Parafermionic bases of standard modules for affine Lie algebras

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Abstract
In this paper we construct combinatorial bases of parafermionic spaces associated with the standard modules of the rectangular highest weights for the untwisted affine Lie algebras. Our construction is a modification of G. Georgiev’s construction for the affine Lie algebra \( \hat{sl}(n+1, \mathbb{C}) \)—the constructed parafermionic bases are projections of the quasi-particle bases of the principal subspaces, obtained previously in a series of papers by the first two authors. As a consequence we prove the character formula of A. Kuniba, T. Nakanishi and J. Suzuki for all non-simply-laced untwisted affine Lie algebras.

Keywords Affine Lie algebras · Parafermionic space · Combinatorial bases

Mathematics Subject Classification Primary 17B67; Secondary 17B69, 05A19

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Introduction

The parafermionic currents present a remarkable class of nonlocal vertex operators with variables in fractional powers. First examples of parafermionic currents were introduced by Zamolodchikov et al. [36] in the context of conformal field theories in two dimensions. Generalizing their work, Gepner [19] constructed a family of solvable parafermionic conformal field theories related to the untwisted affine Kac–Moody Lie algebras at the positive integer levels. Roughly speaking, the main building block for such conformal field theories is the so-called parafermionic space, which is spanned by the monomials of coefficients of the parafermionic currents. By studying a certain correspondence between the aforementioned conformal field theories and the Thermodynamic Bethe Ansatz Kuniba et al. [26] conjectured the character formulas of the parafermionic spaces associated to the untwisted affine Lie algebras at the positive integer levels. These character formulas, which can be expressed as Rogers–Ramanujan-type sums, were proved by Georgiev [18] in the simply laced case. In this paper, we prove the character formulas of Kuniba, Nakanishi and Suzuki in the non-simply laced case, thus completing the verification of their conjecture.

Now we describe the main result of this paper, the construction of the so-called parafermionic bases. Let

\[ \tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \]

be the affine Kac–Moody Lie algebra associated with the simple Lie algebra \( \mathfrak{g} \) of rank \( l \). Denote by \( \alpha_1, \ldots, \alpha_l \) the positive simple roots of \( \mathfrak{g} \). Let \( \Lambda_0, \ldots, \Lambda_l \) be the fundamental weights of \( \tilde{\mathfrak{g}} \). Consider the standard \( \tilde{\mathfrak{g}} \)-modules \( L(\Lambda) \), i.e. the integrable highest weight modules for \( \tilde{\mathfrak{g}} \), of highest weight \( \Lambda \) of the form

\[ \Lambda = k_0 \Lambda_0 + k_j \Lambda_j \quad \text{for} \quad k_0, k_j \in \mathbb{Z}_{\geq 0} \quad \text{such that} \quad k = k_0 + k_j > 0 \quad (1) \]

with \( j \) as in (1.8). Let \( W_{L(\Lambda)} \) be the principal subspace of \( L(\Lambda) \). The notion of principal subspace goes back to Feigin and Stoyanovsky [11–13]. These subspaces of standard modules posses the so-called quasi-particle bases, which we denote by \( \mathfrak{B}_{W_{L(\Lambda)}} \). The bases are expressed in terms of monomials of quasi-particles, i.e. of certain operators

\[ x_{n\alpha_i}(z) = \sum_{r \in \mathbb{Z}} x_{n\alpha_i}(r)z^{-r-n} \in \text{End } L(\Lambda)[[z^{\pm 1}]], \quad n \geq 1, \quad i = 1, \ldots, l. \]

The significance of the quasi-particle bases lies in the interpretation of the sum sides of various Rogers–Ramanujan-type identities which they provide. Such bases were established by Feigin and Stoyanovsky [11–13] for \( \mathfrak{g} \) of type \( A_1 \). Their results were further generalized by Georgiev [17] to \( \mathfrak{g} \) of type \( A_l \) for all principal subspaces \( W_{L(\Lambda)} \) of the highest weight \( \Lambda \) as in (1). Finally, the quasi-particle bases of \( W_{L(\Lambda)} \) for all \( \Lambda \) as in (1) were constructed by
the first author for \( \mathfrak{g} \) of types \( B_l, C_l, F_4, G_2 \) [2–4,6] and by the first and the second author for \( \mathfrak{g} \) of types \( D_l, E_6, E_7, E_8 \) [7].

The major step towards finding the parafermionic bases is the construction of the suitable bases of the standard modules. This construction relies on the quasi-particle bases of the corresponding principal subspaces from \([2–4,6,7,11–13,17]\). Consider the subalgebras \( \mathfrak{h}^\pm = \mathfrak{h} \otimes \mathbb{R}^\pm \mathbb{C} [t] \) of \( \hat{\mathfrak{g}} \), where \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \). We express the bases for \( L(\Lambda) \) as

\[
\mathfrak{B}_{L(\Lambda)} = \left\{ e_\mu h b \mid \mu \in Q^\vee, h \in B_U(\hat{\mathfrak{h}}^-), b \in \mathfrak{B}' \right\},
\]

(2)

where \( e_\mu \) denote the Weyl group translation operators parametrized by the elements of the coroot lattice \( Q^\vee \) of the simple Lie algebra \( \mathfrak{g} \), the set \( B_U(\hat{\mathfrak{h}}^-) \) is the Poincaré–Birkhoff–Witt-type basis of the universal enveloping algebra \( U(\hat{\mathfrak{h}}^-) \) and \( \mathfrak{B}' \) is a certain subset of \( \mathfrak{B}_{L(\Lambda)} \). We verify that set (2) spans \( L(\Lambda) \) by using the relations among quasi-particles and arguing as in \([2–4,6,7]\). On the other hand, our proof of linear independence relies on generalizing Georgiev’s arguments originated in \([17]\) to standard modules for all untwisted affine Lie algebras.

Next, we turn our attention to the vacuum space of the standard module \( L(\Lambda) \),

\[
L(\Lambda) \hat{\mathfrak{h}}^+ = \{ v \in L(\Lambda) \mid \hat{\mathfrak{h}}^+ \cdot v = 0 \}.
\]

The direct sum decomposition of the standard module,

\[
L(\Lambda) = L(\Lambda) \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- U(\hat{\mathfrak{h}}^-) \cdot L(\Lambda) \hat{\mathfrak{h}}^+,
\]

defines the projection

\[
\pi_{\hat{\mathfrak{h}}^+} : L(\Lambda) \to L(\Lambda) \hat{\mathfrak{h}}^+.
\]

In parallel with Georgiev’s construction in the \( \mathfrak{g} = \mathfrak{sl}_{l+1} \) case \([18]\), by considering the image of (2) with respect to the projection, we find the following basis of the vacuum spaces \( L(\Lambda) \hat{\mathfrak{h}}^+ \) for \( \mathfrak{g} \) of other types,

\[
\mathfrak{B}_{L(\Lambda) \hat{\mathfrak{h}}^+} = \left\{ e_\mu \pi_{\hat{\mathfrak{h}}^+} b \mid \mu \in Q^\vee, b \in \mathfrak{B}' \right\}.
\]

(3)

Recall the Lepowsky–Wilson’s \( Z \)-operators

\[
Z_{\alpha_i}(z) = \sum_{r \in \mathbb{Z}} Z_{\alpha_i}(r) z^{-r-n} \in \text{End} L(\Lambda)[[z^{\pm 1}]], \quad n \geq 1, i = 1, \ldots, l,
\]

which commute with the action of the Heisenberg subalgebra \( \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C} c \) on the standard module \( L(\Lambda) \); see \([28,30]\). Their coefficients \( Z_{\alpha_i}(r) \) can be regarded as operators \( L(\Lambda) \hat{\mathfrak{h}}^+ \to L(\Lambda) \hat{\mathfrak{h}}^+ \). It is worth noting that the projection \( \pi_{\hat{\mathfrak{h}}^+} \) maps the formal series \( x_{\alpha_i}(z) v_{L(\Lambda)} \) to \( Z_{\alpha_i}(z) v_{L(\Lambda)} \). In fact, the elements of vacuum space bases (3) are of the form \( e_\mu b' v_{L(\Lambda)} \), where \( \mu \in Q^\vee \) and \( b' \) is a monomial of \( Z \)-operators \( Z_{\alpha_i}(r) \) whose charges and energies satisfy certain constraints.

Finally, we consider the parafermionic spaces. The notion of parafermionic space in the \( \mathfrak{g} = \mathfrak{sl}_{l+1} \) case, which can be directly generalized to the simply laced case, was introduced by G. Georgiev \([18]\). Unfortunately, his definition relies on the lattice vertex operator construction which we do not have at our disposal in the non-simply laced case. Therefore, we had to slightly alter the original approach. Introduce the integers \( k_{\alpha_i} = 2k/\langle \alpha_i, \alpha_i \rangle \) defined with
respect to the suitably normalized nondegenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$. We define the parafermionic space of highest weight $\Lambda$ as

$$L(\Lambda)_{Q(k)}^{\hat{\mathfrak{h}}^+} = \bigoplus_{0 \leq m_1 \leq k_{a_1} - 1} \cdots \bigoplus_{0 \leq m_l \leq k_{a_l} - 1} L(\Lambda)^{\hat{\mathfrak{h}}^+}_{\Lambda + m_1 a_1 + \cdots + m_l a_l}.$$  \hfill (4)

where $L(\Lambda)_{Q(k)}^{\hat{\mathfrak{h}}^+}$ denote the $\mu$-weight subspaces of the vacuum space $L(\Lambda)^{\hat{\mathfrak{h}}^+}$. In the simply laced case, such definition turns into original Georgiev’s notion of parafermionic space. We should mention that, in the non-simply laced case, there exists another, related notion of the parafermionic space which, in contrast with (4), possesses a generalized vertex operator algebra structure; see, e.g., the book by C. Dong [33]. However, as we do not make use of the generalized vertex algebra theory in this paper and the construction in (4) is easy enough to handle, there was no need to adapt our setting to that of [33].

As with the simply laced case [18], we can define the parafermionic projection

$$\pi_{Q(k)}^{\hat{\mathfrak{h}}^+} : L(\Lambda)^{\hat{\mathfrak{h}}^+} \rightarrow L(\Lambda)^{\hat{\mathfrak{h}}^+}_{Q(k)}.$$  

By employing the projection, we obtain our main result, the construction of bases of the parafermionic spaces of highest weights $\Lambda$ as in (1) which are given by

$$\mathfrak{B}_{L(\Lambda)^{\hat{\mathfrak{h}}^+}} = \left\{ \pi_{Q(k)}^{\hat{\mathfrak{h}}^+} \left( \pi_{Q(k)}^{\hat{\mathfrak{h}}^+} b \right) \mid b \in \mathfrak{B}_W(\Lambda) \right\}.$$  \hfill (5)

These bases generalize Georgiev’s construction [18] to the non-simply laced types.

In order to handle the elements of bases (5), we introduce the parafermionic currents of charge $n \geq 1$ by

$$\Psi_{n\alpha_i}(z) = \sum_{r \in \frac{n}{k_{a_i}} + \mathbb{Z}} \psi_{n\alpha_i}(r) z^{-r-n} = Z_{n\alpha_i}(z) z^{-n/\alpha_i k} \in z^{-n/k_{a_i}} \text{ End } L(\Lambda)^{\hat{\mathfrak{h}}^+}[[z^{\pm 1}]]$$

for $i = 1, \ldots, l$, where $z^{-n/\alpha_i k}$ is a certain operator on the vacuum space $L(\Lambda)^{\hat{\mathfrak{h}}^+}$. The coefficients $\psi_{n\alpha_i}(r)$ can be regarded as operators $L(\Lambda)^{\hat{\mathfrak{h}}^+}_{Q(k)} \rightarrow L(\Lambda)^{\hat{\mathfrak{h}}^+}_{Q(k)}$. Moreover, the elements of parafermionic space bases (5) are of the form $b'' v_{L(\Lambda)}$, where $b''$ is a monomial of operators $\psi_{n\alpha_i}(r)$ whose charges and energies satisfy certain constraints and $v_{L(\Lambda)}$ now denotes the image of the highest weight vector with respect to the parafermionic projection. The characters of conformal field theories, as well as their connection to Rogers–Ramanujan-type identities, were extensively studied in physics literature; see, e.g., [1,8,16,25,26] and the references therein. In particular, fermionic formulas related to certain tensor products of standard modules which generalize those in [26], were studied in [21,22] and, more recently, in [9,34], in the setting of quantum affine algebras. The problem of developing the precise mathematical foundation for parafermionic conformal field theories led to important generalizations of the notion of vertex operator algebra; see, e.g., the book by C. Dong and J. Lepowsky [10]. In particular, as demonstrated by H.-S. Li [33], the vacuum space $L(k\Lambda_0)^{\hat{\mathfrak{h}}^+}$ possesses the structure of generalized vertex algebra. Hence it is equipped by a certain grading operator $L_{\Omega}(0)$; see also [10,18]. We show that the corresponding induced operator on the parafermionic space $L(\Lambda)^{\hat{\mathfrak{h}}^+}_{Q(k)}$ is the parafermionic grading operator. More specifically, we check that the elements of parafermionic basis (5) are its eigenvectors. Finally,
we use their so-called conformal energies, i.e. the eigenvalues of the parafermionic basis elements with respect to $L_{\Omega(0)}$, to calculate the characters for the parafermionic spaces, thus recovering the character formulas of A. Kuniba, T. Nakanishi and J. Suzuki [26].

1 Preliminaries

1.1 Modules of affine Lie algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $l$, $\mathfrak{h}$ its Cartan subalgebra and $R$ the system of roots. Let $Q \subset P \subset \mathfrak{h}^*$ and $Q^\vee \subset P^\vee \subset \mathfrak{h}$ be the root, weight, coroot and coweight lattices respectively. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Using the form one can identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via $\langle \alpha, h \rangle = \alpha(h)$ for $\alpha \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. We fix the form so that $\langle \alpha, \alpha \rangle = 2$ if $\alpha$ is a long root.

