Wavefield reconstruction inversion in the frequency domain of on-ground GPR data

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Abstract. Full waveform inversion (FWI) is one of the most promising techniques for the quantitative characterization of the shallow subsurface of the Earth. However, the conventional FWI with the reduced approach is highly nonlinear, a challenging source of which is the cycle skipping. It would drive FWI to spurious local minima as soon as the initial model parameters are not accurate enough. By introducing a non-negative penalty term and a penalty parameter, wavefield reconstruction inversion (WRI) extends the search space of full waveform inversion, relaxes the requirement to satisfy exactly the wave equation at each iteration, and mitigates some of the problems related to poor start models, which leads to the cycle skipping phenomenon. Moreover, WRI breaks down FWI into two linear sub-problems and reconstruct the wavefield and estimate the model parameter in a loop and there is no need to update and store the wavefield. In this paper, we perform the model reconstruction of on-ground GPR using WRI in the frequency domain by inverting data of a limited number of frequency components, and compare the results of it with that of the conventional FWI. With a limited illumination of the subsurface, WRI with small penalty parameters has a positive performance.

1. Introduction

Ground penetrating radar (GPR) is a nondestructive subsurface geophysical prospecting technique [5] and has found wide utilization in many fields [11]. The accurate visualization and quantitative characterization of the distribution of the Earth’s shallow subsurface media is the ultimate goal of exploration technology. However, in the actual detection of GPR, we can only roughly estimate the location and size of the targets through the acquisition data profiles, but have challenge to accurately define the attributes of the target body.

Recently, the full waveform inversion (FWI) becomes one of the most promising techniques for building quantitative, high-resolution images of the subsurface [10]. It seeks to estimate parameters of a medium with a wavelength-scale resolution by matching the recorded and simulated date [2][3][4], and was originally developed for the acoustic and elastic wave equations in the seismic prospecting method [9][16][13][14], then rapidly developed for Maxwell's equations to estimate parameters of subsurface medium in the GPR exploration [6][7][11][10][17][15][12]. It has been proved that, by inverting data at a very limited number of frequency components, the frequency-domain inversion is equivalent to the time-domain inversion, which is conducive to the fast solution of inversion problem and can also avoid the data redundancy when we invert all frequency data components in the frequency-domain approach (or, equivalently, in the time-domain approach) [8].

FWI is a partial differential equation (PDE)-constrained nonlinear optimization problem and is classically solved with the reduced approach (also namely as the adjoint-state method). This reduced approach strictly enforces the constraints (because the wave equation is solved exactly with the current guess of the subsurface parameters) and is computationally tractable [4]. However, the conventional
FWI with the reduced approach is highly nonlinear, a challenging source of which is the cycle skipping. It would drive FWI to spurious local minima as soon as the initial model parameters are not accurate enough or the low-frequency components are missing [2]. For reducing the cycle skipping, long-offset wide-azimuth acquisitions and broadband sources with a richer low frequency component are developed in seismic exploration [14][8][4], but it has no reference significance in GPR due to its characteristics of the simple, nondestructive equipment and the shallow-buried-depth detection object. In addition, although another inverse strategy, the full-space approach, is more resilient to cycle skipping, it needs to simultaneously update wavefields, adjoint wavefields and subsurface model [4] and is typically not feasible for large-scale problems since we cannot afford to simultaneously update and hence store all the variables [17].

Van Leeuwen and Herrmann proposed a wavefield reconstruction inversion (WRI) to mitigate some of the problems related to missing low-frequency data and poor start models, which leads to the cycle skipping phenomenon. In WRI, the PDE-constrained optimization problem underlying FWI is recast as an unconstrained problem by introducing a non-negative penalty term (the L2 norm of the source residuals, or namely, the PDE-constraint) and a penalty parameter [17]. This penalty method extends the search space and relaxes the requirement to satisfy exactly the wave equation at each iteration. Arguably, WRI leads to a less-non-linear problem and is less sensitive to the initial iterate [17]. To make WRI computationally tractable, the wavefield reconstruction and the model parameter estimation are solved in alternating optimization strategy, which means that WRI breaks down FWI into two linear subproblems and solves them in a loop and there is no need to update and store the wavefield. Specifically, the wavefield is first reconstructed by data and current model parameters (the wavefield in conventional reduced approach is obtained by directly solving PDE, which is the main difference between FWI and WRI), then the model parameters are estimated from the previously-reconstructed wavefields by minimizing the source residuals generated by the wave-equation relaxation. This cycle being iterated until convergence. An advantage of the alternating-direction strategy is linearise the parameter-estimation subproblem around the reconstructed wavefield because the wave equation constraint is bilinear [3].

WRI has been used in seismic, while it was rarely reported in GPR. In this paper, we first derive the equations from FWI to WRI for the radar wave in the frequency domain, and perform numerical experiments of FWI and WRI with a layered model, which contains multiple abnormal bodies.

