On the stability of boundary-layer flow over a rotating cone using new solution methods

Dr Zahir Hussain* and Prof. Stephen J. Garrett
1Centre for Mathematical Modelling and Flow Analysis, Manchester Metropolitan University, Manchester, UK, 2School of Mathematics and Actuarial Science, University of Leicester, Leicester, UK
E-mail: *z.hussain@mmu.ac.uk

Abstract. In this study, a new solution is applied to the model problem of boundary-layer flow over a rotating cone in still fluid. The mean flow field is perturbed leading to disturbance equations that are solved via a more accurate spectral numerical method involving Chebyshev polynomials, both of which are compared with previous numerical and analytical approaches. Importantly, favourable comparisons are yielded with existing experiments [17] and theoretical investigations [6] in the literature. Meanwhile, further details will be provided of potential comparisons with new experiments currently in the pipeline.

Physically, the problem represents a model of airflow over rotating machinery components at the leading edge of a turbofan. In such applications, laminar-turbulent transition within the boundary layer can lead to significant increases in drag, resulting in negative implications for fuel efficiency, energy consumption and noise generation. Consequently, delaying transition to turbulent flow is seen as beneficial, and controlling the primary instability may be one route to achieving this. Ultimately, control of the input parameters of such a problem may lead to future design modifications and potential cost savings.

Our results are discussed in terms of existing experimental data and previous stability analyses on related bodies. Importantly, broad-angled rotating cones are susceptible to a crossflow instability [6], visualised in terms of co-rotating spiral vortices, whereas slender rotating cones have transition characteristics governed by a centrifugal instability [9], which is visualised by the appearance of counter-rotating Görtler vortices. We investigate both parameter regimes in this study and comment on the accuracy of the new solution method compared with previous methods of solving the stability equations.

1. Introduction
There has been considerable research on the flow over rotating geometries, particularly the Von Kármán flow over a rotating disk, which has long been used as a model for flow over a swept-wing, since the early work of [8]. Recently, the stability of rotating boundary layer flows has experienced increased attention in the literature and has been investigated within a wide range of flow setups and variations. In particular, the flow over a rotating cone represents a complex generalisation of rotating disk flow, and involves the introduction of a cone half-angle, which is equal to 90° for a rotating disk. Recently, research on this geometry has accelerated, with both theoretical global linear instability [21], numerical simulations [20] and experimental investigation of broad cones [16], which follows on from earlier experimental work on the rotating-disk boundary layer [12]. In this paper, we will build upon the investigation of boundary-layer...
transition over rotating cones. Specifically, we consider a family of rotating cones in still fluid, with major emphasis on the parameter values represented by the crossflow instability, for which the cone half-angle is large, typically larger than 40° and less than 90°. Conversely, cones with half-angles in the parameter regime less than 40° are referred to as slender cones, where the boundary-layer flow is instead susceptible to a centrifugal instability. In particular, following recent findings, we focus on the changing nature of the dominant instability mechanism, which has been observed to transition from crossflow to a centrifugal instability mechanism as the half-angle is reduced from 90°.

The methods used in this study develop a linear stability analysis of the full disturbance equations via a spectral numerical code, which is subsequently used to analyse the energy production and dissipation terms for the crossflow instability modes. The approach follows that of [3] for a compliant rotating disk, and has been verified successfully for related investigations on the rotating disk by [2].

The paper proceeds as follows: we formulate the problem in §2, starting from the modified von Kármán basic flow solution of [6] to represent the full disturbance equations for the crossflow instability. We then move onto outline the spectral analysis in §3 along with the major findings of the solution technique. The results are discussed and compared with theoretical and experimental studies in the literature in §4. Finally, conclusions are drawn in §5.

