A DECOUPLING PROOF OF THE TOMAS RESTRICTION THEOREM

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Abstract. We give a new proof of a classic Fourier restriction theorem for the truncated paraboloid in $\mathbb{R}^n$ based on the $l^2$ decoupling theorem of Bourgain-Demeter. Focusing on the extension formulation of the restriction problem (dual to the original restriction formulation), we find that the $l^2$ decoupling theorem directly implies a local variant of the desired extension estimate incurring an $\varepsilon$-loss. To upgrade this result to the desired global extension estimate, we employ some $\varepsilon$-removal techniques first introduced by Tao. By adhering to the extension formulation, we obtain a more natural proof of the required $\varepsilon$-removal result.

1. Introduction

Fix $n \geq 2$ and define the truncated paraboloid in $\mathbb{R}^n$ by

$$P^{n-1} := \{ (\xi, |\xi|^2) \in \mathbb{R}^n : \xi \in [-1,1]^{n-1} \},$$

which we equip with the surface measure $d\sigma$. We define the extension operator $E$ on $L^p(P^{n-1},d\sigma)$ by

$$Ef(x) = \int_{P^{n-1}} f(\xi)e^{2\pi i x \cdot \xi} d\sigma(\xi),$$

and denote by $R^* (p \to q)$ the statement that $E$ is a bounded operator $L^p(P^{n-1},d\sigma) \to L^q(\mathbb{R}^n)$. Thus, $R^* (p \to q)$ is equivalent to the extension estimate

$$\|Ef\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(P^{n-1},d\sigma)}, \quad \forall f \in L^p(P^{n-1},d\sigma). \quad (1)$$

The notation $R^* (p \to q)$ is motivated by the equivalence of estimate (1), by a standard duality argument, to the corresponding restriction estimate

$$\|\hat{g}|_{P^{n-1}}\|_{L^{p'}(P^{n-1},d\sigma)} \lesssim \|g\|_{L^{p'}(\mathbb{R}^n)}, \quad \forall g \in \mathcal{S}(\mathbb{R}^n), \quad (2)$$

which we denote by $R(q' \to p')$. The restriction problem is rooted in estimates of the form (2), and much of the terminology we will employ is therefore phrased in terms of restriction. Despite this, we will continue to focus on extension estimates.

The restriction conjecture states that $R^* (p \to q)$ holds for all exponents $p$ and $q$ satisfying

$$\frac{1}{q} < \frac{n-1}{2n}, \quad \frac{n+1}{q} \leq \frac{n-1}{p'}.$$

Thus, the conjectured strong-type diagram for $E$ is as follows:
Early progress on the restriction conjecture was made by Tomas [Tom75], using interpolation methods to prove $R^*(2 \to q)$ for all $q > \frac{2(n+1)}{n-1}$, a result which we will refer to as the Tomas restriction theorem. This result of Tomas was improved by Stein in the same year to include the endpoint $q = \frac{2(n+1)}{n-1}$, which we therefore refer to as the Tomas-Stein exponent. This improvement of Stein was first published by Tomas [Tom79], and a proof may also be found in Chapter IX, Section 2.1 of [Ste93]. Further developments, proving restriction theorems of Tomas-Stein type for more general measures, have since been made in [Moc00, Mit02, BS11]; recent progress on the restriction conjecture itself has been made in [Gut16, Gut18, HR19, Wan20, HZ20]. We will present a new proof of the Tomas restriction theorem based on the recent $l^2$ decoupling theorem of Bourgain-Demeter [BD15]. We note that the idea of using decoupling to prove restriction results is not entirely new; an application of decoupling to a discrete restriction theorem is also given in [BD15].

