Hardy spaces $H^p$ over non-homogeneous metric measure spaces and their applications

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Abstract Let $(X,d,\mu)$ be a metric measure space satisfying both the geometrically doubling and the upper doubling conditions. Let $\rho\in(1,\infty)$, $0 < p \leq 1 < q \leq \infty$, $p \neq q$, $\gamma \in [1,\infty)$ and $\epsilon \in (0,\infty)$. In this article, the authors introduce the atomic Hardy space $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$ and the molecular Hardy space $\tilde{H}_{mb}^{p,q,\gamma}(\mu)$ via the discrete coefficient $\tilde{K}_{\beta}^{(\rho),p}$, and prove that the Calderón-Zygmund operator is bounded from $\tilde{H}_{mb}^{p,q,\gamma}(\mu)$ (or $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$) into $L^p(\mu)$, and from $\tilde{H}_{atb,\rho}^{p,q,\gamma+1}(\mu)$ into $\tilde{H}_{mb,\rho}^{p,q,\gamma+1/2(\beta-\frac{\gamma+\epsilon}{\gamma})}(\mu)$. The boundedness of the generalized fractional integral $T_{\beta}(\mu) (\beta \in (0,1))$ from $\tilde{H}_{mb}^{p,q,\gamma}(\mu)$ (or $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$) into $L^p(\mu)$ with $1/p = 1/p_2 - \beta$ is also established. The authors also introduce the $p$-weakly doubling condition, with $p \in (1,\infty)$, of the measure $\mu$ and construct a non-doubling measure $\mu$ satisfying this condition. If $\mu$ is $p$-weakly doubling, the authors further introduce the Campanato space $E_{\rho}^{p,q,\gamma}(\mu)$ and show that $E_{\rho}^{p,q,\gamma}(\mu)$ is independent of the choices of $p$, $\gamma$, $\epsilon$ and $\rho$. The authors then introduce the atomic Hardy space $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$ and the molecular Hardy space $\tilde{H}_{mb}^{p,q,\gamma}(\mu)$, which coincide with each other; the authors finally prove that $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$ is the predual of $E_{\rho}^{1/p-1,1}(\mu)$. Moreover, if $\mu$ is doubling, the authors show that $E_{\rho}^{p,q,\gamma}(\mu)$ and the Lipschitz space $\text{Lip}_{\alpha,\gamma}(\mu)$ ($\alpha = [1,\infty)$), or $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$ and the atomic Hardy space $H_{\rho}^{p,q,\gamma}(\mu)$ ($\rho \in (1,\infty)$) of R. R. Coifman and G. Weiss coincide. Finally, if $(X,d,\mu)$ is an RD-space with $\mu(X) = \infty$, the authors prove that $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$, $\tilde{H}_{mb}^{p,q,\gamma}(\mu)$ and $H_{\rho}^{p,q,\gamma}(\mu)$ coincide for any $\gamma \in (1,\infty)$. In particular, when $(X,d,\mu) := (\mathbb{R}^D,|\cdot|,dx)$ with $dx$ being the $D$-dimensional Lebesgue measure, the authors show that spaces $\tilde{H}_{atb}^{p,q,\gamma}(\mu)$, $\tilde{H}_{mb}^{p,q,\gamma}(\mu)$, $H_{\rho}^{p,q,\gamma}(\mu)$ and $H_{\rho}^{p,q,\gamma}(\mu)$ all coincide with $H^p(\mathbb{R}^D)$ for any $\gamma \in (1,\infty)$.

Keywords geometrically doubling measure, upper doubling measure, $p$-weakly doubling measure, non-homogeneous metric measure space, RD-space, Hardy space, Campanato space, Lipschitz space, Calderón-Zygmund operator, fractional integral, predual, atomic block, molecular block

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1 Introduction

It is well known that the real variable theory of Hardy spaces $H^p(\mathbb{R}^D)$ on the $D$-dimensional Euclidean space $\mathbb{R}^D$ has many important applications in various fields of analysis such as harmonic analysis and partial differential equations; see, for example, [12, 48–50]. When $p \in (1, \infty)$, $L^p(\mathbb{R}^D)$ and $H^p(\mathbb{R}^D)$ are essentially the same; however, when $p \in (0, 1]$, the space $H^p(\mathbb{R}^D)$ is much better adapted to problems arising in the theory of the boundedness of operators, since some of singular integrals (for example, Riesz transforms) are bounded on $H^p(\mathbb{R}^D)$, but not on $L^p(\mathbb{R}^D)$. In 1972, Fefferman and Stein [12] showed that the Hardy space $H^1(\mathbb{R}^D)$ is the predual of the space $\text{BMO}(\mathbb{R}^D)$. Later, Walsh [61] proved that the dual space of the Hardy space $H^p(\mathbb{R}^D)$ is the Campanato space introduced by Campanato [4]. From then on, various characterizations of $H^p(\mathbb{R}^D)$, including the atomic and the molecular characterizations, and their applications were studied extensively in harmonic analysis; see, for example, [5, 8, 9, 16, 32, 38, 40, 51, 64]. Moreover, the atomic and the molecular characterizations enabled the extension of the real variable theory of Hardy spaces on $\mathbb{R}^D$ to spaces of homogeneous type in the sense of Coifman and Weiss [10, 11], which is a far more general setting for function spaces and singular integrals than Euclidean spaces.

Recall that a metric space $(X, d)$ equipped with a non-negative measure $\mu$ is called a space of homogeneous type, if $(X, d, \mu)$ satisfies the measure doubling condition: there exists a positive constant $C(\mu)$ such that, for all balls $B(x, r) := \{y \in X : d(x, y) < r\}$ with $x \in X$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \leq C(\mu)\mu(B(x, r)).$$

(1.1)

This measure doubling condition is one of the most crucial assumptions in the classical harmonic analysis. We point out that a space of homogeneous type in [10, 11] is endowed with a quasi-metric. However, for simplicity, throughout this article, we always assume that a space of homogeneous type is endowed with a metric.

Nevertheless, in recent years, it has been proved that many results in the classical theory of Hardy spaces and singular integrals on $\mathbb{R}^D$ remain valid with the $D$-dimensional Lebesgue measure replaced by a non-doubling measure (see, for example, [6, 24, 41, 53–59]). Recall that a Radon measure $\mu$ on $\mathbb{R}^D$ is called a non-doubling measure, if there exist positive constants $C_0$ and $\kappa \in (0, D]$ such that, for all $x \in \mathbb{R}^D$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq C_0r^\kappa,$$

(1.2)

where $B(x, r) := \{y \in \mathbb{R}^D : |y - x| < r\}$. Tolsa [53, 55] introduced the atomic Hardy space $H_{\text{atb}}^{1, q}(\mu)$, for $q \in (1, \infty]$, and its dual space, $\text{RBMO}(\mu)$, the space of functions with regularized bounded mean oscillation, with respect to $\mu$ as in (1.2), and proved that Calderón-Zygmund operators are bounded from $H_{\text{atb}}^{1, q}(\mu)$ into $L^1(\mu)$. Later, Chen, Meng and Yang [6] showed that Calderón-Zygmund operators are bounded on $H_{\text{atb}}^{1, q}(\mu)$. In [24], Hu, Meng and Yang established an equivalent characterization of $H_{\text{atb}}^{1, q}(\mu)$ to obtain the $L^q(\mu)$-boundedness of commutators and their endpoint estimates. More research on function spaces, mainly on Morrey spaces, and their applications related to non-doubling measures can be found in [20, 42–47]. We point out that the analysis on such non-doubling context plays a striking role in solving several long-standing problems related to the analytic capacity, like Vitushkin’s conjecture or Painlevé’s problem; see [56–59].

However, as was pointed out by Hytönen in [27], the measure satisfying (1.2) is different from, but not more general than, the doubling measure. In [27], Hytönen introduced a new class of metric measure spaces satisfying the so-called geometrically doubling and the upper doubling conditions (see, respectively, Definitions 2.1 and 2.3 below), which are also simply called non-homogeneous metric measure spaces. This new class of non-homogeneous metric measure spaces includes both spaces of homogeneous type and metric spaces with non-doubling measures as special cases. It is already known that singular integrals on non-homogeneous metric measure spaces arise naturally in the study of complex and harmonic analysis questions in several complex variables (see [29, 60] for the details).

In this new setting, Hytönen [27] introduced the space $\text{RBMO}(\mu)$ and established the corresponding John-Nirenberg inequality. Later, Hytönen et al. [30], and Bui and Duong [3], independently, introduced
the atomic Hardy space $\tilde{H}^{1,q}_{\text{atb}}(\mu)$ and proved that the dual space of $H^{1,1}_{\text{atb}}(\mu)$ is RBMO($\mu$). Hytönen et al. [28] and Liu et al. [37] established some equivalent characterizations for the boundedness of Calderón-Zygmund operators on $L^q(\mu)$ with $q \in (1, \infty)$ and their endpoint boundedness. Fu et al. [13] introduced a version of the atomic Hardy space $\tilde{H}^{1,q}_{\text{atb}}(\mu) \subset H^{1,q}_{\text{atb}}(\mu)$ via the discrete coefficients $K_B^{(\rho)}$, and showed that the Calderón-Zygmund operator is bounded on $\tilde{H}^{1,q}_{\text{atb}}(\mu)$ via establishing a molecular characterization of $\tilde{H}^{1,q}_{\text{atb}}(\mu)$ in this context. Recently, Fu et al. [15] introduced generalized fractional integrals and established the boundedness of generalized fractional integrals and their commutators in this setting. More research on the boundedness of various operators on non-homogeneous metric measure spaces can be found in [1, 26, 34-36]. We refer the reader to the survey [62] and the monograph [63] for more progress on the theory of Hardy spaces and singular integrals over non-homogeneous metric measure spaces.

We point out that the space $\tilde{H}^{1,q}_{\text{atb}}(\mu)$ seems to be more useful in the study on the boundedness of operators, since it was shown in [13, Theorem 1.4] that Calderón-Zygmund operators are bounded on $\tilde{H}^{1,q}_{\text{atb}}(\mu)$, but the method does not work for the boundedness of Calderón-Zygmund operators on $H^{1,q}_{\text{atb}}(\mu)$ over general non-homogeneous metric measure spaces defined via the continuous coefficients (see [13, Remark 2.4] or Remark 4.2(iv) below).

To the best of our knowledge, the theory of the Hardy space $H^p$ on non-homogeneous metric measure spaces is still unknown, even on Euclidean spaces endowed with non-doubling measures. Let $(\chi, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [27]. The main purposes of this article are two-fold. First, via the discrete coefficients $K_B^{(\rho)}$, we introduce the atomic Hardy space $\tilde{H}^{p,q,\rho}_{\text{atb},\rho}(\mu)$ and the molecular Hardy space $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$, and give their applications to the boundedness of Calderón-Zygmund operators and generalized fractional integrals. However, it is still unknown whether $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ is independent of the choices of $\rho, \gamma$ and $q$ or not even under some additional condition, called the $\rho$-weakly doubling condition (see Definition 6.1 below). Moreover, the dual space of $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ and the equivalence between $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ and $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$ are also unclear. Thus, we are forced to turn to the other goal: introduce another atomic Hardy space $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ and another molecular Hardy space $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$, and then show that $H^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ is independent of the choices of $\rho, \gamma$ and $q$ under the $\rho$-weakly doubling condition. Then we study the Campanato space $\mathcal{E}^{\alpha,q,\rho,\gamma}_{\text{atb},\rho}(\mu)$, the dual space of $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$, and the equivalence between $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ and $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$ if $\mu$ is $\rho$-weakly doubling. Moreover, if $\mu$ is doubling, we show that $\mathcal{E}^{\alpha,q,\rho,\gamma}_{\text{atb},\rho}(\mu)$ and the Lipschitz space $Lip_{\alpha,q}(\mu)$ coincide for any $q \in (1, \infty)$, or $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ and the atomic Hardy space $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ coincide for any $q \in (1, \infty)$ introduced by Coifman and Weiss in [11] coincide with equivalent quasi-norms. Finally, if $\mu$ is RD-space with $\mu(X) = \infty$, we prove that $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$, $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$ and $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ coincide for any $q \in (1, \infty)$, which is still unknown if $q \in (2, \infty]$. In particular, when $(\chi, d, \mu) := ([R^D], \cdot, dx)$ with $dx$ being the $D$-dimensional Lebesgue measure, we show that the spaces $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$, $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$, $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ and $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$ all coincide with $H^p(R^D)$ for any $q \in (1, \infty)$.

The organization of this article is as follows.

In Section 2, we first recall some necessary notation and notions, including the discrete coefficient $K_B^{(\rho)}$, and give out some fundamental properties on $K_B^{(\rho)}$, which are crucial to the succeeding content.

In Section 3, we introduce the atomic Hardy space $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ via the discrete coefficient $K_B^{(\rho)}$, where the dominating function of the considered measure appears in the size condition of the atomic block, which seems to be well adapted to the study of the boundedness of Calderón-Zygmund operators and generalized fractional integrals, and establish a useful property. The key innovation in this section is the definition of $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$ as the completeness of a subspace of $L^2(\mu)$, $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$, which is a suitable substitute of the classical fact that the set of all Schwartz functions having infinite order vanishing moments is dense in the Hardy space $H^p(R^D)$.

In Section 4, we introduce the notion of the molecular Hardy space $H^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$, and prove that the Calderón-Zygmund operator is bounded on $H^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$ (or $\tilde{H}^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho}(\mu)$) into $L^p(\mu)$ by borrowing some ideas from [30, Theorem 4.2] with much more complicated arguments, and from $H^{p,q,\rho,\gamma,\epsilon}_{\text{atb},\rho,\rho+1}(\mu)$ into $H^{p,q,\rho,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$ by using a method similar to that used in the proof of [13, Theorem 1.14] with
some technical modifications.

In Section 5, we establish the boundedness of the generalized fractional integral $T_{\beta}$ ($\beta \in (0,1)$) from $\hat{H}_{mb,\rho}^{p, q, \gamma, \delta} (\mu)$ (or $\hat{H}_{ath,\rho}^{p, q, \gamma} (\mu)$) into $L^{p_{2}} (\mu)$ with $1/p_{2} = 1/p_{1} - \beta$. The proof of the above result is parallel to that of the conclusion for Calderón-Zygmund operators in Section 4 with slight modifications. For the sake of the clearness, we present the full details there.

Section 6 is mainly devoted to the theory of Campanato spaces. We first introduce an additional assumption, called the $\rho$-weakly doubling condition (see (6.1) below), which is satisfied by spaces of homogeneous type. We also construct a non-trivial example to show that there exist some non-homogeneous metric measure spaces satisfying the $\rho$-weakly doubling condition (6.1); see Example 6.3 below. However, it turns out that there exist many non-homogeneous metric measure spaces which do not satisfy the $\rho$-weakly doubling condition; see Example 6.4 below. Then we introduce the Campanato space $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ and show that $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ is independent of the choices of $\rho, \eta, \gamma$ and $q$ under the assumption of $\rho$-weakly doubling conditions. Precisely, via a useful property of $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ (see Proposition 6.7(a) below) and the geometrically doubling condition, we prove that $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ is independent of the choices of $\rho$ and $\eta$, where the $\rho$-weakly doubling condition plays a decisive role. Then, by establishing an equivalent characterization of $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu) := E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ and a useful lemma (see Lemma 6.12 below), which is analogous to [30, Lemma 2.7], and by borrowing some ideas from the proof of [30, Proposition 2.5], we show that $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ is independent of the choice of $\gamma$. Next, by the above equivalent characterization of $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ and the $\rho$-weakly doubling condition, we establish the John-Nirenberg inequality for $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu) := E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$, which further implies that $E^{\alpha, q}_{\rho} (\mu)$ is independent of the choice of $q$. We point out that, on spaces of homogeneous type, the independence of $q$ of $E^{\alpha, q}_{\rho} (\mu)$ is due to the coincidence between $E^{\alpha, q}_{\rho} (\mu)$ and the Lipschitz space $\text{Lip}_{\rho} (\mu)$; see [39]. However, this coincidence is unknown on non-homogeneous metric measure spaces, even under the $\rho$-weakly doubling condition. Alternatively, we adopt the method developed by Hytönen for the proof of the John-Nirenberg inequality for the BMO type space in [27]; see also [53]. At the end of this section, we establish another useful characterization of $E^{\alpha}_{\rho} (\mu) := E^{\alpha, 1}_{\rho} (\mu)$, which plays important roles in the later context.

In Section 7, we introduce the atomic Hardy space $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ and the molecular Hardy space $\hat{H}_{mb, \rho}^{p, q, \gamma, \epsilon} (\mu)$ and investigate their relation under the $\rho$-weakly doubling condition. By using a method similar to that used in the proof of [13, Theorem 1.11], together with some technical modifications, we prove that $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ and $\hat{H}_{mb, \rho}^{p, q, \gamma, \epsilon} (\mu)$ coincide with equivalent quasi-norms. It is still unclear whether the above result holds true or not on general non-homogeneous metric measure spaces, even on Euclidean spaces with non-doubling measures.

Section 8 is mainly devoted to investigating the dual space of $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ under the $\rho$-weakly doubling condition. To this end, we first show that $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ is independent of the choices of $\rho$ and $\gamma$. Precisely, by the $\rho$-weakly doubling condition and borrowing some ideas from the proof of [30, Proposition 3.3(ii)], we first prove that $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ is independent of the choice of $\rho$. By establishing the corresponding result (see Lemma 6.11 below) to [53, Lemma 9.2] and constructing a sequence of $(\rho, \beta_{\rho})$-doubling balls which is a refinement of that appears in the proof of [53, Lemma 9.3], we further show that $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ is independent of $\gamma$. Finally, via the independence of $\rho$ for $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ and the equivalent characterization of $E^{\alpha}_{\rho} (\mu) := E^{\alpha, 1}_{\rho} (\mu)$ established in Section 6 (see Proposition 6.18 below), we show that $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ is the predual of $E^{p, 1}_{\rho} (\mu)$. It is still unknown whether the above results hold true or not on general non-homogeneous metric measure spaces, even on Euclidean spaces with non-doubling measures.

In Section 9, let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss. We investigate the relations between the Campanato space $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ and the Lipschitz space $\text{Lip}_{\alpha, q} (\mu)$, or between $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ and the atomic Hardy space $\hat{H}_{\text{atb}, \rho}^{p, q} (\mu)$ introduced by Coifman and Weiss [11]. By carefully dividing the situation into several parts, constructing a sequence of balls via using a method similar to that used in the proof of the independence of $\gamma$ for $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ in Section 6 and adopting some ideas from [27, Proposition 4.7], we show that, if $q \in [1, \infty)$, then $E^{\alpha, q}_{\rho, \eta, \gamma} (\mu)$ and $\text{Lip}_{\alpha, q} (\mu)$ coincide with equivalent norms. By a method similar to that used in the proof of this result, we also establish the coincidence of $\hat{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu)$ and $\hat{H}_{\text{atb}, \rho}^{p, q} (\mu)$ for any $q \in (1, \infty]$ directly.
In Section 10, suppose that $(\mathcal{X}, d, \mu)$ is an RD-space with $\mu(\mathcal{X}) = \infty$ and $q \in (1, 2]$. We show that $\tilde{H}_{\text{at}, \rho}^{p, q, \gamma, \epsilon}(\mu)$, $\tilde{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)$ and $\tilde{H}_{\text{st}, \rho}^{p, q, \gamma, \epsilon}(\mu)$ coincide. Let $\tilde{H}_{\text{at}, \rho}^{p, q, \gamma, \epsilon}(\mu)$ and $\tilde{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)$ be dense subspaces of $\tilde{H}_{\text{at}, \rho}^{p, q, \gamma, \epsilon}(\mu)$ and $\tilde{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)$, respectively (see Definitions 3.2 and 4.1 below). We prove that

$$(H_{\text{at}}^{p, q}(\mu) \cap L^2(\mu)) \subset \tilde{H}_{\text{at}, \rho}^{p, q, \gamma, \epsilon}(\mu) \subset \tilde{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu) \subset (H_{\text{st}}^{p, q}(\mu) \cap L^2(\mu))$$

by two steps. In Step 1, to show that $(H_{\text{at}}^{p, q}(\mu) \cap L^2(\mu)) \subset \tilde{H}_{\text{at}, \rho}^{p, q, \gamma, \epsilon}(\mu)$ for any $q \in (1, 2]$, we first establish a technical lemma (see Lemma 10.2 below). Then we establish a useful criteria for the boundedness of some integral operators, it is plays an essential role in the proof of Step 1. In Step 2, via the fact that $\tilde{H}_{\text{at}, \rho}^{p, q, \gamma, \epsilon}(\mu) \subset \tilde{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)$ (see Proposition 4.3 below) and establishing the boundedness of the Littlewood-Paley $g$-function on $L^2(\mu)$ obtained in [21], we give out a key atomic decomposition for all functions from $H_{\text{at}}^{p, q}(\mu)$ in $L^2(\mu)$ (see (10.5) below), which plays an essential role in the proof of Step 1.

Finally, we make some conventions on notation. Throughout this article, $C$ stands for a positive constant which is independent of the choices of the main parameters, but it may vary from line to line. Constants with subscripts, such as $C_0$, do not change in different occurrences. Furthermore, we use $C_{(\rho, \alpha, \ldots)}$ to denote a positive constant depending on parameters $\rho, \alpha, \ldots$. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any ball $B$, the center and the radius of $B$ are denoted, respectively, by $c_B$ and $r_B$. For any subset $E$ of $\mathcal{X}$, we use $\chi_E$ to denote its characteristic function.

## 2 Preliminaries

In this section, we recall some necessary notation and notions, including the discrete coefficient $\tilde{K}_{B, S}^{(\rho), p}$ and give out some fundamental properties on $\tilde{K}_{B, S}^{(\rho), p}$ in the non-homogeneous context.

The following notion of the geometrically doubling is well known in analysis on metric spaces, which was originally introduced by Coifman and Weiss in [10, pp. 66-67] and is also known as metrically doubling (see, for example, [23, p. 81]).

**Definition 2.1.** A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_{i,}$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0$.

**Remark 2.2.** Let $(\mathcal{X}, d)$ be a metric space. In [27], Hytönen showed that the following statements are mutually equivalent:

(i) $(\mathcal{X}, d)$ is geometrically doubling.

(ii) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, cr)\}_{i}$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n_0}$, here and hereafter, $N_0$ is as in Definition 2.1 and $n_0 := \log_2 N_0$.

(iii) For every $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most $N_0 \epsilon^{-n_0}$ centers of disjoint balls $\{B(x_i, cr)\}_{i}$.

(iv) There exists $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$ contains at most $M$ centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, r/4)\}_{i=1}^M$. 
Recall that spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [10, pp. 66-68].

The following notion of upper doubling metric measure spaces was originally introduced by Hytönen [27] (see also [28, 37]).

**Definition 2.3.** A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if $\mu$ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_\lambda$, depending on $\lambda$, such that, for each $x \in \mathcal{X}$, $r \mapsto \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty),$

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2).$$

(2.1)

A metric measure space $(\mathcal{X}, d, \mu)$ is called a non-homogeneous metric measure space if $(\mathcal{X}, d)$ is geometrically doubling and $(\mathcal{X}, d, \mu)$ is upper doubling.

**Remark 2.4.** (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where we take the dominating function $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. On the other hand, the $D$-dimensional Euclidean space $\mathbb{R}^D$ with any Radon measure $\mu$ as in (1.2) is also an upper doubling space by taking $\lambda(x, r) := C_0 r^\alpha$ for all $x \in \mathbb{R}^D$ and $r \in (0, \infty).

(ii) Let $(\mathcal{X}, d, \mu)$ be upper doubling with $\lambda$ being the dominating function on $\mathcal{X} \times (0, \infty)$ as in Definition 2.3. It was proved in [30] that there exists another dominating function $\overline{\lambda}$ such that $\lambda \leq \overline{\lambda}$, $C_\overline{\lambda} \leq C_\lambda$ and, for all $x, y \in \mathcal{X}$ with $d(x, y) \leq r$,

$$\overline{\lambda}(x, r) \leq C_\overline{\lambda} \overline{\lambda}(y, r).$$

(2.2)

(iii) It was shown in [52] that the upper doubling condition is equivalent to the weak growth condition: there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$, with $r \mapsto \lambda(x, r)$ non-decreasing, positive constants $C_\lambda$, depending on $\lambda$, and $\epsilon$ such that

(iii)$_1$ for all $r \in (0, \infty)$, $t \in [0, r]$, $x, y \in \mathcal{X}$ and $d(x, y) \in [0, r],$

$$|\lambda(y, r + t) - \lambda(x, r)| \leq C_\lambda \left[\frac{d(x, y)}{r} + t\right]^{\epsilon} \lambda(x, r);$$

(iii)$_2$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$, $\mu(B(x, r)) \leq \lambda(x, r)$.

Based on Remark 2.4(ii), from now on, we always assume that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space with the dominating function $\lambda$ satisfying (2.2).

Though the measure doubling condition is not assumed uniformly for all balls in the non-homogeneous metric measure space $(\mathcal{X}, d, \mu)$, it was shown in [27] that there still exist many balls which have the following $(\alpha, \beta)$-doubling property.

**Definition 2.5.** Let $\alpha, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be $(\alpha, \beta)$-doubling if $\mu(\alpha B) \leq \beta \mu(B)$, where, for any ball $B := B(c_B, r_B)$ and $\rho \in (0, \infty)$, $\rho B := B(c_B, \rho r_B)$.

To be precise, it was proved in [27, Lemma 3.2] that, if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\alpha, \beta \in (1, \infty)$ with $\beta > [C_\lambda]^{\log_2 \alpha} =: \alpha^\nu$, then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+$ such that $\alpha^j B$ is $(\alpha, \beta)$-doubling. Moreover, let $(\mathcal{X}, d)$ be geometrically doubling, $\beta > \alpha^{n_0}$ with $n_0 := \log_2 N_0$ and $\mu$ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [27, Lemma 3.3] also showed that, for $\mu$-almost every $x \in \mathcal{X}$, there exist arbitrary small $(\alpha, \beta)$-doubling balls centered at $x$. Furthermore, the radii of these balls may be chosen to be of the form $\alpha^{-j} r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this article, for any $\alpha \in (1, \infty)$ and ball $B$, the smallest $(\alpha, \beta_\alpha)$-doubling ball of the form $\alpha^j B$ with $j \in \mathbb{Z}_+$ is denoted by $\overline{B}^\alpha$, where

$$\beta_\alpha := \alpha^{3(\max\{n_0, \nu\})} + [\max\{5\alpha, 30\}]^{n_0} + [\max\{3\alpha, 30\}]^\nu.$$

Before we introduce the discrete coefficient $F_{B, S}^{(p)}$, we first give an assumption on the relation between two balls $B$ and $S$, which is supposed to hold true throughout the whole article:

**A** If $B = S$, then $c_B = c_S$ and $r_B = r_S$. 
Then we claim that, if \( B \subset S \), then \( r_B \leq 2r_S \). Indeed, assume that \( r_B > 2r_S \). By this and \( B \subset S \), together with the triangle inequality satisfied by \( d \), we see that \( S \subset B \). Thus, \( B = S \), which, together with the assumption (A), implies that \( r_B = r_S \). This contradicts to \( r_B > 2r_S \), which completes the proof of the above claim.

On the other hand, we give a simple example to illustrate that, if \( B \subsetneq S \), then it may happen that \( r_B > r_S \). Let \( (X, d) := (\{-1, 1, 3\}, \cdot \cdot \cdot) \), \( B := \{x \in \{-1, 1, 3\} : |x + 1| < 3\} \) and
\[
S := \left\{ x \in \{-1, 1, 3\} : |x - 1| < \frac{5}{2} \right\}.
\]
Obviously, \( r_B > r_S \) and \( B = \{-1, 1\} \subsetneq \{-1, 1, 3\} = S \).

**Definition 2.6.** For any \( \rho \in (1, \infty) \), \( p \in (0, 1) \) and any two balls \( B \subset S \subset X \), let
\[
\tilde{K}^{(p)}_{B, S} := 1 + \sum_{k=-\lfloor \log_{\alpha} 2 \rfloor}^{N_{B, S}^{(p)}} \left( \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)} \right)^p, \tag{2.3}
\]
here and hereafter, for any \( a \in \mathbb{R} \), \( \lfloor a \rfloor \) represents the biggest integer which is not bigger than \( a \), and \( N_{B, S}^{(p)} \) is the smallest integer satisfying \( \rho^{N_{B, S}^{(p)}} r_B \geq r_S \).

**Remark 2.7.** (i) By a change of variables and (2.1), we easily conclude that
\[
\tilde{K}^{(p)}_{B, S} \sim \left( 1 + \sum_{k=-\lfloor \log_{\alpha} 2 \rfloor}^{N_{B, S}^{(p)} + \lfloor \log_{\alpha} 2 \rfloor + 1} \left( \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)} \right)^p \right)^{1/p},
\]
where the implicit equivalent positive constants are independent of balls \( B \subset S \subset X \), but depend on \( \rho \) and \( p \).

(ii) A continuous version, \( K_{B, S} \), of the coefficient in Definition 2.6 when \( p = 1 \) was introduced in [27, 30] as follows: for any two balls \( B \subset S \subset X \),
\[
K_{B, S} := 1 + \int_{(2S) \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \, d\mu(x). \tag{2.4}
\]
It was proved in [30, Lemma 2.2] that \( K_{B, S} \) has all properties similar to those for \( \tilde{K}^{(p)}_{B, S} \) as in Lemma 2.8 below. Unfortunately, \( K_{B, S} \) and \( \tilde{K}^{(1)}_{B, S} \) are usually not equivalent, but this is true for \((\mathbb{R}^d, | \cdot |, \mu)\) with \( \mu \) as in (1.2); see [13] for more details on this.

Now we give some simple properties of \( \tilde{K}^{(p)}_{B, S} \) defined by (2.3) adapted from [14, Lemma 3.5], in which \( \rho = 6 \) and \( p = 1 \). The arguments therein are still valid for the present case. For the sake of reader’s convenience, we present some details. In what follows, for any \( a \in \mathbb{R} \), \( \lfloor a \rfloor \) represents the smallest integer which is not smaller than \( a \).

**Lemma 2.8.** Let \((X, d, \mu)\) be a non-homogeneous metric measure space and \( p \in (0, 1] \).

(i) For any \( \rho \in (1, \infty) \), there exists a positive constant \( C(p) \), depending on \( \rho \), such that, for all balls \( B \subset R \subset S \), \( [\tilde{K}^{(p)}_{B, R}]^{p} \leq C(p) [\tilde{K}^{(p)}_{B, S}]^{p} \).

(ii) For any \( \alpha \in (1, \infty) \) and \( p \in (1, \infty) \), there exists a positive constant \( C(\alpha, p) \), depending on \( \alpha \) and \( p \), such that, for all balls \( B \subset S \) with \( r_S \leq \alpha r_B \), \( [\tilde{K}^{(p)}_{B, S}]^{p} \leq C(\alpha, p) \).

(iii) For any \( \rho \in (1, \infty) \), there exists a positive constant \( C(p, \rho) \), depending on \( \rho \), such that, for all balls \( B \subset R \subset S \), \( [\tilde{K}^{(p)}_{B, R}]^{p} \leq C(p, \rho) [\tilde{K}^{(p)}_{B, S}]^{p} \).

(iv) For any \( \rho \in (1, \infty) \), there exists a positive constant \( c(p, \rho) \), depending on \( \rho \), such that, for all balls \( B \subset R \subset S \),
\[
[\tilde{K}^{(p)}_{B, S}]^{p} \leq [\tilde{K}^{(p)}_{B, R}]^{p} + c(p, \rho) [\tilde{K}^{(p)}_{B, S}]^{p}.
\]
(v) For any $\rho \in (1, \infty)$, there exists a positive constant $\bar{c}_{(\rho, p, \nu)}$, depending on $\rho$, $p$ and $\nu$, such that, for all balls $B \subset R \subset S$, $[\tilde{K}_{B, R}^{(\rho)}]^{p} \leq \bar{c}_{(\rho, p, \nu)} [\tilde{K}_{B, S}^{(\rho)}]^{p}$.

Proof. Fix $p \in (0, 1]$, $\rho \in (1, \infty)$ and $\alpha \in [1, \infty)$. We first show (i). By $R \subset S$, we have $r_{R} \leq 2r_{S}$. Hence, $r_{R} \leq 2r_{S} \leq 2\rho^{N_{B, S}^{(\rho)}, s_{R}B} \leq \rho^{[\log_{2} \rho] + N_{B, S}^{(\rho)}, s_{R}B}$. Thus, $N_{B, R}^{(\rho)} \leq [\log_{2} \rho] + N_{B, S}^{(\rho)}$. By this and (2.1), we see that

\[
\left[\tilde{K}_{B, R}^{(\rho)}, p\right]^{p} \leq 1 + \sum_{k=-[\log_{2} \rho]}^{[\log_{2} \rho] + N_{B, S}^{(\rho)}} \frac{\mu(\rho^{k}B)}{\lambda(c_{B}, \rho^{k}r_{RB})}^{p} \leq 1 + \sum_{k=-[\log_{2} \rho]}^{N_{B, S}^{(\rho)}} \frac{\mu(\rho^{k}B)}{\lambda(c_{B}, \rho^{k}r_{RB})}^{p} + [\log_{2} \rho] \leq (1 + [\log_{2} \rho]) \left[\tilde{K}_{B, S}^{(\rho)}, p\right]^{p}.
\]

This shows (i).

Next we prove (ii). By the fact that $\rho^{N_{B, S}^{(\rho)} - 1} r_{B} \leq r_{S} \leq \alpha r_{B}$, we have $N_{B, S}^{(\rho)} \leq [\log_{\rho} \alpha]$. Thus, $N_{B, S}^{(\rho)} - 1 \leq [\log_{\rho} \alpha]$. From this and (2.1), we deduce that

\[
\left[\tilde{K}_{B, S}^{(\rho)}, p\right]^{p} \leq 1 + N_{B, S}^{(\rho)} + [\log_{2} \rho] \leq 2 + [\log_{\rho} \alpha] + [\log_{2} \rho].
\]

Thus, (ii) holds true.

Let us now prove (iii). To this end, let $N$ be the first integer such that $\rho^{N} B$ is $(\rho, \beta_{D})$-doubling. If $B$ is $(\rho, \beta_{D})$-doubling, the conclusion of (iii) holds true trivially. Thus, without loss of generality, we may assume that $B$ is non-$(\rho, \beta_{D})$-doubling, which implies that $N \geq 1$. For any $k \in \{-\lfloor \log_{2} \rho \rfloor, \ldots, N - 1\}$, we have $\mu(\rho^{k+1}B) > \beta_{D} \mu(\rho^{k}B)$. Thus, for any $k \in \{-\lfloor \log_{2} \rho \rfloor, \ldots, N - 1\}$, $\mu(\rho^{k}B) < \frac{\mu(\rho^{N}B)}{\beta_{D}}$. By this, together with (2.1) and the fact that $\beta_{D} = [C(\sqrt{\rho})]^{2}2^{(2\rho)}$, we conclude that

\[
\left[\tilde{K}_{B, R}^{(\rho)}, p\right]^{p} = \left[\tilde{K}_{B, S}^{(\rho)}, p^{N_{B, S}^{(\rho)}} B\right]^{p} \leq 1 + \sum_{k=-[\log_{2} \rho]}^{N_{B, S}^{(\rho)}} \frac{\mu(\rho^{k}B)}{\lambda(c_{B}, \rho^{k}r_{RB})}^{p} \leq 1 + \sum_{k=-\infty}^{\infty} \left[\frac{2(2\rho)^{\nu}}{\beta_{D}}\right]^{pk} \lesssim 1,
\]

where the implicit positive constant only depends on $\rho$, $p$ and $\nu$. Similarly, the other part of (iii) holds true, the details being omitted. This proves (iii).

Next we show (iv). By (i), $N_{B, R}^{(\rho)} \leq N_{B, S}^{(\rho)} + [\log_{2} \rho]$. If $N_{B, S}^{(\rho)} \leq N_{B, S}^{(\rho)} + [\log_{2} \rho]$, then there is nothing to prove. If $N_{B, R}^{(\rho)} < N_{B, S}^{(\rho)}$, from the facts that $N_{B, S}^{(\rho)} \leq N_{B, R}^{(\rho)} + N_{R, S}^{(\rho)}$ (since $\rho^{-N_{B, R}^{(\rho)}, s_{R}B} \geq \rho^{-N_{B, S}^{(\rho)}, s_{R}B} \geq r_{R}$), $\rho^{N_{B, R}^{(\rho)}}, s_{R}B \geq r_{R}$, $\rho^{k+N_{B, R}^{(\rho)} + [\log_{2} \rho]} B \subset \rho^{k+2+[\log_{2} \rho]} B$ for all $k \in \mathbb{Z} \cap \{-\lfloor \log_{2} \rho \rfloor, \infty\}$, and (2.1), it follows that

\[
\left[\tilde{K}_{B, R}^{(\rho)}, p\right]^{p} = \left[\tilde{K}_{B, S}^{(\rho)}, p\right]^{p} + \sum_{k=N_{B, R}^{(\rho)} + \lfloor \log_{2} \rho \rfloor}^{N_{B, S}^{(\rho)} + [\log_{2} \rho] + 1} \frac{\mu(\rho^{k}B)}{\lambda(c_{B}, \rho^{k}r_{RB})}^{p} \leq \left[\tilde{K}_{B, R}^{(\rho)}, p\right]^{p} + \sum_{k=-\infty}^{N_{B, S}^{(\rho)}} \frac{\mu(\rho^{k+1}B)^{p}}{\lambda(c_{B}, \rho^{k+N_{B, R}^{(\rho)} + [\log_{2} \rho]} R)}^{p} \leq \left[\tilde{K}_{B, R}^{(\rho)}, p\right]^{p} + \sum_{k=-\infty}^{N_{B, S}^{(\rho)}} \frac{\mu(\rho^{k+1}B)^{p}}{\lambda(c_{B}, \rho^{k+1+[\log_{2} \rho]} R)}^{p}.
\]
Proof. We first prove that $N_{B,R}^{(\rho)} + N_{R,S}^{(\rho)} \leq N_{B,S}^{(\rho)} + 1$. Since

$$r_R = \rho^{-N_{R,S}^{(\rho)} + 1} r_S \leq \rho^{-N_{R,S}^{(\rho)} + 1} r_B \leq \rho^{-N_{R,S}^{(\rho)} + 1} r_B = \rho^{-N_{B,R}^{(\rho)} + N_{R,S}^{(\rho)} + 1} r_B,$$

we obtain $N_{B,R}^{(\rho)} \leq N_{B,S}^{(\rho)} - N_{R,S}^{(\rho)} + 1$. From this, $r_R > \rho^{-N_{R,S}^{(\rho)} + 1} r_B$, $\rho R \subset \rho^{k + [\log_2 \rho] + N_{R,B}^{(\rho)}}$ for all $k \in \mathbb{Z} \cap [-[\log_2 \rho], \infty)$, (2.1) and (2.2), it follows that

$$\left[ \tilde{K}_{R,S}^{(\rho)} \right]^p \leq 1 + \sum_{k = -[\log_2 \rho]}^{N_{R,S}^{(\rho)}} \left[ \frac{\mu(\rho k^2 + [\log_2 \rho] + N_{R,B}^{(\rho)})}{\lambda(c_R, \rho_2 k + N_{R,B}^{(\rho)} \rho_2 k + N_{R,B}^{(\rho)})} \right]^p \sim \left[ \tilde{K}_{B,S}^{(\rho)} \right]^p,$$

where the implicit positive constants depend only on $\rho$, $p$ and $\nu$. This finishes the proof of (v) and hence Lemma 2.8.

Now we show that, for any $\rho_1, \rho_2 \in (1, \infty)$ and $p \in (0, 1]$, $\tilde{K}_{B,S}^{(\rho_1)} \sim \tilde{K}_{B,S}^{(\rho_2)}$ for all balls $B \subset S$.

**Lemma 2.9.** Let $(X, d, \mu)$ be a non-homogeneous metric measure space, $\rho_1, \rho_2 \in (1, \infty)$ and $p \in (0, 1]$. Then there exist positive constants $c(\rho_1, \rho_2, p, \nu)$ and $C(\rho_1, \rho_2, p, \nu)$ depending on $\rho_1, \rho_2, \nu$ and $p$, such that, for all balls $B \subset S$,

$$c(\rho_1, \rho_2, p, \nu) \tilde{K}_{B,S}^{(\rho_1)} \leq \tilde{K}_{B,S}^{(\rho_2)} \leq C(\rho_1, \rho_2, p, \nu) \tilde{K}_{B,S}^{(\rho_1)}.$$

**Proof.** For the sake of simplicity, we only prove this lemma for $p = 1$. With some slight modifications, the arguments here are still valid for $p \in (0, 1)$. For any $\rho_1, \rho_2 \in (1, \infty)$, without loss of generality, we may assume that $\rho_1 > \rho_2 > 1$. For any two fixed balls $B \subset S$, let $N_j := N_{B,S}^{(\rho_j)}$ and $\tilde{K}_{B,S}^{(\rho_j)} := \tilde{K}_{B,S}^{(\rho_j)} (j \in \{1, 2\})$. It is obvious that $N_1 \leq N_2$. Now we consider the following two cases:

**Case i)** $\rho_1 N_1 \leq \rho_2 N_2$. It is easy to see that $\rho_2 N_2 - 1 \leq \rho_1 N_1$. We first prove that $\tilde{K}_{B,S}^{(\rho_1)} \leq \tilde{K}_{B,S}^{(\rho_2)}$.

Indeed, for any $n_1 \in \{-[\log_2 \rho_1], \ldots, N_1\}$, let $n_2$ be the smallest integer such that $\rho_2^{n_2} \geq \rho_1^{n_1}$. Then we have

$$n_2 \in \{-[\log_2 \rho_2], \ldots, N_2\} \quad \text{and} \quad \rho_2^{n_2 - 1} < \rho_1^{n_1} \leq \rho_2^{n_2}.$$

Consequently, $\rho_2^{n_2 - 1} B \subset \rho_1^{n_1} B \subset \rho_2^{n_2} B$. By some simple calculations, we see that, for any $n_2 \in \{-[\log_2 \rho_2], \ldots, N_2\}$, there exists at most one $n_1$ satisfying (2.5). By the above facts, $-\log_2 \rho_1 \geq -\log_2 \rho_2$ and (2.1), we obtain

$$\tilde{K}_{B,S}^{(\rho_1)} \leq 1 + \sum_{n_1 = -\log_2 \rho_1}^{N_1} \frac{\mu(\rho_2^{n_2 - 1} B)}{\lambda(c_B, \rho_2^{n_2 - 1} r_B)} \leq 1 + \sum_{n_2 = -\log_2 \rho_2}^{N_2} \frac{\mu(\rho_2^{n_2} B)}{\lambda(c_B, \rho_2^{n_2} r_B)} \leq \tilde{K}_{B,S}^{(\rho_2)}.$$
where the implicit positive constants depend only on $\rho_1$, $\rho_2$ and $\nu$.

On the other hand, for the case $N_2 < 1$, it is obvious that $\widetilde{K}_{B,S}^{(\rho_2)} \lesssim 1 \lesssim K_{B,S}^{(\rho_1)}$, which completes the proof of Case i). Thus, without loss of generality, we may assume that $N_2 \geq 1$. We notice that

$$\widetilde{K}_{B,S}^{(\rho_2)} \lesssim 1 + \sum_{n_2 = -\lceil \log_{\rho_2} 2 \rceil}^{N_2 - 1} \frac{\mu(\rho_2^{n_2}B)}{\lambda(cB, \rho_2^{n_2}r_B)}.$$  

For any fixed $n_2 \in \{-\lceil \log_{\rho_2} 2 \rceil, \ldots, N_2 - 1\}$, let $n_1$ be the smallest positive integer such that $\rho_1^{n_1} \geq \rho_2^{n_2}$. Then we have

$$n_1 \in \{-\lceil \log_{\rho_2} 2 \rceil, \ldots, N_1\} \quad \text{and} \quad \rho_1^{n_1-1} \leq \rho_2^{n_2} \leq \rho_1^{n_1}. \quad (2.6)$$

Consequently, $\rho_1^{n_1-1}B \subset \rho_2^{n_2}B \subset \rho_1^{n_1}B$. By some simple calculations, we see that, for any $n_1 \in \{-\lceil \log_{\rho_2} 2 \rceil, \ldots, N_1\}$, the number of $n_2$ satisfying (2.6) does not exceed $\lceil \frac{\ln \rho_1}{\ln \rho_2} \rceil$. By the above facts and (2.1), we know that

$$\widetilde{K}_{B,S}^{(\rho_2)} \lesssim 1 + \sum_{n_2 = -\lceil \log_{\rho_2} 2 \rceil}^{N_2 - 1} \frac{\mu(\rho_2^{n_2}B)}{\lambda(cB, \rho_2^{n_2}r_B)} \sim 1 + \sum_{n_1 = -\lceil \log_{\rho_1} 2 \rceil}^{N_1} \sum_{n_2 : \rho_1^{n_1-1} \leq \rho_2^{n_2} \leq \rho_1^{n_1}} \frac{\mu(\rho_2^{n_2}B)}{\lambda(cB, \rho_1^{n_1}r_B)} \sim \widetilde{K}_{B,S}^{(\rho_1)},$$

where the implicit positive constants depend only on $\rho_1$, $\rho_2$ and $\nu$. This finishes the proof of Case i).

Case ii) $\rho_2^{N_2} < \rho_1^{N_1}$. The proof of this case is similar to that of Case i), the details being omitted. This finishes the proof of Lemma 2.9. \hfill\square

3 Atomic Hardy Spaces $\widetilde{H}^{p,q,\gamma}_{atb,\rho}(\mu)$

In this section, we introduce the atomic Hardy space $\widetilde{H}^{p,q,\gamma}_{atb,\rho}(\mu)$ and establish a useful property. Before introducing the notion of $\widetilde{H}^{p,q,\gamma}_{atb,\rho}(\mu)$, we first recall some notions related to quasi-Banach spaces; see, for example, [19].

Definition 3.1. (i) A quasi-Banach space $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_\mathcal{B}$ which is non-negative, non-degenerate (namely, $\|f\|_\mathcal{B} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, namely, there exists a constant $K \in [1, \infty)$ such that, for all $f, g \in \mathcal{B}$,

$$\|f + g\|_\mathcal{B} \leq K(\|f\|_\mathcal{B} + \|g\|_\mathcal{B}).$$

(ii) Let $r \in (0, 1]$. A quasi-Banach space $\mathcal{B}_r$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_r}$ is called a $r$-quasi-Banach space if $\|f + g\|_{\mathcal{B}_r} \leq \|f\|_{\mathcal{B}_r} + \|g\|_{\mathcal{B}_r}$ for all $f, g \in \mathcal{B}_r$. Hereafter, $\|\cdot\|_{\mathcal{B}_r}$ is called the $r$-quasi-norm of the $r$-quasi-Banach space $\mathcal{B}_r$.

Then we introduce the notion of $\widetilde{H}^{p,q,\gamma}_{atb,\rho}(\mu)$ over general non-homogeneous metric measure spaces.

