ON THE HOMOTOPY DIRICHLET PROBLEM FOR
p-HARMONIC MAPS

STEFANO PIGOLA AND GIONA VERONELLI

Abstract. In this paper we deal with the relative homotopy Dirichlet problem for p-harmonic maps from compact manifolds with boundary to manifolds of non-positive sectional curvature. Notably, we give a complete solution to the problem in case the target manifold is either compact, rotationally symmetric or two dimensional and simply connected. The proof of the compact case uses some ideas of White to define the relative d-homotopy type of Sobolev maps, and the regularity theory by Hardt and Lin. To deal with non-compact targets we introduce a periodization procedure which permits to reduce the problem to the previous one. Also, a general uniqueness result is given.

Contents

Introduction 1
1. Compact targets 8
1.1. Lipschitz approximation in relative homotopy class 8
1.2. p-Minimizing tangent maps 10
1.3. Extending relative d-homotopies 12
1.4. The d-homotopy type of $W^{1,p}$ maps 13
1.5. Proof of Theorem B 15
2. Complete targets 17
2.1. Gluing model manifolds keeping $\text{Sect} \leq 0$ 18
2.2. Convex exhaustion functions on glued manifolds 19
2.3. Compact hyperbolic manifolds with large injectivity radii 22
2.4. A maximum principle for $p$-harmonic maps 24
2.5. Proof of Theorem D 25
2.6. A general uniqueness result 26
References 29

Introduction

Let $(M, g)$ and $(N, h)$ be Riemannian manifolds of dimensions $m$ and $n$ respectively. Let $u : M \to N$ be a $C^1$ map. The $p$-energy density $e_p(u) :
$M \to \mathbb{R}$ is the non-negative function defined on $M$ as

$$e_p(u)(x) = \frac{1}{p} |du|_{HS}^p(x).$$

Here the differential $du$ is considered as a section of the $(1, 1)$-tensor bundle along the map $u$, i.e. $du \in \Gamma(T^*M \otimes u^{-1}TN)$ is a vector valued differential $1$-form. Moreover $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product. If $\Omega \subset M$ is a compact domain, we define the $p$-energy of $u|_{\Omega} : \Omega \to N$ by

$$E^\Omega_p(u) = \int_{\Omega} e_p(u) dV_M.$$

Let $X$ be a $C^1$ vector field along $u$, i.e. a section of the bundle $u^{-1}TN$, supported in $\Omega$. Then

$$u_t(x) = N \exp_{u(x)} tX(x).$$

defines a variation of $u$ which preserves $u$ on $\partial\Omega$. The map $u : M \to N$ is said to be $p$-harmonic if, for each compact domain $\Omega \subset M$, it is a stationary point of the $p$-energy functional, that is

$$\frac{d}{dt} \bigg|_{t=0} E^\Omega_p(u_t) = \int_M \langle |du|^{p-2} du, dX \rangle_{HS} dV_M = 0.$$

The latter equality corresponds to the weak formulation of the $p$-laplacian equation

$$\Delta_p u = \text{div}(|du|^{p-2} du) = 0. \tag{1}$$

Here $- \text{div} = \delta$ is the formal adjoint of the exterior differential $d$, with respect to the standard $L^2$ inner product on vector-valued differential $1$-forms on $M$. In local coordinates, (1) takes the expression

$$(\Delta_p u)^A = g^{ij} \left( \frac{\partial}{\partial x^j} \left( |du|^{p-2} \frac{\partial u^A}{\partial x^i} \right) - M \Gamma^k_{ij} \frac{\partial u^A}{\partial x^k} |du|^{p-2} \right.$$

$$\left. + N \Gamma^A_{BC} \frac{\partial u^B}{\partial x^i} \frac{\partial u^C}{\partial x^j} |du|^{p-2} \right) = 0,$$

which, in turn, can be written in the compact form

$$(\Delta_p u)^A = \text{div} \left( |du|^{p-2} \nabla u^A \right) + |du|^{p-2} \Gamma^A(du, du) = 0.$$

It's worthwhile to observe that, in case $N \hookrightarrow \mathbb{R}^q$ is isometrically embedded in some Euclidean space $\mathbb{R}^q$ with second fundamental form $\mathcal{A}$, the above definition of $p$-harmonicity for a $C^1$ map $u : M \to \mathbb{R}^q$ is equivalent to the standard notion of weak $p$-harmonicity, that is

$$\int |Du|^{p-2} \left\{ Du \cdot D\varphi + \mathcal{A}(Du, Du) \cdot \varphi(x) \right\} = 0, \quad \forall \varphi \in C^\infty_c(\Omega, \mathbb{R}^q) \tag{2}$$

where we have set

$$Du \cdot D\varphi = g^{ij} \delta_{AB} \frac{\partial u^A}{\partial x^i} \frac{\partial \varphi^B}{\partial x^j}.$$
and
\[ A(Du, Du) \cdot \varphi = g^{ij} \delta_{CD} A^C_{AB} \frac{\partial u^A}{\partial x^i} \frac{\partial u^B}{\partial x^j} \varphi^D. \]

To see this, it’s enough to take
\[ X_x = D (\Pi_N)|_{u(x)} \cdot \varphi (x) \in T_{u(x)} N \subset \mathbb{R}^q, \]
where \( \Pi_N \) is the nearest point projection from a tubular neighborhood of \( N \) in \( \mathbb{R}^q \).

In this paper we address the problem of the unique solvability of the homotopy Dirichlet problem for \( p \)-harmonic maps into a geodesically complete, non-compact manifold \( N \) of non-positive curvature.

**Problem A.** Let \((M, g)\) be a compact, \( m \)-dimensional Riemannian manifold with smooth boundary \( \partial M \neq \emptyset \) and let \( N \) be a complete, possibly compact, \( n \)-dimensional Riemannian manifold without boundary. Assume also that \( N \) has non-positive sectional curvature or, more generally, that the universal covering of \( N \) supports a strictly convex exhaustion function. For any \( p \geq 2 \) and any given \( f \in C^0(M, N) \), consider the \( p \)-Dirichlet problem
\[
\begin{align*}
\Delta_p u &= 0 \quad \text{on } M \\
u &= f \quad \text{on } \partial M.
\end{align*}
\]

Has this \( p \)-Dirichlet problem a (unique) solution \( u \in C^{1, \alpha}(\text{int}(M), N) \cap C^0(M, N) \) in the homotopy class of \( f \) relative to \( \partial M \)?

Actually one expects Problem A to have a positive answer. A first evidence in this direction is given by the classical harmonic case. When \( p = 2 \), the Dirichlet problem for harmonic maps into non-positively curved manifolds has been solved by R. Schoen and S.T. Yau, who extended to non-compact targets a previous result due to R. Hamilton; see [Ham] and Theorem 8.5 in Chapter IX of [SY2]. Schoen and Yau’s proof makes use of Hamilton’s heat flow for harmonic maps and it is unlikely that this can be extended to the \( p > 2 \) case.

In the \( p \)-harmonic realm, the first interesting result is due to S.W. Wei [We2] who considered, for a compact target \( N \) and boundary datum \( f \in C^0(M, N) \cap \text{Lip}(\partial M, N) \), a weaker version of Problem A. More precisely, in Theorem 7.1 of [We2] Wei looked for solutions to the Dirichlet problem in the free homotopy class of the initial datum \( f \). Wei’s proof is based on a minimization procedure which permits to find a \( p \)-energy minimizer \( u \in W^{1,p} \) in the free homotopy class of \( f \). Namely, he minimizes the \( p \)-energy among all the maps \( v \in W^{1,p} \) which coincide with \( f \) on the boundary \( \partial M \) (in the trace sense), and such that \( v \) and \( f \) induce conjugate homomorphisms \( v_* \) and \( f_* \) between the corresponding fundamental groups \( \pi_1(M) \) and \( \pi_1(N) \).

The fact that the definition of the induced homomorphisms for, a priori non-continuous, \( W^{1,p} \) maps is well posed follows by the work of B. White [Wh]. Moreover, the regularity of \( u \) is provided by a theorem of R. Hardt and F.-H. Lin [HL], which extends results of R. Schoen and K. Uhlenbeck.
dealing with 2-energy minimizers in the assumption that $N$ is non-positively curved and the minimizer $u$ has image $u(M)$ bounded in $N$. By the way, we recall that a similar procedure was previously applied by F. Burstall to complete non-compact domain manifolds $M$, to obtain the existence of smooth harmonic maps in the free homotopy class of a finite-energy initial datum [Bu, We].

If we try to solve Problem A even for a compact target $N$ of non-positive curvature, it is easily seen that the proof proposed by Wei can not work without changes. In fact the induced homomorphism is not enough to determine completely the relative homotopy type of a given map. In this spirit, an easy counterexample can be constructed by considering the 2-dimensional torus $N = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the compact manifold with boundary $M \subset T^2$ given by $M = \{([x_1, x_2]) \in T^2 : 0 \leq x_1 \leq 1/2\}$. Then, choosing $f : M \to N$ defined by $f([x_1, x_2]) = ([3x_1, x_2])$, one has that the inclusion map $i : M \to N$ is a harmonic map with $i|_{\partial M} = f|_{\partial M}$, which induces a homomorphism $i_*: \pi_1(M) \to \pi_1(N)$ conjugated to $f_*$, but $f$ and $i$ are not homotopic relative to $\partial M$.

Nevertheless, since in [Wh] the relative $[p-1]$-homotopy type of a $W^{1,p}$ map is defined, as already observed by White one can minimize the $p$-energy among maps preserving such a relative $[p-1]$-homotopy type, and show that the regularity theory of [HL] applies also in this case. Thus, using this strategy we are able to prove the following theorem which represents the first main result of the paper.

**Theorem B.** Let $M$ be a compact manifold with boundary $\partial M \neq \emptyset$, and $N$ be a compact manifold whose universal covering supports a strictly convex exhaustion function. Let $f \in C^0(M, N) \cap \text{Lip}(\partial M, N)$. Then, for any $p \geq 2$, there exists a $p$-harmonic map $u \in C^{1,\alpha}(\text{int}(M), N) \cap C^0(M, N)$ which minimizes the $p$-energy among all the $W^{1,p}$ maps in the homotopy class of $f$ relative to $\partial M$. In particular $u$ is the unique solution of Problem A when $N$ Sect. $\leq 0$.

In the attempt to generalize Theorem B to non-compact manifolds $N$, some problems arise. First, the technique proposed in [Wh] to define the (relative) $[p-1]$-homotopy type of a map requires the target manifold $N$ to have an isometric embedding $i : N \to \mathbb{R}^d$ in some Euclidean space $\mathbb{R}^d$ with a uniform tubular neighborhood $\mathcal{T}$, $\overline{N} \subset \mathcal{T} \subset \mathbb{R}^d$, such that a nearest point retraction $\mathcal{T} \to N$ is well defined. In fact, given maps $u, v : M \to N$, White considers the $d$-homotopy type of the composed $W^{1,p}$ maps $i \circ u, i \circ v : M \to \mathcal{T} \subset \mathbb{R}^d$ and uses the uniform tubular neighborhood to retract the homotopy $H : [0, 1] \times M^d \to \mathcal{T}$ between $i \circ u$ and $i \circ v$ to a homotopy $H' : [0, 1] \times M^d \to N$ between $u$ and $v$, $M^d$ being a $d$-skeleton of $M$ for $d \leq [p-1]$. In general, for a negatively curved non-compact manifold $N$, we are not able to prove the existence of such a uniform tubular neighborhood. Second, the regularity theory developed by Schoen and Uhlenbeck and Hardt and Lin [SU1, SU2, HL] requires the map to have bounded image in $N$. As remarked below, satisfactory regularity results for non-compact
target have not been obtained yet. In a forthcoming work we will discuss how to face at least the first of these obstructions, in order to define the relative 1-homotopy type of a $W^{1,p}$ map when the target is non-compact.

In the present paper we propose a completely different, and somewhat more geometric, strategy which permits, in some pretty general situations, to overcome these obstructions. In this direction, our first remark is that the only interesting case involves target manifolds without compact quotients for, otherwise, the non-compact problem can be reduced to the compact one where the machinery alluded to above can be applied without changes.

**Proposition C.** Let $(M, g)$ be a compact, $m$-dimensional Riemannian manifold with smooth boundary $\partial M \neq \emptyset$ and let $(N, h)$ be a complete, Riemannian manifold of dimension $n$ such that its universal cover supports a strictly convex exhaustion function. Assume that there exists a subgroup $\Gamma$ of isometries of $N$ acting freely, properly and co-compactly on $N$. Then, for any $p \geq 2$ and for every $f \in C^0(M, N) \cap \text{Lip}(\partial M, N)$, the homotopy $p$-Dirichlet problem has a solution $u \in C^{1,\alpha}(\text{int}(M), N) \cap C^0(M, N)$. Moreover, the solution is unique provided $N$ has non-positive sectional curvature.

