FREE PRODUCTS WITH AMALGAMATION OVER CENTRAL C*-SUBALGEBRAS

KRISTIN COURTNEY AND TATIANA SHULMAN

ABSTRACT. Let $A$ and $B$ be C*-algebras whose quotients are all RFD, and let $C$ be a central C*-subalgebra in both $A$ and $B$. Then the full amalgamated product $A*_C B$ is RFD. This generalizes Korchagin’s result [13] that amalgamated free products of commutative C*-algebras are RFD.

1. Introduction

A C*-algebra is residually finite-dimensional (RFD) if its finite-dimensional representations separate its points. Various characterizations of the class RFD C*-algebras have been obtained over the years (notably [1], [8], [12], [6]), and numerous classes of C*-algebras have been shown to be RFD. The class of RFD C*-algebras is obviously closed under taking C*-subalgebras, and it is also closed under free products as was proved by Exel and Loring [8] (see also [10]). However the situation with amalgamated free products is far from being understood. When the amalgam is finite-dimensional, necessary and sufficient conditions for the free product to be RFD were found in [15]. In [14] it was proved that amalgamated free products of commutative C*-algebras are RFD. In this paper we substantially generalize this statement. Let us say that a C*-algebra is strongly RFD if all its quotients are RFD. Here we prove

**Theorem** Let $A$ and $B$ be (unital) separable strongly RFD C*-algebras and let $C$ be a central C*-subalgebra in both $A$ and $B$. Then the (unital) amalgamated free product $A*_C B$ is RFD.

In particular, since all commutative C*-algebras are clearly strongly RFD, this gives the result of [14] with a different and shorter proof.

The property of being RFD can be considered as a C*-anologue of maximal almost periodicity for groups. Recall that a group is maximally almost periodic (MAP) if its finite-dimensional representations separate points. Clearly if $C^*(G)$ is RFD then $G$ is MAP but the opposite is not true (see [14] for counterexamples). Known examples of groups with RFD C*-algebras include full group C*-algebras of nonabelian free groups [5], amenable maximally periodic groups [4], surface groups and fundamental groups of closed hyperbolic 3-manifolds that fiber over the circle [16], and many 1-relator groups with non-trivial center [13]. By Exel and Loring’s result, the class of groups with RFD full C*-algebras is closed under free products. However it is not closed under amalgamated free products. Indeed amalgamated free products of MAP groups need not even be MAP ([2]) and hence need not have RFD C*-algebras.

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As a corollary to our main theorem, we give sufficient conditions for amalgamated free products of groups to have RFD C*-algebras.

**Corollary** Let \(G_1\) and \(G_2\) be virtually abelian discrete groups and let \(H\) be a central subgroup in both \(G_1\) and \(G_2\). Then the full group C*-algebra of the amalgamated free product \(G_1 *_H G_2\) is RFD.

### 2. Proofs

**Definition 1.** A C*-algebra is called strongly RFD if all its quotients are RFD.

Here are 2 examples of classes of C*-algebras that are strongly RFD:

1) C*-algebras whose irreducible representations are all finite-dimensional (In [6], we call such C*-algebras FDI.)

Since irreducible representations separate points, any FDI C*-algebra is RFD. Moreover any quotient of an FDI C*-algebra is again FDI since an irreducible representation of a quotient of a C*-algebra gives rise to an irreducible representation of the C*-algebra.

Particular cases of FDI C*-algebras are subhomogeneous C*-algebras (i.e. those whose irreducible representations are all of dimension no more than some fixed \(n < \infty\)) and continuous fields of finite-dimensional C*-algebras (see [7]).

2) RFD just-infinite C*-algebras

In [9], Grigorchuk, Musat, and Rørdam defined a just-infinite C*-algebra to be an infinite-dimensional C*-algebra whose proper quotients are all finite-dimensional. In the same paper, they demonstrate the existence of just-infinite RFD C*-algebras. Moreover, they prove that there are examples of non-exact, just-infinite RFD C*-algebras, which means, in particular, that strongly RFD C*-algebras need not be nuclear. It is worth noting that no C*-algebra can be both FDI and just-infinite.

