A Littlewood-Paley-Rubio de Francia inequality for bounded Vilenkin systems

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Abstract. Rubio de Francia proved a one-sided Littlewood-Paley inequality for the square function constructed from an arbitrary system of disjoint intervals. Later, Osipov proved a similar inequality for Walsh systems. We prove a similar inequality for more general Vilenkin systems.

Bibliography: 11 titles.

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§ 1. Introduction

Let \( \{I_j\}_{j \in \mathbb{Z}} \) be a family of pairwise disjoint intervals in \( \mathbb{Z} \), and let \( f \) be a function defined on \( \mathbb{T} \). We let \( P_j \) denote the operator defined by \( (P_j f) \hat{f} = \chi_{I_j} \hat{f} \), where \( \hat{f} \) is the Fourier transform of \( f \) (that is, simply, the sequence of Fourier coefficients of \( f \)). Rubio de Francia [6] showed that, for \( p \geq 2 \),

\[
\left\| \left( \sum_j |P_j f|^2 \right)^{1/2} \right\|_p \lesssim \| f \|_p.
\]

In this inequality, the notation \( A \lesssim B \) means that the left-hand side of the inequality is majorized by the right-hand side multiplied by some constant. Here this constant is independent of the choice of the intervals \( \{I_j\} \) and the function \( f \).

By duality, it is easily seen that this inequality is equivalent to

\[
\left\| \sum_j f_j \right\|_p \lesssim \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_p, \quad 1 < p \leq 2,
\]

where the functions \( f_j \) are such that \( \text{supp} \hat{f}_j \subset I_j \).

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We prove an analogous inequality for Vilenkin systems. First, we introduce the necessary notation and definitions.

By $L^p$ we will always mean the space $L^p[0,1]$; $L^p(\ell^2) = L^p([0,1];\ell^2)$ is the space of $p$-summable $\ell^2$-valued functions. In many of the arguments which follow it will be immaterial whether we consider scalar or $\ell^2$-valued functions. (For an $\ell^2$-valued function $f$, $|f|$ is defined to be $\|f(\cdot)\|_{\ell^2}$.)

Let $\{p_i\}_{i=1}^\infty$ be a sequence of natural numbers each of which is not smaller than 2. The Vilenkin system corresponding to this sequence is defined as the set of characters (that is, continuous homomorphisms into the unit circle $\{z \in \mathbb{C}: |z| = 1\}$) on the group $G = \prod_{i=1}^\infty \mathbb{Z}_{p_i}$.

Such systems were first introduced and studied by Vilenkin [8].

We let $m_l$ denote the product $p_1p_2\cdots p_l$ (taking $m_0 = 1$). It will be convenient to assume that the functions in a Vilenkin system are defined on the closed interval $[0,1]$ (sometimes they are defined on a half-open interval; for example, see [2] and [10], but in the setting considered here this plays no role) — the set $G$ (equipped with the Haar measure) can be identified with the closed interval $[0,1]$ (apart from a countable number of points) via the mapping $G \ni (a_1,a_2,\ldots) \mapsto \sum_{i=1}^\infty \frac{a_i}{m_i}$.

Here we assume that $0 \leq a_i \leq p_i - 1$. It is easily seen that this mapping preserves the measure.

We partition the interval $[0,1]$ into $p_1$ equal subintervals and let $r_1$ denote the function equal to $e^{2\pi ik/p_1}$ on the $k$th interval (we number the intervals starting from zero). Next, we perform the same operation with each of the resulting intervals: we partition it into $p_2$ parts and denote the function equal to $e^{2\pi ik/p_2}$ on the $k$th part by $r_2$. Repeating this process, we obtain a sequence of functions $r_i$ analogous to the Rademacher functions.

All functions from a Vilenkin system are products of the generalized Rademacher functions thus constructed. Namely, for each $n \in \mathbb{N}_0$ ($\mathbb{N}_0$ is the set of nonnegative integers), we consider its ‘base-$(m)$ representation’, that is, $n = \alpha_1 + \alpha_2m_1 + \cdots + \alpha_km_{k-1}$, where $0 \leq \alpha_i \leq p_i - 1$. For this $(m)$-ary representation, it is convenient to use the following notation:

$$n \sim \begin{pmatrix} m_{k-1} & \cdots & m_1 & m_0 \\ \alpha_k & \cdots & \alpha_2 & \alpha_1 \end{pmatrix}.$$ 

In this case, the Vilenkin function $w_n$ can be written as $r_1^{\alpha_1}r_2^{\alpha_2}\cdots r_k^{\alpha_k}$. For the above representation and numbering of Vilenkin systems see [9], for example. A Vilenkin system is a complete orthonormal system in the space $L^2$. Note that if all the $p_i$ are equal to 2, then we have the classical Rademacher and Walsh functions as a special case.

