The formulations of Classical Mechanics by Lagrange and Hamilton are the modern foundation of classical physics [Ar]. Not only do these theories describe the motion of systems of particles, but Maxwell’s theory of electromagnetism, as well as other field theories, can also be formulated in Lagrangian and Hamiltonian terms. A Lagrangian field theory is defined by a local functional of the fields, called the lagrangian, and its integral over spacetime, called the action. The classical solutions of the field theory are the critical points of the action. In particular, the minima satisfy the “least action principle” of Maupertius. The Hamiltonian theory is defined by a function, the hamiltonian, on phase space, or more generally on a symplectic manifold. The classical motion of the system is then described by Hamilton’s equations, whose solutions are integral curves of the symplectic gradient vector field of the hamiltonian. For many mechanical systems of particles, which should be regarded as 0 + 1 dimensional field theories, there is both a Lagrangian and Hamiltonian formulation. Then the relationship between them is expressed by the Legendre transform, if the lagrangian is nondegenerate. A typical example is the motion of a particle on a Riemannian manifold $M$. The action of the Lagrangian theory is the energy of a path in $M$. The hamiltonian is the norm square function on the tangent bundle $TM$, which obtains a symplectic structure from the metric. For higher dimensional field theories the Hamiltonian formulation only makes sense on spacetimes which are globally a product of space and time, that is, on $(d + 1)$-dimensional manifolds which are the product of a $d$-manifold with a 1-manifold. A field on this product manifold is a path of fields in space. Symmetry plays an important role in classical physics; many mechanical systems admit groups.
of symmetries. Noether discovered that symmetries are linked to conservation principles. This leads to the study of Noether currents in the Lagrangian theory and moment maps in the Hamiltonian theory.

The Chern-Simons form of a connection, introduced in [CS], can be viewed as the Lagrangian of a classical field theory. In this paper we study the $3 = 2 + 1$ dimensional case. We are principally motivated by recent work of Witten [W], who demonstrates that the quantization of this classical theory leads to new topological invariants. Our study of the classical theory clarifies certain features which are reflected in the quantum theory. In the course of our study we found that the classical Chern-Simons theory has many subtle aspects which provoke some refinements of the usual mathematical formulations of field theories. However, one should be aware that the Chern-Simons theory differs from most field theories in two important respects. First, the theory is topological (generally covariant) in that it is defined on oriented manifolds without any choice of metric. Hence it admits diffeomorphism groups as symmetries, whereas the usual examples admit only isometries. Second, the Lagrangian (1.26) involves only first powers of first derivatives of the fields, which leads to a first order Euler-Lagrange equation (3.4); the usual examples involve second derivatives or first derivatives squared, and so lead to second order equations of motion.

Everything is derived from the action. This principle, articulated by many theoretical physicists, places the Lagrangian formulation ahead of the Hamiltonian formulation. In fact, the latter is derived from the former. In many field theories one writes down the action and verifies its fundamental properties immediately; by contrast in §2 we take many pages to define the Chern-Simons action. If $X$ is a closed, oriented 3-manifold, then the action is a complex number with unit norm. This is the exponential of $2\pi i$ times the usual action, which is a real number determined modulo the integers. The fact that the action not a real number, but is only defined up to integers, leads to the interesting geometry of the theory. The reality of the action, which implies that the exponential has unit norm, reflects the unitarity of the theory. If $X$ has a nonempty boundary, then the action is an element of unit norm in an abstract metrized complex line which depends on the restriction of the field to $\partial X$. We call this line the Chern-Simons line.\footnote{Instead of complex lines with inner products we can restrict to the elements of unit norm, which form a principal homogeneous space for the circle group. Many discussions of classical and quantum Chern-Simons theory replace the Chern-Simons line bundle (over the moduli space of flat connections; see below) with a determinant line bundle. Although these two bundles may be isomorphic if appropriately chosen, the Chern-Simons action leads directly to the Chern-Simons line bundle, not the determinant line bundle. From this perspective the determinant line bundle is extraneous to the theory. Observe that the Chern-Simons line bundle is defined by a cohomology class in $H^4(BG)$, whereas the determinant line bundle is defined by a representation of $G$. However, the determinant line bundle most likely enters into another theory, whose action is the $\eta$ invariant of Atiyah-Patodi-Singer.} Thus the Lagrangian theory assigns a metrized line to every field on a space (closed, oriented 2-manifold) $Y$, and a unit element in the line of the boundary field for every field on a spacetime (compact, oriented 3-manifold) $X$. In most field theories these lines are canonically trivial; the nontriviality of these lines in the Chern-Simons theory is why this theory illuminates the mathematical structure of classical field theory so vividly. We remark that it is crucial to construct these lines precisely, and not just up to isomorphism. The characteristic properties of the Lagrangian theory are spelled out in Theorem 2.19, which is an axiomatization of topological classical
field theory. The Chern-Simons action is invariant under a large symmetry group (of gauge transformations), which we discuss below.

The most important property of a Lagrangian field theory is \textit{locality}. Fields are local; they can be cut and pasted. (A formal statement looks very similar to the definition of a sheaf.) The action is local in that it adds when we glue fields together. We call this a \textit{gluing law}; it is stated precisely in Theorem 2.19(d). All of our constructions in this paper are local, and so lead to gluing laws: Theorem 2.19(d), Proposition 4.4(d), Proposition 4.11, Theorem 4.26(d), and Proposition 5.29.

So far we have discussed general properties of field theories. Chern-Simons theory is a gauge theory, so in §1 we review the geometry of connections on principal bundles with compact structure group $G$. Our exposition is slanted toward field theory. For example, we emphasize cutting and pasting of connections. At the end of this section we introduce the Chern-Simons 3-form of a connection, which is our lagrangian. Chern-Simons forms are parametrized by invariant bilinear forms on the Lie algebra of $G$. To get an action which is well-defined up to integers, we must restrict to a certain lattice of \textit{integral} forms. For simple, simply connected groups like $SU(2)$ these forms (or cohomology classes) are parametrized by the integers (6.2). This integer is usually called the “level” of the theory. This is sufficient to define the Chern-Simons action if $G$ is connected and simply connected, as we assume throughout this paper. But for a general compact Lie group $G$ we need to choose a refinement of this bilinear form to define the Chern-Simons action. This is (up to isomorphism) a cohomology class in $H^4(BG) = H^4(BG; \mathbb{Z})$, where $BG$ is the classifying space of $G$; the integral bilinear forms are in natural 1:1 correspondence with the image of $H^4(BG)$ in $H^4(BG; \mathbb{R})$. The theories for arbitrary compact Lie groups are the subject of Part 2 of this paper [F2]. The importance of the general case in the study of rational conformal field theories has been emphasized by Moore and Seiberg [MS]. The definition of the action for the general case was sketched by Dijkgraaf and Witten [DW]. However, they use singular cochains, and it is not possible to discuss smoothness (of the Chern-Simons line bundle, for example) in their framework.

The classical solutions in Chern-Simons theory are flat connections. We derive the Euler-Lagrange equations at the beginning of §3, where we also discuss a little of the geometry of the space of flat connections. Mostly in §3 we study the Chern-Simons action on product manifolds $X = [0, 1] \times Y$; that is, we study the Hamiltonian theory. Then the action can be interpreted as the parallel transport of a unitary connection on the Chern-Simons line bundle formed over the space of fields on the boundary (Proposition 3.17). The abstract argument of Appendix B shows that the existence of this connection depends only on the basic properties of the action, in particular the gluing law, not on particular features of this theory, and so should exist in any Lagrangian field theory. However, our proof in §3 proceeds by direct calculation. The curvature of this connection (times $i/2\pi$) is the symplectic structure (3.18) of the Hamiltonian theory.$^5$ Notice that the symplectic form automatically satisfies an integrality condition, the integrality condition in “geometric quantization” theory, as it is the curvature of a line bundle. Since the Chern-Simons

$^4$We write $' [0, 1]'$ first in this product so that the orientations work out properly. This should be done quite generally for homotopies and bordisms.

$^5$This geometric point of view on the relationship between the Lagrangian and Hamiltonian theories appears to be well-known (see [Ax], for example), though it is not often expressed in the literature.
theory is topological, the Hamiltonian function is identically zero—there is no local motion. The symmetries in the theory—gauge transformations—lift to the Chern-Simons line bundle over the space of fields; this lift is constructed directly from the Chern-Simons action. From this lift one computes a moment map (3.19) for the symmetry on the space of fields. Not only does the Lagrangian formulation guarantee the existence of a moment map, but it also constructs one (via this action on the line bundle) in situations where the moment map is not unique.6 The space of classical solutions on the infinite cylinder \([0, \infty) \times Y\) is also a symplectic manifold, in this case the symplectic quotient of the space of fields. The space of classical solutions on an arbitrary 3-manifold with boundary, which is not necessarily a product, maps onto a Lagrangian submanifold of the symplectic manifold attached to the boundary (Proposition 3.27). Not only the symplectic form, but also the Chern-Simons line bundle with connection are trivial on this submanifold. The Hamiltonian theory is then the assignment of a symplectic manifold to every space \(Y\) and a Lagrangian submanifold to every spacetime \(X\).

One feature of the Chern-Simons theory which is not usually considered in standard field theories is the geometry of the theory on spaces (as opposed to spacetimes) with boundary, here 2-manifolds with boundary. In the quantum theory [W] these are especially important and lead to an “exact solution” of the path integral. In §4 and §5 we construct the classical theory on surfaces with boundary. Our constructions depend on a closely related \((1+1)\) dimensional field theory, the Wess-Zumino-Witten theory, which is discussed in Appendix A. In these constructions we generalize the theory on closed surfaces and so construct complex lines associated to connections on surfaces with boundary. To accomplish this we must introduce certain trivializing data on the boundary of the surface. Hence in §4 we make a digression into the geometry of connections on the circle. In particular, we discuss the universal bundle and properties of the holonomy map (Lemma 4.18). We then construct the Chern-Simons lines, but only if we suitably restrict the values of the boundary holonomies (Theorem 4.26). These lines fit together to form a (Chern-Simons) line bundle over the moduli space of flat connections on the surface with boundary, now with fixed boundary holonomies. We discuss this moduli space at the end of §4 and derive a formula for its dimension (4.44). In §5 we construct a connection on the Chern-Simons line bundle and compute its curvature (Proposition 5.9). This line bundle and connection depends on the trivializing boundary data. This dependence will be more clear in Part 2, where we also construct geometric objects similar to line bundles with connection which exist without restriction on the boundary holonomies.

In many ways the classical Chern-Simons theory unifies the gauge-theoretic geometrical objects in low dimensions that are associated with a compact Lie group \(G\) and an integral bilinear form on its Lie algebra;7 a characteristic class of \(G\) bundles over a closed oriented 4-manifold, the Chern-Simons invariant of a connection over a closed oriented 3-manifold, and the line bundle with connection over the moduli space of flat connections on a closed oriented 2-manifold. There are corresponding geometric objects over manifolds with boundary, as we discussed earlier. There is even something one can say in 1 dimension, but this is more complicated.

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6 This occurs in Chern-Simons theory if the center of the gauge group \(G\) has positive dimension, which does not happen in the groups we consider here, but certainly there are compact Lie groups (e.g. \(G = T\), the circle group) where this occurs. They will be considered in Part 2.

7 More precisely, a class in \(H^4(BG)\).
constructions are all fit in nicely with smooth Deligne cohomology. In fact, they can be constructed by integration of Deligne cocycles over these various manifolds. We develop this integration theory in [F1], and discuss the Chern-Simons theory from this point of view in Part 2 of this paper [F2]. Finally, then, the Chern-Simons theory looks like standard field theories—the action is an integral over spacetime.

A different integration theory (for singular cocycles) can be used when the gauge group is finite [FQ]. There is probably an analogous integration theory for $K$-theory and superconnections. The outstanding question is to construct integration theories which give the corresponding objects in quantum Chern-Simons theory. (When the gauge group is finite, the standard path integral is a finite sum, and the theory can be constructed directly [FQ].)

We conclude with a final comment about the symmetries in Chern-Simons theory, or more generally in any classical gauge theory. Namely, the symmetries form a groupoid rather than a group. A groupoid is a “group with states”. In gauge theory the states are the various principal bundles over a given manifold, and the group elements are isomorphisms between principal bundles (which cover the identity map on the base). For each principal bundle the automorphisms form a group—the group of gauge transformations. However, the larger symmetry of the groupoid enters since there is no canonical identification of two isomorphic bundles with connections; in other words, connections have automorphisms. This is particularly important when we glue together bundles and connections. This extra symmetry has nontrivial consequences, some of which are manifest in the quantum theory, and we find that they justify our use of categorical language.

Several closely related manuscripts appeared while this work was in progress. Among them is Axelrod’s thesis [Ax], which also treats the classical Chern-Simons theory (and other topics). He considers more examples of classical theories, and he goes further towards formulating axioms for classical topological field theories in general. Daskalopoulos and Wentworth [DasW] also construct line bundles with connection over the moduli spaces of flat connections on a surface with boundary. Their connection differs from ours; they found a relationship between boundary holonomies and representations that our connection does not exhibit (cf. §5). Chang [Ch] constructs a determinant line bundle with connection over these moduli spaces.

Over the two years this work has been in progress, I talked to many people about the contents of this paper. I thank all of them, and especially single out Scott Axelrod, Joseph Bernstein, Sheldon Chang, George Daskalopoulos, Frank Quinn, Karen Uhlenbeck, Alan Weinstein, and Richard Wentworth for illuminating conversations. This paper grew out of lectures given at the University of Texas in the spring of 1990. I thank the audience for their feedback. I’d also thank the Aspen Center for Physics and the Regional Geometry Institute in Park City for their hospitality while some of this work was carried out. Finally, my thinking about field theories has been stimulated greatly by Graeme Segal’s work in this area.

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8Gawędzki [Ga] treated the action in Wess-Zumino-Witten theory using smooth Deligne cohomology. Related ideas also appear in work of Brylinski.
§1 Connections on principal bundles

In this section we review some basics about Lie groups and connections on principal bundles. We consider arbitrary compact Lie groups. Although subsequent sections of this paper only treat connected and simply connected groups, in Part 2 [F2] we take up the general case. Beyond textbook material, which we review to establish notation, we discuss carefully how to cut and paste connections (Proposition 1.24). This establishes the fact that connections can serve as local fields in a field theory. Then we introduce the lagrangian, the Chern-Simons form (1.26), and state its basic properties (Proposition 1.27). This form is only defined on the total space of a bundle, not on the base, and this leads to some of the rich geometry of the theory. Since connections have automorphisms, we use the language of categories to describe the space of all connections.

Let $G$ be a compact Lie group. The identity component $G_0$ is a normal subgroup with quotient the finite group $\Gamma = \pi_0 G$ of components:

(1.1) \[ 1 \to G_0 \to G \to \Gamma \to 1. \]

Then there is a finite covering

(1.2) \[ 1 \to A \to \tilde{G}_0 \to G_0 \to 1 \]

for some finite abelian group $A$ such that

(1.3) \[ \tilde{G}_0 \cong T \times K_1 \times \cdots \times K_s, \]

where $T$ is a torus and $K_i$ are connected, simply connected simple groups. The basic examples of compact Lie groups are finite groups, the circle group $\mathcal{T}$, and the connected, simply connected simple groups (of type $A,B,C,D,E,F,G$); equations (1.1) and (1.2) show that all others are products of these up to finite covers and finite extensions.

The Lie algebra $\mathfrak{g}$ of $G$ is the vector space of left invariant vector fields on $G$ endowed with the Lie bracket. The Lie algebras of $G$, $G_0$, and $\tilde{G}_0$ coincide. There is a natural $\mathfrak{g}$-valued 1-form $\theta$ on $G$ which assigns to each vector its left invariant extension.\footnote{It is often denoted $g^{-1} dg$ for matrix groups.} This Maurer-Cartan form satisfies

(1.4) \[ L_g^* \theta = \theta, \]
\[ R_g^* \theta = \text{Ad}_{g^{-1}} \theta, \]

where $L_g : G \to G$ is left multiplication by $g \in G$, the map $R_g : G \to G$ is right multiplication, and $\text{Ad}_g = L_g \circ R_{g^{-1}}$ is the adjoint map. Its differential satisfies the Maurer-Cartan equation

\[ d\theta + \frac{1}{2} [\theta \wedge \theta] = 0; \]

it is a simple consequence of the relationship of $d$ to the Lie bracket.

A principal $G$ bundle $P \to X$ is a manifold $P$ with a free right $G$ action; the quotient $X$ is then also a manifold. Notice that if $G$ is finite, then $P$ is a regular
covering space of $X$.\footnote{However, the categories of principal bundles and regular covering spaces differ: Maps of principal bundles commute with the group action, whereas maps of covering spaces need not.} In general, the fiber $P_x$ over $x$ is a space on which $G$ acts simply transitively by right multiplication. Hence for each $p \in P_x$ there is an isomorphism of right $G$-spaces

$$
\tau_p : G \to P_x \\
g \mapsto p \cdot g.
$$

The “coordinate change” $\tau_p^{-1} \tau_p = L_g$, where $g \in G$ is the unique element with $p' = p \cdot g$. Hence left invariant tensors on $G$ induce tensors on $P_x$. Thus we identify $T_p P_x \cong g$ for all $p \in P_x$. Dually, the Maurer-Cartan form $\theta$ induces a form $\theta_x$ on $P_x$ which satisfies

$$
R_g^* \theta_x = \text{Ad}_{g^{-1}} \theta_x, \\
d\theta_x + \frac{1}{2} [\theta_x \wedge \theta_x] = 0.
$$

(1.5)

Here $R_g : P_x \to P_x$ is the right action of $g \in G$. If there is a global section $s : X \to P$ then $P$ is trivial; the map

$$
\tau_s : X \times G \to P \\
\langle x, g \rangle \mapsto s(x) \cdot g
$$

is a trivialization. Any principal $G$ bundle is locally trivial.

Suppose $h : G \hookrightarrow G'$ is an inclusion of compact Lie groups and $P \to X$ a principal $G$ bundle. Then the induced $G'$ bundle $P_G' \to X$ is the quotient

$$
P_G' = P \times_G G' = P \times G' \big/ \{ p, g' \} \sim \langle p \cdot g, h(g^{-1})g' \rangle, \\
p \in P, \quad g' \in G', \quad g \in G.
$$

(1.6)

There is a natural inclusion

$$
h_P : P \to P_G' \\
p \mapsto \langle p, e \rangle
$$

(1.7)

which covers the identity map on $X$.

A map of principal $G$ bundles $\varphi : P' \to P$ is a smooth map of manifolds which commutes with the $G$ action. Hence it induces a map $\bar{\varphi} : X' \to X$ on the quotient spaces. When $X' = X$ and $\bar{\varphi}$ is the identity we term $\varphi$ a \textit{morphism} of principal bundles over $X$. If, in addition, $P' = P$ then $\varphi$ is termed an automorphism of $P$, or a \textit{gauge transformation}. In this case there is an associated map

$$
g_{\varphi} : P \to G
$$

defined by the equation

$$
\varphi(p) = p \cdot g_{\varphi}(p).
$$

(1.8)
Clearly maps of principal $G$ bundles compose, and the set of gauge transformations of $P$ forms a group $\mathcal{G}_P$. If $h: G \hookrightarrow G_1$ is an inclusion, and $\varphi: P' \to P$ a map, then there is an induced map $P'_{G_1} \to P_{G_1}$ on the $G_1$ extensions. Also, if $P \to X$ is a principal $G$ bundle and $\tilde{\varphi}: X' \to X$ a smooth map, then there is an induced bundle $\tilde{\varphi}^*P \to X'$ and a bundle map $\varphi: \tilde{\varphi}^*P \to P$ covering $\tilde{\varphi}$.

A connection on $P \to X$ is a $\mathfrak{g}$-valued 1-form $\Theta$ such that

\begin{align}
(1.9) & \quad i_x^*\Theta = \theta_x, \\
(1.10) & \quad R^*_g \Theta = \text{Ad}_{g^{-1}} \Theta,
\end{align}

where $i_x: P_x \hookrightarrow P$ is the inclusion of the fiber over $x \in X$ and $R_g: P \to P$ is the right translation by $g \in G$. Connections can be constructed locally using local trivializations. Since (1.9) and (1.10) are convex conditions, local connections patch together via a partition of unity into global connections. Hence connections exist. In fact, (1.9) and (1.10) are affine conditions, so the set $\mathcal{A}_P$ of all connections on $P$ is an affine subspace of $\Omega^1_P(\mathfrak{g})$, the vector space of $\mathfrak{g}$-valued 1-forms on $P$. A tangent vector to the space of connections is a $\mathfrak{g}$-valued 1-form $\hat{\Theta}$ on $P$ such that

\begin{align}
(1.11) & \quad i_x^*\hat{\Theta} = 0, \\
(1.12) & \quad R^*_g \hat{\Theta} = \text{Ad}_{g^{-1}} \hat{\Theta}.
\end{align}

The vector space of forms satisfying (1.11) and (1.12) is denoted $\Omega^1_X(\mathfrak{g}_P)$, since $\hat{\Theta}$ is the lift to $P$ of a 1-form on $X$ transforming in the adjoint bundle $\mathfrak{g}_P = P \times_G \mathfrak{g}$. The curvature $\Omega$ of a connection $\Theta$ is the $\mathfrak{g}$-valued 2-form

$$
(1.13) \quad \Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta];
$$

it satisfies

\begin{align}
(1.14) & \quad R^*_g \Omega = \text{Ad}_{g^{-1}} \Omega, \\
(1.15) & \quad i_x^* \Omega = 0.
\end{align}

(This last equation follows from (1.5).) Hence the curvature is an element of $\Omega^2_X(\mathfrak{g}_P)$.

Differentiating (1.13) we obtain the Bianchi identity

$$
(1.16) \quad d\Omega + [\Theta \wedge \Omega] = 0.
$$

Introduce the covariant derivative

$$
\delta \Theta = d + \text{ad}(\Theta).
$$

Then the Bianchi identity can be rewritten as

$$
\delta \Theta \Omega = 0.
$$

Suppose $h: G \to G'$ is an inclusion and $\Theta$ a $G$ connection on $P \to X$. Then there is an induced $G'$ connection $\Theta_{G'}$ on $P_{G'} \to X$. Namely, define the $\mathfrak{g}'$-valued 1-form

$$
(1.17) \quad (\Theta_{G'})_{(p,g')} = h(\text{Ad}_{(g')^{-1}} \Theta_p) + (\theta')_{g'}
$$
on $P \times G'$, where $\theta'$ is the Maurer-Cartan form on $G'$, and $\hat{h}: g \mapsto g'$ the induced inclusion of Lie algebras. From (1.4) and (1.10) we see that (1.17) vanishes along the orbits of the right $G$ action on $P \times G'$ and is invariant under that action, so passes to a $g'$-valued 1-form on $P_G' = P \times G' / G$ (cf. (1.6)). Properties (1.9) and (1.10) are easily verified, so $\Theta_{G'}$ is a connection. The pullback of $\Theta_{G'}$ under the natural map $h_P: P \to P_{G'}$ (1.7) can be identified with the original connection $\Theta$.

