Quantifying galaxy shapes: Sérsiclets and beyond

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ABSTRACT
Parametrisation of galaxy morphologies is a challenging task, for instances in shear measurements of weak gravitational lensing or investigations of formation and evolution of galaxies. The huge variety of different morphologies requires a parametrisation scheme that is highly flexible and that accounts for certain morphological observables, such as ellipticity, steepness of the radial light profile and azimuthal structure. In this article, we revisit the method of sérisclets, where galaxy morphologies are decomposed into a set of polar basis functions that are based on the Sérsic profile. This approach is justified by the fact that the Sérsic profile is the first-order Taylor expansion of any real light profile. We show that sérisclets indeed overcome the modelling failures of shapelets in the case of early-type galaxies. However, sérisclets implicate an unphysical relation between the steepness of the light profile and the spatial scale of the polynomial oscillations, which is not necessarily obeyed by real galaxy morphologies and can therefore give rise to modelling failures. Moreover, we demonstrate that sérisclets are prone to undersampling, which restricts sérisclet modelling to highly resolved galaxy images. Analysing data from the weak-lensing Great08 challenge, we demonstrate that sérisclets should not be used in weak-lensing studies. We conclude that although the sérisclet approach appears very promising at first glance, it suffers from conceptual and practical problems that severely limit its usefulness. In particular, sérisclets do not provide high precision results in weak-lensing studies. Finally, we show that the Sérsic profile can be enhanced by higher-order terms in the Taylor expansion, which can drastically improve model reconstructions of galaxy images. When orthonormalised, these higher-order profiles can overcome the problems of sérisclets while preserving their mathematical justification. However, this method is computationally expensive.

Key words: Galaxies: general – Methods: data analysis, statistical – Techniques: image processing – Gravitational lensing: weak.

1 INTRODUCTION
There are two scientific key diagnostics in modern investigations of galaxy formation and evolution, namely star-formation rates and galaxy morphologies. Galaxy morphologies are known to correlate to different degrees with other more physical parameters such as star formation (e.g.,Kennicutt[1998]) or environment (e.g., van der Wel et al.[2010]). Although morphology by itself is not a fundamental parameter of a galaxy, it provides a direct observable. Conversely, e.g., star-formation rates are derived quantities based on additional assumptions and extrapolations (e.g., Rosa-Gonzalez et al.[2002]). Studies of galaxy morphologies are therefore an important complementary means for studying the physics of galaxy formation. Furthermore, certain investigations are solely based on morphologies, e.g., weak-lensing measurements (e.g., Bernstein & Jarvis[2002]) or studies of angular-momentum alignments of spiral galaxies (e.g., Slosar et al.[2009]; Andrae & Jahnke[prep.]). Concerning weak gravitational lensing, the investigation of shape-measurement algorithms currently is a very active and important field of research since established and well known procedures fail to measure object shapes accurately enough to fully exploit the potential of future large-scale imaging surveys (e.g., Bridle et al.[2010]; Bernstein[2010]). Concerning morphological classification, galaxies were traditionally described mostly by visual classifications. However, given the enormous amount of galaxies known from today’s sky surveys, it is evident that an automated parametrisation of galaxy morphologies is required. Consequently, there has
been substantial effort to define automated parametrisation schemes for galaxy morphologies (e.g., Sér&egrave;e 1968; Abraham et al. 1996, 2003; Bershady et al. 2000; Conselice 2003; Lotz et al. 2004, 2008) to name just a few. Unfortunately, these parametrisation schemes usually invoke rather restrictive assumptions, such that they lack the flexibility to describe the huge variety of different galaxy morphologies present in modern databases (Andrae et al. 2010a). The Sér&egrave; profile is an example in this respect: The Sér&egrave; model is the first-order Taylor expansion of any real radial light profile (cf. Appendix A1) and therefore provides a good first-order approximation to real galaxies. However, second-order effects &mdash; e.g., deviations of the radial profile from the Sér&egrave; form or the appearance of azimuthal structures such as star-forming regions, galactic bars or spiral-arm patterns &mdash; are typically not negligible, and hence there have been several attempts to enhance the Sér&egrave; profile:

- Mixtures of two or more Sér&egrave; profiles (e.g., Simmat et al. 2010).
- Modification of the Sér&egrave; profile by Fourier modes in order to describe azimuthal structures (e.g., GALFIT 3. Peng et al. 2010).
- Orthogonalisation of the Sér&egrave; profile in order to describe azimuthal structures (Ngan et al. 2009).

The idea of using basis functions in order to parametrise galaxy morphologies is not new. For instances, “shapelets” (Réfrégier 2003) are a set of basis functions based on the Gaussian profile. The appealing properties of shapelets gave rise to a number of investigations of galaxy morphologies (e.g., Kelly & McKay 2004, 2005; Andrae et al. 2010b) and weak-lensing studies (e.g., Massey et al. 2007; Ferry et al. 2008). However, as was recently shown by Melchior et al. (2010), shapelets can suffer from strong biases that originate from the radial light profiles of galaxies potentially being much steeper than a Gaussian profile, in particular for early-type galaxies. These biases manifest themselves as ring-like artefacts in the models and the residual maps (cf. Sect. 3.2). They are able to wash out virtually all the information about ellipticity of an object and are likely to affect other morphological quantities in a similar way. Hence, it was obvious to introduce a set of basis functions based on the Sér&egrave; profile (Ngan et al. 2009), called “sér&egrave;cilts”. As these basis functions explicitly account for the steepness of radial profiles, they are highly flexible and seem to be promising candidates to provide a method for describing the huge variety of different galaxy morphologies.

This article is organised as follows. We define sér&egrave;cilts before we then give our definition.

2 POLAR SÉRISICLETS

We briefly comment on a prior attempt to introduce sér&egrave;cilts before we then give our definition.

2.1 The first attempt to introduce sér&egrave;cilts

Given the aforementioned problems of shapelets, it was obvious to orthonormalise the Sér&egrave; profile. Moreover, this choice is “natural” because the Sér&egrave; profile is the first-order Taylor expansion of any real light profile (cf. Appendix A1). This is likely to remove the strong biases of shapelets. The resulting basis functions are called sér&egrave;cilts and were first investigated by Ngan et al. (2009).

However, Ngan et al. (2009) faced a severe problem with sér&egrave;cilts. In a simple test case, Ngan et al. (2009) observed that the polar sér&egrave;cilts were incapable of fitting a given object. Even when increasing the number of basis functions used for the decomposition, the model did not converge to the given image data. This directly implies that the basis functions constructed by Ngan et al. (2009) are not complete. We found out that this problem originates from the fact that Ngan et al. (2009) orthonormalised their basis functions out to one half-light radius &mdash; not on the interval $r \in [0, \infty]$ &mdash; but then extended their basis functions beyond this range. Figure 1 shows what happens in this case. In the inner region, the basis functions may be orthonormal, but for larger radii the leading order monomial in the basis functions dominates over the Sér&egrave; profile. This creates an artificial bump outside the region of orthonormality, which appears in all radial modes of order $l > 0$ and becomes the dominant feature. Consequently, these basis functions loose their linear independence, which explains the non-convergence observed by Ngan et al. (2009).

