Production of twisted particles in magnetic fields

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Abstract

The use of a (quasi)uniform magnetic field opens new possibilities for the production of twisted particles having orbital angular momenta. We ascertain these possibilities. Quantum states suitable for the creation of charged particles in a uniform magnetic field are determined. The particle penetration from a solenoid to vacuum or another solenoid is analyzed in detail. It is shown that a previously proposed approach can be utilized for the successful production of twisted positrons and positroniums. We also find a new effect of increasing the uncertainty of the phase of the particle rotation with the distance passed by the particle in the solenoid. This effect, based on the fundamentals of quantum mechanics, leads to exciting new possibilities for the production of twisted particles in real solenoids without additional particle sources therein.

Keywords: twisted particles, orbital angular momentum, uncertainty principle

1. Introduction

Twisted (vortex) particles having orbital angular momenta (OAM) now have an important place in contemporary physics (see [1]). Twisted photons and electrons were discovered in 1992 [2] and 2010 [3], respectively. The discovery of twisted neutrons [4] has been put into question in [5]. Nevertheless, only four such neutrons have been probably observed [6]. In [7], a more advanced and precise method of the generation of twisted neutrons has been developed and grating methods have also been used. Methods of producing twisted neutrons using special geometries of magnetic fields have been elaborated on in [8]. Vortex beams of atoms and molecules have been obtained in [9]. The generation of relativistic positrons carrying intrinsic OAM has been considered in [10].

The wonderful method of producing twisted particles in a uniform magnetic field based on the quantum Busch theorem [11, 12] has opened up new exciting possibilities in the discovery of new twisted particles. We confirm the main conclusions made in [11–13] and present a more detailed analysis of the physical effects accompanying the production of twisted particles in real solenoids.

Nevertheless, the necessity to put a needed target into an appropriate solenoid restricts experimental possibilities. In the present paper, we prove that twisted particle beams can also be produced thanks to the passing of initial plane waves through real solenoids (see figure 5). This possibility can extraordinarily simplify the production of twisted particles.

We use the Foldy–Wouthuysen (FW) [14] transformation. The results are exact for relativistic particles in static uniform and nonuniform magnetic fields [15]. The FW representation in relativistic quantum mechanics (QM) is equivalent to the Schrödinger representation in nonrelativistic QM and
the operators in these representations (but not in the Dirac one) are quantum-mechanical counterparts of the corresponding classical variables (see [16]). A great advantage of the FW representation is the simple form of operators corresponding to classical observables. In this representation, the Hamiltonian and all operators are even, i.e. block-diagonal (diagonal in two spinors). The passage to the classical limit usually reduces to a replacement of operators in quantum-mechanical Hamiltonians and equations of motion with the corresponding classical quantities. The possibility of such a replacement, explicitly or implicitly used in, practically, all works devoted to the FW transformation, has been rigorously proven for the stationary case in [17].

In the present study, we analyze the problem of the production of particles with OAMs in detail. We consider the penetration of twisted particle beams produced in a solenoid into a vacuum or another solenoid. This penetration is necessary for the generation of new twisted particles. Unlike previous studies, our analysis is mostly based on gauge-invariant quantum-mechanical and classical equations of motion. Our approach also uses conservation laws while they need to be taken into account for a gauge. Kinetic OAMs are not conserved in a magnetic field due to the torque causing the Larmor precession. We also propose experiments for the generation of twisted positrons and positroniums and prove that twisted particle beams can also be produced thanks to the passage of initial plane waves or beams carrying no OAM through real solenoids.

We assume that \( h = 1, c = 1 \) but include \( h \) and \( c \) into some formulas when this inclusion clarifies the problem.

The paper is organized as follows. In section 3, we solve the important problem of quantum states suitable for the production of charged particles in a uniform magnetic field. Canonical and kinetic OAMs of particles produced in real solenoids are considered in section 4. The key problem of a particle penetration from a solenoid to vacuum or another solenoid is expounded in section 5. In section 6, we show the existence of new possibilities for the production of twisted particles in real solenoids without additional devices like particle sources. Experiments for the generation of twisted positrons and positroniums are developed in sections 7 and 8, respectively. The results obtained are discussed and summarized in section 9.

2. Used approach and fundamental properties of Laguerre–Gauss wave beams

In this section, we give wave functions and consider main unusual properties of free twisted particle beams. Such beams are now widely used not only in optics but also in electron physics. One of the most important kinds of a twisted beam is a paraxial Laguerre–Gauss (LG) wave beam [18–21]. This beam is one of the known solutions of the paraxial equation which is obtained in optics and relativistic QM in the paraxial approximation (\( p_\perp \ll p \)):

\[
\left( \nabla_\perp^2 + 2ik \frac{\partial}{\partial z} \right) \Psi = 0, \quad \nabla_\perp^2 = \nabla^2 - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}.
\]

The LG wave beam is given by

\[
\Psi = A \exp(i\Phi), \quad A = \frac{C_{nl} }{w(z)} \left( \sqrt{\frac{2r}{w(z)}} \right)_{\ell} \left( \frac{2r^2}{w^2(z)} \right)_{n} \times \exp \left( -\frac{r^2}{w^2(z)} \right),
\]

\[
\Phi = \ell \phi + \frac{k_0^2}{2R(z)} - \Phi_G(z), \quad C_{nl} = \sqrt{\frac{2n!}{\pi (n + |\ell|)!}},
\]

\[
w(z) = w_0 \sqrt{1 + \frac{4z^2}{k_0^2 w_0^2}}, \quad R(z) = z + \frac{k_0^2 w_0^2 z}{4z},
\]

\[
\Phi_G(z) = (2n + |\ell| + 1) \arctan \left( \frac{2z}{k_0 w_0} \right), \quad \int | \Psi^\dagger \Psi r dr d\phi | = 1,
\]

where the real functions \( A \) and \( \Phi \) define the amplitude and phase of the beam, \( k \) is the beam wavenumber, \( w_0 \) is the minimum beam width for \( n = \ell = 0 \), \( L_{nl}^{(\ell)} \) is the generalized Laguerre polynomial, \( h/2 \) is the OAM, \( R(z) \) is the radius of curvature of the wave front, and \( n = 0, 1, 2, \ldots \) is the radial quantum number. The beam is unlocalized in the longitudinal direction \( z \) and transversely 2D-localized. For nonzero \( n, \ell \), the minimum beam width is equal to \( w_0 \sqrt{2n + |\ell| + 1} \) (see [22]).

Some properties of twisted beams are rather unusual. The recent discovery of the subluminality of light beams [23–26] and its explanation in [23, 25–28] change perceptions of structured light. When any partial wave moves with the light velocity, \( c \), the velocity of the resulting interference pattern is subluminal. A similar situation takes place when one measures the velocity of a plane wave with respect to an axis non-parallel to the wave vector [23, 25–27]. The QM predicts a change of observable properties of semiclassical Einstein quanta [28]. Semiclassical quanta of structured light are subluminal and massive. Their inertial masses coincide with kinematic ones. The quantum-mechanical and semiclassical descriptions of twisted and other structured electrons lead to similar results. The velocity and the effective mass of the structured photon and electron are quantized [28]. This effect also takes place for electrons in a uniform magnetic field [29].

Since twisted electrons have large OAMs, they also possess large magnetic momenta. Additionally, massive particles in twisted states are characterized by a tensor magnetic polarizability [30] and a measurable (spectroscopic) electric quadrupole moment [30, 31]. In a uniform magnetic field, the effect of the radiative orbital polarization of a twisted electron beam resulting in a nonzero average projection of the intrinsic OAM in the field direction should take place [32]. We should also mention specific effects in the scattering of twisted particles [33, 34].

3. Quantum states suitable for a production of charged particles in a uniform magnetic field

In this section, we study quantum states suitable for the production of charged particles in a uniform magnetic field. We suppose that \( B = Be_z, |B| = B \).
It is instructive to compare the particle production in a uniform magnetic field with the vacuum. Hereinafter, basic and nonbasic states denote quantum states eligible and ineligible for such a production. Basic states represent a complete set of quantum states and can be used to quantize the field and describe a particle. Such states are used in statistical physics. In the vacuum, a basic state cannot characterize a twisted particle because the particle momentum $p$ is fixed and the wave function has the simplest form. Certainly, particles can also be created in nonbasic states but only at specific conditions. In particular, the production of photons and electrons in states with nonzero OAMs described by LG beams requires the use of holograms, gratings or other appropriate devices. As an example, we can also note an emission of twisted photons at a helical motion of charged particles [35–40]. Therefore, the quantum states described by the LG beams in vacuum are nonbasic. We can conclude that basic states in vacuum describe only plane waves.

For a charged particle in a uniform magnetic field, the simplest quantum states are the Landau ones. For these states, only plane waves.

For the same problem, a transition to the paraxial equation generalizing the nonrelativistic Landau solution are presented, e.g. in [29, 41]. These wave functions are given by

$$\Psi = A \exp (i\ell \Phi) \exp (ip_z z) \int \Psi^\dagger \Psi \, \mathrm{d}r \mathrm{d}\phi = 1,$$

$$A = \frac{\sqrt{2\pi}}{w_m} \left( \frac{2\pi}{w_m} \right)^{\ell/2} L_{\ell}^{(0)} \left( \frac{2\pi}{w_m} \right) \exp \left( -\frac{r^2}{w_m^2} \right) \eta,$$

$$C_{\ell m} = \sqrt{\frac{2\pi}{(\pi + 1) \eta}}, \quad w_m = \frac{2}{\sqrt{\eta} \mu}.$$  \hspace{1cm} (3)

The spin function $\eta$ is an eigenfunction of the Pauli operator $\sigma_z$:

$$\sigma_z \eta^+ = \pm \eta^+, \quad \eta^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (4)

The distinctive feature of the Landau solution is the trivial (exponential) dependence of the particle wave function on $z$. Values of the operator $p_z$ are fixed and $\Psi$ is an eigenfunction of $p_z$.

For the same problem, a transition to the paraxial equation commonly used in beam theory replaces the exponential factor $\exp (ip_z z)$ in equation (3) with $\exp [-i\Phi_G (z)]$, where [29, 41]

$$\Phi_G (z) = (2n + 1 + |\ell| + \ell + 2z_k) \frac{z}{z_m}, \quad z_m = \frac{kw_m^2}{2}.$$  \hspace{1cm} (5)

Here $\Phi_G (z)$ is the Gouy phase, $z_k$ is the spin projection onto the $z$ direction, and $k = p/\hbar$. $\sigma_z$ is the spin projection on the $z$ direction, and $\Phi_G (z)$ is the paraxial approximation changes the phase. While, as a rule, the waist (focal spot size) of the beam is $z$-dependent under the paraxial approximation, $w_m$ remains constant. In the considered case, the $z$-independent waist follows from the fact that the paraxial approximation cannot change the observable properties of the Landau beam.

The paraxial approximation, however, has allowed us to find the solution defining LG beams with a $z$-dependent waist. Finally, we present the wave functions for a charged particle in a uniform magnetic field obtained in [29] and having the form of LG beams. These wave functions are defined by equation (2), but the parameters $w(z), R(z)$, and $\Phi_G (z)$ are different:

$$w(z) = w_0 \sqrt{1 + \left( \frac{w_m^2}{w_0^2} - 1 \right) \cos \frac{z}{z_m}},$$

$$R(z) = kw_m^2 \cos \frac{z}{z_m} + \frac{1}{m^2} \sin^2 \frac{z}{z_m},$$

$$\Phi_G (z) = (2n + |\ell| + 1) \arctan \left( \frac{w_m^2 \tan \frac{z}{z_m}}{w_0^2} \right) + \frac{(\ell + 2z_k)z}{z_m}.$$  \hspace{1cm} (6)

In the special case of $w_0 = w_m$, the LG beams are equivalent to the Landau beams with the same quantum numbers. Equations (1), (4) and (6) describe particles with negative and positive charges like electrons and positrons.

