Discrete dynamical systems associated with the configuration space of 8 points in $\mathbb{P}^3(\mathbb{C})$

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Abstract

A 3 dimensional analogue of Sakai’s theory concerning the relation between rational surfaces and discrete Painlevé equations is studied. For a family of rational varieties obtained by blow-ups at 8 points in general position in $\mathbb{P}^3$, we define its symmetry group using the inner product that is associated with the intersection numbers and show that the group is isomorphic to the Weyl group of type $E_7^{(1)}$. By normalizing the configuration space by means of elliptic curves, the action of the Weyl group and the dynamical system associated with a translation are explicitly described. As a result, it is found that the action of the Weyl group on $\mathbb{P}^3$ preserves a one parameter family of quadratic surfaces and that it can therefore be reduced to the action on $\mathbb{P}^1 \times \mathbb{P}^1$.

1 Introduction

Relations between Painlevé equations and rational surfaces were first studied by Okamoto[12]. He showed that for each Painlevé equation, by elimination of the singularity of the equation, the solutions can be regularly extended into a family of rational surfaces. Such a family of rational surfaces is called the space of initial conditions for the Painlevé equation. Conversely, it is clarified by Saito and Takano et al. [16, 14] that for a given space of initial conditions the Hamilton system of a Painlevé equation can be determined.

Since the singularity confinement method was introduced by Grammaticos et al. [6], the discrete Painlevé equations have been studied extensively ([13] for example). Emphasizing the fact that each discrete Painlevé equation preserves a family of rational surfaces, Sakai constructed the discrete Painlevé equations from families of rational surfaces (called generalized Halphen surfaces) and subsequently classified them. Such a family of rational surfaces is called the space of initial conditions for that discrete Painlevé equation. A generalized Halphen surface can be seen to be isomorphic to a surface obtained by 9 blow ups from $\mathbb{P}^2$. Sakai’s classification also shows that the spaces of initial conditions for the discrete Painlevé equations include those for continuous ones. Furthermore, the largest symmetry arises when the 9 points are in general position; all other symmetries are in the case where the points are in some special position. Each Painlevé equation is obtained as a translation associated with the corresponding affine Weyl group (extended by the automorphisms of the associated Dynkin diagram). Moreover, if the space of initial conditions is that of a continuous one, its Weyl group coincides with the group of that equation’s Bäcklund transformations.

The aforementioned results all concern non-autonomous dynamical systems. There also exist some studies that deal with autonomous ones. For example, in the continuous case Adler and van Moerbeke have studied Painlevé manifolds [1] and in the discrete case
various authors have studied the relations between dynamical systems and the automorphism groups of manifolds ([4, 5] for example).

Sakai's procedure for describing discrete Painlevé equations is closely related to the studies on the Cremona isometry carried out by Coble et al. [2, 3] and these two approaches coincide in the case of 9 points in general position. Whereas in the case of points in general position the Weyl group is generated by the standard Cremona transformation and exchanges of the points, in the degenerate case its generators can be constructed by changing the blow down structures. Concerning this point one has to cite the pioneering research by Looijenga [10].

Dolgachev and Ortland also studied the case of 3 (or more) -dimensional rational varieties [3]: here (for example) the affine Weyl group of type $E_7^{(1)}$ appears in the case where 8 generic points in $\mathbb{P}^3$ are blown-up. If the number of points is less than 8 the Weyl group is finite and it is indefinite if the number of points is larger than 8. However, in the 3-dimensional case the action of each element of the Weyl group cannot be lifted to an isomorphism between rational varieties, obtained by blow-ups at some points. Dolgachev and Ortland call such a map a pseudo-isomorphism. Note that although they described the roots of the Weyl group by means of a Neron-Severi bi-lattice, the geometric origin of the dual bases were not clear.

In this paper we study the symmetry, the normalization of the configuration space and the associated discrete dynamical systems for the family of rational varieties obtained by blow-up at 8 ordered points in general position in $\mathbb{P}^3$.

In section 2, we reconstruct the argument of Dolgachev and Ortland. We consider birational automorphisms of the family of varieties such that (i) each of them acts as an automorphism on the configuration space (ii) for rational varieties on the configuration space it preserves the “inner product” of the Picard group Pic($X$). Here, the inner product is defined by using the intersection numbers and the canonical divisor $K_X$ as $(D, D') := D \cdot D' \cdot (-\frac{1}{2}K_X)$ for $D, D' \in \text{Pic}(X)$. It is shown that the resulting symmetry group is the Weyl group of type $E_7^{(1)}$. This group coincides with that of Dolgachev and Ortland.

In section 3, the normalization of the configuration space is discussed. Although there is a straightforward normalization, using this it is difficult to describe the action of the Weyl group and to see the properties of the resulting dynamical systems. In this paper we therefore use a normalization in terms of elliptic curves. Quadratic surfaces passing through the 8 points we consider form at least one parameter family. Here the 8 points are on the intersection curve of the pencil of surfaces. Normalizing the pencil, one obtains a normalization of the configuration space.

In section 4, we describe the action of the Weyl group obtained in section 2 in normalized coordinates. In order to calculate the concrete action we apply a 3-dimensional analogue of the period map which is introduced by Looijenga-Sakai for surfaces.

In section 5, we construct a birational dynamical system in $\mathbb{P}^3$ by using the action obtained in section 4. Such systems are obtained corresponding to translations associated to the Weyl group. We describe one of them explicitly.

In section 6, it is shown that the action of the Weyl group preserves each element of the pencil of quadratic surfaces used for the normalization and that it can therefore be reduced to an action on $\mathbb{P}^1 \times \mathbb{P}^1$. The reduced action of the Weyl group of type $E_7^{(1)}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ coincides with the action of a sub-group of the Weyl group of type $E_8^{(1)}$, which is the symmetry of the family of (the most) general Halphen surfaces.

Section 7 is devoted to conclusions and discussions.
2 Symmetry

Let $X(4,8)$ denote the configuration space of ordered 8 points in $\mathbb{P}^3(\mathbb{C})$ such that every 4 points are not on the same plane:

$$PGL(4,\mathbb{C}) \begin{array}{c}
\left\{ \begin{pmatrix} x_1 & x_2 & \cdots & x_8 \\ y_1 & y_2 & \cdots & y_8 \\ z_1 & z_2 & \cdots & z_8 \\ w_1 & w_2 & \cdots & w_8 \end{pmatrix} \in (\mathbb{C})^8 \right. \\
\text{every 4 } \times \text{ 4 minor determinant} \\
is \text{ nonzero} \end{array} \bigg/ (\mathbb{C}^*)^8, (1)$$

where two configurations are identified if one can be transformed to the other by a projective transformation. We also denote the 3-dimensional rational variety obtained by successive blowing-up at distinct 8 points $P_i(x_i : y_i : z_i : w_i)$ by $X_{P_1,\ldots,P_8}$ (or simply by $X$) and the family of all $X_{P_1,\ldots,P_8}$'s, where $\{P_1,\ldots,P_8\} \in X(4,8)$, by $\{X_{P_1,\ldots,P_8}\}$. $X(4,8)$ is called the parameter space.

Let $Pic(X)$ be the Picard group of the variety $X$ (the additive group of isomorphism classes of invertible sheaves $\simeq$ the additive group of linear equivalence classes of divisors).

We have

$$Pic(X) = ZE \oplus ZE_1 \oplus ZE_2 \oplus ZE_3 \oplus ZE_4 \oplus ZE_5 \oplus ZE_6 \oplus ZE_7 \oplus ZE_8,$$

$$K_X = -4E + 2(E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8),$$

where $E$ denotes the total transform of the divisor class of the plane in $\mathbb{P}^3$ and $E_i$ denotes the total transform of the exceptional divisor generated by blow-up at $P_i$. We define $\delta$ as

$$\delta := \frac{-1}{2}K_X$$

for the following argument.

We consider the group (written as $Gr(\{X\})$) of birational transformations on the family $\{X_{P_1,\ldots,P_8}\}$ such that i) $\varphi : X(4,8) \to X(4,8)$ is an automorphism; here, we denote $\varphi(\{P_1,\ldots,P_8\})$ by $\{P'_1,\ldots,P'_8\}$ ii) for any $\{P_1,\ldots,P_8\} \in X(4,8)$ the map $\varphi : X_{P_1,\ldots,P_8} \to X_{P'_1,\ldots,P'_8}$ is a birational map preserving the inner product of $Pic(X)$ which is defined by using the intersection numbers as $(D,D') := D \cdot D' - \frac{1}{2}K_X$ for $D, D' \in Pic(X)$, where $D, D', D'' \in Pic(X)$, $D \cdot D' \cdot D''$ denotes the intersection number $Pic(X) \times Pic(X) \times Pic(X) \to \mathbb{Z}$.

Theorem 2.1. $Gr(\{X\})$ is the affine Weyl group of type $E_7^{(1)}$.

Before the proof we describe some formulae for the intersection numbers. For elements in $Pic(X)$ the intersection numbers are given by

$$E \cdot E \cdot E = 1, \quad E \cdot E \cdot E_i = 0,$$

$$E \cdot E_i \cdot E_i = 0, \quad E_i \cdot E_i \cdot E_i = 1,$$

$$E_i \cdot E_j \cdot D = 0 \quad (i \neq j, \forall D \in Pic(X))$$

and their linear combinations. Hence the inner product is given by

$$(E, E) = 2, \quad (E, E_i) = 0, \quad (E_i, E_j) = -\delta_{i,j}$$

and their linear combinations.

Lemma 2.1. If a birational map $\varphi : X_{P_1,\ldots,P_8} \to X_{P'_1,\ldots,P'_8}$ preserves the inner product, then $\varphi_* : Pic(X_{P_1,\ldots,P_8}) \to Pic(X_{P'_1,\ldots,P'_8})$ is an isomorphism of lattices.
Proof.\ Note that $\varphi_*\ (= (\varphi^{-1})^*)$ can be considered to be a linear transformation to itself. Assume that there exists a nonzero divisor $D \in \text{Pic}(X_{P_1, \ldots, P_8})$ such that $\varphi_*(D) = 0$. Since $D$ is nonzero, there exists a divisor $D' \in \text{Pic}(X_{P_1, \ldots, P_8})$ such that $(D, D') \neq 0$. On the other hand, $\varphi_*(D) = 0$ and hence $(\varphi_*(D), \varphi_*(D')) = 0$, which contradicts the assumption that $\varphi$ preserves the inner product. \hfill $\Box$

By this lemma, each $\varphi_* : \text{Pic}(X_{P_1, \ldots, P_8}) \to \text{Pic}(X_{P'_1, \ldots, P'_8})$ is an isomorphism and preserves
a) the inner product;
b) the anti-canonical divisor $-K_X$, i.e. $\varphi_*(-K_{X_{P_1, \ldots, P_8}}) = -K_{X_{P'_1, \ldots, P'_8}}$, since $\varphi$ is birational;
c) the effectiveness of divisors.

