Moments of Directed Paths in a Wedge

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Abstract. A directed path from the origin in the square lattice, and confined to a wedge, exerts a net entropic force on the wedge. If the wedge is formed by the $Y$-axis and the line $Y = rX$, then the moment of the force on the line $Y = rX$ about the origin is given by

$$M_\alpha = -\frac{\log \cot \alpha}{(1 + \cot \alpha)^2}$$

if $0 \leq \alpha \leq \pi/4$,

where $\alpha$ is the vertex angle of the wedge formed by the lines $X = 0$ and $Y = rX$ in the square lattice. If $\alpha \in \{\pi/4, \pi/2\}$, then the moment about the origin is zero. This model is closely related to a model of a descending directed path crossing a wedge from the point $(0, N)$ to the point $(pM, qM)$ on the line $Y = (q/p)X$. If lengths in this model are rescaled by $pM$, while $N = \lfloor \beta qM \rfloor$ and $(q/p) \to r$, where $r$ is an irrational number, then a limiting model of a path crossing the wedge from the point $(0, \beta)$ to the point $(1, r)$ on the line $Y = rX$ is obtained. The limiting path exerts a force on the line $Y = rX$, and the moment of this force about the origin is

$$M_\alpha = \frac{-\log((\beta - 1) \cot \alpha)}{(1 + (\beta - 1) \cot \alpha)^2}$$

if $\beta > 1$ and where $\alpha \in [0, \pi/2]$ is the vertex angle of the wedge.

1. Introduction

The phase diagram of polymers in dilute solution includes critical points and lines which separate phases of expanded polymers, collapsed polymers, adsorbed polymers, and other phases. These phases are the consequences of the free energy of the polymer. Critical behaviour close to the critical lines and points have been studied for decades, and considerable progress have been made in describing the basic scaling and thermodynamic behaviour of polymers near critical lines or points.

There is a recent and renewed interest in the effects of polymers on solvents and colloids. It is known that steric stabilization of colloidal dispersions by polymers is the result of the adsorption of polymer chains on the surfaces of the colloidal particles [21]. The polymer chains exert a repulsive entropic force between approaching colloidal particles and stabilizes the dispersion. The repulsive force is due to a loss in the configurational entropy of the polymer: Approaching particles confine the polymer to a smaller region; the resulting loss in conformational entropy induces a repulsive force between the particles. This effect is the mechanism of sensitized flocculation of the colloid.

This situation has been modeled by a directed path confined to a slit in two dimensions [2], and by the numerical simulation of self-avoiding walks in a slab in three dimensions [20]. Models such as these were considered first by Guttmann and Whittington in 1978 [12]. The polymer is modeled either as a directed path in the square lattice, or as a self-avoiding walk in the cubic lattice.
Figure 1. Two limiting models of directed path models of a polymer in a \( r \)-wedge formed by the \( Y \)-axis and the line \( Y = rX \). These models are obtained by taking the path length to infinity, and by rescaling the lengths in the model suitably. (a) A limiting model of a directed path in the \( r \)-wedge with step-length \( 1/n \) for large \( n \). (b) A limiting model of a directed path crossing an \( r \)-wedge from the point \((0, \beta r)\) to the point \((1, r)\) on the boundaries of the \( r \)-wedge.

Lattice models of polymers, which include directed lattice path models, and self-avoiding walk models, have long been used to model the entropic nature of linear polymers. These models go back to the work of Flory [5, 6] and Hammersley [13]. Tony Guttmann pioneered a combinatorial approach to lattice models [7, 8, 9, 10, 11, 23] of polymers and other discrete objects.

Lattice paths and walks interacting with a wall are models of a polymer adsorbing on the surface of a particle or an interface. These models have received considerable attention in the literature [15, 16, 24, 25, 26, 27]. Conformal invariance techniques allowed the prediction of some critical exponents describing the adsorption transition of polymers [4, 22]. Directed lattice path models of polymers adsorbing on a wall have also received considerable attention [1, 3, 5]. These directed models are closely related to combinatorial models of lattice paths, which are known in combinatorial mathematics.

In this paper a brief review and some new results on the simplest models of a linear polymer in a wedge geometry are presented. A linear polymer with its first monomer fixed at the apex of a wedge may be modeled by a self-avoiding walk from the origin in the square lattice, and confined to a two dimensional wedge with its apex at the origin and boundaries the \( Y \)-axis and the \( Y = rX \) as illustrated in figure 1. The geometric constraints imposed by the wedge geometry on the walk reduce the conformational entropic contribution to the free energy. Thus, one may expect that the walk will exert a net entropic force on the boundaries of the wedge. In the self-avoiding walk model this net force is known to be identically zero for any vertex-angle of the wedge: This result is a consequence of a theorem that the connective constant of the self-avoiding walk in a linear wedge is independent of the vertex-angle of the wedge [14].