2. Methods

2.1. Inversion of GPR in the frequency domain

Frequency-domain inversion of GPR, inferring the subsurface parameters from observed data, can be written as a partial-differential equation (PDE) -constrained optimization problem:

$$\min_{u, m} \frac{1}{2}\|Pu - d\|^2$$

subject to \(A(m)u = q\) \hspace{1cm} (1)

where \(m = [m_1, m_2, \ldots, m_M]^T\) denotes the vector of discrete model parameters (here, \(M=NE\), \(NE\) is the number of degrees of freedom in the spatial computational mesh), \(u\) is the wavefield vector, \(q\) is the source vector, \(d = [d_1, d_2, \ldots, d_N]^T\) is the recorded wavefield (data) at receiver locations (here, \(N\) is the number of data), \(P\) is a projection operator that samples the modeled wavefield at the receiver positions, and the matrix \(A\) represents the discretized PDE Helmholtz operator.

2.2. From FWI to WRI

The conventional approach to solve the constrained optimization problem is to eliminate the constraints by enforcing the PDE constraint at each iteration, yielding the reduced problem (unconstrained optimization problem), which is the general forms of FWI.

The data misfit function of FWI is defined as
where the symbol $^T$ denotes the transpose $(\cdot)^{-}\text{-conjugate (}^*\text{)}$ operator (introduced to ensure the misfit function is a true(real-valued) norm for complex-valued data), $N_w$ is the number of frequencies, and $N_s$ is the number of sources. $A(m)^{-1}q$ represents the modeling wavefield $u$.

The gradient $\nabla \Phi_p(m)$ can be calculated by

$$\nabla \Phi_p(m) = \sum_{N_w} J^T \Delta d$$

where $\Re$ denotes the real part operator, $J$ denotes Frechét partial derivative matrix (Jacobian matrix), and the error vector $\Delta d = d^{\text{obs}} - d^{\text{calc}}$ (here $d^{\text{obs}}$ is the observation data and $d^{\text{calc}} = Pu$ is the calculation data). We calculate the Jacobian matrix via the adjoint state method for this work and (3) can be written as

$$\nabla \Phi_p(m) = \sum_{N_w} \Re \{ u^{-1} (-G)^{-1} v \}$$

where $G$ represents the diffraction matrix (or sensitivity kernel) characterizes the sensitivity to the parameters, and $v$ is defined as the adjoint wavefield:

$$v = (A(m)^{-1})^T P^T (Pu - d)$$

It shows that the gradient can be efficiently recast as a product between the incident wavefield $u$ and the back-propagated wavefield $v$, using residuals at receiver positions as a composite source. Therefore, it is required to explicit solve both the forward $A(m)u = q$ and adjoint wave equation $A(m)^{-1}v = P^T (Pu - d)$ per frequency and source location at each iteration of the optimization.

The gains, of course, are that we no longer have to optimize over $u$ and that we can compute the $u$ and $v$ independently in parallel. However, these gains are perhaps a bit short-lived, because it is computationally expensive to update $u$ and $v$ at each iteration and the reduced approach dramatically reduces the search-space by projecting the full space onto the parameter search space, making it more difficult to find an appropriate minimizer. Note that the conventional FWI strictly enforces the constraint at each iteration, possibly leading to a very non-linear problem in $m$, and empirically tends to get stuck in the spurious local minima if starting from a poor initial model.

To mitigate the nonlinearity issue of FWI, van Leeuwen and Herrmann (2013) recast the PDE-constrained optimization problem as a multi-variate unconstrained quadratic penalty problem for $u$ and $m$, which is known as WRI methodology. The penalty misfit function of WRI is defined as:

$$\Phi_{\rho}(u, m) = \frac{1}{2} \sum_{N_w} \sum_{N_s} (Pu - d)^T (Pu - d) + \frac{\lambda_{\rho}}{2} \sum_{N_w} \sum_{N_s} (A(m)u - q)^T (A(m)u - q)$$

where $u$ is the forward wavefield vector, the scalar $\lambda_{\rho} > 0$ is the penalty parameter.

The optimization in the full $(u, m)$-space is not feasible for large-scale problem. van Leeuwen and Herrmann (2013) use an alternating optimization method to solve this optimization problem while avoiding storage of the wavefields. First, keeping $m$ fixed, minimize $\Phi_{\rho}$ with respect to $u$ for eliminating it at each iteration by solving:

$$\nabla_u \Phi_{\rho} = P^T (Pu - d) + \lambda_{\rho} A(m)^T (A(m)u - q) = 0$$
where $T$ denotes a (non-conjugate) transpose. The optimal estimation-wavefield $\bar{u}$ is given by

$$
\bar{u}(m) = \left( \lambda_p A(m)^\dagger A(m) + P^\dagger \lambda \right)^{-1} \left( \lambda_p A(m)^\dagger q + P^\dagger d \right)
$$

where $\bar{u}$ is the estimation-wavefield vector. Next, keeping the reconstructed wavefields $\bar{u}$ fixed, update $m$ by minimizing (10) as FWI. And the gradient $\nabla \Phi_p(m)$ can be calculated by

$$
\nabla \Phi_p(m) = \sum_{n=1}^{N_s} \sum_{n=1}^{N_s} \left\{ \lambda_p \bar{u}^\dagger G^\dagger (A(m)\bar{u} - q) \right\}
$$

where $\bar{u}$ is the estimation-wavefield vector as (9), $G$ is the diffraction matrix (or sensitivity kernel) characterizes the sensitivity to the parameters and the expression is as same as (5).

