Peculiarities of electron energy spectrum in Coulomb field of super heavy nucleus

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Just after the Dirac equation was established, a number of physicists tried to comment on and solve the spectral problem for the Dirac Hamiltonian with the Coulomb field of arbitrarily large charge \( Z \), especially with \( Z \) that is more than the critical value \( Z_c = \alpha^{-1} \approx 137.04 \), making sometimes contradictory conclusions and presenting doubtful solutions. It seems that there is no consensus on this problem up until now and especially on the way of using corresponding solutions of the Dirac equation in calculating physical processes. That is why in the present article, we turn once again to discussing peculiarities of electron energy spectrum in the Coulomb field of superheavy nucleus. In the beginning, we remind the reader of a long story with a wrong interpretation of the problem in the case of a point nucleus and its present correct solution. We then turn to the spectral problem in the case of a regularized Coulomb field. Under a specific regularization, we derive an exact spectrum equation determining the point spectrum in the energy interval \((-m, m)\) and present some of its numerical solutions. We also derive an exact equation for charges \( Z \) providing bound states with energy \( E = -m \). Its analytical and numerical analysis shows that there exists an infinite number of such charges; in this connection, we discuss the notion of supercritical charge.

To our mind, their existence does not mean that the one-particle relativistic quantum mechanics based on the Dirac Hamiltonian with the Coulomb field of such charges is mathematically inconsistent. In any case, it is physically unacceptable because the spectrum of the Hamiltonian is unbounded from below, which requires the secondary Fermi–Dirac quantization and transition to many-particle quantum field theory. The consequences of the existence of such charges for quantum electrodynamics with the corresponding Coulomb field remain to be established in the process of constructing such a theory.

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DG and BV devote this paper to their friend and permanent coauthor Igor Tyutin

1. INTRODUCTION

Relativistic quantum effects, in particular, electron-positron pair creation, in superstrong Coulomb field attract attention of physicists already for a long time. However, their qualitative and especially quantitative description is lacking up until now. We believe that such a description can be done only in the framework of a nonperturbative quantum electrodynamics (QED) with superstrong Coulomb field as an external background. Unfortunately, such a version of QED does not exist at present. Our experience in quantum field theory (QFT) with different backgrounds, see Refs. [1, 2], allows us to expect that constructing this version of QED needs at least a complete and clear mathematical solution of the spectral problem for the Dirac Hamiltonian with the Coulomb field of arbitrarily large charge \( Ze \) \((e > 0\) is the absolute value of the electron charge) of a nucleus (in what follows, we call \( Z \) simply the charge of the nucleus). We also realize that a solution of the latter problem marks only the beginning of constructing QED with superstrong Coulomb field. It should be noted that just after the Dirac equation was established, a number of physicists tried to comment on and solve this spectral problem making sometimes contradictory conclusions. It seems that there is no consensus on this problem even at present. That is why we turn once again to a discussion of peculiarities of the energy spectrum of an electron in the Coulomb field of a superheavy nucleus. The paper is organized as follows. In Sec² we recall a long story with controversial interpretations of the problem in the case of a point nucleus and its present correct solution. Then in Sec. ³ we turn to the spectral problem in the case of a
regularized Coulomb field with a specific cutoff, which allows an exact solution. We analyze the problem in the part concerning the point (discrete) spectrum located in the segment \([-m, m]\) and the corresponding bound states. In contrast to the earlier works, we do not use the approximation of small cutoff radius. In Sec. 4 we derive an exact spectrum equation determining the point spectrum in the energy interval \((-m, m)\) and present some of its numerical solutions related to different \(Z\). In Sec. 5 we derive exact equations for the charges providing the bound states with energies \(E = -m\) and show that there exists an infinite number of such charges, generally not integer-valued; the first of these charges are calculated numerically. In this connection, we discuss a controversial notion of supercritical charge.

2. SPECTRAL PROBLEM WITH COULOMB FIELD OF POINT NUCLEUS

The spectral problem for the Dirac Hamiltonian with the Coulomb field of a point nucleus has a long story. The electronic structure of an atom with \(Z \leq Z_c = \alpha^{-1} \simeq 137.04\), where \(\alpha\) is the fine structure constant, and \(Z_c\) is the critical charge, was described by the Dirac equation, which gives relativistic electron energy spectrum (the Sommerfeld spectrum) in agreement with experiment. It was commonly believed that the Dirac equation with nucleus charges \(Z > Z_c\) meets insuperable difficulties. However a short time ago, it was demonstrated that the common belief that the Dirac Hamiltonian with the Coulomb field of a point nucleus is consistent only at \(Z < Z_c\) is erroneous, see [8,10]. The known difficulties with its spectrum for \(Z > Z_c\) do not arise if the Dirac Hamiltonian \(\hat{H}(Z)\) is correctly defined as a self-adjoint (s.a.) operator. An important remark concerning admissible values of charge \(Z\) is appropriate here. Only integer-valued \(Z\), \(Z \in \mathbb{N}\), have a physical meaning, but from the standpoint of the spectral analysis of the Dirac Hamiltonian, it is useful, and is commonly adopted, to consider \(Z\) as a parameter taking arbitrary nonnegative values, \(Z \in \mathbb{R}_+\).

It was demonstrated that from a mathematical standpoint, a definition of the Dirac Hamiltonian as a s.a. operator presents no problem for arbitrary \(Z\). The Dirac Hamiltonian \(\hat{H}(Z)\) with any \(Z\) can be correctly defined as a s.a. operator in the Hilbert space of bispinors.

For \(Z < Z_s = (\sqrt{3}/2) \alpha^{-1} \simeq 118.7\), where \(Z_s\) is the lower critical charge, the Dirac Hamiltonian \(\hat{H}(Z)\) is defined uniquely. For \(Z \geq Z_s\), there exists a family \(\{\hat{H}^{(\nu)}(Z)\}\) of possible s.a. Dirac Hamiltonians \(\hat{H}^{(\nu)}(Z)\) specified by additional boundary conditions at the origin, \(\nu\) is generally a certain \(Z\) dependent set of parameters. The spectrum and inversion formulas were found for any \(\hat{H}^{(\nu)}(Z)\). The eigenfunctions of the discrete spectrum and generalized eigenfunctions of the continuous spectrum form a complete orthonormalized system in the Hilbert space of bispinors. The continuous spectrum is the union of the two semiaxis \(E \leq -m\) and \(E \geq m\), while the discrete spectrum \(\{E_n^{(\nu)}(Z)\}\) is located in the interval \(|E| \leq m\). The position of discrete energy levels \(E_n^{(\nu)}(Z)\) essentially depends on \(\nu\), in particular, for any \(Z \geq Z_s\), there exist parameters \(\nu = \nu_m\), for which the lowest energy level coincides with the upper boundary \(-m\) of the negative branch \((-\infty, -m]\) of the continuous spectrum, \(E_0^{(-m)}(Z) = -m\). For \(Z < Z_s\), the Sommerfeld spectrum is generated by the Dirac Hamiltonian \(\hat{H}(Z)\), while for \(Z_c > Z \geq Z_s\), it is generated by the Dirac Hamiltonian \(\hat{H}^{(0)}(Z)\), see [9].

