One-loop corrections to $\eta/s$ in AdS$_4$/CFT$_3$

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Abstract

We study quantum corrections at one-loop order to the shear viscosity to entropy ratio by implementing the Vilkovisky-Barvinsky effective action in asymptotically Anti-de Sitter spacetimes. The shear viscosity is shown to receive no corrections at this order, but the entropy acquires a logarithmic correction. The coefficient of this logarithm turns out to depend on the spin of the particles running in the loop and it can be either positive or negative. On the basis of this result, we argue that the Kovtun–Son–Starinets bound cannot be seen as a fundamental property of nature beyond the classical regime.
1 Introduction

Dualities in physics have become a major subject of investigation since the prominent work by Maldacena [1]. His foundational paper sets the ground for the equivalence between gauge theory and gravity, which led to a breakthrough in theoretical physics, as it promoted a novel way of relating different fields that turned out to be not so different at all. This intriguing conjecture, although having its roots in string theory, has spread out to all arXiv research areas and has become one of the most popular ideas in high-energy physics.

Although theoretically very appealing, the AdS/CFT correspondence has not made its way into the real world yet. It has failed to predict the exact values for the shear viscosity to entropy ratio $\eta/s$ of the quark-gluon plasma measured in the Relativistic Heavy Ion Collider (RHIC). Some attribute this difference to the fact that QCD does not possess conformal symmetry. Furthermore, the correspondence is established for $N \to \infty$, which conflicts with the finite number of colors $N = 3$ in QCD. Deformations on the gauge side of the correspondence allows one to make progress towards more realistic scenarios, including the finite $N$ case. This corresponds to quantum corrections on the gravitational side.

In this paper, we look at one-loop corrections to general relativity in AdS$_4$ and study their effects on the corresponding three-dimensional gauge theory. We take an effective field theory approach to investigate the model-independent infrared portion of quantum general relativity [2, 3]. This differs from the usual local higher curvature corrections which requires knowledge of the UV and thus corresponds to the unreliable, from a bottom-up perspective, high-energy part of loops. It is important to make a distinction between our bottom-up construction, typical of effective theories, and the top-down $\alpha'$ corrections due to the string tension commonly found in the literature [4–18]. In the former, local terms encode in their coefficients the unknown high-energy information, which must be fixed by observations at some energy scale rather than computed from first principles as in the latter. Our primary interest is, however, in the infrared portion of quantum gravity, where both quantum general relativity and string theory should agree.

We are mainly interested in the shear viscosity to entropy ratio $\eta/s$ as it is one of the most important observables used to corroborate the AdS/CFT correspondence. We show that the shear viscosity does not receive corrections at one-loop order. The entropy, on the other hand, is modified by a logarithm term. This logarithmic correction has already been identified in other contexts [19–33]. Due to the spin dependence of the quantum action, we argue that the ratio $\eta/s$ violates the Kovtun–Son–Starinets (KSS) bound even within quantum general relativity. Our result might be important to alleviate the difference observed at the RHIC with respect to the classical prediction of $\eta/s$. 
This paper is organized as follows. In Sect. 2, we review the general formalism due to Vilkovisky et al for the construction of a gauge-independent quantum action for gauge theory and quantum gravity. Sect. 3 is devoted to the calculation of one-loop corrections to the shear viscosity of the gauge theory. In Sect. 4, we employ the Euclidean formalism to calculate quantum corrections to the entropy of the Schwarzschild-AdS (SAdS) black hole, finally leading to the ratio \( \frac{\eta}{s} \). We then assess and discuss our results in Sec. 5.

2 Vilkovisky-Barvinsky effective action

The quantum action is a central object in quantum field theory. It contains, in a single place, all the information regarding correlation functions, which is of utmost importance for experimental physics, and yet provides the dynamics of the mean field due to the backreaction of quantum modes. A general scheme for the computation of the quantum action in arbitrary backgrounds for a gauge theory of arbitrary spin has been introduced in [34–37]. In the following, we give a brief overview of this formalism.