In order to solve this problem, they suggested to discard all modes with azimuthal “quantum number” $m \neq 0$, i.e., all mode with azimuthal structure, and only maintain the radial modes. While this approach may be viable for weak-lensing applications, where azimuthal structures are rarely visible, it does not solve the actual problem, which is the extension...
beyond the orthonormalisation interval. Moreover, the resulting basis functions may still represent an expansion into radial modes, but it is not only radial but also azimuthal structure of galaxies that is interesting, e.g., in morphological classification. In fact, it is the ability to describe azimuthal structure in a (theoretically) well defined way that makes basis-function expansions of galaxy morphologies such a compelling approach. Consequently, Jiménez-Teja & Benítez (2011) criticised that this implementation of s´ersiclets is incapable of describing irregular galaxies.

2.2 Definition of polar s´ersiclets

We now define the s´ersiclet basis functions. In the case of shapelets, we can define both Cartesian and polar shapelets due to the mathematical features of the Gaussian weight function. However, for general S´ersic profiles, we cannot meaningfully define Cartesian basis functions because the ground states hardly resemble any known galaxy morphology, as is shown in Fig. 2. Therefore, we have to introduce s´ersiclets as polar basis functions.

The crucial idea to overcome the problems faced by Ngan et al. (2009) is to realise that we cannot expect a galaxy’s structure being well captured within one half-light radius. For instances, disc galaxies may exhibit spiral-arm patterns in their outskirts. Therefore, we require orthonormality on the full interval $r \in [0, \infty]$. Apart from solving the problems reported by Ngan et al. (2009), this also has the advantage that the orthonormal polynomials exist analytically. The s´ersiclet basis functions are derived in Appendix B and are given by

$$R_i(r) \propto L_i^{2n_S-1} \left[ b (r/\beta)^{1/n_S} \right] \exp \left[ -\frac{b}{2} (r/\beta)^{1/n_S} \right],$$  

where the normalisation factor is also derived in Appendix B. Apart from the linear expansion coefficients, s´ersiclets have the following nonlinear model parameters:

- The maximum order $N_{\text{max}}$, 
- The S´ersic index $n_S$, 
- The scale radius $\beta$, 
- The centroid position $\vec{x}_0$, which enters via $r = |\vec{x} - \vec{x}_0|$, 
- The complex ellipticity $\epsilon$, which enters via a shear transformation (e.g. Bartelmann & Schneider 2001) thereby affecting $\beta$.

The coefficient $b$ is defined by (Graham & Driver 2005)

$$\Gamma(2n_S) = 2\gamma(2n_S, b),$$  

where $\Gamma$ and $\gamma$ denote the complete and incomplete gamma functions. We explain in Appendix B how to fit these model parameters for given imaging data. In the case of $n_S = 0.5$ and $b = 1$ the polar s´ersiclets reduce to the special case of polar shapelets. Furthermore, for $n_S = 1$ and $b = 1$, we get a set of basis function that could be called “discllets”, since they have an exponential profile as weight function. Similarly, we could define “devaucouleurlets”. All these sets are special cases of Eq. (1). Finally, we emphasise that when fitting a s´ersiclet model, we always also fit a S´ersic model, which is the ground state of the s´ersiclets. Figure 3 displays an example of s´ersiclet basis functions.

2.3 Interpretation of the S´ersic index

In the case of s´ersiclet basis functions, the S´ersic index $n_S$ changes its interpretation. First, $n_S$ regulates the steepness of the weight function, which is a normal S´ersic profile. Second, via the steepness of the weight function, $n_S$ also regulates the spatial scale on which the associated Laguerre polynomials oscillate. In simple words, there is a fixed relation between steepness of the weight function and oscillation scale of the polynomials. However, real morphologies do not necessarily obey such a relation. For instances, the steepness of the bulge is not related to the positions of spiral arms or star-forming regions. In practice, this can lead to modelling problems, if the steepness of the profile and the range of oscillation scales required for a faithful description of a given galaxy morphology do not match.

3 TESTING S´ERSICLETS

Before we can apply s´ersiclets to scientific questions, we need to investigate the performance and fidelity of s´ersiclets.

3.1 Completeness

First, we need to test the completeness of s´ersiclets in order to proof that we indeed overcome the problems reported by Ngan et al. (2009). We decompose three real galaxy images from the Sloan Digital Sky Survey into s´ersiclets with increasing maximum order. If our basis functions were not linearly independent, completeness would break down and the $\chi^2$-values would not decrease with increasing maximum order as in the case of Ngan et al. (2009). Figure 4 shows the results. First and foremost, the residuals of s´ersiclets are decreasing with increasing maximum order, i.e., we indeed set up a set of basis functions that are linearly independent. Furthermore, it is interesting to compare the residuals of s´ersiclets with those of circular shapelets. In the angle and axis ratio. Using the orientation angle as parameter would cause severe problems with convergence in the case of nearly spherical objects.

Note that the polar functions defined by Massey & Réfrégier (2005) are not orthogonal. A correct functional form of polar shapelets is derived in Appendix B.
case of the spiral galaxy, sérisclets yield residuals comparable to shapelets. This is not surprising, because shapelets excel in modelling extended objects with lots of substructure, such as a face-on spiral galaxy. However, in the case of the edge-on disc and the elliptical galaxy, the $\chi^2$-values of sérisclets are substantially lower. Evidently, sérisclets outperform shapelets in modelling galaxies of these types, as expected from the steep light profiles.

### 3.2 Image decompositions

We have seen in the previous section that sérisclet decompositions produce substantially lower residuals than (circular) shapelet decompositions when it comes to modelling galaxies that exhibit steep profiles, such as elliptical galaxies or edge-on discs. However, we still have to check whether sérisclets indeed overcome the ring-like artefacts produced by shapelets.

Figure 5 compares the best-fitting models and residuals of (circular) shapelets and (elliptical) sérisclets using $N_{\text{max}} = 8$ for the three test galaxies. As expected from the similar $\chi^2$-values, in the case of the spiral galaxy, both shapelets and sérisclets perform well in modelling the spiral-arm patterns. Sérisclets fit the central region better than shapelets, whereas shapelets tend to describe the outskirts better. As expected from a sérisclet model with $n_S > 0.5$, the polynomial oscillations appear on smaller scales than for the shapelet model with $n_S = 0.5$. In the case of the edge-on disc, sérisclets are clearly superior. First, the ring-like artefacts of shapelets are gone. Second, the intrinsic ellipticity of the sérisclet model allows the basis functions to also describe the outermost regions of the disc, whereas these regions go almost unfitted by circular shapelets. In the case of the elliptical galaxy, the ring-like artefacts are very prominent in the shapelet residuals. Conversely, the sérisclet residuals do not exhibit any artefacts of this kind, i.e., the steepness of the light profile is indeed described properly.

