Dynamical chiral symmetry breaking in QED$_3$ at finite density and impurity potential

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We study the effects of finite chemical potential and impurity scattering on dynamical fermion mass generation in (2+1)-dimensional quantum electrodynamics. In any realistic systems, these effects usually cannot be neglected. The longitudinal component of gauge field develops a finite static length produced by chemical potential and impurity scattering, while the transverse component remains long-ranged because of the gauge invariance. Another important consequence of impurity scattering is that the fermions have a finite damping rate, which reduces their lifetime staying in a definite quantum state. By solving the Dyson-Schwinger equation for fermion mass function, it is found that these effects lead to strong suppression of the critical fermion flavor $N_c$ and the dynamical fermion mass in the symmetry broken phase.

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I. INTRODUCTION

The dynamical chiral symmetry breaking (DCSB) in (2+1)-dimensional quantum electrodynamics (QED$_3$) has been investigated intensively for more than twenty years \cite{2,4,9,11,14}. On the one hand, these investigations may help to gain deeper understanding of DCSB in QCD. On the other hand, this non-perturbative phenomenon gives an elegant field theoretic description for the formation of long-range antiferromagnetic order in two-dimensional quantum Heisenberg antiferromagnet \cite{15,16,17}.

Using the Dyson-Schwinger (DS) equation approach, Appelquist et al. first found that DCSB caused by fermion mass generation takes place only when the fermion flavor $N$ is below some critical value $N_c$ \cite{3}. When $N > N_c$, the fermions remain massless and the chiral symmetry is preserved. Although there was some controversy about the existence and precise value of $N_c$ in the literature, most analytical and numerical computations confirmed that $N_c \approx 3.5$ \cite{3,6,8,11,14}.

The DCSB is realized by forming fermion-anti-fermion pairs mediated by strong gauge interaction. In order to trigger this low-energy phenomenon, an essential requirement is that the gauge interaction between fermions should be sufficiently long-ranged. If the gauge field develops an effective screening length, then the gauge interaction is significantly weakened. We have considered the effect of a finite gauge boson mass $m_a$ on DCSB previously \cite{17}, and found that the increasing $m_a$ leads to a significant suppression of DCSB.

When the action contains only massless Dirac fermions and non-compact gauge field, the gauge interaction is always long-ranged since the gauge invariance ensures that no photon mass term can be generated by the radial correction from fermions. If the gauge field also couples to an additional Abelian Higgs model, then the photon can acquire a finite mass via the Anderson-Higgs mechanism in the superconducting state \cite{17}. In the context of high temperature superconductor, the suppression of DCSB by photon mass $m_a$ can be used to qualitatively understand the competition between antiferromagnetism and superconductivity, which is one of the most prominent phenomena in high temperature superconductors \cite{17}.

Apart from Anderson-Higgs mechanism, the gauge field may also be screened by other physical effects. For example, when the fermion density is nonzero, the chemical potential $\mu$ should be explicitly taken into account. In addition, the fermions may be scattered by some impurity atom (defect or imperfection) in realistic interacting system described by QED$_3$. Although generally chemical potential and impurity scattering have distinct influences on the physical properties of the interacting fermion system, there is a common feature that they can both induce a finite density of states at the Fermi level. This screens the time component of the gauge field, which then becomes short-ranged. However, the transverse gauge field remains long-ranged in the sense that the static screening length still diverges as at zero chemical potential in the clean limit ("clean" means there is completely no impurity scattering). In principle, the long-range nature of transverse gauge field is guaranteed by the gauge invariance. However, though the transverse part is unscreened, the effective gauge interaction strength is reduced by the fermion density caused by chemical potential and/or impurity scattering process, which certainly affects the critical flavor $N_c$ as well as the dynamical fermion mass. Besides generating finite fermion density, the impurity scattering leads to the damping of fermionic quantum states by producing a finite scattering rate $\Gamma_0$. Such damping effect reduces the lifetime of a fermion staying at a definite quantum state, specified by such quantum numbers as momentum and/or energy. This will also have important effects on DCSB.

While there appeared several papers discussing the effect of chemical potential on DCSB in QED$_3$ \cite{18}, to our knowledge the effect of impurity scattering has never been considered in previous work. Indeed, in any realistic applications of QED$_3$ to condensed matter physics, such as high temperature superconductor, the chemical potential and impurity scattering are usually important and thus should not be ignored. The purpose of this paper is to study the dependence of DCSB on finite den-
sity and impurity scattering.

The paper is organized as follows. In Sec. III we set up the Lagrangian and then discuss the screening effect of gauge interaction by including chemical potential and impurity scattering rate into the vacuum polarization function. We then solve the Dyson-Schwinger equation and present the critical flavor and fermion mass function in Sec. IV. The discussion is presented in Sec. V.