The situation is more interesting if directed paths are confined in a wedge; in some cases there is a net force, and thus a net moment of the force about the origin [18, 19]. The first model is illustrated in figure 1(a). This is a fully directed path confined in the \( r \)-wedge formed by the \( Y \)-axis and the line \( Y = rX \) in the first quadrant in the square lattice. By taking the length of the path to infinity (and by suitably rescaling the lengths in the model), a model of a polymer in a wedge is obtained. This model is reviewed in section 2. By examining the generating function of the path in this model, one may determine the forces exerted by the path on the line \( Y = rX \). I show that the net entropic force about the origin is

\[
F_r = \begin{cases} 
-\log \frac{r}{(1 + r)^2}, & \text{if } r \geq 1 \\
0, & \text{if } 0 \leq r < 1.
\end{cases} 
\]  

(1.1)

This force is directed along the line \( X = 1 \) in the negative \( Y \) direction in the \( XY \)-plane. In terms of the
vertex-angle $\alpha$ of the wedge, the moment of the force about the origin is

$$M_\alpha = \begin{cases} 
-\log \cot \alpha 
\frac{1}{(1 + \cot \alpha)^2}, & \text{if } \alpha \leq \pi/4 \\
0, & \text{if } \alpha \in [\pi/4, \pi/2].
\end{cases}$$

(1.2)

An alternative model of a descending lattice path crossing a wedge form the $Y$-axis to the line $Y = rX$ is considered in section 3 and illustrated in figure 1(b). This is the limiting model of a lattice path taking East and South steps and crossing the wedge from the point $(0, \beta r)$ to the point $(1, r)$ on the boundaries of the $r$-wedge. The limiting model induces a force

$$F_r = \frac{-(\beta - 1) \log((\beta - 1)r)}{(1 + (\beta - 1)r)^2}$$

(1.3)

about the origin, for $r > 0$, if $\beta > 1$. This result reduces to equation (1.1) if both $r > 1$ and $\beta = 2$, but the models are very differently defined.

Since $F_r$ includes both positive and negative contributions of equal magnitude, the total work done by closing the wedge from $\alpha = \pi/2$ to $\alpha = 0$ is zero in this model. There is a curve of zero force along $r = \frac{1}{2(\beta - 1)} W(1/\sqrt{4e})$.

(1.4)

where $W(t)$ is the Lambert-W function, and $W(1/\sqrt{4e}) = 0.2388350311 \ldots$.

The moment of the force in equation (1.3) about the origin is given by

$$M_\alpha = \frac{-(\beta - 1) \log((\beta - 1) \cot \alpha)}{(1 + (\beta - 1) \cot \alpha)^2}.$$ 

(1.5)

As a function of $\beta$ for fixed $\alpha$ the moment about the origin is first positive (anti-clockwise) and then negative (clockwise) about the origin. It has a minimum and maximum values at solutions of

$$(1 + (\beta - 1)r) + (1 - (\beta - 1)r) \log((\beta - 1)r) = 0$$

(1.6)

for a given value of $r = \cot \alpha$.

The paper is concluded with some final remarks in section 4.

2. Directed Paths in a Wedge

The simplest directed model of a two-dimensional polymer is a directed path from the origin which takes unit length steps in the North (N) and East (E) directions in the square lattice. Define the $r$-wedge to be the subset of the square lattice between the line $Y = rX$, where $r \geq 0$, and the $Y$-axis in the first quadrant (see figure 2). If the directed path is confined to the $r$-wedge, then we have a model of a polymer confined to the $r$-wedge.

Let $t$ be the generating variable of edges in the directed path in figure 2, and let $g_r \equiv g_r(t)$ be the generating function of directed paths in an $r$-wedge. For some values of $r$, $g_r$ is known. For example,

$$g_1 = \frac{1 - 2t - \sqrt{1 - 4t^2}}{2t(2t - 1)}$$

(2.1)

and $g_0 = 1/(1 - 2t)$. 

Figure 2. A fully directed path in a wedge formed by the $Y$-axis and the line $Y = rX$. The angle $\alpha$ is related to $r$ by $\cot \alpha = r$. This is a model of a polymer confined to a wedge. If $r$ is a rational number, then the path can pass through vertices in $Y = rX$. If $r$ is irrational, then the path cannot pass through points in $Y = rX$.

Figure 3. A $q/p$-Dyck path. Such a path ends in a vertex with coordinates $(Np, Nq)$, for some integer $N \geq 0$. The path has a first return to the line $Y = (q/p)X$. The path may continue as an arbitrary $q/p$-Dyck path after the return at $v$. The part of the path from the origin to the first return is primitive, or is a $q/p$-excursion.

More generally, $g_r$ is not known explicitly. It is possible to determine the radius of convergence $t_r$ of $g_r$ [19]. Since $g_r$ is a non-decreasing function with increasing $r$, $t_r$ is a non-decreasing function of $r$. Thus, $t_r$ is continuous (and differentiable) almost everywhere, and it is finite since the number of paths of length $n$ is at most $2^n$. By equation (2.1) one notes that

$$t_r = 1/2, \quad \text{if } 0 \leq r \leq 1$$

(2.2)

since $t_0 = t_1 = 1/2$. In the next section we determine $t_r$ for larger values of $r$.

2.1. Determining $t_r$

Consider the line $Y = (q/p)X$, where $r = (q/p)$ is a rational number and where $(p, q)$ is a pair of relative primes. A $q/p$-Dyck path is a fully directed path from the origin to a point with coordinates $(Np, Nq)$ for some integer $N \geq 0$ which avoids taking steps below the line $Y = (q/p)X$ (see figure 3).

