Entangling atoms in bad cavities

Anders Søndberg Sørensen and Klaus Mølmer
QUANTOP, Danish quantum optics center
Institute of Physics and Astronomy,
University of Aarhus, DK-8000 Aarhus C, Denmark
(Dated: October 24, 2018)

We propose a method to produce entangled spin squeezed states of a large number of atoms inside an optical cavity. By illuminating the atoms with bichromatic light, the coupling to the cavity induces pairwise exchange of excitations which entangles the atoms. Unlike most proposals for entangling atoms by cavity QED, our proposal does not require the strong coupling regime \( g^2/\kappa \gg 1 \), where \( g \) is the atom cavity coupling strength, \( \kappa \) is the cavity decay rate, and \( \Gamma \) is the decay rate of the atoms. In this work the important parameter is \( Ng^2/\kappa \Gamma \), where \( N \) is the number of atoms, and our proposal permits the production of entanglement in bad cavities as long as they contain a large number of atoms.

PACS numbers: 03.65.Ud, 03.67.-a, 42.50.-p

I. INTRODUCTION

To obtain a large coherent coupling of individual quantum systems while at the same time maintaining a low decoherence rate is the main challenge in the experimental exploration of entanglement. An example of this is cavity QED which was one of the first proposals for the construction of a quantum computer and creation of entanglement of atoms [1,2], but where the experimental progress has been hampered by decoherence caused by cavity decay and spontaneous emission from the atoms. To overcome these problems experiments have resorted to very small optical cavities where the small cavity volume increases the interaction strength [3,4], or Rydberg atoms in superconducting microwave cavities, where the decoherence rates are low [5,6].

Most cavity QED schemes which have been proposed so far require the cavity to be in a strong coupling regime \( g^2/\kappa \GG 1 \), where \( g \) is the atom cavity coupling strength, \( \kappa \) is the cavity decay rate, and \( \Gamma \) is the decay rate of the atoms, and this limit is very hard to achieve experimentally. In this paper we propose a scheme where the created entanglement depends on the parameter \( Ng^2/\kappa \Gamma \), where \( N \) is the number of atoms. With this scheme it is in principle possible to produce entanglement in any cavity, but in practise the entanglement becomes unmeasurable if \( Ng^2/\kappa \Gamma \ll 1 \). On the other hand, if \( Ng^2/\kappa \Gamma \GG 1 \) a measurable entanglement is produced. Compared to the requirement \( g^2/\kappa \GG 1 \) the present approach thus allows a substantially reduction in the requirements for the cavity if a large number of atoms is used. We note that the requirement \( Ng^2/\kappa \Gamma \GG 1 \) is equivalent to the criterion for optical bistability and squeezing in cavity QED as studied experimentally in Refs. [7,8,9].

We propose to produce so-called spin squeezed states [10]. The collective properties of \( N \) two level atoms are conveniently described by pseudo angular momentum operators defined by \( J_z = \sum_k (|a\rangle_k \langle a| - |b\rangle_k \langle b|)/2 \) and \( J_+ = \sum_k |a\rangle_k \langle b| \), where the sum is over the individual atoms, and where \( |a\rangle \) and \( |b\rangle \) are the two internal states of the atoms. The state where all atoms are in the \( a \) state is an eigenstate of the \( J_z \) operator with eigenvalue \( N/2 \). If the \( J_x \) operator is measured in this state, the result will fluctuate around the mean value of zero with a variance \( N/4 \). By entangling the atoms it is possible to maintain a large value of the mean spin \( \langle J_z \rangle = N/2 \) while considerably reducing the noise in a spin component \( J_\theta = \cos(\theta)J_z + \sin(\theta)J_y \) perpendicular to the mean spin. A state with this property is called a spin squeezed state.

The experimental generation of spin squeezed states has potential applications in high precision spectroscopy and atomic clocks. For spectroscopy on a collection of two-level atoms Wineland et al. [1] have shown that the possible gain in precision by using a spin squeezed state is given by the quantity

\[
\xi^2 = \min_\theta \left( \frac{\langle J_z^2 \rangle}{\langle J_z \rangle^2} \right).
\]

It has also been shown that \( \xi^2 < 1 \) indicates that the atom are in an entangled state [12,13], and here we shall use \( \xi^2 \) to characterize the entanglement of the atoms.

Several schemes for the production of spin squeezed states have been proposed, and recently the first weakly squeezed states have been produced experimentally by absorption of squeezed light [14], by QND-detection [15], and by collisional interaction [16]. The possibility to produce spin squeezed states by having a large number of atoms in a bad cavity was already proposed in Ref. [17], but the method proposed here is more efficient; with the same cavity parameters our proposal enables a stronger squeezing of the spin. During the preparation of this work we became aware that a scheme very similar to ours has recently been proposed [18]. Compared to that work we apply a simpler level scheme in a different regime, but the fundamental ideas and the results are very similar. Also the results reached in Ref. [17] are similar to ours, but the mechanism employed in that paper is very different from what we propose here.

