Simple PTAS’s for families of graphs excluding a minor

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Abstract

We show that very simple algorithms based on local search are polynomial-time approximation schemes for Maximum Independent Set, Minimum Vertex Cover and Minimum Dominating Set, when the input graphs have a fixed forbidden minor.

1 Introduction

In this paper we present very simple PTAS’s (polynomial-time approximation schemes) based on greedy local optimization for Maximum Independent Set, Minimum Vertex Cover and Minimum Dominating Set in minor-free families of graphs. The existence of PTAS’s for such problems was shown by Grohe [8], and better time bounds were obtained using the framework of bidimensionality; see the survey [5] and references therein. The advantage of our algorithms is that they are surprisingly simple and do not rely on deep structural results for minor-free families.

A graph \(H\) is a minor of \(G\) if \(H\) can be obtained from a subgraph of \(G\) by edge contractions. We say that \(G\) is \(H\)-minor-free if \(H\) is not its minor. A family of graphs is \(H\)-minor-free if all the graphs in the family are \(H\)-minor-free. It is well-known that the family of planar graphs is \(K_{3,3}\)-minor-free and \(K_5\)-minor-free, and similar results hold for graphs on surfaces. Thus, minor-free families is a vast extension of the family of planar graphs and, more generally, graphs on surfaces. We will restrict our attention to \(K_h\)-minor-free graphs, where \(K_h\) is the complete graph on \(h\) vertices, because \(H\)-minor-free graphs are also \(K_{|V(H)|}\)-minor-free.

The development of PTAS’s for graphs with a forbidden fixed minor is often based on a complicated theorem of Robertson and Seymour [13] describing the structure of such graphs. In fact, one needs an algorithmic version of the structural theorem and much work has been done to obtain simpler and faster algorithms finding the decomposition. See Grohe, Kawarabayashi and Reed [9] for the latest improvement and a discussion of previous work. Even those simplifications are still very complicated and, in fact, the description of the structure of \(K_h\)-minor-free graphs is cumbersome in itself. Obtaining a PTAS for Maximum Independent Set restricted to minor-free families is easier and can be done through the computation of separators, as shown by Alon, Seymour, and Thomas [1]. However, the approach does not work for Minimum Vertex Cover and Minimum Dominating Set. Baker [2] developed a technique to obtain PTAS for planar graphs using more elementary tools. In fact, much of the work for minor-free families is a vast, complex generalization of the approach by Baker.

To show how simple is our approach, look at the algorithm Independent\((h, G, \varepsilon)\) for Maximum Independent Set shown in Figure 1. The algorithms for Minimum Vertex Cover and

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Algorithm 1: \textsc{Independent}(h, G, \varepsilon)

\textbf{Input}: An integer \(h > 0\), a \(K_h\)-minor-free graph \(G = (V, E)\), and a parameter \(\varepsilon \in (0, 1)\)

\textbf{Output}: An independent set \(U\) of \(G\)

\begin{enumerate}
    \item \(r = C_h / \varepsilon^2\), where \(C_h\) is an appropriate constant depending on \(h\)
    \item \(U = \emptyset\)
    \item \textbf{while} \(\exists U_1 \subseteq U, V_1 \subseteq V\) with \(|U_1| < |V_1| \leq r\) and \((U \setminus U_1) \cup V_1\) is an independent set \textbf{do}
    \item \(U = (U \setminus U_1) \cup V_1\)
    \item \textbf{return} \(U\)
\end{enumerate}

Figure 1: PTAS for Maximum Independent Setset for \(K_h\)-minor-free graphs.

\textbf{Minimum Dominating Set} are similar and provided in Section 3. In the algorithms we use a constant \(C_h\) that depends only on the size of the forbidden minor. Its actual value is in \(\Theta(h^3)\), as we shall see.

We see that, for any fixed \(h\), the algorithm is a very simple local optimization that returns an independent set that is \(O(\varepsilon^{-2})\)-locally optimal, in the sense that it cannot be made larger by substituting any \(O(\varepsilon^{-2})\) of its vertices. The algorithm runs in time \(n^{O(\varepsilon^{-2})}\), for any fixed \(h\).

