Elementary Derivations of the Real Composition Algebras
Adapted from Gadi Moran’s last paper

Tomer Moran∗ Shay Moran† Shlomo Moran‡
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Abstract

"Real Normed Algebras Revisited", the last paper of the late Gadi Moran, attempts to re-
construct the discovery of the complex numbers, the quaternions and the octonions, as well as
proofs of their properties, using only what was known to 19th century mathematicians. In his
research, Gadi had discovered simple and elegant proofs of the above mentioned classical results
using only basic properties of the geometry of Euclidean spaces and tools from high school
geometry. Although his reconstructions are different from the original derivations by Hamilton
and others [van der Waerden 1976], they underline an interesting connection between Euclidean
geometry and these algebras. The goal of this manuscript is to present Gadi’s derivations in a
way which is accessible to a wide audience of math readers.

1 Introduction

Inspired by how complex numbers could be represented as points in the plane, Sir William Rowan
Hamilton attempted to create an algebra in 3-dimensional space. Unaware of the fact that his
requirements were impossible to meet, it took him some time and effort before he eventually
realized, in 1843, that such an algebra required a fourth dimension. Hamilton then discovered
the quaternions, a non-commutative 4-dimensional algebra, and he devoted the remainder of his
life to studying and teaching them [6, 5]. Inspired by Hamilton’s discovery, Graves discovered
shortly later the octonions [3], and at about the same time they were discovered by Cayley [1].
Subsequently it was proved that the complex numbers, the quaternions and the octonions
are the only possible extensions of the real numbers to algebras which satisfy certain natural
properties, called real composition algebras. Additional information on these discoveries and
their applications can be found eg in [7].

In an attempt to reconstruct Hamilton’s discovery of the quaternions using only what was
then known to Hamilton and to other mathematicians of the 19th century, the late Gadi Moran
developed an alternative derivation of real composition algebras. Interestingly, Gadi’s proofs
only rely on elementary properties of the Euclidean norm (distance), which can make them more
tangible and easier to appreciate by students who first encounter the topic. As Gadi mentions [5],
it turns out that his derivation significantly diverges from Hamilton’s, as described in [6]. Thus
it leads to an alternative proof of Hurwitz’ Theorem [4], which states that real composition
algebras can only have dimension 1, 2, 4, or 8; and of Frobenius’ Theorem [2], which states that
only the 1, 2, and 4 dimensional cases are associative.

Unfortunately, Gadi passed away on January 1, 2016 in a train accident, before he was able
to publish his work. As his close relatives (grandson, nephew, and brother, respectively), we
have taken it upon ourselves to publish Gadi’s most recent version posthumously [5]. Since

∗McGill University
†Google AI, Princeton
‡CS Department, Technion Israel

1An algebraic structure which supports addition and multiplication, to be defined soon.
Gadi was always enthusiastic to expose the beauty of mathematics to a wide audience, we have written this paper with the aim of making Gadi’s proofs accessible to people who master high school mathematics. The definitions of the relevant algebraic concepts - e.g. field, linear space, norm, algebra are usually studied in a first year course of college mathematics.

Organization. We first formally state our requirements for real composition algebras in Section 2. Then, we present Gadi’s derivation of complex numbers in Section 3; his derivation of quaternions in Section 4; his derivation of octonions and his proof of Frobenius’ Theorem in Section 5; and finally his proof of Hurwitz’s Theorem in Section 6.

2 Premises

Geometry of Numbers. Our starting point is \( \mathbb{R} \), the field of real numbers, which (like any field) is equipped with addition and multiplication. Geometrically, \( \mathbb{R} \) is naturally identified with the one-dimensional Euclidean space \( \mathbb{E}^1 \) (i.e. the real line), denoted by \( \mathbb{E} \). A paramount extension of \( \mathbb{R} \) is \( \mathbb{C} \), the field of complex numbers. Analogously, \( \mathbb{C} \) is identified with the two-dimensional Euclidean space \( \mathbb{E}^2 \) (i.e. the real plane). The latter identification follows by associating the complex number \( x + y \cdot i \in \mathbb{C} \) with the point \( (x, y) \in \mathbb{E}^2 \). This suggests the following question, which motivates the study of real composition algebras:

For which values of \( n \in \mathbb{N} \) can the \( n \)-dimensional Euclidean space \( \mathbb{E}^n \) be equipped with multiplication?

