DUAL CURVATURE MEASURES ON CONVEX FUNCTIONS AND ASSOCIATED MINKOWSKI PROBLEMS

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Abstract. In this paper, the $q$-th dual curvature measure is extended to convex functions and the associated Minkowski problem is posed. A special case includes the $q$-th dual curvature measure of convex bodies which defined by Huang, Lutwak, Yang and Zhang. Existence for the functional dual Minkowski problem is showed when $q \leq 0$ and the uniqueness part is obtained with some assumptions.

1. Introduction

The classical Brunn-Minkowski theory originated with the thesis of Brunn in 1887 and is in its essential parts with the creation of Minkowski around the turn of the century. Mixed volumes are important geometric concepts, which include quermass-integrals, surface area measures, curvature measures and mixed area measures. An interesting problem related to surface area measures is the classical Minkowski problem that is crucial important in the Brunn-Minkowski theory:

What are necessary and sufficient conditions for a finite Borel measure $\mu$ on the unite sphere $S^{n-1}$ so that $\mu$ is the surface area measure of a convex body in $\mathbb{R}^n$?

The Minkowski problem was solved in multiple ways by Minkowski, Aleksandrov, and Fenchel and Jessen [55, 62]. Landmark contributions to establishing regularity for the Minkowski problem are due to Lewy [40], Nirenberg [56], Pogorelov [57, 58], Cheng and Yau [12], Caffarelli [9], Guan and Ma [32] and others.

If the measure has a density $f$ with respect to the Lebesgue measure of the unit sphere $S^{n-1}$, the Minkowski problem is equivalent to the study of solution to the following Monge-Ampère equation on the unit sphere

$$\det(h_{ij} + \delta_{ij}h) = f,$$

where $h$ is the unknown function on $S^{n-1}$ to be found, $h_{ij}$ is the covariant derivative of $h$ with respect to an orthonormal frame on $S^{n-1}$ and $\delta_{ij}$ is the Kronecker delta.

The $L_p$ Brunn-Minkowski theory, a generalization of the classical Brunn-Minkowski theory, attracted increasing interest in the recent years. Lutwak [48] introduced the...
$L_p$ surface area measure, a Borel measure on convex body in $\mathbb{R}^n$ with the origin in its interior. The $L_p$ surface area measure is an important concept in the $L_p$ Brunn-Minkowski theory, and it provides an integral representation for the $L_p$ mixed volume. By the $L_p$ cosine transform of the $L_p$ surface area measure, Lutwak, Yang and Zhang [49] defined the $L_p$ Petty projection body and established the well-known $L_p$ Petty projection inequality. Finding the necessary and sufficient conditions for a given measure to guarantee that it is the $L_p$ surface area measure is the existence problem called the $L_p$ Minkowski problem posed in [48]. Solving the $L_p$ Minkowski problem requires solving a degenerate and singular Monge-Ampère type equation on the unit sphere. The $L_p$ Minkowski problem has been solved for $p \geq 1$ [13, 38, 48, 51], and for the critical case $p < 1$ [7, 63, 64, 72–75]. The solution of the $L_p$ Minkowski problem and the $L_p$ Petty projection inequality are the key tools used for establishing the $L_p$ affine Sobolev inequality and analogues [14, 33, 34, 50, 68].

A theory analogous to the Brunn-Minkowski theory was introduced in [45]. It demonstrates a remarkable duality in convex geometry, and thus is called the dual Brunn-Minkowski theory. The important work of the duality between projection bodies and intersection bodies exhibited in [46]. The dual Brunn-Minkowski theory has attracted extensive attention since the intersection body helped achieving a major breakthrough in the solution of the celebrated Busemann-Petty problem. In the past decades, the dual Brunn-Minkowski theory has a rapid growth [28–30, 47, 67, 69, 77].

Very recently, Huang, Lutwak, Yang and Zhang [35] introduced dual curvature measures. This new family of measures miraculously connects the cone-volume measure and Aleksandrov’s integral curvature. The associated Minkowski problems of dual curvature measures are called dual Minkowski problems. Since the groundbreaking work of Huang, Lutwak, Yang and Zhang, the intrinsic partial differential equations that arise within the dual Brunn-Minkowski theory which wait a full 40 years after the birth of the dual theory to emerge. Some significant breakthroughs of dual Minkowski problems have been achieved [6, 8, 11, 31, 36, 37, 41, 52, 65, 66, 70, 71, 76] but many cases remain open.

Motivated by works of Huang, Lutwak, Yang and Zhang [35], we will introduce the functional dual curvature measures on the setting of convex functions. The new functional dual curvature measures include dual curvature measures introduced in [35] as a special case. All works in this paper are characterized as the functional dual Brunn-Minkowski theory, a new branch in convex geometric analysis.

It is well known that the classical isoperimetric inequality is equivalent to the Sobolev inequality. This equivalence shows a close connection between geometry and analysis. The geometric inequalities and geometric problems of functions received considerable attention. Actually an analytic inequality contains more information than its corresponding geometric inequality. During the past decades, many analytic inequalities with geometric background were found [1–4, 10, 15, 16, 22–24, 26, 27, 42, 54, 59] basically from the viewpoint of integral and convex geometry.
The algebraic structure (i.e., “Minkowski sum” and “scalar multiplication”) on the set of convex bodies is the cornerstone in the classical Brunn-Minkowski theory. By using the infimal convolution and right scalar multiplication of convex functions, Colesanti and Frągala [17] introduced the “sum” and “scalar multiplication” of log-concave functions. Colesanti and Frągala’s work belongs to the functional Brunn-Minkowski theory. We recall that the infimal convolution and the right scalar multiplication. Let \( \varphi, \psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be convex functions. The infimal convolution of \( \varphi \) and \( \psi \) is defined by

\[
\varphi \square \psi(x) = \inf_{y \in \mathbb{R}^n} \{ \varphi(x - y) + \psi(y) \} \quad \forall x \in \mathbb{R}^n,
\]

and the right scalar multiplication, \( \varphi t \), is defined by,

\[
(\varphi t)(x) = t \varphi \left( \frac{x}{t} \right), \quad \text{for} \quad t > 0.
\]

In order to study the functional dual Minkowski problems, the crucial step is to properly define the functional dual quermassintegrals. For a non-negative convex function \( \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and \( q \in \mathbb{R} \), the \((n - q)\)-th dual quermassintegral \( \tilde{W}_{n-q}(\varphi) \) is defined by

\[
\tilde{W}_{n-q}(\varphi) = \int_{\mathbb{R}^n} \varphi(x)^{-q} d\gamma_n(x), \tag{1.1}
\]

where \( d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx \) denotes the Gaussian probability measure. We will prove that the \((n - q)\)-th dual quermassintegral, \( \tilde{W}_{n-q}(\varphi) \), of the convex function \( \varphi \) includes the \((n - q)\)-th dual quermassintegral of a convex body.

For \( q \in \mathbb{R} \setminus \{0\} \), the \((n - q)\)-th dual mixed quermassintegral, \( \tilde{W}_{n-q}(\varphi, \psi) \), of non-negative convex functions \( \varphi \) and \( \psi \) is defined by

\[
\tilde{W}_{n-q}(\varphi, \psi) = \frac{1}{q} \lim_{t \to 0^+} \frac{\tilde{W}_{n-q}(\varphi \square (\psi t)) - \tilde{W}_{n-q}(\varphi)}{t}. \tag{1.2}
\]

The Minkowski type inequality for the functional dual mixed quermassintegrals can be stated as follows:

**Minkowski type inequality.** If \( q \leq 0 \) and \( \varphi, \psi \) are non-negative convex functions in \( \mathbb{R}^n \), then

\[
\tilde{W}_{n+1-q}(\varphi, \psi) \geq \frac{\tilde{W}_{n+1-q}(\psi) \tilde{W}_{n+1-q}(\varphi)}{\tilde{W}_{n+1-q}(\varphi) \tilde{W}_{n+1-q}(\varphi) q^{-q}} - \tilde{W}_{n+1-q}(\varphi, \varphi), \tag{1.3}
\]

with equality if and only if \( \varphi(x) = \psi(x - x_0) \) for some \( x_0 \in \mathbb{R}^n \) when \( q = 0 \), and \( \varphi(x) = \psi(x) \) when \( q < 0 \).

It should be pointed out that \( \tilde{W}_{n+1-q}(\varphi, \varphi) \) does not agree with \( \tilde{W}_{n+1-q}(\varphi) \) but it satisfies

\[
q \tilde{W}_{n-q}(\varphi, \varphi) = (n - q) \tilde{W}_{n-q}(\varphi) - \int_{\mathbb{R}^n} |x|^2 \varphi(x)^{-q} d\gamma_n(x),
\]

for \( q \in \mathbb{R} \setminus \{0\} \).
If $q \leq 0$ and $\varphi, \psi$ satisfy some additional conditions (see details in Theorem 3.1), we prove that the functional dual mixed quermassintegral has the following integral representation:

$$
\frac{d}{dt} \tilde{W}_{n+1-q}(\varphi \square (\psi t)) \bigg|_{t=0+} = \int_{\mathbb{R}^n} \psi^*(x) d\tilde{C}_q(\varphi, x), \quad (1.4)
$$

where $\psi^*$ is the Legendre transform of $\psi$ (defined in Section 2) and the Borel measure $\tilde{C}_q(\varphi, \cdot)$ on $\mathbb{R}^n$ is defined by

$$
\tilde{C}_q(\varphi, \cdot) = (\nabla \varphi)(\varphi^{-q} d\gamma_n).
$$

When limited to a subclass of convex functions, $\tilde{C}_q(\varphi, \cdot)$ reduces to the $q$-th dual curvature measure of a convex body. The $q$-th dual curvature measure of the convex function $\varphi$ is also called the functional $q$-th dual curvature measure. It leads naturally to the following Minkowski problem involving the functional $q$-th dual curvature measure.

**The functional dual Minkowski problem.** Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$ and $q \in \mathbb{R}$. Find the necessary and sufficient conditions on $\mu$ so that it is the $q$-th dual curvature measure $\tau \tilde{C}_q(\varphi, \cdot)$ of a convex function $\varphi$ in $\mathbb{R}^n$, where the constant $\tau \in \mathbb{R}$.

We will show that when the measure $\mu$ has a density function $g : \mathbb{R}^n \to \mathbb{R}$, the partial differential equation for the functional dual Minkowski problem is a Monge-Ampère type equation:

$$
(2\pi)^{-\frac{N}{2}} f(\nabla f^*)^{-q} e^{-\frac{|\nabla f^*|^2}{2}} \det(\nabla^2 f^*) = g \quad \text{on} \quad \Omega, \quad (1.5)
$$

where $f$ is the unknown convex function in $\mathbb{R}^n$ to be found, $\nabla f$ and $\nabla^2 f$ denote respectively the gradient vector and the Hessian matrix of $f$ with respect to an orthonormal frame in $\mathbb{R}^n$, and $f^*$ denotes the Legendre transform of $f$ (see the detailed definition in Section 2), $\Omega = \{ x \in \mathbb{R}^n : f(x) < +\infty \} \cap \{ x \in \mathbb{R}^n : f^*(x) < +\infty \}$.

We solve the functional dual Minkowski problem when $q \leq 0$.

**Existence of solution to the functional dual Minkowski problem.** Let $\mu$ be a non-zero finite Borel measure with $\int_{\mathbb{R}^n} |x| d\mu(x) < \infty$ on $\mathbb{R}^n$ and $q \leq 0$. If $\mu$ is not supported in a lower-dimensional subspace, then there exists a constant $\tau \in \mathbb{R}$ and a non-negative, convex function $\varphi$ with finite $\int_{\mathbb{R}^n} \varphi(x)^{1-q} d\gamma_n(x)$ such that $\mu(\cdot) = \tau \tilde{C}_q(\varphi, \cdot)$. Moreover, if $\varphi$ is essentially-continuous then $\tilde{C}_q(\varphi, \cdot)$ is not supported in a lower-dimensional subspace.

