On $q$- and $h$-deformations of 3d-superspaces

SALİH ÇELİK∗

Department of Mathematics, Yıldız Technical University, İstanbul, Turkey

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Abstract: In this paper, we introduce nonstandard deformations of (1+2)- and (2+1)-superspaces via a contraction using standard deformations of them. This deformed superspaces are denoted by $A_{h}^{1|2}$ and $A_{h'}^{2|1}$, respectively. We find a two-parameter $R$-matrix satisfying quantum Yang–Baxter equation and thus obtain a new two-parameter nonstandard deformation of the supergroup $GL(1|2)$. Finally, we get a new superalgebra derived from the Hopf superalgebra of functions on the quantum superspace $A_{p,q}^{1|2}$.

Key words: Quantum superspace, Hopf superalgebra, quantum supergroup, quantum Lie superalgebra, super $\star$-algebra

1. Introduction

There are two distinct deformations for general Lie (super)groups as standard and nonstandard (or Jordanian). One of them is the well-known quantum ($q$-deformed) group and the other is the so-called Jordanian ($h$-deformed) one. Specially, quantum groups $GL_q(2)$ [10] and $GL_h(2)$ [9] have been obtained by deforming the coordinates of a plane to be noncommutative objects. In [1], the authors showed that the $h$-deformed group can be obtained from the $q$-deformed Lie group through a singular limit $q \to 1$ of a linear transformation. This method is known as the contraction procedure. Using this method, one- and two-parameter $h$-deformations of supergroup $GL(1|1)$ were obtained in [7] and [2], respectively.

In this paper, we give some standard (as $q$-deformation) deformations of (1+2)-superspace using the Hopf superalgebra structure of $O(A^{1|2})$ and nonstandard (as $h$-deformation) deformations using standard deformations via a contraction. We also introduce an $(h, h')$-deformed supergroup acting on these two-parameter $h$-deformed superspaces. Finally, we define involutions on $h$-deformed superspaces and use the generators of $(p, q)$-deformed superalgebra $O(A_{p,q}^{1|2})$ to get a new Lie superalgebra.

Throughout the paper, we will fix a base field $K$. The reader may consider it as the set of real numbers, $R$, or the set of complex numbers, $C$. We will denote by $G$ the Grassmann numbers and by $K'$ the set $K \cup G$.

2. On $(p, q)$-deformation of superspaces $A_{h}^{1|2}$ and $A_{h'}^{2|1}$

In order to define superalgebras and Hopf superalgebras, some minor changes are made in familiar definitions. These are briefly mentioned in the following.

A supervector space $\mathcal{X}$ over a field $K$ is a $\mathbb{Z}_2$-graded vector space $\mathcal{X}$ together with two subspaces $\mathcal{X}_0$ and $\mathcal{X}_1$ of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$. If a space $\mathcal{X}$ is a superspace, then we denote by $\tau(a)$ the $\mathbb{Z}_2$-grade of

∗Correspondence: sacelik@yildiz.edu.tr
the element \( a \in X \). If \( \tau(a) = 0 \), then we will call the element \( a \) even and if \( \tau(a) = 1 \), it is called odd.

If \( f : X \rightarrow Y \) is a linear map of supervector spaces and it satisfies
\[
\tau(f(v)) = \tau(f) + \tau(v) \pmod{2}
\]
for all \( v \in X \), then \( f \) is called a supervector space homomorphism.

A superalgebra (or \( \mathbb{Z}_2 \)-graded algebra) \( A \) over \( K \) is a supervector space over \( K \) with a map \( A \times A \rightarrow A \) such that \( A_i \cdot A_j \subseteq A_{i+j} \) for \( i, j = 0, 1 \). The superalgebra \( A \) is called supercommutative if
\[
ab = (-1)^{\tau(a)\tau(b)}ba
\]
for homogeneous elements \( a, b \in A \).

Let \( f : A \rightarrow B \) be a map of definite degree of superalgebras. If it is a supervector space homomorphism and it obeys
\[
f(ab) = (-1)^{\tau(a)\tau(f)}f(a)f(b), \quad \forall a, b \in A,
\]
then \( f \) is called a superalgebra homomorphism.

2.1. The algebra of polynomials on the quantum superspace \( A_{q}^{1|2} \)

Let \( \mathbb{K} \langle X, \Theta_1, \Theta_2 \rangle \) be a free algebra with unit generated by \( X, \Theta_1, \) and \( \Theta_2 \), where the coordinate \( X \) is even and the coordinates \( \Theta_1 \) and \( \Theta_2 \) are odd.

**Definition 2.1** [11] Let \( I_q \) be the two-sided ideal of \( \mathbb{K} \langle X, \Theta_1, \Theta_2 \rangle \) generated by the elements \( X\Theta_1 - q\Theta_1X, X\Theta_2 - q\Theta_2X, \Theta_1\Theta_2 + q^{-1}\Theta_2\Theta_1, \Theta_1^2, \) and \( \Theta_2^2 \). The quantum superspace \( A_{q}^{1|2} \) with the function algebra
\[
\mathcal{O}(A_{q}^{1|2}) = \mathbb{K} \langle X, \Theta_1, \Theta_2 \rangle / I_q
\]
is called \( \mathbb{Z}_2 \)-graded quantum space (or quantum superspace).

This associative algebra over the complex number is known as the algebra of polynomials over quantum \((1+2)\)-superspace. In accordance with the above definition, we have
\[
X \Theta_i = q \Theta_i X, \quad \Theta_i \Theta_j = -q^{-j} \Theta_j \Theta_i, \quad (i, j = 1, 2)
\]
(2.1)
where \( q \in \mathbb{K} - \{0\} \).

