Abstract

A rigorous Path Integral construction for a wide class of Weyl evolution operators is based on a pseudo-differential $Ω$-calculus on flat phase spaces of finite and infinite dimensions.

1 Introduction

Among the various uses of the Path Integrals the earliest and foremost is the application to quantum evolution. The Path Integral formalism is appreciated because of its compact operational notation with all the gamut of the finite dimensional Integral Calculus, such as integration by parts, repeated integration, canonical substitutions, analytic continuation (Wick rotation), stationary phase approximations etc.

However, in R.Feynman words, the Path Integral is “an intuitive leap at mathematical formalism”. A natural justification would be by suitable integral approximations, the route chosen originally by Feynman himself in the 40’s via time-slicing and discretization. Unfortunately the discretization ambiguities along with the convergence problems have plagued the deed from the start (the notable exception [15] is for rather special hamiltonians).

Nevertheless we propose a rigorous time-slicing construction of the (flat) phase space Path Integral for propagators both in Quantum Mechanics and Quantum Field Theory for a fairly general class of quasi-dissipative quantum observables (e.g. the Schrödinger hamiltonians
with smooth scalar potentials of any power growth). Moreover we allow time-dependent hamiltonians and a great variety of discretizations, in particular, the standard, Weyl, and normal ones.

**Abstract Cauchy Problem.** Consider the Initial Value Problem

\[
\frac{d\psi}{dt}(t) + A(t)\psi(t) = 0, \quad \psi(0) = \psi_0, \quad 0 \leq t \leq T, \quad \psi(t) \in \mathcal{H},
\]

wherein \(\mathcal{H}\) is a Hilbert space and \(A(t)\) is a family of (usually) unbounded operators on \(\mathcal{H}\).

The Cauchy Problem is called proper relative a dense subspace \(S\) in \(\mathcal{H}\) if there is the unique solution \(\psi\) for every \(\psi_0 \in S\) and the **Evolution Operator**

\[
U(t'', t')\psi(t') = \psi(t''), \quad \psi(t') \in S, \quad t'' > t',
\]

is bounded on \(\mathcal{H}\).

The Evolution Operator may be sought in the time-slicing form of the strong operator limit **Product Integral**:

\[
U(t'', t') = \prod_{t'' \geq t \geq t'} \exp[-A(t)dt] := \lim_{|P| \to 0} U^P(t'', t') := \lim_{|P| \to 0} \prod_{t_{j+1} > t_j} \exp[-A(t_j)\Delta t_j].
\]

Here \(P\) is a finite partition \(0 \leq t' = t_0 < \cdots < t_j < t_{j+1} < \cdots < t'' = t_P \leq T\) of the interval \(t' \leq t \leq t''\), \(\Delta t_j = t_{j+1} - t_j\), \(|P| = \max_j |\Delta t_j|\).

**From Product to Path Integrals on the Phase Space.** Consider a quantum evolution equation on \(L^2(\mathbb{R}^d)\)

\[
\frac{\partial \psi}{\partial t}(t, q) + \frac{i}{\hbar}f(t, q, \frac{i}{\hbar} \frac{\partial}{\partial q})\psi(t, q) = 0, \quad \psi(0, q) = \psi_0,
\]

with a pseudodifferential operator \(f(t, q, \frac{i}{\hbar} \frac{\partial}{\partial q})\) in the standard form of the \(qp\)-quantization of a complex-valued function \(f(t, q, p)\) on the phase space \(\mathbb{R}^{2d}\), the standard symbol of \(f(t, q, \frac{i}{\hbar} \frac{\partial}{\partial q})\).

The Tobocman’s version of the Dirac-Feynman Ansatz: For small \(\Delta t\) the standard symbol \(<p|U(t + \Delta t, t)|q>\) of the propagator \(U(t + \Delta t, t)\)
is approximately equal to \( \exp[-\frac{i}{\hbar}f(t,q,p)\Delta t] \). Then according to the product rule for the standard symbols, the standard symbol of \( U^P \) is approximately equal to the distributional multiple integral

\[
\int \prod_{j=1}^{d-1} d\lambda(t_j) \exp \frac{i}{\hbar} \sum_{j=0}^{d-1} [p(t_{j+1})\Delta q(t_j) - f(t_j, q(t_j), p(t_{j+1})\Delta t_j]
\]

where \( d\lambda(t_j) = (2\pi\hbar)^{-d} dq(t_j) dp(t_j) \) is the Lebesgue-Liouville measure on the phase space \( \mathbb{R}^{2d} \).

