GEOMETRIC REALIZATION AND K-THEORETIC DECOMPOSITION OF C*-ALGEBRAS

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Abstract. Suppose that $A$ is a separable C*-algebra and that $G_*$ is a (graded) subgroup of the $\mathbb{Z}/2$-graded group $K_* (A)$. Then there is a natural short exact sequence

(*) \[ 0 \to G_* \to K_*(A) \to K_*(A)/G_* \to 0. \]

In this note we demonstrate how to geometrically realize this sequence at the level of C*-algebras. As a result, we KK-theoretically decompose $A$ as

\[ 0 \to A \otimes K \to A_{f} \to SA_{t} \to 0 \]

where $K_*(A_{t})$ is the torsion subgroup of $K_*(A)$ and $K_*(A_{f})$ is its torsionfree quotient. Then we further decompose $A_{t}$: it is KK-equivalent to $\oplus_p A_p$ where $K_*(A_p)$ is the $p$-primary subgroup of the torsion subgroup of $K_*(A)$. We then apply this realization to study the Kasparov group $K^*(A)$ and related objects.

In Section 1 we produce the basic geometric realization. For any separable C*-algebra $A$ and group $G_*$ we produce associated C*-algebras $A_s$ (s for subgroup) and $A_q$ (q for quotient group) and, most importantly, a short exact sequence of C*-algebras

\[ 0 \to A \otimes K \to A_{q} \to SA_{s} \to 0 \]

whose associated $K_*$-long exact sequence is (*). In the case where $G_*$ is the torsion subgroup of $K_*(A)$ we use the notation $A_t$ (t for torsion) and $A_f$ (f for torsionfree) respectively. We further decompose $A_t$ into its $p$-primary summands $A_p$ for each prime $p$.

Section 2 deals with the following question: may calculations of the Kasparov groups $KK_*(A, B)$ be reduced down to the four cases $(A_t, B_t), (A_t, B_{f}), (A_f, B_t)$ and $(A_f, B_f)$? We show that this is indeed possible in a wide variety of situations. Sections 3 and 4 deal with these special cases.

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Geometric realization as a general technique was introduced to topological K-theory of spaces by M. F. Atiyah [1] in his proof of the Küneth theorem for $K^*(X \times Y)$. We adapted the technique [6] to prove the corresponding theorem for the $K$-theory for $C^*$-algebras and used it with J. Rosenberg in our proof of the Universal Coefficient Theorem (UCT) [4].

1. Geometric Realization

In this section we produce the main geometric realization and we extend the result to give a $p$-primary decomposition for a $C^*$-algebra.

Let $\mathcal{N}$ denote the bootstrap category [6, 4].

**Theorem 1.1.** Suppose that $A$ is a separable $C^*$-algebra. Let $G_*$ be some subgroup of $K_*(A)$. Then there is an associated $C^*$-algebra $A_s \in \mathcal{N}$, a separable $C^*$-algebra $A_q$, and a short exact sequence

\begin{equation}
0 \to A \otimes K \to A_q \to S A_s \to 0
\end{equation}

whose induced $K$-theory long exact sequence fits into the commuting diagram

\begin{equation}
\begin{array}{cccccc}
0 & \to & K_*(A_s) & \to & K_*(A \otimes K) & \to & K_*(A_q) & \to & 0 \\
\downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\
0 & \to & G_* & \to & K_*(A) & \to & K_*(A)/G_* & \to & 0.
\end{array}
\end{equation}

If $A$ is nuclear then so is $A_q$. If $A \in \mathcal{N}$ then so is $A_q$. If $A \in \mathcal{N}$ and if $G_*$ is a direct summand of $K_*(A)$ then $A$ is $KK$-equivalent to $A_s \oplus A_q$.

Note that we think of $A_s$ as realizing the subgroup $G_*$ and $A_q$ as realizing the quotient group $K_*(A)/G_*$, hence the notation.

**Proof.** Let $A_s$ denote any $C^*$-algebra in $\mathcal{N}$ with

$K_*(A_s) \cong G_*$. Such $C^*$-algebras exist and are unique up to $KK$-equivalence by the UCT [4]. Let

$\theta : K_*(A_s) \to K_*(A)$

be the corresponding homomorphism. Since $A_s \in \mathcal{N}$, the UCT holds for the pair $(A_s, A)$, and so $\theta$ is in the image of the index map

$\gamma : KK_*(A_s, A) \to Hom_Z(K_*(A_s), K_*(A))$.

