A New Classification of Hemirings through Double-Framed Soft h-Ideals
(Pengelasan Baru Hemirings melalui h-Ideals Lembut-Dual Kerangka)

FAIZ MUHAMMAD KHAN, NIE YUFENG, HIDAYAT ULLAH KHAN & ASGHAR KHAN

ABSTRACT
Due to lack of parameterization, various ordinary uncertainty theories like theory of fuzzy sets, and theory of probability cannot solve complicated problems of economics and engineering involving uncertainties. The aim of the present paper was to provide an appropriate mathematical tool for solving such type of complicated problems. For the said purpose, the notion of double-framed soft sets in hemirings is introduced. As h-ideals of hemirings play a central role in the structural theory, therefore, we developed a new type of subsystem of hemirings. Double-framed soft left (right) h-ideal, double-framed soft h-bi-ideals and double-framed soft h-quasi-ideals of hemirings are determined. These concepts are elaborated through suitable examples. Furthermore, we are bridging ordinary h-ideals and double-framed soft h-ideals of hemirings through double-framed soft including sets and characteristic double-framed soft functions. It is also shown that every double-framed soft h-quasi-ideal is double-framed soft h-bi-ideal but the converse inclusion does not hold. A well-known class of hemirings i.e. h-hemiregular hemirings is characterized by the properties of these newly developed double-framed soft h-ideals of hemirings.

Keywords: DFS h-bi-ideal; DFS h-hemiregularhemirin; DFS h-quasi-idealg; DFS sets; h-ideal

ABSTRAK
Disebabkan oleh kekurangan pemparameteran, pelbagai teori ketidakpastian biasa seperti teori set kabur dan teori kebarangkalian tidak boleh menyelesaikan masalah ekonomi dan kejuruteraan yang rumit yang melibatkan ketidakpastian. Tujuan penulisan kertas ini adalah untuk menyediakan satu alat matematik yang sesuai untuk menyelesaikan masalah rumit yang sedemikian. Untuk tujuan tersebut, satu tanggapan set lembut dual kerangka dalam hemirings diperkenalkan. Oleh kerana h-ideals hemiring memainkan peranan utama dalam teori struktur, maka kami telah membangunkan satu jenis subsistem hemiring baru. h-ideals lembut kiri (kanan) dual kerangka, h-dwi-ideal lembut dual kerangka dan h-separa-ideal lembut dual kerangka hemirings ditentukan. Konsep ini dihuraikan melalui contoh yang sesuai. Selain itu, kami menghubungkan h-ideals biasa dan h-ideals lembut dual kerangka hemirings melalui set lembut dual kerangka dan pencirian fungsi lembut dual kerangka. Kajian ini juga menunjukkan bahawa setiap h-quasi-ideal lembut dual bangkai adalah h-dwi-ideal lembut dual kerangka tetapi rangkuman akas tidak dapat bertahan. Satu kelas hemirings terkenal iaitu h-hemisekata hemirings dicirikan oleh sifat h-ideals dua bangkai lembut daripada yang baru dibangunkan ini.

Kata kunci: Set DFS; DFS h-dwi-ideal; DFS h-hemisekata hemiring; h-ideal; DFS h-separa-ideal

INTRODUCTION
In modern era, economic and technological advancement plays a remarkable role in the development of any particular country. Due to the high-quality research in advanced fields like control engineering, data analysis, computer science, error correcting codes, economics, decision making, forecasting and robotics, most of the countries are left behind. These advanced countries are spending a huge part of budget on these domains. On the other hand, the aforementioned fields are facing some complicated problems involving uncertainties. These complicated problems cannot be handled through classical methods. There are certain types of theories such as theory of probability, theory of fuzzy sets and theory of rough sets which can be used in aforementioned problems. However, all of these theories have their significance as well as inherent limitations. One major problem faced by these theories is their incompatibility with the parameterization tools. To overcome such type of difficulties, in 1999, Molodtsov initiated the ice breaking concept of soft set theory. The notion of soft sets is a new mathematical approach for dealing with uncertainties. This new approach is free from the difficulties pointed out in the other theories of uncertainties which usually use membership function. Soft sets gain reputation from the last decade due to its parameterization nature and which is free of membership function. Due to its dynamical nature, soft sets are nowadays extensively used in various applied fields. More precisely, soft sets are used in decision making problems (Çagman & Enginoglu 2010a, 2010b; Feng 2011; Feng et al. 2010; Maji et al. 2002; Roya & Maji 2007), soft derivatives, soft integrals and soft numbers along with their applications are thoroughly discussed (Molodtsov et al. 2006), in international trade, soft sets are used for
forecasting the export and import volumes (Sezer 2014). Simultaneously, this theory is very much useful due to its applications in information sciences with intelligent systems, approximate reasoning, expert and decision support systems and decision making (Acar et al. 2010; Atagun & Sezgin 2011; Cagman & Enginoğlu 2011; Feng et al. 2008, 2011; Jun et al. 2010, 2009a, 2009b, 2008a, 2008b; Majumdar & Samanta 2008; Sezgin & Atagun 2011; Xiao et al. 2010; Yin Li; 2008; Zhan et al. 2010; Zou & Xiao 2008).

It is also important to note that, soft set are used in algebraic framework which successfully leads to the applications of algebraic structures in aforementioned advanced applied fields. Keeping this motivation in view, Maji et al. (2003) presented several operations of algebraic structures in terms of soft sets which is further extended (Ali et al. 2011, 2009).

Presently, among other algebraic structures, semirings (Vandiver 1934), are also used in diverse fields like computer programming, coding theory, fuzzy automata, optimization, formal languages, graph theory and much more (Aho & Ullman 1976; Benson 1989; Conway 1971; Golon 1998; Hebisch & Weinert 1998; Henriksen 1958; Iizuka 1959; Kuich & Salomma 1986; Torre 1965). Among these, several fields such as theory of automata, formal languages and computer sciences used special type of semirings known as hemirings (Benson 1989; Golon 1999; Hebisch & Weinert 1998). Hemirings are those semirings which are commutative with addition and having zero element. Further, ideals of hemirings play a key role in structure theory for many purposes. In 1965, Torre determined h-ideals and h-ideals in hemirings with several classification of hemirings are discussed in terms of these ideals. The h-hemiregularity are investigated (Yin & Li 2008). They also determined h-intra hemiregular hemirings and presented various characterization theorems of hemirings in terms of these notions. In 2013, Droste and Kuich discuss hemirings in automata domain. Moreover, Ma and Zhan (2014) characterized hemiregular hemirings by the properties of new type of soft union sets. For other applications of soft union sets in hemirings, the readers refer to (Ma et al. 2016; Zhan & Maji 2014). The concept of cubic h-ideals along with several characterization theorems in hemirings is presented (Khan et al. 2015).

Recently, the notion of union and intersectional soft sets is further extended (Jun et al. 2012) to double-framed soft sets and defined double-framed soft subalgebra of a BCK/BCI-algebra. Beside this, Jun et al. (2013) also determined double-framed soft ideals of BCK/BCI-algebra. In 2017a, Khan et al. applied the notion of double-framed soft sets to AG-groupoids and investigated various results. Moreover, double-framed soft sets are further elaborated in LA-semigroups (Khan et al. 2017b).

