The Spherical Landau Problem

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Abstract

The magnetization for electrons on a two-dimensional sphere, under a spherically symmetrical normal magnetic field has been studied in the large field limit. This allows us to use an Euclidean approximation for low energies electron states getting an analytical solution for the problem and avoiding the difficulties of quantization on a curved manifold. At low temperatures our results are exact and allow direct comparison with the planar Landau case. In this temperature limit we compute the magnetization and show it exhibit an oscillatory de Hass-Van Alphen type of behaviour.

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1. Introduction

Landau studied the problem of the magnetization of a system of free-electrons in the presence of a perpendicular magnetic field in Euclidean three-space([1]). It is well known that, at low temperatures and in the limit of a strong magnetic field $B$, many magnetic properties of the system show an oscillatory behaviour as a function of $1/B([5])$. Typical examples are the de Haas-van Alphen effect and the Shubnikov-de Haas effect. The first one refers to the magnetic susceptibility while the other to the magnetoresistance. Those effects are described with the simple model of free electrons with an effective mass $\mu$. The electron spin coupling with the external magnetic field and spin-orbit contributions are approximately incorporated through an effective scalar giromagnetic $g$ factor. Boundary effects have been considered in several papers (see references within([4])) and the effects of confining potentials were studied by Kubo ([2]) at the low temperature limit. More recently ([4]) dealt with the problem of current distribution under harmonic confining potentials.

In this work our main goal is to study the effects of the curvature of a two-dimensional substrate in the orbital magnetic properties in the limits of strong field and low temperature. For the sake of simplicity we assume that the curvature is constant and positive, that is, we consider the case of a normal magnetic field to a 2-dimensional sphere. The high field limit allows one to work in a semilocal approximation. This is due to the fact that the classical motion of the electron in the mentioned limit is, for small values of energy, confined to a small neighborhood of its initial position. The classical canonical quantization for the Euclidean plane can then be invoked as a first order approximation for the problem. As it will be shown, the first order terms of our results coincides with the standards results for the planar Landau problem([5]).

The paper is organized as follows. In section(2) we built our model and compute the energy eigenvalues. In section(3) we compute the free-energy. In section(4) we study the magnetization of our system and show that a de Hass-Van Alphen effect is present. In section(5) we draw our final conclusions.

2. The Quantization

The classical hamiltonian of a charged particle of charge $e$, with effective mass $\mu$ on a sphere of radius $r$ under a constant normal magnetic field is given by

$$H = \frac{(p_\theta - eA_\theta)^2}{2\mu r^2} + \frac{(p_\phi - eA_\phi)^2}{2\mu r^2 \cos^2(\theta)}.$$  \hspace{1cm} (1)

Here $(\phi, \theta)$ are standard spherical coordinates and the magnetic field is given by

$$\vec{B}(\theta, \phi) = \nabla \times \vec{A}(\theta, \phi),$$

and $A_\theta, A_\phi$ are such that $\vec{B} = b\hat{r}$. We work in the gauge

$$A_\theta = 0 \text{ and } A_\phi = b r \theta.$$  

The main reason for this gauge choice is that the Hamiltonian becomes $\phi$ independent and therefore separable. By multiplying the Hamiltonian ([4]) by $r^2$ one redefines the system’s energy $E$. For a fixed energy in the limit of large $b$, and considering $\theta(0) \approx 0$ it follows from the energy equation that

$$|\theta(t)| < \frac{|p_\phi| + \sqrt{2\mu E}}{ebr}.$$  \hspace{1cm} (2)
Therefore for large $b$ and small $E$ we have that $|\theta(t)|$ is small. Expanding the Hamiltonian in $\theta$ we obtain

$$\tilde{H} = \frac{p_\theta^2}{2\mu} + \frac{(p_\phi - ebr)^2}{2\mu \cos^2(\theta)} = \frac{p_\theta^2}{2\mu} + \frac{(p_\phi - ebr)^2}{2\mu} \left(1 + \theta^2 + O(|\theta^3|)\right),$$

where $\tilde{H} = r^2 H$. We define the cyclotron frequency $\omega_c = \frac{e b}{\mu}$. Neglecting third order terms we quantize $\tilde{H}$ following the standard canonical procedure, obtaining the time independent Schrödinger equation

$$\left\{ -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial \theta^2} + \frac{1}{2\mu} \left(i\hbar \frac{\partial}{\partial \phi} + ebr\right)^2 \left(1 + \theta^2\right) \right\} \Psi(\theta, \phi) = E \Psi(\theta, \phi).$$