We remark that a convex function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is essentially-continuous if $\varphi$ is lower semi-continuous and if the set of points where $\varphi$ is discontinuous has zero $\mathcal{H}^{n-1}$-measure, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

By the Minkowski type inequality (1.3), we establish the uniqueness of the solution to the functional dual Minkowski problem.
Uniqueness of the solution to the functional dual Minkowski problem. Let $q \leq 0$ and let $\varphi_1, \varphi_2$ be non-negative convex functions in $\mathbb{R}^n$ with $\varphi_1(0) = \varphi_2(0) = 0$. If there exist $c_1, c_2 > 0$ such that $\varphi_1^* - c_2 \varphi_2^*$ and $\varphi_2^* - c_1 \varphi_1^*$ are convex, then $\tilde{C}_q^{-1}(\varphi_1, \cdot) = \tilde{C}_q^{-1}(\varphi_2, \cdot)$ implies that $\varphi_1$ and $\varphi_2$ agree up to a translation when $q = 0$, and $\varphi_1 = \varphi_2$ almost everywhere when $q < 0$.

In [35], authors proved that the 0-th dual curvature measure is the Aleksandrov’s integral curvature, and the related Minkowski problem is called Aleksandrov problem. Therefore, the existential part can be viewed as the solution to the functional Aleksandrov problem when $q = 0$. Similarly, the functional $n$-th dual curvature measure, $\tilde{C}_n(\varphi, \cdot)$, is the functional cone-volume measure, and the corresponding Minkowski problem is the functional log-Minkowski problem. Unfortunately, the uniqueness part does not imply these two special cases. In this paper, we only consider the functional dual Minkowski problem for the case $q \leq 0$.

2. Preliminaries

2.1. The setting of convex functions.

A function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if for every $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$

$$\varphi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\varphi(x) + \lambda \varphi(y).$$

Let

$$\text{dom}(\varphi) = \{x \in \mathbb{R}^n : \varphi(x) \in \mathbb{R}\}.$$ We say that $\varphi$ is proper if $\text{dom}(\varphi) \neq \emptyset$. The Legendre transform of $\varphi$ is the convex function defined by

$$\varphi^*(y) = \sup_{x \in \mathbb{R}^n} \left\{ \langle x, y \rangle - \varphi(x) \right\} \quad \forall y \in \mathbb{R}^n. \quad (2.1)$$

Clearly, $\varphi(x) + \varphi^*(y) \geq \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$. There is an equality if and only if $\varphi(x) < +\infty$ and $y \in \partial \varphi(x)$, the subdifferential of $\varphi$ at $x$ define by

$$\partial \varphi(x) = \{z \in \mathbb{R}^n : \varphi(y) \geq \varphi(x) + \langle z, y - x \rangle \text{ for all } y \in \mathbb{R}^n\}. \quad (2.2)$$

Hence,

$$\varphi^*(\nabla \varphi(x)) + \varphi(x) = \langle x, \nabla \varphi(x) \rangle.$$ The following elementary property was proved in [60],

$$\varphi^*(0) = -\inf \varphi. \quad (2.3)$$

For convenient, we set

$$\mathcal{L} = \left\{ \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \mid \varphi \text{ proper, convex, } \lim_{|x| \to +\infty} \varphi(x) = +\infty \right\},$$

and $\mathcal{L}_o$, all non-negative convex functions of the subset of $\mathcal{L}$.

A function $\varphi \in \mathcal{L}$ is lower semi-continuous, if the subset $\{x \in \mathbb{R}^n : \varphi(x) > t\}$ is an open set for any $t \in (-\infty, +\infty]$. The function $\varphi^*$ is always convex and lower semi-continuous. If $\varphi$ is a lower semi-continuous convex function, then $\varphi^*$ is also a
lower semi-continuous convex function, and \( \varphi^{**} = \varphi \). It should be noted that for a general function \( \varphi \) one always has \( \varphi^{**} \leq \varphi \).

For \( \varphi, \psi \in \mathcal{L} \), the infimal convolution is defined by
\[
\varphi \Box \psi(x) = \inf_{y \in \mathbb{R}^n} \{ \varphi(x - y) + \psi(y) \} \quad \forall x \in \mathbb{R}^n,
\]
and the right scalar multiplication is defined by,
\[
(\varphi t)(x) = t\varphi \left( \frac{x}{t} \right), \quad \text{for} \quad t > 0.
\]

A function \( f : \mathbb{R}^n \to \mathbb{R} \) is log-concave if \( f = e^{-\varphi} \) with \( \varphi \in \mathcal{L} \). Then we have the following Prékopa-Leindler inequality [62]:

Let \( 0 < t < 1 \) and \( f, g, h \) be non-negative integrable functions in \( \mathbb{R}^n \) satisfying
\[
h((1-t)x + ty) \geq f(x)^{1-t} g(y)^t,
\]
for all \( x, y \in \mathbb{R}^n \). Then
\[
\int_{\mathbb{R}^n} h(x)dx \geq \left( \int_{\mathbb{R}^n} f(x)dx \right)^{1-t} \left( \int_{\mathbb{R}^n} g(x)dx \right)^t.
\]
Equality holds in (2.6) if and only if \( f \) and \( g \) are log-concave functions, and \( f(x) = g(x - x_0) \) for some \( x_0 \in \mathbb{R}^n \).

The following properties will be useful late [17, 60].

**Proposition 2.1.** Let \( \varphi, \psi \in \mathcal{L} \). Then
1. \( (\varphi \Box \psi)^* = \varphi^* + \psi^* \);
2. \( (\varphi t)^* = t\varphi^* \), \( t > 0 \).

**Theorem A.** [60, Theorem 10.9] Let \( C \) be a relatively open convex set, and let \( f_1, f_2, \cdots \), be a sequence of finite convex functions on \( C \). Suppose that the real number sequence \( f_1(x), f_2(x), \cdots \), is bounded for each \( x \in C \). It is then possible to select a subsequence of \( f_1, f_2, \cdots \), which converges uniformly on closed bounded subsets of \( C \) to some finite convex function \( f \).

### 2.2. Dual curvature measures.

Let \( \mathbb{R}^n \) be equipped with the usual Euclidean norm \( | \cdot | \). A subset \( K \) of \( \mathbb{R}^n \) is called a convex body if it is a compact convex set with non-empty interior. The set of all convex bodies that contain the origin in their interiors is denoted by \( \mathcal{K}_o^n \). For \( x, y \in \mathbb{R}^n \), let \( \langle x, y \rangle \) denote the standard inner product of \( x \) and \( y \). For a convex body \( K \) in \( \mathbb{R}^n \), we write \( \partial K \) for the boundary of \( K \).

For \( K, L \in \mathcal{K}_o^n \) and real \( t, s > 0 \), the Minkowski combination, \( tK + sL \), is defined by
\[
tK + sL = \{ tx + sy : x \in K, y \in L \}.
\]
The support function \( h_K(\cdot) = h(K, \cdot) : \mathbb{R}^n \to \mathbb{R} \) of a convex body \( K \) is defined by
\[
h_K(x) = \max \{ \langle x, y \rangle : y \in K \}, \quad x \in \mathbb{R}^n.
\]
The convex body $K$ is uniquely determined by its support function $h(K, \cdot)$. It is obvious that for $t > 0$, the support function of the convex body $tK = \{tx : x \in K\}$ satisfies
\[ h(tK, \cdot) = th(K, \cdot). \]
For $K \in K^n_o$, its radial function $\rho(K, \cdot) : \mathbb{R}^n \setminus \{o\} \to [0, \infty)$ is defined by
\[ \rho(K, x) = \max\{t \geq 0 : tx \in K\}. \]
The radial function is denoted also by $\rho_K(x)$.

For a convex body $K$, its gauge function $\| \cdot \|_K$ is defined by
\[ \|x\|_K = \min\{t \geq 0 : x \in tK\}. \tag{2.7} \]

The polar body $K^\circ$ of $K$ is defined by
\[ K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}. \]
If $K \in K^n_o$, then
\[ \|x\|_K = h_{K^\circ}(x) = \frac{1}{\rho_K(x)}. \tag{2.8} \]
It is obvious that
\[ \|x\|_K = 1 \text{ whenever } x \in \partial K. \tag{2.9} \]
Let $K \in K^n_o$ and $q \in \mathbb{R}$. The $q$-th dual curvature measure $\widetilde{C}_q(K, \cdot)$ is defined by (see, e.g., [35])
\[ \widetilde{C}_q(K, \omega) = \frac{1}{n} \int_{\alpha_K^\circ(\omega)} \rho_K^q(u) du, \tag{2.10} \]
for each Borel set $\omega \subset S^{n-1}$. Here $\alpha_K^\circ(\omega)$ denotes the set of directions $u \in S^{n-1}$ such that the boundary point $\rho_K(u)u$ belongs to $\nu_K^{-1}(\omega)$, where $\nu_K$ is the Gaussian map which defined on the boundary $\partial K$ of $K$, and $\nu_K^{-1}$ is the inverse Gaussian map. Equivalently, for a bounded Borel function $g : S^{n-1} \to \mathbb{R}$,
\[ \int_{S^{n-1}} g(v) d\widetilde{C}_q(K, v) = \frac{1}{n} \int_{S^{n-1}} g(\alpha_K(u)) \rho_K^q(u) du, \tag{2.11} \]
where $\alpha_K(u) = \nu_K(\rho_K(u)u)$. Moreover, if $K$ is strictly convex, then
\[ \int_{S^{n-1}} g(v) d\widetilde{C}_q(K, v) = \frac{1}{n} \int_{\partial K} g(\nu_K(x)) h_K(\nu_K(x)) |x|^{q-n} d\mathcal{H}^{n-1}(x), \tag{2.12} \]
where $\mathcal{H}^{n-1}$ is the $(n - 1)$-dimensional Hausdorff measure. We note that the total measure of the $q$-th dual curvature measure is the $(n - q)$-th dual quermassintegral $\widetilde{W}_{n-q}(K)$, i.e.,
\[ \widetilde{W}_{n-q}(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^q(u) du. \tag{2.13} \]

It was proved in [35] that the $n$-th dual curvature measure of a convex body is the cone volume measure of this convex body, while the zeroth dual curvature measure
of a convex body is Aleksandrov’s integral curvature of the polar body of this body, i.e.,
\[
\tilde{C}_n(K, \cdot) = \frac{1}{n} h_K(\cdot) dS_K(\cdot),
\]
(2.14)
\[
\tilde{C}_0(K, \cdot) = \mathcal{H}^{n-1}(\alpha_{K^\circ}(\cdot)).
\]
(2.15)

The associated Minkowski problems of the cone volume measure and the Aleksandrov’s integral curvature measure are called log-Minkowski problem (see, e.g., [7, 63, 64, 72]) and Aleksandrov problem (see, e.g., [62]), respectively.

3. Mixed dual quermassintegrals of convex functions

In this section, we will define the dual quermassintegrals for convex functions that include the dual quermassintegrals for convex bodies. Inspired by works of [35], we investigate the functional dual mixed quermassintegrals.

3.1. Integral formula of the functional dual mixed quermassintegrals.

Let \( q \in \mathbb{R} \setminus \{0\} \), and \( \varphi \in \mathcal{L}_o \). The \((n - q)\)-th dual quermassintegral is defined as
\[
\tilde{W}_{n-q}(\varphi) = \int_{\mathbb{R}^n} \varphi(x)^{-q} d\gamma_n(x).
\]
(3.1)

Let \( q < n \) and \( K \in K_o^n \). By the polar coordinates of \( x \), (2.13) and (2.8), we have
\[
\tilde{W}_{n-q}(\|x\|_{K^\circ}) = \int_{\mathbb{R}^n} \|x\|_{K^\circ}^{-q} d\gamma_n(x)
= (2\pi)^{-\frac{n}{2}} \int_0^{+\infty} e^{-\frac{r^2}{2}} r^{n-q-1} dr \int_{S^{n-1}} \|u\|_{K^\circ}^{-q} du
= n(2\pi)^{-\frac{n}{2}} 2^{\frac{n-q}{2}} \Gamma\left(\frac{n-q}{2}\right) \tilde{W}_{n-q}(K).
\]

In this sense, our definition (3.1) extended dual quermassintegrals of convex bodies to the functional version.

For real \( q \neq 0 \), the normalized dual quermassintegral \( \overline{W}_{n-q}(\varphi) \) is defined as
\[
\overline{W}_{n-q}(\varphi) = \left( \int_{\mathbb{R}^n} \varphi(x)^{-q} d\gamma_n(x) \right)^{\frac{1}{q}},
\]
(3.2)

and, for \( q = 0 \), by
\[
\overline{W}_n(\varphi) = \exp\left( -\int_{\mathbb{R}^n} \log \varphi(x) d\gamma_n(x) \right).
\]
(3.3)

The functional dual mixed quermassintegrals are also introduced.

