A New Approach for Higher Order Difference Equations and Eigenvalue problems via Physical Potentials

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Abstract

In this study, we give the variation of parameters method from a different viewpoint for the \textit{Nth} order inhomogeneous linear ordinary difference equations with constant coefficient by means of delta exponential function $e_p (t, s)$. Advantage of this new approachment is to enable us to investigate the solution of difference equations in the closed form. Also, the method is supported with three difference eigenvalue problems; the second-order Sturm-Liouville problem, which is called also one dimensional Schrödinger equation, having Coulomb potential, hydrogen atom equation, and the fourth-order relaxation difference equations. We find sum representation of solution for the second order discrete Sturm-Liouville problem having Coulomb potential, hydrogen atom equation, and analytical solution of the fourth order discrete relaxation problem by the variation of parameters method via delta exponential and delta trigonometric functions.

Keywords: Schrödinger equation, energy level, difference equation, variation of parameters method, Coulomb potential, hydrogen atom, relaxation equation.

AMS Subject Classification: 39A06; 39A12; 39A70; 35A25.

1. Introduction

Variation of parameters method, that is a general method in the solution methods of inhomogeneous linear ordinary differential equations, firstly was introduced by Euler and Lagrange while they were studying the celestial bodies and orbital elements \cite{15} \cite{16}.

Later, this method was adapted to solve inhomogeneous linear ordinary difference equations for the following equation types in \cite{12} \cite{13}

\begin{equation}
x_{n+N} + p_{N-1}x_{n+N-1} + \cdots + p_0 x_n = q (n) .
\end{equation}

Recently, delta exponential function has been described in \cite{11} similarly to exponential function in the continuous case. $e^p t$, $p \in \mathbb{R}$, is the solution of the following problem

\begin{align*}
y' (t) &= py (t) , \\
y (0) &= 1 ,
\end{align*}

and delta exponential function $e_p (n, s)$ is the solution of the following problem

\begin{align*}
\Delta x (n) &= p (n) x (n) , \\
x (s) &= 1 .
\end{align*}

Also, delta exponential function $e_p (n, s)$ is used to find the homogeneous solution of linear difference equations with constant coefficient as with differential equations and besides, variation of constants formula is given for the first order linear difference equations in \cite{11}.

In this study, we generalize the method for \textit{Nth} order inhomogeneous linear ordinary difference equations with constant coefficient by means of delta exponential function $e_p (t, s)$ and consider the following equation in a closed form differently from the equation \cite{11}

\begin{equation}
\Delta^N x (n) + r_{N-1} \Delta^{N-1} x (n) + \cdots + r_0 x_n = q (n) .
\end{equation}
We present a different approach to the variation of parameters method by means of delta exponential function for higher order inhomogeneous linear ordinary difference equations with constant coefficient. This approach enables us to investigate the solution of difference equations in the closed form as given in (2), otherwise equation (2) has to be expanded by using binomial expansion of the difference operator

\[ \Delta^N x(t) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} x(t + N - k). \]

In the last section, we give three examples for explaining the method, also specifically we find the representation of solution of discrete Sturm-Liouville problem, which is called also one dimensional Schrödinger equation, having Coulomb potential, discrete hydrogen atom equation, and the fourth-order discrete relaxation equation.

Sturm-Liouville equations play an important role in mathematical physics. Lately, Sturm-Liouville differential and difference equations have been considered similarly to the continuous counterpart [1–10, 14].

Hydrogen atom equation is studied by [19, 20, 17, 18]. Hydrogen atom equation is used in quantum mechanics for determining energy levels of hydrogen atom [19]. Hydrogen atom equation is defined as follows

\[ y'' + \left( \lambda - \frac{l(l+1)}{x^2} + \frac{2}{x} - q(x) \right) y = 0. \]

Then, let’s introduce discrete hydrogen atom equation

\[ \Delta^2 x(n-1) + \left( \lambda - q(n) + 2 n - \frac{l(l+1)}{n^2} \right) x(n) = 0, \quad n = 1, ..., b, \]

where \( l \) is a positive integer or zero, \( v(n) \in l^2 [0, b], q(n), b, \lambda \) and \( n \) is as defined above, \( -q(n) + \frac{2}{n} - \frac{l(l+1)}{n^2} \) are called potential function.

