $\Lambda_0, \Lambda_1 \subset X$ be smoothly embedded positive Lagrangians, and let $N$ be a connected smooth manifold of dimension $n-1$. Write $L := N \times [0,1]$. A Lagrangian cylinder of type $L$ between $\Lambda_0$ and $\Lambda_1$ is an immersed Lagrangian $Z = [\chi : L \to X]$ satisfying the following conditions.

1. The Lagrangian immersion $\chi$ carries the boundary components $N \times \{i\}$, $i = 0, 1$, to $\Lambda_i$, $i = 0, 1$, respectively.
2. For $i = 0, 1$ and $p \in N \times \{i\}$ we have $d\chi_p(T_p L) \neq T_{\chi(p)}\Lambda_i$.

We let $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$ denote the space of Lagrangian cylinders of type $L$ between $\Lambda_0$ and $\Lambda_1$ and $\mathcal{SCLC}(N; \Lambda_0, \Lambda_1)$ the subspace of Lagrangian cylinders. In the present work, we assume all Lagrangian cylinders have embedded boundary. That is, for $Z = [\chi : L \to X] \in \mathcal{LC}(N; \Lambda_0, \Lambda_1)$, we assume $\chi|_{\partial L}$ is an embedding. We denote by $\mathcal{SCLC}(\Lambda_0, \Lambda_1)$ the union of the spaces $\mathcal{SCLC}(N; \Lambda_0, \Lambda_1)$ as $N$ varies.

When $N$ is closed, we describe the natural Fréchet structure of $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$ via Lemmas 2.25 and 2.26 below. Lemma 2.25 is an adaptation of the Weinstein neighborhood theorem to Lagrangian cylinders proved in greater generality in Lemma 2.8 of [33]. For a smooth manifold $M$ and a submanifold $Q \subset M$, we let $\nu_Q \subset T^*M$ denote the conormal bundle of $Q$.

**Lemma 2.25.** Let $X$ be Calabi-Yau and let $\Lambda_0, \Lambda_1 \subset X$ be smoothly embedded positive Lagrangians intersecting transversally. Let $N$ be a closed connected smooth manifold, write $L := N \times [0,1]$, and let $Z = [\chi : L \to X] \in \mathcal{LC}(N; \Lambda_0, \Lambda_1)$. Identify $L$ with the zero section in $T^*L$. Then there exist an open neighborhood $L \subset V \subset T^*L$ and a local symplectomorphism $\varphi : V \to X$ with the following properties.

1. $\varphi|_L = \chi$.
2. For $i = 0, 1$, a point $p \in N \times \{i\}$ and a covector $\xi \in T^*_p L$, we have $\varphi(p, \xi) \in \Lambda_i \iff \xi \in \nu_{N \times \{i\}}$.

A pair $(V, \varphi)$ with the above properties is called a Weinstein neighborhood of $Z$ compatible with $\Lambda_0$ and $\Lambda_1$.

The following is Lemma 4.3 of [33].
Lemma 2.26. Let $N$ be a connected smooth manifold and let $\alpha$ be a closed 1-form on the cylinder $N \times [0, 1]$ annihilating the boundary component $N \times \{0\}$. Then $\alpha$ is exact.

We continue with the above notation. For a smooth manifold $M$ with boundary, let $\Omega_B^1(M)$ denote the Fréchet space consisting of closed 1-forms on $M$ annihilating the boundary. Given a compact Lagrangian cylinder $Z \in \mathcal{LC}(N; \Lambda_0, \Lambda_1)$, any Weinstein neighborhood of $Z$ compatible with $\Lambda_0$ and $\Lambda_1$ identifies an open neighborhood $Z \in U \subset \mathcal{LC}(N; \Lambda_0, \Lambda_1)$ with an open neighborhood $0 \in V \subset \Omega_B^1(Z)$. The space $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$ is thus a smooth Fréchet manifold. For a cylinder $Z$ as above, let $C_i$, $i = 0, 1$, denote the boundary components corresponding to $N \times \{i\}$. Let $C_{\text{COB}}^\infty(Z)$ denote the space of smooth functions on $Z$ which vanish on $C_0$ and are constant on $C_1$. By Lemma 2.26, an open neighborhood of $Z$ in $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$ is diffeomorphic to a neighborhood of 0 in $C_{\text{COB}}^\infty(Z)$. We remark that the identification $T_Z \mathcal{LC}(N; \Lambda_0, \Lambda_1) \cong C_{\text{COB}}^\infty(Z)$ does not depend on the choice of a Weinstein neighborhood.

For an imaginary special Lagrangian cylinder $Z$, define a differential operator

$$\Delta_\rho : C^\infty(Z) \to C^\infty(Z), \quad u \mapsto *\delta(\rho * du),$$

where $*$ denotes the Hodge star operator of the Kähler metric and $\rho$ is the function given by (1). The following lemma is a restatement of Lemmas 4.5 and 4.6 of [33]. Let $C^\infty(L; \partial L)$ denote the space of smooth functions on $L$ which vanish on the boundary.

Lemma 2.27. Let $N$ be a closed connected manifold, let $Z = [f : L \to X]$ be an immersed imaginary special Lagrangian of type $L := N \times [0, 1]$, and let $\Delta_\rho$ be as in (1).

(a) Let $f_t : L \to X$, $t \in (-\epsilon, \epsilon)$, be a smooth family of Lagrangian immersions with $f_0 = f$. Write $v := \frac{d}{dt}\bigg|_{t=0} f_t$ and suppose we have $i_\cdot \omega = du$ for some $u \in C^\infty(L)$. Then

$$\frac{d}{dt}\bigg|_{t=0} (*f_t^* \text{Re} \Omega) = \Delta_\rho u.$$

(b) The linear map

$$\Delta_\rho|_{C^\infty(L; \partial L)} : C^\infty(L; \partial L) \to C^\infty(L)$$

is an isomorphism. In particular, the intersection $\ker \Delta_\rho \cap C^\infty_{\text{COB}}(L)$ is 1-dimensional.

Definition 2.28. Let $Z = [f : N \times [0, 1] \to X]$ be an immersed imaginary special Lagrangian cylinder. For $i = 0, 1$, let $C_i$ denote the boundary component of $Z$ corresponding to $N \times \{i\}$.

(a) The fundamental harmonic of $Z$ is the unique function $\sigma \in C^\infty_{\text{COB}}(Z)$ such that

$$\Delta_\rho \sigma = 0, \quad \sigma|_{C_1} \equiv 1.$$

(b) We say that $Z$ has regular harmonics if the fundamental harmonic of $Z$ has no critical points.

(c) The immersion $f$ representing $Z$ is said to be adapted to the harmonics of $Z$ if

$$\sigma \circ f(p, t) = t, \quad (p, t) \in N \times [0, 1].$$

In this case, it follows immediately that $Z$ has regular harmonics.

The following is Proposition 4.7 of [33], which is an adaptation of McLean’s result on the deformation theory of closed special Lagrangians [23].
Proposition 2.29. Let Λ₀, Λ₁ ⊂ X be smoothly embedded positive Lagrangians intersecting transversally, and let N be a closed connected smooth manifold. Then the space $SLC(N; Λ₀, Λ₁)$ is a smooth manifold of dimension 1. For an imaginary special Lagrangian cylinder $Z \in SLC(N; Λ₀, Λ₁)$ we have

$$T_Z SLC(N; Λ₀, Λ₁) = \ker \Delta_\rho \cap C^\infty_{COB}(Z).$$

By Proposition 2.29 imaginary special Lagrangian cylinders naturally appear in 1-parameter families. The following proposition shows that 1-parameter families of imaginary special Lagrangian cylinders arise naturally from geodesics. The proposition is an immediate corollary of Propositions 5.1 and 5.3 of [33].

Proposition 2.30. Let $(X, \omega, J, \Omega)$ be a Calabi-Yau manifold.

(a) Let $(\Lambda_t)_{t \in [0,1]}$ be a geodesic of positive Lagrangians in X with Hamiltonian $(h_t)_t$. Let $(\Psi_t : L \to X)_{t \in [0,1]}$ be a horizontal lifting of $(\Lambda_t)_t$, and let $h : L \to \mathbb{R}$ denote the Hamiltonian of the geodesic with respect to $(\Psi_t)_t$, that is, $h := h_t \circ \Psi_t$. For $c \in \mathbb{R}$, let

$$L_c := (h^{-1}(c) \setminus \text{Crit}(h)) \times [0,1].$$

Then, the mapping

$$\Phi_c : L_c \to X, \quad (p, t) \mapsto \Psi_t(p),$$

is an imaginary special Lagrangian immersion.

(b) The mapping $\Phi_c$ is adapted to the harmonics of the cylinder $Z = [\Phi_c]$. In particular, $Z$ has regular harmonics.

The following is an amalgamation of Definitions 5.2 and 1.4 of [33].

Definition 2.31. Let $(X, \omega, J, \Omega)$ be a Calabi-Yau manifold and let $(\Lambda_t)_{t \in [0,1]}$ be a geodesic of positive Lagrangians in X with derivative $(h_t)_t$. Let $(\Psi_t : L \to X)_{t \in [0,1]}$ be a horizontal lifting of $(\Lambda_t)_t$, and let $h : L \to \mathbb{R}$ denote the Hamiltonian of the geodesic with respect to $(\Psi_t)_t$.

(1) For $c \in \mathbb{R}$, the cylinder of $c$ level sets associated to the geodesic is the immersed imaginary special Lagrangian cylinder represented by the map $\Phi_c$ of Proposition 2.30(a).

(2) The cylindrical transform of the geodesic $(\Lambda_t)_{t \in [0,1]}$ is the subset of the space of imaginary special Lagrangian cylinders $SLC(\Lambda_0, \Lambda_1)$ parameterized by the family of imaginary special Lagrangian immersions $\Phi_c : L_c \to X$ from Proposition 2.30(a) for $c \in \mathbb{R}$ such that $L_c \neq \emptyset$.

Let $\Lambda_0, \Lambda_1 \subset X$ be Lagrangian submanifolds and let $N$ be a connected closed manifold of dimension $n - 1$. Let $(Z_s)_{s \in [s_0, s_1]}$ be a family of Lagrangian cylinders in $LC(N; \Lambda_0, \Lambda_1)$. For $s \in [s_0, s_1]$, write

$$h_s := \frac{d}{ds} Z_s \in C^\infty_{COB}(Z_s).$$

For $i = 0, 1$, let $C_{i,s}$ denote the boundary component of the cylinder $Z_s$ corresponding to $N \times \{i\}$ and let $A_s \in \mathbb{R}$ be the unique constant such that

$$h_s|_{C_{i,s}} \equiv A_s.$$

The following is Definition 4.11 of [33].

Definition 2.32. The relative Lagrangian flux of the family $(Z_s)_{s \in [s_0, s_1]}$ is given by

$$\text{RelFlux} \left((Z_s)_{s \in [s_0, s_1]} \right) := -\int_{s_0}^{s_1} A_s ds.$$
More generally, if $I \subset \mathbb{R}$ is an interval, possibly open or half open, with endpoints $a < b$, and $(Z_s)_{s \in I}$ is a path in $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$, we write
\[ \text{RelFlux} \left( (Z_s)_{s \in I} \right) := \lim_{s_0 \to a} \lim_{s_1 \to b} \text{RelFlux} \left( (Z_s)_{s \in [s_0, s_1]} \right) \]
whenever the limit exists.

\textbf{Remark 2.33.}

1. It is clear from the definition that $\text{RelFlux}((Z_s)_{s \in I})$ is a symplectic invariant.
2. The relative Lagrangian flux $\text{RelFlux}((Z_s)_{s \in [s_0, s_1]})$ depends only on the homotopy class of the path $(Z_s)_s$ relative to its endpoints. See Remark 4.12 of [33].

The following is a reformulation of Lemma 5.5 of [33]. It asserts that the Hamiltonian of a geodesic can be recovered from the relative Lagrangian flux of the associated cylinders of $c$-level sets.

\textbf{Lemma 2.34.} Let $(\Lambda_t)_{t \in [0,1]}$ be a geodesic with Hamiltonian $(h_t)_{t}$ and assume the functions $h_t$ are proper. For $c$ in the image of $h_t$, let $Z_c$ denote the associated cylinder of $c$ level sets. Since the functions $h_t$ are proper, the cylinder $Z_c$ is compact when $c$ is a regular value of $(h_t)_t$. Let $c_0 < c_1 \in \mathbb{R}$ be such that the interval $(c_0, c_1)$ consists of regular values of $(h_t)_t$. For $b_0 < b_1 \in (c_0, c_1)$ we have
\[ \text{RelFlux} \left( (Z_c)_{c \in [b_0, b_1]} \right) = b_1 - b_0. \]

Recall that the Euler vector field on a real vector space is the radial vector field that integrates to rescaling by $e^t$. The following is Lemma 4.15 from [33].

\textbf{Lemma 2.35.} Equip $\mathbb{C}^n$ with the standard Calabi-Yau structure, let $\Lambda_0, \Lambda_1 \subset \mathbb{C}^n$ be positive Lagrangian linear subspaces, and let $Z \in \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_1)$. Then $Z$ has regular harmonics if and only if $Z$ is nowhere tangent to the Euler vector field.

The following simplified notion of ends for 1-dimensional manifolds is given in Section 4.3 of [33]. Let $C$ be a connected non-compact 1-dimensional manifold. That is, $C$ is a curve diffeomorphic to the real line. A ray in $C$ is a connected open proper subset $U \subset C$ with non-compact closure. Two rays, $U, V \subset C$ are said to be equivalent if $U \subset V$ or $V \subset U$. Finally, an end is an equivalence class of rays. Every curve $C$ as above has exactly two ends. The following is taken from Definition 4.17 of [33].

\textbf{Definition 2.36.}

1. Let $U \subset \mathcal{SLC}(N; \Lambda_0, \Lambda_1)$ be open and connected. An \textit{interior-regular parameterization} of $U$ is a smooth immersion $\Phi : N \times [0,1] \times (a,b) \to X$ satisfying the following conditions:
   a. The restriction of $\Phi$ to a boundary component $\Phi|_{N \times \{1\} \times (a,b)}$ is an embedding for $i = 0, 1$.
   b. For $s \in (a,b)$, the restricted immersion $\Phi_s := \Phi|_{N \times [0,1] \times \{s\}}$ represents an element of $U$.
   c. The map $\chi : (a,b) \to U, \ s \mapsto [\Phi_s]$, is a diffeomorphism.
   The subset $U$ is said to be \textit{interior-regular} if it admits an interior-regular parameterization.

2. Let $Z \subset \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_1)$ be a connected component, let $E$ be an end of $Z$, and let $q \in \Lambda_0 \cap \Lambda_1$. A \textit{regular parameterization} of $E$ about $q$ is a smooth map $\Phi : S^{n-1} \times [0,1] \times [0, \epsilon) \to X$ satisfying the following conditions:
   a. For $(p,t) \in S^{n-1} \times [0,1]$ we have $\Phi(p,t,0) = q$.
   b. The restricted map $\Phi|_{S^{n-1} \times [0,1] \times (0,\epsilon)}$ is an interior-regular parameterization of $U$, for some ray $U \subset Z$ representing $E$. 

