Abstract—Many real world practical problems can be formulated as $\ell_0$-minimization problems with nonnegativity constraints, which seek the sparsest nonnegative signals to underdetermined linear systems. They have been widely applied in signal and image processing, machine learning, pattern recognition and computer vision. Unfortunately, this $\ell_0$-minimization problem with nonnegativity constraint is computational and NP-hard because of the discrete and discontinuous nature of the $\ell_0$-norm. In this paper, we replace the $\ell_0$-norm with a non-convex fraction function, and study the minimization problem of this non-convex fraction function in recovering the sparse nonnegative signals from an underdetermined linear system. Firstly, we discuss the equivalence between $(P_{0,0}^2)$ and $(FP_{a,0}^2)$, and the equivalence between $(FP_{0,0}^2)$ and $(FP_{a,0}^2)$. It is proved that the optimal solution of the problem $(P_{0,0}^2)$ could be approximately obtained by solving the regularization problem $(FP_{a,0}^2)$ if some specific conditions satisfied. Secondly, we propose a nonnegative iterative thresholding algorithm to solve the regularization problem $(FP_{a,0}^2)$ for all $a > 0$. Finally, some numerical experiments on sparse nonnegative signal recovery problems show that our method performs effective in finding sparse nonnegative signals compared with the linear programming.

Index Terms—Compressed sensing, Sparse nonnegative signal, Non-convex fraction function, Equivalence, Nonnegative iterative thresholding algorithm.

I. INTRODUCTION

Many real world practical problems can be formulated as $\ell_0$-minimization problems with nonnegativity constraints, which seek the sparsest nonnegative signals to underdetermined linear systems. They have been widely applied in signal and image processing (see, e.g., [4], [8], [9], [10], [11], [12], [13], [14]), machine learning (see, e.g., [15], [16], [17], [18], [19]), pattern recognition and computer vision (see, e.g., [13], [20]), and so on. The $\ell_0$-minimization problem with the nonnegativity constraint can be modeled into the following minimization

\[ (P_{0,0}^2) \min_{x \in \mathbb{R}^n} \|x\|_0 \text{ subject to } Ax = b, \quad x \geq 0 \]  

where $A$ is a $m \times n$ real matrix of full row rank with $m \ll n$, $b$ is a nonzero real column vector of $m$-dimension, and \( \|x\|_0 \) is the so-called $\ell_0$-norm of real vector $x$, which counts the number of the non-zero entries in $x$ (see, e.g., [1], [2], [3]). In general, the problem \( (P_{0,0}^2) \) is computational and NP-hard because of the discrete and discontinuous nature of the $\ell_0$-norm. A large amount of recent attention is attracted to the following minimization problem

\[ (P_{FP}^2) \min_{x \in \mathbb{R}^n} \|x\|_1 \text{ subject to } Ax = b, \quad x \geq 0. \]

The problem $(P_{FP}^2)$ has shown to be efficient for solving $(P_{0,0}^2)$ in many situations (see, e.g., [21], [22], [23], [24], [25]), especially, evidence in [23], assuming the range space property (RSP) is adopted, the problem $(P_{FP}^2)$ can really make an exact recovery, and any linear programming solver can be used to solve it. However, as the compact convex relaxation of the problem \( (P_{0,0}^2) \), the problem \( (P_{FP}^2) \) may be suboptimal for recovering a real sparse signal.

Inspired by the good performances of the fraction function in image restoration and compressed sensing (see, e.g., [7], [8]), in this paper, we replace the discontinuous $\ell_0$-norm $\|x\|_0$ with a continuous sparsity promoting penalty function

\[ P_a(x) = \sum_{i=1}^{n} \rho_a(x_i), \quad a > 0 \]

where

\[ \rho_a(t) = \frac{a |t|}{a |t| + 1} \]

is the fraction function, and it is increasing and concave in $t \in [0, +\infty]$.}

![Fig. 1. The behavior of the fraction function $\rho_a(t)$ for various values of $a$.](image)

With the change of parameter $a > 0$, the non-convex function $P_a(x)$ interpolates the $\ell_0$-norm

\[ \lim_{a \to +\infty} \rho_a(x_i) = \begin{cases} 0, & \text{if } x_i = 0; \\ 1, & \text{if } x_i \neq 0. \end{cases} \]
Then, we translate problem \((P^2_0)\) into the following minimization problem
\[ (FP^2_a) \quad \min_{x \in \mathbb{R}^n} P_a(x) \text{ subject to } Ax = b, \ x \geq 0 \] (6)
for the constrained form and
\[ (FP^2_{a,\lambda}) \quad \min_{x \geq 0} \left\{ \|Ax - b\|^2 + \lambda P_a(x) \right\} \] (7)
for the regularization form.

The paper is organized as follows. In Section II we establish the equivalences of \((P^2_0)\), \((FP^2_a)\) and \((FP^2_{a,\lambda})\). In section III the nonnegative iterative thresholding algorithm is proposed to solve the regularization problem \((FP^2_{a,\lambda})\) for all \(a > 0\). And the convergence of our algorithm is established in Section IV. In Section V, a series of experiments on some sparse nonnegative signal recovery problems are demonstrated. We conclude this paper in Section VI.