The paper is organized as follows. In Sec. 1 we present...
our scheme in the ideal case where no dissipation takes place. In Sec. II we analyse the scheme in the more realistic situation where cavity decay and decay of the excited atomic states affect the preparation of the entangled state. In Sec. III we summarize our proposal, and we discuss how some states are more robust than others against dissipation and loss. The derivation of the evolution in the presence of dissipation uses the standard method of adiabatic elimination and produces quite complicated equations. These expressions are not essential for the understanding of the functioning of the proposal and we have put the technical derivation in the Appendix.

II. IDEAL CASE

The energy levels of the atoms and the laser couplings are depicted in Fig. 1 (a). We consider a Λ type three level atom with two stable ground states |a⟩ and |b⟩ with an energy difference \( \omega_{ab} \) and an excited state |e⟩ with energy difference \( \omega_{ae} \) to the ground states |a⟩ (\( \hbar = 1 \)). The state |a⟩ is coupled to the excited state |e⟩ by a laser with a resonant Rabi frequency \( \Omega_1 \) and a frequency \( \omega_1 \) which is detuned from the excited state. Similarly the state |b⟩ is also coupled to the excited state by another detuned laser with resonant Rabi frequency \( \Omega_2 \) and frequency \( \omega_2 \). The two frequencies of the lasers are chosen such that their difference is exactly twice the energy splitting between the two ground states \( \omega_1 - \omega_2 = 2 \omega_{ab} \). With this choice of frequencies all transitions involving only a single atom are off-resonant, but a transition which transfers pairs of atoms from state |a⟩ to |b⟩ is resonant. A similar choice of resonance conditions has also been proposed for trapped ions [20, 21], and recently this scheme has allowed the first experimental production of four particle entangled states [22]. To produce the pairwise excitations of the atoms, it is not sufficient that the process is resonant; it is also necessary that there exists a physical mechanism which enables an interaction between the atoms. In [20, 21] this was done by the Coulomb interaction between the ions. Here we assume that the quantized field in an optical cavity couples both the states |a⟩ and |b⟩ to the excited state |e⟩ with coupling constants \( g_a \) and \( g_b \) respectively. With this coupling to the cavity there exists a transition path for the pairwise transition, as shown in Fig. 1 (b), and the matrix element for the transition becomes non-zero. In the remainder of this section we show that this coupling leads to a spin squeezed state if we apply the coupling to a state where all atoms are initially in the |a⟩ state.

If we assume all fields to be propagating in the same direction, the experimental situation is described by the Hamiltonian

\[
H = \omega_0 \hat{c}^\dagger \hat{c} + \sum_{k=1}^N \omega_{ae} |e\rangle_k \langle e| + \omega_{ab} |b\rangle_k \langle b| + H_{\text{int},k}
\]

\[H_{\text{int},k} = \left( \frac{\Omega_1}{2} e^{-i \omega_1 t} + g_a \hat{c} \right) |e\rangle_k \langle a| + \left( \frac{\Omega_2}{2} e^{-i \omega_2 t} + g_b \hat{c} \right) |e\rangle_k \langle b| + H.C.,\]

where \( \hat{c} \) and \( \omega_0 \) denote the annihilation operator and frequency of the relevant cavity mode.

If we are in a regime where the laser power is sufficiently weak that we do not transfer any population to the excited atomic state we may adiabatically eliminate this state and obtain an effective Hamiltonian for the coupled state of the ground states and the cavity [22]. Assuming further that the lasers are also sufficiently weak that we do not create a significant photon excitation in the cavity we may also adiabatically eliminate the cavity field and we are left with an effective Hamiltonian for the atoms [22]

\[
H = \frac{1}{\delta} \left( \frac{|\Omega_1|^2 |g_b|^2}{4 \Delta_1^2} J_+ J_- + \frac{|\Omega_2|^2 |g_a|^2}{4 \Delta_2^2} J_+ J_- + \frac{\Omega_1^* g_a g_b \Omega_2}{4 \Delta_1 \Delta_2} J_+ J_+ + \frac{\Omega_2^* g_a g_b \Omega_1}{4 \Delta_1 \Delta_2} J_+ J_- \right),
\]

where we have omitted some unimportant energy shifts, and we have introduced the detunings from the excited state \( \Delta_1 = \omega_{ae} - \omega_1 \) and \( \Delta_2 = \omega_{ae} - \omega_{ab} - \omega_2 = \Delta_1 + \omega_{ab} \), and the detuning from the cavity mode \( \delta = \omega_1 - \omega_{ab} - \omega_0 = \omega_2 + \omega_{ab} - \omega_0 \). These detunings are also defined in Fig. 1. The angular momentum operators are defined as the angular momentum operators in Sec. I. The origin of each of the terms in this Hamiltonian can be understood from processes like the one shown in Fig. 1 (b) which gives the term with \( J_+ J_- \), i.e., a double Raman process which takes two atoms from |a⟩ to |b⟩ by absorption by \( \Omega_1 \), emission into the cavity by \( g_b \), reabsorption of the cavity photon by \( g_a \), and emission by \( \Omega_2 \).

