Evolution of gluon TMD at low and moderate $x$

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We study how the rapidity evolution of gluon transverse momentum dependent distribution changes from nonlinear evolution at small $x \ll 1$ to linear double-logarithmic evolution at moderate $x \sim 1$.

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1. Introduction

A TMD factorization \cite{1, 2} generalizes the usual concept of parton density by allowing PDFs to depend on intrinsic transverse momenta in addition to the usual longitudinal momentum fraction variable. These transverse-momentum dependent parton distributions (also called unintegrated parton distributions) are widely used in the analysis of semi-inclusive processes like Drell-Yan process or single-inclusive deep inelastic scattering (SIDIS) or dijet production in hadron-hadron collisions (for a review, see Ref. \cite{2}). However, the analysis of TMD evolution in these cases are mostly restricted to the evolution of quark TMDs, whereas at high collider energies the majority of produced particles will be small-$x$ gluons. In this case one has to understand the transition between non-linear dynamics at small $x$ and presumably linear evolution of gluon TMDs at intermediate $x$, which is not clear at present.

To make things even more complicated, there are two non-equivalent definitions of gluon TMDs in small-$x$ and “medium $x$” communities. In the small-$x$ literature the Weizsacker-Williams (WW) unintegrated gluon distribution \cite{3} is defined in terms of the matrix element

$$\sum_X \text{tr}(p|U\partial_i U^\dagger(z_\perp)|X\rangle \langle X|U\partial_i U^\dagger(0_\perp)|p\rangle$$

between target states (typically protons). Here $\sum_X$ denotes the sum over full set of hadronic states and $U_z$ is a Wilson-line operator - infinite gauge link ordered along

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the light-like line

\[ U(z_\perp) = [\infty n + z_\perp, -\infty n + z_\perp], \quad [x, y] \equiv P e^{i g \int \! du \, (x-y)^\mu A^\mu_n (ux + (1-u)y) } \] (2)

In the spirit of rapidity factorization, Bjorken \( x \) enters this expression as a rapidity cutoff for Wilson-line operators. Roughly speaking, each gluon emitted by Wilson line has rapidity restricted from above by \( \ln x_B \).

One can rewrite the above matrix element (up to some trivial factor) in the form

\[ \alpha_s D(x_B, z_\perp) = -\frac{\alpha_s}{2\pi (p \cdot n) x_B} \int \! du \sum_X \langle p | \tilde{F}_\xi (z_\perp + un) | X \rangle \langle X | F^{u\xi}(0) | p \rangle \]

\[ F^\xi(z_\perp + un) = [\infty n + z_\perp, un + z_\perp]^{am} n^\mu F_m^{\mu n}(un + z_\perp) \]

\[ \tilde{F}_\xi(z_\perp + un) = n^\mu F_m^{\mu n}(un + z_\perp)[un + z_\perp, \infty n + z_\perp]^{ma} \] (3)

and define the “WW unintegrated gluon distribution”

\[ D(x_B, k_\perp) = \int d^2 z_\perp e^{-i (k, z)_\perp} D(x_B, z_\perp) \] (4)

where \((k, z)_\perp\) denotes the scalar product in 2-dim transverse Euclidean space. It should be noted that since Wilson lines are renorm-invariant \( \alpha_s D(x_B, z_\perp) \) does not depend on the renormalization scale \( \mu \).

