Total edge irregularity strength of \((n,t)\)-kite graph

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Abstract. Let \(G(V,E)\) be a simple, connected, and undirected graph with vertex set \(V\) and edge set \(E\). A total \(k\)-labeling is a map that carries vertices and edges of a graph \(G\) into a set of positive integer labels \(\{1,2,\ldots,k\}\). An edge irregular total \(k\)-labeling \(\lambda : V(G) \cup E(G) \rightarrow \{1,2,\ldots,k\}\) of a graph \(G\) is a labeling of vertices and edges of \(G\) in such a way that for any different edges \(e\) and \(f\), weights \(w(e)\) and \(w(f)\) are distinct. The weight \(w(e)\) of an edge \(e = xy\) is the sum of the labels of vertices \(x\) and \(y\) and the label of the edge \(e\). The total edge irregularity strength of \(G\), \(tes(G)\), is defined as the minimum \(k\) for which a graph \(G\) has an edge irregular total \(k\)-labeling. An \((n,t)\)-kite graph consist of a cycle of length \(n\) with a \(t\)-edge path (the tail) attached to one vertex of a cycle. In this paper, we investigate the total edge irregularity strength of the \((n,t)\)-kite graph, with \(n \geq 3\) and \(t \geq 1\). We obtain the total edge irregularity strength of the \((n,t)\)-kite graph is \(tes((n,t)\text{-kite}) = \lceil \frac{n+t+2}{3} \rceil\).

Keywords: total \(k\)-labeling, edge irregular total \(k\)-labeling, total edge irregularity strength, \(tes(G)\), \((n,t)\)-kite graph.

1. Introduction

Let \(G(V,E)\) be a connected, simple, and undirected graph with vertex set \(V\) and edge set \(E\). In [7], Wallis defined a labeling of a graph \(G\) is a mapping that carries a set of graph elements into a set of integers. The domain of mapping is a vertex set, or an edge set, or an union of vertex and edge sets. If the domain is vertex set or edge set, the labeling is called respectively vertex labeling or edge labeling. If the domain is union of vertex and edge set, then labeling is called total labeling. There are various kinds of labeling on graph and many ways to label graphs, one of them is an irregular total labeling (Gallian [2]).

A total \(k\)-labeling is a map that carries vertices and edges of a graph \(G\) into a set of positive integer labels \(\{1,2,\ldots,k\}\). Baća et al. [1] defined an edge irregular total \(k\)-labeling \(\lambda : V(G) \cup E(G) \rightarrow \{1,2,\ldots,k\}\) of a graph \(G\) is a labeling of vertices and edges of \(G\) in such a way that for any different edges \(e\) and \(f\), weights \(e\) and \(f\) are distinct. The weight of an edge \(e = xy\), denoted by \(w(e)\), is the sum of the labels of vertices \(x\) and \(y\) and the label of the edge \(e\), that is \(w(e) = \lambda(x) + \lambda(e) + \lambda(y)\). Moreover, the minimum \(k\) for which the graph \(G\) has
an edge irregular total k-labeling is called the total edge irregularity strength of the graph $G$, denoted by $tes(G)$.

There are various classes of graph have been determined for the total edge irregularity strength. Baća et al. [1], put forward the lower bound and upper bound of total edge irregularity strength for any graph $G(V,E)$ in terms of $|V(G)|$ and $|E(G)|$, which may be state as the equation (1),

$$\left\lfloor \frac{|E|+2}{3} \right\rfloor \leq tes(G) \leq |E|.$$  

(1)

In the same paper, Baća et al. [1] determined the total edge irregularity strength of path $P_n$, cycle $C_n$, star $S_n$, wheel $W_n$, and friendship graph $F_n$, that are $tes(P_n) = tes(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$, $tes(S_n) = \left\lceil \frac{n+1}{2} \right\rceil$, $tes(W_n) = \left\lceil \frac{2n+2}{3} \right\rceil$ for $n \geq 3$, and $tes(F_n) = \left\lceil \frac{3n+2}{3} \right\rceil$, respectively. In 2012, Haque [3] determined $tes(G)$ of generalized Pertersen graphs $P(n,k)$. Then, $tes(G)$ of tree $G$ with edge set $E$ and maximum degree $\Delta$ had been found by Ivančo and Jendrol’ [5], that is $tes(G) = \max\{\left\lceil \frac{\Delta+1}{2} \right\rceil, \left\lceil \frac{|E|+2}{3} \right\rceil\}$.

