Ricci and Matter inheritance collineations of 
Robertson-Walker space-times

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Abstract

It is well known that every Killing vector is a Ricci and Matter collineation. Therefore the metric, the Ricci tensor and the energy-momentum tensor are all members of a large family of second order symmetric tensors which are invariant under a common group of symmetries. This family is described by a generic metric which is defined from the symmetry group of the space-time metric. The proper Ricci and Matter (inheritance) collineations are the (conformal) Killing vectors of the generic metric which are not (conformal) Killing vectors of the space-time metric. Using this observation we compute the Ricci and Matter inheritance collineations of the Robertson-Walker space-times and we determine the Ricci and Matter collineations without any further calculations. It is shown that these higher order symmetries can be used as supplementary conditions to produce an equation of state which is compatible with the Geometry and the Physics of the Robertson-Walker space-times.

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1 Introduction

A geometric symmetry or collineation is defined by a relation of the form:

$$\mathcal{L}_\xi \Phi = \Lambda$$  \hspace{1cm} (1)

where $\xi^a$ is the symmetry or collineation vector, $\Phi$ is any of the quantities $g_{ab}, \Gamma^a_{bc}, R_{ab}, R^a_{bcd}$ and geometric objects constructed from them and $\Lambda$ is a tensor with the same index symmetries as $\Phi$. By demanding specific forms for the quantities $\Phi$ and $\Lambda$ one finds all the well known collineations. For example $\Phi_{ab} = g_{ab}$ and $\Lambda_{ab} = 2\psi g_{ab}$ defines the Conformal Killing vectors (CKV) and specializes to a Special Conformal Killing vector (SCKV) when $\psi_{ab} = 0$, to a Homothetic vector field (HVF) when $\psi = \text{constant}$ and to a Killing vector (KV) when $\psi = 0$.

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When $\Phi_{ab} = R_{ab}$ and $\Lambda_{ab} = 2\psi R_{ab}$ the symmetry vector $\xi^a$ is called a Ricci Inheritance Collineation (RIC) and specializes to a Ricci Collineation (RC) when $\Lambda_{ab} = 0$. When $\Phi_{ab} = T_{ab}$ and $\Lambda_{ab} = 2\psi T_{ab}$, where $T_{ab}$ is the energy momentum tensor, the vector $\xi^a$ is called a Matter Inheritance Collineation (MIC) and specializes to a Matter collineation (MC) when $\Lambda_{ab} = 0$. The function $\psi$ in the case of CKVs is called the conformal factor and in the case of inheriting collineations the inheriting factor.

There are other important families of collineations and in fact most (but not all) of them have been classified in a tree like diagram which exhibits clearly their relative properness [1, 2]. Collineations of a different type are not independent, for example a KV is a RC or MC but not the opposite. Using this simple observation we approach collineations form a different perspective which groups seemingly unrelated collineations in one, thus changing the above classification and in fact collapsing the classification tree considerably. The core of our argument lies in the definition of a metric by its symmetry algebra.

Assume that a manifold $\mathcal{M}$ admits the symmetry (Lie) group $G$ of dimension $r$ whose associated Lie algebra is $\mathcal{G}$. Then there exists on $\mathcal{M}$ a symmetric non-degenerate second order tensor (which we call generic metric) invariant under the group $G$. The action of the symmetry group $G$ is characterized by the dimension $d$ of the group orbits. If $s$ is the dimension of the stability (isotropy) group then $r = d + s$. In the above scenario the type of the orbits (that is the signature of the metric on the orbits) does not appear explicitly and it is fixed, in general, by an extra assumption. This analysis leads one to consider the following "inverse" problem.

Suppose one is given a symmetry group of a space-time metric (that is a group of KVs) of dimension $r$ whose orbits have everywhere dimension $d$. Let $\{X_1, \ldots, X_r\}$ be the KVs generating the Lie algebra of $G$. Find all non degenerate, second order symmetric tensors, $G_{ab}$ say, which satisfy the equations $\mathcal{L}_{X_a} G_{ab} = 0$ where $A = 1, 2, \ldots, r$, that is, their symmetry algebra is isomorphic to $\mathcal{G}$.

Obviously the metric of the space-time manifold belongs to the family of these tensors but, as we shall see below, there are more. In order to solve the problem one observes that the data $\{\text{Symmetry algebra } \mathcal{G}, \text{ dimension of orbits}\}$ define a generic metric (=second order tensor) which can be constructed as follows:

a. The "part" of the metric on the group orbits has the same functional form as the original space-time metric but not a specific signature.

b. Outside the group orbits, only the functional dependence of the components of the metric will be restricted, that is, they will be independent of the coordinates defined on the orbits i.e. if one selects coordinates adapted to the KVs of the metric then the components of the required tensors in the quotient submanifold $\mathcal{M}/\text{orbit}$ do not depend on these coordinates.

The above construction implies that all symmetric tensors we are looking for, are obtained from a generic element provided that the orbits are non null. In the following the set of all second order symmetric tensors which admit the symmetry algebra of a space-time metric as their symmetry algebra we shall call family of metrics (FOM) and the generic element describing this set generic metric. The later we shall denote $G_{ab}$.

One can break the FOM into disjoint subsets by demanding specific signatures on the orbit part of the generic metric. The KVs of the elements in each subset are, in general, functionally different but all belong to the symmetry algebra of the FOM. This implies that in order to compute the complete algebra of KVs of the elements in each subset, it is enough to solve Killing’s equations for the FOM and then select the KVs for various values of the signature. In
this case the complete Lie algebra of KVs contains as a subalgebra the symmetry algebra $\mathcal{G}$ of the original space-time metric.

The above considerations can be useful in two important cases.

**The Ricci Tensor**

Suppose one has a space-time (the argument is independent of the signature and holds for non-space-time metrics as well) metric $g_{ab}$ which admits a certain Lie algebra $\mathcal{G}$ of KVs. Let $G_{ab}$ the generic metric defined by the symmetry algebra $\mathcal{G}$. The associated Ricci tensor $R_{ab}$ of the metric $g_{ab}$ is computed by covariant differentiation (geometry only!) and is a second order symmetric tensor on the space-time manifold. It is well known that the KVs of a metric are RCs, therefore the Ricci tensor inherits the symmetries of the metric. This implies that the Ricci tensor and the original space-time metric $g_{ab}$ belong to the FOM generated by the $G_{ab}$. Furthermore the Ricci tensor must follow from the generic metric $G_{ab}$ for some values of its defining parameters and the signature. In conclusion:

a. The form of the Ricci tensor is restricted by the symmetry (must belong to one of the subsets of the FOM defined by $G_{ab}$).

b. The symmetries of the Ricci tensor are computed from the KVs of $G_{ab}$. The KVs of the generic metric which are not, in general, KVs of the space-time metric $g_{ab}$ are precisely the proper RCs. In the following we shall call these vectors extra KVs of the space-time metric $g_{ab}$. Hence in order to compute the proper RCs of a given space-time metric $g_{ab}$ it is enough one to compute the extra KVs of the generic metric $G_{ab}$ and subsequently express them in terms of the components of $R_{ab}$ (replacing those of the generic metric).

