THE BRAUER GROUP OF AZUMAYA CORINGS AND THE SECOND COHOMOLOGY GROUP

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Abstract. Let $R$ be a commutative ring. An Azumaya coring consists of a couple $(S, C)$, with $S$ a faithfully flat commutative $R$-algebra, and an $S$-coring $C$ satisfying certain properties. If $S$ is faithfully projective, then the dual of $C$ is an Azumaya algebra. Equivalence classes of Azumaya corings form an abelian group, called the Brauer group of Azumaya corings. This group is canonically isomorphic to the second flat cohomology group. We also give algebraic interpretations of the second Amitsur cohomology group and the first Villamayor-Zelinsky cohomology group in terms of corings.

Introduction

Let $k$ be a field, and $l$ a Galois field extension of $k$ with group $G$. The Crossed Product Theorem states that we have an isomorphism $\text{Br}(l/k) \cong H^2(G, l^*)$. The map from the second cohomology group to the Brauer group can be described easily and explicitly: if $f \in Z^2(G, l^*)$ is a 2-cocycle, then the central simple algebra representing the class in $\text{Br}(l/k)$ corresponding to $f$ is

$$A = \bigoplus_{\sigma \in G} Au_\sigma,$$

with multiplication rule

$$(au_\sigma)(bu_\tau) = a\sigma(b)f(\sigma, \tau)u_{\sigma\tau}.$$

From the fact that every central simple algebra can be split by a Galois extension, it follows that the full Brauer group $\text{Br}(k)$ can be described as a second cohomology group

$$\text{Br}(k) \cong H^2(\text{Gal}(k^{\text{sep}}/k), k^{\text{sep}*}),$$

where $k^{\text{sep}}$ is the separable closure of $k$.

The definition of the Brauer group can be generalized from fields to commutative rings (see [2]), or, more generally, to schemes (see [13]). The cohomological description of the Brauer group of a commutative ring is more complicated; first of all, Galois cohomology is no longer sufficient, since not every Azumaya algebra can be split by a Galois extension. More general cohomology theories have to be introduced, such as Amitsur cohomology (over commutative rings) or Čech cohomology (over schemes). The Crossed Product Theorem is replaced by a long exact sequence, called the Chase-Rosenberg sequence. We can introduce the second étale cohomology group $H^2(\text{R}_{\text{et}}, \mathbb{G}_m)$, as the second right derived functor of a global section functor. If $R = k$ is a field, then this group equals the total Galois cohomology

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group $H^2(\text{Gal}(k^{\text{sep}}/k), k^{\text{sep}}^*)$. Then we have a monomorphism
\[ \text{Br}(R) \hookrightarrow H^2(R_{et}, \mathbb{G}_m). \]
In general, this monomorphism is not surjective, as the Brauer group is always torsion, and the second cohomology group is not torsion in general. Gabber [12] proved that the Brauer group is isomorphic to the torsion part of the second cohomology group.

In [24], Taylor introduced a new Brauer group, consisting of equivalence classes of algebras that do not necessarily have a unit. The classical Brauer group is a subgroup, and it is shown in [20] that Taylor’s Brauer group is isomorphic to the full second étale cohomology group. The proof depends on deep results, such as Artin’s Refinement Theorem (see [3]); also the proof does not provide an explicit procedure producing a Taylor-Azumaya algebra out of an Amitsur cocycle.

In this paper, we propose a new Brauer group, and we show that it is isomorphic to the full second flat cohomology group. The elements of this new Brauer group are equivalence classes of corings. Corings were originally introduced by Sweedler [23]; inspired by an observation made by Takeuchi that a large class of generalized Hopf modules can be viewed as comodules over a coring, Brzeziński [5] revived the theory of corings. [5] was followed by a series of papers giving new applications of corings, we refer to [6] for a survey.

Let $S$ be a commutative faithfully flat $R$-algebra. We can define a comultiplication and a counit on the $S$-bimodule $S \otimes S$, making $S \otimes S$ into a coring. This coring, called Sweedler’s canonical coring, can be used to give an elegant approach to descent theory: the category of descent data is isomorphic to the category of comodules over the coring. Our starting observation is now the following: an Amitsur 2-cocycle can be used to deform the comultiplication on $S \otimes S$, such that the new comultiplication is still coassociative. Thus the Amitsur 2-cocycle condition should be viewed as a coassociativity condition rather than an associativity condition. In the situation where $S$ is faithfully projective as an $R$-module, we can take the dual of the coring $S \otimes S$, which is an $S$-ring, isomorphic to $\text{End}_R(S)$. Amitsur 2-cocycles can then be used to deform the multiplication on $\text{End}_R(S)$, leading to an Azumaya algebra in the classical sense; this construction leads to a map $H^2(S/R, \mathbb{G}_m) \to \text{Br}(S/R)$, and we will show that it is one of the maps in the Chase-Rosenberg sequence.

The canonical coring construction can be generalized slightly: if $I$ is an invertible $S$-module, then we can define a coring structure on $I^* \otimes I$. Such a coring will be called an elementary $S/R$-coring. Azumaya $S/R$-corings are then introduced as twisted forms of elementary $S/R$-corings. If $S/R$ is faithfully projective, then the dual of an Azumaya $S/R$-coring is an Azumaya algebra containing $S$ as a maximal commutative subalgebra. The set of isomorphism classes of Azumaya $S/R$-corings forms a group; after we divide by the subgroup consisting of elementary corings, we obtain the relative Brauer group $\text{Br}^c(S/R)$; we will show that $\text{Br}^c(S/R)$ is isomorphic to Villamayor and Zelinsky’s cohomology group with values in the category of invertible modules $H^1(S/R, \text{Pic})$ [25]. As a consequence, $\text{Br}^c(S/R)$ fits into a
Chase-Rosenberg type sequence (even if $S/R$ is not faithfully projective).

An Azumaya coring will consist of a couple $(S,C)$, where $S$ is a (faithfully flat) commutative ring extension of $R$, and $C$ is an $S/R$-coring. On the set of isomorphism classes, we define a Brauer equivalence relation, and show that the quotient set is a group under the operation induced by the tensor product over $R$. This group is called the Brauer group of Azumaya corings, and we can show that it is isomorphic to the full second cohomology group.

If $C$ is an object of a category $C$, then the identity endomorphism of $C$ will also be denoted by $C$.

1. The Brauer Group of a Commutative Ring

1.1. Amitsur cohomology. Let $R$ be a commutative ring, and $S$ an $R$-algebra that is faithfully flat as an $R$-module. Tensor products over $R$ will be written without index $R$: $M \otimes N = M \otimes_R N$, for $R$-modules $M$ and $N$. The $n$-fold tensor product $S \otimes \cdots \otimes S$ will be denoted by $S^\otimes n$. For $i \in \{1, \ldots, n+2\}$, we have an algebra map

$$\eta_i : S^\otimes(n+1) \to S^\otimes(n+2),$$

given by

$$\eta_i(s_1 \otimes \cdots \otimes s_{n+1}) = s_1 \otimes \cdots \otimes s_{i-1} \otimes 1 \otimes s_i \otimes \cdots \otimes s_{n+1}.$$

Let $P$ be a covariant functor from a full subcategory of the category of commutative $R$-algebras that contains all tensor powers $S^\otimes n$ of $S$ to abelian groups. Then we consider

$$\delta_n = \sum_{i=1}^{n+2} (-1)^{i-1}P(\eta_i) : P(S^\otimes(n+1)) \to P(S^\otimes(n+2)).$$

It is straightforward to show that $\delta_{n+1} \circ \delta_n = 0$, so we obtain a complex

$$0 \to P(S) \xrightarrow{\delta_1} P(S^\otimes 2) \xrightarrow{\delta_2} P(S^\otimes 3) \xrightarrow{\delta_3} \cdots,$$

called the Amitsur complex $C(S/R)$. We write

$$H^n(S/R, P) = \text{Ker} \delta_n / \text{Im} \delta_{n-1};$$

$$B^n(S/R, P) = Z^n(S/R, P)/B^n(S/R, P).$$

$H^n(S/R, P)$ will be called the $n$-th Amitsur cohomology group of $S/R$ with values in $P$. Elements in $Z^n(S/R, P)$ are called $n$-cocycles, and elements in $B^n(S/R, P)$ are called $n$-coboundaries.

In this paper, we will mainly look at the following two examples: $P = \text{Pic}$, where $\text{Pic}(S)$ is the Picard group of $S$, consisting of isomorphism classes of invertible $S$-modules, and $P = \mathbb{G}_m$, where $\mathbb{G}_m(S)$ is the group consisting of all invertible elements of $S$.

If $u \in S^\otimes n$, then we will write $u_i = \eta_i(u)$. Observe that $u \in \mathbb{G}_m(S^\otimes 3)$ is then a cocycle in $Z^2(S/R, \mathbb{G}_m)$ if and only if

$$u_1u_2^{-1}u_3u_4^{-1} = 1.$$

Amitsur cohomology was first introduced in [1] (over fields); it can be viewed as an affine version of Čech cohomology. For a more detailed discussion, see for example [4, 10, 11]. We now present some elementary properties of Amitsur cohomology groups. We will adopt the following notation: an element $u \in S^\otimes n$ will be written formally as $u = u^1 \otimes u^2 \otimes \cdots \otimes u^n$, where the summation is understood implicitly.
Proposition 1.1. Let $R$ be a commutative ring, and $f : S \to T$ a morphism of commutative $R$-algebras. $f$ induces maps $f_* : H^n(S/R, P) \to H^n(T/R, P)$. If $g : S \to T$ is a second algebra map, then $f_* = g_*$ (for $n \geq 1$).

Proof. The first statement is obvious. For the proof of the second one, we refer to [10] Prop. 5.1.7.

The following result is obvious.

Lemma 1.2. If $u, v \in Z^n(S/R, S, \mathbb{G}_m)$, then

$$u \otimes v = (u^1 \otimes v^1) \otimes (u^2 \otimes v^2) \otimes \cdots \otimes (u^n \otimes v^n) \in Z^n(S \otimes S/R, S, \mathbb{G}_m).$$

If $u, v \in B^n(S/R, S, \mathbb{G}_m)$, then $u \otimes v \in B^n(S \otimes S/R, S, \mathbb{G}_m)$.

Corollary 1.3. If $u \in Z^n(S/R, S, \mathbb{G}_m)$, then $[u \otimes 1] = [1 \otimes u]$, and $[u \otimes u^{-1}] = 1$ in $H^n(S \otimes S/R, S, \mathbb{G}_m)$.

Proof. Apply Proposition 1.1 to the algebra maps $\eta_1, \eta_2 : S \to S \otimes S/R, \eta_1(s) = 1 \otimes s, \eta_2(s) = s \otimes 1$.

Lemma 1.4. Take a cocycle $u = u^1 \otimes u^2 \otimes u^3 \in Z^2(S/R, S, \mathbb{G}_m)$. $|u| = u^1 u^2 u^3 \in \mathbb{G}_m(S)$ is called the norm of $u$, and

$$u_1 \otimes |u|^{-1} u_2 u_3 = 1 \otimes 1 = |u|^{-1} u_1 u_2 \otimes u_3.$$

Proof. The first equality is obtained after we multiply the second, third and fourth tensor factors in the cocycle condition $u_1 u_2^{-1} u_3 u_4^{-1} = 1$. The second equality is obtained after multiplying the first three tensor factors.

A 2-cocycle $u$ is called normalized if $|u| = 1$.

Lemma 1.5. Every cocycle $u$ is cohomologous to a normalized cocycle.

Proof. First observe that $\Delta_1(|u|^{-1} \otimes 1) = 1 \otimes |u|^{-1} \otimes 1$. The cocycle $u \Delta_1(|u|^{-1} \otimes 1) = u^1 \otimes |u|^{-1} u_2 \otimes u_3$ is normalized and cohomologous to $u$.