The fields \( \mathbb{R} \) and \( \mathbb{C} \) witness that it can be done for \( n = 1, 2 \). To address this question more systematically, one needs to specify which properties should this multiplication satisfy. We address this in the next section.

2.1 Formal Requirements

Our goal is to extend the vector space \( \mathbb{E}^n \) to an algebraic structure which also supports a multiplication. In this section we specify three basic axioms we require from the desired multiplication. The first two axioms are algebraic, and they can be posed on vector spaces over arbitrary fields. The third axiom is geometric in the sense that it hinges on the Euclidean norm, which is special to real vector spaces. All three axioms are trivially satisfied by the complex plane and the real line.

It is natural to require the distributive law – the basic property that ties multiplication and addition:

**Axiom 1** (Left and Right Distributive Law). For all \( x, y, z \in \mathbb{E}^n \),

\[
x(y + z) = xy + xz, \\
(y + z)x = yx + zx.
\]

We further identify the scalar \( c \in \mathbb{R} \) with the vector \( (c, 0 \ldots 0) \in \mathbb{E}^n \) such that multiplying \( x = (x_1, \ldots, x_n) \in \mathbb{E}^n \) with \( (c, 0 \ldots 0) \) agrees with the standard scalar-multiplication:

**Axiom 2** (Homogeneity). For all \( x, y \in \mathbb{E}^n \) and \( a \in \mathbb{R} \):

\[
a(xy) = (ax)y = x(ay),
\]

where \( a \in \mathbb{R} \) is identified with \( (a, 0 \ldots 0) \in \mathbb{E}^n \).

Last but not least, we require that the sought multiplication is compatible with the geometry of \( \mathbb{E}^n \) in the following sense. Recall that \( \mathbb{E}^n \) is equipped with the Euclidean norm, which represents length, and is defined by

\[
\|x\| = \|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}.
\]

The sought multiplication is required to satisfy the following multiplicity of norm rule:

\[\text{By } n\text{-dimensional Euclidean space we mean a vector space over } \mathbb{R}^n \text{ with the standard Euclidean distance.}\]

\[\text{One could identify } \mathbb{R} \text{ with any 1-dimensional subspace - i.e. a line through the origin - of } \mathbb{E}^n.\]
Axiom 3 (Multiplicity of Norm (MoN)). For all \( x, y \in \mathbb{E}^n \),
\[
\| xy \| = \| x \| \| y \| .
\]
In words, the norm of a product is the product of the norms.

Summarizing the above, our task reduces to the following question:

**Question** (Main Question.). For what values of \( n > 1 \) it is possible to define multiplication
over \( \mathbb{E}^n \) which satisfies the following three properties?

(i) left and right distributivity (Axiom 1);
(ii) homogeneity (Axiom 2); and
(iii) Multiplicity of Norm (MoN) rule (Axiom 3).

The complete answer to this question, originally found by Hamilton and his contemporaries
and reproduced here, will show that \( \mathbb{C} \) is the only composition algebra over \( \mathbb{E}^2 \), and that the
only other possible composition algebras are the quaternions over \( \mathbb{E}^4 \), in which multiplication is
not commutative, and the octonions over \( \mathbb{E}^8 \), in which multiplication is neither commutative or
associative. This provides an alternative proof of Frobenius’ Theorem and Hurwitz’s Theorem.

### 2.2 Useful Properties

The presented proofs use elementary geometric arguments. Specifically, they use solely Property 1 and Property 2 below, both of which are implied by the next versions of the triangle inequality and the Pythagorean Theorem.