We aim at facing the general situation where either we have no information on the structure of the isometry group of $N$ or it is known that $N$ has no compact quotients. This latter case occurs, for instance, if its geometry is not bounded at some finite order. As a first step forward in this direction, we decide to focus our attention on Cartan-Hadamard targets, i.e., we add the request that the complete non-positively curved manifold $N$ is simply connected, hence it is diffeomorphic to the Euclidean space via the exponential map (from any reference point). Obviously, in this case, the homotopy condition is trivially satisfied by any continuous solution of the $p$-Dirichlet problem. Also, the presence of a global (geodesic) chart could suggest one to translate the problem into (almost) purely analytic terms as a problem concerning a system of coupled semi-linear elliptic equations. Yet, the original $p$-Dirichlet problem remains interesting and definitely non-trivial. Indeed, as said before, the non-compactness of the target, a-priori prevents the straightforward use of standard tools developed by Schoen-Uhlenbeck, Hardt-Lin and White, [SU1, SU2, HL, Wh]. On the other hand, due to the presence of a first order term like $|Du|^p$ in the local expression of the $p$-harmonicity condition, the regularity theories taken from the analysis of elliptic systems does not apply; see e.g. pp. 25–27 in Ladyzhenskaya-Ural’tseva [LU]. It’s worthwhile to note that, also in the $p = 2$ case, the effective approaches proposed to solve the Dirichlet problem for harmonic maps obtained via elliptic theory require precisely that the non-positively curved manifold $N$ is simply connected [HKW1, KS] [J]. In these works, the proofs depend deeply on some a priori estimates from the standard theory of harmonic maps into Cartan-Hadamard manifolds, linking the energy of the harmonic maps and the convex distance function in $N$. Once again, so
far a suitable counterpart for $p > 2$ has not been obtained yet. See also Section 2.4 below.

Actually, to the best of our knowledge, the $p$-Dirichlet problem has not been solved yet neither for Cartan-Hadamard targets whose metric tensor, in a global polar coordinates system, is rotational symmetric around the origin of the system. Formally, having fixed a smooth function $\sigma : [0, +\infty) \rightarrow [0, +\infty)$ satisfying

$$\sigma^{(2k)}(0) = 0, \forall k \in \mathbb{N}, \quad \sigma'(0) = 1, \quad \sigma(r) > 0, \forall r > 0,$$

we shall denote by $N^n_\sigma$ the smooth $n$-dimensional Riemannian manifold given by

$$(4) \quad ([0, +\infty) \times \mathbb{S}^{n-1}, dr^2 + \sigma^2(r) d\theta^2),$$

where $d\theta^2$ denotes the standard metric on $\mathbb{S}^{n-1}$. Clearly, $N^n_\sigma$ is diffeomorphic to $\mathbb{R}^n$ and geodesically complete for any choice of $\sigma$. Usually, $N^n_\sigma$ is called a model manifold with warping function $\sigma$ and pole 0. The $r$-coordinate in the expression (4) of the metric represents the distance from the pole. Thus, at a given point of $N^n_\sigma$ we distinguish the radial sectional curvatures and the tangential sectional curvatures of the model, according to whether the 2-plane at hand contains the radial vector field $\partial/\partial r$ or not. Standard formulas for warped product metrics reveal that

$$\text{Sect}_{\text{rad}} = -\frac{\sigma''}{\sigma}, \quad \text{Sect}_{\text{tg}} = \frac{1 - (\sigma')^2}{\sigma^2}$$

Thus, in particular, the model manifold $N^n_\sigma$ is Cartan-Hadamard if and only if

$$\sigma'' \geq 0.$$ 

Indeed, since $\sigma'(0) = 1$, the convexity of $\sigma$ always implies $\sigma' \geq 1$. It is worth to point out that there are Cartan-Hadamard model manifolds with bounded (pinched) negative curvature and without compact quotients. An example of special geometric interest was constructed by M. Anderson, [An], to settle in the negative a conjecture due to J. Dodziuk on the $L^2$-cohomology in pinched negative curvature. We also observe that formulas (5) hold in some sense for 2-manifolds even without rotational symmetry condition. Namely, given a 2-dimensional Cartan-Hadamard manifold $(N, h_N)$, in the global geodesic chart $(r, \theta)$ around some fixed pole $o \in N$ the metric $h_N$ can be expressed as

$$h_N|_{(r,\theta)} = dr^2 + \nu^2(r, \theta) d\theta^2.$$

Since the function $\nu > 0$ completely determines the Riemannian structure of $N$, in the following we will use the notation $N = N^2_\nu$. Direct computations show that the only (radial) sectional curvature of $N^2_\nu$ satisfies at any point $(r, \theta)$ the formula

$$\text{Sect}(r, \theta) = \text{Sect}_{\text{rad}}(r, \theta) = -\nu^{-1}(r, \theta) \frac{\partial^2 \nu(r, \theta)}{\partial r^2}.$$
The second main result of the paper is represented by the following

**Theorem D.** Let \((M, g)\) be a compact, \(m\)-dimensional Riemannian manifold with smooth boundary \(\partial M \neq \emptyset\) and let \(N\) be either an \(n\)-dimensional model manifold of non-positive curvature \(N^n_{\sigma}\), \(n > 3\), or a generic Cartan-Hadamard 2-dimensional manifold \(N^2_{\nu}\). Then, for any \(p \geq 2\) and any given \(f \in C^0(M, N) \cap \text{Lip}(\partial M, N)\), the \(p\)-Dirichlet problem

\[
\begin{align*}
\Delta_p u &= 0 &\text{on } M \\
u &= f &\text{on } \partial M,
\end{align*}
\]

has a unique solution \(u \in C^{1,\alpha}(\text{int}(M), N) \cap C^0(M)\).

The approach we propose inspires to the reduction procedure used to obtain Proposition C. This latter implies that Problem A is easily solved when \(N\) has a compact quotient, but, as observed above, this is not the case for a general Cartan-Hadamard model manifold \(N^n_{\sigma}\). The possible lack of discrete, co-compact isometry subgroups is overcome by using a combination of cut&paste and periodization arguments. Namely, we will show that it is possible to perturb the metric of \(N^n_{\sigma}\) in the exterior of a fixed geodesic ball in \(N^n_{\sigma}\) such that the complete manifold thus obtained is again Cartan-Hadamard and has compact quotients. A new maximum principle for the composition of the \(p\)-harmonic map and the convex distance function of \(N^n_{\sigma}\) then gives that this perturbation does not affects the solution to the original problem. The uniqueness part of the theorem can be clearly considered as a bypass product of the reduction to the compact case. However, at the end of the paper, we shall include a very general uniqueness result that works for homotopic (rel \(\partial M\)) \(p\)-harmonic maps into a complete manifold of non-positive curvature. This extends to non-compact targets previous results by W. Wei. [We2].

**Theorem E.** Let \((M, g)\) be a compact, \(m\)-dimensional Riemannian manifold with smooth boundary \(\partial M \neq \emptyset\) and let \(N\) be a complete manifold such that \(\text{Sect} \leq 0\). Let \(f \in C^0(M, N)\). Then, for any \(p \geq 2\), there is at most a \(C^1\) solution to the homotopy \(p\)-Dirichlet Problem A with datum \(f\).

Actually, in view of Theorem D and motivated by the known results in the case \(p = 2\), [HKW1, HKW2, HKW3], one may wonder whether the strategy outlined above can be adapted when the rotationally symmetric target is replaced by a general Cartan-Hadamard manifold or even by a convex ball. In this respect, it is worth to point out that the perturbation procedure outside a large ball can be carried out in such a way that the resulting manifold supports a strictly convex exhaustion function and possesses a compact quotient. However, we are not able to guarantee that the new manifold is still Cartan-Hadamard and this prevents us to complete the argument. See also Remark 2.12 below.
1. Compact targets

In this section we prove the existence and uniqueness of $C^{1,\alpha}$ solutions to Problem A when the target manifold $N$ is compact. The proof will require some preliminary results of very different nature that are collected in the next three subsections. The proof of Theorem B will be presented in Subsection 1.5.

1.1. Lipschitz approximation in relative homotopy class. According to Section 4 in [Wh] a boundary datum $f \in C^0(M,N) \cap \text{Lip}(\partial M,N)$ has a $C^0(M,N) \cap W^{1,p}(M,N)$ representative in its homotopy class relative to $\partial M$. This fact will be extensively used in the proof of Theorem B. Actually, the representative can be chosen to be $\text{Lip}(M,N)$; see Proposition 1.3 below. This follows by combining the standard Whitney approximation result with the next Lemma which is implicitly contained in [Wh], see especially the proof of Theorem 4.1 there.

Lemma 1.1. Let $(M,g)$ be a compact $m$-dimensional Riemannian manifold with boundary $\partial M \neq \emptyset$. Then, there exists a Lipschitz map $u : M \to M$ satisfying the following conditions:

(a) $u$ is homotopic to the identity map $\text{id}_M$ relative to $\partial M$.

(b) $u$ is a smooth retraction of a collar neighborhood $V$ of $\partial M$ onto $\partial M$.

Moreover, the neighborhood $V$ can be chosen as small as desired.

(c) $u$ is a diffeomorphism of $M \setminus V$ onto $M$.

Proof. Fix an open collar neighborhood $W$ of $\partial M$ and let $\alpha : \partial M \times [0, +\infty) \to W$ be a diffeomorphism satisfying $\alpha(x,0) = x$. Note that, if we set $\alpha^{-1}(x) = (\alpha_1(x), \alpha_2(x))$, the map

$$r(x) = \alpha(\alpha_1(x), 0) : W \to \partial M$$

is a natural smooth retraction of $W$ onto $\partial M$. Now, for any $0 \leq s < t$, let

$$\mathcal{M}_s^t = \alpha(\partial M \times [s,t])$$

$$\mathcal{M}_t = M \setminus \mathcal{M}_0^t$$

$$\mathcal{B}_t = \alpha(\partial M \times t)$$

so that

$$\mathcal{M}_t \cap \mathcal{M}_s^t = \mathcal{B}_t.$$ 

Observe that $\mathcal{M}_0 = M$ can be realized as the obvious gluing

$$M = \mathcal{M}_2 \cup_{\text{id}_2} \mathcal{M}_0^2$$

whereas

$$\mathcal{M}_1 = \mathcal{M}_2 \cup_{\text{id}_2} \mathcal{M}_1^2.$$ 

On the other hand, $\mathcal{M}_1^2$ is diffeomorphic to $\mathcal{M}_0^2$ via

$$\beta(x) = \alpha(\alpha_1(x), 2\alpha_2(x) - 2)$$
which keeps $B_2$ fixed, i.e., $\beta = \text{id}_{B_2}$ on $B_2$. It follows that the homeomorphism $\gamma : M_1 \to M$ such that

\begin{equation}
\gamma (x) = \begin{cases} 
\text{id}_{M_2} (x), & x \in M_2 \\
\beta (x), & x \in M_1^2
\end{cases}
\end{equation}

smooths out along the closed submanifold $B_2 = M_2 \cap M_1^2$ and gives rise to a global diffeomorphism $\Gamma : M_1 \to M$ satisfying $\Gamma = \gamma$ outside a small neighborhood of $B_2$; see e.g. Theorem 1.9 of [Hi]. In particular,

$$\Gamma (x) = r (x), \text{ on } B_1.$$

To conclude, we put

$$V = \alpha (\partial M \times [0,1)) \subset W$$

and define $u : M \to M$ by setting

$$u (x) = \begin{cases} 
\Gamma (x), & x \in M_1^1 = \bar{V} \\
r (x), & x \in M_1.
\end{cases}$$

\[\Box\]

**Remark 1.2.** The same proof works if $M$ is a non-compact manifold with compact boundary $\partial M$. Clearly, in this case, $u$ is only $\text{Lip}_\text{loc} (M, N)$. In fact, note that the assumption that $\partial M$ is compact is just used to smoothing out the homeomorphism $\gamma$ along the submanifold $B_2$ and this is needed to obtain $u$ satisfying the further condition $(c)$ in the statement of Lemma. If we are not interested in smooth regularity, we can use directly $\gamma$, whose construction does not require any compactness assumption on $\partial M$. In this case, condition $(c)$ has to be replaced by

$(c)' \; u$ is a $\text{BiLip}_\text{loc}$-homeomorphism of $M \setminus V$ onto $M$.

**Proposition 1.3.** Keeping the notation and assumptions of the previous Lemma, suppose we are given a map $f \in C^0 (M, N) \cap \text{Lip} (\partial M, N)$, where $(N, h)$ is an $n$-dimensional Riemannian manifold without boundary. Then, there exists $F \in \text{Lip} (M, N)$ which is homotopic to $f$ relative to $\partial M$.

