Quench echo and work statistics in integrable quantum field theories

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Quantum quenches

Quantum quench: paradigmatic out of equilibrium protocol

\[ H_0 \] prepares ground state \[ \rightarrow \] \[ H \] evolves system

Experimental progress in ultra cold atoms
- Coherent control, long coherence times
- Can realize paradigmatic model systems (Hubbard, Lieb-Liniger)
- Has become a playground for out of equilibrium quantum physics

One usually looks at evolution of n-point functions
\[ \langle O \rangle(t), \langle O_1(x)O_2(0) \rangle(t) \]

Thermalization? Role of integrability?

Kinoshita, Nature 440 (2006), 900
Quantum thermodynamics

- Treat the quench (or some more generic protocol) as a thermodynamic transformation: tdin variables: work, entropy, heat

- Why is this interesting?
  - e.g. 2nd law $W \geq \Delta F \rightarrow \langle W \rangle \geq \Delta F$
  - $T \neq 0$: fluctuation relations
  - $T = 0$: in TDL: $W$ is extensive and $P(W/V) \rightarrow \delta(\langle W/V \rangle)$

  - interesting part: finite volume
  - quench and system specific information
  - A quantity “close to experiments”

How to define work?

$$H_0 = \sum_n E_n |n\rangle \langle n| \quad H_1 = \sum_m \tilde{E}_m |m\rangle \langle m|$$

- two projective energy meas
- for fast process stochastic

$$P(W) = \sum_{n,m} p_n p(m|n) \delta(W - (\tilde{E}_m - E_n))$$
Results in low D systems

- **Global: done so far:**
  - Essentially free fermion systems (Ising, noisy Ising, Dicke)
    Silva (2008), Paraan-Silva (2009), Smacchia-Silva (2013), Marino-Silva (2014), Gambassi-Silva (2013)
  - Luttinger liquid
    Dora et al. (2012), (2013)
  - Free bosons (weak interaction limit of sine-Gordon, e.g. relative phase of two interacting BEC wires)
    Sotiriadis-Gambassi-Silva (2013)
  - CFT (only Loschmidt)
    Cardy (2014)
  - Spin chains (only Loschmidt)
    Venuti et al. (2011), Fagotti (2013), Pozsgay (2014), De Luca (2014), Andraschko-Sirker (2014)

- **Local quenches**

- **What about interactions? In 1D it can be very important.**
Outline

- Preliminaries
  - Work statistics and Loschmidt echo
  - Partition function, TBA, bTBA
- Analytics
  - multiparticle expansion
  - low energy part
- Numerics
  - avoiding oscillatory integrals
- Conclusions
**Relation to Loschmidt echo**

e.g. in Talkner et al. Phys. Rev. E 75 (2007), 050102(R)

**Characteristic function of the work distribution**

\[
G(t) = \int dW e^{iWt} P(W) = \int dW e^{iWt} \sum_{nm} \delta(W - \bar{E}_m + E_n) \frac{p(m|n)}{|\langle m|U|n\rangle|^2} p_n
\]

\[
= \sum_{nm} \langle m|U e^{-iuH_0} |n\rangle \langle n|U^\dagger e^{iuH_1} |m\rangle p_n = \text{Tr} e^{-iuH_0} e^{iuH_{1,H}} \rho_0 = \langle T e^{iu \int_0^t \frac{\partial H(s)}{\partial \lambda_s} ds} \rangle
\]

Classical result: \( \langle e^{iu \int_0^t \lambda_s \frac{\partial \phi_s(\lambda_s)}{\partial \lambda_s} ds} \rangle_{\text{cl.}} \)

\( \rightarrow \langle \Omega | e^{-iuH_0} e^{iuH_1} |\Omega\rangle = L(u) \)

sudden quench limit, \( T = 0 \)