Say that

$\theta = \gamma(\tau)$

for some

$\tau \in KK_0(A_s, A)$. As $A_s$ is nuclear,

$KK_0(A_s, A) \cong Ext(S A_s, A)$.
and hence $\tau$ corresponds to an equivalence class of extensions of $C^*$-algebras of the form

$$0 \to A \otimes K \to E \to SA_s \to 0.$$ 

Define $A_q = E$. (This choice depends upon the choice of $A_s$ among its $KK$-equivalence class and the choice of $\tau$ modulo the kernel of $\gamma$). Note that $E$ is nuclear/bootstrap if and only if $A$ is nuclear/bootstrap. Then the diagram

$$
\begin{array}{cccc}
K_j(A_q) & \longrightarrow & K_j(SA_s) & \stackrel{\delta}{\longrightarrow} & K_{j-1}(A \otimes K) & \longrightarrow & K_{j-1}(A_q) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
K_{j-1}(A_s) & \stackrel{\theta}{\longrightarrow} & K_{j-1}(A) & & & & 
\end{array}
$$

commutes, and thus $\delta$ is mono and the long exact $K_*$-sequence breaks apart as shown.

If $G_*$ is a direct summand of $K_*(A)$ then

$$K_*(A) \cong G_* \oplus K_*(A)/G_* \cong K_*(A_s) \oplus K_*(A_q) \cong K_*(A_s \oplus A_q)$$

and, replacing algebras by their suspensions as needed, the $KK$-equivalence is obtained.

□

Henceforth we shall regard $A_s$ and $A_q$ as $C^*$-algebras associated to $A$ and $G_*$, with the understanding that these are well-defined only up to $KK$-equivalence modulo the kernel of $\gamma$, as explained above.

The next step is to decompose $A_t$ into its $p$-primary components.

**Theorem 1.4.** Let $A \in \mathcal{N}$ and suppose that $K_*(A)$ is a torsion group, so that $A = A_t$. Then $A$ is $KK$-equivalent to a $C^*$-algebra $\oplus A_p$, where

$$K_*(A_p) \cong K_*(A)_p$$

the $p$-primary torsion subgroup of $K_*(A)$.

**Proof.** For each prime $p$, choose $N_{(p)} \in \mathcal{N}$ with $K_1(N_{(p)}) = 0$ and

$$K_0(N_{(p)}) \cong \mathbb{Z}_{(p)}$$

the integers localized at $p$. Define

$$A_p = A_t \otimes N_{(p)}.$$ 

The Künneth formula [6] implies that

$$K_*(A_p) \cong K_*(A_t \otimes N_{(p)}) \cong K_*(A_t) \otimes K_*(N_{(p)}) \cong K_*(A_t) \otimes \mathbb{Z}_{(p)} \cong K_*(A)_p$$

as desired. Then

$$K_*(\oplus_p A_p) \cong \oplus_p K_*(A_p) \cong \oplus_p K_*(A)_p \cong K_*(A_t)$$

and another use of the UCT implies that $A_t$ is $KK$-equivalent to $\oplus_p A_p$.

□
We summarize:

**Theorem 1.5.** Suppose that $A$ is a separable $C^*$-algebra. Then there is an associated $C^*$-algebra $A_t \in \mathcal{N}$, a separable $C^*$-algebra $A_f$, and a short exact sequence

\begin{equation}
0 \to A \otimes K \to A_f \to SA_t \to 0
\end{equation}

whose induced $K$-theory long exact sequence fits into the commuting diagram

\begin{equation}
\begin{array}{cccccc}
0 & \to & K_*(A_t) & \to & K_*(A \otimes K) & \to & K_*(A_f) & \to & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 & \to & K_*(A)_t & \to & K_*(A \otimes K) & \to & K_*(A)_f & \to & 0.
\end{array}
\end{equation}

