Overdamped quantum phase diffusion and charging effects in Josephson junctions

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Exploiting the recently derived quantum Smoluchowski equation the classical Ivanchenko Zil’berman theory for overdamped diffusive phase motion of low capacitance Josephson junctions is extended to the low temperature quantum domain where charging effects appear. This formulation allows to derive explicit results for the current-voltage characteristics over a broad range of parameters that reduce to known findings in certain limits. In particular, the transparent analytical approach comprises Coulomb blockade physics, coherent Cooper pair transfer, and the precursors of macroscopic quantum tunneling and needs to be supplemented by more sophisticated methods only at very low temperatures.

I. INTRODUCTION

In the last decades the physics of Josephson junctions (JJ) has revealed an extraordinary wealth of phenomena studied theoretically and experimentally as well [1, 2, 3]. The underlying system, two superconducting domains separated by a tunnel barrier, can be found in a variety of realizations, recently e.g. in superconducting atomic contacts [4] and solid-state quantum bits [5]. Basically two parameters determine the dynamics, namely, the Josephson coupling energy $E_J$ and the charging energy $E_c = 2e^2/C$ of a junction with capacitance $C$. It turns out that the competition between these two scales is crucially influenced by the electromagnetic environment surrounding the junction, i.e. its impedance which in the simplest case is given by an ohmic resistor with resistance $R$.

For the classical JJ dynamics charging effects are negligible and the charge transfer cor-

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responds to a phase-coherent Cooper pair current

\[ I_s = I_c \sin(\phi) \quad , \quad \dot{\phi}(t) = \frac{2e}{\hbar} V(t) \]  

(1)

with \( I_c = (2e/h) E_J \), the phase difference across the junction \( \phi \), and where the phase velocity \( \dot{\phi} = d\phi/dt \) is related to the voltage drop \( V(t) \) across the junction. In the opposite limit of dominating \( E_c \), however, charging effects prevail and Coulomb blockade is known to rule the transfer of incoherent tunneling of Cooper pairs [3].

A particular appealing feature of a JJ is the fact that its classical dynamics can be visualized as the diffusive motion of a fictitious classical particle (RSJ model) [1]. Accordingly, for a current biased JJ the case of very small capacitance corresponds to very strong friction (Smoluchowski limit) associated with the phase dynamics

\[ m \gamma \dot{\phi} + dU(\phi)/d\phi = \xi(t) . \]  

(2)

Here the translation rules are: Mass \( m = (h/2e)^2 C \), damping strength \( \gamma = 1/RC \), potential \( U(\phi) = -E_J \cos(\phi) - E_b \phi \) with the energy \( E_b = (h/2e) I \) related to the bias current, and current noise \( \langle \xi(t) \rangle = 0, \langle \xi(t)\xi(t') \rangle = (2\gamma/\beta) \delta(t-t') \) \( (\beta = 1/k_BT) \). The corresponding time evolution equation for the phase distribution (Smoluchowski equation) has been studied in detail already in the 60s [6, 7]. Even in the overdamped limit the charging energy \( E_c \) can be relatively large and the crucial question arises in which way charging effects alter the diffusive phase dynamics of JJs. While this has been analyzed at very low temperatures [8] and for small [9] and large [10] Josephson couplings, much less is known for intermediate temperatures and coupling energies.

The goal of this paper is to fill this gap for overdamped junctions by providing a generalization of the classical Ivanchenko-Zil’berman Theory (IZT) [6]. As we will show, the classical and quantum regimes are characterized by \( \beta E_c/\pi \rho \ll 1 \) and \( \beta E_c/\pi \rho \gg 1 \), respectively, with \( \rho = R/R_Q \) \( (R_Q = h/4e^2) \) meaning that in a mechanical analog where

\[ \beta E_c/\pi \rho = \hbar \beta \gamma \]  

(3)

the appearance of charging effects is associated with the changeover from classical \( (\gamma \hbar \beta \ll 1) \) to quantum \( (\gamma \hbar \beta \gg 1) \) overdamped phase diffusion. While in general the dissipative dynamics of a quantum particle is a formidable task [11], it was shown recently [12] that in the overdamped limit considered here the reduced quantum dynamics can be described by
a general extension of the classical Smoluchowski equation, the so-called Quantum Smoluchowski Equation (QSE). This allows to reveal the relation between classical and Coulomb blocked transport in overdamped JJs in terms of a simple analytical approach.

II. QUANTUM PHASE DIFFUSION IN JJS AT STRONG FRICITION

Classical overdamped motion is based upon a separation of time scales between relaxation of position and relaxation of momentum [13]. Momentum is slaved to position and on a coarse grained time scale can assumed to be in thermal equilibrium with respect to the instantaneous position of the particle.