Definition 3.2. Let $\rho \in (1, \infty)$, $0 \leq p \leq q \leq \infty$, $p \neq q$ and $\gamma \in [1, \infty)$. A function $b$ in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$ is called a $(p, q, \gamma)$-atomic block if

(i) there exists a ball $B$ such that $\text{supp}(b) \subset B$;

(ii) $\int_B b(x) \, d\mu(x) = 0$;

(iii) for any $j \in \{1, 2\}$, there exist a function $a_j$ supported on a ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$ and

$$\|a_j\|_{L^q(\mu)} \leq \left[ \mu(b_{B_j}) \right]^{1/q-1} \left[ \lambda(cB, r_B) \right]^{1-1/p} \left[ \widetilde{K}_{B_j, B}^{(\rho)} \right]^{-\gamma}.$$  

Moreover, let $|b|_{\widetilde{H}^{p,q,\gamma}_{atb,\rho}(\mu)} := |\lambda_1| + |\lambda_2|.$
A function \( f \) is said to belong to the space \( \overline{H}^{p,q,\gamma}(\mu) \) if there exists a sequence of \((p,q,\gamma,\rho)\)-atomic blocks, \( \{b_i\}_{i=1}^{\infty} \), such that \( f = \sum_{i=1}^{\infty} b_i \) in \( L^2(\mu) \) when \( p \in (0,1) \) and in \( L^1(\mu) \) when \( p = 1 \), and

\[
\sum_{i=1}^{\infty} |b_i|^p_{\overline{H}^{p,q,\gamma}(\mu)} < \infty.
\]

Moreover, define

\[
\|f\|_{\overline{H}^{p,q,\gamma}(\mu)} := \inf \left\{ \left( \sum_{i=1}^{\infty} |b_i|^p_{\overline{H}^{p,q,\gamma}(\mu)} \right)^{1/p} \right\},
\]

where the infimum is taken over all possible decompositions of \( f \) as above.

The atomic Hardy space \( \tilde{H}^{p,q,\gamma}(\mu) \) is then defined as the completion of \( \overline{H}^{p,q,\gamma}(\mu) \) with respect to the \( p \)-quasi-norm \( \| \cdot \|_{\overline{H}^{p,q,\gamma}(\mu)} \).

**Remark 3.3.** (i) By the theorem of completion of Yosida [65, p. 56], we see that \( \overline{H}^{p,q,\gamma}(\mu) \) has a completion space \( \overline{H}^{p,q,\gamma}(\mu) \), namely, for any \( f \in \overline{H}^{p,q,\gamma}(\mu) \), there exists a Cauchy sequence \( \{f_k\}_{k=1}^{\infty} \) in \( \overline{H}^{p,q,\gamma}(\mu) \) such that

\[
\lim_{k \to \infty} \|f_k - f\|_{\overline{H}^{p,q,\gamma}(\mu)} = 0.
\]

Moreover, if \( \{f_k\}_{k=1}^{\infty} \) is a Cauchy sequence in \( \overline{H}^{p,q,\gamma}(\mu) \), then there exists a unique \( f \in \overline{H}^{p,q,\gamma}(\mu) \) such that (3.1) holds true.

(ii) When \( p = 1 \), the space \( \overline{H}^{1,q,\gamma}(\mu) \) was introduced in [30] and proved to be a Banach space. Thus, \( \overline{H}^{1,q,\gamma}(\mu) = \overline{L}^{1,q,\gamma}(\mu) \); see also [3].

(iii) Fix \( p, \rho \) and \( \gamma \) as in Definition 3.2. For \( 1 \leq q_1 \leq q_2 \leq \infty \), we easily obtain

\[
\overline{H}^{p,q_2,\gamma}(\mu) \subset \overline{H}^{p,q_1,\gamma}(\mu).
\]

(iv) In Definition 3.2, it seems natural to assume \( b \in L^q(\mu) \) and to require \( f = \sum_{i=1}^{\infty} b_i \) also holds true in \( L^q(\mu) \). However, if so, then it is unclear whether (iii) of this remark still holds true or not, which is crucial in applications (see, for example, Remark 10.11(i)).

Now we show that any element in \( \tilde{H}^{p,q,\gamma}(\mu) \) has a decomposition in terms of some \((p, q, \gamma, \rho)\)-atomic blocks, \( \{b_i\}_{i=1}^{\infty} \), in \( \overline{H}^{p,q,\gamma}(\mu) \).

**Proposition 3.4.** Let \((\mathcal{X}, d, \mu)\) be a non-homogeneous metric measure space, \( \rho \in (1, \infty) \), \( 0 < p \leq q \leq \infty \), \( p \neq q \) and \( \gamma \in [1, \infty) \). Then \( f \in \tilde{H}^{p,q,\gamma}(\mu) \) if and only if there exist \((p,q,\gamma,\rho)\)-atomic blocks \( \{b_i\}_{i=1}^{\infty} \) such that

\[
f = \sum_{i=1}^{\infty} b_i \quad \text{in} \quad \tilde{H}^{p,q,\gamma}(\mu),
\]

and \( \sum_{i=1}^{\infty} |b_i|^p_{\tilde{H}^{p,q,\gamma}(\mu)} < \infty \). Moreover,

\[
\|f\|^p_{\tilde{H}^{p,q,\gamma}(\mu)} = \inf \left\{ \sum_{i=1}^{\infty} |b_i|^p_{\tilde{H}^{p,q,\gamma}(\mu)} \right\},
\]

where the infimum is taken over all possible decompositions of \( f \) as in (3.2).

**Proof.** We first assume that \( f \in \tilde{H}^{p,q,\gamma}(\mu) \). Observe that, if (3.2) holds true, it is easy to see that

\[
\|f\|^p_{\tilde{H}^{p,q,\gamma}(\mu)} \leq \inf \left\{ \sum_{i=1}^{\infty} |b_i|^p_{\tilde{H}^{p,q,\gamma}(\mu)} \right\},
\]

where the infimum is taken over all possible decompositions of \( f \) as in (3.2). It remains to prove (3.2) and the reverse inequality of (3.3). For any \( f \in \tilde{H}^{p,q,\gamma}(\mu) \), we consider the following two cases.
**Case i)** $f \in \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$. By Definition 3.2, there exists a sequence of $(p, q, \gamma, \rho)_{\lambda}$-atomic blocks, \( \{b_i\}_{i=1}^{\infty} \), such that $f = \sum_{i=1}^{\infty} b_i$ in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$ and $\sum_{i=1}^{\infty} |b_i|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} < \infty$.

Now we claim that (3.2) holds true.

Indeed, for any $M \in \mathbb{N}$, $f - \sum_{i=1}^{M} b_i = \sum_{i=M+1}^{\infty} b_i$ in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$. Then we know that

$$
\left\| f - \sum_{i=1}^{M} b_i \right\|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} \leq \sum_{i=M+1}^{\infty} |b_i|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} \to 0 \quad \text{as} \quad M \to \infty.
$$

Then the claim holds true. Again, by Definition 3.2 and (3.3), we obtain the desired result for **Case i**.

**Case ii)** $f \in \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu) \setminus \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$. By Remark 3.3(i), there exists a Cauchy sequence \( \{f_k\}_{k=1}^{\infty} \) in $\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$ such that

$$
\|f - f_k\|_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} \leq 2^{-k-2}\|f\|_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)}.
$$

It is easy to see that $f = \sum_{k=1}^{\infty} (f_k - f_{k-1})$ in $\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$, where we let $f_0 := 0$. Since $f_k - f_{k-1} \in \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$ for all $k \in \mathbb{N}$, by Definition 3.2 and **Case i**, we see that, for any $\epsilon \in (0, \infty)$ and any $k \in \mathbb{N}$, there exists a sequence of $(p, q, \gamma, \rho)_{\lambda}$-atomic blocks, \( \{b_{k, i}\}_{i=1}^{\infty} \), such that

$$
f_k - f_{k-1} = \sum_{i=1}^{\infty} b_{k, i} \quad \text{in both} \quad L^2(\mu) \text{when} \quad p \in (0, 1), \quad \text{or} \quad L^1(\mu) \text{when} \quad p = 1, \quad \text{and} \quad \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)
$$

and

$$
\sum_{i=1}^{\infty} |b_{k, i}|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} < \|f_k - f_{k-1}\|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} + \epsilon 2^{-k}.
$$

From this and $f = \sum_{k=1}^{\infty} (f_k - f_{k-1})$ in $\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$, we further deduce that

$$
f = \sum_{k=1}^{\infty} (f_k - f_{k-1}) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} b_{k, i} \quad \text{in} \quad \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)
$$

and

$$
\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |b_{k, i}|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} \leq \sum_{k=1}^{\infty} \|f_k - f_{k-1}\|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} + \sum_{k=1}^{\infty} \epsilon 2^{-k}
$$

$$
\leq \sum_{k=1}^{\infty} \left[ \|f_k - f\|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} + \|f_k - f\|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} \right] + \epsilon
$$

$$
\leq \sum_{k=1}^{\infty} 2^{-k} \|f\|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} + \epsilon = \|f\|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} + \epsilon,
$$

which, together with the arbitrariness of $\epsilon$, completes the proof of **Case ii** and hence the necessity of Proposition 3.4.

Conversely, let $f = \sum_{i=1}^{\infty} b_i$ in $\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$ and

$$
\sum_{i=1}^{\infty} |b_i|^{p}_{\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)} < \infty.
$$

Then, for each $k \in \mathbb{N}$, $f_k = \sum_{i=1}^{k} b_i \in \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$ and $\lim_{k \to \infty} f_k = f$ in $\tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$. Thus, $f \in \tilde{H}_{atb, \rho}^{p, q, \gamma}(\mu)$, which completes the proof of the sufficiency and hence Proposition 3.4. 

\( \Box \)
4. Boundedness of Calderón-Zygmund Operators

In this section, we introduce the notion of the molecular Hardy space $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$ and prove that the Calderón-Zygmund operator $T$ is bounded from $\tilde{H}_{mb,\rho}^{p,q,\gamma,\delta}(\mu)$ (or $\tilde{H}_{mb,\rho}^{p,q,\gamma}(\mu)$) into $L^p(\mu)$, and from $\tilde{H}_{mb,\rho}^{p,q,\gamma+1}(\mu)$ into $\tilde{H}_{mb,\rho}^{p,q,\gamma}(\mu)$, where $\delta$ is some positive constant depending on $T$; see Definition 4.6 below.

We first introduce the notion of molecular Hardy spaces in a non-homogeneous metric measure space.

**Definition 4.1.** Let $\rho \in (1, \infty)$, $0 < p < 1 \leq q \leq \infty$, $p \neq q$, $\gamma \in [1, \infty)$ and $\epsilon \in (0, \infty)$. A function $b$ in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$ is called a $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular block if

(i) $\int_X b(x) d\mu(x) = 0$;

(ii) there exist some ball $B := B(c_B, r_B)$, with $c_B \in X$ and $r_B \in (0, \infty)$, and some constants $\tilde{M}, M \in \mathbb{N}$ such that, for all $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$ with $M_k := \tilde{M}$ if $k = 0$ and $M_k := M$ if $k \in \mathbb{N}$, there exist functions $m_{k,j}$ supported on some balls $B_{k,j} \subset U_k(B)$ for all $k \in \mathbb{Z}_+$, where $U_0(B) := \rho^2 B$ and $U_k(B)$ is some positive constant depending on $T$; see Definition 4.6 below.

Moreover, define

$$\|m_{k,j}\|_{L^\infty(\mu)} \leq \rho^{-\lambda} [\mu(\rho B_{k,j})]^{1/q-1} \left[ \lambda(c_B, \rho^{k+2} r_B) \right]^{1-1/p} \left[ K_{B_{k,j},\rho^{k+2} B} \right]^{-\gamma} \quad (4.1)$$

and

$$|b|^p_{\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)} := \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_k|^p < \infty.$$

A function $f$ is said to belong to the space $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$ if there exists a sequence of $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular blocks, $(b_i)_{i=1}^\infty$, such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0, 1)$ and in $L^1(\mu)$ when $p = 1$, and

$$\sum_{i=1}^\infty |b_i|^p_{\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)} < \infty.$$

Moreover, define

$$\|f\|_{\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)} := \inf \left\{ \left( \sum_{i=1}^\infty |b_i|^p_{\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)} \right)^{1/p} \right\},$$

where the infimum is taken over all possible decompositions of $f$ as above.

The molecular Hardy space $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$ is then defined as the completion of $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$ with respect to the $p$-quasi-norm $\|f\|_{\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)}$.

**Remark 4.2.** (i) From the theorem of completion of Yosida [65, p. 56], it follows that $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$ has a completion space $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$, namely, for any $f \in \tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$, there exists a Cauchy sequence $\{f_k\}_{k=1}^\infty$ in $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$ such that

$$\lim_{k \to \infty} \|f_k - f\|_{\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)} = 0. \quad (4.2)$$

Moreover, if $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $\tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$, then there exists a unique $f \in \tilde{H}_{mb,\rho}^{p,q,\gamma,\epsilon}(\mu)$ such that (4.2) holds true.

(ii) It was proved, in [13, Proposition 2.2(i)], that $\tilde{H}_{mb,\rho}^{1,q,\gamma,\epsilon}(\mu)$ is a Banach space and hence

$$\tilde{H}_{mb,\rho}^{1,q,\gamma,\epsilon}(\mu) = \tilde{H}_{mb,\rho}^{1,q,\gamma,\epsilon}(\mu).$$

(iii) Fix $p$, $\rho$, $\gamma$ and $\epsilon$ as in Definition 4.1. For $1 \leq q_1 \leq q_2 \leq \infty$, we easily have

$$\tilde{H}_{mb,\rho}^{p,q_2,\gamma,\epsilon}(\mu) \subset \tilde{H}_{mb,\rho}^{p,q_1,\gamma,\epsilon}(\mu).$$

(iv) We point out that, via replacing the discrete coefficient $\tilde{K}_{B,S}^{(\rho)}$ in Definitions 3.2 and 4.1 by the continuous coefficient $K_{B,S}$ as in (2.4), the atomic Hardy space $\tilde{H}_{ab,S}^{p,q,\gamma}(\mu)$ and the molecular Hardy space...
By Definitions 3.2 and 4.1, it is easy to see that, for any \((\rho, \gamma, q)\), constants \(\epsilon\) which completes the proof of Proposition 4.3.

Let \(p, q, \gamma, \rho\) be injection and Proposition 4.3. By \([13, \text{Theorem } 1.11]\), we see that \(\|Tf\|_{L^p(\mu)} \leq \|Tf\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)} \leq C \|f\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)}\).

Proof. Let \(\rho, p, q, \gamma, \epsilon\) be as in Proposition 4.3. By the Fatou lemma, it suffices to prove that, for any \((p, q, \gamma, \rho)\lambda\)-atomic block \(b, \rho\) is also a \((p, q, \gamma, \epsilon, \rho)\lambda\)-molecular block which belongs to \(L^p(\mu)\) and

\[
\|b\|_{L^p(\mu)} \lesssim \|b\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)} \lesssim \|b\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)}.
\]

By Definitions 3.2 and 4.1, it is easy to see that, for any \((p, q, \gamma, \rho)\lambda\)-atomic block \(b, \rho\) is also a \((p, q, \gamma, \epsilon, \rho)\lambda\)-molecular block and \(\|b\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)} \lesssim \|b\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)}\).

On the other hand, for any \((p, q, \gamma, \epsilon, \rho)\lambda\)-molecular block \(b\) with the same notation as in Definition 4.1, by the Fatou lemma, the H"older inequality, (4.1), \(B_{k,j} \subset \rho^{k+2}B\) and (2.1), we see that

\[
\|b\|_{L^p(\mu)} \leq \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|^p \|m_{k,j}\|_{L^p(\mu)} \leq \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|^p \|m_{k,j}\|_{L^p(\mu)} \mu(B_{k,j})^{1-p/q}
\]

\[
\lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|^p \mu(B_{k,j})^{1-p/q} \rho^{-k} \leq \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|^p \mu(B_{k,j})^{1-p/q} \rho^{-k} \leq \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|^p \sim \|b\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)}.
\]

which completes the proof of Proposition 4.3.

Remark 4.4. Let \(p \in (0, 1)\), and \(p, q, \gamma, \epsilon\) be as in Definition 4.1. By Proposition 4.3, we easily conclude that there exists a map \(T\) from \(\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\) to \(\mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\) such that, for any \(f \in \mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\), there is a unique element \(\tilde{f} \in \mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\) satisfying \(T\tilde{f} = f\) and \(\|\tilde{f}\|_{\mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)} \lesssim \|f\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)}\), where the implicit positive constant is independent of \(f\). In this sense, we say that \(\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu) \subset \mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\), which is different from the classical inclusion relation of spaces, since it is still unclear whether \(T\) is an injection and \(\|\tilde{f}\|_{\mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)} \sim \|f\|_{\mathcal{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon}(\mu)}\) or not.

Now we show that any element in \(\mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\) can be decomposed into a sum of a sequence of \((p, q, \gamma, \rho)\lambda\)-atomic blocks, \(\{b_j\}_{j=1}^\infty\), in \(\mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\). The proof is similar to that of Proposition 3.4, the details being omitted.

Proposition 4.5. Suppose that \((X, d, \mu)\) is a non-homogeneous metric measure space. Let \(p, \rho, q, \gamma, \epsilon\) be as in Definition 4.1. Then \(f \in \mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)\) if and only if there exist \((p, q, \gamma, \epsilon, \rho)\lambda\)-molecular blocks \(\{b_i\}_{i=1}^\infty\) such that

\[
f = \sum_{i=1}^\infty b_i \quad \text{in} \quad \mathcal{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon}(\mu)
\]
and
\[ \sum_{i=1}^{\infty} |b_i|_{\tilde{H}^{p,q,\gamma,\delta}_{mb,\rho}}^p < \infty. \]

Moreover,
\[ \|f\|_{\tilde{H}^{p,q,\gamma,\delta}_{mb,\rho}}^p = \inf \left\{ \sum_{i=1}^{\infty} |b_i|_{\tilde{H}^{p,q,\gamma,\delta}_{mb,\rho}}^p \right\}, \]
where the infimum is taken over all possible decompositions of \( f \) as in (4.4).

Now we consider the boundedness of Calderón-Zygmund operators on these atomic and molecular Hardy spaces. To this end, we first recall the following notion of Calderón-Zygmund operators from [29].

**Definition 4.6.** A function \( K \in L_{loc}^1((\mathcal{X} \times \mathcal{X}) \setminus \{(x,x): x \in \mathcal{X}\}) \) is called a **Calderón-Zygmund kernel** if there exists a positive constant \( C(K) \), depending on \( K \), such that

(i) for all \( x, y \in \mathcal{X} \) with \( x \neq y \),
\[ |K(x,y)| \leq C(K) \frac{1}{\lambda(x,d(x,y))}; \quad (4.5) \]

(ii) there exist positive constants \( \delta \in (0,1) \) and \( c(K) \), depending on \( K \), such that, for all \( x, \tilde{x}, y \in \mathcal{X} \) with \( d(x,y) \geq c(K)d(x,\tilde{x}) \),
\[ |K(x,y) - K(\tilde{x},y)| + |K(y,x) - K(y,\tilde{x})| \leq C(K) \frac{d(x,\tilde{x})^\delta}{[d(x,y)]^\delta \lambda(x,d(x,y))}. \quad (4.6) \]

A linear operator \( T \) is called a **Calderón-Zygmund operator** with kernel \( K \) satisfying (4.5) and (4.6) if, for all \( f \in L^p_{\ast}(\mu) := \{ f \in L^\infty(\mu) : \text{supp}\ f \text{ is bounded}\}, \)
\[ Tf(x) := \int_{\mathcal{X}} K(x,y)f(y) d\mu(y), \quad x \notin \text{supp}\ f. \quad (4.7) \]

A new example of operators with kernel satisfying (4.6) and (4.7) is the so-called Bergman-type operator appearing in [60]; see also [29] for an explanation.

We first recall the following useful lemma from [28].

**Lemma 4.7.** Suppose that \((\mathcal{X},d,\mu)\) is a non-homogeneous metric measure space. Let \( T \) be a Calderón-Zygmund operator defined by (4.7) associated with kernel \( K \) satisfying (4.5) and (4.6). Then the following statements are equivalent:

(i) \( T \) is bounded on \( L^2(\mu) \);

(ii) \( T \) is bounded on \( L^q(\mu) \) for all \( q \in (1,\infty) \);

(iii) \( T \) is bounded from \( L^1(\mu) \) to weak-\( L^1(\mu) \).

Now we prove the boundedness of Calderón-Zygmund operators from \( \tilde{H}^{p,q,\gamma,\delta}_{mb,\rho}(\mu) \) into \( L^p(\mu) \). Hereafter, let \( \nu := \log_2 C(\lambda) \), and \( \delta \) be as in Definition 4.6.

**Theorem 4.8.** Suppose that \((\mathcal{X},d,\mu)\) is a non-homogeneous metric measure space. Let \( \rho \in (1,\infty) \), \( \frac{\nu}{\nu+\delta} < p \leq 1 < q < \infty \) and \( \gamma \in (1,\infty) \). Assume that the Calderón-Zygmund operator \( T \) defined by (4.7) associated with kernel \( K \) satisfying (4.5) and (4.6) is bounded on \( L^2(\mu) \). Then \( T \) is bounded from \( \tilde{H}^{p,q,\gamma,\delta}_{mb,\rho}(\mu) \) into \( L^p(\mu) \).

**Proof.** Let \( \rho, p, q \) and \( \gamma \) be as in the assumptions of Theorem 4.8. For the sake of simplicity, we take \( \rho = 2 \) and \( \gamma = 1 \). With some slight modifications, the arguments here are still valid for general cases. We first reduce the proof to showing that, for all \((p,q,1,\delta,2)_{\lambda}\)-molecular blocks \( b \),
\[ \|Tb\|_{L^p(\mu)} \lesssim |b|_{\tilde{H}^{p,q,1,\delta}_{mb,2}(\mu)}. \quad (4.8) \]

Indeed, assume that (4.8) holds true. For any \( f \in \tilde{H}^{p,q,1,\delta}_{mb,2}(\mu) \), there exists a sequence \( \{b_i\}_{i \in \mathbb{N}} \) of \((p,q,1,\delta,2)_{\lambda}\)-molecular blocks such that \( f = \sum_{i=1}^{\infty} b_i \) in \( L^2(\mu) \) when \( p \leq 0 \) and in \( L^1(\mu) \) when \( p = 1 \).
and
\[ \sum_{i=1}^{\infty} |b_i|^{p} H_{\text{mb},2}^{p,q,1,\delta}(\mu) \sim \|f\|_{L^p}^{p} H_{\text{mb},2}^{p,q,1,\delta}(\mu). \]

If \( f = \sum_{i=1}^{\infty} b_i \) in \( L^2(\mu) \), then, by the boundedness of \( T \) on \( L^2(\mu) \), we see that, for any \( N \in \mathbb{N} \),
\[ \left\| \sum_{i=1}^{N} T(b_i) - TF \right\|_{L^2(\mu)} = \left\| T \left( \sum_{i=1}^{N} b_i - f \right) \right\|_{L^2(\mu)} \lesssim \left\| \sum_{i=1}^{N} b_i - f \right\|_{L^2(\mu)} \to 0 \text{ as } N \to \infty, \]
which further implies that, for all \( \eta \in (0, \infty) \),
\[ \mu \left( \left\{ x \in \mathcal{X} : \left| \sum_{i=1}^{N} T(b_i)(x) - TF(x) \right| > \eta \right\} \right) \to 0 \text{ as } N \to \infty. \quad (4.9) \]

If \( f = \sum_{i=1}^{\infty} b_i \) in \( L^1(\mu) \), then, by the boundedness of \( T \) from \( L^1(\mu) \) to weak-\( L^1(\mu) \), we still know that (4.9) holds true. Thus, by the Riesz theorem, we know that there exists a subsequence of partial sums, \( \{\sum_{i=1}^{N_k} T(b_i)\}_k \), such that
\[ Tf = \lim_{k \to \infty} \sum_{i=1}^{N_k} T(b_i) \mu - \text{almost everywhere on } \mathcal{X}, \]
which, together with the Fatou lemma and (4.8), implies that
\[ \|Tf\|_{L^p(\mu)} \leq \liminf_{k \to \infty} \int_{\mathcal{X}} \left| \sum_{i=1}^{N_k} T(b_i)(x) \right|^p d\mu(x) \leq \sum_{i=1}^{\infty} \|T(b_i)\|_{L^p(\mu)}^p \]
\[ \lesssim \sum_{i=1}^{\infty} |b_i|^{p} H_{\text{mb},2}^{p,q,1,\delta}(\mu) \sim \|f\|_{L^p}^{p} H_{\text{mb},2}^{p,q,1,\delta}(\mu). \]

Moreover, by a standard density argument, we extend \( T \) to be a bounded linear operator from \( H_{\text{mb},2}^{p,q,1,\delta}(\mu) \) into \( L^p(\mu) \), which is the desired result.

Now we prove (4.8). Let \( b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j} \) be a \( (p,q,1,\delta,2)_\lambda \)-molecular block, where, for any \( k \in \mathbb{Z}_+ \) and \( j \in \{1, \ldots, M_k\} \), \( \text{supp}(m_{k,j}) \subset B_{k,j} \subset U_k(B) \) for some balls \( B \) and \( B_{k,j} \) as in Definition 4.1. Without loss of generality, we may assume that \( M = M \) in Definition 4.1.

By the linearity of \( T \), we write
\[ \|Tb\|_{L^p(\mu)}^{p} \leq \sum_{\ell=5}^{\infty} \int_{U_{\ell}(B)} \left| T \left( \sum_{k=0}^{M} \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j} \right) (x) \right|^p d\mu(x) \]
\[ + \sum_{\ell=5}^{\infty} \int_{U_{\ell}(B)} \left| T \left( \sum_{k=\ell+4}^{M} \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j} \right) (x) \right|^p d\mu(x) \]
\[ + \sum_{\ell=5}^{\infty} \int_{U_{\ell}(B)} \left| T \left( \sum_{k=\ell+5}^{M} \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j} \right) (x) \right|^p d\mu(x) + \sum_{\ell=0}^{4} \int_{U_{\ell}(B)} |Tb(x)|^p d\mu(x) \]
\[ =: I + II + III + IV. \]

Now we first estimate III. By (4.5), (2.1), (2.2), the Hölder inequality and (4.1), we obtain
\[ III \leq \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{M} \sum_{j=1}^{M_k} |\lambda_{k,j}|^{p} \int_{U_{\ell}(B)} \left[ \int_{B_{k,j}} |m_{k,j}(y)| |K(x,y)| d\mu(y) \right]^p d\mu(x) \]
\[ \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{M} \sum_{j=1}^{M_k} |\lambda_{k,j}|^{p} \int_{U_{\ell}(B)} \left[ \int_{B_{k,j}} \frac{|m_{k,j}(y)|}{\lambda(x,d(x,y))} d\mu(y) \right]^p d\mu(x)\]
For I, from (4.6), (2.2), (2.1), the Hölder inequality, (4.1) and the fact that in order to estimate I, write

\[
I \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left( \int_{U_{\ell/2}^{(B)}} \frac{1}{\lambda(c_B, d(x, c_B))} d\mu(x) \right)^p m_{k,j}^p_{L^1(\mu)}
\]

\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \frac{\mu(2^{\ell+2}B)}{\lambda(c_B, d(x, c_B))} \left( \mu(B_{k,j}) \right)^{p'/q'} \|m_{k,j}\|_{L^q(\mu)}^p
\]

\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[ \mu \left( 2^{\ell+2}B \right) \right]^{1-p} \left( \mu(B_{k,j}) \right)^{p'/q'} \times 2^{-k\delta p} \left( \mu(2B_{k,j}) \right)^{-p'/q'} \left[ \lambda \left( c_B, 2k^2r_B \right) \right]^{p-1}
\]

\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} \sum_{m=10}^{\infty} k^2 2^{-k\delta p} |\lambda_{k,j}|^p \lesssim \sum_{k=0}^{M} \sum_{j=1}^{M} |\lambda_{k,j}|^p \sim |b|^p_{B^{p,q,1,\delta}(\mu)}.
\]

In order to estimate I, write

\[
I \lesssim \sum_{\ell=5}^{\infty} \int_{U_{\ell/2}^{(B)}} \left( \int_{K(x, c_B)} \left( \int_{K(x, c_B)} K(x, y) \right)^p \right) d\mu(x)
\]

\[
+ \sum_{\ell=5}^{\infty} \int_{U_{\ell/2}^{(B)}} \left( \int_{K(x, c_B)} K(x, y) \right)^p \right) d\mu(x) =: I_1 + I_2.
\]

From (4.6), (2.2), (2.1), the Hölder inequality, (4.1) and the fact that \( p \in \left( \frac{q}{p+q}, 1 \right) \), it follows that

\[
I_1 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left( \int_{U_{\ell/2}^{(B)}} \left| m_{k,j}(y) \right|^s \left( \frac{d(y, c_B)}{\lambda(c_B, d(x, c_B))} \right)^\delta \right)^p d\mu(y).
\]

\[
I_2 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left( \int_{U_{\ell/2}^{(B)}} \left| m_{k,j}(y) \right|^s \left( \frac{d(y, c_B)}{\lambda(c_B, d(x, c_B))} \right)^\delta \right)^p d\mu(y).
\]

For \( I_2 \), the vanishing moment of \( b \), together with (4.5), (2.1) and (2.2), implies that

\[
I_2 = \sum_{\ell=5}^{\infty} \int_{U_{\ell/2}^{(B)}} \left( \int_{K(x, c_B)} \left( \int_{K(x, c_B)} K(x, y) \right)^p \right) d\mu(x)
\]

\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{\ell-4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left( \int_{U_{\ell/2}^{(B)}} \left| m_{k,j}(y) \right|^s \left( \frac{d(y, c_B)}{\lambda(c_B, d(x, c_B))} \right)^\delta \right)^p d\mu(y).
\]
By the Hölder inequality, $L^2(\mu)$-boundedness of $T$, Lemma 4.7, (4.1) and (2.1), we see that

\[
\Pi_1 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[ \mu(2B_{k,j}) \right]^{1-p/q} \|m_{k,j}\|_{L^q(\mu)}^p
\]

\[
\Pi_2 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B)} \left[ \int_{B_{k,j}} \frac{|m_{k,j}(y)|}{\lambda(c_{B_{k,j}}, d(x,c_{B_{k,j}}))} \, d\mu(y) \right]^p \, d\mu(x)
\]

Combining $I_1$ and $I_2$, we conclude that $I \lesssim |b|_{\tilde{H}^{p,q}_{mb,2,1}(\mu)}^p$.

Then we turn to estimate $\Pi$. We further write

\[
\Pi \leq \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B)} |Tm_{k,j}(x)|^p \, d\mu(x)
\]

\[
\leq \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{2B_{k,j}} |Tm_{k,j}(x)|^p \, d\mu(x)
\]

\[
+ \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B) \setminus 2B_{k,j}} \cdots =: \Pi_1 + \Pi_2.
\]

By the Hölder inequality, $L^2(\mu)$-boundedness of $T$, Lemma 4.7, (4.1) and (2.1), we see that

\[
\Pi_1 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[ \mu(2B_{k,j}) \right]^{1-p/q} \|Tm_{k,j}\|_{L^q(\mu)}^p
\]

\[
\Pi_2 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} \left[ \int_{B_{k,j}} \frac{|m_{k,j}(y)|}{\lambda(c_{B_{k,j}}, d(x,c_{B_{k,j}}))} \, d\mu(y) \right]^p \, d\mu(x)
\]

\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+M} \sum_{j=1}^{M} \left[ \frac{1}{\lambda(c_{B_{k,j}}, d(x,c_{B_{k,j}}))} \right]^p \|m_{k,j}\|_{L^1(\mu)}^p
\]

For $\Pi_2$, from (4.5), $d(x,y) \geq d(x,c_{B_{k,j}}) - d(y,c_{B_{k,j}}) \geq \frac{1}{2}d(x,c_{B_{k,j}})$ for $x \notin 2B_{k,j}$ and $y \in B_{k,j}$, (2.2), (2.1), the Hölder inequality and (4.1), we deduce that
\[
\sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p |m_{k,j}|^p \left[ \mu \left( 2^{N_1(2) B_{k,j}, 2^{k+5} B} \right) \right]^{1-p} \left[ \tilde{K}_{B_{k,j}, 2^{k+5} B}^{(2), p} \right]^p \\
\sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[ \mu (B_{k,j}) \right]^{p/q} |m_{k,j}|^p \left[ \mu (2^{k+q} B) \right]^{1-p} \left[ \tilde{K}_{B_{k,j}, 2^{k+5} B}^{(2), p} \right]^p \\
\sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} -2^{-k\delta p} |\mu (2^{k+9} B)|^{-p/q} \left[ \lambda (\epsilon B, 2^{k+2} T_B) \right]^{p-1} \left[ \tilde{K}_{B_{k,j}, 2^{k+5} B}^{(2), p} \right]^p \\
\sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} 2^{-k\delta p} |\lambda_{k,j}|^p \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \sim |b|_{H_{mb, p, \gamma, \epsilon}}^p(\mu),
\]
which, together with the estimate for II, implies that II \( \lesssim |b|_{H_{mb, p, \gamma, \epsilon}}^p(\mu) \).

To estimate IV, observe that

\[
IV \lesssim \sum_{t=0}^{4} \int_{U_t(B)} \left| T \left( \sum_{k=0}^{M} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right) \right|^p \, d\mu(x) \\
+ \sum_{t=0}^{4} \int_{U_t(B)} \left| T \left( \sum_{k=\ell-5}^{\ell} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right) \right|^p \, d\mu(x) =: IV_1 + IV_2.
\]

By some arguments similar to those used in the estimates for II and III, we respectively obtain

\[
IV_1 \lesssim |b|_{H_{mb, p, 1, \delta}}^p(\mu) \quad \text{and} \quad IV_2 \lesssim |b|_{H_{mb, 1, \delta}}^p(\mu),
\]
which, together with the estimates for III, I and II, completes the proof of Theorem 4.8.

Now we show the boundedness of Calderón-Zygmund operators from \( \tilde{H}_{p, \rho, \gamma}(\mu) \) into \( L^p(\mu) \).

**Corollary 4.9.** Suppose that \((X, d, \mu)\) is a non-homogeneous metric measure space. Let \( \rho \in (1, \infty) \), \( \frac{1}{\rho} < p < q < \infty \) and \( \gamma \in [1, \infty) \). Assume that the Calderón-Zygmund operator \( T \) defined by (4.7) associated with kernel \( K \) satisfying (4.5) and (4.6) is bounded on \( L^2(\mu) \). Then \( T \) is bounded from \( H_{p, \rho, \gamma}(\mu) \) into \( L^p(\mu) \).

**Proof.** Let \( \rho, \ p, \ q, \ \gamma \) and \( \delta \) be as in assumptions of Corollary 4.9. For the sake of simplicity, we take \( \rho = 2 \) and \( \gamma = 1 \). By an argument similar to that used in the proof of Theorem 4.8, it suffices to show that, for any \((p, q, 1, 2)\)-atomic block \( b \),

\[
\| T b \|_{L^p(\mu)} \lesssim |b|_{H_{mb, p, 1, \delta}}^p(\mu),
\]
which is an easy consequence of the facts that \( b \) is also a \((p, q, 1, \delta, 2)\)-atomic block and \( |b|_{H_{mb, p, 1, \delta}}^p(\mu) \sim |b|_{H_{mb, 1, \delta}}^p(\mu) \) (see (4.3)), together with (4.8) from the proof of Theorem 4.8. This finishes the proof of Corollary 4.9.

**Remark 4.10.** When \( p = 1 \), Theorem 4.8 or Corollary 4.9 is a special case of [30, Theorem 4.1], since, for any \( q \in (1, \infty) \), \( \rho \in (1, \infty) \), \( \gamma \in [1, \infty) \) and \( \epsilon \in (0, \infty) \), \( H_{1, q, \gamma}(\mu) \subset H_{1, \rho, \gamma}(\mu) \), where \( H_{1, q, \gamma}(\mu) \) is introduced in [30] (see [13, Remark 1.9(i)]), and, by [13, Theorem 1.11 and Remark 1.9(i)], we know that \( H_{1, q, \gamma}(\mu) \) is independent of the choices of \( q, \rho \) and \( \gamma \), and \( H_{1, q, \gamma}(\mu) \) and \( H_{1, q, \epsilon}(\mu) \) coincide with equivalent norms.

Now we establish the \( (\tilde{H}^p_{p, \rho, \gamma}, 1 + \frac{1}{p}(6 - \frac{\rho}{q} + \epsilon)) \)-boundedness of Calderón-Zygmund operators. In what follows, for \( T \) as in Corollary 4.9, \( T \) is said to satisfy \( T^* 1 = 0 \) if, for all \( h \in L^\infty(\mu) \),

\[
\int_X T h(y) \, d\mu(y) = 0,
\]

where \( T h \) is defined as

\[
T h(y) := \sum_{x \in \text{atoms}} T h(x) \chi_{B(x, \rho \|T h\|_X)}(y).
\]
Observe that, for such $T$ and $h$, by Corollary 4.9, we have $T h \in L^1(\mu)$.

**Theorem 4.11.** Suppose that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space. Let $\rho \in [2, \infty)$, $\frac{\nu}{\nu + \delta} < p \leq q < \infty$ and $\gamma \in [1, \infty)$. Assume that the Calderón-Zygmund operator $T$ defined by (4.7) associated with kernel $K$ satisfying (4.5) and (4.6) is bounded on $L^2(\mu)$ and $T^{*1} = 0$. Then $T$ is bounded from $\overline{H}_{\text{atb}, \rho(p+1)}^p, q, \gamma, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu)(\mu)$ into $\overline{H}_{\text{mb}, \rho}^p, q, \gamma, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu)(\mu)$.

**Proof.** Observe that, when $p = 1$, Theorem 4.11 is a special case of [13, Theorem 1.14], since it was shown, by [13, Theorem 1.11 and Remark 1.9(i)], that, for any $q \in (1, \infty)$, $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$ and $\epsilon \in (0, \infty)$, $\overline{H}_{\text{atb}, \rho}^1, q, \gamma, \epsilon(\mu)$ and $\overline{H}_{\text{mb}, \rho}^1, q, \gamma, \epsilon(\mu)$ coincide with equivalent norms, and $\overline{H}_{\text{atb}, \rho}^1, q, \gamma(\mu)$ is independent of the choices of $q$, $\rho$ and $\gamma$. Thus, to show Theorem 4.11, we only need to consider the case $p \in (\frac{\nu}{\nu + \delta}, 1)$. Moreover, for the sake of simplicity, we assume that $\gamma = 1$ and $\rho = 2$. Via some slight modifications, the arguments here are still valid for general cases. We first reduce our proof to showing that, for any $(p, q, 2, 6)_\lambda$-atomic block, $T b$ is a $(p, q, 1, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu), 2)_\lambda$-molecular block and

$$
|T b|_{\overline{H}_{\text{atb}, \rho}^p, q, \gamma, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu)(\mu)} \leq |b|_{\overline{H}_{\text{atb}, \rho}^p, q, 2(\mu)}.
$$

(4.10)

Indeed, assume that (4.10) holds true. For any $\overline{H}_{\text{atb}, \rho}^p, q, 2(\mu)$, there exists a sequence $\{b_i\}_{i \in \mathbb{N}}$ of $(p, q, 2, 6)_\lambda$-atomic blocks such that $f = \sum_{i = 1}^{\infty} b_i$ in $L^2(\mu)$ and

$$
\sum_{i = 1}^{\infty} |b_i|_{\overline{H}_{\text{atb}, \rho}^p, q, 2(\mu)} \sim \|f\|_{\overline{H}_{\text{atb}, \rho}^p, q, 2(\mu)}.
$$

By the boundedness of $T$ on $L^2(\mu)$, we see that

$$
\left\| \sum_{i = 1}^{N} T(b_i) - T f \right\|_{L^2(\mu)} \leq \left\| T \left( \sum_{i = 1}^{N} b_i - f \right) \right\|_{L^2(\mu)} \leq \sum_{i = 1}^{N} \left\| b_i - f \right\|_{L^2(\mu)} \to 0 \quad \text{as} \quad N \to \infty.
$$

Thus, $T f = \sum_{i = 1}^{\infty} T(b_i)$ in $L^2(\mu)$. Moreover, by (4.10) and $T(b_i)$ is a $(p, q, 1, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu), 2)_\lambda$-molecular block for any $i \in \mathbb{N}$, we know that

$$
\|T f\|_{\overline{H}_{\text{atb}, \rho}^p, q, \gamma, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu)(\mu)} \leq \sum_{i = 1}^{\infty} \|T(b_i)\|_{\overline{H}_{\text{atb}, \rho}^p, q, \gamma, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu)(\mu)} \leq \sum_{i = 1}^{\infty} |b_i|_{\overline{H}_{\text{atb}, \rho}^p, q, 2(\mu)} \sim \|f\|_{\overline{H}_{\text{atb}, \rho}^p, q, 2(\mu)}.
$$

Furthermore, by a standard density argument, we extend $T$ to be a bounded linear operator from $\overline{H}_{\text{atb}, \rho}^p, q, 2(\mu)$ into $\overline{H}_{\text{atb}, \rho}^{\frac{\nu}{\nu + \delta}, \frac{1}{2}(\delta - \frac{\nu}{p} + \nu)}(\mu)$.

Now we show that (4.10) holds true. Let $b$ be a $(p, q, 2, 6)_\lambda$-atomic block. Then $b := \sum_{j = 1}^{2} \lambda_j a_j$, where, for any $j \in \{1, 2\}$, supp $(a_j) \subset B_j \subset B$ for some balls $B_j$ and $B$ as in Definition 3.2. Let $B_0 := 8B$. We write

$$
Tb = (Tb) \chi_{B_0} + \sum_{k = 1}^{\infty} (Tb_k) \chi_{2^k B_0 \backslash 2^{k-1} B_0} =: A_1 + A_2.
$$

We first estimate $A_1$. Since $B_j \subset B$, we have $3B_j \subset B = B_0$. Let $N_j := N_j^{(2)}(B_j, B_0)$. Obviously, $N_j \geq -1$. Without loss of generality, we may assume that $N_j \geq 3$. For the case $N_j \in [-1, 3)$, we easily observe that $2B_j \subset B_0 \subset 2^2 B_j$, which can be reduced to the case $N_j \geq 3$. We further decompose

$$
A_1 = \sum_{j = 1}^{2} \lambda_j (T a_j) \chi_{2B_j} + \sum_{j = 1}^{N_j - 3} \lambda_j (T a_j) \chi_{2^j+1 B_j \backslash 2^j B_j} + \sum_{j = 1}^{2} \lambda_j (T a_j) \chi_{B_0 \backslash 2^{N_j - 3} B_j} =: A_{1,1} + A_{1,2} + A_{1,3}.
$$

To estimate $A_{1,1}$, by Definition 3.2(iii), the boundedness of $T$ on $L^2(\mu)$, Lemma 4.7, (v), (iv) and (ii) of Lemma 2.8, Lemma 2.9 and $\overline{K}_{3B_j, B_0}(\mu) \geq 1$, we see that, for any $j \in \{1, 2\}$,

$$
\|T a_j\|_{L^2(\mu)} \leq \|a_j\|_{L^2(\mu)} \leq |\mu(6B_j)|^{1/q - 1} |\lambda(c_B, r_B)|^{1 - 1/p} \left( \overline{K}_{3B_j, B_0}^{(6)} \right)^{-\frac{1}{p}}.
$$
Let \( c_B \) and \( c_{B_0} \) denote the centers of \( B \) and \( B_0 \), respectively. For any \( j \)

\[
\sigma_j := c_B \text{ and } n_{j,1} := c_B^{-1}(T_a) x_{2B_0}.
\]

Then \( A_{1,1} = \sum_{j=1}^{2} \sigma_j n_{j,1} \), supp \( (n_{j,1}) \subset 3B_j \subset B_0 \) and

\[
\|n_{j,1}\|_{L^q(\mu)} \lesssim [\mu(6B_j)]^{1/q-1} [\lambda(c_B, 8r_B)]^{1-1/p} \left[ \bar{K}_{3B_j, 8B}^{(2), p} \right]^{-2}.
\]

Here and hereafter, \( c_B \) and \( c_{B_0} \) denote the centers of \( B \) and \( B_0 \), respectively. Let \( c_1 \), independent of \( a_j \) and \( j \), be the implicit positive constant of the above inequality, \( \sigma_j,1 := c_1 \) and \( n_{j,1} := c_B^{-1}(T_a) x_{2B_0} \). Then \( A_{1,1} = \sum_{j=1}^{2} \sigma_j n_{j,1} \), supp \( (n_{j,1}) \subset 3B_j \subset B_0 \) and

\[
\|n_{j,1}\|_{L^q(\mu)} \lesssim [\mu(6B_j)]^{1/q-1} [\lambda(c_{B_0}, r_{4B_0})]^{1-1/p} \left[ \bar{K}_{3B_j, 4B_0}^{(2), p} \right]^{-1}.
\]

For \( A_{1,3} \), we observe that \( r_{B_0} \sim r_{2^{N_j-1}B_j} \), where \( r_j \) and \( r_{2^{N_j-1}B_j} \) denote the radii of \( B_0 \) and \( 2^{N_j-1}B_j \), respectively. For any \( j \in \{1, 2\} \), let \( x_j \) and \( y_j \) be the center and the radius of \( B_j \), respectively. By (4.5), (2.2), (2.1), the Hölder inequality, Definition 3.2(iii), \( \bar{K}_{B_j, B}^{(6), p} \geq 1 \), \( B_0 \subset 2^{N_j+3}B_j \), Lemma 2.8(ii) and Lemma 2.9, we obtain

\[
\left\| (T_a) x_{B_0 \setminus 2^{N_j-1}B_j} \right\|_{L^q(\mu)} \lesssim \left\{ \int_{8B_j \setminus 2^{N_j-1}B_j} \left[ \int_{B_j} \frac{|a_j(y)|}{\lambda(x, d(x, y))} \, d\mu(y) \right]^q \, d\mu(x) \right\}^{1/q}.
\]

Let \( c_2 \), independent of \( a_j \) and \( j \), be the implicit positive constant of the above inequality, \( \sigma_j,3 := c_2 \lambda_j \) and \( n_{j,3} := n_{B_0 \setminus 2^{N_j-1}B_j} \). Then \( A_{1,3} = \sum_{j=1}^{2} \sigma_j n_{j,3} \), supp \( (n_{j,3}) \subset 16B = 2B_0 \) and

\[
\|n_{j,3}\|_{L^q(\mu)} \lesssim [\mu(4B_0)]^{1/q-1} [\lambda(c_{B_0}, r_{4B_0})]^{1-1/p} \left[ \bar{K}_{2B_0, 4B_0}^{(2), p} \right]^{-1}.
\]

We now estimate \( A_{1,2} \). By (4.5), (2.2), (2.1), Definition 3.2(iii), the Hölder inequality, (v), (iv) and (ii) of Lemma 2.8, and Lemma 2.9, we conclude that

\[
\left\| (T_a) x_{2^{i+1}B_j \setminus 2^iB_j} \right\|_{L^q(\mu)} \lesssim \left\{ \int_{2^{i+1}B_j \setminus 2^iB_j} \left[ \int_{B_j} \frac{|a_j(y)|}{\lambda(x, d(x, y))} \, d\mu(y) \right]^q \, d\mu(x) \right\}^{1/q}.
\]

Let \( c_3 \), independent of \( a_j \) and \( i \), be the implicit positive constant of the above inequality,

\[
\sigma_j^{(i)} := c_3 \lambda_j \frac{\mu(2^{i+3}B_j)}{\lambda(x_j, 2^r_j)} \left[ \bar{K}_{B_j, B}^{(2), p} \right]^{-1}.
\]
Thus, \( k \in \{ 1 \} \), \( \mu \left( 2^{j+2} B_j \right) = 0 \) and Lemma 2.8(ii), it follows that, for any \( 0 \leq \mu \), \( \Omega \left( cB_0, r_4B_0 \right) \) \( \left[ K_{2^{j+2}B_j, 4B_0} \right]^{-1} \) .