On the other hand, at moderate \( x_B \) the unintegrated gluon distribution is defined as

\[ D(x_B, k_\perp, \eta) = \int d^2 z_\perp e^{-i (k, z)_\perp} D(x_B, z_\perp, \eta), \] (5)

\[ \alpha_s D(x_B, z_\perp, \eta) = -\frac{-x_B^{-1} \alpha_s}{2\pi (p \cdot n)} \int \! du \, e^{-i x_B u (pm)} \sum_X \langle p | \tilde{F}_\xi (z_\perp + un) | X \rangle \langle X | F^{u\xi}(0) | p \rangle \]

There are more involved definitions with Eq. (5) multiplied by some Wilson-line factors [2] but we will discuss the “primordial” TMD [5]. The Bjorken \( x \) is now introduced explicitly in the definition of gluon TMD. However, because light-like Wilson lines exhibit rapidity divergencies, we need a separate cutoff \( \eta \) (not necessarily equal to \( \ln x_B \)) for the rapidity of the gluons emitted by Wilson lines. In addition, the matrix elements (5) will have double-logarithmic contributions of the type \((\alpha_s \eta \ln x_B)^n\) while the WW distribution (3) has only single-log terms \((\alpha_s \ln x_B)^n\) described by the BK evolution [5, 6].

In the present paper we study the connection between rapidity evolution of two gluon TMDs (3) and (5) in the kinematic region where \( s \gg k^2_\perp \) and \( k^2_\perp \geq \) few GeV\(^2\). The latter condition means that we can use perturbative QCD while the former one indicates that TMD factorization, rather than the collinear factorization, is applicable [2]. In this kinematic region we will vary Bjorken \( x \) and look how non-linear evolution at small \( x \) transforms into linear evolution at moderate \( x_B \).

It should be emphasized that we consider gluon TMD with Wilson links going to \(+\infty\) in the longitudinal direction. Another gluon TMD with links going to \(-\infty\)
arises in the problems with exclusive particle production, see the discussion in Ref. [7]. We plan to study it in future publications.

The paper is organized as follows. In Sec. 2 we remind the general logic of rapidity factorization and rapidity evolution. In Sec. 3 we calculate the Lipatov vertex of the gluon production by TMD operator and in Sec. 4 we obtain the so-called virtual factorization and rapidity evolution. In Sec. 5 we calculate the Lipatov vertex of the

2. Rapidity factorization and evolution

In the spirit of high-energy OPE, the rapidity of the gluons is restricted from above by the “rapidity divide” \( \eta \) separating the impact factor and the matrix element so the proper definition of \( U_x \) is

\[
U_x^\eta = \text{Pexp}\left[ ig \int_{-\infty}^{\infty} du \, p_1^\alpha A_\mu^\eta (u p_1 + x_\perp) \right],
\]

\[
A_\mu^\eta (x) = \int \frac{d^4 k}{16 \pi^4} \theta (e^\eta - |\alpha|) e^{ik \cdot x} A_\mu (k)
\]

(6)

where the Sudakov variable \( \alpha \) is defined as usual, \( k = \alpha p_1 + \beta p_2 + k_\perp \). We define the light-like vectors \( p_1 \) and \( p_2 \) such that \( p_1 = n \) and \( p_2 = p - \frac{m^2}{s} n \) where \( p \) is the momentum of the target particle of mass \( m \). We use metric \( g^{\mu \nu} = (1, -1, -1, -1) \) so \( p \cdot q = (\alpha p_1 \beta_q + \alpha_q \beta_p) \frac{x}{2} - (p, q) \perp \). For the coordinates we use the notations \( x_* = x p_1^\mu \) and \( x_* = x p_2^\mu \) related to the light-cone coordinates by \( x_+ = \sqrt{\frac{s}{2}} x_+ \) and \( x_- = \sqrt{\frac{s}{2}} x_- \). It is convenient to define Fourier transform of the operator \( F_i^a \)

\[
F_i^a (k_\perp, \beta_B) = \int d^2 z_\perp e^{-i(k,z)_\perp} F_i^a (z_\perp, \beta_B),
\]

\[
F_i^a (z_\perp, \beta_B) \equiv \frac{2}{s} \int dz_+ e^{i \beta_B z_+} [\infty, z_\perp] a m F_i^a (z_+, z_\perp)
\]

