On dispersion of small energy solutions of the nonlinear Klein Gordon equation with a potential

Dario Bambusi, Scipio Cuccagna

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Abstract

In this paper we study small amplitude solutions of nonlinear Klein Gordon equations with a potential. Under smoothness and decay assumptions on the potential and a genericity assumption on the nonlinearity, we prove that all small amplitude initial data with finite energy give rise to solutions asymptotically free. In the case where the linear system has at most one bound state the result was already proved by Soffer and Weinstein: we obtain here a result valid in the case of an arbitrary number of possibly degenerate bound states. The proof is based on a combination of Birkhoff normal form techniques and dispersive estimates.

1 Introduction

In this paper we study small amplitude solutions of the nonlinear Klein Gordon equation (NLKG)

\[ u_{tt} - \Delta u + Vu + m^2 u + \beta'(u) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 \] (1.1)

with \(-\Delta + V(x) + m^2\) a positive short range Schrödinger operator, and \(\beta'\) a smooth function having a zero of order 3 at the origin and growing at most like \(u^3\) at infinity. Under suitable smoothness and decay properties on the potential \(V\) and on \(\beta\), and under a genericity assumption to be discussed below, we prove that all initial data with small enough energy give rise to asymptotically free solutions. Thus in particular the system does not admit small amplitude periodic or quasiperiodic solutions with finite energy, in contrast with what happens in bounded domains where KAM theory can be used to prove existence of quasiperiodic solutions [Kuk93, CW93, Way90, Bou05, EK06].

A crucial role in our discussion is played by the spectrum of the Schrödinger operator \(-\Delta + V(x)\). If \(-\Delta + V(x)\) does not have eigenvalues, then the asymptotic freedom of solutions follows from a perturbative argument based on a theorem by Yajima [Y]. If \(-\Delta + V + m^2\) has just one nondegenerate eigenvalue lying close to the continuous spectrum, then the result is proved by [SW1]. We generalize this result, easing most restrictions on the spectrum of \(-\Delta + V + m^2\).
From a technical standpoint, the key is to prove that, due to nonlinear coupling, there is leaking of energy from the discrete modes to the continuous ones. The continuous modes should disperse by perturbation, because of the linear dispersion. In [SW1] this leaking occurs because the discrete mode equation has a key coefficient of positive sign, which yields dissipation. In [SW1] this coefficient is of the form \( \langle DF, F \rangle \) for \( D \) a positive operator and \( F \) a function. Assuming the generic condition \( \langle DF, F \rangle \neq 0 \) (which is called nonlinear Fermi golden rule or FGR), then such a quantity is strictly positive. This gives rise to dissipative effects leading to the result. The presence of terms of the form \( \langle DF, F \rangle \) was first pointed out and exploited for nonlinear problems in [Si], which proves that periodic and quasiperiodic solutions of the linear equation are unstable with respect to nonlinear perturbations. In the problem treated in [Si], this coefficient appears directly. In our case, to exploit the coefficient it is first necessary to simplify the equations by means of normal form expansions. The normal forms argument was first introduced in [BP2], later by [SW1], (see also [GS, CM] and for further references [CT]).

In the case when the eigenvalues of \(-\Delta + V + m^2\) are not close to the continuous spectrum, the crucial coefficients in the equations of the discrete modes are of the form \( \langle DF, G \rangle \) for \( F \) and \( G \) not obviously related, if one follows the scheme in [BP2, SW1, GS, CM]. The argument in [CM] shows indirectly that, in the case of just one simple eigenvalue, this coefficient is semidefinite positive. But this is not clear any more in the case of multiple eigenvalues of possibly high multiplicity, if one follows the scheme in [BP2, SW1, GS, CM]. In the present paper we fill this gap. Using the Hamiltonian structure of (1.1) and the Birkhoff normal form theory, we show that dissipativity is a generic feature of the problem. Here lies the novelty of this paper: previous references perform normal form expansions losing sight of the Hamiltonian structure of (1.1). It turns out that the Hamiltonian structure is crucial.

We recall that Birkhoff normal form theory has been recently extended to a quite large class of Hamiltonian partial differential equations (see for example [BN98, Bam03, BG06]). However here we need to deal with two specific issues. The first one is that we need to produce a normal form which keeps some memory of the fact that the original Hamiltonian is local, since locality is a fundamental property needed for the dispersive estimates used to prove dissipation. The second issue is that the Hamiltonian function (and its vector field) of the NLKG has only finite regularity, so it is not a priori obvious how to put the system in normal form at high order. This problem is here solved by noticing that our normal form is need only to simplify the dependence on the discrete modes and to decouple the discrete modes from the continuous ones. This can be obtained by a coherent recursive construction yielding analytic canonical transformations.

We end this introduction by recalling the related problem of asymptotic stability of solitary waves for the NLS, which has been studied in a substantial number of papers. We reference only the seminal papers [SW2, BP1, BP2, GS] and the paper [CM] from which we draw for our proof. See [CT] for further references. Here too positive answers are known only if either the spectrum of the linearized operator at the solitary wave has only one eigenvalue (besides
those related to the symmetries of the problem), or if a more restrictive version of FGR is true, see [CM]. We expect our methods to be relevant also to this problem.

2 Statement of the main result

We begin by stating our assumptions.

(H1) $V(x)$ is real valued and $|\partial^\alpha_x V(x)| \leq C(x)^{-5-\sigma}$ for $|\alpha| \leq 2$, where $C > 0$ and $\sigma > 0$ are fixed constants and $\langle x \rangle := \sqrt{1 + |x|^2}$; $V(x)$ is smooth with $|\partial^\alpha_x V(x)| \leq C_\alpha < \infty$ for all $\alpha$;

(H2) $0$ is not an eigenvalue or a resonance for $-\Delta + V$, i.e. there are no nonzero solutions of $\Delta u = V u$ in $\mathbb{R}^3$ with $|u(x)| \lesssim \langle x \rangle^{-1}$.

It is well known that (H1)–(H2) imply that the set of eigenvalues $\sigma_\ell (-\Delta + V) \equiv \{-\lambda^2_j\}_{j=1}^n$ is finite, contained in $(-\infty, 0)$, with each eigenvalue of finite multiplicity. We take a mass term $m^2$ such that $-\Delta + V + m^2 > 0$ and we assume that indexes have been chosen so that $-\lambda^2_1 \leq \cdots \leq -\lambda^2_n$. We set $\omega_j = \omega_j(m) := \sqrt{m^2 - \lambda^2_j}$. Notice that the $-\lambda^2_j$ are not necessarily pairwise distinct. We assume that $m$ is not a multiple of any of the $\omega_j$’s:

(H3) for any $\omega_j$ there exists $N_j \in \mathbb{N}$ such that $N_j \omega_j < m < (N_j + 1)\omega_j$.

Notice that $N_1 = \sup_j N_j$. Hypothesis (H3) is a special case of the following hypothesis:

(H4) there is no multi index $\mu \in \mathbb{Z}^n$ with $|\mu| := |\mu_1| + \cdots + |\mu_k| \leq 2N_1 + 3$ such that $\mu \cdot \omega = m$.

We furthermore require:

(H5) if $\omega_{j_1} < \cdots < \omega_{j_k}$ are $k$ distinct $\omega$’s, and $\mu \in \mathbb{Z}^k$ satisfies $|\mu| \leq 2N_1 + 3$, then we have

$$\mu_1 \omega_{j_1} + \cdots + \mu_k \omega_{j_k} = 0 \iff \mu = 0.$$  

Remark 2.1. There exists a discrete set $D \subset (-\lambda^2_1, \infty)$, such that for $m \notin D$ hypotheses (H3-H5) are true.

Assumptions (H1)–(H5) refer to the properties of the linear part of the equation. Concerning the nonlinear part $\beta'(u)$ we assume the following hypothesis:

(H6) We denote by $\beta(u)$ the antiderivative with $\beta(0) = 0$. We assume that there exists a smooth function $\tilde{\beta} \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\beta(u) = u^4\tilde{\beta}(u)$ and, for any $j \geq 0$ there exists $C_j > 0$ such that $|\tilde{\beta}^{(j)}(u)| \leq C_j (u)^{-j}$.

Finally there is a nondegeneracy hypothesis relating the linear operator $-\Delta + V + m^2$ and the nonlinearity $\beta(u)$. Specifically it relates to resonance between discrete and continuous modes. Its precise statement requires some notations and preliminaries, so it is deferred to section 5.1.
(H7) We assume that (5.30) or, equivalently, (5.33) holds.

(H7) is the most significant of our hypotheses. It should hold quite generally. By way of illustration, in Section 5.1 we prove the following result:

**Proposition 2.2.** Assume that $V$ satisfies (H1)-(H2), decreases exponentially as $|x| \to \infty$, and all its eigenvalues are simple. Then there exist a finite set $\mathcal{M} \subset (-\lambda_1^2, +\infty)$, for any $m \in (-\lambda_1^2, +\infty) \setminus \mathcal{M}$ a finite set $\tilde{\mathcal{M}}(m) \subset \mathbb{Z}^n$ locally constant in $m$, functions $f_{\mu,m} \in C^\infty(\mathbb{R}^{\mid \mu \mid - 4}, \mathbb{R})$ for $\mu \in \tilde{\mathcal{M}}(m)$, such that (H7) holds if the following is true: $m \in (-\lambda_1^2, +\infty) \setminus \mathcal{M}$ and

$$\beta_{\mid \mu \mid} \neq f_{\mu,m}(\beta_4, ..., \beta_{\mid \mu \mid - 1}) \text{ for all } \mu \in \tilde{\mathcal{M}}(m) \text{ and } \beta_j := \beta_j(0)/j!.$$ 

Now we state the main result of this paper. Denote $K_0(t) = \frac{\sin(t\sqrt{-\Delta + m^2})}{\sqrt{-\Delta + m^2}}$.

Then we prove:

**Theorem 2.3.** Assume hypotheses (H1)-(H7). Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $\| (u_0, v_0) \|_{H^1 \times L^2} \leq \varepsilon < \varepsilon_0$ the solution of (1.1) with $(u(0), v(0)) = (u_0, v_0)$ is globally defined and there are $(u_\pm, v_\pm)$ with $\| (u_\pm, v_\pm) \|_{H^1 \times L^2} \leq C\varepsilon$

$$\lim_{t \to -\infty} \| u(t) - K_0(t)u_\pm - K_0(t)v_\pm \|_{H^1} = 0. \quad (2.1)$$

It is possible to write $u(t, x) = A(t, x) + \tilde{u}(t, x)$ with $|A(t, x)| \leq C_N(t)(x)^{-N}$ for any $N$, with $\lim_{t \to -\infty} C_N(t) = 0$ such that for any pair $(r, p)$ which is admissible, by which we mean that

$$2/r + 3/p = 3/2, \quad 6 \geq p \geq 2, \quad r \geq 2, \quad (2.2)$$

we have

$$\| \tilde{u} \|_{L^r_t W^{1/2}_x + 1/p} \leq C \| (u_0, v_0) \|_{H^1 \times L^2}. \quad (2.3)$$

**Remark 2.4.** Theorem 2.3 is well known in the particular case $V = 0$. In this case $\tilde{u} = u$. If the operator $-\Delta + V$ does not have eigenvalues and satisfies the estimates in Lemma 6.1, then Theorem 2.3 continues to hold. Work by Yajima [Y] guarantees that this indeed is the case for operators satisfying (H1)-(H2) such that $\sigma_d(-\Delta + V)$ is empty, see Lemma 6.3. These results are obtained thinking the nonlinear problem as a perturbation of the linear problem.

**Remark 2.5.** Theorem 2.3 can thought as an asymptotic stability result of the $0$ solution. Stability is well known, see Theorem 3.1 below.

**Remark 2.6.** Theorem 2.3 in the case when $\sigma_d(-\Delta + V)$ consists of a single eigenvalue can be proved following a simpler version of the argument in [CM].

**Remark 2.7.** Theorem 2.3 in the case when $\sigma_d(-\Delta + V)$ consists of a single eigenvalue $-\lambda^2$ such that for $\omega = \sqrt{m^2 - \lambda^2}$ we have $3\omega > m$ is proved in [SW1] assuming $\| (u_0, v_0) \|_{(H^1 \cap W^{2,1}) \times (H^1 \cap W^{1,1})}$ small. Notice that formula (1.10) [SW1] contains a decay rate of dispersion of the various components of $u(t)$. For the initial data in the larger class considered in Theorem 2.3, such kind of decay rates cannot be proved. Restricting initial data to the class in [SW1], it is possible to prove appropriate decay rates also for the solutions in Theorem 2.3.
Definition. Theorem 2.3 is stated only for $\mathbb{R}^d$ with $d = 3$. Versions of this theorem can be proved for any $d$. In particular, the crux of the paper, that is the normal form expansion in Theorem 4.9 and the discussion of the discrete modes, are not affected by the spatial dimension. For $d \leq 2$ the absence of the endpoint Strichartz estimate can be offset with appropriate smoothing estimates.

In view of the above remarks, we focus our attention to the case when $-\Delta + V$ admits eigenvalues, especially the case of many eigenvalues.

We end this section with some notation. Given two functions $f, g : \mathbb{R}^3 \to \mathbb{C}$ we set $(f, g) = \int_{\mathbb{R}^3} f(x)g(x)dx$. For $k \in \mathbb{R}$ and $1 < p < \infty$ we denote for $K = \mathbb{R}, \mathbb{C}$

$$W^{k,p}(\mathbb{R}^3, K) = \{ f : \mathbb{R}^3 \to K \ s.t. \| f \|_{W^{k,p}} := \|(-\Delta + 1)^{k/2}f\|_{L^p} < \infty \}$$

In particular we set $H^k(\mathbb{R}^3, K) = W^{k,2}(\mathbb{R}^3, K)$ and $L^p(\mathbb{R}^3, K) = W^{0,p}(\mathbb{R}^3, K)$. For $p = 1, \infty$ and $k \in \mathbb{N}$ we denote by $W^{k,p}(\mathbb{R}^3, K)$ the functions such that $\partial_\alpha^j f \in L^p(\mathbb{R}^3, K)$ for all $|\alpha| \leq k$ (we recall that for $1 < p < \infty$ the two definitions of $W^{k,p}$ yield the same space). For any $s \in \mathbb{R}$ we set

$$H^{k,s}(\mathbb{R}^3, K) = \{ f : \mathbb{R}^3 \to K \ s.t. \| f \|_{H^{k,s}} := \|(x)^s(-\Delta + 1)^{k/2}f\|_{L^2} < \infty \}.$$ 

In particular we set $L^{2,s}(\mathbb{R}^3, K) = H^{0,s}(\mathbb{R}^3, K)$. Sometimes, to emphasize that these spaces refer to spatial variables, we will denote them by $W^{k,p}_x$, $L^p_x$, $H^k_x$, $H^{k,s}_x$ and $L^{2,s}_x$. For $I$ an interval and $Y_x$ any of these spaces, we will consider Banach spaces $L^p_I(I, Y_x)$ with mixed norm $\| f \|_{L^p_I(I, Y_x)} := \| \| f \|_{Y_x} \|_{L^p(I)}$. Given an operator $A$, we will denote by $R_A(z) = (A - z)^{-1}$ its resolvent. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We will consider multi indexes $\mu \in \mathbb{N}_0^n$. For $\mu \in \mathbb{Z}^n$ with $\mu = (\mu_1, ..., \mu_n)$ we set $|\mu| = \sum_{j=1}^n |\mu_j|$.