Now we formulate the main result in this paper. We assume that the sequence $p_i$ is bounded: $p_i \leq M$ for some $M > 2$. Such Vilenkin systems are called bounded.
We will see below that this assumption of boundedness is important for our analysis. We denote the sequence of coefficients of the expansion of \( f \) in the Vilenkin system by \( \hat{f} \): \( \hat{f}(n) = (f, w_n) = \int_{\mathbb{R}} f \overline{w}_n \). It is clear that \( f = \sum_{n \in \mathbb{N}_0} \hat{f}(n) w_n \) (for \( f \in L^2 \)).

In what follows, we fix a bounded sequence \( \{p_i\}_{i \in \mathbb{N}} \) and the corresponding Vilenkin system \( \{w_n\}_{n \in \mathbb{N}_0} \). We prove the following result.

**Theorem 1.** Let \( \{I_s\} \) be a family of pairwise disjoint finite intervals in \( \mathbb{N}_0 \) and let functions \( f_s \) be such that \( \text{supp} \hat{f}_s \subset I_s \) (so that each function \( f_s \) is a Vilenkin polynomial). Then for all \( 1 < p \leq 2 \),

\[
\left\| \sum_s f_s \right\|_p \lesssim \left( \sum_s |f_s|^2 \right)^{1/2}.
\]

Note that the corresponding result for Walsh functions was proved in [5]. However, the available argument for Walsh functions cannot be extended to Vilenkin systems directly—it turns out that to do this we have to use a special (nonmartingale) square function and modify the combinatorial arguments in [5] suitably.

It is also worth mentioning that Rubio de Francia’s classical inequality (1.1) was later proved by Bourgain [1] for \( p = 1 \) and by Kislyakov and Parilov [4] for all \( p \in (0, 2] \). For more on this in the context of our paper, see §4.

**§ 2. Auxiliary results**

Let \( k, l \in \mathbb{N}_0 \) be numbers with \((m)\)-base representations

\[
k = \alpha_1 + \alpha_2 m_1 + \cdots + \alpha_j m_{j-1} \quad \text{and} \quad l = \beta_1 + \beta_2 m_1 + \cdots + \beta_j m_{j-1}.
\]

Then it is easily seen that the product of the functions \( w_k \) and \( w_l \) is the Vilenkin function \( w_{k+l} \), where \( k + l \) is the number with the \((m)\)-base representation

\[
\begin{pmatrix}
m_{j-1} & \cdots & m_1 & m_0 \\
(\alpha_j + \beta_j) \mod p_j & \cdots & (\alpha_2 + \beta_2) \mod p_2 & (\alpha_1 + \beta_1) \mod p_1
\end{pmatrix}.
\]

We denote the inverse of \( k \) with respect to the operation \( \hat{+} \) by \( -k \).

Let \( \mathcal{F}_k \) be the \( \sigma \)-algebra generated by the intervals \([jm_k^{-1}, (j + 1)m_k^{-1})\), \( 0 \leq j \leq m_k - 1 \). Then the operator \( \mathbb{E}_k \),

\[
\mathbb{E}_k f = \sum_{n=0}^{m_k-1} (f, w_n) w_n,
\]

is the operator of conditional expectation with respect to \( \mathcal{F}_k \) and the corresponding martingale differences have the form

\[
\Delta_k f = \mathbb{E}_k f - \mathbb{E}_{k-1} f = \sum_{n=m_k-1}^{m_k-1} (f, w_n) w_n.
\]
We set $\Delta_0 f$ to be the function $(f, w_0)w_0$ (that is, simply the function equal to $\int f$ on the interval $[0, 1]$). Note that the boundedness of the sequence $\{p_i\}$ is equivalent to saying that the filtration $\mathcal{F}_k$ is regular (that is, for any set $e \in \mathcal{F}_k$ there exists a set $e' \in \mathcal{F}_{k-1}$ containing $e$ and such that the measure of $e'$ is at most $M$ fold greater than that of $e$).