(c) The derivative
\[
\frac{\partial}{\partial s} \bigg|_{s=0} \Phi(\cdot,\cdot, s) : S^{n-1} \times [0, 1] \to T_qX
\]
is an immersion and the restriction \(\frac{\partial}{\partial s} \bigg|_{s=0} \Phi(\cdot,\cdot, s)|_{S^{n-1} \times \{1\}}\) is an embedding for \(i = 0, 1\).

(d) The Euler vector field on \(T_qX\) is nowhere tangent to the immersion \(\frac{\partial}{\partial s} \bigg|_{s=0} \Phi(\cdot,\cdot, s)\).

In this case, we also say that \(\Phi\) is a regular parameterization of \(U\) about \(q\).

We say the end \(E\) or the ray \(U\) converges regularly to the intersection point \(q\) if it admits a regular parameterization about \(q\). We may use a half-open interval with arbitrary endpoints, open either from below or above, in place of the half-open interval \([0, \epsilon)\).

The following is Corollary 4.20 from [33].

**Lemma 2.37.** If \(\Phi : S^{n-1} \times [0, 1] \times [0, \epsilon) \to X\) satisfies conditions 2(a), 2(b), and 2(c) of Definition 2.36, then possibly after diminishing \(\epsilon\), it also satisfies condition 2(d) and thus it is a regular parameterization.

The following are Definition 5.6 and Lemma 5.8 from [33].

**Definition 2.38.** Let \(\Lambda_0, \Lambda_1 \subset X\) be smooth Lagrangians and let \(N\) be a closed connected smooth manifold of dimension \(n-1\). Let \(U \subset SL(N; \Lambda_0, \Lambda_1)\) be open, connected and interior-regular. Let \(\Phi : N \times [0,1] \times (0,1) \to X\) be an interior regular parameterization of \(U\). For \(i = 0, 1\), the submanifold of \(\Lambda_i\) swept by \(U\) is the image of the embedding \(\Phi|_{N \times \{i\} \times (0,1)}\). This is independent of \(\Phi\). Similarly, suppose \(U \subset SL(S^{n-1}; \Lambda_0, \Lambda_1)\) is a ray and \(\Phi : S^{n-1} \times [0,1] \times [0, \epsilon) \to X\) is a regular parameterization of \(U\) about an intersection point \(q \in \Lambda_0 \cap \Lambda_1\). For \(i = 0, 1\), the unpunctured submanifold of \(\Lambda_i\) swept by \(U\) is the image of the restricted map \(\Phi|_{S^{n-1} \times \{i\} \times (0, \epsilon)}\).

**Lemma 2.39.** Let \(\Lambda_0, \Lambda_1 \subset X\) be smoothly embedded positive Lagrangians intersecting transversally at a point \(q\). Suppose there exists a connected component \(Z \subset SL(S^{n-1}; \Lambda_0, \Lambda_1)\) with an end \(E\) converging regularly to \(q\). Let \(U \subset Z\) be a ray representing \(E\) admitting a regular parameterization about \(q\). For \(i = 0, 1\), let \(\Lambda_i^U\) denote the unpunctured submanifold of \(\Lambda_i\) swept by \(U\). Then, there exists a geodesic of positive Lagrangians between \(\Lambda_0^U\) and \(\Lambda_1^U\) with cylindrical transform \(U\). This geodesic is unique up to reparameterization and has critical locus \(\{q\}\).

Recall Notation 2.21.

**Setting 2.40.** Let \((\Lambda_t)_{t \in [0,1]}\) be a geodesic of positive Lagrangians in \(X\) with Hamiltonian \((h_t)_t\). Suppose the endpoints \(\Lambda_0\) and \(\Lambda_1\) are smoothly embedded and let \(q \in \text{Crit}((\Lambda_t)_t)\) be a transverse intersection point of \(\Lambda_0\) and \(\Lambda_1\). Moreover, for \(t \in [0,1]\) assume \(q_t\) is an absolute minimum or maximum of \(h_t\). Let \((\Psi_t : L \to X, S)_t\) be a horizontal lifting. Let \(h = h_t \circ \Psi_t\) denote the Hamiltonian with respect to \((\Psi_t)\). Let \(p \in L\) be such that \(\Psi_t(p) = q\) for \(t \in [0,1]\). By Lemma 2.23, \(p\) is a non-degenerate critical point of \(h\). By Lemma 2.19, choose a positive \(\epsilon\) and a polar coordinate map \(\kappa : S^{n-1} \times (0, \epsilon) \to L\) centered at \(p\) such that for each \(s \in (0, \epsilon)\) the restricted map \(\kappa|_{S^{n-1} \times \{s\}}\) parameterizes a level set of \(h\).

The following is Lemma 5.11 from [33].

**Lemma 2.41.** In Setting 2.40, let \(Z\) denote the cylindrical transform of \((\Lambda_t)_t\). Then, one end of \(Z\) converges regularly to \(q\). In fact, the map
\[
\Phi : S^{n-1} \times [0,1] \times [0, \epsilon) \to X, \quad (c, t, s) \mapsto \Psi_t(\kappa(c, s))
\]
is a regular parameterization of $Z$ about $q$.

The following are Definitions 5.13 and 6.6 from [33].

**Definition 2.42.** Let $\Lambda_0, \Lambda_1 \subset X$ be smooth positive Lagrangian submanifolds intersecting at two points $q_0$ and $q_1$. Let $Z \subset SLC(S^{n-1}; \Lambda_0, \Lambda_1)$ be a connected component. A regular parameterization of $Z$ is a smooth map $\Phi : S^{n-1} \times [0,1] \rightarrow X$ satisfying the following conditions:

1. The restricted map $\Phi|_{S^{n-1} \times [0,1] \times (0,1)}$ is an interior-regular parameterization of $Z$.
2. The restricted maps $\Phi|_{S^{n-1} \times [0,1] \times (0,1/2)}$ and $\Phi|_{S^{n-1} \times [0,1] \times (1/2,1)}$ are regular parameterizations of the two ends of $Z$ about the intersection points $q_0$ and $q_1$, respectively.

We say $Z$ is regular if it admits a regular parameterization.

**Definition 2.43.** Let $\mathcal{O}$ be a Hamiltonian isotopy class of positive Lagrangian spheres. For $\Lambda_0, \Lambda_1 \in \mathcal{O}$, we write $\Lambda_0 \pitchfork_2 \Lambda_1$ if $\Lambda_0$ and $\Lambda_1$ intersect transversally at exactly two points. Let

$$\mathcal{Z}_\mathcal{O} := \left\{ (\Lambda_0, \Lambda_1, Z) \mid \Lambda_i \in \mathcal{O}, \ i = 0, 1, \ \Lambda_0 \pitchfork_2 \Lambda_1, \ Z \subset SLC(\Lambda_0, \Lambda_1) \text{ a regular component} \right\}.$$  

We define the strong and weak $C^{k,\alpha}$ topologies on $\mathcal{Z}_\mathcal{O}$ as follows. For $\mathcal{V} \subset C^\infty(S^{n-1} \times [0,1], X)$, $\mathcal{U} \subset C^\infty(S^{n-1} \times [0,1], TX)$, open subsets in the $C^{k,\alpha}$ topology, write

$$\mathcal{T}_{\mathcal{U}, \mathcal{V}} := \left\{ (\Lambda_0, \Lambda_1, Z) \in \mathcal{Z}_\mathcal{O} \mid \begin{array}{l} \forall Z \in \mathcal{Z}, \ \exists f : S^{n-1} \times [0,1] \rightarrow X \text{ representing } Z \\
\text{ such that } f \in \mathcal{V}, \\
\forall E \text{ an end of } Z, \ \exists \Phi : [0,\epsilon) \rightarrow X \text{ a regular parameterization of } E \text{ such that } \frac{\partial \Phi}{\partial s}|_{s=0} \in \mathcal{U} \end{array} \right\}$$

and

$$\mathcal{X}_{\mathcal{V}} = \left\{ (\Lambda_0, \Lambda_1, Z) \in \mathcal{Z}_\mathcal{O} \mid \exists Z \in \mathcal{Z}, \ \exists f : S^{n-1} \times [0,1] \rightarrow X \text{ representing } Z \\
\text{ such that } f \in \mathcal{V} \right\}.$$  

Then, a basis for the strong $C^{k,\alpha}$ topology on $\mathcal{Z}_\mathcal{O}$ is given by sets of the form $\mathcal{T}_{\mathcal{U}, \mathcal{V}}$ and a sub-basis for the weak $C^{k,\alpha}$ topology on $\mathcal{Z}_\mathcal{O}$ is given by sets of the form $\mathcal{X}_{\mathcal{V}}$.

Let

$$\mathcal{G}_\mathcal{O} := \left\{ (\Lambda_t)_{t \in [0,1]} \mid (\Lambda_t)_{t \in [0,1]} \text{ is a geodesic with } \Lambda_0, \Lambda_1 \in \mathcal{O}, \ \Lambda_0 \pitchfork_2 \Lambda_1 \right\}$$

denote the space of geodesics with endpoints in $\mathcal{O}$ intersecting transversally at two points. By Theorem 1.5 of [33], the cylindrical transform gives a bijection

$$\mathcal{G}_\mathcal{O} \simeq \mathcal{Z}_\mathcal{O}.$$  

So, the strong and weak $C^{k,\alpha}$ topologies on $\mathcal{Z}_\mathcal{O}$ give rise to topologies on $\mathcal{G}_\mathcal{O}$, which we also call the strong and weak $C^{k,\alpha}$ topologies respectively.

3. The positive Lagrangian Grassmannian

3.1. The linear positive Lagrangian connection. Let $(C^n, \omega, J, \Omega)$ denote the standard Calabi-Yau structure on $C^n$. Let $\mathcal{LG}(n)$ denote the Lagrangian Grassmannian,

$$\mathcal{LG}(n) := \{ \Lambda \in \text{Gr}_R(n, C^n) \mid \omega|_\Lambda = 0 \},$$

where $\text{Gr}_R(n, C^n)$ denotes the space of real $n$-dimensional linear subspaces of $C^n$. Then $\mathcal{LG}(n)$ is a smooth manifold of dimension $\frac{n(n+1)}{2}$. The following well-known lemma provides a convenient description of the tangent bundle $T\mathcal{LG}(n)$.
Lemma 3.1. For $\Lambda \in \mathcal{L}G(n)$ there is a canonical isomorphism $T_\Lambda \mathcal{L}G(n) \cong Q(\Lambda)$, the space of quadratic forms on $\Lambda$.

Proof. As always when working with spaces of Lagrangian submanifolds, the desired isomorphism is obtained via contraction with $\omega$. Let $(\Lambda_t)_{t \in (-\epsilon, \epsilon)}$ be a smooth path in $\mathcal{L}G(n)$ and let $\Psi_t: \mathbb{R}^n \to \mathbb{C}^n$, $t \in (-\epsilon, \epsilon)$, be a smooth linear lifting. As a 1-form, the time-derivative of $(\Lambda_t)_t$ is given by

$$
\sigma = \frac{d}{dt} \bigg|_{t=0} \Lambda_t := (\Psi_0)_* \left( i \frac{\pi}{\pi} \psi_t \omega \right) \in \Omega^1(\Lambda_0).
$$

As the path $(\Lambda_t)_t$ is Lagrangian, $\sigma$ is closed and thus equal to the derivative of a unique function, $h \in C^\infty(\Lambda_0)$, satisfying $h(0) = 0$. As the lifting $(\Psi_t)_t$ is linear, the linear functional $\sigma_x : T_x \Lambda_t \to \mathbb{R}$ depends linearly on $x$ under the canonical isomorphism $T_x \Lambda_t \simeq \Lambda_t$. It follows that $h$ is a quadratic form.

As every Lagrangian $\Lambda \in \mathcal{L}G(n)$ is in particular totally real and $\Omega$ is of type $(n,0)$, we have $\Omega|_\Lambda \neq 0$ (see [17]). Let $\text{phase} : \mathcal{L}G(n) \to \mathbb{R}/\pi\mathbb{Z}$ denote the Lagrangian phase,

$$
\text{phase}(\Lambda) := \arg (\Omega|_\Lambda), \quad \Lambda \in \mathcal{L}G(n).
$$

The phase is only well-defined in $\mathbb{R}/\pi\mathbb{Z}$, rather than $\mathbb{R}/2\pi\mathbb{Z}$, as the elements of $\mathcal{L}G(n)$ are not oriented. We define the positive Lagrangian Grassmannian to be the set of positive Lagrangian linear subspaces,

$$
\mathcal{L}G^+(n) := \left\{ \Lambda \in \mathcal{L}G(n) \mid \text{phase}(\Lambda) \neq \frac{\pi}{2} \pmod{\pi\mathbb{Z}} \right\}.
$$

We equip each $\Lambda \in \mathcal{L}G^+(n)$ with the orientation making $\text{Re} \Omega|_\Lambda$ a positive volume form. So, the phase lifts to a well-defined real-valued function on $\mathcal{L}G^+(n)$, which by abuse of notation we continue to denote by

$$
\text{phase} : \mathcal{L}G^+(n) \to \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).
$$

As shown in [29], every path of closed positive Lagrangians in a Calabi-Yau admits horizontal liftings. We show the same holds true for paths in $\mathcal{L}G^+(n)$.

Lemma 3.2. Let $(\Lambda_t)_{t \in [0,1]}$ be a smooth path in $\mathcal{L}G^+(n)$ and let $\Psi_0 : \mathbb{R}^n \to \Lambda_0$ be an isomorphism of real vector spaces. Then $\Psi_0$ extends uniquely to a horizontal lifting, $\Psi_t : \mathbb{R}^n \to \mathbb{C}^n$, $t \in [0,1]$, of the path $(\Lambda_t)_t$. Moreover, the unique lifting $(\Psi_t)_t$ is linear.

Proof. The argument is similar to the one presented in [29]. Let $(\tilde{\Psi}_t)_{t \in [0,1]}$ be a linear lifting of $(\Lambda_t)_t$ with $\tilde{\Psi}_0 = \Psi_0$. For $t \in [0,1]$, let $w_t$ be the vector field on $\mathbb{R}^n$ given by

$$
i_{w_t} \left( \tilde{\Psi}_t \right)^* \text{Re} \Omega = -i \tilde{\Psi}_t \text{Re} \Omega.
$$

Then $w_t$ is a linear time-dependent vector field and hence integrable. Its flow, denoted by $\varphi_t$, is also linear. The desired lifting is given by

$$
\Psi_t := \tilde{\Psi}_t \circ \varphi_t.
$$

Remark 3.3. By virtue of Lemma 3.2, all horizontal liftings of paths in $\mathcal{L}G^+(n)$ are henceforth assumed to be linear.
Lemma 3.4. Let $(\Lambda_t)_{t \in [0,1]}$ be a smooth path in $\mathcal{LG}^+(n)$ and let $(\Psi_t)_{t \in [0,1]}$ be a linear lifting. Let $(h_t)_{t \in [0,1]}$ be a vector field along the path $(\Lambda_t)$. Then, $h_t$ is a quadratic form on $\Lambda_t$ for $t \in [0,1]$. Then for $t_0 \in [0,1]$, the function

$$\frac{d}{dt} \bigg|_{t=t_0} (h_t \circ \Psi_t) : \mathbb{R}^n \to \mathbb{R}$$

is a quadratic form.