3 Global well posedness and Hamiltonian structure

In $H^1(\mathbb{R}^3, \mathbb{R}) \times L^2(\mathbb{R}^3, \mathbb{R})$ endowed with the standard symplectic form, namely

$$\Omega((u_1, v_1); (u_2, v_2)) := (u_1, v_2)_{L^2} - (u_2, v_1)_{L^2}$$

we consider the Hamiltonian

$$H = H_L + H_P$$

$$H_L := \int_{\mathbb{R}^3} \left( \frac{1}{2} (v^2 + |\nabla u|^2 + Vu^2 + m^2 u^2) \right) dx$$

$$H_P := \int_{\mathbb{R}^3} \beta(u) dx.$$ 

The corresponding Hamilton equations are \( \dot{v} = -\nabla_u H, \ \dot{u} = \nabla_v H \), where $\nabla_u H$ is the gradient with respect to the $L^2$ metric, explicitely defined by

$$\langle \nabla_u H(u), h \rangle = d_u H(u)h, \quad \forall h \in H^1,$$
and \( d_u H(u) \) is the Frechét derivative of \( H \) with respect to \( u \). It is easy to see that the Hamilton equations are explicitly given by

\[
\dot{v} = \Delta u - V u - m^2 u - \beta'(u), \quad \dot{u} = v
\]  

(3.3)

First we recall that the NLKG (1.1) is globally well posed for small initial datum.

**Theorem 3.1.** Assume \( V \in L^p_x \) with \( p > 3/2 \). Then there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for any \( \|(u_0, v_0)\|_{H^1_t L^2_x} \leq \varepsilon < \varepsilon_0 \) equation (1.1) admits a unique weak solution with \( (u(0), v(0)) = (u_0, v_0) \). This solution is globally defined and it is a strong solution, that is

\[
u \in C^0(\mathbb{R}, H^1_x) \cap C^1(\mathbb{R}, L^2_x).
\]  

(3.4)

The map \( (u_0, v_0) \to (u(t), v(t)) \) is a continuous from the ball \( \|(u_0, v_0)\|_{H^1_t L^2_x} < \varepsilon_0 \) to \( C^1(I, H^1_x) \times C^0(I, L^2_x) \) for any bounded interval \( I \). The Hamiltonian \( H(u(t), v(t)) \) is constant, and

\[
\|(u(t), v(t))\|_{H^1_t L^2_x} \leq C\|(u_0, v_0)\|_{H^1_t L^2_x}.
\]  

(3.5)

We have the equality

\[
u(t) = K_0(t)u_0 + K_0(t)v_0 - \int_0^t K_0(t-s)(Vu(s) + \beta'(u(s)))ds.
\]  

(3.6)

For the proof see §6.2 and 6.3 [CH].

We associate to any \(-\lambda_j^2\) an \( L^2 \) eigenvector \( \varphi_j(x) \), real valued and normalized. We have \( \varphi_j \in H^{k,s}(\mathbb{R}^3, \mathbb{R}) \) for all \( s \) and \( k \). Set \( P_{\varphi} u = \sum_j \langle u, \varphi_j \rangle \varphi_j \) and set \( P_c = 1 - P_d \), the projector in \( L^2 \) associated to the continuous spectrum. Denote

\[
u = \sum_j q_j \varphi_j + P_c u, \quad v = \sum_j p_j \varphi_j + P_c v.
\]  

(3.7)

We have

\[
H_P = \int_{\mathbb{R}^3} \beta \left( \sum_j q_j \varphi_j + P_c u \right) dx.
\]  

(3.8)

Introduce the operator

\[
B := P_c (-\Delta + V + m^2)^{1/2} P_c,
\]  

(3.9)

and the complex variables

\[
\xi_j := \frac{q_j \sqrt{\omega_j} + i \frac{p_j}{\sqrt{\omega_j}}}{\sqrt{2}}, \quad f := \frac{B^{1/2} u + iB^{-1/2} v}{\sqrt{2}}.
\]  

(3.10)

By Theorem 6.2 or Lemma 3.3 below, (3.10) this defines an isomorphism between the phase space \( H^1(\mathbb{R}^3, \mathbb{R}) \times L^2(\mathbb{R}^3, \mathbb{R}) \) and the space \( \mathcal{P}^{1/2,0} := \mathbb{C}^n \oplus \mathbb{R}^n \).
which from now on will be our phase space. We will often represent functions (and maps) on the phase space as functions of the variables \(\xi_j, \bar{\xi}_j, f, \bar{f}\). By this we mean that a function \(F(\xi, \bar{\xi}, f, \bar{f})\) is actually the composition of the maps

\[
(\xi, f) \mapsto (\xi, \bar{\xi}, f, \bar{f}) \mapsto F(\xi, \bar{\xi}, f, \bar{f}).
\]

Correspondingly we define \(\partial_{\xi_j} = \frac{1}{2}(\partial_{\text{Re}} \xi_j - i \partial_{\text{Im}} \xi_j)\) and \(\partial_{\bar{\xi}_j} = \frac{1}{2}(\partial_{\text{Re}} \xi_j + i \partial_{\text{Im}} \xi_j)\), and analogously \(\nabla_f := \frac{1}{2}(\nabla_{\text{Re}} f - i \nabla_{\text{Im}} f)\), \(\nabla_{\bar{f}} := \frac{1}{2}(\nabla_{\text{Re}} f + i \nabla_{\text{Im}} f)\).

In terms of these variables the Hamilton equations take the form

\[
\dot{\xi}_j = -i \frac{\partial H}{\partial \bar{\xi}_j}, \quad \dot{\bar{\xi}}_j = -i \nabla_f H . \tag{3.11}
\]

The Hamiltonian vector field \(X_H\) of a function is given by

\[
X_H(\xi, \bar{\xi}, f, \bar{f}) = \left( -i \frac{\partial H}{\partial \bar{\xi}_j}, i \frac{\partial H}{\partial \xi_j}, -i \nabla_f H, i \nabla_{\bar{f}} H \right). \tag{3.12}
\]

We recall that the Poisson bracket of two functions is the Lie derivative of the second with respect to the Hamiltonian vector field of the first one. In these variables we have, formally,

\[
\{H, K\} := i \sum_j \left( \frac{\partial H}{\partial \xi_j} \frac{\partial K}{\partial \bar{\xi}_j} - \frac{\partial H}{\partial \bar{\xi}_j} \frac{\partial K}{\partial \xi_j} \right) + i \langle \nabla_f H; \nabla_{\bar{f}} K \rangle - i \langle \nabla_{\bar{f}} K; \nabla_f H \rangle \tag{3.13}
\]

We emphasize that if \(H\) and \(K\) are real valued, then \(\{H, K\}\) is real valued. Later we will consider Hamiltonians for which (3.13) makes sense.

We introduce now some further notations that we will use in the following.

- We denote the phase spaces \(P_{k,s} = \mathbb{C}^n \times P_c H^{k,s}(\mathbb{R}^3, \mathbb{C})\) with the spectral decomposition associated to \(-\Delta + V\).
- We will denote \(\zeta_j := \bar{\xi}_j, \quad j > 0\)
  \(\zeta_j := \bar{\xi}_{-j}, \quad j < 0,\)
- \(f := (f, \bar{f}),\) and we will denote by \(\Phi := (\Phi, \Psi)\) a pair of functions each of which is in \(H^{k,s}, \forall k, s > 0.\)
- Given \(\mu = (\mu_{-n}, \ldots, \mu_{-1}, \mu_1, \ldots, \mu_n) \in \mathbb{N}^{2n}\) we denote \(\zeta^\mu := \prod_j \zeta_j^{\mu_j}.\)
- A point of the phase space will usually be denoted by \(z = (\xi, f)\).

The form of \(H_L\) and of \(H_P\) are respectively

\[
H_L = \sum_{j=1}^n \omega_j |\xi_j|^2 + \langle \bar{f}, Bf \rangle. \tag{3.14}
\]

\[
H_P(\xi, f) = \int_{\mathbb{R}^3} \beta \sum_j \frac{\xi_j + \bar{\xi}_j}{\sqrt{2\omega_j}} \varphi_j(x) + U(x) dx \tag{3.15}
\]
where we wrote for simplicity $U = B^{-\frac{1}{2}}(f + \tilde{f}) \equiv P_{c}u$.

Actually we will need something more from the structure of the nonlinearity. Consider the Taylor expansion

$\beta(\sum \frac{\xi_j + \xi_j}{\sqrt{2\omega_j}} \varphi_j + U) = \sum_{l=0}^{3} F_l(\xi, x)U^l + F_4(\xi, x, U)U^4$

with

$F_l(\xi, x) = \frac{1}{l!} \beta^{(l)}(\sum \frac{\xi_j + \xi_j}{\sqrt{2\omega_j}} \varphi_j)$, \quad $l = 0, 1, 2, 3$ \quad (3.16)

$F_4(\xi, x, U) = \int_{0}^{1} (1-\sigma)^{l} \beta^{(4)}(\sum \frac{\xi_j + \xi_j}{\sqrt{2\omega_j}} \varphi_j + \tau U) d\tau$. \quad (3.17)

**Lemma 3.2.** The following holds true.

1. For $l \leq 3$, the functions $\xi \to F_l(\xi, \cdot)$ are in $C^\infty(\mathbb{R}^n, H^{k,s})$ for any $k, s$, and

$$H_l(\xi, U) = \int_{\mathbb{R}^3} F_l(\xi, x)U^l dx$$

are $H_l \in C^\infty(\mathbb{R}^n \times H^1, \mathbb{R})$. In particular we have derivatives

$$\partial_\xi^\alpha d_{\xi}^l H_l[\otimes_{j=1}^l \xi_j] = l \cdots (l - \ell + 1) \int_{\mathbb{R}^3} \partial_\xi^\alpha F_l(\xi, x)U^{l-\ell}(x) \prod_{j=1}^{m} g_j(x) dx.$$

2. $F_l$ has a 0 of order $4 - l$ at $\xi = 0$:

$$||F_l(\xi, \cdot)||_{H^{k,s}} \leq C ||\xi||^{4-l}.$$

3. The map $\mathbb{R}^n \times \mathbb{R}^3 \times \mathbb{R} \ni (\xi, x, Y) \mapsto F_4(\xi, x, Y) \in \mathbb{R}$ is $C^\infty$; for any $k > 0$ there exists $C_k$ such that $|\partial_\xi F_4(x, \xi, Y)| \leq C_k$. Denote

$$H_4(\xi, U) = \int_{\mathbb{R}^3} F_4(x, \xi, U(x))U^4(x) dx.$$

Then the map $\mathbb{R}^n \ni \xi \mapsto H_4(\xi, \cdot) \in C^2(H^1)$ is $C^\infty$. In particular

$$\partial_\xi^\alpha d_{\xi} H_4[\cdot] = \int_{\mathbb{R}^3} \partial_\xi^\alpha \partial_\xi \Psi(\xi, x, U(x))g(x) dx$$

where $\Psi(\xi, x, Y) = F_4(\xi, x, Y)Y^4$.

**Proof.** The result follows by standard computations and explicit estimates of the remainder, see p. 59 [Ca].

**Lemma 3.3.** For any $(k, a, s) \in \mathbb{R}^3$ we have $\| B^a \|_{H^{k,s} \to H^{k-a,s}} < \infty$.

**Proof.** $B^a = p_n(x, D) + S_n$ with $p_n(x, D)$ a pseudo differential operator with $|\partial_\xi^\alpha \partial_\xi^\beta p_n(x, y)| \leq C_{n, \beta}(y)^{a-|\beta|}$ and with $S_n$ an operator which maps tempered distributions into Schwartz functions, see p. 296 [T1]. For $p_n(x, D)$ and $S_n$ the statement is true.
4 Normal form

4.1 Lie transform

We will iteratively eliminate from the Hamiltonian monomials simplifying the part linear in \( f \) and \( \bar{f} \). We will use canonical transformations generated by Lie transform, namely the time 1 flow of a suitable auxiliary Hamiltonian function. Consider a function \( \chi \) of the form

\[
\chi(z) \equiv \chi(\xi, f) = \chi_0(\xi, \bar{\xi}) + \sum_{|\mu|=M_0+1} \zeta^\mu \int_{\mathbb{R}^3} \Phi_\mu \cdot f dx
\]  

(4.1)

where \( \Phi_\mu \cdot f := \Phi_\mu f + \Psi_\mu \bar{f} \) with \( \Phi_\mu, \Psi_\mu \in H^{k,\tau}(\mathbb{R}^3, \mathbb{C}) \) for all \( k \) and \( \tau \). where \( \chi_0 \) is a homogeneous polynomial of degree \( M_0 + 2 \). The Hamiltonian vector field satisfies

\[
\|X_{\chi}(z)\|_{P^k,\tau} \leq C_{k,s,\kappa,\tau} \|z\|_{P^{k_0+1}}^{M_0+1}. \tag{4.2}
\]

Since \( X_{\chi} \) is a smooth polynomial it is also analytic. Denote by \( \phi^t \) its flow. For fixed \( \kappa, s \), \( \phi^t \) is defined in \( P^{-\kappa,-s} \) up to any fixed time \( \bar{t} \), in a sufficiently small neighborhood \( U^{-\kappa,-s} \) of the origin. For \( P^{k,\tau} \hookrightarrow P^{-\kappa,-s} \) by (4.2) the flow \( \phi^t \) is defined for \( 0 \leq t \leq \bar{t} \) in \( U^{-\kappa,-s} \cap P^{k,\tau} \). Set \( \phi := \phi^t \equiv \phi^1 \big|_{t=1} \)

**Definition 4.1.** The canonical transformation \( \phi \) will be called the Lie transform generated by \( \chi \).

**Remark 4.2.** The function \( \chi \) extends to an analytic function on the complexification of the phase space, namely the space in which \( \xi \) is independent of \( \bar{\xi} \) and \( f \) is independent of \( \bar{f} \). If the original function \( \chi \) is real valued (as in our situation), then \( \chi \) takes real values when \( f \) is the complex conjugated of \( \bar{f} \) and \( \xi \) the complex conjugated of \( \bar{\xi} \). In this case, by the very construction, the Lie transform generated by \( \chi \) leaves invariant the submanifold of the complexified phase space corresponding to the original real phase space.

