A family of diameter perfect constant-weight codes from Steiner systems*

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Abstract

If $S$ is a transitive metric space, then $|C| \cdot |A| \leq |S|$ for any distance-$d$ code $C$ and a set $A$, "anticode", of diameter less than $d$. For every Steiner $S(t, k, n)$ system $S$, we show the existence of a $q$-ary constant-weight code $C$ of length $n$, weight $k$ (or $n - k$), and distance $d = 2k - t + 1$ (respectively, $d = n - t + 1$) and an anticode $A$ of diameter $d-1$ such that the pair $(C, A)$ attains the code–anticode bound and the supports of the codewords of $C$ are the blocks of $S$ (respectively, the complements of the blocks of $S$). We study the problem of estimating the minimum value of $q$ for which such a code exists, and find that minimum for small values of $t$.

**Keywords:** diameter perfect codes, anticodes, constant-weight codes, code–anticode bound, Steiner systems.

1. Introduction

Diameter perfect codes are natural generalizations of perfect codes, fascinating structures in coding theory. They can be defined for different metrics,
especially, for the Hamming metric. In this paper, we discuss diameter perfect codes in the subspaces of the Hamming metric space restricted by the words whose weight is constant.

The concept of a diameter perfect codes \([1]\) is based on the code–anticode bound, which is a generalization of the sphere-packing bound; the most famous referenced result with that bound is Delsarte’s [14, Theorem 3.9]. However, the main importance of [14, Theorem 3.9] is that the code–anticode bound is proved there for any distance-regular graph (more generally for an association scheme), without any assumption on its symmetries. For symmetric metric spaces, such as Hamming and Johnson schemes, the bound has a much simpler proof, and probably was known earlier (in particular, the key lemma in the proof of the Elias–Bassalygo bound [4] is based on the same generalized pigeonhole principle).

Lemma 1 (code–anticode bound). Let \(\mathcal{V}\) be a finite metric space whose automorphism group \(\text{Iso}(\mathcal{V})\) acts transitively on its points. Let \(\mathcal{C}\) be a subset of \(\mathcal{V}\) (a code) with minimum distance \(d\) between elements, and let \(\mathcal{A}\) be a subset of \(\mathcal{V}\) of diameter \(D\) (an anticode), where \(D < d\). Then

\[
|\mathcal{C}| \cdot |\mathcal{A}| \leq |\mathcal{V}|.
\]

(1)

Proof. Denote \(\overline{\mathcal{A}} = \{\pi(\mathcal{A}) : \pi \in \text{Iso}(\mathcal{V})\}\). Because of the transitivity, each point of \(\mathcal{V}\) belongs to exactly \(s\) sets in \(\overline{\mathcal{A}}\), where \(s = |\overline{\mathcal{A}}| \cdot |\mathcal{A}| / |\mathcal{V}|\). From the hypothesis on the diameter of \(\mathcal{A}\) and minimum distance of \(\mathcal{C}\), we see that each set from \(\overline{\mathcal{A}}\) contains at most one element of \(\mathcal{C}\). On the other hand, each element of \(\mathcal{C}\) is contained in \(s\) elements of \(\overline{\mathcal{A}}\). Therefore, \(|\overline{\mathcal{A}}| \geq s|\mathcal{C}|\).

Substituting \(s = |\overline{\mathcal{A}}| \cdot |\mathcal{A}| / |\mathcal{V}|\), we get (1). \(\square\)

If (1) holds with equality, then \(\mathcal{C}\) is called a diameter perfect distance-\(d\) code or \(D\)-diameter perfect code. It is easy to see that in this case \(\mathcal{A}\) is a maximum diameter-\(D\) anticode. The space \(\mathcal{J}_q(n, w)\) of weight-\(w\) words of length \(n\) over the \(q\)-ary alphabet, as was shown in [17], belongs to the class of spaces for which the code–anticode bound holds. In [17] (see also [18, Ch. 9]), Etzion classified known families of non-binary diameter perfect constant-weight codes as follows (below, \(q\) is the alphabet size, \(n\) is the length, \(w\) is the weight and \(d\) is the minimum distance):

F1 Non-binary diameter-perfect constant-weight codes for which \(w = n\).

Essentially, they are diameter-perfect codes in the Hamming space \(\mathcal{H}(q - 1, n)\) (perfect codes, extended perfect codes, MDS codes).
F2 Non-binary diameter perfect constant-weight codes for which \( w = n - 1 \). Known constructions [19], [23], [24], [25], [29], [31] are related to perfect codes in the Hamming space \( \mathcal{H}(q-1, n-1) \), \( q = 2^m + 1 \), and their extensions, with one exception: \( q = 3, n = w + 1 = 6, d = 4 \) [26].

F3 Generalized Steiner systems [16] (see also [6] and references there), for which every codeword of weight \( t, t < w \), is at distance \( w - t \) from exactly one codeword and \( d = 2(w-t) + 1 \) (see, e.g., [35], where \( q = w = 4, t = 2 \)).

F4,F6 \( d = w \) or \( d < w \), respectively. Codes of size \( \binom{n}{w}(q-1)^{w-d+1} \), where all codewords of one of \( \binom{n}{w} \) possible supports form an unrestricted distance-\( d \) MDS code of length \( w \) over the \((q-1)\)-ary alphabet \( \{1, \ldots, q-1\} \).

F5 Codes of size \( \binom{n}{w} \) with \( d = w + 1 \), where the supports of the codewords form the complete design (all \( w \)-subsets of \( \{1, \ldots, n\} \)).

Actually, family F5 also falls in the definition of F4 and F6 with \( d = w + 1 \). However, we keep it separate in the list because our goal is to extend it. In this paper, we further explore diameter perfect constant-weight codes over non-binary alphabets and maximum anticodes corresponding to those codes. We consider two families of codes, F5’ and F5”, generalizing F5 (F5 corresponds to F5’ with \( t = w \) and to F5” with \( n - t = w \)):

F5’ Codes of size \( \binom{n}{t}/\binom{n}{w} \) with \( d = 2w - t + 1 \), where the supports of the codewords form a Steiner system \( S(t, w, n) \), \( w \leq n/2 \).

F5” Codes of size \( \binom{n}{t}/\binom{n-w}{t} \) with \( d = n - t + 1 \), where the complements of the supports of the codewords form a Steiner system \( S(t, n - w, n) \), \( w \geq n/2 \).

It is notable that F5’ and F5” are related to the two known families of binary diameter perfect constant-weight codes, constructed from Steiner systems [17, Sect. III]. While in the binary case, the codes from the two families are connected by a trivial operation (swapping the role of 0 and 1), the codes from F5’ and F5” are essentially different, except for the special case \( w = n/2 \).

It is worth noting that diameter perfect codes form a proper subclass of the class of cardinality-optimal codes (that is, codes with the maximum
cardinality among all codes with the same distance in the same space). There are many constructions of optimal non-binary constant-weight codes, most of which are not diameter perfect, see, e.g., [7], [8], [9], [10], [11], [13], [20], [22], [26], [34], [36], [35], [37]. However, in the most of cases, a small alphabet or a small code distance is considered, while the codes we focus on have the minimum distance larger than the weight and a relatively large alphabet size. Intersections of the families of codes known before with family F5′ (including F5) are mentioned in Section 4.1 (Remarks 2, 3, 4).

