On Motzkin-Straus Type of Results and Frankl-Füredi Conjecture for Hypergraphs

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Abstract

A remarkable connection between the order of a maximum clique and the Graph-Lagrangian of a graph was established by Motzkin and Straus in 1965. This connection and its extension were useful in both combinatorics and optimization. Since then, Graph-Lagrangian has been a useful tool in extremal combinatorics. In this paper, we give a parametrized Graph-Lagrangian for non-uniform hypergraphs and provide several Motzkin-Straus type results for nonuniform hypergraphs which generalize results from [1] and [2]. Another part of the paper concerns a long-standing conjecture of Frankl-Füredi on Graph-Lagrangians of hypergraphs. We show the connection between the Graph-Lagrangian of \{1, r_1, r_2, \ldots, r_l\}-hypergraphs and \{r_1, r_2, \ldots, r_l\}-hypergraphs. Some of our results provide solutions to the maximum value of a class of polynomial functions over the standard simplex of the Euclidean space.

Keywords: Graph-Lagrangians of hypergraphs Extremal combinatorics Polynomial optimization

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1 Introduction

In 1965, Motzkin and Straus [3] established a connection between the order of a maximum clique and the Graph-Lagrangian of a graph. This connection and its extensions were successfully employed in optimization to provide heuristics for the maximum clique problem [4, 5, 6, 7, 8]. This connection provided another proof of Turán’s theorem [9] which pushed the development of extremal graph theory. More generally, the connection between Graph-Lagrangians and Turán densities can be used to give another proof of the fundamental result of Erdős-Stone-Simonovits on Turán densities of graphs; see Keevash’s survey paper [10]. However, the obvious generalization of Motzkin and Straus’ result to \(r\)-uniform hypergraphs is false. i.e., the Graph-Lagrangian of a hypergraph is not always the same as the Graph-Lagrangian of its maximum cliques. There are many examples of \(r\)-uniform hypergraphs other than complete \(r\)-uniform hypergraphs that do not achieve their Graph-Lagrangian on any proper subhypergraph. In spite of this, Graph-Lagrangians has been a useful tool in extremal problems in combinatorics. In 1980’s, Sidorenko [11] and Frankl and Füredi [12] developed the method of applying Graph-Lagrangians in determining hypergraph Turán densities. More recent applications of Graph-Lagrangians can be found in Keevash’s survey paper [10], [13] and [14]. In most applications in extremal combinatorics, we need an upper bound for the Graph-Lagrangians of hypergraphs. In the course of estimating Turán densities of hypergraphs by applying

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the Graph-Lagrangians of related hypergraphs, Frankl and Füredi [12] asked the following question: Given \( r \geq 3 \) and \( m \in \mathbb{N} \) how large can the Graph-Lagrangian of an \( r \)-graph with \( m \) edges be? They proposed the following conjecture: The \( r \)-graph with \( m \) edges formed by taking the first \( m \) sets in the colex ordering of \( N^{(r)} \) has the largest Graph-Lagrangian of all \( r \)-graphs with \( m \) edges. Mozklin-Straus result implies that this conjecture is true for \( r = 2 \). For \( r \geq 3 \), this conjecture seems to be very challenging. Talbot first confirmed this conjecture for some cases in [15]. Later Tang et al. confirmed this conjecture for some more cases in [16, 17, 18].

Recently, the study of Turán densities of non-uniform hypergraphs has been motivated by the study of extremal poset problems [19, 20]. In [21], Johnston and Lu gave a generalization of the concept of Turán density of a non-uniform hypergraph. In [1], Peng et al. introduced the Graph-Lagrangian of a non-uniform hypergraph, and gave an extension of Erdős-Stone-Simonovits theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices by applying Graph-Lagrangians of non-uniform hypergraphs (this extension of Erdős-Stone-Simonovits theorem to non-uniform hypergraphs was given in [21] by a different method). In this paper, we study a more generalized question for non-uniform hypergraphs and provide several results related to this question (Theorems 2.10, 2.11, 2.12 and 2.13). Although the truth of Conjecture 2.3 of Frankl and Füredi is not known in general even for \( r \)-uniform hypergraphs, we propose that a similar result is true for non-uniform hypergraphs (Problem 2) and provide some partial results (Theorem 2.15).

Our main results provide solutions to the maximum value of a class of polynomial functions in several variables.

## 2 Definitions, notations and main results

A hypergraph is a pair \( H = (V(H), E(H)) \) consisting of a vertex set \( V(H) \) and an edge set \( E(H) \), where each edge is a subset of \( V(H) \). The set \( T(H) = \{|e| : e \in E\} \) is called the set of edge types of \( H \). We also say that \( H \) is a \( T(H) \)-graph. For example, if \( T(H) = \{1, 3\} \), then we say that \( H \) is a \( \{1, 3\} \)-graph. If all edges have the same cardinality \( r \), then \( H \) is an \( r \)-uniform hypergraph, which is simply written as \( r \)-graph. A 2-uniform hypergraph is a simple graph. A hypergraph is non-uniform if it has at least two edge types. Write \( H^2_r \) for a hypergraph \( H \) on \( n \) vertices with \( T(H) = T \). For any \( r \in T(H) \), the \( r \)th-level hypergraph \( H'^r \) is the hypergraph consisting of all edges containing \( r \) vertices of \( H \). For \( Q \subset T \), let \( H^Q \) denote the hypergraph \( \cup_{r \in Q} H'^r \). We also use \( E'^r \) to denote the set of all edges with \( r \) vertices of \( H \). For convenience, an edge \( \{i_1, i_2, \ldots, i_r\} \) in a hypergraph is simply written as \( i_1i_2\ldots i_r \) throughout the paper.

For an integer \( n \), let \( [n] \) denote the set \( \{1, 2, \ldots, n\} \). For a set \( V \) and a positive integer \( i \), let \( \binom{V}{i} \) be the set of all subsets of \( V \) with \( i \) elements. The complete hypergraph \( K^2_n \) is a hypergraph on vertex set \( [n] \) with edge set \( \bigcup_{i \in T} \binom{[n]}{i} \). For example, \( K^{(r)}_n \) is the complete \( r \)-uniform hypergraph on \( n \) vertices. \( K_n^{[r]} \) is the non-uniform hypergraph with all possible edges of cardinality at most \( r \). Let \( [n]^T \) represent the complete \( T \)-type hypergraph on vertex set \( [n] \). For example, \( [n]^{(1,3)} \) represents the complete \( \{1, 3\} \)-hypergraph on vertex set \( [n] \). We also let \( [n]^{(r)} \) represent the complete \( r \)-uniform hypergraph on vertex set \( [n] \). A hypergraph \( H \) is a subgraph of a hypergraph \( G \), denoted by \( H \subseteq G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). A complete subhypergraph of a hypergraph \( H \) with the same edge type as \( T(H) \) is called a clique of \( H \). If \( W \subseteq V(H) \), then the subhypergraph of \( H \) induced by \( W \) is denoted by \( H[W] \), i.e. the vertex set of \( H[W] \) is \( W \) and the edge set of \( H[W] \) is the set of all edges in \( H \) whose vertices are in \( W \).
**Remark 2.1** Let $S := \{ \vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \}$. The Graph-Lagrangian of $H$, denoted by $\lambda(H)$, is the maximum of the above homogeneous multilinear polynomial of degree $r$ over the standard simplex $S$. Precisely, 

$$\lambda(H) := \max \{ \lambda(H, \vec{x}) : \vec{x} \in S \}.$$ 

The value $x_i$ is called the weight of the vertex $i$. A vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for $H$ if and only if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called an optimal weighting for $H$ if and only if $\lambda(H, \vec{y}) = \lambda(H)$.

**Remark 2.1** $\lambda(H)$ was called Lagrangian of $H$ in literature $[12, 15, 22, 27]$. The terminology ‘Graph-Lagrangian’ was suggested by Franco Giannessi.