Let $\varphi: P' \to P$ be a map of principal bundles and $\Theta$ a connection on $P$; then $\varphi^*(\Theta)$ is a connection on $P'$. In particular, the group $G_P$ of gauge transformations acts (on the right) on the space of connections $A_P$. Suppose $\varphi: P \to P$ is a gauge transformation with associated map $g_\varphi: P \to G$ (1.8). Let $\phi_\varphi = g_\varphi^{-1}(\theta)$ be the pullback of the Maurer-Cartan form. Then a basic equation in the theory of connections asserts

\begin{equation}
\varphi^*(\Theta) = Ad_{g_\varphi^{-1}} \Theta + \phi_\varphi.
\end{equation}

By contrast the curvature transforms as a tensor:

\begin{equation}
\varphi^*(\Omega) = Ad_{g_\varphi^{-1}} \Omega.
\end{equation}

An automorphism of a connection $\Theta$ is a gauge transformation which satisfies $\varphi^*(\Theta) = \Theta$. Equation (1.18) is then a first order differential equation for $\varphi$, which asserts that $g_\varphi$ is parallel. Since parallel sections over connected manifolds are determined by their value at a single point, we have proved the following.

**Proposition 1.20.** If $\varphi$ is an automorphism of $\Theta$, then $g_\varphi$ is parallel. In particular, if $X$ is connected and $\varphi$ is the identity at some point of $X$, then $\varphi$ is the identity on $X$.

We will often encounter connections over cylinders. The following lemma gives a standard form for such connections.

**Lemma 1.21.** Suppose $Q \to Y$ is a $G$ bundle over a manifold $Y$, and $[0, \infty) \times Q \to [0, \infty) \times Y$ the pullback to the cylinder over $Y$. Let $\Theta$ be a connection on $[0, \infty) \times Q$, which we write as

\[ \Theta = \eta_t + \xi_t \, dt, \quad t \in [0, \infty), \quad \eta_t \in \Omega^1_Q(g), \quad \xi_t \in \Omega^0_Q(g). \]

Then there exists a unique gauge transformation $\varphi$ of $[0, \infty) \times Q$ which satisfies

\begin{align}
\varphi\big|_{[0) \times Q} &= \text{id}, \\
\varphi^*(\Theta) &= \tilde{\eta}.
\end{align}

In other words, the transformed connection has no $dt$ component.\footnote{A section $s$ of $[0, \infty) \times Q$ with the property that $s^* \Theta$ has no $dt$ component is called a temporal gauge; it exists if $Q$ is trivializable.}

**Proof.** Let $g_t: Q \to G$ be the map associated to $\varphi\big|_{[t) \times Q}$ and $\phi_t = g_t^{-1}(\theta)$. Then (1.22) asserts $g_0 = \text{id}$, and by (1.18) we see that (1.23) is equivalent to

\[ \text{Ad}_{g_t^{-1}} \xi_t + \phi_t \left( \frac{\partial}{\partial t} \right) = 0. \]
In more usual notation this reads

\[ g^{-1} \xi g + g^{-1} \frac{\partial g}{\partial \tau} = 0, \]

which has a unique solution with initial condition \( g_0 = \text{id} \) by the standard theory of first order ordinary differential equations.

We apply this lemma to glue connections over manifolds with boundary along boundaries where the connections agree.

**Proposition 1.24.** Suppose \( P \to X \) is a \( G \) bundle over an oriented manifold \( X \), and \( Y \to X \) is an oriented codimension one submanifold. Let \( X^{\text{cut}} \) be the manifold obtained by cutting \( X \) along \( Y \). There is a gluing map \( \bar{g}: X^{\text{cut}} \to X \) which is a diffeomorphism off of \( Y \) and maps two distinct submanifolds \( Y_1, Y_2 \) of \( \partial X^{\text{cut}} \) diffeomorphically onto \( Y \). Let \( P^{\text{cut}} = \bar{g}^* P \) be the cut bundle and \( g: P^{\text{cut}} \to P \) the gluing map. Now suppose \( \Theta^{\text{cut}} \) is a connection on \( P^{\text{cut}} \) such that there exists a connection \( \eta \) on \( P \big|_Y \) with \( \bar{g}^*(\eta) = \Theta \big|_{Y_1 \cup Y_2} \). Then \( \eta \) extends to a connection \( \Theta \) on \( X \) such that \( \bar{g}^*(\Theta) \) is gauge equivalent to \( \Theta^{\text{cut}} \). The gauge equivalence class of \( \Theta \) is uniquely determined.

The hypothesis asserts that \( \Theta^{\text{cut}} \big|_{Y_1} \) and \( \Theta^{\text{cut}} \big|_{Y_2} \) agree under the given identification of the bundles. Just as a smooth function on \( X^{\text{cut}} \) whose restrictions to \( Y_1 \) and \( Y_2 \) agree does not necessarily glue into a smooth function on \( X \), since the resulting function may not be smooth in the transverse direction, so too the connection \( \Theta^{\text{cut}} \) does not glue directly. Proposition 1.24 asserts that we can glue smoothly if we make a gauge transformation.

**Proof.** Choose tubular neighborhoods \( N_i \cong [0, \infty) \times Y_i \) of \( Y_i \) in \( X^{\text{cut}} \) and bundle isomorphisms \( P \big|_{N_i} \cong [0, \infty) \times P \big|_{Y_i} \). Let \( \varphi_i \) be the gauge transformations over \( N_i \) guaranteed by Lemma 1.21. Let \( \rho_1: [0, 1] \to [0, 1] \) be a monotone increasing smooth function with \( \rho_1(t) = t \) for \( 0 \leq t \leq 0.5 \) and \( \rho_1([0.9, 1]) = 1 \). Let \( \rho_2: [0, 1] \to [0, 1] \) be a monotone increasing smooth function with \( \rho_2([0, 0.1]) = 0 \) and \( \rho_2([0.9, 1]) = 1 \). Define \( \hat{\varphi}_i \) by

\[
\hat{\varphi}_i(t, y) = \begin{cases} 
\varphi_i(\rho_1(t), y), & 0 \leq t \leq 1; \\
\varphi_i(2 - \rho_2(t), y), & 1 \leq t \leq 2; \\
\text{id}, & 2 \leq t,
\end{cases}
\]

and extend \( \hat{\varphi}_i \) to a smooth gauge transformation \( \hat{\varphi} \) on \( X^{\text{cut}} \) which is the identity outside of \( N_i \). Then \( \hat{\varphi}^*(\Theta^{\text{cut}}) \) glues to form a smooth connection \( \Theta \) on \( X \), since near \( Y_i \) the transverse component of \( \hat{\varphi}^*(\Theta^{\text{cut}}) \) vanishes.

Let \( \langle \cdot \rangle: g \otimes g \to \mathbb{R} \) be an Ad-invariant symmetric bilinear form on the Lie algebra \( g \). Since

\begin{equation}
(1.25) \quad g \cong t \oplus t_1 \oplus \cdots \oplus t_s,
\end{equation}

is a direct sum of an abelian Lie algebra \( t \) with simple Lie algebras \( t_i \), from (1.1)–(1.3), the invariant form also decomposes as a direct sum. Any invariant form on \( t_i \) is a multiple of the Killing form \( b_i \), and any symmetric form on \( t \) is invariant. Thus the space of forms on \( g \) is \( S^2t^* \oplus \mathbb{R}b_1 \oplus \cdots \oplus \mathbb{R}b_s \).
Suppose $\Theta$ is a $G$ connection on $P \to X$ with curvature $\Omega$. Let $\langle \Omega \wedge \Omega \rangle$ be the Chern-Weil 4-form associated with the bilinear form $\langle \cdot \rangle$. It follows from (1.14) and (1.15) that this form on $P$ is the lift of a 4-form on $X$, which we also denote $\langle \Omega \wedge \Omega \rangle$. Furthermore, a simple computation using (1.16) shows that this form is closed. The fundamental result of Chern-Weil theory is that the de Rham cohomology class of this form is a certain characteristic class of $P$. The form $\langle \Omega \wedge \Omega \rangle$ is gauge invariant by (1.19) and the fact that $\langle \cdot \rangle$ is Ad-invariant.

The Chern-Simons form $[CS]$ is an antiderivative of $\langle \Omega \wedge \Omega \rangle$ on $P$. Set

\begin{equation}
\alpha = \alpha(\Theta) = \langle \Theta \wedge \Omega \rangle - \frac{1}{6} \langle \Theta \wedge [\Theta \wedge \Theta] \rangle.
\end{equation}

**Proposition 1.27.** The 3-form $\alpha$ satisfies:

(a) $i^*_x \alpha = -\frac{1}{6} \langle \theta_x \wedge \theta_x \wedge \theta_x \rangle$;
(b) $d\alpha = \langle \Omega \wedge \Omega \rangle$;
(c) $R^*_x \alpha = \alpha$;
(d) If $\varphi: P' \to P$ is a bundle map and $\Theta$ a connection on $P$, then $\alpha(\varphi^* \Theta) = \varphi^* \alpha(\Theta)$.
(e) If $\varphi: P \to P$ is a gauge transformation with associated map $g = g_\varphi: P \to G$, and $\phi = \phi_\varphi = \varphi^* (\theta)$, then

\begin{equation}
\varphi^* \alpha = \alpha + d(\langle \text{Ad}_{g_{-1}} \Theta \wedge \phi \rangle - \frac{1}{6} \langle \phi \wedge [\phi \wedge \phi] \rangle).
\end{equation}

**Proof.** (a) and (c) follow easily from (1.9), (1.10), (1.14), (1.15), and the Ad-invariance of $\langle \cdot , \cdot \rangle$. (d) is also trivial. (b) and (e) require some calculation; we illustrate (b) and leave (e) to the reader:

\[
d\alpha = \langle d\Theta \wedge \Omega \rangle - \langle \Theta \wedge d\Omega \rangle - \frac{1}{2} \langle d\Theta \wedge [\Theta \wedge \Theta] \rangle
\]
\[
= \langle (\Omega - \frac{1}{2} [\Theta \wedge \Theta]) \wedge \Omega \rangle + \langle \Theta \wedge [\Theta \wedge \Theta] \rangle - \frac{1}{2} \langle \Omega \wedge [\Theta \wedge \Theta] \rangle
\]
\[
= \langle \Omega \wedge \Omega \rangle.
\]

Here we use

\begin{equation}
\langle [\Theta \wedge \Theta] \wedge [\Theta \wedge \Theta] \rangle = \langle [[\Theta \wedge \Theta] \wedge \Theta] \wedge \Theta \rangle = 0
\end{equation}

which follows from Ad-invariance and the Jacobi identity.

It is useful for us to think about all connections at once. In a technical sense we shouldn’t think of the set of connections since the set of “all” of anything leads to contradictions. Also, we want to account for the automorphisms of connections, since that is our basic symmetry. Thus we consider the category of all connections and morphisms of connections. As the word ‘category’ is anathema for many mathematicians and most physicists, but is a useful concept for us, we insert a few general remarks. The simplest algebraic structure on a set $S$ is a binary associative composition law $S \times S \to S$. Such a structure is termed a monoid.\(^{12}\) The simplest

\(^{12}\)If there is an identity element as well, $S$ is termed a semigroup. If, in addition, there are inverses, then $S$ is a group, which is perhaps the most familiar algebraic type. Categories are usually defined as generalizations of semigroups, but we will not insist on identity elements.
example is the set of natural numbers under addition. A category $\mathcal{C}$ is nothing more than a “monoid with states”. Thus there is a collection $\text{Obj}(\mathcal{C})$ of objects (states) and for each pair of objects $C_1, C_2$ there is a set of morphisms $\text{Mor}(C_1, C_2)$. A morphism is represented by an arrow $C_1 \xrightarrow{\phi} C_2$ from the initial state to a final state. Two morphisms $C_1 \xrightarrow{\phi} C_2$ and $C'_1 \xrightarrow{\phi'} C'_2$ compose if and only if $C_2 = C'_1$, and the composition law is assumed associative. A monoid is then a category with a single object. Notice that we don’t assume that $\text{Obj}(\mathcal{C})$ is a set, but we do assume that $\text{Mor}(C_1, C_2)$ is a set for every pair of morphisms.

Let $X$ be a fixed manifold. Define the category $\mathcal{C}_X = C^G_X$ of $G$ connections as follows. An object in $\mathcal{C}_X$ is a connection $\Theta$ on a principal $G$ bundle $P \to X$. A morphism $\Theta' \xrightarrow{\phi} \Theta$ is a bundle map $\phi: P' \to P$ covering the identity map on $X$ (i.e., a bundle morphism) such that $\Theta' = \phi^* \Theta$. Notice that the category $\mathcal{C}_X$ contains identity maps $\Theta \xrightarrow{\text{id}} \Theta$ and that every morphism is invertible. Such a category is termed a groupoid. The category $\mathcal{C}_X$ carries a topology.\(^{13}\) The objects form a union of affine spaces

\begin{equation}
\text{Obj}(\mathcal{C}_X) = \bigsqcup_P A_P,
\end{equation}

where $\{P\}$ is the collection of all principal $G$ bundles over $X$. The set of morphisms between two connections is also a topological space in a natural way.

There is an obvious equivalence relation on $\mathcal{C}_X$—two connections $\Theta, \Theta'$ are equivalent if and only if there exists a morphism $\Theta \xrightarrow{\text{id}} \Theta'$. We denote the set of equivalence classes by $\overline{\mathcal{C}}_X$. Let $\{P_i\}$ be a set of representatives of topological equivalence classes of $G$ bundles over $X$. Then there is a noncanonical isomorphism

\[ \overline{\mathcal{C}}_X \cong \bigsqcup_{\{P_i\}} A_{P_i} / G_{P_i} \]

as a disjoint union.

Heuristically, a groupoid is a “group with states”. In gauge theory this notion provides an illuminating formalism in which to keep track of the symmetry. If all $G$ bundles over $X$ are isomorphic (which occurs if $G$ is connected and simply connected and $\dim X \leq 3$, for example), then we might fix $P \to X$ and consider the space of fields to be $A_P$ with the group of gauge transformations $G_P$ acting as symmetries. This is the usual picture. Or more generally we might fix a set of representatives for the different topological types. However, this is not adequate. Consider a situation in which we have two isomorphic bundles $P \to X$ and $P' \to X'$ we wish to identify, as when we glue connections. The glued connection, perhaps even its equivalence class, depends on the particular identification we choose. As there is no canonical identification, we need to keep track of all possible identifications. This is an additional symmetry and justifies our consideration of the category of bundles rather than simply a set of representatives.

A functor $\mathcal{F}: \mathcal{C}_1 \to \mathcal{C}_2$ generalizes the notion of a homomorphism of monoids. Namely, $\mathcal{F}$ is a map $\text{Obj}(\mathcal{C}_1) \to \text{Obj}(\mathcal{C}_2)$, and $\mathcal{F}$ assigns to each morphism $C_1 \xrightarrow{\phi} C_2$...
a morphism $F(C_1) \xrightarrow{F(\varphi)} F(C_2)$ such that composition is preserved. (If the categories have identity maps, then these must also be preserved.) Let $h: G \hookrightarrow G'$ be an inclusion of groups. Then our previous remarks about extensions of structure group are summarized by the statement that $h$ induces a functor

$$h_X: C^G_X \longrightarrow C^{G'}_X$$

for any manifold $X$. 

In this section we integrate the Chern-Simons lagrangian (1.26) over spacetime (a compact, oriented 3-manifold) to construct the Chern-Simons action. Since the Chern-Simons form lives on the total space of a bundle, and not on the base, we choose a section of the bundle to define the action. On closed 3-manifolds the integral is independent of the section, up to an integer, if we make the appropriate integrality hypothesis on the bilinear form (Hypothesis 2.5). If the 3-manifold has a boundary, then the integral depends only on the restriction of the section over the boundary, and the dependence is encoded in a cocycle (2.15). The Wess-Zumino-Witten action (2.13) enters in the formula for this cocycle; we develop its properties in Appendix A. The cocycle determines a hermitian complex line which only depends on the restriction of the connection to the boundary. We term this line the “Chern-Simons line”. These lines vary smoothly, so form a line bundle over smooth families of connections (Proposition 2.17). This line bundle and cocycle were also considered in [RSW] In Theorem 2.19 we state carefully the properties of the Chern-Simons action. This can be viewed as an axiomatization of a local (topological) classical Lagrangian field theory, in the spirit of Segal’s axioms for conformal field theory [S] and Atiyah’s axioms for topological quantum field theory [A]. The crucial gluing law, which is an assertion of the locality of the action, is (2.27).

From now on we fix a connected, simply connected, compact Lie group $G$ and an invariant form $\langle \cdot \rangle$ on its Lie algebra $\mathfrak{g}$. We single out simply connected groups because of the following topological fact.

**Lemma 2.1.** If $G$ is simply connected, then any principal $G$ bundle over a manifold of dimension $\leq 3$ admits a global section, hence is trivializable.

The proof uses $\pi_0 G = \pi_1 G = \pi_2 G = 0$ and elementary obstruction theory. Suppose $\Theta$ is a connection on a principal $G$ bundle $P \to X$, where $X$ is a closed, oriented 3-manifold. Let $p: X \to P$ be a section. Recall the Chern-Simons form $\alpha = \alpha(\Theta) \in \Omega^3_P$ (1.26). Define

\begin{equation}
S_X(p, \Theta) = \int_X p^* \alpha(\Theta).
\end{equation}

**Proposition 2.3.** Let $\varphi: P \to P$ be a gauge transformation with associated map $g_{\varphi}: P \to G$. Set $g = g_{\varphi} p: X \to G$ and let $\phi_g = g^* \theta$ be the pullback of the Maurer-Cartan form. Then

\begin{equation}
S_X(\varphi p, \Theta) = S_X(p, \varphi^* \Theta) = S_X(p, \Theta) - \int_X \frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle.
\end{equation}

**Proof.** The first equality is immediate from (2.2). The second equality follows from (1.28) and Stokes’ theorem.

We now make the following integrality hypothesis on the bilinear form $\langle \cdot \rangle$.  

\footnote{In this paper we restrict ourselves to trivializable bundles (cf. Lemma 2.1). The extension to nontrivializable bundles is carried out in Part 2 [F2].}
Hypothesis 2.5. Assume that the closed form $-\frac{1}{6}(\theta \wedge [\theta \wedge \theta])$ represents an integral class in $H^3(G; \mathbb{R})$.

Then the integral in (2.4) is an integer. Since any two sections of $P$ are related by a gauge transformation, it follows that

\begin{equation}
S_X(\Theta) = S_X(p, \Theta) \quad (\text{mod } 1)
\end{equation}

is independent of the section $p$. This is the Chern-Simons action on closed manifolds.

We can also deduce the independence of (2.6) on the section from the fact that the Chern-Simons form is closed (cf. Proposition 1.27(b)). Thus (2.2) doesn’t change under homotopies of $p$. Then the integrality of the restriction of the cohomology class of the Chern-Simons form to the fiber gives the rest.

Proposition 2.7. Let $X$ be a closed oriented 3-manifold. The Chern-Simons action

\[ S_X : C_X \longrightarrow \mathbb{R}/\mathbb{Z} \]

is smooth\(^{15}\) and satisfies:

(a) (Functoriality) If $\varphi : P' \rightarrow P$ is any bundle map covering an orientation preserving diffeomorphism $\bar{\varphi} : X' \rightarrow X$, and $\Theta$ is a connection on $P$, then

\[ S_{X'}(\varphi^*\Theta) = S_X(\Theta). \]

(b) (Orientation) Let $-X$ denote $X$ with the opposite orientation. Then

\[ S_{-X}(\Theta) = -S_X(\Theta). \]

(c) (Additivity) If $X = X_1 \sqcup X_2$ is a disjoint union, and $\Theta_i$ are connections over $X_i$, then

\[ S_{X_1 \sqcup X_2}(\Theta_1 \sqcup \Theta_2) = S_{X_1}(\Theta_1) + S_{X_2}(\Theta_2). \]

It follows from (a) that there is an induced action

\[ S_X : \mathcal{C}_X \longrightarrow \mathbb{R}/\mathbb{Z} \]

defined on the fields modulo symmetries. We will often write the action as $e^{2\pi i S_X(\Theta)}$, in which case (c) appears as a multiplicative property.

Proof. The smoothness follows since $\alpha(\Theta)$ is a smooth function of $\Theta$ (1.26). For (a), let $p' : X' \rightarrow P'$ be a section and $p = \varphi p' \bar{\varphi}^{-1}$ the induced section of $P$. Then by Proposition 1.27(d)

\[ S_X(p, \Theta) = \int_X (\bar{\varphi}^{-1})^* p'^* \varphi^* \alpha(\Theta) = \int_{X'} p'^* \alpha(\varphi^* \Theta) = S_{X'}(p', \varphi^* \Theta). \]

Assertions (b) and (c) follow from standard properties of integration of differential forms on oriented manifolds.

\(^{15}\)We will not bother with technicalities regarding the smooth structure of infinite dimensional manifolds here; they are well-understood. Rather, we will usually verify smoothness for a family of connections parametrized by a smooth manifold $U$, which the reader can take to be finite dimensional.
The Chern-Simons action of a connection which extends over a 4-manifold can be computed by integrating the Chern-Weil form. Let $\Theta$ be a connection on a principal $G$ bundle $\tilde{P} \to W$ over a compact oriented 4-manifold $W$, and denote the curvature of $\Theta$ by $\tilde{\Omega}$. Then the action of the restriction $\partial \tilde{\Theta}$ to the boundary is

$$S_{\partial W}(\partial \tilde{\Theta}) = \int_W \langle \tilde{\Omega} \wedge \tilde{\Omega} \rangle \pmod{1}. \tag{2.8}$$

This follows from Proposition 1.27(b) and Stokes' theorem.

The Chern-Simons functional also behaves well under extension of structure group.

**Proposition 2.9.** Let $k: G \to G'$ be an inclusion of connected, simply connected, compact Lie groups, and $X$ a closed oriented 3-manifold. Suppose $\langle \cdot \rangle'$ is an integral form on $g'$ which restricts to $\langle \cdot \rangle$ on $g$. Then for any $G$ connection $\Theta$ over $X$, its extension $\Theta'$ to a $G'$ connection satisfies $S_X(\Theta') = S_X(\Theta)$.

In fancier language, Proposition 2.9 asserts that the diagram

$$C_X(G_1) \xrightarrow{h_X} C_X(G'_1) \quad \xrightarrow{S_X} \quad \mathbb{R}/\mathbb{Z}$$

commutes, where $h_X$ is the functor which extends $G$ connections to $G'$ connections (1.31).

**Proof.** Let $P \to X$ be the $G$ bundle carrying $\Theta$ and $P' \to X$ its $G'$ extension. There is an inclusion $h_P: P \hookrightarrow P'$. Recall from (1.17) that $h_P^*(\cdot) = h(\cdot)$. Since $h^*(\langle \cdot \rangle) = \langle \cdot \rangle$ it follows from (1.26) that $h_P^*\alpha(\Theta') = \alpha(\Theta)$. Fix a section $p: X \to P$; then $h_P \circ p$ is a section of $P'$. Now

$$S_X(h_P \circ p, \Theta') = \int_X p^*h_P^*\alpha(\Theta') = \int_X p^*\alpha(\Theta) = S_X(p, \Theta).$$

Next, we consider compact oriented 3-manifolds $X$ with nonempty boundary. Suppose $\Theta$ is a $G$ connection on $P \to X$. We retain the definition (2.2). But now Proposition 2.3 is replaced by

**Proposition 2.10.** Let $\varphi: P \to P$ be a gauge transformation with associated map $g_\varphi: P \to G$. Set $g = g_\varphi p: X \to G$ and let $\phi_g = g^*\theta$ be the pullback of the Maurer-Cartan form. Then

$$S_X(\varphi p, \Theta) = S_X(p, \varphi^*\Theta) = S_X(p, \Theta) + \int_{\partial X} \langle \text{Ad}_{g^{-1}} p^*\Theta \wedge \phi_g \rangle - \int_X \frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle. \tag{2.11}$$

It is no longer true that the last two terms in (2.11) vanish modulo integers, but rather
Lemma 2.12. With Hypothesis 2.5 the functional

\[(2.13) \quad W_{\partial X}(g) = \int_X -\frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle \pmod{1}\]

depends only on the restriction of \(g: X \to G\) to \(\partial X\).