Indeed, by [9, Lemma 5.4], no RFD just-infinite C*-algebra is of type I; while, on the other hand, all FDI algebras are type I.

The following result of Hadwin will be crucial for the proof of the main theorem.

**Lemma 2.** (Hadwin [11]) Let \(\{e_n\}\) be an orthonormal basis in a Hilbert space \(H\), \(A, B, C, D \in B(H)\) and \(T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\). Then for any unitary \(w_n : H \to H \oplus H\) such that \(w_n e_k = (e_k, 0), 1 \leq k \leq n\), \(A = WOT - \lim w_n^* T w_n\).

Moreover \(A = SOT - \lim w_n^* T w_n\) if and only if \(C = 0\), and \(A = * - SOT - \lim w_n^* T w_n\) if and only if \(C = B = 0\).

We will need one more lemma, which is essentially the statement 5 in [12] Lemma 1], where it is formulated in slightly different terms. For the reader’s convenience we give a proof of it here.

**Lemma 3.** (Hadwin [12]) Let \(u \in B(H)\) be a unitary operator and let \(P_n \in B(H)\), \(n \in \mathbb{N}\), be a sequence of finite rank projections such that \(P_n \uparrow 1\) in *-strong operator topology. Then for each \(n \in \mathbb{N}\) there is a unitary operator \(u_n \in B(P_n H)\) such that \(u_n \to u\) in *-strong operator topology.

**Proof.** Write \(u\) as \(u = e^{2\pi i S}\), where \(-1 \leq S \leq 1\), and let \(u_n = e^{2\pi i P_n S P_n}\). Since \(P_n S P_n \to S\) *-strongly and since the functional calculus is continuous with respect to the *-strong operator topology, one has \(u_n \to u\) *-strongly. \(\square\)
Theorem 4. Let $A$ and $B$ be unital separable strongly RFD $C^*$-algebras and let $C$ be a central $C^*$-subalgebra in both $A$ and $B$. Then the unital amalgamated free product $A*_{C} B$ is RFD.

Proof. Let $0 \neq x \in A *_{C} B$. We construct a finite-dimensional representation $\sigma$ of $A *_{C} B$ such that $\sigma(x) \neq 0$. There exist an irreducible representation $\rho$ of $A *_{C} B$ on a Hilbert space $H$ and a unit vector $\xi \in H$ such that $||\rho(x)\xi|| \geq \frac{2}{3}$. Let $i_A$ and $i_B$ denote the standard embeddings of $A$ and $B$ into $A *_{C} B$. There exist $K \in \mathbb{N}$, $a_i^{(k)} \in A, b_i^{(k)} \in B, i = 1, \ldots, N^{(k)}, k = 1, \ldots, K$ such that for

$$
\hat{x} = \sum_{k=1}^{K} i_A(a_i^{(k)})i_B(b_i^{(k)}) \cdots i_A(a_N^{(k)})i_B(b_N^{(k)}),
$$

we have

$$
||x - \hat{x}|| \leq \frac{||x||}{8}.
$$

Then

$$
||\rho(\hat{x})\xi|| \geq \frac{3}{8}||x||.
$$

We denote the representations of $A$ and $B$ induced by $\rho$ with $\rho_A$ and $\rho_B$ respectively, that is $\rho_A(a) = \rho(i_A(a))$, $\rho_B(b) = \rho(i_B(b))$, for each $a \in A, b \in B$. Let

$$
N = \max_{1 \leq k \leq K} N^{(k)},
$$

$$
E = \{a_i^{(k)} \mid k = 1, \ldots, K, i = 1, \ldots, N^{(k)}\},
$$

$$
F = \{b_i^{(k)} \mid k = 1, \ldots, K, i = 1, \ldots, N^{(k)}\},
$$

and

$$
G = \{\xi \mid \rho_B(b_i^{(k)})\rho_A(a_{i+1}^{(k)})\rho_B(b_{i+1}^{(k)}) \cdots \rho_A(a_{N_i^{(k)}}^{(k)})\rho_B(b_{N_i^{(k)}}^{(k)})\xi \mid i = 1, \ldots, N^{(k)}, k = 1, \ldots, K\}
$$

