Research Article

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Optimality of Serrin type extension criteria to the Navier-Stokes equations

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Abstract: We prove that a strong solution $u$ to the Navier-Stokes equations on $(0, T)$ can be extended if either $u \in L^\theta(0, T; \dot{U}_{\infty,1}^a(\mathbb{R}^n))$ for $2/\theta + \alpha = 1$, $0 < \alpha < 1$ or $u \in L^2(0, T; \dot{V}_0^{\infty,\infty,2})$, where $\dot{U}_{p,\beta,\alpha}$ and $\dot{V}_p^{\infty,\infty,2}$ are Banach spaces that may be larger than the homogeneous Besov space $\dot{B}_p^s$. Our method is based on a bilinear estimate and a logarithmic interpolation inequality.

Keywords: Serrin type extension criterion; Navier-Stokes equations; bilinear estimate; logarithmic interpolation inequality

MSC: 35Q30 (primary), 35B65, 46E35, 76D05

1 Introduction

The motion of a viscous incompressible fluid in $\mathbb{R}^n$, $n \geq 2$, is governed by the Navier-Stokes equations:

\[
\begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi &= 0, \quad x \in \mathbb{R}^n, \ t > 0, \\
\text{div} \ u &= 0, \quad x \in \mathbb{R}^n, \ t > 0, \\
u|_{t=0} &= u_0,
\end{aligned}
\]

(N-S)

where $u = (u_1(x, t), \cdots, u_n(x, t))$ and $\pi = \pi(x, t)$ denote the velocity vector field and the pressure of the fluid at the point $x \in \mathbb{R}^n$ and time $t > 0$, respectively, while $u_0 = u_0(x)$ is the given initial vector field for $u$.

It is known that for every $u_0 \in H^2 \equiv W^{2,2}(\mathbb{R}^n)$, there exists a unique solution $u \in C([0, T); H^2)$ to (N-S) for some $T > 0$. Such a solution is in fact smooth in $\mathbb{R}^n \times (0, T)$. See, for instance Fujita-Kato [9]. It is an important open question whether $T$ may be taken as $T = \infty$ or $T < \infty$. In this direction, Giga [10] gave a Serrin type criterion, i.e., if the solution $u$ satisfies the condition

\[
\int_0^T \|u(t)\|_{L^p}^\theta \ dt < \infty, \quad 2/\theta + n/p = 1, \ n < p \leq \infty,
\]

then $u$ can be extended to the solution in the class $C([0, T'); H^2)$ for some $T' > T$. Later on, the condition (1.1) was relaxed from the $L^p$-criterion to

\[
\int_0^T \|u(t)\|_{B_{\infty,\infty}^\alpha}^\theta \ dt < \infty, \quad 2/\theta + \alpha = 1, \ 0 < \alpha < 1
\]

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by Kozono-Ogawa-Taniuchi [16] and Kozono-Shimada [17]. In a recent work, Nakao-Taniuchi [22] gave a new criterion, instead of (1.1) and (1.2) with \( p = \infty \) and \( \alpha = 0 \) (\( \theta = 2 \)), in such a way that

\[
\int_0^T \| u(t) \|_{V_{1/2}}^2 \, dt < \infty.
\]

(1.3)

Here, \( V_\beta, \beta > 0 \), is introduced by

\[
V_\beta := \{ f \in \mathcal{S} : \| f \|_{V_\beta} < \infty \},
\]

\[
\| f \|_{V_\beta} := \sup_{N=1,2,...} \| \psi^*_N * f \|_{N^\beta},
\]

where \( \psi \in \mathcal{S} \) is a radially symmetric function with \( \hat{\psi}(\xi) = 1 \) in \( B(0, 1) \) and \( \hat{\psi}(\xi) = 0 \) in \( B(0, 2) \) and \( \psi_N(x) := 2^{nN} \psi(2^Nx) \). This function space \( V_\beta \) is called the Vishik space and admits a continuous embedding \( L^\infty \subset V_\beta \) for each \( \beta > 0 \). The above three criteria are important from a viewpoint of scaling invariance. Indeed, it is easy to show that if \( (u, \pi) \) satisfies (N-S), then so does \( (u_A, \pi_A) \) for all \( \lambda > 0 \), where \( u_A(x, t) := \lambda u(\lambda x, \lambda^2 t) \) and \( \pi_A(x, t) := \lambda^3 \pi(\lambda x, \lambda^2 t) \). We call a Banach space \( X \) scaling invariant for the velocity \( u \) with respect to (N-S) if

\[
\| u_\lambda \|_X = \| u \|_X
\]

holds for all \( \lambda > 0 \). In fact, the spaces \( L^\theta(0, \infty; L^p) \) with \( 2/\theta + n/p = 1 \), \( L^\theta(0, \infty; \dot{B}^{n,\infty}_{\infty,\infty}) \) with \( 2/\theta + \alpha = 1 \) and \( L^2(0, \infty; V_{1/2}) \) are scaling invariant for \( u \) with respect to (N-S).

On the other hand, Beale-Kato-Majda [1] and Beirão da Veiga [2] gave a criterion by means of the vorticity, i.e., if the solution \( u \) satisfies the condition

\[
\int_0^T \| \text{rot } u(t) \|_{L^p}^\theta \, dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 2, \quad \frac{n}{2} < p \leq \infty,
\]

(1.4)

then \( u \) can be extended to a solution in the class \( C([0, T'); H^s(\mathbb{R}^n)) \) for some \( T' > T \). Later on, the condition (1.4) was relaxed from the \( L^p \)-criterion to

\[
\int_0^T \| \text{rot } u(t) \|_{B^0_{p,\infty}}^\theta \, dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 2, \quad n \leq p \leq \infty
\]

(1.5)

by Kozono-Ogawa-Taniuchi [16]. Moreover, Nakao-Taniuchi [21] gave a similar type of the criterion as (1.3), instead of (1.4) and (1.5) with \( p = \infty \) (\( \theta = 1 \)), in such a way that

\[
\int_0^T \| \text{rot } u(t) \|_{V_1} \, dt < \infty.
\]

Note that \( V_\beta \) admits the following continuous embeddings in the case \( \beta = 1 \):

\[
L^\infty \subset bmo \subset B^{n,\infty}_{\infty,\infty} \subset V_1.
\]

Futhermore, the author [12] improved the \( B^{n,\infty}_{\infty,\infty} \)-criterion (1.5) to

\[
\int_0^T \| \text{rot } u(t) \|_{B^{n,\infty}_{\infty,\infty}}^\theta \, dt < \infty, \quad \frac{2}{\theta} + \frac{n}{p} = 2, \quad r \leq p \leq \infty
\]

(1.6)

for \( L'(n < r < \infty) \) strong solutions to (N-S). Here, \( V^S_{p,q,\theta} \) is a Banach space introduced by Definition 2.1 and has a continuous embedding \( B^{n,\infty}_{\infty,\infty} \subset V^S_{p,\infty,\theta} \). The above criteria by means of the vorticity are also important from a viewpoint of scaling invariance. Indeed, since \( u_A = \lambda^2 \text{rot } u(\lambda x, \lambda^2 t) \), the spaces \( L^\theta(0, \infty; L^p), L^\theta(0, \infty; \dot{B}^0_{p,\infty}), L^\theta(0, \infty; \dot{V}^0_{p,\infty,\theta}) \) with \( 2/\theta + n/p = 2 \) and \( L^1(0, \infty; V_1) \) are scaling invariant for the vorticity with respect to (N-S).
The aim of this paper is to improve the extension criterion (1.2) to the Navier-Stokes equations by means of Banach spaces which are larger than $\dot{B}_{\infty,\infty}^{\alpha}$ in the same way that the condition (1.5) was relaxed to (1.6). In fact, we prove that if the solution $u$ to (N-S) on $(0, T)$ satisfies the condition either

$$\int_0^T \|u(t)\|_{\dot{B}_{\infty,\infty}^{\theta}}^2 \, dt < \infty, \quad \frac{2}{\theta} + \alpha = 1, \ 0 < \alpha < 1$$

(1.7)

or

$$\int_0^T \|u(t)\|_{\dot{B}_{\infty,1}^{\theta}}^2 \, dt < \infty,$$

(1.8)

then $u$ can be extended to a solution in the class $C([0, T'); H^s(\mathbb{R}^n))$ for some $T' > T$. Here, $\dot{U}_{p, \beta, \sigma}^s$ is a Banach space introduced by Definition 2.2 and has the following continuous embeddings:

$$\dot{B}_{\infty,\infty}^\alpha \subset \dot{V}_{\infty,\infty, \theta}^\alpha \subset \dot{U}_{\infty,1/\theta, \infty}^\alpha \quad \frac{2}{\theta} + \alpha = 1, \ 0 < \alpha < 1.$$