As the mesh \( |P| \to 0 \), the multiple integrals are presumed to converge to a Hamiltonian Path Integral for the standard symbol of the evolution operator \( U(t'', t') \)

\[
\int \prod_{t'' \geq t \geq t'} d\lambda(t) \exp \frac{i}{\hbar} \int_{t'}^{t''} [p(t)\dot{q}(t) - f(t, q(t), p(t)] dt
\]

where \( d\lambda(t) = (2\pi\hbar)^{-d} dq(t) dp(t) \) is "the Feynman-Liouville measure" on the space of paths from \( q(t') = q \) to \( p(t'') = p \) in \( \mathbb{R}^{2d} \) and

\[
\int_{t'}^{t''} [p(t)\dot{q}(t) - f(t, q(t), p(t)] dt
\]

is the hamiltonian symplectic action functional on that space.

The standing physical presumption: all calculus rules are valid in the limit. However there are two fundamental mathematical problems: The validity of the DFT-ansatz and the existence of the limit. So far both problems have been settled only for special \( f \) (c.f. [15, 14, 9]).

**Euler Detour.** Euler polygonal approximation for the solution of the quantum Cauchy Problem

\[
\frac{d\psi}{dt} + \hat{f}(t)\psi(t) = 0
\]

is the finite difference approximation

\[
\frac{\psi(t_{j+1}) - \psi(t_j)}{\Delta t_j} + \hat{f}(t_j)\psi(t_j) = 0, \quad \psi(t_0) = \psi_0,
\]
or
\[ \psi(t_{j+1}) = (1 - \hat{f}(t_j) \Delta t_j) \psi(t_j), \]
so that the Evolution Operator might be the strong operator limit
\[ U(t'', t') = \lim_{|\mathcal{P}| \to 0} \prod_j (1 - \hat{f}(t_j) \Delta t_j) := \prod_{t'' \geq t' \geq t} [1 - \hat{f}(t) dt]. \]

If \( \hat{f}(t) \) is a standard pseudodifferential operator then the partial product approximations are pseudodifferential operators again. However, if the order of \( \hat{f}(t) \) is positive, then the order of these pseudodifferential operator approximations increases to infinity and the convergence of their symbols is out of the control.

Fortunately, the backward Euler approximation
\[ \frac{\psi(t_{j+1}) - \psi(t_j)}{\Delta t_j} + A(t_{j+1}) \psi(t_{j+1}) = 0 \]
suggests
\[ U(t'', t') = \prod_{t'' \geq t' \geq t} (1 + \hat{f}(t) dt)^{-1} \]
with zero order approximation symbols.

Our main result is that for apt functions \( f \) this backward approximation entails (in the spirit of the DFT ansatz)
\[ U(t'', t') = \prod_{t'' \geq t' \geq t} [(1 - f(t) dt)^{-1}] \]
leading to a Path Integral representation of the symbol \( \langle p | U(t'', t') | q \rangle \).

Incidentally, the Green function (the coordinate propagator) can be easily expressed via the symbol:
\[ \langle q'' | U(t'', t') | q' \rangle = \int dp \langle q'' | p \rangle \langle p | U(t'', t') | q' \rangle . \]
2 Rigorized $\Omega$-symbolic calculus

This section provides necessary technical tools.

For $z = (q, p) \in \mathbb{R}^d \times \mathbb{R}^d$ introduce the complex coordinates $z^+ = 2^{-1/2}(q + ip)$, $z^- = 2^{-1/2}(q - ip)$ so that the standard symplectic form $[(p_1, q_1), (p_2, q_2)] = p_1 q_2 - p_2 q_1$ on $\mathbb{R}^{2d}$ becomes

$$\frac{1}{i} [z_1, z_2] = \frac{1}{i} (z_1^+ z_2^- - z_1^- z_2^+).$$

The $\hbar$-Symplectic Fourier transform is defined as

$$\hat{f}(\zeta) = \int f(z) e^{\frac{i}{\hbar} [z, \zeta]} d\lambda_\hbar(z), \quad d\lambda_\hbar(z) = \frac{1}{(\pi \hbar)^d} dz^+ dz^-.$$

Heisenberg Canonical Commutation Relations $z \mapsto \hat{z}$ over $\mathbb{R}^{2d}$ in a Hilbert space $\mathcal{H}$ is a linear map of $\mathbb{R}^{2d}$ to essentially self-adjoint operators on a common invariant subspace $\mathcal{G}$ of $\mathcal{H}$ (the Gårding domain) such that $[\hat{z}_1, \hat{z}_2] = \hbar [z_1, z_2] 1$.