Quantum dissipative dynamics is much more complicated and usually has to be formulated in terms of the path integral formalism [11]. This approach has been exploited very successfully in the past to describe low temperature properties of JJs in particular [8, 9, 10] and of dissipative particles in periodic potentials in general [11, 14]. Explicit calculations require involved techniques though, and analytic expressions are available only in certain ranges of parameter space. The problem is that quantum fluctuations acting on time scales of order $\hbar\beta$ lead, particularly at low temperatures, to long range interactions in time [11]. Recently it was shown, however, that in the overdamped domain the exact quantum time evolution can be mapped onto a quantum generalization of the classical Smoluchowski equation for the position probability distribution. The resulting QSE reads

$$\frac{\partial P(\phi, t)}{\partial t} = \frac{1}{\gamma m} \frac{\partial}{\partial \phi} \left[ U'_{\text{eff}}(\phi) + \frac{1}{\beta} \frac{\partial}{\partial \phi} D(\phi) \right] P(\phi, t)$$

with $' = d/d\phi$ and the translation rules as specified above. Quantum fluctuations appear as a Josephson potential with an effective Josephson coupling

$$U_{\text{eff}}(\phi) = -E^*_J \cos(\phi) - E_b \phi , \quad E^*_J = E_J \left( 1 - \frac{\Lambda}{2} \right)$$

and a phase dependent diffusion coefficient capturing quantum noise [15]

$$D(\phi) = \left[ 1 - \theta \cos(\phi) \right]^{-1}, \quad \theta = \Lambda \beta E_J .$$

The changeover from classical to quantum dynamics is governed by a function $\Lambda$ that contains $\gamma\hbar\beta$ as the essential quantity. With (3) it takes the explicit form

$$\Lambda = 2\rho \left[ c + \frac{2\pi^2 \rho}{\beta E_c} + \Psi \left( \frac{\beta E_c}{2\pi^2 \rho} \right) \right]$$
FIG. 1: Effective potential $U_{\text{eff}}(\phi)$ for $E_b = 0.4$ (solid, in units of $E_J$) and diffusion coefficient $D(\phi)$ (dashed, for $\beta E_J = 1$) together with the bare Josephson potential $U(\phi)$ (dotted, in units of $E_J$). Other parameters are $\beta E_c = 20, \rho = 0.04$.

where $c = 0.5772\ldots$ is Euler’s constant and $\Psi(\cdot)$ denotes the logarithmic derivative of the gamma function. In the high temperature limit $\beta E_c/\rho \ll 1$ quantum fluctuations become small and independent of $\rho$, i.e., $\Lambda \approx \beta E_c/\pi^2$, and one regains classical Smoluchowski dynamics. In the low temperature range $\beta E_c/\rho \gg 1$ one has $\Lambda \approx 2\rho \log(\beta E_c/2\pi^2 \rho)$ leading to a substantial influence of quantum fluctuations. Qualitatively, the effects of $E_J^*$ and $\theta$ on the dynamics of the fictitious particle are easily understood (cf. fig. 1): $E_J^* < E_J$ lowers the potential barrier in $U_{\text{eff}}$, $\theta$-dependent terms in the diffusion coefficient [see (5,6)] enhance the noise strength in the wells ($D > 1$) and lower it around the barrier tops ($D < 1$). It turns out that this causes a moving particle to experience a larger mobility where in limiting cases barrier ($\beta E_c \ll 1$) or diffusion ($\beta E_J \gg 1$) related quantum fluctuations dominate.

Let us now first address the range of applicability of (4) for the physics of JJs. The QSE is valid if (i) a separation of time scales $\hbar \beta, 1/\gamma \ll \gamma/\omega_J^2$ with the plasma frequency $\omega_J = (2e/\hbar) R I_c$ is guaranteed. Classically, for the McCumber parameter this condition reduces to $\beta_1 = \gamma/\omega_J \gg 1$. Further, (ii) in order for the momentum $m \dot{\phi}$ to relax within the $RC = 1/\gamma$-time to a Boltzmann distribution around $\langle \phi \rangle$, the external voltage $V$ is restricted by $eV \ll \hbar \gamma$. We note in passing that (ii) also ensures that the actual non-ohmic impedance seen by the junction can effectively be treated as ohmic. By combining (i) and (ii) and re-expressing them in junction parameters we thus expect the QSE to describe quantum
phase diffusion in JJ if (see fig. 2)
\[ \frac{E_c}{E_J 2\pi^2 \rho^2} \gg 1, \pi \rho^2 E_J, \frac{V}{RI_c}. \] (8)

Since typically \( \rho \ll 1 \) the above condition allows for a broad range of values for \( E_c/E_J, \beta E_J \), and also large voltages \( V/RI_c \). It is important to note that for realistic JJ with finite \( \rho \) the classical description is always restricted to small and moderate values of \( \beta E_J \), while with increasing \( \beta E_J \) one approaches the quantum Smoluchowski range with a finite \( \Lambda \) [see (7) and fig. 2].