We identify the lattices $\text{Pic}(X_{P_1, \ldots, P_8})'$s and call an automorphism of $\text{Pic}(X)$ which preserves a), b), c) a (3-dimensional) Cremona isometry.

**Proof of Theorem 2.1**

As in the 2-dimensional case, the theorem is proved by investigating the group of Cremona isometries and realizing the corresponding birational transformations.

Let $\varphi \in \text{Gr}(\{X\})$. Since $\varphi_*$ preserves $K_X$ and the inner product, it also preserves the orthogonal complement of $K_X$

$$Q(\alpha) := \langle \alpha_0, \alpha_1, \ldots, \alpha_7 \rangle_{\mathbb{Q}},$$

where

$$\alpha_0 = E_1 - E_2, \quad \alpha_1 = E_2 - E_3, \quad \ldots, \quad \alpha_6 = E_7 - E_8, \quad \alpha_7 = E - E_1 - E_2 - E_3 - E_4. \quad (5)$$

**Claim 2.1.** The basis $\langle \alpha_0, \alpha_1, \ldots, \alpha_7 \rangle$ of the linear space $K_X^\perp$ generates the lattice $\langle \alpha_0, \alpha_1, \ldots, \alpha_7 \rangle_{\mathbb{Z}}$, i.e.

$$\langle \alpha_0, \alpha_1, \ldots, \alpha_7 \rangle_{\mathbb{Q}} \cap \text{Pic}(X) = \langle \alpha_0, \alpha_1, \ldots, \alpha_7 \rangle_{\mathbb{Z}}$$

holds.

**Proof.** We show that the left hand side includes the right hand side. Let $a_0\alpha_0 + \cdots + a_7\alpha_7 \in \text{Pic}(X)$. Since the coefficient of $E_8$ is an integer, $a_6$ is also an integer. Since the coefficient of $E_7$ is an integer, $-a_5 + a_6$ is also an integer, and so is $a_5$. Along the same line all $a_i$'s are integers. \hfill $\Box$

From this claim, $\varphi_*$ is an automorphism of the sub-lattice $Q(\alpha)$ and preserves the inner product. The matrix defined by using the inner product as

$$(c_{i,j})_{i,j} := 2\langle \alpha_i, \alpha_j \rangle / (\alpha_i, \alpha_i) \quad (6)$$

is the affine Cartan matrix of type $E_7^{(1)}$. We denote the affine Weyl group generated by

$$r_j(\alpha) := r_{\alpha_j}(\alpha) = \alpha - 2\langle \alpha_j, \alpha \rangle / (\alpha_j, \alpha_j)\alpha_j \quad (\alpha \in Q(\alpha)) \quad (7)$$

$(i = 0, 1, \ldots, 7)$ by $W(E_7^{(1)})$. By the following proposition by Kac we have the fact that the group of isometries of $Q(\alpha)$ each of which preserves the inner product is $\pm \text{Aut(Dynkin)} \times \tilde{W}(E_7^{(1)})$ (which is written as $\pm \tilde{W}$).
Proposition 2.1. ([8] §5.10) If the generalized Cartan Matrix $c_{ij}$ is a symmetric matrix of finite, affine, or hyperbolic type, then the group of all automorphisms of $Q(\alpha)$ preserving the bilinear form is $\pm \tilde{W}$.

Note that

$$\delta = -\frac{1}{2} K_X = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 \quad (8)$$

is preserved by $W(E_7^{(1)})$ and the automorphism of the Dynkin diagram. In our case, since $\varphi_*$ preserves the anti-canonical divisor $-K_X$, each element of $-\text{Aut(Dynkin)} \ltimes W(E_7^{(1)})$ is not a Cremona isometry.

Claim 2.2. The automorphism of the Dynkin diagram is not a Cremona isometry.

Proof. There exist only one automorphism of the Dynkin diagram, which is the involution exchanging $\alpha_0, \alpha_1, \alpha_2$ and $\alpha_6, \alpha_5, \alpha_4$ respectively.

Denoting the actions of these involutions on $\text{Pic}(X)$ by $D \mapsto \overline{D}$, we have

$$\overline{E} = E + 3E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 + 4E_8$$
$$\overline{E_1} = E_1 - E_8 + \overline{E_8}$$
$$\overline{E_2} = E_1 - E_7 + \overline{E_8}$$
$$\vdots$$
$$\overline{E_7} = E_1 - E_2 + \overline{E_8}.$$

Set $\overline{E_8} = eE + e_1 E_1 + \cdots + e_8 E_8$. From

$$\langle \overline{E}, \overline{E_8} \rangle = \langle E, E_8 \rangle = 2e - 3e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 - 4 = 0$$
$$\langle \overline{E_1}, \overline{E_8} \rangle = \langle E_1, E_8 \rangle = -e_1 + e_8 - 1 = 0$$
$$\vdots$$
$$\langle \overline{E_7}, \overline{E_8} \rangle = \langle E_7, E_8 \rangle = -e_1 + e_2 - 1 = 0,$$

we have $e = -2e_1 - 3/2$, which contradicts the assumption that $e$ is an integer. \qed

Now we have the fact that the actions of Cremona isometries on $Q(\alpha)$ are included by $W(E_7^{(1)})$.

Claim 2.3. The action of $W(E_7^{(1)})$ are uniquely extended onto $\text{Pic}(X)$ as

$$r_j(D) = D - 2 \frac{\langle \alpha_j, D \rangle}{\langle \alpha_j, \alpha_j \rangle} \alpha_j \quad (D \in \text{Pic}(X)) \quad (9)$$
Proof. Let $s$ and $s'$ be Cremona isometries such that the action of $s$ is identical to that of $s'$ on $Q(\alpha)$. We show $s' \circ s^{-1} = \text{Identity}$ on $\text{Pic}(X)$. Since $\{E_1, \alpha_0, \alpha_1, \cdots, \alpha_7\}$ is a basis of $\text{Pic}(X)$, we can set

$$s' \circ s^{-1}(E_1) = e_1 E_1 + a_0 \alpha_0 + a_1 \alpha_1 + \cdots + a_7 \alpha_7.$$  

From $(s' \circ s^{-1}(E_1), \delta) = (E_1, \delta) = 1$ and $(\alpha_i, \delta) = 0$ (for $\forall i$), we have $e_1 = 1$. Since

$$(s' \circ s^{-1}(E_1), \alpha_i) = (E_1, \alpha_i) \iff (s' \circ s^{-1}(E_1) - E_1, \alpha_i) = 0$$

holds,

$$-(c_{i,j})_{0 \leq i, j \leq 7} a = 0$$

holds, where $(c_{i,j})_{0 \leq i, j \leq 7}$ is the Cartan matrix of type $E_7^{(1)}$. Hence we have $s' \circ s^{-1}(E_1) = E_1 + z \delta$ ($z \in \mathbb{Z}$). Finally, from $(s' \circ s^{-1}(E_1), s' \circ s^{-1}(E_1)) = -1 + 2z = -1$, we have $z = 0$. Hence it has been shown that $s' \circ s^{-1}$ does not change the basis of $\text{Pic}(X)$ and therefore $s' \circ s^{-1} = \text{Identity}$. 

By this claim, the actions of simple reflections on $\text{Pic}(X)$ are given by

$$r_i : \quad E_{i+1} \mapsto E_{i+2}$$

$$E_{i+2} \mapsto E_{i+1} \quad (\text{for } 0 \leq i \leq 6)$$

$$r_7 : \quad E \mapsto 3E - 2E_1 - 2E_2 - 2E_3 - 2E_4$$

$$E_k \mapsto E - E_1 - E_2 - E_3 - E_4 + E_k \quad (\text{for } 1 \leq k \leq 4),$$

where preserved elements are omitted. Finally, we have the claim of the theorem by the following claim.

**Claim 2.4.** The action of each element of $W(E_7^{(1)})$ on $\text{Pic}(X)$ is uniquely realized as an element of $\text{Gr}(\{X\})$.

**Proof.** It is enough to show for the simple reflections $r_i$'s.

i) in the case of $0 \leq i \leq 6$. Since $E \mapsto E$, the action on $\mathbb{P}^3$ is linear. This is nothing but the projective transformation $PGL(4)$. Hence, without loss of generality, we can assume that the action is the identity. Since $r_i$ exchanges $E_{i+1}$ and $E_{i+2}$, we have

$$r_i : \quad ((P_1, \cdots, P_{i+1}, P_{i+2}, \cdots, P_8); x) \in X(4.8) \times \mathbb{P}^3$$

$$\mapsto ((P_1, \cdots, P_{i+2}, P_{i+1}, \cdots, P_8); x). \quad (10)$$

Here, we have described the action $r_i : X_{P_1, \cdots, P_8} \to X_{P'_1, \cdots, P'_8}$ in terms of the coordinate $x$ of $\mathbb{P}^3$.