Now, let’s define briefly Sturm-Liouville operator having Coulomb potential. Motion of electrons moving under the Coulomb potential has importance in quantum theory. This problem is used for finding energy levels for hydrogen atom and single valence electron atoms. Time-dependent Schrödinger equation is as follows

\[ i\hbar \frac{\partial \omega}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \omega}{\partial x^2} + U(x, y, z) \omega, \quad \int_{\mathbb{R}^3} |\omega|^2 dx dy dz = 1, \]

from here, in consequence of some transformations, we obtain Sturm-Liouville equation having Coulomb potential

\[ -y'' + \left[ \frac{A}{x} + q(x) \right] y = \lambda y, \]

where \( \lambda \) is a parameter which corresponds to the energy [21]. The following problem

\[ -\Delta^2 x(n-1) + \left( \frac{A}{n} + q(n) \right) x(n) = \lambda u(n), \quad n = a, ..., b, \]

\[ x(a-1) + \hbar x(a) = 0, \]

is called discrete Sturm-Liouville problem having Coulomb potential.
2. Preliminaries

**Definition 1.** Let’s define regressive functions,\[\mathbb{R} = \{p : \mathbb{N}_a \to \mathbb{R} \text{ such that } 1 + p(n) \neq 0 \text{ for } n \in \mathbb{N}_a\}.\]

**Theorem 2.** Let’s define delta exponential function. Suppose that \(p \in \mathbb{R}\) and \(s \in \mathbb{N}_a\), then
\[e_p(n, s) = \begin{cases} \prod_{\tau=s}^{n-1} [1 + p(\tau)], & n \in \mathbb{N}_s \\ \prod_{\tau=n}^{s-1} [1 + p(\tau)]^{-1}, & n \in \mathbb{N}_a^n \end{cases},\]
where \(\prod_{\tau=a}^b = 1\), if \(a > b\).

**Theorem 3.** Suppose that \(p, q \in \mathbb{R}\) and \(n, s \in \mathbb{N}_a\). Then
(i) \(e_0(n, s) = 1\);
(ii) \(\Delta e_p(n, s) = p(n) e_p(n, s)\);
(iii) \(e_p(n, s) e_q(n, s) = e_{p+q}(n, s)\), where \(p \oplus q = p + q + pq\).

**Definition 4.** Let’s define delta sine and cosine functions as follows,
\[\cos_p(n, a) = \frac{e_{ip}(n, a) + e_{-ip}(n, a)}{2}, \quad \sin_p(n, a) = \frac{e_{ip}(n, a) - e_{-ip}(n, a)}{2i},\]
where \(n \in \mathbb{N}_a, \pm ip \in \mathbb{R}\).

**Definition 5.** Let’s define delta integral. Suppose \(f : \mathbb{N}_a \to \mathbb{R}\) and \(c \leq d, c, d \in \mathbb{N}_a\), then
\[\int_c^{d+1} f(n) \Delta n = \sum_{n=c}^{d} f(n),\]
where \(\sum_{n=c}^{d} . = 0\), if \(c > d\).

**Definition 6.** If \(F(n)\) is delta integral of \(f(n)\), then
\[\int_a^b f(n) \Delta n = F(b) - F(a).\]

**Let’s consider the second order linear homogeneous ordinary difference equation with constant coefficients as follows,**
\[\Delta^2 y(n) + p\Delta y(n) + qy(n) = 0, \quad n \in \mathbb{N}_a,\]
where \(p, q \in \mathbb{R}\) hold \(p \neq 1 + q\).

**Theorem 7.** Characteristic equation of (3), by the help of delta exponential function, is given by
\[m^2 + pm + q = 0,\]
let \( m_1, m_2 \) are distinct characteristic roots of the characteristic equation, so
\[
y (n) = c_1 e_{m_1} (n, a) + c_2 e_{m_2} (n, a), \tag{4}
\]
where \( c_1, c_2 \) are constants, is a general solution of (3).

