The Euler Scheme for Feller Processes

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We consider the Euler scheme for stochastic differential equations with jumps, whose intensity might be infinite and the jump structure may depend on the position. This general type of SDE is explicitly given for Feller processes and a general convergence condition is presented.

In particular, the characteristic functions of the increments of the Euler scheme are calculated in terms of the symbol of the Feller process in a closed form. These increments are increments of Lévy processes and, thus, the Euler scheme can be used for simulation by applying standard techniques from Lévy processes.

Keywords Euler scheme; Feller process; Jump processes; SDE; Stochastic differential equations with jumps.

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1. Introduction and Main Result

The most general stochastic differential equation (SDE) defining a time homogeneous Markov process \((X_t)_{t \geq 0}\) taking values in \(\mathbb{R}^d, d \geq 1\) is of the form

\[
X_t = X_0 + \int_0^t a(X_s) \, ds + \sum_{j=1}^n \int_0^t b^{(j)}(X_s) \, dW_s^{(j)}
+ \int_0^t \int_{|u| \leq 1} k(X_s, u) \, q(\cdot ; ds, du)
+ \int_0^t \int_{|u| > 1} k(X_s, u) \, p(\cdot ; ds, du)
\]

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where \( a, b, k \) are the coefficients, \( W^{(j)} \) are independent Brownian motions, \( p \) is a Poisson random measure and \( q \) is the corresponding compensated Poisson random measure (cf. [23]). This equation even includes time inhomogeneous Markov processes, since one can transform any time inhomogeneous Markov process by extending the state space into a time homogeneous Markov process (cf. [26], 8.5.5).

Note that letting \( k \equiv 0 \) in (1) yields a diffusion equation, the classical setting for the Euler-Maruyama scheme. A Lévy driven SDE is also just a special case of this equation. To see this let \( f \) be a \( d \times n \) valued function, \( l \in \mathbb{R}^{n \times n} \) a positive semi-definite matrix in \( \mathbb{R}^{n \times n} \) and \( N \) an \( n \)-dimensional Lévy measure and set \( a(X_s) = f(X_s)l, b(X_s) = f(X_s)\sigma, k(X_s, u) = f(X_s) \) and let \( p \) have the dual predictable projection \( ds N(du) \). Then equation \( X_t = X_0 + \int_0^t f(X_s) \, dZ_s \) where \( Z_s \) is the Lévy process on \( \mathbb{R}^n \) with triplet \( (l, \sigma^2, N) \). In contrast to these two examples \( k \) may depend on \( u \) in this note. For an overview of numerical approximation schemes for this case with finite jump intensity see, for example, Bruti-Liberatia and Platen [6].

The Euler approximation with step size \( h \) for an SDE of form (1) is given by

\[
\overline{X}_t := X_0 + \int_0^t a(\overline{X}_s) \, ds + \sum_{i=1}^d \int_0^t b^{(i)}(\overline{X}_s) \, dW^{(i)}_s
\]

and for \( m \in \mathbb{N}_0 \)

\[
\overline{X}_{(m+1)} := \overline{X}_m + \int_m^{m+1} a(\overline{X}_s) \, ds + \sum_{i=1}^d \int_m^{m+1} b^{(i)}(\overline{X}_s) \, dW^{(i)}_s
\]

For the convergence of the Euler scheme it is necessary that small changes of \( X_0 \) only cause comparable small changes of the distribution of \( X_t \) for fixed \( t > 0 \) and for \( t \downarrow 0 \) the distribution of \( X_t \) should converge to the Dirac distribution with point mass at \( X_0 \). A natural choice of processes satisfying these conditions are Feller processes (cf. Section 2).

Stroock [24] uses an Euler scheme approach to construct Feller processes as solutions to (1), although the SDE is not mentioned explicitly. Therein conditions are formulated in terms of the coefficients, they are related to the usual Lipschitz conditions which ensure the existence of a solution to (1) (see [23]).

Contrary to this, the conditions in the theorem below will be given in terms of the symbol of the generator rather than in terms of the coefficients of the SDE. This is motivated by the following facts:

(i) For construction and analysis of Feller processes the generator is the natural object to start with, see, for example, Ethier and Kurtz [12], chapter 1. Furthermore using formula (3.13) of Courrège [11] it is possible to calculate the symbol without knowledge of the SDE. Nevertheless, it is a natural question, if the process can be described by an SDE and then approximated (and simulated) by an Euler scheme. Our main theorem gives an affirmative answer to this question.
(ii) If the coefficients of an SDE of type (8) below are given, one can directly write down the symbol by formula (9) and check if the assumptions of the theorem are fulfilled.

