A quantum Kolmogorov-Arnold-Moser theorem in the anisotropic Dicke model and its possible implications in the hybrid Sachdev-Ye-Kitaev models

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The classical Kolmogorov-Arnold-Moser (KAM) theorem provides the underlying mechanism for the stability of the solar system under some small chaotic perturbations. Despite many previous efforts, any quantum version of the KAM theorem remains elusive. In this work, we provide a quantum KAM theorem in the context of the anisotropic Dicke model which is the most important quantum optics model. It describes a single mode of photons coupled to \( N \) qubits with both a rotating wave (RW) term and a counter-RW (CRW) term. As the ratio of the CRW over the RW term increases from zero to one, the systems evolves from quantum integrable to quantum chaotic.

We establish a quantum KAM theorem to characterize such a evolution quantitatively by both large \( N \) expansion and Random Matrix Theory and find agreement from the two complementary approaches. Connections and differences between the Dicke models and Sachdev-Ye-Kitaev (SYK) or hybrid SYK models are examined. Possible Quantum KAM theorem in terms of other quantum chaos criterion such as quantum Lyapunov exponent is also discussed.

I. INTRODUCTION

In classical chaos, the Kolmogorov-Arnold-Moser (KAM) theorem\textsuperscript{1,2} describes how an integrable Hamiltonian \( H_0 \) respond to a chaotic perturbation \( \Delta H \), which makes the total Hamiltonian \( H = H_0 + \Delta H \) non-integrable. It states that if the two conditions are satisfied: (a) \( \Delta H \) is sufficiently small (b) the frequencies \( \omega_i \) of \( H_0 \) are in-commensurate, then the system remains quasi-integrable. The classical KAM theorem has played important roles in the stability of the solar system and many other classical chaotic systems. It remains an outstanding problem to find a quantum analogue of the KAM theorem for a quantum many-body system. Here, we will try to achieve such a goal in the context of the anisotropic ( \( J - U(1)/Z_2 \) ) Dicke model\textsuperscript{3,4} Eq.\textsuperscript{[1]} which is the most important model in quantum optics.

There are previous works\textsuperscript{5-7} studying both quantum phase transitions (QPT) and quantum chaos in several extreme limits of the \( J - U(1)/Z_2 \) Dicke model Eq.\textsuperscript{[1]}. It describes a single mode of photons coupled to \( N \) qubits with both a rotating wave (RW) \( g \) term and a counter-RW (CRW) \( g' \) term at any ratio \( \beta = g/g' \). For example, the authors in\textsuperscript{5} studied the \( J - Z_2 \) Dicke model with \( \beta = 1 \) at the thermodynamic limit \( J = \infty \) and also its energy level statistics (ELS)\textsuperscript{10,11} at a finite \( J = N/2 \). In the \( J = \infty \) limit, as the atom-photon coupling strength increases above a critical value, it displays a QPT from the normal to the superradiant phase\textsuperscript{12,13}. However, the system becomes non-integrable at any finite \( J \). By studying its ELS by exact diagionization (ED) at finite sizes \( N \geq 10 \) at a given parity sector, they found that in the normal phase, it is Possionian \( P_p(s) = e^{-s} \), but in the superradiant phase, becomes Wigner-Dyson (WD) distribution in the Gaussian orthogonal ensemble (GOE) class \( P_v(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} \) in the RMT classification\textsuperscript{10,11}. This fact suggests that the quantum Chaotic to Integrable transition (CIT) at a finite \( N \) may be associated to the QPT at \( N = \infty \). On the other limit, the \( U(1) \) Dicke model\textsuperscript{5,6} with \( \beta = 0 \) is always integrable and still undergoes a QPT from the normal to the super-radiant phase at \( N = \infty \). This fact indicates that a QPT may not be related to a CIT. In\textsuperscript{5,6}, we evaluate the whole energy spectrum of the \( J - U(1) \) Dicke model (with \( J = N/2 \)) with \( \beta = 0 \) by a 1/\( J \) expansion and find nearly perfect agreements with those found from the ED when \( N \) is even as small as \( N = 2 \).