### 3.3 Orthonormality and sampling

Let $\Sigma$ denote the pixel covariance matrix of the noise in the imaging data and $X$ the design matrix (see Appendix C). In order to get the maximum-likelihood estimate of the ex-
Figure 5. Models and residual maps resulting from Sérsiclets and shapelets for the spiral galaxy (top row), the edge-on disc (central row), and the elliptical galaxy (bottom row). All models used $N_{\text{max}} = 8$. For the sake of visualisation, the colour code of the model and data maps is nonlinear and ranges from $-4.5\sigma$ to the maximum value. The colour code of the residual maps is linear and ranges from $-4.5\sigma$ to $+4.5\sigma$, where $\sigma$ denotes the standard deviation of the background noise. The peak significance of the imaging data is $132\sigma$ for the spiral galaxy, $101\sigma$ for the edge-on disc, and $610\sigma$ for the elliptical galaxy.

Pansion coefficients from Eq. (C3), the matrix $X^T \cdot \Sigma^{-1} \cdot X$ or $X^T \cdot X$ for uncorrelated pixel noise – needs to be invertible. If it were not for pixellation, finite image grids, and the PSF, $X^T \cdot X$ should be an identity matrix due to the orthonormality of the basis functions. We now investigate the orthonormality of Sérsiclets in two tests.

First, we compare the orthonormality of polar shapelets (Sérsiclets with $n_S = 0.5$ and $b = 1$) and Cartesian shapelets of Melchior et al. (2007). We sample polar and Cartesian shapelets of constant scale radius $\beta = 5$ on pixel grids of sizes $50 \times 50$, $100 \times 100$, and $500 \times 500$ and compute the determinant of $X^T \cdot X$ for maximum orders $0 \leq N_{\text{max}} \leq 16$, where $\det(X^T \cdot X) < 1$ indicates a violation of orthonormality. Figure 6 shows the results of this test. In the case of the $50 \times 50$ grid (panel (a)) both Cartesian and polar shapelets suffer from non-orthonormality for increasing $N_{\text{max}}$. This problem can be caused either by boundary truncation or undersampling of higher-order modes (with rapidly oscillating polynomials). Therefore, in panel (b) we increase the pixel grid from $50 \times 50$ to $100 \times 100$. The non-orthonormality of Cartesian shapelets is now cured, i.e., it has indeed been caused by boundary truncation. However, polar shapelets still exhibit non-orthonormality. Therefore, in panel (c) we increase the pixel grid again now from $100 \times 100$ to $500 \times 500$. The behaviour of polar shapelets is unchanged, i.e., the non-orthonormality is not caused by boundary truncation. In order to demonstrate that the non-orthonormality is caused by undersampling, we increase in panel (d) the object size from $\beta = 5$ to $\beta = 25$ while keeping the $500 \times 500$ pixel grid. The non-orthonormality of polar shapelets is now almost cured, which verifies that this was an undersampling effect. Evidently, undersampling has a larger impact on polar shapelets than on Cartesian shapelets. The reason is that polar basis functions exhibit most of their structure in the central region (cf. Fig. 3) and therefore suffer strongly from pixellation. Loosely speaking, polar basis functions do not appreciate being sampled on a Cartesian pixel grid.

Second, having attested problems of polar shapelets with orthonormality due to undersampling, we now investigate the orthonormality of (polar) Sérsiclets in general. We

3 In particular, for polar shapelets and $N_{\text{max}} \geq 8$ the determinant of $X^T \cdot X$ is zero, i.e., $X^T \cdot X$ is not invertible and the estimator of the expansion coefficients (Eq. (C3)) breaks down.
Figure 7. Smallest (solid line) and largest (dashed line) diagonal elements of matrix $X^T \cdot X$ as a function of $\beta$ at $N_{\text{max}} = 12$. Deviation from unity (dotted line) indicates the loss of orthonormality. The image size was $100 \times 100$ pixels (left columns) and $1000 \times 1000$ pixels (right column). We used $b = 2 \log 3$, such that at one scale radius the weight function drops to one third of its central value.

Compare the value of the largest and smallest diagonal element of $X^T \cdot X$, respectively, repeating the same test as Berry et al. (2004) did for shapelets. We evaluate sérsiclet models of maximum order $N_{\text{max}} = 12$, Sérsic indices $n_S = 0.5, 1, 2, 4$ and axis ratios $q = 1, 0.8, 0.6, 0.4$ on a $100 \times 100$ pixel grid for varying scale radii $\beta$. There is no PSF in this test. Figure 7 shows test results. For the moment, we only consider the first row in Fig. 7, which corresponds to polar shapelets. The results agree with Fig. 6, revealing undersampling effects for small $\beta$ and boundary truncation for large $\beta$ (cf. Melchior et al. 2007, and discussion therein). Furthermore, we can see that the axis ratio has only a mild impact on the orthonormality. However, inspecting the other rows corresponding to steeper profiles ($n_S = 1, 2, 4$), Fig. 7 reveals serious violations of orthonormality. Obviously, for $n_S \geq 1$ the undersampling regime is not overcome before the boundary effects set in. This is confirmed by the right-most column in Fig. 7 which used an enlarged pixel grid while keeping the resolution (scale radius $\beta$) constant. In this case, sérsiclets with $n_S = 1$ exhibit decent orthonormality on this larger grid, while the cases with $n_S > 1$ still to not reach an acceptable level of orthonormality. The reason for this peculiar behaviour is the argument $b(\tau/\beta)^{1/n_S}$ of the associated Laguerre polynomials in Eq. (1). It implies that the polynomials are only slowly varying with $\tau$ for large values of $n_S$. Consequently, polar sérsiclets have a serious problem with orthonormality, especially for large Sérsic indices.

What is the impact of undersampling? If $\det(X^T \cdot X)$ is too close to zero, numerical inaccuracies will dominate its inversion and the estimate of the expansion coefficients (Eq. (C3)) will catch up random errors. Consequently, this increases the uncertainty in the model parameters and may even lead to completely random parameter values.

The impact of undersampling could be alleviated by sampling the model several times within each pixel. In the limit of infinitely fine sampling this amounts to a convolution of the model with the pixel-response function. As the undersampling problem stems from a key feature of the employed sérsiclet model, namely rapid polynomial oscillations, an oversampled model may better approximate the image data. However, the information about the galaxy morphology contained in the expansion coefficients has been lost and is not restored by oversampling. Instead, excessive oversampling generates artificial information that compromises the model parameters. As we require meaningful information from the expansion coefficients, we will not further pursue
this approach. If one wants to employ this approach, one has to bear in mind, that the additional convolution will lead to covariances among the coefficients and therefore render the fitting procedure more complicated.

### 3.4 Analogy to Nyquist frequency

In order to avoid the undersampling problem in practice, e.g., in morphological classification, we want to derive a lower limit to the scale radius \( \beta \) of a Sérsiclet model. This limit is given by comparing the pixel size to the scale on which the radial components of Sérsiclets – essentially the Laguerre polynomials – vary. In fact, this is an analogy to the Nyquist frequency in the case of Fourier transform.

