Meissner Effect from Landau Problem

Paola Arias, J. Gamboa, F. Méndez and David Valenzuela

1 Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago, Chile
2 Departamento de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
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The Landau problem for inhomogeneous magnetic fields is examined in a very general context and several interesting analogies with the Nielsen-Olesen vortices are established. Firstly we show that the Landau problem with non-homogeneous magnetic fields exhibits Meissner effect that is unstable unless two-body interactions are added and vortices emerge. Using the scaling freedom we can write the Schrödinger equation in terms of the scales ratio \( \kappa = E/m \propto 1 - T/T_c \), where the last identification is realised simply by using the Ginzburg-Landau theory. We find our equations are valid in the superconducting regime, and it is not possible for the Cooper pairs amplitude to reach to a constant, non-zero value, and therefore the theory is unstable. The supersymmetric quantum mechanics version, by completeness, is also considered.

I. INTRODUCTION

The Meissner effect is a remarkable experimental result that shows that the force lines of an external magnetic field are expelled when penetrate a superconductor sample \cite{1}. From a theoretical point of view this effect is explained by the fact that the photons generate a mass \( m \), which is the inverse of the penetration depth in the superconductor.

Technically speaking, this effect is obtained when the current \( \mathbf{J} \) – which is the source of Ampere’s law – acquires a piece \( \mathbf{J}_L = \alpha \mathbf{A} \) where \( \alpha \) is identified \textit{a posteriori} with the mass of the photon and the relation between \( \mathbf{J}_L \) and \( \mathbf{A} \) is called the London equation.

The London equation is an \textit{ad hoc} relationship that explains an experimental fact but its conceptual content is weak unless one demonstrates that it comes from a fundamental explanation.

On the other hand, although the Ginzburg-Landau theory is made on purely intuitive basis, its predictive validity is widely proven not only in the field of superconductivity but also in the modern approach of Bose-Einstein condensation in statistical systems \cite{2}, particle physics \cite{3} and cosmology \cite{4}.

In spite of the fact that the relationship between superconductivity and the quantum Hall effect is widely discussed in connection with anyons, it seems that a discussion in between superconductivity and the Landau problem \cite{5}, \textit{i.e.} the motion of charged particles in (non)homogeneous magnetic fields, is less explored problem.

This point of view is very interesting, we think, because it helps to clarify the relationship between the Ginzburg-Landau mean field theory \cite{6} and the cornerstone Landau problem.

This “just in between” region is also interesting because corresponds to the transition between two non-perturbative sectors where, in the first case, the role of Cooper pairs is dominant while in the second case the presence of the external magnetic field instead of the Cooper pairs becomes more important.

In this paper we will discuss the connection between the points outlined above and we will show how in the Landau problem the Meissner effect emerges, and that this effect would be destabilised by the absence of two-body interactions.

The stabilisation of the Meissner effect requires spontaneous symmetry breaking, or in other words, it requires the Hartree term

\[
H_H = \int dy \, \psi^*(x) \psi^*(y) W(x-y) \psi(y) \psi(x).
\]

where one identifies \( W(x-y) \), the two-body interaction, with the contact term, \textit{i.e.} \( W(x-y) = \lambda \delta(x-y) \) with \( \lambda \) the coupling constant.

In the absence of two-body interaction the spontaneous symmetry breaking does not occur and the mass term for \( \psi(x) \) has a sign ambiguity associated with one of the two phases of the system. These conclusions will be reached analytic and numerically.

The effects due to fermions will be also discussed using supersymmetric quantum mechanics techniques and we will show that, except for the Pauli’s term, the results of the previous sections are maintained.