II. DYSON-SCHWINGER EQUATION IN THE PRESENCE OF $\mu$ AND $\Gamma_0$

The Lagrangian for (2+1)-dimensional QED with $N$ flavors of massless fermions is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^{N} \bar{\psi}_a (i \partial_{\mu} - e A_{\mu}) \psi_a. \quad (1)$$

In (2+1)-dimensional space-time, the lowest rank spinorial representation is two-component spinor whose $2 \times 2$ representation may be chosen as the Pauli matrices $\gamma_\mu = (\sigma_2, i\sigma_3, i\sigma_1)$. However, it is impossible to define a $2 \times 2$ matrix that anticommutes with all these matrices. Therefore, there is no chiral symmetry in this representation. The fermion can be described by a four-component spinor field $\psi$, whose conjugate spinor field being defined as $\bar{\psi} = \psi^\dagger \gamma_0$. The $4 \times 4$ $\gamma$-matrices can be defined as $\gamma_3 = (\sigma_3, i\sigma_1, i\sigma_2) \otimes \sigma_3$, satisfying the standard Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, with metric $g_{\mu\nu} = \text{diag}(1, -1, -1)$. It is easy to verify that there are two $4 \times 4$ matrices

$$\gamma_3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

which anticommute with all $\gamma_\mu$. The massless Lagrangian (Eq. 1) preserves a continuous U(2N) chiral symmetry $\psi \rightarrow e^{i\alpha \gamma_5} \psi$. The mass term generated by fermion-anti-fermion pairing will break this global chiral symmetry dynamically to subgroup $U(N) \times U(N)$. In the following we consider a general large $N$ and perform the $1/N$ expansion. For convenience, we work in units where $\hbar = k_B = 1$.

In the Euclidian space, the full gauge field propagator $D_{\mu\nu}(q)$ is given by the equation

$$D_{\mu\nu}^{-1}(q) = D_{\mu\nu}^{(0)}^{-1}(q) + \Pi_{\mu\nu}(q), \quad (2)$$

with the free photon propagator being

$$D_{\mu\nu}^{(0)}(q) = \frac{1}{q^2}(g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2}), \quad (3)$$

in the Landau gauge. To the leading order of $1/N$ expansion, the one-loop contribution to vacuum polarization tensor $\Pi_{\mu\nu}$ is

$$\Pi_{\mu\nu}(q) = -\alpha \int \frac{d^3k}{(2\pi)^3} \frac{\text{Tr}[\gamma_{\mu} k_{\nu}(q + k)]}{k^2(q + k)^2}, \quad (4)$$

where $\alpha = Ne^2$. The QED$_3$ can be treated using the $1/N$ expansion, with the product $\alpha = Ne^2$ being fixed when $N \rightarrow \infty$ and $e^2 \rightarrow 0$. This tensor can also be written as $\Pi_{\mu\nu}(q) = (g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2})\Pi(q^2)$ according to the gauge invariance. It is easy to find the polarization function: $\Pi(q) = \frac{\alpha q^2}{8\pi}$. Now the propagator of gauge field has the form

$$D_{\mu\nu}^{-1}(q) = \frac{1}{q^2 + \frac{\alpha q^2}{8\pi}}(g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2}). \quad (5)$$

In order to study the dynamical fermion mass generation, one can write the following DS equation for fermions propagator

$$S_F^{-1}(p) = S_F^{(0)-1}(p) - e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{k^2 + \Sigma^2(k)} \text{Tr}[\gamma_{\mu} D_{\mu\nu}(p-k)\Gamma_{\nu}(k,p)]. \quad (6)$$

To the leading order in $1/N$ expansion, the vertex function $\Gamma_\nu$ is replaced by the bare matrix $\gamma_{\nu}$. The inverse full propagator of fermion is

$$S_F(p)^{-1} = i\not p + \Sigma(p), \quad (7)$$

where the wave-function renormalization is neglected. Taking trace on both sides of the DS equation, we get an integral equation

$$\Sigma(p) = \frac{e^2}{4} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{k^2 + \Sigma^2(k)} \frac{1}{(p-k)^2 + \frac{\alpha(p-k)}{8\pi}}, \quad (8)$$