ii) in the case of $r_7$. Using $PGL(4)$, we may assume

$$P_1 = P'_1 = (1 : 0 : 0 : 0), \cdots, P_4 = P'_4 = (0 : 0 : 0 : 1) \quad (11)$$

without loss of generality. Since $E \mapsto 3E - 2E_1 - 2E_2 - 2E_3 - 2E_4$, the action on $\mathbb{P}^3$: $(x : y : z : w) \mapsto (x' : y' : z' : w')$ is in the 3rd degree. Moreover, from $E_1 \mapsto E - E_2 - E_3 - E_4$, we have $(1 : 0 : 0 : 0) \iff x' = 0$. Since similar facts hold for the case of $E_2, E_3$, and $E_4$, considering the degree with respect to $x, y, z, w$, we have $(x' : y' : z' : w') = (ayzw : bxwz : cuxyz : dxyz)$, where $a, b, c, d \in \mathbb{C}^\times$. Furthermore, we can normalize it as $a = b = c = d = 1$ preserving $\square$. This is nothing but the
standard Cremona transformation of \( \mathbb{P}^3 \). Hence the action of \( r_7 \) on \( \{X\} \) is given by the composition of maps:

\[
((P_1, P_2, P_3, P_4), (P_5, P_6, P_7, P_8); x) \in X(4,8) \times \mathbb{P}^3
\]

\[
\mapsto \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad P^{-1}_{1234}((P_5, P_6, P_7, P_8); P^{-1}_{1234}(x))
\]

\[
:= ((P'_1, \ldots, P'_8); x')
\]

\[
\mapsto \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
x'_{2,5}x'_{3,5}x'_{4,5} & \cdots & x'_{2,8}x'_{2,8}x'_{4,8} \\
x'_{3,5}x'_{4,5}x'_{5,5} & \cdots & x'_{3,8}x'_{2,8}x'_{4,8} \\
x'_{4,5}x'_{1,5}x'_{2,5} & \cdots & x'_{4,8}x'_{1,8}x'_{2,8} \\
x'_{1,5}x'_{2,5}x'_{3,5} & \cdots & x'_{1,8}x'_{2,8}x'_{3,8}
\end{pmatrix}, \quad \begin{pmatrix}
y'z'x' \\
z'w'x' \\
w'x'y' \\
x'y'z'
\end{pmatrix}
\]

(12)

where \( P_{1234} \) denotes the square matrix \((P_1, P_2, P_3, P_4)\).

\[\square \]

**Proof of Theorem 2.1**

### 3 Normalization of the configuration space

In this section we discuss normalization of the configuration space \( X(4,8) \) by \( PGL(4) \), i.e. how to choose representative elements. Notice that without good normalization it is difficult to see the concrete action of the group and properties of associated dynamical systems. For example, although \( X(4,8) \) is easily normalized as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & y_6 & y_7 & y_8 \\
0 & 0 & 1 & 0 & z_6 & z_7 & z_8 \\
0 & 0 & 0 & 1 & w_6 & w_7 & w_8
\end{pmatrix}
\]

(13)

the action of the Weyl group on this coordinate becomes complicated. We normalize it in terms of elliptic curves.

Note the following lemma.

**Lemma 3.1.** Let \( P_1, P_2, \ldots, P_8 \) be 8 points in \( \mathbb{P}^3 \). Quadratic surfaces passing through \( P_1, P_2, \ldots, P_8 \) form at least one parameter family.

**Proof.** It is clear from the fact that quadratic surfaces passing through \( P_1, P_2, \ldots, P_8 \) and arbitrary point \( P_9 \) in \( \mathbb{P}^3 \) is given by the equation

\[
\begin{vmatrix}
x^2 & y^2 & z^2 & w^2 & xy & yz & zw & wx & xz & yw \\
x_1^2 & y_1^2 & z_1^2 & w_1^2 & x_1y_1 & y_1z_1 & z_1w_1 & w_1x_1 & x_1z_1 & y_1w_1 \\
x_2^2 & y_2^2 & z_2^2 & w_2^2 & x_2y_2 & y_2z_2 & z_2w_2 & w_2x_2 & x_2z_2 & y_2w_2 \\
x_3^2 & y_3^2 & z_3^2 & w_3^2 & x_3y_3 & y_3z_3 & z_3w_3 & w_3x_3 & x_3z_3 & y_3w_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{vmatrix} = 0,
\]

where \( P_i = (x_i : y_i : z_i : w_i) \) (if the left hand side is identically zero, exchange one of the points for a generic point).

\[\square \]

The pencil of quadratic surfaces passing through \( P_1, P_2, \ldots, P_8 \) can be written as

\[
x^t(\alpha A + \beta B)x = 0 \quad (x \in \mathbb{P}^3)
\]

(14)
by $4 \times 4$ complex symmetric matrices $A, B$ and $(\alpha : \beta) \in \mathbb{P}^1$. Normalizing (14) by $PGL(4)$ (cf. [18]), we have the next theorem, which provides a normalization of $X(4, 8)$. The proof of this theorem will be given in the last part of this section.

**Theorem 3.1.** Each element of $X(4, 8)$ can be normalized so that $P_1, P_2, \ldots, P_8$ are on the intersection curve(s) of one of the following 3 type pencils of quadratic surfaces.

(i)

\[
\begin{align*}
(E) & \quad x^2 - zw = 0 \\
(F) & \quad y^2 - 4xw + g_2xz + g_3z^2 = 0
\end{align*}
\]

(ii)

\[
\begin{align*}
x y - zw & = 0 \\
xw - z^2 & = 0
\end{align*}
\]

(iii)

\[
\begin{align*}
x y - zw & = 0 \\
4x^2 - 2zw + w^2 & = 0.
\end{align*}
\]

Moreover,

(i-1) in the case where the intersection curve is non-singular ($\Delta = 27g_3^2 - g_2^3 \neq 0$), $P_i$ can be parameterized as

\[P_i = (\wp(u_i) : \wp'(u_i) : 1 : \wp^2(u_i)), \quad u_i \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \quad (15)\]

where $\wp(u)$ is Wierstrass $\wp$ function with the periods $(1, \tau)$.

(i-2) in the case where $\Delta = 0$ and $g_2 \neq 0$, the intersection curve can be renormalized to the intersection curve of

\[
\begin{align*}
x^2 - zw & = 0 \\
y^2 - 4w(x + hz) & = 0
\end{align*}
\]

and $P_i$ can be parameterized as

\[P_i = \left( \frac{4hu_i}{(1 - u_i)^2} : \frac{-8h^{3/2}u_i(1 + u_i)}{(1 - u_i)^3} : 1 : \frac{(4hu_i)^2}{(1 - u_i)^4} \right), \quad u_i \in \mathbb{P}^1 \setminus \{0, \infty\}. \quad (16)\]

(i-3) in the case where $g_2 = g_3 = 0$, the intersection curve can be renormalized to the intersection curve of

\[
\begin{align*}
x^2 - zw & = 0 \\
y^2 - 4xw & = 0
\end{align*}
\]

and $P_i$ can be parameterized as

\[P_i = (u_i^{-2} : -2u_i^{-3} : 1 : u_i^{-4}), \quad u_i \in \mathbb{P}^1 \setminus \{\infty\}. \quad (17)\]

(ii) the intersection consists of 2 curves \{(0 : s : 0 : t) | s : t \in \mathbb{P}^1\} and \{(s^3 : t^3 : s^2t : st^2) | s : t \in \mathbb{P}^1\}.

(iii) the intersection consists of 2 curves \{(0 : s : 0 : t) | s : t \in \mathbb{P}^1\} and \{(2st^2 : s(s^2 + 4t^2) : t(s^2 + 4t^2) : 2s^2t) | s : t \in \mathbb{P}^1\}.

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Remark 3.1. The parameterizations of (i-2) and (i-3) of Theorem 3.1 are chosen so that the period map becomes simple.

From the proof in case (i-1) of this theorem we have the next corollary.

**Corollary 3.1.** If the intersection curve of surfaces passing through \(P_1, P_2, \ldots, P_8\) is non singular, \(X(4,8)\) can be normalized as \(P_1, P_2, \ldots, P_8\) are on the intersection curve of 2 surfaces

\[
\theta_{00}^2 x^2 - \theta_{01}^2 y^2 - \theta_{10}^2 z^2 = 0 \\
\theta_{10}^2 y^2 - \theta_{01}^2 z^2 - \theta_{00}^2 u^2 = 0,
\]

(18)

where we write \(\theta_{ij} := \theta_{ij}(0)\) for the theta function \(\theta_{ij}(u)\) with the fundamental periods \((1, \tau)\) and therefore \(P_i\) can be parameterized as

\[
P_i = (\theta_{00}(2u_i) : \theta_{01}(2u_i) : \theta_{10}(2u_i) : \theta_{11}(2u_i)).
\]

(19)

**Corollary 3.2.** \(X(4,8)\) restricted to case (i -1), (i -2) or (i -3) of Theorem 3.1 is isomorphic to

(i-1)

\[
\{(u_1, \ldots, u_8) \in (\mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau))^8, \tau \in \mathbb{H}/SL(2, \mathbb{Z}) \mid u_i + u_j + u_k + u_l \neq 0\},
\]

(20)

where \(1 \leq i, j, k, l \leq 8\) are different each other and \(\mathbb{H}\) is the upper half of the complex plane;

(i-2)

\[
\{(u_1, \ldots, u_8) \in (\mathbb{P}^1 \setminus \{0, \infty\})^8 \mid u_i u_j u_k u_l \neq 1\};
\]

(21)

(i-3)

\[
\{(u_1, \ldots, u_8) \in (\mathbb{P}^1 \setminus \{\infty\})^8 \mid u_i + u_j + u_k + u_l \neq 0\}.
\]

(22)

**Proof.** It is enough to show that \(P_i, P_j, P_k, P_l\) are on the same plane if and only if \(u_i + u_j + u_k + u_l = 0\) holds \((u_i u_j u_k u_l = 1\) holds in case (i-2)). Notice that \(P_i, P_j, P_k, P_l\) are on the same plane if and only if

\[
\begin{vmatrix}
x_i & y_i & z_i & w_i \\
x_j & y_j & z_j & w_j \\
x_k & y_k & z_k & w_k \\
x_l & y_l & z_l & w_l
\end{vmatrix} = 0
\]

(23)

holds.

- In case (i-1). When (21) is considered to be a rational function of \(u_i\), the origin is the unique pole of order 4. By Abel’s Theorem the sum of zero points are 0. Here, \(u_i = u_j, u_k, u_l\) are zero points and hence the other zero point is \(u_i = -u_j - u_k - u_l\).

- In case (i-2) or (i-3). When (21) is considered to be a rational function of \(u_i, u_i = 1\) the origin in case (i-3)) is the unique pole of order 4 and therefore there are 4 zero points. It is easily shown that \(u_i = u_j, u_k, u_l, (u_j u_k u_l)^{-1}\), \((u_i = u_j, u_k, u_l, -u_j - u_k - u_l\) in case (i-3)) are those 4 points by substitution. 

A similar argument leads the following theorem concerning the dimension of linear system \(|-\frac{1}{2}K_X|\) of rational variety \(X = X_{P_1, \ldots, P_8}\).
**Theorem 3.2.** In case (i) of theorem 3.1, \( \dim(-\frac{1}{2}K_X) \geq 2 \) if and only if, in case (i-1) or (i-2)

\[
u_1 + u_2 + \cdots + u_8 = 0
\]

holds and in case (i-1)

\[
u_1u_2 \cdots u_8 = 1
\]

holds.