Hence, we see that (1.7) and (1.8) may be regarded as a weaker condition than (1.2). Moreover, note that the spaces $L^2(0, \infty; \dot{U}_{\infty,1/\theta, \infty}^\alpha)$ with $2/\theta + \alpha = 1$ and $L^2(0, \infty; \dot{V}_{\infty,\infty, \theta}^\alpha)$ are also scaling invariant for solutions $u$ to (N-S). In order to obtain our extension principle, we need a logarithmic interpolation inequality by means of $\dot{U}_{p, \beta, \sigma}^s$:

$$\|f\|_{\dot{U}_{p, \beta, \sigma}^s} \leq C \left( 1 + \|f\|_{\dot{U}_{p, \beta, \sigma}^s} \log^{\beta} (e + \|f\|_{\dot{U}_{p, \beta, \sigma}^s}) \right).$$

This is related to the Brezis-Gallouet-Wainger inequality given in Brezis-Gallouet [5] and Brezis-Wainger [6]. Several inequalities of Brezis-Gallouet-Wainger type were established in [1], [7], [8], [11], [12], [15], [16], [19], [20], [21], [22], [23], [24], [25]. Moreover, we prove that $\dot{U}_{p, \beta, \sigma}^s$ is the weakest normed space that satisfies such a logarithmic interpolation inequality. Thus, roughly speaking, new conditions (1.7) and (1.8) may be regarded as optimal Serrin type criteria that guarantee a priori estimates of $H^s$ strong solutions to (N-S) with double exponential growth form.

The present paper is organized as follows. In the next section, we shall state our main results. In section 3 and 4, proofs of our main results are established.

## 2 Results

### 2.1 Function spaces

We first introduce some notation. Let $S = S(\mathbb{R}^n)$ be the set of all Schwartz functions on $\mathbb{R}^n$, and $\mathcal{S}'$ the set of tempered distributions. The $L^p$-norm on $\mathbb{R}^n$ is denoted by $\| \cdot \|_p$. We recall the Littlewood-Paley decomposition and use the functions $\psi, \phi_j \in \mathcal{S}, \ j \in \mathbb{Z}$, such that

$$\hat{\psi}(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases}$$

$$\hat{\phi}(\xi) := \hat{\psi}(\xi) - \hat{\psi}(2\xi), \ \hat{\phi}_j(\xi) := \hat{\phi}(\xi/2^j).$$

Let $\mathcal{Z} := \{ f \in \mathcal{S}; D^a \hat{f}(0) = 0 \text{ for all } a \in \mathbb{N}^n \}$ and $\mathcal{Z}'$ denote the dual space of $\mathcal{Z}$. We note that $\mathcal{Z}'$ can be identified with the quotient space $\mathcal{S}' / \mathcal{P}$ of $\mathcal{S}'$ with respect to the space of polynomials, $\mathcal{P}$. Furthermore, the homogeneous Besov space $\dot{B}_{p,q}^s := \{ f \in \mathcal{Z}; \| f \|_{\dot{B}_{p,q}^s} < \infty \}$ is defined by the norm

$$\|f\|_{\dot{B}_{p,q}^s} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^js^q \| \phi_j \ast f \|_p^q \right)^{\frac{1}{q}}, & q \neq \infty, \\ \sup_{j \in \mathbb{Z}} 2^js^q \| \phi_j \ast f \|_p, & q = \infty. \end{cases}$$


See Bergh-Löfström [3, Chapter 6.3] and Triebel [26, Chapter 5] for details. Let \( C^\infty_0(\mathbb{R}^n) \) denote the set of all \( C^\infty \) functions with compact support in \( \mathbb{R}^n \) and \( C^\infty_{0, \alpha} := \{ \phi \in (C^\infty_0(\mathbb{R}^n))^n; \text{div} \phi = 0 \} \). Concerning Sobolev spaces we use the notation \( H^p(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \). Then \( H^p_\alpha \) is the closure of \( C^\infty_{0, \alpha} \) with respect to \( H^p \)-norm. In Section 4 we will also use homogeneous Sobolev spaces \( H^p(\mathbb{R}^n) \) and note that \( H^p = B^0_{2, 2} \) for all \( s \in \mathbb{R} \).

We now introduce Banach spaces \( \dot{V}^s_{p, q, \alpha} \) and \( \dot{U}^s_{p, q, \alpha} \) which are larger than the homogeneous Besov spaces \( B^s_{p, q} \). These spaces may be regarded as modified versions of spaces defined by Nakao-Taniuchi [22] and Vishik [27].

**Definition 2.1.** Let \( s \in \mathbb{R} \), \( 1 \leq p, q, \theta \leq \infty \) and let \( \{ \phi_j \}_{j=-\infty}^\infty \) be the Littlewood-Paley decomposition. Then, \( \dot{V}^s_{p, q, \alpha}(\mathbb{R}^n) := \{ f \in \mathcal{Z}^\prime; \| \cdot \|_{\dot{V}^s_{p, q, \alpha}} < \infty \} \) is introduced by the norm

\[
\| f \|_{\dot{V}^s_{p, q, \alpha}} := \begin{cases} 
\sup_{N=1, 2, \ldots} \left( \frac{1}{N^{s-\frac{\alpha}{\theta}}} \left( \sum_{|\alpha| \leq N} 2^{j\alpha s} \| \phi_j \ast f \|_p \right)^{\frac{1}{\theta}} \right), & \theta \neq \infty, \\
\max_{|\alpha| \leq N} 2^j \| \phi_j \ast f \|_p, & \theta = \infty.
\end{cases}
\]

**Definition 2.2.** Let \( s, \beta \in \mathbb{R} \), \( 1 \leq p, \alpha \leq \infty \) and let \( \{ \phi_j \}_{j=-\infty}^\infty \) be the Littlewood-Paley decomposition. Then, \( \dot{U}^s_{p, \beta, \alpha}(\mathbb{R}^n) := \{ f \in \mathcal{Z}^\prime; \| \cdot \|_{\dot{U}^s_{p, \beta, \alpha}} < \infty \} \) is equipped with the norm

\[
\| f \|_{\dot{U}^s_{p, \beta, \alpha}} := \begin{cases} 
\sup_{N=1, 2, \ldots} \left( \frac{1}{N^\beta} \left( \sum_{|\alpha| \leq N} 2^{j\alpha s} \| \phi_j \ast f \|_p \right)^{\frac{1}{\alpha}} \right), & \alpha \neq \infty, \\
\max_{|\alpha| \leq N} 2^j \| \phi_j \ast f \|_p, & \alpha = \infty.
\end{cases}
\]

We see from the following proposition that \( \dot{V}^s_{p, q, \alpha} \) and \( \dot{U}^s_{p, \beta, \alpha} \) are extensions of \( B^s_{p, q} \) and \( \dot{V}^s_{p, q, \alpha} \), respectively.

**Proposition 2.3.**

(i) Let \( s \in \mathbb{R} \), \( 1 \leq p, q \leq \infty \) and \( 1 \leq \theta_1 \leq \theta_2 \leq q < \theta_3 \). Then, it holds that

\[ \{ 0 \} = \dot{V}^s_{p, q, \theta_1} \subset \dot{B}^s_{p, q} = \dot{V}^s_{p, q, \theta_2} \subset \dot{V}^s_{p, q, \theta_3} \subset \dot{V}^s_{p, q, \theta_1}. \]

(ii) Let \( s \in \mathbb{R} \), \( 1 \leq p, \alpha \leq \infty \) and \( \beta_1 < 0 \leq \beta_2 < \beta_3 \). Then, it holds that

\[ \{ 0 \} = \dot{U}^s_{p, \beta_1, \alpha} \subset \dot{B}^s_{p, \beta} = \dot{U}^s_{p, \beta_2, \alpha} \subset \dot{U}^s_{p, \beta_3, \alpha} \subset \dot{U}^s_{p, \beta_1, \alpha}. \]

(iii) Let \( s, \beta \in \mathbb{R} \), \( 1 \leq p, q, \theta \leq \infty \), \( \beta = \frac{1}{\beta} - \frac{1}{q} \) and \( 1 \leq \alpha_1 \leq \alpha_2 \leq \infty \). Then, it holds that

\[ \dot{V}^s_{p, q, \theta} = \dot{U}^s_{p, \beta, \theta} \quad \text{and} \quad \dot{U}^s_{p, \beta, \alpha_1} \subset \dot{U}^s_{p, \beta, \alpha_2}. \]

**Proof.** We easily prove \( \dot{V}^s_{p, q, \alpha_1} \subset \dot{V}^s_{p, q, \alpha_2} \) in (i) by the standard and the reverse Hölder’s inequality. The others follow from the definitions of \( \dot{B}^s_{p, q} \), \( \dot{V}^s_{p, q, \alpha} \) and \( \dot{U}^s_{p, \beta, \alpha} \).