The length of a $q/p$-Dyck path is always a multiple of $(p + q)$. Let $E_{q/p}$ be the number of $q/p$-Dyck paths of length exactly $(p + q)$ and final vertex $(p, q)$. Consider a sequence $(q_n/p_n)$ of rational numbers, and suppose that

$$\lim_{n \to \infty} \frac{q_n}{p_n} = r$$

(2.3)

where $r > 0$ is an irrational number. Then $E_{q_n/p_n}^{1/(p_n+q_n)}$ converges as $n \to \infty$. The following theorem was proven in reference [17].
Theorem 2.1. Suppose that \((p_n, q_n)\) are pairs of positive and relative prime integers, and that the ratios 
\[ \frac{q_n}{p_n} \rightarrow r, \]  
where \(r\) is a non-negative irrational number. Then the sequence \(\left\langle E_{q_n/p_n}^{1/(p_n+q_n)} \right\rangle\) has a limit, 
and moreover,
\[
\lim_{n \to \infty} E_{q_n/p_n}^{1/(p_n+q_n)} = \frac{1 + r}{r/(1+r)}.
\]

Since the limit in theorem 2.1 is defined for any irrational number \(r > 0\), and since it is monotonic in the intervals \([0, 1]\) and \([1, \infty]\), its definition may be extended to include all rational numbers as well: If \(r > 0\) is rational, let \(\langle r_i \rangle\) be a sequence of irrational number such that \(r_i \rightarrow r\). Define
\[
\lim_{n \to \infty} E_{q_n/p_n}^{1/(p_n+q_n)} = \lim_{i \to \infty} \frac{1 + r_i}{r_i/(1+r_i)} = \frac{1 + r}{r/(1+r)} \quad (2.4)
\]
if the sequence \(\langle q_n/p_n \rangle\) converges to a rational number \(r\). In that case the limit in theorem 2.1 is defined for all real numbers \(r \geq 0\).

Next, let \((p, q)\) be a pair of relative prime integers, and consider \(q/p\)-Dyck paths. A \(q/p\)-Dyck path with only two vertices in the line \(Y = (q/p)X\) is primitive or is a \(q/p\)-excursion. Each \(q/p\)-Dyck path (except the empty path consisting of one vertex) has a first return to the line \(Y = (q/p)X\) (see figure 3). After this first return, the path may either terminate, or it may continue as an arbitrary \(q/p\)-Dyck path. This renewal property allows one to write down a recurrence for the generating function. However, the part of the path from the origin to the first return is primitive, and the recurrence is only useful if the generating function of primitive paths can be determined.

The arguments in figure 3 shows that each \(q/p\)-Dyck path is either the empty path, or is a \(q/p\)-excursion followed by an arbitrary \(q/p\)-Dyck path. In other words, if the generating function of \(q/p\)-Dyck paths is \(g_{q/p}\) and \(e_{q/p}\) is the generating function of primitive \(q/p\)-Dyck paths, then
\[
g_{q/p} = 1 + e_{q/p} g_{q/p}. \quad (2.5)
\]
The generating function for \(q/p\)-excursions is not known, but one can show that the recurrence
\[
g_{q/p} = 1 + E_{q/p}[t g_{q/p}]^{p+q} \quad (2.6)
\]
generates a subclass of \(q/p\)-Dyck paths, where \(E_{q/p}\) is defined before equation (2.3). One may also show that the recurrence
\[
g_{q/p} = 1 + \left( \binom{p+q}{q} \right) [t g_{q/p}]^{p+q} \quad (2.7)
\]
generates a class of objects that includes all \(q/p\)-Dyck paths.

Following the arguments in reference [17], one may instead consider a recurrence of the kind
\[
g_{q/p} = 1 + F_{q/p}[t g_{q/p}]^{p+q} \quad (2.8)
\]
where \(F_{q/p}\) is a constant and bounded by
\[
E_{q/p} \leq F_{q/p} \leq \binom{p+q}{q}. \quad (2.9)
\]

Consider then the recurrence in equation (2.8). The radius of convergence \(t_{q/p}\) of \(g_{q/p}\) can be determined as follows. Fix \(t\) and choose \(g_{q/p}[0]\) as an initial guess of \(g_{q/p}\). Solve for \(g_{q/p}\) from equation (2.8) in two ways to set up the recurrences
\[
g_{q/p}[N + 1] = F_{q/p} \left[ t g_{q/p}[N] \right]^{p+q} + 1, \quad \text{for } N = 0, 1, 2, \ldots; \quad (2.10)
\]
and
\[ g_{q/p}[N + 1] = \frac{1}{t} \left[ (g_{q/p}[N] - 1)/F_{q/p} \right]^{1/(p+q)}, \quad \text{for } N = 0, 1, 2, \ldots; \] (2.11)
given \( g_{q/p}[0] \). These recurrences may be written in simplified fashion by defining the two functions
\[ f_-(g) = F_{q/p}(gt)^{p+q} + 1, \quad \text{and} \quad f_+(g) = \frac{1}{t} \left( \frac{g - 1}{F_{q/p}} \right)^{1/(p+q)}, \] (2.12)
in which case the recurrences are \( g_{q/p}[N + 1] = f_{\pm}(g_{q/p}[N]) \) for \( N = 0, 1, 2, \ldots \) for given \( g_{q/p}[0] \) and fixed values of \( t \in (0, \infty) \).