The aim of the present paper was to apply the idea of double-framed soft sets to hemirings and to investigate double-framed soft h-ideals of a hemiring We define double-framed soft h-ideals, double-framed soft h-bi-ideals and double-framed soft h-quasi-ideals of hemiring R. Further, these notions are elaborated through suitable examples. DFS Soft h-sum and h-product are developed and several results are determined by these notions. On the other hand, we are also bridging ordinary h-ideals and double-framed soft h-ideals of hemirings through double-framed soft including sets and characteristic double-framed soft functions which is the key milestone of the present paper. It is also shown that every double-framed soft h-quasi-ideal is double-framed soft h-bi-ideal but for the converse inclusion, the counter example is provided that it does not hold in general. Lastly, h-hemiregular hemirings are characterized by the properties of these newly developed double-framed soft h-ideals of R.

PRELIMINARIES

This section presents the fundamental concepts of hemirings which will be used throughout this paper.

An algebraic system \((R, +, \cdot)\) consists of a non-empty set \(R\) with two binary operations addition and multiplication is known as a semiring, if \((R, +)\) and \((R, \cdot)\) are semigroups with the following distributive laws are satisfied \(a \cdot (b + c) = a \cdot b + a \cdot c\) and \((a + b) \cdot c = a \cdot c + b \cdot c\) for all \(a, b, c \in R\).

An element \(0 \in R\) is called zero of a semiring \((R, +, \cdot)\), if \(0 \cdot x = x \cdot 0 = 0\) and \(0 + x = x + 0 = x\) for all \(A\) unit of a semiring is an element \(1 \in R\) such that \(1 \cdot x = x = 1 \cdot x\) for all \(x \in R\). A semiring \(R\) with zero element and in which \((R, +)\) is a commutative semigroup is known as hemiring. Throughout the paper, \(ab\) will be used instead of \(a \cdot b\) such that \(a, b \in R\) for the sake of simplicity.

Since the objectives of the present research was to discussed several classifications of hemirings by the properties of various types of ideals, therefore, the basic types of ideals in hemirings are necessary to be coated over here. A subhemiring of \(R\) is a subset \(A\) of \(R\) which is both closed under addition and multiplication. A subset \(A\) of \(R\) is called a left (right) ideal of \(R\) if \(A\) is closed under addition and \(RA \subseteq A\) (resp. \(AR \subseteq A\)). A subset \(A\) of \(R\) is called an ideal of \(R\) if it is both left and right ideal of \(R\) A subset \(B\) of \(R\) is called a bi-ideal of \(R\) if \(B\) is closed under addition and multiplication such that \(BRB \subseteq B\). A subset \(Q\) of \(R\) is called a quasi-ideal of \(R\) if \(Q\) is closed under addition and \(RQ \cap QR \subseteq Q\). A subhemiring (left ideal, right ideal, bi-ideal) \(A\) of \(R\) is called an h-subhemiring (left h-ideal, right h-ideal, h-ideal, h-bi-ideal), respectively, if for any \(x, z \in R, a, b \in A, x + a + z = b + z \rightarrow x \in A\).

The h-closure \(\overline{A}\) of a subset is defined as

\[ \overline{A} = \{ x \in R \mid x + a + z = b + z \text{ for some } a, b \in A, z \in R \}. \]

A quasi-ideal \(Q\) in a hemiring \(R\) is called a h-quasi-ideal of \(R\) if \(RQ \cap QR \subseteq Q\) and for any \(x, z \in R, a, b \in Q\), \(x + a + z = b + z \rightarrow x \in Q\).

Note that, for subsets \(A, B\) and \(C\) of a hemiring \(R\), \(A \subseteq \overline{A}, AB = \overline{AB}, AB \subseteq A \cup B\) and \(\overline{A} - A\). A subset \(I\) in a
Definition 4. If \( f_A, f_B \in S(U) \), then the intersection of \( f_A \) and \( f_B \), denoted by \( f_A \cap f_B \), is defined by \( f_A \cap f_B = f_{A \cap B} \), where \( f_{A \cap B}(x) = f_A(x) \cap f_B(x) \) for all \( x \in E \).

Throughout this paper, \( R \) will denote a hemiring unless otherwise stated.

Definition 5 A double-framed soft set of \( A \) over \( U \) is a pair \( \langle f_A^e, f_A^\gamma \rangle : A \rangle \), where \( f_A^e \) and \( f_A^\gamma \) both are mappings from \( A \) to \( P(U) \). It is denoted by \( DFS(U) \).

The set of all \( DFS \)-set of \( A \) over \( U \) is denoted by \( DFS(U) \).

\( \gamma \)-inclusive set: If \( \langle f_A^e, f_A^\gamma \rangle : A \rangle \) be a DFS-set of \( A \) and \( \gamma \) be a subset of \( U \), then the \( \gamma \)-inclusive set is denoted by \( i_{\gamma}(f_A^\gamma) \) and defined as

\[
i_{\gamma}(f_A^\gamma) = \{ x \in A | f_A^\gamma(x) \supseteq \gamma \}
\]

\( \delta \)-exclusive set: If \( \langle f_A^e, f_A^\gamma \rangle : A \rangle \) be a DFS-set of \( A \) and \( \delta \) be a subset of \( U \), then the \( \delta \)-exclusive set is denoted by \( e_{\delta}(f_A^\gamma) \) and defined as

\[
e_{\delta}(f_A^\gamma) = \{ x \in A | f_A^\gamma(x) \subseteq \delta \}.
\]

A double-framed soft including set is of the form

\[
\text{DF}(f_A^e, f_A^\gamma : A) = \{ x \in A | f_A^\gamma(x) \supseteq \gamma, f_A^\gamma(x) \subseteq \delta \}
\]

clearly, \( \text{DF}(f_A^e, f_A^\gamma : A) = i_{\gamma}(f_A^\gamma) \cap e_{\delta}(f_A^\gamma) \).

In the following, the double-framed soft sum briefly \( h \)-sum and int-uniform product \( (h \)-product) for two double-framed soft sets of hemirings are introduced.

Definition 6 Let \( f_A = \langle f_A^e, f_A^\gamma \rangle : A \rangle \) and \( g_A = \langle g_A^e, g_A^\gamma \rangle : A \rangle \) be two double-framed soft sets of a hemiring \( R \) over \( U \). Then the \( h \)-sum is denoted by \( f_A \oplus g_A = \langle f_A^e \oplus g_A^e, f_A^\gamma \cup g_A^\gamma \rangle : A \rangle \) is defined to be a double-framed soft set of \( R \) over \( U \), in which \( f_A^\gamma \oplus g_A^\gamma \) and \( f_A^\gamma \cup g_A^\gamma \) are soft mappings from \( R \) to \( P(U) \) given as:

\[
f_A^\gamma \oplus g_A^\gamma : x \mapsto \begin{cases} \bigcup_{x \in \delta} \bigcup_{a \in A, b \in \delta} f_A^\gamma(a) \cap (f_A^\gamma(a) \cap g_A^\gamma(b)) \cap g_A^\gamma(b) & \text{if } x \text{ can be expressed as } x = a + b + z \text{, with } a, b, z \in A, B, Z, \\ \emptyset & \text{if } x \text{ can be expressed as } x = a + b + z \text{, with } a, b, z \notin A, B, Z. \end{cases}
\]