The martingale square function $Sf$ is defined by

$$Sf = \sqrt{\sum_{j=0}^{\infty} |\Delta_j f|^2}.$$

It is well known that $\|Sf\|_p \precsim \|f\|_p$ for $p > 1$ (for example, see §2.2 in [10]).

However, when dealing with Vilenkin systems it is frequently more convenient to use another square function. Consider the operators $\Delta_{k,l}$ defined by

$$\Delta_{k,l} f = \sum_{n=lm_{k-1}}^{(l+1)m_{k-1}-1} (f, w_n)w_n, \quad 1 \leq l \leq p_k - 1.$$

The square function, which we will employ, has the form

$$\tilde{S}f = \left(|\Delta_0 f|^2 + \sum_{k=1}^{\infty} \sum_{l=1}^{p_k-1} |\Delta_{k,l} f|^2\right)^{1/2}.$$

This square function has already been considered in [9], where it was shown that its $L^p$-norm can be estimated in terms of the $L^p$-norm of the function $f$ itself. However, for completeness, we give a short proof of this fact (in addition, this proof also applies to $\ell^2$-valued functions $f$).

**Lemma 1.** If $1 < p < \infty$ and $f \in L^p$, then $\|\tilde{S}f\|_p \precsim \|f\|_p$.

**Proof.** The pointwise estimate $Sf \precsim \tilde{S}f$ is clear, and so it suffices to verify the inequality $\|\tilde{S}f\|_p \precsim \|Sf\|_p$. To do this, for each $k$ we fix $l_k$, $1 \leq l_k \leq p_k - 1$, and prove the estimate

$$\left\|\left(\sum_{k} |\Delta_{k,l_k} f|^2\right)^{1/2}\right\|_p \precsim \|Sf\|_p.$$

This inequality implies the required result, because $\tilde{S}f$ can be represented as the root of the sum of squares of some (at most $M$) square functions from the left-hand side of the inequality.

Consider the family of functions $(\Delta_1 f, \Delta_2 f, \ldots) = (f_1, f_2, \ldots)$. Note that

$$\Delta_{k,l_k} f = w_{l_km_{k-1}} E_{k-1} [w_n^{-1} w_{l_km_{k-1}} f_k].$$

This is indeed so, because $w_n^{-1} w_m = w_{m-n}$ and

$$[l_k m_{k-1}, (l_k + 1)m_{k-1} - 1] \div l_k m_{k-1} = [0, m_{k-1} - 1].$$
So we have
\[
\left\| \left( \sum_k |\Delta_{k,l} f|^2 \right)^{1/2} \right\|_p = \left\| \left( \sum_k |\mathbb{E}_{k-1}[w_{l_km_{k-1}}^{-1}f]|^2 \right)^{1/2} \right\|_p
\leq \left\| \left( \sum_k |w_{l_km_{k-1}}^{-1}f|^2 \right)^{1/2} \right\|_p = \|Sf\|_p.
\]

Here we have used the inequality
\[
\|\{\mathbb{E}_{nk}g_k\}\|_{L^p(\ell^2)} \lesssim \|\{g_k\}\|_{L^p(\ell^2)},
\]
for any natural number \(n_k\), which follows from the fact that the square function (martingale) of an \(\ell^2\)-valued function \(\{\mathbb{E}_{nk}g_k\}\) is majorized by the square function of \(\{g_k\}\). Indeed, we have
\[
S(\{g_k\}) = \left( \sum_k \sum_{j=0}^\infty |\Delta_jg_k|^2 \right)^{1/2} \geq \left( \sum_k \sum_{j=0}^{n_k} |\Delta_jg_k|^2 \right)^{1/2} = S(\{\mathbb{E}_{nk}g_k\}).
\]

Lemma 1 is proved.

As we already pointed out, this lemma holds for both scalar and \(\ell^2\)-valued functions \(f\). We also note that the boundedness of the Vilenkin system is used here in an essential way — without this assumption the conclusion of the lemma is false, as shown in [9].

We will need one more simple property of the operators \(\Delta_{k,l}\). Note that if the support of a function \(f\) lies in the set \(e_k \in \mathcal{F}_{k-1}\), then the support of the function \(\Delta_kf\) also lies in \(e_k\) (as \(\Delta_kf = \mathbb{E}_kf - \mathbb{E}_{k-1}f\) and the operators \(\mathbb{E}_k\) and \(\mathbb{E}_{k-1}\) are the averaging operators of the function \(f\) over intervals from \(\mathcal{F}_k\) and \(\mathcal{F}_{k-1}\), respectively). We verify this property also for \(\Delta_{k,l}f\).