We note that the states defined by equations (1), (2) and (6) are scalar forms that cannot be solutions of the spinor Dirac equation. To obtain such solutions in the FW representation, we present the wave functions of these equations in the spinor form

$$\Psi = A \exp (i\Phi) \eta,$$  \hspace{1cm} (7)

where $\eta$ is defined by equation (4).

Importantly, all twisted states defined by equations (2), (3) and (6) are independent of the particle mass. Nonrelativistic Landau solutions possess the same property. The energies of these states depend on the classical angular frequency of particle revolution $\omega = |e| B/m$ and are proportional to $m^{-1}$, where $m$ is the particle mass. However, the corresponding wave functions contain only $w_m = 2/\sqrt{m\omega}$ and do not depend on $m$ [42]. All wave functions of the above-mentioned equations are also mass-independent because, like Landau wave functions, they are eigenfunctions of the operator $\pi^2$.

Both the states (5) and (6) satisfy the paraxial equation (for electrons) [29, 41]

$$\left( \pi^2 _\perp - e \Sigma \cdot B - 2iK \frac{\partial}{\partial z} \right) \Phi_{FW} = 2k^2 \Phi_{FW},$$

$$\pi^2 _\perp = -\nabla^2 _\perp + ieB \frac{\partial}{\partial \phi} + \frac{e^2 B^2 r^2}{4},$$  \hspace{1cm} (8)

have the same transverse structure (LG modes) and energy levels (see equation (10) below). Nevertheless, these states are nonequivalent. This fact should be commented on. Attentive consideration shows that the wave functions (5) are also eigenfunctions of the squared transverse and longitudinal kinetic momenta separately, while the wave functions (6) do not possess this property and are only eigenfunctions of the squared total kinetic momentum $\pi^2$. This is the reason for their nonequivalence.

We add that the oscillatory behavior of $w(z)$ in equation (6) follows from the standard description of a beam in a constant
focusing channel by means of classical electrodynamics. With the use of a different denotation,
\[ \beta(z) = \beta_0 \cos^2 \left( \sqrt{Kz} \right) + \frac{1}{\beta_0 K} \sin^2 \left( \sqrt{Kz} \right), \]  
(9)
where \( \beta \) is the optical \( \beta \) function \([43]\) and \( K \) is the focusing strength of the channel.

For relativistic particles with a negative charge, the Landau energy levels are defined by (see \([41]\) and references therein)
\[ E = \sqrt{m^2 + p_0^2 + (2n + 1 + |\ell| + |\ell + 2m|) eB}, \]
(10)
where \( n = 0, 1, 2, \ldots \) is the radial quantum number and \( \ell \) is an eigenvalue of the OAM operator projected on the z axis, \( \ell \equiv L_z/\hbar = (r \times p)_z/\hbar \). A symmetric gauge \( \mathbf{A} = B \times r/2, \mathbf{A}_\phi = Br/2, \mathbf{A}_r = A_z = 0 \) is used, where \( r \) is the cylindrical coordinate and \( r = 0 \) corresponds to the symmetry axis of the particle states. The transverse energy of the Landau levels is defined by the sum of quantum numbers \( N = n + (|\ell| + \ell)/2 + z \). Equation (10) shows that these energies depend on \( N \) and the Landau levels are degenerate. For relativistic particles with a positive charge, \( \ell \) in this equation should be replaced with \(-\ell\). In the general case,
\[ E = \sqrt{m^2 + p_0^2 + (2n + 1 + |\ell| - sgn(e)\ell - 2sgn(e)\ell j) |eB|.} \]
(11)
The energy of all states with \( sgn(e)\ell \equiv 0, 1, 2, \ldots \) and coinciding \( n \) is the same. Owing to the states with \( sgn(e)\ell > 0 \), the degeneracy of the Landau levels is infinite for any \( n \). We underline that the energy levels (11) are common for the Landau and LG beams.

However, the fact that the Landau states are the simplest ones does not mean that all Landau states are basic. This situation is not extraordinary. LG beams in a uniform magnetic field, as well as in the vacuum, are solutions of the corresponding quantum-mechanical equations but are not basic. It has been noted in \([41]\) that the Landau states with \( sgn(e)\ell = 0, -1, -2, \ldots \) and \( sgn(e)\ell = 1, 2, 3, \ldots \) substantially differ. It can be appropriately proven that the former states are basic and the latter ones are nonbasic. The proof utilizes the definite connection between QM and classical physics. It has been shown in \([17]\) with the use of the Wentzel–Kramers–Brillouin method that the classical limit of relativistic quantum-mechanical equations is reduced to the replacement of operators in the Hamiltonian and quantum-mechanical equations of motion by the respective classical quantities. This allows us to utilize classical electrodynamics for the identification of basic and nonbasic quantum states.

Hereinafter, \( \pi = p - e\mathbf{A} \) is the kinetic momentum. In the classical (non-quantum) limit of no radial motion in a uniform field, the connection between the kinetic (\( \mathcal{L} = r \times \pi \)) and canonical (\( \mathbf{L} = r \times p \)) OAMs agree with the corresponding quantum-mechanical connection and is given by (see \([11, 44]\))
\[ \mathcal{L} = L - \frac{e}{2} r \times (B \times r) = L - \frac{eB}{2} r^2, \]
(12)
where \( r = |r| \) is the radial coordinate in the plane orthogonal to \( B \). In the quantum-mechanical picture, \( L_z = \hbar \ell \) and \( \hbar \ell \) should be close to \( -(e/2)Br^2 \) when \( \ell \gg 1 \). The kinetic OAM, unlike the canonical one, is gauge-invariant (see \([44]\)). Well-known relativistic classical formulas defining the particle dynamics lead to the following relations:
\[ r = |r| = -\frac{\pi \rho}{eB}, \quad \pi_\perp = 2p_\perp = -2eA, \quad L_z = 2L_z = -eB\ell^2 = -\frac{\pi B}{eB}, \quad r^2 = -\frac{2L_z}{eB}. \]
(13)
The corresponding formula for the angular velocity of particle revolution reads
\[ \omega = -\frac{eB}{E}. \]
(14)

Some fundamental properties of the conjugate variables \( L_z = \hbar \ell \) and \( \phi \) become essentially different in the presence of a uniform magnetic field. The evolution of the canonical OAM depends on \( \ell \). The evolution of the phase (i.e. the angular position) \( \phi \) is gauge-invariant because it is defined by the quantity \( \omega = \nu_\phi/\nu_0 = \phi/|E| \).

For the symmetric gauge \( \mathbf{A} = B \times r/2 \), the following relations are valid:
\[ \frac{d\pi}{dt} = \omega \times \pi, \quad \frac{d\mathbf{A}}{dt} = \omega \times \mathbf{A}, \quad \frac{dp}{dt} = \omega \times p. \]
(15)
The Landau state radii with respect to the synchrotron motion center are defined by \([22, 45]\)
\[ \langle r^2 \rangle = \frac{2(2n + |\ell| + 1)}{|e|B}. \]
(16)
The corresponding root-mean-square radius
\[ r_{rms} = \sqrt{\frac{2(2n + |\ell| + 1)}{|e|B}}, \]
is a quantum-mechanical analog of the classical value \( r \) defined by equation (13). Therefore, the connection between the kinetic and canonical OAM operators in the FW representation is given by (see equation (12) where \( L \) is an invariant)
\[ \langle \mathcal{L}_z \rangle = (\hbar - sgn(e)(2n + |\ell| + 1)\hbar = L_z - sgn(e)[(2n + 1)\hbar + |L_z|], \]
\[ \langle \mathcal{L}_z \rangle = 2L_z = -sgn(e)(2n + |\ell| + sgn(e)\ell + 1)\hbar. \]
(17)
Due to the absence of any radial motion, the classical limit of equation (17) does not take into account the radial quantum number \( n \) and reduces to
\[ \langle \mathcal{L}_z \rangle = L_z - sgn(e)|L_z|. \]
(18)
Evidently, the classical limit of the relativistic extension of the Landau result is equivalent to the classical equation (13) when \( sgn(e)\ell \equiv 0 \) and disagrees with it when \( sgn(e)\ell > 0 \).

The degeneracy order of the Landau energy levels (11) is equal to infinity owing to the states with \( sgn(e)\ell > 0 \). For basic degenerate states, the probability of particle production can vary for different states. However, the infinite degeneracy
of the states with \( \text{sgn}(e)\ell > 0 \) hinders their inclusion into the basic states. We should also note that the quantum-mechanical description should correspond to the classical one for \( |\ell| \gg 1 \). However, equations (13) and (17) show that this is the case only for levels with \( \text{sgn}(e)\ell < 0 \). Therefore, the Landau states with \( \text{sgn}(e)\ell > 0 \) and \( |\ell| \gg 1 \) are not basic for the particle production.

There is also a serious argument showing that all states with \( \text{sgn}(e)\ell > 0 \) are nonbasic. In relativistic QM, the operator of the canonical momentum describing a charged particle in a magnetic field rotates in a definite direction (see equation (27) below). For states with \( \text{sgn}(e)\ell > 0 \), the sign of \( \ell \) does not correspond to the real direction of particle revolution. This fact unambiguously demonstrates that \( \ell \) is not connected with a particle revolution in a magnetic field and is caused by other reasons. For example, twisted particle states in free space are not basic but twisted particle beams can be obtained with special devices (see, e.g., [1]). Similarly, these states with \( \text{sgn}(e)\ell > 0 \) are also nonbasic but they can be obtained by special methods. The subsequent analysis shows that such states can be filled up due to the particle penetration from a solenoid with the same \( B \) but the antiparallel field direction (see section 5).

The new quantum-mechanical results obtained in [29] make it possible to explain the origin of the states with \( \text{sgn}(e)\ell > 0 \). It has been shown [29] that general solutions for charged particles in a uniform magnetic field are LG beams with an oscillating beam width. Oscillations do not take place when the beam waist \( w_0 \) is equal to the transverse width of the Landau levels \( w_m \) [29].

Thus, our analysis demonstrates that the Landau states can be separated into basic and nonbasic. Particles can be produced only in basic states.

Remarkably, equations (2) and (3) and the general results obtained in [29] show that the radial structure of the wave functions describing the Landau and LG beams in uniform magnetic fields and the LG beams in the vacuum are the same. The wave function should be continuous. We therefore conclude that crossing the sharp boundary between the two areas does not change the width of any Landau beam penetrating from a uniform magnetic field [46]. As a result, the beam width immediately after crossing the boundary remains equal to \( w_m \). However, increasing the distance from the boundary leads to widening of the beam i.e. increasing \( w(z) \) in the vacuum and the circular motion of the beam as a whole (see equation (23) below) in a different magnetic field. The corresponding picture in classical particle physics differs from this quantum-mechanical one due to the absence of intrinsic OAMs. For beams penetrated into the vacuum, classical particle physics predicts a beam momentum spread contrary to non-spreading twisted beams in QM. We add that multivave states are appropriately described by classical wave physics. Equation (3) and [29] demonstrate that there is also a special case. The beam width does not oscillate and the circular motion of the beam does not take place in the field \( \dot{\mathbf{B}} = B_\ell \mathbf{e}_z = -B_e \mathbf{e}_z \) because the beam waist in the two fields, \( \dot{\mathbf{B}} \) and \( \mathbf{B} \), is the same and is equal to \( w_m \). In this case, the OAM direction which is unnatural in the field \( \mathbf{B} \) (\( \text{sgn}(e)\ell > 0 \)) becomes natural in the field \( \dot{\mathbf{B}} = -\mathbf{B} \).

