Twisted behavior of dipolar BECs: Dipole-dipole interaction beyond the self-consistent field approximation and exchange electric dipole interaction

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We will show below that the self-consistent field approximation might be applied. In most cases it is correct, but dipolar BECs reveal a surprise. Structure of the self-consistent field term requires that interacting particles are in different quantum states, while in BECs all particles in a single quantum state. This fact requires to consider the two-particle polarisation, which describes dipole-dipole interaction, in more details. We present this consideration and show an astonishing result that the two-particle quantum correlation in dipolar BECs reveals in the same form as the self-consistent field term.

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I. INTRODUCTION

Classic papers on dipolar BECs [1], [2], [3], suggesting a generalisation of the Gross–Pitaevskii equation, do not describe approximation for dipole-dipole interaction in BECs. They just note that long-range and anisotropic nature of dipole-dipole interaction is highly interesting. One of the following papers [4] discusses justification of the model in terms of scattering that does not fully satisfy picture of the long-range interaction developed in plasmas physics in 30-th of XX century [6].

After many applications the physical picture giving understanding of the approximation suggested in Refs. [1], [2], [3] has not appeared. There are a lot of reviews on dipolar BECs, see for instance [6] and [7].

Recently we have derived models of dipolar BECs in different approximations: align dipoles [8], [9], dipoles with evolution of dipole directions (see [10], [11] for BECs, and [12] for ultracold fermions). All these papers are based on earlier papers on dynamics of dipolar charged particles at finite temperatures [13], [14], [15].

In Refs. [8]-[15] the dipole-dipole interaction is considered as a long-range interaction. Consequently the self-consistent field approximation was applied. Formally the self-consistent field approximation means that two-particle hydrodynamic functions appearing in the force field are represented as product of corresponding one-particle functions, for instance the two-particle concentration \( n_2(r, r', t) \) is replaced by the product of concentrations \( n(r, t)n(r', t) \).

We will show below that the self-consistent field approximation requires that interacting particles are in different quantum states. Consequently this approximation works well for systems with thermally distributed particles, and ultracold fermions distributed over the large number low laying quantum states due to the Pauli principle. However the self-consistent field approximation can not be applied to BECs, where all particles are located in a single quantum state. Same conclusion is correct for magnetic dipolar BECs.

Similar picture appears for the Coulomb interaction of Cooper pair of electrons in superconductors. Interaction of charges arises in the form similar to the self-consistent field, but its nature lays in the exchange Coulomb interaction of bosons located in the BEC state, and the self-consistent field equals to zero.

A big step towards consideration of quantum Bose and Fermi gases was made in Ref. [16], where authors developed a kinetic model for finite temperature gas. This model appears to have complicate structure. However this paper does not contain exhaustive analysis of the zero temperature limit, which is essential for physics of ultracold gases. Hence authors of Ref. [16] did not uncover unusual behavior of the model of dipolar BECs described in our paper.

In this paper we discuss background of minimal coupling model for dipolar BECs, but we can point out on resent generalisations of this model. For instance in Ref. [17], author considered the dipole-dipole interaction as a scattering process and suggested a way of generalisation of the standard model [1], [2], [3] going beyond the first Born approximation. The presence of particles in states with non-minimal energy (quantum fluctuations) due to interparticle interactions, and their influence on the properties of BECs, were considered in Ref. [18], leading to an apparatus for generalising the GP equation for dipolar, fully polarised, BECs. Following by Popov’s steps, authors of Ref. [19] considered the Landau damping in a collisionless dipolar Bose gas. Dipolar BECs with the Rashba spin-orbit interaction was considered in Ref. [20].

The standard model also find a lot of new applications presented in Refs. [21]-[34]. Speaking about application of the standard model of dipolar BECs we should mention that we explicitly apply the full potential of the electric dipole interaction (see formula (5) below, or papers [8], [9], [11]), whereas the standard model contains the shorted potential with no delta function term.

Presented in this paper conclusions about the nature
of the dipole-dipole interaction do not depend on the explicit form of the dipole-dipole potential. However, the non-integral form of equations and explicit form of equations of field are directly related to the explicit form of the potential of dipole-dipole interaction.

Two dimensional dipolar BECs were considered in Refs. [37, 38]. It seems that problem of presence of the delta function in the dipole-dipole interaction potential was ignored there. Hence we briefly debate this problem below in this paper.