(3)

We stress that, since we are working in a non-flat, non-contractible manifold, the quantization approach is not well defined\(^{(5)}\). Therefore, our choice for this procedure is based on the classical locality argument given above. As it will be discussed later, our results will show, in the large magnetic field limit, a good qualitative agreement with the flat Landau problem. Using the geometry of the sphere and considering the fact that $\phi$ is a cyclic variable we impose,

$$\Psi(\theta, \phi - \frac{\pi}{2}) = \Psi(\theta, \phi + \frac{\pi}{2}),$$

and write

$$\Psi(\theta, \phi) = \sum_{n \in \mathbb{Z}} \Theta_m(\theta) e^{im\phi}.$$

The equation for $\Theta_m(\theta)$ is given by,

$$\frac{d^2}{d\theta^2} \Theta_m(\theta) - \left(\frac{\mu^2 \omega_m^2}{\hbar^2} \theta^2 + \frac{2ebr}{\hbar^2} \theta\right) \Theta_m(\theta) = -\frac{2\mu}{\hbar^2} \tilde{E}_m \Theta_m(\theta)$$

where

$$\begin{cases}
\omega_m &= \sqrt{\omega_c^2 + \frac{m^2}{\mu^2}} \\
\tilde{E}_m &= E - \frac{\hbar^2 m^2}{2\mu}. \\
\end{cases}$$

(5)

Writting $\theta = \rho_m y$ for $\rho_m = \sqrt{\frac{\hbar}{\mu \omega_m}}$, equation (4) becomes

$$Y_m''(y) - \left(y^2 + \lambda_m y\right) Y_m(y) = -\frac{2\tilde{E}_m}{\hbar \omega} Y_m(y)$$

where

$$\lambda_m = \frac{2e m b r}{\hbar^2} \left(\frac{\hbar}{\mu \omega_m}\right)^{\frac{3}{2}}.$$ 

Doing $x = y + \frac{1}{2}$ we obtain

$$X_m''(x) - x^2 X_m(x) = -\tilde{\epsilon}_m X_m(x),$$

with

$$\tilde{\epsilon}_m = \frac{2\tilde{E}_m}{\hbar \omega} + \frac{\lambda_m^2}{4}.$$
This is the 1-dimensional harmonic oscillator equation and therefore one can determine the energy eigenvalues of our approximated Hamiltonian obtaining

\[ E_{m,l} = \frac{\hbar^2 m^2}{2\mu} + \hbar\omega \left( l + \frac{1}{2} \right) - \hbar\omega_m \frac{\lambda^2}{8}. \]

We observe that for large \( b \) the levels \( E_{m,l} \) are always positive since the last term tends to zero. One can also note that we have lost the typical degeneracy of the Landau problem on the plane. Due to the compactness of \( S^2 \) the eigenvalues of our Hamiltonian are now labeled by two quantum numbers. According to our approximation this expression is valid only on the limit of low-energy levels i.e. only for a finite number of eigenstates. Therefore we assume \( \{m, l\} < \{m_{\text{max}}, l_{\text{max}}\} \). The levels \( m_{\text{max}} \) and \( l_{\text{max}} \) depend on the magnetic field value.

**Remark:** We observe that in the limit \( b \rightarrow \infty \) our eigenvalues tend to the eigenvalues of the Landau problem. In fact one can compute that \( \lim_{b \rightarrow \infty} \lambda_m = 0 \). Comparing the magnitudes of the remaining terms for the large field limit we obtain the result. This has a simple explanation: Since the classical solutions of the Landau problem are circles on the sphere, as the field increases the circles corresponding to the classical solutions contract to points. Therefore, all classical solutions with small energy concentrate on a neighborhood of its initial position and therefore the local flat approximation of the sphere works better higher is the magnetic field. This is our main justification for the use of the canonical quantization procedure in this limit.