It is instructive to mention that in this case

\[
\pi^2_\perp = -\nabla^2_\perp + ieB \frac{\partial}{\partial \phi} + e^2 B^2 r^2 \frac{1}{4},
\]

\[
\pi^2_\parallel - \pi^2_\perp = e \left( B + |\dot{\mathbf{B}}| \right) \ell + e^2 \left( B^2 - B^2_\perp \right) r^2 \frac{1}{4} = 2eB\ell,
\]

\[
\mathbf{p}^2 - p^2_\perp = -e \left( B + |\dot{\mathbf{B}}| \right) (\ell + 2x_z) - e^2 \left( B^2 - B^2_\perp \right) r^2 \frac{1}{4} = -2eB(\ell + 2x_z).
\]

The longitudinal momentum remains independent of \( r \) and fixed after the penetration into the oppositely directed field only when \( \dot{B} = -B \) (\( B = -B \)). This property confirms that the Landau levels with \( \text{sgn}(e)\ell > 0 \) are nonbasic but can be occupied thanks to the penetration of beams with appropriate parameters.

When a particle penetrates from the vacuum to a solenoid, \( w(z) \) in the vacuum may not be equal to \( w_m \) in the magnetic field, and the structure of the wave function could be changed. Certainly in this case, the wave function should also be continuous. At any moment of time, \( w_m \) should be compared with \( w(z_0) \), where \( z_0 \) is a coordinate of the boundary between the vacuum and the solenoid. In the solenoid, the wave function has the form of an LG beam with the same energy and an oscillating beam width described in [29]. The quantity \( w(z_0) \) is a minimum beam width when \( w(z_0) < w_m \) and a maximum one when \( w(z_0) > w_m \).

Similar conclusions can be made on particle production in significantly nonuniform magnetic fields. In this case, the classical relation \( \mathbf{L} \cdot \mathbf{L} = 2L_z \) remains invalid. However, equation (12) shows that \( |L_z| > |L| \) and \( \mathbf{L} \cdot \mathbf{L} > 0 \) in any case for basic states. In addition, the above-mentioned correspondence between the commutators of the angular momentum and spin components and the related Poisson brackets in classical mechanics as well as between quantum-mechanical and classical equations of motion leads to a wide agreement between quantum-mechanical and classical results. For quantum states with \( \text{sgn}(e)\ell > 0 \) in nonuniform magnetic fields, \( \mathbf{L} \cdot \mathbf{L} \leq 0 \) in the classical limit. Therefore, these states are not eligible for particle production.

We conclude that the Landau beams with the OAM direction \( \text{sgn}(e)\ell > 0 \) can be disregarded in a study of the particle production. However, they are not nonphysical and can exist due to the penetration of beams with appropriate parameters. For real solenoids, \( \nabla \cdot \mathbf{B} = 0 \), \( (1/R)\partial(RB_\phi)/\partial(R) = -\partial B_\ell/(\partial z) \), where \( R \) is the distance from the solenoid axis. To determine the magnetic field near a boundary between the two areas, we can use known formulas given, e.g., in [11]. We suppose that \( R \ll \mathcal{L} \), where \( \mathcal{L} \) is the solenoid radius. In the needed approximation,

\[
B_\phi(R, \phi, z) = \frac{-R}{2} \frac{\partial B_\ell}{\partial z} \equiv -\frac{R}{2} B_\ell', \quad B_\phi(R, \phi, z) = 0,
\]

\[
B_z(R, \phi, z) = B_z(0, \phi, z) - \frac{R^2}{4} \frac{\partial^2 B_\ell}{\partial z^2} \equiv B_z(0, \phi, z) - \frac{R^2}{4} B_\ell''.
\]
The same coordinate system can be used for an antiparallel magnetic field. In this case, reversing the coordinate axis results in $z \rightarrow -z, \phi \rightarrow -\phi$.

The exact relativistic FW Hamiltonian is given by [15, 47–49]

$$i \frac{\partial \Psi_{FW}}{\partial t} = H_{FW} \Psi_{FW}, \quad H_{FW} = \beta \sqrt{m^2 + \pi^2} - e \Sigma \cdot B, \quad (21)$$

where $\beta$ and $\Sigma$ are the Dirac matrices and the magnetic field can be uniform or nonuniform. This Hamiltonian acts on the bispinor $\Psi_{FW} = ( \Psi_{FW}^+ \Psi_{FW}^- )$. The lower spinor is nonzero only for states with negative total energy (virtual states).

The operator equation of motion in the FW representation is similar to the corresponding classical equation, defined by the operator Lorentz force, and is given by [49] (see also [29] and references therein)

$$\frac{d\pi}{dt} = \beta \frac{e}{4} \left\{ \frac{1}{e'}, (\pi \times B - B \times \pi) \right\}, \quad \epsilon' = \sqrt{m^2 + \pi^2} - e \Sigma \cdot B, \quad (22)$$

where $\Sigma$ is the spin operator and $\beta$ is the Dirac matrix. Curly brackets denote anticommutators. If the total particle energy is positive and only the upper FW spinor is used, $\beta$ matrix can be omitted:

$$\frac{d\pi}{dt} = \frac{e}{4} \left\{ \frac{1}{e'}, (\pi \times B - B \times \pi) \right\}, \quad (23)$$

The velocity operator is given by

$$v \equiv \frac{d\pi}{dt} = \frac{1}{2} \left\{ \frac{1}{e'}, \pi \right\}. \quad (24)$$

Noncommutativity of the operators $\epsilon'$ and $\pi$ can be neglected in the weak-field approximation. Moreover, the nonuniformity of the magnetic field in solenoids is usually small and the operator equation (23) formally coincides with the corresponding classical equation:

$$\frac{d\pi}{dt} = ev \times B, \quad (25)$$

The nonzero value of $d\pi_B/(dt)$ is caused only by the centripetal force conditioning a circular motion of a charge in the transversal plane. This centripetal force orthogonal to $\pi_\perp$ rotates the vector $\pi_\perp$ but does not change its absolute value, $\pi_\perp$.

Other relativistic quantum-mechanical equations of motion are similar to the classical equations (14) and (15). Equations (23) and (25) can be written in the form

$$\frac{d\pi}{dt} = \frac{1}{2} (\omega \times \pi - \pi \times \omega), \quad \omega = -\frac{eB}{c^2} \cdot (26)$$

When we use the general operator relation $v_\phi = \{ \omega, r \}/2$,

$$\frac{d\mathbf{A}}{dt} = \frac{B \times v}{2} = \frac{1}{2} (\omega \times \mathbf{A} - \mathbf{A} \times \omega), \quad \frac{dp}{dt} = \frac{1}{2} (\omega \times p - p \times \omega). \quad (27)$$

4. Canonical and kinetic OAM of particles produced in real solenoids

In the most important applications, an external magnetic field is not perfectly uniform but smoothly depends on coordinates and has axial symmetry. A change of such a field at a distance of the order of $\sqrt{r^2}$ (see equation (16)) is often small and can even be neglected. In particular, such a situation takes place in the important case of a particle production in a real solenoid, which has been considered in [11, 12]. We will analyze the particle production in such devices in more detail. Evidently, the symmetric gauge should be changed as compared with the previous section. In the general case, the symmetry axis of the external magnetic field (z axis), i.e. the solenoid axis, does not coincide with that of particle states. Let us determine the canonical and kinetic OAMs relative to any point $O$ on the former axis. The particle states relative to the latter axis are defined by the usual quantum-mechanical equations where the magnetic field is equal to the local magnetic field $B(R_0)$. Let us choose the plane normal to $B$ and suppose that the point $O$ and the vectors $R$ and $R_0$ belong to this plane. When the radius of the circular motion of the particle is much smaller than the solenoid radius and the considered particle position is far from the solenoid edge, one can use the approximation $B(R) \approx B(R_0)$. In this case, it is convenient to choose the symmetric gauge of the vector potential as follows:

$$\mathbf{A} = A_0(R_0) + \mathbf{A}, \quad A_0 = \frac{1}{2} B(R_0) \times R_0, \quad \mathbf{A} = \frac{1}{2} B(R_0) \times r, \quad R = R_0 + r. \quad (28)$$

Evidently, $B(R_0)$ is a constant. When the particle position is far from the solenoid edge, $B_{\perp}'$ is relatively small while $B_{\parallel}'$ can be taken into account. As a result, $B_{\parallel} = B$ and $A_{\parallel,0} = B R_0/2$.

In this section, we consider the particle production far from the solenoid edge. We can immediately check that $(r) = 0, (\mathbf{A}) = 0, (\pi_\parallel) = 0$ and

$$R \times A(R) = \frac{1}{2} B_{\parallel}(R_0) \left( R_0^2 + 2 R_0 \cdot r + r^2 \right),$$

$$\langle R \times A(R) \rangle = \frac{1}{2} B_{\parallel}(R_0) \left( R_0^2 + (r^2) \right). \quad (29)$$

Consideration of the canonical and kinetic OAM at particle production is straightforward. When only the vector $\mathbf{A}$ is taken into account, one can define the intrinsic canonical OAM $(r \times \pi^{(i)} = 0)$:

$$L^{(i)}_z = \left( r \times p^{(i)} \right)_z = -e (r \times \mathbf{A})_z = -\text{sgn}(e) \frac{|r| \pi_\phi|}{2}. \quad (30)$$

The total canonical OAM is given by

$$L_z = \langle R \times p \rangle_z = \langle R \times \pi \rangle_z + e \langle R \times A \rangle_z. \quad (31)$$

Since $p^{(i)} = \pi + eA, \pi = -2eA, \text{and } R \times A = r \times A_0 + r \times \mathbf{A} + R_0 \times A_0 + R_0 \times \mathbf{A}, \text{we obtain}$
\[ L - L^{(1)} = R_0 \times \pi + e (r \times A_0 + R_0 \times A_0 + R_0 \times \mathbf{A}) \]
\[ = e (r \times A_0 + R_0 \times A_0 - R_0 \times \mathbf{A}) = e R_0 \times A_0 = \frac{e B}{2} R_0^2. \]

Evidently, the intrinsic and total canonical OAMs are conserved and do not depend on time. In a uniform magnetic field, this property is valid for any \( R_0 \). Equation (15) shows that the canonical and kinetic momenta, \( p \) and \( \pi \), are nonconserved. Taking into account equation (13), we obtain (see [44, 50, 51])
\[ L = \frac{e B}{2} \left( R_0^2 - r^2 \right). \] (32)

We suppose that the charge \( e \) can be positive and negative.

The kinetic OAM is non-conserved and its evolution is defined by that of \( R \times A (R) \). Specifically,
\[ L_\zeta = L_\zeta - \frac{e}{2} B \zeta (R_0) \left( R_0^2 + 2 R_0 r + r^2 \right) \]
\[ = L_\zeta - \frac{e}{2} B \zeta (R_0) \left[ R_0^2 + r^2 + 2 R_0 r \cos (\omega t + \phi) \right], \] (33)
where \( \omega \) is the particle rotation frequency (cyclotron frequency).