**Lemma 4.3.** Consider a functional \( \chi \) of the form (4.1). Assume \( \Phi_\mu, \Psi_\mu \in H^{k,\tau} \) for all \( \mu \)'s and for all \( \tau > 0 \) and \( k > 0 \). Let \( \phi \) be its Lie transform. Denote \( z = (z', z \equiv (\xi, f) \) and similarly for \( z' \). Then there exist functions \( G_\mu(z'), G_j(z') \) with the following three properties.

1. \( G_j, G_\mu \in C^\infty(U^{-\kappa,-s}, \mathbb{C}) \) with \( U^{-\kappa,-s} \subset P^{-\kappa,-s} \) an appropriately small neighborhood of the origin.

2. The transformation \( \phi \) has the following structure:

\[
\begin{align*}
\xi_j &= \xi_j' + G_j(z') \\
\bar{f} &= f' + \sum_{\mu} G_\mu(z') \Phi_\mu.
\end{align*}
\]  

(4.3)  

(4.4)
3. There are constants $C_{\tau,k}$ such that

$$\|z - T(z)\|_{\mathcal{P}^{\kappa,s}} \leq C_{\tau,k} |\xi|^{M_0} (|\xi| + \|f\|_{H^{-\kappa,-s}}).$$

Furthermore there are constants $c_{\kappa,s}$ such that

$$|G_j(\xi, f)| \leq c_{\kappa,s} |\xi|^{M_0} (|\xi| + \|f\|_{H^{-\kappa,-s}}),$$

$$|G_{\mu}(\xi, f)| \leq c_{\kappa,s} |\xi|^{M_0+1}.$$  

Proof. The Hamilton equations of $\chi$ have the structure

$$\dot{f} = -i \sum_{\mu} \xi^\mu \Phi_{\mu}, \quad \dot{\xi}_j = P_j(\xi) + \sum_{\mu} \tilde{P}_{\mu}(\xi) \int_{\mathbb{R}^3} \Phi_{\mu} \cdot f \, dx$$

with suitable polynomials $P_j(\xi)$ homogenous of degree $M_0 + 1$ and $\tilde{P}_\mu(\xi)$ homogenous of degree $M_0$. A similar equation holds for $\tilde{f}$. By the existence and uniqueness theorem for differential equations the solution exists up to time 1, provided that the initial data are small enough. Then inequality (4.5) is an immediate consequence of the obvious equality

$$\phi(z) - z = \int_0^1 X_\chi(z(s)) ds .$$

Any map $(\xi', f') \to \xi$ can be written in the form (4.3). From the first of eq.(4.8), equation (4.4) holds with

$$G_{\mu}(\zeta(0), f(0)) := -i \int_0^1 \xi^\mu(s, \zeta(0), f(0)) ds .$$

The $G_j$ in (4.3) and the $G_{\mu}$ in (4.4) are analytic by the analiticity of flow $\phi^t(\xi, f)$, which is a consequence of the analiticity of $X_\chi$ as a function defined in $\mathcal{P}^{-\kappa,-s}$.

The proof of the next two lemmas is elementary.

**Lemma 4.4.** Let $K \in C^k(\mathcal{P}^{1/2,0}, \mathbb{R})$, $k \geq 3$ be a function fulfilling $|K(z)| \leq C \|z\|^{M_1}$, $M_1 \geq 2$. Let $\phi$ be the lie transform generated by a function $\chi$ fulfilling the assumptions of Lemma 4.3. Then $K \circ \phi \in C^k(\mathcal{P}^{1/2,0}, \mathbb{R})$ and $\{K, \chi\} \in C^{k-1}(\mathcal{P}^{1/2,0}, \mathbb{R})$. Furthermore one has

$$|K(\phi(z))| \leq C \|z\|^{M_1}.$$  

$$|K(\phi(z)) - K(z)| \leq C \|z\|^{M_1+M_2}.$$  

**Lemma 4.5.** Let $K \in C^\infty(\mathcal{U}^{-k,-s})$, where $\mathcal{U}^{-k,-s} \subset \mathcal{P}^{-k,-s}$, with some $s \geq 0$, $k \geq 0$. Then one has $X_K \in C^\infty(\mathcal{U}^{-k,-s}, \mathcal{P}^{k,s})$. 

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4.2 Normal form

**Definition 4.6.** A polynomial \( Z \) will be said to be in normal form if it has the structure

\[
Z = Z_0 + Z_1
\]  

where: \( Z_1 \) is a linear combination of monomials of the form

\[
\xi^\mu \xi'^{\nu} \int \Phi(x) f(x) dx \,, \quad \xi^\mu \xi'^{\nu} \int \Phi(x) \tilde{f}(x) dx
\]

with indexes satisfying

\[
\omega \cdot (\nu - \mu) > m \,, \quad \omega \cdot (\nu' - \mu') < -m \,, \quad \omega \cdot (\mu - \nu) > m \,, \quad \omega \cdot (\mu' - \nu') < -m
\]

and \( \Phi \in H^{k,s} \) for all \( k, s \); \( Z_0 \) is independent of \( f \) and is a linear combination of monomials \( \xi^\mu \xi'^{\nu} \) satisfying

\[
\{ H \varphi \xi^\mu \xi'^{\nu} \} = 0 \quad (4.15)
\]

**Remark 4.7.** Equation (4.15) is equivalent to \( \omega \cdot (\mu - \nu) = 0 \).

**Remark 4.8.** By (H5) \( \omega \cdot (\mu - \nu) = 0 \) implies \( |\mu| = |\nu| \).

**Theorem 4.9.** Fix \( k > 0 \) and \( s > 0 \). For any integer \( r \) there exist open neighbourhoods of the origin \( U_r \subset \mathcal{P}^{1/2,0} \), and \( \mathcal{U}_{r - k, -s} \subset \mathcal{P}^{-k,-s} \), and an analytic canonical transformation \( T_r : \mathcal{U}_r \rightarrow \mathcal{P}^{1/2,0} \), which puts the system in normal form up to order \( r + 4 \), namely such that

\[
H^{(r)} := H \circ T_r = H_L + Z^{(r)} + R^{(r)}
\]  

where:

(i) \( Z^{(r)} \) is a polynomial of degree \( r + 3 \) which is in normal form; furthermore, when we expand

\[
Z_1^{(r)}(\xi, f) = \sum_{\mu, \nu} \xi^\mu \xi'^{\nu} \int \Phi_{\mu, \nu} f dx + \sum_{\mu, \nu} \tilde{\xi}^\mu \xi'^{\nu} \int \tilde{\Phi}_{\mu, \nu} \tilde{f} dx
\]

we have, for \( \beta_{|\mu|} := \beta(|\mu|)(0), \varphi^\mu = \prod_j \varphi_j^{\mu_j} \) and similarly \( \omega^\mu = \prod_j \omega_j^{\mu_j} \),

\[
\Phi_{\mu, 0} = \frac{2^{-|\mu|}}{\mu!} \beta_{|\mu|+1} B_{\frac{1}{2}}(\varphi^\mu(x)) + \tilde{\Phi}_{\mu, 0}
\]

where \( \tilde{\Phi}_{\mu, 0} = \tilde{\Phi}_{\mu, 0}(m, \beta_1, ..., \beta_{|\mu|}) \) is a piecewise smooth function of \( m \) which is \( C^\infty \) in the other arguments.

(ii) the transformation \( T_r \) has the structure (4.3), (4.4), has the property that \( 1 - T_r \) extends to an analytic map from \( \mathcal{U}_{r - k, -s} \) to \( \mathcal{P}^{1/2,0} \), and fulfills the inequality

\[
\|z - T_r(z)\|_{\mathcal{P}^{k,-s}} \leq C \|z\|^3_{\mathcal{P}^{k,-s}} \quad (4.19)
\]
(iii) we have $\mathcal{R}^{(r)} = \sum_{d=0}^{4} \mathcal{R}^{(r)}_d$ with the following properties

(iii.1) For $d = 0$, $\mathcal{R}^{(r)}_0 \in C^\infty(U^{-k,-s}, \mathbb{R})$;

(iii.2) for $d = 1, ..., 3$, $\mathcal{R}^{(r)}_d$ has the structure

$$\mathcal{R}^{(r)}_d = \int_{\mathbb{R}^3} F^r_d(x,z)[U(x)]^d dx,$$

where $U = B^{-1/2}(f + \bar{f})$; $F^r_d \in C^\infty(U^{-k,-s}, H^{k,s})$ and for $d = 1, 2, 3$

satisfies

$$\|F^r_d(.,z)\|_{H^{k,s}} \leq C \|z\|_{\mathcal{P}^{-k,-s}}^{4-d};$$

(iii.3) for $d = 4$ we have

$$\mathcal{R}^{(r)}_4 = \int_{\mathbb{R}^3} F^r_4(x,T^r(z))[U(x)]^4 dx,$$

where $F_4(x,z) = F_4(x,\xi, U)$ is the function in Lemma 3.2;

(iii.4) we have, for $\nabla f = (\nabla f, \nabla \bar{f})$ and for $\mathcal{H}^{k,s} = H^{k,s} \times H^{k,s}$,

$$|\mathcal{R}^{(r)}_0(\xi,0)| + \|\nabla f \mathcal{R}^{(r)}_0(\xi,0)\|_{\mathcal{H}^{k,s}} \leq C \|\xi\|^{r+4}$$

$$\|\nabla f \mathcal{R}^{(r)}_1(\xi,0)\|_{\mathcal{H}^{k,s}} \leq C \|\xi\|^{r+3}. $$

### 4.3 The Homological Equation

Let $K(\xi, \bar{\xi}, f, \bar{f})$ be a homogeneous polynomial of degree $M_1$ having the form

$$K = \sum_{|\mu|+|\nu|=M_1} K_{\mu\nu} \xi^\mu \bar{\xi}^\nu + \sum_{|\mu'|+|\nu'|=M_1-1} \xi^{\mu'} \bar{\xi}^{\nu'} \int \Phi_{\mu'\nu'} f$$

$$+ \sum_{|\mu''|+|\nu''|=M_1-1} \xi^{\mu''} \bar{\xi}^{\nu''} \int \Psi_{\mu''\nu''} \bar{f},$$

with functions $\Phi_{\mu'\nu'}, \Psi_{\mu''\nu''} \in H^{k,s}$ for all $k, s \geq 0$, then a key step in the proof of theorem 4.9 consists in solving the so called Homological equation,

$$\{H_L, \chi\} + Z = K,$$

namely in determining two functions $\chi$ and $Z$ such that $Z$ is in normal form and (4.25) holds. First we define $Z$ to be the expression obtained by considering the r.h.s. of (4.24) and restricting the sum to the indexes such that

$$\omega \cdot (\mu - \nu) = 0, \quad \omega \cdot (\nu' - \mu') > m, \quad \omega \cdot (\mu'' - \nu'') > m,$$

which satisfy the normal form condition.

Then, in order to come to the construction of $\chi$, define the homological operator $\mathcal{L}$ acting on polynomials, by

$$\mathcal{L} \chi := \{H_L, \chi\}$$

To start with we consider the restriction of $\mathcal{L}$ to polynomials independent of $f, \bar{f}$. One has the following lemma.
Lemma 4.10. Each monomial $\xi^\mu \bar{\xi}^\nu$ is an eigenvector of the operator $\mathcal{L}$; the corresponding eigenvalue is $\imath(\omega \cdot (\nu - \mu))$.

Proof. Indeed

$$\mathcal{L} \xi^\mu \bar{\xi}^\nu = \{H_0, \xi^\mu \bar{\xi}^\nu\} = \left(\imath \sum_j \omega_j \left(\bar{\xi}_j \frac{\partial}{\partial \bar{\xi}_j} - \xi_j \frac{\partial}{\partial \xi_j}\right)\right) \xi^\mu \bar{\xi}^\nu = \imath \omega \cdot (\nu - \mu) \xi^\mu \bar{\xi}^\nu,$$

where we used

$$\bar{\xi}_j \frac{\partial}{\partial \bar{\xi}_j} \xi^\mu = \nu_j \xi^\mu \bar{\xi}^\nu.$$

Recall that $\xi^\mu \bar{\xi}^\nu$ is not normal form exactly when $\omega \cdot (\nu - \mu) \neq 0$. Turning to polynomials linear in $f, \bar{f}$ we have the following formulas

$$\mathcal{L} \left(\xi^\mu \bar{\xi}^\nu \int \Phi f\right) = -\imath \xi^\mu \bar{\xi}^\nu \int (\omega \cdot (\mu - \nu) + B) \Phi f,$$

$$\mathcal{L} \left(\xi^\mu \bar{\xi}^\nu \int \Phi \bar{f}\right) = -\imath \xi^\mu \bar{\xi}^\nu \int (\omega \cdot (\mu - \nu) - B) \Phi \bar{f},$$

obtained by a simple variant of the previous lemma and by the fact that $B$ is $L^2$ symmetric.

Recall that, if the monomial is not in normal form then the resolvent $R_{\mu\nu} := (\omega \cdot (\mu - \nu) - B)^{-1}$ is well defined. Thus it is easy to obtain the following key lemma.

Lemma 4.11. Let $K$ be a polynomial of the form (4.24); define $Z$ as above and $\chi$ by

$$\chi := \sum_{\mu, \nu} \frac{\imath K_{\mu\nu}}{\omega \cdot (\mu - \nu)} \xi^\mu \bar{\xi}^\nu + \sum_{\mu, \nu} (\xi^\mu \bar{\xi}^\nu \int f R_{\mu\nu} \Phi_{\mu\nu} - \imath \xi^\mu \bar{\xi}^\nu \int \bar{f} R_{\mu\nu} \Phi'_{\mu\nu})$$

where the indexes in the sum are such that

$$\omega \cdot (\mu - \nu) \neq 0, \quad \omega \cdot (\nu' - \mu') > m, \quad \omega \cdot (\nu'' - \nu''') > m.$$

Then the following equation holds

$$\{H_L, \chi\} + Z = K.$$

Furthermore, if $K_{\mu\nu} = \overline{K_{\nu\mu}}$ and $\Phi'_{\mu\nu} = \overline{\Phi_{\nu\mu}}$, also the coefficients in (4.31) satisfy this property.
4.4 Proof of Theorem 4.9

Proof of Theorem 4.9. By Lemma (3.2), $H$ satisfies assumptions and conclusions of Theorem 4.9 with $r = 0$, $T_0 \equiv 1$, $\mathcal{R}(0) := H_{P}$, $Z^{(0)} = 0$.