**Proof.** Let $u : M \to M$ be the map defined in Lemma 1.1 and consider $\overline{f} = f \circ u : M \to M$. Then, $\overline{f} \in C^0 (M) \cap \text{Lip}_\text{loc} (V)$ where $V$ is a collar neighborhood of $\partial M$ which retracts to $\partial M$ via $u$. In particular $f = \overline{f}$ on $\partial M$. Let $W$ be a smaller collar neighborhood of $\partial M$ such that $\overline{W} \subset V$. Then, we can apply the standard approximation procedure by H. Whitney keeping $\overline{W}$ fixed (see e.g. [Le]) and obtain a Lipschitz map $F : M \to N$ with the desired properties. More precisely: (i) $F$ is smooth on $M \setminus V$; (ii) $F = \overline{f}$ on $\overline{W}$ and (iii) $F$ is homotopic to $\overline{f}$, relative to $\partial M$. \[\Box\]

**Remark 1.4.** Using the version of Lemma 1.1 observed in Remark 1.2, we can skip the assumption that $M$ is compact and obtain that any $f \in C^0 (M, N) \cap \text{Lip}_\text{loc} (\partial M, N)$ has a representative $F \in \text{Lip}_\text{loc} (M, N)$ in its homotopy class relative to $\partial M$. 

1.2. $p$-Minimizing tangent maps. Another key ingredient in the proof of Theorem 1.4 is the fact that a manifold does not contain any $p$-minimizing tangent sphere provided its universal covering supports a smooth strictly convex function. This is the content of Proposition 1.6 that will be vital to apply the full regularity theory by Hardt-Lin. The same conclusion for $p = 2$ was first obtained in [SU1] while the general case $p > 2$ was observed in [WY]. Due to its importance, we shall provide a detailed and complete proof.

First, we introduce the Sobolev spaces of maps that will be used throughout the paper. Since $N$ is compact, according to the Nash embedding theorem, we can assume that there is an isometric embedding $i : N \rightarrow \mathbb{R}^q$ of $N$ into some Euclidean space. For all maps $u : M \rightarrow N$ we define $\bar{u} := i \circ u : M \rightarrow \mathbb{R}^q$. For $p > 1$, we denote by $W^{1,p}_0(M,\mathbb{R}^q)$ (resp. $W^{1,p}(M,\mathbb{R}^q)$) the Sobolev space of maps $v : M \rightarrow \mathbb{R}^q$ whose component functions and their first weak derivatives are in $L^p(M)$ (resp. in $L^p(M)$). Moreover we define

$$W^{1,p}_{loc}(M,N) := \{ v \in W^{1,p}_{loc}(M,\mathbb{R}^q) : v(x) \in N \text{ for a.e. } x \in M \},$$

$$W^{1,p}(M,N) := \{ v \in W^{1,p}(M,\mathbb{R}^q) : v(x) \in N \text{ for a.e. } x \in M \}.$$  

Finally we will say that $v \in W^{1,p}(M,N)$ has boundary trace $f$ if $\bar{v} - \bar{f} \in W^{1,p}_0(M,\mathbb{R}^q)$, where $W^{1,p}_0(M,\mathbb{R}^q)$ denotes the closure of $C^\infty_c(M,\mathbb{R}^q)$ in the $W^{1,p}(M,\mathbb{R}^q)$ norm.

Following [HL, p. 572] we say that a map $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ is a $p$-minimizing tangent map ($p$-MTM) from $\mathbb{R}^{l+1}$ to $N$ if $\psi$ minimizes the $p$-energy on compact sets and $\bar{\psi}$ is homogeneous of degree 0, that is, $\partial \bar{\psi}/\partial r = 0$ a.e., $r$ being the radial coordinate. Clearly, here we are thinking of $\bar{\psi}$ as an $\mathbb{R}^q$-valued map once $N$ is isometrically embedded in the Euclidean space $\mathbb{R}^q$. Note that $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ is homogeneous of degree 0 if and only if there exists $\psi \in W^{1,p}(\mathbb{S}^l,N)$ such that

$$\bar{\psi}(x) := \psi \left( \frac{x}{|x|} \right), \quad \forall x \neq 0.$$  

Indeed, condition (5) clearly implies that $\partial \bar{\psi}/\partial r = 0$ a.e. On the other hand, if $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ is homogeneous of degree 0, since by Fubini’s theorem $\bar{\psi}(\cdot,\theta) \in W^{1,p}_{loc}(\mathbb{R}_{>0},N)$ for a.e. $\theta \in \mathbb{S}^l$, we deduce that $\bar{\psi}(\cdot,\theta)$ is constant a.e. Therefore, $\psi(\theta) = \bar{\psi}(\cdot,\theta)$ satisfies (5) and, again by Fubini, it is $W^{1,p}(\mathbb{S}^l,N)$.

**Lemma 1.5.** Assume that $\bar{\psi} \in W^{1,p}_{loc}(\mathbb{R}^{l+1},N)$ satisfies (5) for some $\psi \in W^{1,p}(\mathbb{S}^l,N)$. Then $\psi : \mathbb{R}^{l+1} \rightarrow N$ is weakly $p$-harmonic (in the sense of (2)) if and only if $\psi : \mathbb{S}^l \rightarrow N$ is weakly $p$-harmonic.
Proof. Let \((r, \theta) \in \mathbb{R}_{>0} \times S^l\) be local polar coordinates on \(\mathbb{R}^{l+1}\). Namely, we suppose to have chosen local angular coordinates \(\{\theta^1, \ldots, \theta^l\}\) on \(S^l\) so that \(\{r, \theta^1, \ldots, \theta^l\}\) is a local coordinates system for \(\mathbb{R}^{l+1} \setminus \{0\}\).

Having fixed an isometric embedding \(i : N \to \mathbb{R}^q\) with second fundamental form \(\mathcal{A}\), we let \(\check{\psi} = iv \check{\varphi} : \mathbb{R}^{l+1} \to \mathbb{R}^q\) and \(\check{\psi} = i\psi : S^l \to \mathbb{R}^q\). By assumption \(\check{\psi}(r, \theta) = \psi(\theta)\), where \(\psi \in W^{1,p}(S^l, N)\).

Let \(\check{\varphi} \in C_c^\infty(\mathbb{R}^{l+1}, \mathbb{R}^q)\). Then, we have

\[
(D\check{\psi} \cdot D\check{\varphi})(r, \theta) = r^{-2}(D\check{\psi} \cdot D(\check{\varphi}(r, \cdot)))(\theta),
\]

and

\[
|D\check{\psi}|^2 (r, \theta) = r^{-2}|D\check{\psi}|^2(\theta).
\]

Whence, it follows that, for every \(R > 0\),

\[
(9) \quad E_p(\bar{\psi} |_{B_R(0)}) = E_p(\psi) \int_0^R r^{l-p} dr
\]

and

\[
(10) \quad \int_{\mathbb{R}^{l+1}} |D\check{\psi}|^{p-2} \left\{ D\check{\psi} \cdot D\check{\varphi} + \mathcal{A}(D\check{\psi}, D\check{\psi}) \cdot \check{\varphi} \right\} dx
\]

\[
= \int_0^\infty r^{l-p} \int_{S^l} |D\check{\psi}|^{p-2}(\theta) \left\{ D\check{\psi} \cdot D(\check{\varphi}(r, \cdot))(\theta) + \mathcal{A}(D\check{\psi}, D\check{\psi}) \cdot \check{\varphi}(r, \theta) \right\} d\sigma(\theta)dr.
\]

If \(p \geq l + 1\), from (9) we must conclude that \(E_p(\psi) = 0\) and, therefore, that \(\psi\) and \(\check{\psi}\) are constant. In particular, they are both trivially \(p\)-harmonic.

Suppose that \(p < l + 1\). Since, for each \(r > 0\), \(\check{\varphi}(r, \cdot) \in C^\infty(S^l, \mathbb{R}^q)\), recalling the extrinsic definition of (weak) \(p\)-harmonicity given in (2), from (10) we deduce that if \(\psi : S^l \to N\) is weakly \(p\)-harmonic, then \(\check{\psi} : \mathbb{R}^{l+1} \to N\) is weakly \(p\)-harmonic. On the other hand, assume that \(\check{\varphi}\) has the form \(\varphi(r, \cdot) = \varphi(\theta)\nu(r)\), where \(\varphi \in C^\infty(S^l, \mathbb{R}^q)\) and \(\nu \in C_c^\infty([0, \infty))\) is such that \(\nu^{2k+1}(0) = 0\) for all \(k \geq 0\). Then \(\check{\varphi} \in C_c^\infty(\mathbb{R}^{l+1}, \mathbb{R}^q)\) and (10) becomes

\[
\int_{\mathbb{R}^{l+1}} |D\check{\psi}|^{p-2} \left\{ D\check{\psi} \cdot D\check{\varphi} + \mathcal{A}(D\check{\psi}, D\check{\psi}) \cdot \check{\varphi} \right\} dx
\]

\[
= \left\{ \int_0^\infty r^{-p} \nu(r) dr \right\} \left\{ \int_{S^l} |D\check{\psi}|^{p-2} |D\check{\psi} \cdot D\varphi + \mathcal{A}(D\check{\psi}, D\check{\psi}) \cdot \varphi| d\sigma \right\},
\]

proving that \(\psi : S^l \to N\) is weakly \(p\)-harmonic whenever \(\check{\psi} : \mathbb{R}^{l+1} \to N\) is weakly \(p\)-harmonic.

\[\square\]

Proposition 1.6. Suppose that \(N\) is compact and that \(\xi \in W^{1,p}_{loc}(\mathbb{R}^l, N)\) is a \(p\)-MTM. If \(l \leq [p]\) then \(\xi\) is constant and, if \(l > [p]\) and \(N\) does not support any non-constant \(p\)-MTM from \(\mathbb{R}^j\) into \(N\), \(j = 1, \ldots, l-1\), then \(\xi\) has at most an isolated singularity at the origin. In particular, \(\xi_{l-1} \in C^{1,\alpha}(S^{l-1}, N)\).

Moreover, if the universal cover \(\tilde{N}\) of \(N\) supports a strictly convex function, then every \(p\)-MTM from \(\mathbb{R}^l\) to \(N\) is constant, for every \(l \geq 1\).
Proof. Indeed, the case $l \leq \lceil p \rceil$ follows directly from Theorem 4.5 of [HL]. For the general case we proceed by induction. Let $l > \lceil p \rceil$ and suppose that every $p$-MTM from $\mathbb{R}^d$ to $N$ is trivial for every $j = 1, \ldots, l - 1$. Let $\xi : \mathbb{R}^l \to N$ be a $p$-MTM. Then, by Theorem 4.5 in [HL] the set of singular points of $\xi$ is discrete (possibly empty) and by homogeneity it reduces to the sole origin. In particular, by Corollary 2.6 and Theorem 3.1 in [HL], we deduce that $\xi|_{\mathbb{R}^{l-1}} : \mathbb{R}^{l-1} \to N$ is $C^{1,\alpha}$ and it is $p$-harmonic thanks to Lemma 1.5. To conclude, in case $\tilde{N}$ supports a convex function, we can apply Theorem 1.4 in [WY] and obtain that $\xi$ is constant. \hfill $\Box$

1.3. Extending relative $d$-homotopies. Fix a triangulation of $\partial M$ and extend it to a triangulation of $M$. Thus $M$ is a CW-complex and $\partial M$ is a subcomplex of $M$, see [Whd, Mu]. Let $M^d$ denote that $d$-skeleton of $M$.

Two continuous maps $v, f : M \to N$ are said to be $d$-homotopic relative to $M^d \cap \partial M$ (or, equivalently, they have the same $d$-homotopy type) if there exists a continuous map $H^d : [0, 1] \times M^d \to N$ such that $H^d(0, x) = v(x)$, $H^d(1, x) = f(x)$ for all $x \in M^d$ and $H^d(\cdot, x) = f(x) = v(x)$ for all $x \in M^d \cap \partial M$. Clearly, when $d \geq \dim M$ the relative $d$-homotopy type of maps is nothing but the usual homotopy type relative to $\partial M$.

By the homotopy extension property of the couple $(M, M^d)$ we already know that if $v$ and $f$ have the same $d$-homotopy type, then $H^d$ extends to a full homotopy $H : M \to N$ such that $H(0, x) = v(x)$. In this subsection, under the assumption that the target manifold $N$ is aspherical, we construct a special extension $H$ of $H^d$ satisfying the further requirements $H(1, x) = f(x)$ for every $x \in M$ and $H(\cdot, x) = f(x) = v(x)$ for every $x \in \partial M$.

Recall that $N$ is said to be aspherical if each homotopy group $\pi_k(N)$ of $N$ is trivial for $k \geq 2$.

Proposition 1.7. Let $v, f \in C^d(M, N)$ and assume that $N$ is aspherical. If $v$ and $f$ have the same relative $d$-homotopy type, $d \geq 1$, then they have the same relative homotopy type.

Proof. By assumption, we know that there exists a continuous map $H^d : [0, 1] \times M^d \to N$ such that $H^d(0, x) = v(x)$, $H^d(1, x) = f(x)$ for all $x \in M^d$ and $H^d(\cdot, x) = f(x) = v(x)$ for all $x \in M^d \cap \partial M$.