This paper is organized as follows. In Sec. II approximations for model of dipolar BECs with aligned dipoles is examined. In Sec. III we examine the model of dipolar BECs with the dipole direction evolution. In Sec. IV we summarize results of the model analysis presented in Sec. II and III.

II. MODEL

At derivation of equations describing collective evolution of quantum systems from the many-particle Schrodinger equation we obtain the continuity and Euler equations:
\[ \partial_t n + \nabla \cdot (nv) = 0, \tag{1} \]
and
\[ mn(\partial_t + v \cdot \nabla)v - \frac{\hbar^2}{4m} \nabla \left( \frac{\Delta n}{n} - \frac{(\nabla n)^2}{2n^2} \right) = -gn\nabla n + P^\beta \nabla \beta e_{ext} + \int d\mathbf{r}' (\nabla G_{\beta}(|\mathbf{r} - \mathbf{r}'|)) P_2^{\beta}(\mathbf{r}, \mathbf{r}', t), \tag{2} \]
where the particle concentration \( n \) is defined in terms of many-particle wave function
\[ n(\mathbf{r}, t) = \int d\mathbf{R}_N \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t), \tag{3} \]
where \( d\mathbf{R}_N = \prod_{p=1}^N dr_p \) is an element of \( 3N \) volume. The particle current \( \mathbf{j} = \mathbf{n} \cdot \mathbf{v} \) also has an explicit definition via the wave function (see for instance formula 4 in Ref. [13], formula 3 in Ref. [14], and formula 10 in Ref. [8]). The polarisation (density of electric dipole moment) \( \mathbf{P}(\mathbf{r}, t) \) arises in the second term on the right-hand side of equation (2). Definition of polarisation is
\[ \mathbf{P}(\mathbf{r}, t) = \int d\mathbf{R}_N \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \mathbf{d}_i \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t). \tag{4} \]
The second term on the right-hand side of equation (2) describes action of an inhomogeneous external electric field on dipolar BECs. The third term on the right-hand side presents the dipole-dipole interaction. It contains the Green function of interaction of electric dipoles
\[ G^{\alpha \beta}(\mathbf{r}, \mathbf{r}') = \partial^\alpha \partial^\beta \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \]
and
\[ n_2(\mathbf{r}, \mathbf{r}', t) = \frac{8\alpha^2}{3\pi^3} \delta(\mathbf{r} - \mathbf{r}') + \frac{1}{3} \delta^{\alpha \beta} \Delta \frac{1}{r}. \tag{5} \]
Equation (2) also contains the two-particle polarisation
\[ P_2^{\alpha \beta}(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{R}_N \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \]
\[ \times d_i^\alpha d_j^\beta \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t), \tag{6} \]
which is a second rank tensor. The two-particle concentration was calculated in Refs. [35], [36]. It seems that problem of presence of the delta function in the dipole-dipole interaction potential was ignored there. Hence we briefly debate this problem below in this paper.

The particle current
\[ \mathbf{j} = \psi^*(\mathbf{R}, t) \mathbf{v}(\mathbf{R}, t) \psi(\mathbf{R}, t) \]
and
\[ \mathbf{n} \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t) \]

In Sec. IV we examine the model of dipolar BECs with the dipole direction evolution. In Sec. IV we summarize results of the model analysis presented in Sec. II and III.

\[ \frac{\partial \mathbf{j}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{j} = -\frac{\nabla n}{n} \psi^*(\mathbf{R}, t) \psi(\mathbf{R}, t) \]
and
\[ \frac{\partial \mathbf{n}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{n} = \frac{\hbar^2}{2m} \nabla \left( \frac{\Delta n}{n} - \frac{(\nabla n)^2}{2n^2} \right) \]

_The second term on the right-hand side of equation (2) describes action of an inhomogeneous external electric field on dipolar BECs. The third term on the right-hand side presents the dipole-dipole interaction. It contains the Green function of interaction of electric dipoles G^{\alpha \beta}(\mathbf{r}, \mathbf{r}') = \partial^\alpha \partial^\beta \frac{1}{|\mathbf{r} - \mathbf{r}'|}._
Here, the particle concentration

\[ n(r, t) = \sum_g n_g \varphi_g^*(r, t) \varphi_g(r, t), \]  

(11)

and the density matrix

\[ \rho(r, r', t) = \sum_g n_g \varphi_g^*(r, t) \varphi_g(r', t), \]

(12)

in terms of the arbitrary single-particle wave functions \( \varphi_g(r, t) \).