The key to set this lower limit is to identify the shortest scale on which a Laguerre polynomial varies. This scale can be inferred from the roots of the Laguerre polynomial.

An associated Laguerre polynomial \( L_k^l(x) \) of order \( l \) real-valued, positive roots within the interval \( (0, l + k + (l - 1)/\sqrt{l + k}) \). The smallest scale that can be resolved by this polynomial is given by the distance between \( x = 0 \) (the peak of the radial component) and the first root \( x_1 > 0 \). Unfortunately, the roots of associated Laguerre polynomials are not known analytically, so the first root \( x_1 \) has to be inferred numerically. As the argument of the associated Laguerre polynomial is \( x = b(r/\beta)^{1/n_S} \), given \( x_1 \), we can infer the radius of the first root to be \( r_1 = \beta (x_1/b)^{n_S} \). Hence, the distance between \( r = 0 \) and the first root is given by

\[
\Delta r = r_1 - r_0 = r_1 = \beta (x_1/b)^{n_S}. \tag{3}
\]

This scale \( \Delta r \) should be well resolved by the pixel grid in order to avoid undersampling. An optimistic lower limit is \( \Delta r = 1 \), i.e., the radial component drops from its central peak to its first root within a single pixel. Setting \( \Delta r = 1 \) and solving for the minimal \( \beta \), we obtain,

\[
\beta \geq \beta_{\text{min}} = (b/x_1)^{n_S}, \tag{4}
\]

where \( \beta \) and \( \beta_{\text{min}} \) are given in units of pixels. Table 1 provides values of \( \beta_{\text{min}} \) for some realistic values of \( n_S \) and maximum polynomial order \( N_{\text{max}} = l \). Considering this table, we also have to keep in mind that the image grid has to be large enough such that several scale radii fit into it. Evidently, as \( n_S \) increases, the limit becomes larger and larger. This can also be inferred qualitatively from Fig. 7, but Eq. (4) provides a quantitative result. Obviously, for \( n_S \geq 2 \), the lower limit becomes extraordinarily large. In particular, \( n_S = 4 \) would require an extremely well resolved object with radius of several thousand pixels.

In the case of shapelets, it is standard practice to choose a maximum order \( N_{\text{max}} \) for a decomposition according to signal-to-noise ratio and resolution. For Sérsiclets, the resolution constraint plays a much more important role and restricts models with large \( n_s \) to low \( N_{\text{max}} \), with which com-

| \( N_{\text{max}} \) | \( n_S = 0.5 \) | \( n_S = 1.0 \) | \( n_S = 2.0 \) | \( n_S = 3.0 \) | \( n_S = 4.0 \) |
|-------------|-------------|-------------|-------------|-------------|-------------|
| 1           | 0.8165      | 0.8333      | 0.8403      | 0.8424      | 0.8435      |
| 2           | 1.0668      | 1.3145      | 1.7599      | 2.0242      | 2.0658      |
| 3           | 0.3256      | 1.7810      | 2.9324      | 4.3311      | 6.0286      |
| 4           | 1.4377      | 2.4223      | 4.9625      | 7.3782      | 11.537      |
| 5           | 1.5904      | 2.7011      | 6.0513      | 11.499      | 19.888      |
| 6           | 1.7296      | 3.1586      | 7.9993      | 16.848      | 31.873      |
| 7           | 1.8584      | 3.6151      | 10.207      | 23.577      | 48.375      |
| 8           | 1.9787      | 4.0712      | 12.674      | 31.841      | 70.372      |
| 9           | 2.0921      | 4.5268      | 15.400      | 41.792      | 98.931      |
| 10          | 2.1996      | 4.9821      | 18.386      | 53.584      | 135.22      |
| 11          | 2.3021      | 5.4373      | 21.632      | 67.371      | 180.49      |
| 12          | 2.4002      | 5.8922      | 25.137      | 83.305      | 236.06      |
| 13          | 2.4944      | 6.3471      | 28.902      | 101.54      | 303.46      |
| 14          | 2.5853      | 6.8018      | 32.296      | 122.23      | 384.13      |
| 15          | 2.6730      | 7.2565      | 37.210      | 143.55      | 479.75      |
| 16          | 2.7579      | 7.7110      | 41.754      | 171.59      | 592.02      |
| 17          | 2.8403      | 8.1656      | 46.557      | 200.57      | 722.76      |
| 18          | 2.9204      | 8.6201      | 51.620      | 232.61      | 873.87      |
| 19          | 2.9983      | 9.0745      | 56.943      | 267.88      | 1047.4      |
| 20          | 3.0742      | 9.5289      | 62.525      | 306.52      | 1245.3      |

Table 1. Lower limits of scale radii \( \beta \) in units of pixels for different maximum radial orders \( N_{\text{max}} \) according to Eq. (4). Here we used \( b = 2n_S - 1/3 \).

Figure 6. Determinant of \( X^T \cdot X \) for Cartesian (dashed lines with squares) and polar (solid lines with dots) shapelets for increasing maximum order \( N_{\text{max}} \), different scale radii \( \beta \) and grid sizes. Panel a: At large \( N_{\text{max}} \) both Cartesian and polar shapelets deviate from \( \det(X^T \cdot X) = 1 \). Panel b: Cartesian shapelets maintain \( \det(X^T \cdot X) = 1 \) for all \( N_{\text{max}} \), proving that boundary effects are now negligible. Panel c: Polar shapelets still differ from \( \det(X^T \cdot X) = 1 \) at large \( N_{\text{max}} \) on this enlarged grid, i.e., this is not a boundary effect. Panel d: Polar shapelets now also maintain \( \det(X^T \cdot X) = 1 \), i.e., the remaining effect was due to undersampling.
plicated morphological features could receive improper description. As elliptical galaxies rarely exhibit such features though, this restriction may not be overly problematic.

4 APPLICATION TO WEAK-LENSING DATA

Given the fact that shapelets were also intended for shear measurements in weak-gravitational lensing (e.g., Refrégier & Bacon 2003; Chang et al. 2004), we now study the potential of sersiclets in this field of research. From the simulated GREAT08 data (Bridle et al. 2009) we selected ten images with artificial galaxies, whose shear values are given in Table 2. Every set contains 10,000 objects sampled on a 40×40 pixel grid. Within such a set the applied shear is constant and the aim is to retrieve its value. Furthermore, all objects have been convolved with a PSF that is a Moffat profile with FWHM= 2.85, β = 3.5, and intrinsic ellipticity ε = 0.019 – 0.007i (cf. Bridle et al. 2010).

We decompose all objects into sersiclets with maximum orders Nmax = 0, 2, 4, 6, taking into account the PSF by forward modelling. For each set, the shear is estimated via the mean ellipticity $\epsilon$ from the 10,000 artificial galaxies via the ellipticity parameter of the sersiclet model, which theoretically provides an unbiased estimator of the gravitational shear (cf. Bartelmann & Schneider 2001). Figure 8 shows the test results as input vs. estimated shear. The goodness of the shear estimates is parametrised by a straight-line model

$$f(x) = a + bx.$$  (5)

A perfect shear estimator yields $a = 0$ and $b = 1$. If a real shear estimator yields an offset $a \neq 0$, i.e., a shear is detected although the input shear was zero, this typically implies that the PSF is not properly corrected for. If $b < 1$, the true shears are underestimated.

