Collective Coherence Resonance in Networks of Optical Neurons

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The dynamical properties of an optical neuron formed by a quantum dot semiconductor laser model subjected to optical injection and optical feedback are analyzed. The parameter space spanned by the injection strength and frequency detuning of the optical injection is systematically scanned and modulations of the bifurcation boundaries that induce complex scenarios are found, which enable new opportunities to introduce the optical setup as an optical neuron. The counterintuitive behavior of coherence resonance for different setups of a single-driven optical neuron under optical feedback is also found. Following the results, the microscopically motivated quantum dot laser rate equation model is reduced to the normal form of a saddle-node infinite period (SNIPER) bifurcation for low injection strengths and a network of four such SNIPER systems in a globally coupled setup is studied. A new phenomenon is observed, which is called collective coherence resonance. This novel dynamical behavior is connected to the coexistence of a network-wide SNIPER bifurcation and a change in stability for the synchronized manifold, analyzed via the master stability function.

1. Introduction

The analysis of nonlinear systems has shown surprising and significant dynamics from relatively simple governing rules. Especially in the field of neuroscience, many new behavioral characteristics emerged out of complex dynamics. One of these effects is coherence resonance,[1–4] where a noisy driven nonlinear system close to a bifurcation with a periodic solution exhibits more regular pulsing for a nonzero noise amplitude.

This has been observed for type I (close to a saddle-node infinite period [SNIPER]-bifurcation)[1,3] and for type II (close to a Hopf-bifurcation)[2,6,7] excitable systems.

Since the first analysis of neuronal dynamics by Hodkin and Huxley,[8] a deep and broad insight into the information processing of biological-inspired neural networks has been gained.[9–11] One of these is the information encoding in the digital and analog signal processing.[12] There, coherence resonance is a candidate for a critical dynamical property of the human brain in processing unclear, inaccurate, or new impulses via the influence of noise. Even though coherence resonance has been studied widely for a range of different nonlinear oscillatory systems, investigations with a focus on networks are sparse.[13–15] In this article, we extend the notion of coherence resonance to networks of coupled semiconductor lasers, i.e., a chip-based realization of a neuronal network. To avoid unwanted instabilities, we choose injected quantum dot (QD) lasers as individual nodes. These nanostructured lasers are known for their high tolerance to optical perturbations.[16–20] QDs with optical injection show highly complex behavior[21–26] and noise-induced spiking[27–31] and thus are promising candidates for implementing optical neurons[32–35] for optical computing.[36] We show that a collective coherence resonance is achieved when in parameter space a network-wide SNIPER bifurcation aligns with a neutral stability line obtained via the master stability function, i.e., the largest Lyapunov exponent of transversal stability of the synchronized solution is close to zero.

Our results yield insight into important characteristics of integrated photonics for the application of optical computing[36] based on semiconductor hardware implementation.[32–35] We explore to what extent a QD laser network can be seen as an optical neural network and to what limit the approximations of the governing equation to a simple SNIPER normal form are valid. This article extends our earlier work[37] by the very intriguing case of collective coherence resonance in a network.

The article is structured as follows. In Section 2 and 3, we will introduce the model for the QD laser with optical injection and feedback and the model for network dynamics, respectively. In Section 4, the results for an optically driven QD laser and its ability to experience coherence resonance are presented. In Section 5, we elaborate on the influence of a network on the dynamics, calculate the stability of the synchronized...
manifold through the master stability function, and show that collective coherence resonance is achievable at special points, where SNIPER bifurcations coalign with the neutral stability line of the master stability function.

2. QD Laser and Network

For the dynamics of the QD laser with injection and feedback (see Figure 1c), we use the microscopically based semiclassical rate-equation model introduced in the study by Lingnau et al. The equations of motion are given by

\[ E = (g - i\omega - \kappa)E + K_{\infty}|E|^2e^{-i(\omega_0t + \Phi)} + K_b|E|^2E(t - \tau) + D_\xi \]  

(1)

\[ \rho_{GS,b}^{\text{act}} = -\frac{g}{\hbar}E^f\delta_{GS,b} + \rho_{GS,b}^{\text{cap,act}} + \rho_{b}^{\text{rel,act}} \]  

(2)

\[ \rho_{ES,b}^{\text{act}} = -\frac{g}{\hbar}E^{\text{inj}}\delta_{ES,b} + \rho_{GS,b}^{\text{cap,act}} + \rho_{b}^{\text{rel,act}} \]  

(3)

\[ \rho_{ES,b}^{\text{cap}} = -\frac{g}{\hbar}\left(\rho_{ES,b}^{\text{cap}} - \frac{1}{2}(f^{\text{act}}\rho_{b}^{\text{rel,act}} + f^{\text{inj}}\rho_{b}^{\text{rel,act}})\right) \]  

(4)

\[ \alpha_b = -2N_{QD}(f^{\text{act}}\rho_{GS,b}^{\text{cap}} + f^{\text{inj}}\rho_{GS,b}^{\text{cap}}) - 4N_{QD}\rho_{ES,b}^{\text{cap}} - R_{\text{loss}}w_iw_h + j \]  

(5)

with the slowly varying amplitude of the complex electric field \( E \), the occupation probabilities of the resonant QDs in the ground state \( \rho_{GS,b}^{\text{act}} \), the off-resonant QDs in the ground state \( \rho_{GS,b}^{\text{inact}} \), the excited state \( \rho_{ES,b}^{\text{act}} \), and the 2D carrier densities \( \alpha_b \) as dynamic variables with \( b = \{ e, h \} \) describing electrons and holes, respectively (Figure 1a shows the corresponding band structure).