There is a good reason to believe that these s.a. Dirac Hamiltonians provide an initial mathematical tool for constructing QED with external strong Coulomb field of a point charge. The question is how to use this tool and does such QED exist in principle.

Usually when constructing QFT with an external background, we decompose the Heisenberg operator of the Dirac field into an adequately chosen complete set of solutions of the Dirac equation. Our previous experience tells us that to have a secondarily quantized formulation in terms of relatively stable quasiparticles, the gap between the lowest discrete energy level and the upper boundary \(-m\) of the negative branch of continuous spectrum has to be big enough. In other words, the discrete energy spectrum has to be isolated enough from the negative branch \((-\infty, -m]\) of the continuous spectrum. Otherwise, a desirable secondarily quantized theory cannot be constructed in full analogy with already known cases [1,2]. At least, it is very likely that such a construction is impossible for s.a. Dirac Hamiltonians \(\hat{H}^{(\nu)}(Z)\) with parameters \(\nu = \nu_m\).

3. SPECTRAL PROBLEM WITH REGULARIZED COULOMB FIELD

3.1. General

Before the works [8,10], the difficulties with the energy spectrum of an electron in the Coulomb field of a point nucleus, and with the spectral problem in general, were explained by a strong singularity at the origin of the Coulomb field of a nucleus with \(Z > Z_c\), see [1,2] and many other articles and books. It was believed that these difficulties
can be eliminated if a nucleus of some finite radius \( r_0 \) is considered. Some calculations were done in support of the conjecture that with cutting off the Coulomb potential at a finite small radius \( r_0 \), the Dirac Hamiltonian has a physically meaningful spectrum for charges \( Z \) not exceeding the so-called supercritical charge \( Z_{sc} \). Its value depends on the cutoff model and approximations made for its evaluation. Mention can be made of the following values of the supercritical charges: \( Z_{sc} = 200 \) (13), \( Z_{sc} = 170 \) (13, 16), \( Z_{sc} = 172 \) (11), and some other values from the interval \((170 - 177)\).

According to the above-listed authors even in the presence of a cutoff, the lowest discrete level passes into the lower continuum for \( Z \geq Z_{sc} \). And again the applicability of the Dirac equation now for nonpoint nuclei with charges \( Z \geq Z_{sc} \) was called into question. It was supposed that the new difficulties are connected with a many-particle character of the problem under consideration, in particular, with a possible \( e^+e^- \) pair creation by a nucleus with the charge \( Z \geq Z_{sc} \). Since that time almost all researchers in this area repeated this point of view in their publications. However, recently, there appeared a publication [17] where this conclusion was recognized to be wrong. For us, after a rehabilitation of the electron spectrum in the Coulomb field of a point nucleus, it would be very strange to accept the fact that a removal of the singularity of the Coulomb potential at the origin (after a cutoff) makes a situation with the spectrum not better, but worse. In view of a great importance of all the details of the spectral problem for the point spectrum located in the interval \([-m, m]\) and the corresponding bound states, with minimum references to the previous works on the subject. We plan to present a detailed comparison of our approach and results with those of numerous previous papers in a subsequent publication.

### 3.2. Radial equations

Recall that a behavior of an electron in a regularized Coulomb field of charge \( Z \) is governed by the Dirac Hamiltonian

\[
\hat{H}(Z) = \gamma^0 (\gamma \hat{p} + m) + V(r), \quad \hat{p} = -i\nabla, \quad r = |r|,
\]

where \( V(r) \) is the potential energy of the electron in the regularized Coulomb field (it is supposed to be spherically symmetric, bounded, and real valued) and \( \gamma^\mu \) are the Dirac gamma matrices.

The Hamiltonian \( \hat{H}(Z) \) with any \( Z \) is a uniquely defined s.a. operator in \( \mathcal{H} \) because it is a sum of the uniquely defined s.a. free Dirac Hamiltonian and the bounded s.a. operator of multiplication by the bounded real-valued function \( V(r) \): an addition of a bounded s.a. operator to any s.a. operator yields a new s.a. operator with the same domain. In contrast to this, in the case of the nonregularized Coulomb field of a point nucleus, where a potential is an unbounded operator, a s.a. Dirac Hamiltonian is defined nonuniquely for \( Z > Z_s = (\sqrt{3}/2)a^{-1} \approx 118,7 \), and the nonuniqueness is growing with increasing \( Z \), see [8, 10].

The stationary Schrödinger equation \( \hat{H}(Z) \Psi(r) = E \Psi(r) \) defines the point energy spectrum of the electron, which is the main subject of our interest.

Choosing solutions of the stationary Schrödinger equation in the well-known form

\[
\Psi_{j,M,\zeta}(r) = \frac{1}{r} \left( \frac{\Omega_{j,M,\zeta}(\theta, \varphi)f(r)}{i \Omega_{j,M,-\zeta}(\theta, \varphi)g(r)} \right),
\]

where \( \Omega_{j,M,\zeta} \) are the normalized spherical spinors, so that bispinors \( \Psi_{j,M,\zeta} \) are common eigenvectors of three commuting s.a. operators \( \hat{J}^2, \hat{J}_z, \) and \( \hat{K} \), where \( \hat{J} \) is the total angular momentum and \( \hat{K} \) is the so-called spin operator,

\[
\hat{J}^2 \Psi = j(j + 1) \Psi, \quad \hat{J}_z \Psi = M \Psi, \quad \hat{K} \Psi = \pm \zeta \Psi, \quad \zeta = \zeta(j + 1/2),
\]

\[
\hat{J} = \hat{L} + \Sigma/2, \quad \hat{L} = [r \times \hat{p}], \quad \hat{K} = \gamma^0 \left[ 1 + (\Sigma \hat{L}) \right],
\]