**Provides a way to measure** \( P(W) \)  
Dorner et al. PRL 110 (2013) 230601
Loschmidt amplitude

\[ L(t) = \langle \Omega | e^{iHt} | \Omega \rangle \]

\[ Z(\tau) = \langle \Omega | e^{-H\tau} | \Omega \rangle \]

Idea: try to find \( Z \) and continue it:

\[ L(t) = Z(-it) \]

E.g. used for dynamical phase transitions by Heyl et al. (2013)
Partition function in 1d

- Finite volume $L$, finite temperature $1/\tau$ (Eucl. time)

$$Z_L(\tau) = \text{Tr} e^{-\tau H_L} = \text{Tr} e^{-L H_{\tau}} = Z_{\tau}(L)$$

two equivalent quantization schemes

TDL in $L$: 

$$e^{-L f(\tau)} \approx \sum_i e^{-\tau E_i(L)} = \sum_j e^{-L \bar{E}_j(\tau)} \approx e^{-L \bar{E}_0(\tau)}$$

free energy density

at finite temperature $\tau$ in TDL

Casimir energy

in finite volume $\tau$

- With boundary: 

$$Z_L(\tau) = \langle \Omega | e^{-\tau H_L} | \Omega \rangle \approx e^{-L \bar{E}_0 \Omega \Omega}(\tau), \quad L \to \infty$$

- Split $f(\tau)$: 

$$f(\tau) = f_b \tau + 2f_s + f_C(\tau)$$

shift norm. shape

$$P(W) = \frac{1}{2\pi} \int dt e^{-iWt} e^{-L f(-it)}$$
Integrability

- Factorized scattering

\[
S(p_2, p_3)S(p_3, p_1)S(p_1, p_2) = S(p_1, p_2)S(p_1, p_3)S(p_2, p_3)
\]

- 1+1 dimensions
- Infinite number of conserved charges
- Particle picture
- Elasticity
- Factorization

- Faddeev-Zamolodchikov operators

Asymptotic states:

\[
|\theta_1 \ldots \theta_n\rangle = Z^\dagger(\theta_1) \ldots Z^\dagger(\theta_n)|0\rangle
\]

\[
E = m \cosh \theta, \quad p = m \sinh \theta, \quad E^2 - p^2 = m^2
\]

\[
Z(\theta_1)Z(\theta_2) = S(\theta_1 - \theta_2)Z(\theta_2)Z(\theta_1)
\]

\[
Z(\theta_1)Z(\theta_2)^\dagger = S(\theta_2 - \theta_1)Z(\theta_2)^\dagger Z(\theta_1) + \delta(\theta_1 - \theta_2)
\]

(free bosons: $S=1$.)
Thermodynamic Bethe Ansatz

- Bethe-Yang equation (constraint):
  \[ e^{i\pi L} \prod_{j \neq i} S(\theta_i - \theta_j) = \pm 1, \quad L \to \infty \]

- Partition function:
  \[ Z = \int [d\rho_r] \exp \left\{ -L \int m\tau \cosh(\theta) \rho_r(\theta) d\theta + S([\rho_r]) \right\} \approx e^{-L f_C(\tau)} \]

  with
  \[ f_C(\tau) = -\frac{m}{2\pi} \int \cosh(\theta) H(\theta), \quad H(\theta) = \log(1 + e^{-\varepsilon(\theta)}), \quad e^{-\varepsilon} = \frac{\rho_r}{\rho_n} \]

  \[ \varepsilon(\theta) = \tau m \cosh \theta + \Phi \ast H(\theta) \]

  \[ y(\theta) = \Phi \ast H(\theta) \]

  a nonlinear integral equation
Boundary TBA

- Integrability preserving boundaries:
  \[ |\Omega\rangle \sim e^{\int K(\theta) Z^\dagger(-\theta) Z^\dagger(\theta) d\theta} |0\rangle \]

- Such initial states can be incorporated into TBA as a rapidity dependent chemical potential