For the sake of notation and convenience, the dependence of the dynamic variables on time is omitted, and the time derivatives of the quantities are denoted with a dot above the variable. \( K_{\infty}|E|^2e^{-i(\omega_0t + \Phi)} \) models the injection of the QD laser via an optical driving laser, with \( K_{\infty} \) being the injection strength normalized to the amplitude output \( |E|^2 \) of the solitary QD laser, \( c^{-i(\omega_0t + \Phi)} \) describes the phase shift due to the frequency detuning \( \Delta\omega_{\text{inj}} \) between driven and driving laser. \( K_b|E|^2E(t - \tau) \) denotes delayed optical self-feedback with the feedback strength \( K_b \) and the feedback phase shift \( e^{i\Phi} \), \( \kappa \) is the cavity loss rate (2\( \kappa \) is the inverse photon lifetime), and \( D_\xi \) denotes the contributions of spontaneous emission, with \( \xi \) modeling Gaussian white noise with zero mean \((\langle \xi(t) \rangle = 0) \) and delta correlation, \((\langle \xi(t)\xi(t') \rangle = \delta(t - t') \)).

The complex linear gain is given by an amplitude gain \( g \) and the change in the instantaneous frequency \( \delta\omega \). Both are described by linear equations that model their dependencies on the charge carrier occupation probabilities in the QD and carrier densities in the surrounding well. Note that the instantaneous frequency shift models an amplitude phase coupling that dynamically depends on the laser state (a nonconstant \( \alpha \)-factor).

\[ g = g_{\text{GS}}(\rho_{GS,e}^{\text{act}} + \rho_{GS,h}^{\text{act}} + 1) \]  

(6)

\[ \delta\omega = \delta\omega_{\text{ES}}(\rho_{ES,e}^{\text{act}} + \rho_{ES,h}^{\text{act}}) + \delta\omega_{Q}^{\text{QW}}\omega_h + \delta\omega_{Q}^{\text{QW}}\omega_h \]  

(7)

\( R_{\text{cap,act}}^{(\text{inj})} = W_{\text{m}}r_{\text{m}}^{(\text{inj})}r_{\text{m}}^{(\text{inj})} \) describes the spontaneous recombination losses of the QD carriers as a phenomenological rate, with \( m = \{ \text{GS, ES} \} \). \( R_{\text{loss}}\omega_{\text{inj}}\omega_{\text{inj}} \) is a phenomenological rate that takes into account spontaneous emission losses as well as defect.

![Figure 1](https://www.advancedsciencenews.com/articles/2100345/72100345_2.png)

**Figure 1.** (a) Sketch of the energy band structure of the QD laser and the corresponding dynamical variables. \( \rho_{\text{GS,b}} \) describes the occupation probability of the ground state; \( \rho_{\text{ES,b}} \) the excited state where \( b = \{ h, e \} \) describes holes and electrons; and \( \omega_{\text{QW}}, \omega_h \) the quantum well (QW) valence band. (b) Sketch of the QD laser under optical injection with the injection strength and detuning \( K_{\text{inj}} \) and \( \Delta\omega_{\text{inj}} \), respectively. (c) Sketch of the QD laser with optical injection and feedback with the feedback parameters \( \tau, \Phi, \) and \( K_b \) modeling the delay time, the feedback phase, and the feedback strength, respectively. (d) Sketch of the all-to-all coupled network of optical neurons without self-feedback. The coupling parameters \( \tau, \Phi, \) and \( K \) correspond to the network coupling parameters.
and Auger recombination in the QW. \( J \) represents the external pump current density per QW layer. The microscopically based Boltzmann scattering rates between the different energy states are given by

\[
S^{\text{cap, in act}}_{\alpha, \beta} = S^{\text{cap, in}}_{\alpha, \beta}(1 - \rho_{\alpha, \beta}^{\text{in act}}) - S^{\text{cap, out}}_{\alpha, \beta} \rho_{\alpha, \beta}^{\text{in act}} \quad (8)
\]

\[
S^{\text{rel, in act}}_{\alpha} = S^{\text{rel, in}}_{\alpha}(1 - \rho_{\alpha, \beta}^{\text{in act}}) - S^{\text{rel, out}}_{\alpha}(1 - \rho_{\alpha, \beta}^{\text{in act}}) \rho_{\alpha, \beta}^{\text{in act}} \quad (9)
\]

They contain all in- and out-scattering processes between the charge carriers. The nonlinear in-scattering rates are approximated by their full microscopically calculated rates to reduce the computational effort; the parameters can be taken from Appendix Table C1. For information, see Lingnau et al.\(^{[38]}\)

For the case of weak injection and no feedback, we can simplify the QD model to a SNIPER normal form, where only two dynamical variables remain, i.e., the rescaled real and imaginary parts of the electric field amplitude. This is done for all four lasers in the network later on. A sketch of the phase space projection before and after the bifurcation as well as the intermitent hybrid state at the SNIPER point is shown in Figure 2a–c together with example time-series in (d) and (e) below and above the bifurcation, respectively. If the system is in the regime of a saddle node or a saddle (slightly before the bifurcation point Figure 2a), a perturbation can kick it across the saddle, yielding an excursion through phase space along the unstable manifold of the saddle which connects to the stable direction of the node. As a result, a noise-induced spike is detected in the amplitude of the dynamical variables (Figure 2d).

