Self-Similar Blowup Solutions to the

2-Component Degasperis-Procesi Shallow Water System

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Revised 13-Aug-2010

Abstract

In this article, we study the self-similar solutions of the 2-component Degasperis-Procesi water system:

\[
\begin{align*}
\rho_t + k_2 u \rho_x + (k_1 + k_2) \rho u_x &= 0 \\
u_t - u_{xxx} + 4u u_x - 3u_x u_{xx} - uu_{x22} + k_3 \rho \rho_x &= 0.
\end{align*}
\]

(1)

By the separation method, we can obtain a class of self-similar solutions,

\[
\begin{align*}
\rho(t, x) &= \max(f(\eta), a(4t) \eta, 0), \\
u(t, x) &= \frac{\dot{a}(4t)}{a(4t)} x \\
a(s) &= \frac{\dot{a}(4t)}{a(4t)} = 0, \\
f(\eta) &= \frac{k_3}{\kappa} \sqrt{-\frac{2}{9}\eta^2 + \left(\frac{\xi}{\alpha}\right)^2} \\
\end{align*}
\]

(2)

where \( \eta = \frac{x}{a(s)^{\frac{1}{4}}} \) with \( s = 4t; \kappa = \frac{k_1}{2} + k_2 - 1, \alpha \geq 0, \xi < 0, a_0 \) and \( a_1 \) are constants.

which the local or global behavior can be determined by the corresponding Emden equation.

The results are very similar to the one obtained for the 2-component Camassa-Holm equations. Our analytical solutions could provide concrete examples for testing the validation and

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stabilities of numerical methods for the systems. With the characteristic line method, blowup phenomenon for \( k_3 \geq 0 \) is also studied.

Mathematics Subject Classification (2010): 35B40, 35B44, 35C06, 35Q53

Key Words: 2-Component Degasperis-Procesi, Shallow Water System, Analytical Solutions, Blowup, Global, Self-Similar, Separation Method, Construction of Solutions, Moving Boundary, 2-Component Camassa-Holm Equations

1 Introduction

The 2-component Degasperis-Procesi shallow water system \([20]\) and \([10]\), can be expressed in the following form

\[
\begin{align*}
\rho_t + k_2 \rho u_x + (k_1 + k_2) \rho u_x &= 0, \quad x \in \mathbb{R} \\
u_t - u_{xxx} + 4u u_x - 3u_x u_{xx} - uu_{xxx} + k_3 \rho u_x &= 0.
\end{align*}
\]

Here \( k_1, k_2, k_3 \) are constants. \( u = u(x, t) \in \mathbb{R} \) is the velocity of fluid. And \( \rho = \rho(t, x) \geq 0 \) is the density of fluid. For \( \rho = 0 \), the system returns to the Degasperis-Procesi equation \([7]\), \([17]\) and \([26]\).

In this article, we adopt an alternative approach (method of separation) to study some self-similar solutions of 2-component Camassa-Holm equations \([3]\). Indeed, we observe that the isentropic Euler, Euler-Poisson, Navier-Stokes and Navier-Stokes-Poisson systems are written by:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0 \\
\rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \nabla P &= -\rho \nabla \Phi + vis(\rho, \vec{u}) \\
\Delta \Phi(t, x) &= \alpha(N)\rho
\end{align*}
\]

where \( \alpha(N) \) is a constant related to the unit ball in \( \mathbb{R}^N \): \( \alpha(1) = 2; \alpha(2) = 2\pi \) and for \( N \geq 3 \),

\[
\alpha(N) = N(N - 2)Vol(N) = N(N - 2)\frac{\pi^{N/2}}{\Gamma(N/2 + 1)}
\]

where \( Vol(N) \) is the volume of the unit ball in \( \mathbb{R}^N \) and \( \Gamma \) is a Gamma function. And as usual, \( \rho = \rho(t, \vec{x}) \) and \( \vec{u} = \vec{u}(t, \vec{x}) \in \mathbb{R}^N \) are the density and the velocity respectively. \( P = P(\rho) = K\rho^\gamma \) is the pressure, the constant \( K \geq 0 \) and \( \gamma \geq 1 \). And \( vis(\rho, \vec{u}) \) is the viscosity function.