\(W_{\partial X}(g)\) is called the Wess-Zumino-Witten functional; it is the action of a 1+1 dimensional field theory. In Appendix A we sketch the development of this theory.

Proof. If \(g': X' \to G\) and there is an orientation reversing diffeomorphism \(\partial X \cong \partial X'\) under which the restrictions of \(g\) and \(g'\) coincide, then these maps patch together into a closed oriented (singular) 3-chain \(\tilde{g}: X' - X \to G\). Then

\[(2.14) \quad W_{\partial X}(g') - W_{\partial X}(g) = \int_{X' - X} -\frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle \in \mathbb{Z}\]

is an integer by Hypothesis 2.5.

Therefore, the Chern-Simons action (2.2) depends in a controlled manner on the restriction of \(p\) to \(\partial X\). We make sense of this by defining a metrized complex line \(L_{\partial \Theta}\) attached to the restriction \(\partial \Theta\) of \(\Theta\) to \(\partial X\). The Chern-Simons invariant of \(\Theta\) takes values in \(L_{\partial \Theta}\). (Compare with the discussion at the beginning of this section.) Notice that we are only concerned with the elements in \(L_{\partial \Theta}\) of unit norm, which form a principal homogeneous space for the circle group \(T\). It is easy to go back and forth between the \(T\)-space and the metrized line, which we do freely (usually without changing the notation).

We remark that we could also treat this dependence on boundary values by examining homotopies of sections and using the fact that the Chern-Simons form is closed; we leave this approach to the reader.

First, we abstract the construction of vector spaces (in this case lines) in situations where one must make a choice (here the choice of a section on the boundary). We suppose that the set of possible choices and isomorphisms of these choices form a groupoid \(C\). Let \(L\) be the category whose objects are metrized complex lines and whose morphisms are unitary isomorphisms. \(^{16}\) Suppose we have a functor \(F: C \to L\). Define \(V_F\) to be the inner product space of invariant sections of the functor \(F\): An element \(v \in V_F\) is a collection \(\{v(C) \in F(C)\}_{C \in \text{Obj}(C)}\) such that if \(C_1 \xrightarrow{\psi} C_2\) is a morphism, then \(F(\psi)v(C_1) = v(C_2)\). Suppose \(C\) is connected, that is, there exists a morphism between any two objects. Then \(\dim V_F = 0\) or \(\dim V_F = 1\), the latter occurring if and only if \(F\) has no holonomy, i.e., \(F(\psi) = \text{id}\) for every automorphism \(C \xrightarrow{\psi} C\). The terminology comes from the example of a metrized line bundle \(L \to M\) with unitary connection over a manifold \(M\). Form a category \(C(M)\) whose objects are the points of \(M\) and whose morphisms are piecewise smooth unparametrized paths in \(M\). The category \(C(M)\) is connected if and only if \(M\) is connected. The functor \(F\) assigns the fiber \(L_m\) to \(m\) and parallel transport to paths. The inner product space \(V_F\) is then the space of flat sections. The functor has no holonomy if and only if the connection has no holonomy.

\(^{16}\)Alternatively we can think of \(L\) as the category whose objects are principal homogeneous \(T\)-spaces, where \(T\) is the circle group. The elements of unit norm in a metrized line form a principal homogeneous \(T\)-space. The tensor product of complex lines is then a product on the category of \(T\)-spaces.
Fix a $G$ connection $\eta$ on $Q \to Y$, where $Y$ is an oriented closed 2-manifold. By Lemma 2.1 the bundle $Q$ is trivializable. Let $\mathcal{C}_Q$ be the category whose objects are sections $q: Y \to Q$. For any two sections $q, q'$ there is a unique morphism $q \xrightarrow{\psi} q'$, where $\psi: Q \to Q$ is the gauge transformation such that $q' = \psi q$. Define the functor $\mathcal{F}_\eta: \mathcal{C}_Q \to \mathcal{L}$ by $\mathcal{F}_\eta(q) = \mathbb{C}$ for all $q$, where $\mathbb{C}$ has its standard metric, and $\mathcal{F}_\eta(q \xrightarrow{\psi} q')$ is multiplication by $c_Y(q^* \eta, g \psi q)$, where $g \psi: Q \to G$ is the map associated to $\psi$, and $c_Y$ is the cocycle $y$ 

\[
c_Y(a, g) = \exp(2\pi i \int_Y \langle \text{Ad}_g^{-1} a \wedge \phi_g \rangle + W_Y(g)), \quad a \in \Omega_Y^1, \quad g: Y \to G.
\]

That $\mathcal{F}_\eta$ is a functor follows from the cocycle identity

\[
c_Y(a, g_1 g_2) = c_Y(a, g_1) c_Y(a^g_1, g_2),
\]

where $a^g = \text{Ad}_g^{-1} a + \phi_g$.

We leave the verification of (2.16) to the reader, who may wish to consult (A.10) and (A.8). Since $\mathcal{C}_Q$ is connected and $\mathcal{F}_\eta$ has no holonomy (there are no nontrivial automorphisms), we obtain the desired metrized line $L_\eta = L_{Y, \eta}$ of invariant sections. From the construction we see that a section $q: Y \to Q$ induces a trivialization $L_\eta \cong \mathbb{C}$.

It is important to observe that these “Chern-Simons lines” vary smoothly in smooth families of connections.

**Proposition 2.17.** Suppose $\eta_u$ is a smooth family of connections on $Q \to Y$ for $u$ varying over a smooth manifold $U$. Then the lines $L_{\eta_u}$ form a smooth hermitian line bundle over $U$.

The line bundle is trivialized by a section $q: Y \to Q$. Transition functions between different trivializations are smooth since the map

\[
u \mapsto c_Y(q^* \eta_u, g \psi q)
\]
is smooth, as formula (2.15) clearly shows. Notice that the elements of unit norm form a circle bundle over $U$.

Return now to the connection $\Theta$ over the 3-manifold $X$. Equation (2.11) shows that

\[
\partial p \mapsto e^{2\pi i S_X(p, \Theta)}
\]
defines an invariant section of unit norm

\[
e^{2\pi i S_X(\theta)} \in L_{\partial \Theta},
\]

where $\partial \Theta$ is the restriction of $\Theta$ to $\partial X$ and $\partial p$ is the restriction of the section $p$ to $\partial X$. This is the Chern-Simons invariant for manifolds with boundary. In good Bourbaki style we allow either $X$ or $\partial X$ to be the empty set $\emptyset$ in (2.18): Set $L_\emptyset = \mathbb{C}$ and $S_\emptyset \equiv 0$. The following theorem generalizes Proposition 2.7 and expresses what we mean by the statement “$S_X$ is the action of a local Lagrangian field theory”.

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18
Theorem 2.19. Let $G$ be a connected, simply connected compact Lie group and $\langle \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ an invariant form on its Lie algebra $\mathfrak{g}$ which satisfies the integrality condition Hypothesis 2.5. Then the assignments

\begin{align*}
(2.20) \quad \eta & \mapsto L_\eta, \quad \eta \in C_Y, \\
\Theta & \mapsto e^{2\pi i S_X(\Theta)}, \quad \Theta \in C_X,
\end{align*}

defined above for closed oriented 2-manifolds $Y$ and compact oriented 3-manifolds $X$ are smooth and satisfy:

(a) (Functoriality) If $\psi : Q' \to Q$ is a bundle map covering an orientation preserving diffeomorphism $\overline{\psi} : Y' \to Y$, and $\eta$ is a connection on $Q$, then there is an induced isometry

\begin{equation}
(2.21) \quad \psi^* : L_\eta \longrightarrow L_{\psi^* \eta}
\end{equation}

and these compose properly. If $\varphi : P' \to P$ is a bundle map covering an orientation preserving diffeomorphism $\varphi : X' \to X$, and $\Theta$ is a connection on $P$, then

\begin{equation}
(2.22) \quad (\partial \varphi)^* \left( e^{2\pi i S_X(\Theta)} \right) = e^{2\pi i S_X(\varphi^* \Theta)},
\end{equation}

where $\partial \varphi : \partial P' \to \partial P$ is the induced map over the boundary.

(b) (Orientation) There is a natural isometry

\begin{equation}
(2.23) \quad L_{-Y, \eta} \cong L_{Y, \eta},
\end{equation}

and

\begin{equation}
(2.24) \quad e^{2\pi i S_{-X}(\Theta)} = e^{2\pi i S_X(\Theta)}.
\end{equation}

(c) (Additivity) If $Y = Y_1 \sqcup Y_2$ is a disjoint union, and $\eta_i$ are connections over $Y_i$, then there is a natural isometry

\begin{equation}
(2.25) \quad L_{\eta_1 \sqcup \eta_2} \cong L_{\eta_1} \otimes L_{\eta_2}.
\end{equation}

If $X = X_1 \sqcup X_2$ is a disjoint union, and $\Theta_i$ are connections over $X_i$, then

\begin{equation}
(2.26) \quad e^{2\pi i S_{X_1 \sqcup X_2}(\Theta_1 \sqcup \Theta_2)} = e^{2\pi i S_{X_1}(\Theta_1)} \otimes e^{2\pi i S_{X_2}(\Theta_2)}.
\end{equation}

(d) (Gluing) Suppose $Y \to X$ is a closed, oriented submanifold and $X^{\text{cut}}$ is the manifold obtained by cutting $X$ along $Y$. Then $\partial X^{\text{cut}} = \partial X \sqcup Y \sqcup -Y$. Suppose $\Theta$ is a connection over $X$, with $\Theta^{\text{cut}}$ the induced connection over $X^{\text{cut}}$, and $\eta$ the restriction of $\Theta$ to $Y$. Then

\begin{equation}
(2.27) \quad e^{2\pi i S_X(\Theta)} = \text{Tr}_\eta \left( e^{2\pi i S_{X^{\text{cut}}}(\Theta^{\text{cut}})} \right),
\end{equation}

where $\text{Tr}_\eta$ is the contraction

\begin{equation*}
\text{Tr}_\eta : L_{\partial \Theta^{\text{cut}}} \cong L_{\partial \Theta} \otimes L_{\eta} \otimes L_{\eta} \to L_{\partial \Theta}
\end{equation*}
using the hermitian metric on \( L_\eta \).

Several comments are in order. From a functorial point of view: (a) implies that \( \eta \mapsto L_\eta \) defines a functor

\[
(2.28) \quad C_Y \to \mathcal{L}
\]

and that each \( X \) determines an invariant section \( e^{2\pi i S_X(\cdot)} \) of the composite functor \( C_X \to C_{\partial X} \to \mathcal{L} \), where the first arrow is restriction to the boundary. From down here on earth: To each bundle \( Q \to Y \) over a closed, oriented 2-manifold there is associated a smooth line bundle \( L_Q \to A_Q \) over the space of connections, and the action of gauge transformations \( \mathcal{G}_Q \) on \( A_Q \) lifts to \( L_Q \). Furthermore, a bundle \( P \to X \) over a compact, oriented 3-manifold determines a restriction map \( A_P \to A_{\partial P} \) and so an induced line bundle \( L_P \to A_P \); the action of gauge transformations \( \mathcal{G}_P \) on \( A_P \) lifts to \( L_P \). Then \( e^{2\pi i S_X(\cdot)} \) is an invariant section of \( L_P \). These line bundles and sections are smooth, by Proposition 2.17. Although (c) looks like a multiplicative property, it expresses the additivity of the classical action \( S_X \). However, \( S_X \) is not defined if \( \partial X \neq \emptyset \), which is why we use the exponential notation \( e^{2\pi i S_X(\cdot)} \).

Theorem 2.19 expresses in a (necessarily) complicated way the fact that \( S_X \) is a local functional of local fields defined as the integral of a local expression (c), (d); is invariant under symmetries of the fields (a); and is unitary (b).

**Proof.** The smoothness of \( \eta \mapsto L_\eta \) is Proposition 2.17. Suppose \( \Theta_u \) is a smooth family of connections parametrized by a smooth manifold \( U \). Fix a section \( q: Y \to Q \); its restriction to \( \partial X \) determines a smooth trivialization of the lines \( L_{\partial \Theta_u} \). Relative to this trivialization \( e^{2\pi i S_X(\cdot)} \) is the function \( L_q \mapsto \psi q \phi^{-1}: Y \to Q \) induces an isometry \( L_q \cong \mathbb{C} \). The induced section \( q = \psi q \phi^{-1}: Y \to Q \) induces an isometry \( L_q \cong \mathbb{C} \). Relative to these trivializations we define \( \psi^*: L_\eta \to L_{\psi^* \eta} \) to be the identity map \( \mathbb{C} \to \mathbb{C} \). A routine check shows that this definition is independent of the choice of section \( q^1 \). Equation (2.22) is proved similarly using a trivialization of \( P' \). For (2.23) we observe that the cocycle (2.15) changes sign when the orientation of \( Y \) is reversed. Similarly, the integral in (2.2) changes sign if the orientation of \( X \) is reversed, from which (2.24) follows (after fixing a trivialization). The statements in (c) follow from the fact that the integral over a disjoint union is the sum of the integrals, and in (d) from the fact that \( \int_{X_{cut}} = \int_X \). We leave details to the reader.

Proposition 2.9 extends to manifolds with boundary.

**Proposition 2.29.** Let \( k: G \to G' \) be an inclusion of connected, simply connected, compact Lie groups. Suppose \( \langle \cdot \rangle^{1} \) is an integral form on \( g' \) which restricts to \( \langle \cdot \rangle \) on \( g \). Then if \( \eta \) is a \( G \) connection over a closed, oriented 2-manifold \( Y \), and \( \eta' \) its \( G' \) extension, there is a natural isometry

\[
(2.30) \quad k_\eta: L_\eta \to L_{\eta'}.
\]

If \( \Theta \) is a \( G \) connection over a compact, oriented 3-manifold \( X \), and \( \Theta' \) its \( G' \) extension, then

\[
k_{\partial \Theta} \left( e^{2\pi i S_X(\cdot)} \right) = e^{2\pi i S_X(\cdot')}.
\]

\[\text{17} \] For this we need the functoriality of the Wess-Zumino-Witten action, which is proved by (2.14).
In categorical terms, $k$ induces a natural transformation from the functor $C^G \to \mathcal{L}$ to the functor $C^G \xrightarrow{k_Y} C^G \to \mathcal{L}$ (cf. (2.28), (1.31)). For each $X$ this induces a natural transformation from $C^G_X \to C^G_{\partial X} \to \mathcal{L}$ to $C^G_X \to C^G_{\partial X} \to \mathcal{L}$ which preserves the invariant sections $e^{2\pi i S_X(\cdot)}$. Though the language is complicated, the proof is straightforward, and we omit it.
Classical Solutions and the Hamiltonian Theory

The classical solutions in a Lagrangian field theory are the critical points of the action, which are computed by a standard calculus of variations argument. In the Chern-Simons theory these are the flat connections (up to equivalence). A standard result in differential geometry (Proposition 3.5) shows that this space is determined by the fundamental group. The Hamiltonian theory pertains to spacetimes which are globally products of a space (here a closed, oriented 2-manifold) with time (the positive reals, say). In the Chern-Simons theory the classical solutions are constant in time, i.e., the Hamiltonian function vanishes (Proposition 3.16). So the space of classical solutions is the moduli space of flat connections on a bundle over a 2-manifold, which has been the object of much study recently. The Chern-Simons lines defined in §2 descend to form a hermitian line bundle over this moduli space. Furthermore, the Chern-Simons action defines a connection on this circle bundle whose curvature is a symplectic form (Proposition 3.17). This is the phase space of the Hamiltonian theory. We also prove some simple facts about the action of the stabilizer of a connection on the Chern-Simons line (Proposition 3.25 and Proposition 3.26).

Our first order of business is to calculate the differential of the action $S_X$ on a closed oriented 3-manifold. Recall that the configuration space of fields is a disjoint union of affine spaces (1.30).

**Proposition 3.1.** Let $\Theta_t$ be a path in $C_X$. Denote $\Theta = \Theta_0$ and $\dot{\Theta} = \frac{d}{dt} \big|_{t=0} \Theta_t$. Then

$$
\left. \frac{d}{dt} \right|_{t=0} S_X(\Theta_t) = 2 \int_X \langle \Omega \wedge \dot{\Theta} \rangle,
$$

where $\Omega$ is the curvature of $\Theta$.

The reader should check that the integrand, which is ostensibly a 3-form on $P$, is the lift of a 3-form on $X$.

**Proof.** The connections $\Theta_s$ form a single connection $\Theta$ on each cylinder $[0, t] \times X$ with curvature $\Omega_s + ds \wedge \dot{\Theta}_s$, where $\Omega_s$ is the curvature of $\Theta_s$. Then (2.8) implies that for each $t$,

$$
S(\Theta_t) - S(\Theta_0) = 2 \int_0^t dt \int_X \langle \Omega_s \wedge \dot{\Theta}_s \rangle.
$$

The proposition follows by differentiating (3.3) at $t = 0$.

It follows that $dS_X(\Theta) = 0$ if and only if $\Omega = 0$, i.e., if and only if $\Theta$ is flat. Notice that the Euler-Lagrange equation

$$
\Omega = 0
$$

is first order.\(^\text{18}\) Since the action is invariant under symmetries, by Proposition 2.7(a), so is the space of solutions to the Euler-Lagrange equation. In this case that fact is also obvious directly. Let

$$
\mathcal{M}_X \subset C_X
$$

\(^\text{18}\)A typical field theory has an action of the form $S_X(f) = \int_X |df|^2$, and then the Euler-Lagrange equation is second order.
be the space of equivalence classes of solutions to (3.4).

Since (3.4) is a local condition and is defined in any dimension, the space $\mathcal{M}_X$ of equivalence classes of flat connections is defined for any manifold $X$. Nevertheless, we derive these spaces directly from the Chern-Simons action for certain 2- and 3-manifolds, as this is the procedure which generalizes to arbitrary field theories. First, we note some general facts about flat connections, which in the context of Chern-Simons theory reflect the topological nature of the action.

**Proposition 3.5.** Let $X$ be any smooth manifold. Choose a basepoint $x_i$ in each component of $X$. Then the holonomy provides a natural identification

$$(3.6) \quad \mathcal{M}_X = \prod_i \text{Hom}(\pi_1(X, x_i), G)/G$$

which is independent of the basepoints.

Here $G$ acts on $\text{Hom}(\pi_1(X, x_i), G)/G$ by conjugation. We omit the proof, which is standard.

The space $\mathcal{M}_X$ is typically not a manifold. If $Y$ is a compact oriented 2-manifold, then $\mathcal{M}_Y$ is a stratified space whose stratum of top dimension is a smooth manifold $[G]$. Suppose $\Theta$ is a flat connection on $P \to X$ (in any dimension), and consider the covariant derivative $d\Theta$ in the adjoint representation. It defines a complex

$$(3.7) \quad 0 \to \Omega^0_X(\mathfrak{g}_P) \xrightarrow{d\Theta} \Omega^1_X(\mathfrak{g}_P) \to \cdots$$

since $d^2\Theta = 0$. We denote the cohomology groups of this complex by $H^\bullet(X; \mathfrak{g}(\Theta))$.

If $\Theta$ represents a smooth point of $\mathcal{M}_X$, then the tangent space at that point is

$$(3.8) \quad T_\Theta \mathcal{M}_X \cong H^1(X; \mathfrak{g}(\Theta)).$$

If $Y$ is a compact oriented 2-manifold, then for a flat connection $\eta$ there is a nondegenerate pairing

$$H^0(Y; \mathfrak{g}(\eta)) \otimes H^2(Y; \mathfrak{g}(\eta)) \to \mathbb{R}.$$

The zeroth cohomology $H^0(Y; \mathfrak{g}(\eta))$ is the Lie algebra of the centralizer $Z_\eta$ of $\eta$ in the group of gauge transformations. For example, it vanishes if $\eta$ is irreducible, since then $Z_\eta$ is isomorphic to the center of $G$, which is finite. In any case the index theorem gives

$$(3.9) \quad \dim \mathcal{M}_Y = \dim H^1(Y; \mathfrak{g}(\eta)) = -\dim G \cdot \chi(Y) + 2 \dim Z_\eta.$$

Of course, this is a formula for the dimension of the moduli space only at smooth points $\eta$; at other points this is a formal dimension. At smooth points $\eta$ the right hand side gives a formula for $\dim \mathcal{M}_Y$. Note in particular that $\mathcal{M}_Y$ has even dimension. This is explained by the theorem of Narasimhan and Seshadri [NS], which identifies $\mathcal{M}_Y$ as a complex manifold (of stable bundles) when $Y$ is endowed with a complex structure.

The following lemma also reflects the topological nature of flat connections.
Lemma 3.10. Suppose $\Theta$ is a flat connection on $P \to X$, for some manifold $X$, and $\varphi_i : P' \to P$, $i = 0, 1$ are bundle maps. Then if $\tilde{\varphi}_0 : X' \to X$ is (pseudo)isotopic to $\tilde{\varphi}_1 : X' \to X$, the connections $\varphi_0^*\Theta$ and $\varphi_1^*\Theta$ are gauge equivalent.

Proof. This follows directly from Proposition 3.5, since $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ induce the same map on $\pi_1$. Alternatively, let $\varphi : [0, 1] \times P' \to [0, 1] \times P$ be a bundle map covering a pseudoisotopy $\tilde{\varphi} : [0, 1] \times X' \to [0, 1] \times X$. Then by Lemma 1.21 there is a unique gauge transformation $\tilde{\varphi}_0 : [0, 1] \times X' \to [0, 1] \times P'$ such that $\tilde{\varphi}_0 = \text{id}$ and $\tilde{\varphi}_1^*\varphi^*\Theta = \eta_t$ has no $dt$ component, where $t$ is the coordinate on $[0, 1]$. The curvature of $\tilde{\varphi}_1^*\varphi^*\Theta$ is

\begin{equation}
\Omega_t = \eta_t \wedge dt,
\end{equation}

where $\Omega_t$ is the curvature of $\eta_t$. But (3.11) vanishes, since $\Theta$ is flat. Hence $\eta_0 = \eta_1$. In other words $\varphi_0^*\Theta = \eta_0 = \eta_1 = \tilde{\varphi}_1^*(\varphi_1^*\Theta)$, which proves the lemma.

Now consider a compact oriented 3-manifold $X$, possibly with nonempty boundary. For $\eta \in \mathcal{C}_\partial X$ we define

$$
\mathcal{C}_X(\eta) = \{ \Theta \in \mathcal{C}_X : \partial \Theta = \eta \}
$$

as the category of fields with boundary value $\eta$. Morphisms in $\mathcal{C}_X(\eta)$ are required to be the identity over $\partial X$.

Proposition 3.12. Let $\Theta_t$ be a path in $\mathcal{C}_X(\eta)$. Then

$$
\frac{d}{dt} \bigg|_{t=0} S_X(\Theta_t) = 2 \int_X (\Omega \wedge \dot{\Theta}).
$$

We omit the derivation, which is similar to the derivation of (3.2), except for two points: (i) the cylinder $[0, t] \times X$ needs to be smoothed at the corners; and (ii) the action on $[0, t] \times \partial X$ is zero since the connection is constant there. The critical points form the moduli space of flat connections $\mathcal{M}_X(\eta) \subset \mathcal{C}_X(\eta)$ which restrict to $\eta$ on the boundary. This space is nonempty only if $\eta$ is flat.