In the background field formalism, the quantum effective action is determined by the following functional integro-differential equation in the Euclidean formalism

\[
e^{-\Gamma[\phi]} = \int \mathcal{D}\Phi \exp^{-S[\Phi]+\int dx (\Phi(x)-\phi(x))\frac{\delta\Gamma[\phi]}{\delta\phi}},
\]

where \( S[\Phi] \) comprises the classical Einstein-Hilbert action in the presence of arbitrary matter fields \( S_m \) and counter-terms \( S_{ct} \) used for renormalization

\[
S[\Phi] = -\int d^4x \sqrt{-g} \frac{1}{16\pi G} (R - 2\Lambda) + S_m + S_{ct},
\]

where \( G \) is the Newton’s constant, which will be set to unity in the next sections. At one-loop level, the divergences are proportional to terms containing up to fourth-order derivatives [38], thus the counter-terms will be given by

\[
S_{ct} = \int d^4x \sqrt{-g} \left[ a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + a_4 \Box R \right],
\]

where the \( a_i \) are bare coefficients. The path integral is over the quantum field \( \Phi \) and depends parametrically on the mean field \( \phi(x) = \langle \Phi(x) \rangle \). The notation \( \Phi = \Phi^A(x) \) is used to denote collectively fields of arbitrary spin. To perform the above path integral, one writes \( \Gamma \) as a loop expansion

\[
\Gamma[\phi] = \sum_n \Gamma^{(n)}[\phi],
\]
where $\Gamma^{(0)}$ denotes the classical Einstein-Hilbert action with a cosmological constant and $\Gamma^{(1)}$ is the one-loop contribution, which is given by

$$\Gamma^{(1)} = \frac{1}{2} \log \det F(\nabla) = \frac{1}{2} \text{Tr} \log F(\nabla),$$

$$F(\nabla) \delta(x, y) = \frac{\delta^2 S[\phi]}{\delta \phi(x) \delta \phi(y)},$$

Using the Schwinger proper time method, the one-loop contribution to the quantum action can be written as

$$\Gamma^{(1)} = -\frac{1}{2} \int_0^\infty ds \frac{s}{s} \text{Tr} K(s),$$

where $K(s) = e^{sF(\nabla)}$ is the heat kernel. The calculation of $\Gamma^{(1)}$ relies on approximate solutions to the heat equation satisfied by $K(s)$. The Schwinger–DeWitt method, for example, consists of a time asymptotic expansion of the heat kernel $K(s)$ at small $s$. Our interest is the covariant perturbation theory approach, whose approximation scheme is an expansion in powers of the curvature.

The operator $F(\nabla)$ can generically be written as

$$F(\nabla) = \Box + \hat{P} - \frac{1}{6} R,$$

where $\Box = g^{\mu \nu} \nabla_\mu \nabla_\nu$ and $\hat{P}$ is an arbitrary potential term. The metric $g_{\mu \nu}$ and the connection $\nabla_\mu$ are characterized by the Riemann curvature $R^{\mu \nu \rho \sigma}$ and the fiber bundle curvature $R_{\mu \nu} = R^A_{\mu \nu}$, respectively:

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\beta \mu \nu} V^\beta,$$

$$[\nabla_\mu, \nabla_\nu] \Phi^A = R^A_{B \mu \nu} \Phi^B,$$

for some vector field $V^\alpha$. We denote by $\mathcal{R} = \{ \hat{P}, R^{\mu \nu \rho \sigma}, R_{\mu \nu} \}$ the set of curvatures that characterizes the operator $F(\nabla)$. The purpose of covariant perturbation theory is to obtain all quantities of interest as an expansion in $\mathcal{R}$. After a lengthy calculation, the resultant quantum effective action in arbitrary dimensions $2\omega$ to second order in curvature reads

$$\Gamma[\phi] = -\frac{\Gamma(2 - \omega) \Gamma(\omega + 1) \Gamma(\omega - 1)}{2(4\pi)^\omega \Gamma(2\omega + 2)} \int dx g^{1/2}(x) \text{tr} \left\{ \hat{R}_{\mu \nu} (-\Box)^{-2} R^{\mu \nu} \hat{1} \\
- \frac{(4 - \omega)(\omega + 1)}{18} R (-\Box)^{-2} R \hat{1} - \frac{2(2 - \omega)(2\omega + 1)}{3} \hat{P} (-\Box)^{-2} R \\
+ 2(4\omega^2 - 1) \hat{P} (-\Box)^{-2} \hat{P} + (2\omega + 1) \hat{R}_{\mu \nu} (-\Box)^{-2} \hat{R}^{\mu \nu} \right\} + O[\mathcal{R}^3],$$

where $\Gamma(z)$ is the Gamma function\(^3\). The effective action (11) is the most general result valid for any gauge field, including gravitons, in arbitrary dimensions up to second order in