If we choose the strength of the two Raman processes to be identical \( \Omega_1 g_b^0 / \Delta_1 = \Omega_2 g_a^0 / \Delta_2 = \Omega g^0 / \Delta \), Eq.
proximately

The squeezing arising from this

Hamiltonian can be calculated analytically [10]. Starting

from an initial state where all atoms are in the $a$ state

and propagating with this Hamiltonian, squeezing by a

factor of $\xi^2 \approx N^{-2/3}$ is produced (in the limit $N \gg 1$),

and this is a significant noise reduction if a large number

of atoms is present. In the following section we show that

a significant squeezing is produced even in the presence

dissipation.

The time it takes to produce a spin squeezed state

of many atoms is very short. To squeeze the spin by

a constant factor, a constant number of atoms has to

be transferred into the $b$ state. With increasing $N$ a de-

creasing fraction of the atoms has to be transferred, and

thus a shorter time (scaling as $1/N$) is necessary to make

the squeezing. The different decoherence mechanisms

therefore have less time to affect the preparation of the

squeezed states, and as we show below, this reduces the

experimental requirement for the production of squeezed

states.

III. ANALYSIS INCLUDING DISSIPATION

AND NOISE

The main purpose of this paper is to demonstrate that

it is possible to use a cavity to entangle atoms even in

situations where substantial dissipation is present. In

this section we analyse the performance of our proposal

in the presence of the two main decoherence mechanisms:

spontaneous emission and cavity decay.

Before making a quantitative analysis of the effect of

dissipation we first make a few simple estimates of the
decoherence. For simplicity we shall here assume that

the lasers have approximately the same Rabi frequen-
cies ($\Omega_i \sim \Omega$) and detunings ($\Delta_i \sim \Delta$) and also

that the cavity couplings are similar $g_i \sim g$. The num-

ber of spontaneously emitted photons is estimated to be

approximately $N_\Gamma \sim N\Gamma/\Omega^2/\Delta^2$, where $\Gamma$ is the
total decay rate. The time required to produce squeezing by

a constant factor is given by $t \sim \Delta^2\delta/(N\gamma^2\Omega^2)$,

and by inserting this expression we find that the total number of

decayed atoms is

$$N_\Gamma \sim \frac{\Gamma\delta}{\gamma^2}. \quad (4)$$

The number of photons decaying out of the cavity during the

same time is estimated to be $N_\gamma \sim N\kappa i\Omega^2\gamma^2/(\Delta^2\delta^2)$

which reduces to

$$N_\gamma \sim \frac{\kappa}{\delta}. \quad (5)$$

when the expression for the time is inserted.

Because the spin squeezed state are only weakly entan-
gled they are quite insensitive to the spontaneous emis-

sion of the atoms. If we assume that we are near the

initial state where $J_z \approx N/2$, $J_x$ and $J_y$ may be replaced

by the canonical conjugate position $x = J_x\sqrt{2/N}$ and

momentum $p = J_y\sqrt{2/N}$ operators of a harmonic oscil-
lator. The relaxation rate for the harmonic oscillator is

then the same as the relaxation rate for a single atom,

and from the well known properties of squeezing of har-
monic oscillators we find that the squeezing is not com-
pletely degraded as long as the number of decayed atoms

is much less than the total number of atoms. The de-
cay of photons out of the cavity is more severe than the
decay of a single atom. Because the cavity couples to a

collective degree of freedom, the decoherence of the cav-
it will also affect the collective degree of freedom. If

$|\Omega_1\gamma_b|/\Delta_1 = |\Omega_2\gamma_a|/\Delta_2$ we estimate that the first decay of a photon out of the cavity increases the variance in all
directions perpendicular to the mean spin by a factor of three. To obtain a large squeezing we therefore require that at most a few photons are scattered out of the cavity. From the expression in Eqs. (4) and (5) we see that we can fulfill both $N_\Gamma \ll N$ and $N_\gamma \ll 1$ if the cavity param-

eters fulfill $N\gamma^2 \gg \kappa^2$ and we thus expect to be able
to produce substantial squeezing in this regime. This is
confirmed by our more accurate treatment of dissipation

below.

Dissipation is described by the master equation for the
density matrix $\rho$

$$\frac{d}{dt}\rho = -i[H, \rho] + \frac{1}{2} \sum_m \left(2\hat{d}_m\rho\hat{d}_m^\dagger - \hat{d}_m^\dagger\hat{d}_m\rho - \rho\hat{d}_m^\dagger\hat{d}_m\right), \quad (6)$$