and \( j = 1/2, 3/2, ... \), \( M = -j, -j + 1, ..., j \), \( \zeta = \pm 1 \), we reduce the above equation to the radial Schrödinger equations

\[
\hat{h}(Z,j,\zeta) F(r) = E(Z,j,\zeta) F(r), \quad F \in L^2(\mathbb{R}^+) = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+),
\]
where $\hat{h}(Z,j,\zeta)$ are s.a. partial radial Hamiltonians acting in the Hilbert space $L^2(\mathbb{R}^+)$ of doublets $F(r)$,

$$F(r) = \begin{pmatrix} f(r) \\ g(r) \end{pmatrix}$$  \hspace{1cm} (5)

by the radial differential operations

$$\hat{h}(Z,j,\zeta) = -i\sigma_2 \frac{d}{dr} + \zeta r^{-1} \sigma_1 + V(r) + m\sigma_3,$$  \hspace{1cm} (6)

where $\sigma_k$, $k = 1, 2, 3$, are the Pauli matrices, see \[5, 6\] and \[8, 9\]. The domain $D$ of each of the operators $\hat{h}$ consists of doublets $F(r)$ that are absolutely continuous on $(0, \infty)$, are vanishing at zero, $f(0) = g(0) = 0$, and are square integrable together with $\hat{h}F(r)$ on $(0, \infty)$ (actually, at infinity). This is the so-called natural domain for $\hat{h}$, see \[9\].

Because the potential $V(r)$ vanishes at infinity, the spectrum of each of $\hat{h}$ consists of a continuous part that is the union ($-\infty, -m] \cup [m, \infty)$ of two semiaxis, negative and positive, and a point spectrum $\{E_n(Z,j,\zeta), n \in \mathbb{Z}^+\}$ located in the segment $[-m, m]$. The total point spectrum of the Dirac Hamiltonian $\hat{H}(Z)$ is the union of partial point spectra of the radial Hamiltonians $\hat{h}(Z,j,\zeta)$.

The radial Schrödinger equation (4) with fixed $Z,j,\zeta$ implies the system of equations for the radial functions $f$ and $g$:

$$\frac{df(r)}{dr} + \frac{\zeta}{r} f(r) - k_+(r) g(r) = 0 \Rightarrow g(r) = \frac{1}{k_+(r)} \left[ \frac{df(r)}{dr} + \frac{\zeta}{r} f(r) \right],$$

$$\frac{dg(r)}{dr} - \frac{\zeta}{r} g(r) + k_-(r) f(r) = 0, \hspace{1cm} k_\pm(r) = E - V(r) \pm m.$$  \hspace{1cm} (7)

In what follows, we consider the regularized Coulomb potential of the form

$$V(r) = -q \begin{cases} r_0^{-1}, & r \leq r_0 \\ r^{-1}, & r \geq r_0 \end{cases}, \hspace{1cm} q = Z\alpha.$$  \hspace{1cm} (8)

It corresponds to the field of the positive charge $Ze$ distributed uniformly on a nucleus spherical surface of radius $r_0$, see FIG.1.

The cutoff radius $r_0$ is usually considered a universal $Z$ independent constant which defines a model. But in accordance with real nuclear physics, it is natural to consider $r_0$ as a $Z$ dependent parameter, $r_0 = r_0(Z)$. Under the approximation that the number of protons and neutrons in a nucleus are equal, this $Z$ dependence is given by

$$r_0 = r_0(Z) = R_0(2.5Z)^{1/3}, \hspace{1cm} R_0 = 1.25 \times 10^{-15}m = 0.635 \times 10^{-8}eV^{-1},$$  \hspace{1cm} (9)

see \[12\]. It should be noted that this approximation becomes more and more rough with increasing $Z$.

In formulas to follow, we write simply $r_0$ for brevity, which allows applying the formulas to any $r_0$, but in numerical calculations, we use \[9\].

In finding point spectra $\{E_n(Z,j,\zeta)\}$, we have to consider the open energy interval $-m < E < m$ and its end points $E = m$ and $E = -m$ separately by technical reasons explained below in the beginning of Sec. \[8\].
4. OPEN ENERGY INTERVAL \((-m, m)\), SPECTRUM EQUATION

4.1. Solving radial equations in region \(0 \leq r \leq r_0\)

In the internal region \(0 \leq r \leq r_0\), where we set \(f(r) = f_{in}(r)\) and \(g(r) = g_{in}(r)\), the functions \(k_{\pm}(r)\) in (7) become constants. The substitution of the representation for the function \(g_{in}(r)\) from the first row in (7) into the second equation in (7) then results in the following second-order differential equation for the function \(f_{in}(r)\):

\[
\frac{d^2 f_{in}(r)}{dr^2} + \left( \eta^2 - \frac{\nu^2 - 1/4}{r^2} \right) f_{in}(r) = 0,
\]

\[
\eta = \sqrt{k_+ k_-}, \quad k_{\pm} = E \pm m + \frac{q}{r_0}, \quad \nu = j + \zeta + \frac{1}{2} = \begin{cases} j, & \text{if } \zeta = -1 \\ j + 1, & \text{if } \zeta = 1 \end{cases}.
\]  

(10)

The equation is complemented by the condition \(E \in (-m, m)\) and the boundary condition \(f_{in}(0) = 0\). We note that \(\nu\) takes positive half-integer values as well as \(j\) does. The substitution

\[
f_{in}(r) = \sqrt{r}w(z), \quad z = \eta r
\]  

(11)

reduces eq. (10) to the well-known Bessel equation, see [20],

\[
\frac{d^2 w(z)}{dz^2} + \frac{1}{z} \frac{dw(z)}{dz} + \left( 1 - \frac{\nu^2}{z^2} \right) w(z) = 0,
\]

(12)

complemented by the boundary condition \(\sqrt{r}w(z) \to 0\) as \(z \to 0\). The general solution of Eq. (12) with \(\nu \geq 1/2\) under this boundary condition is \(w(z) = cJ_{\nu}(z)\), where \(J_{\nu}(z)\) is the well-known Bessel function, see [20]. Using the representation for the function \(g_{in}(r)\) in the first row in (7) and the relation \(J'_{\nu}(z) + (\zeta \nu)/z J_{\nu}(z) = \zeta J_{\nu-\zeta}(z)\), see [20], we obtain that the general solution of system (7), (8) in the region \(0 \leq r \leq r_0\) and under the above-mentioned conditions is given by

\[
f_{in}(r) = c \sqrt{r}J_{\nu}(\eta r), \quad g_{in}(r) = c \sqrt{r}J_{\nu-\zeta}(\eta r).
\]  

(13)

The formulas (13) give two forms of representation for the functions \(f_{in}(r)\) and \(g_{in}(r)\): the condensed form in terms of \(\nu, \zeta\) and the expanded form in terms of \(j, \zeta = -1\) and \(j, \zeta = 1\).

