Canonical Quantization of 2d Gravity Coupled to $c < 1$ Matter

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ABSTRACT

We study 2d gravity coupled to $c < 1$ matter through canonical quantization of a free scalar field, with background charge, coupled to gravity. Various features of the theory can be more easily understood in the canonical approach, like gauge independence of the path-integral results and the absence of the local physical degrees of freedom. By performing a non-canonical transformation of the phase space variables, we show that the theory takes a free-field form, i.e. the constraints become the free-field Virasoro constraints. This implies that the David-Distler-Kawai results can be derived in a gauge independent way, and also proves the free-field assumption which was used for obtaining the spectrum of the theory in the conformal gauge. A discussion of the physical spectrum of the theory is presented, with an analysis of the unitarity of the discrete momentum states.

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1. Introduction

Understanding the two dimensional quantum gravity is important because of its relevance for the non-critical string theory, statistical mechanics of random surfaces, and as a toy model of quantum gravity in four dimensions. The theory so far has been mainly analyzed in the path-integral quantization scheme [1, 2]. Although many important results have been achieved in this scheme, it is also important to understand the theory in the Dirac canonical quantization approach [3]. First, the path integral quantization of a gauge invariant system requires gauge fixing, so that the questions of gauge independence and relation of the results in different gauges inevitably appear. This has been automatically taken care of in the Dirac approach, since it is a gauge independent quantization method. Second, understanding the relation between the path-integral and the Dirac quantization results is important question in its own right, especially if one is interested in possible relations to 4d quantum gravity, where the Dirac quantization is much better understood than the path-integral quantization.

The first exact results in 2d quantum gravity were obtained by Knizhnik, Polyakov and Zamolodchikov (KPZ), who studied 2d quantum gravity coupled to matter in a chiral gauge [1]. They concluded that the theory is free of anomalies and solvable for $c_M \leq 1$ and $c_M \geq 25$, where $c_M$ is the matter central charge. Subsequently, their results were rederived in the conformal gauge by David, Distler and Kawai (DDK) [2]. The structure of the physical Hilbert space was studied in a series of papers [4, 5, 6, 7, 8]. In all these papers, the physical Hilbert space was defined as a cohomology of a BRST charge, which was postulated from the beginning, without a simple and direct relation to a particular action. The choice of the BRST charge in [4, 5, 6] was motivated by the results of the KPZ analysis, while the choice of the BRST charge in [7, 8] was motivated by the results of the DDK analysis. The BRST analysis in the conformal gauge also requires an additional assumption that the Liouville and the matter sector can be described as free-field theories with background charges [8]. Another puzzling feature is that the physical spectrum is like that of a system with finitely many degrees of freedom, although the starting point is a field theory coupled to gravity.

In this paper we show that all these features of the theory can be easily understood in the canonical quantization approach, if one starts from the action for a free scalar field with background charge, coupled to gravity. In section 2 we describe the canonical structure and analyze the constraints of such an action. In section 3 we discuss some general features of the Gupta-Bleuler and the BRST quantization, which are relevant for our case. In section 4 we analyze the theory in terms of the
SL(2, R) Kac-Moody variables. A discussion of the issue of hermiticity is presented, with an emphasis on the matter sector. In section 5 we introduce variables which transform the theory into a free-field form, by using the Wakimoto construction for the Kac-Moody variables. We then discuss the relation between the chiral and the conformal gauge spectrum. This is followed by a discussion about the problem of the complex momentum of the discrete states and their unitarity. We present our conclusions in section 6.

2. Canonical Analysis

Since we are interested in quantizing 2d gravity coupled to \( c < 1 \) matter, a natural choice for the classical action is

\[
S = -\frac{1}{2} \int_M d^2x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \alpha R \phi + \Lambda),
\]

(2.1)

where \( g_{\mu\nu} \) is a 2d metric, \( \phi \) is a scalar field, \( \alpha \) is the background charge, \( R \) is the 2d curvature scalar and \( \Lambda \) is the cosmological constant. In the canonical approach the 2d manifold \( M \) must have a topology of \( \Sigma \times \mathbb{R} \), where \( \Sigma \) is the spatial manifold and \( \mathbb{R} \) is the real line corresponding to the time direction. In two dimensions \( \Sigma \) can be either a real line or a circle \( S^1 \). Since we are interested in string theory, we will analyze the compact case. We will label the time coordinate \( x^0 = \tau \) and the space coordinate \( x^1 = \sigma \).