Assume now that the theorem is true for $r$, we prove it for $r + 1$.

Consider the Taylor expansion of $\mathcal{R}^{(r)}$ and $\mathcal{R}^{(r)}_1$ in $f$ up to order 1. Further expand such quantities in Taylor series in $\xi$ keeping only the main term. One thus gets

\[
\mathcal{R}^{(r)}_0(\xi, f) = \sum_{|\mu| + |\nu| = r + 4} \mathcal{R}^{(r)}_{0\mu\nu} \xi^\mu \bar{\xi}^\nu \quad (4.34)
\]

\[
+ \sum_{|\mu| + |\nu| = r + 3} \xi^\mu \bar{\xi}^\nu \left( \int_{\mathbb{R}^3} \left[ \Phi^{(1)}_{0\mu\nu}(x) f + \Phi^{(2)}_{0\mu\nu}(x) \bar{f} \right] dx \right) \quad (4.35)
\]

\[
+ \mathcal{R}^{(r)}_1(\xi, f) = \sum_{|\mu| + |\nu| = r + 3} \xi^\mu \bar{\xi}^\nu \int_{\mathbb{R}^3} \left[ \Phi^{(1)}_{1\mu\nu}(x) f + \Phi^{(2)}_{1\mu\nu}(x) \bar{f} \right] dx \quad (4.36)
\]

\[
+ \mathcal{R}^{(r)}_{1,2}(\xi) \quad (4.38)
\]

where $\Phi^{(r)}_{\mu\nu} \in H^{k,s}$ and where $\mathcal{R}^{(r)}_1 \in C^\infty(\mathcal{P}^{-k,-s})$ satisfy

\[
\left| \mathcal{R}^{(r)}_{1,2}(z) \right| \leq C \left( \|\xi\|^r + \|\xi\|^r \|f\|_{H^{-k,-s}} + \|z\|^2_{\mathcal{P}^{-k,-s}} \|f\|^2_{H^{-k,-s}} \right) \quad (4.39)
\]

Thus the quantity

\[
K_{r+1} := (4.34) + (4.35) + (4.37) \quad (4.40)
\]

has the structure (4.24) so we can apply Lemma 4.11. Let $\chi_{r+1}$, $Z_{r+1}$ be the solutions of the homological equation

\[
\{ H_L, \chi_{r+1} \} + Z_{r+1} = K_{r+1} ,
\]

and let $\phi_{r+1}$ be the Lie transform generated by $\chi_{r+1}$. Let $U_{r+1}$, $U_{r+1}^{-k,-s}$ be such that $\phi_{r+1}(U_{r+1}) \subset U_r$ and $\phi_{r+1}(U_{r+1}^{-k,-s}) \subset U_r^{-k,-s}$.

Denote $(\xi, f) = \phi_{r+1}(\xi', f')$, then $f = f' + \sum_{\mu} \Phi^{(r+1)}_{\mu}(z') G^{(r+1)}_{\mu}(z)$, with $G^{(r+1)}_{\mu}$ described by Lemma 4.3. Denote

\[
G'_{U} := B^{-1/2} \sum_{\mu} (\Phi^{(r+1)}_{\mu} G^{(r+1)}_{\mu} + \Phi^{(r+1)}_{\mu} \bar{G}^{(r+1)}_{\mu}) .
\]

Recall that

\[
\left| G_{U}(z) \right|_{H^{k,s}} \leq C \left( \|z\|^r_{\mathcal{P}^{-k,-s}} \right) \quad (4.41)
\]

We will now prove that

\[
H^{(r+1)} := H^{(r)} \circ \phi_{r+1} \equiv H \circ (T_r \circ \phi_{r+1}) \equiv H \circ T_{r+1} ,
\]
has the desired structure. Write

\[
H^{(r)} \circ \phi_{r+1} = H_0 + Z^{(r)} + Z_{r+1} 
\]

\[
+ (Z^{(r)} \circ \phi_{r+1} - Z^{(r)}) 
\]

\[
+ K_{r+1} \circ \phi_{r+1} - K_{r+1} 
\]

\[
+ H_L \circ \phi_{r+1} - (H_L + \{\chi_{r+1}; H_L\}) 
\]

\[
+ (R_{02}^{(r)} + R_{12}^{(r)}) \circ \phi_{r+1} 
\]

\[
+ R_3^{(r)} \circ \phi_{r+1} 
\]

\[
+ R_4^{(r)} \circ \phi_{r+1}. 
\]

Define \(Z^{(r+1)} := Z^{(r)} + Z_{r+1}\).

We begin by studying (4.47) and (4.48). For \(d = 2, 3\), one has

\[
\left( R_d^{(r)} \circ \phi_{r+1} \right)(z) = \int_{R^3} F_d(x, \phi_{r+1}(z))[U + G_U(z)]^d 
\]

\[
= \sum_{j=0}^{d} \binom{d}{j} \int_{R^3} F_d(x, \phi_{r+1}(z))[G_U(z)]^{d-j}[U(x)]^j = \sum_{j=0}^{d} H_{d_j}. 
\]

Each of the functions \(H_{d_j}\) has the structure (iii.1-iii.2). Condition (iii.4) is an immediate consequence of (4.41) and of (4.21).

Consider now

\[
(R_4^{(r)} \circ \phi_{r+1})(z) = \int_{R^3} F_4(x, T_{r+1}(z))[U + G_U(z)]^4 
\]

\[
= \sum_{d=0}^{4} \binom{4}{d} \int_{R^3} F_d(x, \phi_{r+1}(z))[G_U(z)]^{4-d}[U(x)]^d 
\]

Then each term with \(d \leq 3\) can be absorbed in \(R_d^{(r+1)}\) because they satisfy the desired inequalities. For \(d = 4\) we get (iii.3).

Consider now (4.46). Since \(R_{02}^{(r)} \in C^\infty(U_{r-k-s}^c)\), since smoothness is preserved under composition with \(\phi_{r+1}\) (restricting the domain), this is a term which can be included in \(R_0^{(r+1)}\). The vanishing properties (4.23) are guaranteed by (4.39).

Since \(K_{r+1} \in C^\infty(U_{r-k-s}^c)\), the term (4.44) can be included in \(R_0^{(r+1)}\). The vanishing properties (4.23) are guaranteed by (4.11). The term (4.43) can be treated exactly in the same way.

It remains to study (4.43). To this end, write

\[
H_L \circ \phi_{r+1} - (H_L + \{\chi_{r+1}; H_L\}) = \int_0^1 \frac{t^2}{2!} \frac{d^2}{dt^2} (H \circ \phi_{r+1}) \, dt 
\]

\[
= \int_0^1 \frac{t^2}{2!} \{\chi_{r+1}, \{\chi_{r+1}, H_L\}\} \circ \phi_{r+1}^t \, dt = \int_0^1 \frac{t^2}{2!} \{\chi_{r+1}, \{Z_{r+1} - K_{r+1}\}\} \circ \phi_{r+1}^t \, dt 
\]
which shows that such a term is in $C^\infty(\mathcal{U}_{-1}^{-k-\varepsilon})$, and also allows to verify that the vanishing properties (4.23) hold.

We come to the proof of equation (4.18). Consider $\Phi_{\mu_0}$ with $|\mu| = r + 3$. Then
\[
\mu!\Phi_{\mu_0} = \partial_\xi^\mu \nabla_f H^{(r)}(0) = \partial_\xi^\mu \nabla_f H^{(r_0)}(0).
\]
We have
\[
\partial_\xi^\mu \nabla_f H^{(r)}(0) = \partial_\xi^\mu \nabla_f H^{(0)}(0) + \partial_\xi^\mu \nabla_f \left[ H^{(0)}(0) \circ T_r - H^{(0)}(0) \right](0) = 2^{-\frac{r-3}{2}} \beta^{(r+4)}(0) \frac{B_{\frac{r}{2}}(\gamma)}{\omega^r} + \partial_\xi^\mu \nabla_f \left[ H^{(0)}(0) \circ T_r - H^{(0)}(0) \right](0) \tag{4.53}
\]
where the first term in the right hand side is obtained by Lemma 3.2. So we need to show that the last term in (4.53) is like the reminder in (4.18). First of all notice that if we consider the embedding $I_k: \mathcal{P}^{k,0} \hookrightarrow \mathcal{P}^{k+1,0}$ for $k > 1/2$ with $I_k(z) = z$, we have $\partial_\xi^\mu \nabla_f H^{(r)}(0) = \partial_\xi^\mu \nabla_f H^{(r)}(0)$ for any $\mu$. In other words, it is enough that we prove our formula restricting the Hamiltonians on $\mathcal{P}^{k,0}$ for $k$ large. Actually we prove that $d^{r+4} \left[ H^{(0)}(0) \circ T_r - H^{(0)}(0) \right](0)$ is a smooth function of $(m, \beta_4, ..., \beta_{r+3})$, where $\beta_\xi := \beta^{(1)}(0)$. We can apply the chain rule and obtain the standard formula
\[
d^{r+4}(H^{(0)} \circ T_r)(0) = \sum_{\alpha} c_\alpha (d^{|\alpha|} H^{(0)})(0) \left( \otimes_{\sum_{j=1}^{r+4} \alpha_j = r + 4} (d^j T_r)(0)^{\alpha_j} \right) \tag{4.54}
\]
with $\sum_{j=1}^{r+4} j \alpha_j = r + 4$ and $c_\alpha$ appropriate universal constants. Insert the decomposition $T_r = I + \tilde{T}_r$ into (4.54). Then $d^{r+4}(H^{(0)} \circ T_r)(0) = d^{r+4}H^{(0)} + \mathcal{E}$ where $\mathcal{E}$ is a sum of terms of the form
\[
c_\alpha (d^{|\alpha|} H^{(0)})(0) \left( \otimes_{\sum_{j=1}^{r+4} \alpha_j = r + 4} (d^j \tilde{T}_r)(0)^{\alpha_j} \right) \tag{4.55}
\]
with at least one $\tilde{\alpha}_j > 0$ and for some $\alpha_0 \geq 0$. By $d^j \tilde{T}_r(0) = 0$ for $0 \leq j \leq 2$ we have $\tilde{\alpha}_1 = \tilde{\alpha}_2 = 0$ and so $\alpha_j = \tilde{\alpha}_j > 0$ for some $j \geq 3$. Hence the terms in (4.55) are such that $|\alpha| < r + 4$. $(d^j H^{(0)})(0)$ for $j < r + 4$ is a smooth function of $(m, \beta_4, ..., \beta_{r+3})$. Indeed, if we reverse the change of variables (3.10), $d^j H^{(0)}(0) = \beta_j$ for all $j$. By induction it is elementary to show that $\tilde{T}_r(z) = \tilde{T}_r(z, m, \beta_4, ..., \beta_{r+3})$ is a smooth function of all its arguments. Indeed $\tilde{T}_0 \equiv 0$, $\tilde{T}_r$ depends on the vector field $K_r$ which in turn is a smooth function of
\[
\partial_\xi^\nu \nabla_f^{(r-1)}(0) \text{ with } |\nu| + j = r + 3 \text{ and } j \leq 1. \tag{4.56}
\]
By induction, (4.56) is a smooth function of $(m, \beta_4, ..., \beta_{r+3})$. Hence we have also proved property (i) of Theorem 4.9.

5 Dynamics of the normal form

Before proceeding with the proof of Theorem 2.3 we give a qualitative description of the behavior of the normalized system and we discuss the nondegeneracy
assumption. This type of argument appeared for the first time in [Si]. In the sequel we assume that the time \( t \) is positive. This is by no means restrictive. We will show later that \( R^{(r)} \) for \( r \geq 2N \) does not modify the dynamics. So we neglect \( R^{(r)} \) and consider the Hamiltonian of the normal form

\[
H_{nf} := H_0(\zeta, f) + Z_0(\zeta) + Z_1(\zeta, f),
\]

with \( Z_0 \) and \( Z_1 \) as in Definition 4.6, where

\[
Z_1(\zeta, \bar{\zeta}, f, \bar{f}) := \langle G, f \rangle + \langle \bar{G}, \bar{f} \rangle,
\]

\[
G := \sum_{\mu, \nu} \xi^\mu \xi^\nu \Phi_{\mu \nu}, \quad \bar{G} = \sum_{\mu, \nu} \xi^\mu \bar{\xi}^\nu \Phi_{\mu \nu},
\]

\( \Phi_{\mu \nu} \in H^{k,s} \) for all \( k, s \) and where the indexes \( \mu, \nu \) fulfill the relations

\[
2 \leq |\mu| + |\nu| \leq r + 4, \quad \omega \cdot (\mu - \nu) < -m.
\]

The Hamilton equations of this system are given by

\[
\dot{f} = -i(Bf + G),
\]

\[
\dot{\xi}_k = -i\omega_k \xi_k - i \frac{\partial Z_0}{\partial \xi_k} - i \left\langle \frac{\partial G}{\partial \xi_k}, f \right\rangle - i \left\langle \frac{\partial \bar{G}}{\partial \xi_k}, \bar{f} \right\rangle.
\]

We prove later that \( f \) is asymptotically free. However we need to examine more in detail \( f \) in order to extract its main contribution to the equations of the \( \xi_k \). With this in mind we decouple further the dynamics of the discrete modes and the continuous ones. This cannot be done by the previous procedure, since by the resonance between continuous and discrete spectrum the Hamiltonian is not well defined in terms of the new decoupled variables. So, we work at the level of vector fields and look for a function \( Y = Y(\xi, \bar{\xi}) \) such that the new variable

\[
g := f + \bar{Y}
\]

is decoupled up to higher order terms from the discrete variables. Substituting in the equation (5.5) one gets

\[
\dot{g} = -iBg - i \left\{ \bar{G} - \left[ B + \sum_k \left( \omega_k \xi_k \frac{\partial}{\partial \xi_k} - \omega_k \bar{\xi}_k \frac{\partial}{\partial \bar{\xi}_k} \right) \right] \bar{Y} \right\} + \text{h.o.t.}
\]

where \text{h.o.t.} denotes terms which are at least linear in \( f \) and of sufficiently high degree in \( \zeta \) (in a sense explained later). Thus we want to determine \( Y \) in such a way that the curly bracket vanishes. Write

\[
\bar{Y} := \sum_{2 \leq |\mu| + |\nu| \leq r + 4, \omega \cdot (\mu - \nu) > m} \bar{Y}_{\mu \nu}(x) \xi^\mu \bar{\xi}^\nu.
\]

Then, the vanishing of the curly brackets in (5.8) is equivalent to

\[
(B - \omega \cdot (\mu - \nu)) \bar{Y}_{\mu \nu} = \Phi_{\nu \mu}
\]
Since $\omega \cdot (\mu - \nu) \in \sigma(B)$ we have to regularize the resolvent. We set

$$R_{\mu \nu}^\pm := \lim_{\epsilon \to 0^+} (B - (\mu - \nu) \cdot \omega \mp i\epsilon)^{-1}. \quad (5.11)$$

Now, in the sequel it is important that $t \geq 0$. Then we define

$$\bar{Y}_{\mu \nu} = R_{\mu \nu}^+ \bar{\Phi}_{\nu \mu} \quad \text{and} \quad Y_{\mu \nu} = \overline{R_{\mu \nu}^- \Phi_{\nu \mu}} = \overline{R_{\mu \nu}^- \Phi_{\nu \mu}}. \quad (5.12)$$

We prove now that with this choice one has $Y_{\mu \nu} \in L^{2; s}$ for all $s > 1/2$ and thus the same holds for $g$.

