Spectral Characterization by the Koopman Linearly-Time-Invariant Analysis: Constitutive Fluid-Structure Correspondence

Cruz Y. Li\(^1\) (李雨桐), Zengshun Chen\(^1\)* (陈增顺), Tim K.T. Tse\(^3\)** (谢锦添), Asiri Umenga Weerasuriya\(^4\), Xuelin Zhang\(^5\) (张雪琳), Yunfei Fu\(^6\) (付云飞), Xisheng Lin\(^7\) (蔺习升)

\(^1\) Department of Civil Engineering, Chongqing University, Chongqing, China
\(^2,3,4,6,7\) Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Hong Kong SAR, China
\(^5\) School of Atmospheric Sciences, Sun Yat-sen University, Zhuhai, China.

\(^1\) yliht@connect.ust.hk; ORCID 0000-0002-9527-4674
\(^2\) zchenba@connect.ust.hk; ORCID 0000-0001-5916-1165
\(^3\) timkttse@ust.hk; ORCID 0000-0002-9678-1037
\(^4\) asiriuw@connect.ust.hk; ORCID 0000-0001-8543-5449
\(^5\) zhangxlin25@mail.sysu.edu.cn; ORCID 0000-0003-3941-4596
\(^6\) yfuar@connect.ust.hk; ORCID 0000-0003-4225-081X
\(^7\) xlinbl@connect.ust.hk; ORCID 0000-0002-1644-8796

* Co-first author with equal contribution.

** Corresponding author

All correspondence is directed to Dr. Tim K.T. Tse.
Abstract

This work delineates a new architecture of the Koopman operator theory called the Koopman Linearly-Time-Invariant (Koopman-LTI) analysis. The Koopman-LTI is formulated to explain fluid-structure interactions by providing a constitutive relationship between fluid excitation and structural pressure response. Results from a demonstrative rendering on a prototypical fluid-structure system, the subcritical prism wake, showed that the Koopman-LTI globally linearizes and approximates the original flow into a linear superposition of temporally orthogonal Ritz descriptors. The approximation proved close to exact and perennially stable. The subsequent analysis of 18 Koopman-LTI systems also alluded to the a posteriori existence of a configuration-wise universal Koopman system. Spectral characterization of the system suggested the complex morphology of the prism wake during the shear layer transition II consists of only six dominant and constitutively interactive excitation-response mechanisms. Merely two excitations at $St=0.1242$ and $0.0497$ primarily dictate the behaviours of the upstream and crosswind walls, and four other collectively distinct mechanisms at $St=0.0683$, $0.1739$, $0.1925$, and $0.2422$ overshadow the downstream wall. The success with a free-shear configuration and inhomogeneous and anisotropic turbulence attests to the applicability of the Koopman-LTI to a range of fluid-structure systems. The Koopman-LTI also stands as a new system identification method of vast practical significance because, through it, one can pinpoint the exact fluid origin of a specific pressure pattern of structural response, or vice versa. It also illuminates one visionary possibility to reduce the intricacies of fluid-structure interaction into constitutive correspondences that are identifiable by inspection—making the complicated matter ever so straightforward.
1. Introduction

Fluid-structure interaction (FSI) is omnipresent in nearly every aspect of life. However, our current understanding of FSI is far from complete. Vast gaps in knowledge await the brilliant sparks of science. The gravest difficulty is the volatility of fluids. Fluids are highly complex, nonlinear, and extremely sensitive to even the slightest perturbations at sufficiently high speeds (Kundu, 2004; Pope, 2000). Therefore, even when men have possessed the deterministic equations of fluid motion, the Navier-Stokes equations, the motion of fluids, especially with turbulence, is still considered stochastic, unpredictable, and one of the greatest quandaries of science. Until the solution of the Navier-Stokes, the analytical path to tackling FSI is destined to be drenched with thorns.

Fortunately, recent advancements in computer science illuminated an alternative route: the data-driven solution of fluid problems (Lusch et al., 2018; Mezić, 2005; Raissi et al., 2019; Ren et al., 2019). To this end, the Koopman analysis emerges as a brilliant method to overcome the aforementioned difficulties (Budišić et al., 2012; Mezić, 2013; Rowley et al., 2009). In essence, the Koopman theory assumes an infinite-dimensional Koopman operator that globally linearizes the dynamics of a system. For a fluid system, the Koopman operator is quintessentially an embodiment of the Navier-Stokes equations in the infinite-dimensional Hilbert space.

However, although many works have applied the Koopman analysis, or its algorithmic subordinates, the dynamic variants of the Dynamic Mode Decomposition (DMD) or the Koopman Mode Decomposition (KMD), to various fluid-structure systems (Carlsson et al., 2014; Kutz et al., 2016; Muld et al., 2012a, 2012b; Rowley & Dawson, 2017; Schmid, 2010), the foci on their interactive mechanisms are only marginal. To the best of the authors’ knowledge, none has specifically targeted establishing a constitutive relationship between fluid excitation and structural response. The constitutive fluid-structure inter-relationship is of tremendous value. With it, one may pinpoint the exact fluid phenomenon that triggers a specific pattern of structural response or vice versa. It, too, is the indispensable core of physics for any interpolation, extrapolation, and prediction of fluid-structure systems, regardless of the technological finesse. Finally, its implications extend to essentially every scientific discipline. Examples include mitigation measures for flow-induced acoustics or vibrations, resilience reinforcement against natural hazards, aeronautics and space engineering, and medical science concerning bodily fluid.
We propose in this article a novel time-invariance architecture of the Koopman analysis, the Koopman Linearly-Time-Invariant (Koopman-LTI) analysis, to establish the constitutive relationship between fluid excitation and structural response through spectral characterization. The novelty of the Koopman-LTI is clarified: it is not another newly invented decomposition or algorithm——it stands on the shoulders of the mathematical giants who have made the Koopman operator theory and the DMD possible (Budišić et al., 2012; Kutz et al., 2016; Mauroy & Mezić, 2013; Mezić, 2005; Rowley et al., 2009; Schmid, 2010). Instead, it appeals to a deliberate rendering, a unique interpretation, of the Koopman theory, which yields revealing insights into fluid-structure interactions. The present work demonstrates the Koopman-LTI, its pragmatic notion of time-invariance, an a posteriori allusion of configuration-wise universality, and the fluid-structure correspondence through a canonical prism wake in the subcritical regime. The DMD is also deployed herein as one of many substitutable algorithms to approximate the Koopman eigen tuples.

The compositional sequence can be summarised as follows. We first proposed and delineated the Koopman-LTI architecture in section 2, in which the theoretical possibilities of a configuration-wise universal Koopman operator and time invariance are established. After, we employed the prototypical prism wake to demonstrate in section 3 the pragmatic notion of time invariance and details about the turbulent flow. Subsequently, we analysed the intra- and inter-group dynamics of 18 flow field and structural observables before pinpointing the constitutive fluid-structure correspondence in section 4. Finally, we summarised the findings and forecasted the upcoming work in section 5.

2. The Koopman Linearly-Time-Invariant analysis

2.1 The Koopman operator theory

In 1932, the works of B. O. Koopman (1931; 1932) cradled the birth of the Koopman operator theory. They outlined the possibility of representing a nonlinear dynamical system in terms of an infinite-dimensional linear operator, which acts on a Hilbert space of measurement functions of the system’s state (Brunton, 2019). Nevertheless, it was not until much later that the mathematical wonder was administered an injection of practicality. The works of Mezić (2005) and several others (Budišić et al., 2012; Froyland et al., 2014; Mauroy & Mezić, 2013; Rowley et al., 2009; Schmid, 2010; Tu et al., 2014) formulated data-driven pathways to acquire tangible
approximations of the Koopman operator on finite-dimensional subspaces---the practical realization of the promising theory.

Following Mezić (2005) and Rowley et al. (2009), one may consider a dynamical system in discrete time,

\[ y_{i+1} = f(y_i), \]  
\[ i \in \mathbb{Z} \]  

where \( i \in \mathbb{Z} \) and \( f \) is a map from a manifold \( M \) to itself. The Koopman operator \( U \) is linear, infinite-dimensional, and acts on scalar-valued functions on \( M \). For any scalar-valued function \( g: M \to \mathbb{R} \), \( U \) maps \( g \) into a new function (Rowley et al., 2009),

\[ Ug(y) = g(f(y)). \]  

The acquisition of \( U \) signals one’s possession of all the information contained in a dynamical system. For a linear system, \( U \) is exact. For a nonlinear system, it is a global and best-fit approximation of the nonlinear dynamics. Conceptually, one may think of \( U \) as a discretization of the underlying functions. Given its infinite-dimensionality, the discretization is infinitesimally fine, hence the approximation infinitely close to exact.

One may imagine, if there are \( k \) dynamical systems, or \( k \) realizations of a dynamical system, then essentially \( k \) sets of

\[ y_{i+1,k} = f(y_{i,k}) \]  

exist with the corresponding Koopman operators

\[ U_k g(y_k) = g(f(y_k)), \]  

where \( k \in \mathbb{Z}^{0+} \). Hypothetically, if the dynamics or spatiotemporal content of the \( k \) systems are identical, then a universal Koopman operator \( \mathcal{U} \) shall exist. If the dynamics of the \( k \) realizations (ensembles) of a system are identical, \( \mathcal{U} \) is then configuration-wise universal. For a stochastic process, in which the dynamics are random but not chaotic, and often governed by a deterministic set of equations, one may expect an infinite-dimensional operator \( \mathcal{U} \) that tends to \( \mathcal{U} \) as time approaches infinity.
\[ \lim_{t \to \infty} \mathcal{U}(t) = \mathcal{U}. \]  

(2.5)

For fluid systems, \( \mathcal{U} \) is quintessentially an implicit, linearized representation of the equations of motion (EOM) of fluids, or the Navier-Stokes equations given the validity of the continuum hypothesis.

So far, the logic is intuitive. Now, we invite the readers to a stretch of the intellect. What if the system(s) for which \( \mathcal{U} \) exists is a fluid-structure system, that is, it contains the simultaneous motions of fluids and structures? Analytically, it is cumbersome to combine the EOMs of the structures with the Navier-Stokes, particularly when the system state becomes large and complex. Furthermore, acquiring the solution for the fluid-structure integrated EOM is a task of Hercules because the fluid portion has yet to find its analytical answers.

This is where the Koopman analysis becomes extremely promising. Its elegance exudes precisely as it produces linearized dynamics of a system with stellar accuracy while completely bypassing the hindrance of the governing equations. This ability has been proven by a class of nonlinear systems in, for example, turbulence (Mezić, 2013; Rowley & Dawson, 2017; Schmid, 2010), neuroscience (Kutz et al., 2016), epidemiology (Kutz et al., 2016), highway traffic (Avila & Mezić, 2020), and even the financial market (Mann & Kutz, 2016). By no exaggeration, our knowledge about some of these systems is so limited, that the problem no longer rests on whether solutions are possible, but the frail belief that some underlying governing equations exist at all. To this end, the data-driven Koopman analysis provides a practical answer.

