Two-oscillator model of trapped-modes interaction in a nonlinear bilayer fish-scale metamaterial

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Abstract
We discuss a similarity between resonant oscillations in two nonlinear systems; namely, a chain of coupled Duffing oscillators and a bilayer fish-scale metamaterial. In such systems two different resonant states arise which differ in their spectral lines. The spectral line of the first resonant state has a Lorentzian form, whereas the second one has a Fano form. This difference leads to a specific nonlinear response of the systems which manifests itself in the appearance of closed loops in spectral lines and bending and overlapping of resonant curves. Conditions for achieving bistability and multistability are determined.

Keywords: coupled oscillators, Duffing oscillator, metamaterial, trapped mode, bistability

(Some figures may appear in colour only in the online journal)

1. Introduction

Resonant phenomena are inherent in all types of vibrations or waves, and a number of resonant states in mechanic, acoustic, electromagnetic, and quantum systems are well known. Importantly, there are standard approaches for describing such diverse resonant phenomena in various branches of physics which are primarily developed within the oscillation theory framework. Within this theory, despite the differences in the nature of the resonant phenomena, they are described from a unified standpoint with similar or even the same equations and techniques.

A resonance is thought to be an enhancement of the response of a system to an external excitation at a particular frequency. It is referred to as the resonant frequency or natural frequency of a system. From the oscillation theory standpoint a resonance is introduced by means of a harmonic oscillator under the action of a periodic driving force. When the frequency of the driving force is close to the eigenfrequency of the oscillator, the amplitude of the oscillations grows toward its maximal value. In addition, many physical systems may also exhibit the opposite phenomenon when their response is suppressed under certain resonant conditions; this phenomenon is sometimes called antiresonance [1]. The simplest example can be illustrated using two coupled harmonic oscillators, where one of them is driven by a periodic force. Remarkably, such a system of two coupled harmonic oscillators simultaneously supporting resonant–antiresonant states is considered an intuitive and popular model to describe features of many resonant phenomena, including...
electromagnetically induced transparency (EIT) [2, 3], the stimulated resonant Raman effect [4], level repulsion [5], conditions for adiabatic and diabatic transitions [6, 7], and quantum coherence–decoherence [8].

Thus, in the system of two coupled oscillators, in general, two resonances are located close to certain eigenfrequencies of each oscillator [9]. One of the resonances of the forced oscillator demonstrates the standard amplitude growing near its eigenfrequency, and it has a symmetric spectral line, described by the Lorentzian function. At the same time the other resonance demonstrates an unusually sharp peak in the amplitude, and it has an asymmetric spectral line, known as the Fano profile. In the antiresonant state there is total suppression of the amplitude of the forced oscillator at the eigenfrequency of the second oscillator, which is one of the basic properties, originating from the resonant destructive interference, that distinguishes the Fano resonance from other resonances [9].

Resonant–antiresonant states were originally studied in quantum physics in relation to asymmetrically shaped ionization spectral lines of atoms and molecules. However, in recent years they have attracted appreciable attention in the field of plasmonic nanoparticles, photonic crystals, and then electromagnetic metamaterials [1, 10–13]. This interest is stimulated by promising applications of resonances with asymmetric spectral lines in sensors, lasing, switching, and nonlinear and slow light devices, due to the steep dispersion of their profile. Despite the fact that the nature of such resonances in photonic devices is quite complicated and is explained by the interference effect between a certain non-radiative mode and a continuum of radiative electromagnetic waves, the simple two-oscillator model is still widely used to reveal the main resonant features of optical systems [14–18].

Another important characteristic of structures supporting resonant–antiresonant states is their potential to provide enhanced storing of energy. Remarkably, the simultaneous presence of both a steep resonant feature and strong field localization brings about the possibility of optimal bistable switching in nonlinear systems. In particular, in optical systems, the main idea of using the resonant–antiresonant states for all-optical switching and bistability is to introduce an element with nonlinear characteristics and achieve a stepwise nonlinearly induced shift of the resonant frequency [19–25]. Thus, by employing such a nonlinear shift one can reach bistability in many devices suggested for the plasmonic, photonic crystal, and metamaterial platforms.