\[
f_A^\gamma \cup g_A^\gamma : x \mapsto \begin{cases} \bigcup_{x \in \delta} \bigcup_{a \in A, b \in \delta} f_A^\gamma(a) \cup (f_A^\gamma(a) \cup g_A^\gamma(b)) \cup g_A^\gamma(b) & \text{if } x \text{ can be expressed as } x = a + b + z \text{, with } a, b, z \in A, B, Z, \\ \emptyset & \text{if } x \text{ can be expressed as } x = a + b + z \text{, with } a, b, z \notin A, B, Z. \end{cases}
\]
Definition 7 Let \( f_A = \{ (f_A^a, f_A^b); A \} \) and \( g_A = \{ (g_A^a, g_A^b); A \} \) be two double-framed soft sets of a hemiring \( R \) over \( U \). Then the \( h \)-product is denoted by \( f_A \circ\circ g_A = \{ (f_A^a \circ g_A^a, f_A^b \circ g_A^b); A \} \) is defined to be a double-framed soft set of \( R \) over \( U \), in which \( f_A \circ g_A^a \) and \( f_A \circ g_A^b \) are soft mappings from \( R \) to \( \mathcal{P}(U) \) given as

\[
\begin{align*}
\forall x, & \quad \circ^A g_A^a : x \mapsto \bigcup_{x \in a, b, \circ = a, b, + z} \{ f_A^a(a) \cap f_A^b(b) \cup g_A^b(b) \} \\
& \quad \circ^A g_A^b : x \mapsto \bigcap_{x \in a, b, \circ = a, b, + z} \{ f_A^a(a) \cup f_A^b(b) \cup g_A^b(b) \}
\end{align*}
\]

Definition 8 Let \( f_A = \{ (f_A^a, f_A^b); A \} \) and \( g_A = \{ (g_A^a, g_A^b); B \} \) be two double-framed soft sets over \( U \). Then \( f_A \subset g_A \) is called a double-framed soft subset of \( g_A \) denoted by \( f_A = g_A \) if \( A \) is the subset of \( B \), \( f_A(x) \subseteq g_A(x) \) and \( f_A(x) \supseteq g_A(x) \) for all \( x \in A \). Also two double-framed soft sets \( f_A = \{ (f_A^a, f_A^b); A \} \) and \( g_A = \{ (g_A^a, g_A^b); B \} \) are equal denoted by \( f_A = g_A \) if \( f_A(x) = g_A(x) \) for all \( x \in A \) both hold.

Definition 9 Let \( f_A = \{ (f_A^a, f_A^b); A \} \) and \( g_A = \{ (g_A^a, g_A^b); A \} \) be two double-framed soft sets of a hemiring \( R \) over \( U \). Then the DFS int-uni set of \( f_A \) and \( g_A \) is to be defined as a DFS set \( \{ (f_A \circ g_A^a, f_A \circ g_A^b); A \} \) where \( f_A \circ g_A^a \) and \( f_A \circ g_A^b \) are mappings from \( A \) to \( \mathcal{P}(U) \) such that \( (f_A \circ g_A^a)(x) = f_A^a(x) \cap g_A^a(x) \) and \( (f_A \circ g_A^b)(x) = f_A^b(x) \cup g_A^b(x) \). It is denoted by \( f_A \circ g_A^a \cap f_A \circ g_A^b \).

Lemma 10 Suppose \( f_A = \{ (f_A^a, f_A^b); A \} \) and \( g_A = \{ (g_A^a, g_A^b); B \} \) and \( h = \{ (h^a, h^b); C \} \) be double-framed soft sets in a hemiring \( R \), then the following hold.

1. \( f_A \circ (g_A \cap h_A) = (f_A \circ g_A) \cap (f_A \circ h_A) \)
2. \( f_A \circ (g_A \cap h_A) = (f_A \circ g_A) \cap (f_A \circ h_A) \)

Proof Let \( x \) be an arbitrary element of a hemiring \( R \) which cannot be expressed as \( x + a, b, + z = a, b, + z \). Then, \( f_A \circ (g_A \cap h_A)(x) = 0 = (f_A \circ g_A) \cap (f_A \circ h_A)(x) \) and \( f_A \circ (g_A \cap h_A)(x) = 0 = (f_A \circ g_A) \cap (f_A \circ h_A)(x) \). Therefore, \( f_A \circ (g_A \cap h_A)(x) = (f_A \circ g_A)(x) \cap (f_A \circ h_A)(x) \). Let \( x \) can be expressed as \( x + a, b, + z = a, b, + z \), then

\[
\begin{align*}
f_A \circ (g_A \cap h_A)(x) &= (f_A \circ g_A)(x) \cap (f_A \circ h_A)(x) \\
&= \bigcup_{x \in a, b, \circ = a, b, + z} \{ f_A^a(a) \cap f_A^b(b) \cap (g_A^a \cap h_A)(b) \} \\
&= \bigcup_{x \in a, b, \circ = a, b, + z} \{ f_A^a(a) \cap f_A^b(b) \cap (g_A^a \cap h_A)(b) \} \\
&= \bigcup_{x \in a, b, \circ = a, b, + z} \{ f_A^a(a) \cap f_A^b(b) \cap h_A^a(b) \}
\end{align*}
\]
Also, \( f_\alpha \bowtie (g_\beta \bowtie h_\gamma)(x) = \bigcup_{v_\lambda + a_\mu + b_\nu + z = a_\sigma + b_\tau + z} \{(f_\alpha(a_\lambda) \cup f_\alpha(a_\mu) \cup g_\beta(b_\nu) \cup g_\beta(b_\tau) \cup h_\gamma(b_\sigma) \cup h_\gamma(b_\tau)) \} \)
\[= \bigcup_{x \in A, y \in B, z \in R} \{f_\alpha(a_\lambda) \cup f_\alpha(a_\mu) \cup g_\beta(b_\nu) \cup g_\beta(b_\tau) \cup h_\gamma(b_\sigma) \cup h_\gamma(b_\tau) \} \]
\[= (f_\alpha \bowtie g_\beta)(x) \cap (f_\alpha \bowtie h_\gamma)(x) \]
\[= (f_\alpha \bowtie g_\beta)(x) \cap (f_\alpha \bowtie h_\gamma)(x) \]
\[= (f_\alpha \bowtie g_\beta)(x) \cap (f_\alpha \bowtie h_\gamma)(x) \]
\[= (f_\alpha \bowtie g_\beta)(x) \cap (f_\alpha \bowtie h_\gamma)(x) \]

Thus, \( f_\alpha \bowtie (g_\beta \bowtie h_\gamma)(x) = (f_\alpha \bowtie g_\beta)(x) \cap (f_\alpha \bowtie h_\gamma)(x) \).

