We discuss the definitions of standard clocks in theories of gravitation. These definitions are motivated by the invariance of actions under different gauge symmetries. We contrast the definition of a standard Weyl clock with that of a clock in general relativity and argue that the historical criticisms of theories based on non-metric compatible connections by Einstein, Pauli and others must be considered in the context of Weyl’s original gauge symmetry. We argue that standard Einsteinian clocks can be defined in non-Riemannian theories of gravitation by adopting the Weyl group as a local gauge symmetry that preserves the metric and discuss the hypothesis that atomic clocks may be adopted to measure proper time in the presence of non-Riemannian gravitational fields. These ideas are illustrated in terms of a recently developed model of gravitation based on a non-Riemannian space-time geometry.
1. Introduction

The strong and electroweak interactions between matter are described by an action principle that encodes the strong and electroweak charges into fields carrying representations of certain Lie groups. The fields are coupled together in a manner that renders an action functional invariant under local gauge transformations. To this end additional Yang-Mills gauge connection fields are introduced as sections of principal bundles over space-time. By comparison Einstein’s classical theory of gravitation is formulated in the language of pseudo-Riemannian geometry. The interaction of gravitation and matter is encoded into a field theory in which matter is described by tensor and spinor fields and their derivatives and the gravitational field is expressed in terms of a space-time metric and its derivatives. These fields are usually required to make some action functional extremal. Unlike the Yang-Mills action principle in which the Yang-Mills gauge fields are dynamically independent, the connection used to describe Einsteinian gravity is constrained to be metric-compatible and torsion-free.

The experimental evidence for Einstein’s theory received considerable enhancement with the recent observation of the rate of slowing of the binary pulsar PSR 1913+16. However it appears that certain other astrophysical observations do not rest so easily with classical gravitation. In particular some velocity distributions of stars in galaxies are hard to reconcile with the observed matter distributions if they follow from Newtonian dynamics. Since Einstein’s theory reproduces Newtonian gravity in a non-relativistic weak-field limit this has led some to conjecture that such galaxies may contain significant amounts of dark matter. An alternative explanation is that Newtonian dynamics requires modifications in this context. If this alternative is taken seriously it invites one to consider alternatives to Einstein’s metric theory that may be testable in an astrophysical domain.

Some of the earliest generalisations to Einstein’s theory were entertained by Cartan and Weyl. The former suggested that the Levi-Civita connection used by Einstein remained metric-compatible but relaxed to admit torsion while the latter made an attempt to unify electromagnetism with gravity in terms of a theory based on a non-metric-compatible connection with zero torsion. Although Weyl’s efforts proved abortive, modern string-inspired low energy effective actions for gravity and matter can be formulated in terms of a non-Riemannian connection with prescribed torsion and non-metricity, [1], [2], [3], [4]. It is difficult at present to confront such effective theories with data since they are plagued by an excess of unobservable scalar fields.

A less ambitious approach is to contemplate simple modifications of Einstein’s theory that do not a-priori constrain the geometry to be pseudo-Riemannian and seek actions with extrema that dynamically prescribe the metric and connection for the gravitational field. There exists a large literature that attempts to pursue this programme. One of the earliest efforts to ameliorate the difficulties that arise in the perturbative quantisation of Einstein’s theory of gravity contemplated a connection constrained to be metric-compatible but with torsion expressed in terms of a hypothetical gravitino field. Theories based on actions in which the connection is not constrained in the variational principle have been studied but the actions have often suffered from being either ad-hoc or lacking in motivation. Furthermore, following Weyl’s earlier attempts, theories with a non-metric compatible connection have often been viewed with suspicion, partly as a result of the arguments made against Weyl’s theory by Einstein, Pauli and others [5], [6]. Since a number of authors are actively pursuing non-Riemannian prescriptions [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18] and there is the possibility of confronting these with the predictions of Einstein’s theory in the context of astrophysical data it is worthwhile investigating the nature of these historic criticisms against non-Riemannian descriptions of gravitation. It is the purpose of this note to put the arguments of Einstein and Pauli into perspective and argue that such criticisms need to be used with caution when discussing theories based on non-metric connections. We
exploit the language of modern differential geometry since this offers the most precise way to express the basic notions involved.