The main idea in the proof of the correctness of our algorithm is dividing the input graph into not-too-many pieces with \(O(\varepsilon^{-2})\) vertices and small boundary, as defined in Section 2. For this we use the existence of separators [1] in the same way as Frederickson [6] did for planar graphs. A similar division has been used in other works; see for example [14]. The division is useful for the following fact: changing the solution \(U\) within one of the pieces cannot result in a better solution because \(U\) is \(O(\varepsilon^{-2})\)-locally optimal. Using this, we can infer (after some work) that, if \(G\) is \(K_h\)-minor-free, then

\[ \text{opt} - |U| \leq \varepsilon \cdot \text{opt}. \]

For \textbf{Minimum Vertex Cover} and \textbf{Minimum Dominating Set} one has to make the additional twist of considering a division in a graph that represents the locally optimal solution and the optimal solution.

It is important to note that the analysis of the algorithm uses separators but the algorithm does not use them. Thus, all the difficulty is in the proof that the algorithm is a PTAS, not in the description of the algorithm. In any case, our proofs only rely on the existence of separators and is dramatically simpler than previous proofs of existence of PTAS’s for \textbf{Minimum Vertex Cover} and \textbf{Minimum Dominating Set}. In particular, we do not need any of the tools developed for the Graph Minor Theorem. A drawback of our method is that the running time is \(n^{O(\varepsilon^{-2})}\), while previous, more complicated methods require \(O(f(\varepsilon)n^c)\), for some constant \(c > 0\) and function \(f\).

Another drawback of our method is that it works only in unweighted problems.

The idea of using separators to show that a local-optimization algorithm is a PTAS was presented by Chan and Har-Peled [3] and independently by Mustafa and Ray [12]. The technique has been used recently to provide PTAS’s for some geometric problems; see for example [4, 7, 11]. However, the use for minor-free families of graphs has passed unnoticed.

2 Dividing minor-free graphs

In this section we present a way of dividing a graph into subgraphs with special properties. We will not use this division in our algorithms, but it will be the main tool for their analysis.

Let \(G\) be a graph and let \(S = \{S_1, \ldots, S_k\}\) be a collection of subsets of vertices of \(G\). We define the \textit{boundary} of a piece \(S_i \in S\) (with respect to \(S\)), denoted by \(\partial S_i\), as those vertices of
A division of a graph \( G \) is a collection \( S = \{S_1, S_2, \ldots, S_k\} \) of subsets of vertices of \( G \) satisfying the following two properties:

- \( G = \bigcup_i G[S_i] \), that is, each edge and each vertex of \( G \) appears in some induced subgraph \( G[S_i] \), and
- for each \( S_i \in S \) and \( v \in \text{int}(S_i) \), all neighbours of \( v \) are in \( S_i \).

We refer to each subset \( S_i \in S \) as a piece of the division. (It may be useful to visualize a piece as the induced subgraph \( G[S_i] \), since we actually use \( S_i \) as a proxy to \( G[S_i] \).)

We want to find a division of a \( K_h \)-minor-free graph \( G \) where, for some parameter \( r \) that we can choose, each piece has roughly \( r \) vertices and all pieces together have roughly \( |V(G)|/\sqrt{r} \) boundary vertices, counted with multiplicity. For technical reasons explained below, we will consider only the case when \( r \geq \Omega(h^3) \). We will prove the following, without trying to optimize the constants involved.

**Lemma 2.1.** For each \( K_h \)-free-minor graph \( G \) with \( n \) vertices and any \( r \) with \( 36h^3 \leq r \leq n \), there exists a division \( \{S_1, \ldots, S_k\} \) of \( G \) satisfying the following two properties:

- \(|S_i| \leq r \) for \( i = 1, \ldots, k \), and
- \( \sum_i |\partial S_i| \leq \frac{36h^3n}{\sqrt{r}} \).

For planar graphs, a stronger lemma was proven by Frederickson [6]. We refer to the draft by Klein and Mozes [10] for a careful treatment. The proof for the more general \( K_h \)-minor-free case is very similar. However, we have not been able to find a careful treatment for \( K_h \)-minor-free graphs and thus decided to include a proof where the dependency on \( h \) is explicit. The main tool in the proof is the separator theorem for \( K_h \)-minor-free graphs proven by Alon, Seymour and Thomas [1]. It states that in every \( K_h \)-minor-free graph \( G \) with \( n \) vertices, there exists a partition of vertices of \( G \) into three sets \( A, B \) and \( X \) such that \(|A|, |B| \leq \frac{3}{4}n \), \(|X| \leq h^3/2 \sqrt{n} \) and no edge connects a vertex from \( A \) to a vertex from \( B \). The set \( X \) is called the separator.

