A revision of the subtract-with-borrow random number generators

Alexei Sibidanov∗

University of Victoria, Victoria, BC, Canada V8W 3P6

Abstract

The most popular and widely used subtract-with-borrow generator, also known as RANLUX, is reimplemented as a linear congruential generator using large integer arithmetic with the modulus size of 576 bits. Modern computers, as well as the specific structure of the modulus inferred from RANLUX, allow for the development of a fast modular multiplication – the core of the procedure. This was previously believed to be slow and have too high cost in terms of computing resources. Our tests show a significant gain in generation speed which is comparable with other fast, high quality random number generators. An additional feature is the fast skipping of generator states leading to a seeding scheme which guarantees the uniqueness of random number sequences.

Keywords: Linear congruential generator; Subtract-with-borrow generator; RANLUX; GMP;

PROGRAM SUMMARY/NEW VERSION PROGRAM SUMMARY

Program Title: RANLUX++
Licensing provisions: GPLv3
Programming language: C++, C, Assembler

∗E-mail address: sibid@uvic.ca
1. Introduction

The well known Linear Congruential Generator (LCG) is a recurrent sequence of numbers calculated as follows:

\[ x_{i+1} = (a \cdot x_i + c) \mod m, \]  

where \( x_0 \) is the initial state or seed, \( a \) – the multiplier, \( c \) – the increment and \( m \) – the modulus. The particular choice of the parameters \( a, c \) and \( m \) with period \( q \) – the minimal number when \( x_q = x_0 \), can be found in the literature \[1\]. Commonly used LCGs are limited to \( m \leq 2^{64} \), and have poor statistical properties. Thus they are not used for Monte-Carlo physical simulations.

This situation can be mitigated when \( m \) reaches several hundreds or even thousand bits. The cost of the increased range of \( m \) is to deal with arbitrary precision integer arithmetic which was believed to be prohibitively expensive for practical purposes. In the last two decades there has been tremendous progress in modern central processor units (CPU) especially for personal computers (PC) which can be employed for long arithmetic.

We have explored the possibility to use the long arithmetic in LCG to improve the quality of generated random numbers and found that, despite a substantial increase in calculations, the time to generate a single random number is not proportionally risen. In fact for some parameters, the computational time decreased compared to ordinary LCGs with machine word modulus size.

2. Subtract-with-borrow generator

At this point no specific constraints on \( a, c \) and \( m \) parameters of LCG have been applied. As a good starting point we choose the subtract-with-borrow generator first introduced in \[2\] and the intimate connection with LCG has been shown as a part of the period calculation. The algorithm has been extensively studied in \[3\] to improve statistical quality of generated numbers. Based on this study the generator RANLUX \[4\] was developed and now it is widely used in physics simulations as well as in other fields where random numbers with high statistical quality are required. However the current method employed by RANLUX to achieve the high quality makes it one of the slowest generators on the market.
The definition of the subtract-with-borrow generator is the following: let \( b \) some integer greater than 1 also called the base and vector \( Y = (y_1, \ldots, y_r, k) \) with the length \( r + 1 \), where \( 0 \leq y_i < b \) and \( k \) or the carry equals 0 or 1. Then define a recursive transformation of the vector \( Y_i \) with the rule:

\[
Y_{i+1} = \begin{cases} 
(y_2, \ldots, y_r, \Delta, 0), & \text{if } \Delta \geq 0 \\
(y_2, \ldots, y_r, \Delta + b, 1), & \text{otherwise} 
\end{cases}
\]

(2)

where \( \Delta = y_{r-s+1} - y_1 - k \) and \( r \) and \( s \) also called the lags. As shown in the work [5], this recursion is equivalent to LCG with the modulus \( m = b^r - b^s + 1 \), the multiplier \( a = m - (m - 1)/b \) and \( c = 0 \) with the relation:

\[
x_i = x(Y_i) \equiv \sum_{j=1}^{r} y_j b^{j-1} - \sum_{j=1}^{s} y_{r-s+j} b^{j-1} + k
\]

(3)