Notice that if the equivalent conditions of Theorem 3.2 is satisfied, \( X \) is an elliptic variety.

**Proof.** Case (i-1).

\((\Rightarrow)\). Since \( \dim(-\frac{1}{2}K_X) \geq 2 \), there exists a surface \( D \in (-\frac{1}{2}K_X) \) which does not include the intersection curve of (i-1). Let \( P_9 \) and \( P_{10} \) be generic points on \( D \). \( D \) is described as

\[
\begin{bmatrix}
x_1^2 & y_1^2 & z_1^2 & w_1^2 & xy_1 & yz_1 & zw_1 & wx_1 & xz_1 & yw_1 \\
x_2^2 & y_2^2 & z_2^2 & w_2^2 & xy_2 & yz_2 & zw_2 & wx_2 & xz_2 & yw_2 \\
\vdots \\
x_9^2 & y_9^2 & z_9^2 & w_9^2 & xy_9 & yz_9 & zw_9 & wx_9 & xz_9 & yw_9 \\
x_{10}^2 & y_{10}^2 & z_{10}^2 & w_{10}^2 & xy_{10} & yz_{10} & zw_{10} & wx_{10} & xz_{10} & yw_{10}
\end{bmatrix} = 0.
\]

By Bézout’s theorem the intersection of \( D \) and the curve of (i-1) is the 8 points \( P_1, P_2, \cdots, P_8 \), which are given by the zero points of (24) with \( (x : y : z : w) = (\varphi(u) : \varphi(u)^3 : 1 : \varphi(u)^2) \). The left hand side of the equation is an elliptic functions of \( u \) and has the unique pole \( u = 0 \) of order 8. Since the 8 points \( u = u_1, u_2, \cdots, u_8 \) are zero points, by Abel’s theorem we have \( u_1 + u_2 + \cdots + u_8 = 0 \).

\((\Leftarrow)\). Assume \( u_1 + u_2 + \cdots + u_8 = 0 \). Considering the zero points of the same elliptic function, it is shown that the intersection of a generic quadratic surface \( D \) passing through 7 points \( P_1, P_2, \cdots, P_7 \) and the curve of (i-1) is the 8 points \( P_1, P_2, \cdots, P_8 \). Hence \( D \) is an element of \( (-\frac{1}{2}K_X) \). The dimension to choose such \( D \) is 2 or more. In case (i-2) or (i-3), it is enough to change the argument about Abel’s theorem as the proof of Corollary 3.2.

**Proof of Theorem 3.1**

Since the latter part is easy, we show only the former part.

We consider normalization of (14) by \( PGL(4) \). Note that \( P^{-1} \) in \( PGL(4) \) acts on the pencil as

\[
x^4(\alpha A + \beta B)x = 0 \rightarrow x^4(\alpha P^4AP + \beta P^4BP)x = 0.
\]

Note also the following facts.

**Fact 1.** Any complex symmetric matrix can be diagonalized to the form \( \text{diag}(1, \cdots, 1, 0, \cdots, 0) \) by \( PGL \).

**Fact 2.** The \( n \times n \) identity matrix is not changed by orthonormal matrices.

**Fact 3.** Since “two complex symmetric matrices are similar if and only if they are similar via a complex orthonormal similarity,” if two matrices have the same Jordan normal form, they are mapped each other by some complex orthonormal matrix. (pp.212 in [7]).
Assume that there exists \((s : t)\) such that \(\text{rank}(sA + tB) = 2\), by Fact 1 the matrix is normalized to \(\text{diag}(1, 1, 0, 0)\) and the defining equation can be factorized. Hence, 4 or more points in the 8 points on \(x'(\alpha A + \beta B)x = 0\) are on the same plane, which contradicts the assumption of the configuration space. So we have \(\text{rank}(sA + tB) \geq 3\) for all \((s : t) \in \mathbb{P}^1\).

**Lemma 3.2.** If \(\text{rank}(sA + tB) \geq 3\) for all \((s : t) \in \mathbb{P}^1\), there exists \((s : t) \in \mathbb{P}^1\) such that \(\text{rank}(sE + tF) = 4\).

**Proof.** Without loss of generality we can assume \(A = \text{diag}(1, 1, 1, 0)\). We normalize \(B\) by \(3 \times 3\) matrix. Since the sizes of Jordan blocks of the \(3 \times 3\) submatrix of \(B\) should be \((1, 1, 1), (2, 1)\) or \((3, 1)\), by Fact 3, we can set \(B\) as

\[
\begin{pmatrix}
    a & 0 & 0 & e \\
    0 & b & 0 & f \\
    0 & 0 & c & g \\
    e & f & g & d
\end{pmatrix},
\begin{pmatrix}
    a + 1 & \sqrt{-1} & 0 & e \\
    \sqrt{-1} & a - 1 & 0 & f \\
    0 & 0 & b & g \\
    e & f & g & d
\end{pmatrix},
\begin{pmatrix}
    a & 1 & \sqrt{-1} & e \\
    1 & a & 0 & f \\
    \sqrt{-1} & 0 & a & g \\
    e & f & g & d
\end{pmatrix}.
\]

Assume \(\text{rank}(sA + tB) = 3\) for all \((s, t) \in \mathbb{P}^1\). In the first case, we have \(d = e = f = g = 0\) and therefore the rank becomes 2 or less for some \(t\), which is a contradiction. Along the same line, in the second or the third case it can be shown that the rank becomes 2 or less, which is a contradiction. Hence there exists \((s : t) \in \mathbb{P}^1\) such that \(\text{rank}(sA + tB) = 4\).

From the above lemma, we may assume \(A = \text{Identity}\) and \(\text{rank}B = 3\). The Jordan normal form of \(B\) has the following 5 possibilities:

1. \((i - 1)\) \(\begin{pmatrix}
    a & 0 & 0 & 0 \\
    0 & b & 0 & 0 \\
    0 & 0 & c & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix},
\)
2. \((i - 2)\) \(\begin{pmatrix}
    a & 1 & 0 & 0 \\
    0 & a & 0 & 0 \\
    0 & 0 & b & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix} \sim \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & a + 1 & \sqrt{-1} & 0 \\
    0 & \sqrt{-1} & a - 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix},
\)
3. \((i - 3)\) \(\begin{pmatrix}
    a & 1 & 0 & 0 \\
    0 & a & 1 & 0 \\
    0 & 0 & a & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix} \sim \begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 1 & 1 & 0 \\
    0 & 0 & 1 & 1 \\
    0 & 0 & 0 & 1
\end{pmatrix},
\)
4. \((ii)\) \(\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0
\end{pmatrix} \sim \begin{pmatrix}
    0 & 0 & 1 & \sqrt{-1} \\
    0 & 0 & \sqrt{-1} & -1 \\
    \sqrt{-1} & 1 & \sqrt{-1} & 1 \\
    \sqrt{-1} & -1 & \sqrt{-1} & 1
\end{pmatrix},
\)
5. \((iii)\) \(\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & a & 1 \\
    0 & 0 & 0 & a
\end{pmatrix} \sim \begin{pmatrix}
    1 & \sqrt{-1} & 0 & 0 \\
    \sqrt{-1} & -1 & 0 & 0 \\
    0 & 0 & 2 & \sqrt{-1} \\
    0 & 0 & \sqrt{-1} & 0
\end{pmatrix},
\)

where the right hand side matrices are similar to \(B\) except the proportional constants. We replace \(B\) by the right hand side matrices.

*Case (i-1).* Since \(\text{rank}(sA + tB) \geq 3\), \(a, b, c\) are not zero and different each other. By replacing the basis of pencil \(x'(\alpha A + \beta B)x = 0\), we may assume \(A = \text{diag}(0, a, b, c)\) and \(B = \text{diag}(d, e, f, 0)\). Moreover, by the actions of diagonal matrices, we can set \(A = \)
diag(0, 1, 1, 1) and $B = \text{diag}(a, a, b, 0)$. Finally, by considering constant $\times B$, we can set $A = \text{diag}(0, 1, 1, 1), B = \text{diag}(1, 1, a, 0)$ ($a \neq 0, 1$). On the other hand if $\Delta = g_3^2 - 27g_3^2$ is not zero, using $e_1 = \phi(w_1/2), e_2 = \phi(w_2/2), e_3 = \phi((w_1 + w_2)/2)$, $(F)$ can be written as

$$y^2 - 4xw + 4(e_1 + e_2 + e_3)x^2 - 4(e_1e_2 + e_1e_3 + e_2e_3)xz + 4e_1e_2e_3z^2 = 0.$$ 

The matrices

$$E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2} & 0
\end{pmatrix},$$

$$F = \begin{pmatrix}
4(e_1 + e_2 + e_3) & 0 & -2(e_1e_2 + e_1e_3 + e_2e_3) & -2 \\
0 & 1 & 0 & 0 \\
-2(e_1e_2 + e_1e_3 + e_2e_3) & 0 & 4e_1e_2e_3 & 0 \\
-2 & 0 & 0 & 0
\end{pmatrix}$$

are normalized to $E' = \text{diag}(0, 1, 1, 1), F' = \text{diag}(1, 1, \lambda, 0)$ via

$$P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & \sqrt{-1} & -1 \\
0 & 0 & \sqrt{-1} & 1
\end{pmatrix},$$

where

$$\lambda = \frac{e_2 - e_3}{e_1 - e_3} = \frac{3^{1/3}(3\sqrt{-1} + \sqrt{3})g_2 - (-3\sqrt{-1} + \sqrt{3})(9g_3 + \sqrt{-3g_3^2 + 81g_3^2})^{2/3}}{-3^{1/3}(-3\sqrt{-1} + \sqrt{3})g_2 + (3\sqrt{-1} + \sqrt{3})(9g_3 + \sqrt{-3g_3^2 + 81g_3^2})^{2/3}}.$$  (27)

is the $\lambda$ function. By suitably replacing $P$, $\lambda$ can be changed to $1/\lambda, 1 - \lambda, 1/(1 - \lambda), \lambda/(\lambda - 1), (\lambda - 1)/\lambda$ (by using the fact that $E'$ and $F'$ are simultaneously decomposed to the eigenspaces, it can also be shown that $\lambda$ cannot be other than these.) Note that $\lambda$ is invariant under the action $(w_1, w_2) \mapsto (sw_1, sw_2) \mapsto (g_2, g_3) \mapsto (g_2/s^4, g_3/s^6)$. We show that there exist corresponding $g_2$ and $g_3$ when $\lambda \neq 0, 1, \infty$ ($\Delta = 0$ if $\lambda = 0, 1$ or $\infty$). Setting

$$y = \frac{\lambda - 1/2 - \sqrt{3}\sqrt{-1}/2}{\lambda - 1},$$

we have

$$y = \frac{3\sqrt{-1}(1/2 - \sqrt{3}/2)g_2}{g_2 - (3\sqrt{3}g_3 + \sqrt{-g_3^2 + 27g_3^2})^{2/3}}.$$ 

We show that there exist corresponding $g_2$ and $g_3$ when $y \neq 1/2 + \sqrt{3}/2, 1, \infty$. Setting $g_2 = a \left(3\sqrt{3}g_3 + \sqrt{-g_3^2 + 27g_3^2}\right)^{(2/3)} = 3b^2$, we have

$$y = \frac{3\sqrt{-1}(1/2 - \sqrt{3}/2)a}{a - 3b^2}.$$ 

Hence, there exist corresponding $a, b \in \mathbb{C}$ for arbitrary $y \in \mathbb{C}$. Since $g_2 = a, g_3 = a^3/(54b^3) + b^3/2$, we can find the corresponding $g_2$ and $g_3$ when $b \neq 0$. If $b = 0$, we
have \( g_2 = g_3 = 0 \), which is not in the case considered here.