**Definition 11** Suppose be a non-empty subset of a hemiring \( R \), then the characteristic double-framed soft mapping of \( A \) is a double-framed soft set denoted by \( C_A = \{(C_x^1, C_x^2) : A\} \) where \( C_x^1, C_x^2 \) are soft mappings from \( R \) to \( P(U) \) and defined as
\[ C_x^1 : x \mapsto \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases} \]
and
\[ C_x^2 : x \mapsto \begin{cases} \emptyset & \text{if } x \in A, \\ U & \text{if } x \notin A. \end{cases} \]

It is important to note that the identity double-framed soft mapping is denoted by \( C_0 = \left\{ (C_x^1, C_x^2) : x \mapsto U \cup A \right\} \) where \( C_x^1 : x \mapsto U \) and \( C_x^2 : x \mapsto \emptyset \) for all \( x \in R \).

**Theorem 12** Suppose \( A \) and \( B \) be two non-empty subsets of a hemiring \( R \), then the following axioms for characteristic double-framed soft mapping are holds:

1. If \( A \subseteq B \) and \( A \subseteq C_0(x) \subseteq C^1(x) \) and \( C^2(x) \supseteq C^0(x) \) for all \( x \in A \).
\[= \bigcap_{x_0 \in A, y_0 \in B} \{C_x^a(a_0) \cup C_y^b(a_0) \cup C_y^b(b_0) \cup C_y^b(b_0)\}\]
\[= \bigcap_{x_0 \in A, y_0 \in B} \{\emptyset \emptyset \emptyset \emptyset \emptyset \}\]
\[= \emptyset\]
\[= C_{\delta A}(x)\]

Hence, \(C_{\delta A} \circ C_B = C_{\delta A}\)

**DOUBLE-FRAMED SOFT LEFT (RIGHT) h-IDEALS**

Double-framed soft structures, are newly developed structures. Comparatively to other structures these can comprehensively discuss and characterized hemirings. The contributions of the present research will play a key role in the structure theory. In this section, hemirings are classified by the properties of double-framed soft left (resp. right) \(h\)-ideals. Several important results are determined by the above said notions. Note that onward, double-framed soft left (resp. right) \(h\)-ideal will simply be denoted by DFS left (resp. right) \(h\)-ideal.

**Definition 13** A DFS-set \((\{I_x^r, I_y^l\}; A)\) of a hemiring \(R\) is said to be a double-framed soft left (resp. right) \(h\)-ideal of if for \(a, b \in R\), the following conditions hold.

1. \(I_x^r(a + b) \supseteq I_x^r(a) \cap I_x^r(b)\)
2. \(I_x^r(a + b) \subseteq I_x^r(a) \cup I_x^r(b)\)
3. \(I_x^r(ab) \supseteq I_x^r(b)(\text{resp.} I_x^r(ab) \supseteq I_x^r(a))\)

Note that a Double-framed soft left \(h\)-ideal \((\{I_x^r, I_y^l\}; A)\) of a hemiring \(R\) with zero element satisfies the inequalities \(I_x^r(0) \supseteq I_x^r(a)\), \(I_x^r(0) \supseteq I_x^r(a)\) for all \(a \in R\).

**Example 14** Suppose \(R = \{0, a, b, c\}\) be a set with addition and multiplication defined in the following tables:

| +   | 0   | a   | b   | c   |
|-----|-----|-----|-----|-----|
| 0   | 0   | 0   | 0   | 0   |
| a   | a   | a   | a   | a   |
| b   | b   | b   | b   | b   |
| c   | c   | c   | c   | c   |

Define a double-framed soft \((\{I_x^r, I_y^l\}; A)\) in \(R\) over \(U = Z\) as follows:

| \(R\) | 0   | a   | b   | c   |
|-------|-----|-----|-----|-----|
| \(I_x^r(a)\) | \(-2\) | \(-2\) | \(-2\) | \(-2\) |
| \(I_x^l(b)\) | \(-2\) | \(-2\) | \(-2\) | \(-2\) |

Using Definition 13, \((\{I_x^r, I_y^l\}; A)\) is a double-framed soft \(h\)-ideal of \(R\) over \(U\).

In the following Lemma, double-framed soft including sets are used to connect ordinary left \(h\)-ideals with DFS left \(h\)-ideals of hemiring \(R\).

**Lemma 15** If \(f_1 = (\{I_x^r, I_y^l\}; A)\) is a double-framed soft set of a hemiring \(R\), then a non-empty double-framed soft including set \(DF_\delta(\{I_x^r, I_y^l\}; A)\) is left \(h\)-ideal of \(R\) if and only if \(f_1 = (\{I_x^r, I_y^l\}; A)\) is DFS left \(h\)-ideal of \(R\).

**Proof** Suppose \(\emptyset \neq DF_\delta(\{I_x^r, I_y^l\}; A) \subseteq R\) (be a left \(h\)-ideal, if there exist \(a, b \in R\) such that \(I_x^r(a + b) \supseteq I_x^r(a) \cap I_x^r(b)\).

Note that \((a + b) \supseteq (a) \cup (b)\) for some \(a, b\) are subsets of \(U\). Then \(a, b \in DF_\delta(\{I_x^r, I_y^l\}; A)\), but \(a + b \not\in DF_\delta(\{I_x^r, I_y^l\}; A)\), which is contradiction to the fact that \(DF_\delta(\{I_x^r, I_y^l\}; A)\) is left \(h\)-ideal of \(R\). Hence \(f_1 = (\{I_x^r, I_y^l\}; A)\) is DFS left \(h\)-ideal of \(R\).

Conversely, suppose that \(f_1 = (\{I_x^r, I_y^l\}; A)\) is double-framed soft left \(h\)-ideal of \(R\);

Suppose \(x, a, b, z \in R\) with the expression \(x + a + z = b + z\) such that \(f_1(x) \subseteq f_1(a) \cap f_1(b)\). Similarly, for \(a, b \in R\), \(r \not\in R\) which is contradiction to the hypothesis so, \(f_1(a) \cap f_1(b)\) hold for all \(a, b \in R\). Consequently, \(f_1 = (\{I_x^r, I_y^l\}; A)\) is double-framed soft left \(h\)-ideal of \(R\) rounding the case for the right \(h\)-ideal can be shown accordingly.

**Theorem 16** Suppose \(A\) is a non-empty subset of a hemiring \(R\) and \(C_A = \{C_x^a, C_y^a\}\) is a double-framed soft set on \(R\) defined by

\[C_x^a(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \not\in A. \end{cases}\]

\[C_y^a(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \not\in A. \end{cases}\]
where $\emptyset \subseteq \delta_1 \subseteq \gamma_1 \subseteq U$ and $\emptyset \subseteq \delta_2 \subseteq \gamma_2 \subseteq R$. Then show that $C_\lambda$ is a double-framed soft left $h$-ideal of $R$.

**Proof** The proof of the theorem is similar to the proof of Lemma 15.

For a hemiring $R$, the double-framed soft sets denoted by $R = \langle (R^s, R^t) : R \rangle$ and $\emptyset = \langle (\emptyset^s, \emptyset^t) : R \rangle$ where $\{R^s, R^t\}$ and $\{\emptyset^s, \emptyset^t\}$ are soft mappings from $R$ to defined by

$$
R^s : x \mapsto R^s(x) = U,
R^t : x \mapsto R^t(x) = \emptyset,
\emptyset^s : x \mapsto \emptyset^s(x) = \emptyset,
\emptyset^t : x \mapsto \emptyset^t(x) = U,
$$

for all $x \in R$.