which was first obtained by Appelquist et al. [3]. They found that DCSB takes place only when the fermion flavor is less than a critical value $N_c = 32/\pi^2$. Motivated by this interesting prediction, a great many attentions have been paid on this issue. After taking into account the high order corrections to wave function renormalization, the critical flavor was found to change to $N_c = 128/3\pi^2$ [4]. Pennington et al [8] used a more careful truncation of the fermion DS equation and found that DCSB can occur for all values of $N$. When the DS equations of fermion self-energy is coupled self-consistently to those of gauge field propagator, Maris [9] showed that $N_c \simeq 3.3$. In a recent publication, Fischer et al. [11] found that $N_c \simeq 4$ after detailed analysis of vertex corrections. Some numerical simulations on lattice QED$_3$ claim that there is no DCSB for $N > 2$ [12]. However, Gusynin et al. found that the absence of DCSB can be attributed to the large infrared cutoff used in lattice studies and the smallness of the generated mass scale [13]. In summary, most analytical and numerical computations seem to agree that the critical fermion flavor should be $N_c \approx 3.5$ [4, 8, 11, 14], close to the original value obtained by Appelquist et al. within the lowest order of $1/N$ expansion.
The above investigation and result are valid at zero chemical potential in clean fermion system. Till now, little attention has been paid to the case with finite fermion density and finite impurity potential. Since the chemical potential and impurity scattering are generally very important in realistic applications, it is important to examine their effects on DCSB. The aim of this paper is to study this problem.

First of all, at nonzero chemical potential the fermions have a finite density at the Fermi level, which screens the temporal component of gauge field and weakens the gauge interaction. The influence of chemical potential on DCSB was previously discussed by Feng et al. [18]. They derived an explicit equation to include the chemical potential into the DS equation. One crucial assumption in their work is the neglecting of chemical potential in the vacuum polarization function. The numerical results of Feng et al. show that the chemical potential leads to only insignificant change of $N_c$ and fermion mass. We think that the screening effect caused by chemical potential is very important and hence will pay special attention to the screening of gauge field induced by chemical potential by studying the vacuum polarization. As will be shown in the context, the chemical potential suppresses both $N_c$ and fermion mass strongly.

The effect of impurity scattering is more complicated than chemical potential. Generally, the scattering of fermions by impurity potential has two important effects. First, it generates a finite density of states at low energy, which also screens the gauge field. This effect is expected to weaken the gauge interaction, analogous to the role played by the chemical potential. Second, the impurity scattering produces a finite damping of fermion quantum states and thus reduces the time for massless Dirac fermions to interact with their anti-particles. These two effects are both very important.

The behavior of massless Dirac fermions in a random potential is of great interest in the context of condensed matter physics since the low-energy properties of some many-body systems, such as $d$-wave high temperature superconductor and graphene, are largely controlled by the interaction of Dirac fermion with impurities. Unfortunately, at present the problem of random Dirac fermion has not been fully understood even when there is no direct interaction between fermions. When the impurity scattering and gauge interaction are both important, the problem is basically out of theoretical control. If we consider a single impurity atom, then the impurity scattering can be treated by the self-consistent Born approximation. Within this approximation, the retarded fermion self-energy function develops a finite imaginary part, which is usually represented by a constant scattering rate $\Gamma_0$. To study the problem about impurity scattering, it is most convenient to work in the Matsubara formalism and write the fermion propagator as

$$S_F(i\omega_n, p) = \frac{1}{i\omega_n\gamma_0 - \gamma \cdot p}.$$  \hfill (10)

where the frequency is $i\omega_n = i\frac{(2n+1)\pi}{\beta}$ with $\beta = \frac{1}{T}$. Once the impurity scattering rate is taken into account, the fermion frequency should be replaced by

$$i\omega_n \rightarrow i\omega_n + i\Gamma \text{sgn}(\omega_n).$$  \hfill (11)

The fermion damping effect can be intuitively understood as follows. After analytical continuation,

$$i\omega_n \rightarrow \omega + i\delta,$$  \hfill (12)

the retarded fermion propagator has the form

$$S_F^{ct}(\omega, p) = \frac{1}{\omega + i\Gamma_0}$$  \hfill (13)

After Fourier transformation, it becomes

$$S_F(t, r) = e^{i\omega t - \Gamma_0 t} e^{i p \cdot r}.$$  \hfill (14)

in real space. Apparently, the parameter $\Gamma_0$ measures the decaying rate of the fermionic state characterized by such quantum numbers as $(\omega, p)$, which is known as the Landau damping effect. Starting from the propagator Eq. (13), various physical quantities can be calculated and compared with experiments. Intuitively, the damping effect is at variance with DCSB since a fermion may be scattered into another state before it combines with its anti-particle to form a stable pair.

Before we set up the DS equation for fermion self-energy, we need first to calculate the photon propagator and discuss the screening effect caused by chemical potential and impurity scattering within the Matsubara formalism. With finite chemical potential $\mu$ and finite damping rate $\Gamma_0$, the fermion propagator reads

$$S_F(i\omega_n, p) = \frac{1}{(i\omega_n + i\Gamma_0 \text{sgn}(\omega_n) - \mu)\gamma_0 - \gamma \cdot p}.$$  \hfill (15)

The energy shift caused by chemical potential and the damping effect caused by impurity scattering are both reflected in this propagator. The screening of gauge interaction induced by them can be seen from the corresponding vacuum polarization functions. We will use this propagator to calculate the vacuum polarization functions and to construct the DS equation for fermion mass.