The configuration space variables are the metric \( g_{\mu\nu}(\sigma, \tau) \) and the scalar field \( \phi(\sigma, \tau) \). The corresponding canonically conjugate momenta are defined as

\[
p^\mu_\nu = \frac{\partial \mathcal{L}}{\partial \dot{g}^\mu_\nu}, \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}},
\]

(2.2)

where \( \mathcal{L} \) is the Lagrangian density of (2.1), and \( \dot{\cdot} \) is the time derivative. We define the canonical Poisson brackets as

\[
\{g_{\mu\nu}(\sigma, \tau), p^{\rho\lambda}(\sigma', \tau)\} = \delta^{(\mu}_{(\rho} \delta^{\nu)}_{\lambda)} \delta(\sigma - \sigma') \quad \{\phi(\sigma, \tau), \pi(\sigma', \tau)\} = \delta(\sigma - \sigma').
\]

(2.3)

Since the action (2.1) is invariant under the 2d diffeomorphisms, this implies that constraints will appear in the canonical formulation. By performing the canonical analysis [9], one can show that (2.2) gives two primary constraints, corresponding to vanishing of \( p^{00} \) and \( p^{01} \). This means that the corresponding coordinates \( g_{00} \) and \( g_{01} \) are non-dynamical. They are the Lagrange multipliers, analogous to the \( A_0 \) component of the gauge field in the case of the Yang-Mills theory. The secondary
constraints are the diffeomorphism constraints

\[
G_0(\sigma) = \frac{1}{2}(\phi')^2 - \frac{2}{\alpha^2}(g_{11}p^{11})^2 - \frac{2}{\alpha}(g_{11}p^{11})\pi - \frac{\alpha g_{11}'}{2 g_{11}}\phi' + \alpha\phi'' - \frac{\alpha}{2}\Lambda g_{11}
\]
\[
G_1(\sigma) = \pi\phi' - 2g_{11}(p^{11})' - p^{11}g_{11}'
\]  

(2.4)

which are first class and irreducible, and obey the diffeomorphism algebra. There are no further constraints, and the action (2.1) can be rewritten as

\[
S = \int d\sigma d\tau (p^{11}\dot{g}_{11} + \pi\dot{\phi} - n^\mu G_\mu)
\]  

(2.5)

where

\[
n^0 = -\sqrt{-g} \frac{g_{01}}{g_{11}}, \quad n^1 = \frac{g_{01}}{g_{11}}
\]  

(2.6)

are the Lagrange multipliers, imposing the constraints (2.4).

Since we are dealing with a reparametrization invariant system, the Hamiltonian vanishes on the constraint surface (i.e. it is proportional to the constraints). Therefore the dynamics is determined by the constraints only. A straightforward consequence of (2.5) is that (2.1) does not have any local physical degrees of freedom since there are as many constraints per space point \(\sigma\), two, as the number of the configuration variables. This means that there is enough gauge invariance to gauge away all the \(\sigma\) dependence of \(g\) and \(\phi\), and only the zero modes may remain. Therefore (2.1) describes a theory without local physical degrees of freedom. As the subsequent analysis will show, only certain global degrees of freedom will remain, i.e. the zero-modes, and in that sense one can think of (2.1) as a topological theory.

When quantizing a gauge theory, anomalies may appear, which prevent us to gauge away all of the non-physical degrees of freedom. In our case we have to examine the quantum theory and see under what conditions the anomalies cancel, so that the quantum theory remains topological.

### 3. Quantum Theory

In order to quantize a constrained system, one can adopt the Dirac quantization procedure [3]. Given the basic canonical variables \((p_j, q^j)\), we promote them into hermitian operators \((\hat{p}_j, \hat{q}^j)\), satisfying the Heisenberg algebra

\[
[\hat{p}_j, \hat{q}^k] = -i\delta^k_j
\]  

(3.1)

A representation of (3.1) defines the Hilbert space of states \(\mathcal{H}\). The constraint conditions \(G_\alpha(p, q) = 0\) are promoted into the operatorial conditions

\[
\hat{G}_\alpha(\hat{p}, \hat{q}) |\psi\rangle = 0
\]  