Given the test results shown in Fig. 8 the sersiclets perform best for $N_{\text{max}} = 0$, where the slope $b$ is consistent with 1 (at a fairly large offset $a$). This is not surprising, because for $N_{\text{max}} = 0$ sersiclets reduce to pure Sersic profiles, which were used to simulate this subset of the GREAT08 data.

However, the real lesson from Fig. 8 is that for $N_{\text{max}} > 0$ the slopes are significantly below 1, i.e., the shear is significantly underestimated. The higher-order sersiclet modes, which should account for any deviation the actual galaxy shape exhibits with respect to the axisymmetric Sersic profile, do more harm than good as they become increasingly prone to undersampling. Low resolution is a generic feature of weak-lensing data, i.e., shear estimates based on sersiclet decompositions always suffer from substantial undersampling effects. Oversampling of the model within each pixel could cure this, but is – at least in our implementation – computationally infeasible. Finally, we note that we observed a bias in the shear estimates based on Sersiclets. The first component of the shear is systematically underestimated whereas the second component is systematically underestimated. We do not yet understand the precise origin of this bias but we may speculate that this is a pixellation effect (cf. Sect. 5.3 of Massey et al. 2007).

5 OUTLOOK: ORTHONORMALISING HIGHER-ORDER TAYLOR EXPANSIONS

As discussed in Appendix A, the Sérsic profile is the first-order Taylor expansion of any light profile. This naturally leads us to the expectation that with improving imaging quality the Sérsic profile will not be a good match anymore, because it is “only” a first-order expansion. Consequently, an obvious strategy to enhance the Sérsic profile is to allow for higher orders in the Taylor expansion of Eq. (A2). Such higher-order radial profiles then can also be orthonormalised in order to describe azimuthal structures. This approach appears to be very promising for investigations of galaxy morphologies, e.g., in the context of classification. However, it
disc type, i.e. having Sérsic index of $n_S = 1$ or $n_S = 4$, respectively.

| Set   | $g_1$         | $g_2$         |
|-------|---------------|---------------|
| 0007  | 0.0005405     | 0.0069236     |
| 0026  | -0.0527875    | -0.0090224    |
| 0035  | 0.0160867     | -0.0045223    |
| 0048  | 0.0708862     | -0.0377040    |
| 0056  | 0.0325078     | 0.0978346     |
| 0091  | -0.0246346    | -0.0488837    |
| 0126  | 0.0170977     | -0.1383142    |
| 0135  | 0.0596913     | 0.0416342     |
| 0268  | -0.0653126    | -0.0883511    |
| 0281  | -0.0431769    | 0.0462176     |

Table 2. Data sets chosen from the GREAT08 sample and their complex shears $g = g_1 + ig_2$. 

Figure 8. Offset $a$ (top) and deviation from unity slope $b − 1$ (bottom) of GREAT08 data sets given in Table 2 for different maximum orders, $N_{\text{max}}$, of sersiclet decomposition. Horizontal dashed lines indicate the ideal case of $a = 0$ and $b = 1$. Errors of $a$ and $b$ were estimated by bootstrap fitting of $f(x) = a + bx$ to the estimated and input shear values.
5.1 Third-order profiles

As explained in Appendix A2, the next useful higher-order expansion which exhibits the correct boundary behaviour beyond the Sérsic profile is a third-order expansion,

\[ p(r) \approx \exp \left[ -e^{A+B \log(r/\beta) + C \log^2(r/\beta) + D \log^3(r/\beta)} \right] , \tag{6} \]

where \( D > 0 \) and \( A, B, \) and \( C \) are arbitrary. Figure 9 shows an example of such a third-order profile in comparison to a normal Sérsic profile. Evidently, the third-order profile can overcome two essential problems of pure Sérsic profiles, since it (a) can exhibit a central cusp, which is also observed in real galaxies, and, (b) approaches zero for increasing radii faster than the pure Sérsic profile. Therefore, we may speculate that such higher-order Taylor expansions could provide a reasonable generalisation, if deviations from the normal Sérsic profile are observed while azimuthal structures are still absent. For instance, this may help to describe the light profiles of elliptical galaxies or unbarred S0 galaxies.

In Fig. 10 we compare the performances of the third-order profile and the Sérsic profile by fitting an elliptical galaxy from the SDSS database. Clearly, the residual map of the third-order profile is almost perfectly random noise whereas the residual map of the Sérsic profile reveals systematic mismodelling. Correspondingly, the ratio of \( \chi^2 \)-values of the third-order profile over Sérsic fit is \( \chi^2_3/\chi^2_S \approx 0.48 \). In fact, the third-order profile fits the data so well that we should ensure that it is not an overfit. For this purpose, Fig. 11 displays the distributions of normalised residuals for both models in comparison to the unit Gaussian (e.g. see Andrae et al. 2010c). Evidently, the normalised residuals of the Sérsic profile have a broader distribution than the unit Gaussian which is indicative of underfitting the data. The normalised residuals of the third-order profile are closer to the unit Gaussian, i.e., they are closer to the truth. However, their distribution does not peak sharper than the unit Gaussian, i.e., the third-order profile does not overfit the data.

5.2 Numerical orthonormalisation

Third-order profiles such as Eq. (6) can be orthonormalised, too. Unfortunately, it is not possible to do this orthonormalisation analytically. Therefore, the orthonormalisation has to be performed numerically. We start with the radial monomials \( (r^0, r^1, r^2, r^3, \ldots) \) which are linearly independent but not orthonormal. Using the definition of the scalar product according to Eq. (31) and Dirac notation, we first normalise the ground state,

\[ |0\rangle = \frac{|r^0\rangle}{\langle r^0|r^0\rangle} . \tag{7} \]

Second, we compute the first-order state

\[ |1\rangle = (\hat{r} - \langle 0|\hat{r}|0\rangle) |0\rangle , \tag{8} \]

which is then also normalised such that \( \langle 1|1\rangle = 1 \). Here, we used the following definition,

\[ \langle k|l\rangle = \int_0^{\infty} dr \, B_k(r) r B_l(r) , \tag{9} \]

where \( B_l \) denotes the radial basis function of order \( l \). The basis functions of order \( l \geq 2 \) are then computed using the three-term recurrence relation

\[ |l\rangle = (\hat{r} - (l-1)|l-1\rangle) |l-1\rangle - (l-1)\langle l-1|l-2\rangle |l-2\rangle . \tag{10} \]

This requires less numerical integrations than the brute-force Gram-Schmidt algorithm. Hence, it is faster and numerically more stable than the Gram-Schmidt algorithm.