Proof. For $t \in [0,1]$, the composition $h_t \circ \Psi_t$ is a quadratic form on $\mathbb{R}^n$. The space $Q(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ is a closed linear subspace. The lemma follows. □

By Lemmas 3.2 and 3.4, we can define the positive Lagrangian connection on $\mathcal{LG}^+(n)$ just as in Definition 2.3. On the other hand, since $\mathbb{R}^n$ is not compact, the Riemannian metric (2) makes no sense in $\mathcal{LG}^+(n)$. In fact, the positive Lagrangian connection in this context is not a metric connection. In what follows, all covariant derivatives, geodesics, and exponential maps in $\mathcal{LG}^+(n)$ are taken with respect to the positive Lagrangian connection.

3.2. Linear geodesics. In this section we study geodesics in $\mathcal{LG}^+(n)$ and prove Theorem 1.5. We equip $\mathbb{R}^n$ with the standard inner product and $\mathbb{C}^n$ with the standard real inner product. Every element in $\mathcal{LG}^+(n)$ thus has an induced inner product. We start by stating the following well-known observation.

Lemma 3.5. Let $\Lambda_0, \Lambda_1 \in \mathcal{LG}(n)$. Then, for some $k \leq n$, there exist distinct elements $\alpha_1, \ldots, \alpha_k \in \mathbb{R}/\pi\mathbb{Z}$ and orthogonal decompositions,

$$\Lambda_t = \bigoplus_{j=1}^k V_{i,j}, \quad i = 0, 1,$$

satisfying

$$e^{\alpha_j}x_j^TV_{0,j} = V_{1,j}, \quad j = 1, \ldots, k.$$  

The arguments $\alpha_j$, $j = 1, \ldots, k$, and decompositions are unique up to order.

Proposition 3.7 below is key in the understanding of geodesics in $\mathcal{LG}^+(n)$. We use the following notion.

Definition 3.6. Let $(\Lambda_t)_{t \in [0,1]}$ be a geodesic in $\mathcal{LG}^+(n)$ with derivative $h_t \in Q(\Lambda_t)$, $t \in [0,1]$. A horizontal lifting $(\Psi_t)_{t \in [0,1]}$ of $(\Lambda_t)_{t}$ is said to be compatible with $h_0$ if $\Psi_0 : \mathbb{R}^n \to \Lambda_0$ is an inner product preserving isomorphism such that for some $a_1, \ldots, a_n \in \mathbb{R}$, we have

$$h_t \circ \Psi_t(x) = \sum_j a_j x_j^2, \quad x \in \mathbb{R}^n, \quad t \in [0,1].$$

The real numbers $a_1, \ldots, a_n$, which are independent of $(\Psi_t)$, up to order, are called the corresponding coefficients.

Below, we use the Einstein summation convention unless one of the indices is specified from the context.

Proposition 3.7. Let $(\Lambda_t)_{t \in [0,1]}$ be a geodesic in $\mathcal{LG}^+(n)$ with derivative $(h_t)_t$. Let $e_1, \ldots, e_n \in \mathbb{R}^n$ denote the standard basis. Let $(\Psi_t)_{t \in [0,1]}$ be a horizontal lifting of $(\Lambda_t)_t$ compatible with $h_0$, and let $a_1, \ldots, a_n$, denote the corresponding coefficients. For $j = 1, \ldots, n$ and $t \in [0,1]$, write $g_j(t) := (\Psi_t(e_j), \Psi_t(e_j))$ and $g^j(t) := (g_j(t))^{-1}$. Then we have the following.

(a) $$\frac{d}{dt} g_j(t) = -4a_j \tan \text{phase}(\Lambda_t), \quad j = 1, \ldots, n, \quad t \in [0,1].$$
(b) \[
\langle \Psi_t(e_j), \Psi_t(e_k) \rangle = 0, \quad j, k = 1, \ldots, n, \quad j \neq k, \quad t \in [0, 1].
\]

(c) For \( j = 1, \ldots, n \), there exists a unique smooth function \( \theta_j : [0, 1] \to \mathbb{R} \) such that
\[
\theta_j(0) = 0, \quad \Psi_t(e_j) \in e^{\theta_j(t)\sqrt{-1}}\mathbb{R}\langle \Psi_0(e_j) \rangle, \quad t \in [0, 1].
\]
The function \( \theta_j \) satisfies
\[
\frac{d}{dt}\theta_j(t) = -2a_jg^j(t).
\]

Proof. Recall the following identity, which is proved in [29, Remark 5.6]:
\[
\frac{d}{dt}\Psi_t(x) = -J\nabla h_t(\Psi_t(x)) - \tan\text{phase}(\Lambda_t)\nabla h_t(\Psi_t(x)), \quad x \in \mathbb{R}^n.
\]
Set \( g_{jk}(t) := \langle \Psi_t(e_j), \Psi_t(e_k) \rangle \), \( j, k = 1, \ldots, n \), \( t \in [0, 1] \), and let \( (g^{jk})_{j,k} \) denote the inverse matrix of \( (g_{jk})_{j,k} \). Let \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \) denote the tangent frames of \( \Lambda_t \), \( t \in [0, 1] \), induced by the parameterizations \( \Psi_t \). By (6) we have
\[
\nabla h_t(\Psi_t(x)) = 2a_jg^{jk}(t)\frac{\partial}{\partial x^j}, \quad j = 1, \ldots, n.
\]
Equation (6) can thus be rewritten as
\[
\frac{d}{dt}\Psi_t(e_j) = -2a_jg^{jk}(t)J\frac{\partial}{\partial x^k} - 2a_j\tan\text{phase}(\Lambda_t)g^{jk}(t)\frac{\partial}{\partial x^j}, \quad j = 1, \ldots, n.
\]
From the Leibniz rule we deduce
\[
\frac{d}{dt}g_{jk}(t) = -2\tan\text{phase}(\Lambda_t)\left(a_\kappa g_{jk}g^{\kappa l} + a_jg_{kl}g^{jl}\right)
\]
\[
= -4a_j\tan\text{phase}(\Lambda_t)\delta_{jk},
\]
establishing parts (a) and (b). Equation (7) can now be rewritten as
\[
\frac{d}{dt}\Psi_t(e_j) = -2a_jg^{jk}(t)J\frac{\partial}{\partial x^j} - 2a_j\tan\text{phase}(\Lambda_t)g^{jk}(t)\frac{\partial}{\partial x^j}, \quad j = 1, \ldots, n.
\]
Part (c) follows. \( \square \)

Definition 3.8. For a geodesic \( (\Lambda_t) \) in \( \mathcal{LG}^+(n) \), we call the functions \( \theta_j \) of Proposition 3.7(c) the partial phase differences.

Remark 3.9. It follows from the complex linearity of \( \Omega \) that if \( (\Lambda_t) \) is a geodesic in \( \mathcal{LG}^+(n) \) and \( \theta_j \) are the partial phase differences, then
\[
\frac{d}{dt}\text{phase}(\Lambda_t) = \sum_{j=1}^{n} \frac{d\theta_j}{dt}(t).
\]
In particular,
\[
\text{phase}(\Lambda_1) - \text{phase}(\Lambda_0) = \sum_{j=1}^{n} \theta_j(1).
\]
The following is an immediate consequence of Proposition 3.7.

Corollary 3.10. Let \( (\Lambda_t)_{t \in [0, 1]} \) be a geodesic in \( \mathcal{LG}^+(n) \) with derivative \( (h_t)_{t \in [0, 1]} \). Then the orthogonal decompositions of \( \Lambda_t \), \( i = 0, 1 \), associated to the quadratic forms \( h_t \) and the induced inner product coincide with the decompositions of Lemma 3.5. Moreover, letting \( (\Psi_t)_t \) be a horizontal lifting of \( (\Lambda_t) \), the isomorphism \( \Psi_1 \circ \Psi_0^{-1} : \Lambda_0 \to \Lambda_1 \) preserves these decompositions.

In light of Remark 3.9, Proposition 3.7(c) also provides a simple proof of the following statement, which is proved in greater generality in [30] and [34].
Corollary 3.11. Let $(\Lambda_t)_{t}$ be a geodesic in $\mathcal{LG}^+(n)$ with derivative $(h_t)_t$. Then we have
\[
\frac{d}{dt} \text{phase}(\Lambda_t) = \Delta h_t,
\]
where $\Delta$ denotes the geometer’s Laplacian with respect to the induced metric on $\Lambda_t$. In particular, if the quadratic forms $h_t$ are positive/negative semi-definite, then phase$(\Lambda_t)$ is a monotone function of $t$.

Let $\Lambda_0, \Lambda_1 \in \mathcal{LG}^+(n)$ and suppose we wish to find a geodesic between $\Lambda_0$ and $\Lambda_1$. In other words, we wish to find a tangent vector $h_0 \in T_{\Lambda_0} \mathcal{LG}^+(n) \cong Q(\Lambda_0)$ with $\exp_{\Lambda_0}(h_0) = \Lambda_1$. In view of Proposition 3.7 and Corollary 3.10, the following is a natural approach. Identify $\Lambda_0$ with $\mathbb{R}^n$ via an orthonormal basis compatible with the orthogonal decomposition
\[
\Lambda_0 = \bigoplus_j V_{0,j}
\]
of Lemma 3.5. With respect to this identification, the desired $h_0$, if it exists, has to be given by
\[
h_0(x) = \sum_j a_j x_j^2
\]
for some $a_1, \ldots, a_n \in \mathbb{R}$. The coefficients $a_j$, $j = 1, \ldots, n$, determine the time-evolution of the partial phase differences, $\theta_1, \ldots, \theta_n$, and thus need to be chosen so that $\theta_j(1)$ represents the argument $\alpha_j$ of Lemma 3.5 for $j = 1, \ldots, n$. In particular, each of the functions $\theta_j$ is monotone, where the sign of $a_j$ determines whether it is increasing or decreasing. It thus makes sense that the Maslov index defined below is related to geodesics in $\mathcal{LG}^+(n)$. We recall the definition given in [31].

Definition 3.12. Let $\Lambda_0, \Lambda_1 \in \mathcal{LG}^+(n)$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}/\pi\mathbb{Z}$ be the arguments of Lemma 3.5 with the right multiplicities. For $j = 1, \ldots, n$, let $\beta_j \in [0, \pi)$ be the unique representative of $\alpha_j$. The Maslov index of the pair $(\Lambda_0, \Lambda_1)$ is the integer given by
\[
m(\Lambda_0, \Lambda_1) := \frac{1}{\pi} \left( \sum_j \beta_j + \text{phase}(\Lambda_0) - \text{phase}(\Lambda_1) \right).
\]
For a given $\Lambda_0 \in \mathcal{LG}^+(n)$, we set
\[
m_0(\Lambda_0) := \{ \Lambda_1 \in \mathcal{LG}^+(n) \mid m(\Lambda_0, \Lambda_1) = 0 \}
\]
and
\[
m_0(\Lambda_0) := \left\{ \Lambda_1 \in \overline{m_0(\Lambda_0)} \mid \Lambda_0 \cap \Lambda_1 = \{0\} \right\}.
\]
As implied by the notation, $m_0(\Lambda_0)$ is indeed the closure of $m_0(\Lambda_0)$. We remark that both spaces are connected. We let $\mathcal{G}(\Lambda_0) \subset m_0(\Lambda_0)$ and $\mathcal{G}(\Lambda_0) \subset \overline{m_0(\Lambda_0)}$ denote the subsets consisting of Lagrangians that can be connected with $\Lambda_0$ by a geodesic with negative semi-definite derivative.

More generally, if $(X, \omega, J, \Omega)$ is a Calabi-Yau manifold and $\Lambda_0, \Lambda_1 \subset X$ are positive Lagrangian submanifolds, the Maslov index of an intersection point $q \in \Lambda_0, \Lambda_1$ is given by
\[
m(q; \Lambda_0, \Lambda_1) := m(T_q\Lambda_0, T_q\Lambda_1).
\]
We divide the proof of Theorem 1.5 into three steps. Fix $\Lambda_0 \in \mathcal{LG}^+(n)$. In Corollary 3.14 we show that $\mathcal{G}(\Lambda_0)$ is non-empty. In Proposition 3.16 we show that $\mathcal{G}(\Lambda_0)$ is a closed subset of $\overline{m_0(\Lambda_0)}$. In Proposition 3.18 we show that $\mathcal{G}(\Lambda_0)$ is an open subset of $m_0(\Lambda_0)$.

Lemma 3.13. Let $(\Lambda_t)_{t \in [0,1]}$ be a geodesic in $\mathcal{LG}^+(n)$ with derivative $(h_t)_t$. Suppose the quadratic forms $h_t$ are negative semi-definite.
(a) We have \( m(\Lambda_0, \Lambda_1) = 0 \).

(b) If \( \Lambda_0 \cap \Lambda_1 = \{0\} \), then the quadratic forms \( h_t \) are negative definite.

**Proof.** Choose a horizontal lifting of \( (\Lambda_t) \), compatible with \( h_0 \), let \( a_1, \ldots, a_n \) denote the corresponding coefficients and let \( \theta_1, \ldots, \theta_n \) denote the partial phase differences. By assumption, the coefficients \( a_j, \ j = 1, \ldots, n \), are all non-positive. It thus follows from Proposition 3.7(c) that

\[
\theta_j(1) \geq 0, \quad j = 1, \ldots, n,
\]

with equality only when \( a_j = 0 \). If \( \theta_j = 0 \) for some \( j \), then \( \Lambda_0 \cap \Lambda_1 \neq \{0\} \). So, part [b] of the lemma follows.

As \( \text{phase}(\Lambda_0), \text{phase}(\Lambda_1) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), Remark 3.9 and inequality (9) imply that

\[
\theta_j(1) < \pi, \quad j = 1, \ldots, n.
\]

The partial phase differences \( \theta_j(1), \ j = 1, \ldots, n \), thus coincide with the representatives \( \beta_j, \ j = 1, \ldots, n \), of Definition 3.12 which implies part [a] of the lemma. \( \square \)

**Corollary 3.14.** For \( \Lambda_0 \in \mathcal{LG}^+(n) \), the space \( \mathcal{G}(\Lambda_0) \) is non-empty.

**Proof.** As \( \mathcal{LG}^+(n) \) is finite-dimensional, the exponential map \( \exp_{\Lambda_0} \) is well-defined on an open neighborhood \( 0 \in U \subset T_{\Lambda_0} \mathcal{G}(n) = Q(\Lambda_0) \). Let \( h \in U \) be negative definite. By Lemma 3.13(a) we have \( \exp_{\Lambda_0}(h) \in m_0(\Lambda_0) \). By Proposition 3.7(c) the partial phase differences are strictly increasing, implying that

\[
\Lambda_0 \cap \exp_{\Lambda_0}(h) = \{0\}.
\]

It follows that \( \exp_{\Lambda_0}(h) \in \mathcal{G}(\Lambda_0) \). \( \square \)

The main ingredient in the proof of Proposition 3.16 is the following a priori estimate.

