Self Organized Critical Dynamics of a Directed Bond Percolation Model

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Abstract

We study roughening interfaces with a constant slope that become self organized critical by a rule that is similar to that of invasion percolation. The transient and critical dynamical exponents show Galilean invariance. The activity along the interface exhibits non-trivial power law correlations in both space and time. The probability distribution of the activity pattern follows an algebraic relation.

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1 Introduction

Self organized critical systems a la invasion percolation and interface dynamics have generated good deal of interest in recent times [1 - 7]. Most of these systems show self affine or self similar structures. They show interesting time time power law correlation. The critical exponents of several of these systems were studied in the said references.

We study a model of roughening interface proposed by Barma [8] to investigate the dynamical exponents at the transient and saturation regions of interface widths. The exponents for time time correlation of interface width, and the time height correlations are found. The time evolution of the activity center, the point at which the movement begins, is determined.

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Unlike most other cases, there is no 'power law' and the distribution of 'active center movement' with time follows an algebraic relation, a linear relation with a negative slope.

The model is defined on a tilted square lattice where each bond (k) on the lattice is assigned a quenched, uncorrelated random number \( f_k \) drawn from the interval \([0, 1]\). In the one dimensional version, a discrete interface \( h(x) \) is defined on a chain \( x = 1, 2, \ldots, L \), \( L \) being the system size. We use cylindrical boundary condition. The chain is updated by finding the smallest \( f_k \) on the chosen interface. In order to preserve the length of the interface, the directed walk character is maintained by local re-adjustments. If the chosen minimal bond has a positive (negative) slope, the sequence of links with negative (positive) slope just below and on the left, also advance as shown in figure 1.

The interface dynamics is mapped onto a system of hard core particles on a ring. A positive slope link of the interface is represented by a particle \( n_k = 1 \), and a negative slope link by a hole \( n_k = 0 \), see figure 1. The difference in height of the interface between sites \( J_1 \) and \( J_2 \) is:

\[
h_{j_2} - h_{j_1} = \sum_{j=j_1}^{j_2} (2n_j - 1) \tag{1}
\]

Each site \( k \) on the ring carries a random number \( f_k \) assigned to the bond in the actual lattice. In each time step activity is initiated at a site with a minimum \( f_k \). If this site contains a particle (hole) it exchanges place with the first hole (particle) on the left (right). A new set of \( f_k \)s

Figure 1: Interface profile for the present model on a tilted lattice
Figure 2: Variation of width $W(L,t)$ with size $L$ for different time step
Figure 3: Variation of width $W(L,t)$ with size $L$ for different time step

are assigned to the particle and hole that are exchanged. All sites hopped over in this process are also assigned new $f_k$s. The overall particle density $\rho$ is conserved throughout the process. $\rho$ determines the mean slope of the interface $m = 2\rho - 1$. Here progress of the interface is in the forward direction only.

2 Results and Discussion

Sneppen [9] investigated numerically the power law dependence of interface width with time $t$ and system size $L$. We studied the interface width or the time time correlation of height defined in terms of the standard deviation

$$W(L, t) = < (|h(x, t + \tau) - h(x, \tau)| - < h(x, t + \tau) - h(x, \tau) >)^2 >^{\frac{1}{2}}$$ (2)
where $\tau$ determines the various starting times. The average $< >$ is done over $x \in [1, L]$ and members of the simulation ensemble. We plot in figure 2 the temporal growth behaviour of the interface width, the transient growth and the saturation region, for a surface starting from a flat interface ($m = 0$). The saturation is identified in terms of saturation of slope. The model shows different scaling behaviour in these two regions.

In table 1 we present the values of the transient and saturation exponents of different lengths $L$. In the transient region we found the power law behaviour

$$W(L, t) \sim t^{\beta_{\text{trans}}} \quad \text{with} \quad \beta_{\text{trans}} = .4341$$

(3)

In the saturation region we get

$$W(L, t) \sim t^{\beta_{\text{sat}}} \quad \text{with} \quad \beta_{\text{sat}} = .3485$$

(4)
The temporal behaviour of time time correlation of height in terms of the range is also studied in the literature

\[ H(L, t) = \langle |\max_x (h(x, t+\tau) - h(x, \tau)) - |\min_x (h(x, t+\tau) - h(x, \tau))| \rangle > \tau \tag{5} \]

The average \( \langle \rangle \) is done over \( x \in [1, L] \) and members of the simulation ensemble. However this model and its algorithm dictates its value to be almost always a constant, namely 2.