The inverse photon propagator for frequency $q_0 = \frac{2m\pi}{\beta}$ and spatial momentum $|q|$ is given by

$$\Delta_{\mu
u}^{-1}(q_0, q, \beta) = \Delta_{\mu\nu}^{(0)}^{-1}(q_0, q, \beta) + \Pi_{\mu\nu}(q_0, q, \beta),$$  \hfill (16)

where the $\Delta_{\mu\nu}^{(0)}$ is the free photon propagator. Here, we use different symbols $D_{\mu\nu}$ and $\Delta_{\mu\nu}$ to denote the gauge boson propagator at zero temperature and finite temperature, respectively. Taking advantage of the transverse condition, $q_\mu \Pi_{\mu\nu}(q) = 0$, the vacuum polarization tensor defined by Eq. (19) can be decomposed in terms of two independent transverse tensors $19$,

$$\Pi_{\mu\nu}(q_0, q, \beta) = \Pi_A(q_0, q, \beta) A_{\mu\nu} + \Pi_B(q_0, q, \beta) B_{\mu\nu}.$$  \hfill (17)
where
\[ A_{\mu\nu} = \left( \delta_{\mu0} - \frac{q_0 q_\nu}{q^2} \right) q^2 \left( \delta_{\nu0} - \frac{q_0 q_\mu}{q^2} \right), \] (18)
\[ B_{\mu\nu} = \delta_{\mu i} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) \delta_{\nu j}. \] (19)

They are orthogonal and related by the relationship
\[ A_{\mu\nu} + B_{\mu\nu} = \delta_{\mu\nu} - \frac{q_0 q_\mu q_\nu}{q^2}. \] (20)

The functions \( \Pi_A(q_0, \mathbf{q}, \beta) \) and \( \Pi_B(q_0, \mathbf{q}, \beta) \) are related to the temporal and spatial components of vacuum polarization tensor \( \Pi_{\mu\nu} \) by the following expressions
\[ \Pi_A = \frac{q_0^2}{q^2} \Pi_{00}, \quad \Pi_B = \Pi_{ii} - \frac{q_0^2}{q^2} \Pi_{00}. \] (21)

Now the full finite temperature photon propagator \( \Delta_{\mu\nu}(q_0, \mathbf{q}, \beta) \) can be written as
\[ \Delta_{\mu\nu}(q_0, \mathbf{q}, \beta) = \frac{A_{\mu\nu}}{q^2 + \Pi_A(q_0, \mathbf{q}, \beta)} + \frac{B_{\mu\nu}}{q^2 + \Pi_B(q_0, \mathbf{q}, \beta)}. \] (22)

At finite temperature, the fermion contribution to vacuum polarization functions should be

\[ \Pi_{\mu\nu}(q, \beta) = -\frac{\alpha}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2 \mathbf{K}}{(2\pi)^2} \text{Tr} \left[ \gamma_\mu k_\nu \gamma_\nu (\gamma^+ + k) \right] \] 
\[ = -\frac{4\alpha}{\beta} \int_0^1 dx \sum_{n=-\infty}^{\infty} \int \frac{d^2 L}{(2\pi)^2} \left[ L_{\mu\nu} + (1 - 2x)(L_{\mu \nu} + q_{\mu \nu} - q \cdot \delta_{\mu\nu}) + 2x(1 - x)(q^2 \delta_{\mu\nu} - q_{\mu \nu}) \right] \] 
\[ \times \frac{-\delta_{\mu\nu}}{[l^2 + x(1 - x)q^2]} \] (23)

where \( q = (q_0, \mathbf{q}), Q = |\mathbf{q}|, q_0 = \frac{2m_\beta}{\beta} \) for gauge boson and \( k = (k_0, \mathbf{k}), K = |\mathbf{k}|, k_0 = \frac{(2n+1)\pi}{\beta} \) for fermion. Here, a new momentum variable is defined by \( l = k + xq \) with
\[ l = (l_0, 1), \quad L = |\mathbf{L}|, \quad l_0 = \frac{2\pi}{\beta} (n + x + m + \frac{1}{2}). \] (24)

In the presence of impurity scattering rate \( \Gamma_0 \) and chemical potential \( \mu \), the variable \( l_0 \) should be replaced by
\[ l_0(n) = \frac{2\pi}{\beta} (n + x + \frac{1}{2} + \frac{\beta}{2\pi} l_0 + \frac{\beta}{2\pi} \Gamma_0 \text{sgn}(\omega_n)). \] (25)