We may seek the radial solutions

\[
\rho(t, \vec{x}) = \rho(t, r) \text{ and } \vec{u} = \frac{\vec{x}}{r} V(t, r) =: \frac{\vec{x}}{r} \overrightarrow{V}
\]
Self-Similar Solutions to 2-Component Degasperis-Procesi Equations

with \( r = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2} \).

By the standard computation, the Euler equations in radial symmetry can be written in the following form:

\[
\begin{align*}
\rho_t + V \rho_r + \rho V_r + \frac{N-1}{r} \rho V &= 0 \\
\rho (V_t + V V_r) + K \frac{\partial}{\partial r} \rho^\gamma &= 0.
\end{align*}
\] (7)

For the mass equation in radial symmetry, \( (7) \), we well know the solutions’ structure (Lemma 3, [24]):

\[
\begin{align*}
\rho(t, r) &= f \left( \frac{r}{a(t)} \right),
\end{align*}
\] (8)

As the 2-component Degasperis-Procesi equations \( (3) \), are very similar to the Euler system \( (7) \), we can apply the separation method ([13], [18], [16], [22], [24]) to the systems \( (3) \). We note that for the 2-component Camassa-Holm equations \([14], [1], [12] \) and \([9]\),

\[
\begin{align*}
\rho_t + u \rho_x + \rho u_x &= 0 \\
m_t + 2u m_x + um_x + \sigma \rho \rho_x &= 0
\end{align*}
\] (9)

with

\[
m = u - \alpha^2 u_{xx}.
\] (10)

By the separation method, we can obtain a class of blowup or global solutions for \( \sigma = 1 \) or \(-1\) \([25]\). In particular, for the integrable system with \( \sigma = 1 \), we have the global solutions:

\[
\begin{align*}
\rho(t, x) &= \begin{cases} 
\frac{f(\eta)}{a(3t)^{2/3}}, & \text{for } \eta^2 < \xi \alpha^2 \\
0, & \text{for } \eta^2 \geq \xi \alpha^2
\end{cases},
\end{align*}
\] (11)

\[
u(t, x) = \begin{cases} 
\frac{a(3t)}{a(0)x}, & \text{for } \eta^2 < \xi \alpha^2 \\
0, & \text{for } \eta^2 \geq \xi \alpha^2
\end{cases},
\]

\[
\dot{a}(s) - \frac{\xi}{3a(s)^{2/3}} = 0, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1
\]

\[f(\eta) = \frac{1}{\xi} \sqrt{-\xi \eta^2 + (\xi \alpha)^2}\]

where \( \eta = \frac{x}{a(s)^{1/3}} \) with \( s = 3t; \xi > 0 \) and \( \alpha \geq 0 \) are arbitrary constants.

As the two component of Camassa-Holm equations are very similar to the two component of Degasperis-Procesi equations, In this article, we continue to apply the separation method, to deduce the nonlinear partial differential equations into much simpler ordinary differential equations. Therefore, we can contribute a new class of self-similar solutions in the following corresponding result:
**Theorem 1** We define the function \( a(s) \) is the solution of the Emden equation:

\[
\begin{aligned}
\ddot{a}(s) - \frac{\xi}{4a(s)} &= 0 \\
a(0) = a_0 > 0, \quad \dot{a}(0) = a_1
\end{aligned}
\]  

(12)

and

\[
f(\eta) = \frac{k_3}{\xi} \sqrt{-\frac{\xi^2}{k_3} \eta^2 + \left( \frac{\xi}{k_3} \alpha \right)^2}
\]

(13)

where \( \eta = \frac{x}{a(s)^{k_2/4}} \) with \( s = 4t; \ \kappa = \frac{k_1}{2} + k_2 - 1, \ \alpha \geq 0, \ \xi \neq 0, \ a_0 \) and \( a_1 \) are constants.

For the 2-component Degasperis-Procesi equations \((3)\), there exists a family of solutions, those are:

(1) for \( k_3 = 0 \), and \( \xi = 0 \), or

\[
\rho(t, x) = \frac{\rho_0(\eta)}{a(4t)^{(k_1+k_2)/4}}, \quad u(t, x) = \frac{\dot{a}(4t)}{a(4t)} x
\]

(14)

where \( \rho_0 \geq 0 \), is an arbitrary \( C^1 \) function.