The paper is organised as follows: in section II we discuss the motion of a charged particle in no-homogeneous magnetic field and we discuss the Meissner effect in this context; in section III the role of fermions is discussed by using supersymmetric quantum mechanics techniques. In the last section we discuss our results and we present the conclusions.
II. MOTION OF CHARGED PARTICLES IN A HOMOGENEOUS MAGNETIC FIELD

The strategy of the calculation is similar to the motion of a charged particle in a homogeneous magnetic field (Landau problem). For purposes of comparison with standard results (particularly with the Nielsen-Olesen vortex solution) it is convenient to describe the system with cylindrical coordinates \( x = \{ \rho, \varphi, z \} \), and to take the vector potential as follows

\[
A_\rho = 0, \quad A_\varphi = \frac{\alpha(\rho)}{\rho}, \quad A_z = 0. \tag{1}
\]

The Hamiltonian describing a particle of mass \( m \) and electric charge \( q \) in the presence of a magnetic field \( B = (\alpha'/\rho) \hat{\rho} \) turns out to be

\[
H = \frac{1}{2m} \left( \vec{p} - \vec{A} \right)^2 = \frac{1}{2m} \left( \hat{p}_\rho^2 + \left( \frac{L_z}{\rho} - A_\varphi \right)^2 + \hat{p}_z^2 - 2 \frac{\hat{L}_z A_\varphi}{\rho} + A_\varphi^2 \right). \tag{2}
\]

with the operators \( \hat{p}_\rho^2 = -\partial_\rho^2 \), \( \hat{L}_z = -i \partial_\varphi \) and \( \hat{p}_z = -\rho^{-1} \partial_\rho (\rho p_\rho) \). Since \( z \) and \( \varphi \) are cyclic variables, we take the following ansatz for the wave function \( \psi(\rho, \varphi, z) \)

\[
\psi(\rho, \varphi, z) = e^{-iL_z \varphi} e^{i\rho z} f(\rho). \tag{3}
\]

where \( p_z \) and \( L_z \) are the conserved quantities corresponding to the cyclic variables. Will be useful to move to dimensionless variables, so let us define \( x \to x m, \quad A_\varphi \to A_\varphi / m, \quad p \to p / m \) and \( \psi \to \psi / m^{3/2} \). Correspondingly, the magnetic field has to be scaled as \( B \to B / m^2 \) and the current source as \( J \to J / m^3 \). Therefore, the Schrödinger equation becomes

\[
\left[ \hat{p}_\rho^2 + \left( \frac{L_z}{\rho} - A_\varphi \right)^2 + \hat{p}_z^2 \right] f(\rho) = 2\kappa^2 f(\rho), \tag{4}
\]

where \( \kappa^2 = \frac{E}{m} \).

Since the wave function must be single-valued for \( \varphi \in [0, 2\pi] \), then \( L_z = n \) with \( n \in \mathbb{Z} \). The Schrödinger equation reads

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \left[ 2\kappa^2 - \frac{\left( n / \rho - A_\varphi \right)^2 - \hat{p}_z^2}{\rho^2} \right] f = 0. \tag{5}
\]

This equation can be solved, in principle, if \( A_\varphi(\rho) \) is given. In order to find this potential in a consistent way, we demand that the source of the magnetic field \( B = (\alpha'/\rho) \hat{\rho} \) in the Ampere’s law, to be the conserved current for the Schrödinger equation \(H\), that is, the conserved current of probability

\[
J = \frac{1}{2i} (\psi^* \nabla \psi - \psi \nabla \psi^*) - A \psi^* \psi. \tag{6}
\]

For the ansatz (3), we find

\[
J = \left[ \left( \frac{L_z}{\rho} - A_\varphi \right) f^2 \right] e_x + p_z^2 f^2 e_z, \tag{7}
\]

with \( \{ e_x, e_z \} \) unit vectors. It is possible to choose without loss of generality a frame such that \( p_z = 0 \). The Ampere’s law

\[
\nabla \times \mathbf{B} = \mathbf{J}, \tag{8}
\]

then yields to

\[
-\nabla^2 A_\varphi = \left( \frac{n}{\rho} - A_\varphi \right) f^2, \tag{9}
\]

or

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial A_\varphi}{\partial \rho} \right) - A_\varphi + \left( \frac{n}{\rho} - A_\varphi \right) f^2 = 0. \tag{10}
\]

We can write now the set of equations for \( f \) and \( A_\varphi \) (namely, the Schrödinger equation and the Ampere’s law) in terms of \( \alpha \), using the ansatz of (1) and we find

\[
f'' + \frac{f'}{\rho} + \left[ 2\kappa^2 - \frac{1}{\rho^2} (n - \alpha)^2 \right] f = 0, \tag{11}
\]

\[
\alpha'' - \frac{\alpha'}{\rho} + (n - \alpha) f^2 = 0. \tag{12}
\]

This set of equations describes the motion of our system. In order to discuss the solutions of this set, we will consider the case \( n = 1 \) in the rest of the text.

