Towards a Gravitational Analog to S-duality in Non-abelian Gauge Theories

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Abstract

For non-abelian non-supersymmetric gauge theories, generic dual theories have been constructed. In these theories the couplings appear inverted. However, they do not possess a Yang-Mills structure but rather are a kind of non-linear sigma model. It is shown that for a topological gravitational model an analog to this duality exists.

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1. INTRODUCTION

Recently $S$-duality in supersymmetric Yang-Mills theories and in superstring theories, has become a most powerful non-perturbative technique to describe and to compute strong coupling dynamics in these theories.

It is well known that even perturbatively, superstring theories describe a consistent (ultraviolet finite and unitary) theory of quantum gravity. However, very important gravitational issues remain behind the non-perturbative sector. For instance, the structure of the microstates used in the computation of the entropy of extremal and near extremal black holes [1].

Roughly speaking, for the effective low energy action of string theories, $S$-duality symmetry is realized at the level of the axion and dilaton moduli. The gravitational sector appears dynamical with respect to this symmetry. In fact, it is well known that gravitational corrections are required in order to test string duality [2], and to check some consistency conditions in $M$-theory [3].

For the heterotic string theory in ten dimensions, toroidal compactification, to four dimensions on the six-torus, gives for its low energy limit, $\mathcal{N} = 4$ super Yang-Mills theory on $R^4$. Thus $S$-duality in the four dimensional effective theory comes as a consequence of superstring dualities in ten dimensions. The four-dimensional effective theory is decoupled from gravitational effects, and gravity enters only as a spectator (not dynamical). From this theory a twisted topological field theory can be constructed on a curved four-manifold, which it is shown to be $S$-dual (according to the Montonen-Olive conjecture), by using different formulas of four-manifolds well known by the topologists [4].

On the side of non-supersymmetric gauge theories in four dimensions, the subject has been explored recently in the abelian as well as in the non-abelian cases [5–10], see also [11–13]. In the abelian case, one considers $CP$ non-conserving Maxwell theory on a curved compact four-manifold $X$ with Euclidean signature or, in other words, $U(1)$ gauge theory with a $\theta$ vacuum coupled to four-dimensional gravity. The manifold $X$ is basically described by its associated classical topological invariants: the Euler characteristic $\chi(X) = \frac{1}{16\pi^2} \int_X \text{tr} R \wedge \tilde{R}$ and the signature $\sigma(X) = -\frac{1}{24\pi^2} \int_X \text{tr} R \wedge R$.

In the Maxwell theory, the partition function $Z(\tau)$ transforms as a modular form under a finite index subgroup $\Gamma_0(2)$ of $\text{SL}(2, \mathbb{Z})$ [5,6], $Z(-1/\tau) = \tau^u \bar{\tau}^v Z(\tau)$, with the modular
weight \((u, v) = (\frac{1}{4}(\chi + \sigma), \frac{1}{4}(\chi - \sigma))\). In the above formula \(\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\), where \(g\) is the U(1) electromagnetic coupling constant and \(\theta\) is the usual theta angle.

Witten has shown [5] that, at the quantum level, in order to cancel the modular anomaly in abelian theories, one has to choose certain holomorphic couplings \(B(\tau)\) and \(C(\tau)\) in the topological gravitational (non-dynamical) sector, through the action

\[
I^{TOP} = \int_X \left( B(\tau) \text{tr} R \wedge \tilde{R} + C(\tau) \text{tr} R \wedge R \right),
\]

(1.1)
i.e., which is proportional to the appropriate sum of the Euler characteristic \(\chi(X)\) and the signature \(\sigma(X)\). These couplings must satisfy the condition that \(\exp(B(\tau))\) and \(\exp(C(\tau))\) should be modular forms and their associate weights are chosen just to cancel the anomaly [5]. Therefore, we must consider, for instance, \(CP\) non-conserving Maxwell theory with action \(I_{M,\theta}\) coupled to \(I^{TOP}\) such that

\[
I = I_{M,\theta} + \int_X \left( B(\tau) \text{tr} R \wedge \tilde{R} + C(\tau) \text{tr} R \wedge R \right).
\]

(1.2)

\(I_{M,\theta}\) consists in a dynamical plus a \(\theta\)-topological term and \(I^{TOP}\) works as a counterterm which cancels the modular anomaly. Furthermore, if one considers the path integration over all phase space of the theory, then it can be shown [8] that the path integral phase space measure \(\int \mathcal{D}A_\alpha \mathcal{D}\pi^\alpha\), where \(A_\alpha\) represents the abelian connection and \(\pi^\alpha\) its conjugate momenta, codifies the modular anomaly in such a way that the phase space partition function \(Z_{ps}\) is in fact a modular invariant, \(Z_{ps}(\tau) = Z_{ps}(-1/\tau)\).