Now the spatial and temporal component of polarization tensor \( \Pi_{\mu\nu} \) can be expressed as
\[ \Pi_{ij}(q_0, \mathbf{q}, \beta) = \frac{4\alpha}{\beta} \int_0^1 dx \int \frac{d^2 L}{(2\pi)^2} \] 
\[ \times \left[ x(1 - x)(q^2 \delta_{ij} - q_i q_j)S_2 - (1 - x)\delta_{ij} q_0 S^* \right], \] (26)
\[ \Pi_{00}(q_0, \mathbf{q}, \beta) = \frac{4\alpha}{\beta} \int_0^1 dx \int \frac{d^2 L}{(2\pi)^2} \] 
\[ \times \left[ S_1 - 2(L^2 + x(1 - x)q_0^2)S_2 + (1 - x)q_0 S^* \right] \] (27)

where, following Ref. [19], we defined
\[ S_i = \sum_{n=-\infty}^{\infty} \frac{1}{l_0(n)} \frac{l_0^2(n) + L^2 + x(1 - x)q^2}{[l_0^2(n) + L^2 + x(1 - x)G]_n^2}, \] (28)
\[ S^* = \sum_{n=-\infty}^{\infty} \frac{l_0(n)}{l_0(n)} \frac{l_0^2(n) + L^2 + x(1 - x)q^2}{[l_0^2(n) + L^2 + x(1 - x)G]_n^2}. \] (29)

It is not easy to calculate Eq. (26) and Eq. (27) analytically. Nevertheless, within the widely used instantaneous approximation \( q_0 = 0 \), the integration can be performed by the methods presented in Ref. [19] and [20]. After tedious but straightforward computation, we obtain the following expressions for polarization functions:
\[ \Pi_A(q, \beta) = \frac{2\alpha}{\pi \beta} \int_0^1 dx \frac{10^{ln} X_\Gamma + 1}{2} \] 
\[ \times \ln \left[ 2 \left( \cosh \left( \frac{2\pi X_\Gamma(x)}{10^{ln} X_\Gamma + 1} \right) + \cosh \left( \frac{2\pi X_\mu}{10^{ln} X_\Gamma + 1} \right) \right) \right], \]
\[ \Pi_B(q, \beta) = \frac{4\alpha}{\beta} \int_0^1 dx X_\mu(x) \] 
\[ \times \frac{\cosh \left( \frac{2\pi X_\Gamma(x)}{10^{ln} X_\Gamma + 1} \right) + \cosh \left( \frac{2\pi X_\mu}{10^{ln} X_\Gamma + 1} \right)}{\cosh \left( \frac{2\pi X_\Gamma(x)}{10^{ln} X_\Gamma + 1} \right) + \cosh \left( \frac{2\pi X_\mu}{10^{ln} X_\Gamma + 1} \right)}, \] (30)

where \( X_\Gamma(x) = \frac{\beta}{2} \sqrt{x(1 - x)}q, \quad X_\mu = \frac{\beta}{2} \Gamma_0, \quad X_\mu = \frac{\beta}{2} \mu. \) In either of the limits \( \beta \to \infty \) (\( T \to 0 \)) or \( q \to \infty \), both the above functions reduce to \( a \alpha / 8 \) and then the photon propagator reduces to the zero-temperature result (Eq. (5)). However, in the limit \( q \to 0, \Pi_A(q, \beta) \) becomes a function of temperature, chemical potential, and impurity scattering rate, \( M_{00}(\beta, X_\Gamma, X_\mu) \), and in the zero energy and zero momentum limit, the photon propagator is
\[ \Delta_{\mu\nu}(q_0 = 0, \mathbf{q} \to 0, \beta) = \frac{A_{\mu\nu}}{q^2 + M_{00}^2} + \frac{B_{\mu\nu}}{q^2 + M_{00}^2}, \] (31)

Apparently, the temporal component of the propagator
now acquires an effective mass and becomes
\[
M_{00} = \sqrt{\frac{2\alpha}{\pi\beta}(10^{10}X_T + 1)} \ln \left[ 2 \cosh \left( \frac{\pi X_\mu}{10^{10}X_T + 1} \right) \right].
\]
(32)

The mass appearing in the longitudinal photon implies that the electric field acquires a static screening length determined by the chemical potential and impurity scattering rate. The transverse photon, however, remains massless and hence the corresponding magnetic field is still long-ranged, albeit dynamically screened.

We now study the problem of DCSB at finite temperature. As in the case of zero temperature, this problem can be studied by the DS equation approach. The DS equation for fermion propagator \( S_F \) at finite temperature is given by
\[
S_F^{-1}(p_0, \mathbf{p}, \beta) = S_F^{(0)-1}(p_0, \mathbf{p}, \beta) - \frac{e^2}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \gamma_\mu S_F(k_0, \mathbf{k}, \beta) \Gamma_\nu \delta_{\mu\nu} (q_0, \mathbf{q}, \beta),
\]
(33)

where \( \Gamma_\nu \) is the fermion-photon vertex and \( q = p - k \).