Motzkin and Straus in $[3]$ proved the following result for the Graph-Lagrangian of a 2-graph. It shows that the Graph-Lagrangian of a graph is determined by the order of its maximum cliques.

**Theorem 2.2** $[3]$ If $G$ is a 2-graph in which a largest clique has order $t$, then,

$$\lambda(G) = \lambda(K_r^{(2)}) = \lambda([t]^{(2)}) = \frac{1}{2} \left( 1 - \frac{1}{r} \right).$$

The Motzkin-Straus result and its extension had many applications in extremal problems in graphs and hypergraphs $[10]$. However, the obvious generalization of Motzkin and Straus’ result to $r$-uniform hypergraphs is false. i.e., the Graph-Lagrangian of a hypergraph is not always the same as the Graph-Lagrangian of its maximum cliques. In spite of this, there are still applications of Graph-Lagrangians of hypergraphs in determining hypergraph Turán densities $[10, 11, 22]$. In most applications, we need an upper bound for the Graph-Lagrangians of hypergraphs. Frankl and Füredi $[12]$ asked the following question: Given $r \geq 3$ and $m \in \mathbb{N}$ how large can the Lagrangian of an $r$-graph with $m$ edges be? In order to state their conjecture on this problem we require the following definition. For distinct $A, B \in \mathcal{N}^{(r)}$ we say that $A$ is less than $B$ in the colex ordering if $\max(\Lambda \cup B) \in B$, where $A \cup B = (A \setminus B) \cup (B \setminus A)$. For example we have $246 < 156$ in $\mathbb{N}^{[3]}$ since $\max(\{2, 4, 6\} \cup \{1, 5, 6\}) = \{5\} \in \{1, 5, 6\}$. In Colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \cdots$. Let $C_{m,r}$ denote the $r$-graph with $m$ edges formed by taking the first $m$ elements in the colex ordering of $\mathbb{N}^{(r)}$. When $m = \binom{r}{1}$, the $r$-graph $C_{1,r}$ is $[1]^{(r)}$. The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the above question.

**Conjecture 2.3** (Frankl and Füredi $[12]$) The $r$-graph formed by taking the first $m$ sets in the colex ordering of $\mathcal{N}^{(r)}$ has the largest Lagrangian of all $r$-graphs with $m$ edges. In other words, if $H$ is an $r$-graph with $m$ edges, then $\lambda(H) \leq \lambda(C_{m,r})$.

Motzkin-Straus’s Theorem (Theorem $[22]$) implies that this conjecture is true when $r = 2$ by Theorem $[22]$. For the case $r = 3$, Talbot in $[15]$ proved the following.
Theorem 2.4 (Talbot [15]) Let m and t be integers satisfying
\[
\left( \frac{t}{3} \right) - 2 \leq m \leq \left( \frac{t}{3} \right) + \left( \frac{t-1}{2} \right) - t.
\]
Then Conjecture 2.3 is true for \( r = 3 \) and this value of \( m \).

Recently, Tang et al. verified this conjecture for more cases.

Theorem 2.5 [15, 16, 18] Let m and t be integers satisfying
\[
\left( \frac{t}{3} \right) - 7 \leq m \leq \left( \frac{t}{3} \right) + \left( \frac{t-1}{2} \right) - \frac{1}{2}t.
\]
Then Conjecture 2.3 is true for \( r = 3 \) and this value of \( m \).

Let
\[
\lambda^r_{(m,t)} = \max \{ \lambda(H) : H = (V,E) \text{ is an } r-\text{graph}, |V| = n, |E| = m \}.
\]
For \( r \geq 4 \), the only known results are

Theorem 2.6 (Talbot [15]) For any \( r \geq 4 \) there exists constants \( \gamma_r \) and \( \kappa_0(r) \) such that if \( m \) satisfies
\[
\left( \frac{t}{r} \right) \leq m \leq \left( \frac{t}{r} \right) + \left( \frac{t-1}{r-1} \right) - \gamma_r(r-2),
\]
with \( t \geq \kappa_0(r) \), then \( \lambda^r_{(m,t+1)} = \lambda(C_{m,r}) = \lambda([r]^r) \).

Theorem 2.7 [17] Let \( m, r \) and \( t \) be integers satisfying
\[
\left( \frac{t}{r} \right) - 4 \leq m \leq \left( \frac{t}{r} \right).
\]
Then \( \lambda^r_{(m,t)} = \lambda(C_{m,r}) \).

Recently, the study of Turán densities of non-uniform hypergraphs has been motivated by the study of extremal poset problems [19, 20]. In [21], Johnston and Lu gave a generalization of the concept of Turán density to a non-uniform hypergraph. In [1], Peng et al. generalized the concept of Graph-Lagrangians to non-uniform hypergraphs, gave a generalization of Mozkin-Straus result to non-uniform hypergraphs, and consequently applied it obtaining a result on Turán densities of {1, 2}-graphs similar to Erdős-Stone-Simonovits classical result on Turán densities of graphs. In this paper, we study the following general optimization problem for non-uniform hypergraphs which generalizes the concept of Graph-Lagrangians.

Problem 2.8 Let \( H \) be an \( \{r_0, r_1, r_2, \ldots, r_t\} \)-graph, with vertex set \( V(H) = [n] \) and edge set \( E(H) \). Let \( \alpha_1, \alpha_2, \ldots, \alpha_t \) be non-negative constants. For \( \vec{x} \in S \), let
\[
L(\alpha_1, \alpha_2, \ldots, \alpha_t)(H,\vec{x}) := \sum_{\vec{v} \in E(H^{[0]})} x_{i_0} \cdot x_{i_1} \cdot \ldots \cdot x_{i_0} + \alpha_1 \sum_{i_1 \leq i_2 \leq \ldots \leq i_t} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_t}
\]
The polynomial optimization problem of $H$ is

$$L_{\{\alpha_1, \alpha_2, \ldots, \alpha_l\}}(H) := \max \{L(H, \vec{x}) : \vec{x} \in S\}. \tag{1}$$

We sometimes simply write $L_{\{\alpha_1, \alpha_2, \ldots, \alpha_l\}}(H, \vec{x})$ and $L_{\{\alpha_1, \alpha_2, \ldots, \alpha_l\}}(H)$ as $L(H, \vec{x})$ and $L(H)$ if there is no confusion. The value $x_i$ is called the weight of the vertex $i$. A vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a feasible solution to (1) if and only if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called a solution to optimization problem (1) if and only if $L(H, \vec{y}) = L(H)$.

**Remark 2.9** If $G$ is a subhypergraph of $H$, then $L_{\{\alpha_1, \alpha_2, \ldots, \alpha_l\}}(G) \leq L_{\{\alpha_1, \alpha_2, \ldots, \alpha_l\}}(H)$.

The characteristic vector of a set $U$, denoted by $\vec{x}^U = (x_1^U, x_2^U, \ldots, x_n^U)$, is the vector in $S$ defined as:

$$x_i^U = \frac{1_{i \in U}}{|U|}$$

where $|U|$ denotes the cardinality of $U$ and $1_P$ is the indicator function returning 1 if property $P$ is satisfied and 0 otherwise.

In this paper, we show the following result to Problem 2.8 for $\{1, 2\}$-graphs which generalizes a result in [1].

**Theorem 2.10** Let $\alpha_2 > 0$ be a constant. If $H$ is a $\{1, 2\}$-graph with $n$ vertices and the order of its maximum clique is $t$, where $t \geq \alpha_2$, then $L_{\{\alpha_2\}}(H) = L_{\{\alpha_2\}}(K_1^{1,2}) = 1 + \frac{\alpha_2}{t} - \frac{\alpha_2}{2t^2}$. Furthermore, the characteristic vector of a maximum clique is a solution to optimization problem (1).