In Section V we shall apply the above conclusions and we shall compute all proper RCs of the RW space-times in a straightforward and simple manner.

**The Energy Momentum Tensor**

Let us consider a space-time metric $g_{ab}$ which is created by a matter distribution described by the energy-momentum tensor $T_{ab}$. From Einstein field equations (Geometry combined with Physics!) we have that the symmetries of the metric pass over to the energy-momentum tensor. Therefore what has been said for the Ricci tensor also applies to the energy momentum tensor.

This has, among others, the following important implications:

a. In a given space-time admitting a group of KVs, the energy momentum tensor is not arbitrary but it must be an element in the FOM defined by that symmetry group. This implies that the symmetry assumptions do not fix only the geometry of space-time but also its dynamics, an interplay much desired in General Relativity.

b. In order to compute the proper MCs of a given space-time one simply expresses the components of the extra KVs of the generic metric $G_{ab}$ in terms of the components of $T_{ab}$.

It must be emphasized that the Ricci tensor and the energy-momentum tensor although formally the same from the point of view of symmetry are different in two important aspects. The first is the energy conditions which apply directly to the energy-momentum tensor and indirectly on the Ricci tensor via the Einstein field equations. The second is the physical interpretation of the energy-momentum tensor in terms of dynamic variables (energy density, pressure etc.) which is achieved by the introduction of the fluid four velocity $u^a$ ($u^a u_a = -1$) and the subsequent well known 1+3 decomposition [4]:

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab}$$  \hspace{1cm} (2)
where \( h_{ab} = g_{ab} + u_a u_b \) is the projection tensor. We shall use these observations later when we discuss the application of matter collineations in RW space-times.

What we have said about the KVs of the space-time metric extends without any change to CKVs which also form a group of symmetries (the conformal symmetry group). This means again that one can compute the proper RICs or MICs from the extra CKVs of the generic metric, that is, the CKVs which are not CKVs of the space-time metric.

In order to demonstrate the validity and the importance of the above considerations and, at the same time, produce useful practical results, in subsequent sections we work with the Robertson-Walker (RW) space-times. Indeed, besides the fact that the RW space-time metric is used as the standard model for cosmological observations, very few of the RCs and the MCs it admits are known. Green et al. \[5\] and more recently Nunez et al. \[6, 7\] found some RCs. As far as we are aware there do not exist results concerning the MCs in RW space-times and only recently Carot et al. \[8\] have studied MCs in a general context. Inherited collineations other than CKVs are relatively new and have not been considered much in the literature. Most of the existing work has been done by Duggal et al. \[9, 10, 11, 12\] Because inherited collineations are more general than simple collineations one expects that perhaps they could possibly be useful in various ways.

One might ask the further question: Why should we be interested in these collineations? What could be their use? Let us answer this question in the case of the standard cosmological model. Using the symmetry assumptions of the RW metric and Einstein Field Equations (EFE) one ends up with a differential equation involving the scale factor and one of the two dynamic quantities \( \mu, p \). This equation cannot be solved unless one introduces an equation of state \( p = p(\mu) \). The first (and logical choice) is a linear equation of state to which one gives physical meaning by considering rather “ideal” matter forms. It would be preferable for one to use non-linear equations of state, which could express more complex matter situations. But how would these equations be constructed?

The above results suggest such a method: Use a geometric condition to get the desired equation and then study its physical significance posteriori. As we have shown the symmetries couple the geometry \( R_{ab} \) and the matter \( T_{ab} \) to the gravitational field \( g_{ab} \) in an inherent manner. Therefore one can use these ”higher” symmetries and let the geometry produce the Physics, if any. In other words one requires that the space-time admits (e.g.) a RC and produces from that demand an equation of state which leads to a solution of the equation defining the scale factor, therefore to a definite cosmological model.

An outline of the paper is as follows: In section II we determine the conformal algebra of a general space which is conformally related to a \( 1 + (n - 1) \) decomposable space and whose \( (n - 1) \) subspaces are spaces of constant curvature \( (n > 2) \). In section III we apply the general results to the case \( n = 4 \). Demanding the signature to be -2 we obtain the conformal algebra of the RW space-time regaining in a natural manner known results \[13\]. For the case where the signature of the \( 1 + 3 \) metric is 4 we determine the conformal algebra of the resulting 4-dimensional Euclidean space. The main results of the paper are in Sections IV, V, and VI which give the complete algebra of RICs of the RW space-times both in the degenerate and in the non-degenerate cases (Section IV) and the proper RCs, MICs and MCs (Sections V and VI respectively). In order to get a physical sense of the results we consider, in Section VII, the standard comoving observers and we study the evolution of the flat RW space-times which admit MCs. Finally Section VIII concludes the paper.
2 The conformal algebra of metrics conformal to a 1 + (n − 1) decomposable metric

In the determination of the RICs and MICs of the RW space-times we shall need the complete conformal algebra of a 4-dimensional metric which is conformally related to a 1+3 decomposable metric whose 3-spaces are spaces of constant curvature (and signature 3).

The conformal algebra of the standard RW space-times has been derived before [13] using mainly the Lie algebra property of CKVs. In this work we compute this algebra using a different and more general approach. The presentation will be concise and most of the results will be taken directly from the existing literature. We work in an n—dimensional space and in the next section we restrict the results to the case n = 4 .

Consider two metrics (in an n—dimensional space) which are conformally related:

$$\hat{g}_{ab} = U^2(x^c)g_{ab}.$$  

(3)

It is well known [14, 15] that these metrics have the same contravariant CKVs $X^a$ and that the covariant vectors $\hat{X}_a = \hat{g}_{ab}X^b$, $X_a = g_{ab}X^b$ are related as follows $\hat{X}_a = U^2(x^c)X_a$. This implies for the conformal factor and the bivector of these vector fields the relations:

$$\hat{\phi} = X^a(\ln U)_a + \phi$$  

(4)

$$\hat{F}_{ab} = U^2F_{ab} - 2UU_{[a}X_{b]}$$  

(5)

where we have used an obvious notation.

A CKV is a gradient CKV if the corresponding bivector vanishes. Let us assume that the second metric is flat (of arbitrary sign). Then it is well known [16] that the CKVs are:

$$X_c = a_c + a_c dx^d + bx_c + 2(b_d dx^d)x_c - b_c (x^d x_d)$$  

(6)

where the constants $a_c, a_c = -a_{dc}, b, b_c$ specify respectively the type (i.e. KV, HVF, SCKV) of the CKV. If we further assume the metric $\hat{g}_{ab}$ to be the metric of a space of constant curvature then $U(x^c) = \left(1 + \frac{k}{4} x^c x_c \right)^{-1}$ where $k = \frac{R}{n(n-1)}$, $R$ is the (constant) scalar curvature of the metric and n is the dimension of the space. In this case it can be shown [17] that the metric $\hat{g}_{ab}$ admits $(n + 1)$ gradient CKVs whose conformal factors satisfy the equation:

$$\hat{\phi}_{;ab} = \frac{k}{n} \hat{\phi} \hat{g}_{ab}.$$  

(7)