**Lemma 3.15.** Let \(-\frac{\pi}{2} < \alpha_0 < \alpha_1 < \frac{\pi}{2}\).

(a) There exists a positive constant \( N = N(\alpha_1) \) with the following property. Let \( (\Lambda_t)_{t \in [0,1]} \) be a geodesic with negative semi-definite derivative \( (h_t)_t \), such that \( \text{phase}(\Lambda_0), \text{phase}(\Lambda_1) \in [\alpha_0, \alpha_1] \). Let \( (\Psi_t)_t \) be a horizontal lifting of \( (\Lambda_t) \), compatible with \( h_0 \) and let \( a_1, \ldots, a_n \) denote the corresponding coefficients. Let \( e_1, \ldots, e_n \) denote the standard basis of \( \mathbb{R}^n \) and write

\[
g_j(t) := \langle \Psi_t(e_j), \Psi_t(e_j) \rangle, \quad g^j(t) := (g_j(t))^{-1}, \quad j = 1, \ldots, n, \quad t \in [0,1].
\]

Then we have

\[
g_j(t) \leq N, \quad j = 1, \ldots, n, \quad t \in [0,1].
\]

(b) There exists a positive constant \( M = M(\alpha_0, \alpha_1) \) such that, in the setting of part [a], we have

\[a_j > -M, \quad j = 1, \ldots, n.\]

**Proof.** Let \( (\Lambda_t)_t, (h_t)_t, (\Psi_t)_t, a_j, j = 1, \ldots, n, \) and \( g_j, j = 1, \ldots, n \), be as in [a]. We assume \( h_t \neq 0 \), otherwise there is nothing to prove. By Corollary 3.11 phase(\( \Lambda_t \)) is strictly increasing with time-derivative

\[
\frac{d}{dt} \text{phase}(\Lambda_t) = \Delta h_t = -2g^j(t)a_j.
\]

In particular, we have

\[
\text{phase}(\Lambda_t) \in [\alpha_0, \alpha_1], \quad t \in [0,1].
\]

Switching the roles of phase(\( \Lambda_t \)) and \( t \), we write

\[
\frac{dt}{d \text{phase}(\Lambda_t)} = -\frac{1}{2g^j(t)a_j}.
\]
Let \( j \in \{1, \ldots, n\} \). If we have \( a_j = 0 \), it follows from Proposition 3.7(a) that \( g_j(t) \equiv 1 \). Hence, we assume \( a_j < 0 \). By Proposition 3.7(a) and (11) we have
\[
\frac{dg_j}{d\text{phase}(\Lambda_t)} = 2 a_j \tan \text{phase}(\Lambda_t) g^j a_k.
\]
If \( \text{phase}(\Lambda_t) \leq 0 \), it follows that
\[
\frac{dg_j}{d\text{phase}(\Lambda_t)} \leq 0.
\]
If \( \text{phase}(\Lambda_t) > 0 \), equation (12) implies
\[
\frac{dg_j}{d\text{phase}(\Lambda_t)} \leq 2 \tan \text{phase}(\Lambda_t) g_j a_j.
\]
The differential inequalities (13) and (14) imply that, if \( \alpha_1 \leq 0 \), we have
\[
g_j(t) \leq 1, \quad t \in [0, 1],
\]
whereas if \( \alpha_1 > 0 \), we have
\[
g_j(t) \leq e^{\pi \tan \alpha_1}, \quad t \in [0, 1],
\]
establishing part (a).

By (a) we have
\[
g^j(t) \geq \frac{1}{N}, \quad j = 1, \ldots, n, \quad t \in [0, 1].
\]
Integrating equation (10), we obtain
\[
\alpha_1 - \alpha_0 = -2 a_j \int_0^1 g^j(t) dt \geq -\frac{2}{N} \sum_j a_j.
\]
Part (b) follows.

Proposition 3.16. Let \( \Lambda_0 \in \mathcal{L}G^+(n) \). Then \( \mathcal{G}(\Lambda_0) \) is a closed subset of \( m_0(\Lambda_0) \).

Proof. Let \( (\Lambda_{1,k})_{k \in \mathbb{N}} \) be a sequence in \( \mathcal{G}(\Lambda_0) \) converging to \( \Lambda_{1,\infty} \in m_0(\Lambda_0) \). For \( k \in \mathbb{N} \), let \( (\Lambda_{t,k})_{t \in [0,1]} \) be a geodesic between \( \Lambda_0 \) and \( \Lambda_{1,k} \) with negative semi-definite derivative. Write \( \text{phase}(\Lambda_0) =: \alpha_0 \). As the sequence \( (\Lambda_{1,k})_k \) is convergent, there exists some \( \alpha_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \) with
\[
\text{phase}(\Lambda_{1,k}) < \alpha_1, \quad k \in \mathbb{N}.
\]
It follows that all the geodesics \( (\Lambda_{t,k})_t \) have images contained in the compact set
\[
\mathcal{L}G^+_{\alpha_0,\alpha_1}(n) := \{ \Lambda \in \mathcal{L}G^+(n) \mid \text{phase}(\Lambda) \in [\alpha_0, \alpha_1] \}.
\]
Also, by Lemma 3.15(b), there exists a compact set \( K \subset T_{\Lambda_0} \mathcal{G}(n) \) containing all the time-derivatives \( \frac{d\Lambda_t}{dt} \mid_{t=0} \Lambda_{t,k} \). It follows that for \( l \in \mathbb{N} \), the derivatives of all geodesics in question up to order \( l \) are uniformly bounded. By the Arzelà-Ascoli theorem, there exists a subsequence of \( ((\Lambda_{t,k})_l)_k \) converging in the \( C^2 \)-sense to a path \( (\Lambda_{t,\infty})_{t \in [0,1]} \), which is a geodesic between \( \Lambda_0 \) and \( \Lambda_{1,\infty} \). As the space of negative semi-definite quadratic forms is a closed subset in the space of general quadratic forms, the geodesic \( (\Lambda_{t,\infty})_{t \in [0,1]} \) has negative semi-definite derivative.
\square
The proof of Proposition 3.18 below is based on the relation between geodesics and imaginary special Lagrangian cylinders, as summarized in Section 2.5. In addition, it relies on the following elementary observation concerning geodesics and Jacobi fields with respect to arbitrary connections.

**Lemma 3.17.** Let $M$ be a smooth finite-dimensional manifold equipped with an affine connection. Let $\gamma : [0,1] \to M$ be a geodesic, and let $J \in \Gamma([0,1], \gamma^*TM)$ be a Jacobi field tangent to $\gamma$. That is, we have $J = f\dot{\gamma}$ for some $f : [0,1] \to \mathbb{R}$. If we have $J(0) = 0$ and $J(1) = 0$, then $J$ is the trivial Jacobi field.

For $\Lambda_0 \in \mathcal{LG}^+(n)$, let $T_{\Lambda_0}\mathcal{LG}^+(n) \subset T_{\Lambda_0}\mathcal{LG}^+(n)$ denote the cone consisting of negative definite quadratic forms.

**Proposition 3.18.** Let $(\Lambda_i)_{i \in [0,1]}$ be a geodesic segment in $\mathcal{LG}^+(n)$ with negative semi-definite derivative and $\Lambda_0 \cap \Lambda_1 = \{0\}$. Then, there exist open neighborhoods $\Lambda_i \subset U_i \subset \mathcal{LG}^+(n), \quad i = 0, 1,$ and a smooth map

$$
\Gamma : [0,1] \times U_0 \times U_1 \to \mathcal{LG}^+(n),
$$

such that for $(u_0, u_1) \in U_0 \times U_1$ the path

$$
t \mapsto \Gamma(t, u_0, u_1)
$$

is a geodesic from $u_0$ to $u_1$ and

$$
\Gamma(t, \Lambda_0, \Lambda_1) = \Lambda_t, \quad t \in [0,1].
$$

In particular, the space $\mathcal{G}(\Lambda_0)$ is an open subset of $m_0(\Lambda_0)$.

**Proof.** It suffices to show that $\Lambda_1$ is not a conjugate point of $\Lambda_0$ along the geodesic $(\Lambda_t)$. Thus, keeping in mind Lemma 3.13(b) it suffices to prove the following.

**Claim:** Let $(\Lambda_t)_{t \in [0,1]}$ be a geodesic with negative definite derivative, and let $J = (J_t)_{t \in [0,1]}$ be a Jacobi field along $(\Lambda_t)_t$ satisfying

$$
J_0 = 0, \quad J_1 = 0.
$$

Then we have $J_t = 0$ for $t \in [0,1]$.

In view of Lemma 3.17 it suffices to prove that, under the assumptions of the above claim, the Jacobi field $J$ is tangent to the geodesic $(\Lambda_t)$. Let the two-parameter family $(\Lambda_{t,s})_{(t,s) \in [0,1] \times (-\epsilon,\epsilon)}$ be a variation realizing the Jacobi field $J$. In other words, the path $(\Lambda_{t,s})_{t \in [0,1]}$ is a geodesic with $\Lambda_{0,s} = \Lambda_0$ for $s \in (-\epsilon,\epsilon)$, and we have

$$
\left. \frac{\partial}{\partial s} \right|_{s=0} \Lambda_{t,s} = J_t, \quad t \in [0,1].
$$

For $(t, s) \in [0,1] \times (-\epsilon,\epsilon)$, write

$$
h_{t,s} := \frac{\partial}{\partial t} \Lambda_{t,s} \in Q(\Lambda_{t,s}).
$$

By assumption, diminishing $\epsilon$ if necessary, all quadratic forms $h_{t,s}$ are negative definite and thus have regular level sets diffeomorphic to $S^{n-1}$. Recalling Definition 2.31 for $s \in (-\epsilon,\epsilon)$, let $Z_s \in SLC(S^{n-1}; \Lambda_0, \Lambda_{1,s})$ be the cylinder of $-1$-level sets associated to the geodesic $(\Lambda_{t,s})$. Abbreviate $L = S^{n-1} \times [0,1]$.

We construct a smooth map

$$
\Upsilon : L \times (-\epsilon,\epsilon) \to \mathbb{C}^n
$$

such that for $s \in (-\epsilon,\epsilon)$ the restricted map $\Upsilon_s := \Upsilon_{|L \times \{s\}}$ is an immersion representing the cylinder $Z_s$ and adapted to its harmonics. The construction is carried
out as follows. Choose a linear parameterization \( \Psi_0 : \mathbb{R}^n \to \Lambda_0 \) and for \( s \in (-\epsilon, \epsilon) \), let \((\Psi_t, s)_{t \in [0, 1]} \) be the horizontal lifting of the geodesic \((\Lambda_t, s)\) with \( \Psi_0, s = \Psi_0 \). Let

\[ h^s := h_{0, s} \circ \Psi_0 : \mathbb{R}^n \to \mathbb{R} \]

and let

\[ \Theta : S^{n-1} \times (-\epsilon, \epsilon) \to \mathbb{R}^n \]

be a smooth map such that \( \Theta_s = \Theta|_{S^{n-1} \times \{s\}} \) carries the sphere \( S^{n-1} \times \{s\} \) diffeomorphically onto the level set \( (h^s)^{-1}(-1) \). Take

(15)

\[ \Upsilon(p, t, s) := \Psi_t, s(\Theta_s(p)). \]

By definition, \( \Upsilon_s \) represents the cylinder of \(-1\) level sets \( Z_s \). Proposition 2.30(b) asserts that \( \Upsilon_s \) is adapted to the harmonics of \( Z_s \).

Define a map \( v : L \to \mathbb{C}^n \) by

\[ v(p, t) := \frac{d}{ds}_{s=0} \Upsilon(p, t, s), \quad (p, t) \in S^{n-1} \times [0, 1] = L. \]

As all the cylinders \( Z_s \) are Lagrangian, the 1-form \( \tau := i_o \omega \in \Omega^1(L) \) is closed. For \( i = 0, 1 \), since \( J_i = 0 \), the vector \( v(q) \) is tangent to \( \Lambda_i \) for \( q \in S^{n-1} \times \{i\} \). It follows that the 1-form \( \tau \) annihilates the boundary component \( S^{n-1} \times \{i\} \subset L \).

By Lemma 2.26, the 1-form \( \tau \) is exact. Let \( u \in C^\infty(\Lambda_0) \) be the unique function satisfying

\[ du = \tau. \]

As the cylinders \( Z_s \) are imaginary special Lagrangian, Lemma 2.27(a) implies that \( \Delta u = 0 \). Since \( \Upsilon_0 \) is adapted to the harmonics of \( Z_0 \), Lemma 2.27(b) implies that \( u|_{S^{n-1} \times \{t\}} \) is constant for \( t \in [0, 1] \).

For \( t \in [0, 1] \), the tangent vector \( J_t \) is a quadratic form on \( \Lambda_t \). It follows from equation (15) that \( \Upsilon_0 \) carries \( S^{n-1} \times \{t\} \) into \( \Lambda_t, 0 \). By construction, for \( t \in [0, 1] \), a point \( q \in S^{n-1} \times \{t\} \) and a tangent vector

\[ \xi \in T_q(S^{n-1} \times \{t\}), \]

we have

\[ 0 = du_q(\xi) \]

\[ = \omega(v(q), d(\Upsilon_0)_q(\xi)) \]

\[ = d(J_t)\Upsilon_0(q)(d(\Upsilon_0)_q(\xi)) \]

\[ = d(J_t \circ \Upsilon_0|_{S^{n-1} \times \{t\}})_q(\xi). \]

Thus, for \( t \in [0, 1] \) the function \( J_t \circ \Upsilon_0|_{S^{n-1} \times \{t\}} \) is constant. By the definition of the cylinder of \(-1\) level sets,

\[ h_{t, 0} \circ \Upsilon_0|_{S^{n-1} \times \{t\}} = -1. \]

Since \( J_t \) and \( h_{t, 0} \) are both quadratic forms on \( \Lambda_t, 0 \), and they share the level set \( \Upsilon_0(S^{n-1} \times \{t\}) \), it follows that they agree up to a constant multiple possibly depending on \( t \). Thus, the Jacobi field \( J \) is tangent to the geodesic \( (\Lambda_t)_t \). The claim follows.

Proof of Theorem 1.5. Since \( m_0(\Lambda_0) \) is connected, the theorem follows from Corollary 3.14 Proposition 3.16 and Proposition 3.18.
4. Geodesics of Lagrangian cones

A smooth immersed Lagrangian cone in \( \mathbb{C}^n \) is a cone-smooth Lagrangian submanifold of \( \mathbb{C}^n \) that can be represented by a cone-immersion \( f : \mathbb{R}^n \to \mathbb{C}^n, 0 \) that is 1-homogeneous. Let \( LCO(n) \) denote the space of smooth immersed Lagrangian cones in \( \mathbb{C}^n \) and \( LCO^+(n) \subset LCO(n) \) the open subspace consisting of positive Lagrangian cones. Then \( LCO^+(n) \) is a smooth Fréchet manifold containing \( LG^+(n) \). The goal of this section, accomplished in Proposition \ref{proposition:geodesics}, is to show that, under some assumptions, geodesics of positive Lagrangian cones are in fact linear. The following description of the tangent bundle \( TLCO(n) \) is analogous to that of Lemma \ref{lemma:linear}.

**Lemma 4.1.** For \( \Lambda \in LCO(n) \) there is a canonical isomorphism \( T_{\Lambda} LCO(n) \cong \mathcal{H}(\Lambda) \), the space of cone-smooth degree-2 homogeneous functions on \( \Lambda \).