**Theorem 8.** \([11]\) Let the characteristic roots are complex pair, \( m_{1,2} = \alpha \pm i\beta, \alpha \neq -1, \beta > 0 \), so
\[
y (n) = c_1 e_\alpha (n, a) e^{\gamma n} \cos \gamma (n, a) + c_2 e_\alpha (n, a) e^{\gamma n} \sin \gamma (n, a), \tag{5}
\]
where \( \gamma = \frac{\beta}{1+\alpha} \), is a general solution of (3).

**Theorem 9.** \([11]\) Let the characteristic roots are double roots, \( m_1 = m_2 = r \), so
\[
y (n) = c_1 e_r (n, a) + c_2 (n - a) e_r (n, a), \tag{6}
\]
is a general solution of (3).

### 3. Main Results

#### 3.1. Analysis of the Method

Let’s reconsider the equation (2):
\[
\Delta^N x (n) + r_{N-1} \Delta^{N-1} x (n) + \cdots + r_0 x_n = q (n).
\]
If we change the variable \( x (n) = e_m (n, 0) \) and consider the homogeneous part of (2), then we have the following characteristic equation
\[
m^N + r_{N-1} m^{N-1} + \cdots + r_0 = 0,
\]
and let its roots are \( m_1, m_2, \ldots, m_N \). Hence, we have the homogeneous solution as follows,
\[
x (n) = c_1 x_1 (n) + c_2 x_2 (n) + \cdots + c_N x_N (n),
\]
where \( x_1 (n) = e_{m_1} (n, 0), x_2 (n) = e_{m_2} (n, 0), \ldots, x_N (n) = e_{m_N} (n, 0) \) is a linearly independent set of solutions.

From here, let’s take a set of new solution functions for the variation of parameters method, \( v_1 (n), v_2 (n), \ldots, v_N (n) \) and so, let’s assume that following equation
\[
X (n) = v_1 (n) x_1 (n) + v_2 (n) x_2 (n) + \cdots + v_N (n) x_N (n)
\]
is a solution of nonhomogeneous part of (2).

For finding the parameters \( v_1 (n), v_2 (n), \ldots, v_N (n) \), firstly let’s take
\[
\Delta X (n) = [\Delta v_1 (n) x_1 (n+1) + \Delta v_2 (n) x_2 (n+1) + \cdots + \Delta v_N (n) x_N (n+1)] + [v_1 (n) \Delta x_1 (n) + v_2 (n) \Delta x_2 (n) + \cdots + v_N (n) \Delta x_N (n)],
\]
let’s assume that the first bracketed part at the right hand side of the equation above equals to zero,
\[
\Delta v_1 (n) x_1 (n+1) + \Delta v_2 (n) x_2 (n+1) + \cdots + \Delta v_N (n) x_N (n+1) = 0,
\]
and so we have,
\[
\Delta X (n) = v_1 (n) \Delta x_1 (n) + v_2 (n) \Delta x_2 (n) + \cdots + v_N (n) \Delta x_N (n),
\]
then, let’s take the difference of the equality above

\[ \Delta^2 X (n) = [\Delta v_1 (n) \Delta x_1 (n + 1) + \Delta v_2 (n) \Delta x_2 (n + 1) + \cdots + \Delta v_N (n) \Delta x_N (n + 1)] + [v_1 (n) \Delta^2 x_1 (n) + v_2 (n) \Delta^2 x_2 (n) + \cdots + v_N (n) \Delta^2 x_N (n)], \]

let’s assume that

\[ \Delta v_1 (n) \Delta x_1 (n + 1) + \Delta v_2 (n) \Delta x_2 (n + 1) + \cdots + \Delta v_N (n) \Delta x_N (n + 1) = 0, \]

so we have

\[ \Delta^2 X (n) = v_1 (n) \Delta^2 x_1 (n) + v_2 (n) \Delta^2 x_2 (n) + \cdots + v_N (n) \Delta^2 x_N (n), \]

proceeding in this fashion, we have

\[ \Delta^N X (n) = v_1 (n) \Delta^N x_1 (n) + v_2 (n) \Delta^N x_2 (n) + \cdots + v_N (n) \Delta^N x_N (n). \]