To complete this discussion we also specify the corresponding quantum Langevin equation (in the Ito sense) which generalizes the classical one to
\[ m\gamma \dot{\phi} + U'_{\text{eff}}(\phi) = \xi(t) \sqrt{D(\phi)}. \] (9)

Now, the Eqs. (11) and (12) are the starting point to study in a very elegant way all aspects of the phase dynamics of JJs within the range (8). While in the quantum domain very low temperatures \( (T \to 0) \) have been explored in detail, finite temperature results exists only for small and large Josephson couplings, respectively. In the sequel we will show that the QSE bridges between these limiting cases.

III. STEADY STATE CURRENT AND CURRENT–VOLTAGE CHARACTERISTICS

For finite bias current \( I_b > 0 \) the phase dynamics approaches a stationary non-equilibrium state at longer times. This is associated with a stationary current \( J_{\text{st}} = \lim_{t \to \infty} \langle \hat{\phi}(t) \rangle/2\pi \) given by
\[ J_{\text{st}} = \frac{1 - e^{-\beta \phi}}{\gamma m \phi} \left[ \int_0^{2\pi} d\phi \frac{e^{-\beta \psi(\phi)}}{D(\phi)} \int_\phi^{\phi+2\pi} d\phi' e^{\beta \psi(\phi')} \right]^{-1} \] (10)
with
\[ \psi(\phi) = -E_j^* \cos(\phi) - E_b \phi + \theta E_b \sin(\phi) - \frac{1}{2} \theta E_j^* \sin^2(\phi). \] (11)

According to \( \langle V \rangle = \langle h/2e \rangle J_{\text{st}} \) one then finds the current voltage–characteristics of a current biased junction to read
\[ \langle V \rangle = \frac{\rho \pi}{\beta_e} 1 - e^{-2\pi \beta E_b} \frac{1}{T_{\text{qm}}} \] (12)
FIG. 2: Range of the QSE for a JJ with $\rho \ll 1$, $V/RI_c < 1$. The classical range (shaded) and the domains of Coulomb blockade (CB) and macroscopic quantum tunneling (MQT) are indicated. The QSE is applicable above the thick line, see (3), and the arrows illustrate various changeovers discussed in the text. The slope of the CB boundary decreases with increasing $V/RI_c$ so that for $V/RI_c > 1$ CB dominates in most parts of parameter space.

where the nominator $T_{qm}$ results from normalizing the steady state phase distribution to 1 and can be written as

$$T_{qm} = \frac{1}{2\pi} \int_0^{2\pi} d\phi' \int_0^{2\pi} d\phi \, e^{-\beta E_b \phi} e^{-2\beta E_{J}^* \cos(\phi') \sin(\phi/2) \left[1 - \theta \sin(\phi' - \phi/2)\right]} e^{2\beta \xi(\phi, \phi')} \tag{13}$$

with

$$\xi(\phi, \phi') = \sin(\phi') \sin(\phi/2) \left[E_b + E_{J}^* \cos(\phi') \cos(\phi/2)\right]. \tag{14}$$

The expression (12) together with (13) is the central result and will be discussed in the following.

For $\beta E_c/\rho \ll 1$ the function $T_{qm}$ reduces to the classical result

$$T_{cl}(E_J) = \int_0^{2\pi} d\phi' e^{-\beta E_b \phi'} I_0[2\beta E_J \sin(\phi'/2)] \tag{15}$$

with $I_0$ the modified Bessel function [1] and IZT [6, 7] is recovered. On the other hand, in the low temperature domain $\beta E_c/\rho \gg 1$ two distinct ranges must be distinguished (see fig. 2). For smaller couplings, $\beta E_J < 1$, where Coulomb blockade (CB) transfers charges incoherently, we have $\theta \ll 1$ so that diffusion related quantum fluctuations are negligible. Then, (13) takes the classical form with quantum fluctuations incorporated in terms of a renormalized Josephson energy, i.e. $T_{qm} \approx T_{cl}(E_{J}^*)$. This important extension of IZT has
first been derived in [9] based on a direct evaluation of the real-time path integral expression. However, since diffusion related quantum fluctuations are disregarded, this simple result is (for finite $\rho \ll 1$) indeed restricted to $\beta E_J \lesssim 1$. In contrast, (13) also holds for much larger $E_J$ [fig. 2, arrow (a)] associated with coherent Cooper pair tunneling. An explicit formula can be obtained from (12) for sufficiently large $\beta E_J > 1$ and $\alpha = E_b/E_J = I/I_c < 1$. Then, occasional phase slips lead to the voltage