\(^3\)Be aware of the difference in notation between the Gamma function $\Gamma(z)$ and the quantum action $\Gamma[\phi]$. 


curvature $\mathcal{R}$. It is a functional of all background fields. Note that structures involving the Riemann curvature, such as $R_{\mu\nu\rho\sigma}f(\Box)R_{\mu\nu\rho\sigma}$, are not displayed in (11). This happens because, due to the second Bianchi identity, the Riemann tensor satisfies a differential equation sourced by the Ricci tensor, thus the Riemann tensor can be determined in terms of the Ricci tensor up to boundary conditions. For asymptotically flat spaces, one can impose a trivial boundary condition such that the Riemann tensor vanishes for a Ricci-flat spacetime \[36\]. However, no such trivial condition can be imposed on asymptotically AdS or dS spaces. While one can still solve the differential equation for non-trivial boundary conditions (see Appendix A), for the time being we choose a more direct approach and leave the Riemann piece in the action below. This will also make the matching with the known results in the literature, such as the values in Table 1, more transparent. In the next sections, however, we make use of the result in the Appendix A to ease our calculations.

Going back to Lorentzian signature and specializing to four dimensions $\omega = 2$, the quantum action (11) for the mean metric $g_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle$ with vanishing mean matter fields becomes $[39,40]$

$$\Gamma = \Gamma_L + \Gamma_{NL},$$

where the local part reads

$$\Gamma_L = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R - 2\Lambda) + b_1 R^2 + b_2 R_{\mu\nu} R^{\mu\nu} + b_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right],$$

and the non-local one reads

$$-\Gamma_{NL} = \int d^4x \sqrt{-g} \left[ c_1 R \log \left( \frac{-\Box}{m^2} \right) R + c_2 R_{\mu\nu} \log \left( \frac{-\Box}{m^2} \right) R_{\mu\nu} 
+ c_3 R_{\mu\nu\rho\sigma} \log \left( \frac{-\Box}{m^2} \right) R_{\mu\nu\rho\sigma} \right],$$

where $m^2 = -2\Lambda$ is the effective mass in the presence of a cosmological constant. We are assuming that $-\Box/m^2 \gg 1$, in which case the form factor in the non-local piece of the action is dominated by $\log(-\Box/m^2)$. Other contributions, such as $-m^2/\Box$, are thus suppressed. The action (12) accounts for one-loop quantum corrections from both matter and gravitons running in the loops. A very useful way of dealing with non-local operators is via their spectral decomposition, which for the log is given by

$$\log \frac{-\Box}{m^2} = \int_0^\infty ds \left( \frac{1}{m^2 + s} - \frac{1}{-\Box + s} \right),$$

where the second term above can be written in terms of the Green function of $-\Box + s$. Note that the integration variable $s$ in (15) does not denote the proper time.

We stress that the non-local piece represents the infrared portion of quantum gravity, which is insensitive to the UV. The coefficients $c_i$ are then genuine predictions of the quantum
theory of gravity. They are determined once the collection of fields \( \Phi \) in (1) and their respective spins are specified; see Table 1. The total contribution to each coefficient is given by simply summing the contribution from each field species. The local action, on the other hand, represents the high energy portion of quantum loops. As a result, the coefficients \( b_i \) cannot be determined from first principles. They are renormalized parameters which must be fixed by observations or by matching with a UV completion. They satisfy the renormalization group equation

\[
\mu \partial_\mu b_i = \beta_i, \tag{16}
\]

for the beta functions \( \beta_i = -2c_i \).