E.g., for the Schrödinger (position) representation on $L^2(\mathbb{R}^d)$

$$\hat{z} \psi(x) = (qx) \psi(x) + \frac{\hbar}{i} \frac{\partial \psi}{\partial x}(x),$$

the Schwartz space $S(\mathbb{R}^d)$ may be chosen as a Gårding domain.

Other examples are the momentum or mixed momentum-position representations, holomorphic Bargmann-Segal representation (conducive to the coherent states Path Integral), Gelfand-Zak representation in the Solid State Physics, the Cartier compact representation in $\theta$-functions.

By the von Neumann-Stone theorem, for given $\hbar > 0$ any Heisenberg Canonical Commutation Relations is unitary equivalent to a direct sum of the Schrödinger representations. Thus we may chose the Gårding domain $\mathcal{G}(\mathcal{H})$ to be unitary equivalent to a direct sum of the spaces $S(\mathbb{R}^d)$.

Correspondingly, the dual space $\mathcal{G}'(\mathcal{H})$ is unitary equivalent to a direct sum of the spaces $S'(\mathbb{R}^d)$. 
Weyl operators $\hat{f}$ on $\mathcal{H}$ associated with generalized functions $f \in \mathbb{R}^{2d}$ are continuous linear operators from $\mathcal{G}(\mathcal{H})$ to $\mathcal{G}'(\mathcal{H})$:

$$\hat{f} = \int \tilde{f}(\zeta) \exp \frac{1}{\hbar} [\zeta, \hat{z}] d\lambda_\hbar(\zeta)$$

wherein

$$[\zeta, \hat{z}] = \zeta^+ \hat{z}^- - \zeta^- \hat{z}^+.$$

A version of the Schwartz Kernel Theorem states that a linear operator from $\mathcal{G}(\mathcal{H})$ to $\mathcal{G}'(\mathcal{H})$ is continuous if and only if it is a Weyl operator $\hat{f}$.

$\Omega$-symbols. Consider a formal power series over $\mathbb{C}$

$$\Omega(\zeta) = 1 + \sum_{|\alpha| > 0} c_\alpha z^\alpha.$$

A formal $\Omega$-symbol of $f \in \mathcal{S}'(\mathbb{R}^{2d})$ is the formal power series over $\mathcal{S}'(\mathbb{R}^{2d})$ defined via

$$\tilde{f}^\Omega(\zeta) = \tilde{f} / \Omega(\zeta).$$

Obviously, this has sense for polynomial $f(z)$ when various $\Omega$ provide common ordering rules according to the following table (c.f. [1]):
| Name                        | $\Omega(\zeta)$ | Ordering $(d = 1)$ |
|-----------------------------|-----------------|-------------------|
| Weyl                        | 1               | $q^n p^m \leftrightarrow \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} \hat{q}^{n-j} \hat{p}^m \hat{q}^j$ |
| Standard (qp or Kohn-Nirenberg) | $e^{\frac{1}{4}((\zeta^+)^2 - (\zeta^-)^2)}$ | $q^n p^m \leftrightarrow \hat{q}^n \hat{p}^m$ |
| Antistandard (or pq)        | $e^{-\frac{1}{4}((\zeta^+)^2 - (\zeta^-)^2)}$ | $q^n p^m \leftrightarrow \hat{p}^m \hat{q}^n$ |
| Normal (or Wick)            | $e^{\frac{1}{2}\zeta^+ \zeta^-}$ | $(z^+)^n (z^-)^m \leftrightarrow (\hat{z}^+)^n (\hat{z}^-)^m$ |
| Antinormal (or Anti-Wick)   | $e^{-\frac{1}{2}\zeta^+ \zeta^-}$ | $(z^+)^n (z^-)^m \leftrightarrow (\hat{z}^-)^m (\hat{z}^+)^n$ |
| Symmetric                   | $\cos\frac{1}{4}[(\zeta^+)^2 - (\zeta^-)^2] \leftrightarrow \frac{1}{2}(\hat{q}^n \hat{p}^m + \hat{p}^m \hat{q}^n)$ |
| Born-Jordan                 | $\sin\frac{1}{4}[(\zeta^+)^2 - (\zeta^-)^2] \leftrightarrow \frac{1}{m+1} \sum_{j=0}^{m} \hat{p}^{m-j} \hat{q}^n \hat{p}^j$ |
Suppose now that $\Omega(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^{2d}$. Then $\tilde{f}(\zeta)/\Omega(\zeta)$ is meaningful for $f \in S'(\mathbb{R}^{2d})$ if and only if $1/\Omega$ is a multiplier in $S(\mathbb{R}^{2d})$. In such a case $f^\Omega$ is called the strict $\Omega$-symbol of the distribution $f \in S'(\mathbb{R}^{2d})$. E.g. every $f \in S'(\mathbb{R}^{2d})$ has strict standard, antistandard and normal symbols. More generally $f^\Omega$ is called the strict $\Omega$-symbol of $f$ if only $\Omega(\zeta)f^\Omega(\zeta) \in S'(\mathbb{R}^{2d})$.