Using the above information one can derive easily the $(n + 1)(n + 2)/2$ CKVs together with their bivector and conformal factor listed in Table I.
TABLE I. The \((n+1)(n+2)/2\) CKVs of an \(n\)-space of constant non-vanishing curvature. \(\delta^{\alpha\lambda}_{[\alpha\beta]} \equiv \delta^{\alpha}_{[\alpha\beta]}\) and the non-tensorial indices \(\alpha, \beta, \lambda, \nu = 1, \ldots, n\) count vector fields.

| Number of CKVs | CKVs of the n-metric | \(F_{\alpha\beta}\) | \(\phi\) | \(\phi,_{\alpha}\) |
|----------------|----------------------|------------------|---------|-------------|
| \(n\)          | \(I_{\nu} = \frac{1}{\nu!} (2U - 1) \delta_{\alpha\nu} + \frac{1}{2} kU x_{\nu} x_{\alpha} \mid \partial_{\alpha}\) | \(2kU^{3} x_{[\alpha} \delta_{\beta]\nu}\) | 0       | 0           |
| \(n(n-1)/2\)   | \(M_{\lambda\nu} = \delta^{\alpha\beta}_{\lambda\nu} x_{\alpha} \partial_{\beta}\) | \(U^{2} \delta^{\lambda\nu}_{\alpha\beta} + kU^{3} x_{[\alpha} \delta^{\lambda\nu}_{\beta]\sigma} x^{\sigma}\) | 0       | 0           |
| 1              | \(H = x^{\alpha} \partial_{\alpha}\) | \(0\) | \(1 - \frac{kU(x^{\alpha} x^{\alpha})}{2}\) | \(-kH_{\alpha}\) |
| \(n\)          | \(C_{\nu} = \frac{1}{\nu!} \delta^{\alpha}_{\nu} - \frac{kU x_{\nu} x^{\alpha}}{2} \mid \partial_{\alpha}\) | \(0\) | \(-kU x_{\nu}\) | \(-kC_{(\nu)\alpha}\) |

A metric space is \(1 + (n-1)\) decomposable if it can be written in the form:

\[
ds^{2} = \epsilon (dx^{1})^{2} + g_{\alpha\beta}(x^{\gamma}) dx^{\alpha} dx^{\beta}
\]

where \(\epsilon = \pm 1\), \(g_{\alpha\beta}(x^{\gamma})\) is the metric in the hypersurface \(x^{1} = \text{const.}\) and Greek indices take the values 2, 3, ..., \(n\). The following general result holds for the conformal algebra of these spaces [17].

**Proposition 1** The proper CKVs of a \(1 + (n-1)\) metric are of the form:

\[
X^{\alpha} = f(x^{b}) \delta^{\alpha}_{\delta} + K^{\alpha} \delta^{\alpha}_{\delta}.
\]

In this formula \(K^{\alpha}\) is a proper CKV of the \((n-1)\) metric given by:

\[
K^{\alpha} = \frac{1}{p} m(x^{1}) \phi^{\alpha}(x^{b}) + L^{a}(x^{j})
\]

where:

1. \(m(x^{1})\) satisfies the equation:

\[
m_{11} + \epsilon pm = 0.
\]

2. The vector \(\phi^{\alpha}(x^{j})\) is a gradient CKV of the \((n-1)\) metric \(g_{\alpha\beta}\) and satisfies the condition:

\[
\phi_{\alpha\beta}(x^{\gamma}) = -p \phi g_{\alpha\beta}
\]

where \(|\) denotes covariant differentiation wrt the \((n-1)\) metric \(g_{\alpha\beta}\).

3. \(L^{a}(x^{b})\) is a non-gradient KV or HVF of the \((n-1)\)-metric and the \(f(x^{b})\) is a \(C^{\infty}\) function given by the formula:

\[
f(x^{b}) = -\frac{\epsilon}{p} m_{1} \phi(x^{b}).
\]

Furthermore the HVF \(H^{a}\) of the \(n\) metric (if it exists) is given by:

\[
H^{a} = bx^{1} \delta^{a}_{1} + L^{a} \delta^{a}_{\delta}
\]

where \(L^{a}\) is the HVF (if it exists) vector of the \((n-1)\)-metric. The last equation implies that the KVs of the \((n-1)\)-metric are identical with those of the \(1 + (n-1)\) decomposable space.

Using Proposition 1 and the previous considerations on the CKVs of conformally related spaces we are able to compute the conformal algebra of a metric conformal to a \(1 + (n-1)\) decomposable metric whose \((n-1)\) part is the metric of a space of constant curvature. In particular for \(n = 4\) we obtain the RW-like metrics which we study in the next section.
3 The case of RW-like metrics

The isotropic RW space-time admits a six dimensional symmetry group acting on 3D spacelike orbits. According to the construction described in the Introduction this symmetry group defines the generic metric:

\[ ds^2 = S^2(\tau) \left( \epsilon d\tau^2 + U^2(k, x^\alpha) d\sigma_3^2 \right) \] (15)

where \( \epsilon = \pm 1 \), \( U(k, x^\alpha) = \left( 1 + \frac{k}{4} x^\alpha x_\alpha \right)^{-1} \), \( k = 0, \pm 1 \) and \( d\sigma_3^2 = dx^2 + dy^2 + dz^2 \). The generic metric (15) we shall call RW-like metric. The standard RW space-time is a member of the FOM generated by the RW-like metric and the variable \( \tau \) is related to the standard variable \( t \) (cosmic time in RW space-time) by the relation:

\[ dt = S(\tau) d\tau. \] (16)

The RW-like metric is conformally related to Einstein space-time (\( \epsilon = -1, k = 1 \)), the anti-Einstein space-time (\( \epsilon = -1, k = -1 \)), the Minkowski space-time (\( \epsilon = -1, k = 0 \)) and to a positive definite 1+3 decomposable metric (\( \epsilon = 1 \)). In each case the 3-space is a space of constant curvature \( R = 6k \).

By applying the results of the previous section for \( n = 4 \) one computes easily the complete conformal algebra of the metric (15). The results of the calculations are collected in Table II for the cases \( k = \pm 1 \) and in Table III for the case \( k = 0 \). In both tables the quantities \( \phi_k \), \( H \) and \( C_\mu \) are defined in Table I and for convenience we have set \( (c_+, c_-) = (\cosh \tau, \cos \tau) \) and \( (s_+, s_-) = (\sinh \tau, \sin \tau) \). Furthermore we give the forms of the "scale factor" \( S(\tau) \) in order the generic metric (15) to admit extra KVs. We note that there are either one or four extra KVs, a result which is in agreement with well known theorems [15].