The effective action \( \Gamma \) is a functional of the arbitrary mean field \( g_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle \), which is not necessarily a solution of the classical equations of motion. In fact, \( g_{\mu\nu} \) carries the information concerning the backreaction of the quantum fields integrated out in (1), thus describing the evolution of the background due to quantum fluctuations. When \( g_{\mu\nu} \) does not satisfy the classical Einstein’s equations, the quantum action is gauge independent [41], but depends parametrically on the gauge fixing and on the parametrization of the quantum field. This issue has been solved by Vilkovisky by introducing a metric and a connection in the configuration space [42]. Nonetheless, both the calculation of the shear viscosity and of the entropy involves the evaluation of the on-shell action in the AdS instanton

\[
ds_{\text{AdS}}^2 = -\left(\frac{r}{b}\right)^2 dt^2 + \frac{dr^2}{\left(\frac{r}{b}\right)^2} + \left(\frac{r}{b}\right)^2 d\vec{x}_2^2, \tag{17}
\]

where \( b \) is the AdS radius, or in asymptotically AdS spaces, such as the SAdS black hole of mass \( M \)

\[
ds_{\text{SAdS}}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2, \quad f(r) = 1 - \frac{2M}{r} + \frac{r^2}{b^2}, \tag{18}
\]

| Field Type | \( c_1 \) | \( c_2 \) | \( c_3 \) |
|-----------|-----------|-----------|-----------|
| real scalar | \( \frac{5(6\xi - 1)^2}{(11520\pi^2)} \) | \( -\frac{2}{(11520\pi^2)} \) | \( \frac{2}{(11520\pi^2)} \) |
| Dirac spinor | \( -\frac{5}{(11520\pi^2)} \) | \( \frac{8}{(11520\pi^2)} \) | \( \frac{7}{(11520\pi^2)} \) |
| vector | \( -\frac{50}{(11520\pi^2)} \) | \( \frac{176}{(11520\pi^2)} \) | \( -\frac{26}{(11520\pi^2)} \) |
| graviton | \( \frac{430}{(11520\pi^2)} \) | \( -\frac{1444}{(11520\pi^2)} \) | \( \frac{424}{(11520\pi^2)} \) |

Table 1: Values of the coefficients \( c_i \) for each spin (\( \xi \) is the non-minimal coupling coefficient of scalars to gravity) extracted from [39]. Each value must be multiplied by the number of fields of its category present in the action \( S[\Phi] \). The total value of each coefficient is then given by summing up all contributions.
thus the parametrization and gauge fixing dependence will not be a concern for us \([43–45]\).

We shall now see how to use the quantum action \((12)\) to calculate the shear viscosity. In the rest of this paper we set \(G = 1\) for convenience.

### 3 One-loop corrections to the shear viscosity

The calculation of the shear viscosity of the gauge theory can be performed in many different ways \([46–48]\). The most usual method employs the Kubo formula

\[
\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \Im G_{R}^{xy,xy}(\omega, \vec{k} = 0),
\]

(19)

which relates the shear viscosity to the imaginary part of the retarded Green’s function for the response of the \(xy\) component of the energy-momentum tensor:

\[
G_{R}^{xy,xy}(\omega, \vec{k}) = -i \int d^3 x e^{i\omega t - i\vec{k} \cdot \vec{x}} \theta(t) \left\langle \left[ T^{xy}(t, \vec{x}), T^{xy}(0, \vec{0}) \right] \right\rangle,
\]

(20)

where \(\theta(t)\) is the Heaviside step function. The Green function \(G_{R}^{xy,xy}\) is then determined with the aid of the AdS/CFT correspondence \(Z_{\text{gauge}} = Z_{\text{AdS}}\), which translates into the GKP–Witten relation in real time

\[
\left\langle \exp \left( i \int d^3 x h_{xy}^{(0)} T^{xy} \right) \right\rangle = \exp \left( i \Gamma[h_{xy}^{(0)}] \right),
\]

(21)

where \(h_{xy}^{(0)} = h_{xy}|_{r=\infty}\) denotes the gravitational perturbation polarized parallel to the brane at the AdS boundary and \(\Gamma[h_{xy}^{(0)}]\) is the on-shell quantum action \((12)\) for the gravitational perturbation in AdS. Under functional variations, one can find from \((21)\) all the correlation functions of the gauge theory. In particular, the one-point function \(\langle T^{xy} \rangle = -G_{R}^{xy,xy} h_{xy}^{(0)}\) is given by

\[
\langle T^{xy} \rangle = \frac{\delta \Gamma[h_{xy}^{(0)}]}{\delta h_{xy}^{(0)}}.
\]

(22)

Finding the the shear viscosity \(\eta\) thus amounts on the calculation of the dynamics of the component \(h_{xy}\) of the bulk perturbation. We shall now see how the one-loop correction in \(\Gamma\) affects its evolution.