Of course not every Weyl operator has either antinormal, or symmetric, or Born-Jordan symbol.

**Quasi-polynomials.** Define (c.f. [19], Appendix 2) for $m = (m_1, m_2)$, $r = (r_1, r_2)$, $r_1 \geq 0$, $r_2 < 1/2$, the class $S(m, r)$ of quasi-polynomial $f =$ \{(f_\hbar(z), \ 0 < \hbar \leq \hbar(f) )\} in $S(\mathbb{R}^{2d})$ such that

$$\partial z^\alpha f = \mathcal{O}_\alpha(1)(1 + |z|)^{m_1 - r_1|\alpha|}\hbar^{m_2 - r_2|\alpha|}$$

wherein $\partial z = \partial/\partial z$ and $\alpha$ is a multiindex.

As usual, $S(-\infty) = \cap S(m, r), \ S(\infty) = \cup S(m, r)$.

A quasi-polynomial $f$ is said to be asymptotic to a series $\sum_{\alpha \geq \mu} f^\alpha$

$$f \simeq \sum_{\alpha \geq \mu} f^\alpha$$

with $f^\alpha \in S(m_\alpha, r_\alpha), \ m_\alpha \searrow -\infty, \ r_\alpha \searrow -\infty$ if for all $\nu$

$$f - \sum_{\alpha < \nu} f^\alpha \in S(m_\nu, r_\nu).$$

The classical Borel-Hörmander construction leads to the following

**Proposition.** For every $f \in S(m, r)$ and a formal $\Omega$ there is a $g \in S(m, r)$ asymptotic to $[1/\Omega(\hbar \partial_z)]f$.

Such function $g$ is called an asymptotic symbol $f^\Omega$ of $f$. It is defined mod $S(-\infty)$.

**$\Omega$-products of quasi-polynomials.** If $f_j$ are quasi-polynomials then $\hat{f}_j$ act from $\mathcal{G}(\mathcal{H})$ to $\mathcal{G}(\mathcal{H})$ and therefore from $\mathcal{G}'(\mathcal{H})$ to $\mathcal{G}'(\mathcal{H})$, so that $\hat{f} = \hat{f}_1\hat{f}_2\ldots\hat{f}_N$ is well defined. Actually $f$ is quasi-polynomial, and

$$f^\Omega(z) = \int K^\Omega(z - z_1, \ldots z - z_N) \prod_{j=1}^N f_j^\Omega(z_j)d\lambda_hz_j,$$
wherein
\[
\tilde{K}^\Omega(\zeta_1, \ldots, \zeta_n) \simeq \prod_j \frac{\Omega(\zeta_j)}{\Omega(\sum \zeta_j)} \exp \left\{ \frac{1}{2} \sum_{j<k} [\zeta_j, \zeta_k] \right\},
\]
(=, if \( \Omega \) is strict). Generally, the integral is distributional, but absolutely converges for some \( \Omega \), e.g., the normal one.

The integral representation entails the asymptotic expansion
\[
f^\Omega(z) \simeq \tilde{K}^\Omega \left( \frac{\hbar}{i} \frac{\partial}{\partial z^+}, -\frac{\hbar}{i} \frac{\partial}{\partial z^-} \right) \prod_j f^\Omega(z_j)|_{z_j=z},
\]
wherein \( \frac{\partial}{\partial z^+} = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial q} + \frac{1}{i} \frac{\partial}{\partial p}) \), \( \frac{\partial}{\partial z^-} = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial q} - \frac{1}{i} \frac{\partial}{\partial p}) \).