Values of \( L_\zeta \) are extremal when \( \omega t + \phi \) is equal to 0 and \( \pi \). In classical physics, \( r = \pi_+ / |e B| \) and equation (33) takes the form of equation (13) at \( R_0 = 0 \). For the electron production \( (e < 0, B_\perp = B, L_\zeta > 0) \), we obtain
\[ L_\zeta = L_\zeta + \frac{|e| B}{2} \left[ R_0^2 + \frac{\pi_+^2}{e B^2} + 2 R_0 \frac{\pi_+}{|e| B} \cos (\omega t + \phi) \right]. \] (34)

When \( R = 0 \) \( (r = -R_0) \) at the moment of particle production \( (t = 0) \), \( L_\zeta = (R \times \pi)_0 = 0 \). In the classical picture, \( L_\zeta = 0 \) but in the quantum-mechanical one \( L_\zeta \neq 0 \) [11, 12] (see also equations (17) and (18)). In the specific case \( R = 0 \), the source of twisted particles is very close to the solenoid axis and, in addition, the average kinetic OAM \( \langle L_\zeta \rangle \) is defined by \( \pi_+ \), is equal to \( L_\zeta + \pi_+ \), and does not depend on \( R_0 \).

We have considered the classical picture of production of twisted particles. The quantum-mechanical picture differs only by the presence of radial motion.

5. Particle penetration from a solenoid to vacuum or another solenoid and from vacuum to a solenoid

LG beams can penetrate from the free space into a magnetic field and the other way round [11, 12, 29] and between two solenoids. For a description of these effects, the penetration of a charged particle from a (quasi)uniform magnetic field \( B \) into free space or a solenoid with another (quasi)uniform magnetic field \( B \) antiparallel to \( B (B = -\hat{B} e_\zeta = -|\hat{B}| e_\zeta) \) can be studied in detail. We also analyze the penetration of a charged particle from the vacuum to a solenoid. However, the solution of the most important problem of the evolution of a quantum state of a charged particle during its passage through a solenoid is postponed until section 6.

We first consider the case when the symmetry axes of both solenoids coincide with the symmetry axis of particle states. We need not assume the boundary between the magnetic field and the free space or between two magnetic fields with opposite directions \( (B = -|\hat{B}| e_\zeta) \) to be sharp.

Classical electrodynamics (specifically, beam optics of charged particles) adequately describe effects taking place when a particle passes through the solenoid [52–57]. The asymmetric particle rotation can be considered and the non-oscillatory solution of a beam in an adiabatically varying focusing channel can be treated. Additional effects like acceleration, additional external transverse forces or space charge forces in a solenoid channel are also considered in classical beam optics of charged particles. Naturally, the cases of a beam starting with a centroid offset relative to the solenoid axis, a beam starting with an initial centroid angle and matching the beam from the vacuum to the solenoid or vice versa belong to questions addressed in classical beam optics as well. We restrict ourselves only to classical and quantum-mechanical descriptions of these and related problems in an approximation sufficient for the correct achievement of desired results. The relativistic quantum-mechanical description being the goal of the present study is substantially simplified in the FW representation because of the deep similarity between the classical and quantum-mechanical equations of motion [17, 49]. We note the necessity to pay attention to the canonical OAM rather than the kinetic one.

Let us consider the case when the symmetry axes of both solenoids coincide but differ from the symmetry axis of particle states. The radial magnetic field defined by equation (20) does not affect \( \pi_R \) when the particle crosses the boundary. As a result, the radial component of the momentum \( \pi_R = p_R \) conserves. In contrast, a change of the azimuthal component of the kinetic momentum takes place. Not only in the semiclassical approximation but also in QM, it is given by
\[ \frac{d\pi_\phi}{dt} = \frac{e B_R \pi_\phi}{e} = \frac{e B}{2} \frac{dz}{dz} \] (35)
Here \( e \phi_0 \perp e_R \) and \( A_0 = RB/2 \).

We disregard the orbital motion. As a result of the integration,
\[ \Delta \phi = \int \frac{e B_R \pi_\phi}{e} dt = -e R \frac{1}{2} \int \frac{\partial B}{\partial \zeta} dz = \frac{e R}{2} \left( B + |\hat{B}| \right) \] (36)
in classical electrodynamics and QM. It follows from equation (36) that the azimuthal component of the kinetic momentum changes. It can be checked that the azimuthal component of the canonical one remains unchanged. The latter property provides for the conservation of the canonical (but not kinetic) OAM. If the spin-dependent interaction is neglected, we obtain that the total momentum is turned through a definite angle in the plane orthogonal to \( e_R \). The result obtained shows that the internal canonical momentum \( p_\phi \) conserves immediately after the penetration. Indeed, \( \Delta p_\phi = \Delta \pi_\phi + e \Delta A_\phi \) and the change of the vector potential is defined by
\[ \Delta A = \frac{\Delta B \times R}{2} = -\frac{R}{2} \left( B + |\hat{B}| \right) e_\phi. \] (37)
As a result, $\Delta p_{\phi} = 0$ and $p_{\perp}$ remains unchanged. However, $\pi = p_{z}$ is changed due to the conservation of $\pi^2$. We underline that the Lorentz force in classical electrodynamics and QM is always orthogonal to $\pi$. This circumstance provides for the energy conservation.

Of course, the fields of the solenoids can be parallel. In this case, $\vec{B} = |\vec{B}|e_{z}$ and the change $B + |\vec{B}| \rightarrow B − |\vec{B}|$ should be made in equations (36) and (37). For vacuum, $\vec{B} = 0$. In any case, the penetration does not change the quantum numbers of the beam.

The classical description of the particle passing from the vacuum through the solenoid to the vacuum should be given in detail. Wonderfully, the corresponding quantum-mechanical description leads to the same results. Since we assume that the orbital motion at the penetration can be disregarded, the canonical OAM relative to the solenoid axis conserves at the penetration from the solenoid to the vacuum and from the vacuum to the solenoid. Taking into account the conservation of the canonical OAM during passing of the particle through the solenoid, we obtain the amazing coincidence of the initial and final canonical OAMs relative to the solenoid axis. We underline that this result (as well as our analysis) is gauge-invariant because all changes in the kinetic momentum are caused by field induction but not by the vector potential. Only the constancy of $p_{\perp}$ during the penetration is a gauge-dependent effect. Equations (15) and (28) show that the transverse canonical momentum has a rotating and constant part.

The transverse kinetic momentum has only the rotating part. In the vacuum, $p = \pi$. Due to the rotation of $p_{\perp}$ and $\pi_{\perp}$ in the solenoid similarly described in classical electrodynamics and QM, the final direction of $p_{\perp}$ depends on the solenoid length. It can be easily shown that the coincidence of the initial and final OAMs (in the vacuum, $L = L$) takes place only relative to the solenoid axis. It is helpful to note that the change of the initial kinetic momentum at the particle penetration to the solenoid is gauge-invariant and can be presented in the form

$$\Delta \pi_{\perp} = \Delta \pi_{\phi}e_{\phi} = -\frac{eB \times (r + R_0)}{2} = \Delta \pi_{\perp}^{(i)} + \Delta \pi_{\perp}^{(e)},$$

(38)

where the transversal vectors $\Delta \pi_{\perp}^{(i)}$ and $\Delta \pi_{\perp}^{(e)}$ are not parallel. This form allows us to separate intrinsic $(i)$ and extrinsic $(e)$ motions even for the de Broglie wave modeled by a point-like particle with the corresponding momentum. The two contributions define changes in the transversal intrinsic and extrinsic kinetic momenta. Inside the solenoid, the former momentum characterizes the particle revolution (see equation (15)). The latter vanishes in the solenoid and defines the motion of the beam as a whole in the vacuum. The vector $r$ has the same absolute value and different directions for the initial and final penetrations. In the solenoid, its direction defines the phase of the particle revolution. In the general case, the evident relation for the initial and final momenta in the vacuum, $(p_{\perp}^{(fin)})^2 = (p_{\perp}^{(ini)})^2$, does not lead to the equality of the squared transversal initial and final momenta. For the particle penetration from the magnetic field $\vec{B}$ into the vacuum or the solenoid with the magnetic field $\vec{B} = |\vec{B}|e_{z}$, we obtain

$$\Delta \pi_{\perp}^{(i)} = \Delta \pi_{\phi}e_{\phi} = \frac{eB \times (r + R_0)}{2},$$

$$\Delta \pi_{\perp}^{(e)} = \frac{eB \times r}{2}, \quad \Delta \pi_{\perp}^{(e)} = \Delta \pi_{\perp}^{(e)} = \frac{eB \times R_0}{2}. \quad (39)$$

When the particle penetrates from the vacuum to the solenoid or from the solenoid to the vacuum, the particle momentum in the vacuum can be expressed in terms of the parameters of the particle revolution and the field in the solenoid:

$$p_{\perp} = \pi_{\perp} + \frac{eB \times R}{2} = p_{\perp}^{(i)} + p_{\perp}^{(e)}, \quad \pi_{\perp} = -\frac{eB \times r}{2},$$

(40)

As a result, particles are accelerated or decelerated in the longitudinal direction when getting in or out the solenoid [54]. When we consider the penetration of LG or Gaussian beams into the solenoid, $p_{\perp}^{(i)} = 0$ and the extrinsic momentum in the vacuum, $p_{\perp}^{(e)}$, defines the transversal motion of the beam as a whole.

As a result, the final and initial OAMs relative to the axis displaced by the vector $d$ of the solenoid one differ:

$$L^{(fin)} - L^{(ini)} = d \times \left(p_{\perp}^{(fin)} - p_{\perp}^{(ini)}\right) = \frac{e}{2}d \times \left(B \times \left(r^{(fin)} - r^{(ini)}\right)\right) \neq 0. \quad (41)$$

Amazingly, even the initial and final OAMs relative to the axis of particle rotation in the solenoid (symmetric axis of particle states) are different. In this case, the OAM conservation is conditioned by the noncoincidence of $r^{(ini)} \times p_{\perp}^{(fin)}$ and $r^{(fin)} \times p_{\perp}^{(ini)}$. This property and equation (41) clearly show that the OAM conservation in the presence of a magnetic field is strongly restricted. The conclusions made are valid in classical electrodynamics and QM.

In QM, the results obtained can be supplemented. The relativistic QM in the FW representation and the paraxial QM use the Schrödinger quantum-mechanical picture. One of the fundamental properties of Schrödinger QM is a continuity of wave functions. When a charged beam in any Landau state crosses the boundary between the solenoid and the vacuum, the corresponding wave function is continuous. It follows from equations (2)–(5) that the final particle state is an LG beam (or a Gaussian beam if $f = 0$). Thus, our detailed analysis confirms the main conclusion made in [11]. The intrinsic OAM is conserved after the penetration [11] but the beam begins to move (as a whole) with the additional transverse momentum $p_{\perp}^{(e)}$. Using classical formulas obtained in the present study, we show that the Landau state can be presented by a continuum of noninteracting charged partial de Broglie waves with definite momenta and such a wave can be modelled by a point-like
particle with the corresponding momentum. Any partial wave forming the particle state freely passes through the boundary between the two areas and cannot disappear at the boundary. The additional transversal kinetic momentum $\Delta \pi^{(\ell)}$ is the same for any partial beam. Since $\langle r \rangle = 0$, this momentum does not influence the intrinsic OAM of the Landau beam (defined relatively to the axis of symmetry of the quantum state) after its penetration to the vacuum, the intrinsic OAM conserves.