It follows by Proposition 2.3 (i) and (iii) that

\[
\dot{B}^s_{\infty, \infty} \subset \dot{V}^s_{\infty, \infty, \theta} \subset \dot{U}^s_{\infty, 1/\theta, \infty}
\]

for \( s \in \mathbb{R} \) and \( 1 \leq \theta < \infty \). We observe from the following examples that the continuous embeddings (2.1) are proper if \( s > -n \) and \( 1 \leq \theta < \infty \), which is important in terms of Theorem 2.9.
Example 2.4. (1) The continuous embedding $\tilde{B}^{s}_{\infty, \infty} \subset \tilde{V}^{s}_{\infty, \infty, \theta}$ is proper if $s > -n$ and $1 \leq \theta < \infty$. We now introduce a distribution $f \in \tilde{V}^{s}_{\infty, \infty, \theta} \setminus \tilde{B}^{s}_{\infty, \infty}$ for $s > -n$ and $1 \leq \theta < \infty$. Let $f \in \mathcal{Z}'$ defined as

$$\hat{f}(\xi) := \begin{cases} 2^{-(n+s)(k^{\theta+1})}, & 2^{k^{\theta+1}+1} \leq |\xi| \leq 2^{k^{\theta+1}+1} \ (k = 1, 2, \cdots), \\ 0, & \text{otherwise}. \end{cases}$$

Indeed, since $\hat{f} \in L^\infty$ holds, we obtain $f \in \mathcal{Z}'$. We easily see that

$$\|\phi_j \ast f\|_\infty = \int_{\mathbb{R}^n} \hat{\phi}_j(\xi) \hat{f}(\xi) \, d\xi = \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \hat{\phi}_j(\xi) \hat{f}(\xi) \, d\xi$$

$$= 2^{-s(k^{\theta+1})} k\|\hat{\phi}\|_1 \quad \text{for } j = [k^{\theta+1}] \ (k = 1, 2, \cdots),$$

$$\leq 2^{-s(k^{\theta+1})} 2^nk\|\hat{\phi}\|_1 \quad \text{for } j = [k^{\theta+1}] + 1 \ (k = 1, 2, \cdots),$$

$$= 0 \quad \text{for } j \in \mathbb{Z} \setminus \{ [k^{\theta+1}] + 1 \}.$$

Hence, it holds that

$$\|f\|_{\tilde{B}^{s}_{\infty, \infty}} \geq \sup_{k=1,2,\ldots} 2^{s(k^{\theta+1})} \|\phi_j \ast f\|_\infty = \sup_{k=1,2,\ldots} k \|\hat{\phi}\|_1 = \infty. \quad (2.2)$$

On the other hand, for any $N = 1, 2, \cdots$, there exists $k_N \in \mathbb{N}$ such that $k_N^{\theta+1} \leq N < (k_N + 1)^{\theta+1}$. Therefore, we obtain

$$\sum_{|j| \leq N} 2^{j\theta} \|\phi_j \ast f\|_\infty \leq \sum_{k=1}^{k_N^{\theta+1}} \sum_{j=[k^{\theta+1}]-1}^{[k^{\theta+1}]+1} 2^{j\theta} \|\phi_j \ast f\|_\infty \leq \sum_{k=1}^{k_N^{\theta+1}} \sum_{j=[k^{\theta+1}]-1}^{[k^{\theta+1}]+1} 2^{j\theta}(2^{-s(k^{\theta+1})} 2^nk\|\hat{\phi}\|_1)^\theta = C \sum_{k=1}^{k_N^{\theta+1}} k^\theta \leq C(k_N + 1)^{\theta+1} \leq Ck_N^{\theta+1} \leq CN,$$

where $C$ is dependent only on $n$, $s$ and $\theta$. Thus, it follows that

$$\|f\|_{\tilde{V}^{s}_{\infty, \infty, \theta}} = \sup_{N=1,2,\ldots} \left( \frac{\sum_{|j| \leq N} 2^{j\theta} \|\phi_j \ast f\|_\infty}{N^{\frac{1}{\theta}}} \right)^{\frac{1}{\theta}} \leq \sup_{N=1,2,\ldots} \frac{C N^{\frac{1}{\theta}}}{N^{\frac{1}{\theta}}} < \infty. \quad (2.3)$$

From (2.2) and (2.3), we get $f \in \tilde{V}^{s}_{\infty, \infty, \theta} \setminus \tilde{B}^{s}_{\infty, \infty}$.

(2) The continuous embedding $\tilde{V}^{s}_{\infty, \infty, \theta} = \tilde{U}^{s}_{\infty, 1/\theta, \infty} \subset \tilde{U}^{s}_{\infty, 1/\theta, \infty}$ is also proper if $s > -n$ and $1 \leq \theta < \infty$. We now introduce a distribution $g \in \tilde{U}^{s}_{\infty, 1/\theta, \infty} \setminus \tilde{V}^{s}_{\infty, \infty, \theta}$ for $s > -n$ and $1 \leq \theta < \infty$. Let $g \in \mathcal{Z}'$ defined as

$$\tilde{g}(\xi) := \begin{cases} k^{\frac{n}{\theta+1}}2^{-(n+s)(k^{\theta+1})}, & 2^{k^{\theta+1}+1} \leq |\xi| \leq 2^{k^{\theta+1}+1} \ (k = 1, 2, \cdots), \\ 0, & \text{otherwise}. \end{cases}$$

Indeed, since $\tilde{g} \in L^\infty$ holds, we obtain $g \in \mathcal{Z}'$. We easily see that

$$\|\phi_j \ast g\|_\infty = \int_{\mathbb{R}^n} \hat{\phi}_j(\xi) \tilde{g}(\xi) \, d\xi = \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \hat{\phi}_j(\xi) \tilde{g}(\xi) \, d\xi$$

$$= 2^{-s(k^{\theta+1})} k^{\frac{n}{\theta+1}} \|\hat{\phi}\|_1 \quad \text{for } j = [k^{\theta+1}] \ (k = 1, 2, \cdots),$$

$$\leq 2^{-s(k^{\theta+1})} 2^nk^{\frac{n}{\theta+1}} \|\hat{\phi}\|_1 \quad \text{for } j = [k^{\theta+1}] + 1 \ (k = 1, 2, \cdots),$$

$$= 0 \quad \text{for } j \in \mathbb{Z} \setminus \{ [k^{\theta+1}], [k^{\theta+1}] + 1 \}.$$
For any $N = 1, 2, \ldots$, we take $k_N \in \mathbb{N}$ such that $k_N^{\theta+1} \leq N < (k_N + 1)^{\theta+1}$. Then, it holds that
\[
\sum_{|j| \leq N} 2^{js} \| \bar{\phi}_j \ast g \|_\infty \geq \sum_{1 \leq k \leq k_N} 2^{s|k^{n+1}|} \| \bar{\phi}_{k^{n+1}} \ast g \|_\infty = C_1 \sum_{1 \leq k \leq k_N} k^{\theta+1} \geq C_1 k_N^{\theta+2} \geq C_1 (k_N + 1)^{\theta+2} \geq C_1 N^{\theta/2},
\]
where $C_1$ is dependent only on $n$ and $\theta$. Hence, we have
\[
\| f \|_{\mathcal{V}_{\infty,\infty,\beta}} = \sup_{N=1,2,\ldots} \left( \frac{1}{N} \sum_{|j| \leq N} 2^{js} \| \bar{\phi}_j \ast f \|_\infty \right)^{\frac{\beta}{s}} \geq \sup_{N=1,2,\ldots} C_1^N N^{\frac{\beta}{s}+1} = \infty. \tag{2.4}
\]
On the other hand, it follows that
\[
\max_{|j| \leq N} 2^{js} \| \bar{\phi}_j \ast g \|_\infty \leq \max_{1 \leq k \leq k_N} \max_{j=[k^{n+1}],|k^{n+1}|s} 2^{js} \| \bar{\phi}_j \ast g \|_\infty \leq \max_{1 \leq k \leq k_N} \max_{j=[k^{n+1}],|k^{n+1}|s} 2^{js} 2^{-s|k^{n+1}|} 2^n k^{-1} \| \bar{\phi}_j \|_1 \leq C_2 \max_{1 \leq k \leq k_N} \max_{j=[k^{n+1}],|k^{n+1}|s} k^{-1} = C_2 (k_N + 1)^{\theta+1} \leq C_2 k_N^{\theta+1} \leq C_2 N^{\frac{\theta+1}{s}},
\]
where $C_2$ is dependent only on $n$ and $s$. Thus, we obtain
\[
\| g \|_{\mathcal{V}_{\infty,\infty,\beta}} = \sup_{N=1,2,\ldots} \max_{|j| \leq N} 2^{js} \| \bar{\phi}_j \ast f \|_\infty \leq \sup_{N=1,2,\ldots} C_2^N N^{\frac{\theta+1}{s}} < \infty. \tag{2.5}
\]
From (2.4) and (2.5), we get $g \in \bigcup_{n=1}^s \mathcal{V}_{\infty,\infty,\beta} \setminus \mathcal{V}_{\infty,\infty,\beta}^s$.