The recurrence \( g_{q/p}[N + 1] = f_- (g_{q/p}[N]) \) generates a power-series in \( t \), and it is convergent if \( t < t_{q/p}^- \), where \( t_{q/p}^- \) is the radius of convergence of the generating function \( g_{q/p} \). On the other hand, the recurrence \( g_{q/p}[N + 1] = f_+(g_{q/p}[N]) \) generates a power-series in \( 1/t \), and it is convergent for values of \( t > t_{q/p}^+ \). If these recurrences are convergent for a given choice of \( t \), then there are fixed points \( g_{q/p}^- \) and \( g_{q/p}^+ \) respectively:
\[ g_{q/p}^- = f_-(g_{q/p}^-), \quad \text{and} \quad g_{q/p}^+ = f_+(g_{q/p}^+). \] (2.13)
By the fixed point theorem, the recurrences will converge to fixed points if for both \( f_- \) and \( f_+ \)
\[ \left| \frac{df_\pm}{dg} \right|_{g=g_{q/p}^\pm} < 1; \] (2.14)
for appropriate choices of the initial guess of \( g_{q/p}[0] \). This inequality implies certain conditions on the choice for \( t \) in the recurrences. In particular, it shows that for \( f_- \),
\[ t^{p+q} < t_{q/p}^- = \frac{1}{F_{q/p}(p+q)[g_{q/p}^-]^{p+q-1}}, \] (2.15)
and for \( f_+ \),
\[ t^{p+q} > t_{q/p}^+ = \frac{1}{F_{q/p}(p+q)[g_{q/p}^+]^{p+q-1}[g_{q/p}^- - 1]^{p+q-1}}. \] (2.16)
These bounds on \( t \) gives critical values \( t_{q/p}^\pm \) for \( t \). If \( t > t_{q/p}^+ \), then the first recurrence is not convergent, and if \( t < t_{q/p}^- \), then the second recurrence is not convergent. Representative curves of the critical values \( t_{q/p}^\pm \) are plotted for the case that \( p = 2 \) and \( q = 3 \) in figure 4 against \( g \). Curve B corresponds to the critical value \( t_{q/p}^- \) of \( t \) in equation (2.15); for values of \( t \) in the region below this curve the recurrence \( g_{q/p}[N + 1] = f_- (g_{q/p}[N]) \) will converge. Curve A corresponds to the critical value \( t_{q/p}^+ \) of \( t \) in equation (2.16). Values of \( t \) above this curve will give convergence in the recurrence \( g_{q/p}[N + 1] = f_+(g_{q/p}[N]) \).

At the intersection of the two critical curves in figure 4 it is the case that \( t_{q/p}^- = t_{q/p}^+ = t_{q/p} \), and this gives the point \((g^*, t^*)\), where \( t^*=t_{q/p}^-=t_{q/p}^+=t_{q/p} \), a potential fixed point for the functional recurrence in equation (2.8). Solving directly for \( g_{q/p}^* \) in
\[ t_{q/p}^{p+q} = \frac{1}{F_{q/p}(p+q)[g_{q/p}^*]^{p+q-1}} = \frac{1}{F_{q/p}(p+q)[g_{q/p}^* - 1]^{p+q-1}} \] (2.17)
gives the result that
\[ g_{q/p}^* = \frac{p + q}{p + q - 1}. \] (2.18)
If the value of $t$ (denoted along the dotted line) approaches the critical point $N$, it follows that there is also a bound on the critical value $t$. The curves $A$ and (2.16) in this case $F_{1/2}(p+q)$ converges as a power series in $t$ if the value of $p+q$ is below the curve marked by A. If $t = t_{q/p}$, then both recurrences have the same fixed point $g = g^*_{q/p}$. This can be verified by substituting $t_{q/p}$ and $g^*_{q/p}$ into equation (2.8). Since the recurrence $g_{N+1} = f_-(g_N)$ generates a power series in $t$, and it is divergent if $g = g^*$ and $t > t_{q/p}$, the radius of convergence of $g_{q/p}$ is $t_{q/p}$. The numerical solutions of $g_{N+1} = f_-(g_N)$ are denoted along the dotted line. This line approaches the critical point $(g^*_{q/p}, t_{q/p})$.

One may then determine the corresponding value for $t^* = t_{q/p}$:

$$t_{p+q} = \frac{(p + q - 1)^{p+q-1}}{F_{q/p}(p + q)^{p+q}}.$$  \hspace{1cm} (2.19)

Direct substitution of $(g^*_{q/p}, t_{q/p})$ in the functional recurrence (2.8) proves that for $t = t_{q/p}$, the fixed point is $g^*_{q/p}$ as given in equation (2.18). Since $g = g^*_{q/p}$ is the fixed point when $t$ is in the critical curve in equation (2.15), we conclude that $t_{q/p}$ is the radius of convergence of $g_{q/p}$.

This gives the following lemma:

**Lemma 2.2.** Let the generating function of paths generated by the functional recurrence

$$g_{q/p} = 1 + F_{q/p}[t_{q/p}]^{p+q}$$

be $g_{q/p}(t)$. The radius of convergence of $g_{q/p}(t)$ is

$$t_{q/p} = \frac{(p + q - 1)^{(p+q-1)/(p+q)}}{F_{q/p}}.$$  \hspace{1cm} (2.20)

and moreover, $g^*_{q/p} = g_{q/p}(t_{q/p}) = (p + q)/(p + q - 1).$

Observe that $g^*_{q/p}$ is independent of the number $F_{q/p}$, and since $F_{q/p}$ is bounded as in equation (2.9), it follows that there is also a bound on the critical value $t_{q/p}$.