**Lemma 2.** Let \(\text{supp } f \subset e_k \in \mathcal{F}_{k-1}\). Then \(\text{supp } \Delta_{k,l}f \subset e_k, 1 \leq l \leq p_k - 1\).

**Proof.** We can assume without loss of generality that \(e_k\) is one of the intervals of length \(m_{k-1}\) that generate the \(\sigma\)-algebra \(\mathcal{F}_{k-1}\), that is, \(e_k = [jm_{k-1}^{-1}, (j + 1)m_{k-1}^{-1})\). We consider another such interval \(e\) and prove that \(\Delta_{k,l}f = 0\) on \(e\). According to the above, the function
\[
\Delta_kf = \sum_{l=1}^{p_k-1} \Delta_{k,l}f
\]
is zero on \(e\). Hence, to prove the lemma it suffices to show that the functions \(\Delta_{k,l}f\), \(1 \leq l \leq p_k - 1\), are pairwise orthogonal in the space \(L^2(e)\).

To do this it is enough to verify that the functions \(w_{n_1}\) and \(w_{n_2}\) are orthogonal in \(L^2(e)\) if \(n_1 \in [l_1m_{k-1}, (l_1 + 1)m_{k-1} - 1]\) and \(n_2 \in [l_2m_{k-1}, (l_2 + 1)m_{k-1} - 1]\). The functions \(w_{n_1}\) and \(w_{n_2}\) have the form
\[
w_{n_1} = r_{\alpha_1}^{\alpha_2} \cdots r_{k-1}^{\alpha_{k-1}} r_{l_1}^{l_1}, \quad w_{n_2} = r_{\beta_1}^{\beta_2} \cdots r_{k-1}^{\beta_{k-1}} r_{l_2}^{l_2},
\]
where \(\alpha_1, \ldots, \alpha_{k-1}\) and \(\beta_1, \ldots, \beta_{k-1}\) are some nonnegative integers. By construction, the functions \(r_1, \ldots, r_{k-1}\) are constant on \(e_k \in \mathcal{F}_{k-1}\), and \(r_{l_1}^{l_1}\) and \(r_{l_2}^{l_2}\) are
orthogonal on $e$ because

$$
\int_e r_k^{l_1} r_k^{l_2} = \int_e r_k^{l_1-l_2} = m_k^{-1} \sum_{s=0}^{p_k-1} \exp \left( \frac{2\pi i (l_1 - l_2) s}{p_k} \right) = 0.
$$

Lemma 2 is proved.

We require an analogue of Gundy’s theorem for Vilenkin systems. Thanks to Lemma 2, we can reformulate this result in a convenient form. A proof of Gundy’s theorem for vector-valued martingales is given in [3].

**Proposition 1.** Let $T$ be a linear operator that sends $\ell^2$-valued functions on $[0, 1]$ to scalar functions. Next, let the domain of $T$ contain all ‘Vilenkin polynomials’, that is, functions $f$ such that $\mathbb{E}_n f = f$ for all sufficiently large $n$. Assume also that the following conditions are satisfied.

1. $\|Tf\|_2 \lesssim \|f\|_2$.
2. If the function $f$ is such that $\Delta_0 f = 0$ and $\text{supp} \Delta_k f \subset e_k$, where $e_k \in \mathcal{F}_{k-1}$, then $\{Tf \neq 0\} \subset \bigcup_{k \geq 1} e_k$.

Then the operator $T$ has the weak type $(1, 1)$, and therefore acts from $L^p(\ell^2)$ into $L^p$ for $1 < p \leq 2$.

According to Lemma 2, in the second condition of this theorem the operators $\Delta_k$ can be replaced by $\Delta_{k,l}$.

**Lemma 3.** Condition 2 in the previous proposition can be replaced by the following.

2$'$. If a function $f$ is such that $\Delta_0 f = 0$ and $\text{supp} \Delta_{k,l} f \subset e_k$ for all $1 \leq l < p_k - 1$, then $\{Tf \neq 0\} \subset \bigcup_{k \geq 1} e_k$.