If $A$ is nuclear then so is $A_f$. If $A \in \mathcal{N}$ then so is $A_f$. Further, the $C^*$-algebra $A_t$ has a $p$-primary decomposition: it is $KK$-equivalent to a $C^*$-algebra $\bigoplus_p A_p$, where $A_p \in \mathcal{N}$ for all $p$ and

$$K_*(A_p) \cong K_*(A)_p$$

the $p$-primary torsion subgroup of $K_*(A)$. Finally, if $A \in \mathcal{N}$ and $K_*(A)_t$ is a direct summand of $K_*(A)$ then $A$ may be replaced by the $KK$-equivalent $C^*$-algebra $A_t \oplus A_f$.

\[\square\]

2. Splitting the Kasparov Groups

If $A$ and $B$ are in $\mathcal{N}$ and their $K$-theory torsion subgroups $K_*(A)_t$ and $K_*(B)_t$ are direct summands then the final conclusion of Theorem 1.5 implies that we may reduce the computation of $KK_*(A, B)$ to the calculation of the four groups, namely

1. $KK_*(A_t, B_t)$
2. $KK_*(A_t, B_f)$
3. $KK_*(A_f, B_t)$
4. $KK_*(A_f, B_f)$.

We discuss the calculation of those groups in subsequent sections. In this section we see what can be done *without* assuming that the torsion subgroups are direct summands.

**Theorem 2.1.** Suppose that $A \in \mathcal{N}$ and $K_*(B)$ is torsionfree. Then there is a short exact sequence

\begin{equation}
0 \to KK_*(A_f, B) \to KK_*(A, B) \to KK_*(A_t, B) \to 0.
\end{equation}

In particular, letting $K^*(A) = KK_*(A, \mathbb{C})$ there is a short exact sequence

\begin{equation}
0 \to K^*(A_f) \to K^*(A) \to K^*(A_t) \to 0.
\end{equation}

If $K_*(B)$ is not necessarily torsionfree, then sequence 2.2 is exact if and only if the natural map...
(2.4) \[ \theta^*_h : \text{Hom}_\mathbb{Z}(K_*(A), K_*(B)) \to \text{Hom}_\mathbb{Z}(K_*(A_t), K_*(B)) \]

is onto, where \( \theta : K_*(A_t) \to K_*(A) \) is the canonical inclusion.

Note that the map \( \theta^*_h \) in (2.4) is frequently onto. This is the case, for instance, if \( K_*(A_t) \) is a direct summand of \( K_*(A) \).

The map \( \theta \) is, up to isomorphism, the boundary homomorphism in the \( K_* \)-sequence associated to the short exact sequence

\[ 0 \to A \otimes K \to A_f \to SA_t \to 0 \]

and hence

\[ \theta(x) = x \otimes_{A_t} \delta \]

where \( \delta \in KK_1(A_t, A) \) by [9]. Thus the map \( \theta^*_h \) of (2.4) is induced from a \( KK \)-pairing.

**Proof.** Consider the commuting diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathbb{Z}(K_*(A_t), K_*(B)) & \to & \text{Ext}^1_\mathbb{Z}(K_*(A_f), K_*(B)) \\
\downarrow \beta & & \downarrow \theta^* \\
\text{Hom}_\mathbb{Z}(K_*(A_t), K_*(B)) & \to & \text{Ext}^1_\mathbb{Z}(K_*(A_t), K_*(B))
\end{array}
\]

The three middle rows are exact by the UCT, the middle column is exact by the exactness of \( KK \), and the two outer columns are exact by the standard \( Hom-Ext \)-sequence.

Suppose that \( K_*(B) \) is torsionfree. Then

(2.5) \[ \text{Hom}_\mathbb{Z}(K_*(A_t), K_*(B)) = 0 \]

since \( K_*(A_t) \) is a torsion group, and the surjectivity of \( \theta^*_e \) implies the surjectivity of \( \theta^* \).