Using the aspherical structure of $N$ in a classical manner (see e.g [Hat]), we are going to show that $H^d$ extends to a homotopy $H^{d+1}$ between $v$ and $f$ on $M^{d+1}$ relative to $M^{d+1} \cap \partial M$, i.e., $H^{d+1} : [0, 1] \times M^{d+1} \to N$ is a continuous function such that $H^{d+1}(0, \cdot) = v(\cdot)$, $H^{d+1}(1, \cdot) = f(\cdot)$ and $H^{d+1}(\cdot, x) = f(x) = v(x)$ for all $x \in M^{d+1} \cap \partial M$. Clearly we can assume $1 \leq d < m$ for otherwise there is nothing to prove.

The $(d+1)$-skeleton $M^{d+1}$ is obtained as $M^{d+1} = M^d \cup (\cup_{\alpha} e^{d+1}_\alpha)$, where $e^{d+1}_\alpha$ are open $(d+1)$-cells with attaching maps $\psi_\alpha : S^d \to M^d$ and corresponding characteristic maps $\tilde{\psi}_\alpha : \mathbb{D}^{d+1} \to M^{d+1}$. Note that $[0, 1] \times M^d \subseteq ([0, 1] \times M)^{d+1}$ and $H^d$ extends to a continuous function $H^{d+1} :
ON THE HOMOTOPY DIRICHLET PROBLEM FOR \( p \)-HARMONIC MAPS

1.4. The \( d \)-homotopy type of \( W^{1,p} \) maps. Let \( d \) be the greatest integer less than or equal to \( p - 1 \) and let \( M^d \) be a \( d \)-dimensional skeleton of \( M \). Clearly, here we mean \( M^d \equiv M \) for \( p - 1 > m \). Recall from the previous subsection that the \( d \) homotopy type of a continuous map from \( M \) to \( N \) is the homotopy type of its restriction to \( M^d \). According to the work of White [Wh], each \( u \in W^{1,p}(M, N) \) with boundary trace \( f \) has a \( d \)-homotopy type \( u_2[M^d(\text{rel } \partial M)] \). This \( d \)-homotopy type is a homotopy class (relative to \( \partial M \)) of continuous mappings from \( M^d \) into \( N \) such that:

1. If \( \{u_i\} \subset W^{1,p}(M, N) \) have boundary trace \( h \), \( ||\bar{u}_i - \bar{u}||_p \to 0 \), and \( ||du_i||_p \) is uniformly bounded, then
\[
(u_i)_2[M^d(\text{rel } \partial M)] = u_2[M^d(\text{rel } \partial M)]
\]
for sufficiently large \( i \).
2. If \( u \in W^{1,p}(M, N) \) has boundary trace \( f \) and is continuous at each \( x \in M^d \), then
\[
u_2[M^d(\text{rel } \partial M)] = [(u|_{M^d})(\text{rel } \partial M)].
\]
3. The set
\[
\{u_2[M^d(\text{rel } \partial M)] : u \in W^{1,p}(M, N) \text{ has boundary trace } f \}
\]
is equal to
\[
\{[(\varphi|_{M^d})(\text{rel } \partial M)] : \varphi \in C^0(M^{d+1}, N), \varphi(x) = f(x) \text{ for } x \in M^d \cap \partial M \}.
\]
The purpose of this subsection is to point out the following property of the $d$-homotopy type whose application will be basic in the proof of Theorem 1.8 to apply the regularity theory of [HL].

**Proposition 1.8.** Let $M$ be a compact $m$-dimensional manifold with (possibly empty) boundary $\partial M$ and let $N$ be a compact manifold. Let $f \in \text{Lip}(\partial M, N)$ and let $v \in W^{1,p}(M, N)$ be a map with boundary trace $f$. For every $x \in M$ there exists an open set $x \in \Omega_x \subset M$ (independent of $v$) with smooth boundary $\partial \Omega_x$, which satisfies the following property: for any other map $w \in W^{1,p}(M, N)$ such that $w|_{\partial M} = v|_{\partial M}$ in the trace sense and $w \equiv v$ on $M \setminus \Omega_x$ it holds

\begin{equation}
(11) \quad w_{\sharp}[M^d(\text{rel } \partial M)] = v_{\sharp}[M^d(\text{rel } \partial M)].
\end{equation}

**Proof.** We consider three cases.

First suppose that $x \in \text{int}(M)$ and that $x \notin M^d$. Since $M^d$ is closed in $M$ we can choose the open set $\Omega_x$ such that $x \in \Omega_x \subset \subset \text{int}(M) \setminus M^d$ and $\partial \Omega_x$ is smooth. By the construction of the $d$-homotopy type given in Section 3 of [Wh], and up to choosing $\delta > 0$ small enough in (the version for manifolds with boundary of) Proposition 3.2 therein, it is clear that in this case perturbing a $W^{1,p}$ map in $\Omega_x$ does not affect the $d$-homotopy type of the map.

Now, let $x \in M^d \cap \text{int}(M)$. Consider, for $\epsilon > 0$ small enough, a normal closed geodesic ball $\bar{B}_{\epsilon}(x)$ centered at $x$. Choose a triangulation $T_x$ of $\bar{B}_{\epsilon}(x)$ such that $x$ is not contained in the $d$-skeleton $T_x^d$ of $T_x$ (to this purpose, one can for instance take such a construction on the Euclidean unit closed ball, and make use of the diffeomorphism with $\bar{B}_{\epsilon}(x)$ given by the normal coordinates).

Note that $T_x$ induces a triangulation of $\partial \bar{B}_{\epsilon}(x)$. Choosing a triangulation of $\partial M$ gives, together with $T_x|_{\partial \bar{B}_{\epsilon}(x)}$, a triangulation of $\partial M \cup \partial \bar{B}_{\epsilon}(x) = \partial (M \setminus B_{\epsilon}(x))$. A classical result ensures us that this triangulation of the boundary can be extended to a triangulation of all of $M \setminus B_{\epsilon}(x)$ [Wh] [Mu].

This latter, together with $T_x$, forms a new triangulation of $M$ whose $d$-skeleton $(M^d)'$ does not contain $x$. According to the previous case there exists an open set with smooth boundary $\Omega_x$ such that, given maps $v$ and $w$ as in the statement, we have

$$w_{\sharp}[(M^d)'(\text{rel } \partial M)] = v_{\sharp}[(M^d)'(\text{rel } \partial M)].$$

To conclude this case, we recall that, thanks to Proposition 3.5 of [Wh], the $d$-homotopy type of a $W^{1,p}$ map does not depend on the choice of the triangulation.

Finally, suppose that $x \in \partial M$. Using the notation of Lemma 1.1 let $y \in \partial V$ be a point satisfying $u(y) = x$. Since $y \in \text{int}(M)$, according to previous paragraphs there exists an open set $\Omega_y'$, $y \in \Omega_y' \subset \subset \text{int}(M)$, such that perturbing a $W^{1,p}$ map inside $\Omega_y'$ does not change the $d$-homotopy type of the map. Let $\Omega_x = u(\Omega_y' \cap (M \setminus V))$. Since $u|_{M \setminus V}$ is a diffeomorphism onto $M$, then $\Omega_x$ is an open set of $M$ containing $x$. Suppose that $v, w$ are two
Let \( \{ I \} \) be a set of indexes. If we keep the same set of indexes each time we will extract a subsequence from a given sequence.

This latter, in turn, implies (11) as aimed. To conclude, observe that since \( \partial M \) is smooth, up to possibly restrict the set \( \Omega_\varepsilon \), we can require that \( \Omega_\varepsilon \) has smooth boundary.

1.5. Proof of Theorem B

For the sake of clarity, we will divide the proof in four steps.

Step 1. Existence of a minimizer in the \( d \)-homotopy class of \( f \).

Define \( \mathcal{H}^d_f \) as the space of maps \( u \in W^{1,p}(M, N) \) such that \( u|_{\partial M} = f|_{\partial M} \) in the trace sense and \( f \) and \( u \) have the same relative \( d \)-homotopy type, i.e.

\[
\mathcal{H}^d_f := \{ u \in W^{1,p}(M, N) : \tilde{u} - \tilde{f} \in W^{1,p}_0(M, \mathbb{R}^q) \text{ and } u_{\sharp} [M^d(\mathrm{rel} \partial M)] = f_{\sharp} [M^d(\mathrm{rel} \partial M)] \}.
\]

According to Proposition 1.3, there is no loss of generality if we assume that \( f \in \text{Lip}(M, N) \). In particular \( f \in \mathcal{H}^d_f \) and, therefore,

\[
\mathcal{I}^d_f := \inf_{u \in \mathcal{H}^d_f} E_p(u) < +\infty.
\]

Let \( \{ v_j \}_{j=1}^\infty \subset \mathcal{H}^d_f \) be a sequence minimizing the \( p \)-energy in \( \mathcal{H}^d_f \), i.e. \( E_p(v_j) \to \mathcal{I}^d_f \) as \( j \to \infty \). For the ease of notation, throughout all the proof we will keep the same set of indexes each time we will extract a subsequence from a given sequence.

Since \( N \) is compact, then \( \{ \tilde{v}_j \}_{j=1}^\infty \) is bounded in \( W^{1,p}(M, \mathbb{R}^q) \) and, up to choosing a subsequence, \( \tilde{v}_j \) converges to some \( \tilde{v} \in W^{1,p}(M, \mathbb{R}^q) \) weakly in \( W^{1,p} \). Since \( M \) is compact, \( \{ \tilde{v}_j \}_{j=1}^\infty \) is bounded in \( W^{1,p'}(M, \mathbb{R}^q) \) for every \( p' \leq p \) which satisfies also \( p' < m \). By the Kondrachov theorem, \( \text{[A]} \) p.55, \( \tilde{v}_j \) converges strongly in \( L^s(M, \mathbb{R}^q) \) for any \( 1 < s < (mp')/(m-p') \), notably for \( s = p \), and hence pointwise almost everywhere. Since \( N \) is properly embedded, this implies \( \tilde{v}(x) \in N \) for a.e. \( x \in M \), so that we can define \( v \in W^{1,p}(M, N) \) by \( v = \tilde{v} \). Since \( \tilde{v}_j - \hat{f} \in W^{1,p}_0(M, \mathbb{R}^q) \) for all \( j \), the weak
limit \( \tilde{v} - \tilde{f} \in W^{1,j}_0(M, \mathbb{R}^q) \).

By the lower semicontinuity of \( E_p \) we have

\[
E_p(v) \leq \liminf_{j \to \infty} E_p(v_j) = T_f^d.
\]

Since \( \{\tilde{v}_j\}_{j=1}^\infty \) is bounded in \( W^{1,j}(M, \mathbb{R}^q) \) and \( \|\tilde{v}_j - \tilde{v}\|_p \to 0 \) as \( j \to \infty \), by the property (1) of the \( d \)-homotopy type of maps we deduce that

\[
v_\sharp[M^d(\text{rel } \partial M)] = (v_j)_\sharp[M^d(\text{rel } \partial M)]=f_\sharp[M^d(\text{rel } \partial M)]
\]

for \( j \) large enough, which implies that \( v \in \mathcal{H}^d_f \). It then follows from (12) that

\[
T_f^d \leq E_p(v) \leq T_f^d,
\]

so that \( E_p(v) = T_f^d \), i.e. \( v \) minimizes the energy in \( \mathcal{H}^d_f \).

**Step 2. Regularity of the minimizer.** We show that the regularity theory of [HL] applies to \( v \), as already remarked on page 3 of [Wh]. Clearly, the only interesting case is \( d := |p| - 1 \leq m - 1 \).

First of all, we note that, for every \( x \in M \), there exists an open set with smooth boundary \( \Omega_x \ni x \) such that \( v|_{\Omega_x} \) is a minimizer for the \( p \)-energy among all the maps \( w \in W^{1,p}(\Omega_x, N) \) which have the same trace boundary of \( v \) on \( \partial \Omega_x \), that is \( v|_{\partial \Omega_x} = w|_{\partial \Omega_x} \) in the trace sense. To see this, let \( \Omega_x \) be the open set given by Proposition L8. We can extend \( w \) to \( \tilde{w} \in W^{1,p}(M, N) \) by setting \( \tilde{w} = v \) on \( M \setminus \Omega_x \). An application of Proposition L8 gives that \( \tilde{w} \in \mathcal{H}^d_f \). Then, \( E_p(v) \leq E_p(\tilde{w}) \). To conclude, we note that

\[
E_p(v|_{B_x}) + E_p(v|_{M \setminus B_x}) = E_p(v) \leq E_p(\tilde{w}) = E_p(w) + E_p(v|_{M \setminus B_x}).
\]

This minimizing property enables us to apply the partial interior regularity and deduce that the singular set \( S(v) \) of \( v \) is empty if \( p > m \) and it is a relatively closed subset of zero \((m - p)\)-Hausdorff dimension if \( p \leq m \). Moreover, \( v \) is \( C^{1,\alpha} \) on \( \text{int}(M) \setminus S(v) \); see Corollary 2.6 and Theorem 3.1 in [HL].