(7)

and similarly

\[
\tilde{F}_i^a (k_\perp, \beta_B) = \int d^2 z_\perp e^{i(k,z)_\perp} \tilde{F}_i^a (z_\perp, \beta_B),
\]

\[
\tilde{F}_i^a (z_\perp, \beta_B) \equiv \frac{2}{s} \int dz_+ e^{-i \beta_B z_+} F_i^m (z_+, z_\perp) [\infty, z_\perp] a m
\]

(8)

in the complex-conjugate part of the amplitude. Here we introduced the “Bjorken \( \beta_B \)” to decouple this notation from \( x_B \) in the denominator in the r.h.s. of Eq. \( [5] \). Also, hereafter we use the notation \( [\infty, z_*] \equiv [\infty p_1 + z_\perp, \frac{2}{s} z_* p_1 + z_\perp] \) where \( [x, y] \) stands for the straight-line gauge link connecting points \( x \) and \( y \) as defined in Eq. \( [2] \).

In this notations the unintegrated gluon TMD \( [5] \) can be represented as

\[
\langle p | F_i^a (k_\perp, \beta_B) F_i^a (k_\perp, \beta_B) | p \rangle \equiv \sum_X \langle p | F_i^a (k_\perp, \beta_B') | X \rangle \langle X | F_i^a (k_\perp, \beta_B) | p \rangle
\]

\[
= -2 \pi \delta (\beta_B - \beta_B') (2 \pi)^2 \delta^{(2)} (k_\perp - k_\perp') 2 \pi x_B D (\beta_B = x_B, k_\perp, \eta)
\]

(9)
Hereafter we use a short-hand notation

$$\langle p|\hat{O}_1...\hat{O}_mO_1...O_n|p\rangle = \sum_X \langle p|\hat{T}\{\hat{O}_1...\hat{O}_m\}|X\rangle \langle X|T\{O_1...O_n\}|p\rangle$$  \hspace{1cm} (10)$$

where tilde on the operators in the l.h.s. of this formula stands as a reminder that they should be inverse time ordered as indicated by inverse-time ordering $\tilde{T}$ in the r.h.s. of the above equation.

As discussed e.g. in Refs [8] such matrix element can be represented by a double functional integral

$$\langle \hat{O}_1...\hat{O}_mO_1...O_n \rangle = \int D\hat{\psi} D\bar{\psi} e^{-iS_{QCD}(\hat{A},\hat{\psi})} \int DAD\bar{\psi} e^{iS_{QCD}(A,\bar{\psi})} \langle \hat{O}_1...\hat{O}_mO_1...O_n \rangle$$  \hspace{1cm} (11)$$

with the boundary condition $\hat{A}(\vec{x},t=\infty) = A(\vec{x},t=\infty)$ (and similarly for quark fields) reflecting the sum over all intermediate states $X$. Due to this condition, the matrix element (22) can be made gauge-invariant by connecting the endpoints of Wilson lines at infinity with the gauge link

$$\langle p|\hat{F}^a_{i}(z'_{\perp},\beta_{B})\hat{F}^a_i(z_{\perp},\beta_B)|p\rangle \rightarrow \langle p|\hat{F}^a_i(z'_{\perp},\beta_{B})[z_{\perp} + \infty p_1, z_{\perp} + \infty p_1]|\hat{F}^a_i(z_{\perp},\beta_B)|p\rangle$$  \hspace{1cm} (12)$$

This gauge link is important if we use the light-like gauge $p_\mu A_\mu = 0$ for calculations [9], but in all other gauges it can be neglected. We will not write it down explicitly but will always assume it in our formulas.