The positive Lagrangian connection on \( LG^+(n) \) extends naturally to \( LCO^+(n) \). Indeed, we have the following analog of Lemma \ref{lemma:connection}:

**Lemma 4.2.** Let \( (\Lambda_t)_{t \in [0,1]} \) be a smooth path in \( LCO^+(n) \). Let \( (\Psi_t : \mathbb{R}^n \to \mathbb{C}^n, 0) \) be a cone-smooth 1-homogeneous immersion representing \( \Lambda_0 \). Then \( \Psi_t \) extends uniquely to a cone-smooth horizontal lifting, \( (\Psi_t : \mathbb{R}^n \to \mathbb{C}^n, 0)_{t \in [0,1]} \), of the path \( (\Lambda_t)_{t} \). Moreover, the unique lifting \( (\Psi_t)_{t} \) is 1-homogeneous.

This lemma enables us to parallel transport homogeneous functions along paths in \( LCO^+(n) \). Hence, the notion of geodesics in \( LCO^+(n) \) makes sense. The following lemma describes the relation between geodesics in \( LCO^+(n) \) and special Lagrangian cylinders. For the purposes of this work, we may restrict the discussion to geodesics with linear endpoints. For a Lagrangian cone \( \Lambda \) and a homogeneous function \( h \in \mathcal{H}(\Lambda) \), we say \( h \) is positive (negative) if we have \( h(p) > 0 \) (\( h(p) < 0 \)) for \( p \in \Lambda \setminus \{0\} \). For \( \Lambda_0, \Lambda_1 \in LG^+(n) \), the positive real numbers \( \mathbb{R}_{>0} \) act on the space of Lagrangian cylinders \( SLC(S^{n-1}; \Lambda_0, \Lambda_1) \) by rescaling \( [\chi] \mapsto [s\chi] \), \( s \in \mathbb{R}_{>0} \).

**Lemma 4.3.** Let \( \Lambda_0, \Lambda_1 \in LG^+(n) \). Then the cylindrical transform gives a bijection from geodesics in \( LCO^+(n) \) between \( \Lambda_0 \) and \( \Lambda_1 \) with positive/negative derivative to \( \mathbb{R}_{>0} \)-orbits in \( SLC(S^{n-1}; \Lambda_0, \Lambda_1) \) consisting of imaginary special Lagrangian cylinders nowhere tangent to the Euler vector field.

**Proof.** Let \( (\Lambda_t)_{t \in [0,1]} \) be a geodesic in \( LCO^+(n) \) with positive derivative, denoted by \( (h_t)_{t} \), between \( \Lambda_0 \) and \( \Lambda_1 \). As the homogeneous functions \( h_t \) are positive, they admit the regular value \( c \) for each \( c > 0 \). Moreover, the level set \( h^{-1}(c) \) is diffeomorphic to \( S^{n-1} \). Let \( Z \) be the cylinder of \( c \)-level sets. By Proposition \ref{proposition:level-set}, the cylinder \( Z \) is stationary special Lagrangian with regular harmonics. By Lemma \ref{lemma:stationary}, the cylinder \( Z \) is nowhere tangent to the Euler vector field. By the homogeneity of \( (h_t)_{t} \), the cylinders of \( c \)-level sets for different values of \( c \) are related by rescaling.

Conversely, let \( Z \in SLC(S^{n-1}; \Lambda_0, \Lambda_1) \) be a cylinder nowhere tangent to the Euler vector field. Let \( \chi : S^{n-1} \times [0,1] \to \mathbb{C}^n \) be an immersion representing \( Z \) adapted to the harmonics of \( Z \). For \( s \in (0, \infty) \), let

\[
Z_s := [s\chi]
\]

be the rescaling of \( Z \) by \( s \). Let

\[
U = \{Z_s\}_{s \in (0, \infty)} \subset SLC(S^{n-1}; \Lambda_0, \Lambda_1)
\]

Define

\[
\Phi : S^{n-1} \times [0,1] \times [0, \infty) \to \mathbb{C}^n
\]

by

\[
\Phi(p, t, s) = s\chi(p, t).
\]
Recalling Definition 2.39 since \( Z \) is nowhere tangent to the Euler vector field, it follows that \( \Phi \) is a regular parameterization of \( U \) about \( 0 \in \mathbb{C}^n \). So, Lemma 2.39 gives us a unique geodesic \((\Lambda_t)_t\) of positive Lagrangians from \( \Lambda_0 \) to \( \Lambda_1 \) with cylindrical transform \( U \). By uniqueness, this geodesic must be invariant under rescaling by \( s \in (0, \infty) \), so it lies in \( \mathcal{LCO}^+(n) \). Let \( h_t : \Lambda \to \mathbb{R} \) be given by \( h_t = \frac{d\Lambda_t}{dt} \). By definition of the cylindrical transform, the level sets of \( h_t \) are spheres of the form \( \Phi(S^{n-1} \times \{t\} \times \{s\}) \) for \( s \in (0, \infty) \), and in particular, compact. Hence, \( h_t \) is either positive or negative.

The following definition is a simplified version of Definition 2.43 for the space of geodesics in \( \mathcal{LCO}^+(n) \) with linear endpoints.

**Definition 4.4.** Let 
\[
\mathcal{L}^{\mathcal{L}G^+}_{\mathcal{L}G^+} = \{(\Lambda_0, \Lambda_1, Z) \mid \Lambda_i \in \mathcal{L}G^+(n), \ Z \in \mathcal{SLC}(\Lambda_0, \Lambda_1) \text{ an } \mathbb{R}_{>0} \text{ orbit}\}.
\]

We define the \( C^{k,\alpha} \) topology on \( \mathcal{L}^{\mathcal{L}G^+}_{\mathcal{L}G^+} \) as follows. For \( \mathcal{V} \subset C^{\infty}(S^{n-1} \times [0, 1], \mathbb{C}^n) \), an open subset in the \( C^{k,\alpha} \) topology, write
\[
\mathcal{X}_\mathcal{V} = \{(\Lambda_0, \Lambda_1, Z) \in \mathcal{L}^{\mathcal{L}G^+}_{\mathcal{L}G^+} \mid \exists Z \in \mathcal{Z}, \exists f : S^{n-1} \times [0, 1] \to \mathbb{C}^n \text{ representing } Z \text{ such that } f \in \mathcal{V}\}.
\]

Then, a basis for the \( C^{k,\alpha} \) topology on \( \mathcal{L}^{\mathcal{L}G^+}_{\mathcal{L}G^+} \) is given by sets of the form \( \mathcal{X}_\mathcal{V} \). Let
\[
\mathcal{G}^{\mathcal{L}G^+}_{\mathcal{L}G^+} = \{(\Lambda_t)_{t \in [0, 1]} \mid (\Lambda_t)_{t \in [0, 1]} \text{ is a geodesic in } \mathcal{LCO}^+(n) \text{ with positive/negative derivative, } \Lambda_0, \Lambda_1 \in \mathcal{L}G^+(n), \ \Lambda_0 \cap \Lambda_1 = \{0\}\}
\]
denote the space of geodesics in \( \mathcal{LCO}^+(n) \) with linear endpoints intersecting transversally. By Lemma 4.3, the cylindrical transform gives a bijection
\[
\mathcal{G}^{\mathcal{L}G^+}_{\mathcal{L}G^+} \simeq \mathcal{L}^{\mathcal{L}G^+}_{\mathcal{L}G^+}.
\]

So, the \( C^{k,\alpha} \) topology on \( \mathcal{L}^{\mathcal{L}G^+}_{\mathcal{L}G^+} \) gives rise to a topology on \( \mathcal{G}^{\mathcal{L}G^+}_{\mathcal{L}G^+} \), which we also call the \( C^{k,\alpha} \) topology.

The following setting will be referred to in Lemmas 4.7 and 4.8 below.

**Setting 4.5.** Let \((\Lambda_{0,r})_{r \in [0, 1]}\) and \((\Lambda_{1,r})_{r \in [0, 1]}\) be smooth paths in \( \mathcal{L}G^+(n) \) with \( \Lambda_{0,r} \cap \Lambda_{1,r} = \{0\}, \ r \in [0, 1] \). Let \((\Lambda_{t,0})_{t \in [0, 1]}\) be a geodesic in \( \mathcal{LCO}^+(n) \) with negative derivative between \( \Lambda_{0,0} \) and \( \Lambda_{1,0} \).

**Notation 4.6.** In Setting 4.5 abbreviate \( L := S^{n-1} \times [0, 1] \). Let
\[
Z_0 = \left\{ f : L \to \mathbb{C}^n \right\} \in \mathcal{SLC} \left( S^{n-1}; \Lambda_{0,0}, \Lambda_{1,0} \right)
\]
denote the cylinder of \(-1\)-level sets associated to \((\Lambda_{t,0})_t\). Recalling Lemma 2.25, choose a Weinstein neighborhood \((V, \psi)\) of \( Z_0 \) compatible with \( \Lambda_{0,0} \) and \( \Lambda_{1,0} \), where \( V \subset T^*L \) and \( \psi : V \to \mathbb{C}^n \) with \( \psi|_L = f \). Let \( \pi_L : T^*L \to L \) denote the projection. Let \( \alpha \in (0, 1) \). For \( u \in C^{2,\alpha}_{\mathcal{COB}}(L) \), let \( \text{Graph}(du) \subset T^*L \) denote the graph. For \( u \) small enough that \( \text{Graph}(du) \subset V \), let \( j_u : L \to \mathbb{C}^n \) be given by
\[
j_u = \psi \circ (\pi_L|_{\text{Graph}(du)})^{-1}.
\]

Let \((\varphi_r : \mathbb{C}^n \to \mathbb{C}^n)_{r \in [0, 1]}\) be a smooth Hamiltonian isotopy such that \( \varphi_r \) carries \( \Lambda_{0,0} \) to \( \Lambda_{0,r} \) and \( \Lambda_{1,0} \) to \( \Lambda_{1,r} \).
Lemma 4.7. In Setting [4.3] suppose the geodesic of positive Lagrangian cones \((\Lambda_t)_{t \in [0,1]}\) extends to a family \((\Lambda_t,r)_{t \in [0,1]}\) \((s,r)\) satisfying the following:

- For \(r \in [0,\epsilon)\), the path \((\Lambda_t,r)_{t \in [0,1]}\) is a geodesic in \(\mathcal{LCO}^+(n)\) with negative derivative between \(\Lambda_0,r\) and \(\Lambda_1,r\).
- The geodesic \((\Lambda_t,r)_{t \in [0,1]}\) depends continuously on \(r\) with respect to the \(C^{1,\alpha}\) topology on \(\mathcal{LCO}^+(n)\).

For \(r \in [0,\epsilon)\) let \(Z_r\) denote the cylinder of \(-1\)-level sets associated to the geodesic \((\Lambda_t,r)_{t \in [0,1]}\). Then, after possibly diminishing \(\epsilon\), for each \(r \in [0,\epsilon)\), there exists a unique function \(u_r \in C^\infty(L;\partial L)\) such that Graph(\(du\)) \(\subset V\) and

\[
\varphi_r \circ j_{u_r} : L \to \mathbb{C}^n
\]

represents the imaginary special Lagrangian cylinder \(Z_r\).

Proof. Let \((f_r : L \to \mathbb{C}^n)_{r \in [0,\epsilon)}\) be a \(C^{1,\alpha}\) continuous family of smooth imaginary special Lagrangian immersions representing \(Z_r\). After possibly shrinking \(\epsilon\), we claim that

\[
\varphi_r^{-1} \circ f_r(L) \subset \psi(V).
\]

Here, the set \(\psi(V)\) is not open in \(\mathbb{C}^n\) because \(V\) is a manifold with boundary. Indeed, \(V\) is an open neighborhood of the zero section of \(T^*L\) and \(L\) is a manifold with boundary. Nonetheless, for \(i = 0,1\), we have \(\varphi_r^{-1} \circ f_r(S^{n-1} \times \{i\}) \subset \Lambda_{i,0}\). Moreover, \(\varphi_r^{-1} \circ f_r\) is close in the \(C^1\) topology to \(f_0\) and \(f_0(L) \subset \psi(V)\). So, the claim follows.

For \(r \in [0,\epsilon)\), let

\[
\kappa_r = \pi_L \circ \psi^{-1} \circ \varphi_r^{-1} \circ f_r : L \to L.
\]

Since \(\kappa_0 = \id_L\), after possibly shrinking \(\epsilon\), we may assume that \(\kappa_r\) is a diffeomorphism for \(r \in [0,\epsilon)\). By Lemma [2.20], there exists a unique \(u_r \in C^\infty_{\text{C}^{1,\alpha}}(L)\) such that

\[
\varphi_r^{-1} \circ j_{u_r} \circ \kappa_r^{-1}.
\]

We claim that for \(r \in [0,\epsilon)\) we have \(u_r \in C^\infty(L;\partial L)\). Indeed, let \(f_{s,r} : L \to \mathbb{C}^n\) be given by

\[
f_{s,r}(p) := sf_r(p), \quad s \in [0,1], \quad r \in [0,\epsilon), \quad p \in L.
\]

Since for \(r \in [0,\epsilon)\) the Hamiltonian of the geodesic \((\Lambda_{t,r})_{t \in [0,1]}\) is 2-homogeneous, it follows that for \(s \in [0,1]\),

\[
Z_{s,r} := [f_{s,r}] \in \mathcal{SLC}(S^{n-1};\Lambda_{0,r},\Lambda_{1,r})
\]

is the cylinder of \(-s^2\) level sets of the geodesic \((\Lambda_{t,r})_{t \in [0,1]}\). By Lemma [2.34] we have

\[
\text{RelFlux}((Z_{s,r})_{s \in (0,1)}) = -1.
\]

Consider the two-parameter family of Lagrangian cylinders,

\[
Z_{s,r}' := \{\varphi_r^{-1} \circ f_{s,r} \} \in \mathcal{LC}(S^{n-1};\Lambda_{0,0},\Lambda_{1,0}), \quad (s,r) \in (0,1] \times [0,\epsilon).
\]

By Remark [2.33] and equation [16], we have

\[
\text{RelFlux}((Z_{s,r}')_{s \in (0,1)}) = -1, \quad r \in [0,\epsilon).
\]

Since \(f_{0,r} = 0\) for \(r \in [0,\epsilon)\), it follows from Remark [2.33] that

\[
\text{RelFlux}((Z_{0,r}')_{r \in [0,\epsilon)}) = 0, \quad r_1 \in [0,\epsilon).
\]

Let \(A_r \in \mathbb{R}\) be the unique constant such that \(\frac{du}{dt}|_{S^{n-1} \times \{1\}} \equiv A_r\). By Definition [2.32] we have

\[
\text{RelFlux}((Z_{0,r}')_{r \in [0,\epsilon)}) = \int_0^{r_1} A_r.
\]

It follows from the fundamental theorem of calculus that \(A_r = 0\). Since \(u_0 = 0\), we obtain \(u_r \in C^\infty(L;\partial L)\) as desired. □
The following lemma is a simple variant of [33] Theorem 1.6.

**Lemma 4.8.** In Setting 4.7, there exists an \( \epsilon > 0 \) such that the geodesic \( (\Lambda_t, 0) \), extends to a smooth family \( (\Lambda_t, r) \in [0,1] \times [0,\epsilon) \) such that for \( r \in [0, \epsilon) \), the path \( (\Lambda_t, r) \in [0,1] \times [0,\epsilon) \) is a geodesic in \( \mathcal{LCO}^+(n) \) with negative derivative between \( \Lambda_0, r \) and \( \Lambda_1, r \). Moreover, if \( (\Lambda'_t, r) \in [0,1] \times [0,\epsilon) \) is another such family of geodesics, but with the dependence on \( r \) being a priori only continuous with respect to the \( C^{1,\alpha} \) topology on \( \mathcal{LCO}^+(n) \), there exists \( \epsilon' > 0 \) such that \( \Lambda'_t, r = \Lambda_t, r \) for \( (t, r) \in [0,1] \times [0, \epsilon') \).