5.3 Revisiting the problems of s´erusclets

The conceptual problem of s´erusclets was the postulated relation of steepness of the weight function and scale of polynomial oscillations that is not obeyed by real galaxies. This fixed relation is loosened by the third-order profiles. Of course, the Sérsic index – or rather \( B = \frac{1}{2n_S} \) – still has an influence on the oscillation scale of the radial polynomials. However, now there are two further model parameters \( C \) and

\[ \chi^2 = \frac{1}{\sigma^2} \int \left( f(r) - \langle f(r) \rangle \right)^2 \, dr , \]

\[ f(r) = \frac{p(r)}{\langle p(r) \rangle} , \]

\[ \langle p(r) \rangle = \int_0^{\infty} dr \, p(r) , \]

\[ (r/\beta)^{-n_S} \exp \left[ -e^{A+B \log(r/\beta) + C \log^2(r/\beta) + D \log^3(r/\beta)} \right] , \tag{6} \]

\[ p(r) \approx \exp \left[ -e^{A+B \log(r/\beta) + C \log^2(r/\beta) + D \log^3(r/\beta)} \right] , \tag{6} \]

\[ p(r) \approx \exp \left[ -e^{A+B \log(r/\beta) + C \log^2(r/\beta) + D \log^3(r/\beta)} \right] , \tag{6} \]
Figure 10. Fitting Sérsic and third-order profile to an elliptical galaxy. The panels denote residual map of Sérsic fit (a), Sérsic model (b), original data (c), best-fitting third-order profile (d), and residual map from third-order profile (e). Panels, b, c, and d have identical scaling. The residual maps (panels a and e) both use plot ranges from $-5\sigma$ to $5\sigma$.

Figure 12. Undersampling test for third-order profiles. We compare basis sets of the Sérsic (dashed lines) and third-order profile (solid lines) shown in Fig. 9. The test is performed on a 100×100 pixel grid and the weight function has no intrinsic ellipticity. In contrast to Fig. 7, we have chosen $b = 4 - 1/3$ in this case. Panel a: Determinant of coefficient covariance matrix of $X^T \cdot X$ which should be 1. Panel b: Largest and smallest diagonal elements of of $X^T \cdot X$ which should be 1. Panel c: Largest absolute value of off-diagonal elements of of $X^T \cdot X$ which should be 0.

D which also influence the polynomials. Consequently, basis functions based on third-order profiles are more flexible and do not impose such a rigid relation as sérsclets do.

The practical problem of sérsclets was undersampling, in particular for Sérsic indices $n_S \geq 1$. In fact, Fig. 6 already suggests that third-order profiles can overcome this problem because they can exhibit central cusps instead of peaks. Therefore, the weights are not as highly concentrated and the polynomials may not oscillate as rapidly. We demonstrate this by repeating the orthonormality test of Fig. 7 for the third-order profile with parameters given in Fig. 9 and its set of basis functions. Figure 12 shows the test results. In direct comparison to sérsclets with $n_S = 1/2 = 2$, the orthonormalised third-order profile indeed suffers less strongly from undersampling, which is most obvious for the determinant of $X^T \cdot X$. In particular, there is a regime $5 \leq \beta \leq 15$ where undersampling is overcome and boundary truncation has not yet set in. Such a regime does not exist for the corresponding set of sérsclets shown here. Consequently, third-order profile can indeed overcome the undersampling problem of sérsclets. Nevertheless, there are parameter choices for third-order profiles where the undersampling problem is as bad as for sérsclets, e.g., when $C = D = 0$ and the third-order profile reduces to a Sérsic profile. However, by choosing appropriate priors it might be possible to exclude such parameter values in order to avoid undersampling.

5.4 Computational feasibility

We have shown that orthonormalisations of third-order profiles can overcome the limitations of sérsclets while preserving their mathematical justification. However, we have not yet mentioned a serious practical limitation of this novel approach: The numerical orthonormalisation of third-order profiles is computationally highly expensive. Fitting a galaxy on a, say, 100×100 pixel grid while freely adjusting all model parameters is infeasible on a standard computer. A problematic work-around could be to first fit a simple third-order profile to the galaxy and then only orthonormalise the best-fit profile in order to model the data’s deviation from the mean profile. However, this step-wise approach by construction is very unlikely to find the best-fitting basis-function expansion because the first step produces a biased fit and not a mean profile. Therefore, we do not recommend this approach. Another work-around would be to set up a library of basis functions for all realistic parameter values of third-order profiles, such that numerical orthonormalisation has not to be conducted on-the-fly during the fit. This approach is hampered by the fact that third-order profiles have three free parameters such that a decently detailed sampling of this three-dimensional parameter space – which is necessary in order to provide reliable fits – would produce an extensive library. The best solution is certainly to maintain the numerical orthonormalisation during the fit and to employ Graphics Processing Units (GPUs) instead of normal CPUs. For such a pure “number crunching” like numerical orthonormalisation, GPUs impressively outperform CPUs (e.g. Fluke et al. 2011).
6 CONCLUSIONS
In this article, we reinvestigated the method of s´ersiclets in order to overcome the limitations of shapelets reported by Melchior et al. (2010) in parametrising galaxy morphologies. Orthogonalising the Sérsic profile is justified by the fact that the Sérsic profile is the first-order Taylor expansion of any real light profile (Appendix A1). We obtain the following results:

- From general radial light profiles like the Sérsic profile, we can only construct polar basis function because Cartesian basis functions do not exhibit the azimuthal symmetry required in the context of galaxy morphologies. Shapelets are the only exception to this rule.
- Sérsiclet basis functions exist analytically, such that no numerical orthonormalisation is required and models are simple to evaluate. Polar shapelets are a special case of the s´ersiclet basis functions.
- S´ersiclets outperform shapelets in the case of edge-on discs and elliptical galaxies, providing models with substantially lower residuals. In particular, s´ersiclets overcome the ring-like artefacts introduced by shapelets when fitting galaxies with steep light profiles. However, in the case of extended objects with lots of substructure, e.g., face-on spiral galaxies, shapelets are competitive.
- S´ersiclets indeed solve the problems of shapelets. However, we revealed two new problems of this approach:
  - S´ersiclets are prone to undersampling effects, if galaxies are not well-resolved. Table 1 provides an estimate of the required resolution in order to avoid undersampling. This undersampling problem stems from the polar basis functions being evaluated on a Cartesian pixel grid. The problem is therefore most prominent in the central regions of the model, where unfortunately also most of the galactic light is concentrated. Undersampling gives rise to an increased uncertainty of the model parameters. For decreasing resolution, increasing parameter uncertainties render s´ersiclet models less informative, which beyond some point obstructs most if not any application. If computationally feasible, the undersampling problem may be alleviated by oversampling the model, although oversampling inevitably compromises the interpretation of the expansion coefficients.
  - S´ersiclets postulate a relation between steepness of the weight function and spatial scale of polynomial oscillations (cf. Sect. 2.3). However, real galaxy morphologies do not necessarily obey this relation, e.g., in the steepness of their bulge vs. the size and distribution of star-formation knots. Therefore, we have to expect modelling problems for s´ersiclets, which may not have a generally predictable impact on a given application.
- We tested the performance of s´ersiclets by analysing simulated weak-lensing data from the GREAT08 challenge (Bridle et al. 2009). We observed that higher-order modes do more harm than good. Given the typically low resolution of realistic weak-lensing images and the undersampling problem of s´ersiclets, we have to conclude that s´ersiclets are inappropriate for shear measurements in weak lensing.