where $\hat{d}_m$ are relaxation operators. We assume that the

separation of the atoms in the cavity is much larger than

the wavelength of the spontaneously emitted photons. In

this limit the decay of the atoms is uncorrelated and can

be described by independent relaxation operators for

each atom. The excited state $|e\rangle$ is assumed to have

three independent decay channels: it may decay to the two

lower states in the $\Lambda$-system $|a\rangle$ and $|b\rangle$ with decay

rates $\gamma_a$ and $\gamma_b$ respectively, and it may decay to some

other state $|c\rangle$ with a decay rate $\gamma_c$. The total effect of

the spontaneous emission is described by $3N$ relaxation operators

$$\hat{d}_{a,k} = \sqrt{\gamma_a}|a\rangle\langle k|,$$

$$\hat{d}_{b,k} = \sqrt{\gamma_b}|b\rangle\langle k|,$$

$$\hat{d}_{c,k} = \sqrt{\gamma_c}|c\rangle\langle k|,$$

where $k = 1, ..., N$. To describe the decay of the cavity with a rate $\kappa$ we introduce a relaxation operator

$$\hat{d}_c = \sqrt{\kappa}. \quad (8)$$

To derive the evolution of the spin squeezing we first

adiabatically eliminate the excited state assuming

$$\frac{|\Omega|^2}{4} \ll \Delta^2 + \frac{\Gamma^2}{4},$$

$$\delta, \kappa' \ll \omega_{ab}. \quad (9)$$
where $l = 1, 2$, and $\Gamma$ is the total decay rate of the excited state $\Gamma = \gamma_a + \gamma_b + \gamma_c$. $\kappa'$ is an effective decay rate of the cavity, which is slightly larger than $\kappa$ due to the scattering of cavity photons by the atoms, cf. Eq. (A4). We then adiabatically eliminate the cavity from the equations assuming

$$N \frac{|\Omega_1 g_b|^2}{4} \ll \left( \Delta_1^2 + \frac{\Gamma^2}{4} \right) \left( \delta^2 + \frac{\kappa'^2}{4} \right). \quad (10)$$

The equations resulting from the adiabatic elimination are quite complicated and we leave the derivation to Appendix A. If we assume that the initial state is almost unaffected by the interaction so that $J_z \approx N/2$, the matrix elements which are quadratic in the angular momentum operators do not couple to higher order terms and we obtain closed equations for the expectation values

$$\frac{d}{dt} \begin{bmatrix} \langle J_z \rangle \\ \langle N_a + N_b \rangle \\ \langle J_a J_b \rangle \\ \langle J_a J_+ \rangle \\ \langle J_- J_b \rangle \\ \langle J_+ J_+ \rangle \end{bmatrix} = M \cdot \begin{bmatrix} \langle J_z \rangle \\ \langle N_a + N_b \rangle \\ \langle J_a J_b \rangle \\ \langle J_a J_+ \rangle \\ \langle J_- J_b \rangle \\ \langle J_+ J_+ \rangle \end{bmatrix}, \quad (11)$$

where $N_l$ is the number operator for atoms of type $l = a, b$ ($\langle N_a + N_b \rangle$ is not a conserved quantity because of the decay to the state $|o\rangle$). The precise form of the matrix $M$ is given by the expressions in Eqs. [A5,A9].

Due to the complicated structure of the matrix $M$ it is difficult to describe the evolution of the squeezing analytically, but it is straightforward to find the evolution numerically, where the solution at a given time $t$ may be found by taking the exponential of the matrix $Mt$. In Fig. 2 we show the evolution of squeezing in a situation where $g^2/\Gamma \ll 1$ but $Ng^2/\Gamma \gg 1$. With the realistic parameters used in the figure we are able to produce squeezing by approximately an order of magnitude after a very short interaction time. The time required to make the squeezing in the figure is less than 1$\mu$s if we chose a realistic cavity coupling parameter $g = (2\pi)100$kHz.

In principle the proposed scheme can be used to produce entanglement in any cavity. The field $\Omega_2$ only couples to the ground state $|b\rangle$ that is initially unpopulated, so that a large value of $\Omega_2$ increases the rate of the coherent transfer of atoms from state $a$ to $b$ but does not affect the initial decoherence rate. The amount of squeezing, however, depends on the cavity parameters. To investigate the obtainable squeezing we have performed a numerical optimization of the coupling strengths and detunings for a number of different cavity parameters. With fixed values of the dissipation rates and energy difference $\omega_{ab}$, we vary $\Omega_2/\Omega_1$, $\delta$, and $\Delta_1$ and search for the values which give the minimal $\xi^2$ (because all terms in $M$ involve the square of the field strength, the minimum only depends on the ratio between the two fields). In the limit $N \gg 1$ and for fixed ratios between the decay rates $\gamma_l$ ($l = a, b, o$) the results of the optimization indicate that the optimal squeezing parameter $\xi^2_{\text{min}}$ is only a function of the parameter $Ng^2/\Gamma \kappa$. In Fig. 3 we show $\xi^2_{\text{min}}$ for different cavity parameters. As expected from our simple estimates, the figure confirms that strong squeezing can be produced in the limit $Ng^2/\Gamma \kappa \gg 1$. In the calculations we have assumed $g_a = g_b = g$ and $\gamma_a = \gamma_b = \gamma_o = \Gamma/3$, and with these values the optimal squeezing is approximately given by $\xi^2_{\text{min}} = 0.7/\sqrt{Ng^2/\Gamma \kappa}$ for $Ng^2/\Gamma \kappa \gtrsim 1$. This is indicated by the dashed line in the figure.