The first two terms in formula (10) represents the particles situating in two different quantum states, while the third term is referred to particles in the same quantum state. Therefore, for the particles in the BEC state, it is sufficient to take into account the third term in formula (10). In consideration of the system of bosons with the temperature differing from zero, where the certain number of the particles is out of the condensate, the first two summands of formula (10) gives the contribution both in the case of interaction of excited particles with each other and in the case of their interaction with the particles appearing in the BEC state. In this case, the third term of formula (10) gives the contribution in the interaction both between the particles appearing in the BEC state and between the excited particles appearing in the same quantum state.

The full concentration can be separated on two parts \( n = n_B + n_n \), where we have used the notions \( n_B(r, t) \) for the concentration of particles situating in the BEC state and \( n_n(r, t) \) for the concentration of excited particles. Hence the product of concentration in formula (10), formally, appears to be

\[ n(r, t)n(r', t) = \left[ n_B(r, t)n_B(r', t) + n_B(r, t)n_n(r', t) \right. \]

\[ + n_n(r, t)n_B(r', t) + n_n(r, t)n_n(r', t) \left. \right]_{\text{formal}}, \]

(13)

but we cannot have product of functions describing particles in the same quantum state. Hence we should exclude the first term on the right-hand side of formula (13). Finally we obtain

\[ n(r, t)n(r', t) = n_B(r, t)n_n(r', t) \]

\[ + n_n(r, t)n_B(r', t) + n_n(r, t)n_n(r', t). \]

(14)

Similar picture we have for the second term in formula (10)

\[ |\rho(r, r', t)|^2 = \rho(r, r', t)\rho(r', r, t) \]

\[ = \rho_B^* \rho_n + \rho_n^* \rho_B + \rho_n^* \rho_n, \]

(15)

where we have used \( \rho(r', r, t) = \rho^*(r, r', t) \).

The first term in formula (10) corresponds to the self-consistent field approximation. Second and third terms of formula (10) represent the part of the force field caused by the exchange interaction. Substituting expression (10) into the force field of dipole-dipole interaction of the boson system, we derive the formula

\[ F(r, t) = \int dr' (\nabla G_{zz}(|r - r'|)) P_{zz}^B(r', t) \]

\[ = d^2 \int dr' (\nabla G_{zz}(|r - r'|)) \left( n_B(r, t)n_n(r', t) + n_n(r, t)n_B(r', t) \right. \]

\[ + n_n(r, t)n_B(r', t) + n_n(r, t)n_n(r', t) + \rho_B^* (r', t)\rho_n(r, t) + \rho_n^* (r, t)\rho_B(r, t) \]

\[ + \rho_n^* (r, t)\rho_n(r, t) + \sum_g n_g (n_g - 1) |\varphi_g(r, t)|^2 |\varphi_g(r', t)|^2 \right), \]

(16)

where \( G_{zz}(\xi) = G_{\alpha\beta}(\xi) \delta_{\alpha\alpha} \delta_{\beta\beta} = \left( -\frac{\delta_{\alpha\beta}}{\xi^2} + \frac{3\delta_{\alpha\beta}}{\xi^4} - \frac{4\delta_{\alpha\beta}\delta(\xi)}{\xi^6} \right) \delta_{\alpha\alpha} \delta_{\beta\beta} = -\frac{1}{\xi^2} + \frac{3\xi^2}{\xi^4} - \frac{4\delta(\xi)}{\xi^6} \) is an explicit form of the \( zz \) matrix element of the Green function of the electric dipole interaction.

Dropping contribution of the excited states we obtain the force field of dipole-dipole interaction for boson systems in the BEC state

\[ F(r, t) = d^2 \int dr' (\nabla G_{zz}(|r - r'|)) \]

\[ \times \sum_g n_g (n_g - 1) |\varphi_g(r, t)|^2 |\varphi_g(r', t)|^2, \]

(17)

where \( g_0 \) is an index of the state with lower energy.

Formula (17) appears from the last term of formula (10). To be more precise we should note that formula (10) corresponds to a single term in the sum presented by the last term of formula (10).