**Definition 3.1.** Let \( q \in \mathbb{R} \setminus \{0\} \), and let \( \varphi, \psi \in \mathcal{L}_o \). The \((n - q)\)-th dual mixed quermassintegral of \( \varphi \) and \( \psi \) is defined as
\[
\tilde{W}_{n-q}(\varphi, \psi) = \lim_{t \to 0^+} \frac{\tilde{W}_{n-q}(\varphi^\square(\psi t)) - \tilde{W}_{n-q}(\varphi)}{t}.
\]
In particular, when $\varphi = \psi$, we have

**Lemma 3.1.** Let $q \in \mathbb{R} \setminus \{0\}$, and let $\varphi \in \mathcal{L}_o$. If $\widetilde{W}_{n-q}(\varphi)$ is finite, then

$$q\widetilde{W}_{n-q}(\varphi, \varphi) = (n-q)\widetilde{W}_{n-q}(\varphi) - \int_{\mathbb{R}^n} |x|^2 \varphi(x)^{-q} d\gamma_n(x).$$

**Proof.** From the definitions of infimal convolution (2.4) and right scalar multiplication (2.5), we have

$$\varphi \Box[\varphi t](x) = \inf_{y \in \mathbb{R}^n} \{ \varphi(x-y) + t\varphi \left( \frac{y}{t} \right) \} \leq (1+t)\varphi \left( \frac{x}{1+t} \right),$$

and the convexity of $\varphi$ deduces

$$\frac{1}{1+t}\varphi(x-y) + t\varphi \left( \frac{y}{t} \right) \geq \varphi \left( \frac{x}{1+t} \right),$$

for any $x, y \in \mathbb{R}^n$ and $t > 0$. This means that $\varphi \Box(\varphi t) = \varphi(1+t)$. A direct calculation yields

$$\widetilde{W}_{n-q}(\varphi \Box(\varphi t)) - \widetilde{W}_{n-q}(\varphi)$$

$$= \frac{(2\pi)^{-\frac{n}{2}}}{t} \left[ \int_{\mathbb{R}^n} (1+t)^{-q} \varphi \left( \frac{x}{1+t} \right)^{-q} e^{-\frac{|x|^2}{2}} dx - \int_{\mathbb{R}^n} \varphi(x)^{-q} e^{-\frac{|x|^2}{2}} dx \right]$$

$$= \frac{(2\pi)^{-\frac{n}{2}}}{t} \left[ \int_{\mathbb{R}^n} (1+t)^{-q} \varphi(x)^{-q} e^{-\frac{(1+t)^2|x|^2}{2}} dx - \int_{\mathbb{R}^n} \varphi(x)^{-q} e^{-\frac{|x|^2}{2}} dx \right]$$

$$= (2\pi)^{-\frac{n}{2}} \left[ \frac{(1+t)^{n-q} - 1}{t} \right] \int_{\mathbb{R}^n} \varphi(x)^{-q} e^{-\frac{(1+t)^2|x|^2}{2}} dx$$

$$+ (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x)^{-q} e^{-\frac{|x|^2}{2}} \left[ \frac{e^{-\frac{(1+t)^2|x|^2}{2}} - 1}{t} \right] dx.$$

By the monotone convergence theorem, let $t \to 0^+$, we obtain

$$q\widetilde{W}_{n-q}(\varphi, \varphi) = (n-q)\widetilde{W}_{n-q}(\varphi) - \int_{\mathbb{R}^n} |x|^2 \varphi(x)^{-q} d\gamma_n(x).$$

□

Let

$$\mathcal{L}' = \left\{ \varphi \in \mathcal{L} \mid \varphi \in C^2_{+}, \lim_{|x| \to +\infty} \frac{\varphi(x)}{|x|} = +\infty \right\}.$$ 

Here $C^2_+$ used for the set of functions whose Hessian matrix are positive definite at each point. Let $\mathcal{L}_o'$ denote the set of the non-negative convex function $\varphi \in \mathcal{L}'$ and there exists constants $a, b > 0$ such that $\varphi(x) \leq b|x|^{1+a}$ when $|x|$ is big enough for $x \in \mathbb{R}^n$. 

**DUAL CURVATURE MEASURES ON CONVEX FUNCTIONS**

9
In order to give the integral representation of the functional dual mixed quermass-integral we need the following three lemmas.

Lemma 3.2. [17, Lemma 2.5] Let \( \varphi \in \mathcal{L} \). There exist constants \( a, b \in \mathbb{R} \) with \( a > 0 \) such that

\[
\varphi(x) > a|x| + b,
\]

for \( x \in \mathbb{R}^n \).

Lemma 3.3. [17, Lemma 4.10] Let \( \varphi, \psi \in \mathcal{L}' \) and, for any \( t > 0 \), set \( \varphi_t = \varphi \Box (\psi t) \). Assume that \( \psi(0) = 0 \), then

1. \( \varphi_1(x) \leq \varphi_t(x) \leq \varphi(x) \), \( \forall x \in \mathbb{R}^n \), \( \forall t \in [0, 1] \);
2. \( \forall x \in \{ x \in \mathbb{R}^n : \varphi(x) < +\infty \} \), \( \lim_{t \to 0^+} \varphi_t(x) = \varphi(x) \);
3. \( \forall E \subset \{ x \in \mathbb{R}^n : \varphi(x) < +\infty \} \), \( \lim_{t \to 0^+} \nabla \varphi_t(x) = \nabla \varphi(x) \) uniformly on \( E \).

Lemma 3.4. [17, Lemma 4.11] Let \( \varphi, \psi \in \mathcal{L}' \) and, for any \( t > 0 \), set \( \varphi_t = \varphi \Box (\psi t) \). Assume that \( \psi(0) = 0 \), then for \( x \in \text{int}(\{ x \in \mathbb{R}^n : \varphi(x) < +\infty \}) \)

\[
\frac{d}{dt} \varphi_t(x) = -\psi^* (\nabla \varphi_t(x)).
\]

The next lemma will be used later.

Lemma 3.5. Let \( q < 0 \) and \( \varphi \in \mathcal{L}_o' \). If \( 0 < \int_{\mathbb{R}^n} \varphi^{2-q}(x)d\gamma_n(x) < +\infty \), then

\[
\int_{\mathbb{R}^n} \varphi^*(\nabla \varphi(x)) \varphi^{-q}(x)d\gamma_n(x) < +\infty.
\]

Proof. By a direct calculation, we have

\[
\int_{\mathbb{R}^n} \varphi^*(\nabla \varphi(x)) \varphi^{-q}(x)d\gamma_n(x)
= \int_{\mathbb{R}^n} (x, \nabla \varphi(x)) - \varphi(x) \varphi^{-q}(x)d\gamma_n(x)
= \frac{1}{1-q} \int_{\mathbb{R}^n} (x, \nabla \varphi^{1-q}(x))d\gamma_n(x) - \int_{\mathbb{R}^n} \varphi^{1-q}(x)d\gamma_n(x)
= \frac{(2\pi)^{-\frac{n}{2}}}{1-q} \int_{\mathbb{R}^n} \text{div}(\varphi^{1-q} e^{-\frac{|x|^2}{2}} x)dx + \frac{1}{1-q} \int_{\mathbb{R}^n} |x|^2 \varphi(x)^{1-q}d\gamma_n(x)
- \frac{n+q-1}{1-q} \int_{\mathbb{R}^n} \varphi(x)^{1-q}d\gamma_n(x).
\]
Since there exist constants $a > 0$, $b > 0$ such that $\varphi(x) \leq b|x|^{1+a}$ when $|x|$ is big enough for $x \in \mathbb{R}^n$, hence

$$\int_{\mathbb{R}^n} \text{div}(\varphi^{1-q} e^{-\frac{|x|^2}{2}} x) dx = \lim_{r \to +\infty} \int_{rB} \text{div}(\varphi^{1-q} e^{-\frac{|x|^2}{2}} x) dx = \lim_{r \to +\infty} r e^{-\frac{r^2}{2}} \int_{\partial(rB)} \varphi^{1-q}(x) d\mathcal{H}^{n-1}(x) = 0.$$ 

The Hölder inequality yields

$$\int_{\mathbb{R}^n} |x|^2 \varphi(x)^{1-q} d\gamma_n(x) \leq \left[ \int_{\mathbb{R}^n} \varphi(x)^{2-q} d\gamma_n(x) \right]^{\frac{1}{2-q}} \left[ \int_{\mathbb{R}^n} |x|^{2(2-q)} d\gamma_n(x) \right]^{\frac{1}{2-q}}.$$ 

The finiteness of $\int_{\mathbb{R}^n} \varphi^* (\nabla \varphi(x)) \varphi^{-q}(x) d\gamma_n(x)$ follows from the above facts, $\widetilde{W}_{n+1-q}(\varphi)$ is finite and $\int_{\mathbb{R}^n} |x|^{2(2-q)} d\gamma_n(x)$ is a constant depend on $n$ and $q$. \(\square\)

A convex function $\varphi$ is an admissible perturbation for the convex function $\psi$ if there exists a constant $c > 0$ such that the function

$$\varphi^* - c\psi^*$$

is convex.

The following theorem provides an integral formula of the $(n + 1 - q)$-th functional dual mixed quermassintegral.

**Theorem 3.1.** Let $q \leq 0$ and $\varphi, \psi \in \mathcal{C}^1_0$ and assume that $\varphi$ is an admissible perturbation for $\psi$. If $\varphi(0) = \psi(0) = 0$ and $\widetilde{W}_{n+2-q}(\varphi)$ is finite, then $\widetilde{W}_{n+1-q}(\varphi, \psi) \in [0, +\infty]$ and

$$\widetilde{W}_{n+1-q}(\varphi, \psi) = \int_{\mathbb{R}^n} \psi^* (\nabla \varphi(y)) \varphi(y)^{-q} d\gamma_n(y) = \int_{\mathbb{R}^n} \psi^*(x) d\widetilde{C}_q(\varphi, x).$$

**Proof.** The proof is divided as several steps.

**Step 1:** $\widetilde{W}_{n+1-q}(\varphi, \psi) \in [0, +\infty]$.

Let $\varphi, \psi$ be non-negative convex functions in $\mathbb{R}^n$ and, for any $t > 0$, set $\varphi_t = \varphi \Box (\psi t)$. Since $\psi(0) = 0$, then Lemma 3.3 (1) yields that the function $\varphi_t$ is pointwise decreasing with respect to $t$.

Because of Lemma 3.3 (1) and (2), so that for every $x \in \mathbb{R}^n$ there exists $\varphi(x) = \lim_{t \to 0^+} \varphi_t(x)$ and it holds $\varphi(x) \leq \varphi(x)$. By the monotone convergence theorem and $q \leq 0$, we have

$$\widetilde{W}_{n+1-q}(\varphi) = \lim_{t \to 0^+} \widetilde{W}_{n+1-q}(\varphi_t) \leq \widetilde{W}_{n+1-q}(\varphi).$$
Let us consider separately the two cases $\tilde{W}_{n+1-q}(\tilde{\varphi}) < \tilde{W}_{n+1-q}(\varphi)$ and $\tilde{W}_{n+1-q}(\tilde{\varphi}) = \tilde{W}_{n+1-q}(\varphi)$.

If $\tilde{W}_{n+1-q}(\tilde{\varphi}) < \tilde{W}_{n+1-q}(\varphi)$, then
$$\lim_{t \to 0^+} \frac{\tilde{W}_{n+1-q}(\tilde{\varphi}_t) - \tilde{W}_{n+1-q}(\varphi)}{t} = -\infty.$$ 

If $\tilde{W}_{n+1-q}(\tilde{\varphi}) = \tilde{W}_{n+1-q}(\varphi)$, we further distinguish the following two subcases:

$\exists \ t_0 > 0$ such that $\tilde{W}_{n+1-q}(\tilde{\varphi}_{t_0}) = \tilde{W}_{n+1-q}(\varphi)$,

or

$\tilde{W}_{n+1-q}(\tilde{\varphi}_t) < \tilde{W}_{n+1-q}(\varphi)$ for all $t > 0$.