II. EQUIVALENCES OF \((P^2_0)\), \((FP^2_a)\) AND \((FP^2_{a,\lambda})\)

In this section, we first discuss the equivalence between \((FP^2_a)\) and \((P^2_0)\), and then we study the equivalence between \((FP^2_{a,\lambda})\) and \((FP^2_a)\).

A. Equivalence between \((FP^2_a)\) and \((P^2_0)\)

Before our discussion, we give some notions and preliminary results that are used in later analysis.

Definition 1. \((27)\) Given the set of \(m\) simultaneous linear equations in \(n\) unknowns

\[ Ax = b. \] (8)

Let \(B\) be any nonsingular \(m \times m\) sub-matrix made up of columns of \(A\). Then, if all \(n - m\) components of \(x\) not associated with columns of \(B\) are set equal to zero, the solution to the resulting set of equations is said to be a basic solution to \((8)\) with respect to the basis \(B\). The components of \(x\) associated with columns of \(B\) are called basic variables.

Definition 2. \((26)\) Given a \(m \times n\) matrix \(A\) and a \(m\)-dimensional vector \(b\), we define the linear problem is to find non-negative solution \(x \in \mathbb{R}^n\) such that

\[ Ax = b, \ x \geq 0. \] (9)

We denote the problem by \(LP(A, b)\), its solution set by \(SOL(A, b)\) and its feasible set by \(FEA(A, b) = \{x | Ax = b, x \geq 0\}\). A feasible solution to the constraints \((9)\) that is also basic is said to be a basic feasible solution. The solution set \(SOL(A, b)\) often has an infinite number of solutions when it is nonempty.

Definition 3. \((27)\) If one or more of the basic variables in a basic solution has value zero, that solution is said to be a degenerate basic solution.

Definition 4. \((27)\) A point \(x\) in a convex set \(C\) is said to be an extreme point of \(C\) if there are no two distinct points \(x_1\) and \(x_2\) in \(C\) such that \(x = \eta x_1 + (1 - \eta)x_2\) for some \(\eta, 0 < \eta < 1\).

An extreme point is thus a point that does not lie strictly within a line segment connecting two other points of the set. The extreme points of a triangle, for example, are its three vertices (see, e.g., [27]).

Theorem 1. \((27)\) (Equivalence of extreme points and basic solutions) Let \(A\) be an \(m \times n\) matrix of rank \(m\) and \(b\) an \(m\)-dimensional vector. Let \(D\) be the convex polytope consisting of all \(n\)-dimension vectors \(x\) satisfying \((9)\). Then, a vector \(x\) is an extreme point of \(D\) if and only if \(x\) is a basic feasible solution to \((9)\).

Corollary 1. \((27)\) If the convex set \(D\) corresponding to \((9)\) is nonempty, it has at least one extreme point.

Corollary 2. \((27)\) If there is a finite optimal solution to a linear programming problem, there is a finite optimal solution which is an extreme point of the constraint set.

Corollary 3. \((27)\) The constraint set \(D\) corresponding to \((9)\) possesses at most a finite number of extreme points.

Equipped above preliminary results, we shall establish the equivalence of the problems \((FP^2_a)\) and \((P^2_0)\).

By Definition 2 the problems \((P^2_{-1,\lambda})\) and \((FP^2_{a,\lambda})\) could be rewritten as

\[ (SOLP^2_0) \quad \min_{x \in \mathbb{R}^n} \|x\|_0 \text{ subject to } x \in SOL(A, b) \] (10)

and

\[ (SOLFP^2_a) \quad \min_{x \in \mathbb{R}^n} P_a(x) \text{ subject to } x \in SOL(A, b). \] (11)

In particular, we call a solution of \((SOLFP^2_a)\) a least fraction solution.

Full rank assumption: The \(m \times n\) matrix \(A\) has \(m < n\), and the \(m\) rows of \(A\) are linearly independent. Otherwise, we make row transformations simultaneously in both sides of the equation \(Ax = b\), resulting in an equivalent equation \(A_1 x = \hat{b}\) with \(A_1\) being of full row rank.

Lemma 1. All least fraction solutions of the \(LP(A, b)\) are extreme points of \(SOL(A, b)\).

Proof. Let \(x^*\) be a least fraction solution. Suppose there exist \(y, z \in SOL(A, b)\) such that \(x^* = \eta y + (1 - \eta)z\) for some \(0 < \eta < 1\). Recall that \(\rho_0(t)\) is strictly concave for \(t \geq 0\). Then it follows

\[ P_a(x^*) = \sum_{i=1}^{n} \rho_a(x_i^*) \]

\[ = \sum_{i=1}^{n} \rho_a(\eta y_i + (1 - \eta)z_i) \]

\[ \geq \eta \sum_{i=1}^{n} \rho_a(y_i) + (1 - \eta) \sum_{i=1}^{n} \rho_a(z_i) \]

\[ = \eta P_a(y) + (1 - \eta)P_a(z) \]

\[ \geq P_a(x^*) \]

where the last inequality uses that \(x^*\) is a least fraction solution. Furthermore, the above equalities hold if and only if \(y = z = x^*,\) which indicates that \(x^*\) is an extreme point of \(SOL(A, b).\)
By Lemma[1] $x^\ast$ is a extreme point of the polytope set $D$. We denote by $E(D)$ the set of extreme points of the polytope set $D$, and define two constants $r(A,b)$ and $R(A,b)$ as follows 
\[
  r(A,b) = \min_{z \in E(D), z_i > 0, 1 \leq i \leq n} z_i. \tag{12}
\]
\[
  R(A,b) = \max_{z \in E(D), z_i > 0, 1 \leq i \leq n} z_i. \tag{13}
\]
Clearly, the defined constant $r(A,b)$ and $R(A,b)$ are finite and positive due to the finiteness of $E(D)$ and positive of $z_i$.