The full interior regularity is now obtained from Theorem 4.5 of [HL] because, according to Proposition L6 above, every \( p \)-minimizing tangent map \( \xi : \mathbb{R}^{l+1} \to N \) is constant, for every \( l \geq 1 \).

Finally, we observe that the boundary regularity theory developed in Section 5 of [HL] works for a Lipschitz boundary datum \( f \). Therefore we can conclude that the minimizer \( v \) is \( C^{0,\alpha} \) on \( M \).

**Step 3. On the relative homotopy class of the minimizer.** It remains to prove that the minimizer \( v \) is homotopic to the datum \( f \) relative to \( \partial M \). To this end, recall that \( M \) is realized as a polyhedral complex, hence a CW complex, in such a way that \( \partial M \) is a subcomplex. By construction, we know that \( v \) has the same \( d(\geq 1) \)-homotopy type of \( f \) relative to \( M^d \cap \partial M \). Note also that \( N \) is aspherical. Indeed, since its universal covering \( \tilde{N} \) supports a strictly convex exhaustion function, by standard Morse theory \( \tilde{N} \)
is diffeomorphic to $\mathbb{R}^n$. The desired conclusion now follows from a direct application of Proposition 1.7.

**Step 4. Non-positively curved targets.** Suppose now that the compact manifold $N$ has non-positive sectional curvature so that, in particular, its universal covering $\tilde{N}$ is a Cartan-Hadamard manifold. By the Hessian comparison theorem, the square of the distance function on $\tilde{N}$ is a strictly convex exhaustion function. Therefore, by the preceding steps, the homotopy $p$-Dirichlet problem has a solution $v \in C^{1,\alpha}(\text{int}(M)) \cap C^0(M)$. Applying Theorem 8.5 (1) of [We2] we conclude that such a solution is unique.

This completes the proof of the Theorem.

2. Complete targets

We begin this section by proving Proposition C, which permits to solve directly Problem A for targets admitting a compact quotient. Even if the proof is pretty elementary, the ideas contained there will be the basis for the periodization procedure developed in the following subsections to prove Theorem D.

**Proof (of Proposition C).** By assumption, $N' = N/\Gamma$ is a compact, aspherical Riemannian manifold covered by $N$ via the quotient projection $P : N \to N'$. The original datum $f$ projects to a new function $P(f) : M \to N'$ which, in turn, can be used to state the corresponding $p$-Dirichlet problem

$$\begin{cases}
\Delta_p u' = 0 & \text{on } M \\
u' = P (f) & \text{on } \partial M.
\end{cases}$$

By Theorem B this problem admits a solution $u' \in C^{1,\alpha}(\text{int}(M),N') \cap C^0(M,N')$ in the homotopy class of $P(f)$ relative to $\partial M$. Let $H' : [0,1] \times M \to N'$ be such a homotopy. The classical theory of fibrations (see e.g. [Hat]) then tells us that $H'$ lifts to a homotopy $H : [0,1] \times M \to N$ satisfying $H(1,x) = f(x)$. The homotopy $H$ is relative to $\partial M$ because, for every $y \in \partial M$, $H([0,1] \times \{y\})$ is contained in the (discrete) fibre over $P(f)(y)$. Let $u(x) = H(0,x)$. Since $P$ is a local isometry and $P(u) = u'$, then $u$ is $p$-harmonic in $M$ of class $C^{1,\alpha}(\text{int}(M),N) \cap C^0(M,N)$. On the other hand, using the fact that $H$ is relative to $\partial M$ we deduce that $u = f$ on $\partial M$. This proves that the original homotopy $p$-Dirichlet problem has a solution. In case $N$ Sect $\leq 0$, uniqueness follows easily from the following few facts: (a) solutions of the homotopy $p$-Dirichlet problem with target $N$ projects to solutions of the corresponding problem with target $N'$; (b) in case of compact targets, the solution is unique; (c) liftings are uniquely determined by their values at a single point.

The remaining part of this section aims to prove Theorem D by reproducing a similar argument even in case the target Cartan-Hadamard space $N$ does not possess compact quotients. This will be done in Subsection 2.5. The proof relies on the preliminary results collected in Subsections 2.2–2.4.
In particular, in 2.1 and 2.2 we discuss a procedure to glue a large ball of \( N \) with a hyperbolic space of sufficiently small curvature \(-k \ll -1\). In 2.3 we record that hyperbolic spaces have discrete, co-compact groups of isometries with arbitrarily large fundamental domains. Finally, in 2.4 we introduce a new maximum principle for the composition of a \( p \)-harmonic map and a convex function.

2.1. **Gluing model manifolds keeping** Sect \( \leq 0 \). In this subsection we show that, in some sense, it is possible to prescribe a hyperbolic infinity to a Cartan-Hadamard model, as well as to a generic Cartan-Hadamard 2-manifold, without violating the non-positive curvature condition. It is convenient to put the following

**Definition 2.1.** By a model of hyperbolic type we mean a model manifold \( H^m_\sigma \) whose warping function satisfies the following further requirements:

(i) \( \sigma'' \geq 0 \).

(ii) Let \( \sigma_k (r) = k^{-1/2} \sigma \left( k^{1/2} r \right) \). Then, for every \( r > 0 \),

\[
\sigma'_k (r) \geq \sigma_k (r) \to +\infty, \text{ as } k \to +\infty.
\]

Note that a hyperbolic type model is Cartan-Hadamard and has, at least, an exponential volume growth. Clearly, the choice \( \sigma (r) = \sinh (r) \) is admissible and the corresponding model is the standard hyperbolic spaceform. Whence, the choice of the name. Note also that \( H^m_\sigma = k^{-1} H^m_\sigma \) in the Riemannian sense.

We are going to show that every compact ball centered at the pole of a model of non-positive curvature can be glued to a hyperbolic type model thus giving a new model manifold with non-positive curvature.

**Theorem 2.2.** Let \( N^n_\rho \) be a Cartan-Hadamard model and let \( H^m_\rho \) be of hyperbolic type. Fix \( R > 0 \). Then, for every \( R > R \) there exist a \( k = k(R) \gg 1 \) and a Cartan-Hadamard model \( M^n_\tau \) such that:

(i) \( B^N_R (0) \subset M^n_\tau \).

(ii) \( M^n_\tau \setminus B^M_R (0) = H^m_{\sigma_k} \setminus B^H_R (0) \).

**Proof.** Thanks to (5), it is enough to produce a warping function \( \tau : [0, +\infty) \to [0, +\infty) \) satisfying the following requirements:

(a) \( \tau = \rho \) on \([0, R] \).

(b) \( \tau = \sigma_k \) on \((R, +\infty) \), \( R \gg 1 \).

(c) \( \tau' \geq 1 \) and \( \tau'' \geq 0 \) on \([0, +\infty) \).

To this end, let \( \bar{R} < R_1 < R_2 \). By the assumptions on \( \sigma \), we can choose \( k = k (R_1, R_2) > 0 \) large enough so that

\[
\rho' (R_1) \leq \frac{\sigma_k (R_2) - \rho (R_1)}{R_2 - R_1} \leq \sigma'_k (R_2).
\]
Define
\[
\tau_1(r) = \begin{cases} 
\rho(r) & \text{on } [0, R_1) \\
\rho(R_1) + \frac{\sigma_k(R_2) - \rho(R_1)}{R_2 - R_1} r & \text{on } [R_1, R_2] \\
\sigma_k(r) & \text{on } (R_2, +\infty)
\end{cases}
\]
Then, \( \tau_1 \) is a piecewise smooth, convex function with \( \tau_1' \geq 1 \). To complete the construction of \( \tau \), it remains to smoothing out the angles with a convex function. This can be done using the approximation procedure described by M. Ghomi in [Gh].

Thanks to the explicit formula (6), in a completely analogous way we can obtain also the following

**Theorem 2.3.** Let \( N^2_\nu \) be a 2-dimensional Cartan-Hadamard manifold and let \( H^2_\sigma \) be a 2-dimensional manifold of hyperbolic type. Fix \( \bar{R} > 0 \). Then, for every \( R > \bar{R} \) there exist a \( k = k(R) >> 1 \) and a Cartan-Hadamard manifold \( M^2_\tau \) such that:

(i) \( B^N_R(0) \subset M^n_\tau \).

(ii) \( M^2_{\tau_k} \setminus B^M_R(0) = H^2_{\sigma_k} \setminus B^H_R(0) \).

**Remark 2.4.** As it is clear from the proof, Theorem 2.2 and Theorem 2.3 hold for a class of “external” manifolds wider than the class of models of hyperbolic type. Namely, the condition (ii) in Definition 2.1 is stronger than necessary, since what is only needed is relation (13) to hold. For instance, one can chose \( \sigma(r)|_{(R_2, +\infty)} = \alpha r - C \), for large enough constants \( \alpha = \alpha(\rho) > 1 \) and \( C = C(\rho) > 0 \). This non-trivial example has linear volume growth and its sectional curvatures satisfy \( \text{Sect}_{\text{rad}} = 0 \) and \( \text{Sect}_{tg} = -\frac{\alpha - 1}{\alpha^2} r^{-2} \) for \( r > R_2 \).

2.2. Convex exhaustion functions on glued manifolds. As a consequence of the Hessian comparison theorem, the square of the distance function of a generic Cartan-Hadamard manifold \( M \) is a smooth, strictly convex exhaustion function. This subsection aims to show how it is possible to prescribe an hyperbolic infinity to \( M \) in such a way that the resulting space supports again a strictly convex, exhaustion function. The lack of rotational symmetry of the source metric of \( M \) will prevent us to guarantee that the new space is Cartan-Hadamard and, therefore, this construction will be not used in the proof of the main theorem of the paper. However, we feel that it is interesting in its own and represents a first important indication that the \( p \)-Dirichlet problem can be solved in the more general situation where the target space is simply Cartan-Hadamard.

Let \( (N_j, \langle \cdot, \cdot \rangle_{N_j}) \) be a \( n \)-dimensional Cartan-Hadamard manifold and fix a pole \( o \in N_j \). Consider \( \mathbb{R}^n \) with polar coordinates \( (t, \Theta) \) around \( o \). Namely, in a neighborhood of each point \( x \in \mathbb{R}^n \) we have a local coordinate system \( (t, \theta^2, \ldots, \theta^n) \), where \( t \) is the radial coordinate and \( (\theta^i)_{i=2}^n \) are local angular coordinates. Since \( N_j \) is Cartan-Hadamard, by the Gauss Lemma we can
write
\[(N_j, \langle \cdot, \cdot \rangle_{N_j}) = (\mathbb{R}^n, dt^2 + j_{il}(t, \Theta)d\theta^id\theta^l).\]
Similarly, we consider the hyperbolic space \(\mathbb{H}^n_k\) of constant curvature \(-k\), and we write \(\mathbb{H}^n_k = (\mathbb{R}^n, dt^2 + h_{il}^{(k)}d\theta^id\theta^l).

**Theorem 2.5.** Let \(N^n_j\) be an \(n\)-dimensional Cartan-Hadamard manifold. Fix \(0 < R_1 < R_2 < \infty\). Then, there exist \(k > 0\) depending on \(j, R_1\) and \(R_2\), and a manifold \((N^n_j, \langle \cdot, \cdot \rangle_{N_j}) = (\mathbb{R}^n, dt^2 + \hat{j}_{il}(t, \Theta)d\theta^id\theta^l)\) such that:

(i) \(B_{R_1}^N(o) \subset N^n_j\).

(ii) \(N^n_j \setminus B_{R_2}^N(\hat{o}) = \mathbb{H}^n_k \setminus B_{R_2}^{n-k}(o')\) for some poles \(\hat{o} \in N_j^\circ\) and \(o' \in \mathbb{H}^n_k\).

(iii) \(N^n_j\) supports a global strictly convex function.