**Example 2.2** If we consider the generators of the algebra \( \mathcal{O}(A_{q}^{1|2}) \) as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation \( \rho : \mathcal{O}(A_{q}^{1|2}) \rightarrow M(3, \mathbb{K}') \) such that matrices
\[
\rho(X) = \begin{pmatrix} q & 0 & 0 \\
0 & q & 0 \\
0 & 0 & q^2 \end{pmatrix}, \quad \rho(\Theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad \rho(\Theta_2) = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & \varepsilon_2 \\
0 & 0 & 0 \end{pmatrix}
\]
(2.2)
representing the coordinate functions satisfy relations (2.1) for all \( \varepsilon_1, \varepsilon_2 \).

**Remark 2.3** In the next section, we will assume that \( \varepsilon_1 \) and \( \varepsilon_2 \) are two Grassmann numbers.
The following definition gives the product rule for tensor product of $\mathbb{Z}_2$-graded algebras.

**Definition 2.4** The product rule is defined by

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = (-1)^{\tau(a_2)\tau(a_3)}(a_1a_3 \otimes a_2a_4)$$

in the $\mathbb{Z}_2$-graded algebra $A \otimes A$, where $A$ is the $\mathbb{Z}_2$-graded algebra and $a_i$'s are homogeneous elements in $A$.

A Hopf superalgebra is a supervector space $A$ over $K$ with two algebra homomorphisms $\Delta : A \to A \otimes A$, called the coproduct, $\epsilon : A \to K$, called the counit, and an algebra antihomomorphism $S : A \to A$, called the antipode, such that

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,$$

$$m \circ (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = m \circ (\text{id} \otimes \epsilon) \circ \Delta,$$

$$m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\text{id} \otimes S) \circ \Delta,$$

and $\Delta(1) = 1 \otimes 1$, $\epsilon(1) = 1$, $S(1) = 1$, where $m$ is the multiplication map, $\text{id}$ is the identity map and $\eta : K \to A$.

**Note.** An element of a Hopf superalgebra $A$ is expressed as a product on the generators and its antipode $S$ is calculated with the property

$$S(ab) = (-1)^{\tau(a)\tau(b)}S(b)S(a), \quad \forall a, b \in A.$$

We denote the unital extension of $O(\mathbb{A}_q^{1\,2})$ by $F(\mathbb{A}_q^{1\,2})$ adding the unit and $x^{-1}$, the inverse of $x$, which obeys $xx^{-1} = 1 = x^{-1}x$. The following theorem says that the superalgebra $F(\mathbb{A}_q^{1\,2})$ has a Hopf algebra structure [4]:

**Theorem 2.5** [4] The superalgebra $F(\mathbb{A}_q^{1\,2})$ is a Hopf superalgebra with the defining coproduct, counit, and antipode on the algebra $F(\mathbb{A}_q^{1\,2})$ as follows:

1. The coproduct $\Delta : F(\mathbb{A}_q^{1\,2}) \to F(\mathbb{A}_q^{1\,2}) \otimes F(\mathbb{A}_q^{1\,2})$ is defined by

$$\Delta(X) = X \otimes X, \quad \Delta(\Theta_1) = \Theta_1 \otimes X + X \otimes \Theta_1, \quad \Delta(\Theta_2) = \Theta_2 \otimes X^2 + X^2 \otimes \Theta_2. \quad (2.3)$$

2. The counit $\epsilon : F(\mathbb{A}_q^{1\,2}) \to K$ is given by

$$\epsilon(X) = 1, \quad \epsilon(\Theta_i) = 0, \quad (i = 1, 2).$$

3. The algebra $F(\mathbb{A}_q^{1\,2})$ admits a $K$-algebra antihomomorphism (antipode) $S : F(\mathbb{A}_q^{1\,2}) \to F(\mathbb{A}_{q^{-1}}^{1\,2})$ defined by

$$S(X) = X^{-1}, \quad S(\Theta_1) = -X^{-1}\Theta_1X^{-1}, \quad S(\Theta_2) = -X^{-2}\Theta_2X^{-2}.$$
Definition 2.6 [5] Let $\Lambda(A_q^{1|2})$ be the algebra with the generators $\Phi$, $Y_1$, and $Y_2$ satisfying the relations

$$\Phi^2 = 0, \quad \Phi Y_1 = qp^{-1}Y_1 \Phi, \quad \Phi Y_2 = pq Y_2 \Phi, \quad Y_1 Y_2 = pq^{-1} Y_2 Y_1.$$  \hfill (2.4)

We call $\Lambda(A_q^{1|2})$ exterior algebra of the $\mathbb{Z}_2$-graded space $A_q^{1|2}$.

Remark 2.7 The exterior algebra $\Lambda(A_q^{1|2})$ of the superspace $A_q^{1|2}$ can be thought of as a two-parameter deformation of the $(2+1)$-superspace $A^{2|1}$. Thus, we denote this algebra by $\mathcal{O}(A_p^{2|1})$.

Example 2.8 If we consider the generators of the algebra $\mathcal{O}(A_p^{2|1})$ as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation $\rho: \mathcal{O}(A_p^{2|1}) \to M(3, K')$ such that matrices

$$\rho(\Phi) = \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(Y_1) = \begin{pmatrix} q & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \quad \rho(Y_2) = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

representing the coordinate functions satisfy relations (2.4) for all $c, \epsilon$.

3. Two-parameter $h$-deformation of the superspaces

In this section, we introduce a two-parameter $h$-deformation of the superspace $A^{1|2}$ (and its dual) from the $(p, q)$-deformation via a contraction similar to the method of [1].