An important difference between classical physics and QM is the impossibility to present a particle as a point-like object. When a particle in a plane-wave state penetrates a solenoid, the point of penetration is unknown. Therefore, the acquired transversal kinetic momentum of the particle defined by equation (38) is not definite because it depends on the point of penetration. As a result, the corresponding Landau states are characterized by different quantum numbers $n, \ell$ and the final particle state is a coherent superposition of such states (the initial particle energy is conserved). While these states can have different transverse energies, their total energy is the same.

This result should be taken into account when considering the particle penetration from a solenoid to another solenoid. In this case, the acquired transverse kinetic momentum of the particle is defined by equation (39). However, the symmetry axis of a quantum state characterizing a particle with a definite kinetic momentum cannot have a fixed position in space. Since $\Delta \pi_\perp$ depends on this position, the final particle state in the second solenoid is also a coherent superposition of states with different quantum numbers $n', \ell'$ and the same energy. There are some techniques achieving a determination of OAMs (see [1] and references therein).

We underline that all partial waves remain coherent in any magnetic field because their total energies do not change after passing through the boundary. A certain complication takes place for spin-polarized particle beams owing to the interaction of magnetic moments of particles with the magnetic field. We should take into account that crossing the boundary conserves the quantity $\pi^2 - e \Sigma \cdot B$, where $\Sigma$ is the spin operator (see [29] and references therein). In this case, penetrating particles are affected by an additional small spin-dependent force proportional to $B'$. This force slightly changes the longitudinal momentum and the kinetic energy but does not influence the transversal momentum and the total energy.

As a result, the particle in one of the Landau states cannot be in a plane wave state or in one of the Landau states after penetration from a solenoid to the vacuum and to another solenoid, respectively. For a penetration between two solenoids, the case of $B = -B$ is an exclusion. The particle state should be an appropriate twisted beam in vacuum and a spatially periodic beam (described in detail in [29]) in the second solenoid. In classical physics, the particle penetration to vacuum leads to a particle spread. Its analog in QM is an increase in the beam width $w(z)$. Therefore, the beam width at the boundary between the solenoid and vacuum is minimum and is equal to $w_0$. For the penetration of the Landau beam, $w_0 = w_m$. When $B = -B$ and two solenoids are coaxial, the Landau beam passes to the second solenoid without a change of its shape. When $B \neq -B$, the Landau beam transforms into a beam with a spatially oscillating width described in [29].

The quantity $\langle \mathcal{L}_z \rangle$ is changed after the penetration despite the conservation of $\pi^2 - e \Sigma \cdot B$. It has been noted in section 3 that the total momentum is turned in the plane orthogonal to $e_r$. The consideration of the momentum dynamics in a real solenoid explains all related problems. This consideration does not require the approximation of the sharp boundary between the solenoid and vacuum or another solenoid.

The beam penetrated into the vacuum can come from different parts of the solenoid and its motion as a whole can result in a significant particle momentum spread. This effect can be undesirable. We underline that such a spread of momenta of twisted beams should not be confused with the spread of twisted states. Any particle in a twisted state moves as a whole and is a stable object whose motion is non-spreading (see [11, 12] and our precedent explanation). The particle momentum spread vanishes when the axes of symmetry of the quantum state and the solenoid coincide. Therefore, the experimental conditions proposed in [11, 12] are optimal only on condition that $\pi_\perp = 0$ at $t = 0$. If $\pi_\perp$ is nonzero at $t = 0$, a target containing produced particles should be displaced relative to the solenoid axis by the distance $R = |\pi_\perp|/|eB|$, where $\pi_\perp$ is taken at $t = 0$.

It is instructive to analyze the more general case when the axes of symmetry of two solenoids are parallel but do not coincide. Even in the simplest case of coinciding magnetic fields in the solenoids ($B = B_0$), there is a new effect of a change of the intrinsic OAM after the penetration of the second solenoid. In this case, the radial fields in the two solenoids are not collinear and are given by

$$B_R = -R \frac{\partial B_z}{\partial z} e_R, \quad \tilde{B}_R = -\frac{\partial B_z}{\partial z} e_R, \quad R = R_0 + r, \quad \tilde{R} = \tilde{R}_0 + r.$$  \hspace{1cm} (42)

Equation (42) shows that the radial magnetic field defined by equation (20) does not influence the vector potentials of solenoid fields. If the term proportional to $B'_1$ can be neglected and the distance between the solenoid axes is defined by the vector $d$ (see figure 1), we obtain $B(R_0) = B(R_0) = B$ and $\tilde{R} = R - d$.

The particle penetration to the second solenoid leads to the appearance of the transverse component of the kinetic momentum of the beam as a whole, $\Delta \pi_\perp^{(e)}$, and to a turn of the total particle momentum in the vertical plane orthogonal to $\Delta \pi_\perp^{(e)}$. Here $\Delta \pi_\perp^{(e)} = 0$. This follows from equation (23) and precedent explanations,

$$\Delta \pi_\perp^{(e)} = \frac{e}{2} \left( B - B^{(0)} \right) \times R = \frac{e}{2} \left( B - B^{(0)} \right) \times \tilde{R}$$

$$= \frac{e}{2} \left( B - B^{(0)} \right) \times d,$$  \hspace{1cm} (43)

where $B^{(0)}$ is the magnetic field between the solenoids. The distribution of currents can be properly calculated because they are always well defined.

The horizontal motion of the beam as a whole in the second solenoid leads to an extra orbital motion changing the kinetic
and canonical OAMs of the twisted particle in this solenoid. We assume that this motion is classical and underlines that the magnetic field remains unchanged. Evidently, $\Delta \pi_\perp = 0$ and the kinetic OAM existing in the first solenoid is conserved after the penetration. The new intrinsic kinetic OAM additionally includes the term

$$R \times \Delta \pi^{(e)}_\perp,$$

where $R$ is the classical radius vector of the orbit corresponding to the extra orbital motion and is defined by equation (13).

The quantum-mechanical picture only qualitatively corresponds to this classical one. The quantum-mechanical picture is demonstrated on figures 2 and 3. Figure 2 shows how the Landau beam moving in the first solenoid and possessing some OAM is transformed into the Landau (or LG) beam with a different OAM in the second solenoid. This transformation takes place due to the acquisition of the transverse kinetic momentum $\Delta \pi^{(e)}_\perp$ by the initial beam as a whole after its penetration to the second solenoid. As a result, the OAM and quantum numbers of the beam in the second solenoid are not equal to their initial parameters. The evolution of the quantum state demonstrated in figure 3 is caused by the uncertainty of the angular velocity of particle revolution described in the next section. As a result, the uncertainty of the phase of this revolution increases with increasing the distance from the solenoid boundary. The distribution of the probability density substantially differs for the initial and final states. The left plot shows that promptly after the penetration of the second solenoid the initial beam state is described by the Landau wave function of the first solenoid. The final state demonstrated in the right plot is a Landau or LG beam with other quantum numbers.

Thus, two new wonderful effects take place in the considered case. The first effect is a change of the intrinsic OAM of the twisted particle after the penetration to another solenoid with the same magnetic field and the shifted symmetry axis (see figure 1). The second effect consists in a shift of the axis of symmetry of the quantum state due to an additional beam motion with the kinetic momentum $\Delta \pi^{(e)}_\perp$.

We can now pass to the rather general case of penetration, when the second solenoid, in addition to shifting the symmetry axis, has a different magnetic field $B \neq B_0$. In this case, the total change of the particle momentum is determined similarly to equations (36), (37) and (43). The beam dynamics of the two solenoids is described by two integrals which generalize the integrals used in equation (43). Their calculation results in

$$\Delta \pi^{(\text{total})}_\perp = \frac{e}{2} \left( B - B^{(0)} \right) \times R - \frac{e}{2} \left( B - B^{(0)} \right) \times \dot{R},$$

$$\Delta \pi^{(i)}_\perp = \frac{e}{2} \left( B - B^{(0)} \right) \times (R_0 + r),$$

$$\Delta \pi^{(e)}_\perp = \frac{e}{2} \left( B - B^{(0)} \right) \times \dot{R} + \frac{e}{2} \left( B - B^{(0)} \right) \times d. \quad (44)$$
The internal canonical momentum and, therefore, the intrinsic canonical OAM is conserved immediately after the penetration (see equation (37)). However, there is also an additional external canonical momentum \( \Delta p_{\perp} \). It is converted into the internal one because the twisted particle rotates as a whole in the magnetic field \( \tilde{B} \).

The corresponding part of the intrinsic kinetic OAM is given by

\[
\langle \tilde{L}_z \rangle = \langle L_z \rangle + \frac{e}{2} \langle \tilde{B} \times \tilde{R} \rangle \langle r^2 \rangle = \hbar \ell - \frac{e}{2} \tilde{B}_z \langle r^2 \rangle.
\]

(45)

The rest of \( \Delta \pi_{\perp}^{(\text{total})} \) defining the second contribution reads

\[
\Delta \pi_{\perp}^{(\text{total})} = \frac{e}{2} \left( \tilde{B} \times \tilde{R} - \tilde{R} \times \tilde{R}_0 \right) + \frac{e}{2} \left( \tilde{B} - \tilde{B}_0 \right) \times d.
\]

Certainly, the vector \( \Delta \pi_{\perp}^{(\text{total})} \) rotates in the second solenoid. According to equation (13), the classical radius of the corresponding circular orbit is equal to

\[
\mathcal{R} = \left| \frac{\Delta \pi_{\perp}^{(\text{total})}}{e \tilde{B}} \right|.
\]

(46)

The radius vector of this orbit \( \mathcal{R} \) is orthogonal to \( \Delta \pi_{\perp}^{(\text{total})} \) and \( \tilde{B} \). The direction of rotation is unambiguously defined by the Lorentz force. The corresponding kinetic OAM \( \Delta L_z^{(\text{kin})} \) should satisfy the relation \( \text{sgn}(e \tilde{B}_z) \Delta L_z^{(\text{kin})} \leq 0 \). The point \( \mathcal{C} \) denoting the center of the circle formed by the beam rotating in the second solenoid is characterized by the radius vector \( \mathcal{R}_0 \) (see figure 1). As a result, the axis of symmetry of the quantum state in the second solenoid is shifted at the distance \( \mathcal{R} \) as compared with the related axis in the first solenoid. However, the classical formula (46) is satisfied only approximately because the orbit radius cannot be introduced in the quantum theory. The penetration changes the intrinsic kinetic and canonical OAMs. As follows from equations (12) and (13), the classical formula for the contributions caused by \( \Delta \pi_{\perp}^{(\text{total})} \) is given by

\[
\langle \Delta L_z^{(\text{kin})} \rangle = 2 \Delta \pi_{\perp}^{(\text{total})} = -e \tilde{B}_z \mathcal{R}^2. \]

(47)

The corresponding quantum-mechanical formula reads

\[
\Delta L_z^{(\text{kin})} = \hbar l, \quad \langle \Delta L_z^{(\text{kin})} \rangle = \Delta L_z^{(\text{kin})} - \frac{e}{2} \tilde{B}_z \mathcal{R}^2,
\]

(48)

where \( l \) is integer. When \( l \gg 1 \), \( \hbar l \) should be close to \(- (e/2) \tilde{B}_z \mathcal{R}^2 \). The total intrinsic kinetic and canonical OAMs are given by

\[
\tilde{L}_z^{(i)} = \hbar (\ell + 1), \quad \langle \tilde{L}_z^{(i)} \rangle = \tilde{L}_z^{(i)} - \frac{e}{2} \tilde{B}_z \langle \mathcal{R}^2 + r^2 \rangle
\]

(49)

The picture of the evolution of the beam state is partially based on the results obtained in the next section. Promptly after the penetration, the OAMs \( \hbar l \) and \( \hbar l \) can be detached. The axis of symmetry of the quantum state in the first solenoid defines the phase of rotation of the initial beam in the second solenoid. The effect of increasing the phase uncertainty of the particle rotation with the distance passed by the particle in the second solenoid results in the growth of the phase uncertainty (see section 6). When the beam is observed far from the boundary, the canonical OAMs \( \hbar l \) and \( \hbar l \) cannot be detached and only the total intrinsic canonical OAM \( \tilde{L}_z^{(i)} \) can be defined (see figure 1).