The structure of the paper is as follows. In Section 2, we give main definitions and basic results. In Section 3, we construct a class of antcodes corresponding to families F5′, F5″ and show that they are maximum for large $q$ (Theorems 1 and 2), also showing the same for a more general class of antcodes (Proposition 1). In Section 4, we observe that the existence of a Steiner system (with the corresponding parameters) implies the existence, for sufficiently large $q$, of diameter perfect codes that attain the code–anticode bound with the new antcodes (Theorem 3), and define $q_0'$ and $q_0''$ as the smallest possible $q$ (corresponding to families F5′ and F5″, respectively) for which such codes exist.

In Section 4.1, we find $q_0'$ in the case of Steiner systems of strength 2 (Theorems 4) and 3 (Theorems 5, Corollaries 2 and 3). Further, in Sections 4.2 and 4.3 we derive an upper bound on $q_0'$ in the general case (Theorems 6) and consider special cases related to the Steiner systems $S(4, 5, 11)$ (Proposition 2), $S(3, 4, 14)$ and $S(3, 4, 20)$ (Proposition 3). In Section 4.4, we consider $q_0''$; we derive a lower bound (Proposition 4, Corollary 5) and solve the special cases related to affine and projective planes (Corollaries 6 and 7). Section 5 contains concluding remarks and conjectures.

2. Preliminaries

In this section, we present definitions and notations, as well as several results to facilitate understanding throughout the paper.

By $\mathbb{Z}_q$, we denote the set $\{0, \ldots, q - 1\}$ (we do not need to endow it by any algebraic operation, such as addition or multiplication); we will also use the notation $[n]$ for $\{1, \ldots, n\}$. Given a positive integer $n$, let $\mathbb{Z}_q^n$ be the set of all words of length $n$ over the alphabet $\mathbb{Z}_q$; for each word $\bar{x} \in \mathbb{Z}_q^n$, we have $\bar{x} = (x_i)_{i \in [n]}$ and $x_i \in \mathbb{Z}_q$ for any $i \in [n]$. The support of a word $\bar{x} \in \mathbb{Z}_q^n$ is the set $\text{supp}(\bar{x}) = \{i \in [n] : x_i \neq 0\}$. The weight of a word $\bar{x} \in \mathbb{Z}_q^n$, ...
denoted by \( \text{wt}(\bar{x}) \), is equal to \( |\text{supp}(\bar{x})| \). The distance (Hamming distance) \( d(\bar{x}, \bar{y}) \) between two words \( \bar{x}, \bar{y} \) in \( \mathbb{Z}_q^n \) is defined to be the number of positions in which \( \bar{x} \) and \( \bar{y} \) differ. By \( \mathcal{J}_q(n, w) \), we denote the subset of \( \mathbb{Z}_q^n \) that is restricted by words of weight \( w \), and the corresponding metric subspace of the Hamming space.

**Definition 1** (codes). Given a metric space, a **code** as a subset with at least two vertices, referred to as **codewords**. A code \( C \) is said to have **distance** \( d \) if the distance between any two distinct codewords is not less than \( d \). A code \( C \) is a \( q \)-ary code of length \( n \) if \( C \subseteq \mathbb{Z}_q^n \). If, additionally, \( C \subseteq \mathcal{J}_q(n, w) \), then it is a constant-weight, or constant-weight-\( w \), code. For brevity, we call a \( q \)-ary constant-weight-\( w \) code of distance \( d \) an \( (n, |C|, d; w)_q \) code, or an \( (n, \cdot, d; w)_q \) code if the cardinality is not specified or important.

**Definition 2** (anticodes and diameter perfect codes). An **anticode** of diameter \( D \) in a metric space \( \mathcal{V} \) is a nonempty subset \( \mathcal{A} \) of \( \mathcal{V} \) such that the distance between any \( \bar{x} \) and \( \bar{y} \) in \( \mathcal{A} \) is not greater than \( D \). A distance-\( d \) code \( C \) in \( \mathcal{V} \) is called **diameter perfect** if it attains the code–anticode bound (1) for some anticode \( \mathcal{A} \) of diameter smaller than \( d \).

**Definition 3** (code matrix). For a code \( C \) in \( \mathbb{Z}_q^n \), we define the **code matrix** whose rows are the codewords of \( C \). To be explicit, we can require that the rows are ordered lexicographically; however, we will never use this order in our considerations.

**Definition 4** (Steiner systems). A (Steiner) system \( S(t, k, n) \), \( 0 < t \leq k < n \), is a pair \( S = (N, B) \), where \( N \) is an \( n \)-set (whose elements are often called **points**) and \( B \) is a set of \( k \)-subsets (called **blocks**) of \( N \), where each \( t \)-subset of \( N \) is contained in exactly one block of \( B \). Systems \( S(2, 3, n) \) are also denoted STS\( (n) \) and called **Steiner triple systems**, STS. Systems \( S(3, 4, n) \) are also denoted SQS\( (n) \) and called **Steiner quadruple systems**, SQS.

**Definition 5** (derived systems). For a Steiner \( S(t, k, n) \) system \( S = (N, B) \) and a set of points \( \alpha \subseteq N \), \( 0 < |\alpha| < t \), the system \( (N \setminus \alpha, \{\beta \setminus \alpha : \alpha \subseteq \beta \in B\}) \) is a Steiner \( S(t - |\alpha|, k - |\alpha|, n - |\alpha|) \) system, called **derived** from \( S \).

**Definition 6** (resolvable systems). A Steiner \( S(t, k, n) \) system \( S = (N, B) \) is called \( l \)-**resolvable**, \( 1 \leq l < t \), (if \( l = 1 \), just **resolvable**) if the set of blocks \( B \) can be partitioned into subsets each of which forms an \( S(l, k, n) \) on the same point set \( N \). If \( l = 1 \), those subsets are referred to as **parallel classes**.
Definition 7 (minimum-distance graph and chromatic number). The minimum-distance graph of an $S(t, k, n)$ is a graph whose vertices are the blocks of the system, with two vertices joined by an edge if and only if the corresponding blocks intersect in $t - 1$ points. For a Steiner system $S = (N, B)$, we denote by $\chi(S)$ the chromatic number of its minimum-distance graph (i.e. the minimum number of cells in a partition of $B$ such that two distinct blocks from the same cell intersect in at most $t - 2$ points).

In the rest of this section, we list some known facts in the form of lemmas, to be utilized in the proofs of our results.

Lemma 2 (see, e.g., [12]). The number of blocks in a Steiner system $S(t, k, n)$ is $\binom{n}{t}/\binom{k}{t}$ and every point in $N$ is contained in $\binom{n-1}{t-1}/\binom{k-1}{t-1}$ blocks.

Lemma 3 ([12 Sect. 5.4]). For a Steiner system $S(t, k, n)$, the number $\Lambda_i(t, k, n)$ of blocks that intersect a fixed block $B$ in exactly $i$ elements does not depend on the choice of $B$ and equals $\binom{k}{i}\lambda_{i,k}$, where $\lambda_{i,j}$, $0 \leq i + j \leq k$, are recursively defined as follows:

$$
\lambda_{i,0} = \binom{n-i}{t-i}/\binom{k-i}{t-i}, \quad 0 \leq i \leq t,
$$

$$
\lambda_{i,0} = 1, \quad t < i \leq k,
$$

$$
\lambda_{i,j} = \lambda_{i,j-1} - \lambda_{i+1,j-1}, \quad 0 \leq i < i+j \leq k.
$$

Lemma 4. Assume positive integers $m$ and $s$ are given, and we consider representations of $m$ as the sum of $s$ nonnegative integers: $m = m_1 + \ldots + m_s$. The value $\sum_i \binom{m_i}{2}$ reaches its minimum if and only if $m_1, \ldots, m_s \in \{a, a+1\}$, where $a = \lfloor m/s \rfloor$.