In [2], Gu et al. give some Motzkin-Straus type results to non-uniform hypergraphs. In a similar way, we give Motzkin-Straus type results to $\{1, r\}$-graphs and $\{1, 2, 3\}$-graphs regarding Problem 1.

**Theorem 2.11** Let $\alpha_1 > 0$ be a constant. Let $H$ be a $\{1, r\}$-graph. If both the order of its maximum complete $\{1, r\}$-subgraphs and the order of its maximum complete $\{1\}$-subgraphs are $t$, where $t \geq \frac{[\alpha_1 - (r - 2)]t^2 - \alpha_1}{(r - 2)!\alpha_1^2}$, then

$$L_{\{\alpha_1\}}(H) = L_{\{\alpha_1\}}(K_1^{1,r}) = 1 + \alpha_1 \frac{\prod_{i=1}^{t-1} (t - i)}{t!t^{t-1}}. $$

Furthermore, the characteristic vector of a maximum clique is a solution to optimization problem (1).

**Theorem 2.12** Let $\alpha_1, \alpha_2 > 0$ be constants. Let $H$ be a $\{1, 2, 3\}$-graph. If both the order of its maximum complete $\{1, 2, 3\}$-subgraph and the order of its maximum complete $\{1\}$-subgraph are $t$, where $t \geq \frac{[\alpha_1 + \alpha_2]t^2 - \alpha_1 + \alpha_2}{\alpha_1 + \alpha_2}$, then

$$L_{\{\alpha_1, \alpha_2\}}(H) = L_{\{\alpha_1, \alpha_2\}}(K_1^{1,2,3}) = 1 + \alpha_1 \frac{t - 1}{2t} + \alpha_2 \frac{(t - 1)(t - 2)}{6t^2}. $$

Furthermore, the characteristic vector of a maximum clique is a solution to optimization problem (1).

A result to $\{1, r_1, r_2, \ldots, r_l\}$-graph will be also given.

**Definition 2.2** Let $H$ be an $\{1, r_1, r_2, \ldots, r_l\}$-graph, $1 < r_1 < r_2 < \ldots < r_l$, with vertex set $V(H) = [n]$ and edge set $E(H)$. For $i \in V(H^1)$, $i$ is isolated in $H$ if and only if there is no edge $e \in E(H^\cap)$ ($i = 1, \ldots, l$) such that $i \in e$. The set of all isolated vertices of $H$ is denoted by $D(H)$.

For example, if $H = \{1, 2, 3, 4, 5\} \cup \{12, 13\} \cup \{123, 356\}$, then vertex 4 is isolated in $H$ and vertices 4 and 5 are isolated in $H[V(H^1)]$, so $D(H[V(H^1)]) = \{4, 5\}$. 

5
Theorem 2.13 Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be non-negative constants. Let $H$ be a $\{1, r_1, r_2, \ldots, r_t\}$-graph, where $1 < r_1 < \ldots < r_t$ and $\sum_{i=1}^{t} \frac{\alpha_i}{(r_i-1)!} \leq 1$. If $V(H^1) \neq D[H[V(H^1)]]$, then $L(\alpha_1, \alpha_2, \ldots, \alpha_t) \geq L(\alpha_1, \alpha_2, \ldots, \alpha_t)(H[V(H^1)] \setminus \{H[V(H^1)]\})$, and if $V(H^1) = D(H[V(H^1)])$, then $L(\alpha_1, \alpha_2, \ldots, \alpha_t)(H) = 1$.

Although the truth of Conjecture 2.1 is not known in general even for 3-uniform hypergraphs, we propose a similar question for hypergraphs. Similarly, for distinct sets $A, B \subset \mathbb{N}$, we say that $A$ is less than $B$ in the colex ordering if $\max(A \cup B) \subset B$. Let $C_{m,T}$ denote the hypergraph with edge type $T$ and $m$ edges formed by taking the first $m$ elements in the colex ordering.

Problem 2.14 Let $H$ be a hypergraph with edge type $T = \{r_0, r_1, r_2, \ldots, r_t\}$ and $m$ edges. For what conditions on $\alpha_i > 0$, the inequality

$$L(\alpha_1, \alpha_2, \ldots, \alpha_t)(H) \leq L(\alpha_1, \alpha_2, \ldots, \alpha_t)(C_{m,T})$$

holds?

Theorems 2.2 and 2.10 provided some results to Problem 2.14 for $T = \{2\}$ or $T = \{1, 2\}$. For $T = \{3\}$, Theorems 2.3 and 2.5 provided partial results to this problem. We show the following connection between $\{1, r_1, r_2, \ldots, r_t\}$-hypergraphs and $\{r_1, r_2, \ldots, r_t\}$-hypergraph concerning this question.

Theorem 2.15 Let $r_1, \ldots, r_t$ be positive integers satisfying $1 < r_1 < \ldots < r_t$. Let $\alpha_i (i = 1, \ldots, l)$ be positive constants satisfying $\sum_{i=1}^{l} \frac{\alpha_i}{(r_i-1)!} \leq 1$. Let $m$ and $t$ be positive integers satisfying $t + \sum_{i=1}^{l} \binom{t}{r_i} \leq m + t + \sum_{i=1}^{l} \binom{t+1}{r_i}$. Let $H$ be a $\{1, r_1, r_2, \ldots, r_t\}$-hypergraph with $m$ edges and $n$ vertices. If for an $\{r_1, r_2, \ldots, r_t\}$-hypergraph $G$ with $m - t - 1$ edges and $n$ vertices, $L(\alpha_1, \alpha_2, \ldots, \alpha_t)(G) \leq L(\alpha_1, \alpha_2, \ldots, \alpha_t)(C_{m-t-1, \{r_1, r_2, \ldots, r_t\}})$ holds, then $L(\alpha_1, \alpha_2, \ldots, \alpha_t)(H) \leq L(\alpha_1, \alpha_2, \ldots, \alpha_t)(C_{m, \{1, r_1, r_2, \ldots, r_t\}})$ holds.

Combining this theorem and the known results given in Theorem 2.5, we can get corresponding results for $\{1, 3\}$-hypergraphs.

3 Some preliminaries

In this section, we give some preliminary results to be applied in the proof.

The support of a vector $\bar{x} \in S$, denoted by $\sigma(\bar{x})$, is the set of indices corresponding to positive components of $\bar{x}$, i.e.,

$$\sigma(\bar{x}) = \{i : x_i > 0, 1 \leq i \leq n\}.$$ 

We will impose an additional condition on a solution $\bar{x} = (x_1, \ldots, x_n)$ to optimization problem (1).

(*) $|\sigma(x)|$ is minimal, i.e., if $\bar{y}$ is a legal weighting for $H$ satisfying $|\sigma(y)| < |\sigma(x)|$, then $L(H, \bar{y}) < L(H)$.

For a hypergraph $H = (V, E)$, $i \in V$, and $r \in T(H)$, let $E_i^r = \{A \in V^{(r-1)} : A \cup \{i\} \in E^r\}$. For a pair of vertices $i, j \in V$, let $E_{ij}^r = \{B \in V^{(r-2)} : B \cup \{i, j\} \in E^r\}$. Let $E_i^r = \{A \in V^{(r-1)} : A \cup \{i\} \in V \setminus E^r\}$, $(E_i^r)^c = \{B \in V^{(r-2)} : B \cup \{i, j\} \in V \setminus E^r\}$, and $E_{ij}^r = E_i^r \cap (E_i^r)^c$. Let $L(E_i^r, \bar{x}) = \alpha_i \lambda(E_i^r, \bar{x})$, $L(E_{ij}^r, \bar{x}) = \alpha_i \lambda(E_{ij}^r, \bar{x})$, and $L(E_{i,j}^r, \bar{x}) = \alpha_i \lambda(E_{i,j}^r, \bar{x})$, where $\alpha_0 = 1$. Let $E_i = \bigcup_{r \in T(H)} E_i^r$, $E_{i,j} = \bigcup_{r \in T(H)} E_{ij}^r$, and $E_{i,j} = \bigcup_{r \in T(H)} E_{i,j}^r$. Let $L(E_i, \bar{x}) = \sum_{r \in T(H)} L(E_i^r, \bar{x})$, $L(E_{i,j}, \bar{x}) = \sum_{r \in T(H)} L(E_{ij}^r, \bar{x})$, and $L(E_{i,j}, \bar{x}) = \sum_{r \in T(H)} L(E_{i,j}^r, \bar{x})$. Note that $L(E_i, \bar{x}) = \frac{\partial L(H, \bar{x})}{\partial \bar{x}_i}$ and $L(E_{i,j}, \bar{x}) = \frac{\partial L(H, \bar{x})}{\partial \bar{x}_{i,j}}$. 