(3.2)
and the set of states $|\psi\rangle$ satisfying (3.2) defines the physical Hilbert space $\mathcal{H}^*$. The standard difficulty of the Dirac procedure is how to define the $\hat{G}_\alpha$ operators. This difficulty arises because of the ordering ambiguities. A related problem is that $\hat{G}_\alpha$ often do not form a closed commutator algebra, which is the source of the anomalies. The anomalies make the conditions (3.2) inconsistent, and one has to use the Gupta-Bleuler conditions instead, which require that only the expectation values $\langle \hat{G}_\alpha \rangle$ vanish. This is usually equivalent to requiring that only a subset of $\hat{G}_\alpha$, which forms a closed subalgebra, annhilates the physical states. Although a consistent scheme, it is often hard to see what happens with the anomalies in the Gupta-Bleuler approach. A more suitable approach is the BRST canonical quantization (for a review and references see [1]). In this approach one enlarges the Hilbert space $\mathcal{H}$ by introducing additional canonical variables $(c_\alpha, b_\alpha)$, i.e. the ghosts and their canonical conjugate momenta (antighosts). Ghosts are of the opposite statistics to $G_\alpha$, and satisfy
\[
\{b_\alpha, c_\beta\} = -i\delta^\beta_\alpha \quad , \tag{3.3}
\]
where $\{,\}$ is the graded anticommutator. In the space $\mathcal{H} \otimes \mathcal{H}_{gh}$, where $\mathcal{H}_{gh}$ is a representation of (3.3), one defines an operator
\[
\hat{Q} = c^\alpha \hat{G}_\alpha - \frac{1}{2} i : f_{\alpha \beta \gamma} c^\alpha c^\beta b_\gamma : + \cdots \quad , \tag{3.4}
\]
where $f_{\alpha \beta \gamma}$ are the structure constants of the algebra $G$, and $\cdots$ are determined from the requirement of nilpotency
\[
\hat{Q}^2 = 0 \quad . \tag{3.5}
\]
Condition (3.5) guarantees the absence of the anomalies in the quantum theory, and often gives conditions on the free parameters of the theory. The physical state conditions (3.2) are replaced with a single condition
\[
\hat{Q} |\Phi\rangle = 0 \quad . \tag{3.6}
\]
Only the non-trivial solutions of (3.6) are considered as physical, where $|\Phi\rangle = \hat{Q} |\chi\rangle$ is trivial. In other words, the physical Hilbert space is the cohomology of the operator $\hat{Q}$. For the systems of interest, the condition (3.6) is equivalent to the Gupta-Bleuler conditions if
\[
|\Phi\rangle = |\psi\rangle \otimes |ghv\rangle \quad , \tag{3.7}
\]
where $|ghv\rangle$ is the ghost-vacuum [14]. The states (3.7) form the “zero” ghost number cohomology. One can also have physical states of non-zero ghost number, which correspond to some other consistent choice of the Gupta-Bleuler conditions. For example, in the bosonic string case, the usual choice $L_n \psi = 0, n \geq 0$ corresponds to
\(N_{gh} = -\frac{1}{2}\), while \(L_n \psi = 0, n \geq -1\) corresponds to \(N_{gh} = -\frac{3}{2}\), where \(N_{gh}\) denotes the ghost number and \(L_n\) are the Virasoro generators. The other possible non-trivial cohomologies arise for \(N_{gh} = \frac{1}{2}, \frac{3}{2}\) [19]. Note that the consistency of the Gupta-Bleuler conditions in the case \(N_{gh} = -\frac{3}{2}\) requires vanishing of the string intercept \(a\). Given that \(a = (D - 2)/24\) [14], this explains why this cohomology class is empty for the critical string \((D = 26)\), while it is non-trivial for a \(D = 2\) string, which is the case relevant for us. The BRST formalism is more restrictive than the Gupta-Bleuler formalism, and the well known example is the bosonic string [14], while a less known example is that of the Siegel superparticle [15].

In our case, the ordering difficulties arise if we use \((g, p)\) and \((\phi, \pi)\) as our basic canonical variables, since the form of \(G_\mu\) is such that \(\hat{G}_\mu\) will be plagued with ordering ambiguities. This could be avoided by choosing a more suitable set of canonical variables. For example, by performing a canonical transformation [11, 13]

\[
\chi = \phi - \frac{\alpha}{2} \ln |g_{11}| \quad , \quad \pi_\chi = \pi \\
\xi = \frac{\alpha}{2} \ln |g_{11}| \quad , \quad \pi_\xi = \frac{2}{\alpha} g_{11} p^{11} + \pi 
\]

the constraints become

\[
G_0 = \frac{1}{2} \pi_\chi^2 + \frac{1}{2} (\chi')^2 + \alpha \pi_\chi'' - \frac{1}{2} \pi_\xi^2 - \frac{1}{2} (\xi')^2 + \alpha \xi'' + \frac{1}{2} \Lambda e^{\frac{2\pi}{\alpha} \xi} \\
G_1 = \pi_\chi \chi' + \alpha \pi_\chi' + \pi_\xi \xi' - \alpha \pi_\xi'
\]

If we neglect the cosmological constant term and the background charges, expressions (3.9) have the same form as the constraints of a \(D = 2\) string. In analogy to the string case we define the left/right movers

\[
h_\pm = \frac{1}{\sqrt{2}}(\pi_\chi \pm \chi) \quad , \quad j_\pm = \frac{1}{\sqrt{2}}(\pi_\xi \mp \xi)
\]

and redefine the constraints as

\[
T_\pm = \frac{1}{2}(G_0 \pm G_1) = \frac{1}{2} h_\pm^2 \pm \frac{\alpha}{\sqrt{2}} h_\pm' \mp \frac{\alpha}{\sqrt{2}} j_\pm' \mp \frac{1}{4} \Lambda e^{-\frac{2\pi}{\alpha} (j_+ - j_-)}
\]

which now obey the Virasoro algebra. In the string case the Virasoro anomaly is \(c = D = 2\), while \(\hat{Q}^2 = 0\) requires \(c = 26\), and therefore the anomaly cannot be removed. In our case the presence of the background charges and the cosmological term may change the formula \(c = D\) and hence allow for \(c = 26\) to be satisfied. Actually, by exploiting the similarity of \(T_\pm\) with the energy-momentum tensor of the Liouville theory (for a review and references see [22]), one can show that \(c = 26\) can be satisfied. However, the \((h, j)\) variables are not convenient for analyzing the spectrum of the theory. Therefore we are going to look for a more convenient set of variables.
4. \textit{SL}(2, \mathbb{R}) \textit{Variables}