(i) The triangle inequality: \( \| x - y \| + \| y - z \| \geq \| x - z \| \) for all \( x, y, z \in \mathbb{E}^n \), with equality if and only if \( y \) lies on the line segment connecting \( x \) and \( z \), and (ii) Pythagorean Theorem: \( \| x \|^2 + \| y \|^2 = \| x + y \|^2 \) if and only if \( x \perp y \), i.e. \( x \) and \( y \) are orthogonal, or \( x \) is perpendicular to \( y \).

**Property 1** (Equality Statement). Let \( x, y \in \mathbb{E}^n \), then
\[
\left( \| x + y \| = \| x \| + \| y \| \quad \text{and} \quad \| x \| = \| y \| \right) \iff (x = y). 
\]

Note that Property 1 is a corollary of the above stated version of the triangle inequality.

**Property 2** (Orthogonality Statement). If \( \| x \| = \| y \| = 1 \), then:
\[
x \perp y \iff \| x + y \| = \sqrt{2}.
\]

Note that Property 2 is a corollary of the Pythagorean Theorem.

### Enough to Define a Multiplication Over a Basis.

Since bilinearity (i.e. the left and right distributive laws) and homogeneity are always required, a multiplication in composition algebras over \( \mathbb{E}^n \) spaces is determined entirely by its definition over any basis of \( \mathbb{E}^n \). Indeed, given a basis \( e_1, e_2, \ldots, e_n \), any \( x \in \mathbb{E}^n \) is a linear combination of the form:
\[
x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n, \quad \text{where} \ \alpha_i \in \mathbb{R} \ \text{are real coefficients}.
\]

For the rest of the paper, elements of \( \mathbb{E}^n \) are represented by the standard orthonormal basis \( \{ e_1, \ldots, e_n \} \), where \( e_i \) has 1 in the \( i^{th} \) entry and 0 otherwise. When there is a composition algebra over \( \mathbb{E}^n \), the basis element \( e_1 = (1, 0, \ldots, 0) \) corresponds to the unit element of this algebra.

With these postulates in hand we are now ready to extend the real numbers to algebras of higher dimensions in a systematic way. We will derive the necessary conditions for these extensions. The sufficiency of these conditions (specifically checking that the MoN rule is satisfied by each given extension) can be verified by brute-force calculations.

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4Note that the Pythagorean Theorem is implied by the Euclidean norm.
3 The Complex Numbers

To demonstrate our approach we show how the complex numbers are naturally and easily derived from Property 2 and MoN (Axiom 3). We start with a general useful result for composition algebras over $\mathbb{E}^n, n \in \mathbb{N}$:

Let $\mathcal{U}_n = \mathbb{R}^n$ denote the orthogonal complement of $\mathbb{R}$ in $\mathbb{E}^n$. That is, $\mathcal{U}_n$ is the set of numbers in $\mathbb{E}^n$ that are orthogonal to all real numbers, i.e. the subspace of $\mathbb{E}^n$ spanned by $\{e_2, \ldots, e_n\}$.

**Proposition 1.** In any composition algebra over $\mathbb{E}^n$, if $u \in \mathcal{U}_n$ and $\|u\| = 1$, then $u^2 = -1$.

**Proof.** Consider the product $(u+1)(u-1)$. By MoN (Axiom 3) and Property 2, we can evaluate its norm:

$$\|(u+1)(u-1)\| = \|u+1\|\|u-1\|$$

$$= \sqrt{2}\sqrt{2} = 2.$$  

(MoN (Axiom 3))

The product can also be simplified by distributing: $(u+1)(u-1) = u^2 - 1$. Therefore we have

$$\|u^2 - 1\| = 2 = 1 + 1 = \|u^2\| + \|-1\|$$

where the last step uses MoN to evaluate $\|u^2\| = \|u\|^2 = 1$. We can rewrite the above as:

$$\|u^2 + (-1)\| = \|u^2\| + \|-1\|$$

Given that $\|u^2\| = 1$, we can apply Property 2 and deduce that $u^2 = -1$. \(\square\)

We next explain how Proposition 1 easily implies the existence and uniqueness of a composition algebra over the Euclidean plane $\mathbb{E}^2$, namely the well known algebra of complex numbers $\mathbb{C}$.