4.2. Solving radial equations in region \(r_0 \leq r < \infty\)

In the external region \(r_0 \leq r < \infty\), where we set \(f(r) = f_{out}(r)\), \(g(r) = g_{out}(r)\), system (7), (8) with \(E \in (-m, m)\) coincides with the system of the point Coulomb problem. Solutions of such a system are well-known, see, for example [3]. The general square-integrable at infinity solution of this system is given by

\[
f_{out}(r) = B \sqrt{m - E} \left(2\beta r\right)^m e^{-\beta r} \left[b_+ \Phi(a + 1, c; 2\beta r) + \Psi(a, c; 2\beta r)\right],
\]

\[
g_{out}(r) = B \sqrt{m + E} \left(2\beta r\right)^m e^{-\beta r} \left[b_- \Phi(a + 1, c; 2\beta r) - \Psi(a, c; 2\beta r)\right],
\]

(14)

where

\[
\beta = \sqrt{m^2 - E^2}, \quad \mu = \sqrt{\mu^2 - q^2}, \quad a = \mu - \frac{qE}{\beta}, \quad c = 1 + 2\mu, \quad b_- = \zeta + \frac{qE}{\beta},
\]  

(15)

and \(\Psi\) is a symbol of one of the standard confluent hypergeometric functions which vanishes at infinity (it is sometimes called the Tricomi function).

We recall, that there are two standard confluent hypergeometric functions \(\Phi(a; c; x)\) and \(\Psi(a, c; x)\), the linearly independent solutions of the confluent hypergeometric equation, see [19].
\[ \Phi(a, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad c \notin \mathbb{Z}_-, \]
\[ \Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} \Phi(a-c+1, 2-c; x). \quad (16) \]

Rewritten in terms of the Whittaker functions \( W \),
\[ W_{\lambda, \mu}(x) = e^{-x/2} x^{\mu/2} \Psi(a, c; x), \quad \lambda = \frac{c}{2} - a, \quad \mu = \frac{c}{2} - \frac{1}{2}, \]
see [19], solution [14], [15] takes the form
\[ f_{\text{out}}(r) = \frac{B m}{\sqrt{m-E}} (2\beta r)^{-1/2} [b_- W_{\lambda, \mu}(2\beta r) + W_{\lambda, \mu}(2\beta r)], \]
\[ g_{\text{out}}(r) = \frac{B m}{\sqrt{m+E}} (2\beta r)^{-1/2} [b_- W_{\lambda, \mu}(2\beta r) - W_{\lambda, \mu}(2\beta r)], \quad \lambda' = \frac{q E}{\beta} - \frac{1}{2}, \quad \lambda = \lambda' + 1. \quad (17) \]

Introducing a new energy variable \( \varepsilon \) by \( E = m \cos \varepsilon \), \( \varepsilon = \arccos \frac{k}{m} \in (0, \pi) \), we rewrite Eqs. (17) as
\[ f_{\text{out}}(r) = B \sec \left( \frac{\varepsilon}{2} \right) (2\beta r)^{-1/2} [(q \csc \varepsilon + \kappa) W_{\lambda, \mu}(2\beta r) + W_{\lambda, \mu}(2\beta r)], \]
\[ g_{\text{out}}(r) = B \sec \left( \frac{\varepsilon}{2} \right) (2\beta r)^{-1/2} [(q \csc \varepsilon + \kappa) W_{\lambda, \mu}(2\beta r) - W_{\lambda, \mu}(2\beta r)], \quad (18) \]
with
\[ \beta = m \sin \varepsilon, \quad \lambda = q \cot \varepsilon + \frac{1}{2}, \quad \lambda' = q \cot \varepsilon - \frac{1}{2}, \quad (19) \]
which we take as the final form for the solution of system [17], [8] in the region \( r \geq r_0 \).

4.3. Continuity conditions and spectrum equation

After the general solution of system [17], [8] is found independently in the respective regions \( 0 \leq r \leq r_0 \) and \( r_0 \leq r < \infty \), it remains to satisfy the basic continuity condition for the solution as a whole (to sew the partial solutions together smoothly), which reduces to the requirement of continuity of the solution at the point \( r = r_0 \):
\[ f_{\text{in}}(r_0) = f_{\text{out}}(r_0), \quad g_{\text{in}}(r_0) = g_{\text{out}}(r_0). \quad (20) \]

The compatibility of these conditions with \( c \neq 0 \) and \( B \neq 0 \) yields the transcendental equation, which determines the discrete energy spectrum in the interval \( -m < E < m \) in terms of the variable \( \varepsilon \), \( E = m \cos \varepsilon \), \( k_\pm = m(\cos \varepsilon \pm 1) + \frac{q}{r_0} \),
\[ J_\nu(\eta r_0) \sec \left( \frac{\varepsilon}{2} \right) \left[ (\varepsilon + q \csc \varepsilon) W_{\lambda, \mu}(2\beta r_0) - W_{\lambda, \mu}(2\beta r_0) \right] \]
\[ - \sqrt{\frac{k_-}{k_+}} \zeta J_{\nu-\zeta}(\eta r_0) \csc \left( \frac{\varepsilon}{2} \right) \left[ (q \csc \varepsilon + \kappa) W_{\lambda, \mu}(2\beta r_0) + W_{\lambda, \mu}(2\beta r_0) \right] = 0. \quad (21) \]

We call this basic equation the spectrum equation for the interval \((-m, m)\). Strictly speaking, we deal with a series of exact spectrum equations for given \( Z, \nu \) and \( \zeta \).

It is evident from [21] that a cutoff removes the degeneracy of the discrete spectrum in \( \zeta \), which is characteristic for a point charge. After the spectrum equation is solved, the corresponding bound states are obtained by substituting the evaluated values of bound state energies \( E_\mu(Z, \nu, \zeta) \) for \( E \) in the respective [13] and [14], [15] with due regard to continuity condition (20), according to which only normalization factors of the wave eigenfunctions (dublets) remain undetermined. An analytical solution of the spectrum equation [21] with any \( Z, \nu, \zeta \) is beyond the scope of our possibilities.