From the viewpoint of the oscillation theory, a study of resonant–antiresonant states in nonlinear systems causes derivation of the particular model of a chain of two non-linearly coupled oscillators [26]. In a mathematical form such a system can be described by a set of two coupled Duffing equations, which is the basic model for illustrating the synchronization phenomenon and related effects [28–30]. It is known that, despite the apparent simplicity of the Duffing equations, there are no easy ways to find an exact analytical solution for the corresponding system of nonlinear equations. In this regard, for this solution some asymptotic approaches are traditionally used. However an obtained solution can comprise a number of peculiarities and stand out by the presence of hysteresis, several stable cycles, complex dynamics, and chaotic regimes.

The complete study of the system of nonlinear equations supposes involving a concept of dynamical systems [31]. However, in this paper we intend to restrict ourselves to extending the results of [9] by adding weak nonlinearity to the system of two coupled oscillators and solving it with the slowly varying amplitude approximation in the frequency domain. The main purpose of this paper is to reveal general changes in the spectral line of the resonant–antiresonant states when such weak nonlinearity is introduced into the system.

Then, on the basis of the results obtained from the nonlinear two-oscillator model, we propose an example of a nonlinear metamaterial configuration whose operating regimes qualitatively resemble characteristics of the mentioned oscillating system. As such a structure we consider a special class of metamaterials which involves planar meta-surfaces supporting so-called ‘trapped modes’. A trapped mode is a specific resonant state that appears in the metamaterials made of subwavelength metallic or dielectric particles (inclusions) with a certain asymmetric form [11, 12, 32, 33]. Trapped modes are the result of antiphase oscillations of fields on the particles parts (arcs) and are excited by an external electromagnetic field. In the literature such trapped-mode metamaterials are sometimes also referred to as EIT-like metamaterials [34]. This is due to the fact that their response is a direct classical analog of EIT because the weak coupling of the antiphased local fields to free space is reminiscent of the weak probability for photon absorption in EIT observed in an atomic system. In this paper we consider a particular configuration of such a metamaterial, namely, a nonlinear bilayer fish-scale structure [35].

2. Two-oscillator model: set of coupled Duffing equations

Our objective here is to study the main spectral features of a chain of two coupled nonlinear oscillators. For this reason we consider the two-harmonic oscillator system which models classically the Fano resonance [9]. If it is supplemented by the cubic nonlinear terms, we arrive at the set of two coupled Duffing equations related to the coordinates $x_1$ and $x_2$ in the form

\[
\begin{aligned}
\frac{d^2x_1}{dt^2} + 2\delta_1 \frac{dx_1}{dt} + \omega_1^2 x_1 + \omega_1^2 \beta_1 x_1^3 - c_1 x_2 &= R_0 \cos(\omega t), \\
\frac{d^2x_2}{dt^2} + 2\delta_2 \frac{dx_2}{dt} + \omega_2^2 x_2 + \omega_2^2 \beta_2 x_2^3 - c_2 x_1 &= 0,
\end{aligned}
\]

(1)

where $\delta_1$ and $\delta_2$ are the damping coefficients, $\omega_1$ and $\omega_2$ are...
the natural frequencies, $\beta_1$ and $\beta_2$ are the nonlinear coefficients, and the coupling between the oscillators is characterized by the coefficients $c_1$ and $c_2$. The first oscillator ($x_1$) is submitted to the action of an external harmonic force with amplitude $P_0$ and frequency $\omega$.

As is generally characteristic of actual nonlinear systems, weak nonlinearity ($\beta_1 \ll 1$, $\beta_2 \ll 1$), weak coupling ($c_1 \ll 1$, $c_2 \ll 1$), and low damping ($\delta_1 \ll 1$, $\delta_2 \ll 1$) can be supposed. Additionally we assume that the driving harmonic force has a small amplitude ($P_0 \ll 1$) and that the resonant frequencies of the system and the frequency of the driving force are closely spaced ($\omega_1 \sim \omega_2 \sim \omega$). Under such quasi-linear conditions the method of slowly varying amplitude [36] can be applied to solve the set of nonlinear equation (1). In the framework of this method the solution to system (1) is sought in the form $x_1(t) = A \cos (\omega t + \theta)$ and $x_2(t) = B \cos (\omega t + \phi)$, where $A$ and $B$ are the slowly varying amplitudes and $\theta$ and $\phi$ are the slowly varying phases.