We next form the union $\mathcal{M}_X$ of the $\mathcal{M}_X(\eta)$.\footnote{Of course, $\mathcal{M}_X$ is simply the moduli space of flat connections on $X$. The algebraic exercise in this paragraph is our understanding of the role that symmetries play in the classical theory on manifolds with boundary. Similar considerations also enter in the quantum theory.} For this we need to divide out the symmetries on the boundary. Let $\eta' \xrightarrow{\tilde{\varphi}} \eta$ be a morphism in $\mathcal{C}_\partial X$. We claim that there is an induced functor $\mathcal{C}_X(\eta') \to \mathcal{C}_X(\eta)$ which maps $\mathcal{M}_X(\eta')$ isomorphically onto $\mathcal{M}_X(\eta)$. For suppose $\Theta$ is a connection on $P \to X$ with $\partial \Theta = \eta$. Choose a bundle $P' \to X$ and a morphism $\varphi : P' \to P$ such that $\partial \varphi = \psi$. Then the induced functor maps the equivalence class of $\varphi^*\Theta$ (in $\mathcal{C}_X(\eta')$) to the equivalence class of $\Theta$. A routine check shows that this is independent of the choice of $\varphi$. The functor is defined in a similar manner on morphisms in $\mathcal{C}_X(\eta')$. Therefore, there is a functor $\eta \mapsto \mathcal{M}_X(\eta)$ from $\mathcal{C}_\partial X$ to the category of (singular) manifolds. Then $\mathcal{M}_X$ is the space of invariant sections of this functor. We repeat that $\mathcal{M}_X$ is simply the moduli space of flat connections on $X$, so is given by (3.6); its topology
and manifold structure should be understood from that point of view. Finally, the restriction to $\partial X$ gives a diagram

\[
\begin{array}{ccc}
\mathcal{M}_X & \longrightarrow & \mathcal{C}_X \\
\downarrow & & \downarrow \\
\mathcal{M}_{\partial X} & \longrightarrow & \mathcal{C}_{\partial X}
\end{array}
\]  

(3.13)

If $\eta$ is a flat connection over $\partial X$, then the fiber of (3.13) over the equivalence class of $\eta$ in $M_{\partial X}$ is isomorphic to $M_X(\eta)$.

If $Y$ is a closed oriented 2-manifold, then $M_Y \subset C_Y$ is the moduli space of flat connections. Again we wish to “derive” this space from the Chern-Simons action. Thus we consider the Hamiltonian formulation of the theory, i.e., study the action on the cylinder $X = [0, \infty) \times Y$. This “spacetime” $X$ is the product of “space” $Y$ and “time” $[0, \infty)$. In the Hamiltonian formulation we rewrite the space of fields as a space of paths.

**Proposition 3.14.** Let $\{Q\}$ be a set of representatives for the equivalence classes of $G$ bundles over $Y$. Then there is an identification

\[
\overline{C}_{[0, \infty) \times Y} = \bigsqcup_{\{Q\}} \text{Map}([0, \infty), A_Q) / G_Q.
\]

**Proof.** If $\Theta \in C_{[0, \infty) \times Y}$ is a connection on $P \to [0, \infty) \times Y$, then $\partial P \cong Q$ for a unique $Q \in \{Q\}$. Since $[0, \infty)$ is contractible there is an isomorphism $P \cong [0, \infty) \times Q$, and by Lemma 1.21 we can choose an isomorphism which takes $\Theta$ to a path $\eta_t$ of connections on $Q$. As this is unique up to an overall gauge transformation on $Q$, the assertion follows.

The classical solutions on $[0, \infty) \times Y$ are completely determined by their initial value.\(^{20}\)

**Proposition 3.16.** The restriction to the boundary $M_{[0, \infty) \times Y} \subset \overline{C}_{[0, \infty) \times Y} \to C_{[0, \infty) \times Y}$ is an isomorphism of $M_{[0, \infty) \times Y}$ onto the moduli space $M_Y$ of flat connections over $Y$.

**Proof.** Suppose $\Theta$ is a flat connection over $[0, \infty) \times Y$. After a gauge transformation this is a path $\eta_t$ in $C_Y$. But the proof of Lemma 3.10 shows that $\eta_t$ is constant. Thus $\eta_0$ determines the gauge equivalence class of $\Theta$.

One fundamental idea in the Hamiltonian formulation of classical mechanics, is that the fields over a cylinder are paths in a symplectic manifold.\(^{21}\) Furthermore, the space of classical solutions also carries a symplectic structure. For Chern-Simons theory the identification (3.15) indicates that the purported symplectic structure on each $A_Q$ is $G_Q$ invariant. These assertions form the next proposition. Recall that the lines $L_\eta$ attached to $\eta \in C_Y$ vary smoothly with $\eta$. So for each $Q$ they form a smooth hermitian line bundle $L_Q \to A_Q$.

---

\(^{20}\)Contrast with the usual case of a second order Euler-Lagrange equation, where both an initial value and an initial derivative must be specified.

\(^{21}\)Many physicists who subscribe to the “symplectic philosophy” emphasize that in a local field theory the symplectic structure is derived from the lagrangian. That is our approach. (See [GS] for a survey of symplectic geometry in physics.)
Proposition 3.17. Fix a $G$ bundle $Q \to Y$ over a closed oriented 2-manifold. Then the Chern-Simons action defines a unitary connection on the hermitian line bundle $L_Q \to \mathcal{A}_Q$. The curvature of this connection times $i/2\pi$ is

\begin{equation}
\omega(\dot{\eta}_1, \dot{\eta}_2) = -2 \int_Y \langle \dot{\eta}_1 \wedge \dot{\eta}_2 \rangle, \quad \dot{\eta}_1, \dot{\eta}_2 \in TA_Q.
\end{equation}

If $\langle \cdot \rangle$ is nondegenerate, then $\omega$ is a symplectic form. The action of $G_Q$ on $\mathcal{A}_Q$ lifts to $L_Q$, and the lifted action preserves the metric and connection. The induced moment map is

\begin{equation}
\mu_\xi(\eta) = 2 \int_Y \langle \Omega(\eta) \wedge \xi \rangle,
\end{equation}

where $\xi \in \Omega^0_Y(g_Q)$ is an infinitesimal gauge transformation. Over subsets of $\mathcal{A}_Q$ where $G_Q$ acts freely, then (a subset of) the moduli space $\mathcal{M}_Q$ of flat connections is the symplectic quotient $\mathcal{A}_Q//G_Q$. There is an induced line bundle $L_Q \to \mathcal{M}_Q$ with metric and connection, and $i/2\pi$ times its curvature is the induced symplectic form on $\mathcal{M}_Q$.

That a nondegeneracy hypothesis on the lagrangian is necessary to obtain a non-degenerate symplectic structure is standard in classical mechanics [Ar]. The symplectic structure of $\mathcal{A}_Q$ is well-known [AB], and the symplectic form on $\mathcal{M}_Q$ can be defined using cohomology [G]. Here, though, we see both structures arising directly from the Chern-Simons action in 3 dimensions.

Proof. Let $\eta_t$ be a smooth path in $\mathcal{A}_Q$. Then $\eta_t$ determines a connection $\eta$ on $[0,1] \times Q \to [0,1] \times Y$. Now $\partial(\eta) = \eta_1 \oplus \eta_0$ is a connection on $Q \sqcup Q \to Y \sqcup -Y$. Then $L_{\partial(\eta)} \cong \underline{L}_{\eta_0} \otimes L_{\eta_1}$ by the properties in Theorem 2.19. Using the metric on $L_{\eta_0}$, we identify $L_{\eta_0} \otimes L_{\eta_1}$ as the line of linear maps $L_{\eta_0} \to L_{\eta_1}$. Let

\begin{equation}
\text{PT}(\eta_t) = e^{2\pi i S_{[0,1] \times Y}(\eta)} : L_{\eta_0} \to L_{\eta_1}
\end{equation}

be the Chern-Simons action of $\eta$. Note that PT($\eta_t$) is unitary, since the corresponding element of $\underline{L}_{\eta_0} \otimes L_{\eta_1}$ has unit norm. We claim that PT($\eta_t$) is the parallel transport of a connection on $L_Q \to \mathcal{A}_Q$.

Fix a section $q: Y \to Q$. Then from the construction of lines in §2 (see (2.15)) this induces a (unitary) trivialization $s_q: \mathcal{A}_Q \to L_Q$. Consider the connection $\eta$ over $[0,1] \times Y$. Using the formula (3.11) for its curvature, we see that the Chern-Simons form is

$q^* \alpha(\eta) = -q^* \langle \eta_t \wedge \dot{\eta}_t \rangle \wedge dt.$

Hence from (2.2) the action relative to the section $q$ is

\begin{equation}
S_{[0,1] \times Y}(q, \eta) = - \int_0^1 dt \int_Y q^* \langle \eta_t \wedge \dot{\eta}_t \rangle.
\end{equation}

This motivates us to introduce the (connection) 1-form $\theta_q$ on $\mathcal{A}_Q$ defined by\footnote{Notice that parallel transport is the integral of minus the connection form relative to some trivialization.}  

\begin{equation}
(\theta_q)_\eta(\dot{\eta}) = 2\pi i \int_Y q^* \langle \eta \wedge \dot{\eta} \rangle.
\end{equation}
We must check that this transforms properly under gauge transformations. If \( \psi : Q \to Q \) is a gauge transformation, then

\[
s_{\psi q}(\eta) = c_Y(q^* \eta, g_\psi q)^{-1} s_q(\eta)
\]

according to (2.15). On the other hand, we easily compute from (1.18) that

\[
(\theta_{\psi q})_\eta(\dot{\eta}) = 2\pi i \int_Y q^*(\psi^* \eta \wedge \psi^* \dot{\eta})
\]

\[
= 2\pi i \int_Y q^*(\text{Ad}_{g_\psi^{-1}} \eta \wedge \text{Ad}_{g_\psi^{-1}} \dot{\eta}) + 2\pi i \int_Y q^*(\phi_{g_\psi} \wedge \text{Ad}_{g_\psi^{-1}} \dot{\eta})
\]

\[
= (\theta_q)_\eta - 2\pi i \int_Y \langle \text{Ad}_{g^{-1}}(q^* \dot{\eta}) \wedge \phi_g \rangle,
\]

where \( g = g_\psi q \). But from (2.15) we see that the logarithmic differential of the cocycle with respect to \( \eta \),

\[
\frac{d c_Y(q^* \eta, g_\psi q)}{c_Y(q^* \eta, g_\psi q)}(\dot{\eta}) = 2\pi i \int_Y \langle \text{Ad}_{g^{-1}}(q^* \dot{\eta}) \wedge \phi_g \rangle,
\]

is minus the last term in (3.23). It follows that the set of forms \( \{ \theta_q \} \), where \( q \) ranges over trivializations of \( Q \to Y \), defines a unitary connection on \( L_Q \to A_Q \). Furthermore, as the logarithm of the parallel transport is the integral of minus the connection form, we see from (3.21) that the Chern-Simons action (3.20) is the parallel transport of this connection. The curvature times \( i/2\pi \) is

\[
\omega(\dot{\eta}_1, \dot{\eta}_2) = \frac{i}{2\pi} (d\theta_q)(\dot{\eta}_1, \dot{\eta}_2) = -2 \int_Y \langle \dot{\eta}_1 \wedge \dot{\eta}_2 \rangle.
\]

If \( \langle \cdot \rangle \) is nondegenerate, then so is \( \omega \). Since the curvature of a line bundle is also closed, it follows that \( \omega \) is a symplectic form.

Now (2.21) is a lift of the \( G_Q \) action on \( A_Q \) to \( L_Q \). Furthermore, (2.22) shows that this lift preserves the parallel transport (3.20), hence the connection. (It also preserves the metric, so acts on the associated circle bundle with connection.) It follows that the \( G_Q \) action on \( A_Q \) preserves the curvature \( \omega \), so is symplectic.

We compute a moment map for this action using the lift to \( L_Q \). In general, if \( L \to M \) is a hermitian line bundle (or circle bundle) over a symplectic manifold, and \( \rho : G \to \text{Aut}(L) \) is a \( G \) action on \( L \) preserving the metric and connection, then the moment map of the quotient \( G \) action on \( M \) is

\[
(3.24) \quad \mu_\xi(m) = -\frac{i}{2\pi} \cdot \text{vert}(\hat{\rho}(\xi)\ell), \quad \xi \in \mathfrak{g},
\]

where \( \ell \in L_m \) is a point of unit norm, \( \hat{\rho}(\xi) \) is the vector field on \( L \) corresponding to \( \xi \in \mathfrak{g} \), and \( \text{vert}(\cdot) \) is the vertical part of a vector computed with respect to the connection on \( L \). Notice that it is exactly the obstruction to descending the

\[\text{For a general symplectic group action there may be many moment maps, if one exists at all. Here the lift of the action to } L_Q, \text{ which is ultimately derived directly from the Chern-Simons action, singles one out.}\]
connection form to the quotient \( L/G \). We compute this in our situation using the trivialization \( s_q : A_Q \to L_Q \) given by a section \( q : Y \to Q \). Set \( a = q^* \eta \). Then a gauge transformation \( \psi : Q \to Q \) corresponds to a map \( g = g_\psi : Y \to G \). It acts as multiplication by \( e^{2\pi ic(a,g)} \) from the (trivialized) fiber at \( \eta \) to the fiber at \( \psi^* \eta \).

An infinitesimal gauge transformation corresponds to a map \( \xi : Y \to g \), and it acts as multiplication by the derivative, which we compute from (2.15) to be

\[
2\pi i \int_Y \langle a \wedge d\xi \rangle.
\]

On the other hand, the connection form in this trivialization is (3.22), and using this we compute the infinitesimal parallel transport in the direction \( \xi \) to be multiplication by

\[
-2\pi i \int_Y \langle a \wedge d\xi \rangle,
\]

where \( d\xi = d\xi + [a \wedge \xi] \) is the covariant derivative. The difference times \(-i/2\pi\) is

\[

\int_Y \langle a \wedge (2d\xi + [a \wedge \xi]) \rangle = 2 \int_Y \langle \Omega(\eta) \wedge \xi \rangle,
\]

where \( \Omega(\eta) \) is the curvature of \( \eta \). Hence by (3.24) the moment map is (3.19), as desired. Since the flat connections are the zeros of \( \mu \), the space of equivalence classes \( M_Q \) of flat connections is the symplectic quotient \( A_Q \// G_Q \), if \( G_Q \) acts freely.

The line bundle, together with its metric and connection, pass to a bundle \( L_Q \to M_Q \) on the quotient. The symplectic form (3.18) also passes to the quotient. Its de Rham cohomology class is an integral element of \( H^2(M_Q; \mathbb{R}) \), since it is the first Chern class of \( L_Q \).

The connection can also be deduced from the gluing law and diffeomorphism invariance of the action. We give this argument in the proof of Proposition 5.9, where we also calculate the curvature and holonomy directly from the action. The simplest case of the Chern-Weil theorem asserts that \( i/2\pi \) times the curvature of a line bundle represents the first Chern class in real cohomology. This applies here to \( L_Q \) over the symplectic quotient. We refine this in Part 2 to calculate the first Chern class in integral cohomology.

We remark that this connection is consistent with the isomorphism (2.30) associated to a change of gauge group; this follows from Proposition 2.9.

In general \( G_Q \) does not act freely on \( A_Q \); the stabilizer at \( \eta \in A_Q \) is the subgroup \( Z_\eta \) of parallel gauge transformations (cf. Proposition 1.20). The value of such a gauge transformation at \( y \in Y \), which lies in \( \text{Aut}(Q_y) \), commutes with the holonomy group at \( y \). There is an action of \( Z_\eta \) on the line \( L_\eta \).

**Proposition 3.25.** The action of \( Z_\eta \) on \( L_\eta \) is constant on components of \( Z_\eta \), and so factors through an action of the finite group \( \pi_0(Z_\eta) \) on \( L_\eta \).

**Proof.** If \( \psi \in Z_\eta \) then we form the bundle \( Q_\psi \to S^1 \times Y \) with connection \( \eta_\psi \) by gluing the ends of \([0, 1] \times Q \) using \( \psi \). According to (3.20) the action of \( \psi \) on \( L_\eta \) is \( e^{2\pi is_{S^1 \times Y}(\eta_\psi)} \). But if \( \psi \) and \( \psi' \) are connected by a path in \( Z_\eta \), then \( \eta_\psi \cong \eta_{\psi'} \).

Along these lines we also have the following.

\[24\] Thus the symplectic form satisfies the integrality constraint in the geometric quantization theory.
Proposition 3.26. The center $Z \subset G$ acts trivially on $L_\eta$ for all $\eta \in A_Q$.

Combining these two propositions we see that the action of $Z_\eta$ on $L_\eta$ factors through an action of $\pi_0(Z_\eta/Z)$.

Proof. Suppose $g \in Z$, which we identify as a constant function on $Y$. Let $q : Y \to Q$ be any section and $a = q^*g\eta$. Then the action of $g$ on $L_\eta$ is multiplication by the cocycle $c_Y(a, g)$ in (2.15), which is easily seen to be the identity.

Next, let $X$ be an arbitrary compact oriented 3-manifold with boundary, and consider the diagram (3.13).

Proposition 3.27. The image of $r_X : M_X \to M_{\partial X}$ is a Lagrangian submanifold. More precisely, the action $e^{2\pi iS_X}(-)$ is a flat section of the pullback $r_X^*L_{\partial X} \to M_{\partial X}$, and therefore the induced symplectic form $r_X^*\omega$ vanishes.

Just as the symplectic form $\omega$ satisfies an integrality condition—it is $i/2\pi$ times the curvature of a line bundle—so too does the Lagrangian submanifold $M_X \to M_{\partial X}$ satisfy an “integrality condition”—the pullback line bundle with connection is trivial. (This gives a restriction on the holonomy, for example.) We use the term “Lagrangian submanifold” imprecisely. Rather, we should assert that $M_X \to M_{\partial X}$ is “a Lagrangian map”. Jeffrey and Weitsman also obtain this result [JW]; they use the term “Bohr-Sommerfeld orbits” for these special Lagrangian submanifolds.

Proof. If $S^1 \to M_X$ is a loop of flat connections $\Theta_t$, there is a corresponding connection $\Theta$ over $S^1 \times X$. Now the log holonomy of $r_X^*L_{\partial X}$ around $S^1$ is the action $S_{S^1 \times \partial X}(\Theta)$, by (3.20). The curvature of $\Theta$ is (3.11)

$$\Omega = -\dot{\Theta}_t \wedge dt.$$  

Hence from (2.8),

$$S_{S^1 \times \partial X}(\Theta) = \int_{S^1 \times X} \left< \Omega \wedge \Omega \right> = 0.$$  

So $r_X^*L_{\partial X}$ is flat. The more precise statement that $e^{2\pi iS_X}(-)$ is a flat section is slightly more intricate. Suppose $\Theta_t$ is a path of flat connections on $X$, and consider the induced connection on $[0, 1] \times X$. The action of its restriction to $[0, 1] \times \partial X$ determines the parallel transport (3.20), whereas the restriction to $\{t\} \times X$ is $S_X(\Theta_t)$.

The curvature is still given by (3.28), so that $\left< \Omega \wedge \Omega \right> \equiv 0$. Then equation (2.8) applied to $[0, 1] \times X$ implies that $e^{2\pi iS_X(\Theta_0)}$ is obtained from $e^{2\pi iS_X(\Theta_0)}$ by parallel transport. This proves that the section is flat.

It remains to show that $2\dim(\text{image } r_X) = \dim M_{\partial X}$. Suppose $\Theta$ is a flat connection on $X$ representing a smooth point of $M_X$ and whose restriction $\partial \Theta$ to $\partial X$ represents a smooth point of $M_{\partial X}$. Then according to (3.8) we must show

$$2\dim \text{image } (r_X)_* : H^1(X; g(\Theta)) \to \dim H^1(\partial X; g(\Theta)) = \dim H^1(\partial X; g(\Theta)).$$

In fact, the image of $(r_X)_*$ is a Lagrangian subspace. When $\Theta$ is trivial the argument is standard. Consider the commutative diagram

$$\begin{array}{cccc}
H^1(X) & \xrightarrow{(r_X)_*} & H^1(\partial X) & \xrightarrow{\cong} & H^2(X, \partial X) \\
\cong & & \cong & & \cong \\
H_2(X, \partial X) & \xrightarrow{i} & H_1(\partial X) & \xrightarrow{\iota} & H_1(X) \\
\end{array}$$

29
where the coefficients are in a vector space. The rows are exact and the vertical isomorphisms are Poincaré duality. By exactness we have \( \text{image}(r_X)_* \cong \ker i \). Also, \( i \) and \( (r_X)_* \) are adjoint maps, from which \( \text{image}(r_X)_* \cong (\ker i)^\circ \) is the annihilator of \( \ker i \). It follows that \( \text{image}(r_X)_* \) is Lagrangian, as claimed. The same argument works if \( \Theta \) is nontrivial, provided we prove that the complex (3.7) on \( \partial X \) defined by \( d_{\partial \Theta} \) also satisfies Poincaré duality. This is shown in [G] using group cohomology. There is also a proof in de Rham theory modeled after [BT,§5].

In summary, the Hamiltonian theory consists of the assignments

\[
\begin{align*}
Y & \mapsto \mathcal{M}_Y \\
X & \mapsto (\mathcal{M}_X \xrightarrow{r_X} \mathcal{M}_{\partial X})
\end{align*}
\]

of a symplectic manifold to each closed oriented 2-manifold and a Lagrangian map to each compact oriented 3-manifold. These assignments obey functoriality, orientation, additivity, and gluing laws analogous to those in Theorem 2.19. More precisely, one should consider instead the hermitian line bundle with connection \( L_Y \to \mathcal{M}_Y \) and the section \( e^{2\pi i S_X(\cdot)} : \mathcal{M}_X \to r_X^* L_{\partial X} \). Then there is little difference between the Hamiltonian theory and the Lagrangian theory.\(^{25}\) We leave the precise formulation of these assertions to the reader (cf. [Ax,§3]).

\(^{25}\)This somewhat startling conclusion is only valid for topological theories. In theories with local motion there is a nonzero hamiltonian function which must be incorporated into the formalism.
In the quantum Chern-Simons theory \([W]\) a key role is played by surfaces with boundary. Ultimately, explicit computations depend on their use. In this section we develop the classical analogue of this structure. Based on Theorem 2.19 we might guess that there exists a metrized line \(L_\eta\) depending smoothly on a connection \(\eta\) over a surface with boundary, and that these lines satisfy a gluing law which constructs the line of a closed surface by cutting and pasting. However, there is a topological obstruction which obstructs such a construction.\(^{26}\) This will be elucidated in Part 2 of the paper \([F2]\).\(^{27}\)

Here we add some “trivializing data” on the boundary of the surface which allows us to construct the desired lines; these lines then obey a gluing law. There is also a gluing law for the Chern-Simons action where we cut along manifolds with corners (Proposition 4.11). The corresponding quantum gluing law \([S]\), \([Wal]\) is quite powerful. In the middle of this section we include a long digression about the geometry of connections over the circle. This is essentially a universal choice of trivializing data for each isomorphism class of bundle over the circle. Then we construct a smooth line bundle over the space of connections (and basepoints) on the surface with boundary, provided we suitably restrict the boundary holonomies (Theorem 4.26). As in \(\S 2\) these line bundles descend to various moduli spaces of flat connections. One of these, by a theorem of Mehta and Seshadri \([MeS]\), can be identified with the moduli space of stable parabolic bundles when the surface carries a complex structure. We compute the dimension of these moduli spaces (4.43), (4.44) even at reducible connections. The index calculation in Lemma 4.40, Corollary 4.41, and Proposition 4.42 may be of independent interest.