Indeed, it follows from Lemma 2 that if $\Delta_k f \subset e_k$, then we also have $\Delta_{k,l} f \subset e_k$. Applying this lemma we can prove that operators of a certain special form, which we need in what follows, are bounded in $L^p$. Set $\delta_{k,l} = [(lm_k - 1) + (l + 1)m_k] - 1$.

**Corollary.** Let $A \subset \mathbb{N}_0^2$ and $h = \{h_{j,k}\}_{(j,k) \in \mathbb{N}_0^2} \subset L^p(\ell^2)$. For each element $(j, k) \in A$ fix a set $\Lambda_A \subset [1, p_k - 1]$. Next, let $\{a_{j,k} + \delta_{k,l}\}_{(j,k) \in A}$ be a tuple of non-negative integers such that $\{a_{j,k} + \delta_{k,l}\}_{(j,k) \in A}$ is a family of disjoint subsets of $\mathbb{N}_0$. Let the operator $G$ be defined by

$$
Gh = \sum_{(j,k) \in A} w_{a_{j,k}} \Delta_{k,l} h_{j,k}.
$$

Then $\|Gh\|_p \lesssim \|h\|_{L^p(\ell^2)}$ for $1 < p \leq 2$.

**Proof.** The sets $a_{j,k} + \delta_{k,l}$ are disjoint, and hence the functions $w_{a_{j,k}} \Delta_{k,l} h_{j,k}$ are pairwise orthogonal. Therefore,

$$
\|Gh\|_2^2 = \sum_{(j,k) \in A} \|w_{a_{j,k}} \Delta_{k,l} h_{j,k}\|_2^2 = \sum_{(j,k) \in A} \|\Delta_{k,l} h_{j,k}\|_2^2 \leq \|h\|^2_2.
$$

Moreover, it is clear that $G$ satisfies condition 2$'$ in Lemma 3, hence $G$ is an operator of weak type $(1, 1)$ which acts from $L^p(\ell^2)$ to $L^p$ for $1 < p \leq 2$. This proves the corollary.

Now we are ready to prove Theorem 1.
§ 3. Proof of Theorem 1

Recall that we are given disjoint intervals $I_s = [a_s, b_s) \subset \mathbb{N}_0$. The proof of Theorem 1 consists of two parts: partitioning each interval into smaller ones (close to what is done in [5]) and applying this partition to estimate the $L^p$-norm of the function $\sum_s f_s$.

3.1. Constructing partitions of intervals. We omit the index $s$ and describe the partition of the interval $I = [a, b)$. Let $b$ have the base-$(m)$ representation:

$$b = \beta_{k+1} m_k + \beta_k m_{k-1} + \cdots + \beta_1.$$

We first split the interval $[0, b)$ as follows:

$$[0, b) = \bigcup_{j=1}^{k+1} J_j,$$

where $J_j = [\beta_{k+1} m_k + \cdots + \beta_j m_j, \beta_{k+1} m_k + \cdots + \beta_j m_{j-1}]$.

In particular, $J_{k+1} = [0, \beta_{k+1} m_k - 1]$. If $\beta_j = 0$ for some $j$, then we take the interval $J_j$ to be empty. Note that $J_j$ consists of all numbers with the following base-$(m)$ representation:

$$J_j \sim \left( \frac{m_k}{\beta_{k+1}} \cdots \frac{m_j}{\beta_{j+1}} \left[ \frac{m_{j-1}}{0}, \frac{\beta_j - 1}{1} \right] \ast \cdots \ast \right).$$  

(3.1)

This notation means that numbers in the interval $J_j$ are represented as $\beta_{k+1} m_k + \cdots + \beta_j m_j + \gamma_j m_{j-1} + \varepsilon_{j-1} m_{j-2} + \cdots + \varepsilon_1$ in the $(m)$-ary system, where $\gamma_j \in [0, \beta_j - 1]$, and $\varepsilon_i$ is any number in $[0, p_i - 1]$, $1 \leq i \leq j - 1$. We will use this notation in what follows without further explanation.