**Lemma 5.1.** For $\Phi \in H^{2, s}$ for $s > 1/2$ and $\lambda > m$, then $R_B^\pm(\lambda)\Phi$ are well defined and belong to $L^{2, s}$.

**Proof.** The argument is standard, we recall it here for the sake of completeness.

We first claim that if we set $\Psi = (B \in \omega)$ we need to show that $\Psi \in L^{2, s}$.

By (5.15) we obtain that

$$\kappa \in \left(1, \infty\right), \quad \text{so} \quad \sup_{t > 0} \kappa \in \left(1, \infty\right).$$

We have $B\Phi = (-\Delta + V(x)) P_\mu \Phi(x) \in L^{2, s}$. On the other hand, the integral kernel $(B^2 + \tau)^{-1}(x, y)$ satisfies for some fixed $C > 0$ and $b > 0$

$$|(B^2 + \tau)^{-1}(x, y)| \leq C \frac{e^{-b\sqrt{\tau + m}|x - y|}}{|x - y|}, \quad (5.14)$$

see Lemma 3.4.3 [D]. Then $B\Phi(x) = \int_{\mathbb{R}^3} K(x, y)(B^2 \Phi)(y)dy$ with

$$|K(x, y)| \leq \frac{e^{-bx|x - y|}}{|x - y|^\frac{1}{2}} \frac{1}{\pi} \int_0^\infty \tau^{-\frac{1}{2}} \frac{1}{\sqrt{\tau + m}} e^{-\frac{b\sqrt{\tau + m}}{|x - y|}} d\tau = C_0 \frac{e^{-bx|x - y|}}{|x - y|^2}. \quad (5.15)$$

By (5.15) we obtain that $T_s(x, y) := \langle x \rangle^{s}\langle y \rangle^{-s}|K(x, y)|$ is for any $s$ the kernel of an operator bounded in $L^2$. Indeed the Young inequality holds:

$$\sup_x \|T_s(x, y)\|_{L^1_y} + \sup_y \|T_s(x, y)\|_{L^1_x} < C_s < \infty,$$

see (1.33) [Y].

We substitute (5.7) in the equations for $\xi$, namely (5.6). Then we get

$$\dot{\xi}_k = -i \omega_k \xi_k - i \frac{\partial G}{\partial \xi_k} + i \left\langle \frac{\partial G}{\partial \xi_k}, \bar{Y} \right\rangle + i \left\langle \frac{\partial G}{\partial \xi_k}, Y \right\rangle \quad (5.16)$$

$$-i \left\langle \frac{\partial G}{\partial \xi_k}, \bar{g} \right\rangle - i \left\langle \frac{\partial G}{\partial \xi_k}, g \right\rangle \quad (5.17)$$
We will show in the next section that $g$ is negligible. So we neglect the last line. We show that what is left in (5.16) is generically a dissipative system. A simple explicit computation shows that the system (5.16) has the form

$$
\dot{\xi}_k = -i \omega_k \xi_k - i \frac{\partial Z_0}{\partial \xi_k} + i \sum_{\mu \mu' \nu' \nu} \xi_{\mu} \tilde{\xi}_{\nu} \nu_k c_{\mu \nu \mu' \nu'} + \sum_{\mu \mu' \nu' \nu} \xi_{\mu} \tilde{\xi}_{\nu} \nu_k c_{\mu \nu \mu' \nu'},
$$

(5.18)\hspace{1cm}(5.19)\hspace{1cm}(5.20)

where the pairs of indexes $(\mu, \nu)$ and $(\nu', \mu')$ satisfy (5.4). In (5.19) $c_{\mu \nu \mu' \nu'} := \langle \Phi_{\mu \nu}, R^+_{\mu' \nu'} \tilde{\Phi}_{\nu' \mu} \rangle$. (5.21)

We further simplify by extracting the main terms. In (5.19) all the terms which do not satisfy $\mu = \nu' = 0$ are negligible, see (7.28), Appendix B,(B.4). In particular, for any of them there is in (5.19) a term such that $\mu = \nu' = 0$ which is, clearly, larger. All the terms in (5.20) are negligible, see (7.28), Appendix B,(B.5).

We simplify our expression neglecting all terms deemed negligible. We write

$$
\dot{\xi}_k = -i \omega_k \xi_k - i \frac{\partial Z_0}{\partial \xi_k} + i \sum_{\nu \mu} \xi_{\mu} \tilde{\xi}_{\nu} \nu_k c_{0 \nu \mu 0}.
$$

(5.22)

We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and we consider

$$
M := \{ \mu \in \mathbb{N}_0^n : \mu \cdot \omega > m, \quad 2 \leq |\mu| \leq r + 4 \}.
$$

(5.23)

We apply once more normal form theory in order to further simplify the system (5.22): the change of variables

$$
\eta_j = \xi_j + \sum_{\mu \in M, \nu' \in M, \omega \cdot (\mu - \nu') \neq 0} \frac{1}{i \omega \cdot (\mu - \nu')} \frac{\xi_{\mu} \tilde{\xi}_{\nu}}{\xi_j} \nu_k c_{0 \nu \mu 0}
$$

(5.24)

reduces (5.18)-(5.19) to a perturbation of the system

$$
\dot{\eta}_k = \Xi_k(\eta, \bar{\eta}) := -i \omega_k \eta_k - i \frac{\partial Z_0}{\partial \eta_k} + i \sum_{\mu \in M, \nu \in M, \omega \cdot (\mu - \nu) = 0} \frac{\eta_{\mu} \bar{\eta}_{\nu}}{\eta_k} \nu_k c_{0 \nu \mu 0}.
$$

(5.25)

Since $H_{0L} \equiv \sum_k \omega_k |\eta_k|^2$ is a conserved quantity for the system in which the last term of (5.25) is neglected it is natural to compute the Lie derivative $L_{\Xi} H_{0L} \equiv \sum_j \omega_j (\bar{\eta}_j \dot{\eta}_j + \dot{\eta}_j \bar{\eta}_j)$. We compute $L_{\Xi} H_{0L}$ using Plemelj formula \( \frac{1}{\pm 0} = PV \frac{1}{2} \pm i \pi \delta(x) \), from which one has $R^+_{\mu 0} = PV (B - \omega \cdot \mu)^{-1} \pm i \pi \delta(B - \omega \cdot \mu)$ (where the distributions...
in $B$ are defined by means of the distorted Fourier transform associated to $-\Delta + V$). Define

$$\Lambda := \bigcup_{\mu \in M} \{ \omega \cdot \mu \} \quad (5.26)$$

$$M_{\lambda} := \{ \mu \in M : \omega \cdot \mu = \lambda \} \text{ for } \lambda \in \Lambda \quad (5.27)$$

$$F_{\lambda} := \sum_{\mu \in M_{\lambda}} \bar{\eta}^\mu \Phi_0 \mu, \quad B_{\lambda} := \pi \delta(B - \lambda). \quad (5.28)$$

**Lemma 5.2.** The following formula holds

$$\mathcal{E}_{\Xi} H_{0L} = - \sum_{\lambda \in \Lambda} \lambda \langle F_{\lambda}; B_{\lambda} \bar{F}_{\lambda} \rangle \quad (5.29)$$

Moreover, the right hand side is semidefinite negative.

**Proof.** We have by (5.24) and (5.21)

$$\mathcal{E}_{\Xi} H_{0L} = - \operatorname{Im} \left[ \sum_{\mu, \nu \in M, \omega \cdot \mu = \nu \wedge \omega \cdot \nu = 0} \omega \cdot \nu \eta^\mu \bar{\eta}^\nu \langle \Phi_0 \nu, (B - \omega \cdot \mu - i0)^{-1} \Phi_0 \mu \rangle \right]$$

$$= - \sum_{\lambda \in \Lambda} \lambda \operatorname{Im} \left[ \langle F_{\lambda}, (B - \lambda - i0)^{-1} \bar{F}_{\lambda} \rangle \right].$$

Plemelji formula yields (5.29). For $\Psi_{\lambda} = (B + \lambda)F_{\lambda}$ we have for $k^2 = \lambda^2 - m^2$

$$\langle F_{\lambda}, (B - \lambda - i0)^{-1} \bar{F}_{\lambda} \rangle = \langle F_{\lambda}, R^\Delta - V (k^2) \bar{\Psi}_{\lambda} \rangle.$$

The latter is well defined, as seen above in Lemma 5.1. By Proposition 2.2 ch. 9 [T2] or by Lemma 7 ch. XIII [RS],

$$\operatorname{Im} \left[ \langle F_{\lambda}, R^\Delta - V (k^2) \bar{\Psi}_{\lambda} \rangle \right] = \pi \langle F_{\lambda}, \delta(-\Delta + V - k^2) \bar{\Psi}_{\lambda} \rangle =$$

$$= \frac{k}{16\pi} \int_{|\xi|=k} \bar{F}_{\lambda}(\xi) \overline{\psi}_{\lambda}(\xi) d\sigma(\xi) = \frac{2\lambda k}{16\pi} \int_{|\xi|=k} |\bar{F}_{\lambda}(\xi)|^2 d\sigma(\xi),$$

where by $\hat{w}$ we mean the distorted Fourier transform of $w$ associated to $-\Delta + V$, see Appendix A.1, ch. 9 [T2] or section XI.6 [RS].

**5.1 The nondegeneracy assumption**

The nondegeneracy assumption we mentioned in the introduction pertains the r.h.s. of (5.29): it is required in order to make possible the use of (5.29) for ensuring a decay rate of the different $\eta$ variables. Specifically, we assume

(H7) There exists a positive constant $C$ and a sufficiently small $\delta_0 > 0$ such that such that for all $|\eta| < \delta_0$

$$\sum_{\lambda \in \Lambda} \lambda \langle F_{\lambda}; B_{\lambda} \bar{F}_{\lambda} \rangle \geq C \sum_{\mu \in M} |\eta^\mu|^2. \quad (5.30)$$
Notice that $M$ and $\Lambda$ depend on $r$. Set now

$$\tilde{M} = \{ \mu \in M : \nu_j \leq \mu_j \ \forall \ j \ \text{and} \ \nu \neq \mu \Rightarrow \nu \notin M \}$$

$$\tilde{\lambda} := \bigcup_{\mu \in \tilde{M}} \{ \omega : \mu \}$$

$$\tilde{M}_\lambda := \{ \mu \in \tilde{M} : \omega \cdot \mu = \lambda \} \ \text{for} \ \lambda \in \tilde{\lambda}.$$  

It is easy to show that (H7) is equivalent to:

$$(H7') \ \text{For any} \ \lambda \in \tilde{\lambda} \ \text{the following matrix is invertible:}$$

$$\{(\bar{\Phi}_{\mu 0}, B_\lambda \Phi_{\mu 0})\}_{\mu, \mu' \in \tilde{M}_\lambda}.$$  

Remark 5.3. The set $\tilde{\lambda}$ depends on $m$; $\tilde{M}_\lambda$ is piecewise constant in $m$.

In the case where $j \neq l$ implies $-\lambda_2^2 \neq -\lambda_1^2$ (this can be easily arranged picking $V(x)$ generic, by elementary methods in perturbation theory), the assumption (H7) can be further simplified. Indeed (H5) implies that for any $\lambda \in \tilde{\lambda}$ there exists a unique $\mu \in \tilde{M}_\lambda$. Then (H7') reduces to

$$(H7'') \ \text{For any} \ \mu \in \tilde{M}_\lambda \ \text{one has} \ \gamma_\mu := \langle \bar{\Phi}_{\mu 0}, B_\omega \Phi_{\mu 0} \rangle \neq 0.$$  

We are now ready to give the proof of Proposition 2.2.

Proof of Proposition 2.2. We use equation (4.18) in order to compute the quantities (5.33) as functions of $m$ and of the Taylor coefficients $\beta_\xi$ of $\beta$. Set $c = c_\mu = \frac{2 - \mu}{\mu}$ and $\Psi_\mu := B^{-1/2} \varphi^\mu$. Then, (4.18) implies

$$\gamma_\mu(m, \beta_1, ..., \beta_{|\mu|+1}) = \gamma_\mu(m, \beta_4, ..., \beta_{|\mu|}, 0) + 2c \beta_{|\mu|+1} \text{Re}(\bar{\Phi}_{\mu 0}(m, \beta_4, ..., \beta_{|\mu|}, 0), B_{\omega \mu} \Psi_\mu)$$

$$+ c^2 \beta_{|\mu|+1}^2 \langle \bar{\Psi}_\mu, B_{\omega \mu} \Psi_\mu \rangle.$$  

We conclude that either (5.34) is independent of $\beta_{|\mu|+1}$ or there exists at most 1 value of $\beta_{|\mu|+1}$ for any choice of $(m, \beta_4, ..., \beta_{|\mu|})$ such that (5.34) vanishes. We show now that, except for at most a finite number of values of $m$ in any compact interval, (5.34) depends on $\beta_{|\mu|+1}$. We have, see the proof of (5.29),

$$\langle \bar{\Psi}_\mu, B_{\omega \mu} \Psi_\mu \rangle = \frac{1}{16\pi} \int_{|\xi| = \sqrt{(\omega \mu)^2 - m^2}} |\tilde{\varphi}^\mu(\xi)|^2 d\sigma(\xi)$$  

where we are using the distorted Fourier transform associated to $-\Delta + V$. Since the $\varphi_j(x)$ are smooth functions decaying like $e^{-|x||\lambda|}$ with all their derivatives, and $V(x)$ is chosen exponentially decreasing as well, by Paley Wiener theory applied to the distorted Fourier transform associated to $-\Delta + V$, the functions $\tilde{\varphi}^\mu(\xi)$ are analytic, see Remark A.1. If the set where $\tilde{\varphi}^\mu(\xi) = 0$ does not contain any sphere then the proof is completed. If $\tilde{\varphi}^\mu(\xi) = 0$ on a sphere, say $|\xi| = a_0$, then, by analyticity, $\tilde{\varphi}^\mu(\xi)$ does not vanish identically on nearby spheres. We eliminate values of $m$ such that $\omega(m) \cdot \mu = a_0$. Since $\omega(m) \cdot \mu$ is a nontrivial
analytic function this can be obtained by removing at most a finite number of values of \(m\). Repeating the operation for all \(\mu \in \hat{M}\) (a finite set) one gets that, apart from a finite set of values of \(m\), the quantity in (5.35) is different from 0. Thus removing at most one value of \(\beta_{|\mu|+1}\) for each \(\mu \in \hat{M}\), one gets \(\gamma_\mu > 0\) \(\forall \mu \in \hat{M}\).