We consider the $q$-deformed algebra of functions on the quantum superspace $A_q^{1|2}$ generated by $X$, $\Theta_1$, and $\Theta_2$ with the relations (2.1) and we introduce new even coordinate $x$ and odd coordinates $\theta_1$, $\theta_2$ with the change of basis in the coordinates of the $q$-superspace using the following $g$ matrix:

$$X = \begin{pmatrix} X \\ \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tilde{h}' \\ 0 & 1 & 0 \\ \tilde{h} & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \theta_1 \\ \theta_2 \end{pmatrix} = g x, \quad \tilde{h} = \frac{h}{q-1}, \quad p, q \

where $h$ and $h'$ ($h \neq 0 \neq h'$) are two new deformation parameters that will be replaced with $q$ and $p$ ($q \neq 1 \neq pq$) in the limits $q \to 1$ and $p \to 1$.

We now assume that the parameters $h$ and $h'$ are both Grassmann numbers ($h^2 = 0 = h'^2$, $hh' = -h'h$) and anticommute with $\theta_i$ for $i = 1, 2$. When the relations (2.1) are used, one gets

$$x \theta_1 = q \theta_1 x, \quad x \theta_2 = q \theta_2 x + hx^2, \quad \theta_2 \theta_1 = -q \theta_1 \theta_2, \quad \theta_1^2 = 0, \quad \theta_2^2 = -h \theta_2 x.$$  \hfill (3.2)

Note that the parameter $h'$ does not enter the above relations. By taking the limit $q \to 1$, we obtain the following exchange relations, which define the $h$-superspace $A_h^{1|2}$:

Definition 3.1 [4] Let $\mathcal{O}(A_h^{1|2})$ be the algebra with the generators $x$, $\theta_1$, and $\theta_2$ satisfying the relations

$$x \theta_1 = \theta_1 x, \quad x \theta_2 = \theta_2 x + hx^2, \quad \theta_1 \theta_2 = -\theta_2 \theta_1, \quad \theta_1^2 = 0, \quad \theta_2^2 = -h \theta_2 x.$$  \hfill (3.3)

We call $\mathcal{O}(A_h^{1|2})$ the algebra of functions on the $\mathbb{Z}_2$-graded quantum space $A_h^{1|2}$.
Example 3.2 Let us assume that $\varepsilon_1$ and $\varepsilon_2$ are two Grassmann numbers. If the $g$ matrix in (3.1) is used, the matrix representation in (2.2) takes the following form:

$$\rho(x) = q \begin{pmatrix} 1 - \tilde{h}h' & 0 & 0 \\ 0 & 1 - \tilde{h}h' & -q^{-1}\tilde{h}'\varepsilon_2 \\ 0 & 0 & q(1 - \tilde{h}h') \end{pmatrix}, \quad \rho(\theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\theta_2) = -\begin{pmatrix} q\tilde{h} & 0 & 0 \\ 0 & q\tilde{h} & -(1 + \tilde{h}h')\varepsilon_2 \\ 0 & 0 & q^2\tilde{h} \end{pmatrix}.$$ (3.4)

These matrices satisfy the relations (3.2), for all $\varepsilon_1$ and $\varepsilon_2$.

Proof Existing claims come from the fact that $\rho$ is an algebra homomorphism. \hfill \Box

In the case of dual (exterior) $h'$-superspace, we use the transformation

$$\hat{X} = g\hat{x}$$ (3.5)

with the components $\varphi$, $y_1$, and $y_2$ of $\hat{x}$. The definition is given below.

Definition 3.3 Let $\mathcal{O}(\mathbb{A}_h^{2|1}) := \Lambda(\mathbb{A}_h^{1|2})$ be the algebra with the generators $\varphi$, $y_1$, and $y_2$ satisfying the relations

$$\varphi y_1 = y_1\varphi, \quad \varphi y_2 = y_2\varphi + h'y_2^2, \quad y_1 y_2 = y_2 y_1, \quad \varphi^2 = h'y_2\varphi$$ (3.6)

where $\tau(\varphi) = 1$ and $\tau(y_1) = 0 = \tau(y_2)$. We call $\Lambda(\mathbb{A}_h^{1|2})$ the quantum exterior algebra of the $\mathbb{Z}_2$-graded quantum space $\mathbb{A}_h^{1|2}$.

Remark 3.4 The parameter $h$ does not enter the relations (3.6). The exterior algebra $\Lambda(\mathbb{A}_h^{1|2})$ of the superspace $\mathbb{A}_h^{1|2}$ can be thought of as an $h'$-deformation of the (2+1)-superspace $\mathbb{A}_{2|1}$.

4. An $R$-matrix and its properties

The relations in (2.1) can be written in a compact form as follows:

$$pX \otimes X = \hat{R}_{p,q} X \otimes X$$ (4.1)

with an $R$-matrix given by [6]

$$\hat{R}_{p,q} = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p-1 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & pq & 0 & 0 \\ 0 & pq^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -pq^{-1} & 0 \\ 0 & q^{-1} & 0 & 0 & 0 & p-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 0 & p-1 & 0 \end{pmatrix}$$

where $p, q \in K - \{0\}$. This matrix satisfies the graded braid equation and the matrix $R_{p,q} = P\hat{R}_{p,q}$ satisfies the graded Yang–Baxter equation where $P$ is the super permutation matrix.
It can be considered that a change of basis in the quantum superspaces leads to a two-parameter $R$-matrix. The corresponding $R$-matrix can be obtained as
\[
\hat{R}_{h,h'} = \lim_{(p,q) \to (1,1)} \left[ (g \otimes g)^{-1} \hat{R}_{p,q} (g \otimes g) \right]
\]
where it is assumed that $\otimes$ is graded. As a result, we obtain the following $R$-matrix
\[
\hat{R}_{h,h'} = \begin{pmatrix}
1 + hh' & 0 & h' & 0 & 0 & 0 & -h' & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
h & 0 & hh' & 0 & 0 & 0 & 1 & 0 & -h' \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-h & 0 & 1 & 0 & 0 & 0 & hh' & 0 & -h' \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -h & 0 & 0 & 0 & -h & 0 & hh' - 1
\end{pmatrix}.
\]

The equation in (4.1) with the new $R$-matrix $\hat{R}_{h,h'}$ takes the form
\[
x \otimes x = \hat{R}_{h,h'} x \otimes x,
\]
that is, the relations (3.3) are equivalent to this equation.