Further manipulations with the force of electric dipole interaction of dipolar particles in the BEC state (17) give

\[ F(r, t) = d^2 \int dr' (\nabla G_{zz}(|r - r'|)) \]

\[ \times n_{g_0}(n_{g_0} - 1) |\varphi_{g_0}(r, t)|^2 |\varphi_{g_0}(r', t)|^2 \approx d^2 \int dr' (\nabla G_{zz}(|r - r'|)) n_{g_0} |\varphi_{g_0}(r, t)|^2 |\varphi_{g_0}(r', t)|^2 \]

\[ \approx d^2 \int dr' (\nabla G_{zz}(|r - r'|)) n(r, t)n(r', t). \]

(18)
Calculations in formula (18) are in accordance with formula (11), which gives $n_B = n_0 |\varphi_{g}\rangle(r, t)|^2$ in the case of all particles located in the BEC state.

Finally we have

$$\mathbf{F}(r, t) = d^2 n(r, t) \int dr' \langle \nabla G^{zz}(|r - r'|) \rangle n(r', t).$$

(19)

This result looks like outcome of the formal application of the self-consistent field approximation, but it has different meaning. Behavior of the exchange correlations in ultracold fermions is rather different, analysis of the Coulomb exchange interaction in quantum plasmas of degenerate electrons was presented in Ref. [40].

The self-consistent field approximation cannot be applied to the long-range interacting particles in the BEC state. However we can use the similarity of the force field (19) with the formal application of the self-consistent field approximation. This similarity allows to introduce the internal electric field created by the dipoles of the medium, so the force field (19) reappear in the following form

$$\mathbf{F}(r, t) = d\nabla \int dr' \langle G^{zz}(|r - r'|) \rangle n(r', t).$$

(21)

The internal electric field (21) satisfies the Maxwell equations

$$\nabla \cdot \mathbf{E}(r, t) = -4\pi \nabla \cdot \mathbf{P}(r, t) = -4\pi (\mathbf{d} \cdot \nabla) n(r, t),$$

(22)

and

$$\nabla \times \mathbf{E}(r, t) = 0.$$  

(23)

The Euler equation for dipolar BECs (2) now can be presented as follows

$$mn(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} - \frac{\hbar^2}{4m} n \nabla \left( \frac{\Delta n}{n} - \frac{(\nabla n)^2}{2n^2} \right)$$

$$= -gn\nabla n + d\nabla (1 \cdot \mathbf{E}).$$

(24)

Equations (1), (22), (23), and (24) form a closed set of equations, which can be applied to the analysis of dipolar BECs.

Formulae (19) and (24) are among the main results of this paper. We should stress attention readers that they are obtained at quantum exchange interaction. These formulae do not related to the self-consistent field approximation, but their form coincides with the results of application of the self-consistent field approximation. This twisted behavior of the model of dipolar BECs allows to justify equations applied in Refs. [3] and [10]. It also gives partial justification of papers [10], [11], and [39], where the evolution of the dipole directions is also considered. The full justification of these papers will be given in the next section of this paper, but general idea is the same as we have in this section.

Since all particles are located in the same quantum state, so they are described by the same single particle wave function. Consequently their velocity field is potential $\mathbf{v}(r, t) = \nabla \phi(r, t)$. Hence we can derive corresponding non-linear Schrodinger equation for the macroscopic wave function (the order parameter, or the wave function in the medium) from the continuity equation (11) the Euler equation (24).

The macroscopic wave function is defined in terms of hydrodynamic variables

$$\phi(r, t) = \sqrt{n(r, t)} \exp(i\phi(r, t)/\hbar),$$

(25)

where $\phi(r, t)$ is the potential of the velocity field $\mathbf{v}(r, t)$.

Differentiating function (25) with respect to time we find the non-linear Schrodinger equation

$$i\hbar \partial_t \Phi(r, t) = \left( -\frac{\hbar^2}{2m} \Delta + g |\Phi(r, t)|^2 - \mathbf{d} \cdot \mathbf{E} \right) \Phi(r, t),$$

(26)

which is the Gross-Pitaevskii equation for dipolar BECs. Equation (26) is an equivalent of the set of quantum hydrodynamic equations (11) and (24). As we can see equation (26) is a non-integral Gross-Pitaevskii equation.

Similar result appears for the force field of Coulomb interaction between Cooper pairs of electrons in superconductors:

$$\mathbf{F}(r, t) = (2e)^2 n(r, t) \int dr' \langle |r - r'| \rangle n(r', t),$$

(27)

where $G = 1/(r - r')$ is the Green function of the Coulomb interaction. Formula arises as the result of calculation of the two-particle concentration (3) and (10) with no references to the self-consistent field approximation. The force field (27) corresponds to the Landau-Ginzburg equation (11).