Let $i$ be the (number associated with the) unit vector $e_2 = (0, 1) \in \mathbb{E}^2$. Then $i \in \mathcal{U}_2$ and hence, by Proposition 1, $i^2 = -1$ in any composition algebra over $\mathbb{E}^2$. In addition $\{1, i\} = \{e_1, e_2\}$ is a basis of $\mathbb{E}^2$. Hence any number $Z$ in such an algebra can be represented as $Z = \alpha + \beta i$ for some $\alpha, \beta \in \mathbb{R}$. Thus, the only multiplication over $\mathbb{E}^2$ which satisfies the formal requirements in Section 2 is the well known multiplication over $\mathbb{C}$ defined by

$$(\alpha + \beta i)(\gamma + \delta i) = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)i.$$  

This proves that $\mathbb{C}$ is the only composition algebra over $\mathbb{E}^2$, as claimed.

4 The Quaternions

We now aim to extend the complex numbers to composition algebras over $\mathbb{E}^n$ for $n > 2$, or to prove that no such extension exists for a given $n$. We do so by defining multiplication over $\mathbb{E}^n$, or show that such a definition is impossible.

A composition algebra over $\mathbb{E}^n$ for $n > 2$ must contain numbers $u, v \in \mathcal{U}_n$ such that $\{1, u, v\}$ is an orthonormal set. For concreteness, we may assume that $u = (0, 1, 0, \ldots)$ and $v = (0, 0, 1, 0, \ldots)$, but this is not necessary. By Proposition 1 we get that $u^2 = v^2 = -1$. We next realize that $uv = -vu$, which implies that commutativity does not hold for composition algebras over $\mathbb{E}^n$ if $n > 2$.

**Proposition 2.** If $u, v \in \mathcal{U}_n$ are orthogonal and $\|u\| = \|v\| = 1$, then $uv = -vu$.

**Proof.** Since $u, v \in \mathcal{U}_n$ and $\|u\| = \|v\| = 1$, Proposition 1 implies that $u^2 = v^2 = -1$, hence $(u - v)(u + v) = uv - vu$. Also since $u$ and $v$ are orthonormal, Property 2 implies that $\|u + v\| = \|u - v\| = \sqrt{2}$. Thus we get

$$2 = \|(u - v)(u + v)\| = \|uv - vu\| = \|uv\| + \|-vu\|$$

Equation 1 implies that $\|uv + (-vu)\| = \|uv\| + \|-vu\|$, and we apply Property 1 to deduce that $uv = -vu$. \(\square\)
Note that the above is still true even if the norm of \( u, v \) is not 1 (so long as it is non-zero). Next we show that if \( \{1, u, v\} \) is an orthonormal set in \( E^n \), then \( uv \) must be a unit vector orthogonal to all three vectors. Since there is no such vector when \( n = 3 \), we conclude that there is no composition algebra over \( E^3 \).

**Proposition 3.** If \( u, v \in U_n \) are orthonormal, then \( uv \in U_n \) and \( uv \perp 1, u, v \).

**Proof.** By Proposition 1 we get that \( uv - 1 = u(v + u) \). Thus the norm of \( uv - 1 \) can be evaluated using MoN (Axiom 3) and Property 2:

\[
\|uv - 1\| = \|u(v + u)\| = \|u\|\|v + u\| = 1\sqrt{2} = \sqrt{2}. \tag{2}
\]

Since \( \|uv + (-1)\| = \sqrt{2} \), Property 2 implies that \( uv \perp -1 \). This means that \( uv \in U_n \), and hence by Proposition 1 that \( (uv)^2 = -1 \).