The $R$-matrix $\hat{R}_{h,h'}$ has some interesting properties. Some of them are listed below, where sometimes we write $\hat{R} = \hat{R}_{h,h'}$ for simplicity.

1. The matrix $\hat{R}_{h,h'}$ satisfies the graded braid equation $\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}$, where $\hat{R}_{12} = \hat{R} \otimes I_3$ and $\hat{R}_{12} = I_3 \otimes \hat{R}$.

2. The matrix $R_{h,h'} = P \hat{R}_{h,h'}$ satisfies the graded Yang–Baxter equation
\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},
\]
where $R_{13}$ acts both on the first and third spaces.

3. The matrix $\hat{R}_{h,h'}$ holds $\hat{R}_{h,h'}^2 = I_9$; thus, it has two eigenvalues $\pm 1$.

4. If we set $hh' = 0$, then the matrix $R_{h,h'}$ can be decomposed in the form
\[
R_{h,h'} = R(h) R(h')
\]
where
\[
R(h) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-h & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
h & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & h & 0 & 0 & 0 & h & 0 & 1
\end{pmatrix}, \quad R(h') = R^* (h) |_{h = h'}.
\]

It can be checked that these matrices satisfy the graded Yang–Baxter equation.
5. If $P_{\pm}$ are the projections onto the eigenspaces $\pm 1$ of $\hat{R}_{h,h'}$, then we have

$$\hat{R}_{h,h'} = P_+ - P_-.$$ 

Let $\mathcal{O}(\mathbb{A}_h^{1|2})$ and $\mathcal{O}(\mathbb{A}_h^{2|1})$ be the quotients of algebras generated by $x, \theta_1, \theta_2$ and $\varphi, y_1, y_2$ modulo the two-sided ideals generated by $\text{Ker}P_-$ and $\text{Ker}P_+$, respectively. Then $\mathcal{O}(\mathbb{A}_h^{1|2})$ and $\mathcal{O}(\mathbb{A}_h^{2|1})$ are isomorphic to $\mathcal{O}(\mathbb{A}_h^{1|2})$ with defining relations (3.3) and $\mathcal{O}(\mathbb{A}_h^{2|1})$ with defining relations (3.6), respectively. That is, we can write

$$P_- x \otimes x = 0 \quad \text{and} \quad (-1)^{\tau(x)} P_+ \hat{x} \otimes \hat{x} = 0.$$

5. The quantum superbialgebra $\mathcal{O}(M_{h,h'}(1|2))$

Let $T$ be a 3x3 matrix in $\mathbb{Z}_2$-graded space given by

$$T = \begin{pmatrix} a & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix} = (t_{ij})$$

where $a, b, c, d, e$ are even and $\alpha, \beta, \gamma, \delta$ are odd. The coordinate ring of such matrices over a field $\mathbb{K}$ is simply the polynomial ring in nine variables, that is $\mathcal{O}(\mathbb{M}(1|2)) = \mathbb{K}[a, b, c, d, e, \alpha, \beta, \gamma, \delta]$.

In this section, we will assume that the matrix entries of $T$ belong to a free superalgebra and define a two-parameter $h$-analogue of $\mathcal{O}(\mathbb{M}(1|2))$. To do so, let $x, \theta_1, \theta_2$ be elements of the superalgebra $\mathcal{O}(\mathbb{A}_h^{1|2})$ subject to the relations (3.3) and $\varphi, y_1, y_2$ be elements of $\mathcal{O}(\mathbb{A}_h^{2|1})$ subject to the relations (3.6), and $t_{ij}$ be nine generators which supercommute with the elements of $\mathcal{O}(\mathbb{A}_h^{1|2})$ and $\mathcal{O}(\mathbb{A}_h^{2|1})$. It is well known that the supermatrix $T$ defines the linear transformations $T : \mathbb{A}_h^{1|2} \rightarrow \mathbb{A}_h^{1|2}$ and $T : \mathbb{A}_h^{2|1} \rightarrow \mathbb{A}_h^{2|1}$. Let $x = (x, \theta_1, \theta_2)^t$ and $\hat{x} = (\varphi, y_1, y_2)^t$. Thus, we can give the following theorem.

**Theorem 5.1** Under the above hypotheses, the following conditions are equivalent:

(i) $Tx = x' \in \mathbb{A}_h^{1|2}$ and $T\hat{x} = \hat{x}' \in \mathbb{A}_h^{2|1}$,
(ii) the relations are satisfied

