On pseudo-periodic perturbations of planetary orbits, and oscillations of Earth’s rotation and revolution: Lagrange’s formulation.

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ABSTRACT

Earth rotation is determined by polar motion (PM) and length of day (lod). The excitation sources of PM are torques linked to fluid circulations ("geophysical excitations"), and those of lod to luni-solar tides ("astronomical excitations"). We explore the links between the rotations and revolutions of planets, following Lagrange’s (1853) presentation of mechanics. The energy of a planet in motion in a central field is the sum of kinetic, centrifugal (planet dependent) and centripetal (identical for all planets) energies. For each planet, one can calculate a “constant of gravitation” $\mathbf{g}_p$. For the giant planets, $\mathbf{g}_p$ decreases as a function of a phelia $a$. There is no such organized behavior for the terrestrial planets. The perturbing potential of other planets $\gamma/r^2$ generates a small angular contribution to the displacement: this happens to be identical to Einstein’s famous formula for precession when $\gamma/(4\pi a)$ is replaced by $1/c^2$. Delays in the planet’s perihelion follow a $(-5/2)$ power law of $a_p$. The differences in delays are negligible from Mars to Neptune. For the three telluric planets the situation is different. This is readily understood in the Lagrange formalism (the centrifugal term takes over for small distances). The telluric planets have lost energy, probably transferred to the planets rotations. The ratio of areal velocities to rotation obeys a $-5/2$ power law of $a_p$. The ratio of areal velocity to integrated period $R$ also fits a $-5/2$ power dependence, implying linearity of the energy exchange between revolution and rotation. For Einstein deformation of space-time by the Sun is the origin of the field perturbation. For Lagrange the perturbation could only be due to the interactions of torques. The perihelion delays, the areal velocities and the planetary rotations display power laws of a phelia, whose behavior contrasts with that of the kinetic moment. The areal velocity being linearly linked to the kinetic moment of planets, this must be the level at which the transfer is achieved. Any torque acting on the Sun is returned to Earth, whose rotation axis is perturbed. The law of rotation periods as a function of a phelia gives the variations of inclination of the rotation axis. Since we are in a closed system, this ratio should be constant for each planet: all planets do follow the same power law of a phelia. Finally, we should expect to find astronomical signals in most terrestrial geophysical phenomena, such as a preponderance of Jupiter’s period of revolution. Such is indeed the case.

Key words. Kepler Law, planetary orbits, Lagrange theory, length of day

1. Introduction

The rotation of Earth in space can be determined by a set of two components in three dimensions: polar motion (PM), that is the evolution in latitude and longitude of the polar coordinates $m_1$ and $m_2$ of the point where the rotation axis intersects the Earth surface, and the length of day (lod) or rotation velocity of coordinate $m_3$ along the rotation axis. The duration of the sidereal Earth day, that is the duration after which a point on the Earth’s surface returns to the same position (ie. a full rotation), is 0.99726949 days (cf. (Aoki et al. 1982)), corresponding to a mean angular velocity of $7.292115 \times 10^{-5}$ rad/s, a constant following the International Association of Geodesy (IAG, 1999). Coordinates $m_1$, $m_2$ and $m_3$ are the solutions of a system of three linear differential equations, named after Liouville and Euler (eg. (Lambeck 2005)), that links them to the Earth’s principal moments of inertia (A, B and C) and to all the torques of forces that act on it. When $(m_1, m_2, m_3)$ are projected on a N-S/E-W coordinate system with its origin at the rotation pole (eg. (Lopes et al. 2022) Figure 01), PM and lod become mathematically separated. This is taken to imply geophysical separation in recent works, as illustrated by two quotations from (Lambeck 2005)’s reference book, chap. 3, pages 34 and 36: "$m_1$ and $m_2$ are the components of the polar motion or wobble and is nearly the acceleration in diurnal rotation" (being the Euler angle associated with $m_3$). "Equations (3.2.6) clearly separate the astronomical and geophysical problems".