6
Lemma 3.1  Let $H = (V, E)$ be an $r$-graph and $\bar{x} = (x_1, \ldots, x_n)$ be an optimal legal weighting for $H$ satisfying (*). Then for $i, j \in \sigma(\bar{x})$,
(a) $\lambda(E_i, \bar{x}) = \lambda(E_j, \bar{x}) = r\lambda(G)$,
(b) there is an edge in $E$ containing both $i$ and $j$.

We give a similar result for a non-uniform hypergraph below.

Lemma 3.2  Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be a solution to the polynomial programming (*) satisfying (*). Then for $i, j \in \sigma(\bar{x})$,
(a) $\frac{\partial L(H, \bar{x})}{\partial x_i} = \frac{\partial L(H, \bar{x})}{\partial x_j}$. This is equivalent to $L(E_i, \bar{x}) = L(E_j, \bar{x})$,
(b) there exists an edge $e \in E(H)$ such that $\{i, j\} \subseteq e$.

Proof. (a) Suppose, for a contradiction, that there exist $i, j \in \sigma(\bar{x})$ such that $\frac{\partial L(H, \bar{x})}{\partial x_i} > \frac{\partial L(H, \bar{x})}{\partial x_j}$. We define a new feasible solution $\bar{y}$ to (1) as follows. Let $y_q = x_q$ for $q \neq i, j$, $y_i = x_i + \delta$ and $y_j = x_j - \delta \geq 0$, then

$$L(H, \bar{y}) - L(H, \bar{x}) = \delta \left( \frac{\partial L(H, \bar{x})}{\partial x_i} - x_i \frac{\partial^2 L(H, \bar{x})}{\partial x_i \partial x_j} \right) - \delta \left( \frac{\partial L(H, \bar{x})}{\partial x_j} - x_j \frac{\partial^2 L(H, \bar{x})}{\partial x_i \partial x_j} \right) + (\delta x_j - \delta x_i - \delta^2) \frac{\partial^2 L(H, \bar{x})}{\partial x_i \partial x_j} = \delta \left( \frac{\partial L(H, \bar{x})}{\partial x_i} - \frac{\partial L(H, \bar{x})}{\partial x_j} \right) - \delta^2 \frac{\partial^2 L(H, \bar{x})}{\partial x_i \partial x_j} > 0$$

for some small enough $\delta$, contradicting to that $\bar{x}$ is a solution to optimization problem (1).

(b) Suppose, for a contradiction, that there exist $i, j \in \sigma(\bar{x})$ such that $\{i, j\} \not\subseteq e$ for any $e \in E(H)$. We define a new feasible solution $\bar{y}$ to (1) as follows. Let $y_q = x_q$ for $q \neq i, j$, $y_i = x_i + \delta$ and $y_j = x_j - \delta = 0$, then $\bar{y}$ is clearly a feasible solution for $H$, and

$$L(H, \bar{y}) - L(H, \bar{x}) = x_j \left( \frac{\partial L(H, \bar{x})}{\partial x_i} - \frac{\partial L(H, \bar{x})}{\partial x_j} \right) - x_j \frac{\partial^2 L(H, \bar{x})}{\partial x_i \partial x_j} = 0.$$

So $\bar{y}$ is a solution to optimization problem (1) and $|\sigma(\bar{y})| = |\sigma(\bar{x})| - 1$, contradicting the minimality of $|\sigma(\bar{x})|$. \hfill \Box

In [14], Talbot introduced the definition of a left-compressed $r$-uniform hypergraph. This concept is generalized to non-uniform hypergraphs in [2].

Let $H = ([n], E)$ be a $T(H)$-graph, where $n$ is a positive integer. For $e \in E$, and $i, j \in [n]$ with $i < j$, define

$$C_{i \rightarrow j}(e) = \begin{cases} (e \setminus \{j\}) \cup \{i\} & \text{if } i \notin e \text{ and } j \in e, \\ e & \text{otherwise.} \end{cases}$$

And

$$C_{i \rightarrow j}(E) = \{C_{i \rightarrow j}(e) : e \in E\} \cup \{e : e, C_{i \rightarrow j}(e) \in E\}.$$  \hfill (2)

Note that $|C_{i \rightarrow j}(E)| = |E|$ from the definition of $C_{i \rightarrow j}(E)$.

We say that $E$ or $H$ is left-compressed if and only if $C_{i \rightarrow j}(E) = E$ for every $1 \leq i < j$. If a $T(H)$-hypergraph $H$ is left-compressed, then for every $r \in T(H)$, the $r$-level hypergraph $H'$ is left-compressed. An equivalent
perhaps more intuitive definition of left-compressed hypergraph is that a $T(H)$-hypergraph $H = ([n], E)$ is left-compressed if and only if for any $r \in T(H)$, $j_1, j_2, \ldots, j_r \in E$ implies $i_1, i_2, \ldots, i_r \in E$ provided $i_p \leq j_p$ for every $p$, $1 \leq p \leq r$. Moreover, if $H$ is a left-compressed $T(H)$-hypergraph and $i < j$, then for every $r \in T(H)$, $E_{j, i} = \emptyset$.

The following lemma is similar to a result given in [15].

**Lemma 3.3** Let $H = ([n], E)$ be a $T(H)$-graph, $i, j \in [n]$ with $i < j$ and $\bar{x} = (x_1, \ldots, x_n)$ be an optimal legal weighting of $H$. Write $H_{i < j} = ([n], \mathcal{C}_{i < j}(E))$. Then,

$$L(H, \bar{x}) \leq L(H_{i < j}, \bar{x}).$$

**Proof:** If $1 \notin T(H)$, then,

$$L(H_{i < j}, \bar{x}) - L(H, \bar{x}) = \sum_{r \in T(H)} \sum_{e \in E^r, C_{i < j}(e) \notin E'} L(e \setminus \{j\}, \bar{x}) (x_i - x_j),$$

and if $1 \in T(H)$, then,

$$L(H_{i < j}, \bar{x}) - L(H, \bar{x}) = \sum_{r \geq 2} \sum_{e \in E^r, C_{i < j}(e) \notin E'} L(e \setminus \{j\}, \bar{x}) (x_i - x_j) + (x_i - x_j) I,$$

where $I$ satisfies that $I = 1$, if $i \notin E^1, j \in E^1$, and otherwise $I = 0$. Hence $L(H_{i < j}, \bar{x}) - L(H, \bar{x})$ is nonnegative in any case, since $i < j$ implies that $x_i \geq x_j$. So this lemma holds.

**Remark 3.4** Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be a solution to the optimization problem [1]. Let $i, j \in \sigma(x)$ with $i < j$.

(a) Lemma 3.2 part (a) implies that

$$x_j L(E_{ij}, \bar{x}) + L(E_{\bar{i}j}, \bar{x}) = x_i L(E_{ij}, \bar{x}) + L(E_{j, i}, \bar{x}).$$

In particular, if $H$ is left-compressed, then

$$(x_i - x_j)L(E_{ij}, \bar{x}) = L(E_{\bar{i}j}, \bar{x})$$

since $E_{j, i} = \emptyset$.