Next, one may determine $t_r$ as follows. Let $r$ be irrational and suppose that $(p_n, q_n)$ is a sequence of positive and relative prime integers such that both $q_n/p_n > r$ and $\lim_{n \to \infty} q_n/p_n = r$. Define $c_{n}(r)$ the number of fully directed paths from the origin confined in the $r$-wedge and with length $n$. Define the generating function $g_r$ (see above) by

$$g_r = \sum_{n=0}^{\infty} c_{n}(r) t^n.$$  \hspace{1cm} (2.20)
It is clearly the case that $g_{q_n/p_n} \leq g_r$ since $q_n/p_n > r$. Thus, the radius of convergence of $g_r$ is less than or equal to $t_{q_n/p_n}$, in other words $t_r \leq t_{q_n/p_n}$ for each value of $n$. Observe that $t_r \geq 1/2$ for any $r$, since the number of (unrestricted) directed paths grows as $2^n$.

Each path counted by $c_n^{(r)}$ has at most $\lceil n/(1 + r) \rceil$ horizontal edges. Thus

$$c_n^{(r)} \leq \sum_{m=0}^{\left\lceil \frac{n}{1+r} \right\rceil} \binom{n}{m}. \quad (2.21)$$

Take the $1/n$-power of this and let $n \to \infty$. This shows that

$$\lim_{n \to \infty} \left[ c_n^{(r)} \right]^{1/n} \leq \begin{cases} 2, & \text{if } 0 \leq r \leq 1; \\ \frac{1 + r}{r/r + 1 + r}, & \text{if } r > 1. \end{cases} \quad (2.22)$$

Therefore, $t_r \geq 1/2$ if $r \in [0, 1]$, and $t_r \geq r^r/(1 + r)/(1 + r)$ if $r > 1$. The result of these calculations is the following theorem.

**Theorem 2.3.** Let $r \geq 0$ be a real number. The radius of convergence of the generating function $g_r$ of fully directed paths confined in an $r$-wedge is

$$t_r = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq r \leq 1; \\ \frac{r^r}{1 + r}, & \text{if } r > 1. \end{cases}$$

**Proof.** $t_r$ is increasing with $r$. Since $g_1 = \sum_{n \geq 0} c_n^{(1)} p_n$ is given in equation (2.1), one may directly check that $t_1 = 1/2$. Since $t_r \geq 1/2$, the result is that $t_r = 1/2$ for all $r \in [0, 1]$.

Consider the irrational number $r > 1$ and suppose that $(p_n, q_n)$ is a sequence of positive and relative prime integers such that both $q_n/p_n > r$ and $\lim_{n \to \infty} q_n/p_n = r$. Then $t_r \leq t_{q_n/p_n}$, and by lemma 2.2,

$$t_r \leq t_{q_n/p_n} = \frac{(p_n + q_n - 1)(p_n + q_n - 1)/(p_n + q_n)}{F_{q_n/p_n}^{1/(p_n + q_n)}(p_n + q_n)},$$

provided that $F_{q_n/p_n}$ is chosen equal to its lower bound in equation (2.9).

Take $n \to \infty$, and use the result in theorem 2.1 and in equation (2.9) to compute the limit of $F_{q_n/p_n}^{1/(p_n + q_n)}$. This gives

$$t_r \leq \frac{r^r/(1 + r)}{1 + r}$$

for any irrational $r > 0$. On the other hand, by equation (2.22) the opposite inequality is also valid if $r > 1$.

These arguments fix the value of $t_r$ if $r > 0$ is irrational. Thus, $t_r$ may be extended to a measurable function defined on real numbers by defining $\lim_{t_r \to \infty} t_r = t_r$ for a sequence $(t_{r_i})$ converging to $t_r$. Since $t_r$ is non-decreasing it is continuous and differential almost everywhere. Thus a continuous and differentiable function is obtained for all $r > 0$. This proves the theorem. \[\square\]
2.2. Forces in Directed Paths in a Wedge

The limiting free energy per vertex of the infinite length directed path can be computed from \( t_\tau \) in theorem 2.3. The result is

\[
F_\tau = -\log t_\tau = \log(1 + r) - \frac{r \log r}{1 + r}
\]  \( (2.23) \)

explicitly as a function of \( r \) if \( r > 1 \). The derivative of the \( F_\tau \) gives the entropic "spring" force of the path as the wedge is squeezed by increasing \( r \). In this model, the repulsive force is given as a function of \( r \) by

\[
F_\tau = \begin{cases} 
0, & \text{if } 0 \leq r < 1; \\
-\frac{\log r}{1 + r}, & \text{if } r \geq 1,
\end{cases}
\]  \( (2.24) \)

and it is directed along the line \( X = 1 \) in the negative \( Y \)-direction in the square lattice.