**Proof.** Otherwise, $m_l \leq m_j + 2$ for some $l, j$. In this case, obviously, $\binom{m_l}{2} + \binom{m_j}{2} > \binom{m_l+1}{2} + \binom{m_j-1}{2}$, and $\sum \binom{m_i}{2}$ is not as small as possible.

3. Anticodes

Generalizing the anticode $A^m(n, w, w)$ from [17 IV.DE], we define these two sets:

$$
A^t_q(n, w, t) = \{(a_1, \ldots, a_t, a_{t+1}, \ldots, a_n) \in \mathbb{Z}_q^n : \text{wt}((a_1, \ldots, a_t)) = t, \text{wt}((a_{t+1}, \ldots, a_n)) = w - t\},
$$
where \( t \leq w \) and \( 2w - t \leq n \), and

\[
A_q''(n, w, t) = \{(0, \ldots, 0, a_{t+1}, \ldots, a_n) \in \mathbb{Z}_q^n : \text{wt}(a_{t+1}, \ldots, a_n) = w\},
\]

where \( t \leq n - w \) and \( 2w + t \geq n \). The main theorem in this section establishes that \( A_q'(n, w, t) \) and \( A_q''(n, w, t) \) are maximum anticodes in \( J_q(n, w) \), assuming some conditions on the parameters hold.

**Theorem 1.** The set \( A_q'(n, w, t) \), \( t \leq 2w - t \leq n \), \( q \geq 3 \), satisfies the following:

1. \( A_q'(n, w, t) \) has \( \binom{n-1}{w-t}(q-1)^w \) words;
2. \( A_q'(n, w, t) \) is a diameter-\( (2w - t) \) anticode;
3. if \( n \geq (w - t + 1)(t + 1) \) and \( q \) is large enough, then \( A_q'(n, w, t) \) is a maximum diameter-\( (2w - t) \) anticode in \( J_q(n, w) \).

**Theorem 2.** The set \( A_q''(n, w, t) \), \( t + w \leq n \leq 2w + t \), \( q \geq 3 \), satisfies the following:

1. \( A_q''(n, w, t) \) is a diameter-\( (n - t) \) anticode;
2. \( A_q''(n, w, t) \) has \( \binom{n-1}{w}(q-1)^w \) words;
3. if \( n \geq (n - w - t + 1)(t + 1) \) (i.e., \( n \leq (w + t - 1)(t + 1)/t \)) and \( q \) is large enough, then \( A_q''(n, w, t) \) is a maximum diameter-\( (n - t) \) anticode in \( J_q(n, w) \).

Claims (1'), (1''), (2'), and (2'') are obvious. To show (3') and (3''), we firstly prove a more general result, of independent interest.

**Proposition 1.** Let \( T \) be a maximum family of different \( w \)-subsets of \([n]\) such that every two subsets from \( T \) intersect in at least \( s \) elements. Let \( A_q \) be the set of all \( |T|(q-1)^w \) words over \( \mathbb{Z}_q \) with supports in \( T \).

1. If \( q = 2 \), then \( A_2 \) is a maximum diameter-(2w - 2s) anticode.
2. If \( q \) is large enough, then \( A_q \) is a maximum diameter-(2w - s) anticode.
3. If \( C_q \) is a diameter perfect \((n, |J_q(n, w)|/|A_q|, 2w - s + 1; w)_q\)-code, then \( C_2 \) is a diameter perfect \((n, |J_2(n, w)|/|A_2|, 2w - 2s + 2; w)_q\)-code, where \( C_2 \) is obtained from \( C_q \) by replacing all non-zero symbols by 1 in all codewords.
Proof. Claim (1) is trivial and follows from the well-known correspondence
between \( s \)-intersecting families of \( w \)-subsets of \([n] \) and sets of diameter-(\( 2w - 2s \)) in \( J_2(n, w) \) (we only note that for binary constant-weight codes, it is
more usual to deal with the Johnson distance, which is twice smaller than
the Hamming distance we consider).

To show (2), let us consider a diameter-(\( 2w - s \)) anticode \( B \).

If for every two words \( \bar{a}, \bar{b} \) in \( B \) it holds \( |\text{supp}(\bar{a}) \cap \text{supp}(\bar{b})| \geq s \), then the
words from \( B \) have at most \( |T| \) different supports, and thus \( |B| \leq |A_q| \).

Assume \( B \) contains \( \bar{a} \) and \( \bar{b} \) such that \( |\text{supp}(\bar{a}) \cap \text{supp}(\bar{b})| < s \). Denote by \( B' \) the set of all supports of words in \( B \); we divide it into two disjoint subsets \( B', B'' \):

\[
B' = \{ \beta \in B : |\beta \cap \alpha| < s \text{ for some } \alpha \in B \},
\]
\[
B'' = \{ \beta \in B : |\beta \cap \alpha| \geq s \text{ for all } \alpha \in B \}.
\]

We state two claims.

- For every \( \beta \) in \( B' \), the number of words in \( B \) with support \( \beta \) is smaller
  than \( (q - 1)^w - (q - 2)^w \). Indeed, by the definition of \( B' \), there is a
  word \( \bar{x} \) in \( B \) such that \( |\beta \cap \text{supp}(\bar{x})| < s \). Any word with support \( \beta \)
  that differs with \( \bar{x} \) in all positions of \( \beta \cap \text{supp}(\bar{x}) \) cannot belong to \( B \)
because of the diameter of \( B \). The number of such words is larger than
  \( (q - 2)^w \), and the claim follows.

- It holds \( |B''| \leq |T| - 1 \). Indeed, by the hypothesis, \( B \) contains at least
  one element \( \alpha \) not in \( |B''| \). By the definition of \( B'' \), each two distinct
  sets in \( B'' \cup \{ \alpha \} \) intersect in at least \( s \) points. Since \( T \) is a maximum
  family with this property, we get \( |B'' \cup \{ \alpha \}| = |B''| + 1 \leq |T| \).

Therefore, we have

\[
|B| < |B''|(q - 1)^w + |B'|((q - 1)^w - (q - 2)^w)
< (|T| - 1)(q - 1)^w + \binom{n}{w}((q - 1)^w - (q - 2)^w).
\]

Since the degree of the polynomial \( (q - 1)^w - (q - 2)^w \) is less than \( w \), we have \( |B| < |T|(q - 1)^w = |A_q| \) for sufficiently large \( q \), and (2) follows.

It remains to prove (3). If \( C_q \) is a distance-(\( 2w - s + 1 \)) code, then the union
of the supports of two distinct codewords has at least \( 2w - s + 1 \) points; in
particular, those supports have at most \( s - 1 \) points in common and hence
the distance between the corresponding codewords of \( C_q \) (defined as in (3)) is
at least \( 2w - 2s + 2 \). It remains to note that \( |J_q(n, w)|/|A_q| \) does not depend
on \( q \), so \( |C_q| \cdot |A_q| = |J_q(n, w)| \) implies \( |C_2| \cdot |A_2| = |J_2(n, w)| \).
The next lemma implies that the set of supports of the words from $A'_q(n, w, t)$ or $A''_q(n, w, t)$ satisfies the hypothesis of Proposition 1.

**Lemma 5.** [Erdős–Ko–Rado theorem [15, 32] for $s$-intersecting families] Let $0 \leq s \leq w \leq n$, and let $T$ be a family of $w$-subsets of $[n]$ such that every two subsets from $T$ intersect in at least $s$ elements.

(i) If $n \geq (w - s + 1)(s + 1)$, then $|T| \leq \binom{n-s}{w-s}$.

(ii) If $2w - s \leq n \leq (w - s + 1)(2w - s - 1)$, then $|T| \leq \binom{2w-s}{w-s}$.