$$
\cup \rho_A(a_i^{(k)})\rho_B(b_i^{(k)})\rho_A(a_{i+1}^{(k)}) \cdots \rho_A(a_{N_i^{(k)}}^{(k)})\rho_B(b_{N_i^{(k)}}^{(k)})\xi \mid i = 2, \ldots, N^{(k)}, k = 1, \ldots, K\}.
$$

Since $\rho$ is irreducible and $C$ is a central subalgebra in $A *_{C} B$, for each $c \in C$ there is $\lambda(c) \in \mathbb{C}$ such that

$$
\rho(c) = \lambda(c).1.
$$

Since $A$ and $B$ are strongly RFD, $\rho_A(A)$ and $\rho_B(B)$ are RFD. Let $\tilde{\pi}_1, \tilde{\pi}_2, \ldots$ be a countable separating family of unital finite-dimensional representations of $\rho_A(A)$ and let $\pi_i = \tilde{\pi}_i \circ \rho_A$. Let $\tilde{\pi}_1', \tilde{\pi}_2', \ldots$ be a countable separating family of unital finite-dimensional representations of $\rho_B(B)$ and let $\pi'_i = \tilde{\pi}'_i \circ \rho_B$. Let $N_i = \dim \pi_i, N'_i = \dim \pi'_i$. Let

$$
\tilde{\pi}_i = \pi_i^{(N'_i)}, \tilde{\pi}'_i = \pi'_i^{(N_i)}.
$$

Then $\dim \tilde{\pi}_i = \dim \tilde{\pi}'_i$. Notice that for each $c \in C$,

$$
\tilde{\pi}_i(c) = \lambda(c).1. \tilde{\pi}'_i(c).
$$

Let $\pi$ be a direct sum of all $\tilde{\pi}_i$’s where each one is repeated infinitely many times, say

$$
\pi = \tilde{\pi}_1 \oplus (\tilde{\pi}_1 \oplus \tilde{\pi}_2) \oplus (\tilde{\pi}_1 \oplus \tilde{\pi}_2 \oplus \tilde{\pi}_3) \oplus \ldots.
$$
We consider \( \pi \) as a representation of \( A \) on \( B(H) \) with respect to some decomposition

\[
H = H_1 \oplus H_2 \oplus H_3 \oplus \ldots
\]

where \( H_1 \cong \mathbb{C}^{N_1 N_1'}, H_2 \cong \mathbb{C}^{N_1 N_1' + N_2 N_2'}, \ldots \). Let \( \pi' : B \to B(H) \) be defined by

\[
\pi' = \pi_1' \oplus (\pi_1' \oplus \pi_2') \oplus (\pi_2' \oplus \pi_3') \oplus \ldots
\]

with respect to the same decomposition of \( H \). Let \( P_i \in B(H) \) be the orthogonal projection onto \( H_1 \oplus \ldots \oplus H_i \).

Now, for any \( a \in A \), we have that \( \pi(a) = 0 \) iff \( \rho_A(a) = 0 \), and \( \text{rank}(\pi(a)) = \infty \) when \( \pi(a) \neq 0 \). So by Voiculescu’s theorem, \( \rho_A \oplus \pi \) is approximately unitarily equivalent to \( \pi \). Hence there exists a unitary \( u : H \oplus H \to H \) such that for all \( d \in E \) we have

\[
\| \left( \begin{array}{c} \rho_A(d) \\ \pi(d) \end{array} \right) - u^* \pi(d) u \| \leq \delta,
\]

where

\[
\delta = \frac{\|x\|}{80NK}.
\]

By Lemma 2 there exist unitaries \( w_m : H \to H \oplus H \) such that for all \( a \in A \)

\[
\rho_A(a) = \ast - \text{SOT lim } w_m^* \left( \begin{array}{c} \rho_A(a) \\ \pi(a) \end{array} \right) w_m.
\]