(iii) If the process under consideration is given by a different type of SDE (cf. [16], chapter 8) it is sometimes hard to transform it into the Skorokhod-type. The symbol on the other hand can occasionally be written down directly and in a neat way: In [21], it is shown that the symbol of the solution of the Lévy driven SDE

$$X_t = X_0 + \int_0^t f(X_s) dZ_s$$

is

$$\psi(f(x)^2)$$

where $\psi$ is the characteristic exponent of the driving Lévy process $Z_t$.

(iv) While the coefficients depend on the choice of the SDE-type and the truncation function (in (2) we have chosen $1_{|x|\leq 1}$; someone interested in limit theorems would probably choose a continuous function), this is not the case for the symbol. In this sense the symbol is a canonical object.

Thus, the conditions in our main theorem are stated in terms of the generator and its symbol:

**Theorem.** Let $(X_t)_{t \geq 0}$ be a Feller process with generator $A$. Assume that

$$C_c^\infty(\mathbb{R}^d)$$

is an operator core of $A$, i.e., the closure of $A|_{C_c^\infty(\mathbb{R}^d)}$ is $A$. (A1)

Let $q(x, \xi)$ be the symbol of $A|_{C_c^\infty(\mathbb{R}^d)}$ and assume that

$$\exists c > 0 : |q(x, \xi)| \leq c(1 + |\xi|^2) \text{ for all } x \text{ and } \xi,$$

(A2)

$$q(x, 0) = 0 \text{ for all } x.$$ (A3)

Then the Euler scheme (2) for the corresponding SDE converges to $(X_t)_{t \geq 0}$ weakly in $D([0, \infty), \mathbb{R}^d)$, moreover given that $\overline{X}_{(m+1)}h = x$ the next step of the Euler scheme $\overline{X}_{(m+1)}\cdot$ has the characteristic function

$$e^{ix\xi} e^{-h q(x, \xi)}.$$ (3)

Since a Feller process is a time homogeneous Markov process it is the solution of an SDE of the form (1), this SDE is meant by the corresponding SDE. The SDE will be explicitly given in Section 3 and the definition of the other terms and objects appearing in the theorem can be found in the next section.

**Remark.** For simulations formula (3) is the key. It shows that starting at $x$ the next position of the scheme is the sum of $x$ and the increment (over time $h$) of a Lévy process with characteristic exponent $\xi \mapsto q(x, \xi)$. Thus, the simulation of Feller processes using the Euler scheme is obvious, if one knows how to simulate Lévy increments. For the latter several techniques are well known, see, for example, Cont and Tankov [9].

An example of a simulation is given in Figure 1. It shows a simulated sample path of a one dimensional stable-like process with generator $-(-\Delta)^{\alpha/2}$ where $\alpha(x) = ((0.9 + x) \wedge 1.9) \vee 0.9$, that is, the process behaves almost like Brownian motion (with double speed) if $X_t > 1$ and almost like a Cauchy process if $X_t < 0$. The state space dependent behavior can be nicely observed in the figure. For the existence of the process and its properties see, for example, [1].
Further properties of the scheme as speed of convergence and error estimates are part of ongoing research. A practitioners guide to simulation of Feller processes can be found in [4].

Note that in the theorem the existence of the Feller process is assumed. It is clearly desirable to find conditions for the family of symbols \( q(x, \xi) \) which already ensure the existence of the limit. This is part of ongoing research, for a survey see, for example, [13]. It is remarkable that the conditions given in the usual constructions are much stronger than the requirements of the theorem above. Thus, all these processes can be approximated by the Euler scheme. One obvious idea to find further sufficient conditions on the symbol would be to translate the conditions on the coefficients for example given by Stroock [24] into conditions on \( q(x, \xi) \) but so far no general criterion for this is known, compare [24] 3.2.2. and Tsuchiya [25]. In this context note that (A2) reflects the assumption of bounded coefficients in the SDE setting.

In the next section, we give the necessary definitions and in Section 3 the proof of the Theorem is presented.