Actually, the classic Erdős–Ko–Rado theorem, proved in [15] for $n \geq s + (w - s)(\binom{w}{s})^2$ and in [32] for $n \geq (w - s + 1)(s + 1)$, does not state (ii), which is a direct corollary of (i), obtained by replacing the sets from $T$ by their complements with respect to $[n]$. All possible maximum $s$-intersecting families of $w$-subsets of $[n]$ are described in [2]. Note that each of them, by Proposition 1, produces a sequence of maximum constant-weight anticode.s over nonbinary alphabet. We focus on only two types of such antico.des, $A'_q(n, w, t)$ and $A''_q(n, w, t)$, because corresponding diameter perfect codes can be constructed from Steiner systems $S(t, k, n)$, which correspond to the two known classes of binary diameter perfect constant-weight codes (with respect to the antico.des $A'_2(n, k, t)$ and $A''_2(n, n - k, t)$). For the other types of maximum $s$-intersecting families, it is conjectured [17, Conjecture 1] that corresponding binary diameter perfect constant-weight codes do not exist, which implies (if the conjecture is true) nonexistence of corresponding non-binary codes, by Proposition 1(3).

**Proof of Theorem 1(3’) and Theorem 2(3”).** From Lemma 5(i) we see that the set of supports from $A'_q(n, w, t)$, under the conditions of Theorem 1, forms a maximum $t$-intersecting family; hence, (3’) follows from Proposition 1

Further, denote $s = 2w - n + t$. We see that the set of supports from $A''_q(n, w, t)$ forms an $s$-intersecting family. The hypothesis $n \leq \frac{(w-s+1)(2w-s-1)}{w-s}$ of Lemma 5(ii) is equivalent to the condition $n \geq (n - w - t + 1)(t + 1)$ of Theorem 2, while $2w - s \leq n$ is equivalent to the trivial $0 \leq t$. Therefore, we can apply Lemma 5(ii) and conclude that we have a maximum $s$-intersecting family. Proposition 1 completes the proof of (3”).}
4. Codes and bounds for the smallest \( q \)

**Theorem 3.** Let \( t, k, \) and \( n \) be integers such that \( 0 < t \leq k < n \). If there exists a Steiner system \( S(t, k, n) \), then the following assertions hold.

(i) For \( w = k \) and \( q \geq \binom{n-1}{t-1} + 1 \), there is an \( (n, \binom{n}{t}, 2w-t+1; w)_q \) code, which attains the code–anticode bound with the anticode \( A_q^w(n, w, t) \).

(ii) For \( w = n - k \) and \( q \geq \binom{n}{t} - \frac{\binom{n-1}{t-1}}{\binom{n-1}{w-1}} + 1 \), there is an \( (n, \binom{n}{t}/(n-w), n-t+1; w)_q \) code, which attains the code–anticode bound with the anticode \( A_q^w(n, w, t) \).

If there is no \( S(t, k, n) \), then \( (n, \binom{n}{t}/(k), 2k-t+1; k)_q \) codes and \( (n, \binom{n}{t}/(k), n-t+1; n-k)_q \) codes do not exist for any \( q \).

**Proof.** (i) Assume there exists a Steiner system \( S(t, w, n) \) with point set \([n]\) and blocks \( \Delta_1, \Delta_2, \ldots, \Delta_M, M = \binom{n}{w}/\binom{w}{\binom{n}{t}} \). Let \( x^i, i = 1, \ldots, M, \) be the length-\( n \) word containing 0 in the \( j \)th position if \( j \notin \Delta_i \) and containing a positive value \( a \) in the \( j \)th position if \( \Delta_i \) is the \( a \)th block with \( j \) (that is, if \( j \in \Delta_i \) and \( a = \{l \in \{1, \ldots, i\} : j \in \Delta_i\} \)). According to this definition, \( \text{wt}(x^i) = w \) and two different words \( x^i, x^j \) coincide only in positions where they both have zeros. By the definition of Steiner system, the supports of such \( x^i \) and \( x^j \) have no more than \( t - 1 \) common elements; hence their union has size at least \( 2w - (t - 1) \), and \( d(x^i, x^j) \geq 2w - t + 1 \). It remains to note that by Lemma 2 all codewords involve \( \binom{n-1}{t-1}/\binom{w-1}{t-1} \) nonzero symbols. We conclude that \( \{\bar{x^1, \ldots, \bar{x^M}\} \) can be treated as an \( (n, \binom{n}{t}/(w), 2w-t+1; w)_q \) code for every \( q \geq \binom{n-1}{t-1}/\binom{w-1}{t-1} + 1 \).

(ii) Similarly, assume there exists a Steiner system \( S(t, n-w, n) \) with point set \([n]\) and blocks \( \Delta_1, \Delta_2, \ldots, \Delta_M, M = \binom{n}{n-w}/\binom{n-w}{\binom{n}{t}} \). Similarly to (i), we can construct a code such that the codeword supports are the complements of the Steiner system blocks and for each coordinate all the nonzero elements of the codewords in that coordinate are distinct. All codewords have weight \( w \), and any two different codewords \( \bar{x}, \bar{y} \) coincide only in positions where they both have zeros. Since such \( \bar{x} \) and \( \bar{y} \) have no more than \( t - 1 \) common zero positions, their union has size at least \( n - (t - 1) \), and \( d(\bar{x}, \bar{y}) \geq n - t + 1 \). It remains to note that by Lemma 2 all codewords involve \( \binom{n}{t}/\binom{w}{t} - \binom{n-1}{t-1}/\binom{w-1}{t-1} \) nonzero symbols. We conclude that \( \{\bar{x^1, \ldots, \bar{x^M}\} \) can be treated as an \( (n, \binom{n}{t}/(n-w), n-t+1; w)_q \) code for every \( q \geq \binom{n-1}{t-1}/\binom{w-1}{t-1} + 1 \).
It remains to show the necessity of a Steiner system. It follows from Proposition 1(3), because the supports of the codewords of an \((n, \binom{n}{t}/\binom{k}{t}, 2k-2t+2; k)\_2\) code, as well as the complements of the codeword supports of an \((n, \binom{n}{t}/\binom{k}{t}, 2k-2t+2; n-k)\_2\) code \((w = n-k\) and \(s = n-2k+t\) in Proposition 1(3)), form an \(S(t, k, n)\).

Remark 1. The special cases \(w = t\) and \(n-w = t\) of Theorem 3 (cases (i) and (ii), respectively) correspond to [17, Theorem 16], where the code distance is \(w+1\). If \(w > t\) and \(n-w > t\), then the code distances in Theorem 3 are larger than \(w+1\).

The bounds for \(q\) given in Theorem 3 are not best possible. For each \(t, k,\) and \(n\) such that there is a Steiner system \(S(t, k, n)\), we denote

- by \(q'_0(t, k, n)\) the smallest \(q\) such that an \((n, \binom{n}{t}/\binom{k}{t}, 2k-t+1; k)\_q\) code exists
- by \(q''_0(t, k, n)\) the smallest \(q\) such that an \((n, \binom{n}{t}/\binom{k}{t}, n-t+1; n-k)\_q\) code exists

We emphasize that in the notation \(q'_0(t, k, n)\) the parameter \(k\) coincides with the weight of the corresponding constant-weight codes; that is why we will often see \(w\) at the place of that parameter. The situation is different with \(q''_0(t, k, n)\), where the corresponding weight is \(n-k\). From Theorem 3 we get the following bounds for \(q'_0\) and \(q''_0\), which will be called trivial and can be further improved.