Remark 5.4. (5.35) with \(\mu = 3\) and \(\ker(-\Delta + V + \lambda^2) = \text{span}\{\varphi\}\) is the condition necessary in the special case in [SW1]. If \(\hat{\varphi}_3(\xi) = \hat{\varphi}_3(|\xi|)\), then the fact that (5.35) is nonzero reduces to \(\hat{\varphi}_3(\sqrt{9\omega^2 - m^2}) \neq 0\), which is the condition written in (1.8) [SW1].

6 Review of linear theory

We collect here some well known facts needed in the paper. First of all, for our purposes the following Strichartz estimates for the flat equation will be sufficient, see [DF]:

Lemma 6.1. There is a fixed \(C\) such that for any admissible pair \((r, p)\), see (2.2), we have

\[
\|K'_0(t)u_0 + K_0(t)v_0\|_{L^r_tW^{\frac{1}{p} + \frac{1}{2}, r}_x} \leq C\|u_0, v_0\|_{H^1 \times L^2}.
\] (6.1)

Furthermore, for any other admissible pair \((a, b)\),

\[
\|\int_{s<t} K_0(t-s)F(s)ds\|_{L^a_tW^{\frac{1}{a'} - \frac{1}{b} + \frac{1}{2}, a'}_x} \leq C\|F\|_{L^a_tW^{\frac{1}{a'} - \frac{1}{b} + \frac{1}{2}, b'}_x},
\] (6.2)

where given any \(p \in [1, \infty]\) we set \(p' = \frac{p}{p-1}\).

We next consider the linearization of (1.1). Notice that under (H1) for any \(k \in \mathbb{N} \cup \{0\}\) and \(p \in [1, \infty]\) the functionals \(\langle \cdot, \varphi_j \rangle\) are bounded in \(W^{m,p}\). Let \(W^k_p, H^k_c\) if \(p = 2\), be the intersection of their kernels in \(W^{k,p}\). We recall the following result by [Y]

Theorem 6.2. Assume: (H2); \(|\partial_x^\alpha V(x)| \leq C|x|^{-\sigma}\) for \(|\alpha| \leq k\), for fixed \(C\) and \(\sigma > 5\). Consider the strong limits

\[
W_\pm = \lim_{t \to \pm \infty} e^{it(-\Delta + V)} e^{it\Delta}, \quad Z_\pm = \lim_{t \to \pm \infty} e^{-it\Delta} e^{it(\Delta - V)} P_c.
\] (6.3)

Then \(W_\pm : H^k \to H^k_c\) are isomorphic isometries which extend into isomorphisms \(W_\pm : W^{k,p} \to W^{k,p}_c\) for all \(p \in [1, \infty]\). Their inverses are \(Z_\pm\). For any Borel function \(f(t)\) we have, for a fixed choice of signs,

\[
f(-\Delta + V)P_c = W_\pm f(-\Delta)Z_\pm, \quad f(-\Delta)Z_\pm = Z_\pm f(-\Delta + V)P_c W_\pm.
\] (6.4)

By the fact that for admissible pairs \((r, p)\) the function \(\frac{1}{p} - \frac{1}{2} + \frac{1}{p} \leq 2\) and \(\frac{1}{p} - \frac{1}{p} + 1 \leq \frac{1}{2}\), by Theorem 6.2 we have the following transposition of Lemma 6.1 to our non flat case:
Lemma 6.3. Set $K(t) = \sin(tB)/B$. Then, if we assume (H1)–(H2) there is a fixed constant $C_0$ such that for any two admissible pairs $(r, p)$ and $(a, b)$ we have
\[
\|K'(t)u_0 + K(t)v_0\|_{L^1_tW^\frac{1}{2} + \frac{1}{2} + \frac{1}{p}} \leq C_0\|(u_0, v_0)\|_{H^1 \times L^2}. \tag{6.5}
\]
and
\[
\|\int_{s<t} K(t-s)F(s)ds\|_{L^1_tW^\frac{1}{2} + \frac{1}{2} + \frac{1}{p}} \leq C_0\|F\|_{L^1_tW^\frac{1}{2} + \frac{1}{2} + \frac{1}{p} + \nu}. \tag{6.6}
\]
Notice that Lemma 6.3 is equivalent to the following:

Lemma 6.4. Under the hypotheses of Lemma 6.3 there is a fixed constant $C_0$ such that for any two admissible pairs $(r, p)$ and $(a, b)$ we have
\[
\|e^{-itB}P_0u_0\|_{L^1_tW^\frac{1}{2} + \frac{1}{2} + \frac{1}{p}} \leq C_0\|u_0\|_{L^2}
\]
\[
\|\int_{s<t} e^{i(s-t)B}P_0F(s)ds\|_{L^1_tW^\frac{1}{2} + \frac{1}{2} + \frac{1}{p}} \leq C_0\|F\|_{L^1_tW^\frac{1}{2} + \frac{1}{2} + \frac{1}{p} + \nu}. \tag{6.7}
\]

Sketches of proofs of Lemmas 6.5 and 6.6 are at the end.

Lemma 6.5. Assume (H1)–(H2) and consider $m < a < b < \infty$. Then for any $\gamma > 9/2$ there is a constant $C = C(\gamma)$ such that we have
\[
\|e^{-itB}R_B(\mu + i0)g\|_{L^2_tL^{2,\gamma}} \leq C\|\langle t\rangle^{-\frac{\gamma}{2}}\|g\|_{L^2_tL^{a,\gamma}} \text{ for any } \mu \in [a, b] \text{ and } t \in \mathbb{R}. \tag{6.8}
\]

Lemma 6.6. Assume (H1)–(H2). Then for any $s > 1$ there is a fixed $C_0 = C_0(s)$ such that for any admissible pair $(r, p)$ we have
\[
\left\|\int_0^t e^{i(t'-t)B}P_tF(t')dt'\right\|_{L^1_tW^\frac{1}{2} + \frac{1}{2} + \frac{1}{p}} \leq C_0\|F\|_{L^1_tL^2}. \tag{6.9}
\]

7 Nonlinear estimates

We apply Theorem 4.9 for $r = 2N$ (recall $N = N_1$ where $N_j \omega_j < m < (N_j + 1)\omega_j$). Any $r \geq 2N$ is also ok. We will show:

Theorem 7.1. There is a fixed $C > 0$ such that for $\epsilon_0 > 0$ sufficiently small and for $\epsilon \in (0, \epsilon_0)$ we have
\[
\|f\|_{L^1_t([0,\infty) \times W^\frac{1}{2} + \frac{1}{2} + \frac{1}{p})} \leq C\epsilon \text{ for all admissible pairs } (r, p) \tag{7.1}
\]
\[
\|\xi^\mu\|_{L^1_t([0,\infty) \times W^0)} \leq C\epsilon \text{ for all multi indexes } \mu \text{ with } \omega \cdot \mu > m \tag{7.2}
\]
\[
\|\xi_j\|_{W^1} \leq C\epsilon \text{ for all } j \in \{1, \ldots, n\}. \tag{7.3}
\]

Theorem 7.1 implies (2.3). The existence of $(u_\pm, v_\pm)$ is instead a consequence of Lemma 7.8 below.

Remark 7.2. By (3.5) one has $\|\xi\|_{L^\infty} + \|f\|_{L^p_t([0,\infty) \times W^\frac{1}{2} + \frac{1}{2} + \frac{1}{p})} \lesssim \epsilon$. Also (7.3) is easy by (3.5) and (3.11), so it will be assumed.
Remark 7.3. It is not restrictive to prove Theorem 7.1 with \( \mathbb{R} \) replaced by \([0, \infty)\), so in the sequel we will consider \( t \geq 0 \) only.

Remark 7.4. We have for any bounded interval \( I \)

\[
f \in L^1_t(I, W^{1/p-1/r, p}_x) \quad \text{for all admissible pairs} \ (r,p).
\]  

(7.4)

This can be seen as follows. \( u \in L^\infty_t(\mathbb{R}, H^1_x) \), implies \( u^3 \in L^\infty_t(\mathbb{R}, L^2_x) \) and \( \|\beta'(u)\|_{L^2_t} \leq \|u\|^3_{L^2_t} \lesssim \|u\|^3_{H^1_t} \). By Lemma 6.3 and (3.6), this implies \( u \in L^1_t(I, W^{1/p-1/r+1/2p}_x) \) over any bounded interval \( I \) for any admissible pair \((r,p)\). Then, the estimate (4.19) implies that the property persists also after the normalizing transformation.

We prove Theorem 7.1 by means of a standard continuation argument, spelled out for example in formulas (2.6)–(2.8) [So]. We know that \( \|f(0)\|_{H^{1/2}} + |\xi(0)| \leq C_0 \epsilon \). We can consider a fixed constant \( C_3 \) valid simultaneously for Lemmas 6.4–6.6. Suppose that the following estimates hold

\[
\|f\|_{L^1_t([0,T], W^{1/p-1/r, p}_x)} \leq C_1 \epsilon \quad \text{for all admissible pairs} \ (r,p)
\]

(7.5)

\[
\|\xi^\mu\|_{L^2_t([0,T])} \leq C_2 \epsilon \quad \text{for all multi indexes} \ \mu \quad \text{with} \ \omega \cdot \mu > m
\]

(7.6)

for fixed large multiples \( C_1, C_2 \) of \( C_0 C_3 \). Then we will prove that, for \( \epsilon \) sufficiently small, (7.5) and (7.6) imply the same estimate but with \( C_1, C_2 \) replaced by \( C_1/2, C_2/2 \). Then (7.5) and (7.6) hold with \([0, T]\) replaced by \([0, \infty)\).

7.1 Estimate of the continuous variable \( f \)

Consider \( H^{(r)} = H_L + Z^{(r)} + R^{(r)} \). We set \( Z = Z^{(r)} \) and \( R = R^{(r)} \). Then we have

\[
if - Bf = \nabla f Z_1 + \nabla f R
\]

(7.7)

Lemma 7.5. Assume (7.5), and (7.6), and fix any \( s \geq 0 \). Then there exists a constant \( C = C(C_1, C_2) \) independent of \( \epsilon \) such that the following is true: we have \( \nabla f R = R_1 + R_2 \) with

\[
\|R_1\|_{L^1_t([0,T], H^\frac{1}{2-s}_x)} + \|R_2\|_{L^2_t([0,T], H^\frac{1}{2-s}_x)} \leq C(C_1, C_2) \epsilon^2.
\]

(7.8)

Proof. Recall (iii) Theorem 4.9. For \( d \leq 1 \) we have \( \nabla f R_d \in H^{\frac{1}{2-s}} \) for arbitrary fixed \( s \). By (4.23)

\[
\|\nabla f R_0\|_{H^{\frac{1}{2-s}}} + \|\nabla f R_1\|_{H^{\frac{1}{2-s}}} \leq C |\xi|^{2N+3}
\]

By (7.6) and Remark (7.2) we get

\[
\|\nabla f (R_0 + R_1)\|_{L^2_t([0,T], H^{\frac{1}{2-s}}_x)} \lesssim \|\xi\|_{L^\infty_t}^{N+2} \|\xi\|_{L^\infty_t}^{N+2} \leq C_2 C^{N+2} \epsilon^{N+3}.
\]

(7.9)
By (iii.2)–(iii.3), the remaining terms we need to bound are of the form
\[ \xi B^{-\frac{1}{2}} \left( \Phi(x)B^{-\frac{1}{2}}f \right) + B^{-\frac{1}{2}} \left( \Phi(x)(B^{-\frac{1}{2}}f)^2 \right) + B^{-\frac{1}{2}}(B^{-\frac{1}{2}}f)^3. \]  \hspace{1cm} (7.10)
for some generic \( \Phi(x) \) continuous and rapidly decreasing at infinity. By Lemma 3.3, Schwartz inequality, Lemma 3.3, (7.6) and Remark 7.2
\[ \| \xi B^{-\frac{1}{2}} \left( \Phi(x)B^{-\frac{1}{2}}f \right) \|_{L^2_t H^\frac{1}{2}} \lesssim \| \xi \|_{L_t^\infty} \| B^{-\frac{1}{2}}f \|_{L^2_t L^6_x} \leq \] \[ \leq C\epsilon \| f \|_{L^2_t L^6_x}^2 \lesssim CC_1 \epsilon. \] \hspace{1cm} (7.11)
Similarly, by Lemma 3.3, by Hölder and Sobolev inequalities, by Theorem 6.2 and by (7.6)
\[ \| B^{-\frac{1}{2}} \left( \Phi(x)(B^{-\frac{1}{2}}f)^2 \right) \|_{L^1_t H^\frac{1}{2}} \lesssim \| B^{-\frac{1}{2}}f \|_{L^2_t L^6_x} \leq \] \[ \lesssim \| f \|_{L^2_t W^{-1/3,6}_x} \leq C_2 \epsilon. \] \hspace{1cm} (7.12)
Similarly, by Lemma 3.3, by Hölder and Sobolev inequalities, by Theorem 6.2 and by Remark 7.2
\[ \| B^{-\frac{1}{2}}(B^{-\frac{1}{2}}f)^3 \|_{L_t^1 H^\frac{1}{2}} \lesssim \| (B^{-\frac{1}{2}}f)^3 \|_{L^1_t L^6_x} \lesssim \| B^{-\frac{1}{2}}f \|_{L_t^\infty L^6_x} \| B^{-\frac{1}{2}}f \|_{L^2_t L^6_x} \] \[ \lesssim \| f \|_{L_t^\infty H^\frac{1}{2}} \| f \|_{L^2_t W^{-1/3,6}_x} \leq C_2 \epsilon. \] \hspace{1cm} (7.13)
Collecting (7.9)–(7.13) the result follows. \( \square \)

**Remark 7.6.** By
\[ |\nabla \xi R| \lesssim |\xi| r+4 + |\xi| r+3 \| B^{-\frac{1}{2}}f \|_{L_t^2 L^{-6}}, \] \[ + \| B^{-\frac{1}{2}}f \|_{L^2_t L^{-6}} + \| B^{-\frac{1}{2}}f \|_{L^2_t L^{-6}} \| B^{-\frac{1}{2}}f \|_{L^2_t H^\frac{1}{2}}; \] \hspace{1cm} (7.14)
and the same method one can easily prove
\[ \left\| \partial_x \mathcal{R}^{(r)} \right\|_{L^1_t} \leq CC_1 (C_2 + C_1 + C^2_1) \epsilon. \] \hspace{1cm} (7.15)

One also has the easier estimate
\[ \left\| \int_0^t e^{iB(s-t)} \nabla f Z_1 \right\|_{L_t^1 W^{-\frac{1}{2}, \frac{1}{2}, 1}_x} \leq C_0 \| \nabla f Z_1 \|_{L_t^2 W^{\frac{1}{2}, \frac{1}{2}, 1}_x} \leq CC_0 C_2 \epsilon. \] \hspace{1cm} (7.16)
The important fact is that (7.16) is independent of \( C_1. \)

It is now easy to use Duhamel formula to get the following
Proposition 7.7. Assume (7.5) and (7.6). Then there exist constants \( C = C(C_1, C_2), K_1 \), with \( K_1 \) independent of \( C_1 \), such that, if \( C(C_1, C_2) \epsilon < C_0 \), with \( C_0 \) the constant in Lemma 6.4, then we have

\[
\|f\|_{L^r_t([0,T], W^{1/p-1/r,p}_x)} \leq K_1 \epsilon \text{ for all admissible pairs } (r,p).
\] (7.17)

Proof. Write

\[
f = e^{-iBt}f(0) - i \int_0^t e^{iB(s-t)} \nabla \bar{f} Z ds - i \int_0^t e^{iB(s-t)} \nabla \bar{f} R ds.
\] (7.18)

The first term at r.h.s. is estimated using (6.7). Consider now the last term at r.h.s.; it is the sum of

\[
\int_0^t e^{iB(s-t)} R_i(s) ds, \quad i = 1, 2.
\]

The term with \( i = 1 \) is estimated using the second of (6.7) with \( a = \infty, b = 2 \) and thus \( a' = 1, b' = 2 \), while the term with \( i = 2 \) is estimated using (6.9). The term containing \( Z \) is estimated in the same way.