Lemma 3.3 (1) declares that $\tilde{W}_{n+1-q}(\varphi_t)$ is a monotone decreasing function of $t$, so that $\tilde{W}_{n+1-q}(\varphi_t) = \tilde{W}_{n+1-q}(\varphi)$ for every $t \in [0, t_0]$ in the former subcase. This means
$$\lim_{t \to 0^+} \frac{\tilde{W}_{n+1-q}(\varphi_t) - \tilde{W}_{n+1-q}(\varphi)}{t} = 0.$$

Now, we consider the latter subcase. Since $\tilde{W}_{n+1-q}(\varphi_t)$ is a decreasing convex function of $t$ when $q \leq 0$, hence
$$\lim_{t \to 0^+} \frac{\tilde{W}_{n+1-q}(\varphi_t) - \tilde{W}_{n+1-q}(\varphi)}{t} \in [-\infty, 0]. \ (3.6)$$

Combining the above facts and the definition of functional dual quermassintegrals, we conclude that $\tilde{W}_{n+1-q}(\varphi, \psi) \in [0, +\infty]$.

**Step 2:** For $t > 0$, it holds
$$\tilde{W}_{n+1-q}(\varphi_t) - \tilde{W}_{n+1-q}(\varphi) = \int_0^t \Phi(s)ds, \ (3.7)$$

where
$$\Phi(s) = \int_{\mathbb{R}^n} \psi^*(x)d\tilde{C}_q(\varphi_s, x). \ (3.8)$$

By Lemma 3.4, it holds
$$\lim_{h \to 0} \frac{\varphi_{t+h}^{1-q}(x) - \varphi_t^{1-q}(x)}{h} = (q - 1)\psi^*(\nabla \varphi_t(x))\varphi_t^{-q}(x), \ (3.9)$$

for $x \in \mathbb{R}^n$. From the facts $q \leq 0$ and $\psi(0) = 0$, Lemma 3.3 (1) and (3), we infer that for every $s \in [0, 1]$, the non-negative function $\psi^*(\nabla \varphi_s(x))\varphi_s^{-q}(x)$ are bounded above.
by some continuous function independent of $s$. Then, by the pointwise convergence in (3.9) and dominated convergence theorem, we have

$$
\frac{1}{q-1} \lim_{h \to 0} \frac{\tilde{W}_{n+1-q}(\varphi_{t+h}) - \tilde{W}_{n+1-q}(\varphi_t)}{h} = \frac{1}{q-1} \lim_{h \to 0} \int_{\mathbb{R}^n} \frac{\varphi_{t+h}^{1-q}(x) - \varphi_t^{1-q}(x)}{h} d\gamma_n(x)
$$

Moreover,

$$
\tilde{W}_{n+1-q}(\varphi_t) - \tilde{W}_{n+1-q}(\varphi) = \int_0^t \int_{\mathbb{R}^n} \psi^*(\nabla \varphi_t(x)) \varphi_t^{-q}(x) d\gamma_n(x).
$$

**Step 3:** The function $\Phi$ defined in (3.8) takes finite values at every $s > 0$.

Let $s > 0$. Since $\varphi(0) = 0$, hence $\varphi^* \geq 0$ and

$$
s \Phi(s) \leq \int_{\mathbb{R}^n} (\varphi^*(x) + s \psi^*(x)) d\tilde{C}_q(\varphi, x)
$$

$$
= \int_{\mathbb{R}^n} \varphi_s^*(x) d\tilde{C}_q(\varphi, x).
$$

Lemma 3.5 ensures $\Phi(s) < +\infty$ when $s > 0$.

Let now $s = 0$. Since $\psi$ is an admissible perturbation for $\varphi$, by its definition (3.4) and Lemma 3.2, there exist constants $a, c > 0$ and $b \in \mathbb{R}$ such that

$$(\varphi^* - c \psi^*)(y) \geq a |y| + b,$$  \hspace{1cm} (3.10)

for $y \in \mathbb{R}^n$. Hence,

$$
\Phi(0) = \int_{\mathbb{R}^n} \psi^*(x) d\tilde{C}_q(\varphi, x)
$$

$$
\leq c^{-1} \int_{\mathbb{R}^n} \varphi^*(x) d\tilde{C}_q(\varphi, x) - \frac{b}{c} \tilde{W}_{n-q}(\varphi) - \frac{a}{c} \int_{\mathbb{R}^n} |\nabla \varphi(x)||\varphi^* q(x) d\gamma_n(x)
$$

$$
\leq c^{-1} \int_{\mathbb{R}^n} \varphi^*(x) d\tilde{C}_q(\varphi, x) - \frac{b}{c} \tilde{W}_{n-q}(\varphi).
$$

The finiteness of $\Phi(0)$ follows from Lemma 3.5, and $a > 0, c > 0$.

**Step 4:** The function $\Phi$ defined in (3.8) is continuous at every $s > 0$, and it is continuous from the right at $s = 0$.

Let $y = \nabla \varphi_s(x)$. Then we have

$$
\Phi(s) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \psi^*(y) \varphi_s(\nabla \varphi_s^*(y)) e^{-\frac{|
abla \varphi_s^*(y)|^2}{2}} \det(\nabla^2 \varphi_s^*(y)) dy.
$$
By the convexity of $\phi = \varphi^* - c\varphi^*$ for some $c > 0$, one gets
\[ \nabla^2 \phi = \nabla^2 \varphi^* - c \nabla^2 \psi^*, \]
and it is a symmetric positive semi-definite matrix. This implies that
\[ \nabla^2 \psi^* = c^{-1} \nabla^2 \varphi^* - c^{-1} \nabla^2 \phi. \]

According to the fact that
\[ \det(A + B)^{1/n} \geq \det(A)^{1/n} + \det(B)^{1/n}, \]
for symmetric positive semi-definite matrices $A$ and $B$, we infer that
\[
\det(\nabla^2 \varphi^*_s(y)) = \det(\nabla^2 \varphi^*(y) + s \nabla^2 \psi^*(y)) = \det((1 + c^{-1}s) \nabla^2 \varphi^*(y) - c^{-1}s \nabla^2 \phi) \leq (1 + c^{-1}s)^n \det(\nabla^2 \varphi^*(y)). \tag{3.11}
\]

Obviously,
\[ |\nabla \varphi^*_s(y)|^2 = |\nabla \varphi^*(y) + s \nabla \psi^*(y)|^2 \]
is a continuous function with respect to $s$. Hence,
\[ \lim_{s \to 0} |\nabla \varphi^*_s(y)|^2 = \lim_{s \to 0} |\nabla \varphi^*(y)|^2, \]
i.e., for any $0 < \varepsilon < 1$ there exists an $s_0 > 0$ small enough such that
\[ -\varepsilon < |\nabla \varphi^*_s(y)|^2 - |\nabla \varphi^*(y)|^2 < \varepsilon \tag{3.12} \]
when $0 < s < s_0$. On the other hand, by the convexity of $\phi = \varphi^* - c\varphi^*$ for some $c > 0$ and $\varphi(0) = \psi(0) = 0$, we have
\[
\varphi_s(\nabla \varphi^*_s(y)) = \langle y, \nabla \varphi^*_s(y) \rangle - \varphi_s(y) \\
= (1 + \frac{s}{c}) \left( \langle y, \nabla \varphi^*(y) \rangle - \varphi^*(y) \right) - \frac{s}{c} \left( \langle y, \nabla \phi(y) \rangle - \phi(y) \right) \\
= (1 + \frac{s}{c}) \varphi(\nabla \varphi^*(y)) - \frac{s}{c} \varphi^*(\nabla \phi(y)) \\
\leq (1 + \frac{1}{c}) \varphi(\nabla \varphi^*(y)). \tag{3.13}
\]

From (3.11), (3.12), (3.13) and (3.10) for $s \in (0, \min\{s_0, 1\})$ we have
\[
\psi^*(y) \varphi_s(\nabla \varphi^*_s(y))^{-q} \det(\nabla^2 \varphi^*_s(y)) e^{-\frac{|\nabla \varphi^*_s(y)|^2}{2}} \\
\leq (1 + c^{-1})^{-q} c \psi^*(y) \varphi(\nabla \varphi^*(y))^{-q} \det(\nabla^2 \varphi^*(y)) e^{-\frac{|\nabla \varphi^*(y)|^2}{2}} \\
\leq (1 + c^{-1})^{-q} c^{-1} \psi^*(y) \varphi(\nabla \varphi^*(y))^{-q} \det(\nabla^2 \varphi^*(y)) e^{-\frac{|\nabla \varphi^*(y)|^2}{2}} \\
- \frac{b}{c} \varphi(\nabla \varphi^*(y))^{-q} \det(\nabla^2 \varphi^*(y)) e^{-\frac{|\nabla \varphi^*(y)|^2}{2}}.
\]
Let \( h_1(y) = (2\pi)^{-\frac{n}{2}}(1+c^{-1})^n c^{-1} e\phi^*(y)\phi(\nabla \phi^*(y))^{-q} \det(\nabla^2 \phi^*(y))e^{-\frac{|\nabla \phi^*(y)|^2}{2}} \) and \( h_2(y) = \max\{0, -\frac{1}{c}\}(2\pi)^{-\frac{n}{2}}(1+c^{-1})^n c^{-1} e\phi^*(y)\phi(\nabla \phi^*(y))^{-q} \det(\nabla^2 \phi^*(y))e^{-\frac{|\nabla \phi^*(y)|^2}{2}}. \) Then
\[
\int_{\mathbb{R}^n} h_1(y)dy = (2\pi)^{-\frac{n}{2}}(1+c^{-1})^n c^{-1} \int_{\mathbb{R}^n} \phi^*(y)\phi(\nabla \phi^*(y))^{-q} \det(\nabla^2 \phi^*(y))e^{-\frac{|\nabla \phi^*(y)|^2}{2}} dy
\]
\[
= (1+c^{-1})^n c^{-1} \int_{\mathbb{R}^n} \phi^*(\nabla \phi(x))\phi(x)^{-q} d\gamma_n(x),
\]
and
\[
\int_{\mathbb{R}^n} h_2(y)dy = \max\{0, -\frac{1}{c}\} \tilde{W}_{n+1-q}(\phi).
\]
From Lemma 3.5, we obtain the integrability of the non-negative function \( h_1 + h_2 \) with respect to the Gaussian probability measure \( \gamma_n \). Therefore, the desired claim follows from dominated convergence theorem.

**Step 5: Equality (3.5) holds.**

Combining with (3.7), the finiteness (which proved in Step 3) and continuity (which proved in Step 4) of \( \Phi(s) \) for \( s > 0 \), we have
\[
\Phi(s) = \frac{d}{dt} \tilde{W}_{n+1-q}(\phi_t) \bigg|_{t=s}.
\] (3.14)
The continuity from the right of \( \Phi \) at \( s = 0 \) implies
\[
\lim_{s \to 0^+} \Phi(s) = \Phi(0) = \int_{\mathbb{R}^n} \psi^*(x) d\tilde{C}_q^*(\phi, x).
\] (3.15)
Then, equality (3.5) follows from (3.7), (3.14) and (3.15). \( \square \)

### 3.2. Minkowski version inequality.

There is a Minkowski inequality for the functional dual mixed quermassintegral.

**Lemma 3.6.** Let \( \phi, \psi \in \mathcal{L} \). For any \( t \in (0, 1) \) and \( x, y \in \mathbb{R}^n \), then
\[
[\phi(1-t)[\phi|\psi]]((1-t)x + ty) \leq (1-t)\phi(x) + t\psi(y),
\] (3.16)
with equality if and only if \( \phi(x) = \psi(x - x_0) \) for some \( x_0 \in \mathbb{R}^n \).

**Proof.** For any \( x, y \in \mathbb{R}^n \) and \( t \in (0, 1) \), we have
\[
[\phi(1-t)[\phi|\psi]]((1-t)x + ty)
\]
\[
= \inf_{z \in \mathbb{R}^n} \left\{ (1-t)\phi \left( \frac{(1-t)x + ty - z}{1-t} \right) + t\psi \left( \frac{z}{t} \right) \right\}
\]
\[
\leq (1-t)\phi(x) + t\psi(y),
\]
where in the last inequality we have used \( z = ty \). This is the desired inequality.