A proof goes like this: \( h(x, t+\tau) - h(x, \tau) \) has integer values 0 and 2, or in multiples of 2 if sufficient time steps are allowed. However the maximum over a long string is almost always 2 (or its multiple if time is large), the minimum is 0. Hence \( H(L, t) \) is almost always 2 (or a multiple thereof if we are looking at large time differences).

At any time \( t \) in the development of the interface the width \( w = \langle (h - < h >)^2 \rangle^{1/2} \) of a saturated interface scales with the system size

\[ w(L, t) \sim L^{\alpha} \tag{6} \]

where \( \alpha \) is the roughness exponent. This scaling is interpretable as a self organization of interface towards a critical attractor. In this model the attractor is a string of sites having high value of \( f_k \)s. The important feature is that the temporal drift of the interface towards the saturation state required no fine tuning of any parameter. Thus this model may be viewed as an example of self organized criticality in one dimension.

The variation of \( w \) with length \( L \) in log-log scale for several time steps is found, a representative part of which is reported in figure 3. Table 2 shows the relevant results for \( \alpha \) for two time steps. Mean \( \alpha \) was found to be .5 to the first place of decimal. This is in excellent agreement with the standard Kardar-Parisi-Zhang scaling \( \alpha = 1/2 \). We also find that the interface motion maintains the Galilean invariance \( \alpha + \alpha/\beta_{sat} = 1.92 \sim 2 \).

Since our model is a one dimensional interface with no overhangs, the study of the distribution of activity along this interface is simple. Activity is described in terms of events on a string. An event begins by finding the minimum probability \( f_k \) on an interface. Let \( x(\tau) \) and \( x(t+\tau) \) denote the positions of the events on the interface at time \( \tau \) and \( (t+\tau) \) respectively. We calculate the probability distribution function \( P(X_t) \) in the variable \( X_t = |x(\tau) - x(t+\tau)| \). In figure 4 this distribution function is shown for different values of \( X_t \). The probability density depends algebraically on \( X_t \). This is in distinction with power law dependence found in other cases, e.g., Sneppen and Jensen [10].

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Tables

Table 1. Transient and saturation exponents of time time correlation of width at different length scales.

| Length | $\beta_{\text{trans}}$ | $\beta_{\text{sat}}$ | $< \beta >$ |
|--------|----------------------|----------------------|-------------|
|        | $\tau = 30; \ t = 120$ | $\tau = 300; \ t = 100$ |             |
| 100    | 0.348509412823739     | 0.289099642351271     |             |
| 200    | 0.395537573475355     | 0.320416123901019     |             |
| 300    | 0.417463014858253     | 0.28759592598634      |             |
| 400    | 0.432676615665537     | 0.361634369182171     | $< \beta_{\text{trans}} > = 0.434103$ |
| 500    | 0.465140826654886     | 0.369840157779528     | $< \beta_{\text{sat}}> = 0.348462$ |
| 600    | 0.447692227261676     | 0.342850618167307     |             |
| 700    | 0.44819146054204      | 0.35239206283316      |             |
| 800    | 0.432095028834515     | 0.335543843346805     |             |
| 900    | 0.446593525738461     | 0.43783326989437      |             |
| 1000   | 0.507132066113331     | 0.387414162344172     |             |
Table 2. Roughness exponent of interface width at two saturation times.
3000 simulation steps
100 - 1000 string length

| Time | $\alpha$  | $<\alpha>$ |
|------|-----------|----------|
| 200  | 0.495378  | 0.494865 |
| 250  | 0.494351  |          |

Figures

- Fig.1. Interface profile for the present model on a tilted square lattice
- Fig.2. Temporal growth behaviour of width $W(L,t)$ for different sizes
- Fig.3. Variation of width $W(L,t)$ with size $L$ for different time step
- Fig.4. Probability distribution of active center displacement