3. Network

In this section, we will introduce a general description of a network of dynamical nodes.

As the QD lasers will be operated in a network, we write a general notation of a network of dynamical units via

\[
x_n(t) = f_n[x_n(t)] + \sum_{m=1}^{N} K_{nm} \omega_{nm} x_m(t - \tau_{nm})
\]

with the current node (QD laser) state of the \( n \)th dynamical node \( x_n(t) \), the nonlinear vector field \( f_n \) given by the right-side of the QD laser rate equation model, describing the local dynamics of the \( n \)th node, the feedback-coupling \( K_{nm} \), the number of nodes (in our case \( N = 4 \)), the entries \( \omega_{nm} \) of the adjacency matrix \( A \in \mathbb{R}^{N \times N} \), the optical coupling scheme \( H_{\text{opt}} \in \mathbb{R}^{d \times d} \), with \( d \) being the number of dynamical variables, and the time delayed \( \tau_{nm} \) node state of the \( m \)th laser \( x_m(t - \tau_{nm}) \), which is directly coupled into the \( n \)th laser. Examplarily, the dynamical state of the 10D (\( d = 10 \)) system for the \( n \)th node is given by Equation (1)–(5) and reads

\[
x_n^T = (\Re(E_n), \Im(E_n), \rho_{GS,n}^{\text{act}}, \rho_{GS,n}^{\text{inact}}, \rho_{ES,n}^{\text{act}}, \rho_{ES,n}^{\text{inact}}, \omega_{x_n} \rho_{x_n})
\]

where the additional subscript \( n \) of each quantity denotes that these are dynamical variables belonging to the \( n \)th dynamical node of the network and the superscript \( T \) denotes the transpose of the vector \( x_n \).

In our setup, we choose all nodes to be identical dynamical systems, which are identically coupled, thus \( f_n \equiv f, H_{\text{opt}} \equiv H, \tau_{nm} \equiv \tau, K_{nm} \equiv K \). The network adjacency matrix \( A \) of our setup of coupled neurons is given by

\[
A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\]
describing a coupling of the electric fields via a rotational matrix and no coupling between the dynamical variables of the carriers, where $\Phi$ is the rotation of the $n$th electric field relative to the $n$th electric field, given via the coupling $e^{i\Phi}$ in Equation (1).

4. Optical Neuron Described by a QD Laser with Optical Injection and Self-Feedback

We will now look at the dynamics of an optically driven solitary QD laser under the influence of optical feedback and the impact of noise on a state close to a SNIPER bifurcation, showing the statistical property of coherence resonance.

First, we give some insight into the behavior of an optically driven solitary laser without any optical feedback. A 2D bifurcation diagram in the parameter plane spanned by the injection strength $K_{\text{inj}}$ and the detuning $\Delta_{\text{inj}}$ of the optical driven system is shown in Figure 3. Figure 3a shows the dynamical properties of cw intensity output or, in the case of more complex dynamical behavior, the number of unique maxima. Figure 3b shows the period of periodic solutions if existent, while Figure 3c shows example time-series and phase space plots for all dynamical regimes. They are indicated by numbers in Figure 3a. We see that even for the optically driven solitary QD laser without any feedback, a rich zoo of dynamical behaviors emerges and yields many possibilities of operating regimes, from cw solutions over periodic solutions to chaotic dynamics. An important point for the injected setup is that a locking cone exists, where the frequencies of the driving laser and the driven laser lock and thus a stable constant intensity output exists.$^3$ Two bifurcations bound this locking cone, one being a Hopf-bifurcation for positive detunings $\Delta_{\text{inj}}$ and larger $K_{\text{inj}}$ and the other being a SNIPER-bifurcation, which will be of interest (marked by the dashed line in Figure 3). The bifurcations were detected by analyzing the periods of the dynamical solutions in the parameter plane.