Now we turn to estimate \( A_2 \). For any \( k \in \mathbb{N} \), by the geometrically doubling condition, there exists a ball covering \( \{ B_{k,j} \}_{j=1}^{M_0} \), with uniform radius \( 2^{k-3} r_{B_0} \), of \( \Omega_k(B_0) := 2^{k} B_0 \setminus 2^{k-1} B_0 \) such that the cardinality \( M_0 \leq N_0 \). Without loss of generality, we may assume that the centers of the balls in the covering belong to \( \Omega_k(B_0) \).

Let \( C_{k,1} := B_{k,1} \), \( C_{k,l} := B_{k,l} \setminus \bigcup_{m=1}^{l-1} B_{k,m} \), \( l \in \{ 2, \ldots, M_0 \} \) and \( D_{k,l} := C_{k,l} \cap \Omega_k(B_0) \) for all \( l \in \{ 1, \ldots, M_0 \} \). Then we know that \( \{ D_{k,l} \}_{l=1}^{M_0} \) is pairwise disjoint, \( \Omega_k(B_0) = \bigcup_{l=1}^{M_0} D_{k,l} \) and, for any \( l \in \{ 1, \ldots, M_0 \} \),

\[
D_{k,l} \subset 2B_{k,l} \subset \Omega_k(B_0) := 2^{k+2} B_0 \setminus 2^{k-2} B_0.
\]

Thus,

\[
A_2 = \sum_{k=1}^{\infty} \sum_{i=1}^{M_0} \chi_{D_{k,i}} = \sum_{k=1}^{\infty} \sum_{l=1}^{M_0} \mathbb{1}(Tb) \chi_{D_{k,l}}.
\]

From \( \int_X b(y) \mu(y) = 0 \), (4.6), (2.2), (2.1), the Hölder inequality, Definition 3.2(iii), \( K_{2^{j+2}B_j, 4B_0} \geq 1 \), \( 4B_{k,l} \subset 2^{k+1} B_0 \) and Lemma 2.8(ii), it follows that, for any \( k \in \mathbb{N} \), \( j \in \{ 1, 2 \} \) and \( l \in \{ 1, \ldots, M_0 \} \),

\[
\| (Tb) \chi_{D_{k,l}} \|_{L^q(\mu)} \leq \left\{ \int_{D_{k,l}} \left( \int_B |b(y)||K(x,y) - K(x,c_B)| \mu(y) \right)^q d\mu(x) \right\}^{1/q}
\leq \left\{ \int_{D_{k,l}} \left( \int_B |b(y)| \left[ \frac{d(y,c_B)^{\delta}}{d(x,c_B)^{\delta} \lambda(c_B,d(x,c_B))} \right] \mu(y) \right)^q d\mu(x) \right\}^{1/q}
\leq \frac{r_B^{\delta} \mu(D_{k,l})^{1/q}}{\lambda(c_B,r^{k+1}B_0)} \left( r^{k+1}B_0 \right)^{\delta} \int_B |b(y)| \mu(y)
\leq 2^{-k\delta} \left[ \mu \left( 2^{k+1} B_0 \right) \right]^{1/q-1} \sum_{j=1}^{2} |\lambda_j| |\mu(B_j)|^{1/q} \left\| a_j \right\|_{L^q(\mu)}
\leq 2^{-k\delta} \left[ C(\lambda) \right]^{(k+2) \left( \frac{1}{p} - 1 \right)} \sum_{j=1}^{2} |\lambda_j| |\mu(4B_{k,l})|^{1/q-1} \left[ \lambda(c_B, r_{2k+2}B_0) \right]^{1-1/p}
\leq 2^{-k\delta} \left[ C(\lambda) \right]^{(k+2) \left( \frac{1}{p} - 1 \right)} \sum_{j=1}^{2} |\lambda_j| |\mu(4B_{k,l})|^{1/q-1} \left[ \lambda(c_B, r_{2k+2}B_0) \right]^{1/q-1}
\times \left[ \lambda(c_B, r_{2k+2}B_0) \right]^{1-1/p} \left[ K_{2^kB_{k,l}, 2^{k+2}B_0} \right]^{-1}.
\]

Let \( c_4 \), independent of \( b \) and \( k \), be the implicit positive constant of the above inequality,

\[
\lambda_{k,l} := c_4 2^{-k\delta \left( \frac{1}{p} - 1 \right)} \sum_{j=1}^{2} |\lambda_j|
\]
and \( m_{k,t} := \lambda_{k,t}^{-1}(Tb)\chi_{B_{k,t}} \). Then
\[
A_2 = \sum_{k=1}^{\infty} \sum_{l=1}^{M_0} \lambda_{k,l} m_{k,l},
\]
\( \text{supp}(m_{k,l}) \subset 2B_{k,t} \subset U_{k}(B_0) \) and
\[
\|m_{k,l}\|_{L^q(\mu)} \leq 2^{-\frac{k}{2}(\delta-\frac{\theta}{p})} \left[ \mu(2(2B_{k,t})) \right]^{1/q-1} \left[ \lambda(c_B, r_{2k+2}B_0) \right]^{1-1/p} \left[ K_{2B_{k,t}, 2k+2B_0}^{(2)} \right]^{-1}.
\]
Combining the estimates for \( A_1 \) and \( A_2 \), we see that \( Tb \) is a \((p, q, 1, 1, \frac{1}{2}(\delta - \frac{\theta}{p}) + \nu), 2)\)-\( \chi \)-molecular block and
\[
|Tb|_{B_{3,2}^{p,q,1,1,0} \left( \frac{1}{2}(\delta - \frac{\theta}{p}) + \nu, \mu \right)} \lesssim \sum_{j=1}^{2} |\sigma_j|^{p} + \sum_{j=1}^{N_j} \lambda_j |\lambda_j|^{p} \left[ \frac{\mu(2j+3B_j)}{\lambda(x_j, 2r_j)} \right]^{\nu} \left[ K_{B_j, B_j}^{(2)} \right]^{-p}
+ \sum_{k=1}^{M_0} \sum_{l=1}^{2} b^{(\delta - \frac{\theta}{p})} \mu \lesssim \sum_{j=1}^{2} |\lambda_j|^{p} \sim |b|_{B_{3,2}^{p,q,1,1,\nu,0} \left( \frac{1}{2}(\delta - \frac{\theta}{p}) + \nu, \mu \right)},
\]
which completes the proof of Theorem 4.11.

**Remark 4.12.** It is still unclear whether the range of \( \rho \) in Theorem 4.11 is sharp or not.

## 5 Boundedness of Generalized Fractional Integrals

In this section, we establish the boundedness of the generalized fractional integral \( T_\beta \) \((\beta \in (0, 1))\) from \( H_{mb, p}^{\alpha, \gamma, \theta}(\mu) \) (or \( H_{atb, p}^{\alpha, \gamma, \theta}(\mu) \)) into \( L^p(\mu) \) with \( 1/p_2 = 1/p_1 - \beta \), where \( \theta \) is some positive constant depending on \( T_\beta \). To this end, we first recall the notion of generalized fractional integrals from [15].

**Definition 5.1.** Let \( \beta \in (0, 1) \). A function \( K_\beta \in L^1_{\text{loc}}(X \times X \setminus \{(x, x) : x \in X\}) \) is called a **generalized fractional integral kernel** if there exists a positive constant \( C_{(K_\beta)} \), depending on \( K_\beta \), such that

(i) for all \( x, y \in X \) with \( x \neq y \),
\[
|K_\beta(x, y)| \leq C_{(K_\beta)} \frac{1}{|\lambda(x, d(x, y))|^{1-\beta}};
\]

(ii) there exist positive constants \( \theta \in (0, 1) \) and \( c_{(K_\beta)} \), depending on \( K_\beta \), such that, for all \( x, \bar{x}, y \in X \) with \( d(x, y) \geq c_{(K_\beta)} d(x, \bar{x}) \),
\[
|K_\beta(x, y) - K_\beta(\bar{x}, y)| + |K_\beta(y, x) - K_\beta(y, \bar{x})| \leq C_{(K_\beta)} \frac{|d(x, \bar{x})|^{\theta}}{|d(x, y)|^{\theta} |\lambda(x, d(x, y))|^{1-\beta}}.
\]

A linear operator \( T_\beta \) is called a **generalized fractional integral** with kernel \( K_\beta \) satisfying (5.1) and (5.2) if, for all \( f \in L^\infty_0(\mu) \) and \( x \notin \text{supp}(f) \),
\[
T_\beta f(x) := \int_X K_\beta(x, y) f(y) \, d\mu(y).
\]

**Remark 5.2.** It was shown in [15, Remark 1.10(iii)] that there exists a specific example of the generalized fractional integral, which is a natural variant of the so-called Bergman-type operator; see [15] for the details.
Now we show that the generalized fractional integral $T_\beta$ is bounded from $\widetilde{H}^{p_1,q,\gamma,\theta}_{mb,\rho}(\mu)$ into $L^{p_2}(\mu)$ for $1/p_2 = 1/p_1 - \beta$. Recall that $\nu := \log_2 C(\lambda)$ and $\theta$ is as in Definition 5.1.

**Theorem 5.3.** Suppose that $(X,d,\mu)$ is a non-homogeneous metric measure space. Let $\beta \in (0, 1/2)$, $\rho \in (1, \infty)$, $\frac{\nu}{\nu + \beta} < p_1 < p_2 \leq 1 < q < 1/\beta$, $1/p_2 = 1/p_1 - \beta$ and $\gamma \in [1, \infty)$. Assume that the generalized fractional integral $T_\beta$ defined by (5.3) associated with kernel $K_\beta$ satisfying (5.1) and (5.2) is bounded from $L^q(\mu)$ into $L^1(\mu)$, where $1/q := 1/q - \beta$. Then $T_\beta$ is bounded from $\widetilde{H}^{p_1,q,\gamma,\theta}_{mb,\rho}(\mu)$ into $L^{p_2}(\mu)$.

**Proof.** Let $\beta$, $\rho$, $p_1$, $p_2$, $q$, $\bar{q}$ and $\gamma$ be as in assumptions of Theorem 5.3. For the sake of simplicity, we take $\rho = 2$ and $\gamma = 1$. With some minor modifications, the arguments here are still valid for general cases.

Since $T_\beta$ is bounded from $L^q(\mu)$ to $L^1(\mu)$ for $q \in (1, 1/\beta)$ and $1/q = 1/\beta - \beta$, by [15, Theorem 1.13], we know that $T_\beta$ is also bounded from $L^1(\mu)$ to weak-$L^{1/(1-\beta)}(\mu)$. By the boundedness of $T_\beta$ from $L^2(\mu)$ to weak-$L^{2/(1-2\beta)}(\mu)$ or from $L^1(\mu)$ into weak-$L^{1/(1-\beta)}(\mu)$ and an argument similar to that used in the proof of Theorem 4.8, it suffices to show that, for all $(p_1, q, 1, \theta, 2)_\lambda$-molecular blocks $b$,

$$
\|T_\beta b\|_{L^{p_2}(\mu)} \lesssim \|b\|_{\widetilde{H}^{p_1,q,\gamma,\theta}_{mb,\rho}(\mu)}.
$$

Let $b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j}$ be a $(p_1, q, 1, \theta, 2)_\lambda$-molecular block, where, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $\text{supp} (m_{k,j}) \subset B_{k,j} \subset U_k(B)$ for some balls $B$ and $B_{k,j}$ as in Definition 4.1. Without loss of generality, we may assume that $M = M$ in Definition 4.1.

By the linearity of $T_\beta$, we write

$$
\|T_\beta b\|_{L^{p_2}(\mu)}^2 \leq \sum_{\ell = 5}^{\infty} \int_{U_\ell(B)} \left| T_\beta \left( \sum_{k=0}^{M} \sum_{j=1}^{1} \lambda_{k,j} m_{k,j} \right)(x) \right|^2 \mu(dx)
$$

$$
+ \sum_{\ell = 5}^{\infty} \int_{U_\ell(B)} \left| T_\beta \left( \sum_{k=0}^{M} \sum_{j=1}^{1} \lambda_{k,j} m_{k,j} \right)(x) \right|^2 \mu(dx)
$$

$$
+ \sum_{\ell = 5}^{\infty} \int_{U_\ell(B)} \left| T_\beta \left( \sum_{k=0}^{M} \sum_{j=1}^{1} \lambda_{k,j} m_{k,j} \right)(x) \right|^2 \mu(dx)
$$

$$
=: I + II + III + IV.
$$

Now we first estimate III. By (5.1), (2.1), (2.2), the Hölder inequality, (4.1) and $1/p_2 = 1/p_1 - \beta$, we obtain

$$
III \leq \sum_{\ell = 5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{1} \int_{U_\ell(B)} \left| \int_{B_{k,j}} |m_{k,j}(y)||K_\beta(x, y)| d\mu(y) \right|^2 \mu(dx)
$$

$$
\leq \sum_{\ell = 5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{1} \int_{U_\ell(B)} \left| \int_{B_{k,j}} \frac{|m_{k,j}(y)|}{|x - d(x, y)|^{1 - \theta}} d\mu(y) \right|^2 \mu(dx)
$$

$$
\leq \sum_{\ell = 5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{1} \int_{U_\ell(B)} \frac{1}{|x - d(x, y)|^{1 - \theta}} d\mu(x)|m_{k,j}|_{L^1(\mu)}^2
$$

$$
\leq \sum_{\ell = 5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{1} \frac{\mu(2^\ell + 2B)}{|x - d(x, y)|^{1 - \theta}} \left[ \mu(B_{k,j}) \right]^{p_2/q'} \|m_{k,j}\|_{L^1(\mu)}^2
$$

$$
\leq \sum_{\ell = 5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{1} \frac{\mu(2^\ell + 2B)}{|x - d(x, y)|^{1 - \theta}} \left[ \mu(B_{k,j}) \right]^{p_2/q'} \|m_{k,j}\|_{L^1(\mu)}^2
$$

$$
\times 2^{-k\theta p_2} \left[ \mu(2B_{k,j}) \right]^{p_2/q'} \left[ \mu(B_{k,j}) \right]^{p_2/q'}
$$

$$
\leq \sum_{\ell = 5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{1} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2} \sim \sum_{j=1}^{\infty} \sum_{k=0}^{M} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2}
$$
\[
\sum_{j=1}^{M} \sum_{k=1}^{\infty} k^{2-k\theta p_2} |\lambda_{k,j}|^{p_2} \lesssim \left\{ \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_1} \right\}^{p_2/p_1} \sim |b|_{P_{\text{mb,2}}^{1,q,\theta}}^{p_2}.
\]

In order to estimate I, write
\[
I \leq \sum_{\ell=5}^{\infty} \int_{U_{\ell}(B)} \left[ \int_{X} \left\{ \sum_{k=0}^{M} \sum_{j=1}^{\infty} \lambda_{k,j} m_{k,j}(y) \right\} \left[ K_\beta(x,y) - K_\beta(x,c_B) \right] d\mu(y) \right] d\mu(x)
+ \sum_{\ell=5}^{\infty} \int_{U_{\ell}(B)} \left[ \int_{X} \left\{ \sum_{k=0}^{M} \sum_{j=1}^{\infty} \lambda_{k,j} m_{k,j}(y) \right\} K_\beta(x,c_B) d\mu(y) \right] d\mu(x) =: I_1 + I_2.
\]

From (5.2), (2.2), (2.1), the Hölder inequality, (4.1) and \(p_1 \in (\frac{2}{\nu+n}, 1]\) and \(1/p_2 = 1/p_1 - \beta\), it follows that
\[
I_1 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} \int_{U_{\ell}(B)} \left\{ \int_{B_{k,j}} \frac{|m_{k,j}(y)| d(y,c_B)^{\theta} |d(y,c_B)|^{1-\theta} \mu(c_B,d(x,c_B))^{\gamma} d\mu(y) \right\}^{p_2} d\mu(x)
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} \frac{2^{(k+2)\theta p_2} \mu^{(2\ell+2)B}}{2^{(\ell-2)\theta p_2} \mu^{(2\ell+2)B}} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(k-\ell)\theta p_2} \left[ \mu(2\ell+2B) \right]^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(\ell-k)\theta p_2} \left[ \mu(2\ell+2B) \right]^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(\ell-k)\theta p_2} \left[ \mu(2\ell+2B) \right]^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(\ell-k)\theta p_2} \mu(2\ell+2B)^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(\ell-k)\theta p_2} \mu(2\ell+2B)^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(\ell-k)\theta p_2} \mu(2\ell+2B)^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(\ell-k)\theta p_2} \mu(2\ell+2B)^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} 2^{-(\ell-k)\theta p_2} \mu(2\ell+2B)^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]

For \(I_2\), the vanishing moment of \(b\), together with (5.1), (2.1), (2.2) and \(1/p_2 = 1/p_1 - \beta\), implies that
\[
I_2 = \sum_{\ell=5}^{\infty} \int_{U_{\ell}(B)} \left[ \int_{X} \left\{ \sum_{k=0}^{M} \sum_{j=1}^{\infty} \lambda_{k,j} m_{k,j}(y) \right\} K_\beta(x,c_B) d\mu(y) \right] d\mu(x)
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} \int_{U_{\ell}(B)} \left\{ \int_{B_{k,j}} \frac{1}{|\lambda(c_B,d(x,c_B))|^{1-\gamma}} d\mu(y) \right\}^{p_2} d\mu(x)
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} \frac{\mu(2\ell+2B)}{\mu(2\ell+2B)^{1-\gamma}} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} \left[ \mu(2\ell+2B) \right]^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\times 2^{-k\theta p_2} \left[ \mu(2\ell+2B) \right]^{1-p_2(1-\beta)} \|m_{k,j}\|_{L^\theta(\mu)}^{p_2}
\]
\[
\times \sum_{\ell=5}^{\infty} \sum_{k=0}^{M} \sum_{j=1}^{\infty} |\lambda_{k,j}|^{p_2} \sim \sum_{k=1}^{M} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} 2^{-k\theta p_2} |\lambda_{k,j}|^{p_2}
\]
Combining $I_1$ and $I_2$, we conclude that 

$$
I \lesssim |b|_{\tilde{B}^{1+1, \theta}_{\text{mb}, 2}}^{p_2}
$$

Then we turn to estimate $II$. We further write

$$
II \leq \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{U_1(B)} |T_\beta(m_{k,j})(x)|^{p_2} \ d\mu(x)
$$

By the Hölder inequality, $(L^q(\mu), L^\theta(\mu))$-boundedness of $T_\beta$, (4.1), (2.1), $1/p_2 = 1/p_1 - \beta$ and $1/q = 1/q - \beta$, we see that

$$
II_1 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} |\mu(2B_{k,j})|^{1-p_2/q} \|T_\beta(m_{k,j})\|^{p_2}_{L^q(\mu)}
$$

For $II_2$, from (5.1),

$$
d(x,y) \geq d(x,c_{B_{k,j}}) - d(y,c_{B_{k,j}}) \geq \frac{1}{c}d(x,c_{B_{k,j}}) \quad \text{for } x \notin 2B_{k,j} \text{ and } y \in B_{k,j}, \ (2.2), \ (2.1), p_2 (1-\beta) < p_1, \text{ the Hölder inequality, } (4.1) \text{ and } 1/p_2 = 1/p_1 - \beta, \text{ we deduce that}
$$

$$
II_2 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{U_1(B)\setminus 2B_{k,j}} \left\{ \int_{B_{k,j}} \frac{|m_{k,j}(y)|}{\|\lambda(c_{B_{k,j}}, d(x,c_{B_{k,j}}))\|^{1-\beta}} \ d\mu(y) \right\}^{p_2} \ d\mu(x)
$$

$$
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \int_{2^{k+6+B}B_{k,j}} \frac{1}{\|\lambda(c_{B_{k,j}}, d(x,c_{B_{k,j}}))\|^{p_2(1-\beta)} \ d\mu(x)} |m_{k,j}|^{p_2}_{L^1(\mu)}
$$

$$
\times \left[ \mu(2^{k+6+B}) \right]^{p_1-2p_2(1-\beta)} \ |m_{k,j}|^{p_2}_{L^1(\mu)}
$$

$$
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \left\{ \sum_{i=0}^{N^{(2)}_{B_{k,j},2^{k+6+B}}} \frac{\mu(2^{i+1}B_{k,j})}{\|\lambda(c_{B_{k,j}}, 2^{i+1}B_{k,j})\|^{p_1}} \right\}^{p_2(1-\beta)} \ |m_{k,j}|^{p_2}_{L^1(\mu)}
$$

$$
\times \left[ \mu(2^{k+6+B}) \right]^{p_1-2p_2(1-\beta)} \ |m_{k,j}|^{p_2}_{L^1(\mu)}
$$

$$
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_2} \left[ \mu \left( 2^{N^{(2)}_{B_{k,j},2^{k+6+B}+2}B_{k,j}} \right) \right]^{(1-p_1)(1-\beta)} \ |m_{k,j}|^{p_2}_{L^1(\mu)}
$$
× \left\{ \frac{N^{(2)}_{B_{k,j}}}{B_{k,j}} \right\}^{2^{k+5}+\rho+1} \sum_{i=0}^{p_{1}(1-\beta)} \left[ \frac{\mu(2^{i+1}B_{k,j})}{\lambda(cB_{k,j})(2^{i+1}B_{k,j})} \right]^{p_{1}} \left[ \frac{\mu(2^{k+6}B)}{\lambda(cB_{k,j})(2^{k+6}B)} \right]^{p_{1}} \left\| m_{k,j} \right\|_{L^{p_{2}}(\mu)}^{p_{2}}

\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{M} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_{2}} \left\| m_{k,j} \right\|_{L^{p_{2}}(\mu)}^{p_{2}} \left[ \mu \left( 2^{N^{(2)}_{B_{k,j}}}, B_{k,j} \right) \right]^{p_{1}-p_{2}(1-\beta)} \left( K_{B_{k,j}, 2^{k+5}B} \right)^{p_{1}} \left( K_{B_{k,j}, 2^{k+5}B} \right)^{p_{1}}

\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{M} \sum_{j=1}^{M} |\lambda_{k,j}|^{p_{2}} \left[ \mu \left( B_{k,j} \right) \right]^{p_{2}/q'} \left\| m_{k,j} \right\|_{L^{p_{2}}(\mu)}^{p_{2}} \left[ \mu \left( 2^{k+9}B \right) \right]^{p_{1}-p_{2}(1-\beta)} \left( K_{B_{k,j}, 2^{k+5}B} \right)^{p_{1}} \left( K_{B_{k,j}, 2^{k+5}B} \right)^{p_{1}}

\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{M} \sum_{j=1}^{M} \left( K_{B_{k,j}, 2^{k+5}B} \right)^{p_{1}} \left( K_{B_{k,j}, 2^{k+5}B} \right)^{p_{1}}

which, together with the estimate for II, implies that

\[ II \lesssim \| b \|_{H^{p_{2}}_{\text{atb}, \gamma, 1, \theta}(\mu)}^{p_{2}}. \]

To estimate IV, observe that

IV \leq \sum_{\ell=0}^{4} \int_{U_{\ell}(B)} \left| T_{\beta} \left( \sum_{k=0}^{M} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right)(x) \right|^{p_{2}} \, d\mu(x)

+ \sum_{\ell=0}^{4} \int_{U_{\ell}(B)} \left| T_{\beta} \left( \sum_{k=\ell+5}^{M} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right)(x) \right|^{p_{2}} \, d\mu(x) =: IV_{1} + IV_{2}

By some arguments similar to those used in the estimates for II and III, we respectively obtain

\[ IV_{1} \lesssim \| b \|_{\dot{H}^{p_{2}}_{\text{atb}, \gamma, 1, \theta}(\mu)}^{p_{2}} \quad \text{and} \quad IV_{2} \lesssim \| b \|_{\dot{H}^{p_{2}}_{\text{atb}, \gamma, 1, \theta}(\mu)}^{p_{2}}, \]

which, together with the estimates for III, I and II, completes the proof of Theorem 5.3.  \[ \square \]

Similar to Corollary 4.9, by the proof of Theorem 5.3, we also obtain the following boundedness of the generalized fractional integral \( T_{\beta} \) from \( \dot{H}^{p_{1}, q, \gamma}_{\text{atb}, \rho}(\mu) \) into \( L^{p_{2}}(\mu) \) for \( 1/p_{2} = 1/p_{1} - \beta \), the details being omitted.

**Corollary 5.4.** Suppose that \( (X, d, \mu) \) is a non-homogeneous metric measure space. Let \( \beta \in (0, 1/2), \rho \in (1, \infty), \frac{\rho}{\nu+\beta} < p_{1} < p_{2} \leq 1 < q < 1/\beta, 1/p_{2} = 1/p_{1} - \beta \) and \( \gamma \in [1, \infty) \). Assume that the generalized fractional integral \( T_{\beta} \) defined by (5.3) associated with kernel \( K_{\beta} \) satisfying (5.1) and (5.2) is bounded from \( L^{q}(\mu) \) into \( L^{q}(\mu) \), where \( 1/q := 1/q - \beta \). Then \( T_{\beta} \) is bounded from \( \dot{H}^{p_{1}, q, \gamma}_{\text{atb}, \rho}(\mu) \) into \( L^{p_{2}}(\mu) \).

**Remark 5.5.** (a) When \( p_{1} = 1 \), Theorem 5.3 or Corollary 5.4 is a special case of [15, Theorem 1.13] by the same reasons as those used in Remark 4.10.

(b) For all \( \beta \in (0, 1) \), \( f \in L_{c}^{\infty}(\mu) \) and \( x \in X \), the fractional integral \( I_{\beta}f(x) \) is defined by

\[ I_{\beta}f(x) := \int_{X} \frac{f(y)}{|x(y, d(x, y))|^{1-\beta}} \, d\mu(y). \]

From [15, Section 4], we deduce that, for some \( \epsilon \in (0, \infty) \), under the weak growth condition as in Remark 2.4(iii) and the following \( \epsilon \)-weak reverse doubling condition on the dominating function \( \lambda \): for all
$r \in (0, 2 \text{diam}(\mathcal{X}))$ and $a \in (1, 2 \text{diam}(\mathcal{X})/r)$, there exists a number $C_{(a)} \in (1, \infty)$, depending only on $a$ and $\mathcal{X}$, such that, for all $x \in \mathcal{X}$, $\lambda(x, ar) \geq C_{(a)} \lambda(x, r)$ and $\sum_{k=1}^{\infty} \frac{1}{r^{\beta k}} < \infty$, the following statements hold true:

(b): the fractional integral $I_\beta$ is a special case of the generalized fractional integral, which is bounded from $L^q(\mu)$ into $L^q(\mu)$ for all $q \in (1, 1/\beta)$ and $1/q = 1/q - \beta$;

(b) all conclusions of Theorem 5.3 and Corollary 5.4 hold true, if $T_\beta$ is replaced by $I_\beta$, where $I_\beta$ has the same assumptions as those of $T_\beta$ in Theorem 5.3 and Corollary 5.4, respectively.

6 Campanato Spaces $\mathcal{E}^{a,q}_{\rho, \eta, \gamma}(\mu)$

In this section, we introduce the Campanato space $\mathcal{E}^{a,q}_{\rho, \eta, \gamma}(\mu)$ and show that $\mathcal{E}^{a,q}_{\rho, \eta, \gamma}(\mu)$ is independent of the choices of $\rho$, $\eta$, $\gamma$ and $\eta$ under the following assumption of the $\rho$-weakly doubling condition.

Definition 6.1. Let $\rho \in (1, \infty)$. The Borel measure $\mu$ is said to satisfy the $\rho$-weakly doubling condition if, for all balls $B \subset \mathcal{X}$, there exists a positive constant $\tilde{C}_1$, depending on $\rho$ but independent of $B$, such that

$$N_{B, \tilde{B}^\rho}^{(\rho)} \leq \tilde{C}_1,$$  \hspace{1cm} (6.1)

where $N_{B, \tilde{B}^\rho}^{(\rho)}$ is defined as in Definition 2.6.

Remark 6.2. (i) Recall that $\tilde{B}^\rho$ is totally determined by $\mu$ and $\rho$. Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type and $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. Then $(\mathcal{X}, d, \mu)$ satisfies (6.1), since $N_{B, \tilde{B}^\rho}^{(\rho)} \sim 1$ for all balls $B$ with equivalent positive constants depending only on $\rho \in (1, \infty)$. However, by Example 6.3 below, there exists a non-doubling measure $\mu$ on a subset of $\mathbb{R}$ satisfying (6.1); by Example 6.4 below, there exists a non-doubling measure not satisfying (6.1). In this sense, a measure satisfying (6.1) is said to be $\rho$-weakly doubling.

(ii) From the fact that $\rho N_{B, \tilde{B}^\rho}^{(\rho)} B = \tilde{B}^\rho$ and (6.1), it follows that there exists a positive constant $C_{(\rho, \tilde{C}_1)}$, depending on $\rho$ and $\tilde{C}_1$, such that, for any ball $B$,

$$r_{\tilde{B}^\rho} \leq C_{(\rho, \tilde{C}_1)} r_B,$$

where $r_B$ and $r_{\tilde{B}^\rho}$ denote the radii of balls $B$ and $\tilde{B}^\rho$, respectively. Obviously, we always have $r_B \leq r_{\tilde{B}^\rho}$.

In the remainder of this section, we always assume that the Borel measure $\mu$ satisfies the $\rho$-weakly doubling condition.

The following example shows that there exist some non-trivial non-doubling measures satisfying (6.1).

Example 6.3. Let

$$\mathcal{X} := [0, 1] \cup \left( \bigcup_{k=1}^{\infty} \left[ \frac{2}{k} \sum_{j=0}^{k-1} e^{-j^2}, \frac{2}{k} \sum_{j=0}^{k-1} e^{-j^2} + e^{-k^2} \right] \right).$$

Denote $[0, 1]$ by $D_0$ and $[2\sum_{j=0}^{k-1} e^{-j^2}, 2 \sum_{j=0}^{k-1} e^{-j^2} + e^{-k^2}]$ by $D_k$ for $k \in \mathbb{N}$. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, we use $B := B(x, r) := \{y \in \mathcal{X} : |y - x| < r\}$ to denote a ball of $\mathcal{X}$. Let $\mu$ be the Lebesgue measure restricted to $\mathcal{X}$. Notice that $\mu(B(x, r)) \leq 2r$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$. Thus, $\mu$ is an upper doubling measure with $\lambda(x, r) := 2r$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$.

Then we claim that $\mu$ is a non-doubling measure. Indeed, notice that

$$\sum_{j=k+1}^{\infty} e^{-j^2} = e^{-k^2} \sum_{j=1}^{\infty} e^{-2kj-j^2} \leq \begin{cases} \sqrt{\frac{\pi}{2}} e^{-k^2} & \text{for all } k \in \mathbb{Z}_+; \\ \sqrt{\frac{\pi}{e^2}} e^{-k^2} & \text{for all } k \in \mathbb{N}. \end{cases} \hspace{1cm} (6.2)$$

Let $x_k := 2 \sum_{j=0}^{k-1} e^{-j^2}$ and $r_k := e^{-(k-1)^2}$. Then

$$\mu(B(x_k, r_k)) \leq \left( 1 + \frac{\sqrt{\pi}}{2} \right) e^{-k^2} \quad \text{and} \quad \mu(B(x_k, 2r_k)) \geq e^{-(k-1)^2}.$$
Thus,
\[
\frac{\mu(B(x_k, 2r_k))}{\mu(B(x_k, r_k))} \geq \frac{1}{1 + \frac{\sqrt{\pi}}{2}} e^{2k-1} \to \infty \quad \text{as} \quad k \to \infty,
\]
which implies that \( \mu \) is non-doubling.

For any ball \( B \), the smallest doubling ball of the form \( 2^j B \) with \( j \in \mathbb{Z}_+ \) is denoted by \( \tilde{B}^{(2)} \). Let \( N_{B, \tilde{B}^{(2)}}^{(2)} \) be the integer such that \( 2^{N_{B, \tilde{B}^{(2)}}^{(2)}} B = \tilde{B}^{(2)} \). We claim that
\[
N_{B, \tilde{B}^{(2)}}^{(2)} \leq 2. \tag{6.3}
\]
To prove this, we consider the following two cases for \( k \).

**Case (I) \( k \in \{0, 1, 2\} \).** In this case, for all \( x \in D_k \), we have
\[
\begin{align*}
\mu(B(x, r)) &\sim r \sim \mu(B(x, 2r)) \quad \text{for} \quad r \in (0, 2 + \sqrt{\pi}]; \\
\mu(B(x, r)) &\sim 1 \sim \mu(B(x, 2r)) \quad \text{for} \quad r \in (2 + \sqrt{\pi}, \infty).
\end{align*}
\]

From this, it is easy to deduce that \( B(x, r) \) with \( x \in D_k \) and \( r \in (0, \infty) \) is a doubling ball and hence \( N_{B, \tilde{B}^{(2)}}^{(2)} = 0 \) in this case.

**Case (II) \( k \in \mathbb{N} \cap (2, \infty) \).** In this case, for all \( x \in D_k \), we have

(i) for \( r \in (0, e^{-k^2}] \), \( \mu(B(x, r)) \sim r; \)

(ii) for \( r \in (e^{-k^2}, e^{-(k-1)^2}] \), \( \mu(B(x, r)) \sim e^{-k^2}; \)

(iii) for \( r \in (e^{-(k-1)^2}, 2e^{-(k-1)^2}] \), \( e^{-k^2} \leq \mu(B(x, r)) < 2e^{-(k-1)^2}; \)

(iv) for \( r \in (2 \sum_{i=1}^{j} e^{-(k-i)^2}, 2 \sum_{i=1}^{j+1} e^{-(k-i)^2} + e^{-(k-j-1)^2}] \) with \( j \in \{1, \ldots, k-2\} \),
\[
\mu(B(x, r)) \sim e^{-(k-j)^2};
\]

(v) for \( r \in (2 \sum_{i=1}^{j} e^{-(k-i)^2} + e^{-(k-j)^2}, 2 \sum_{i=1}^{j+1} e^{-(k-i)^2}] \) with \( j \in \{1, \ldots, k-2\} \),
\[
e^{-(k-j)^2} \leq \mu(B(x, r)) < 2e^{-(k-j-1)^2};
\]

(vi) for \( r \in (2 \sum_{i=1}^{k-1} e^{-(k-i)^2}, \infty) \), \( \mu(B(x, r)) \sim 1. \)

Now we show that (6.3) holds true in cases (i) through (vi).

Indeed, in the case (vi), it is easy to see that \( B(x, r) \) is a doubling ball and hence \( N_{B, \tilde{B}^{(2)}}^{(2)} = 0 \) in this case. That is, (6.3) holds true in this case. Therefore, we only need to show that (6.3) holds true in cases (i) through (v).

In the case (i), since \( k \geq 3 \), we have \( 2r \in (0, e^{-(k-1)^2}] \). If \( 2r \in (0, e^{-k^2}] \), then we have
\[
\mu(B(x, r)) \sim \mu(B(x, 2r)) \sim r;
\]

if \( 2r \in (e^{-k^2}, e^{-(k-1)^2}] \), then \( r \in (\frac{e^{-k^2}}{2}, e^{-k^2}] \), which, together with (ii), leads to
\[
\mu(B(x, r)) \sim \mu(B(x, 2r)) \sim e^{-k^2}.
\]

It then follows that \( B(x, r) \) is a doubling ball, which shows that \( N_{B, \tilde{B}^{(2)}}^{(2)} = 0 \) in this case. That is, (6.3) holds true.

In the case (ii), if \( 2r \in (e^{-k^2}, e^{-(k-1)^2}] \), then we have \( \mu(B(x, r)) \sim \mu(B(x, 2r)) \sim e^{-k^2} \), which implies that \( \mu(B(x, r)) \) is a doubling ball; if \( 2r \in (e^{-(k-1)^2}, 2e^{-(k-1)^2}] \), then, for sufficiently large \( k \), \( B(x, r) \) may be a non-doubling ball, for example, \( x = 2 \sum_{j=0}^{k-1} e^{-j^2} \) and \( r = e^{-(k-1)^2} \). In the latter case, if \( B(x, r) \) is a non-doubling ball, we consider the ball \( B(x, 2r) \). If \( B(x, 2r) \) is a doubling ball, namely, there exists a
positive constant $C$, independent of $x$ and $r$, such that $\mu(B(x, 4r)) \lesssim C\mu(B(x, 2r))$, then there is nothing to prove. Otherwise, we consider the ball $B(x, 4r)$. Notice that $2r \in (e^{-(k-1)^2}, 2e^{-(k-1)^2}]$. Then we have $r \in \left(\frac{e^{-(k-1)^2}}{2}, e^{-(k-1)^2}\right]$, which, together with $k \geq 3$, shows that
\[
2e^{-(k-1)^2} < 4r \leq 4e^{-(k-1)^2} < 2e^{-(k-1)^2} + e^{-(k-2)^2}
\]
and
\[
2e^{-(k-1)^2} < 8r \leq 8e^{-(k-1)^2} < 2e^{-(k-1)^2} + e^{-(k-2)^2}.
\]
It then follows, from (iv) with $j = 1$, that $\mu(B(x, 4r)) \sim \mu(B(x, 8r)) \sim e^{-(k-1)^2}$, which implies that $B(x, 4r)$ is a doubling ball. From the above estimate, we conclude that $N_{B, \tilde{B}}^{(2)} \leq 2$ in this case.

The argument of the case (iii) is similar to that used in the case of $2r \in (e^{-(k-1)^2}, 2e^{-(k-1)^2}]$ of (ii). Moreover, we have $N_{B, \tilde{B}}^{(2)} \leq 1$ in this case.

Before we deal with the case (iv), we first consider the case (v). In the case (v), if $B(x, r)$ is a doubling ball, then there is nothing to prove. Otherwise, we consider the following two cases for $j$. If $j = k - 2$, we have
\[
4r > 2r > 4 \sum_{i=1}^{j} e^{-(k-1)^2} + 2e^{-(k-j-1)^2} > 2 \sum_{i=1}^{j+1} e^{-(k-1)^2} = 2 \sum_{i=1}^{k-1} e^{-(k-i)^2},
\]
which, together with (vi), shows that $\mu(B(x, 4r)) \sim \mu(B(x, 2r)) \sim 1$. Thus, $B(x, 2r)$ is a doubling ball. If $j \leq k - 3$, then, by (6.2), we see that
\[
6 \sum_{i=1}^{j+1} e^{-(k-i)^2} < e^{-(k-(j+1)-1)^2}.
\]
It then follows that
\[
2 \sum_{i=1}^{j+1} e^{-(k-i)^2} < 2r < 4r \leq 8 \sum_{i=1}^{j+1} e^{-(k-i)^2} < 2 \sum_{i=1}^{j+1} e^{-(k-i)^2} + e^{-(k-(j+1)-1)^2}.
\]
This via (iv) shows that $\mu(B(x, 2r)) \sim \mu(B(x, 4r)) \sim e^{-(k-j-1)^2}$, which implies that $B(x, 2r)$ is a doubling ball and hence $N_{B, \tilde{B}}^{(2)} \leq 1$ in this case.

In the case (iv), we see that $2r \in (4 \sum_{i=1}^{j} e^{-(k-i)^2}, 4 \sum_{i=1}^{j} e^{-(k-i)^2} + 2e^{-(k-j-1)^2}]$. Notice that, by (6.2) and $j \leq k - 2$, we see that
\[
2 \sum_{i=1}^{j} e^{-(k-i)^2} = 2 \sum_{i=k-j}^{k-1} e^{-i^2} \leq \frac{2\sqrt{\pi}}{2e^2} e^{-(k-j)^2} < e^{-(k-j-1)^2}.
\]
Thus, we consider the following three cases for $2r$.

**Case (a)** $2r \in (4 \sum_{i=1}^{j} e^{-(k-i)^2}, 2 \sum_{i=1}^{j} e^{-(k-i)^2} + e^{-(k-j-1)^2}]$. In this case, it is easy to see that
\[
\mu(B(x, 2r)) \sim \mu(B(x, r)) \sim e^{-(k-j)^2},
\]
which implies that $B(x, r)$ is a doubling ball.

**Case (b)** $2r \in (2 \sum_{i=1}^{j} e^{-(k-i)^2} + e^{-(k-j-1)^2}, 2 \sum_{i=1}^{j+1} e^{-(k-i)^2}]$. In this case, by an argument similar to that used in (v), we conclude that $N_{B, \tilde{B}}^{(2)} \leq 2$.

**Case (c)** $2r \in (2 \sum_{i=1}^{j+1} e^{-(k-i)^2}, 4 \sum_{i=1}^{j} e^{-(k-i)^2} + 2e^{-(k-j-1)^2}]$. In this case, if $j = k - 2$, we have
\[
4r > 2r > 2 \sum_{i=1}^{k-1} e^{-(k-i)^2},
\]
which, together with (vi), implies that $B(x, 2r)$ is a doubling ball; if $j \leq k - 3$, by (6.2), we know that
\[
2 \sum_{i=1}^{j+1} e^{-(k-i)^2} < 2r < 4r \leq 8 \sum_{i=1}^{j} e^{-(k-i)^2} + 4e^{-(k-j-1)^2} < 2 \sum_{i=1}^{j+1} e^{-(k-i)^2} + 6 \sum_{i=1}^{j} e^{-(k-i)^2}
\]
\[ < 2 \sum_{i=1}^{j+1} e^{-(k-i)^2} + e^{-|k-(j+1)|^2}. \]

This via (iv) shows that \( \mu(B(x, 2r)) \sim \mu(B(x, 4r)) \sim e^{-(k-j)^2} \), which implies that \( B(x, 2r) \) is a doubling ball and hence \( N^{(2)}_{B, \tilde{B}}(\rho) \leq 1 \) in the case (iv).

Combining the above estimates, we obtain (6.3), which completes the proof of our claim and hence the example.

On the other hand, it turns out that there exist many non-homogeneous metric measure spaces which do not satisfy the \( \rho \)-weakly doubling condition (6.1). We give the following Gauss measure on \( \mathbb{R} \) as an example.

**Example 6.4.** Let \((\mathcal{X}, | \cdot |, \mu) := (\mathbb{R}, | \cdot |, \mu)\), where \(| \cdot |\) denotes the Euclidean distance and \( \mu \) the Gauss measure on \( \mathbb{R} \), that is, \( d\mu(x) := \pi^{-\frac{1}{2}} e^{-x^2} dx \) for all \( x \in \mathbb{R} \). As in Example 6.3, for any \( x \in \mathbb{R} \) and \( r \in (0, \infty) \), we use \( B := B(x, r) := \{ y \in \mathbb{R} : |y - x| < r \} \) to denote a ball of \( \mathbb{R} \). First, we show that \( \mu \) is a non-doubling measure with the dominating function \( \lambda(x, r) := 2\pi^{-\frac{1}{2}} r = \lambda(x, r) \).

Indeed, for all \( x \in \mathbb{R} \) and \( r \in (0, \infty) \),

\[
\mu(B(x, r)) = \pi^{-\frac{1}{2}} \int_{x-r}^{x+r} e^{-y^2} dy \leq 2\pi^{-\frac{1}{2}} r = \lambda(x, r).
\]

On the other hand, let \( x_k = 2^k \) and \( r_{k, j} = 2^j \) with \( k \in \mathbb{N} \) and \( j \in \{1, \ldots, k\} \). Then, for all \( k \in \mathbb{N} \) and \( j \in \{1, \ldots, k\} \), we observe that

\[
\mu(B(x_k, r_{k, j})) = \pi^{-\frac{1}{2}} \int_{2^{k+2j}}^{2^{k+2j+2}} e^{-x^2} dx \leq \pi^{-\frac{1}{2}} e^{-(2^{k+2j})^2} 2j+1,
\]

and

\[
\mu(B(x_k, 2r_{k, j})) = \pi^{-\frac{1}{2}} \int_{2^{k+2j+2}}^{2^{k+2j+4}} e^{-x^2} dx \geq \pi^{-\frac{1}{2}} e^{-(2^{k+2j+2})^2} 2j+1.
\]

Thus,

\[
\frac{\mu(B(x_k, 2r_{k, j}))}{\mu(B(x_k, r_{k, j}))} \geq \frac{1}{4} e^{2^{k+2j+2}} \geq \frac{1}{4} e^{2^{k+1} - 2^{2k}} \geq \frac{1}{4} e^{2^{k+1} - 2^{2k}} \to \infty \text{ as } k \to \infty,
\]

which implies that \( \mu \) is non-doubling.

Now we claim that, for any \( \rho \in (1, \infty) \), there exists a ball \( B \) such that the number \( N^{(\rho)}_{B, \tilde{B}} \) can be arbitrarily large. For the sake of simplicity, we only prove our claim for \( \rho = 2 \). With some simple modifications, the arguments here are still valid for all \( \rho \in (1, \infty) \). Recall that a ball \( B \subset \mathbb{R} \) is said to be \((2, \beta_2)\)-doubling if \( \mu(2B) \leq \beta_2 \mu(B) \) and \( \tilde{B}^{(2)} \) is the smallest \((2, \beta_2)\)-doubling ball of the form \( 2^j B \) with \( j \in \mathbb{Z}_+ \). Let \( k_0 \) be the smallest positive integer such that \( \frac{1}{4} \pi^2 e^{2k_0} > \beta_2 \). Then, for all \( k \in \mathbb{N} \cap \{k_0, \infty\} \), we have \( \frac{1}{4} \pi^2 e^{2k_0} \geq \beta_2 \). Let \( B_k := B(2^{2k}, 2) \). By (6.4), it is easy to see that, for all \( j \in \{0, \ldots, k-1\} \), \( 2^j B_k \) is not a \((2, \beta_2)\)-doubling ball. It then follows, from the definition of \( N^{(2)}_{B_k, \tilde{B}_k} \), that

\[
N^{(2)}_{B_k, \tilde{B}_k} > k - 1,
\]

which implies our claim and completes the proof of Example 6.4.

We now state the definition of the Campanato space \( \mathcal{E}^{\alpha, q}_{\rho, \eta, \gamma}(\mu) \).