In the spirit of rapidity factorization, in order to find the evolution of the operator $\hat{F}^a_i(z'_{\perp},\beta_{B})\hat{F}^a_i(z_{\perp},\beta_B)$ with respect to rapidity cutoff $\eta$, see Eq. (6), one should integrate in the matrix element (22) over gluons and quarks with rapidities $\eta > Y > \eta'$ and temporarily “freeze” fields with $Y < \eta'$ to be integrated over later. (For a review, see Refs. [10] [11]). In this case, we obtain functional integral of Eq. (11) type over fields with $\eta > Y > \eta'$ in the “external” fields with $Y < \eta'$. In terms of Sudakov variables we integrate over gluons with $\alpha$ between $\sigma_1 = e^\eta$ and $\sigma_2 = e^{\eta'}$ and, in the leading order, only the diagrams with gluon emissions are relevant - the quark diagrams will enter as loops at the next-to-leading (NLO) level.

As discussed in Ref. [5], the interaction of gluons with large $\alpha$ with small $\alpha$ fields is described by eikonal gauge factors. The typical longitudinal size of small $\alpha$ fields is $\sigma_* \sim \frac{\sigma_{2s}}{\pi^2}$, which is much smaller than the typical distances $\sim \frac{\sigma_{2s}}{\pi^2}$ traveled by large-$\alpha$ gluons. Effectively, large-$\alpha$ gluons propagate in the external field of the small-$\alpha$ shock wave. In the leading order there is only one extra gluon and we get the typical diagrams of Fig. 1 type. The real part of the kernel can be obtained as a square of a Lipatov vertex - the amplitude of the emission of a real gluon by the operator $F^a_i$.

2.1. Lipatov vertex

In this Section we will present the real contribution corresponding to square of Lipatov vertex describing the emission of a gluon from the operator $F^a_i$. The Lipatov
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Fig. 1. Typical diagrams for real (a) and virtual (b) contributions to the evolution kernel. The dashed lines denote gauge links.

The vertex is defined as

\[ L_{\mu}^{ab}(k, z_\perp) = i \lim_{k^2 \to 0} k^2 \langle T \{ A_\mu^a(k) F^b_i(z_\perp, \beta_B) \} \rangle \]  

(13)

It is convenient to use light-like gauge \( p_\mu^2 A_\mu = 0 \), cf. Ref [13]. The three corresponding diagrams are shown in Fig. 2.

The actual calculations will be published elsewhere and here we present only the final results in the \( p_\mu^2 A_\mu = 0 \) gauge

\[
L_{\mu}^{ab}(k, z_\perp) = i \lim_{k^2 \to 0} k^2 \langle T \{ A_\mu^a(k) F^b_i(z_\perp, \beta_B) \} \rangle \\
= \alpha \beta_B s \left( g_{\mu}^\perp - \frac{2}{s \alpha} p_\mu^\perp k_\perp^\perp \right) (k_\perp \perp) \frac{1}{p_\perp^2 + \alpha \beta_B s} - U \frac{1}{p_\perp^2 + \alpha \beta_B s} U'^\perp (z_\perp)^{ab} \\
+ 2(k_\perp |U| \left( \frac{p_\mu}{p_\perp} + \frac{2}{\alpha s} p_\mu (p, k)^\perp \right) U'^\perp - p_\perp \left( \frac{p_\mu}{p_\perp} + \frac{2}{\alpha s} p_\mu (p, k)^\perp \right) \right) (z_\perp)^{ab} \\
+ 2(k_\perp |U(\frac{p_\mu}{p_\perp} + 2 p_\mu \alpha s) F_i(\beta_B) (z_\perp)^{ab} \)  

(14)

where we use Schwinger’s notations

\[ (x_\perp | f(p_\perp) | y_\perp) = \int d^2 p_\perp e^{i(p_\perp - y_\perp) \cdot p} f(p), \quad (x_\perp | p_\perp) = e^{i(p_\perp x_\perp)} \]
and the space-saving notation $d^n p \equiv \frac{d^n p}{(2\pi)^n}$. It is easy to check that $k_\mu L_\mu^b(k, z_\perp) = 0$ as required by gauge invariance.