In the following theorem, the necessary and sufficient conditions for double-framed soft left $h$-ideal is provided.

**Theorem 17** Suppose $f_\lambda = \langle f^s, f^t \rangle : A \to R$ is a double-framed soft left $h$-ideal of a hemiring $T$. Then the following are the necessary and sufficient conditions for $f_\lambda = \langle f^s, f^t \rangle : A \to R$ to be a DFS left $h$-ideal of $R$ are:

1. $(\forall x, y \in R)(f^s(a + b) \supseteq f^s(a) \cap f^s(b)$ and $f^t(a + b) \subseteq f^t(a) \cup f^t(b))$,
2. $C_\lambda \delta f_\lambda \subseteq f_\lambda$,
3. $(\forall a, b, x, z \in R)(x + a + z = b + z \rightarrow f^s(a) \cap f^s(b)$ and $f^t(a) \cup f^t(b))$.

**Proof** $(\Rightarrow)$ Suppose that $f_\lambda = \langle f^s, f^t \rangle : A \to R$ is a double-framed soft left $h$-ideal of $R$. Then, conditions (1) and (3) directly follows from the Definition 13. For Condition (2), if $x \in R$ can not be expressed as $x + a_1 b_1 + z = a_2 b_2 + z$, then $(C^s \otimes f^s_\lambda)(x) = \emptyset \subseteq f^s_\lambda(x)$ and $(C^t \boxdot f^t_\lambda)(x) = U \supseteq f^t_\lambda(x)$. Hence, $C_\lambda \delta f_\lambda \subseteq f_\lambda$. Now assume that $x$ can be expressed as $x + a_1 b_1 + z = a_2 b_2 + z$, since $f_\lambda = \langle f^s_\lambda, f^t_\lambda \rangle : A \to R$ is a DFS left $h$-ideal, therefore, we have,

$$
\begin{align*}
(C^s \otimes f^s_\lambda)(x) & = \bigcup_{i \times k \times j \times m} \{C^s(a_i) \cap C^s(a_j) \cap f^s_\lambda(b_k) \cap f^s_\lambda(b_m)\} \\
& = \bigcup_{i \times k \times j \times m} \{U \cup \cup f^s_\lambda(b_k) \cap f^s_\lambda(b_m)\} \\
& \subseteq f^s_\lambda(b_k) \cap f^s_\lambda(b_m) \\
& \subseteq f^s_\lambda(a_i b_i) \cap f^s_\lambda(a_j b_j) \\
& = f^s_\lambda(x),
\end{align*}
$$

and

$$
\begin{align*}
(C^t \boxdot f^t_\lambda)(x) & = \bigcap_{i \times k \times j \times m} \{C^t(a_i) \cap C^t(a_j) \cap f^t_\lambda(b_k) \cap f^t_\lambda(b_m)\} \\
& = \bigcap_{i \times k \times j \times m} \{\emptyset \cap \cap f^t_\lambda(b_k) \cap f^t_\lambda(b_m)\} \\
& = f^t_\lambda(x),
\end{align*}
$$

Hence, $C_\lambda \delta f_\lambda \subseteq f_\lambda$.

$(\Leftarrow)$ now assume that Conditions (1)-(3) holds, let $x, z \in R$, such that $x + (a_1 b_1) + z = (a_2 b_2) + z$, then using Condition 2, we have,

$$
\begin{align*}
f^s_\lambda(x) & \supseteq (C^s_\lambda \otimes f^s_\lambda)(x y) \\
& = \bigcup_{i \times k \times j \times m} \{C^s(a_i) \cap C^s(a_j) \cap f^s(b_k) \cap f^s(b_m)\} \\
& = \bigcup_{i \times k \times j \times m} \{U \cup \cup f^s(b_k) \cap f^s(b_m)\} \\
& \supseteq f^s(b_k) \cap f^s(b_m) \\
& \supseteq \bigcup_{i \times k \times j \times m} f^s(b_i) \cap f^s(b_j) \\
& = f^s_\lambda(y),
\end{align*}
$$

also,

$$
\begin{align*}
f^t_\lambda(x) & \subseteq (C^t_\lambda \boxdot f^t_\lambda)(x y) \\
& = \bigcap_{i \times k \times j \times m} \{C^t(a_i) \cup C^t(a_j) \cup f^t(b_k) \cup f^t(b_m)\} \\
& = \bigcap_{i \times k \times j \times m} \{U \cup \cup f^t(b_k) \cup f^t(b_m)\} \\
& \subseteq f^t(b_k) \cup f^t(b_m) \\
& \subseteq \bigcup_{i \times k \times j \times m} f^t(b_i) \cup f^t(b_j) \\
& = f^t_\lambda(y).
\end{align*}
$$

Thus, $f_\lambda = \langle f^s_\lambda, f^t_\lambda \rangle : A \to R$ is double-framed soft left $h$-ideal of $R$.

**Lemma 18** Suppose $f_\lambda = \langle f^s_\lambda, f^t_\lambda \rangle : A \to R$ is a double-framed soft right $h$-ideal of a hemiring $T$. Then the following are the necessary and sufficient conditions for $f_\lambda = \langle f^s_\lambda, f^t_\lambda \rangle : A \to R$ to be a DFS right $h$-ideal of $R$ are:

1. $(\forall x, y \in R)(f^s(a + b) \supseteq f^s(a) \cap f^s(b)$ and $f^t(a + b) \subseteq f^t(a) \cup f^t(b))$,
2. $f_\lambda \delta C_\lambda \subseteq f_\lambda$,
3. $(\forall a, b, x, z \in R)(x + a + z = b + z \rightarrow f^s(a) \cap f^s(b)$ and $f^t(a) \cup f^t(b))$.

**Proof** The proof of the lemma follows from Theorem 17.

DOUBLE-FRAMED SOFT $h$-BI-IDEALS AND $h$-QUASI-IDEALS

An important milestone of the present section is to develop the connection between ordinary $h$-bi-ideals ($h$-quasi-ideals) with double-framed soft $h$-bi-ideals (double-framed soft $h$-quasi-ideals). For this purpose, double-framed soft including set are used. Further, it is shown that every
double-framed soft $h$-quasi-ideal of a hemiring is a double-framed soft $h$-bi-ideal but the converse is not true in general.

**Definition 19** A DFS-set $\left(\mathcal{F}, \mathcal{F}^*_x, \mathcal{F}^*_y; A\right)$ of a hemiring $R$ is said to be a double-framed soft $h$-bi-ideal of $R$ if the following conditions hold.