**Definition 6.5.** Let \( \alpha \in [0, \infty) \), \( \eta \in (1, \infty) \), \( \rho \in [\eta, \infty) \) and \( q, \gamma \in [1, \infty) \). A function \( f \in L^1_{loc}(\mu) \) is said to belong to the Campanato space \( \mathcal{E}^{\alpha, q}_{\rho, \eta, \gamma}(\mu) \) if

\[
\|f\|_{\mathcal{E}^{\alpha, q}_{\rho, \eta, \gamma}(\mu)} := \sup_B \left\{ \frac{1}{\mu(\eta B)} \frac{1}{|\mu(B)|^{\alpha q}} \int_B |f(y) - m_B(f)|^q \, d\mu(y) \right\}^{1/q}
\]
Thus, by the above two inequalities, Remark 6.2(ii) and (2.1), we have

\[ |m_{\tilde{B}^0}(f) - m_{\tilde{S}^0}(f)| \leq |m_{\tilde{B}^0}(f) - m_{S_0}(f)| + |m_{\tilde{S}^0}(f) - m_{S_0}(f)| \]

\[ \lesssim \left\{ \left[ K_{\tilde{B}^0, S_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma + \left[ K_{\tilde{S}^0, S_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma \right\} [\lambda(c_{S_0}, r_{S_0})]^\alpha \| f \|_{\varepsilon_{\rho, q, \gamma}^0, (\mu)} \]

\[ \lesssim \left[ K_{\tilde{B}^0, S_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma [\lambda(c_{S_0}, r_{S_0})]^\alpha \| f \|_{\varepsilon_{\rho, q, \gamma}^0, (\mu)} \]

This finishes the proof of Case (i).

Case (ii) \( r(\tilde{S}^0) \leq r(\tilde{B}^0) \). Obviously, \( \tilde{S}^0 \subset 2\tilde{B}^0 \). Let \( B_0 := (2\tilde{B}^0) \). From (iv), (ii) and (iii) of Lemma 2.8 with \( p = 1/(\alpha+1) \), we deduce that

\[ \left[ K_{\tilde{B}^0, B_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma \lesssim \left[ K_{\tilde{B}^0, 2\tilde{B}^0}^{(\rho), 1/(\alpha+1)} \right]^\gamma + \left[ K_{\tilde{B}^0, B_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma \lesssim 1. \]

By this, \( \tilde{S}^0 \subset S \subset B \) and (v), (iv) and (iii) of Lemma 2.8 with \( p = 1/(\alpha+1) \), we know that

\[ \left[ K_{\tilde{S}^0, B_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma \lesssim \left[ K_{\tilde{B}^0, B_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma \lesssim \left[ K_{\tilde{B}^0, B_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma + \left[ K_{\tilde{B}^0, B_0}^{(\rho), 1/(\alpha+1)} \right]^\gamma \lesssim 1. \]

Thus, combining the above two inequalities, (2.2), (2.1) and Remark 6.2(ii), we have

\[ |m_{\tilde{B}^0}(f) - m_{\tilde{S}^0}(f)| \leq |m_{\tilde{B}^0}(f) - m_{B_0}(f)| + |m_{\tilde{S}^0}(f) - m_{B_0}(f)| \]
It is obvious that \( E \) is an equivalent norm.

Let \( B \) be the ball centered at \( x \) with radius \( \rho \). For any \( \rho \), \( \eta \), and \( \gamma \), we have

\[
\|h\|_{E^{\alpha,q},\gamma} \lesssim \|h\|_{E^{\alpha,q},\gamma}.
\]

This finishes the proof of Case (ii) and hence (a).

To prove (b), since \( \max\{f, g\} = \frac{f + g + |f - g|}{2} \) and \( \min\{f, g\} = \frac{f + g - |f - g|}{2} \), it suffices to show that, for any real-valued function \( h \in E^{\alpha,q},\gamma(\mu) \), \( |h| \in E^{\alpha,q},\gamma(\mu) \) and

\[
\|\|h\|\|_{E^{\alpha,q},\gamma} \lesssim \|h\|_{E^{\alpha,q},\gamma}.
\]

To this end, by Definition 6.5, the Hölder inequality, Remark 6.2(ii) and (2.1), we see that, for any ball \( B \),

\[
\left\{ \frac{1}{\mu(\eta B)} \int_B |h(y)| - m_{B,\rho}(|h|)^q \, dm(y) \right\}^{1/q} \leq \left\{ \frac{1}{\mu(\eta B)} \int_B |h(y)| - m_{B,\rho}(|h|)^q \, dm(y) \right\}^{1/q} + \|m_{B,\rho}(|h|) - m_{\eta B}(|h|)\|
\]

\[
\leq \left\{ \frac{1}{\mu(\eta B)} \int_B |h(y)| - m_{B,\rho}(|h|)^q \, dm(y) \right\}^{1/q} + m_{\eta B}(|h - m_{B,\rho}(h)|)
\]

\[
\lesssim \{\lambda(c_B, r_B)\}^\alpha + \|\lambda(c_B, r_B)\| \|h\|_{E^{\alpha,q},\gamma} \lesssim [\lambda(c_B, r_B)]^\alpha \|h\|_{E^{\alpha,q},\gamma}.
\]

On the other hand, by Definition 6.5, (2.1) and (2.2), we find that, for all \((\rho, \beta, \mu)\)-doubling balls \( B \subset S \),

\[
|m_B(|h|) - m_S(|h|)\|
\]

\[
\lesssim |m_B(|h|) - m_B(|h|)| + ||m_B(|h|) - m_S(|h|)| + ||m_S(|h|) - m_S(|h|)|
\]

\[
\lesssim m_B(|h - m_B(\rho)|) + m_B(|h - m_S(\rho)|) + m_S(|h - m_S(\rho)|)
\]

\[
\lesssim \{\lambda(c_B, r_B)\}^\alpha + \|\lambda(c_B, r_B)\| \left[ K^{(\rho),1/(\alpha+1)}_{\beta,\rho} \right] [\lambda(c_B, r_B)]^\alpha \|h\|_{E^{\alpha,q},\gamma} \lesssim [\lambda(c_B, r_B)]^\alpha \|h\|_{E^{\alpha,q},\gamma}.
\]

The above two inequalities imply that \( |h| \in E^{\alpha,q},\gamma(\mu) \) and \( \|h\|_{E^{\alpha,q},\gamma} \lesssim \|h\|_{E^{\alpha,q},\gamma} \), which completes the proof of (b) and hence Proposition 6.7.

We now show that the space \( E^{\alpha,q},\gamma(\mu) \) is independent of the choices of \( \rho \) and \( \eta \).

Proposition 6.8. Suppose that \((\mathcal{X}, d, \mu)\) is a non-homogeneous metric measure space satisfying (6.1).

Let \( \alpha \in [0, \infty) \) and \( q, \gamma \in [1, \infty) \). The following statements hold true:

(i) for any \( \eta_1, \eta_2 \) and \( \rho \) satisfying \( 1 < \eta_1 < \eta_2 \leq \rho < \infty \), \( E^{\alpha,q},\gamma(\mu) \) and \( E^{\alpha,q},\gamma(\mu) \) coincide with equivalent norms;

(ii) for any \( \rho_1, \rho_2 \) and \( \eta \) satisfying \( 1 < \rho_1 < \rho_2 < \infty \), \( E^{\alpha,q},\gamma(\mu) \) and \( E^{\alpha,q},\gamma(\mu) \) coincide with equivalent norms.

Proof. We first prove (i). Fix \( \alpha \in [0, \infty) \) and \( q, \gamma \in [1, \infty) \). Let \( \eta_1, \eta_2 \) and \( \rho \) satisfy \( 1 < \eta_1 < \eta_2 \leq \rho < \infty \).

It is obvious that \( E^{\alpha,q},\gamma(\mu) \subset E^{\alpha,q},\gamma(\mu) \) and, for all \( f \in E^{\alpha,q},\gamma(\mu) \), \( \|f\|_{E^{\alpha,q},\gamma(\mu)} \leq \|f\|_{E^{\alpha,q},\gamma(\mu)} \).

Conversely, let \( f \in E^{\alpha,q},\gamma(\mu) \). We show that \( f \in E^{\alpha,q},\gamma(\mu) \) and \( \|f\|_{E^{\alpha,q},\gamma(\mu)} \leq \|f\|_{E^{\alpha,q},\gamma(\mu)} \). To this end, it suffices to show that, for any ball \( B \),

\[
\left\{ \frac{1}{\mu(\eta B)} \int_B |f(y) - m_{B,\rho}(f)|^q \, dm(y) \right\}^{1/q} \leq \{\lambda(c_B, r_B)\}^\alpha \|f\|_{E^{\alpha,q},\gamma(\mu)}.
\]

To do so, for any \( x \in B \), let \( B_x \) be the ball centered at \( x \) with radius \( \frac{\eta_1 - 1}{\eta_2} r_B \). Then \( r_{\eta_2 B_x} = \frac{\eta_1 - 1}{\eta_2} r_B \) and \( \eta_2 B_x \subset \eta_1 B \).
By the geometrically doubling condition and Remark 2.2(ii), we see that there exist $N_1 \in \mathbb{N}$, depending on $\eta_1$, $\eta_2$ and $(X, d, \mu)$, and a finite sequence $\{B_i\}_{i=1}^{N_1} = \{B_i\}_{i=1}^{N_1}$ of balls such that $x_i \in B$ for all $i \in \{1, \ldots, N_1\}$ and $B \subseteq \bigcup_{i=1}^{N_1} B_i$. By this, $\eta_2 B_i \subset \eta_1 B$ for all $i \in \{1, \ldots, N_1\}$, (2.1), (2.2), Proposition 6.7(a) and Lemma 2.8(ii), we conclude that

\[
\frac{1}{\mu(\eta_1 B)} \int_B |f(y) - m_{\eta_1 B}(f)|^q \, d\mu(y) \\
\leq \frac{1}{\mu(\eta_1 B)} \sum_{i=1}^{N_1} \int_{B_i} |f(y) - m_{\eta_1 B}(f)|^q \, d\mu(y) \\
\leq \frac{1}{\mu(\eta_1 B)} \sum_{i=1}^{N_1} \left( |\lambda(c_{B_i}, r_{B_i})\| \|f\|_{L^{q, \eta_1 B}(\mu)}^q + |m_{\eta_1 B}(f) - m_{\eta_2 B_i}(f)|^q + |m_{\eta_2 B_i}(f) - m_{\eta_1 B}(f)|^q \right) \\
\leq \left[ \lambda(c_{B}, r_{B}) \| \|f\|_{L^{q, \eta_1 B}(\mu)} \right] \left( \sum_{i=1}^{N_1} \left[ K_{B_i, \eta_1 B}^{(\rho_1), 1/\alpha_1} \right] \gamma \|f\|_{L^{q, \eta_1 B}(\mu)} \right) \leq \left[ \lambda(c_{B}, r_{B}) \right] \|f\|_{L^{q, \eta_1 B}(\mu)}
\]

which completes the proof of (6.5) and hence (i).

To show (ii), fix $\alpha \in [0, \infty)$ and $q, \gamma \in [1, \infty)$. Let $\rho_1, \rho_2$ and $\eta$ satisfy $1 < \eta \leq \rho_1, \rho_2 < \infty$. By the symmetry of $\rho_1$ and $\rho_2$, it suffices to show that $E_{\rho_2, \eta, \gamma}(\mu) \subset E_{\rho_1, \eta, \gamma}(\mu)$ and $\|f\|_{L^{q, \rho_1, \eta, \gamma}(\mu)} \leq \|f\|_{L^{q, \rho_2, \eta, \gamma}(\mu)}$ for all $f \in E_{\rho_2, \eta, \gamma}(\mu)$. Assume that $f \in E_{\rho_2, \eta, \gamma}(\mu)$. From the Minkowski inequality, the Hölder inequality, Proposition 6.7, $\rho_1 \geq \eta$, Remark 6.2(ii), Lemma 2.9 and Lemma 2.8(iii), we deduce that

\[
\left\{ \frac{1}{\mu(\eta B)} \int_B |f(y) - m_{\eta B}(f)|^q \, d\mu(y) \right\}^{1/q} \\
\leq \left\{ \frac{1}{\mu(\eta B)} \int_B |f(y) - m_{\eta B}(f)|^q \, d\mu(y) \right\}^{1/q} + |m_{\eta B}(f) - m_{\eta \beta B_2}(f)| \\
\leq |\lambda(c_{\beta B_2}, r_{\beta B_2})\| \|f\|_{L^{q, \eta B_2}(\mu)}^q + |m_{\eta B}(f) - m_{\eta \beta B_2}(f)| + |m_{\eta \beta B_2}(f) - m_{\eta B}(f)| \\
\leq |\lambda(c_{\beta B_2}, r_{\beta B_2})\| \|f\|_{L^{q, \eta B_2}(\mu)} + \left( \|f\|_{L^{q, \eta B_2}(\mu)} \right)^q \left( \sum_{i=1}^{N_1} \left[ K_{\beta B_2, \eta B}^{(\rho_1), 1/\alpha_1} \right] \gamma \|f\|_{L^{q, \eta B_2}(\mu)} \right) \leq |\lambda(c_{\beta B_2}, r_{\beta B_2})\| \|f\|_{L^{q, \eta B_2}(\mu)}
\]

On the other hand, for all $(\rho_1, \rho_2)$-doubling balls $B \subset S$, by the Hölder inequality, Proposition 6.7, $\rho_1 \geq \eta$, Lemma 2.9, (2.1) and (2.2), we have

\[
|m_B(f) - m_S(f)| \\
\leq |m_B(f) - m_{\beta B}(f)| + |m_{\beta B}(f) - m_{\beta S}(f)| + |m_{\beta S}(f) - m_S(f)| \\
\leq \left\{ \frac{1}{\mu(B)} \int_B |f(y) - m_{\beta B}(f)|^q \, d\mu(y) \right\}^{1/q} + \left[ K_{B, \beta B}^{(\rho_2), 1/\alpha_1} \right] \gamma \|f\|_{L^{q, \eta B_2}(\mu)} \\
+ \left\{ \frac{1}{\mu(S)} \int_S |f(y) - m_{\beta S}(f)|^q \, d\mu(y) \right\}^{1/q} \\
\leq |\lambda(c_{\beta B}, r_{\beta B})\| \|f\|_{L^{q, \eta B_2}(\mu)} + \left[ K_{B, \beta B}^{(\rho_1), 1/\alpha_1} \right] \gamma \|f\|_{L^{q, \eta B_2}(\mu)} \\
\leq \left[ K_{B, \beta B}^{(\rho_1), 1/\alpha_1} \right] \gamma \|f\|_{L^{q, \eta B_2}(\mu)}
\]

which completes the proof of (ii) and hence Proposition 6.8.
Remark 6.9. (i) By Proposition 6.8, we know that the space $E_{p, \eta, \gamma}^\alpha(\mu)$ is independent of the choices of $\rho$ and $\eta$. From now on, unless explicitly pointed out, we always assume that $\rho = \eta$ in $E_{p, \eta, \gamma}^\alpha(\mu)$ and write $E_{p, \eta, \gamma}^\alpha(\mu)$ simply by $E_{p, \gamma}^\alpha(\mu)$ and its norm $\| \cdot \|_{E_{p, \eta, \gamma}^\alpha(\mu)}$ simply by $\| \cdot \|_{E_{p, \gamma}^\alpha(\mu)}$.

(ii) It is still unknown whether $E_{p, \eta, \gamma}^\alpha(\mu)$ is independent of the choices of $\rho$ and $\eta$ or not on general non-homogeneous metric measure spaces without the assumption (6.1).

Before we show that $E_{p, \eta, \gamma}^\alpha(\mu)$ is independent of the choices of $\gamma$ and $q$, we first give a useful characterization of $E_{p, \gamma}^\alpha(\mu)$ which is a variant of [30, Proposition 2.10].

Proposition 6.10. Suppose that $(X, d, \mu)$ is a non-homogeneous metric measure space satisfying (6.1). Let $\rho \in (1, \infty)$, $\alpha \in [0, \infty)$ and $\gamma \in [1, \infty)$. The following statements are equivalent:

(a) $f \in E_{p, \gamma}^\alpha(\mu)$;

(b) there exists a sequence $\{f_B\}_B$ of complex numbers associated with balls $B := B(c_B, r_B)$, with $c_B \in X$ and $r_B \in (0, \infty)$, such that

$$
\|f\|_{E_{p, \gamma}^\alpha(\mu)} := \sup_B \left\{ \frac{1}{\mu(B)} \int_B \frac{1}{|\lambda(c_B, r_B)|^\alpha} |f(y) - f_B|^q \ d\mu(y) \right\}^{1/q} < \infty,
$$

where $c_B \in X$ and $r_B \in (0, \infty)$ denote, respectively, the center and the radius of the ball $S$, and the first supremum is taking all balls $B \subset X$ and the second one all balls $B \subset S \subset X$.

Moreover, the norms $\| \cdot \|_{E_{p, \gamma}^\alpha(\mu)}$ and $\| \cdot \|_{E_{p, \gamma}^\alpha(\mu)}$ are equivalent.

Proof. Fix $\rho \in (1, \infty)$, $\alpha \in [0, \infty)$ and $q, \gamma \in [1, \infty)$. Let $f \in E_{p, \gamma}^\alpha(\mu)$. We first show that $\|f\|_{E_{p, \gamma}^\alpha(\mu)} \leq \|f\|_{E_{p, \gamma}^\alpha(\mu)}$. Indeed, for any ball $B$, let $f_B := m_{B, r_B}(f)$. Then Proposition 6.7 implies that, for any two balls $B \subset S$,

$$
|f_B - f_S| \leq |\lambda(c_S, r_S)|^\alpha (K_{B, S}^{\rho, \beta}(1/\alpha + 1)) \|f\|_{E_{p, \gamma}^\alpha(\mu)}.
$$

This, together with the fact that, for any ball $B$,

$$
\left\{ \frac{1}{\mu(B)} \int_B \frac{1}{|\lambda(c_B, r_B)|^\alpha} |f(y) - f_B|^q \ d\mu(y) \right\}^{1/q} \leq \|f\|_{E_{p, \gamma}^\alpha(\mu)},
$$

implies that $\|f\|_{E_{p, \gamma}^\alpha(\mu)} \leq \|f\|_{E_{p, \gamma}^\alpha(\mu)}$.

Conversely, assume that $\|f\|_{E_{p, \gamma}^\alpha(\mu)} < \infty$. If $B$ is a $(\rho, \beta)$-doubling ball, then, by the Hölder inequality, we have

$$
|f_B - m_B(f)| \leq \left( \frac{1}{\mu(B)} \int_B |f(y) - f_B|^q \ d\mu(y) \right)^{1/q} \leq |\lambda(c_B, r_B)|^\alpha \|f\|_{E_{p, \gamma}^\alpha(\mu)},
$$

which, together with the Minkowski inequality, implies that

$$
\left\{ \frac{1}{\mu(B)} \int_B |f(y) - m_B(f)|^q \ d\mu(y) \right\}^{1/q} \leq \left( \frac{1}{\mu(B)} \int_B |f(y) - f_B|^q \ d\mu(y) \right)^{1/q} + |f_B - m_B(f)| \leq |\lambda(c_B, r_B)|^\alpha \|f\|_{E_{p, \gamma}^\alpha(\mu)}.
$$

Thus, by this, the Minkowski inequality, (6.6), Remark 6.2(ii) and Lemma 2.8(iii), we obtain

$$
\left\{ \frac{1}{\mu(B)} \int_B |f(y) - m_{B'}(f)|^q \ d\mu(y) \right\}^{1/q} \leq \left( \frac{1}{\mu(B)} \int_B |f(y) - f_{B'}|^q \ d\mu(y) \right)^{1/q} + |f_B - f_{B'}| + |f_{B'} - m_{B'}(f)| \leq \left( |\lambda(c_B, r_B)|^\alpha + |\lambda(c_{B'}, r_{B'})|^\alpha \left[ K_{B, B'}^{\rho, \beta}(1/\alpha + 1) \right] \right) \|f\|_{E_{p, \gamma}^\alpha(\mu)}.
$$
\[ \lesssim \| \lambda(c_B,r_B) \|_\alpha \| f \|^{(q)}_{s,p}. \]

Moreover, by (6.6), (2.1) and (2.2), we conclude that, for all \((\rho, \beta)\)-doubling balls \(B \subset S\),
\[ |m_B(f) - m_S(f)| \leq |m_B(f) - f_B| + |f_B - f_S| + |f_S - m_S(f)| \]
\[ \lesssim \left\{ \lambda(c_B,r_B) \right\}^\alpha \left[ \lambda(c_S,r_S) \right]^\alpha \frac{1}{\gamma} \| f \|^{(q)}_{s,p} \| \tilde{K}_{B,S}^{(p),1/(\alpha+1)} \|^{\gamma}, \]
which completes the proof of Proposition 6.10.

To show that \(E^\alpha, \gamma, (\mu)\) is independent of the choice of \(\gamma \in [1, \infty)\), we need the following technical lemma, which is similar to [30, Lemma 2.6] (see also [53, Lemma 9.2]).

**Lemma 6.11.** Suppose that \((X, d, \mu)\) is a non-homogeneous metric measure space. Let \(m \in \mathbb{N} \cap (1, \infty)\), \(\rho \in (1, \infty)\), \(p \in (0, 1)\) and \(B := B_1 \subset \ldots \subset B_m\) be concentric balls with center \(c_B\) and radii of the form \(\rho^N r_B\), where \(N \in \mathbb{Z}_+\). If \(K_{B_i,B_{i+1}}^{(p)} > (3 + \left\lfloor \log_2 \| f \|^{(q)}_{s,p} \right\rfloor)^{1/p} \) for any \(i \in \{1, \ldots, m - 1\}\), then
\[ \sum_{i=1}^{m-1} \left[ K_{B_i,B_{i+1}}^{(p)} \right]^p < (3 + \left\lfloor \log_2 \| f \|^{(q)}_{s,p} \right\rfloor)^{1/p}. \]

**Proof.** Fix \(m \in \mathbb{N}, \rho \in (1, \infty)\) and \(p \in (0, 1)\). Assume that, for any \(i \in \{1, \ldots, m\}\), \(r_{B_i} := \rho^N r_B\) for some \(N_i \in \mathbb{Z}_+\). For any \(i \in \{1, \ldots, m - 1\}\), by \(K_{B_i,B_{i+1}}^{(p)} > (3 + \left\lfloor \log_2 \| f \|^{(q)}_{s,p} \right\rfloor)^{1/p} \), it is easy to see that \(N_i + 1 - N_i = N_i^{(p)}(B_i,B_{i+1}) \geq 1\), \(1 < \sum_{k=1}^{N_i^{(p)}(B_i,B_{i+1})} \frac{\mu(r^k B_i)}{\lambda(c_B, r^k T B_i)} \), and \(N_m = N_m^{(p)}(B_m,B_m)\). From these facts and (2.1), we deduce that, for any \(i \in \{1, \ldots, m - 1\}\),
\[ \left[ K_{B_i,B_{i+1}}^{(p)} \right]^p \leq 2 + \left\lfloor \log_2 \| f \|^{(q)}_{s,p} \right\rfloor + \sum_{k=1}^{N_i^{(p)}(B_i,B_{i+1})} \left[ \frac{\mu(r^k B_i)}{\lambda(c_B, r^k T B_i)} \right]^p \]
\[ < (3 + \left\lfloor \log_2 \| f \|^{(q)}_{s,p} \right\rfloor)^{1/p} \sum_{k=1}^{N_i^{(p)}(B_i,B_{i+1})} \left[ \frac{\mu(r^k B_i)}{\lambda(c_B, r^k T B_i)} \right]^p \]
\[ = (3 + \left\lfloor \log_2 \| f \|^{(q)}_{s,p} \right\rfloor)^{1/p} \sum_{k=1}^{N_i^{(p)}(B_i,B_{i+1})} \left[ \frac{\mu(r^k B_i)}{\lambda(c_B, r^k T B_i)} \right]^p. \]
Notice that \(p \in (0, 1)\) and \(N_m = N_m^{(p)}(B_m,B_m)\). It then follows that
\[ \sum_{i=1}^{m-1} \left[ K_{B_i,B_{i+1}}^{(p)} \right]^p < (3 + \left\lfloor \log_2 \| f \|^{(q)}_{s,p} \right\rfloor)^{1/p}. \]

which implies (6.7) and hence completes the proof of Lemma 6.11.

The following lemma is an analogue of [30, Lemma 2.7], whose proof needs to use Lemma 6.11, the details being omitted.

**Lemma 6.12.** Suppose that \((X, d, \mu)\) is a non-homogeneous metric measure space. Let \(\alpha \in [0, \infty)\), \(\rho \in (1, \infty)\) and \(p \in (0, 1)\). For a large positive constant \(C\), the following statement holds true: let \(x \in X\) be a fixed point, and \(\{f_B\}_{B \ni x}\) some collection of complex numbers associated with balls \(B \ni x\). If there exists a positive constant \(C_x\), depending on \(x\), such that, for all balls \(B\) and \(S\) with \(x \in B \subset S\) and \(K_{B,S}^{(p), \alpha} \leq C, \| f_B - f_S \| \leq C \| K_{B,S}^{(p), \alpha} \| \lambda(c_s, r_s)\| \), then, for all balls \(B\) and \(S\) with \(x \in B \subset S\),
\[ \| f_B - f_S \| \leq CC_x K_{B,S}^{(p), \alpha} \lambda(c_s, r_s)\|. \]
By Lemma 6.12, now we are ready to state the result that $E_{p,\gamma}^{\alpha,q}(\mu)$ is independent of the choice of $\gamma$, whose proof is similar to that of [30, Proposition 2.5], the details being omitted.

**Proposition 6.13.** Suppose that $(X,d,\mu)$ is a non-homogeneous metric measure space satisfying (6.1). Let $\alpha \in [0, \infty)$, $\rho, \gamma \in (1, \infty)$ and $q \in [1, \infty)$. Then $E_{p,\gamma}^{\alpha,q}(\mu)$ and $E_{p,\ast}^{\alpha,q}(\mu)$ coincide with equivalent norms.

**Remark 6.14.** (i) By Proposition 6.13, we know that the space $E_{p,\gamma}^{\alpha,q}(\mu)$ is independent of the choice of $\gamma$. From now on, unless explicitly pointed out, we always assume $\gamma = 1$ in $E_{p,\gamma}^{\alpha,q}(\mu)$ and write $E_{p,\ast}^{\alpha,q}(\mu)$ simply by $E_{p,\ast}^{\alpha,q}(\mu)$.

(ii) It is still unknown whether $E_{p,\ast}^{\alpha,q}(\mu)$ is independent of the choice of $\gamma$ or not on general non-homogeneous metric measure spaces without the assumption (6.1), even on Euclidean spaces endowed with non-doubling measures.

In order to show that $E_{p,\ast}^{\alpha,q}(\mu)$ is independent of $q$, we establish the following John-Nirenberg type inequality which is a generalization of [27, Proposition 6.1]. Hereafter, $E_{p,\ast}^{\alpha,1}(\mu)$ is simply denoted by $E_{p}^{\alpha}(\mu)$ and its equivalent norm $\| \cdot \|_{\ast,p}$ simply by $\| \cdot \|_{\ast}$. 

**Proposition 6.15.** Suppose that $(X,d,\mu)$ is a non-homogeneous metric measure space satisfying (6.1). Let $\alpha \in [0, \infty)$ and $\rho \in (1, \infty)$. Then there exists a positive constant $c$ such that, for any $f \in E_{p}^{\alpha}(\mu)$, $t \in (0, \infty)$ and every ball $B_{0} := B(x_{0}, r)$ with $x_{0} \in X$ and $r \in (0, \infty)$,

$$
\mu \left( \left\{ x \in B_{0} : \frac{|f(x) - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} > t \right\} \right) \leq 2 \mu(\rho B_{0}) e^{-\frac{C_{1}a}{t^{\rho}L}},
$$

where $f_{B_{0}}$ is as in Proposition 6.10(ii) with $B$ replaced by $B_{0}$.

**Proof.** Fix $\alpha \in [0, \infty)$ and $\rho \in (1, \infty)$. Let $\sigma := \beta_{p}$, $f \in E_{p}^{\alpha}(\mu)$ and $L$ be a large positive constant whose value will be determined later. We first claim that, for $\mu$-almost every $x \in B_{0}$ with $\frac{|f(x) - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} > 2L$, there exists a $(\sigma, \beta_{\sigma})$-doubling ball $B_{x}^{\sigma}$ of the form $B(x, \sigma^{-r})$, $i \in \mathbb{N}$, satisfying

$$
\hat{B}_{x}^{\sigma} \subset \sqrt[p]{\rho} B_{0} \quad \text{and} \quad \frac{|f_{\hat{B}_{x}^{\sigma}} - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} > L. \quad (6.8)
$$

Indeed, from $\frac{|f(x) - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} > 2L$ and [27, Corollary 3.6], it follows that there exists a $(\sigma, \beta_{\sigma})$-doubling ball $\hat{B}_{x}^{\sigma}$ of the form $B(x, \sigma^{-r})$, $i \in \mathbb{N}$, such that $\hat{B}_{x}^{\sigma} \subset \sqrt[p]{\rho} B_{0}$ and $\frac{|m_{\hat{B}_{x}}(f) - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} > 2L$. Thus, by this, Propositions 6.8 and 6.10, (2.1) and (2.2), we conclude that

$$
\frac{|f_{\hat{B}_{x}^{\sigma}} - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} \geq \frac{|m_{\hat{B}_{x}^{\sigma}}(f) - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} \geq \frac{|f_{\hat{B}_{x}^{\sigma}} - m_{\hat{B}_{x}^{\sigma}}(f)|}{|\lambda(x_{0}, r)|^{\alpha}} \geq 2L - \frac{1}{|\lambda(x_{0}, r)|^{\alpha}} \mu(\hat{B}_{x}^{\sigma}) \int_{\hat{B}_{x}^{\sigma}} |f(y) - f_{\hat{B}_{x}^{\sigma}}| d\mu(y) \geq 2L - C_{1} \| f \|_{\ast}, \rho \geq L,
$$

provided that $L \geq C_{1} \| f \|_{\ast,\rho}$ and $C_{1}$ is a positive constant, which implies the claim.

Now we let $\hat{B}_{x}^{\sigma}$ be the biggest $(\sigma, \beta_{\sigma})$-doubling ball of the form $B(x, \sigma^{-r})$, $i \in \mathbb{N}$, satisfying (6.8). By (6.8), (2.1) and (2.2), we know that

$$
\frac{1}{\mu(\hat{B}_{x}^{\sigma})} \int_{\hat{B}_{x}^{\sigma}} \frac{|f(y) - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} d\mu(y) \geq \frac{|f_{\hat{B}_{x}^{\sigma}} - f_{B_{0}}|}{|\lambda(x_{0}, r)|^{\alpha}} - \frac{1}{\mu(\hat{B}_{x}^{\sigma})} \int_{\hat{B}_{x}^{\sigma}} \frac{|f(y) - f_{\hat{B}_{x}^{\sigma}}|}{|\lambda(x_{0}, r)|^{\alpha}} d\mu(y) \geq L - \frac{\lambda(x, \sqrt[p]{\rho} r)^{\alpha}}{|\lambda(x_{0}, r)|^{\alpha}} \beta_{\sigma} \| f \|_{\ast} \sqrt[p]{\rho} \geq L - C_{1} \| f \|_{\ast,\rho} \geq L/2, \quad (6.9)
$$
provided that \(L \geq 2C_1 \|f\|_{*, \rho} \).

Then we show that the ball \( \widehat{B}_x^\sigma \) equals 0 provided that
\[
\|f_{\tilde{B}_x^\sigma} - f_{B_0}\|_{\lambda(x_0, r)} \leq L.
\]

(6.10)

Indeed, it suffices to prove that, if \( \tilde{B}_x \subset \sqrt{\rho} B_0 \), then
\[
\|f_{\tilde{B}_x} - f_{B_0}\|_{\lambda(x_0, r)} \leq L.
\]
From Lemma 2.9, \(B(x, r) \subset 2\sqrt{\rho} B_0\), (v), (iv), (iii) and (ii) of Lemma 2.8, it follows that, if \(\tilde{B}_x \subset \sqrt{\rho} B_0 \subset 2\sqrt{\rho} B_0\), then

\[
\|f_{\tilde{B}_x} - f_{B_0}\|_{\lambda(x_0, r)} \leq \frac{2L}{\lambda(x_0, r)}.
\]

Moreover, if \(\tilde{B}_x \not\subset \sqrt{\rho} B_0\), let \(\sigma_j \tilde{B}_x^\sigma\) be the smallest ball of the form \(\sigma^k \tilde{B}_x^\sigma\) (\(k \in \mathbb{N}\)) satisfying \(\sigma^j \tilde{B}_x^\sigma \not\subset \sqrt{\rho} B_0\). We easily obtain

\[
r_{\sigma_j \tilde{B}_x^\sigma} \sim r_{B_0} \quad \text{and} \quad \tilde{B}_x = \left(\sigma_j \tilde{B}_x^\sigma\right)^\sigma,
\]

where \(r_{\sigma_j \tilde{B}_x^\sigma}\) and \(r_{B_0}\) denote the radii of balls \(\sigma^j \tilde{B}_x^\sigma\) and \(B_0\), respectively. By this, \(\sigma^j \tilde{B}_x^\sigma \subset 3\sigma \sqrt{\rho} B_0\), \(\hat{B}_x^\sigma \subset \sqrt{\rho} B_0\), Remark 6.2(ii), (2.1), (2.2), Lemma 2.9, (ii) and (iii) of Lemma 2.8, we have

\[
\|f_{\sigma^j \tilde{B}_x^\sigma} - f_{B_0}\|_{\lambda(x_0, r)} \leq \frac{2L}{\lambda(x_0, r)}.
\]

provided that \(L \geq C_3 \|f\|_{*, \rho}\) and \(C_3\) is a positive constant.

Thus, in any case, we have

\[
\|f_{\tilde{B}_x^\sigma} - f_{B_0}\|_{\lambda(x_0, r)} \leq L.
\]

(6.11)

provided that \(L \geq \max\{C_2, C_3\} \|f\|_{*, \rho}\).

Furthermore, by [23, Theorem 1.2] and [27, Lemma 2.5], we see that there exists a sequence \(\{\tilde{B}_{x_i}^\sigma\}_{i \in I}\) of disjoint balls such that \(x_i \in B_0\) for any \(i \in I\) and \(B_0 \subset \cup_{x_i \in B_0} \tilde{B}_{x_i}^\sigma \subset \cup_{i \in I} 5\tilde{B}_{x_i}\). Let \(B^{(i)} := 5\tilde{B}_{x_i}^\sigma\) for any \(i \in I\). Observe that, for any \(n \in \mathbb{N} \cap [2, \infty]\), if \(x \in B_0\) and \(\|f(x) - f_{B_0}\|_{\lambda(x_0, r)} > 2L\), then there exists \(i \in I\) such that \(x \in B^{(i)}\) and, from (6.11), Remark 6.2(ii), (2.1), Lemma 2.9, (v), (iv), (ii) and (iii) of Lemma 2.8, it follows that

\[
\|f(x) - f_{B_0}\|_{\lambda(x_0, r)} \geq \|f(x) - f_{B_0}\|_{\lambda(x_0, r)} - \|f_{B_0} - f_{\tilde{B}_x^\sigma}\|_{\lambda(x_0, r)} - \|f_{\tilde{B}_x^\sigma} - f_{5\tilde{B}_{x_i}^\sigma}\|_{\lambda(x_0, r)}
\]

\[
> nL - \frac{\|f(x) - f_{B_0}\|_{\lambda(x_0, r)}}{\lambda(x_0, r)} - \|f_{\tilde{B}_x^\sigma} - f_{5\tilde{B}_{x_i}^\sigma}\|_{\lambda(x_0, r)} - \frac{nL - \|f_{\tilde{B}_x^\sigma} - f_{5\tilde{B}_{x_i}^\sigma}\|_{\lambda(x_0, r)}}{\lambda(x_0, r)}
\]

\[
> (n - 1)L - C_4 \|f\|_{*, \rho} \geq (n - 2)L,
\]

(6.12)
provided that \( L \geq C_4 \| f \|_{s, \rho} \) and \( C_4 \) is a positive constant.

By (6.9), the disjointness of \( \{ \tilde{B}_i^\alpha \}_{i \in I}, \tilde{B}_i^\alpha \subset \sqrt{\rho}B_0 \) for all \( i \in I \), Proposition 6.8(ii), Proposition 6.10(b), Lemma 2.8(ii) and (2.1), we further see that

\[
\sum_{i \in I} \mu(\rhoB(i)) \leq \beta_\rho \sum_{i \in I} \mu(\tilde{B}_i^\alpha) \leq \frac{2\beta_\rho}{L} \sum_{i \in I} \int_{\tilde{B}_i^\alpha} \frac{|f(y) - f_{B_0}|}{|\lambda(x_0, r)|} \, d\mu(y) \\
\leq \frac{2\beta_\rho}{L} \left\{ \int_{\sqrt{\rho}B_0} \frac{|f(y) - f_{\sqrt{\rho}B_0}|}{|\lambda(x_0, r)|} \, d\mu(y) + \frac{|f_{\sqrt{\rho}B_0} - f_{B_0}|}{|\sqrt{\rho}B_0|} \mu(\rhoB_0) \right\} \\
\leq \frac{1}{L} \left\{ \frac{|\lambda(x_0, \sqrt{\rho}r)|}{|\lambda(x_0, r)|} \| f \|_{s, \rho} \mu(\rhoB_0) + \mu(\rhoB_0) \tilde{K}(\rho B_0, \lambda B_0, 1/(\alpha+1)) \right\} \\
\leq \frac{C_5}{L} \| f \|_{s, \rho} \mu(\rhoB_0) \leq \frac{1}{2} \mu(\rhoB_0),
\]

provided that \( L \geq 2C_5 \| f \|_{s, \rho} \) and \( C_5 \) is a positive constant.

Moreover, for any \( t \in (0, \infty) \), there exists \( n \in \mathbb{Z}_+ \) such that \( 2nL \leq t < 2(n+1)L \). By this and (6.12), we know that

\[
\left\{ x \in B_0 : \left| \frac{f(x) - f_{B_0}}{|\lambda(x_0, r)|} \right| > t \right\} \subset \left\{ x \in B_0 : \left| \frac{f(x) - f_{B_0}}{|\lambda(x_0, r)|} \right| > 2nL \right\} \\
\subset \bigcup_{i \in I} \left\{ x \in B(i) : \left| \frac{f(x) - f_{B(i)}}{|\lambda(x_0, r)|} \right| > 2(n-1)L \right\}.
\]

Finally, by (6.13), (6.14), iterating with the balls \( B(i) \) in place of \( B_0 \) and an argument similar to that used in the proof of [27, Proposition 6.1], we conclude that

\[
\mu \left( \left\{ x \in B_0 : \left| \frac{f(x) - f_{B_0}}{|\lambda(x_0, r)|} \right| > t \right\} \right) \leq 2\mu(\rhoB_0) e^{-\frac{c t}{\| f \|_{s, \rho} \lambda(x_0, r)}},
\]

with \( c := \frac{\ln 2}{2L} \| f \|_{s, \rho} \) and \( L := 2 \max \{ C_i : i \in \{1, \ldots, 5\} \} \). This finishes the proof of Proposition 6.15.

By Proposition 6.15, we easily obtain the following conclusion.

**Corollary 6.16.** Suppose that \( (\mathcal{X}, d, \mu) \) is a non-homogeneous metric measure space satisfying (6.1). Let \( \alpha \in [0, \infty), \rho \in (1, \infty) \) and \( q \in [1, \infty) \). Then there exists a positive constant \( C \) such that, for any \( f \in \mathcal{E}_\rho^\alpha(\mu) \) and every ball \( B := B(c_B, r_B) \) with \( c_B \in \mathcal{X} \) and \( r_B \in (0, \infty) \),

\[
\left\{ \frac{1}{\mu(\rhoB)} \int_B \left| f(x) - f_B \right|^q \, d\mu(x) \right\}^{1/q} \leq C \| f \|_{s, \rho} \lambda(c_B, r_B) \alpha,
\]

where \( f_B \) is as in Proposition 6.10(ii).

**Remark 6.17.** (i) By Corollary 6.16 and Proposition 6.10, together with the Hölder inequality, we know that \( \mathcal{E}_\rho^\alpha q(\mu) \) is independent of the choice of \( q \), the details being omitted. From now on, unless explicitly pointed out, we always assume that \( q = 1 \) in \( \mathcal{E}_\rho^\alpha q(\mu) \) and write \( \mathcal{E}_\rho^\alpha 1(\mu) \) simply by \( \mathcal{E}_\rho^\alpha(\mu) \) and its norm \( \| \cdot \|_{\mathcal{E}_\rho^\alpha 1(\mu)} \) simply by \( \| \cdot \|_{\mathcal{E}_\rho^\alpha(\mu)} \).

(ii) It is still unknown whether \( \mathcal{E}_\rho^\alpha q(\mu) \) is independent of the choice of \( q \) or not on general non-homogeneous metric measure spaces without the assumption (6.1), even on Euclidean spaces endowed with non-doubling measures.

We establish another characterization of \( \mathcal{E}_\rho^\alpha \) which is needed in the later context. To this end, we first recall the so-called median value of a function on balls in [25,30]. Precisely, let \( f \) be a measurable function. The median value of \( f \) on any ball \( B \), denoted by \( m_f(B) \), is defined as follows. If \( f \) is real-valued, then, for any ball \( B \) with \( \mu(B) \neq 0 \), let \( m_f(B) \) be some real number such that \( \inf_{c \in \mathbb{R}} \frac{1}{\mu(B)} \int_B |f(x) - c| \, d\mu(x) \) is attained. It is known that \( m_f(B) \) satisfies

\[
\mu(\{ x \in B : f(x) > m_f(B) \}) \leq \mu(B)/2.
\]
and
\[ \mu(\{ x \in B : f(x) < m_f(B) \}) \leq \mu(B)/2. \]
For all balls \( B \) with \( \mu(B) = 0 \), let \( m_f(B) := 0 \). If \( f \) is complex-valued, we take
\[ m_f(B) := |m\Re f(B)| + |m\Im f(B)|, \]
where \( i^2 = -1 \) and, for any complex number \( z \), denote by \( \Re z \) and \( \Im z \) its real part and the imaginary part, respectively.

Let \( \alpha \in [0, \infty) \), \( \rho \in [2, \infty) \) and \( q, \gamma \in [1, \infty) \). The norm \( \| f \|_{\alpha, \rho} \) of a suitable function \( f \) is defined by
\[
\| f \|_{\alpha, \rho} := \sup_{B: \text{B} \text{ is \((\rho, \beta_\rho)\)-doubling ball}} \frac{1}{\mu(B)} \frac{1}{\lambda(c_B, r_B)^{\alpha}} \int_B |f(y) - m_f(B)| \, d\mu(y) + \sup_{B \subset S: B, S \text{ are \((\rho, \beta_\rho)\)-doubling balls}} \frac{|m_f(B) - m_f(S)|}{\lambda(c_S, r_S)^{\gamma} [K^{(\rho), 1/(\alpha+1)}_{B, S}]^{\gamma}}.
\]

Then we have the following equivalent characterization of \( \mathcal{E}_\rho^\alpha(\mu) \).

**Proposition 6.18.** Suppose that \((X, d, \mu)\) is a non-homogeneous metric measure space satisfying (6.1). Let \( \alpha \in [0, \infty) \), \( \rho \in [6/5, \infty) \) and \( q, \gamma \in [1, \infty) \). Then the norms \( \| \cdot \|_{\alpha, \rho} \) and \( \| \cdot \|_{\mathcal{E}_\rho^\alpha(\mu)} \) are equivalent.

**Proof.** Fix \( \alpha \in [0, \infty) \) and \( \rho \in [6/5, \infty) \). For the sake of the simplicity, we assume that \( \gamma = 1 \). The arguments here are still valid for the general case with some minor modifications. Let \( f \in \mathcal{E}_\rho^\alpha(\mu) \). Now we show that \( \| f \|_{\alpha, \rho} \lesssim \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} \). For any \((\rho, \beta_\rho)\)-doubling ball \( B \), by the definition of \( m_f(B) \), we conclude that
\[
|m_f(B) - m_B(f)| \leq \frac{1}{\mu(B)} \int_B |f(y) - m_f(B)| \, d\mu(y) \leq \frac{1}{\mu(B)} \int_B |f(y) - m_B(f)| \, d\mu(y) \lesssim \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} [\lambda(c_B, r_B)]^{\alpha},
\]
which implies that, for any \((\rho, \beta_\rho)\)-doubling ball \( B \),
\[
\frac{1}{\mu(B)} \int_B |f(y) - m_f(B)| \, d\mu(y) \lesssim \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} [\lambda(c_B, r_B)]^{\alpha}.
\]
On the other hand, by (6.15), (2.1) and (2.2), we know that, for all \((\rho, \beta_\rho)\)-doubling balls \( B \subset S \),
\[
|m_f(B) - m_f(S)| \leq |m_f(B) - m_B(f)| + |m_B(f) - m_S(f)| + |m_S(f) - m_f(S)| \lesssim \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} [\lambda(c_S, r_S)]^{\alpha} K^{(\rho), 1/(\alpha+1)}_{B, S} + \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} [\lambda(c_B, r_B)]^{\alpha} \lesssim \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} [\lambda(c_S, r_S)]^{\alpha} K^{(\rho), 1/(\alpha+1)}_{B, S}.
\]
Combining these two inequalities, we conclude that \( \| f \|_{\alpha, \rho} \lesssim \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} \).

Conversely, let \( \| f \|_{\alpha, \rho} < \infty \). We now prove that \( \| f \|_{\mathcal{E}_\rho^\alpha(\mu)} \lesssim \| f \|_{\alpha, \rho} \). For any ball \( B \), if \( B \) is \((\rho, \beta_\rho)\)-doubling, we see that
\[
\frac{1}{\mu(B)} \int_B |f(y) - m_B(f)| \, d\mu(y) \leq \frac{1}{\mu(B)} \int_B |f(y) - m_f(B)| \, d\mu(y) + |m_f(B) - m_B(f)| \lesssim \frac{1}{\mu(B)} \int_B |f(y) - m_f(B)| \, d\mu(y) \lesssim \| f \|_{\alpha, \rho} [\lambda(c_B, r_B)]^{\alpha}.
\]
Thus, we only need to consider the case that \( B \) is non-\((\rho, \beta_\rho)\)-doubling.