The number $a$ lies in one of the intervals $J_j$; assume that $a \in J_t$, $1 \leq t \leq k + 1$. In this case, in the $(m)$-ary system the number $a$ can be expressed as

$$a = \beta_{k+1} m_k + \cdots + \beta_t m_t + \alpha_t m_{t-1} + \alpha_{t-1} m_{t-2} + \cdots + \alpha_1,$$

where $\alpha_t < \beta_t$. For convenience, set $\alpha_{k+1} = \beta_{k+1}, \cdots, \alpha_{t+1} = \beta_{t+1}$. Now we split the interval $[a, \beta_{k+1} m_k + \cdots + \beta_t m_t + \beta_t m_{t-1}] = [a, +\infty) \cap J_t$ as follows:

$$[a, +\infty) \cap J_t = \{a\} \cup \bigcup_{j=1}^{t} \tilde{J}_j,$$

where, for $1 \leq j \leq t - 1$,

$$\tilde{J}_j = [\alpha_{k+1} m_k + \cdots + \alpha_{j+1} m_j + (\alpha_j + 1) m_{j-1}, \alpha_{k+1} m_k + \cdots + \alpha_{j+1} m_j + p_j m_{j-1}],$$

and the interval $\tilde{J}_t$ has the form

$$\tilde{J}_t = [\alpha_{k+1} m_k + \cdots + \alpha_{t+1} m_t + (\alpha_t + 1) m_{t-1}, \alpha_{k+1} m_k + \cdots + \alpha_{t+1} m_t + \beta_{t, m_{t-1}}].$$

If $\alpha_j = p_j - 1$, $1 \leq j \leq t - 1$, then we take the interval $\tilde{J}_j$ to be empty. Similarly, if $\alpha_t = \beta_{t-1}$, then $\tilde{J}_t = \emptyset$. We note again that for $1 \leq j \leq t - 1$ the interval $\tilde{J}_j$ consists of the numbers with the following $(m)$-ary representation:

$$\tilde{J}_j \sim \left( \frac{m_k}{\alpha_{k+1}} \cdots \frac{m_j}{\alpha_{j+1}} \left[ \frac{m_{j-1}}{\alpha_j + 1, p_j - 1} \right] \ast \cdots \ast \right).$$  

(3.2)
The interval $\tilde{J}_t$ can be written in this form as follows:

$$\tilde{J}_t \sim \begin{pmatrix} m_k & \cdots & m_t & m_{t-1} & m_{t-2} & \cdots & m_0 \end{pmatrix}.$$

So we have constructed the following partition of the interval $I = [a, b)$:

$$I = \{a\} \cup \bigcup_{j=1}^{t} \tilde{J}_j \cup \bigcup_{j=1}^{t-1} J_j.$$  

3.2. Completing the proof. Now we use the subscript $s$ for intervals obtained by partitioning $I_s$:

$$I_s = \{a_s\} \cup \bigcup_{j=1}^{t_s} \tilde{J}_{j,s} \cup \bigcup_{j=1}^{t_s-1} J_{j,s}.$$  

We also set $\{a_s\} =: \tilde{J}_{0,s}$. We represent each function $f_s$ as the corresponding sum

$$f_s = \sum_{j=0}^{t_s} \tilde{f}_{j,s} + \sum_{j=1}^{t_s-1} f_{j,s},$$

where the functions $\tilde{f}_{j,s}$ and $f_{j,s}$ are defined by

$$\tilde{f}_{j,s} = \sum_{n \in J_{j,s}} (f_s, w_n)w_n, \quad 0 \leq j \leq t_s, \quad \text{and} \quad f_{j,s} = \sum_{n \in J_{j,s}} (f_s, w_n)w_n, \quad 1 \leq j \leq t_s-1.$$  

Consider the following functions:

$$\tilde{g}_{j,s} = w_{a_s}^{-1} \tilde{f}_{j,s}, \quad 0 \leq j \leq t_s, \quad g_{j,s} = w_{b_s}^{-1} f_{j,s}, \quad 1 \leq j \leq t_s-1.$$  

In this case the functions $f_s$ can be written as

$$f_s = w_{a_s} \sum_{j=0}^{t_s} \tilde{g}_{j,s} + w_{b_s} \sum_{j=1}^{t_s-1} g_{j,s}.$$  

(3.4)

Note that the nonzero Vilenkin coefficients of the functions $\tilde{g}_{j,s}$ and $g_{j,s}$ lie in the closed intervals $\tilde{J}_{j,s} \sim a_s$ and $J_{j,s} \sim b_s$, respectively. From (3.1)–(3.3) it follows that these sets have the form (in the formulae that follow we assume that $1 \leq j \leq t_s-1$):

$$\tilde{J}_{j,s} \sim a_s \sim \begin{pmatrix} m_k & \cdots & m_j & m_{j-1} & m_{j-2} & \cdots & m_0 \end{pmatrix},$$

$$J_{j,s} \sim b_s \sim \begin{pmatrix} m_k & \cdots & m_j & m_{j-1} & m_{j-2} & \cdots & m_0 \end{pmatrix}.$$  