### 2.2 Logarithmic interpolation inequalities and optimality

**Theorem 2.5.** (i) Let $s_0, s_1, s_2 \in \mathbb{R}$ satisfy $s_1 < s_0 < s_2$, let $0 \leq \beta < \infty$ and $1 \leq p, \sigma \leq \infty$. Then there exists a positive constant $C$ depending only on $s_0, s_1, s_2$, but not on $p, \beta, \sigma$ such that
\[
\| f \|_{\mathcal{B}^{s_0}_{p,\sigma}} \leq C \left( 1 + \| f \|_{\mathcal{V}_{\infty,\infty,\beta}} \log^\beta (e + \| f \|_{\mathcal{B}^{s_1}_{p,\beta} \cap \mathcal{B}^{s_2}_{p,\sigma}}) \right) \tag{2.6}
\]
for all $f \in \mathcal{B}^{s_1}_{p,\beta} \cap \mathcal{B}^{s_2}_{p,\sigma}$.

(ii) Let $s_0 \in \mathbb{R}$, $0 \leq \beta < \infty$ and $1 \leq p, \sigma \leq \infty$, and let $X$ be a normed space of distributions on $\mathbb{Z}$. Assume that $X$ satisfies the following conditions:

(C1) $X \to \mathbb{Z}$;

(C2) there exists a constant $K_1 > 0$ such that
\[
\| f(\cdot - y) \|_X \leq K_1 \| f \|_X \quad \text{for all } f \in X \text{ and all } y \in \mathbb{R}^n;
\]

(C3) there exists a constant $K_2 > 0$ such that
\[
\| \rho \ast f \|_X \leq K_2 \| \rho \|_1 \| f \|_X \quad \text{for all } \rho \in \mathbb{Z} \text{ and all } f \in X;
\]

(C4) there exist $s_1, s_2 \in \mathbb{R}$ satisfy $s_1 < s_0 < s_2$ and $K_3 > 0$ such that
\[
\| f \|_{\mathcal{B}^{s_0}_{p,\beta}} \leq K_3 \left( 1 + \| f \|_X \log^\beta (e + \| f \|_{\mathcal{B}^{s_1}_{p,\beta} \cap \mathcal{B}^{s_2}_{p,\sigma}}) \right) \quad \text{for all } f \in X \cap \mathbb{Z}.
\]
Then, \( X \to \dot{U}_{p,\beta,\alpha}^{s_0} \) holds.

**Remark 2.6.** (1) In the first part of Theorem 2.5, the assumption \( s_1 < s_0 < s_2 \) is essential. If either of \( s_1 \) or \( s_2 \) tends to \( s_0 \), then the constant \( C \) appearing on the right hand side diverges to infinity.

(2) By Proposition 2.3 (ii), we observed that the following continuous embeddings hold for \( s_1 < s_0 < s_2 \) and \( \beta \geq 0 \):

\[
\dot{B}_{p,\infty}^{s_1} \cap \dot{B}_{p,\infty}^{s_2} \subset \dot{B}_{p,\alpha}^{s_0} \subset \dot{U}_{p,\beta,\alpha}^{s_0}.
\]

Thus, (2.6) may be regarded as an interpolation inequality.

(3) From Theorem 2.5 (i), we see that \( \dot{U}_{p,\alpha,\beta}^{s_0} \) satisfies conditions (C1)-(C4). Hence, Theorem 2.5 (ii) implies that \( \dot{U}_{p,\alpha,\beta}^{s_0} \) is the weakest normed space that satisfies (C1)-(C4).

(4) By Proposition 2.3 (iii), we see that Theorem 2.5 covers the result given by the author [12]. Indeed, by setting \( \beta = \frac{1}{q} - \frac{1}{q'} \), \( \alpha = \theta \left( 1 \leq q \leq \infty, 1 \leq \theta \leq q \right) \) in (2.6), it holds that

\[
\|f\|_{\dot{B}_{p,\theta}^{s_0}} \leq C \left( 1 + \|f\|_{\dot{B}_{p,q}^{s_0}} \log^{\frac{1}{q'} - \frac{1}{q}} \left( e + \|f\|_{\dot{B}_{p,q}^{s_0}} \right) \right)
\]

for all \( f \in \dot{B}_{p,\theta}^{s_0} \cap \dot{B}_{p,\infty}^{s_2} \).

### 2.3 Serrin type regularity criteria for Navier-Stokes systems

**Definition 2.7.** Let \( s > n/2 - 1 \) and let \( u_0 \in H_0^s \). A measurable function \( u \) on \( \mathbb{R}^n \times (0, T) \) is called a strong solution to (N-S) in the class \( \mathcal{C}_L \) if

(i) \( u \in \mathcal{C}((0, T); H_0^s) \cap \mathcal{C}^1((0, T); H_0^s) \cap \mathcal{C}((0, T); \mathcal{H}_0^{s+2}) \);

(ii) \( u \) satisfies (N-S) with some distribution \( \pi \) such that \( \nabla \pi \in \mathcal{C}((0, T); \mathcal{H}_0^s) \).

**Remark 2.8.** For \( s > n/2 - 1 \), the existence of a strong solution to (N-S) in the class \( \mathcal{C}_L \) has been proven in Fujita-Kato [9], Kato [14] and Giga [10].

Our result on extension of strong solutions now reads as follows:

**Theorem 2.9.** (i) Let \( 0 < \alpha < 1 \), \( s > n/2 - \alpha \) and let \( u_0 \in H_0^s \). Assume that \( u \) is a strong solution to (N-S) in the class \( \mathcal{C}_L \). If the solution \( u \) satisfies

\[
\int_0^T \|u(t)\|_{\dot{U}_{\infty,1;0,\infty}^\alpha}^\beta \, dt < \infty, \quad \frac{2}{\beta} + \alpha = 1,
\]

then \( u \) can be extended to a strong solution to (N-S) in the class \( \mathcal{C}_L \) for some \( T' > T \).

(ii) Let \( s > n/2 \) and let \( u_0 \in H_0^s \). Assume that \( u \) is a strong solution to (N-S) in the class \( \mathcal{C}_L \). If the solution \( u \) satisfies

\[
\int_0^T \|u(t)\|_{\dot{U}_{\infty,1}^{s_0}}^2 \, dt < \infty,
\]

then \( u \) can be extended to a strong solution to (N-S) in the class \( \mathcal{C}_L \) for some \( T' > T \).

**Remark 2.10.** (1) Let \( 0 < \alpha < 1 \). As is mentioned Example 2.4, we have proper embeddings \( \dot{B}_{\infty,\infty}^{s_0} \subset \dot{U}_{\infty,\infty}^{a,0} \subset \dot{U}_{\infty,1;0,\infty}^{a} \) and hence Theorem 2.9 (i) covers the extension criterion in \( \dot{B}_{\infty,\infty}^{s_0} \) given by Kozono-Shimada [17] for \( s > n/2 - \alpha \). Indeed, if the solution \( u \) satisfies either

\[
\int_0^T \|u(t)\|_{\dot{B}_{\infty,\infty}^a}^\beta \, dt < \infty, \quad \frac{2}{\beta} + \alpha = 1,
\]
or
\[
\int_0^T \|u(t)\|_{B_{\infty,\infty}^\alpha}^2 \, dt < \infty, \quad \frac{2}{\theta} + \alpha = 1,
\]
then the estimate (2.8) is easily obtained, so that the solution can be extended beyond \(t = T\).

(2) From Example 2.4, the proper embeddings \(\dot{B}_{\infty,\infty}^s \subset \dot{V}_{\infty,\infty}^0 \subset \dot{U}_{\infty,1/2,\infty}^0\) hold. Hence, Theorem 2.9 (ii) may be regarded as an extension of the \(\dot{B}_{\infty,\infty}^s\)-criterion given by Kozono-Ogawa-Taniuchi [16] for \(s > n/2\). On the other hand, it seems to be difficult to obtain the same result as in Theorem 2.9 (ii) under the condition
\[
\int_0^T \|u(\tau)\|_{\dot{U}_{\infty,1/2,\infty}^0}^2 \, d\tau < \infty.
\]
This stems from inapplicability of Lemma 4.1 with \(\alpha = 0\).