**Theorem 2.** There exist some constants $\hat{a} > 0$ such that the optimal solution to the problem ($FP_a^{\geq}$) also solves ($P_0^{\geq}$).

**Proof.** Let $\{a_i | i = 0, 1, 2, \cdots\}$ be a increasing infinite sequence with $\lim_{i \rightarrow \infty} a_i = \infty$ and $a_0 = 1$. For each $a_i$, by Lemma[1] the optimal solution $\hat{x}_i$ to the problem ($SOLP_a^{\geq}$) is an extreme point of the polytope set $D$. Since the polytope set $D$ has a finite number of extreme points (see Theorem[1] Corollary[2][3]), one extreme point, named $\hat{x}$, will repeatedly solve the problem ($SOLP_a^{\geq}$) for some subsequence $\{a_{i_k} | k = 1, 2, \cdots\}$ of $\{a_i\}$. For any $a_{i_k} \geq a_i$, and $x \in R^n$, we have 
\[
  P_{a_{i_k}}(x_{i_k}) = \min_{a_{i_k}} P_{a_{i_k}}(x) \leq \|x\|_0.
\]
Letting $i_k \rightarrow \infty$, we have 
\[
  \|\hat{x}\|_0 \leq \|x\|_0.
\]
Hence $\hat{x}$ is the optimal solution to the problem ($SOLP_0^{\geq}$). This proves that there exists some constant $\hat{a} > 0$ such that the optimal solution to the problem ($FP_a^{\geq}$) also solves ($P_0^{\geq}$). \hfill \qed

Furthermore, we have:

**Theorem 3.** There exists a constant $a^\ast > 0$ such that, whenever $a > a^\ast$, every optimal solution to the problem ($FP_a^{\geq}$) also solves ($P_0^{\geq}$), where $a^\ast$ depends on $A$ and $b$.

**Proof.** Let $x^\ast$ be the optimal solution to the problem ($SOLP_a^{\geq}$) and $\bar{x}^\ast$ be the optimal solution to the problem ($SOLP_0^{\geq}$). By Lemma[1] we know that $x^\ast$ is an extreme point of the polytope set $D$.

Therefore, we have 
\[
  \min_{x \in SOL(A,b)} \|x\|_0 = \|x^0\|_0 \geq \sum_{i \in \text{supp}(x^0)} \frac{|a_i|^0}{1 + a_i |x^0_i|} \geq \sum_{i \in \text{supp}(x^\ast)} \frac{|a_i|^\ast}{1 + a_i |x^\ast_i|} \geq \|x^\ast\|_0 \frac{|a_i|^\ast}{1 + a_i |x^\ast_i|},
\]
which implies that 
\[
  \|x^\ast\|_0 \leq (1 + \frac{1}{ar})\|x^0\|_0 = (1 + \frac{1}{ar}) \min_{x \in SOL(A,b)} \|x\|_0.
\]
Because $\|x^\ast\|_0$ is an integer number, from the inequality above, it follows that $\|x^\ast\|_0 = \min_{x \in SOL(A,b)} \|x\|_0$ (that is, $x^\ast$ solves ($SOLP_0^{\geq}$)) when 
\[
  (1 + \frac{1}{ar}) \min_{x \in SOL(A,b)} \|x\|_0 < \min_{x \in SOL(A,b)} \|x\|_0 + 1 \tag{14}
\]
Obviously, the inequality (14) is true whenever 
\[
  a > \min_{x \in SOL(A,b)} \frac{\|x\|_0}{r(A,b)}. \tag{15}
\]
Therefore, with $a^\ast$ denoting the right side of the inequality (15), we conclude that when $a > a^\ast$, every solution $x^\ast$ to the problem ($SOLP_a^{\geq}$) also solves ($SOLP_0^{\geq}$). This proves that whenever $a > a^\ast$, every solution to the problem ($FP_a^{\geq}$) also solves ($P_0^{\geq}$). \hfill \qed

**B. Equivalence between ($FP_{a,\lambda}^{\geq}$) and ($FP_a^{\geq}$)**

In this subsection, we study the equivalence of the regularization problem ($FP_{a,\lambda}^{\geq}$) with the constrained problem ($FP_a^{\geq}$).

**Theorem 4.** Let $\{\lambda_n\}$ be a decreasing sequence of positive numbers with $\lambda_n \rightarrow 0$, and $x_{\lambda_n}$ be a global minimizer of the problem ($FP_{a,\lambda_n}^{\geq}$) with $\lambda = \lambda_n$. If the problem ($FP_a^{\geq}$) is feasible, then the sequence $\{x_{\lambda_n}\}$ is bounded and any of its accumulation points is a global minimizer of the problem ($FP_a^{\geq}$).