**Corollary 1.** \(q'_0(t, w, n) \leq \binom{n-1}{t-1} + 1; \quad q''_0(t, k, n) \leq \binom{n}{t} - \binom{n-1}{t-1} + 1.\)

Below, we find some values of \(q'_0\) and \(q''_0\) for small \(t\), improve the trivial bound for arbitrary \(t\), and consider in details some concrete parameter collections \((t, w, n)\).

### 4.1. Bounds for \(q'_0\): \(t = 2\) and \(t = 3\)

**Theorem 4.** If there exists an \(S(2, w, n)\), then \(q'_0(2, w, n) = \frac{n-1}{w-1} + 1.\)

**Proof.** According to Corollary 1, \(q'_0(2, w, n) \leq \frac{n-1}{w-1} + 1.\)
On the other hand, assume we have an \((n, \frac{n(n-1)}{w(w-1)}, 2w - 1; w)_q\) code \(C\) constructed from \(S(2, w, n)\). Take any two codewords \(\bar{x}, \bar{y}\) in \(C\). By the code parameters, \(wt(\bar{x}) = wt(\bar{y}) = w\) and \(d(\bar{x}, \bar{y}) \geq 2w - 1\). Let
\[
k = |\{i \in [n] \mid x_i = y_i \neq 0\}|.
\]
We have \(2w - 1 \leq d(\bar{x}, \bar{y}) \leq 2(w - k)\); hence \(k = 0\). By Lemma 2 at any position, there are \(\frac{n-1}{w-1}\) codewords in \(C\) that have a non-zero symbol at this position. From \(k = 0\), we see that all those non-zero symbols are distinct. Therefore, \(q'_0(2, w, n) \geq \frac{n-1}{w-1} + 1\).

**Remark 2.** For \(t = w = 2\), we have \(q'_0(2, 2, n) = n\); this case was considered in [17, Corollary 11], see also [11, Theorem 9], where the maximum cardinality of an \((n, \cdot, 3; 2)_q\) code was determined for any \(q\) and \(n\). Diameter perfect codes with \(t = 2\) and \(w = 3\) occur in [10, Theorem 6.6] as a part of the classification of optimal \((n, \cdot, 5; 3)_q\) codes.

**Example 1.** From the Steiner system \(S(2, 3, 7)\), with the blocks
\[
\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\},
\]
we can construct the following \((7, 7; 5; 3)_q\) code, \(q \geq 4:\)
\[
\{1110000, 2001100, 3000011, 0202020, 0300202, 0023003, 0030330\}.
\]
The anticode \(A'_q(7, 3, 2)\) has diameter 4 and size \(5(q - 1)^3\). It follows from the existence of the code that \(A'_q(7, 3, 2)\) is a maximum diameter-4 anticode of length 7 and weight 3 over \(\mathbb{Z}_q\), \(q \geq 4\).

**Theorem 5.** If there exists an \(S(3, w, n)\) system \(S\), then
\[
q'_0(3, w, n) = \min_{S \in S(3, w, n)} \max_{D \in S'} \chi(D) + 1,
\]
where \(S'\) is the set of all \(S(2, w - 1, n - 1)\) derived from \(S\).

**Proof.** The upper bound \(q'_0(3, w, n) \leq \min_{S \in S(3, w, n)} \max_{D \in S'} \chi(D) + 1\) is proved in Theorem 6 below for more general settings. It remains to show the lower bound. Assume we have an \((n, \binom{n}{3}/\binom{w}{3}, 2w - 2; w)_q\) code \(C\). From the code distance, we see that the supports of two different codewords cannot intersect in more than 2 points, and from the code cardinality we conclude that the
codeword supports form an $S(3, w, n)$. Next, if two different codewords $\bar{x}$ and $\bar{y}$ have the same nonzero value in some position $i$, then $d(\bar{x}, \bar{y}) = 2w - 2$ and the corresponding two blocks supp($\bar{x}$), supp($\bar{y}$) intersect in only one point, $i$. It follows that supp($\bar{x}$)$\setminus\{i\}$, supp($\bar{y}$)$\setminus\{i\}$ are disjoint blocks of the derived $S(2, w - 1, n - 1)$ system $D$ on the point set $[n]\setminus\{i\}$. Thus, if we now color the blocks of $D$ with the value of the corresponding codeword in the position $i$, the coloring will be proper for the minimum-distance graph of $D$. It follows that $q \geq \chi(D) + 1$. \hfill $\Box$

**Corollary 2.** If there is an $S(3, w, n)$ such that each its derived $S(2, w - 1, n - 1)$ is resolvable, then it holds

$$q'_0(3, w, n) = \frac{n - 2}{w - 2} + 1.$$  \hfill (3)

**Proof.** For a resolvable $S(2, w - 1, n - 1)$ system $D$, we have

$$\chi(D) = \frac{\binom{n-1}{2}}{\binom{w-1}{2}} = \frac{n - 2}{w - 2},$$

and (2) turns to (3). \hfill $\Box$

**Remark 3.** The case $t = w = 3$ was solved in [17, Theorem 17]: if $n$ is odd then $q'_0(3, 3, n) = n - 1$, and $q'_0(3, 3, n) = n$ if even. The corresponding code parameters are also found in [7], as a part of the classification of optimal constant-weight codes with weight 3.

Trivially, all $S(t - 1, w - 1, n - 1)$ systems derived from a 2-resolvable $S(t, w, n)$ are resolvable. It is proved in [3] that there are 2-resolvable $S(3, 4, 4^m)$ for each $m \geq 1$. In [30], 2-resolvable Steiner systems $S(3, 4, n)$ are constructed for $n = 2 \cdot p^m + 2$, $p \in \{7, 31, 127\}$, $m \geq 1$. Moreover, from Keevash’s theory [21] we know that for given $w$ and $n$ large enough, 2-resolvable $S(3, w, n)$ systems exist if some divisibility conditions are satisfied. In particular, for $w = 4$, those conditions are equivalent to $n \equiv 4 \mod 12$. However, for Corollary 2 the condition $n \equiv 4 \mod 12$ (as well as the 2-divisibility of $S(3, 4, n)$) is not necessary; for example all $S(2, 3, 9)$ derived from $S(3, 4, 10)$ are resolvable, while $10 \not\equiv 4 \mod 12$ and $S(3, 4, 10)$ is not 2-resolvable.

**Corollary 3.** If $n = 4^m$ or $n = 2 \cdot p^m + 2$, $p \in \{7, 31, 127\}$, $m \in \{1, 2, \ldots\}$, or $n$ large enough satisfying $n \equiv 4 \mod 12$, or $n = 10$, then $q'_0(3, 4, n) = n/2$.  

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Remark 4. In the case of $S(3, 4, n)$, (3) turns to $q_0(3, 4, n) = n/2$. In [34, Lemma 2.12], 2-resolvable $S(3, 4, n)$ are used to construct optimal $(n, (q - 1)n(n - 1)/12, 6; 4)_q$ codes for every $q$ in $\{3, \ldots, n/2\}$. For $q = n/2$, this gives the same result as in Corollary 3 except the case $n = 10$. However, even for $w = 4$, Corollary 2 can be more general, see Conjecture 1 in the Conclusion.

4.2. Bounds for $q'_0$: arbitrary $t$

In this section, we generalize Theorem 5 to arbitrary $t$. It is hard to expect that the generalization gives a good evaluation of the real value of $q'_0$ for $t > 3$; however, it is important to conclude that the trivial bound from Corollary 1 is not tight. It is also a little step towards the research direction pointed in [17] as Problem 4.

