AN ALGEBRAIC ANALYSIS OF THE GRAPH MODULARITY

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Abstract. One of the most relevant tasks in network analysis is the detection of community structures, or clustering. Most popular techniques for community detection are based on the maximization of a quality function called modularity, which in turn is based upon particular quadratic forms associated to a real symmetric modularity matrix $M$, defined in terms of the adjacency matrix and a rank one null model matrix. That matrix could be posed inside the set of relevant matrices involved in graph theory, alongside adjacency and Laplacian matrices. In this paper we analyze certain spectral properties of modularity matrices, that are related to the community detection problem. In particular, we propose a nodal domain theorem for the eigenvectors of $M$; we point out several relations occurring between graph’s communities and nonnegative eigenvalues of $M$; and we derive a Cheeger-type inequality for the graph modularity.

Key words. Graph partitioning, community detection, nodal domains, graph modularity, spectral partitioning.

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1. Introduction. For the sake of conciseness, we say that a complex network is a graph occurring in real life. Relevant examples include the Internet and the world wide web, biological and social systems like food webs, economic networks, social networks, communication and distribution networks, and many others [17]. Various mathematical disciplines collaborate in the analysis and treatment of such complex systems; and matrix analysis often plays an important role beside e.g., discrete mathematics and computer science. Here we consider a clear example of this collaboration, namely, the subdivision of a network into “clusters” (typically connected subnetworks) having certain qualitative properties, a task which is required in a number of applications. Two main research directions can be easily recognized within that topic, both having a considerable scientific literature: the graph partitioning and the community detection (or clustering).

Graph partitioning is the problem of dividing the vertices of a graph into a given number of disjoint subsets of given sizes such that the overall number or weight of edges between such sets is minimized. The important point here is that the number and sizes of the subsets are, at least roughly, prescribed. For instance, the probably best known example of a graph partitioning problem is the problem of dividing an unweighted graph into two subsets of comparable size, such that the number of edges between them is minimized.

Community detection problems differ from graph partitioning in that the number and size of the subsets into which the network is divided are generally not apriori specified. Instead it is assumed that the graph is intrinsically structured into communities or groups of vertices which are more or less evidently delimited, the aim being to reveal the presence and the consistency of such groups. In particular it should be taken into account the possibility that no significant subdivisions exist for a given graph. A comprehensive review of methods for the solution of partitioning and clustering problems can be found in [6, Ch. 8] and [17, Ch. 11]; see also [13] for a good survey.

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The question that mainly motivated the present work is indeed related with evaluating the quality of a particular division of a network into communities and providing efficient, mathematically sound methods and estimates to locate them. As underlined in [16] and [18], “a good division of a network into communities is not merely one in which there are few edges between communities; it is one in which there are fewer than expected edges between communities”. Newman and Girvan therefore introduced a measure of the quality of a particular division of a network, which they call modularity. Despite several other quality functions have been proposed in the last ten years for analogous purposes, the modularity is by far the most popular quality function for evaluating the quality of a graph partitioning, and is currently adopted by various successful partitioning algorithms, e.g., the so-called Louvain method [5]. The interesting fact here, and the issue that has drawn our attention to this topic, is that the modularity, as well as other related graph-oriented topological invariants, is defined in terms of certain quadratic forms associated to a matrix $M$, called modularity matrix. That matrix can be considered as one of the relevant matrices naturally associated to a graph, together with adjacency and Laplacian matrices.

The main aim of this paper is to analyze certain spectral properties of modularity matrices that are relevant to the community detection problem. In the subsequent part of this Introduction we provide the notational and conceptual background for the subsequent discussion. Sections 2 and 3 introduce the modularity matrix of a graph, its relationships with the modularity of a (sub-)graph and with the Laplacian matrix, and outline the special role of one of its eigenvalues. In Section 4 we present a nodal domain theorem for the eigenvectors of modularity matrices. The subsequent sections are devoted to the analysis of various connections between optimal partitions of a graph and nonnegative eigenvalues of its modularity matrix. Main results are summarized in the concluding Section 8, which comprises also our final comments and possible directions for further research.

1.1. Notations and preliminary definitions. To avoid any ambiguity we fix here our notations and some preliminary definitions. We give a brief review of standard concepts from algebraic graph theory that we will use extensively throughout the paper, referring the reader to e.g., [6, Ch. 2] of [17, Ch. 6] for a careful and succinct introduction to the topic.

From a purely algebraic point of view a graph $G$ consists of a triple $G = (V, E, \omega)$ where $V$ is the set of vertices (or nodes), $E$ is the set of edges and formally is a subset of $V \times V$, and $\omega : E \to \mathbb{R}^+$ is a nonnegative weight function defined over $E$, representing the strength of the relation modeled by the edges. We shall always assume that a graph $G$ is finite, simple, connected, not oriented. We always identify $V$ with $\{1, \ldots, n\}$. We use the simpler notation $G = (V, E)$ when $\omega(ij) = 1$, that is, edges are not weighted.

If not otherwise specified, the symbol $A$ will always denote the adjacency matrix of $G$, that is, $A \equiv (a_{ij})$ such that $a_{ij} = \omega(ij)$ iff $ij \in E$, and $a_{ij} = 0$ otherwise. In particular, $A$ is a symmetric, irreducible, componentwise nonnegative matrix. For the sake of clarity, further definitions are listed hereafter:

- If $ij \in E$ we write $i \sim j$ and say that $i$ and $j$ are adjacent.

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Our use of the term “community” hereafter makes no reference to its meaning in social sciences and other disciplines. We limit ourselves to its common meaning in the network analysis context. Accordingly, we may also use the term “cluster” as alternative to “community”: “Clustering is a synonym for the decomposition of a set of entities into natural groups” [6].
For any $i \in V$, $d_i$ denotes its (generalized) degree, $d_i = \sum_{j, ij \in E} \omega(ij)$. Moreover, we let $d = (d_1, \ldots, d_n)^T$ and $D = \text{Diag}(d_1, \ldots, d_n)$.

For any $S \subseteq V$ we denote by $\overline{S}$ the complement $V \setminus S$, and let $\text{vol } S = \sum_{i \in S} d_i$ be the volume of $S$. Correspondingly, $\text{vol } G = \sum_{i \in V} d_i$ denotes the volume of the whole graph.

A partition of $V$ is a collection of subsets $\mathcal{P} = \{S_1, \ldots, S_k\}$ such that $\bigcup_i S_i = V$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

For $S \subseteq V$, we denote by $A(S)$ the principal submatrix of $A$ made by the rows and columns whose indices belong to $S$. Moreover, we denote by $G(S)$ the subgraph induced by the vertices in $S$, that is the subgraph of $G$ whose adjacency matrix is $A(S)$.

1 denotes the vector of all ones whose dimension depends on the context.

The cardinality of a set $S$ is denoted by $|S|$. In particular, $|V| = n$.

For any $S \subseteq \{1, \ldots, n\}$ we let $\mathbf{1}_S$ be its characteristic vector, defined as $(\mathbf{1}_S)_i = 1$ if $i \in S$ and $(\mathbf{1}_S)_i = 0$ otherwise.

For any subsets $S, T \subseteq V$ let $E(S, T)$ be the set of edges joining vertices in $S$ with vertices in $T$; and let

$$e(S, T) = \mathbf{1}_S^T A \mathbf{1}_T = \sum_{i \in S} \sum_{j \in T} \omega(ij).$$

Note that, if $G = (V, E)$ is unweighted and loopless then $e(S, T) = 2|E(S, T)|$.

For simplicity, we use the shorthands $e_{\text{in}}(S) = e(S, S)$ and $e_{\text{out}}(S) = e(S, \overline{S})$, so that we have also

$$\text{vol } S = e_{\text{in}}(S) + e_{\text{out}}(S).$$

For a matrix $A$ and a vector $x$, we write $A \geq 0$ or $x \geq 0$ (resp. $A > 0$ or $x > 0$) to denote componentwise nonnegativity (resp., positivity); and $\rho(A)$ denotes the spectral radius of $A$.

If $X$ is a symmetric matrix then its eigenvalues are ordered as $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$, unless otherwise specified.

We will freely use familiar properties of eigenvalues of symmetric matrices, and fundamental results in Perron-Frobenius theory; see e.g., [3, 4]. For completeness, we recall hereafter some important facts concerning the symmetric eigenvalue problem:

- Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $Z \in \mathbb{R}^{n \times (n-k)}$ be a matrix with orthonormal columns. Then, for all $i = 1, \ldots, n-k$,

$$\lambda_i(A) \geq \lambda_i(Z^T A Z) \geq \lambda_{i+k}(A). \quad (1.1)$$

- Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and let $B \in \mathbb{R}^{(n-k) \times (n-k)}$ be a principal submatrix of $A$. Then, for all $i = 1, \ldots, n-k$,

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{i+k}(A). \quad (1.2)$$

- Let $A$ be a real symmetric matrix of order $n$ and $v \in \mathbb{R}^n$. Then, for $i = 1, \ldots, n-1$,

$$\lambda_i(A) \geq \lambda_{i+1}(A + vv^T) \geq \lambda_{i+1}(A). \quad (1.3)$$
1.1.1. The modularity matrix. The modularity matrix of the graph is defined as follows:

\[ M = A - \frac{1}{\text{vol } G} \bar{d} \bar{d}^T. \]  

(1.4)

Modularity matrices have been introduced originally for unweighted graphs \( G = (V, E) \); in that case, the number \( a_{ij} \) indicates the presence of an edge between nodes \( i \) and \( j \), whereas \( \bar{d}_i \bar{d}_j / \text{vol } G \) estimates the expected number of edges between vertices \( i \) and \( j \), if edges in the graph were placed with a uniformly random distribution, according to the given degree sequence \( d_1, \ldots, d_n \). Therefore the \((i,j)\)-entry \( m_{ij} = a_{ij} - (\bar{d}_i \bar{d}_j / \text{vol } G) \) of \( M \) measures the disagreement between the expected number and the actual number of edges joining \( i \) and \( j \). It is a common practice to extend rather informally this definition to any weighted graph \( G = (V, E, \omega) \). In the next paragraph, we outline a formal justification of this rather natural extension.