(4a). $\forall a, b \in R (f^*_a(a + b) \supseteq f^*_a(a) \cap f^*_b(b))$

(4b). $\forall a, b \in R (f^*_a(a + b) \subseteq f^*_a(a) \cup f^*_b(b))$

(5a). $\forall a, b \in R (f^*_a(ab) \supseteq f^*_a(a) \cap f^*_b(b))$

(5b). $\forall a, b \in R (f^*_a(ab) \subseteq f^*_a(a) \cup f^*_b(b))$

(6a). $\forall a, b, c \in R (f^*_a(abc) \supseteq f^*_a(a) \cap f^*_b(b) \cap f^*_c(c))$

(6b). $\forall a, b, c \in R (f^*_a(abc) \subseteq f^*_a(a) \cup f^*_b(b) \cup f^*_c(c))$

(7a). $\forall a, b, c, z \in R (x + a + z = b + z \rightarrow f^*_a(x) \supseteq f^*_a(a) \cap f^*_b(b))$

(7b). $\forall a, b, c, z \in R (x + a + z = b + z \rightarrow f^*_a(x) \subseteq f^*_a(a) \cup f^*_b(b))$

Note that for a double-framed soft $h$-bi (quasi)-ideal $f_a^*$ = $\left[\left\{f^*_a, f^*_b; A\right\} \atop \mathcal{D} \right]$ the inequality $f^*_a(0) \supseteq f^*_a(x)$ and $f^*_a(0) \subseteq f^*_a(x)$ are hold for all $x \in R$.

**Example 20** Let $U = Z^+$ (positive integers) be the universal set. Consider a parameter set $A = \{0, 1, 2, 3\}$, the set of non-negative integers modulo 4 is a hemiring.

Define a double-framed soft $\left(\left\{f^*_a, f^*_b; A\right\} \atop \mathcal{D} \right)$ in $R$ over $U = Z^+$ as follows:

| $R$ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| $f^*_a(x)$ | $Z^*$ | (2,3) | (1,2,3,5) | (2,3) |
| $f^*_b(x)$ | (3,4) | $Z^*$ | (1,3,4,6) | $Z^*$ |

using Definition 19, $\left[\left\{f^*_a, f^*_b; A\right\} \atop \mathcal{D} \right]$ is a double-framed soft $h$-bi-ideal of $R$ over $Z^+$. If for the same universal set $Z^+$, we consider another parameter set:

$B = \left\{\left\{x, y \atop x, y \in \mathbb{Z}_7 \cap \{0, 1\}\right\} \atop \mathcal{D} \right\}$

where $\mathbb{Z}_7$ is the set of non-negative integers modulo 4.

Define a double-framed soft $\left(\left\{f^*_a, f^*_b; B\right\} \atop \mathcal{D} \right)$ over $Z^+$ as follows:

| $g$ | [0] | [0] | [1] | [1] | [1] | [0] |
|-----|----|----|----|----|----|----|
| $f^*_a(g)$ | $Z^*$ | (2,3,4,5) | (1,2,3,4,5,7) | (1,2,3,4,5,7,8) | (1,2,3,4,5) | (1,2,3,4,5) |
| $f^*_b(g)$ | (3,4) | $Z^*$ | (1,3,4,5,6,7,8) | (1,3,4,5,7) |

then by Definition 19, $\left(\left\{f^*_a, f^*_b; B\right\} \atop \mathcal{D} \right)$ is a double-framed soft $h$-bi-ideal of $Z^+$.

**Theorem 21** A DFS-set $f_a = \left(\left\{f^*_a, f^*_b; A\right\} \atop \mathcal{D} \right)$ of a hemiring $R$ is a double-framed soft $h$-bi-ideal of $R$ if and only if

1. $f_a \cup f_b \subseteq f_a$

2. $f_a \cap f_b \subseteq f_a$

3. $f_a \cup f_b \subseteq f_b$

**Proof** Let $f_a = \left(\left\{f^*_a, f^*_b; A\right\} \atop \mathcal{D} \right)$ be a double-framed soft $h$-bi-ideal of $R$, and $x \in R$ be such that it can not be expressed in the form $x + (a_1 + b_1) + z = (a_2 + b_2) + z$, then $(f_a \cup f_b)(x) = \emptyset \subseteq f_a(x)$ and $(f_a \cap f_b)(x) = U \supseteq f_a(x)$. So $f_a \cup f_b \subseteq f_a$. Now if $x$ can be expressed in the form $x + (a_1 + b_1) + z = (a_2 + b_2) + z$, then

$(f_a \cup f_b)(x) =$}

$$
\bigcup_{x \in f_a(x)} f_a(x) \cap f_b(x) \subseteq f_a(x) \cap f_b(x) \supseteq f_a(x) \cap f_b(x) \subseteq f_a(x)
$$

also,

$(f_a \cap f_b)(x) =$}

$$
\bigcap_{x \in f_a(x)} f_a(x) \cap f_b(x) \subseteq f_a(x) \cap f_b(x) \subseteq f_a(x)
$$

Hence, $f_a \cup f_b \subseteq f_a$.

Now, if $x \in R$ can not be expressed in the form $x + (a_1, b_1)$ + $z = (a_2, b_2)$ + $z$, then $(f_a \cup f_b)(x) = \emptyset \subseteq f_a(x)$ and $(f_a \cap f_b)(x) = U \supseteq f_a(x)$. So $f_a \cap f_b \subseteq f_a$. If can be written in the form $x + (a_1, b_1) + z = (a_2, b_2) + z$, then

$(f_a \cup f_b)(x) =$}

$$
\bigcup_{x \in f_a(x)} f_a(x) \cap f_b(x) \subseteq f_a(x) \cap f_b(x) \supseteq f_a(x) \cap f_b(x) \subseteq f_a(x)
$$

also,

$(f_a \cap f_b)(x) =$}

$$
\bigcap_{x \in f_a(x)} f_a(x) \cap f_b(x) \subseteq f_a(x) \cap f_b(x) \subseteq f_a(x)
$$


\[ f'_0(x) \supseteq (f'_0 \oplus f'_2)(x + y) \]
\[ \supseteq \bigcup_{(a, b) \in \mathcal{D}_0} \{ f'_0(a) \cap f'_0(b) \cap f'_2(h) \cap f'_2(b) \} \]
\[ \supseteq \{ f'_0(0) \cap f'_0(0) \cap f'_0(x) \cap f'_0(y) \} \]
\[ = f'_0(x) \cap f'_0(y) \]
\[ \text{also,} \]
\[ f'_0(x + y) \subseteq (f'_0 \boxdot f'_0)(x + y) \]
\[ \subseteq \bigcup_{(a, b) \in \mathcal{D}_0} \{ f'_0(a) \cup f'_0(b) \cup f'_0(h) \cup f'_0(b) \} \]
\[ \subseteq \{ f'_0(0) \cup f'_0(0) \cup f'_0(x) \cup f'_0(y) \} \]
\[ = f'_0(x) \cup f'_0(y) \]

Further,
\[ f'_0(x) \supseteq (f'_0 \oplus f'_2)(x) \]
\[ \supseteq \bigcup_{(a, b) \in \mathcal{D}_0} \{ f'_0(a) \cap f'_0(b) \cap f'_2(h) \cap f'_2(b) \} \]

If \( x + a + z = b + z \), then \( x + a + 0 + z = b + 0 + z \), therefore,
\[ f'_0(x) \supseteq \{ f'_0(a) \cap f'_0(b) \cap f'_0(0) \} = \{ f'_0(a) \cap f'_0(b) \} \]
similarly,
\[ f'_0(x) \subseteq (f'_0 \boxdot f'_0)(x) \]
\[ = \bigcup_{(a, b) \in \mathcal{D}_0} \{ f'_0(a) \cup f'_0(b) \cup f'_0(h) \cup f'_0(b) \} \]
\[ \subseteq \{ f'_0(a) \cup f'_0(0) \cup f'_0(b) \cup f'_0(0) \} \]
\[ = f'_0(x) \cup f'_0(y) \]

The rest of the conditions can be proved in similar manner. Consequently, \( f' = (f'_0, f'_2) \) is a double-framed soft \( h \)-ideal of \( R \).