· Case (i-2). Set

\[
P_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{\sqrt{a}}{\sqrt{\alpha}} & 0 & 0 \\
0 & 0 & -\frac{1}{(a-1)\sqrt{\alpha}} & 0 \\
0 & 0 & 0 & \frac{2(1-a)}{\sqrt{\alpha}}
\end{pmatrix},
\]

\[
A' = P_2^t(P_1 AP_1 - P_1 BP_1)P_2,
\]

\[
B' = \frac{1-a}{a} P_2^t P_1 BP_1 P_2,
\]

Then \( A' = \text{diag}(1, 1, 1, 0) \) and the Jordan normal form of \( B' \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(28)

On the other hand, if \( \Delta = 0 \), setting

\[
P_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & \sqrt{-1} & -1 \\
0 & 0 & \sqrt{-1} & 1
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
\sqrt{2\sqrt{3}g_2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
E' = P_2 P_1^t AP_1 P_2,
\]

\[
F' = -\frac{2}{3} P_2 (P_1^t BP_1 \frac{\sqrt{3}}{4\sqrt{g_2}} - P_1^t AP_1) P_2,
\]

we have \( E' = \text{diag}(1, 1, 1, 0) \) and \( \text{(28)} \) as the Jordan normal form of \( F' \). Hence it has been shown that the two pencils are equivalent modulo \( PGL(4) \).

· Case (i-3). If \( g_2 = g_3 = 0 \), setting

\[
P = \begin{pmatrix}
0 & 1 & \frac{2\sqrt{-1}}{\sqrt{\alpha}} & \frac{2}{\sqrt{\alpha}} \\
1 & 0 & 0 & 0 \\
0 & 0 & \frac{2\sqrt{-1}}{\sqrt{\alpha}} & \frac{-2}{\sqrt{\alpha}} \\
0 & 0 & \frac{2\sqrt{-1}}{\sqrt{\alpha}} & \frac{1}{\sqrt{\alpha}}
\end{pmatrix},
\]

\[
E' = P^t (A + B) P, \quad F' = P^t AP,
\]

we have \( E' = \text{Identity} \) and

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

as the Jordan normal form of \( F' \). Hence it has been shown that the two pencils are equivalent modulo \( PGL(4) \).
· Case (ii). Setting
\[
P = \begin{pmatrix}
1 & 1/2 & 0 & 0 \\
-\sqrt{-1} & \sqrt{-1}/2 & 0 & 0 \\
0 & 0 & -1 & 1/2 \\
0 & 0 & -\sqrt{-1} & -\sqrt{-1}/2
\end{pmatrix},
\]
\[
A' = P^t AP, \\
B' = P^t BP,
\]
we have
\[
A' = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \\
B' = \begin{pmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & -4 & 0 \\
2 & 0 & 0 & 0
\end{pmatrix}.
\]

· Case (iii). Setting \(P, A'\) and \(B'\) as in case (ii), we have
\[
A' = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \\
B' = \begin{pmatrix}
4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix}.
\]

\[\square\] Proof of Theorem 3.1

4 Period map and the action of the Weyl group

In this section we describe the action of \(W(E_7^{(1)})\) in case (i) of Theorem 3.1 in normalized coordinates. For this purpose, it is enough to normalize the simple reflections obtained in Claim 2.4, but it is not easy calculus. Thus, first, we define a linear map \(\chi_X\) from the lattice \(Q(\alpha)\) to \(\mathbb{C}\) (which has ambiguity corresponding to the periods as discussed later). Next, we compute the action by using the fact that \(\chi\) is invariant under the action of \(W(E_7^{(1)})\), i.e. \(\chi_X(\alpha) = \chi_{w(X)}(w(\alpha))\) for \(\alpha \in Q(\alpha)\) and \(w \in W(E_7^{(1)})\). This method is an analogue of that of the period map essentially introduced by Looijenga for surfaces.

Period map and the action on the intersection curve

In the following, we shall discuss only in case (i-1) of Theorem 3.1. For case (i-2) and (i-3), we shall write the results only. Replace \(x/z, y/z, w/z\) by \(x, y, w\). Let \(D_1, D_2 \in -\frac{1}{2}K_X\) the divisors determined by \((E), (F)\) in Theorem 3.1. We denote the set of piece-wise smooth singular 3-chains in \(X - D_1 - D_2\) by \(S(X - D_1 - D_2)\). Using the holomorphic 3-form on \(X \setminus (D_1 \cup D_2)\)

\[
\omega = \frac{c \ dx \wedge dy \wedge dw}{(x^2 - w)(y^2 - 4xw + g_2x + g_3)}
\]

(29)

(the constant \(c \in \mathbb{C}^\times\) is determined later), we define the map \(\chi_X : S(X - D_1 - D_2) \to \mathbb{C}\) by the paring \(\int_\Gamma \omega, \ (\Gamma \in S(X - D_1 - D_2))\).

Let \(C\) denote the elliptic curve \(D_1 \cap D_2\). We define a map \(Q(\alpha) \to S(X - D_1 - D_2)/H_1(C, \mathbb{Z})\). For this purpose, it is enough to define for the basis \(\alpha_i\)’s.
In the case of \( 0 \leq i \leq 6 \). We have \( \alpha_i = E_{i+1} - E_{i+2} \). Let \( C^\tau_i \) be a real curve on \( C \) from \( C \cap E_{i+1} \) (which is expressed as \( u = u_{i+1} \) in the coordinate \( u \)) to \( C \cap E_{i+2} \) (\( u = u_{i+2} \)). Here, \( C \) has ambiguity of \( H_1(C, \mathbb{Z}) \simeq \mathbb{Z} + \mathbb{Z} \tau \). Let \( \varepsilon > 0 \) be a sufficiently small number and \( \Gamma_i \in S(X - D_1 - D_2) \) the set of points such that \( |x^2 - w| = \varepsilon, |y^2 - 4xw + g_2x + g_3| = \varepsilon \) and \( x \) is in the projection of \( C^\tau_i \) to the \( x \) coordinate.

In the case of \( i = 7 \). We have \( \alpha_7 = (E - E_1 - E_2 - E_3) - E_4 \). Let \( C^\tau_i \) be a real curve on \( C \) from \( C \cap (E - E_1 - E_2 - E_3) \) (\( u = -u_1 - u_2 - u_3 \)). Let \( \Gamma_7 \in S(X - D_1 - D_2) \) be the set of points such that \( |x^2 - w| = \varepsilon, |y^2 - 4xw + g_2x + g_3| = \varepsilon \) and \( x \) is in the projection of \( C^\tau_i \) to the \( x \) coordinate.

Remark 4.1. As in 2-dimensional case, we can take \( \Gamma_i \) from \( H_3(X - D_1 - D_2, \mathbb{Z}) \). Let \( F_1, F_2 \) be divisors such that \( \alpha_i \) is written as \( \alpha_i = F_1 - F_2 \) as above. For our purpose, it is enough to add singular 3-chains in \( F_1 \) and \( F_2 \) to the above \( \Gamma_i \). Here, the extended part is included by 2-dimensional algebraic subvariety and hence the effect for the integration is zero.

By the composition \( Q(\alpha) \rightarrow S(X - D_1 - D_2)/H_1(C, \mathbb{Z}) \rightarrow \mathbb{C} \), the map \( \chi_X: Q(\alpha) \rightarrow \mathbb{C} \) is determined modulo the image of \( H_1(C, \mathbb{Z}) \). Let \( \pi_x \) denote the projection to the \( x \) coordinate. By residue theorem, we have

\[
\chi_X(\alpha) = \int_{\Gamma_i} \int_{x \in \pi_x(C^\tau_i)} \frac{dx \wedge dy \wedge dw}{(x^2 - w)(y^2 - 4xw + g_2x + g_3)}
\]

\[
= \int_{x \in \pi_x(C^\tau_i)} \frac{dx \wedge dy}{(y^2 - 4x^2 + g_2x + g_3)}
\]

\[
= \int_{x \in \pi_x(C^\tau_i)} \frac{dx}{y}
\]

\[
= \int_{C^\tau_i} \frac{du}{x} \quad (x = \varphi(u), y = \varphi'(u))
\]

\[
= \left\{ \begin{array}{ll}
\int_{u_{i+1} - u_{i+2}}^{u_{i+1} - u_{i+2}} & \quad (0 \leq i \leq 6) \\
\int_{-u_1 - u_2 - u_3 - u_4}^{u_{i+1} - u_{i+2} - u_3 - u_4} & \quad (i = 7)
\end{array} \right.
\]

where \( C^\tau_{i*} \) denotes the pullback of \( C^\tau_i \) to the space of \( u \) and the last result should be considered modulo \( c''(\mathbb{Z} + \mathbb{Z} \tau) \). Since the constant \( c \in \mathbb{C}^\times \) has been arbitrary, we can determine it so that \( c'' = 1 \).