It was well known that quantum dynamics are inherently encoded in any quantum many body systems, one effective way to characterize any possible quantum chaos in such an intrinsic quantum dynamics is through Random matrix theory (RMT)\textsuperscript{10,11,15}. We first propose a quantum version of KAM theorem to describe the quantum Chaotic to Integrable transition (CIT) in terms of the RMT. Then we derive the effective Hamiltonian Eq.\textsuperscript{[1]} at any ratio \( 0 < \beta = g'/g < 1 \) by the 1/\( N \) expansion. By using the effective Hamiltonian, we investigate the analytic scaling form of the quantum KAM theorem at a finite \( N \) near its quantum integrable \( U(1) \) limit \( \beta = g'/g < 1 \). By carefully identifying the chaotic source leading to the energy level correlations near the quantum integrable \( U(1) \) limit, we show that the quantum KAM theorem Eq.\textsuperscript{[2]} holds in the strong coupling limit \( N > g/g_c \gg 1 \), so the system remains quasi-integrable.
We stress the important roles played by the Berry phase in its establishment. By using the RMT, we evaluate the energy level statistic (ELS) at $N = 10, 20$ at a given parity sector by exact diagonalizations (ED) at any $0 < \beta < 1$. The ED data in the RMT fit well the analytic quantum KAM theorem Eq[12] achieved by the $1/N$ expansion. We explore the intrinsic connections between the QPT characterized by $1/N$ expansions and the CIT characterized by the RMT. Contrasts to the Sachdev-Ye-Kitaev (SYK) and hybrid SYK models are made. Possible quantum KAM in terms of the Lyapunov exponent is discussed.

\section{The $1/N$ Expansion of the J–U(1)/Z$_2$ Dicke Model in the Super-Radiant Phase and QPT.}

In the $J$–U(1)/Z$_2$ Dicke model, a single mode of photons couple to $N$ two level atoms projected in the total angular momentum $J = N/2$ state $\hat{a}^\dagger \hat{a}$. 

\begin{align}
H_J &= \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{J}_Z + \frac{g}{\sqrt{2}J}(\hat{a}^\dagger \hat{a} \hat{J}_- + \hat{a} \hat{J}_+ ) \\
&+ \frac{g'}{\sqrt{2J}}(\hat{a} \hat{J}_+ + \hat{a}^\dagger \hat{J}_- )
\end{align}

(1)

where $\omega_a, \omega_b$ are the energy of the cavity photons and the two atomic levels respectively, $g = \sqrt{N}g', g' = \sqrt{N}g''$ are the collective photon-atom rotating wave (RW) coupling and counter-rotating wave (CRW) term respectively. If $\beta = g/g' = 0$, Eq[1] reduces to the U(1) Dicke model with the U(1) symmetry $a \rightarrow ae^{i\theta}, \sigma^- \rightarrow \sigma^e^{i\theta}$ leading to the conserved quantity $P = a^\dagger a + J_Z$ where $J^2 = \frac{1}{2} \sum_i \sigma_i^2$. The CRW $g'$ term breaks the U(1) to the Z$_2$ symmetry $a \rightarrow -a, \sigma^- \rightarrow -\sigma^-$ with the conserved parity operator $\Pi = e^{i\pi(a^\dagger a + J_Z)}$. If $\beta = 1$, it becomes the Z$_2$ Dicke model. If $\beta = \infty$, it can be mapped to the static version of Landau-Zener (LZ) model. In this work, we fix the ratio to be $0 < g/g' = \beta < 1$. The other case with $1 < \beta < \infty$ need a different treatment and will be discussed in a separate publication.