Fix $C \geq 1$. For each $\xi \in \mathbb{R}^{n-1}$ and each $\delta > 0$, let $P_{\xi, \delta} \subset \mathbb{R}^n$ denote the region

$$P_{\xi, \delta} = \{ (\xi', \xi_n) : \xi' \in \xi + (-\delta, \delta)^{n-1}; |\xi_n - |\xi|^2 - 2\xi \cdot (\xi' - \xi)| < C\delta^2 \}.$$  

This can be thought of as the hyperplane tangent to the paraboloid at $(\xi, |\xi|^2)$, thickened to scale $\sim \delta^2$ in the $\xi_n$ direction and lying above a cube of scale $\sim \delta$ about $\xi$. The $l^2$ decoupling theorem of Bourgain and Demeter may be reformulated as follows:

**Theorem 1.1** (The $l^2$ decoupling theorem: Theorem 1.1, [BD15]). Let $0 < \delta \leq 1$, and let $\Sigma \subset [-1, 1]^{n-1}$ be $\delta$-separated. If for each $\xi \in \Sigma$, $f_\xi \in S(\mathbb{R}^n)$ has Fourier support in $P_{\xi, \delta}$, then

$$\left\| \sum_{\xi \in \Sigma} f_\xi \right\|_{L^{2(n+1)/n-1}(\mathbb{R}^n)} \lesssim \delta^{-\varepsilon} \left( \sum_{\xi \in \Sigma} \|f_\xi\|_{L^{2(n+1)/n-1}(\mathbb{R}^n)}^2 \right)^{1/2}$$

for all $\varepsilon > 0$.

Note that the decoupling exponent $\frac{2(n+1)}{n-1}$ afforded to us by Theorem 1.1 is the same as the Tomas-Stein exponent. This is the primary fact we will exploit to give a new proof of the Tomas restriction theorem based on decoupling.
Inherent to the decoupling theorem is the $\varepsilon$-loss in the factor $\delta^{-\varepsilon}$. Consequently, we may only hope to use the decoupling theorem to directly prove a version of the extension estimate $R^*(p \to q)$ which also incurs an $\varepsilon$-loss. We therefore introduce a “local” variant of extension estimates.

**Definition 1.2.** Let $\varepsilon > 0$. If we have
\[ \|Ef\|_{L^q(B(0,R))} \lesssim R^\varepsilon\|f\|_{L^p(P^{n-1},d\sigma)} \]
for all $R \geq 1$ and all $f \in L^p(P^{n-1},d\sigma)$, we say that $R^*(p \to q; \varepsilon)$ holds.

There is a general principle known as $\varepsilon$-removal which states that the local extension estimates $R^*(p \to q; \varepsilon)$ for all $\varepsilon > 0$ should imply a global extension estimate in which the $\varepsilon$-loss is transferred to one or both of the exponents $p,q$. This idea was first explored by Tao [Tao99], and an adaptation of Tao’s methods yields the following concrete result to be explored in the final section:

**Theorem 1.3** ($\varepsilon$-removal). The local extension estimates $R^*(2 \to \frac{2(n+1)}{n-1}; \varepsilon)$ for all $\varepsilon > 0$ imply $R^*(2 \to q)$ for all $q > \frac{2(n+1)}{n-1}$.

Our goal is therefore to use the $L^2$ decoupling theorem to prove the local extension estimates $R^*(2 \to \frac{2(n+1)}{n-1}; \varepsilon)$ for all $\varepsilon > 0$. Do to so, it is helpful to reformulate the local extension estimates, so we will use the following well-known equivalent condition for $R^*(p \to q; \varepsilon)$. In what follows, we let $\mathcal{N}_\delta$ denote the $\delta$-neighbourhood of $P^{n-1}$ for any $\delta > 0$.

**Lemma 1.4.** Let $1 \leq p,q \leq \infty$ and $\varepsilon > 0$. Then, $R^*(p \to q; \varepsilon)$ holds if and only if for all $R \geq 1$ and all $g \in \mathcal{S}(\mathbb{R}^n)$ with Fourier support in $\mathcal{N}_{R^{-1}}$, we have $\|g\|_{L^q(B(0,R))} \lesssim R^{\varepsilon/p'}\|\hat{g}\|_{L^p(\mathbb{R}^n)}$.

2. A decoupling proof of the Tomas restriction theorem

In light of Theorem 1.3 and Lemma 1.4, to prove the Tomas Restriction theorem, it is sufficient to prove that for all $R \geq 1$ and all $g \in \mathcal{S}(\mathbb{R}^n)$ with Fourier support in $\mathcal{N}_{R^{-1}}$, we have
\[ \|g\|_{L^{\frac{2(n+1)}{(n-1)}}(B(0,R))} \lesssim R^{\varepsilon/2}\|\hat{g}\|_{L^2(\mathbb{R}^n)} \] (3)
for all $\varepsilon > 0$. In addition to the decoupling theorem, the main ingredient will be a variant of Bernstein’s inequality specific to the regions $P_{\xi,\delta}$.

