Seismic barriers filled with granular metamaterials: Mathematical models for granular metamaterials

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Abstract. The problem of seismic protection from the main types of surface acoustic waves and shear–pressure (SP) evanescent waves emanating from vicinity of an epicenter of an earthquake is discussed. Herein, SP waves represent a kind of the evanescent waves arising at critical angles of incident of bulk shear waves. The proposed seismic protection method utilizes vertical trenches (vertical barriers) filled with the specially constructed granular metamaterials. Some of nonlinear hyperelastic models along with nonlinear and inelastic models are analyzed for applications using granular metamaterials in case of cyclic dynamic loadings that correspond to arrival of the large intensity surface acoustic and evanescent waves. The main attention is paid to arrival of the large intensity Rayleigh, Rayleigh–Lamb and SP waves, as the most frequent waves and the most dangerous waves for engineering structures. Some of the new constitutive equations for metamaterials exhibiting different elastic moduli at tension and compression phases are proposed and discussed.

1. Introduction

Granular metamaterials used as filler for seismic barriers and seismic cushions are extensively studied in various laboratories around the world [1–25]. For analyzing mechanical properties of the discussed granular metamaterials at cyclic dynamic loadings, several methods are proposed, including elastic, hyperelastic and hypoelastic equations of state [26–29], elastic-plastic [1–3, 6, 7, 30], viscoelastic-plastic [19, 20], hydrodynamic equations of state [21, 22], etc. Herein, various nonlinear hyperelastic equations of state are analyzed.

The problem of seismic protection from the main types of surface acoustic waves and shear–pressure (SP) evanescent waves emanating from vicinity of the epicenter of an earthquake is discussed. The proposed seismic protection method utilized vertical trenches (vertical barriers) filled in with the specially constructed granular metamaterials. Herein, the following notation is applied, the term vertical barrier in contrast to the horizontal barrier, is referred to a vertical formation in the upper layers of the Earth crust. The typical vertical dimension for a vertical
where in view of (1) introducing Cauchy relation for the infinitesimal strain tensor and the displacement field [24]

\[ \sigma = \lambda(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon)I + 2\mu(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon)\varepsilon, \]

where \( I \) denotes the unit diagonal matrix; Lame’s \( \lambda \) and \( \mu \) are functions of the strain (or stress) invariants, that can be written in the following form

\[ I_\varepsilon \equiv \text{tr}(\varepsilon), \quad \Pi_\varepsilon \equiv \frac{1}{3}(I_\varepsilon^2 - \varepsilon \cdot \varepsilon), \quad \PiI_\varepsilon \equiv \det(\varepsilon). \]  

Note that equation (1) ensures existence of the scalar potential, that will be introduced below in (6). It will be assumed that the strain energy that relates to equation (1), ensures that the strong ellipticity condition of the corresponding elastic tensor is satisfied [26,30]. The assumed strong ellipticity condition requires the following inequalities: \( \mu > 0, 3\lambda + 2\mu > 0 \).

2. Hyperelastic potentials

2.1. Equations of state

At the assumption of the infinitesimally small deformations, the stress–strain relation for the hyperelastic material takes the form

\[ \sigma = \lambda(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon)I + 2\mu(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon)\varepsilon, \]  

where \( I \) denotes the unit diagonal matrix; Lame’s \( \lambda \) and \( \mu \) are functions of the strain (or stress) invariants, that can be written in the following form

\[ I_\varepsilon \equiv \text{tr}(\varepsilon), \quad \Pi_\varepsilon \equiv \frac{1}{3}(I_\varepsilon^2 - \varepsilon \cdot \varepsilon), \quad \PiI_\varepsilon \equiv \det(\varepsilon). \]  

The gradient \( \nabla_x \lambda \) is defined similarly.

In addition to equation (1) for a hyperelastic material it is assumed the potential \( \Psi(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon) \) exists, such that [27]

\[ \sigma = \nabla_\varepsilon \Psi(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon) \]  

Accounting relations (2), the condition (6) can be rewritten as [36]

\[ \sigma = \frac{\partial \Psi}{\partial I_\varepsilon} I_\varepsilon + \frac{\partial \Psi}{\partial \Pi_\varepsilon}(I_\varepsilon - I) + \frac{\partial \Psi}{\partial \PiI_\varepsilon} (\varepsilon - \varepsilon - I_\varepsilon + \Pi_\varepsilon I). \]  