Similarly, applying MoN (Axiom 3) and Property 2 to the expression \( uv + u \) we get:

\[
\|uv + u\| = \|u(v + 1)\| = \|u\|\|v + 1\| = 1\sqrt{2} = \sqrt{2},
\]

which implies by Property 2 that \( uv \perp u \). The same reasoning can be applied with \( w + v \) to finally show that \( uv \perp v \). \( \square \)

We have shown that if there exists a composition algebra over \( E^n \), then products in \( U_n \) are anti-commutative, and that no real composition algebra exists for \( n = 3 \). Indeed, Proposition 3 implies that introducing a second imaginary dimension implicitly adds a third one, as shown by the fact that the number \( uv \) is perpendicular to 1, \( u \), and \( v \). We claim that these four vectors \( B_{\mathbb{H}} = \{1, u, v, uv\} \) form an orthonormal basis for the 4-dimensional Euclidean space \( E^4 \) - to complete the proof that this indeed forms a composition algebra, it remains to show that \( B_{\mathbb{H}} \) is closed under multiplication; that is, the product of elements in \( E^4 \) is also in \( E^4 \). To prove the latter we first derive the following useful identity.

**Proposition 4.** If \( x, y \in U_n \) are orthonormal, then \( (xy)x = x(xy) = y \).

**Proof.** Consider the product \( ((1 + x)(1 + y))(1 + x) \):

\[
((1 + x)(1 + y))(1 + x) = (1 + x + y + xy)(1 + x) = 1 + x + y + xy + x^2 + yx + (xy)x = (x + y) + (1 + xy)x. \tag{3}
\]

where the last step uses \( xy = -yx \) and \( x^2 = -1 \). By MoN (Axiom 3) and Equation 3 above we get:

\[
2\sqrt{2} = \sqrt{2}\sqrt{2} = \|((1 + x)(1 + y))(1 + x)\| = \|(x + y) + (1 + xy)x\|. \]

Next observe that \( \|x + y\| = \sqrt{2} \) by Property 2 and \( \|(1 + xy)x\| = \sqrt{2} \) by Property 2 and MoN. Thus we get

\[
\|x + y\| + \|(1 + xy)x\| = 2\sqrt{2} = \|(x + y) + (1 + xy)x\|. \tag{4}
\]

By Property 1 equation 4 implies that the two terms are equal:

\[
x + y = (1 + xy)x = x + (xy)x,
\]

which directly implies that \( y = (xy)x \). Finally, two applications of Proposition 2 yield \( x(yz) = -(yx)x = (xy)x \). \( \square \)

From Proposition 4 we can easily show that \( E^4 \) is closed under multiplication. Indeed, Proposition 3 (combined with Proposition 2 and Proposition 1) imply that the product of any two elements in the basis \( B_{\mathbb{H}} = \{1, u, v, uv\} \) is in \( \{\pm 1, \pm u, \pm v, \pm uv\} \subseteq E^4 \); by bilinearity and homogeneity the closure extends to the entire space. This implies a 4-dimensional real composition algebra, which we denote as \( \mathbb{H} \). To show that this corresponds precisely to the quaternions, we first show that multiplication in \( \mathbb{H} \) is associative.

**Proposition 5.** Products in \( \mathbb{H} \) are associative. That is, for any \( x, y, z \in \mathbb{H} \), \( x(yz) = (xy)z \).
Proof. As discussed at the end of Section\[2\] due to the bilinearity it suffices to prove associativity for products of elements in an orthonormal basis of $\mathbb{H}$, such as $B_{\mathbb{H}} = \{1, u, v, uv\}$. The proof is trivial when either $x, y, z$ equal 1, by the properties of scalar multiplication (Axiom\[2\]). Thus, we consider only triples in $\{u, v, uv\}$. For triples where a factor is repeated:

- $x(xx) = x(-1) = (-1)x = (xx)x$ (by Proposition\[1\])
- $(xy)x = -(yx)x = -(yx)x = x(yx)$ (by Propositions\[2\] and\[4\])
- $x(xy) = -(yx)x = -y = (xx)y$ (by Propositions\[2\] and\[4\])

Of the $3! = 6$ possible remaining products, we only prove 3; the other proofs follow essentially the same structure. All of the following derivations use Propositions\[1\] to\[4\]. To avoid ambiguity, we use $[uv]$ to denote the element $uv$.