We have illustrated the strange behavior of the dipolar BECs on an example of the electric dipolar BECs, but we have similar picture for the magnetic dipolar BECs. Let us also mention that difference in behavior of the electric and magnetic dipolar BECs was described in Refs. [9] and [10].

A. Two dimensional dipolar BECs of aligned dipoles

Two dimensional dipolar BECs were recently considered in Refs. [32] and [36]. The standard model of dipolar BECs does not employ the delta function term in the potential of electric dipole interaction (1), (2), (3). However it creates some problem at transition to two-dimensional samples.

Since the two dimensional plane-like structure of dipoles is located in three dimensional space. Hence
dipoles do not have to be parallel to the plane, but they can be directed at an angle to the plane, or they can be directed perpendicular to the plane. Consequently the potential of dipole-dipole interaction in two-dimensional sample can be written as

\[ U_{dd}(2D) = -d^3d^3[\partial^3G^3 \frac{1}{|r-r'|}]_{z=z'=0} \]

\[ = -[d^3d^3]|_{inplane}[\partial^3G^3 \frac{1}{|r-r'|}]_{z=z'=0} \]

\[ + d^2(\frac{1}{|r-r'|^3} + \frac{1}{3} \frac{1}{r^2})_{z=z'=0} \]

Let us consider the last term containing the projection of the dipoles on z-axis, which is perpendicular to the plane. As a possible way to deal with two-dimensional samples we can rewrite this term as follows

\[ U_{d,d}(2D) = d^2 \left( \frac{1}{|r_{2D} - r_{2D}'|^3} + \frac{1}{3} \frac{1}{r_{2D}^2} \right), \] (28)

where we have substituted \( z = z' = 0 \) before differentiating \( \frac{1}{r_{2D}} \). We can notice that in the two-dimensional case \( \Delta \frac{1}{r_{2D}} = \frac{1}{r_{2D}} \), where we included \( \nabla r_{2D} = 2 \). Consequently we can present a short form of potential of dipole-dipole interaction of dipoles perpendicular to the plane[25] in the next form

\[ U_{d,d}(2D) = -2d^2 \frac{1}{|r_{2D}|}, \] (29)

which differs by multiplier 2/3 from the similar result obtain with no application of the delta function term in the original potential of the electric dipole interaction.

The Fourier transform of the inplane part of the electric dipole potential reads \( U_{\text{inplane}}(k) = (dk)^2 \frac{2}{k} \). The Fourier transform of the part of the electric dipole potential related to dipoles perpendicular to the plane is \( U_{d,d}(k) = 4\pi k/3 \).

**III. DIPOLAR BEC WITH DIPOLE DIRECTION EVOLUTION**

If we do not consider approximation of the aligned dipoles and include the evolution of the dipole directions we need to calculate the two-particle polarisation \( P_{2}^{\alpha \beta}(r, r', t) \) instead of the two-particle concentration \( n_{2}(r, r', t) \).

Our calculation of the two-particle polarisation \( P_{2}^{\alpha \beta}(r, r', t) \) gives

\[ P_{2}^{\alpha \beta}(r, r', t) = P^{\alpha}(r, t)P^{\beta}(r', t) \]

\[ + \frac{1}{2} \left( \Gamma^{\alpha}(r, r', t)(\Gamma^{\beta}(r, r', t))^* + c.c. \right) \]

\[ + \sum_{g} n_{g}(n_{g} - 1) d_{g}^{\alpha} d_{g}^{\beta} |\varphi_{g}(r, t)|^{2} |\varphi_{g}(r', t)|^{2}, \] (30)

where

\[ P(r, t) = \sum_{g} n_{g} d_{g} \varphi_{g}^{*}(r, t) \varphi_{g}(r, t), \] (31)

and

\[ \Gamma^{\alpha}(r, r', t) = \sum_{g} n_{g} d_{g}^{\alpha} \varphi_{g}^{*}(r, t) \varphi_{g}(r', t), \] (32)

with application of the arbitrary single-particle wave functions \( \varphi_{g}(r, t) \) presented above. Dipole moments have subindex \( g \), because they different in different quantum states.

Dropping contribution of the excited states we obtain the force field of dipole-dipole interaction for boson systems in the BEC state

\[ \mathbf{F}(r, t) = \int dr' (\nabla G^{\gamma}(|r - r'|)) \times \]

\[ \times \sum_{g=g_{0}} n_{g}(n_{g} - 1) d_{g}^{\beta} d_{g}^{\gamma} |\varphi_{g}(r, t)|^{2} |\varphi_{g}(r', t)|^{2}, \] (33)

where \( g_{0} \) is an index of the state with lower energy.