2. Formulation
We model a cone of half-angle $\psi$ rotating in a fluid of kinematic viscosity $\nu^*$ with an angular velocity $\Omega^*$ in an anti-clockwise direction around the streamwise coordinate axis $x^*$ (where a * denotes a dimensional quantity). The flow obeys the Navier–Stokes equations, as described in [6], where the governing parameter $\psi$ models a family of rotating cones, which we reference as $\psi = 20^\circ$–90° in 10° increments. Physically, the flow exhibits a boundary layer of thickness $\delta^*$ close to the cone surface characterised by the local Reynolds number $R_L$, which, for a particular
streamwise location along the cone \( x'_L \), is given by:

\[
R_L = \frac{x'_L \Omega^* \sin \psi \delta^*}{\nu^*}.
\]

The mean flow profiles in the streamwise, \( x^* \), azimuthal, \( \theta \), and surface normal, \( z^* \), directions are \( U(\eta; \psi), V(\eta; \psi) \) and \( W(\eta; \psi) \), where \( \eta = z^*/\delta^* \) is a re-scaled surface-normal coordinate and the boundary layer thickness is given by \( \delta^* = (\nu^*/\Omega^* \sin \psi)^{\frac{1}{2}} \). A diagram of the flow setup is shown in figure 1 and the mean flows are obtained by following the numerical analysis in §4 of [6] and originally [5] in order to obtain the governing Von Kármán basic flow equations for rotating cones of varying values of \( \psi \). Following [6], we proceed to substitute the perturbation quantities above into the governing linearised Navier-Stokes equations, reducing the system of PDEs to the following sixth-order system of ODEs, which is characterised by eigenfunctions dependent on \( \eta \) and normal mode expansions in the remaining spatial and temporal variables, \( x, \theta, t \), respectively. We then proceed to rearrange the perturbation equations to obtain the governing eigenvalue problem for \( \alpha \), given by:

\[
\begin{align*}
\dot{u} &= u(\eta; \alpha, n, \gamma, R_L, \psi) \exp(i(\alpha x \sin \psi + n\theta - \gamma t)), \\
\dot{v} &= v(\eta; \alpha, n, \gamma, R_L, \psi) \exp(i(\alpha x \sin \psi + n\theta - \gamma t)), \\
\dot{w} &= w(\eta; \alpha, n, \gamma, R_L, \psi) \exp(i(\alpha x \sin \psi + n\theta - \gamma t)), \\
\dot{p} &= p(\eta; \alpha, n, \gamma, R_L, \psi) \exp(i(\alpha x \sin \psi + n\theta - \gamma t)),
\end{align*}
\]

where \( \alpha = \alpha_r + i\alpha_i \) is the complex wavenumber in the streamwise direction, while \( n \) and \( \gamma \) are the wavenumber in the circumferential direction and the frequency for time \( t \), respectively. Following [6], we proceed to substitute the perturbation quantities above into the governing linearised Navier-Stokes equations, reducing the system of PDEs to the following sixth-order system of ODEs, which is characterised by eigenfunctions dependent on \( \eta \) and normal mode expansions in the remaining spatial and temporal variables, \( x, \theta, t \), respectively. We then proceed to rearrange the perturbation equations to obtain the governing eigenvalue problem for \( \alpha \), given by:

\[
\begin{align*}
iu\alpha + \sin \psi &\frac{\sin \psi}{R_L} u + \nu v + \frac{\cos \psi}{R_L} w + w' = 0, \\
\frac{u}{R_L} \alpha^2 + i(U u + p)\alpha + \left(M_{n,\gamma} + \frac{U \sin \psi}{R_L}\right) u &+ \frac{W}{R_L} u'' - \frac{1}{R_L} u'' - \frac{2(V + 1) \sin \psi}{R_L} v + U' w = 0, \\
v \frac{R_L}{\alpha^2} + iU v \alpha + \left(M_{n,\gamma} + \frac{U \sin \psi}{R_L}\right) v &+ \frac{(2(V + 1) \sin \psi)}{R_L} u + \frac{W}{R_L} v' - \frac{1}{R_L} v'' + p' = 0, \\
w \frac{R_L}{\alpha^2} + iU w \alpha + \left(M_{n,\gamma} + \frac{W''}{R_L}\right) w &- \left(\frac{2(V + 1) \cos \psi}{R_L}\right) v + \frac{W}{R_L} w' - \frac{1}{R_L} w'' + p' = 0,
\end{align*}
\]