\[
\begin{align*}
  a\alpha &= (1 + hh')\alpha a - h'(a\delta + da), \\
  a\beta &= \beta a + h'(a^2 - ea - \beta\delta) - h\beta^2, \\
  a\gamma &= (1 + hh')\gamma a + h(\gamma\beta - ea), \\
  ac &= ca - hc\beta - h'\gamma a + hh'\gamma\beta, \\
  a\delta &= \delta a + h(a^2 - ea + \delta\beta) + h'\delta^2, \\
  ad &= da + haoa + h'd\delta - hh'\alpha\delta, \\
  ae &= ea + h\beta(a - e) + h'(e - a)\delta, \\
  a\alpha\beta &= -(1 + hh')\beta\alpha + h'(\beta d + ea), \\
  a\alpha\gamma &= -\gamma\alpha, \\
  ac &= ca, \\
  a\delta &= -\delta a - hao + h'\delta d - hh'ad, \\
  ad &= da + h'd^2, \\
  ae &= ea + h\beta\alpha + h'ed - hh'd\beta, \\
  \beta\gamma &= -\gamma\beta + hc\beta - h'\gamma a - hh'ca, \\
  \beta c &= (1 - hh')c\beta - h'(\gamma\beta + ca), \\
  \beta\delta &= -\delta\beta + (h\beta + h'\delta)(e - a), \\
  \beta d &= d\beta + h\alpha\beta + h'de - hh'ea, \\
  \beta e &= e\beta + h'(e^2 - ea - \delta\beta) - h\beta^2, \\
  \gamma c &= c\gamma + he^2, \\
  \gamma\delta &= -(1 + hh')\delta\gamma + h(e\gamma + \delta c), \\
  \gamma d &= d\gamma, \\
  \gamma e &= e\gamma + hec - h'\delta\gamma - hh'\epsilon, \\
  cd &= dc, \\
  ce &= (1 - hh')ec + h'(e\gamma - \delta c), \\
  \delta d &= (1 - hh')d\delta + h(\alpha\delta - da), \\
  \delta e &= e\delta + h(e^2 - ea + \beta\delta) + h'\delta^2, \\
  de &= (1 - hh')ed + h(\beta d - ea), \\
  \alpha^2 &= h'\alpha d, \\
  \beta^2 &= h'\beta(e - a), \\
  \gamma^2 &= h\gamma c, \\
  \delta^2 &= h\delta(e - a), \\
  b t_{ij} &= t_{ij}b, \\
  a(h\beta + h'\delta) &= (h\beta + h'\delta)a, \\
  e(h\beta + h'\delta) &= (h\beta + h'\delta)e.
\end{align*}
\]  

(5.1)

**Proof** A direct verification shows that the relations (5.1) respect the ideals defining \(A_h^{1|2}\) and \(A_{h'}^{2|1}\). □

Standard FRT construction [8], namely, the relations (5.1), is obtained via the matrix \(\hat{R}_{h,h'}\) given in Section 4:

**Theorem 5.2** A 3x3-matrix \(T\) is a \(\mathbb{Z}_2\)-graded quantum supermatrix if and only if

\[
\hat{R}_{h,h'}T_1T_2 = T_1T_2\hat{R}_{h,h'}
\]

where \(T_1 = T \otimes I_3\) and \(T_2 = PT_1P\).

**Definition 5.3** The superalgebra \(\mathcal{O}(M_{h,h'}(1|2))\) is the quotient of the free algebra \(\mathbb{K}\{a,b,c,d,e,\alpha,\beta,\gamma,\delta\}\) by the two-sided ideal \(J_{h,h'}\) generated by the relations (5.1) of Theorem 5.1.

**Remark 5.4** The quantum matrix space \(M_{p,q}(1|2)\) is obtained in [6]. It is clear that a change of basis in the quantum superspace leads to the similarity transformation \(T = g^{-1}T'g\), where \(T' \in M_{p,q}(1|2)\). Therefore, the entries of the transformed quantum matrix \(T\) fulfill the commutation relations (5.1) of the matrix elements of the matrix \(T\) in \(M(1|2)\).

**Theorem 5.5** The superalgebra \(\mathcal{O}(M_{h,h'}(1|2))\) with the following two algebra homomorphisms of superalgebras (1) the coproduct \(\Delta : \mathcal{O}(M_{h,h'}(1|2)) \rightarrow \mathcal{O}(M_{h,h'}(1|2)) \otimes \mathcal{O}(M_{h,h'}(1|2))\) determined by \(\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj}\),
(2) the counit $\epsilon : \mathcal{O}(M_{h,h'}(1|2)) \rightarrow \mathbb{K}$ determined by $\epsilon(t_{ij}) = \delta_{ij}$ becomes a super bialgebra.

**Proof** It can be easily checked the properties of the costructures hold:
(i) The coproduct $\Delta$ is coassociative in the sense of

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$$

where id denotes the identity map on $M_{h,h'}(1|2)$ and $\Delta(ab) = \Delta(a)\Delta(b)$, $\Delta(1) = 1 \otimes 1$.

(ii) The counit $\epsilon$ has the property

$$m \circ (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = m \circ (\text{id} \otimes \epsilon) \circ \Delta$$

where $m$ stands for the algebra product and $\epsilon(ab) = \epsilon(a)\epsilon(b)$, $\epsilon(1) = 1$. □

It is well known that $\mathcal{O}(A_{1|2})$ is comodule algebra over the bialgebra $\mathcal{O}(M(1|2))$. The following theorem gives a quantum version of this fact.

**Theorem 5.6** There exist algebra homomorphisms

$$\delta_L : \mathcal{O}(A_{h|1}) \rightarrow \mathcal{O}(M_{h,h'}(1|2)) \otimes \mathcal{O}(A_{h|1}), \quad \delta_L(x_i) = \sum_{k=1}^{3} t_{ik} \otimes x_k,$$

$$\tilde{\delta}_L : \mathcal{O}(A_{h'}^{2|1}) \rightarrow \mathcal{O}(M_{h,h'}(1|2)) \otimes \mathcal{O}(A_{h'}^{2|1}), \quad \tilde{\delta}_L(\hat{x}_i) = \sum_{k=1}^{3} t_{ik} \otimes \hat{x}_k$$

where $x_i \in \{x, \theta_1, \theta_2\}$ and $\hat{x}_i \in \{\varphi, y_1, y_2\}$.