If we formulate our assumptions above, we have

\[ \Delta^k X (n) = v_1 (n) \Delta^k x_1 (n) + v_2 (n) \Delta^k x_2 (n) + \cdots + v_N (n) \Delta^k x_N (n), \quad k = 1, 2, \ldots, N - 1, \]

\[ \Delta^k v_1 (n) \Delta x_1 (n + 1) + \Delta^k v_2 (n) \Delta x_2 (n + 1) + \cdots + \Delta^k v_N (n) \Delta x_N (n + 1) = 0, \quad k = 0, 1, \ldots, N - 2, \]

if obtained equalities above is written in \( \{2\} \), then we have the following equation system

\[ \begin{align*}
\Delta v_1 (n) & x_1 (n + 1) + \Delta v_2 (n) x_2 (n + 1) + \cdots + \Delta v_N (n) x_N (n + 1) = 0, \\
\Delta v_1 (n) & \Delta x_1 (n + 1) + \Delta v_2 (n) \Delta x_2 (n + 1) + \cdots + \Delta v_N (n) \Delta x_N (n + 1) = 0, \\
& \vdots \\
\Delta v_1 (n) & \Delta^{N-2} x_1 (n + 1) + \Delta v_2 (n) \Delta^{N-2} x_2 (n + 1) + \cdots + \Delta v_N (n) \Delta^{N-2} x_N (n + 1) = 0, \\
\Delta v_1 (n) & \Delta^{N-1} x_1 (n + 1) + \Delta v_2 (n) \Delta^{N-1} x_2 (n + 1) + \cdots + \Delta v_N (n) \Delta^{N-1} x_N (n + 1) = q (n). 
\end{align*} \]

If we solve this system by Cramer rule, we have the Casoratian,

\[ \det \begin{vmatrix}
 x_1 (n + 1) & x_2 (n + 1) & \cdots & x_N (n + 1) \\
\Delta x_1 (n + 1) & \Delta x_2 (n + 1) & \cdots & \Delta x_N (n + 1) \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{N-1} x_1 (n + 1) & \Delta^{N-1} x_2 (n + 1) & \cdots & \Delta^{N-1} x_N (n + 1)
\end{vmatrix} = W (x_1, x_2, \ldots, x_N) (n + 1) \]

and assuming the Casoratian is different from zero. Let \( W_i \) correspond the determinant of the \( i \)th column of the Casoratian with the column \((0, 0, 0, \ldots, 0, 1)\) and so, solution of the system as follows

\[ \Delta v_1 (n) = \frac{q (n) W_1 (n)}{W (n + 1)}, \quad \Delta v_2 (n) = \frac{q (n) W_2 (n)}{W (n + 1)}, \ldots, \quad \Delta v_N (n) = \frac{q (n) W_N (n)}{W (n + 1)}, \]

from here we have the parameters as follows

\[ v_1 (n) = \int \frac{q (n) W_1 (n)}{W (n + 1)} \Delta n, \quad v_2 (n) = \int \frac{q (n) W_2 (n)}{W (n + 1)} \Delta n, \ldots, \quad v_N (n) = \int \frac{q (n) W_N (n)}{W (n + 1)} \Delta n. \]

Finally, the particular solution is as follows

\[ X (n) = x_1 (n) \int \frac{q (n) W_1 (n)}{W (n + 1)} \Delta n + x_2 (n) \int \frac{q (n) W_2 (n)}{W (n + 1)} \Delta n + \cdots + x_N (n) \int \frac{q (n) W_N (n)}{W (n + 1)} \Delta n. \]
3.2. Numerical Results and Discussions of Some Discrete Eigenvalue Problems Having Physical Potential

In this section, a new version of the variation of parameters method is applied by using delta exponential function. First of all, we consider the second-order Sturm-Liouville problem, which is called also one dimensional Schrödinger equation, having Coulomb potential, hydrogen atom equation, and the fourth-order relaxation difference equations. We find sum representation of solution for the second order discrete Sturm-Liouville problem having Coulomb potential, hydrogen atom equation, and analytical solution of the fourth order discrete relaxation problem by means of variation of parameters method by using delta exponential function.