Making the standard change of variables [36] and with subsequent averaging we arrive at the following system of reduced equations:

$$\frac{dA}{d\tau} = -A - k_1 B \sin (\theta - \phi) - P \sin \theta, \quad A \frac{d\theta}{d\tau} = -\Omega A + \gamma_1 A^3 - k_1 B \cos (\theta - \phi) - P \cos \theta, \quad B \frac{d\phi}{d\tau} = -(\Omega - \eta) B + \gamma_2 B^3 - k_2 A \cos (\phi - \theta), \quad \frac{dB}{d\tau} = -\delta B - k_2 A \sin (\phi - \theta),$$

(2)

where $\tau = \delta t$ is the ‘slow’ time, $\gamma_1 = 3 \omega \beta_1 / 8 \delta_1$ and $\gamma_2 = 3 \omega \beta_2 / 8 \delta_2$ are the normalized nonlinear coefficients, $k_1 = c_1 / 2 \omega_1 \delta_1$ and $k_2 = c_2 / 2 \omega_2 \delta_2$ are the normalized coupling coefficients, $\delta = \delta_2 / \delta_1$ is the relative damping, $\Omega = (\omega_2^2 - \omega_1^2) / 2 \omega_1 \delta_1$ is the frequency mismatch, $\eta = (\omega_2^2 - \omega_1^2) / 2 \omega_2 \delta_2$ is the frequency difference, and $P = P_0 / 2 \omega_1 \delta_1$ is the normalized amplitude of the driving force. The set of equation (2) has a steady-state solution as $\tau \to \infty$. We should note that the same approach has been used to study the coupled system of Duffing and harmonic oscillators in [37].

In practice, to find the steady-state solution we use the conditions of constant amplitudes and phases; i.e., we set $dA / d\tau = dB / d\tau = d\theta / d\tau = d\phi / d\tau = 0$. Thus, the steady-state solution to system (2) satisfies the following system of algebraic equations:

$$A^2 = \frac{B^2}{k_2} \left[ \delta^2 + \left( \Omega - \eta - \gamma_2 B^2 \right)^2 \right],$$

$$A^2 + \frac{k_1}{k_2} \left[ \Omega A^2 - \gamma_1 A^4 \right] - \frac{k_1}{k_2} \left[ \Omega - \eta - \gamma_2 B^2 \right]^2 = A^2 P^2.$$

(3)

This defines the steady-state amplitudes of small nonlinear oscillations existing in the set of two coupled Duffing oscillators (1).

As is well known [9], if the nonlinearity in the system (1) is absent ($\beta_1 = \beta_2 = 0$), each of the two linear oscillators has two resonant states at certain frequencies. In such a linear system the amplitudes can be calculated exactly using the complex amplitude method. Assuming the notations used here, they have the following form:

$$A = \frac{(-\Omega + \eta + i \delta) P}{(\Omega - i) (\Omega - \eta - i \delta - k_1 k_2)},$$

$$B = \frac{k_2 P}{(\Omega - i) (\Omega - \eta - i \delta - k_1 k_2)}.$$

(4)

It should be noted that to simplify derivation of the amplitudes (4), we use a complex harmonic exponent instead of a real cosine function on the right-hand side of the first equation in (1).

Typical dependences of the amplitudes (4) on the frequency $\Omega$ are presented in figure 1. The spectral line of the second linear oscillator (which is unforced) has two resonances of symmetrical Lorentzian shape positioned near the corresponding eigenfrequencies (see the dashed red line in figure 1). At the same time the spectral line of the first oscillator (which is forced) has two resonances with both symmetrical Lorentzian and asymmetrical Fano shapes (see the solid blue line in figure 1). The Fano resonance is a result of the composition (interference) of two oscillations from the driving force and the second coupled oscillator. If the phase of the oscillator changes monotonically when the driving frequency passes through the resonance, the Lorentzian resonance is observed. On the other hand, if the phase dependence has a gap, the Fano resonance appears [9]. In particular for the considered system, the antiresonant state takes place at the frequency $\Omega = \eta = -5$. Note that the presence of damping in the system ($\delta \neq 0$) reduces the quality factor of both resonances and causes the condition that the amplitude at the antiresonance does not reach zero [38].