2. Non-Riemannian geometry

Let us briefly recall that a non-Riemannian space-time geometry is defined by a pair \((g, \nabla)\) where \(g\) is a metric tensor with Lorentzian signature and \(\nabla\) is a general linear (Koszul) connection. From this pair one can construct \(S = \nabla g\) the gradient of \(g\), \(T\), the torsion tensor and \(R_{X,Y,Z}\), the curvature operator. If \(X, Y, Z\) are arbitrary vector fields on space-time then

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad \text{and} \quad R_{X,Y,Z} = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

If \(\nabla\) is chosen so that \(S = 0\) and \(T = 0\) the geometry is pseudo-Riemannian and the gravitational field is associated with the Riemann curvature tensor \(R\) where \(R(X, Y, Z, \beta) = \beta(R_{X,Y,Z})\) for an arbitrary 1-form \(\beta\). This tensor is then determined solely by the metric and the connection is called the Levi-Civita connection. Just as the Levi-Civita connection of a pseudo-Riemannian geometry can be expressed in terms of \(g\) alone a general \(\nabla\) can be expressed in terms of \(g\), \(T\) and \(S\):

\[
2g(Z, \nabla_X Y) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [X, Y]) - g(X, T(Y, Z)) - g(Y, T(X, Z)) - g(Z, T(Y, X)) - S(X, Y, Z) - S(Y, Z, X) + S(Z, X, Y) \quad (1)
\]

A geometry with non-zero torsion and a non-metric compatible connection is also referred to as a “metric-affine” geometry by some authors [19].

3. Weyl’s Theory

In 1918 Weyl [20], [21], [22], [23] proposed a theory based on an action functional \(S[g, A]\) where \(\nabla\) was a non-Riemannian connection constrained to have

\[
T = 0 \quad (2)
\]

\[
S = A \otimes g \quad (3)
\]

for some 1-form \(A\). Since his action was invariant under the substitutions:

\[
g \rightarrow e^\lambda g \quad (4)
\]

\[
A \rightarrow A + d\lambda \quad (5)
\]

for any 0-form \(\lambda\), this theory determined a class of solutions \([g, A]\); elements being equivalent under what Weyl termed the gauge transformations (4), (5). Classical observables predicted by this theory should be gauge invariant. In Einstein’s pseudo-Riemannian description of gravitation a standard clock is modeled by any time-like curve \(C\) parametrised with a tangent vector \(\dot{C}\) of constant length \(\sqrt{-g(\dot{C}, C)}\). Such a clock can be calibrated to measure proper time \(\tau\) with a standard rate independent of \(C\), by fixing the parametrisation of \(C\) so that

\[
g(\dot{C}, \dot{C}) = -1 \quad (6)
\]

(in a metric with signature \((-\,+,\,+,\,)\)). The notion of a standard clock makes precise the notion of a freely falling observer, namely an affinely parametrised autoparallel (geodesic) integral curve of the Levi-Civita connection. Since such a connection is compatible with a prescribed metric, \((S = 0)\), the normalisation of \(\dot{C}\) is preserved for any \(C\). Thus although the elapsed proper time between events connected by \(C\) is path dependent, any particular standard Einsteinian clock admits a proper time parametrisation independent of its world line. In Weyl’s geometry no particular \(g\) in the class \([g, A]\) is preferred so the identification of a clock as a device for measuring proper time requires more care. The condition (6) is not
invariant under Weyl’s gauge group. However in Weyl’s geometry $T = 0$ and it then follows from (1) that under the transformations (4), (5) Weyl’s connection $\nabla$ remains invariant. Thus if $g$ and $A$ determine $\nabla[g, A]$ and $g = e^\lambda g_1$, $A = A_1 + d\lambda$ then for any $X, Y, Z$:

$$g(Z, \nabla_X[g, A]_1 Y) = g(Z, \nabla_X[g, A]_1 Y).$$  \hfill (7)

A definition of a standard clock should then refer to the gauge invariant connection rather than the gauge non-invariant condition (6). Thus one may model a Weyl standard clock to be a time-like (with respect to any $g$ in $[g, A]$) curve $C$ such that

$$g(\nabla \dot{C}^e, \dot{C}^e) = 0.$$  \hfill (8)

This condition is manifestly gauge invariant under (4) and (5). It follows that for each time-like curve there exists a standard clock parametrisation of $C$ that is unique up to the affine reparametrisation

$$\tau \mapsto a\tau + b$$  \hfill (9)

with real constants $a$ and $b$ [24]. However if $dA \neq 0$ one cannot choose $a = 1$ for all such curves. (If this were possible one could construct a Weyl parallel normalised tangent vector on any closed curve. That this is impossible with $dA \neq 0$ follows by differentiating (6) with $\nabla$.) Thus the relative rates of two such standard clocks depend on their relative histories in general. (This effect should not be confused with the dependence of elapsed time between events produced by the difference in paths linking such events.)