We are going to use the separator theorem recursively. We restrict our attention to the case \( r \geq \Omega(h^3) \) because, when we get to subgraphs with \( h^3 \) vertices, the size of the separator guaranteed by the separator theorem is also \( h^3 \), and thus we cannot benefit from recursion anymore.

**Proof of Lemma 2.1.** Let \( G \) be a \( K_h \)-minor free graph with \( n \) vertices and assume that \( r \geq 36h^3 \).

Consider the following algorithm to compute a division into pieces of size \( r \). We start setting \( S = \{V(G)\} \). While \( S \) has some piece \( S \) with more than \( r \) vertices, we remove \( S \) from \( S \), use the separator theorem on the induced subgraph \( G[S] \) to obtain sets \( A, B \) and \( X \), and put the pieces \( A \cup X \) and \( B \cup X \) in \( S \). This finishes the description of the construction.

Whenever we apply the separator theorem to a piece \( S \) with more than \( r \geq 36h^3 \) vertices, the sets \( A \cup X \) and \( B \cup X \) are strictly smaller than \( S \). Thus, the algorithm finishes. Since in each iteration of the construction the separator goes to both subpieces, we maintain a division. Formally, one could show by induction on the number of iterations that \( S \) is always a division. By construction, each of the pieces in the resulting division has at most \( r \) vertices. It remains to bound the sum of the size of the boundaries.

Let \( S \) be any of the pieces considered through the algorithm, and assume that the construction subdivides \( S \) into pieces \( S_1, \ldots, S_t \). We define

\[
\beta(S) := \left( \sum_{i=1}^{t} |S_i| \right) - |S|.
\]
Thus \( \beta(S) \) is the sum of the sizes of the final pieces obtained through the recursive partitioning of \( S \), minus the size of \( S \). Let \( \beta(m) := \max \beta(S) \), where the maximum is taken over all pieces \( S \) with \( m \) vertices that appear through the construction. We want to bound \( \beta(V(G)) = \beta(n) \).

When we break a piece \( S \) with \( m \) vertices into two pieces \( S_1 \) and \( S_2 \) using a separator \( X \) of size \( h^{3/2} \sqrt{m} \), we have \( |S_1| + |S_2| \leq |S| + h^{3/2} \sqrt{m} \) and therefore \( \beta(S) \leq \beta(S_1) + \beta(S_2) + h^{3/2} \sqrt{m} \). Thus, for every \( m \), there exist \( m_1 \) and \( m_2 \) such that we have the recurrence

\[
\beta(m) \leq \begin{cases} 
\beta(m_1) + \beta(m_2) + h^{3/2} \sqrt{m} & \text{if } m > r, \\
0 & \text{if } m \leq r,
\end{cases}
\]

where \( m_1, m_2 \geq m/3 \) and \( m_1 + m_2 \leq m + h^{3/2} \sqrt{m} \). It follows by induction that

\[
\beta(m) \leq \begin{cases} 
\left(\frac{10h^{3/2}m}{\sqrt{r/3}} - \left(10h^{3/2}/2\right)\sqrt{m/3} \right) & \text{if } m \geq \frac{r}{3} \\
0 & \text{otherwise}.
\end{cases}
\]

The proof is a standard computation and we include it in Appendix A for completeness.

Consider the division \( S = \{S_1, \ldots, S_k\} \) constructed by the algorithm, and define \( \partial = \bigcup_i \partial S_i \). Thus \( \partial \) is the set of all vertices that are boundary of some piece \( S_i \in S \). Since each vertex in \( \partial \) is boundary in at least 2 pieces, we have \( 2 : |\partial| \leq \sum_i |\partial S_i| \) and therefore

\[
\sum_i |\partial S_i| \leq 2 \left( \sum_i |\partial S_i| - |\partial| \right) = 2 \left( \sum_i |S_i| - n \right) = 2 \cdot \beta(V(G))
\]

\[
\leq 2 \cdot \beta(n) \leq 2 \frac{\left(10h^{3/2}/3\right)n}{\sqrt{r/3}} < \frac{36h^{3/2}n}{\sqrt{r}}.
\]

We conclude that the algorithm has constructed a division with the desired properties. \( \square \)

### 3 Algorithms and analyses

In this section we present and analyze PTAS’s for the problems MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER and MINIMUM DOMINATING SET restricted on \( K_h \)-minor-free graphs. All algorithms will be simple local optimizations, as discussed in the introduction.