With this formulation, the excitation sources of PM are the torques of forces linked to circulations in the core, mantle, ocean and atmosphere (that is "geophysical excitations"), whereas those of lod are the torques associated with luni-solar tides (that is "astronomical excitations").
It has been known for over two and a half centuries that the length of day is not a constant (d’Alembert 1749; Laplace 1799; Lagrange 1833; Poincaré 1893; Spencer Jones 1932, 1939; Brouwer 1952; Martin 1969; Melchior 1972; Stephenson et Morrison 1984; Gross 2001; Le Mouël et al. 2019; Lopes et al. 2022). Various types of spectral analyses have identified a number of cycles or quasi-cycles, such as 13.66 and 27.54 days (e.g. (Ray et Erofeeva 2014)) due to the joint actions of the Moon and Sun (0.73 and 0.39 ms respectively, (Le Mouël et al 2019)), as predicted by models (e.g. Jobert (1973)). There are also much longer cycles, such as the quasi-biennial oscillation (QBO; 0.08 ms, (Baldwin et al. 2001)), the 11 yr cycle (0.46 ms, Schwabe (1844)), and the great lunar nutation (18.6 yr, 1.3 ms). Their amplitudes are given above (from Le Mouël et al (2019)). Some of these same cycles are found in PM (Lopes et al. 2017, 2021).

Lod is measured daily by satellite since 1962 and reported by the International Earth Rotation and Reference System Service (IERS1). The historical analyses of Stephenson et Morrison (1984) and Gross (2001) allow us to go back to 1832. The two series and their combined trend are shown in Figure 1.

Over the past 180 years, lod has fluctuated between a minimum of -3 ms reached in 1870 and a maximum of +4 ms around 1900-1910. The trend has been negative since 1970, falling by 3 ms to reach a minimum of -1.2873 on July 12, 2020 (Trofimov et al. 2021). This recent acceleration of Earth rotation has been publicized in the media, though it was surpassed in around 1900-1910. The trend has been negative since 1970, passing in 1870. The longer, multi-decadal oscillations seen in Figure 1 are also found in PM (Lopes et al 2022). Various types of spectral analyses have identified a number of cycles or quasi-cycles, such as 13.66 and 27.54 days (e.g. (Ray et Erofeeva 2014)) due to the joint actions of the Moon and Sun (0.73 and 0.39 ms respectively, (Le Mouël et al 2019)), as predicted by models (e.g. Jobert (1973)). There are also much longer cycles, such as the quasi-biennial oscillation (QBO; 0.08 ms, (Baldwin et al. 2001)), the 11 yr cycle (0.46 ms, Schwabe (1844)), and the great lunar nutation (18.6 yr, 1.3 ms). Their amplitudes are given above (from Le Mouël et al (2019)). Some of these same cycles are found in PM (Lopes et al. 2017, 2021).

2. A momento of Lagrangian mechanics

We first start with a summary of the basic equations of mechanics, involving gravitational potential, planetary torques and centrifugal forces, as elegantly presented by Lagrange (1853).

In a Galilean reference system, in which time is uniform and the physical laws are homogeneous and isotropic, any mass in motion in that reference system conserves three quantities: energy (E), impetus (P) and moment (M). Let a planet with mass m and cylindrical coordinates (r, ϕ) be in motion in a central field \( U(r) \) of the form . The symmetry axis is labeled z. As implied by the law of transformation of the kinetic moment, mechanical properties of the system do not change under any rotation about this axis. The moment must be defined with respect to a point located on that same axis. The total energy of the system can be written as:

\[
E = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) + U(r) = \frac{m \dot{r}^2}{2} + \frac{\mathcal{L}^2}{2mr^2} + U(r)
\]

that is the sum of the kinetic, centrifugal and centripetal energies. The two other primary integrals are given by:

\[
\mathcal{P} = m \dot{r} \quad (1b) \\
M = r \times \mathcal{P} \\n\]

In a closed system or in a central field, total energy \( E \) and Lagrangian \( \mathcal{L} \) are univocally linked and (1a) can be used to fully write the equations of motion of the planet. Equation (1c) introduces the kinetic revolution moment \( M \) (in kg m² s⁻¹) of planet m. In (1a) the quantity \( \frac{\mathcal{L}^2}{2mr^2} \) is called the centrifugal energy. Here, impetus \( \mathcal{P} \) is reduced to:

\[
\dot{\mathcal{P}} = m \dot{r} \dot{\phi} \quad (2a)
\]

Given the law that links \( M \) and \( \mathcal{P} \) and the conservation of kinetic moment,

\[
M = m r^2 \dot{\phi} = C^\alpha 
\]

From a geometrical stand point, (2b) \( \frac{1}{2} r \cdot rd\phi \) represents the area \( df \) a sector of the orbit formed by the two infinitesimal vectors and the element of arc of the orbit/trajetory of \( m \) (Figure 2). Thus, one can write:

\[
M = 2mf \\
\]

This is Kepler’s second law (cf. Warrain (1942), pages 78–79); is called the areolar velocity.