As we will shortly show with empirical results, evidence alludes to the existence of a configuration-wise universal \( \mathcal{U} \). In this case, \( \mathcal{U} \) is an implicit representation of the EOM-Navier-Stokes integrated equations and contains all the interactive mechanisms of the fluid-structure system.

### 2.2 Data-driven approximation of the Koopman operator

Acquiring the full-order Koopman operator is only a theoretical possibility. Practically, any measurement of a continuous system inevitably reduces its dimension from infinite to finite.
both spatially and temporally. Due to this inherent reduction of order, the finite approximation of $\mathcal{U}$ is possible by the data-driven approach.

As pointed out by Williams et al. (2015), several algorithms exist for computing the finite-dimensional Koopman approximation and its eigen tuples (i.e., eigenfunctions, eigenvalues, Koopman modes), for example, the generalized Laplace analysis (GLA) (Mauroy & Mezić, 2012, 2013, 2016), the Ulam Galerkin method (Bollt & Santitissadeekorn, 2013; Froyland et al., 2014), and the Dynamic Mode Decomposition (DMD) (Rowley et al., 2009; Schmid, 2010). The GLA, with a prescription of eigenvalues, approximates the Koopman modes and eigenfunctions (Budišić et al., 2012; Mauroy & Mezić, 2013). The Ulam Galerkin does so for the eigenfunctions and eigenvalues in the approximation of the Perron-Frobenius operator, which is the adjoint Koopman operator (Froyland et al., 2014). The DMD also approximates the eigenvalues and Koopman modes with algorithmic simplicity and robustness (Tu et al., 2014; Williams et al., 2015).

For this reason, the DMD is one of the most used algorithms to approximate the Koopman modes. The DMD also has several variants, for example, the Arnoldi-based formulation, also known as the Koopman Mode Decomposition (Rowley et al., 2009), the companion-matrix formulation (Schmid, 2010), and similarity-matrix formulation (Tu et al., 2014). The present work selects the similarity-matrix formulation because of its tractability with high-dimensional data (Kutz et al., 2016). The procedure for computing the so-called exact DMD modes is equivalent to an algorithm for finding the Koopman eigen tuples with linearly independent data snapshots (Tu et al., 2014). Given that most fluid applications uphold this condition, the DMD suits the analytical agenda of the present work.

2.3 The Similarity-Expression Dynamic Mode Decomposition

To preserve literary concision, we retain only the quintessence of the similarity-expression DMD in the upcoming presentation. Readers may refer to Appendix I for details and Tu et al. (2014) for derivations.

The key is to assume a mapping matrix $A$ that connects two time-shifted snapshot sequences, $X_1$ and $X_2$, of a particular variable of interest or observable. $X_1$ and $X_2$ have the spatial dimension $n$ (rows) and temporal dimension $m$ (columns),
\[ X_2 = AX_j. \] (2.6)

Evidently, \( A \) is an unknown matrix that mimics the map \( f \), hence that of the Koopman operator \( U \). Intuitively, the accuracy of \( A \) increases with the dimensionality, so too is the computational expense of \( A \).

By a Singular-Value-Decomposition (SVD)-based algorithm, one arrives at a similar-matrix \( \tilde{A} \) that replaces \( A \) in equation (2.6). \( \tilde{A} \) is the data-driven, reduced-order, and globally optimal approximation of \( A \), hence the Koopman operator \( U \). An attractive feature of this approach is that the deduction of \( \tilde{A} \) is exclusively implicit, meaning it does not require explicit knowledge of a system’s underlying dynamics, which is particularly useful for fluid analysis.

Concomitantly, an equivalent data-driven approximation \( \tilde{\mathcal{A}} \) for \( \mathcal{U} \) or \( \tilde{\mathcal{U}} \) also exists. The \textit{a posteriori} observations deduced from the fluid-structure system herein allude to the existence of \( \tilde{\mathcal{A}} \). Nonetheless, the \textit{a priori} deduction remains an area for exploration.

The arrival at \( \tilde{A} \) signals one’s tangible possession of all the spatiotemporal content. What can be done to the \( \tilde{A} \) matrix is essentially limited only by the borders of linear algebra. Nevertheless, this work adopts the default procedure of the DMD, which characterizes the modal features of \( \tilde{A} \) by an eigendecomposition.

An eigendecomposition by the Ritz method yields

\[ \tilde{A}W = WA, \] (2.7)

where \( W \) contains the eigenvectors (Ritz vectors) \( w_j \), and \( A \) contains the corresponding discrete-time eigenvalues (Ritz values) \( \lambda_j \).

The eigen tuples yield the \textit{exact} DMD modes (Tu et al., 2014) or the Koopman modes (Rowley et al., 2009) as

\[ \Phi = X_2 V \Sigma^{-1} W, \] (2.8)

where \( \Phi \) contains the Koopman/DMD mode \( \phi_j \), \( \Sigma \) and \( V \) are outcomes of the SVD and contains the singular values \( \sigma_j \) and temporally orthogonal modes \( v_j \), respectively.

Every mode \( \phi_j \) corresponds to a physical frequency \( \omega_j \) in continuous time.
\begin{equation}
\omega_j = \Im \{ \log(\lambda_j) \}/t^*,
\end{equation}

and a growth/decay rate \( g_j \)

\begin{equation}
g_j = \Re \{ \log(\lambda_j) \}/t^*,
\end{equation}

where \( t^* \) is the uniform step of shift between \( X_1 \) and \( X_2 \).

\section*{2.4 Linear-Time-Invariance}

\subsection*{2.4.1 Linearity}

Following the preceding procedure, one shall arrive at a Koopman system that constitutes a set of linear descriptors,

\begin{equation}
\mathbf{x}_{\text{Koopman},i} = \sum_{j=1}^{r} \phi_j \exp(\omega_j t_i) \alpha_j,
\end{equation}

which together offers the Koopman approximation of the input data at instant \( i \). \( r \) denotes the truncation order of \( \hat{A} \) and \( \alpha_j \) denotes the coefficient of weight, or the modal amplitude.

Instead of the conventional, evolution-wise static Koopman/DMD modes, we define an evolutionary mode called the Ritz descriptors

\begin{equation}
\mathbf{M}_j = \phi_j \exp(\omega_j t) \alpha_j,
\end{equation}

in which

\begin{equation}
t = \{ t^*, 2t^*, 3t^*, \ldots, ht^* \},
\end{equation}

where \( h \) is the temporal dimensions of \( X_1 \).

\( \mathbf{M}_j \), while each contains a fragment of the total information that has been harmonically averaged over a prescribed frequency span (Rowley et al., 2009), together describe the globally linearized dynamics of the input system. As pointed out by Towne et al. (2018), \( \phi_j \) of the DMD are not spatially orthogonal. Therefore, \( \phi_j \) and \( \mathbf{M}_j \) of a particular flow realization may slightly differ from those of another. One can achieve the orthogonality by a weighted ensemble
averaging, or by an optimally averaging with respect to the second-order space-time flow statistics. The resultant linear combination of Koopman/DMD modes is known as the Spectral Proper Orthogonal Decomposition (SPOD). Nevertheless, for this rendering, the most important outcome is the sets of temporally orthogonal Ritz descriptors that optimally approximate the nonlinear dynamics of the input system, thus justifying the linearity in LTI. The SPOD can be considered as another algorithm to approximate the Koopman mode, through which spatiotemporal orthogonality is obtained.

### 2.4.2 Time-Invariance: Statistical Stationarity

On top of linearity, time-invariance is the next critical aspect. Perhaps some readers already see the resemblance between the Koopman analysis and the Fourier, or in a more general sense, the Laplace transform. The snapshot formulation appeals more to their discrete counterparts, the Discrete Fourier Transform (DTC) and the Z-transform, respectively. Chen et al. (2012) have formally demonstrated that the Koopman/DMD modes are equivalent to the DFT modes with zero-mean data. Any incongruence between the Koopman/DMD and DFT modes is attributed to the artifact of the approximation of the Koopman operator (Rowley et al., 2009). Therefore, data curation often recommends mean-subtraction to ensure the incongruence is as small as possible (Towne et al., 2018).

Moreover, statistical stationarity is indispensable. Although the Koopman analysis does not limit its scope to stationary flows, as one can conceptually imagine, the sweep of the Z-transform and its subsequent labelling of the frequency content into sinusoids and exponentials,

\[ Z[x[k]] = \sum_{k=0}^{\infty} x[k]z^k, \quad (2.14) \]

where \( k \in \mathbb{Z}^+ \) yields a unilateral Z-transform and \( z=be^{j\theta} \), are more suitable for oscillatory dynamics. Therefore, stationary data substantially elevates the stability of Koopman systems.

Statistical stationarity is also critical to characterizing the predominant contributors of a flow. As pinpointed by Williams et al. (2015), on a particular subspace of the Koopman operator, one may capture the long-term dynamics of an input observable in a span of eigenfunctions associated with eigenvalues near the unit circle in discrete time. The span is known as the slow subspace. The opposite, as the fast subspace, captures the transient dynamics that quickly
emerge and dissipate. In the case of a fluid system, the substance of the slow subspace represents the predominant features of a flow in the steady or stationary state, while the subspace *per se* enables a low-dimensional approximation of the Koopman operator, or, equivalently, the Navier-Stokes. As such, stationarity is the first prerequisite of time-invariance. In what follows, the focus is exclusively on stationary flows. The extension to transience awaits the brilliance of future efforts.

### 2.4.3 Time-Invariance: Convergence of Sampling

On a different note, readers are reminded that apart from the universal Koopman operator $\mathcal{U}$, which supposedly captures every spatial and temporal variation of a system, none of the other quantities are strictly time-independent. For a stochastic system, the stringent independence of time can only be achieved by

$$\lim_{t \to \infty} \tilde{\mathbf{A}}(t) = \tilde{\mathbf{A}} \approx \tilde{\mathbf{U}}(t) \approx \mathbf{U},$$

where the spatial dimension of $\tilde{\mathbf{A}}$, $n$, must also be sufficiently large. One may also rationalize time-invariance from the perspective of the DFT. The stringent independence of time means an infinitesimal bin width over which some frequency content is averaged, or equivalently, an infinite number of bins to resolve the full spectrum. It is clearly impractical.