Fix simple roots $\alpha_1, \ldots, \alpha_l$ and let $R_+$ ($R_-$) be the set of positive (negative) roots. Denote by $\theta$ the highest root. We take labelings of the Dynkin diagrams as in Fig. 1.

Simple roots span over $\mathbb{Z}$ the root lattice $Q$. The coweight lattice $P^\vee$ consists of elements on which the simple roots take integer values. The coroot lattice $Q^\vee$ is spanned by the simple coroots $\alpha^\vee_i, i = 1, \ldots, l$, and the the weight lattice $P$ is spanned by the fundamental weights $\omega_i, i = 1, \ldots, l$, defined by $\langle \omega_i, \alpha^\vee_r \rangle = \delta_{ir}$ for each $i, r = 1, \ldots, l$, (cf. [23]).

We have a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{n}_\alpha$, and $\mathfrak{n}_\alpha$ is the root subspace for $\alpha \in R$. For every positive root $\alpha$, we fix $x_{\pm \alpha} \in \mathfrak{n}_{\pm \alpha}$ and $\alpha^\vee \in \mathfrak{h}$ such that

\[
[x_\alpha, x_{-\alpha}] = \alpha^\vee, \quad [\alpha^\vee, x_\alpha] = 2x_\alpha \quad \text{and} \quad [\alpha^\vee, x_{-\alpha}] = -2x_\alpha.
\]
That is, $\mathfrak{sl}_2(\alpha) := \mathbb{C}x_\alpha \oplus \mathbb{C}a^\vee \oplus \mathbb{C}x_{-\alpha}$ is isomorphic to $\mathfrak{sl}_2$.

Consider the untwisted affine Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

with the canonical central element $c$, where

$$[x(m), y(n)] = [x, y](m + n) + \langle x, y \rangle m\delta_{m+n, 0} c,$$

for all $x, y \in \mathfrak{g}, m, n \in \mathbb{Z}$ (cf. [24]). Here, traditionally, we denote $x \otimes t^m$ by $x(m)$. For an element $x \in \mathfrak{g}$ denote the generating function of $x(m)$ by

$$x(z) = \sum_{m \in \mathbb{Z}} x(m)z^{-m-1}.$$

If we adjoin the degree operator $d$, where

$$[d, x(m)] = mx(m) \quad \text{and} \quad [d, c] = 0,$$

we obtain the affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}d$.

Define the following subalgebras of $\mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$:

$$\widehat{\mathfrak{h}}^+ = \mathfrak{h} \otimes t\mathbb{C}[t], \quad \widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}] \quad \text{and} \quad \mathfrak{s} = \bigsqcup_{n \in \mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c. \quad (1.6)$$

Then $\mathfrak{s}$ is a Heisenberg subalgebra of $\widehat{\mathfrak{g}}$. Denote by $M(k)$ the irreducible $\mathfrak{s}$-module with $c$ acting as a scalar $k$ (cf. [24]). As a vector space, $M(k)$ is isomorphic to the $U(\widehat{\mathfrak{h}}^-)$.

The form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$ extends to $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ using this form we identify $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with its dual $(\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$. The simple roots of $\widehat{\mathfrak{g}}$ are $\alpha_0, \alpha_1, \ldots, \alpha_l$ and the simple coroots are $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee$. For each $\alpha$ denote by $\widehat{\mathfrak{sl}}_2(\alpha)$ the affine Lie subalgebra of $\widehat{\mathfrak{g}}$ of type $A^{(1)}_l$ with the canonical central element $c_\alpha = \frac{2c}{\langle \alpha, \alpha \rangle}$.

Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_l$ be the fundamental weights of $\widehat{\mathfrak{g}}$. Denote by $L(\Lambda)$ the standard $\widehat{\mathfrak{g}}$-module with the highest weight vector $\nu_{\Lambda}$ of the rectangular highest weight $\Lambda$, i.e. a weight of the form

$$\Lambda = k_0\Lambda_0 + k_l\Lambda_l \quad \text{for} \quad k_0, k_l \in \mathbb{Z}_{\geq 0} \quad \text{such that} \quad k = k_0 + k_l > 0, \quad (1.7)$$

where

$$\begin{cases}
  j = 1, \ldots, l & \text{for } \widehat{\mathfrak{g}} \text{ of type } A^{(1)}_l \text{ or } C^{(1)}_l, \\
  j = 1, l & \text{for } \widehat{\mathfrak{g}} \text{ of type } B^{(1)}_l, \\
  j = 1, l - 1, l & \text{for } \widehat{\mathfrak{g}} \text{ of type } D^{(1)}_l, \\
  j = 1, 6 & \text{for } \widehat{\mathfrak{g}} \text{ of type } E^{(1)}_6, \\
  j = 1 & \text{for } \widehat{\mathfrak{g}} \text{ of type } E^{(1)}_7, \\
  j = 4 & \text{for } \widehat{\mathfrak{g}} \text{ of type } F^{(1)}_4, \\
  j = 2 & \text{for } \widehat{\mathfrak{g}} \text{ of type } G^{(1)}_2. 
\end{cases} \quad (1.8)$$

Note that the standard module $L(\Lambda)$ of level $k$ can be regarded as an integrable $\widehat{\mathfrak{sl}}_2(\alpha)$-module of level

$$k_\alpha = \frac{2k}{\langle \alpha, \alpha \rangle}.$$

Consider the following subalgebras of $\widehat{\mathfrak{g}}$

$$\widehat{n} = n_+ \otimes \mathbb{C}[t, t^{-1}] \quad \text{and} \quad \widehat{n}_\alpha = n_\alpha \otimes \mathbb{C}[t, t^{-1}].$$
We define the principal subspace $W_{L(\Lambda)}$ of $L(\Lambda)$ as

$$W_{L(\Lambda)} = U(\tilde{\mathfrak{g}}) v_\Lambda,$$

(cf. [11–13]). In [2–7,17] the bases of the principal subspaces $W_{L(\Lambda)}$ for the affine Lie algebras $\tilde{\mathfrak{g}}$ of different types were described in terms of quasi-particles, which we introduce in the following subsection.

### 1.2 Quasi-particles

Recall that $L(k\Lambda_0)$ is a vertex operator algebra (cf. [15,27]) with the vacuum vector $v_{L(k\Lambda_0)}$ generated by $x(-1)v_{L(k\Lambda_0)}$ for $x \in \mathfrak{g}$ such that

$$Y(x(-1)v_{L(k\Lambda_0)}, z) = x(z).$$

The level $k$ standard $\tilde{\mathfrak{g}}$-modules are modules for this vertex operator algebra.

We will consider the vertex operators

$$x_{\alpha_1}(z) = \sum_{m \in \mathbb{Z}} x_{\alpha_1}(m) z^{-m} - \sum_{r \geq 1} x_{\alpha_1}(z) \cdots x_{\alpha_1}(z)^r$$

associated with the vector $x_{\alpha_1}(-1)^r v_{L(k\Lambda_0)} \in L(k\Lambda_0)$. Now, as in [17], for a fixed positive integer $r$ and a fixed integer $m$ define the quasi-particle of color $i$, charge $r$ and energy $-m$ as the coefficient $x_{\alpha_1}(m)$ of (1.9).

### 1.3 Quasi-particle bases of the principal subspaces of standard modules

Denote by $M_{QP}$ the set of all quasi-particle monomials of the form

$$b = b(\alpha_i) \cdots b(\alpha_1)$$

$$= x_{n_{i_1}^{r_{i_1}}} \cdots x_{n_{i_l}^{r_{i_l}}} (m_{r_{i_1}}^{s_{i_1}}) \cdots x_{n_{1,1}} a_1 (m_{1,1}),$$

where $1 \leq n_{i_j}^{r_{i_j}} \leq \ldots \leq n_{1,i}$ for all $i = 1, \ldots, l$ and $b(\alpha_i)$ denotes the submonomial of $b$ consisting of all color $i$ quasi-particles. We allow the indices $r_{i_j}^{s_{i_j}} = 0$ for some $i = 1, \ldots, l$ this means that the monomial $b$ does not contain any quasi-particles of color $i$ and we have $b(\alpha_i) = 1$. We now introduce some terminology (cf. [2–4,17]). The charge-type of the monomial $b$ is defined as the $l$-tuple

$$\mathcal{R}' = (\mathcal{R}'_1, \ldots, \mathcal{R}'_l),$$

where $\mathcal{R}'_i = (n_{i_1}^{r_{i_1}}, \ldots, n_{1,i})$ for $i = 1, \ldots, l$.

The dual-charge-type of $b$ is defined as the $l$-tuple

$$\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_l),$$

where $\mathcal{R}_i = (r_{i_1}^{s_{i_1}}, \ldots, r_{i_l}^{s_{i_l}})$ for $i = 1, \ldots, l$, (1.11)

where $r_{i_j}^{(n)}$ denotes the number of quasi-particles of color $i$ and of charge greater than or equal to $n$ in the monomial $b$ and $s_{j} = n_{1,j}$. The charge-type and the dual-charge-type of monomial (1.10) can be represented by the $l$-tuple of diagrams, so that the $i$-th diagram corresponds to $b(\alpha_i)$. In the $i$-th diagram, the number of boxes in the $n$-th row equals $r_{i_j}^{(n)}$ and the number of boxes in the $p$-th column equals $n_{p,i}$, where the rows are counted from the bottom and the columns are counted from the right. Clearly, the charge-type and the dual-charge-type of monomial (1.10) can be easily recovered from such diagrams; see Fig. 2.
The energy-type of $b$ (see [7]) is given by
\[ E = (m_{r_1^{(1)},1}, ..., m_{r_1^{(s-1)},1}; ..., m_{r_1^{(1)},l}, ..., m_{1,1}) \, . \]
Denote by $e_n b$ the total energy of monomial $b$,
\[ e_n b = -m_{r_1^{(1)},1} - \cdots - m_{1,l} - \cdots - m_{r_1^{(1)},1} - \cdots - m_{1,1} \, . \]
For every color $i$, $1 \leq i \leq l$, we introduce the total charge of color $i$
\[ \text{chg}_i b = \sum_{p=1}^{r_1^{(1)}} n_{p,i} = \sum_{t=1}^{r_i} n_{t,i} \, , \quad (1.12) \]
and also the total charge of the monomial $b$
\[ \text{chg} b = \sum_{i=1}^{l} \text{chg}_i b \, . \]
Following [17] (see also [2–4,7]) we say that the monomial $b$ is of color-type
\[ \mathcal{C} = (\text{chg}_1 b, ..., \text{chg}_l b) \, . \]
Let $b$ and $\bar{b}$ be any two monomials of the same total charge and let $\mathcal{C} = (c_1, ..., c_l)$ and $\bar{\mathcal{C}} = (\bar{c}_1, ..., \bar{c}_l)$ be the color-types of $b$ and $b'$ respectively. We write $\mathcal{C} \prec \bar{\mathcal{C}}$ if there exists $u = 1, ..., l$ such that
\[ c_1 = \bar{c}_1, \, c_2 = \bar{c}_2, ..., c_{u-1} = \bar{c}_{u-1} \, \text{and} \, c_u < \bar{c}_u \, . \]
On the set of monomials in $M_{QP}$ of the fixed total energy and $h$-weight we introduce the linear order “$<$” as follows. We write
\[ b < \bar{b} \quad \text{if} \quad \mathcal{R} < \mathcal{R}' \quad \text{or} \quad \mathcal{R}' = \mathcal{R} \quad \text{and} \quad E < \bar{E} \, , \]
where $\mathcal{R}'$ and $\mathcal{E}$ denote the charge-type and the energy-type of the monomial $\bar{b}$ and we write $\mathcal{R}' < \mathcal{R} \, \text{with} \, \mathcal{R}' = (\bar{n}_{r_1^{(1)},1}, ..., \bar{n}_{l,1})$, if there exist $i = 1, ..., l$ and $u = 1, ..., r_i^{(1)}$ such...
that
\[ r_j^{(1)} = \bar{r}_j^{(1)} \quad \text{and} \quad n_{1,j} = \bar{n}_{1,j}, \quad n_{2,j} = \bar{n}_{2,j}, \ldots, \quad n_{r_j^{(1)},j} = \bar{n}_{r_j^{(1)},j} \quad \text{for} \quad j = 1, \ldots, i - 1, \]
\[ u \leq r_i^{(1)} \quad \text{and} \quad n_{1,i} = \bar{n}_{1,i}, \quad n_{2,i} = \bar{n}_{2,i}, \ldots, \quad n_{u-1,i} = \bar{n}_{u-1,i}, \quad n_{u,i} < \bar{n}_{u,i} \quad \text{or} \]
\[ u = r_i^{(1)} < \bar{r}_i^{(1)} \quad \text{and} \quad n_{1,i} = \bar{n}_{1,i}, \quad n_{2,i} = \bar{n}_{2,i}, \ldots, \quad n_{u,i} = \bar{n}_{u,i}. \]
In a similar way we define the order “<” for energy-types.