Assume that \( B \) is a non-\((\rho, \beta_\rho)\)-doubling ball. For any \( x \in B \), let \( B_x \) be the biggest \((5\rho, \beta_\rho)\)-doubling ball centered at \( x \) with radius \((5\rho)^{-k} r_B \) for some \( k \in \mathbb{N} \) (since \( \beta_\rho > (5\rho)^{\alpha} \) and \([27, \text{Lemma 3.3}]\)). From \( \rho \geq 6/5 \), it follows easily that \( 5B_x \subset 2B \subset (6/5)B \subset \bar{B}^\rho \). Moreover, by \([23, \text{Theorem 1.2}] \) and \([27, \text{Lemma 2.5}] \), we see that there exists a countable disjoint subfamily \( \{ B_x \}_i = \{ B_i \}_i \) of \( \{ B_x \}_x \) such that \( x_i \in B \) for
all $i$ and $B \subset \bigcup_{x \in B} B_x \subset \bigcup \{5B_i\}$. For any $i$, by $5B_i \subset B(x_i, r_B) \subset (6/5)B \subset \rho B \subset \tilde{B}_\rho$, and (iv), (iii), (ii) and (v) of Lemma 2.8, we see that
\[
\widehat{K}^{(p),(1/(\alpha+1))}_{\tilde{B}_\rho, \tilde{B}_\rho} \lesssim \widehat{K}^{(p),(1/(\alpha+1))}_{5B_i, B(x, r_B)} + \widehat{K}^{(p),(1/(\alpha+1))}_{B(x, r_B), \rho B} + \widehat{K}^{(p),(1/(\alpha+1))}_{\rho B, \tilde{B}_\rho} \lesssim 1.
\]
From this, together with the fact that $5B_i$ is a $(\rho, \beta_\rho)$-doubling ball for any $i$, (2.1) and (2.2), Remark 6.2(ii), $5B_i \subset (6/5)B \subset \rho B \subset \tilde{B}_\rho$ for any $i$ and the disjointness of \{\{B_i\}i\}, it follows that
\[
\int_B |f(y) - m_{\tilde{B}_\rho}(f)| \, d\mu(y) \\
\leq \sum_i \int_{5B_i} |f(y) - m_f(5B_i)| \, d\mu(y) + \sum_i \mu(5B_i) \left[ |m_f(5B_i) - m_f(\tilde{B}_\rho)| + |m_f(\tilde{B}_\rho) - m_{\tilde{B}_\rho}(f)| \right] \\
\lesssim \|f\|_{\infty, \rho} \sum_i \mu(5B_i) \{ |\lambda(c_B, r_B)|^\alpha + \sum_i \mu(5B_i) \left( \|f\|_{\infty, \rho} [\lambda(c_B, r_{\tilde{B}_\rho})]^\alpha \right) \}
\]
\[
\lesssim \|f\|_{\infty, \rho} \sum_i \mu(5B_i) |\lambda(c_B, r_B)|^\alpha \lesssim \|f\|_{\infty, \rho} \mu(\rho B) |\lambda(c_B, r_B)|^\alpha.
\]
On the other hand, for all $(\rho, \beta_\rho)$-doubling balls $B \subset S$, by (2.1) and (2.2), we have
\[
|m_B(f) - m_S(f)| \\
\leq |m_B(f) - m_f(5B_i)| + |m_f(B) - m_f(S)| + |m_f(S) - m_S(f)| \\
\lesssim \left\{ |\lambda(c_B, r_B)|^\alpha + |\lambda(c_S, r_S)|^\alpha \widehat{K}^{(p),(1/(\alpha+1))}_{\tilde{B}_\rho, \tilde{B}_\rho} + |\lambda(c_S, r_S)|^\alpha \right\} \|f\|_{\infty, \rho} \\
\lesssim \|f\|_{\infty, \rho} |\lambda(c_S, r_S)|^\alpha \widehat{K}^{(p),(1/(\alpha+1))}_{\tilde{B}_\rho, \tilde{B}_\rho}.
\]
These two inequalities show that $\|f\|_{\varepsilon_p(\mu)} \lesssim \|f\|_{\infty, \rho}$, which completes the proof of Proposition 6.18. \qed

We point out that it is still unclear whether the range of $\rho$ in Proposition 6.18 is sharp or not.

7 Atomic Hardy Spaces $\widehat{H}^{p,q,\gamma}_{\text{ath},\rho}(\mu)$ and Molecular Hardy Spaces $\widehat{H}^{p,q,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$

In this section, under the assumption of $\rho$-weakly doubling conditions, we introduce the atomic Hardy space $\widehat{H}^{p,q,\gamma}_{\text{ath},\rho}(\mu)$ and the molecular Hardy space $\widehat{H}^{p,q,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$, and show that the spaces $\widehat{H}^{p,q,\gamma}_{\text{ath},\rho}(\mu)$ and $\widehat{H}^{p,q,\gamma,\epsilon}_{\text{mb},\rho}(\mu)$ coincide with equivalent quasi-norms.

**Definition 7.1.** Let $\rho \in (1, \infty)$, $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$ and $\gamma \in [1, \infty)$. A function $b \in L^1(\mu)$ is called a $(p, q, \gamma, \rho)\lambda, 1$-atomic block if $b$ satisfies (i), (ii) and (iii) of Definition 3.2. Moreover, let $|b|_{\widehat{H}^{p,q,\gamma}_{\text{ath},\rho}} := |\lambda_1| + |\lambda_2|$.

**Remark 7.2.** It is easy to see that any $(1, q, \gamma, \rho)\lambda, 1$-atomic block is also a $(1, q, \gamma, \rho)\lambda$-atomic block and vice versa. We point out that the difference between the $(p, q, \gamma, \rho)\lambda$-atomic block and the $(p, q, \gamma, \rho)\lambda, 1$-atomic blocks exists in that the former is an $L^2(\mu)$ function when $p \in (0, 1)$, while the latter is only an $L^1(\mu)$ function.

Observe that, for any $(p, q, \gamma, \rho)\lambda, 1$-atomic block $b$, there exist some balls $B_j$ $(j \in \{1, 2\})$ and $B$, and some numbers $\lambda_j \in \mathbb{C}$ $(j \in \{1, 2\})$ such that $\text{supp}(b) \subset B$, $b = \lambda_1 a_1 + \lambda_2 a_2$ and $\text{supp}(a_j) \subset B_j \subset B$, $j \in \{1, 2\}$. By $\int_X b(x) \, d\mu(x) = 0$, Proposition 6.7(a), Definition 3.2(iii), (2.1) and (2.2), we know that
\[
\int_X |f(x)b(x)| \, d\mu(x)
\]
However, when $H_{\text{over general non-homogeneous metric measure spaces}}$.

\[
\begin{align*}
&= \int_X [f(x) - m_{\hat{B}_r}(f)] b(x) \, d\mu(x) \\
&\leq \sum_{j=1}^2 |\lambda_j| \int_{B_j} [f(x) - m_{\hat{B}_r}(f)] |a_j(x)| \, d\mu(x) \\
&\leq \sum_{j=1}^2 |\lambda_j| \left[ \int_{B_j} |a_j(x)|^q \, d\mu(x) \right]^{1/q} \left[ \int_{B_j} |f(x) - m_{\hat{B}_r}(f)|^{q'} \, d\mu(x) \right]^{1/q'} \\
&\leq \sum_{j=1}^2 |\lambda_j| \|\mu(\rho B_j)\|^{1/q-1} [\lambda(cB, rB)]^{1-1/p} \left[ K_{\hat{B}_j, B}^{(p), p} \right]^{-\gamma} \\
&\times \left\{ \left[ \int_{B_j} |f(x) - m_{\hat{B}_r}(f)|^{q'} \, d\mu(x) \right]^{1/q'} + |\mu(B_j)|^{1/q'} \left| m_{\hat{B}_r}(f) - m_{\hat{B}_r}(f) \right| \right\} \\
&\leq \sum_{j=1}^2 |\lambda_j| \|f\|_{E_{\rho, p}^{1/q-1}(\mu)} \sim \|b\|_{H_{\text{atb}, \rho}^{p, q, \gamma}(\mu)} \|f\|_{E_{\rho, p}^{1/q-1}(\mu)}. \quad (7.1)
\end{align*}
\]

Thus, a $(p, q, \gamma, \rho)_{\lambda, 1}$-atomic block $b$ can be seen as an element in the dual space $(E_{\rho, p}^{1/q-1}(\mu))^*$ of $E_{\rho, p}^{1/q-1}(\mu)$.

**Definition 7.3.** Let $0 < p \leq 1 < q \leq \infty$ and $p \neq q$. The atomic Hardy space $H_{\text{atb}, \rho}^{p, q, \gamma}(\mu)$ is defined as the subspace of $(E_{\rho, p}^{1/q-1}(\mu))^*$ when $p < 1$ and of $L^1(\mu)$ when $p = 1$, consisting of those linear functional admitting an atomic decomposition

\[
f = \sum_{i=1}^\infty b_i \tag{7.2}
\]

in $(E_{\rho, p}^{1/q-1}(\mu))^*$ when $p < 1$ and in $L^1(\mu)$ when $p = 1$, where $\{b_i\}_{i=1}^\infty$ are $(p, q, \gamma, \rho)_{\lambda, 1}$-atomic blocks such that

\[
\sum_{i=1}^\infty |b_i|_{H_{\text{atb}, \rho}^{p, q, \gamma}(\mu)} < \infty.
\]

Moreover, define

\[
\|f\|_{H_{\text{atb}, \rho}^{p, q, \gamma}(\mu)} := \inf \left\{ \left[ \sum_{i=1}^\infty |b_i|_{H_{\text{atb}, \rho}^{p, q, \gamma}(\mu)}^p \right]^{1/p} \right\},
\]

where the infimum is taken over all possible decompositions of $f$ as above.

**Remark 7.4.** (i) It follows from Remark 7.2 that $H_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$ is the atomic Hardy space defined via the discrete coefficients $K_{B, S}^{(p)}$ introduced in [13], where it was shown that $H_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$ is independent of the choices of $q$, $\rho$ and $\gamma$. Hereafter, $H_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$ is simply denoted by $H_{\text{atb}, \rho}^{1, q, \gamma}(\mu)$.

(ii) Let $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$ and $q \in (1, \infty)$. By Remarks 3.3(ii) and (i) of this remark, we know that

\[
H_{\text{atb}, \rho}^{1, q, \gamma}(\mu) = H_{\text{atb}, \rho}^{1, q, \gamma}(\mu)
\]

over general non-homogeneous metric measure spaces.

(iii) Fix $p$, $\rho$ and $\gamma$ as in Definition 7.1. For $1 \leq q_1 \leq q_2 \leq \infty$ and $q_1 > p$, we notice that $H_{\text{atb}, \rho}^{p, q_2, \gamma}(\mu) \subset H_{\text{atb}, \rho}^{p, q_1, \gamma}(\mu)$.

(iv) By the results in [13], we know that the Calderón-Zygmund operator is bounded on $H_{\text{atb}}^{1, \gamma}(\mu)$. However, when $p \in (0, 1)$, it is still unclear whether the Calderón-Zygmund operator is bounded on $H_{\text{atb}, \rho}^{p, q, \gamma}(\mu)$ or not.
We now introduce the notion of the molecular Hardy space $\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$ in the non-homogeneous setting by first presenting the following notion of $(p, q, \gamma, \epsilon, \rho)_{\lambda, 1}$-molecular blocks.

**Definition 7.5.** Let $\rho \in (1, \infty)$, $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $\gamma \in [1, \infty)$ and $\epsilon \in (0, \infty)$. A function $b \in L^1(\mu)$ is called a $(p, q, \gamma, \epsilon, \rho)_{\lambda, 1}$-molecular block if

1. \( \int_X b(x) \, d\mu(x) = 0; \)
2. there exist some ball $B$ and some constants $\tilde{M}$, $M \in \mathbb{N}$ such that, for all $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$ with $M_k := \tilde{M}$ if $k = 0$ and $M_k := M$ if $k \in \mathbb{N}$, there exist functions $m_{k,j}$ supported on some balls $B_{k,j} \subset U_k(B)$ for all $k \in \mathbb{Z}_+$, where $U_0(B) := \rho^2 B$ and $U_k(B) := \rho^{2k+2} B \setminus (k-2) B$ with $k \in \mathbb{N}$, and $\lambda_{k,j} \in \mathbb{C}$ such that $b = \sum_{k=0}^\infty \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j}$ in $L^1(\mu)$ when $p = 1$ and in both $L^1(\mu)$ and $(\mathcal{E}_p^{1/p-1}(\mu))^*$ when $p \in (0, 1)$,

\[
\|m_{k,j}\|_{L^q(\mu)} \leq \rho^{-k\gamma} [\rho (\rho B_{k,j})]^{1/q-1} \left[ \lambda \left(c_{B}, \rho^{k+2} r_B\right)\right]^{-1/p} \left[ K^{(\rho)}_{B_{k,j}, \rho^{k+2}}\right]^{-\gamma} \tag{7.3}
\]

and

\[
|b|^p_{\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)} := \sum_{k=0}^\infty \sum_{j=1}^{M_k} |\lambda_{k,j}|^p < \infty.
\]

**Remark 7.6.** Observe that any $(1, q, \gamma, \epsilon, \rho)_{\lambda, 1}$-molecular block is also a $(1, q, \gamma, \epsilon, \rho)_{\lambda, 1}$-molecular block and vice versa.

**Definition 7.7.** Let $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$ and $\epsilon \in (0, \infty)$. The molecular Hardy space $\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$ is defined as the subspace of $(\mathcal{E}_p^{1/p-1}(\mu))^*$ when $p < 1$ and of $L^1(\mu)$ when $p = 1$, consisting of those linear functional admitting a molecular decomposition

\[
f = \sum_{i=1}^\infty b_i \tag{7.4}
\]

in $(\mathcal{E}_p^{1/p-1}(\mu))^*$ when $p < 1$ and in $L^1(\mu)$ when $p = 1$, where $\{b_i\}_{i=1}^\infty$ are $(p, q, \gamma, \epsilon, \rho)_{\lambda, 1}$-molecular blocks such that

\[
\sum_{i=1}^\infty |b_i|^p_{\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)} < \infty.
\]

Moreover, define

\[
\|f\|_{\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)} := \inf \left\{ \sum_{i=1}^\infty |b_i|^p_{\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)} \right\}^{1/p},
\]

where the infimum is taken over all possible decompositions of $f$ as above.

**Remark 7.8.** (i) It follows from Remark 7.6 that $H_{mb, \rho}^{1, q, \gamma, \epsilon}(\mu)$ is the molecular Hardy space defined via the discrete coefficients $\tilde{K}_{B, \rho}$ introduced in [13].

(ii) Let $\rho$, $p$, $q$, $\gamma$ and $\epsilon$ be as in Definition 7.5. Then each $(p, q, \gamma, \rho)_{\lambda, 1}$-atomic block is a $(p, q, \gamma, \epsilon, \rho)_{\lambda, 1}$-molecular block and hence $\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu) \subset H_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$ and, for all $f \in \tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$,

\[
\|f\|_{\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)} \leq \|f\|_{H_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)}.
\]

Moreover, we have the following relation between $\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$ and $\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$.

**Theorem 7.9.** Suppose that $(X, d, \mu)$ is a non-homogeneous metric measure space satisfying (6.1). Let $\rho \in (1, \infty)$, $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, $\gamma \in [1, \infty)$ and $\epsilon \in (0, \infty)$. Then $\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$ and $\tilde{H}_{mb, \rho}^{p, q, \gamma, \epsilon}(\mu)$ coincide with equivalent quasi-norms.

**Proof.** When $p = 1$, the conclusion of Theorem 7.9 was obtained in [13, Theorem 1.11] without the assumption (6.1). Thus, we only need to consider the case $p \in (0, 1)$. Fix $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$, $\epsilon \in (0, \infty)$, $0 < p < 1 \leq q \leq \infty$ and $p \neq q$. Let $I_B^{(\rho)} := N_{B, B^c}^{(\rho)}$ for any ball $B$. By Remark 7.8(ii),...
to show Theorem 7.9 in this case, it suffices to prove that $\tilde{H}^{p,q,\gamma,\epsilon}_{mb,\rho}(\mu) \subset \tilde{H}^{p,q,\gamma}_{ab,\rho}(\mu)$ and that, for any $f \in \tilde{H}^{p,q,\gamma,\epsilon}_{mb,\rho}(\mu)$, $f \in \tilde{H}^{p,q,\gamma}_{ab,\rho}(\mu)$ and $\|f\|_{\tilde{H}^{p,q,\gamma,\epsilon}_{mb,\rho}(\mu)} \lesssim \|f\|_{\tilde{H}^{p,q,\gamma}_{ab,\rho}(\mu)}$. To this end, we first show that any $(p, q, \gamma, \rho, \lambda)$, 1-molecular block $b$ can be decomposed into a sum of some $(p, q, \gamma, \rho, \lambda)$, 1-atomic blocks and $(p, \infty, \gamma, \rho, \lambda)$, 1-atomic blocks and $|b|_{\tilde{H}^{p,q,\gamma}_{ab,\rho}(\mu)} \lesssim |b|_{\tilde{H}^{p,q,\gamma}_{mb,\rho}(\mu)}$.

For any $(p, q, \gamma, \epsilon, \rho, \lambda)$, 1-molecular block $b$, by Definition 7.5, we know that

$$b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j} \text{ in } L^1(\mu) \text{ and } \left(\mathcal{E}_{\rho}^{1/p-1}(\mu)\right)^*,$$

where, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $\lambda_{k,j} \in \mathbb{C}$ and $\text{supp}(m_{k,j}) \subset B_{k,j} \subset U_k(B)$ with the same notation as in Definition 7.5. Moreover, observe that, by (7.5), the Hölder inequality and (7.3), we have

$$\sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \|\lambda_{k,j} m_{k,j}\|_{L^1(\mu)} \leq \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}| |\rho|^{-kr} \left[\lambda \left(\epsilon_{B_k,\rho}^{k+2} r_B\right)\right]^{1-1/p} \left[\sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \|\lambda_{k,j}\|^p\right]^{1/p} \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \|\lambda_{k,j}\|^p \left[\lambda(\epsilon_{B_k,\rho} r_B)\right]^{1-1/p} < \infty.$$  

(7.6)

For each $k \in \mathbb{Z}_+$, let $b_k := \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j}$, $B_{k+2}^p := \rho^{k+2} B$ and $\tilde{B}_{k+2}^p := (\rho^{k+2} B)^{2p}$. By (7.5), we write

$$b = \sum_{k=0}^{\infty} \left[ b_k - \frac{\chi\tilde{B}_{k+2}^p}{\mu(B_{k+2}^p)} \int_X b_k(y) \, d\mu(y) \right] + \sum_{k=0}^{\infty} \frac{\chi\tilde{B}_{k+2}^p}{\mu(B_{k+2}^p)} \int_X b_k(y) \, d\mu(y)$$

$$= \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \lambda_{k,j} \left[ m_{k,j} - \frac{\chi\tilde{B}_{k+2}^p}{\mu(B_{k+2}^p)} \int_{B_{k,j}} m_{k,j}(y) \, d\mu(y) \right] + \sum_{k=0}^{\infty} \frac{\chi\tilde{B}_{k+2}^p}{\mu(B_{k+2}^p)} \int_X b_k(y) \, d\mu(y)$$

$$=: \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} b_{k,j} + \sum_{k=0}^{\infty} \chi_k \tilde{M}_k = I + II,$$

where, for all $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$,

$$b_{k,j} := \lambda_{k,j} \left[ m_{k,j} - \frac{\chi\tilde{B}_{k+2}^p}{\mu(B_{k+2}^p)} \int_{B_{k,j}} m_{k,j}(y) \, d\mu(y) \right],$$

$$\chi_k := \frac{\chi\tilde{B}_{k+2}^p}{\mu(B_{k+2}^p)} \text{ and } \tilde{M}_k := \int_X b_k(y) \, d\mu(y).$$

From (7.6), it follows that $\sum_{k=0}^{\infty} \sum_{j=1}^{M_k} b_{k,j}$ and $\sum_{k=0}^{\infty} \chi_k \tilde{M}_k$ both converge in $L^1(\mu)$.

To estimate $I$, we first show that, for any $k \in \mathbb{Z}_+$ and $j \in \{1, \ldots, M_k\}$, $b_{k,j}$ is a $(p, q, \gamma, \rho, \lambda)$, 1-atomic block. Noticing that $\text{supp}(b_{k,j}) \subset 2\tilde{B}_{k+2}^p$ and $\int_X b_{k,j}(y) \, d\mu(y) = 0$, to show this, it only needs to show that $b_{k,j}$ satisfies Definition 3.2(iii). To this end, we further decompose $b_{k,j}$ into

$$b_{k,j} = \lambda_{k,j} \left[ m_{k,j} - \frac{\chi\text{supp}(m_{k,j})}{\mu(B_{k+2}^p)} \int_{B_{k,j}} m_{k,j}(y) \, d\mu(y) \right]$$

$$- \lambda_{k,j} \frac{\chi\tilde{B}_{k+2}^p \text{supp}(m_{k,j})}{\mu(B_{k+2}^p)} \int_{B_{k,j}} m_{k,j}(y) \, d\mu(y) =: A_{k,j}^{(1)} - A_{k,j}^{(2)}.$$

By the Minkowski inequality, the Hölder inequality, (7.3), (2.1), Remark 6.2(ii), (iv) and (iii) of Lemma 2.8, we know that

$$\left\|A_{k,j}^{(1)}\right\|_{L^q(\mu)} \leq |\lambda_{k,j}| \left\{ |m_{k,j}|_{L^q(\mu)} + \frac{\left[\mu(\text{supp}(m_{k,j}))\right]^{1/q}}{\mu(B_{k+2}^p)} \left[\int_{B_{k,j}} m_{k,j}(y) \, d\mu(y)\right] \right\}.$$
\[ \begin{aligned} \lesssim |\lambda_{k,j}| \left\{ \|m_{k,j}\|_{L^q(\mu)} + \frac{[\mu(\text{supp}(m_{k,j}))]^{1/q}[\mu(B_{k,j})]^{1/q^*}}{\mu(B_{k+2}^p)} \right\} \\
\lesssim |\lambda_{k,j}| \|m_{k,j}\|_{L^q(\mu)} \lesssim |\lambda_{k,j}|\rho^{-ke} [\mu(pB_{k,j})]^{1/q-1} \left[ \lambda \left( c_B, r_{B_{k+2}^p} \right) \right]^{1-1/p} \left[ \tilde{K}_{B_{k,j}, B_{k+2}^p}^{(\rho), p} \right]^{-\gamma} \\
\lesssim |\lambda_{k,j}| \rho^{-ke} [\mu(pB_{k,j})]^{1/q-1} \left[ \lambda \left( c_B, r_{2\tilde{B}_{k+2}^p} \right) \right]^{1-1/p} \left[ \tilde{K}_{B_{k,j}, 2\tilde{B}_{k+2}^p}^{(\rho), p} \right]^{-\gamma}. \end{aligned} \]

Let \( c_5 \), independent of \( k \) and \( j \), be the implicit positive constant of the above inequality,
\[ \mu_{k,j}^{(1)} := c_5|\lambda_{k,j}|\rho^{-ke} \]
and \( a_{k,j}^{(1)} := \frac{1}{\mu_{k,j}}A_{k,j}^{(1)} \). Then \( A_{k,j}^{(1)} = \mu_{k,j}^{(1)} a_{k,j}^{(1)} \), \( \text{supp}(a_{k,j}^{(1)}) \subset B_{k,j} \subset 2\tilde{B}_{k+2}^p \) and
\[ \left\| a_{k,j}^{(1)} \right\|_{L^q(\mu)} \leq \left[ \mu(pB_{k,j}) \right]^{1/q-1} \left[ \lambda \left( c_B, r_{2\tilde{B}_{k+2}^p} \right) \right]^{1-1/p} \left[ \tilde{K}_{B_{k,j}, 2\tilde{B}_{k+2}^p}^{(\rho), p} \right]^{-\gamma}. \]

From the Hölder inequality, (7.3), the fact that \( \tilde{K}_{B_{k,j}, B_{k+2}^p}^{(\rho), p} \geq 1 \), the \( (2\rho, 2\rho)-\)doubling property of \( \tilde{B}_{k+2}^p \), Remark 6.2(ii) and Lemma 2.8(ii), it follows that
\[ \left\| A_{k,j}^{(2)} \right\|_{L^q(\mu)} = |\lambda_{k,j}| \frac{[\mu(\tilde{B}_{k+2}^p \setminus \text{supp}(m_{k,j}))]^{1/q}}{\mu(\tilde{B}_{k+2}^p)} \int_{B_{k,j}} m_{k,j}(y) \, d\mu(y) \]
\[ \lesssim |\lambda_{k,j}| \left[ \mu \left( \tilde{B}_{k+2}^p \right) \right]^{1/q-1} \left[ \mu(B_{k,j}) \right]^{1/q} \|m_{k,j}\|_{L^q(\mu)} \]
\[ \lesssim |\lambda_{k,j}| \left[ \mu \left( \tilde{B}_{k+2}^p \right) \right]^{1/q-1} \left[ \mu(B_{k,j}) \right]^{1/q} \rho^{-ke} [\mu(pB_{k,j})]^{1/q-1} \left[ \lambda \left( c_B, r_{B_{k+2}^p} \right) \right]^{1-1/p} \]
\[ \lesssim |\lambda_{k,j}| \left[ \mu \left( \tilde{B}_{k+2}^p \right) \right]^{1/q-1} \rho^{-ke} \left[ \lambda \left( c_B, r_{B_{k+2}^p} \right) \right]^{1-1/p} \left[ \tilde{K}_{B_{k,j}, B_{k+2}^p}^{(\rho), p} \right]^{-\gamma}. \]

Let \( c_6 \), independent of \( k \) and \( j \), be the implicit positive constant of the above inequality,
\[ \mu_{k,j}^{(2)} := c_6|\lambda_{k,j}|\rho^{-ke} \]
and \( a_{k,j}^{(2)} := \frac{1}{\mu_{k,j}}A_{k,j}^{(2)} \). Then \( A_{k,j}^{(2)} = \mu_{k,j}^{(2)} a_{k,j}^{(2)} \), \( \text{supp}(a_{k,j}^{(2)}) \subset 2\tilde{B}_{k+2}^p \) and
\[ \left\| a_{k,j}^{(2)} \right\|_{L^q(\mu)} \leq \left[ \mu \left( 2\tilde{B}_{k+2}^p \right) \right]^{1/q-1} \left[ \lambda \left( c_B, r_{2\tilde{B}_{k+2}^p} \right) \right]^{1-1/p} \left[ \tilde{K}_{2\tilde{B}_{k+2}^p, 2\tilde{B}_{k+2}^p}^{(\rho), p} \right]^{-\gamma}. \]

Thus, \( b_{k,j} = \mu_{k,j}^{(1)} a_{k,j}^{(1)} + \mu_{k,j}^{(2)} a_{k,j}^{(2)} \) is a \((p, q, \gamma, \rho)\)-1-atomic block and
\[ |b_{k,j}|_{\tilde{B}_{k,j}^{p, q, \gamma, \rho}(\mu)} \lesssim |\lambda_{k,j}|\rho^{-ke}. \]

Moreover, we have
\[ \left\| |I|^{p}_{\tilde{B}_{k,j}^{p, q, \gamma, \rho}(\mu)} \right\| \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|\rho^{-kpe} \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|\rho \sim |b|^{p}_{\tilde{B}_{k,j}^{p, q, \gamma, \rho}(\mu)}. \quad (7.7) \]

Now we turn to estimate II. Observe that, by (7.6) and the Hölder inequality, we have
\[ \sum_{k=0}^{\infty} M_k \leq \sum_{k=0}^{\infty} |b_k|_{L^q(\mu)} < \infty. \]
For each $k \in \mathbb{Z}_+$, let $N_k := \sum_{i=k}^{\infty} \tilde{M}_i$. From the Hölder inequality and (7.3), it follows that
\[
\sum_{k=0}^\infty \| \chi_k N_k \|_{L^1(\mu)} \leq \sum_{k=0}^\infty \sum_{i=k}^\infty \| \chi_i \tilde{M}_i \|_{L^1(\mu)} \leq \sum_{k=0}^\infty \sum_{i=k}^\infty \| b_i \|_{L^1(\mu)}
\]
\[
\leq \sum_{k=0}^\infty \sum_{i=k}^\infty \sum_{j=1}^{M_i} | \lambda_{i,j} | \| m_{i,j} \|_{L^{q}(\mu)[\mu(B_{i,j})]^{1/q'}}
\]
\[
\leq \sum_{k=0}^\infty \sum_{i=k}^\infty \sum_{j=1}^{M_i} | \lambda_{i,j} | \rho^{-\kappa} [ \lambda (c_B, \rho^{i+2} r_B) ]^{1-1/p}
\]
\[
\leq \sum_{i=0}^\infty \rho^{-\kappa} \sum_{j=1}^{M_i} | \lambda_{i,j} | [ \lambda (c_B, r_B) ]^{1-1/p}
\]
\[
\leq \left( \sum_{i=0}^\infty \sum_{j=1}^{M_i} | \lambda_{i,j} |^p \right)^{1/p} [ \lambda (c_B, r_B) ]^{1-1/p} < \infty.
\]
Similarly, $\sum_{k=0}^\infty \| \chi_k N_{k+1} \|_{L^1(\mu)} < \infty$. By the above facts, we have
\[
\sum_{k=0}^\infty \chi_k \tilde{M}_k = \sum_{k=0}^\infty \chi_k (N_k - N_{k+1}) = \sum_{k=0}^\infty (\chi_{k+1} - \chi_k) N_{k+1} + \chi_0 N_0
\]
\[
= \sum_{k=0}^\infty (\chi_{k+1} - \chi_k) N_{k+1}
\]
\[
= \sum_{k=0}^\infty \sum_{i=k+1}^\infty \sum_{j=1}^{M_i} \lambda_{i,j} (\chi_{k+1} - \chi_k) \int_{B_{i,j}} m_{i,j}(y) \, d\mu(y)
\]
\[
= \sum_{i=0}^\infty \sum_{j=1}^{M_i} b_{k,j,i},
\]
where the summation in the last equality holds in $L^1(\mu)$.

Now we prove that, for any $k \in \mathbb{Z}_+$, $i \in \{k+1, k+2, \ldots \}$ and $j \in \{1, \ldots, M_k\}$, $b_{k,j,i}$ is a $(p, \infty, \gamma, \rho)$-atomic block. Observing that $\text{supp } (b_{k,j,i}) \subset 2B_{k+3}^i$ and $\int_{B_{k,j,i}}(y) \, d\mu(y) = 0$, we only need to show that $b_{k,j,i}$ satisfies Definition 3.2(iii).

To this end, we further write
\[
b_{k,j,i} = \lambda_{i,j} \chi_{k+1} \int_{B_{i,j}} m_{i,j}(y) \, d\mu(y) - \lambda_{i,j} \chi_k \int_{B_{i,j}} m_{i,j}(y) \, d\mu(y) =: A^{(1)}_{k,j,i} - A^{(2)}_{k,j,i}.
\]
From the Hölder inequality, (7.3), the fact that $K^{(p),p}_{B_{k+3}^i, B_{k+4}^i} \geq 1$, Remark 6.2(ii) and Lemma 2.8(ii), we deduce that
\[
\left\| A^{(1)}_{k,j,i} \right\|_{L^\infty(\mu)} \leq | \lambda_{i,j} | \left[ \frac{1}{\mu(B_{k+3}^i)} \right]^{1/q'} | m_{i,j} \|_{L^{q}(\mu)}
\]
\[
\leq | \lambda_{i,j} | \left[ \frac{1}{\mu(B_{k+3}^i)} \right]^{1/q'} \left[ \mu(B_{k+4}^i) \right]^{1/q-1} \left[ \lambda (c_B, r_{B_{k+2}^i}) \right]^{1-1/p} \left[ K^{(p),p}_{B_{k+3}^i, B_{k+4}^i} \right]^{-\gamma}
\]
\[
\leq | \lambda_{i,j} | \rho^{-\kappa} \left[ \mu (2B_{k+3}^i) \right]^{1-1/p} \left[ \lambda (c_B, r_{B_{k+2}^i}) \right]^{1-1/p} \left[ K^{(p),p}_{2B_{k+3}^i, 2B_{k+4}^i} \right]^{-\gamma},
\]
Let $c_7$, independent of $k$ and $j$, be the implicit positive constant of the above inequality,
\[
\mu^{(1)}_{k,j,i} := c_7 | \lambda_{i,j} | \rho^{-\kappa}
\]
and $a_{k,j,i}^{(1)} := \frac{1}{\mu_{k,j,i}} A_{k,j,i}^{(1)}$. Then we see that $A_{k,j,i}^{(1)} = \mu_{k,j,i} a_{k,j,i}^{(1)}$, $\operatorname{supp}(a_{k,j,i}^{(1)}) \subset 2\tilde{B}^{p}_{k+3}$ and

$$
\left\| a_{k,j,i}^{(1)} \right\|_{L^{\infty}(\mu)} \leq \left[ \mu \left( 2\rho \tilde{B}^{p}_{k+3} \right) \right]^{-1} \left[ \lambda \left( \rho \tilde{B}^{p}_{k+3} \right) \right]^{1-1/p} \left[ \tilde{K}_{p, \frac{1}{2} + \frac{1}{2}}(\rho, p, 2\tilde{B}^{p}_{k+3}, 2\tilde{B}^{p}_{k+3}) \right]^{-\gamma}.
$$

By an argument similar to that used in the estimate for $A_{k,j,i}^{(1)}$, we conclude that

$$
\left\| A_{k,j,i}^{(2)} \right\|_{L^{\infty}(\mu)} \leq c_8 |\lambda_{i,j}|^{|\rho|^{-i}} \left[ \mu \left( 2\rho \tilde{B}^{p}_{k+2} \right) \right]^{-1} \left[ \lambda \left( \rho \tilde{B}^{p}_{k+2} \right) \right]^{1-1/p} \left[ \tilde{K}_{p, \frac{1}{2} + \frac{1}{2}}(\rho, p, 2\tilde{B}^{p}_{k+2}, 2\tilde{B}^{p}_{k+2}) \right]^{-\gamma},
$$

where $c_8$ is a positive constant independent of $k, j$ and $i$. Let $\mu_{k,j,i}^{(2)} := c_8 |\lambda_{i,j}|^{|\rho|^{-i}}$ and

$$
a_{k,j,i}^{(2)} := \frac{1}{\mu_{k,j,i}^{(2)}} A_{k,j,i}^{(2)}.
$$

Then $A_{k,j,i}^{(2)} = \mu_{k,j,i} a_{k,j,i}^{(2)}$, $\operatorname{supp}(a_{k,j,i}^{(2)}) \subset 2\tilde{B}^{p}_{k+2} \subset 2\tilde{B}^{p}_{k+3}$ and

$$
\left\| a_{k,j,i}^{(2)} \right\|_{L^{\infty}(\mu)} \leq \left[ \mu \left( 2\rho \tilde{B}^{p}_{k+2} \right) \right]^{-1} \left[ \lambda \left( \rho \tilde{B}^{p}_{k+2} \right) \right]^{1-1/p} \left[ \tilde{K}_{p, \frac{1}{2} + \frac{1}{2}}(\rho, p, 2\tilde{B}^{p}_{k+2}, 2\tilde{B}^{p}_{k+2}) \right]^{-\gamma}.
$$

Thus, $B_{k,j,i} = b_{k,j,i}^{(1)} A_{k,j,i}^{(1)} + \mu_{k,j,i}^{(2)} A_{k,j,i}^{(2)}$ is a $(p, \infty, \gamma, \rho)\lambda, 1$-atomic block and

$$
|b_{k,j,i}|_{\tilde{H}^{p, \infty, \gamma}_{\lambda, \rho}(\mu)} \lesssim |\lambda_{i,j}|^{|\rho|^{-i}}.
$$

Moreover, we have

$$
\|\Pi\|_{\tilde{H}^{p, \infty, \gamma}_{\lambda, \rho}(\mu)} \lesssim \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \sum_{j=1}^{M_i} b_{k,j,i} \|b_{k,j,i}\|_{\tilde{H}^{p, \infty, \gamma}_{\lambda, \rho}(\mu)}
\lesssim \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \rho^{-i/p} \sum_{j=1}^{M_i} |\lambda_{i,j}|^{|\rho|^{-i}} \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M_i} |\lambda_{i,j}|^{|\rho|^{-i}} \|b_{\tilde{H}^{p, \infty, \gamma, \rho}}\|_{\lambda, \rho}(\mu).
$$

From this fact and (7.7), we deduce that both $\sum_{k=0}^{\infty} \sum_{j=1}^{M_i} b_{k,j,i}$ and $\sum_{k=0}^{\infty} \sum_{i=k+1}^{M_i} \sum_{j=1}^{M_i} b_{k,j,i}$ converge in $(\mathcal{E}_{\rho}^{1/p-1}(\mu))^\ast$.

Now we claim that

$$
b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_i} b_{k,j} \quad \text{in} \quad (\mathcal{E}_{\rho}^{1/p-1}(\mu))^\ast.
$$

Indeed, by $b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_i} \lambda_{k,j} m_{k,j}$ in $(\mathcal{E}_{\rho}^{1/p-1}(\mu))^\ast$, we see that, for any $g \in \mathcal{E}_{\rho}^{1/p-1}(\mu)$,

$$
\int_X b(x)g(x) \, d\mu(x) = \lim_{K \to \infty} \sum_{k=0}^{M_k} \sum_{j=1}^{M_i} \int_X \lambda_{k,j} m_{k,j}(x)g(x) \, d\mu(x)
\quad = \lim_{K \to \infty} \sum_{k=0}^{M_k} \sum_{j=1}^{M_i} \int_X b_{k,j}(x)g(x) \, d\mu(x) + \lim_{K \to \infty} \sum_{k=0}^{K} \tilde{M}_k \int_X \chi_k(x)g(x) \, d\mu(x).
$$

Moreover, by the fact that $|N_k| < \infty$ for any $k \in \mathbb{Z}_+$ and $N_0 = 0$, we further write

$$
\lim_{K \to \infty} \sum_{k=0}^{K} \tilde{M}_k \int_X \chi_k(x)g(x) \, d\mu(x)
$$
Thus, to prove the above claim, it suffices to show that \( A = 0 \). To this end, by the H"older inequality and (7.3), we conclude that, for any \( K \in \mathbb{N} \),

\[
|N_{K+1}| \leq \sum_{i=K+1}^{\infty} |\widetilde{M}_i| \leq \sum_{i=K+1}^{\infty} \int_X |b_i(y)| \, d\mu(y)
\]

\[
\leq \sum_{i=K+1}^{\infty} \sum_{j=1}^{M_i} |\lambda_{i,j}| \int_{B_{i,j}} |m_{i,j}(y)| \, d\mu(y)
\]

\[
\leq \sum_{i=K+1}^{\infty} \sum_{j=1}^{M_i} |\lambda_{i,j}| |\mu(B_{i,j})|^{1/q} \|m_{i,j}\|_{L^q(\mu)}
\]

\[
\leq \sum_{i=K+1}^{\infty} \sum_{j=1}^{M_i} |\lambda_{i,j}| |\mu(B_{i,j})|^{1/q} \rho^{-i\epsilon} |\mu(\rho B_{i,j})|^{1/q-1} \lambda(c_B, \rho^{i+2} r_B)^{1-1/p}
\]

\[
\leq \sum_{i=K+1}^{\infty} \sum_{j=1}^{M_i} |\lambda_{i,j}| \rho^{-K\epsilon} \lambda(c_B, \rho^{K+2} r_B)^{1-1/p}
\]

\[
\leq \rho^{-K\epsilon} \lambda(c_B, \rho^{K+2} r_B)^{1-1/p} \left( \sum_{i=K+1}^{\infty} \sum_{j=1}^{M_i} |\lambda_{i,j}|^p \right)^{1/p} . \tag{7.8}
\]

We write

\[
\int_X \chi_K(x) N_{K+1} g(x) \, d\mu(x) = |N_{K+1}| \left| m_{\widetilde{\beta}_K}^0(g) \right|
\]

\[
\leq |N_{K+1}| \left| m_{\widetilde{\beta}_K}^0(g) - m_{\beta_0}^0(g) \right| + |N_{K+1}| \left| m_{\beta_0}^0(g) \right|
\]

\[
= I_K + II_K .
\]

By (7.8), \( g \in L^1_{\text{loc}}(\mu) \) and \( \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{i,j}|^p < \infty \), we know that

\[
II_K \leq \left| m_{\beta_0}^0(g) \right| \rho^{-K \epsilon} \lambda(c_B, r_B)^{1-1/p} \left( \sum_{i=K+1}^{\infty} \sum_{j=1}^{M_i} |\lambda_{i,j}|^p \right)^{1/p} \to 0 \quad \text{as} \quad K \to \infty .
\]

From Proposition 6.7(a), \( g \in \mathcal{E}_p^{1/p-1}(\mu) \), (7.8) and \( \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{i,j}|^p < \infty \), we further deduce that

\[
I_K \leq |N_{K+1}| \left[ \widetilde{\beta}_K(g, \rho^{K+2} r_B) \right]^{\gamma} \lambda(c_B, \rho^{K+2} r_B)^{1/p-1} \left\| g \right\|_{\mathcal{E}_p^{1/p-1}(\mu)}
\]

\[
\lesssim |N_{K+1}| K^{\gamma/p} \left[ \lambda(c_B, \rho^{K+2} r_B) \right]^{1/p-1} \left\| g \right\|_{\mathcal{E}_p^{1/p-1}(\mu)}
\]
Let \( b \in H^{p, q, \gamma}_\text{atb, } \rho (\mu) \) and
\[
\| b \|_{H^{p, q, \gamma}_\text{atb, } \rho (\mu)} \leq \| f \|_{\tilde{H}^{p, q, \gamma}_\text{atb, } \rho (\mu)} + \| f \|_{H^{p, q, \gamma}_\text{atb, } \rho (\mu)} + \| f \|_{H^{p, \infty, \gamma}_\text{atb, } \rho (\mu)} \leq \| b \|_{H^{p, q, \gamma}_\text{atb, } \rho (\mu)} ,
\]
which, together with some standard arguments, then completes the proof of Theorem 7.9.

\[ \Box \]

Remark 7.10. (i) As was pointed out in the proof of Theorem 7.9, if \( \rho \in (1, \infty) \), \( q \in (1, \infty) \), \( \gamma \in [1, \infty) \) and \( \epsilon \in (0, \infty) \), then \( \tilde{H}^{1, q, \gamma}_\text{atb, } \rho (\mu) \) and \( \tilde{H}^{1, q, \gamma, \epsilon}_\text{atb, } \rho (\mu) \) coincide with equivalent norms, which is just [13, Theorem 1.11]; namely, in this case, the assumption (6.1) is superfluous. However, when \( p \in (0, 1) \), without (6.1), it is still unclear whether Theorem 7.9 holds true or not.

(ii) By Theorem 7.9, we see that \( \tilde{H}^{p, q, \gamma, \epsilon}_\text{mb, } \rho (\mu) \) is independent of the choice of \( \epsilon \) under the assumption (6.1).

The following result is an easy consequence of Theorem 7.9 and Remark 6.2(i), the details being omitted.

Corollary 7.11. Let \((X, d, \mu)\) be a space of homogeneous type with the dominating function
\[
\lambda(x, r) := \mu(B(x, r)) \quad \text{for all } x \in X \text{ and } r \in (0, \infty),
\]
and \( \rho, p, q, \gamma \) and \( \epsilon \) be as in Theorem 7.9. Then the conclusions in Theorem 7.9 and Remark 7.10 also hold true in this setting.

8 Duality between \( \tilde{H}^{p, q, \gamma}_\text{atb, } \rho (\mu) \) and \( \mathcal{E}^{1/p-1}_\rho (\mu) \)

In this section, we show that \( \mathcal{E}^{1/p-1}_\rho (\mu) \) is the dual space of \( \tilde{H}^{p, q, \gamma}_\text{atb, } \rho (\mu) \). To this end, assuming that \((X, d, \mu)\) satisfies the assumption (6.1), we show that \( \tilde{H}^{p, q, \gamma}_\text{atb, } \rho (\mu) \) is independent of the choices of \( \rho \) and \( \gamma \).

We point out that all conclusions in this section hold true for the case \( p = 1 \) without the assumption (6.1); see [13, 30] for the details. Thus, we mainly focus on \( p \in (0, 1) \) in this section.

Proposition 8.1. Suppose that \((X, d, \mu)\) is a non-homogeneous metric measure space satisfying (6.1). Let \( \rho \in (1, \infty) \), \( 0 < p < 1 \leq q \leq \infty \) and \( \gamma \in [1, \infty) \). Then the space \( \tilde{H}^{p, q, \gamma}_\text{atb, } \rho (\mu) \) is independent of the choice of \( \rho \in (1, \infty) \).

Proof. Let \( 0 < p < 1 \leq q \leq \infty \) and \( \gamma \in [1, \infty) \). Assume that \( \rho \geq \rho_1 > \rho_2 > 1 \). It is easy to see that \( \tilde{H}^{p, q, \gamma}_\text{atb, } \rho_1 (\mu) \subset \tilde{H}^{p, q, \gamma}_\text{atb, } \rho_2 (\mu) \) and, for all \( f \in \tilde{H}^{p, q, \gamma}_\text{atb, } \rho_1 (\mu) \),
\[
\| f \|_{\tilde{H}^{p, q, \gamma}_\text{atb, } \rho_2 (\mu)} \leq \| f \|_{\tilde{H}^{p, q, \gamma}_\text{atb, } \rho_1 (\mu)}.\]

On the other hand, to show that \( \tilde{H}^{p, q, \gamma}_\text{atb, } \rho_1 (\mu) \subset \tilde{H}^{p, q, \gamma}_\text{atb, } \rho_2 (\mu) \), let
\[
b = \sum_{j=1}^{2} \lambda_j a_j \in \tilde{H}^{p, q, \gamma}_\text{atb, } \rho_2 (\mu)
\]
be a \((p, q, \gamma, \rho_2)\)-atomic block, where, for any \( j \in \{1, 2\} \), \( a_j \) is a function supported on \( B_j \subset B \) for some balls \( B_j \) and \( B \) as in Definition 7.1.

Now we claim that, without loss of generality, we may assume that \( B \) is \((\rho^2, \beta, \rho^2)\)-doubling. The reasons are as follows: if \( B \) is non-\((\rho^2, \beta, \rho^2)\)-doubling, by Lemma 2.9, (iv) and (ii) of Lemma 2.8, (2.1) and Remark 6.2(ii), we see that
\[
\| a_j \|_{L^q(\mu)} \leq [\mu(\rho_2 B_j)]^{1/q-1} \| \lambda(e_B, r_B) \|^{1-1/p} \left[ \frac{\mu(B_j)}{\mu(B)} \right]^{-\gamma}.
\]
Thus, we can replace $B$ by $\tilde{B}^{p_2}$, which shows the claim.