The results of the work done in \cite{12, 13} on the \textit{SL}(2, \mathbb{R}) symmetry of the induced 2d gravity imply that the corresponding gauge indepident variables exist. Following \cite{13}, let us introduce non-canonical phase space variables

\[
\begin{align*}
J^+ &= \frac{\sqrt{2}}{g_{11}} T_- + \frac{\Lambda}{2\sqrt{2}} \\
J^0 &= -\left[ g_{11} p_{11} + \frac{\alpha}{2} \left( \pi - \frac{\alpha g_{11}}{2 g'_{11}} \right) \right] + \frac{1}{\sqrt{2}} x^- J^+ \\
J^- &= -\frac{\alpha^2}{\sqrt{2}} (g_{11} + 1) + \sqrt{2} x^0 J^0 + \frac{1}{\sqrt{2}} (x^-)^2 J^+ \\
P_M &= \frac{1}{\sqrt{2}} \left( \pi + \phi' - \frac{\alpha g_{11}}{2 g'_{11}} \right).
\end{align*}
\]

Note that $(J, P_M)$ variables cover the whole phase space. Another important point is that $x^-$ has to be considered as a constant parameter, independent of $\sigma$, because otherwise the $J$’s from (4.1) will not be periodic functions in $\sigma$, and one could not use the Fourier modes of $J$ to define the quantum theory. The $J$’s satisfy an \textit{SL}(2, \mathbb{R}) current (Kac-Moody) algebra

\[
\{ J^a(\sigma_1), J^b(\sigma_2) \} = f^{ab}_{\ c} J^c(\sigma_2) \delta(\sigma_1 - \sigma_2) + \frac{\alpha^2}{2} \eta^{ab} \delta' (\sigma_1 - \sigma_2) ,
\]

where $f^{ab}_{\ c} = 2 \epsilon^{abcd} \eta_{dc}$ and

\[
\eta^{ab} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}.
\]

$P_M$ has the Poisson bracket of a free scalar field ($P_M = \partial_+ \phi_M$ in the conformal gauge)

\[
\{ P_M(\sigma_1), P_M(\sigma_2) \} = \delta'(\sigma_1 - \sigma_2)
\]

and $\{ J, P \} = 0$. Note that the constraints become

\[
J^+ - \lambda = 0
\]

\[
T \equiv T_g + T_M = 0 ,
\]

where $\lambda = \frac{\Lambda}{2\sqrt{2}}$ and

\[
T_g = \frac{1}{\alpha^2} \eta_{ab} J^a J^b + (J^0)' , \quad T_M = \frac{1}{2} P_M^2 + \frac{\alpha}{\sqrt{2}} P_M'.
\]

can be interpreted as the gravity and the matter energy-momentum tensors, respectively.
To construct the Hilbert space $\mathcal{H}$, we promote $J$ and $P_M$ into hermitian operators, satisfying the operator version of (4.2-3)

\[
[J^a(\sigma_1), J^b(\sigma_2)] = i f_{abc} J^c (\delta(\sigma_1 - \sigma_2) - \frac{i k}{2} \eta^{ab} \delta'(\sigma_1 - \sigma_2)),
\]

\[
[P_M(\sigma_1), P_M(\sigma_2)] = i \delta'(\sigma_1 - \sigma_2).
\] (4.6)

We introduce a new constant $k$, which is different from $\alpha^2$ due to ordering ambiguities. It will be determined from the requirement of anomaly cancellation. Now one can follow the standard way of constructing $\mathcal{H}$ as a Fock space built on the vacuum state annihilated by the positive Fourier modes of $J$ and $P$. Let $f(\sigma) = \sum_n e^{i\epsilon n\sigma} f_n$, where $\epsilon = \pm 1$, and let $J_n$, $\alpha^M_n$ and $L_n$ denote the Fourier modes of $J$, $P_M$ and $T$, respectively. We represent the Fock space vacuum as $|j, m\rangle \otimes |p_M\rangle$, where $|j, m\rangle$ is the vacuum for the Kac-Moody sector, while $|p_M\rangle$ is the vacuum for the matter sector. The Kac-Moody vacuum states satisfy

\[
J^a_n |j, m\rangle = 0 \quad n \geq 1
\]

\[
J^a_0 |j, m\rangle = j^a |j, m\rangle ,
\] (4.7)

where the last condition means that the vacuum states form an $SL(2, \mathbb{R})$ representation. Unitary $SL(2, \mathbb{R})$ representations are infinite dimensional (since $SL(2, \mathbb{R})$ is a non-compact group), and can be labeled with a complex number $j$, which can take the following values

\[
j = \frac{1}{2} + ir \quad r \in \mathbb{R} \quad 0 < j < 1 \quad \text{or} \quad j \text{ is a half-integer},
\] (4.8)

where $j(j - 1)$ is an eigenvalue of $j^a j_a$, while the second label $m \in \mathbb{Z}$, is an eigenvalue of $j^0 [16]$.