In \(\S 5\) we construct connections on the line bundles over these moduli spaces.

Suppose \(Y\) is a compact oriented 2-manifold and \(Q \to Y\) a principal \(G\) bundle. Let \(\mathcal{S}_{\partial Q}\) be the space of sections \(r: \partial Y \to \partial Q\) of \(\partial Q \to \partial Y\). It is a principal homogeneous space for the action of the group of gauge transformations \(G_{\partial Q}\). Then for each connection \(\eta \in \mathcal{A}_{Q}\) we construct a metrized line \(L_{\eta,r}\) which varies smoothly in \(\eta\) and \(r\). To do this define the cocycle

\[
    c_Y(a,g) = \exp\left(2\pi i \int_Y \langle \text{Ad}_{g^{-1}} a \wedge \varphi_g \rangle \right) e^{2\pi i W_Y(g)} \in K_{\partial g}, \quad a \in \Omega^1_Y(g), \quad g: Y \to G
\]

as in (2.15). Notice that this cocycle takes values in the Wess-Zumino-Witten line \(K_{\partial g}\) (cf. Proposition A.1), where \(\partial g: \partial Y \to G\) is the restriction of \(g\) to the boundary. The cocycle identity

\[
    c_Y(a,g_1g_2) = c_Y(a,g_1)c_Y(a^{g_1},g_2)
\]

---

\(^{26}\) I thank George Daskalopoulos and Sheldon Chang for (separately) pointing this out to me.

\(^{27}\) The analogy with Theorem 2.19 is illuminating. For a closed 3-manifold the Chern-Simons invariant is a complex number, whereas for a 3-manifold with boundary it is an element in a complex line. Now for a closed 2-manifold the Chern-Simons functional constructs a complex line, whereas for a 2-manifold with boundary it constructs a \ldots The differential-geometric object which replaces the dots in the previous sentence together with its generalizations form the subject of \([F1]\).
holds for these cocycles, as the reader may verify. (Equation (A.7) enters this calculation.) Let $\mathcal{C}_Q$ be the category whose objects are sections $q: Y \to Q$. We do not require that these sections restrict to $r$ on the boundary; rather the quotient by $r$ is a map $r^{-1}\partial q: \partial Y \to G$. In other words, $\partial q = r \cdot (r^{-1}\partial q)$, where $\cdot$ is the action of $G$ on the principal bundle $Q$. Now for any two sections $q, q'$ there is a unique morphism $q \xrightarrow{\psi} q'$ which is the gauge transformation $\psi: Q \to Q$ with $q' = \psi q$. Define a functor $F_{q,r}: \mathcal{C}_Q \to \mathcal{L}$ by

\begin{equation}
F_{q}(q) = K_{r^{-1}\partial q}
\end{equation}

and $F_{q}(q \xrightarrow{\psi} q')$ equals multiplication by $c_Y(q^*\eta, q^{-1}q')$. Here $q^{-1}q': Y \to G$ is the unique function so that $q' = q \cdot (q^{-1}q')$. The multiplication is via the isomorphism (cf. (A.2))

\[K_{r^{-1}\partial q} \otimes K_{(\partial q)^{-1}((\partial q')^{-1})} \to K_{r^{-1}\partial q'}.
\]

That $F_q$ is a functor follows from the cocycle identity (4.2) and the associativity law (A.3). Define $L_{q,r}$ to be the line of invariant sections of $F_q$.

**Proposition 4.4.** Let $G$ be a connected, simply connected compact Lie group and $\langle \cdot \rangle$ an invariant form on its Lie algebra $\mathfrak{g}$ which satisfies the integrality condition. Suppose $Y$ is a compact oriented 2-manifold and $Q \to Y$ a principal $G$ bundle. Then the assignment

\[\eta, r \mapsto L_{\eta, r}, \quad \eta \in \mathcal{A}_Q, \quad r \in \mathcal{S}_Q,
\]

of a metrized line to a connection $\eta$ on a $G$ bundle $Q \to Y$ and section $r: \partial Y \to \partial Q$ is smooth, agrees with the corresponding assignment (2.20) if $\partial Y = \emptyset$, and satisfies:

(a) (Functoriality) If $\psi: Q' \to Q$ is a bundle map covering an orientation preserving diffeomorphism $\psi: Y' \to Y$, then there is an induced isometry

\begin{equation}
\psi^*: L_{\eta, r} \longrightarrow L_{\psi^*\eta, \psi^{-1}r}
\end{equation}

and these compose properly.

(b) (Orientation) There is a natural isometry

\begin{equation}
L_{-Y, \eta, r} \cong L_{Y, \eta, r}.
\end{equation}

(c) (Additivity) If $Y = Y_1 \sqcup Y_2$ is a disjoint union, $\eta_i$ are connections over $Y_i$, and $r_i$ are sections over $\partial Y_i$, then there is a natural isometry

\begin{equation}
L_{\eta_1 \sqcup \eta_2, r_1 \sqcup r_2} \cong L_{\eta_1, r_1} \otimes L_{\eta_2, r_2}.
\end{equation}

(d) (Gluing) Suppose $S \hookrightarrow Y$ is a closed oriented codimension one submanifold and $Y^\text{cut}$ the manifold obtained by cutting along $S$. Then $\partial Y^\text{cut} = \partial Y \sqcup S \sqcup -S$. Furthermore, if $Q \to Y$ is a bundle, and $Q^\text{cut} \to Y^\text{cut}$ the cut bundle, then there is an identification of $Q^\text{cut} \big|_S$ and $Q^\text{cut} \big|_{-S}$. Fix trivializations $r_S, r_{-S}$ over these boundary pieces which agree under this identification. Suppose $\eta$ is a connection on $Q$ and $\eta^\text{cut}$ the induced connection on $Q^\text{cut}$. Then for any trivialization $r_{\partial Y}$ of $\partial Q$ there is an isometry

\begin{equation}
L_{\eta^\text{cut}, r_{\partial Y} \sqcup r_S \sqcup r_{-S}} \cong L_{\eta, r_{\partial Y}}.
\end{equation}
(c) (Change of Trivialization) If \( r' \) is another trivialization over \( \partial Y \), then there is an isometry

\[
L_{\eta,r'} \cong K_{r' - 1}(\cdot) \otimes L_{\eta,r}.
\]

These compose in a natural way for three trivializations \( r, r', r'' \).

Proof. For smoothness note that a section \( q: Y \to Q \) induces an isomorphism \( L_{\eta,r} \cong K_{r - 1} \otimes q \), and the lines \( K_{r - 1} \) vary smoothly in \( r \) by Proposition A.1. Now if \( q' \) is any other section then the patching function is \( cy(q'^{-1} q) \), which is independent of \( r \) and varies smoothly in \( \eta \) by the explicit formula (4.1). Also, note that if \( \partial Y = 0 \) then the construction given here for \( L_{\eta,r} \) reduces to the construction of lines in §2.

Next, if \( \psi: Q' \to Q \) is the bundle map in (a), fix a section \( q: Y \to Q \). This induces the section \( \psi^{-1} q: Y \to Q' \). Notice that \( (\psi^{-1} r)^{-1} \partial(\psi^{-1} q) = r^{-1} \partial q \), since \( \psi^{-1} \) is a gauge transformation. Then relative to the resulting isomorphisms \( L_{\eta,r} \cong K_{r - 1} \partial q \) and \( L_{\psi^* \eta, \psi^{-1} r} \cong K_{(\psi^{-1} r)^{-1} \partial(\psi^{-1} q)} = K_{r^{-1} \partial q} \) we define (4.5) to be the identity map. It is easy to check that this is independent of the choice of \( \eta \).

Equations (4.6) and (4.7) are routine. For (4.8) fix a section \( q: Y \to Q \), which induces isomorphisms

\[
\begin{align*}
L_{\eta,\text{cut},r|Y \cup S \cup \partial Y - r} &\cong K_{(r \cup S \cup \partial Y - r)^{-1} \partial q} \otimes K_{(r \cup S \cup \partial Y - r)^{-1} \partial q}, \\
L_{\eta,\text{cut},r|Y} &\cong K_{(r \cup S \cup \partial Y - r)^{-1} \partial q}.
\end{align*}
\]

(In the first equation \( \partial q \) stands for the restriction of \( q \) to the appropriate piece of the boundary.) Now the orientation and additivity properties in Proposition A.1 give a trivialization of \( K_{(r \cup S \cup \partial Y - r)^{-1} \partial q} \). Then the isometries in (4.10) combine to give the isometry (4.8). It is easy to check that this is independent of the section \( q: Y \to Q \) using the gluing law for the Wess-Zumino-Witten functional (Proposition A.9). Finally, relative to a section \( q: Y \to Q \) the isometry (4.9) becomes the isometry

\[
K_{r^{-1} \partial q} \cong K_{r'^{-1} \partial q} \otimes K_{r - 1 \partial q}
\]

of (A.2), which again transforms properly under change of trivializing section.

The following gluing law for the Chern-Simons action generalizes Theorem 2.19(d) to “manifolds with corners”. Such manifolds can be smoothed out uniquely up to diffeomorphism by “straightening the angle” [CF].

**Proposition 4.11.** Let \( X \) be a compact oriented 3-manifold and \( Y \) a compact oriented 2-manifold with boundary. Suppose \( Y \hookrightarrow X \) is an embedding onto a neat submanifold of \( X \); that is, \( Y \cap \partial X = \partial Y \) and \( Y \) is transverse to \( \partial X \). Let \( X^\text{cut} \) be the manifold with corners obtained by cutting \( X \) along \( Y \). Then \( \partial X^\text{cut} = \partial X \cup Y \cup -Y \) where the union is over \( \partial Y \cup -\partial Y \). Suppose \( \Theta \) is a connection over \( X \). Let \( \Theta^\text{cut} \) be the induced connection over \( X^\text{cut} \), let \( \eta \) be the restriction of \( \Theta \) to \( Y \), and let \( (\partial \Theta)^\text{cut} \) be \( \partial \Theta \) cut along \( \partial Y \hookrightarrow \partial X \). Then for any trivialization \( r \) over \( \partial Y \), we have

\[
e^{2 \pi i S_X(\Theta)} = \text{Tr}_\eta \left( e^{2 \pi i S_{X^\text{cut}}(\Theta^\text{cut})} \right),
\]

where \( \text{Tr}_\eta \) is the contraction

\[
\text{Tr}_\eta: L_{\partial \Theta^\text{cut}} \cong L_{(\partial \Theta)^\text{cut}} \otimes L_{\eta,r} \otimes L_{\eta,r} \to L_{(\partial \Theta)^\text{cut},r} \cong L_{\partial \Theta}.
\]
using the hermitian metric on $L_{n,r}$ and the isomorphism (4.8).

We omit the proof, which contains no new ideas.

Let $A_Q$ be the space of all connections on a fixed bundle $Q \to Y$, let $G_Q$ be the space of gauge transformations, and let $S_Q$ be the space of sections. Assume that $dY \neq \emptyset$. Then the group $G_Q \times \operatorname{Map}(dY, G)$ acts on $A_Q \times S_Q$ via the action

$$\langle \psi, \gamma \rangle \cdot \langle \eta, r \rangle = \langle \psi \star \eta, \psi^{-1} r \star \gamma \rangle, \quad \psi \in G_Q, \quad \gamma \in \operatorname{Map}(dY, G), \quad \eta \in A_Q, \quad r \in S_Q.$$ 

Proposition 4.4 asserts that the lines $L_{n,r}$ form a smooth line bundle $L \to A_Q \times S_Q$, but the action (4.12) only lifts to an action of the central extension $\tilde{G}_Q \times \operatorname{Map}(dY, G)$ defined in (A.5). So we do not obtain a line bundle over the quotient $A_Q/G_Q$. From this point of view the central extension is the obstruction to the existence of that line bundle. In fact, there are topological obstructions to the existence of a line bundle over $A_Q/G_Q$ which obeys the desired gluing law. So we must content ourselves with a bundle over a subset of this quotient. Furthermore, we introduce additional choices on the boundary to construct the line bundle.

We first digress to discuss connections over the circle. Without extra effort we consider connections for arbitrary compact Lie groups $G$. Let $S^1 = [0, 1] / 0 \sim 1$ be the standard circle with basepoint $0 \in S^1$. We consider principal bundles $R \to S^1$ with a fixed basepoint in $R$ over the basepoint in $S^1$. There is a category $\mathcal{C}'_{S^1}$ of connections on pointed bundles; morphisms are required to preserve the basepoints. Then there are no nontrivial automorphisms of a connection in this category, since any automorphism which fixes the basepoint is the identity, according to Proposition 1.20. It follows that if $\theta_1 \cong \theta_2$ are isomorphic connections, there is a unique (pointed) isomorphism between them. It is precisely to obtain this rigidity property that we introduce basepoints. Now the basepoint determines a holonomy map $\mathcal{C}'_{S^1} \to G$. Since isomorphic connections have the same holonomy, there is an induced map on the equivalence classes $\mathcal{C}_{S^1} \to G$. It is an easy consequence of Lemma 1.21 that this latter map is $1 : 1$; that is, a connection with basepoint over $S^1$ is determined up to isomorphism by its holonomy. There is a universal bundle and connection. Define

$$\mathbf{R} = G \times \mathbb{R} \times G / \langle h, s, g \rangle \sim \langle h, s + 1, h^{-1} g \rangle.$$ 

This is a $G$ bundle $\mathbf{R} \to G \times S^1$ via projection onto the first two factors. Better, it is a family of pointed $G$ bundles $\mathbf{R}_h \to S^1$ over $S^1$ parametrized by $h \in G$; the basepoint in $\mathbf{R}_h$ is $\langle h, 0, e \rangle$, where $e \in G$ is the identity. The Maurer-Cartan form $\theta$ on $G$, lifted to $\langle h \rangle \times \mathbb{R} \times G$, drops to a connection $\theta_{\mathbf{R}_h}$ on $\mathbf{R}_h$ with holonomy $h$. There exist connections $\theta_{\mathbf{R}}$ on $\mathbf{R}$ which restricts to $\theta_{\mathbf{R}_h}$ on each slice $\mathbf{R}_h$; we construct one explicitly below (4.14). Summarizing, if $\theta$ is any (pointed) connection over $S^1$, then there is a unique isomorphism $\theta \cong \theta_{\mathbf{R}_h}$ for some $h$.

We now explicitly construct the connection $\theta_{\mathbf{R}}$. Let $\theta$ be the Maurer-Cartan form on $G$, lifted to $G \times (-0.1, 1.1) \times G$ via projection onto the third factor, and $\theta_h$ the Maurer-Cartan form lifted via projection onto the first factor. Fix a smooth cutoff function $\rho: (-0.1, 1.1) \to [0, 1]$ so that $\rho((-0.1, 0.1]) = 0$ and $\rho([0.9, 1.1]) = 1$. Set

$$\langle \theta_{\mathbf{R}} \rangle_{h,s,g} = \theta + \rho(s) \operatorname{Ad}_{g^{-1} h} \theta_h = g^{-1} d g + \rho(s) g^{-1} d h h^{-1} g,$$ 

(4.14)
where the second expression makes sense for matrix groups. Then an easy calculation with the gluing function in (4.13) shows that (4.14) determines a connection form on \( \mathbf{R} \). The restriction to a slice \( \mathbf{R}_h \) is \( \theta_{\mathbf{R}_h} \). From (1.13) we compute the curvature 2-form on \( G \times S^1 \):

\[
(4.15) \quad \Omega_{\mathbf{R}} = \rho'(s) ds \wedge \text{Ad}_{g^{-1}h} \theta_h + \frac{1}{2} (\rho(s)^2 - \rho(s)) \text{Ad}_{g^{-1}h} [\theta_h \wedge \theta_h].
\]

For an integral form \( \langle \cdot \rangle \) on the Lie algebra \( \mathfrak{g} \), we compute the Chern-Weil 4-form on \( G \times S^1 \) as

\[
\langle \Omega_{\mathbf{R}} \wedge \Omega_{\mathbf{R}} \rangle = \rho'(s) (\rho(s)^2 - \rho(s)) ds \wedge \langle [\theta_h \wedge [\theta_h \wedge \theta_h] \rangle,
\]

in view of (1.29). Finally, note that

\[
(4.16) \quad \int_{S^1} \langle \Omega_{\mathbf{R}} \wedge \Omega_{\mathbf{R}} \rangle = \int_0^1 ds \rho'(s) (\rho(s)^2 - \rho(s)) \wedge \langle [\theta_h \wedge [\theta_h \wedge \theta_h] \rangle
\]

\[
= -\frac{1}{6} \langle [\theta_h \wedge [\theta_h \wedge \theta_h] \rangle.
\]

By Hypothesis 2.5 this 3-form on \( G \) represents an integral cohomology class in \( H^3(G; \mathbb{R}) \).

**Proposition 4.17.** The bundle \( \mathbf{R} \rightarrow G \times S^1 \) is nontrivial if \( G \neq 1 \).

**Proof.** According to (1.25) the Lie algebra \( \mathfrak{g} \) decomposes as a sum of an abelian algebra and simple algebras. If there is a simple summand, then we choose \( \langle \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R} \) to be its Killing form. It follows from (4.16) that \( \langle \Omega_{\mathbf{R}} \wedge \Omega_{\mathbf{R}} \rangle \) represents a nonzero cohomology class in \( H^4(G \times S^1) \). By Chern-Weil theory this is a characteristic class of \( \mathbf{R} \rightarrow G \times S^1 \), and so the bundle is nontrivial. If \( \mathfrak{g} \) is abelian, then a similar argument can be made using a 2-form \( \langle \cdot \rangle : \mathfrak{g} \rightarrow \mathbb{R} \) on \( \mathfrak{g} \). If \( G \) is a finite group, then the slice \( \mathbf{R}_h \rightarrow S^1 \) is nontrivial if \( h \neq 1 \).

We also record the following lemma about holonomy. Let \( R \rightarrow S \) be a principal \( G \)-bundle over an oriented, connected 1-manifold. Though \( S \) is diffeomorphic to a circle, we do not need a fixed parametrization. Fix a basepoint \( s \in S \). Then the space of connections and basepoints in the fiber is \( \mathcal{A}_R \times R_s \), and the holonomy is a smooth map

\[
\text{hol} : \mathcal{A}_R \times R_s \rightarrow G.
\]

**Lemma 4.18.**

(i) \( \text{hol} \) is a submersion, i.e., \( d\text{hol}_{(\eta,b)} \) is surjective for all \( \eta \in \mathcal{A}_R \), \( b \in R_s \).

(ii) If \( \hat{b} \in T_b(R_s) \cong \mathfrak{g} \), and \( \text{hol}(\eta,b) = h \), then

\[
(4.19) \quad h^{-1} d\text{hol}_{(\eta,b)}(0,\hat{b}) = (1 - \text{Ad}_{h^{-1}})(\hat{b}).
\]

(iii) Let \( \mathfrak{g}_R \) be the adjoint bundle of \( R \), and suppose \( \xi \in \Omega^0_S(\mathfrak{g}_R) \). With respect to the basepoint \( b \in R_s \) we identify \( \xi(s) \) as an element of \( \mathfrak{g} \). Then if \( \text{hol}(\eta,b) = h \),

\[
(4.20) \quad h^{-1} d\text{hol}_{(\eta,b)}(d\eta \xi,0) = (1 - \text{Ad}_{h^{-1}})(-\xi(s)).
\]

(iv) Suppose \( \text{hol}(\eta,b) = h \) and \( \zeta \in \Omega^0_S(\mathfrak{g}_R) \) with \( d\eta \xi = 0 \). Then

\[
(4.21) \quad \langle \xi(s) \otimes h^{-1} d\text{hol}_{(\eta,b)}(\hat{\eta},\hat{b}) \rangle = -\int_s \langle \zeta \wedge \hat{\eta} \rangle.
\]
Of course, (i) implies that $\text{hol}^{-1}(h)$ is a manifold for each $h \in G$. In (iii) note that the covariant derivative is a map

$$\tag{4.22} \Omega^0_S(\mathfrak{g} R) \xrightarrow{d_n} \Omega^1_S(\mathfrak{g} R).$$

The domain $\Omega^0_S(\mathfrak{g} R)$ is the Lie algebra of the group of gauge transformations $G_R$, the codomain $\Omega^1_S(\mathfrak{g} R)$ is the tangent space to the affine space $A_R$ of connections, and $d_n$ is the infinitesimal action of gauge transformations on connections. Here $d_n$ is Fredholm; its kernel is the space of covariant constant sections, which can be identified with the invariants of $\text{Ad}_h$ by evaluation at the basepoint. Assertions (i) and (iii) imply that $h^{-1}d\text{hol}_{(\eta, r)}$ induces an isomorphism of the cokernel of (4.22) with the coinvariants of $\text{Ad}_h$, i.e., an isomorphism $\text{coker} d_n \cong \text{coker}(1 - \text{Ad}_h)$.

**Proof.** Cut the circle $S$ at the basepoint $s$ to obtain an interval $S^{\text{cut}}$ and a bundle $R^{\text{cut}} \to S^{\text{cut}}$. Then a connection on $R$ determines and is determined by a (horizontal) section of $R^{\text{cut}}$. Since the fibers over the endpoints of $S^{\text{cut}}$ are identified, we can compare the section at these two endpoints by an element of $G$; this is the holonomy. Relative to a local trivialization near the terminal endpoint, (i) translates to the obvious statement that evaluation at the endpoint is a submersion from the space of paths in a connected manifold $A$ to $A$. Notice that we can fix the basepoint, which is the value of the section at the initial endpoint, for this argument.

For (ii) suppose $b_t$ is a curve of basepoints with tangent $\dot{b}$. Write $b_t = bg_t$ for a curve $g_t \in G$. Then the holonomy of $\eta$ with respect to $b_t$ is $g_t^{-1}h g_t$. Differentiate to deduce (4.19).

For (iii) let $\varphi_t$ be a curve in $G_R$ with tangent $\zeta$. Then $\varphi_t^* \eta$ is a path of connections with tangent $d_{\eta} \zeta$. But

$$\text{hol}(\varphi_t^* \eta, b) = \text{hol}(\eta, \varphi_t^{-1} b),$$

and the tangent to the path $\varphi_t^{-1}(b)$ is $-\zeta(s)$. So (iii) follows from (ii).

To prove (iv) we first note that (4.19) implies

$$\langle (s) \otimes h^{-1} d\text{hol}_{(\eta, b)} (0, b) \rangle = \langle (s) \otimes (1 - \text{Ad}_{h^{-1}}) (\dot{b}) \rangle = \langle (1 - \text{Ad}_{h^{-1}})(\zeta(s)) \otimes \dot{b} \rangle = 0.$$

In the last step we use the fact that $\zeta$ is parallel, so commutes with the holonomy. Hence we may set $b = 0$. Let $\eta_t$ be a path of connections on $R$ with initial velocity $\dot{\eta}$, and $b_t$ a path in $R$, with initial velocity $\dot{b}$. We cut at the basepoint $s \in S$ to obtain a path of connections on $R^{\text{cut}} \to S^{\text{cut}}$. Let $r$ be a flat section of $R^{\text{cut}}$ relative to $\eta$ with initial value $b$, and define $g_t: S^{\text{cut}} \to G$ so that $rg_t$ is a flat section of $R^{\text{cut}}$ relative to $\eta_t$ with initial value $b$. Then from (1.18) and the fact that $(rg_t)^* \dot{\eta}_t = 0$ we deduce $r^* \dot{\eta}_t = -d_{r^* \eta} \dot{g}$. Therefore, by Stokes’ theorem we have

$$\tag{4.23} \int_S \langle \zeta \wedge \dot{\eta} \rangle = \int_{S^{\text{cut}}} (r^* \zeta \wedge r^* \dot{\eta}) = -\int_{S^{\text{cut}}} (r^* \dot{\zeta} \wedge d_{r^* \eta} \dot{g}) = -\langle r^* \dot{\zeta} \otimes \dot{g} \rangle \big|_{\partial S^{\text{cut}}}.$$

At the initial point of $S^{\text{cut}}$ we have $\dot{g} = 0$, whereas at the terminal point $\dot{g} = h^{-1} d\text{hol}_{(\eta, b)} (\dot{b}, b)$. Thus (4.23) implies (4.21).