We conclude that the complete quantum-mechanical picture of the penetration is very nontrivial due to the effect described in [11, 12]. The intrinsic canonical and kinetic OAMs of the first solenoid are defined by equations (12) and (17). The intrinsic canonical OAM \( L_z \) existing in the first solenoid remains unchanged in the second solenoid, but it is added by the quantity \( \Delta L_z^{(c)} \) of this effect is rather nontrivial because even the direction of \( L_z \) is unnatural when the field directions in the solenoids are antiparallel. We show that the penetration also leads to the appearance of additional beam rotation and shifting the axis of symmetry of the quantum state in the second solenoid.

6. Evolution of quantum states of charged particles at the penetration through a solenoid

The production of new twisted particles in a magnetic field is strongly connected with the evolution of quantum states of charged particles moving through a solenoid. Such a production is possible only because a particle penetrating from the solenoid to the vacuum and being in one of the twisted Landau states conserves its intrinsic OAM (see [11, 12] and section 5). All twisted Landau states are multiwave and phases \( \phi \) of partial waves differ. The partial waves do not interact with each other, penetrate independently, and conserve their transversal canonical momenta relative to the solenoid axis after the penetration (see section 5).

Our results underline that there is a fundamental difference between the classical and quantum-mechanical pictures of the evolution of particle states. Classical physics is based on mechanical determinism. In this case, the inversion of time \( t \to -t \) leads to a correct mechanical process where the initial and final states are swapped. Therefore, the acquisition of a nonzero intrinsic OAM by a plane wave or a beam carrying no OAM and flying through the solenoid is an effect forbidden in the classical theory. In contrast, the main specific features of QM (e.g. in the Copenhagen interpretation [58]) is its indeterminism [59]. In the quantum-mechanical picture, the inversion of time does not lead to the initial state. Therefore, the problem of the evolution of quantum states of charged particles during penetration through a solenoid should be reconsidered. The indeterminism and uncertainty of QM make the appearance of the intrinsic OAM of the charged particle passing through the solenoid possible. Both classical and quantum-mechanical pictures satisfy the strongly restricted law of conservation of only the canonical OAM relative to the solenoid axis.

For a particle flying into a solenoid with definite transversal momentum \( p_{\perp} \), classical physics allows one to determine fixed values of the orbit radius \( r \) and the OAM \( L_z \) (see
The quantum-mechanical picture substantially differs. If the initial state of a particle in the vacuum is described by a de Broglie wave, neither the axis of symmetry of a quantum state inside the solenoid nor the OAM can be defined. While the classical formula (13) gives \( L_z \gg \hbar \), real quantum-mechanical \( L_z \) cannot substantially differ from the classical value defined by this formula. Different particles are not coherent because they are in different quantum states due to the Pauli principle. However, we consider a single particle. The energy remains unchanged and all partial de Broglie waves corresponding to fixed particle momenta and characterizing the quantum state are coherent. We can choose a quantum state with any fixed axis of symmetry and \( L_z \). A determination of the evolution of this quantum state allows us to make some general conclusions applicable for arbitrary positions of the axis of symmetry and OAMs. Evidently, the velocity operator \( dr/dt \) does not commute with the Hamiltonian (21) and, therefore, the velocity is not definite. Thus, the particle trajectory is a more or less blurred band but not a circle with a definite radius. After some revolutions, the particle ‘forgets’ its initial state. As a result, the probability density to find the particle at a fixed spatial point becomes (almost) equal to the value defined by the Landau theory. We underline that this analysis is based only on the fundamentals of QM.

We can now pass to a detailed consideration of the evolution of the phase characterizing the rotational motion of the particle.

It is necessary to analyze the situation when a particle penetrating from the solenoid to the vacuum has a definitely directed nonzero transversal momentum and, therefore, a definite phase. This situation has been noted in [11]. This is realized when a particle carrying no OAM and described by a de Broglie wave passes through a rather short solenoid placed in the vacuum. Of course, the rather short solenoid with a uniform magnetic field and sharp boundaries with the vacuum is the only approximation needed to simplify the explanation. The initial phase and canonical momentum of the particle are changed by the magnetic field but remain definite up to the penetration of the vacuum. As a result, the final particle state is a de Broglie wave and the particle does not acquire an OAM. In classical physics, such a situation always takes place not only for a short but also for an arbitrarily long solenoid because the angular velocity of the particle revolution is well defined by equation (14).

The quantum-mechanical and classical descriptions of this situation in a long solenoid substantially differ. To determine the quantum state of the particle at the moment of penetration to the vacuum, we need to solve the problem of the evolution of the rotational phase \( \phi \). The evolution of the phase of the particle revolution in the transverse plane is defined only if the angular velocity of this revolution

\[
\omega = \frac{d\phi}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \phi]
\]

commutes with the Hamiltonian and, therefore, is fixed. However, equation (21) shows that this is not the case and

\[
\omega = \frac{1}{2} \left\{ \frac{1}{\epsilon^2} \pi_{\phi}, r \right\} = \frac{\pi_{\phi}}{r} = \frac{p_{\phi}}{r} - eB = \frac{-i\hbar}{r^2} \partial \phi - \frac{eB}{2}.
\]

We consider positive-energy states and omit the matrix \( \beta \). Evidently, the operator \( r \) does not commute with the FW Hamiltonian and \([\mathcal{H}_{FW}, \omega] \neq 0\). Specifically,

\[
\left\{ \pi_{\phi}, \frac{1}{r^2} \right\} = -i\hbar \left[ \pi \cdot \nabla \left( \frac{1}{r^2} \right) + \nabla \left( \frac{1}{r^2} \right) \cdot \pi \right] = 2i \left\{ r, \frac{1}{r^3} \right\},
\]

\[
i \left[ \mathcal{H}_{FW}, \frac{1}{r^2} \right] = -\frac{1}{2} \left\{ \frac{1}{\epsilon^2}, \left\{ r, \frac{1}{r^3} \right\} \right\},
\]

and

\[
\frac{d\omega}{dt} = i \left[ \mathcal{H}_{FW}, \omega \right] = -\frac{1}{4} \left\{ \frac{1}{\epsilon^2}, \left\{ L_z, \frac{1}{r^3} \right\} \right\}, L_z = -\frac{i}{\hbar} \frac{\partial \phi}{\partial \phi},
\]

(51)

The operators \( L_z \) and \( \epsilon^2 \) commute with the FW Hamiltonian. Certainly,

\[
\left\{ r, \frac{1}{r^3} \right\} \Psi_{FW} = -\frac{i}{2} \frac{\partial \Psi_{FW}}{\partial r} + \frac{3}{4} \Psi_{FW}.
\]

The phase defined by the direction of \( \pi_{\perp} \) remains unchanged for the beam in the vacuum but becomes uncertain in a magnetic field. We shall add that the radial motion defined by the operator \( r^2 \) and vanishing in the classical limit is always nonzero in the quantum-mechanical picture.

The equations presented in this section clarify the quantum-mechanical picture. While the radius \( r \) (see equation (13)) and the angular velocity \( \omega \) are well defined in classical electrodynamics, they become undefined in QM. In contrast, the OAM is fixed. As a result, the particle penetration into the solenoid leads to an uncertainty of the phase of the particle rotation.

Equation (11) shows that the OAM is restricted and satisfies the condition

\[
|L_z^{\text{max}}| = \frac{\left( p_{\perp}^{(i) \text{vac}} \right)^2}{2|e|B} = \frac{\hbar}{2} (2n + 1 + 2x),
\]

where \( p_{\perp}^{(i) \text{vac}} \) is the transversal momentum in the vacuum. A decrease of \( |L_z| \) leads to the corresponding increase of \( n \).

As a result, \( \omega \) is not fixed even in the semiclassical case when \(|\ell| \gg 1, n \sim 1\). Its dispersion is defined by \( D(\omega) = \langle (\omega - \langle \omega \rangle)^2 \rangle \) and is nonzero. In particular, the analytical derivation presented in the appendix shows that the average value and dispersion of \( \omega \) are given by

\[
\langle \omega \rangle = -\frac{eB}{E}, \quad D(\omega) = \frac{e^2B^2}{4E^2} \frac{(2n + 1)|\ell| + 1}{(\ell^2 - 1)},
\]

(52)
where \( E \) is defined by equation (11). The relative root-mean-square deviation reads

\[
\delta = \frac{\sqrt{\langle \omega^2 \rangle}}{\langle \omega \rangle} = \frac{\sqrt{(2n + 1) |\ell| + 1}}{4(\ell^2 - 1)}. \tag{53}
\]

In particular, \( \delta \approx 0.087 \) when \( |\ell| = 100, n = 1; \) \( \delta \approx 0.027 \) when \( |\ell| = 1000, n = 1; \) and \( \delta \approx 0.042 \) when \( |\ell| = 1000, n = 3. \) These values have been confirmed by numerical calculations. Thus, the root-mean-square uncertainty of the phase is equal to \( 2\pi \) after approximately 11.5, 37.0, and 23.8 revolutions, respectively. While our derivation is valid for any direction of \( L_z, \) we consider only the basic states \( \text{sgn}(\ell)\ell \leq 0. \)

An approximate description of the growth of the phase uncertainty can be done with the assumption that the permissible phases of the beam revolution belong the interval \( [\omega_1 t, \omega_2 t], \) where \( \omega_1 = (1 - \delta/2)\langle \omega \rangle, \omega_2 = (1 + \delta/2)\langle \omega \rangle. \) It is convenient to use the angular distance \( \phi = (\omega) t \) instead of \( t. \) In this case, the phase spread is approximated by the quantity \( (\omega_2 - \omega_1)t = \delta \phi. \) The situation is illustrated by figure 4. Figure 5 shows an evolution of a quantum state. The initial state of the particle in the solenoid (the first ring) is characterized by a definite phase and momentum and is described by a single wave. The last ring in the solenoid denotes the final state of the particle being a multiwave Landau state. After the particle penetration to the vacuum, it is transformed to an LG state. In both states, the probability density of finding the particle is the same for any azimuth.