**Proof.** Recall Notation 4.6. Let \( 0 \in W \subset C^{2,\alpha}(L; \partial L) \), be an open neighborhood such that for \( u \in W \) we have \( \text{Graph}(du) \subset V \). Define the differential operator

\[
\mathcal{F} : W \times [0,1] \rightarrow C^\alpha(L), \quad (u, r) \mapsto s_{j_u}^L \varphi^*_u \Re \Omega.
\]

Then \( \mathcal{F} \) is smooth with \( \mathcal{F}(0,0) = 0 \) and \( \mathcal{F}(u, r) = 0 \) if and only if the immersion \( \varphi_r \circ j_u : L \rightarrow \mathbb{C}^n \) is imaginary special Lagrangian. By Lemma 2.27 the linearization of \( \mathcal{F} \) at \( (0, 0) \) in the directions of \( W \) is equal to the Riemannian Laplacian, which is an isomorphism \( C^{2,\alpha}(L; \partial L) \rightarrow C^\alpha(L) \). By the implicit function theorem, shrinking \( \epsilon \) if necessary, for some \( \epsilon > 0 \), there exists a smooth map \( k : [0, \epsilon) \rightarrow W \) such that, for \( r \in [0, \epsilon) \), the function \( k(r) \) is the unique element in \( W \) satisfying \( \mathcal{F}(k(r), r) = 0 \). By elliptic regularity (e.g. [13] Chapter 17), all the functions \( k(r) \) are smooth. For \( r \in [0, \epsilon) \), we have \( Z_r := [\varphi_r \circ j_u(r)] : L \rightarrow \mathbb{C}^n \in \mathcal{SCLC} (S^{n-1}; \Lambda_0, r, \Lambda_1, r) \). Diminishing \( \epsilon \) if necessary, the cylinders \( (Z_r)_{r \in [0,\epsilon]} \) are nowhere tangent to the Euler vector field. For \( r \in [0, \epsilon] \), let \( (\Lambda_t, r)_{t \in [0,1]} \) denote the geodesic associated to the cylinder \( Z_r \) by Lemma 4.7. Then \( (\Lambda_t, r)_{t \in [0,1]} \) is the desired family.

To prove the uniqueness claim, let \( u_\epsilon \in C^\infty(L; \partial L), r \in (0, \epsilon') \), be the family of functions associated \( \Lambda'_t, r \) as in Lemma 4.7. Then \( \mathcal{F}(u_\epsilon, r) = 0 \), so by the uniqueness of \( k(r) \), we have \( u_\epsilon = k(r) \).

**Proposition 4.9.** Let \( (\Lambda_0, r)_{r \in [0,1]} \) and \( (\Lambda_1, r)_{r \in [0,1]} \) be smooth paths in \( \mathcal{LCO}^+(n) \) with \( \Lambda_0, r \cap \Lambda_1, r = \{0\} \), \( r \in [0,1] \). Let \( (\Lambda_t, r)_{(t,r) \in [0,1] \times [0,1]} \) be a family in \( \mathcal{LCO}^+(n) \) such that for \( r \in [0,1] \) the path \( (\Lambda_t, r)_{t \in [0,1]} \) is a geodesic with negative derivative between \( \Lambda_0, r \) and \( \Lambda_1, r \) depending continuously on \( r \) with respect to the \( C^{1,\alpha} \) topology on \( \mathcal{LCO}^+(n) \). Suppose the initial geodesic \( (\Lambda_t, 0) \) lies in \( \mathcal{LCO}^+(n) \). Then the entire given family lies in \( \mathcal{LCO}^+(n) \).

**Proof.** Let \( A \subset [0,1] \) denote the set consisting of values of \( r \) such that the geodesic \( (\Lambda_t, r) \) lies in \( \mathcal{LCO}^+(n) \). By assumption, \( A \) is non-empty. As \( \mathcal{LCO}^+(n) \subset \mathcal{LCO}^+(n) \) is a closed subset, \( A \) is closed in \([0,1] \). By Lemma 3.13 (a) we have \( m(\Lambda_0, 0, \Lambda_1, 0) = 0 \), and it follows that \( m(\Lambda_0, r, \Lambda_1, r) = 0 \) for \( r \in [0,1] \). By Proposition 3.18 and the uniqueness part of Lemma 4.8 the set \( A \) is open in \([0,1] \).

5. **Blowing up at a critical point of a geodesic**

Throughout this section, let \( (X, \omega, J, \Omega) \) denote a Calabi-Yau manifold as in Definition 2.2. We give a blowup procedure that is used to establish \( C^{1,1} \) regularity of geodesics of positive Lagrangians in \( X \).

### 5.1. Geodesic of tangent cones

The following lemma, needed for the proof of Theorem 1.3, is the original observation behind this article. We make use of Notation 2.21.

**Lemma 5.1.** Let \( (\Lambda_t)_{t \in [0,1]} \) be a geodesic of positive Lagrangians in \( X \) with derivative \( (h_t)_{t} \). Suppose the endpoints \( \Lambda_0 \) and \( \Lambda_1 \) are smoothly embedded and let
$q \in \text{Crit}((\Lambda_t)_t)$. Then, the path $(TC_{q_t}\Lambda_t)_{t\in[0,1]}$ is a geodesic of positive Lagrangian cones in $T_qX$ with derivative given by
\[
\left(\frac{d}{dt}TC_{q_t}\Lambda_t\right)(v) = \frac{1}{2} \nabla_v dh_t(v), \quad t \in [0,1], \ v \in TC_{q_t}\Lambda_t.
\]
Moreover, if $(\Psi_t : L \to X, S)_t$ is a horizontal lifting of $(\Lambda_t)_t,$ and $p \in L$ with $\Psi_t(p) = q,$ then the family of cone derivatives
\[
(\partial \Psi_t)_p : T_pL \to T_qX, \quad t \in [0,1],
\]
is a horizontal lifting of $(TC_{q_t}\Lambda_t)_{t\in[0,1]}$.

**Definition 5.2.** In the situation of Lemma 5.1, we call $(TC_{q_t}\Lambda_t)_{t\in[0,1]}$ the geodesic of tangent cones of $(\Lambda_t)_t$ at $q.$

The proof of Lemma 5.1 relies on the following elementary observation (compare with [24, Lemma 2.1]).

**Lemma 5.3.** Let $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ be smooth with $f(x,0) = 0$, $x \in \mathbb{R}^k$. Then we have $f(x,s) = s \cdot g(x,s)$ for $(x,s) \in \mathbb{R}^k \times \mathbb{R}$, where $g : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ is smooth and satisfies
\[
g(x,0) = \frac{\partial f}{\partial s}(x,0), \quad \frac{\partial g}{\partial s}(x,0) = \frac{1}{2} \frac{\partial^2 f}{\partial s^2}(x,0), \quad x \in \mathbb{R}^k.
\]

**Proof of Lemma 5.1** Identify a neighborhood $q \in U \subset X$ and a ball centered at zero $V \subset \mathbb{C}^n$ via a Darboux parameterization
\[
X : V \to U
\]
such that $X(0) = q$ and $X^{-1}(\Lambda_0)$ and $X^{-1}(\Lambda_1)$ are contained in real linear subspaces of $\mathbb{C}^n$. For $s \geq 0$, let $M_s : \mathbb{C}^n \to \mathbb{C}^n$ denote multiplication by $s,$ and write
\[
V_s := M_s^{-1}(V).
\]
For $s > 0$, define a one-parameter family, $\Omega_s := s^{-n}M_s^*X^*\Omega,$ of differential forms on $V_s.$ Then there is a smooth form $\Omega_0$ on $V_0 = \mathbb{C}^n$ such that
\[
\Omega_s \underset{s \to 0}{\to} \Omega_0,
\]
where the convergence is in the $C^\infty$-sense on compact subsets. In fact, $\Omega_0$ is obtained by extending the alternating multi-linear form $(X^*\Omega)_0$ on $T_0V_0$ to a constant coefficient differential form on $V_0$.

Choose a horizontal lifting $(\Psi_t : L \to X, S)_t$ of the geodesic $(\Lambda_t)_t.$ Let $p \in L$ satisfy $\Psi_t(p) = q.$ Let $p \in W \subset L$ be a neighborhood such that $\Psi_t(W) \subset U$, $t \in [0,1].$

After perhaps shrinking $W$, identify $W$ with a ball $B \subset \mathbb{R}^n$ centered at zero by a diffeomorphism $Y : B \to W$ such that $Y(0) = p.$ For $s \geq 0,$ let $\mu_s : \mathbb{R}^n \to \mathbb{R}^n$ denote multiplication by $s$ and write
\[
B_s := \mu_s^{-1}(B).
\]
For $s > 0$, define a family of Lagrangian cone-immersions
\[
\chi_{s,t} : B_s \to V_s
\]
by
\[
\chi_{s,t} = M^{-1}_s \circ X^{-1} \circ \Psi_t \circ Y \circ \mu_s.
\]
By Lemma 5.3 since
\[
X^{-1} \circ \Psi_t \circ Y \circ \mu_0(x) = 0, \quad x \in B_0,
\]
Thus, identifying $\mathbb{C}^n, \mathbb{R}^n$, with their respective tangent spaces at zero, we have

\begin{equation}
\frac{d}{ds}X_0(\chi_{0,t}(x)) = (d\Psi_t)_p(dY_0(x)).
\end{equation}

In particular,

\begin{equation}
dX_0 \circ \chi_{0,t} : \mathbb{R}^n \to TC_{\hat{q}, t} \Lambda_t
\end{equation}
is a cone smooth parameterization.

We proceed to compute the Hamiltonian $k_{s,t}$ of the family of Lagrangian cone-immersions $\chi_{s,t}$. Without loss of generality, we may assume that $h_t(q_t) = 0$. Since $X$ is a symplectomorphism, and $M^{-1} \chi$ rescales the symplectic form by a factor of $s^{-2}$, it follows that $k_{s,t}$ is given by

\begin{equation}
k_{s,t} \circ \chi_{s,t} = s^{-2} h_t \circ \Psi_t \circ Y \circ \mu_s, \quad s > 0.
\end{equation}

Remark 2.22 asserts that $\hat{q}_t = [\![ (\Psi_t, p) ]\!]$ is a critical point of $h_t$, so Lemma 5.3 implies that $k_{s,t} \circ \chi_{s,t}$ extends smoothly to $s = 0$. Moreover,

\begin{equation}
k_{0,t} \circ \chi_{0,t}(x) = \frac{1}{2} \left( \frac{\partial^2}{\partial s^2} \right) h_t \circ \Psi_t \circ Y(sx) \bigg|_{s=0} = \frac{1}{2} \left| \nabla_v dh_t(v) \right|,
\end{equation}

where

\begin{equation}
v = d(\Psi_t \circ Y)_0(x) \in TC_{\hat{q}, t} \Lambda_t.
\end{equation}

Since $(\Psi_t, S)_t$ is a horizontal lifting of $(\Lambda_t)_t$, it follows that $(\chi_{s,t}, 0)_t$ is a horizontal lifting of the path of cone-immersed Lagrangians $\{(\chi_{s,t}, 0)\}_t$ with respect to the $n$-form $\Omega_t$ for $s > 0$. By continuity, $(\chi_{s,t}, 0)_t$ is a horizontal lifting of the path of cone-immersed Lagrangians $\{(\chi_{s,t}, 0)\}_t$ with respect to the $n$-form $\Omega_t$. Since $\Lambda_t$ is a geodesic, $(k_{s,t})_t$ is constant in $t$ and thus $\{(k_{s,t})_t\}_t$ is a geodesic for $s \geq 0$. Since $(X^\ast \Omega)_0 = (\Omega)_0$, it follows that $dX_0 \circ \chi_{0,t}$ is a horizontal lifting of the path of Lagrangian cones $(TC_{\hat{q}, t} \Lambda_t)_t$. Moreover, $(TC_{\hat{q}, t} \Lambda_t)_t$ is a geodesic with Hamiltonian as claimed. Finally, it follows from equation (17) that $(d\Psi_t)_p : T_p L \to T_q X, t \in [0, 1]$, is a horizontal lifting of $(TC_{\hat{q}, t} \Lambda_t)_{t \in [0, 1]}$.

5.2. Cylindrical transform commutes with blowup.

**Lemma 5.4.** Let $\Lambda_0, \Lambda_1$, be smoothly embedded positive Lagrangians in $X$ that intersect transversally at a point $q$ of Maslov index 0. Let $Z \subset SL\mathcal{C}(S^{n-1}; \Lambda_0, \Lambda_1)$ be a family of imaginary special Lagrangian cylinders converging regularly to $q$. Let $\Phi : S^{n-1} \times [0, 1] \times [0, 1) \to X$ be a regular parameterization of $Z$. Then,

\begin{equation}
\frac{\partial \Phi}{\partial s} \bigg|_{s=0} : S^{n-1} \times [0, 1] \to T_q X
\end{equation}
is an imaginary special Lagrangian immersion with respect to the induced Calabi-Yau structure on $T_q X$.

**Proof.** Identify a neighborhood $q \in U \subset X$ and a ball centered at zero $V \subset \mathbb{C}^n$ via a Darboux parameterization

\begin{equation}
X : V \to U
\end{equation}
such that $X(0) = q$ and $X^{-1}(\Lambda_0)$ and $X^{-1}(\Lambda_1)$ are contained in real linear subspaces of $\mathbb{C}^n$. For $s \geq 0$, let $M_s : \mathbb{C}^n \to \mathbb{C}^n$ denote multiplication by $s$, and write

\begin{equation}
V_s := M_s^{-1}(V).
\end{equation}
For $s > 0$, define a one-parameter family, $\Omega_s := s^{-n}M^n_x X^*\Omega$, of differential forms on $V_s$. Then there is a smooth form $\Omega_0$ on $V_0 = \mathbb{C}^n$ such that

$$\Omega_s \xrightarrow{s \to 0} \Omega_0,$$

where the convergence is in the $C^\infty$-sense on compact subsets. In fact, $\Omega_0$ is obtained by extending the alternating multi-linear form $(X^*\Omega)_0$ on $T_0 V_0$ to a constant coefficient differential form on $V_0$.