If \( K_*(B) \) is not necessarily torsionfree, then the Snake Lemma [11] implies that there is an exact sequence

\[ 0 = \text{Coker}(\theta^*_e) \to \text{Coker}(\theta^*) \to \text{Coker}(\theta^*_h) \to 0 \]

and hence \( \theta^*_h \) is onto if and only if \( \theta^* \) is onto. The theorem then follows immediately, for the middle column of the diagram degenerates to (2.2) if and only if \( \theta^* \) is onto. \( \square \)

Next we consider the dual situation, when \( K_*(A) \) is a torsion group.
Theorem 2.6. Suppose that $A \in \mathcal{N}$ and that $K_*(A)$ is a torsion group. Then there is a natural exact sequence

$$0 \to KK_*(A, B_i) \to KK_*(A, B) \to KK_*(A, B_f) \to 0.$$ 

If $K_*(A)$ is not a torsion group then sequence (*) is exact if and only if the natural map

$$\pi_* : \text{Hom}_\mathbb{Z}(K_*(A), K_*(B)) \to \text{Hom}_\mathbb{Z}(K_*(A), K_*(B_f))$$

is onto, where $\pi : B \otimes K \to B_f$ is the natural map.

The proof of this result is dual to that of Theorem 2.1 and is omitted for brevity. □

3. Computing $KK_*(A_f, B)$

In this section we consider the case where $K_*(A)$ is torsionfree (so that $A = A_f$). Recall [2, 12] that a subgroup $H$ of an abelian group $K$ is pure if for each positive integer $n$,

$$nH = H \cap nG,$$

and an extension of groups

$$0 \to H \to K \to G \to 0$$

is pure if $H$ is a pure subgroup of $K$. For abelian groups $G$ and $H$, $\text{Pext}_\mathbb{Z}^1(G, H)$ is the subgroup of $\text{Ext}_\mathbb{Z}^1(G, H)$ consisting of pure extensions.

Recall [5, 8] that there is a natural topology on the Kasparov groups and that with respect to this topology the UCT sequence splittings constructed in [4] are continuous, so that the splitting is a splitting of topological groups [9].

Theorem 3.1. Suppose that $A \in \mathcal{N}$ and that $K_*(A)$ is torsionfree. Then there is a natural sequence of topological groups

$$0 \to \text{Pext}_\mathbb{Z}^1(K_*(A), K_*(B)) \to KK_*(A, B) \to \text{Hom}_\mathbb{Z}(K_*(A), K_*(B)) \to 0$$

The group $\text{Pext}_\mathbb{Z}^1(K_*(A), K_*(B))$ is the closure of zero in the natural topology on the group $KK_*(A, B)$ and thus the group $\text{Hom}_\mathbb{Z}(K_*(A), K_*(B))$ is the Hausdorff quotient of $KK_*(A, B)$.

Proof. The UCT gives us the sequence

$$0 \to \text{Ext}_\mathbb{Z}^1(K_*(A), K_*(B)) \to KK_*(A, B) \to \text{Hom}_\mathbb{Z}(K_*(A), K_*(B)) \to 0$$

which splits unnaturally. If $K_*(A)$ is torsionfree then

$$\text{Pext}_\mathbb{Z}^1(K_*(A), K_*(B)) \cong \text{Ext}_\mathbb{Z}^1(K_*(A), K_*(B)).$$

The remaining part of the theorem holds since we have shown in general [10] that the group $\text{Pext}_\mathbb{Z}^1(K_*(A), K_*(B))$ is the closure of zero in the natural topology on $KK_*(A, B)$ in the presence of the UCT. □

We note that the resulting algebraic problems are frequently very difficult. If $G$ is a torsionfree abelian group then $\text{Hom}_\mathbb{Z}(G, H)$ is unknown in general, though there is much known in special cases (cf. [2, 3]). The group $\text{Pext}_\mathbb{Z}^1(G, H)$ is also difficult, though the case $\text{Pext}_\mathbb{Z}^1(G, \mathbb{Z})$ is known (cf. [3]). We discuss $\text{Pext}$ in detail in [12].
4. Computing $KK_*(A_t, B)$

In this section we concentrate upon the situation when $K_*(A)$ is a torsion group. Before beginning, we digress slightly to recall [7] in more detail how one introduces coefficients into $K$-theory.