In addition, the shape of the linear resonances does not depend on the amplitude of the driving force. In other words,
changing the amplitude $P$ in equation (4) leads only to scaling along the ordinate axis in figure 1. But this is not the case with a nonlinear system ($\beta_1 \neq 0, \beta_2 \neq 0$). Indeed, one can see the complicated dependences of the steady-state amplitudes on the driving force in equation (3). The peculiarity of the nonlinear resonance consists in the presence of significant dependence of the resonant shapes on the amplitude of the driving force. Such frequency dependences of the steady-state amplitudes (3) on the driving force $P$ are represented in figure 2. The curves illustrate the transformations of the coupled Lorentzian and Fano resonances in system (1) having weak nonlinearity. For the small amplitudes of the driving force there is no significant difference between the features of the nonlinear resonances and those of the linear resonances (figure 2(a)). Further increasing the amplitude of the driving force leads to some deformations of the resonant curves. The peaks of the Lorentzian resonances become frequency shifted and their shape becomes bent. The form of the spectrum line of the left resonant state transforms into a closed loop, whereas the frequency of the antiresonant state acquires a shift (figures 2(b) and (c)).

It is known that for each nonlinear resonance there is some critical amplitude of the driving force. If the amplitude of the driving force is greater than the critical one, points with a vertical tangent line appear on the resonant curve. It should be noted that the left and right resonances of the same resonant curve have different critical amplitudes. But the corresponding resonances of different curves have the same critical amplitudes. When the amplitude of the driving force is high enough, the frequency bands appear where the steady-state amplitudes have an ambiguous dependence (hysteresis) on the frequency of the driving force. Some parts of the resonant curves within these bands become unstable sets. Therefore, continuous variation of the driving force frequency leads to jumping of the steady-state amplitudes on the boundaries of the unstable regions, which results in the appearance of bistability.

For certain parameters of system (1), overlapping of the two different resonances can be observed. It is evident in this case that the spectral curves acquire more than two stable states (which are marked in figure 2(c) with circles), i.e., the effect of multistability arises in the system.

3. Electromagnetic analog: nonlinear bilayer fish-scale metamaterial

Subsequently, in this section, we confirm the predictions of our nonlinear two-oscillator model in an optical system. As such a system we consider a particular configuration of a metamaterial in which resonant–antiresonant states can be excited effectively using trapped modes. The configuration consists of equidistant arrays of continuous meander metallic strips placed on both sides of a thin dielectric substrate (bilayer fish-scale structures [35]). In such a fish-scale structure the trapped-mode resonances can be excited if the incident field is polarized along the strips and when the form of these strips is slightly different from the straight line. Furthermore, in the bilayer structure, in addition to the trapped-mode resonance excited within each grating, another trapped-mode resonance can appear due to a specific interaction of the antiphase current oscillations between two adjacent gratings. Thus our structure supports two distinct resonant states; this corresponds to the characteristic of the two-oscillator model.

A sketch of the studied structure is presented in figure 3. The structure consists of two gratings of planar perfectly
conducting infinite wavy-line strips placed on both sides of a dielectric slab with thickness \( h \) and permittivity \( \varepsilon \). The elementary translation cell of the structure under study is a square with sides \( d = d_x = d_y \). The full length of the strip within the elementary translation cell is \( S \). Suppose the thickness \( h \) and size \( d \) are less than the wavelength \( \lambda \) of the incident electromagnetic radiation (\( h \ll \lambda, d < \lambda \)). The width of the metal strips and their deviation from the straight line are \( 2w \) and \( \Delta \), respectively.

Assume that the normally incident field is a plane monochromatic wave polarized parallel to the strips (x-polarization) and that the amplitude of the primary field is \( A_0 \).

In the frequency domain we use the method of moments to solve the problem of electromagnetic wave scattering by the bilayer fish-scale metamaterial [39, 40]. It involves solving the integral equation related to the surface currents which are induced in the metal pattern by the field of the incident wave. In the framework of the method of moments, the metal pattern is treated as a perfect conductor, whereas the substrate is assumed to be a lossy dielectric (\( \varepsilon = \varepsilon' + i\varepsilon'' \)).

In the bilayer configuration the method of solution rigorously takes into account an electromagnetic coupling between two adjacent gratings via evanescent partial spatial waves. The metamaterial response can be expressed through the induced currents \( J_1 \) and \( J_2 \) which flow along the strips of the corresponding grating and the reflection \( R \) and transmission \( T \) coefficients as functions of normalized frequency (\( \omega = d/\lambda \)), permittivity \( \varepsilon \), and other parameters of the structure.