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1 https://www.iers.org/IERS/EN/DataProducts/EarthOrientationData/eop.html

**Fig. 1:** Daily evolution of lod measured by satellites since 1962 (gray vertical bars). Annual evolution of lod from 1832 to 1997 (Gross (2001); black curve). In red, the concatenation of the trends from the two data series (Lopes et al. 2022).

**Fig. 2:** Infinitesimal displacement of a planet of mass m along its elliptical orbit.
If one integrates (2c) along the full ellipse of revolution with period \( T \), (2c) becomes:

\[
2mf = TM
\]  

(3a)

The surface \( f \) is equal to \( ab \), where \( a \) and \( b \) are the semi-major and semi-minor axes of the ellipse. \( p \) and \( e \) being respectively the ellipse’s parameter and eccentricity, one gets (cf. Chandrasekhar (1969)):

\[
a = \frac{p}{1 - e^2} = \frac{a}{2e}, \quad b = \frac{p}{\sqrt{1 - e^2}} = \frac{M}{2m[E]}
\]  

(3b)

We point out at this stage that when \( m \) is far from the Sun the semi-major axis depends only on the field, but when it is closer, the displacement takes over. Injecting (3b) in (3a), one gets:

\[
T = 2\pi a^{3/2} \sqrt{\frac{m}{\alpha}} = \pi a \sqrt{\frac{m}{2[E]}}
\]  

(3c)

This is Kepler’s third law, \( T^2 \) proportional to \( a^3 \). The constant ratio \( \frac{T^2}{a^3} = K \) is given by Newton’s law:

\[
K = \frac{GM_m}{4\pi^2}
\]  

(3d)

with the gravitational constant \( G = 6.67384 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \) and the Sun’s mass \( M_s = 1.98892 \times 10^{30} \text{ kg} \).

Using equations (3) and the planetary values listed in Table A (Appendix A), one can calculate a value \( \mathcal{G}_p \) of \( \mathcal{G} \). We have done it in the following way. For a given planet, the Sun’s attraction is given by \( \frac{GM_m}{r^2} \) and derives from the potential \( \frac{GM_m}{r} \) which has the form \( \frac{\alpha}{r} \) being given by (3c). So, for each planet one can write:

\[
\alpha = 4\pi^2 \frac{a^3}{T^2} \ast m = \frac{\mathcal{G}_p M_s \ast m}{4\pi^2}
\]  

(3e)

and finally:

\[
\mathcal{G}_p = \frac{16\pi^4 T^2}{Msa^3}
\]  

(3f)

Results are shown in Figure 3. The first four points correspond to the terrestrial (inner) planets (Mercury, Venus, Earth and Mars). Their \( \mathcal{G}_p \) values are similar with no specific behavior as a function of distance to the Sun. In contrast, the points corresponding to the giant (outer) planets (Jupiter, Saturn, Uranus and Neptune) follow a regular pattern with \( \mathcal{G}_p \) decreasing regularly as a function of distance to the Sun (aphelia), and tending towards \( \mathcal{G} \) (dashed line). This is readily understood: Kepler’s third law ensures that the ratio \( \frac{T^2}{a^3} \) be constant, implying that with each full revolution the planet spans the same surface of the ellipse of revolution, but it does not imply that after a full revolution in a central field the planet returns to its initial position in the universe. The trajectory of a revolution is not closed and precession results. In a Galilean system, total energy \( E \) is conserved and the energy corresponding to the central field \( U(r) \) is a constant, and the same for all planets. The balance between the kinetic energy and the centrifugal energy (that takes the moment into account) ensures that the planet retain the same trajectory.