Set \( \mu_i = k_{\alpha_i}/k_{\alpha_j} \) for \( i = 2, \ldots, l, \) where
\[
i' = \begin{cases} l - 2, & \text{if } i = l \text{ and } g = D_l, \vspace{2mm} \\ 3, & \text{if } i = l \text{ and } g = E_6, E_7, \vspace{2mm} \\ 5, & \text{if } i = l \text{ and } g = E_8, \vspace{2mm} \\ i - 1, & \text{otherwise.} \end{cases}
\]
For any \( j = 1, \ldots, l \) as in (1.8) let
\[
j_t = \begin{cases} j, & \text{if } v_jk_0 + (v_j - 1)k_j + 1 \leq t \leq k_{\alpha_j}, \quad \text{where} \quad v_j = k_{\alpha_j}/k. \vspace{2mm} \\ 0, & \text{otherwise,} \end{cases}
\]
We have (see [2–7,17])

**Theorem 1.1** For any integer \( k > 0 \) the set
\[ \mathfrak{B}_{W_{L(\Lambda)}} = \{ b v_\Lambda \mid b \in B_{W_{L(\Lambda)}} \} \]
is a basis of the principal subspace \( W_{L(\Lambda)} \), where

\[
B_{W_{L(\Lambda)}} = \bigcup_{\begin{subarray}{c} r_1^{(1)}, \ldots, r_l^{(1)} \geq 0 \\ 1 \leq n_{r_1^{(1)},1} \leq \ldots \leq n_{1,1} \leq k_{\alpha_1} \\ \vspace{2mm} \vdots \vspace{2mm} \\ 1 \leq n_{r_l^{(1)},1} \leq \ldots \leq n_{1,1} \leq k_{\alpha_l} \end{subarray}} \left( \bigcup_{\begin{subarray}{c} r_1^{(1)} \geq \ldots \geq r_l^{(1)} \geq 0 \vspace{2mm} \\ \vspace{2mm} \vdots \vspace{2mm} \\ r_l^{(1)} \geq \ldots \geq r_1^{(1)} \geq 0 \end{subarray}} \right) \left| \begin{array}{c} b = b(\alpha_1) \ldots b(\alpha_l) \\ = x_{n_{r_1^{(1)},1} \alpha_1 (m_{r_1^{(1)},1} \alpha_1 (m_{1,1} \ldots x_{n_{r_l^{(1)},1} \alpha_1 (m_{r_1^{(1)},1} \ldots x_{n_{1,1} \alpha_1 (m_{1,1})} } \end{array} \right| \\
| m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_i^{(1)}} \min \left\{ \mu_i n_{q,i}, n_{p,i}, 2(p - 1)n_{p,i} - \sum_{t=1}^{n_{p,i}} \delta_{ij} \right\} \right. \\
\left. \quad \text{for all } 1 \leq p \leq r_i^{(1)}, \quad 1 < i \leq l; \right.
\]
\[ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i}, \quad 1 \leq p \leq r_i^{(1)} - 1, \quad 1 < i \leq l \]

Denote by \( M'_{QP} \subset M_{QP} \) the set of all quasi-particle monomials as in (1.10) with no quasi-particles \( x_{k_{\alpha_i}(r)} \) for \( i = 1, \ldots, l \). Finally, we introduce the set
\[ \mathfrak{B}'_{W_{L(\Lambda)}} = \{ b v_\Lambda \mid b \in B'_{W_{L(\Lambda)}} \}, \quad \text{where} \]
\[ B'_{W_{L(\Lambda)}} = B_{W_{L(\Lambda)}} \cap M'_{QP} = \{ b \in B_{W_{L(\Lambda)}} \mid n_{1,i} < k_{\alpha_i} \text{ for } i = 1, \ldots, l \}. \]
1.4 Character formula for principal subspaces

We write

\[(a; q)_r = \prod_{i=1}^{r} (1 - aq^{i-1}) \quad \text{for } r \geq 0 \quad \text{and} \quad (a; q)_\infty = \prod_{i \geq 1} (1 - aq^{i-1}).\]

Set \(n_i = \sum_{t=1}^{k_{ai}} r_i^{(t)}\) for \(i = 1, \ldots, l\). Notice that \(n_i = \text{chg}_i b\); recall (1.12). For any dual-charge-type \(\mathcal{R}\), as given by (1.11), which satisfies

\[r_i^{(1)} \geq \ldots \geq r_i^{(k_{ai})} \geq 0 \quad \text{for all } i = 1, \ldots, l,\]

define the expressions

\[F_{\mathcal{R}_i}(q) = \frac{q^{\sum_{t=1}^{k_{ai}} r_i^{(t)^2}} + \sum_{t=1}^{k_{ai}} r_i^{(t)\delta_{ij}}}{(q; q)_{r_i^{(1)}-r_i^{(2)} \ldots - (q; q)_{r_i^{(k_{ai})}}}} \quad \text{and} \quad I_{\mathcal{R}_i, \mathcal{R}_i'}^{ij}(q) = q^{-\sum_{t=1}^{k} t_{ij} \sum_{p=0}^{q-1} (n_i^{(t-p)})},
\]

where for \(i = 1\) we set \(I_{\mathcal{R}_i, \mathcal{R}_i'}^{ij}(q) = 1\).

Let \(\delta = \sum_{i=0}^{l} a_i \alpha_i\) be the imaginary root as in [24, Chapter 5]. Then the integers \(a_i\) are the labels in the corresponding Dynkin diagram; see [24, Table Aff]. Define the character \(\text{ch} W_{L(A)}\) of the principal subspace \(W_{L(A)}\) by

\[\text{ch} W_{L(A)} = \sum_{n,n_1,\ldots,n_l \geq 0} \dim (W_{L(A)})_{(n; n_1, \ldots, n_l)} q^n y_1^{n_1} \ldots y_l^{n_l},\]

where \(q, y_1, \ldots, y_l\) are formal variables and \((W_{L(A)})_{(n; n_1, \ldots, n_l)}\) is the weight subspace of \(W_{L(A)}\) of the weight \(\Lambda - n\delta + n_1 \alpha_1 + \ldots + n_l \alpha_l\) with respect to \(\mathfrak{h} \oplus \mathbb{C} d\). Theorem 1.1 implies the following character formulas:

**Theorem 1.2** For any integer \(k \geq 1\) we have

\[\text{ch} W_{L(A)} = \sum_{i=1}^{l} F_{\mathcal{R}_i}(q) I_{\mathcal{R}_i, \mathcal{R}_i'}^{ij}(q) \prod_{p=1}^{l} y_p^{n_p},\]

where the sum in (1.14) goes over all dual-charge-types \(\mathcal{R}\) satisfying (1.13).

Recall that \(v_i = \frac{k_{ai}}{k} = \frac{2}{(a_i, \alpha_i)}\) for \(i = 1, \ldots, l\). Define

\[G_{mn}^{rr} = \min \{ v_r m, v_i n \} \cdot \langle \alpha_i, \alpha_r \rangle \quad \text{for all } i, r = 1, \ldots, l \quad \text{and} \quad m, n \geq 1.
\]

For dual-charge-type (1.11) with \(s_i = k_{ai}\) define the elements \(\mathcal{P}_i = (p_i^{(1)}, \ldots, p_i^{(k_{ai})})\) by

\[\mathcal{P}_i = (r_i^{(1)} - r_i^{(2)}, r_i^{(2)} - r_i^{(3)}, \ldots, r_i^{(k_{ai} - 1)} - r_i^{(k_{ai})}, r_i^{(k_{ai})}), \quad \text{where } i = 1, \ldots, l,
\]

so that \(p_i^{(t)}\) denotes the number of quasi-particles of color \(i\) and charge \(t\) in monomial (1.10).

Note that the integers \(n_i = \sum_{t=1}^{k_{ai}} r_i^{(t)}\) can be expressed in terms of \(p_i^{(t)}\) as \(n_i = \sum_{t \geq 1} t p_i^{(t)}\).

Organize all elements \(\mathcal{P}_i\) into the \(l\)-tuple \(\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l)\). Using the notation

\[D_\mathcal{P}(q) = \frac{1}{\prod_{i=1}^{l} \prod_{r=1}^{k_{ai}} (q; q)_{p_i^{(r)}}}, \quad G_\mathcal{P}(q) = q^{1/2} \sum_{r=1}^{l} \sum_{m=1}^{k_{ai}} \sum_{n=1}^{k_{ai}} G_{mn}^{rr} p_i^{(m)} p_r^{(n)},\]

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\(B_{\mathcal{P}}(q) = q^{\sum_{i=1}^{l} k_{i}}\) one can express character formula (1.14) as

\[
\text{ch } W_{L(\Lambda)} = \sum_{\mathcal{P}} D_{\mathcal{P}}(q) G_{\mathcal{P}}(q) B_{\mathcal{P}}(q) \prod_{i=1}^{l} y_{i}^{n_{i}},
\]

where the sum goes over all finite sequences \(\mathcal{P} = (\mathcal{P}_{1}, \ldots, \mathcal{P}_{l})\) of \(k_{\alpha_{1}} + \cdots + k_{\alpha_{l}}\) nonnegative integers. In particular, observe that for \(\Lambda = k_{0} \Lambda_{0} = k \Lambda_{0}\) we have \(B_{\mathcal{P}}(q) = 1\), so the character formula takes the form

\[
\text{ch } W_{L(k \Lambda_{0})} = \sum_{\mathcal{P}} D_{\mathcal{P}}(q) G_{\mathcal{P}}(q) \prod_{i=1}^{l} y_{i}^{n_{i}^{'}}.
\]  

In the end, note that

\[
\text{ch span } \mathcal{B}_{W_{L(\Lambda)}} = \sum_{\mathcal{P}} D'_{\mathcal{P}}(q) G'_{\mathcal{P}}(q) B'_{\mathcal{P}}(q) \prod_{i=1}^{l} y_{i}^{n_{i}^{'}},
\]

where

\[
D'_{\mathcal{P}}(q) = \frac{1}{\prod_{i=1}^{l} \prod_{r=1}^{k_{i}-1} (q; q)_{P_{i}^{(r)}}}, \quad G'_{\mathcal{P}}(q) = q^{\frac{1}{2} \sum_{r=1}^{l} \sum_{m=1}^{k_{i}-1} \sum_{n=1}^{k_{i}-1} G_{rr}^{m} p_{i}^{(m)} p_{i}^{(n)}},
\]

\[
B'_{\mathcal{P}}(q) = q^{\sum_{i=1}^{l} k_{i} - \sum_{r=1}^{k_{i}-1} (t-v_{j}k_{0}+(v_{j}-1)k_{j}) p_{i}^{(r)}}.
\]  

The sum in (1.16) goes over all finite sequences \(\mathcal{P} = (\mathcal{P}_{1}, \ldots, \mathcal{P}_{l})\) of \(k_{\alpha_{1}} + \cdots + k_{\alpha_{l}} - l\) nonnegative integers, with \(\mathcal{P}_{i} = (p_{1}, \ldots, p_{l}^{(k_{i}-1)})\) and with \(n_{i}^{'} = \sum_{t=1}^{l} p_{i}^{(t)}\).

### 1.5 Weyl group translation operators

For every root \(\alpha \in R\) we define on the standard module \(L(\Lambda)\) of level \(k\) the Weyl group translation operator \(e_{\alpha^{\vee}}\) by

\[
e_{\alpha^{\vee}} = \exp x_{-\alpha}(1) \exp x_{-\alpha}(1) \exp x_{\alpha}(0) \exp (-x_{\alpha}(0)) \exp x_{\alpha}(0),
\]

(cf. [24]). The map \(\alpha^{\vee} \mapsto e_{\alpha^{\vee}}\) extends to a projective representation of the root lattice \(Q^{\vee}\) on \(L(\Lambda)\) such that \(e_{\alpha^{\vee}} e_{-\alpha^{\vee}} = 1\).

We will use the following relations on the standard \(\mathfrak{g}\)-module \(L(\Lambda)\), which are consequence of the adjoint action of \(e_{\alpha^{\vee}}\) on \(\mathfrak{g}\) (cf. [14,24], see also [35]):

\[
e_{\alpha^{\vee}} c e_{\alpha^{\vee}} = c, \quad \alpha^{\vee} = \frac{1}{2} \{ \alpha^{\vee}, \alpha^{\vee} \} c, \quad h c e_{\alpha^{\vee}} = h - \{ \alpha^{\vee}, h \} c, \quad e_{\alpha^{\vee}} h(j)e_{\alpha^{\vee}} = h(j) \text{ for } j \neq 0, \quad e_{\alpha^{\vee}} x_{\beta}(j)e_{\alpha^{\vee}} = x_{\beta}(j - \beta(\alpha^{\vee})),
\]

where \(h \in \mathfrak{h}, \beta \in R\) and \(j \in \mathbb{Z}\).
\section*{1.6 Operators $E^{\pm}(h, z)$}

For $h \in \mathfrak{h}$ set

$$E^{\pm}(h, z) = \exp\left(\sum_{n \geq 1} h(\pm n) \frac{z^{\pm n}}{\pm n}\right).$$

On the highest weight level $k\hat{\mathfrak{g}}$-module the formal power series $E^-(h, z)E^+(h, z)$ is well defined. We have

$$[h'(i), E^-(h, z)E^+(h, z)] = -\{h', h\}kz^i E^-(h, z)E^+(h, z) \quad \text{for } h' \in \mathfrak{h}, \ i \in \mathbb{Z}.$$ 

Note that

$$[h'(i), x_\alpha(z)] = \alpha(h')z^ix_\alpha(z) \quad \text{for } h' \in \mathfrak{h}, \ i \in \mathbb{Z}, \ \alpha \in \mathbb{R}. \quad (1.24)$$

Hence on the highest weight level $k\hat{\mathfrak{g}}$-module Lepowsky–Wilson’s $Z$-operators

$$Z_\alpha(z) = E^-(\alpha, z)^{1/k} x_\alpha(z) E^+(\alpha, z)^{1/k}$$

commute with the action of the Heisenberg subalgebra $s$ defined by (1.6).