As an immediate consequence of the above Theorem 2.9, we have the following blow-up criteria of strong solutions:

**Corollary 2.11.** (i) Let \(0 < \alpha < 1, \ s > n/2 - \alpha\) and let \(u_0 \in H^s\). Assume that \(u\) is a strong solution to (N-S) in the class \(CL_s(0, T)\). If \(T\) is maximal, i.e., \(u\) cannot be extended in the class \(CL_s(0, T')\) for any \(T' > T\), then it holds that
\[
\int_0^T \|u(t)\|_{\dot{U}_{\infty,1/2,\infty}^0}^{\theta} \, dt = \infty, \quad \frac{2}{\theta} + \alpha = 1.
\]
In particular, we have \(\lim\sup_{t \to T} \|u(t)\|_{\dot{U}_{\infty,1/2,\infty}^0}^{\theta} = \infty\).

(ii) Let \(s > n/2\) and let \(u_0 \in H^s\). Assume that \(u\) is a strong solution to (N-S) in the class \(CL_s(0, T)\). If \(T\) is maximal, then it holds that
\[
\int_0^T \|u(t)\|_{\dot{V}_{\infty,\infty}^0}^2 \, dt = \infty.
\]
In particular, \(\lim\sup_{t \to T} \|u(t)\|_{\dot{V}_{\infty,\infty}^0}^2 = \infty\).

### 3 Proof of Theorem 2.5

We first prove Theorem 2.5 (i). To this aim, we use arguments given in Kozono-Ogawa-Taniuchi [16], Nakao-Taniuchi [21] and Kanamaru [12].

**Proof of Theorem 2.5 (i).** We first consider the case \(1 \leq \sigma < \infty\). By the definition of the Besov space, we obtain
\[
\|f\|_{B_{\infty,\infty}^\sigma} = \left( \sum_{j \in \mathbb{Z}} 2^{js_0\sigma} \|\phi_j * f\|_q^\sigma \right)^{\frac{1}{\sigma}}
\]
\[
\leq \sum_{j < -N} 2^{js_0} \|\phi_j * f\|_p + \sum_{j = N} 2^{js_0} \|\phi_j * f\|_p + \left( \sum_{j \in \mathbb{Z}} 2^{js_0} \|\phi_j * f\|_p^\sigma \right)^{\frac{1}{\sigma}}
\]
\[
=: S_1 + S_2 + S_3
\]
Concerning \( S_1 \), it holds that
\[
S_1 \leq \sum_{j<-N} 2^{fs} \| \phi_j \| \| f \|_p \| a \|^{2(s_0-s_1)} \\
\leq \| f \|_{\B^{s_0}_{p,\infty}} \sum_{j<-N} 2^{fs} \\
\leq C_1 2^{-(s_0-s_1)N} \| f \|_{\B^{s_0}_{p,\infty}},
\]
where \( C_1 \) is dependent only on \( s_0 \) and \( s_1 \). For \( S_2 \), in the same way as (3.2), we have
\[
S_2 \leq C_2 2^{-(s_1-s_0)N} \| f \|_{\B^{s_1}_{p,\infty}},
\]
where \( C_2 \) is dependent only on \( s_0 \) and \( s_2 \).

We finally estimate \( S_3 \). By Definition 2.2, it clearly follows that
\[
S_3 \leq N^\beta \| f \|_{\I^{s_0}_{p,\alpha}}.
\]
Combining (3.2), (3.3) and (3.4) with (3.1), we obtain
\[
\| f \|_{\B^{s_0}_{p,\alpha}} \leq C \left( 2^{-sN} \| f \|_{\B^{s_0}_{p,\infty}}, \| f \|_{\I^{s_0}_{p,\alpha}} + N^\beta \| f \|_{\I^{s_0}_{p,\alpha}} \right)
\]
for \( s_* := \min(s_0-s_1, s_2-s_0) \) and \( C = C(s_1, s_2-s_1) \). In the case \( \| f \|_{\B^{s_0}_{p,\infty}}, \| f \|_{\I^{s_0}_{p,\alpha}} \leq 1 \), we take \( N = 1 \) in (3.5). Then it holds that
\[
\| f \|_{\B^{s_0}_{p,\alpha}} \leq C \left( 1 + \| f \|_{\I^{s_0}_{p,\alpha}} \right) \leq C \left( 1 + \| f \|_{\I^{s_0}_{p,\alpha}} \log(\epsilon + \| f \|_{\B^{s_0}_{p,\infty}}, \| f \|_{\B^{s_0}_{p,\alpha}}) \right);
\]
this is the desired estimate (2.6). In the case \( \| f \|_{\B^{s_0}_{p,\infty}}, \| f \|_{\I^{s_0}_{p,\alpha}} > 1 \), we take \( N = 1 + \left[ \log(\epsilon + \| f \|_{\B^{s_0}_{p,\infty}}, \| f \|_{\B^{s_0}_{p,\alpha}})/s_* \log 2 \right] \) in (3.5), where \([\cdot]\) denotes the Gauß symbol. Then, we get (2.6) again.

In the case \( \sigma = \infty \), we obtain, instead of (3.1),
\[
\| f \|_{\B^{s_0}_{p,\alpha}} \leq \sup_{j<-N} 2^{js} \| \phi_j \| \| f \|_p + \sup_{j>N} 2^{js} \| \phi_j \| \| f \|_p + \max_{j \leq N} 2^{js} \| \phi_j \| \| f \|_p
\]
\[
= : \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3
\]
(3.6)
Therefore, using the same argument as in the previous case \( 1 \leq \sigma < \infty \), we get (2.6).

In order to prove the second part of Theorem 2.5, we use the following Lemma.

**Lemma 3.1.** Let \( \rho \in \mathcal{Z} \) and Let \( X \) be a normed space. Assume that \( X \) satisfies conditions (C1) and (C2) given in Theorem 2.5 (ii). Then, it holds that
\[
\rho * g \in L^\infty \quad \text{for all } g \in X.
\]

**Proof.** By (C1), we get that for all \( \phi \in \mathcal{Z} \), there exists a constant \( C = C(\phi) > 0 \) such that
\[
\| g(\phi) \| \leq C \| g \|_X \quad \text{for all } g \in X.
\]
(3.8)

Assume that (3.8) does not hold. Then, there is a \( \phi_0 \in \mathcal{Z} \) with the following property: for each positive integer \( N \), there is a \( g_N \in X \) such that
\[
\| g_N(\phi_0) \| > N \| g_N \|_X.
\]
(3.9)

Letting \( h_N := \frac{g_N}{N^\sigma \| g_N \|_X} \), we obtain \( \| h_N \|_X = N^{-\frac{\sigma}{2}} \to 0 \) as \( N \to \infty \), which implies \( h_N \to 0 \) in \( X \). By (C1), this convergence holds in \( \mathcal{Z} \). On the other hand, by (3.9),
\[
\| h_N(\phi_0) \| = \frac{|g_N(\phi_0)|}{N^\sigma \| g_N \|_X} \to N^{\frac{\sigma}{2}} \to \infty \quad \text{as } N \to \infty,
\]
which contradicts \( h_N \to 0 \) in \( \mathcal{Z} \). Thus we get (3.8).
Concerning the second term on the right-hand side of (3.13), we obtain

$$
|\rho \ast g(x)| \leq C(\rho)\|\tau_{-x}g\|_X \leq C(\rho, K_1)\|g\|_X \quad \text{for all } x \in \mathbb{R}^n,
$$

which means (3.7).

We are now in position to prove the second part of Theorem 2.5 and follow arguments given by Nakao-Taniuchi [21] and the author [12].

**Proof of Theorem 2.5 (ii).** Substituting \( f = \frac{h}{\epsilon \|h\|_{L^p_{\mu,\nu} \cap L^p_{\mu,\nu}^{\sigma}}} \) into the inequality given in (C4), we obtain

$$
\|h\|_{L^p_{\mu,\nu} \cap L^p_{\mu,\nu}^{\sigma}} \leq K_3 \left( \epsilon \|h\|_{L^p_{\mu,\nu} \cap L^p_{\mu,\nu}^{\sigma}} + \|h\|_X \log^p \left( e + \frac{1}{\epsilon} \right) \right) \tag{3.10}
$$

for all \( h \in X \cap \mathcal{Z} \) and all \( \epsilon > 0 \). Let \( g \in X \) and \( \Phi_N := \sum_{|j| \leq N} \phi_j \in \mathcal{Z} \) for \( N = 1, 2, \ldots \). By Lemma 3.1, \( \Phi_N \ast g \in L^\infty \).