In the case of Yang-Mills theories, the lack of a generalized Poincaré lemma means that there is no dual theory in the same sense as for abelian theories [14]. For generic Yang-Mills theories, one can follow the same procedure as in the abelian case [8][10]. Usually one constructs an intermediate Lagrangian from which one recuperates the original Lagrangian and its dual, as different limits (for a recent review, see [15])

\[
L = -\alpha G^\alpha_{\mu\nu} G^{\mu\nu}_\alpha + G^\alpha_{\mu\nu} F^{\alpha}_{\mu\nu}(A),
\]

(1.3)
\(\alpha\) is the constant coupling, \(F^\alpha_{\mu\nu}\) is a Lie algebra-valued tensor field and \(G^\alpha_{\mu\nu}\) is a Lie algebra-valued Lagrange multiplier tensor field. Obviously, if the variables \(G\) are integrated out, we get the usual Yang-Mills Lagrangian \(L = \frac{1}{4A} F^\alpha_{\mu\nu} F^{\alpha\mu\nu}\), where \(F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + f^{\alpha}_{\beta\gamma} A^\beta_\mu A^\gamma_\nu\) and \(f^{\alpha}_{\beta\gamma}\) are the structure constants of the Lie algebra of the gauge group.
Thus, the Euclidean partition function of (1.3), after partial integration, will be given by

\[ Z = \int \mathcal{D}G \exp \left( - \int \alpha G^a_{\mu \nu} G^{\mu \nu} a \right) \int \mathcal{D}A \exp \left[ \int (2 \partial_{\mu} G^{a}_{\mu} A^{a}_{\nu} - M^{ab}_{\mu} A^{b}_{\mu} A^{a}_{\nu}) dx \right], \tag{1.4} \]

where \( M^{ab}_{\mu} = f^{c}_{ab} G^{c \mu} \) is the adjoint transformed of \( G \). Furthermore, this partition function can be rewritten as

\[ Z = \int \mathcal{D}G \exp \left[ - \int (\alpha G^a_{\mu \nu} G^{\mu \nu} a - M^{1ab}_{\mu} \partial_{\rho} G^{\mu \rho}_{b} \partial_{\tau} G^{\tau \rho}_{a}) dx \right] \int \mathcal{D}A \exp (\int M^{ab}_{\mu} A^{a}_{\mu} A^{b}_{\mu} dx) \]

\[ = \sqrt{\pi} \int \mathcal{D}G \sqrt{\det(M^{-1})} \exp \left[ - \int (\alpha G^a_{\mu \nu} G^{\mu \nu} a - M^{1ab}_{\mu} \partial_{\rho} G^{\mu \rho}_{b} \partial_{\tau} G^{\tau \rho}_{a}) dx \right], \tag{1.5} \]

which represents in some sense the dual of the starting Yang-Mills theory. Obviously, this is a complicated “massive” non-linear sigma model which could not be well defined, depending on if the “metric” \( M \) is singular or not. Gannor and Sonnenschein show how to regain a Yang-Mills theory from this, in such a way that it apparently leads to the dual theory. Following them, let us define \( \bar{A}^{a}_{\mu} = -(M^{-1})^{ab}_{\mu} \partial_{\rho} G^{\mu \rho}_{b} \), that is, \( G \) satisfies the equations of motion \( \partial_{\nu} G^{\mu \nu}_{b} + M^{ab}_{\mu} \bar{A}^{a}_{\nu} = 0 \). In abelian theories, the Poincaré lemma gives the solution to this equation in terms of a vector potential for the dual of \( G \). However, for non-abelian theories, although \( F^{a}_{\mu \nu}(\bar{A}) = \partial_{\mu} \bar{A}^{a}_{\nu} - \partial_{\nu} \bar{A}^{a}_{\mu} + f^{a}_{bc} \bar{A}^{b}_{\mu} \bar{A}^{c}_{\nu} \) is a solution for \( G^{a}_{\mu \nu} \), it is not the most general solution.