In the RANLUX generator the lags \( r = 24 \) and \( s = 10 \) with the base \( b = 2^{24} \) are chosen among other suggested parameters in [2], and thus the modulus \( m = b^{24} - b^{10} + 1 \) is a prime number and the multiplier \( a = m - (m - 1)/b \) and \( c = 0 \) with the relation:

Due to the selected base \( b \) the natural choice to keep the generator state is a vector of length 24 composed of 24-bit numbers. This implementation uses the properties of the modulus \( m \) to avoid long arithmetic calculations, and a single step equivalent to one modular multiplication \( (a \cdot x \mod m) \) that requires only subtraction of two 24-bit numbers and carry propagation. In the original FORTRAN implementation, 24-bit numbers were stored as floats to avoid at that time, a high cost integer-to-float conversion.

2.1. Remainder

The simple structure of the modulus \( m \) allows us to calculate the remainder using only additions, subtractions and bit shifts. The modulus \( m \) and thus the generator state \( x \) have size of \( 24 \cdot 24 = 576 \) bits and fits into 9 64-bit machine words. The result of the product \( z = a \cdot x \) fits into 18 64-bit machine words which can be represented as a 48 element array of 24-bit numbers: \( z = [z_0, z_1, \ldots, z_{46}, z_{47}] \). The number \( r \) obtained by the procedure shown in Algorithm 1 is congruent to \( z \mod m \) and \( r < b^{24} \). Note the product \( c \cdot m \) is also only bit shifting due to the simple structure of \( m \). The calculation of \( c \) is a sum of carry bits of each arithmetic operation.
**Algorithm 1** Calculating remainder using only additions, subtractions and bit shifts for the modulus \( m = b^{24} - b^{10} + 1 \).

```
1: procedure Remainder(z) \( \triangleright 0 \leq z < b^{38} \)
2:     \( t_0 \leftarrow [z_0, \ldots, z_{23}] \) \( \triangleright 0 \leq t_0 < b^{24} \)
3:     \( t_1 \leftarrow [z_{24}, \ldots, z_{47}] \) \( \triangleright 0 \leq t_1 < b^{24} \)
4:     \( t_2 \leftarrow [z_{38}, \ldots, z_{47}] \) \( \triangleright 0 \leq t_2 < b^{10} \)
5:     \( t_3 \leftarrow [z_{24}, \ldots, z_{37}] \) \( \triangleright 0 \leq t_3 < b^{14} \)
6:     \( r \leftarrow t_0 - (t_1 + t_2) + (t_3 + t_2) \cdot b^{10} \)
7:     \( c \leftarrow [r/b^{24}] \) \( \triangleright \text{floor function rounds to } -\infty \)
8:     \( r \leftarrow r - c \cdot m \) \( \triangleright 0 < r < b^{24} \)
9:     return \( r \)
10: end procedure
```

### 2.2. Skipping

Examining the result of a single step of Eq. 1 one can note that the main part of the number \( x_i \) is preserved in its successor \( x_{i+1} \) which is just rotated by 24 bits. This strong correlation is the reason of the poor statistical quality of the original subtract-with-borrow generator [2]. The bright idea developed in [3] is to apply the transformation (2) many times to break correlations between nearby states before using the state for actual physical simulation. The drawback of this method is obvious – all intermediate states have to be explicitly calculated even if they are not needed. Despite the single step being simple with small resource consumption, good statistical quality requires several hundred steps thus in total, the skipping requires a lot of time. This is a luxury to spend resources and not use the results. Thus so-called luxury levels were introduced as aliases for how many generated numbers have to be wasted.

Using Eq. 1 we can efficiently skip numbers since all \( p \) recurrent steps collapse to a single multiplication:

\[
\underbrace{a \cdot (a \cdot (\ldots) \mod m)}_{p \text{ times}} \mod m = (a^p \mod m) \cdot x \mod m = A \cdot x \mod m, \quad (4)
\]