Then, for each $j \in \{1, 2\}$, we have

$$\|a_j\|_{L^q(\mu)} \leq [\mu(p_2 B_j)]^{1/q-1} [\lambda(c_B, r_B)]^{1-1/p} \left[ \tilde{K}^{(p_2)}_{(B_j, B)} \right]^{-\gamma}. \quad (8.1)$$

From Remark 2.2(ii), it follows that there exists a sequence $\{B_{k, j}\}_{k=1}^N$ of balls such that

$$B_j \subset \bigcup_{k=1}^N B_{k, j} = \bigcup_{k=1}^N \left( c_{B_{k, j}}, \frac{p_2 - 1}{10r_0(p_1 + 1)} r_B \right)$$

and $c_{B_{k, j}} \in B_j$ for all $k \in \{1, \ldots, N\}$, where $r_0 \in (1, p_1)$. Observe that $p_1 r_0 B_{k, j} \subset p_2 B_j$. For any $k \in \{1, \ldots, N\}$, define $a_{k, j} := a_j \frac{\lambda_{B_{k, j}}}{\sum_{k=1}^N \lambda_{B_{k, j}}}$ and $\lambda_{k, j} := \lambda_j$. Then we have

$$\text{supp} (a_{k, j}) \subset r_0 B_{k, j} \quad \text{and} \quad b = \sum_{j=1}^2 \lambda_j a_j = \sum_{j=1}^2 \sum_{k=1}^N \lambda_{k, j} a_{k, j}.$$ 

Moreover, by (8.1), the fact that $p_2 B_j \subset 3r_B$, (2.1), Lemma 2.9, (i), (ii), (iv) and (v) of Lemma 2.8 and the fact that $p_0 B_{k, j} \subset \rho B$, we know that

$$\|a_{k, j}\|_{L^q(\mu)} \leq \|a_j\|_{L^q(\mu)} \leq [\mu(p_2 B_j)]^{1/q-1} [\lambda(c_B, r_B)]^{1-1/p} \left[ \tilde{K}^{(p_2)}_{(B_j, B)} \right]^{-\gamma}$$

$$\leq [\mu(p_1, p_0 B_j)]^{1/q-1} [\lambda(c_B, r_B)]^{1-1/p} \left[ \tilde{K}^{(p_1)}_{(B_j, 3r_B)} \right]^{-\gamma}$$

$$\leq [\mu(p_1, p_0 B_j)]^{1/q-1} [\lambda(c_B, r_B)]^{1-1/p} \left[ \tilde{K}^{(p_1)}_{(p_0, r_B)} \right]^{-\gamma}. \quad (8.2)$$

Let $C_{k, j} := \lambda_{k, j} (a_{k, j} + \gamma_j x B)$, where $\gamma_j := -\frac{1}{(p_2)^{1/q}} \int_X a_{k, j}(x) \, d\mu(x)$. Now we claim that $C_{k, j}$ is a $(p, q, \gamma, p_1)$-atomic block. Indeed, $\text{supp} (C_{k, j}) \subset \rho B$ and $\int_X C_{k, j}(x) \, d\mu(x) = 0$. Moreover, since $B_{k, j} \subset \rho B$, by the Hölder inequality, (8.2), $B$ is $(p^2, \beta, p)$-doubling, $\rho > p_1$, (2.1) and Lemma 2.8(ii), we conclude that

$$\|\gamma_j x B\|_{L^q(\mu)} \leq [\mu(B)]^{1/q-1} \|a_{k, j}\|_{L^q(\mu)} \|\mu(B_{k, j})\|^{1-1/q}$$

$$\leq [\mu(p_1, p_0 B_j)]^{1/q-1} [\lambda(c_B, r_B)]^{1-1/p} \left[ \tilde{K}^{(p_1)}_{(B_j, p_0 r_B)} \right]^{-\gamma}$$

$$\leq [\mu(p_1, p_0 B_j)]^{1/q-1} [\lambda(c_B, r_B)]^{1-1/p} \left[ \tilde{K}^{(p_1)}_{(p_0, r_B)} \right]^{-\gamma}. \quad (8.3)$$

This, together with (8.2), (2.1), Lemma 2.8(ii), implies that $|C_{k, j}|_{H^{p, q, \gamma}_B(x, r_1, p_1)} \lesssim |\lambda_{k, j}|$. Thus, the claim holds true.

By the above claim and $\int_X b(x) \, d\mu(x) = 0$, we see that

$$b = \sum_{j=1}^2 \sum_{k=1}^N C_{k, j} \in \tilde{H}^{p, q, \gamma}_{atb}(\mu) \quad (8.3)$$

and

$$\|b\|_{H^{p, q, \gamma}_{atb}(\mu)} \lesssim \sum_{j=1}^2 \sum_{k=1}^N |C_{k, j}|_{H^{p, q, \gamma}_{atb}(\mu)} \lesssim \sum_{j=1}^2 |\lambda_j|^p \sim \|b\|_{H^{p, q, \gamma}_{atb}(\mu)} \quad (8.4)$$
For all \( f \in \hat{H}^{p,q,\gamma}_{\text{atb},\rho_2} (\mu) \), by Proposition 6.8, we know that there exists a sequence \( \{b_i\}_1 \) of \((p,q,\gamma,\rho_2)\)-atomic blocks such that \( f = \sum_{i=1}^{\infty} b_i (E^{\frac{1}{p}-1}_{\rho_2})^* = (E^{\frac{1}{p}-1}_{\rho_1})^* \) and
\[
\sum_{i=1}^{\infty} \|b_i\|^p_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho_2} (\mu)} \lesssim \|f\|^p_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho_2} (\mu)}.
\]
From this fact, (8.3) and (8.4), we further deduce that \( f = \sum_{i=1}^{\infty} \sum_{j=1}^{2} \sum_{k=1}^{N} C_{k,j}^i \) in \((E^{1/p-1}_{\rho_2})^*\), where \{\(C_{k,j}^i\)\} are all \((p,q,\gamma,\rho_1)\)-atomic blocks as in (8.3) satisfying
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{2} \sum_{k=1}^{N} \|C_{k,j}^i\|^p_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho_1} (\mu)} \lesssim \sum_{i=1}^{\infty} \|b_i\|^p_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho_2} (\mu)} \lesssim \|f\|^p_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho_2} (\mu)}.
\]
which implies that \( f \in \hat{H}^{p,q,\gamma}_{\text{atb},\rho_1} (\mu) \) and
\[
\|f\|^p_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho_1} (\mu)} \lesssim \|f\|^p_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho_2} (\mu)}.
\]
This finishes the proof of Proposition 8.1.

**Proposition 8.2.** Let \( \rho \in (1, \infty) \), \( 0 < p < 1 \leq q \leq \infty \) and \( \gamma \in [1, \infty) \). Then the space \( \hat{H}^{p,q,\gamma}_{\text{atb},\rho} (\mu) \) is independent of the choice of \( \gamma \) in \([1, \infty)\).

**Proof.** Assume that \( 1 \leq \gamma_1 < \gamma_2 \). Notice that \( \left[ K_{B,S}^{(\rho),p} \right]^{-\gamma_2} \subseteq \left[ K_{B,S}^{(\rho),p} \right]^{-\gamma_1} \) for all balls \( B \subset S \). From this, we deduce that \( \hat{H}^{p,q,\gamma_2}_{\text{atb},\rho} (\mu) \subseteq \hat{H}^{p,q,\gamma_1}_{\text{atb},\rho} (\mu) \) and, for all \( f \in \hat{H}^{p,q,\gamma_2}_{\text{atb},\rho} (\mu) \) and \( f \in \hat{H}^{p,q,\gamma_1}_{\text{atb},\rho} (\mu) \)
\[
\|f\|^p_{\hat{H}^{p,q,\gamma_1}_{\text{atb},\rho} (\mu)} \lesssim \|f\|^p_{\hat{H}^{p,q,\gamma_2}_{\text{atb},\rho} (\mu)}.
\]
Now we consider the following converse inclusion that \( \hat{H}^{p,q,\gamma_1}_{\text{atb},\rho} (\mu) \subseteq \hat{H}^{p,q,\gamma_2}_{\text{atb},\rho} (\mu) \). Let
\[
b = \sum_{j=1}^{2} \lambda_j a_j \in \hat{H}^{p,q,\gamma_1}_{\text{atb},\rho} (\mu)
\]
be a \((p,q,\gamma_1,\rho)\)-atomic block, where, for any \( j \in \{1,2\} \), \( a_j \) is a function supported on \( B_j \subset B \) for some balls \( B_j \) and \( B \) as in Definition 7.1. We first show that any \((p,q,\gamma_1,\rho)\)-atomic block can be decomposed into a sum of some \((p,q,\gamma_2,\rho)\)-atomic blocks and
\[
\|b\|^p_{\hat{H}^{p,q,\gamma_2}_{\text{atb},\rho} (\mu)} \lesssim \|b\|^p_{\hat{H}^{p,q,\gamma_1}_{\text{atb},\rho} (\mu)}.
\]
To prove (8.5), we consider the following four cases:

**Case (I)** For any \( j \in \{1,2\} \), \( \tilde{K}_{B_j,B}^{(\rho),p} \leq \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^1/p \), where \( C_{(\rho)} \) is as in Lemma 2.8(i);

**Case (II)** \( K_{B_j,B}^{(\rho),p} \geq \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^1/p \) and \( \tilde{K}_{B_j,B}^{(\rho),p} \leq \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^1/p \);

**Case (III)** \( K_{B_j,B}^{(\rho),p} \leq \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^1/p \) and \( \tilde{K}_{B_j,B}^{(\rho),p} \geq \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^1/p \);

**Case (IV)** For any \( j \in \{1,2\} \), \( \tilde{K}_{B_j,B}^{(\rho),p} > \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^1/p \).

In **Case (I)**, for any \( j \in \{1,2\} \), we have
\[
\left[ \hat{K}_{B_j,B}^{(\rho),p} \right]^{-\gamma_2} < 1 = \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^{\gamma_2/p} \cdot \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^{-\gamma_2/p} \leq \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^{\gamma_2/p} \left[ \hat{K}_{B_j,B}^{(\rho),p} \right]^{-\gamma_2} \cdot 2.
\]
For any \( j \in \{1,2\} \), let \( \tilde{x}_j := \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^{\gamma_2/p} \lambda_j \) and \( \tilde{a}_j := \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^{-\gamma_2/p} a_j \). Then \( b = \tilde{x}_1 \tilde{a}_1 + \tilde{x}_2 \tilde{a}_2 \). From this, it is easy to see that \( b \) is a \((p,q,\gamma_2,\rho)\)-atomic block, which implies that \( b \in \hat{H}^{p,q,\gamma_2}_{\text{atb},\rho} (\mu) \) and
\[
\|b\|^p_{\hat{H}^{p,q,\gamma_2}_{\text{atb},\rho} (\mu)} \leq \left( 3 + [\log_2 2] \right) C_{(\rho)} \right)^{\gamma_2/p} (|\lambda_1| + |\lambda_2|) \sim \|b\|^p_{\hat{H}^{p,q,\gamma_1}_{\text{atb},\rho} (\mu)}.
\]
in this case.

The proofs of Case (II), Case (III) and Case (IV) are similar. For brevity, we only prove Case (II).

In Case (II), we have $K_{B_1, B}^{(p), p} > [(3 + [\log_p 2])C(\rho)]^{1/p}$. We now choose a sequence $\{B_1^{(i)}\}_{i=0}^m$ of balls with certain $m \in \mathbb{N}$ as follows. Let $B_1^{(0)} := B_1$ and $B_0 := \rho_{B_1}^{-1/\nu} B_1$. To choose $B_1^{(1)}$, let $N_1$ be the smallest positive integer satisfying $K_{B_1^{(0)}, \rho N_1 B_1^{(0)}}^{(p), p} > [(3 + [\log_p 2])C(\rho)]^{1/p}$. If $r_{\rho N_1 B_1^{(0)}, B_0} \geq r_{B_0}$, then we let $B_1^{(1)} := B_0$ and the selection process terminates. Otherwise, we let $B_1^{(1)} := B_1^{(0)}$. To choose $B_1^{(1)}$, we have the process will stop after $m$. For any $N \in \mathbb{N}$, $K_{B_1^{(0)}, \rho N B_1^{(0)}}^{(p), p} \leq [(3 + [\log_p 2])C(\rho)]^{1/p}$, let $B_1^{(2)} := B_0$ and the selection process terminates. Otherwise, let $N_2$ be the smallest positive integer satisfying $K_{B_1^{(1)}, \rho N_2 B_1^{(1)}}^{(p), p} > [(3 + [\log_p 2])C(\rho)]^{1/p}$. If $r_{\rho N_2 B_1^{(1)}, B_0} \geq r_{B_0}$, then we let $B_1^{(2)} := B_0$ and the selection process terminates. Otherwise, we let $B_1^{(2)} := \rho N_2 B_1^{(1)}$. We continue as long as this selection process is possible; clearly, finally the condition $r_{\rho^2 N B_1^{(m)}, B_0} \geq r_{B_0}$ is violated after finitely many steps. Without loss of generality, we may assume that the process will stop after $m$ ($m \in \mathbb{N} \cap (1, \infty)$) steps. Now we conclude that $\{B_1^{(i)}\}_{i=0}^m$ have the following properties:

(i) $B_1^{(0)} := B_1$, $B_1^{(i)} := \rho_{N_i} B_1^{(i-1)}$ for any $i \in \{1, \ldots, m-1\}$, and $B_1^{(m)} := B_0$;

(ii) for any $i \in \{1, \ldots, m-1\}$, by Lemma 2.8(i) and the definition of $N_i$, we have

$$K_{B_1^{(i-1)}, B_1^{(i)}}^{(p), p} \geq [C(\rho)]^{-1/p} \rho_{N_i B_1^{(i-1)}}^{(p), p} > (3 + [\log_p 2])^{1/p};$$

(iii) there exists a positive constant $C$ such that, for any $i \in \{1, \ldots, m\}$, $K_{B_1^{(m-1)}, B_1^{(i)}}^{(p), p} \leq C$. Indeed, if, for any $N \in \mathbb{N}$, $K_{B_1^{(m-1)}, \rho N_i B_1^{(i-1)}}^{(p), p} \leq [(3 + [\log_p 2])C(\rho)]^{1/p}$, then, from the choice of $B_1^{(m)}$, we have $K_{B_1^{(m-1)}, B_1^{(m)}} \leq 1$. Otherwise, by Lemma 2.9, (iv), (ii) and (iii) of Lemma 2.8 and the definition of $N_i$, we see that, for any $i \in \{1, \ldots, m\}$,

$$K_{B_1^{(i-1)}, B_1^{(i)}}^{(p), p} \leq 2^{1-p} \left[ K_{B_1^{(i-1)}, \rho N_i B_1^{(i-1)}}^{(p), p} + C(\rho, p, \nu) K_{B_1^{(i-1)}, \rho N_i B_1^{(i-1)}}^{(p), p} \right] \leq C;$$

(iv) by (ii), Lemma 6.11, the fact that $B_1^{m-1} \subset B_1^{(m)} \subset 2\rho \tilde{C}_{1+1} B$, and (i), (iv) and (ii) of Lemma 2.8, where $\tilde{C}_1$ is as in (6.1), we know that

$$m = (m - 2) + 2 \leq \sum_{i=1}^{m-2} \left[ K_{B_1^{(i)}, B_1^{(i+1)}}^{(p), p} \right] + 2 < (3 + [\log_p 2]) \left[ K_{B_1, B_1^{m-1}}^{(p), p} \right] + 2$$

$$\leq \left[ K_{B_1, 2\rho^{2\tilde{C}_{1+1} B}}^{(p), p} \right] + \left[ K_{B_1, 2\rho^{2\tilde{C}_{1+1} B}}^{(p), p} \right] \leq \left[ K_{B_1, B}^{(p), p} \right].$$

Let $C$ be the implicit positive constant of the above inequality, $\tilde{C}_0)^{p} := C[\tilde{K}_{B_1, B}^{(p), p}]^{p}$ and $\tilde{c}_0 := \tilde{C}_0 a_1$. For any $i \in \{1, \ldots, m\}$, let

$$\tilde{c}_i := \frac{\lambda_{B_1^{(i)}}}{\mu(B_1^{(i)})} \int_X \tilde{c}_{i-1}(y) \mu(y).$$

If $i = 0$, by Definition 3.2(iii), (2.2), $r_{B_1^{(i)}, B_0} \leq \rho^{2\tilde{C}_{1+1} B}$, (2.1) and (iii), we have

$$\|\tilde{c}_0\|_{L^q(\mu)} \leq \mu(B_1) \tilde{c}_0^{1/q-1} \lambda(\tilde{C}_0) \tilde{c}_0^{1-1/p} \left[ K_{B_1, B}^{(p), p} \right]^{-\gamma_{i+1}}.$$
\[
\lesssim \left[ \mu \left( \rho B_1^{(0)} \right) \right]^{1/q-1} \left[ \lambda \left( c_{B_1}, r_{B_1} \right) \right]^{1-1/p}
\lesssim \left[ \mu \left( \rho B_1^{(0)} \right) \right]^{1/q-1} \left[ \lambda \left( c_{B_1}, r_{\rho B_1^{(1)}} \right) \right]^{1-1/p} \left[ \tilde{K}_{B_1^{(0)}, \rho B_1^{(1)}}^{(p), p} \right]^{-\gamma_2},
\]  

(8.6)

where the implicit positive constant is independent of \( \tilde{K}_{B_1^{(0)}, B'}^{(p), p} \). For \( i = 1 \), by the Hölder inequality, Definition 3.2(iii), (2.2), the facts that \( r_{B_1^{(1)}} \leq \rho \tilde{c}_{B_1^{(1)}} \) and \( B_1^{(1)} \) is doubling, (2.1) and Lemma 2.8(ii), we conclude that

\[
\| \tilde{c}_i \|_{L^q(\mu)} \lesssim \left[ \mu \left( \rho B_1^{(1)} \right) \right]^{1/q-1} \left[ \mu(B_1) \right]^{1-1/q} \left\| \tilde{C}_0 a_1 \right\|_{L^q(\mu)}
\lesssim \left[ \mu \left( \rho B_1^{(1)} \right) \right]^{1/q-1} \left[ \mu(B_1) \right]^{1-1/q} \left[ \lambda(c_{B_1}, r_{B_1}) \right]^{1-1/p} \left[ \tilde{K}_{B_1^{(0)}, B_1}^{(p), p} \right]^{-\gamma_1+1}
\lesssim \left[ \mu \left( \rho B_1^{(1)} \right) \right]^{1/q-1} \left[ \lambda(c_{B_1}, r_{B_1}) \right]^{1-1/p}
\lesssim \left[ \mu \left( \rho^2 B_1^{(1)} \right) \right]^{1/q-1} \left[ \lambda(c_{B_1}, r_{\rho B_1^{(1)}}) \right]^{1-1/p} \left[ \tilde{K}_{\rho B_1^{(1)}, \rho B_1^{(1)}}^{(p), p} \right]^{-\gamma_2}.
\]  

(8.7)

Similar to (8.6) and (8.7), respectively, for any \( i \in \{2, \ldots, m\} \), we have

\[
\| \tilde{c}_{i-1} \|_{L^q(\mu)} \lesssim \left[ \mu \left( \rho^2 B_1^{(i-1)} \right) \right]^{1/q-1} \left[ \lambda \left( c_{B_1}, r_{\rho B_1} \right) \right]^{1-1/p} \left[ \tilde{K}_{\rho B_1^{(i-1)}, \rho B_1^{(i)}}^{(p), p} \right]^{-\gamma_2}
\]  

(8.8)

and

\[
\| \tilde{c}_i \|_{L^q(\mu)} \lesssim \left[ \mu \left( \rho^2 B_1^{(i)} \right) \right]^{1/q-1} \left[ \lambda \left( c_{B_1}, r_{\rho B_1^{(i)}} \right) \right]^{1-1/p} \left[ \tilde{K}_{\rho B_1^{(i)}, \rho B_1^{(i)}}^{(p), p} \right]^{-\gamma_2}.
\]  

(8.9)

For any \( i \in \{1, \ldots, m\} \), let \( c_i := \frac{\lambda}{C_b} (\tilde{c}_{i-1} - \tilde{c}_i) \). Then \( \text{supp}(c_i) \subset \rho B_1^{(i)} \) and

\[
\int_X c_i(x) \, d\mu(x) = 0,
\]

which, together with (8.8) and (8.9), implies that \( c_i \) is a \( (p, q, \gamma_2, \rho) \lambda_1 \)-atomic block associated with the ball \( \rho B_1^{(i)} \) and

\[
|c_i|_{B^{p, q, \gamma_2} \sigma, \rho} (\mu) \lesssim \frac{|\lambda_1|}{C_b}.
\]  

(8.10)

Now we see that

\[
b = \sum_{i=1}^{m} c_i + \frac{\lambda_1}{C_b} \tilde{c}_m + \lambda_2 a_2.
\]

Notice that \( \int_X b(x) \, d\mu(x) = 0 \) and \( \int_X c_i(x) \, d\mu(x) = 0 \). It then follows that

\[
\int_X \left[ \frac{\lambda_1}{C_b} \tilde{c}_m(x) + \lambda_2 a_2(x) \right] \, d\mu(x) = 0.
\]

On the other hand, we have \( \text{supp}(\tilde{c}_m) \subset \rho B_1^{(m)} \subset 2 \rho \tilde{c}_{i+2} B : B', r_{B'} = 2 \rho \tilde{c}_{i+2} r_{B_1^{(1)}} \leq 2 \rho \tilde{c}_{i+2} B_1^{(m)} \) and \( \text{supp}(a_2) \subset B_2 \subset B' \). An argument similar to that used in the estimate of (8.7) shows that

\[
\| \tilde{c}_m \|_{L^q(\mu)} \lesssim \left[ \mu \left( \rho B_1^{(m)} \right) \right]^{1/q-1} \left[ \lambda \left( c_{B_1}, r_{B_1} \right) \right]^{1-1/p} \left[ \tilde{K}_{B_1^{(m)}, B_1}^{(p), p} \right]^{-\gamma_2}.
\]

From Definition 3.2(iii), (2.1), \( \tilde{K}_{B_1^{(m)}, B_1}^{(p), p} \leq \left[ (3 + |\log_\rho 2|) C_{\rho} \right]^{1/p} B' = 2 \rho \tilde{c}_{i+2} B \) and (iv) and (ii) of Lemma 2.8, it follows that

\[
\| a_2 \|_{L^q(\mu)} \lesssim \left[ \mu(\rho B_2) \right]^{1/q-1} \left[ \lambda(c_{B_1}, r_{B_1}) \right]^{1-1/p} \left[ \tilde{K}_{B_2^{(1)}, B_1}^{(p), p} \right]^{-\gamma_2}.
\]
\[ \|f\|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu) \lesssim \|f\|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu). \]

From this fact and (8.5), we further deduce that \( f = \sum_{j=1}^{\infty} b_j \) in \( (E^{\alpha,q}_{\rho,\gamma_1}(\mu))^* \), where \( \{c_{j,i}\}_{j,i} \) are all \( (p,q,\gamma_2,\rho)_{\lambda,1} \)-atomic blocks as in (8.5) satisfying

\[ \sum_{j=1}^{\infty} \sum_{i=1}^{m_j+1} |c_{j,i}|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu) \lesssim \sum_{j=1}^{\infty} |b_j|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu) \lesssim \|f\|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu), \]

which implies that \( f \in \tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu) \) and

\[ \|f\|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu) \lesssim \|f\|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu). \]

This finishes the proof of Proposition 8.2.

Now we are ready to show that \( E^{1/p-1}_{\rho}(\mu) \) is the dual space of \( \tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu) \).

**Theorem 8.3.** Let \( p \in (0,1], \rho \in (1,\infty), \gamma \in [1,\infty) \) and \( q \in (1,\infty) \). Then

\[ E^{1/p-1}_{\rho}(\mu) = (\tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu))^*. \]

**Proof.** When \( p = 1 \), by [13, Remark 2.6(iii)], the conclusion of Theorem 8.3 holds true without the assumption (6.1). Thus, it remains to consider the case when \( p \in (0,1) \). Let \( p \in (0,1), \rho \in (1,\infty), \gamma \in [1,\infty) \) and \( q \in (1,\infty) \). We first show that \( E^{1/p-1}_{\rho}(\mu) \subset (\tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu))^* \). To this end, let \( f \in E^{1/p-1}_{\rho}(\mu) \).

Recall that any \( h \in \tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu) \) is, by Definition 7.3, a continuous linear functional in \( (E^{1/p-1}_{\rho}(\mu))^* \). Let us write \( (h,f) \) to denote the value of the linear functional \( h \) at \( f \) in \( E^{1/p-1}_{\rho}(\mu) \). Then the mapping \( \ell_f : h \to (h,f) \) is a well defined linear functional on \( \tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu) \). If \( h = \sum_{i=1}^{\infty} |b_i|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}}(\mu) \), then we have

\[ \|\ell_f(h)\| \lesssim \|f\|_{E^{1/p-1}_{\rho}(\mu)} \|h\|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu)}. \]

Therefore, we conclude that

\[ \|\ell_f(h)\| \lesssim \|f\|_{E^{1/p-1}_{\rho}(\mu)} \|h\|_{\tilde{H}^{p,q,\gamma}_{atb,\rho}(\mu)}. \]
Hence, by the Riesz representation theorem, there exists a unique
\[ \mathcal{E}^{1/p-1}_\rho(\mu) \subset (\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^*. \]

To see the converse, for any \( q \in (1, \infty) \), we first claim that, if \( \ell \in (\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^* \), then there exists a function \( f \in L^{q'}_{\text{loc}}(\mu) \) such that, for all \( g \in \cup_B L^q_{\text{loc}}(B) \),
\[
\ell(g) = \int_X f(x)g(x) \, d\mu(x),
\]
where, for all balls \( B \subset X \), \( L^q_{\text{loc}}(B) \) denotes the subspace of \( L^q(B) \) consisting of functions having integral zero. Indeed, let \( \{B_k\}_k \) be an increasing sequence of balls which exhausts \( X \). For each \( k \), let \( \mathbb{C}(B_k) \) denote the space of functions those are constants on \( B_k \). Suppose that \( q \in (1, \infty) \) and \( \ell \in (\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^* \).
Then \( \ell \in (L^q_{\text{loc}}(B_k))^* = L^{q'}(B_k)/\mathbb{C}(B_k) \). Indeed, if \( g \in L^q_{\text{loc}}(B_k) \), then \( g \in \hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu) \) and
\[
\|g\|_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu)} \leq \|\mu(\rho B_k)\|^{1/q} \lambda(c_{B_k}, r_{B_k})^{1/p-1} \|g\|_{L^q(\mu)}.
\]
We further see that
\[
|\ell(g)| \leq \|\ell\|_{(\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^*} \|g\|_{\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu)} \\
\leq \|\ell\|_{(\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^*} \|\mu(\rho B_k)\|^{1/q} \lambda(c_{B_k}, r_{B_k})^{1/p-1} \|g\|_{L^q(\mu)}.
\]
Hence, by the Riesz representation theorem, there exists a unique \( f_k \in L^{q'}(B_k)/\mathbb{C}(B_k) \) such that, for every \( g \in L^q_{\text{loc}}(B_k) \),
\[
\ell(g) = \int_{B_k} f_k(x)g(x) \, d\mu(x).
\]
Since \( \{B_k\}_k \) is increasing, by a standard argument, we see that there exists a unique function \( f \in L^{q'}_{\text{loc}}(\mu) \) such that, for all \( g \in \cup_B L^q_{\text{loc}}(B) \),
\[
\ell(g) = \int_X f(x)g(x) \, d\mu(x).
\]
This proves the claim.

We now show that, if \( f \in L^{q'}_{\text{loc}}(\mu) \) such that \( \ell \in (\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^* \), then \( f \in \mathcal{E}^{1/p-1}_\rho(\mu) \) and
\[
\|f\|_{\mathcal{E}^{1/p-1}_\rho(\mu)} \lesssim \|\ell\|_{(\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^*}.
\]
To this end, by Proposition 6.18, it suffices to show that, for any \((\rho, \beta, \beta, \rho)\)-doubling ball \( B \),
\[
\frac{1}{\mu(\rho B)} |\lambda(c_B, r_B)|^{1/p-1} \int_B |f(x) - m_f(B)| \, d\mu(x) \lesssim \|\ell\|_{(\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^*}.
\]
and, for all \((\rho, \beta, \beta, \rho)\)-doubling balls \( B \subset S \),
\[
|m_f(B) - m_f(S)| \lesssim \|\ell\|_{(\hat{H}^{p,q,\gamma}_{\text{atb},\rho}(\mu))^*} \left[ \mathcal{K}_{B,\rho}^{\beta,\gamma,\rho}(\mu) \right]^{1/p-1} |\lambda(c_S, r_S)|^{1/p-1}.
\]
We first prove (8.11). Let \( B \) be a \((\rho, \beta, \rho)\)-doubling ball. Assume that
\[
\int_{\{x \in B: f(x) > m_f(B)\}} |f(x) - m_f(B)|^{q'} \, d\mu(x) \\
\geq \int_{\{x \in B: f(x) < m_f(B)\}} |f(x) - m_f(B)|^{q'} \, d\mu(x).
\]
Consider
\[
a(x) := \begin{cases} 
|f(x) - m_f(B)|^{q'-1}, & \text{if } x \in \{x \in B: f(x) > m_f(B)\}; \\
0, & \text{otherwise},
\end{cases}
\]
and
where $\tilde{C}_B$ denotes the constant such that $\int_B a(x) \, d\mu(x) = 0$. By the definition of $m_f(B)$, we have
\[
\mu(\{x \in B : f(x) > m_f(B)\}) \leq \mu(B)/2 \leq \mu(\{x \in B : f(x) \leq m_f(B)\}).
\] (8.14)

From this fact, we deduce that $\text{supp}(a) \subset \sqrt{p}B$, $a$ is a $(p, q, \gamma; \sqrt{p})\lambda$-atomic block and
\[
\|a\|_{\hat{H}^{p,q,\gamma}(\mu)} \leq \|a\|_{L^q(\mu)} \mu(\sqrt{p} \times \sqrt{p}B)^{1/q} |\lambda(c_B, r_B)|^{1/p-1} 
\leq [\mu(\rho B)]^{1/q} |\lambda(c_B, r_B)|^{1/p-1} \left[ \int_{\{x \in B : f(x) > m_f(B)\}} |f(x) - m_f(B)|^q \, d\mu(x) \right]^{1/q} 
+ \int_{\{x \in B : f(x) \leq m_f(B)\}} |\tilde{C}_B|^q \, d\mu(x)^{1/q}.
\]

By (8.14), the definition of $\tilde{C}_B$ and the Hölder inequality, we have
\[
\int_{\{x \in B : f(x) \leq m_f(B)\}} |\tilde{C}_B|^q \, d\mu(x)^{1/q} 
= \left| \int_{\{x \in B : f(x) \leq m_f(B)\}} \tilde{C}_B \, d\mu(x)^{1/q} \right| [\mu(\{x \in B : f(x) \leq m_f(B)\})]^{1-q} 
\leq \int_{\{x \in B : f(x) > m_f(B)\}} |f(x) - m_f(B)|^{q-1} \, d\mu(x)^{1/q} [\mu(B)]^{1-q} 
\leq \int_{\{x \in B : f(x) > m_f(B)\}} |f(x) - m_f(B)|^q \, d\mu(x).
\]

From this, $\text{supp}(a) \subset \sqrt{p}B$ and Proposition 8.1, it follows that
\[
\|a\|_{\hat{H}^{p,q,\gamma}(\mu)} \sim \|a\|_{\hat{H}^{p,q,\gamma}(\mu)} 
\leq [\mu(\rho B)]^{1/q} |\lambda(c_B, r_B)|^{1/p-1} \left[ \int_{\{x \in B : f(x) > m_f(B)\}} |f(x) - m_f(B)|^q \, d\mu(x) \right]^{1/q}.
\] (8.15)

On the other hand, by the definition of $a$ and (8.13), we see that
\[
\int_B f(x)a(x) \, d\mu(x) = \int_B [f(x) - m_f(B)]a(x) \, d\mu(x) 
\geq \int_{\{x \in B : f(x) > m_f(B)\}} |f(x) - m_f(B)|^q \, d\mu(x) 
\geq \frac{1}{2} \int_B |f(x) - m_f(B)|^q \, d\mu(x),
\]

which, together with (8.15), implies that
\[
\left[ \int_B |f(x) - m_f(B)|^q \, d\mu(x) \right]^{1/q} \|a\|_{\hat{H}^{p,q,\gamma}(\mu)} 
\leq [\mu(\rho B)]^{1/q} |\lambda(c_B, r_B)|^{1/p-1} \int_B |f(x) - m_f(B)|^q \, d\mu(x) 
\leq [\mu(\rho B)]^{1/q} |\lambda(c_B, r_B)|^{1/p-1} \int_B f(x)a(x) \, d\mu(x) 
\leq [\mu(\rho B)]^{1/q} |\lambda(c_B, r_B)|^{1/p-1} \|\ell\|_{\hat{H}^{p,q,\gamma}(\mu)}^2 \|a\|_{\hat{H}^{p,q,\gamma}(\mu)}.
\]

From this and the Hölder inequality, it then follows that
\[
\frac{1}{\mu(\rho B)} \frac{1}{|\lambda(c_B, r_B)|^{1/p-1}} \int_B |f(x) - m_f(B)| \, d\mu(x).
\]
Thus, (8.11) holds true.
To show (8.12), for all $(\rho, \beta_\rho)$-doubling balls $B \subset S$, let
\[ a_1 := \frac{|f - m_f(S)|^{\theta'}}{|f - m_f(S)} \chi_{\{x \in S: f(x) \neq m_f(S)\}} \]
and $a_2 := \tilde{C}_S \chi_S$, where $\tilde{C}_S$ denotes the constant such that $\int_S [a_1(x) + a_2(x)] d\mu(x) = 0$. Observe that
\begin{equation}
|\tilde{C}_S| \leq |\mu(S)|^{-1} |\mu(B)|^{-1/\theta'} \left[ \int_B |f(x) - m_f(S)|^{\theta'} d\mu(x) \right]^{1/\theta'}.
\end{equation}
From this, together with the fact that $B$ and $S$ are $(\rho, \beta_\rho)$-doubling and Proposition 8.1, it follows that $\text{supp}(b) \subset (2\sqrt{\rho} + 1)S$,
\[ b := \lambda_1 \tilde{a}_1 + \lambda_2 \tilde{a}_2 \in \hat{H}_{\text{atb}, \rho}^{p, q, \gamma}(\mu) \subset \hat{H}_{\text{atb}, \rho}^{p, q, \gamma}(\mu) \]
and
\begin{equation}
|b| \hat{P}_{\text{atb}, \rho}^{p, q, \gamma}(\mu) \sim |b| \hat{H}_{\text{atb}, \rho}^{p, q, \gamma}(\mu) \leq |\mu(\sqrt{\rho} \times \sqrt{\rho}B)|^{-1/\theta'} \left[ \lambda \left(c_S, r(2\sqrt{\rho} + 1)S\right) \right]^{1/p-1}
\times \left[ \frac{\hat{K}^{(\rho), p}_{\sqrt{\rho}B, (2\sqrt{\rho} + 1)S}}{\sqrt{\rho}S, (2\sqrt{\rho} + 1)S} \right]^{\gamma} \left[ \int_B |f(x) - m_f(S)|^{\theta'} d\mu(x) \right]^{1/\theta'}.
\end{equation}
where
\begin{align*}
\tilde{a}_1 &:= a_1 |\mu(\rho B)|^{-1} \left[ \int_B |f(x) - m_f(S)|^{\theta'} d\mu(x) \right]^{-1/\theta'} \times \left[ \hat{K}^{(\rho), p}_{\sqrt{\rho}B, (2\sqrt{\rho} + 1)S} \right]^{-\gamma} \left[ \lambda(c_S, r(2\sqrt{\rho} + 1)S) \right]^{1/p-1}, \\
\tilde{a}_2 &:= |\mu(\rho S)|^{-1} \left[ \lambda \left(c_S, r(2\sqrt{\rho} + 1)S\right) \right]^{1/p-1} \times \left[ \hat{K}^{(\rho), p}_{\sqrt{\rho}S, (2\sqrt{\rho} + 1)S} \right]^{-\gamma} \chi_S, \\
\lambda_1 &:= \left[ \int_B |f(x) - m_f(S)|^{\theta'} d\mu(x) \right]^{1/\theta'} \hat{K}^{(\rho), p}_{\sqrt{\rho}B, (2\sqrt{\rho} + 1)S} \left[ \mu(\rho B) \right]^{-1/\theta'} \times \left[ \lambda \left(c_S, r(2\sqrt{\rho} + 1)S\right) \right]^{1/p-1} \\
\lambda_2 &:= \tilde{C}_S |\mu(\rho S)| \left[ \lambda(c_S, r(2\sqrt{\rho} + 1)S) \right]^{1/p-1} \times \left[ \hat{K}^{(\rho), p}_{\sqrt{\rho}S, (2\sqrt{\rho} + 1)S} \right]^{-\gamma} \chi_S.
\end{align*}
Then $\text{supp}(\tilde{a}_1) \subset \sqrt{\rho}B \subset (2\sqrt{\rho} + 1)S$ and $\text{supp}(\tilde{a}_2) \subset \sqrt{\rho}S \subset (2\sqrt{\rho} + 1)S$. By the definition of $a_1$, the $(\rho, \beta_\rho)$-doubling property of $B$ and $S$, the vanishing moment of $b$, (8.16), (8.11) and (8.17), we see that
\begin{align*}
\int_B |f(x) - m_f(S)|^{\theta'} d\mu(x) \\
= \int_B a_1(x) |f(x) - m_f(S)| d\mu(x) \\
\leq \int_B f(x) b(x) d\mu(x) + |\tilde{C}_S| \int_B |f(x) - m_f(S)| d\mu(x) \\
\leq \|b\| \hat{P}_{\text{atb}, \rho}^{p, q, \gamma}(\mu) + |\tilde{C}_S| \|b\| \hat{H}_{\text{atb}, \rho}^{p, q, \gamma}(\mu) \| \mu(\rho S) |\lambda(c_S, r_S)|^{1/p-1} \\
\leq \|b\| \hat{P}_{\text{atb}, \rho}^{p, q, \gamma}(\mu) \left[ \int_B |f(x) - m_f(S)|^{\theta'} d\mu(x) \right]^{1/\theta'} \left[ \hat{K}^{(\rho), p}_{B, S} \right]^{-\gamma} \left[ \mu(B) \right]^{-1/\theta'} |\lambda(c_S, r_S)|^{1/p-1}.
\end{align*}
This implies that
\[
\left[ \int_B |f(x) - m_f(S)|^{q'} \, d\mu(x) \right]^{1/q'} \lesssim \|\ell\|_{(\hat{H}^{p,q}_{\text{atb},\rho}(\mu))} \left[ \tilde{K}^{(p),p}_{B,S} \right]^{\gamma} \|\mu(B)\|^{1-1/q'[\lambda(c_s, r_S)]^{1/p-1}}.
\]
Thus, from this, (8.11), the \((\rho, \beta_\mu)\)-doubling property of \(B\) and \(S\), and the Hölder inequality, it follows that
\[
|m_f(B) - m_f(S)| = \frac{1}{\mu(B)} \int_B |m_f(B) - m_f(S)| \, d\mu(x)
\]
\[
\leq \frac{1}{\mu(B)} \int_B |f(x) - m_f(B)| \, d\mu(x) + \frac{1}{\mu(B)} \int_B |f(x) - m_f(S)| \, d\mu(x)
\]
\[
\leq \left[ \frac{1}{\mu(B)} \int_B |f(x) - m_f(B)|^{q'} \, d\mu(x) \right]^{1/q'} + \left[ \frac{1}{\mu(B)} \int_B |f(x) - m_f(S)|^{q'} \, d\mu(x) \right]^{1/q'}
\]
\[
\lesssim \|\ell\|_{(\hat{H}^{p,q}_{\text{atb},\rho}(\mu))} \|\lambda(c_b, r_B)\|^{1/p-1} + \|\lambda(c_s, r_S)\|^{1/p-1} \left[ \tilde{K}^{(p),p}_{B,S} \right]^{\gamma} \|\ell\|_{(\hat{H}^{p,q}_{\text{atb},\rho}(\mu))}^{1/p-1}
\]
which implies (8.12), and hence completes the proof of Theorem 8.3. □

**Remark 8.4.** It is still unclear whether Theorem 8.3 holds true or not for \(q = 1\) and \(p \in (0, 1)\), or \(q = \infty\) and \(p \in (0, 1]\) on non-homogeneous metric measure spaces satisfying the \(\rho\)-weakly doubling condition (6.1).

### 9 Relations between \(E^\alpha_p(\mu)\) and \(\text{Lip}_{\alpha,q}(\mu)\) or between \(\hat{H}^{p,q}_{\text{atb}}(\mu)\) and \(H^{p,q}_{\text{at}}(\mu)\)

In this section, we investigate the relations between \(E^\alpha_p(\mu)\) and \(\text{Lip}_{\alpha,q}(\mu)\), and between \(\hat{H}^{p,q}_{\text{atb}}(\mu)\) and \(H^{p,q}_{\text{at}}(\mu)\) introduced by Coifman and Weiss [11] over spaces of homogeneous type.

Let \((\mathcal{X}, d, \mu)\) be a space of homogeneous type with \(\lambda(x, r) := \mu(B(x, r))\) for all \(x \in \mathcal{X}\) and \(r \in (0, \infty)\). Recall that (6.1) holds true in spaces of homogeneous type. Thus, all the results obtained in Sections 6, 7 and 8 are still valid in this setting and we denote \(E^\alpha_{p,q,\gamma}(\mu)\) simply by \(E^\alpha_p(\mu)\). We first establish an equivalent characterization of \(E^\alpha_p(\mu)\). To this end, we recall the notions of spaces \(\text{Lip}_{\alpha,q}(\mu)\) and \(\text{Lip}_\alpha(\mu)\) in [39]. To be precise, let \(\alpha \in [0, \infty)\), \(q \in [1, \infty)\), \(\rho \in (1, \infty)\) and \(\delta\) be a quasi-distance on \(\mathcal{X}\). A function \(\phi\) is said to be in the Lipschitz space \(\text{Lip}_{\alpha,q}(\mu; \delta)\) if
\[
\|\phi\|_{\alpha,q}^{(\delta)} := \sup_{B_\delta} \left\{ \frac{1}{|\mu(B_\delta)|^{1+\alpha q}} \int_{B_\delta} |f(y) - m_{B_\delta}(f)|^q \, d\mu(y) \right\}^{1/q} < \infty,
\]
where the supremum is taken over all balls \(B_\delta\) from \((\mathcal{X}, \delta, \mu)\), and a function \(\psi\) is said to be in the space \(\text{Lip}_\alpha(\mu; \delta)\) if
\[
\|\psi\|_{\alpha,\delta} := \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|\delta(x, y)|^{\alpha}} < \infty.
\] (9.1)
Then we let \(\text{Lip}_{\alpha,q}(\mu) := \text{Lip}_{\alpha,q}(\mu; d)\) and \(\text{Lip}_\alpha(\mu) := \text{Lip}_\alpha(\mu; d)\), respectively.

**Remark 9.1.** By [39, Theorem 5], we see that, for any \(\alpha \in (0, \infty)\), there exists a quasi-distance \(\delta\) on \(\mathcal{X}\), defined by setting, for all \(x, y \in \mathcal{X}\),
\[
\delta(x, y) := \inf \{\mu(B) : B \text{ is a ball containing } x \text{ and } y\}
\]
such that \((\mathcal{X}, \delta, \mu)\) is a normal space. Namely, there exist two positive constants, \(c_9\) and \(c_{10}\), such that
\[
c_9 \delta \leq \mu(B_\delta(x, r)) \leq c_{10} \delta
\]
for every \(x \in \mathcal{X}\), \(r \in (\mu(x), \mu(\mathcal{X}))\) and
\[
B_\delta(x, r) := \{x \in \mathcal{X} : \delta(x, y) < r\}
and, for any $q \in [1, \infty)$, a function $\phi$ is in the Lipschitz space $\text{Lip}_{\alpha,q}(\mu)$ of $(X, d, \mu)$ if and only if there exists a function $\psi$ in the space $\text{Lip}_\alpha(\mu)$ such that $\phi = \psi$ for $\mu$-almost every $x \in X$. Moreover, $\|\phi\|^{(d)}_{\alpha,q} \sim \|\psi\|^{(d)}_\alpha$.

Hereafter, we always let $\delta$ be as in Remark 9.1.

Now we discuss the relation between $E_\rho^\alpha(\mu)$ and $\text{Lip}_{\alpha,q}(\mu)$.

**Proposition 9.2.** Suppose that $(X, d, \mu)$ is a space of homogeneous type, $\alpha \in [0, \infty)$, $\rho \in (1, \infty)$ and $q \in [1, \infty)$. Then $E_\rho^\alpha(\mu)$ and $\text{Lip}_{\alpha,q}(\mu)$ coincide with equivalent norms.

**Proof.** Fix $\alpha \in [0, \infty)$ and $q \in [1, \infty)$. By Proposition 6.8(ii), without loss of generality, we may assume that $\rho = 2$. By Definition 6.5, we know that $E_\rho^\alpha(\mu) \subset \text{Lip}_{\alpha,q}(\mu)$ and, for all $f \in E_\rho^\alpha(\mu)$, $\|f\|_{\alpha,q} \leq \|f\|_{E_\rho^\alpha(\mu)}$.

Conversely, by Definition 6.5, it suffices to prove that, for all $f \in \text{Lip}_{\alpha,q}(\mu)$ and balls $B \subset S$,

$$\|m_B(f) - m_S(f)\| \leq \|f\|_{\alpha,q} \bar{K}^{(\rho)}_{B,S}[\mu(S)]^\alpha.$$  \hfill (9.2)

To this end, we consider the following two cases.

**Case (I)** $\mu(S) \leq 4\mu(\mu)\mu(B)$, where $\mu(\mu)$ is as in (1.1). Thus, by this and the Hölder inequality, we have

$$\|m_B(f) - m_S(f)\| \leq \frac{1}{\mu(B)} \int_B |f(x) - m_S(f)| \, d\mu(x) \leq \frac{1}{\mu(S)} \int_S |f(x) - m_S(f)| \, d\mu(x) \leq \frac{1}{\|f\|_{\alpha,q}^{1/q}} \left( \int_S |f(x) - m_S(f)|^q \, d\mu(x) \right)^{1/q} \leq \|f\|_{\alpha,q} \mu(S).$$

which implies (9.2) in **Case (I)**.

**Case (II)**

$$\mu(S) > 4\mu(\mu)\mu(B).$$  \hfill (9.3)

Now we show (9.2). Let $N$ be the smallest integer such that $2^N r_B \geq r_S$. Let $B_* := 2^{N+1} B$. Then $S \subset B_* \subset 6S$, which implies that

$$\mu(S) \leq \mu(B_*) \leq \mu(6S) \leq [C(\mu)]^3 \mu(S).$$  \hfill (9.4)

Furthermore, let $B^{(0)} := B$. By (9.3) and (9.4), we see that

$$\mu \left( 2^{N+1} B \right) = \mu(B_*) \geq \mu(S) > 4\mu(\mu)\mu(B) > 2\mu \left( B^{(0)} \right).$$

Thus, let $B^{(1)} := 2^{N_1} B^{(0)}$, $N_1 \in \mathbb{N}$, be the smallest ball in the form of $2^k B^{(0)}$ ($k \in \mathbb{N}$) such that $\mu(2^k B^{(0)}) > 2\mu(B^{(0)}))$. Moreover, by (9.3) and (9.4), we know that $r_{B^{(1)}} \leq r_{B_*}$ and

$$\mu \left( B^{(1)} \right) \leq C(\mu) \mu \left( 2^{-1} B^{(1)} \right) \leq 2C(\mu)\mu(B^{(0)}).$$

We further consider the following two subcases.

**Subcase i).** There exists $k \in \mathbb{N}$ such that $\mu(2^k B^{(1)}) > 2\mu(B^{(1)})$. In this case, we let $2^{N_2} B^{(1)}$, $N_2 \in \mathbb{N}$, be the smallest ball in the form of $2^k B^{(1)}$, $k \in \mathbb{N}$, such that $\mu(2^k B^{(1)}) > 2\mu(B^{(1)})$. Now we divide this subcase into two parts:

(a) $r_{2^{N_2} B^{(1)}} \leq r_{B_*}$. Let $B^{(2)} := 2^{N_2} B^{(1)}$. Then $\mu(B^{(2)}) \leq C(\mu)\mu(2^{-1} B^{(2)}) \leq 2C(\mu)\mu(B^{(1)})$;
(b) $r_{2^{N_2} B^{(1)}} > r_{B_*}$. Let $B^{(2)} := B_*$. Then $\mu(B^{(2)}) \leq 2\mu(B^{(1)})$, where we terminate the construction in this subcase.