\[ \text{TABLE II. The nine proper CKVs of the RW-like metrics for the case } k = \pm 1 \text{ together with their conformal factors. The last column gives } S(\tau) \text{ for extra KVs.} \]

| # | Proper CKV X | Conformal Factor \( \psi \) | \( S(\tau) \) for \( k \text{ extra KVs} \) |
|---|-------------|----------------|-------------------------------|
| 1 | \( Y = \partial_\tau \) | \( [\ln S(\tau)]_\tau \) | \( A \) |
| 1 | \( H^k_1 = \epsilon k \phi_k(H)c_\epsilon k \partial_\tau + Hs_\epsilon k \) | \( \epsilon k \frac{\phi_k(H)}{R(\tau)} [S(\tau)c_\epsilon k]_\tau \) | \( A/c_\epsilon k \) |
| 1 | \( H^k_2 = \phi_k(H)s_\epsilon k \partial_\tau + Hc_\epsilon k \) | \( \phi_k(H) \frac{[S(\tau)s_\epsilon k]}{R(\tau)} \) | \( A/s_\epsilon k \) |
| 3 | \( Q^k_\mu = \epsilon k \phi_k(C_\mu)c_\epsilon k \partial_\tau + C_\mu s_\epsilon k \) | \( \epsilon k \frac{\phi_k(C_\mu)}{R(\tau)} [S(\tau)c_\epsilon k]_\tau \) | \( A/c_\epsilon k \) |
| 3 | \( Q^k_{\mu+3} = \phi_k(C_\mu)s_\epsilon k \partial_\tau + C_\mu c_\epsilon k \) | \( \phi_k(C_\mu) \frac{[S(\tau)s_\epsilon k]}{R(\tau)} \) | \( A/s_\epsilon k \) |

\[ \text{TABLE III. The nine proper CKVs of the RW-like spaces } (k = 0) \text{ together with their conformal factors. The last column shows } S(\tau) \text{ for extra KVs (whenever they exist).} \]
Proper CKV $X$ | Conformal Factor $\psi$ | $S(\tau)$ for extra KVs
--- | --- | ---
1 $P_\tau = \partial_\tau$ | $(\ln S)_\tau$ | $A$
3 $M_{\tau\alpha} = x_\alpha \partial_\tau - \epsilon \tau \partial_\alpha$ | $x_\alpha (\ln S)_\tau$ | $A$
1 $H = x^\alpha \partial_\alpha$ | $H(\ln S) + 1$ | $1/A\tau$
1 $K_\tau = 2\epsilon \tau H - (x^c x_c) \partial_\tau$ | $-(\ln S)_\tau (-\epsilon \tau^2 + x^2 + y^2 + z^2) + 2\epsilon \tau$ | $---$
3 $K_\mu = 2x_\mu H - (x^c x_c) \partial_\mu$ | $2x_\mu [\tau (\ln S)_\tau + 1]$ | $1/A\tau$

In both cases the remaining eight or five vector fields are proper CKVs. If we set $\epsilon = -1$ we find the CKVs of the standard RW metrics. These vectors coincide with those found earlier [13].

4 Ricci Inheriting Collineations of RW space-times

As it has been explained in the Introduction the proper RICs and the proper RCs of a given space-time metric $g_{ab}$ are the extra CKVs and KVs respectively of the generic metric $G_{ab}$ defined by the symmetry group of the space-time metric $g_{ab}$. In fact in this approach one essentially sees the Ricci tensor $R_{ab}$ of $g_{ab}$ as a "metric" on the manifold $M$. This metric is different from the space-time metric $g_{ab}$ because:

1. It can be degenerate i.e. of rank $< 4$.
2. Its signature can change from point to point on the manifold.

Due to the strong relation between the two metrics $g_{ab}$, $R_{ab}$ in the case of non-degeneracy, the results on the isometries (KVs) of $g_{ab}$ extend naturally to the RCs. For example the following result on RCs due to Hall et al. [13] is a direct consequence of the corresponding result on KVs of the original metric $g_{ab}$:

If the Ricci tensor is of rank 4, at every point of the space-time manifold, then the smooth ($C^2$ is enough) RCs form a Lie algebra of smooth vector fields whose dimension is $\leq 10$ and $\neq 9$. This Lie algebra contains the proper RCs and their degeneracies.

Concerning the RICs of the RW metric we expect that, due to the conformal flatness of the RW-like metric, the maximum number of expected RICs is 15, of which 6 are KVs and 9 proper RICs (one of which reduces to a KV or to extra 4 KVs for certain forms of the conformal factor as we shall show below in Table II and Table III). These vectors have already been found and are given in the second column of Table II and Table III. What it remains to be done is to compute the Ricci tensor and express their components in terms of those of the Ricci tensor. **No further computations are needed!**

The Ricci tensor of the standard RW metric is:

$$R_{ab} = \text{diag} \left[ R_0(\tau), R_1(\tau)U^2(k, x^\alpha), R_1(\tau)U^2(k, x^\alpha), R_1(\tau)U^2(k, x^\alpha) \right]$$

(17)

where:
\[ R_0(\tau) = \frac{3 \left( \dot{S}^2 - S \ddot{S} \right)}{S} \quad (18) \]
\[ R_1(\tau) = \frac{S \dddot{S} + 2 \dot{S}^2 + 2kS^2}{S^2} \quad (19) \]

and \( U(k, x^\alpha) = \left( 1 + \frac{k}{4} x^\alpha x_\alpha \right)^{-1} \).

We note immediately that this belongs to the family of metrics \((15)\), as expected. Because there is no guarantee that this tensor is non-degenerate and of constant signature one has to consider the cases that the Ricci tensor is degenerate and non-degenerate and in the latter case consider further the two possible subcases of Euclidean and Lorentzian signature.

### 4.1 Ricci Inheritance Collineations of RW space-times in the non-degenerate case

In this case we write:

\[ ds^2_R = R_0(\tau) d\tau^2 + R_1(\tau) U^2(k, x^\mu) d\sigma^2_3. \quad (20) \]

Using the transformation \((R_0(\tau)R_1(\tau) \neq 0)\):

\[ d\tau = \sqrt{\frac{R_0(\tau)}{R_1(\tau)}} d\tau \Rightarrow \partial_\tau = \sqrt{\frac{R_1(\tau)}{R_0(\tau)}} \partial_\tau \quad (21) \]

we find the metric in the form:

\[ ds^2_R = R_1(\tau) \left[ \epsilon d\bar{\tau}^2 + U^2(k, x^\alpha) d\sigma^2_3 \right] \quad (22) \]

where:

\[ \epsilon = \text{sign} \left( \frac{R_0(\tau)}{R_1(\tau)} \right). \quad (23) \]

The metric \((22)\) is a non-degenerate metric of the form \((15)\) therefore its \((C^\infty)\) RICs are the CKVs of the RW-like FOM \((15)\). Furthermore the proper RCs of the RW metric are the extra KVs of the generic metric (which occur for special forms of the "conformal" factor \(R_1(\tau)\)).