Hence, since \( \Phi_N \ast g = \Phi_{N+1} \ast \Phi_N \ast g \), we have \( \Phi_N \ast g \in \mathcal{Z} \). On the other hand, it holds from (C3) that

$$
\|\Phi_N \ast g\|_X \leq K_2 \|\Phi_N\|_1 \|g\|_X \leq K_2 (\|\psi_N\|_1 + \|\psi_{N-1}\|_1) \|g\|_X \leq K_2 \|\psi_1\|_1 \|g\|_X, \tag{3.11}
$$

where \( \psi_j(x) := 2^{jn} \psi(2^j x) \). Thus, we also get \( \Phi_N \ast g \in X \). Substituting \( h = \Phi_N \ast g \in X \cap \mathcal{Z} \) into (3.10), we obtain

$$
\|\Phi_N \ast g\|_{L^p_{\mu,\nu} \cap L^p_{\mu,\nu}^{\sigma}} \leq K_3 \epsilon \|\Phi_N \ast g\|_{L^p_{\mu,\nu} \cap L^p_{\mu,\nu}^{\sigma}} + K_3 \|\Phi_N \ast g\|_X \log^p \left( e + \frac{1}{\epsilon} \right). \tag{3.12}
$$

We first consider the case \( 1 \leq \sigma < \infty \).

The left-hand side of (3.12) can be estimated from below as follows. Noting that \( \text{supp} \hat{\Phi}_N \subset \{ 2^{-N-1} \leq |\xi| \leq 2^{N+1} \} \), we get

$$
\|\Phi_N \ast g\|_{L^p_{\mu,\nu} \cap L^p_{\mu,\nu}^{\sigma}}^\sigma = \sum_{|j| \leq N+1} 2^{(s_0 + \sigma)} \|\phi_j \ast \Phi_N \ast g\|_p^\sigma \tag{3.13}
$$

Concerning the second term on the right-hand side of (3.13), we obtain

$$
\sum_{j=N,N+1} 2^{(s_0 + \sigma)} \|\phi_j \ast \Phi_N \ast g\|_p^\sigma \geq 2^{-(s_0 + \sigma)} 2^{N s_0} \sum_{j=N,N+1} \|\phi_j \ast \Phi_N \ast g\|_p^\sigma \geq 2^{-(s_0 + \sigma)} 2^{N s_0} \left( \sum_{j=N,N+1} \|\phi_j \ast \Phi_N \ast g\|_p \right)^\sigma \tag{3.14}
$$

As in (3.14), similar estimates hold when replacing \( N \) and \( N + 1 \) by \( -N \) and \( -N - 1 \), respectively. Summarizing (3.13), (3.14) we obtain that

$$
\|\Phi_N \ast g\|_{L^p_{\mu,\nu} \cap L^p_{\mu,\nu}^{\sigma}} \geq 2^{-(s_0 + \sigma)} \left( \sum_{|j| \leq N} 2^{(s_0 + \sigma)} \|\phi_j \ast g\|_p \right)^\sigma. \tag{3.15}
$$
Next, we estimate the first term on the right-hand side of (3.12). From Young’s inequality and Hölder’s inequality, it holds that

\[
\|\Phi_N * g\|_{L^{p_0}_x} = \sup_{|\alpha| \leq s_N} 2^{j_0} \|\phi_\alpha \ast \Phi_N * g\|_p \\
\leq \sup_{|\alpha| \leq s_N} 2^{j_0} \|\phi_\alpha\|_1 \|\Phi_N * g\|_p \\
\leq C_1 2^{j_0} \sum_{|\alpha| \leq N} 2^{j_0} \|\phi_\alpha \ast g\|_p \\
\leq C_1 2^{j_0} \left( \sum_{|\alpha| \leq N} \frac{1}{s_N} \left( \sum_{|\alpha| \leq N} 2^{j_0} \|\phi_\alpha \ast g\|_p^\alpha \right)^\frac{1}{\alpha} \right) \\
\leq C_1 2^{j_0} \left( \sum_{|\alpha| \leq N} \frac{1}{s_N} \|\phi_\alpha \ast g\|_p^\alpha \right)^\frac{1}{\alpha},
\tag{3.16}
\]

where \(C_1\) depends only on \(n\) and \(s_1\). In the same way as (3.16), we have

\[
\|\Phi_N * g\|_{L^{p_0}_x} \leq C_2 2^{j_0} \left( \sum_{|\alpha| \leq N} \frac{1}{s_N} \|\phi_\alpha \ast g\|_p^\alpha \right)^\frac{1}{\alpha},
\tag{3.17}
\]

where \(C_2\) depends only on \(n\) and \(s_2\). In the end, from (3.16) and (3.17), we get that

\[
\|\Phi_N * g\|_{L^{p_0}_x} \leq C_3 2^{j_0} \left( \sum_{|\alpha| \leq N} \frac{1}{s_N} \|\phi_\alpha \ast g\|_p^\alpha \right)^\frac{1}{\alpha},
\tag{3.18}
\]

for \(s^* := |s_0| + \max(|s_1|, |s_2|) + 1\) and \(C_3 = C_3(n, s_1, s_2)\).

Thus, combining (3.11), (3.15) and (3.18) with (3.12), we obtain

\[
\left( \sum_{|\alpha| \leq N} 2^{j_0} \|\phi_\alpha \ast g\|_p^\alpha \right)^\frac{1}{\alpha} \leq Ce^{2s^* N} \left( \sum_{|\alpha| \leq N} 2^{j_0} \|\phi_\alpha \ast g\|_p^\alpha \right)^\frac{1}{\alpha} + C\|g\|_X \log^\beta \left( e + \frac{1}{e} \right)
\]

for all \(N = 1, 2, \ldots, \) all \(e > 0\) and \(C = C(n, s_0, s_1, s_2, K_2, K_3)\). Taking \(e = \frac{1}{2e^{2s^*}}\), from the above inequality, we get

\[
\left( \sum_{|\alpha| \leq N} 2^{j_0} \|\phi_\alpha \ast g\|_p^\alpha \right)^\frac{1}{\alpha} \leq C\|g\|_X \text{ for all } N = 1, 2, \cdots.
\]

This implies

\[
\|g\|_{L^{p_0}_{\tilde{p}, 0}} \leq C\|g\|_X \text{ for all } g \in X,
\]

i.e., the embedding \(X \hookrightarrow L^{p_0}_{p, 0}\).

In the case \(s = \infty\), we obtain, instead of (3.13),

\[
\|\Phi_N * g\|_{L^{p_0}_x} = \max \left( \max_{|\alpha| \leq N-1} 2^{j_0} \|\phi_\alpha \ast g\|_p, \max_{j=N-1} 2^{j_0} \|\phi_\alpha \ast \Phi_N * g\|_p \right) \\
\max_{j=-N,-N-1} 2^{j_0} \|\phi_\alpha \ast \Phi_N * g\|_p.
\]

Therefore, by using the same argument as in the case \(1 \leq s < \infty\), we get

\[
\|g\|_{L^{p_0}_x} \leq C\|g\|_X \text{ for all } g \in X.
\]

This proves Theorem 2.5 (ii).
4 Proof of Theorem 2.9

In order to prove Theorem 2.9, we need bilinear estimates which are related to Leibniz’ rule. Therefore, we first recall the following two lemmata.

**Lemma 4.1** ([13], Proposition 2.2). Let $1 \leq p, q \leq \infty$, $s_0 > 0$, $\alpha > 0$ and $\beta > 0$. Moreover, assume that $1 \leq p_1, p_2, p_3, p_4 \leq \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Then, there exists a constant $C(n, s_0, \alpha, \beta) > 0$ such that

$$\|f \cdot g\|_{B^{s_0}_{p_4,q}} \leq C\left( \|f\|_{B^{s_0}_{p_1,q}} \|g\|_{B^{s_0}_{p_2,q}} + \|f\|_{B^{s_0}_{p_3,q}} \|g\|_{B^{s_0}_{p_4,q}} \right)$$

for all $f \in B^{s_0+\epsilon}_{p_1,q} \cap B^{-\beta}_{p_3,q}$ and $g \in B^{-\alpha}_{p_2,q} \cap B^{s_0+\beta}_{p_4,q}$.