where the factor \( A \equiv (a^p \mod m) \) is precomputed and thus the cost to calculate the next state with or without skipping is the same. Any state in the entire period \( q = (m - 1)/48 \approx 10^{171} \) can be calculated in no more than \( 2 \times \log_2(q) \approx 1140 \) long multiplications using fast exponentiation by squaring which takes order of tens of \( \mu \text{sec} \) on modern CPUs.
In the Table 1 the precomputed values of $A \equiv (a^p \mod m)$ where the values of $p$ is taken from [4] are shown for illustrative purposes. In the initial rows, long chains of 0 or 1 in binary representation are clearly visible and this can be interpreted such that for each bit of the state $x_{i+p}$ only a few bits of the state $x_i$ contributes. Even at the highest luxury level 4 there are still some patterns observable and a demanding user maybe not be completely satisfied. For such user the two last rows would be more attractive especially since it is for free! Such chaotic multipliers mean that if any single bit of the state $x_i$ is changed in the next step the altered state will be absolutely different from the unaltered one.

With explicit long multiplication, there is no need to keep the multiplier $A$ as a power of $a$, it can be adjusted to get the full period, $m - 1$. As an example the number $(a^{2^{48}} + 13 \mod m)$ is a primitive root modulo $m$ and with this multiplier all numbers in the range $1 \ldots m - 1$ will appear in the sequence only once with any initial $x_0$ from the same range.

The fast skipping also provides a seeding scheme which guarantees non-colliding sequences of random numbers – what is needed is to just skip a large enough sequence. This is suggested in [5] using the seed number as a power of the multiplier:

$$x_s = (a^n)^s \cdot x_0 \mod m,$$

where $s$ is the seed, $x_0$ is the initial state which is the same for all seeds, $x_s$ is the starting state to deliver random numbers for the seed $s$ and $n$ is the seed skipping factor to guarantee non-colliding sequences for any Monte Carlo simulation. Lets take $n = 2^{96} \approx 10^{29}$ which is so big that for any modern computer the required time to visit all states within the range exceeds the age of the Universe, even if it can generate a new state on every CPU clock cycle. In this case the maximum seed number is $s < q/n \approx 2^{474} \approx 10^{143}$.

3. Implementation

The core procedure for efficient implementation of LCG is fast long multiplication. This provides the infrastructure to deliver random numbers in an efficient as well as convenient form to users. Programming languages such as C++, C or FORTRAN which are usually used in Monte Carlo physics simulations, have no native support of the long arithmetic despite all desktop grade CPUs providing hardware instructions suitable for it. The support
Table 1: The multiplier $A \equiv a^p \mod m$ which corresponds to the RANLUX strategy to waste $p - 24$ random numbers before next 24 numbers will be delivered to a user as well as the associated luxury levels.

| luxury p level | $a^p \mod m$ |
|----------------|---------------|
| 0 24           | ffffffffffffffffefffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffefffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffffflli
of long arithmetic is provided by external libraries. We choose the GMP library \cite{6} for proof of concept. This library is highly optimized for basic arithmetic operations with long numbers and supports a large number of CPU architectures. It was found that GMP is a great tool but some operations with long numbers still do not fit properly to the interface provided and the conversion to a proper format and back adds a significant overhead. This fact and for the sake of all inclusive package without external library dependencies, the core functions are written in the assembly language for the AMD64 CPU architecture. This should fit to virtually all computers used for physics simulations. For other architectures the long multiplication can be easily reimplemented. The GMP version of the generator was used to validate the results. The problem size is known in advance and this knowledge has been exploited to write the fast and efficient procedures.

Since inception of AMD64 CPU architecture the instruction set has been significantly expanded and new instructions profitable for long arithmetic were introduced. Specifically the instruction \texttt{mulx} which streams the result of the multiplication to arbitrary registers and then the instructions \texttt{adox} and \texttt{adcx} the addition instructions which affect only corresponding overflow and carry flags of the status register and are suitable to break the dependency chain in long summations. To use advantages of the new instructions where it is possible, three versions of 576 by 576 bit multiplication were written and the best suitable version is selected and executed in run-time by a CPU dispatcher. In all cases the method to multiply numbers is a simple schoolbook method which requires \(81 \times 64\times 64\) bit multiplications.