If one assumes that a standard clock in Weyl’s geometry $[g, A]$ corresponds to an atom emitting light of a definite frequency then two identical atoms that diverged from a unique space-time event and returned to any later event, could not have the same frequency at such an event if $\int_{\Sigma} dA \neq 0$, where $\Sigma$ is any world sheet bounded by the world lines of the two atoms. Weyl attempted to identify $F = dA$ with the Maxwell electromagnetic field before the $U(1)$ nature of the coupling to charged fields was recognised. Hence the spectra emitted by atoms in an ambient electromagnetic field would be predicted to depend on their histories contrary to observation. This was the reason that his unified theory of gravitation and electromagnetism fell prey to the early criticisms by Einstein and Pauli. Note however that such criticisms remain valid whether or not $F$ is identified with the electromagnetic field. They rely only on the gauge invariant definition of the time parametrisation of a Weyl standard clock and the assumed correspondence of an atomic spectral line with the rate associated with such a clock.

4. Einsteinian Clocks

It is of interest to compare the above identification of a Weyl standard clock with an atomic clock with that used in Einstein-Maxwell relativity. Since this theory works with a unique classical pair $\{g, F\}$ satisfying the Einstein-Maxwell field equations, where $F$ is the Maxwell 2-form, one identifies the angular frequency $\omega[C_e]$ of an emitted electromagnetic wave from any timelike world-line $C_e$, with the scalar product

$$\omega[C_e] = -g(\dot{C}_e, \dot{\Gamma}_p)|C_e$$  \hfill (10)

where $g(\dot{C}_e, \dot{C}_e) = -1$ and $\dot{\Gamma}_p$ is tangent to an integral curve of the null field $\Gamma$ that intersects $C$ at the event $p$:

$$g(\dot{\Gamma}_p, \dot{\Gamma}_p) = 0.$$  \hfill (11)

If the vector field $\Gamma$ is identified with $g(d\phi, -)$ where $\phi$ is the phase of the emitted electromagnetic wave it follows from the identity

$$(\nabla_X d\phi)(Y) = (\nabla_Y d\phi)(X)$$  \hfill (12)
for any vector fields $X, Y$ and Levi-Civita $\nabla$ that
\[ \nabla_\dot{\Gamma} \dot{\Gamma} = 0 \] (13)
i.e. the integral curves of $\Gamma$ are affinely parametrised null geodesics. It is a basic assumption in Einsteinian relativity that the frequency of an atomic clock does not depend on its world line and so one chooses a parametrisation of $\Gamma$ that keeps the emitted frequency constant. If the emitted electromagnetic wave is detected by some observer world-line $C_o$ with $g(\dot{C}_o, \dot{C}_o) = -1$ the observed angular frequency is
\[ \omega[C_o] = -g(\dot{C}_o, \dot{\Gamma})|_{C_o} \] (14)
which may not be constant along $C_o$. It is traditional to measure the ratio
\[ 1 + z = \frac{g(\dot{C}_e, \dot{\Gamma}_e)}{g(\dot{C}_o, \dot{\Gamma}_o)} \] (15)
since any affine parametrisation of the null geodesic joining the emission event to the observation event can then be employed.

Thus the criticism of Weyl’s theory is essentially based upon the notion used for identifying standard clocks. By contrast to Einstein’s theory which works with a well defined metric, the necessity of making observables class invariant necessitates an alternative definition of a standard clock. The identification of atomic clocks with the parametrisation of curves defined by (8) is responsible for the criticisms made by Einstein, Pauli and others. Such criticisms remain in force however one identifies the metric-gradient field $S$ in Weyl’s theory. We stress that the essence of these criticisms lies in the fundamental gauge symmetry associated with Weyl’s action principle not with the identification of $F$ with the Maxwell field. This symmetry, in turn, follows from Weyl’s particular choice of a non-Riemannian geometry having zero torsion and $S = A \otimes g$.

5. Non-Riemannian Gravitational Fields

Given the success of the gauge description of the Yang-Mills interactions in which the connection associated with any Yang-Mills gauge group is unconstrained in a variational principle, a more natural approach to a non-Riemannian description of gravitation is to seek a purely gravitational action $S[g, \nabla]$ that gives field equations determining a unique metric $g$ without constraining $T$ and $S$. As with the Yang-Mills action principles that dynamically prescribe a class of Yang-Mills connections one may expect such an action to determine a class of Koszul connections $[\nabla]$ rather than a class of pairs $[g, A]$ as in the Weyl’s approach.