Before we look at the delay-coupled network of optical neurons, we will shortly discuss the effect of delayed self-feedback on the solitary QD laser. This is helpful to understand the new dynamics induced by the delay and to decide for a suitable delay length. The influence of optical feedback is shown in Figure 4, again plotted in the same 2D bifurcation diagram as in Figure 3. The panels show the results for increasing delay times from $\tau = 0.08$ ns up to $\tau = 0.8$ ns. The constant intensity output is encoded in blue if a cw regime is present and otherwise the number of unique intensity maxima indicated in yellow red color code. With increasing $\tau$ more stable intensity solutions (external cavity modes [ECMs]) are born and increase the level of multistability.$^{18,41}$ The observed regions of complex dynamics emerge when the first ECM looses its stability in a Hopf bifurcation. As soon as a new ECM is available, the laser chooses this solution and stable emission is observed again. The center of the different cones coincides with the frequency of an ECM. If compared with the solitary laser frequency their frequency is modified which is why the locking condition is reached for negative detunings (the blue cone moved downward). With increasing feedback strength $K_B$, the stability of the ECMs decreases$^{42}$ which narrows the locking tongues, while their number increases (they are born in saddle node bifurcations due to the feedback, see previous studies$^{42-45}$). For recent reviews on the feedback-induced dynamic, see Lüdge and Lingnau.$^{43}$

![Figure 3](image-url)  
**Figure 3.** Dynamics of the injected QD laser: a,b) 2D bifurcation diagrams, where the color codes the steady-state intensity of continuous wave (cw) solutions (blue color), the number of unique maxima of periodic solutions (yellow to red), the period of the oscillating solutions (violet), and complex or aperiodic dynamics with many unique intensity maxima (gray). The black dot at $(K_{\text{inj}} = 0.1, \Delta_{\text{inj}} \approx -1.2588 \text{ GHz})$ is close to a SNIPER bifurcation, while the red dot at $(K_{\text{inj}} = 0.3, \Delta_{\text{inj}} \approx 2.647 \text{ GHz})$ is close to a Hopf bifurcation. The labels 1–5) at the blue crosses mark the positions at which the time-series in panel (c) are computed. The dashed lines in panel (a) and (b) show the SNIPER bifurcation. Parameters otherwise can be found in Table C1.
We will concentrate on low to intermediate feedback times of $\tau = 8 \text{ ps}$ up to $\tau = 80 \text{ ps}$ corresponding to a delay length of $l = 2.4 \text{ mm}$ up to $l = 24 \text{ mm}$ in air, where $l$ is the external cavity length. We do this because this gives a high probability for finding bifurcations that can induce coherence resonance phenomena without inducing to highly complex dynamical regimes.

To show that already the optically driven solitary laser without feedback exhibits coherence resonance, we simulated the QD laser in the cw regime close to the SNIPER bifurcation. At these parameter values, the laser behaves like an excitable optical neuron that emits self-similar spikes after optical perturbations (see time-series in Figure 2d). The results are shown in Figure 5. Three different noise strengths $D$ are applied with values of $D \approx 10^{-3}$, $D \approx 10^{-2}$, and $D \approx 10^{-1}$. Their corresponding time-series are shown in Figure 5a–c. Because of perturbations that kick the systems state beyond the saddle point, excursions of the phase space trajectory result in spiking amplitudes of the intensity. The higher the noise strengths, the more spikes are observed because higher noise values have a higher probability of kicking the system beyond the saddle point. It is a known idea that information encoding in the brain is done digital via the spike itself and analog via the period between spike arrival times.[12] Thus, an interesting observation is the evolution of the regularity of spike interval times $T_{isi}$ (see Figure 2d for its definition) as a function of the noise strength $D$. We measure this regularity via the normalized standard deviation $R_T$ given by

![Figure 4. 2D bifurcation diagrams for an optically injected QD laser with optical self-feedback at a feedback phase $\Phi = 0$ and a feedback strength $K_{fb} = 0.05$ for different self-feedback delay times $\tau$, in a) $\tau = 0.08 \text{ ns}$, b) $\tau = 0.3 \text{ ns}$, c) $\tau = 0.8 \text{ ns}$. The areas with blue color code represent the steady-state intensity of cw solutions, while the discrete yellow red color code refers to the number of unique intensity maxima of oscillating solutions. Parameters are given in Table C1. Masked line indicates the SNIPER bifurcation, evaluated by the period of the solutions in the parameter plane.](image)

![Figure 5. a–c) Noise-induced spiking time-series of an optically injected QD laser in the excitable regime, with $D = 10^{-3}$, $D = 10^{-2}$, and $D = 10^{-1}$, respectively (blue lines). The time-series are taken near the SNIPER bifurcation, marked with a black dot in Figure 3 at an injection strength of $K_{inj} = 0.1$ and for three different spontaneous emission noise strengths at an injected detuning of $\Delta \nu_{inj,crit} \approx -1.25866 \text{ GHz}$, $\Delta \nu_{inj,crit} \approx -1.25865 \text{ GHz}$, and $\Delta \nu_{inj,crit} \approx -1.2571 \text{ GHz}$. The red line shows the deterministic trajectory. (d) Normalized standard deviation $R_T$ of the interspike intervals for the solitary injected QD laser plotted over the noise strength $D$. Parameters are given in Table C1.](image)
\[ R_T = \sqrt{\frac{\langle T_{int}^2 \rangle - \langle T_{int} \rangle^2}{\langle T_{int} \rangle}}. \] (14)

The results as a function of \( D \) are shown in Figure 5d. The typical coherence resonance phenomenon can be seen, i.e., that the regularity of the interspike intervals \( T_{int} \) is highest (lowest normalized standard deviation \( R_T \)) at a finite noise strength \( D \).