**Proof** Using the relations (3.3) and (3.6) together with (5.1), it is enough to check that

$$\delta_L(x_1\theta_1 - \theta_1 x) = \delta_L(x)\delta_L(\theta_1) - \delta_L(\theta_1)\delta_L(x) = 0,$$

etc., in $\mathcal{O}(M_{h,h'}(1|2)) \otimes \mathcal{O}(A_{h|1})$. To see that $\delta_L$ defines a comodule structure we check that

$$(\Delta \otimes \text{id}) \circ \delta_L = (\text{id} \otimes \delta_L) \circ \Delta, \quad m \circ (\epsilon \otimes \text{id}) \circ \delta_L = \text{id}.$$ □

A quantum supergroup (Hopf superalgebra) can be regarded as a generalization of the notion of a supergroup. It is defined by

$$\mathcal{O}(GL_{h,h'}(1|2)) = \mathcal{O}(M_{h,h'}(1|2))[t]/(ts\det_{h,h'} - 1).$$

This case is also inviting to generalize the corresponding notions of differential geometry [12]. A differential calculus on $\mathcal{O}(GL_{h,h'}(1|2))$ will be discussed in the next work.

6. A Lie superalgebra derived from $\mathcal{F}(A_{p,q}^{1|1})$

It is known that an element of a Lie group can be represented by exponential of an element of its Lie algebra. In [3], by virtue of this fact, using the generators of the superalgebra $\mathcal{F}(A_{p,q}^{1|1})$, a new superalgebra is obtained
from this algebra. In this section, we will obtain a new superalgebra from \( F(A_{p,q}^{1/2}) \). Thus, let us begin with the definition of \( F(A_{p,q}^{1/2}) \) which is an extension to two parameters of \( F(A_{q}^{1/2}) \).

**Definition 6.1** Let \( I_{p,q} \) be the two-sided ideal of \( K\langle X, \Theta_1, \Theta_2 \rangle \) generated by the elements \( X\Theta_1 - q\Theta_1X, X\Theta_2 - p\Theta_2X, \Theta_1\Theta_2 + pq^{-2}\Theta_2\Theta_1, \Theta_1^2, \) and \( \Theta_2^2 \). The quantum superspace \( A_{p,q}^{1/2} \) with the function algebra

\[
O(A_{p,q}^{1/2}) = K\langle X, \Theta_1, \Theta_2 \rangle / I_{p,q}
\]

is called quantum superspace.

In accordance with this definition, we have

\[
X\Theta_1 = q\Theta_1X, \quad X\Theta_2 = p\Theta_2X, \quad \Theta_1\Theta_2 = -pq^{-2}\Theta_2\Theta_1, \quad \Theta_1^2 = 0, \quad \Theta_2^2 = 0
\]

(6.1)

where \( p,q \in K \setminus \{0\} \).

**Example 6.2** If we consider the generators of the algebra \( O(A_{p,q}^{1/2}) \) as linear maps, then we can find the matrix representations of them. In fact, it can be seen that there exists a representation \( \rho : O(A_{p,q}^{1/2}) \to M(3, K') \) such that matrices

\[
\rho(X) = \begin{pmatrix} q & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & pq \end{pmatrix}, \quad \rho(\Theta_1) = \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(\Theta_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \\ 0 & 0 & 0 \end{pmatrix}
\]

representing the coordinate functions satisfy relations (6.1) for all \( \varepsilon_1, \varepsilon_2 \).

Let \( K\langle u, \xi_1, \xi_2 \rangle \) be a free algebra generated by \( u, \xi_1, \xi_2 \), where \( \tau(u) = 0, \tau(\xi_1) = 1 = \tau(\xi_2) \). Let \( L \) be the quotient of the free algebra \( K\langle u, \xi_1, \xi_2 \rangle \) by the two-sided ideal \( J_0 \) generated by the elements \( u\xi_k - \xi_ku \), \( \xi_1\xi_2 + \xi_2\xi_1, \xi_k^2 \) for \( k = 1, 2 \).

Now, we will show that the Hopf superalgebra of Theorem 2.5 can be embedded into the enveloping superalgebra of a Lie superalgebra, with Lie structure and a deformed coproduct. Thus, let us define the generators of the algebra \( F(A_{p,q}^{1/2}) \) as

\[
X := e^u, \quad \Theta_k := e^{\xi_k}u_k
\]

for \( k = 1, 2 \). The first equality implies that the generator \( X \) is invertible. Then, by direct calculations we can prove the following lemma.

**Lemma 6.3** The generators \( u, \xi_1, \xi_2 \) have the following commutation relations (Lie (anti-)brackets), for \( j, k = 1, 2 \)

\[
[u, \xi_k] = i\hbar_k \xi_k, \quad [\xi_j, \xi_k]_+ = 0, \quad (6.2)
\]

where \( q = e^{i\hbar_1}, \quad p = e^{i\hbar_2} \) with \( i = \sqrt{-1} \) and \( \hbar_1, \hbar_2 \in \mathbb{R} \).

We denote the algebra for which the generators obey the relations (6.2) by \( L_{\hbar_1,\hbar_2} := L(A_{p,q}^{1/2}) \). Let \( U(L_{\hbar_1,\hbar_2}) \) be the algebra defined by (6.2). The Hopf superalgebra structure of \( U(L_{\hbar_1,\hbar_2}) \) can be read off from Theorem 2.5.

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Theorem 6.4 The superalgebra $U(\mathcal{L}_{h_1,h_2})$ is a Hopf superalgebra with coproduct, counit, and antipode on the algebra $\mathcal{L}_{h_1,h_2}$ defined by
\[
\Delta(u_i) = u_i \otimes 1 + 1 \otimes u_i, \hspace{1cm} \epsilon(u_i) = 0, \hspace{1cm} S(u_i) = -u_i.
\]
for $u_i \in \{u, \xi_1, \xi_2\}$.