2. Preliminaries

Let \( C_\alpha(\mathbb{R}^d) \) and \( C^\infty(\mathbb{R}^d) \) be the continuous functions vanishing at infinity and the arbitrary often differentiable functions with compact support respectively. \( \mathcal{B}_0(1) \) is the closed unitball in \( \mathbb{R}^d \) and we use the notation \( 1_{|y| \leq 1} \) for \( 1_{\mathcal{B}_0(1)}(y) \).

We consider \( \mathbb{R}^d \) valued Markov processes in the sense of Blumenthal and Getoor [3] and denote such a process by \( X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \mathbb{P}^x)_{x \in \mathbb{R}^d} \). The expectation with respect to \( \mathbb{P}^x \) is denoted by \( \mathbb{E}^x \).

A stochastic process \( X \) is called Feller process if the family of operators \( \{T_t\}_{t \geq 0} \) defined by

\[
T_t u(x) := \mathbb{E}^x(u(X_t)), \quad u \in C^\infty(\mathbb{R}^d),
\]
is a Feller semigroup. This is a strongly continuous contraction semigroup on $C_c(\mathbb{R}^d)$ which is positivity preserving. The semigroup and thus the corresponding process is called conservative, if $T_1 = 1$.

The infinitesimal generator $(A, \mathcal{D}(A))$ of a Feller semigroup is defined by

$$Au := \lim_{t \to 0} \frac{T_t u - u}{t} \quad \text{on } \mathcal{D}(A) := \left\{ u \in C_c(\mathbb{R}^d) : \lim_{t \to 0} \frac{T_t u - u}{t} \text{ exists strongly} \right\}.$$

If the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the generator $A$ of a Feller semigroup, Courrège [10] showed that this integral is meant componentwise if $A$ is meant as matrix-vector multiplication, that is, for a $d \times d$-matrix and a vector $u$ we write $A u := \lim_{t \to 0} \frac{T_t u - u}{t}$.

Further we set $\mathcal{F} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ where $\mathcal{P}$ is the predictable $\sigma$-algebra and $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel sets of $\mathbb{R}^d$. An integral with respect to a vector of processes is meant as matrix-vector multiplication, that is, for a $d \times d$-matrix valued process $Y$ and a $d$-dimensional vector valued process $X$ we write

$$\int_0^t Y_{s \cdot} \, dX_s = \left( \sum_{j=1}^d \int_0^t Y_{s \cdot}^{(j)} \, dX_s^{(j)} \right).$$

Integrals with respect to random measures are denoted by $H \ast \mu$ (cf. [14] Section 2.1) and this integral is meant componentwise if $H$ is a vector.
If random variables $X$ and $Y$ are equal in distribution we write $X \overset{D}{=} Y$. Finally for a measure $N$ and a measurable function $g$ the image measure (push forward) is denoted by $N(g(\bullet) \in dy)$.

3. Proof of the Theorem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \mathbb{P})_{\omega \in \Omega}$ be a càdlàg Feller process with generator $A$ such that $C_c^\infty(\mathbb{R}^d) \subset D(A)$. The process admits the symbol $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by (4) with $c(x) = 0$ for all $x$, since (A3) holds.

Note that $A(C_c^\infty(\mathbb{R}^d)) \subset C_\infty(\mathbb{R}^d)$ and, thus, (A3) implies by Schilling [19], theorem 4.4, that

$$x \mapsto q(x, \xi)$$ is continuous for all $\xi$. (6)

Furthermore $X_t$ is conservative by theorem 5.2. in [19] using (A1)–(A3).

Now the proof will be divided into three parts. First, a result about Markov chain approximation of Feller processes is recalled and afterwards the SDE of a Feller process is calculated explicitly. Finally the characteristic functions of the increments of the Euler scheme to the SDE are calculated and the Markov chain approximation result is applied.

Given the assumptions the main theorem of Böttcher and Schilling [5] is applicable. This shows that

$$Y^h \left( \lfloor \frac{\cdot}{h} \rfloor \right) \xrightarrow{h \to 0} X \quad \text{in } D([0, \infty), \mathbb{R}^d)$$

where $(Y^h(k))_{k \in \mathbb{N}}$ is for each $h > 0$ a Markov chain with initial value $Y^h(0) := X_0$ and transition kernel $\mu_{s,h}(dy)$ defined by

$$\int_\mathbb{R}^d e^{iy \cdot \xi} \mu_{s,h}(dy) = e^{i\xi \cdot \hat{q}(s, \xi)}.$$