Return now to the case of a connected and simply connected group $G$. Since the bundle $R \to G \times S^1$ is not trivial, we cannot continuously trivialize the bundles
\( R_h \to S^1 \) simultaneously for all \( h \in G \). Hence restrict to a subset \( V \subset G \) where this is possible, and thus choose a smooth trivialization

\[
(4.24) \quad r : V \times S^1 \to R,
\]
i.e., a smooth family of trivializations \( r_h : S^1 \to R_h \). Our constructions in the next paragraphs and in §5 depend on the trivializations \( r_h \) through (4.9).

To use these standard trivializations on 2-manifolds we must identify boundary components with the standard circle. Let \( Y \) be a compact oriented 2-manifold, and fix a diffeomorphism \( S^1 \to (\partial Y)_i \) for each component of \( \partial Y \). We do not require that these boundary parametrizations preserve orientation, and so we distinguish ‘+’ boundary components and ‘−’ boundary components according to whether the parametrization preserves or reverses the orientation. The images of the basepoint in \( S^1 \) give a basepoint on each component of \( \partial Y \). We consider connections \( \eta \) on bundles \( Q \to Y \) together with basepoints \( b_i \in Q \mid_{(\partial Y)_i} \) over the basepoints on the \( (\partial Y)_i \).

We denote the collection of basepoints as \( b = \{b_1, \ldots, b_k\} \). Bundle morphisms are required to preserve the basepoints. These connections on bundles with basepoints together with these morphisms form a category \( C_Y \) of pointed connections.

Now suppose \( (\eta, b) \in C_Y \) is a pointed connection on \( Q \to Y \). Then the restriction of \( \eta \) to any boundary component \( (\partial Y)_i \), is isomorphic to a standard connection \( \tilde{\eta} \), where \( h_i \in G \) is the holonomy of \( \eta \) around \( (\partial Y)_i \), computed with respect to the basepoint. Hence there is a unique pointed isomorphism \( Q \mid_{(\partial Y)_i} \cong R_{h_i} \) which pulls \( \tilde{\eta} \) back to the restriction of \( \eta \). Assume that all of the boundary holonomies fall within the distinguished subset \( V \subset G \). Then the trivializations \( r_{h_i} \) chosen above induce a trivialization \( r(\eta, b) : \partial Y \to \partial Q \) of \( Q \) over the boundary. Now let

\[
(4.25) \quad L_{\eta, b} = L_{\eta, r(\eta, b)}
\]
be the line defined above relative to this distinguished trivialization.

**Theorem 4.26.** Let \( G \) be a connected, simply connected compact Lie group and \( \langle \cdot \rangle \) an invariant bilinear form on its Lie algebra \( \mathfrak{g} \) which satisfies the integrality Hypothesis 2.5. Fix a subset \( V \subset G \) over which we choose a smooth trivialization (4.24). Suppose \( Y \) is a compact oriented 2-manifold with parametrized boundary. Let \( C'_Y(V) \) denote the subset of pointed connections whose boundary holonomies all lie in \( V \). Then the assignment

\[
(4.27) \quad \langle \eta, b \rangle \mapsto L_{\eta, b}, \quad (\eta, b) \in C'_Y(V),
\]
of a metrized line to such pointed connections is smooth and agrees with the corresponding assignment (2.20) if \( \partial Y = \emptyset \). It satisfies the functoriality property (2.21) for bundle maps which preserve the basepoints and the boundary parametrizations. Furthermore, it satisfies the orientation property (2.23) and the additivity property (2.25). Finally, it satisfies:

\(d\) (Gluing) Suppose \( S \hookrightarrow Y \) is a closed, oriented codimension one submanifold and \( Y^\text{cut} \) the manifold obtained by cutting along \( S \). Then \( \partial Y^\text{cut} = \partial Y \cup S \sqcup -S \) and we use parametrizations which agree on \( S \) and \( -S \). Suppose \( \langle \eta, b \rangle \) is a pointed connection over \( Y \) and \( \langle \eta^\text{cut}, b^\text{cut} \rangle \) the induced pointed connection over \( Y^\text{cut} \). (We choose basepoints which agree over \( S \) and \( -S \) to form \( b^\text{cut} \) from \( b \).) Then there is a natural isometry

\[
L_{\eta, b} \cong L_{\eta^\text{cut}, b^\text{cut}}.
\]

37
Proof. For smoothness, it suffices to remark that the holonomy is a smooth function of a connection. For the functoriality suppose \( \psi : Q' \to Q \) is a bundle map preserving the basepoints and the boundary parametrizations. Let \( R', R \) be the restrictions of \( Q', Q \) to some boundary components which correspond under \( \psi \), and let \( \psi_h : R \to R_h \) be the unique pointed map with \( \psi_h^* (\theta_{R_h}) = \eta \mid_{R'} \). Then \( \psi_h \psi : R' \to R_h \) satisfies \( (\psi_h \psi)^* (\theta_{R_h}) = \psi^* \eta \mid_{R'} \). Now the functoriality follows from (4.5). The other properties follow directly from the corresponding properties in Proposition 4.4.

Fix a bundle \( Q \to Y \) together with parametrizations of the components of \( \partial Y = \bigsqcup_{i=1}^k (\partial Y)_i \). Let \( y_i \) be the basepoint of \( (\partial Y)_i \). The holonomy is a map

\[
\text{hol}_Q : \mathcal{A}_Q \times Q_{y_1} \times \cdots \times Q_{y_k} \to G \times \cdots \times G.
\]

Here \( Q_{y_i} \) is the fiber of \( Q \) over \( y_i \), the space of basepoints at \( y_i \).

Theorem 4.26 asserts the existence of a smooth line bundle

\[
L_{Q; V, \ldots, V} \to \mathcal{A}_{Q; V, \ldots, V}.
\]

Furthermore, the map (4.28) is invariant under the \( G_Q \) action, and Theorem 4.26 asserts that this action lifts to the line bundle \( L \). Notice that this \( G_Q \) action is free, since we include basepoints.

There is a larger symmetry group which acts when the boundary holonomies are fixed. Fix \( h_1, \ldots, h_k \in V \subset G \) and let

\[
\mathcal{A}_{Q; h_1, \ldots, h_k} = \text{hol}_Q^{-1} (h_1, \ldots, h_k);
\]

by Lemma 4.18(i) this is a smooth manifold. Consider the restriction

\[
L_{Q; h_1, \ldots, h_k} \to \mathcal{A}_{Q; h_1, \ldots, h_k}
\]

of \( L_{Q; V, \ldots, V} \) to this space. Let \( Z_{h_i} \subset G \) be the centralizer of \( h_i \). A result of Bott and Taubes [BTau, Prop. 10.2] asserts that \( Z_{h_i} \) is connected.\(^{28}\) If \( h_i \) is a regular element of \( G \), then \( Z_{h_i} \) is a maximal torus in \( G \). Let \( Z_{h_i} \) act on the fiber \( Q_{y_i} \) via the right principal bundle action of \( G \). Then the group

\[
\mathcal{G}_{Q; h_1, \ldots, h_k} = \mathcal{G}_Q \times Z_{h_1} \times \cdots \times Z_{h_k}
\]

acts on \( \text{hol}_Q^{-1} (h_1, \ldots, h_k) \) via the formula

\[
\langle \eta; b_1, \ldots, b_k \rangle \cdot \langle \psi; g_1, \ldots, g_k \rangle = \langle \psi^* \eta; \psi^{-1} (b_1) \cdot g_1, \ldots, \psi^{-1} (b_k) \cdot g_k \rangle,
\]

where \( \eta \in \mathcal{A}_Q \), \( \psi \in \mathcal{G}_Q \), \( b_i \in Q_{y_i} \), and \( g_i \in Z_{h_i} \). In general this action is neither free nor effective. We determine the stabilizer at \( \langle \eta; b_1, \ldots, b_k \rangle \) in case \( Y \) has no closed components. Any gauge transformation that preserves \( \eta \) is parallel, by Proposition 1.20, and its value in \( \text{Aut}(Q_y) \) at \( y \in Y \) commutes with the holonomy

\(^{28}\) This depends on the fact that \( G \) is connected and simply connected.
group at $y$. Let $Z_\eta \subset G_\eta$ be the subgroup of all such $\psi$. Evaluation at the basepoint $b_i$ gives an embedding $Z_\eta \hookrightarrow G$, and it is easy to see that the image is contained in $Z_{\eta_i}$. This gives an embedding of $Z_\eta$ in $G_{Q; b_1,\ldots, b_k}$, and the image is the stabilizer of $\langle \eta; b_1,\ldots, b_k \rangle$. Notice that the kernel of the action, which is the intersection of these stabilizers, contains the center $Z \subset G$, which sits in $G_\eta$ as the subgroup of \textit{global} gauge transformations.

Let $A_{Q; b_1,\ldots, b_k}^{\text{flat}}$ denote the space of flat connections in $A_{Q; b_1,\ldots, b_k}$. This is not necessarily a manifold, but the subset of \textit{irreducible} flat connections $A_{Q; b_1,\ldots, b_k}^{\text{flat}}$ does form a submanifold. In any case the group of gauge transformations $M$ acts freely, if $Y$ has no closed components, and we denote the quotient by

$$
M_{Y; b_1,\ldots, b_k} = A_{Q; b_1,\ldots, b_k}^{\text{flat}} / G_\eta.
$$

(4.33)

The prime reminds us of the basepoints. This quotient is independent of the choice of bundle $Q \to Y$ up to canonical diffeomorphism, since all $G$ bundles over $Y$ are isomorphic (they are all trivial). It is easy to see that up to noncanonical diffeomorphism this moduli space only depends on the conjugacy classes of the $h_i$. Thus we can restrict $h_i$ to be in a maximal torus of $G$. If we divide out by the full symmetry group (4.31), we obtain a smaller moduli space

$$
M_{Y; \bar{h}_1,\ldots, \bar{h}_k} = A_{Q; \bar{h}_1,\ldots, \bar{h}_k}^{\text{flat}} / G_{Q; \bar{h}_1,\ldots, \bar{h}_k}.
$$

(4.34)

This moduli space depends only on the conjugacy classes $\bar{h}_i$ of the $h_i$. It is the space of equivalence classes of flat connections on $Q$ whose boundary holonomies are conjugate to the $h_i$. Notice that no basepoints are needed to describe this moduli space. There is a map

$$
M_{Y; \bar{h}_1,\ldots, \bar{h}_k} \to M_{Y; h_1,\ldots, h_k}.
$$

(4.35)

The fiber over a connection $\eta$ is the set of basepoints with respect to which the boundary holonomies equal the $h_i$. The group

$$
Z_h = Z_{h_1} \times \cdots \times Z_{h_k}
$$

(4.36)

acts on the fiber with kernel $Z_\eta$. If $\eta$ is irreducible, then $Z_\eta \cong Z$ is the group of global gauge transformations, which is isomorphic to the center of $G$; it is a normal subgroup of $Z_h$. Thus over the subspace

$$
M_{Y; \bar{h}_1,\ldots, \bar{h}_k} \subset M_{Y; h_1,\ldots, h_k}
$$

(4.37)

of irreducible connections, (4.35) is a principal bundle with structure group $Z_h / Z$.

To determine the tangent space to $M_{Y; \bar{h}_1,\ldots, \bar{h}_k}$ at $\langle \eta; b_1,\ldots, b_k \rangle$ we examine the differential of the equations which assert that $\eta$ is flat and that the boundary holonomies are $h_1,\ldots, h_k$. This leads to the complex

$$
0 \to \Omega^0 \left( \mathfrak{g}_Q \right) \xrightarrow{\text{ev}_b} \Omega^1 \left( \mathfrak{g}_Q \right) \oplus \mathfrak{g}^k \xrightarrow{d_\eta \oplus h^{-1} d_{\text{hol}}(\eta; b)} \Omega^2 \left( \mathfrak{g}_Q \right) \oplus \mathfrak{g}^k \to 0.
$$

(4.38)

Here $\text{ev}_b$ is evaluation at the basepoints. The fact that (4.38) is a complex follows from (4.19) and (4.20). We identify the cohomology of (4.38) with the \textit{compactly
supported cohomology of the interior $\tilde{Y}$ with coefficients in $g_Q$, i.e., with the cohomology of the complex

\[(4.39) \quad 0 \rightarrow \Omega^0_{\tilde{Y}}(g_Q)_c \xrightarrow{d_0} \Omega^1_{\tilde{Y}}(g_Q)_c \xrightarrow{d_0} \Omega^2_{\tilde{Y}}(g_Q)_c \rightarrow 0\]

of compactly supported differential forms on $\tilde{Y}$ with coefficients in $g_Q$. Let $\tilde{H}^\bullet = \tilde{H}^\bullet(Y; \eta, b)$ denote the cohomology of (4.38) and $H^\bullet_c = H^\bullet_c(\tilde{Y}; g(\eta))$ the cohomology of (4.39).

**Lemma 4.40.** The inclusion of (4.39) into (4.38) induces an isomorphism $\tilde{H}^\bullet \cong H^\bullet_c$ on cohomology.

A more direct approach might be to show that $\tilde{H}^\bullet$ is a de Rham model for the twisted relative cohomology $H^{\bullet\bullet}(Y, \partial Y; g(\eta))$. The following would be a de Rham proof of excision in that context.

**Proof.** A simple check shows that the inclusion commutes with the differentials. We first show that the induced map on cohomology is injective. At degree 0 there is nothing to check. For degree 1 suppose $\dot{\eta} \in \Omega^1_{\tilde{Y}}(g_Q)_c$ is closed and maps to zero in $\tilde{H}^1$. Then there exists $\zeta \in \Omega^0_Y(g_Q)$ such that $\dot{\eta} = d_0\zeta$ and $\zeta_b = 0$. Since $\dot{\eta} = 0$ near $\partial Y$ we see that $\zeta$ is parallel near $\partial Y$, and then the condition $\zeta_b = 0$ implies $\zeta = 0$ near $\partial Y$. Thus $\zeta$ has compact support, and so $\dot{\eta}$ vanishes in $H^1_c$.

For degree 2 suppose $\tau \in \Omega^2_{\tilde{Y}}(g_Q)_c$ is closed and maps to zero in $\tilde{H}^2$. Then there exists $(\dot{\eta}, \dot{b}) \in \Omega^1_Y(g_Q) \oplus g^\oplus k$ such that $d_2\tau = \tau$ and $h^{-1}d\text{hol}_{(\eta, b)}(\dot{\eta}, \dot{b}) = 0$. Now (4.19) implies that $h^{-1}d\text{hol}_{(\eta, b)}(\dot{\eta}, 0)$ is in the image of $1 - \text{Ad}_{h^{-1}}$, and then the remarks following Lemma 4.18 imply that $\partial \dot{\eta}$ is in the image of $d_2$ on $\partial Y$. Extending away from the boundary we construct $\zeta \in \Omega^0_Y(g_Q)$ such that $\dot{\eta} = d_0\zeta$ near $\partial Y$. So $\tau = d_2(\dot{\eta} - d_0\zeta)$ and $\dot{\eta} - d_0\zeta \in \Omega^1_{\tilde{Y}}(g_Q)_c$, whence $\tau$ vanishes in $H^2_c$.

It remains to show that the inclusion of (4.39) into (4.38) induces a surjection on cohomology. The only possible cohomology of (4.38) in degree 0 occurs on closed components of $Y$, since a flat section of $g_Q$ which vanishes at a basepoint vanishes identically on the component of $Y$ containing the basepoint. The compact cohomology and ordinary cohomology agree on closed components, so we are done. For degree 1 suppose $(\dot{\eta}, \dot{b}) \in \Omega^1_Y(g_Q) \oplus g^\oplus k$ satisfies

$$d_2\dot{\eta} = 0$$

$$h^{-1}d\text{hol}_{(\eta, b)}(\dot{\eta}, \dot{b}) = 0.$$ 

As above we can find $\zeta \in \Omega^0_Y(g_Q)$ so that $\dot{\eta} = d_0\zeta$ near $\partial Y$. We can alter any choice of $\zeta$ by a parallel section near the boundary, and in this way we arrange that $\zeta_b = \dot{b}$. Then $(\dot{\eta} = d_0\zeta, 0)$ is cohomologous to $(\dot{\eta}, \dot{b})$, and $\dot{\eta} - d_0\zeta \in \Omega^1_{\tilde{Y}}(g_Q)_c$ has compact support. This completes the proof for degree 1. There is nothing to prove in degree 2.

Since the compact cohomology and the ordinary cohomology are in (Poincaré) duality, we deduce the following.

**Corollary 4.41.** If $Y$ has no closed components, then

$$\tilde{H}^0(Y; \eta, b) = 0$$

$$\dim \tilde{H}^2(Y; \eta, b) = \dim H^3(Y; g(\eta)).$$
Note that $H^0(Y; g(\eta))$ is the Lie algebra of $Z_\eta$, the subgroup of gauge transformations which fix $\eta$.

Lemma 4.40 implies that the index of (4.38) is the index of (4.39), which by Poincaré duality is the Euler characteristic of $Y$ with coefficients in $g_Q$. (Since (4.38) differs from the twisted de Rham complex by a finite dimensional operator of index 0, this also follows directly.)

**Proposition 4.42.** The index of (4.38) is

$$\dim \tilde{H}^0(Y; \eta, b) - \dim \tilde{H}^1(Y; \eta, b) + \dim \tilde{H}^2(Y; \eta, b) = \dim G \cdot \chi(Y).$$

**Proof.** By the de Rham theorem this twisted Euler characteristic can be computed using other models of cohomology, e.g. cellular theory. It suffices to consider the case where $Y$ is connected. Fix a basepoint $y \in Y$ and a trivialization of the fiber of $g_Q$ at $y$. Choose a finite CW structure on $Y$, and let $\tilde{C}_* (Y)$ be the cellular chain complex of the universal cover of $Y$; it is a finite dimensional module over the group algebra $\mathbb{R}[\pi_1(Y, y)]$. Since the flat connection $\eta$ determines a representation of $\pi_1(Y, y)$ on $\mathfrak{g}$, we can form a cochain complex $\text{Hom}_{\pi_1(Y, y)} (\tilde{C}_*(Y), \mathfrak{g})$. Its cohomology is the cohomology of $Y$ with coefficients in $g_Q$ (twisted by the flat connection $\eta$). By counting dimensions we see that its Euler characteristic, which equals the Euler characteristic of its cohomology, is $\dim G \cdot \chi(Y)$.

Combining Proposition 4.42 with Corollary 4.41 we obtain a formula for the dimension of the moduli space $\mathcal{M}_{Y; h_1, \ldots, h_k}$ (4.33) at a smooth point, assuming that $Y$ has no closed components:

$$\dim \mathcal{M}_{Y; h_1, \ldots, h_k} = \dim \tilde{H}^1(Y; \eta, b) = - \dim \chi(Y) + \dim Z_\eta.$$  

(4.43)

From the discussion following (4.35) we see that the dimension of $\mathcal{M}_{Y; \tilde{h}_1, \ldots, \tilde{h}_k}$ (4.34) is

$$\dim \mathcal{M}_{Y; \tilde{h}_1, \ldots, \tilde{h}_k} = - \dim \chi(Y) - \sum_i \dim Z_{h_i} + 2 \dim Z_\eta$$

(4.44)

at the connection $\eta$. Here $Y$ is the compactification of $Y$ obtained by gluing in standard disks. The second equation in (4.44) makes clear that $\dim \mathcal{M}_{Y; h_1, \ldots, h_k}$ is an even number. This is explained by the theorem of Mehta and Seshadri [MeS], which identifies $\dim \mathcal{M}_{Y; \tilde{h}_1, \ldots, \tilde{h}_k}$ as a complex manifold (of *stable parabolic bundles*) when $Y$ is endowed with a complex structure.30

The passage from (4.43) to (4.44) and the map (4.35) are illuminated by the exact cohomology sequence of the pair $(Y, \partial Y)$, modified using excision:

$$\begin{array}{cccccc}
0 & \longrightarrow & H^0(Y; g(\eta)) & \longrightarrow & H^0(\partial Y; g(\eta)) & \longrightarrow & H^1_c(\tilde{Y}; g(\eta)) & \longrightarrow & H^1(Y; g(\eta)) \\
& & \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \\
0 & \longrightarrow & \text{Lie } Z_\eta & \longrightarrow & \oplus \text{Lie } Z_{h_i} & \longrightarrow & T_\eta \mathcal{M}_{Y; h_1, \ldots, h_k} & & \\
\end{array}$$

29I thank Frank Quinn for pointing out this proof. My original proof, based on Atiyah-Patodi-Singer, was much less elementary.

30There is a recent gauge theoretic proof of the Mehta-Seshadri theorem by Jonathan Poritz [Po].
We still assume that $Y$ has no closed components. The formal tangent space $T_\eta \mathcal{M}_{Y;\bar{h}_1,\ldots,\bar{h}_k}$ is the quotient of the formal tangent space $T_\eta \mathcal{M}_{Y;h_1,\ldots,h_k}$ by the image of $\delta$.

Theorem 4.26 states that the bundle (4.30) descends to the quotient $\mathcal{M}_{Y;\bar{h}_1,\ldots,\bar{h}_k}$. In fact, this bundle also descends further to the moduli space $\mathcal{M}_{Y;\bar{h}_1,\ldots,\bar{h}_k}$.

**Proposition 4.46.** The action of $G_{Q;h_1,\ldots,h_k}$ on $A_{\text{flat}}^{Q;h_1,\ldots,h_k}$ lifts to the line bundle $L_{Q;h_1,\ldots,h_k} \to A_{\text{flat}}^{Q;h_1,\ldots,h_k}$. The center $Z \hookrightarrow G_{Q;h_1,\ldots,h_k}$ acts trivially.

**Proof.** If $g \in Z_{h_i}$, then the change of basepoint $g_1 \to b_i g$ changes the fiber $L_\eta$ over a connection $\eta$ according to (4.9). The Wess-Zumino-Witten line $K_g$ which enters is canonically trivial (A.4) since $g$ is constant, and this trivialization is compatible with multiplication in $Z_{h_i}$. This gives a lift of the $Z_{h_i}$ action to $L_{Q;h_1,\ldots,h_k}$, as desired.

An element of the center $Z$ is represented as a constant map $g: Y \to G$. Fix a section $q: Y \to Q$ and let $a = q^* \eta$. Then the action of $g$ on $L_\eta$ is multiplication by $c_Y(a, g)$ (4.1), which is trivial. (We use the trivialization (A.4) here.)
§5 Hamiltonian Theory on Surfaces with Boundary

As in §3 we use the Chern-Simons functional on paths of connections over a surface, now possibly with boundary, to define a connection on a line bundle \( (4.30) \) over the space of connections (Proposition 5.9). Rather than proceed by direct calculation, we use the abstract construction of a connection from its holonomy explained in Appendix B. Just as the construction of the bundle depends on the choice of trivializing section \( (4.24) \), so too does the connection.