The matter vacuum satisfies a $U(1)$ version of (4.7)

\[
\alpha^M_n |p_M\rangle = 0 \quad n \geq 1
\]

\[
\alpha^M_0 |p_M\rangle = p_M |p_M\rangle .
\] (4.9)

$|p_M\rangle$ is the usual momentum state, so that $p_M$ is real and continuous eigenvalue. The $\alpha^M_n$ modes satisfy

\[
[\alpha^M_n, \alpha^M_m] = -\epsilon n \delta_{n+m,0} ,
\] (4.10)

which gives for the matter central charge

\[
c_M = 1 - \epsilon 12 Q_M^2 .
\] (4.11)
where \( Q_M = \sqrt{\pi} \alpha \). Hence we will chose \( \epsilon = 1 \) in order to have \( c_M < 1 \). If we want the \( L_n \)'s to satisfy the usual Virasoro algebra

\[
[L_n, L_m] = (n - m)L_{n+m} + A_n \delta_{n+m,0} ,
\]

where \( A_n \) is the anomaly, we have to change the sign of the \( L_n \)'s coming from (4.4), i.e. define \(-T\) as our energy momentum tensor. Hence

\[
-T = \frac{1}{2\pi} \sum_n L_n e^{in\sigma} ,
\]

where

\[
L_n^g = \frac{1}{k+2} \sum_m \eta^{ab} J^a_{n-m} J^b_m - inJ^0_n
\]

and

\[
L_n^M = -\frac{1}{2} \sum_m \alpha^M_{n-m} \alpha_m + iQ_M \alpha^M_n .
\]

Note that the factor \( \frac{1}{\alpha^2} \) in the classical expression for \( T_g \) in (4.5) has become \(-\frac{1}{k+2} \).

The BRST quantization requires the enlarged Hilbert space \( \mathcal{H} \otimes \mathcal{H}_{gh} \), and the whole set up is equivalent to the starting point of the chiral gauge analysis of [4, 5]. The only difference is that our expressions are gauge independent. Our choice of the Kac-Mody modes is the same as that of [4], while the choice of the matter modes is different from [4, 5]. Their modes, which we denote as \( \tilde{\alpha}^M_n \), are related to ours as

\[
\alpha^M_n = \tilde{\alpha}^M_n + iQ_M \delta_{n,0} ,
\]

and their relation to our modes is analogous to the relation of the Kac-Moody modes of [4] to the Kac-Moody modes of [4]. The barred modes are often used in conformal field theory, and arise from mapping the cylinder \( S^1 \times \mathbb{R} \) onto the complex plane. The corresponding mode expansion is given by

\[
\tilde{P}_M(z) = \frac{1}{\sqrt{2\pi}} \sum_n \tilde{\alpha}^M_n z^{-n-1} ,
\]

where \( z\tilde{P}_M(z) \) coincides with \( P_M(\sigma) \) for \( z = e^{-i\sigma} \).

Note that our choice of representation for the matter \( L_n \)'s is not the one used in the conformal field theory (CFT). Namely, in order to have the usual hermitian conjugacy rules

\[
\alpha_n^\dagger = \alpha_{-n} \rightarrow L_n^\dagger = L_{-n}
\]

and \( c_M < 1 \), we had to introduce “negative” \( L_n \)'s, given by (4.15). As a consequence, the matter Fock space has a negative norm, since \( \epsilon = 1 \) in (4.10). This is not a
problem, since the positivity of the norm is only required for the physical Fock space. One could have chosen the CFT representation \[17\], where \(L_n\)’s are “positive”

\[
L_n^M = \frac{1}{2} \sum_m \alpha_{n-m}^M \alpha_m^M + nQ_M \alpha_n^M \tag{4.19}
\]

and the \(\alpha\)’s satisfy (4.10) with \(\epsilon = -1\). But then in order to have \(c_M < 1\), the background charge has to be imaginary, and the \(L_n\)’s will not be hermitian under the usual scalar product represented by the rules (4.18). A modified scalar product can be introduced, which acts on \(F^* \times F\), where \(F^*\) is the dual of the matter Fock space \(F\) \[17\]. The duality relation has a property that \(F^*_{2Q-p}\) is isomorphic to \(F_p\), where \(p\) is the momentum of the vacuum state. However, we are going to use the representation (4.15), to which we are going to refer as the string representation, since it is more convenient for our approach.

The BRST charge can be constructed from the equation (3.4)

\[
\hat{Q} = c_0(\Lambda_0 - a) + \sum_{n \neq 0} c_n \Lambda_{-n} + \sum_n c_n^+ J_n^+ + \cdots \tag{4.20}
\]

where \(a\) is the intercept. The nilpotency condition requires vanishing of the total central charge, which includes the matter, Kac-Moody and the ghosts contributions

\[
c_M + \frac{3k}{k+2} - 6k - 26 - 2 = 0 \tag{4.21}
\]

and the intercept must satisfy

\[
a = 1 + \frac{k}{4} + \frac{1}{2} Q_M^2 \tag{4.22}
\]

Expression (4.22) differs from the corresponding expression in \[4\] because we used the string representation modes \(\alpha_n\). Note that the equation (4.21) implies

\[
k + 3 = \frac{1}{12} \left( c_M - 1 \pm \sqrt{(1-c_M)(25-c_M)} \right) \tag{4.23}
\]

and therefore \(k\) is real if \(c_M \leq 1\) or \(c_M \geq 25\), which justifies our choice \(\epsilon = 1\).