Furthermore, Jendrol’ et al. [6] have determined the total edge irregularity strength of complete graphs and complete bipartite graphs, that are $tes(K_n) = \left\lceil \frac{n^2-n-4}{6} \right\rceil$, $n \neq 5$ and $tes(K_{m,n}) = \left\lceil \frac{m-n+2}{3} \right\rceil$ for $m, n \geq 2$. Indirati et al. [4] determined the total edge irregularity strength of generalized helm, that is $tes(H^1_n) = \left\lceil \frac{4n+4}{3} \right\rceil$, $tes(H^2_n) = \left\lceil \frac{5n+2}{3} \right\rceil$, and $tes(H^m_n) = \left\lceil \frac{(m+3)n+2}{3} \right\rceil$ for $n \geq 3$ and $m \equiv 0 \pmod{3}$. In this paper, we investigate the total edge irregularity strength of $(n,t)$-kite graph, with $n \geq 3$ and $t \geq 1$.

2. Main Result

An $(n,t)$-kite graph consist of a cycle of length $n$ with a $t$-edge path (the tail) attached to one vertex of a cycle (Wallis et al. [8]). An $(n,t)$-kite graph has $n+t$ vertices and $n+t$ edges. Let the vertex set of $(n,t)$-kite be $V((n,t)-\text{kite}) = \{v_i : 1 \leq i \leq n\} \cup \{u_j : 1 \leq j \leq t\}$ and the edge set of $(n,t)$-kite be $E((n,t)-\text{kite}) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{u_ju_{j+1} : 1 \leq j \leq t-1\} \cup \{u_tv_1\}$. The $(n,t)$-kite graph illustrated by Figure 1.

![Figure 1. The $(n,t)$-kite graph](image)

In the following theorem, we present the total edge irregularity strength of $(n,t)$-kite graph for $n \geq 3$ and $t \geq 1$. 


Theorem 2.1 Every \((n,t)\)-kite with \(n \geq 3\) and \(t \geq 1\) satisfies \(tes((n,t)\text{-kite}) = \lceil \frac{n+t+2}{3} \rceil\).

Proof. From the lower bound of total edge irregularity strength we have that \(tes((n,t)\text{-kite}) \geq \lceil \frac{n+t+2}{3} \rceil\), for \(n \geq 3\) and \(t \geq 1\). To prove the equality, it is sufficient to show the existence of an edge irregular total \(k\)-labeling, with \(k = \lceil \frac{n+t+2}{3} \rceil\). We describe a total \(k\)-labeling \(\lambda : V((n,t)\text{-kite}) \cup E((n,t)\text{-kite}) \rightarrow \{1, 2, \ldots, k\}\) as follows:

\[
\lambda(u_j) = \begin{cases} 
1, & \text{if } 1 \leq j \leq 2; \\
\lceil j + 1 \rceil, & \text{if } 1 \leq j \leq t - 1;
\end{cases}
\]

\[
\lambda(u_ju_{j+1}) = \left\lceil \frac{j}{3} \right\rceil, \quad \text{for } 1 \leq j \leq t - 1;
\]

\[
\lambda(u_tv_1) = \left\lceil \frac{t}{3} \right\rceil.
\]

For \(t \equiv 0 \pmod{3}\)

\[
\lambda(v_i) = \begin{cases} 
\lceil \frac{i}{3} \rceil + \left\lceil \frac{i}{2} \right\rceil, & \text{if } 1 \leq i \leq n + 1 - \left\lceil \frac{n+t+1}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil; \\
n + 2 + \left\lceil \frac{i}{3} \right\rceil - i, & \text{if } n + 2 - \left\lceil \frac{n+t+1}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \leq i \leq n;
\end{cases}
\]

\[
\lambda(v_iv_{i+1}) = \begin{cases} 
\lceil \frac{i}{3} \rceil + \left\lceil \frac{i+1}{2} \right\rceil, & \text{if } 1 \leq i \leq n + 1 - \left\lceil \frac{n+t+1}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil; \\
n + 1 + \left\lceil \frac{i}{3} \right\rceil - i, & \text{if } n + 2 - \left\lceil \frac{n+t+1}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \leq i \leq n - 1;
\end{cases}
\]

\[
\lambda(v_nv_1) = 1 + \left\lceil \frac{t}{3} \right\rceil.
\]