LeClair et al. Nucl. Phys. B 453 (1995), 581

\[ f_C(\tau) = -\frac{m}{4\pi} \int d\theta \cosh(\theta) H(\theta) \]
\[ H(\theta) = \log(1 + |K(\theta)|^2 e^{-2\tau m \cosh\theta - y(\theta)}) \]
Initial states

\[ f_C(\tau) = -\frac{m}{4\pi} \int \cosh(\theta) \log(1 + |K(\theta)|^2 e^{-2\tau m \cosh \theta - y(\theta)}) \]

Depends on \[ Z_0(p)|\Omega\rangle = 0 \] for free theories: \[ |\Omega\rangle \sim e^{\int K(\theta) Z^\dagger(-\theta) Z^\dagger(\theta) d\theta} |0\rangle \]

Sotiriadis et al. Phys. Rev. E 87, 052129 (2013)
Initial states relative to QQs

Sotiriadis et al. Phys. Lett. B 734 (2014), 52

- Quenching from infinite mass:
  no field fluctuations, Dirichlet state:

  \[ |\Omega\rangle \sim e^{\int K_D(\theta) Z^{\dagger}(-\theta) Z^{\dagger}(\theta) d\theta} |0\rangle \]

- In general:
  \[ Z_0(p)|\Omega\rangle = 0 \]
  for zero prequench interaction:

  \[ (E_{0p}\phi(p) + [\phi(p), H]) |\Omega\rangle = 0 \]
  - multiparticle expansion
  - infinite system of integral equations involving form factors
  - \( m_0 \to \infty : K \to K_D K_{\text{free}} \)
  \[ K(0) = 0 \] like fermions
Results
Analytics: multiparticle expansion

bTBA

\[ y(\theta) = \Phi * H(\theta) \]

\[ y(\theta, it) = \int d\theta' \Phi(\theta - \theta') \log(1 + |K|^2 e^{-2mit \cosh \theta' - y(\theta', it)}) \]

\[ = \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{(-k)^\ell}{k!} \int d\theta' \Phi(\theta - \theta') |K(\theta')|^{2k} e^{-2kmit \cosh \theta'} y(\theta', it) = \sum_{n=1}^{\infty} y_n(\theta, it) \]

\[ \mathcal{F}[y_n(\theta, it)](\theta, \omega) \sim \Theta(\omega - 2nm) \]

- multiparticle expansion

Iterative solution

\[ y(0) \equiv 0 \]

\[ y(n) \equiv \Phi * H[y(n-1)] \]

\[ y_1 = y(1),_1 = y(2),_1 = \cdots \]

\[ y_2 = y(2),_2 \]

\[ y_3 = y(3),_3 \]

Lower terms become exact after a few iterations

Similar multiparticle expansions for \( P(W), f_C(W) \)

\[ P(W) = e^{-2f_s} \left[ \delta(W) + \sum_{n=1}^{\infty} p_n(W) \Theta(W - 2nm) \right] \]
Low energy part of $P(W)$

$$P(W) = e^{-2f_s} \left[ \delta(W) + \sum_{n=1}^{\infty} p_n(W) \Theta(W - 2nm) \right]$$

$p_1(W) = \frac{mL}{4\pi} \int d\theta \cosh \theta |K(\theta)|^2 \delta(W - 2m \cosh \theta)$

$p_2(W) = \frac{m^2L^2}{32\pi^2} \int d\theta \int d\theta' \cosh \theta \cosh \theta' |K(\theta)|^2 |K(\theta')|^2 \left[ 1 + \frac{4\pi\delta(\theta - \theta')}{mL \cosh \theta'} - \frac{8\pi\Phi(\theta - \theta')}{mL \cosh \theta'} \right] \delta(W - 2m \cosh \theta - 2m \cosh \theta')$

$W < 4m$: independent of $\Phi$:
2 particles never scatter off each other in infinite volume