In the realm of effective field theory, the quantum action has to be treated perturbatively. The lowest order determines the degrees of freedom and their interactions, while higher order terms make contributions only to the latter, i.e. to vertices of Feynman diagrams. This is the standard lore of effective field theory. We thus need to linearize the quantum action
around some fixed background by performing the transformation $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background metric. The quantum action then becomes

$$\Gamma[\bar{g} + h] = \Gamma[\bar{g}] + \int d^4x \delta \Gamma^{(1)}[\bar{g}] h_{\mu\nu}(x) + \frac{1}{2} \int d^4xd^4y \frac{\delta^2 \Gamma^{(0)}[\bar{g}]}{\delta g_{\mu\nu}(x) \delta g^{\rho\sigma}(y)} h_{\mu\nu}(x) h^{\rho\sigma}(y) + \cdots,$$

(23)

where we made a loop expansion $\Gamma = \sum_n \Gamma^{(n)}$, denoting $\Gamma^{(0)}$ as the classical Einstein-Hilbert action with a cosmological constant and $\Gamma^{(1)}$ as the one-loop contribution. Note that the $\Gamma^{(0)}$ does not contribute to the linear term because the linearization is around the AdS instanton, thus the variation of $\Gamma^{(0)}$ vanishes by the classical equations of motion. The radiative correction $\Gamma^{(1)}$ does not contribute to the quadratic order either as it is suppressed in this perturbative treatment.

Let us focus on the non-local part $\Gamma_{NL}$ of the effective action as it represents the infrared regime that we are interested in; the result for the local part $\Gamma_L$ can be obtained analogously. The equations of motion for the perturbation around some arbitrary background is then given by

$$\square \bar{h}_{\mu\nu} = -(c_1 - c_3) \bar{R} \log \left( -\frac{\square}{\mu^2} \right) \bar{R}_{\mu\nu} + \frac{1}{2}(c_2 + 4c_3) \bar{g}_{\mu\nu} \bar{R}_{\rho\sigma} \log \left( -\frac{\square}{\mu^2} \right) \bar{R}^{\rho\sigma},$$

(24)

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} h$. Note that, for both AdS (17) and SAdS (18), the metric $\bar{g}_{\mu\nu}$ and the Ricci tensor $\bar{R}_{\mu\nu}$ are diagonal, thus the equation of motion for $h_{xy}$ reads

$$\square h_{xy} = 0.$$

(25)

We conclude that, at least at one-loop order, the evolution of $h_{xy}$ as much as the value of the shear viscosity are exactly as in classical general relativity. The shear viscosity is then given by the classical result [49–53]

$$\eta = \frac{\sigma_{abs}(\omega = 0)}{16\pi},$$

(26)

where the absorption cross-section $\sigma_{abs}(\omega = 0) = A_+$ equals the horizon area.

## 4 One-loop corrections to the entropy

In this section, we use the Euclidean method to calculate the entropy of the SAdS black hole, following the usual procedure of transforming to the Euclidean time $\tau = it$, for $\tau \in (0, \beta)$, and imposing periodic boundary conditions so to avoid conical singularities. In asymptotically AdS spaces, one can use the canonical ensemble to compute thermodynamical quantities, where the black hole is put in contact with a thermal bath. To get rid of potential divergences in the entropy, one must subtract the AdS entropy from the SAdS one, which corresponds
to normalizing the partition function with respect to AdS. This has the effect of eliminating the contribution from the thermal bath.

In Euclidean time, the normalized partition function reads

$$Z(\beta) = e^{-\Delta \Gamma},$$

(27)

where $\Delta \Gamma \equiv \Gamma_{SAdS} - \Gamma_{AdS}$ is the difference of the on-shell actions evaluated at SAdS and AdS, respectively. Differently from the standard Schwarzschild case, where the contribution to the partition function comes solely from the Gibbons-Hawking-York boundary terms, we can disregard any boundary terms because they are canceled out in the difference $\Gamma_{SAdS} - \Gamma_{AdS}$.