- $u([uv]) = u(u) = -1 = [uv][uv] = (uv)[uv]$
- $[uv](vu) = -[uv](uv) = 1 = -(uvu) = -(uv)[v]u = ([uv]v)u$
- $v([uv]u) = vv = -1 = uu = ([v]u)u$

We have thus completed the discovery of quaternions $\mathbb{H}$, and have proved that it is the only composition algebra over the 4-dimensional Euclidean space. By relabeling $i = u, j = v, k = uv$, we recover Hamilton’s familiar equations of quaternions:

$$i^2 = j^2 = k^2 = ijk = -1$$

5 The Octonions

Any composition algebra over $\mathbb{E}^n$ for $n > 4$ contains the quaternions, and in addition it contains also a unit vector $w$ which is perpendicular to all vectors in $B_{\mathbb{H}}$. The next Proposition shows that such an algebra must contain 8 orthonormal vectors, i.e. its dimension is at least 8.

Proposition 6. Let $w$ be a unit vector which is orthogonal to all vectors in $B_{\mathbb{H}}$. Then the set $B_0 = \{1, u, v, w, u, uv, uv, w\}$ is orthonormal.

Proof. We use the following observation, implied by MoN, Property\[2\] and the distributive laws:

For any three unit vectors $x, y, z$:

$$x \perp y \Leftrightarrow xz \perp yz \Leftrightarrow zx \perp yz$$

(5)

Let $B_{01} = B_{\mathbb{H}} = \{1, u, v, w\}$ and $B_{02} = \{w, uv, uv, (uv)w\}$. We already knew that $B_{01}$ is orthonormal, which immediately implies by Eq.\[5\] that $B_{02}$ is orthonormal as well.

It remains to show that each vector in $B_{01}$ is perpendicular to all vectors in $B_{02}$, namely that for any pair $x, y \in B_{01}, x \perp yw$. So let $x, y \in B_{01}$ be given. It is easily verified that for some $z \in B_{01}$, it holds that $x = yz$ or $x = -yz$, so assume without loss of generality that $x = yz$ (the other case is similar). Since $z \perp w$, by Eq.\[5\] we get that $x = yz \perp yw$.

It turns out that $B_0 = \{1, u, v, w, uw, uv, uv, (uv)w\}$, the orthonormal set defined in Proposition\[7\] above, forms a basis for a composition algebra over the 8-dimensional Euclidean space $\mathbb{E}^8$. This algebra is called octonions and denoted by $\mathbb{O}$. To see that $\mathbb{O}$ is indeed a composition algebra it suffices to show that it is closed under multiplication. The product between two elements in $B_{01}$ is inherited from the quaternions. For the other products we use the following useful identity:

Proposition 7. Let $x, y, z \in U_n$ be orthonormal vectors such that $xy \perp z$. Then $(xy)(yz) = xz$.$\,$\footnote{One can show that Eq.\[5\] holds also when $x, y, z$ are not necessarily unit vectors.}
Proof. The following equalities hold for any set of orthonormal vectors \( \{x, y, z\} \):
\[
(xy - z)(x + yz) = (xy)x + (xy)(yz) - xx - z(yz) \\
= y + (xy)(yz) + xx - y \\
= (xy)(yz) + xx
\]
where the first and last terms were simplified using Propositions 2 and 4. Since \( xy \perp z \) we also have by Proposition 3 that \( x \perp yz \). Thus by MoN (Axiom 3) and Property 2:
\[
\| (xy - z)(x + yz) \| = \sqrt{2} \sqrt{2} = 2
\]
Thus we conclude from Eq. (6) that:
\[
2 = \| (xy)(yz) + xz \| = \| (xy)(yz) \| + \| xz \|
\]
which by Property 1 implies that \( (xy)(yz) = xz \).

We now show that the product of two distinct elements in \( B_{32} \), or its negation, is in \( B_{31} \). Let these elements be \( xw, yw \), for some distinct \( x, y \in B_{31} \). If \( x = 1 \) or \( y = 1 \) then this follows by Proposition 4. Otherwise, \( x, w \) and \( y \) satisfy the premises of Proposition 7, hence \( (xw)(yw) = -(xw)(wy) = -xy \) and we are done.