Inside the super-radiant phase, it is convenient to write both the photon and atom in the polar coordinates $a = \sqrt{\lambda_a^2 + \delta \rho_a e^{i\theta_a}}, b = \sqrt{\lambda_b^2 + \delta \rho_b e^{i\theta_b}}$. We first minimize the ground state energy at the order $J$ and found the saddle point values of $\lambda_a$ and $\lambda_b$:

\begin{align}
\lambda_a &= \frac{g + g'}{\omega_a} \sqrt{\frac{j}{2}(1 - \mu^2)}, \quad \lambda_b = \sqrt{\frac{j}{2}(1 - \mu)}
\end{align}

(2)

where $\mu = \omega_a \omega_b / (g + g')^2$. In the superradiant phase, $\mu < 1$, so that $g + g' > g''_c = \sqrt{2 \omega_a \omega_b}$. In the normal phase $g + g' < g''_c$, one gets back to $\lambda_a = \lambda_b = 0$. At a fixed $\beta$, the QPT happens at $g_c = \frac{\omega_a \omega_b}{\beta}$.

Well inside the super-radiant phase, $\lambda_a^2 \sim \lambda_b^2 \sim J$, it is convenient to introduce the $\pm$ modes: $\theta_{\pm} = (\theta_a \pm \theta_b)/2, \delta \rho_{\pm} = \delta \rho_a \pm \delta \rho_b, \lambda_a^2 = \lambda_a^2 \pm \lambda_b^2$. The Berry phase in the $+ \text{sector}$ can be defined as

\begin{align}
\lambda_a^2 = P + \alpha
\end{align}

where $P = 1, 2, \cdots$ is the closest integer to $\lambda_a^2$, so $-1/2 < \alpha < 1/2$.

After shifting $\theta_{\pm} \rightarrow \theta_{\pm} + \pi/2$, we reach the Hamiltonian to the order of $1/J$:

\begin{align}
\mathcal{H}[\delta \rho_{\pm}, \theta_{\pm}] &= \frac{D}{2} (\delta \rho_{+} - \alpha)^2 + D_-(\delta \rho_{-} + \gamma (\delta \rho_{+} - \alpha))^2 \\
&+ 4 \omega_a \lambda_a^2 \left[ \frac{1}{1 + \beta} \sin^2 \theta_{-} + \frac{\beta}{1 + \beta} \sin^2 \theta_{+} \right]
\end{align}

(4)

where $D = \frac{2 \omega_a (g + g')^2}{E_H N}$ is the phase diffusion constant in the + sector, $D_- = E_H^2/16 \omega_a^2$ with $E_H = (\omega_a + \omega_b)^2 + 4(g + g')^2 \lambda_a^2 / N$. The $\gamma = \frac{\omega_b^2}{E_H^2} (1 - (g + g')^2)$ is the coupling between the $+$ and $-$ sector. Due to the large gap in the $\theta_-$ sector when $0 < \beta < 1$, it is justified to drop the Berry phase in the $-$ sector.

It is instructive to re-write Eq[4] as:

\begin{align}
\mathcal{H}[\delta \rho_{\pm}, \theta_{\pm}] &= H_{U(1)} + 4 \omega_a \lambda_a^2 \beta \frac{1}{1 + \beta} \sin^2 \theta_{+}
\end{align}

(5)

where the $H_{U(1)}$ is the Hamiltonian of the $J$–U(1) model:

\begin{align}
H_{U(1)} &= \frac{D}{2} (\delta \rho_{+} - \alpha)^2 \\
&+ D_-(\delta \rho_{-} + \gamma (\delta \rho_{+} - \alpha))^2 + 4 \omega_a \lambda_a^2 \frac{1}{1 + \beta} \sin^2 \theta_{-}
\end{align}

(6)

which conserves $\delta \rho_+$. Its eigen-energies and eigen-states are listed in and also reviewed in the appendix A. Eq[5] naturally separates the chaotic perturbation $H'_c$ from those integrable ones. Near the integrable U(1) limit $\beta \ll 1$, the second term $H'_c$ in Eq[5] can be treated as the small chaotic perturbation [$H'_c, H_{U(1)} \neq 0$, it violates the conservation of $\delta \rho_+$, but still keeps the parity $\Pi = e^{i\pi(P + \delta \rho_+)}$. Despite Eq[5] explicitly contains $\beta$ dependencies, they still keep the integrability, so do not change the ELS. This observation will be analyzed further in the following section.