The independence of time must adopt a pragmatic approach, which translates to the convergence of sampling of the algorithm through which the Koopman operator is approximated. That is, the algorithm, being practically indifferent to changes of sample size, shall generate a sufficiently consistent set of Koopman modes or Ritz descriptors to describe the dynamics of the input data. Our previous work (Li, Chen, et al., 2021b) has established the convergence of sampling for the DMD and offered a set of practical suggestions. The upcoming section will present key evidence for the convergence of sampling. To this end, pragmatically achieving the convergence of sampling is the second prerequisite of time-invariance.
3. Practical rendering of the Koopman-LTI

3.1 Flow overview

The fluid-structure system we consider is a subcritical free-shear flow over an infinitely long and stiff square prism. In the stationary state, a free-shear flow is closely self-similar, and many characteristics are shared within the free-shear family (i.e., jet, mixing layers, etc.), thus boosting the pertinence of the present findings. The prototypical flow-over-prism configuration is also geometrically simplistic but phenomenologically sophisticated, involving stagnation, forced separation, reattachment, roll-up, entrainment, etc. (Bai & Alam, 2018; Kundu, 2004). The four prism walls are behaviourally complex and distinct (Z. Chen et al., 2021; Lander et al., 2016), and the infinite spanwise length prevents undesired complications due to the end effects (Z. Chen et al., 2018; White, 2006). As we impose an infinite stiffness, we simplify the bi-directional feedback loop of fluid-structure interaction into a mono-direction case, in which the structure receives fluid excitation without altering the field by its own motion. The simplification is ubiquitously adopted for external flows, for example, in large-scale or civil applications (Z. Chen et al., 2018, 2020; Portela et al., 2017; Rodi, 1997; Tse et al., 2020). On this note, wall pressure characterises structural response.

In essence, this work aims to present the most fundamental yet sufficiently sophisticated case to encourage intellectual empathy with the readership. Without a structure, the most canonical jet flow is not appropriate. The boundary layer flow over a flat plate has only one fluid-structure interface, which is less demonstrative. The flow-over-cylinder case collapses down to a single curvilinear interface, too. To this end, we have a relatively fair understanding of the phenomenology of the prism wake (Kundu, 2004; Lander et al., 2016; Paidoussis et al., 2010; Rastan et al., 2021), which has also been widely adopted to study inhomogeneous anisotropic turbulence (Lander et al., 2018; Portela et al., 2017). Furthermore, we selected the turbulence regime, instead of the more predictable laminar regime, because success herein will greatly substantiate the technique’s generality for a class of stochastic processes, perhaps even beyond those under fluid mechanics.

We deliberately chose the subcritical range with the Reynolds Number $Re = U_\infty D/\nu = 2.2 \times 10^4$, where $U_\infty$ is the free-stream speed, $D$ is the prism side length, and $\nu$ is the kinematic viscosity of the fluid. This $Re$ is representative of a neighbourhood of phenomenological similitude, in which the shear layer undergoes the turbulence transition II (Bai & Alam, 2018). It also ensures
the incompressibility of air ($M<0.3$), so the divergence-free condition applies when solving the Navier-Stokes equations. Finally, the $Re$ permits high-fidelity numerical simulation with an acceptable expense and existing literature for validation (Cao et al., 2020; Portela et al., 2017; Trias et al., 2015).

### 3.2 Large-eddy simulation

The turbulent flow is simulated by the Large-Eddy Simulation with Near-Wall Resolution (LES-NWR) as defined by Pope (2000). The simulation adopts the DNS domain from Portela et al. (2017), except using $4D$ for the spanwise length instead of $\pi D$ for computational ease in the Euclidean space (see figure 1). The LES-NWR solves the spatially-filtered incompressible Navier-Stokes equations on structured grids, while a dynamic-stress Smagorinsky model simulates the subgrid-scale dynamics. Inlet perturbation is spared to avoid synthetic turbulence.

We employed a finite-volume, segregated, pressure-based solution algorithm for this low-Mach-number incompressible flow. With second-order schemes for both spatial and temporal discretization, and a stringent convergence criterion of $O^6$, the effects of numerical dissipation and dispersion are minimized, respectively. Moreover, the simulation evolved by a non-dimensional time-interval $t^*$,

$$ t^* = \frac{\Delta t U_{\infty}}{D} = 1.61 \times 10^{-3}, \quad (3.1) $$

where $\Delta t$ is the physical time step, so the Courant-Friedrichs-Lewy (CFL) convergence condition is always satisfied, eliminating the possible time marching issues when solving partial differential equations.

The accuracy of the LES-NWR is critical to the subsequent analysis with the Koopman-LTI. To avoid repetition, we direct the readers to our previous work (Li, Chen, et al., 2021b), which contains comprehensive simulation details, grid resolution assessment, and case validation. The simulation resolves at least 80% of the turbulent kinetic energy, qualifying itself as proper LES-NWR. It also achieves an accuracy comparable to several DNS renderings (Cao et al., 2020; Portela et al., 2017; Trias et al., 2015). Given the Koopman-LTI targets only the slow subspace, thus the most dominant fluid dynamics preceding the subgrid-scale, numerical results from LES-NWR are appropriate for the present investigation.
3.3 Statistical stationarity

Per aforenoted discussion in section 2.4, stationarity is a critical prerequisite of time-invariance. This section presents the practical assurance of stationarity. We first examine the lift force coefficient as an indicator of global stationarity (figure 2). The time history shows that while the instantaneous lift exhibits evident fluctuations, the time-dependent mean and root-mean-square (r.m.s.) lift are constant at 0 and 1.173, respectively. The constancy of the global statistics attests to the flow’s stationarity. Accordingly, all measurements are sampled in the range $t^* > 1.2 \times 10^5$. 

Figure 1: Schematic illustration of the computational domain of the numerical simulation.
Figure 2: Time history of the instantaneous, mean, and root-mean-square lift coefficient.

On top of global statistics, we further ensure stationarity with local statistics. Throughout the simulation, seven points that characterize stagnation, separation, and the prism base were selected to monitor local statistics. As illustrated in figure 3, the normalized mean of the velocity magnitude at all seven points exhibits substantial variations in the early stage of the simulation. Nonetheless, the variations gradually subdue before uniformly reaching their respective asymptotes in the range $t^* > 1.0 \times 10^5$. The stabilization is evident even near in shear layers and near-wake---regions where the most tumultuous turbulence takes place. From both global and local statistics, the stationarity of the flow is lucid.

(a) (b)
Figure 3: (a) Schematic illustration of monitor points 1-7; (b)-(h) normalized mean of velocity magnitude versus time step at monitor points 1-7.

3.4 Sampling convergence: index of quantification

Fulfilling the prerequisite of stationarity, the second requirement is the algorithmic independence of sampling. In practice, the greatest difficulty for assessing the sampling convergence is finding the optimal quantification index. Our previous work found the accuracy of Koopman reconstruction, hence the error between original and reconstructed sequences, a revealing index, because it directly reflects how well the underlying dynamics are represented.
To explain the rationale in brevity, we found the ensuing metaphorical example particularly suiting. Suppose the nonlinear dynamics of a physical system are metaphorically embodied by the Burj Khalifa in Dubai, UAE. The Koopman operator can be considered as an infinite-resolved and infinite-large photograph of the Burj, because it globally translates the dynamics into a combination of linearized, discrete constituents, or the photo’s pixels (see figure 4). Although the Koopman linearization does not produce the purported 3D-to-2D reduction, the figurative implication is to some extent similar: the level of complexity (nonlinearity) is significantly reduced.

Thereafter, any sampling or modeling, whether field, empirical or numerical, inevitably reduces $U$ into the finite-dimensional $A$. It is equivalent to building a numerical model of the Burj based on the photograph, which contains, no matter how large, a finite number of nodes. The process produces the first component of the approximation error

$$U \rightarrow A \rightarrow \epsilon_{UA}.$$  \hspace{1cm} (3.2)

The numerical model also carves the unimportant background and retains only the core building.

![Figure 4: A metaphorical illustration showing the source of reconstruction error in each step of the Koopman approximation. Electronic art (left to right) by Donaldytong (2012), Simscale (2021), Lego (n.d.), and Chia (2007).](image)

(Li, Chen, et al., 2021b).
Then, the decomposition of $A$ into Koopman modes, whether by the similarity approach $\tilde{A}$, other variants of the DMD, the GLA, or any other algorithm, introduces the second component of the approximation error

$$A \rightarrow e_{AT}$$  \hspace{1cm} (3.3)

The step intends to, via computationally tractable and robust algorithms, build a further reduced model. It is metaphorically analogous to building a Lego model of the Burj. The Lego tower is less accurate than the photograph or numerical model, but sufficient to capture all the significant features of the Burj. Most importantly, it is easy and robust to assemble and decompose. Of course, the accuracy of the Lego tower depends on the total number and the individual shapes of the Lego pieces, which is equivalent to the size and resolution of the input sample.

Finally, dismantling the Lego tower into individual pieces metaphorically describes the decomposition of $\tilde{A}$ into $\phi_j$ or $M_j$. The Koopman modes are linear and have different spatiotemporal weights, like how the Lego pieces are all rectangular but vary in size. Users may also choose to neglect some smaller pieces without excessively affecting the overall architecture. The third component of the approximation error is therefore introduced

$$\tilde{A} \rightarrow e_{AT_{\text{rer}}}$$  \hspace{1cm} (3.4)

Until one obtains the individual pieces, there is no way to understand completely which shapes, colours, and combinations of the Lego blocks together make up the exquisite appearance of the Burj. Most beautifully, assembling the Lego Burj does not require esoteric knowledge in architecture, which alludes to the bypassing of the governing equations in a Koopman analysis.

Then, the best way to evaluate the accuracy of a Koopman analysis is to reassemble the Lego pieces and compare its difference with the original. This is the essence and purpose of reconstruction. The overarching reconstruction error can be quantified as

$$e = e_{UA} + e_{AT} + e_{AT_{\text{rer}}} = \frac{X_{\text{Koopman}} \cdot X}{X}.$$  \hspace{1cm} (3.5)

One may also assess the accuracy for each spatial and temporal dimension. This work adopts the instantaneous, spatially-averaged, and $l_2$-normalized reconstruction error, $\|e\|_{2,i}$ or $\|e\|_{2,\text{ins}} \in \mathbb{R}^+$.
\[ \|e\|_{2,i} = \|e\|_{2,\text{ins}} = \frac{1}{n} \sum_{k=1}^{n} \left[ \frac{X_{\text{Koopman},k,i} - x_{k,i}}{x_{k,i}} \right]^2 \]  \( \text{(3.6)} \)

and its rms value \( \|e\|_{2,\text{rms},i} \in \mathbb{R}^+ \)

\[ \|e\|_{2,\text{rms},i} = \left[ \frac{1}{n} \sum_{k=1}^{n} \left( \frac{X_{\text{Koopman},k,i} - x_{k,i}}{x_{k,i}} \right)^2 \right]^{1/2} \]  \( \text{(3.7)} \)

where \( k \in \mathbb{Z}^+ \), as the indices of quantification.