Before moving on to the analysis and solving of these coupled equations, a comment about the physics of this model is in order. From equation (13), with \( p_z = 0 \), we can find the energy functional that leads to such equation. We call \( \mathcal{E} \) the (dimensionless) total energy per unit of length of the system, and we have

\[
\mathcal{E} = \int d^2 x \left[ \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} \left( -i \nabla - \mathbf{A} \right) \psi \lambda - \kappa^2 |\psi|^2 \right]. \tag{13}
\]

Using the chosen vector potential given in (1), it can be easily checked that the above functional gives both, Schrödinger and Ampere’s law. We can compare this energy with the one of a superconducting sample, which is usually written as

\[
\mathcal{E} = \int d^2 x \left[ \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} \left( -i \nabla - \mathbf{A} \right) \psi \lambda^2 + \frac{a_0}{4} \left( \frac{T}{T_c} - 1 \right) |\psi|^2 + \frac{g}{4} |\psi|^4 \right], \tag{14}
\]

The \( |\psi|^4 \) term gives the spontaneous symmetry breaking that makes possible the second order phase transition, and \( a_0 \) and \( g \) are dimensionless constants. The important point for us is that we can make the following identification

\[
\kappa^2 = \frac{a_0}{2} \left( 1 - \frac{T}{T_c} \right), \tag{15}
\]

\[1\] Through all the text we use natural units.
which implies that our equations are valid in the superconducting phase, meaning $T \leq T_c$. We note that $\kappa \ll 1$ should represent a sample in the vicinity of the phase transition point, whereas a greater $\kappa$ corresponds to one deep into the superconducting regime.

Coming back to our set of equations (11) and (12), they can be solved for a set of boundary conditions. Such conditions will be obtained from physical requirements in the present approach.

The first requirement refers to the magnetic field. We demand the following values at boundaries (in the rescaled dimensionless variables)

$$B = \begin{cases} 1 & \text{for } \rho \to 0, \\ 0 & \text{for } \rho \to \infty, \end{cases} \quad (16)$$

with $B = |\mathbf{B}|$ and $\mathbf{B} = \nabla \times \mathbf{A} = \frac{\partial}{\partial \rho} \hat{z}$.

Equation (16) imposes Meissner effect and for $\alpha$ it is equivalent to

$$\alpha(\rho) = \begin{cases} \frac{\kappa^2}{\rho} & \text{for } \rho \ll 1, \\ 1 & \text{for } \rho \gg 1. \end{cases} \quad (17)$$

In particular, this behaviour sets the following initial conditions $\alpha(0) = 0$, and $\alpha'(0) = 1$ or, equivalently, the following boundary conditions $\alpha(0) = 0$ and $\alpha(\infty) = 1$, where the symbol ‘$\infty$’ stands for $\rho \gg 1$.

The boundary conditions for $f$ are determined by our second requirement that current is finite in the near region $\rho \approx 0$, while it should vanish in the far region $\rho \gg 1$. From (11), it is direct to check that both conditions hold if

$$f(\rho) = \begin{cases} 0 & \text{for } \rho \ll 1, \\ \text{constant} & \text{for } \rho \gg 1. \end{cases} \quad (18)$$

### A. Asymptotic behaviour of the fields

In order to see if the assumed asymptotic behaviour of $\alpha$ and $f$ given in (17) and (18) are consistent with our set of equations (11)-(12), let us perform an analysis in both limits.