5. Network Dynamics and Collective Coherence Resonance

Next, we discuss whether noise can also have a positive effect on the regular spiking of a network of optical neurons. Consequently, we will take a look at a network of coupled injected QD lasers, which we operate close to the SNIPER bifurcations and which thus behave like optical neurons. First, we will give a few examples of observed network dynamics.

Figure 6a is a 2D bifurcation diagram in the coupling strength \( K \) and coupling phase \( \Phi \) parameter plane, indicating the emergence of complex behavior for phases in between \( \Phi = 1/2\pi \) and \( \Phi = 3/2\pi \). As before, blue color encodes the cw intensity, while the yellow red colors show the number of unique maxima for periodic or aperiodic solutions. Figure 6b shows the different network states the system can exhibit. These are additionally portrayed in Figure 6c1–c6. Figure 6c1 shows a synchronized equilibrium state, which indicates that all four lasers emit a constant intensity. Figure 6c2 shows a “cluster 13” state, where all four lasers operate on a limit cycle with periodically varying intensity. At the same time, three are synchronized on the same limit cycle, while the fourth laser exhibits a higher intensity varying limit cycle solution. Figure 6c3 is a “cluster 22” state, very similar to the “cluster 13” state, where now two lasers operate in the same limit cycle regime. In contrast, the other two synchronously operate at a different limit cycle regime. Figure 6c4 is a chimera state in this very small network system, where two lasers are synchronous while the other two are desynchronized.\(^{[46]}\) Chimera states are a known, and much looked into subject of networks of identical nodes with strong coupling symmetry.\(^{[47]}\) Figure 6c5 shows a splay state, where all four lasers operate on the same limit cycle, synchronized with a phase shift of \( \pi/2 \). Finally, Figure 6c6 is a complete desynchronized more complex solution. These rich and intriguing dynamics observed in a network of coupled not-injected lasers will now be additionally enhanced by the zoo of interesting and complex dynamics with optical injection.

To corroborate our findings we give again the 2D bifurcation diagram in the parameter space spanned by the injection strength \( K_{\text{inj}} \) and frequency detuning \( \Delta f_{\text{inj}} \) for a delay time of \( \tau = 0.08\text{ns} \), a coupling phase of \( \Phi = \pi \) and a coupling strength of \( K = 0.05 \), as shown in Figure 7. For the network of the optically driven QD lasers, we see a similar bifurcation diagram, as we have already seen for an optically driven QD laser with optical feedback in Figure 4. In addition, the locking cone shifted to positive detuning frequencies and a broad range of complex behavior emerged, especially for negative detunings. Figure 7b shows insight into the complex network behavior that was already introduced in Figure 6. We see a wide variety and fast changing dynamical properties of the network for small changes in the parameter plane. Figure 7b was achieved by starting the network in a synchronized state and after a transient period a small perturbation was applied to the system. Giving the system a little bit time to react to the perturbation, the system state was analyzed and plotted, showing either stable or unstable synchronization. Figure 7c shows the largest Lyapunov exponent for the transversal stability of the synchronized manifold, indicating the

![Figure 6](image_url)
Figure 7. Dynamics of the injected QD laser network for a coupling strength $K = 0.05$, a coupling phase $\Phi = \pi$ and a delay time $\tau = 0.08$ns: a,b) 2D bifurcation diagrams, where the color codes the steady-state intensity of cw solutions (blue color code), the number of unique maxima of periodic solutions (yellow, red), the period of the oscillating solutions (violet), and complex or chaotic dynamics with many unique intensity maxima (gray). c) Largest transversal Lyapunov exponent of the synchronized manifold, describing the stability of the synchronized solution toward small perturbations. The dashed lines in panel (a,b) show the SNIPER bifurcation. Parameters are given in Table C1.

stability of the synchronized manifold toward small perturbations. The largest transversal Lyapunov exponent is calculated via the master stability function. For details, see Appendix B. Mostly the locking cone with cw output is stable (green colors in Figure 7c), while also a small range of more complex behavior can exist as a stable synchronized solution.

From all these different cases, we see that the network of four globally coupled optically driven QDs exhibits very rich dynamics. Out of these dynamical regimes we are interested in the excitable regions because our focus lies on collective coherence resonance, where we assume constant carrier densities. In the final part of the article, we reduce our QD laser model for small injection strength $K_{inj}$. This leads us to equations that are similar to a SNIPER normal form,\(^1\) whose dynamical variables are given by the real and imaginary parts of the complex electric field.

The SNIPER normal form is given by

$$
\dot{x} = [1 - (x^2 + y^2)]x + (x - b)y
$$

$$
\dot{y} = [1 - (x^2 + y^2)]y + (x - b)x
$$

where $x$ and $y$ describe the dynamical variables and $b$ is a control parameter. As already introduced in Section 2 and shown in Figure 2, a SNIPER bifurcation is a global bifurcation describing the collision of a saddle and a stable node, resulting in a period solution.