**Remark 1.1.** In any unweighted graph, the number \( d_i d_j / \text{vol } G \) is always an upper bound on the probability that \( ij \in E \), assuming that edges are placed in \( G \) independently at random, conditionately to the given degrees. In fact, that number is the first term in a sign-alternating series expressing the actual probability, where successive terms represent the probability that \( i \) and \( j \) are connected by multiple edges. If \( d_i d_j / \text{vol } G \ll 1 \) then the alternating series is rapidly convergent, and the bound is a good approximation to the true value. On the other hand, graphs of practical interest may contain node pairs with \( d_i d_j / \text{vol } G > 1 \) and weighted edges. In any case, a more principled motivation of the rank-one correction in the modularity matrix, which carries over the weighted graph case, relies on the so-called Chung-Lu random graph model, to be recalled in the next paragraph.

1.1.2. The Chung-Lu random graph model. The Chung-Lu random graph model is one of the most widespread and successful models for the analysis of large graphs with general degree distributions. Let \( w = (w_1, \ldots, w_n)^T > 0 \) be a vector fulfilling the condition \( \max_i \bar{w}_i^2 < \sum_{i=1}^n w_i \). We say that a graph \( G = (V, E) \) follows the Chung-Lu random graph model with parameter \( w \), denoted by \( G(w) \), if the existence of the edge \( ij \in E \) is determined by an independent Bernoulli trial with probability \( p_{ij} = w_i w_j / (\sum_{i=1}^n w_i) \). That model has been popularized in [7, Ch. 5]; and various statistical properties have been described e.g., in [1, 20]. A basic and very useful property of this model is that, if \( G = (V, E) \) is a random graph drawn from \( G(w) \), then the expected degree of \( i \in V \) is exactly \( w_i \). Consequently, if only the degree vector \( d \) is known, it is reasonable to assume \( w = d \). Actually, this equality leads to an asymptotically unbiased estimator of \( w \) [1]. Hereafter, we propose a generalization of the Chung-Lu model which is convenient for working with weighted graphs.

**Definition 1.2.** Let \( w = (w_1, \ldots, w_n)^T > 0 \), and let \( X(p) \) be a nonnegative random variable parametrized by the scalar parameter \( p \in [0, 1] \), whose expectation is \( \mathbb{E}(X(p)) = p \). We say that a weighted graph \( G = (V, E, \omega) \) follows the \( X \)-weighted Chung-Lu random graph model \( G(w, X) \) if, for all \( i, j \in V \), \( \omega(ij) \) are independent random variables distributed as \( X(p_{ij}) \) where \( p_{ij} = w_i w_j / \sum_{i=1}^n w_i \), with the convention that \( ij \in E \iff \omega(ij) > 0 \), that is, edges with zero weight are removed from \( G \).

We point out that \( G(w) \) is the special case of \( G(w, X) \) where \( X(p) \) is the Bernoulli trial with success probability \( p \). On the other hand, if \( X(p) \) has a continuous part, then \( G(w, X) \) may contain graphs with generic weighted edges. In any case, as in the original Chung-Lu model, if \( G \) is a random graph drawn from \( G(w, X) \) then the
expected degree of node $i$ is

$$
E(d_i) = \sum_{j=1}^{n} E(\omega(ij)) = \sum_{j=1}^{n} p_{ij} = w_i.
$$

1.1.3. The Laplacian matrix. The modularity matrix (1.4) is a rank one perturbation of the adjacency matrix, which is still symmetric but loses the nonnegativity of its entries. The kernel of $M$ is nontrivial and, indeed, 1 always is a nonzero element in $\ker M$. This is reminiscent of another key matrix associated to a graph $G$: the Laplacian matrix. Such matrix is defined as $L = D - A$, where $D$ denotes the diagonal matrix with diagonal entries $d_1, \ldots, d_n$. A huge literature has been developed around $L$, its spectral properties, and their connections with combinatorial and topological properties of $G$, see e.g., [8, 14] and references therein; in fact, this matrix can be thought as a discrete version of the Laplacian differential operator, under many respects.

The bilinear form associated to $L$ admits the expression

$$
v^T L v = \sum_{ij \in E} \omega(ij)(v_i - v_j)^2, \quad (1.5)
$$

where the sum ranges over all edges in the graph, each edge being counted only once. Thus, $L$ is symmetric and positive semidefinite; zero always is an eigenvalue of $L$, with associated eigenvector 1, and that eigenvalue is simple if and only if $G$ is connected. Conventionally the eigenvalues of $L$ are ordered from smallest to largest; for a connected graph, $0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_n(L)$.

1.1.4. Nodal domains. The study of the spectral properties of the Laplacian matrix has originated one of the best known methods for graph partitioning, the spectral partitioning [17, §11.5]. The idea was pioneered by Fiedler in [11, 12], where he observed that a strong relation exists among connectivity properties of $G$, the second smallest eigenvalue of $L$ (the smallest one being zero), and the changes of signs of the entries of any eigenvector relative to such eigenvalue. Following Fiedler’s works, the number $\lambda_2(L)$ is usually called algebraic connectivity of $G$ and denoted by $a(G)$; furthermore, it is a well established practice to call Fiedler vector any eigenvector associated to it.

Let us recall a couple of definitions and relevant results. Inspired by Courant’s nodal domains theorem (which bounds the number of nodal domains of eigenfunctions of the Laplacian operator on smooth Riemannian manifolds), nodal domains induced by a real vector $u$ are commonly defined as follows:

**Definition 1.3.** Let $0 \neq u \in \mathbb{R}^n$. A subset $S \subseteq V$ is a strong nodal domain of $G$ induced by $u$ if the subgraph $G(S)$ induced on $G$ by $S$ is a (maximal) connected component of either $\{i: u_i > 0\}$ or $\{i: u_i < 0\}$.

**Definition 1.4.** Let $0 \neq u \in \mathbb{R}^n$. A subset $S \subseteq V$ is a weak nodal domain of $G$ induced by $u$ if the subgraph $G(S)$ induced on $G$ by $S$ is a (maximal) connected component of either $\{i: u_i \geq 0\}$ or $\{i: u_i \leq 0\}$ and contains at least one node $i$ where $u_i \neq 0$.

Actually, the previous definitions are a slight modification of the terminology used in e.g., [9, 10], but their meaning is unchanged. For any connected graph $G$, $\lambda_1(L) = 0$

\[\text{\footnotesize 2 Unlike their continuous analogous, in the present context nodal domains are located by sign variations rather than zero values. Therefore some authors call them sign domains [9]. We prefer to maintain the “classical” terminology.}\]
is simple and has $1$ as associated eigenvector. It clearly follows that the only possible nodal domain for $\lambda_1(L)$ is $G$ itself. On the other hand, since $L$ is real and symmetric, each other eigenvector of $L$ can be chosen to be real and orthogonal to $1$, that is, any eigenvector $u$ of $L$ that is not constant has at least two components of different signs. Therefore any such $u$ has at least two nodal domains. Fiedler noted in [12, Cor. 3.6] that the weak nodal domains induced by any eigenvector associated to $a(G)$ are at most two, and thus are exactly two. Many authors derived analogous results for the other eigenvalues of $L$ afterward [9, 10, 19]. The following nodal domain theorem summarizes their work:

**Theorem 1.5.** Let $L$ be the Laplacian matrix of a connected graph. Let $\lambda$ be an eigenvalue of $L$ and let $u$ be an associated eigenvector. Let $\ell$ and $\ell'$ be the number of eigenvalues of $L$ that are not larger than $\lambda$ and strictly smaller than $\lambda$, respectively, counted with their multiplicity. Then $u$ induces at most $\ell$ strong nodal domains and at most $\ell' + 1$ weak nodal domains.

2. **Modularity of a subgraph.** A central problem in graph clustering is to look for a quantitative definition of community. Although all authors agree that a community should be a connected group of nodes that is more densely connected among each other than with the rest of the network, as a matter of fact no definition is universally accepted. A variety of merit functions to quantify the strength of a subset $S \subseteq V$ as a community in $G$ is listed in [6, Ch. 8]; all of them are essentially based on a trade-off between the total weight of edges insisting on vertices in $V$ (which should be “large”) and the one of the edges connecting vertices in $V$ with vertices outside $V$ (which should be “small”, for a “good” community).

Fortunato in its comprehensive report [13] classifies various definitions of community according to whether they are based on graph-level properties, subgraph-level properties, or vertex similarity, and underlines that the global definition based on the modularity quality function introduced by Newman and Girvan in [18] is by far the most popular definition. Their definition can be informally stated as follows: A subset of vertices $S \subseteq V$ forms a community if the subgraph $G(S)$ contains a larger number of edges than expected. Obviously, such statement is not rigorous, until one defines the probability distribution underlying the concept of “expected number”. Doubtless, the most simple and natural guess is to assign an equal probability to the connection between any two nodes in the network. The corresponding random graph model is known as Erdős-Rényi model. That model is at the basis of various successful approaches to community detection [2, 21, 22]. In this work, we follow [15, 16, 18] and assume, instead, the Chung-Lu random graph model with parameter $d$ as reference.