Considering the state \(x\) as 576 bit entropy pool, to produce floating point numbers, 24 or 52 bits are fetched from the pool for single or double precision number correspondingly. The latter number has full precision of 53 bits but from the 576 bits, only 10 53-bit numbers can be produced, wasting the remaining 46 bits of entropy which are expensive. A good compromise is to have random first 52 bits and waste only 4 bits of entropy. Conversion of bit strings to the floating point representation is done in batch into intermediate buffer from which the numbers delivered to a user. This strategy significantly improves overall performance.

4. Benchmarks

The benchmarks were conducted on three types of CPU – Core2 (Intel(R) Core(TM)2 Duo CPU E8400 @ 3.00GHz), Haswell (Intel(R) Core(TM) i7-
Table 2: CPU clocks needed to multiply two 576 bit numbers for generic functions from GMP (mul\_basecase\_*) and the author’s 9 × 9 limbs multiplications for various CPU families.

| Function         | Core2  | Haswell | Skylake |
|------------------|--------|---------|---------|
| mul\_basecase\_core2 | 370.9  | 218.5  | 197.1  |
| mul\_basecase\_coreihwl | n.a.  | 205.6  | 203.7  |
| mul\_basecase\_coreibwl | n.a.  | n.a.   | 163.0  |
| mul9x9          | 534.7  | 198.8  | 192.0  |
| mul9x9mulx      | n.a.   | 162.1  | 154.5  |
| mul9x9mulxadox  | n.a.   | n.a.   | 119.4  |

4790K CPU @ 4.00GHz) and Skylake(Intel(R) Core(TM) i5-6200U CPU @ 2.30GHz). Since data for all benchmarks fits to L1 cache and there is not much dependence on main memory we present results of benchmarks in CPU clocks. It was observed that within CPU family benchmark results are stable despite different CPUs can have largely different clock frequencies. All measurements were done by the Linux perf program which provides convenient access to internal CPU performance counters.

The benchmark results of the specialized 9 by 9 64-bit limbs multiplications together with the generic functions of GMP are shown in Table 2. It is seen that modern CPUs provide significant improvement in integer arithmetic especially in the straightforward implementation of the long multiplication the author made. Apriori knowledge of operand sizes gives additional performance boost compared to the GMP functions which work with arbitrary size vectors.

Algorithm 1 to calculate the remainder does not fit properly into the GMP function set and no elegant and performance wise ways were found, so only the author’s implementations are tested. The benchmarks of the modular multiplication are shown in Table 3 where the multiplication and the remainder calculation are joined into a single function to keep intermediate results in CPU registers. About 6.5-8.3 and 14-18 clocks is needed to produce 24 and 52 bits of high quality entropy correspondingly for single and double precision numbers.

It is interesting to compare different implementations to perceive how they can vary. Fair comparison of different implementations of random number generators requires special attention and microbenchmark results on modern
Table 3: CPU clocks needed for multiplication of two 576 bit numbers modulo $2^{576} - 2^{240} + 1$.

| Function             | CPU type | Haswell | Skylake |
|----------------------|----------|---------|---------|
| mulmod9x9mulx        |          | 201.2   | 191.6   |
| mulmod9x9mulxadox    |          | n.a.    | 155.4   |

CPUs have to be cautiously interpreted due to out-of-order execution of instructions, memory accesses in real applications and so on. The small benchmark conducted is to sum $10^9$ random numbers uniformly distributed from 0 to 1. In this procedure we ensured that the explicit call instruction is emitted to produce the random number in order to avoid inlining of a function body into the summation loop. The results of testing various random number generators are shown in Table 4.

The dummy function calculates nothing and simply immediately returns 0.5 demonstrating the overall overhead of the benchmark as well as its uncertainty. The “std” prefix shows functions from the recent C++11 standard described in the (random) header implemented in GNU C++ compiler. For the subtract-with-borrow functions implemented in the C++11 standard the double precision numbers, for the benchmarking purpose, have explicitly only 48 bits of randomness to avoid the situation when 96 bits of entropy produced and only 53 bits of them are used. The “gsl” prefix shows functions from the GNU Scientific Library. TRandom1 is the ROOT implementation of RANLUX. RANLUX is the original FORTRAN implementation. The functions ranlxs and ranlxdo are SIMD implementations for single and double precision numbers correspondingly of M. Lüscher, the author of the skipping approach. RANLUX++ is the suggested implementation using the long modular multiplication. The “array” comment shows another way to fetch produced random numbers – first fill a large array of numbers to avoid boundary checks for each number and then sum numbers from the array.