To combine nodal and instantaneous error into a single spatiotemporal index, we define the grand mean \( l_2 \)-norm of reconstruction error as

\[ G\|e\|_2 = \frac{1}{m} \sum_{i=1}^{m} \|e\|_{2,\text{ins},i} \]  \( \text{(3.8)} \)

3.5 Sampling convergence: the universal states

Using the reconstruction error as an index of quantification, this section presents the achievement of sampling convergence in practice. Our previous work (Li, Chen, et al., 2021b) discovered four convergence states through the prism wake: Initialization, Transition, Stabilization, and Divergence (figure 5). In short, the Stabilization is preferred state in which the Koopman approximation is the most accurate and the spatiotemporal dominance of the Koopman modes is the most consistent. In the Stabilization state, the Koopman system is practically indifferent to the sampling range, so the size of \( X_1 \) and \( X_2 \). We then confirmed the universality of the convergence states by replicating the analysis (Li, Chen, et al., 2021a) on a cylinder wake (figure 5 bottom), which is phenomenological different and twice in computational cells. Results corroborated the existence of the Initialization, Transition, Stabilization states, proving sampling convergence a general possibility. On this note, the Divergence state occurs when the tacit \( m \ll n \) condition of the DMD is violated and originates from the SVD formulation. In contrast to common belief, the discovery of Divergence advises against extensive data sampling.
Figure 5: The universal convergence states for the DMD in the prism wake (top) and cylinder wake (bottom).

To this end, reaching the *Stabilization* state is a requirement for pragmatic time-invariance. Moreover, a subsequent bi-parametric investigation revealed that the sampling resolution, so the inter-snapshot step of $X_1$ and $X_2$, is independent of the sampling range, as exemplified by figure 6. The convergence of sampling resolution is also a mode-specific behaviour. Therefore, we established a practical guideline to resolve the oscillation cycle of an activity of interest by at least 15 snapshots.
3.6 Inventory of observables

The present work sampled 500 snapshots of the entire 3D LES domain with the inter-snapshot step \( t^* = 400\Delta t^* \). The sample size, consisting of 20 oscillations cycles, meets the convergence of the sampling range. The time step resolves the configuration’s shedding cycle at \( St=0.127 \) by 25 frames, ensuring the convergence of sampling resolution.

In total, 18 observables have been sampled as independent realizations of the fluid system (see table 1). The observables include wall and flow field variables, both primary and secondary. Each data sequence has also been sampled, curated, and post-processed independently without any algorithmic cross-communication. The DMD algorithm is also physics-uninformed, so no prior connection has been established between the 18 data sequences. Readers may find details about the observables in Appendix II. The subsequent text refers to the upstream (AB), top (BC), downstream (CD), and bottom (DA) walls according to the definition in figure 7. After Liu (2019), we also refer to the vorticity-based vortex identification criterion, namely \( |\omega| \), as...
the first-generation, the eigenvalue-based criteria, namely $q$ and $\lambda_2$, as the second-generation, and the ratio-based criteria, namely $\Omega$ and $\tilde{\Omega}_R$, as the third-generation.

Figure 7: Schematic illustration of the prism walls.

| Primary Observable | Secondary Observable |
|--------------------|----------------------|
| **Wall**           | **Flow Field**       | **Turbulence** | **Vortex Identification** |
| BC [Top wall]      | $P$ [Total pressure] | $\langle u'v' \rangle$ | $|\omega|$ [Vorticity magnitude] (Helmholtz, 1858) |
| DA [Bottom wall]   | $u$ [x-velocity]     | $\langle u'w' \rangle$ | $q$ [q-criterion] (Hunt et al., 1988) |
| AB [Upstream wall] | $v$ [y velocity]     | $\langle v'w' \rangle$ | $\lambda_2$ [\(\lambda_2\)-criterion] (Jeong & Hussain, 1995) |
| CD [Downstream wall]| $w$ [z-velocity]     | $\langle k \rangle$ | $\Omega$ [\(\Omega\)-criterion] (C. Liu et al., 2016) |
|                    | [$U$] [Velocity magnitude] | \(\tilde{\Omega}_R\) [\(\Omega\)-Liutex criterion] (C. Liu et al., 2018) |

Table 1: Summary of the inventory consisting of 18 observables.

4. **Constitutive fluid-structure relationship**

With the guarantee of pragmatic time-invariance, the present investigation passes the inventory into the DMD algorithm and generates 18 independent Koopman-LTI systems. We first examine the accuracy and stability of the systems.
4.1 Accuracy and stability

The ensuing figure 8 presents the instantaneous mean and r.m.s. reconstruction error of the 18 Koopman-LTI systems. The maximum mean and r.m.s. errors are $O^9$ and $O^6$, respectively. Excluding the outliers, the maxima further reduce to $O^{12}$ and $O^9$, which are simply numerical zeros. The virtually non-existent reconstruction error is extraordinary by any standard and shows that the Koopman-LTI approximation of the original flow is infinitely close to exact. Compared with the Koopman systems generated in our previous work (Li et al., 2020; Li, Chen, et al., 2021b; Li, Tse, et al., 2021), the notion of linearly-time-invariance substantially improves the reconstruction accuracy by several orders of magnitude. The a posteriori realization herein also demonstrates the practical possibility of the Koopman-LTI to perform high-fidelity linearization for stochastic systems.

On the other hand, stability is a measure of how well a system behaves with regularity. It also reflects how well periodic descriptors, through the sinusoids and exponentials of the $Z$-transform, depict a systems’ dynamics. To assess the stability of the Koopman-LTI systems, their respective Regions of Convergence (ROC), or the so-called DMD spectra, are presented in figure 9. In a perfectly oscillatory or marginally stable system, the modes or the poles lie exactly on the $\Re^2 + \Im^2 = 1$ unit circle.

Three key observations are derived from the DMD spectra. First, all poles lie infinitely close to the unit circle, showing their near-perfect oscillatory nature, hence stellar stability. The stability attests to the quality of the Koopman-LTI decomposition, and the adequacy of the Ritz descriptors for describing the physics of fluid-structure interaction. Moreover, the growth/decay rates presented in figure 10 are numerical indices that quantify the proximity of the poles to the unit circle, or their stability. The maximum $g_j$ is in the order of $O^8$. Excluding the case of the upstream wall AB, the maximum reduces to $O^{12}$. Again, the numerical zero testifies the near-perennial oscillation of the Koopman modes, hence the extraordinary stability of the Koopman-LTI systems.
Figure 8: (a) Mean reconstruction error and (b) root-mean-square reconstruction error of the 18 Koopman-LTI systems versus nondimensional time \( t^* \).

The second observation is that the ROCs (in mint green) are characteristic of acausal systems. Acausality guarantees that the behaviours of a system do not depend on any past input but only on future ones (Oppenheim et al., 1997). A natural interpretation is that the current sample sufficiently captures all the major (or even minor) contributors, so the entire slow subspace, of the input fluid-structure system. Again, this is concrete evidence for the independence of algorithmic (DMD) sampling, or equivalently, the pragmatic time-invariance (relative to the past) of the Koopman-LTI systems.
Figure 9: The DMD spectra showing the Region of Convergence of the 18 Koopman-LTI systems (left) and zoomed-in (right) near $\Re(\lambda_j)=1$ and $\Im(\lambda_j)=0$.

Figure 10: Growth/decay rate of the Koopman-LTI modes of the 18 Koopman-LTI systems versus mode number $M_j$.

The third observation is that the poles of all 18 systems superimpose exactly onto one another at all characteristic frequencies. The indication is that the systems are composed of the same set of Ritz descriptors $M_j$. This is a critical revelation. Given the consistency in the spatial dimension $n$ and temporal dimension $m$, the Koopman-LTI essentially distributes the spatiotemporal spectrum of the systems across the same set of equidistant frequency bins. Consequently, and most importantly, if the input observables correspond to measurements of...
the structure and flow field, then a direct constitutive relationship can be established for structural response and fluid excitation. The Ritz descriptors also provide temporally-orthogonal, thus frequency-wise disentangled, descriptions of fluid-structure interaction. To this end, the merit of the Koopman-LTI is well defined. In the sections to come, we will demonstrate the constitutive relationship by analysing both intra- and inter-group dynamics, and the insights it reveals into the prism wake and the Koopman-LTI architecture.

4.2 Intra-Group dynamics

Inspection indicates that although the frequency bins $St_j$ are universal for all 18 Koopman-LTI systems, the leading coefficients $\alpha_j$ are vastly different, implying disparities in their spatiotemporal content. The assessment of the disparity fosters the definition of the normalized modal amplitude, as

$$-1 \leq |\tilde{\alpha}_j| \in \mathbb{R} \leq 1.$$ (4.1)

The ranking of $M_j$ in their respective Koopman-LTI systems directly reflects the spatiotemporal disparity, as summarized in table 2. Readers are reminded of several other criteria (Jovanović et al., 2014; Kou & Zhang, 2017; Sayadi et al., 2014) besides the $\alpha_j$ to be selected by user preference. The selection of the 10 most dominant modes of each observable (highlighted) results in exactly 30 across the entire inventory. The upcoming analysis first groups the observables by their origin and subsequently investigates their intra-group dynamics. For each group, $|\tilde{\alpha}|$ of the most dominant mode is anchored at 1 for comparative ease.

Prism walls

As shown below, figure 11 presents $|\tilde{\alpha}_j|$ versus $St_j$ of the four wall observables. $|\tilde{\alpha}_j|$ of the top (BC), bottom (DA), and upstream (AB) walls, or collectively referred to as the on-wind walls, share similar distributions. The most dominant concentration of energy, or referred to as a peak, appears at $St=0.1242$. Its broadband content also spreads across several frequency bins in the neighbourhood $St=0.1-0.15$. A secondary narrowband peak, which is particularly clear on the log scale, appears at $St=0.0497$ and has $\sim 25\%|\tilde{\alpha}|$. Here $\sim$ denotes the approximate range of multiple observables, whereas the percentage of a single observable is exact. Other peaks are also faintly visible, for example, at $St=0.2422$ with $\sim 10\%|\tilde{\alpha}|$, but are deemed trivial under the shadow of the two primary peaks. For clarification, each peak on the spectrum corresponds to
a natural structure of the flow (Hussain, 1986), and $|\tilde{a}_j|$ allegorically represents the energy associated with it. Equivalently speaking, a peak inversely represents the energy required to excite a natural structure---the greater the $|\tilde{a}_j|$, the more natural the structure.