where

\[
M_{n,\gamma} = \left(\nu + \frac{n^2}{R_L} - i\gamma\right),
\]

and the boundary conditions are given by

\[
\begin{align*}
u &= v = w = p' = 0, \quad \eta = 0, \\
u &= v = w = p' = 0, \quad \eta \to \infty.
\end{align*}
\]
Subsequently, in §3 we describe details of the convective stability analysis, which proceeds to solve the equations (1)–(4), subject to the boundary conditions (5) and (6), via the spectral method.

3. Spectral stability analysis

In order to solve the quadratic eigenvalue problem defined by the equations (1)–(4), we employ a spectral method based on the Chebyshev-tau technique. This technique has been used by [3] to solve for the spatial eigenvalues observed in rotating disk flow over a compliant wall, and previously by [19] for the temporal eigenvalues of the rotating disk boundary-layer. More recently, [1] have used the method to obtain the spatial eigenvalues for the BEK family of flows in the presence of surface roughness. The technique has been shown to achieve measurable gains in computational efficiency and accuracy in computing the neutral stability curves, for example when compared to related techniques, such as Gram-Schmidt orthonormalisation, and, in particular, the Orr–Sommerfeld solution method. Following the analysis of [3], we transform from the physical surface-normal coordinate $z \in [0, \infty)$ to the computational domain $y \in [-1, 1]$. Here, we provide a brief outline of the technique, which involves employing Chebyshev polynomial expansions in trigonometric form and parameterised by typically 100 Gauss-Lobatto collocation points. Full details of the discretisation and polynomials employed are omitted here but have been adapted from the method outlined by [1]. The technique involves obtaining the derivatives of the Chebyshev polynomials, as well as expansions of the velocity and pressure perturbations, with the boundary conditions (5) and (6) being imposed at $z = 0$ and $z = z_{\text{max}}$, where we use $z_{\text{max}} = 20$ on the computational domain.

Upon substituting the calculated expansions into the system of equations (1)–(4), we obtain the governing quadratic eigenvalue matrix equation for $\alpha$, which we solve using a QZ algorithm. We are able to iterate through the values of $\alpha$ for varying $R_L$ in order to identify the neutral points ($\alpha_i = 0$) and construct the curves of neutral stability for each $\psi$. Subsequently, these are calculated for $\alpha_r$ and $n$ for each $R_L$. Starting from $\psi = 90^\circ$, we employ a continuation technique of smoothly merging between neutral curves for small decrements in $\psi$, as we move towards $\psi = 20^\circ$.