In particular there exists a unitary \( w : H \to H \oplus H \) such that for all \( d \in E, \eta \in G \)

\[
\| \rho_A(d) \eta - w^* \left( \begin{array}{c} \rho_A(d) \\ \pi(d) \end{array} \right) w \eta \| < \delta.
\]

Hence

\[(4) \quad \| \rho_A(d) \eta - w^* u^* \pi(d) uw \eta \| \leq 2\delta.\]

Similarly we find unitaries \( u', u' \) such that

\[(5) \quad \| \rho_B(d) \eta - w'^* u'^* \pi'(d) u'w' \eta \| \leq 2\delta.\]

for all \( d \in F, \eta \in G \). Applying Lemma 3 to \( uw \) and \( u'w' \), we find \( M \in \mathbb{N} \) and unitaries \( v \) and \( v' \) on \( P_M H \) such that

\[
\| \eta - P_M \eta \| \leq \delta,
\]

\[
\| (uw - v) P_M \eta \| \leq \delta,
\]

\[
\| (u'w' - v') P_M \eta \| \leq \delta,
\]

for all \( \eta \in G \),

\[
\| (w^* u^* - v^*) \pi(d) uw P_M \eta \| \leq \delta,
\]

for all \( d \in E, \eta \in G \) and

\[
\| (w'^* u'^* - v'^*) \pi'(d) u'w' P_M \eta \| \leq \delta,
\]

for all \( d \in F, \eta \in G \).

Now we define finite-dimensional representations \( \sigma_A : A \to P_M B(H) P_M \) and \( \sigma_B : B \to P_M B(H) P_M \) by

\[
\sigma_A(a) = v^* (P_M \pi(a) P_M) v, \quad \sigma_B(b) = v'^* (P_M \pi'(b) P_M) v',
\]

for any \( a \in A, b \in B \). Then by 3

\[
\sigma_A(c) = \lambda(c) 1_{P_M H} = \sigma_B(c),
\]
for any \( c \in C \). As \( \sigma_A \) and \( \sigma_B \) agree on \( C \), we obtain a finite-dimensional representation \( \sigma : A \ast_C B \to P_M B(H) P_M \).

Since \( P_M H \) is invariant subspace for \( \pi \), we have \( P_M \pi(a) P_M v = \pi(a) v \), for each \( a \in A \). Using this we obtain

\[
\begin{align*}
(6) \quad \|w^* u \pi(d)uw\eta - \sigma_A(d)P_M \eta\| &= \|w^* u \pi(d)uw\eta - v^* (P_M \pi(d)P_M) v P_M \eta\| \\
&\leq \|w^* u \pi(d)uw(\eta - P_M \eta)\| + \|w^* u \pi(d)uwP_M \eta - v^* \pi(d) v P_M \eta\| \\
&\leq \|w^* u \pi(d)uw(\eta - P_M \eta)\| + \|(w^* u - v^*) \pi(d)uwP_M \eta\| + \|v^* \pi(d)(uw - v)P_M \eta\| \\
&\leq 3\delta,
\end{align*}
\]

for any \( d \in E, \eta \in G \). By (4) and (6), we have

\[
\begin{align*}
(7) \quad \|\rho_A(d)\eta - \sigma_A(d)P_M \eta\| &\leq 5\delta,
\end{align*}
\]

for any \( d \in E, \eta \in G \). Similarly we obtain

\[
\begin{align*}
(8) \quad \|\rho_B(d)\eta - \sigma_B(d)P_M \eta\| &\leq 5\delta,
\end{align*}
\]