### 3. Free Energy Calculation

Using the energy eigenstates just computed we calculate the free-energy (\[ F \]), defined by

\[ F = N \nu - \frac{1}{\beta} \sum_{m,l} \ln(1 + e^{\beta (\nu - E_{m,l})}) \]

where \( \beta = \frac{1}{K T} \) and \( \nu \) is the chemical potential. We redefine the energy in order to consider spin effects by writing

\[ \tilde{E}_{m,l} = E_{m,l} \pm \frac{g\hbar\omega_0}{4}, \]

where \( \omega_0 = \frac{eb}{m_0} \), with \( m_0 \) representing the free electron mass, \( g \) is the effective giromagnetic factor and the signs \( \pm \) result from different spin directions. We review briefly how to compute \( F \). Introduce the classical partition function

\[ Z(\beta) = \sum_{m,l} e^{-\beta \tilde{E}_{m,l}} \]

and the auxiliary function

\[ g(E) = (1 + e^{\beta (\nu - E)}). \]

Let \( \phi(s) \) denote the Laplace transform of \( g(E) \). Define \( z(E) \) as the inverse Laplace transform of \( Z(\beta)/\beta^2 \) i.e.

\[ \frac{Z(\beta)}{\beta^2} = \int_0^\infty z(E) e^{-\beta E} dE. \]
Therefore

\[ z(E) = \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} e^{E s} s^{-2} Z(s) \, ds \]  

(7)

and

\[ g(E) = \frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} \phi(s) e^{E s} \, ds. \]

\[ c \]

must be chosen so that all the singularities of the integrals are on the left of the integration path. With these functions the free-energy can be written as

\[ F = N \nu - \beta \sum_{l,m} g(E_{l,m}) = N \nu - \beta \int_{c-i \infty}^{c+i \infty} \frac{Z(-s)}{s^2} s^2 \phi(s) \, ds \]

since \( s^2 \phi(s) \) is the Laplace transform of \( \frac{\partial^2 g}{\partial E^2} \) we obtain finally that

\[ F = N \nu - \beta \int_0^\infty z(E) \frac{\partial^2 g}{\partial E^2} \, dE, \]

(8)

where

\[ \frac{\partial^2 g}{\partial E^2} = -\beta \frac{\partial f}{\partial E} \]

for

\[ f(E - \nu) = \frac{1}{1 + e^{\beta (E - \nu)}}, \]

the fermi function. Therefore

\[ F = N \nu + \int_0^\infty z(E) \frac{df}{dE} \, dE. \]

We can compute now the unit area partition function for our system:

\[ Z(\beta) = \frac{eb}{4 \pi \hbar^2} \sum_{\text{spin}} \sum_{l=0}^{l_{\text{max}}} \sum_{m=0}^{m_{\text{max}}} e^{\pm \beta \frac{\hbar \omega_0}{4}} e^{-\beta (l + \frac{1}{2}) \hbar \omega_m} e^{-\beta \left( \frac{\hbar^2 m^2}{2 \mu} + \frac{\lambda^2}{2} \hbar \omega_m - E \right)} \cdot \cosh\left( \frac{\beta \hbar \omega_0}{4} \right). \]

Using that

\[ \sum_{l \in \mathbb{Z}} e^{-\beta (l + \frac{1}{2}) \hbar \omega_m} = \frac{1}{\sinh(\frac{\beta \hbar \omega_m}{2})}, \]

\[ \sum_{\text{spin}} e^{\pm \frac{\beta \hbar \omega_0}{4}} = 2 \cosh(\frac{\beta \hbar \omega_0}{4}), \]

along with equation (7) we obtain

\[ z(E) = \frac{1}{2 \pi i} \frac{eb}{4 \pi \hbar^2} \sum_{l \in \mathbb{Z}} \int_{c-i \infty}^{c+i \infty} e^{-\beta \left( \frac{\hbar^2 m^2}{2 \mu} + \frac{\lambda^2}{2} \hbar \omega_m - E \right)} \cosh(\frac{\beta \hbar \omega_m}{4}) \beta^2 \sinh(\frac{\beta \hbar \omega_m}{2}) \, d\beta \]

(9)

As this integral has poles at

\[ \beta \hbar \omega_m = n \pi i \]