**Proof.** By 
\[
  \lambda_n P_a(x) \leq \|Ax - b\|_2^2 + \lambda_n P_a(x),
\]
we can see that the objective function in the problem ($FP_{a,\lambda_n}^{\geq}$) with $\lambda = \lambda_n$ is bounded from below and is coercive, i.e., 
\[
  \|Ax - b\|_2^2 + \lambda_n P_a(x) \rightarrow +\infty \text{ as } \|x\|_2 \rightarrow +\infty,
\]
and hence the set of global minimizers of ($FP_{a,\lambda_n}^{\geq}$) with $\lambda = \lambda_n$ is nonempty and bounded.

By assumption, we suppose that the problem ($FP_a^{\geq}$) is feasible and $\bar{x}$ is any feasible point, then 
\[
  A\bar{x} = b.
\]
Since $\{x_{\lambda_n}\}$ is a global minimizer of the problem ($FP_{a,\lambda_n}^{\geq}$) with $\lambda = \lambda_n$, we have 
\[
  \lambda_n P_a(x_{\lambda_n}) \leq \|Ax_{\lambda_n} - b\|_2^2 + \lambda_n P_a(x_{\lambda_n}) \leq \|Ax - b\|_2^2 + \lambda_n P_a(\bar{x}) = \lambda_n P_a(\bar{x}). \tag{16}
\]
Hence, the sequence $\{P_a(x_{\lambda_n})\}_{\lambda_n \in \Lambda^+}$ is bounded, and the sequence $\{x_{\lambda_n}\}$ has at least one accumulation point. In addition, by inequality (16), we can get that 
\[
  \|Ax_{\lambda_n} - b\|_2^2 \leq \lambda_n P_a(\bar{x}) \text{ for any } \lambda_n \rightarrow 0.
\]
If we set $x^\ast$ be any accumulation point of the sequence $\{x_{\lambda_n}\}$, we can derive that 
\[
  Ax^\ast = b.
\]
That is, \( x^* \) is a feasible point of the problem \( (FP_{a,\lambda}^0) \). Combined with \( P_0(x^*) \leq P_0(\bar{x}) \) and the arbitrariness of \( \bar{x} \), we can get that \( x^* \) is a global minimizer of \( (FP_{a,\lambda}^0) \).

Theorem 2 and 3 demonstrate that the optimal solution to the problem \( (P_0^0) \) can be exactly obtained by solving \( (FP_{a,\lambda}^0) \) if some specific conditions satisfied. Theorem 4 displays that the optimal solution to the problem \( (FP_{a,\lambda}^0) \) can be approximately obtained by solving \( (FP_{a,\lambda}^0) \) for some proper smaller \( \lambda > 0 \).

III. NONNEGATIVE ITERATIVE THRESHOLDING (NIT) ALGORITHM FOR SOLVING \( (FP_{a,\lambda}^0) \)

In this section, the nonnegative iterative thresholding (NIT) algorithm is proposed to solve the problem \( (FP_{a,\lambda}^0) \) for all \( a \geq 0 \). Before we introduce the NIT algorithm, there are some results need to be prepared.

A. Export the NIT algorithm

Lemma 2. Define three threshold values

\[
t^*_1 = \frac{\sqrt{\frac{2}{\pi}} \sqrt{\lambda a^2} - 1}{a}, \quad t^*_2 = \frac{\lambda}{2a}, \quad t^*_3 = \sqrt{\lambda} - \frac{1}{2a}
\]

for any positive parameters \( \lambda \) and \( a \), then the inequalities \( t^*_1 \leq t^*_3 \leq t^*_2 \) hold. Furthermore, they are equal to \( \frac{\lambda}{2a} \) when \( \lambda = \frac{1}{\lambda} \).

Theorem 5. Given any vector \( v \in \mathbb{R}^n \), the thresholding operator \( T_{a,\lambda} : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
T_{a,\lambda}(v) := \arg \min_{x \in \mathbb{R}^n} \{ \| x - v \|_2^2 + \lambda P_a(x) \},
\]

can be expressed as

\[
T_{a,\lambda}(v_i) = \begin{cases} g_{a,\lambda}(v_i), & \text{if } |v_i| > t^*; \\ 0, & \text{if } |v_i| \leq t^*. \end{cases}
\]

where \( g_{a,\lambda}(v_i) \) is defined as

\[
g_{a,\lambda}(v_i) = \text{sgn}(v_i) \left( \frac{1 + \sqrt{4 |v_i|^2 + 3}}{3} - \frac{1}{a} \right),
\]

\[
\phi(t) = \arccos \left( \frac{27 \lambda^2 a^2}{4(1 + a|t|)^2} - 1 \right)
\]

and the threshold function satisfies

\[
t^* = \begin{cases} t^*_2 = \frac{\lambda}{2a}, & \text{if } \lambda \leq \frac{1}{\lambda}; \\ t^*_3 = \sqrt{\lambda} - \frac{1}{2a}, & \text{if } \lambda > \frac{1}{\lambda}. \end{cases}
\]

The proof of Theorem 5 used the Cartans root-finding formula expressed in terms of hyperbolic functions and it is a special case of the reference [6], and the detailed proof can be seen in [5].