Now we consider the Amitsur complex $C(S \otimes S/R \otimes S)$. We have a natural isomorphism

$$(S \otimes S)^{\otimes_R S_n} \cong S^{\otimes (n+1)}, \quad (s_1 \otimes t_1) \otimes \cdots \otimes (s_n \otimes t_n) \mapsto s_1 \otimes \cdots \otimes s_n \otimes t_1 \cdots t_n.$$ 

The augmentation maps ($i = 1, 2, 3$)

$$\eta_i : (S \otimes S)^{\otimes_R S^2} \to (S \otimes S)^{\otimes_R S^3}$$

can then be viewed as maps

$$\eta_i : S^{\otimes 3} \to S^{\otimes 4},$$

and we find, for $u \in Z^2(S/R, S, \mathbb{G}_m)$ and $i = 1, 2, 3$ that $\eta_i(u) = u_i$. Consequently $u \otimes 1 = u_4 = u_1 u_2^{-1} u_3 = \Delta_1(u) \in B^2(S \otimes S/R \otimes S, S, \mathbb{G}_m)$.

Lemma 1.6. If $u \in Z^2(S/R, S, \mathbb{G}_m)$, then $u \otimes 1 \in B^2(S \otimes S/R \otimes S, S, \mathbb{G}_m)$.
1.2. Derived functor cohomology. Let $R$ be a commutative ring. $\text{cat}(R_{\text{fl}})$ is the full subcategory of commutative flat finitely presented $R$-algebras. A covariant functor $P : \text{cat}(R_{\text{fl}}) \to \text{Ab}$ is called a presheaf on $R_{\text{fl}}$. The category of presheaves on $R_{\text{fl}}$ and natural transformations will be denoted by $\mathcal{P}(R_{\text{fl}})$.

A presheaf $P$ is called a sheaf if $H^q(S'/S, P) = P(S')$ for every faithfully flat $R$-algebra homomorphism $S \to S'$. The full subcategory of $\mathcal{P}(R_{\text{fl}})$ consisting of sheaves is denoted by $\mathcal{S}(R_{\text{fl}})$. $\mathcal{P}(R_{\text{fl}})$ and $\mathcal{S}(R_{\text{fl}})$ are abelian categories having enough injective objects.

$\mathbb{G}_a$ and $\mathbb{G}_m$ are sheaves on $R_{\text{fl}}$. The embedding functor $i : \mathcal{S}(R_{\text{fl}}) \to \mathcal{P}(R_{\text{fl}})$ has a left adjoint $a : \mathcal{P}(R_{\text{fl}}) \to \mathcal{S}(R_{\text{fl}})$.

The “global section” functor $\Gamma : \mathcal{S}(R_{\text{fl}}) \to \text{Ab}$ is left exact, so we can consider its $n$-th right derived functor $R^n\Gamma$. We define the $n$-th flat cohomology group by

$$H^n(R_{\text{fl}}, \mathbb{G}_m) = R^n\Gamma(\mathbb{G}_m).$$

Fix a faithfully flat $R$-algebra $S$, and consider the functor

$$g = H^0(S/R, -) : \mathcal{P}(R_{\text{fl}}) \to \text{Ab}.$$  

Then $\Gamma = g \circ i$, and $i$ takes injective objects of $\mathcal{S}(R_{\text{fl}})$ to $g$-acyclics (see [7, lemma 5.6.6]), and we have exact long sequences, for every sheaf $F$, and for every $q \geq 0$ (see [7, 25]):

$$(1) \quad 0 \longrightarrow H^1(S/R, C^q) \longrightarrow H^{q+1}(R_{\text{fl}}, F) \longrightarrow H^0(S/R, H^{q+1}(\bullet, F)) \longrightarrow H^2(S/R, C^q) \longrightarrow H^1(S/R, C^{q+1}) \longrightarrow \cdots \longrightarrow H^{q+1}(S/R, C^q) \longrightarrow H^0(S/R, H^{q+1}(\bullet, F)) \longrightarrow \cdots,$$

The sheaf $C^i$ is the $i$-th syzygy of an injective resolution $0 \to F \to X^0 \to X^1 \to \cdots$ of $F$ in $\mathcal{S}(R_{\text{fl}})$, that is, $C^i = \text{Ker}(X^i \to X^{i+1})$.

A morphism $f : S \to T$ of commutative faithfully flat $R$-algebras induces a map between the corresponding sequences [11], namely we have a commutative diagram

$$0 \to H^1(S/R, C^q) \to H^{q+1}(R_{\text{fl}}, F) \to H^0(S/R, H^{q+1}(\bullet, F)) \to \cdots$$

$$(2) \quad \begin{array}{ccc} f_* & = & f_* \\ 0 \to H^1(T/R, C^q) \to H^{q+1}(R_{\text{fl}}, F) \to H^0(T/R, H^{q+1}(\bullet, F)) \to & \cdots \end{array}$$

It is known that $H^1(R_{\text{fl}}, \mathbb{G}_m) = \text{Pic}(R)$, the group of rank one projective $R$-modules. Writing down [11] for $F = \mathbb{G}_m$ and $q = 0$, we find the exact sequence

$$0 \to H^1(S/R, \mathbb{G}_m) \to \text{Pic}(R) \to H^0(S/R, \text{Pic}) \to H^2(S/R, \mathbb{G}_m) \to H^1(S/R, C^1) \to H^3(S/R, \mathbb{G}_m) \to \cdots$$

Let $\mathcal{R}$ be the category with faithfully flat commutative $R$-algebras as objects. The set of morphisms between two objects $S$ and $T$ is a singleton if there exists an algebra morphism $S \to T$ (then we write $S \leq T$), and is empty otherwise. Then $\mathcal{R}$ is a directed preorder, that is a category with at most one morphism between two objects, and such that every pair of objects $(S, T)$ has a successor, namely $S \otimes T$. 

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(1) 0 \to H^1(S/R, C^q) \to H^{q+1}(R_{\text{fl}}, F) \to H^0(S/R, H^{q+1}(\bullet, F)) \to \\
(2) \begin{array}{ccc} f_* & = & f_* \\ 0 \to H^1(T/R, C^q) \to H^{q+1}(R_{\text{fl}}, F) \to H^0(T/R, H^{q+1}(\bullet, F)) \to & \cdots \end{array} \\
(3) 0 \to H^1(S/R, \mathbb{G}_m) \to \text{Pic}(R) \to H^0(S/R, \text{Pic}) \to H^2(S/R, \mathbb{G}_m) \to H^1(S/R, C^1) \to H^3(S/R, \mathbb{G}_m) \to \cdots
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Let $P$ be a presheaf on $R_\et$. It follows from Proposition 1.1 that we have a functor $$H^n(\bullet/R, P) : \mathcal{R} \to \text{Ab},$$
and we can consider the colimit $$\check{H}^n(R_\et, P) = \text{colim} H^n(\bullet/R, P).$$
Now let $F$ be a sheaf. Using the exact sequences (1) and the commutative diagrams (2), we find a homomorphism of abelian groups

$$\check{H}^n(R_\et, F) \to H^n(R_\et, F).$$

If $n = 1$, this map is an isomorphism. In particular, we have

$$H^2(R_\et, \mathbb{G}_m) \cong H^1(R_\et, C^1) \cong \check{H}^1(R_\et, C^1).$$

The category $\text{cat}(R_\et)$ can be replaced by $\text{cat}(R_\et)$, the category of étale $R$-algebras. All results remain valid, and, moreover, we have

$$\check{H}^n(R_\et, F) \cong H^n(R_\et, F).$$

The proof of this result is based on Artin’s Refinement Theorem [3].

1.3. Amitsur cohomology with values in $\text{Pic}$. Let $R$ be a commutative ring. The category of invertible $R$-modules and $R$-module isomorphisms is denoted by $\text{Pic}(R)$. The Grothendieck group $K_0\text{Pic}(R)$ is the Picard group $\text{Pic}(R)$. The inverse of $[I] \in \text{Pic}(R)$ is represented by $I^* = \text{Hom}_R(I, R)$. If $I \in \text{Pic}(R)$, then the evaluation map $e_I : I \otimes I^* \to R$ is an isomorphism, with inverse the coevaluation map $e_I^* : R \to I \otimes I^*$. If $e_I(1) = \sum_i e_i \otimes e_i^*$, then $\{(e_i, e_i^*) \mid i = 1, \ldots, n\}$ is a finite dual basis for $I$.

Let $S$ be a commutative faithfully flat $R$-algebra. For every positive integer $n$, we have a functor

$$\delta_{n-1} : \text{Pic}(S^\otimes n) \to \text{Pic}(S^\otimes (n+1)),$$

given by

$$\delta_{n-1}(I) = I_1 \otimes_S S^\otimes (n+1) I_2^* \otimes_S S^\otimes (n+1) \cdots S^\otimes (n+1) J_{n+1},$$

$$\delta_{n-1}(f) = f_1 \otimes_S S^\otimes (n+1) (f_2^*)^{-1} \otimes_S S^\otimes (n+1) \cdots S^\otimes (n+1) (g_n)^{\pm 1},$$

with $J = I$ or $I^*$, $g = f$ or $f^*$ depending on whether $n$ is odd or even. Here $I_i = I \otimes_S S^\otimes n$, $S^\otimes n+1$, where $S^\otimes n+1$ is a left $S^\otimes n$-module via $\eta_i : S^\otimes n \to S^\otimes n+1$ (see Section 1.1). We easily compute that

$$\delta_n \delta_{n-1}(I) = \bigotimes_{j=1}^{n+2} \bigotimes_{i=1}^{j-1} (I_{ij} \otimes_S S^\otimes (n+2) I_{ij}^*),$$

so we have a natural isomorphism

$$\lambda_I = \bigotimes_{j=1}^{n+2} \bigotimes_{i=1}^{j-1} e_{I_{ij}} : \delta_n \delta_{n-1}(I) \to S^\otimes (n+2).$$

$Z_{n-1}(S/R, \text{Pic})$ is the category with objects $(I, \alpha)$, with $I \in \text{Pic}(S^\otimes n)$, and $\alpha : \delta_{n-1}(I) \to S^\otimes (n+1)$ an isomorphism of $S^\otimes (n+1)$-modules such that $\delta_n(\alpha) = \lambda_I$. A morphism $(I, \alpha) \to (J, \beta)$ is an isomorphism of $S^\otimes$-modules $f : I \to J$ such that $\beta \circ \delta_{n-1}(f) = \alpha$. $Z_{n-1}(S/R, \text{Pic})$ is a symmetric monoidal category, with tensor
Lemma 1.7. Let \((I, \alpha) \in \mathbb{Z}^1(S/R, \text{Pic})\). Then
\[
(I \otimes S, \alpha \otimes S) \cong \delta_0(I) \text{ in } \mathbb{Z}^1(S \otimes S/R \otimes S, \text{Pic}),
\]
and consequently \([I \otimes S, \alpha \otimes S] = 1\) in \(H^1(S \otimes S/R \otimes S, \text{Pic})\).

Proof. The isomorphism \(\alpha : I_1 \otimes_{S \otimes S} I_1^* \otimes_{S \otimes S} I_1 \to S^{\otimes 3}\) induces an isomorphism
\[
\beta : I_3 = I \otimes S \to I_1^* \otimes_{S \otimes S} I_2 = (S \otimes I)^* \otimes_{(S \otimes S) \otimes R \otimes S} (S \otimes S) (S \otimes I).
\]
The fact that \(\delta_2(\alpha) = \lambda_I\) implies that \(\beta\) is an isomorphism in \(\mathbb{Z}^1(S \otimes S/R \otimes S, \text{Pic})\).