To make our argumentation more self contained we note that this can also be seen in the following way: let $U_h$ be the transition operator corresponding to the kernel $\mu_{s,h}(dy)$. By (A1) we can apply theorem 17.28 of Kallenberg [15]. Using the mean value theorem twice with suitable intermediate values $r, s \in (0, h)$ and applying (A2) and (5) it follows that

$$\left| \frac{e^{-hq(s, \xi)} - 1}{h} + q(x, \xi) \right| = \left| -q(x, \xi) (e^{-sq(s, \xi)} - 1) \right|$$

$$= |sq(x, \xi)e^{-rq(s, \xi)}| \leq c^2h(1 + |\xi|^2)^2.$$

Thus, for $f \in C_c^\infty(\mathbb{R}^d)$ we obtain

$$\left| \frac{U_h f(x) - f(x)}{h} - Af(x) \right| = \left| \int e^{i\xi \cdot \hat{\xi}} \left( \frac{e^{-hq(s, \xi)} - 1}{h} + q(x, \xi) \right) \hat{f}(\xi) \, d\xi \right|$$

$$\leq c^2h \int (1 + |\xi|^2)|\hat{f}(\xi)| \, d\xi \to 0$$
for $h \downarrow 0$. This convergence is uniformly in $x$ and thus the $i) \Rightarrow iv)$ part of theorem 17.25 [15] implies that the Markov chain approximates the Feller process in $D([0, \infty), \mathbb{R}^d)$.

In the following we will show that the approximation defined in (2) for the SDE corresponding to $X_t$ coincides with the Markov chain in distribution, that is,

$$\overline{X}_{(m+1)h} \text{ has the characteristic function } e^{ix^Tz}e^{-hq(x,z)} \text{ given that } \overline{X}_{mh} = x. \tag{7}$$

For this, we have first to find the SDE corresponding to $X_t$ explicitly.

By Schnurr [22], Theorem 3.14 (see also [20]), using the assumption of conservativeness, one obtains that $(X_t)_{t \geq 0}$ is an Itô process in the sense of Cinlar et al. [8], that is, it is a strong Markov process which is a semimartingale with respect to every $\mathbb{P}^x$ and its characteristics are

$$B^j_t(\omega) = \int_0^t \ell^j(X_s(\omega))ds \quad j \in \{1, \ldots, d\},$$

$$C^k_t(\omega) = \int_0^t Q^k(X_s(\omega))ds \quad j, k \in \{1, \ldots, d\},$$

$$\nu(\omega; ds, dy) = N(X_t(\omega), dy)ds,$$

with respect to the truncation function $h(y) := y1_{\overline{\mathbb{R}}^d}(y)$. By Cinlar and Jacod [7], theorem 3.33, we obtain that on a suitable enlargement of the stochastic basis, the so called Markov extension, the process $(X_t)_{t \geq 0}$ is the solution of the following SDE. Let the Markov extension be

$$(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \overline{\mathbb{P}}^x)_{x \in \mathbb{R}^d}.$$ On this space we have

$$X_t = x + \int_0^t \ell(X_s)ds + \int_0^t \sigma(X_s)d\overline{W}_s$$

$$+ \int_0^t \int_{z \neq 0} k(X_s, z)1_{[|k(X_s, z)| \leq 1]}(\overline{\mu}(\cdot; ds, dz) - ds\overline{N}(dz))$$

$$+ \int_0^t \int_{z \neq 0} k(X_s, z)1_{[|k(X_s, z)| > 1]}\overline{\mu}(\cdot; ds, dz) \tag{8}$$

where $\overline{W}$ is a $d$-dimensional Brownian motion, $\overline{\mu}$ is a Poisson random measure on $[0, \infty] \times \mathbb{R} \setminus \{0\}$ with dual predictable projection $dt\overline{N}(dz)$. Furthermore $\ell : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and $k : \mathbb{R}^d \times \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^d$ are measurable functions and such that

$$\overline{N}(k(X_s(\omega), \bullet) \in dy)ds = \nu(\omega; ds, dy)$$

$\overline{\mathbb{P}}^x$-a.s. for every $x \in \mathbb{R}^d$ on the Markov extension (compare in this context Cinlar and Jacod [7] (3.9) and their remark following theorem 3.7). Note that

$$\int_0^t \int_{z \neq 0} k(X_s, z)(1_{[|k(X_s, z)| \leq 1]} - 1_{[|z| \leq 1]})\overline{N}(dz)ds$$