\section{A Quantum KAM Theorem: General Statement.}

Inspired by the classical KAM theorem, we expect a quantum analogue of the KAM theorem exists near an integrable quantum many-body system whose eigen-energies are in-commensurate. In Eq[3] it is the frustration due to the Berry phase $\alpha$ which make its eigen-energies in-commensurate except at $\alpha = 0, \pm 1/2$ which have zero measures anyway. We state the general form of a quantum KAM theorem from the RMT point of view: Near the integrable limit of an incommensurate quantum many-body system, when the energy level repulsion caused by a small chaotic perturbation is less
than the average many-body energy spacing of the integrable system, the system remains quasi-integrable, so its ELS remains to be Possionian. This is justified, because all the energy levels in the integrable side are un-correlated. Specifically, taking two nearest neighbour (NN) bulk energy states with the NN energy level spacing \( s_0 = E^0_2 - E^0_1 \), one can write down the \( 2 \times 2 \) quantum chaotic matrix within the two NN energy levels subspaces as \( \Delta_{ij} = \langle |iH'| |j \rangle \). By a bulk energy level \( E_B \), we mean \( \lim_{N \to \infty} \frac{E-B}{\sqrt{N}} \neq 0 \) where \( E_0 \) is the ground state energy. Then the perturbed NN energy level spacing is:

\[
S = \sqrt{s^2 + |\Delta_{12}|^2}
\]

where \( s = s_0 + \Delta_{22} - \Delta_{11} \) is the diagonal energy shift due to the chaotic perturbation, the \( \Delta_{12} \) is the off-diagonal one. It is important to observe that if setting \( \Delta_{12} = 0 \) in Eq.7 then the ELS still stays Possionian. This is because the diagonal shift does not change the ELS, only the off-diagonal does. Indeed, near any integral limit \( \mathcal{H}_c \), if one adds a perturbation \( \mathcal{H}_1 \) which commutes with the integral Hamiltonian \( \{ \mathcal{H}_0, \mathcal{H}_1 \} = 0 \), \( \mathcal{H}_1 \) is a conserved quantity, the system remains integrable. However, \( \mathcal{H}_1 \) still induces a diagonal energy shift, but not the off-diagonal one. Obviously, it is the off-diagonal one which introduces the level repulsion between the two NN levels, which, in turns, leads to the change of ELS to WD. So we conclude that when \( |\Delta_{12}| < s \) in Eq.7 the ELS stays Possionian, the quantum KAM applies. In the following, instead of giving a rigorous mathematical proof of this quantum KAM theorem, we derive its analytic finite size scaling form by \( 1/N \) expansion in Eq.8 and compare with our ED at various available values of \( N \).

IV. THE QUANTUM KAM THEOREM: THE 1/N EXPANSION ON THE DICKE MODEL

Let’s take two NN states as \( |B1\rangle = |l\rangle_m |m\rangle \) and \( |B2\rangle = |l\rangle_{m+2} |m+2\rangle \) where \( l \sim N/2 \) and \( m \sim 1 \) with a typical Berry phase \(-1/2 < a < 1/2 \). Then \( \lim_{N \to \infty} \frac{E_{|B2\rangle} - E_{|B1\rangle}}{2m} = \hbar \omega / 2. \) One can immediately evaluate the diagonal matrix element \( \langle B1|H'|B1\rangle = \langle B2|H'|B2\rangle = 2\omega\lambda^2 \beta / 1+\beta \), So the chaotic perturbation does not change the diagonal energy level spacing

\[
s = E_0(l, m + 2) - E_0(l, m) = 2D(m - \alpha + 2)
\]

which is independent of the Landau level index \( l \sim N/2 \). This fact simplifies the computation considerably.