The full fermion propagator can be written as
\[
S_F^{-1}(p_0, \mathbf{p}, \beta) = A(p_0, \mathbf{p}, \beta) S_F^{(0)-1} + \Sigma(p_0, \mathbf{p}, \beta),
\]
(34)

where \( A(p_0, \mathbf{p}, \beta) \) is the wave-function renormalization. Substituting Eq. (13) and Eq. (34) into Eq. (33) and then taking trace on both sides of this equation, we get the following couple of closed integral equations
\[
A(p_0, \mathbf{p}, \beta) = 1 + \frac{e^2}{4\beta} \sum_{n} \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[ S_F^{(0)} \gamma_\mu S_F \Gamma_\nu \delta_{\mu\nu} \right],
\]
(35)

Here, \( \Delta_{\mu\nu} \) is the photon propagator as defined by Eq. (10) and the \( \Gamma_\nu \) is the full vertex function. To treat this equation, a number of approximations should be made.

At present, there is no well-controlled way to choose the vertex function \( \Gamma_\nu \). In order to satisfy the Ward-Takahashi identity, Maris et al. [9] studied several different Ansätze for the full vertex function and compared the results. For the sake of simplicity, they assumed that \( \Gamma_\nu = f(A(p), A(k), A(p - k)) \gamma_\nu \), with \( A(p) \equiv 1 \) corresponding to the bare vertex. They found that, within a range of vertex functions, the critical behavior of DCSB is almost independent of the precise form of \( f \)-function and the bare vertex is actually a good approximation. Once \( \Gamma_\nu = \gamma_\nu \) is assumed, then the Ward-Takahashi identity requires that the wave function renormalization should be \( A(p) \equiv 1 \).

In the present case, it is much more difficult to solve the DS equations Eq. (35) with finite chemical potential and impurity scattering than in the clean system. Unlike the zero temperature case, the particle energy and momentum should be treated separately in the finite temperature perturbation theory. Now, the energy becomes discrete frequencies proportional to the temperature \( T \). To simplify the theoretical and numerical analysis, we keep only the leading order of \( 1/N \) expansion by replacing the vertex function \( \Gamma_\nu \) by \( \gamma_\nu \) and taking \( A(p) = 1 \). The DS equation of fermion Eq. (34) can now be simplified as
\[
S_F^{-1}(p_0, \mathbf{p}, \beta) = \left[ (\mathbf{i} \omega_n + i \Gamma_0 \text{sgn}(\omega_n) - \mu) \gamma_0 - \gamma \cdot \mathbf{p} \right] + \Sigma(p_0, \mathbf{p}, \beta),
\]
(36)

The fermion mass function \( \Sigma(p_0, \mathbf{p}, \beta) \) satisfies the following integral equation
\[
\Sigma(p_0, \mathbf{p}, \beta) = \frac{e^2}{\beta} \sum_{n} \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[ \gamma_\mu S_F \Gamma_\nu \delta_{\mu\nu} (q_0, \mathbf{q}, \beta) \right]
\]
\[
\times 4 \left[ (\mathbf{i} \omega_n + \Gamma_0 \text{sgn}(\omega_n) + i \mu)^2 + \mathbf{k}^2 + \Sigma^2(k_0, \mathbf{k}, \beta) \right]
\]
\[
\times \left[ (n + \frac{1}{2}) + X_T \text{sgn}(\omega_n) + i X_\mu \right]^2 + M^2,\]
(37)

where
\[
X_T = \frac{\beta}{2\pi}, \quad M = X_T \sqrt{\mathbf{k}^2 + \Sigma^2(k, \beta)}.
\]
(38)

Note that chemical potential and impurity scattering rate both appear in two places: in the occupation number and in the polarization function. The energy shift induced by chemical potential and the fermion damping effect induced by impurity scattering are both represented in the former place. The screening of gauge interaction caused by chemical potential and impurity scattering is reflected in the vacuum polarization function. In order to perform the frequency summation in Eq. (37), we utilize the approximation \( \Sigma(p_0, \mathbf{p}, \beta) \approx \Sigma(p_0 = 0, \mathbf{p}, \beta) \). At \( \omega_n = q_0 = 0 \), the summation over \( \omega_n \) in Eq. (37) yields the equation
\[
\Sigma(p, \beta) = \frac{\beta \alpha}{4N\pi^2} \int \frac{d^2k}{(2\pi)^2} \frac{\Sigma(k, \beta)}{M} \Delta_{\mu\nu} (q_0 = 0, \mathbf{q}, \beta)
\]
\[
\times \Im \left[ \psi \frac{1}{2} + X_T \pm i X_\mu + i M \right].
\]
(39)