(b) If $H$ is left-compressed, then

$$x_i - x_j = \frac{L(E_{\bar{i}j}, \bar{x})}{L(E_{ij}, \bar{x})}$$

holds. If $H$ is left-compressed and $E_{\bar{i}j} = \emptyset$, then $x_i = x_j$.

(c) If $H$ is left-compressed, then

$$x_1 \geq x_2 \geq \ldots \geq x_n \geq 0.$$

A result similar to Lemma 2.4 in [15] is also true for non-uniform hypergraphs.

**Lemma 3.5** For any positive integers $m, t, r_1, r_2, \ldots, r_l$ satisfying

$$\sum_{i=1}^{l} \binom{t}{r_i} \leq m \leq \sum_{i=1}^{l} \binom{t}{r_i} + \sum_{i=1}^{l} \binom{t-1}{r_i-1},$$

we have $L(C_{m, (r_1, r_2, \ldots, r_l)}) = L([t]^{\{r_1, r_2, \ldots, r_l\}})$. 

8
Proof. Note that the vertex set of $C_{m,[r_1,r_2,...,r_t]}$ is $[t+1]$ and $[t][r_1,r_2,...,r_t] \subset C_{m,[r_1,r_2,...,r_t]}$. So $L(C_{m,[r_1,r_2,...,r_t]}) \geq L([t][r_1,r_2,...,r_t])$. Let $\bar{x} = (x_1,x_2,...,x_{t+1})$ be an optimal weighting of $C_{m,[r_1,r_2,...,r_t]}$. Note that the range of $m$ guarantees that there is no edge in $C_{m,[r_1,r_2,...,r_t]}$ containing both $r$ and $r+1$. By Lemma 3.2, $x_{t+1} = 0$. Therefore,

$$L(C_{m,[r_1,r_2,...,r_t]}) \leq L([t][r_1,r_2,...,r_t]).$$

This completes the proof of this lemma.  

We will need the following lemma in the proof of our main results.

Lemma 3.6 [24] There exists a left-compressed extremal $T$-hypergraph $H$ for $L_T^{(m,n)}$.

Proof. Let $H = (V,E)$ be an extremal $r$-graph for $L_T^{(m,n)}$. Let $\bar{x} = (x_1,x_2,...,x_n)$ be an optimal weight of $H$. We can assume that $x_i \geq x_j$ when $i < j$ since otherwise we can just relabel the vertices of $H'$ and obtain another extremal $r$-graph with an optimal weight $\bar{x} = (x_1,x_2,...,x_n)$ satisfying $x_i \geq x_j$ when $i < j$. If $H$ is not left compressed, performing a sequence of left-compressing operations (i.e. replace $E$ by $E'$ if $E'$ is not left compressed), we will get a left-compressed $r$-graph $H'$ with the same number of edges, the same number of vertices, and $\lambda(H') \geq \lambda(H)$. So $H'$ is a left-compressed extremal $r$-graph for $L_T^{(m,n)}$.  

4 Proof of Theorem 2.10

Let $H$ be a $\{1,2\}$-graph on $[n]$. In this case,

$$L_{\{\alpha_1\}}(H,\bar{x}) = \sum_{\bar{x} \in E(H^1)} x_i + \alpha_1 \sum_{\bar{x} \in E(H^2)} x_i x_j,$$

and

$$L_{\{\alpha_1\}}(H) = \max \{ L(H,\bar{x}) : \bar{x} \in S \}. \quad (6)$$

In the proof, we simply write $L_{\{\alpha_1\}}(H,\bar{x})$ and $L_{\{\alpha_1\}}(H)$ as $L(H,\bar{x})$ and $L(H)$.

Proof of Theorem 2.10 Applying Lemma 3.2(a) and a direct calculation, we get a solution $\bar{y}$ to (6) when $H = K^{(1,2)}$, which is given by $y_i = 1/t$ for each $i$, $1 \leq i \leq t$. Then $L(K^{(1,2)}) = 1 + \frac{n}{t} - \frac{n}{2}$. Since $K^{(1,2)} \subset H$, then $L(H) \geq L(K^{(1,2)})$.

Now we proceed to show that $L(H) \leq L(K^{(1,2)}) = 1 + \frac{n}{t} - \frac{n}{2}$. Let $\bar{x} = (x_1,x_2,...,x_n)$ be a solution to (6) satisfying (*) with $k$ positive weights. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = x_{k+2} = \ldots = x_n = 0$. By Lemma 3.2(b), $\forall 1 \leq i < j \leq k, ij \in E(H^2)$.

Claim 4.1 $\forall 1 \leq i < j \leq k$, if $i \in E(H)$ but $j \notin E(H)$, then $x_i - x_j = \frac{1}{\alpha_i}$.

Proof of Claim 1. By Lemma 3.2(a), $\frac{\partial L(H,\bar{x})}{\partial x_j} = \frac{\partial L(H,\bar{x})}{\partial x_j}$. By Lemma 3.2(b), $\forall 1 \leq i < j \leq k, ij \in E(H^2)$, therefore $1 + \alpha_i (1-x_i) = \alpha_j (1-x_j)$, i.e. $x_i - x_j = \frac{1}{\alpha_i}$.  


Let’s continue the proof of Theorem 5.5. Let \( p = [\alpha_1] \). Assume that there are \( q \) 1-sets of \( \{1, 2, 3, \ldots, k\} \) in \( H^1 \). If \( q = k \), then \( i \in E(H^1) \) for all \( 1 \leq i \leq k \), then \( K_k^{(1,2)} \) is a subgraph of \( H \). Since \( t \) is the order of the maximum complete \( \{1, 2\} \)-graph of \( H \), then \( k \leq t \). We have

\[
L(H, \bar{x}) = L(K_k^{(1,2)}) = 1 + \frac{\alpha_i}{2} - \frac{\alpha_i}{2k} = 1 + \frac{\alpha_i}{2} - \frac{\alpha_i}{2r}.
\]

Therefore we can assume that \( q = k - 1 \). Without loss of generality, assume that \( i \in E(H^1) \) for \( 1 \leq i \leq q \) and \( j \notin E(H^1) \) for \( q + 1 \leq j \leq k \), by Claim 1, \( x_i = x_j + \frac{1}{\alpha_i} \), \( \forall 1 \leq i \leq q \) and \( q + 1 \leq j \leq k \).

Case 1. \( 0 < \alpha_i < 1 \).

Note that \( q = 0 \). Otherwise, by Claim 1, we have \( x_i = \frac{1}{\alpha_i} + x_k > 1 \), which is a contradiction. So \( \forall 1 \leq i \leq k \), \( i \notin E(H^1) \). Therefore,

\[
L(H, \bar{x}) = L(K_k^{(2)}) = \frac{\alpha_i}{2} - \frac{\alpha_i}{2k} = 1 + \frac{\alpha_i}{2} - \frac{\alpha_i}{2r}.
\]

Case 2. \( \alpha_i \geq 1 \).