Let us now calculate the on-shell action. Note that both AdS and SAdS metrics satisfy the classical equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu},$$

(28)

$$R = 4\Lambda.$$  (29)

The main difficulty is then to calculate the action of the log operator on the metric and on the cosmological constant. Using the spectral representation of the log, we find

$$\log \left( -\frac{\Box}{m^2}\right) g_{\mu\nu} = \int_{M_c^2}^{\infty} ds \left[ \frac{1}{s + m^2} - \frac{1}{s - \Box + m^2} \right] g_{\mu\nu}$$

$$= \int_{M_c^2}^{\infty} ds \left[ \frac{1}{s + m^2} - \frac{1}{m^2} \right] g_{\mu\nu}$$

$$= \log \left( \frac{M_c^2}{m^2} \right) g_{\mu\nu},$$

(30)

where $M_c$ is some mass scale of $g_{\mu\nu}$ used to regulate the divergence appearing at $s = 0$. For SAdS, we take $M_c = M$ as the mass of the black hole, while for AdS we take $M_c = \varepsilon \to 0$ as a temporary cut-off. The final result will turn out to be independent of $\varepsilon$. Note that we used the metric compatibility $\nabla_{\mu} g_{\rho\sigma} = 0$ in the second equality above. The action on the cosmological constant gives the same result as in Eq. (30) for obvious reasons.

Using Eqs. (28), (29) and (30) in (14), gives

$$\Gamma_{SAdS} = \left[-\Lambda + \frac{4\Lambda^2}{3} \log \left( \frac{M^2}{m^2} \right) \right] \left(12c_1 + 3c_2 + 2c_3\right) V_{SAdS},$$

(31)

$$\Gamma_{AdS} = \left[-\Lambda + \frac{4\Lambda^2}{3} \log \left( \frac{M_c^2}{m_1^2} \right) \right] \left(12c_1 + 3c_2 + 2c_3\right) V_{AdS},$$

(32)

where $m_1 = -2\Lambda_1$ is the arbitrary effective mass of AdS and

$$V_{SAdS} = \frac{4\pi}{3} \beta (L^3 - r_+^3),$$

(33)

$$V_{AdS} = \frac{4\pi}{3} \beta_1 L^3,$$

(34)
are the volume of SAdS and AdS, respectively, where we have introduced an infrared cut-off $L$ and $r_+$ is the horizon radius obtained by solving $f(r_+) = 0$. The period $\beta$ of SAdS is fixed so to avoid the conical singularity

$$\beta = \frac{4\pi b^2 r_+}{b^2 + 3r_+^2}. \quad (35)$$

On the other hand, the period $\beta_1$ of AdS is a priori arbitrary. However, since the two metrics must coincide at $L \to \infty$, the time coordinate must have the same period:

$$\beta_1 \sqrt{1 + \frac{L^2}{b^2}} = \beta \sqrt{1 - \frac{2M}{L} + \frac{L^2}{b^2}} \implies \frac{\beta_1}{\beta} \approx 1 - \frac{Mb^2}{L^3}. \quad (36)$$

Therefore, the difference of the on-shell actions $\Delta \Gamma \equiv \Gamma_{\text{SAdS}} - \Gamma_{\text{AdS}}$ reads

$$\Delta \Gamma = \Delta \Gamma^{(0)} + \Delta \Gamma^{(1)}, \quad (37)$$

where

$$\Delta \Gamma^{(0)} = \frac{\pi r_+^2 (b^2 - r_+^2)}{b^2 + 3r_+^2} \quad (38)$$

is the usual general relativistic result and

$$\Delta \Gamma^{(1)} = \frac{16\pi}{9} \Lambda^2 (12c_1 + 3c_2 + 2c_3) \beta \left[ \log \left( \frac{M^2}{m^2} \right) (L^3 - r_+^3) - \log \left( \frac{M^2}{m^1} \right) (L^3 - Mb^2) \right] \quad (39)$$