Given a graph $G = (V, E, \omega)$, consider a subset of vertices $S \subseteq V$. For graphs following the (weighted) Chung-Lu model with parameter $d$, the overall weight of edges joining vertices in $S$ can be estimated by

$$\sum_{i \in S} \sum_{j \in S} \frac{d_i d_j}{\text{vol}G} = \frac{(\text{vol}S)^2}{\text{vol}G}.$$ 

Consequently, we define the modularity of $S$ as

$$Q(S) = e_{in}(S) - \frac{(\text{vol}S)^2}{\text{vol}G}. \quad (2.1)$$

If that difference is positive then there is a clear indication that the subgraph $G(S)$ contains “more edges” than expected from the reference model. This fact can be
considered as a clue (apart from connectedness) that $S$ is a closely knit set of vertices and as such, a possible community inside $G$.

An easy computation exploiting the identities $\text{vol } S = e_{in}(S) + e_{out}(S)$ and $\text{vol } G - \text{vol } S = \text{vol } \overline{S}$ reveals that

$$Q(S) = \text{vol } S - e_{out}(S) - \frac{(\text{vol } S)^2}{\text{vol } G} = \text{vol } S(1 - \frac{\text{vol } S}{\text{vol } G}) - e_{out}(S) - \frac{\text{vol } S \cdot \text{vol } \overline{S}}{\text{vol } G} - e_{out}(S).$$  \hspace{1cm} (2.2)

Such relation shows that $Q(S) = Q(\overline{S})$. Therefore, modularity is a quality of the cut $\{S, \overline{S}\}$ rather than of $S$ itself. Moreover it reveals that $Q(S)$ is large when both $S$ and its complement $\overline{S}$ have comparable volumes (in fact $\text{vol } S \cdot \text{vol } \overline{S}/\text{vol } G$ is large when $\text{vol } S \approx \text{vol } \overline{S} \approx \frac{1}{2} \text{vol } G$) and the overall weight of edges elapsing between $S$ and $\overline{S}$ is small. Consequently, (2.2) bares that the modularity $Q(S)$ shares the structure of virtually all reasonable clustering indices [6, Ch. 8], consisting of the difference between $\text{vol } S \cdot \text{vol } \overline{S}/\text{vol } G$, which is a term measuring the density of the “clusters” $S$ and $\overline{S}$, and $e_{out}(S)$, which quantifies the sparsity of their connection. Furthermore, the resulting equalities $Q(\emptyset) = Q(V) = 0$ formalize the common understanding that neither the empty set nor the whole graph constitute a community.

It is almost immediate to recognize that $e_{in}(S) = \frac{1}{T} S A S$ and $\text{vol } S = \frac{1}{T} d$. Hence, we can express the modularity (2.1) in terms of the modularity matrix (1.4) as follows:

$$Q(S) = \frac{1}{T} S M S.$$  \hspace{1cm} (2.3)

**Remark 2.1.** In principle other vectors can be chosen in place of $d$ inside (1.4), depending on the null model one is assuming for the distribution of the edges in $G$. For example, if $G$ is unweighted and the null model assumed is the Erdős-Rényi random graph model, in which every edge has probability $p$ to appear, then the appropriate definition for the modularity matrix of $G$ would be $M = A - p I$ with $p = \text{vol } G / n^2$, so that $Q(V) = \frac{1}{T} M I = 0$. In this case, the resulting modularity matrix allows us to express by means of a formula analogous to (2.3) certain modularity-type merit functions based on 2-state Potts Hamiltonian functions adopted in, e.g., [21, 22].

In a somehow heuristic way at this stage, we see from (2.3) that the existence of a subset $S \subseteq V$ having positive modularity is related with the positive eigenvalues of $M$ and their corresponding eigenspaces. In fact, if $\mathbb{F}_n = \{0, 1\}^n$ is the set of binary $n$-tuples, the search of a maximal modularity subgraph is formalized by the optimization problem

$$\max_{x \in \mathbb{F}_n} x^T M x.$$  \hspace{1cm} (2.4)

The problem as is stated is clearly NP-complete, so a standard and widely used procedure is to move to a continuous relaxation, for example,

$$\max_{x \in \mathbb{R}^n} x^T M x,$$  \hspace{1cm} (2.5)

which is solved by an eigenvector associated to the largest eigenvalue of $M$, properly normalized. Once a solution $\hat{x}$ for the latter problem (2.5) is computed, the sign vector $s = \text{sign}(\hat{x})$ is chosen as an approximate solution for (2.4). Note that such $s$ realizes the best approximation to $\hat{x}$ in the $L^p$ sense, that is $\|s - \hat{x}\|_p = \min_{x \in \mathbb{F}_n} \|x - \hat{x}\|_p$, for
$p \in [1, \infty]$. The spectral analysis of $M$ and of the maximal subgraphs induced by the change of signs in its eigenvectors (nodal domains) is the central topic of Sections 4.

In what follows, we adopt from [16, 18] the following definitions:

**Definition 2.2.** A module in a given graph $G$ is a subgraph having positive modularity. A graph is indivisible if it has no modules, and divisible otherwise.

Probably, the main reason of the success of modularity as a quantitative measure of community strength is the fact that modules having significant size and modularity are typically decent indicators of community structure.

**Remark 2.3.** Cliques and star graphs are indivisible graphs. On the other hand, indivisible graphs are rather scarce. Indeed, a simple computation based on the formula (2.1) shows that, if $i, j \in V$ are two vertices joined by an edge, and

$$d_i + d_j < \sqrt{2} \omega(ij) \text{vol}(G),$$

then $Q(\{i, j\}) > 0$. Consequently, a graph is divisible if it has at least one edge fulfilling the previous inequality, a condition which is easily met in practice.

### 3. The algebraic modularity of a graph

Since the pioneering works by Fiedler [11, 12] the algebraic connectivity of a connected graph $G$ is classically defined as the smallest positive eigenvalue of its Laplacian matrix:

$$\lambda_1(G) = \min_{x, \|x\|_1 = 1} \frac{x^T L x}{x^T x}.$$  

Analogously, we can define the algebraic modularity of $G$ as

$$m(G) = \max_{x, \|x\|_1 = 1} \frac{x^T M x}{x^T x}. \tag{3.1}$$

Differently to (2.5), any vector $x$ attaining the maximum in (3.1) must have entries with opposite signs. We will see afterward that $m(G)$ plays a relevant role in the community detection problem, exactly in the same way as $a(G)$ with respect to the partitioning problem. Furthermore, in tandem with Definition 2.2, it is rather natural to say that $G$ is algebraically indivisible if its modularity matrix has no positive eigenvalues. For example, cliques and star graphs are algebraically indivisible graphs.

**Remark 3.1.** The number $m(G)$ is the largest eigenvalue of $M$ after deflation of the subspace $1$, which is an invariant subspace associated to the eigenvalue 0. More precisely, we have $\lambda_1(M) = \max\{m(G), 0\}$. Hence, we can say that $G$ is algebraically indivisible if and only if $m(G) \geq 0$.

We point out that any algebraically indivisible graph is indivisible as well. Indeed, the existence of a subgraph $S$ having positive modularity implies that $M$ has at least one positive eigenvalue: $\lambda_1(M) \geq 1^T S M 1_S / 1^T S 1_S = Q(S) / |S| > 0$. We shall explore in greater detail in Section 6 the relationship between divisibility of $G$ and positive eigenvalues of $M$. For the moment, the following argument shows that a better bound than $Q(S) \leq |S| \lambda_1(M)$ can be derived:

**Lemma 3.2.** For any $S \subseteq V$ we have $Q(S) \leq m(G) |S||\bar{S}| / n$.

**Proof.** Let $\alpha = |S| / n$. Then, the vector $1_S - \alpha 1$ is orthogonal to $1$ and moreover,

$$(1_S - \alpha 1)^T (1_S - \alpha 1) = (1_S - \alpha 1)^T 1_S = |S| - \alpha |S| = \frac{|S||\bar{S}|}{n}.$$  

Recalling that $M 1 = 0$ and the definition (3.1) we have

$$Q(S) = 1^T S M 1_S = (1_S - \alpha 1)^T M (1_S - \alpha 1) \leq m(G) (1_S - \alpha 1)^T (1_S - \alpha 1),$$
and we complete the proof. □

It is worth noting that the modularity matrix \( M \) can be expressed as the difference of two Laplacian matrices. Indeed,

\[
M = A - D + D - dd^T/(1^T d) = L_0 - L, \tag{3.2}
\]

where \( L = D - A \) is the Laplacian matrix of \( G \) and \( L_0 = D - dd^T/(1^T d) \) can be regarded as the Laplacian matrix of the complete graph \( G_0 = (V, V \times V, \omega_0) \) where the weight \( \omega_0(ij) = d_id_j/1^T d \) is placed on the edge \( ij \). Thus, in some sense, \( G_0 \) represents the “average graph” in the Chung-Lu model with parameter \( d \).

The formula (3.2) yields a decomposition of \( M \) in terms of two positive semidefinite matrices. A noticeable consequence of the Courant-Fischer theorem is the following set of inequalities, relating algebraic connectivity and modularity of \( G \), and whose simple proof is omitted for brevity:

\[
d_{\min} - a(G) \leq a(G_0) - a(G) \leq m(G) \leq d_{\max} - a(G),
\]

where \( d_{\min} \) and \( d_{\max} \) denote the smallest and largest degree of vertices in \( G \), respectively. Consequently, a necessary condition for \( G \) being algebraically indivisible is \( a(G_0) \leq a(G) \). By a result by Fiedler [11], whose proof extends immediately to weighted graphs, \( a(G) \leq \lfloor n/(n-1) \rfloor d_{\min} \). Hence, \( m(G) \geq -d_{\min}/(n-1) \). This lower bound is attained by a clique, thus it is sharp.