Only the zero ghost number cohomology is non-trivial \[4, 5\], which corresponds to the usual Gupta-Bleuler conditions

\[
(L_n - a\delta_{n,0}) |\psi\rangle = 0 \quad , \quad (J_n^+ - \lambda \delta_{n,0}) |\psi\rangle = 0 \quad , \quad n \geq 0 \tag{4.24}
\]

It consists of the vacuum states of the Fock space \(\mathcal{H}\)

\[
|\psi_0\rangle = |j\rangle \otimes |p_M\rangle \tag{4.25}
\]
where $|j\rangle$ satisfies $j^+|j\rangle = \lambda |j\rangle$. $j$ and $p_M$ are related through the ground state on-shell condition

$$
(L_0 - a) |\psi_0\rangle = 0 \rightarrow 1 = j(j - 1) - \frac{k}{k+2} - \frac{k}{4} - \Delta(p_M) .
$$

(4.26)

$\Delta(p_M)$ plays the role of the matter conformal dimension, and can be expressed as

$$
\Delta(p_M) = \frac{1}{2}(p_M^2 + Q_M^2) = \frac{1}{2}\bar{p}_M(\bar{p}_M + 2iQ_M) ,
$$

(4.27)

where $\bar{p}$ is the eigenvalue of the CFT mode $\bar{a}_0^M$. This coincides with the usual formula for $\Delta(p_M)$ if $\bar{p}$ is imaginary. Neglecting for the moment the problem of the imaginary momenta, which we are going to discuss in the next section, the formula (4.9) coincides with the expression given in [4]. Therefore the quantum analysis confirms our classical picture of only the zero modes being physical.

5. Free-field Variables

The existence of the $SL(2,\mathbb{R})$ variables (4.1) is a strong indication that the conformal gauge variables used in [4, 5] should also have a gauge independent realization. The results of Itoh’s work in the chiral gauge [6] imply that the new variables can be determined from the Wakimoto’s construction [18]. Let us introduce three new variables $\beta$, $\gamma$ and $P_L$ such that

$$
\begin{align*}
J^+(\sigma) &= \beta(\sigma) \\
J^0(\sigma) &= \beta(\sigma)\gamma(\sigma) + k_1 P_L(\sigma) \\
J^-(\sigma) &= \beta(\sigma)\gamma^2(\sigma) + 2k_1\gamma(\sigma)P_L(\sigma) + k_2\gamma'(\sigma) ,
\end{align*}
$$

(5.1)

and require

$$
\{\beta(\sigma_1), \gamma(\sigma_2)\} = \delta(\sigma_1 - \sigma_2) , \quad \{P_L(\sigma_1), P_L(\sigma_2)\} = -\delta'(\sigma_1 - \sigma_2) ,
$$

(5.2)

whith the other Poisson brackets being zero. From the requirement that the expressions (5.1) satisfy the Poisson algebra (4.2), one can easily see that

$$
k_1 = \frac{\alpha}{\sqrt{2}} , \quad k_2 = \alpha^2 .
$$

(5.3)

In terms of the new variables $T_g$ becomes

$$
-T_g = -\beta'\gamma + \frac{1}{2}P^2_L + \frac{Q_L}{\sqrt{2\pi}} P_L' ,
$$

(5.4)
where $Q_L = -\sqrt{\pi} \alpha$. In the quantum case, $\beta$, $\gamma$ and $P_L$ become hermitian operators, satisfying
\[
[\beta(\sigma_1), \gamma(\sigma_2)] = i\delta(\sigma_1 - \sigma_2) \quad [P_L(\sigma_1), P_L(\sigma_2)] = -i\delta'(\sigma_1 - \sigma_2). \quad (5.5)
\]
The quantum analog of (5.1) is
\[
J^+(\sigma) = \beta(\sigma) \quad J^0(\sigma) = :\beta(\sigma)\gamma(\sigma) + k_1 P_L(\sigma) \quad J^-(\sigma) = :\beta(\sigma)\gamma^2(\sigma) + 2k_1 \gamma(\sigma)P_L(\sigma) + k_2 \gamma'(\sigma), \quad (5.6)
\]
where now $k_1$ and $k_2$ have acquired new quantum values
\[
k_1^2 = \frac{1}{\alpha^2} = -(k + 2) \quad k_2 = -k, \quad (5.7)
\]
due to the normal ordering effects. The normal ordering in (5.6) is with respect to the vacuum $|\text{vac}\rangle$
\[
\beta_n |\text{vac}\rangle = \gamma_n |\text{vac}\rangle = 0 \quad n \geq 1, \quad (5.8)
\]
where $\beta_n$ and $\gamma_n$ are the Fourier modes of $\beta$ and $\gamma$. The expression for $T_g$ retains the classical form (5.4), with the appropriate normal ordering. However, $Q_L$ acquires the quantum value $Q_L = -2\alpha - \alpha$.