31 This line bundle with connection descends to the moduli spaces \( M_{Y,h_1,...,h_k} \) \((4.33)\) and \( M_{Y,\bar{h}_1,...,\bar{h}_k} \) \((4.37)\).

Because our constructions are local, these line bundles obey a gluing law (Proposition 5.29). The analogue of this gluing law in the quantum theory leads to a modular functor \([S] \), from which one obtains the Verlinde algebra \([V] \). For details in the case of a finite gauge group, see \([FQ] \).

One can twist the line bundle over \( M_{Y,\bar{h}_1,...,\bar{h}_k} \) by representations of the centralizers of the boundary holonomies. These twisted line bundles \((5.31)\) are considered by Daskalopoulos and Wentworth \([DasW] \). They only construct them for certain conjugacy classes of the fixed boundary holonomies, but we find no such restriction on the holonomy.32

Let \( Y \) be a compact oriented 2-manifold with parametrized boundary. Fix a \( G \) bundle \( Q \rightarrow Y \). Now a path of connections and basepoints \( \langle \eta_t, b_t \rangle \) on \( Q \) determines a connection \( \eta \) and a path \( b \) of basepoints on \([0,1] \times Q \). The connection \( \eta \) has no \( dt \) component. Restricting to the boundary we obtain a unique map to the universal bundle

\[
(5.1) \quad [0,1] \times \partial Q \xrightarrow{\varphi} \mathbb{R} \quad \text{and} \quad [0,1] \times \partial Y \xrightarrow{\varphi} G \times S^1
\]

which restricted to a boundary component \( \langle \partial Q \rangle \) and a fixed \( t \) satisfies \( \varphi^*(\theta_{R,h_i(t)}) = \partial \eta_i(t) \) and preserves the basepoints. Here \( h_i(t) \) is the holonomy around \( \langle \partial Q \rangle_i \). The pullback of the universal connection is

\[
(5.2) \quad \varphi^*(\theta_R) = \partial \eta + \xi dt
\]

for some \( \xi : [0,1] \times \partial Q \rightarrow g \). An easy argument shows that for fixed \( t \) the infinitesimal gauge transformation \( \xi_t \in \Omega^0_{\partial Y}(g_Q) \) depends only on the first derivative \( \langle \dot{\eta}_t, \dot{b}_t \rangle \) of \( \langle \eta_t, b_t \rangle \). So we have a function

\[
(5.3) \quad \xi : T(A_Q \times Q_{y_1} \times \cdots \times Q_{y_k}) \rightarrow \Omega^0_{\partial Y}(g_Q).
\]

Now from (5.2) we compute the curvature of \( \varphi^*(\theta_R) \) to be

\[
dt \wedge (\partial \dot{\eta} - d_\eta \xi),
\]

31 We can also construct a connection on \((4.29)\) where the boundary holonomies are allowed to vary over a subset \( V \subset G \) on which the universal bundle is trivialized. Then the connection also depends on a choice of universal connection \((4.14)\). We omit that here, but return to it in Part 2 where we can make better sense of the dependence on these choices.

32 If we did not include the 1-form \((5.8)\) in our connection, then we would find their restriction.
where $d_\eta$ is the covariant derivative. On the other hand this curvature is the pullback of (4.15) under (5.1). In particular, if the boundary holonomies are constant in $t$, then this curvature vanishes and

\begin{equation}
\partial \hat{\eta} = d_\eta \xi.
\end{equation}

Fix a subset $V \subset G$ and a trivialization (4.24) of the universal bundle over $V$. (In a moment we will fix the boundary holonomies.) Then for a path $(\eta_t, b_t) \in A_{Q; V, \ldots, V}$ with boundary holonomies in $V$ we obtain from (5.1) a section

\begin{equation}
r(\eta, b): [0, 1] \times \partial Y \longrightarrow [0, 1] \times \partial Q.
\end{equation}

These sections glue properly when we glue together paths. According to the construction (4.25) and (4.3) of the lines (4.27), and the trivialization (A.4), the section (5.5) induces an isometry

\begin{equation}
L_{[0, 1] \times \partial Y; \partial \eta, b_0, b_1} \cong \mathbb{C}
\end{equation}

These isometries are compatible with gluing and gauge transformations.

Now fix the boundary holonomies to be $h_1, \ldots, h_k \in V$. Then the pullback

\begin{equation}
a = r(\eta, b)^* \partial \eta \in \Omega^1_{\partial Y}(g), \quad (\eta, b) \in A_{Q; h_1, \ldots, h_k},
\end{equation}

is independent of $(\eta, b)$. Define the 1-form $\alpha \in \Omega^1_{A_{Q; h_1, \ldots, h_k}}$ by

\begin{equation}
\alpha(\dot{\eta}, \dot{b}) = 2\pi i \int_{\partial Y} (\xi(\dot{\eta}, \dot{b}) \wedge a).
\end{equation}

This 1-form enters in the following extension of Proposition 3.17.

**Proposition 5.9.** Fix a $G$ bundle $Q \to Y$ over a compact oriented 2-manifold with parametrized boundary $\partial Y = \bigsqcup_{i=1}^k (\partial Y)_i$. Choose also a universal connection (4.14) and a trivialization (4.24) of the universal bundle over points $h_1, \ldots, h_k \in G$. Then the Chern-Simons action, modified by the 1-form (5.8), defines a unitary connection on the hermitian line bundle (4.30). The curvature of this connection times $i/2\pi$ is

\begin{equation}
\omega_{(\eta, b)}(\dot{\eta}_1, \dot{b}_1; \dot{\eta}_2, \dot{b}_2) = -2 \int_Y (\eta_1 \wedge \eta_2) + 2 \int_{\partial Y} (\xi_1 \wedge d_\eta \xi_2),
\end{equation}

where $\xi_i = \xi_{(\eta, b)}(\dot{\eta}_i, \dot{b}_i)$ is defined by (5.3). The action of $G_Q$ on $A_{Q; h_1, \ldots, h_k}$ lifts to $L_{Q; h_1, \ldots, h_k}$, and the lifted action preserves the metric and connection. The induced “moment map” is

\begin{equation}
\mu_\xi(\eta) = 2 \int_Y (\Omega(\eta) \wedge \xi),
\end{equation}

where $\xi \in \Omega^0_Y(g_Q)$ is an infinitesimal gauge transformation.

We do not have a good geometric explanation of the “correction” 1-form (5.8). Even if $\langle \cdot \rangle$ is nondegenerate, the form $\omega$ is degenerate; the kernel consists of variations
where $\dot{\eta} = 0$ and $\dot{b}$ is arbitrary. The “moment map” (5.11) is still well-defined as the obstruction to descending the connection to the quotient by $G_Q$.

We remark that Chang [Ch] constructs a connection on a determinant line bundle with similar formulas for the curvature and moment map.

One can give a proof of Proposition 5.9 by calculations similar to those in the proof of Proposition 3.17. For this one needs detailed geometry of the Wess-Zumino-Witten line bundle, explained in Appendix A. We opt for an easier route, based on the gluing law in Proposition 4.11 and the theorem in Appendix B which constructs a connection from the parallel transport.

**Proof.** Let $\langle \eta_t, b_t \rangle$ be a smooth path in $A_{Q; h_1, \ldots, h_k}$. As above we obtain a connection $\eta$ on $[0, 1] \times Q \rightarrow [0, 1] \times Y$. The boundary $\partial(\eta) = \eta_1 \cup \eta_0 \cup \partial \eta_t$ is a connection on $Q \cup Q \cup [0, 1] \times \partial Q \rightarrow Y \sqcup -Y \sqcup -[0, 1] \times \partial Y$. Using the inverse of the trivialization (5.6) we identify the Chern-Simons action on $\eta$ as a map

$$\text{PT}(\eta_t) = e^{2\pi i S_{[0, 1] \times Y}(\eta)} : L_{\eta_0, b_0} \rightarrow L_{\eta_1, b_1}. \quad (5.12)$$

$\text{PT}(\cdot)$ is a smooth function of the path, since the Chern-Simons action is smooth. For constant paths (5.12) is the identity map, since then the connection $\eta$ bounds a connection on $D^2 \times Y$ which is constant in the $D^2$ factor. Furthermore, $\text{PT}(\cdot)$ is reparametrization invariant by the functoriality property (2.22). The generalized gluing law Proposition 4.11 and the gluing law for the trivializations (5.6) imply that (5.12) composes properly when we glue paths; in the terminology of Appendix B, it is an additive function. Hence the hypotheses of Corollary B.6 are satisfied, and this theorem asserts the existence of a connection on $L_{Q; h_1, \ldots, h_k} \rightarrow A_{Q; h_1, \ldots, h_k}$ with parallel transport (5.12). Notice that (B.3) is an explicit formula for the connection form.

However, this is not the connection we want. The connection on $L_{Q; h_1, \ldots, h_k}$ we consider is this connection plus the 1-form $\alpha$ defined in (5.8). We will now show that the curvature and moment map of this modified connection are (5.10) and (5.11).

To verify the curvature formula (5.10), fix $x, y$ small and consider the connection

$$\eta + sx\dot{\eta}_1 + ty\dot{\eta}_2 \quad (5.13)$$
on the bundle $[0, 1]_s \times [0, 1]_t \times Q$. Choose basepoints which agree with $b$ at $s = t = 0$, have derivative $\dot{b}_1$ and $\dot{b}_2$ along the $s$ and $t$ directions and have constant boundary holonomies. Let

$$\gamma_{x,y} : \partial([0, 1] \times [0, 1]) \rightarrow A_{Q; h_1, \ldots, h_k}$$

be the resulting map on the boundary of the $s$,$t$ rectangle. Then by the calculus of differential forms we see that the curvature of the connection on $L_{Q; h_1, \ldots, h_k}$ evaluated in the directions $\langle \dot{\eta}_1, \dot{b}_1 \rangle$, $\langle \dot{\eta}_2, \dot{b}_2 \rangle$ is

$$- \frac{d}{dx} \bigg|_{x=0} \frac{d}{dy} \bigg|_{y=0} \log \text{hol}(\gamma_{x,y}) + d\alpha(\dot{\eta}_1, \dot{b}_1; \dot{\eta}_2, \dot{b}_2), \quad (5.14)$$

Clearly this argument generalizes to an axiomatic framework, where one deduces the Hamiltonian theory (symplectic structure) from the general properties of the action. We will not attempt this here (but see [Ax, §3]).
where $\text{hol}(\gamma_{x,y})$ is the holonomy around the loop. Equation (5.12) defines this holonomy as the exponential of $2\pi i$ times the Chern-Simons action, which we compute using (2.8) applied to $W = [0, 1] \times [0, 1] \times Y$. Note

$$
\partial W = \partial([0, 1] \times [0, 1]) \times Y \cup [0, 1] \times [0, 1] \times \partial Y.
$$

Now the curvature of (5.13) is

$$
\Omega = \Omega_{s,t} + x \, ds \wedge \dot{h}_1 + y \, dt \wedge \dot{h}_2,
$$

where $\Omega_{s,t}$ is the curvature of (5.13) restricted to $Y \times \{s\} \times \{t\}$. A straightforward computation shows

$$
\int_{[0,1] \times [0,1] \times Y} (\Omega \wedge \Omega) = -2xy \int_Y (\dot{h}_1 \wedge \dot{h}_2).
$$

It remains to calculate the action on $[0, 1] \times [0, 1] \times \partial Y$ (c.f. (5.15)) and the contribution from $da$ in (5.14). The definition (5.12) instructs us to use the trivialization (5.6), and so to calculate the action on $X = [0, 1] \times [0, 1] \times \partial Y$ relative to the canonical section $r$ on $\partial X$. In fact, we may as well use $r$ over all of $X$. Consider the connection (5.13) on $[0, 1] \times [0, 1] \times \partial Q$. (We now omit the ‘$\partial$’ from the notation for the connection.) As in (5.2) we have

$$
r^*(\eta + sx\dot{h}_1 + ty\dot{h}_2) = a - x\xi_1 ds - y\xi_2 dt,
$$

where $a$, defined in (5.7), is independent of $s$ and $t$, and $\xi_1, \xi_2 : X \to \mathfrak{g}$ are defined by (5.3). From (5.16) we see that

$$
r^*(\Omega) = xds \wedge \dot{h}_1 + ydt \wedge \dot{h}_2.
$$

The action relative to the trivialization $r$ is computed from (1.26) and (2.2):

$$
S_X(r, \eta + sx\dot{h}_1 + ty\dot{h}_2) = xy \int_{[0,1] \times [0,1]} ds \wedge dt \int_{\partial Y} -\langle \xi_1 \wedge \dot{h}_2 \rangle + \langle \xi_2 \wedge \dot{h}_1 \rangle + \langle [\xi_1, \xi_2] \wedge a \rangle.
$$

Combining (5.17) and (5.19), substituting (5.4), and using Stokes’ theorem we see that

$$
- \frac{i}{2\pi} \frac{d}{dx} \bigg|_{x=0} \frac{d}{dy} \bigg|_{y=0} \log \text{hol}(\gamma_{x,y}) = -2 \int_Y (\dot{h}_1 \wedge \dot{h}_2) + \int_{\partial Y} 2\langle \xi_1 \wedge d_q \xi_2 \rangle - \int_{\partial Y} \langle [\xi_1, \xi_2] \wedge a \rangle.
$$

It is easy to see that $\frac{i}{2\pi} da$ in (5.14) cancels the last term, thereby proving (5.10).

Next, we verify (5.11). Suppose $\xi \in \Omega^1_Y(\mathfrak{g}_Q)$ and $\psi_t \in \mathcal{G}_Q$ is a path of gauge transformations with $\psi_0 = \text{id}$ and $\psi_1 = \xi$. Consider the path $\langle \psi_t \eta, \psi^{-1}_t b \rangle \in A_{Q, h_1, \ldots, h_k}$. Now the path $\psi_t^* \eta$ forms a connection $\eta$ on $[0, x] \times Y$. Fix a section $q : Y \to Q$ which has $\partial q = r(\eta, b)$, and let $q : [0, x] \times Y \to [0, x] \times Q$ be the section $\psi_t^{-1} q$. According to (3.24) and (5.12) the moment map is

$$
\mu_\xi(\eta) = -\frac{d}{dx} \bigg|_{x=0} S_{[0,x] \times Y}(q, \eta) - \frac{i}{2\pi} \alpha(d_q \xi, \xi)_{\text{basepoints}}.
$$
Now
\[ q^*\eta = q^*\eta - \xi dt, \]
from which the pullback of the curvature \( \Omega \) of \( \eta \) is
\[ q^*\Omega = \Omega + dt \wedge d_a \xi, \]
where \( \Omega = \Omega(\eta) \) is the curvature of \( \eta \). A short computation shows
\[ S_{[0,x]}(q, \eta) = \int_0^x dt \left[ -2 \int_Y (\Omega \wedge \xi) + \int_{\partial Y} (\xi(t) \wedge a) \right]. \]

We deduce (5.11) easily from (5.20) and (5.21). This completes the proof of Proposition 5.9.

**Corollary 5.22.** The line bundle with connection in Proposition 5.9 descends to a line bundle with connection
\[ L_{[0,1] \times Y} = \mathcal{M}_{[0,1] \times Y} / \mathcal{G}_Q \]
over the moduli space (4.33). The curvature times \( i/2\pi \) is given by (5.10).

This follows directly from the fact that the “moment map” (5.11), which is the obstruction to descending the connection, vanishes on the space \( \mathcal{A}^{\text{flat}}_{Q,h_1,\ldots,h_k} \) of flat connections. The quotient line bundle with connection is independent of \( Q \) up to canonical diffeomorphism.

We next show that this connection descends to \( \mathcal{M}_{[0,1] \times Y} \), the moduli space of flat connections whose boundary holonomies are conjugate to the \( h_i \) (cf. (4.34)), at least over the irreducible connections.

**Proposition 5.24.** The lift of \( \mathcal{G}_{Q,h_1,\ldots,h_k} \) to \( L_{[0,1] \times Y} \rightarrow \mathcal{A}^{\text{flat}}_{Q,h_1,\ldots,h_k} \) defined in Proposition 4.46, is parallel with respect to the connection on \( L_{[0,1] \times Y} \) defined in Proposition 5.9 and preserves that connection as well.

**Proof.** By Proposition 5.9 we need only consider the action of the subgroup \( Z_h \) (4.36). An easy argument shows that it preserves the holonomy along paths, hence preserves the connection, so we need only check that the action is parallel.

Suppose \( \langle \eta, b \rangle \in \mathcal{A}_{Q,h_1,\ldots,h_k} \). Fix a section \( q: Y \rightarrow Q \) such that \( \partial q = r(\eta, b) \). Now let \( \lambda \in \text{Lie}(Z_h) \) for some fixed \( i \), and consider the path \( g_t = e^{t\lambda} \in Z_h \). Recall that \( Z_h \) is connected, so every element can be written \( e^{\lambda} \) for some \( \lambda \). Acting on \( \langle \eta, b \rangle \) we obtain the path
\[ t \mapsto \langle \eta; b_1, \ldots, b_i g_t, \ldots, b_k \rangle \]
in \( \mathcal{A}_{Q,h_1,\ldots,h_k} \). The connection on \( [0,1] \times Q \) is independent of the first factor, but the basepoint on the \( i \)-th boundary component varies. Now the constant section \( q: [0,1] \times Y \rightarrow [0,1] \times Q \) does not induce the (inverse) trivialization (5.6) on \( [0,1] \times (\partial Y)_1 \), as required by (5.12). Instead, we need to use the section \( qg_t \). Note that \( e^{2\pi i W_{[0,1] \times (\partial Y)_1}(g_t)} \) induces the trivialization (A.4) of \( K_{g_t} \) on the boundary. Hence
the parallel transport (5.12) of the lift of (5.25) to \( L_{Q; h_1, \ldots, h_k} \), computed relative to the trivialization of \( L_{Q; h_1, \ldots, h_k} \) induced by \( q \), is (cf. (4.1))

\[
c_{[0,1] \times (\partial Y), (r^* (\partial \eta), g_t)} = c_{[0,1] \times (\partial Y), ((rg_t)^* (\partial \eta), g_t^{-1})} = \exp \left( -2 \pi i \int_0^1 dt \int_{(\partial Y)_t} \langle \operatorname{Ad}_{g_t} a \wedge g_t^{-1} \dot{g}_t \rangle \right)
\]

This is canceled by the correction 1-form (5.8).

The space \( A^\ast_{Q; h_1, \ldots, h_k} \) of irreducible flat connections is a manifold, and \( G_{Q; h_1, \ldots, h_k} \) acts with constant stabilizer \( Z \) (cf. the discussion following (4.32)). Since \( Z \) acts trivially on the line bundle \( L_{Q; h_1, \ldots, h_k} \) over \( A^\ast_{Y; \bar{h}_1, \ldots, \bar{h}_k} \), and \( M^\ast_{Y; \bar{h}_1, \ldots, \bar{h}_k} \) acts trivially on the line bundle \( L_{Y; h_1, \ldots, h_k} \) over the moduli space of irreducible flat connections with boundary holonomies conjugate to the \( h_i \).

**Proposition 5.27.** If \( \langle \cdot \rangle \) is a nondegenerate form on \( g \), then the curvature of (5.26) is a symplectic form on \( M^\ast_{Y; \bar{h}_1, \ldots, \bar{h}_k} \).

**Proof.** By Lemma 4.40 we can realize tangent vectors at \( \eta \) to \( M^\ast_{Y; \bar{h}_1, \ldots, \bar{h}_k} \) by compactly supported forms \( \dot{\eta} \), and evaluated on two such forms the curvature (5.10) is

\[
\omega_\eta (\dot{\eta}_1, \dot{\eta}_2) = -2 \int_Y \langle \dot{\eta}_1 \wedge \dot{\eta}_2 \rangle.
\]

If \( \langle \cdot \rangle \) is nondegenerate, then Poincaré duality (for twisted coefficients) implies that \( \omega_\eta \) induces a nondegenerate pairing

\[
H^1_c \left( Y; g(\eta) \right) \otimes H^1 (Y; g(\eta)) \to \mathbb{R};
\]

here we do not require that \( \dot{\eta}_2 \) in (5.28) have compact support. The exact sequence (4.45) implies that \( \omega_\eta \) is also nondegenerate as a bilinear pairing on \( H^1_c \left( \tilde{Y}; g(\eta) \right) \) if \( \delta \equiv T_\eta \mathcal{M}^\ast_{Y; \bar{h}_1, \ldots, \bar{h}_k} \).

As with all of our constructions there is a gluing law for the connection constructed in Proposition 5.9. One consequence of the basepoints, which rigidify the connection over the boundary, is that gluing is well-defined for the pointed moduli space (5.23). It is not, however, well-defined once we remove the basepoint (5.26).

**Proposition 5.29.** Suppose \( Q \to Y \) is a \( G \) bundle over a compact oriented 2-manifold with parametrized boundary. Fix boundary holonomies, which we collectively denote \( h_{\partial Y} \). Now suppose \( S \to Y \) is a closed oriented codimension one
submanifold, and $Q^{\text{cut}} \to Y^{\text{cut}}$ the cut bundle. Fix holonomies $h_S$ for the components of $S$, and denote by $h_{-S}$ their inverses. Then (4.8) leads to a diagram of maps

$$
\begin{array}{ccc}
L_{Q^{\text{cut}}; h_{Y}, h_{S}, h_{-S}} & \longrightarrow & L_{Q; h_{Y}} \\
\downarrow & & \downarrow \\
\mathcal{M}_{Y^{\text{cut}}; h_{Y}, h_{S}, h_{-S}} & \longrightarrow & \mathcal{M}_{Y; h_{Y}}
\end{array}
$$

(5.30)

which is compatible with the connection constructed in Corollary 5.22.

There is a corresponding gluing law for the connection in Proposition 5.9; it is proved using the gluing law Proposition 4.11 for the action and the definition (5.12), (5.8) of the connection. Then (5.30) is the quotient by $G_Q$.

We also remark that this connection is compatible with change of gauge group, as in Proposition 2.9 and Proposition 2.29. However, we must be careful to use compatible trivializations of the universal bundles of connections over the circle.

Finally, we construct twistings of the line bundle (5.26). Recall that the map (4.35), restricted to the subspace of irreducible connections, is a principal bundle with structure group $Z_h/Z$, where $Z_h$ is defined in (4.36). Suppose

$$
\lambda: Z_h/Z \longrightarrow \mathbb{T}
$$

is a unitary representation. Then there is an induced hermitian line bundle\footnote{We did not succeed in finding a unitary connection on $L_\lambda$. Perhaps one should be induced from a connection on (4.35), but we did not find a connection there either.} $L_\lambda \longrightarrow \mathcal{M}_{Y; h_1, \ldots, h_k}^*$. In the quantum Chern-Simons theory one considers the tensor product bundles

$$
(5.31)
L_{Y; h_1, \ldots, h_k} \otimes L_\lambda \longrightarrow \mathcal{M}_{Y; h_1, \ldots, h_k}^*
$$

for various $\lambda$. We do not find a relationship between the holonomies $h_i$ and the representation $\lambda$, as was found in [DasW, §5].
§6 Special Cases

We first note some special features of the Hamiltonian theory (§3) on surfaces of genus 0 and genus 1. We also state the usual formula (6.3) for the Chern-Simons functional when $G = SU(2)$.

The 2-sphere $Y = S^2$ is simply connected, so according to Proposition 3.5 the moduli space $\mathcal{M}_Y$ consists of a single point. Of course, the dimension formula (3.9) predicts that $\dim \mathcal{M}_Y = 0$. The line corresponding to a trivial connection is canonically trivial (over any surface).