Proposition 1.8. Let \(f : S \to T\) be a morphism of commutative faithfully flat \(R\)-algebras. \(f\) induces group morphisms \(f_* : H^n(S/R, \text{Pic}) \to H^n(T/R, \text{Pic})\). If \(g : S \to T\) is a second algebra morphism, then \(f_* = g_*\).

Proof. We have a functor \(f_* : \mathbb{Z}^{n-1}(S/R, \text{Pic}) \to \mathbb{Z}^{n-1}(T/R, \text{Pic})\), given by
\[
f_*(I, \alpha) = (I \otimes_{S \otimes S} T^{\otimes n}, \alpha \otimes_{S \otimes S} T^{\otimes n+1}).
\]
\(f_*\) induces maps \(f_* : H^n(S/R, \text{Pic}) \to H^n(T/R, \text{Pic})\).
\(f\) and \(g\) induce maps \(f_*\) and \(g_*\) between the exact sequence \([4]\) and its analog with \(S\) replaced by \(T\). We have seen in Proposition \([14]\) that these maps coincide on \(H^n(S/R, \mathbb{G}_m)\) and \(H^n(S/R, \text{Pic})\). It follows from the five lemma that they also coincide on \(H^n(T/R, \text{Pic})\).
It follows from Proposition 1.8 that we have a functor

\[ H^1(\bullet/R, \text{Pic}) : \mathcal{R} \to \text{Ab}, \]

so we can consider the colimit

\[ \hat{H}^n(R_{fl}, \text{Pic}) = \text{colim} H^1(\bullet/R, \text{Pic}). \]

If \( f : S \to T \) is a morphism of commutative faithfully flat \( R \)-algebras, then the maps \( f_* \) establish a map between the corresponding exact sequences \((5)\). This implies that the isomorphisms \((6)\) fit into commutative diagrams

\[ \begin{array}{ccc}
H^n(S/R, \text{Pic}) & \xrightarrow{\cong} & H^n(S/R, C^1) \\
\downarrow f_* & & \downarrow f_* \\
H^n(T/R, \text{Pic}) & \xrightarrow{\cong} & H^n(T/R, C^1)
\end{array} \]

Consequently, the functors \( H^n(\bullet/R, \text{Pic}) \) and \( H^n(\bullet/R, C^1) \) are isomorphic, and

\[ (7) \quad \hat{H}^1(R_{fl}, \text{Pic}) \cong \hat{H}^1(R_{fl}, C^1) \cong H^2(R_{fl}, \mathbb{G}_m). \]

1.4. The Brauer group. Let \( R \) be a commutative ring. An \( R \)-algebra \( A \) is called an Azumaya algebra if there exists a commutative faithfully flat \( R \)-algebra \( S \) such that \( A \otimes S \cong \text{End}_S(P) \) for some faithfully projective \( S \)-module \( P \). There are several equivalent characterizations of Azumaya algebras, we refer to the literature \[7, 10, 16\]. An Azumaya algebra over a field is nothing else then a central simple algebra.

Two \( R \)-Azumaya algebras \( A \) and \( B \) are called Brauer equivalent if there exist faithfully projective \( R \)-modules \( P \) and \( Q \) such that \( A \otimes \text{End}(P) \cong B \otimes \text{End}(Q) \) as \( R \)-algebras. This induces an equivalence relation on the set of isomorphism classes of \( R \)-Azumaya algebras. The quotient set \( \text{Br}(R) \) is an abelian group under the operation induced by the tensor product. The inverse of a class represented by an algebra \( A \) is represented by the opposite algebra \( A^{\text{op}} \).

If \( i : R \to S \) is a morphism of commutative rings, then we have an associated abelian group map

\[ \text{Br}(i) : \text{Br}(R) \to \text{Br}(S), \quad i[A] = [A \otimes S]. \]

The kernel \( \text{Ker}(\text{Br}(i)) = \text{Br}(S/R) \) is called the part of the Brauer group of \( R \) split by \( S \).

If \( S/R \) is faithfully flat, then we have an embedding \( \text{Br}(S/R) \to H^1(S/R, C^1) \). This embedding is an isomorphism if \( S \) is faithfully projective as an \( R \)-module. Consequently, we have an embedding

\[ \text{Br}(S/R) \to H^2(R_{fl}, \mathbb{G}_m), \]

and

\[ \text{Br}(R) \to H^2(R_{fl}, \mathbb{G}_m). \]

Since every \( R \)-Azumaya algebra can be split by an étale covering, \( H^2(R_{fl}, \mathbb{G}_m) \) can be replaced by \( H^2(R_{et}, \mathbb{G}_m) \) in the two formulas above. If \( R \) is a field, or, more generally, if \( R \) is a regular ring, then we have an isomorphism

\[ \text{Br}(R) \cong H^2(R_{et}, \mathbb{G}_m). \]
In general, we do not have such an isomorphism, because the Brauer group is torsion, and the second cohomology group is not (see [13]). Gabber (12, see also [17]) showed that
\[ Br(R) \cong H^2(R_{et}, G_m)_{tors}, \]
for every commutative ring \( R \). Taylor [24] introduced a Brauer group \( Br'(R) \) consisting of classes of algebras that have not necessarily a unit, but satisfy a weaker property. \( Br'(R) \) contains \( Br(R) \) as a subgroup, and we have an isomorphism [20]
\[ Br'(R) \cong H^2(R_{et}, G_m). \]
The proof is technical, and relies on Artin's refinement Theorem [3]. It provides no explicit description of the Taylor-Azumaya algebra that corresponds to a given cocycle.

2. SOME ADJOINTNESS PROPERTIES

We start this technical Section with the following elementary fact. For any morphism \( \eta : R \to S \) of rings, we have an adjoint pair of functors \( (F = - \otimes_R S, G) \) between the module categories \( M_R \) and \( M_S \). \( F \) is called the induction functor, and \( G \) is the restriction of scalars functor. For every \( M \in M_R, N \in M_S \), we have a natural isomorphism
\[ \text{Hom}_R(M, G(N)) \cong \text{Hom}_S(M \otimes_R S, N). \]
\( f : M \to G(N) \) and the corresponding \( \tilde{f} : M \otimes_R S \to N \) are related by the following formula:
\[ \tilde{f}(m \otimes_R s) = f(m)s. \]

Now assume that \( R \) and \( S \) are commutative rings, and consider the ring morphisms \( \eta_i : S \otimes_R S \to S \otimes_R S \otimes_R S \) \((i = 1, 2, 3)\) introduced at the beginning of Section 1.1. The corresponding adjoint pairs of functors between \( M_{S^{\otimes 2}} \) and \( M_{S^{\otimes 3}} \) will be written as \((F_i, G_i)\). \( M \in M_{S^{\otimes 2}} \) will also be regarded as an \( S \)-bimodule, and we will denote \( M_i = F_i(M) \). For \( m \in M \), we write
\[ m_i = (M \otimes_{S^{\otimes 2}} \eta_i)(m). \]
In particular, \( m_3 = m \otimes 1 \) and \( m_1 = 1 \otimes m \).

**Lemma 2.1.** Let \( M \in M_{S^{\otimes 2}} \). Then we have an \( S \)-bimodule isomorphism
\[ G_2(M_3 \otimes_{S^{\otimes 3}} M_1) \cong M \otimes_S M, \]
and an isomorphism
\[ s\text{Hom}_S(M, M \otimes_S M) \cong \text{Hom}_{S^{\otimes 3}}(M_2, M_3 \otimes_{S^{\otimes 3}} M_1). \]

**Proof.** The map
\[ \alpha : M_3 \otimes M_1 \to M \otimes_S M, \quad \alpha((m \otimes s) \otimes (t \otimes n)) = tm \otimes_S ns \]
induces a well-defined map
\[ \alpha : M_3 \otimes_{S^{\otimes 3}} M_1 \to M \otimes_S M. \]
Indeed, for all \( m, n \in M \) and \( s, t, u, v, w \in S \), we easily compute that
\[ \alpha((m \otimes s)(u \otimes v \otimes w) \otimes (t \otimes n)) = \alpha((umv \otimes sw) \otimes (t \otimes n)) \]
\[ = tumv \otimes_S ns = utm \otimes_S vnsw \]
\[ = \alpha((m \otimes s) \otimes (ut \otimes vns)) = \alpha((m \otimes s) \otimes (u \otimes v \otimes w)(t \otimes n)). \]
The map
\[ \beta : M \otimes M \to M_3 \otimes_{S \otimes S} M_1, \quad \beta(m \otimes n) = m_3 \otimes_{S \otimes S} m_1 \]
induces a well-defined map
\[ \beta : M \otimes_S M \to M_3 \otimes_{S \otimes S} M_1. \]
Indeed,
\[ \beta(ms \otimes n) = (ms \otimes 1) \otimes_{S \otimes S} (1 \otimes n) = (m \otimes 1)(1 \otimes s \otimes 1) \otimes_{S \otimes S} (1 \otimes n) = (m \otimes 1) \otimes_{S \otimes S} (1 \otimes sn) = \beta(m \otimes sn). \]
It is clear that \( \alpha \) and \( \beta \) are inverse \( S \)-bimodule maps. Finally, the adjunction cited above tells us that
\[ s\text{Hom}_S(M, M \otimes_S M) \cong \text{Hom}_{S \otimes S}(G_2(M_3 \otimes_{S \otimes S} M_1)) \cong \text{Hom}_{S \otimes S}(M_2, M_3 \otimes_{S \otimes S} M_1). \]

Using \( i \), we can write an explicit formula for the map \( \hat{f} : M_2 \to M_3 \otimes_{S \otimes S} M_1 \) corresponding to \( f : M \to M \otimes_S M \). To this end, we first introduce the following Sweedler-type notation:
\[ f(m) = m_{(1)} \otimes_S m_{(2)}, \]
where summation is understood implicitly. Then we have
(9) \[ \hat{f}(m_2) = \beta(f(m)) = m_{(1)3} \otimes_{S \otimes S} m_{(2)1}. \]

For \( i = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \), we now consider the ring morphisms
\[ \eta_{ij} = \eta_i \circ \eta_j : S \otimes_R S \to S \otimes_R S \otimes_R S \]
and the corresponding pairs of adjunct functors \((F_{ij}, G_{ij})\) between the categories \( \mathcal{M}_{S \otimes S} \) and \( \mathcal{M}_{S \otimes S} \).