**Proof.** Consider a smooth partition of unity \(\phi_j, \phi_h \in C^\infty((0, +\infty))\) such that
\[0 \leq \phi_j(t) \leq 1, \quad \phi_j|_{(0, R_1]} \equiv 1, \quad \phi_j|_{[R_2, \infty)} \equiv 0, \quad \phi_j' \leq 0\]
and
\[\phi_j(t) + \phi_h(t) = 1, \quad \forall t \in (0, +\infty).\]
Define a new Riemannian manifold \(N^n_j\), by endowing \(\mathbb{R}^n\) with a new metric \(\langle \cdot, \cdot \rangle_{N_j}\). Namely, in polar coordinates, we set
\[N^n_j = (\mathbb{R}^n, dt^2 + \hat{j}_{il}(t, \Theta)d\theta^id\theta^l),\]
where
\[\hat{j}_{il}(t, \Theta) := \phi_j(t)j_{il}(t, \Theta) + \phi_h(t)h_{il}^{(k)}(t, \Theta).\]
Note that \(N_j\) is a well defined \(n\)-dimensional Riemannian manifold and conditions (i) and (ii) of the statement are automatically satisfied by construction. Define the coordinate function \(T : \mathbb{R}^n \to \mathbb{R}\) as \(T(t, \Theta) = t\) and observe that \(T \in C^\infty(\mathbb{R}^n \setminus \{0\})\). Moreover, since \(N^n_j\) and \(\mathbb{H}^n_k\) are Cartan-Hadamard manifolds, we have that \(T\) on \(N^n_j\) and \(\mathbb{H}^n_k\) is the Riemannian radial function. In particular \(T\) is convex both on \(N^n_j\) and \(\mathbb{H}^n_k\), strictly convex off the radial direction, and \(T^2\) is smooth and strictly convex on both \(N^n_j\) and \(\mathbb{H}^n_k\). To prove the theorem we will show that \(T : N^n_j \to \mathbb{R}\) is convex (strictly except for the radial direction). This will imply that \(T^2 : N^n_j \to \mathbb{R}\) is strictly convex and, because of (i), smooth on all of \(N^n_j\). To this end, we use the following lemma.

**Lemma 2.6.** Consider a Riemannian manifold structure \(N_j\) on \(\mathbb{R}^n \setminus \{0\}\), i.e. \(N_j = (\mathbb{R}^n \setminus \{0\}, \langle \cdot, \cdot \rangle_{N_j})\), and suppose that, in (local) polar coordinates and with notation as above, \(\langle \cdot, \cdot \rangle_{N_j}\) can be expressed as
\[\langle \cdot, \cdot \rangle_{N_j}(t, \Theta) = dt^2 + j_{il}(t, \Theta)d\theta^id\theta^l.\]
Define \(T : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}\) as \(T(t, \Theta) = t\). Then
\[N^n_j \text{Hess } T|_{(t, \Theta)}(X, X) = \frac{1}{2} X^i X^l \frac{\partial}{\partial t} \hat{j}_{il}(t, \Theta),\]
for all vector fields on $\mathbb{R}^n$.

**Proof (of Lemma 2.6).** By definition of hessian, it holds

$$N_j \text{Hess} T_{(t, \Theta)}(X, X) = \langle N_j \nabla_{N_j} T, X \rangle_{N_j}$$

$$= \frac{1}{2} N_j \nabla T \langle X, X \rangle_{N_j} + \langle X, [X, N_j \nabla T] \rangle_{N_j},$$

for any vector field $X$ on $\mathbb{R}^n$. Since $T$ is defined as the coordinate function $t$ and the metric has the expression (15), we have $N_j \nabla T = \frac{\partial}{\partial t} = \partial_t$. Given a vector field $X$, this can be expressed in coordinates as $X = X^0 \partial_t + X^i \partial_i$, where $\partial_i := \frac{\partial}{\partial x^i}$ for $i = 2, \ldots, n$. Whence, we get

$$\frac{1}{2} N_j \nabla T \langle X, X \rangle_{N_j} = \frac{1}{2} \partial_t \left( (X^0)^2 + j_{il}(t, \Theta) X^i X^l \right)$$

$$= X^0 \partial_t X^0 + j_{il}(t, \Theta) X^i \partial_t X^l + \frac{1}{2} X^i X^l \frac{\partial}{\partial t} j_{il}(t, \Theta).$$

Moreover

$$[X, N_j \nabla T] = [X, \partial_t] = -\partial_t X^0 \partial_t - \partial_t X^i \partial_i,$$

which gives

$$\langle X, [X, N_j \nabla T] \rangle_{N_j} = \langle X^0 \partial_t + X^i \partial_i, -\partial_t X^0 \partial_t - \partial_t X^i \partial_i \rangle_{N_j}$$

$$= -X^0 \partial_t X^0 - j_{il}(t, \Theta) X^i \partial_t X^l.$$

Summing (16) and (17) concludes the proof. $\square$

According to Lemma 2.6 we thus have

$$N_j \text{Hess} T_{(t, \Theta)}(X, X) = \frac{1}{2} X^i X^l \partial_l j_{il}(t, \Theta)$$

$$= \frac{1}{2} X^i X^l \partial_l \left[ \phi_j(t) j_{il}(t, \Theta) + \phi_h(t) h^{(k)}_{il}(t, \Theta) \right]$$

$$= \phi_j(t) \left[ \frac{1}{2} X^i X^l \partial_l j_{il}(t, \Theta) \right]$$

$$+ \phi_h(t) \left[ \frac{1}{2} X^i X^l \partial_l h^{(k)}_{il}(t, \Theta) \right]$$

$$+ \partial_t \phi_j \left[ \frac{1}{2} X^i X^l j_{il}(t, \Theta) \right] + \partial_t \phi_h \left[ \frac{1}{2} X^i X^l h^{(k)}_{il}(t, \Theta) \right].$$

Applying again Lemma 2.6 and recalling (14), this latter gives

$$N_j \text{Hess} T_{(t, \Theta)}(X, X) = \phi_j(t)^N_j \text{Hess} T_{(t, \Theta)}(X, X)$$

$$+ \phi_h(t)^{N_j} \text{Hess} T_{(t, \Theta)}(X, X)$$

$$+ \frac{1}{2} \phi_h(t) X^i X^l \left[ h^{(k)}_{il}(t, \Theta) - j_{il}(t, \Theta) \right].$$

Since $\phi_h \geq 0$ and recalling the above considerations, in order to conclude the proof it’s enough to show that for $k$ large enough it holds

$$X^i X^l h^{(k)}_{il}(t, \Theta) \geq X^i X^l j_{il}(t, \Theta).$$
for all \((t, \Theta) \in \mathbb{B}_{R_2} \setminus \mathbb{B}_{R_1} \subset \mathbb{R}^n\). Since \(\bar{B}_{R_2} \setminus \mathbb{B}_{R_1} \subset \mathbb{R}^n\), there exists a constant \(c_2 = c_2(R_1, R_2) > 0\) such that

\[
X^i X^j h_{il}^{(1)}(t, \Theta) \geq c_2 X^i X^j j_{il}(t, \Theta)
\]

for all vector fields \(X\) and all \((t, \Theta) \in \mathbb{B}_{R_2} \setminus \mathbb{B}_{R_1}\). Finally, since the coordinate system is fixed, we have that

\[
h_{il}^{(k)} = \frac{\sinh^2 \left( \sqrt{k}t \right)}{k \sinh^2 t} h_{il}^{(1)},
\]

so that it is enough to choose \(k\) in such a way that

\[
\sinh^2 \left( \sqrt{k}R_1 \right) \geq c_2^{-1} k \sinh^2 R_1.
\]

\[\square\]

**Remark 2.7.** In the proof of Theorem 2.5 the assumption \(N_j \text{Sect} \leq 0\) was required in order to guarantee that the metric of \(N_j\) has the form (15) and that \(T\) is strictly convex in \(B_{R_2} \subset N_j\). Accordingly, it is clear that the Theorem works as well when \(N_j\) is the interior of a convex geodesic ball without curvature assumptions.

**Theorem 2.8.** Let \(B_R \subset N^n\) be a convex geodesic ball of radius \(R\) in an \(n\)-dimensional manifold. Then, for every \(0 < R_1 < R_2 < R\) there exist a \(k > 0\) depending on \(N, R_1\) and \(R_2\), and a manifold \((N^n_j, \langle \cdot, \cdot \rangle_{N_j}) = (\mathbb{R}^n, dt^2 + \hat{j}_{il}(t, \Theta) d\theta^i d\theta^l)\) such that:

(i) \(B_{R_1}^N (o) \subset N^n_j\).

(ii) \(N^n_j \setminus B_{R_2}^N (\hat{o}) = \mathbb{H}^n_k \setminus B_{R'}^N (o')\) for some poles \(\hat{o} \in N_j\) and \(o' \in \mathbb{H}^n_k\).

(iii) \(N_j\) supports a global strictly convex exhaustion function.

### 2.3. Compact hyperbolic manifolds with large injectivity radii.

It is intuitively clear that actions of small discrete groups on a complete Riemannian manifold give rise to large fundamental domains. The intuition is confirmed in the next simple result.

**Lemma 2.9.** Let \((N, h)\) be a complete Riemannian manifold. Suppose that there exists a filtration

\[
\Gamma_0 \triangleright \Gamma_2 \triangleright \Gamma_3 \triangleright \cdots \triangleright \Gamma_k \triangleright \cdots \triangleright \{1\}
\]

of discrete groups \(\Gamma_k \subset \text{Iso}(N)\) acting freely and properly on \(N\). Then, for every arbitrarily large ball \(B_R (p)\), there exists \(K > 0\) such that the following holds: for every \(k > K\) we find a fundamental domain \(\Omega_k\) of \(\Gamma_k\) containing \(p\) and satisfying

\[
B_R^N (p) \subset \subset \Omega_k.
\]
Proof. Let $D_k (p)$ be the Dirichlet domain of $\Gamma_k$ centered at $p$. Recall that $D_k (p) = \cap_{\gamma \in \Gamma_k} H_\gamma (p)$ where

$$H_\gamma (p) = \{ x \in N : d_N (x, p) < d_N (x, \gamma \cdot p) \}.$$ 

One can easily verify that if $B^N_R (p) \cap (N \setminus D_k (p)) \neq \emptyset$ then

$$(19) \quad B^N_R (p) \cap \gamma \cdot B^N_R (p) \neq \emptyset,$$

for some $\gamma \in \Gamma_k \subset \Gamma_0$. Since $\Gamma_0$ acts properly on $N$ it follows that $(19)$ can be satisfied for at most a finite number of $\gamma_1, ..., \gamma_N \in \Gamma_0$. To conclude the validity of $(18)$, we now use that $\cap \Gamma_k = \{1\}$ and, therefore, $\gamma_1, ..., \gamma_N \notin \Gamma_k$, for every large enough $k$. \qed

There are situations where condition $(18)$ has an immediate interpretation in terms of the injectivity radius of the corresponding quotient space. Let $N$ be a Cartan-Hadamard manifold and let $\Gamma \subset \text{Iso} (N)$ be a discrete group acting freely and co-compactly on $N$. Then, the orbit space $N_\Gamma = N/\Gamma$ is a smooth manifold universally covered by the quotient projection $P : N \to N_\Gamma$ and the metric of $N$ descends to a complete metric on $N_\Gamma$. Moreover $\Gamma \simeq \pi_1 (N_\Gamma)$. By the Cartan-Hadamard theorem, another way to define the universal covering of $N_\Gamma$ is to use the exponential map $\exp_q : T_q N_\Gamma \to N_\Gamma$ from a fixed point $q \in N_\Gamma$. The universal covering property yields that there exists a fiber-preserving diffeomorphism $I : N \to T_q N_\Gamma$. Therefore, we can always identify $N = T_q N_\Gamma$ and $P = \exp_q$. Fix $p \in N$ and let $q = P (p)$. Let also $E_q (N_\Gamma) = N_\Gamma \setminus \text{cut} (q)$ and $\Omega = \exp_q^{-1} E_q$. Then $\Omega$ is a fundamental domain for the action of $\Gamma$ on $N$ and $p \in \Omega$. In particular, from the equality $P (B^N_R (p)) = B^N_R (q)$ we deduce that, for every $R < \text{inj}_{N/\Gamma} (q)$, it holds $B^N_R (p) \subset \subset \Omega$. Summarizing, on a Cartan-Hadamard manifold, the existence of a co-compact discrete group of isometries with large fundamental domain follows from the existence of a quotient manifold with large injectivity radius. The converse also holds because $\text{inj} (N) = +\infty$. Therefore, if $\text{inj} (q) \geq R$. A case of special interest is obtained by taking $N = \mathbb{H}^n_{-k^2}$, the standard hyperbolic spaceform of constant curvature $-k^2 < 0$. If $\Gamma$ is a co-compact discrete group of isometries acting freely and properly on $\mathbb{H}^n_{-k^2}$, the corresponding Riemannian orbit space $\mathbb{H}^n_{-k^2}/\Gamma$ is named a compact hyperbolic manifold (of constant curvature $-k^2$). The following result was first observed in [Fa], see p.74.