Theorem 6. If there exists an $S(t, w, n)$ system $S$, $t \geq 3$, then

$$q'_0(t, w, n) \leq \frac{(n-1)}{(t-1)} - \frac{(n-t+2)}{2}$$

$$\min_{s \in S(t, w, n)} \max_{D \in S'} \chi(D) + 1,$$  

where $S'$ is the set of all $S(2, w-t+2, n-t+2)$ derived from $S$.

Proof. Assume there exists an $S(t, w, n)$ system $S$ with point set $[n]$ and block set $B = \{\Delta_1, \ldots, \Delta_M\}$, $M = \binom{n}{w}/\binom{w}{t}$. For convenience, we consider all operations over the point set modulo $n$. Denote by $\Gamma_i$ the set $[i - t + 3, i]$ of $t - 2$ consequent (modulo $n$) points, by $S_i$ the $S(2, w-t+2, n-t+2)$ system on the point set $[n]\setminus \Gamma_i$ derived from $S$, and by $\lambda_i$ its chromatic number $\chi(S_i)$, $i = 1, \ldots, n$. Next, we denote by $B_i$ the subset of $B$ consisting of all blocks that include $\Gamma_i$. By the definition of $\chi(S_i)$, we can partition $B_i$ into $\lambda_i$ disjoint subsets, cells:  

$$B_i = B^1_i \cup B^2_i \cup \ldots \cup B^\lambda_i,$$

such that the intersection of any two blocks from the same cell is $\Gamma_i$. Let $\bar{x}_j = (x^j_1, \ldots, x^j_n)$, $j = 1, \ldots, M$, be the length-$n$ word defined by assigning the exact value in the $i$th position, $i = 1, \ldots, n$:  

$$x^j_i = \begin{cases} 0, & \text{if } i \notin \Delta_j, \\ s, & \text{if } i \in \Delta_j \in B^s_i \text{ for some } s \in [\lambda_i], \\ \lambda_i + |\{l \in \{1, \ldots, j\} : i \in \Delta_l \notin B_i\}|, & \text{if } i \in \Delta_j \notin B_i. \end{cases}$$
We see that \(\text{supp}(\bar{x}^j) = \Delta_j\); in particular, \(\text{wt}(\bar{x}^j) = w\). Any two different codewords \(\bar{x}^j\) and \(\bar{x}^l\) coincide either only in positions where both have zeros, or also in positions where they both have nonzeros. In the first case, the distance between \(\bar{x}^j\) and \(\bar{x}^l\) is at least \(2w - t + 1\), as desired. Consider the second case. In that case, \(\bar{x}^j\) and \(\bar{x}^l\) both have a positive value \(s\) in the \(i\)th position, i.e., \(\Delta_j, \Delta_l\) are in the same cell \(B_i^s\), which means \(\Delta_j \cap \Delta_l = \Gamma_i\). It is easy to see that \(\bar{x}^j\) and \(\bar{x}^l\) coincide only in the \(i\)th position and in the positions where both have zero; thus \(d(\bar{x}^j, \bar{x}^l) = 2w - t + 1\). Besides, according to the definition rule of \(\bar{x}^j\), we know that all codewords involve

\[
\left(\frac{n-1}{t-1}\right) / \left(\frac{w-1}{t-1}\right) - \left(\frac{n-t+2}{2}\right) / \left(\frac{w-t+2}{2}\right) + \max_{D \in S} \chi(D) \tag{4}
\]

nonzero symbols, and \(\bar{x}^1, \ldots, \bar{x}^M\) form an \((n, n^t / (t^w); 2w - t + 1; w)_q\) code for any \(q\) larger than \([4]\).

4.3. Bounds for \(q'_0\): S(4,5,11), SQS(14), and SQS(20)

In this section, we evaluate \(q'_0(4,5,11)\), related to a unique \(S(4,5,11)\), derived from the small Witt design \(S(5,6,12)\) \([33]\), and find \(q'_0(3,4,14)\) and \(q'_0(3,4,20)\).

Proposition 2. It holds \(9 \leq q'_0(4,5,11) \leq 11\).

Proof. Computing \(\chi(S')\) for the unique (up to equivalence) \(S(3,4,10)\), which is derived from \(S(4,5,11)\), gives \(q'_0(4,5,11) \geq \chi(S') + 1 = 9\).

An example of a \((11,66,7;5)_{11}\) code, which shows that \(q'_0(4,5,11) \leq 11\), is the cyclic code with representatives \(000a100615, 00080901087, 00103009406, 00022013005, 00070040726, 0000035480a\) (each representative generates 11 codewords by cyclic shifting).

Proposition 3. The values \(q'_0(3,4,14)\) and \(q'_0(3,4,20)\) equal 9 and 11, respectively.

Proof. There are two STS(13), up to permutation of points, and each of them has chromatic number 8 (a computational result). By Theorem \([5]\), we have \(q'_0(3,4,14) = 8 + 1\).

The number of blocks in an STS(19) \(S'\) is 57, and the maximum number of mutually disjoint blocks is \(\lceil 19/3 \rceil = 6\). So, \(\chi(S') \geq \lceil 57/6 \rceil = 10\) and hence \(q'_0(3,4,20) \geq 11\) by Theorem \([5]\).
Consider the cyclic SQS(20) (II.A.1) from [27]. It consists of the blocks 
\{1, 6, 11, 16\}, \{1, 3, 11, 13\}, \{1, 5, 11, 15\}, \{1, 4, 9, 17\}, \{1, 4, 7, 14\}, \{1, 2, 3, 12\}, 
\{1, 5, 7, 10\}, \{1, 3, 7, 8\}, \{1, 3, 4, 10\}, \{1, 6, 7, 9\}, \{1, 3, 9, 14\}, \{1, 2, 6, 10\}, 
\{1, 2, 9, 18\}, \{1, 2, 7, 13\}, \{1, 2, 5, 14\}, \{1, 4, 6, 8\} and all their cyclic shifts 
\(a_1, a_2, a_3, a_4 \rightarrow a_1 + i, a_2 + i, a_3 + i, a_4 + i\), where + is modulo 20. 
Because of the cyclicity, all derived STS are isomorphic, and we consider the 
derived STS \(S'\) on the point set \(\{1, \ldots, 19\}\). The block set of \(S'\) is partitioned 
into 10 cells with mutually disjoint blocks in each cell:

\[
\begin{align*}
\{\{1, 2, 11\}, \{3, 5, 7\}, \{4, 10, 14\}, \{6, 17, 19\}, \{8, 12, 15\}, \{13, 16, 18\}\}, \\
\{\{1, 3, 15\}, \{2, 10, 12\}, \{5, 13, 17\}, \{6, 11, 18\}, \{7, 16, 19\}, \{8, 9, 14\}\}, \\
\{\{1, 4, 13\}, \{2, 15, 18\}, \{3, 12, 19\}, \{5, 6, 8\}, \{7, 14, 17\}, \{9, 10, 11\}\}, \\
\{\{1, 5, 9\}, \{2, 8, 13\}, \{3, 14, 18\}, \{4, 7, 12\}, \{6, 10, 16\}, \{11, 15, 17\}\}, \\
\{\{1, 6, 12\}, \{2, 4, 17\}, \{3, 8, 16\}, \{5, 10, 15\}, \{9, 18, 19\}, \{11, 13, 14\}\}, \\
\{\{1, 7, 18\}, \{2, 5, 16\}, \{3, 10, 17\}, \{4, 8, 19\}, \{6, 14, 15\}, \{9, 12, 13\}\}, \\
\{\{1, 8, 17\}, \{2, 14, 19\}, \{3, 6, 13\}, \{4, 5, 18\}, \{7, 9, 15\}, \{11, 12, 16\}\}, \\
\{\{2, 6, 7\}, \{3, 4, 11\}, \{5, 12, 14\}, \{8, 10, 18\}, \{9, 16, 17\}, \{13, 15, 19\}\}, \\
\{\{1, 10, 19\}, \{2, 3, 9\}, \{4, 15, 16\}, \{7, 8, 11\}, \{12, 17, 18\}\}, \\
\{\{1, 14, 16\}, \{4, 6, 9\}, \{5, 11, 19\}, \{7, 10, 13\}\}.
\end{align*}
\]

So, \(\chi(S') = 10\) and \(q_0'(20, 4, 3) = \chi(S') + 1 = 11.\)

\[\square\]

### 4.4. Bounds for \(q_0''\)

The trivial upper bound on \(q_0''(t, k, n)\) in Corollary 11 comes from a construction 
where in each column of the code matrix all nonzero symbols are distinct. 
Merging some symbols results in decreasing the distance between some 
codewords. However, this does not always lead to decreasing the code distance, 
and in general a code with the same parameters and a smaller number of 
symbols used can be found. This is not the case if \(t = 1\) because the distance 
between any two codewords in the trivial construction already coincides with 
the minimum distance of the code.