**Lemma 4.2** ([18], Lemma 1). Let $1 < p < \infty$ and $a, \beta \in \mathbb{N}^n$. Then, there exists a constant $C(n, p, a, \beta) > 0$ such that

$$\|\partial^a f \cdot \partial^\beta g\|_p \leq C \left( \|f\|_{B^{a(p)}_{\infty,q}} \|g\|_p + \||-\Delta\|_p \|f\| \|g\|_{BMO} \right)$$

for all $f, g \in BMO \cap W^{a(p)}[\beta]_p$.

We are now in a position to prove Theorem 2.9 and follow arguments given by Kozono-Ogawa-Taniuchi [16], Kozono-Shimada [17], Kozono-Taniuchi [18] and the author [12].

**Proof of Theorem 2.9.** (i) It is well-known that the local existence time $T_*$ of the strong solution to (N-S) can be estimated from below as

$$T_* \geq \frac{C(n, s)}{\|u_0\|_{H^{s+1}}},$$

see e.g. [10] and [14]. Hence by the standard argument of continuation of local solutions, it suffices to establish the following *a priori* estimate:

$$\sup_{\epsilon_0 \leq t \leq T} \|u(t)\|_{H^{s+1}} \leq C \left( n, s, \alpha, T, \|u(\epsilon_0)\|_{H^{s+1}}, \int_{\epsilon_0}^T \|u(\tau)\|_{\dot{B}^{a(p)}_{\infty,q}} \, d\tau \right)$$

for some $\epsilon_0 \in (0, T)$, where $[\cdot]$ denotes the Gauß symbol.

Applying $\partial^k$ with $|k| = 0, 1, \cdots, [s] + 1$ to (N-S), we have

$$\partial_t v_k - \Delta v_k + \nabla q_k = F_k,$$

where $v_k := \partial^k u$, $q_k := \partial^k \pi$ and $F_k := -\partial^k (\nu \cdot v) u = -\partial^k \nabla \cdot u \cdot u$. Taking the inner product in $L^2$ between (4.4) and $2v_k$, and then integrating the resulting identity on the time interval $(\epsilon_0, t)$, we obtain

$$\frac{\|v_k(t)\|^2_2}{2} + \int_{\epsilon_0}^t \|\nabla v_k\|^2_2 \, d\tau \leq \|v_k(\epsilon_0)\|^2_2 + 2 \int_{\epsilon_0}^t (F_k, v_k) \, d\tau, \quad \epsilon_0 \leq t < T,$$

where

$$|(F_k, v_k)| = \langle |(-\Delta)^{-\frac{s}{2}} \partial^k \nabla \cdot u \cdot u, (-\Delta)^{-\frac{s}{2}} v_k \rangle \leq C \|u \cdot u\|_{\dot{B}^{1|1|,-s\cdot\alpha}_{2,2}} \|v_k\|_{H^s}.$$

By the bilinear estimate Lemma 4.1 (4.1) with $p = q = 2$, $p_1 = p_4 = 2$, $p_2 = p_3 = \infty$, $s_0 = 1 + |k| - \alpha, \beta = \alpha$, it follows that

$$\|u \cdot u\|_{\dot{B}^{1|1|,-s\cdot\alpha}_{2,2}} \leq C \|u\|_{\dot{B}^{s\cdot\alpha}_{2,2}} \|u\|_{\dot{B}^{s\cdot|k|}_{2,2}}.$$

Together with an interpolation inequality applied to $\|v_k\|_{H^s}$, we conclude from Young’s inequality that

$$|(F_k, v_k)| \leq C \|u\|_{\dot{B}^{s\cdot\alpha}_{2,2}} \|u\|_{\dot{B}^{s\cdot|k|}_{2,2}} \|v_k\|_{H^s} \|v_k\|_{H^s}^{1-s} \leq C \|u\|_{\dot{B}^{s\cdot\alpha}_{2,2}} \|\nabla v_k\|_{H^s}^{1-s} \|v_k\|_{H^s}^{1-s} \leq C \|u\|_{\dot{B}^{s\cdot\alpha}_{2,2}} \|v_k\|_{H^s}^{2} + \frac{1 + \alpha}{2} \|\nabla v_k\|_{H^s}^{2}.$$
where $\theta = \frac{2}{1 - p}$. $C$ depends on $n, s, a$. Inserting (4.6) to the right-hand side of (4.5), summing for $|k| = 0, 1, \cdots, [s] + 1$, and absorbing the terms $\|\nabla v_k\|_2^2$ from the right-hand side by the left-hand side, we obtain

$$\|u(t)\|_{H^0}^2 \leq \|u(\varepsilon_0)\|_{H^0}^2 + C \int_{\varepsilon_0}^t \|u(\tau)\|_{B^s_{\infty,\infty}}^\theta \|u(\tau)\|_{H^0}^2 \, d\tau,$$

for all $\varepsilon_0 \leq t < T$. By using Gronwall’s inequality, we get

$$\|u(t)\|_{H^0} \leq \|u(\varepsilon_0)\|_{H^0} \exp \left( C \int_{\varepsilon_0}^t \|u(\tau)\|_{B^s_{\infty,\infty}}^\theta \, d\tau \right), \tag{4.7}$$

Now, applying the logarithmic interpolation inequality (2.6) with $s_0 = -1, s_1 = -n/2$ and $s_2 = s - n/2 (> -a)$, $\beta = 1/\theta$, $p = \sigma = \infty$ to $f = u(\tau)$, it follows that

$$\|u(\tau)\|_{B^s_{\infty,\infty}} \leq C \left( 1 + \|u(\tau)\|_{U^{1,1/\sigma}} \log \|u(\tau)\|_{B^s_{\infty,\infty}} \right). \tag{4.8}$$

By the embeddings $B^0_{2,2} \subset B^{-n/2}_{2,\infty} \subset B^{s-n/2}_{2,\infty}$ and $H^s \subset B^s_{2,2} \subset B^s_{2,\infty} \cap B^s_{2,\infty}$, we have

$$\|u(\tau)\|_{B^s_{\infty,\infty}} \leq C \|u(\tau)\|_{B^s_{2,2}} \leq C \|u(\tau)\|_{B^s_{2,\infty}} \leq C \|u(\tau)\|_{H^s}. \tag{4.9}$$

Hence, by (4.7), (4.8) and (4.9), it holds that

$$\|u(t)\|_{H^0} \leq \|u(\varepsilon_0)\|_{H^0} \exp \left( C \int_{\varepsilon_0}^t \left( 1 + \|u(\tau)\|_{U^{1,1/\sigma}} \log (1 + \|u(\tau)\|_{H^0}) \right) \, d\tau \right),$$

where $C = C(n, s, a)$. Therefore, with $g(t) \equiv \log (1 + \|u(t)\|_{H^0})$, we obtain

$$g(t) \leq g(\varepsilon_0) + C \int_{\varepsilon_0}^t \left( 1 + \|u(\tau)\|_{U^{1,1/\sigma}} \right) g(\tau) \, d\tau.$$

Then Gronwall’s inequality implies that

$$g(t) \leq g(\varepsilon_0) \exp \left( C \int_{\varepsilon_0}^t \left( 1 + \|u(\tau)\|_{U^{1,1/\sigma}} \right) \, d\tau \right)$$

for all $\varepsilon_0 \leq t < T$. Thus, we get the estimate (4.3) in the form

$$\sup_{\varepsilon_0 \leq t < T} \|u(t)\|_{H^0} \leq \left( e + \|u(\varepsilon_0)\|_{H^0} \right) \exp \left( C_T + C \int_{\varepsilon_0}^t \|u(\tau)\|_{U^{1,1/\sigma}} \, d\tau \right).$$

(ii) By the same argument as in the above proof, it suffices to establish the following a priori estimate:

$$\sup_{\varepsilon_0 \leq t < T} \|u(t)\|_{H^0} \leq C \left( n, s, T, \|u(\varepsilon_0)\|_{H^0}, \int_{\varepsilon_0}^T \|u(\tau)\|_{U^{1,1/\sigma}}^2 \, d\tau \right) \tag{4.10}$$

for some $\varepsilon_0 \in (0, T)$.