The cross section of gluon emission will be proportional to the square of Lipatov vertex (14). The terms $\sim p_2\mu$ do not contribute to the square so one can use the Lipatov vertex of transverse gluon emission in the form

$$L_{\mu\perp}^{ab}(k, q_\perp) = (k_\perp | U \frac{p_i^2 g_{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} U^\dagger - \frac{p_i^2 g_{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} + 2\frac{p_i}{p_\perp} F_i(\beta_B) | z_\perp)^{ab}$$ (15)

It is worth noting that at $\beta_B = 0$ this results agrees with the Lipatov vertex obtained in Ref. [12]. The real-production part of the evolution kernel for operator (22) is the square of the Lipatov vertex (15):

$$\langle \tilde{F}^{ai}(z_\perp', \beta_B) F^{ai}(z_\perp, \beta_B) \rangle_{\text{real}} = -\frac{g^2}{4\pi} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \text{Tr}(z_\perp' | \tilde{U} \frac{p_i^2 g_{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} U^\dagger - \frac{p_i^2 g_{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} + 2\tilde{F}_i(\beta_B) \frac{p_j}{p_\perp} | \tilde{U} \frac{p_i^2 g_{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} U^\dagger - \frac{p_i^2 g_{ij} + 2p_i p_j}{p_\perp^2 + \alpha \beta_B s} + 2\tilde{F}_i(\beta_B) \frac{p_j}{p_\perp} | z_\perp)$$ (16)

where $\text{Tr}$ stands for the trace in the adjoint representation and the summation goes over the transverse Latin indices $i, j = 1, 2$. This expression is UV finite and the IR-divergent contribution has the form

$$\frac{g^2}{\pi} N_c \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \tilde{F}^{ai}(\beta_B, z_\perp') (z_\perp' | \frac{1}{p_\perp} | z_\perp) F^{ai}(\beta_B, z_\perp)$$ (17)

3. Virtual corrections

Apart from the “real correction” part of the evolution kernel proportional to square of Lipatov vertex (16) we should take into account virtual corrections coming from diagrams shown in Fig. 3. The result for these diagrams is

Fig. 3. Virtual gluon corrections.
\[ \langle \mathcal{F}_i^a(\beta_B, z_\perp) \rangle = \frac{g^2}{2\pi} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \left\{ -N_c \mathcal{F}_i^a(\beta_B, z_\perp) \left( \frac{1}{p_1^2} \right) \right\} \] (18)

\[ + i f^{abc} \left[ (z_\perp \mid \frac{1}{p_1^2} \mathcal{F}_i(\beta_B) U \frac{p_1^2}{\alpha_B s + p_1^2} U^\dagger |z_\perp \rangle)^{bc} - (z_\perp \mid \frac{1}{p_1^2} \partial_\perp^2 U \frac{p_1}{\alpha_B s + p_1^2} U^\dagger |z_\perp \rangle)^{bc} \right] \}

\[ = - \frac{g^2}{2\pi} f^{abc} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} (z_\perp \mid \frac{1}{p_1^2} \mathcal{F}_i(\beta_B) U \frac{\alpha_B s}{\alpha_B s + p_1^2} U^\dagger + \frac{1}{p_1^2} \partial_\perp^2 U \frac{p_1}{\alpha_B s + p_1^2} U^\dagger |z_\perp \rangle)^{bc} \]

It should be emphasized that the UV divergence over \( p_1^2 \) in the r.h.s. of this equation is absent due to the cutoff \( \alpha < \sigma_1 \).