Using the results of Table II and Table III we write without any further computations all proper RICs by changing \(S(\tau) \leftrightarrow R_1(\tau)\) and \(\tau \leftrightarrow \bar{\tau}\). The results for \(k = \pm 1\) are listed in Table IV and those for \(k = 0\) in Table V. The quantities \(\phi_k\) and the vector fields \(H, C_\mu\) of the 3-space of constant curvature are defined in Table I. We have set again \((c_+, c_-) \equiv [\cosh \bar{\tau}(\tau), \cos \bar{\tau}(\tau)]\), \((s_+, s_-) \equiv [\sinh \bar{\tau}(\tau), \sin \bar{\tau}(\tau)]\).
TABLE IV. The complete algebra of proper RICs of the RW space-times for $k = \pm 1$.

| # | RICs X                                                                 | Conformal factor $\psi$                                                                 |
|---|------------------------------------------------------------------------|----------------------------------------------------------------------------------------|
| 1 | $Y = \left| \frac{R_{1}(\tau)}{R_{0}(\tau)} \right|^{1/2} \partial_{\tau}$ | $\left| R_{1}(\tau)R_{0}(\tau) \right|^{-1/2} \left( \ln R_{1}(\tau) \right)_{\tau}$ |
| 1 | $H_{1}^{k} = \epsilon k \phi_{k}(H) \left| \frac{R_{1}(\tau)}{R_{0}(\tau)} \right|^{1/2} c_{ek} \partial_{\tau} + Hs_{ek}$ | $\epsilon k \phi_{k}(H) \left| R_{1}(\tau)R_{0}(\tau) \right|^{-1/2} \left[ \frac{R_{1}(\tau)^{1/2} c_{ek}}{R_{0}(\tau)} \right]_{\tau}$ |
| 1 | $H_{2}^{k} = \phi_{k}(H) \left| \frac{R_{1}(\tau)}{R_{0}(\tau)} \right|^{1/2} s_{ek} \partial_{\tau} + Hc_{ek}$ | $\phi_{k}(H) \left| R_{1}(\tau)R_{0}(\tau) \right|^{-1/2} \left[ \frac{R_{1}(\tau)^{1/2} s_{ek}}{R_{0}(\tau)} \right]_{\tau}$ |
| 3 | $Q_{\mu}^{k} = \epsilon k \phi_{k}(C_{\mu}) \left| \frac{R_{1}(\tau)}{R_{0}(\tau)} \right|^{1/2} c_{ek} \partial_{\tau} + C_{\mu}s_{ek}$ | $\epsilon k \phi_{k}(C_{\mu}) \left| R_{1}(\tau)R_{0}(\tau) \right|^{-1/2} \left[ \frac{R_{1}(\tau)^{1/2} c_{ek}}{R_{0}(\tau)} \right]_{\tau}$ |
| 3 | $Q_{\mu+3}^{k} = \phi_{k}(C_{\mu}) \left| \frac{R_{1}(\tau)}{R_{0}(\tau)} \right|^{1/2} s_{ek} \partial_{\tau} + C_{\mu}c_{ek}$ | $\phi_{k}(C_{\mu}) \left| R_{1}(\tau)R_{0}(\tau) \right|^{-1/2} \left[ \frac{R_{1}(\tau)^{1/2} s_{ek}}{R_{0}(\tau)} \right]_{\tau}$ |

To demonstrate how these results are obtained from those of Table II and Table III let us consider the case of the RIC $P_{\tau}$. This corresponds to the CKV $P_{\tau}$ of the RW-like metrics $[\text{III}]$ whose conformal factor is $\psi(P_{\tau}) = \left[ \ln S(\tau) \right]_{\tau}$. To compute $P_{\tau}, \psi(P_{\tau})$ we make the correspondence $S(\tau) \leftrightarrow R_{1}(\tau)$ and $\tau \leftrightarrow \bar{\tau}$ in (21) and write down the result immediately. For the other vectors we work in a similar manner being careful to keep the unknown function $\bar{\tau}(\tau)$ wherever it occurs.

4.2 Ricci inheritance collineations of the RW space-times in the degenerate case

In order to obtain results comparable with the ones in the existing literature we consider the RW space-time metric in spherical coordinates:

$$ds^{2} = -dt^{2} + S^{2}(t) \left( \frac{dr^{2}}{1 - kr^{2}} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2} \right).$$

The Ricci tensor is again diagonal with elements:

$$R_{00} = -3S_{,\mu}/S \quad \text{(24)}$$

$$R_{\alpha\beta} = \frac{\Delta}{S^{2}S_{,\alpha\beta}} \quad \text{(25)}$$
where $g_{\alpha\beta}$ is the metric of the 3-space of constant curvature and the quantity:

$$\Delta = S\ddot{S} + 2\dot{S}^2 + 2k.$$  

(26)

The condition for the Ricci tensor to be degenerate is $\det R_{ab} = 0$. This implies that there are two cases to consider $R_{00} = 0$, $R_{\alpha\beta} \neq 0$ ($\Delta \neq 0$) and $R_{00} \neq 0$, $R_{\alpha\beta} = 0$ ($\Delta = 0$). We have the following result.

**Proposition 2** The RW space-times with a degenerate Ricci tensor admit infinite RICs as follows:

(a) Case $R_{00} = 0$, $\Delta \neq 0$.

The scale factor is $S(t) = at + b$ where $a, b$ are constants and the RICs are of the form

$$X = f(x^a)\partial_t + X^\mu(x^a)\partial_\mu$$  

(27)

where $X^\mu(x^a)$ is a CKV of the 3-metric $g_{\alpha\beta}$ and $f(x^a)$ is an arbitrary smooth function.

(b) Case $R_{00} \neq 0$, $\Delta = 0$.

The form of the RICs is:

$$X = \frac{h(t)}{|R_{00}|^{1/2}}\partial_t + X^\mu(x^a)\partial_\mu$$  

(28)

where $X^\mu(x^a)$ are arbitrary (but smooth) functions of the space-time coordinates.

**Proof**

We write the Ricci Inheritance equations $\mathcal{L}_\xi R_{ab} = 2\psi R_{ab}$ in detailed form:

$$R_{00,0}X^0 + 2R_{00}X^0_0 = 2\psi R_{00}$$  

(29)

$$R_{00}X^0_\mu + R_{\mu\mu}X^\mu_0 = 0$$  

(30)

$$R_{\mu\mu,0}X^0 + \sum_\nu R_{\mu\nu,\nu}X^\nu_0 + 2R_{\mu\nu}X^\nu_\mu = 2\psi R_{\mu\mu}$$  

(31)

$$R_{\mu\mu}X^\mu_\nu + R_{\nu\nu}X^\nu_\mu = 0 \text{ (no sum over } \mu, \nu)$$  

(32)

(a) Case $R_{00} = 0$, $\Delta \neq 0$.

Equation (29) implies $X^0 = f(x^a)$ where $f(x^a)$ is an arbitrary (smooth) function of the co-ordinates. Furthermore eq. (30) gives $X^\mu = X^\mu(x^\mu)$ which together with $S(t) = at + b$ means that $\Delta = \text{const}$. Hence the space-time is $1 + 3$ decomposable. Using eq. (15) it easy to show that equations (31) and (32) can be written in concise form as:

$$\mathcal{L}_{X_\perp}g_{\mu\nu} = 2\psi(x^a)g_{\mu\nu}$$  

(33)

where $X_\perp = X^\mu(x^\mu)\partial_\mu$ is the spatial part of the symmetry vector and $g_{\mu\nu}$ is the 3-metric of the hypersurfaces $t = \text{const.}$ of constant curvature:

$$g_{\mu\nu} = U^2(x^\mu)\delta_{\mu\nu}.$$  

(34)

The last equation implies that $X_\perp$ is a CKV of the 3-metric $g_{\mu\nu}$ and can be found from the results of Section II (Table I) setting $n = 3$. (This result agrees with the general result of Hall
et al. on $1 + 3$ decomposable space-times [13]). However due to the fact that the component $X^0$ of the RIC is arbitrary the Lie algebra is infinite dimensional.