Further manipulations give

\[ \mathbf{F}(r, t) = \int dr' (\nabla G^{\gamma}(|r - r'|)) \times \]

\[ \times n_{g_{0}} (n_{g_{0}} - 1) d_{g_{0}}^{\beta} d_{g_{0}}^{\gamma} |\varphi_{g_{0}}(r, t)|^{2} |\varphi_{g_{0}}(r', t)|^{2} \approx \int dr' (\nabla G^{\gamma}(|r - r'|)) \times \]

\[ \times d_{g}^{\beta} d_{g}^{\gamma} (n_{g_{0}} |\varphi_{g_{0}}(r, t)|^{2})(n_{g_{0}} |\varphi_{g_{0}}(r', t)|^{2}) \]

\[ \approx \int dr' (\nabla G^{\gamma}(|r - r'|)) P_{2}^{\beta}_{B}(r, t)P_{2}^{\gamma}_{B}(r', t), \] (34)

in accordance with formula [31], which gives \( P_{B} = n_{g_{0}} d_{g_{0}} |\varphi_{g_{0}}(r, t)|^{2} \) in the case of all particles located in the BEC state.

This calculation leads to the following force field

\[ \mathbf{F}(r, t) = P^{\beta}(r, t) \int dr' (\nabla G^{\gamma}(|r - r'|)) P^{\gamma}(r', t). \] (35)

This result coincides with the formal application of the self-consistent field approximation in the Euler equation.
for dipolar BECs with the dipolar direction evolution, but it is based on the dipole-dipole exchange interaction.

Similarly to the aligned dipoles (see formulae [20]-[24]) we can use similarity of our result [36] with the selfconsistent field approximation and introduce the electric field created by dipoles

$$\mathbf{E}^\alpha(r, t) = \int d\mathbf{r}' (G^{\alpha\beta}(|\mathbf{r} - \mathbf{r}'|)) \mathbf{P}^\beta(r', t).$$

(36)

In this case the force field [35] can be rewritten as $F = P^\beta \nabla \mathbf{E}^\beta$. So we can rewrite the Euler equation as follows

$$mn(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} - \frac{\hbar^2}{4m} \nabla \left( \frac{\Delta n}{n} - \frac{(\nabla n)^2}{2n^2} \right)$$

$$= -gn \nabla n + P^\beta \nabla \mathbf{E}^\beta.$$  

(37)

The electric field satisfies the Maxwell equation

$$\nabla \cdot \mathbf{E}(r, t) = -4\pi \nabla \cdot \mathbf{P}(r, t),$$

(38)

and

$$\nabla \times \mathbf{E}(r, t) = 0.$$  

(39)

### A. Polarisation evolution

In this section we consider the dipole direction evolution. Consequently we can not give simple representation of the polarisation $\mathbf{P}$ in terms of the particle concentration $n$. Hence the set of continuity [1], Euler [37], and Maxwell [38], [39] equations is not a closed set. Now we need to find equation for polarisation evolution. This equation can be derived by differentiating the definition of polarisation [4] with respect to time, analogously to derivation of the continuity equation by the differentiating of the particle concentration [3] with respect to time. Let us mention that the time derivative of the many-particle wave function can be taken from the Schrodinger equation [9]. After some straightforward calculations one can find the equation of polarization evolution

$$\partial_t \mathbf{P}^\alpha(r, t) + \partial^\beta \mathbf{R}^{\alpha\beta}(r, t) = 0,$$

(40)

where $\mathbf{R}^{\alpha\beta}(r, t)$ is the current of polarization [10], [11], [12], [15], [39]. Its explicit form is

$$\mathbf{R}^{\alpha\beta}(r, t) = \int d\mathbf{R} \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \frac{d_i^\alpha}{2m_i} \times$$

$$\times \left( \psi^\ast(\mathbf{R}, t)(D_i^\beta \psi(\mathbf{R}, t)) + (D_i^\beta \psi(\mathbf{R}, t))^\ast \psi(\mathbf{R}, t) \right).$$

(41)