#### 3.1 Independent set

Consider the algorithm INDEPENDENT\((h, G, \varepsilon)\) for MAXIMUM INDEPENDENT SET that was given in Figure 1. We will show that, for any fixed constant \( h \), the algorithm is a PTAS.

**Theorem 3.1.** For any fixed integer \( h > 0 \), the algorithm INDEPENDENT\((h, G, \varepsilon)\) is a PTAS for the problem MAXIMUM INDEPENDENT SET restricted to \( K_h \)-minor-free graphs with running time \( n^{O(1/\varepsilon^2)} \). The constant hidden in the big O in the running time is polynomial in \( h \).

**Proof.** Let \( G \) be a \( K_h \)-minor-free graph, let \( U^* \) be a largest independent set of \( G \), and let \( U \) be the independent set returned by the algorithm. We have to show that

\[
|U^*| - |U| \leq \varepsilon |U^*|.
\]

Consider the induced subgraph \( \hat{G} = G[U \cup U^*] \), which is also \( K_h \)-minor-free. Set \( C_h = 144^2 h^3 \) in the algorithm INDEPENDENT\((h, G, \varepsilon)\). Let \( \{S_1, \ldots, S_k\} \) be the division of \( G \) guaranteed by
Algorithm 2: VERTEX($h, G, \varepsilon$)

Input: An integer $h > 0$, a $K_h$-minor-free graph $G = (V, E)$ and a parameter $\varepsilon \in (0, 1)$

Output: A vertex cover $U$ of $G$

1. $r = C_h / \varepsilon^2$, where $C_h$ is an appropriate constant depending on $h$
2. $U = V$
3. while $\exists U_1 \subseteq U, V_1 \subseteq V$ with $|V_1| < |U_1| \leq r$ and $(U \setminus U_1) \cup V_1$ is a vertex cover do
   4. $U = (U \setminus U_1) \cup V_1$
5. return $U$

Figure 2: PTAS for MINIMUM VERTEX COVER for $K_h$-minor-free graphs.

Lemma 2.1 for $r = C_h / \varepsilon^2$, as set in INDEPENDENT($h, G, \varepsilon$). Note that $r \geq 36h^3$ satisfies the requirements for Lemma 2.1.

Note that a subset of $U \cup U^*$ is independent in $G$ if and only if it is independent in $\tilde{G}$. Therefore,

$$\forall i \in [k] : (U \setminus S_i) \cup (U^* \cap \text{int}(S_i))$$

is an independent set in $G$.

By the algorithm, the independent set $U$ can not be made larger by any such a transformation, thus we have

$$\forall i \in [k] : |U| \geq |U| - |U \cap S_i| + |U^* \cap \text{int}(S_i)|,$$

or alternatively

$$\forall i \in [k] : |U \cap S_i| \geq |U^* \cap \text{int}(S_i)|.$$

We can use this inequality, summed over all $i \in [k]$, to get

$$|U^*| \leq \sum_{i} |U^* \cap \text{int}(S_i)| + \sum_{i} |\partial S_i| \leq \sum_{i} |U \cap S_i| + \sum_{i} |\partial S_i| \leq |U| + 2 \sum_{i} |\partial S_i|.$$ 

Using the bound $\sum_{i} |\partial S_i| \leq \frac{36h^{3/2}|U \cup U^*|}{\sqrt{r}}$ from Lemma 2.1 and substituting $r$, we get

$$|U^*| - |U| \leq \frac{2 \cdot 36h^{3/2} \cdot |U \cup U^*|}{\sqrt{r}} \leq \frac{72h^{3/2} \cdot 2 \cdot |U^*|}{\sqrt{144h^3/\varepsilon^2}} = \varepsilon |U^*|.$$ 

The running time is $n^{O(r)} = n^{O(C_h/\varepsilon^2)} = n^{O(h^3/\varepsilon^2)}$.