The inequality (3.16) is equivalent to
\[
e^{-[\phi(1-t)[\phi|\psi]]((1-t)x + ty)} \geq (e^{-\phi(x)})^{1-t} (e^{-\psi(y)})^t.
\]
Then, the Prékopa-Leindler inequality (2.6) deduces
\[
\int_{\mathbb{R}^n} e^{-[\varphi(1 - t)]\Box[\psi t](x)} \, dx \geq \left( \int_{\mathbb{R}^n} e^{-\varphi(x)} \, dx \right)^{1-t} \left( \int_{\mathbb{R}^n} e^{-\psi(y)} \, dx \right)^t. \tag{3.17}
\]
If there is an equality in (3.16), then Hölder inequality yields
\[
\int_{\mathbb{R}^n} e^{-[\varphi(1 - t)]\Box[\psi t](x)} \, dx = \int_{\mathbb{R}^n} (e^{-\varphi(x)})^{1-t} (e^{-\psi(x)})^t \, dx.
\]
\[
\leq \left( \int_{\mathbb{R}^n} e^{-\varphi(x)} \, dx \right)^{1-t} \left( \int_{\mathbb{R}^n} e^{-\psi(y)} \, dx \right)^t.
\]
This means that there is an equality in the Prékopa-Leindler inequality (3.17). The equality condition of Prékopa-Leindler inequality guarantees that convex functions \( \varphi \) and \( \psi \) agree up to a translation. Conversely, it is not hard to check that there is an equality in (3.16) if \( \varphi(x) = \psi(x - x_0) \) for some \( x_0 \in \mathbb{R}^n \).

We are now in the position to obtain the following Brunn-Minkowski inequality for the functional dual quermassintegrals.

**Theorem 3.2.** Let \( \varphi, \psi \in \mathcal{L}_o \). If \( \overline{W}_{n-q}(\varphi) \) and \( \overline{W}_{n-q}(\psi) \) are finite and \( q \leq -1 \), for any \( t \in (0, 1) \), then
\[
\overline{W}_{n-q}([\varphi(1-t)]\Box[\psi t]) \leq (1-t)\overline{W}_{n-q}(\varphi) + t\overline{W}_{n-q}(\psi) \tag{3.18}
\]
with equality if and only if \( \varphi(x) = \psi(x - x_0) \) with \( x_0 \in \mathbb{R}^n \) when \( q = -1 \) and \( \varphi = \psi \) when \( q < -1 \).

**Proof.** Since \( q \leq -1 \), (3.16) and Minkowski’s inequality, we have
\[
\overline{W}_{n-q}([\varphi(1-t)]\Box[\psi t]) \leq \left( \int_{\mathbb{R}^n} ((1-t)\varphi(x) + t\psi(y))^{-q} \, d\gamma_n(x) \right)^{\frac{1}{q}}
\]
\[
\leq (1-t)\overline{W}_{n-q}(\varphi) + t\overline{W}_{n-q}(\psi).
\]
The equality conditions follow from the equality conditions of inequality (3.16) and Minkowski’s inequality.

Similar to the proof of Theorem 3.2, we also obtain the case of \( q > -1 \) and \( q \neq 0 \).

**Corollary 3.1.** Let \( \varphi, \psi \in \mathcal{L}_o \). If \( \overline{W}_{n-q}(\varphi) \) and \( \overline{W}_{n-q}(\psi) \) are finite and \( q > -1 \) and \( q \neq 0 \), then
\[
\overline{W}_{n-q}([\varphi(1-t)]\Box[\psi t]) \geq (1-t)\overline{W}_{n-q}(\varphi) + t\overline{W}_{n-q}(\psi) \tag{3.19}
\]
with equality if and only if \( \varphi = \psi \).

The following lemma is useful.

**Lemma 3.7.** Let \( q \leq 0 \) and \( \varphi, \psi \in \mathcal{L}_o \). Then
\[
\lim_{t \to 0^+} \frac{\overline{W}_{n+1-q}([\varphi(1-t)]\Box[\psi t]) - \overline{W}_{n+1-q}(\varphi)}{t}
\]
\[
= (q-1)\overline{W}_{n+1-q}(\varphi, \psi) - (q-1)\overline{W}_{n+1-q}(\varphi, \varphi). \]
Proof. Let \( a(t) = \frac{t}{1-t} \), for \( t \in (0, 1) \). From the definition of infimal convolution (2.4), we have

\[
[\varphi \Box [\psi a(t)]] (1-t) (x) = (1-t) [\varphi \Box [\psi a(t)]] (x) = (1-t) \inf_{y \in \mathbb{R}^n} \left\{ \varphi \left( \frac{x - (1-t)y}{1-t} \right) + t \psi \left( \frac{(1-t)y}{t} \right) \right\} = \inf_{y \in \mathbb{R}^n} \left\{ (1-t) \varphi \left( \frac{x-y}{1-t} \right) + t \psi \left( \frac{y}{t} \right) \right\} = [\varphi (1-t) \Box [\psi t]] (x),
\]

and

\[
\tilde{W}_{n+1-q} \left( [\varphi (1-t) \Box [\psi t]] \right) - \tilde{W}_{n+1-q} (\varphi) = \frac{\tilde{W}_{n+1-q} \left( [\varphi \Box [\psi a(t)]] (1-t) \right) - \tilde{W}_{n+1-q} (\varphi \Box [\psi a(t)])}{t} + \frac{\tilde{W}_{n+1-q} (\varphi \Box [\psi a(t)] - \tilde{W}_{n+1-q} (\varphi)}{t}. \tag{3.20}
\]

We set \( \varphi_a(t) = \varphi \Box [\psi a(t)] \), and

\[
b_t(s) = \tilde{W}_{n+1-q} (\varphi_a(t)(1-s)),
\]

for \( s \in (0, 1) \). By a direct calculation,

\[
\frac{d}{ds} b_t(s) = (2\pi)^{-\frac{n}{2}} \frac{d}{ds} (1-s)^{1-q} \int_{\mathbb{R}^n} \left( \varphi_a(t) \left( \frac{x}{1-s} \right) \right)^{1-q} e^{-\frac{|x|^2}{2}} dx
\]

\[
= (2\pi)^{-\frac{n}{2}} \frac{d}{ds} (1-s)^{n+1-q} \int_{\mathbb{R}^n} \left( \varphi_a(t) (x) \right)^{1-q} e^{-(1-s)^2 |x|^2} dx
\]

\[
= -(n + 1 - q) (2\pi)^{-\frac{n}{2}} (1-s)^{n+1-q} \int_{\mathbb{R}^n} \left( \varphi_a(t) (x) \right)^{1-q} e^{-(1-s)^2 |x|^2} dx
\]

\[
+ (2\pi)^{-\frac{n}{2}} (1-s)^{n+1-q} \int_{\mathbb{R}^n} \left( \varphi_a(t) (x) \right)^{1-q} e^{-(1-s)^2 |x|^2} (1-s) |x|^2 dx.
\]
Then for every fixed \( t \in (0, 1) \), by Lagrange theorem we have that there exists a \( \bar{s} \in (0, 1) \) such that

\[
\frac{\tilde{W}_{n+1-q}(\lfloor \varphi \lfloor [\psi a(t)](1-t)\rfloor) - \tilde{W}_{n+1-q}(\varphi \lfloor [\psi a(t)]\rfloor)}{t} \\
= b'_t(\bar{s}) \\
= -(n + 1 - q)(2\pi)^{-\frac{n}{2}}(1 - \bar{s})^{n-q} \int_{\mathbb{R}^n} (\varphi a(t)(x))^{1-q} e^{-\frac{(1-s)^2|x|^2}{2}} dx \\
+ (2\pi)^{-\frac{n}{2}}(1 - \bar{s})^{n+1-q} \int_{\mathbb{R}^n} (\varphi a(t)(x))^{1-q} e^{-\frac{(1-s)^2|x|^2}{2}} (1 - \bar{s})|x|^2 dx.
\]

By Lemma 3.3, as \( t \to 0^+ \) functions \( \varphi a(t) \) converge decreasing to a convex function \( \varphi \). Then, by monotone convergence theorem, \( \bar{s} \to 0^+ \) and \( a(t) \to 0^+ \) as \( t \to 0^+ \), we obtain

\[
\lim_{t \to 0^+} \frac{\tilde{W}_{n+1-q}(\lfloor \varphi \lfloor [\psi a(t)](1-t)\rfloor) - \tilde{W}_{n+1-q}(\varphi \lfloor [\psi a(t)]\rfloor)}{t} \\
= -(n + 1 - q)\tilde{W}_{n+1-q}(\varphi) + \int_{\mathbb{R}^n} \tilde{\varphi}(x)^{1-q} |x|^2 d\gamma_n(x). \tag{3.21}
\]

Concerning the second addendum in (3.20), Definition 3.1 shows immediately that

\[
\lim_{t \to 0^+} \frac{\tilde{W}_{n+1-q}(\varphi \lfloor [\psi a(t)]\rfloor) - \tilde{W}_{n+1-q}(\varphi)}{t} = (q - 1)\tilde{W}_{n+1-q}(\varphi, \psi). \tag{3.22}
\]

Recall that the functions \( \varphi a(t) \) converge decreasing to some convex positive function \( \bar{\varphi} \) as \( t \to 0^+ \). In order to obtain the desired equality, we may distinguish the two cases of \( \tilde{W}_{n+1-q}(\bar{\varphi}) \leq \tilde{W}_{n+1-q}(\varphi) \) and \( \tilde{W}_{n+1-q}(\bar{\varphi}) = \tilde{W}_{n+1-q}(\varphi) \). If \( \tilde{W}_{n+1-q}(\bar{\varphi}) \leq \tilde{W}_{n+1-q}(\varphi) \), the limit in (3.22) becomes \( -\infty \), and the limit in (3.21) is finite, hence

\[
\lim_{t \to 0^+} \frac{\tilde{W}_{n+1-q}(\lfloor \varphi (1-t)\lfloor [\psi a(t)](1-t)\rfloor) - \tilde{W}_{n+1-q}(\varphi)}{t} = -\infty,
\]

and the result of the lemma holds true. If \( \tilde{W}_{n+1-q}(\bar{\varphi}) = \tilde{W}_{n+1-q}(\varphi) \), then \( \bar{\varphi} = \varphi \) almost everywhere, so that the right hand side of (3.21) is \( (1-q)\tilde{W}_{n+1-q}(\varphi, \varphi) \). The lemma follows by summing up (3.21) and (3.22). \( \Box \)

By Lemma 3.7, we obtain the following Minkowski inequality for the functional dual mixed quermassintegrals.

**Theorem 3.3.** Let \( \varphi, \psi \in \mathcal{L}_n \). If \( q \leq 0 \), then

\[
\tilde{W}_{n+1-q}(\varphi, \psi) \geq \frac{\tilde{W}_{n+1-q}(\psi) - \tilde{W}_{n+1-q}(\varphi)}{\tilde{W}_{n+1-q}(\varphi)^q} + \tilde{W}_{n+1-q}(\varphi, \varphi), \tag{3.23}
\]

with equality if and only if \( \varphi(x) = \psi(x - x_0) \) with \( x_0 \in \mathbb{R}^n \) when \( q = 0 \) and \( \varphi(x) = \psi(x) \) when \( q < 0 \).
Proof. Let
\[ \Upsilon(t) = \widetilde{W}_{n+1-q}(\varphi(1-t)\square[\psi t])^{\frac{1}{1-q}}. \]
From the Brunn-Minkowski style inequality (3.19), we noticed that \( \Upsilon(t) \) is convex on \([0, 1]\), hence
\[ \Upsilon(t) \leq \Upsilon(0) + t[\Upsilon(1) - \Upsilon(0)], \tag{3.24} \]
for \( t \in [0, 1] \). As a consequence, the derivative of the function \( \Upsilon \) at \( t = 0 \) satisfies
\[ \Upsilon'(0) \leq \Upsilon(1) - \Upsilon(0). \tag{3.25} \]
Lemma 3.7 deduces that
\[ \Upsilon'(0) = \widetilde{W}_{n+1-q}(\varphi)^{\frac{2}{2-q}} \left[ \widetilde{W}_{n+1-q}(\varphi, \varphi) - \widetilde{W}_{n+1-q}(\varphi, \psi) \right]. \]
Inequality (3.25) can be rewritten as
\[ \widetilde{W}_{n+1-q}(\varphi)^{\frac{2}{2-q}} \left[ \widetilde{W}_{n+1-q}(\varphi, \varphi) - \widetilde{W}_{n+1-q}(\varphi, \psi) \right] \leq \widetilde{W}_{n+1-q}(\varphi)^{\frac{2}{2-q}} - \widetilde{W}_{n+1-q}(\varphi)^{\frac{2}{2-q}}. \tag{3.26} \]
Finally, assume that \( \varphi(x) = \psi(x - x_0) \) with \( x_0 \in \mathbb{R}^n \) when \( q = 0 \) and \( \varphi(x) = \psi(x) \) when \( q < 0 \), then (3.23) holds with equality sign. Conversely, assume that (3.23) holds with equality sign. By inspection of the above proof one sees immediately that also inequality (3.25), and hence inequality (3.24) and inequality (3.19), must hold with equality sign. This entails that (3.19) holds as an equality, and therefore, \( \varphi \) and \( \psi \) agree up to a translation when \( q = 0 \) and \( \varphi(x) = \psi(x) \) when \( q < 0 \). \( \square \)

4. Dual curvature measures for convex functions

Theorem 3.1 provides a suitable way to define dual curvature measures for convex functions, and we will investigate the functional dual curvature measure in this section.