The magnitude of \( F_\tau \) is a maximum when \( r = 2.09349 \ldots \) This value is obtained by solving for the extremum in \( F_\tau \) using Maple 9; the solution is \( \log r = 1/2 + W(1/\sqrt{4e}) \) where \( W \) is the Lambert-W function. This value of \( r \) corresponds to the angle \( \alpha = 0.445624612 \ldots \). Note that \( \pi/7 = 0.448798950 \ldots \) Thus, the force (in the \( r \)-direction) is nearly a maximum if the vertex angle of the wedge is \( \alpha \approx \pi/7 \). The force is plotted as a function of \( r \) in figure 5.

One may also consider the force \( F_\tau \) as a function of the vertex angle \( \alpha \) of the wedge. In terms of \( \alpha \), the force is

\[
F_\alpha = \begin{cases} 
\left[ \frac{1 + \cot^2 \alpha}{(1 + \cot \alpha)^2} \right] \log(\cot \alpha), & \text{if } 0 \leq \alpha < \pi/4; \\
0, & \text{if } \alpha \geq \pi/4.
\end{cases}
\]  \( (2.25) \)

In other words, the repulsive force decreases with increasing wedge-angle until it reaches zero strength at \( \alpha = \pi/4 \). Thereafter, the path exerts no force on the line.

Since these forces are conservative, one may compute the work performed as a function of the wedge angle. Direct integration of (2.25) or (2.24) shows that the amount of work is \( \sqrt{2} \) units to close the angle from any angle larger than \( \pi/4 \) to zero.

2.3. Discussion

The explicit formula for the magnitude of the force of a directed path in an \( r \)-wedge is given by equation (2.24). This force acts along the line \( X = 1 \) on the line \( Y = rX \), and its magnitude is plotted in figure 5.
5. The net force is zero whenever $0 \leq r \leq 1$; this corresponds to wedges with vertex-angles larger than $\pi/4$.

Forces in fully directed paths in confined geometries have also been determined by Brak et al [2]. The net entropic force of a fully directed path confined to a strip of width $w$ is given by

$$F_w = \frac{\pi \tan(\pi/(w+2))}{(w+2)^2}$$

and it falls off as an inverse cube of $w$ as $w \to \infty$: $F_w = \pi^2/w^3 + O(w^{-4})$ for large $w$.

In a wedge of vertex-angle $\alpha$, the behaviour of $F_\alpha$ in equation (2.25) is also of interest as $\alpha \to 0^+$. Examination of our results show that the magnitude of the force diverges logarithmically with the vertex-angle $\alpha$ as $\alpha \to 0^+$:

$$F_\alpha = -\log \alpha + 2\alpha \log\alpha + O(\alpha^2 \log\alpha), \quad \text{as } \alpha \to 0^+. \tag{2.27}$$

Similar results have been determined for Motzkin paths in a wedge [18]. In particular, the first few terms of the expansion of the force about the origin is given by

$$F_\alpha = -\log(\alpha/2) - (\alpha/2)(1 - 2\log(\alpha/2)) + O(\alpha^2 \log(\alpha/2)), \quad \text{as } \alpha \to 0^+ \tag{2.28}$$

for Motzkin paths [18]. In other words, the logarithmic dependence of the force on $\alpha$ is present in both models.

The moment $\mu_\alpha$ of the force about the origin induced by the directed path in the wedge is given by

$$\mu_\alpha = \begin{cases} -\log(\cot \alpha) & \text{if } 0 \leq \alpha < \pi/4; \\ \frac{-\log(\cot \alpha)}{(1 + \cot \alpha)^2}, & \text{if } \alpha \geq \pi/4. \end{cases} \tag{2.29}$$

For small values of $\alpha$, this is negative and clockwise about the origin, and can be expanded to

$$\mu_\alpha = -\left[1 - 2\alpha + O(\alpha^2)\right] \alpha^2 \log \alpha. \tag{2.30}$$

A similar result is known for Motzkin paths in a wedge [18].
3. Directed Paths Crossing a Wedge

In this section I consider an alternative model of directed paths in a wedge. The basic model is illustrated in figure 7(a). This model is a directed path which may only step in the East (E) or South (S) directions, starting from the vertex \((0, N)\) on the \(Y\)-axis and terminating in the vertex \((Mp, Mq)\) on the line \(Y = (q/p)X\). This is a model of a polymer attached to opposite sides of a wedge, and crossing if from one side to other side.

Consider the model in figure 7(a). The number of paths between the vertices \((0, N)\) and \((Mp, Mq)\) is given by the binomial coefficient

\[
\binom{N + M(p - q)}{Mp}
\]  \hspace{1cm} (3.1)

If \(t\) is the generating variable of edges in the path, then a model of paths crossing the \(q/p\)-wedge from the \(Y\)-axis to the line \(Y = (q/p)X\) is obtained by summing over all \(N\) and \(M\). This gives the generating function

\[
G_{q/p} = \sum_{M=0}^{\infty} \sum_{N=Mq}^{\infty} \binom{N + M(p - q)}{Mp} t^{N+M(p-q)}
\]

\[
= \frac{1 - t^p}{1 - t^{p - q}}.
\]  \hspace{1cm} (3.2)

The result is only dependent on \(p\), and a little reflection shows that this should be expected; for any fixed \(p\) the summation over \(N\) integrates over all values of \(q\). Secondly, the radius of convergence of \(G_{q/p}\) is \(t_c = 1/2\) for any \(q/p\). If this is extended to all \(r\)-wedges, then the resulting model is not very interesting: The path exerts no net force on the line \(Y = rX\). The model is more interesting if \(N\) is fixed as a multiple of \(Mq\), and then by taking the limit \(M \to \infty\).