We end this subsection by proving asymptotic flatness of \( f \) if the bound (7.17) holds. This implies the existence of the \((u_\pm, v_\pm)\) and their properties in Theorem 2.3.

Lemma 7.8. Assume (7.17). Then there exists \( f_+ \in H^{1/2}_x \) such that

\[
\lim_{t \to \pm \infty} \|f(t) - e^{-iBt} f_+\|_{H^{1/2}_x} = 0.
\] (7.19)

Proof. We have

\[
e^{iB} f(t) = f(0) - i \int_0^t e^{iB} \nabla f R ds
\]

and so for \( t_1 < t_2 \)

\[
e^{it_2B} f(t_2) - e^{it_1B} f(t_1) = -i \int_{t_1}^{t_2} e^{it'B} \nabla f R dt'.
\]

By Lemmas 6.4, 6.6 and 7.5 we get:

\[
\|e^{it_2B} f(t_2) - e^{it_1B} f(t_1)\|_{H^{1/2}_x} = \| \int_{t_1}^{t_2} e^{it'B} \nabla f R dt'\|_{H^{1/2}_x} \leq \|\nabla f R\|_{L^2([t_1,t_2], H^{1/2}_x)} \to 0 \text{ for } t_1 \to \infty \text{ and } t_1 < t_2.
\] (7.20)

Then \( f_+ = \lim_{t \to \infty} e^{iB} f(t) \) satisfies the desired properties. \(\square\)
7.2 Estimate of $g$

Before estimating the discrete degrees of freedom we need an estimate for the $g$ defined in (5.7) with $Y$ given by (5.9), (5.12). Then, if $f$, $\xi$ fulfill the Hamilton equations of (4.16), $g$ satisfies

$$i\dot{g} -Bg = \nabla \bar{f} R + \sum_k \left[ \partial_{\xi_k} \bar{Y} \partial_{\xi_k} (Z + R) - \partial_{\xi_k} \bar{Y} \partial_{\xi_k} (Z + R) \right]$$  \hfill (7.21)

Remark 7.9. We do not substitute $g$ in place of $f$ in the r.h.s., but we keep the original variable $f$. In this way we reuse (7.8) and we can use (7.5). We also avoid the trouble of having to estimate nonlinearities in terms of $Y$.

We have:

Lemma 7.10. For $\epsilon$ sufficiently small and for $C_0$ the constant in Lemma 6.4, we have

$$\|g\|_{L^2_t H^{-\frac{1}{2}}_{x,-s}} \leq C_0 \epsilon + O(\epsilon^2).$$  \hfill (7.22)

Proof. We can apply Duhamel formula and write

$$g(t) = e^{-iBt}g(0) - i \int_0^t e^{iB(t'-t)}[\nabla \bar{f} R + (7.21)] dt'. \hfill (7.23)$$

First of all we prove $\|e^{-iBt}g(0)\|_{L^2_t H^{-\frac{1}{2}}_{x,-s}} \leq C_0 \epsilon + O(\epsilon^2)$. To this end recall that $g(0) = f(0) + \bar{Y}(0)$. By Schwartz and the Strichartz inequalities in Lemma 6.4, we have

$$\|e^{-iBt}f(0)\|_{L^2_t H^{-\frac{1}{2}}_{x,-s}} \lesssim \|e^{-iBt}f(0)\|_{L^2_t W^{-\frac{1}{2}}_{x,6}} \leq C_0 \epsilon.$$

The estimate of $\|e^{-iBt}\bar{Y}(0)\|_{L^2_t H^{-\frac{1}{2}}_{x,-s}}$ follows from

$$\|e^{-iBt}\xi(0)\bar{\xi}(0)\|_{L^2_t H^{-\frac{1}{2}}_{x,-s}} \lesssim |\xi(0)\bar{\xi}(0)| \|\bar{\Phi}_{\nu\mu}\|_{L^2_{x,6}} \lesssim \epsilon^{n+1},$$

which in turn follows from Lemma 6.5. The contribution to the retarded terms in (7.23) from $\nabla \bar{f} R$ are easily shown to be $O(\epsilon^2)$ using (7.8). (7.21) contributes various terms to (7.23), we consider the main ones (for the others the argument is simpler). Consider in particular contributions from $Z_0$. For $\mu_j \neq 0$ we have by Lemma 6.5

$$\| \int_0^t e^{i(t'-t)B} \xi(0)\bar{\xi}(0) Z_0 R^{+}_{\nu\mu} \bar{\Phi}_{\nu\mu} dt' \|_{L^2_t H^{-\frac{1}{2}}_{x,-s}} \leq C \|\xi(0)\bar{\xi}(0)\|_{L^2_t H^{-\frac{1}{2}}_{x,-s}} \|\bar{\Phi}_{\nu\mu}\|_{L^2_{x,6}}.$$

We need to show

$$\|\frac{\xi(0)\bar{\xi}(0)}{\xi_j} \bar{\Phi}_{\xi_j} Z_0\|_{L^2_t H^{-\frac{1}{2}}_{x,-s}} = O(\epsilon^2).$$  \hfill (7.24)
By (5.4) and (5.12) we have
\[
\omega \cdot (\mu - \nu) > m. \tag{7.25}
\]
Let \( \xi^\alpha \bar{\xi}^\beta \) be a generic monomial of \( Z_0 \). The nontrivial case is \( \beta_j \neq 0 \). Then \( \partial_{\xi_j} (\xi^\alpha \bar{\xi}^\beta) = \beta_j \xi_j \xi^\beta \). By Definition 4.6 we have \( \omega \cdot (\alpha - \beta) = 0 \), and by Remark 4.8, \(|\alpha| = |\beta| \geq 2\). Thus in particular one has
\[
\omega \cdot \alpha \geq \omega_j \impliedby \omega \cdot (\mu + \alpha) - \omega_j > m. \tag{7.26}
\]
So the following holds
\[
\left\| \frac{\xi^\mu \bar{\xi}^\nu \xi^\alpha \bar{\xi}^\beta}{\xi_j} \right\|_{L^2_t} \leq \left\| \frac{\xi^\mu \bar{\xi}^\nu}{\xi_j} \right\|_{L^\infty_t} \left\| \frac{\xi^\alpha}{\xi_j} \right\|_{L^2_t} \leq C_2 C_\epsilon^2 |\nu| + |\beta| \leq C C_2 \epsilon^2 \tag{7.27}
\]
This completes the proof of Lemma 7.10. \( \square \)

### 7.3 Estimate of the discrete variables \( \xi \)

We now conclude the estimates by estimating the various remainders in the equation for the \( \eta \) variables. More precisely in the equations for the time derivative of \( H_{0L} = \sum_j \omega_j |\eta_j|^2 \).

So, we consider the complete equations for the variable \( \xi_j \) as deduced from the Hamiltonian (4.16); they are a perturbation of (5.6). In the terms contained in (5.6) we insert the variable \( g \) as defined in (5.7) (and estimated by (7.22)). Thus we get a perturbation of (5.16), (5.17). Then, we split the terms (5.19) into a dominant part and a remainder (according to the discussion of section 5). Then we introduce the variables \( \eta \) according to (5.24). In order to write the equation for such variables, introduce the notation

\[
\Delta_j(\xi) := \sum_{\mu, \nu \in M, \omega(\mu, \nu) \neq 0} \frac{1}{\omega \cdot (\mu - \nu)} \frac{\xi^\mu \bar{\xi}^\nu}{\xi_j} \xi_j c_{0\mu \nu} \tag{7.28}
\]

\[
G_{0,j} := i \sum_{\mu, \nu \in M, \omega(\mu, \nu) \neq 0} \frac{\xi^\mu \bar{\xi}^\nu}{\xi_j} \xi_j c_{0\mu \nu},
\]

\[
G_{1,j} := i \sum_{\mu, \nu \in M, \omega(\mu, \nu) \neq 0} \frac{\xi^{\mu'} \bar{\xi}^{\nu'} + \nu \mu'}{\xi_j} \xi_j c_{\mu \nu \mu' \nu'} + i \sum_{\mu, \nu \in M, \omega(\mu, \nu) \neq 0} \frac{\xi^{\mu+p} \bar{\xi}^{\nu+p}}{\xi_j} \xi_j c_{\mu \nu \mu' \nu'}
\]

\[
N_j(\xi) := i \omega_j \Delta_j(\xi) - i \sum_k \frac{\partial \Delta_j(\xi)}{\partial \xi_k} \omega_k \xi_k + G_{0,j}.
\]

\( N_j \) coincides with the last term in (5.25), namely the part in normal form coming from the interaction between discrete and continuous spectrum. Then we get
\[
\dot{\eta}_j = -i \omega_j \eta_j - i \frac{\partial Z_0}{\partial \xi_j}(\eta) + i N_j(\eta) + \mathcal{E}_j \tag{7.29}
\]
where, for the $G$ in the first two lines defined in (5.2),

$$
E_j := i \sum_k \frac{\partial \Delta_j}{\partial \xi_k}(\xi) \left[ -i \frac{\partial Z_0}{\partial \xi_k}, f \right] - i \left\langle \frac{\partial G}{\partial \xi_k}, f \right\rangle - i \left\langle \frac{\partial \tilde{G}}{\partial \xi_k}, \bar{g} \right\rangle - i \frac{\partial R}{\partial \xi_k} + G_{1,k} \right] (7.30)
$$

Equation (7.29) is a perturbation, through $E$, of equation (5.25). The estimate we need for $E$ is given by the following lemma

**Lemma 7.11.** Provided $\epsilon$ is small enough, the following estimate holds

$$
\sum_j \|\eta_j E_j\|_{L^1} \leq CC_2 \epsilon^2 (7.31)
$$

As we will see the important fact is that the right hand side is only linear in $C_2$. The somewhat technical proof of this lemma is postponed to Appendix B.

### 7.4 End of the proof

We can now easily conclude the proof. Using the notations of section 5, we have that along the solutions of the system (7.29) the following equation holds

$$
\frac{dH_0}{dt} = - \sum_{\lambda \in \Lambda} \langle F_\lambda; B_\lambda \bar{F}_\lambda \rangle + \sum_j \omega_j (\eta_j \tilde{E}_j + \bar{\eta}_j E_j) \right] (7.32)
$$

Integrating and reorganizing the terms one has

$$
H_{0L}(t) + \sum_{\lambda} \int_0^t \langle F_\lambda; B_\lambda \bar{F}_\lambda \rangle(s)ds = H_{0L}(0) + \int_0^t \sum_j \omega_j (\eta_j \tilde{E}_j + \bar{\eta}_j E_j)(s)ds \right] (7.33)
$$

From which, using assumption (H5) we immediately get

$$
\sum_{\mu \in M} \int_0^T |\eta^\mu| dt \leq (C + CC_2) \epsilon^2 (7.34)
$$

which clearly implies

$$
\sum_{\mu \in M} \int_0^T |\xi^\mu| dt \leq (C + CC_2) \epsilon^2 (7.35)
$$

We have thus proved the following final step of the proof:
Theorem 7.12. The inequalities (7.5) and (7.6) imply
\[
\|f\|_{L^2_t([0,T],W^{1/p-1/r,p}_x)} \leq K_1(C_2)\epsilon \text{ for all admissible pairs } (r,p) \tag{7.36}
\]
\[
\|\xi^n\|_{L^2_t([0,T])} \leq C\sqrt{C_2}\epsilon \text{ for all multi indexes } \mu \text{ with } \omega \cdot \mu > m \tag{7.37}
\]

Thus, provided that \(C_2/2 > C\sqrt{C_2}\) and \(C_1/2 > K_1(C_2)\), we see that (7.5)–(7.6) imply the same estimates but with \(C_1, C_2\) replaced by \(C_1/2, C_2/2\). Then (7.5) and (7.6) hold with \([0,T]\) replaced by \([0,\infty)\). This yields Theorem 7.1.