**Definition 4.1.** Let \( q \in \mathbb{R} \) and \( \varphi \in \mathcal{L}_o \). If \( 0 < \int_{\mathbb{R}^n} \varphi(x)^{-q} d\gamma_n(x) < +\infty \), then \( \widetilde{C}_q(\varphi, \cdot) \), the \( q \)-th dual curvature measure of \( \varphi \), is defined as
\[ \int_{\mathbb{R}^n} g(z) d\widetilde{C}_q(\varphi, z) = \int_{\mathbb{R}^n} g(\nabla \varphi(x)) \varphi(x)^{-q} d\gamma_n(x), \tag{4.1} \]
for every bounded continuous function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \). Here \( d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx \) is the Gaussian probability measure.

If \( \varphi \) is a \( C^2 \)-smooth and strictly convex function, then the map
\[ \nabla \varphi : \{ x \in \mathbb{R}^n : \varphi(x) < +\infty \} \rightarrow \{ x \in \mathbb{R}^n : \varphi^*(x) < +\infty \} \]
is smooth and bijective. By the formulas (see, e.g., [53]),
\[ x = \nabla \varphi^*(\nabla \varphi(x)) \quad \text{and} \quad \nabla^2 \varphi^*(\nabla \varphi(x)) = (\nabla^2 \varphi(x))^{-1}. \tag{4.2} \]
Let \( \varphi(x) < +\infty \) at which its Hessian \( \nabla^2 \varphi \) exists and is invertible, we have

\[
\int_{\Omega} g(x) d\tilde{C}_q(\varphi, x) = (2\pi)^{-\frac{n}{2}} \int_{\Omega} g(\nabla \varphi(x)) \varphi(x)^{-q} e^{-\frac{|x|^2}{4}} dx
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{\Omega} g(\nabla \varphi(x)) \varphi(\nabla \varphi^*(\nabla \varphi(x)))^{-q} e^{-\frac{|\nabla \varphi^*(\nabla \varphi(x))|^2}{2}} dx
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{\Omega} g(z) \varphi(\nabla \varphi^*(z))^{-q} e^{-\frac{|\nabla \varphi^*(z)|^2}{2}} \det(\nabla^2 \varphi^*(z)) dz,
\]

where \( \Omega = \{ x \in \mathbb{R}^n : \varphi(x) < +\infty \} \cap \{ x \in \mathbb{R}^n : \varphi^*(x) < +\infty \} \). Hence, if \( \varphi \) is a \( C^2 \)-smooth and strictly convex function, then \( \tilde{C}_q(\varphi, \cdot) \) is absolutely continuous with respect to the \( n \)-dimensional Hausdorff measure and the Radon-Nikodym derivative is

\[
\frac{d\tilde{C}_q(\varphi, x)}{dx} = (2\pi)^{-\frac{n}{2}} \varphi(\nabla \varphi^*(x))^{-q} e^{-\frac{|\nabla \varphi^*(x)|^2}{2}} \det(\nabla^2 \varphi^*(x)) \text{ on } \Omega.
\]

The following lemma shows that the Borel measure \( \tilde{C}_q(\varphi, \cdot) \) includes the Borel measure \( \tilde{C}_q(K, \cdot) \) of [35].

**Lemma 4.1.** Let \( K \in \mathcal{K}_n^0 \) be strictly convex and \( q < n \). Then the Borel measure \( \tilde{C}_q(\|x\|_K, \cdot) \) includes the \( q \)-th dual curvature measure, \( \tilde{C}_q(K, \cdot) \), of \( K \).

**Proof.** Let \( K \in \mathcal{K}_n^0 \) be strictly convex and \( \varphi(x) = \|x\|_K \). Let \( \nabla_K \) denote the normalized cone measure of \( K \), which is given by

\[
\frac{d\nabla_K(z)}{d\mathcal{H}^{n-1}(z)} = \frac{\langle z, \nu_K(z) \rangle}{nV(K)} \text{ for } z \in \partial K.
\]

Here \( V(K) \) denotes the \( n \)-dimensional volume of \( K \). For \( x \in \mathbb{R}^n \), we write \( x = rz \), with \( z \in \partial K \), then \( dx = nV(K)r^{n-1} dr d\nabla_K(z) \). Since the map \( x \mapsto \nabla \|x\|_K \) is 0-homogeneous, hence

\[
\int_{\mathbb{R}^n} g(x) d\tilde{C}_q(\varphi, x)
\]

\[
= \int_{\mathbb{R}^n} g(\nabla \varphi(x)) \varphi(x)^{-q} d\gamma_n(x)
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\nabla \|x\|_K) \|x\|_K^{-q} e^{-\frac{|x|^2}{4}} dx
\]

\[
= n(2\pi)^{-\frac{n}{2}} V(K) \int_0^\infty \int_{\partial K} g(\nabla \|z\|_K) r^{n-q-1} e^{-\frac{r^2}{2}} d\nabla_K(z) dr
\]

\[
= (2\pi)^{-\frac{n}{2}} 2^{\frac{n-q-2}{2}} nV(K) \int_0^\infty e^{-r} r^{\frac{n-q-1}{2}} \int_{\partial K} g(\nabla \|z\|_K) |z|^{q-n} d\nabla_K(z)
\]

\[
= (2\pi)^{-\frac{n}{2}} 2^{\frac{n-q-2}{2}} \int_0^\infty e^{-r} r^{\frac{n-q-1}{2}} \int_{\partial K} g(\nabla \|z\|_K) |z|^{q-n} \nu_K(\nu_K(z)) d\mathcal{H}^{n-1}(z).
\]
Because of (see, e.g., [62]),
\[ \nabla \|z\|_K = \frac{\nu_K(z)}{\|\nu_K(z)\|_{K^*}}, \]
for \( z \in \partial K \), then
\[ \int_{\mathbb{R}^n} g(x) d\widetilde{C}_q(\varphi, x) = (2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-r} r^{\frac{n-q-2}{2}} dr \int_{\partial K} g \left( \frac{\nu_K(z)}{\|\nu_K(z)\|_{K^*}} \right) |z|^{q-n} h_K(\nu_K(z)) d\mathcal{H}^{n-1}(z). \]
This means that the Borel measure \( \widetilde{C}_q(\|x\|_K, \cdot) \) takes the following form: For any Borel subsets \( I \subseteq [0, \infty) \) and \( \Omega \subseteq \partial K \),
\[ \widetilde{C}_q(\varphi, I \times \Omega) = \nu_1(I) \nu_2(\Omega). \]
The measure \( \nu_1 \) is independent of the choice of \( K \) and it is a measure with density
\[ (2\pi)^{-\frac{n}{2}} \int_0^\infty e^{-r} r^{\frac{n-q-2}{2}} dr. \]
The measure \( \nu_2 \) is a measure on \( \partial K \) which implies the \( q \)-th dual curvature measure, \( \widetilde{C}_q(K, \cdot) \), of \( K \). In fact, if \( R(x) = \frac{x}{|x|} \) for \( x \in \mathbb{R}^n \setminus \{0\} \), then
\[ R_* (\nu_2) \]
is referred to as the dual curvature measure, \( \widetilde{C}_q(K, \cdot) \), of \( K \). This shows that the dual curvature measure of a convex body can be recovered from the functional dual curvature measure of a particular convex function. \( \square \)

The next lemma will be used later.

**Lemma 4.2.** [21, Lemma 3] Let \( \varphi : \mathbb{R}^n \to [0, +\infty] \) be an essentially-continuous, convex function. Fix a vector \( 0 \neq \theta \in \mathbb{R}^n \) and let \( H = \theta^\perp \subset \mathbb{R}^n \) be the hyperplane orthogonal to \( \theta \). Then, for \( \mathcal{H}^{n-1} \)-almost every \( y \in H \), the function
\[ \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \]
\[ t \mapsto \varphi(y + t\theta), \]
is continuous on \( \mathbb{R} \), and locally-Lipschitz in the interior of the interval in which it is finite.

The support \( \text{Supp}(\mu) \) of a measure \( \mu \) in \( \mathbb{R}^n \) is a closed set that contains all points \( x \in \mathbb{R}^n \) with \( \mu(U) > 0 \) for any open set \( U \) containing \( x \).

**Lemma 4.3.** Let \( \varphi : \mathbb{R}^n \to [0, +\infty] \) be an essentially-continuous, convex function and \( q \leq 0 \). If
\[ 0 < \int_{\mathbb{R}^n} \varphi^{1-q}(x) d\gamma_n(x) < +\infty \quad \text{and} \quad \varphi(0) = 0, \]
then the \( q \)-th dual curvature measure of \( \varphi \) is not supported in any hyperplane.
Proof. Assume that \( \text{Supp}(\tilde{C}_q(\varphi, \cdot)) \subseteq \theta^\perp \) for a unit vector \( \theta \in S^{n-1} \). Without loss of generality, assume \( \theta = e_n \), where \( e_n = (0, \cdots, 0, 1) \). For \( x \in \mathbb{R}^n \), we write \( x = (y, t) \) with \( y \in \mathbb{R}^{n-1} \) and \( t = x_n \in \mathbb{R} \). Suppose \( \varphi(x) > 0 \) for \( x \in \mathbb{R}^n \), then

\[
0 = \int_{\mathbb{R}^n} |z_n| d\tilde{C}_q(\varphi, z) \\
= \int_{\mathbb{R}^n} \left| \frac{\partial \varphi(x)}{\partial x_n} \right| \varphi(x)^{-q} d\gamma_n(x) \\
= \frac{1}{1-q} \int_{\mathbb{R}^n} \left| \frac{\partial \varphi_{1-q}(x)}{\partial x_n} \right| d\gamma_n(x).
\]

For almost every \((y, t) \in \mathbb{R}^n, \)

\[
\frac{\partial \varphi_{1-q}(y, t)}{\partial t} = 0.
\]

Since \( q \leq 0 \), the function \( \varphi_{1-q} \) is also convex. According to Lemma 4.2, for almost any \( y \in \mathbb{R}^{n-1} \), the function \( t \to \varphi_{1-q}(y, t) \) is continuous in \( \mathbb{R} \) and locally-Lipschitz in the interior of the interval \( \{ t : \varphi(y, t) < +\infty \} \). Therefore, for almost any \( y \in \mathbb{R}^{n-1} \), the convex function \( t \to \varphi_{1-q}(y, t) \) is constant in \( \mathbb{R} \). The condition \( \varphi(0) = 0 \) implies that \( \varphi \equiv 0 \) in \( \mathbb{R} \) and so that the integral \( \int \varphi_{1-q}(y, t)e^{-t^2/2} dt \) equals to 0. By Fubini theorem, the function \( \varphi_{1-q} \) can not have a finite, non-zero integral. This is a contradiction. \( \square \)

From Jensen’s inequality we know that

\[
-q \mapsto \left( \int_{\mathbb{R}^n} \varphi(x)^{-q} d\gamma_n(x) \right)^{-\frac{1}{q}}
\]

is strictly monotone increasing, unless \( \varphi \) is constant on \( \mathbb{R}^n \). Hence, the condition in Lemma 4.3 includes the assumption in the Definition 4.1 when \( q \leq 0 \).

A function \( Z \) defined on a lattice \((C, \lor, \land)\) and taking values in an abelian semi-group is called a valuation if

\[
Z(f \lor g) + Z(f \land g) = Z(f) + Z(g),
\]
(4.3)

for all \( f, g \in C \). A function \( Z \) defined on a set \( S \subset C \) is called a valuation if (4.3) holds whenever \( f, g, f \lor g, f \land g \in S \).

