The Exact Solution of Born-Infeld Theory
in Two Dimension

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Abstract

We obtain the exact operator solution of two-dimensional quantum Born-Infeld theory. This theory has a Lagrangian density non-polynomial in the fundamental fields. So this analysis might shed some light on the analysis of non-perturbative effects of field theories. We find the new exact soluble class of quantum field theories.

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1 Introduction

The Born-Infeld theory is recently investigated in the aspect of the string theory. However, in this paper, we analyze this theory in view of quantum field theory for point particles. It is necessary to analyze non-renormalizable field theories in detail, because non-renormalizability is one of the biggest difficulties of quantum field theories. For example, four dimensional Einstein gravitational theory is not renormalizable in the conventional sense.

The non-polynomiality is a common feature of the gravitational theories, which makes the analysis of the gravitational theory difficult.

The Lagrangian density of the Born-Infeld theory is non-polynomial in the fundamental fields, and it is not renormalizable in the sense of the naive power counting. Thus the analysis of the Born-Infeld theory might give us a key to investigate the above problems.

In this paper, we exactly solve two-dimensional Born-Infeld theory in the light-cone gauge. The method to solve it is not the conventional perturbative method, because in this theory, we cannot obtain the exact solutions by the usual perturbation based on the free field. The approach employed in our paper is based on the new method proposed in [4].

In [4], Abe and Nakanishi have proposed the new method to solve quantum field theory by the Heisenberg picture. The procedure is the following.

First we calculate equal-time commutation relations of the fundamental fields from the canonical commutation relations of canonical conjugate quantities. From equal-time commutation relations and the equations of motion, we set up the Cauchy problems for two-dimensional commutation relations. By solving the Cauchy problems with operator coefficients, we obtain the two-dimensional commutation relations of fundamental fields. Finally, we construct Wightman functions as they are compatible with multiple commutation relations for fundamental fields and energy positivity requirement.

We calculate all the exact multiple commutation relations and all n-point Wightman functions for the electromagnetic field.
2 The Solution

The action of the Born-Infeld theory in two dimension is written as

\[ S = \int d^2x \mathcal{L}, \]

\[ \mathcal{L} = \frac{1}{\lambda^2} \left[ -\sqrt{-\det(\eta_{\mu\nu} - \lambda F_{\mu\nu})} + \sqrt{-\det \eta_{\mu\nu}} \right] \]

\[ = \frac{1}{\lambda^2} \left[ -\sqrt{1 + \frac{\lambda^2}{2} F_{\mu\nu} F^{\mu\nu}} + 1 \right], \quad (1) \]

We rewrite the coupling constant as \( \lambda^2 = 2\kappa \) for simplicity. Then we can rewrite (1) as

\[ \mathcal{L} = \frac{1}{2\kappa} \left[ -\sqrt{1 + \kappa F_{\mu\nu} F^{\mu\nu}} + 1 \right]. \quad (2) \]

Expanding the Lagrangian density in power of \( \kappa \), we obtain the Maxwell theory at zeroth order:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \cdots. \quad (3) \]

However the Lagrangian density becomes the infinite series of the fundamental fields. The mass dimension of \( \kappa \) is \( [M]^{-1} \). Hence this theory is non-renormalizable in the naive sense.

In order to carry out canonical quantization, we fix the gauge of \( U(1) \) symmetry. We take the light-cone gauge: \( A^- = 0 \), where

\[ A^\pm = A_0 \pm A_1. \quad (4) \]

Then (4) is written as

\[ \mathcal{L} = \frac{1}{2\kappa} \left[ -\sqrt{1 - 2\kappa(\partial_- A_+ - \partial_+ A_-)^2} + 1 \right] + B A_-, \quad (5) \]

where \( B \) is the Nakanishi-Lautrup field and \( x^\pm = x^0 \pm x^1 \). If we eliminate \( A_- \) explicitly, (5) becomes

\[ \mathcal{L} = \frac{1}{2\kappa} \left[ -\sqrt{1 - 2\kappa(\partial_- A_+)^2} + 1 \right]. \quad (6) \]

The equations of motion are derived from (6) as follows:

\[ \partial_- \left[ \frac{\partial_- A_+}{\sqrt{1 - 2\kappa(\partial_- A_+)^2}} \right] = 0, \quad (7) \]

\[ \partial_+ \left[ \frac{\partial_+ A_+}{\sqrt{1 - 2\kappa(\partial_- A_+)^2}} \right] = B, \quad (8) \]

\[ A_- = 0. \quad (9) \]
The canonical conjugate momentum of $A_+^-$ is
\[
\pi_{A_+} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_0 A_+)} = \frac{\partial_- A_+}{2\sqrt{1 - 2\kappa(\partial_- A_+)^2}}.
\] (10)

We calculate a solution which is
\[
\sqrt{1 - 2\kappa(\partial_- A_+)^2} \neq 0.
\] (11)