Nevertheless, it can be easily seen that the second term in the exponential of (1.5) can be rewritten as \( M^{ab}_{\mu} \bar{A}^{a}_{\nu} \bar{A}^{b}_{\mu} \), as well as \( \partial_{\rho} G^{\mu \rho}_{b} \bar{A}^{b}_{\nu} \) plus a total derivative term. Thus, the partition function turns out to be

\[ Z = \sqrt{\pi} \int \mathcal{D}G \sqrt{\det M^{-1}} \int \mathcal{D}A \exp \left\{ - \int [\alpha G^a_{\mu \nu} G^{\mu \nu} a - G^{a}_{\mu \nu} F^{a}_{\mu \nu}(\bar{A})] dx \right\} \delta(2 \bar{A}^{a}_{\mu} + 2M^{-1ab}_{\nu \mu} \partial_{\rho} G^{\rho \nu}_{b}), \tag{1.6} \]

where the factor 2 in the delta function was introduced for convenience. If now in this expression the square root of the determinant and the Dirac delta function are written as exponentials, after a partial integration we get

\[ Z = \sqrt{\pi} \int \mathcal{D}G \mathcal{D} \bar{A} \mathcal{D} \Omega \mathcal{D} \Lambda \exp (\int \{G^{a}_{\mu \nu} [\alpha G^a_{\mu \nu} + F^{a}_{\mu \nu} (\bar{A}) - 2 \mathcal{D}^{(\bar{A})} \Lambda^{a}_{\nu}] - M^{-1ab}_{\mu \nu} \Omega^{a}_{\mu} \Omega^{b}_{\nu} \} dx), \tag{1.7} \]

which after some manipulations turns out to be

\[ Z = \pi \int \mathcal{D}G \mathcal{D} \bar{A} \exp \left( \int G^{a}_{\mu \nu} [\alpha G^a_{\mu \nu} + F^{a}_{\mu \nu} (\bar{A})] dx \right), \tag{1.8} \]
where $\tilde{A} = \bar{A} - \Lambda$. This result shows the way back to the model we started with, as well the covariance of the partition function (1.3) [9].

Now we follow the procedure stated above, to find (1.5) for the non-Abelian case with a $\text{CP}$-violating $\theta$-term, on the manifold $X$ described as above. The action can be written as

$$I_{Y,M,\theta} = \frac{1}{8\pi} \int_X d^4x \left( \frac{4\pi}{g_{YM}} \text{tr}[F_{\mu\nu} F^{\mu\nu}] + \frac{i\theta}{4\pi} \epsilon_{\mu\nu\rho\sigma} \text{tr}[F^{\mu\nu} F^{\rho\sigma}] \right),$$  
(1.9)

$\theta$ is the non-Abelian theta-vacuum and $g_{YM}$ is the Yang-Mills coupling constant. Equivalently,

$$I_{Y,M,\theta} = \frac{i}{8\pi} \int_X d^4x \left( \bar{\tau} \text{tr}[F^+_{\mu\nu} F^{+\mu\nu}] - \tau \text{tr}[F^-_{\mu\nu} F^{-\mu\nu}] \right),$$  
(1.10)

here $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}}$ and $\bar{\tau}$ its complex conjugate, $F = dA - A \wedge A$, and $F^\pm_{mn}$ are the self-dual and anti-self-dual field strengths respectively. We will obtain a Lagrangian which seems to be the corresponding S-dual Lagrangian to (1.9) employing, as before, an appropriate version of the Roček-Verlinde procedure [13]. As already shown, one finds the dual Lagrangian of (1.10) [8–10] to be of the form

$$\tilde{I}_{Y,M,\theta} = \frac{i}{8\pi} \int_X d^4x \left( - \frac{1}{7} G_{\mu\nu}^+ G^{+\mu\nu} + \frac{1}{7} G^{-a}_{\mu\nu} G^{-\mu\nu} + 2(M^+)_{\nu\mu}^{-1ab} \partial_\rho G^+_{\rho\mu} a_\sigma G^{+\nu\sigma}_b - 2(M^-)_{\nu\mu}^{-1ab} \partial_\rho G^-_{\rho\mu} a_\sigma G^{-\nu\sigma}_b \right),$$  
(1.11)

where $G^\pm$ are, as mentioned, arbitrary two-forms on $X$ and $M^\pm$ are, as previously, the adjoints of $G^\pm$, correspondingly.