![FIG. 2: Evolution of squeezing for $N = 10^6$ atoms in a bad cavity (full line). The parameters used in the simulation are $g_a = g_b = g$, $\Gamma = \kappa = 100g$, $\gamma_a = \gamma_b = \gamma_o$, $\Omega_1 = \Omega_2 = 10^4g$, $\Delta_1 = 10^5g$, $\delta = 5 \cdot 10^2g$, and $\omega_{ab} = 10^4g$ corresponding to $g^2/\Gamma \ll 1$ but $Ng^2/\Gamma \gg 1$. For comparison we also show the evolution with the same parameters but without dissipation, $\Gamma = \kappa = 0$ (dashed line).](image)

![FIG. 3: Minimum squeezing parameter obtained by a numerical optimization. The points (+) are the results of the minimalization and the dashed line $0.7/\sqrt{Ng^2/\Gamma \kappa}$ approximates $\xi^2_{\text{min}}$ for $Ng^2/\Gamma \kappa \gtrsim 1$. In the calculation we have assumed $N = 10^6$, $g_a = g_b = g$, $\gamma_a = \gamma_b = \gamma_o$, and $\omega_{ab} = 10^5$. The same minimum is obtained for three different ratios $\kappa/\Gamma = 10^{-2}, 1$, and $10^2$, and the results are independent of $N$ in the limit $N \gg 1$.](image)
same behaviour but with a slightly different constant has also been found in [13]. The obtained results only change slightly if we vary the ratio between the coupling constants or between the decay rates.

The results of the numerical simulations agree very well with the behaviour expected from the simple estimates. Our assumptions of weak excitation of the atoms and of the cavity field mode imply the necessary condition \( \Delta_1 \gg \Gamma \). It turns out, however, that the detuning \( \delta \) from the intermediate state in Fig.1 (b) with one cavity photon excited, does not need to be large. If only the coupling is weak enough \( \sqrt{N}[\Omega_1 g_b/\Delta_1] \ll \kappa \), and if the process that absorbs cavity photons is stronger than the one producing them, \( |\Omega_2 g_b|/(\Delta_2^2 + \Gamma^2/4) > |\Omega_1 g_b|/(\Delta_1^2 + \Gamma^2/4) \), the photon excited state can be eliminated, and we find good squeezing for all values of \( \delta \). The minimum value of \( \xi^2 \) is found for \( \delta = 0 \).

Finally, let us briefly comment on a few experimental aspect of our proposal. The analysis above shows that our proposal is robust against the spontaneous emission caused by the laser coupling and the decay of photons out of the cavity. The coupling of the lasers, however, introduces other possible decoherence mechanisms if we are not able to control the lasers with a high enough accuracy. In the treatment so far we have ignored the AC-Stark shifts caused by the lasers because they can be compensated by a small change in the frequencies. But the magnitude of the AC-Stark depends on the power of the lasers so that fluctuations in the power has a detrimental effect on the squeezing. To suppress this effect we propose to adjust the relative strengths of the two fields so that \( |\Omega_1|^2 \Delta_1/(\Delta_1^2 + \Gamma^2/4) = |\Omega_2|^2 \Delta_2/(\Delta_2^2 + \Gamma^2/4) \). With this choice the AC-Stark shifts of the two ground states become identical and have no effect on the internal state preparation. The problem of stabilizing the power can thus be reduced to the problem of stabilizing the relative frequency and intensity of the two fields which is much easier experimentally if the two fields are derived from the same source.

The efforts to entangle atoms through cavity QED have so far concentrated on the strong coupling regime \( g^2/\kappa \Gamma \gg 1 \). To achieve this limit it has been desirable to use very small standing wave cavities where the coupling constant varies sinusoidally along the cavity axis with a period of half the optical wavelength. A controlled evolution in these cavities therefore requires that the atoms are localized in regions smaller than the optical wavelength. Since our proposal puts much less stringent requirement on the cavity parameters it should not be necessary to use such small cavities and it could for instance be implemented with a ring cavity. Then, the magnitude of the coupling constant does not depend on the position of the atom along the cavity axis, and if the classical fields are co-propagating with the cavity field it is no longer necessary to localize the atoms within a wavelength. The atoms only need to be confined within the waist of the cavity mode, and this can be done with cold atoms trapped in a far detuned optical dipole trap or optical lattice, or even with atoms in a glass cell at room temperature.

IV. CONCLUSION

We have shown that it is possible to observe significant spin squeezing of atoms coupled to the field mode in a lossy cavity. The loss of quantum correlations between the particles which is caused by atomic decay and cavity loss is balanced by the strong non-linear coupling achievable in the limit of very many atoms. It has been shown that spin squeezing implies entanglement, i.e., a separable state cannot lead to values of \( \xi^2 \) smaller than unity. We have thus created entangled states which are fairly robust against dissipation and loss.