In contrast to the similitude of the on-wind walls, the downstream wall (CD) exhibits a fundamentally different distribution. Although the most prominent broadband peak still appears at $St=0.1242$, the secondary peak at $St=0.0497$ with 28.2$|\tilde{a}_j|$ is buried. Instead, several other peaks at $St=0.0683$, 0.0745, 0.1739, 0.1925, 0.2422, and 0.3664 with 37.6, 42.0, 55.6, 40.6, 32.6, and 37.2$|\tilde{a}_j|$, respectively, overtake the dominance. A direct interpretation suggests the dynamics of the downstream wall underlie more complications and are utterly different from those of the on-wind walls. Moreover, compared to that of the on-wind walls, the difference between $|\tilde{a}_j|$ and $|\tilde{a}_j|$ of other non-broadband peaks is significantly reduced from ~75% to 44.5%, implying dramatically more interwoven physics therein. The observations allude to the findings of several works on the negative base pressure, which results from the tumultuous vortical activities and entrainment in both the stream- and span-wise directions inside a turbulent wake (Lander et al., 2016; Luo et al., 1994; Unal & Rockwell, 1988). Based on this analysis, we refer to $St=0.1242$ as the primary peak, $St=0.0497$ as the secondary peak, and all others as the ancillary peaks in the subsequent text.

![Figure 11: $|\tilde{a}_j|$ versus $St$ of the wall observables (BC, DA, AB, CD) in linear scale and $St=0-0.3$ (left), linear scale and $St=0-1.5$ (right top), and log scale and $St=0-1.5$ (right bottom).](image)
| Mode - Frequency | Primary Observable | Input Data | Turbulence | Vortex Identification Criteria |
|------------------|-------------------|------------|------------|-------------------------------|
|                  | Wall ($P$)        | Reynolds Stress | TKE | 1st Gen. | 2nd Gen. | 3rd Gen. |
|                  | BC | DA | AB | CD | $P$ | $u$ | $v$ | $w$ | $U$ | $\langle u'v' \rangle$ | $\langle u'w' \rangle$ | $\langle v'w' \rangle$ | $\langle k \rangle$ | $|\omega|$ | $q$ | $\lambda_2$ | $\Omega$ | $\tilde{\Omega}_R$ |
| $M_1$ - $St_1=0.1242$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 26 | 1 | 11 | 20 | 19 | 1 | 1 | 1 | 1 | 1 |
| $M_2$ - $St_2=0.1180$ | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 11 | 2 | 14 | 19 | 20 | 3 | 2 | 4 | 4 | 3 | 3 |
| $M_3$ - $St_3=0.2422$ | 14 | 13 | 19 | 9 | 5 | 3 | 5 | 42 | 4 | 19 | 40 | 41 | 10 | 4 | 2 | 2 | 2 | 2 |
| $M_4$ - $St_4=0.1304$ | 3 | 3 | 3 | 11 | 3 | 4 | 3 | 31 | 5 | 23 | 21 | 23 | 5 | 8 | 6 | 7 | 5 | 7 |
| $M_5$ - $St_5=0.0497$ | 4 | 4 | 4 | 14 | 4 | 5 | 8 | 1 | 3 | 5 | 8 | 8 | 2 | 3 | 5 | 5 | 7 | 5 |
| $M_6$ - $St_6=0.0745$ | 10 | 17 | 15 | 4 | 8 | 6 | 13 | 4 | 6 | 13 | 12 | 12 | 19 | 5 | 15 | 16 | 14 | 10 |
| $M_7$ - $St_7=0.0683$ | 11 | 21 | 17 | 6 | 10 | 7 | 12 | 14 | 8 | 10 | 11 | 11 | 31 | 7 | 8 | 8 | 11 | 9 |
| $M_8$ - $St_8=0.1428$ | 5 | 6 | 5 | 8 | 6 | 8 | 4 | 6 | 7 | 27 | 23 | 22 | 4 | 9 | 17 | 18 | 8 | 11 |
| $M_9$ - $St_9=0.1739$ | 17 | 11 | 21 | 2 | 9 | 11 | 9 | 24 | 11 | 25 | 28 | 27 | 7 | 6 | 3 | 3 | 10 | 8 |
| $M_{10}$ - $St_{10}=0.1118$ | 6 | 5 | 6 | 10 | 7 | 10 | 6 | 25 | 10 | 21 | 18 | 18 | 6 | 16 | 20 | 22 | 13 | 18 |
| $M_{11}$ - $St_{11}=0.1366$ | 7 | 8 | 7 | 18 | 11 | 11 | 7 | 21 | 9 | 24 | 22 | 21 | 23 | 15 | 25 | 28 | 15 | 19 |
| $M_{12}$ - $St_{12}=0.1056$ | 8 | 7 | 8 | 38 | 12 | 13 | 10 | 22 | 13 | 20 | 17 | 17 | 27 | 17 | 31 | 30 | 19 | 24 |
| $M_{13}$ - $St_{13}=0.1925$ | 23 | 27 | 29 | 5 | 13 | 15 | 9 | 28 | 15 | 22 | 31 | 31 | 11 | 11 | 7 | 6 | 6 | 6 |
| $M_{14}$ - $St_{14}=0.1553$ | 9 | 16 | 9 | 24 | 16 | 16 | 20 | 17 | 21 | 26 | 25 | 26 | 21 | 18 | 10 | 11 | 29 | 25 |
| $M_{15}$ - $St_{15}=0.0559$ | 20 | 9 | 10 | 12 | 15 | 19 | 28 | 23 | 16 | 9 | 9 | 9 | 17 | 10 | 26 | 25 | 25 | 21 |
| $M_{16}$ - $St_{16}=0.3664$ | 37 | 33 | 46 | 7 | 25 | 34 | 15 | 64 | 27 | 45 | 59 | 60 | 16 | 39 | 29 | 26 | 23 | 15 |
| $M_{17}$ - $St_{17}=0.0994$ | 24 | 10 | 13 | 29 | 17 | 14 | 21 | 18 | 17 | 18 | 16 | 16 | 12 | 20 | 37 | 37 | 21 | 20 |
| $M_{18}$ - $St_{18}=0.0373$ | 13 | 44 | 20 | 25 | 24 | 20 | 24 | 2 | 14 | 7 | 6 | 6 | 9 | 19 | 34 | 34 | 26 | 29 |
| $M_{19}$ - $St_{19}=0.0311$ | 25 | 30 | 30 | 19 | 29 | 21 | 32 | 3 | 19 | 6 | 5 | 5 | 14 | 13 | 36 | 36 | 40 | 35 |
| $M_{20}$ - $St_{20}=0.1677$ | 26 | 32 | 32 | 21 | 33 | 25 | 22 | 5 | 32 | 28 | 27 | 28 | 32 | 27 | 60 | 58 | 34 | 28 |
| $M_{21}$ - $St_{21}=0.0807$ | 16 | 12 | 11 | 30 | 14 | 12 | 16 | 7 | 12 | 15 | 13 | 13 | 8 | 12 | 16 | 15 | 16 | 14 |
| $M_{22}$ - $St_{22}=0.0248$ | 35 | 20 | 26 | 27 | 30 | 26 | 42 | 8 | 23 | 4 | 4 | 4 | 22 | 21 | 48 | 48 | 41 | 34 |
| $M_{23}$ - $St_{23}=0.0124$ | 33 | 42 | 34 | 43 | 42 | 32 | 41 | 9 | 31 | 2 | 2 | 2 | 26 | 29 | 58 | 63 | 37 | 36 |
| $M_{24}$ - $St_{24}=0.0435$ | 28 | 19 | 27 | 41 | 26 | 17 | 27 | 10 | 20 | 8 | 7 | 7 | 20 | 14 | 23 | 24 | 18 | 17 |
| $M_{25}$ - $St_{25}=0.0062$ | 45 | 39 | 36 | 53 | 43 | 29 | 30 | 12 | 29 | 1 | 1 | 1 | 28 | 37 | 63 | 64 | 38 | 42 |
| $M_{26}$ - $St_{26}=0.0186$ | 34 | 24 | 25 | 36 | 35 | 31 | 34 | 13 | 28 | 3 | 3 | 3 | 25 | 28 | 49 | 52 | 36 | 39 |
| $M_{27}$ - $St_{27}=0.0621$ | 12 | 38 | 16 | 22 | 19 | 22 | 26 | 20 | 22 | 12 | 10 | 10 | 13 | 22 | 30 | 29 | 31 | 31 |
| $M_{28}$ - $St_{28}=0.4161$ | 50 | 40 | 55 | 45 | 52 | 53 | 56 | 67 | 52 | 62 | 66 | 68 | 53 | 43 | 9 | 9 | 54 | 51 |
| $M_{29}$ - $St_{29}=0.2236$ | 27 | 22 | 23 | 15 | 23 | 30 | 14 | 36 | 33 | 37 | 36 | 37 | 34 | 31 | 12 | 10 | 9 | 12 |
| $M_{30}$ - $St_{30}=0.2484$ | 32 | 35 | 37 | 26 | 22 | 27 | 25 | 40 | 25 | 29 | 39 | 39 | 18 | 32 | 22 | 21 | 4 | 4 |

Table 2: Summary of 30 dominant modes and their respective $|\tilde{a}|$ ranking in each Koopman-LTI system (Highlighted: 10 most dominant).
Flow field

This section investigates the intra-group dynamics of the field observables $|U|$, $P$, $u$, $v$, and $w$. Figure 12 presents $|\tilde{a}_j|$ versus $St$ of the five Koopman-LTI systems. Except for $w$, the distributions of $|U|$, $P$, $u$, and $v$ resemble those of the on-wind walls by exhibiting the predominance of primary and secondary peaks $\sim 30\% |\tilde{a}_1|$. The ancillary peaks of considerably sharper $|\tilde{a}_i|$ also appear at $St=0.0683$, $0.0745$, $0.1739$, and $0.1925$, increasing from $\sim 8-13\% |\tilde{a}_1|$ of the on-wind walls to $\sim 15-30\% |\tilde{a}_i|$ herein. $St=0.2422$ even increased from $\sim 10\%$ to $\sim 25\% |\tilde{a}_i|$. The statement is solid: a constitutive relationship exists between structural response and fluid excitation. Nevertheless, the flow field strongly and directly reflects the energy concentration of the constitutive excitation mechanisms, whereas the manifestation on the walls is less distinct. In the upcoming section, we will demonstrate how straightforward the identification of the constitutive relationship can be with the Koopman-LTI.

![Figure 12: $|\tilde{a}_j|$ versus $St$ of the Reynold stresses ($|U|$, $P$, $u$, $v$, $w$) in linear scale and $St=0-0.3$ (left), linear scale and $St=0-1.5$ (right top), and log scale and $St=0-1.5$ (right bottom).](image)

Furthermore, among the three velocity components, the distributions of $u$ and $v$ are comparable to that of $|U|$, though the resemblance of the former is better than the latter. The observation is attributed to the convection-dominance of free-shear flows (Pope, 2000). $u$, being the component in alignment with the free-stream, embodies the preponderance of convection. It
also contributes the most to the substance of $|U|$, explaining why, even with $w$ displaying a fundamentally different distribution, $|U|$ is practically insensitive to the disparity and akin to the distribution of $u$.