3.2 Vertex cover

For MINIMUM VERTEX COVER consider the greedy local optimization algorithm VERTEX($h, G, \varepsilon$) given in Figure 2. Its structure is very similar to the algorithm for MAXIMUM INDEPENDENT SET.

Theorem 3.2. For any fixed integer $h > 0$, the algorithm VERTEX($h, G, \varepsilon$) is a PTAS for the problem MINIMUM VERTEX COVER restricted to $K_h$-minor free graphs with running time $n^{O(1/\varepsilon^2)}$. The constant hidden in the big O in the running time is polynomial in $h$. 

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Proof. The proof is very similar to the proof for Maximum Independent Set. We do not attempt to shorten it and follow very much the same structure.

Let \( G \) be a \( K_h \)-minor-free graph, let \( U^* \) be a smallest vertex cover of \( G \), and let \( U \) be the vertex cover returned by the algorithm. We have to show that

\[
|U| - |U^*| \leq \epsilon |U^*|.
\]

Consider the induced subgraph \( \tilde{G} = G[U \cup U^*] \), which is also \( K_h \)-free-minor. Set \( C_h = 4 \cdot 144^2 h^3 \) in the algorithm \textsc{Vertex}(\( h \), \( G \), \( \epsilon \)). Let \( \{ S_1, \ldots, S_k \} \) be the division of \( \tilde{G} \) guaranteed by Lemma 2.1 for \( r = C_h/\epsilon^2 \), as set in \textsc{Vertex}(\( h \), \( G \), \( \epsilon \)). Note that \( r \geq 36h^3 \) satisfies the requirements for Lemma 2.1.

Consider an edge \( uv \) of \( G \) and assume that \( u \in U \cap \text{int}(S_i) \). If \( u \notin U^* \), then \( v \in U^* \) and \( uv \in E(\tilde{G}) \), which implies that \( v \in U^* \cap S_i \) because \( u \in \text{int}(S_i) \). We conclude that \( u \) or \( v \) are in \( U^* \cap S_i \). Therefore

\[
\forall i \in [k] : (U \cap \text{int}(S_i)) \cup (U^* \cap S_i) \text{ is a vertex cover}.
\]

By the algorithm, the vertex set \( U \) can not be made smaller by any such a transformation, thus we have

\[
\forall i \in [k] : |U| \leq |U| - |U \cap \text{int}(S_i)| + |U^* \cap S_i|,
\]

or alternatively

\[
\forall i \in [k] : |U \cap \text{int}(S_i)| \leq |U^* \cap S_i|.
\]

We can use this inequality, summed over all \( i \in [k] \), to get

\[
|U^*| \geq \sum_i |U^* \cap S_i| - \sum_i |\partial S_i|
\]

\[
\geq \sum_i |U \cap \text{int}(S_i)| - \sum_i |\partial S_i|
\]

\[
\geq |U| - 2 \sum_i |\partial S_i|.
\]

Using the bound \( \sum_i |\partial S_i| \leq \frac{36h^3/2 |U \cup U^*|}{\sqrt{r}} \) from Lemma 2 and substituting \( r \), we get

\[
|U| - |U^*| \leq 2 \frac{36h^3/2 |U \cup U^*|}{\sqrt{r}} \leq \frac{72 \cdot 2 |U|}{\sqrt{4 \cdot 144^2/\epsilon^2}} = \frac{\epsilon}{2} |U|,
\]

which implies

\[
|U| \leq \frac{1}{1 - \epsilon/2} \cdot |U^*| \leq (1 + \epsilon) \cdot |U^*|
\]

for \( \epsilon \in (0, 1) \). The running time is \( n^{O(r)} = n^{O(C_h/\epsilon^2)} = n^{O(h^3/\epsilon^2)} \).

3.3 Dominating set

The PTAS for the problem Minimum Dominating Set on \( K_h \)-minor-free families of graphs is practically the same as the algorithm \textsc{Vertex}(\( h \), \( G \), \( \epsilon \)). We call it \textsc{Dominating}(\( h \), \( G \), \( \epsilon \)) and include it in Figure 3 to reference to it.