3.2.1. Discrete Hydrogen Atom Equation and Discrete Sturm-Liouville Equation Having Coulomb Potential

First of all, let’s consider the following second order hydrogen atom equation,

\[-\Delta^2 x(n-1) + \left(\frac{A}{n} + q(n)\right) x(n) = \lambda x(n), \]  

with the boundary conditions,

\[x(0) = x(b) = 0, \]  

has a unique solution \(x(n)\) as follows, \(n \in [0, b],\ n \text{ is a finite integer},\ x(n) \in l^2 [0, b]\)

\[x(n, \lambda) = \sin n\theta + \frac{1}{\sin \theta} \int_1^{n+1} \left(\frac{A}{s} + q(s)\right) x(s) \sin(n-s) \theta \Delta s.\]

Secondly, let’s consider the following second order Sturm-Liouville difference equation having Coulomb potential,

\[-\Delta^2 x(n-1) + \left(-q(n) + \frac{2}{n^2} - \frac{l(l+1)}{n^2}\right) x(n) = \lambda x(n), \]  

with the boundary conditions, has a unique solution \(x(n)\) as follows, \(n \in [0, b],\ n \text{ is a finite integer},\ x(n) \in l^2 [0, b]\)

\[x(n, \lambda) = \sin n\theta + \frac{1}{\sin \theta} \int_1^{n+1} \left(-q(s) + \frac{2}{s - \frac{l(l+1)}{s^2}}\right) x(s) \sin(n-s) \theta \Delta s.\]
Fig 3: Comparison of the eigenfunctions for the problems (7)–(8) and (9)–(8)

Fig 4: Comparison of datas in Table 1

Table 1: Eigenfunctions correspond to the eigenvalue $\lambda = 1$.

| $n$ | $x(n)$ for Hydrogen | $x(n)$ for Coulomb |
|-----|---------------------|--------------------|
| 1   | 0.866025            | 0.866025           |
| 2   | 2.59808             | 5.19615            |
| 3   | 4.86821             | 10.6024            |
| 4   | 6.70353             | 11.5276            |
| 5   | 8.6296              | 5.24802            |
| 6   | 4.60124             | -4.77228           |
| 13  | 6.18061             | -7.29527           |
| 14  | 4.53803             | -11.2966           |
| 15  | -0.105595           | -5.75247           |
| 16  | -4.67793            | 4.67243            |
| 21  | 5.26773             | -9.75089           |
| 23  | -3.99289            | 8.78504            |
| 24  | -6.01193            | 10.8038            |
| 25  | -3.49672            | 3.43627            |

Fig 5: Comparison of datas in Table 2

Table 2: Eigenfunctions correspond to the eigenvalue $\lambda = 2 - \sqrt{2}$.

| $n$ | $x(n)$ for Hydrogen | $x(n)$ for Coulomb |
|-----|---------------------|--------------------|
| 1   | 0.707107            | 0.707107           |
| 2   | 2.41421             | 4.53553            |
| 3   | 5.62132             | 11.182             |
| 4   | 10.6548             | 17.7341            |
| 5   | 17.4379             | 20.5481            |
| 6   | 25.2922             | 17.227             |
| 13  | -7.27102            | 6.52242            |
| 14  | -22.8474            | 16.3393            |
| 15  | -32.7782            | 19.1177            |
| 16  | -34.1566            | 13.5942            |
| 21  | 32.7366             | -17.9177           |
| 23  | 19.0974             | 2.89076            |
| 24  | 0.835118            | 19.0276            |
| 25  | -17.7111            | 15.2384            |