**Trace.** A density operator \( \hat{\rho} : \mathcal{G}' \to \mathcal{G} \) is a Weyl operator with \( \rho \in S(\mathbb{R}^d) \). The operator trace of \( f^\Omega \rho^\Omega \) is well defined and may be evaluated for strict \( \Omega \) via the **Trace formula:**
\[
\text{Tr}(\hat{f}^\Omega \hat{\rho}^\Omega) = \langle f^\Omega | \rho^\Omega \rangle.
\]

### 3 Main Theorem.

A quasi-polynomial \( f \in S(m, r), \ m > 0 \), is called apt if for sufficiently small \( \hbar \) it satisfies the following three conditions uniformly:

- **Quasi-dissipativity:** \( \text{Re}(if) > \delta \), a constant.
- **Hypoellipticity:** for all multi-indices \( \alpha \) and \( 0 \leq t', t'' \leq T \)
  \[
  \partial_{t''}^\alpha f(t'' , z) = O_\alpha(1)|if(t') - \delta|(1 + |z|)^{-r_1|\alpha|}\hbar^{-r_2|\alpha|}.
  \]
- **t-Continuity** of \( f(t, \cdot) \) in \( S(m, r) \).

**Law of Inertia:** If \( f \) is apt then all its asymptotic symbols \( f^\Omega \) are apt as well, albeit on different intervals of \( \hbar \).

Also if \( f \) is hypoelliptic and real then \( \hat{f}(t) \) are essentially self-adjoint (c.f. [19], Proposition A2.1) in \( \mathcal{H} \).
Main Theorem. If $f$ is an apt quasi-polynomial, then for sufficiently small $\hbar$

(1) The Cauchy Problem
\[
\frac{d\psi}{dt} + \hat{f}(t)\psi(z,t) = 0, \quad \psi(z,0) = \psi_0, \quad 0 \leq t \leq T,
\]
is proper on $\mathcal{H}$ relative $\mathcal{G}(H)$.

(2) The evolution operator is the strong product integral
\[
U(t'',t') = \prod_{t'' \geq t \geq t'} [(1 + \frac{idt}{\hbar} f(t, \cdot))^{-1}] \hat{}.
\]

(3) A strict $\Omega$-symbol $u^\Omega(t'',t',z)$ of the evolution operator is the limit in $S'(\mathbb{R}^d)$ of the strict $\Omega$-symbols $u^P(t'',t',z)$ of the partial operator products $\prod_{t'' \geq t_j \geq t'} [(1 + \frac{i\Delta t_j}{\hbar} f(t_j, z))^{-1}] \hat{}$ as $|P| \to 0$.

Proof (outline). We apply the $\Omega$-calculus along with the theory of Abstract Cauchy Problems (c.f. [6]) and the theory of Finite Difference Methods for Initial Value Problems (c.f. [16]) with the terminology thereof.

The following statements hold for various intervals of positive $\hbar$.

By the Law of Inertia, the anti-normal symbol of $f$ is quasi-dissipative. Then (c.f. [19], Proposition 24.1) the real part of $\langle \psi|\delta_1 \mathbf{1} + i\hat{f}(t)|\psi \rangle$ is greater than $\gamma < \psi|\psi >$ with some constants $\gamma > 0$ and $\delta_1$. It is safe to assume that $\delta_1 = 0$. Together with the hypoellipticity (c.f. [13], Theorem 25.4) this entails that $\|[(\lambda \mathbf{1} + i\hat{f}(t))^{-1}] \| < 1/\lambda$ for positive $\lambda$ so that the operators $\hat{f}(t)$ are a $(1,0)$-stable family in $\mathcal{H}$.

When both $\psi$ and $\hat{f}(t)\psi$ belong to $\mathcal{H}$ for some $t = t_0$ then it is so for all $t$ by the virtue of the hypoellipticity. The space $\mathcal{F}$ of all such $\psi$ is dense in $\mathcal{H}$ and is a Hilbert space relative the new Hermitean product $\langle \psi|\psi >_0 = < \hat{f}(t_0)\psi|\hat{f}(t_0)\psi >$. Now the $\hat{f}(t) : \mathcal{F} \to \mathcal{H}$ form a $t$-continuous family of bounded operators. Moreover, $\langle \hat{f}(t)\psi|\psi >_0 = < \hat{g}(t)\psi|\psi >$ with $\hat{g}(t) = \hat{f}(t_0)\hat{f}(t)$ so that $g$ is apt again and thus (as above) is $(1,0)$-stable in $\mathcal{F}$. By the Hille-Yosida theorem $\hat{f}(t)$ generates for every $t$ a contractive operator semi-group in $\mathcal{F}$. Since
the family \( \hat{f}(t) - \hat{f}(T) \) has similar properties, the theorem 7.7.13 of [3] establishes (1), the *properness* of the Cauchy problem.