The full dynamical system describing the time evolution of the real and imaginary values of the electric field in the network is given by

$$
\dot{E}_{x,n} = (1 - E_{x,n}^2 - E_{y,n}^2)(E_{x,n} - \hat{b}E_{y,n}) + \mathcal{K} + E_{y,n}\Delta
$$

$$
+ D\delta_{x,n} + K\sum_{j\neq n}\cos(\Phi - \kappa\tau\Delta)E_{x,m}(t - \tau)
$$

$$
- \sin(\Phi - \kappa\tau\Delta)E_{y,m}(t - \tau)
$$

$$
\dot{E}_{y,n} = (1 - E_{x,n}^2 - E_{y,n}^2)(E_{y,n} + \hat{b}E_{x,n}) - E_{x,n}\Delta + D\delta_{y,n}
$$

$$
+ K\sum_{j\neq n}\sin(\Phi - \kappa\tau\Delta)E_{x,m}(t - \tau)
$$

$$
+ \cos(\Phi - \kappa\tau\Delta)E_{y,m}(t - \tau)
$$

where $E_{x,n}$ and $E_{y,n}$ are the real and imaginary parts of the electric field for the $n$th node, $\hat{b}$ and $\mathcal{K}$ are given by the approximation of the local dynamics of the injected QD laser, $E_{x,n}, E_{y,n}$ are a real Gaussian random variable, $D$ is the noise amplitude, $K$ is the coupling strength $\Phi - \Delta\kappa\tau$ is a phase shift resulting from the laser-network setup and the travel time between two lasers, where $\Delta$ is the rescaled frequency detuning between injecting laser and QD laser. Details on the model reduction can be found in.\(^{[37]}\) In short, the new parameters for this reduced model are

$$
\mathcal{K} = \frac{K_{inj}|E_0|}{\tau_{inj}k}, \quad \Delta = \frac{2\pi\Delta_{inj}}{k}, \quad \delta = \frac{\delta_{inj}}{k}
$$

where $\delta_{inj}$ is the steady-state instantaneous frequency shift and $|E_0|$ is the amplitude of the electric field for the nonperturbed solitary QD laser and $\tau_{inj}$ is one cavity round trip in the solitary QD laser system.

In Figure 8 we plot the same parameter plane already introduced in Figure 7, for the reduced network model. Remembering that our approximation was done for small injection strengths $K_{inj}$ and comparing the results in Figure 8 with the ones in Figure 7, we see that we have similar behavior close to the small
locking cone starting at around \( \Delta \nu_{\text{inj}} \approx 1 \text{GHz} \) and \( K_{\text{inj}} \approx 0.01 \). The small choice of the delay time for the reduced model is given by the fact that it is crucial to avoid multistability, which otherwise could result in perturbations kicking the system to another attractor, destroying the possibility for collective coherence resonance. This was often observed for other parameters, especially for higher delays; thus, we chose the setup given in Figure 8 as a proof of concept. In Figure 8 we additionally plotted the red dashed contour line, showing where the synchronized manifold changes its stability. We see in Figure 8a that this neutrally stable line coaligns with the network-wide SNIPER bifurcation (dashed blue line).

We will now operate the full network at the parameter values shown with a red star in Figure 8 while adding noise to the system. An example time-series is shown in Figure 9a. We see the spiking behavior resulting from the kick of the system beyond the saddle point, from which an excursion in the phase space follows. The exciting part at especially this parameter setup is the combination of perturbation-induced spiking behavior with the on–off switching of the stability of the synchronous manifold. From this, we see a high spike for one specific laser, while all other three exhibit similarities to an inhibitory behavior; thus, desynchronization is achieved for a short time, while synchronization returns after the pulsing behavior ends (we color-coded one time-series as black and the other as red, blue, and orange to highlight this behavior). Interestingly, such a setup exhibits a combination of collective spiking while not giving the trivial case of complete synchronization for the full system for all time, which would be identical to a single dynamical system.

Figure 8. Dynamic regimes found for the reduced model, Equation (18) and (17), in the injection parameter space \((K,\Delta)\) with a coupling strength \( K = 0.05 \), a coupling phase shift of \( \Phi = \pi \) and a time delay of \( T = 8 \text{ps} \). a) \( \text{cw} \) intensity in blue color code or number of unique maxima in yellow to red. b) Different system states numerically measured after perturbations to the synchronized state. c) Value of the largest transversal Lyapunov exponent. In all three plots, the neutrally stable line is additionally depicted as a red contour line showing the border where the synchronized manifold changes its stability. The red star indicates the parameters used for the coherence resonance simulation shown in Figure 9. The blue dashed line indicates the SNIPER bifurcation detected from the evolution of the period.