Note that \( q \leq p \). Otherwise, \( x_1 = x_k + \frac{1}{\alpha_i} + \ldots + x_q = x_k + \frac{q}{\alpha_i} > 1 \), conflicts to \( \sum_{i=1}^{k} x_i = 1 \) and \( x_i > 0 \) for \( 1 \leq i \leq k \). Then \( x_1 = \ldots = x_q = \frac{1-q/\alpha_i}{1/\alpha_i} + 1/\alpha_i \), \( x_{q+1} = \ldots = x_k = \frac{1-q/\alpha_i}{k} \).

\[
L(H, \bar{x}) = x_1 + \ldots + x_q + \alpha_i \sum_{1 \leq i \neq j \leq k} x_i x_j
\]

\[
= q x_1 + \alpha_i \sum_{1 \leq i \neq j \leq q} x_i x_j + \alpha_i \sum_{1 \leq i \neq j \leq q} x_i x_j + \alpha_i \sum_{q+1 \leq i \neq j \leq k} x_i x_j
\]

\[
= q x_1 + \alpha_i \left( \frac{q}{2} \right) x_1^2 + \alpha_i q (k-q) x_1 x_k + \alpha_i \left( \frac{k-q}{2} \right) x_k^2
\]

\[
= \frac{q + \alpha_i^2}{2 \alpha_i} - \frac{(\alpha_i - q)^2}{2 \alpha_i k}.
\]

Next, we show \( \frac{q + \alpha_i^2}{2 \alpha_i} - \frac{(\alpha_i - q)^2}{2 \alpha_i k} < 1 + \frac{\alpha_i}{2} - \frac{\alpha_i}{2r} \).

\[
[1 + \frac{\alpha_i}{2} - \frac{\alpha_i}{2r}] - \left[ \frac{q + \alpha_i^2}{2 \alpha_i} - \frac{(\alpha_i - q)^2}{2 \alpha_i k} \right] = 1 - \frac{q}{2 \alpha_i} + \frac{(\alpha_i - q)^2}{2 \alpha_i k} - \frac{\alpha_i}{2r}
\]

\[
> 1 - \frac{q}{2 \alpha_i} - \frac{\alpha_i}{2r}.
\]

Since \( p < \alpha_i \leq p+1, q \leq p \) and \( t \geq \alpha_i \), then \( \frac{q}{2 \alpha_i} + \frac{\alpha_i}{2r} < 1 \). This completes the proof.

### 5.5 Proof of Theorem 5.5

Let \( H \) be a \( \{1,r\} \)-graph with vertex set \( V(H) = [n] \) and edge set \( E(H) \). In this case,

\[
L_{(\alpha_i)}(H, \bar{x}) = \sum_{i \in E(H^1)} x_i + \alpha_i \sum_{i_1, i_2, \ldots, i_t \in E(H^1)} x_{i_1} x_{i_2} \cdots x_{i_t},
\]

and

\[
L_{(\alpha_i)}(H) = \max\{L(H, \bar{x}) : \bar{x} \in S\}.
\]

In the proof, we simply write \( L_{(\alpha_i)}(H, \bar{x}) \) and \( L_{(\alpha_i)}(H) \) as \( L(H, \bar{x}) \) and \( L(H) \).
Proof of Theorem 2.11. Applying Lemma 3.2 (a) and a direct calculation, we get a solution \( \bar{y} \) to (7) when \( H = K_t^{(1,r)} \) which is given by \( y_i = 1/t \) for each \( i \) (1 \( \leq i \leq t \)) and \( y_i = 0 \) else. So \( L(K_t^{(1,r)}) = 1 + \alpha_t \sum_{i=1}^{t} (r-1) \). Since \( K_t^{(1,r)} \subseteq H \), then \( L(H) \geq L(K_t^{(1,r)}) \). Now we need to prove that \( L(H) \leq L(K_t^{(1,r)}) \). Denote \( M_{t,(1,r)} = \max \{ L(H) : H \text{ is a } \{1,r\}-\text{graph, } H \text{ contains a maximum complete subgraph } K_t^{(1,r)} \} \) and a maximum complete subgraph \( K_t^{(1)} \). If \( M_{t,(1,r)} \leq L(K_t^{(1,r)}) \), then \( L(H) \leq L(K_t^{(1,r)}) \). Hence we can assume that \( H \) is an extremal hypergraph, i.e., \( L(H) = M_{t,(1,r)} \). If \( H \) is not left-compressed, performing a sequence of left-compressing operations (i.e., replace \( E \) by \( \mathcal{G}_i(E) \) if \( \mathcal{G}_i(E) \neq E \), we will get a left-compressed \( \{1,r\}-\text{graph} H' \) with the same number of edges. The condition that the order of a maximum complete \( \{1\} \)-subgraph of \( H \) is \( t \) guarantees that both the order of a maximum complete \( \{1,r\} \)-subgraph of \( H' \) and the order of a maximum complete \( \{1\} \)-subgraph of \( H' \) are still \( t \). By Lemma 3.3 \( H' \) is an extremal graph as well. So we can assume that the edge set of \( H \) is left-compressed, \( H_t = [r] \) and \( [r] \subseteq H' \). Let \( \vec{x} = (x_1, \ldots, x_n) \) be a solution to (7), where \( x_1 \geq x_2 \geq \ldots \geq x_k = x_{k+2} = \ldots = x_n = 0 \). If \( k \leq t \), then \( L(H) \leq L([k]^{(1,r)}) \leq L([r]^{(1,r)}) \). So it suffices to show that \( x_{t+1} = 0 \).

Let \( 1 \leq i \leq t \). If \( x_{t+1} > 0 \), then by Lemma 3.2 there exists \( e \in E(H') \) such that \( \{i, t+1\} \subseteq e \) and \( \frac{\partial L(H, \vec{x})}{\partial x_i} = \frac{\partial L(H, \vec{x})}{\partial x_t} - \frac{\partial L(H, \vec{x})}{\partial x_{t+1}} \). If \( x_{t+1} > 0 \), then \( x_i \geq \frac{1}{k} + x_{t+1} \). Note that \( E_{t+1}^{(r)} \) is a \((r-2)\)-graph on \([n]\setminus\{i, t+1\} \), so \( 0 < A \leq \frac{1}{\alpha_t (1-x_t-x_{t+1})^{r-2}} \). Then

\[
\frac{(r-2)!}{\alpha_t (1-x_t-x_{t+1})^{r-2}} + x_{t+1}.
\]

(8)

The above inequality clearly implies that \( x_i > \frac{(r-2)!}{\alpha_t} \). If \( \alpha_t \leq (r-2)! \), then \( x_i > 1 \) which is a contradiction. So what left is to consider \( \alpha_t > (r-2)! \). Combining \( x_i > \frac{(r-2)!}{\alpha_t} \) with (8), we have

\[
x_i > \frac{(r-2)!\alpha_t^{-3}}{\alpha_t - (r-2)!\alpha_t^{-3}}.
\]

(9)

Recall that \( t \geq \left\lfloor \frac{\alpha_t - (r-2)!\alpha_t^{-3}}{(r-2)!\alpha_t^{-3}} \right\rfloor \), with the aid of (9), \( \sum_{i=1}^{t} x_i > 1 \), a contradiction. So \( x_{t+1} = 0 \). The proof is thus complete.

6 Proof of Theorem 2.12

Let \( H \) be a \( \{1,2,3\} \)-graph with vertex set \( V(H) = [n] \) and edge set \( E(H) \). In this case,

\[
L_{(\alpha, \beta)}(H, \vec{x}) = \sum_{i \in E(H^1)} x_i + \alpha_1 \sum_{i \in E(H^2)} x_i x_{i+1} + \alpha_2 \sum_{i \in E(H^3)} x_i x_{i+1} x_{i+2},
\]

and

\[
L_{(\alpha, \beta)}(H) = \max \{ L(H, \vec{x}) : \vec{x} \in S \}.
\]

(10)
In the proof, we simply write $L_{(a_1, a_2)}(H, \bar{x})$ and $L_{(a_1, a_2)}(H)$ as $L(H, \bar{x})$ and $L(H)$.