We straightforwardly obtain the following quantum-mechanical picture. We cannot define the moment of time of the particle penetration from the vacuum to the solenoid because the energy is fixed. Therefore, we choose any point on the boundary between the vacuum and the solenoid and check the particle motion on a helix. To simplify the analysis, we use the approximation of sharp boundaries while the described effect takes place in all cases. Very close to the boundary, the transversal kinetic momentum of the particle has the well-defined direction \( \pi_\perp = p_\perp^0_{\text{vac}}. \) However, \( r \) and, therefore, \( \omega \) are not well defined and increasing the distance from the boundary leads to the appearance of an infinite set of different directions. In contrast, the OAM operator \( -i\hbar \partial/\partial \phi \) commutes with the Hamiltonian and \( \ell \) is fixed. Since the particle cannot be characterized by a definite transversal momentum and a definite phase, the particle state is multiwave (any partial wave has a definite phase). The root-mean-square uncertainty of the phase increases with increasing the distance from the boundary. When this uncertainty reaches \( 2\pi \) (or a bigger value), the average transversal kinetic momentum of the particle is approximately or exactly equal to zero. This result is quite natural because the OAM is fixed in any Landau state and the uncertainty relation between the OAM and the phase \([60-66]\) forbids the phase to be definite. Equations (52) and (53) describe the smooth transition from classical to quantum physics provided by the semiclassical approximation.

There are also minor reasons accelerating the phase spread. One of them follows from the more precise formula for the radial magnetic field \([11, 53]\):

\[
B_R(R, \phi, z) = -\frac{R}{2}B'_z + \frac{R^3}{16}B''_z - \ldots. \tag{54}
\]

The nonzero \( B''_z \) influences the longitudinal and transversal components of the particle momentum. A change of the latter component affects the quantity \( \omega \) and, therefore, contributes to the phase spread. The nonlinear transverse force caused by the nonzero \( B''_z \) increases the phase space volume. The first term in equation (54) influences the longitudinal and transversal components as well, but this term preserves the phase space volume.

As it has been mentioned above, the partial waves do not interact with each other, the particle state after the penetration
to the vacuum is also multivariate and the average transversal momentum of the particle is equal to zero. Due to the conservation of the intrinsic OAM after the penetration to the vacuum (see [11] and section 5), the final particle state in the vacuum is a twisted beam. The Landau state in the magnetic field transforms to the LG beam in the vacuum.

We should also take into account the degeneracy of the energy of the Landau states. The particle state after the penetration of the magnetic field is a superposition of Landau quantum states with different \( n \) and \( | \ell | \) but fixed \( 2n + | \ell | - \text{sgn}(e) \ell \). All these states have definite initial phases and are characterized by the same \( \langle \omega \rangle \) and different \( \delta \). The phase spread increases with the growth of \( \phi \) and reaches \( 2\pi \) for any state, if the solenoid is long enough. For Landau states with different quantum numbers, the quantities \( p_{\perp} \) and \( p_{\parallel} \), but not \( p_{\perp} \) and the axes of symmetry can differ.

A similar situation takes place when the charged particle penetrates from another solenoid and is in a multistate wave. In such states, there is an additional factor in the difference of longitudinal momenta of different partial waves accelerating the appearance of the uncertainty in the phase of the particle rotation. When the solenoid length is rather small, this uncertainty can be neglected and the particle exits the solenoid with the same intrinsic OAM, \( L_{i}^{(1)} \), as in the initial state. When this length is large enough, the intrinsic OAM of the exiting particle, \( L_{e}^{(1)} \), can significantly differ from the intrinsic OAM of the entering particle. Evidently, the intrinsic OAM \( L_{i}^{(1)} \) defined by equation (49) does not depend on the phase. The phase uncertainty is not changed during the rotation and is followed by the stable conserved OAM.

We conclude that the particle motion into the solenoid leads to an uncertainty in the phase of the particle rotation. This effect follows from the fundamentals of QM. This can open exciting new possibilities for the production of twisted particles in real solenoids. There are many useful methods of production of twisted particles. One can use spiral phase plates, fork grating holograms, a magnetic needle, and an immersed cathode being a target (11, 12) (see also [1] and references therein). A space-variant Wien filter converts the spin angular momentum into the OAM of the beam [67]. Our method is much simpler and requires only a real solenoid without additional particle sources there.

7. Production of charged twisted particles and a measurement of their OAM in magnetic fields

The fulfilled theoretical analysis allows us to propose a further development of the method of production of charged twisted particles in magnetic fields proposed in [11, 12]. This method is based on the quantum Busch theorem (see [11, 12]). The Busch theorem [68] states that the canonical angular momentum (OAM) of a particle moving in a region with an axially symmetric magnetic field is conserved. Due to the conservation of the canonical angular momentum, charged particle beams which are generated inside a solenoid field acquire OAMs outside of the solenoid field. Busch’s theorem relates these OAMs to the field strength and the beam sizes on the cathode. The quantum Busch theorem states that the canonical angular momentum is quantized as follows [11]:

\[
L_{z} = \hbar \ell = -\frac{e}{2} B_{z}(0) \langle \tilde{r}^2 \rangle,
\]

where \( B_{z}(0) \) is the field on the axis of the solenoid. In the present study, only the production of twisted positrons and positroniums is considered while the developed method can be applied to other charged twisted particles. The method is also based on the quantum Busch theorem (see [11, 12]) but uses a specific experimental design. We propose to utilize a positron source placed in a quasi-uniform magnetic field of a real solenoid. The results presented in section 3 show that charged particles produced in uniform and even nonuniform magnetic fields have definite directions of the OAMs. Their orbital polarization can be rather large. In this case, the spin polarization of the produced particles can be neglected. The connection between the canonical and kinetic OAMs of created particles is analyzed in section 4. The variation of the direction and value of the magnetic induction in the solenoid changes the direction and values of the OAMs.

We assume that the emitted positrons and other particles penetrate from the solenoid and are then governed by a quadrupole magnetic field. This field acts on the large magnetic moments of twisted particles caused by their large OAMs [69]:

\[
\mu = \frac{e(L + 2s)}{2\gamma m},
\]

where \( \gamma \) is the Lorentz factor. As a result, the particle motion is not rectilinear. A narrow source outlet decreases the beam momentum spread after penetration. Let the \( z \) axis of the Cartesian coordinate system coincide with the symmetry axes of the solenoid and source of particles. Similar to the Stern–Gerlach experiment [70], one should deflect particles with different orbital polarizations. Since the OAMs of all particles are parallel and antiparallel to the \( z \) axis, one needs to use a magnet with maximum \( |B_{z}| \), minimum \( |B_{z}| \), and \( B_{z} = 0 \). In this case,

\[
\frac{\partial B_{z}}{\partial x} = \frac{\partial B_{x}}{\partial z}, \quad \tilde{B}_{x} = \frac{\partial B_{z}}{\partial x} z.
\]

We can present this magnetic field as the sum of the uniform field \( \tilde{B} = \tilde{B}_{z}(x = y = 0) e_{z} \equiv \tilde{B} e_{z} \) and the quadrupole magnetic field

\[
\tilde{B}_{x} = \kappa z, \quad \tilde{B}_{y} = 0, \quad \tilde{B}_{z} = \kappa x, \quad \kappa = \frac{\tilde{B}_{z}}{\partial x}.
\]

We suppose that \( \tilde{B}_{x}(x = y = 0) > 0 \) and the magnetic field in the solenoid section, \( \tilde{B} \), can be parallel and antiparallel to the \( z \) direction.

It is convenient to apply the magnetic field line tangent to the vector \( \tilde{B} \). These lines are defined by the equation

\[
\frac{dx}{dz} = \frac{\tilde{B}_{x}}{\tilde{B}_{z}} = \frac{\kappa z}{B + \kappa x}.
\]
The line containing the point \((x_0, y_0 = 0, z_0)\) and tangent to \(\vec{B}\) has the form
\[
z^2 = \left( x + \frac{2\vec{B}}{\kappa} \right) x - \left( x_0 + \frac{2\vec{B}}{\kappa} \right) x_0 + z_0^2.
\] (57)

Since the inversion \(z \rightarrow -z\) takes place, it is convenient to choose the points \((x_0, y_0 = 0, z_0 = 0)\) and vary \(x_0\). When \(\kappa\) is small, the magnetic field lines are parabolas:
\[
z^2 = \frac{2\vec{B} (x - x_0)}{\kappa}.
\] (58)

Evidently, needed magnet parameters in the proposed experiment and the Stern–Gerlach one [70] substantially differ. The force acting on twisted particles is given by
\[
f = \nabla \left( \mathbf{\mu} \cdot \vec{B} \right), \quad f_\ell = \frac{eL_\ell}{2\gamma m} \frac{\partial B_y}{\partial x}.
\] (59)

It deflects twisted particles in the \(x\) direction. Importantly, the Lorentz force acting due to the weak field \(B_y\) on charges leads to a deflection in the \(y\) direction. The deflection is independent of \(L_\ell\).

It is very important that twisted particles created in the uniform magnetic field remain twisted and non-spreading after the penetration to the vacuum [11, 12] or to a different magnetic field (see section 5). In the latter case, the beam width spatially oscillates [29]. When the axes of symmetry of the solenoid and the quantum state of a twisted particle coincide, the particle penetrating into the vacuum or a different uniform magnetic field moves in the same direction (the \(z\) axis). In a nonuniform magnetic field, the required deflection is provided by the force (59). When the above-mentioned axes do not coincide, the penetrating particle does not move along the solenoid axis (i.e. \(x \neq 0\) or \(y \neq 0\)). Certainly, such a behavior which reason is expounded in section 5 is rather unwanted. As a result, the diameter of the outlet of the particle source should be small enough in comparison with the deflection of the twisted particle. This is very possible because the deflection is proportional to its OAM, which can be rather large. Thus, some twisted particles can be successfully produced.

We propose the use of the developed experimental design for the production of twisted positrons (see figure 6). We suggest applying a solenoid with a variable (quasi)uniform field for the production of twisted positron beams which then penetrate into magnet 7, giving the sum of uniform and quadrupole magnetic fields. The total field satisfies equation (56). The beam OAMs can be measured because beams with different OAMs are registered in different parts of the target. The blue and green lines demonstrate that the beams are produced at opposite field directions in the solenoid. The field direction shown on figure 6 corresponds to the OAM \(L_1\).

Section 5 shows that the beam penetration between two solenoids with noncoincident fields (including the case of \(\vec{B} = -\vec{B}\)) leads to a nonzero transversal motion of the beam with the momentum \(p_{\perp}^{(c)} = \Delta \kappa^{(c)}\). Therefore, the use of the same (single) solenoid in the two sections is also eventual. In this case, the main field in these sections is the same \((\vec{B} = \vec{B})\) and is reversed for the OAMs \(L_1\) and \(L_2\). The deflecting quadrupole fields remain unchanged.

The standard positron source features a very broad energy spread (2–10 MeV) and very large transverse momenta. The transverse momenta are directed in all directions. This circumstance is rather unwanted. Better beam conditions are found in so-called slow-position sources (see, e.g. [71, 72]) which can also be used for the generation of positronium. An in-depth discussion of the source parameters can be considered as an outlook.

Possibly, the easiest way to create twisted positrons is through the optical transformation of an asymmetric Hermite-like beam from a storage ring into an LG-like beam by means of three skew-quadrupoles. This method does not require a solenoid with a positron source.