In this paper we attempt to explore a possible analog to these kind of “$S$-dual” theories for the gravitational sector. We will show that these types of theories in fact exist. We will consider only the non-dynamical gravitational sector as a first step to gain some insight about the much more involved dynamical sector.

The paper is organized as follows. In Sec. 2 we propose an intermediate gravitational Lagrangian which interpolates between our original topological gravitational Lagrangian and its dual. This dual gravitational Lagrangian is computed following the procedure given in [13]. Sec. 3 is devoted to make a similar procedure but now including a $BF$ non-abelian gravitational Lagrangian. Sec. 4 is devoted to final remarks.
2. THE GRAVITATIONAL ANALOG

We will first show our procedure to define a gravitational “$S$-dual” Lagrangian, by beginning with the non-dynamical topological gravitational action of the general form (1.1)

\[ I^{\text{TOP}} = \frac{\Theta_G^E}{2\pi} \int_X \text{tr} R \wedge \tilde{R} + \frac{\Theta_G^P}{2\pi} \int_X \text{tr} R \wedge R, \quad (2.12) \]

$X$ is a four dimensional closed lorentzian manifold, i.e compact, without boundary $\partial X = 0$ and withlorentzian signature. In this action, the coefficients are the gravitational analogues of the $\theta$-vacuum in QCD \[17,18\]. Another actions including gravitational $\Theta$-terms have been analyzed recently in \[19\].

This action can be written in terms of the self-dual and anti-self-dual parts of the Riemann tensor as follows

\[ I^{\text{TOP}} = \int_X \left( \tau^+ \text{tr} R^+ \wedge R^+ - \tau^- \text{tr} R^- \wedge R^- \right), \quad (2.13) \]

with $\tau^\pm = \frac{1}{2\pi}(\Theta_G^E \mp \Theta_G^P)$. In local coordinates on $X$, this action is written as

\[ I^{\text{TOP}} = \int_X d^4x \varepsilon^{\mu\nu\rho\sigma} \left( \tau^+ R^+_{\mu\nu} R^+_{\rho\sigma} - \tau^- R^-_{\mu\nu} R^-_{\rho\sigma} \right), \quad (2.14) \]

where $R^\pm_{\mu\nu}{}^{cd} = \frac{1}{2}(R_{\mu\nu}{}^{ab} \mp \frac{i}{2} \varepsilon^{ab}_{\ CD} R_{\mu\nu}{}^{cd})$ and satisfies

\[ \varepsilon^{ab}_{\ CD} R^\pm_{\mu\nu}{}^{cd} = \pm 2i R^\pm_{\mu\nu}{}^{ab}. \quad (2.15) \]

Self-dual (anti-self-dual) Riemann tensors are defined as well in terms of the self-dual (anti-self-dual) component of the spin connection $\omega^\pm_\mu := \frac{1}{2}(\omega^a_\mu \mp \frac{i}{2} \varepsilon^a_{\ CD} \omega^{cd}_\mu)$ as

\[ R^\pm_{\mu\nu}{}^{ab} = \partial_\mu \omega^\pm_{\nu}{}^{ab} - \partial_\nu \omega^\pm_{\mu}{}^{ab} + \frac{1}{2} f^{[ab]}_{[cd][ef]} \omega^\pm_{\mu}{}^{cd} \omega^\pm_{\nu}{}^{ef}, \quad (2.16) \]

with

\[ f^{[ab]}_{[cd][ef]} = \frac{1}{2} \left( \eta_{ce} \delta^a_d \delta^b_f - \eta_{cf} \delta^a_d \delta^b_e + \eta_{df} \delta^a_e \delta^b_c - \eta_{de} \delta^a_f \delta^b_c - (a \leftrightarrow b) \right). \quad (2.17) \]

The Lagrangian (2.14) can be written obviously as $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$, where $\mathcal{L}_\pm = \varepsilon^{\mu\nu\rho\sigma} (\tau^\pm R^\pm_{\mu\nu}{}^{ab} R^-_{\rho\sigma}{}^{cd})$.