There is no precise quantitative measure for the entanglement of a large collection of particles, but a natural qualitative measure is to consider the possible gain, e.g., in spectroscopic resolution, that the entangled states offers with respect to a disentangled state. By binding the \( N \) atoms together in \( N/P \) maximally entangled \( P \)-particle states \( \langle iaa...a \rangle + \langle bbb...b \rangle/\sqrt{2} \), one obtains a spectroscopic resolution corresponding to a state with \( \xi^2 = 1/P [2] \). Hence in terms of spectroscopic resolution the spin squeezed states are as powerful as if the atoms had been divided into groups of maximally entangled states of \( 1/\xi^2 \) particles.

From a practical perspective, however, the squeezed states offer a significant advantage compared to a collection of highly entangled states. In spin squeezed states the relevant observables are collective operators involving all the atoms. There is no need to address the atoms individually, and the manipulation and detection of the squeezing can therefore be achieved by lasers addressing all atoms collectively. Furthermore the spin squeezed states are also easier to produce: In an ideal spin squeezed state the one-particle density matrix \( \rho_1 \) is very close to the initial pure state projection operators \( \rho_1 = |a\rangle |a\rangle + O(1/N) \), and the state of each atom is thus almost disentangled from the state of the other atoms. This means that we only need to perturb the initial state slightly to turn it into a squeezed state. This is a significant advantage in any experimental attempt to produce entanglement because (a) the states are more robust against decoherence than more highly entangled states and (b) the interaction time required to make the desired state is much shorter than for the highly entangled states. The large number of atoms increases the decoherence rates, i.e., more photons are scattered, but our calculations show that the two advantages (a) and (b) outweigh the increased decoherence rate and enable the construction of entangled states in situations where the experimental capabilities do not permit the construction of entangled states of a few atoms.

In a broader context our proposal fits into the field of ensemble quantum information processing, where the quantum information is encoded into the collective de-
degrees of freedom of a collection of atoms. A number of papers [23, 26, 27] have proposed schemes for the processing of information encoded in such a way and the first experimental implementation of these concepts has recently been reported [28]. By combining these ideas with the present work one may for example imagine a quantum computer with several separate and individually addressable atomic clouds which communicate in a controlled manner via cavity modes.

Acknowledgments

We are grateful to Michael Drewsen and Eugene Polzik for useful discussions about possible experimental realizations of the scheme. This work was supported by the Danish National Research Foundation through QUANTOP, the Danish Quantum Optics Center, and by CAUAC (contract no. HPRN-CT-2000-00165).

APPENDIX A: DERIVING THE EQUATIONS OF MOTION

In this section we derive the equations describing the time evolution of squeezing in the presence of dissipation. We first consider only the Hamiltonian describing a single atom by assuming that the population of that state is negligible. In this approximation the equations for the ground state density matrix elements are equivalent to the evolution by a Hamiltonian

\[
H = - \left( \frac{\Delta_1 |\Omega_1|^2}{4(\Delta_1^2 + \frac{\Gamma_1}{2})} + \frac{\Delta_2 |g_2|^2 \hat{c} \hat{c}^\dagger}{\Delta_2^2 + \frac{\Gamma_2}{2}} \right) |a\rangle \langle a| \\
- \left( \frac{\Delta_2 |\Omega_2|^2}{4(\Delta_2^2 + \frac{\Gamma_2}{2})} + \frac{\Delta_1 |g_1|^2 \hat{c} \hat{c}^\dagger}{\Delta_1^2 + \frac{\Gamma_1}{2}} \right) |b\rangle \langle b| \\
- \frac{\Delta_1}{\Delta_1^2 + \frac{\Gamma_1}{2}} \left( \frac{\Omega_1^* g_b^*}{2} |b\rangle \langle a| \hat{c} \hat{c}^\dagger e^{-i\delta t} + \frac{\Omega_1 g_b}{2} |a\rangle \langle b| \hat{c} \hat{c}^\dagger e^{-i\delta t} \right) \\
- \frac{\Delta_2}{\Delta_2^2 + \frac{\Gamma_2}{2}} \left( \frac{\Omega_2^* g_a^*}{2} |b\rangle \langle a| \hat{c} \hat{c}^\dagger e^{i\delta t} + \frac{\Omega_2 g_a}{2} |a\rangle \langle b| \hat{c} \hat{c}^\dagger e^{i\delta t} \right)
\]

(A1)

and six relaxation operators \( \hat{d}_{k,l} \) (\( k = a, b, o \) and \( l = 1, 2 \)) describing the combined excitation with a detuning \( \Delta_l \) and decay to a state \( |k\rangle \), e.g.,

\[
\hat{d}_{a,1} = \frac{\sqrt{\gamma_a}}{\Delta_1 - i \frac{\Gamma_1}{2}} \left( \frac{\Omega_1}{2} |a\rangle \langle a| + g_b |a\rangle \langle b| \hat{c} \hat{c}^\dagger e^{i\delta t} \right). \tag{A2}
\]

To derive these results we have assumed that \( \delta \ll \Delta_1, \Delta_2 \), and we have used the second relation in Eq. (9) to neglect processes which creates photons without a change in the atomic state.