We have (cf. [29,31,32])

\textbf{Lemma 1.3} For any simple root $\alpha$ and $h \in \mathfrak{h}$ we have

$$E^+(h, z_1)x_\alpha(z_2) = (1 - z_2/z_1)^{-\alpha(h)} x_\alpha(z_2) E^+(h, z_1), \quad (1.25)$$

$$x_\alpha(z_2)E^-(h, z_1) = (1 - z_1/z_2)^{-\alpha(h)} E^-(h, z_1)x_\alpha(z_2). \quad (1.26)$$

\section*{1.7 Vertex operator formula}

From $x_{(k_\alpha+1)\alpha}(z) = 0$ on $L(\Lambda)$ follows that $\exp(z x_\alpha(z))$ is well defined. Then we have a generalization of the Frenkel–Kac [14] vertex operator formula

$$\exp(z x_\alpha(z)) = E^-(\alpha^\vee, z)\exp(z x_{-\alpha}(z)) E^+(\alpha^\vee, z)e_{\alpha^\vee} z^{\epsilon_{\alpha} + \alpha^\vee}, \quad (1.27)$$

where $z^{\epsilon_{\alpha} + \alpha^\vee}$ is defined by

$$z^{\epsilon_{\alpha} + \alpha^\vee} v_\mu = v_\mu z^{k_\alpha + \mu(\alpha^\vee)}, \quad (1.28)$$

for a vector $v_\mu$ of $\mathfrak{h}$-weight $\mu$ (see [28,35]). The $\mathfrak{h}$-weight components of the vertex operator formula (1.27) give relations

$$\frac{1}{p!} (z x_\alpha(z))^p = \frac{1}{q!} E^-(\alpha^\vee, z)(-z x_{-\alpha}(z))^q E^+(\alpha^\vee, z)e_{\alpha^\vee} z^{\epsilon_{\alpha} + \alpha^\vee}, \quad (1.29)$$

where $k_\alpha = p + q$, $p, q \geq 0$.

\section*{2 Quasi-particle bases of standard modules}

\subsection*{2.1 Spanning sets for standard modules}

We consider standard modules $L(\Lambda)$ of the highest weight $\Lambda$ given by (1.7).

\textbf{Lemma 2.1} For any standard module we have:

(i) $L(\Lambda) = U(\tilde{\mathfrak{h}}^-) Q^\vee W_{L(\Lambda)}$.
(ii) $L(\Lambda) = Q^\vee W_{L(\Lambda)}$.

Proof Since $\hat{g}$ is generated by the Chevalley generators $e_i$ and $f_i$, $i = 0, \ldots, l$, the standard module $L(\Lambda)$ is clearly the span of noncommutative monomials in $x_{\pm a_i}(m)$, $i = 1, \ldots, l$ and $m \in \mathbb{Z}$, acting on the highest weight vector $v^\Lambda$. By using the vertex operator formula (1.27) on $L(\Lambda)$ we can express $x_{-a_i}(m)$ in terms of $x_{a_i}(m')$, the Weyl group translation operator $e_{-a_i}$ and a polynomial in $U(\hat{h})$. By using the relations (1.19)–(1.23) we see that (i) holds. Since monomials in coefficients of $E^\pm(-\alpha_i, \varepsilon)$ span $U(\hat{h}^-)$, one can prove that (i) implies (ii) by using relation (1.29) for $q = 0$ and relations (1.19)–(1.23).

Remark 2.2 The second statement in Lemma 2.1 implies that

$$\{e_\mu v \mid \mu \in Q^\vee, v \in \mathcal{B}_{W_L(\Lambda)}\}$$

is a spanning set of $L(\Lambda)$. However this is not a basis. For example, in the $\mathfrak{sl}_2$-module $L(\Lambda_0)$ we have the relation

$$x_{\alpha}(-1)v_{\Lambda_0} = -e^{\alpha v^\Lambda}v_{\Lambda_0}.$$ 

Consider the basis $B_{U(\hat{h}^-)}$ of the irreducible $\mathfrak{s}$-module $M(k)$ of level $k$,

$$B_{U(\hat{h}^-)} = \{ h_{a_1} \cdots h_{a_l} \mid h_{a_i} = (\alpha'_i(-j_{i+1,i}))^{r_{i+1,i}} \cdots (\alpha'_i(-j_{1,i}))^{r_{1,i}}, \ r_{p,i}, j_{p,i} \in \mathbb{N}, \ j_{1,i} \leq \cdots \leq j_l,i, \ t_i \in \mathbb{Z}_{\geq 0}, \ p = 1, \ldots, t_i, \ i = 1, \ldots, l \}. $$

For the elements $h = h_{a_1} \cdots h_{a_l}, \ h = \tilde{h}_{a_1} \cdots \tilde{h}_{a_l} \in B_{U(\hat{h}^-)}$ of fixed degree, we will write $h < \tilde{h}$ if

1. $(r_{l+1,1}, \ldots, r_{1,1}) < (\overline{r}_{l+1,1}, \ldots, \overline{r}_{1,1})$ or
2. $(r_{l+1,1}, \ldots, r_{1,1}) = (\overline{r}_{l+1,1}, \ldots, \overline{r}_{1,1})$ and $(j_{l+1,1}, \ldots, j_{1,1}) < (\overline{j}_{l+1,1}, \ldots, \overline{j}_{1,1}),$

where the order “$<$” in (1) and (2) is defined in the same way as the linear order on charge-types in Sect. 1.3. For the purpose of proving the next lemma we now generalize the linear order on the quasi-particle monomials, as defined in Subsection 1.3, as well as the linear order on the elements of $B_{U(\hat{h}^-)}$, to the set

$$\{e_\mu hbv^\Lambda \mid \mu \in Q^\vee, h \in B_{U(\hat{h}^-)}, b \in M'_{QP}\}$$

as follows. For any two vectors $e_\mu h bv^\Lambda, e_\mu \overline{h} bv^\Lambda$ in (2.1) of fixed degree and $\hat{h}$-weight denote the color-types of $b$ and $\overline{b}$ by $\mathcal{C}$ and $\overline{\mathcal{C}}$ respectively. Now, we write $e_\mu h bv^\Lambda < e_\mu \overline{h} bv^\Lambda$ if one of the following conditions holds:

1. $\text{chg} \ b > \text{chg} \ \overline{\mathcal{C}}$
2. $\text{chg} \ b = \text{chg} \ \overline{\mathcal{C}}$ and $\mathcal{C} < \overline{\mathcal{C}}$
3. $\mathcal{C} = \overline{\mathcal{C}}$ and $\text{en} \ b < \text{en} \ \overline{b}$
4. $\mathcal{C} = \overline{\mathcal{C}}$, $\text{en} \ b = \text{en} \ \overline{b}$ and $b < \overline{b}$
5. $b = \overline{b}$ and $h < \overline{\mathcal{C}}$.

Introduce the set

$$\mathcal{B}_{L(\Lambda)} = \{e_\mu h bv^\Lambda \mid \mu \in Q^\vee, h \in B_{U(\hat{h}^-)}, b \in B'_{W_L(\Lambda)}\}.$$ 

Now, we have

**Lemma 2.3** For any positive integer $k$ the set $\mathcal{B}_{L(\Lambda)}$ spans $L(\Lambda)$.
Proof Suppose that $\Lambda = k \Lambda_0$. By Lemma 2.1 (i), the set of vectors
\[\left\{e_\mu h b v_\Lambda \mid \mu \in Q^\vee, h \in B_U(\hat{\mathfrak{h}}^-), \ b \in B_{W_L(\Lambda)}\right\}\]
spans $L(\Lambda)$. If a quasi-particle monomial $b$ contains a quasi-particle $x_{k,\alpha}(m)$, we use the relation (1.29) to replace it with $e_{\alpha^\vee}$ and monomials of the Heisenberg subalgebra elements. By using relations (1.23) and (1.24), we move $e_{\alpha^\vee}$ and the elements of $\hat{\mathfrak{h}}^-$ to the left, and the elements of $\hat{\mathfrak{h}}^+$ to the right. As a result, we get a linear combination of monomials $e_{\mu'} h' b' v_\Lambda$, where $\mu' \in Q^\vee$, $h' \in B_U(\hat{\mathfrak{h}}^-)$, and quasi-particle monomials $b'$ contain one quasi-particle $x_{k,\alpha}(m)$ less than $b$ and do not necessarily satisfy the difference conditions. Hence we can reduce the spanning set of $L(\Lambda)$ to set (2.1).

We can now prove the lemma by using the linear order “$<$” and applying almost verbatim Georgiev’s arguments in the proof of [18, Theorem 5.1]. More precisely, we use the relations in [5, Lemmas 1 and 2] and relation (1.29) to reduce the spanning set (2.1) to its subset $\mathcal{B}_{L(\Lambda)}$ with elements satisfying combinatorial difference and initial conditions, and the order “$<$” is designed in such a way that, by applying the relations, vectors in (2.1) do not satisfy combinatorial difference and initial conditions can be expressed in terms greater elements in (2.1).

The aforementioned lemmas from [5], which are a consequence of the commutation relations in $\hat{\mathfrak{g}}$, were used in [2–4, 7] to reduce the spanning set of the principal subspace of the generalized Verma module of highest weight $\Lambda$ (cf. [5, Theorem 1], [7, Theorem 3.1]). As for the principal subspace $W_{L(\Lambda)}$ of level $k$, we also need the following relations on $L(\Lambda)$:
\[x_{p \alpha}(z) = 0 \quad \text{for} \quad p > k_\alpha.\]
We can combine these relations with (1.29) in order to eliminate all vectors whose quasi-particle monomials contain $x_{k,\alpha}(j)$ from spanning set (2.1).

By applying [5, Lemma 1] to eliminate all vectors $e_\mu h b v_\Lambda$ such that the quasi-particle monomial $b$ does not satisfy the difference conditions, we obtain vectors with quasi-particle monomial factors of the form $x_{(n'+1)\alpha}(m')$, which are, by definition of order “$<$”, greater than $e_\mu h b v_\Lambda$. We now proceed by closely following the argument from the proof of [7, Theorem 3.1]. More specifically, the vectors $e_\mu h b' v_\Lambda$, whose factors satisfy $n' + 1 < k_\alpha$, are greater than $e_\mu h b v_\Lambda$ with respect to the linear order “$<$”, with $b < b'$ having the same color-charge-type and total energy, as in the proof of [7, Theorem 3.1]. On the other hand, if $n' + 1 = k_\alpha$, we use relation (1.29) to express $x_{(n'+1)\alpha}(m')$ in terms of $e_{\alpha^\vee}$ and Heisenberg Lie algebra elements, thus obtaining vectors $e_\mu h' b' v_\Lambda$ such that $\text{chg} b' < \text{chg} b$, so again the resulting vectors are greater with respect to “$<$”.

Finally, since [5, Lemma 2] relates quasi-particle monomials of the same total charge and total energy, we can eliminate the vectors $e_\mu h b v_\Lambda$ such that the quasi-particle monomials $b$ do not satisfy the difference conditions by arguing as in the proof of [7, Theorem 3.1].

The general case $\Lambda = k_0 \Lambda_0 + k_j \Lambda_j$ with $k_j > 0$ is verified analogously. However, the argument employs two additional results, [6, Lemmas 2.0.1 and 2.0.2].

\[2.2 \text{ The main theorem}\]

Consider the decomposition
\[L(\Lambda) = \coprod_{r \in \mathbb{Z}} L(\Lambda)_r, \quad \text{where} \quad L(\Lambda)_r = \coprod_{r_2, \ldots, r_1 \in \mathbb{Z}} L(\Lambda)_{k\Lambda|\tilde{h} + r_1 \alpha_1 + \cdots + r_2 \alpha_2 + r \alpha_1}.\]
By complete reducibility of tensor products of standard modules we have

\[
L(\Lambda) \subset L(\Lambda_{jk}) \otimes \cdots \otimes L(\Lambda_{j1})
\]  

(2.2)

with the highest weight vector

\[
v_{L(\Lambda)} = v_{L(\Lambda_{jk})} \otimes \cdots \otimes v_{L(\Lambda_{j1})}, \quad \text{where} \quad j^r = \begin{cases} 0, & \text{if } r = 1, \ldots, k_0, \\ j, & \text{if } r = k_0 + 1, \ldots, k. \end{cases}
\]

In the proof of Theorem 2.4 we will use the Georgiev-type projection

\[
\pi_{\mathcal{R}_{c_1}} : (L(\Lambda_j)^{\otimes k_j} \otimes L(\Lambda_0)^{\otimes k_0})_{r_1} \to L(\Lambda_{jk})_{r_1}^{(1)} \otimes \cdots \otimes L(\Lambda_{j1})_{r_1}^{(k)},
\]  

(2.3)

where \( \mathcal{R}_{c_1} = (r_1^{(1)}, r_1^{(2)}, \ldots, r_1^{(k_1)}) \) is a fixed dual-charge type for the color 1 and \( r_1 = \sum_{t=1}^{k_1} r_1^{(t)}. \) The projection can be generalized to the space of formal series with coefficients in (2.2). We denote this generalization by \( \pi_{\mathcal{R}_{c_1}} \) as well. Recall that the image of the generating function

\[
e_\mu (\alpha_{i}^\vee (-j_{t_1}, t)) r_{t_1}^{(1)} \cdots (\alpha_{1}^\vee (-j_{1}, 1)) r_{1}^{(1)} x_{r_{1}^{(1)}, 1} a_{1} (m_{r_{1}^{(1)}, 1}) \cdots x_{n_{1}, a_{1}} (m_{a_{1}, 1}) v_{L(\Lambda)}
\]

with respect to \( \pi_{\mathcal{R}_{c_1}} \), where the monomial \( b(\alpha_1) = x_{r_{1}^{(1)}, 1} a_{1} (m_{r_{1}^{(1)}, 1}) \cdots x_{n_{1}, a_{1}} (m_{a_{1}, 1}) \) is of dual-charge-type \( \mathcal{R}_{c_1} \), coincides with the coefficient of the corresponding projection of the generating function

\[
e_\mu (\alpha_{r_{1}^{(1)}, 1} (w_{r_{1}^{(1)}, 1})) r_{r_{1}^{(1)}, 1}^{(1)} \cdots (\alpha_{1}^\vee (w_{1}, 1)) r_{1}^{(1)} x_{r_{1}^{(1)}, 1} a_{1} (z_{r_{1}^{(1)}, 1}) \cdots x_{n_{1}, a_{1}} (z_{a_{1}, 1}) v_{L(\Lambda)},
\]

where \( \alpha_{r_{1}^{(1)}, 1}^\vee (z) = \sum_{m<0} \alpha_{r_{1}^{(1)}, 1}^\vee (m) z^{-m-1} \). For more details see for example [7, Section 5.2].