On a different note, the distribution of $w$ is a singularity. Though still visible, the dominance of the primary peak $St=0.1242$ is much less prevailing compared to its peers. The z-component velocity also weakly reflects the convection-driven activities pertaining to the secondary and ancillary peaks. It is to say $w$ has only marginal contributions to the flow’s dominant excitations. The observation is more lucid on the broadband spectrum ($St=0-1.5$), particularly in the log scale. Peaks are buried amidst an overarching trend that allegorically appeals to the classic wavenumber spectrum of the Richardson-Kolmogorov notion (Pope, 2000). To this end, the bleak presence of $w$ justifies the selection of several previous efforts that had studied the three-dimensional prism wake by their planar counterpart (Braza et al., 2006; Ong & Wallace, 1996).

Reynolds stresses

The distributions of the Reynolds stresses display no resemblance to those of the prism walls and flow field, but a remarkable consistency across the three components (figure 13). $|\tilde{\sigma}_{ij}|$ unanimously displays an inverse proportionality with $St_j$ in a largely steady exponential decay. To rationalize the observation, readers are reminded of the implication of the Reynolds stresses---they are the averaged deviatoric components of the stress tensor that accounts for the turbulent fluctuations of fluid momentum. Conceptually, they constitute a measure of the distortion of an infinitesimal fluid element, therefore an index quantifying the tendency of that element towards turbulence, or inversely, laminarization. This knowledge motivates a critical revelation. The spatiotemporal content of the Reynolds stresses metaphorically depicts the imagery of the energy cascade. The largest eddies, corresponding to the smallest wavenumbers, extract kinetic energy from the mean-field and channel it into turbulence through production processes. They are, however, the most unstable and vulnerable to distortion, granting them the highest tendency to turbulence hence the greatest dominance to the Reynolds stresses. Although the nonlinear inter-scale transfer can be both forward and inverse, the overall energy balance is negative (Pope, 2000; Portela et al., 2017). So, as the large, more energetic eddies break down into smaller, more abundant ones in the forward cascade, $|\tilde{\sigma}_{ij}|$ decays exponentially. To this end, the depiction of the cascade by the Koopman-LTI is macroscopically accurate and illustrative. Interestingly, amidst the overwhelming trend, evident peaks, though far less standout, still indicate the presence of the dominant excitations of the configuration.
Figure 13: $|\vec{a}_j|$ versus $St$ of the primary field observables ($\langle u'v' \rangle$, $\langle u'w' \rangle$, $\langle v'w' \rangle$) in linear scale and $St=0-0.3$ (left), linear scale and $St=0-1.5$ (right top), and log scale and $St=0-1.5$ (right bottom).

Vortex structures

The distribution of the turbulence kinetic energy $\langle k \rangle$ displays more similarity with those of the vortex identification criteria than the Reynolds stresses (figure 14). Fluid theories define $\langle k \rangle$ as the mean kinetic energy per unit mass of eddies (Kundu, 2004; Pope, 2000; White, 2006). One also derives it through the isotropic components of the stress tensor. So, the empirical observation here is vastly interesting: the spatiotemporal content of the dilatory components alludes more to those of vortical structures than their deviatoric cousins. Conversely, dilation drives the behaviours of vortices more intensely than distortion. On this note, the prominent peak at $St=0.1242$ of $\langle k \rangle$ is less acute, and the ancillary ones are 15-30% more energy-potent compared to those of the prism walls and flow field.

The distributions of the first- and second-generation vortex identification criteria are energetically weaker than that of $\langle k \rangle$, while those of the third-generation criteria are more. Or, inversely, the primary peak is more and less prevailing relative to its peers, respectively. An emanating remark indicates the dynamics of the third-generation criteria are more entwined than those of the others. Another interesting observation is, except for $|\omega|$, all other observables exhibit the secondary peak at $St=0.2422$ in lieu of $St=0.0497$. Inspection reveals that the
difference originates from $u$ (figure 12). In addition, all vortex criteria exhibit mild peaks at $St=0.0683$ and $0.0745$, while they are particularly distinctive in $\langle k \rangle$. Finally, despite minor differences, the vortex criteria and flow field observables generally agree with spectral locations of the energy concentration, namely at $St=0.0497, 0.0683, 0.0745, 0.1242, 0.1739, 0.1925$, and $0.2422$.

![Graph](image-url)

**Figure 14:** $|\tilde{\alpha}_j|$ versus $St$ of the turbulence kinetic energy and vortex identification criteria ($\langle k \rangle, |\omega|, q, \lambda_2, \Omega, \tilde{Q}_R$) in linear scale and $St=0-0.3$ (left), linear scale and $St=0-1.5$ (right top), and log scale and $St=0-1.5$ (right bottom).

### 4.3 Inter-group constitution

Analysis of the intra-group dynamics assessed the spatiotemporal content of the observables relative to their peers. While pinpointing the singularities of $w$ and the Reynolds stresses, the remaining observables suggested a consensus between the natural structures of fluid excitation and structural response or, equivalently, direct fluid-structure correspondence. By studying the inter-group dynamics, this section confirms the constitutive relationship, thus underscoring a significant step towards understanding fluid-structure interaction. The upcoming discussion also demonstrates how one can straightforwardly, as easy as by inspection, establish the constitutive fluid-structure relationship by the Koopman-LTI. To this end, the upcoming analysis is more qualitative, through which we offer our interpretation of the empirical results.
**Fluid-structure correspondence**

In supplement to table 2, figure 15 presents the ten most dominant modes of the 18 systems versus $St$. In figure 15, while different classes of colour (blue, orange, green, and maroon) distinguish the four groups, the darkness of the colour and the radius of the circle figuratively illustrate the dominance of a Ritz descriptor. Previous conclusions are reaffirmed: except for $w$ and the Reynolds stresses, all other observables reflect the primary peak at $St=0.1242$, the secondary peak at $St=0.0497$, and the ancillary peaks to different degrees. Most importantly, the direct constitutive relationship between the fluid and structure is lucid by mere inspection. The responses of the on-wind walls are primarily excited by the broadband primary peak at $St=0.1242$ and the narrowband secondary peak at $St=0.0497$. The correspondence is highlighted in sky blue and named *Class 1*. The structural response of the downstream wall, while still dominated by the primary peak, reflects the effects of several subsidiary excitations, namely those depicted by the ancillary peaks of descending dominance at $St=0.1739$, $St=0.0683$, $St=0.1925$, and $St=0.2422$. The correspondence is highlighted in lavender and named *Class 2*.

![Figure 15: 10 modes with the highest $|\tilde{\alpha}_i|$ versus $St$ of the 18 Koopman-LTI systems.](chart.png)
Commentary on implications, significance, and $\hat{A}$

More significantly, unearthing from the observations is a critical revelation of the mechanics of the fluid-structure system. In the moderately subcritical regime during the shear layer transition II, the complex morphology of a prism wake consists of only six dominant mechanisms (two from Class 1 and four from Class 2). Specifically, the three geometrically distinct on-wind walls are excited by two predominant mechanisms, while the downstream wall is excited by a whole different class of four fluid phenomena. The $a$ posteriori evidence also facilitates two notable discoveries about the Koopman-LTI. First, the clear existence of fluid-structure correspondences per se is an avid manifest of the capacity of the Koopman-LTI. Not only did it prove the tangibility of the integrated governing equations, which certainly exist but are difficult to conjure and impossible to solve, it also illuminated a data-driven path to acquire the equations’ spatiotemporal quintessence. Second, although sampled as independent flow realizations, analysed as standalone data sequences, and processed by an entirely physics-uninformed algorithm, the wall and flow field observables conduce Koopman-LTI systems of remarkable congruence and informational consistency, underpinning the existence of a configuration-wise universal Koopman system $\hat{A}$, hence operator $\hat{U}$, which omnisciently describes the fluid-structure system.

Several other interesting observations arise from figure 15. The flow field observables, especially $P$ and $u$, generally capture the dominant modes of the prism walls. Flow field observables, $\langle k \rangle$ and $|\omega|$ miss, though to different degrees, some ancillary peaks, such as those at $St=0.1739$ or $0.1925$. However, the second- and third-generation vortex identification criteria adequately capture all the pertinent modes. Besides making the criteria the optimal indicators of the fluid-structure correspondence, the observation also suggests that structure responses are closely associated with, if not directly instigated by, vortical activities in the flow field. Furthermore, the observation translates to a set of practical guidelines. When using the Koopman-LTI to analyse fluid-structure systems, after establishing $\hat{A}$ from an inventory of observables, one shall deploy the second- or third-generation criteria to identify the direct correspondence. If mathematical manoeuvres via the velocity gradient tensor are deemed cumbersome, then one shall at least employ the velocity magnitude $|U|$ or pressure $P$ for the identification. The adequacy of $u$ in the present work is attributed to the dominance of convection, therefore, case-specific.
Some may argue that the user-defined threshold of ten modes might oversimplify the dynamics, thus artificially constraining the configuration to only six dominant mechanisms. The doubt is reasonable and deserves clarification. Figure 12 presents the frequency spectra when 20 and 50 dominant modes are selected. Per an increase from 10 to 20, the most apparent change is the widening of existing clusters, symbolizing the enhanced description of their broadband content (figure 16a). Moreover, some Class 2 mechanisms begin to show, though minor in extent, effects on the on-wind walls, for example, at St=0.0683, 0.1739 and 0.2422. The new additions are highlighted in gold. Perhaps only the response of the downstream wall at St=0.2236 barely qualifies as a solid correspondence because four vortex identification criteria faintly illustrate its flow field correspondence. The peaks at St=0.2987 and 0.3664 are only sporadically identified by some observables without consistency. To this end, the six mechanisms are the most overarching excitations of the prism wake.

On the other hand, the analysis exposes that some excitation mechanisms like that at St=0.3664, though deemed only as a subsidiary peak by |U| (ranks the 27th), q (ranks the 29th), and $\tilde{\Omega}_R$ (ranks the 15th), projects an overwhelming influence (ranks the 6th) on the downstream wall. The opposite is also true. For example, the excitation mechanism at St=0.2484 is deemed prominent (ranks the 4th) by both $\Omega$ and $\tilde{\Omega}_R$, but instigates only trivial responses from the structure (ranks >25 for all walls). Even after a global linearization, the interactive mechanisms between fluid and structure are still anything but straightforward. The Koopman-LTI merely offers a new scope of lenses through which they can be incisively scrutinized.