We conclude that although the now formally correct s´ersiclet approach appears very reasonable at first glance, it suffers from substantial problems in practice. Therefore, for every usage of s´ersiclets, we recommend a critical assessment of the impact of undersampling and the postulated unphysical relation between steepness and oscillation scales onto the results.

Finally, we demonstrated that third-order Taylor expansions of galaxy light profiles can drastically improve the modelling fidelity in comparison to Sérsic profiles. If orthonormlised, third-order profiles can overcome the undersampling problem of s´ersiclets and loosen the tight relation between steepness of the weight function and spatial scale of polynomial oscillations. Furthermore, they preserve the mathematical justification of s´ersiclets. However, such basis functions are computationally highly demanding. In fact, they may only be feasible using appropriate computer hardware such as GPUs.

Jiménez-Teja & Benítez (2011) introduce a set of basis functions based on Chebyshev rational functions. The authors demonstrate that their method is capable of providing excellent model reconstructions of galaxies of very different morphological types with very little computational cost. Orthonormalised higher-order profiles may provide similarly good models but certainly not at a similarly low computational cost. Hence, the method of Jiménez-Teja & Benítez (2011) appears to be superior. In general, basis-function expansions can describe azimuthal structures of galaxy morphologies very accurately. However, the astrophysical interpretation of the expansion coefficients is not obvious and has to be laboriously deduced. This is a generic problem of basis-function expansions. Conversely, GALFIT3 (Peng et al. 2010) allows to directly model the azimuthal structures exhibited by galaxy morphologies, such as spiral-arm patterns. The modelling and choice of components are “guided” by astrophysical intuition. Consequently, results obtained from GALFIT3 are easier to interpret. As discussed by Jiménez-Teja & Benítez (2011), this requires substantial manual interaction of the user which is conceptually hard to automatisate. The above mentioned methods highlight a fundamental conflict for any attempt of modelling galaxies. One can either seek to first optimally describe their shapes and then interpret the abstract model parameters later – or attempt to immediately explain galaxy shapes as mixtures of a limited set of physically motivated features. The s´ersiclets approach provides a compromise between these two extremes, as it incorporates the flexibility and completeness of basis-function approaches and the intuition of using a weight function, which is well-adapted to galactic morphologies.

The s´ersiclet code is implemented in C++ and is available on request.

8 Nevertheless, Chebyshev rational functions still have a long way to go. First, the choice of scale radius suggested by Jiménez-Teja & Benítez (2011) is a work-around that may erase discriminative information when it comes to classifying galaxy morphologies. Second, the authors yet have to investigate how their basis functions fare in the regime of poorly resolved galaxies which constitute the vast majority of galaxies in surveys.

7 In fact, this problem also applies to shapelets.
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Appendix A: Sértec Profile as First-Order Taylor Expansion

We reveal the nature of the Sértec profile as a first-order Taylor expansion. Our emphasis is on the justification of the choice of the Sértec profile for orthogonalisation. Furthermore, we briefly outline an alternative approach to generalise Sértec profiles by higher-order expansions.

A1 First-order expansion

Let us consider the real, two-dimensional light profile of a galaxy $I(\vec{x})$ projected onto the sky, which may exhibit arbitrary radial and azimuthal structures. Due to observational effects, e.g., noise and resolution, the observed light profile $I_{\text{obs}}(\vec{x})$ will be different. If noise and resolution have a strong impact, we cannot identify azimuthal structures anymore and only the radial decline is left, i.e., $I_{\text{obs}}(\vec{x}) \approx I_{\text{obs}}(r)$. Let us further consider a rescaling of the observed light profile $p(r) = I_{\text{obs}}(r)/I_{\text{obs}}(0)$, such that $0 < p(r) \leq 1$. Then we can take the logarithm of $p(r)$ and due to $\log(p(r))$ we can also take the logarithm a second time, introducing

$$\tilde{p}(r) = \log(-\log(p(r))) \quad (A1)$$

We now Taylor expand this function $\tilde{p}(r)$ in $\log r$ at a characteristic radius $\log \beta$ to first order, i.e., a constant plus the first nontrivial term, which yields,

$$\tilde{p}(r) \approx A + B(\log r - \log \beta) = A + \log(r/\beta)^B \quad (A2)$$

Let us transform backwards now,

$$\log p(r) = -e^{\tilde{p}(r)} \approx -e^A(r/\beta)^B$$

and hence

$$p(r) \approx \exp \left[-e^A(r/\beta)^B\right] \quad (A4)$$

Ciotti & Bertin (1999) also investigate a Taylor-expansion of the Sértec profile, but they expand in powers of Sértec index and not in powers of radius.
All we need to do now is to identify the coefficients $A$ and $B$ of the Taylor expansion in Eq. (A2). If we simply “rename” these constants by $B = 1/n_S$ and $A = \log(b_n)$, then we have arrived at the definition of the Sérsic profile

$$p_S(r) = \exp \left[ -b_n \left( \frac{r}{\beta} \right)^{1/n_S} \right].$$

(A5)

Evidently, the Sérsic profile is a first-order Taylor expansion at $r_0 = \beta > 0$. In other words, in the limit of low resolution and low signal-to-noise ratios (where substructures are negligible) any radial light profile is approximately a Sérsic profile. This naturally explains why the Sérsic profile is such a good fit to real galaxies in this regime. Furthermore, this also leads us to the expectation that with improving imaging quality the Sérsic profile will not be a good match anymore, because it is “only” a first-order expansion.

### A2 Higher-order expansions

Given the fact that the Sérsic profile is the first-order Taylor expansion of a real light profile, an obvious strategy to enhance the Sérsic profile is to allow for higher orders in the Taylor expansion of a real light profile, an obvious strategy to enhance the Sérsic profile is to allow for higher orders in the Taylor expansion of Eq. (A2). However, a realistic profile is now of the form $u^ke^{-u}$, where $k = 2n_S - 1$, and the corresponding set of orthogonal polynomials are the associated Laguerre polynomials

$$L^k_l(u) = \frac{e^u u^{-k} d^k}{k!} \left( e^{-u} u^{1+k} \right).$$

(B5)

The $L^k_l$ exist if and only if $k > -1$. This is guaranteed, since $k = 2n_S - 1$ and the Sérsic index satisfies $n_S > 0$. The normalisation factor is

$$\int_0^\infty du L^k_l(u) L^k_l(u) e^{-u} = \frac{\Gamma(l+k+1)}{l!},$$

(B6)

where $\Gamma$ denotes the Gamma function. Consequently, the radial parts of the sérsclets read

$$R_l(r) = \frac{1}{N_l} L_{2n_S-1}^l \left[ b(r/\beta)^{1/n_S} \right] \exp \left[ -\frac{b}{2} (r/\beta)^{1/n_S} \right]$$

(B7)

with normalisation factor

$$N_l = \sqrt{\frac{\beta^2 n_S}{2^{2n_S}} \Gamma(l+2n_S) / l!}.$$ (B8)

Mind the factor of 1/2 in the exponent of Eq. (B9), which arises from our definition that the Sérsic profile is the weight function in Eq. (B1). We could also have defined Eq. (B9) with a pure Sérsic profile, resulting in a factor 2 arising in Eq. (B1). Fit results obtained from both definitions will not differ, since these factors are absorbed in the parameter $b$.