Figure 7. (a) A model of a polymer crossing a \(q/p\)-wedge. A directed path with East and South steps crosses the wedge from \((0, N)\) on the \(Y\)-axis to the point \((Mp, Mq)\) on the line \(Y = (q/p)X\). The length of the path is \(n = N + M(p - q)\), and note that \(N \geq Mq\). By taking \(n \to \infty\) and at the same time rescaling all lengths in (a) by the factor \(1/n\) and by taking the ratio \(q/p \to r\), the limiting model in (b) is obtained. This is a directed polymer crossing an \(r\)-wedge.

3.1. The Limiting Model

Let \(r > 0\) be an irrational number, and consider the (increasing) sequence \((q_n/p_n)\) where \(q_n/p_n \to r\) as \(n \to \infty\), and \((q_n, p_n)\) are relative prime. Observe that the wedge-angle \(\alpha\) in figure 7(a) is given by \(\cot \alpha = q_n/p_n\). Fix the value of \(M\), and let \(\beta \geq 1\) be a constant. Put \(N = [\beta q_n M]\).

If \(n \to \infty\), then \(N \to \infty\). If lengths in figure 7(a) are rescaled by \(p_nM\) as \(n \to \infty\), then \(\cot \alpha \to r\), while the endpoints of the path converges to \((0, \beta r)\) and \((1, r)\). These are indicated in figure 7(b).
For finite \( n \), the generating function is given by
\[
G_n(\beta) = \sum_{M=0}^{\infty} \left( \left[ \beta q_n M \right] + \left( p_n - q_n \right) M \right) \left| \beta q_n M \right| \left( p_n - q_n \right) M.
\] (3.3)

The radius of convergence of this generating function is given by
\[
t_n = \frac{p_n / (p_n + (\beta - 1)q_n) \left( (\beta - 1)q_n \right) / (p_n + (\beta - 1)q_n)}{p_n + (\beta - 1)q_n}.
\] (3.4)

Taking \( n \to \infty \) gives the radius of convergence
\[
t_r = \frac{[(\beta - 1)r][(\beta - 1)r]/(1+(\beta - 1)r)}{1+(\beta - 1)r}
\] (3.5)
of the generating function in the limit as \( n \to \infty \). Since \( -\log t_n \) is the free energy of the model for any finite value of \( n \), \( -\log t_r \) is the free energy of the limiting model in figure 7(b). This is given by
\[
F_r = -\log t_r = \log(1 + (\beta - 1)r) - \frac{(\beta - 1)r \log((\beta - 1)r)}{1 + (\beta - 1)r}.
\] (3.6)

This relation gives the free energy as a function of \( \beta \) and \( r \), where \( r \) is an irrational number. Since this function is a continuous function on \( r \), and rational numbers have zero measure, the limiting free energy can be defined for all real numbers \( r \geq 0 \).

The entropic force in this model can be determined by taking the derivative of the free energy to \( r \). This force is given by
\[
F_r = \frac{-(\beta - 1) \log((\beta - 1)r)}{(1 + (\beta - 1)r)^2}.
\] (3.7)

The sign of the force depends on the value of \( \beta \). If \( (\beta - 1)r > 1 \), then this force is negative, and it acts along the line \( X = 1 \) in the negative \( Y \)-direction. The moment of the force is clockwise about the origin (negative). If \( (\beta - 1)r < 1 \), then the force is positive, and it acts along the line \( X = 1 \) in the positive \( Y \)-direction. The moment of the force is anti-clockwise about the origin (positive).

Consider the force as a function of \( \beta \geq 1 \) for \( r \geq 0 \). Along the curve \( \beta = 1 + 1/r \) the force is zero, and there is no net moment about the origin. The force is maximal along the curve \( 1 + 1/(2\pi W(1/\sqrt{4e})) = 1 + 1/(0.47767 \ldots \times r) \), and it is directed along the line \( X = 1 \) in figure 7(b). By noting that \( r = \cot \alpha \), where \( \alpha \) is the vertex angle of the wedge at the origin, an explicit formula for the force about the origin can be written down. In terms of the vertex angle \( \alpha \) the force along the line \( X = 1 \) is given by
\[
F_\alpha = \frac{(\beta - 1)(1 + \cot^2 \alpha) \log((\beta - 1) \cot \alpha)}{(1 + (\beta - 1) \cot \alpha)^2}.
\] (3.8)

This force vanishes in the \((\alpha, \beta)\)-plane along the curve \( (\beta - 1) \cot \alpha = 1 \). For values of \( \beta \) in the interval \((1, 1 + 1/\cot \alpha)\), the force will be negative, and it will induce a moment clockwise about the origin. This force would act to increase the value of \( \alpha \). If \( \beta > 1 + 1/\cot \alpha \), then the force will be positive, and induce a moment anti-clockwise about the origin. This force would act to decrease the value of \( \alpha \).

Similarly, if \( \alpha \) is a small angle \( (r \) is large), then the force is positive, and it would act to decrease the value of \( \alpha \) even more. When \( \alpha \) approaches \( \pi/2 \), the force will be negative, and it will act to increase the value of \( \alpha \) to approach \( \pi/2 \).