A Proofs of Lemmas 6.5 and 6.6

A.1 Proof of Lemma 6.5

By a simple argument as in p.24 [SW1] which uses Theorem 6.2, it is enough to prove bounds
\[
\|\chi(B)e^{-iBt}R_B(\mu + i0)g\|_{L^2_t} \leq C(t)^{-\frac{n}{2}}\|g\|_{L^2_t}. \tag{A.1}
\]
with \(\chi \in C_0^\infty((m,\infty),\mathbb{R})\) with \(\chi \equiv 1\) in [\(a, b]\). We have
\[
\langle x \rangle^{-\gamma} \chi(B)e^{-iBt}R^+ (\mu) \langle y \rangle^{-\gamma} = e^{-i\omega t} \langle x \rangle^{-\gamma} \int_\mathbb{R} e^{-i(B-\mu-\omega)s} \chi(B)ds \langle y \rangle^{-\gamma}. \tag{A.2}
\]
Using the distorted plane waves \(u(x,\xi)\) associated to the continuous spectrum of \(-\Delta + V\), we can write
\[
\langle x \rangle^{-\gamma} \chi(B)e^{-iBt} \langle y \rangle^{-\gamma} = \langle x \rangle^{-\gamma} \int_{\mathbb{R}^3} u(x,\xi)e^{-i\sqrt{\xi^2 + m^2} + i\mu - \gamma s}\chi(\sqrt{\xi^2 + m^2})\tilde{u}(y,\xi)d\xi \langle y \rangle^{-\gamma}. \tag{A.3}
\]
We have \(u(x,\xi) = e^{ix\cdot\xi} + e^{ix\cdot\xi}w(x,\xi)\), with \(w(x,\xi)\) the unique solution in \(L^2, -s\), \(s > 1/2\), of the integral equation
\[
w(x,\xi) = -F(x,\xi) - \int_{\mathbb{R}^3} w(y,\xi)V(y)\frac{e^{i|\xi||y-x|}}{4\pi|y-x|}e^{i(y-x)\cdot\xi}dy, \tag{A.4}
\]
with
\[
F(x,\xi) = \int_{\mathbb{R}^3} V(y)\frac{e^{i|\xi||z-x|}}{4\pi|y-x|}e^{i(y-x)\cdot\xi}dy. \tag{A.5}
\]
It is elementary to show that for \(\xi\) in the support of \(\chi(\sqrt{\xi^2 + m^2})\) and for \(|\beta| \leq 3\) then \(|\partial_x^\alpha \partial_\xi^\beta F(x,\xi)| \leq \tilde{c}_{\alpha\beta}(x)|\beta|\). Using standard arguments from stationary scattering theory it is possible for \(|\beta| \leq 3\) to conclude correspondingly \(|\partial_x^\alpha \partial_\xi^\beta w(x,\xi)| \leq c_{\alpha\beta}(x)|\beta|\). This implies that, after integration by parts (i.e. using \(e^{-i\sqrt{\xi^2 + m^2}} = \frac{i\sqrt{\xi^2 + m^2}}{|\xi|} \frac{d|\xi|}{d|\xi|} e^{-i\sqrt{\xi^2 + m^2}}\) etc., see [SW1] p. 25)
\[
|\langle x \rangle| \leq c(x)^{-\gamma+r}(y)^{-\gamma+r}e^{-rt} \langle x \rangle^{-\gamma+r}t^{-r+1} \text{ and so } |\langle x \rangle| \leq c(x)^{-\gamma+r}(y)^{-\gamma+r}t^{-r+1}.
\]
For \(\gamma > r + 3/2\) and \(r = 3\), we obtain the conclusion.
Remark A.1. Notice that when $|V(y)| \leq Ce^{-a|y|}$ for $a > 0$, equations (A.4)–(A.5) make sense with $i|\xi|$ replaced by $\sqrt{-\xi_1^2 - \xi_2^2 - \xi_3^2}$ with $\xi$ in an open neighborhood $U$ of $\mathbb{R}^3 \setminus \{0\}$ in $\mathbb{C}^3 \setminus \{0\}$. Then we get solutions $w(x, \xi)$ bounded and analytic in $\xi$. Correspondingly we obtain $u(x, \xi)$ for $\xi \in U$ analytic in $U$ and with $|u(x, \xi)| \leq Ce^{|x|\sum_{j=1}^3 |\text{Im } \xi_j|}$. Consequently, if $|v(x)| \leq c_0 e^{-b|x|}$ for $b > 0$ and for the distorted plane wave transformation

$$\hat{v}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \overline{\varphi(y, \xi)} v(y) dy,$$

then $\hat{v}(\xi)$ extends into an holomorphic function in some open neighborhood of $\mathbb{R}^3 \setminus \{0\}$ in $\mathbb{C}^3 \setminus \{0\}$.

### A.2 Proof of Lemma 6.6

The proof originates from [M], but here we state the steps of a simplification in [CT]. We first state Lemmas A.2–A.3. They imply Lemma 6.6 by an argument in [M]. First of all we need some estimates on the resolvent, for the proof see Lemma 2.8 [DF]:

**Lemma A.2.** For any $s > 1$ there is a $C > 0$ such that for any $z$ with $\text{Im } z > 0$ we have

$$\|R_B(z)P_c\|_{B(L^2_{s^*},L^2_{-s^*})} \leq C.$$  \hspace{1cm} (A.7)

Estimates (A.7) yield a Kato smoothness [K1] result, see the proof of Lemma 3.3 [CT]:

**Lemma A.3.** Under the hypotheses of Lemma A.2 for any $s > 1$ there is a $C$ s.t. for all Schwartz functions $u_0(x)$ and $g(t, x)$ we have

$$\|e^{-iBt}P_c u_0\|_{L^2_{s^*}} \leq C\|u_0\|_{L^2_s}$$  \hspace{1cm} (A.8)

$$\|\int_0^t e^{itB}P_c g(t, \cdot) dt\|_{L^2_s} \leq C\|g\|_{L^2_{s^*}}.$$  \hspace{1cm} (A.9)

Now we are ready to prove Lemma 6.6. For $g(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ set

$$Tg(t) = \int_0^{+\infty} e^{-i(t-s)B} P_c g(s) ds.$$  \hspace{1cm} (A.9)

(A.9) implies $f := \int_0^{+\infty} e^{ixB} P_c g(s) ds \in L^2_x$. By Lemma 6.4 for all $(r, p)$ admissible we have

$$\|Tg(t)\|_{L^r W^{\frac{3}{2}, \frac{3}{2}, p}_{t, x}} \lesssim \|f\|_{L^r_x} \lesssim \|g\|_{L^r_x L^2_{s^*}}.$$  \hspace{1cm} (A.9)

The following well known result by Christ Kiselev, see Lemma 3.1 [SmS], yields Lemma 6.6.
Lemma A.4. Consider two Banach spaces and $X$ and $Y$ and $K(s, t)$ continuous function valued in the space $B(X, Y)$. Let

$$T_K f(t) = \int_{-\infty}^{\infty} K(t, s) f(s) ds \quad \text{and} \quad \tilde{T}_K f(t) = \int_{-\infty}^{t} K(t, s) f(s) ds.$$ 

Then we have: Let $1 \leq p \leq q \leq \infty$ and $I$ an interval. Assume that there exists $C > 0$ such that

$$\| T_K f \|_{L^q(I, Y)} \leq C \| f \|_{L^p(I, X)}.$$

Then

$$\| \tilde{T}_K f \|_{L^q(I, Y)} \leq C' \| f \|_{L^p(I, X)}$$

where $C' = C'(C, p, q) > 0$.

B Proof of Lemma 7.11.

First of all (7.6) immediately implies the a estimate

$$\| \eta^\nu \|_{L^2_t([0, T])} \leq C_2 \epsilon \quad \text{for all multi indexes } \mu \text{ with } \omega \cdot \mu > m \quad \text{(B.1)}$$

Then $\| \partial \xi R \|_{L^1_t} \lesssim \epsilon^2$, see Remark 7.6. We look at other terms.

We start by studying the product of $\eta_j$ with the remaining terms of the second line of (7.34).

Lemma B.1. Provided $\epsilon$ is small enough one has

$$\left\| \sum_j \eta_j \left( \delta_{jk} + \frac{\partial \Delta}{\partial \xi_k} (\xi) \right) \left[ -1 \left\langle \frac{\partial G}{\partial \xi_k}, g \right\rangle - i \left\langle \frac{\partial \tilde{G}}{\partial \xi_k}, \bar{g} \right\rangle + G_{1, k} \right] \right\|_{L^1_t} \leq C \epsilon^2 \quad \text{(B.2)}$$

Proof. We begin by the simplest term, i.e. the one of the form $\sum_j \eta_j \left\langle \frac{\partial G}{\partial \xi_j}, g \right\rangle$. $g$ is already estimated by (7.22), so it is enough to estimate (for each $j$)

$$\left\| \eta_j \frac{\partial G}{\partial \xi_j} \right\|_{L^2_t H^{4, s}} = \left\| \xi_j \frac{\partial G}{\partial \xi_j} \right\|_{L^2_t H^{4, s}} + O(\epsilon^3) \leq C \sum_{\omega \cdot (v - \mu) > m} \left\| \xi \xi^\nu \phi_{\mu} \right\|_{L^2_t H^{4, s}} \leq C \epsilon.$$

$|\xi_j G_{1, j}|$ is bounded by the absolute values of terms of the form either

$$\xi^{\mu + \mu'} \xi^{\nu + \nu'}, \mu \in M, \nu \in M \text{, } (\mu, \nu') \neq (0, 0), \quad \text{(B.4)}$$

or

$$\xi_j \xi^{\mu} \xi^{\nu}, \mu \in M, \nu \in M \quad \text{(B.5)}$$

In case (B.4)

$$\| \xi^{\mu + \mu'} \xi^{\nu + \nu'} \|_{L^1_t} \leq \| \xi^\nu \|_{L^2_t} \| \xi^{\mu'} \|_{L^2_t} \leq \| \xi \|_{L^\infty_t} \| \xi \|_{L^\infty_t} \| \xi \|_{L^\infty_t} \leq C \epsilon \quad \text{(B.6)}$$
Similarly, in case (B.5)

$$\|\xi_j \xi^\mu \xi^\nu\|_{L^1_t} \leq \|\xi^\nu\|_{L^2_t} \|\xi^\mu\|_{L^2_t} \|\xi_j\|_{L^\infty_t} \leq CC_2^2 \epsilon^3.$$  

We come to the terms involving $\Delta$. Notice that

$$\xi_j \frac{\partial \Delta_j}{\partial \xi_k} \sim \frac{\xi^\mu \xi^\nu}{\xi_k}$$  

with $\mu, \nu$ in $M$, $\mu_k \neq 0$.

Then we have

$$\left\| \frac{\xi^\mu \xi^\nu}{\xi_k} \right\|_{L^2_t} \leq \|\xi^\nu\|_{L^2_t} \left\| \frac{\xi^\mu}{\xi_k} \right\|_{L^\infty_t} \leq CC_2 \epsilon^3.$$  

(B.6)

Here we used the fact that $\xi^\mu = \xi^\beta$ for some $\beta \neq 0$, for otherwise $\xi^\mu = \xi_k$, which by $\mu \in M$ would imply $\omega_k > m$, which is not true. (B.6) can be easily combined with estimates on $G_{1,k}$ and $g$ to complete the proof of (B.2).

Lemma B.2. We have

$$\left\| \left[ -\frac{\partial Z_0}{\partial \xi_j}(\xi) + \mathcal{N}_j(\xi) + \frac{\partial Z_0}{\partial \xi_j}(\eta) - \mathcal{N}_j(\eta) \right] \eta_j \right\|_{L^1_t} \leq C \epsilon^3.$$  

(B.7)

Proof. For definiteness we focus on $\| (\partial_j Z_0(\xi) - \partial_j Z_0(\eta)) \eta_j \|_{L^1_t}$. It is enough to consider quantities $\eta^\alpha \eta^\beta - \xi^\alpha \xi^\beta$ with $\omega \cdot \alpha = \omega \cdot \beta$ and $\beta_j > 0$. By Taylor expansion these are

$$\sum_k \partial_k \left( \frac{\eta^\alpha \eta^\beta}{\eta_j} \right) (\xi_k - \eta_k) \eta_j + \sum_k \tilde{\partial}_k \left( \frac{\eta^\alpha \eta^\beta}{\eta_j} \right) (\xi_k - \bar{\eta}_k) \bar{\eta}_j + \tilde{\eta}_j O(||\xi - \eta||^2).$$

The reminder term is the easiest, the other two terms similar. Substituting (5.24), a typical term in the first summation is $\frac{\eta^\alpha \eta^\beta \xi^A \xi^B}{||\xi||^2}$, with all four $\alpha, \beta, A$ and $B$ in $M$ and with $\alpha_k \neq 0 \neq B_k$. (H5) and $\omega \cdot \alpha = \omega \cdot \beta$ imply that there is at least one index $\beta_l \neq 0$ such that $\omega_l = \omega_k$. Then

$$\left\| \frac{\eta^\alpha \eta^\beta \xi^A \xi^B}{||\xi||^2} \right\|_{L^1_t} \leq \left\| ||A||_{L^2_t} \left\| \frac{\xi^B \xi^A}{\xi_k} \right\|_{L^2_t} \|\xi^\alpha \xi^\beta\|_{L^\infty_t} \leq C_2^2 \epsilon^{|\alpha| + |\beta|} \leq C_2^2 \epsilon^4$$  

(B.8)

by the fact that monomials $\xi^\alpha \xi^\beta$ in $Z_0$ are such that $|\alpha| = |\beta| \geq 2$. Other terms can be bounded similarly.

Finally, the proof of Lemma 7.11 is completed with the following:

Lemma B.3. The following estimate holds

$$\sum_j \left\| \xi_j \frac{\partial \Delta_j}{\partial \xi_l} \frac{\partial Z_0}{\partial \xi_l} \right\|_{L^1_t} \leq CC_2^2 \epsilon^3.$$  

(B.9)
Proof. The quantity under consideration is the sum of terms of the form

\[ \frac{\xi^\mu \xi^\nu \xi^\alpha \xi^\beta}{\xi^I} \]  \tag{B.10}

where the indexes are such that

\[ \mu, \nu \in M, \omega \cdot (\alpha - \beta) = 0, \mu_I \neq 0 \neq \beta_I. \]

By (H5) there is \( \alpha_k \neq 0 \) such that \( \omega_k = \omega_I \). Then

\[ \left\| \frac{\xi^\mu \xi^\nu \xi^\alpha \xi^\beta}{\xi^I} \right\|_{L^1} \leq \left\| \xi^\omega \right\|_{L^2} \left\| \frac{\xi^\mu \xi^\nu}{\xi^I} \right\|_{L^2} \left\| \frac{\xi^\alpha \xi^\beta}{\xi^k \xi^l} \right\|_{L^\infty} \lesssim C^2 |\alpha| + |\beta| \leq C^2 \epsilon^4 \]

by the fact that monomials \( \xi^\alpha \xi^\beta \) in \( Z_0 \) are such that \( |\alpha| = |\beta| \geq 2. \) \hfill \( \square \)

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Dipartimento di Matematica “Federico Enriques”, Università degli Studi di Milano, Via Saldini 50, 20133 Milano, Italy.

*E-mail Address:* dario.bambusi@unimi.it

DIMSI University of Modena and Reggio Emilia, Via Amendola 2, Padiglione Morselli, Reggio Emilia 42100, Italy.

*E-mail Address:* ciccagna.scipio@unimore.it