The first two lines in Eq. (A1) represent AC-Stark shifts of the ground states. The first part of the shifts containing the classical fields \( \Omega_1 \) and \( \Omega_2 \) can be compensated if we make a change in the frequency of the fields. The second part containing the quantum field \( \hat{c} \) is much smaller than the first and by inserting the approximate time and Eq. (A3) below, we find that this term gives a negligible phase shift if \( g^2/\delta \Delta \ll 1 \) and we shall neglect these terms.

We then adiabatically eliminate the cavity in the Heisenberg picture. Setting \( d(\hat{c} \hat{c}^\dagger)/dt = 0 \) we obtain

\[
\hat{c} \hat{c}^\dagger = - \frac{1}{\delta + i \frac{\kappa'}{2}} \left( \frac{\Omega_1 g_b^*}{\Delta_1 - i \frac{\Gamma_1}{2}} J_+ + \frac{\Omega_2 g_a^*}{\Delta_2 - i \frac{\Gamma_2}{2}} J_+ \right) + \text{noise}, \tag{A3}
\]

where the noise ensures the commutation relation of the operator. Here we have introduced an effective decay rate for the cavity

\[
\kappa' = \kappa + \frac{N |g_a|^2}{\Delta_2^2 + \frac{\Gamma_2}{2}} \tag{A4}
\]

which takes into account that the cavity photons may be scattered by the atoms. In Eq. (A4) we have assumed that essentially all atoms remain in the \( a \) state.

The adiabatic elimination of the cavity requires that the atoms are disentangled from the cavity, i.e., that \( \langle \hat{c} \hat{c}^\dagger \rangle \ll 1 \). At \( t = 0 \) this reduces to Eq. (10), and at later times our numerical simulation indicate that \( \langle \hat{c} \hat{c}^\dagger \rangle \) typically changes slowly, so that the condition is fulfilled if it is fulfilled at \( t = 0 \).

Finally, we insert Eq. (A3) into the time derivatives of the angular momentum operators, and by assuming that the initial state only changes slightly so that \( J_z \approx N/2 \), we obtain the following expressions
\[
\frac{d}{dt}\langle J_+ \rangle = -\frac{\gamma_0 + \gamma_0/2}{\Delta_a^2 + \frac{\Gamma}{4}} \langle \hat{N}_a \rangle + \frac{\gamma_a + \gamma_0/2}{\Delta_b^2 + \frac{\Gamma}{4}} \langle \hat{N}_b \rangle - \frac{1}{\delta^2 + \frac{\kappa^2}{4}} \left[ \frac{|\Omega_1|^2|g_b|^2}{4(\Delta_a^2 + \frac{\Gamma}{4})^2} \left( -\delta \Delta_1(2\gamma_b + \gamma_a) + \kappa' \Delta_a^2 + \frac{\kappa'T(\gamma_a - \gamma_b)}{4} \right) \langle J_+J_- \rangle + \frac{|\Omega_2|^2|g_a|^2}{4(\Delta_b^2 + \frac{\Gamma}{4})^2} \left( \delta \Delta_2(2\gamma_a + \gamma_b) - \kappa' \Delta_b^2 + \frac{\kappa'T(\gamma_b - \gamma_a)}{4} \right) \langle J_-J_+ \rangle + \frac{\Omega_1 \Omega_2^* g_a g_b}{4(\Delta_a^2 + \frac{\Gamma}{4})(\Delta_b^2 + \frac{\Gamma}{4})} \left( \delta \Delta_1 \Delta_2 - i \kappa' \Delta_1(\gamma_a + \frac{\gamma_b}{2}) - i \frac{\kappa' \Delta_2(\gamma_b + \frac{\gamma_a}{2})}{2} - \delta \Delta_1 \left( \frac{\gamma_a + \gamma_0}{2} + \delta \Delta_2 \left( \frac{\gamma_a + \gamma_0}{2} + \frac{\kappa'T(\gamma_a - \gamma_b)}{4} \right) \right) \langle J_-J_- \rangle + \frac{\Omega_1 \Omega_2^* g_a g_b}{4(\Delta_b^2 + \frac{\Gamma}{4})(\Delta_a^2 + \frac{\Gamma}{4})} \left( \delta \Delta_1 \Delta_2 + i \kappa' \Delta_1(\gamma_a + \frac{\gamma_b}{2}) + i \frac{\kappa' \Delta_2(\gamma_b + \frac{\gamma_a}{2})}{2} - \delta \Delta_1 \left( \frac{\gamma_b + \gamma_0}{2} + \delta \Delta_2 \left( \frac{\gamma_b + \gamma_0}{2} + \frac{\kappa'T(\gamma_b - \gamma_a)}{4} \right) \right) \langle J_+J_+ \rangle \right].
\]