One may integrate the force \( F_\alpha \) to determine the total work done as the wedge is squeezed closed. In this model, this work is zero, since there are positive and negative contributions to the work that cancels. If the integration is done from the curve of zero force at \( r = 1/((\beta - 1) \log((\beta - 1)r)/(1 + (\beta - 1)r)^2) \), then the amount of work is
\[
\left| \int_{1/(\beta - 1)}^{\infty} \frac{(\beta - 1) \log((\beta - 1)r)}{(1 + (\beta - 1)r)^2} \right| \, dr
\]
The moment of the force about the origin as a function of $\alpha$ for $\beta = 20$. The moment is negative, and is directed clockwise about the origin. For larger angles it is positive, and is directed anti-clockwise about the origin. The curve passes through the zero force point when $\cot^{-1}(1/19) = 1.518213265\ldots$

$$\int_1^\infty \left[ \log u \right] \frac{du}{(1+u)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \text{Catalan's Constant} = 0.915965594\ldots$$

The moment of the force about the origin may also be determined: It is given by

$$M_\alpha = -\frac{(\beta - 1) \log ((\beta - 1) \cot \alpha)}{(1 + (\beta - 1) \cot \alpha)^2}.$$ (3.9)

In figure 8 the moment of the force is plotted against $\alpha$ for $\beta = 20$. The curve passes through the zero force point at $\cot \alpha = 1/19$, or when $\alpha = 1.518213265\ldots = \pi/2.069269665\ldots$. For angles smaller than this, the moment of the force is negative and acts clockwise about the origin. For angles larger than this, it is positive, and acts anti-clockwise about the origin. Expanding the moment for small values of $\alpha$ and for $\beta > 1$ and close to 1 gives

$$M_\alpha = -\frac{1}{\alpha} + 2\alpha(\beta - 1) + 3\alpha^2(\beta - 1)^2 + O(\alpha^3) \log ((\beta - 1)\alpha).$$ (3.10)

There is a logarithmic divergence of $M_\alpha$ as $\alpha \to 0^+$. For any given $\alpha$ there is a $\beta$ which maximizes the moment. This value of $\beta$ is given by

$$\beta_m = 1 + \frac{1}{2\alpha W(1/\sqrt{4\epsilon})}.$$ (3.11)

where $W$ is the Lambert-W function.

One may also instead find the maximum in the moment for given and fixed values of $r$. In this model, there are extrema in the moment if $\beta$ equals the positive solutions of

$$(1 + (\beta - 1)r) + (1 - (\beta - 1)r) \log((\beta - 1)r) = 0.$$ (3.12)
The Moment and its Derivative

Figure 9. The moment of a path crossing a wedge as a function of $\beta$ (curve A). The (scaled) derivative of this curve is plotted as curve B, and its roots give the extreme values of the moment. In this figure, $r = 1.125$; other values of $r > 0$ give a similar result. Curve B has been reduced by a factor of 10 to give a suitably scaled curve for plotting in this diagram.

For small values of $\beta$, the moment about the origin in figure 9 is positive. For large $\beta$, it is negative. The solutions of the last equation thus gives those values of $\beta$ at which the moments about the origin are extreme. The curve $M_r$ is plotted against $\beta$ for $r = 1.125$ in figure 9 (curve A). The extreme values of $M_r$ is obtained by solving equation (3.12); a scaled version of this function is plotted against $\beta$ in figure 9 as well (curve B), and its roots $\beta_m$ are denoted by $\beta_m$. The moment has a maximum for small $\beta$ (positive, anti-clockwise about the origin), and a minimum for a larger value of $\beta$ (negative, or clockwise about the origin).

4. Conclusions
This paper is a short summary of the calculation of forces in models of directed paths confined in a wedge. These models are more interesting than the corresponding model of a self-avoiding walk in a wedge. It is known that the limiting free energy of a self-avoiding walk in a wedge is independent of the vertex angle of the wedge [14]. Thus, the walk does not exert a net entropic force on the boundaries of the wedge.

The basic result in models of directed paths in a wedge is that the entropic force diverges as a logarithm in the vertex angle $\alpha$ of the wedge as $\alpha$ approaches zero when the wedge is squeezed closed. This logarithmic divergence is found in both the models examined in this paper, and also in a model of Motzkin paths confined in a wedge examined in reference [18].

In section 2 a model of a directed path in a wedge was reviewed. The properties of the generating function was determined, and in particular, I showed that the radius of convergence of the generating function can be obtained directly from the recurrence relation, without having to solve explicitly for...
the generating function (in general the generating function remains unknown). I used the fixed point theorem to determine convergences for recurrences that generates the generating function, and these results enabled the direct determination of the radius of convergence. The free energy of the infinite length (rescaled) path can be determined from this, and by taking derivatives of the free energy, the entropic force can be determined. This is done in section 2.2, where an explicit formula for the force is given.

In section 3 a model of directed paths crossing a wedge is considered instead. It may appear at first that this model is fundamentally different from the model in section 2, but the resulting entropic force in section 3 is a generalization of the model in section 2. While there is no net work in this model when the wedge is squeezed closed, there is a line of zero force, and the amount of work on squeezing the wedge closed from the line of zero force is given by Catalan’s constant.

Acknowledgments
EJJvR is supported by a grant from NSERC (Canada).

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