The equation of motion of \( m \) can be derived from (1). For the radial coordinate one obtains by integration from (1a):

\[
\dot{r} \equiv \frac{dr}{dt} = \sqrt{\frac{2}{m}[E - U(r)] - \frac{M^2}{m^2 r^2}}
\]  

(4a)

Separating variables and integrating:

\[
t = \int \frac{dr}{\sqrt{\frac{2}{m}[E - U(r)] - \frac{M^2}{m^2 r^2}}} + C''
\]  

(4b)

From (2b) one derives:

\[
d\varphi = \frac{M}{mr^2} dt
\]  

(4c)

Integrating (4c) in (4a):

\[
\varphi = \int \frac{M}{r^2} dt + C''
\]  

(4d)

Equations (4c) and (4e) are the full equations of motion of a planet \( m \) in a central field. The second provides the link between \( r \) and \( \varphi \). Equation (1) shows that the radial part of the motion can be considered as a linear motion in a central field with “effective” potential energy:

\[
U_{eff} = U(r) + \frac{M^2}{2mr^2}
\]  

(4e)

This equation underlines the reason why the revolution momentum is so important for geophysicists. For instance, one may ask what are the boundaries of the domain covered by the planet, that is when (1) reduces to:

\[
E = \frac{M^2}{2mr^2} + U(r)
\]  

(5a)
Radial velocity (4a) vanishes, but tangential velocity ((4c)) does not. At the singular points where ((5a) holds, the function changes from growing to decreasing (and vice-versa). The domain is bounded by two circles $r_{\text{min}}$ and $r_{\text{max}}$. The trajectory is finite but not necessarily closed. Let $\delta\varphi$ be the angle covered by the planet as $r$ decreases from $r_{\text{max}}$ and $r_{\text{min}}$ hen grows back to $r_{\text{max}}$. From (4e):

$$\Delta\varphi = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{M}{r^2} \, dr = \int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{2m(E - U)} - \frac{M^2}{r^2} \, dr$$  \hspace{1cm} (5b)$$

In order for the trajectory to be closed, $\Delta\varphi$ must be of the form $2\pi n/m$ where $n$ and $m$ are integers, respectively the number of repeats per period $T$ and the number of full revolutions. This may happen when there is a single planet revolving about the Sun, but with 8 planets the correction term is never a rational fraction and the trajectories are not closed and therefore precess (Figure 4).

**Fig. 4:** Schematic representation of a precessing planet on a non-closed quasi elliptical trajectory.

Let $\delta U = \frac{\gamma}{r^3}$ be the (small) perturbing potential from a second planet, and let $U(r) = -\frac{\alpha}{r} + \delta U$, then integrate (4e). One gets (recall that $(\cos u) = \frac{-u}{\sqrt{1 - u^2}}$):

$$\varphi = \arccos \frac{M}{r} - \frac{ma}{M} + C$$  \hspace{1cm} (6a)$$

From (3b) one has $p = \frac{M^2}{ma}$ and $e = \sqrt{1 + \frac{2EM}{ma^2}}$ and (6a) can be written:

$$p/r = 1 + e \cos \varphi$$  \hspace{1cm} (6b)$$

One can write the perturbing field $r^2 \delta U = \frac{\gamma}{r}$. (5b) can now be written as:

$$\Delta\varphi = -2 \frac{\partial}{\partial \mathcal{M}} \int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{2m(E - U)} - \frac{M^2}{r^2}$$  \hspace{1cm} (6c)$$

With $U(r) = -\frac{\alpha}{r} + \delta U$ and developing the expression under the integral in successive powers of $\delta U$, the order 0 term of the displacement is $2\pi$ and the order 1 term is:

$$\delta\varphi = -\frac{6\gamma}{\alpha p^2}$$  \hspace{1cm} (6d)$$

Given (3b) and (3c), (6d) becomes:

$$\delta\varphi = \frac{24\pi^2 a^2}{T^2(1 - e^2)} + \frac{\gamma}{\alpha a}$$  \hspace{1cm} (6e)$$

If one writes $\frac{\gamma}{\alpha a} = \frac{1}{c^2}$, then (6e) becomes Einstein's formula for the precession of perihelion:

$$\delta\varphi_{\text{einstein}} = \frac{24\pi^2 a^2}{T^2c^2(1 - e^2)} = \frac{6\pi McG}{c^2a(1 - e^2)}$$  \hspace{1cm} (6f)$$

Parameter $\alpha = mAcG$ can be fully determined from the values listed in Table A, through Kepler’s law (3c). $G$ follows Figure 3. We next calculate the delay in the planets’ perihelia, using Table A and Einstein’s formula under two cases:

1. the values of gravitation (black diamonds) are as in Figure 3;
2. one assumes a constant $G$ (red dots).