is the one-loop contribution. The divergence $L \to \infty$ in $\Delta \Gamma^{(1)}$ is not automatically removed as in the classical part $\Delta \Gamma^{(0)}$, but we can exploit the arbitrariness of the effective mass $m_1$ to cancel out this divergence. This is achieved with the choice $m_1^2 = \frac{M^2}{M_+^2 m^2}$, which makes the logarithms in Eq. (39) equal and ultimately leads to

$$\Delta \Gamma^{(1)} = \frac{96\pi^2}{3} (12c_1 + 3c_2 + 2c_3) \left[ \log \left( \frac{r_+^2}{m^2} \right) + 2 \log \left( 1 + \frac{r_+^2}{b^2} \right) \right] \frac{r_+^2 (b^2 - r_+^2)}{b^2 (b^2 + 3r_+^2)}. \quad (40)$$

The entropy is finally given by

$$S = (\beta \partial_\beta - 1) \Delta \Gamma = A_+ + 8\pi (12c_1 + 3c_2 + 2c_3) A_+ \frac{b^2}{b_+^2} \log \left( \frac{M^2}{m^2} \right) + \Xi \left( \frac{r_+}{b} \right), \quad (41)$$

where $A_+ = 4\pi r_+^2$ is the horizon area and

$$\Xi \left( \frac{r_+}{b} \right) = 64\pi^2 (12c_1 + 3c_2 + 2c_3) \frac{r_+^2 (1 - r_+^2/b^2)(1 + 3r_+^2/b^2)}{b^2 (1 + r_+^2/b^2)(1 - 3r_+^2/b^2)}. \quad (42)$$

The logarithmic correction to the entropy seems to be a universal feature of quantum gravity [21, 27]. It has been obtained in different contexts using different techniques [19–32]. Our result is yet another instance of this apparent universality.
We are finally able to calculate the shear viscosity to entropy ratio, which is the main concern of this paper. Combining Eqs. (26) and (41), we obtain
\[
\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + \frac{32\pi(12c_1 + 3c_2 + 2c_3)}{b^2} \log \left( \frac{M^2}{m^2} \right) + \frac{4}{A_+} \Xi \left( \frac{r_+}{b} \right) \right]^{-1}.
\] (43)

Eq. (43) has been obtained for SAdS with spherical horizon. For the planar horizon case of a black brane, which can be seen as the limiting case of a large black hole with constant \( r_+ \), this result simplifies to
\[
\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + \frac{32\pi(12c_1 + 3c_2 + 2c_3)}{b^2} \log \left( \frac{M^2}{m^2} \right) \right]^{-1}.
\] (44)

For completeness, we include the result for the local action even though its contribution is suppressed as \( M/m \gg 1 \):
\[
\frac{\eta}{s} = \frac{1}{4\pi} \left\{ 1 + \frac{32\pi(12c_1 + 3c_2 + 2c_3)}{b^2} \log \left( \frac{M^2}{m^2} \right) + (12b_1 + 3b_2 + 2b_3) \right\}^{-1}.
\] (45)

Note that the combination of coefficients \( 12c_1 + 3c_2 + 2c_3 \) can be either positive or negative depending on the spin of particles which had been integrated out to obtain the quantum action (see Table 1). Therefore, the KSS bound
\[
\frac{\eta}{s} > \frac{1}{4\pi}
\] (46)
is not necessarily satisfied by all kinds of integrated particles and does not seem to represent a fundamental bound that holds beyond the classical level. We must stress that, contrary to other violations of the KSS bound [4–16], the above result has been obtained within general relativity by using effective field theory techniques to identify the infrared portion of quantum gravity, which permitted the evaluation of the one-loop contribution to \( \eta/s \). The quantum action (12) is not supposed to be seen as a modification of gravity, after all the degrees of freedom and the interactions are the ones of general relativity, but the latter receives radiative corrections due to quantum fields running in the loops.

5 Conclusions

In this paper, we have studied one-loop corrections of matter fields and gravitons to the shear viscosity to entropy ratio in AdS4/CFT3. Although the former does not receive any correction at one-loop order, the latter gets corrected by a term proportional to the logarithm of the black hole mass. The coefficient of this correction does not have a definite sign because of its spin dependence. We thus argued that the celebrated KSS bound cannot be seen as
a fundamental relation beyond the tree level. We should emphasize, once again, that our result has been obtained within general relativity by using effective field theory to calculate the leading order corrections. The aforementioned violation is entirely due to the quantum nature of fields, including the graviton excitations, in the SAdS background.