This leads to a preliminary Product Integral representation

\[
U(t'', t') = \prod_{t'' \geq t \geq t'} \left[ 1 + \frac{i\hbar}{\hbar} \hat{f}^{-1} \right]
\]

(c.f. the proof of the theorem 7.7.5 of [3]). It implies the Product Integral representation (2) via the Lax Equivalence Theorem [16] whereby the required consistency is checked via the Weyl calculus.

The last statement (3) follows from the *trace formula*.

4 Path Integrals in Quantum Field Theory.

Infinite dimensional phase spaces. In the case of \( d = \infty \) there are non-isomorphic phase spaces and the symplectic structures usually appear with extra features.

Our phase space is based on a separable Frechet nuclear space \( \mathcal{Z} \) over \( \mathbb{C} \) with a “dotless” hermitian product \( zw^* \).

If \( \mathcal{H} \) is the corresponding Hilbert space completion of the \( \mathcal{Z} \), and \( \mathcal{Z}^* \) is the corresponding anti-dual of \( \mathcal{Z} \) then

\[
\mathcal{Z} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{Z}^*
\]

is a Gelfand nuclear triplet.

The phase space is \( \mathcal{Z} \) taken over \( \mathbb{R} \) with the symplectic form \(-\text{Im}(zw^*)\).

It is also a pre-Hilbert space with the scalar product \( \text{Re}(zw^*) \).

Complex Gaussian rigging. (C.f. [8].) The Gaussian measure \( \gamma_h \) of covariance \( 1/\hbar \) is defined via its characteristic function

\[
\int_{\mathcal{Z}^*} e^{i\text{Re}(z\zeta^*)} d\gamma_h(\zeta^*) = e^{-zz^*/2\hbar}, \quad z \in \mathcal{Z},
\]

so it stands for the non-existent \( (\frac{\hbar}{2\pi})^\infty \exp(-h\zeta^*\zeta^*) d\zeta^* \).

The Bargmann-Segal space \([\mathcal{H}]\) is the closure of the subspace of the continuous complex analytic polynomials on \( \mathcal{Z}^* \) in \( L^2(\mathcal{Z}^*) \). Its elements are entire functions \( h(z^*) \) of order 2 and type \( < h/2 \):

\[
h(z^*) = \mathcal{O}(e^{h|z^*|^2/2})
\]
for some dual semi-norm $p$ on $Z^*$.

Let $[Z]$ denote the space of entire functions $h(z^*)$ of order 2 and minimal type, and $[Z]^*$ denote the space of entire functions $h(z^*)$ of order 2 and maximal type. Then $[Z]$ is naturally a separable nuclear Frechet space, and $[Z]^*$ its anti-dual. Thus

$$[Z] \hookrightarrow [H] \hookrightarrow [Z]^*$$

is another Gelfand triplet, the Complex Gaussian rigging of the triplet $Z \hookrightarrow H \hookrightarrow Z^*$.

The coherent states $e_w(z^*) := \exp(-\hbar wz^*)$, $w \in Z$, form a total (overcomplete) set in $[Z]$.

**Free bosonic field over $Z$ in $[Z]^*$**. Let $z \rightarrow \bar{z}$ be an antilinear conjugation on $Z$ and correspondingly on $Z^*$. Set

$$z^+ = z/\sqrt{2} \in Z, \quad z^- = \bar{z}/\sqrt{2} \in \bar{Z}.$$  

The operators $\hat{z}^+$ and $\hat{z}^-$ are defined on $f \in [Z]^*$ as

$$\hat{z}^+ f(\zeta^*) = (z\zeta^*)f(\zeta^*), \quad \hat{z}^- = \hbar \partial_z f(\zeta^*).$$

They represent the Canonical Commutation Relations (CCR):

$$[\hat{z}^-, \hat{z}^+] = \hbar 1.$$  

The coherent states are entire vectors for the CCR.