**sinh-Gordon model**

$$H = \frac{1}{2} \pi^2(x) + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{m^2}{b^2} (\cosh b\phi - 1)$$

$$S(\theta) = \frac{\sinh \theta - i \sin \frac{\pi B}{2}}{\sinh \theta + i \sin \frac{\pi B}{2}}, \quad B = \frac{b^2}{1 + b^2}$$

- Genuine interaction
- Sine-Gordon with imaginary $b$
- One particle, simplest Toda
Numerics: oscillatory integrals

- $W$ greater, exact solution becomes tedious. What about non-analyticities?
- A numerical approach is needed.
- Complex $\tau$, integrals highly oscillatory
  \[
  y(\theta, \tau) = \int d\theta' \Phi(\theta - \theta') \log(1 + |K|^2 e^{-2m\tau \cosh \theta} - y(\theta', \tau))
  \]
  Alternative formula:
  \[
  f_C(\tau) = -\frac{m}{4\pi} \int d\theta \cosh(\theta) \log(1 + |K|^2 e^{-2m\tau \cosh \theta} \cdot y(\theta, \tau))
  \]
  \[
  \tilde{y} = \tilde{\Phi} \cdot \tilde{H}
  \]
  \[
  \tilde{f}(x) \sim e^{-bx}
  \]
  \[
  \tilde{f}(z) \text{ closest pole}
  \]
- Avoiding this integral:
  \[
  \Phi(\theta) = \Phi_1 e^{-\theta} + \ldots
  \]
  \[
  H(\theta) = H_1 e^{-2\theta} + \ldots
  \]
  \[
  y(\theta) = \Phi_1 \tilde{H}(-i) e^{-\theta} + \ldots
  \]
  \[
  \int d\theta \cosh \theta H(\theta)
  \]
- Alternative formula:
  \[
  f_C(\tau) = -\frac{m}{4\pi} \lim_{\theta \to \infty} \frac{y(\theta, \tau)}{\Phi(\theta)}
  \]
Numerics: sinh-Gordon model

- Trivial phase diagram, no branch cuts?
  No pole contributions picked up? Etc.
- Numerical approach: solve the original bTBA
- Free energy: continuous, Cauchy-Riemann satisfied
  - True analytic continuation

\[
H = \frac{1}{2} \pi^2(x) + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{\pi^2}{b^2} (\cosh b\phi - 1)
\]
\[
S(\theta) = \frac{\sinh \theta - i \sin \frac{\pi B}{2}}{\sinh \theta + i \sin \frac{\pi B}{2}}, \quad B = \frac{b^2}{1 + b^2}
\]

- Genuine interaction
- Sine-Gordon with imaginary b
- One particle, simplest Toda

\[\text{Re } f(\tau) \quad \text{(a)} \quad \text{Im } f(\tau) \quad \text{(b)}\]

\[L = \frac{5}{m}, \ m_0 = 10m\]
Loschmidt echo

- Loschmidt echo per volume

![Graph of Loschmidt echo over time](image)

- $l(t)$
Work statistics - globally

- Extensivity of $W$
  $L = 5, 10, 20, 30, 40, 70, 100$

- Departure from free boson result
Work statistics - details

- Features of free fermion statistics appear
- Exponents change due to interaction, $-1/2$ to $+1/2$
- Not fermionic: there are multiple edges, however less pronounced

$L = 5/m, m0 = 10m$
Conclusions

- Quench performed on an integrable QFT, calculated Loschmidt echo, work statistics
- Nice application of boundary TBA - solving for imaginary temperature
- Low-energy part is given analytically through a multiparticle expansion and is model independent (for identical initial states)
- To perform numerical calculations proved a property of (b)TBA and used it to avoid oscillatory integrals for complex temperatures
- Investigated effects of turning on a genuine interaction term - free boson to sinh-Gordon
- Observed that the details of the statistics change considerably, developing fermionic features