The constraints can be now written as
\[
J^+ = \beta = 0 \quad -T = -\beta'\gamma + \frac{1}{2} P_L^2 + \frac{Q_L}{\sqrt{2\pi}} P_L' - \frac{1}{2} P_M^2 - \frac{Q_M}{\sqrt{2\pi}} P_M' = 0, \quad (5.9)
\]
where $\beta$ in (5.9) is shifted by the constant $\lambda$. Vanishing of $\beta$ means that we can drop that variable, together with its canonically conjugate variable $\gamma$, and we are left with $P_L$ and $P_M$ variables, obeying only one constraint
\[
-T \approx T_{cf} \equiv \frac{1}{2} P_L^2 + \frac{Q_L}{\sqrt{2\pi}} P_L' - \frac{1}{2} P_M^2 - \frac{Q_M}{\sqrt{2\pi}} P_M' = 0. \quad (5.10)
\]
But this is precisely the starting point of the conformal gauge analysis [7, 8]. The only difference is that the expression (5.10) is gauge independent, and in the conformal gauge reduces to the expression used in [7, 8].

If we introduce a notation
\[
X^\mu = (X^L, X^M), \quad X \cdot Y = \eta_{\mu\nu} X^\mu Y^\nu, \quad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.11)
\]
then
\[
L_n = \frac{1}{2} \sum_m : \alpha_{n-m} \cdot \alpha_m : + i nQ \cdot \alpha_n. \quad (5.12)
\]
The BRST charge is then given by the usual expression
\[ \hat{Q} = \sum_n c_n L_{-n} + \frac{1}{2} \sum_{m,n} (m - n) : c_m c_n b_{-m-n} : - c_0 a \]  
(5.13)

The normal ordering is with respect to the vacuum \(|\text{vac}\rangle = |p\rangle \otimes |0\rangle\)
\[ \alpha_n |\text{vac}\rangle = c_n |\text{vac}\rangle = b_n |\text{vac}\rangle = 0 \quad , \quad n \geq 1 \]  
(5.14)

where \(|p\rangle\) is the \(\alpha\)-modes vacuum \((\alpha_0 |p\rangle = p |p\rangle)\), while \(|0\rangle\) is the ghost vacuum, satisfying \(b_0 |0\rangle = 0\) (the other possibility \(c_0 |0\rangle = 0\) gives symmetric results). Nilpotency of \(\hat{Q}\) implies
\[ Q^2 = Q^2_L - Q^2_M = 2 \quad , \quad a = 0 \]  
(5.15)

The results of the BRST analysis in [7, 8] can be now understood in the following way. The zero-ghost number cohomology corresponds to the usual Gupta-Bleuler conditions
\[ L_n |\psi\rangle = 0 \quad , \quad n \geq 0 \]  
(5.16)

where \(|\psi\rangle\) belongs to the \(\alpha\)-modes Fock space \(F(\alpha)\). Clearly, the ground state \(|p\rangle\) is a solution of (5.16) if
\[ p^2 = p^2_L - p^2_M = 0 \]  
(5.17)

In terms of the CFT modes, (5.17) translates into
\[ \Delta(\bar{p}_L) - \Delta(\bar{p}_M) = 1 \]  
(5.18)

Note that (5.17) is equivalent to the \(SL(2, \mathbb{R})\) condition (4.26), and \(|p\rangle\) is the same as \(|j, p_M\rangle\). This implies the relation
\[ \frac{j(j - 1)}{k + 1} - \frac{k}{4} = \frac{1}{2} Q^2_M + \frac{1}{2} p^2_L \geq 0 \]  
(5.19)

which is satisfied if \(j\) belongs to any of the continuous series of representations from (4.8), and if \(k\) is given by the negative root of (4.23). The negative root is taken because then \(k + 3\) coincides with the string susceptibility coefficient \(\gamma_{str}\) [2]. If \(j\) belongs to the discrete series, then (5.19) is satisfied for \(j_-(Q_M) \leq j \leq j_+(Q_M)\), where \(j_{\pm}(Q_M)\) are the roots of the equation (5.19).

As far as the excited states are concerned, the results of the BRST analysis imply that they are physical only for discrete values of the momenta [7, 8]. Furthermore, when translated into our conventions, these discrete values of the momenta are purely imaginary
\[ p_L = \frac{i}{2}(r + s)Q_L - \frac{i}{2}(r - s)Q_M \]  
\[ p_M = \frac{i}{2}(r - s)Q_L - \frac{i}{2}(r + s)Q_M \]  
(5.20)
where \( r, s \in \mathbb{Z} \), and \( rs \) is the excitation level number.

The states in the \( \pm 1 \) cohomology sector have only discrete values of the momenta. They are of the form\(^8\)

\[
|\psi\rangle \otimes b_{-n} |0\rangle \quad \text{or} \quad |\psi\rangle \otimes c_{-n} |0\rangle , \quad n \geq 1 ,
\]

where \( |\psi\rangle \in F(\alpha) \). Absence of the continuous momentum states in this case can be understood on the example of \( |\psi\rangle \otimes b_{-1} |0\rangle \), since then (3.6) implies \( L_n |\psi\rangle = 0 \) for \( n \geq -1 \), which for the ground state \( |p\rangle \) implies \( p = 0 \). Similarly to the zero ghost number case, the excited states are physical only for complex discrete values of the momenta, given by the equation (5.20).