The Euclidean partition function is defined as \[20\]

\[ Z(\tau) = Z(\tau^+, \tau^-) = \int \mathcal{D}\omega^+ \mathcal{D}\omega^- \exp \left( - \int_X (\mathcal{L}_+ + \mathcal{L}_-) \right), \quad (2.18) \]
which satisfies a factorization $Z(\tau^+, \tau^-) = Z_+(\tau^+)Z_-(\tau^-)$ where

$$Z_{\pm}(\tau^\pm) = \int D\omega_{\pm}^* \exp \left( - \int_X L_{\pm} \right). \tag{2.19}$$

It is an easy matter to see from action (2.14) that the partition function (2.18) is invariant under combined shifts of $\Theta^E_G$ and $\Theta^P_G$ for $\tau^\pm \to \tau^\pm \mp 2$, in the case of spin manifolds and for non-spin manifolds $\tau^\pm \to \tau^\pm \mp 5$.

In order to find a S-dual theory to (2.14) we follow references [9,10]. The procedures are similar to those presented in the Introduction for Yang-Mills theories. We begin by proposing an intermediate Lagrangian

$$L = L_+ + L_- = \epsilon^{\mu\nu\rho\sigma} \left[ \tau^+ R_{\mu\nu}^{ab} R_{\rho\sigma}^{+ab} + i G_{\mu\nu}^{+ab} \left( R_{\rho\sigma}^{+ab} - (\partial_\rho \omega^+_{\sigma ab} - \partial_\sigma \omega^+_{\rho ab} + [\omega^+_{\rho}, \omega^+_{\sigma}]_{ab}) \right) \right] + \epsilon^{\mu\nu\rho\sigma} \left[ \tau^- R_{\mu\nu}^{-ab} R_{\rho\sigma}^{-ab} + i G_{\mu\nu}^{-ab} \left( R_{\rho\sigma}^{-ab} - (\partial_\rho \omega^-_{\sigma ab} - \partial_\sigma \omega^-_{\rho ab} + [\omega^-_{\rho}, \omega^-_{\sigma}]_{ab}) \right) \right], \tag{2.20}$$

where $R^\pm_{ab}$ is of course treated as a field independent on $\omega^\pm$.

For simplicity, we focus on the partition function of the self-dual part of this Lagrangian, for the anti-self-dual part $L_-$ one can follow the same procedure, the corresponding partition function is

$$Z^*_+(\tau^+) = \int DR^+D\omega^+DG^+ \exp \left( - \int_X L^*_+ \right). \tag{2.21}$$

First of all we would like to regain the original Lagrangian (2.14). Thus, we should integrate over the Lagrange multiplier $G^+$

$$\exp \left( - \int_X L^*_+ \right) = \int DG^+ \exp \left( - \int_X L_+ \right). \tag{2.22}$$

The relevant part of the integral is

$$\int DG^+ \exp \left\{ - i \int_X \epsilon^{\mu\nu\rho\sigma} G^{\pm ab}_{\mu\nu} \left( R_{\rho\sigma}^{\pm ab} - (\partial_\rho \omega^{\pm}_{\sigma ab} - \partial_\sigma \omega^{\pm}_{\rho ab} + [\omega^\pm_{\rho}, \omega^\pm_{\sigma}]_{ab}) \right) \right\}. \tag{2.23}$$

Using the formula $\delta(x) = \int \frac{d\omega}{2\pi} \exp(i\omega x)$ we find

$$Z^*_+(\tau^+) = -2\pi \int DR^+D\omega^+ \delta[\epsilon(R^\pm - \ldots)] \exp(- \int_X \epsilon^{\mu\nu\rho\sigma} \tau^\pm R_{\mu\nu}^{\pm ab} R_{\rho\sigma}^{\pm ab}). \tag{2.24}$$

Thus we get the full original Lagrangian $L^{**} = L^{**}_+ + L^{**}_- = \epsilon^{\mu\nu\rho\sigma} (\tau^\pm R_{\mu\nu}^{\pm ab} R_{\rho\sigma}^{\pm ab} + \tau^- R_{\mu\nu}^{-ab} R_{\rho\sigma}^{-ab})$, where now $R_{\mu\nu}^{\pm ab}$ depend on $\omega^{\pm ab}$ as usual.
Obviously, as before, factorization holds, \( Z^*(\tau^+, \tau^-) = Z^*_+(\tau^+)Z^*_-(\tau^-) \).