**Corollary 4.** \(q_0''(1, k, n) = n/k, \text{ i.e., an } (n, n/k, n; n − k)\)

**Corollary.** An \((n, n/k, n; n − k)\) code exists if and only if \(k\) divides \(n\) (necessary and sufficient condition for the existence of \(S(1, k, n)\)) and \(q \geq n/k\).

In the following proposition, we estimate the number of symbol pairs that 
can be merged if \(t > 1\). The formula there looks a bit complicated, but as we 
see from the corollaries below, for special cases it becomes much simpler.
Proposition 4. If there exists an $(n, M, n-t+1; w)_q$ code $C$, where $w = n-k$ and $M = \binom{n}{t}/\binom{k}{t}$, then

(i) $C$ is a diameter perfect code in $J_q(n, w)$;

(ii) the supports of the codeword of $C$ are the complements of the blocks of an $S(t, k, n)$ system $S$;

(iii) it holds

$$(q - 1) \left( \frac{a}{2} \right) + ba \leq \left\lfloor \tilde{P} \right\rfloor,$$

where

$$a = \left[ R/(q-1) \right], \quad b = R - a(q-1),$$

$$R = \left( \begin{array}{c} n \\ t \end{array} \right) \binom{k}{t} - \left( \begin{array}{c} n-1 \\ t-1 \end{array} \right) \binom{k-1}{t-1},$$

$$\tilde{P} = \frac{M}{2n} \sum_{i=0}^{t-2} (t-1-i) \Lambda_i(t, k, n),$$

and $\Lambda_i(t, k, n) = \binom{k}{i} \lambda_{i,k-i}$ is from Lemma 3;

(iii') in particular, $q - 1 \geq R - \left\lfloor \tilde{P} \right\rfloor$.

Proof. (i) Since $C$ meets the code–anticode bound with the anticode $A'_q(n, n-k, t)$, it is diameter perfect.

(ii) The cardinality of the code coincides the number of blocks in $S(t, k, n)$. Each codeword has exactly $k$ zeros. Each two codewords have at most $t - 1$ common position with a zero. The claim is now straightforward.

(iii) The code matrix is an $M \times n$ matrix. Each row contains $k$ zeros and $n - k$ nonzeros. Each column contains $\binom{n-1}{t-1}/\binom{k-1}{t-1}$ zeros (Lemma 2) and

$$R = \left( \begin{array}{c} n \\ t \end{array} \right) \binom{k}{t} - \left( \begin{array}{c} n-1 \\ t-1 \end{array} \right) \binom{k-1}{t-1} = \frac{(n-1)!(k-t)!(n-k)}{(n-t)!k!}$$

nonzeros. The set of zero positions of each row corresponds to some block of $S$.

We will say that we have a collision $(a, b)$ if two different cells $a, b$ in the same column contain the same nonzero value. Let us evaluate the maximum number of collisions. If two rows correspond to blocks with $i$ common positions, then they have at most $t - 1 - i$ collisions because the code distance
is \( n - t + 1 \). We denote the number of pairs of such blocks by \( P_i \). Utilizing Lemma 3 we find that \( P_i = \frac{M}{2} \cdot \Lambda_i(t, k, n) \). So, the number of collisions does not exceed \( P = \sum_{i=1}^{t-1} (t - 1 - i) P_i \). The average number of collisions in one column does not exceed \( \tilde{P} \), where \( \tilde{P} = P/n \) and there is a column with at most \( \lfloor \tilde{P} \rfloor \) collisions.

On the other hand, the number of nonzeros in each column is \( R \), while the number of different nonzero symbols is \( q - 1 \). With those constants, the minimum number of collisions is possible if each nonzero symbol occurs \( a \) or \( a + 1 \) times (see Lemma 4), where \( a = \lfloor R/(q - 1) \rfloor \). This gives the left part of the inequality in (iii). (iii') coincides with (iii) if \( a = 1 \) (i.e., when \( \tilde{P} \) is not more than the half of the number of ones in a column of the code matrix); otherwise (iii') is weaker.

Corollary 5. In the case \( t = 2 \), Proposition 4 holds with

\[
R = \frac{(n - 1)(n - k)}{k(k - 1)} \quad \text{and} \quad \tilde{P} = \frac{(n - 1)((n - k^2)(n - 1) + k(k - 1)^2)}{2k^2(k - 1)^2}.
\]

Proof. Substituting \( t = 2 \), we get

\[
M = \frac{n(n - 1)}{k(k - 1)}, \quad \text{and} \quad \tilde{P} = \frac{M}{2n} \cdot \lambda_{0,k}.
\]

From the definition of \( \lambda_{i,j} \) in Lemma 3, we find that for \( j \geq 1 \)

\[
\lambda_{i,j} = \lambda_{i,j-1} - \lambda_{i+1,j-1} = 0 \quad \text{if} \quad i \geq t;
\]

\[
\lambda_{t-1,j} = \lambda_{t-1,j-1} - \lambda_{t,j-1} = \frac{n-t+1}{k-t+1} - 1;
\]

\[
\lambda_{t-2,j} = \lambda_{t-2,j-1} - \lambda_{t-1,j-1} = \frac{2n+t^2(n-t+1)}{(k-t+2)(k-t+1)} - j \cdot \frac{n-t+1}{k-t+1} - (j - 1).
\]

Substituting \( t = 2 \) and \( j = k \), we get

\[
\lambda_{0,k} = \frac{n(n - 1)}{k(k - 1)} - k \cdot \frac{n - 1}{k - 1} - (k - 1) = \frac{n(n - 1) - k^2(n - 1) - k(k - 1)^2}{k(k - 1)}.
\]

It is easy to check now that \( \frac{M}{2n} \lambda_{0,k} \) is exactly the expression for \( \tilde{P} \) from the claim of the corollary. \( \square \)
The next claim is about Steiner systems related to affine planes.

**Corollary 6.** If $k$ is a prime power, then $q''_0(2, k, k^2) = k^2 - \lfloor \frac{k-1}{2} \rfloor$.

**Proof.** For $n = k^2$, $\bar{P}$ turns to $\frac{k^2-1}{2k}$, and $[\bar{P}] = \lfloor \frac{k}{2} - \frac{1}{2k} \rfloor = \lfloor \frac{k-1}{2} \rfloor$. At the same time, $\frac{(n-1)(n-k)}{k(k-1)} = k^2 - 1$. We see that (iii') turns to $q \geq k^2 - \lfloor \frac{k-1}{2} \rfloor$.