The equation of polarisation evolution [10] does not contain any information about the influence of the interaction on the polarisation evolution. So, following Refs. [10], [11], [12], [15] we derive the polarisation current $R^{\alpha\beta}(r, t)$ evolution equation

$$\partial_t R^{\alpha\beta} + \partial^\gamma \left( R^{\alpha\beta} v^\gamma + R^{\alpha\gamma} v^\beta - P^{\alpha\beta} v^\gamma \right)$$

$$- \frac{\hbar^2}{4m^2} \partial^\beta \Delta P^\alpha + \frac{\hbar^2}{8m^2} \partial^\gamma \left( \frac{\partial_\beta P^\alpha \cdot \partial_\gamma n}{n} + \frac{\partial_\gamma P^\alpha \cdot \partial_\beta n}{n} \right)$$

$$= - \frac{1}{2m} g \partial^\beta P^\alpha(r, r, t)$$

$$+ \frac{1}{m} \int d\mathbf{r}' (\partial^\beta G^\gamma(\mathbf{r} - \mathbf{r}')) D_2^{\alpha\gamma\delta}(\mathbf{r}, \mathbf{r}', t),$$

(42)

where we have included the explicit form of representation of the third-rank tensor of the flux of polarisation current. This tensor splits on three parts. The first of them is presented by the second group of terms on the left-hand side of equation [12]. It is related to the motion of the local centre of mass and contain the velocity field $\mathbf{v}$. It is an analog of $(\mathbf{v} \cdot \nabla) \mathbf{v}$ in the Euler equation. The second part of the flux of polarisation current is the quantum part presented by the third and fourth terms in the polarisation current evolution equation. These terms are proportional to the square of the Planck constant. The third part is related to the thermal motion. In the zero temperature limit, corresponding to the BEC dynamics, this term equals to zero, so it is not presented in equation [12].

Equation [12] contains two two-particle hydrodynamic functions, here we present their definitions arising during derivation of equation [12]

$$P^\alpha(r, r, t) = Tr(P^\alpha(r, r', t)),$$

(43)

with

$$P^\alpha(r, r', t) = \int d\mathbf{R}_N \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \times$$

$$\times d_i^\alpha \psi^\ast(\mathbf{R}, t) \psi(\mathbf{R}, t),$$

(44)

and

$$D_2^{\alpha\beta\gamma}(r, r', t) = \int d\mathbf{R}_N \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \times$$

$$\times d_i^\alpha d_j^\beta d_k^\gamma \psi^\ast(\mathbf{R}, t) \psi(\mathbf{R}, t).$$

(45)

Function $D_2^{\alpha\beta\gamma}(r, r', t)$ can be considered similarly to the two-particle concentration $n_2(r, r', t)$ (see formulae [8] and [10]), and the two-particle polarisation $P_2^{\alpha\beta}(r, r', t)$ (see formulae [6] and [39]). Hence we can
find that for particles collected in the BEC state function
\[ D_{2}^{\beta\gamma}(r', t) \] arises in the following form
\[ D_{2}^{\alpha\beta}(r, r', t) = \sum_{g=g_{0}} n_{g}(n_{g} - 1)d_{g}^{\alpha}d_{g}^{\beta}\phi_{g}(r, t)^{2}|\phi_{g}(r', t)|^{2}. \quad (46) \]

To interpret this formula we need to consider the first term of the expansion of function \( D_{2}^{\alpha\beta}(r, r', t) \) corresponding to the self-consistent field approximation, which is \( D_{2}^{\alpha\beta}(r, r', t) \rightarrow D_{\alpha\beta}(r, t) \cdot P^{\gamma}(r', t) \). Definition of function \( D_{\alpha\beta} \) occurs here is
\[ D_{\alpha\beta}(r, t) = \int dR \sum_{i} \delta(r - r_{i})d_{i}^{\alpha}d_{i}^{\beta}\psi(r, t)\psi(R, t). \quad (47) \]

An approximate formula for this function has been used in literature \([10, 12, 15, 39]\)
\[ D_{\alpha\beta}(r, t) = \sigma \frac{P^{\alpha}(r, t)}{n(r, t)} D_{\alpha\beta}(r, t). \quad (48) \]

We can analyze formula (48) for dipolar BECs. In terms of functions \( \phi_{g}(r, t) \) function \( D_{\alpha\beta} \) can be written as
\[ D_{\alpha\beta}(r, t) = \sum_{g} n_{g}d_{g}^{\alpha}d_{g}^{\beta}\phi_{g}(r, t)\phi_{g}(r, t) \quad (49) \]
(for polarisation \( P^{\gamma}(r', t) \) see formula (31)).