**Example 23** The set of all non-negative integers \( \mathbb{U} = \mathbb{N}_0 \), is a hemiring with respect to usual addition and multiplication. Suppose \( u_1, u_2, u'_1, u'_2 \in \mathbb{P}(\mathbb{U}) \) be such that \( \emptyset \neq u_1 \subset u_1 \) and \( \emptyset \neq u'_2 \subset u'_2 \), where is power set of define a double-framed soft set over as follows:

| \( u_1 \) | \( u_2 \) |
|---|---|
| \( u_1 \) | \( u_2 \) |

then by Definition 22, \( \{ f'_0, f'_0 \} \) is a double-framed soft \( h \)-quasi-ideal of \( N_0 \).

**Theorem 24** If \( f'_0 = (f'_0, f'_0) \) is a double-framed soft set of a hemiring \( R \), then a non-empty double-framed soft including set \( DF_0(f'_0, f'_0) \) is \( h \)-bi-ideal (resp. \( h \)-quasi-ideal) of \( R \) if \( f'_0 = (f'_0, f'_0) \) and only if is DFS \( h \)-bi-ideal (resp. \( h \)-quasi-ideal) of \( R \).

**Proof** The proof of the theorem follows from Lemma 15. Using characteristic double-framed soft sets, ordinary \( h \)-ideals (\( h \)-left, \( h \)-right) ideals, \( h \)-bi-ideals, \( h \)-quasi-ideals) in a hemiring \( R \) are linked with DFS \( h \)-ideals (DFS \( h \)-left (right)-ideals, DFS \( h \)-bi-ideals, DFS \( h \)-quasi-ideals) in the following result.
Corollary 25 If A is any non-empty subset of a hemiring \( R \), then, characteristic double-framed soft set \( C_A = \left( \{a, b\}; A \right) \) is a DFS \( h \)-ideal (resp. DFS \( h \)-bi-ideal, DFS \( h \)-quasi-ideals) of \( R \) if and only if \( A \) is an \( h \)-ideal (resp. \( h \)-bi-ideal, \( h \)-quasi-ideal) of \( R \).

Proof Follows from Lemma 15 and Theorem 24.

Theorem 26 If \( f_A = \left( \{f_A, A\}; \tilde{A} \right) \) is DFS right \( h \)-ideal and \( g_B = \left( \{g_B, B\}; \tilde{B} \right) \) is DFS left \( h \)-ideal of a hemiring \( R \), then \( f_A \cap g_B \) is a double-framed soft \( h \)-quasi-ideal of \( R \).

Proof Assume \( x, y \in R \), then

\[
\begin{align*}
(f_A \cap g_B)(x + y) &= f_A(x + y) \cap g_B(x + y) \\
&\supseteq \{f_A(x) \cap g_B(x) \cap \{f_A(x) \cap g_B(y)\}
\]

and

\[
(f_A \cap g_B)(x + y) = f_A(x) \cap g_B(x) \cap \{f_A(y) \cap g_B(y)\}
\]

Now let \( x, a, b, z \in R \) with the expression \( x + a + z = b + z \), then

\[
(f_A \cap g_B)(x + y) = f_A(x + y) \cap g_B(x + y)
\]

and

\[
(f_A \cap g_B)(x) \cap (f_A \cap g_B)(y)
\]

Also, \( (f_A \cap g_B)(x + y) \cap \{f_A(x) \cap g_B(y)\} \supseteq \{f_A(x) \cap g_B(y)\} \)

By similar way, we can show that \( f_A(ab) \supseteq f_A(a) \cap f_A(b) \) and

\[
(f_A \cap g_B)(xy) = (f_A(x) \cap g_B(x) \cap \{f_A(y) \cap g_B(y)\})
\]

The converse of the above proposition is not true in general, as shown in the following example.

Example 28 (Yin & Li 2008). Let \( R \) be the set of all \( 2 \times 2 \) matrices i.e.,

\[
R = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \text{where } a_{ij} \in \mathbb{N}_+(\text{non-negative integers})
\]

then \( R \) is a hemiring with usual addition and multiplication of matrices. Let \( Q = \begin{bmatrix} a & 0 \\ 0 & \cdots & 0 \end{bmatrix} \) \( a \in \mathbb{N}_+ \), then \( Q \) is an \( h \)-quasi-ideal of \( R \) but not an \( h \)-ideal of \( R \). Hence, \( Q \) is not an \( h \)-bi-ideal of \( R \). Using Corollary 25, the double-framed
soft characteristic function $C_h$ is a double-framed soft $h$-quasi-ideal but not a double-framed soft $h$-ideal of $R$. Therefore, it is not a double-framed soft $h$-bi-ideal of $R$.

**Definition 29** (Zhan & Dudek 2007). A hemiring $R$ is $h$-hemiregular if for all $x \in R$, there exist $a_1, a_2, z \in R$ such that $x + x_a x + z = x a x + z$.

**Lemma 30** (Zhan & Dudek 2007). If $R$ is a hemiring, then the following conditions are equivalent:

(i) $R$ is $h$-hemiregular hemiring.

(ii) $M \cap N$, where $M$ is right $h$-ideal and $N$ is left $h$-ideal of $R$.

**Lemma 31** (Yin & Li 2008). If $R$ is a hemiring, then the following conditions are equivalent:

(i) $R$ is $h$-hemiregular.

(ii) $B = \text{DFS}$, where $B$ is $h$-bi-ideal of $R$.

(iii) $Q = \text{DFS}$, where $Q$ is $h$-quasi-ideal of $R$.

**Theorem 32** If $R$ is a hemiring, then the following conditions are equivalent:

(1). $R$ is $h$-hemiregular hemiring.

(2). If $f'_x = \{f'_a \in \text{DFS} \mid a$ is $h$-ideal of $R\}$ and $f'_x \otimes g'_y = \{f'_a \otimes g'_y \mid a \in A, y \in B\}$

Proof (1)$\Rightarrow$(2): Let $R$ be $h$-hemiregular hemiring and $f'_x = \{f'_a \in \text{DFS} \mid a$ is $h$-ideal of $R\}$ and $f'_x \otimes g'_y = \{f'_a \otimes g'_y \mid a \in A, y \in B\}$.

Thus, $f'_x \otimes g'_y = \{f'_a \otimes g'_y \mid a \in A, y \in B\}$.