**Theorem 2.1.** Let $\xi \in \mathbb{R}^{n-1}$ and $\delta > 0$, and suppose $g \in \mathcal{S}(\mathbb{R}^n)$ has Fourier support in $P_{\xi,\delta}$. Then, for all $1 \leq p \leq q \leq \infty$, we have
\[ \|g\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q} |P_{\xi,\delta}|^{1/p-1/q}\|g\|_{L^p(\mathbb{R}^n)}. \]

The proof of this version of Bernstein’s inequality is a simple adaptation of the proof of Bernstein’s inequality for discs: one produces a family of invertible affine transformations $\{T_{\xi,\delta} \mid \xi \in \mathbb{R}^{n-1}, \delta > 0\}$ such that each $T_{\xi,\delta}$ maps $P_{\xi,\delta}$ to the cube $[-1,1]^n$. Fixing $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $[-1,1]^n$, we see that if $g \in \mathcal{S}(\mathbb{R}^n)$ has Fourier support in $P_{\xi,\delta}$, then $g = g*(\varphi \circ T_{\xi,\delta})^\circ$, from which Young’s convolution inequality implies the result.

We are now ready to give a new proof of the Tomas restriction theorem based on decoupling.
Proof of the Tomas restriction theorem. Let $R \geq 1$, and suppose $g \in \mathcal{S}({\mathbb{R}}^n)$ has Fourier support in $\mathcal{N}_{R^{-1}}$. Let $\Sigma_R = R^{-1/2}Z^{n-1} \cap [-1, 1]^{n-1}$, and note that $\mathcal{N}_{R^{-1}}$ is covered by the collection $\mathcal{P}_R = \{ P_{\xi,R^{-1/2}} : \xi \in \Sigma_R \}$ provided the constant $C > 0$ in the definition of the regions $P_{\xi,\delta}$ was chosen large enough to begin with. Let $(\eta_P)_{P \in \mathcal{P}_R}$ be a partition of unity subordinate to this cover, and for each $P \in \mathcal{P}_R$, let $g_P = (\hat{\eta}_P)\hat{g}$. Each $g_P$ is Schwartz as the inverse Fourier transform of a Schwartz function, and by Fourier inversion, $\widehat{g_P} = \hat{\eta}_P \hat{g}$ is supported in $P$. It follows from the property $\sum_{P \in \mathcal{P}_R} \eta_P = 1$ that $\sum_{P \in \mathcal{P}_R} \widehat{g_P} = \hat{g}$ and hence, upon Fourier inversion, we have $\sum_{P \in \mathcal{P}_R} g_P = g$. The $l^2$ decoupling theorem therefore gives
\begin{equation}
\|g\|_{L^\frac{2(n+1)}{n-r}({\mathbb{R}}^n)} = \left\| \sum_{P \in \mathcal{P}_R} g_P \right\|_{L^\frac{2(n+1)}{n-r}({\mathbb{R}}^n)} \lesssim \varepsilon R^e \left( \sum_{P \in \mathcal{P}_R} \|g_P\|_{L^\frac{2(n+1)}{n-r}({\mathbb{R}}^n)}^2 \right)^{1/2},
\end{equation}
for all $\varepsilon > 0$. Since $\widehat{g_P}$ is supported in $P$, Theorem 2.1 and Plancherel’s theorem give
\begin{align*}
\|g_P\|_{L^\frac{2(n+1)}{n-r}({\mathbb{R}}^n)} & \lesssim |P|^{\frac{1}{2} - \frac{n-1}{2(n+1)}} \|g_P\|_{L^2({\mathbb{R}}^n)} \\
& \lesssim (R^{-(n+1)/2})^{\frac{1}{2} - \frac{n-1}{2(n+1)} \|\widehat{g_P}\|_{L^2({\mathbb{R}}^n)}} \\
& = R^{-1/2} \|\widehat{g_P}\|_{L^2({\mathbb{R}}^n)},
\end{align*}
from which (4) gives
\begin{equation}
\|g\|_{L^\frac{2(n+1)}{n-r}({\mathbb{R}}^n)} \lesssim \varepsilon R^e \left( \sum_{P \in \mathcal{P}_R} \|\widehat{g_P}\|_{L^2({\mathbb{R}}^n)}^2 \right)^{1/2},
\end{equation}
for all $\varepsilon > 0$. Now, each point in $\mathbb{R}^n$ lies in at most $O(1)$ of the regions $P$, and it follows that $\sum_{P \in \mathcal{P}_R} |\eta_P|^2 = O(1)$. We therefore have
\begin{align*}
\sum_{P \in \mathcal{P}_R} \|\widehat{g_P}\|_{L^2({\mathbb{R}}^n)}^2 = \sum_{P \in \mathcal{P}_R} \int_{\mathbb{R}^n} |\hat{\eta}_P|^2 |\hat{g}| dx &= \int_{\mathbb{R}^n} |\hat{g}|^2 \left( \sum_{P \in \mathcal{P}_R} |\eta_P|^2 \right) dx \\
& \lesssim \|\hat{g}\|_{L^2({\mathbb{R}}^n)}^2,
\end{align*}
which together with (5) gives
\begin{equation}
\|g\|_{L^\frac{2(n+1)}{n-r}({\mathbb{R}}^n)} \lesssim \varepsilon R^e \|\hat{g}\|_{L^2({\mathbb{R}}^n)}.
\end{equation}
In particular, inequality (3) holds, from which the Tomas restriction theorem follows.