$$\langle V \rangle_{RI_c} = \frac{\sqrt{1 - \alpha^2}}{2\pi} e^{-2\beta E_J (1-\alpha^2)^{3/2}/(3\alpha^2)} e^{2\theta \sqrt{1-\alpha^2}}$$  \hspace{1cm} (16)$$

which via $\theta$ is affected by diffusion related quantum fluctuations. Accordingly, quantum effects cannot always be captured by a renormalization of the Josephson coupling, rather such a simplification occurs for CB dominated transport only. As can be observed in fig. 2, the result (16) tends for $\theta \to 0$ to classical thermal activation over the barriers of the washboard potential $U(\phi)$ where quantum corrections are of the known damping independent form [16]. At lower temperatures, i.e. for finite $\theta$, they show a complicated dependence on $\rho$ and capture the precursors of macroscopic quantum tunneling (MQT) found at very low temperatures [10]. Thus, the central result (12) fills the gap between established results in different transport domains: On the one hand for fixed $\beta E_c/\rho \gg 1$ it leads with increasing $\beta E_J$ from Coulomb blockade to coherent Cooper pair tunneling [fig. 2, arrow (a)]. On the other hand for fixed $\beta E_J > 1$ it connects with varying $\beta E_c/\rho$ classical thermal activation with MQT [fig. 2, arrow (b)]. Apart from limiting cases, the result (12) must be evaluated numerically as illustrated in fig. 3 (left).

What is left, is the domain of small Josephson coupling $\beta E_J < 1$ and increasing $\beta E_c/\rho$ [fig. 2, arrow (c)] that has been discussed in the literature [8, 9] for a voltage biased junction. This changeover leads from classical coherent IZT [6] to CB [17]. Here, we demonstrate that the QSE recovers this scenario, too, and hence gives a complete description throughout the QSE range sketched in fig. 2. Results for a voltage biased JJ are gained from the current biased case by means of $I = V/R$ and $\langle V \rangle = V - RI_c$. The interesting observable then is the supercurrent $I_s$. From (12) we obtain to order $(\beta E_J)^2$

$$\frac{I_s}{I_c} = \frac{(\beta E_J)^2}{2} \frac{V}{RI_c} \frac{1}{1 + [\beta E_J V/(RI_c)]^2} \hspace{1cm} (17)$$

with the characteristic renormalized $E_J$ discussed above. This result coincides with that of [9] and connects quite different transport regimes [see figs. 2,3 (right)]. Namely, in the
classical limit $\beta E_c/\rho \ll 1$ it gives the small-$\beta E_J$-limit of the IZT \cite{6}, while in the opposite quantum domain $\beta E_c/\rho \gg 1$ it agrees with the small $\rho, \Lambda$ expansion of the CB expression \cite{17}

$$\frac{I_s}{I_c} = \frac{\beta E_J e^{-\Lambda}}{4\pi} \frac{\Gamma(\rho - i\beta eV/\pi)^2}{\Gamma(2\rho)} \sinh(\beta eV).$$ \tag{18}

Thus, we conclude that the quantum Smoluchowski result \cite{17} interpolates in the domain of small Josephson couplings between phase-coherent and incoherent charge transport (with $E_J \to E_{J*}$) where the strength of charging effects controls the changeover from one regime to the other (see fig. 2). Note that \cite{17} yields the supercurrent reasonably well also for $\beta E_c \gg 1$ ($\Lambda$ of order 1) \cite{fig. 3 (right), bottom].

IV. CONCLUSIONS

Based on the analogy between JJ physics and the diffusive motion of a fictitious particle in a washboard-type of potential we generalized for strong friction (low capacitance) the IZT to the low temperature quantum domain. Quantum fluctuations affect the potential profile as well as the diffusion constant and physically are associated with charging phenomena. This formulation reduces to known results derived in certain limits and thus gives a complete description over broad domains of parameter space. Our main expression is specified in \cite{12}.
together with (13). It contains incoherent charge transfer (CB) and coherent Cooper pair tunneling as well as the precursors of MQT. In particular, the incorporation of quantum fluctuations in terms of a renormalized Josephson junction turned out to be characteristic for CB only. Eventually, our findings confirm the recently developed quantum Smoluchowski theory to be a powerful tool for studying low temperature quantum diffusion in mesoscopic physics.

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