Definition 5. Define the thresholding operator \( T_{a,\lambda} \) as a nonlinear analytically expressive operator, and can be specified by

\[
T_{a,\lambda}(x) = (g_{a,\lambda}(x_1), \ldots, g_{a,\lambda}(x_n))^T,
\]

where \( g_{a,\lambda}(x_i) \) is defined in Theorem 5.
(iii) \[ \|x_i - \|_2^2 = 0 \Leftrightarrow x_i = 0 \quad \forall i \in I^-, \]

where the second relation follows from relation (i) and the fact that

\[ (\nabla_+(v))_i = v_i \]

for any \( i \in I^+ \) and

\[ (\nabla_+(v))_i = 0 \]

for any \( i \in I^- \).

From the above facts (i)-(iii), we thus have that \( \bar{x} \in \mathcal{T}_{\alpha,\lambda}(\nabla_+(v)) \) if and only if

\[
\bar{x} = \arg \min_{x \geq 0} \{ \|x - v\|_2^2 + \lambda P_a(x) \}
\]

\[
\begin{aligned}
&= \arg \min_{x \geq 0} \{ \| (x - v)_+ \|_2^2 - \| (x - v)_- \|_2^2 \} + \lambda P_a(x) \\
&= \arg \min_{x \geq 0} \{ \| (x - v)_+ \|_2^2 + \| x - \|_2^2 - 2 \sum_{i \in I^-} x_i v_i \} \\
&\quad + \lambda P_a(x) \\
&= \arg \min_{x \in \mathbb{R}^n} \{ \| (x - v)_+ \|_2^2 + \lambda P_a(x) : x_i = 0 \quad \forall i \in I^- \} \\
&= \arg \min_{x \in \mathbb{R}^n} \| (x - v)_+ \|_2^2 + \lambda P_a(x) : \| x - \|_2^2 = 0 \} \\
&= \arg \min_{x \in \mathbb{R}^n} \{ \| (x - v)_+ \|_2^2 + \| x - \|_2^2 \} + \lambda P_a(x) \\
&= \mathcal{T}_{\alpha,\lambda}(\nabla_+(v)).
\end{aligned}
\]

Now, we show that the optimal solution to the problem \((FP_{\alpha,\lambda}^\geq)\) can be expressed as a thresholding operation.

For any fixed positive parameters \( \lambda > 0, \mu > 0, \alpha > 0 \) and \( x, z \in \mathbb{R}^n \), let

\[ C_1(x) = \|Ax - b\|_2^2 + \lambda P_a(x) \tag{21} \]

and its surrogate function

\[ C_2(x, z) = \mu [C_1(x) - \|Ax - Az\|_2^2] + \| x - z \|_2^2 \tag{22} \]

where \( \mu > 0 \) is a balancing parameter. Clearly, \( C_2(x, x) = \mu C_1(x) \).

**Theorem 7.** For any fixed positive parameters \( \lambda > 0, \mu > 0 \) and \( z \in \mathbb{R}^n \), \( \min_{x \geq 0} C_2(x, z) \) equivalents to

\[ \min_{x \in \mathbb{R}^n} \{ \| x - \nabla_+(B_\mu(z)) \|_2^2 + \lambda \mu P_a(x) \} \tag{23} \]

where \( B_\mu(z) = z + \mu A^T (b - Az) \).

**Proof.** We first notice that, \( C_\mu(x, z) \) can be rewritten as

\[
\begin{aligned}
C_2(x, z) &= \| x - (z - \mu A^T Az + \mu A^T b)_+ \|_2^2 + \lambda P_a(x) \\
&\quad + \mu [b\|_2^2 + \| \|_2^2 - \| A z \|_2^2 - \| z - \mu A^T A z \|_2^2 \\
&\quad + \| A b \|_2^2] \\
&= \| x - B_\mu(z) \|_2^2 + \lambda P_a(x) + \mu [b\|_2^2 + \| \|_2^2 - \| A z \|_2^2 \\
&\quad - \| A b \|_2^2 - \| B_\mu(z) \|_2^2]
\end{aligned}
\]

Combined with Theorem 6, we can get that \( \min_{x \geq 0} C_2(x, z) \), for any fixed \( \mu, \lambda \) and \( z \in \mathbb{R}^n \), equivalents to

\[ \min_{x \in \mathbb{R}^n} \{ \| x - \nabla_+(B_\mu(z)) \|_2^2 + \lambda \mu P_a(x) \}. \]

**Corollary 4.** Let \( x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \) be an optimal solution of \( \min_{x \geq 0} C_2(x, z) \) if and only if, for any \( i, x_i^* \) solves the problem

\[ \min_{x_i \in \mathbb{R}} \{ (x_i - (\nabla_+(B_\mu(z)))_i)^2 + \lambda \mu P_a(x_i) \} \]

**Theorem 8.** For any fixed \( \lambda > 0 \) and \( 0 < \mu < \frac{1}{\|A\|_2^2} \). If \( x^* \) is an optimal solution of \( \min_{x \geq 0} C_1(x) \), then \( x^* \) is also an optimal solution of \( \min_{x \geq 0} C_2(x, x^*) \), that is

\[ C_2(x^*, x^*) \leq C_2(x, x^*) \]

for any \( x \geq 0 \).