3. $\varepsilon$-removal

We now outline a new approach to proving Theorem 1.3. In fact, we will prove the following result for a more general range of exponents, yielding Theorem 1.3 as a simple corollary:

**Theorem 3.1.** Given $2 \leq p \leq q \leq \infty$ and $\varepsilon > 0$ sufficiently small, $R^e(p \to q; \varepsilon)$ implies $R^e(p \to r)$ whenever
\begin{equation}
\frac{1}{r} < \frac{1}{q_0} := \frac{1}{q} - \frac{4\log 2}{q \log(1/\varepsilon q)}.
\end{equation}
Observe that if the weak-type bound
\[ \|Ef\|_{L^{q_0,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(P^{n-1}, d\sigma)}, \quad \forall f \in L^p(P^{n-1}, d\sigma) \quad (6) \]
is known, then Theorem 3.1 follows by interpolation with the known estimate \( R^*(p \to \infty) \). Our goal is therefore to prove the weak-type estimate (6). It is via this reduction that we obtain a more natural proof of Theorem 3.1 than that given for the original \( \varepsilon \)-removal result of Tao (Theorem 1.2, [Tao99]). The argument of Tao was phrased in terms of the restriction formulation, in which case equation (6) was replaced by a dual Lorentz space estimate of the form
\[ \|g\|_{P^{n-1}L^{p_0}(P^{n-1}, d\sigma)} \lesssim \|g\|_{P^{0,1}L^{n_0}(\mathbb{R}^n)}, \quad \forall g \in \mathcal{S}(\mathbb{R}^n). \]

To prove this estimate, Tao sketched some further reductions of a highly technical nature which are quite laborious when fully elaborated. By adhering to the extension formulation, we obtain a more natural proof which we can more easily elaborated.

Though we adopt a different formulation to Tao, our proof will use the same key components. In particular, we borrow the following definition and key lemmas from [Tao99] (with minor modifications):

**Definition 3.2.** A collection \( \{B(x_i, R)\}_{i=1}^N \) of \( R \)-balls is said to be sparse if \( |x_j - x_i| \gtrsim (RN)^{1-\varepsilon} \) for all \( i \neq j \).

The first key lemma allows us to bootstrap the local extension estimate \( \|Ef\|_{L^p(B(0,R))} \lesssim R^\varepsilon \|f\|_{L^p(P^{n-1}, d\sigma)} \) to an analogous estimate for multiple \( R \)-balls \( \bigcup_i B(x_i, R) \), provided the collection of balls is sparse.