Example 6.5 There exists a Lie algebra homomorphism $\mu$ from $\mathcal{L}_{h_1,h_2}$ into $\text{M}(3, \mathbb{K}')$.

Proof We see that there exists an algebra homomorphism $\rho$ from $\mathcal{F}(\mathbb{A}_{p,q}^{1|2})$ into $\text{M}(3, \mathbb{K}')$ such that the relations (6.1) hold. As a consequence of this fact, there exists a Lie algebra homomorphism $\mu$ from $\mathcal{L}_{h_1,h_2}$ into $\text{M}(3, \mathbb{K}')$. The action of $\mu$ on the generators of $\mathcal{L}_{h_1,h_2}$ is of the form
\[
\mu(u) = \begin{pmatrix} \text{i} h_2 & 0 & 0 \\ 0 & \text{i} h_1 & 0 \\ 0 & 0 & \text{i}(h_1 + h_2) \end{pmatrix}, \hspace{1cm} \mu(\xi_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-\text{i}(h_1+h_2) \varepsilon_1} & 0 & 0 \end{pmatrix}, \hspace{1cm} \mu(\xi_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-2\text{i}(h_1+h_2) \varepsilon_2} & 0 \end{pmatrix}
\]
where $\varepsilon_1$ and $\varepsilon_2$ are two Grassmann numbers. To see that the relations (6.2) are preserved under the action of $\mu$, we use the fact that $\mu[a, b] = [\mu(a), \mu(b)]$ for all $a, b \in \mathcal{L}_{h_1,h_2}$. \hfill $\square$

7. $\star$-Structures on the algebras $\mathcal{O}(\mathbb{A}_{h}^{1|2})$ and $\mathcal{O}(\mathbb{A}_{h}^{2|1})$

It is possible to define the star operation (or involution) on the Grassmann generators. However, there are two possibilities to do so*. If $\alpha$ and $\beta$ are two Grassmann generators and $\lambda$ is a complex number and $\bar{\lambda}$ its complex conjugate, the star operation, denoted by $\star$, is defined by
\[
(\lambda \alpha)^\star = \bar{\lambda} \alpha^\star, \hspace{1cm} (\alpha \beta)^\star = \beta^\star \alpha^\star, \hspace{1cm} (\alpha^\star)^\star = \alpha
\]
and the superstar operation, denoted by $\#$, is defined by
\[
(\lambda \alpha)^\# = \bar{\lambda} \alpha^\#, \hspace{1cm} (\alpha \beta)^\# = \alpha^\# \beta^\#, \hspace{1cm} (\alpha^\#)^\# = -\alpha.
\]

It is easily shown that there exists a star operation on the algebra $\mathcal{O}(\mathbb{A}_{q}^{1|2})$ if $q$ is a complex number of modulus one:

Proposition 7.1 (i) If $\bar{q} = q^{-1}$ then the algebra $\mathcal{O}(\mathbb{A}_{q}^{1|2})$ equipped with the involution determined by
\[
X^\star = X, \hspace{1cm} \Theta_i^\star = \Theta_i \hspace{1cm} (i = 1, 2)
\]
becomes a $\star$-algebra.

(ii) If $\bar{p} = p^{-1}$ and $\bar{q} = q^{-1}$ then the algebra $\mathcal{O}(\mathbb{A}_{p,q}^{2|1})$ equipped with the involution determined by
\[
\Phi^\star = \Phi, \hspace{1cm} Y_i^\star = -Y_i \hspace{1cm} (i = 1, 2)
\]
becomes a $\star$-algebra.

*arXiv.org e-Print archive (1996). Dictionary on Lie Superalgebras [online]. Website https://arxiv.org/abs/hep-th/9607161 [18 July 1996].

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7.1. $\ast$-Structures on the algebra $\mathcal{O}(\mathbb{A}^{1|2}_h)$

As noted in Section 3, the relations in (3.3) do not include the parameter $h'$. Thus, we can rearrange the change of basis in the coordinates (see, equation (3.1)) as

$$
\begin{pmatrix}
X \\
\Theta_1 \\
\Theta_2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{h}{q-1} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
\theta_1 \\
\theta_2
\end{pmatrix}.
$$

(7.3)

This case can help us to define a star operation on the algebra $\mathcal{O}(\mathbb{A}^{1|2}_h)$ by a coordinate transformation using the generators of the algebra $\mathcal{O}(\mathbb{A}^{1|2}_q)$ and to prove the following lemma.

Lemma 7.2 For a certain special choice of $h$, there exists an involution on the algebra $\mathcal{O}(\mathbb{A}^{1|2}_h)$.

Proof Using the equation (7.3), we introduce the coordinates $x$, $\theta_1$, and $\theta_2$ with the change of basis in the coordinates of the superspace $\mathbb{A}^{1|2}_q$ as follows:

$$x = X, \quad \theta_1 = \Theta_1, \quad \theta_2 = \Theta_2 - \frac{h}{q-1} X.$$

Then, with $|q| = 1$ and (7.1)

$$\theta_2^* = \Theta_2^* - \frac{q\bar{h}}{1-q} X^* = \theta_2 + \frac{h + q\bar{h}}{q-1} x$$

so that, if we demand that $\bar{h} = -h$, we obtain $\theta_2^* = \theta_2 - hx$. Note that

$$(x^*)^* = x, \quad (\theta_1^*)^* = \theta_1, \quad (\theta_2^*)^* = \theta_2,$$

for all $h$. □

Proposition 7.3 If $\bar{h} = -h$, then the algebra $\mathcal{O}(\mathbb{A}^{1|2}_h)$ supplied with the involution determined by

$$x^* = x, \quad \theta_1^* = \theta_1, \quad \theta_2^* = \theta_2 - hx$$

(7.4)

becomes a $\ast$-algebra.