Also,

$$f'_x \otimes g'_y(x) = \bigcap_{a \in A, y \in B} \{f'_a(a) \otimes f'_y(x) \mid (a, x) \in A \otimes B, y \in B\}$$

$$\subseteq \bigcup_{a \in A, y \in B} \{f'_a(x) \cap f'_y(x) \mid (a, x) \in A \otimes B, y \in B\}$$

$$\subseteq \bigcup_{a \in A, y \in B} \{f'_a(x) \cap f'_y(x) \mid (a, x) \in A \otimes B, y \in B\}$$

$$= f'_x \cap f'_y(x)$$

$$= (f'_x \otimes g'_y)(x).$$
also
\[ (f_n \circ C_n \circ f_n^*) (x) = \bigcap_{x=a_1+a_2=b_1+b_2} \{(f_n \circ C_n \circ f_n^*) (a_1) \cup (f_n \circ C_n \circ f_n^*) (a_2) \cup f_n^* (b_1) \cup f_n^* (b_2) \} \]
\[ \subseteq (f_n \circ C_n \circ f_n^*) (xa) \cup (f_n \circ C_n \circ f_n^*) (xa) \cup f_n^* (x) \]
\[ \subseteq \left\{ \bigcap_{x=a_1+a_2=b_1+b_2} \{(f_n^* (a_1) \cup f_n^* (a_2) \cup f_n^* (b_1) \cup f_n^* (b_2) \} \right\} \cup f_n^* (x) \]
\[ \subseteq \{ f_n^* (xa, x) \cup f_n^* (xa, x) \cup f_n^* (xa, x) \cup f_n^* (xa, x) \cup f_n^* (x) \] 
\[ \subseteq \{ f_n^* (x) \cup f_n^* (x) \cup f_n^* (x) \cup f_n^* (x) \} = f_n^* (x). \]

(2) \Rightarrow (3). This implication holds using Remark 27.

(3) \Rightarrow (1). If Q is any h-ideale of R, then by Corollary 25, the characteristic function \( f_n \) of a DFS h-ideale of R. Therefore, by hypothesis, \( f_n \subseteq C_n \circ C_n \circ f_n^* \). Now by Theorem 12, \( C_n \circ C_n \circ C_n = C_{n+1} \). Thus, \( C_n \subseteq C_{n+1} \) implies Q \( \subseteq \text{QRQ} \) and reverse inclusion hold because Q is a h-ideale of R i.e., \( \text{QRQ} \subseteq Q \), which implies that \( \text{QRQ} = Q \). Hence by Lemma 31, is h-hemiregular.

**Theorem 34** If R is a hemiring, then the following conditions are equivalent:

1. R is a h-hemiregular hemiring.
2. \( f_n \cap g_n \subseteq f_n \circ g_n \circ f_n \) for every DFS h-bi-ideal \( f_n \) and every DFS h-ideale \( g_n \) of R.
3. \( f_n \cap g_n \subseteq f_n \circ g_n \circ f_n \) for every DFS h-ideale \( f_n \) and every DFS h-ideale \( g_n \) of R.

**Proof** (1) \Rightarrow (2). Consider R is an h-hemiregular. Suppose \( f_n \) is a DFS h-bi-ideal and \( g_n \) is a DFS h-ideale of R. Let \( x \in R \), then there exist \( a_1, a_2, z \in R \) such that \( x = xa_1 + xa_2 + za_1 + za_2 \). Therefore, \( f_n \cap g_n \subseteq f_n \circ g_n \circ f_n \)
\[ \subseteq \left\{ \bigcap_{x=a_1+a_2=b_1+b_2} \{(f_n^* (a_1) \cup f_n^* (a_2) \cup f_n^* (b_1) \cup f_n^* (b_2) \} \right\} \cup f_n^* (x) \]
\[ \subseteq \{ f_n^* (xa, x) \cup f_n^* (xa, x) \cup f_n^* (xa, x) \cup f_n^* (xa, x) \cup f_n^* (x) \] 
\[ \subseteq \{ f_n^* (x) \cup f_n^* (x) \cup f_n^* (x) \cup f_n^* (x) \} = f_n^* (x). \]

(2) \Rightarrow (3). This implication can be shown simply.

(3) \Rightarrow (1). Let \( f_n \) is any DFS h-ideale of R, since \( C_n \) is a DFS h-ideale, \( f_n \circ g_n \cap f_n \circ g_n \circ f_n \) for every DFS h-bi-ideal \( f_n \) and every DFS h-ideale \( g_n \) of R.

**Lemma 35** (Yin & Li 2008). If is a hemiring, then the following conditions are equivalent:

(i) R is a h-hemiregular hemiring.
(ii) Both right h-ideale M and left h-ideale N of R are idempotent and \( MN \) is an h-ideale of R.

**Theorem 36** If R is a hemiring, then the following conditions are equivalent:

(i) R is a h-hemiregular hemiring.
(ii) If \( f_n \) is a DFS right h-ideale and \( g_n \) is a DFS left h-ideale, then both \( f_n \) and \( g_n \) are idempotent and \( f_n \circ g_n \circ f_n \) is a DFS h-ideale of R.

**Proof** (i) \Rightarrow (ii). If R is a h-hemiregular hemiring and \( f_n \) is a DFS right h-ideale of R, then \( f_n \circ g_n \cap f_n \circ g_n \circ f_n \) is a DFS h-ideale by Lemma 18). Also, by Theorem 32, \( f_n \subseteq f_n \circ g_n \circ f_n \). Thus \( f_n \circ g_n \cap f_n \circ g_n \circ f_n \) implies that \( f_n \) is an idempotent. Similarly, \( g_n \) is an idempotent. Now using Theorem 32, \( f_n \circ g_n \circ f_n \) is a DFS h-ideale of R, therefore \( f_n \circ g_n \) is a DFS h-ideale of R.

(ii) \Rightarrow (i). Suppose A is any right h-ideale, then by Corollary 25, \( C_n \) is a DFS right h-ideale of R. Therefore, by hypothesis \( C_n = C_n \circ C_n \circ C_n \), so by Theorem 12, \( C_n = C_{n+1} \), hence A is idempotent. Similarly, B is also idempotent. Also, \( C_n \circ C_n \circ C_n \)
Due to the diverse application of both hemirings and soft sets, the new investigations using soft structures in hemirings are becoming the central focus for researchers. The present research achieved another milestone in the hemiring theory by developing double-framed soft h-ideal theory in hemirings. More precisely, this research introduced double-framed soft left h-ideals, DFS right h-ideals, DFS h-bi-ideals and DFS h-quasi-ideals of hemirings. Double-framed soft including sets and characteristic double-framed soft functions are used to provide the bridge between ordinary h-ideals and double-framed soft h-ideals of hemirings. An important class of hemirings i.e. h-hemiregular hemirings are characterized by the properties of the aforementioned double-framed soft h-ideals of hemirings which yields several characterization theorems of hemirings. The research at hand will further motivate the researcher to apply the concept of double-framed soft sets in other algebraic structures which will ultimately be applied in various applied fields of science.

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Faiz Muhammad Khan* & NieYufeng
Department of Applied Mathematics
School of Natural and Applied Sciences
Northwestern Polytechnical University Xi’an, Shaanxi
PR China

Hidayat Ullah Khan
Department of Mathematics
University of Malakand
Lower Dir Chakdara, KP
Pakistan

Asghar Khan
Department of Mathematics
Abdul Wali Khan University Mardan
Mardan, KP
Pakistan

Faiz Muhammad Khan
Department of Mathematics and Statistics
University of Swat, KP
Pakistan

*Corresponding author; email: faiz_zady@yahoo.com

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