One can also compute the splitting (NN energy level repulsion) in terms of the coherent state:

\[
\Delta_{12} = \langle B1|H'|B2\rangle = \omega\lambda^2 \beta \frac{1}{1+\beta} m \langle l | l \rangle_{m+2}
\]

where one can evaluate the matrix element

\[
f(l, l) = m \langle l | l \rangle_{m+2} = \langle l | D(iG) | l \rangle = e^{-G^2/2} \! \sum_{r=0}^{1} \! \frac{(-1)^{1-r} G^{2l-2r}}{(l-r)! r!}
\]

where \( G = g_{m+2} - g_m = \frac{\sqrt{2} \pi}{\beta} \). One can find \( f(0, 0) = e^{-G^2/2}, f(1, 1) = e^{-G^2/2} (1 - G^2), \)... In fact, more straightforwardly, in terms of the wavefunction of a harmonic oscillator, \( f(l, l) = \int d\theta |\psi_\theta(l)_m|^2 e^{2\gamma \theta}, l \in 1, 2, 3, \ldots \), one reach the same results as Eq.8.

One can write down the general expressions of the three quantities \( \lambda^2 \) in Eq.8 the diffusion constant \( D \) in Eq.9 and \( G^2/2 \) in Eq.10 in terms of \( g/g_e \) inside the super-radiant phase. They simplify dramatically in the strong coupling limit \( N > g/g_e \gg 1 \)

\[
\lambda^2 \propto \frac{(\frac{g}{g_e})^2}{N}, \quad D = 2\omega \frac{g}{g_e} \frac{1}{N} \sim 2\omega / \lambda^2 , \quad G^2/2 = \frac{\sqrt{1 + \beta}}{2} \frac{1}{N}
\]

In the large \( N \) limit, the polynomial in \( f(l = N/2, l = N/2) \) multiplying the exponential \( e^{-G^2/2} \) becomes \( 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{6} = \cdots \). So we conclude \( f(N/2, N/2) \sim e^{-G^2/2} \) as \( N \to \infty \).

Applying the general criterion to the two typical bulks states in Eq.8 and Eq.9 leads to scaling form of the quantum KAM theorem:

\[
\beta_{c1} \sim N^{-1/2} e^{1/2N} (g/g_e)^{-4}
\]
V. THE ENERGY LEVEL STATISTICS IN RMT AND THE CIT.

Now we look at RMT classification of Eq.1. The Time reversal symmetry is nothing but the complex conjugate operator \( \mathcal{T} = K \) which acts as \( K a K^{-1} = a, K a^\dagger K^{-1} = a^\dagger, K i K^{-1} = -i \). Obviously, \( K J_a K^{-1} = J_a, a = \pm, z. \) It keeps the commutation relations \([a, a^\dagger] = 1, [J_z, J_\pm] = \pm J_\pm. \) So the many-body Hamiltonian Eq.1 has the Time-reversal symmetry, also \( K^2 = 1 \), so it satisfies GOE.

As shown in \ref{15}-\ref{19} and reviewed in the appendix B, the most effective way to characterize an ELS is to study the distribution of the ratio of two NN energy level spacings \( \langle \tilde{r} \rangle \). We plot \( \langle \tilde{r} \rangle \) vs \( \ln \beta \) at a fixed \( g/g_c = 3, 2, 1 \) and \( \langle \tilde{r} \rangle \) vs \( g/g_c \) at a fixed \( \beta = 0.1, \cdots, 0.8 \) in Fig.2a and Fig.2b respectively for two different sizes \( N = 10, 20 \). The data \( \beta_{c1} / \beta_{c2} / \beta_{c3} \) at the two different sizes \( N = 10, 20 \) match qualitatively the scaling in Eq.12. A small discrepancy maybe attribute to the cutoff introduced in the ED (See appendix C).