(b) $R_{00} \neq 0$, $\Delta = 0$.

Equations (29), (30) imply that the quantities $X^0(t)$ and $\psi(t)$ depend only on the timelike coordinate $t$. We introduce the positive function $h(t)$ as follows:

$$\psi(t) = 2X^0(t) [\ln h(t)]_t$$

Then (29) gives $X^0 = \frac{h(t)}{\sqrt{|R_{00}|}}$. The rest of the proof follows easily.

For degenerate Ricci tensor, RICs are infinite therefore mathematically they are of no interest. From the physical point of view they do lead to restrictions on the scale factor and consequently on the matter fluid. Thus it has been shown [6] that in the case $R_{00} = 0$ the equation of state is $\rho + 3p = 0$ whereas for $\Delta = 0$ the equation of state is $\rho = p$ (stiff matter). However both equations of state are of limiting physical interest.

## 5 Ricci collineations of the RW space-time

We consider the non-degenerate case because in the degenerate case we have infinite RCs which is of no interest. The proper RCs of the RW space-time are obtained form Table II and Table III by setting the conformal factor of each vector equal to zero. The results are given in the last column of Table IV and Table V from which it follows that there are two cases to be considered: $R_1(\tau) = \text{const}$ and $R_1(\tau) \neq \text{const}$. The results of the computations for the cases $k = \pm 1$ are collected in Table VI and those of $k = 0$ in Table VII. We have introduced for convenience the new time coordinate $\tilde{\tau}$ by the relation:

$$\tilde{\tau}(\tau) = \int |R_0(\tau)|^{1/2} d\tau. \quad (35)$$

TABLE VI. The proper RCs of the RW space-times for $k = \pm 1$ and the expression of $R_1$ for which the corresponding collineations are admitted. $A$ is an integration constant.

| # | RCs X ($k = \pm 1$) | $R_1(\tau)$ |
|---|---------------------|-------------|
| 1 | $Y = A\partial_{\tilde{\tau}}$ | $A$ |
| 1 | $H_1^k = \epsilon k\phi_k(H)A\partial_{\tilde{\tau}} + Hta_{-ek}(\frac{t}{A})$ | $A^2c_{-ek}^2(\frac{t}{A})$ |
| 1 | $H_2^k = \phi_k(H)A\partial_{\tilde{\tau}} - H\coth(\frac{t}{A})$ | $A^2\sinh^2(\frac{t}{A})$ |
| 3 | $Q_{\mu}^k = \epsilon k\phi_k(C_{\mu})A\partial_{\tilde{\tau}} + C_{\mu}ta_{-ek}(\frac{t}{A})$ | $A^2c_{-ek}^2(\frac{t}{A})$ |
| 3 | $Q_{\mu+3}^k = \phi_k(C_{\mu})A\partial_{\tilde{\tau}} - C_{\mu}\coth(\frac{t}{A})$ | $A_k^2\sinh^2(\frac{t}{A})$ |
TABLE VII. The proper RCs of the RW space-times for \(k = 0\). \(A\) is an integration constant.

| \# | RCs X (\(k = 0\)) | \(\hat{R}_1(\tau)\) |
|----|-------------------|------------------|
| 1  | \(P_\tau = |A|^{1/2} \partial_\tau\) | \(A\) |
| 2  | \(M_{\alpha\tau} = x^\alpha |A|^{1/2} \partial_\tau - \epsilon \tilde{\tau}(\tau) |A|^{-1/2} \partial_\alpha\) | \(A\) |
| 3  | \(H = A \partial_\tau + x^\alpha \partial_\alpha\) | \(\epsilon |A|^{-2} e^{-2\tilde{\tau}(\tau)/A}\) |
| 3  | \(K_\alpha = 2x_\alpha H - (\epsilon e^{2\tilde{\tau}/A} + x^\beta x_\beta) \partial_\alpha\) | \(\epsilon |A|^{-2} e^{-2\tilde{\tau}(\tau)/A}\) |

From Table VI and Table VII we have the following result:

**Proposition 3** RW space-times with \(k = \pm 1\) admit exactly one proper RC when the spatial component \(R_1 = A \neq 0\) and four when \(R_1\) is of the form \(A^2 \sinh^2(\frac{2\tau}{A})\) or \(A^2 e^{2\epsilon x_k (\frac{\tilde{\tau}(\tau)}{A})}\). The flat RW space-times \((k = 0)\) admit four proper RCs when the spatial component \(R_1\) of the Ricci tensor equals \(R_1 = A \neq 0\) and \(R_1 = \epsilon A^2 e^{-2\tilde{\tau}(\tau)/A}\). In both cases \(A\) is a real constant and \(\tilde{\tau}(\tau)\) is defined in (32).

### 6 Matter inheritance collineations of the RW space-times

MICs and MCs of the RW metric are calculated in the same manner as the RICs and RCs. Replacing \(R_{ab}\) from (13), (18) in Einstein field equations:

\[
T_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{36}
\]

and the generic metric (13) for \(\epsilon = -1\) we compute:

\[
T_{ab} = \text{diag} \left[ T_0(\tau), T_1(\tau)U^2(x^\mu), T_1(\tau)U^2(x^\mu), T_1(\tau)U^2(x^\mu) \right]
\]

where:

\[
T_0 = \frac{3(S_{\tau\tau})^2}{S^2}, \quad T_1 = \frac{-2SS_{\tau\tau} + (S_{\tau\tau})^2}{S^2} \tag{37}
\]

are functions of the time coordinate \(\tau\). As expected this is also a RW-like metric. In the case of non-degeneracy \(T_0(\tau)T_1(\tau) \neq 0\) we write:

\[
T_{ab} = T_1(\tau)\text{diag} \left[ \frac{T_0(\tau)}{T_1(\tau)}, U^2(x^\mu), U^2(x^\mu), U^2(x^\mu) \right]. \tag{38}
\]

Therefore the MICs and MCs can be written down immediately from the results of Table II and Table III, if one makes the replacements \(S(\tau) \leftrightarrow T_1(\bar{\tau})\) and \(\tau \leftrightarrow \bar{\tau}\) where \(\bar{\tau}(\tau)\) is defined as follows:

\[
d\bar{\tau} = \sqrt{\frac{T_0(\tau)}{T_1(\tau)}} d\tau \Rightarrow \partial_\tau = \sqrt{\frac{T_1(\tau)}{T_0(\tau)}} \partial_\tau. \tag{39}
\]

We note that the results are similar with those of RICs and RCs collected in tables IV, V and VI, VII respectively provided that one makes the replacements \(R_0(\tau) \leftrightarrow T_0(\tau), R_1(\tau) \leftrightarrow T_1(\tau)\). Obviously there is no need to write explicitly these tables again. Furthermore Proposition 3 applies equally well to MICs and MCs.
Consider the standard RW cosmological model with vanishing cosmological constant and co-moving observers \( u^a = S^{-1}(\tau)\delta^a_0 \), where \( \tau = \int \frac{dt}{S(t)} \) and \( t \) being the standard cosmic time. For these observers the energy momentum tensor has a perfect fluid form i.e. \( T_{ab} = \mu u_a u_b + \rho \delta_{ab} \) where \( \mu, \rho \) are the energy density and the isotropic pressure measured by the observers \( u^a \).