Thereafter, figure 16b illustrates the changes when we further increased the threshold to 50. The inventory unanimously includes the low-frequency range of the spectrum. The implication is a consensus that the low-wavenumber, more energetic eddies have higher dominance in this turbulent flow. It, of course, meets expectations. Once again, the unanimous consensus justifies the existence of the configuration-wise universal Koopman system $\tilde{\mathcal{A}}$: with sufficient inclusion, and even with outliers like w and the Reynolds stresses, all 18 systems collapse down to a universal dynamical picture. Nonetheless, the over-inclusion significantly downplays the portrayal of energy concentration, burying much insightful information amidst the sheer mass, much like how truth is often overlooked amidst collective fervours.
Comparison with the power spectral density analysis

Perhaps some readers already empathize that the preceding analysis suggests connections between the Koopman analysis and the Fourier transform. For zero-mean data taken from linearly independent snapshots, Chen et al. (2012) had formally formulated the mathematical relationship between the DMD and the Discrete-Fourier transform (DFT). Although this is not strictly true in this work, the stationary flow yields near-zero growth/decay rates, thus sufficiently approaching the condition. As pinpointed by Towne et al. (2018):

Mezić (2005) showed that for any dynamical system with a Borel probability measure, the growth/decay rate is zero and Koopman modes are equivalent to Fourier modes. Stationary flows possess an ergodic measure by definition, so their Koopman modes are simply Fourier modes.

Therefore, a DMD implementation with mean-subtracted data herein mimics the perennially oscillatory Koopman modes. One may also consider a DMD mode’s non-zero growth/decay rate as the artifact of the DMD approximation. Nonetheless, the diminutive growth/decay rate in $O^{-12}$ indicates that the DMD, Koopman, and DFT modes are practically identical.

The mathematical connection motivated the subsequent empirical comparison between the Koopman-LTI and the power spectral density (PSD) analysis in figure 17. The stacked-up $|\tilde{a}i|$ exhibits the apparent congruence between the on-wind walls and $u$. The uniqueness of the downstream wall (CD) and $\langle u'v' \rangle$ are also lucid. Accurate representations of the broadband content are also depicted by several sufficiently resolved frequency bins.

For comparison, the present work performs the PSD with seven data series. It selects four pressure series at the geometric centre of each prism wall and another three $u$-velocity series at Points 2, 5, and 6, which characterize the locations of stagnation, entrainment, and separation, respectively (figure 3). In some respects, the process of data selection per se is disadvantageous. Extracting pointwise series is cumbersome with a full-scale sample of the three-dimensional field. The PSD characterization is also local, so the capture of a system-wise representative trend is somewhat fortuitous. Apart from BC and DA, the primary peak at $St = 0.1242$ is unclear in all other six series, while foregoing the Koopman-LTI analysis indicates its unwaveringly
overwhelming, configuration-wise universal predominance. As readers may also empathize, the practical implementation of the PSD is often trial-and-error based and depends heavily on the selection of the sampling series, hence a user’s analytical experience.

Due to these limitations, one can hardly deduce any fluid-structure correspondence from the PSD portrayal in figure 17. However, the Koopman-LTI completely bypasses the constraints of locality and esotericism by generating a globally optimal characterization. With it, one may straightforwardly establish, as easy as by mere inspection of figures 15 and 17, a constitutive relationship between fluid excitation and structural response. On this note, we shall clarify that we do not intend to adjudicate the superiority of the Koopman-LTI over the PSD, nor the other way around. As both are valuable tools for fluid analysis, we merely advise users to make informed decisions about the compromise between locality and globality.

Figure 17: Modal amplitude versus the frequency of selected Koopman-LTI systems on log scale(left), power-spectral analysis of selected data series of prism walls and u-velocity field (right).

5. Conclusions

At the conclusive stage, the merit of the present work emerges with clarity. We offered a new architecture of the Koopman operatory theory, the Koopman Linearly-Time-Invariant analysis, that provides direct constitutive fluid-structure correspondences by globally linearizing and
modally characterizing a fluid-structure system. A demonstrative rendering implements the Koopman-LTI on a subcritical prism wake. Results proved that the Koopman-LTI approximation of the original flow, as a linear superposition of temporally orthogonal Ritz descriptors, is close to exact and perennially stable. The notion of time-invariance has also been pragmatically defined and accomplished via statistical stationarity and convergence of sampling.

The subsequent analysis of 18 Koopman-LTI systems alluded to the existence of a configuration-wise universal Koopman system, through which a critical revelation ensued: For a prism wake undergoing the shear layer transition II, the complex wake and structural morphologies consist of only six primarily dominant and constitutively interactive excitation-response mechanisms. Through their constitutive relationships, the behaviours of the upstream and crosswind walls are attributed to merely two flow field excitations at $St=0.1242$ and $0.0497$. By contrast, the downstream wall behaviours are instigated by another four collectively distinct mechanisms at $St=0.0683, 0.1739, 0.1925,$ and $0.2422$.

The practical rendering of a turbulence-rampant free-shear flow attests to the generality of the Koopman-LTI to a class of fluid-structure systems, if not other stochastic ones beyond fluid mechanics. The success herein bears practical significance as fluid applications can now pinpoint the exact source of an observed structural pattern or vice versa. If we were to summarize the merit of the Koopman-LTI into a single sentence, it would be *an architecture that illuminates one visionary possibility to reduce the intricacies of fluid-structure interaction into constitutive correspondences that are identifiable by inspection---making the complicated matter ever so straightforward.*

In our following work, we will further demonstrate the phenomenological relationship atop the fluid-structure constitution. By proposing the dynamic visualization of the Ritz descriptors, we will unveil the origins of the six mechanisms identified herein.

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Appendix I

Mathematical Formulation of the Similarity-Expression Dynamic Mode Decomposition

Suppose one has a snapshot sequence of a particular variable of interest, or observable, sampled from a fluid system. The sequence can be arranged into two matrices, $X_1$ and $X_2$, that are separated by a uniform time step,

\[
X_1 = \{x_j, x_2, x_3, \ldots, x_{m-1}\} \quad (A1.1)
\]

\[
X_2 = \{x_2, x_3, x_4, \ldots, x_m\} \quad (A1.2)
\]

where $x_i \in \mathbb{C}^n$ are individual data snapshots in the vector form. $n$ denotes the spatial dimension of the data sequence, which is capped by the maximum spatial resolution of the measurement apparatus or numerical grid. This work considers a fixed spatial domain. Readers may refer to Erichson et al. (2019) for techniques dealing with an unfixed spatial domain. $m$ denotes the temporal dimension of the data sequence, which is capped by the sample size of the data sequence. Reader may refer to Kutz et al. (2016) for techniques dealing with non-uniform sampling. Additionally, readers are reminded of a tacit assumption of the DMD, $m << n$.

The beauty of the DMD exudes precisely from the path it provides to forthrightly compute an approximation of the infinite-dimensional Koopman operator on a finite-dimensional subspace. The key is to assume a mapping matrix $A$ that connects $X_1$ and $X_2$,

\[
X_2 = AX_1. \quad (A1.3)
\]

Evidently, $A$ is an unknown matrix that mimics the map $f$, hence that of the Koopman operator $U$. Intuitively, the accuracy of $A$ increases with the dimensionality, so too is the computational expense to obtain $A$.

By the similarity-matrix expression, one may impose another matrix, the so-called similarity matrix $\tilde{A}$. The imposition of $\tilde{A}$ provides a computationally straightforward and algebraically tractable way to approximate $A$ by a singular-value-based algorithm. Readers may find the mathematical derivation and algorithmic procedure from Tu et al. (2014) and Kutz et al. (2016). Sparing repetition, this section only outlines the plainest mathematics deployed in this implementation.
A Singular-Value Decomposition (SVD) is performed on $Y_1$,

$$X_1 = U \Sigma V^T, \quad (AI.4)$$

where $U \in \mathbb{C}^{n \times r}$ contains spatially orthogonal Proper Orthogonal Decomposition (POD) modes $u_j$ on an optimal subspace; $\Sigma \in \mathbb{C}^{r \times r}$, a diagonal matrix, contains singular values $\sigma_j$ that describe the energy of $u_j$; $V \in \mathbb{C}^{m \times r}$ contains temporally orthogonal modes $v_j$ pertaining to the evolution of $u_j$; the superscript $^T$ denotes the conjugate transposition; $r$ denotes the truncation rank, which controls the order of $\tilde{A}$.

The POD-projected $\tilde{A} \in \mathbb{C}^{r \times r}$ relates to $A$ by

$$A = U \tilde{A} U^T. \quad (AI.5)$$

Minimizing the difference between $X_2$ and $AX_1$, one obtains the expression

$$\minimize_A \|X_2 - AX_1\|_F^2, \quad (AI.6)$$

where $\| \|_F$ denotes the Frobenius normalization. Then, equations (A.4) and (A.5) are substituted into equation (A.6) in place of $X_1$ and $A$, respectively, obtaining

$$\minimize_A \|X_2 - U \tilde{A} \Sigma V^T\|_F^2 \quad (AI.7)$$

According to Tu et al. (2014), the expression reduces to the approximation

$$A \approx \tilde{A} = U^T X_2 \Sigma^{-1}. \quad (AI.8)$$

Therefore, one may write the similarity-matrix-approximated expression of equation (A.3) as

$$X_2 = \tilde{A} X_1, \quad (AI.9)$$

in which $\tilde{A}$ is the data-driven, reduced-order, and globally optimal approximation of $A$, hence the Koopman operator $U$. An attractive feature of this approach is that the deduction of $\tilde{A}$ is exclusively implicit, meaning it does not require explicit knowledge of a system’s underlying dynamics, which is particularly useful for fluid analysis.

Intuitively, an equivalent data-driven approximation $\mathcal{A}$ for the universal Koopman operator $\mathcal{U}$, as well as the finite-dimensional $\tilde{U}$, also exists. The $a \ posteriori$ observations deduced from
the fluid-structure system herein, as will be presented in the subsequent sections, allude to the existence of $\tilde{A}$. Nonetheless, the a priori deduction remains an area for exploration.

*Modal characterization of the reduced order Koopman operator*

The acquisition of $\tilde{A}$ by the similarity-matrix expression, or any other forms of $A$ by other methods, signals one’s comprehensive possession of a system’s underlying dynamics. What can be done to this tangible representation in the form of a matrix is only limited by the bounds of linear algebra. The default procedure for both the DMD and KMD, however, characterizes the modal features of $\tilde{A}$ by an eigen decomposition, which is what this work has adopted.

An eigen-Ritz decomposition yields

$$\tilde{A}W = WA,$$  \hspace{1cm} (AI.10)

where $W$ contains the eigenvectors (Ritz vectors) $w_j$, and $A$ contains the corresponding eigenvalues (Ritz values) $\lambda_j$.

The eigen tuples yield the exact DMD modes or the Koopman modes as

$$\Phi = X_2 V \Sigma^{-1} W,$$  \hspace{1cm} (AI.11)

where $\Phi$ contains the mode shape $\phi_j$.

