Nonlocal total variation regularization with Shape Adaptive Patches for image denoising via Split Bregman method

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Abstract. Non-Local or patch-based approaches are used in most of the state-of-the-art methods for image denoising. Denoising with regular square patches may cause noise halos around edges, particularly high contrasted edges. In order to overcome this drawback, this work presents an extension of the Non-Local TV framework that effectively exploits the potential local geometric features of the image by replacing the simple square patches with different oriented shapes. We first solve the denoised models of ROF-NLTV with different oriented patch shapes by Split Bregman algorithm. Then, use exponentially weighted aggregation method based on Stein’s unbiased risk estimate to combine the estimators obtained in the first step. The numerical results show our method outperforms some previous ones. Moreover, common noise halos around edges usually observed by denoising with Non-Local TV method are reduced thanks our approach.

Key words: Image denoising; Nonlocal total variation; Split Bregman algorithm; aggregation

1. Introduction

Image denoising is a fundamental and challenging problem that has been studied for decades. In the classical Rudin-Osher and Fatemi (ROF) denoising model \cite{1}, total variation (TV) regularization term is introduced to stabilize the solution. This model has proved to be capable of handing properly edges and removing noise in a given image. However, TV based methods do not preserve the fine structures and textures.

In recent years, Non-Local or patch-based approaches are used in most of the state-of-the-art methods for image denoising such as NL-Means \cite{2}, BM3D \cite{3}, NL-Bayes \cite{4}. These methods are quite efficient at dealing with smooth and textures. Inspired by the Nonlocal denoising method based on patch-distances which proposed by Buades, B. Coll, and J.-M. Morel in \cite{2}, G. Gilboa and S. Osher proposed Non-Local TV regularization framework \cite{5}, and changed the TV regularization in ROF model into Non-Local TV. In practical application, NLTV was proved better than TV for improving the signal-to-noise ratio \cite{6, 7, 8}, and it not only inherits the advantages of preserving the edge and detail information of TV, but also reduces the block effect and degradation of image quality caused by TV regularization.

Most of Non-Local methods (including NLTV method) consist in averaging similar pixels by measuring their similarity over the whole image through fixed square shape and fixed scale. Denoising with regular square patches may cause noise halos around edges, particularly high contrasted edges. In order to overcome this drawback in NLM denoising, Charles-Alban Deledalle, V. Duval, and J.
Salmon [9] replace square patches with various shapes (discs, band, half-discs, quarte-discs) and combine their estimations using Stein’s unbiased risk estimatme. Vadim Fedorov, C Ballester [10] use the affine invariant patch similarity measure to detect more similar patches in NLM denoising, this in turn leads to well-suited for denoising along contracted edges. Thus, the two methods above do not suffer from the halo artifacts.

The aim of this paper is to reduce the noise halos artifacts in ROF-NLTV denoising model by effectively exploits the potential local geometric features of the image. The patches fixed in the original NLTV for measuring similarity are changed into small half-pie sets with various directions. Then, the images restored under different patches by the Split-Bregman algorithm have no halo effect at the edges of their different geometrical features. Finally, the weighted aggregation methods can be used to restore more accurate images than any of the estimators obtained with different shape patches.

The remainder of the paper is organized as follows. In Section 2, basic works. In Section 3, we give the proposed method. In Section 4, we present the experimental results and analysis. In the final section, conclusion.

2. Basic works

2.1. Non-Local TV denoising model

We use the additive white Gaussian noise (AWGN) model for the noisy image \( u : u(i) = f(i) + \epsilon(i) \), where \( \epsilon \) is independent Gaussian noise with known variance \( \delta^2 \), \( f \) is noise free image, and any \( i \) in the grid \( \Omega \) \((\Omega \subset \mathbb{Z}^2)\). The Non-Local ROF denoising model proposed by G. Gilboa and S. Osher as follows:

\[
 u^* = \arg\min_u \| u \|_{NLTV} + \mu \| u - f \|_2^2
\]  

(1)