It seems that only numerical solution of these equations is realizable at present.
An equivalent expanded form of the spectrum equation \((21)\), maybe more suitable for numerical calculations, is

\[
\sqrt{k_- J_{j+1}(\eta r_0)} - \tan \left(\frac{\varepsilon}{2}\right) \frac{q \csc \varepsilon - j - \frac{1}{2}}{q \csc \varepsilon - j + \frac{1}{2}} W_{\lambda,\mu}(2\beta r_0) = 0, \quad \zeta = 1,
\]

\[
\sqrt{k_+ J_j(\eta r_0)} + \tan \left(\frac{\varepsilon}{2}\right) \frac{q \csc \varepsilon - j + \frac{1}{2}}{q \csc \varepsilon - j - \frac{1}{2}} W_{\lambda,\mu}(2\beta r_0) + W_{\lambda,\mu}(2\beta r_0) = 0, \quad \zeta = -1,
\]

What concerns a qualitative analysis of the spectrum equation, we can say no more than the following. It can be shown that the l.h.s. of the spectrum equation \((21)\) infinitely oscillates around zero as \(E \to m\), i.e. \(\beta \to 0\) (a proof of the statement appears to be rather nontrivial). This implies that there exists an infinite set \(\{E_n(Z,j,\zeta)\}\) of roots of the spectrum equation, bound-state energies, with any fixed \(Z,j,\zeta\), which are accumulated at the point \(E = m, E_n(Z,j,\zeta) \to m\) as \(n \to \infty\). If \(qmr_0 \ll 1\) and \(\mu > 0\), the asymptotic behavior of the binding energy \(\varepsilon_n = m - E_n\) as \(n \to \infty\) is given by

\[
\varepsilon_n(Z,j,\zeta) = \frac{q^2}{2(n + \mu + \Delta(Z,j,\zeta))^2}, \quad \Delta(Z,j,\zeta; r_0) = (qmr_0)^{2\mu} c(Z,j,\zeta), \quad n \to \infty,
\]

or roughly speaking, \(\varepsilon_n = (q^2 n^{-2}/2) \left[1 + O\left(n^{-1}\right)\right]\), \(n \to \infty\), which reproduces the well-known result for a nonrelativistic electron in the Coulomb field of a point charge. In particular, the Zommerfeld spectrum is restored in the limit \(r_0 \to 0\).

In this paper, we restrict ourselves to the special case of \(j = 1/2\) and \(\zeta = -1\), which produces the lowest energy levels. In this case, we have

\[
J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad \mu = \sqrt{1 - q^2}, \quad J_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left(-\cos z + \frac{\sin z}{z}\right),
\]

and the spectrum equation in form \((22)\) becomes

\[
\sqrt{k_- J_{j+1}(\eta r_0)} - \frac{\cot \eta r_0 - \frac{1}{\eta r_0}}{\tan \left(\frac{\varepsilon}{2}\right)} \frac{q \csc \varepsilon - 1}{q \csc \varepsilon - 1} W_{\lambda,\mu}(2\beta r_0) - W_{\lambda,\mu}(2\beta r_0) = 0.
\]

We solve this spectrum equation numerically for a series of \(Z\) assuming that the cutoff radius \(r_0\) is \(Z\) dependent according to Eq. \((9)\). Results of the numerical calculations are presented in FIG. 2.
5. ON BOUND STATES WITH $E = \pm m$, SUPERCRITICAL CHARGE

The preceding consideration is not directly applicable to the points $E = m$ and $E = -m$. The reason is that formulas (14), (15) break down at these points because of vanishing the variable $\beta = \sqrt{m^2 - E^2}$ and the respective blowing up of the factors $\frac{1}{\sqrt{m - E}}$ and $\frac{1}{\sqrt{m + E}}$, $b_- = \kappa + \frac{qE}{\beta}$ and the parameter $a = \mu - \frac{qE}{\beta}$. Each of these points requires a separate consideration.

5.1. Point $E = m$

Although it seems evident that there is no bound state with energy $E = m$, for completeness, we consider this point and show that what seems evident really holds (actually, an absence of bound states with zero binding energy for an electron in the attractive Coulomb field is by no means a trivial fact, see a discussion in the end of the subsection). For this purpose, it is sufficient to consider system of radial equations (7) and (8) for bound states with $E = m$ in the external region $r \geq r_0$ where the system becomes

\[
\frac{df(r)}{dr} + \frac{\kappa}{r} f(r) - \left[2m + \frac{q}{r}\right] g(r) = 0,
\]
\[
\frac{dg(r)}{dr} - \frac{\kappa}{r} g(r) + \frac{q}{r} f(r) = 0 \implies f(r) = \frac{1}{q} \left[ -r \frac{dg(r)}{dr} + \kappa g(r) \right].\tag{26}
\]

It is complemented by the conditions that the both functions $f(r)$ and $g(r)$ are absolutely continuous and square integrable together with their derivatives on $(r_0, \infty)$.

Substituting the representation for the function $f(r)$ from the second row in (26) into the first equation in (26), we obtain that the function $g(r)$ satisfies the second-order differential equation

\[
r \frac{d^2 g(r)}{dr^2} + \frac{dg(r)}{dr} + 2qmg(r) - \frac{\kappa^2}{r} g(r) = 0.\tag{27}
\]

The substitution $g(r) = w(z)$, $z = 2\sqrt{2qm\kappa}$, reduces Eq. (27) to Bessel equation (12) with $\nu = \tilde{\nu} = 2\mu$, $2\sqrt{2qm}\kappa r_0 \leq z < \infty$. The general solution of this equation is given by

\[
w(z) = c_1 H_{\nu}^{(1)}(z) + c_2 H_{\nu}^{(2)}(z),\tag{28}
\]

where $H_{\nu}^{(1)}(z)$ and $H_{\nu}^{(2)}(z)$ are the respective first and second Hankel functions, see [19]. Its asymptotic behavior at infinity is given by

\[
w(z) = c_1 \sqrt{\frac{2}{\pi z}} \exp \left[ \frac{i}{4} (4z - 2\pi \tilde{\nu} - \pi) \right] \left[ 1 + O \left( \frac{1}{z} \right) \right] + c_2 \sqrt{\frac{2}{\pi z}} \exp \left[ -\frac{i}{4} (4z - 2\pi \tilde{\nu} - \pi) \right] \left[ 1 + O \left( \frac{1}{z} \right) \right], z \to \infty,\tag{29}
\]
see \[10\]. It follows that the asymptotic behavior of the both functions \(f(r)\) and \(g(r)\) at infinity is estimated as \(f(r), g(r) = O(r^{-1/4}), \ r \to \infty\), so that the both functions are not square integrable at infinity. This means that system \([20]\) has no square-integrable solutions, and therefore, there are no bound states with energy \(E = m\), i.e., with zero binding energy, for an electron in the Coulomb field of any charge \(Z\) with cutoff \([8]\), as well as in the Coulomb field of a point charge.