$$= \left[ \int_0^t \int_{0 < |z| \leq 1} k(X_s, z)(\overline{\mu}(\cdot; ds, dz) - ds\overline{N}(dz)) + \int_0^t \int_{|z| > 1} k(X_s, z)\overline{\mu}(\cdot; ds, dz) \right]$$
\[- \left[ \int_0^t \int_{\mathbb{R}^d} k(X_{s-}, z) 1_{|k(X_{s-}, z)| \leq 1}(\tilde{\mu}(\cdot; ds, dz) - ds\tilde{N}(dz)) \right. \\
+ \left. \int_0^t \int_{|z| > 1} k(X_{s-}, z) 1_{|k(X_{s-}, z)| > 1}(\tilde{\mu}(\cdot; ds, dz) \right]. \]

The integral on the left hand side exists since representation (8) is valid and, therefore, either \( k(X_{s-}, z) \underset{z \to 0}{\to} 0 \) or \( \tilde{N} \) integrates constants at the origin. Thus by a change of the cutoff function (8) is the same as

\[ X_t = x + \int_0^t \tilde{\ell}(X_s) \, ds + \int_0^t \sigma(X_s) \, d\tilde{W}_t \\
+ \int_0^t \int_{0 < |z| \leq 1} k(X_{s-}, z) (\tilde{\mu}(\cdot; ds, dz) - ds\tilde{N}(dz)) \]

\[ + \int_0^t \int_{|z| > 1} k(X_{s-}, z) \tilde{\mu}(\cdot; ds, dz) \]

where \( \tilde{\ell}(x) = \ell(x) - \int_{|z| \neq 0} k(x, z) \left( 1_{|k(x, z)| \leq 1} - 1_{|z| \leq 1} \right) \tilde{N}(dz) \). Thus, \( (X_t)_{t \geq 0} \) is the solution of an SDE of form (1).

In [22], theorem 5.7 (see also [21]), it is shown that, given (6), the Itô process \( (X_t)_{t \geq 0} \) has the symbol \( p: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) given by

\[ p(x, \xi) = -i\ell(x)'\xi + \frac{1}{2} \xi' \sigma(x) \sigma(x)' \xi \\
- \int_{|z| \neq 0} (e^{ik(x, z)\xi} - 1 - ik(x, z)'\xi, 1_{\mathbb{R}^d\setminus0}(k(x, z))) \tilde{N}(dz). \quad (9) \]

that is,

\[ p(x, \xi) := -\lim_{t \to 0} \mathbb{E}^x \frac{e^{i(X_{t-} - x)'\xi} - 1}{t} \]

for every first exit time \( T \) of a compact set containing \( x \). The symbol \( p(x, \xi) \) and the symbol \( q(x, \xi) \) coincide for Feller processes by corollary 4.5 of [22]. For every fixed \( x \in \mathbb{R}^d \) both functions are continuous and negative definite in the co-variable \( \xi \). Since the Lévy triplet of such a function is unique for a fixed cut-off function (cf. [2], theorem 10.8), we obtain \( Q(x) = \sigma(x)\sigma(x)' \) and

\[ N(x, dy) = \tilde{N}(k(x, \bullet) \in dy). \quad (10) \]

Now we define for fixed \( t \geq 0 \) and \( x \in \mathbb{R}^d \) a process \( (Y_t)_{t \geq 0} \) by

\[ Y_t = x + \int_0^t \tilde{\ell}(x) \, ds + \int_0^t \sigma(x) \, d\tilde{W}_s \\
+ \int_0^t \int_{|z| \neq 0} k(x, z) 1_{|k(x, z)| \leq 1}(\tilde{\mu}(\cdot; ds, dz) - ds\tilde{N}(dz)) \]

\[ + \int_0^t \int_{|z| > 1} k(x, z) 1_{|k(x, z)| > 1}(\tilde{\mu}(\cdot; ds, dz) \]. \quad (11) \]

\[ \]
Lemma. For the process \( Y := (Y_h)_{h \geq 0} \) defined above we obtain:

(a) \( Y \) is a Lévy process.