**Wick Operators** are, by definition, the continuous linear operators $W$ from $[G]$ to $[G]^*$. The Wick symbol of $W$ is

$$w(z^+, z^-) := e^{-z^+z^-} \int_{\bar{Z}^*} [W e_{z^+}(\zeta^-)]e_{z^-}(-\zeta^+)d_\gamma(\zeta), \quad z \in Z.$$  

The Wick symbol $w$ is an entire function on $Z \times Z$, so that the operator $\hat{w} := w(-\hat{z}^+, \hat{z}^-)$ is well defined on the coherent states and $W = \hat{w}$ and $W$ is its closure.
Ω-symbols. Consider a formal complex power series $1 + \sum_{|\alpha| > 0} z^\alpha$ on $\mathbb{Z}^*$. The formal \(\Omega\)-symbol of \(w(\zeta)\) is

$$w^\Omega := \left[ \frac{1}{\Omega(\frac{\hbar}{i} \partial_z)} \right] w(\zeta).$$

Quasi-polynomial \(w(\zeta)\) is the family \(\{w_h(\zeta) : 0 < h \leq h(w)\}\) such that for a dual semi-norm \(p\) on $\mathbb{Z}^*$

$$\partial^\alpha w(\zeta) = O_{\alpha,p}(1) (1 + p(\zeta))^{m_1 - r_1|\alpha|} h^{m_2 - r_2|\alpha|}.$$

The class of such families is denoted \(S(m, r)\), \(m = (m_1, m_2), r = (r_1, r_2)\).

Weyl symbols and operators. The Weyl symbols \(f(z)\) of \(w(z)\) correspond to \(\Omega(z) = \exp(-\frac{1}{2} z^+ z^-)\) (c.f. the table above):

$$f(z) = \exp \left[ \frac{\hbar^2}{2} \frac{\partial^2}{\partial z^+ \partial z^-} \right] w(z).$$

Conversely

$$w(z) = \int_{\mathbb{Z}^*} f(z - \zeta) d\gamma_{\hbar^2/2}(\zeta).$$

(Note: not every Wick operator has a strict Weyl symbol.)

The corresponding Wick operators are the Weyl operators \(\hat{f}\). The Borel-Hörmander constructions for a countable fundamental family of dual gaussian semi-norms followed by the Cantor diagonal trick imply that for every \(\Omega\) and Weyl \(f \in S(m, r)\) there is \(g \in S(m, r)\) asymptotic to \(f^\Omega\).

The \(\Omega\)-symbols of the operator product \(\hat{f}_1 \hat{f}_2 \cdots \hat{f}_N\) have the asymptotic expansions just as in the case \(d < \infty\). However their integral representations are known rarely. Fortunately, for the normal symbols \(w\) of \(\hat{w}_1 \hat{w}_2 \cdots \hat{w}_N\)

$$w(z) = \int \prod_{j=1}^N e^{z_j^+ z_j^-} w_j(z_j^-, z_j^+) \prod_{j=1}^{N-1} d\gamma_{\hbar}(z_j^-, z_j^+)$$

\(z_0^+ := z^+, \quad z_N^- := z^-\).
Finally, as in the finite-dimensional case, the Main Theorem holds in infinite dimensions (with the same proof) at least for the strict Wick symbols of evolution operators in $G = [\mathcal{H}]$. In the latter case the symbol approximations are absolutely convergent multiple integrals with respect to $d\gamma_\hbar$ over the infinite-dimensional phase space $\mathcal{Z}$.

5 Conclusion and outlook.

1. The phase space Path Integral (according to L.Shulman, ”a difficult form” of the path integral) was originated in different ways by Feynman himself [7] in 1951 and by Tobocman [20] in 1956. The Coherent State discretization was introduced in 1960 by Klauder [12] in the Schrödinger representation and in 1962 by Schweber [17] in the Bargmann-Fock representation. In the 70’s Berezin [1] considered various discretizations on the basis of pseudo-differential analysis. However no convergence of the discretizations has been proved until now.

On the other hand Daubeshies & Klauder [5] have established in 1984 that a wide class of coherent state path integrals (essentially with self-adjoint polynomial hamiltonians) on a flat finite-dimensional phase space may are limits of Wiener Integrals on the space of paths in the phase space. They even suggested that Feynman type time-slicing construction is impossible for the phase space Path Integrals.

2. We have presented a rigorous time-slicing Phase Space Path Integral construction for the symbols of the Evolution Operators with wide variety of smooth hamiltonians both in finite and infinite degrees of freedom. The convergence is established only for small $\hbar$, in agreement with the postulated semi-classical nature of the Path Integral which relates the classical and quantum dynamics.