From this decomposition of \( T_{ab} \) and in the coordinates \( (\tau, x^\mu) \) one has:

\[
T_{00} = \mu S^2(\tau), T_{11} = \rho S^2(\tau) U^2(k, x^\mu). \tag{40}
\]

On the other hand using Einstein equations one computes the components \( T_{00}(S, S, S, \tau, \tau, U) \) and \( T_{11}(S, S, S, \tau, \tau, U) \). These two results allow us to compute \( \mu, \rho \) in terms of \( k \) and \( S(\tau) \). Therefore the field equations leave one variable (the \( S(\tau) \)) free and one has to supplement an extra condition to solve the model. This extra equation is a barotropic equation of state \( p = p(\mu) \). The obvious choice is a linear equation of state \( p = (\gamma - 1)\mu \). There are several solutions for this simple choice which are of cosmological interest. For example \( \gamma = 1 \ (p = 0) \) implies degeneracy of the energy momentum-tensor (dust) and the value \( \gamma = \frac{4}{3} \ (p = \frac{4}{3}\mu) \) implies radiation dominated matter. Both states of matter are extreme and they have been relevant at certain stages of the evolution of the Universe. For other values of \( \gamma \) one obtains intermediate states which cannot be excluded as unphysical (see e.g. [19] for a thorough review). However it appears that there does exist some uncertainty in the choice on the value of \( \gamma \) and of course a bigger one in the choice of a linear equation of state. That is, one would be interested to use a non-linear equation of state and deal with more complex forms of matter, but there does not seem to exist an "objective" criterion for writing down such an equation.

One such criterion can be established by the "higher" symmetries we considered in the previous sections. Indeed one can look upon the RICs/RCs and MICs/MCs as "aesthetic" symmetries which can provide via the equation defining them, an extra equation generating an equation of state and consequently a definite RW cosmological model. The reason for the use of these symmetries is twofold:

a. They solve the model completely, that is, they allow us to compute all kinematic and dynamic quantities involved in the RW cosmology

b. As we have shown, they do not violate either the geometry or the coupling of the geometry to Physics.

In the following we shall follow this point of view and we shall use MCs to define the equation of state, that is, we shall determine all flat \((k = 0)\) RW cosmological models which admit a MC. The reason for using MCs is that they are directly related to the components of the energy momentum tensor and one expects that they will have immediate and stronger physical implications.

Writing Table VII in terms of the energy momentum tensor as explained in section 6 we see that there are two cases to consider i.e. \( T_1 = T_{11} = A = const \) and \( T_1 = \varepsilon A^2 e^{-2\tilde{\tau}(\tau)A} \) where the "time" coordinate \( \tilde{\tau} \) is defined as follows:

\[
\tilde{\tau}(\tau) = \int |T_0(\tau)|^{1/2} d\tau = \int \sqrt{\mu} S d\tau \tag{41}
\]

and we have used equations (16) and (40).

From eqs. (16) and (37) we obtain:
\[ \mu = \frac{3(S, \tau)^2}{S^4}, \quad p = \frac{-2SS, \tau + (S, \tau)^2}{S^4} \] (42)

which express the dynamic variables \( \mu, p \) in terms of the scale factor \( S(\tau) \). Two other important kinematic quantities in the RW universe are the Hubble "constant" \( H \) and the deceleration parameter \( q \) defined as follows:

\[ H = \frac{1}{3} \theta = \frac{S, \tau}{S^2}, \quad q = 1 - \frac{SS, \tau}{(S, \tau)^2}. \] (43)

Case I: \( T_1(\tau) = A \equiv \varepsilon_1 a^2 (\varepsilon_1 = \pm 1, a \in \mathbb{R}) \)

The constraint \( T_1(\tau) = \varepsilon_1 a^2 \) leads to the condition:

\[ pS^2(\tau) = \varepsilon_1 a^2 \] (44)

which by means of the second of (42) gives the equation:

\[ -2SS, \tau + (S, \tau)^2 = \varepsilon_1 a^2 S^2. \] (45)

The solution of the differential equation (45) provides the unknown scale factor \( S(\tau) \) and describes the RW model completely. To solve equation (45) we write it in the form:

\[ 2 \left( \frac{S, \tau}{S} \right)_\tau + \left( \frac{S, \tau}{S} \right)^2 = -\varepsilon_1 a^2 \] (46)

which can be integrated easily. In Table VIII we present all four solutions of (46) together with the physical variables of the model that is, energy density (\( \mu \)), isotropic pressure (\( p \)), Hubble constant (\( H \)) and deceleration parameter (\( q \)).

| Case | \( S(\tau) \) | \( \mu(\tau) \) | \( p(\tau) \) | \( H(\tau) \) | \( q(\tau) \) | Restrictions |
|------|---------------|---------------|---------------|---------------|---------------|--------------|
| 1    | \( B^{\varepsilon_1 \tau} \) | \( -\frac{3A}{B^{2(1+\cos \alpha \tau)}} \) | \( \frac{A}{B^{2(1+\cos \alpha \tau)}} \) | \( \frac{3A}{B^{2(1+\cos \alpha \tau)}} \) | 0            | \( \varepsilon_1 = -1, A < 0 \) |
| 2    | \( B \cos^{2} \alpha \tau \) | \( \frac{12A(1-\cos \alpha \tau)}{B^{2(1+\cos \alpha \tau)}} \) | \( \frac{4A}{B^{2(1+\cos \alpha \tau)}} \) | \( \frac{2a \sin \alpha \tau}{B^{2(1+\cos \alpha \tau)}} \) | \( \frac{1+\cos \alpha \tau}{\sin^2 \alpha \tau} \) | \( \varepsilon_1 = 1, A > 0 \) |
| 3    | \( B \sinh^{2} \alpha \tau \) | \( \frac{-3A \cosh^{2} \alpha \tau}{B^{2} \sinh^{4} \alpha \tau} \) | \( \frac{A}{B^{2}} \sinh^{-4} \alpha \tau \) | \( \frac{a \cosh \alpha \tau}{B \sinh^{4} \alpha \tau} \) | \( \frac{1}{2 \cosh^{2} \alpha \tau} \) | \( \varepsilon_1 = -1, A < 0 \) |
| 4    | \( B \cosh^{2} \alpha \tau \) | \( \frac{-3A \tanh^{2} \alpha \tau}{B^{2} \cosh^{4} \alpha \tau} \) | \( \frac{A}{B^{2}} \cosh^{-4} \alpha \tau \) | \( \frac{a \tanh \alpha \tau}{B \cosh^{4} \alpha \tau} \) | \( -\frac{1}{2 \sinh^{2} \alpha \tau} \) | \( \left( \frac{S, \tau}{S} \right)^2 > a^2 \) |

It is a straightforward matter (e.g. by using any algebraic computing program) to check that indeed all four solutions of RW space-times of Table VIII admit the MCs given in Table VII. A detailed study shows that all MCs are proper, except the \( P_{\tau} \) for case 1 which degenerates to a HVF. Furthermore all the energy conditions are satisfied.