(2) for \( k_3 > 0 \) and \( \xi > 0 \), or

(3) for \( k_3 < 0 \) and \( \xi < 0 \),

\[
\rho(t, x) = \max\left(\frac{f(\eta)}{a(4t)^{(k_1+k_2)/4}}, 0\right), \quad u(t, x) = \frac{\dot{a}(4t)}{a(4t)} x.
\]

(15)

**Remark 2** The structure of solutions \((14)\) of the 2-component Degasperis-Procesi equations are very similar the one \((11)\), \((22)\) of the Camassa-Holm equations.

## 2 Separation Method

The mass of the solution is not conserved except on \( k_1 = 0 \) and \( k_2 = 1 \) in equation \((3)_1\). However, we can also design a nice functional structure for the mass equation:

**Lemma 3** For the 1-dimensional equation of mass \((3)_1\):

\[
\rho_t + k_2u\rho_x + (k_1 + k_2)\rho u_x = 0
\]

(16)

there exist solutions,

\[
\rho(t, x) = \frac{f\left(\frac{x}{a(4t)^{k_2/4}}\right)}{a(4t)^{(k_1+k_2)/4}}, \quad u(t, x) = \frac{\dot{a}(4t)}{a(4t)} x
\]

(17)

with the form \( f(\eta) \geq 0 \in C^1 \) with \( \eta = \frac{x}{a(4t)^{k_2/4}} \), and \( a(4t) > 0 \in C^1 \).
Proof. We just plug (17) into (16) to check:

\[ \rho_t + k_2 \rho_x + (k_1 + k_2) \rho u_x = \frac{\partial}{\partial t} \left( \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \right) + k_2 \frac{\partial}{\partial x} \left( \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \right) + (k_1 + k_2) \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \frac{\partial}{\partial x} \left( \frac{a(t)}{a(4t)^{k_1+k_2/4}} \right) \]

\[ = \frac{1}{a(4t)^{k_1+k_2/4+1}} \left( -\frac{(k_1 + k_2)}{4} \right) \cdot \dot{a}(t) \cdot 4 \cdot f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right) + \frac{1}{a(4t)^{k_1+k_2/4}} \frac{\partial}{\partial t} \left( \frac{x}{a(4t)^{k_2/4}} \right) \frac{\partial}{\partial t} \left( \frac{x}{a(4t)^{k_2/4}} \right) \]

\[ + k_2 \dot{a}(4t) \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \frac{\partial}{\partial x} \left( \frac{x}{a(4t)^{k_2/4}} \right) + \frac{(k_1 + k_2)}{a(4t)^{k_1+k_2/4}} \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \dot{a}(4t) \]

\[ = -\frac{(k_1 + k_2)}{a(4t)^{k_1+k_2/4+1}} \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \frac{\partial}{\partial x} \left( \frac{x}{a(4t)^{k_2/4}} \right) - \frac{1}{a(4t)^{k_1+k_2/4+1}} \frac{\partial}{\partial t} \left( \frac{x}{a(4t)^{k_2/4}} \right) \frac{x}{a(4t)^{k_2/4}} \frac{k_2}{4} \dot{a}(4t) \cdot 4 \]

\[ + k_2 \dot{a}(4t) \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \frac{1}{a(4t)^{k_2/4}} + \frac{(k_1 + k_2)}{a(4t)^{k_1+k_2/4}} \frac{f\left(\frac{x}{a(4t)^{k_1+k_2/4}}\right)}{a(4t)^{k_1+k_2/4}} \dot{a}(4t) \]

\[ = 0. \]

The proof is completed. ■

On the other hand, in [5] and [23], the qualitative properties of the Emden equation

\[ \begin{align*}
\ddot{a}(s) - \frac{\xi}{a(s)^{\kappa}} &= 0 \\
a(0) &= a_0 \neq 0, \quad \dot{a}(0) = a_1
\end{align*} \]

where for \( \kappa \geq 1 \) in [5] and [23] and \( \kappa = \frac{1}{3} \) in [23], we studied. Therefore, the similar local existence of the Emden equations [12],

\[ \begin{align*}
\ddot{a}(s) - \frac{\xi}{a(s)^{\kappa}} &= 0 \\
a(0) &= a_0 \neq 0, \quad \dot{a}(0) = a_1
\end{align*} \]

can be proved by the standard fixed point theorem [5] and [23]. To additionally show the blowup property of the time function \( a(s) \), the following lemmas are needed.