**Theorem 2.4** For any highest weight \( \Lambda \) as in (1.7) the set \( \mathcal{B}_{L(\Lambda)} \) is a basis of \( L(\Lambda) \).

**Proof** We prove linear independence of the spanning set \( \mathcal{B}_{L(\Lambda)} \) by slightly modifying the arguments in [2–4, 6, 7]. We consider a finite linear combination of vectors in \( \mathcal{B}_{L(\Lambda)} \),

\[
\sum c_{\mu, h, b} e_{\mu} h b v_{L(\Lambda)} = 0
\]  

(2.4)

of the fixed degree \( n \) and \( \mathfrak{h} \)-weight \( \rho \). Our goal is to show that all coefficients \( c_{\mu, h, b} \) are zero. Since \( e_{\nu}, \nu \in Q^\vee \), is a linear bijection and since (1.21) implies that for any \( \mathfrak{h} \)-weight \( \psi \) the image of \( V_\psi \) under the action of \( e_{\alpha_{1}^\vee} \) is in \( V_{\psi + k_0 a_1} \), we may assume that for all summands in (2.4) with the monomial \( b \) of the maximal chg \( b \) the corresponding \( \mu \) has \( \alpha_{1}^\vee \) coordinate zero. That is, we assume that in (2.4) appear summands of the form.

\[ \text{Springer} \]
\[ e_{\mu} h b v_{L(A)}, \text{ with } \text{chg}_{1} b = r_1 \text{ and } \mu = c_1 \alpha_1^{\vee} + \cdots + c_2 \alpha_2^{\vee}, \quad \text{or} \quad (A) \]

\[ e_{\mu'} h' b' v_{L(A)}, \text{ with } \text{chg}_{1} b' = r' < r_1 \text{ and } \mu' = c'_1 \alpha_1^{\vee} + \cdots + c'_1 \alpha_1^{\vee}, \quad \text{where} \quad c'_1 > 0. \quad (B) \]

In the sum (2.4), among the monomials \( b \) with \( \text{chg}_{1} b = r_1 \), choose a monomial \( b_0 \) with the maximal charge type \( \mathcal{R}_{\alpha_1} \) and the corresponding dual-charge-type

\[ \mathcal{R}_{\alpha_1} = (r_1^{(1)}, r_1^{(2)}, \ldots, r_1^{(p)}) \]

for the color \( i = 1 \), where \( p < k_{\alpha_1} \) and \( r_1 = r_1^{(1)} + \cdots + r_1^{(p)} \). Denote by \( \pi_{\mathcal{R}_{\alpha_1}} \) the Georgiev-type projection (2.3), where \( r_1^{(t)} = 0 \) for \( t > p \). Our key observation is that for the vectors of the form (B) we have \( \pi_{\mathcal{R}_{\alpha_1}} (e_{\mu'} h' b' v_{L(A)}) = 0 \), since

\[ e_{\alpha_1^\vee} (v_{L(A,j)})^{(1)} \otimes \cdots \otimes (v_{L(A,j)})^{(2)} = e_{\alpha_1^\vee} v_{L(A,j)}^{(1)} \otimes \cdots \otimes e_{\alpha_1^\vee} v_{L(A,j)}^{(2)}, \]

and hence

\[ e_{\mu'} h' b' v_{L(A)} \in \bigoplus_{r_1, \ldots, r_{k-1} \in \mathbb{Z}} \mathbb{Z} \quad L(\Lambda_{j_1})^{(1)} \otimes \cdots \otimes L(\Lambda_{j_1})^{(2)}. \]

This means that the \( \pi_{\mathcal{R}_{\alpha_1}} \) projection of the sum (2.4) contains only the projections of the summands of the form (A).

Hence we can proceed with Georgiev-type arguments, i.e. with iterated use of the simple current operator, the Weyl group translation operator and the intertwining operators as in [2–4,6,7] (see, e.g., [7, Sect. 5.3]) and briefly outlined in [5, Sect. 4], until we reduce (2.4) to a linear combination of vectors \( e_{\mu} h b v_{L} \) such that charge \( \text{chg}_{1} b = 0 \), i.e. of vectors with no quasi-particles of color \( \alpha_1 \). Then we start with a similar argument for the color \( \alpha_2 \) by choosing the monomials with the maximal 2-charge and the corresponding Georgiev-type projection.

It should be noted that the Georgiev-type procedure for \( \alpha_1 \) changes in some vectors (2.4) the energies of the (projected) quasi-particle monomials for some \( \alpha_2, \ldots, \alpha_1 \), but their dual-charge types \( \mathcal{R}_{\alpha_2}, \ldots, \mathcal{R}_{\alpha_1} \) are not changed and, moreover, the changed vectors satisfy the combinatorial initial and difference conditions; see, e.g., [2, Proposition 3.4.1].

At some point of our argument we shall have to consider the linear combination of vectors projected from (2.4) such that

\[ \text{chg}_{1} b = \cdots = \text{chg}_{s-1} b = 0 \]

for all \( b \) and \( \text{chg}_{s} b \neq 0 \) for some \( b \) and a short root \( \alpha_{s} \). Then \( k_{\alpha_{s}} = k \) in the case \( \tilde{\mathfrak{g}} \) is of type \( D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, \text{and} E_8^{(1)} \), \( k_{\alpha_{s}} = 2k \) in the case of \( B_l^{(1)}, C_l^{(1)} \text{and} F_4^{(1)} \), and \( k_{\alpha_{s}} = 3k \) in the case of \( G_2^{(1)} \). Again, in the same way, we choose the monomials with the maximal \( s \)-charge \( \text{chg}_{s} \), and then, among them, we find the monomial \( b \) with the maximal charge-type for the color \( \alpha_{s} \),

\[ \mathcal{R}_{\alpha_{s}} = (r_{s}^{(1)}, r_{s}^{(2)}, \ldots, r_{s}^{(p)}), \]

where \( p < k_{\alpha_{s}} \) and \( r_{s} = r_{s}^{(1)} + \cdots + r_{s}^{(p)} \). For the first short simple root \( \alpha_{s} \) we consider monomial vectors with \( k_{\alpha_{s}} > k \) quasi-particles of color \( s \), so for \( k_{\alpha_{s}} = 2k \) we consider the modified Georgiev-type projection

\[ \pi_{\mathcal{R}_{\alpha_{s}}} : (L(\Lambda_{j})^{(1)} \otimes L(\Lambda_{0})^{(2k)})^{r_{s}} \to L(\Lambda_{j,k})^{r_{s}^{(1)} + r_{s}^{(2)} \otimes \cdots \otimes L(\Lambda_{j,1})^{(2k-1)} + r_{s}^{(2k)}, \]

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where \( r_s^{(t)} = 0 \) for \( t > p \) (see [7, Eqs. (5.7) and (5.8)] and [6, Sect. 3.1]), and in a similar way we consider the modified Georgiev-type projection \( \pi_{\mathcal{R}_{\alpha_s}} \) for \( G_2^{(1)} \) when \( k_{\alpha_2} = 3k \); cf. [4, Sect. 3.3] and [6, Sect. 3.1].

For short simple roots \( \alpha_s, \ldots, \alpha_l \) we have the associated affine Lie subalgebra of \( \hat{\mathfrak{g}} \) of type \( A_{l-3}^{(1)} \) (see Fig. 1) and the restriction of a level \( k \) standard \( \hat{\mathfrak{g}} \)-module to this subalgebra is a direct sum of level \( k_{\alpha_s} \) standard \( A_{l-3}^{(1)} \)-modules. In the final stage of our proof (essentially) only monomials of quasi-particles for colors \( s, \ldots, l \) appear, and by applying the restriction of standard \( \hat{\mathfrak{g}} \)-modules to \( A_{l-3}^{(1)} \) we use another Georgiev-type projections for short roots—for \( \alpha_s \) essentially the mapping \( \pi'_{\mathcal{R}_{\alpha_s}} \) from

\[
L(\Lambda_1)_{r_s^{(1)}} \otimes \cdots \otimes L(\Lambda_1)_{r_s^{(2)}}
\]

to

\[
L(\Lambda_1)_{r_s^{(1)}} \otimes L(\Lambda_1)_{r_s^{(2)}} \otimes \cdots \otimes L(\Lambda_1)_{r_s^{(2k-1)}} \otimes L(\Lambda_1)_{r_s^{(2k)}}
\]

when \( k_{\alpha_s} = 2k \) (see [2, Sect. 3.1] and [6, Section 3.4]). Another Georgiev-type projection \( \pi'_{\mathcal{R}_{\alpha_s}} \) for \( G_2^{(1)} \) is obtained in a similar way by realizing level 3 standard \( A_1^{(1)} \)-modules within a tensor product of three level 1 standard \( A_1^{(1)} \)-modules; see [4, Eqs. (3.26) and (3.27)] and [6, Sect. 3.4].

The action of the group element \( e^{\alpha_s} \) increases the weight by \( \alpha_s \) on each tensor factor in (2.5) and, in particular, it increases the weight by \( \alpha_s \) on the last tensor factor. Therefore, our key observation again holds:

\[
\pi'_{\mathcal{R}_{\alpha_s}}(e^{\alpha_s} h' b' v_{L(\Lambda)}) = 0
\]

when \( \text{chg}_s b' < r_s \), so we can proceed as in [2–4,6,7]; see, e.g., [6, Sect. 3.4].

\[\square\]

3 Parafermionic bases

3.1 Vacuum space and \( \mathcal{Z} \)-algebra projection

Let \( \Lambda \) be the highest weight of level \( k \) as in (1.7). Denote by \( L(\Lambda)_{\mathcal{Z}}^+ \) the vacuum space of the standard module \( L(\Lambda) \), i.e.

\[
L(\Lambda)_{\mathcal{Z}}^+ = \{ v \in L(\Lambda) \mid \mathcal{Z}^+ v = 0 \}.
\]

By the Lepowsky–Wilson theorem [31,32] we have the canonical isomorphism of \( d \)-graded linear spaces

\[
U(\mathfrak{h}^-) \otimes L(\Lambda)_{\mathcal{Z}}^+ \xrightarrow{\cong} L(\Lambda)
\]

\[
h \otimes u \longmapsto h \cdot u,
\]

where

\[
S(\mathfrak{h}^-) \cong U(\mathfrak{h}^-) \cong M(k)
\]

is the Fock space of level \( k \) for the Heisenberg Lie algebra \( \mathfrak{s} = \mathfrak{h}^- \oplus \mathfrak{h}^+ \oplus \mathbb{C} c \) with the action of \( c \) as the multiplication by scalar \( k \). We consider the projection

\[
\pi_{\mathcal{Z}}^+: L(\Lambda) \to L(\Lambda)_{\mathcal{Z}}^+
\]
given by the direct sum decomposition
\[ L(\Lambda) = L(\Lambda)\hat{h}^+ \oplus \hat{h}^- U(\hat{h}^-) \cdot L(\Lambda)\hat{h}^+. \]

By (1.22) we have the action of the Weyl group translations \( e_{\alpha^\vee} \) on the vacuum space
\[ e_{\alpha^\vee} : L(\Lambda)\hat{h}^+ \to L(\Lambda)\hat{h}^+. \]

We recall Lepowsky–Wilson’s construction of \( Z \)-operators which commute with the action of the Heisenberg subalgebra \( \mathfrak{h} \) on the level \( k \) standard module \( L(\Lambda) \) ([30], see also [18,28]):
\[ Z_\alpha(z) = E^-(\alpha, z)^{1/k}x_\alpha(z)E^+(\alpha, z)^{1/k}. \]

We also need \( Z \)-operators for quasi-particles of higher charge
\[ Z_{\alpha_{\alpha_0}}(z) = E^-(\alpha, z)^{n/k}x_{\alpha_{\alpha_0}}(z)E^+(\alpha, z)^{n/k} \]
and, even more general, for quasi-particle monomials of charge type \( R' = (n_{r_1^{(1)}}, \ldots, n_{1,1}) \)
\[ Z_{R'}(z_{r_1^{(1)}}, \ldots, z_{1,1}) = E^-(\alpha_1, z_{r_1^{(1)}})^{n_{r_1^{(1)}}/k} \ldots E^-(\alpha_1, z_{1,1})^{n_{1,1}/k}x_{R'}(z_{r_1^{(1)}}, \ldots, z_{1,1}) \]
\[ \times E^+(\alpha_1, z_{r_1^{(1)}})^{n_{r_1^{(1)}}/k} \ldots E^+(\alpha_1, z_{1,1})^{n_{1,1}/k}, \]
where
\[ x_{R'}(z_{r_1^{(1)}}, \ldots, z_{1,1}) = x_{n_{r_1^{(1)}}}(z_{r_1^{(1)}}) \ldots x_{n_{1,1}}(z_{1,1}). \]

As usual, we write this formal Laurent series as
\[ Z_{R'}(z_{r_1^{(1)}}, \ldots, z_{1,1}) = \sum_{m_{r_1^{(1)}}^{(1)}, \ldots, m_{1,1} \in \mathbb{Z}} Z_{R'}(m_{r_1^{(1)}}^{(1)}, \ldots, m_{1,1})z_{r_1^{(1)}}^{-m_{r_1^{(1)}}} \ldots z_{1,1}^{-m_{1,1}}, \]
and the coefficients act on the vacuum space
\[ Z_{R'}(m_{r_1^{(1)}}^{(1)}, \ldots, m_{1,1}) : L(\Lambda)\hat{h}^+ \to L(\Lambda)\hat{h}^+. \]