The nature of this phenomenon is a long-range character of the Coulomb potential, which generates an infinite set of bound states with energy levels accumulated at the point \(E = m\), but not reaching this point. This picture is stable under changing the charge: all these levels go down with increasing \(Z\), but no bound state with zero binding energy appears. A completely different type of situation occurs in the case of short-range attractive potentials which can generate bound states with zero binding energy. For example, an electron in an attractive electric square-well potential can have such states under certain relations between the radius \(\ell_0\) of the well and its depth \(V_0\). For \(\zeta = 1\) and for any \(j\), these relations look rather simple being given by \(\sqrt{V_0(2m + V_0)}\ell_0 = \zeta_n(j)\), \(n \in \mathbb{N}\), where \(\zeta_n(j)\) are zeroes of the Bessel functions, \(J_j(\zeta_n(j)) = 0\). As is well known, these \(\zeta_n(j)\) form an infinite sequence going to infinity almost periodically with increasing \(n\), \(\zeta_n(j) \to \infty\) as \(n \to \infty\), \(\zeta_{n+1}(j) - \zeta_n(j) \to \pi\); in particular, \(\zeta_n(1/2) = n\pi\). Accordingly, at fixed radius \(\ell_0\), the bound states with given angular momentum \(j\) and zero binding energy appears sequentially and almost periodically with increasing depth \(V_0\).

5.2. Point \(E = -m\)

The system of radial equations \([17]\) and \([8]\) for bound states with energy \(E = -m\), i.e., with binding energy \(2m\), of a relativistic electron in the Coulomb field with cutoff radius \([8]\) becomes

\[
\frac{df(r)}{dr} + \frac{\varkappa}{r}f(r) + V(r)g(r) = 0, \quad \frac{dg(r)}{dr} - \frac{\varkappa}{r}g(r) = [2m + V(r)]f(r) = 0, \tag{30}
\]

it is complemented by the conditions that the both functions \(f(r)\) and \(g(r)\) are absolutely continuous together with their first derivatives and square-integrable on \((0, \infty)\) and are vanishing at zero, \(f(0) = g(0) = 0\).

5.2.1. Solving radial equations in region \(0 \leq r \leq \ell_0\)

The general solution of eqs. \([17]\) with \([11]\) in the internal region \(0 \leq r \leq \ell_0\), where we set \(f(r) = f_{in}(r), \ g(r) = g_{in}(r)\), under the above-mentioned conditions is given by

\[
\begin{align*}
  f_{in}(r) &= c\sqrt{r} J_{\zeta}(\eta_0 r) = c\sqrt{r} \begin{cases} J_{\zeta}(\eta_0 r), & \zeta = -1 \\ J_{\zeta+1}(\eta_0 r), & \zeta = 1 \end{cases}, \quad \eta_0 = \frac{q}{\ell_0} \sqrt{1 - \frac{2m\ell_0}{q}}, \\
  g_{in}(r) &= c\sqrt{r} \sqrt{1 - \frac{2m\ell_0}{q}} \zeta J_{\zeta-1}(\eta_0 r) = c\sqrt{r} \sqrt{1 - \frac{2m\ell_0}{q}} \begin{cases} -J_{\zeta+1}(\eta_0 r), & \zeta = -1 \\ J_{\zeta}(\eta_0 r), & \zeta = 1 \end{cases}, \tag{31}
\end{align*}
\]

it is sufficient to put \(E = -m\) and \(\ell_0 = \ell_0(Z)\) in \([12]\).

5.2.2. Solving radial equations in region \(\ell_0 \leq r < \infty\)

In the external region \(\ell_0 \leq r < \infty\), where we set \(f(r) = f_{out}(r), \ g(r) = g_{out}(r)\), system \([10]\) becomes

\[
\begin{align*}
  \frac{df_{out}(r)}{dr} + \frac{\varkappa}{r}f_{out}(r) - \frac{q}{r}g_{out}(r) = 0 &\Rightarrow g_{out}(r) = \frac{1}{q} \left[ \frac{df_{out}(r)}{dr} + \varkappa f_{out}(r) \right], \\
  \frac{dg_{out}(r)}{dr} - \frac{\varkappa}{r}g_{out}(r) + \frac{q}{r}f_{out}(r) - 2mf_{out}(r) &= 0, \tag{32}
\end{align*}
\]

it is complemented by the conditions that the both functions \(f_{out}(r)\) and \(g_{out}(r)\) are absolutely continuous and square integrable together with their derivatives on \((\ell_0, \infty)\).

Substituting the representation for the function \(g_{out}(r)\) from the first row in \([12]\) into the second equation in \([12]\), we obtain that the function \(f_{out}(r)\) satisfies the second-order differential equation

\[
\frac{d^2 f_{out}(r)}{dr^2} + \frac{df_{out}(r)}{dr} - 2qm f_{out}(r) - \frac{q^2 - q^2}{r} f_{out}(r) = 0. \tag{33}
\]
The substitution  \( f_{\text{out}}(r) = w(z), \ z = 2\sqrt{2qm}r \), reduces eq. (33) to the equation for the modified Bessel functions (Bessel functions of pure imaginary argument), see [20]:

\[
\frac{d^2w(z)}{dz^2} + \frac{1}{z} \frac{dw(z)}{dz} - \left( 1 + \frac{\tilde{\nu}^2}{z^2} \right) w(z) = 0, \ \tilde{\nu} = 2\mu, \ 2\sqrt{2qm}r_0 \leq z < \infty.
\]

The requirement for  \( f_{\text{out}}(r) \) to be square-integrable at infinity then yields  \( f_{\text{out}}(r) = AK_{\tilde{\nu}}(z) \), where  \( K_{\tilde{\nu}}(z) \) is the MacDonald function,

\[
K_{\tilde{\nu}}(z) = \frac{\pi}{2\sin(\pi\tilde{\nu})} [I_{-\tilde{\nu}}(z) - I_{\tilde{\nu}}(z)], \ \tilde{\nu} \neq n \in \mathbb{Z}_+,
\]

\[
I_{\tilde{\nu}}(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\tilde{\nu}}}{m!\Gamma(m+\tilde{\nu}+1)}, \ K_{\tilde{\nu}}(z) = K_{-\tilde{\nu}}(z).
\]

For  \( \tilde{\nu} = n \in \mathbb{Z}_+ \), the functions  \( K_n(z) \) contain terms with a logarithmic factor, see [20].