Now we would like to find the dual Lagrangian \( \tilde{L}^{**} \). To do this we have first to integrate out the variables \( R^+ \) and \( \omega^+ \), thus

\[
\exp \left( -\int_X \tilde{L}^{**} \right) = \int DR^+ D\omega^+ \exp \left( -\int_X L^+ \right).
\]

We consider first the integration over \( R^+ \)

\[
\exp \left( -\int_X \tilde{L}^{**}_{\omega^+} \right) = \int DR^+ \exp \left( -\int_X \epsilon^{\mu\nu\rho\sigma} \left( \tau^+ R^{+\mu\nu} R^{+\rho\sigma} + i R^{+\mu\nu} C^{+\rho\sigma}_{\rho\sigma} \right) \right). \tag{2.25}
\]

The functional integral over \( R^+ \) is of the Gaussian type and defines the following Lagrangian

\[
L^+ = \epsilon^{\mu\nu\rho\sigma} \left[ \frac{1}{4\tau^+} G^{+\mu\nu}_{\mu\nu} G^{+\rho\sigma}_{\rho\sigma} - i G^{+\mu\nu}_{\mu\nu} R^{+\rho\sigma}_{\rho\sigma} (\omega^+) \right], \tag{2.26}
\]

where \( R^{+\rho\sigma}_{\rho\sigma} (\omega^+) \) is given by (2.14). Therefore, another intermediate Lagrangian emerges, which otherwise would result from adding the degrees of freedom \( G^{\pm\rho\sigma}_{\rho\sigma} \).

Now we integrate in the variable \( \omega^+ \); using the fact \( \partial X = 0 \) and after some manipulations, the relevant part of the above Lagrangian necessary for the integration in \( \omega^+ \) is

\[
\tilde{L}^{**}_{\omega^+} = \ldots - 2i \epsilon^{\mu\nu\rho\sigma} \omega^{+\mu\rho}_{\omega^+} \partial_\nu G^{+\rho\sigma}_{\rho\sigma} + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} f_{[cd][ef]}^{[ab]} \omega^{+\mu\rho}_{\omega^+} \omega^{+\nu\sigma}_{\omega^+} G^{+\rho\sigma}_{\rho\sigma} + \ldots. \tag{2.27}
\]

Before performing the functional integration it is convenient to define

\[
M^{+\rho\sigma}_{\rho\sigma} \equiv i \epsilon^{\mu\nu\rho\sigma} f_{[cd][ef]}^{[ab]} G^{+\rho\sigma}_{\rho\sigma}. \tag{2.28}
\]

Inserting (2.28) into (2.25) and integrating out with respect to \( \omega^+ \), we finally find

\[
\tilde{L}^{**} = + 2\epsilon^{\mu\nu\rho\sigma} \partial_\nu G^{+\rho\sigma}_{\rho\sigma}(M^+)_{\mu\lambda}^{-1} abcd \epsilon^{\lambda\theta\alpha\beta} \partial_\theta G^{+\alpha\beta}_{\alpha\beta} + \ldots. \tag{2.29}
\]

Therefore, the complete dual Lagrangian is

\[
\tilde{L}^{**} = \tilde{L}^{**}_+ + \tilde{L}^{**}_- = \epsilon^{\mu\nu\rho\sigma} \left[ - \frac{1}{4\tau^+} G^{+\rho\sigma}_{\rho\sigma} G^{+\mu\nu}_{\mu\nu} + \frac{1}{4\tau^-} G^{-\rho\sigma}_{\rho\sigma} G^{-\mu\nu}_{\mu\nu} + 2 \partial_\nu G^{+\rho\sigma}_{\rho\sigma}(M^+)_{\mu\lambda}^{-1} abcd \epsilon^{\lambda\theta\alpha\beta} \partial_\theta G^{+\alpha\beta}_{\alpha\beta} - 2 \partial_\nu G^{-\rho\sigma}_{\rho\sigma}(M^-)_{\mu\lambda}^{-1} abcd \epsilon^{\lambda\theta\alpha\beta} \partial_\theta G^{-\alpha\beta}_{\alpha\beta} \right]. \tag{2.30}
\]

Of course, the condition of factorization for the above Lagrangian still holds.
3. INCLUDING A BF GRAVITATIONAL TERM

In the previous section we found a dual Lagrangian for a gravitational topological model. Therefore, we have worked out therefore a non-dynamical gravitational system. We would like to extend these results to a theory which contains dynamical gravity. This problem is not easy, so we shall try first a toy model. In this section we consider a four-dimensional non-abelian BF-theory \[21,22\], coupled to our topological field theory \( L_{TOP} \)

\[ L = L_{BF} + L_{TOP}, \]  