The Non-Local gradient-based functional \( \| u \|_{NLTV} \) is defined as:

\[
\| u \|_{NLTV} = \sum_i \sqrt{\sum_j w_{ij}(u(i) - u(j))},
\]  

(2)

where \( w_{ij} \) is the weight between the two square patches \( v_i, v_j \) of size \( p \times p \) (\( p \) is an odd number) centered on pixel coordinates \( i, j \) \((i, j \in \Omega)\). \( w_{ij} = \frac{1}{z_i} e^{-\frac{|v_i - v_j|^2}{\alpha^2}} \), \( z_i = \sum_j e^{-\frac{|v_i - v_j|^2}{\alpha^2}} \) is the normalizing constant. \( \| v_i - v_j \|^2_{L_2} \) is the distance between the two patches with a Gaussian kernel \( G_\alpha(\tau) = \frac{1}{4\pi\alpha^2} e^{-\frac{\|\tau\|^2}{2\alpha^2}} \) \((\alpha > 0 \text{ is the standard deviation})\):

\[
\| v_i - v_j \|^2_{L_2} = \sum_{\tau} G_\alpha(\tau)|v_i(i+\tau) - v_j(j+\tau)|^2.
\]

In non-local TV denoising model, \( w_{ij} \) is computed based on square patch. Under the NLTV constraint, the pixel value at coordinates \( i \) is \( \sum_j (u(i) - u(j))w_{ij} \). Details about \( w_{ij} \) are described below according to Figure 1. First, add borders to the noisy image as shown in Figure 1(b). Then, the patch \( v_i \) is the black square region of the noise map which is centered on the coordinates \( i \) (red dot) as shown in Figure 1(c) and \( v_i = \left[ u(i+\tau), \tau \in \left[ -\frac{p-1}{2}, \frac{p-1}{2} \right] \right] \), where \( p \times p \) is the patch sizes. The search window is the white square area as shown in Figure 1(d), and the patch \( v_j \) is centered on the pixel point in the search window with coordinates \( j \) \((j = 1, 2, \cdots, t \times t)\), search window sizes: \( t \times t, t > p \). For example, the gray area shown in Figure 1(e) is the first \( v_j \) patch in the search.
window. In the special case of edge, as shown in Figure 1(f), (g), (h), no details here. So, the pixels with a similar grey level neighborhood to \( v_i \) have larger weights in the average obviously.

Figure 1. Non-Local Operation with square patches.

2.2. Split Bregman Algorithm for Non-Local ROF model
Goldstein Osher introduced the Split Bregman iteration method and applied it to the ROF image denoising model [11]. Zhang-Burger-Bresson-Osher extended the Split bregman technique for Non-Local TV regularization problems in [12], and Xavier Bresson showed that this method is better than Steepest Descent [5], projection algorithm of Chambolle [13]. A Non-Local Split Bregman version for problem (1) is as shown in Table 1.

Table 1. Split Bregman Algorithm for Non-Local ROF model, see [14] for more details.

| Inputs:    | noisy image \( f \). Parameters: Search window width \( t \), patch width \( p \), \( \mu \) and \( \beta \). Output: estimated image \( u \). Initialization: \( k = 0 \); \( n = 0 \); set \( u \), \( d \) and \( b \) as zero matrices of image size. |
| For (External iteration \( k = 0,1,\ldots,K \)); For (Internal iteration \( n = 0,1,\ldots,N \)); Update the solution: |
| \( u_{k+1} = \frac{1}{\mu + \beta \sum_j w_{ij}} (\beta \sum_j w_{ij} k_{ij} + \mu f_i + \beta \sum_j \sqrt{w_{ij} (d_{ij} k_{ij} - d_{ij}^{k+1} - b_{ij}^{k+1} + b_{ij}^{k+1})}, u_k^{k+1} = u_k \) |
| End for; Update \( d \), which is the approximation of \( \nabla_N u = \sum_j \left( u(i) - u(j) \right) \sqrt{w_{ij}} : |
| \( d_{ij}^{k+1} = -\frac{\sqrt{w_{ij} (u_{ij}^{k+1} - u_{ij})} + b_i^{k} + \sum_j w_{ij} \left( u_{ij}^{k+1} - u_{ij} \right)^2 + (b_{ij}^{k})^2 - \beta, 0} \) |
| Update the Bregman variable \( b : b_{ij}^{k+1} = b_{ij}^{k} + \sqrt{w_{ij} (u_{ij}^{k+1} - u_{ij}^{k+1}) - d_{ij}^{k+1}} \) |
| End for; Note: In experiments, \( N = 2 \) is enough to determine a good approximation. |