The 1-dimensional Weyl group offers perhaps the simplest gauge symmetry for a non-Riemannian action principle. In this approach the action $S[g, \nabla]$ is invariant under
\[ g \rightarrow g \] (16)
\[ \nabla \rightarrow \nabla + df \otimes \] (17)
where $f$ is any 0-form and the action of $\nabla$ on an arbitrary tensor is induced from that given by the action on 1-forms in (17). Note, by contrast to the gauge transformations (4), (5), that the metric tensor remains inert under this transformation. In a fixed local coframe the connection 1-forms $A^c_b$ of $\nabla$ transform according to
\[ A^c_b \rightarrow A^c_b + \delta^c_b df \] (18)
which may be recognised as Einstein’s $\lambda$-transformation [25]. It follows that the curvature scalar $R$ associated with $\nabla$ is invariant under this gauge symmetry so the Einstein-Hilbert
action remains a candidate contribution to such an action. However since the connection is not constrained to be metric-compatible the gauge-invariant 2-form defined by
\[ \text{ric}(X,Y) = \text{Trace}(\mathbf{R}_{X,Y}). \] (19)
is not necessarily zero. In the absence of matter a simple modification of the Einstein-Hilbert action is
\[ S[g, \nabla] = \int (\kappa_1 \mathcal{R} \ast 1 + \kappa_2 \text{ric} \wedge \ast \text{ric}) \] (20)
in terms of the Hodge map \( \ast \) of \( g \) and coupling constants \( \kappa_1, \kappa_2 \). The field equations that render this action extremal \[26\] generate a non-Riemannian geometry with
\[ S = \frac{1}{4} \mathbf{A} \otimes g, \] (21)
\[ T = -\frac{1}{4} (\mathbf{A} \wedge e^a) \otimes X_a \] (22)
where the Weyl 1-form \( \mathbf{A} \) satisfies
\[ d \ast d \mathbf{A} = 0 \] (23)
and \( \{e^a\} \) and \( \{X_b\} \) is a pair of locally dual bases.

Unlike Weyl’s theory however, the metric \( g \) is uniquely determined in any chart by the Einstein equation
\[ \kappa_1 \mathbf{Ein} = T^{\text{ric}} \] (24)
with a stress tensor \( T^{\text{ric}} \) of purely geometric origin:
\[ T^{\text{ric}}_{ab} = -\kappa_2 (4 \text{ric}_{ac} \text{ric}^c_b + g_{ab} \text{ric}_{cd} \text{ric}^{cd}). \] (25)
In fact it follows from (21) and (22) that the Einstein tensor \( \mathbf{Ein} \) in (24) coincides with the Einstein tensor constructed with the Levi-Civita connection and that any traditional electro-vac solution to the Einstein-Maxwell equations will also solve (24). Although the additional gravitational fields \( T \) and \( S \) are not zero the whole theory can be recast in terms of the metric-compatible torsion-free Levi-Civita connection. The gauge symmetry (17), (16) is compatible with the definition of a standard Einsteinian clock to define proper time.

6. Conclusion

In general relativity, the identification of proper time with the time indicated by an atomic clock is often regarded as due to the “local position invariance” that is embodied in the Einstein equivalence principle \[27\]. The mathematical expression of this principle is well known: the laws of physics that include the effects of Einsteinian gravity exploit only the metric of a pseudo-Riemannian geometry. Modulo interactions with curvature they can often be obtained from their special relativistic form by replacing the Minkowski metric by a general pseudo-Riemannian metric and expressing derivatives as covariant derivatives with respect to the associated Levi-Civita connection. However in a generalised theory of gravitation this principle of “universal coupling to the metric” must be abandoned. The nature of the gravitational coupling to matter is dictated solely by an action principle. As a consequence, some matter fields may couple not only to the metric but also to torsion and the metric gradient, violating the above “local position invariance”. In Weyl’s original action principle the gauge symmetry (4), (5) prohibits the identification of the parametrisation (6) with measured time along a worldline since it is not a gauge invariant condition. A standard clock in Weyl’s theory can, however, be constructed with the aid of the gauge invariant condition (8) but such a clock cannot in general be identified with a unique parametrisation for its world line. The criticism’s leveled at Weyl’s theory arose from the hypothesis that standard Weyl clocks could be identified with atomic clocks. In an action principle based on
invariance under the gauge transformations (16), (17) it is the metric that remains invariant rather than the connection so there is nothing to prevent one from defining a standard clock as in Einstein’s theory. If the matter composing an atomic clock has no significant interaction with the torsion or the metric gradient of the non-Riemannian gravitational field then it would be reasonable to identify such a clock with a standard Einsteinian clock.

In both Weyl’s theory and one based on the gauge symmetry (16), (17) an appropriate gauge invariant definition of a standard clock can be made. Only when this is done can the predictions of any theory be compared with the results of measurements made with physical clocks. In Weyl’s theory the adoption of standard Einsteinian clocks is prohibited. On the other hand experiment suggests that an atomic clock offers a robust device for the measurement of time. If space-time admits non-Riemannian gravitational fields based on the gauge symmetry (16), (17) one may experimentally check whether such a device is insensitive to the torsion and metric gradient fields of its local environment and whether it may consequently be modelled by a standard Einsteinian clock. The relevance of these additional gravitational fields to large scale astrophysical problems and their interaction with matter in general is under current investigation.

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