Concerning the determination of the equation of state we use the energy conservation equation (i.e. \( T^{\gamma \alpha}_{\alpha} = 0 \)) which in the coordinates \((\tau, x^h)\) gives:

\[ \mu, \tau = -3H(\mu + p)S. \] (47)
From the symmetry condition \((44)\) we compute:

\[
2HSp = -p_{,\tau}.
\]  
(48)

Eliminating \(S(\tau)\) from the last two relations we find:

\[
\frac{dp}{d\mu} = \frac{p_{,\tau}}{\mu_{,\tau}} = \frac{2}{3} \frac{p}{p + \mu}.
\]  
(49)

This equation has two solutions:

\[
p = -\frac{1}{3} \mu
\]  
(50)

and:

\[
\mu - \frac{3B}{a} |p|^{3/2} + 3p = 0.
\]  
(51)

For \(B = 0\) we obtain the first solution, which is a linear equation of state with \(\gamma = \frac{2}{3}\). It corresponds to the solution of case 1 of Table VIII whose metric is:

\[
ds^2 = -dt^2 + t^{\frac{4}{3}(dx^2 + dy^2 + dz^2)}.
\]  
(52)

This space-time admits a HVF \([20]\) represented by the vector \(P_{\tau}\). The rest three vector fields are proper MCs.

The other solution of \((49)\) \((B \neq 0)\) leads to a non-linear equation of state and concerns the cases 2,3,4 of Table VIII.

Case II: \(T_1(\tau) = \varepsilon A^2 e^{-2f(\tau)} A\) \((A \equiv \varepsilon_1 a^2 , \varepsilon_1 = \pm 1 , a \in R)\)

From eqs \((41)\) and \((42)\) we compute \(\tilde{\tau}\) in terms of the scale factor:

\[
\tilde{\tau}(\tau) = \int \sqrt{3} [\ln S(\tau)]_{,\tau} d\tau = \ln[S^{\sqrt{3}}].
\]  
(53)

Using the last equation and the second of \((40)\) \((k = 0)\) we obtain:

\[
p = p_0 S^{-3B}
\]  
(54)

where \(B = \frac{2}{3} \left(1 + \frac{\sqrt{3}}{A}\right)\) and \(p_0 = \varepsilon A^2\). Furthermore eqs \((42)\) and \((54)\) give:

\[
2SS_{,\tau\tau} - (S_{,\tau})^2 = -p_0 S^{4-3B}
\]  
(55)

whose solution is not simple to find.

However we can find the equation of state. From \((54)\) we compute:

\[
p_{,\tau} = -3BpSH
\]  
(56)

which, when combined with \((47)\), gives the following equation among the dynamic variables \(\mu , p:\)

\[
\frac{dp}{d\mu} = \frac{Bp}{\mu + p}.
\]  
(57)

We consider two subcases.
\[ B = 1 \iff A = 2\sqrt{3} \]

The solution of (57) is:

\[ \mu = p \ln |C_p|, C = \text{const}, C_p > 0 \]  \hspace{1cm} (58)

which is always a non-linear equation.

\[ B \neq 1 \iff A \neq 2\sqrt{3} \]

In this case the solution of (57) is:

\[ p - Dp^{1/3} = (B - 1)\mu, D = \text{const.}, B \neq 0, 1. \]  \hspace{1cm} (59)

Note that \( D \neq 0 \) and we have always a non-linear barotropic equation of state.

From the above we conclude that:

**Proposition 4** The only perfect fluid and flat RW universe with a linear equation of state which admits proper MCs is the RW model (52) for \( \gamma = \frac{2}{3} \).

### 8 Discussion

Nearly all known solutions of General Relativity involve some symmetry assumption for the space-time metric. The symmetry is propagated in an intrinsic manner to the higher levels of geometry (by covariant differentiation) and to the Physics (by Einstein field equations). These facts are in complete agreement with the fundamental approach of General Relativity and perhaps they have also helped Einstein in the development of the theory. The most important quantities effected by the symmetry assumption are the Ricci tensor and the energy momentum tensor.

One usually treats the Ricci tensor and the energy momentum tensor as independent of the metric without making use of the fact that these tensors admit the isometry group \( G \) of the metric, therefore they all are members of the same family of second order tensors defined by the group \( G \). As we have shown it is the Lorentzian signature and the non-degeneracy which distinguish the metric from the other two tensors. A further differentiation between them is done by the energy conditions and the direct kinematic and dynamic interpretation of the energy momentum tensor by a congruence of observers.

One interesting question is if there are metrics whose Ricci or Matter tensor is both non-degenerate and has Lorentzian character, therefore it can be used equally well as a space-time metric with the same symmetries as the original space-time metric! This is indeed possible. For example the RW metric:

\[ ds^2 = -dt^2 + \cosh^2 t (dx^2 + dy^2 + dz^2) . \]  \hspace{1cm} (60)

has the Ricci tensor:

\[ R_{00} = -3 , \quad R_{\alpha\beta} = \frac{3}{2} \cosh 2t - \frac{1}{2} \]  \hspace{1cm} (61)

which is again a RW metric.

The results obtained in this paper can be summarized as follows.

a. The determination of the proper symmetries of \( R_{\alpha\beta}, T_{\alpha\beta} \) does not require any new calculations because they follow as the extra symmetries of the generic metric (if they exist) which
is defined by the KVs of the space-time metric. Indeed in Sections V and VI we computed all RCs and MCs collineations of RW space-times using the KVs of the generic RW like metric (15).

b. The collineation tree introduced by Katzin et al. should be reconsidered. That is (for space-time metrics with symmetry) the essential part of the tree ends in the Projective collineations, the rest of them being subcases (except perhaps very few peculiar or uninteresting collineations) which follow as special cases of the basic collineations.

c. RCs, MCs etc. are useful in supplying external constraints to the field equations in order to make the system of field equations autonomous. Because the equations resulting from these constraints are compatible with the assumed symmetry of space-time and the restrictions imposed on Geometry and Physics one expects that they can lead to physically viable models. Indeed in Section VII we have shown that the geometric assumption that the flat RW space-time admits a MC leads to a concrete non-linear (in general) equation of state and consequently to definite RW cosmological models which satisfy the energy conditions and are physically acceptable.

d. The energy momentum tensor attains a physical interpretation in terms of energy density, pressure etc. only after a congruence of observers has been selected. For example, if one considers tilted observers [21, 22] one obtains a RW cosmological model with a non-perfect fluid. This means that equations of state imposed on the dynamical quantities \( \mu, p \) etc. hold for this concrete congruence of observers and do not hold necessarily for other observers. However equations of state defined by means of a symmetry assumption (e.g. a MC) are independent of the observers and apply to the space-time Physics universally.

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