2.3. Aggregation method by SURE
Multiple denoising results can get if square patches changed to multiple shapes. Then how to aggregate these estimates and preserve their unique performance is an aggregation optimization problem. The problem can be described as a linear combinatorial optimization problem [15]: Let \( u^* \) be the closest to \( \tilde{f} \), \( \Lambda \subset \mathbb{R}^n \), and \( u_1, u_2, \ldots, u_n \) are given \( n \) denoising estimates, linear combination of functions \( u_1, u_2, \ldots, u_n \) with coefficients from \( \lambda \in \Lambda \):

\[
\lambda^* = \arg \min_{\lambda} \left\| f - \sum_{k=1}^n \lambda_k u_k \right\|_2^2
\]

(4)

Our goal is to find, given \( u_1, u_2, \ldots, u_n \), a combination \( \sum_{k=1}^n \lambda_k u_k \) with \( \lambda \in \Lambda \) which is nearly as close to \( \tilde{f} \) as \( u^* \). There are many aggregation methods to get the approximate solution of this problem, see [16, 17, 18], Dimitri V. proposed to use Stein’s unbiased risk estimate (SURE) to monitor the MSE of the NLM algorithm for restoration of an image corrupted by additive white Gaussian noise. Charles-
Alban et al used the SURE method to the aggregation of NLM-SAP denoiser in [9], they compute the risk map based on Stein’s unbiased risk estimate as follow:

\[ r_k(i) = (u_k(i) - u(i))^2 + 2\delta \frac{\partial u_k(i)}{\partial \varepsilon(i)} - \delta^2, \]  

where \( r_k(i) \) is an unbiased estimate of the risk at pixel \( i \) for the \( k \)-th shape-based estimate \( u_k \). The closed-form of expression of \( \frac{\partial u_k(i)}{\partial \varepsilon(i)} \) for NLM is

\[ \frac{\partial u_k(i)}{\partial \varepsilon(i)} = \frac{1}{\sum_j \omega_j} \left( \omega_i + \sum_j \omega_j \frac{\partial w_i(j)}{\partial \varepsilon(j)} - \sum_j \omega_j \frac{\omega_j}{\omega_i} \frac{\partial w_i(j)}{\partial \varepsilon(j)} \right), \]  

and

\[ \frac{\partial w_i(j)}{\partial \varepsilon(j)} = \frac{1}{h^2} (S_k(i)[u(i) - u(j)] + S_k(2i - j)[u(i) - u(2i - j)]). \]  

The distance between the two patches with a Gaussian kernel mentioned in section 2.1 can be restated if the shapes are square:

\[ \| v_i - v_j \|^2_{L^2} = \sum_{\tau} S(\tau) \| v_i(i+\tau) - v_j(j+\tau) \|^2, \quad S(\tau) = \begin{cases} \frac{1}{4\pi\alpha^2} e^{-\frac{|\tau|^2}{4\alpha^2}} & \text{if } \|\tau\|_\infty \leq \frac{p-1}{2}, \\ 0 & \text{otherwise} \end{cases} \]  

Lastly, according to exponentially weighted aggregation method proposed in [19], the coefficients \( \lambda_k \) in problem (4) are:

\[ \frac{\exp(-\text{risk}_k(i)/T)}{\sum_{k=1}^n \exp(-\text{risk}_k(i)/T)}, \]  

where \( T > 0 \) is the smoothing parameter.