Given a countable abelian group $G$, select some $C^*$-algebra $N_G \in \mathcal{N}$ with

$$K_0(N_G) = G \quad K_1(N_G) = 0.$$ 

The $C^*$-algebra $N_G$ is unique up to $KK$-equivalence, by the UCT. Then for any $C^*$-algebra $A$, define

$$(4.1) \quad K_j(A; G) \cong K_j(A \otimes N_G).$$ 

The Künneth Theorem [6] implies that there is a natural short exact sequence

$$(4.2) \quad 0 \to K_j(A) \otimes G \xrightarrow{\alpha} K_j(A; G) \to Tor_1^\mathbb{Z}(K_{j-1}(A), G) \to 0$$ 

which splits unnaturally. If $G$ is torsionfree then $\alpha$ is an isomorphism

$$\alpha : K_j(A) \otimes G \xrightarrow{\cong} K_j(A; G).$$ 

Let $X(G) = \text{Hom}(G, \mathbb{R}/\mathbb{Z})$ denote the Pontryagin dual of the group $G$.

**Theorem 4.3.** Suppose that $A \in \mathcal{N}$ with $K_*(A)$ a torsion group and suppose that $K_*(B)$ is torsionfree, so that $A = A_t$ and $B = B_f$. Then:

1. $KK_*(A, B) \cong \text{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*-1}(B))$.
2. $KK_*(A, B) \cong \text{Hom}_{\mathbb{Z}}(K_*(A), K_{*-1}(B) \otimes \mathbb{Q}/\mathbb{Z})$.
3. The group $KK_*(A, B)$ is reduced and algebraically compact.
4. $K^j(A) \cong X(K_{j-1}(A))$.
5. More generally, if $K_*(B)$ is finitely generated free, then

$$KK_*(A, B) \cong \bigoplus_n X(K_{j-1}(A))$$ 

where $n$ is the number of generators of $K_*(B)$.

**Proof.** Part 1) follows at once from the UCT and the fact that there are no non-trivial homomorphisms from a torsion group to a torsionfree group. Part 2) follows from Part 1) by elementary homological algebra. Part 3) follows easily from a deep result of Fuchs and Harrison [cf. 2, 46.1]: if $G$ is a torsion group then any group of the form $\text{Hom}_{\mathbb{Z}}(G, H)$ is reduced and algebraically compact. Part 4) follows from part 3) by setting $B = \mathbb{C}$ and observing that for any torsion group $G$, we have

$$X(G) = \text{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$$ 

$\square$

There is one additional case that fits into the present discussion and which partially overlaps with the result above.
Theorem 4.4. Suppose that $A \in \mathcal{N}$ and that $K_*(A)$ has no free direct summand. Then there is a natural short exact sequence of topological groups

\begin{equation}
0 \to \Hom_{\mathbb{Z}}(K_*(A), \mathbb{R}) \to \mathbf{X}(K_*(A)) \xrightarrow{\chi} K^*(A) \to 0.
\end{equation}

The map $\chi : \mathbf{X}(K_*(A)) \to K^*(A)$ is a degree one continuous open surjection. It is a homeomorphism if and only if $K_*(A)$ is a torsion group.

To be explicit about the grading,

$$\chi : \mathbf{X}(K^j(A)) \to K^{j-1}(A)$$

which is the usual parity shift as torsion phenomena move from homology to cohomology.

Proof. The UCT for $K^*(A)$ has the form

$$0 \to \Ext^1_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \xrightarrow{\delta} K^*(A) \to \Hom_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to 0$$

with $\delta$ of degree one, so it suffices to compute $\Ext$. In general the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

yields a long exact sequence

$$\Hom_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to \Hom_{\mathbb{Z}}(K_*(A), \mathbb{R}) \to \mathbf{X}(K_*(A)) \to \Ext^1_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to 0.$$

The fact that $K_*(A)$ has no free direct summand implies that $\Hom_{\mathbb{Z}}(K_*(A), \mathbb{Z}) = 0$, so the sequence degenerates to

$$0 \to \Hom_{\mathbb{Z}}(K_*(A), \mathbb{R}) \to \mathbf{X}(K_*(A)) \to \Ext^1_{\mathbb{Z}}(K_*(A), \mathbb{Z}) \to 0.$$

Applying the UCT one obtains the sequence 4.5 as desired. The map $\chi$ is the composite of the UCT map and a natural homeomorphism. The rest of the Theorem is immediate.

$\square$
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