Comparing equations (1) and (7) yields the following representation of Lame’s constants in terms of the potential:

\[ \lambda(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon) = \frac{\partial \Psi}{\partial I_\varepsilon} I_\varepsilon^{-1} + \frac{\partial \Psi}{\partial \Pi_\varepsilon} I_\varepsilon^{-1} + \frac{\partial \Psi}{\partial \PiI_\varepsilon} I_\varepsilon^{-1}, \quad 2\mu(I_\varepsilon, \Pi_\varepsilon, \PiI_\varepsilon) = -\frac{\partial \Psi}{\partial I_\varepsilon} + \frac{\partial \Psi}{\partial \PiI_\varepsilon} (\varepsilon^{-1} - I_\varepsilon). \]
Equations (8) impose some restrictions on behavior of the potential $\Psi$. In particular, since Lame’s constants assumed to be continuous with respect to strain invariants, should be bounded at $I_\varepsilon \to 0$, equations (8) yields

$$\frac{\partial \Psi}{\partial I_\varepsilon} = O(I_\varepsilon), \quad I_\varepsilon \to 0; \quad \frac{\partial \Psi}{\partial \Pi_\varepsilon} = O(I_\varepsilon), \quad \Pi_\varepsilon \to 0; \quad \frac{\partial \Psi}{\partial \Pi_\varepsilon} = O(\Pi_\varepsilon), \quad \Pi_\varepsilon \to 0.$$  

At modeling of both statics and dynamics of granular materials, the hyperelastic constitutive equations are applied quite often [24]. It should be noted that in most of these works a concept of the multi-moduli media, actually, bi-modulus material, was applied [37] with a simple hyperelastic potential that is homogeneous of degree 2 with respect to the infinitesimal strain tensor

$$\Psi(I_\varepsilon, \Pi_\varepsilon^\sim) \equiv \alpha I_\varepsilon^2 + \beta \Pi_\varepsilon^\sim + \gamma I_\varepsilon \sqrt{\Pi_\varepsilon^\sim}.$$  

However, the discussed potential unfortunately, becomes irregular at vanishing second invariant. In the above equation, $\alpha, \beta, \gamma$ are the corresponding elastic material constants, independent of the invariants $I_\varepsilon, \Pi_\varepsilon^\sim$

$$\Pi_\varepsilon^\sim = -2I_\varepsilon + I_\varepsilon^2.$$  

Introducing parameter $\gamma$ allows one to account dependence of material properties on sign of the first invariant.

It should also be noted that with introduction [23] of the potential

$$\Psi(I_\varepsilon, \Pi_\varepsilon^\sim) = \Psi_1(I_\varepsilon, \Pi_\varepsilon^\sim)(1 - \exp(-\chi(\Pi_\varepsilon^\sim))); \quad \chi(\Pi_\varepsilon^\sim) \to 0, \quad \Pi_\varepsilon^\sim \to 0, \quad \chi(\Pi_\varepsilon^\sim) \to \infty, \quad \Pi_\varepsilon^\sim \to \infty,$$  

where $\Psi_1(I_\varepsilon, \Pi_\varepsilon^\sim)$ is an arbitrary potential, media with the dropdown (softening) diagrams can be modeled.

3. Elastic models

3.1. General equations

Elastic models are described by the following equation of state

$$\sigma = \lambda(I_\sigma, \Pi_\sigma, \Pi_\sigma)I_\varepsilon I + 2\mu(I_\sigma, \Pi_\sigma, \Pi_\sigma)\varepsilon.$$  

Compare this equation for the general nonlinear elastic and isotropic media at the infinitesimal deformations with equation (1) for the hyperelastic isotropic media.

3.2. Equations of motion

By analogy with equation (4), the linearized equation of motion can be represented in a form

$$\frac{\lambda + 2\mu}{\rho} \nabla_x \text{div}_x u - \frac{\mu}{\rho} \text{rot}_x \text{rot}_x u + \frac{1}{\rho} \left[ \nabla_x \lambda \text{div}_x u + \nabla_x \mu (\nabla_x u + \nabla_x u^T) \right] + b = \ddot{u}.$$  

Despite the apparent more generality, the elastic models are rarely used for modeling granular materials. In [5, 17] problems related to the determination of velocities of acoustic waves in a granular media modeled by a system of elastic balls, interacting by the Hertz theory, were considered.

4. Hypoelastic models

4.1. General equations

According to Trusedell [27] the time derivative of the stress tensor $\dot{\sigma}$ for a hypoelastic medium is determined by the time derivative of the strain tensor $\dot{\varepsilon}$. Assuming infinitesimal strains, the constitutive relation for an isotropic hypoelastic material can be written in a form

$$\dot{\sigma} = \lambda(I_\sigma, \Pi_\sigma, \Pi_\sigma)I_\varepsilon I + 2\mu(I_\sigma, \Pi_\sigma, \Pi_\sigma)\dot{\varepsilon}.$$  

(15)
where $\dot{\sigma} = \frac{\partial \sigma}{\partial t}$; $\lambda$ and $\mu$ are functions of the corresponding invariants. Comparing the stress–strain relations for hypoelastic (15) and elastic media (13) reveals, the only difference is in the incremental form of the constitutive relation for the hypoelastic medium.