Next, we are going to construct a $(k^2, k^2+k, k^2-1; k^2-k)_q$ code such that $q = k^2 - \lfloor \frac{k-1}{2} \rfloor$ and the supports of codewords form an $S(2, k, k^2)$ system. For the role of the system, we take the affine plane over the finite field $\mathbb{F}_k$ of order $k$. The points are the pairs from $\mathbb{F}_k^2$ (to treat the points as indices, we identify them with the integers from $[k^2]$, in an arbitrary fixed order); the lines (blocks of the $S(2, k, k^2)$) are sets of points $(x, y)$ satisfying one of the equations $x + ay = b$, where $a, b \in \mathbb{F}_w$, or $y = b$, where $a \in \mathbb{F}_w$. We want the code matrix to have $k^2 \cdot \lfloor \frac{k-1}{2} \rfloor$ collisions in total, each column to have $\lfloor \frac{k-1}{2} \rfloor$ pairwise disjoint collisions, and each two rows to have no more than one collision. We consider two cases, depending on the parity of $k$.

**Odd** $k$, $\lfloor \frac{k-1}{2} \rfloor = \frac{k-1}{2}$. Let $S$ be a subset of $\mathbb{F}_k$ of size $\frac{k-1}{2}$ such that $0, 1 \not\in S$ and $|S \cap \{s, -s\}| = 1$ for every $s$ from $\mathbb{F}_k^\times$.

For each $x_0, y_0$ from $\mathbb{F}_k$ and $s$ from $S$, we make a collision in the column corresponding to the point $(x_0, y_0)$ and the lines

$$\{ (x, y) : x + (y_0 + s)y = x_0 + (y_0 + s)y_0 - 1 \}, \quad (5)$$

$$\{ (x, y) : x + (y_0 + s)y = x_0 + (y_0 + s)y_0 - 1 + s \}. \quad (6)$$

We now check the required properties.

- First of all, we see that the point $(x_0, y_0)$ does not lie on any of the two lines (5), (6); hence the code matrix has nonzeros in the corresponding two cells and we can make a collision there.

- Next, each point corresponds to $|S| = \lfloor \frac{k-1}{2} \rfloor$ collisions and the collisions are pairwise disjoint (for different $s$, the coefficients at $y$ are different, and hence the lines are different and the corresponding rows are different too). So, after making all these collisions, the number of symbols used in each column becomes $k^2 - \lfloor \frac{k-1}{2} \rfloor$, as required.

- Finally, for any two parallel lines $x+ay = b$ and $x+ay = c$, we uniquely find $s$ from $s \in \{b - c, c - b\}$, then $y_0$ from $y_0 + s = a$, and $x_0$ from $c$ or $b$. This means that the rows corresponding to such two lines have exactly one collision.
Additionally, any two intersecting lines have no collisions, and we conclude that the code distance is \( k^2 - 1 \), as required.

Even \( k \), \( \lfloor \frac{k-1}{2} \rfloor = \frac{k}{2} - 1 \). Let \( S_{x_0}, S_{x_0} \subseteq \mathbb{F}_k \) be defined for each \( x_0 \) from \( \mathbb{F}_k \) in such a way that:

(i) \( 0, 1 \notin S_{x_0} \),

(ii) for every \( s \) from \( \mathbb{F}_k \setminus \{0, 1\} \), it holds \( |S_{x_0} \cap \{s, s + 1\}| = 1 \),

(iii) if \( s \in S_{x_0} \), then \( s \notin S_{x_0 + s} \).

Let us show that such \( S_{x_0}, x_0 \in \mathbb{F}_k \), exists. For every pair \( \{s, s + 1\} \), \( s \in \mathbb{F}_k \setminus \{0, 1\} \), we divide \( \mathbb{F}_k \) into \( k/4 \) quadruples of form \( \{z, z + s, z + 1, z + s + 1\} \). Then, for each such quadruple, we include \( s \) in \( S_z \) and \( S_{z + 1} \), but not in \( S_{z + s} \) or \( S_{z + s + 1} \), and include \( s + 1 \) in \( S_{z + s} \) and \( S_{z + s + 1} \), but not in \( S_z \) or \( S_{z + 1} \). This rule depends on the choice of the representative \( z \) in each quadruple, but once the representatives are chosen, it uniquely specifies for each \( x_0 \), which element of \( \mathbb{F}_k \setminus \{0, 1\} \) belongs to \( S_{x_0} \) and which element does not belong. Moreover, we see that (ii) and (iii) are satisfied automatically and we can require (i).

Now we proceed similarly to the case of odd \( k \), but with \( S_{x_0} \) instead of \( S \). For each \( x_0, y_0 \) from \( \mathbb{F}_k \) and \( s \) from \( S_{x_0} \), we make a collision in the column corresponding to the point \( (x_0, y_0) \) and lines (5), (6). The properties are also checked similarly, except the last one, verifying of which differs a bit:

- For any two parallel lines \( x + ay = b \) and \( x + ay = c \), we uniquely find \( s \) from \( s = b - c \), then \( y_0 \) from \( y_0 + s = a \) and finally \( x_0 \) from \( x_0 + (y_0 + s)y_0 - 1 \in \{c, b\} \) and (iii).

\[ \square \]

**Remark 5.** We did not make any collisions with lines of form \( y = b \). Potentially, they can also be used to decrease the number of used symbols in some columns, but it is not enough to decrease the alphabet in all columns.

Another special case \( S(2, k, k^2 - k + 1) \) is known as *projective plane* of order \( k - 1 \). In a projective plane, every two blocks (lines) have a common point. It follows that in a column of the code matrix all nonzero symbols are distinct (alternatively, one can see from Corollary 5 that \( P = n\overline{P} = 0 \) if \( n = k^2 - k + 1 \)).

**Corollary 7.** If there exists a projective plane of order \( s \), then

\[ q''_0(2, s + 1, s^2 + s + 1) = s^2 + 1. \]
5. Conclusion

In this paper, we explore two new classes of diameter perfect constant-weight nonbinary codes and estimated the smallest values $q_0'$ and $q_0''$ of the alphabet size for which such codes exist. Further studying these values, for special parameters, is related to some problems in the design theory, regarding some weak form of resolvability. Below, we formulate those problems in the form of conjectures. Another interesting connection of weakly resolvable designs with optimal non-binary constant-weight codes is shown in [5].

It is well known that resolvable STS($v$) exist if and only if $v \equiv 3 \mod 6$ [28]. To show that $q_0'(3, 4, n) = n/2$, (Corollary 2) we need the following:

**Conjecture 1.** If $n \equiv 4 \mod 6$, then there is an SQS($n$) such that all its derived STS are resolvable.

Up to now, we only know that the conjecture is true for $n = 10$ and in the cases when 2-resolvable SQS($n$) are known to exist: for $n = 4^m$ [3]; for $n = 2 \cdot p^m + 2$, $p \in \{7, 31, 127\}$ [30]; for sufficiently large $n$, $n \equiv 4 \mod 12$ [21].

If $v \equiv 1 \mod 6$, then an STS($v$) cannot be resolvable, and the number of mutually disjoint blocks in such a system $S'$ does not exceed $(v-1)/3$. So, $\chi(S') \geq \lceil \frac{v(v-1)}{6} \rceil = \frac{v+1}{2}$.

**Conjecture 2.** If $20 \leq n \equiv 2 \mod 6$, then there is an SQS($n$) such that all its derived STS have chromatic number $n/2$, i.e., can be partitioned into $n/2$ partial parallel classes (equivalently, $q_0'(3, 4, n) = n/2 + 1$).

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Data availability**

No data was used for the research described in the article.
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