Stationary modes

We proceed to analyse the stationary modes of instability, which correspond to $\gamma = 0$ in the perturbation equations and model vortices that remain fixed on the cone surface. The results are shown in tables 1 and 2, where we present the critical Reynolds number ($R_{L,c}$), the critical vortex wavenumbers in the streamwise ($\alpha_{r,c}$) and azimuthal ($n$) directions, as well as the vortex waveangle, $\phi = \pi/2 - \arctan(\alpha_r x/n)$. From table 1, we see that the type I critical values of $n$ and $\phi$ reduce as $\psi$ is decreased, with $\alpha_{r,c}$ increasing. Meanwhile, in general, $R_{L,c}$ exhibits a decreasing trend, following a slight initial increase. Consistent with [6], the results indicate that the type I mode is destabilised as $\psi$ is reduced. Interestingly, the type II mode undergoes a stabilisation for reducing $\psi$, and eventually disappears completely, in contrast to [6] where the type II mode was observed to destabilise approximately in parallel with the type I mode. However, while we observe the type I mode to be more dominant when compared with the type II mode, it is important to note that previous studies for the rotating disk [4, 13, 14, 15] have observed the type II mode experimentally. Furthermore, [22] has stated that these type II modes may correspond to a lower Reynolds number. Therefore, further investigation of the behaviour of the type II mode may reveal useful insights into the transition characteristics of such flows and should not be discounted when compared with the type I mode. Interestingly, we see good comparison at $\psi = 60^\circ$ with the results of [20], who conduct a weakly divergent stability analysis. From their results measured specifically at $n = 17$, we obtain an estimate of $R_{L,c} \approx 290$ as the location of
ψ  |  $R_{L,c}$  |  $a_{r,c}$  |  $n$  |  $\phi$  \\
---|---|---|---|---
90° | 286.07  |  0.38123  |  22  |  11.42°  \\
80° | 289.35  |  0.38519  |  22  |  11.16°  \\
70° | 288.41  |  0.38969  |  21  |  10.61°  \\
60° | 283.24  |  0.40221  |  20  |  9.73°   \\
50° | 273.83  |  0.42547  |  18  |  8.55°   \\
40° | 260.23  |  0.45525  |  15  |  7.14°   \\
30° | 242.49  |  0.51119  |  12  |  5.48°   \\
20° | 221.93  |  0.61181  |  9   |  3.65°   \\

Table 1. Critical parameters at the onset of the more dangerous type I mode for a range of cone half-angles $\psi$. The type I mode arises from a crossflow mechanism based on an inflexion point in the effective velocity profile and dominates for $\psi > 40°$.

ψ  |  $R_{L,c}$  |  $a_{r,c}$  |  $n$  |  $\phi$  \\
---|---|---|---|---
90° | 461.59  |  0.13123  |  21  |  19.32°  \\
80° | 478.52  |  0.13024  |  22  |  19.06°  \\
70° | 490.08  |  0.12953  |  21  |  18.36°  \\
60° | 497.49  |  0.12927  |  20  |  17.19°  \\
50° | 502.56  |  0.13776  |  19  |  15.03°  \\

Table 2. Critical parameters at the onset of the type II mode for a range of cone half-angles $\psi$. The viscous-dominated type II mode is based on zero effective shear at the wall and is less dangerous than the type I mode.

The most dangerous mode with maximum growth rate, which agrees reasonably with our type I mode estimate of $R_{L,c} = 283.24$ at $n \approx 20$. We note a small discrepancy due to a difference in the measured numbers of vortices.

Figures 2 and 3 show the curves of neutral-stability in the $R_L$–$\alpha_r$ and $R_L$–$n$ planes for rotating cones varying with $\psi = 20°$–$90°$. Both plots show that the rotating disk case $\psi = 90°$ is the most stable case, with the flow becoming more unstable for reducing $\psi$.

We observe the gradual disappearance of the type II mode below $\psi = 40°$ is illustrated graphically on both lower branches. As the type II mode arises from a viscous-Coriolis balance, pertaining to zero effective shear at the wall, we hypothesise that its disappearance may be related to the emergence of the centrifugal instability modelled in [9] as the dominant instability mechanism. This transition essentially acts to suppress the type II crossflow mode, leading to its disappearance from the neutral stability curve. In fact, it appears that the general behaviour of the type II mode is somewhat affected by the solution method. In particular, the type II critical value varies by about 10 units in $R_L$ between the current approach and the Gram–Schmidt orthonormalisation technique detailed in [6]. Such complex behaviour of the type II mode has been observed previously by [3] for the rotating disk boundary-layer flow with a compliant wall. We suspect that the use of primitive variables and the more precise spectral method yields more accurate results when compared with those of [6].

4. Comparisons between spectral analysis and the literature

In this section, we develop comparisons between the results of the spectral analysis presented in §3 and the existing literature. In particular, we focus on the critical Reynolds number, $R_{L,c}$ and new findings pertaining to the spectral analysis approach of solving for the type I and II modes, as well as providing discussion and explanation of the results, where possible.