**Theorem 3.3.** For any fixed integer \( h > 0 \), the algorithm \textsc{Dominating}(\( h \), \( G \), \( \epsilon \)) is a PTAS for the problem Minimum Dominating Set restricted to \( K_h \)-minor-free graphs with running time \( n^{O(1/\epsilon^2)} \). The constant hidden in the big \( O \) in the running time is polynomial in \( h \).\]
Algorithm 3: Dominating\((h, G, \varepsilon)\)

**Input:** An integer \(h > 0\), a \(K_h\)-minor-free graph \(G = (V, E)\) and a parameter \(\varepsilon \in (0, 1)\)

**Output:** A dominating set \(U\) of \(G\)

1. \(r = C_h/\varepsilon^2\), where \(C_h\) is an appropriate constant depending on \(h\)
2. \(U = V\)
3. while \(\exists U_1 \subseteq U, V_1 \subseteq V\) with \(|V_1| < |U_1| \leq r\) and \((U \setminus U_1) \cup V_1\) is a dominating set do
4. \(\text{U} = (U \setminus U_1) \cup V_1\)
5. return \(U\)

Figure 3: PTAS for Minimum Dominating Set for \(K_h\)-minor-free graphs.

Although the algorithms \(\text{Vertex}(h, G, \varepsilon)\) and \(\text{Dominating}(h, G, \varepsilon)\) are almost identical, we need an additional idea in the analysis of the latter.

**Proof.** Let \(G\) be a \(K_h\)-minor-free graph, let \(U^*\) be a smallest dominating set of \(G\), and let \(U\) be the dominating set returned by the algorithm. We have to show that

\[ |U| - |U^*| \leq \varepsilon |U^*|. \]

If we would take a division \(\{S_1, \ldots, S_k\}\) of the induced graph \(G[U \cup U^*]\), as in the vertex cover case, then the sets

\[ (U \setminus \text{int}(S_i)) \cup (U^* \cap S_i) \]

would not necessarily be dominating; see Figure 4. This is crucial for the argument to go through, so we need to proceed differently.

![Figure 4](image.png)

Figure 4: If \(\{S_1, \ldots, S_k\}\) is a division of the graph \(G[U \cup U^*]\), then the vertex \(v\) might be dominated by \(U\) but not by \((U \setminus \text{int}(S_i)) \cup (U^* \cap S_i)\).

For every vertex \(v \in V \setminus (U \cup U^*)\), choose an edge that connects this vertex to \(U\) and contract it. Such an edge exists because \(U\) is a dominating set. Let \(\hat{G}\) be the resulting graph. Its vertex set is \(U \cup U^*\). It is clear that \(\hat{G}\) is \(K_h\)-minor-free, since it is a minor of \(G\). Set \(C_h = 4 \cdot 144^2 h^3\) in algorithm \(\text{Dominating}(h, G, \varepsilon)\), and let \(\{S_1, \ldots, S_k\}\) be the division of \(\hat{G}\) guaranteed by Lemma 2.1 for \(r = C_h/\varepsilon^2\).

We claim that, for each index \(i\), the set

\[ U_i = (U \setminus \text{int}(S_i)) \cup (U^* \cap S_i) \]

is dominating in \(G\). We do not know of a better way to verify this than by a systematic case-by-case analysis over all vertices \(v \in V\).
Case \( v \in V \setminus (U \cup U^*) \). Because \( U \) and \( U^* \) are dominating sets, there exist vertices \( u \in U \) and \( u^* \in U^* \) that are neighbours of \( v \). Without loss of generality we may assume that the edge \( uv \) was contracted when \( \tilde{G} \) was constructed. If \( u \notin U_i \), it must be that \( u \in \text{int}(S_i) \), which implies that \( u^* \in S_i \), since \( u^* \) is a neighbor of \( u \) in \( \tilde{G} \). Hence, \( u \in U_i \) or \( u^* \in U_i \), which implies that \( v \) is dominated by \( U_i \).

Case \( v \in U \). Because \( U^* \) is a dominating set, there exist a vertex \( u^* \in U^* \) that dominates \( v \) in \( G \). If \( v \notin U_i \), then \( v \in \text{int}(S_i) \) which implies \( u^* \in S_i \), thus \( u^* \in U_i \). Hence, \( v \) is dominated by \( U_i \).

Case \( v \in U^* \). Because \( U \) is a dominating set, there exist a vertex \( u \in U \) that dominates \( v \) in \( G \). If \( u \notin U_i \), then \( u \in \text{int}(S_i) \) which implies \( v \in S_i \), thus \( v \in U_i \). Hence, \( v \) is dominated by \( U_i \).