(3.32)

where \( L_{BF} = \alpha \text{tr}(\Sigma \wedge R) + \beta \text{tr}(\Sigma \wedge \Sigma) \) and \( L_{TOP} = \gamma \text{tr}(R \wedge \tilde{R}) + \delta \text{tr}(R \wedge R) \) with \( \alpha \) and \( \beta \) parameters of the BF theory, and \( \gamma \) and \( \delta \) proportional to the gravitational \( \Theta \) angles, defined in the previous section. Thus, we can see this Lagrangian as the non-abelian BF-theory to which we add a topological (non-dynamical) gravitational Lagrangian, in a similar spirit as we dealt with the Yang-Mills case in Eq. (1.2).

Now, proceeding as in the last section, we consider only the self-dual terms

\[ \mathcal{L}_+ = \epsilon^{\mu\nu\rho\sigma} \left( AR^{+\mu\nu} R^{+\rho\sigma} + B\Sigma^{+\mu\nu} R^{+\rho\sigma} + C\Sigma^{+\mu\nu} \Sigma^{+\rho\sigma} \right), \]  

(3.33)

where \( A, B \) and \( C \) are appropriate constants. This action looks similar to the Plebański-Ashtekar dynamical action \[23,24\]. Hence, by studying the action (3.33), we hope to learn how to deal with the dynamical case.

In order to dualize \( \mathcal{L}_+ \), let us propose, as before, the intermediate self-dual Lagrangian

\[ L_+ = \epsilon^{\mu\nu\rho\sigma} \left( AR^{+\mu\nu} R^{+\rho\sigma} + B\Sigma^{+\mu\nu} R^{+\rho\sigma} + C\Sigma^{+\mu\nu} \Sigma^{+\rho\sigma} + EG^{+\mu\nu} \right) \]

\[ + (R^{+\rho\sigma} - \partial_\rho \omega^{+\sigma} + \partial_\sigma \omega^{+\rho} + \frac{1}{2} f_{abcde} \omega^{+\rho} \omega^{+\sigma} \omega^{+\lambda} f), \]  

(3.34)

where \( G^{+\mu\nu} \) is a Lagrange multiplier and \( E \) is a constant. The self-dual sector of the Euclidean partition function is

\[ Z_+^* = \int \mathcal{D}R^+ \mathcal{D}\omega^+ \mathcal{D}\Sigma^+ \mathcal{D}G^+ \exp \left( - \int_X L_+ \right), \]  

(3.35)

where \( R^+ \) is of course treated as an independent field.

Following the same procedure as in section 2, we will compute \( \tilde{L}_+^* \), the dual of the Lagrangian (3.33). It is defined by
\[
\exp\left(-f_X \tilde{L}^*\right) = \int \mathcal{D} \omega^+ \mathcal{D} \Sigma^+ \mathcal{D} R^+ \exp\left(-f_X L_+\right)
\]

\[
= \int \mathcal{D} \omega^+ \mathcal{D} \Sigma^+ \exp\left(-f_X L_{\Sigma^+}\right)
\]

\[
= \int \mathcal{D} \omega^+ \exp\left(-f_X L_{\omega^+}\right).
\]

(3.36)

The first integration respect to \(R^+\) gives

\[
L_{\Sigma^+} = \varepsilon^{\mu \nu \rho \sigma} \left[ -\frac{1}{4A}(B \Sigma_{\mu \nu}^{+ab} + EG_{\mu \nu}^{+ab})(B \Sigma_{\rho \sigma}^{+ab} + EG_{\rho \sigma}^{+ab}) 
+ C \Sigma_{\mu \nu}^{+ab} \Sigma_{\rho \sigma}^{+ab} + EG_{\mu \nu}^{+ab} (\partial_\rho \omega_{\sigma \rho}^+ - \partial_\sigma \omega_{\rho \rho}^+ + \frac{1}{2} f_{[\alpha \beta \gamma \delta]} \omega_{\alpha \beta \gamma \delta}^+) \right].
\]

(3.37)

The relevant part of this Lagrangian for the integration in \(\Sigma^+\) is

\[
L_{\Sigma^+} = \varepsilon^{\mu \nu \rho \sigma} \left[ \ldots + \left(C - \frac{B^2}{4A}\right) \Sigma_{\mu \nu}^{+ab} \Sigma_{\rho \sigma}^{+ab} - \frac{BE}{2A} G_{\mu \nu}^{+ab} \Sigma_{\rho \sigma}^{+ab} + \ldots \right],
\]