3. Numerical example

The test discusses the differences between WRI and the conventional FWI. For the numerical example, forward modelling is performed with regular-quadrilateral-mesh-based finite element method and PML absorbing boundary conditions, and the inversion is performed by inverting several selected frequency data with an L-BFGS optimization method and a line search procedure for step length estimation (that satisfies the strong Wolfe conditions). The source signature is a Ricker wavelet with a center frequency of 400 MHz, and 10 frequencies (50, 75, 100, 120, 140, 160, 180, 190, 200, and 400 MHz) we re used in the inversion process.

Figure 1(a) shows the true models of subsurface relative permittivity. The subsurface model contains three layered media and several anomalies that is embedded in the background medium. The calculated area of the models is 3.25 m long and 5.50 m deep and is discretized with a 0.025 m grid interval. Model parameters are not updated in the PML, which consist of 10 grids around the computed region. Above the layered medium is the loaded air layer, in which the regular surface acquisition is deployed with a total of 20 sources (as depicted with yellow circles) and 100 receivers (as depicted with red ×, recording the signal of all the sources). The relative permittivity of layer media from top to bottom, are 1, 5, 7, and 9, respectively, and the anomalies’ relative permittivity are 12, 1, and 6, respectively. In the experiment, we compare the performance of FWI and WRI with different penalty parameters. The starting models for relative permittivity is the shown as Figure 1 (b), which are the Gaussian smoothing of the layered background media. In this test, we select 3 penalty parameters for WRI, which are 1e0, 1e2, and 1e4 respectively.

Figure 1 (c) shows the media distribution of FWI and Figures 1 (d) - 1 (f) are that of WRI with three different penalty factors. When the penalty parameter is 1e4, the inversion results of WRI are similar to that of FWI, while the resolution of WRI with small penalty parameters is relatively high. For example, in Figures 1 (e) and 1 (f), the reconstructed model profiles can identify the detail outline approaching the real model. To accurately compare the inversion results of FWI and WRI, we select 3 vertical and 2 horizontal profiles across the abnormal bodies of the model to depict the inversion results of relative permittivity. As shown in Figure 2 and Figure 3, WRI with bigger penalty parameter and FWI reconstruct the variation trend of the media and approximate positions of the anomalies. And WRI with smaller penalty parameter can reconstruct better the shape of the anomaly with however more obvious numerical fluctuation (as the red dotted lines). Especially, in the depth ranges from 1.375 m to 1.625 m
in Figure 2 (a) and the coordinates between 1.25 m and 1.75 m in Figure 3 (b), the values presented by red dotted lines (WRI with smaller penalty parameter) are closer to the values of real model. However, red dotted lines (WRI with smaller penalty parameter) present a great value fluctuation in depth than the red solid lines (WRI with bigger penalty parameter) and blue solid lines (FWI).

Figure 1. True model, initial model, and reconstructed models of FWI and WR.

Figure 4 shows the comparisons of the objective function convergence curves and model reconstruction error curves during the inversion process. Figure 4 (a) demonstrates that the misfit value is directly related to the value of the penalty parameter of WRI, which can be theoretically supported from (10). With the increase of the penalty parameter, the objective function of WRI is gradually approaches to that of FWI, and when the penalty parameter equal to 1e4, the objective function of WRI is basically consistent with that of FWI. As shown in Figure 4 (b), the model reconstruction error of WRI with smaller penalty parameter decreases faster in the later iterations (about after 10 iteration) and is smaller at the last iteration than that of conventional FWI. At the last iteration, as noted in Figure 5, although the time spending in WRI is roughly similar to that in FWI, in general, the numbers of PDE solutions in WRI are approximately 1/2 of that in FWI.

Figure 2. Vertical profiles in the true model, initial model, and estimated models of FWI and WRI.
Figure 3. Horizontal profiles in the true model, initial model, and estimated models of FWI and WRI.

4. Conclusions
In this paper, we derive the formula system of WRI for GPR in the frequency domain and perform the model reconstruction by WRI using multi-offset data of on-ground measurements. The tests show that as the penalty parameter increases, the result of WRI approaches that of FWI. And when the penalty parameter is small, the result of WRI gets better, the errors are smaller at final iteration, and the resolution of inversion is higher. However, the small penalty parameter in WRI would increase the oscillation artifacts to a certain extent, which needs further research.

Figure 4. Misfit and Error convergence curves of FWI and WRI.

Figure 5. Final PDE solves and time of FWI and WRI.

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