Suppose that $q(n) = \frac{1}{\sqrt{n}}$, $A = 1$, $l = 2$ in the figures and tables above.
Proof. Firstly, we study to find the general solution of the equation (7) by the variation of parameters method without using the boundary conditions. Homogenous part of (7) is as follows,
\[ \Delta^2 x(n-1) + \lambda x(n) = 0. \]
By using delta exponential function, we have the characteristic equation,
\[ \frac{m^2}{1+m} + \lambda = 0, \]
and characteristic roots are as follows,
\[ m_{1,2} = \frac{-\lambda \pm \sqrt{\lambda(\lambda - 4)}}{2}, \]
where \( m_{1,2} \in \mathbb{R} \). So, the homogeneous solution is found from (4) as follows,
\[ x_h(n) = c_1 e_{m_1}(n,0) + c_2 e_{m_2}(n,0). \] (11)
Then, if we apply the variation of parameters method, linearly independent solutions is found as \( e_{m_1}(n,0) \), \( e_{m_2}(n,0) \).
From here, if we take the constants as parameters \( v_1(n) \) and \( v_2(n) \), then we find the variables by Cramer rule, let \( q_1(n) = \frac{A}{n} + q(n) \)
\[ \Delta v_1(n-1) = \frac{-q_1(n) x(n) \Delta e_{m_2}(n,0)}{W(e_{m_1}(n,0), e_{m_2}(n,0))}, \]
where \( W \) is Casoratian,
\[ W(e_{m_1}(n,0), e_{m_2}(n,0)) = e_{m_1:\oplus m_2}(n,0) (m_2 - m_1), \]
\[ = -\sqrt{\lambda(\lambda - 4)}. \]
Hence,
\[ v_1(n) = \frac{1}{\sqrt{\lambda(\lambda - 4)}} \int_0^{n+1} q_1(s) x(s) e_{m_2}(s,0) \Delta s. \]
Similarly,
\[ v_2(n) = -\frac{1}{\sqrt{\lambda(\lambda - 4)}} \int_0^{n+1} q_1(s) x(s) e_{m_1}(s,0) \Delta s. \]
Finally, the general solution is found by
\[ x(n) = c_1 e_{m_1}(n,0) + c_2 e_{m_2}(n,0) + \frac{1}{\sqrt{\lambda(\lambda - 4)}} \int_0^{n+1} q_1(s) x(s) [e_{m_1}(s,0) e_{m_2}(n,0) - e_{m_2}(s,0) e_{m_1}(n,0)] \Delta s. \]
Now, let’s continue to the proof by using the boundary conditions (8). Homogeneous solution is as in the equality (11). For finding untrivial solution, we analyze the eigenvalue \( \lambda \) in four cases, these are
i) \( \lambda = 0 \),
ii) \( \lambda = 4 \),
iii) \( \lambda > 0 \) and \( \lambda < 4 \),
iv) \( 0 < \lambda < 4 \).
We have trivial solutions for the first two cases, we arrive at a contradiction for the third case and finally, if $0 < \lambda < 4$, then we have untrivial solution. The characteristic roots are complex pair and taking $\lambda = 2 - 2 \cos \theta$, then we have

$$m_{1,2} = (-1 + \cos \theta) \pm i \sin \theta$$

So, homogeneous solution is as follows by (15)

$$x(n) = c_1 e^{\alpha} (n, 0) \cos \gamma (n, 0) + c_2 e^{\alpha} (n, 0) \sin \gamma (n, 0),$$

where $\alpha = -1 + \cos \theta$, $\beta = \sin \theta$, $\gamma = \tan \theta$. If we insert $\alpha$, $\beta$, $\gamma$ in the equality above and use Theorem 2 and Definition 4, then we have the homogeneous solution of (7)

$$x(n) = c_1 \cos n\theta + c_2 \sin n\theta.$$

Then, it is easily found the general solution by the variation of parameters method.

Similarly, representation of solution is obtained for (9).

### 3.2.2. Fourth Order Relaxation Difference Equation

Let’s consider the following the fourth order relaxation difference equation,

$$\Delta^4 x(n) - \lambda x(n) = q(n),$$

with the initial conditions

$$x(0) = x(1) = x(2) = x(3) = 1.$$ (13)