4. Modularity nodal domains. As recalled in Definition 1.3 and Definition 1.4, any vector \( u \in \mathbb{R}^n \) induces some nodal domains over \( G \), that is some maximal connected subsets of the vertices \( V \) related with sign changes inside \( u \). Hereafter, we consider nodal domains induced by eigenvectors of the modularity matrix of the graph, which we call modularity nodal domains. The aim of this section is to derive a nodal domain theorem analogous to Theorem 1.5 for the modularity nodal domains, contributing to the analysis and the improvement of the spectral-based methods for community detection, proposed by Newman and Girvan [18] and well summarized in [13] and [17, Ch. 11].

We will say that a nodal domain \( S \subset V \) induced by a vector \( u \) is positive or negative, according to the sign of \( u \) over \( S \). If \( S_1 \) and \( S_2 \) are two nodal domains, we say that \( S_1 \) is adjacent to \( S_2 \), in symbols \( S_1 \approx S_2 \), if there exists \( i \in S_1 \) and \( j \in S_2 \) such that \( i \sim j \). The maximality of the nodal domains therefore implies that a necessary condition for \( S_1 \approx S_2 \) is that \( S_1 \) and \( S_2 \) have different signs.

Given a real vector \( u \neq 0 \) the following properties on the nodal domains it induces are not difficult to be observed; some of them are borrowed from [9]:

P1. In any nodal domain there exists at least one node where \( u \) is nonzero. Moreover, if \( S_1 \) and \( S_2 \) are weak nodal domains such that \( S_1 \cap S_2 \neq \emptyset \) then \( S_1 \) and \( S_2 \) have opposite sign and \( u_i = 0 \) for any \( i \in S_1 \cap S_2 \).

P2. Let \( A \) be the adjacency matrix of \( G \). If \( S \subset V \) is a (strong or weak) nodal domain, then \( G(S) \) is connected and the principal submatrix \( A(S) \) is irreducible. Therefore, since two nodal domains of the same sign can not be adjacent, for any vector \( u \) there exists a labeling of the vertices of \( V \) such that the adjacency matrix \( A \) of \( G \) has the form

\[
A = \begin{pmatrix}
A_+ & B & C \\
B^T & A_- & D \\
C^T & D^T & A_0
\end{pmatrix}
\]

(4.1)
where rows and columns of $A_+$, $A_-$, and $A_0$ correspond to entries in $u$ that are positive, negative, and zero, respectively, and $A_+$ and $A_-$ are the direct sum of overall $s$ irreducible matrices, $s$ being the number of strong nodal domains.

**P3.** If $S_1$ and $S_2$ are adjacent weak nodal domains, then there exists $i \in S_1$ and $j \in S_2 \setminus S_1$ such that $i \sim j$ and $u_j \neq 0$. In fact, if $S_1 \cap S_2 = \emptyset$ then the assertion follows by definition. (If $i \sim j$ and $u_j = 0$ then $j \in S_1 \cap S_2$.) Whereas if $S_1 \cap S_2 \neq \emptyset$ then, by property P1, there must be at least a pair of vertices $i, j \in S_1 \cup S_2$ for which $i \in S_1 \cap S_2$ (whence $u_i = 0$), $i \sim j$, and $j \in S_2 \setminus S_1$ (so that $u_j \neq 0$); otherwise, there would be no edge joining $S_1 \cap S_2$ and $S_2 \setminus S_1$, contradicting the hypothesis that $G(S_2)$ is connected.

The following theorem, which is a slight generalization of [12, Thm. 2.1] and [19, Thm. 1], is the key for deriving a nodal domain theorem for the modularity eigenvectors. We stress that such theorem and its corollaries hold for any undirected simple graph, that is, loops and weighted edges are possibly allowed.

**Theorem 4.1.** Let $A$ be the adjacency matrix of a simple, connected graph $G$. Let $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n$ be such that at least two entries of $u$ have opposite signs and $Au \geq \lambda u$, in the componentwise sense. Let $\ell$ and $\ell'$ be respectively the number of eigenvalues of $A$ that are greater than or equal to $\lambda$ and the number of eigenvalues that are strictly greater than $\lambda$, counted with their multiplicity. Then $u$ induces at most $\ell$ positive strong nodal domains and at most $\ell'$ positive weak nodal domains.

**Proof.** Let $s \geq 1$ be the number of positive strong nodal domains induced by $u$. Due to property P2 above, we can assume without loss in generality that the vector $u$ can be partitioned into $s + 1$ subvectors, $u = (u_1, \ldots, u_s, u_{s+1})^T$ such that $u_i > 0$, for $i = 1, \ldots, s$, $u_{s+1} \leq 0$ and $A$ is conformally partitioned as

$$
A = \begin{pmatrix}
A_1 & B_1 \\
\vdots & \vdots \\
A_s & B_s \\
B_1^T & \cdots & B_s^T & B_{s+1}
\end{pmatrix},
$$

where $A_i$ are nonnegative and irreducible, since they are the adjacency matrices of connected graphs. By hypothesis, $A_i u_i + B_i u_{s+1} \geq \lambda u_i$ for $i = 1, \ldots, s$. Therefore $A_i u_i \geq \lambda u_i - B_i u_{s+1} \geq \lambda u_i$ and, by Perron-Frobenius theorem we have

$$
\rho(A_i) = \max_{x \neq 0} \frac{x^T A_i x}{x^T x} \geq \frac{u_i^T A_i u_i}{u_i^T u_i} \geq \lambda.
$$

This implies that $A_i$ has at least one eigenvalue not smaller than $\lambda$, for $i = 1, \ldots, s$. By eigenvalue interlacing inequalities (1.2) we conclude that $A$ has at least $s$ eigenvalues greater than or equal to $\lambda$, whence $s \leq \ell$. This proves the first inequality in the claim.

The second one can be proved analogously. As for the strong domains, two positive weak nodal domains can not overlap, therefore there exists a labeling of $V$ such that $A$ admits the block form

$$
A = \begin{pmatrix}
A_1 & B_1 \\
\vdots & \vdots \\
A_w & B_w \\
B_1^T & \cdots & B_w^T & B_{w+1}
\end{pmatrix},
$$
where \( w \) is the number of weak positive nodal domains, and the vector \( u \) is partitioned conformally as \( u = (u_1, \ldots, u_w, u_{w+1}) \) where \( u_i \geq 0 \) for \( i = 1, \ldots, w \) and \( u_{w+1} \leq 0 \). In fact, the entries in \( u_{w+1} \) correspond to nodes belonging to the complement of the union of all positive weak nodal domains, and \( u \) may vanish also on some of those nodes. Nevertheless, property \( P3 \) above imply that each \( B_i \) contains at least one nonzero entry, and \( B_i u_\pm \leq 0 \) with strict inequality in at least one entry. For any fixed \( i = 1, \ldots, w \) let \( x_i \) be a Perron eigenvector of \( A_i \), \( A_i x_i = \rho(A_i) x_i \), with positive entries. Hence \( x_i^T u_i > 0 \) and \( x_i^T B_i u_{w+1} < 0 \). From the inequality \( A_i u_i \geq \lambda u_i - B_i u_{w+1} \) we obtain

\[
\rho(A_i) x_i^T u_i = x_i^T A_i x_i \geq \lambda x_i^T u_i - x_i^T B_i u_{w+1} \;
\]

for \( i = 1, \ldots, w \). Again by the eigenvalue interlacing \((1.2)\) we see that \( A \) has at least \( w \) eigenvalues strictly greater than \( \lambda \), concluding that \( w \leq \ell' \). \( \square \)

Note that in the preceding theorem \( \lambda \) may not be an eigenvalue of \( A \), in which case \( \ell = \ell' \). If \( \lambda \) is an eigenvalue of \( A \) then the difference \( \ell - \ell' \) equals its algebraic/geometric multiplicity.

### 4.1. A modularity nodal domain theorem

A direct consequence of Theorem 4.1 is the following result concerning the nodal domains of eigenvectors of modularity matrices, as announced:

**Theorem 4.2.** Let \( \lambda \) be an eigenvalue of \( M \) and let \( u \) be an associated eigenvector, oriented so that \( d^T u \geq 0 \). Let \( \ell \) and \( \ell' \) be respectively the number of eigenvalues of \( M \) which are greater than or equal to \( \lambda \) and the number of eigenvalues which are strictly greater than \( \lambda \), counted with their multiplicity. If at least two entries of \( u \) have opposite signs then \( u \) induces at most \( \ell + 1 \) positive strong nodal domains and at most \( \ell' + 1 \) positive weak nodal domains.

**Proof.** By hypotheses, \( \ell \geq \ell' + 1 \) and the eigenvalues of \( M \) fulfill

\[
\lambda_{\ell}(M) > \lambda_{\ell+1}(M) = \ldots = \lambda_{\ell}(M) > \lambda_{\ell+1}(M),
\]

the first (last) inequality being missing if \( \ell' = 0 (\ell = n, \text{ respectively}) \). Since \( A - M \) is a positive semidefinite rank-one matrix, inequalities \((1.3)\) imply the following interlacing between the eigenvalues of \( A \) and of \( M \):

\[
\lambda_1(A) \geq \lambda_1(M) \geq \lambda_2(A) \geq \lambda_2(M) \geq \ldots \geq \lambda_n(M).
\]

By inspecting the preceding inequalities we get that

- \( \lambda_\ell(A) \geq \lambda > \lambda_{\ell+2}(A) \); thus \( \ell \leq |\{i: \lambda_i(A) \geq \lambda\}| \leq \ell + 1 \).
- \( \lambda_\ell(A) > \lambda \geq \lambda_{\ell+2}(A) \); thus \( \ell' \leq |\{i: \lambda_i(A) > \lambda\}| \leq \ell' + 1 \).