(3.38)

which is of the Gaussian type. Integration gives

\[
L_{\omega^+} = \frac{E^2}{4A} \left(\frac{B^2}{\Delta} - 1\right) \varepsilon^{\mu \nu \rho \sigma} G_{\mu \nu}^{+ab} G_{\rho \sigma}^{+ab} + 2E \varepsilon^{\mu \nu \rho \sigma} \partial_\rho G_{\mu \nu}^{+ab} \omega_{\sigma \rho}^+ 
+ \frac{1}{2} E \varepsilon^{\mu \nu \rho \sigma} f_{[\alpha \beta \gamma \delta]} G_{\mu \nu}^{+ab} \omega_{\alpha \beta \gamma \delta}^+ \omega_{\sigma \rho}^+,
\]

(3.39)

where \(\Delta = B^2 - 4AC\). Finally, the computation of \(\int \mathcal{D} \omega^+ \exp\left(-f_X L_{\omega^+}\right)\) gives

\[
\tilde{L}^{**} = \frac{E^2}{4A} \left(\frac{B^2}{\Delta} - 1\right) \varepsilon^{\mu \nu \rho \sigma} G_{\mu \nu}^{+ab} G_{\rho \sigma}^{+ab} - 2E \varepsilon^{\mu \nu \rho \sigma} \partial_\rho G_{\mu \nu}^{+ab} (M^{+1})_{\mu \nu \lambda \delta} \varepsilon^{\lambda \rho \sigma} \partial_\theta G_{\rho \sigma}^{+ ab} \ v_{\lambda \delta},
\]

(3.40)

where \(M^{+1}_{\mu \nu \lambda \delta} \equiv \varepsilon^{\mu \nu \rho \sigma} f_{[\alpha \beta \gamma \delta]} G_{\mu \nu}^{+ab} \ v_{\lambda \delta} \varepsilon^{\rho \sigma} \).

The full action can be finally constructed from

\[
\tilde{L}^{**} = \tilde{L}^{**} + \tilde{L}^{**}_-. \]

(3.41)

Obviously, Lagrangian (3.41) contains terms of the form \(\Sigma \tilde{\Sigma}\) and \(\tilde{R} \Sigma\) that will add to (3.32), giving then a generalized \(BF\)-theory. The coefficient of the \(G^2\)-term in (3.40) can be adjusted to be the inverse of \(A\) or \(B\) in (3.33). We can observe that in this case the dual transformation involves a more complicated law of transformation for the coupling constants.
4. FINAL REMARKS

In this paper we have defined a gravitational analog of $S$-duality following similar procedures to those well known in non-abelian non-supersymmetric Yang-Mills theories \[9,10\].

We have shown, how to construct ‘dual’ Lagrangians for pure topological gravity and for a $BF$ theory coupled to topological gravity. In the computation of ‘dual’ Lagrangians we have not considered fermionic determinants of zero modes \[20\]. We expect that the computation of these determinants will give the manner of transforming the partition function at the quantum level. This should give rise to some analog of ‘modular weights’ \[3\].

We are aware that this analogy only was carried out at the level of the structure group of the frame bundle over $X$ and not over all the genuine symmetries which arise in Einstein gravity theories, such as $\text{Diff}(X)$. Related to this, it is interesting to note that in our ‘dual’ Lagrangians in Eqs. (2.31), and (3.41) only partial derivatives of the $G$'s fields appear instead of covariant derivatives. However, it can be shown that following a procedure similar to that presented in the introduction, one can get the covariance of the partition functions. As it has been pointed out by Atiyah \[25\], $S$-duality symmetry in field theory is a duality between the fundamental homotopy group of the circle and the space of group characters of representation theory of the circle. That means a sort of ‘identification’ between algebraic properties of a topological space. It would be very interesting to investigate what is the mathematical interpretation of the ‘gravitational $S$-duality’. The way how this ‘gravitational $S$-duality’ can be mixed with the usual $S$-duality in field theory and string theory, are now under current investigation. Finally, in a forthcoming paper we will exhibit a “dual” theory to dynamical gravity. For this purpose, the MacDowell-Mansouri gauge theory of gravity \[26\] is worked out.

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