a) for $\rho \ll 1$, we can decouple the set of equations, considering $\alpha(\rho) \ll 1$ and $f(\rho) \ll 1$ and we obtain

$$f'' + \frac{f'}{\rho} - \frac{f}{\rho^2} = 0, \quad (19)$$

$$\alpha'' - \frac{\alpha'}{\rho} = 0, \quad (20)$$

which accounts for a behaviour of $\alpha(\rho) \sim c_1 \left( \frac{\rho}{\kappa} \right)^2$ and $f(\rho) \sim c_2 \frac{\rho}{\kappa}$. Where $c_1, c_2$ are constants.

b) For $\rho \gg 1$ we consider $\alpha(\rho) \sim 1 + \epsilon_1(\rho)$, with $\epsilon \ll 1$, in order to have attenuation of the magnetic field at greater distances (Meissner effect), therefore the equation (12) gets replaced by

$$\epsilon_1'' - \frac{\epsilon_1'}{\rho} - \epsilon_1 f^2 = 0. \quad (21)$$

We see that if $f \approx \text{const}$ in the last equation, we obtain for $\alpha(\rho) \sim 1 + \mu K_1(f\rho)$, however, the replacement of this expression back into eq. (11), for $f \sim \text{const}$, shows that $f \approx 0$. Thus, we have to instead consider the following asymptotic equation

$$\epsilon_1'' - \frac{\epsilon_1'}{\rho} = 0, \quad (22)$$

whose solutions are $\epsilon \sim 1 + b_1 + b_2\rho^2$, with $b_1, b_2$ constants, which is consistent with the Meissner effect for $b_2 \rightarrow 0$. Therefore, the asymptotic behaviour of $f(\rho)$ that is consistent with expulsion of the magnetic field in the sample is $f(\rho) \rightarrow 0$ when $\rho \rightarrow \infty$.

In order to find numerical solutions for our sets of differential equations we implement a shooting method to match the solution near $\rho \rightarrow 0$ and $\rho \rightarrow \infty$. In order to do so, we consider for $\rho \ll 1$ a polynomial behaviour $f$ and $\alpha$ and for $\rho \gg 1$ we can linearise the equations as we did above, considering $\alpha(\rho) \sim 1 + \epsilon_1(\rho)$ and $f(\rho) \sim \epsilon_2(\rho) \ll 1$. Thus,

$$\epsilon_2'' + \frac{\epsilon_1'}{\rho} + 2\kappa^2 \epsilon_2 = 0, \quad (23)$$

$$\epsilon_1'' - \frac{\epsilon_1'}{\rho} = 0, \quad (24)$$

with solutions

$$\alpha(\rho) = 1 + b_1 + b_2\rho, \quad (25)$$

$$f(\rho) = d_1 J_0(\sqrt{2\kappa} \rho) + d_2 Y_0(\sqrt{2\kappa} \rho), \quad (26)$$

where $J_0$ and $Y_0$ are Bessel functions and $b_1, b_2, d_1$ and $d_2$ are constants of integration. In figure 1 we show the magnetic field and Cooper pair amplitude as a function of the distance for several values of $\kappa$. For smaller $\kappa$ the attenuation of the magnetic field is almost negligible, and also the growing of the Cooper pair amplitude, this corresponds to a temperature very near the phase transition, $T = T_c$. As $\kappa$ gets larger (and therefore the temperature goes much below $T_c$) the effects on the magnetic field are more significant. We can understand this behaviour since, for $\rho < 1$, the missing $f^4$ term that appears in the usual Ginzburg-Landau theory can be neglected in comparison to the linear term $\mathbf{H}$. Thus, near the border of the superconductor, our model is a good approximation of the superconductivity theory. But as $\rho > 1$, the cubic term is needed to stabilise the Meissner effect. This last statement can be used to understand why the amplitude of the Cooper pairs fails to reach a constant asymptotic behaviour, since the contact term between electrons is missing. This translates to a magnetic flux inside the sample which is not quantised and therefore, no vortices can appear using this theory.
The superpotential and with $B$, the covariant derivative defined as $D = \partial + \nabla$, results in the Pauli term from the beginning the Pauli term $\sigma \mu \nu F_{\mu \nu}$ that for the static magnetic field this is reduced to $\sigma \cdot B$.