**Proof.** Based on the definition of \( C_2(x, z) \), we have

\[
\begin{aligned}
C_2(x, x^*) &= \mu [C_1(x) - \|Ax - Ax^*\|_2^2] + \| x - x^* \|_2^2 \\
&= \mu [\|Ax - b\|_2^2 + \lambda P_a(x)] + \| x - x^* \|_2^2 \\
&\quad - \|Ax - Ax^*\|_2^2 \\
&\geq \mu [\|Ax - b\|_2^2 + \lambda P_a(x)] \\
&= \mu C_1(x) \\
&\geq \mu C_1(x^*) \\
&= C_2(x^*, x^*).
\end{aligned}
\]

Theorem 8 shows that \( x^* \) is an optimal solution to \( \min_{x \in \mathbb{R}^n} C_\mu(x, x^*) \) as long as \( x^* \) is an optimal solution of the problem \((FP_{\alpha,\lambda}^\geq)\). Combined with Theorem 7, we derive the most important conclusion in this paper, which underlies the algorithm to be proposed.

**Corollary 5.** Let \( x^* \) be an optimal solution of the problem \((FP_{\alpha,\lambda}^\geq)\). Then \( x^* \) is also an optimal solution of the following minimization problem

\[ \min_{x \in \mathbb{R}^n} \{ \| x - B_\mu(x^*) \|_2^2 + \lambda \mu P_a(x) \} \]

for any \( x \in \mathbb{R}^n \).

Combining Corollary 5 and Theorem 8, we can immediately conclude that the thresholding operation of the problem \((FP_{\alpha,\lambda}^\geq)\) can be given by

\[ x^* = \mathcal{T}_{\alpha,\mu}(\nabla_+(B_\mu(x^*))) \tag{24} \]

where \( \mathcal{T}_{\alpha,\mu} \) is obtained by replacing \( \lambda \) with \( \lambda \mu \) in \( \mathcal{T}_{\alpha,\lambda} \).

With the thresholding representation (24), the procedure of the NIT algorithm can be naturally defined as

\[ x^{k+1} = \mathcal{T}_{\alpha,\mu}(\nabla_+(B_\mu(x^k))) \tag{25} \]

where \( B_\mu(x^k) = x^k + \mu A^T (b - Ax^k) \).
B. Adjusting values for the regularization parameter

In our algorithm, the cross-validation method is accepted to select the proper regularization parameter. Schwarz in [28] demonstrated that when some prior information is known for a regularization problem, this selection is more reasonably and intelligently.

We suppose that the vector \( x^* \) of sparsity \( r \) is the optimal solution of the problem \((FP_{\alpha}^2)\), without loss of generality, we set

\[
(\nabla_+(B_\mu(x^*))_1 \geq \cdots \geq (\nabla_+(B_\mu(x^*))_r \geq (\nabla_+(B_\mu(x^*))_{r+1} = \cdots = 0. \tag{26}
\]

By Theorem 5 we have

\[
(\nabla_+(B_\mu(x^*))_i > t^* \iff i \in \{1, 2, \cdots, r\},
(\nabla_+(B_\mu(x^*))_j \leq t^* \iff j \in \{r + 1, r + 2, \cdots, n\},
\]

where \( t^* \) is the threshold value defined in (19) obtained by replacing \( \lambda \) with \( \lambda \mu \).

According to \( t_2^* \leq t_2^* \), we can get that

\[
\frac{2(\nabla_+(B_\mu(x^*)))_{r+1}}{a \mu} \leq \lambda \leq \frac{(2a(\nabla_+(B_\mu(x^*)))_{r+1})^2}{4a^2 \mu}. \tag{28}
\]

The inequality (28) helps us to set the strategy in selecting the best regularization parameter, and we denote \( \lambda_1 \) and \( \lambda_2 \) as the left and the right of above inequality respectively:

\[
\begin{cases} 
\lambda_1 = \frac{2(\nabla_+(B_\mu(x^*)))_{r+1}}{a \mu}; \\
\lambda_2 = \frac{(2a(\nabla_+(B_\mu(x^*)))_{r+1})^2}{4a^2 \mu}.
\end{cases}
\]

A choice of \( \lambda \) is

\[
\lambda = \begin{cases} 
\lambda_1, & \text{if } \lambda_1 \leq \frac{t_2^*}{\sigma^2}; \\
\lambda_2, & \text{if } \lambda_2 > \frac{t_2^*}{\sigma^2}. \tag{29}
\end{cases}
\]

Since \( x^* \) is unknown, and \( x^k \) can be viewed as the best available approximation to \( x^* \), a proper choice for the value of \( \lambda \) at \( k \)-th iteration is given by

\[
\lambda = \begin{cases} 
\lambda_1^k = \frac{2(\nabla_+(B_\mu(x^k)))_{r+1}}{a \mu}, & \text{if } \lambda_1^k \leq \frac{t_2^*}{\sigma^2}; \\
\lambda_2^k = \frac{(2a(\nabla_+(B_\mu(x^k)))_{r+1})^2}{4a^2 \mu}, & \text{if } \lambda_2^k > \frac{t_2^*}{\sigma^2}. \tag{30}
\end{cases}
\]

That is, (30) can be used to adjust the value of the regularization parameter \( \lambda \) during iteration.