VI. CONTRAST ANISOTROPIC DICKE MODELS WITH HYBRID SYK MODELS

So far we focused on the \( U(1)/\mathbb{Z}_2 \) anisotropic Dicke model which is the most important model in quantum optics. The quantum chaos in the Dicke model is investigated here by both \( 1/N \) expansion and RMT. Similar approaches have also been used to explore the quantum chaos in the Sachdev-Ye-Kitaev (SYK) model which maybe dual to a quantum black hole\ref{16}-\ref{19},\ref{21}-\ref{24}. It is constructive to contrast the (anisotropic) Dicke model to the (hybrid) SYK models. (a) Both have an infinite-range interaction, so are effectively \( 0 + 1 \) dimensional systems. (b) Both show quantum chaos and quantum information scramblings which can be studied by \( 1/N \) expansion and Random matrix theory (RMT) respectively. But the mechanism leading to the quantum chaos is very much different. The former is due to both interactions and disorders. The latter is due to the atom-photon interactions (c) The former is an interacting fermionic model with quenched disorders, while the latter is a clean interacting bosonic one consisting of \( N \) qubits interacting with photons. So when performing the ED in RMT, the Hilbert space in the former is automatically finite \( \sim 2^N \), while the latter is infinite, so a finite cut-off need to be introduced (see the appendix C). (d) The ground state of SYK is a conformally invariant gapless QSL which leads to the maximal Lyapunov exponent \( \lambda_L = 2\pi/\beta \) by \( 1/N \) expansion when \( 1 < \beta J < N \). In the super-radiant phase in the Dicke model, there is a symmetry breaking in the
N \to \infty$ limit, but the symmetry breaking is restored at a finite $N$ by the quantum tunneling process. So it has a finite gap $\sim \omega$ at a given parity sector in the strong coupling $g/g_c \gg 1$ limit, the quantum Lyapunov exponent $\lambda_\perp = 0$ in the low temperature range $1 < \beta \omega < N$. However, we expect $\lambda_\perp = g[(1-\beta) + \cdots]$ when $T \gg \omega$ and reach the maximum at the $Z_2$ limit $\beta = 1$ and vanish when $\beta < \beta_{c1}$. (e) Various hybrid SYK model also hold various CITs from the quantum integrable side of the $q = 2$ SYK to the quantum chaotic side of the $q = 4$ SYK tuned by the ratio of the couplings of the two sides. We expect the general statement on a KAM theorem in Sec.III still apply to the hybrid SYK models. Some preliminary results on the scaling forms of a quantum KAM in the hybrid SYK contexts are presented in. However, due to the quenched disorders, constructing a rigorous quantum KAM to characterize the CIT in the hybrid SYK models maybe more challenging, but important to pursue.

VII. CONCLUSIONS

In this work, we take a $2 \times 2$ matrix spanned by two typical NN bulk energy levels to find the quantum KAM scaling Eq. This is justified, because all the energy levels in the integrable side are un-correlated. In principle, one may also take all the bulk energy levels. However, as shown in and reviewed in the appendix B, by considering a $L \times L = 2 \times 2$ matrices, Wigner derived a simple approximate expression for the distribution function $P(s)$ of the NN spacing for GOE, GUE and GSE respectively. Despite this so called Wigner surmise was achieved only for $2 \times 2$ matrices, it is in very good agreement with the exact large-$L$ expressions. We expect the similar thing happen here: the exact $L \to \infty$ calculation will only slightly modify the pre-factor in Eq. The qualitative agreement between our analytical KAM scaling Eq. with the ED data in the RMT supports this claim. It is important to extend the concepts and methods developed here for interacting bosonic systems to interacting fermionic systems with quenched disorders such as the hybrid SYK models.