**Lemma 2.2.** Let \( M \in \mathcal{M}_{S \otimes S} \). Then we have a natural isomorphism of \( S \)-bimodules
\[ G_{23}(M_{34} \otimes_{S \otimes S} M_{14} \otimes_{S \otimes S} M_{12}) \cong M \otimes_S M \otimes_S M, \]
and an isomorphism
\[ s\text{Hom}_S(M, M \otimes_S M \otimes_S M) \cong \text{Hom}_{S \otimes S}(M_{23}, M_{34} \otimes_{S \otimes S} M_{14} \otimes_{S \otimes S} M_{12}). \]
The map \( \hat{f} \) corresponding to \( f \in s\text{Hom}_S(M, M \otimes_S M \otimes_S M) \), with \( f(m) = m_{(1)} \otimes_S m_{(2)} \otimes_S m_{(3)} \) is given by the formula
(10) \[ \hat{f}(m_{23}) = m_{(1)34} \otimes_{S \otimes S} m_{(2)14} \otimes_{S \otimes S} m_{(3)12}. \]

**Proof.** The map
\[ \alpha : M_{34} \otimes_{S \otimes S} M_{14} \otimes_{S \otimes S} M_{12} \to M \otimes_S M \otimes_S M \]
and
\[ \beta : M \otimes_S M \otimes_S M \to M_{34} \otimes_{S \otimes S} M_{14} \otimes_{S \otimes S} M_{12} \]
given by the formulas
\[ \alpha \left( (m \otimes s \otimes t) \otimes_{S \otimes S} (s' \otimes n \otimes t') \otimes_{S \otimes S} (s'' \otimes t'' \otimes p) \right) = s'' s' m \otimes_S t'' ns \otimes_S p t t' \]
and
\[ \beta(m \otimes n \otimes p) = m_{34} \otimes_{S \otimes S} n_{14} \otimes_{S \otimes S} p_{12} \]
are well-defined inverse \(S\)-bimodule maps. Verification of the details goes precisely as in the proof of Lemma 2.1. Then, using the adjunction from the beginning of this Section, we find

\[
\text{Hom}_S(M, M \otimes_S M \otimes_S M) \cong \text{Hom}_{S^{\otimes 2}}(M, G_{23}(M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12})) \\
\cong \text{Hom}_{S^{\otimes 4}}(M_{23}, M_{34} \otimes_{S^{\otimes 4}} M_{14} \otimes_{S^{\otimes 4}} M_{12}).
\]

Using \(\Delta\), we find that \(\hat{f}(m_{23}) = \beta(f(m))\), and \((\ref{2.1})\) then follows easily. \(\square\)

Let \(S\) be a commutative faithfully flat \(R\)-algebra. We have an algebra morphism \(m : S^{\otimes n} \to S\), \(m(s_1 \otimes \cdots \otimes s_n) = s_1 \cdots s_n\), and the corresponding induction functor

\[- \otimes_{S^{\otimes n}} S = | - | : \mathcal{M}_{S^{\otimes n}} \to \mathcal{M}_S,
\]

which is strongly monoidal since \(|S^{\otimes n}| = S\), and

\[|M \otimes_{S^{\otimes n}} N| = M \otimes_{S^{\otimes n}} N \otimes_{S^{\otimes n}} S \cong (M \otimes_{S^{\otimes n}} S) \otimes_{S} (N \otimes_{S^{\otimes n}} S) \cong |M| \otimes_{S} |N|.
\]

Recall from \([14\text{ IX.4.6}]\) that an \(R\)-module \(M\) is faithfully projective if and only if there exists an \(R\)-module \(N\) such that \(M \otimes N \cong R^n\). This implies that \(|-|\) sends faithfully projective (resp. invertible) \(S^{\otimes n}\)-modules to faithfully projective (resp. invertible) \(S\)-modules.

**Lemma 2.3.** Let \(M_1, \ldots, M_n \in \mathcal{M}_S\). Then

\[|M_1 \otimes \cdots \otimes M_n| \cong M_1 \otimes_S \cdots \otimes_S M_n.
\]

**Proof.** The natural epimorphism \(\pi : M_1 \otimes \cdots \otimes M_n \to |M_1 \otimes \cdots \otimes M_n|\) factors through \(M_1 \otimes_S \cdots \otimes_S M_n\) since

\[
\pi(m_1 \otimes \cdots \otimes sm_i \otimes \cdots \otimes m_n) = (m_1 \otimes \cdots \otimes sm_i \otimes \cdots \otimes m_n) \otimes_{S^{\otimes n}} 1 \\
= (m_1 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_n) \otimes_{S^{\otimes n}} s \\
= (m_1 \otimes \cdots \otimes sm_j \otimes \cdots \otimes m_n) \otimes_{S^{\otimes n}} 1,
\]

for all \(i, j\), so we have a map

\[\alpha : M_1 \otimes_S \cdots \otimes_S M_n \to |M_1 \otimes \cdots \otimes M_n|.
\]

In a similar way, the quotient map \(M_1 \otimes \cdots \otimes M_n \to M_1 \otimes_S \cdots \otimes_S M_n\) factors through \(|M_1 \otimes \cdots \otimes M_n|\), so we have a map

\[\beta : |M_1 \otimes \cdots \otimes M_n| \to M_1 \otimes_S \cdots \otimes_S M_n,
\]

which is inverse to \(\alpha\). \(\square\)

3. Corings

Let \(S\) be a ring. Recall that an \(S\)-coring is a coalgebra (or comonoid) \(C\) in the category \(\mathcal{S}_{\mathcal{M}_S}\). This means that \(C\) is an \(S\)-bimodule, together with two \(S\)-bimodule maps \(\Delta_C : C \to C \otimes S C\) and \(\varepsilon_C : C \to S\), satisfying the usual coassociativity and counit conditions:

\[(C \otimes S \Delta_C) \circ \Delta_C = (\Delta_C \otimes S C) \circ \Delta_C ; (C \otimes S \varepsilon_C) \circ \Delta_C = (\varepsilon_C \otimes S C) \circ \Delta_C = C.
\]

For the comultiplication \(\Delta_C\), we use the following Sweedler type notation:

\[\Delta_C(c) = c_{(1)} \otimes_S c_{(2)}.
\]
A right $C$-comodule $M$ is a right $S$-module together with a right $S$-linear map $\rho : M \to M \otimes_S C$ such that
\[
(M \otimes_S \Delta_C) \circ \rho = (\rho \otimes S M) \circ \rho ; \quad (M \otimes_S \varepsilon_C) \circ \rho = S.
\]
If $C$ is an $S$-comodule, then $\text{sHom}(C, S)$ is an $S$-ring. This means that $\text{sHom}(C, S)$ is a ring, and that we have a ring morphism $j : S \to \text{sHom}(C, S)$. The multiplication on $\text{sHom}(C, S)$ is given by the formula
\[
(g \# f)(c) = f(c(1))g(c(2)).
\]
The unit is $\varepsilon_C$, and $j(s)(c) = \varepsilon_C(c)s$, for all $s \in S$ and $c \in C$. In a similar way, $\text{Hom}_S(C, S)$ is an $S$-ring. The multiplication is now given by the formula
\[
(f \# g)(c) = f(g(c(1))c(2)).
\]
For a detailed discussion of corings and their applications, we refer to [6].

Example 3.1. Take an invertible $S$-module $I$. Then $I$ is finitely projective as an $S$-module, and we have a finite dual basis $\{(e_i, f_i) \in I \times I^* \mid i = 1, \ldots, n\}$ of $I$. Then $\sum_i e_i \otimes_S f_i = 1 \in I \otimes_S I^* \cong S$. We have an $S/R$-coring $C = \text{Can}_R(I; S) = I^* \otimes_R I$,

with structure maps
\[
\Delta_C : I^* \otimes_R I \to I^* \otimes_R I \otimes_S I^* \otimes_R I \cong I^* \otimes_R S \otimes_R I
\]
\[
\varepsilon_C : I^* \otimes_R I \to S
\]
given by
\[
\Delta_C(f \otimes x) = \sum_i f \otimes e_i \otimes_S f_i \otimes x = f \otimes 1 \otimes x ; \quad \varepsilon_C(f \otimes x) = f(x).
\]
We call $C$ an elementary coring. If $I = S$, then we obtain Sweedler’s canonical coring, introduced in [23]; in general, $\text{Can}_R(I; S)$ is an example of a comatrix coring, as introduced in [11]. We also compute that
\[
\text{sHom}(C, S) = \text{sHom}(I^* \otimes_R I, S) \cong \text{RHom}(I, I) = \text{REnd}(I).
\]
$\text{REnd}(I)$ is an $R$-algebra (under composition) and an $S$-ring, and we find an isomorphism of $S$-rings
\[
\text{sHom}(C, S) \cong \text{REnd}(I)^{\text{op}}.
\]

Lemma 3.2. Let $S$ and $T$ be commutative $R$-algebras. Then we have a strongly monoidal functor
\[
F = - \otimes_R T : \mathcal{M}_{S \otimes_R S} \to \mathcal{M}_{(S \otimes_R T) \otimes_T (S \otimes_R T)} = \mathcal{M}_{S \otimes_R S \otimes_R T}.
\]
Consequently, if $C$ is an $S/R$-coring, then $F(C) = C \otimes_R T$ is an $S \otimes_R T/T$-coring.
Lemma 4.1. Let $S$ be an $S$-algebra, and assume that its corresponding map $\Delta : I_2 \to I_3 \otimes S \otimes I_1$ in $\mathcal{M}_{S \otimes S}$ is an isomorphism. Then we have an isomorphism of $S \otimes S$-modules
\begin{equation}
\alpha^{-1} = (\Delta \otimes S) (I_2) \otimes \text{coev}_{I_3} : S \otimes S \to I_2 \otimes S \otimes I_1 \otimes S \otimes I_2.
\end{equation}

$\Delta$ is coassociative if and only if $(I, \alpha) \in Z^1(S \otimes R, \text{Pic})$.

Proof. We have the following isomorphisms of $S \otimes S$-modules:
\begin{align*}
\tilde{\Delta}_1 : I_{21} &= I_{13} \to I_{31} \otimes S \otimes I_{11} = I_{14} \otimes S \otimes I_{12}; \\
\tilde{\Delta}_2 : I_{22} &= I_{23} \to I_{32} \otimes S \otimes I_{12} = I_{24} \otimes S \otimes I_{12}; \\
\tilde{\Delta}_3 : I_{23} &= I_{33} \otimes S \otimes I_{13} = I_{34} \otimes S \otimes I_{13}; \\
\tilde{\Delta}_4 : I_{24} &= I_{34} \otimes S \otimes I_{14}.
\end{align*}

$(I, \alpha) \in Z^1(S \otimes R, \text{Pic})$ if and only if the composition
\begin{equation}
I_{23} \xrightarrow{I_{23} \otimes \text{coev}_{I_{13}}} I_{23} \otimes S \otimes I_{13} \otimes S \otimes I_{13} \xrightarrow{\Delta \otimes S \otimes I_3} I_{34} \otimes S \otimes I_{13} \otimes S \otimes I_{14} \otimes S \otimes I_{12} \xrightarrow{\text{coev}_{I_{14}} \otimes I_{12}} I_{24} \otimes S \otimes I_{24} \otimes S \otimes I_{23} \xrightarrow{\Delta \otimes I_{24} \otimes I_{23}} I_{34} \otimes S \otimes I_{14} \otimes S \otimes I_{12}.
\end{equation}

Let $\{(e_i, e_i^* ) \mid i = 1, \ldots, n\}$ be a finite dual basis of $I$. For all $c \in I$, we compute
\begin{align*}
\left( I_{34} \otimes \text{ev}_{I_{13}} \otimes I_{14} \otimes I_{12} \right) \circ \left( \Delta_3 \otimes I_{13} \otimes \tilde{\Delta}_1 \right) \circ \left( I_{23} \otimes \text{coev}_{I_{13}} \right)(c_{23}) &= \left( \left( I_{34} \otimes \text{ev}_{I_{13}} \otimes I_{14} \otimes I_{12} \right) \circ \left( \Delta_3 \otimes I_{13} \otimes \tilde{\Delta}_1 \right) \right) \left( \sum_i c_{i13} \otimes e_i^* \otimes e_i e_i^* \right) \\
&= \left( I_{34} \otimes \text{ev}_{I_{13}} \otimes I_{14} \otimes I_{12} \right) \left( \sum_i c_{(1)34} \otimes c_{(2)13} \otimes e_i^* \otimes \tilde{\Delta}_1(e_i e_i^*) \right) \\
&= \sum_i c_{(1)34} \otimes \tilde{\Delta}_1((c_{(2)} e_i^*) e_i)_{13} = c_{(1)34} \otimes \tilde{\Delta}_1(c_{(2)13}) \\
&= c_{(1)34} \otimes c_{(2)(1)14} \otimes c_{(2)(2)12}.
\end{align*}
and
\[
\left( (I_{34} \otimes I_{14} \otimes \text{ev}_{I_{24}} \otimes I_{12}) \circ \left( \tilde{\Delta}_4 \otimes J_{24}^* \otimes \tilde{\Delta}_2 \right) \circ (\text{coev}_{I_{24}} \otimes I_{23}) \right)(c_{23}) = \left( (I_{34} \otimes I_{14} \otimes \text{ev}_{I_{24}} \otimes I_{12}) \circ \left( \tilde{\Delta}_4 \otimes J_{24}^* \otimes \tilde{\Delta}_2 \right) \right) \left( \sum_i e_{i24} \otimes e_{i24}^* \otimes c_{124} \otimes c_{121} \right).
\]

From Lemma 4.2, it follows that \((I, \alpha) \in Z^1(S/R, \text{Pic})\) if and only if the maps in Hom\(_{S \otimes S}(I_{24}, I_{34} \otimes S \otimes I_{14} \otimes S \otimes I_{12})\) associated to \((\Delta \otimes I) \circ \Delta\) and \((I \otimes \Delta) \circ \Delta\) in \(\text{Hom}_S(I, J \otimes I \otimes S)\) are equal. This is equivalent to the coassociativity of \(\Delta\).