IV. THE CONVERGENCE OF NIT ALGORITHM

In this section, we present the convergence of NIT algorithm under some specific conditions.

Theorem 9. Let \( \{x^k\} \) be the sequence generated by iteration [25] with the step size \( \mu \) satisfying \( 0 < \mu < \frac{1}{\|A\|_2^2} \). Then

1. The sequence \( \{C_1(x^k)\} \) is decreasing;
2. \( \{x^k\} \) is asymptotically regular, i.e., \( \lim_{k \to \infty} \|x^{k+1} - x^k\|_2 = 0 \);
3. Any accumulation point of \( \{x^k\} \) is a stationary point of the problem \((FP_{\alpha}^2)\).

Algorithm 1 : NIT algorithm

1. Initialize: Choose \( x^0, \mu_0 = \frac{1-\theta}{\|A\|_2^2} \) and \( \alpha \);
2. while not converged do
   \[ z^k : = B_\mu(x^k) = x^k + \mu A^T(y - Ax^k) \]
   \[ \lambda_1^k = \frac{2B_\mu(x^k)_{r+1}}{a \mu}; \quad \lambda_2^k = \frac{(2B_\mu(x^k))_{r+1}^2}{4a^2 \mu}; \]
   if \( \lambda_1^k \leq \frac{t_2^*}{\sqrt{\mu}} \)
      \[ \lambda = \lambda_1^k; \quad t = \frac{\lambda \mu}{2} \]
   else
      \[ \lambda = \lambda_2^k; \quad t = \max\{\sqrt{\mu} - \frac{1}{\sqrt{\mu}}, 0\} \]
   \[ \text{for } i = 1 : \text{length}(x) \]
   1. \( (\nabla_+(B_\mu(x^k)))_i \geq t, x_i^{k+1} = g\lambda \mu (\nabla_+(B_\mu(x^k)))_i; \)
   2. \( (\nabla_+(B_\mu(x^k)))_i \leq t, x_i^{k+1} = 0; \)
   \[ \text{end} \]
   \[ k \rightarrow k + 1 \]
   \[ \text{end while} \]
3. return: \( x^{k+1} \)

Proof. (1) By the proof of Theorem [7] we have

\[ C_2(x^{k+1}, x^k) = \min_{x \geq 0} C_2(x, x^k). \tag{31} \]

Moreover, according to the definition of \( C_2(x, z) \), we have

\[ C_1(x^{k+1}) = \frac{1}{\mu}(C_2(x^{k+1}, x^k) - \|x^{k+1} - x^k\|_2) \]
\[ + ||Ax^{k+1} - Ax^k||_2. \tag{32} \]

Since \( 0 < \mu < \frac{1}{\|A\|_2^2} \), we can get that

\[ C_1(x^{k+1}) = \frac{1}{\mu}(C_2(x^{k+1}, x^k) - ||x^{k+1} - x^k||_2) \]
\[ + ||Ax^{k+1} - Ax^k||_2 \leq \frac{1}{\mu}(C_2(x^k, x^k) - ||x^{k+1} - x^k||_2) \]
\[ + ||Ax^{k+1} - Ax^k||_2 \]
\[ = C_1(x^k) - \frac{1}{\mu}||x^{k+1} - x^k||_2 \]
\[ + ||Ax^{k+1} - Ax^k||_2 \]
\[ \leq C_1(x^k). \tag{33} \]

That is, the nonnegative sequence \( \{x^k\} \) is a minimization sequence of function \( C_1(x) \) for the constraint \( x \geq 0 \), and

\[ C_1(x^{k+1}) \leq C_1(x^k) \]

for all \( k \geq 0 \).

(2) Let \( \theta = 1 - \mu \|A\|_2^2 \) and by the assumption about \( \mu \), we have \( \theta \in (0, 1) \), and

\[ \mu \|A(x^{k+1} - x^k)\|_2 \leq (1 - \theta) \|x^{k+1} - x^k\|_2. \tag{34} \]

By the inequality (33), we can get that

\[ \frac{1}{\mu}||x^{k+1} - x^k||_2 - \|A(x^{k+1}) - A(x^k)\|_2 \leq C_1(x^k) - C_1(x^{k+1}). \tag{35} \]
Combining the inequalities (33) and (35), we have
\[
\sum_{k=1}^{N} \|x^{k+1} - x^k\|_2^2 \leq \frac{1}{\theta} \sum_{k=1}^{N} \|x^{k+1} - x^k\|_2^2 - \frac{\mu}{\theta} \sum_{k=1}^{N} \|Ax^{k+1} - Ax^k\|_2^2 \\
\leq \frac{\mu}{\theta} \sum_{k=1}^{N} (C_1(x^k) - C_1(x^{k+1})) \\
= \frac{\mu}{\theta} (C_1(x^1) - C_1(x^{N+1})) \\
\leq \frac{\mu}{\theta} C_1(x^1)
\]
where the last inequality holds by the fact that the sequence \( \{C_1(x^k)\} \) is decreasing. Thus, the series \( \sum_{k=1}^{\infty} \|x^{k+1} - x^k\|_2^2 \) is convergent, which implies that
\[
\|x^{k+1} - x^k\|_2^2 \to 0 \text{ as } k \to \infty.
\]