We solve the equations of motion. From (7), we can write
\[
\frac{\partial_- A_+}{\sqrt{1 - 2\kappa(\partial_- A_+)^2}} = f(x^+),
\] (12)
where $f(x^+)$ is a function depending only on $x^+$. Then (12) is rewritten as
\[
\partial_- A_+ = \frac{f(x^+)}{\sqrt{1 + 2\kappa f^2(x^+)}}.
\] (13)

So we can solve $A_+(x)$ as
\[
A_+(x) = \frac{f(x^+)}{\sqrt{1 + 2\kappa f(x^+)^2}} x^- + g(x^+),
\] (14)
where $g(x^+)$ is a function depending only on $x^+$. From (13), we obtain
\[
\sqrt{1 - 2\kappa(\partial_- A_+)^2} = \frac{1}{\sqrt{1 + 2\kappa f^2(x^+)}}.
\] (15)

$\kappa = \frac{\lambda^2}{2} > 0$. Then if $f(x^+)$ is hermitian, $\sqrt{1 - 2\kappa(\partial_- A_+)^2}$ in the action is not zero.

We can express $\pi_{A_+}$ in terms of $f(x^+)$ from (10) and (12):
\[
\pi_{A_+} = \frac{1}{2} f(x^+).
\] (16)

In order to quantize the theory, we set up the canonical commutation relations of the canonical quantities as follows:
\[
[\pi_{A_+}, A_+]_0 = -i\delta(x^1 - y^1),
\]
\[
[A_+, A_+]_0 = 0,
\]
\[
[\pi_{A_+}, \pi_{A_+}]_0 = 0,
\] (17)
where \([\cdot, \cdot]_0\) denotes the equal-time commutation relations at \(x^0 = y^0\). If we substitute (14) and (16) to (17), we obtain the equal-time commutation relations of \(f\) and \(g\). Since \(f\) and \(g\) depend on only \(x^+\), two dimensional commutation relations of \(f\) and \(g\) are calculated as follows:

\[
[f(x^+), f(y^+)] = [g(x^+), g(y^+)] = 0,
\]

\[
[f(x^+), g(y^+)] = -2i\delta(x^+ - y^+).
\] (18)

Since \(\partial_- f = \partial_- g = 0\), \(f\) and \(g\) are the currents which generate the residual gauge symmetries. From (18) and (14), we can derive the two-dimensional commutation relation of \(A_+\) as follows:

\[
[A_+(x), A_+(y)] = -2i \frac{1}{(1 + 2\kappa f(x))^2} (x^- - y^-) \delta(x^+ - y^+)
\]

\[
= -2i [1 - 2\kappa (\partial_- A_+(x))]^2 \frac{1}{2} (x^- - y^-) \delta(x^+ - y^+)
\]

\[
= -\frac{i}{\pi} [1 - 2\kappa (\partial_- A_+(x))]^2 \frac{4}{3} D(x - y),
\] (19)

where \(D(x)\) is defined as

\[
D(x) = 2\pi x^- \delta(x^+).
\] (20)

We obtain an infinite series when we expand (19) in power of the coupling constant \(\kappa\). We cannot determine the exact \(\kappa\) dependence by the usual perturbation theory. We derive multiple commutation relations of \(A_+\) recursively:

\[
[\cdots, [A_+(x_1), A_+(x_2)], \cdots, A_+(x_n)]
\]

\[
= (-2i)^{n-1} \left[ \left( \frac{d}{dz} \right)^{n-1} \left( \frac{z}{\sqrt{1 + 2\kappa z^2}} \right) \right]_{z=f(x_1)} (x_1^- - x_2^-)
\]

\[
\times \delta(x_1^+ - x_2^+) \delta(x_1^+ - x_3^+) \cdots \delta(x_1^+ - x_n^+),
\]

\[
= \left( -\frac{i}{\pi} \right)^{n-1} \left[ \left( \frac{d}{dz} \right)^{n-1} \left( \frac{z}{\sqrt{1 + 2\kappa z^2}} \right) \right]_{z=f(x_1)}
\]

\[
\times D(x_1 - x_2) \partial_{z_1} D(x_1 - x_3) \cdots \partial_{z_1} D(x_1 - x_n),
\] (21)

Next, we derive the Wightman functions of this theory. We set the vacuum expectation values of \(f\) and \(g\) as

\[
\langle f(x^+) \rangle = F(x^+),
\]

\[
\langle g(x^+) \rangle = G(x^+),
\] (22)
where \( F \) and \( G \) are some c-number functions. If these one-point functions are non-vanishing, Lorentz invariance is broken, but we dare to include nonzero expectation values to consider general situations. As is considered later, if \( F \) is a constant, the vacuum satisfies the subsidiary condition which define the physical space. Since

\[
[f(x), f(y)] = 0,
\]

we can trivially calculate \( n \)-point Wightman functions of \( f \). For example, the two-point Wightman function of \( f(x) \) is calculated as

\[
\langle f(x)f(y) \rangle = \langle f(x) \rangle \langle f(y) \rangle = F(x)F(y).
\] (24)