**Proposition 2.10.** Let $n \geq 0$, $R > 0$ and $p \in \mathbb{H}^n_{-k^2}$. Then, there exists a co-compact, discrete group $\Gamma$ of isometries of $\mathbb{H}^n_{-k^2}$ acting freely and properly on $\mathbb{H}^n_{-k^2}$ and whose fundamental domain $\Omega$ containing $p$ satisfies

$$B_R (p) \subset \subset \Omega.$$
Equivalently,
\[ \text{inj}(\mathbb{H}^n_{-k^2}/\Gamma) \geq R. \]

Proof. By a result of A. Borel [Bo], \( \mathbb{H}^n_{-k^2} \) has a co-compact, discrete group of isometries \( \Gamma_0 \) acting freely and properly. According to a result by A. Malcev, \( \Gamma_0 \) is residually finite, i.e., there exists a filtration
\[ \Gamma_0 \triangleright \Gamma_2 \triangleright \Gamma_3 \triangleright \cdots \triangleright \Gamma_k \triangleright \cdots \triangleright \{1\} \]
satisfying \( [\Gamma_k : \Gamma_{k-1}] = |\Gamma_k/\Gamma_{k-1}| < +\infty \). To conclude, we now apply Lemma 2.9 and recall the previous discussion on the injectivity radius. \( \Box \)

2.4. A maximum principle for \( p \)-harmonic maps. It is well known, and an easy consequence of the composition law of the Hessians, that by composing a harmonic map \( u : M \to N \) with a convex function \( h : N \to \mathbb{R} \) gives a subharmonic function \( v = h \circ u : M \to \mathbb{R} \), i.e., \( \Delta v \geq 0 \). In particular, if \( M \) is compact with smooth boundary \( \partial M \neq \emptyset \) and \( N \) is Cartan-Hadamard, we can choose \( h(x) = d_N^2(x,o) \) and apply the usual maximum principle to conclude that the image \( u(M) \subset N \) is confined in a ball \( B_R^N(o) \) of radius \( R > 0 \) depending only on the values of \( u \) on \( \partial \Omega \), namely, \( R = \max_{\partial \Omega} d_N(u,o) \). Very recently, it was proved in [Ve1] that, in general, the nice composition property of harmonic maps does not extend to \( p \)-harmonic maps, \( p > 2 \). Nevertheless, we are able to recover the above conclusion thus establishing a new maximum principle for the composition of a \( p \)-harmonic map and a convex function.

Theorem 2.11 (weak maximum principle). Let \( M \) be a compact Riemannian manifold with boundary \( \partial M \neq \emptyset \), and let \( u \in C^1(M,N) \) be a \( (\geq 2) \)-harmonic map. Assume that \( N \) supports a smooth convex function \( f : N \to \mathbb{R} \). Set \( w = f \circ u : M \to \mathbb{R} \). Then
\[ \sup_{M} w = \sup_{\partial M} w. \]

Proof. Let \( w^* = \sup_{\partial M} w \) and, by contradiction, suppose that \( w(x_0) > w^* \) for some \( x_0 \in \text{int}(M) \). Fix \( 0 < \varepsilon << 1 \) so that \( w(x_0) - w^* > 2\varepsilon \). Let \( \lambda : \mathbb{R} \to [0,1] \) satisfy \( \lambda' \geq 0 \), \( \lambda' > 0 \) on \((\varepsilon, +\infty)\), \( \lambda = 0 \) on \((-\infty, \varepsilon]\). Define the vector field
\[ Z = |du|^{p-2} \lambda(w - w^*) \nabla w \]
and note that \( \text{supp} Z \subset \text{int}(M) \). Direct computations show that
\[ \text{div} Z = \lambda' \circ (w - w^*) |du|^{p-2} |\nabla w|^2 + \lambda \circ (w - w^*) \text{tr Hess}(f) \left(|du|^{p-2} du, du\right) + \lambda \circ (w - w^*) df(\Delta_p u) \geq |\nabla w|^2 |du|^{p-2} \lambda' \circ (w - w^*), \]
and applying the divergence theorem we get
\[ 0 \leq \int_M |\nabla w|^2 |du|^{p-2} \lambda' \circ (w - w^*) \leq \int_M \text{div} Z = 0. \]
This proves that
\[(20) \quad |\nabla w|^2 |du|^{p-2} = 0 \text{ on } M_\varepsilon,\]
where we have denoted with $M_\varepsilon$ the connected component containing $x_0$ of the open set
\[
\{x \in M : w - w^* - \varepsilon > 0\}.
\]
Since, by (20), $dw = df(du) = 0$ where $du \neq 0$ and $dw = df(du) = 0$ where $du = 0$, it follows that $w$ is constant on $M_\varepsilon$ and this easily gives the desired contradiction. \(\square\)

2.5. Proof of Theorem D. In this last subsection we put all the previous ingredients together to get a proof of the second main result of the paper.

The boundary datum $f$ has image confined in a ball $B_{R_0}(0)$ of $N^n$. Using Theorem 2.2 we glue $B_{R_0}(0)$ to the exterior of a large ball in the hyperbolic spaceform $H_n^{k_2}$ of sufficiently small curvature $-k^2 << -1$, say $H_n^{k_2} \setminus B_{R_1}(0), R_1 >> R_0$, thus obtaining a new Cartan-Hadamard model manifold $(N', h')$. On the other hand, by Proposition 2.10, $H_n^{k_2}$ has compact quotients with arbitrarily large injectivity radii. Accordingly, we can choose a discrete subgroup $\Gamma$ of isometries acting freely and co-compactly on $H_n^{k_2}$ in such a way that $B_{R_1}(0)$ is contained in a relatively compact, fundamental domain of the action, say $B_{R_1}(0) \subset \subset \Omega$. Making use of $\Gamma$ we extend the deformed metric of $\bar{\Omega}$ periodically thus obtaining a new Riemannian manifold $N''$ diffeomorphic to $H_n^{k_2}$. More precisely, the metric $h''$ of $N''$ is defined by setting
\[
h''_{\gamma p} = (\gamma^{-1})^*_{\gamma p} h'.
\]
Since $h'$ is hyperbolic in a neighborhood of $\partial \Omega$, the definition of $h''$ is well posed. Moreover, $(N'', h'')$ has non-positive curvature, hence it is Cartan-Hadamard, and, by construction, $\Gamma$ acts freely and co-compactly by isometries on $N''$. In particular, each copy of $\Omega$ contains an isometric image of $B_{R_0}(0)$. Now, we take the quotient manifold $N''/\Gamma$ which is compact and covered by $N''$ via the quotient projection $P : N'' \to N''/\Gamma$. By construction, the original datum $f$ well defines $f'' = f : M \to N''$. Applying Proposition C we get a unique solution $u'' \in C^0(M, N'') \cap C^{1,\alpha}(\text{int}(M), N'')$ to the Dirichlet problem
\[
\begin{align*}
\Delta_p u'' &= 0 & \text{on } M \\
v'' &= f'' & \text{on } \partial M.
\end{align*}
\]
To complete the argument, it remains to show that, actually, $u''$ gives rise to a solution of the original problem. This clearly follows if we are able to show that its image is confined in $B_{R_0}(0) \subset N''$. To prove that this is the case, we recall that $N''$ is Cartan-Hadamard and, therefore, the function $d_{N''}^2(y, 0)$ is smooth and strictly convex. By means of Theorem 2.11 we deduce that
$d^2_{N'}(u'',0)$ achieves its maximum on $\partial M$. To conclude, it suffices to recall that $f(M) \subset B^N_{R_0}(0)$ and to use the equality $u'' = f$ on $\partial M$.

**Remark 2.12.** In view of known results in the harmonic case, [HKW1, HKW2, HKW3], it is an interesting problem to extend the conclusion of Theorem D to the case where $N$ is a general Cartan-Hadamard manifold or it is replaced by a regular ball in a complete manifold. Apparently, the above strategy cannot be readily adapted to these situations. One of the main obstructions is that, despite of the use of Theorem 2.5 to obtain $N'$ supporting a strictly convex exhaustion function, we are not able to show that such a nice function can be constructed also on $N''$. This latter is the complete, simply connected manifold which gives rise to a compact quotient and, therefore, that enables us to apply Theorem E. Once this problem is solved, we should also relate the convex function on $N''$ to the size of the unperturbed ball $B^N_{R_0}$ in such a way that, applying the maximum principle, we can conclude that the lifted solution is confined in $B^N_{R_0}$.

2.6. **A general uniqueness result.** In Theorem D the uniqueness property enjoyed by solutions of the $p$-Dirichlet problem can be obtained as a consequence of a result by W. Wei for compact targets, proceeding as in the uniqueness part of the proof of Theorem B. In this subsection we extend Wei’s result to solutions of the homotopic $p$-Dirichlet problem in case the target manifold is non-compact. In particular, our result applies directly in the situation of Theorem D. The construction via the quotient manifold $\hat{N}$ proposed in the proof below comes back to Schoen and Yau [SY1], which studied the moduli space of harmonic maps when $M$ is a complete non-compact manifold with finite volume. Subsequently, in [PRS] it was observed that it is enough for $M$ to be parabolic, while a generalization of Schoen and Yau’s uniqueness results to $p$-harmonic maps has been obtained in [Ve2]. In particular, in [Ve2] there were introduced the convexity result stated below as Lemma 2.13 and the “mixed” vector field $X$ used here, which in turn inspires to [PRS] and [HPV].

**Proof (of Theorem E).** Suppose $u$ and $v$ are two $C^1(M,N)$ solutions to Problem A. Let $P_M : \hat{M} \to M$ and $P_N : \hat{N} \to N$ be the universal Riemannian covers of $M$ and $N$, respectively. Note that $\hat{M}$ is a simply connected manifold with non-empty boundary $\partial \hat{M}$ (which is in general neither compact nor simply connected) such that $P_M(\partial \hat{M}) \equiv \partial M$.

The fundamental groups $\pi_1(M,\ast)$ and $\pi_1(N,\ast)$ act as groups of isometries on $\hat{M}$ and $\hat{N}$ respectively, so that $M = \hat{M}/\pi_1(M,\ast)$ and $N = \hat{N}/\pi_1(N,\ast)$. Let $\text{dist}_{\hat{N}} : \hat{N} \times \hat{N} \to \mathbb{R}$ be the distance function on $\hat{N}$. Since $\hat{N}$ Sect $\leq 0$, we know that $\text{dist}_{\hat{N}}$ is smooth on $(\hat{N} \times \hat{N}) \setminus \hat{D}$, where $\hat{D}$ is the diagonal set $\{(\hat{x},\hat{x}) : \hat{x} \in \hat{N}\}$, and $\text{dist}^2_{\hat{N}}$ is smooth on $\hat{N} \times \hat{N}$. Now $\pi_1(N,\ast)$ acts on $\hat{N} \times \hat{N}$ as a group of isometries by 

$$\beta(\hat{x},\hat{y}) = (\beta(\hat{x}),\beta(\hat{y})) \quad \text{for } \beta \in \pi_1(N,\ast).$$
Thus $\text{dist}_N^2$ induces a smooth function
\[
\hat{r}^2 : \hat{N} \to \mathbb{R},
\]
where we have defined
\[
\hat{N} := (\hat{N} \times \check{N}) / \pi_1(N, \ast).
\]
Let $U : M \times [0,1] \to N$ be a continuous relative homotopy between $u$ and $v$. Since $\hat{M}$, hence $\hat{M} \times [0,1]$, is simply connected, $U$ lifts to a homotopy $\check{U}$ between $\check{u}(\cdot) := \check{U}(\cdot,0)$ and $\check{v}(\cdot) := \check{U}(\cdot,1)$ relative to $\partial \hat{M}$. Clearly, $P_N(\check{u}) = u(P_M)$ and $P_N(\check{v}) = v(P_M)$. Since Riemannian coverings are local isometries, $\check{u}$ and $\check{v}$ are $p$-harmonic maps and
\[
|d\check{u}|(\check{q}) = |du|(P_M(\check{q})), \quad |d\check{v}|(\check{q}) = |dv|(P_M(\check{q})).
\]
Now, $\pi_1(M, \ast)$ acts as a group of isometries on $\check{M}$ and we have
\[
(21) \quad \check{u}(\gamma(\check{q})) = u_\ast(\gamma)\check{u}(\check{q}), \quad \check{v}(\gamma(\check{q})) = v_\ast(\gamma)\check{v}(\check{q}), \quad \forall \check{q} \in \hat{M}, \gamma \in \pi_1(M, \ast),
\]
where $u_\ast, v_\ast : \pi_1(M, \ast) \to \pi_1(N, \ast)$ are the induced homomorphism and $u_\ast \equiv v_\ast$ since $u$ is homotopic to $v$.
Thus, the map $\check{j} : \hat{M} \to \hat{N} \times \check{N}$ defined by $\check{j}(\check{x}) := (\check{u}(\check{x}), \check{v}(\check{x}))$ induces via $\check{j}$ a map
\[
\hat{j} : M \to \hat{N}.
\]
Furthermore, we can construct a vector valued 1-form $J \in T^*M \otimes \check{j}^{-1}T\hat{N}$ along $\hat{j}$ by projecting via $(21)$ the vector valued 1-form $\hat{J}$ along $\check{j}$ defined as
\[
\hat{J} := (\mathcal{K}_p(\check{u}), \mathcal{K}_p(\check{v})) \in T^*\hat{M} \otimes \check{j}^{-1}T\left(\hat{N} \times \check{N}\right).
\]
Here and on, the symbol $\mathcal{K}_p(\check{u})$ stands for
\[
\mathcal{K}_p(\check{u}) := |d\check{u}|^{p-2}d\check{u}.
\]
Consider the vector field on $M$ given by
\[
X|_q := [d\hat{r}^2|_{\check{j}(q)} \circ J|_q]^\sharp.
\]
Note that
\[
(22) \quad X|_q := dP_M|_{\check{q}} \circ \check{X}|_{\check{q}},
\]
where
\[
\check{X}|_{\check{q}} := \left[ d \left( \text{dist}^2_{\check{N}} \right)|_{\check{j}(\check{q})} \circ \hat{J}|_{\check{q}} \right]^\sharp.
\]
We claim that $(22)$ is well defined. To this end, let $S_\check{q} \in T_{\check{q}}\hat{M}$ be an arbitrary vector and let $q' \in P_M^{-1}(q) \subset T\hat{M}$. If $q' \neq \check{q}$, there exists $\gamma \in \pi_1(M, \ast)$ such that $q' = \gamma q$. Then,
\[
\hat{J}|_{\gamma \check{q}}(d\gamma(S_\check{q})) = (d[u_\ast(\gamma)](\mathcal{K}_p(\check{u})(S_\check{q})), d[v_\ast(\gamma)](\mathcal{K}_p(\check{v})(S_\check{q}))).
\]
Since $u$ is homotopic to $v$, $u_s = v_s$. Moreover $\text{dist}_K$ is equivariant with respect to the action of $\pi_1(N)$ on $\hat{N} \times \hat{N}$, i.e.