Observe that the map \(\tilde{\Delta}\) can be recovered from \(\alpha\) using the following formula
\[
(15) \quad \tilde{\Delta} = (I_3 \otimes I_1 \otimes \text{ev}_{I_2}) \circ (\alpha^{-1} \otimes I_2).
\]

**Lemma 4.2.** Let \(I, \Delta, \tilde{\Delta}, \alpha\) be as in Lemma 4.1, and take \(J \in \text{Pic}(S)\). Then we have an isomorphism of bimodules with coassociative comultiplication \(I \cong \text{Can}_{R(J; S)}\) if and only if \((I, \alpha) \cong \delta_0(J)\) in \(Z^1(S/R, \text{Pic})\).

**Proof.** Take \(J \in \text{Pic}(S)\). Then \(\delta_0(J) = J_1 \otimes S \otimes J_2 = J^* \otimes J\), and
\[
\delta_1 \delta_0(J) = \delta_0(J)_1 \otimes S \otimes \delta_0(J)_3 \otimes S \otimes \delta_0(J)^*_2 = J_{11} \otimes S \otimes J_{21}^* \otimes S \otimes J_{13} \otimes S \otimes J_{23}^* \otimes S \otimes J_{12} \otimes S \otimes J_{22} = J_{12} \otimes S \otimes J_{13}^* \otimes S \otimes J_{13} \otimes S \otimes J_{23} \otimes S \otimes J_{23}^* \otimes S \otimes J_{23}.
\]

The map \(\lambda_J\) is obtained by applying the evaluation map on tensor factors 1 and 2, 3 and 4 and 6. Let \(\{(e_i, e_i^*) \mid i = 1, \cdots, n\}\) be a finite dual basis of \(J^*\) as an \(S\)-module. Then
\[
\lambda_J^{-1}(1 \otimes 1 \otimes 1) = \sum_{i,j,k} e_{i12} \otimes e_{j13}^* \otimes e_{j13} \otimes e_{k23}^* \otimes e_{i12}^* \otimes e_{k23}.
\]

Take \(x^* \otimes 1 \otimes x = x_{12} \otimes S \otimes x_{23}^* \in (J^* \otimes J)_2 = J^* \otimes S \otimes J = J_{12} \otimes S \otimes J_{23} \). We then compute, using (14)
\[
\tilde{\Delta}(x^* \otimes 1 \otimes x) = \left( \delta_0(J) \otimes S \otimes \delta_0(J)_3 \otimes S \otimes \text{ev}_{\delta_0(J)_2}(\lambda_J^{-1} \otimes S \otimes \delta_0(J)_2) \right) (x_{12} \otimes x_{23}^*) = \sum_{i,j,k} e_{i12} \otimes e_{j13}^* \otimes e_{j13} \otimes e_{k23}^* \otimes e_{i12}^* \otimes x_{23}^* \otimes (x_{12} \otimes x_{23}^*) = \sum_{j} x_{12} \otimes e_{j13} \otimes e_{j13}^* \otimes x_{23}^* = \sum_{j} x^* \otimes (e_j^* \otimes S \otimes e_j) \otimes x \in J^* \otimes (J^* \otimes S \otimes J) \otimes J.
\]

Consequently
\[
\Delta(x^* \otimes x) = \sum_{j} x^* \otimes (e_j^*, e_j) \otimes x = x^* \otimes 1 \otimes x
\]
is the comultiplication on \(\text{Can}_{R(J; S)}\). In a similar way, starting from the comultiplication \(\Delta\) on \(\text{Can}_{R(J; S)}\), we find that the map \(\alpha\) defined in (14) is precisely \(\lambda_J\). \(\square\)
Theorem 4.3. Let $C$ be a faithfully projective $S \otimes S$-module, and $\Delta : C \to C \otimes_S C$ an $S$-bimodule map. We consider the corresponding map $\tilde{\Delta} : C_2 \to C_3 \otimes_{S \otimes S} C_1$ in $\mathcal{M}_{S \otimes S}$ (cf. Lemma 2.7). Then the following assertions are equivalent.

1) $\Delta$ is coassociative and $\tilde{\Delta}$ is an isomorphism in $\mathcal{M}_{S \otimes S}$;
2) $C \in \text{Pic}(S \otimes S)$ and $(C, \alpha) \in \mathbb{Z}^1(S/R, \text{Pic})$, with $\alpha$ defined by (4);
3) $C \in \text{Pic}(S \otimes S)$ and $C \otimes S$ is isomorphic to $\text{Can}_{R \otimes S}(C; S \otimes S)$ as bimodules with coassociative comultiplication;
4) there exists a faithfully flat commutative $R$-algebra $T$ such that $(C \otimes_R T, \tilde{\Delta} \otimes_R T)$ is isomorphic to $\text{Can}_T(I; S \otimes T)$, for some $I \in \text{Pic}(S \otimes T)$, as a bimodule with a coassociative comultiplication;
5) $(C, \Delta)$ is a coring and $\tilde{\Delta}$ is an isomorphism in $\mathcal{M}_{S \otimes S}$.

Proof. 1) $\to$ 2). From the fact that $\tilde{\Delta}$ is an isomorphism, it follows that $C_2 \cong C_3 \otimes_{S \otimes S} C_1$. Applying the functor $| - | : \mathcal{M}_{S \otimes S} \to \mathcal{M}_S$, we find that $|C| \cong |C| \otimes_S |C|$. $C$ is a faithfully projective $S \otimes S$-module, so $|C|$ is a faithfully projective $S$-module. Its rank is an idempotent, so it is equal to one, and $|C| \in \text{Pic}(S)$. Now switch the second and third tensor factor in $C_2 \cong C_3 \otimes_{S \otimes S} C_1$, and then apply $| - |$ to the first and second factor. We find that $|C| \otimes S \cong C \otimes S \otimes S$, with $\tau(C)$ equal to $C$ as an $R$-module, with newly defined $S \otimes S$-action $c \cdot (s \otimes t) = c(t \otimes s)$. Now $|C| \otimes S \in \text{Pic}(S \otimes S)$, and it follows that $C \in \text{Pic}(S \otimes S)$. It follows now from Lemma 4.1 that $(C, \alpha) \in \mathbb{Z}^1(S/R, \text{Pic})$.

2) $\to$ 3). It follows from Lemma 1.2 that $(C \otimes S, \alpha \otimes S) \cong \delta_0(C) \in \mathbb{Z}^1(S \otimes S \otimes S \otimes S, \text{Pic})$. From Lemma 1.2 it follows that $C \otimes S \cong \text{Can}_{R \otimes S}(C; S \otimes S)$ as bimodules with coassociative comultiplication.

3) $\to$ 4) is obvious.

4) $\Rightarrow$ 1). After faithfully flat base extension, $\Delta$ becomes coassociative, and $\tilde{\Delta}$ becomes an isomorphism. Hence $\Delta$ is coassociative and $\tilde{\Delta}$ is an isomorphism.

1) $\Rightarrow$ 5). We have an isomorphism of $S \otimes S$-modules $\alpha : C_2 \otimes_{S \otimes S} C_1 \otimes_{S \otimes S} C_3 \to S \otimes S$. Applying the functor $| - |$, we find an isomorphism of $S$-modules $|\alpha| : |C| \to S$. Now we consider the composition $\varepsilon = |\alpha| \circ \pi : C \to S$. In the situation where $C = \text{Can}_R(I; S)$, $\varepsilon$ is the counit of $C$. By 4), $\varepsilon$ has the counit property after a base extension. Hence $\varepsilon$ has itself the counit property. So $(C, \Delta, \varepsilon)$ is a coring. 5) $\Rightarrow$ 1) is obvious.

If $(C, \Delta, \varepsilon)$ satisfies the equivalent conditions of Theorem 4.3, then we call $C$ an Azumaya $S/R$-coring. The connection to Azumaya algebras is discussed in the following Proposition.

Proposition 4.4. Let $S$ be a faithfully projective commutative $R$-algebra, and $C$ an Azumaya $S/R$-coring. Then $S \text{Hom}(C, S)$ and $\text{Hom}_S(C, S)$ are Azumaya $R$-algebras split by $S$.

Proof. Using Theorem 4.3 and (13), we find the following isomorphisms of $S$-algebras:

$$S \text{Hom}(C, S) \otimes S = S \text{Hom}(C, S) \otimes S \text{Hom}(S, S) \cong S \otimes S \text{Hom}(C \otimes S, S \otimes S) \cong S \otimes S \text{Hom}(\text{Can}_{R \otimes S}(C; S \otimes S), S \otimes S) \cong R \otimes S \text{End}(C)^{op}.$$
Theorem 4.5. Let $(C, \Delta)$ and $(C', \Delta')$ be Azumaya $S/R$-corings, and consider the corresponding $(C, \alpha), (C', \alpha') \in \mathbb{Z}^1(S/R, \text{Pic})$. Let $f : C \rightarrow C'$ be an isomorphism in $\text{Pic}(S \otimes S)$. Then $f$ is an isomorphism of corings if and only if $f$ defines an isomorphism in $\mathbb{Z}^1(S/R, \text{Pic})$.

Proof. $f$ is an isomorphism of corings if and only if the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes S C \\
\downarrow{f} & & \downarrow{f \otimes_S f} \\
C' & \xrightarrow{\Delta'} & C' \otimes S C'.
\end{array}
$$

This is equivalent to commutativity of the diagram

$$
\begin{array}{ccc}
C_2 & \xrightarrow{\Delta} & C_3 \otimes_{S \otimes S} C_1 \\
\downarrow{f_2} & & \downarrow{f_3 \otimes_{S \otimes S} f_1} \\
C_2' & \xrightarrow{\Delta'} & C_3' \otimes_{S \otimes S} C_1'.
\end{array}
$$

This is equivalent to commutativity of the right square in the next diagram

$$
\begin{array}{ccc}
S \otimes S & \xrightarrow{\text{coev}_{C_2}} & C_2' \otimes_{S \otimes S} C_2 \\
\downarrow{\text{coev}_{C'}_{2}} & & \downarrow{(f_2')^{-1} \otimes_{S \otimes S} f_2} \\
S & \xrightarrow{\Delta} & C_3 \otimes_{S \otimes S} C_1 \\
\downarrow{\Delta} & & \downarrow{(f_2')^{-1} \otimes_{S \otimes S} f_3 \otimes_{S \otimes S} f_1} \\
S & \xrightarrow{\Delta} & C_3' \otimes_{S \otimes S} C_1'.
\end{array}
$$

The left square is automatically commutative. Commutativity of the full diagram is equivalent to $\alpha' \circ \delta_1(f) = \alpha$, as needed.

Let $\mathcal{A}(S/R)$ be the category of Azumaya $S/R$-corings and isomorphisms of corings.

Proposition 4.6. $(\mathcal{A}(S/R), \otimes_{S \otimes S}, \text{Can}_R(S; S))$ is a monoidal category.