It remains outstanding to construct a quantum KAM theorem through the Lyapunov exponent or any other criterion characterizing the quantum chaos in the context of both Dicke and SYK models. It is also interesting to establish some quantum-classical correspondence between the classical KAM and the quantum KAM in the context of driven non-equilibrium Dicke or driven non-equilibrium SYK models.

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Appendix A: The eigen-energy and eigen-functions in the $1/N$ expansion.

Let us start with Eq.M5. Obviously $[H_{U(1)}, \delta \rho_+] = 0$, the $\delta \rho_+$ is a conserved quantity, so one can find the simultaneous eigenstates of $H_{U(1)}$ and $\delta \rho_+$ which will be achieved in the following.

As explained in Sec.M2, the $\theta_-$ is very massive, after pinning $\theta_-$ around $\theta_- \sim 0$, one can approximate $\sin^2 \theta_- \sim \theta_-^2$, therefore one may ignore the Berry phase in the $\theta_-$ sector, then the above equation can be simplified to:

$$H_{U(1)} = \frac{D}{2} (\delta \rho_+ - \alpha)^2 + D_- [\delta \rho_+ + \gamma (\delta \rho_+ - \alpha)]^2 + 2 \omega_\alpha \lambda^2 \frac{2}{1 + \beta} (\theta_-)^2 \quad (A1)$$

whose wavefunctions can be written as $|m\rangle l_m$ where

$$\langle \theta_+ | m \rangle = \frac{1}{\sqrt{2 \pi}} e^{im\theta_+}, \langle \theta_- | m \rangle = e^{i(m-\alpha)\theta_-} \psi_l(\theta_-) \quad (A2)$$

where the $\psi_l(\theta_-)$ is just the $l$-th wavefunction of a harmonic oscillator. So the total wavefunction $\psi_{l,m}(\theta_+, \theta_-) = \langle \theta_+ | \theta_- | m \rangle |l\rangle m$ is

$$\psi_{l,m}(\theta_+, \theta_-) = \frac{1}{\sqrt{2 \pi}} e^{i(m\theta_+ + \gamma(m-\alpha)\theta_-)} \psi_l(\theta_-) \quad (A3)$$

where the wavefunction is only periodic in $0 < \theta_+ < 2\pi$, the $-\infty < \theta_- < \infty$ is treated as a continuous variable. The corresponding eigen-energy is:

$$E(l, m) = (l + 1/2) \hbar \omega_\alpha D + \frac{D}{2} (m - \alpha)^2 \quad (A4)$$

where the $\omega_\alpha = E_H / \sqrt{1 + \beta}$ where $E_H^2 = (\omega_a + \omega_b)^2 + 4(g + g')^2 \lambda^2 / N$ is defined below Eq.M4.

It is important to observe that Eq.(A1) still contains $\beta$ dependence. Only setting $\beta = 0$ in Eq.(A1) recovers the $1/N$ expansion of the original Eq.M1. As stressed below Eq.M6, despite Eq.(A1) explicitly contains $\beta$ dependencies, they still keep the integrability, so do not change the ELS. So it is a good starting point to look at the effects of a chaotic perturbation.

Note that the Landau level index $l = 0, 1, \ldots, N$ ( $N + 1$ Landau levels ) denotes the high energy Higgs type of excitation, while the magnetic number $m = -P + l, -P + l + 1, \ldots$ ( no upper bounds ) denotes the low energy Goldstone types of excitations. The total parity is $I = (-1)^{P+m}$ at the sector $P$ where $P \geq l + 1$ has no upper bound either. At a given Landau level $l$ and a given sector...
agreement with the exact large-N expressions. It was achieved only for 2
GSE respectively. Despite this so called Wigner surmise
where
\[ r_n = \frac{s_n}{s_{n+1}} \]
which distributes around 1. This quantity has the advantage that it requires no un-
folding since ratios of consecutive level spacings are inde-
pendent of the local density of states.