By further blow-up along lines, the simple reflection \( r_i \) can be considered to be an exchange of the blow-down structure of \( X = X_{P_1, \ldots, P_8} \) and that of \( r_i(X) = X_{r_i(P_1, \ldots, P_8)} \), i.e. it just changes how to blow-down corresponding to the change of basis of \( \text{Pic}(X) \) (cf. Remark 1.2). Let \( u = u_0 \) denote the intersection point of a effective divisor \( D \) and the curve \( C \). Since the curve \( C \) is preserved by \( r_i \) (because the modulus of \( C \) is not changed), \( r_i(D) \) and \( C \) also intersect at \( u = u_0 \). Since \( D \) is arbitrary, we have

\[
\chi_X(\alpha) = \chi_{r_i(X)}(r_i(\alpha)) \quad \alpha \in Q(\alpha).
\]

Considering the composition, we have

\[
\chi_X(\alpha) = \chi_{w(X)}(w(\alpha)) \quad (30)
\]

for all \( w \in W(E^{(1)}_i) \).
Remark 4.2. This fact can also be considered to be that \( r_i \) is a pseudo-isomorphism from \( X \) to \( r_i(X) \) and determines an exchange of their blow-up points.

From (7), (30), we have
\[
\begin{align*}
    r_i : (u_{i+1}, u_{i+2}) &\mapsto (u_{i+2}, u_{i+1}) \\
    r_7 : (u_1, \cdots, u_8) &\mapsto (u_1 - \lambda_1, \cdots, u_4 - \lambda_1, u_5 + \lambda_1, \cdots, u_8 + \lambda_1),
\end{align*}
\]
where \( \lambda_1 = \frac{1}{2}(u_1 + u_2 + u_3 + u_4) \) (preserved elements are omitted). Moreover, since \( r_i \) acts on the elliptic curve \( C \) birationally and therefore it is a translation for the points on \( C \) except \( P_j \) and \( r_i(P_j) \) (1 \( \leq j \leq 8 \)), we have
\[
\begin{align*}
    r_i : u = 0 &\mapsto u = 0 \quad (0 \leq i \leq 6) \\
    r_7 : u = 0 &\mapsto u = \lambda_1 .
\end{align*}
\]

In case (i-2)

We have
\[
\chi_X(\alpha_i) = c'' \int_{C_{i}^{\text{reg}}} \frac{du}{u} = \begin{cases} 
  c'\log \frac{u_{i+2}}{u_{i+1}} & (0 \leq i \leq 6) \\
  -c''\log(u_1u_2u_3u_4) & (i = 7)
\end{cases}
\]

\[
\begin{align*}
    r_i : (u_{i+1}, u_{i+2}) &\mapsto (u_{i+2}, u_{i+1}) \quad (0 \leq i \leq 6) \\
    r_7 : (u_1, \cdots, u_8) &\mapsto (u_1\lambda_1^{-1}, \cdots, u_4\lambda_1^{-1}, u_5\lambda_1, \cdots, u_8\lambda_1),
\end{align*}
\]
where \( \lambda_1 = (u_1u_2u_3u_4)^{1/2} \), and we have
\[
\begin{align*}
    r_i : u = 1 &\mapsto u = 1 \quad (0 \leq i \leq 6) \\
    r_7 : u = 1 &\mapsto u = \lambda_1 .
\end{align*}
\]

In case (i-3)

It is the same with case (i-1).

The action on \( \mathbb{P}^3 \)

We investigate the action to generic points in \( X_{P_1, \cdots, P_8} \).

The action of \( r_i \) (0 \( \leq i \leq 6 \))

Since the simple reflection \( r_i \) just exchanges the blow-up points, it acts on \( \mathbb{P}^3 \) as the identical map
\[
    r_i : \mathbf{x} \mapsto \mathbf{x} .
\]

The action of \( r_7 \)

We write \( P_i = (f_x(u_i), f_y(u_i), 1, f_w(u_i))^t \) by the parametric representation of \( C \): \([15], [16]\) or \([17]\). Let \( \overline{\pi} \) denote the image of \( \pi \) by \( r_7 \) and set
\[
((P_1, P_2, P_3, P_4), (P_5, P_6, P_7, P_8)) = (A, B) .
\]
For a matrix $P$, let $1/P$ denote a matrix whose elements are the reciprocal number of corresponding elements of $P$. The action of $r_7$ is given by

$$
\begin{align*}
&A^{-1} \\
&SCT \\
&\text{diag}(PGL(4)) \\
&\pi
\end{align*}
$$

where $\text{SCT}$ denotes the standard Cremona transformation. Setting $\mathbf{x}' = \text{diag}(b_1, b_2, b_3, b_4)(1/(A^{-1}\mathbf{x}))$, we have

$$
\mathbf{x}' = \left(\frac{b_1}{|x, P_2, P_3, P_4|}, \frac{-b_2}{|x, P_3, P_4, P_1|}, \frac{b_3}{|x, P_4, P_1, P_2|}, \frac{-b_4}{|x, P_1, P_2, P_3|}\right)
$$

and

$$
\mathbf{x}' = \overline{A}^{-1}\mathbf{x} = (|x, P_2, P_3, P_4|, -|x, P_3, P_4, P_1|, |x, P_4, P_1, P_2|, -|x, P_1, P_2, P_3|).
$$

On the other hand, from (32), in case (i-1) and case (i-2) we have

$$
\mathbf{p} = (f_x(\lambda_1) : f_y(\lambda_1) : 1 : f_w(\lambda_1))^t
$$

for $\mathbf{p} = (f_x(0) : f_y(0) : 1 : f_w(0))^t = (0 : 0 : 0 : 1)^t$ (in case (i-3) we have $\mathbf{p} = (f_x(1) : f_y(1) : 1 : f_w(1))^t = (0 : 0 : 0 : 1)^t$). Using these, we can obtain $b_i$ explicitly. Consequently, we have

$$
\mathbf{x} = \begin{pmatrix}
 f_x(\bar{u}_1) & f_x(\bar{u}_2) & f_x(\bar{u}_3) & f_x(\bar{u}_4) \\
 f_y(\bar{u}_1) & f_y(\bar{u}_2) & f_y(\bar{u}_3) & f_y(\bar{u}_4) \\
 1 & 1 & 1 & 1 \\
 f_w(\bar{u}_1) & f_w(\bar{u}_2) & f_w(\bar{u}_3) & f_w(\bar{u}_4)
\end{pmatrix} \begin{pmatrix}
 l_{2,3,4}(\mathbf{x}) \\
 -l_{3,4,1}(\mathbf{x}) \\
 l_{1,1,2}(\mathbf{x}) \\
 -l_{1,2,3}(\mathbf{x})
\end{pmatrix},
$$

where $\bar{u}_k := \overline{u_k} = u_k - \lambda_1$ ($1 \leq k \leq 4$) ($\bar{u}_k := \overline{u_k} = u_k\lambda_1^{-1}$ ($1 \leq k \leq 4$) in case (i-2)) and

$$
l_{i,j,k}(\mathbf{x}) = \begin{vmatrix}
 f_x(\lambda_1) & f_x(\bar{u}_i) & f_x(\bar{u}_j) & f_x(\bar{u}_k) & 0 & f_x(\bar{u}_i) & f_x(\bar{u}_j) & f_x(\bar{u}_k) \\
 f_y(\lambda_1) & f_y(\bar{u}_i) & f_y(\bar{u}_j) & f_y(\bar{u}_k) & 0 & f_y(\bar{u}_i) & f_y(\bar{u}_j) & f_y(\bar{u}_k) \\
 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 f_w(\lambda_1) & f_w(\bar{u}_i) & f_w(\bar{u}_j) & f_w(\bar{u}_k) & 1 & f_w(\bar{u}_i) & f_w(\bar{u}_j) & f_w(\bar{u}_k)
\end{vmatrix}.
$$

\section{Dynamical systems}

In this section we consider a dynamical system corresponding to a translation of the Weyl group $W(E_7^{(1)})$. Note that although one can consider dynamical systems for all translations, many of them are generated by birational conjugates of one of them.

Notice that

$$
r_w(\alpha_i)(\beta) := \beta - 2\left[\frac{w(\alpha_i)}{\langle \alpha_i, \alpha_i \rangle}\right]w(\alpha_i) = w^{-1} \circ r_i \circ w(\beta)
$$

$$
= \left[\frac{w(\alpha_i)}{\langle \alpha_i, \alpha_i \rangle}\right]f_{w(\alpha_i)}(\beta).
$$

17
holds for $w \in W(E_7^{(1)})$, a simple reflection $\alpha_i$ and $\beta \in Q(\alpha)$. Since the map

$$T := r_{E - E_5 - E_6 - E_7 - E_8} \circ r_{E - E_1 - E_2 - E_3 - E_4}$$

acts on the root basis as

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5, \alpha_6, \alpha_7 - 2\delta),$$

$T$ is a translation. Therefore $T^n$ defines a birational dynamical system on $X(4,8) \times \mathbb{P}^3$. It can also be considered to be a dynamical system on $\mathbb{P}^3$ with the parameters $u_i$'s (or $P_i$'s).

In case (i-1) or (i-3), similar to the above section, we have the action of $T$ on the parameter space $\{u_i\}$ as

$$T : (u_1, \cdots, u_8) \mapsto (u_1 + \lambda, u_2 + \lambda, u_3 + \lambda, u_4 + \lambda, u_5 - \lambda, \cdots, u_8 - \lambda),$$

where $\lambda = \frac{1}{2} \sum_{i=1}^{8} u_i$.

Since the explicit action of the transformation on $\mathbb{P}^3$ is complicated, we give it using a decomposition. Although it is enough to compose $r_{E - E_5 - E_6 - E_7 - E_8}$ and $r_{E - E_1 - E_2 - E_3 - E_4}$ of course, here we use the fact $T$ can be written as $T = S^2$ by

$$S := r_{E_4 - E_6} \circ r_{E_3 - E_7} \circ r_{E_2 - E_6} \circ r_{E_1 - E_3} \circ r_{E - E_1 - E_2 - E_3 - E_4}$$

and describe the action of $S$. $S$ acts on $\{u_i\}$ as

$$S : (u_1, \cdots, u_8) \mapsto (u_5 + \lambda_1, u_6 + \lambda_1, u_7 + \lambda_1, u_8 + \lambda_1, u_1 - \lambda_1, \cdots, u_4 - \lambda_1)$$

and the action on $\mathbb{P}^3$ is given by (35) and (36), where $\overline{u}_k := u_k - \frac{1}{2} \lambda_1 (1 \leq k \leq 4)$.