Proof. Take two Azumaya $S/R$-corings $(C, \Delta)$ and $(C', \Delta')$, and let $\tilde{D}$ be the following composition

$$(C \otimes_{S \otimes S} C')_2 = C_2 \otimes_{S \otimes S} C'_2 \xrightarrow{\Delta \otimes \Delta'} C_3 \otimes_{S \otimes S} C_1 \otimes_{S \otimes S} C'_3 \otimes_{S \otimes S} C'_1 \xrightarrow{\text{can}_R \otimes \text{can}_R'} C_3 \otimes_{S \otimes S} C'_3 \otimes_{S \otimes S} C_1 \otimes_{S \otimes S} C'_1 = (C \otimes_{S \otimes S} C')_3 \otimes_{S \otimes S} (C \otimes_{S \otimes S} C')_1.
$$

The comultiplication on $C \otimes_{S \otimes S} C'$ is the corresponding map

$$
D : C \otimes_{S \otimes S} C' \rightarrow (C \otimes_{S \otimes S} C') \otimes_S (C \otimes_{S \otimes S} C').
$$

Observe that the $S$-bimodule structure on $C \otimes_{S \otimes S} C'$ is given by the formulas

$$
s(c \otimes c') = sc \otimes c' = c \otimes sc' : (c \otimes c')t = c \otimes (c't) = ct \otimes c'.
$$
We have that
\[ \tilde{D}(c \otimes c')_2 = (c_{(1)} \otimes S^2 c'_{(1)}) \otimes (c_{(2)} \otimes S^2 c'_{(2)})_1, \]
hence
\[ D(c \otimes c') = (c_{(1)} \otimes S^2 c'_{(1)}) \otimes (c_{(2)} \otimes S^2 c'_{(2)}). \]
It is then easy to see that \( D \) is coassociative, and that
\[ C \otimes S^2 \text{Can}_R(S; S) \cong C \cong \text{Can}_R(S; S) \otimes S^2 C. \]
\[ \square \]

**Corollary 4.7.** We have a monoidal isomorphism of categories
\[ H : \mathcal{A}_c(S/R) \to \mathbb{Z}^1(S/R, \text{Pic}). \]
Consider the subgroup \( \text{Can}^c(S/R) \) of \( K_0 \mathcal{A}_c(S/R) \) consisting of isomorphism classes represented by an elementary coring \( \text{Can}_R(I; S) \) for some \( I \in \text{Pic}(S) \). The quotient
\[ \text{Br}^c(S/R) = K_0 \mathcal{A}_c(S/R)/\text{Can}^c(S/R) \]
is called the relative Brauer group of Azumaya \( S/R \)-corings.

**Corollary 4.8.** We have an isomorphism of abelian groups
\[ \text{Br}^c(S/R) \cong H^1(S/R, \text{Pic}). \]
Consequently, we have an exact sequence
\[ 0 \to H^1(S/R, \mathbb{G}_m) \to \text{Pic}(R) \to H^0(S/R, \text{Pic}) \to H^2(S/R, \mathbb{G}_m) \to \text{Br}^c(S/R) \to H^1(S/R, \text{Pic}) \to H^3(S/R, \mathbb{G}_m). \]

Let \( f : S \to T \) be a morphism of faithfully flat commutative \( R \)-algebras. Then we have a functor \( \tilde{f} : \mathcal{A}_c(S/R) \to \mathcal{A}_c(T/R) \) such that the following diagram commutes
\[ \begin{array}{ccc}
\mathcal{A}_c(S/R) & \xrightarrow{H} & \mathbb{Z}^1(S/R, \text{Pic}) \\
\tilde{f} \downarrow & & \downarrow f_* \\
\mathcal{A}_c(T/R) & \xrightarrow{H} & \mathbb{Z}^1(T/R, \text{Pic}).
\end{array} \]
\[ \tilde{f}(C) = C \otimes_{S^2} \text{Can}_R(T; T), \] with comultiplication \( \Delta_C \otimes_{S^2} \Delta \), where \( \Delta \) is the comultiplication on the canonical coring \( \text{Can}_R(T; T) \). This induces a commutative diagram
\[ \begin{array}{ccc}
\text{Br}^c(S/R) & \xrightarrow{\cong} & H^1(S/R, \text{Pic}) \\
\tilde{f} \downarrow & & \downarrow f_* \\
\text{Br}^c(T/R) & \xrightarrow{\cong} & H^1(T/R, \text{Pic}).
\end{array} \]

Otherwise stated, the isomorphisms in Corollary 4.8 define an isomorphism of functors
\[ \text{Br}^c(\_ / R) \cong H^1(\_ / R, \text{Pic}) : \mathcal{R} \to \text{Ab}. \]
It will be convenient to use the canonical identification
\[ \mathrm{End}_R(C_S) = \mathrm{Can}_R(S; S)_u, \]
which is equal to \( S \otimes S \) as an \( S \)-bimodule, with comultiplication
\[ \Delta_u : S \otimes S \rightarrow S \otimes S \otimes S \otimes S \cong S \otimes S \otimes S, \quad \Delta_u(s \otimes t) = u^1 s \otimes u^2 \otimes u^3 t. \]
The coassociativity follows immediately from the cocycle condition; the counit \( \varepsilon \) is given by the formula (see Lemma 1.4)
\[ \varepsilon(s \otimes t) = |u|^{-1} st. \]
The counit property follows from Lemma 1.4. If \( u \) is normalized, then the counit coincides with the counit in \( \mathrm{Can}_R(S; S) \).
Let us compute the right dual \( \mathrm{Hom}_S(C, S) \). As an \( R \)-module, \( \mathrm{Hom}_S(C, S) = \mathrm{End}_R(S) \). We transport the multiplication on \( \mathrm{Hom}_S(C, S) \) to \( \mathrm{End}_R(S) \) as follows: take \( \varphi, \psi \in \mathrm{End}_R(S) \), and define \( f, g \in \mathrm{Hom}_S(S \otimes S, S) \) by
\[ f(s \otimes t) = \varphi(s)t ; g(s \otimes t) = \psi(s)t. \]
Then we find, using (12),
\[ (\varphi * \psi)(s) = (f \# g)(s \otimes 1) = f(\psi(su^1))u^2 \otimes u^3 = \varphi(\psi(su^1))u^2 u^3, \]
or
\[ \varphi * \psi = u^3 \varphi^2 \psi. \]
In a similar way, we find that \( \mathrm{Hom}_S(C, S) \cong \mathrm{End}_R(S) \), with twisted multiplication
\[ \varphi * \psi = u^1 \varphi^2 \psi^3. \]
If \( S \) is faithfully projective as an \( R \)-module, then it is well-known that there exists a morphism
\[ \alpha : H^2(S/R, G_m) \rightarrow \mathrm{Br}(S/R). \]
More precisely, we can associate an Azumaya algebra \( A(u) \) to any cocycle \( u \in Z^2(S/R, G_m) \). The construction of \( A(u) \) was given first in [21 Theorem 2]. It is explained in [16 V.2] and [15 7.5] using descent theory. Let us summarize the construction of \( A(u) \), following [16]. Take a cocycle \( u = u^1 \otimes u^2 \otimes u^3 = U^1 \otimes U^2 \otimes U^3 \) with inverse \( u^{-1} = v^1 \otimes v^2 \otimes v^3 \), and consider the map
\[ \Phi : S \otimes S \otimes \mathrm{End}_R(S) \rightarrow S \otimes \mathrm{End}_R(S) \otimes S, \quad \Phi(s \otimes t \otimes \varphi) = su^1 v^1 \otimes u^3 \varphi v^3 \otimes tu^2 v^2. \]
Then
\[ A(u) = \{ x \in S \otimes \mathrm{End}_R(S) \mid x \otimes 1 = \Phi(1 \otimes x) \}. \]
It will be convenient to use the canonical identification \( \mathrm{End}_R(S) \cong S^* \otimes S \). Then \( x = \sum_i s_i \otimes t_i^* \otimes t_i \in S \otimes S^* \otimes S \) lies in \( A(u) \) if and only if
\[ \sum_i s_i \otimes t_i^* \otimes t_i \otimes 1 = \sum_i u^1 v^1 \otimes t_i^* v^3 \otimes u^3 t_i \otimes u^2 v^2 s_i, \]
or
\[ \sum_i s_i \otimes 1 \otimes t_i^* \otimes t_i = \sum_i u^1 v^1 \otimes u^2 v^2 s_i \otimes t_i^* v^3 \otimes u^3 t_i, \]
or
\[(22) \quad x_2 = x_1 u_3 u_4^{-1} \text{ or } x_2 u_4 = x_1 u_3.\]
Let $\text{End}_R(S)_u$ be equal to $\text{End}_R(S)$, with twisted multiplication given by $(22)$. We know from Proposition 4.4 that $\text{End}_R(S)_u$ is an Azumaya algebra split by $S$.

**Theorem 4.9.** Let $S$ be a faithfully projective commutative $R$-algebra, and $u \in Z^2(S/R, \mathbb{G}_m)$. Then we have an isomorphism of $R$-algebras $\gamma : \text{End}_R(S)_u \rightarrow A(u)$.

**Proof.** We define $\gamma$ by the following formula:
\[
\gamma(\varphi) = u^1 \otimes u^1 \varphi u^2,
\]
or
\[
\gamma(t^* \otimes t) = u^1 \otimes t^* u^2 \otimes u^3 t.
\]
We have to show that $x = \gamma(t^* \otimes t)$ satisfies $(22)$. Indeed,
\[
x_2 u_4 = (1 \otimes 1 \otimes t^* \otimes t) u_2 u_4 = (1 \otimes 1 \otimes t^* \otimes t) u_1 u_3 = x_1 u_3.
\]
Let us next show that $\gamma$ is multiplicative. We want to show that
\[
\gamma(\psi) \circ \gamma(\varphi) = \gamma(\psi \ast \varphi)
\]
or
\[
u^1 U^1 \otimes u^3 \psi u^2 U^3 \varphi U^2 = U^1 \otimes U^3 u^3 \psi u^2 \varphi u^1 U^2.
\]
It suffices that
\[
u^1 U^1 \otimes u^3 \otimes u^2 U^3 \otimes U^2 = U^1 \otimes U^3 u^3 \otimes u^2 \otimes u^1 U^2,
\]
or
\[
u^1 U^1 \otimes U^2 \otimes u^2 U^3 \otimes u^3 = U^1 \otimes U^2 \otimes u^2 \otimes U^3 u^3.
\]
This is precisely the cocycle condition $u_2 u_4 = u_1 u_3$.
The inverse of $\gamma$ is given by
\[
\gamma^{-1}(\sum_i s_i \otimes t_i^* \otimes t_i) = \sum_i t_i^* v^2 \otimes v^1 v^3 s_i t_i,
\]
for all $x = \sum_i s_i \otimes t_i^* \otimes t_i \in A(u)$. We compute that
\[
\gamma(\gamma^{-1}(x)) = \gamma(\sum_i t_i^* v^2 \otimes v^1 v^3 s_i t_i) = u^1 \otimes t_i^* v^2 u^2 \otimes u^3 v^3 s_i t_i.
\]
It follows from $(22)$ that
\[
x_2 = x_1 u_3 u_4^{-1} = x_1 u_2 u_4^{-1} = u^1 \otimes s_i v^1 \otimes t_i^* v^2 u^2 \otimes t_i u^3 v^3.
\]
Multiplying the second and the fourth tensor factor, we obtain that
\[
\gamma(\gamma^{-1}(x)) = u^1 \otimes t_i^* v^2 u^2 \otimes u^3 v^3 s_i t_i = x.
\]
Finally
\[
\gamma^{-1}(\gamma(t^* \otimes t)) = \gamma^{-1}(u^1 \otimes t^* u^2 \otimes u^3 t) = t^* u^2 v^2 \otimes v^1 v^3 u^1 u^1 t = t^* \otimes t.
\]
\qed
5. A Normal Basis Theorem

Let $S$ be a faithfully flat commutative $R$-algebra. We say that an $S \otimes S$-module with coassociative comultiplication has normal basis if it is isomorphic to $S \otimes S$ as an $S$-bimodule. Examples are the Azumaya $S/R$-corings $\text{Can}_R(S;S)_u$, with $u \in Z^2(S/R, G_m)$, as considered above. The category of $S/R$-corings (resp. $S \otimes S$-modules with coassociative comultiplication) with normal basis will be denoted by $FAz(S/R)$ (resp. $FAz'(S/R)$), $FAz(S/R, \otimes_{S \otimes S}, \text{Can}_R(S;S))$ and $FAz'(S/R, \otimes_{S \otimes S}, \text{Can}_R(S;S))$ are monoidal categories, and the sets of isomorphism classes of $S/R$-Azumaya corings with normal basis. We have inclusions

$$FAz(S/R) \subset F(S/R) \subset F'(S/R).$$

We will give a cohomological description of these monoids.