(b) The following identity holds in distribution

\[
Y_h \overset{d}{=} x + h\ell(x) + \sigma(x)W_h + \int_0^h \int_{y \neq 0} y \cdot 1_{\{|y| \leq 1\}}(\tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy)) - dsN(x, dy)
+ \int_0^h \int_{y \neq 0} y \cdot 1_{\{|y| > 1\}}\tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy)
\]

(c) The characteristic function of \( Y_h \) is

\[
\mathbb{E}^t(e^{ix\tilde{\gamma}}) = e^{ix\tilde{\gamma}}e^{-hq(x, \tilde{\gamma})}.
\]

Proof. Fix \( x \in \mathbb{R}^d \) and \( t \geq 0 \). The four integral terms in (11) are stochastically independent and we will treat them separately. For the first two integrals all statements of the Lemma are easily obtained, since for every \( t \geq 0 \)

\[
\int_t^{t+h} \ell(x) \, ds + \int_t^{t+h} \sigma(x) \, d\tilde{W}_t = \ell(x)h + \sigma(x)(\tilde{W}_{t+h} - \tilde{W}_t),
\]

where \( \tilde{W}_{t+h} - \tilde{W}_t \) is again a Brownian motion having the same distribution as \((\tilde{W}_h)_{h \geq 0}\). For the integrals with respect to the (compensated) random measures we will have to proceed step-by-step.

(a) First we show that the random measure \( dsN(x, dy) \) is the dual predictable projection of the measure \( \tilde{\mu}(\omega; ds, k(x, \bullet) \in dy) \). To this end let \( H : (\Omega, \mathcal{F}, \mathbb{R}^4) \to \mathbb{R} \) be positive. Then we have

\[
\mathbb{E}^t(H(\cdot, s, y) * \tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy)) = \mathbb{E}^t(H(\cdot, s, k(x, z)) * \tilde{\mu}(\cdot; dz))
= \mathbb{E}^t(H(\cdot, s, k(x, z)) * ds\tilde{N}(dz))
= \mathbb{E}^t(H(\cdot, s, y) * ds\tilde{N}(k(x, \bullet) \in dy))
= \mathbb{E}^t(H(\cdot, s, y) * dsN(x, dy))
\]

where we have used that \( ds\tilde{N}(dz) \) is the dual predictable projection of \( \tilde{\mu}(\cdot; ds, dz) \) and (10) for the last equality. By theorem II.1.8 of [14], we obtain that \( dsN(x, dy) \) is the dual predictable projection of \( \tilde{\mu}(\omega; ds, k(x, \bullet) \in dy) \).

Now we are in the position to deal with the third integral term:

\[
\int_t^{t+h} \int_{z \neq 0} k(x, z)1_{\{|k(x, z)| \leq 1\}}(\tilde{\mu}(\cdot; ds, dz) - ds\tilde{N}(dz))
= \lim_{n \to \infty} \int_t^{t+h} \int_{|k(x, z)| \leq 1/n} k(x, z)1_{\{|k(x, z)| \leq 1\}}(\tilde{\mu}(\cdot; ds, dz) - ds\tilde{N}(dz))
= \lim_{n \to \infty} \int_t^{t+h} \int_{|k(x, z)| \leq 1/n} y1_{\{|y| \leq 1\}}(\tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy) - dsN(x, dy))
= \int_t^{t+h} \int_{|y| \neq 0} y1_{\{|y| \leq 1\}}(\tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy) - dsN(x, dy)).
\]
Let us emphasize that the integrals in the second and third line can be written as the difference of two integrals with respect to the respective random measures. Therefore, these measures can be transformed one-by-one. The limit in third line exists and therefore does the limit in the second line, too.

In particular \( \tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy) \) is a Poisson random measure by the structure of its compensator, because \( N(x, dy) \) is a \( \text{Lévy} \) measure for every fixed \( x \in \mathbb{R}^d \).

The fourth term can now be written as

\[
\int_t^{t+h} \int_{z \neq 0} k(x, z) 1_{\{|k(x,z)| > 1\}} \tilde{\mu}(\cdot; ds, dz) = \int_t^{t+h} \int_{y \neq 0} y 1_{\{|y| > 1\}} \tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy).
\]

Putting the four terms together we obtain a Lévy-Itô decomposition (cf. chapter 4 of [18]) of the process \( Y = (Y_t)_{t \geq 0} \), although the third and fourth integral are still ‘shifted’. In particular \( Y \) is a Lévy process.