Figure 9. a) Example time-series showing the collective behavior for the network of four globally coupled QD laser (see Figure 1) operated close to a SNIPER bifurcation at \( D = 10^{-4} \). b) Normalized variance \( R_T \) and autocorrelation time \( \tau_{\text{corr}} \) for an injection strength of \( K = 0.129 \), a detuning of \( \Delta = 0.145 \), a coupling strength of \( K = 0.05 \), a phase shift of \( \Phi = \pi \), and a time delay of \( T = 8 \text{ps} \) plotted in dependence of the noise strength \( D \). Different markers indicate the four different lasers of the network. The collective coherence resonance curves can be seen with an optimum nonzero noise strength of \( D = 10^{-4} \).
Starting from the dynamics in Figure 9a we will look at the regularity of spiking intervals $T_{\text{isi}}$ by checking the standard deviation $R_I$ introduced in Equation (14) for each laser separately (red markers and lines) of the spiking intervals and additionally the autocorrelation time $\tau_{\text{corr}}$ (black markers and lines) for all four lasers, where the autocorrelation time is given by

$$\tau_{\text{corr}} = \int_0^\infty \left| \frac{(I(t) - \langle I \rangle)(I(t + \tau) - \langle I \rangle)}{(I(t) - \langle I \rangle)^2} \right| \, d\tau$$

where $\langle I \rangle$ is the average of the intensity, i.e., $I = |E|^2$. Both quantities are shown in Figure 9b plotted over the noise strength $D$, where all four different lasers are plotted with different markers. We see an apparent minimum of the normalized standard deviation $R_I$, i.e., maximum of the autocorrelation time $\tau_{\text{corr}}$ for a finite value of the noise strength $D$. We thus achieve a collective coherence resonance of the network of dynamical nodes without setting up the system in the trivial state of complete synchronization. This is a very intriguing detail setting the operation point of the optically driven QD lasers close to the SNIPER bifurcation in analogy to the operation of optical neurons.

### 6. Conclusion

We performed numerical simulations of a microscopically motivated optically driven QD laser subjected to optical feedback. We have found that the well-known locking region of the optically driven QD laser is shifted and deformed in its parameter range by the influence of the optical feedback and we showed that the phenomenon of coherence resonance can be found close to the SNIPER bifurcation. We further investigated a system of four such optically driven QD lasers connected in a global network setup and have investigated the different dynamical regimes. We found highly complex behavior ranging from synchronized cw solutions to desynchronized aperiodic behavior. By operating a single injected QD laser at small injection strengths, we were able to reduce the whole network setup to four coupled SNIPER normal forms, showing similar results compared with the full system. We observed a new collective phenomenon which we call collective coherence resonance. There, all four lasers of the network exhibit the effect of coherence resonance, without being either completely synchronized or completely unsynchronized. This effect of noise induced optimal coherent spiking is found at parameter values where a network wide SNIPER bifurcation coaligns with a stability change in the synchronized manifold. The reduction of the full 10D QD laser model to the normal form of a SNIPER bifurcation supports the notion of an optical neuron. It also provides an explanation for the observed occurrence of counterintuitive dynamical phenomenon of coherence resonance. In fact, coherence resonance was first reported in the SNIPER model subject to white noise. Over the years, coherence resonance has received much attention in studies of neuronal, excitable systems. Investigations on coherence resonance have also been extended in the presence of delay and to coupled elements and neuronal networks. Moreover, this nonlinear dynamical effect has been linked to biological relevance and direct implications for perception.

The realization of the neuronal dynamics by means of optically active QD lasers offers new experimental pathways for excitatory dynamics mimicking activity in the brain, but on a much faster time scale than in the nervous system. For networked systems, this opens the door for further investigations of the impact of delayed coupling and time-delayed feedback control on coherence resonance both experimentally and numerically.

### Appendix A: QD Scattering Rates

The QD scattering rates, i.e., the Coulomb charge carrier scattering processes from Lingnau, are fitted by

$$S^{\text{cp,lin}}(\omega_b) = \frac{A_{m,b} \omega_b^2}{B_{m,b} + \omega_b}$$  \hspace{1cm} (A1)

$$S^{\text{cel,lin}}(\omega_b) = \frac{C_{m,b} \omega_b}{D_b + \omega_b}$$  \hspace{1cm} (A2)

with the fit parameters $A_{m,b}$, $B_{m,b}$, $C_b$, and $D_b$ given in Table C1. The out-scattering rates are given by the detailed balance relationships. They depend on the confinement energies $E_{m,b}$, and the quasi-Fermi energies of electrons and holes $E_{\text{Fb}}$ are given in

$$S^{\text{cp, out}}_{m,b} = S^{\text{cp, in}}_{m,b} \exp \left( - \frac{E_{Fm,b}^2 - E_{b}}{k_b T} \right)$$  \hspace{1cm} (A3)

$$S^{\text{cel, out}}_{b} = S^{\text{cel, in}}_{b} \exp \left( - \frac{E_{\text{ES},b} - E_{\text{GS},b}}{k_b T} \right)$$  \hspace{1cm} (A4)

$$E_{\text{Fb}}(\omega_b) = k_b T \log \left[ \exp \left( \frac{10^{12} \omega_b}{\rho_b k_b T} \right) - 1 \right]$$  \hspace{1cm} (A5)

where $E_{\text{Fb}}$ is the quasi-Fermi energy. See Figure 1 for the definition of the energies and the scattering processes.