Remarkably, due to the bilayer configuration of the structure under study, there are two possible current distributions which cause the trapped-mode resonances. The first distribution is the antiphase current oscillations in arcs of each grating. The currents flow in the same manner on both gratings, and the resonance exists due to the curvilinear form of the strips. This resonance is inherent in both single-layer and bilayer structure configurations [39, 40]. The resonant frequency is labeled in figure 4(a) with the letter \( \omega_1 \), and it corresponds to the first resonant frequency of our two-oscillator model. The second distribution is the antiphase current oscillations excited between two adjacent gratings. Obviously this resonance can be excited only in the bilayer structure configuration. The resonant frequency is labeled in figure 4(b) with the letter \( \omega_2 \), and hence it corresponds to the second resonant frequency of the two-oscillator model.

Thus the current oscillations on the upper and bottom gratings are characterized by two resonant states whose amplitudes are presented in figure 5. In contrast with the spectral line of the current amplitude related to the bottom grating, the corresponding spectral line related to the upper grating has a specific asymmetric form with antiresonant state that is in full compliance with predictions of the two-oscillator model. However, a small dip in the amplitude of the Lorentzian resonance related to the bottom grating is explained by its incomplete screening by the upper grating.

For a particular bilayer fish-scale structure, two resonant states correspond to two peaks of reflectivity, whereas the antiresonant state corresponds to the maximum of transmissivity. At the same time these two resonant states have different quality factors. The quality factor of the first resonance depends on the form of the strips and is practically independent of the substrate permittivity \( \varepsilon \). Although on the straight strips both resonant states are not excited at all, for the first resonant state the less the form of the strips is different from a straight line, the greater its quality factor. On the other hand, the quality factor of the second resonance depends crucially on both the distance between gratings and the permittivity of the substrate. Also, the presence of ohmic losses (\( \varepsilon'' \neq 0 \)) in the substrate reduces the quality factor of both resonances and restricts achievement of complete transmission at the antiresonance due to partial absorption of electromagnetic waves within the structure.

Thus, varying the distance between gratings and substrate permittivity changes the trapped-mode resonant conditions, and this changing manifests itself in the current amplitudes \( J_1 \) and \( J_2 \). We argue that due to such current distributions the field turns out to be localized between the gratings, i.e., directly in the substrate, which can sufficiently enhance the nonlinear effects if the substrate is made of a field intensity dependent (nonlinear) material.

In this case, permittivity of the substrate \( \varepsilon \) becomes dependent on the intensity of the electromagnetic field inside it (\( \varepsilon = \varepsilon_1 + \varepsilon_2 |E_{\text{int}}|^2 \)). In [23–25, 41] an approximate treatment was proposed to solve such a nonlinear problem. It is obtained by introducing two approximations. The first postulates that the inner field intensity is directly proportional to the square of the current amplitude averaged over a metal pattern extent, \( I_m \sim J^2 \), where \( J = (J_1 + J_2)/2 \). The second approximation assumes that, in view of the smallness of the elementary translation cell of the array (\( d < \lambda \)), the nonlinear substrate remains a homogeneous dielectric slab under the action of intensive light.

At the trapped-mode resonance, the electromagnetic energy is confined in a very small region between the strips. This allows us to apply the transmission line theory to estimate the intensity of the inner field which is localized within the system. In greater detail, according to this theory, the reference meander wire is considered a conducting wire periodically loaded by short-circuited sections of transmission
is 2d is the current magnitude 2V ΔZJ
the length of the equivalent line section. For the equivalent
λ
is the magnetic field magnitude 2d 0 is the impedance

(4)
As previously mentioned, such a form of lines is

Figure 5. The frequency dependences (x = d/λ) of current
amplitudes induced on the upper and bottom gratings; ϵ0 = 3,
e′ = 0.01, 2w/d = 0.05, h/d = 0.2, Δ/d = 0.25.