**Lemma 3.3** (Lemma 3.2, [Tao99]). Let \( 2 \leq p \leq q \leq \infty \) and \( \varepsilon > 0 \), and suppose that \( R^*(p \to q; \varepsilon) \) holds. Then for all \( R \geq 1 \) and all \( f \in L^p(P^{n-1}, d\sigma) \), we have
\[ \|Ef\|_{L^p(\bigcup_{i=1}^N B(x_i, R))} \lesssim R^\varepsilon \|f\|_{L^p(P^{n-1}, d\sigma)} \]
whenever \( \{B(x_i, R)\}_{i=1}^N \) is a sparse collection of \( R \)-balls.

The second key lemma allows us to cover a set which is a union of unit cubes by a reasonably small number of sparse collections of balls of sufficiently small radius.

**Lemma 3.4** (Lemma 3.3, [Tao99]). Let \( A \) be a union of unit cubes. For any \( N \geq 1 \), \( A \) may be covered by \( O(N|A|^1/N) \) sparse collections of balls of radius \( O(|A|^{2N}) \).

In the forthcoming proof, we will have frequent need to consider various superlevel sets, so we introduce some notation here for convenience:

**Notation 3.5.** Let \( f : \mathbb{R}^n \to \mathbb{C} \) be measurable. Given \( t > 0 \), let \( \eta_f(t) \) denote the superlevel set
\[ \eta_f(t) = \{x \in \mathbb{R}^n : |f(x)| \geq t\}. \]

Given \( r > 0 \), we will also denote by \( Q_r \) the cube \([−r, r]^n\).

**Proof of Theorem 3.1.** Let \( 2 \leq p \leq q \leq \infty \), \( \varepsilon > 0 \), and suppose \( R^*(p \to q; \varepsilon) \) holds. Recall that it suffices to prove the estimate (6).

Let \( f \in L^p(P^{n-1}, d\sigma) \), and assume without loss of generality (by scale invariance) that \( \|f\|_{L^p(P^{n-1}, d\sigma)} = 1 \). We begin by bounding the left-hand side of (6) by the \( L^{q_0,\infty}(\mathbb{R}^n) \)
norm of the average \(|Ef| \ast \chi_{Q_{1/4}}\), motivated by the heuristic that the superlevel sets of \(|Ef| \ast \chi_{Q_{1/4}}\) resemble unions of unit cubes, so Lemma 3.4 will become applicable.

By Fubini’s theorem, we may compute

$$\widehat{\chi_{Q_{1/4}}} (\xi) = \prod_{i=1}^n \frac{\sin(\pi \xi_i/2)}{\pi \xi_i},$$

so \(\widehat{\chi_{Q_{1/4}}} \sim 1\) on \(Q_{3/2}\). Let \(\varphi \in C_c^\infty (Q_{3/2})\) be a bump function with \(\varphi \equiv 1\) on \(Q_1\), and define

$$\psi := \begin{cases} \varphi/\widehat{\chi_{Q_{1/4}}} & \text{on } Q_{3/2}; \\ 0 & \text{on } \mathbb{R}^n \setminus Q_{3/2}. \end{cases}$$

Since \(\varphi \equiv 1\) on \(P^{n-1} \subset Q_1\), Fubini’s theorem gives \(Ef = Ef \ast \hat{\varphi}\). But clearly, \(\varphi = \widehat{\chi_{Q_{1/4}}} \psi\), from which we see by Fourier inversion that \(\hat{\varphi} = \chi_{Q_{1/4}} \ast \hat{\psi}\), hence \(Ef = Ef \ast \chi_{Q_{1/4}} \ast \hat{\hat{\psi}}\).

Young’s convolution inequality for weak \(L^p\) spaces therefore gives \(\|Ef\|_{L^{p0, \infty} (\mathbb{R}^n)} \lesssim \|Ef \ast \chi_{Q_{1/4}}\|_{L^{p0, \infty} (\mathbb{R}^n)} \lesssim 1\). So by our normalisation, it suffices to prove \(\|Ef \ast \chi_{Q_{1/4}}\|_{L^{p0, \infty} (\mathbb{R}^n)} \lesssim 1\). That is, it suffices to prove

$$|\eta_{Ef} \ast \chi_{Q_{1/4}} (t)|^{1/q_0} \lesssim \frac{1}{t}, \quad \forall t > 0. \quad (7)$$