Lemma 4 For the Emden equation [12],

\[ \begin{align*}
\ddot{a}(s) - \frac{\xi}{a(s)^{\kappa}} &= 0 \\
a(0) &= a_0 > 0, \quad \dot{a}(0) = a_1
\end{align*} \]
the solution exists locally.

After obtaining the nice structure of solutions (17), we just use the techniques of separation of variable ([13], [18], [16], [22], [24] and [25]), to prove the theorem:

**Proof of Theorem 1.** From Lemma 3, it is clear to see our functions (15) fit well into the mass equation, (3)\textsubscript{1}, except for two boundary points.

The second equation of 2-component Camassa-Holm equations (3)\textsubscript{2}, becomes:

\begin{equation}
\dot{u}_t - u_{xx} + 4uu_x - 3u_xu_{xx} - uu_{xxx} + k_3\rho_x
= u_t + 4uu_x + k_3\rho_x
\end{equation}

As the velocity \( u \), in the solutions (15) is a linear flow:

\begin{equation}
u = \frac{\dot{a}(4t)}{a(4t)}x \tag{30}\end{equation}

we have

\begin{equation}
u_{xx} = 0. \tag{31}\end{equation}

The equation (29) becomes:

\begin{equation}
= u_t + 4uu_x + k_3\rho_x
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} \left( \frac{\dot{a}(4t)}{a(4t)} \right) x + 4 \left( \frac{\dot{a}(4t)}{a(4t)} \right) x \frac{\dot{a}(4t)}{a(4t)} + k_3 \frac{f(\frac{\dot{a}(4t)^{1/2}}{a(4t)^{1/2}})}{a(4t)^{(k_1+k_2)/4}} \left( \frac{f(\frac{\dot{a}(4t)^{1/2}}{a(4t)^{1/2}})}{a(4t)^{(k_1+k_2)/4}} \right)_x
\end{equation}

\begin{equation}
= \left( \frac{\dot{a}(4t)}{a(4t)} - \frac{\dot{a}(4t)^2}{a(4t)^2} \right) x + 4 \left( \frac{\dot{a}(4t)}{a(4t)} \right) \frac{\dot{a}(4t)}{a(4t)} x + k_3 \frac{f(\frac{\dot{a}(4t)^{1/2}}{a(4t)^{1/2}})}{a(4t)^{(k_1+k_2)/4}} \frac{f(\frac{\dot{a}(4t)^{1/2}}{a(4t)^{1/2}})}{a(4t)^{(k_1+k_2)/4}} \frac{1}{a(4t)^{k_2/4}}
\end{equation}

\begin{equation}
= \frac{k_3}{a(4t)^{k_2/4 + \frac{k_2}{2}}} \left( \frac{\dot{a}(4t)}{a(4t)} \right) f(\frac{\dot{a}(4t)^{1/2}}{a(4t)^{1/2}})
\end{equation}

\begin{equation}
= \frac{k_3}{a(4t)^{k_2/4 + \frac{k_2}{2}}} \left( \frac{\dot{a}(4t)}{a(4t)} \right) \left( \frac{\dot{a}(4t)^{1/2}}{a(4t)^{1/2}} \right)
\end{equation}

\begin{equation}
= \frac{k_3}{a(4t)^{k_2/4 + \frac{k_2}{2}}} \left( \frac{\dot{a}(4t)}{a(4t)} \right) \left( \frac{\dot{a}(4t)^{1/2}}{a(4t)^{1/2}} \right)
\end{equation}

for \( k_3 \neq 0 \), with the Emden equation:

\[\begin{align*}
\dot{a}(s) - \frac{s}{4\dot{a}(s)^2} &= 0 \\
a(0) &= a_0 \neq 0, \quad \dot{a}(0) = a_1
\end{align*}\]
by defining the variables \( s := 4t, \eta := x/a(s)^{k_3/4} \) and \( \kappa = \frac{k_1}{2} + k_2 - 1 \).

(1) For \( k_3 = 0 \), we have \( \xi = 0 \):

\[
\rho(t, x) = \frac{\rho_0(\eta)}{a(4t)^{(k_3+k_3)/4}}, \quad u(t, x) = \frac{\dot{a}(4t)}{a(4t)}x
\]

(38)

where \( \rho_0 \) is an arbitrary \( C^1 \) function.