In theoretical works [26, 30] it was demonstrated that the special triggering mechanism can be incorporated into equation of state (15) allowing to account different states for active and unloading cases; thus, the general elastic-plastic behavior can be modeled within the hypoelastic models.

4.2. Equations of motion

For a hypoelastic medium the equation of motion can be written in the form

$$\text{div} \dot{\sigma} + \rho \dot{b} = \rho \ddot{v},$$

(16)

where $\rho$ is the material density; it is assumed that $\dot{\rho} = 0$; $\dot{b}$ is the field of body forces. Substituting equation of state (15) into equation of motion (16) with account of the linearized Cauchy relations

$$\dot{\epsilon} = \frac{1}{2} \left( \nabla_x v + \nabla_x v^T \right)$$

(17)

yields

$$\frac{\lambda + 2\mu}{\rho} \nabla_x \text{div} x v - \frac{\mu}{\rho} \text{rot} x \text{rot} x v + \nabla_x \frac{\lambda}{\rho} \text{div} x v + \nabla_x \mu (\nabla_x v + \nabla_x v^T) + \dot{b} = \ddot{v},$$

(18)

where

$$\nabla_x \frac{\lambda}{\rho} = \frac{1}{\rho} \left( \frac{\partial \lambda}{\partial I_\sigma} \nabla_x I_\sigma + \frac{\partial \lambda}{\partial II_\sigma} \nabla_x II_\sigma + \frac{\partial \lambda}{\partial III_\sigma} \nabla_x III_\sigma \right).$$

(19)

The gradient $\nabla_x \mu$ is defined analogously.

Despite the obvious generality, the hypoelastic media are rarely used for modeling granular materials; in this regard it should be mentioned that the hypoelastic models were used for analyzing propagation of the impact bulk wave fronts propagating in granular materials [38], and the horizontally polarized surface acoustic waves; see [4].

5. Some inelastic models

5.1. General considerations

Along with various elastic models, there is a large number of works accounting inelastic behavior of granular metamaterials. Apparently, one of the simplest inelastic models applicable for static and quasi static modeling of granular metamaterials are based on various variants of the Mohr–Coulomb and Drucker–Prager theories.

Remaining within a more traditional approach based on the Mohr–Coulomb plasticity model, several approaches can be mentioned that are used for applications in the mechanics of granular media; see for example [7]. The Mohr–Coulomb plasticity model is also applied to analyzing known effects of arising and developing inelastic strain prior to the extensive flow of the avalanches; see the experimental work [18].

5.2. Specific inelastic models for dynamics of granular metamaterials

For the considered inelastic models used for dynamics of granular metamaterials apparently, the most widespread is the cam–clay (CC), the modified cam–clay (MCC) and the related models; see [1–3], along with some more recent works [6, 9]. For example, the ellipsoidal yield surface for the MCC model can be written as [9]

$$f(p, q_s, p_c) \equiv \frac{1}{\beta} \left( \frac{p}{a} - 1 \right)^2 + \left( \frac{q_s}{Ma} \right)^2 - 1 = 0,$$

(20)
where $\beta$ is a dimensionless parameter specifying the ellipsoid shape: in a subcritical zone $\beta = 1$ (left side), in a supercritical zone $\beta \leq 1$ (right side); the dimensionless parameter $M$, known as the critical cone tangent, specifies ellipsoid dimension along $q_s$-axis; $a$ is the “central” point of the ellipsoid, this parameter defines ellipsoid dimension along $p$-axis:

$$a = \frac{p_c}{1 + \beta^2}$$ (21)

where $p_c$ is the current yield pressure value, note, that at $\beta = 1$ parameter $a$ takes value $p_c/2$. Actually, parameter $p_c$ specifies evolution of the ellipsoidal surface.

6. Concluding remarks

As the presented review shows, the hyperelastic equations of state are presumably, the most widespread for the use in characterization of the granular metamaterials behavior at the cyclic dynamic loadings.

Meanwhile, equation of state (10) for hyperelastic models is not the only equation of state used for characterization of the metamaterials having different moduli at the tension and compression phases. At the uniaxial motions some other potentials may be used, e.g., Morse and Lennard–Jones potentials; see [25].

The other problem of characterizing the analyzed granular metamaterials at acoustic wave propagation, associates with formation of the shock waves formation at the interfaces between tension and compression phases, where the bulk elastic moduli (and quite often shear moduli) become different.

One more problem is the structural heterogeneity, which is deliberately created to increase the energy dissipation by granular metamaterials. The corresponding phenomena are discussed in [10, 39, 40].