For \(t \equiv 1 \pmod{3}\)

\[
\lambda(v_i) = \begin{cases} 
\left\lceil \frac{i}{3} \right\rceil - 1 + \left\lceil \frac{i+1}{2} \right\rceil, & \text{if } 1 \leq i \leq n - \left\lceil \frac{n+t+1}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil; \\
n + 1 + \left\lceil \frac{i}{3} \right\rceil - i, & \text{if } n + 1 - \left\lceil \frac{n+t+1}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \leq i \leq n;
\end{cases}
\]

\[
\lambda(v_iv_{i+1}) = \begin{cases} 
\left\lceil \frac{i}{3} \right\rceil - 1 + \left\lceil \frac{i+1}{2} \right\rceil, & \text{if } 1 \leq i \leq n - \left\lceil \frac{n+t+2}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil; \\
n + 1 + \left\lceil \frac{i}{3} \right\rceil - i, & \text{if } n + 1 - \left\lceil \frac{n+t+2}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \leq i \leq n - 1;
\end{cases}
\]

\[
\lambda(v_nv_1) = 1 + \left\lceil \frac{t}{3} \right\rceil.
\]
For $t \equiv 2 \pmod{3}$

$$
\lambda(v_i) = \begin{cases} 
\left\lceil \frac{t}{3} \rightceil + \left\lceil \frac{i}{2} \rightceil, & \text{if } 1 \leq i \leq n + 1 - \left\lceil \frac{n+t+2}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil; \\
n + 2 + \left\lceil \frac{t}{3} \right\rceil - i, & \text{if } n + 2 - \left\lceil \frac{n+t+2}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \leq i \leq n; 
\end{cases}
$$

$$
\lambda(v_i v_{i+1}) = \begin{cases} 
\left\lfloor \frac{t}{3} \right\rfloor - 1 + \left\lceil \frac{i+1}{2} \right\rceil, & \text{if } 1 \leq i \leq n + 1 - \left\lceil \frac{n+t+2}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil; \\
n + \left\lceil \frac{t}{3} \right\rceil - i, & \text{if } n + 2 - \left\lceil \frac{n+t+2}{3} \right\rceil + \left\lceil \frac{t}{3} \right\rceil \leq i \leq n - 1; 
\end{cases}
$$

$$
\lambda(v_n v_1) = \left\lceil \frac{t}{3} \right\rceil.
$$

Observe that under $k$-labeling $\lambda$ the weights of the edges of $(n,t)$-kite graph are:

$$
wt(u_j u_{j+1}) = j + 2, \text{ for } 1 \leq j \leq t - 1;
$$

$$
wt(u_t v_1) = t + 2;
$$

$$
wt(v_i v_{i+1}) = \begin{cases} 
t + \left\lceil \frac{3(i+1)}{2} \right\rceil, & \text{if } 1 \leq i \leq \left\lfloor \frac{2n-1}{3} \right\rfloor; \\
t + 3(n - i) + 4, & \text{if } \left\lceil \frac{2n+2}{3} \right\rceil \leq i \leq n - 1; 
\end{cases}
$$

$$
wt(v_n v_1) = t + 4.
$$

The weights of the edges of $(n,t)$-kite graph under the labeling $\lambda$ constitute the set \{3, 4, 5, \ldots, n + t + 2\}, that is different for every edge irregular total $k$-labeling. The function $\lambda$ is mapping from $V((n,t)-kite) \cup E((n,t)-kite)$ into \{1, 2, \ldots, \left\lceil \frac{n+t+2}{3} \right\rceil\}.

The total $k$-labeling $\lambda$ has the required properties on an edge irregular total labeling. We then have $tes((n,t)-kite) \leq \left\lceil \frac{n+t+2}{3} \right\rceil$. However, by equation (1), $tes((n,t)-kite) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{n+t+2}{3} \right\rceil$. Then, we proved the theorem. \qed

We give an example of an edge irregular total labeling of $(n,t)$-kite graph for $n = 5$ and $t = 4$ in Figure 2.

![Figure 2. An edge irregular total 4-labeling of (5,4)-kite graph](image-url)
3. Conclusion

According to the discussion above it can be concluded that the total edge irregularity strength of \((n,t)-kite\) graph, \(tes((n,t)-kite)=[\frac{n+t+2}{3}]\), for \(n \geq 3\) and \(t \geq 1\). Furthermore, we propose the following open problem.

**Open Problem.** Let \(G\) be a lollipop graph \(L_{m,n}\) with \(m \geq 3\) and \(n \geq 1\) that obtained by joining a complete graph \(K_m\) to a path graph \(P_n\) with a bridge. Determine the total edge irregularity strength of graph \(G\), \(m \neq 5\).

Acknowledgments

We gratefully acknowledge the support from Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sebelas Maret. Then, we wish to thank the referees for their valuable suggestions and references, which helped to improve the paper.

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