For \( S \) the space of convex bodies, \( K^n \), in \( \mathbb{R}^n \) with \( \lor \) denoting union and \( \land \) intersection, the notion of valuation is classical. The classical valuation played a critical role in Dehn’s solution of Hilbert’s Third Problem and have been a central focus in convex geometric analysis. In the 1930s, Blaschke classified the real-valued valuations on convex bodies that are \( \text{SL}(n) \) invariant. Blaschke’s work was greatly extended by Hadwiger in his famous classification of continuous, rigid motion invariant valuations and characterization of elementary mixed volumes.

In [35], authors proved that dual curvature measures are valuations, i.e.,

\[
\tilde{C}_q(K, \cdot) + \tilde{C}_q(L, \cdot) = \tilde{C}_q(K \cap L, \cdot) + \tilde{C}_q(K \cup L, \cdot),
\]
for each \( K, L \in \mathcal{K}^n_o \) such that \( K \cup L \in \mathcal{K}^n_o \).

Recently, valuations were defined on function spaces. For a space \( S \) of real-valued functions we denote by \( f \lor g \) the pointwise maximum of \( f \) and \( g \) while \( f \land g \) denotes their pointwise minimum. The characteristic functions \( \chi_K, \chi_L \) of convex bodies \( K \) and \( L \) satisfy

\[
\chi_{K \cup L} = \max\{\chi_K, \chi_L\} \quad \text{and} \quad \chi_{K \cap L} = \min\{\chi_K, \chi_L\},
\]

for all \( K, L \in \mathcal{K}^n_o \) such that \( K \cup L \in \mathcal{K}^n_o \). Valuations on function spaces can be seen as a generalization of valuations on \( \mathcal{K}^n_o \). For the classification of valuations on function spaces one can refer to [18–20, 43, 44].

We are ready to prove that the \( q \)-th dual curvature measure \( \widetilde{C}_q(\varphi, \cdot) \) is a valuation.

**Lemma 4.4.** Let \( \varphi, \psi \in \mathcal{L}_o \), and \( q \in \mathbb{R} \). If \( \min\{\varphi, \psi\} \) is convex, then

\[
\widetilde{C}_q(\varphi, \cdot) + \widetilde{C}_q(\psi, \cdot) = \widetilde{C}_q(\min\{\varphi, \psi\}, \cdot) + \widetilde{C}_q(\max\{\varphi, \psi\}, \cdot). \tag{4.4}
\]

**Proof.** If \( \min\{\varphi, \psi\} \) is convex, then for almost every \( x \in \mathbb{R}^n \),

\[
\nabla(\max\{\varphi, \psi\})(x) = \begin{cases} 
\nabla\varphi(x) & \text{when } \varphi(x) > \psi(x) \\
\nabla\psi(x) & \text{when } \varphi(x) < \psi(x) \\
\nabla\varphi(x) = \nabla\psi(x) & \text{when } \varphi(x) = \psi(x)
\end{cases}
\]

and

\[
\nabla(\min\{\varphi, \psi\})(x) = \begin{cases} 
\nabla\varphi(x) & \text{when } \varphi(x) < \psi(x) \\
\nabla\psi(x) & \text{when } \varphi(x) > \psi(x) \\
\nabla\varphi(x) = \nabla\psi(x) & \text{when } \varphi(x) = \psi(x)
\end{cases}
\]
Therefore,
\[
\int_{\mathbb{R}^n} g(y) d\tilde{C}_q(\max\{\varphi, \psi\}, y) + \int_{\mathbb{R}^n} g(y) d\tilde{C}_q(\min\{\varphi, \psi\}, y) \\
= \int_{\mathbb{R}^n} g(\nabla (\max\{\varphi, \psi\})(y)) (\max\{\varphi, \psi\})^{-q} d\gamma_n(y) \\
+ \int_{\mathbb{R}^n} g(\nabla (\min\{\varphi, \psi\})(y)) (\min\{\varphi, \psi\})^{-q} d\gamma_n(y) \\
= \int_{\mathbb{R}^n \cap \{\varphi \geq \psi\}} g(\nabla (\max\{\varphi, \psi\})(y)) (\max\{\varphi, \psi\})^{-q} d\gamma_n(y) \\
+ \int_{\mathbb{R}^n \cap \{\varphi < \psi\}} g(\nabla (\max\{\varphi, \psi\})(y)) (\max\{\varphi, \psi\})^{-q} d\gamma_n(y) \\
+ \int_{\mathbb{R}^n \cap \{\varphi \geq \psi\}} g(\nabla (\min\{\varphi, \psi\})(y)) (\min\{\varphi, \psi\})^{-q} d\gamma_n(y) \\
+ \int_{\mathbb{R}^n \cap \{\varphi < \psi\}} g(\nabla (\min\{\varphi, \psi\})(y)) (\min\{\varphi, \psi\})^{-q} d\gamma_n(y)
\]
for all continuous integrable function \(g\). Hence, we obtain the desired valuation property (4.4).

\[\blacksquare\]

5. Dual Minkowski problems for convex functions

5.1. A minimization problem.

We write \(L_1(\mu)\) to denote the class of \(\mu\)-integrable functions \(f : \mathbb{R}^n \to [0, \infty]\), i.e.,
\[
L_1(\mu) = \left\{ f : \int_{\mathbb{R}^n} f(x) d\mu(x) < +\infty \right\}.
\]
Let \(\mu\) be a finite Borel measure on \(\mathbb{R}^n\) and \(q \in \mathbb{R}\). We set
\[
\Psi_\mu(f) = \frac{1}{|\mu|} \int_{\mathbb{R}^n} f(x) d\mu(x) - e^{-W_{n+1-q}(f^*)},
\]
for \(f \in L_1(\mu)\) with \(f(0) = 0\). Here \(|\mu|\) denotes the total measure of \(\mu\) and \(W_{n+1-q}(f^*)\) is the normalized dual quermassintegral, namely,
\[
W_{n+1-q}(f^*) = \left[ \int_{\mathbb{R}^n} (f^*(x))^{1-q} d\gamma_n(x) \right]^\frac{1}{1-q}.
\]
We should point out that the restriction \( f(0) = 0 \) guarantees that \( f^*(x) \geq 0 \), since (2.3) and the fact \( f^{**} \leq f \).

We consider the minimization problem,

\[
\inf_f \left\{ \Psi_\mu(f) : 0 < \overline{W}_{n+1-q}(f^*) < \infty, f \in L_1(\mu) \text{ and } f(0) = 0 \right\}. \tag{5.2}
\]

If the convex function \( \varphi_0 \) solves the infimum in (5.2), then the pre-given measure \( \mu \) is exactly the \( q \)-th dual curvature of \( \varphi_0^* \). We need the following result which has been provided in [5] and a special case one can find in [17]. For a complete proof one can see in the recent paper [61].

**Lemma 5.1.** Let \( \varphi \) be a lower semi-continuous convex function in \( \mathbb{R}^n \) and let \( g \) be a bounded continuous function in \( \mathbb{R}^n \). If \( \varphi_t(x) = \varphi(x) + tg(x) \) for sufficiently small \( t \) (i.e., \( |t| < a \) for sufficiently small \( a \)) and \( x \in \mathbb{R}^n \), then

\[
\frac{d}{dt} \varphi_t^*(x) \bigg|_{t=0} = -g(\nabla \varphi^*(x))
\]

at any point \( x \in \mathbb{R}^n \) in which \( \varphi^* \) is differentiable.

The next shows that if \( \varphi_0 \) solves the optimal problem (5.2) then the pre-given measure \( \mu \) is the \( q \)-th dual curvature measure of \( \varphi_0^* \).

**Lemma 5.2.** Let \( \mu \) be a finite Borel measure on \( \mathbb{R}^n \) that is not supported in a lower dimensional subspace with \( \int_{\mathbb{R}^n} |x|d\mu(x) < \infty \) and \( q \leq 0 \). If there exists a lower semi-continuous and convex \( \mu \)-integrable function \( \varphi_0 : \mathbb{R}^n \to \mathbb{R} \) such that

\[
\Psi_\mu(\varphi_0) = \inf_f \left\{ \Psi_\mu(f) : f \in L_1(\mu), 0 < \overline{W}_{n+1-q}(f^*) < \infty, f(0) = 0 \right\},
\]

then

\[
\mu = \tau \overline{C}_q(\varphi_0^*, \cdot),
\]

with \( \tau = \frac{|\mu|}{\overline{W}_{n-q}(\varphi_0^*)} \).

**Proof.** We assume that there exists a \( \varphi_0 \) with \( 0 < \overline{W}_{n+1-q}(\varphi_0^*) < \infty \) such that it is a minimizer for \( \Psi_\mu(f) \), i.e.,

\[
\Psi_\mu(\varphi_0) = \inf_f \left\{ \Psi_\mu(f) : f \in L_1(\mu), 0 < \overline{W}_{n+1-q}(f^*) < \infty, f(0) = 0 \right\}. \tag{5.3}
\]

For any \( \mu \)-integrable continuous, bounded function \( g : \mathbb{R}^n \to \mathbb{R} \) with \( g(0) = 0 \), we set \( \varphi_t(x) = \varphi_0(x) + tg(x) \). Since \( g \) is bounded, if \( |g| < M \) then we can select a \( t_0 > 0 \) such that

\[
\varphi_0 - t_0 M < \varphi_t < \varphi_0 + t_0 M, \tag{5.4}
\]

and

\[
\frac{1}{|\mu|} \int_{\mathbb{R}^n} \varphi_t(x)d\mu(x) \leq \frac{1}{|\mu|} \int_{\mathbb{R}^n} \varphi_0(x)d\mu(x) + t_0 M,
\]

for all \( t \in [-t_0, t_0] \). This means that there exists \( t_0 > 0 \) such that \( \varphi_t \) is \( \mu \)-integrable on \( \mathbb{R}^n \) and \( \varphi_t(0) = 0 \) for all \( t \in [-t_0, t_0] \). Inequalities (5.4) imply that

\[
\varphi_0^* - t_0 M < \varphi_t^* < \varphi_0^* + t_0 M, \tag{5.5}
\]
for all \( t \in [-t_0, t_0] \). Hence, the Minkowski inequality deduces
\[
W_{n+1-q}(\varphi_0^*) \leq W_{n+1-q}(\varphi_0^*) + t_0 M < +\infty.
\]
Because of \( \varphi_0 \) is a minimizer of the optimal problem (5.3),
\[
\Psi_\mu(\varphi_0) \leq \Psi_\mu(\varphi_t) \quad \text{for all} \quad t \in [-t_0, t_0].
\]
Hence,
\[
\frac{d}{dt}\frac{d}{dt}\Psi_\mu(\varphi_t)\bigg|_{t=0} = 0.
\]

By Lemma 5.1 deduces
\[
\frac{d}{dt}\varphi_t^*(x)\bigg|_{t=0} = -g(\nabla \varphi_0^*(x))
\]
at any point \( x \in \mathbb{R}^n \) in which \( \varphi_0^* \) is differentiable. By the dominated convergence theorem, we have
\[
\frac{d}{dt}\Psi_\mu(\varphi_t)\bigg|_{t=0} = \frac{d}{dt}\left(\frac{1}{|\mu|} \int_{\mathbb{R}^n} \varphi_t(x)d\mu(x)\right)\bigg|_{t=0} - \frac{d}{dt}\left(e^{-W_{n+1-q}(\varphi_t^*)}\right)\bigg|_{t=0}
\]
\[
= \frac{1}{|\mu|} \int_{\mathbb{R}^n} g(x)d\mu(x) - W_{n+1-q}(\varphi_0^*) \int_{\mathbb{R}^n} g(\nabla \varphi_0^*(x)) (\varphi_0^*(x))^{-q} d\gamma_n(x),
\]
for any \( \mu \)-integrable continuous, bounded function \( g : \mathbb{R}^n \to \mathbb{R} \). This means
\[
\mu(\cdot) = |\mu|W_{n+1-q}(\varphi_0^*) e^{-W_{n+1-q}(\varphi_0^*)} \tilde{C}_q(\varphi_0^*, \cdot).
\]
By taking the integration from both sides of (5.6), one sees that
\[
\frac{1}{W_{n-q}(\varphi_0^*)} = W_{n+1-q}(\varphi_0^*) e^{-W_{n+1-q}(\varphi_0^*)}.
\]
This finishes the proof. \( \square \)

Next, we prove that the infimum in (5.2) is attained. The following lemma can be found in [21].