Substituting the full photon propagator Eq. (22) into the mass equation and use the same approximative method as that presented in Ref. [20], we have
\[
\Sigma(p, \beta) = \frac{\beta \alpha}{4N\pi^2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{1}{q^2 + \Pi_A(q, \beta)} + \frac{1}{q^2 + \Pi_B(q, \beta)} \right]
\]
\[
\times \left\{ \frac{\Sigma(k, \beta)}{M} \Im \left[ \psi \frac{1}{2} + X_T \pm i X_\mu + i M \right] \right\}
\]
\[ \approx \frac{\beta \alpha}{4N\pi^2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{1}{q^2 + \Pi_A(q, \beta) + \frac{1}{q^2 + \Pi_B(q, \beta)}} \right] \times \left\{ \frac{\Sigma(k, \beta)}{M} \frac{\pi}{2} \frac{\tanh \left[ \frac{\pi (M \pm x_{\mu})}{10^{n/2} X_T + 1} \right]}{10^{n/2} X_T + 1} \right\}. \]  

(40)

If this nonlinear integral equation develops a nontrivial solution, then the massless fermion acquires a finite mass.

III. SOLUTION OF THE DYSON-SCHWINGER EQUATION

The nonlinear integral equation can be solved by the bifurcation theory and parameter imbedding method [21, 22]. The basic idea and detailed computation procedures are presented in previous papers [17, 21, 22]. To determine the bifurcation point that separates the chiral symmetric phase and symmetry broken phase, we need to find the eigenvalues of the associated linearized equation.

Taking the Fréchet derivative of the nonlinear integral equation Eq. (40), we have

\[ \Sigma(p, \beta) = \frac{\beta \alpha}{4N\pi^2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{1}{q^2 + \Pi_A(q, \beta) + \frac{1}{q^2 + \Pi_B(q, \beta)}} \right] \times \Sigma(k, \beta) \frac{\partial}{\partial \Sigma} \left\{ \frac{\Sigma(k, \beta)}{M} \frac{\pi}{2} \frac{\tanh \left[ \frac{\pi (M \pm x_{\mu})}{10^{n/2} X_T + 1} \right]}{10^{n/2} X_T + 1} \right\} \right|_{\Sigma=0} \]

\[ = \frac{\beta}{4N\pi^2} \int \frac{d^2k}{(2\pi)^2} \left[ \frac{1}{q^2 + \Pi_A(q, \beta) + \frac{1}{q^2 + \Pi_B(q, \beta)}} \right] \times \Sigma(k, \beta) \frac{\pi}{2} \frac{\tanh \left[ \frac{\pi (X_T |k| \pm X_{\mu})}{10^{n/2} X_T + 1} \right]}{10^{n/2} X_T + 1}. \]

(41)

To facilitate numerical calculation, we divide the momenta, temperature, fermion mass, chemical potential and impurity scattering rate by parameter \(\alpha\) and make them dimensionless. The UV cutoff for momentum now is replaced by \(\Lambda/\alpha\), which should be properly chosen to ensure the results insensitive to UV cutoff. As pointed out by Appelquist et al. [2, 3], the integral in the zero temperature DS equation damps rapidly for momenta greater than \(\alpha\), it is thus natural to set \(\Lambda/\alpha \simeq 1\). This is also true for DS equation at finite temperature [5]. In the following we take advantage of this fact and assume that \(\Lambda/\alpha = 1\).

The dependence of critical flavor number \(N_c\) on chemical potential \(\mu\) and impurity scattering rate \(\Gamma_0\) are shown in Fig. 1 for a list of values of temperature \(T = 10^{-8}, 10^{-6}, 10^{-4}\). It is easy to see that the critical flavor \(N_c\) is a decreasing function of \(T, \mu\), and \(\Gamma_0\), implying a strong suppression of DCSB by thermal fluctuation, fermion density, and impurity scattering.

The fermion mass function \(\Sigma(p, \beta)\) can be obtained after solving the DS equation Eq. (40) using the straightforward iteration method. The results are presented in Fig. 2 and Fig. 3. In Fig. 2 we show the re-scaled zero-momentum fermion mass \(\Sigma(0)\) for several different values of fermion flavor \(N\) at a fixed temperature \(T = 10^{-10}\), with \(\mu = \Gamma_0 = 0\). Apparently, the fermion mass \(\Sigma(0)\) decreases rapidly as the fermion flavor \(N\) increases. This is easy to understand since the inverse of fermion flavor \(1/N\) serves as the effective coupling strength of gauge interaction.