**Subcase ii).** For any $k \in \mathbb{N}$, $\mu(2^k B^{(1)}) \leq 2\mu(B^{(1)})$. Let $B^{(2)} := B_*$. Then $\mu(B^{(2)}) \leq 2\mu(B^{(1)})$ and we terminate the construction in this subcase.

We continue to choose the balls $\{B^{(i)}\}$ in this way. Clearly, finally the condition $r_{2^{N_{i+1}} B^{(i)}} \leq r_{B_*}$ ($i \in \mathbb{N}$) is violated after finitely many steps. Without loss of generality, we may assume that the process stops after $m$ ($m \in \mathbb{N} \cap (1, \infty)$) steps. Then we obtain a sequence of balls, $\{B^{(i)}\}_{i=0}^m$, such that

(i) $B := B^{(0)} \subset \cdots \subset B^{(m)} := B_*$. 
(ii) for any \( i \in \{1, \ldots, m-1\} \), \( 2\mu(B^{(i-1)}) < \mu(B^{(i)}) \leq 2C_m\mu(B^{(i-1)}) \);

(iii) \( \mu(B^{(m)}) \leq 2C_m\mu(B^{(m-1)}) \) and \( \mu(S) \leq \mu(B^{(m)}) \leq \lceil C_m \rceil \mu(S) \).

Observe that \( m \leq N + 1 \). Thus, by the fact that \( S \subset B_x \subset 6S \), the Hölder inequality, (i), (ii) and (iii), we have

\[
|m_B(f) - m_S(f)| 
\leq \sum_{i=1}^{m} |m_{B^{(i-1)}}(f) - m_{B^{(i)}}(f)| + |m_{B^{(m)}}(f) - m_S(f)| 
\leq \sum_{i=1}^{m} \frac{1}{\mu(B^{(i-1)})} \int_{B^{(i-1)}} |f(x) - m_{B^{(i)}}(f)| \, d\mu(x) + \frac{1}{\mu(S)} \int_S |f(x) - m_{B^{(m)}}(f)| \, d\mu(x) 
\leq \sum_{i=1}^{m} \frac{1}{\mu(B^{(i)})} \int_{B^{(i)}} |f(x) - m_{B^{(i)}}(f)| \, d\mu(x) + \frac{1}{\mu(B^{(m)})} \int_{B^{(m)}} |f(x) - m_{B^{(m)}}(f)| \, d\mu(x) 
\leq \sum_{i=1}^{m} \left[ \frac{1}{\mu(B^{(i)})} \int_{B^{(i)}} |f(x) - m_{B^{(i)}}(f)|^q \, d\mu(x) \right]^{1/q} + \left[ \frac{1}{\mu(B^{(m)})} \int_{B^{(m)}} |f(x) - m_{B^{(m)}}(f)|^q \, d\mu(x) \right]^{1/q} 
\leq \sum_{i=1}^{m} \left[ \mu(B^{(i)}) \right]^\alpha \|f\|_{\alpha,q} + \left[ \mu(B^{(m)}) \right]^\alpha \|f\|_{\alpha,q} 
\leq (1 + m) \left[ \mu(B^{(m)}) \right]^\alpha \|f\|_{\alpha,q} \lesssim (1 + m) \|\mu(6S)\|^{\alpha} \|f\|_{\alpha,q} 
\lesssim (1 + N) \|\mu(6S)\|^{\alpha} \|f\|_{\alpha,q} \sim [K_B^{(p),q}]^p \|\mu(S)\|^{\alpha} \|f\|_{\alpha,q}.
\]

This finishes the proof of (9.2) in Case (II) and hence Proposition 9.2. \( \square \)

**Remark 9.3.** When \( \alpha = 0 \), Proposition 9.2 is just [27, Proposition 4.7] with \( \lambda(x,r) = \mu(B(x,r)) \) for all \( x \in \mathcal{X} \) and \( r \in (0,\infty) \).

Now we recall the notion of the atomic Hardy space \( H_{at}^{p,q}(\mu) \) from [11]. Suppose that \( p \in (0,1) \) and \( q \in [1,\infty) \cap (p,\infty) \). A function \( a \) on \( \mathcal{X} \) is called a \((p,q)\)-atom if

(i) \( \text{supp}(a) \subset B \) for some ball \( B \subset \mathcal{X} \);

(ii) \( \|a\|_{L^q(\mu)} \leq \mu(B)^{1/q-1/p} \);

(iii) \( \int_B a(x) \, d\mu(x) = 0 \).

A function \( f \in L^1(\mu) \) or a linear functional \( f \in (\text{Lip}_{1/p-1}(\mu))^* \) when \( p \in (0,1) \) is said to be in the Hardy space \( H_{at}^{p,q}(\mu) \) when \( p = 1 \) or \( H_{at}^{p,q}(\mu) \) when \( p \in (0,1) \) if there exist \((p,q)\)-atoms \( \{a_j\}_{j=1}^\infty \) and \( \{\lambda_j\}_{j=1}^\infty \subset \mathbb{C} \) such that

\[
f = \sum_{j \in \mathbb{N}} \lambda_j a_j,
\]

which converges in \( L^1(\mu) \) when \( p = 1 \) or in \( (\text{Lip}_{1/p-1}(\mu))^* \) when \( p \in (0,1) \), and

\[
\sum_{j \in \mathbb{N}} |\lambda_j|^p < \infty.
\]

Moreover, the norm of \( f \) in \( H_{at}^{p,q}(\mu) \) with \( p \in (0,1] \) and \( q \in [1,\infty) \cap (p,\infty) \) is defined by

\[
\|f\|_{H_{at}^{p,q}(\mu)} := \inf \left\{ \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p} \right\},
\]

where the infimum is taken over all possible decompositions of \( f \) as above.

Coifman and Weiss [11] proved that \( H_{at}^{p,q}(\mu) \) and \( H_{at}^{p,\infty}(\mu) \) coincide with equivalent norms for all \( p \in (0,1] \) and \( q \in [1,\infty) \cap (p,\infty) \). Thus, we denote \( H_{at}^{p,q}(\mu) \) simply by \( H_{at}^{p}(\mu) \).
Let \( p \in (0, 1], q \in (1, \infty], \gamma \in [1, \infty) \) and \( \rho \in (1, \infty) \). Recall that the space \( \hat{H}^{p,q,\gamma}_{atb}(\mu) \) is independent of the choices of \( \gamma \in [1, \infty) \) and \( \rho \in (1, \infty) \); see Propositions 8.1 and 8.2. Denote \( \hat{H}^{p,q,\gamma}_{atb}(\mu) \) simply by \( \hat{H}^{p,q}_{atb}(\mu) \). Moreover, without loss of generality, we may let \( \gamma = 1/p \) and \( \rho = 2 \).

Now we show that \( \hat{H}^{p,q}_{atb}(\mu) \) and \( H^p_{at}(\mu) \) coincide with equivalent quasi-norms.

**Theorem 9.4.** Let \( (X, d, \mu) \) be a space of homogeneous type, \( p \in (0, 1] \) and \( q \in (1, \infty] \). Then the spaces \( \hat{H}^{p,q}_{atb}(\mu) \) and \( H^p_{at}(\mu) \) coincide with equivalent quasi-norms.

**Proof.** Let \( p \in (0, 1] \) and \( q \in (1, \infty] \). We first show that \( H^p_{at}(\mu) \subset \hat{H}^{p,q}_{atb}(\mu) \). To this end, let \( f \in H^p_{at}(\mu) \). Then there exist sequences of \( (p,q) \)-atoms, \( \{b_k\}_{k=1}^{\infty} \), and numbers, \( \{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{C} \), such that \( f = \sum_{k=1}^{\infty} \lambda_kb_k \) in \( (\mathrm{Lip}_{1/p-1}(\mu; \delta))^* \) and

\[
\sum_{k=1}^{\infty} |\lambda_k|^p \leq \|f\|_{H^p_{at}(\mu)}^p.
\]

We then claim that, for each \( k \), \( \lambda_kb_k \) is a \((p,q,1/p,2)\lambda,1\)-atomic block and

\[
|\lambda_kb_k|_{\hat{H}^{p,q}_{atb}(\mu)} \lesssim |\lambda_k|.
\]

Indeed, let \( \rho \in (1, \infty) \) and \( \gamma \in [1, \infty) \). It suffices to prove that, if \( b \) is a \((p,q)\)-atom, then \( b \) is a \((p,q,1/p,2)\lambda,1\)-atomic block. Suppose that \( \text{supp} (b) \subset B(c_B, r_B) := B \), then

\[
\|b\|_{L^q(\mu)} \lesssim |\mu(B)|^{1/q-1/p}.
\]

Let

\[
a_1 = a_2 := \left[ \frac{\mu(B)}{\mu(\nu)} \right]^{1/q-1} \left[ \frac{K^{(\rho)}_{p,q}}{K^{(\rho)}_{p,q}} \right] b
\]

and

\[
\lambda_1 = \lambda_2 := \frac{1}{2} \left[ \frac{\mu(B)}{\mu(\nu)} \right]^{1/q-1} \left[ \frac{K^{(\rho)}_{p,q}}{K^{(\rho)}_{p,q}} \right].
\]

It then follows that \( \text{supp} (a_1) \subset B_1 := B \), \( \text{supp} (a_2) \subset B_2 := B \), \( b = \lambda_1a_1 + \lambda_2a_2 \) and, for \( j \in \{1, 2\} \),

\[
\|a_j\|_{L^q(\mu)} \lesssim |\mu(\nu)|^{1/q-1} |\mu(B)|^{1-1/p} \left[ \frac{K^{(\rho)}_{p,q}}{K^{(\rho)}_{p,q}} \right]^{-1/p},
\]

which further implies that \( b \) is a \((p,q,1/p,2)\lambda,1\)-atomic block and, moreover,

\[
|b|_{\hat{H}^{p,q}_{atb}(\mu)} = |\lambda_1| + |\lambda_2| \lesssim 1.
\]

This finishes the proof of the above claim. Moreover, from Remark 9.1 and Proposition 9.2, we deduce that \( f = \sum_{k=1}^{\infty} \lambda_kb_k \) in \((E^p_{\rho-1}(\mu))^*\), which, together with (9.5) and (9.6), further implies that \( f \in \hat{H}^{p,q}_{atb}(\mu) \) and

\[
\|f\|_{\hat{H}^{p,q}_{atb}(\mu)} \lesssim \|f\|_{H^p_{at}(\mu)}.
\]

Now we consider the converse inclusion that \( \hat{H}^{p,q}_{atb}(\mu) \subset H^p_{at}(\mu) \). Let \( b = \sum_{j=1}^{2} \lambda_ja_j \) be a \((p,q,1/p,2)\lambda,1\)-atomic block, where, for any \( j \in \{1, 2\} \), \( a_j \) is a function supported on \( B_j \subset B \) for some balls \( B_j \) and \( B \) as in Definition 7.1, and

\[
\|a_j\|_{L^q(\mu)} \lesssim |\mu(B)|^{1/q-1} |\mu(B)|^{1-1/p} \left[ \frac{K^{(\rho)}_{p,q}}{K^{(\rho)}_{p,q}} \right]^{-1/p}.
\]

We consider the following four cases:

**Case (I)** For any \( j \in \{1, 2\} \), \( \frac{\mu(B)}{\mu(B_j)} \leq 4C(\mu) \), where \( C(\mu) \) is as in (1.1);

**Case (II)** \( \frac{\mu(B)}{\mu(B_j)} > 4C(\mu) \) and \( \frac{\mu(B)}{\mu(B_j)} \leq 4C(\mu) \);

**Case (III)** \( \frac{\mu(B)}{\mu(B_j)} \leq 4C(\mu) \) and \( \frac{\mu(B)}{\mu(B_j)} > 4C(\mu) \);

**Case (IV)** For any \( j \in \{1, 2\} \), \( \frac{\mu(B)}{\mu(B_j)} > 4C(\mu) \).

In Case (I), we see that

\[
\|b\|_{L^q(\mu)} \lesssim |\lambda_1| \|a_1\|_{L^q(\mu)} + |\lambda_2| \|a_2\|_{L^q(\mu)} \lesssim [4C(\mu)]^{1-1/q} (|\lambda_1| + |\lambda_2|) |\mu(B)|^{1/q-1/p},
\]
which implies that $\left[4C_{(p)}\right]^{1/q-1}(|\lambda_1| + |\lambda_2|)^{-1}b$ is a $(p, q)$-atom and

$$\left\| \left[4C_{(p)}\right]^{1/q-1}(|\lambda_1| + |\lambda_2|)^{-1}b \right\|_{H^s_{\infty}(\mu)} \leq 1.$$  

The proofs of Case (II), Case (III) and Case (IV) are similar. For brevity, we only prove Case (II).

In Case (II), $\frac{\mu(B)}{\mu(B')} > 4C_{(p)}$ and $\frac{\mu(B)}{\mu(B')} \leq 4C_{(p)}$. We now choose a sequence of balls, $\{B^{(i)}\}_{i=0}^m$ with certain $m \in \mathbb{N}$, as follows. Let $B^{(0)}_1 := B_1$ and $B_0 := 2^N(B_1, \sigma+1)$. Then

$$B \subset B_0 \subset 6B,$$

which, together with (1.1), shows that

$$\mu(B) \leq \mu(B_0) \leq \mu(6B) \leq \left[C_{(p)}\right]^3 \mu(B). \quad (9.8)$$

To choose $B^{(1)}_1$, let $N_1$ be the smallest positive integer satisfying $\mu(2^N, B^{(0)}_1) > 2 \mu(B^{(0)}_1)$. We let $B^{(1)}_1 := 2^N, B^{(0)}_1$. By (9.8), (1.1) and the choice of $B^{(1)}_1$, we have $r_{B^{(1)}_1} < r_{B_0}$ and

$$2 \mu \left( B^{(0)}_1 \right) \leq \mu \left( B^{(1)}_1 \right) \leq C_{(p)} \mu \left( 2^{-1} B^{(1)}_1 \right) \leq 2C_{(p)} \mu \left( B^{(0)}_1 \right).$$

To choose $B^{(2)}_1$, if, for any $N \in \mathbb{N}$, $\mu(2^N, B^{(1)}_1) \leq 2 \mu(B^{(1)}_1)$, let $B^{(2)}_1 := B_0$ and the selection process terminates. Otherwise, let $N_2$ be the smallest positive integer satisfying $\mu(2^N, B^{(1)}_1) > 2 \mu(B^{(1)}_1)$. If $r_{2^N, B^{(1)}_1} \geq r_{B_0}$, then we let $B^{(2)}_1 := B_0$ and the selection process terminates. Otherwise, we let $B^{(2)}_1 := 2^N, B^{(1)}_1$. We continue as long as this selection process is possible; clearly, finally the condition $r_{2^N, B^{(1)}_1} < r_{B_0}$ is violated after finitely many steps. Without loss of generality, we may assume that the process stops after $m \in \mathbb{N}$ steps. Then we obtain a family of balls, $\{B^{(i)}_1\}_{i=1}^m$, such that:

(i) $B^{(i)}_1 := B_1 \subset B$, $B^{(i)}_1 := 2^N, B^{(i-1)}_1 \subset B^{(m)} := B_0 \subset 6B$ for any $i \in \{1, \ldots, m-1\};$

(ii) for any $i \in \{1, \ldots, m-1\}$, by (1.1) and the definition of $N_i$, we have

$$2 \mu \left( B^{(i-1)}_1 \right) \leq \mu \left( B^{(i)}_1 \right) \leq 2C_{(p)} \mu \left( B^{(i-1)}_1 \right);$$

(iii) $\mu(B^{(m)}_1) \leq 2C_{(p)} \mu(B^{(m)}_1)$ and $\mu(B) \leq \mu(B^{(m)}_1) \leq \left[C_{(p)}\right]^3 \mu(B);$ 

(iv) from the above selection process and the definition of $\hat{K}^{(2), p}_{B_1,B}$, we conclude that

$$m \leq N^{(2)}_{B_1,B} + 1 \leq \left[1 + N^{(2)}_{B_1,B}\right]^{1/p} \leq \hat{K}^{(2), p}_{B_1,B} =: \left[\hat{C}_b\right]^p. \quad (9.9)$$

Let $\tilde{c}_0 := \hat{C}_b a_1$. For any $i \in \{1, \ldots, m\}$, let

$$\tilde{c}_i := \frac{\chi_{B^{(i)}_1}}{\mu(B^{(i)}_1)} \int_{X} \tilde{c}_{i-1}(y) d\mu(y).$$

For $i \in \{1, \ldots, m\}$, we claim that

$$\|\tilde{c}_{i-1}\|_{L^s(\mu)} \leq \left[\mu(B^{(i)}_1)\right]^{1/q-1/p} \quad (9.10)$$

and

$$\|\tilde{c}_i\|_{L^s(\mu)} \leq \left[\mu(B^{(i)}_1)\right]^{1/q-1/p}, \quad (9.11)$$

where the implicit positive constant is independent of $\hat{C}_b$ and hence $B_1$, $B_2$ and $B$. Indeed, we prove (9.10) and (9.11) by induction. By (9.7), $B^{(1)}_1 \subset \subset 6B \subset 1$ and (ii), we have

$$\|\tilde{c}_0\|_{L^s(\mu)} \leq \left[\mu \left( 2B^{(0)}_1 \right) \right]^{1/q-1} \left[\mu(B)\right]^{1/q-1/p} \left[\hat{K}^{(2), p}_{B_1,B} \right]^{-1/p+1/p} \leq \left[\mu \left( B^{(1)}_1 \right) \right]^{1/q-1/p}.$$
For $i = 1$, by the Hölder inequality, (9.7), $B_1^{(1)} \subset \not\subset B_0 \subset 6B$, (1.1) and (ii), we conclude that
\[
\|\tilde{c}_i\|_{L^q(\mu)} \lesssim \left[ \mu \left( B_1^{(1)} \right)^{1/q-1} \left[ \mu \left( B_1^{(0)} \right) \right]^{1-1/q} \right]^{1/q-1} \|\tilde{c}_i a_1\|_{L^q(\mu)}
\lesssim \left[ \mu \left( B_1^{(1)} \right)^{1/q-1} \left[ \mu \left( B_1^{(0)} \right) \right]^{1-1/q} \left[ \mu \left( 2B_1^{(0)} \right) \right]^{1/q-1} \left[ \mu(B) \right]^{1-1/p} \left[ \tilde{K}_{B_1,B}^{(2)} \right]^{-1/p+1/p} \right]^{1/q-1/p}.
\]
Moreover, by (ii) (if $m = 2$, we use (iii)), we have
\[
\|\tilde{c}_i\|_{L^q(\mu)} \lesssim \left[ \mu(B_1^{(2)}) \right]^{1/q-1/p}.
\]
Now, we assume that (9.10) and (9.11) hold true for $i \in \mathbb{N} \cap [1, m)$. It then follows, from the Hölder inequality, (9.11) and (ii) (if $i+1 = m$, we use (iii)), that
\[
\|\tilde{c}_{i+1}\|_{L^q(\mu)} \lesssim \left[ \mu \left( B_1^{(i+1)} \right)^{1/q-1} \mu \left( B_1^{(i)} \right)^{1-1/q} \|\tilde{c}_{i}\|_{L^q(\mu)} \right]^{1/q-1/q} \lesssim \left[ \mu \left( B_1^{(i+1)} \right)^{1/q-1} \left[ \mu \left( B_1^{(i)} \right) \right]^{1/q-1/q} \left[ \mu(B) \right]^{1-1/p} \right]^{1/q-1/p} \lesssim \left[ \mu \left( B_1^{(i+1)} \right) \right]^{1/q-1/p}
\]
and, moreover, if $i+1 < m$, we further have
\[
\|\tilde{c}_{i+1}\|_{L^q(\mu)} \lesssim \left[ \mu \left( B_1^{(i+2)} \right) \right]^{1/q-1/p}.
\]
By induction, we conclude that (9.10) and (9.11) hold true.

For $i \in \{1, \ldots, m\}$, let $c_i := \lceil C \rceil^{1/q} (\tilde{c}_i - \tilde{c}_i)$, where $\tilde{C}$ is a positive constant to be fixed later. Then supp $(c_i) \subset 2B_1^{(i)}$ and \(\int_X c_i(x) \, d\mu(x) = 0\), which, together with (9.10), (9.11) and (1.1), implies that $c_i$ is a multiple of a $(p, q)$-atom associated with the ball $2B_1^{(i)}$ provided $\tilde{C}$ large enough. We now write
\[
b = \frac{\tilde{C} \lambda_1}{C_b} \sum_{i=1}^m c_i + \frac{\lambda_1}{C_b} \tilde{c}_m + \lambda_2 a_2.
\]
Let $c_{m+1} := [\tilde{C}(|\lambda_1| + |\lambda_2|)]^{-1} (\frac{\lambda_1}{C_b} \tilde{c}_m + \lambda_2 a_2)$. Notice that \(\int_X b(x) \, d\mu(x) = 0\) and, for each $i \in \{1, \ldots, m\}$, \(\int_X c_i(x) \, d\mu(x) = 0\). It then follows that \(\int_X c_{m+1} \, d\mu(x) = 0\). On the other hand, we have supp $(\tilde{c}_m) \subset 2B_1^{(m)} \subset 12B =: B'$, $\mu(B') \sim \mu(B)$ and supp $(a_2) \subset B_2 \subset B'$. By (9.11), (iii) and $\mu(B') \sim \mu(B)$, we have
\[
\|\tilde{c}_m\|_{L^q(\mu)} \lesssim \left[ \mu \left( B_1^{(m)} \right) \right]^{1/q-1/p} \lesssim \left[ \mu(B') \right]^{1/q-1/p}.
\]
From (9.7), \(\frac{\mu(B')}{\mu(B_2)} \leq 4C(\mu)\) and $\mu(B') \sim \mu(B)$, it follows that
\[
\|a_2\|_{L^q(\mu)} \lesssim \left[ \mu \left( 2B_2 \right) \right]^{1/q-1/q} \left[ \mu(B) \right]^{1-1/p} \left[ \tilde{K}_{B_2,B}^{(2)} \right]^{-1/p} \lesssim \left[ \mu \left( B' \right) \right]^{1/q-1/p}.
\]
Let $\tilde{C}$ be a positive constant, which is independent of $\tilde{C}_b$ and $m$, such that
\[
\|c_i\|_{L^q(\mu)} \lesssim \tilde{C} \left[ \mu \left( B_1^{(i+2)} \right) \right]^{1/q-1/p}
\]
for each $i \in \{1, \ldots, m\}$, and
\[
\left\| \frac{\lambda_1}{C_b} \tilde{c}_m + \lambda_2 a_2 \right\|_{L^q(\mu)} \lesssim \tilde{C} (|\lambda_1| + |\lambda_2|) \left[ \mu \left( B' \right) \right]^{1/q-1/p}.
\]
Then we see that $c_{m+1}$ is a $(p, q)$-atom associated with the ball $B'$. From this and (iv), we conclude that
\[
b = \frac{\tilde{C} \lambda_1}{C_b} \sum_{i=1}^m c_i + \tilde{C} (|\lambda_1| + |\lambda_2|) c_{m+1} \in H^p_{at}(\mu)
\]
and
\[ \|b\|_{H^p_{\text{at}}(\mu)} \lesssim m \frac{|\lambda_1|^p}{|C_b|^p} + (|\lambda_1| + |\lambda_2|)^p \lesssim (|\lambda_1| + |\lambda_2|)^p \sim |b|_{\widetilde{H}^p_{\text{at}}(\mu)} \]  
(9.12)

where the implicit positive constant is independent of \( \tilde{C}_b \) and \( m \).

Moreover, for any \( f \in \widetilde{H}^p_{\text{at}}(\mu) \), by Definition 7.3 with \( \gamma = 1/p \) and \( \rho = 2 \), we know that there exists a sequence \( \{b_k\}_{k \in \mathbb{N}} \) of \((p,q,1/p,2)_{\lambda,1}\)-atomic blocks such that \( f = \sum_{k \in \mathbb{N}} b_k \) in \((\mathcal{E}^{1/p-1}_{\rho}(\mu))^* \) and
\[ \|b_k\|_{\widetilde{H}^p_{\text{at}}(\mu)} \lesssim \|f\|_{\widetilde{H}^p_{\text{at}}(\mu)} \]  
(9.13)

For each \( k \), assume that \( b_k = \lambda_k,1 a_k,1 + \lambda_k,2 a_k,2 \), where \( \text{supp} (b_k) \subset B_k \), \( \text{supp} (a_k,j) \subset B_k,j \) for \( j \in \{1,2\} \). Let \( \tilde{C}_k := \tilde{K}^{(2),p}_{B_k,1,B_k,2} \). From Remark 9.1 and Proposition 9.2, (9.9), (9.12) and (9.13), we deduce that
\[ f = \sum_{k \in \mathbb{N}} b_k = \sum_{k \in \mathbb{N}} \left[ \frac{\sum_{i=1}^{m_k} \tilde{C}_{\lambda_k,1} c_{\lambda_k,i} + \tilde{C} (|\lambda_k,1| + |\lambda_k,2|) c_{m_k+1}}{C_k} \right] \]
in \((\mathcal{E}^{1/p-1}_{\rho}(\mu))^* \) = \((\text{Lip}_{1/p-1}(\mu; \delta))^* \) and
\[ \|b_k\|_{\widetilde{H}^p_{\text{at}}(\mu)} \lesssim \|f\|_{\widetilde{H}^p_{\text{at}}(\mu)} \]
(9.13)

This implies that \( f \in H^p_{\text{at}}(\mu) \) and \( \|f\|_{H^p_{\text{at}}(\mu)} \lesssim \|f\|_{\widetilde{H}^p_{\text{at}}(\mu)} \), which completes the proof of Theorem 9.4.

Remark 9.5. (i) Theorem 9.4 implies that, if \((X,d,\mu)\) is a space of homogeneous type with \( \lambda(x,r) := \mu(B(x,r)) \) for all \( x \in X \) and \( r \in (0,\infty) \), then \( \widehat{H}^p_{\text{at,}}(\mu) \) and \( H^p_{\text{at}}(\mu) \) coincide with equivalent norms. We notice that it is not a paradox of the example given by Tolsa [53, Example 5.6] since \( \lambda(x,r) \sim r \) for all \( x \in X \) and \( r \in (0,\sqrt{2}] \), which is not equivalent to \( \mu(B(x,r)) \sim r^2 \), in that example.

(ii) Let \( \rho \in (1,\infty) \), \( \gamma \in [1,\infty) \) and \( q \in (1,\infty) \). Combining Theorem 9.4 and Remark 7.4(ii), we obtain
\[ \widehat{H}^{1,q,\gamma}_{\text{at,}}(\mu) = H^1_{\text{at}}(\mu) = H^{1,q,\gamma}_{\text{at,}\rho}(\mu) \]
over spaces of homogeneous type.

(iii) From Theorem 9.4 and [11, Theorem A], it follows that \( \widehat{H}^{p,q,\gamma}_{\text{at,}}(\mu) \) is independent of the choice of \( q \) in spaces of homogeneous type with \( \lambda(x,r) := \mu(B(x,r)) \) for all \( x \in X \) and \( r \in (0,\infty) \).

10 Relation between \( \widehat{H}^{p,q,\gamma}_{\text{at,}}(\mu) \) and \( H^p_{\text{at}}(\mu) \) over RD-spaces

In this section, we investigate the relations among \( \widehat{H}^{p,q,\gamma,\epsilon}_{\text{at,}}(\mu), \widehat{H}^{p,q,\gamma,\epsilon}_{\text{mb,}}(\mu) \) and \( H^p_{\text{at}}(\mu) \) over RD-spaces. As a corollary, the relations among \( \widehat{H}^{p,q,\gamma,\epsilon}_{\text{at,}}(\mu), \widehat{H}^{p,q,\gamma,\epsilon}_{\text{mb,}}(\mu), H^p(\mathbb{R}^D), \widehat{H}^{p,q,\gamma,\epsilon}_{\text{at,}}(\mu) \) and \( \widehat{H}^{p,q,\gamma,\epsilon}_{\text{mb,}}(\mu) \) over Euclidean spaces \( (\mathbb{R}^D, |\cdot|) \) endowed with the \( D \)-dimensional Lebesgue measure \( dx \) are also presented.

Let \((X,d,\mu)\) be a space of homogeneous type with \( \lambda(x,r) := \mu(B(x,r)) \) for all \( x \in X \) and \( r \in (0,\infty) \). The following notion of RD-spaces was introduced by Han, Müller and Yang [22]. A space of homogeneous type is called an RD-space if there exist constants \( \kappa \in (0,\nu) \) and \( C \in [1,\infty) \) such that, for all \( x \in X \), \( r \in (0, \text{diam}(X)/2) \) and \( \lambda \in [1, \text{diam}(X)/(2r)] \),
\[ C^{-1}\lambda^{\nu} \mu(B(x,r)) \leq \mu(B(x,\lambda r)) \leq C \lambda^{\nu} \mu(B(x,r)), \]
(10.1)
where \( \text{diam}(X) := \sup_{x,y \in X} d(x,y) \) and \( \nu := \log_2 C(\lambda) \) is as in Section 1. We point out that the RD-space is also a space of homogeneous type. In the remainder of this section, we always assume that \((X,d,\mu)\) is
an RD-space with \(\mu(\mathcal{X}) = \infty\) and let \(V_r(x) := \mu(B(x, r))\) and \(V(x, y) := \mu(B(x, d(x, y)))\) for all \(x, y \in \mathcal{X}\) and \(r \in (0, \infty)\).

The following space of test functions on \(\mathcal{X}\) was introduced by Han, Müller and Yang [21,22]. Throughout this section, we fix \(x_1 \in \mathcal{X}\).

Let \(\beta \in (0, 1]\) and \(\gamma \in (0, \infty)\). A function \(f\) on \(\mathcal{X}\) is said to belong to the space of test functions, \(\mathcal{G}(\beta, \gamma)\), if there exists a non-negative constant \(\bar{C}\) such that

\[
(A6) \text{ for all } x \in \mathcal{X}, \quad |f(x)| \leq \frac{\bar{C}}{V_t(x_1) + V(x_1, x)} \left[ \frac{1}{1 + d(x_1, x)} \right]^{\gamma};
\]

\[
(A7) \text{ for all } x, y \in \mathcal{X}\text{ satisfying } d(x, y) \leq (1 + d(x_1, x))/2, \quad |f(x) - f(y)| \leq \frac{\bar{C}}{V_t(x_1) + V(x_1, x)} \left[ \frac{d(x, y)}{1 + d(x_1, x)} \right]^\beta \left[ \frac{1}{1 + d(x_1, x)} \right]^{\gamma}.
\]

Moreover, for \(f \in \mathcal{G}(\beta, \gamma)\), its norm is defined by

\[
\|f\|_{\mathcal{G}(\beta, \gamma)} := \inf \{ \bar{C} : \bar{C} \text{ satisfies (A6) and (A7)} \}.
\]

The space \(\tilde{\mathcal{G}}(\beta, \gamma)\) is defined as the set of all functions \(f \in \mathcal{G}(\beta, \gamma)\) satisfying \(\int_{\mathcal{X}} f(x) d\mu(x) = 0\). Moreover, we endow the space \(\tilde{\mathcal{G}}(\beta, \gamma)\) with the same norm as the space \(\mathcal{G}(\beta, \gamma)\). Furthermore, \(\tilde{\mathcal{G}}(\beta, \gamma)\) is a Banach space.

For any given \(\epsilon \in (0, 1]\), let \(\mathcal{G}_0(\beta, \gamma)\) be the completion of the set \(\mathcal{G}(\epsilon, \epsilon)\) in \(\mathcal{G}(\beta, \gamma)\) when \(\beta, \gamma \in (0, \epsilon]\). Moreover, if \(f \in \mathcal{G}_0(\beta, \gamma)\), we then define \(\|f\|_{\mathcal{G}_0(\beta, \gamma)} := \|f\|_{\mathcal{G}(\beta, \gamma)}\). We define the dual space \((\mathcal{G}_0(\beta, \gamma))^*\) to be the set of all continuous linear functionals \(L\) from \(\mathcal{G}_0(\beta, \gamma)\) to \(\mathbb{C}\), and endow it with the weak* topology.

Suppose that \(\epsilon_1 \in (0, 1]\) and \(\epsilon_2, \epsilon_3 \in (0, \infty)\). Let \(\{D_t\}_{t \in (0, \infty)}\) be a family of bounded linear operators on \(L^2(\mu)\) such that, for all \(t \in (0, \infty)\), \(D_t(x, y)\), the kernel of \(D_t\), is a measurable function from \(\mathcal{X} \times \mathcal{X}\) to \(\mathbb{C}\) satisfying the following estimates: there exists a positive constant \(L_0\) such that, for all \(t \in (0, \infty)\) and all \(x, \tilde{x}, y \in \mathcal{X}\) with \(d(x, \tilde{x}) \leq [t + d(x, y)]/2\),

\[
(A1) \quad |D_t(x, y)| \leq L_0 \left[ \frac{1}{V_t(x) + V_t(y) + V(x, y)} \right]^{\epsilon_2} \left[ \frac{t}{t + d(x, y)} \right]^{\epsilon_2};
\]

\[
(A2) \quad |D_t(x, y) - D_t(\tilde{x}, y)| \leq L_0 \left[ \frac{d(x, \tilde{x})}{t + d(x, y)} \right]^{\epsilon_1} \left[ \frac{1}{V_t(x) + V_t(y) + V(x, y)} \right]^{\epsilon_3} \left[ \frac{t}{t + d(x, y)} \right]^{\epsilon_3};
\]

\[
(A3) \quad \text{Property (A2) also holds true with the roles of } x \text{ and } y \text{ interchanged};
\]

\[
(A4) \quad \int_{\mathcal{X}} D_t(x, y) d\mu(x) = 0;
\]

\[
(A5) \quad \int_{\mathcal{X}} D_t(x, y) d\mu(y) = 0.
\]

Now we recall the following Calderón reproducing formula which is a continuous variant of [22, Theorem 3.10]. Hereafter, we let \(a \wedge b := \min\{a, b\}\) and \(a \vee b := \max\{a, b\}\) for all \(a, b \in \mathbb{R}\).

**Lemma 10.1.** Let \(\epsilon_1 := 1, \epsilon_2, \epsilon_3 \in (0, \infty), \epsilon \in (0, \epsilon_1 \wedge \epsilon_2)\) and \(\{D_t\}_{t \in (0, \infty)}\) be as above. Then there exists a family of linear operators \(\{\tilde{D}_t\}_{t \in (0, \infty)}\) such that, for all \(f \in \mathcal{G}_0(\beta, \gamma)\) with \(\beta, \gamma \in (0, \epsilon)\),

\[
f = \int_0^{\infty} \tilde{D}_t D_t(f) \frac{dt}{t}
\]

in \(\mathcal{G}_0(\beta, \gamma)\) and in \(L^q(\mu)\) for all \(q \in (1, \infty)\). Moreover, the kernels of the operators \(\tilde{D}_t\) satisfy the conditions \((A1), (A2), (A4) \text{ and } (A5)\) with \(\epsilon_1 \text{ and } \epsilon_2\) replaced by \(\tilde{\epsilon} \in (\epsilon, \epsilon_1 \wedge \epsilon_2)\).

To the best of our knowledge, the following useful property is well known but there exists no complete proof. We present full details here.
Lemma 10.2. Let $\varepsilon_1$ be as in (A2), $\varepsilon \in (0, \varepsilon_1]$, $\beta, \gamma \in (0, \varepsilon]$ and $q \in (1, \infty)$. Then $\mathcal{G}_0(\beta, \gamma)$ is dense in $L^q(\mu)$.

To prove this lemma, we need the following two technical conclusions. Hereafter, for any $\varepsilon \in (0, \infty)$, we denote $\| \cdot \|_\varepsilon^{(d)}$, which is as in (9.1) with $\delta$ and $\alpha$ replaced respectively by $d$ and $\varepsilon$, simply by $\| \cdot \|_\varepsilon$.

Lemma 10.3. Let $\varepsilon \in (0, 1]$, $F$ be a nonempty closed set and $G$ an open set containing $F$. Then there exists $f \in \text{Lip}_\varepsilon(\mu)$ such that

$$f = 1 \text{ on } F; \quad \text{supp } (f) \subset \overline{G}; \quad 0 \leq f \leq 1 \text{ on } \mathcal{X},$$

where, for any set $A \subset \mathcal{X}$, $\overline{A}$ represents the smallest closed set containing $A$.

Proof. For $x \in \mathcal{X}$, let

$$f(x) := \left( \frac{d(x, \overline{G})}{d(x, F) + d(x, G^c)} \right)^\varepsilon,$$

here and hereafter, for any two sets $A, B \subset \mathcal{X}$, $A^c := \mathcal{X} \setminus A$ and

$$d(A, B) := \inf \{d(a, b) : \ a \in A \text{ and } b \in B \}.$$

It is easy to show that $f$ has all the required properties in Lemma 10.3 with $\|f\|_\varepsilon \leq \frac{1}{\|d(F, G^c)\|^\varepsilon}$, which completes the proof of Lemma 10.3.

Moreover, we have the following conclusion.

Lemma 10.4. Let $\varepsilon \in (0, 1]$. For any ball $B(x_0, r_0)$ and $\eta \in (0, \infty)$, there exists

$$h \in \text{Lip}_{\varepsilon, \beta}(\mu) := \{f \in \text{Lip}_\varepsilon(\mu) : \text{supp } (f) \text{ is bounded}\}$$

such that

$$\text{supp } (h) \subset (B(x_0, r_0))^c, \quad \int_{\mathcal{X}} h(x) \, d\mu(x) = 1 \quad \text{and} \quad \|h\|_{L^\eta(\mu)} < \eta.$$

Proof. For any $k \in \mathbb{N}$, let $F_k := \overline{B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1)}$ and

$$G_k := B(x_0, k + r_0 + 2) \setminus B(x_0, r_0).$$

From [22, Remark 1.2(i)], it follows that $G_k \supset F_k \neq \emptyset$ for some sufficiently large $k$. By Lemma 10.3 with $F$ and $G$ replaced by $F_k$ and $G_k$, respectively, we conclude that there exists $h_k \in \text{Lip}_\varepsilon(\mu) (\|h_k\|_\varepsilon \leq 1)$ such that $h_k = 1$ on $F_k$, $\text{supp } (h_k) \subset \overline{G_k}$ and $0 \leq h_k \leq 1$ on $\mathcal{X}$. Let $\tilde{h}_k := \frac{h_k}{\mu(\mathcal{X})}$, where $J_k := \int_{\mathcal{X}} \tilde{h}_k(x) \, d\mu(x)$. Thus, $\int_{\mathcal{X}} h_k(x) \, d\mu(x) = 1$ and

$$J_k \geq \int_{F_k} d\mu(x) \geq \mu(B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1)). \quad (10.2)$$

We only need to show that $\lim_{k \to \infty} \|h_k\|_{L^\eta(\mu)}^\eta = 0$. By (10.2) and $0 \leq \tilde{h}_k \leq 1$ on $\mathcal{X}$, we have

$$\|h_k\|_{L^\eta(\mu)}^\eta \leq \frac{\mu(B(x_0, k + r_0 + 3) \setminus B(x_0, r_0))}{\mu(B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1)) \left[ \mu(B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1)) \right]^{q-1}}.$$

Notice that $\mu(\mathcal{X}) = \infty$, we obtain

$$\lim_{k \to \infty} \frac{1}{\mu(B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1))^{q-1}} = 0,$$

which reduces the proof to the fact that

$$\limsup_{k \to \infty} \frac{\mu(B(x_0, k + r_0 + 3) \setminus B(x_0, r_0))}{\mu(B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1))} \leq C.$$
where $C$ is as in (10.1).

Indeed, from $\mu(\mathcal{X}) = \infty$ and (10.1), we deduce that

$$\lim_{k \to \infty} \sup_{k \to \infty} \frac{\mu(B(x_0, k + r_0 + 3) \setminus B(x_0, r_0))}{\mu(B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1))} \leq 1 + \lim_{k \to \infty} \sup_{k \to \infty} \frac{\mu(B(x_0, k + r_0 + 3) \setminus B(x_0, k + r_0 + 1))}{\mu(B(x_0, k + r_0 + 1) \setminus B(x_0, r_0 + 1))} \leq 1 + \lim_{k \to \infty} \sup_{k \to \infty} \left]\frac{(k + r_0 + 3) - 1}{(k + r_0 + 1) - 1}\right]\mu(B(x_0, k + r_0 + 1)) \leq C,$$

where $\nu$ and $C$ is as in (10.1). This finishes the proof of Lemma 10.4.

Now we are ready to prove Lemma 10.2.

**Proof of Lemma 10.2.** By some arguments as in [22, p. 19], we know that

$$\tilde{\text{Lip}}_{\epsilon, b}(\mu) := \left\{ f \in \text{Lip}_{\epsilon, b}(\mu) : \int_{\mathcal{X}} f(x) d\mu(x) = 0 \right\} \subset \tilde{G}(\epsilon, \epsilon) \subset \tilde{G}_0^\ast(\beta, \gamma) \subset L^2(\mu).$$

Thus, to show Lemma 10.2, it suffices to prove that $\tilde{\text{Lip}}_{\epsilon, b}(\mu)$ is dense in $L^q(\mu)$. By the fact that $\text{Lip}_{\epsilon, b}(\mu)$ is dense in $L^q(\mu)$ (see, for example, [22, Corollary 2.11(ii)]), we know that, for any $\eta \in (0, \infty)$ and $f \in L^q(\mu)$, there exists $g \in \text{Lip}_{\epsilon, b}(\mu)$ such that $\|g - f\|_{L^q(\mu)} < \eta/2$. Now we show that there exists $\tilde{g} \in \tilde{\text{Lip}}_{\epsilon, b}(\mu)$ such that $\|\tilde{g} - f\|_{L^q(\mu)} < \eta$. We consider the following two cases:

**Case (i)** $\int_{\mathcal{X}} g(x) d\mu(x) = 0$. The result holds true immediately.

**Case (ii)** $\int_{\mathcal{X}} g(x) d\mu(x) = A \neq 0$. Let a ball $B(x_0, r_0) \supset \text{supp} (g)$. By Lemma 10.4, there exists $h \in \text{Lip}_{\epsilon, b}(\mu)$ such that $\text{supp} (h) \subset (B(x_0, r_0))$, $\int_{\mathcal{X}} h(x) d\mu(x) = 1$ and $\|h\|_{L^q(\mu)} < \frac{\eta}{2|A|}$, which implies that $\tilde{g} := g - Ah \in \tilde{\text{Lip}}_{\epsilon, b}(\mu)$ and

$$\|f - \tilde{g}\|_{L^q(\mu)} \leq \|f - g\|_{L^q(\mu)} + |A|\|h\|_{L^q(\mu)} < \frac{\eta}{2} + |A|\frac{\eta}{2|A|} = \eta.$$ 

This, together with **Case (i)**, finishes the proof of Lemma 10.2.

Recall that the *Littlewood-Paley S-function* $S(f)(x)$ for any $f \in L^q(\mu)$ with $q \in (1, \infty)$ and $x \in \mathcal{X}$ is defined by

$$S(f)(x) := \left\{ \int_{\Gamma(x)} |D_t(f)(y)|^2 d\mu(y) dt V_t(x) t \right\}^{1/2},$$

where

$$\Gamma(x) := \{(y, t) \in \mathcal{X} \times (0, \infty) : d(y, x) < t\},$$

generalizes the notion of a cone with vertex at $x$ and aperture 1.

In RD-spaces, Han, Müller and Yang [21] introduced the *Hardy space* $H^p(\mu)$ defined by

$$H^p(\mu) := \left\{ f \in (\tilde{G}_0^\ast(\beta, \gamma))^* : S(f) \in L^p(\mu) \right\}$$

equipped with the quasi-norm

$$\|f\|_{H^p(\mu)} := \|S(f)\|_{L^p(\mu)}.$$ 

Moreover, Grafakos, Liu and Yang [19] proved that $H^p_{\text{at}}(\mathcal{X})$ and $H^p(\mu)$ coincide with equivalent quasi-norms.

Before dealing with the relation between $\tilde{H}^{\mu, \eta, \gamma}_0(\mu)$ and $H^p_{\text{at}}(\mu)$, we need the following construction of dyadic cubes on spaces of homogeneous type from [7]; see also [21].
Lemma 10.5. Let $X$ be a space of homogeneous type. Then there exists a collection
\[ \{Q^k_\beta \subset X : k \in \mathbb{Z}, \beta \in I_k\} \]
of open subsets, where $I_k$ is some index set, and positive constants $\delta \in (0, 1)$ and $L_1, L_2$ such that
(i) $\mu(X \setminus \bigcup \beta Q^k_\beta) = 0$ for each fixed $k$, and $Q^k_\beta \cap Q^k_\gamma = \emptyset$ if $\beta \neq \gamma$;
(ii) for any $\beta, \gamma, k$ and $l$ with $l \geq k$, either $Q^l_\gamma \subset Q^k_\beta$ or $Q^l_\gamma \cap Q^k_\beta = \emptyset$;
(iii) for each $(k, \beta)$ and each $l < k$, there exists a unique $\gamma$ such that $Q^k_\beta \subset Q^l_\gamma$;
(iv) diam$(Q^k_\beta) \leq L_1 \delta^k$;
(v) each $Q^k_\beta$ contains some ball $B(z^k_\beta, L_2 \delta^k)$, where $z^k_\beta \in X$.

We further introduce some notation from [21]. Let
\[ \mathcal{R} := \{Q^k_\beta \subset X : k \in \mathbb{Z}, \beta \in I_k\}. \]
For any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in X : S(f)(x) > 2^k\}$ and
\[ \mathcal{R}_k := \{Q \in \mathcal{R} : \mu(Q \cap \Omega_k) > \frac{1}{2} \mu(Q) \text{ and } \mu(Q \cap \Omega_{k+1}) \leq \frac{1}{2} \mu(Q)\}. \]
Moreover, for any $Q^k_\beta \in \mathcal{R}$, let
\[ \hat{Q}^k_\beta := \{(x, t) \in X \times (0, \infty) : x \in Q^k_\beta \text{ and } L_1 \delta^k < t \leq L_1 \delta^{k-1}\}, \]
\[ \mathcal{R}^\text{mc}_k := \{Q \in \mathcal{R}_k : \text{if } \hat{Q} \supset Q \text{ and } \tilde{Q} \in \mathcal{R}, \text{ then } \tilde{Q} \notin \mathcal{R}_k\}, \]
and, for any $Q^\text{mc}_k \in \mathcal{R}^\text{mc}_k$,
\[ \hat{Q}^\text{mc}_k := \bigcup_{Q \in \mathcal{R}_k, Q \subset Q^\text{mc}_k} \hat{Q}. \]
Here and hereafter, “mc” means maximal cubes.

We need the following useful lemma.

Lemma 10.6. Let $k, j \in \mathbb{Z}$ and $k < j$. Then
\[ \left( \bigcup_{Q^\text{mc}_k \in \mathcal{R}^\text{mc}_k} \hat{Q}^\text{mc}_k \right) \cap \left( \bigcup_{Q^\text{mc}_j \in \mathcal{R}^\text{mc}_j} \hat{Q}^\text{mc}_j \right) = \emptyset. \]

Proof. Let $k, j \in \mathbb{Z}$ and $k < j$. To prove this lemma, it suffices to prove that, for any two dyadic cubes $Q$ and $P$ satisfying $Q \in \mathcal{R}_k$, $Q \subset Q^\text{mc}_k \in \mathcal{R}^\text{mc}_k$, $P \in \mathcal{R}_j$ and $P \subset Q^\text{mc}_j \in \mathcal{R}^\text{mc}_j$, it holds true that $\hat{Q} \cap \hat{P} = \emptyset$.