**Proof of Theorem 2.12** Applying Lemma 3.2 (a) and a direct calculation, we get a solution $\bar{y}$ to (10) when $H = K_t^{(1, r)}$ which is given by $y_i = 1/t$ for each $i (1 \leq i \leq t)$ and $y_i = 0$ else. So $L(K_t^{(1, 2, 3)}) = 1 + \alpha_1 \frac{1}{t^2} + \alpha_2 \frac{1}{t(t-1)(t-2)}$. Hence we only need to prove $L(H) = L(K_t^{(1, 2, 3)})$. Since $K_t^{(1, 2, 3)} \subseteq H$, then, $L(H) = L(K_t^{(1, 2, 3)})$. Hence, to prove Theorem 2.12 it suffices to prove that $L(H) \leq L(K_t^{(1, 2, 3)})$. Denote $M_{[r, (1, 2, 3)]} = \max L(H)$. Then $L(H)$ contains a maximum complete subgraph of $K_t^{(1, 2, 3)}$ and a maximum $\{1\}$ complete subgraph $K_{[1]}^{(1)}$. If $M_{[r, (1, 2, 3)]} \leq L(K_t^{(1, 2, 3)})$, then $L(H) \leq L(K_t^{(1, 2, 3)})$. Hence we can assume that $H$ is an extremal hypergraph, i.e., $L(H) = M_{[r, (1, 2, 3)]}$. If $H$ is not left-compressed, performing a sequence of left-compressing operations (i.e. replace $\bar{y}$ by $\bar{y}$ $\bar{y}_i(E)$ if $\bar{y}_i(E) \neq E$), we will get a left-compressed $\{1, 2, 3\}$-graph $H'$ with the same number of edges. The condition that the order of a maximum complete $\{1\}$-subgraph of $H$ is $t$ guarantees that both the order of a maximum complete $\{1, 2, 3\}$-subgraph of $H'$ and the order of a maximum complete $\{1\}$-subgraph of $H'$ are still $t$. By Lemma 3.3, $H'$ is an extremal graph as well. So we can assume that the edge set of $H$ is left-compressed, $H' = [t], [r][2] \subseteq H^2$ and $[r][3] \subseteq H^3$. Let $\bar{y} = (x_1, \ldots, x_n)$ be a solution to (10), where $x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = x_{k+2} = \ldots = x_n = 0$. If $k \leq t$, then $L(H) \leq L(K_t^{(1, 2, 3)}) \leq L([r][1, 2, 3])$. So it suffices to show that $x_{t+1} = 0$.

Let $1 \leq i \leq t$. If $x_{t+1} > 0$, then by Lemma 3.2 there exists $e \in E(H)$ such that $\{i, t+1\} \subset e$ and $\frac{\partial L(H, \bar{x})}{\partial x_{t+1}} = \frac{\partial L(H, \bar{x})}{\partial x_{t+1}}$. Let $L(E_{i(t+1)}, \bar{x}) = \alpha_1$, if $i(t+1) \in E(H^2)$, and let $L(E_{i(t+1)}, \bar{x}) = 0$, if $i(t+1) \notin E(H^2)$. Recall that $i \in E(H^1)$ and $t+1 \notin E(H^1)$. Then

$$0 = \frac{\partial L(H, \bar{x})}{\partial x_i} = \frac{\partial L(H, \bar{x})}{\partial x_{t+1}}$$

Thus $x_i \geq x_{t+1}$, with $0 < A \leq \alpha_1 + \alpha_2(1 - x_i - x_{t+1})$. Hence

$$x_i > \frac{1}{\alpha_1 + \alpha_2(1 - x_i - x_{t+1})} + x_{t+1}.$$  \hspace{1cm} (11)

The above inequality clearly implies that $x_i > \frac{1}{\alpha_1 + \alpha_2}$. If $\alpha_1 + \alpha_2 \leq 1$, then $x_i > 1$ which is a contradiction. Combining $x_i > \frac{1}{\alpha_1 + \alpha_2}$ with (11), we have

$$x_i > \frac{\alpha_1 + \alpha_2}{(\alpha_1 + \alpha_2)^2 - \alpha_2}.$$ \hspace{1cm} (12)

Recall that $t \geq \lceil \frac{\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2} \rceil$, with the aid of (12), $\sum_{i=1}^{t} x_i > 1$, a contradiction. So $x_{t+1} = 0$. The proof is thus complete.

**7 Proof of Theorem 2.13**

Let $H$ be a $\{1, r_1, r_2, \ldots, r_t\}$-graph, $1 < r_1 < r_2 < \ldots < r_t$, with vertex set $V(H) = [n]$ and edge set $E(H)$. In this case,

$$L_{(a_1, \ldots, a_t)}(H, \bar{x}) = \sum_{i \in E(H)} x_i + a_1 \sum_{i_j \neq x} x_{i_1} x_{i_2} \ldots x_{i_{t-1}} + \ldots$$
In the proof, we simply write \(L_{\{a_1,...,a_l\}}(H,\bar{x})\) and \(L_{\{a_1,...,a_l\}}(H)\) as \(L(H,\bar{x})\) and \(L(H)\).

**Proof of Theorem 2.13** Let \(\bar{x} = (x_1, x_2, ..., x_n)\) be a solution to (13) satisfying (*) with \(k\) positive weights. We may assume that \(x_1 \geq x_2 \geq ... \geq x_k > x_{k+1} = ... = x_n\). Clearly, \(L(H) \geq L(H[V(H^1)])\).

**Claim 7.1**

**(a)** For all \(1 \leq i \leq k\), \(i \in E(H^1)\).

**(b)** Either \(i\) is isolated for all \(1 \leq i \leq k\) in \(H[[k]]\) or \(i\) is not isolated for any \(1 \leq i \leq k\) in \(H[[k]]\).

**Proof of Claim 2.** First we prove (a). If (a) fails to hold, then there are two possibilities.

**Case 1.** If \(i \notin E(H^1)\), for all \(1 \leq i \leq k\), then

\[
L(H,\bar{x}) = \alpha_i \sum_{i_1 \leq ... \leq i_k \in E(H^1)} x_{i_1}x_{i_2}...x_{i_{k-1}} + \ldots + \alpha_i \sum_{i_1 \leq ... \leq i_k \in E(H^1)} x_{i_1}x_{i_2}...x_{i_k} \\
\leq \sum_{i=1}^{l} \frac{\alpha_i}{r_i!} < \sum_{i=1}^{l} \frac{\alpha_i}{(r_i - 1)!} \leq 1.
\]

But \(L(H,\bar{x}) = L(H) \geq L(H^1) = 1 > L(H,\bar{x})\), Contradiction!

**Case 2.** If there exists \(1 \leq i \leq k\) and \(1 \leq j \leq k\) such that \(i \in E(H^1)\) and \(j \notin E(H^1)\), then

\[
\frac{\partial L(H,\bar{x})}{\partial x_i} = 1 + \alpha_i \sum_{i_1 \leq ... \leq i_{k-1} \in E_i^1} x_{i_1}x_{i_2}...x_{i_{k-1}} + \ldots + \alpha_i \sum_{i_1 \leq ... \leq i_{k-1} \in E_i^1} x_{i_1}x_{i_2}...x_{i_k} > 1,
\]

and

\[
\frac{\partial L(H,\bar{x})}{\partial x_j} = \alpha_j \sum_{i_1 \leq ... \leq i_{k-1} \in E_j^1} x_{i_1}x_{i_2}...x_{i_{k-1}} + \ldots + \alpha_j \sum_{i_1 \leq ... \leq i_{k-1} \in E_j^1} x_{i_1}x_{i_2}...x_{i_k} < \sum_{i=1}^{l} \frac{\alpha_i}{(r_i - 1)!} \leq 1.
\]

\[
\frac{\partial L(H,\bar{x})}{\partial x_i} > \frac{\partial L(H,\bar{x})}{\partial x_j},\text{ contradiction to Lemma 3.2(a).} \text{ So (a) holds.}
\]

By (a), we have \(L(H) \leq L(H[[k]]) \leq L(H[V(H^1)])\), so \(L(H) = L(H[[k]]) = L(H[V(H^1)])\).