The total virtual correction is the sum of the correction (18) with the similar correction in the complex conjugate amplitude

\[ \langle \mathcal{\tilde{F}}^a_i(\beta_B, z'_\perp) \rangle = - \frac{g^2}{2\pi} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \]

\[ \times \text{Tr} \{ \mathcal{\tilde{F}}_i(\beta_B, z'_\perp) (z_\perp \mid \frac{1}{p_1^2} \mathcal{F}_i(\beta_B) U \frac{\alpha_B s}{\alpha_B s + p_1^2} U^\dagger + \frac{1}{p_1^2} \partial_\perp^2 U \frac{p_1}{\alpha_B s + p_1^2} U^\dagger |z_\perp \rangle) \}

\[ + (z'_\perp \mid \tilde{U} \frac{p_1}{\alpha_B s + p_1^2} \partial_\perp^2 \tilde{U}^\dagger \frac{1}{p_1^2} \tilde{U} \mathcal{\tilde{F}}_i(\beta_B) \frac{1}{p_1^2} \langle z'_\perp | z_\perp \rangle | \mathcal{F}_i(\beta_B, z_\perp) \rangle \}

Note that the IR divergence in the r.h.s. of this equation cancels with that of Eq. (17).

4. Discussion

Adding together the production part (16) and the virtual corrections (18) one obtains

\[ \langle \mathcal{\tilde{F}}^a_i(z'_\perp, \beta_B) \mathcal{F}_i^a(z_\perp, \beta_B) \rangle = \frac{g^2}{4\pi} \text{Tr} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \left\{ \left( z'_\perp \mid \mathcal{\tilde{F}}_i(z'_\perp, \beta_B) \mathcal{\tilde{F}}_i(z_\perp, \beta_B) \right) \right\} \]

\[ - 2 \mathcal{\tilde{F}}_i(z'_\perp, \beta_B) \mathcal{\tilde{F}}_i(z_\perp, \beta_B) \left[ (z_\perp \mid \frac{1}{p_1^2} |z_\perp\rangle + (z'_\perp \mid \frac{1}{p_1^2} |z'_\perp\rangle) \right] + 2 \mathcal{\tilde{F}}_i(z'_\perp, \beta_B) \]

\[ \times (z_\perp \mid \frac{p_1}{p_1^2} U \frac{p_1^2}{\alpha_B s + p_1^2} U^\dagger |z_\perp \rangle + 2(z'_\perp \mid \mathcal{\tilde{U}} \frac{p_1^2}{p_1^2} \mathcal{\tilde{U}}^\dagger \frac{p_1}{p_1^2} \mathcal{\tilde{F}}_i(z'_\perp, \beta_B) \mathcal{\tilde{F}}_i(z_\perp, \beta_B) \right\} \]
It can be rewritten in the form where the cancellation of UV and IR divergencies is obvious

\[
\langle \tilde{F}^{ai}_i(k'_1, \beta_B)^{\alpha}(k_1, \beta_B) \rangle_{\text{frag}} \equiv \sum_X \langle 0 | \tilde{F}^{ai}_i(k'_1, \beta_B)^{\alpha} | p + X \rangle \langle p + X | F^{ai}(k_1, \beta_B) | 0 \rangle = -2 \pi \delta(\beta_B - \beta'_B)(2\pi)^2 \delta^{(2)}(k_1 - k'_1) 2\pi x_B D_{\text{frag}}(\beta_B = x_B, k_1, \eta) 
\]

(22)

It is easy to see that our formula (24) for the evolution kernel smoothly interpolates between the \( k_T \)-factorization and TMD-factorization cases. Indeed, in the framework of the usual small-\( x \) approximation \( \beta_B \) is neglected so the corresponding “small-\( x \)” gluon TMD looks like

\[
F_i^{ai}(z_\perp, 0) = U_i^{ai}(z_\perp) \equiv -2\text{tr}\{t^a U \partial U^\dagger\} 
\]

(23)