for any \( d \in F, \eta \in G \). By repeated uses of (7) and (8),

\[
\begin{align*}
&\|\sigma_A(a_1^{(k)})\sigma_B(b_1^{(k)}) \ldots \sigma_A(a_N^{(k)})\sigma_B(b_N^{(k)})P_M \xi \\
&\quad - \rho_A(a_1^{(k)})\rho_B(b_1^{(k)}) \ldots \rho_A(a_N^{(k)})\rho_B(b_N^{(k)})\xi\| \\
&\leq \|\sigma_A(a_1^{(k)})\sigma_B(b_1^{(k)}) \ldots \sigma_A(a_N^{(k)})\xi\| + \|\sigma_A(a_1^{(k)})\sigma_B(b_1^{(k)}) \ldots \sigma_A(a_N^{(k)})\rho_B(b_N^{(k)})\xi\| \\
&\quad - \rho_A(a_1^{(k)})\rho_B(b_1^{(k)}) \ldots \rho_A(a_N^{(k)})\rho_B(b_N^{(k)})\xi\| \\
&\leq \|\sigma_A(a_1^{(k)})\sigma_B(b_1^{(k)}) \ldots \sigma_A(a_N^{(k)})\rho_B(b_N^{(k)})\xi\| \\
&\quad - \rho_A(a_1^{(k)})\rho_B(b_1^{(k)}) \ldots \rho_A(a_N^{(k)})\rho_B(b_N^{(k)})\xi\| \\
&\leq 5\delta
\end{align*}
\]

+ \[\|\sigma_A(a_1^{(k)})\sigma_B(b_1^{(k)}) \ldots \sigma_A(a_N^{(k)})\rho_B(b_N^{(k)})\xi\| \\
&\quad - \rho_A(a_1^{(k)})\rho_B(b_1^{(k)}) \ldots \rho_A(a_N^{(k)})\rho_B(b_N^{(k)})\xi\| \\
&\leq \ldots \leq 2N^{(k)}5\delta \leq 10N\delta.
\]

Hence

\[
(9) \quad \|\sigma(\tilde{x}) P_M \xi - \rho(\tilde{x}) \xi\| \leq 10N\delta = \frac{\|x\|}{8}.
\]

Combining (9), (2) and (11) we obtain

\[
\|\sigma(x) P_M \xi\| \geq \|\sigma(\tilde{x}) P_M \xi\| - \|\sigma(x - \tilde{x}) P_M \xi\| \\
&\geq \|\rho(\tilde{x}) \xi\| - \|\sigma(\tilde{x}) P_M \xi - \rho(\tilde{x}) \xi\| - \|\sigma(x - \tilde{x}) P_M \xi\| \geq \frac{\|x\|}{8}.
\]

Thus \( \sigma(x) \neq 0 \). \( \square \)

**Remark 5.** The theorem also holds for the non-unital amalgamated product. If the amalgam \( C \) is non-zero, then one does not even need to change anything in the proof. Indeed, then the \( C^* \)-algebras \( \rho_A(A) \) and \( \rho_B(B) \) are unital and hence the
representations $\pi_i$ and $\pi'_i$ are automatically non-degenerate so that we still can use Voiculescu’s theorem. If the amalgam $C$ is zero, one must take $\bar{\pi}_i$ and $\bar{\pi}'_i$ to be non-degenerate representations, and then the rest of the proof goes without change.

**Corollary 6. (Korchagin [14])** Amalgamated free product of commutative $C^*$-algebras is RFD.

**Corollary 7.** Let $G_1$ and $G_2$ be virtually abelian discrete groups and let $H$ be a central subgroup in both $G_1$ and $G_2$. Then $C^*(G_1 \ast_H G_2) \cong C^*(G_1) \ast C^*(H) \ast C^*(G_2)$ is RFD.

*Proof.* It was proved in [17] that $C^*$-algebras of discrete virtually abelian groups are subhomogeneous. Hence they are strongly RFD and Theorem 4 applies. \hfill $\square$

It would be interesting to characterize all discrete groups that have strongly RFD $C^*$-algebras.

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Mathematical Institute, WWU Münster, Einsteinstr. 62, Münster
E-mail address: kcourtne@uni-muenster.de

Department of Mathematical Physics and Differential Geometry, Institute of Mathematics of Polish Academy of Sciences, Warsaw
E-mail address: tshulman@impan.pl