By hypothesis we have \( A u = M u + |d^T u/(\text{vol } G)| d \geq \lambda u \). The claim follows immediately by Theorem 4.1. \( \square \)

A close inspection of the preceding proof reveals that, if \( \lambda \) is not an eigenvalue of \( A \) then we must have \( \ell = \ell' + 1 \) and the previous inequalities become

\[
\lambda_{\ell'}(M) > \lambda_{\ell}(A) > \lambda_{\ell}(M) = \lambda > \lambda_{\ell+1}(A) > \lambda_{\ell+1}(M).
\]

Consequently the bound for the induced positive strong nodal domains in the theorem above becomes simply \( \ell \). The following corollary specializes the content of the preceding theorem to eigenvectors associated to the algebraic modularity:
Corollary 4.3. Let \( u \) be an eigenvector associated to \( m(G) \) and oriented so that \( d^T u \geq 0 \). If \( m(G) \) is simple and is not an eigenvalue of \( A \) then \( u \) induces exactly one positive (strong) nodal domain.

Proof. It suffices to observe that, if \( m(G) = 0 \) then \( u \) must be a multiple of \( 1 \), and the claim is trivial. On the other hand, if \( m(G) \neq 0 \) then \( u^T 1 = 0 \), so \( u \) has at least two entries with different signs, and the claim follows from the aforementioned interlacing inequalities and Theorem 4.2. \( \square \)

Unfortunately there exists no analogous of Theorem 4.2 for the negative nodal domains. This is illustrated by the following example.

Example 4.4. We produce a family of graphs, of arbitrarily large size, to show that the number of (unsigned) nodal domains induced by the leading eigenvector of \( M \) can be arbitrary, whilst there is exactly one positive nodal domain, if signs are chosen as prescribed by Theorem 4.2. Consider a weighted star graph with loops on \( n = m + 1 \) nodes, whose structure and adjacency matrix are as follows:

\[
A = \begin{pmatrix}
\alpha & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 & \beta
\end{pmatrix}
\]

Hence, \( \alpha \) and \( \beta \) are the weights of the loops placed on the leaf nodes \( 1, \ldots, m \) and on the root node \( n = m + 1 \), respectively. In particular, the degree vector is \( d = A1 = (1+\alpha, \ldots, 1+\alpha, \beta+m)^T \) and the volume is \( \text{vol} \ G = d^T 1 = m(2+\alpha) + \beta \). Straightforward computations leads to the conclusion that the eigenvalues of the modularity matrix \( M \) are the following:

- 0, with associated eigenvector \( 1 \);
- \( \alpha \), with multiplicity \( m-1 \) and associated eigenvectors \( i_1 - i_j \) for \( j = 2, \ldots, m \);
- \( \lambda = (\alpha \beta - m)(m+1)/\text{vol} \ G \), with associated eigenvector \( \overline{\nu} = (-1, \ldots, -1, m)^T \).

Observe that, when

\[ \alpha \beta - m > (\alpha + 1)^2 \]

then \( \overline{\nu} \) is positive and dominant. If in addition \( \alpha + 1 \leq \beta + m \) then the vector \( \overline{\nu} \) fulfills the inequality \( d^T \overline{\nu} \geq 0 \), whence the spectral clustering of the graph consists of one positive nodal domain, given by node \( n \), and \( m \) distinct, negative nodal domains, given by the leaf nodes. Both inequalities above are fulfilled when, for example, \( \alpha = 1 \) and \( \beta = m + 5 \).

On the other hand, the Laplacian matrix of the same graph is

\[
L = \begin{pmatrix}
1 & -1 \\
\ddots & \ddots & \ddots \\
-1 & \cdots & 1 & -1 & m
\end{pmatrix}
\]

independently on \( \alpha \) and \( \beta \). Its smallest nonzero eigenvalue is 1 and the associated eigenspace is set of all zero-sum vectors that are orthogonal to the \( n \)-vector \( (1, \ldots, 1, 0) \). Hence, any spectral partitioning induced by a Fiedler vector has exactly two weak
nodal domains (which intersect at the root node), whereas the number of (positive and negative) strong nodal domains can vary in the range $2, \ldots, m$.

Analogous examples can be built up using loopless, unweighted graphs. Indeed, consider a graph with $p + mq$ nodes consisting of one clique with $p$ nodes and $m$ copies of the clique with $q$ nodes. Moreover, add $m$ edges connecting a fixed node of the former subgraph with one node of each of the latter subgraphs. The case with $p = 4$, $m = q = 3$ is shown hereafter:

Under appropriate conditions on the parameters $p$, $q$, and $m$ the leading eigenvector of $M$ splits the graph into the $m + 1$ cliques, each belonging to a different nodal domain; the (unique) positive nodal domain being the clique having order $p$. Computer experiments show that those conditions are met e.g., for $p = 4$, $q = 3$, and $m = 2, \ldots, 11$.

5. Upper bounds on the graph modularity. In the preceding sections we have understood modularity as a functional defined over arbitrary subsets of $V$. For the purposes of community detection problems, it is convenient to extend the previous definition to arbitrary partitions. In fact, Newman and Girvan original definition of the modularity of a partition $\mathcal{P} = \{S_1, \ldots, S_k\}$ of $V$, see Equation (5) in [18], can be expressed in our notations as

$$q(\mathcal{P}) = \frac{1}{\text{vol} G} \sum_{i=1}^{k} Q(S_i) = \frac{1}{\text{vol} G} \sum_{i=1}^{k} 1_{S_i}^T M 1_{S_i}. \quad (5.1)$$

The normalization factor $1/\text{vol} G$ is purely conventional and has been included by the authors for compatibility with previous works, to settle the value of $q(\mathcal{P})$ in a range independent of $G$. That definition has been introduced as a merit function to quantify the strength of the community structure defined by $\mathcal{P}$. In the earliest community detection algorithm, the function $q(\mathcal{P})$ is optimized by a hierarchical clustering method. Subsequent improvements of that algorithm maintain essentially the original approach, see [5]. The use (and the definition itself) of the modularity matrix to compute the modularity of a partitioning has been introduced successively in [15, 16].

As recalled in the Introduction, in the community detection problem one has no preliminary indications on the number and size of possible communities inside $G$. Hence, it is natural to introduce the number

$$q_G = \max_{\mathcal{P}} q(\mathcal{P}),$$

where the maximum is taken over all nontrivial partitions of $V$, and try to bound it in terms of spectral properties of $M$ only.

Remark 5.1. An optimal partition $\mathcal{P}_* = \{S_1, \ldots, S_k\}$, that is, a partition such that $q_G = q(\mathcal{P}_*)$, has the property that if any two subsets are merged then the overall modularity does not increase. This does not imply that $Q(S_i) > 0$ for all $i = 1, \ldots, k$, even if $G$ is divisible. Nevertheless, if $Q(S) > 0$ for some $S \subset V$ then $q_G \geq q(\{S, \overline{S}\}) > 0$, so that the condition $q_G > 0$ is equivalent to say that $G$ is divisible.
In the $k = 2$ case we have $\mathcal{P} = \{S, \overline{S}\}$ for some $S \subset V$. Since $Q(S) = Q(\overline{S})$, for notational simplicity, we can write $q(S)$ in place of $q(\mathcal{P})$. Correspondingly, we also consider the quantity

$$q'_G = \max_{S \subset V} q(S),$$

whose computation corresponds to the identification of a set $S$ or, equivalently, a cut $\{S, \overline{S}\}$ with maximal modularity. We prove hereafter a very general upper bound for $q'_G$ in terms of $m(G)$; a lower bound is considered in the forthcoming Corollary 7.2, under additional hypotheses.