(where \( \text{tr} \) stands for the trace in the fundamental representation) and Eq. (20)
reduces to the non-linear equation
\[
(\tilde{U}^{ai}(z^1_\perp)U^a_\perp(z_\perp)) \equiv 0 \quad \text{(24)}
\]
\[
= \frac{g^2}{\pi} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \left[ - (z^1_\perp|\tilde{U}_p,\tilde{U}^\dagger|\tilde{U}^{\dagger} \frac{p^i}{p^j_\perp} \tilde{U} - \frac{p^i}{p^j_\perp}) (U \frac{p^j}{p^j_\perp} U^\dagger - \frac{p^j}{p^j_\perp}) U p^i|U^\dagger|z_\perp)_{a\alpha} \\
- \frac{N_c}{2} \tilde{U}^{ai}(z^1_\perp)U^{ai}(z_\perp) [\{ (z^1_\perp | p^j_{\perp} | z_\perp) + (z^1_\perp | p^j_{\perp} | p^j_{\perp} | z_\perp) \] \\
+ i f^{abc} \tilde{U}^{ai}(z^1_\perp)(z_\perp | p^j_\perp p^j_\perp U^\dagger | z_\perp)_{bc} + i f^{abc} (z^1_\perp | \tilde{U}^{\dagger} \frac{p^i}{p^j_{\perp}} \tilde{U}^{\dagger} \frac{p^j}{p^j_{\perp}} | z_\perp)_{ab} U^a_\perp(z_\perp) \]
\]
which agrees with the results of Ref. \[10\]. It is convenient to rewrite this equation as follows (cf. Ref. \[14\]):
\[
\frac{d}{d\eta} \tilde{U}^{a_\perp}(z_\perp) = \frac{g^2}{\pi} \text{Tr}(-\frac{1}{\eta^2} \tilde{U}^{a_\perp}) \int d^2 z_3 (\tilde{U}_{3_\perp} \tilde{U}^\dagger_{3_\perp} - 1) \frac{z^2_3}{2^{1/2}} (U_{3_\perp} U^\dagger_{3_\perp} - 1) (i \partial^\perp + U^\dagger_{3_\perp}) \quad \text{(25)}
\]
where all indices are 2-dimensional and Tr stands for the trace in the adjoint representation. It is easy to see that the expression in the square brackets is actually the BK kernel for the double-functional integral for cross sections \[10\] \[15\].

On the other hand, if \(\beta_B \sim 1\) so that \(\alpha \beta_B \gg p^2_\perp\) we get a linear equation
\[
\langle \tilde{F}^{ai}(z^1_\perp, \beta_B) F^a_\perp(z_\perp, \beta_B) \rangle \equiv 0 \quad \text{(26)}
\]
\[
= - \frac{g^2 N_c}{\pi} \int_{\sigma_2}^{\sigma_1} \frac{d\alpha}{\alpha} \int \frac{d^2 p}{p^2} \left[ 1 - e^{i(p, z^1_\perp)} \right] \langle \tilde{F}^{ai}(z^1_\perp, \beta_B) F^a_\perp(z_\perp, \beta_B) \rangle
\]
which can be rewritten as a linear equation
\[
\frac{d}{d\eta} D(x_B, z_\perp, \eta) = - \frac{\alpha_s N_c}{\pi^2} D(x_B, z_\perp, \eta) \int \frac{d^2 p}{p^2} \left[ 1 - e^{i(p, z^\perp)} \right]
\]

We see that the IR divergence at \(p^2_\perp \rightarrow 0\) cancels while the UV divergence in the virtual correction should be cut from above by the condition \(p^2_\perp < \sigma s\) following from Eq. \[20\]. Actually, at \(x_B \sim 1\) there will be logarithmical region \(e^{\pi m \sqrt{5}} \gg p^2_\perp \gg m^2\) so one has to sum up leading logarithms \((\alpha_s \eta)^n\) in the evolution kernel Eq. \[26\] after which the kernel should reproduce the usual Sudakov double logarithms.

From Eq. \[20\] it is clear that the transition between linear evolution \[26\] and the non-linear evolution \[25\] occurs at \(x_B = \beta_B \sim \frac{m^2}{s}\). We plan to study this transition in future publications.

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