The problem (12) - (13) has a unique solution as follows, $n \in [0, b], n$ is a finite integer, $x(n) \in l^2 [0, b]$

$$x(n) = \frac{1}{4s^2(-1 + s^3)}(1 + s)^n \left(s^7 + q_0 - s^3(1 + q_0) + s^2(q_0 - q_1) - 3q_1 + 3q_2 - s(q_0 - 2q_1 + q_2) - q_3\right) -$$

$$i \frac{1}{4s^2(-1 + s^3)}(1 + s)^n \left((-1 + s^2)q_0 - (-3 + s^2)q_1 - 3q_2 + q_3\right) +$$

$$\frac{\frac{1}{4s^2(-1 + s^3)}(1 - s)^n + (1 + is)^n n}{4s^2(-1 + s^3)} \sum_{i=0}^{n} \left( \frac{1}{4s^2} \left( (1 - is)^i + (1 + is)^{-i} \right) \left( (1 - is)^{1+i} - (1 + is)^{1-i} \right) q_i \right)$$

$$+ \frac{\frac{1}{2} \frac{1}{4s^2(-1 + s^3)}(1 - s)^n + (1 + is)^n n}{4s^2(-1 + s^3)} \sum_{i=0}^{n} \left( \frac{1}{4s^2} \left( (1 - is)^i + (1 + is)^{-i} \right) \left( (1 - is)^{1+i} - (1 + is)^{1-i} \right) q_i \right)$$

$$+ \frac{\frac{1}{2} \frac{1}{4s^2(-1 + s^3)}(1 + s)^n \sum_{i=0}^{n} \left( \frac{1}{4s^2} \left( (1 - is)^i - (1 + is)^{1+i} \right) \left( (1 - is)^{1+i} - (1 + is)^{1-i} \right) q_i \right)$$

$$+ \frac{\frac{1}{2} \frac{1}{4s^2(-1 + s^3)}(1 + s)^n \sum_{i=0}^{n} \left( \frac{1}{4s^2} \left( (1 - is)^i - (1 + is)^{1+i} \right) \left( (1 - is)^{1+i} - (1 + is)^{1-i} \right) q_i \right)$$
Fig 6: Eigenfunctions for the problem (11)–(12), \( q(n) = n \)

| \( n \) | \( x(n), q(n) = \frac{1}{n+1} \) | \( x(n), q(n) = \frac{1}{\sqrt{n+1}} \) | \( x(n), q(n) = \frac{1}{n+1} \) |
|-----|----------|----------|----------|
| 0   | 1        | 1        | 0        |
| 1   | 1        | 1        | 1        |
| 2   | 1        | 1        | 1        |
| 3   | 1        | 1        | 1        |
| 4   | 16.9467  | 2.0407   | 4        |
| 5   | 80.7556  | 5.9615   | 5        |
| 6   | 240.321  | 15.2739  | 6        |
| 7   | 559.519  | 33.0829  | 7        |
| 8   | 1373.36  | 63.0968  | 8        |
| 9   | 4308.94  | 109.83   | 9        |
| 10  | 14838.5  | 179.1    | 10       |
| 11  | 47386.4  | 279.4    | 11       |
| 12  | 138351   | 932.08   | 12       |
| 13  | 393074   | 1377.6   | 13       |

Table 3: Eigenfunctions correspond to the eigenvalue \( \lambda = 0.0625 \)

Table 4: Eigenfunctions correspond to the eigenvalue \( \lambda = 0.1296 \)

Proof. Firstly, we study to find the general solution of the equation (12) by the variation of parameters method without using the boundary conditions. Homogenous part of (12) is as follows,

\[ \Delta^4 x(n) - \lambda x(n) = 0. \]

By using delta exponential function, we have the characteristic equation,

\[ m^4 - \lambda = 0, \]

and characteristic roots are as follows,

\[ m_{1,2} = \pm \sqrt[4]{\lambda}, m_{3,4} = \pm i \sqrt[4]{\lambda} \]
where \( m_{1,2,3,4} \in \mathbb{R} \). So, the homogeneous solution is found from (14) as follows,

\[
x_h(n) = c_1e_{m_1}(n, 0) + c_2e_{m_2}(n, 0) + c_3e_{m_3}(n, 0) + c_4e_{m_4}(n, 0).
\]