(3.5)
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So, (3.4) can be written as

\[ f_s = w_{a_s} \left( \Delta_0 \tilde{g}_0,s + \sum_{j=1}^{t_s-1} \sum_{l=1}^{p_j-1} \Delta_j, l \tilde{g}_j,s + \sum_{l=1}^{\alpha_{t_s,s}} \Delta_{t_s, l} \tilde{g}_{t_s,s} \right) \]

\[ + w_{b_s} \sum_{j=1}^{t_s-1} \sum_{l=p_j-\beta_{j,s}}^{p_j-1} \Delta_j, l g_j,s. \]

Employing the corollary to Lemma 3, this establishes that

\[ \left\| \sum_s f_s \right\|_p \lesssim \left\| \left( \sum_s \sum_{j=0}^{t_s} |\tilde{g}_j,s|^2 + \sum_s \sum_{j=1}^{t_s-1} |g_j,s|^2 \right)^{1/2} \right\|_p. \]  \hspace{1cm} (3.8)

This expression is majorized by

\[ \left\| \left( \sum_s \sum_{j=0}^{t_s} |\tilde{g}_j,s|^2 \right)^{1/2} \right\|_p + \left\| \left( \sum_s \sum_{j=1}^{t_s-1} |g_j,s|^2 \right)^{1/2} \right\|_p =: A + B. \]

We estimate \( A \) and \( B \) separately.

Set

\[ \tilde{g}_s = w_{a_s}^{-1} f_s = \sum_{j=0}^{t_s} \tilde{g}_j,s + w_{a_s}^{-1} \sum_{j=1}^{t_s-1} f_j,s. \]

Using (3.5), (3.6) and (3.1), respectively, we find that

\[ \tilde{g}_j,s = \Delta_j \tilde{g}_s, \hspace{1cm} 0 \leq j \leq t_s - 1, \]

\[ \tilde{g}_{t_s,s} = \sum_{l=1}^{\beta_{t_s,s}-1-\alpha_{t_s,s}} \Delta_{t_s, l} \tilde{g}_s, \]

\[ w_{a_s}^{-1} \sum_{j=1}^{t_s-1} f_j,s = \Delta_{t_s, \beta_{t_s,s}-\alpha_{t_s,s}} \tilde{g}_s. \]

As a result,

\[ A \lesssim \left\| \left( |\Delta_0 \tilde{g}_s|^2 + \sum_{j=1}^{\infty} \sum_{l=1}^{p_j-1} |\Delta_j, l \tilde{g}_s|^2 \right)^{1/2} \right\|_p = \| \tilde{S} (\{ \tilde{g}_s \}_s) \|_p, \]

where \( \tilde{S} (\{ \tilde{g}_s \}_s) \) denotes the operator \( \tilde{S} \) applied to the \( \ell^2 \)-valued function \( \{ \tilde{g}_s \}_s \).

It remains to invoke Lemma 1 and note that \( \| \{ g_s \}_s \|_{L^p(\ell^2)} = \| \{ f_s \}_s \|_{L^p(\ell^2)} \), which completes our estimate of the expression \( A \).

The expression \( B \) is estimated similarly. Set

\[ g_s = w_{b_s}^{-1} f_s = w_{b_s}^{-1} \sum_{j=0}^{t_s} \tilde{f}_j,s + \sum_{j=1}^{t_s-1} g_j,s. \]
From (3.2), (3.3) and (3.7) it follows that
\[ w_{b_s}^{-1} \sum_{j=0}^{t_s} \tilde{f}_{j,s} = \Delta t_s g_s, \]
\[ g_{j,s} = \Delta_j g_s, \quad 1 \leq j \leq t_s - 1. \]

Hence
\[ B \lesssim \| S(\{g_s\}_s) \|_p \lesssim \| \{g_s\}_s \|_{L^p(\ell^2)} = \| \{f_s\}_s \|_{L^p(\ell^2)}, \]
completing the proof of Theorem 1.

§ 4. The case \( p \leq 1 \)

In this section we show that, in fact, from the above proof of Theorem 1 we can also derive an inequality for \( p \leq 1 \). To do this we need the martingale Hardy classes \( \mathcal{H}^p \) (the definition and necessary information can be found in [10]). Note that all the results we require hold for both scalar and \( \ell^2 \)-valued Hardy classes.