Using the \( n \)-point functions of \( f(x) \), we obtain the one-point function of \( A_+ \) as follows:

\[
\langle \partial_- A_+(x) \rangle = \frac{F(x)}{\sqrt{1 + 2\kappa F(x)^2}},
\]

\[
\langle A_+(x) \rangle = \frac{F(x)}{\sqrt{1 + 2\kappa F(x)^2}} x^- + G(x),
\] (25)

Since two \( \partial_- A_+ \)'s commute mutually as is seen from (13) and (23):

\[
[\partial_- A_+(x), \partial_- A_+(y)] = 0,
\] (26)

we obtain the two-point truncated Wightman function of \( \partial_- A_+ \) as

\[
\langle \partial_- A_+(x_1)\partial_- A_+(x_2) \rangle_T = 0.
\] (27)

Hence (19) and the energy positivity requirement lead the two-point Wightman function of \( A_+ \) to

\[
\langle A_+(x_1)A_+(x_2) \rangle_T = -\frac{1}{2\pi} \sum_{i=1}^{2} [1 - 2\kappa\langle (\partial_- A_+(x_i)) \rangle^2] \frac{1}{4} D^{(+)}(x_1 - x_2)
\]

\[
= -\frac{1}{2\pi} \sum_{i=1}^{2} \frac{1}{(1 + 2\kappa F(x_i)^2)^{\frac{3}{2}}} D^{(+)}(x_1 - x_2),
\] (28)

where

\[
D^{(+)}(x) = \frac{x^-}{x^+ - i0^+}.
\] (29)
From (21), we can calculate the \( n \)-point Wightman functions of \( A_+ \),

\[
\langle A_+(x_1) A_+(x_2) \cdots A_+(x_n) \rangle_T = \frac{1}{n(n-2)!} \sum_{P(i_1, \ldots, i_n)} \left[ \left( \frac{d}{dZ} \right)^{n-1} \left( \frac{Z}{\sqrt{1+2\kappa Z^2}} \right) \right]_{Z=F(x_{i_1})} \times \left( -\frac{1}{\pi} \right)^n D^{(+)}_<(x_{i_1} - x_{i_2}) \partial^{x_1} D^{(+)}_<(x_{i_1} - x_{i_3}) \cdots \partial^{x_{i_2}} D^{(+)}_<(x_{i_1} - x_{i_n}), \tag{30}\]

where

\[
D^{(+)}_<(x_i - x_j) = \begin{cases} D^{(+)}(x_i - x_j), & \text{if } i < j \\ D^{(+)}(x_j - x_i), & \text{if } i > j \end{cases} \tag{31}\]

and \( P(i_1, \ldots, i_n) \) is a permutation of \((1, \ldots, n)\).

The exact Wightman functions break the equations of motion in some theories. Therefore we discuss the consistency with the above solution (30) and the equations of motion.

We find that if a truncated Wightman function includes \( \partial_- x_k A_+(x_k) \), it does not depend on \( x_k^- \) from (30). And since \( \partial_- A_+ \) commute mutually in two dimension and the truncated \( n \)-point functions of \( \partial_- A_+ \) are zero, the Wightman functions which include any functional of \( \partial_- A_+ \) are non-singular at the same spacetime point. Thus

\[
\left\langle \frac{\partial_- x_1 A_+(x_1)}{\sqrt{1-2\kappa(\partial_- x_1 A_+(x_1))^2}} A_+(x_2) \cdots A_+(x_n) \right\rangle, \tag{32}\]

does not depend on \( x_1^- \). Therefore

\[
\left\langle \partial_- x_1 \left[ \frac{\partial_- x_1 A_+(x_1)}{\sqrt{1-2\kappa(\partial_- x_1 A_+(x_1))^2}} \right] A_+(x_2) \cdots A_+(x_n) \right\rangle = 0, \tag{33}\]

and we can confirm the Wightman functions is consistent with the equation of motion.

We can set up the subsidiary condition to select the physical Fock space as follows:

\[
B^{(+)}(x)|_{ \text{phys} } = 0, \tag{34}\]

where \((+)\) is the positive frequency part of the field \( B \). From the equations of motion, we obtain the following relation:

\[
B(x) = \partial_+ f^+(x^+), \tag{35}\]

so (34) is equivalent to the following one,

\[
\partial_+ f^+(x^+)|_{ \text{phys} } = 0. \tag{36}\]
3 Conclusion and Discussion

We have exactly solved the quantum Born-Infeld theory in two dimension in the light-cone gauge. The usual perturbation method based on the free field does not work. We calculated the exact multiple commutation relations and $n$-point Wightman functions by the non-perturbative method. This method for solving quantum theory will be useful to treat non-perturbative effects of quantum field theories.

The generalization to the non-abelian Born-Infeld theory is straightforward.

The Schwinger model, which is exactly soluble, is an important two-dimensional field theory. So it may be interesting to analyze the Born-Infeld theory coupled with fermion matters.

We can generalize the Lagrangian density (3) to

$$\mathcal{L} = \Phi(F_{\mu\nu}F^{\mu\nu}),$$

where $\Phi(x)$ is a function. If $\Phi(x)$ is satisfied with a certain condition, the theory is also soluble. We will analyze the above generalized theory in detail at the next paper.

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