We first note that (7) is clear for all \(t\) larger than a particular threshold independent of \(f\). Indeed, Young’s convolution inequality followed by the known estimate \(R^* (p \rightarrow \infty)\) and our normalisation \(\|f\|_{L^p (P^{n-1}, d\sigma)} = 1\) gives

$$\|Ef \ast \chi_{Q_{1/4}}\|_{L^{p0, \infty} (\mathbb{R}^n)} \leq \|Ef\|_{L^{p0, \infty} (\mathbb{R}^n)} \|\chi_{Q_{1/4}}\|_{L^{1} (\mathbb{R}^n)} \lesssim 1.$$  

It follows that \(\|Ef \ast \chi_{Q_{1/4}}\|_{L^{p0, \infty} (\mathbb{R}^n)} \leq K\) for some constant \(K\) independent of \(f\), from which we see that \(|\eta_{Ef} \ast \chi_{Q_{1/4}} (t)| = 0\) for \(t > K\). Equation (7) therefore clearly holds for \(t > K\), so we may assume without loss of generality that \(0 < t \leq K\).

Given \(t > 0\), let \(A_t\) be the union of all unit cubes of the form \(z + Q_{1/2}\) for some \(z \in \mathbb{Z}^n\) which have nonempty intersection with the superlevel set \(\eta_{Ef} \ast \chi_{Q_{1/4}} (t)\). Clearly, \(\eta_{Ef} \ast \chi_{Q_{1/4}} (t) \subset A_t\), hence \(|\eta_{Ef} \ast \chi_{Q_{1/4}} (t)|^{1/q_0} \lesssim |A_t|^{1/q_0};\) it therefore suffices to prove

$$|A_t|^{1/q_0} \lesssim \frac{1}{t}, \quad \forall 0 < t \leq K. \quad (8)$$

Since \(A_t\) is a union of unit cubes, Lemma 3.4 allows us to cover \(A_t\) by sparse collections \(C_1, \ldots, C_m\) of balls of radius \(O(|A_t|^{2/3})\) for any \(N\), where \(m = O(N |A_t|^{1/3})\). Since these collections cover \(A_t\), we have

$$A_t = \bigcup_{i=1}^m \left( A_t \cap \bigcup_{B \in C_i} B \right),$$

hence,

$$|A_t| \leq \sum_{i=1}^m \left| A_t \cap \bigcup_{B \in C_i} B \right|. \quad (9)$$

If \(x \in A_t\) then by definition, there exists \(z \in \mathbb{Z}^n\) and \(y \in \mathbb{R}^n\) such that \(x, y \in z + Q_{1/2}\) and \(|Ef \ast \chi_{Q_{1/4}} (y)| \geq t\). It follows that \(|x - y| \leq \sqrt{n}\) and hence, \(|Ef \ast \chi_{Q_{1/4}} \sqrt{n}(x) \geq
\(|Ef| \ast \chi_{Q_{1/4}(y)} \geq t.\) We therefore have \(A_t \subset \eta|Ef| \ast \chi_{Q_{1/4+N}}(t),\) and in particular, equation \([9]\) gives

\[
|A_t| \leq \sum_{i=1}^{m} |A_t \cap \bigcup_{B \in \mathcal{C}_i} B| \leq \sum_{i=1}^{m} |\eta|Ef| \ast \chi_{Q_{1/4+N}}(t) \cap \bigcup_{B \in \mathcal{C}_i} B|.
\]

\[
= \sum_{i=1}^{m} |\eta|Ef| \ast \chi_{Q_{1/4+N}}(t) \chi_{\bigcup_{B \in \mathcal{C}_i} B}(t)|
\]

\[
\leq \frac{1}{t^q} \sum_{i=1}^{m} \|Ef| \ast \chi_{Q_{1/4+N}}\|_{L^q(\bigcup_{B \in \mathcal{C}_i} B)}^q,
\]

where the last line follows by Chebyshev’s inequality. But for all \(x \in \bigcup_{B \in \mathcal{C}_i} B,\) we have

\[
|Ef| \ast \chi_{Q_{1/4+N}}(x) = (|Ef| \chi_{\bigcup_{B \in \mathcal{C}_i} (B+Q_{1/4+N})}) \ast \chi_{Q_{1/4+N}}(x).
\]