We prove it by contradiction. Suppose that $\hat{Q} \cap \hat{P} \neq \emptyset$. Then $Q \cap P \neq \emptyset$. By Lemma 10.5(ii), $Q \subset P$. Without loss of generality, we may assume that $Q \subset P$. Let $Q = Q^m_\beta$ and $P = Q^n_\gamma$ for some $m, n \in \mathbb{Z}$, $\beta \in I_m$, and $\gamma \in I_n$. If $m \neq n$, then $\hat{Q} \cap \hat{P} = \emptyset$, which contradicts to the assumption that $\hat{Q} \cap \hat{P} \neq \emptyset$. Thus, $m = n$, which, together with Lemma 10.5(i), implies that $Q^m_\beta \subset Q^n_\gamma$. Moreover, from $Q \in \mathcal{R}_k$, it follows that $\mu(Q \cap \Omega_k) > \frac{1}{2} \mu(Q)$ and $\mu(Q \cap \Omega_{k+1}) \leq \frac{1}{2} \mu(Q)$. On the other hand, by $P \in \mathcal{R}_j$, we see that $\mu(P \cap \Omega_j) > \frac{1}{2} \mu(P)$ and $\mu(P \cap \Omega_{j+1}) \leq \frac{1}{2} \mu(P)$. Meanwhile, $k < j$ implies that $\Omega_j \subset \Omega_{k+1}$ and hence $\mu(P \cap \Omega_{k+1}) > \frac{1}{2} \mu(P)$. Thus, $P \neq Q$, which contradicts to the fact that $Q = Q^m_\beta = Q^n_\gamma = P$. This finally finishes the proof of Lemma 10.6. $\blacksquare$

Now we introduce some useful decompositions of $D_i(x, y)$ in Lemma 10.1 which are easy consequences of [22, Proposition 2.9], the details being omitted.

Lemma 10.7. Let $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \infty)$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$, $\tilde{\epsilon} \in (\epsilon, \epsilon_1 \wedge \epsilon_2)$ and $\{D_i\}_{t \in (0, \infty)}$ be as in Lemma 10.1. Then, for any $N \in (0, \epsilon_1]$, $t \in (0, \infty)$ and $x, y \in X$,
\[ \tilde{D}_i(x, y) = \sum_{\ell=0}^{\infty} 2^{-\ell N} \varphi_{2^\ell i}(x, y), \]
where \( \varphi_{2^t}(x, y) \) is an adjust bump function in \( x \) associated with the ball \( B(y, 2^t) \), which means that there exists a positive constant \( C \) such that, for all \( t \in (0, \infty) \) and \( y \in \mathcal{X} \),

(i) \( \text{supp}(\varphi_{2^t}(\cdot, y)) \subset B(y, 2^t) \);

(ii) \( |\varphi_{2^t}(x, y)| \leq C \frac{1}{2^{\rho_t(y)}} \) for all \( x \in \mathcal{X} \);

(iii) \( \|\varphi_{2^t}(\cdot, y)\| \leq C (2^t)^{-\gamma} \frac{1}{2^{\rho_t(y)}} \) for all \( 0 < \gamma \leq \varepsilon \);

(iv) \( \int_{\mathcal{X}} \varphi_{2^t}(x, y) \, d\mu(x) = 0 \).

Then we introduce a useful criteria for the boundedness of some integral operators. Hereafter, we denote the inner product of \( L^2(\mu) \) by \( \langle \cdot, \cdot \rangle \).

**Lemma 10.8.** Let \( K_t(\cdot, \cdot) \) for \( t \in (0, \infty) \) be a measurable function from \( \mathcal{X} \times \mathcal{X} \) to \( \mathbb{C} \) and \( \{K_t\}_{t \in (0, \infty)} \) a set of \( L^2(\mu) \)-bounded linear operators defined by

\[
K_t(f)(x) := \int_{\mathcal{X}} K_t(x, y) f(y) \, d\mu(y) \quad \text{for all } t \in (0, \infty), \ x \in \mathcal{X} \text{ and } f \in L^2(\mu).
\]

If there exist positive constants \( \epsilon_1, \epsilon_2 \) and \( C \) such that, for all \( x, y \in \mathcal{X} \) and \( s, t \in (0, \infty) \),

\[
|K_t(x, y)| \leq C \left( \frac{t}{s} \wedge \frac{s}{t} \right)^{\epsilon_1} \frac{1}{V_t(x) + V_t(y) + V(x, y)} \left[ \frac{t}{t + d(x, y)} \right]^{\epsilon_2},
\]

and

\[
|K_t K_s^*(x, y)| \leq C \left( \frac{t}{s} \wedge \frac{s}{t} \right)^{\epsilon_1} \frac{1}{V_t(x) + V_t(y) + V(x, y)} \left[ \frac{t}{t + s + d(x, y)} \right]^{\epsilon_2},
\]

where \( K_s^* \) is the adjoint operator of \( K_t \). Then there exists a positive constant \( \tilde{C} \) such that, for all \( f \in L^2(\mu) \),

\[
\|G(f)\|_{L^2(\mu)} \leq \tilde{C} \|f\|_{L^2(\mu)},
\]

where, for all \( x \in \mathcal{X} \),

\[
G(f)(x) := \left\{ \int_0^\infty |K_t(x)|^2 \, dt \right\}^{1/2}.
\]

**Proof.** For any \( f \in L^2(\mu) \), by the Fubini theorem, we write

\[
\|G(f)\|_{L^2(\mu)}^2 = (G(f), G(f)) = \int_{\mathcal{X}} \int_0^\infty |K_t(f(x)|^2 \, dt \, d\mu(x)
\]

\[
= \int_0^\infty \int_{\mathcal{X}} |K_t(f)(x)|^2 \, d\mu(x) \, dt = \lim_{N \to \infty} \int_{1/N}^N (K_t(f), K_t(f)) \, dt
\]

\[
= \lim_{N \to \infty} \int_{1/N}^N \int_{1/N}^N K_t^* K_t(f)(x) \, f(x) \, d\mu(x) \, dt
\]

\[
= \lim_{N \to \infty} \int_{1/N}^N \int_{1/N}^N K_t^* K_t(f)(x) \, f(x) \, d\mu(x) \, dt
\]

Moreover, from (10.3), (10.4) and the Schur lemma (see [17, p. 457]), we deduce that, for all \( s, t \in (0, \infty) \), \( K_t^* \) and \( K_{s}^* \) are bounded on \( L^2(\mu) \) and

\[
\|K_t^* K_s^* K_s\|_{L^2(\mu) \to L^2(\mu)} \lesssim \|K_t K_s^*\|_{L^2(\mu) \to L^2(\mu)} \lesssim \left( \frac{t}{s} \wedge \frac{s}{t} \right)^{\epsilon_1}.
\]

By the above two inequalities and [18, p. 237, Exercise 8.5.8], we conclude that

\[
\|G(f)\|_{L^2(\mu)}^2 \leq \liminf_{N \to \infty} \int_{1/N}^N K_t^* K_t(f) \, dt \|f\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\mu)}^2,
\]

which completes the proof of Lemma 10.8.
Before showing the main result of this section, we introduce another technical lemma which gives a sufficient condition to the fact that \( f = g \) in \( L^2(\mu) \) for all \( f, g \in (\mathcal{G}_0(\beta, \gamma))^* \).

**Lemma 10.9.** Let \( \epsilon_1 \) be as in (A2) and \( \beta, \gamma \in (0, \epsilon_1) \). If \( f, g \in L^2(\mu) \) and \( f = g \in (\mathcal{G}_0(\beta, \gamma))^* \), then \( f = g \) in \( L^2(\mu) \).

**Proof.** For any \( f, g \in L^2(\mu) \), let \( (f, g) := \int_X f(x)g(x) \, d\mu(x) \). Now we claim that \( (f, \cdot) \) is a bounded functional on \( \mathcal{G}_0(\beta, \gamma) \subset L^2(\mu) \). Indeed, by (A6) and the Hölder inequality, we conclude that, for any \( f \in L^2(\mu) \) and \( h \in \mathcal{G}_0(\beta, \gamma) \),

\[
|f, h| \leq \int_X |f(x)||h(x)| \, d\mu(x)
\]

which implies the claim.

From this, a density argument, Lemma 10.2 and \( f = g \) in \( (\mathcal{G}_0(\beta, \gamma))^* \), it follows that

\[
\|f - g\|_{L^2(\mu)} = \text{sup} \left\{ |(f - g, h)| : \|h\|_{L^2(\mu)} \leq 1 \right\}
\]

\[= \text{sup} \left\{ |(f - g, h)| : h \in \mathcal{G}_0(\beta, \gamma) \text{ and } \|h\|_{L^2(\mu)} \leq 1 \right\}
\]

\[= \text{sup} \left\{ |(f, h) - (g, h)| : h \in \mathcal{G}_0(\beta, \gamma) \text{ and } \|h\|_{L^2(\mu)} \leq 1 \right\} = 0,
\]

which completes the proof of Lemma 10.9. \( \square \)

Now we are ready to prove the following main result of this section.

**Theorem 10.10.** Let \( (X, d, \mu) \) be an RD-space with \( \mu(X) = \infty \), \( \frac{p}{p+1} < p < q < 2 \), \( \rho \in (1, \infty) \), \( \gamma \in [1, \infty) \) and \( \epsilon \in (0, \infty) \), where \( \nu \) is as in (10.1). Then \( \mathcal{H}^{p, q, \gamma, \epsilon}_{\text{mb}, \rho}(\mu) \), \( \mathcal{H}^{p, q, \gamma}_{\text{at}, \rho}(\mu) \) and \( \mathcal{H}^{p, \rho}(\mu) \) coincide with equivalent quasi-norms.

**Proof.** Let \( p, q, \rho, \gamma \) and \( \epsilon \) be as in assumptions of Theorem 10.10. We first claim that \( \mathcal{H}^{p, \rho}_\text{at}(\mu) \cap L^2(\mu) \) is dense in \( \mathcal{H}^{p, \rho}_\text{at}(\mu) \). Indeed, for any \( f \in \mathcal{H}^{p, \rho}_\text{at}(\mu) \), by Theorem 9.4, we have \( f = \sum_{i=1}^{\infty} \lambda_i b_i \in (\text{Lip}_{1/p-1}(\mu))^* \), where \( \{b_i\} \) is a sequence of \( (p, 2) \)-atoms, \( \text{supp}(b_i) \subset B_i \) for some ball \( B_i \) and \( \|b_i\|_{L^2(\mu)} \leq [\mu(B_i)]^{1/2-1/p} \).

Let \( f_N := \sum_{i=1}^{N} \lambda_i b_i \), \( N \in \mathbb{N} \). Then \( f_N \in L^2(\mu) \) for all \( N \in \mathbb{N} \). Meanwhile, \( f - f_N = \sum_{i=N+1}^{\infty} \lambda_i b_i \in (\text{Lip}_{1/p-1}(\mu))^* \), and \( \|f - f_N\|_{L^2(\mu)} \leq \sum_{i=N+1}^{\infty} |\lambda_i|^p \to 0 \) as \( N \to \infty \). Thus, \( \mathcal{H}^{p, \rho}_\text{at}(\mu) \cap L^2(\mu) \) is dense in \( \mathcal{H}^{p, \rho}_\text{at}(\mu) \), which completes the proof of this claim.

We easily observe that \( L^2(\mu) \subset (\mathcal{G}_0(\beta, \gamma))^* \). By [19, Remark 5.5(ii)], we know that \( \mathcal{H}^{p, \rho}_\text{at}(\mu) = \mathcal{H}^{p, \rho}(\mu) \) with equivalent quasi-norms. By this, the above claim and a standard density argument, to show Theorem 10.10, it suffices to prove that

\[
(\mathcal{H}^{p, \rho}_\text{at}(\mu) \cap L^2(\mu)) \subset \mathcal{H}^{p, q, \gamma}_{\text{at}, \rho}(\mu) \subset \mathcal{H}^{p, q, \gamma}_{\text{mb}, \rho}(\mu) \subset (\mathcal{H}^{p, \rho}(\mu) \cap L^2(\mu)).
\]

We show this by two steps.

**Step 1.** Now we show that \( (\mathcal{H}^{p, \rho}_\text{at}(\mu) \cap L^2(\mu)) \subset \mathcal{H}^{p, q, \gamma}_{\text{at}, \rho}(\mu) \). By Remark 3.3(iii) and \( q \in (1, 2) \), it suffices to show that \( (\mathcal{H}^{p, \rho}_\text{at}(\mu) \cap L^2(\mu)) \subset \mathcal{H}^{p, q, \gamma}_{\text{at}, \rho}(\mu) \). To this end, by Lemma 10.1 and an argument similar to that
used in the proof of [22, p. 1524, (2.30)], we know that, for any \( f \in (\mathcal{G}_0'(\beta, \gamma))^* \) with \( 0 < \beta, \gamma < \epsilon < 1 \wedge \epsilon_2, \epsilon_3 \in (0, \infty) \) and \( \epsilon \in (\epsilon_1 \wedge \epsilon_2) \) \((\epsilon_2 \text{ and } \epsilon_3 \text{ are as in (A2))},

\[
f(x) = \sum_{t=0}^{\infty} 2^{-Nt} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{R}^m} \int_{Q_{mc}} \varphi_{2^t}(x, y) D_t(f)(y) \frac{d\mu(y)dt}{t} \quad \text{in} \quad (\mathcal{G}_0'(\beta, \gamma))^*,
\]

where \( \varphi_{2^t}(x, y) \) is as in Lemma 10.7. Then we show that, for any \( f \in H_{\alpha}^m(\mu) \cap L^2(\mu), \)

\[
f(x) = \sum_{t=0}^{\infty} 2^{-Nt} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{R}^m} \int_{Q_{mc}} \varphi_{2^t}(x, y) D_t(f)(y) \frac{d\mu(y)dt}{t} \quad \text{in} \quad L^2(\mu). \quad (10.5)
\]

By Lemma 10.9, it suffices to prove that

\[
\left\| \sum_{t=0}^{\infty} 2^{-Nt} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{R}^m} \int_{Q_{mc}} \varphi_{2^t}(\cdot, y) D_t(f)(y) \frac{d\mu(y)dt}{t} \right\|_{L^2(\mu)} < \infty.
\]

To this end, for any \( x, y \in \mathcal{X}, t, s \in (0, \infty) \) and \( f \in L^2(\mu), \) let

\[
\varphi_{2^t}(f)(x) := \int_{\mathcal{X}} \varphi_{2^t}(x, z) f(z) \, d\mu(z).
\]

Thus

\[
\varphi_{2^t}^s(f)(x) := \int_{\mathcal{X}} \varphi_{2^t}(x, z) f(z) \, d\mu(z)
\]

and

\[
\varphi_{2^t}^s \varphi_{2^t}(x, y) := \int_{\mathcal{X}} \varphi_{2^t}(x, z) \varphi_{2^t}(z, y) \, d\mu(z).
\]

By a duality method, Lemma 10.2, the Fubini theorem, the Hölder inequality, Lemma 10.6 and [21, Proposition 2.14], we obtain
where, in the third equality of the above equation, we used the fact that, for any $Q_{k}^\text{mc} \in \mathcal{R}_{k}^\text{mc}$, $f \in L^2(\mu)$ and $h \in Q_{k}^\text{mc}(\beta, \gamma)$,

$$O := \int_{\mathcal{X}} \int_{Q_{k}^\text{mc}} |\varphi_{2t \ell}(x, y)||h(x)||D_1(f)(y)||d\mu(y)\frac{dt}{\ell}d\mu(x) < \infty.$$ 

Indeed, let $Q_{k}^\text{mc} := Q_{k0}^{\beta_0}$ for some $k_0 \in \mathbb{Z}$ and some $\beta_0 \in I_{k_0}$. By the Fubini-Tonelli theorem, (i) and (ii) of Lemma 10.7, (A6), $Q_{k}^\text{mc} \subset Q_{k0}^{\beta_0} \times (0, 1, 6^{\phi_0^{-1}}]$ (see [21, p. 1524]) and [21, Proposition 2.14], we conclude that

$$O = \int_{Q_{k}^\text{mc}} \left[ \int_{\mathcal{X}} |\varphi_{2t \ell}(x, y)||h(x)||D_1(f)(y)||d\mu(y)\frac{dt}{\ell}d\mu(x) \right] \leq \int_{Q_{k}^\text{mc}} \left[ \int_{B(y, 2t \ell)} \frac{|h(x)|}{V_{2t \ell}(y)}d\mu(x) \right] |D_1(f)(y)||d\mu(y) \int_{Q_{k}^\text{mc}} |D_1(f)(y)||d\mu(y)$$

$$\leq \|h\|Q_{k}^\text{mc}(\beta, \gamma) \int_{Q_{k}^\text{mc}} |D_1(f)(y)||d\mu(y) \int_{Q_{k}^\text{mc}} |D_1(f)(y)||d\mu(y)$$

$$\leq \|h\|Q_{k}^\text{mc}(\beta, \gamma) \int_{Q_{k}^\text{mc}} |D_1(f)(y)||d\mu(y) \int_{Q_{k}^\text{mc}} |D_1(f)(y)||d\mu(y)$$

$$\leq \|h\|Q_{k}^\text{mc}(\beta, \gamma) \int_{Q_{k}^\text{mc}} \left[ \mu \left( Q_{k0}^{\beta_0} \right) L_1 \delta_0 \right]^{1/2} \left[ \int_{\mathcal{X}} |D_1(f)(y)||d\mu(y) \right]^{1/2}$$

which implies the desired result.

Let $\Phi_t(h)(y) := \{ \int_{0}^{\infty} |2^{-Nt/2} \varphi_{2t \ell}(h)(y)|^{2} \frac{dt}{\ell} \}^{1/2}$ for all $y \in \mathcal{X}$. To prove (10.5), we only need to show that

$$\|\Phi_t(h)\|_{L^2(\mu)} \leq \|h\|_{L^2(\mu)},$$

(10.6)

where the implicit positive constant is independent of $\ell$.

By Lemma 10.8, we need to show that $\{ \varphi_{2t \ell} \}_{t \in (0, \infty)}$ satisfy (10.3) and (10.4). From Lemma 10.7, we easily deduce that (10.3) holds true for $\{ \varphi_{2t \ell} \}_{t \in (0, \infty)}$. Thus, it suffices to show that, for all $x, y \in \mathcal{X}$, $s, t \in (0, \infty)$,

$$\left(2^{-Nt/2} \varphi_{2t \ell}(x, y)\right)^{s} \left(2^{-Nt/2} \varphi_{2t \ell}(x, y)\right)^{t} \leq 2^{Nt} \left( \frac{s}{t} \right)^{\eta} \left[ \frac{1}{V_{t+s}(x) + V_{t+s}(y) + V(x, y)} \left[ \frac{t \wedge s}{t \wedge s + d(x, y)} \right] \right]^{N/2}.$$

Due to the symmetry of $s$ and $t$, without loss of generality, we may assume that $s < t$. Thus, we only need to show that, for all $x, y \in \mathcal{X}$, $0 < s < t < \infty$,

$$|\varphi_{2t \ell} \varphi_{2t s}(x, y)| \leq 2^{Nt} \left( \frac{s}{t} \right)^{\eta} \left[ \frac{1}{V_{t}(x) + V_{t}(y) + V(x, y)} \left[ \frac{t}{t + d(x, y)} \right] \right]^{N/2}.$$

By Lemma 10.7(iv), we write

$$|\varphi_{2t \ell} \varphi_{2t s}(x, y)| \leq \int_{\mathcal{X}} |\varphi_{2t \ell}(z, x) - \varphi_{2t \ell}(y, x)| |\varphi_{2t s}(z, y)| d\mu(z)$$

$$\leq \int_{\{ z \in \mathcal{X} : d(z, y) \leq \frac{t \wedge s + d(x, y)}{2} \}} |\varphi_{2t \ell}(z, x) - \varphi_{2t \ell}(y, x)| |\varphi_{2t s}(z, y)| d\mu(z)$$

$$+ \int_{\{ z \in \mathcal{X} : d(z, y) > \frac{t \wedge s + d(x, y)}{2} \}} |\varphi_{2t \ell}(z, x) | |\varphi_{2t s}(z, y)| d\mu(z)$$
Now we turn to estimate $I_1$. Observe that, if $z \in B(x, 2^\ell t)$ and \( \frac{2^{\ell + d(x,y)}}{2} \geq d(y, z) \geq d(x, y) - d(x, z) \), then $y \in B(x, 2^{\ell + 2}t)$; if $d(x,y) < 2^{\ell + 2}t$, then
\[
\left( \frac{t}{t + d(x,y)} \right)^{N/2} \geq \left( \frac{t}{t + d(x,y)} \right)^N \geq 2^{-N\ell}.
\]
From the above two facts, (i), (ii) and (iii) of Lemma 10.7 and Remark 2.4(ii), it follows that
\[
I_1 \lesssim \int_{\{z \in X: \ d(z,y) \leq \frac{2^{\ell + d(x,y)}}{2}\}} \left\| \varphi_{2^t \ell}(\cdot, x) \right\|_\eta \left[ d(y, z) \right]^\eta \frac{\chi_{B(x,2^\ell t)}(y)}{V_{2^\ell t}(x)} \frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)} d\mu(z).
\]
\[
\lesssim \int_{\{z \in X: \ d(z,y) \leq \frac{2^{\ell + d(x,y)}}{2}\}} \frac{1}{(2^\ell)^\eta} \frac{\chi_{B(x,2^\ell t)}(y)}{V_{2^\ell t}(x)} \frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)} d\mu(z).
\]
\[
\lesssim \left( \frac{s}{t} \right)^\eta \int_{\{z \in X: \ d(z,y) \leq \frac{2^{\ell + d(x,y)}}{2}\}} \frac{\chi_{B(x,2^{\ell + 1}t)}(y)}{V_{2^{\ell + 1}t}(x)} \frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)} d\mu(z).
\]
\[
\lesssim 2^{N\ell} \left( \frac{s}{t} \right)^\eta \frac{1}{V_{2^{\ell + 1}t}(x) + V_{2^{\ell + 1}t}(y) + V(x, y)} \left[ \frac{t}{t + d(x,y)} \right]^{N/2}.
\]
Now we turn to estimate $I_2$. Observe that, if $\{z \in X: \ d(z,y) > \frac{2^{\ell + d(x,y)}}{2}\} \neq \emptyset$ and $z \in B(x, 2^\ell t) \cap B(y, 2^\ell s)$, then $y \in B(x, 2^{\ell + 1}t)$ and $1 \lesssim \left( \frac{s}{t} \right)^\eta$. By the above facts, (i) and (ii) of Lemma 10.7 and some arguments similar to those used in the estimate for $I_1$, we further have
\[
I_2 \lesssim \int_{\{z \in X: \ d(z,y) > \frac{2^{\ell + d(x,y)}}{2}\}} \frac{\chi_{B(x,2^\ell t)}(z)}{V_{2^\ell t}(x)} \frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)} d\mu(z).
\]
\[
\lesssim \frac{\chi_{B(x,2^{\ell + 1}t)}(y)}{V_{2^{\ell + 1}t}(x)} \int_{\{z \in X: \ d(z,y) > \frac{2^{\ell + d(x,y)}}{2}\}} \frac{\chi_{B(y,2^\ell s)}(z)}{V_{2^\ell s}(y)} d\mu(z).
\]
\[
\lesssim \left( \frac{s}{t} \right)^\eta \frac{1}{V_{2^{\ell + 1}t}(x) + V_{2^{\ell + 1}t}(y) + V(x, y)} \left[ \frac{t}{t + d(x,y)} \right]^{N/2}.
\]
For $I_3$, by some arguments similar to those used in the estimate for $I_2$, we see that
\[
I_3 \lesssim \int_{\{z \in X: \ d(z,y) > \frac{2^{\ell + d(x,y)}}{2}\}} \frac{\chi_{B(x,2^\ell t)}(z)}{V_{2^\ell s}(y)} d\mu(z)
\]
\[
\lesssim 2^{N\ell} \left( \frac{s}{t} \right)^\eta \frac{1}{V_{2^\ell t}(x) + V_{\ell}(y) + V(x, y)} \left[ \frac{t}{t + d(x,y)} \right]^{N/2}.
\]
Combining the estimates for $I_1$, $I_2$ and $I_3$, we finish the proof of (10.6) and hence (10.5).

Moreover, from the proof of [21, Theorem 2.21], it follows that, for any $f \in H^p_{\text{at}}(\mu) \cap L^2(\mu)$, there exists a positive constant $L_3$ such that
\[
f(x) = \sum_{\ell=0}^{\infty} 2^{-N\ell} \sum_{k \in \mathbb{Z}} \sum_{Q_{\ell} \in \mathcal{R}_{\ell}^k} \chi_{Q_{\ell}} f Q_{\ell} \in L^2(\mu),
\]
where
\[ \lambda^{L_{\infty}}_{Q_k} = L_3 [\mu (B^{mc}_k)]^{1/p-1/2} \left[ \int_{Q_k^{mc}} |D_k (f)(y)|^2 \frac{d\mu(y)dt}{t} \right]^{1/2}, \]

\[ Q_k^{mc} := Q_{\beta_k}, \quad B^{mc}_k := B(\beta_k, (\frac{1}{\delta} + 1)L_1 2^d \delta_k) \] and, for all \( x \in \mathcal{X}, \)

\[ a^{L_{\infty}}_{Q_k}(x) := \frac{1}{\lambda^{L_{\infty}}_{Q_k}} \int_{Q_k^{mc}} \varphi^{2_L} (x, y) D_k (f)(y) \frac{d\mu(y)dt}{t} \]
is a \((p, 2)\)-atom supported on \( B^{mc}_k \). By an argument similar to that used in the proof of (9.6), we further conclude that \( a^{L_{\infty}}_{Q_k} \) is also a \((p, 2, \gamma, \rho)\)-atomic block and \( |a^{L_{\infty}}_{Q_k}| \leq \lambda^{L_{\infty}}_{Q_k} \). Thus, \( f \in H^{p, \gamma, \rho}_m(\mu) \) and

\[ \|f\|_{H^{p, \gamma, \rho}_m(\mu)} \leq \|f\|_{H^{p, \gamma, \rho}_m(\mu)}, \]

which completes the proof of Step 1.

**Step 2.** In this step, we show that \( H^{p, \gamma, \rho}_m(\mu) \subset H^{p, \gamma, \rho}_m(\mu) \subset (H^p(\mu) \cap L^2(\mu)) \) for any \( q \in (1, \infty) \). By Proposition 4.3, we see that \( H^{p, \gamma, \rho}_m(\mu) \subset H^{p, \gamma, \rho}_m(\mu) \) for any \( q \in (1, \infty) \). Thus, to prove the desired conclusion, it suffices to show that \( H^{p, \gamma, \rho}_m(\mu) \subset (H^p(\mu) \cap L^2(\mu)) \) for any \( q \in (1, \infty) \).

We first reduce the proof to showing that, if \( b \) is a \((p, q, \gamma, \rho)\)-molecular block, then

\[ S(b) \in L^p(\mu) \quad \text{and} \quad \|S(b)\|_{L^p(\mu)} \leq \|b\|_{H^{p, q, \gamma, \rho}_m(\mu)}^{\gamma}. \tag{10.7} \]

Indeed, assume that (10.7) holds true. For any \( f \in H^{p, q, \gamma, \rho}_m(\mu) \), by Definition 4.1, we know that there exists a sequence \( \{b_i\}_{i \in \mathbb{N}} \) of \((p, q, \gamma, \rho)\)-molecular blocks such that \( f = \sum_{i=1}^{\infty} b_i \) in \( L^2(\mu) \) and

\[ \sum_{i=1}^{\infty} \|b_i\|_{H^{p, q, \gamma, \rho}_m(\mu)}^{\gamma} \sim \|f\|_{H^{p, q, \gamma, \rho}_m(\mu)}^{\gamma}. \tag{10.8} \]

Notice that \( D_t (y, \cdot) \in L^2(\mu) \) for any \( y \in \mathcal{X} \) and \( t \in (0, \infty) \). Thus, for any \( y \in \mathcal{X} \), we have

\[ |D_t (f)(y)| = |(D_t (y, \cdot), f)| \leq \sum_{i=0}^{\infty} |(D_t (y, \cdot), b_i)| = \sum_{i=0}^{\infty} |D_t (b_i)(y)|. \]

From this, the Fatou lemma, (10.7) and (10.8), we deduce that

\[ \|S(f)\|_{L^p(\mu)} \leq \sum_{i=1}^{\infty} \|S(b_i)\|_{L^p(\mu)} \leq \sum_{i=1}^{\infty} \|b_i\|_{H^{p, q, \gamma, \rho}_m(\mu)} \sim \|f\|_{H^{p, q, \gamma, \rho}_m(\mu)}, \]

which completes the proof of Step 2.

Now we prove (10.7) following the ideas of the proof of Theorem 4.8. For the sake of simplicity, we assume that \( \gamma = 1 \) and \( \rho = 2 \). Let \( b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \lambda_k, j \) be an \((p, q, 1, 2)\)-molecular block as in Definition 4.1 with \( \gamma = 1 \) and \( \rho = 2 \), where, for any \( k \in \mathbb{Z}_+ \) and \( j \in \{1, \ldots, M_k\} \), \( \sum a_{k, j} \subset \mathcal{U}_k(B) \) for some \( \mathcal{U}_k(B) \) as in Definition 4.1. Without loss of generality, we may assume that \( M = M \) in Definition 4.1. Since \( S \) is sublinear, we write

\[ \|S(b)\|_{L^p(\mu)} \leq \sum_{k=0}^{\infty} \int_{\mathcal{U}_k(B)} \left| S \left( \sum_{j=1}^{M_k} \lambda_k, j m_k, j \right)(x) \right|^p d\mu(x) \]

\[ + \sum_{k=0}^{\infty} \int_{\mathcal{U}_k(B)} \left| S \left( \sum_{j=1}^{M_k} \lambda_k, j m_k, j \right)(x) \right|^p d\mu(x) \]

\[ + \sum_{k=0}^{\infty} \int_{\mathcal{U}_k(B)} \left| S \left( \sum_{j=1}^{M_k} \lambda_k, j m_k, j \right)(x) \right|^p d\mu(x) + \sum_{k=0}^{4} \int_{\mathcal{U}_k(B)} |S(b)(x)|^p d\mu(x) \]
For any \( x, y, z \) and \( c \) where \( \delta_B \leq c \leq z \), Moreover, by \( \Gamma(x) \) and \( (r, c) \), denote the center and the radius of \( B \), respectively. Thus, \( d(x, z) \geq d(y, z) - d(y, x) \geq \frac{1}{2} d(x, z) \). It then follows, from this, (A1) and (10.1), that

\[
M_1(x, z) := \left[ \int_{B(x, \delta_B \leq c \leq z)} \int_{\{y,t\in \mathcal{X} \times (0,\infty)\}} |D_t(y, z)|^2 \frac{d\mu(y)dt}{V_t(x)t} \right]^{1/2},
\]

\[
M_2(x, z) := \left[ \int_{B(x, \delta_B \leq c \leq z)} \int_{\{y,t\in \mathcal{X} \times (0,\infty)\}} |D_t(y, z)|^2 \frac{d\mu(y)dt}{V_t(x)t} \right]^{1/2}.
\]

Moreover, by \( z \in B_{k,j} \subset 2^{k+2}B \setminus 2^{k-2}B \), \( k \geq \ell + 5 \), \( x \in 2^{\ell+2}B \setminus 2^{\ell-2}B \), we have \( d(x, c_B) < 2^{\ell+2}r_B \), \( d(z, c_B) \geq 2^{k-2}r_B \geq 2^{k+3}r_B \) and

\[
d(x, z) \geq d(z, c_B) - d(x, c_B) \geq 2^{k+3}r_B - 2^{\ell+2}r_B = 2^{\ell+2}r_B > d(x, c_B),
\]

where \( c_B \) and \( r_B \) denote the center and the radius of \( B \), respectively. Thus, for all \( x \in \mathcal{X} \),

\[
S \left( \sum_{k=\ell+5}^\infty \sum_{j=1}^M \lambda_{k,j}m_{k,j} \right)(x) \lesssim \sum_{k=\ell+5}^\infty \sum_{j=1}^M |\lambda_{k,j}| \frac{1}{V(x, c_B)} \|m_{k,j}\|_{L^1(\mu)}.
\]
From (10.9), the Hölder inequality, (4.1) and (10.1), we deduce that

\[
\text{III} \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_{\ell}(B)} \frac{1}{V(x,cB)^p} d\mu(x) \|m_{k,j}\|_{L^1(\mu)}^p \\
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \frac{V_{2\ell+2r_B}(cB)}{V_{2\ell-r_B}(cB)} \|m_{k,j}\|_{L^q(\mu)}^p |\mu(B_{k,j})|^{p/p'} \\
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p |V_{2\ell-r_B}(cB)|^{1-p} 2^{-kp} \left[\lambda(cB, 2^{k+1}r_B)\right]^{p-1} \\
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell+5}^{\infty} \sum_{j=1}^{M} 2^{-kp} |\lambda_{k,j}|^p \sim \sum_{j=1}^{M} \sum_{k=0}^{\infty} 2^{-kp} |\lambda_{k,j}|^p \lesssim \sum_{j=1}^{M} \sum_{k=0}^{\infty} |b_{k,j}|^p \sim |b|^p_{R_{m,b,2,1+r}\mu}.
\]

In order to estimate I, for all \(x \in \mathcal{X}\), we write

\[
S \left( \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right)(x) \\
\leq \left\{ \int_{\Gamma(x)} \left| \int_{X} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(z) [D_t(y,z) - D_t(y,cB)] d\mu(z) \right|^2 \frac{d\mu(y)dt}{V_t(x)t} \right\}^{1/2} \\
+ \left\{ \int_{\Gamma(x)} \left| \int_{X} \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}(z) D_t(y,z) d\mu(z) \right|^2 \frac{d\mu(y)dt}{V_t(x)t} \right\}^{1/2} =: M_3(x) + M_4(x).
\]

To estimate \(M_3(x)\), by the Minkowski inequality, we further write, for all \(x \in \mathcal{X}\),

\[
M_3(x) \leq \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}| \left\{ \int_{\Gamma(x)} \left| \int_{B_{k,j}} |m_{k,j}(z)| [D_t(y,z) - D_t(y,cB)] d\mu(z) \right|^2 \frac{d\mu(y)dt}{V_t(x)t} \right\}^{1/2} \\
\leq \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}| \int_{B_{k,j}} |m_{k,j}(z)| \left\{ \int_{\Gamma(x)} |D_t(y,z) - D_t(y,cB)|^2 \frac{d\mu(y)dt}{V_t(x)t} \right\}^{1/2} d\mu(z) \\
\leq \sum_{k=0}^{\ell-5} \sum_{j=1}^{M} |\lambda_{k,j}| \int_{B_{k,j}} |m_{k,j}(z)| [M_{3,1}(x,z) + M_{3,2}(x,z)] d\mu(z),
\]

where, for all \(x \in \mathcal{X}\) and \(z \in B_{k,j}\) with \(k \in \mathbb{Z}_+ \cap [0, \ell - 5]\) and \(j \in \{1, \ldots, M\},

\[
M_{3,1}(x,z) := \left[ \int_{\Gamma(x) \cap \{(y,t) \in \mathcal{X} \times (0,\infty) : t \leq \frac{d(z,y,cB)}{s}\} \frac{d\mu(y)dt}{V_t(x)t} \right]^{1/2},
\]

\[
M_{3,2}(x,z) := \left[ \int_{\Gamma(x) \cap \{(y,t) \in \mathcal{X} \times (0,\infty) : t > \frac{d(z,y,cB)}{s}\} \frac{d\mu(y)dt}{V_t(x)t} \right]^{1/2}.
\]

Now we give some observations. For any \(z \in B_{k,j} \subset 2^{k+2}B \setminus 2^{k-2}B\), \(k \in \mathbb{Z}_+ \cap [0, \ell - 5]\), \(j \in \{1, \ldots, M\}\) and \(y \in \Gamma(x)\), we have \(d(z,cB) < 2^{k+2}r_B\) and \(d(y,cB) \geq d(x,cB) - d(y,x) \geq 2^{\ell-2}r_B - t\) and hence

\[
d(y,cB) + t \geq 2^{\ell-2}r_B \geq 2^{k+3}r_B > 2d(z,cB).
\]
Meanwhile, for any \( y \in \Gamma(x) \cap \{(y, t) \in \mathcal{X} \times (0, \infty) : t \leq \frac{d(x,cb)}{8}\} \), we have
\[
d(y, cb) \geq d(x, cb) - d(x, y) \geq d(x, cb) - \frac{d(x, cb)}{8} = \frac{7}{8}d(x, cb).
\]

From these observations, (A3) and (10.1), it follows that, for all \( x \in \mathcal{X} \) and \( z \in B_{k,j} \) with \( k \in \mathbb{Z}_+ \cap [0, \ell - 5] \) and \( j \in \{1, \ldots, M\} \),
\[
\begin{align*}
M_{3,1}(x, z) & \lesssim \frac{2^k r_B}{d(x, cb)} \left( \int_0^{d(x, cb)} \int_{B(x,t)} \frac{1}{|V(c_B, x)|^2 |d(x, cb)|^{r_3}} \frac{t^{r_3-1} \, d\mu(y) \, dt}{V_t(x)} \right)^{1/2} \\
& \lesssim \frac{2^k r_B}{d(x, cb)} V(c_B, x)
\end{align*}
\]
and
\[
\begin{align*}
M_{3,2}(x, z) & \lesssim \frac{2^k r_B}{d(x, cb)} \left( \int_0^{d(x, cb)} \int_{B(x,t)} \frac{1}{|V(c_B, x)|^2 |d(x, cb)|^{r_3}} \frac{t^{r_3-1} \, d\mu(y) \, dt}{V_t(x)} \right)^{1/2} \\
& \lesssim \frac{2^k r_B}{d(x, cb)} V(c_B, x)
\end{align*}
\]
Combining the estimates of \( M_{3,1}(x, z) \) and \( M_{3,2}(x, z) \), we find that, for all \( x \in \mathcal{X} \),
\[
M_3(x) \lesssim \sum_{k=0}^{\ell-5} \sum_{j=1}^M |\lambda_{k,j}| \|m_{k,j}\|_{L^1(\mu)} \frac{2^k r_B}{d(x, cb)} \frac{1}{V(c_B, x)}.
\]

By this, the Hölder inequality, (4.1), (10.1) and \( p > \frac{\nu}{\nu+1} \), we conclude that
\[
\begin{align*}
&\int_{\mathcal{X}} \int_{U_t(B)} |M_3(x)|^p \, d\mu(x) \\
&\lesssim \sum_{k=0}^{\ell-5} \sum_{j=1}^M |\lambda_{k,j}| \|m_{k,j}\|_{L^1(\mu)} \int_{U_t(B)} \frac{2^k r_B}{|d(x, cb)|^p} \frac{1}{|V(c_B, x)|^p} \, d\mu(x) \\
&\lesssim \sum_{k=0}^{\ell-5} \sum_{j=1}^M |\lambda_{k,j}| \|m_{k,j}\|_{L^s(\mu)} \frac{2^k r_B}{2^{(\ell-2)p} r_B} \frac{V_2^{\ell+2r_B}(c_B)}{|V_2^{\ell+2r_B}(c_B)|^p} \\
&\lesssim \sum_{k=0}^{\ell-5} \sum_{j=1}^M |\lambda_{k,j}| 2^{k r_2 (k-\ell)p_2 (\ell-k) (1-p)\nu} \frac{V_2^{\ell+2r_B}(c_B)}{|V_2^{\ell+2r_B}(c_B)|^p} \frac{V_2^{\ell+2r_B}(c_B)}{|V_2^{\ell+2r_B}(c_B)|^p} \\
&\lesssim \sum_{k=0}^{\ell-5} \sum_{j=1}^M |\lambda_{k,j}| \|m_{k,j}\|_{L^1(\mu)} \int_0^{d(x, cb)} \int_{B(x,t)} \frac{1}{|V_t(x)|^2 |d(x, cb)|^{r_3}} \, d\mu(y) \, dt \\
&\lesssim \sum_{k=0}^{\ell-5} \sum_{j=1}^M |\lambda_{k,j}| \|m_{k,j}\|_{L^1(\mu)} \frac{1}{|V_t(x)|^2 |d(x, cb)|^{r_3}} \int_0^{d(x, cb)} \int_{B(x,t)} \frac{1}{|V_t(x)|^2 |d(x, cb)|^{r_3}} \, d\mu(y) \, dt \\
&\lesssim \int_\mathcal{X} b(x) \, d\mu(x) = 0 \quad \text{and some arguments similar to those used in the estimate of (10.9), we see that, for all } x \in \mathcal{X},
\end{align*}
\]
\[
M_4(x) = \left\{ \int_{\mathcal{X}} \left\{ \int_\mathcal{X} \sum_{k=0}^{\ell-4} \sum_{j=1}^M \lambda_{k,j} m_{k,j}(z) D_t(y, cb) \, d\mu(z) \right\}^2 \frac{d\mu(y) \, dt}{V_t(x)} \right\}^{1/2}
\]
that, for any
By this, the Minkowski inequality and an argument similar to that used in the estimate of (10.9), we know that
\[
\sum_{\ell=5}^{\infty} \int_{U_\ell(B)} |M_4(x)|^p \, d\mu(x) \lesssim |b|_{H^{\rho,q,1,1}_m}^p.
\]
Thus,
\[
I \lesssim \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} |M_3(x)|^p \, d\mu(x) + \sum_{\ell=5}^{\infty} \int_{U_\ell(B)} |M_4(x)|^p \, d\mu(x) \lesssim |b|_{H^{\rho,q,1,1}_m}^p.
\]
Then we turn to estimate II. We first write
\[
II \leq \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_k(B)} |S(m_{k,j})(x)|^p \, d\mu(x)
\]
\[
\leq \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{B_k,j} |S(m_{k,j})(x)|^p \, d\mu(x)
\]
\[
+ \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B) \setminus 2B_k,j} |S(m_{k,j})(x)|^p \, d\mu(x) =: II_1 + II_2.
\]
By the Hölder inequality, the \(L^q(\mu)\)-boundedness (\(q \in (1, \infty)\)) of \(S(f)\) and (4.1), we conclude that
\[
II_1 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[\mu(2B_k,j)\right]^{1-\frac{q}{p}} \|S(m_{k,j})\|_{L^q(\mu)}^p.
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[\mu(2B_k,j)\right]^{1-\frac{q}{p}} \|m_{k,j}\|_{L^q(\mu)}^p.
\]
\[
\lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^{p-2/k} \|m_{k,j}\|_{L^q(\mu)}^{1-p} \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p \lesssim |b|_{H^{\rho,q,1,1}_m}^p.
\]
To estimate \(II_2\), fix \(\ell \in \mathbb{N} \cap [5, \infty)\), \(k \in \{\ell-4, \ldots, \ell+4\}\), \(j \in \{1, \ldots, M\}\) and \(x \in U_\ell(B) \setminus 2B_k,j\). Notice that, for any \(z \in B_k,j\) and \(x \notin 2B_k,j\),
\[
d(x, z) \geq d(x, c_{B_k,j}) - d(z, c_{B_k,j}) \geq \frac{1}{2} d(x, c_{B_k,j}).
\]
By this, the Minkowski inequality and an argument similar to that used in the estimate of (10.9), we further obtain
\[
S(m_{k,j})(x) \leq \left\{ \int_{\Gamma(x)} \left[ \int_{B_k,j} |m_{k,j}(z)||D_t(y,z)| \, d\mu(z) \right] \, d\mu(y) \right\}^{1/2}
\]
\[
\leq \int_{B_k,j} |m_{k,j}(z)| \left[ \int_{\Gamma(x)} |D_t(y,z)|^2 \, d\mu(y) \right]^{1/2} \, d\mu(z)
\]
\[
\lesssim \frac{1}{V(x, c_{B_k,j})} \|m_{k,j}\|_{L^1(\mu)}.
\]
From this, the Hölder inequality, (4.1) and (10.1), we deduce that
\[
II_2 \lesssim \sum_{\ell=5}^{\infty} \sum_{k=\ell-4}^{\ell+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \int_{U_\ell(B) \setminus 2B_k,j} \frac{1}{V(x, c_{B_k,j})^p} \, d\mu(x) \|m_{k,j}\|_{L^1(\mu)}^p.
\]
Thus, \( H_{\text{atb}, \rho}^{p, q, \gamma} (\mu) = H_{\text{atb}, \rho}^{p} (\mu) = H_{\text{mb}, \rho}^{p, q, \gamma} (\mu) = H_{\text{mb}, \rho}^{p, q, \gamma, \epsilon} (\mu) \) over an RD-space \((\mathcal{X}, d, \mu)\) with \( \mu(\mathcal{X}) = \infty \).

(ii) It is still unclear whether \( \bar{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu) \) (or \( H_{\text{atb}, \rho}^{p} (\mu) \)) and \( \bar{H}_{\text{mb}, \rho}^{p, q, \gamma} (\mu) \) (or \( H_{\text{mb}, \rho}^{p, q, \gamma, \epsilon} (\mu) \)) coincide or not for any \( q \in (2, \infty] \) over RD-spaces \((\mathcal{X}, d, \mu)\) with \( \mu(\mathcal{X}) = \infty \).

(iii) Let \((\mathcal{X}, d, \mu) := (\mathbb{R}^D, |\cdot|, dx)\) with the \(D\)-dimensional Lebesgue measure \( dx, \rho \in (1, \infty), \gamma \in [1, \infty), \frac{\rho - 1}{\rho - 1} > p \leq 1 < q \leq 2\). Combining Propositions 8.1 and 8.2, and Theorems 10.10 and 9.4, we finally obtain 

\[
\int_{\mathbb{R}^D} a(x) \, dx = 0.
\]

Thus, \( a \) is a \((p, q)\)-atom. From this, Step 2 of the proof of Theorem 10.10 and the proof of [2, Theorem 5.4], we deduce that 

\[
(\mathcal{H}^p(\mathbb{R}^D) \cap L^2(\mathbb{R}^D)) = \bar{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu).
\]

Thus, \( \mathcal{H}^p(\mathbb{R}^D) = \bar{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu) \).

Moreover, by Step 2 of the proof of Theorem 10.10 and (10.10), we know that 

\[
\bar{H}_{\text{atb}, \rho}^{p, q, \gamma} (\mu) \subset \bar{H}_{\text{mb}, \rho}^{p, q, \gamma, \epsilon} (\mu) \subset \mathcal{H}^p(\mathbb{R}^D) \cap L^2(\mathbb{R}^D) = \bar{H}_{\text{atb}, \rho}^{p, q, \gamma, \epsilon} (\mu).
\]
Thus, by this and Theorems 9.4 and 7.9, we have

\[
\tilde{H}^{p,q,\gamma,\epsilon}_{mb,\rho} (\mu) = \tilde{H}^{p,q,\gamma}_{atb,\rho} (\mu) = H^p(D) = \tilde{H}^{p,q,\gamma}_{mb,\rho} (\mu).
\]

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