The fact that all discrete states have complex momenta explains why they were not found in the analysis of \( \text{\cite{1, 2, 3}} \), since they are not defined in the standard framework. According to the standard construction of the free-field Fock space \( \mathcal{H} \), the discrete states do not even belong to \( \mathcal{H} \), because of the complex momentum. However, there are strong indications that the discrete states are physical \( \text{\cite{20}} \), and in order to incorporate them into a Hilbert space, one has to find another free-field realization of the Fock space \( \mathcal{H} \). One can see the difficulty in doing this by considering the zero-modes sector, where a representation of the Heisenberg algebra has to be constructed. The usual momentum states \( |p\rangle \) are constructed as

\[
|p\rangle = e^{ip\hat{q}} |p = 0\rangle .
\]

\[(5.22)\]

The states (5.22) are \( \delta \)-function normalizable for \( \text{Im} p = 0 \), while otherwise cannot be normalized. According to the Stone-von Neumann theorem \( \text{\cite{21}} \), \( \text{Im} p = 0 \) is the only inequivalent unitary irreducible representation of the Heisenberg algebra, which implies that complex momentum states are not unitary. A possible resolution of this problem may be in the fact that the Stone-von Neumann theorem applies to the case when \( -\infty < q < +\infty \). When \( 0 \leq q < +\infty \), a case relevant for the components of the metric, then \( \hat{q} \) is not hermitian with respect to the usual scalar product, and the Stone-von Neumann theorem does not apply any more. This will require a further investigation, in particular a careful treatment of the range of \( q_L \), a coordinate canonically conjugate to the zero-mode of \( P_L \).

Note that Sieberg has found the same phenomenon, i.e. non-normalizability of the complex momentum states, in the Liouville theory approach to 2d quantum gravity \( \text{\cite{22}} \). This indicates a strong connection between our model and the Liouville theory. In order to better understand this connection, a study of our theory in terms of the variables defined by the equation (3.8) will be suitable.
6. Conclusions

We have demonstrated that the conformal gauge results of DDK can be derived in the gauge independent way. To do this, we have used the Dirac quantization procedure, which is gauge independent and therefore convenient for such a task. In order to obtain the free-field variables $(\beta, \gamma, P_L, P_M)$, we went through a series of transformations

$$(g, p, \phi, \pi) \rightarrow (J^a, P_M) \rightarrow (\beta, \gamma, P_L, P_M). \quad (6.1)$$

Note the importance of the sequence (6.1), since it implicitly defines the $(g, p, \phi, \pi) \rightarrow (\beta, \gamma, P_L, P_M)$ transformation. The $J$ variables are also important for understanding the $c_M < 1$ constraint, as well as for the simple calculation of the $Q_L$ renormalization.

Given the free-field variables, one can use the results of the BRST analysis \cite{2, 3} to obtain the physical spectrum of the theory. The analysis of the spectrum confirms the classical picture of only the zero-modes of the gravity and the matter sector propagating, which can be described as states of a $D = 2$ massless relativistic particle. Existence of the discrete states means that the massive states are not completely pure gauge states, and can be physical for specific discrete values of the momenta. However, incorporating the discrete states into a Hilbert space is still an open question, due to their complex momentum, and further work along the lines suggested in section 5 is necessary.

Our results imply the following physical picture: 2d quantum gravity coupled to a scalar field is described by a Liouville-like theory if one uses the variables defined by the equation (3.8). The quantum theory can be transformed into a free-field form for $c_M \leq 1$ or $c_M \geq 25$. For these values of $c_M$ the quantum theory retains its classical topological features. Note that in the case when the scalar field describes a minimal CFT, then the theory looks even more topological, since then $\Delta(\bar{p}_M)$ can take only discrete values, and one is left with only discrete momentum states. This implies that the effective field theory describing the interactions among these states is zero-dimensional, which explains why the zero-dimensional matrix models can be used to describe the minimal models coupled to gravity. Formulating the interacting theory in the canonical approach can be done in a string field theory framework.

The $c_M = 1$ case does not follow from the canonical analysis of (2.1) with $\alpha = 0$. The $\alpha = 0$ case is just a $D = 1$ bosonic string theory. According to the no-ghost theorem \cite{23}, if $D < 26$ then there are $D - 1$ physical degrees of freedom per space point $\sigma$. For the $D = 1$ case this means that only the zero modes are propagating, which formally agrees with the $c_M = 1$ path-integral result. However, $\hat{Q}^2 \neq 0$ in the canonical treatment of the $D = 1$ string, and conformal anomaly is present. A way
to reconcile these results is that $c_M = 1$ case with $\hat{Q}^2 = 0$ can be obtained in the canonical approach from a $D = 2$ string with a dilaton coupling.

As far as the supersymmetric case is concerned, we expect that the canonical treatment of the supersymmetric generalization of the action (2.1) will give the results analogous to the bosonic case, i.e. that only the zero modes of the super-matter and the super-Liouville sector will propagate. This would rigorously prove the results of the super-conformal gauge BRST analysis [24].

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