Our intuition for why dividing \( \tilde{G} \) is better than dividing \( G[U \cup U^*] \) is that \( \tilde{G} \) has all the edges of \( G[U \cup U^*] \) plus some additional ones and hence the division of \( \tilde{G} \) is stronger.

The proof from here on is identical to the one in the vertex cover case. \( \square \)

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We want to show by induction that
\[ \beta(m) \leq \beta_{\text{ind}}(m) := \begin{cases} \frac{(10h^{3/2})m}{\sqrt{r/3}} - (10h^{3/2})\sqrt{m}, & \text{if } m \geq \frac{r}{3} \\ 0, & \text{otherwise.} \end{cases} \]

We first check the base case. When \( m < r/3 \) we have \( \beta(m) = \beta_{\text{ind}}(m) = 0 \). When \( r/3 \leq m \leq r \), we have
\[ \frac{(10h^{3/2})m}{\sqrt{r/3}} \geq (10h^{3/2})\sqrt{m} \]
and therefore
\[ \beta(m) = 0 \leq \frac{(10h^{3/2})m}{\sqrt{r/3}} - (10h^{3/2})\sqrt{m} = \beta_{\text{ind}}(m). \]

When \( m > r \) we use use the recurrence (1), where \( m_1, m_2 \geq m/3 \geq r/3 \), and the induction hypothesis to obtain
\[ \beta(m) \leq \beta(m_1) + \beta(m_2) + h^{3/2}\sqrt{m} \]
\[ \leq \frac{(10h^{3/2})m_1}{\sqrt{r/3}} - (10h^{3/2})\sqrt{m_1} + \frac{(10h^{3/2})m_2}{\sqrt{r/3}} - (10h^{3/2})\sqrt{m_2} + h^{3/2}\sqrt{m} \]
\[ = \frac{(10h^{3/2})(m_1 + m_2)}{\sqrt{r/3}} - (10h^{3/2})(\sqrt{m_1} + \sqrt{m_2}) + h^{3/2}\sqrt{m} \]
\[ \leq \frac{(10h^{3/2})(m + h^{3/2}\sqrt{m})}{\sqrt{r/3}} - (10h^{3/2})(\sqrt{m_1} + \sqrt{m_2}) + h^{3/2}\sqrt{m} \]
\[ = \beta_{\text{ind}}(m) + (10h^{3/2})\sqrt{m} + \frac{(10h^{3/2})h^{3/2}\sqrt{m}}{\sqrt{r/3}} - (10h^{3/2})(\sqrt{m_1} + \sqrt{m_2}) + h^{3/2}\sqrt{m} \]
\[ = \beta_{\text{ind}}(m) + (11h^{3/2})\sqrt{m} + \frac{(10h^{3/2})h^{3/2}\sqrt{m}}{\sqrt{r/3}} - (10h^{3/2})(\sqrt{m_1} + \sqrt{m_2}). \]

To get \( \beta(m) \leq \beta_{\text{ind}}(m) \), it suffices to show that
\[ (11h^{3/2})\sqrt{m} + \frac{(10h^{3/2})h^{3/2}\sqrt{m}}{\sqrt{r/3}} \leq (10h^{3/2})(\sqrt{m_1} + \sqrt{m_2}). \]

Dividing by \( h^{3/2}\sqrt{m} \), using that \( \sqrt{\cdot} \) is concave, and that \( m_1 + m_2 \geq m \) with \( m_1, m_2 \geq m/3 \), we see that it is enough to show that
\[ 11 + \frac{10h^{3/2}}{\sqrt{r/3}} \leq 10 \left( \sqrt{1/3} + \sqrt{2/3} \right) = 13.9384 \ldots , \]
or equivalently
\[
\frac{\frac{h^{3/2}}{\sqrt{r}}}{\sqrt{r}} \leq \frac{10\left(\sqrt{1/3} + \sqrt{2/3}\right) - 11}{10\sqrt{3}} = 0.1696\ldots
\] (2)

Since \( h^{3/2} \leq \sqrt{r/6} = \sqrt{r} \cdot 0.1666\ldots \), the inequality (2) holds and therefore \( \beta(m) \leq \beta_{\text{ind}}(m) \) for \( m \geq r \). This finishes the proof by induction that \( \beta(m) \leq \beta_{\text{ind}}(m) \) for all \( m \).