Now, we can separate the partial differential equations into two ordinary differential equations. Then, we only need to solve for \( \frac{k_3}{2} < 0 \),

\[
\begin{cases}
\frac{k_3}{2} \eta + f(\eta) \dot{f}(\eta) = 0 \\
f(0) = -\alpha \leq 0
\end{cases}
\]

(39)

or for \( \frac{k_3}{2} > 0 \),

\[
\begin{cases}
\frac{k_3}{2} \eta + f(\eta) \dot{f}(\eta) = 0 \\
f(0) = \alpha \geq 0
\end{cases}
\]

(40)

The ordinary differential equations \( (39) \) or \( (40) \) can be solved exactly as

\[
f(\eta) = \frac{k_3}{\xi} \sqrt{-\frac{\xi}{k_3} \eta^2 + \left( \frac{\xi}{k_3} \alpha \right)^2}.
\]

(41)

In fact, we have the self-similar solutions in details:

(2) For \( k_3 > 0 \) and \( \xi > 0 \), or

(3) For \( k_3 < 0 \) and \( \xi < 0 \):

\[
\begin{cases}
\rho(t, x) = \max\left( \frac{f(\eta)}{a(4t)\eta^{(k_3+k_3)/4}}, 0 \right), \quad u(t, x) = \frac{\dot{a}(4t)}{a(4t)}x \\
\dot{a}(s) - \frac{\xi}{4a(s)^2} = 0, \quad a(0) = a_0 \neq 0, \quad \dot{a}(0) = a_1
\end{cases}
\]

(42)

With the assistance of Lemmas 4, we may obtain the local existence of the above solutions.

The proof is completed.

After we construct the solutions \( (15) \), the blowup or global behavior can be analyzed from the Emden equation:

\[
\begin{cases}
\dot{a}(s) - \frac{\xi}{4a(s)^2} = 0 \\
a(0) = a_0 \neq 0, \quad \dot{a}(0) = a_1
\end{cases}
\]

(43)
To additionally show the blowup or global property of the time function \( a(s) \), the following lemma is needed. For the particular case of \( k_1 = 1 \) and \( k_2 = 1 \), we have

\[
\kappa = \frac{k_1}{2} + k_2 - 1 = \frac{1}{2} < 1.
\] (44)

For the case \( \kappa = 1 \) or \( \kappa = 1/3 \), the blowup or global results for the Emden equation are already shown in [23] and [25]. For the case \( 0 < \kappa \leq 1 \), we can show by the energy method [15]. Due to the proof is very similar, we omit the details here to have the lemma:

**Lemma 5** For \( 0 < \kappa \leq 1 \), the Emden equation (12),

\[
\begin{align*}
\ddot{a}(s) - \frac{\xi}{a(s)^{\kappa}} &= 0 \\
\dot{a}(0) &= a_0 > 0, \quad \dot{a}(0) = a_1.
\end{align*}
\] (45)

(1) if \( \xi < 0 \), there exists a finite time \( S \), such that

\[
\lim_{s \to S^-} a(s) = 0.
\] (46)

(2) if \( \xi > 0 \), the solution \( a(t) \) exists globally, such that

\[
\lim_{s \to +\infty} a(s) = -\infty.
\] (47)

After obtaining the above lemma, it is clear to have the following result:

**Corollary 6** For \( 0 < \kappa \leq 1 \) and \( a_0 > 0 \), we have:

(1) \( k_3 > 0 \) and \( \xi > 0 \), the solution (15) exists globally.

(2) \( k_3 < 0 \) and \( \xi < 0 \), the solution (15) blows up on a finite time \( T \).

For the other cases, the interested reader may determine easily by the classical energy method.

In particular, for \( \kappa > 1 \), the blowup or global behavior may be referred [5].

**Remark 7** For \( k_3 < 0 \) and \( \xi < 0 \), the blowup solutions (15) collapse at the origin:

\[
\lim_{t \to T^-} \rho(t, 0) = +\infty
\] (48)
with a finite time $T$;

For $k_3 > 0$ and $\xi > 0$, the global behavior of the solution (15) at the origin is:

$$\lim_{t \to +\infty} \rho(t, 0) = 0.$$  \hfill (49)

**Remark 8** The solutions (15) are only $C^0$ functions, as the function $f(\eta)$ is discontinuous at the two boundary points, for $\alpha > 0$:

$$\lim_{\eta \to |\xi\alpha|} \dot{f}(\eta) \neq 0.$$  \hfill (50)

**Remark 9** Our analytical solutions could provide concrete examples for testing the validation and stabilities of numerical methods for the systems. Additionally, our special solutions can shed some light on the understanding of evolutionary pattern of the systems.