Figure 4 displays the critical Reynolds numbers $R_{L,c}$ for a range of $\psi$. Although some
Figure 2. Neutral stability curves for rotating cones of $\psi = 20^\circ$ (left-most curve) to $\psi = 90^\circ$ (lower-most curve) in 10° increments for the spectral method in the $R_L-\alpha_r$ plane. Type I and type II modes are represented by the upper left and lower right lobes, respectively. We observe the more dangerous type I modes to have lower critical Reynolds numbers than the viscous type II modes, which disappear for $\psi < 50^\circ$.

Figure 3. Neutral stability curves for rotating cones of $\psi = 20^\circ$ (left-most curve) to $\psi = 90^\circ$ (upper-most curve) in 10° increments for the spectral method in the $R_L-n$ plane. Type I and type II modes are represented by the left and right lobes, respectively. We observe the more dangerous type I modes to have lower critical Reynolds numbers than the viscous type II modes, which disappear for $\psi < 50^\circ$. 
Figure 4. Comparison of the critical Reynolds number $R_{L,c}$ for the onset of each instability mode with the findings of [6], [9] and with experimental results in the literature, updated from [6]. We note that there exists a discrepancy between the current results and the experiments for small $\psi$, which is accounted for by an alternative mechanism that dominates in this region. This centrifugal mode is determined mathematically in [9].

Numerical differences exist between the spectral method and the Runge–Kutta integration scheme with Gram–Schmidt orthonormalisation used in [6], the qualitative behaviour of the type I mode remains mostly unchanged. Note that we have used the Reynolds number based on boundary-layer thickness at the location of the stability analysis, $R_L$, as opposed to the experimental Reynolds number ($R^2_L$) used for comparisons in [6], which helps to avoid magnifying any discrepancies unnecessarily. In terms of the results, the type I mode exhibits a gradual destabilisation as $\psi$ is decreased, which is consistent with [6] and the experimental findings of [17] and more recently [16] for $\psi = 60^\circ$, both observing the type I mode for $\psi > 40^\circ$. As discussed in [6] and [9], the comparisons diverge as $\psi$ is reduced owing to the change in dominant instability mechanism from one of crossflow type to centrifugal, which appears to occur at around $\psi = 40^\circ$. Additionally, both the current type I predictions and those calculated in [6], as well as the experimental type I measurements of [17], disagree with the experimental results of [18], which is highlighted in [6] and maybe due to the experimental technique used.

5. Conclusion

In this paper, we have provided new insights into the boundary-layer flow over a rotating cone in still fluid. We have applied a spectral numerical solution technique to solve the crossflow stability problem with improved accuracy and stability when compared with the prior technique of [6]. The results have provided further support to our existing hypothesis posed in [6] and [9] that the rotating cone boundary-layer undergoes a change in dominant instability as $\psi$ is reduced from a crossflow mode to a centrifugal mode, with the crossover critical angle still in the region of $\psi_c = 40^\circ$, as predicted in [9].
Furthermore, the behaviour of the type II mode was found to be qualitatively different to that observed previously in that it undergoes a slight stabilisation before disappearing completely below $\psi = 50^\circ$. The behaviour is consistent with a weakening of the crossflow instability as $\psi$ is decreased. As this crossflow mechanism weakens, we hypothesise that the alternative centrifugal instability mode will in turn strengthen.

Lastly, efforts are currently underway to investigate the applicability of the new technique to the more general and complex problems of broad and slender rotating cones within axial flow. While progress has been made on the linear stability problems, which exhibit the strong interaction of a number of competing instabilities, by [5], [7], [10] and [11], the complete neutral-stability curves for each $\psi$ in a wide range of axial flows remain a significant milestone in this geometry. However, the results presented in this study indicate that the spectral approach successfully models the full disturbance equations, including three-dimensionality, viscous streamline-curvature and Coriolis effects for the still fluid problem. Hence, it has proven to be a strong candidate to achieve the accuracy required in the more complex axial flow case.

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