### 3. Proposed method: the ROF-NLTV-SAP model and its solution

![Figure 2](image-url)  

**Figure 2.** Noise halo in high contrasted edge of the image denoised by ROF-NLTV. The first row: the origin image and the four origin image’s areas (the yellow frame area). The second row: the denoised areas by NLTV-ROF method where noise halos exist more obviously compared with the origin image’s areas.

Our main concern is to address the noise halo (as show in Figure 2) around high contrasted edges on images denoised by ROF-NLTV model. The traditional Non-local methods based on patches takes
pixels as the center of patches (see Figure 3(a)). Eccentric patches method is used in [9] for NLM method to find more similar patches. We introduce it in NLTV regularization methods, the advantages are shown in Figure 3.

![Centered patches](image1)
![Centered weights](image2)
![Eccentric patches](image3)
![Eccentric weights](image4)

**Figure 3.** Centered and eccentric patches around in high contrasted edges. If choose the centered patches (a) to estimate the pixel value at the red dot, the little pixel at blue dots (b) that similar with the pixel at red dot are take into consider. Contrary, there will be more similar blue dot (d) used to estimate the value at red dot if with eccentric patches (c).

![Shapes](image5)

**Figure 4.** Square, disk and half-pie shapes: right-half pie, up-half pie, left-half pie, down-half pie.

There are many shapes to replace the square patch considered in [9], such as disk, half-pies, quarter pies, bands. The simplest patches set of combinations with disk shape, half-pie as shown in Figure 4, which had been showed could make the best solution of the NL-Means model than other shapes. In NLTV methods, we choose the square shapes and the four half-pie shapes for more accurate result according to experiment experience. If the shapes are disks, the $S(\tau)$ in formula (8) without Gaussian kernel can be written as: $S(\tau) = 1$, if $\|r\|_2 \leq \frac{D-1}{2}$, or 0 otherwise.

![Noise Halo](image6)

**Figure 5.** The noise halo in each geometrical direction by ROF-NLTV with different oriented patch shapes. From left to right: the denoised areas with the patch shape of right-half pie, up-half pie, left-half pie and down-half pie by ROF-NLTV respectively, where the noise halos are reduced.

The high contrasted edges on images have many directions, such as lateral, vertical, oblique, edge, etc. We show that the noise halo in each geometrical direction will be reduced by replacing square patches with different oriented patch shapes in ROF-NLTV model (see Figure 5). Based on this, we first compute the ROF-NLTV model based on different shapes:

$$u_1 = \arg \min_u \|u\|_{NLTV-shape_1} + \mu \|u - f\|_2^2, \quad u_2 = \arg \min_u \|u\|_{NLTV-shape_2} + \mu \|u - f\|_2^2, \cdots,$$

$$u_n = \arg \min_u \|u\|_{NLTV-shape_n} + \mu \|u - f\|_2^2.$$

(9)
Then, we use aggregation method based on risk estimates to combine the initial estimators which have different good performance in different direction edges, \( u^* = \sum_{k=1}^{n} \lambda_k u_k \), where \( \lambda_k \) is the adaptive selection coefficient based on the Stein’s unbiased risk map \( r_k \) introduced in section 2.3. Lastly, the ROF-NLTV model based on shape adaptive patches we called it ROF-NLTV-SAP can write as:

\[
(u^*) = \min_{u} \left\{ \frac{1}{2} \sum_{k=1}^{n} \lambda_k \left( \frac{1}{2} \| u_{NLTV-shape_k} - \mu_t f \|_2^2 + \| u_{NLTV-SAP} - \mu_t f \|_2^2 \right) \right\}
\]

The new nonlocal total variation regularization (NLTV-SAP) \( \| u \|_{NLTV-SAP} \) can be used either in image denoising or in image reconstruction. The solution of the new model (9) as shown in Table 2.