For smaller $\kappa$ we expect to be at the beginning of the superconducting phase, $T \sim T_c$, and for bigger $\kappa$, at temperatures below the phase transition $T < T_c$.

### III. THE ROLE OF FERMIONS

In a conductive material the carriers are electrons and therefore one should formulate the above problem including the fermionic character of the carriers. From the Hamiltonian point of view this means that we must put from the beginning the Pauli term $\sigma \mu \nu F_{\mu \nu}$ that for the static magnetic field this is reduced to $\sigma \cdot B$.

In order to explain the previous statement we will proceed following an approach based in supersymmetry proposed by one of us long time ago [10]. Thus, we start by considering the following supersymmetric charges

$$ S = (D_i + \partial_i W) \sigma_1 \otimes \sigma_-, \quad S^\dagger = (-D_i + \partial_i W) \sigma_1 \otimes \sigma_+, $$

(27)

where $\sigma_\pm = \sigma_1 \pm i\sigma_2$ and $\sigma_i$ are Pauli’s matrices, $D_i$ is the covariant derivative defined as $D_i = \partial_i + A_i$, $W$ is the superpotential and

$$ [D_i, D_j] = -iF_{ij}, $$

with $B_k = \frac{1}{2} \epsilon_{kij} F_{ij}$, the magnetic field.

In addition, supercharges must satisfy the algebra

$$ \{S, S^\dagger\} = 2H_s, $$

$$ [H_s, S] = 0 = [H_s, S^\dagger], $$

$$ S^2 = 0 = S^\dagger^2. $$

(28), (29), (30)

The explicit calculation of (28) defines $H_s$ an then, by using (27), we find

$$ H_s = \frac{1}{2} (p - A)^2 - \frac{1}{2} \sigma \cdot B + (\nabla W)^2 - \nabla^2 W \sigma_3, $$

(31)

with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The last term in the previous equation is identified with the potential energy through $(\nabla W)^2 \mp \nabla^2 W = \psi_{\mp}(x)$, defining a Ricatti equation for $W$. The signs $\pm$ are related to the normalization of the ground state and, for example, $\psi_{0^+}$ means that the spinor that corresponds to the ground state is chosen as

$$ \psi_{0^+} = \begin{pmatrix} e^{-W} \\ 0 \end{pmatrix}. $$

The ground state is then related to $W$ through $W = -\ln \psi$ and $W$ is the superpotential. So, since the Hamiltonian is hermitian, $W \in \mathbb{R}$, then in symbolic form we can write the Hamiltonian as

$$ H_s^\pm = \frac{1}{2} (p - A)^2 - \frac{1}{2} \sigma \cdot B + (\nabla W)^2 \mp \nabla^2 W, $$

(32)

understanding, of course, that one and only one component of the spinor is normalizable.

If we choose $B = (0, 0, B_3(x))$, which is what we have been considering, then

$$ H^\pm = \frac{1}{2} (p - A)^2 + (\nabla W)^2 \mp \left( \nabla^2 W - \frac{1}{2} B_3(x) \right), $$

(33)

this is the fermionic extension of the Hamiltonian [2].

### IV. DISCUSSION AND CONCLUSIONS

The motion of charged particles in an external (constant) magnetic field is one of the most interesting problems of contemporary physics, and is present in many other topics such as anyons, cosmic strings, Aharonov-Bohm effect, cosmology, quantum Hall effect and so on.

The mechanism itself is remarkable because it makes use of the gauge invariance (and its topological implications) through the modification of the canonical commutators thus providing a natural link with highly sophisticated mathematics (as for example noncommutative geometry and Poissonian manifolds).

However, we think that the Landau problem, properly modified, can also be useful to understand the physical basis that connects the superconductivity and the quantum Hall effect.

The results that we have presented in this paper are a step towards this direction. We have shown how to approach the critical points on both sides in a phase transition region and we have shown that even not being in the critical phase equally one can find an interesting phenomenon resembling the Meissner effect.

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2 The literature in supersymmetric quantum mechanics is very extensive and in the context of our discussion see [11], [12] and [13].