Using  \( f_{\text{out}}(r) = AK_{\tilde{\nu}}(z) \) and the representation for the function  \( g_{\text{out}}(r) \) in the first row in (32), we finally obtain that the general solution of system (30) under the above-mentioned conditions is given by

\[
f_{\text{out}}(r) = AK_{\tilde{\nu}}(z), \ g_{\text{out}}(r) = \frac{A}{q} \left\{ -\frac{z}{4} K_{\tilde{\nu}+1}(z) + \kappa \tilde{K}_{\tilde{\nu}}(z) \right\}, \ z = 2\sqrt{2qm},
\]

where we use the known formula  \( K_{\tilde{\nu}+1}(z) + K_{\tilde{\nu}-1}(z) = -2K_{\tilde{\nu}}'(z) \) (see [20]).

5.2.3. Charges providing bound states with energy  \( E = -m \), supercritical charge

After the general solution of system (30) is found independently in the respective regions 0 \( \leq r \leq r_0 \) and  \( r_0 \leq r < \infty \), it remains to satisfy the basic continuity condition for the solution as a whole (to sew the partial solutions together smoothly), which reduces to the requirement of continuity of the solution at the point  \( r = r_0 \),

\[
f_{\text{in}}(r_0) = f_{\text{out}}(r_0), \ g_{\text{in}}(r_0) = g_{\text{out}}(r_0).
\]

The compatibility of equalities (37) with  \( c \neq 0 \),  \( A \neq 0 \) yields the relation

\[
\eta_0 r_0 = q \sqrt{1 - \frac{2mr_0}{q}}, \ z_0 = 2\sqrt{2qm}r_0, \ \nu = j + \frac{\zeta + 1}{2}, \ \tilde{\nu} = 2\sqrt{\left( j + \frac{1}{2} \right)^2 - q^2}, \ \kappa = \zeta \left( j + \frac{1}{2} \right),
\]

that can be considered as the (transcendental) equation for charges  \( Z \) providing bound states with energy  \( E = -m \) for an electron with given total angular momentum  \( j \) and spin number  \( \zeta \). We let  \( Z^{(-m)}(j, \zeta) \) denote such charges.

An analytical solution of equation (38) for  \( Z^{(-m)}(j, \zeta) \) with arbitrary  \( j \) and  \( \zeta \) is unlikely to be possible at present. We only try to analyze it qualitatively and solve it numerically.

An equivalent form of equation (38) that seems more suitable for its qualitative analysis and its numerical solution is the equation

\[
\frac{\zeta}{J_{j+\frac{1}{2}}(\eta_0 r_0)} \left\{ \eta_0 J_{\zeta-1}(\eta_0 r_0) \right\} - \frac{z_0}{4} \frac{J_{\zeta-1}(z_0) + J_{\zeta+1}(z_0)}{J_{\zeta}(z_0)} = 0.
\]

What concerns a qualitative analysis of Eq. (39), we can say the following. As follows from the well-known asymptotic behavior of the Bessel function  \( J_{\nu}(z) \) as  \( z \to \infty \), the ratio of the Bessel functions  \( J_{\nu} \) multiplied by  \( \zeta \) in the l.h.s. of (39),  \( \zeta J_{\zeta-1}(\eta_0 r_0)/J_{\nu}(\eta_0 r_0) \), oscillates with increasing  \( Z \) around zero almost periodically, the period is  \( \pi \alpha^{-1} \), and ranges from  \( \infty \) to  \( -\infty \) (more specifically, as  \( \cot[\pi/4(1 - 2j)] \) for  \( \zeta = -1 \) and as  \( -\tan[\pi/4(1 - 2j)] \) for  \( \zeta = 1 \).

A plausible estimate of the asymptotic behavior of the MacDonald function  \( K_{\nu}(a\sqrt{\sigma}) \),  \( a, \sigma \in \mathbb{R} \), as  \( \sigma \to \infty \) allows a conclusion that the behavior of the ratio of the MacDonald functions  \( K \) in the l.h.s. of (39) with increasing  \( Z \)
is similar, a difference is that the oscillation frequency grows logarithmically with \( Z \). It follows that each Eq. (39) with any fixed \( j, \zeta \) has an infinite sequence \( \{ Z_n^{(-m)}(j, \zeta), n \in \mathbb{N} \} \) of solutions, \( Z_n^{(-m)}(j, \zeta) \to \infty \) as \( n \to \infty \), and the difference between the subsequent terms in this sequence decreases with \( n \).

We are now going to discuss the notion of the so-called supercritical charge. It seems that at present, there is no generally excepted understanding of this notion among physicists. In particular, each \( Z_n^{(-m)}(j, \zeta) \) of the whole set \( \cup_{j, \zeta} \{ Z_n^{(-m)}(j, \zeta), n \in \mathbb{N} \} \) is sometimes called the supercritical charge (or sometimes critical charge, as in Refs. [13, 17]), so that there is an infinite set of supercritical charges of nonclear physical meaning. We cannot agree with such a viewpoint for at least two reasons. First, almost all \( Z_n^{(-m)}(j, \zeta) \) are nonintegral and therefore have no direct physical meaning. Second, our standpoint is that the supercritical charge must be unique and integer valued. It remains to describe the charge distribution in a nucleus of finite radius. It is different for a uniformly charged sphere and for a uniformly charged ball. Shortly speaking, the supercritical charge is model dependent.

Returning to equation (39) for \( Z_n^{(-m)}(j, \zeta) \), we restrict ourselves to the case \( j = 1/2, \zeta = -1 \) and, in particular, find the supercritical charge.