**Theorem 5.2.** Let $\langle d \rangle = \text{vol}G/n$ be the average degree in $G$. Then,

$$q'_G \leq \frac{m(G)}{2 \langle d \rangle}.$$

**Proof.** Since $Q(S) = Q(\overline{S})$, for any $S \subseteq V$ we have by definition

$$q(S) = \frac{1}{\text{vol}G} (Q(S) + Q(\overline{S})) = \frac{2Q(S)}{\text{vol}G}. \tag{5.2}$$

Let $S$ be a set maximizing $q(S)$. From Lemma 3.2 we obtain

$$q'_G = \frac{2Q(S)}{\text{vol}G} \leq \frac{2m(G) |S||\overline{S}|}{\text{vol}G \cdot n} \leq \frac{m(G) n}{\text{vol}G 2},$$

since $|S||\overline{S}|$ is upper bounded by $n^2/4$ for any $S$. \[ \square \]

In what follows, we prove a result analogous to the preceding theorem but for the number $q_G$. For clarity of exposition, we derive firstly a preliminary result:

**Lemma 5.3.** Let $A$ and $B$ be two symmetric matrices of order $n$, with eigenvalues $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \ldots \geq \lambda_n(B)$, respectively. Then,

$$\text{trace}(AB) \leq \sum_{i=1}^{n} \lambda_i(A) \lambda_i(B).$$

**Proof.** The claim can be derived easily from the Hoffman-Wielandt inequality [4, Thm. VI.4.1]

$$\sum_{i=1}^{n} (\lambda_i(A) - \lambda_i(B))^2 \leq \|A - B\|_F^2,$$

using the expansion $\|A - B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 - 2 \text{trace}(AB)$ and the equality $\|A\|_F^2 = \sum_{i=1}^{n} \lambda_i(A)^2$. \[ \square \]

Consider an arbitrary partition $\mathcal{P} = \{S_1, \ldots, S_k\}$ of the node set $V$. Assume for simplicity that $|S_i| \geq |S_{i+1}|$ for $i = 1, \ldots, k - 1$. Introduce the $n \times k$ “index matrix” $Z = [1_{S_1} \cdots 1_{S_k}]$ and define $B = ZZ^T = \sum_{i=1}^{k} 1_{S_i} 1_{S_i}^T$. Then $B$ has rank $k$ and the cardinalities $|S_i|$ are its nonzero eigenvalues in nonincreasing order. Recalling that, for arbitrary matrices $A$ and $B$ it holds $\text{trace}(AB) = \text{trace}(BA)$, (5.1) can be rewritten as follows:

$$q(\mathcal{P}) = \frac{1}{\text{vol}G} \text{trace}(Z^T M Z) = \frac{1}{\text{vol}G} \text{trace}(BM).$$
With the help of Lemma 5.3 we immediately get

\[ q(P) \leq \frac{1}{\text{vol}G} \sum_{i=1}^{k} |S_i| \lambda_i(M) \leq \frac{n}{\text{vol}G} \lambda_1(M). \]

The latter bound, which does not depend on \( P \), can be improved as follows:

**Theorem 5.4.** For any graph \( G \),

\[ q_G \leq n - 1 - \frac{1}{\text{vol}G} m(G). \]

**Proof.** Let \( V = \{1\}^\perp \) and let \( V \) be any matrix whose columns form an orthonormal basis of \( V \). Observe that \( W = V V^T \) is the orthogonal projector onto \( V \), that is, \( W = I - \frac{1}{n} 11^T \). Moreover, \( \lambda_1(V^TMV) = m(G) \) by (3.1).

Let \( P_* = \{S_1, \ldots, S_k\} \) be an optimal partition of \( G \), that is \( q_G = q(P_*) \), and let again \( Z = [1_{S_1} \cdots 1_{S_k}] \). Since \( M = WMW \), Lemma 5.3 leads us to

\[
\text{trace}(Z^T M Z) = \text{trace}(Z^T W M W Z) = \text{trace}((V^T Z Z^T V)(V^T M V)) \\
\leq \sum_{i=1}^{n} \lambda_i(V^T Z Z^T V) \lambda_i(V^T M V) \\
\leq \text{trace}(V^T Z Z^T V) m(G) = \text{trace}(Z^T W Z) m(G).
\]

From \( W = I - \frac{1}{n} 11^T \), letting \( z = Z^T 1 = (|S_1|, \ldots, |S_k|)^T \) we obtain

\[
Z^T W Z = Z^T Z - \frac{1}{n} Z^T 11^T Z = \text{Diag}(|S_1|, \ldots, |S_k|) - \frac{1}{n} z z^T.
\]

Owing to the fact that \( \|z\|_2 \geq \|z\|_1 / \sqrt{n} = \sqrt{n} \), we have \( \text{trace}(Z^T W Z) = n - \|z\|_2^2 / n \leq n - 1 \). It suffices to collect terms, and the proof is complete. \( \square \)

6. **How many modules?** Based on rather informal arguments, Newman claims in [15, Sect. B] that the number of positive eigenvalues of \( M \) is related to the number of communities recognizable in the graph \( G \), tightening the connection between spectral properties of \( M \) and the community structure of the network it describes. More precisely, the author argues that the number of positive eigenvalues, plus 1, is an upper bound on the number of communities that can be recognized in \( G \). In this section we prove various results supporting that conclusion, that culminate in the subsequent Corollary 6.3.

Already in Remark 3.1 we noticed that the existence of a subgraph \( S \) having positive modularity implies that \( M \) has at least one positive eigenvalue. By the way, if \( Q(S) > 0 \) then also \( Q(S) > 0 \), therefore two modules (according to Definition 2.2) give rise to one positive eigenvalue. The forthcoming theorem proves that, if \( G \) has \( k \) subgraphs that are well separated and sufficiently rich in internal edges, then \( M \) has at least \( k - 1 \) positive eigenvalues.

**Theorem 6.1.** Let \( S_1, \ldots, S_k \) be pairwise disjoint subsets of \( V \), with \( k \geq 1 \), such that \( \text{vol}(S_i) \leq \frac{1}{2}\text{vol}G \) and \( e_{\text{in}}(S_i) > e_{\text{out}}(S_i) \). Then, \( Q(S_i) > 0 \) and \( M \) has at least \( k - 1 \) positive eigenvalues.

**Proof.** Firstly we observe that, owing to the stated hypotheses, the sets \( S_1, \ldots, S_k \)
have positive modularity. Indeed, for \( i = 1, \ldots, k \) we have \( \text{vol} G \leq 2 \text{vol} S_i \), whence

\[
Q(S_i) \frac{\text{vol} G}{\text{vol} S_i} = \text{vol} S_i - e_{\text{out}}(S_i) \frac{\text{vol} G}{\text{vol} S_i}
= e_{\text{in}}(S_i) - e_{\text{out}}(S_i) \left( \frac{\text{vol} G}{\text{vol} S_i} - 1 \right)
\geq e_{\text{in}}(S_i) - e_{\text{out}}(S_i) > 0.
\]

Let \( Z = [1_{S_1} \cdots 1_{S_k}] \) and consider the \( k \times k \) matrix \( C = Z^T A Z \). For \( i, j = 1, \ldots, k \) we have

\[
C_{ij} = 1_{S_i}^T A 1_{S_j} = \begin{cases} e_{\text{in}}(S_i) & i = j \\ e(S_i, S_j) & i \neq j. \end{cases}
\]

The matrix \( C \) is symmetric, nonnegative, and (strongly) diagonally dominant. Indeed, owing to the fact that the \( S_j \)'s are pairwise disjoint, and \( E(S_i, S_j) \supseteq \cup_{j \neq i} E(S_i, S_j) \), for \( i = 1, \ldots, k \) we have

\[
C_{ii} = e_{\text{in}}(S_i) > e_{\text{out}}(S_i) \geq \sum_{j \neq i} e(S_i, S_j) = \sum_{j \neq i} C_{ij}.
\]

As a result, by Gershgorin theorem, \( C \) is positive definite.

Introduce the diagonal matrix \( \Delta = \text{Diag}(\sqrt{|S_1|}, \ldots, \sqrt{|S_k|})^{-1} \). Owing to the orthogonality of the columns of \( Z \), the matrix \( \tilde{Z} = Z \Delta \) has orthonormal columns. By Sylvester’s law of inertia, also the matrix \( \Delta C \Delta = \tilde{Z}^T A \tilde{Z} \) is positive definite. From eigenvalue interlacing inequalities (1.1),

\[
\lambda_k(A) \geq \lambda_k(\tilde{Z}^T A \tilde{Z}) = \lambda_k(\Delta C \Delta) > 0.
\]

Finally, using (1.3) we conclude \( \lambda_{k-1}(M) \geq \lambda_k(A) > 0 \) and the proof is complete. \( \Box \)

In the subsequent theorem we apply an argument similar to the one in the above-mentioned result directly to the matrix \( M \) instead of \( A \), as intermediate step. Before that, it is convenient to introduce an auxiliary notation.

Let \( S_1 \) and \( S_2 \) two disjoint subsets of \( V \). We define their joint modularity as

\[
Q(S_1, S_2) = e(S_1, S_2) - \frac{\text{vol} S_1 \text{vol} S_2}{\text{vol} G}.
\]

Its absolute value \( |Q(S_1, S_2)| \) is sometimes referred to as discrepancy between \( S_1 \) and \( S_2 \), see e.g., [8, §5.2] and [14]. The following properties are straightforward:

1. Clearly, \( Q(S_1, S_2) = Q(S_2, S_1) \) and \( Q(S) = Q(S, \emptyset) \). Furthermore, we can express the joint modularity of \( S_1 \) and \( S_2 \) equivalently as

\[
Q(S_1, S_2) = 1_{S_1}^T M 1_{S_2}.
\]

Note that \( Q(S_1, S_2) \) is the difference between the overall weight of edges bridging \( S_1 \) and \( S_2 \) and its value as expected by the (weighted) Chung-Lu model.

2. From the equation \((1_{S_1} + 1_{S_2})^T M (1_{S_1} + 1_{S_2}) = 1_{S_1}^T M 1_{S_1} + 1_{S_2}^T M 1_{S_2} + 2 1_{S_1}^T M 1_{S_2}\) we have

\[
Q(S_1 \cup S_2) = Q(S_1) + Q(S_2) + 2Q(S_1, S_2).
\]
In particular, \( Q(S_1, S_2) > 0 \) if and only if \( Q(S_1 \cup S_2) > Q(S_1) + Q(S_2) \). Hence, when looking for an optimal partitioning of \( G \) into modules, it is necessary that the joint modularity of any two subsets is \( \leq 0 \), otherwise, we can increase the overall modularity by merging two subgraphs into one.

The forthcoming theorem proves that, under ample hypotheses, the number of positive eigenvalues of \( M \), plus 1, is actually an upper bound for the cardinality of any partition of \( G \) into modules such that if any two subsets are merged then the overall modularity does not increase.