Function \( D_{\alpha\beta} \) in the limit of BEC is
\[ D_{\alpha\beta}(r, t) = n_{g_{0}}d_{g_{0}}^{\alpha}d_{g_{0}}^{\beta}\phi_{g_{0}}(r, t)\phi_{g_{0}}(r, t). \quad (50) \]

Comparing formulae (48) and (50) we find that numerical coefficient in formula (48) should be equal to one in the limit of the zeroth temperature bosons \( \sigma_{BEC} = 1 \).

Similarly we can write function \( D_{2}^{\beta\gamma}(r', r', t) \) in term of function \( D_{\alpha\beta} \), or in terms of polarisation only, at our choice
\[ D_{2}^{\beta\gamma}(r', r', t) = D_{\alpha\beta}(r, t)P^{\gamma}(r', t) \]
\[ = \frac{1}{n(r, t)}P^{\alpha}(r, t)P^{\beta}(r, t)P^{\gamma}(r', t). \quad (51) \]

We have finished discussion of function \( D_{2}^{\beta\gamma}(r', r', t) \), so we can substitute of final result in the equation of the polarisation current evolution (32)
\[ \partial_{t}R^{\alpha\beta} + \partial_{\gamma}\left(R^{\alpha\beta\gamma} + R^{\alpha\gamma\nu\beta} - P^{\alpha\nu\beta}\gamma\right) \]
\[ - \frac{\hbar^{2}}{4m^{2}}\partial_{\beta}\Delta P^{\alpha} + \frac{\hbar^{2}}{8m^{2}}\partial_{\gamma}\left(\frac{\partial_{\beta}P^{\alpha} \cdot \partial_{\nu}n}{n} + \frac{\partial_{\gamma}P^{\alpha} \cdot \partial_{\beta}n}{n}\right) = - \frac{1}{2m}g\phi_{\beta}(nP^{\alpha}) + \frac{1}{m}D_{\gamma}^{\alpha\beta} \int dr'(G^{\delta}(r-r'))P^{\delta}(r', t), \quad (52) \]
with function \( D_{\alpha\beta} = P^{\alpha}P^{\beta}/n \).

Non-integral form of the polarisation current evolution equation can be found with traditional introduction of the electric field created by dipoles
\[ \partial_{t}R^{\alpha\beta} + \partial_{\gamma}\left(R^{\alpha\beta\gamma} + R^{\alpha\gamma\nu\beta} - P^{\alpha\nu\beta}\gamma\right) \]
\[ - \frac{\hbar^{2}}{4m^{2}}\partial_{\beta}\Delta P^{\alpha} + \frac{\hbar^{2}}{8m^{2}}\partial_{\gamma}\left(\frac{\partial_{\beta}P^{\alpha} \cdot \partial_{\nu}n}{n} + \frac{\partial_{\gamma}P^{\alpha} \cdot \partial_{\beta}n}{n}\right) \]
\[ = - \frac{1}{2m}g\phi_{\beta}(nP^{\alpha}) + \frac{1}{m}P^{\alpha}P^{\gamma}, \quad (53) \]
where the electric field \( E \) obeys the Maxwell equations (38) and (39).

Equation (53) gives the final justification of the model of the dipolar BECs with dipole direction evolution presented in Refs. \([10, 11, 39]\). However, the consideration presented in this paper gives rather different physical picture behind these equations. Since we have shown that these equations arise from the exchange part of dipole-dipole interaction of bosons with electric dipole moments. The self-consistent part of the dipole-dipole interaction appears to be equal to zero for bosons in the BEC state. Nevertheless, the form of final equations coincides with the formal application of the self-consistent field approximation.

**IV. CONCLUSION**

In spite the fact that the self-consistent field approximation can not be applied to the BECs of particles with the long-range interaction we find that the results obtained earlier on the way of the formal application of the self-consistent field approximation do not contradict to the correct theory.

This unusual conclusion arises due to twisted behavior of the quantum exchange correlations in systems of bosons in the limit of the extremely low temperatures, when all particles are located in the BEC state. The term corresponding the self-consistent field approximation appears at consideration of bosons located in different quantum states, therefore it does not exist for particles located in the single quantum state. Part of quantum correlations has same fate.

The part of quantum correlation related to interaction of particles existing in a same quantum state survives in the BEC state and splits in the product of corresponding one-particle hydrodynamic functions, similar to the formal application of the self-consistent field approximation.

We have this picture in the Euler and the polarisation current evolution equations. Hence our conclusion
os correct for both cases of aligned dipoles and dipoles with the dipole direction evolution.

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