Since we can “reverse” (3.7) and express monomials in quasi-particles in terms of the Laurent series \( Z_{R'}(z_{r_1^{(1)}}, \ldots, z_{1,1}), (3.8) \) implies
\[ \pi^\hat{h}^+ : x_{R'}(z_{r_1^{(1)}}, \ldots, z_{1,1})v_{L(\Lambda)} \mapsto Z_{R'}(z_{r_1^{(1)}}, \ldots, z_{1,1})v_{L(\Lambda)}. \]

Now Theorem 2.4 implies:

**Theorem 3.1** The set of vectors
\[ e_{\mu} Z_{R'}(m_{r_1^{(1)}}, \ldots, m_{1,1})v_{L(\Lambda)}, \]
such that \( \mu \in Q' \) and the charge-type \( R' \) and the energy-type \( (m_{r_1^{(1)}}, \ldots, m_{1,1}) \) satisfy difference and initial conditions for \( B'_{W_{L(\Lambda)'}} \), is a basis of the vacuum space \( L(\Lambda)\hat{h}^+ \).
**Proof** The images of elements of the basis $\mathfrak{B}_{L(\Lambda)}$ with respect to the projection $\pi = \pi \hat{h}^+$, form a spanning set for the vacuum space $L(\Lambda)\hat{h}^+$. Furthermore, as the projection annihilates all elements $e_\mu h b v_{L(\Lambda)} \in \mathfrak{B}_{L(\Lambda)}$ with $h \neq 0$, the vectors $e_\mu (\pi(b)v_{L(\Lambda)})$ span the vacuum space. Let $v$ be an arbitrary weight. The theorem now follows from (3.2) by comparing the dimensions of the subspace spanned by all vectors $h \cdot (e_\mu \pi(b)v_{L(\Lambda)})$ of weight $\nu$ and the subspace spanned by all vectors $e_\mu h b v_{L(\Lambda)}$ of weight $\nu$. $\square$

### 3.2 Parafermionic space and parafermionic projection

For $ADE$ type untwisted affine Lie algebras Georgiev in [18] uses lattice vertex operator construction $V_P$ of standard level 1 modules, where

$$ V_P = M(1) \otimes \mathbb{C}[P] \quad \text{and} \quad \mathbb{C}[P] = \text{span} \{ e^\mu \mid \mu \in P \}. $$

Then for a level $k$ standard module $L(\Lambda)$ he uses the embedding

$$ L(\Lambda) \subset V_P \otimes \cdots \otimes V_P. \quad (3.9) $$

This construction gives a diagonal action of the sublattice $kQ \subset Q$ on $V_P^{\otimes k}$:

$$ k\alpha \mapsto \rho(k\alpha) = e^\alpha \otimes \cdots \otimes e^\alpha, \quad \alpha \in Q, \quad \text{such that} \quad \rho(k\alpha) : L(\Lambda)_{\hat{h}^+} \rightarrow L(\Lambda)_{\hat{h}^+}^{\hat{h}+}. $$

Georgiev defines the *parafermionic space of highest weight* $\Lambda$ as the space of $kQ$-coinvariants in the $kQ$-module $L(\Lambda)^{\hat{h}+}$:

$$ L(\Lambda)^{\hat{h}+}_{kQ} = L(\Lambda)^{\hat{h}+} / \text{span} \left\{ (\rho(k\alpha) - 1) \cdot v \mid \alpha \in Q, \ v \in L(\Lambda)^{\hat{h}+} \right\} \quad (3.10) $$

with the canonical projection on the quotient space

$$ \pi_{kQ}^{\hat{h}+} : L(\Lambda)^{\hat{h}+} \rightarrow L(\Lambda)^{\hat{h}+}_{kQ}. $$

Note that in this case we have

$$ L(\Lambda)^{\hat{h}+}_{kQ} \cong \bigoplus_{\mu \in \Lambda^+ / Q / kQ} L(\Lambda)^{\hat{h}+}_{\mu}. $$

In the non-simply laced case we do not have a lattice vertex operator construction at hand nor operators $\rho(k\alpha)$, so we have to alter Georgiev’s construction: the map $\alpha^\vee \mapsto e_{\alpha^\vee}$ extends to a projective representation of $Q^\vee$ on the vacuum space $L(\Lambda)^{\hat{h}+}$ and the action of $e_{\alpha^\vee}$ gives isomorphisms of the $\hat{h}$-weight subspaces

$$ e_{\alpha^\vee} : L(\Lambda)^{\hat{h}+}_v \rightarrow L(\Lambda)^{\hat{h}+}_{v+k_{\alpha^\vee} \alpha}. $$

Denote by

$$ Q(k) = \bigoplus_{i=1}^l \mathbb{Z} k_{\alpha_i} \alpha_i \subset Q. $$
Note that in the simply laced case we have \( Q(k) = kQ \). In the non-simply laced case, we define the parafermionic space of highest weight \( \Lambda \) as

\[
L(\Lambda)_{Q(k)}^{\hat{h}^+} = \bigoplus_{0 \leq m_1 \leq k_{\alpha_1} - 1} L(\Lambda)^{\hat{h}^+}_{\Lambda + m_1 \alpha_1 + \cdots + m_l \alpha_l}.
\]

For an \( h \)-weight \( \mu = (\Lambda + m_1 \alpha_1 + \cdots + m_l \alpha_l) |_h \) there is a unique \( e_{\alpha_1}^{p_1} \cdots e_{\alpha_l}^{p_l} \) such that

\[
e_{\alpha_1}^{p_1} \cdots e_{\alpha_l}^{p_l} : L(\Lambda)^{\hat{h}^+}_{\mu} \cong L(\Lambda)^{\hat{h}^+}_{\Lambda + (m_1 + p_1 k_{\alpha_1}) \alpha_1 + \cdots + (m_l + p_l k_{\alpha_l}) \alpha_l} \subset L(\Lambda)_{Q(k)}^{\hat{h}^+}.
\]

Since \( L(\Lambda)^{\hat{h}^+} \) is a direct sum of \( h \)-weight subspaces \( L(\Lambda)^{\hat{h}^+}_{\mu} \), the above maps define our parafermion projection in the non-simply laced case:

\[
\pi_{Q(k)}^{\hat{h}^+} : L(\Lambda)^{\hat{h}^+} \rightarrow L(\Lambda)_{Q(k)}^{\hat{h}^+}.
\]

Along with this definition of the parafermionic space and the corresponding projection, we keep in mind isomorphisms

\[
e_{\alpha_1}^{p_1} \cdots e_{\alpha_l}^{p_l} : L(\Lambda)^{\hat{h}^+}_{\Lambda + m_1 \alpha_1 + \cdots + m_l \alpha_l} \cong L(\Lambda)^{\hat{h}^+}_{\Lambda + (m_1 + p_1 k_{\alpha_1}) \alpha_1 + \cdots + (m_l + p_l k_{\alpha_l}) \alpha_l}
\]

which allow us to identify the \( h \)-weight subspaces \( L(\Lambda)^{\hat{h}^+}_{\Lambda + \mu} \) and \( L(\Lambda)^{\hat{h}^+}_{\Lambda + \mu'} \) with \( h \)-weights \( \mu \) and \( \mu' \) in the same class \( \mu + Q(k) \in Q(k) \).

We define parafermionic current as in [18]:

\[
\Psi_\alpha(z) = Z_\alpha(z)z^{-\alpha/k}, \quad \Psi_\alpha(z) = \sum_{m \in \frac{1}{k_\alpha} + \mathbb{Z}} \psi_\alpha(m)z^{-m-1},
\]

and the parafermionic currents of charge \( n \):

\[
\Psi_{n\alpha}(z) = Z_{n\alpha}(z)z^{-n\alpha/k}, \quad \Psi_{n\alpha}(z) = \sum_{m \in \frac{n}{k_\alpha} + \mathbb{Z}} \psi_{n\alpha}(m)z^{-m-n}.
\]

For monomials of quasi-particles of charge type \( \mathcal{R}' = (n_{r_1}, \ldots, n_{1,1}) \) we define the corresponding \( \Psi \)-operators

\[
\Psi_{\mathcal{R}'}(z_{r_1}, \ldots, z_{1,1}) = Z_{\mathcal{R}'}(z_{r_1}, \ldots, z_{1,1})z_{r_1}^{-n_{r_1}/k} \cdots z_{1,1}^{-n_{1,1}/k},
\]

\[
\Psi_{\mathcal{R}'}(z_{r_1}, \ldots, z_{1,1}) = \sum_{m_{r_1}, \ldots, m_{1,1}} \psi_{\mathcal{R}'}(m_{r_1}, \ldots, m_{1,1})z_{r_1}^{-m_{r_1}/k} \cdots z_{1,1}^{-m_{1,1}/k},
\]

where the summation in the second equality is over all sequences \( (m_{r_1}, \ldots, m_{1,1}) \) such that \( m_{i,r} \in \frac{n_{i,r}}{k_{\alpha_r}} + \mathbb{Z} \). It is clear that \( \Psi \)-operators commute with the action of the Heisenberg subalgebra \( \mathfrak{h} \).

The following lemma reveals the relation between the coefficients of \( Z \)-operators and the coefficients of \( \Psi \)-operators (cf. [18, Eq. (2.14)]):

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Lemma 3.2 For any simple root $\beta$, $m \in \mathbb{Z}$ and $\mu \in P$ we have
\[
\mathcal{Z}_{\beta}(m)\bigg|_{L(\Lambda)_{\mu}^{\mathbf{h}^+}} = \psi_{\beta}(m + (\beta, \mu)/k)\bigg|_{L(\Lambda)_{\mu}^{\mathbf{h}^+}}.  \tag{3.13}
\]

\textbf{Proof} By applying (3.12) and then restricting to the $\mu$-weight subspace $L(\Lambda)_{\mu}^{\mathbf{h}^+}$ we find
\[
\mathcal{Z}_{\beta}(z)\bigg|_{L(\Lambda)_{\mu}^{\mathbf{h}^+}} = \Psi_{\beta}(z) z^{\beta/k} \bigg|_{L(\Lambda)_{\mu}^{\mathbf{h}^+}} = z^{(\beta, \mu)/k} \psi_{\beta}(z) \bigg|_{L(\Lambda)_{\mu}^{\mathbf{h}^+}},  \tag{3.14}
\]
so by taking the coefficients of $z^{-m-1}$ in (3.14) we obtain (3.13), as required. \hfill \Box

The following two lemmas reveal the relation between different $\Psi$-operators defined above. We have the following lemma (cf. [18]):

Lemma 3.3 For a simple root $\beta$ and a positive integer $n$
\[
\Psi_{n\beta}(z) = \left( \prod_{1 \leq p < s \leq n} \left( 1 - \frac{z_p}{z_s} \right)^{(\beta, \beta)/k} \right) \psi_{\beta}(z_n) \cdots \psi_{\beta}(z_1) \bigg|_{z_n = \cdots = z_1 = z}.  \tag{3.15}
\]

\textbf{Proof} By using (3.5) and (3.12) one can express the parafermionic current as
\[
\psi_{\beta}(z) = E^{-}(\beta/k, z) \psi_{\beta}(z) E^{+}(\beta/k, z) z^{-\beta/k}.  \tag{3.16}
\]

By combining identities (1.25) and (3.16) and the aforementioned fact that the parafermionic currents commute with the action of the Heisenberg subalgebra we find
\[
\psi_{\beta}(z_2) \psi_{\beta}(z_1) = \psi_{\beta}(z_2) E^{-}(\beta/k, z_1) \psi_{\beta}(z_1) E^{+}(\beta/k, z_1) z_1^{-\beta/k} = E^{-}(\beta/k, z_1) \psi_{\beta}(z_2) \psi_{\beta}(z_1) E^{+}(\beta/k, z_1) z_1^{-\beta/k} = E^{-}(\beta/k, z_1) E^{-}(\beta/k, z_2) \psi_{\beta}(z_2) E^{+}(\beta/k, z_1) E^{+}(\beta/k, z_1) z_1^{-\beta/k} = (1 - z_1/z_2)^{-\beta/k} E^{-}(\beta/k, z_1) E^{-}(\beta/k, z_2) \psi_{\beta}(z_2) \psi_{\beta}(z_1) z_1^{-\beta/k} \times z_2^{-\beta/k} E^{+}(\beta/k, z_2) E^{+}(\beta/k, z_1) z_2^{-\beta/k} z_1^{-\beta/k},
\]
where in the last equality we also used (1.28). The statement of the lemma for $n = 2$ now follows by multiplying the equality by $(1 - z_2/z_1)^{(\beta, \beta)/k}$ and then applying the substitution $z_1 = z_2 = z$. The $n > 2$ case is verified by induction on $n$. \hfill \Box

For given simple roots $\beta_r, \ldots, \beta_1$ and the corresponding sequence of charges $n_r, \ldots, n_1$ set
\[
\Psi_{n_r \beta_r, \ldots, n_1 \beta_1}(z_r, \ldots, z_1) = \prod_{s=1}^{r} E^{-}(n_s \beta_s/k, z_s) x_{n_r \beta_r}(z_r) \cdots x_{n_1 \beta_1}(z_1) \times \prod_{s=1}^{r} E^{+}(n_s \beta_s/k, z_s) \prod_{s=1}^{r} z_s^{-n_s \beta_s/k}.
\]

Analogously to Lemma 3.3 we can show:
Lemma 3.4  For any simple roots $\beta_r, \ldots, \beta_1$ and charges $n_r, \ldots, n_1$ we have

$$\Psi_{n_r, \beta_r, \ldots, n_1, \beta_1}(z_r, \ldots, z_1) = \left( \prod_{1 \leq s < r} \left( 1 - \frac{z_p}{z_s} \right)^{\langle n_l \beta_s, n_p \beta_p \rangle / k} \right) \times \Psi_{n_r, \beta_r}(z_r) \ldots \Psi_{n_1, \beta_1}(z_1).$$  \hspace{1cm} (3.17)