(10)
for \( n \in \mathbb{Z} \), we compute the integral in complex plane, by the convenient path shown in figure (1) above. Summing the contribution of each pole in the imaginary axis together with the small contour around zero, we find

\[
z(E) = \frac{eb}{4\pi \hbar^2} \cdot \sum_n \sum_m \left\{ \frac{\left(E - \frac{\hbar^2 m^2}{2\mu} + \frac{\chi^2}{2}\right)}{\hbar \omega_m} \right\}^2 + \left[ \frac{1}{2} \left( \frac{g \omega_0}{2\omega_m} \right)^2 - \frac{1}{6} \right] \hbar \omega_m + \ldots
\]

\[
- \frac{\hbar \omega_m}{2} (-1)^n \cdot \cos \left( \frac{n\pi \omega_0}{2\omega_m} \right) \cdot \cos \left( \frac{2n\pi}{\hbar \omega_m} \left( E - \frac{\hbar^2 m^2}{2\mu} + \frac{\chi^2}{2} \right) \right) \}
\]

where \( n \) labels the \( n \)-th pole in the imaginary axis, and the double sum is consistent with (10). Recalling that the free-energy is given by (8) we get,

\[
F = N\nu + \int_0^\infty z(E) \frac{\partial f(E - \nu)}{\partial E} dE,
\]

and using the Fermi function defined we obtain

\[
F = N\nu - \int_0^\infty z(E) \cdot \frac{\beta}{4 \cosh^2 \left( \frac{\beta(E - \nu)}{2} \right)} dE.
\]

In order to compute \( F \) we proceed in two steps. First we compute the following integral:

\[
\int_0^\infty \cos \left( \frac{2\pi E}{\hbar \omega_m} - \frac{\alpha_{n,l}}{\hbar \omega_m} \right) \frac{\beta}{4 \cosh^2 \left( \frac{\beta(E - \nu)}{2} \right)} dE,
\]

where

\[
\alpha_{m,n} = \frac{\hbar^2 m^2}{2\mu} + \frac{\chi^2}{2}.
\]

Defining \( y = \beta(E - \nu) \) we can write this integral as
\[
\frac{1}{4} \text{Re} \left\{ e^{i \left( \frac{2 \pi m \nu}{\hbar \omega_m} - \frac{\alpha_{m,l}}{\hbar \omega_m} \right)} \int_{-\infty}^{+\infty} \frac{e^{i \frac{2 \pi n \omega}{\hbar \omega_m} y}}{\cosh^2 \left( \frac{y}{\omega_m} \right)} \, dy \right\};
\]

We have extended the integral lower limit from \(-\nu \beta\) to \(-\infty\). The error caused by this replacement is of the order \(e^{-\nu \beta}\), which is negligible, considering the low temperature limit of our model. This integral can be performed in any algebraic software and the result is given by:

\[
- \left( \frac{2 \pi^2 \beta n}{\hbar \omega_m} \right) \cdot \cos \left( \frac{2 \pi n \nu}{\hbar \omega_m} \right) \cdot \sinh \left( \frac{2 \pi \beta n}{\hbar \omega_m} \right).
\]

For the second step we consider the integral of the terms coming from very high temperatures, namely, the contributions to the former integral coming from the small contour around \(\beta = 0\). Observing that the Fermi function tends to a Heavside step function on the limit of \(\beta \to 0\), we write \(\frac{\partial f}{\partial E}(E - \nu) = -\delta(E - \nu)\), and the computations of the high temperature terms are trivial. Collecting those results we finally obtain that

\[
F = N\nu + \frac{eb}{4\pi \hbar^2} \cdot \sum_m \left\{ \frac{\left( \nu - \frac{\alpha_{m,l}}{\hbar \omega_m} \right)}{\hbar \omega_m^2} + \left( \frac{1}{2} \left( \frac{g \omega_0}{2 \omega_m} \right)^2 - \frac{1}{6} \right) \hbar \omega_m \right\} + \frac{eb}{4\pi \hbar^2} \cdot \sum_m \sum_n (-1)^n \frac{4\beta n}{\sinh \left( \frac{2 \pi^2 n}{\beta \hbar \omega_m} \right)} \cdot \sin \left( \frac{2 \pi n \omega_0}{\hbar \omega_m} \right) \cdot \cos \left( \frac{n \omega_0}{\hbar \omega_m} \right) \sinh \left( \frac{2 \pi \beta n}{\hbar \omega_m} \right).
\]

4. Magnetization and the de Haas-Van Alphen effect

The magnetization \(M\) is found differentiating the free-energy (17) with respect to \(b\). In the low temperature limit we can write that

\[
F \approx \frac{eb}{4\pi \hbar^2} \cdot \sum_m \sum_n (-1)^n \frac{4\beta n}{\sinh \left( \frac{2 \pi^2 n}{\beta \hbar \omega_m} \right)} \cdot \sin \left( \frac{2 \pi n \omega_0}{\hbar \omega_m} \right) \cdot \cos \left( \frac{n \omega_0}{\hbar \omega_m} \right)
\]