The two determinations are essentially identical as seen in Figure 5a. Their differences are shown in Figure 5b. A very good fit with a power law is shown as a dashed curve in Figure 5a. It is noteworthy that the exponent is indistinguishable from -5/2. This result, that we have not been able to find in the published literature, is readily understood in Lagrange’s formalism. Torques have their origin in the revolution of planets, that act on the Sun. The Sun’s rotation is modified, as evidenced for instance by sunspots. Since the system is Galilean with uniform time, the modified solar rotation acts instantaneously on the rotation axes of planets. Both the classical and the relativistic interpretations are similar, with the two parameters $1/c^2$ and $\gamma/(\alpha a)$ as the link.

The $r\delta\varphi$ (or $a\delta\varphi$) in Figure 2 can be regarded as the apparent surface of delay (or advance) with respect to closed trajectories (Kepler’s second law). That delay does not depend on time ($t$ does not appear in equation 6e). So $a\delta\varphi$ follows the law of areas and should have dimension $a^{-3/2}$. This is indeed seen in Figure 5c.
Thus $\delta \phi$ behaving as $a^{-3/2}$ implies that $\delta \phi$ behaves as $a^{-5/2}$ as found in Figure 5a. The delay in the periaster is a constant in our system; its value depends only on the distance to the origin of the central field. This was understood by Einstein but may be seen more clearly in the Lagrange formalism behind equation 6e.

In Figure 5b, we see that the differences in delays are constant and negligible from planet Mars to Neptune. For the three telluric planets closest to the Sun, the situation is different. This is not easily understood in Einstein’s formalism but readily so in Lagrange’s formalism, with reference to equation (1). Since we are in a central field and a Galilean reference frame, the system is closed, $E$ is conserved and the centrifugal term takes over (for small distances).

Replacing radial coordinate $r$ by aphelia $a$ in equation (1c), the centrifugal term involves the moment $M = a \times P$ (Figure 7) behaves as $a^{-1/2}$. This is readily understood as, from Kepler’s third law $T^2 = C^m \frac{1}{a^3}$ is dimension-less and $e$ behaves as $a^{-1/2}$.

\[ a = 1.476 \times 10^{29} \pm 5.2605 \times 10^{28} \]
\[ b = -2.558 \pm 0.0145 \]

\[ y = a^* b \]

\[ a = 1.166 \times 10^{19} \pm 5.5245 \times 10^{18} \]
\[ b = -1.549 \pm 0.019 \]

\[ y = a^* b \]

\[ a = 1.149 \times 10^7 \pm 3 \times 10^4 \]
\[ b = -0.4999 \pm 0.0001 \]

\[ y = a^* b \]
telluric planets are essentially negligible whereas those of the giant planets follow a monotonous decreasing trend. In contrast, the areal velocities of all planets follow a $a^{-3/2}$ law (Figure 7a). It therefore seems that the telluric planets have lost energy. We hypothesize that this energy is transferred to the planets rotation axis. Following Laplace (1799) and Lagrange (1853), any modification of the inclination of the planet’s rotation axis leads to a modification of the full rotation.

Laplace (1799) provides the system of linear differential equations (known as Liouville-Euler equations) that link the derivative of rotation velocity to changes in rotation axis inclination (see Lopes et al. (2021, 2022)). Geophysical or astronomical perturbations of a planet’s revolution lead to changes in its rotation. We now attempt to identify the corresponding law that is obeyed by the ratio (revolution period/rotation period).

\[ \gamma = a^2 \delta \]

Figure 7 shows the evolution of the ratio of areal velocity to integrated period $R$ as a function of perihelion delay $\delta \phi$.

3. Discussion

Section 2 of this paper has attempted a rough summary of the main results obtained by Lagrange (1853) on classical mechanics, a theory better known as the theory of the spinning top, that he applied to astronomy. Equation 1 is the fundamental starting point, in that it gives the Lagrangian energy of a planet in motion in a central field, and allows one to describe fully the physics acting on the system. In his Philosophiae naturalis principia mathematica, published in 1687 - the last edition dating from 1726 (see Madame du Châtelet, 1756) - Newton borrowed an optical analogy from Robert Hooke (see Koyré (1952)): he is interested only in centripetal forces acting on planets gravitating about the Sun. Indeed, these forces are the only ones to appear in Newton’s table of contents. Although this is not widely cited, some of Newton’s conclusions were criticized by several of his contemporaries and successors (eg. d’Alembert (1749); Madame du Châtelet (1756); Laplace (1799)). Rightly so, as his theory did not match some of the observations. In the present paper, we have developed the Lagrangian approach that actually underlies the equations of most physics papers and books, often without being explicitly quoted or the prerequisite being mastered (eg. Poincaré (1890, 1893); Landau and Lifshitz (1988)).