3. According to the $\Omega$-calculus, the discretizations of the Path Integral are distributional multiple integrals. E.g., as mentioned in the Introduction, the traditional discretization of the Phase Space Integral comes from the standard $\Omega$-calculus. Similarly, the Coherent State Path Integral discretization is associated with the
normal Ω-symbol in which case the multiple integrals are absolutely convergent.

4. The last statement in the Main Theorem is equivalent to a modified DTF-ansatz: the Ω-symbol of the short time propagator is approximately equal to \([1 + \frac{i}{\hbar} f(t, z) \Delta t]^{-1}\). However, in the case of the normal Ω-calculus (because of the absolute convergence) one may consistently replace it with the more customary ansatz \(\exp[−\frac{i}{\hbar} f(t, q, p) \Delta t]\).

5. Our Path Integrals are “pathless”, in agreement with the Uncertainty Principle: no quantum path in the phase space (c.f. \([7]\) for an illuminating somewhat different point of view). Yet they are semiclassical in the following sense: the principal terms in the \(\hbar\)-expansions of the partial products symbols are the backward Euler approximations of the corresponding classical Hamilton-Jacobi equations.

6. We have rigorized the Ω-calculus of Agarval & Wolf \([1]\) to justify numerous Path Integral discretizations and as an important techniques. However in the infinite degrees we have been able to prove the convergence only for normal (Wick) symbols.

Actually the formal Ω-calculus is a special case of the formal \(*\)-calculus \([3]\). Since on the finite-dimensional flat symplectic space all \(*\)-products are formally equivalent, our results yield a construction of the formal \(*\)-exponential, a solution of a well known problem (c.f \([18]\) for an interpretation of the Evolution Operator symbol as a \(*\)-exponential).

7. Most of the other mathematical interpretations of the Path Integral are primarily in terms of various distributional integrals on the paths in the configuration space: first, by Kac \([11]\) via analytic continuation to a Wiener integral (the Feynman-Kac formula), followed by DeWitt-Morette \([14]\) in terms of prodistributions, by Albeverio and Høeg-Krohn \([2]\) in terms of the Parseval equation for the oscillatory Gaussian integrals, and by Hida & Streit \([8]\) in terms of White Noise distributions. Notably, these Path Integrals are associated only with the Schrödinger hamilt-
nians (essentially) of quadratic growth, with the presumed “Feyn-
man measure” built from the kinetic energy term.

References

[1] Agarwal,G.,and Wolf,E.: *Phys.Rev.D* (1970),2161.

[2] Albeverio,S. and Høeg-Krohn,R.: *Lecture Notes in Math.* 523,Springer-Verlag, Berlin,1976.

[3] Bayen,F.,Flato,M.,Fronsdal,C.,Licnerowicz,A.,and Sternheimer,D: *Ann.Phys.* 111 (1978),61.

[4] Berezin,F: *Soviet Phys.Uspekhi* 23 (1980),763.

[5] Daubechies,I. and Klauder,J.: *Phys.Rev.Letters* 52 1954,1161.

[6] Fattorini,H.: *The Cauchy Problem*, Addison-Wesley,London,1983.

[7] Feynman,R.: *Phys.Rev.* 84 (1951),108.

[8] Hida,T. and Streit: *Stoch.Appl.* 16(1983),55.

[9] Hida,T.,Kuo,H-H.,Potthof,J.,and Streit,L.: *White Noise Analysis: An Infinite Dimensional Calculus*,Kluwer Acad.Publ.,Dodrecht,1993.

[10] Hille,E.and Phillips,R.: *Functional Analysis and Semi-groups*, Amer.Math.Soc.Colloquium Publications,Rhode Island,1957.

[11] Kac,M.: *Trans.Amer.Math.Soc.* 65 (1949),1.

[12] Klauder,J.: *Ann.Phys.(N.Y.)* 11 (1960),123.

[13] Klauder,J.: *quant-ph/9710029*.

[14] Morette-DeWitt,C.: *Com.Math.Phys.* 28(1972),47.

[15] Nelson,E.,: *J.Math.Physics* 5 (1964),332.

[16] Richtmyer,R. and Morton,K.: *Difference Methods for Initial Problems*,Interscience Publishers,New York,1967.
[17] Schweber,S.: *J.Math.Phys.* 3(1962),831.

[18] Sharan,P.: *Phys.Rev.D* 20(1978),414.

[19] Shubin,M.: *Pseudodifferential operators and spectral theory.* Springer-Verlag,Berlin.1987.

[20] Tobocman,W.: *Nuovo Cimento* 3 (1956),1213.