**Lemma 5.3.** Let \( \mu \) be a non-zero finite Borel measure on \( \mathbb{R}^n \) that is not supported in a lower-dimensional subspace, and let \( M \) be the interior of \( \text{conv} (\text{Supp} (\mu)) \). If \( x_0 \in M \), then there exists \( C_{\mu,x_0} > 0 \) with the following property: For any non-negative, \( \mu \)-integrable, convex function \( \varphi : \mathbb{R}^n \to [0, \infty] \),
\[
\varphi(x_0) \leq C_{\mu,x_0} \int_{\mathbb{R}^n} \varphi d\mu.
\]
Proposition 5.1. Let $\mu$ be a finite Borel measure on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} |x| d\mu(x) < \infty$ that is not supported in a lower dimensional subspace. If $q \leq 0$, then there exists a non-negative convex, lower semi-continuous and $\mu$-integrable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\varphi(0) = 0$ and $\overline{W}_{n+1-q}(\varphi^*)$ is finite such that

$$
\Psi_{\mu}(\varphi) = \inf_f \{ \Psi_{\mu}(f) : 0 < \overline{W}_{n+1-q}(f^*) < \infty, f \in L_1(\mu), f(0) = 0 \}. 
$$

(5.7)

Proof. Since $f^{**} \leq f$ and $f^{***} = f^*$ for any function $f$, it yields

$$
\Psi_{\mu}(f^{**}) \leq \Psi_{\mu}(f).
$$

We can just consider the infimum of $\Psi_{\mu}$ by restricting our attention to the non-negative, convex and lower semi-continuous functions.

Let $\tilde{f}(x) = |x|$ and set $c_{\mu} = \Psi_{\mu}(\tilde{f})$. Then $c_{\mu}$ is a finite real number. Let $\varphi_1, \varphi_2, \ldots$ be a minimizing sequence $\mu$-integrable convex functions with $\varphi_i(0) = 0$ ($i = 1, 2, \ldots$), i.e.,

$$
\lim_{i \to \infty} \Psi_{\mu}(\varphi_i) = \inf_f \{ \Psi_{\mu}(f) : 0 < \overline{W}_{n+1-q}(f^*) < \infty, f(0) = 0 \} \leq c_{\mu}.
$$

(5.8)

Furthermore, we may remove finitely many elements from the sequence $\{\varphi_i\}$ and assume that for all $i$,

$$
\Psi_{\mu}(\varphi_i) \leq c_{\mu}.
$$

(5.9)

Since $\overline{W}_{n+1-q}(\varphi_i^*) > 0$ for all $i$, therefore

$$
\frac{1}{|\mu|} \int_{\mathbb{R}^n} \varphi_i d\mu \leq c_{\mu} + 1,
$$

(5.10)

for all $i$. Let $M$ be the interior of $\text{conv}(\text{Supp}(\mu))$. By Lemma 5.3 and (5.10), we have

$$
\varphi_i(x_0) \leq (c_{\mu} + 1) C_{\mu, x_0},
$$

(5.11)

for all $i$ and any $x_0 \in M$. Theorem A ensures that there exists a subsequence $\{\varphi_{i_j}\}_{j=1,2,\ldots}$ that converges pointwise in $M$ to a convex function $\varphi : M \to \mathbb{R}$. Additionally, $\varphi$ is non-negative in $M$ with $\varphi(0) = 0$.

We extend the definition of $\varphi$ by setting $\varphi(x) = +\infty$ for $x \notin \overline{M}$. We still need to define $\varphi$ for $x \in \partial M$. Set

$$
\varphi(x) = \lim_{\lambda \to 1^-} \varphi(\lambda x), \quad x \in \partial M.
$$

(5.12)

Since the function $\lambda \to \varphi(\lambda x)$ is non-decreasing for $\lambda \in (0,1)$, hence the limit in (5.12) always exists in $[0, +\infty]$. We need to show that $\varphi$ is $\mu$-integrable. For any $x \in \overline{M}$, the point $\lambda x \in M$ for $0 < \lambda < 1$, and by the pointwise convergence in $M$,

$$
\varphi(\lambda x) = \lim_{j \to \infty} \varphi_{i_j}(\lambda x),
$$

(5.13)

for any $x \in \overline{M}$. Since the measure $\mu$ does not support in a lower-dimensional subspace, so that $0 \in \text{Supp}(\mu) \subseteq \overline{M}$, then from (5.13), (5.10), for any $0 < \lambda < 1$, by the
convexity and \( \varphi_i(0) = 0 \),
\[
\varphi_i(\lambda x) \leq \lambda \varphi_i(x) + (1 - \lambda) \varphi_i(0) \\
\leq \varphi_i(x),
\]
by Fatou’s Lemma yields
\[
\int_{\mathbb{R}^n} \varphi(\lambda x) d\mu(x) \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \varphi_i(\lambda x) d\mu(x) \\
\leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \varphi_i(x) d\mu(x). 
\tag{5.14}
\]
Recall that we have \( \varphi(\lambda x) \nearrow \varphi(x) \) as \( \lambda \to 1^- \). From the monotone convergence theorem and (5.14),
\[
\int_{\mathbb{R}^n} \varphi(x) d\mu(x) = \lim_{\lambda \to 1^-} \int_{\mathbb{R}^n} \varphi(\lambda x) d\mu(x) \\
\leq \lim_{j \to \infty} \int_{\mathbb{R}^n} \varphi_i(x) d\mu(x) \\
< +\infty.
\]
Hence, \( \varphi \) is \( \mu \)-integrable.

By the continuity of \( \varphi \) in \( M \), we can choose a dense sequence \( x_1, x_2, \cdots \) in \( M \) such that
\[
\varphi^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - \varphi(x) \} \\
= \sup_{x \in M} \{ \langle x, y \rangle - \varphi(x) \} \\
= \sup_{i \geq 1} \{ \langle x_i, y \rangle - \varphi(x_i) \},
\]
for \( y \in \mathbb{R}^n \). For \( j \geq 1 \), we set \( \tilde{\varphi}^*_j(y) = \max_{1 \leq i \leq j} \{ \langle x_i, y \rangle - \varphi(x_i) \} \). We notice that \( \tilde{\varphi}_j^* \nearrow \varphi^* \), then the increment of function \( t^{1-q} \) when \( q \leq 0 \) and the monotone convergence theorem deduce
\[
\overline{W}_{n+1-q}(\varphi^*) = \lim_{j \to \infty} \overline{W}_{n+1-q}(\tilde{\varphi}_j^*).
\]
For any \( \varepsilon > 0 \), there exists \( j_0 \) such that
\[
|\overline{W}_{n+1-q}(\varphi^*) - \overline{W}_{n+1-q}(\tilde{\varphi}_{j_0}^*)| \leq \frac{\varepsilon}{2}.
\]
By the fact \( \varphi_i \to \varphi \) pointwise on the set \( \{ x_1, \cdots, x_{j_0} \} \), then for sufficiently large \( j \), we have
\[
|\varphi^*_i(x) - \tilde{\varphi}^*_{j_0}(x)| \leq \frac{\varepsilon}{2}, \quad (5.15)
\]
for all $x \in \mathbb{R}^n$. Since $q < 0$, then Minkowski’s inequality deduces
\[
\overline{W}_{n+1-q}(\varphi^*) \leq \overline{W}_{n+1-q}(\tilde{\varphi}_{j_0}^*) + \frac{\varepsilon}{2}
\]
\[
\leq \lim inf_{j \to \infty} \left( \int_{\mathbb{R}^n} (\varphi_{ij}^*(x) + \frac{\varepsilon}{2})^{1-q} d\gamma_n(x) \right)^{\frac{1}{1-q}} + \frac{\varepsilon}{2}
\]
\[
\leq \lim inf_{j \to \infty} \overline{W}_{n+1-q}(\varphi_{ij}^*) + \varepsilon.
\]
This means
\[
\overline{W}_{n+1-q}(\varphi^*) \leq \lim inf_{j \to \infty} \overline{W}_{n+1-q}(\varphi_{ij}^*) < +\infty,
\]
and
\[
-e^{-\overline{W}_{n+1-q}(\varphi^*)} \leq \lim inf_{j \to \infty} -e^{-\overline{W}_{n+1-q}(\varphi_{ij}^*)}. \tag{5.16}
\]
Together (5.16) with (5.14), we have
\[
\Psi_\mu(\varphi) \leq \lim inf_{i \to \infty} \Psi_\mu(\varphi_i) \tag{5.17}
\]
Therefore, there exists a non-negative, $\mu$-integrable, convex function $\varphi$ with $\varphi(0) = 0$ such that
\[
\Psi_\mu(\varphi) = \inf_f \{ \Psi_\mu(f) : f \in L_1(\mu), f(0) = 0 \}.
\]
This completes our proof. \qed

We are now in the position to prove the existence of solution to the functional dual Minkowski problem.

**Theorem 5.1.** Let $\mu$ be a non-zero finite Borel measure on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} |x| d\mu(x) < \infty$ which is not supported in a lower-dimensional subspace and $q \leq 0$. There exists a non-negative convex function $\varphi$ with $\varphi(0) = 0$ and $\int_{\mathbb{R}^n} \varphi(x)^{1-q} d\gamma_n(x)$ is finite such that $\mu(\cdot) = \tau \tilde{C}_q(\varphi, \cdot)$.

**Proof.** The proof follows from Lemma 5.2 and Proposition 5.1. \qed

### 5.2. Uniqueness of the dual functional Minkowski problem.

In this subsection we will show that if the dual curvature measures of two convex functions are equal under some assumptions, then the convex functions are also equal almost everywhere in $\mathbb{R}^n$. The tool we will be used is the Minkowski inequality (3.23), similar to the uniqueness part of the $L_p$ Minkowski problem for convex bodies.

We are ready to prove the uniqueness of the solution to the functional dual Minkowski problems.

**Theorem 5.2.** Let $q \leq 0$ and $\varphi_1, \varphi_2 \in \mathcal{L}_0'$ with $\varphi_1(0) = \varphi_2(0) = 0$. Assume that $\varphi_1$ is an admissible perturbation for $\varphi_2$ and $\varphi_2$ is also an admissible perturbation for $\varphi_1$. If $\tilde{C}_{q-1}(\varphi_1, \cdot) = \tilde{C}_{q-1}(\varphi_2, \cdot)$, then $\varphi_1(x) = \varphi_2(x - x_0)$ for some $x_0 \in \mathbb{R}^n$ when $q = 0$ and $\varphi_1(x) = \varphi_2(x)$ when $q < 0$. 
Proof. Firstly notice that the equality \( \tilde{C}_{q-1}(\varphi_1, \cdot) = \tilde{C}_{q-1}(\varphi_2, \cdot) \) implies \( \tilde{W}_{n+1-q}(\varphi_1) = \tilde{W}_{n+1-q}(\varphi_2) \). By the fact \( \tilde{W}_{n+1-q}(\varphi_1) = \tilde{W}_{n+1-q}(\varphi_2) \) and (3.23), we have
\[
\tilde{W}_{n+1-q}(\varphi_1, \varphi_2) \geq \tilde{W}_{n+1-q}(\varphi_1, \varphi_1), \tag{5.18}
\]
and
\[
\tilde{W}_{n+1-q}(\varphi_2, \varphi_1) \geq \tilde{W}_{n+1-q}(\varphi_2, \varphi_2). \tag{5.19}
\]
On the other hand, since \( \tilde{C}_{q-1}(\varphi_1, \cdot) = \tilde{C}_{q-1}(\varphi_2, \cdot) \), then Theorem 3.1 yields
\[
\tilde{W}_{n+1-q}(\varphi_1, \psi) = \tilde{W}_{n+1-q}(\varphi_2, \psi),
\]
for any convex function \( \psi \) with \( \psi(0) = 0 \). In particular, taking \( \psi = \varphi_1 \) or \( \psi = \varphi_2 \), one sees that
\[
\tilde{W}_{n+1-q}(\varphi_1, \varphi_1) = \tilde{W}_{n+1-q}(\varphi_2, \varphi_1), \tag{5.20}
\]
and
\[
\tilde{W}_{n+1-q}(\varphi_2, \varphi_2) = \tilde{W}_{n+1-q}(\varphi_1, \varphi_2). \tag{5.21}
\]
Equalities (5.20) and (5.21) guarantee the inequalities (5.18) and (5.19) both hold with equality. Finally, the equality condition of (3.23) has make sure that \( \varphi_1(x) = \varphi_2(x - x_0) \) for some \( x_0 \in \mathbb{R}^n \) when \( q = 0 \) and \( \varphi_1(x) = \varphi_2(x) \) when \( q < 0 \). \( \square \)

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