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A Elimination of the Riemann tensor for spaces with \( \Lambda \neq 0 \)

It was shown in [36] that the Riemann tensor can be eliminated from the quantum action for asymptotically flat spaces. We generalize the argument to asymptotically AdS and dS spaces. The second Bianchi identity reads

\[
\nabla^\lambda R^{\alpha\beta\mu\nu} + \nabla^\mu R^{\alpha\beta\lambda\nu} + \nabla^\nu R^{\alpha\beta\lambda\mu} = 0,
\]

which can be contracted to give

\[
\nabla_\alpha R^{\alpha\beta\mu\nu} = \nabla^\mu R^{\beta\nu} - \nabla^\nu R^{\beta\mu}.
\]

Contracting (47) with \( \nabla_\lambda \) and using (48) with the aid of the commutation of covariant derivatives gives

\[
\Box R^{\alpha\beta\mu\nu} = \nabla^\mu \nabla_\alpha R^{\nu\beta} - \nabla^\nu \nabla_\alpha R^{\mu\beta} - \nabla^\mu \nabla_\beta R^{\nu\alpha} + \nabla^\nu \nabla_\beta R^{\mu\alpha} - 4R^{\alpha}_\sigma R^{\beta\nu\lambda\sigma} + 2R^{\alpha}_\sigma R^{\beta\lambda\nu\sigma} - R^{\alpha\beta}_\sigma R^{\mu\nu\sigma\lambda}.
\]

Eq. (49) can be solved iteratively for the Riemann tensor, which is determined in terms of the Ricci tensor up to boundary conditions. For asymptotically AdS or dS spaces, one can use a maximally symmetric space as the boundary condition such that

\[
R^{\mu\nu\rho\sigma} = \Lambda (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \quad \text{for} \quad R_{\mu\nu} = \Lambda g_{\mu\nu}.
\]

Therefore, the Riemann tensor can be uniquely determined from Eq. (49) by imposing the boundary condition (50) and using some appropriate Green function for \( \Box^{-1} \). To lowest
order, one finds

\[ R^{\alpha\beta\mu\nu} = \nabla^\mu \nabla^{\alpha} \Box^{-1} R^{\nu\beta} - \nabla^\nu \nabla^{\alpha} \Box^{-1} R^{\mu\beta} - \nabla^\mu \nabla^{\beta} \Box^{-1} R^{\nu\alpha} + \nabla^\nu \nabla^{\beta} \Box^{-1} R^{\mu\alpha} \]

\[ + \frac{\Lambda}{3} (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}) + \mathcal{O}(R^2), \]

(51)

which can be used to eliminate the Riemann tensor from the effective action in favor of the Ricci tensor and the cosmological constant. In particular, the computation of the square of the Riemann tensor gives

\[ R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = 4 R_{\mu\nu} R^{\mu\nu} - R^2 + \frac{8\Lambda^2}{3} + \nabla_\mu \zeta^\mu + \mathcal{O}(R^3), \]

(52)

where

\[ \zeta^\mu = 4 \nabla^\nu \Box^{-1} R^{\alpha\beta} \left( \nabla_\nu \nabla^\mu \Box^{-1} R_{\alpha\beta} - \nabla_\beta \nabla^\mu \Box^{-1} R_{\nu\alpha} + \nabla_\alpha \nabla^\beta \Box^{-1} R^{\mu\nu} - \nabla_\nu \nabla_\alpha \Box^{-1} R^{\mu\beta} \right) \]

\[ + 4 \Box^{-1} R_{\alpha\beta} \nabla^\alpha R^{\mu\beta} - 4 R^{\alpha\beta\mu\nu} \Box^{-1} R_{\alpha\beta} + R \nabla^\mu \Box^{-1} R - 2 \Box^{-1} R^{\mu\nu} \nabla_\nu R. \]

(53)

Note that one recovers the result of [36] for \( \Lambda = 0 \). While the third term on the RHS of (52) does not contribute to the equations of motion (24), it definitely contributes to the entropy (41).

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