### Appendix B: Master Stability Function

The transversal Lyapunov exponent of the master stability function, introduced in Pecora and Carroll, yields the stability of the synchronous manifold of a network of identical dynamical nodes toward small perturbations. We would like to calculate the stability of the synchronous manifold of our network and thus give a short recap on the master stability function here. Writing the dynamics of the synchronized Network in vector notation yields

$$x_i(t) = F[x_i(t)] + aKA \otimes Hx_i(t - \tau)$$  \hspace{1cm} (B1)

where $x_1(t) = \ldots = x_4(t) = x_i(t)$, the row sum of the adjacency matrix $a = 3$, $A$ being the adjacency matrix, $H$ being the coupling scheme, and denoting the Hadamard product. By transforming the system into the eigendirections of the adjacency matrix via

$$A = SDS^{-1} \iff D = S^{-1}AS$$  \hspace{1cm} (B2)

with the matrix $S$ that diagonalizes $A$, formed by the eigenvectors of $A$ as column vectors and the diagonal matrix $D$ with the eigenvalues of $A$ on the main diagonal. Following the scheme introduced in Pecora and Carroll and improved for networks of
delay in Lehnert,\cite{59} we then linearize the system at the synchronous state \( x_s(t) \), with which we arrive at

\[
\delta X(t) = \begin{bmatrix} Df_{x_1}(t) & 0 & 0 & 0 \\ 0 & Df_{x_2}(t) & 0 & 0 \\ 0 & 0 & Df_{x_3}(t) & 0 \\ 0 & 0 & 0 & Df_{x_4}(t) \end{bmatrix} \delta X(t) + aKk \begin{bmatrix} \lambda_0 \mathbf{H} & 0 & 0 & 0 \\ 0 & \lambda_1 \mathbf{H} & 0 & 0 \\ 0 & 0 & \lambda_2 \mathbf{H} & 0 \\ 0 & 0 & 0 & \lambda_3 \mathbf{H} \end{bmatrix} \delta X(t - \tau) \tag{B3}
\]

with the zero matrix \( 0 \in \mathbb{R}^{4 \times 4} \) and the eigenvalues \( \{\lambda_k\}_{k \in \{0,1,2,3\}} \) of the adjacency matrix \( A \), which appear on the main diagonal due to the diagonalization of \( A \). Equation (28) is equivalent to \( n \) blocks of the form

\[
\delta x_k(t) = Df_{x_k}(t) \delta x_k(t) + \lambda_k aKH \delta x_k(t - \tau) \tag{B4}
\]

where \( \{\delta x_k\}_{k \in \{0,\ldots,N\}} \) are \( d \)-dimensional perturbations of the synchronous state in the eigensystem of \( A \) and \( Df_{x_k}(t) \) is the Jacobian of the system evaluated at the synchronous state \( x_s(t) \). The eigenvalues of \( A \), i.e., the topological eigenvalues, are for our system either \( \lambda_0 = 1 \) (longitudinal eigenvalue) or \( \lambda_{1,2,3} = -1 \) (transversal eigenvalues) and the corresponding Lyapunov exponents determine the stability of the synchronization manifold, either longitudinal for \( \lambda = 1 \) or transversal for \( \lambda = -1 \) to the synchronous state. We will therefore analyze the largest Lyapunov exponent in the transversal direction.

### Appendix C: Parameters

Table C1 shows the parameters used for the simulations of the microscopically motivated QD laser rate equation system if not given otherwise in the article.

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### Conflict of Interest

The authors declare no conflict of interest.

### Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

### Keywords

coherence resonance, master stability function, nonlinear dynamics, optical neurons, photonic

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[1] G. Hu, T. Ditzinger, C. Z. Ning, H. Haken, Phys. Rev. Lett. 1993, 71, 807.
[2] A. Pikovsky, J. Kurths, Phys. Rev. Lett. 1997, 78, 775.
[3] E. Mompo, M. Ruiz-Garcia, M. Carretero, H. T. Grahn, Y. Zhang, L. L. Bonilla, Phys. Rev. Lett. 2012, 108, 106805.
[4] Y. Kato, H. Nakao, New J. Phys. 2021, 23, 043018.
[5] R. Aust, P. Hövel, J. Hizanidis, E. Schöll, Eur. Phys. J. Spec. Top. 2010, 187, 77.
[6] N. B. Janson, A. G. Balanov, E. Schöll, Phys. Rev. Lett. 2004, 93, 010601.
[7] A. G. Balanov, N. B. Janson, E. Schöll, Physica D 2004, 199, 1.
[8] A. L. Hodgkin, A. F. Huxley, J. Physiol. 1952, 117, 500.
[9] A. E. Runnova, A. E. Hramov, V. V. Grubov, A. A. Koronovskii, M. K. Kurovskaya, A. N. Pisarchik, Chaos Solitons Fractals 2016, 93, 201.
[10] A. Pisarchik, R. Jaime-Reategui, D. Magallón-García, C. Castillo-Morales, Biol. Cybernet. 2014, 108, 397.
[11] S. Kelso, Dynamic Patterns: The Self-Organization of Brain and Behavior, MIT Press, Cambridge, MA 1995, pp. 257–257.
[12] S. J. Thorpe, in Parallel Processing in Neural Systems and Computers (Eds: G. Hartmann, R. Eckmüller, G. Hauske), Elsevier, North Holland 1990, pp. 91–94.
[13] A. V. Andreev, V. V. Makarov, A. E. Runnova, A. N. Pisarchik, A. E. Hramov, Chaos Solitons Fractals 2018, 106, 80.