Choose $\delta > 0$ small enough that $\Phi(p, t, s) \in U$ for all $(p, t) \in S^{n-1} \times [0, 1]$ and $s < \delta$. By Lemma 5.3 since $\Phi(t, p, 0) = q$ for all $(t, p) \in S^{n-1} \times [0, 1]$, we have

$$X^{-1} \circ \Phi(p, t, s) = s \cdot \Psi(p, t, s), \quad (p, t, s) \in S^{n-1} \times [0, 1] \times [0, \delta),$$

where $\Psi : S^{n-1} \times [0, 1] \times [0, \delta) \to \mathbb{C}^n$ is smooth with

$$\Psi(p, t, 0) = \frac{\partial (X^{-1} \circ \Phi)}{\partial s}(p, t, 0), \quad (p, t) \in S^{n-1} \times [0, 1].$$

For $s \in [0, \delta)$, write

$$\Psi_s := \Psi|_{S^{n-1} \times [0, 1] \times \{s\}}.$$

For $s \in (0, \delta)$, the map $\Psi_s$ is an immersion representing an $\Omega_s$-imaginary special Lagrangian immersion. As $\Phi$ is regular, it follows from Definition 2.36 (2) that the cylinders in $S^{n-1}$ are nowhere tangent to the Euler vector field. Thus,

$$\left.\frac{\partial \Phi}{\partial s}\right|_{s=0} = dX_0 \circ \left.\frac{\partial (X^{-1} \circ \Phi)}{\partial s}\right|_{s=0} = dX_0 \circ \Psi_0$$

is an imaginary special Lagrangian immersion with respect to the induced Calabi-Yau structure on $T_q X$.

**Definition 5.5.** In the setting of Lemma 5.4, let $Z_q^g \in \mathcal{S}\mathcal{L}\mathcal{C}(S^{n-1}; T_q \Lambda_0, T_q \Lambda_1)$ be given by

$$Z_q^g := \left[\left.\frac{\partial \Phi}{\partial s}\right|_{s=0}\right] : S^{n-1} \times [0, 1] \to T_q X,$$

and let $Z_q$ denote the $\mathbb{R}_{\geq 0}$ orbit of $Z_q^g$ in $\mathcal{S}\mathcal{L}\mathcal{C}(S^{n-1}; T_q \Lambda_0, T_q \Lambda_1)$. We call $Z_q$ the tangent family of $Z$ at $q$. One verifies that $Z_q$ is independent of $\Phi$. By Definition 2.36 (2) the cylinders in $Z_q$ are nowhere tangent to the Euler vector field.

In the following, we use Notation 2.21.

**Lemma 5.6.** Let $(\Lambda_t)_{t \in [0, 1]}$ be a geodesic of positive Lagrangians in $X$ with derivative $(\ell_t)_t$. Suppose the endpoints $\Lambda_0$ and $\Lambda_1$ are smoothly embedded and let $q \in \text{Crit}(\Lambda_t)_{t \in [0, 1]}$ be a transverse intersection point of $\Lambda_0$ and $\Lambda_1$. Moreover, for $t \in [0, 1]$ assume $\hat{q}_t$ is an absolute minimum or maximum of $\ell_t$. Let $Z$ denote the cylindrical transform of $(\Lambda_t)_{t \in [0, 1]}$, which converges regularly to $q$ by Lemma 4.4. Then, the tangent family $Z_q \subset \mathcal{S}\mathcal{L}\mathcal{C}(S^{n-1}; T_q \Lambda_0, T_q \Lambda_1)$ coincides with the cylindrical transform of the geodesic of tangent cones $(TC_{\hat{q}_t} \Lambda_t)_{t \in [0, 1]}$.

**Proof.** By definition, the tangent family $Z_q$ is an $\mathbb{R}_{\geq 0}$ orbit in the space of imaginary special Lagrangian cylinders $\mathcal{S}\mathcal{L}\mathcal{C}(S^{n-1}; T_q \Lambda_0, T_q \Lambda_1)$. By Lemma 4.3 so is the cylindrical transform of the geodesic of tangent cones $(TC_{\hat{q}_t} \Lambda_t)_{t \in [0, 1]}$. Thus, to prove the lemma, it suffices to show that the two orbits share a common point.

Recall Setting 2.40 and let $\sigma : \mathbb{F}^+(T_p L) \cong S^{n-1} \to T_p L \setminus \{0\}$ be the section associated with $\kappa$. By Lemma 2.41 the map

$$\Phi : S^{n-1} \times [0, 1] \times [0, \epsilon) \to X, \quad (c, t, s) \mapsto \Psi_t(\kappa(c, s)),$$
is a regular parameterization of $Z$ about $q$. We calculate

$$
\frac{\partial \Phi}{\partial s}(c, t, 0) = (d\Psi_t)_p \left( \frac{\partial h}{\partial s}(c, 0) \right) = (d\Psi_t)_p(\sigma(c)).
$$

Lemma 5.1 asserts that $((d\Psi_t)_p)_t$ is a horizontal lifting of the geodesic of tangent cones $(TC_q, \Lambda_t)_t$. Let

$$
h^T_t = \frac{d}{dt}TC_q, \Lambda_t
$$
denote the Hamiltonian of the geodesic of tangent cones and let $h^T_t = h^T_t \circ (d\Psi_t)_p$ denote the Hamiltonian with respect to the horizontal lifting $((d\Psi_t)_p)_t$. By Lemma 5.1 we have

$$
h^T_t(v) = \frac{1}{2} \nabla_v dh(v), \quad v \in T_p L.
$$

Keeping in mind that $p$ is a critical point of $h$, we have

$$
\nabla_\sigma dh(\sigma(c)) = \frac{\partial^2 (h \circ \kappa)}{\partial s^2}(c, 0).
$$

Thus, since $h \circ \kappa(c, s)$ is independent of $c$, also $a := \nabla_\sigma dh(\sigma(c))$ is independent of $c$. So, it follows from equation (19) that $\sigma$ parameterizes the $\frac{a}{2}$ level set of $h^T$. By equation (18), the cylinder $Z^*_\Psi$ from Definition 5.5 coincides with the cylinder of $\frac{a}{2}$ level sets associated to $(TC_q, \Lambda_t)_t$ as desired. \qed

5.3. $C^{1,1}$ regularity from blowing up. The following lemmas are used in the proof of Theorem 1.3 as well as Theorem 1.1 (b).

**Lemma 5.7.** Let $\Theta$ be a compact manifold with corners and let $((f_t : M \to N, S))_{t \in \Theta}$ be a cone smooth family of maps. Let $p \in S$ such that the cone-derivative $(df_t)_p$ is linear for all $t$. Then, $f_t : M \to N$ is $C^{1,1}$ in a neighborhood of $p$ uniformly in $t$.

**Proof.** It suffices to prove the lemma when $N = \mathbb{R}$. It follows from the definition of the cone derivative that $f_t$ is differentiable at $p$ and its ordinary derivative coincides with its cone derivative. After subtracting a smooth family of functions, we may assume that $(df_t)_p = 0$.

Recall Definition 2.9. We think of the blowup differential $\tilde{df}_t$ as a section of the dual of the blowup tangent bundle $\tilde{TM}^*_S$. It follows from equation (5) that $(\tilde{df}_t)_{\tilde{p}}$ vanishes for all $\tilde{p} \in E_p$. Choose an open neighborhood $E_p \subset U \subset \tilde{M}_S$ and a diffeomorphism $\chi : S^{n-1} \times [0, \epsilon) \to U$. Let $s : U \to \mathbb{R}$ be given by the composition of $\chi^{-1}$ and the projection to $[0, \epsilon)$. By Lemma 5.3, there exists a smooth family of sections $\xi_t$ of $\tilde{TM}^*_S|U$ such that $\tilde{df}_t = s\xi_t$. Let $d : M \times M \to \mathbb{R}$ denote the distance function of a smooth Riemannian metric on $M$ and let $\delta : \tilde{M}_S \times \tilde{M}_S \to \mathbb{R}$ denote the distance function of a smooth Riemannian metric on $\tilde{M}_S$. Then, there exists a constant $C > 0$ such that for

$$
\tilde{x}, \tilde{y} \in U, \quad x = \pi(\tilde{x}), \quad y = \pi(\tilde{y}),
$$

we have

$$
\min(s(\tilde{x}), s(\tilde{y})) \delta(\tilde{x}, \tilde{y}) \leq Cd(x, y), \quad |s(\tilde{x}) - s(\tilde{y})| \leq Cd(x, y).
$$

Choose a local trivialization of $T^*M$ near $p$ and pull it back to obtain a local trivialization of $\tilde{TM}^*_S$ near $E_p$. Working in these trivializations, and assuming without
loss of generality that \( \min(s(\tilde{x}), s(\tilde{y})) = s(\tilde{x}) \), we have

\[
\frac{|(d\tilde{f})_x - (d\tilde{f})_y|}{d(x, y)} = \frac{|(\tilde{d}f)_x - (\tilde{d}f)_y|}{d(x, y)} = \frac{|s(\tilde{x})\xi_t(\tilde{x}) - s(\tilde{y})\xi_t(\tilde{y})|}{d(x, y)} \leq \frac{|s(\tilde{x})\xi_t(\tilde{x}) - s(\tilde{y})\xi_t(\tilde{y})| + |s(\tilde{x}) - s(\tilde{y})|\xi_t(\tilde{y})}{d(x, y)} \leq C\frac{|\xi_t(\tilde{x}) - \xi_t(\tilde{y})|}{d(\tilde{x}, \tilde{y})} + C|\xi_t(\tilde{y})|
\]

which is uniformly bounded because \( \xi_t \) is a smooth family. \(\square\)

**Lemma 5.8.** Let \((\Lambda_t)_{t \in [0, 1]}\) be a geodesic of positive Lagrangians in \(X\) with derivative \((h_t)_t\). Suppose the endpoints \(\Lambda_0\) and \(\Lambda_1\) are smoothly embedded and let \(q \in \text{Crit}(\Lambda_t)\). If the tangent cones \(TC_{\tilde{q}}\Lambda_t\) are linear for \(t \in [0, 1]\), then the geodesic \((\Lambda_t)_t\) is of regularity \(C^{1,1}\) in a neighborhood of \(q\).

**Proof.** Let \((\Psi_t : S^n \to X, S)_{t \in [0, 1]}\) be a horizontal lifting of the geodesic \(\Lambda_t\) with \(\Psi_0\) smooth and let \(p \in S\) such that \(\Psi_t(p) = q\). By Lemma 5.1 the family of cone derivatives \((d\Psi_t)_p : T_pS^n \to T_qX\) is a horizontal lifting of the geodesic of Lagrangian linear subspaces \(TC_{\tilde{q}}\Lambda_t \subset T_qX\). Since \(\Psi_0\) is smooth, the cone derivative \((d\Psi_0)_p\) is linear, so by Lemma 3.2 the cone derivative \((d\Psi_t)_p\) is linear for \(t \in [0, 1]\). Thus, by Lemma 5.7 the cone immersions \((\Psi_t : S^n \to X, S)\) are \(C^{1,1}\) in a neighborhood of \(p\) uniformly in \(t\). Furthermore, the time derivative \(\frac{\partial \Psi_t}{\partial t}\) exists by definition of a cone smooth family.

Next, we claim that the differential \((d\Psi_t)_x\) is \(C^{0,1}\) as a function of \(t\) uniformly in \(x \in S^n\) in a neighborhood of \(p\). Indeed, the blowup derivative \((d\Psi_t)_x\) depends smoothly on \(t\) and \(\tilde{x}\). So, it is \(C^{0,1}\) in \(\tilde{x}\) uniformly in \(\tilde{x}\). On the other hand, for \(\tilde{x}\) such that \(\pi(\tilde{x}) = x\), we have

\[
(d\Psi_t)_x = (d\Psi_1)_x.
\]

Finally, we show the partial derivative \(\frac{\partial \Psi_t}{\partial t}\) is \(C^{0,1}\) in a neighborhood of \(p\). Indeed, since \(\tilde{q}\) is a critical point of \(h_t\), Lemma 5.7 implies that \(h_t\) is \(C^{1,1}\). So, \(dh_t\) is \(C^{0,1}\).

Write

\[
h = h_t \circ \Psi_t,
\]

for the Hamiltonian associated to the horizontal lifting \((\Psi_t)_t\), which is independent of \(t\) because \((\Lambda_t)_t\) is a geodesic. The Hamiltonian \(h\) is \(C^{1,1}\) as the composition of \(C^{1,1}\) maps. Recall that

\[
\theta_{\Lambda_t} : \Lambda_t \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
\]

denotes the Lagrangian angle, and write

\[
\tilde{\theta}_{\Lambda_t} = \theta_{\Lambda_t} \circ \Psi_t : L \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
\]

Write \(\nabla\) for the gradient with respect to the pull-back metric \(\Psi_t^*g\). Let

\[
J : TL \to \Psi_t^*TX
\]

denote the bundle map given by

\[
J(\xi) = J \circ d\Psi_t(\xi).
\]
Since $d\Psi_t$ is $C^{0,1}$ jointly in space and time, it follows that the families $\Psi_t^\ast g, \tilde{\theta}_\Lambda_t$ and $J^t$, are $C^{0,1}$ jointly in space and time. By [23] Remark 5.6 we have for $x \in S^n$,
\[
\frac{\partial \Psi_t}{\partial t}(x) = -J^t \nabla h_t(\Psi_t(x)) - \tan \theta_{\Lambda_t}(\Psi_t(x)) \nabla h_t(\Psi_t(x))
= -J^t \nabla h(x) - \tan \theta_{\Lambda_t}(x) \nabla h(x),
\]
so $\frac{\partial \Psi_t}{\partial t}(x)$ is $C^{0,1}$ as desired.

It follows that $\Psi_t$ is $C^{1,1}$ as a map $S^n \times [0, 1] \to X$ and the geodesic $(\Lambda_t)_t$ is $C^{1,1}$ as claimed.

**Proof of Theorem 1.3.** Theorem 1.3 is the same as Theorem 1.6 of [33] with the exception of the assumption that $(\Lambda_t)_t$ is $C^1$ and the claim that for $\Lambda \in \mathcal{Y}$ the geodesic from $\Lambda_0$ to $\Lambda$ is of regularity $C^{1,1}$. Possibly shrinking $\mathcal{Y}$, we may assume that all $\Lambda \in \mathcal{Y}$ intersect $\Lambda_0$ transversally at exactly two points. After possibly replacing $\mathcal{Y}$ with its connected component containing $\Lambda_1$, we may assume that $\mathcal{Y}$ is connected. For $\Lambda \in \mathcal{Y}$, consider a smooth path $(\Lambda_{1,r})_{r \in [0, 1]}$ with $\Lambda_{1,0} = \Lambda_1$ and $\Lambda_{1,1} = \Lambda$. For $r \in [0, 1]$ let $(\Lambda_{t,r})_{t \in [0, 1]}$ be the unique geodesic in $\mathcal{X}$ from $\Lambda_0$ to $\Lambda_{1,r}$. We prove that $(\Lambda_{t,r})_{t \in [0, 1]}$ is of regularity $C^{1,1}$.

Indeed, for $(t, r) \in [0, 1] \times [0, 1]$, write
\[
h_{t,r}(x) := \frac{d}{dt} \Lambda_{t,r}.
\]
Recalling Remark 2.22 and Notation 2.21 let $\hat{q}_{t,r}$ be the point of $\Lambda_{t,r}$ where $h_{t,r}$ attains its maximum and let $q_r := \text{im} \hat{q}_{t,r} \in \text{Crit}((\Lambda_{t,r})_t)$.