Along the same line, in case (i-2) we have

$$S : (u_1, \cdots, u_8) \mapsto (u_5 \lambda_1, u_6 \lambda_1, u_7 \lambda_1, u_8 \lambda_1, u_1 \lambda_1^{-1}, \cdots, u_4 \lambda_1^{-1})$$

and the action of $S$ on $\mathbb{P}^3$ is given by (35) and (36), where $\overline{u}_k := \overline{u}_k = u_k \lambda_1^{-1} (1 \leq k \leq 4)$.

6 Conservation law

In this section we prove the next theorem.

**Theorem 6.1.** In case (i) of Theorem 3.4, the action of the Weyl group preserves each element of the pencil of quadratic surfaces in Theorem 3.4.

**Remark 6.1.** The pencil of Theorem 3.4 is given by the linear system $| - \frac{1}{2} K_{X} |$ in generic. By Theorem 3.1, if the dimension of $| - \frac{1}{2} K_{X} |$ is 2 or more ($\delta = 0$ by Theorem 3.2), each fiber is preserved by translations associated with the Weyl group $W(E_7^{(1)})$, because (i) each surface is an elliptic surface (ii) each map is birational (hence continuous except at the indefinite points) and preserves the fibration (iii) the modulus of elliptic curve is preserved (iv) at least the intersection curve of the pencil is preserved.

Since for every discrete Painlevé equation the polynomial degree of the $n$-th iterate is in the order $n^2$ as $n \to \infty$ [17], by Theorem 3.4 the next corollary follows.

**Corollary 6.1.** Let $\varphi$ be a map on $\mathbb{P}^3$ associated with a translation of $W(E_7^{(1)})$. The degree of $\varphi^n$ is in the order $n^2$ as $n \to \infty$. 


Since \(-\frac{1}{2}K_X\) is preserved and therefore the pencil itself is preserved, in order to prove Theorem 6.1 it is enough to show that the automorphism of \(\mathbb{P}^1\) defined by this correspondence is the identity. In case (i-2) and (i-3) it is easily shown by direct computation. Moreover, the simple reflection \(r_i(0 \leq i \leq 6)\) acts on \(\mathbb{P}^3\) as the identity. Hence it is enough to show for \(r_7\) in case (i-1). For \(r_7\), to prove by direct calculation seems to be beyond our computational ability. We proof the theorem in this case by means of a birational representation of \(W(E_8^{(1)})\) on \(\mathbb{P}^1 \times \mathbb{P}^1\).

Notice that a smooth quadratic surface is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\) via the Segre map

\[
\mathbb{P}^1 \times \mathbb{P}^1 \ni (x : 1, y : 1) \mapsto (x : y : 1 : xy) \in \{(x : y : z : w) \in \mathbb{P}^3 \mid xy - zw = 0\}.
\]

Here we renormalize the parameter space so that the 8 points are on the intersection curve of the pencil spanned by the 2 quadratic surfaces

\[
\left\{ \begin{array}{l}
xy - zw = 0 \\
(x + y + z)(4w - \frac{g_3}{\varphi^2(2t)}z) = \left(w + x + y + \frac{g_2}{4\varphi^2(2t)}z\right)^2
\end{array} \right. \quad (G) \quad \text{(H)},
\]

where \(t \in (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}t)) \setminus \{0\}\) is an arbitrary extra-parameter.

**Remark 6.2.** The parameter \(\tau = w_2/w_1\) is invariant with respect to \(t\).

By this normalization, \(P_i\) can be parameterized as

\[
P_i\left(\frac{\varphi(t + u_i)}{\varphi(2t)} : \frac{\varphi(t - u_i)}{\varphi(2t)} : 1 : \frac{\varphi(t + u_i)\varphi(t - u_i)}{\varphi^2(2t)}\right).
\]

The action of \(r_7\) on \((u, t)\) and on \(\mathbb{P}^3\) are also given by (31) and (35) respectively, where we set

\[
f_x(u) = \frac{\varphi(t + u)}{\varphi(2t)}, \quad f_y(u) = \frac{\varphi(t - u)}{\varphi(2t)}, \quad f_w(u) = \frac{\varphi(t + u)\varphi(t - u)}{\varphi^2(2t)},
\]

\[
u_k := u_k - \lambda_1 \ (1 \leq k \leq 4)
\]

and

\[
l_{i, j, k}(x) = \begin{vmatrix}
f_x(u_1) & f_x(u_j) & f_x(u_k) & 1 & f_x(u_i) & f_x(u_j) & f_x(u_k) \\
f_y(u_1) & f_y(u_j) & f_y(u_k) & 1 & f_y(u_i) & f_y(u_j) & f_y(u_k) \\
f_w(u_1) & f_w(u_j) & f_w(u_k) & 1 & f_w(u_i) & f_w(u_j) & f_w(u_k) \\
x & f_x(u_i) & f_x(u_j) & f_x(u_k) \\
y & f_y(u_i) & f_y(u_j) & f_y(u_k) \\
w & f_w(u_i) & f_w(u_j) & f_w(u_k)
\end{vmatrix}
\]

Next, we list the necessary results by Murata et al. [11] concerning birational maps on \(\mathbb{P}^1 \times \mathbb{P}^1\) whose space of initial conditions are given by blown-ups of \(\mathbb{P}^1 \times \mathbb{P}^1\) at generic 8 points on the smooth curve of degree (2, 2)

\[
(x + y + 1)(4xy - \frac{g_3}{\varphi^2(2t)}) = \left(xy + x + y + \frac{g_2}{4\varphi^2(2t)}\right)^2.
\]

Let \(H_0, H_1\) and \(E_i\) denote the total transform of \(x = c \in \mathbb{P}^1\), that of \(y = c' \in \mathbb{P}^1\) and that of the exceptional divisor generated by the blow up at the point \(P_i\) respectively. The Picard group and the canonical divisor are

\[
\begin{align*}
\text{Pic}(X) &= \mathbb{Z}H_0 \oplus \mathbb{Z}H_1 \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \cdots \oplus \mathbb{Z}E_8 \\
K_X &= -2H_0 - 2H_1 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8
\end{align*}
\]
and the intersection numbers are given by

\[ H_i \cdot H_j = 1 - \delta_{i,j}, \quad H_i \cdot E_j = 0, \quad (E_i, E_j) = -\delta_{i,j}. \]

The root basis is given by

\[ \alpha_i = E_{7-i} - E_{8-i} \quad (i = 0, 1, \cdots, 5), \]
\[ \alpha_6 = H_1 - E_1 - E_2, \quad \alpha_7 = H_0 - H_1, \quad \alpha_8 = E_1 - E_2, \]

\[
\begin{array}{cccccccc}
\alpha_7 & \alpha_0 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_8 \\
\end{array}
\]

Figure 2: \( E_8^{(1)} \) Dynkin diagram

and each action on the parameter space \((u, t)\) becomes

\[
\begin{align*}
 r_1 : (u_{7-i}, u_{8-i}) & \mapsto (u_{8-i}, u_{7-i}), \quad (i = 0, 1, \cdots, 5) \\
 r_6 : & \left( \begin{array}{cccc}
 u_1 & u_2 & u_3 & u_4 \\
 u_5 & u_6 & u_7 & u_8 \\
 t & & & \\
 \end{array} \right) \\
 & \mapsto \left( \begin{array}{cccc}
 u_1 - 3\lambda_{1,2} & u_2 - 3\lambda_{1,2} & u_3 + \lambda_{1,2} & u_4 + \lambda_{1,2} \\
 u_5 + \lambda_{1,2} & u_6 + \lambda_{1,2} & u_7 + \lambda_{1,2} & u_8 + \lambda_{1,2} \\
 t - \lambda_{1,2} & & & \\
 \end{array} \right), \\
 r_7 : t & \mapsto -t, \\
 r_8 : (u_1, u_2) & \mapsto (u_2, u_1), \\
\end{align*}
\]

where \( \lambda_{1,2} = \frac{1}{4}(2t + u_1 + u_2). \)

**Proof of Theorem 6.1**

As mentioned above, it is sufficient to show for \( r_7 = r_{E-E_1-E_2-E_3-E_4} \) in case (i-1). Let \( G \) and \( H \) denote the matrices corresponding to \((G)\) and \((H)\) of the pencil \([41]\). We write \( (x : y : z : w) := r_7(x : y : z : w) \). Since the image of an element of the pencil \( (x : y : z : w)_{x'(\alpha_0G+\alpha_1H)x=0} \) \((\alpha_0 : \alpha_1) \in \mathbb{P}^1\) is again an element of \(| - \frac{1}{2}K_X |\) and therefore is again an element of the pencil (the intersection curve does not move). Hence, there exists \((\beta_0 : \beta_1) \in \mathbb{P}^1\) such that \( x'(\beta_0G+\beta_1H)x = 0 \). Since this correspondence defines an automorphism of the base space \( \mathbb{P}^1 \) of the pencil, it is enough to show that 3 elements of the pencil are preserved. Hence we may assume rank\((\alpha_0G+\alpha_1H) = \text{rank}(\beta_0G+\beta_1H) = 4\). In the following, we assume \((\alpha_0 : \alpha_1) \neq (\beta_0 : \beta_1) \) and lead a contradiction.

**Lemma 6.1.** Assume \( \text{rank}(\alpha_0G + \alpha_1H) = 4 \). There exist \( P \in PGL(4), v \in \mathbb{C}^\times \) and \( t' \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \) \( \{0\} \) such that

\[ P^t(\alpha_0G + \alpha_1H)P = G, \quad P^tHP = vH(\varphi(2t')) \]

and moreover there exist \( Q_1, Q_2 \in PGL(4), v_1, v_2 \in \mathbb{C}^\times \) and \( c, d \in \mathbb{C} \setminus \{0, 1\} \) such that

\[ Q_1^t(\alpha_0G + \alpha_1H)Q_1 = \text{diag}(1, 1, 1, 1), \quad Q_1^tHQ_1 = v_1\text{diag}(0, 1, c, d), \]
\[ Q_2^tH(\varphi(2t'))Q_2 = \text{diag}(1, 1, 1, 1), \quad Q_2^tHQ_2 = v_2\text{diag}(0, 1, c, d). \]

**Proof.** Since \( \text{rank}H = 3 \), similar to the proof of Theorem 3.1 \( G \) and \( H \) can be transformed to

\[ G' = \text{Id}, \quad H' = \text{diag}(0, a, b, 1) \quad a, b, 1 \text{ are different each other} \]
by $PGL(4)$. Here, $a$ and $b$ are functions of $g_2, g_3$ and $\varphi(2t)$. Conversely, if $a, b, 1$ are different each other, there exist corresponding $g_2, g_3$ and $\varphi(2t)$. On the other hand, they are also transformed to

$$(\alpha_0 G + \alpha_1 H)' = \text{Id. } H' = \text{diag}(0, c, d, 1) \quad c, d, 1 \text{ are different each other by } PGL(4).$$

Hence, there exist corresponding $g_2', g_3', \varphi(2t')$ and $P \in PGL(4)$ such that

$$P : \alpha_0 G + \alpha_1 H, H \mapsto G, H (g_2', g_3', \varphi(2t')).$$

Here, $r_7$ defines a birational map between the intersection curves and therefore the parameter $\tau w_2/w_1$ does not change. As Remark 6.2 if we fix $w_1$ as $w_1 = 1$, only $t'$ can change, which shows that $H'$ is a function only on $\varphi(2t')$.  