Under these approximations a nonlinear equation related
to the current magnitude averaged over the metal pattern
extent within an elementary translation cell is obtained in the
form [23–25]

\[ \bar{J} = \bar{A}_0 F_0 \left[ \alpha, \epsilon_1 + \epsilon_2 \left( I_{in}(\bar{J}) \right) \right], \]

where \( \sim A_0 \) is a dimensionless coefficient which depicts
how many times the incident field magnitude \( A_0 \) is greater
than 1 V cm\(^{-1} \). The input field magnitude \( A_0 \) is a parameter of
this nonlinear equation. So at a fixed frequency \( \alpha \), the solu-
tion of this equation gives us the averaged current magnitude \( \bar{J} \),
which depends on the magnitude of the incident field \( A_0 \).
On the basis of the current \( \bar{J}(A_0) \) found by a numerical
solution to the nonlinear equation, the actual value of
permittivity \( \epsilon \) of the nonlinear substrate is determined and
the reflection \( R \) and transmission \( T \) coefficients are calculated

\[
R = R \left[ \alpha, \epsilon_1 + \epsilon_2 \left( I_{in}(A_0) \right) \right],
T = T \left[ \alpha, \epsilon_1 + \epsilon_2 \left( I_{in}(A_0) \right) \right].
\]

as functions of the frequency \( \alpha \) and the magnitude of the
incident field \( A_0 \). For further details about the method of
solution the reader is referred to [41].

One can see that as the amplitude of the incident field
rises, the frequency dependences of the inner field intensity
acquire a form of bent resonances (figure 6), which com-
pletely confirms the assumption of our nonlinear two-osci-
lator model. As previously mentioned, such a form of lines is
a result of the nonlinearly induced shift of the resonant fre-
quency. In particular, in the optical system, when the fre-
quency of the incident wave is tuned to nearly the resonant
frequency, the field localization produces growing intensity of
the inner field, which can alter the permittivity enough to shift
the resonant frequency [41]. When this shift brings the
excitation closer to the resonant condition, even more field is
localized in the system, which further enhances the shift of
resonance. This positive feedback leads to formation of the
hysteresis loop in the inner field intensity with respect to the
incident field amplitude, and, as a result, under a certain
amplitude of the incident field, the frequency dependences of
the inner field intensity take the form of bent resonances. It is
evident that, in certain frequency bands, the transmission
coefficient acquires two stable states where the effect of
bistability takes place.

An important point is that in our system this bending is
different for distinctive resonances due to differences in their
nature and, respectively, in their current amplitudes [41].
This results in a specific distortion of the curves of the transmission
coefficient amplitude nearly equivalent to the trapped-mode
resonant frequencies. Thus, at the frequency \( \alpha_1 \sim 0.78 \) the
inner field produced by antiphase current oscillations is
confined in the area in the vicinity of each grating, and it
weakly affects the permittivity of the dielectric substrate. In
this case the resonant line acquires a transformation into a
closed loop which is a dedicated characteristic of sharp
nonlinear Fano-shaped resonances and is related to the char-
acteristic of our nonlinear two-oscillator model. The second
resonance \( \alpha_2 \sim 0.82 \) is smooth, but the current oscillations
produce a strong field concentration between two adjacent
gratings directly inside the dielectric substrate. This leads to a
considerable distortion of the transmission coefficient amplitude in a wide frequency range, and at a certain incident field amplitude this resonance reaches the first one and tends to overlap it (figure 6(b)). Thus, in this case, the transmission coefficient acquires more than two stable states; i.e., the effect of multistability arises in complete accordance with the prediction of the two-oscillator model.

4. Conclusions

In this paper a direct analogy of oscillation characteristics of two nonlinear systems is developed. As such systems a chain of coupled Duffing oscillators and an optical structure in the form of bilayer fish-scale metamaterial bearing trapped modes are considered. It is shown that the spectral features of both systems are distinguished by two resonant and single anti-resonant states whose profiles acquire corresponding Lorentzian and Fano forms.

Certain peculiarities of nonlinear impact on spectral lines changing with increasing the amplitude of the driving force and the intensity of the incident field are studied for the two-oscillator model and the planar metamaterial, respectively. In the nonlinear regime, resonance bending, closed loop formation, and effects of bistability and multistability in the spectra of both structures have been demonstrated.

We argue that our nonlinear two-oscillator model can be used to reveal the physical nature of the resonant behavior of such a complicated optical system and can help to identify conditions for the appearance in it of chaotic oscillations, the synchronization phenomenon, and other related nonlinear effects.

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