Moreover, we have

$$\Psi_{n_r, \beta_r, \ldots, n_1, \beta_1}(z_r, \ldots, z_1) = \left( \prod_{p=1}^{r-1} \left( 1 - \frac{z_p}{z_r} \right)^{\langle n_r \beta_r, n_p \beta_p \rangle / k} \right) \times \Psi_{n_r, \beta_r}(z_r) \Psi_{n_{r-1}, \beta_{r-1}, \ldots, n_1, \beta_1}(z_{r-1}, \ldots, z_1).$$  \hspace{1cm} (3.18)

\textbf{Proof}  It is clear that (3.17) follows by successive applications of equality (3.18) so let us prove (3.18). As with the proof of Lemma 3.3, the following calculation relies on identities (1.25), (1.28) and (3.16) and the fact that the parafermionic currents commute with the action of the Heisenberg subalgebra. We have

$$\Psi_{n_r, \beta_r}(z_r) \Psi_{n_{r-1}, \beta_{r-1}, \ldots, n_1, \beta_1}(z_{r-1}, \ldots, z_1)$$

$$= E^-(n_r \beta_r / k, z_r) x_{n_r, \beta_r}(z_r) E^+(n_r \beta_r / k, z_r) z_r^{-n_r \beta_r / k}$$

$$\times \prod_{s=1}^{r-1} E^-(n_s \beta_s / k, z_s) x_{n_r, n_{r-1}, \ldots, n_1, \beta_1}(z_{r-1}) \ldots \times n_1, \beta_1(z_1)$$

$$\times \prod_{s=1}^{r-1} E^+(n_s \beta_s / k, z_s) \prod_{s=1}^{r-1} z_s^{-n_s \beta_s / k}$$

$$= \prod_{s=1}^{r-1} E^-(n_s \beta_s / k, z_s) x_{n_r, \beta_r}(z_r) E^+(n_r \beta_r / k, z_r) z_r^{-n_r \beta_r / k}$$

$$\times x_{n_{r-1}, \beta_{r-1}, \ldots, n_1, \beta_1}(z_{r-1}) \ldots x_{n_1, \beta_1}(z_1) \prod_{s=1}^{r-1} E^+(n_s \beta_s / k, z_s) \prod_{s=1}^{r-1} z_s^{-n_s \beta_s / k}$$

$$= \prod_{p=1}^{r-1} \left( 1 - \frac{z_p}{z_r} \right)^{\langle n_r \beta_r, n_p \beta_p \rangle / k} \times \prod_{s=1}^{r} E^-(n_s \beta_s / k, z_s) x_{n_r, \beta_r}(z_r) \ldots x_{n_1, \beta_1}(z_1) \prod_{s=1}^{r} E^+(n_s \beta_s / k, z_s) \prod_{s=1}^{r} z_s^{-n_s \beta_s / k}$$

$$= \left( \prod_{p=1}^{r-1} \left( 1 - \frac{z_p}{z_r} \right)^{\langle n_r \beta_r, n_p \beta_p \rangle / k} \times \prod_{s=1}^{r} E^-(n_s \beta_s / k, z_s) x_{n_r, \beta_r}(z_r) \ldots x_{n_1, \beta_1}(z_1) \prod_{s=1}^{r} E^+(n_s \beta_s / k, z_s) \prod_{s=1}^{r} z_s^{-n_s \beta_s / k} \right) \Psi_{n_r, \beta_r, \ldots, n_1, \beta_1}(z_r, \ldots, z_1),$$

so equality (3.18) now follows. \qed

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We thank the referee for their comments which improved the presentation of this paper.
Lemma 3.5 The equalities
\[ \Psi_\beta(z) e_{\alpha^\vee} = e_{\alpha^\vee} \Psi_\beta(z) \quad \text{and} \quad \Psi_{\alpha^\vee}(z_{\alpha_{1}}^1, \ldots, z_{1,1}) e_{\alpha^\vee} = e_{\alpha^\vee} \Psi_{\alpha^\vee}(z_{\alpha_{1}}^1, \ldots, z_{1,1}). \]
hold for operators on \( L(\Lambda) \) or \( L(\Lambda)^{\hat{h}^+} \).

Proof Since (1.23) can be written as
\[ x_\beta(z) e_{\alpha^\vee} = z_\beta(\alpha^\vee) e_{\alpha^\vee} x_\beta(z) \]
and (1.21)
\[ z^h e_{\alpha^\vee} = e_{\alpha^\vee} z^{h + (\alpha^\vee, h)}, \]
we have
\[ \Psi_\beta(z) e_{\alpha^\vee} = Z_\beta(z) z^{-\beta/k} e_{\alpha^\vee} = Z_\beta(z) e_{\alpha^\vee} z^{-\beta/k} z^{-\beta(\alpha^\vee)} = e_{\alpha^\vee} Z_\beta(z) z^{-\beta/k} z^{-\beta(\alpha^\vee)} = e_{\alpha^\vee} \Psi_\beta(z). \]
The proof of the second statement is similar. \( \square \)

Since the coefficients \( \psi_\beta(m) \) commute with all \( e_{\alpha^\vee} \), in the simply laced case the subspace
\[ \text{span}\left\{ (\rho(k\alpha) - 1) \cdot v \mid \alpha \in Q, \ v \in L(\Lambda)^{\hat{h}^+} \right\} \]
is invariant for \( \psi_\beta(m) \) and we have the induced operator
\[ \overline{\psi_\beta(m)}: L(\Lambda)^{\hat{h}^+} \rightarrow L(\Lambda)^{\hat{h}^+} \]
on the quotient (3.10).

In the non-simply laced cases we only have a projective representation \( \varphi \mapsto e_\varphi \) of \( Q^\vee \) on a standard module or its vacuum space, i.e.
\[ e_\varphi e_{\varphi'} = c(\varphi, \varphi') e_{\varphi + \varphi'} \]
for some non-zero constant \( c(\varphi, \varphi') \), but we can still define the induced operators \( \overline{\psi_\beta(m)} \) on the parafermionic space (3.11) as
\[ \overline{\psi_\beta(m)} = \pi_{Q(k)}^\hat{h}^+ \circ \psi_\beta(m): \bigsqcup \mu L(\Lambda)^{\hat{h}^+}_\mu \rightarrow \bigsqcup \mu L(\Lambda)^{\hat{h}^+}_\mu, \]
where, as in (3.11), both direct sums go over all weights \( \mu \) of the form \( \mu = \Lambda + m_1 \alpha_1 + \cdots + m_l \alpha_l \) such that \( 0 \leq m_1 \leq k_{\alpha_1} - 1, \ldots, 0 \leq m_l \leq k_{\alpha_l} - 1 \). That is, we apply first
\[ \psi_\beta(m): L(\Lambda)^{\hat{h}^+}_\mu \rightarrow L(\Lambda)^{\hat{h}^+}_{\mu + \beta}, \]
and then compose it with the projection
\[ \overline{\psi_\beta(m)} = \pi_{Q(k)}^\hat{h}^+ \circ \psi_\beta(m). \]
In other words, if \( \mu + \beta \) is not the chosen class in (3.11), but rather \( v \in \mu + \beta + Q(k) \), then we identify with the unique \( e_\varphi \) two subspaces
\[ e_\varphi: L(\Lambda)^{\hat{h}^+}_{\mu + \beta} \rightarrow L(\Lambda)^{\hat{h}^+}_v, \]
i.e.\[
\overline{\psi}_\beta(m) = e_\phi \circ \psi_\beta(m).
\]
A drawback of this construction is that for a composition of \(\overline{\psi}_\beta(m')\) and \(\overline{\psi}_\beta(m)\) we may need two identifications by \(e_{\phi'}\) and \(e_\phi\) and for \(\psi_\beta(m) \overline{\psi}_\beta(m')\) we need only one identification by \(e_{\phi+\phi'}\) so that we get\[
\psi_\beta(m) \circ \overline{\psi}_\beta(m') = c(\phi, \phi')\psi_\beta(m) \circ \overline{\psi}_\beta(m').
\]
Having all this in mind, we omit the “overline” from our notation and consider the induced operators\[
\psi_\beta(m) : L(\Lambda) \hat{h}^+_{Q(k)} \to L(\Lambda) \hat{h}^+_{Q(k)}
\]
on the parafermionic space. With such convention we shall also write\[
\psi_\mathcal{R}'(m_{r_1}^{(1)}, \ldots, m_{1,1}) : L(\Lambda) \hat{h}^+_{Q(k)} \to L(\Lambda) \hat{h}^+_{Q(k)}.
\]
Now Theorem 3.1 implies:

**Theorem 3.6** For any highest weight \(\Lambda\) as in (1.7) the set of vectors\[
\pi_{Q(k)} \mathcal{Z}_{\mathcal{R}'}(m_{r_1}^{(1)}, \ldots, m_{1,1})v_{L(\Lambda)}
\]
\
\[=\psi_\mathcal{R}'(m_{r_1}^{(1)} + n_{r_1}^{(1)}/k, \Lambda)/k, \ldots, m_{1,1} + \langle n_{1,1} \alpha_1, \Lambda \rangle/k)v_{L(\Lambda)},\]

such that the charge-type \(\mathcal{R}'\) and the energy-type \((m_{r_1}^{(1)}, \ldots, m_{1,1})\) satisfy difference and initial conditions for \(B'_W\) is a basis of the parafermionic space \(L(\Lambda) \hat{h}^+_{Q(k)}\).

### 3.3 Parafermionic character formulas

The results of Dong and Lepowsky in [10] on \(\mathcal{Z}\)-algebras and parafermionic algebras for affine Lie algebras of ADE-type have been extended in [33]; in particular it is proved that the vacuum space \(\Omega_V\) of a Heisenberg algebra in a general vertex operator algebra \(V\) has a natural generalized vertex algebra structure and that the vacuum space \(\Omega_W\) of a \(V\)-module \(W\) has a natural \(\Omega_V\)-module structure \((\Omega_W, Y_{\Omega})\) (see Theorem 3.10 in [33]). In our case the so-called parafermionic grading operator \(L_{\Omega}(0) = \text{Res}_z zY(\omega_{\Omega}, z)\), defined by (3.35) in [33]\[
Y_{\Omega}(\omega_{\Omega}, z) = \sum_{n \in \mathbb{Z}} L_{\Omega}(n)z^{-n-2}, \quad \omega_{\Omega} = \omega - \omega_h,
\]
is the difference between the grading operators of the vertex operator algebras \(L(k\Lambda_0)\) and \(M(k)\), as in [10, Chapter 14] and [18]. By using the commutator formula for vertex algebras (cf. (3.1.9) in [27]) and Proposition 3.8 and Theorem 6.4 in [33] we get\[
[L_{\Omega}(0), x_\beta(m)] = \left(-m - \frac{1}{k_\beta}\right)x_\beta(m) \quad \text{for} \quad \beta \in R, \ m \in \mathbb{Z}.
\]
(3.19)

Since \(L_{\Omega}(0)\) commutes with the action of Heisenberg subalgebra \(s\), the assumption \(L_{\Omega}(0)v = \lambda v\) for \(v \in L(\Lambda) \hat{h}^+\) implies\[
L_{\Omega}(0) \pi \hat{h}^+ \cdot x_\beta(m) v = \left(-m - \frac{1}{k_\beta} + \lambda\right) \pi \hat{h}^+ \cdot x_\beta(m) v.
\]
From here we see that the conformal energy of $\psi_\beta(m)$ equals
\[ \text{en } \psi_\beta(m) = -m - \frac{1}{k_\beta}, \text{ i.e. } [L_\Omega(0), \psi_\beta(m)] = \left(-m - \frac{1}{k_\beta}\right) \psi_\beta(m). \quad (3.20) \]

**Lemma 3.7** For a simple root $\beta$ and a charge $n$ we have on $L(\Lambda)^\hat{b}^+$
\[ \text{en } \psi_{n\beta}(m) = -m - \frac{n^2}{k_\beta}. \quad (3.21) \]
Moreover, for simple roots $\beta_r, \ldots, \beta_1$ and charges $n_r, \ldots, n_1$ we have
\[ \text{en } \psi_{n_r\beta_r, \ldots, n_1\beta_1}(m_r, \ldots, m_1) = -\sum_{i=1}^r \left(m_i + \frac{n_i^2}{k_{\beta_i}} + \frac{1}{k} \langle n_i \beta_i, \sum_{s=i}^{i-1} n_s \beta_s \rangle\right). \quad (3.22) \]

**Proof** Consider the right hand side of (3.15). The contribution of the coefficient of the parafermionic current $\Psi_\beta(z_i)$ with $i = 1, \ldots, n$ to the conformal energy of $\psi_{n\beta}(m)$ equals $-m_i - \frac{1}{k_\beta}$, where $m_1 + \cdots + m_n = m$. Moreover, each term $z_s^{(\beta, \beta_\beta)}/k$ decreases the conformal energy by $\langle \beta, \beta \rangle/k$. As the right hand side of (3.15) contains $(n-1)/2$ such terms and $k_\beta \langle \beta, \beta \rangle = 2k$, the conformal energy of $\psi_{n\beta}(m)$ is found by
\[ \text{en } \psi_{n\beta}(m) = -\sum_{i=1}^n \left(m_i + \frac{1}{k_\beta} \right) - \frac{n(n-1)}{2} \cdot \frac{\langle \beta, \beta \rangle}{k} = -m - \frac{n^2}{k_\beta}. \]

We generalize (3.21) to an arbitrary $\psi_{n_r\beta_r, \ldots, n_1\beta_1}(m_r, \ldots, m_1)$. Consider the right hand side of (3.17). By (3.21) we conclude that the contribution of the coefficient of the parafermionic current $\Psi_{n_i\beta_i}(z_i)$ with $i = 1, \ldots, n$ to the total conformal energy equals $-m_i - \frac{n_i^2}{k_{\beta_i}}$. Since each term $z_s^{(n, \beta_s, n_p \beta_p)}/k$ decreases the conformal energy by $\langle n_s \beta_s, n_p \beta_p \rangle/k$, formula (3.22) follows.