8. Potential for a production of twisted positroniums

Another exciting possibility is the production of twisted positronium. A positronium is an atom formed by an electron and a positron. Parapositronium and orthopositronium have spins 0 and 1, respectively. Twisted positrons are emitted in the solenoid by the magnetic field \(\vec{B}\) and then penetrate into the vacuum or a medium. Their effective masses in the vacuum are given by [28]
\[
M = \sqrt{m^2 + <p_{\perp}^2>} = \sqrt{m^2 + \frac{2(2n + |\ell| + 1)}{w_0^2}}.
\] (60)

Here \(w_0 = w_m\) (see section 5) and \(w_m\) is given by equation (3). When \(B = 1\) T, \(w_m = 5.1 \times 10^{-3}\) m and the quantities \(1/w_m\) and \(1/(2mw_0^2)\) are equivalent to 3.9 eV and 1.5 \(\times 10^{-5}\) eV, respectively. When the OAM is relatively large, \(|\ell| \approx 10^4\), and \(n \approx 1, 2(2n + |\ell| + 1)/(2mw_0^2)\) is approximately equivalent to 0.3 eV. The last value defines the difference \(M - m\).
After penetration, the twisted positron can pick up an electron and can form a positronium which should also be twisted. The mass $m_{\text{Ps}}$ is equal to the doubled electron mass corrected for the binding energy (approximately 6.8 eV). The effective mass of the twisted positronium in vacuum is also defined by equation (60) (with $m_{\text{Ps}}$ instead of $m$) but $w_0$ can be different. In the frame, in which the momentum of the twisted positronium vanishes, its total energy is equal to $Mc^2$. To form the positronium, the electron and positron should have a smaller total kinetic energy in their center-of-mass frame than the binding energy. This requirement leads to a need for thermalization (deceleration) of the positron. This goal can be achieved with an appropriate electric field in vacuum or by passing the positron through a medium. The momentum and OAM of the positron are usually collinear. Since the decelerating electric field is antiparallel to the positron momentum, the first method conserves the positron OAM. The second method decreases it because the positron can lose the OAM owing to interaction with the medium. When the difference $M - m_{\text{Ps}}$ is less than the binding energy of the positronium, the spontaneous decomposition of Ps into $e^+$ and $e^-$ is impossible. Therefore, it is preferable when the condition $M - m_{\text{Ps}} < 6.8$ eV is satisfied. The estimate made in the previous paragraph shows that its satisfaction always (or almost always) takes place.

To determine the difference between the effective mass of positronium and $m_{\text{Ps}}$, one can monitor the positronium annihilation. Orthopositronium and parapositronium annihilate with an emission of three and two gamma quanta, respectively. The total momentum of these quanta is equal to zero in the frame in which their total energy is equal to $Mc^2$. Since measurements of positronium annihilation have high precision, a determination of difference $M - m_{\text{Ps}} \sim 0.1$ eV is quite possible.

Thus, the proposed experiment with the positron source within a solenoid allows one a successful production of the twisted positronium.

9. Discussion and summary

Important results obtained by Floettmann and Karlovets [11] and Karlovets [12] have opened wonderful possibilities for discoveries of new twisted particles having OAMs. In the present study, we have considered these possibilities in more detail and have determined the potential for the production of twisted positrons and positroniums.

The new possibilities found in [11, 12] are based on the use of a (quasi)uniform magnetic field of a solenoid. We have specified quantum states suitable for the production of charged particles in a uniform magnetic field. The analysis of canonical and kinetic OAMs of particles produced in a (quasi)uniform field of real solenoids has been fulfilled. We have expounded the particle penetration from a solenoid to the vacuum or another solenoid. This detailed analysis is necessary for the substantiation of our principal proposition to produce twisted particles at passing particles carrying no OAM through the magnetic field of a standard solenoid.

We demonstrate in section 5 that the torque caused by the magnetic field strongly restricts the OAM conservation law and reduces it to the conservation of the total canonical OAM relative to the solenoid axis. This restricted law is satisfied when the final state characterizes a particle with an azimuth of the $p_\phi$ direction or an LG beam with the corresponding intrinsic OAM. The first and second possibilities are realized in the classical and quantum-mechanical pictures, respectively, due to the determinism of classical physics and the indeterminism of QM. In QM, the phase and angular velocity of the particle revolution in the magnetic field do not commute with the Hamiltonian and are not definite. If the particle leaves the solenoid, which is long enough, all phases are equally probable (see section 6). If the partial states characterized by these phases are noncoherent, the particle would fly out of the solenoid with zero OAM. However, such partial states are perfectly coherent and the final quantum state of the particle is a twisted LG beam. The found effect is very important because it can open new possibilities in the production of charged twisted particles without additional devices like particle sources [11, 12], spiral phase plates, and gratings (see [1] and references therein).

We show that the particle penetration from one solenoid to another solenoid usually leads to the appearance of additional beam rotation and shifting the axis of symmetry of the quantum state in the second solenoid. Finally, we have developed experiments allowing a production of twisted positrons and positroniums. Placing a positron source into a solenoid leads to the appearance of twisted positrons and, as a result of their interaction with medium, twisted positroniums. We have also developed methods for checking their twisting. We propose that these methods can also be applicable to other charged particles.

In summary, the main results obtained in the present study are the following. We have specified the basic states of particles in a uniform magnetic field eligible for particle production. We show that not all Landau states are basic. The penetration of a charged particle from a (quasi)uniform magnetic field into the vacuum or a solenoid with another (quasi)uniform magnetic field and from the vacuum to a solenoid has been analyzed in detail. The effect of evolution of a quantum state of a charged particle during its passage through a solenoid has been found and studied. This effect allows us to propose a new method of production of twisting particle beams based on an acquisition of an OAM by a charged particle passing through a (quasi)uniform magnetic field. The production of twisted positrons and positroniums based on the quantum Busch theorem and developments made in [11, 12] has also been proposed.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix. Analytical derivation of particle phase dynamics in the solenoid

The angular velocity operator is defined by equation (50). We use the semiclassical approximation. In this approximation, $|\Delta p| \cdot |\Delta r| \ll |p| \cdot |r|$, $|\Delta L_c| \cdot |\Delta \phi| \ll |L_c| \cdot |\phi|$. Therefore, we can neglect the noncommutation of some operators and the operator $\omega^2$ can be presented as follows:

$$\omega^2 = \left\{ \frac{1}{2e^2} - \frac{F_B}{r^2} \frac{\partial}{\partial \phi} + \frac{ie B}{2} \right\}.$$

We use the Landau wave function (2) and the relations $\frac{1}{\ell} |\Psi\rangle = \frac{1}{\ell} \hat{\Psi}$ and $\langle \Psi | \frac{1}{\ell} = \frac{1}{\ell} \hat{\Psi}$. As a result, we now can calculate the following expectation values:

$$\langle \omega \rangle = -\frac{1}{E} \left( \frac{i\hbar}{r^2} \frac{\partial}{\partial \phi} + \frac{eB}{2} \right).$$

We need to follow the usual integrals of Laguerre:

$$\int_{0}^{\infty} x^{\ell+1} \left( L_n^\ell \right) (x)^2 e^{-x} dx = \frac{(n+|\ell|)!}{n!} (2n+|\ell|+1),$$

$$\int_{0}^{\infty} x^{\ell-1} \left( L_n^\ell \right) (x)^2 e^{-x} dx = \frac{(n+|\ell|)!}{|\ell|n!},$$

$$\int_{0}^{\infty} x^{\alpha-1} L_m^\lambda (px) L_\beta^\gamma (px) e^{-mx} dx = \frac{p^{-\alpha} \Gamma (n+\alpha+\beta+1) \Gamma (n+\lambda+1)}{m^n n! \Gamma (1-\alpha+\beta) \Gamma (1+\lambda)} \Lambda F_2$$

$$\times (-m, \alpha-\alpha; -n+\alpha-\beta, \lambda+1; 1).$$

The last integral is taken from the book [73], where $\Lambda F_2$ is the generalized hypergeometric function, and $\Gamma$ is the Gamma one. We need to calculate the integral

$$\int_{0}^{\infty} x^{\ell-2} \left( L_n^\ell \right) (x)^2 e^{-x} dx.$$

After the substitution $p = 1$, $\lambda = \beta = |\ell|$, $m = n = n$, $\alpha = |\ell|$, we obtain

$$\int_{0}^{\infty} x^{\ell-2} \left( L_n^\ell \right) (x)^2 e^{-x} dx =$$

$$\frac{\Gamma (|\ell| - 1) \Gamma (n-|\ell|+1+|\ell|) \Gamma (n+|\ell|+1)}{n^n n! \Gamma (1-\ell+|\ell|+1) \Gamma (|\ell|+1)} \Lambda F_2$$

$$\times (-n, |\ell| - 1, |\ell| - 1 - |\ell|; -n+|\ell| - 1, |\ell| + 1; 1)$$

$$= \frac{2n+|\ell|+1}{(|\ell|^2 - 1)|\ell|} \frac{1}{\ell} \frac{\Gamma (n+|\ell|+1)}{n!}.$$

We can now calculate the following expectation values:

$$\langle \omega \rangle = -\frac{1}{E} \left( \frac{i\hbar}{r^2} \frac{\partial}{\partial \phi} + \frac{eB}{2} \right).$$

The expectation value of the angular velocity can now be derived:

$$\langle \omega \rangle = -\frac{1}{E} \left( \frac{i\hbar}{r^2} \frac{\partial}{\partial \phi} + \frac{eB}{2} \right).$$

$$= -\frac{1}{E} \left( \frac{eB}{2r^2} + \frac{eB}{2} \right).$$

$$= \frac{eB}{2E} \left( \frac{eB}{2} \right).$$

where $E = \sqrt{m_0^2 + p^2 + (2n+1+|\ell|+\ell+2x_2)|e|} B$ is the relativistic energy of the Landau level. The equation (69) is in line with the results obtained in [44, 74, 75] with the nonrelativistic quantum-mechanical probability current $j_\phi$. For basic states (see section 3) with $\ell \neq 0$, $\ell/e = -|\ell|/|e|$ and
\[ \langle \omega \rangle = -eB/E = \pm \omega_c, \]  
where \( \omega_c = |eB/E \) is the cyclotron frequency. For nonbasic states, \( \ell/\ell = \ell/|e| \) and \( \langle \omega \rangle = 0 \). When \( \ell = 0 \), we obtain \( p_\phi = 0, \pi_\phi/r = -(eB)/2 \), and, therefore, \( \langle \omega \rangle = -eB/(2E) = \pm \omega_c/2 \).

The expectation value of \( \omega^2 \) is given by

\[
\langle \omega^2 \rangle = \left\langle \frac{1}{2e^2} - \frac{\hbar^2}{\ell^2} \right\rangle = \frac{1}{E^2} \left\langle \frac{1}{p^2} - \hbar eB \right\rangle = \frac{1}{E^2} \left\langle \frac{1}{p^2} + eB^2 \right\rangle = \frac{1}{E^2} \left\langle \left( \frac{2n + |\ell| + 1}{\ell^2 - 1} \right) \frac{1}{|\ell|} - \frac{eB^2}{4} \right\rangle = \frac{1}{E^2} \left( \frac{2n + |\ell| + 1}{\ell^2 - 1} \right) \frac{1}{|\ell|} + 1 = \frac{eB^2}{4}. (70)
\]

The semiclassical approximation used here satisfies the conditions \( |\ell| > 1, e = -|\ell|/|e|, n \sim 1 \). As a result,

\[
\langle \omega^2 \rangle = \frac{1}{E^2} \left( \frac{2n + |\ell| + 1}{|\ell|} + 3 \right) = \frac{eB^2}{4}. (71)
\]

We can now obtain equation (52) for \( D(\omega) \):

\[
D(\omega) = \frac{eB^2}{4E^2} \left( \frac{2n + |\ell| + 1}{|\ell|} + 3 \right) = \frac{eB^2}{2E} = \frac{eB^2}{2E} \left( \frac{2n + |\ell| + 1}{|\ell|} + 1 \right) = \frac{eB^2}{4E^2} \left( \frac{2n + |\ell| + 1}{|\ell|} + 1 \right).
\]

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