A function \( f \) is said to lie in the Hardy class \( \mathcal{H}^p \) if \( Sf \in L^p \), \( 0 < p \leq 2 \). For \( p > 1 \) it is known that \( \mathcal{H}^p = L^p \), while for \( p \leq 1 \) the Hardy classes are different from the \( L^p \). By definition, \( \| f \|_{\mathcal{H}^p} = \| Sf \|_p \) (note that \( \| \cdot \|_p \) for \( p < 1 \) is a quasi-norm and not a norm).

The proof of the next theorem is similar to that of Theorem 1.

**Theorem 2.** Under the hypotheses of Theorem 1, the inequality
\[ \left\| \sum_s f_s \right\|_p \lesssim \| \{w_{a_s}^{-1} f_s\}_s \|_{\mathcal{H}^p(\ell^2)} + \| \{w_{b_s}^{-1} f_s\}_s \|_{\mathcal{H}^p(\ell^2)} \]
holds for the intervals \( I_s = [a_s, b_s) \) for \( 0 < p \leq 2 \).

**Proof.** First we note that for \( p > 1 \) this theorem is equivalent to Theorem 1 (it suffices to replace the norms in the Hardy class on the right of the inequality by the \( L^p(\ell^2) \)-norms and note that \( \| \{w_{a_s}^{-1} f_s\}_s \|_{L^p(\ell^2)} = \| \{w_{b_s}^{-1} f_s\}_s \|_{L^p(\ell^2)} = \| \{f_s\}_s \|_{L^p(\ell^2)} \).)

Now we show how to modify the proof of Theorem 1 in order to get this inequality for \( 0 < p \leq 1 \).

First we note that an operator \( T \) satisfying the conditions of Lemma 3 acts from \( \mathcal{H}^p(\ell^2) \) into \( L^p \). This clearly follows from the atomic decomposition for Hardy classes (again, see [10], which presents the corresponding theory for scalar \( \mathcal{H}^p \)-classes; all the corresponding results, including the atomic decomposition, remain valid for the spaces \( \mathcal{H}^p(\ell^2) \)).

From this observation it follows that inequality (3.8) holds also for \( 0 < p \leq 1 \) (since \( \Delta_j \tilde{g}_{j,s} = \tilde{g}_{j,s} \) and \( \Delta_j g_{j,s} = g_{j,s} \), see (3.5)–(3.7)).

Now, \( A \) and \( B \) have been majorized by \( \| \tilde{S}(\{g_s\}_s) \|_p \) and \( \| \tilde{S}(\{g_s\}_s) \|_{p_t} \), respectively. To complete the proof of Theorem 2 it remains to note that \( \| \tilde{S}f \|_p \approx \| Sf \|_p \) for \( 0 < p \leq 1 \) (that is, Hardy classes for Vilenkin systems can also be defined via the square function \( \tilde{S} \)). This is also known, because in view of the inequality \( Sf \lesssim \tilde{S}f \) this simply reformulates the fact that the operator \( \tilde{S} \) acts from \( \mathcal{H}^p \) to \( L^p \),
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and this is indeed so, because \( \tilde{S} \) is a sublinear operator satisfying the hypotheses of Lemma 3 (see [7], for example, where it is also shown that \( \| \tilde{S}f \|_p \preceq \| Sf \|_p \) for \( p = 1 \)). Theorem 2 is proved.

§ 5. Open problems

The results here are proved under the assumption of that the Vilenkin system is bounded. This assumption is natural in many analytic problems on Vilenkin groups (see, for example, [7] and [9]). As already noted, the Vilenkin system being bounded is a necessary condition for the operator \( \tilde{S} \) to be bounded in \( L^p \). Moreover, this assumption implies that the filtration under consideration is regular. Nevertheless, in some problems this assumption can be dropped (for example, see [11]). We do not know if our theorems extend to unbounded Vilenkin systems, but the ‘combinatorial’ approach used above depends substantially on this boundedness.

In addition, we do not know whether the inequality in Theorem 1 is itself true for \( 0 < p \leq 1 \). However, since inequality (1.1) holds also for \( p \leq 1 \), we might hope that the inequality in Theorem 1 also holds for \( p \leq 1 \). Nevertheless, this question is open even in the particular case of Walsh functions.

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