### Table 2. The solution for ROF-NLTV-SAP model.

**Inputs:** noisy image \( f \). **Parameters:** Search window width \( t \), patch width \( p \), \( \mu \) and \( \beta \). **Output:** estimated image \( u^* \).

**Start:** Build different shapes as shown in Figure 4. For \( k = 1, 2, \cdots, K \)

- Calculate the distance (8) with shape \( k \), and calculate \( \partial u_k(i) / \partial e(i) \) \( \partial w_k(j) / \partial e(j) \) used in the risk map by (6), (7).
- Solve \( u_k \) by Split Bregman Algorithm for Non-Local ROF (see Table 1).
- Calculate risk map \( r_k \) by (5). **End for**

Aggregation processing: \( u^* = \sum_{k=1}^{n} \lambda_k u_k \).

### 4. Experiments and analysis

**Figure 6.** The noise-free images of 256 × 256, from left to right: cameraman, windmill and lake\(^1\).

The original image used in this experiment as shown in Figure 6, which present high contrasted edges and suffer from the noise halo in ROF-NLTV method. The results of the proposed work is compared with existing methods like NLM [2], NLM-SAP [9] and ROF-NLTV [14] in terms of visual effect, the peak signal to noise ratio (PSNR) and structural similarity (SSIM) defined in [20]. In order to improve computational time and storage efficiency, we compute \( w_{ij} \) i.e. similarity, with little nearest neighbor in patches and set thresholds to adaptively select patches with larger weights in the search window, and then patches with lower similarity are ignored. The related parameters are set as follows: Pixel values range: 0-1; Noise deviation \( \delta = 0.05 \); Patch sizes: 7 × 7 px\(^2\); Search window sizes: 11 × 11 px\(^2\); The scale parameter \( h = 1/40 \); \( \mu = 2000 \); \( \beta = 4300 \); \( T = 0.5 \); Threshold of \( w_{ij} = 0.65 \); Nearest neighbor number 8.

The PSNR of the denoised results under different shapes using ROF-NLTV for the three images are shown in Table 3. The final result aggregated by the 5 estimates based on SURE method is better.

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\(^1\) Laurent Condat’s image database: http://www.gipsa-lab.grenoble-inp.fr/~laurent.condat/imagebase.html.
than any estimate, which shows the effectiveness of this method. Then we compare our denoising method with other methods. As shown in Figure 7, our approach can effectively reduce the halo effect caused by NLTV denoising. Numerically, our approach is better than other approaches in peak signal-to-noise ratio (PSNR) and structural similarity (SIMM).

Table 3. PSNR values for estimates with different oriented patch shapes and the aggregation results.

| Shape       | Image     | Square  | Right-half pie | Up-half pie | Left-half pie | Down-half pie | Aggregation (PSNR) |
|-------------|-----------|---------|----------------|-------------|---------------|---------------|--------------------|
| Cameraman   | 34.3899   | 34.0845 | 33.9762        | 34.0667     | 34.0079       | 34.6005       |
| Windmill    | 30.6874   | 29.6560 | 29.9889        | 29.9494     | 29.7913       | 30.9993       |
| Lake        | 30.8337   | 30.2453 | 30.6605        | 30.5751     | 30.7677       | 31.0722       |

Figure 7. Comparisons of the four methods. From left to right: the noised image, the denoised images by NLM [2], NLM-SAP [9], NLTV [14] and our NLTV-SAP methods respectively. And from left to right, the PSNR(db)/SSIM values: 28.2246/0.8044, 33.1026/0.9693, 33.3570/0.9707, 34.2311/0.9775 and 34.6005/0.9785.

5. Conclusion
In this paper, we proposed a shape adaptive patches NLTV denoising mode, and provided the solution to this model. First, Split Bregman algorithm has been used to solve the ROF-NLTV denoising model with different oriented patch shapes. Then, the denoising results are obtained by an exponential weighting aggregation method based on risk estimation. Compared with other denoising methods, our method is better than other methods in peak signal to noise ratio, structure similarity and visually reduced the halo effect of high contrast edges.

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