Proof Since $\bar{h} = -h$, we have

$$(x\theta_1 - \theta_1 x)^* = \theta_1 x - x\theta_1,$$

$$(x\theta_2 - \theta_2 x - hx^2)^* = (\theta_2 - hx)x - x(\theta_2 - hx) + hx^2 = (\theta_2 x - x\theta_2 + hx^2),$$

$$(\theta_1\theta_2 + \theta_2\theta_1)^* = \theta_2 x + \theta_1(\theta_2 - hx) = \theta_2\theta_1 + \theta_1\theta_2,$$

$$(\theta_2^2 + h\theta_2 x)^* = \theta_2 x + x(\theta_2 - hx)(-h) = \theta_2^2 + h\theta_2 x.$$

Hence, the ideal $(x\theta_1 - \theta_1 x, x\theta_2 - \theta_2 x - hx^2, \theta_1\theta_2 + \theta_2\theta_1, \theta_1^2, \theta_2^2 + h\theta_2 x)$ is $\ast$-invariant and the quotient algebra

$$\mathbb{K}\langle x, \theta_1, \theta_2 \rangle/(x\theta_1 - \theta_1 x, x\theta_2 - \theta_2 x - hx^2, \theta_1\theta_2 + \theta_2\theta_1, \theta_1^2, \theta_2^2 + h\theta_2 x)$$

becomes a $\ast$-algebra. □

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Remark 7.4 Of course, we can consider the change of basis in the coordinates of the superspace $\mathbb{A}_{q}^{1|2}$ in (3.1). In this case, since

\begin{align*}
x^* &= (1 + \tilde{h} - \tilde{h}')(x + (\tilde{h}' - \tilde{h})\theta_2), \\
\theta'_1 &= \theta_1, \\
\theta'_2 &= (1 - \tilde{h}(\tilde{h}' - \tilde{h}))\theta_2 + (\tilde{h} - \tilde{h})x,
\end{align*}

we have again (7.4) with the choices $\tilde{h} = -\overline{h}$ and $\tilde{h}' = h'$.

7.2. $\ast$-Structure on the algebra $O(\mathbb{A}_{q}^{2|1})$

Since the relations in (3.6) do not include the parameter $\overline{h}$, we can rearrange the change of basis in the coordinates (see, equation (3.1)) as

\begin{equation}
\begin{pmatrix}
\Phi \\
Y_1 \\
Y_2
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 & \frac{h'}{pq-1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\varphi \\
y_1 \\
y_2
\end{pmatrix}.
\end{equation}

There exists a special case, where the algebra $O(\mathbb{A}_{q}^{2|1})$ admits an involution. The proofs of the following lemma and proposition can be done in a similar way to Lemma 7.2 and Proposition 7.3.

Lemma 7.5 If $\tilde{h}' = h'$, there exists an involution on the algebra $O(\mathbb{A}_{q}^{2|1})$.

Proposition 7.6 If $\tilde{h}' = h'$, then the algebra $O(\mathbb{A}_{q}^{2|1})$ supplied with the involution determined by

\begin{equation}
\varphi^* = \varphi - h'y_2, \quad y_i^* = -y_i, \quad (i = 1, 2)
\end{equation}

becomes a $\ast$-algebra.

References

[1] Aghamohammadi A, Khorrami M, Shariati A. $\overline{h}$-deformation as a contraction of $q$-deformation. Jordanian deformation of SL(2) as a contraction of its Drinfeld-Jimbo deformation. Journal of Physics A: Mathematical and General 1995; 28 (8): L225-L231.

[2] Celik S. Two-parametric extension of $h$-deformation of GL(1|1). Letters in Mathematical Physics 1997; 42: 299-308.

[3] Celik S. Differential geometry of the Lie algebra of the quantum superplane. Balkan Physics Letters 2003; 11: 119-127. (arXiv:math/0201170)

[4] Celik Sultan A. Bicovariant differential calculus on the quantum superspace $\mathbb{R}_q(1|2)$. Journal of Algebra and its Applications 2016; 15: 1650172 (17 pages).

[5] Celik Sultan A. Differential calculi on super-Hopf algebra $\mathcal{F}(\mathbb{R}_q(1|2))$. Advanced in Applied Clifford Algebras 2018; 28: 85:1-16.

[6] Celik Sultan A. A two-parameter deformation of supergroup GL(1|2). Journal of the Institute of Science and Technology 2018; 8(4): 271-280.

[7] Dabrowski L, Parashar P. $h$-deformation of GL(1|1). Letters in Mathematical Physisc 1996; 38: 331-336.
[8] Faddeev LD, Reshetikhin NY, Takhtajan LA. Quantization of Lie groups and Lie algebras. Leningrad Mathematical Journal 1990; 1 (1): 193-225.

[9] Kupershmidt BA. The quantum group $GL_h(2)$. Journal of Physics A: Mathematical and General 1992; 25: L1239-L1244.

[10] Manin YI. Quantum groups and noncommutative geometry. Montreal, Canada: Centre de Recherches Mathematiques, Universite de Montreal, 1988.

[11] Manin YI. Multiparametric quantum deformation of the general linear supergroup. Communications in Mathematical Physics 1989; 123 (1): 163-175.

[12] Woronowicz SL. Differential calculus on compact matrix pseudogroups (quantum groups). Communications in Mathematical Physics 1989; 122 (1): 125-170.