In this case, equation (39) for \( Z^{(-m)}(1/2, -1) \) becomes, see (24),

\[
\varphi (Z) = 0, \quad \varphi (Z) = (\eta r_0) \cot (\eta r_0) + \frac{z_0 K_{\nu-1}(z_0) + K_{\nu+1}(z_0)}{K_{\nu}(z_0)}
\]

with

\[
\eta r_0 = q \sqrt{1 - \frac{2m\nu}{q}}, \quad z_0 = 2 \sqrt{2qm\nu}, \quad \nu = 2\sqrt{1 - q^2}.
\]

In examining equation (41), we have to distinguish two regions of \( Z \), namely, the region \( 0 < Z \leq Z_c \) (\( 0 < q \leq 1 \)) and the region \( Z > Z_c \) (\( q > 1 \)). In the first region, the parameter \( \nu \) is real-valued, while in the second region, \( \nu \) becomes pure imaginary, \( \nu = i\sigma, \sigma > 0 \). The variable \( z_0 \) is real-valued and positive in the both regions. As is known, the MacDonald function of positive argument and real index is strictly positive, see [18], while the MacDonald function of positive argument and pure imaginary index is alternating in sign. It follows that in the first region of \( Z \), the second term in Eq. (41) is strictly positive, in fact, with any \( r_0 \),

\[
\frac{z_0 K_{\nu-1}(z_0) + K_{\nu+1}(z_0)}{4 K_{\nu}(z_0)} > 0, \quad 0 < Z \leq Z_c, \forall r_0,
\]

while in the second region of \( Z \), this term is alternating in sign. On the other hand, as is easily verified, in the first region of \( Z \), the first term in Eq. (41) is strictly positive, in fact, with any \( r_0 \),

\[
(\eta r_0) \cot (\eta r_0) > 0, \quad 0 < Z \leq Z_c, \forall r_0.
\]

Really, if \( q < 2m\nu \), we have \( \eta r_0 = i\tau, \tau > 0 \), and \( (\eta r_0) \cot (\eta r_0) = \tau \coth \tau > 0 \), while if \( q > 2m\nu \), we have \( 0 < \eta r_0 < 1 \) and \( \cot (\eta r_0) > \cot 1 > 0 \). In the second region, this term is evidently oscillating function ranging from \( \infty \) to \( -\infty \) with increasing \( Z \),

\[
(\eta r_0) \cot (\eta r_0) \in (-\infty, \infty), \quad Z \in (Z_c, \infty).
\]

It follows from (43) and (44) that in the first \( Z \) region, \( 0 < Z \leq Z_c \), equation (41) for \( Z^{(-m)}(1/2, -1) \) has no solution, or equivalently, under any \( r_0 \), there is no charge in the region \( 0 < Z \leq Z_c \) providing bound state with energy \( E = -m \). On the contrary, in the second \( Z \) region, \( Z > Z_c \), equation (41) has an infinite growing sequence of solutions, as was already stated above. All this allows us to make the assertion that there is an infinitely growing
sequence \( \{Z_n^{(-m)}(1/2, -1), n \in \mathbb{N}, Z_n^{(-m)}(1/2, -1) > Z_c\} \) of charges providing the bound state with energy \( E = -m \). It is interesting to compare this situation with a similar situation in the case of the pure Coulomb field of a point charge (without a cutoff). We know from [9] that for a pure Coulomb field with \( Z < Z_s = (\sqrt{3}/2)Z_c \), in which case the Dirac Hamiltonian is defined uniquely, there is no bound state with \( E = -m \), while for each \( Z \geq Z_s \), such a bound state does exist. More exactly, for each \( Z \geq Z_s \), the s.a. radial Hamiltonian \( \hat{h}(Z, 1/2, -1) \) is defined nonuniquely; instead, there is a one-parameter family of s.a. Hamiltonians specified by certain s.a. boundary conditions at zero, and among these, there is a Hamiltonian, specified by peculiar s.a. boundary conditions at zero, which has a bound state with energy \( E = -m \).

We solve Eq. (41) numerically and find several first terms of the infinite sequence \( \{Z_n^{(-m)}(1/2, -1)\} \). Results of the numerical calculations are presented in FIG. 3 and by Eq. (46).

\[
\{Z_n^{(-m)}(1/2, -1)\} = \{173.92; 245.01; 327.39; 412.15; 496.11; 578.55; 659.82; 740.44; 820.53; 899.93; 978.50; 1056.28; 1133.46; 1210.16; 1286.43; 1362.21; 1437.48; \ldots\}
\]

We note, that as was already stated above, all the presented terms are noninteger and have no direct physical meaning. It is also worth noting that the difference between the subsequent terms of the sequence decreases with \( n \), as was expected. For us, only the first term of the sequence is important. According to our definition, the supercritical charge in our model is \( Z_{scr} = 174 \) with \( r_0 = 9.47 \times 10^{-15} \text{m} \approx 10 \text{F} \) which almost coincides with the result by Popov in [22].

6. CONCLUSION

We share an opinion widespread among physicists that the supercritical charge marks a boundary, after which, i.e., for \( Z \geq Z_{scr} \), the description of a behavior of an electron in strong Coulomb field, even regularized at the origin, in the framework of one-particle relativistic quantum mechanics based on the Dirac Hamiltonian definitely fails. It is believed that if \( Z \geq Z_{scr} \), processes with a fixed number of particles do not exist, in particular, do not exist pure one-particle processes, any process is accompanied by multiple \( e^+e^- \) pair creation. In such a situation, only a consistent QED with strong Coulomb field may provide rules for calculating all the quantum processes. The same concerns the critical charge \( Z_c = \alpha^{-1} \), and maybe even the lower critical charge \( Z_s = (\sqrt{3}/2)\alpha^{-1} \), for the Coulomb field of a point nucleus. It also must be remembered that although one-particle relativistic quantum mechanics with Dirac Hamiltonian with any potential, including the Coulomb field of any charge \( Z \), point or nonpoint, is mathematically consistent, in particular, describes a unitary evolution, it is unsatisfactory from the physical standpoint because of the unboundedness of the electron energy spectrum from below. As is well known, this drawback is overcome by secondary Fermi–Dirac quantization and transition to many-particle QFT. In any case, only the future QED can provide a proper description of a behavior of an electron in the Coulomb field of any strength.

It should be noted that the existing heavy nuclei can imitate time-dependent supercritical Coulomb fields at collisions. Then one can try to calculate the pair creation effect using elements of the well-elaborated QFT with time-dependent external electric fields that are switched on and off at the respective initial and final instants of time, see [7, 21] and citations therein.
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