The dynamical fermion mass as a function of momentum is shown in Fig. 3 for \(N = 2\) at different values of temperatures, chemical potential and impurity scattering rate. We notice that the dynamical fermion mass is significantly suppressed by the increasing chemical potential and impurity scattering rate. It also decreases with the increasing fermion momentum.

We next would like to compare the results presented above with those in QED\(_3\) with finite photon mass gen-
erated by Anderson-Higgs mechanism. Different from the partial screening of gauge interaction, the Anderson-Higgs mechanism induces a complete static screening. In this case, the gauge boson acquires a physical mass by eating the massless Goldstone bosons, so that the propagator (Eq. 22) becomes

$$\Delta_{\mu\nu}(q_0, q, \beta) = \frac{A_{\mu\nu}}{q^2 + \Pi_A(q_0, q, \beta) + m_a^2} + \frac{B_{\mu\nu}}{q^2 + \Pi_B(q_0, q, \beta) + m_a^2}. \quad (42)$$

If $m_a \neq 0$, both temporal and spatial components of gauge field develop a finite screening length, which is given by the gauge boson mass. Such screening length eliminates the contribution of small momenta to the DS equation Eq. (41). It is expected that a large $m_a$ will prevent the DCSB. The dependence of critical flavor $N_c$ on $m_a$ and $\Gamma_0$ is shown in Fig. 4. The gauge boson mass and impurity scattering lead to similar suppression effect of DCSB.

IV. SUMMARY AND DISCUSSION

In summary, we studied the effects of finite chemical potential and impurity scattering on dynamical mass generation in QED$_3$. In the realistic applications of QED$_3$ to condensed matter systems, these effects usually cannot be ignored. By solving the DS equation for fermion mass function, we found that both chemical potential and impurity scattering lead to strong reduction of critical fermion flavor $N_c$ and dynamical fermion mass. In reality, the chemical potential, impurity scattering, and even a finite gauge boson mass may coexist at the same time. When their effects are all important, the DCSB is completely suppressed. These results impose an important constraint on the applicability of QED$_3$ in various physical systems. For a system with large fermion density and impurity potential, it seems impossible that DCSB can indeed take place.

We now would like to briefly discuss the meaning of finite chemical potential and the possible experimental study of its effect on DCSB. In condensed matter physics, the zero temperature ground state (vacuum) of a many-body system is characterized by the presence of a Fermi level which separates the fully occupied and fully empty states. In some condensed matter systems, such as high temperature superconductor and graphene, the valence band and conduction band touch at discrete Dirac points. When the Fermi level lies exactly at Dirac points, the chemical potential is zero and the low-energy excitations

![Figure 2](image1.png) Zero-momentum mass $\Sigma(0)$ for different values of $N$ at $T = 10^{-10}$ and $\mu = \Gamma_0 = 0$.

![Figure 3](image2.png) (a) Zero-momentum mass $\Sigma(0)$ for different values of $\mu$ and $\Gamma_0$ at $T = 10^{-8}$; (b) Dependence of mass $\Sigma(p)$ on momentum $p$ for different values of $\mu$ and $\Gamma_0$ at $T = 10^{-6}$.

![Figure 4](image3.png) Dependence of $N_c$ on $m_a$ and $\Gamma_0$ at $T = 10^{-8}$. 

are massless Dirac fermions. The Fermi level moves upwards (downwards) from the Dirac points once the fermion density increases (decreases). Now the chemical potential is just the quantity that measures how the new Fermi level is far from the original Dirac points. At zero chemical potential, the pairs are formed by fermions slightly above Dirac points and anti-fermions (holes in condensed matter terminology) slightly below the Dirac points. Once the fermions become massive, they are confined. At finite chemical potential, the pairs are formed by fermions slightly above the finite Fermi surface and anti-fermions slightly below Fermi surface. The massive fermions are also confined by the gauge force.

In realistic condensed matter systems, the fermion density or chemical potential can be continually turned by chemical doping or gate voltage. We first have a number of samples, each with different chemical potential, and let them stay at some finite temperature where DCSB is of samples, each with different chemical potential, and let them stay at some finite temperature where DCSB is completely prevented by thermal fluctuations. We then cool down these samples and study whether DCSB indeed take place at very low temperature by measuring some observable quantities, such as specific heat, susceptibility, and electric conductivity. Through this way, in principle the effect of finite chemical potential on the fate of DCSB can be experimentally studied.

We emphasize that our treatment on the random potential applies only to the case of weak impurity potential where the self-consistent Born approximation is valid. When the random potential caused by impurities takes the form of Gaussian white noise, such treatment might be questionable. For this kind of impurities, the random potential has to be averaged by performing proper functional integration, which yields an effective four-fermion interaction in the whole action. This new effective interaction can contribute a new term to the DS equation. At present, it is unclear technically how to study the DCSB and the fermion damping effect in a unified framework. This problem is subject to further investigation.

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