It remains to show that products between elements of \( B_{31} \) and \( B_{32} \), or their negations, are in \( B_0 \). This is done by the following proposition, which as a byproduct demonstrates that certain products in \( \mathcal{O} \) are anti associative.

**Proposition 8.** For any distinct \( x, y \in B_{31} \), \( x(yw) = -(xy)w \).

**Proof.** First observe that \( x, y \) and \( w \) satisfy the premises of Proposition 7. Hence:
\[
(x - xy)(yw + w) = x(yw) + xw - (xy)(yw) - (xy)w \\
= x(yw) + xw - xw - (xy)w \\
= x(yw) - (xy)w.
\]

By MoN (Axiom 3) and Property 2 we evaluate the norm:
\[
\| (x - xy)(yw + w) \| = \sqrt{2} \sqrt{2} = 2
\]
Comparing with the reduced expression:
\[
\| x(yw) - (xy)w \| = \| x(yw) \| + \| -(xy)w \|
\]
By Property 1 we find that \( x(yw) = -(xy)w \). \( \Box \)

Propositions 7 and 8 above complete the definition of products over \( B_{3} \), the 8 base vectors of the octonions. Our derivation demonstrates that the octonions are the (only) composition algebra over the 8-dimensional Euclidean space. During this process, we have also demonstrated that no real composition algebra can exist in 5, 6, or 7 dimensions. We also note that unlike the reals, the complex numbers, and the quaternions, multiplication over the octonions is non-associative. This proves Frobenius’ Theorem, which states that only the 1, 2, and 4 dimensional real composition algebras are associative.

6 Extending the Octonions

We next push our method further to check if there exists a composition algebra over \( \mathbb{E}^n \) for \( n > 8 \). Again, such an algebra must contains the octonions, and in addition also a unit vector \( s \in U_n \) which is orthogonal to all the vectors in \( B_0 \). By repeating the arguments in Proposition 6 we obtain an orthonormal set \( B' \) of 16 vectors, that we (wrongly) assume is a basis for a composition algebra over \( \mathbb{E}^{16} \):
\[
B' = \{ 1, u, v, w, uw, vw, (uw)w, s, us, vs, uw, ws, (uw)s, (uw)s, ((uw)s)s \}
\]
Proposition 9. It is impossible to define a bilinear and homogeneous multiplication which satisfies MoN, Property 1 and Property 2 over the 16-dimensional space spanned by $B'$.

Proof. Assume that it is possible to define such a product, and consider the product:

$$(uv + ws)(sv + wu) = (uv)(sv) + (uv)(wu) + (ws)(sv) + (ws)(wu)$$

$$= -us + uv + wv - su$$  \hspace{1cm} \text{(by Propositions 2 and 2)}

$$= -us - wv + wv + us = 0$$  \hspace{1cm} \text{(by Proposition 2)}

However since $uv \perp ws$ and $sv \perp wu$, the norm of the product at the left-hand side is $\sqrt{2} \sqrt{2} = 2$, a contradiction.

Proposition 9 implies that no real composition algebra exists for $n > 8$. Indeed, any extension of the octonions will break MoN (Axiom 3). This proves Hurwitz’s Theorem, which states that real composition algebras can only have dimension 1, 2, 4, or 8.

7 Conclusion

In this paper we have presented Gadi Moran’s construction of complex numbers, quaternions, and octonions. The proofs used in this construction are based solely on properties of Euclidean geometry, and more specifically on the Pythagorean theorem. As such, this construction is more intuitive and is likely to be more accessible to high-school students than standard constructions which derive real composition algebras from a purely algebraic standpoint.

Interestingly, our construction uses the properties of extended multiplication in $\mathbb{E}^n$ without ever pondering on their geometrical significance. As it is well-known, multiplication by imaginary numbers can be visualized as consisting of rotations in $n$-dimensional space. For instance, multiplication of a complex number by $i$ performs a rotation by 90 degrees in the plane. This idea, when applied to quaternions, results in very useful techniques to model rotations in 3D space, which are most commonly used in computer graphics.

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