Suppose that

\[
\lambda = s^4,
\]

and so, homogeneous solution is as follows by (5)

\[
x(n) = c_1e_s(n, 0) + c_2e_s(n, 0) + c_3e_{\alpha}(n, 0)\cos\gamma(n, 0) + c_4e_{\alpha}(n, 0)\sin\gamma(n, 0),
\]

where \( \alpha = 0, \beta = s, \gamma = s \). If we insert \( \alpha, \beta, \gamma \) in the equality above and using Theorem 2 and Definition 4, then we have the homogenous solution of (12)

\[
x(n) = c_1(1-s)^n + c_2(1+s)^n + c_3\frac{(1+is)^n + (1-is)^n}{2} + c_4\frac{(1+is)^n - (1-is)^n}{2i}.
\]

Then, if we apply the variation of parameters method and if we take the constants as parameters \( v_1(n), v_2(n), v_3(n), v_4(n) \), then we find the variables by Cramer rule,

\[
\begin{align*}
v_1 &= \sum_{i=0}^{n} \left( \frac{-2is^3(1+s)^i(1+s^2)^i(1+s+s^2+s^3)i}{8s^6(1-s^4)^{1+i}} q_i \right), \\
v_2 &= \sum_{i=0}^{n} \left( \frac{-2(1-s)s^3(1+s^2)^i(-1+s-s^2+s^3)i}{8s^6(1-s^4)^{1+i}} q_i \right), \\
v_3 &= \sum_{i=0}^{n} \left( \frac{2s^3(i((1-is)^i-(1+is)^i) + ((1-is)^i + (1+is)^i)s)(1-s^2)^{1+i}}{8s^6(1-s^4)^{1+i}} q_i \right), \\
v_4 &= \sum_{i=0}^{n} \left( \frac{-2s^3((1-is)^i+(1+is)^i) - i((1-is)^i-(1+is)^i)s)(1-s^2)^{1+i}}{8s^6(1-s^4)^{1+i}} q_i \right).
\end{align*}
\]

Applying the initial conditions (13), we have the constants as follows

\[
\begin{align*}
c_1 &= \frac{1}{4s^3(-1+s^4)} \left( s^7 - q_0 - s^3(1+q_0) + 3q_1 + s^2(-q_0 + q_1) - 3q_2 - s(q_0 - 2q_1 + q_2) + q_3 \right), \\
c_2 &= \frac{1}{4s^3(-1+s^4)} \left( s^7 + q_0 - s^3(1+q_0) + s^2(q_0 - q_1) - 3q_1 + 3q_2 - s(q_0 - 2q_1 + q_2) - q_3 \right), \\
c_3 &= \frac{s^6 + q_0 - s^2(1+q_0) - 2q_1 + q_2}{2s^2(-1+s^4)}, \\
c_4 &= \frac{(-1+s^2)q_0 - (-3+s^2)q_1 - 3q_2 + q_3}{2s^3(-1+s^4)}.
\end{align*}
\]

Hence, we obtain the general solution (11).

**Conclusion**

Consequently, the variation of parameters method for higher order linear ordinary difference equations with constant coefficient is considered with a new approachment by using delta exponential function. Analysis of the method is given in detailed and the advantage of this new approachment is to enable us to investigate the solution of difference equations in the closed form given in (22), otherwise it has to be expanded by using binomialal expansion of the difference operator

\[
\Delta^N x(t) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} x(t + N - k)
\]
and the method is supported with two difference eigenvalue problems; the second-order Sturm-Liouville and the fourth-order relaxation difference equations. We find the sum representation of solution of Sturm-Liouville difference problem and also, we find the analytical solution of the fourth-order relaxation difference problem.

Moreover, behaviors of eigenfunctions for the problems and are analyzed and illustrated by graphics and tables. Firstly, we show the behaviors of eigenfunctions while and eigenvalues are continuous in Fig1 and Fig2, and we observe that eigenfunctions are continuous according to the eigenvalues. Also we compare the behaviors of eigenfunctions in Fig3. Then, we analyze the behaviors of eigenfunctions for the specific eigenvalues in Table1, Table2 and Fig4 and Fig5 and we observe that eigenfunctions are discrete. We analyze the similar properties for the problem in Fig6 while and in Fig7 and Table3 while and in Fig8 and Table4 while.

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