(b) It is enough to give the proof for the case \( d = 1 \), because the integrals with respect to the (compensated) random measures are defined componentwise. This time we start with the fourth term of (11). Since \( \tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy) \) is a Poisson random measure we have

\[
\tilde{\mu}(\cdot; [a, b], k(x, \bullet) \in C) \overset{D}{=} \tilde{\mu}(\cdot; [t + a, t + b], k(x, \bullet) \in C)
\]

for \( a < b \) and \( C \in \mathcal{B}(\mathbb{R} \setminus \{0\}) \). Therefore, we obtain

\[
\int_0^\infty \int_{|y| > 1} 1_{[a, b]}(s) \cdot 1_C(y) \tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy) \overset{D}{=} \int_0^\infty \int_{|y| > 1} 1_{[t + a, t + b]}(s) \cdot 1_C(y) \tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy).
\]

In order to keep notation simple, we set

\[
I(a, b; 1_C) := \int_0^\infty \int_{|y| > 1} 1_{[a, b]}(s) \cdot 1_C(y) \tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy).
\]

Now let \( \varphi : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}_+ \) be a simple function, that is, \( \varphi \) can be written as \( \varphi(y) = \sum_{j=1}^m d_j \cdot 1_{C_j}(y) \) where \( m \in \mathbb{N} \) the \( C_j \) are disjoint sets in \( \mathbb{R} \setminus \{0\} \) and \( d_j \geq 0 \) for every \( j \in \{1, \ldots, m\} \). Since the \( C_j \) are disjoint the random variables \( (I(a, b_j; 1_{C_j}))_{j=1,\ldots,m} \) and \( (I(t + a, t + b_j; 1_{C_j}))_{j=1,\ldots,m} \) are independent respectively. And, thus, we obtain

\[
I(a, b; \varphi) \overset{D}{=} I(t + a, t + b; \varphi).
\]

Furthermore there exists a sequence of simple functions \( (\varphi_n)_{n \in \mathbb{N}} \) such that \( \varphi_n \uparrow \text{id} \) on \( (1, \infty) \). In the limit we obtain

\[
I(a, b; \text{id}) \overset{D}{=} I(t + a, t + b; \text{id})
\]

and analogous on \( (-\infty, -1) \). The fact

\[
([a, b] \times \{y > 1\}) \cap ([a, b] \times \{y < -1\}) = \emptyset
\]
implies that the random variables
\[
\int_{[-1]}^{b} y^{\tilde{\mu}}(\cdot; ds, k(x, \bullet) \in dy) \quad \text{and} \quad \int_{[-1]}^{b} y^{\tilde{\mu}}(\cdot; ds, k(x, \bullet) \in dy)
\]
are independent. Therefore, we obtain the representation of the fourth term of b).

To deal with the third term we write
\[
\int_{t}^{t+h} \int_{y \neq 0} y^{1}_{|y| \leq 1} y^{\tilde{\mu}}(\cdot; ds, k(x, \bullet) \in dy) - dsN(x, dy))
\]
and note that by using the same arguments as for the fourth term we get
\[
\int_{t}^{t+h} \int_{|y| > 1/n} y^{1}_{|y| \leq 1} y^{\tilde{\mu}}(\cdot; ds, k(x, \bullet) \in dy) - dsN(x, dy))
\]
The integrals with respect to \( dsN(x, dy) \) are clearly invariant with respect to a shift of \( t \) because of the product structure and the non-randomness. Thus, we obtain
\[
Y_{h} \overset{D}{=} x + h \ell(x) + \sigma(x) \tilde{W}_{h}
\]
\[
+ \int_{0}^{h} \int_{|y| \neq 0} y \cdot 1_{|y| \leq 1} (\tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy) - dsN(x, dy))
\]
\[
+ \int_{0}^{h} \int_{|y| > 1/n} y \cdot 1_{|y| > 1} (\tilde{\mu}(\cdot; ds, k(x, \bullet) \in dy). \quad (12)
\]
(c) The right-hand side of (12) is the Lévy-Itô decomposition of a Lévy process. This process has the same one-dimensional distributions as \( Y \). Therefore, the characteristic functions of the two processes coincide and we obtain
\[
\mathbb{E}^{x}(e^{iy_{t}^{\tilde{\mu}}}) = e^{ix_{t}^{\tilde{\mu}}} \mathbb{E}^{x}(e^{-hq(x, \tilde{\mu})}),
\]
which proves the Lemma. \( \square \)

For \( t = mh \) the Lemma shows that (7) holds and, thus, the theorem is proven.

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