By considering 3 × 3 matrices system, the authors in
obtained the Wigner-like surmises of the ratio of consecu-
tive level spacings distribution
\[ P_p(r) = \frac{1}{(1 + r)^2}, \quad P_w(r) = \frac{1}{Z_\beta} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1 + 3\beta/2}} \]
(B4)
where \( \beta = 1, 2, 4 \) and \( Z_\beta = 8/27, 4\pi/81\sqrt{3}, 4\pi/729\sqrt{3} \)
for GOE, GUE and GSE respectively. The distribution
\( P_w(r) \) has the same level repulsion at small r as \( P_W(s) \),
namely, \( P_w(r) \sim r^\beta \). However, the large r asymptotic
behavior \( P_W(r) \sim r^{-(2+\beta)} \) is dramatically different than
the fast exponential decay of \( P_W(s) \).

One may also compute the distribution of the loga-
ithmic ratio \( P_{\ln}(r) = P(r)\ln r \). Because \( P(\ln r) \) is
symmetric under \( r \leftrightarrow 1/r \), one may confine \( 0 < r < 1 \) and
double the probability density \( P(\tilde{r}) = 2P(r) \). Therefore,
the above two distributions have two different sets of expected
values of \( \bar{r} = \min\{r, 1/r\} \):
\[ \langle \tilde{r} \rangle_p = \int_0^1 2rP_p(r)dr = 2\ln 2 - 1 \approx 0.38629, \]
\[ \langle \tilde{r} \rangle_w = \int_0^1 2rP_{w,\beta=1,2,4}(r)dr \]
(B5)
which is \( 4 - 2\sqrt{3} \approx 0.53590, 2\sqrt{3}/\pi - 1/2 \approx 0.60266, \)
\( 32\sqrt{3}/(15\pi) - 1/2 \approx 0.67617 \) for GOE,GUE and GSE
respectively. These Wigner-like surmises Eq.(B4-B5) were
also shown to be very accurate when compared to nu-
merics and exact calculations in the exact large-N ex-
pressions.

In the main text, the CIT is from the GOE to the
Posssion, so only the two values \( \langle \tilde{r} \rangle_p \approx 0.38629 \) and
\( \langle \tilde{r} \rangle_{GOE} \approx 0.53590 \) are used and plotted in Fig.M2.

Appendix C: The high energy cutoff in the exact
diagonizations (ED).

We do the ED in Fig.M2 on the \( J-U(1)/Z_2 \) Dicke
model Eq.M1 in the Fock basis where the complete basis
is \( |n\rangle_{j,m}, n = 0, 1, 2, ..., \infty, j = N/2, m = -j, ..., j \)
where the \( n \) is the number of photons and the \( |j, m\rangle \) is the
Dicke states. In performing the ED, following Ref.5, one has
to use a truncated basis \( n = 0, 1, ..., n_c \) in the photon
sector where the \( n_c \sim 500 \rightarrow 2000 \gg N \) is the maximum
photon number in the artificially truncated Hilbert space.
The total number of states is \( n_c \times (2j+1) = n_c \times (N+1) \).

\[ P_p(s) = e^{-s} \]
This is also the size of the RMT $L = n_c \times (N + 1)$. The average many body energy level spacing at a given parity sector is $\frac{n_o}{\sum n_c \times (N + 1)} \sim \frac{\omega_0}{2(N + 1)}$. This qualitative estimate is consistent with Eq.M8 achieved by the systematic $1/N$ expansion. As long as the energy levels in Fig.M2 are well below $n_o \omega_0$, then the energy levels should be very close to the exact results without the truncation (namely, sending $n_c \to \infty$). However, the ED may not be precise anymore when $g$ gets too close to the upper cutoff.

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