The result of the benchmark shows that the suggested implementation of LCGs with long modular multiplication is among the fastest random numbers generators. The other RANLUX implementations with the high statistical quality are an order of magnitude slower. In the current benchmark the suggested generator is even faster than the simpliest LCG in the C++ standard std::minstdrand. With already theoretically proven statistical quality also supported by empirical tests as well as the seeding scheme guaran-
Table 4: CPU clocks needed to sum $10^9$ random numbers at Haswell CPU normalized to a single number. All tests were compiled by gcc v6.2.1 with -O3 optimization flag.

| Generator            | Number type | double | float |
|----------------------|-------------|--------|-------|
| dummy                |             | 9.1    | 9.1   |
| std::minstd_rand     |             | 35.1   | 20.2  |
| std::mt19937_64      |             | 36.0   | 37.0  |
| std::ranlux24_base   |             | 47.2   | 26.0  |
| std::ranlux48_base   |             | 24.4   | 26.1  |
| std::ranlux24        |             | 387.0  | 197.7 |
| std::ranlux48        |             | 640.5  | 638.4 |
| gsl_ranlxs0          |             |        | 84.2  |
| gsl_ranlxs1          |             |        | 125.9 |
| gsl_ranlxs2          |             |        | 215.5 |
| gsl_ranlxd1          |             | 213.8  |       |
| gsl_ranlxd2 ($p = 397$) |             | 394.0  |       |
| gsl_ranlux ($p = 223$) |             |        | 185.0 |
| gsl_ranlux389 ($p = 389$) |         | 315.4  |       |
| TRandom1 ($p = 389$) |             | 362.7  |       |
| RANLUX ($p = 389$)   |             | 378.3  |       |
| ranlxs (array, SSE, $p = 397$) |         | 50.4   |       |
| ranlxd (array, SSE, $p = 397$) |         | 95.0   |       |
| RANLUX++             |             | 29.2   | 20.0  |
| RANLUX++ (array)     |             | 24.7   | 15.7  |

For non-colliding sequences this generator might be the best option for large scale parallel physical simulations.

5. Conclusion

We revised the approach that the large moduli in linear congruential generators are unpractical in MC physical simulations. We showed that LCG can be efficiently implemented in case of specially selected modulus such as in the subtract-with-borrow generator. The famous RANLUX random number generator was reimplemented using the long modular multiplication. The test results show an order of magnitude improvement in the generation speed on modern CPUs. The generator has a relatively small state vector of 72 bytes,
Appendix A. An efficient implementation of the subtract-with-borrow generator

The step of the subtract-with-borrow generator is very simple and the current RANLUX algorithm can be implemented in a way to take advantages of modern CPUs. At the default luxury level $p = 223$ about 9 states are skipped so it is more efficient to skip an entire state, not numbers one by one. This immediately gives a performance boost since the array elements can be efficiently prefetched to CPU registers for processing. The carry bit $k$ can be calculated exploiting the two’s compliment format of signed integer number. The carry $k$ in this format is the most significant bit of the difference $\Delta$ which is 1 for $\Delta < 0$ and 0 otherwise. Extracting the last 24 bits of the difference $\Delta$ by bit masking makes the addition of $b = 2^{24}$ in Eq. 2 redundant since it does not change those bits. The optimized version of the step shown in Algorithm 2 can be easily implemented in the C language. The algorithm is also well suited for a SIMD implementation similarly to what was done in [10] running several generators in parallel according to the width of SIMD registers. Several implementations of the algorithm were written and tested. The test results are shown with $p = 17 \times 24 = 408$ (the closest to and greater than $p = 389$ or the highest luxury level 4 of the original RANLUX generator) in Table A.5 for various types of optimization: “scalar” version is the straightforward implementation of Algorithm 2 “scalar(asm)” – main loop is optimized in the assembler language to access the hardware carry propagation, “SSE2” and “AVX2” are the optimizations using vector instructions. The generation speed for the latter case is so large that mainly the overhead of the benchmark itself is seen. There is an order of magnitude speedup compared to the results shown in Table 4.