**Lemma 3.8** On $L(\Lambda)^\hat{b}^+$ we have
\[ [L_\Omega(0), e_\beta^\vee] = 0 \text{ for } \beta \in R. \]

**Proof** Vertex operator formula (1.29) for $q = 0$ gives
\[ e_\beta^\vee v_{L(\Lambda)} = \frac{1}{k_\beta} \partial_{x_{k_\beta}} (-k_\beta - \Lambda(\beta^\vee)) v_{L(\Lambda)}. \]
Since
\[ L_\Omega(0) v_{L(\Lambda)} = c_\Lambda v_{L(\Lambda)} \quad (3.23) \]
for some complex number $c_\Lambda$, from (3.21) it follows
\[ L_\Omega(0) \pi^\hat{b}^+ \cdot x_{k_\beta}(-k_\beta - \Lambda(\beta^\vee)) v_{L(\Lambda)} = L_\Omega(0) \cdot x_{k_\beta}(-k_\beta - \Lambda(\beta^\vee)) v_{L(\Lambda)} \]
\[ = L_\Omega(0) \Psi_{k_\beta}(\Lambda(\beta^\vee)) v_{L(\Lambda)} \]
\[ = \left(k_\beta - k_\beta^2 c_\Lambda\right) \Psi_{k_\beta}(\Lambda(\beta^\vee)) v_{L(\Lambda)} \]
\[ = c_\Lambda \pi^\hat{b}^+ \cdot x_{k_\beta}(-k_\beta - \Lambda(\beta^\vee)) v_{L(\Lambda)}. \]

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Hence we have
\[ L_\Omega(0)e_{\beta^\nu}v_{L(\Lambda)} = e_{\beta^\nu}L_\Omega(0)v_{L(\Lambda)} = c_\Lambda e_{\beta^\nu}v_{L(\Lambda)}. \]  
(3.24)

By Lemma 3.5 the action of parafermionic currents \( \psi_\beta(m) \) on \( v_{L(\Lambda)} \) generates the parafermionic space and commutes with \( e_\alpha^\nu \) for all \( \alpha \in R \). Hence lemma follows from (3.24) and (3.20).

The above lemma implies that \( L_\Omega(0) \) is a parafermionic degree operator on the parafermionic space \( L(\Lambda)^{\hat{G}^+}_{Q(k)} \) and we want to determine a formula for the corresponding parafermionic character
\[ \text{ch } L(\Lambda)^{\hat{G}^+}_{Q(k)} = q^{-c_\Lambda} \text{tr } q^{L_\Omega(0)}, \]  
(3.25)

with \( c_\Lambda \) as in (3.23). Consider an arbitrary quasi-particle monomial
\[ x_{n_1,j}^r \alpha_j(m_{r_1}^{(1)},) \ldots x_{n_1,j}^r \alpha_j(m_{1,1}) \ldots x_{n_1,j}^r \alpha_j(m_{r_1}^{(1)},) \ldots x_{n_1,j}^r \alpha_j(m_{1,1}) \in B_{W_{L(\Lambda)}}'. \]  
(3.26)

Note that (3.26) does not contain any quasi-particles of color \( i \) and charge \( k_{\alpha_i} \) for \( i = 1, \ldots, l \). Denote by

\[ \mathcal{R}' = (n_{r_i}^{(1)}, \ldots, n_{1,1}), \quad \mathcal{R} = (r_i^{(1)}, \ldots, r_1^{(k_{\alpha_i}-1)}) \quad \text{and} \quad \mathcal{E} = (m_{r_i}^{(1)}, \ldots, m_{1,1}) \]

its charge-type, dual-charge-type and energy-type respectively. Next, define the elements
\[ P_i = (p_1^{(1)}, \ldots, p_i^{(k_{\alpha_i}-1)}) \]

by

\[ P_i = (r_1^{(1)} - r_i^{(2)}, r_1^{(2)} - r_i^{(3)}, \ldots, r_1^{(k_{\alpha_i}-2)} - r_i^{(k_{\alpha_i}-1)}, r_1^{(k_{\alpha_i}-1)}), \quad \text{where } i = 1, \ldots, l. \]

Clearly, the numbers \( p_i^{(m)} \) denote the number of quasi-particles of color \( i \) and charge \( m \) in quasi-particle monomial (3.26). Consider the parafermionic space basis, as given by Theorem 3.6. By (3.22), the conformal energy of the basis vector
\[ \psi_{\mathcal{R}'(\mathcal{E})}v_{L(\Lambda)} = \psi((n_{r_i}^{(1)}, \ldots, n_{1,1})(m_{r_i}^{(1)}, \ldots, m_{1,1}))v_{L(\Lambda)}, \]  
(3.27)

which corresponds to quasi-particle monomial (3.26), is equal to

\[ -\sum_{i=1}^{l} \sum_{u=1}^{r_i^{(1)}} m_{u,i} - \sum_{i=1}^{l} \sum_{u=1}^{r_i^{(1)}} \left( \frac{n_{u,i}^2}{k_{\alpha_i}} + \frac{1}{k} \left( n_{u,i} \alpha_i \sum_{s=1}^{u-1} n_{s,i} \alpha_i + \sum_{s=1}^{u-1} \sum_{p=1}^{r_i^{(1)}} n_{s,p} \alpha_p \right) \right) \]
\[ -\frac{k_j}{k_{\alpha_j}} \sum_{t=1}^{k_{\alpha_j}-1} t P_j^{(r)}, \]  
(3.28)

where the third summand is due to the identity
\[ -\frac{1}{k} \left( \sum_{t=1}^{k_{\alpha_j}-1} t P_j^{(r)}, \Lambda \right) = -\frac{k_j}{k_{\alpha_j}} \sum_{t=1}^{k_{\alpha_j}-1} t P_j^{(r)}. \]  
(3.29)

Let
\[ K_p(q) = q^{\frac{1}{2} \sum_{i,r=1}^{l} \sum_{m=1}^{k_{\alpha_i}-1} k_{ir}^m p_i^{(m)} p_r^{(a)}}, \quad \text{where } K_{ir}^{mn} = G_{ir}^{mn} - \frac{mn}{k} \langle \alpha_i, \alpha_r \rangle \]

and the numbers \( G_{ir}^{mn} \) are given by (1.15). Define
\[ C_p(q) = B_p(q) q^{-\frac{k_j}{k_{\alpha_j}} \sum_{t=1}^{k_{\alpha_j}-1} t P_j^{(r)}}, \]
where $B'_P(q)$ is given by (1.18).

**Theorem 3.9** For any highest weight $\Lambda$ as in (1.7) we have

$$\text{ch} L(\Lambda)^{\hat{\mathfrak{h}}^+_Q(k)} = \sum_P D'_P(q) C_P(q) K_P(q),$$  

(3.30)

where the sum goes over all finite sequences $P = (P_1, \ldots, P_l)$ of $k_{\alpha_1} + \cdots + k_{\alpha_l} - l$ nonnegative integers and $D'_P(q)$ is given by (1.17).

**Proof** By (1.16) the product

$$\sum_P D'_P(q) G'_P(q) B'_P(q),$$

where $G'_P(q)$ is given by (1.17), counts all quasi-particle monomials (3.26), i.e., in terms of conformal energy, it corresponds to the first term

$$-\sum_{i=1}^l \sum_{u=1}^{r_i^{(1)}} m_{u,i}$$

(3.31)
in (3.28). Therefore, in order to verify character formula (3.30), it is sufficient to check that

$$\frac{K_P(q) C_P(q)}{G'_P(q) B'_P(q)} = q^{-\frac{1}{2} \sum_{i,r=1}^l \sum_{m=1}^{k_{\alpha_i}-1} \sum_{n=1}^{k_{\alpha_i}-1} \frac{mn}{k} \langle \alpha_i, \alpha_r \rangle p_i^{(m)} p_r^{(n)} \frac{k_i j_i}{k_{\alpha_i}} \sum_{r=1}^{k_{\alpha_i}-1} t_{p_r}^{(j)}}$$

(3.32)
corresponds to the parafermionic shift, i.e. to the remaining terms in (3.28).

We now consider the parafermionic pairs which consist of a parafermion of color $i$ and charge $m$ and a parafermion of color $r$ and charge $n$. More specifically, for fixed

$$i, r = 1, \ldots, l, \quad m = 1, \ldots, k_{\alpha_i} - 1 \quad \text{and} \quad n = 1, \ldots, k_{\alpha_r} - 1$$

(3.33)

we compute the contribution of all such pairs to the conformal energy of basis vector (3.27). In order to prove the theorem, we will demonstrate that, for fixed $P$, the sum of all such contributions for $i, r, m, n$ as in (3.33) coincides with both the power of $q$ in (3.32) and the difference of (3.28) and (3.31).

Fix integers $i, r, m, n$ as in (3.33).

(a) Suppose that $i \neq r$. By (3.28), the contribution to the conformal energy of the basis monomial $\psi_{P'}(E)$ in (3.27) equals

$$-\sum_{u=r_i^{(m)+1}}^{r_i^{(m)+1}} \frac{1}{k} \left\langle n_{u,i} \alpha_i, \sum_{s=r_i^{(n)+1}}^{r_i^{(n)+1}} n_{s,i} \alpha_i \right\rangle = -\frac{mn}{k} \langle \alpha_i, \alpha_r \rangle p_i^{(m)} p_r^{(n)}$$

(3.34)

and the right hand side coincides with the corresponding term in the power of $q$ in (3.32).

(b) Suppose that $i = r$ and $m \neq n$. By (3.28), the contribution of the basis monomial $\psi_{P'}(E)$ to the conformal energy of (3.27) equals

$$-\sum_{u=r_i^{(m)+1}}^{r_i^{(m)+1}} \frac{1}{k} \left\langle n_{u,i} \alpha_i, \sum_{s=r_i^{(n)+1}}^{r_i^{(n)+1}} n_{s,i} \alpha_i \right\rangle = -\frac{mn}{k} \langle \alpha_i, \alpha_i \rangle p_i^{(m)} p_i^{(n)}$$

(3.35)

and the right hand side coincides with the corresponding term in the power of $q$ in (3.32).
Remark 3.10 By definition (3.11) the parafermionic space and (3.37) with the power of \( L \) parafermionic space weight vector \( v/L_{\Lambda 1} \) and the right hand side coincides with the corresponding term in the power of \( q \) in (3.32).

In addition to (3.34)–(3.36), as the monomial \( \psi_{\mathcal{R}'}(\mathcal{E}) \) in (3.27) is applied on the highest weight vector \( v_{\Lambda} \), its terms of color \( j \) contribute to the conformal energy of (3.27) by

\[
- \frac{k_j}{k_{\alpha_j}} \sum_{i=1}^{r_j} \frac{1}{m_i} \left( p_{\lambda_i}^{(m_i)} \right)^2
\]

which coincides with the corresponding term in the power of \( q \) in (3.32). Indeed, as indicated above, this follows from identity (3.29).

Finally, the theorem follows by comparing the sum over all indices (3.33) of (3.34)–(3.36) and (3.37) with the power of \( q \) in (3.32). \( \square \)

Remark 3.10 By definition (3.11) the parafermionic space \( L(\Lambda) \hat{\mathbf{b}}^+ \) is a sum of \( k_{\alpha_1} \cdots k_{\alpha_l} \) \( \eta \)-weight subspaces \( L(\Lambda) \hat{\mathbf{b}}^+ \) of the vacuum space \( L(\Lambda) \hat{\mathbf{b}}^+ \). Since each subspace \( L(\Lambda) \hat{\mathbf{b}}^+ \) has a basis consisting of eigenvectors for \( L_{\Omega}(0) \), we may consider the restriction \( L_{\Omega}(0)|_{L(\Lambda) \hat{\mathbf{b}}^+} \) of \( L_{\Omega}(0) \) on \( L(\Lambda) \hat{\mathbf{b}}^+ \) and the corresponding character

\[
\chi_{\Lambda}^\Lambda = \text{ch} L(\Lambda) \hat{\mathbf{b}}^+ = q^{-c_{\Lambda}} \text{tr} q^{L_{\Omega}(0)|_{L(\Lambda) \hat{\mathbf{b}}^+}},
\]

with \( c_{\Lambda} \) as in (3.23). Formula (18) for \( \chi_{\Lambda}^\Lambda \) in [20] is a generalization of the Kuniba–Nakanishi–Suzuki character for the parafermionic space \( L(k \Lambda_0) \hat{\mathbf{b}}^+ \) to the character of parafermionic space \( L(k \Lambda_0) \hat{\mathbf{b}}^+ \) for rectangular \( \Lambda = k_0 \Lambda_0 + k_j \Lambda_j \), \( j \) is as in (1.8), where \( k_0 \geq 1 \), \( k_j = 1 \) in the cases when \( \Lambda_j \) is the fundamental weight corresponding to the short root \( \alpha_j \) and \( k_j \geq 1 \) in the cases when \( \Lambda_j \) is the fundamental weight corresponding to the long root \( \alpha_j \). Our formula (3.30) in Theorem 3.9 is a generalization of Gepner’s formula to the character of parafermionic space \( L(k \Lambda_0) \hat{\mathbf{b}}^+ \), where \( \Lambda = k_0 \Lambda_0 + k_j \Lambda_j \), \( j \) is as in (1.8) and \( k_0, k_j \geq 1 \).

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