(A5)

\[
\frac{d}{dt}(\hat{N}_a + \hat{N}_b) = -\frac{\gamma_0}{\Delta_a^2 + \frac{\Gamma}{4}} \langle \hat{N}_a \rangle - \frac{\gamma_0}{\Delta_b^2 + \frac{\Gamma}{4}} \langle \hat{N}_b \rangle + \frac{\gamma_a}{\delta^2 + \frac{\kappa^2}{4}} \left[ \frac{|\Omega_1|^2|g_b|^2}{4(\Delta_a^2 + \frac{\Gamma}{4})^2} \left( 2\delta \Delta_1 + \kappa' \Delta_a^2 \right) \langle J_+J_- \rangle + \frac{|\Omega_2|^2|g_a|^2}{4(\Delta_b^2 + \frac{\Gamma}{4})^2} \left( 2\delta \Delta_2 + \kappa' \Delta_b^2 \right) \langle J_-J_+ \rangle + \frac{\Omega_1 \Omega_2^* g_a g_b}{4(\Delta_a^2 + \frac{\Gamma}{4})(\Delta_b^2 + \frac{\Gamma}{4})} \left( \delta \Delta_1 + \Delta_2 \right) (1 + i + \kappa' \Delta_2) \langle J_-J_- \rangle + \frac{\Omega_1 \Omega_2^* g_a g_b}{4(\Delta_b^2 + \frac{\Gamma}{4})(\Delta_a^2 + \frac{\Gamma}{4})} \left( \delta \Delta_1 + \Delta_2 \right) (1 + i + \kappa' \Delta_2) \langle J_+J_+ \rangle \right].
\]

(A6)

\[
\frac{d}{dt}(J_+J_+) = -\frac{\Gamma}{\Delta_a^2 + \frac{\Gamma}{4}} \langle J_+J_+ \rangle - \frac{\Gamma}{\Delta_b^2 + \frac{\Gamma}{4}} \langle J_+J_+ \rangle - \frac{2i\Delta_1}{\delta^2 + \frac{\kappa^2}{4}} \left[ \frac{|\Omega_1|^2|g_b|^2}{4(\Delta_a^2 + \frac{\Gamma}{4})^2} \left( \Delta_2 + \frac{\Gamma}{4} \right) \langle J_+J_+ \rangle + \frac{|\Omega_2|^2|g_a|^2}{4(\Delta_b^2 + \frac{\Gamma}{4})^2} \left( \Delta_2 + \frac{\Gamma}{4} \right) \langle J_+J_+ \rangle + \frac{\Omega_1 \Omega_2^* g_a g_b}{4(\Delta_a^2 + \frac{\Gamma}{4})(\Delta_b^2 + \frac{\Gamma}{4})} \left( \Delta_1 + \frac{\Gamma}{2} \right) \left( \Delta_2 - \frac{\Gamma}{2} \right) \langle J_-J_- \rangle + \left( \Delta_1 + \frac{\Gamma}{2} \right) \left( \Delta_2 - \frac{\Gamma}{2} \right) \langle J_-J_- \rangle \right],
\]

(A7)

and

\[
\frac{d}{dt}(J_+J_-) = \frac{|\Omega_1|^2}{4(\Delta_a^2 + \frac{\Gamma}{4})} \left( \gamma_a \langle \hat{N}_a \rangle - \Gamma \langle J_+J_- \rangle \right) + \frac{|\Omega_2|^2}{4(\Delta_b^2 + \frac{\Gamma}{4})} \left( \gamma_b \langle \hat{N}_b \rangle + \Gamma \langle J_+J_- \rangle \right) - \frac{N}{\delta^2 + \frac{\kappa^2}{4}} A
\]

(A8)
where

\[ A = \frac{|\Omega_1|^2|g_b|^2}{4(\Delta_1^2 + \frac{\Gamma_1^2}{4})^2} (-\kappa') \left( \Delta_1^2 + \frac{\Gamma_1^2}{4} \right) \langle J_+ J_- \rangle + \frac{|\Omega_2|^2|g_a|^2}{4(\Delta_2^2 + \frac{\Gamma_2^2}{4})^2} \left( \Delta_2^2 \kappa' - 2\Delta_2 \delta \Gamma - \frac{\kappa' \Gamma_2^2}{4} \right) \langle J_- J_+ \rangle \]

\[ + \frac{\Omega_1 \Omega_2^* g_a g_b}{4(\Delta_2^2 + \frac{\Gamma_2^2}{4})(\Delta_1^2 + \frac{\Gamma_1^2}{4})} \left( 2i\delta \Delta_1 \Delta_2 - \delta \Delta_2 \Gamma + i\Delta_1 \frac{\Gamma \kappa'}{2} - \frac{\kappa' \Gamma_2^2}{4} \right) \langle J_- J_+ \rangle \]

\[ + \frac{\Omega_1^* \Omega_2 g_a^* g_b}{4(\Delta_2^2 + \frac{\Gamma_2^2}{4})(\Delta_1^2 + \frac{\Gamma_1^2}{4})} \left( -2i\delta \Delta_1 \Delta_2 - \delta \Delta_2 \Gamma - i\Delta_1 \frac{\Gamma \kappa'}{2} - \frac{\kappa' \Gamma_2^2}{4} \right) \langle J_+ J_- \rangle. \]