The throughput of the skipping procedure itself without delivering numbers to the user corresponds to what one would expect from Algorithm 2: about 2 clock/(24 bit) for the scalar case, 0.5 clock/(24 bit) for SSE2 and 0.25 clock/(24 bit) for AVX2. The assembler implementation of the scalar case was written to access the hardware propagation of the carry bit, unfortunately the necessity to clear 8 most significant bits of each number also affects
Algorithm 2 Optimized version of the subtract-with-borrow step updating the element $x_i$ in $x = [x_0, \ldots, x_{23}]$ and the carry $k$.

1: procedure UpdateElement($i, j, k$) \Comment{$0 \leq i, j < 24, 0 \leq k \leq 1$}
2: \hspace{1em} $d \leftarrow x_j - x_i - k$ \Comment{32 bit signed integer in two’s compliment format}
3: \hspace{1em} $k \leftarrow d \gg 31$ \Comment{logical right shift to extract the MSB}
4: \hspace{1em} $x_i \leftarrow d \& 0xfffff$ \Comment{select and store the last 24 bits of $d$}
5: \hspace{1em} return $k$ \Comment{return carry}
6: end procedure

Table A.5: CPU clocks needed to sum $10^9$ single precision random numbers of the subtract-with-borrow algorithm with the skipping at Haswell CPU normalized to a single number. The skipping mode corresponds to the highest luxury level 4 of RANLUX. All tests were compiled by gcc v6.2.1 with -O3 optimization flag.

| scalar | scalar(asm) | SSE2 | AVX2 |
|--------|-------------|------|------|
| 40.3   | 28.5        | 12.0 | 7.95 |

the status register with the carry bit. Thus this requires special treatment in the code and only the throughput about 1.5 clock/(24 bit) was achieved. These improvements can be easily applied in all current implementations of the subtract-with-borrow generator.

References

[1] D. E. Knuth, The Art of Computer Programming, Volume 2 (3rd Ed.): Seminumerical Algorithms, Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1997.

[2] G. Marsaglia, A. Zaman, A new class of random number generators, The Annals of Applied Probability 1 (3) (1991) 462–480. URL [http://www.jstor.org/stable/2959748](http://www.jstor.org/stable/2959748)

[3] M. Lüscher, A portable high-quality random number generator for lattice field theory simulations, Computer Physics Communications 79 (1) (1994) 100 – 110. doi:10.1016/0010-4655(94)90232-1 URL [http://www.sciencedirect.com/science/article/pii/0010465594902321](http://www.sciencedirect.com/science/article/pii/0010465594902321)

[4] F. James, RANLUX: A Fortran implementation of the high-quality pseudorandom number generator, Computer Physics Communications 79 (1) (1994) 111 – 114.
[5] S. Tezuka, P. L’Ecuyer, R. Couture, On the lattice structure of the add-with-carry and subtract-with-borrow random number generators, ACM Trans. Model. Comput. Simul. 3 (4) (1993) 315–331. doi:10.1145/159737.159749.
URL http://doi.acm.org/10.1145/159737.159749

[6] GNU multiple precision arithmetic library.
URL https://gmplib.org/

[7] Draft: C++ international standard.
URL http://www.open-std.org/JTC1/SC22/WG21/docs/papers/2011/n3242.pdf

[8] GNU scientific library.
URL https://www.gnu.org/software/gsl/

[9] Data analysis framework.
URL https://root.cern.ch/

[10] Ranlux: Random number generator.
URL http://luscher.web.cern.ch/luscher/ranlux/

[11] P. L’Ecuyer, R. Simard, Testu01: A C library for empirical testing of random number generators, ACM Trans. Math. Softw. 33 (4) (2007) 22:1–22:40. doi:10.1145/1268776.1268777.
URL http://doi.acm.org/10.1145/1268776.1268777

[12] Source code repository.
URL https://github.com/sibidanov/ranluxpp/