Applying $\partial^k$ with $|k| = 0, 1, \cdots, [s] + 1$ to (N-S), we have

$$\partial_t v_k - \Delta v_k + u \cdot \nabla v_k + \nabla q_k = G_k, \tag{4.11}$$

where $v_k := \partial^k u$, $q_k := \partial^k \pi$ and $G_k := - \sum_{l,k,|l| < |k| - 1} (l) \partial^{k-l} u \cdot \nabla (\partial^l u)$. Testing (4.11) with $v_k$ and integrating the resulting identity on the time interval $(\varepsilon_0, t)$, we obtain

$$\|v_k(t)\|_2^2 + 2 \int_{\varepsilon_0}^t \|\nabla v_k\|_2^2 \, d\tau \leq \|v_k(\varepsilon_0)\|_2^2 + 2 \int_{\varepsilon_0}^t \|G_k, v_k\|_2 \, d\tau, \quad \varepsilon_0 \leq t < T. \tag{4.12}$$
Now the bilinear estimate (4.2) with $p = 2$, $|\alpha| = |k| - |l|$, $|\beta| = |l| + 1$, implies that

$$
\|G_k\|_2 \leq C\|u\|_{BMO}(\|\Delta\|^{k}_{l+1} u\|_2.
$$

(4.13)

From (4.13) and Young’s inequality we conclude that

$$
|\langle G_{k}, v_{k} \rangle| \leq \|G_{k}\|_2 \|v_{k}\|_2 \leq C\|u\|_{BMO}(\|\Delta\|^{k}_{l+1} \|u\|_2 \|v_{k}\|_2
$$

\leq C\|u\|_{BMO}^2 \|v_{k}\|_2^2 + \frac{1}{2}\|\nabla v_{k}\|_2^2,
$$

(4.14)

with $C = C(n, s)$. Inserting (4.14) to the right-hand side of (4.12) and summing for $|k| = 0, 1, \ldots, [s] + 1$, we obtain that

$$
\|u(t)\|_{H^{s+1}}^2 \leq \|u(\varepsilon_0)\|_{H^{s+1}}^2 + \frac{C}{\varepsilon_0} \int_{\varepsilon_0}^t \|u(\tau)\|_{BMO}^2 \|u(\tau)\|_{H^{s+1}}^2 \, d\tau,
$$

for all $\varepsilon_0 \leq t < T$. By using Gronwall’s inequality and then the continuous embedding $\dot{B}_{\infty, 2}^{0} \subset BMO$, we get

$$
\|u(t)\|_{H^{s+1}} \leq \|u(\varepsilon_0)\|_{H^{s+1}} \exp \left( \frac{C}{\varepsilon_0} \int_{\varepsilon_0}^t \|u(\tau)\|_{BMO}^2 \, d\tau \right),
$$

(4.15)

Now, by applying the logarithmic interpolation inequality (2.6) with $s_1 = -n/2 < s_0 = 0 < s_2 = s - n/2$, $\beta = 1/2$, $p = \infty$ and $\sigma = 2$ to $f = u(\tau)$, it follows that

$$
\|u(\tau)\|_{\dot{B}_{\infty, 2}^{0}} \leq C \left( 1 + \|u(\tau)\|_{\dot{B}_{\infty, 2}^{0}} \log^\frac{1}{2} (e + \|u(\tau)\|_{\dot{B}_{\infty, 2}^{0}}) \right).
$$

(4.16)

Here, we note that $\dot{B}_{\infty, 1/2, 2}^{0} = \dot{B}_{\infty, 0, 2}^{0}$ holds due to Proposition 2.3 (iii). Hence, combining (4.15), (4.16) and (4.9), it holds that

$$
\|u(t)\|_{H^{s+1}} \leq \|u(\varepsilon_0)\|_{H^{s+1}} \exp \left( C \int_{\varepsilon_0}^t \left( 1 + \|u(\tau)\|_{\dot{B}_{\infty, 2}^{0}} \log(e + \|u(\tau)\|_{H^{s+1}}) \right) \, d\tau \right),
$$

where $C = C(n, s)$. Therefore, letting $g(t) \equiv \log(e + \|u(t)\|_{H^{s+1}})$, we obtain

$$
g(t) \leq g(\varepsilon_0) + C \int_{\varepsilon_0}^t \left( 1 + \|u(\tau)\|_{\dot{B}_{\infty, 2}^{0}} \right) g(\tau) \, d\tau,
$$

which by Gronwall’s inequality implies that

$$
g(t) \leq g(\varepsilon_0) \exp \left( C \int_{\varepsilon_0}^t \left( 1 + \|u(\tau)\|_{\dot{B}_{\infty, 2}^{0}} \right) \, d\tau \right)
$$

for all $\varepsilon_0 \leq t < T$. Thus, we get the estimate

$$
\sup_{\varepsilon_0 \leq t < T} \|u(t)\|_{H^{s+1}} \leq \left( e + \|u(\varepsilon_0)\|_{H^{s+1}} \right) \exp \left( C T + C \int_{\varepsilon_0}^T \|u(\tau)\|_{\dot{B}_{\infty, 2}^{0}} \, d\tau \right),
$$

which is the desired estimate (4.10).

\[\square\]

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References

[1] Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Comm. Math. Phys. 94, 61-66 (1984)

[2] Beirão da Veiga, H.: A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$. Chinese Ann. Math. Ser. B 16B, 407-412 (1995)

[3] Bergh, J., Löfström, J.: Interpolation spaces. An introduction. Berlin-New York-Heidelberg, Springer-Verlag (1976)

[4] Bony, J. M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. Éc. Norm. Supér. (4) 14, 209-246 (1981)

[5] Brezis, H., Gallouet, T.: Nonlinear Schrödinger evolution equations. Nonlinear Anal. TMA 4, 677-681 (1980)

[6] Brezis, H., Wainger, S.: A note on limiting cases of Sobolev embeddings and convolution inequalities. Comm. Partial Differential Equations 5, 773-789 (1980)

[7] Chae, D.: On the well-posedness of the Triebel-Lizorkin spaces. Comm. Pure Appl. Math. 55, 654-678 (2002)

[8] Engler, H.: An alternative proof of the Brezis-Wainger inequality. Comm. Partial Differential Equations. 14(4), 541-544 (1989)

[9] Fujita, H., Kato, T.: On the Navier-Stokes initial value problem I. Arch. Rational Mech. Anal. 16, 269-315 (1964)

[10] Giga, Y.: Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system. J. Differential Equations 62, 186-212 (1986)

[11] Kanamaru, R.: Brezis-Gallouet-Wainger type inequalities and a priori estimates of strong solutions to Navier-Stokes equations. J. Funct. Anal. 278 (2020). https://doi.org/10.1016/j.jfa.2019.108277

[12] Kanamaru, R.: Optimality of logarithmic interpolation inequalities and extension criteria to the Navier-Stokes and Euler equations in Vishik spaces. J. Evol. Equ. 20, 1381-1397 (2020)

[13] Kaneko, K., Kozono, H., Shimizu, S.: Stationary solution to the Navier-Stokes equations in the scaling invariant Besov space and its regularity. Indiana Univ. Math. J. 68, 857-880 (2019)

[14] Kato, T.: Strong $L^p$-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. Math. Z. 187, 471-480 (1984)

[15] Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier-Stokes equations. Comm. Pure Appl. Math. 41, 891-907 (1988)

[16] Kozono, H., Ogawa, T., Taniuchi, Y.: The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations. Math. Z. 242, 251-278 (2002)

[17] Kozono, H., Shimada, Y.: Bilinear estimates in homogeneous Triebel-Lizorkin spaces and the Navier-Stokes equations. Math. Nachr. 276, 63-74 (2004)

[18] Kozono, H., Taniuchi, Y.: Bilinear estimates in BMO and the Navier-Stokes equations. Math. Z. 235, 173-194 (2000)

[19] Kozono, H., Taniuchi, Y.: Limiting case of Sobolev inequality in BMO, with application to the Euler equations. Comm. Math. Phys. 214, 191-200 (2000)

[20] Kozono, H., Wadade, H.: Remarks on Gagliardo-Nirenberg type inequality with critical Sobolev space and BMO. Math. Z. 259, 935-950 (2008)

[21] Nakao, K., Taniuchi, Y.: Brezis-Gallouet-Wainger type inequalities and blow-up criteria for Navier-Stokes equations in unbounded domains. Comm. Math. Phys. 357, 951-973 (2018)

[22] Nakao, K., Taniuchi, Y.: Brezis-Gallouet-Wainger type inequalities and its application to the Navier-Stokes equations. Contemp. Math. 710, 211-222 (2018)

[23] Ogawa, T., Taniuchi, Y.: On blow-up criteria of smooth solutions to the 3-D Euler equations in a bounded domain. J. Differential Equations 190, 39-63 (2003)

[24] Ogawa, T., Taniuchi, Y.: A note on blow-up criterion to the 3-D Euler equations in a bounded domain. J. Math. Fluid Mech. 5, 17-23 (2003)

[25] Ozawa, T.: On critical cases of Sobolev’s inequalities. J. Funct. Anal. 127, 259-269 (1995)

[26] Triebel, H.: Theory of Function Spaces. Akademische Verlagsgesellschaft Leipzig 1983

[27] Vishik, M.: Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. Ann. Sci. Éc. Norm. Supér. 32, 769-812 (1999)