In the special case of polar shapelets these expressions simplify. First, for $n_S = 0.5$, we note that $k = 2n_S - 1 = 0$, which implies $\Gamma(l+k+1) = \Gamma(l+1) = l!$ and simplifies the associated Laguerre polynomial $L^0_l$ to the “normal” Laguerre polynomial $L^0_l = L_l$. Second, in order to obtain shapelets, we need to set $b = 1$. Therefore, the radial parts of polar shapelets read

$$R_l(r) = \frac{1}{\sqrt{\beta^2/2}} L_l \left[ (r/\beta)^2 \right] \exp \left[ -\frac{r^2}{2\beta^2} \right].$$ (B9)

The basis functions introduced by Massey & Réfrégier (2005) as “polar shapelets” (their Eq. (8)) differ from Eq. (B9) by using $x^{[m]} L_{(l-[m])/2}^{|m|}(x^2)$ instead of $L_l(x^2)$. Their basis functions are not orthogonal, e.g., consider

$$(l = 1, m = 1/l = 2, m = 0) \propto \int_0^\infty dr r e^{-r^2} r^{|m|} L_{0}^{|m|}(r^2) r^{[0]} L_{0}^{|m|}(r^2),$$

(B10)

which equals $-\sqrt{\pi}/8$ and is thus non-zero. Consequently, the “polar shapelets” defined by Massey & Réfrégier (2005) are incorrect. However, the linear independence is preserved because no two states with identical values of $m$ (otherwise the azimuthal parts $e^{im\phi}$ would differ) and different values of $l$ can have identical $x^{[m]} r^{[m]} L_{(l-[m])/2}^{|m|}(x^2)$.

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10 This is not true for radial profiles that are not differentiable, i.e., where no Taylor expansion exists. However, such profiles are unphysical.

11 http://mathworld.wolfram.com/LaguerrePolynomial.html

Quantifying galaxy shapes: Sérsiclets and beyond
APPENDIX C: OPTIMISATION PROCEDURE

Having defined sérésiclet models, we also need to clarify how to fit such models to given imaging data.

C1 Maximum-likelihood solution

For the moment, we assume that we are given estimates of the nonlinear model parameters \((N_{\text{max}}, n_S, \beta, \vec{x}_0, \epsilon)\). We define a vector \(\vec{c}\) of free parameters that form the coefficients. Furthermore, we write the image as a vector \(\vec{y}\) of size \(N\), where \(N\) denotes the number of pixels in the given image. Finally, we define the so-called \(N \times P\) design matrix \(X\), such that \(X_{np}\) is the \(p\)-th basis function evaluated at the \(n\)-th image pixel (after forward convolution with the PSF). With these definitions, we define the residual vector

\[
\vec{R} = \vec{y} - X \cdot \vec{c},
\]

which is an vector of size \(N\) itself. Here “\(\cdot\)” denotes matrix multiplication. As the pixel noise of modern imaging data is usually Gaussian in excellent approximation, we are allowed to define

\[
\chi^2 = \vec{R}^T \cdot \Sigma^{-1} \cdot \vec{R},
\]

where \(\Sigma\) denotes the \(N \times N\) pixel covariance matrix. In the case of uncorrelated Gaussian noise with constant variance \(\sigma^2\) in all pixels, \(\Sigma\) takes the simple form \(\Sigma = \text{diag}(\sigma^2, \ldots, \sigma^2) = \sigma^2 I\). It is straightforward to show that the maximum-likelihood estimate that minimises \(\chi^2\) is

\[
\vec{c} = (X^T \cdot \Sigma^{-1} \cdot X)^{-1} \cdot X^T \cdot \Sigma^{-1} \cdot \vec{y}.
\]

The matrix \(X^T \cdot \Sigma^{-1} \cdot X\) has a special meaning: It is the \(P \times P\) covariance matrix of the coefficients \(\vec{c}\), which would follow from a Fisher analysis. In this formalism, we get this information for free.

The advantage of expressing the maximum-likelihood solution in terms of matrix operations is that we can employ fast and efficient linear-algebra algorithms. In our case, the basis functions are implemented in C++ and we use wrapper classes to import the packages ATLAS\(^{12}\) and LAPACK\(^{13}\).

C2 Optimising the nonlinear parameters for given \(N_{\text{max}}\)

We now assume that we are given an estimate of the maximum order, \(N_{\text{max}}\), and need to get estimates of \((n_S, \beta, \vec{x}_0, \epsilon)\). These estimates are derived from a Simplex algorithm (Nelder & Mead\(^{14}\) 1965) that we incorporate from the GNU Scientific Library (GSL\(^{12}\)). The Simplex algorithm is an iterative optimisation algorithm that does not employ derivatives but only evaluations of \(\chi^2\). We also employ priors in order to restrict the model parameters to reasonable values. These priors in detail ensure that:

- The Sérsic index is in the range \(0.1 \leq n_S \leq 8\).
- The scale radius satisfies \(\beta > 0\).
- The object centroid \(\vec{x}_0\) is within the pixel grid.
- The complex ellipticity satisfies \(|\epsilon| < 1\).

Within these reasonable parameter ranges, we employ flat priors.

C3 Estimating the maximum order

In the case of shapelets, the maximum order, \(N_{\text{max}}\), is estimated via a reduced \(\chi^2\). However, demanding that \(\chi^2\) equals the number of degrees of freedom is justified if and only if the model is purely linear (e.g., Barlow\(^{15}\) 1993, Andrae et al.\(^{14}\) 2010c). In the case of sérésiclets, this assumption is definitely violated, since sérésiclets contain many nonlinear fit parameters. We rather recommend to estimate \(N_{\text{max}}\) by comparing the normalised residuals for models of different maximum orders. The optimal \(N_{\text{max}}\) is then defined by the model whose normalised residuals are closest to a Gaussian with mean zero and variance one (Andrae et al.\(^{14}\) 2010c).

Concerning large samples of galaxy images, one can also decompose all objects using identical \(N_{\text{max}}\). This approach may be favourable because many techniques to analyse the resulting catalogue of models require that all coefficient vectors have the same dimensionality (e.g. clustering analysis, cf. Kelly & McKay\(^{11}\) 2004, 2005, Andrae et al.\(^{14}\) 2010b).

This paper has been typeset from a \(\TeX/\LaTeX\) file prepared by the author.