(3) Let \( \{x^k\} \) be a convergent nonnegative subsequence of \( \{x^k\} \), and denote \( x^* \) as the limit point of \( \{x^k\} \), i.e.,
\[
x^{ki} \to x^* \text{ as } k_i \to \infty.
\]

Since
\[
\|x^{ki+1} - x^*\|_2 \leq \|x^{ki+1} - x^{ki}\|_2 + \|x^{ki} - x^*\|_2
\]
and
\[
\|x^{ki+1} - x^{ki}\|_2 + \|x^{ki} - x^*\|_2 \to 0 \text{ as } k_i \to \infty,
\]
we have
\[
x^{ki+1} \to x^* \text{ as } k_i \to \infty.
\]

Moreover, by iteration (25), it follows that
\[
x^{ki+1} = \mathcal{T}_{\alpha, \lambda \mu} (\nabla^+ (B_{\mu}(x^{ki}))),
\]
and combined with Corollary 5 we have
\[
\|x^{ki+1} - \nabla^+ (B_{\mu}(x^{ki}))\|_2^2 \leq \|x - \nabla^+ (B_{\mu}(x^{ki}))\|_2^2 + \lambda \mu P_a(x^{ki+1}) \\
\leq \|x - \nabla^+ (B_{\mu}(x^{ki}))\|_2^2 + \lambda \mu P_a(x).
\]
Taking the limit of \( X^{ki+1} \) and using the continuity of \( P_a \) as well as (36) and (37), we can immediately get that
\[
\|x^* - \nabla^+ (B_{\mu}(x^*))\|_2^2 \leq \|x - \nabla^+ (B_{\mu}(x^*))\|_2^2 + \lambda \mu P_a(x).
\]
for any \( x \geq 0 \), which implies that \( x^* \) minimizes the following function
\[
\|x - \nabla^+ (B_{\mu}(x^*))\|_2^2 + \lambda \mu P_a(x),
\]
and we can conclude that
\[
x^* = \mathcal{T}_{\alpha, \lambda \mu} (\nabla^+ (B_{\mu}(x^*)�.
\]

V. Numerical experiments

In this section, we carry out a series of simulations to demonstrate the performance of NIT algorithm. To show the success rate of NIT algorithm in recovering a signal with the different cardinality for a given measurement matrix \( A \), we consider a random matrix \( A \) of size \( 100 \times 256 \) with entries independently drawn by random from a Gaussian distribution of zero mean and unit variance, \( N(0,1) \). By randomly generating some sufficiently sparse nonnegative vectors \( x_0 \), we generate vectors \( b \), and we know the sparsest solution to \( Ax_0 = b \). The stopping criterion is usually as following
\[
\frac{\|x^{k+1} - x^k\|_2}{\|x^k\|_2} \leq \text{Tol}
\]
where \( x^{k+1} \) and \( x^k \) are numerical results from two continuous iterative steps and Tol is a given small number.

The success is measured by computing the relative \( \ell_2 \)-error value
\[
\text{RE} = \frac{\|x^* - x_0\|_2}{\|x_0\|_2}
\]
to indicate a perfect recovery of the original sparse nonnegative vector \( x_0 \). In our experiments, we set to \( \text{Tol} = 1e-8 \), and \( \text{RE} = 1e-4 \). For each experiment, we repeatedly perform 100 tests and present average results.

![Fig. 3. The behaviors of the NIT algorithm for various values of \( a > 0 \).](image)

![Fig. 4. The comparison of the NIT algorithm and linear programming (LP) in the recovery of sparse nonnegative signals.](image)

The graphs presented in Fig. 3 show the success rate of NIT algorithm in recovering the true (sparsest) solution with some different \( a > 0 \), and \( a = 5 \) seems to be the best strategy in our simulations. The graphs demonstrated in Fig. 4 show us that

\[
\square
\]
NIT algorithm can exactly recover the ideal signal until \( r \) is around 40, and linear programming (LP) is around 33.

From Fig. 5 we can see that NIT algorithm always has the smallest relative \( \ell_2 \)-error value, and as we can see, the NIT algorithm (\( a = 5 \)) again has the best performance, with LP as the second.

**VI. Conclusion**

In this paper, we replace the \( \ell_0 \)-norm \( \|x\|_0 \) with a non-convex fraction function in the NP-hard problem (\( P_0^2 \)), and translate this NP-hard problem into the problem (\( FP_{a,\lambda}^2 \)). We discussed the equivalence between (\( FP_{a,\lambda}^2 \)) and (\( P_0^2 \)). Moreover, we also proved that the optimal solution of the problem (\( FP_{a,\lambda}^2 \)) could be approximately obtained by solving its regularization problem (\( FP_{a,\lambda}^2 \)) for some proper smaller \( \lambda > 0 \). The NIT algorithm is proposed to solve the regularization problem (\( FP_{a,\lambda}^2 \)) for all \( a > 0 \). Numerical experiments on sparse nonnegative signal recovery problems show that our method performs effective in finding sparse nonnegative signals compared with the linear programming.

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