For $(\alpha_0 : \alpha_1)$ and $(\beta_0 : \beta_1)$ we denote $t'$ determined by Lemma 6.1 by $t_a$ and $t_b$ respectively. First, we show $t_a \neq t_b$. Assume $t_a = t_b$. Since only $PGL(4)$ is needed for $\text{diag}(0, 1, c, d)$ when one normalizes the later form of the basis of pencil in the above lemma to the form $\text{diag}(0, 1, 1, 1, \ldots, 1, \lambda, 0)$, there exist elements $J$ and $J'$ of pencil, where $J \neq J'$, $\text{rank }J = \text{rank }J' = 3$, such that both pairs $J, H$ and $J', H$ can be transformed to the form $\text{diag}(0, 1, 1, 1, \ldots, 1, \lambda, 0)$ by $PGL(4)$. Hence there exist elements $K = \text{diag}(1, 1, 1, 0)$ and $K'$ of the pencil $\{s_0 \text{diag}(0, 1, 1, 1) + s_1 \text{diag}(1, 1, 1, 0)\}$, where $K \neq K'$, $\text{rank }K' = 3$, such that the pair $K, H$ can be transformed to the pair $K', H$ by $PGL(4)$. The elements of this pencil in rank 3 are $K = \text{diag}(0, 1, 1, 1), \text{diag}(1, 1, 1, 0), K_1 := \text{diag}(1, 0, \lambda - 1, -1)$ and $K_2 := \text{diag}(1, 1, 1, 0)$. Therefore, $K'$ should be $K_1$ or $K_2$. Normalizing each by $PGL(4)$, we have that $\lambda = \lambda/(\lambda - 1), (\lambda - 1)/\lambda, 1 - \lambda$ or $1/(1 - \lambda)$. If $\lambda \neq 2, 1, 1 + \sqrt{3}/2$, it does not coincide with $\lambda$. Hence, if $\tau$ does not correspond to these $\lambda$, we have $t_a \neq t_b$ (as shown by (27), $t$ does not depend on $\varphi(2t)$)

Let $P_a, P_b$ denote the elements of $PGL(4)$ which give $t_a, t_b$ in Lemma 6.1. We consider the map defined by $P_b \circ r_7 \circ P_a^{-1}$. Since the period map is conserved by $PGL(4)$, we have $\overline{r_7} = u_i - \lambda_1 + c$ ($1 \leq i \leq 4$), $\overline{U} = u_i + \lambda_1 + c$ ($5 \leq i \leq 8$), where $c$ is a constant. On the other hand, the map

$$x^t Gx = 0 \rightarrow x^t (\alpha_0 G + \alpha_1 H) x = 0 \rightarrow x^t (\beta_0 G + \beta_1 H) x = 0 \rightarrow P_b \rightarrow x^t Gx = 0$$

and Segré map define a birational map on $\mathbb{P}^1 \times \mathbb{P}^1$.

By blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at $\{P_1\}$ and $\{\overline{P_1}\}$ the map is lifted to an isomorphism and included by the Weyl group of type $E_8^{(1)}$. Since the Weyl group conserves $\chi_S(\delta) = -\sum_{i=1}^8 u_i$, we have $c = 0$. Hence $r_7$ can be written as

$$r_7 = \tau_{H_0 + H_1 - E_1 - E_2 - E_3 - E_4} = r_6 \circ r_7 \circ r_5 \circ r_8 \circ r_4 \circ r_5 \circ r_6 \circ r_5 \circ r_4 \circ r_8 \circ r_5 \circ r_7 \circ r_6$$

by the root system of $E_8^{(1)}$, and we have $t_a = t_b$ form (12), which contradicts $t_a \neq t_b$.

We have shown $(\alpha_0 : \alpha_1) = (\beta_0 : \beta_1)$ if $\tau$ does not correspond to $\lambda \neq 2, 1, 1 + \sqrt{3}/2$. When $\lambda = 2, 1, 1 + \sqrt{3}/2$, it can be shown by continuity of $\varphi$ with respect to $\tau$.

\textbf{Proof of Theorem 6.1}

\section{Conclusions and discussions}

In this paper, we defined the inner product for the Picard group of varieties obtained by blow-ups at 8 points in $\mathbb{P}^3$ by means of the intersection numbers and the anti-canonical
divisor and showed that the symmetry group defined by means of the inner product is the Weyl group of type $E_7^{(1)}$. As in 2-dimensional case [15], if the configuration of points is special, the symmetry may become smaller.

This method can be applied to other families of 3-dimensional rational varieties.

**Example 7.1.** Let $X$ be a variety obtained by blow-ups at generic 6 points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $H_i$ and $E_i$ denote the total transform of divisor class of a plane such that one of its coordinate of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is constant and that of the exceptional divisor generated by a blow-up respectively. The symmetry group $Gr(\{X\})$ becomes the Weyl group of type $E_7^{(1)}$ defined by the root system

\[
\begin{align*}
\alpha_0 &= H_0 - H_2, \quad \alpha_1 = H_1 - H_2, \quad \alpha_3 = H_2 - E_1 - E_2, \\
\alpha_i &= E_{i-1} - E_i \ (3 \leq i \leq 6), \quad \alpha_7 = E_1 - E_2,
\end{align*}
\]

where the inner product is given by $(H_i, H_j) = 1 - \delta_{i,j}$, $(H_i, E_j) = 0, (E_i, E_j) = -\delta_{i,j}$.

This $X$ and the variety obtained of blow-ups at generic 8 points in $\mathbb{P}^3$ are not isomorphic. Thus, the relation between these 2 Weyl groups is not trivial.

It may be worth commenting that the space of initial conditions for Kajiwara-Noumi-Yamada’s birational representation of the Weyl group of type $A_1^{(1)} \times A_2^{(1)}$ [9] can be obtained when the points of blow-ups are in special position in the above example. These examples are 3-dimensional but it is expected that our method can be applied to 4(or more)-dimensional cases.

The other results are summarized as follows.

- We normalized the configuration space by normalizing the pencil of quadratic surfaces passing through 8 points ($\in |-\frac{1}{2}K_X|$). The intersection curve is an elliptic curve in generic.
- In order to obtain concrete expression of the action of the Weyl group, we introduced a 3-dimensional analogue of period map.
- We showed the action of the Weyl group preserves each element of $|-\frac{1}{2}K_X|$ and therefore it reduces to the action on $\mathbb{P}^1 \times \mathbb{P}^1$. The reduced action of the Weyl group of type $E_7^{(1)}$ is included by the action of the Weyl group of type $E_8^{(1)}$, which is the symmetry of the family of generic Halphen surfaces.

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**References**

[1] Adler, M. and van Moerbeke, P. The complex geometry of the Kowalewski-Painlevé analysis. Invent. Math. 97, 3–51 (1989)

[2] Coble, A. B. Algebraic geometry and theta functions. American Mathematical Society Colloquium Publications, vol. X. Providence: American Mathematical Society 1929

[3] Dolgachev, I. and Ortland, D. Point sets in projective spaces and theta functions. Astérisque Soc.Math.de France 165, 1988
[4] Diller, J. and Favre, C. Dynamics of bimeromorphic maps of surfaces. Amer. J. Math. 123, 1135–1169 (2001)

[5] Gizatullin, M. H. Rational G-surfaces. Izv. Akad. Nauk SSSR Ser. Mat. 44, 110–144, 239 (1980)

[6] Grammaticos, B., Ramani, A. and Papageorgiou, V. Do integrable mappings have the Painlevé property? Phys. Rev. Lett. 67, 1825–1827 (1991)

[7] Horn, A. and Hohnson, R. Matrix analysis. Cambridge: Cambridge University Press 1985

[8] Kac, V. Infinite dimensional lie algebras, 3rd ed. Cambridge: Cambridge University Press 1990

[9] Kajiwara, K., Noumi, M. and Yamada, Y. Discrete dynamical systems with $W(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ symmetry. Lett. Math. Phys 60, 211–219 (2002)

[10] Looijenga, E. Rational surfaces with an anti-canonical cycle. Annals of Math. 114, 267–322 (1981)

[11] Murata, M., Sakai, H. and Yoneda, J. Riccati solutions of discrete Painlevé equations with Weyl group symmetry of type $E_8^{(1)}$. J. Math. Phys. 44, 1396–1414 (2003)

[12] Okamoto, K. Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé. Japan J. Math. 5, 1–79 (1979)

[13] Ramani, A., Grammaticos, B. and Hietarinta, J. Discrete versions of the Painlevé equations. Phys. Rev. Lett. 67, 1829–1832 (1991)

[14] Saito, M. H., Takebe, T. and Terajima, H. Deformation of Okamoto–Painlevé pairs and Painlevé equations. J. Algebraic Geom. 11, 311–362 (2002)

[15] Sakai, H. Rational surfaces associated with affine root systems and geometry of the Painlevé equations. Commun. Math. Phys. 220, 165–229 (2001)

[16] Shioda, T. and Takano, K. On some Hamiltonian structures of Painlevé systems, I. Funkcial. Ekvac. 40, 271–291 (1997)

[17] Takenawa, T. Algebraic entropy and the space of initial values for discrete dynamical systems. J. Phys. A: Math. Gen. 34, 10533–10545 (2001)

[18] Yoshida, M. Hypergeometric functions, my love. Modular interpretations of configuration spaces. Aspects of Mathematics, E32. Braunschweig: Friedr. Vieweg Sohn 1997