The three Lagrange integrals are “prime” integrals, that is they preserve physical quantities when the system is either closed or open but with a central field $U(r)$, with a single symmetry axis (Lagrange 1853). The most important equations are (1a, 1b, 1c), the rest being their consequences. They apply...
in a Galilean reference, for a physical system that is in that case open with a central field. As we have seen, that central field $U(r)$ is perturbed; it is no more exactly in $1/r$, therefore the orbits are not closed and precession occurs. Fortunately, since this perturbation is very small, one can use Lagrangian mechanics throughout.

As a counter-example, the conservation of kinetic momentum does not apply to an elastic Earth (cf. Lambeck (2005), chapter 3). Indeed, most researchers who discuss an elastic Earth consider a single isolated planet without any central field. In other words, the kinetic moment, which is linked to the order 2 inertia tensor, may not be symmetrical and diagonalizable any more. This is already clearly stated by Laplace (1799), page 347, and Appendix B) or Poincaré (1893). The equations are valid for a solid Earth or one in which deformations are negligible. In equation (1a), there is a competition between 3 energies, kinetic, centrifugal and gravitational attraction (the last one being identical for all planets). The farther the distance to the Sun, the larger the influence of $U(r)$. The only planet that does not fit the overall $y = x$ law as seen in Figure 9 is Neptune, simply because it is so far from the Sun (large $r$) that the leading term in the energy is $U(r)$ and the planet does not need to dissipate its excess energy by altering its rotation.

The important law, and an observational one, established by Kepler in 1619 (cf. Warrain (1942)) is that the ratio of the square of the period of revolution to the cube of aphelia is a constant. Yet, this gives no indication on the equation of motion. A planet can revolve about the Sun without coming back to its initial location in the universe. That is the phenomenon of precession (cf. d’Alembert (1749); Milankovic (1920) and see Figure 4). We have seen that the conditions for trajectory closure are not met. Perihelion can be ahead of time or delayed with respect to the prediction by Newton. Thus, in 1869, Urbain Le Verrier could not find the delay of Mercury despite including attractive forces from all planets (Weinberg 1972). In an attempt to circumvent this problem, several authors included the flattening of the Sun in the attractive forces (Newcomb 1882). The discrepancy between Newtonian theory and observations has led to numerous studies on the possible variations of the “constant”, particularly since the 1970s (eg. Pochoda and Schwarzschild (1964); Shapiro et al. (1971); Wu and Wang (1978); Combos and Tiret (2010); Alvey et al (2020)). Some of these works are still actively pursued, such as the seeds for dark matter (in a universe that Einstein (1915) believed to be empty). Some could have resulted from an incomplete interpretation of the laws of physics.

Einstein and his German School contemporaries (Schwarzschild, Hilbert, Flamm, ...) developed a mathematical metric for astronomical laws. There is no a priori physics behind. In equation (1a), we have seen that $\left(\frac{d}{dt}\right)^2$ can be zero without the planets being motionless (the trajectories become circles). In one of the two remaining terms in equation (1a), there is one, centrifugal energy, that is planet-dependent. All gravitating planets precess according to a $-5/2$ power law of aphelia (Figure 5a). We finally seek a small perturbation of $U(r)$ that will prevent trajectories from closing with a dependence on aphelia that is of a lesser degree than the centrifugal force (or planets would leave their finite trajectory). The solution $\frac{1}{r^3}$ meets these constraints (with $\gamma$ and $\alpha$ constant). The Lagrange and Einstein theories then lead to the same result (Equations 6e and 6f, and Figure 5a). There is however a fundamental difference between the two: for Einstein (1915) deformation of space-time by the Sun is the origin of a field perturbation that it would generate by itself. For Lagrange (with Kepler) the perturbation $\delta U$ could only be due to the interactions of torques, since $U$ is a constant imposed by the mass and immobility of our star, and $\left(\frac{d}{dt}\right)^2$ can be zero without changing drastically the astronomy orbits. The only remaining term is the centrifugal force due to planetary rotations, that is indeed torques.