To compute \(M\) we differentiate \(F\) and observe from (5) that for large \(b\), \(\omega_m \approx \omega_c\). We also note that for large \(b\) the ratio \(\frac{\omega_m}{\omega_c}\) is of order 0 in \(b\). From (14) we obtain that in the large field limit the leading terms in magnetization per unit of area are given by

\[
M = \frac{e}{4\pi \hbar^2} \cdot \sum_m \sum_n (-1)^n \frac{4\beta n}{\sinh \left( \frac{2 \pi^2 n}{\beta \hbar \omega_m} \right)} \cdot \sin \left( \frac{2 \pi n \omega_0}{\hbar \omega_m} \right) \cdot \cos \left( \frac{n \omega_0}{\hbar \omega_m} \right) \times \left( 1 + \frac{e^{2 \pi^2 n \omega_c}}{\mu \hbar^4 \beta} \right).
\]

The magnetization is thus expressed in terms of a superposition of periodic functions on \(\frac{1}{\omega_m}\). For the planar case \(\omega_m = \omega_c\) and therefore only one frequency dominates the oscillatory behaviour of the magnetization. We remark that on the \(\beta \to \infty\)
limit, which corresponds to the low temperature regime, the denominator of (18) tends to a constant and we have a pure superposition of periodic functions. We call this behaviour a de Hass-Van Alphen type effect.

The analogy with the classical Landau problem is better drawn looking at the ground state of our system: \( l = 0, m = 0 \). In this case we have \( \omega_m = \omega_c, \alpha_{m,n} = 0 \) and we obtain for (18)

\[
M = \frac{e}{4\pi\hbar^2} \sum_n \frac{(-1)^n \sin \left( \frac{2\pi n \mu}{\hbar \omega_c} \right)}{4\beta n \sinh \left( \frac{2\pi^2 n}{\beta \hbar \omega_c} \right)} \times \left( 1 + \frac{e \frac{2\pi^2 n}{\hbar \omega_c}}{\tanh \left( \frac{2\pi^2 n}{\beta \hbar \omega_c} \right)} \right)
\]  

considering that in the low temperature limit \( \beta \gg 1 \) the second term in the brackets of (18) tends to \( \frac{1}{n} \), we can write in the large magnetic field limit that

\[
M = \frac{e}{4\pi\hbar^2} \sum_n \frac{(-1)^n \sin \left( \frac{2\pi n \mu}{\hbar \omega_c} \right)}{4\beta n \sinh \left( \frac{2\pi^2 n}{\beta \hbar \omega_c} \right)} \cdot \cos \left( \frac{n \pi \omega_0}{2\omega_c} \right)
\]

that is up to a phase the exact expression for the Landau planar problem in two dimensions.

5. CONCLUSIONS

From (20) we see that the lowest energy state, as expected, ”do not see” the curvature of the sphere. The complete solution of spherical Landau problem will be a superposition of different discrete frequencies, each one corresponding to a different energy eigenstate. We remark that, as explained before, (18) holds only in the limit of low energy states. Also we must stress that this result is valid in the range where the flat de Haas-van Alphen is also observed, i.e. in the low temperature-high magnetic field regime.

A more general treatment of the spherical Landau problem still remains to be done, in the sense that one should consider the other limiting cases. Due to the non-local aspects of those limits, more sophisticated techniques are to be employed. Whatever such approach might be, we expect that our present results arise as an asymptotic, low temperature-high magnetic field, behaviour.

REFERENCES

[1] L.D.Landau, Z.Phys. 64 (1930) 629.
[2] R.Kubo, J. Phys. Soc. Jpn 19 (1964) 2127.
[3] L.D.Landau and E.M.Lifshitz, Statistical Physics (Pergamon, Oxford, 1980).
[4] H.Fukuyama and Y.Ishikawa, cond-mat/9904054.
[5] J. Callaway, Quantum Theory of Solid State, 2nd Edition (Academic Press, Boston, 1991).
[6] S.R. Elliot, The Physics and Chemistry of Solids, (John Willey and Sons Ltd., New York, 1998).
[7] N.M.J.Woodhouse, Geometric Quantization, 2nd Edition, (Clarendon Press, Oxford, 1991)