Take $u = u^1 \otimes u^2 \otimes u^3 \in S^{\otimes 3}$. As usual, summation is implicitly understood. We do not assume that $u$ is invertible. We call $u$ a 2-cosickle if $u_1 u_3 = u_2 u_3$. If, in addition, $u^1 u^2 \otimes u^3$ and $u^1 \otimes u^2 u^3$ are invertible in $S^{\otimes 2}$, then we call $u$ an almost invertible 2-cosickle. This implies in particular that $|u| = u^1 u^2 u^3$ is invertible in $S$. Almost invertible 2-cosickles have been introduced and studied in [13]. Let $S^2(S/R)$ be the set of 2-cosickles and $S^2(S/R)$ the set of almost invertible 2-cosickles. $S^2(S/R)$ and $S^2(S/R)$ are multiplicative monoids, and we have the following inclusions of monoids:

$$B^2(S/R, G_m) \subset Z^2(S/R, G_m) \subset S^2(S/R) \subset S^2(S/R) \subset S^{\otimes 3}.$$ 

We consider the quotient monoids

$$M^2(S/R) = S^2(S/R)/B^2(S/R, G_m); M^2(S/R) = S^2(S/R)/B^2(S/R, G_m).$$

$M^2(S/R)$ is called the second (Hebrew) Amitsur cohomology monoid; the subgroup consisting of invertible classes is the usual (French) Amitsur cohomology group $H^2(S/R, G_m)$ (the Hebrew-French dictionary is explained in detail in [13]). We have the following inclusions:

$$H^2(S/R, G_m) \subset M^2(S/R) \subset M^2(S/R).$$

**Theorem 5.1.** Let $S$ be a commutative faithfully flat $R$-algebra. An $S \otimes S$-module with coassociative comultiplication and normal basis is an Azumaya $S/R$-coring if and only if it represents an invertible element of $F'(S/R)$. Furthermore

$$F'(S/R) \cong M^2(S/R), F(S/R) \cong M^2(S/R) \text{ and } FAz(S/R) \cong H^2(S/R, G_m).$$

**Proof.** We define a map $\alpha': S^2(S/R) \to F'(S/R)$ as follows: $\alpha'(u) = \text{Can}_R(S;S)_u$, with comultiplication given by (18). It is easy to see that $\alpha'$ is a map of monoids. $\alpha'$ is surjective: let $C = S^{\otimes 2}$ with a coassociative comultiplication $\Delta_C$, and take

$$u = u^1 \otimes u^2 \otimes u^3 = \Delta_C(1 \otimes 1) \in S \otimes S \otimes S \otimes S \otimes S \cong S^{\otimes 3}.$$ 

From the coassociativity of $\Delta_C$, it follows that $u_1 u_3 = u_2 u_4$, so $u \in S^2(S/R)$, and $\alpha'(u) = C$.

Take $u \in \text{Ker} \alpha'$. We then have a comultiplication preserving $S$-bimodule isomorphism $\varphi: \text{Can}_R(S;S) \to \text{Can}_R(S;S)_u$. Put $\varphi(1 \otimes 1) = v \in S^{\otimes 2}$. From the fact that
\( \varphi \) is an automorphism of \( S \otimes S \) as an \( S \)-bimodule, it follows that \( v^{-1} = \varphi^{-1}(1 \otimes 1) \).

\( \varphi \) preserves comultiplication, so it follows that

\[
v_1 v_3 = (\varphi \otimes S \varphi)(\Delta_1(1 \otimes 1)) = \Delta_u(\varphi(1 \otimes 1)) = \Delta_u(v) = v^1 u^1 \otimes u^2 \otimes u^3 = v_2 u,
\]

hence \( u = \delta_1(v) \in B^2(S/R) \). It follows that \( F'(S/R) \cong M'^2(S/R) \) as monoids.

If \( u \in S^2(S/R) \), then \( \alpha'(u) = \text{Can}_R(S; S)_u \) has counit given by \( \text{Can}_R(S; S)_u \). Conversely, let \( C \in F(S/R) \), and take \( u = \alpha'^{-1}(C) \). Let \( \varepsilon_C(1 \otimes 1) \). Using the counit property and the fact that \( \varepsilon_C \) is a bimodule map, we then compute that

\[
1 \otimes 1 = \varepsilon_C(1 \otimes 1) = u^1 u^2 \otimes u^3; \\
1 \otimes 1 = u^1 \otimes \alpha'(1 \otimes 1) = u^1 \otimes u^2 \otimes u^3.
\]

It follows that \( u^1 u^2 \otimes u^3 \) and \( u^1 \otimes u^2 u^3 \) are invertible, and that \( v = |u|^{-1} \). Hence \( u \in S^2(S/R) \), and it follows that \( \alpha' \) restricts to an epimorphism of monoids

\[
\alpha : S^2(S/R) \rightarrow F(S/R).
\]

It is clear that \( \ker \alpha = B^2(S/R, \mathbb{G}_m) \), and it follows that \( M^2(S/R) \cong F(S/R) \).

If \( u \in Z^2(S/R, \mathbb{G}_m) \), then \( \alpha'(u) = \text{Can}_R(S; S)_u \) is an Azumaya \( S/R \)-coring. Conversely, let \( C \) be an Azumaya \( S/R \)-coring with normal basis, and \( u = \alpha'^{-1}(C) \). Then \( [u] \) is invertible in \( M^2(S/R) \), so there exists \( v \in S^2(S/R) \) such that \( uv \in B^2(S/R) \).

Since every element in \( B^2(S/R) \) is invertible in \( S^\otimes 3 \), it follows that \( u \in \mathbb{G}_m(S^\otimes 3) \), and \( u \in Z^2(S/R, \mathbb{G}_m) \). So \( \alpha \) restricts to an epimorphism \( \alpha'' : Z^2(S/R, \mathbb{G}_m) \rightarrow FAz(S/R) \). Clearly \( \ker \alpha'' = B^2(S/R, \mathbb{G}_m) \), hence \( B^2(S/R, \mathbb{G}_m) \cong FAz(S/R) \).

\[ \square \]

6. THE BRAUER GROUP

An Azumaya coring over \( R \) is a pair \( (S, C) \), where \( S \) is a faithfully flat finitely presented commutative \( R \)-algebra, and \( C \) is an Azumaya \( S/R \)-coring. A morphism between two Azumaya corings \( (S, C) \) and \( (T, D) \) over \( R \) is a pair \( (f, \varphi) \), with \( f : S \rightarrow T \) an algebra isomorphism, and \( \varphi : C \rightarrow D \) an \( R \)-module isomorphism preserving the bimodule structure and the comultiplication, that is

\[
\varphi(sc's') = f(s)\varphi(c)f(s') \text{ and } \Delta_D(\varphi(c)) = \varphi(c_{(1)}) \otimes_T \varphi(c_{(2)}),
\]

for all \( s, s' \in S \) and \( c \in C \). The counit is then preserved automatically. Let \( \mathcal{C}(R) \) be the category of Azumaya corings over \( R \).

**Lemma 6.1.** Suppose that \( S \) and \( T \) are commutative \( R \)-algebras. If \( M \in \mathcal{M}_{S \otimes S} \) and \( N \in \mathcal{M}_{T \otimes T} \), then \( M \otimes_R N \in \mathcal{M}_{(S \otimes S) \otimes_R (T \otimes T)} \).

If \( C \) is an (Azumaya) \( S/R \)-coring, and \( D \) is an (Azumaya) \( T/R \)-coring, then \( C \otimes_R D \) is an (Azumaya) \( S \otimes T \)-coring.

**Proof.** The proof of the first two assertions is easy; the structure maps are the obvious ones. Let us show that \( C \otimes_R D \) is an Azumaya \( S \otimes T \)-coring.

\[
C \otimes_R D \otimes_R S \otimes_R T \cong C \otimes_R S \otimes_R D \otimes_R T \\
\cong \text{Can}_S(I; S \otimes S) \otimes \text{Can}_T(J; T \otimes T) = (I^* \otimes_S I) \otimes_R (J^* \otimes_T J) \\
\cong (I^* \otimes_R J^*) \otimes_S (I \otimes_R J) = \text{Can}_S(I \otimes_R J; S \otimes_R T \otimes_R S \otimes_R T).
\]

\[ \square \]
Let \((C, \Delta)\) be an Azumaya \(S/R\)-coring, and consider the corresponding \((\mathcal{C}, \alpha) \in Z^2(S/R, \text{Pic})\). Its inverse in \(Z^1(S/R, \text{Pic})\) is represented by \((\mathcal{C}^*, (\alpha^*)^{-1})\). The corresponding coring will be denoted by \((\mathcal{C}^*, \Delta)\).

**Proposition 6.2.** Let \(\mathcal{C} \) be an Azumaya \(S/R\)-coring. Then \(\mathcal{C} \otimes \mathcal{C}^* \) is an elementary coring.

**Proof.** Consider \(H(\mathcal{C}) = (\mathcal{C}, \alpha) \in Z^2(S/R, \text{Pic})\), and the maps \(\eta_1, \eta_2 : S \to S \otimes S\). It follows from Proposition 1.8 that

\[
[\eta_1(\mathcal{C}, \alpha)] = [(\mathcal{C} \otimes \mathcal{S}^2, \alpha \otimes \mathcal{S}^3)] = [\eta_2(\mathcal{C}, \alpha)] = [(\mathcal{S}^2 \otimes \mathcal{C}, \mathcal{S}^3 \otimes \alpha)]
\]

in \(H^1(S \otimes S/R, \text{Pic})\). Consequently

\[
[H^{-1}(\eta_1(\mathcal{C}, \alpha))] = [\mathcal{C} \otimes \mathcal{R}(S; S)] = [H^{-1}(\eta_2(\mathcal{C}, \alpha))] = [\mathcal{R}(S; S) \otimes \mathcal{C}]
\]

in \(Br^c(S \otimes S/R)\). The inverse of \([\mathcal{R}(S; S) \otimes \mathcal{C}]\) in \(Br^c(S \otimes S/R)\) is represented by \([\mathcal{C} \otimes \mathcal{R}(S; S)] = \mathcal{C} \otimes \mathcal{C}^*\). It follows that

\[
(\mathcal{C} \otimes \mathcal{R}(S; S)) \otimes \mathcal{S}^4 \mathcal{R}(S; S) \otimes \mathcal{C}^* \equiv \mathcal{C} \otimes \mathcal{C}^*
\]

is an elementary coring. \(\square\)

Let \((S, \mathcal{C})\) and \((T, \mathcal{D})\) be Azumaya corings over \(R\). We say that \(\mathcal{C}\) and \(\mathcal{D}\) are Brauer equivalent (notation: \(\mathcal{C} \sim \mathcal{D}\)) if there exist elementary corings \(E_1\) and \(E_2\) over \(R\) such that \(\mathcal{C} \otimes E_1 \cong \mathcal{D} \otimes E_2\) as Azumaya corings over \(R\). Since the tensor product of two elementary corings is elementary, it is easy to show that \(\sim\) is an equivalence relation. Let \(Br^c_{\mathcal{H}}(R)\) be the set of equivalence classes of isomorphism classes of Azumaya corings over \(R\).