**Remark 10** We may calculate the mass of

(1) for $a_0 > 0$, the solutions (15) with $(k_3 > 0$ and $\xi > 0)$ or $(k_3 < 0$ and $\xi < 0)$:

$$\text{Mass} = \int_{-\infty}^{+\infty} \rho(t, x)dx < +\infty;$$  \hfill (51)

(2) for $a_0 < 0$ and $(-1)^\kappa = -1$, the solutions (15) with $(k_3 > 0$ and $\xi < 0)$ or $(k_3 < 0$ and $\xi > 0)$:

$$\text{Mass} = +\infty.$$  \hfill (52)

3 Blowup Phenomenon for $k_3 \geq 0$

In this section, we show the blowup phenomenon by the standard method. The system can be converted to a quasi-linear evolution of hyperbolic type:

$$\begin{align*}
&\rho_t = -k_2\rho_x u - (k_1 + k_2)\rho u_x \\
u_t + u \cdot u_x + \partial_x G * (\frac{\alpha}{2}u^2 + \frac{k_3}{2}\rho^2) = 0
\end{align*}$$  \hfill (53)

where the sign $*$ denotes the spatial convolution, $G(x)$ is the associated Green function of the operator $(1 - \partial_x^2)^{-1}$. For the local well-possness and the blowup phenomenon with odd initial
values \((\rho_0, u_0, \ldots)\), are discussed in [10] recently. We may show the blowup by the particle trajectory method. Consider the following problem:

\[
\begin{cases}
q_t = u(q, t), & 0 < t < T \\
q(x, 0) = x.
\end{cases}
\tag{54}
\]

we apply the characteristic curve method of hyperbolic partial differential equations, as the following theorem:

**Theorem 11** Suppose the velocity \(u\) is uniformly bounded at some point \(x_0\), such that \(u(t, x_0) \leq M\) and \(u_x(x_0, 0) < -\sqrt{\frac{3}{2}} |M|\) and the regularity of the solution at \(x_0\) is sufficient enough. The nontrivial solutions blow up before a finite time \(T\).

**Proof.** In general, we show that the \(\rho(t, x(t; x))\) preserves its positive nature as the mass equation (53) can be converted to be

\[
\rho_t + k_2 \rho_x u = -(k_1 + k_2) \rho u_x
\tag{55}
\]

\[
\frac{D\rho}{Dt} + (k_1 + k_2) \rho \frac{\partial}{\partial x} \cdot u = 0
\tag{56}
\]

with the material derivative:

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}.
\tag{57}
\]

We integrate the equation (56):

\[
\rho(t, x) = \rho_0(x_0(0, x_0)) \exp \left(-(k_1 + k_2) \int_0^t \nabla \cdot u(t, x(t; 0, x_0)) dt \right) \geq 0
\tag{58}
\]

for \(\rho_0(x_0(0, x_0)) \geq 0\), along the characteristic curve.

To drive the argument for blowing up, we differentiate equation (53) with respect to \(x\):

\[
u_{xx} = -u_x^2 - u u_{xx} + \frac{3}{2} u^2 + \frac{k_3}{2} \rho^2 - G * \left(\frac{3}{2} u^2 + \frac{k_3}{2} \rho^2\right)
\tag{59}
\]

Along the another characteristic curve with

\[
d \frac{d}{dt} := \frac{\partial}{\partial t} - u \frac{\partial}{\partial x},
\tag{60}
\]

we have for some point \(x_0\):

\[
\frac{du_x(x_0, t)}{dt} = -u_x^2(x_0, t) + \frac{3}{2} u^2(x_0, t) - G * \left(\frac{3}{2} u^2 + \frac{k_3}{2} \rho^2\right)(x_0, t)
\tag{61}
\]

\[
\leq -u_x^2(x_0, t) + \frac{3}{2} M^2.
\tag{62}
\]
The above Racci equation blows up before a finite time $T$, if the initial value,
\[ u_x(x_0, 0) < -\frac{\sqrt{3}}{2} |M|. \]  
(63)

The proof is completed. 

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