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References

[1] Borja R I and Lee S R 1990 Comput. Methods Appl. Mech. Engrg. 178 49–72
[2] Borja R I and Tamagnini C 1998 Comput. Methods Appl. Mech. Engrg. 155 73–95
[3] Borja R I, Sanna K M and Sanz P F 2003 Comput. Methods Appl. Mech. Engrg. 192 1227–58
[4] Chandrasekharan D S 1977 Proc. - Indian Acad. Sci., Sect. A 86 383–91
[5] Coste C and Gilles B 1999 Eur. Phys. J. B 7 166–8
[6] Goldstein R V, Dudchenko A V and Kuznetsov S V 2016 Arch. Appl. Mech. 86 2021–31
[7] Goodman M A and Cowin S C 1972 Arch. Ration. Mech. Anal. 44 250–67
[8] Herbold E B, Nesterenko V F, Benson D J, Cai J, Vecchio K S, Jiang F, Addiss J W, Walley S M and Proud W G 2008 J. Appl. Phys. 104 103903
[9] Ilyashenko A V and Kuznetsov S V 2017 Mechanics and Mechanical Engineering 21 813–21
[10] Ilyashenko A V and Kuznetsov S V 2018 Z. Angew. Math. Phys. 69 17
[11] Massoudi M and Mehrabadi M M 2001 Acta Mech. 152 121–38
[12] Molinari A and Daraio C 2009 Phys. Rev. E: Stat., Nonlinear, Soft Matter Phys. 80 056602
[13] Nariboli G A and Juneja B L 1971 Int. J. Nonlinear Mech. 6 13–25
[14] Nedderman R M 1992 Statics and Kinematics of Granular Materials (Cambridge University Press)
[15] Nesterenko V F 2001 Dynamics of Heterogeneous Materials (New York: Springer-Verlag)
[16] Nesterenko V F, Herbold E B, Benson D J, Kim J and Daraio C 2008 J. Acoust. Soc. Am. 123 3271
[17] Norris A N and Johnson D L 1997 Trans. ASME 64 39–49
[18] Rajchenbach J 1990 Phys. Rev. Lett. 65 2221–4
[19] Roscoe K H, Schofield A N and Wroth C P 1958 Geotechnique 8 22–53
[20] Roscoe K H and Schofield A N 1963 Mechanical behaviour of an idealised wet clay European Conf. on Soil Mechanics and Foundation Engineering, Wiesbaden 1963 vol I pp 47–54
[21] Sun J, Hong J, Bang J, Avalos E and Doney R 2008 Phys. Rep. 462 21–66
[22] Sun J and Sundaresan S 2013 Powder Technol. 242 81–5
[23] Volokh K Y 2007 *J. Mech. Phys. Solids* **55** 2237–64
[24] Wang C C and Truesdell C 1973 *Introduction to Rational Elasticity* (Berlin, Heidelberg, New York: Springer)
[25] Zhen S and Davies G J 1983 *Phys. Status Solidi A* **78** 595–605
[26] Gurtin M E 1983 *Trans. ASME* **50** 894–6
[27] Truesdell C 1955 *J. Math. Mech.* **4** 83–133
[28] Truesdell C 1963 *J. Res. Natl. Bur. Stand., Sect. B* **67** 141–3
[29] Truesdell C and Toupin R A 1960 The classical field theories *Principles of Classical Mechanics and Field Theory* (Encyclopedia of Physics vol III/1) ed Flügge S (Berlin, Heidelberg: Springer) pp 226–793
[30] Green A E 1956 *Proc. Roy. Soc. London* **234** 46–59
[31] De Buhan P, Mangiavacchi R, Nova R, Pellegrini G and Salencon J 1989 *Geotechnique* **39** 189–201
[32] Kuznetsov S V and Nafasov A E 2011 *Adv. Acoust. Vib.* **2011** 150310
[33] Li S, Brun M, Djeran-Maigre I and Kuznetsov S 2019 *Comput. Geotech.* **109** 69–81
[34] Brule S, Javelaud E H, Enoch S and Guenneau S 2014 *Phys. Rev. Lett.* **112** 133901
[35] Infanti S, Papanikolas P and Castellano M G 2003 Seismic protection of the rion-antirion bridge *Proc. of 8th World Seminar on Seismic Isolation, Energy Dissipation and Active Vibration Control of Structures* (Yerevan, Armenia)
[36] Ericksen J L 1960 Tensor fields *Principles of Classical Mechanics and Field Theory* (Encyclopedia of Physics vol III/1) ed Flügge S (Berlin, Heidelberg: Springer) pp 794–858
[37] Lomakin E V and Rabotnov Y N 1978 *Mech. Solids* **6** 29–34
[38] Varley E and Dunwoody J 1965 *J. Mech. Phys. Solids* **13** 17–28
[39] Kuznetsov S V 2019 *Journal of Vibration and Control* **88** 196–204
[40] Kuznetsov S V 2019 *Wave Motion* **25** 11227–32