**Proposition 6.3.** \(Br^c_{\mathcal{H}}(R)\) is an abelian group under the operation induced by the tensor product \(\otimes_R\), with unit element \([R]\).

**Proof.** It follows from Proposition 6.2 that the inverse of \([\mathcal{C}, \Delta]\) is \([\mathcal{C}^*, \Delta]\). \(\square\)

**Lemma 6.4.** Let \(\mathcal{C}, \mathcal{E}\) be Azumaya \(S/R\)-corings, and assume that \(\mathcal{E} = \mathcal{R}(J; S)\) is elementary. Then the Azumaya corings \(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}\) and \(\mathcal{C}\) are Brauer equivalent.

**Proof.** Let \(H(\mathcal{C}) = (\mathcal{C}, \alpha)\). We know that \(H(\mathcal{E}) = (J^* \otimes J, \lambda_J)\), and

\[
[(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}, \alpha \otimes \mathcal{S}^3 \lambda_J)] = [(\mathcal{C}, \alpha)]
\]

in \(H^1(S \otimes S/R, \text{Pic})\). From Proposition 1.8 it follows that

\[
[\eta_1(\mathcal{C}, \alpha)] = [(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}, \alpha \otimes \mathcal{S}^3 \lambda_J)] = [(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}) \otimes \mathcal{S}^2, (\alpha \otimes \mathcal{S}^3 \lambda_J) \otimes \mathcal{S}^3]
\]

in \(H^1(S \otimes S/R, \text{Pic})\). Applying \(H^{-1}\) to both sides, we find that

\[
[\mathcal{R}(S; S) \otimes \mathcal{C} = [(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}) \otimes \mathcal{R}(S; S)]]
\]

in \(Br^c(S \otimes S/R)\). Since the inverse of \([\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}]\) in \(Br^c(S \otimes S/R)\) is \([\mathcal{C} \otimes \mathcal{C}^*]\), we obtain that

\[
[(\mathcal{R}(S; S) \otimes \mathcal{C}^*) \otimes \mathcal{S}^4] = [(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}) \otimes \mathcal{C}^*] = 1
\]

in \(Br^c(S \otimes S/R)\). Consequently \((\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}) \otimes \mathcal{C}^* = \mathcal{F}\) is an elementary coring, and

\[
(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E}) \otimes \mathcal{C}^* \equiv \mathcal{C} \otimes \mathcal{E} = \mathcal{F} \otimes \mathcal{C}.
\]

We have seen in Proposition 6.2 that \(\mathcal{C} \otimes \mathcal{C}^*\) is elementary, and it follows that \(\mathcal{C} \otimes \mathcal{S}^2 \mathcal{E} \sim \mathcal{C}\). \(\square\)
Lemma 6.5. Let \( f : S \to T \) be a morphism of faithfully flat commutative \( R \)-algebras. If \( C \) is an Azumaya \( S/R \)-coring, then \( C \sim \tilde{f}(C) = C \otimes_{S \otimes 2} \text{Can}_R(T; T) \).

Proof. As before, let \( H(C) = (C, \alpha) \). Consider the maps \( \varphi, \psi : S \to S \otimes T \) given by 
\[
\varphi(s) = 1 \otimes f(s) ; \quad \psi(s) = s \otimes 1.
\]
Applying Proposition 6.3, we find that
\[
[\varphi_*(C, \alpha)] = [(S \otimes S \otimes 2, \alpha) \otimes (\alpha \otimes S \otimes 3, T)]
\]
in \( H^1(S \otimes T/R, \text{Pic}) \). Consequently
\[
[H^{-1}(\varphi_*(C, \alpha))] = [\text{Can}_R(S; S) \otimes (C \otimes S \otimes 2, \text{Can}_R(T; T))]
\]
in \( Br^c(S \otimes T/R, \text{Pic}) \). The inverse of \( [C \otimes \text{Can}_R(T; T)] \) in \( Br^c(S \otimes T/R, \text{Pic}) \) is \( [C^* \otimes \text{Can}_R(T; T)] \), and it follows that
\[
\begin{align*}
(C^* \otimes \text{Can}_R(T; T)) & \otimes_{S \otimes S \otimes 3} (\text{Can}_R(S; S) \otimes (C \otimes S \otimes 2, \text{Can}_R(T; T))) \\
& \cong C^* \otimes (C \otimes S \otimes 2, \text{Can}_R(T; T)) \cong \mathcal{E}
\end{align*}
\]
where \( \mathcal{E} \) is an elementary \( S \otimes T/R \)-coring. We then have
\[
C \otimes C^* \otimes (C \otimes S \otimes 2, \text{Can}_R(T; T)) \cong C \otimes \mathcal{E}.
\]
We know from Proposition 6.2 that \( C \otimes C^* \) is elementary, so we can conclude that \( C \sim \tilde{f}(C) = C \otimes_{S \otimes 2} \text{Can}_R(T; T) \). \( \square \)

Proposition 6.6. Let \( S \) be a commutative faithfully flat \( R \)-algebra. We have a well-defined group monomorphism
\[
i_S : Br^c(S/R) \to Br^c(R), \quad i_S([C]) = [C].
\]
If \( f : S \to T \) is a morphism of commutative faithfully flat \( R \)-algebras, then we have a commutative diagram
\[
\begin{array}{ccc}
Br^c(S/R) & \xrightarrow{i_S} & Br^c(R) \\
\downarrow{\tilde{f}} & \searrow{\phi} & \\
Br^c(T/R)
\end{array}
\]

Proof. It follows from Lemma 6.3 that \( i_S \) is well-defined. Let us show that \( i_S \) is a group homomorphism. Consider two Azumaya \( S/R \)-corings \( C \) and \( D \). Then the \( S \otimes S/R \)-coring \( C^* \otimes C = \mathcal{E}_1 \) and the \( S/R \)-coring \( C \otimes S \otimes 2 C^* = \mathcal{E}_2 \) are both elementary. From Lemma 6.3, it follows that
\[
\begin{align*}
C \otimes D & \sim (C \otimes D) \otimes_{S \otimes S \otimes 4} (C^* \otimes C) \\
& \cong (C \otimes S \otimes 2 C^*) \otimes (D \otimes S \otimes 2 C) \\
& \sim D \otimes S \otimes 2 C \cong C \otimes S \otimes 2 D.
\end{align*}
\]
Consequently
\[
i_S[C \otimes S \otimes 2 D] = [C \otimes D] = i_S[C]i_S[D].
\]
It is clear that $i_S$ is injective.
Finally, it follows from Lemma 6.5 that $i_S[C] = [C] = [C \otimes_{S \otimes Z} \text{Can}_R(T; T)] = (i_T \circ \tilde{f})(C)$.  \[\square\]

**Theorem 6.7.** Let $R$ be a commutative ring. Then

$$\text{Br}^c(R) \cong \text{colim} \text{Br}^c(\bullet/R) \cong H^2(\mathcal{R}_R, \mathbb{G}_m).$$

**Proof.** It follows from Proposition 6.6 and the definition of colimit that we have a map

$$i : \text{colim} \text{Br}^c(\bullet/R) \to \text{Br}^c(R).$$

Suppose that $A$ is an abelian group, and suppose that we have a collection of maps $\alpha_S : \text{Br}^c(S/R) \to A$ such that $\alpha_T \circ \tilde{f} = \alpha_S$, for every morphism of faithfully flat commutative $R$-algebras $f : S \to T$. Take $x \in \text{Br}^c(R)$. Then $x$ is represented by an Azumaya $S/R$-coring $C$. We claim that the map $\alpha : \text{Br}^c(R) \to A, \alpha(x) = \alpha_S[C]$ is well-defined. Take an Azumaya $T/R$-coring $D$ that also represents $x$. Then

$$C \otimes \text{Can}_R(T; T) \sim C \sim D \otimes \text{Can}_R(S; S)$$

and it follows from the injectivity of $i_{S \otimes T}$ (see Proposition 6.4) that $[C \otimes \text{Can}_R(T; T)] = [D \otimes \text{Can}_R(S; S)]$ in $\text{Br}^c(S \otimes T/R)$, hence

$$\alpha_S[C] = \alpha_{S \otimes T}[C \otimes \text{Can}_R(T; T)] = \alpha_{S \otimes T}[D \otimes \text{Can}_R(S; S)] = \alpha_T[D],$$

as needed. We have constructed $\alpha$ in such a way that the diagrams

$$\begin{array}{ccc}
\text{Br}^c(S/R) & \xrightarrow{i_S} & \text{Br}^c(R) \\
\downarrow{\alpha_S} & & \downarrow{\alpha} \\
A & & 
\end{array}$$

commute. This means that $\text{Br}^c(R)$ satisfies the required universal property. Finally, apply (17).  \[\square\]

**Corollary 6.8.** Let $S$ be a faithfully flat commutative $R$-algebra. Then

$$\text{Ker}(\text{Br}^c(R) \to \text{Br}^c(S)) = \text{Br}^c(S/R).$$

**Proof.** Applying Corollary 6.8, (9), (11) with $q = 1$ and Theorem 6.7 we find that

$$\begin{align*}
\text{Br}^c(S/R) & \cong H^1(S/R, \mathbb{P}(c)) \cong H^1(S/R, C^1) \\
& \cong \text{Ker}(H^2(\mathcal{R}_R, \mathbb{G}_m) \to H^2(S_R, \mathbb{G}_m)) \\
& \cong \text{Ker}(\text{Br}^c(R) \to \text{Br}^c(S)).
\end{align*}$$

\[\square\]
All our results remain valid if we replace the condition that $S$ is faithfully flat by the condition that $S$ is an étale covering, a faithfully projective extension or a Zarisky covering of $R$ (see e.g. [10] for precise definitions). It follows from Artin’s Refinement Theorem [3] that the (injective) map

$$\hat{H}^2(R_{et}, \mathbb{G}_m) \to H^2(R_{et}, \mathbb{G}_m)$$

is an isomorphism. We will now present an algebraic interpretation of $\hat{H}^2(R_{fl}, \mathbb{G}_m)$ independent of Artin’s Theorem. Consider the subgroup $\text{Br}^\text{cnb}_R(R)$ consisting of classes of Azumaya corings represented by an Azumaya coring with normal basis.

**Theorem 6.9.** Let $R$ be a commutative ring. Then

$$\text{Br}^\text{cnb}_R(R) \cong \hat{H}^2(R_{fl}, \mathbb{G}_m).$$

**Proof.** Let $S$ be a faithfully flat commutative $R$-algebra, and consider the maps

\[
\begin{array}{ccc}
\text{Br}^c(S/R) & \xrightarrow{\gamma} & H^2(S/R, \mathbb{G}_m) \\
& \xrightarrow{\beta} & H^1(S/R, \text{Pic}) \\
\end{array}
\]

If $C$ is an Azumaya $S/R$-coring with normal basis, then $\gamma[C] \in \text{Im} (\beta)$, so the image of $[\gamma]$ in $H^2(R_{fl}, \mathbb{G}_m)$ lies in the subgroup $\hat{H}^2(R_{fl}, \mathbb{G}_m)$. It follows that we have a monomorphism $\kappa : \text{Br}^\text{cnb}_R(R) \to \hat{H}^2(R_{fl}, \mathbb{G}_m)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Br}^\text{cnb}_R(R) & \xrightarrow{\kappa} & \text{Br}^c(R) \\
\end{array}
\]

It is clear that $\gamma$ is surjective. □

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