It follows that

\[
\|Ef| \ast \chi_{Q_{1/4+N}}\|_{L^q(\bigcup_{B \in \mathcal{C}_i} B)}^q \leq \|(|Ef| \chi_{\bigcup_{B \in \mathcal{C}_i} (B+Q_{1/4+N})}) \ast \chi_{Q_{1/4+N}}\|_{L^q(\mathbb{R}^n)}^q
\]

\[
\lesssim \|Ef\|_{L^q(\bigcup_{B \in \mathcal{C}_i} (B+Q_{1/4+N}))}^q,
\]

where the last line follows by Young’s convolution inequality. But clearly, \(B(x, R) + Q_{1/4+N} \subset B(x, R + 2n)\) for any ball \(B(x, R) \subset \mathbb{R}^n.\) Letting \(\tilde{C}_i\) denote the collection of balls obtained by enlarging the radius of each ball in \(C_i\) by \(2n,\) it follows that

\[
\|Ef\|_{L^q(\bigcup_{B \in \mathcal{C}_i} (B+Q_{1/4+N}))} \leq \|Ef\|_{L^q(\bigcup_{B \in \mathcal{C}_i} B)},
\]

which together with \([10]\) and \([11]\) implies

\[
|A_t| \lesssim \frac{1}{t^q} \sum_{i=1}^{m} \|Ef\|_{L^q(\bigcup_{B \in \mathcal{C}_i} B)}^q,
\]

Noting that each of the collections \(\tilde{C}_i\) is also sparse and is comprised of balls of radius \(\lesssim |A_t|^{2N} + 2n,\) Lemma \([3.3]\) and our normalisation give

\[
\|Ef\|_{L^q(\bigcup_{B \in \mathcal{C}_i} B)} \lesssim (|A_t|^{2N} + 2n)^{\varepsilon q}.
\]

Now, if \(|A_t| < 2n, we get \(|A_t|^{1/\varepsilon q} \lesssim 1 \lesssim 1/t\) (recalling that \(0 < t \leq K),\) which is the conclusion \([8].\) On the other hand, if \(2n \leq |A_t|,\) then \([13]\) gives

\[
\|Ef\|_{L^q(\bigcup_{B \in \mathcal{C}_i} B)} \lesssim |A_t|^{\varepsilon q^{2N}}.
\]

Combining \([12]\) and \([14]\) and recalling that \(m = O(N|A_t|^{1/N}),\) we find that

\[
|A_t| \lesssim \frac{1}{t^q} N |A_t|^{\varepsilon q^{2N} + 1/N}.
\]

Setting \(N = \frac{\log(1/\varepsilon q)}{2\log 2},\) we get \(\varepsilon q^{2N} = 1,\) hence \(\varepsilon q^{2N} = 2^{-N} \leq 1/N.\) Noting that \(|A_t| \geq 1, \)

\([15]\) then gives \(|A_t| \lesssim \frac{1}{t^q} N |A_t|^{2/N}\) for our particular choice of \(N.\) Hence, if \(|A_t|\) is finite,
we may rearrange to conclude that $|A_t|^{1/q - 2/Nq} \lesssim \frac{N^{1/q}}{t}$. Substituting our expression for $N$, this is equivalent to

$$|A_t|^{\frac{1}{q} - \frac{2 \log 2}{q \log(1/\varepsilon q)}} \lesssim \frac{1}{t},$$

which is the conclusion (8). To see that $|A_t|$ is indeed finite, we note that by the asymptotics of the Fourier transform of a surface-supported measure and a density argument, $Ef(x)$ decays to 0 as $|x| \to \infty$. It follows that the superlevel set $\eta_{Ef(x) > \varepsilon Q_{1/4}}(t)$ is bounded, from which it is clear that $A_t$ is also bounded, and therefore has finite measure. Our manipulations leading to (16) are therefore justified, and we are done. □

Observe that as $\varepsilon \to 0$, we have $\frac{1}{q} - \frac{4 \log 2}{q \log(1/\varepsilon q)} \to \frac{1}{q}$. When combined with Theorem 3.1, this observation yields the following simple corollary of which Theorem 1.3 is a special case:

**Corollary 3.6.** Given $2 \leq p \leq q \leq \infty$, the local extension estimates $R^*(p \to q; \varepsilon)$ for all $\varepsilon > 0$ imply $R^*(p \to r)$ for all $r > q$.

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