**Theorem 6.2.** Let \( \mathcal{P} = \{S_1, \ldots, S_k\} \) be a partition of \( V \), with \( k \geq 2 \), such that \( Q(S_i) > 0 \) and \( Q(S_i, S_j) \leq 0 \) for \( i \neq j \). Consider the matrix \( C \) such that \( C_{ii} = Q(S_i) \) and \( C_{ij} = Q(S_i, S_j) \) for \( i \neq j \). If \( C \) is irreducible then \( M \) has at least \( k - 1 \) positive eigenvalues.

**Proof.** Consider the matrices \( Z = [1_{S_i} \cdots 1_{S_k}] \), \( \Delta = \text{Diag}(|S_1|, \ldots, |S_k|)^{-1/2} \) and \( \hat{Z} = Z \Delta \). Then, \( C = \hat{Z}^T M \hat{Z} \). Furthermore, \( C \) is weakly diagonally dominant. Indeed,

\[
\sum_{j=1}^{k} C_{ij} = 1_{S_i}^T M \sum_{j=1}^{k} 1_{S_j} = 1_{S_i}^T M 1 = 0.
\]

Using Gershgorin theorem we deduce \( C \) is a symmetric positive semidefinite matrix, with a zero eigenvalue which is associated to the eigenvector \( 1 \). For a sufficient large \( \alpha > 0 \) the matrix \( B = \alpha I - C \) is entrywise nonnegative and irreducible. Hence, by Perron-Frobenius theory, its largest eigenvalue is simple. Since the eigenspaces of \( B \) and \( C \) coincide, the zero eigenvalue of \( C \) must be simple.

We deduce that \( C \) has \( k - 1 \) positive eigenvalues. The same conclusion holds true also for the matrix \( \Delta C \Delta = \hat{Z}^T M \hat{Z} \), by Sylvester’s law of inertia. Finally, eigenvalue interlacing inequalities (1.1) imply \( \lambda_{k-1}(M) \geq \lambda_{k-1}(\hat{Z}^T M \hat{Z}) = \lambda_{k-1}(\Delta C \Delta) > 0 \), and the proof is complete. \( \blacksquare \)

Note that, in the preceding theorem, irreducibility of \( C \) is verified in particular when \( Q(S_i, S_j) < 0 \) for all \( i \neq j \). That condition is fulfilled by any partition maximizing \( q_\mathcal{G} \) which contains the least number of sets among all such partitions (otherwise we can reduce their number by merging pairs whose joint modularity is zero without decreasing the overall modularity). We get an immediate corollary:

**Corollary 6.3.** Let \( \mathcal{P}_* \) be a minimal cardinality partition with \( q_\mathcal{G} = q(\mathcal{P}_*) \), interely made by modules. Then \( \mathcal{P}_* \) contains no more than \( k + 1 \) sets, being \( k \) the number of positive eigenvalues of \( M \).

The following example, which is inspired by a popular benchmark in the community detection literature, shows that this result is optimal:

**Example 6.4 (Circulant ring of clusters).** Given integers \( p > 2 \) and \( q > 2 \), consider the graph consisting of \( n = pq \) vertices, partitioned as \( \mathcal{P} = \{ S_1, \ldots, S_p \} \); every \( G(S_i) \) is a clique of order \( q \); the cliques are arranged circularly, and every node of \( S_i \) is connected to the corresponding node of the two neighboring cliques by an edge whose weight is \( \gamma \in (0, 1) \) (so that the generalized degree of each node is \( q - 1 + 2\gamma \)). In this graph, the \( p \) cliques have positive modularity, and in fact are clearly recognizable as “communities”. We show hereafter that if \( 0 < \gamma \leq 1/2 \) the modularity matrix of this graph has exactly \( p - 1 \) positive eigenvalues.

With a natural numbering of the nodes, the adjacency matrix can be expressed as
a block circulant matrix with circulant blocks,

\[
A = \begin{pmatrix}
B_q & \gamma I & \gamma I \\
\gamma I & B_q & \ddots \\
& \ddots & \ddots & \gamma I \\
\gamma I & & \gamma I & B_q
\end{pmatrix},
\]

where \(B_q = 11^T - I\) is the adjacency matrix of a \(q\)-order clique; and the corresponding modularity matrix is \(M = A - c11^T\), with \(c = (q - 1 + 2\gamma)/n\), which is still block circulant with circulant blocks and, furthermore, simultaneously diagonalizable with \(A\). In fact, let \(C_p \equiv (c_{ij})\) be the \(p \times p\) symmetric circulant matrix such that \(c_{ij} = 1\) if \(|i - j| \equiv 0\) mod \(p\) and \(c_{ij} = 0\) otherwise. Denoting by \(F_k\) the unitary Fourier matrix of order \(k\), we have the spectral factorizations

\[
C_p = F_p \text{Diag}(\lambda_1^{(p)}, \ldots, \lambda_p^{(p)}) F_p^*, \quad \text{with} \quad \lambda_j^{(p)} = 2\cos(2\pi(j - 1)/p),
\]

\[
B_q = F_q \text{Diag}(q - 1, -1, \ldots, -1) F_q^*.
\]

Making use of the Kronecker (tensor) product, the adjacency matrix \(A\) admits the decomposition \(A = I \otimes B_q + \gamma C_p \otimes I\), whence it is diagonalized by \(F_p \otimes F_q\). Consequently, the eigenvalues of \(A\) are readily computed as follows:

a) \(q - 1 + \gamma \lambda_j^{(p)}\) for \(j = 1, \ldots, p\), each of them having multiplicity 1; and

b) \(\gamma \lambda_j^{(p)} - 1\) for \(j = 1, \ldots, p\), each of them having multiplicity \(q - 1\).

A careful observation reveals that, if \(\gamma \leq \frac{1}{2}\) then the \(p\) largest eigenvalues of \(A\) are precisely the numbers in the preceding item a), which are positive; and the remaining eigenvalues are \(\leq 0\).

The eigenvalues of \(M\) coincide with those of \(A\) with the exception of the largest one, which is annihilated by the rank-one correction \(A - M = c11^T\). Consequently, the matrix \(M\) has at least \(p - 1\) positive eigenvalues; for all \(0 < \gamma \leq \frac{1}{2}\), they are exactly \(p - 1\), that is, the number of “communities” minus one. It is interesting to note that eigenvectors associated to these eigenvalues lie in the span of \(\mathbb{R}(f_k) \otimes 1\) and \(\mathbb{R}(f_k) \otimes 1\), where \(f_k\) is the \(k\)-th column of \(F_p\); in particular, they are constant within each clique. Furthermore, for any two distinct integers \(i, j = 1, \ldots, p\), one such eigenvector assumes opposite signs on \(S_i\) and \(S_j\), so that communities in this graph are demarcated precisely by modularity nodal domains associated to positive eigenvalues of \(M\).

7. A Cheeger-type inequality. Let \(G = (V, E)\) an unweighted graph. The number

\[
h_G = \min_{0 < |S| \leq \frac{1}{2}} \frac{|E(S, S)|}{|S|}
\]

is one of best known topological invariants of \(G\), as it establishes a wealth of deep and important relationships with various areas of mathematics [8, 14]. Its connection with graph partitioning, and discrete versions of the isoperimetric problem, is apparent. Hence, it is of no surprise that various relationships have been uncovered between \(h_G\) and \(\alpha(G)\), also under slightly different definitions.

The bound \(h_G \geq \alpha(G)/2\) can be obtained by rather elementary arguments. Various converse inequalities exist and bear the name of Cheeger inequality, analogously to
a classical result in Riemannian geometry that relates the solution of the isoperimetric problem to the smallest positive eigenvalue of the Laplacian differential operator on manifolds. For example, it is known that if $G$ is a $k$-regular graph (that is, $d_i = k$ for $i = 1, \ldots, n$) then $h_G \leq \sqrt{2k\alpha(G)}$ [14, Thm. 4.11]. In the forthcoming Corollary 7.2 we provide a Cheeger-type inequality between modularity and algebraic modularity of a regular graph. Although practical graphs are seldom regular, that hypothesis is important to obtain a converse result to Theorem 5.2.

**Theorem 7.1.** Let $G = (V, E)$ be a connected, $k$-regular graph, and let $f$ be an eigenvector associated to $m(G)$: $Mf = m(G)f$. Let $w_1 \geq w_2 \geq \ldots \geq w_n$ be the values of $f_1, \ldots, f_n$ sorted in nonincreasing order. Introduce the sets

$$S_i = \{j : f_j \leq w_i\}, \quad i = 1, \ldots, n,$$

and let $Q_* = \max_i Q(S_i)$. Then,

$$Q_* \geq \frac{1}{w_1 - w_n} \left( \frac{k}{2} \| f \|_1 - \| f \|_2 \sqrt{(k - m(G)) \frac{kn}{2}} \right).$$

**Proof.** We start by noticing that $f$ is a Fiedler vector of $G$. Indeed, if $G$ is $k$-regular then the matrix $L_0$ in (3.2) becomes $L_0 = kI - (k/n)11^T$ whence $L_0 f = kf$; moreover, from the equation $m(G) = k - a(G)$ and the decomposition (3.2) we obtain $L f = a(G) f$, that is, $f$ is a Fiedler vector.