Every mode $\phi_j$ corresponds to a physical frequency $\omega_j$ in continuous time,

$$\omega_j = \Im \left\{ \log(\lambda_j) \right\} / \Delta t,$$  \hspace{1cm} (AI.12)

and a growth/decay rate $g_j$

$$g_j = \Re \left\{ \log(\lambda_j) \right\} / \Delta t.$$  \hspace{1cm} (AI.13)

The sum of the Ritz descriptors expectedly returns the Koopman approximation of the input sequence, or the Koopman system,
\[ x_{\text{Koopman},i} = \sum_{j=1}^{r} \phi_j \exp(\omega_j t) \alpha_j, \quad (AI.14) \]

where \( \alpha_j \) is the coefficient of weight, or the modal amplitude, of \( \phi_j \). It is conveniently acquired through a mapping with respect to the initial conditions,

\[ \alpha = \Phi^\dagger x_i, \quad (AI.15) \]

where the superscript \( \dagger \) denotes the Moore-Penrose pseudoinverse.

One may easily assess the accuracy of the Koopman system for each spatial and temporal dimension by comparing the difference between the original and Koopman reconstructed sequences. This work adopts the instantaneous, spatially-averaged, and \( l_2 \)-normalized reconstruction error,

\[ \|e\|_{2,i} = \frac{1}{n} \sum_{k=1}^{n} \left[ \frac{(x_{\text{Koopman},k,i} - x_{k,i})}{x_{k,i}} \right]^2 \frac{1}{2}, \quad (AI.16) \]

and its rms value

\[ \|e\|_{2,\text{rms},i} = \left[ \frac{1}{n} \sum_{k=1}^{n} \left( \frac{x_{\text{Koopman},k,i} - x_{k,i}}{x_{k,i}} \right)^2 \right]^{1/2}. \quad (AI.17) \]

as the indices of quantification.
Appendix II

Details of the observables

Fluid – Primary Observables

The static pressure $p$, or gauge pressure relative to the operating pressure ($1 \text{ atm}=101,325 \text{ Pa}$), on the top (BC), bottom (DA), upstream (AB), and downstream (CD) wall make up the four primary observables that describe structure response, as depicted by figure 7.

Fluid – Primary Observables

The total pressure, or the pressure at the thermodynamic state that would exist is the fluid were brought to zero velocity and zero potential, was sampled for the flow field. For this specific low-Mach number flow, air is treated as incompressible, therefore the Bernoulli’s equation,

$$P_o = p + p_{dyn} \quad (\text{AII.1})$$

is used to calculate the total pressure $P_o$, where $p$ is static pressure, $p_{dyn}$ is the local dynamic pressure

$$p_{dyn} = \frac{1}{2}\rho \mathbf{v}^2. \quad (\text{AII.2})$$

For simplicity, the wall static pressure and total pressure field will be denoted by $P$ in the ensuing discussions.

The velocity field generated another four observables, which are the $u$, $v$, and $w$ components of the velocity vector $\mathbf{u}$ along the $x$-, $y$-, and $z$-axis of the Euclidean space, respectively, and the velocity magnitude

$$|\mathbf{U}|=(u^2+v^2+w^2)^{1/2}. \quad (\text{AII.3})$$

$P$, $u$, $v$, $w$, and $|\mathbf{U}|$ make up the five primary observables that describe the flow field.
Fluid – Turbulence

Four turbulence-related observables were sampled for the flow field as well, which include the Reynolds stress $\langle u'v' \rangle$, $\langle u'w' \rangle$, and $\langle v'w' \rangle$, as well as the turbulence kinetic energy $\langle k \rangle$

$$\langle k \rangle = \frac{1}{2}(\langle u'u' \rangle + \langle v'v' \rangle + \langle w'w' \rangle), \quad (AII.4)$$

where $u'$, $v'$, and $w'$ denote the fluctuating components of $u$, $v$, and $w$, respectively. They are obtained by the Reynolds decomposition. Evidently, the turbulence observables correspond to the isotropic and deviatoric components of the Reynolds stress tensor.

Fluid – Vortex Identification Criteria

First Generation: Vorticity-Based

The last five observables are mathematically derived criteria that have been proposed to identify vortical structures in a flow. The first criterion is the most primitive, or the so-called first-generation criterion, vorticity magnitude $|\omega|$, developed according to the concept of vortex filament/tube after the Helmholtz’s theorems (Helmholtz, 1858)

$$|\omega| = \left(\omega_x^2 + \omega_y^2 + \omega_z^2\right)^{1/2} = |\nabla \times u|, \quad (AII.5)$$

Despite common usage of vorticity-based criteria, fundamental differences exist between the vorticity and a vortex. Vorticity is regarded as the spin of an infinitesimal fluid particle and can be calculated for essential every point inside a fluid domain. A vortex, although its definition remains highly controversial till today, can be generally perceived as the rotation of a fluid region. A classic counter-example is the laminar boundary layer, in which the viscous effect of the wall generates substantial vorticity, but no rotational motion (i.e., vortex). The maximum vorticity magnitude also does not necessarily occur in the central region of a vortex (C. Liu et al., 2019).

Second Generation: Eigenvalue-Based

The second-generation criterion, as defined by Liu et al. (C. Liu et al., 2019), is the eigenvalue-based method which stems from the notion that a vortex can be seen as a region of closed or spiralling streamlines (Lugt & Gollub, 1998; Robinson et al., 1989). These criteria are generally based on the velocity gradient, or the rate-of-strain, tensor, $L$. 

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\[
\n\nabla u = \frac{\partial u}{\partial x} = L = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\
\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\
\frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}
\end{bmatrix}
\]  

(AII.6)

Finding the three eigenvalues of \( L \), \( \lambda_1, \lambda_2, \) and \( \lambda_3 \), the characteristic equation with Galilean invariance can be written as

\[
\lambda^3 + P\lambda^2 + q\lambda + R = 0.
\]

(AII.7)

Galilean invariance means the result remains the same if the flow field undergoes a translation or superposition by a uniform field.

The three invariants of \( L \) are

\[
P = -\text{tr}(L)
\]

(AII.8)

\[
q = -\frac{1}{2} \left[ \text{tr}(\nabla u^2) - \text{tr}(L)^2 \right]
\]

(AII.9)

\[
r = -\text{det}(L)
\]

(AII.10)

For incompressible flows, the first invariant \( P = 0 \).

The most common vortex identification criterion is the \( q \)-criterion, which is proposed by (Hunt et al., 1988) and is directed calculated by the second invariant \( q \). The \( q \)-criterion is also conveniently calculated and conceptually developed by the following expression,

\[
q = \frac{1}{2} \left( \frac{1}{2} \|Q\|^2 - \|S\|^2 \right)
\]

(AII.11)

where \( s \) and \( Q \) are the symmetric and antisymmetric parts of \( L \), or the

\[
S = \frac{1}{2} (\nabla u + \nabla u^T)
\]

(AII.12)

\[
Q = \frac{1}{2} (\nabla u - \nabla u^T)
\]

(AII.13)

Theoretically, a region in which \( q > 0 \), and that its pressure is lower than the surrounding, defines a vortex. In practice, the pressure requirement is often overlooked, and a user-defined threshold shall be defined to produce meaningful identification.
Another common criterion is the Galilean invariant \( \lambda_2 \)-criterion proposed by (Jeong & Hussain, 1995). The basis for this criterion is the observation that a local pressure minimum in a plane fails to identify vortices under strong unsteady and viscous effects (C. Liu et al., 2019). This method determines whether an arbitrary point in a flow field is a vortex core, thus finding the connected regions that make up a vortex. Formulation-wise, the \( \lambda_2 \) is calculated by finding the second eigenvalue of \( \mathbf{Q}^2 + \mathbf{s}^2 \), where \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \). Theoretically, a vortex core is identified by \( \lambda_2 < 0 \). However, in practice, a user-defined threshold required.

**Third Generation: Ratio-Based**

The problem of the second-generation criterion is related to the user-defined threshold. As pointed out by Liu et al. (C. Liu et al., 2016), the threshold is case- and resolution-sensitive, so defining a proper one is usually an experience-based task. Improper definition results in issues like incorrect vortex capture and breakdown, the coexistence of weak and strong vortices, etc. To provide a universal index, the original \( \Omega \)-criterion was proposed (C. Liu et al., 2016).

The \( \Omega \)-criterion is established upon a critical understanding that a vortex is a region where the vorticity overtakes deformation. Therefore, \( 0 \leq \Omega \in \mathbb{R}^+ \leq 1 \) is defined as the ratio of vorticity tensor and the sum of the vorticity and deformation tensors,

\[
\Omega = \frac{\| \mathbf{Q} \|^2_F}{\| \mathbf{s} \|^2_F + \| \mathbf{Q} \|^2_F + \epsilon_o},
\]

(AII.14)

where \( \epsilon_o \) is a small number to avoid division by zero. The threshold of the \( \Omega \)-criterion is universal, \( \Omega > 0.51 \) or 0.52 effectively detects a vortex.

On top of the \( \Omega \)-criterion, an improved variant, called the Omega-Liutex, has been proposed (J. Liu & Liu, 2019). The \( \Omega_R \)-criterion adds the capability of defining the local rotational axis by considering the eigenvector of the velocity gradient tensor, readers can refer to (Gao et al., 2019; C. Liu et al., 2018, 2019) for details. The \( \Omega_R \)-criterion \( 0 \leq \Omega_R \in \mathbb{R}^+ \leq 1 \) is defined as

\[
\Omega_R = \frac{\beta^2}{\beta^2 + \zeta^2 + \epsilon_o},
\]

(AII.15)

where

\( \beta \) and \( \zeta \) are

\[
\beta = \frac{\mathbf{Q} \cdot \mathbf{Q}}{\| \mathbf{Q} \|^2_F},
\]

\[
\zeta = \frac{\mathbf{s} \cdot \mathbf{s}}{\| \mathbf{s} \|^2_F}.
\]
\[
\zeta = \frac{1}{2} \left( \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^2 \right)^{1/2},
\] 
(AII.16)

\[
\beta = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right),
\] 
(AII.17)
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Conflict of Interest
The authors declare that they have no conflict of interest.

Availability of Data and Material
The datasets generated during and/or analysed during the current work are restricted by provisions of the funding source but are available from the corresponding author on reasonable request.

Code Availability
The custom code used during and/or analysed during the current work are restricted by provisions of the funding source.

Author Contributions
All authors contributed to the study conception and design. Funding, project management, and supervision were performed by Tim K.T. Tse and Zengshun Chen. Material preparation, data collection, and formal analysis were led by Cruz Y. Li and Zengshun Chen, and assisted by Asiri Umenga Weerasuriya and Yunfei Fu. The first draft of the manuscript was written by Cruz Y. Li and all authors commented on previous versions of the manuscript. All authors read, contributed, and approved the final manuscript.
Compliance with Ethical Standards

All procedures performed in this work were in accordance with the ethical standards of the institutional and/or national research committee and with the 1964 Helsinki declaration and its later amendments or comparable ethical standards.

Consent to Participate

Informed consent was obtained from all individual participants included in the study.

Consent for Publication

Publication consent was obtained from all individual participants included in the study.