$$\text{dist}_K(\alpha\hat{t}_1, \alpha\hat{t}_2) = \text{dist}_K(\hat{t}_1, \hat{t}_2), \quad \forall \alpha \in \pi_1(N), \ x_1, x_2 \in \hat{N}.$$ 

Then

$$dP_M|\hat{q} \circ \left[ d\left(\text{dist}^2_K\right)|_{\hat{q}} \circ \hat{f}\right]$$

does not depend on the choice of $\hat{q} \in P_{M}^{-1}(q)$.

Now, we recall the following “convexity” result of [Ve2].

**Lemma 2.13.** For all $q \in M$ and for any choice of $\hat{q} \in P_{M}^{-1}(q)$ we have

$$\text{tr}_M \cdot \hat{N} \times \hat{N} \cdot \text{Hess} \text{dist}^2_K|_{\hat{q}} (\hat{d}J, \hat{J}) \geq 0$$

Moreover, having fixed an orthonormal frame $\hat{E}_i$ in $T_{\hat{q}}\hat{M}$, with $i = 1, \ldots, m$, the equality holds in (23) if and only if there are parallel vector fields $Z_i$, defined along the unique geodesic $\gamma_{\hat{q}}$ in $\hat{N}$ joining $\bar{u}(\hat{q})$ and $\bar{v}(\hat{q})$, such that $Z_i(\bar{u}(\hat{q})) = d\bar{u}|\hat{q}(\hat{E}_i)$, $Z_i(\bar{v}(\hat{q})) = d\bar{v}|\hat{q}(\hat{E}_i)$ and $\left\langle \hat{N} R(Z_i, \gamma_{\hat{q}}) \gamma_{\hat{q}}, Z_i \right\rangle_{\hat{N}} \equiv 0$ along $\gamma_{\hat{q}}$. Moreover, $d(\text{dist}_K(\hat{q})) = 0$.

In particular, if $N$ Sect $< 0$, $Z_i$ is proportional to $\gamma_{\hat{q}}$ for each $i = 1, \ldots, m$.

By the homotopy assumption, for each $q \in \partial M$ and any $\hat{q} \in P_{M}^{-1}(q)$ we have $\bar{u}(\hat{q}) = \bar{v}(\hat{q})$, i.e., $\hat{J}(\hat{q}) \in \hat{D}$. In particular, this implies that $\hat{r}^2(\hat{J})|_{\partial M} = 0$, and, since $d\hat{r}^2 = 2\hat{r}d\hat{r}$,

$$X|_{\partial M} = 0.$$ 

Then, applying the divergence theorem,

$$\int_M \text{div} X dV_M = 0.$$ 

On the other hand, by the $p$-harmonicity of $u$ and $v$ and by the isometry property of the coverings projections,

$$\text{div} X|_{q} = \text{tr}_M \cdot \hat{N} \cdot \text{Hess} \text{dist}^2_K|_{\hat{q}} (\hat{d}J, \hat{J}) = \text{tr}_M \cdot \hat{N} \cdot \text{Hess} \text{dist}^2_K|_{\hat{q}} (\hat{d}J, \hat{J})$$

for each $q \in M$ and any $\hat{q} \in P_{M}^{-1}(q)$. By Lemma 2.13 we thus get $\text{div} X \geq 0$ and (24) implies $\text{div} X \equiv 0$. Thus (25) holds with the equality sign and the equality conditions in Lemma 2.13 give $d(\text{dist}_K)(d\bar{u}, d\bar{v}) \equiv 0$. Since $\text{dist}_K(\hat{u}, \hat{v})|_{\partial M} = 0$ we get $\hat{u} \equiv \hat{v}$ and, projecting on $M$, $u \equiv v$.

To conclude the proof, let us remark that in general relations (24) and (25) has to be considered in the weak sense. Lemma 7 in [Ve2] proves the weak validity of (25), i.e.

$$-\int_M [d\hat{r}^2|_{\hat{q}} \circ J] \left(\text{Hess} \eta\right) = \int_M \eta \cdot \text{tr}_M \cdot \hat{N} \cdot \text{Hess} \text{dist}^2_K|_{\hat{q}} (\hat{d}J, \hat{J})$$
for all $\eta \in C^\infty_0(M)$. Moreover we can choose a 1-parameter family of smooth cut-off functions $\{\eta_\epsilon\}$ compactly supported in $\text{int}(M)$ such that
\[ \sup_M |\nabla \eta_\epsilon| = O(\epsilon^{-1}) \]
as $\epsilon \to 0$ and $\eta_\epsilon(q) = 1$ for all $q \in M$ satisfying $\text{dist}_M(q, \partial M) > \epsilon$. Since $X|_{\partial M} \equiv 0$, $X$ is continuous and
\[ \text{Vol}_M(\{q \in M : \text{dist}_M(q, \partial M) \leq \epsilon\}) = O(\epsilon) \]
as $\epsilon \to 0$, applying (26) with $\eta = \eta_\epsilon$ and letting $\epsilon \to 0$, we can conclude that the LHS of (26) tends to 0. In some sense this gives a weak version of (21).

On the other hand by Lemma 2.13, we can apply monotone convergence to the RHS of (26) to get
\[ \int_M \text{tr}_M \hat{N} \text{Hess} \hat{r}^2 |_{j(q)} (dj, J) = 0. \]

Acknowledgement. We are indebted to François Fillastre for some conversations concerning closed hyperbolic manifolds which have revealed very useful to the draft of Subsection 2.3.

References

[An] M. Anderson, $L^2$ harmonic forms on complete Riemannian manifolds. Lecture Notes in Math., 1339, Springer, Berlin, 1988.
[An] M. Anderson, $L^2$ harmonic forms on complete Riemannian manifolds. Lecture Notes in Math., 1339, Springer, Berlin, 1988.
[An] M. Anderson, $L^2$ harmonic forms on complete Riemannian manifolds. Lecture Notes in Math., 1339, Springer, Berlin, 1988.
[Au] T. Aubin, Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. xviii+395 pp.
[Bo] A. Borel, Compact Clifford-Klein forms of symmetric spaces. Topology 2 (1963) 111-122.
[Bu] F. Burstall, Harmonic maps of finite energy from non-compact manifolds. Jour. London Math. Soc. 30 (1984), 361-370.
[Gh] M. Ghomi, The problem of optimal smoothing for convex functions. Proc. Amer. Math. Soc. 130 (2002), no. 8, 2255-2259.
[Fa] F.T. Farrell, Lectures on surgical methods in rigidity. Published for the Tata Institute of Fundamental Research, Bombay by Springer-Verlag, Berlin, 1996.
[Ham] R. Hamilton, Harmonic maps of manifolds with boundary. Lecture notes, Mathematics, No. 471 Springer, Berlin, Heidelberg, New York (1975).
[Hi] M.W. Hirsch, Differential topology Graduate Texts in Mathematics, 33 Springer-Verlag, New York, 1994.
[HL] R. Hardt, F.-H. Lin, Mappings minimizing the $L^p$ norm of the gradient. Comm. Pure Appl. Math. 40 (1987), no. 5, 555-588.
[Hat] A. Hatcher, Algebraic topology. Cambridge University Press, Cambridge, 2002. xii+544 pp.
[HKW1] S. Hildebrandt, H. Kaul, K.-O. Widman, Harmonic mappings into Riemannian manifolds with non-positive sectional curvature. Math. Scand. 37 (1975), no. 2, 257-263.
[HKW2] S. Hildebrandt, H. Kaul, K.-O. Widman, Dirichlet’s boundary value problem for harmonic mappings of Riemannian manifolds. Math. Z. 147 (1976), no. 3, 225-236.
[HKW3] S. Hildebrandt, H. Kaul, K.-O. Widman, An existence theorem for harmonic mappings of Riemannian manifolds. Acta Math. 138 (1977), no. 1-2, 116.
[HPV] I. Holopainen, S. Pigola, G. Veronelli Global comparison principles for the $p$-Laplace operator on Riemannian manifolds. Potential Anal. 34 no. 4 (2011), p. 371-384.
30  STEFANO PIGOLA AND GIONA VERONELLI

[J]  J. Jost,  Riemannian geometry and geometric analysis. Fourth edition. Universitext. Springer-Verlag, Berlin, 2005. xiv+566.
[Le]  J. M. Lee,  Introduction to smooth manifolds  Graduate Texts in Mathematics, 218. Springer-Verlag, New York, 2003.
[LU]  O.A. Ladyzhenskaya, N.N. Ural’teva,  Linear and Quasilinear Elliptic Equations, 2nd ed., Nauka Press, Moscow 1973 (in Russian); Academic Press, New York, 1968 (English transl. of the 1st ed.).
[KS]  N.J. Korevaar, R. Schoen,  Sobolev spaces and harmonic maps for metric space targets. Comm. Anal. Geom. 1 (1993), no. 3-4, 561-559.
[Mu]  J.R. Munkres,  Elementary differential topology. Annals of Mathematics Studies, No. 54. Princeton University Press, Princeton, N.J. 1966.
[PRS]  S. Pigola, M. Rigoli, A.G. Setti,  Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds. Math. Z. 258 (2008), no. 2, 347-362.
[SU1]  R. Schoen, K. Uhlenbeck,  A regularity theory for harmonic maps. J. Differential Geom. 17 (1982), no. 2, 307-335.
[SU2]  R. Schoen, K. Uhlenbeck,  Boundary regularity and the Dirichlet problem for harmonic maps. J. Differential Geom. 18 (1983), no. 2, 253-268.
[SY1]  R. Schoen, S.T. Yau,  Compact group actions and the topology of manifolds with nonpositive curvature. Topology 18 (1979), 361-380.
[SY2]  R. Schoen, S.T. Yau,  Lectures on Harmonic Maps. Lectures Notes in Geometry and Topology, Volume II, International Press.
[Ve1]  G. Veronelli,  On p-harmonic maps and convex functions. Manuscripta Math. 131 (2010), no. 3-4, 537-546.
[Ve2]  G. Veronelli,  A global comparison theorem for p-harmonic maps on Riemannian manifolds. J. Math. Anal. Appl. 391 (2012) 335-349
[Wei1]  S.W. Wei,  The minima of the p-energy functional. Elliptic and parabolic methods in geometry (Minneapolis 1994), pp. 171–203. A K Peters, Wellesley (1996).
[Wei2]  S.W. Wei,  Representing homotopy groups and spaces of maps by p-harmonic maps. Indiana Univ. Math. J. 47 (1998), no. 2, 625-670.
[WY]  S.W. Wei, C.-M. Yau,  Regularity of p-energy minimizing maps and p-superstrongly unstable indices. (English summary) J. Geom. Anal. 4 (1994), no. 2, 247–272.
[Wh]  B. White,  Homotopy classes in Sobolev spaces and the existence of energy minimizing maps. Acta Math. 160 (1988), no. 1-2, 1–17.
[Whd]  J. H. C. Whitehead,  On C1-complexes. Ann. of Math. 41 (1940), 809-824.

Dipartimento di Fisica e Matematica, Università dell’Insubria - Como, via Valleggio 11, I-22100 Como, Italy.
E-mail address: stefano.pigola@uninsubria.it

INdAM Fellowships in Mathematics and/or Applications for Experienced Researchers cofunded by Marie Curie, at Département de Mathématiques, Université de Cergy-Pontoise, Site de Saint Martin 2, avenue Adolphe Chauvin 95302 Cergy-Pontoise Cedex France
E-mail address: giona.veronelli@gmail.com