Consider the quantity

$$\sigma = \sum_{i \sim j} |f_i - f_j|,$$

where the sum runs on the edges of $G$, every edge being counted only once. By Cauchy-Schwartz inequality and (1.5),

$$\sigma \leq \sqrt{\sum_{i \sim j} (f_i - f_j)^2} \sqrt{\sum_{i \sim j} 1} = \| f \|_2 \sqrt{\text{vol} G} \frac{\text{vol} G}{2} = \| f \|_2 \sqrt{(k - m(G)) \frac{kn}{2}}.$$

For ease of notation, we re-number the vertices of $G$ so that $f_1 \geq f_2 \geq \ldots \geq f_n$. In this way, the sets $S_1, \ldots, S_n$ introduced in the claim are given by $S_i = \{1, \ldots, i\}$. Furthermore, the edge boundary $\partial S_i$ is the set of all edges having one vertex in $\{1, \ldots, i\}$ and the other in $\{i + 1, \ldots, n\}$. Let $Q_* = \max_i Q(S_i)$. Using

$$|\partial S_i| \geq \text{vol} S_i \frac{\text{vol} S_i}{\text{vol} G} - Q_* = \frac{k i(n - i)}{n} - Q_*$$

we obtain

$$\sigma = \sum_{i \sim j} (f_i - f_j) = \sum_{i \sim j} \sum_{t=1}^{j-1} (f_t - f_{t+1}) = \sum_{i=1}^{n-1} (f_i - f_{i+1}) \cdot |\partial S_i|$$

$$\geq k \sum_{i=1}^{n-1} (f_i - f_{i+1}) \frac{i(n - i)}{n} - Q_* \sum_{i=1}^{n-1} (f_i - f_{i+1})$$

$$= k \sum_{i=1}^{n} f_i \frac{n + 1 - 2i}{n} - Q_* (f_1 - f_n)$$

$$= 2k \sum_{i=1}^{n} \sum_{j=1}^{i} f_j - Q_* (f_1 - f_n).$$

19
The last passages are obtained by collapsing the telescopic sums, rearranging terms, and exploiting the equality \( \sum_{i=1}^{n} f_i = 0 \). Now, let \( m \) be an integer such that

\[
f_1 \geq \ldots \geq f_m \geq 0 \geq f_{m+1} \geq \ldots \geq f_n.
\]

(7.1)

Owing to the fact that \( \sum_i f_i = 0 \) we have

\[
\max_i \sum_{j=1}^{i} f_j = \sum_{j=1}^{m} f_j = \frac{1}{2} \left( f_1 + \cdots + f_m + |f_{m+1}| + \cdots + |f_n| \right) = \frac{1}{2} \|f\|_1.
\]

Introduce the notation \( F_i = \sum_{j=1}^{i} f_j \). By virtue of the inequalities (7.1), for all \( j = 0, \ldots, m \) and \( k = 0, \ldots, n - m \) we have

\[
F_j + F_{m-j} \geq F_m, \quad F_{m+k} + F_{n-k} \geq F_m.
\]

Thus we obtain

\[
\sum_{i=1}^{n} \sum_{j=1}^{i} f_j = \sum_{i=1}^{n} F_i = \frac{1}{2} \sum_{j=1}^{m} (F_j + F_{m-j}) + \frac{1}{2} \sum_{k=0}^{n-m} (F_{m+k} + F_{n-k})
\]

\[
\geq \frac{n}{2} F_m = \frac{n}{4} \|f\|_1.
\]

Putting it all together we get

\[
\frac{k}{2} \|f\|_1 - Q_*(f_1 - f_n) \leq \sigma \leq \|f\|_2 \sqrt{(k - m(G)) \frac{kn}{2}}.
\]

whence we obtain the claim. \( \square \)

**Corollary 7.2.** If \( G = (V, E) \) is a connected, \( k \)-regular graph then

\[
\frac{1}{2n} - \sqrt{\frac{k-m(G)}{2k}} \leq q_G \leq \frac{m(G)}{2k}.
\]

**Proof.** The upper bound directly follows by Theorem 6.1 in the \( k \)-regular case. In the notations of the preceding theorem we observe that \( w_1 - w_n = w_1 + |w_n| \leq 2 \|f\|_\infty \).

Using the inequality \( \|f\|_2 \leq \sqrt{n} \|f\|_\infty \) and \( \|f\|_\infty \leq \|f\|_1 \) we obtain

\[
Q_* \geq k \|f\|_1 \frac{1}{4 \|f\|_\infty} - \frac{\|f\|_2}{2 \|f\|_\infty} \sqrt{(k - m(G)) \frac{kn}{2}} \geq \frac{k}{4} - \sqrt{n} \sqrt{(k - m(G)) \frac{kn}{2}}.
\]

To complete the proof it is sufficient to observe that, in view of (5.2), we have \( q_G \geq 2Q_*/\text{vol} \leq \frac{2Q_*/(kn)}{2} \).

**8. Concluding remarks.** In this paper we have studied the community detection problem through modularity optimisation from an uncommon algebraic point of view. In particular we have tried to propose popular concepts from complex networks and physics literatures in a mathematical formalism involving mainly linear algebra and matrix theory.

We introduce the concept of algebraic modularity of a graph, allowing to clarify the difference between indivisible graph and algebraically indivisible graph, often used with not much attention interchangeably one with the other. We focus our attention
on the nodal domains induced by the eigenvectors of the modularity matrix and we derive a nodal domain theorem for such eigenvectors, in complete analogy with the well known Fiedler vector theorem for the Laplacian matrix [12], and some further developments proposed more recently in [9, 10, 19]. However, unlike in the Laplacian case, nodal domains arising with modularity matrices are naturally endowed by a sign, with different properties for positive and negative nodal domains.

Then we consider the possible relationship between the number of modules in $G$ and the number of positive eigenvalues of its modularity matrix. Newman claimed in [15] that the number of positive eigenvalues of $M$ is related to the number of communities recognizable in the graph $G$, but his claim was based on rather informal arguments. Our analysis of $M$ instead tries to support this claim showing, in particular, that the presence of communities in $G$ implies that the spectrum of $M$ at least partially lies on the positive axis. We would point out here that a reverse implication is realistic and desirable, but is still an open problem.

Finally we focus the attention on Cheeger-type inequalities, discovering that a nice estimate elapses between modularity and algebraic modularity of $G$. At present, our result is limited to regular graphs; its possible extension to more general graphs seems to be a major task and is left as an open problem.

As the importance of the community detection problem is apparent, and modularity-based techniques are by far the most popular in this ambit, we believe that the modularity matrix $M$ could be considered as a relevant matrix in algebraic graph theory, together with adjacency and Laplacian matrices. The results we obtain give rise to a first spectral graph analysis aimed at the problems of existence, estimation and localization of optimal subdivisions of the graph into communities. Our results adhere to modularity-related definitions borrowed from current literature. Probably, modified (maybe, “normalized”) versions of modularity matrices and functions may lead to conclusions different from those presented here.

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REFERENCES

[1] N. Arcolano, K. Ni, B. A. Miller, N. T. Bliss, and P. J. Wolfe. Moments of parameter estimates for Chung-Lu random graph models. In IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP 2012), pages 3961–3964, 2012.
[2] A. Arenas, A. Fernandez, and S. Gomez. Analysis of the structure of complex networks at different resolution levels. New J. Phys., 10:053039, 2008.
[3] A. Berman and R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences, volume 9 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.
[4] R. Bhatia. Matrix Analysis. Graduate Texts in Mathematics. Springer, 1996.
[5] V. D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre. Fast unfolding of communities in large networks. Journal of Statistical Mechanics: Theory and Experiment, 2008(10):P10008, 2008.
[6] U. Brandes and T. Erlebach, editors. Network Analysis. Methodological Foundations, volume 3418 of Lecture Notes in Computer Sciences. Springer, 2005.
[7] F. Chung and L. Lu. Complex Graphs and Networks, volume 107 of CBMS Regional Conference Series in Mathematics. AMS, 2006.
[8] F. R. K. Chung. Spectral Graph Theory, volume 92 of CBMS Regional Conference Series in Mathematics. AMS, 1997.
[9] E. B. Davies, G. M. L. Gladwell, J. Leydold, and P. F. Stadler. Discrete nodal domain theorems. Linear Algebra Appl., 336:51–60, 2001.
[10] A. M. Duval and V. Reiner. Perron-Frobenius type results and discrete versions of nodal domain theorems. *Linear Algebra Appl.*, 294:259–268, 1999.

[11] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23:298–305, 1973.

[12] M. Fiedler. A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory. *Czechoslovak Mathematical Journal*, 25(100):619–633, 1974.

[13] S. Fortunato. Community detection in graphs. *Physics Reports*, 486:75–174, 2010.

[14] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.

[15] M. E. J. Newman. Finding community structure in networks using the eigenvectors of matrices. *Phys. Rev. E*, 69:321–330, 2006.

[16] M. E. J. Newman. Modularity and community structure in networks. *Proc. Natl. Acad. Sci. USA*, 103:8577–8582, 2006.

[17] M. E. J. Newman. *Networks: An Introduction*. OUP Oxford, 2010.

[18] M. E. J. Newman and M. Girvan. Finding and evaluating community structure in networks. *Phys. Rev. E*, 69(026113), 2004.

[19] D. L. Powers. Graph partitioning by eigenvectors. *Linear Algebra Appl.*, 101:121–133, 1988.

[20] N. Przulj and D. J. Higham. Modelling protein-protein interaction networks via a stickiness index. *J. Roy. Soc. Interface*, 3:711–716, 2006.

[21] J. Reichardt and S. Bornholdt. Statistical mechanics of community detection. *Phys. Rev. E*, 74:016110, 2006.

[22] V. A. Traag, P. Van Dooren, and Y. Nesterov. Narrow scope for resolution-limit-free community detection. *Phys. Rev. E*, 84:016114, Jul 2011.