By Lemma 2.23 the point $\hat{q}_{t,r}$ is a non-degenerate critical point of $h_{t,r}$. Thus, for $0 \neq v \in TC_{\hat{q}_{t,r}} \Lambda_{t,r}$ we have
\[
\nabla_v d\Lambda_{t,r}(v) < 0.
\]
Let
\[
(\varphi_r : \mathbb{C}^n \to T_{q_r} X)_{r \in [0, 1]}
\]
be a smooth family of symplectic complex-linear isomorphisms such that for $r \in [0, 1]$ we have
\[
\varphi_r^\ast \Omega = \rho(q_r) \Omega_0,
\]
where $\Omega_0$ denotes the standard holomorphic $n$-form on $\mathbb{C}^n$. For $(t, r) \in [0, 1] \times [0, 1]$, write
\[
C_{t,r} := \varphi_r^{-1}(TC_{\hat{q}_{t,r}} \Lambda_{t,r}).
\]
By Lemma 5.1 and inequality (20), for $r \in [0, 1]$ the path $(C_{t,r})_{t \in [0, 1]}$ is a geodesic in $\mathcal{LCO}^+(n)$ with negative derivative. By definition of the strong $C^{1,\alpha}$ topology on $\mathcal{G}_\mathcal{O}$, the geodesic $(C_{t,r})_{t \in [0, 1]}$ depends continuously on $r$ with respect to the $C^{1,\alpha}$ topology on $\mathcal{G}_\mathcal{O}^+(n)$.

By Assumption, the geodesic $(\Lambda_{t,0})_t$ is linear. By Proposition 4.9 all the cones $C_{t,r}$ are in fact linear subspaces, and so are the tangent cones $TC_{\hat{q}_{t,r}} \Lambda_{t,r}$. Similarly, when $\hat{q}_{t,r}$ is the point of $\Lambda_{t,r}$ where $h_{t,r}$ attains its minimum, the tangent cones $TC_{\hat{q}_{t,r}} \Lambda_{t,r}$ are linear. So, by Lemma 5.8 the geodesic $(\Lambda_{t,r})_{t \in [0, 1]}$ is of regularity $C^{1,1}$.

6. **Geodesics of small open Lagrangians**

In this section we show how to integrate geodesics of positive Lagrangian cones to construct geodesics of small open positive Lagrangians. The section culminates with the proof of Theorem 1.1. In the following we use Notation 2.21.
Lemma 6.1. Let \((X, \omega, J, \Omega)\) be Calabi-Yau, and let \(\Lambda_0, \Lambda_1 \subset X\) be smoothly embedded positive Lagrangians intersecting transversally at a point \(q\). Let \((C_t)_{t \in [0,1]}\) be a geodesic of positive Lagrangian cones with positive/negative derivative in \(T_qX\) such that \(C_i = T_q \Lambda_i\) for \(i = 0, 1\).

(a) There exists an one-parameter family \((Z_s)_{s \in (0,\epsilon)} \subset S\mathcal{L}\mathcal{C}\left(S^{n-1}; \Lambda_0, \Lambda_1\right)\) converging regularly to \(q\) with tangent family the cylindrical transform of \((C_t)\). This family is unique up to reparameterization.

(b) There exist open neighborhoods, \(q \in U_i \subset \Lambda_i, \ i = 0, 1\), which are connected by a geodesic \((U_i)_{t \in [0,1]}\) of open positive Lagrangians with tangent cones \(TC_0 U_i = C_i\). Given \(U_i, i = 0, 1\), such a geodesic is unique up to reparameterization.

Proof. Identify a neighborhood \(q \in U \subset X\) and a ball centered at zero \(V \subset C^n\) via a Darboux parameterization

\[
X : V \to U
\]
such that \(X(0) = q\) and \(X^{-1}(\Lambda_0)\) and \(X^{-1}(\Lambda_1)\) are contained in real linear subspaces \(\hat{\Lambda}_0, \hat{\Lambda}_1 \subset C^n\) respectively. For \(s \geq 0\), let \(M_s : C^n_\omega \to C^n\) denote multiplication by \(s\), and write

\[
V_s := M_s^{-1}(V).
\]
For \(s > 0\), define a complex structure and an \(n\)-form on \(V_s\) by

\[
J_s := M_s^* X^* J, \quad \Omega_s := s^{-n} M_s^* X^* \Omega.
\]
Then, there are a smooth complex structure \(J_0\) and a smooth \(n\)-form \(\Omega_0\) on \(V_0 = C^n\) such that

\[
J_s \underset{s \to 0}{\longrightarrow} J_0, \quad \Omega_s \underset{s \to 0}{\longrightarrow} \Omega_0,
\]
where the convergence is in the \(C^\infty\)-sense on compact subsets. In fact, \(J_0\) is obtained by extending the linear complex structure \((X^*J)_0\) on \(T_0V_0\) to a constant coefficient complex structure on \(V_0\). Similarly, \(\Omega_0\) is obtained by extending the alternating multi-linear form \((X^* \Omega)_0\) on \(T_0V_0\) to a constant coefficient differential form on \(V_0\).

Let \(\omega_0\) denote the standard symplectic form on \(V_0 = C^n\). After possibly rescaling \(\Omega_0\) by a positive constant, the quadruple \((V_0, \omega_0, J_0, \Omega_0)\) is a Calabi-Yau manifold isomorphic to \(C^n\) with the standard structure. Let \(g_0\) denote the corresponding Kähler metric. The derivative of the Darboux chart gives an isomorphism \(dX : V_0 \simeq T_0V_0 \to T_qX\) respecting the Calabi-Yau structure \((\omega_0, J_0, \Omega_0)\) and the induced Calabi-Yau structure on the tangent space \(T_qX\).

So,

\[
\tilde{\Lambda}_t := (dX)^{-1}(\tilde{C}_t), \quad t \in [0,1],
\]
is an \(\Omega_0\)-geodesic of positive Lagrangian cones from \(\tilde{\Lambda}_0\) to \(\tilde{\Lambda}_1\) with positive/negative definite derivative, which we denote by \(\left(\tilde{h}_t\right)_t\). Abbreviate \(L = S^{n-1} \times [0,1]\). Let

\[
\tilde{Z}_0 = [f : L \to V] \in S\mathcal{L}\mathcal{C}(S^{n-1}; \tilde{\Lambda}_0, \tilde{\Lambda}_1)
\]
belong to the \(R_{>0}\) orbit corresponding to the geodesic \(\left(\tilde{\Lambda}_t\right)_{t \in [0,1]}\) by Lemma 4.3. In particular, \(\tilde{Z}_0\) is nowhere tangent corresponding to the Euler vector field. Recalling Lemma 2.25, choose a Weinstein neighborhood \((W, \psi)\) of \(\tilde{Z}_0\) compatible with \(\tilde{\Lambda}_0\) and \(\tilde{\Lambda}_1\), where \(W \subset T^{\ast} L\) and \(\psi : W \to V\) with \(\psi|_L = f\). Let \(\pi_L : T^{\ast} L \to L\) denote the projection. Let \(\alpha \in (0,1)\). For \(u \in C^{2,\alpha}_{C^0}(L)\), let \(Graph(du)\) be an open neighborhood such that for \(u \in W\) we have \(Graph(du) \subset W\). For \(u \in W\), let \(j_u : L \to V\) be given by

\[
j_u = \psi \circ (\pi_L|_{Graph(du)})^{-1}.
\]
Define a differential operator
\[ F : \mathcal{W} \times [0, 1) \to C^\alpha (L), \quad (u, s) \mapsto * j_s^* \text{Re} \Omega, \]
where * denotes the Hodge star operator of the metric \( f^* g_0 \). Since \( \hat{Z}_0 \) is imaginary special Lagrangian, we have \( F(0, 0) = 0 \). The operator \( F \) is smooth and by Lemma 2.27 the linearization in the directions of \( \mathcal{W} \) is equal to the Riemannian Laplacian,
\[ dF_{(0, 0)}(u, 0) = \Delta u, \quad u \in C^{2,\alpha} (L; \partial L), \]
which is an isomorphism \( C^{2,\alpha} (L; \partial L) \to C^\alpha (L) \). By the implicit function theorem, for some \( \epsilon > 0 \) and shrinking \( \mathcal{W} \) if necessary, there exists a smooth map
\[ k : [0, \epsilon) \to \mathcal{W} \]
such that for \( s \in [0, \epsilon) \), the function \( k(s) \) is the unique element of \( \mathcal{W} \) satisfying \( F(k(s), s) = 0 \). By elliptic-regularity (e.g. Chapter 17), \( k \) is in fact a smooth map \([0, \epsilon) \to C^\infty (L; \partial L)\). The cylinder \( \hat{Z}_s := [j_{k(s)} : L \to V] \) is \( \Omega_s \)-imaginary special Lagrangian for \( s \in [0, \epsilon) \).

The one-parameter family of cylinders of part (a) is given by
\[ Z_s := [X \circ M_s \circ j_{k(s)} : L \to X], \quad s \in (0, \epsilon). \]
Indeed, all the cylinders \( Z_s \) are imaginary special Lagrangian with respect to the Calabi-Yau form \( \Omega \), so it remains to show the regularity of the family \( (Z_s) \) about \( q \). Define a map
\[ \Phi : S^{n-1} \times [0, 1) \times [0, \epsilon) \to X, \quad (p, t, s) \mapsto X \circ M_s \circ j_{k(s)}(p, t). \]
We proceed to verify that \( \Phi \) is a regular parameterization of \( (Z_s) \) about \( q \). By construction, \( \Phi_s := \Phi|_{S^{n-1} \times [0, 1) \times \{s\}} \) parameterizes \( Z_s \) for \( s \in (0, \epsilon) \), so \( \Phi_s \) satisfies condition (b) of Definition 2.36. Condition (a) is satisfied because \( X(0) = q \). Furthermore, since \( j_0 = f \), we have
\[ \left. \frac{\partial \Phi}{\partial s} \right|_{s=0} = dX_0 \circ f. \]
So, \( \Phi \) satisfies conditions (c) and (d) of Definition 2.36. So, by Lemma 2.37 possibly after diminishing \( \epsilon \), the map \( \Phi \) is a regular parameterization of \( (Z_s) \) about \( q \). Recalling Definition 5.5, it also follows from equation (21) that the tangent family of \( (Z_s) \) is the cylindrical transform of \( (C_t) \).

Next, we address uniqueness. Let \( (Y_s)_{s \in (0, \epsilon')} \subset \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \) be another family converging regularly to \( q \) with tangent family the cylindrical transform of \( (C_t) \). We show that after reparameterization in \( s \) and possibly shrinking \( \epsilon' \), the families \( (Y_s) \) and \( (Z_s) \) coincide. Indeed, let \( \Phi' : S^{n-1} \times [0, 1) \times [0, \epsilon') \to X \) be a regular parameterization of \( (Y_s) \) about \( q \). Since the tangent families of \( (Z_s) \) and \( (Y_s) \) both coincide with the cylindrical transform of \( (C_t) \), after possibly reparameterizing \( (Y_s) \) and composing \( \Phi' \) with a diffeomorphism of \( S^{n-1} \times [0, 1) \times [0, \epsilon') \), we may assume that
\[ \left. \frac{\partial \Phi'}{\partial s} \right|_{s=0} = \left. \frac{\partial \Phi}{\partial s} \right|_{s=0}. \]
Possibly shrinking \( \epsilon' \), we may assume that \( Y_s \) is contained in \( U \) for \( s \in (0, \epsilon') \). For \( \rho : [0, \epsilon'') \to [0, \epsilon') \) a diffeomorphism with \( \rho'(0) = 1 \), define \( l_0^s : L \to V \) by
\[ l_0^s(p, t) := M_{\rho^{-1}}^{-1} \circ X^{-1} \circ \Phi'(p, t, \rho^{-1}(s)). \]
By Lemma 5.3 equation (21) and equation (22), the family \( l_0^s \) extends smoothly to \( s = 0 \), with
\[ l_0^0 = f_0. \]
Possibly after composing $\Phi'$ with a diffeomorphism of $S^{n-1} \times [0, 1] \times [0, \epsilon')$, for $s \in [0, \epsilon'')$ there exists $u_s^\rho \in C^\infty_{\text{COB}}(L)$ such that $l_s^\rho = ju_s^\rho$. It follows from equation (23) that $u_s^0 = 0$.

We claim that for an appropriate choice of $\rho$, we have $u_s^\rho \in C^\infty(L; \partial L)$. Indeed, for $r \in [0, 1]$ and $s \in [0, \epsilon''),$ define $l_s^\rho : L \to V$ by

$$l_s^\rho(p, t) := rl_s^\rho(p, t).$$

Let

$$\hat{Y}_{r,s} := [l_{r,s}^\rho : L \to V] \in \mathcal{LC}(S^{n-1}; \Lambda_0, \Lambda_1).$$

For $s \in [0, \epsilon'')$, let

$$F_s^\rho := \text{RelFlux} \left( \hat{Y}_{r,s} \right)_{r \in [0, 1]}.$$

Since $M_s\omega_0 = s^2\omega_0$, we have

$$F_s^\rho/F_s^{\text{id}} = s^2/\rho(s)^2.$$

Choosing

$$\rho(s) := s\sqrt{F_s^{\text{id}}} / \sqrt{F_s^{\text{id}}},$$

it follows that $F_s^\rho$ is constant in $s$, and thus $F_s^\rho$ is also constant in $s$. Since $l_0^\rho = 0$ for $s \in [0, \epsilon'')$, it follows from Remark 2.33 [2] that

$$\text{RelFlux} \left( \hat{Y}_{1,s}^\rho \right)_{s \in [0, s_1]} = 0, \quad s_1 \in [0, \epsilon'').$$

Let $A_s \in \mathbb{R}$ be the unique constant such that $\frac{\partial u_s^\rho}{\partial s} \big|_{S^n \setminus \{1\}} \equiv A_s$. By Definition 2.32, we have

$$\text{RelFlux} \left( \hat{Y}_{1,s}^\rho \right)_{r \in [0, s_1]} = \int_{0}^{s_1} A_s.$$

It follows from the fundamental theorem of calculus that $A_s = 0$ for $s \in [0, \epsilon'')$. Since $u_s^0 = 0$, we obtain $u_s \in C^\infty(L; \partial L)$ as desired.

By construction, the cylinder

$$\hat{Y}_s^\rho := [y_{s}^\rho : L \to V_s]$$

is $\Omega_\epsilon$-imaginary special Lagrangian. So, $\mathcal{F}(u_s^\rho, s) = 0$ and by the uniqueness of $k(s)$, it follows that $u_s^\rho = k(s)$. It follows that $Y_{\rho^{-1}(s)} = Z_s$. Thus, we have established part [a] of the lemma.

We construct the geodesic $(U_t)_t$ of open positive Lagrangians of part [b] by combining Lemma 2.39 and part [a]. The tangent cones $TC_{\hat{q}}U_t$ coincide with $C_t$ by Lemma 5.6. The uniqueness claim follows from Lemma 5.8, the uniqueness claim of Lemma 2.39 and the uniqueness claim of part [a].

\[\square\]

Proof of Theorem 1.7 By Theorem 1.5 there exists a geodesic of Lagrangian linear subspaces $(\lambda_t)_{t \in [0, 1]} \subset T_qX$ with $\lambda_t = T_qA_t$ for $i = 0, 1$. Applying Lemma 6.1 with $C_t = \lambda_t$, we obtain a regular family of cylinders as in part [a] of the theorem and a geodesic of open positive Lagrangians $(U_t)_t$ as in part [b] of the theorem. Since $TC_{\hat{q}}U_t = \lambda_t$ is linear, it follows from Lemma 5.8 that $(U_t)_t$ has regularity $C^{1,1}$. \[\square\]
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