Since we are in a Galilean system, $U$ being constant and $E$ being conserved, there will be transfers of energy as a function of aphelia.

In Figures 1 and 7b, we see that there is a difference in amplitude and behavior of the $G$ and kinetic moments of planets, both quantities being observations. Yet, the perihelion delays (Figure 5a), the areal velocities (Figure 7a) and the planetary rotations (Figure 9, top) display power laws of aphelia, whose behavior contrasts for instance with that of the kinetic moment (Figure 7b). The areal velocity being linearly linked to the kinetic moment of planets (equation 2b), this must be the level at which the transfer is achieved. As shown in two volumes by Laplace (1799) and recently addressed by Lopes et al. (2021, 2022), variations in the Earth’s rotation axis (Figure 6) and rotation velocity (or length of day) are connected by a time derivative. Given the instantaneous and reciprocity of Galilean systems, any torque acting on the Sun is returned to Earth, whose rotation axis is perturbed (same as what happens with the weight of a spinning top to its inclination. If one considers the law of rotation periods as a function of aphelia (red dots in Figure 9 top), we obtain – to a small error – the variations of inclination of the rotation axis. Since we are in a closed system, this ratio must be constant for each planet, and all planets should follow the same power law of aphelia ($\mathcal{M}$ being actually "hidden" in the transfer).
Appendix A

| mass (kg)  | Mercury | Venus | Earth | Mars | Jupiter | Saturn | Uranus | Neptune |
|-----------|---------|-------|-------|------|---------|--------|--------|---------|
| 3.301110^26 | 4.875310^24 | 5.976510^27 | 6.418510^26 | 6.046410^27 | 6.031010^27 | 4.92310^27 |
| radius (km)  | 2440 | 6052 | 6378 | 3394 | 71420 | 60206 | 25559 | 24764 |
| aphelion(s) | 57900050 | 19350000 | 19350000 | 19350000 | 19350000 | 19350000 | 19350000 | 19350000 |
| eccentricity | 0.205 | 0.00673 | 0.01671022 | 0.09339 | 0.8439 | 0.0539 | 0.04726 | 0.00059 |
| resolution (day) | 87.906 | 264.467 | 305.236 | 466.885 | 472.011 | 10754 | 36008 | 62168 |
| rotation (day) | 59.646 | 243.123 | 9.997 | 1.0239 | 0.144 | 0.488 | 0.718 | 0.671 |

resolution (m/s^2) (kg.m/s)
| 9.1510^11 | 1.9410^10 | 2.6610^10 | 5.5310^11 | 1.9310^11 | 7.8210^11 | 1.9110^11 | 2.3010^11 |

Appendix B

Laplace (1799), vol. 5, cap. 1, page 347: "Nous avons fait voir (n o 8), que le mouvement de rotation de la Terre est uniforme, dans la supposition que cette planète est entièrement solide, et l’on vient de voir que la fluidité de la mer et de l’atmosphère ne doit point altérer ce résultat. Les mouvements que la chaleur du Soleil excite dans l’atmosphère, et d’où naissent les vents alizés semblent devoir diminuer la rotation de la Terre: ces vents soufflent entre les tropiques, d’occident en orient, et leur action continue sur la mer, sur les continents et les montagnes qu’ils rencontrent, paraît devoir affaiblir insensiblement ce mouvement de rotation. Mais le principe de conservation des aires, nous montre que l’effet total de l’atmosphère sur ce mouvement doit être insensible; car la chaleur solaire dilatant également l’air dans tous les sens, elle ne doit point altérer la somme des aires décrites par les rayons vecteurs de chaque molécule de la Terre et de l’atmosphère, et multipliées respectivement par leurs molécules correspondantes; ce qui exige que le mouvement de rotation ne soit point diminué. Nous sommes donc assurés qu’en même temps que les vents analysés diminuent ce mouvement, les autres mouvements de l’atmosphère qui ont lieu au-delà des tropiques, les accélèrent de la même quantité. On peut appliquer le même raisonnement aux tremblements de Terre, et en général, à tous ce qui peut agiter la Terre dans son intérieur et à sa surface. Le déplacement de ces parties peut seul altérer ce mouvement; si, par exemple un corps placé au pole, était transporté à l’équateur; la somme des aires devant toujours rester la même, le mouvement de rotation en serait un peu diminué; mais pour que cela fût sensible, il faudrait supposer de grands changements dans la constitution de la Terre".

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