Next, we prove (b). By (a), \(i \in E(H^1)\) for all \(1 \leq i \leq k\). If there exists \(1 \leq i \leq k\) and \(1 \leq j \leq k\) such that \(i\) is not isolated in \(H[[k]]\) and \(j\) is isolated in \(H[[k]]\). Then there is some edge in \(E(H[[k]]) \setminus \{1, 2, ... , k\}\) containing \(i\) in but no edge in \(E(H[[k]]) \setminus \{1, 2, ... , k\}\) containing \(j\), so

\[
\frac{\partial L(H,\bar{x})}{\partial x_i} = 1 + \alpha_i \sum_{i_1 \leq ... \leq i_{k-1} \in E_i^1} x_{i_1}x_{i_2}...x_{i_k-1} + \ldots + \alpha_i \sum_{i_1 \leq ... \leq i_{k-1} \in E_i^1} x_{i_1}x_{i_2}...x_{i_k} > 1,
\]

\[
\frac{\partial L(H,\bar{x})}{\partial x_j} = \sum_{i_1 \leq ... \leq i_{k-1} \in E_j^1} x_{i_1}x_{i_2}...x_{i_k-1} + \ldots + \alpha_j \sum_{i_1 \leq ... \leq i_{k-1} \in E_j^1} x_{i_1}x_{i_2}...x_{i_k} \leq 1.
\]

By (b), we have \(L(H) = L(H[[k]]) = L(H[V(H^1)])\), so \(L(H) = L(H[[k]]) = L(H[V(H^1)])\).
but

\[ \frac{\partial L(H, \bar{x})}{\partial x_j} = 1. \]

Contradiction to \( \frac{\partial L(H, \bar{x})}{\partial x_j} = \frac{\partial L(H, \bar{x})}{\partial x_i} \). \( \blacksquare \)

Let’s continue the proof of Theorem 2.13 if \( i \) is isolated for all \( 1 \leq i \leq k \) in \( H[[k]] \), then \( L(H) = 1 \). If \( i \) is not isolated for any \( 1 \leq i \leq k \) in \( H[[k]] \), then \( i \) is not isolated for any \( 1 \leq i \leq k \) in \( H[V(H^1)] \). So \( L(H) = \sum \frac{\alpha_i}{(r_i - 1)!} \leq 1 \).

Proof of Theorem 2.15

Let \( 1 < r_1 < \ldots < r_l \) be positive integers and let \( \alpha_i (i = 1, \ldots, l) \) be positive constants satisfying \( \sum_{i=1}^l \alpha_i \leq 1 \). Let \( m \) and \( t \) be positive integers satisfying \( t + \sum_{i=1}^l \binom{r_i}{1} = m + t + \sum_{i=1}^l \binom{r_i}{1} \). Let \( T = \{r_1, r_2, \ldots, r_l\} \) and \( Q = \{r_1, r_2, \ldots, r_l\} \). Let \( H = (V, E) \) be an extremal \( T \)-graph for \( L_{(m,n)} \). By Lemma 5.6(b), we can assume that \( H \) is left-compressed. Let \( \bar{x} = (x_1, x_2, \ldots, x_n) \) be an optimal weighting of \( H \) and \( k \) be the number of non-zero weights in \( \bar{x} \). Then \( x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \cdots = x_n = 0 \). Since \( \bar{x} \) has only \( k \) positive weights, we can assume that \( G \) is on \([k]\).

It is sufficient to show that \( L(H, \bar{x}) \leq L(C_{m,T}) \). If \( k \leq t \), then \( L(H) \leq L(K^T) \leq L(C_{m,T}) \) since the range of \( m \) guarantees that \( K^T \subset C_{m,T} \). So we can assume that \( k \geq t + 1 \).

We first show the following result.

Lemma 8.1 \( E^1 = [k](1) \).

Proof of Lemma 8.1 If the lemma does not hold, then there are two possible cases. Case 1. For each \( i, 1 \leq i \leq k \), \( i \notin E(H^1) \). In this case,

\[ L(H, \bar{x}) = \alpha_1 \sum_{i_{i_1} \leq \ldots \leq i_{t_1} \in E(H^1)} x_{i_1} x_{i_2} \ldots x_{i_{t_1}} + \cdots + \alpha_l \sum_{i_{i_1} \leq \ldots \leq i_{t_l} \in E(H^1)} x_{i_1} x_{i_2} \ldots x_{i_{t_l}}. \]

Note that for each \( j, 1 \leq j \leq l \),

\[ \sum_{i_{i_1} \leq \ldots \leq i_{t_j} \in E(H^1)} x_{i_1} x_{i_2} \ldots x_{i_{t_j}} \leq \sum_{i_{i_1} \leq \ldots \leq i_{t_j} \in E([k](j))} x_{i_1} x_{i_2} \ldots x_{i_{t_j}} \]

and \( \sum_{i_{i_1} \leq \ldots \leq i_{t_j} \in E([k](j))} x_{i_1} x_{i_2} \ldots x_{i_{t_j}} \) reaches the maximum \( \frac{k^j}{j!} \leq \frac{1}{r_j!} \) when \( \bar{x} \) is the characteristic vector of \([k]\). Therefore,

\[ L(H, \bar{x}) \leq \sum_{i=1}^l \frac{\alpha_i}{r_i!} \leq \sum_{i=1}^l \frac{\alpha_l}{(r_i - 1)!} \leq 1. \]

But \( L(H, \bar{x}) = L(H) \geq L(H^1) = 1 > L(H, \bar{x}) \). Contradiction!

Case 2. There exists \( 1 \leq i \leq k \) and \( 1 \leq j \leq k \) such that \( i \in E(H^1) \) and \( j \notin E(H^1) \). In this case,

\[ \frac{\partial L(H, \bar{x})}{\partial x_i} = 1 + \alpha_1 \sum_{i_{i_1} \leq \ldots \leq i_{t_1} \in E_i^1} x_{i_1} x_{i_2} \ldots x_{i_{t_1}} + \cdots + \alpha_l \sum_{i_{i_1} \leq \ldots \leq i_{t_l} \in E_i^l} x_{i_1} x_{i_2} \ldots x_{i_{t_l}} > 1, \]
and

\[ \frac{\partial L(H, \bar{x})}{\partial x_j} = \alpha_1 \sum_{i_1, i_2, \ldots, i_{r-1} \in E_1'} x_{i_1} x_{i_2} \cdots x_{i_{r-1}} + \cdots \]

\[ + \alpha_\ell \sum_{i_1, i_2, \ldots, i_{r-1} \in E_\ell'} x_{i_1} x_{i_2} \cdots x_{i_{r-1}} < \sum_{i=1, \ldots, \ell} \alpha_i (r_i - 1)! \leq 1. \]

So \( \frac{\partial L(H, \bar{x})}{\partial x_i} > \frac{\partial L(H, \bar{x})}{\partial x_j} \), contradiction to Lemma 3.2. This completes the proof of Lemma 8.1.

Let us continue the proof of the theorem. By Lemma 8.1, \( L(H, \bar{x}) = 1 + L(H^Q, \bar{x}) \), and the number of edges in \( H^Q \) is \( m - k \leq m - t - 1 \). By the assumption, \( L(H^Q) \leq L(C_{m-t-1,Q}) \). Therefore, \( L(H) = L(H, \bar{x}) \leq 1 + L(C_{m-t-1,Q}) \). Note that \( E(C_{m,t}) = [t+1] \cup E(C_{m-t-1,Q}) \). So \( 1 + L(C_{m-t-1,Q}) = L(C_{m,t}) \). Consequently, \( L(H) \leq L(C_{m,t}) \). This completes the proof of this theorem.

9 Conclusion

The classical method of Lagrange multiplier has been applied often in evaluating the Graph-Lagrangian of a hypergraph. However, evaluating the Graph-Lagrangian of a general hypergraph seems to be challenging and very few general results are known for hypergraphs. In the future, we will learn and explore whether modern Lagrange theory [25] will help advance the research.

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