If $Y = S^1 \times S^1$ is a torus, then $\pi_1(S^1 \times S^1, \ast) \cong \mathbb{Z} \times \mathbb{Z}$ for the standard basepoint $\ast \in S^1 \times S^1$. Hence $\text{Hom}(\pi_1(S^1 \times S^1, \ast), G)$ is given by the set of commuting pairs of elements $g_1, g_2 \in G$. We can simultaneously conjugate both elements into a fixed maximal torus $T \subset G$, since they commute. Then the normalizer $N(T)$ acts by conjugation on pairs of elements in $T$, with $T \subset N(T)$ acting trivially. Thus the quotient Weyl group $W = N(T)/T$ acts, and

$$\mathcal{M}_{S^1 \times S^1} \approx T \times T / W.$$  

Note that for a generic connection $\eta$ on the torus the centralizer $Z_\eta$ is isomorphic to the maximal torus $T$. If the holonomies are both singular elements of $G$, then the centralizer is bigger. Thus the $W$ action degenerates at pairs of singular points of $G$ (fixed by the same element of $W$). Away from these singular points (3.9) predicts the proper formula for the dimension of (6.1).

We now specialize to $G = SU_2$. Fix the standard maximal torus $T$ of diagonal matrices. The nontrivial element of the Weyl group $W \cong \mathbb{Z}/2\mathbb{Z}$ acts on $T$ by permuting the elements in a diagonal matrix. The fixed points are $1, -1 \in T$, so there are four singular points in (6.1).

The integral forms on $\mathfrak{g} = \mathfrak{su}_2$ are parametrized by an integer $k$:

$$\langle a \otimes b \rangle = -\frac{k}{8\pi^2} \text{Tr}(ab), \quad a, b \in \mathfrak{su}_2.$$  

Then the Chern-Simons functional (2.2) takes the form

$$S_X(A) = -\frac{k}{8\pi^2} \int_X \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$  

where $A \in \Omega_X^1(\mathfrak{su}_2)$ is the connection form relative to some trivialization.
Appendix: Classical Wess-Zumino-Witten Theory

The Wess-Zumino-Witten functional (2.13) is the action of a $1 + 1$ dimensional topological classical field theory. It arises in the Chern-Simons theory as a term in the cocycle which measures the dependence of the Chern-Simons action on the boundary trivialization. In this appendix we briefly study it as a field theory in its own right. In fact, it is simpler than the Chern-Simons theory and is a nice first example of the geometry of classical field theory. Our main interest, however, is in the metrized lines $K_\gamma$ attached to loops $\gamma: S^1 \to G$, which together form a central extension of the loop group. We restrict ourselves to connected, simply connected, compact Lie groups $G$ endowed with an integral bilinear form $\langle \cdot \rangle: g \otimes g \to \mathbb{R}$ on the Lie algebra. There is a theory for arbitrary compact $G$ endowed with a class in $H^3(G)$; this can be constructed using the techniques of Part 2 (cf. [Gu]). Our constructions of the lines here is quite simple, based on the fact that every field on a “space” is the boundary of a field on some “spacetime”. This procedure is elaborated in [Ax,§4] in a more general context.

For $Y$ a closed oriented 2-manifold, equation (2.13) defines a function $W_Y(g)$ on the space of (piecewise) smooth maps $g: Y \to G$; this is the content of Lemma 2.12. To generalize to 2-manifolds with boundary, we must construct lines attached to the boundary fields.

Proposition A.1. Let $S$ be a closed oriented 1-manifold (union of circles). Then for each smooth map $\gamma: S \to G$ there is attached a metrized complex line $K_\gamma = K_{S,\gamma}$. The assignment $\gamma \mapsto K_\gamma$ is smooth; satisfies functoriality, orientation, and additivity conditions analogous to (2.21), (2.23), and (2.25); and for $\gamma_1, \gamma_2: S \to G$ with pointwise product $\gamma_1 \gamma_2$ there is an isometry

$$K_{\gamma_1} \otimes K_{\gamma_2} \longrightarrow K_{\gamma_1 \gamma_2}. \tag{A.2}$$

Furthermore, for three loops $\gamma_1, \gamma_2, \gamma_3$ we have the commutative diagram

$$\begin{array}{ccc}
K_{\gamma_1} \otimes K_{\gamma_2} \otimes K_{\gamma_3} & \longrightarrow & K_{\gamma_1} \otimes K_{\gamma_2 \gamma_3} \\
\downarrow & & \downarrow \\
K_{\gamma_1 \gamma_2} \otimes K_{\gamma_3} & \longrightarrow & K_{\gamma_1 \gamma_2 \gamma_3} \
\end{array} \tag{A.3}$$

Finally, if $\gamma$ is a constant loop then there is a trivialization

$$K_\gamma \cong \mathbb{C} \tag{A.4}$$

which respects (A.2).

It follows from (A.2) and (A.3) that the set of elements of unit norm in the lines $K_\gamma$ form a group $\text{Map}(S, G)$ which is a central extension of the loop group:

$$1 \to \mathbb{T} \to \text{Map}(S, G) \to \text{Map}(S, G) \to 1. \tag{A.5}$$

This construction of the central extension first appeared in Mickelsson [M]. The last assertion means that this central extension is split over the constant loops. The}

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35I thank Jacek Brodzki for discussions about material in this section.
functoriality means that the group of orientation preserving diffeomorphisms \( \text{Diff}^+(S) \) acts on \( \text{Map}(S,G) \). For more on this central extension, see [PS, §4].

**Proof.** We only sketch the constructions. Fix an oriented 2-manifold \( D \) with \( \partial D = S \). Let \( \mathcal{C}_\gamma \) be the category of extensions \( \Gamma: D \to G \) of \( \gamma: S \to G \); it is nonempty by our assumptions on the topology of \( G \). Endow \( \mathcal{C}_\gamma \) with a unique morphism between any two objects. Define a functor \( \mathcal{F}_\gamma: \mathcal{C}_\gamma \to \mathcal{L} \) by \( \mathcal{F}_\gamma(\Gamma) = \mathcal{C} \) and \( \mathcal{F}_\gamma(\Gamma_0 \to \Gamma_1) \) is multiplication by

\[
(A.6) \quad \exp(2\pi i W_{D\text{double}}(\Gamma_0 \cup -\Gamma_1)).
\]

Here \( D\text{double} \) is the double of \( D \) and \( \Gamma_0 \cup -\Gamma_1 \) is the (piecewise smooth) map which is \( \Gamma_0 \) on one copy of \( D \) and \( \Gamma_1 \) on the other, with the indicated orientations. It is easy to check that \( \mathcal{F}_\gamma \) is a functor; its invariant sections form the line \( \hat{g} \).

To construct the isometry \( (A.2) \) fix extensions \( \Gamma_1, \Gamma_2 \) of \( \gamma_1, \gamma_2 \). Then \( \Gamma_1 \Gamma_2 \) is an extension of \( \gamma_1 \gamma_2 \). These extensions trivialize the lines in \( (A.2) \), and relative to these trivializations the isometry \( (A.2) \) is multiplication by

\[
(A.7) \quad \exp(2\pi i \int_D \langle \Gamma_1^* \theta \wedge \text{Ad}_g \Gamma_2^* \theta \rangle) = \exp(2\pi i \int_D \langle \Gamma_1^{-1} d\Gamma_1 \wedge d\Gamma_2 \Gamma_2^{-1} \rangle),
\]

where \( \theta \) is the Maurer-Cartan form on \( G \). This calculation proceeds from the following formula. For any 3-manifold \( X \) and \( g: X \to G \) a smooth map, we set

\[
\omega(g) = -\frac{1}{6} \langle g^* \theta \wedge [g^* \theta \wedge g^* \theta] \rangle = -\frac{1}{6} \langle g^{-1} dg \wedge [g^{-1} dg \wedge g^{-1} dg] \rangle,
\]

the Wess-Zumino-Witten lagrangian. Then for \( g_1, g_2: X \to G \) we have

\[
(A.8) \quad \omega(g_1 g_2) = \omega(g_1) + \omega(g_2) + d\sigma(g_1, g_2),
\]

where

\[
\sigma(g_1, g_2) = d(g_1^* \theta \wedge \text{Ad}_{g_2} g_2^* \theta) = d(g_1^{-1} dg_1 \wedge dg_2 g_2^{-1}).
\]

For the functoriality, suppose \( f: S' \to S \) is a diffeomorphism, \( \gamma: S \to G \) a map, and \( \gamma' = \gamma \circ f \). Then an extension \( \Gamma': D' \to G \) gives a trivialization \( K_{\gamma'} \cong \mathbb{C} \). Let \( D' \) be the 2-manifold \( D' = S' \cup_f D \). Then \( \partial D' = S' \) and \( \gamma' \) extends to a map \( \Gamma': D' \to G \) which agrees with \( \Gamma \) on \( D \). Now \( \Gamma' \) gives a trivialization \( K_{\gamma'} \cong \mathbb{C} \). Relative to these trivializations we define \( f^*: K_{\gamma} \to K_{\gamma'} \) to be the identity.

The trivialization for constant \( \gamma \) comes from the constant extension \( \Gamma \).

**Proposition A.9.** Let \( Y \) be a compact oriented 2-manifold with boundary and \( g: Y \to G \) a (piecewise) smooth map. Then the action

\[
e^{2\pi i W_Y(g)} \in K_{\partial Y, \partial g}
\]

is defined and satisfies functoriality, orientation, additivity, and gluing conditions analogous to (2.22), (2.24), (2.26), and (2.27). In addition, if \( g_1, g_2: Y \to G \) with pointwise product \( g_1, g_2 \), then

\[
(A.10) \quad e^{2\pi i W_Y(g_1 g_2)} = \exp(2\pi i \int_Y \langle g_1^* \theta \wedge \text{Ad}_{g_2} g_2^* \theta \rangle) e^{2\pi i W_Y(g_1)} \otimes e^{2\pi i W_Y(g_2)}.
\]
To compute the action we choose an oriented 2-manifold $D$ with $\partial D = \partial Y$ and extend $\gamma = \partial g$ to $\Gamma: D \to G$. Then we apply formula (2.13) to $g \cup -\Gamma: Y \cup -D \to G$. The dependence on the extension transforms by the cocycle (A.6).

One consequence of (A.10) is

$$e^{2\pi i W_Y (g^{-1})} = e^{2\pi i W_Y (g)}.$$ 

We record here the differential of the Wess-Zumino-Witten functional. For any manifold $M$ we identify the tangent space to $\text{Map}(M, G)$ at any point with $\text{Map}(M, g) = \Omega^0_M(g)$ via left translation.

**Proposition A.11.** Let $Y$ be a closed oriented 2-manifold. Suppose $g: Y \to G$ and $\Xi \in \Omega^0_Y(g)$. Set $\Phi = g^* \theta \in \Omega^1_Y(g)$, where $\theta$ is the Maurer-Cartan form on $G$. Then

$$d(W_Y)_g(\Xi) = \int_Y \langle \Xi \wedge d\Phi \rangle.$$ 

If $Y$ has nonempty boundary, then the same formula serves to compute the covariant derivative of the action (relative to some trivialization) with respect to the connection defined below.

**Proof.** Let $X$ be an oriented 3-manifold with $\partial X = Y$. We extend $g, \Phi, \Xi$ over $X$ without introducing new notation. Let $\Gamma_t$ be a path of maps $X \to G$ with $\Gamma^{-1} \dot{\Gamma} = \Xi$, where we use the Leibnitz notation for the derivative at $t = 0$. Set $\Phi_t = \Gamma_t^* \theta$. Then we calculate

$$d\Phi = -\frac{1}{2} [\Phi \wedge \Phi]$$
$$d\Xi = \dot{\Phi} + [\Xi \wedge \Phi].$$

Now the Wess-Zumino-Witten lagrangian is

$$-\frac{1}{6} \langle \Phi \wedge [\Phi \wedge \Phi] \rangle = \frac{1}{3} \langle \Phi \wedge d\Phi \rangle.$$ 

Some calculation with (A.13) shows

$$\frac{d}{dt} \bigg|_{t=0} \frac{1}{3} \langle \Phi \wedge d\Phi \rangle = d\langle \Xi \wedge d\Phi \rangle,$$

and then (A.12) follows by Stokes’ theorem.

The analogue of Proposition 3.17 is the following.

**Proposition A.14.** Fix a closed oriented 1-manifold $S$. Then the Wess-Zumino-Witten action defines a unitary connection $\alpha$ on the hermitian line bundle $K \to \text{Map}(S, G)$. The curvature of $\alpha$ times $i/2\pi$ is

$$\tau_\gamma(\xi_1, \xi_2) = -\int_S \langle [\xi_1, \xi_2] \wedge \phi \rangle, \quad \xi_1, \xi_2 \in \Omega^0_S(g),$$

where $\phi = \gamma^* \theta$ is the pullback of the Maurer-Cartan form by $\gamma: S \to G$. The central extension $\text{Map}(S, G)$ acts on $K$ by both left and right multiplication. The action of
the center preserves the connection $\alpha$. Left multiplication by $\gamma_0: S \to G$ changes $\alpha$ by the 1-form

$$\lambda_\gamma(\xi) = 2\pi i \int_S (\gamma_0^* \theta \wedge \text{Ad}_\gamma \xi).$$

Right multiplication by $\gamma_0: S \to G$ changes $\alpha$ by the 1-form

$$\rho_\gamma(\xi) = 2\pi i \int_S (\xi \wedge \text{Ad}_{\gamma_0} \gamma_0^* \theta).$$

Note that since the center of $\Map(S,G)$ preserves $\alpha$, the action of an element in $\Map(S,G)$ depends only on its image in $\Map(S,G)$. Formulas (A.16) and (A.17) give the difference between the pullback of $\alpha$ and $\alpha$. Also, the curvature (A.15) is the transgression of $-\frac{1}{6} (\theta \wedge [\theta \wedge \theta]) \in \Omega^3_G$ to $\Map(S,G)$.

**Proof.** We use ‘$\alpha$’ to denote both the connection (a 1-form on the circle bundle of unit vectors in $K$) and its local expression on $\Map(S,G)$ relative to a trivialization.

We suppose that for each $\gamma: S \to G$ in some open set in the loop group we are given a smoothly varying extension $\Gamma: D \to G$, where $\partial D = S$, and so a trivialization of $K$ over that subset. Note that these extensions determine extensions of tangent vectors as well. Now if $\gamma_1$ is a path in this subset, and $\gamma: [0, 1] \times S \to G$ the resulting map, then

$$W_{[0,1] \times S}^{\gamma} = - \int_0^1 dt \int_D \langle \Gamma^{-1} \dot{\Gamma} \wedge d\Phi \rangle,$$

where $\Phi = \Gamma^* \theta$ is the pullback of the Maurer-Cartan form by the extension $\Gamma_t$ of $\gamma_t$.

Consider the (connection) 1-form

$$\alpha_\gamma(\xi) = 2\pi i \int_D \langle \Xi \wedge d\Phi \rangle, \quad \gamma: S \to G, \quad \xi \in \Omega^0_S(g),$$

where $\Gamma: D \to G$ and $\Xi \in \Omega^0_D(g)$ are the given extensions of $\gamma$ and $\xi$, and $\Phi = \Gamma^* \theta$.

If $\tilde{\Gamma}$ is a different extension of $\gamma$, and it induces the extension $\tilde{\Xi}$ of $\xi$, then the trivialization of $K$ changes by the inverse of the cocycle (A.6). The logarithmic differential of this cocycle is computed by (A.12) as

$$-2\pi i \int_D \langle \tilde{\Xi} \wedge d\tilde{\Phi} \rangle + 2\pi i \int_D \langle \Xi \wedge d\Phi \rangle,$$

which is precisely $-(\tilde{\alpha} - \alpha)$, **minus** the difference of the connection forms relative to the two trivializations. This consistency shows that $\alpha$ defines a unitary connection on $K \to \Map(S,G)$.

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36. The sign in (A.18) caused the author great confusion. It is simply explained by the observation

$$\partial([-0,1] \times D) = \{-1\} \times D \cup \{0\} \times D \cup [0,1] \times S.$$

37. Recall that parallel transport is the integral of **minus** the connection form relative to a trivialization.
Now if \( \xi_1, \xi_2 \in \Omega^0_0(g) \) with extensions \( \Xi_1, \Xi_2 \in \Omega^0_D(g) \), then the derivative of \( \langle \Xi_2 \wedge d\Phi \rangle \) in the direction \( \Xi_1 \) is
\[
\langle \Xi_2 \wedge d\Phi \rangle = \langle \Xi_2 \wedge -d[\Xi_1, \Phi] \rangle \\
= \langle [d\Xi_1, \Xi_2] \wedge \Phi \rangle + \langle [\Xi_1, \Xi_2] \wedge d\Phi \rangle
\]
by (A.13). Hence the curvature times \( i/2\pi \) is
\[
\frac{i}{2\pi} d\alpha_\gamma(\xi_1, \xi_2) = - \int_D \left( \langle [d\Xi_1, \Xi_2] \wedge \Phi \rangle + \langle [\Xi_1, \Xi_2] \wedge d\Phi \rangle \right) - \left( \langle [d\Xi_2, \Xi_1] \wedge \Phi \rangle + \langle [\Xi_2, \Xi_1] \wedge d\Phi \rangle \right) - \langle [\Xi_1, \Xi_2] \wedge d\Phi \rangle
\]
\[
= - \int_D \langle [d\Xi_1, \Xi_2] \wedge \Phi \rangle + \langle [\Xi_1, d\Xi_2] \wedge \Phi \rangle + \langle [\Xi_1, \Xi_2] \wedge d\Phi \rangle
\]
\[
= - \int_S \langle [\xi_1, \xi_2] \wedge \phi \rangle.
\]

The assertion about the action of the center is one of the defining properties of a connection form on a circle bundle—it is invariant under the \( T \) action. Formulas (A.16) and (A.17) follow most easily by computing the parallel transport (A.18) along \( \gamma_0 \gamma \) and \( \gamma \gamma_0 \), using (A.10). Alternatively, compute directly with (A.19), but include a factor (A.7) for the identification of tensor products.
\textbf{§B Appendix: Functions of paths and 1-forms}

Let $N$ be a smooth manifold, possibly infinite dimensional, and $\mathcal{P}(N)$ the space of smooth parametrized paths $\ell: [0, 1] \to N$. Then $\mathcal{P}(N)$ is also a smooth manifold; a tangent to a path $\ell$ is a vector field $v(t) \in T_{\ell(t)}N$ along $\ell$. A function $g: \mathcal{P}(N) \to \mathbb{R}$ is invariant under reparametrization if $g(\tilde{\ell}) = g(\ell)$, where $\tilde{\ell}(t) = \ell(s(t))$ for some diffeomorphism $s: [0, 1] \to [0, 1]$ which fixes the endpoints. Suppose $g$ is such a function, $\ell$ is any path, and $t_i$ are chosen with $0 = t_0 \leq t_1 \leq \cdots \leq t_M = 1$. Let $\ell_i = \ell|_{[t_{i-1}, t_i]}$. Then we call $g$ additive if

$$g(\ell) = \sum_{i=1}^{M} g(\ell_i),$$

where $g(\ell_i)$ makes sense as $g$ is invariant under reparametrization. A point path is a constant function $\ell: [0, 1] \to N$.

**Proposition B.1.** Suppose $g: \mathcal{P}(N) \to \mathbb{R}$ is a smooth function such that: (i) $g$ vanishes on point paths; (ii) $g$ is invariant under reparametrization; and (iii) $g$ is additive. Then there is a smooth 1-form $\theta$ on $N$ such that

$$g(\ell) = \int_{\ell} \theta. \quad (B.2)$$

**Proof.** Let $\pi: T^*N \to N$ be the projection, and define the linear inclusion

$$i: T^*N \to T\mathcal{P}(N)$$

$$v \mapsto (t \mapsto tv),$$

where $t \mapsto tv$ is a tangent vector to the constant path at $\pi(v)$. Now set

$$\theta(v) = dg(i(v)). \quad (B.3)$$

Since $g$ is smooth, so is $\theta$; and $\theta$ is linear since $i$ and $dg$ are. Hence $\theta$ is a smooth 1-form on $N$. Fix a path $\ell$ and a large integer $M$. Set $\ell_i = \ell|_{[t_{i-1}, t_i]}$. Then by additivity,

$$g(\ell) = \sum_{i=1}^{M} g(\ell_i). \quad (B.4)$$

Reparametrize $\ell_i$ by

$$\ell_i(t) = \ell\left(\frac{t+i-1}{M}\right), \quad 0 \leq t \leq 1,$$

and set

$$\ell'_i(t) = \ell\left(\frac{t+i-1}{M}\right), \quad 0 \leq t \leq 1.$$

Then $\epsilon \mapsto \ell', 0 \leq \epsilon \leq 1$, is a smooth path in $\mathcal{P}(N)$ from the point path at $\ell(\frac{i-1}{M})$ to $\ell_i$. The tangent to this path at $\epsilon = 0$ is $t \mapsto \frac{1}{M} \ell'(\frac{i-1}{M})$. By Taylor's theorem and the definition (B.3) of $\theta$,

$$g(\ell_i) = \left. \frac{1}{M} \theta \left(\ell'(\frac{i-1}{M})\right) \right) + O\left(\frac{1}{M^2}\right). \quad (B.5)$$

Finally, combine (B.4) and (B.5) and take $M \to \infty$ to obtain (B.2).
**Corollary B.6.** Let $L \to M$ be a smooth hermitian line bundle. Suppose that for each path $\ell \in \mathcal{P}(M)$ there is given an isometry

\[(B.7) \quad \text{PT}(\ell) : L_{\bar{\ell}(0)} \to L_{\bar{\ell}(1)}\]

depending smoothly on $\ell$ such that: (i) PT is the identity map on point paths; (ii) PT is invariant under reparametrization; and (iii) PT is additive. Then there is a unitary connection on $L$ for which PT is the parallel transport.

**Proof.** Let $N \subset L$ be the vectors of unit norm. Then $N \to M$ is a circle bundle. If $\ell \in \mathcal{P}(N)$ is a smooth path in $N$, then its projection $\bar{\ell}$ is a smooth path in $M$. Using the parallel transport (B.7) we find an element $\text{PT}(\bar{\ell})\ell(0) \in L_{\bar{\ell}(1)}$ of unit norm. Hence it differs from $\ell(1)$ by an element $h(\ell) \in \mathbb{C}$ of unit norm:

\[(B.8) \quad \ell(1) = \text{PT}(\bar{\ell})\ell(0) \cdot h(\ell),\]

where the multiplication $\cdot$ occurs in the fiber $L_{\bar{\ell}(1)}$. In other words, $h$ determines a smooth function

$$h : \mathcal{P}(N) \to \mathbb{T}$$

into the unit complex numbers. The hypotheses imply: (i) $h \equiv 1$ on point paths; (ii) $h$ is invariant under reparametrization; and (iii) $h$ is additive. In addition, we claim that $h$ has a logarithm; that is, there exists a function $g : \mathcal{P}(N) \to i\mathbb{R}$ with $h(\ell) = e^{g(\ell)}$. This is because $\mathcal{P}(N)$ retracts onto the space of constant paths and $h \equiv 1$ on constant paths. Therefore, by Proposition B.1 there exists a smooth (imaginary) 1-form $\theta$ on $N$ such that

$$h(\ell) = e^{\int_{\ell} \theta}.$$

From (B.8) it is obvious that the restriction of $\theta$ to a fiber of the circle bundle $N \to M$ is the standard Maurer-Cartan form on the circle. Similarly, from (B.8) we see that $h$ is invariant under the circle action on $N$, whence $\theta$ is also. Therefore, $\theta$ is a connection on $N$ and (B.7) is the associated parallel transport.
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58