A CLASS OF ANISOTROPIC EXPANDING CURVATURE FLOWS

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ABSTRACT. In this paper, we consider an expanding flow of smooth, closed, uniformly convex hypersurfaces in Euclidean $\mathbb{R}^{n+1}$ with speed $u^\alpha \sigma_k^\beta$ firstly, where $u$ is support function of the hypersurface, $\alpha, \beta \in \mathbb{R}$, and $\beta > 0$, $\sigma_k$ is the $k$-th symmetric polynomial of the principal curvature radii of the hypersurface, $k$ is an integer and $1 \leq k \leq n$. For $\alpha \leq 1 - k\beta$, $\beta > \frac{1}{k}$ we prove that the flow has a unique smooth and uniformly convex solution for all time, and converges smoothly after normalisation, to a sphere centered at the origin. Moreover, for $\alpha \leq 1 - k\beta$, $\beta > \frac{1}{k}$, we prove that the flow with the speed $fu^\alpha \sigma_k^\beta$ exists for all time and converges smoothly after normalisation to a soliton which is a solution of $fu^\alpha - \frac{1}{\sigma_k} = c$ provided that $f$ is a smooth positive function on $S^n$ and satisfies that $(\nabla_i \nabla_j f)^{\frac{1}{1 + k \alpha - \delta}} + \delta_{ij} f^{\frac{1}{1 + k \alpha - \delta}}$ is positive definite. When $\beta = 1$, our argument provides a proof to the well-known $L_p$ Christoffel-Minkowski problem for the case $p \geq k + 1$ where $p = 2 - \alpha$, which is identify with Ivaki’s recent result. Especially, we obtain the same result for $k = n$ without any constraint on smooth positive function $f$. Finally, we also give a counterexample for the two anisotropic expanding flows when $\alpha > 1 - k\beta$.

1. Introduction. Flows of convex hypersurfaces in $\mathbb{R}^{n+1}$ by a class of speed functions which are homogeneous and symmetric in principal curvatures have been studied by many authors. Firey [15] first introduced the Gauss curvature flow as a model for the shape change of tumbling stones. In [23] Huisken considered the mean curvature flow. He showed that the flow has a unique smooth solution and the hypersurface converges to a round sphere if the initial hypersurface is closed and convex. Later, a range of flows with the speed of homogeneous degree one in principal curvatures were established, see [12, 13, 3] etc. for example. For the curvature flow at the speed of $\alpha$-power of the Gauss-Knonecker curvature, it was conjectured that the solution will converge to a round point along the flow for $\alpha > \frac{1}{n+2}$. Chow [12], Andrews [4], Choi and Daskalopoulos [9] gave the partial answers respectively. In [8], Brendle et al. finally resolved the conjecture for all $\alpha > \frac{1}{n+2}$ in all dimensions recently.

For the problem on the existence of the prescribed polynomial of the principal curvature radii of the hypersurface, Urbas[29], Chow and Tsai[14], Gerhardt [16],

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Xia [31] studied the convergence for the flow with the speed of $F(\lambda_1, \ldots, \lambda_n)$, where $F$ is symmetric polynomial of the principal curvature radii $\lambda_i$ of the hypersurface. Guan and Ma [18], Hu et al. [21], and Guan-Xia [20] etc. gave the proofs for a class of $L_p$ Christoffel-Minkowski problems. On the other hand, as a nature extension, anisotropic flows usually provide some proofs of smooth solutions existence to PDEs arising in convex geometry, see [30, 10, 19, 26, 24] etc.. In this paper, we consider two expanding flows of the convex hypersurfaces at the speeds of $u^\alpha \sigma_k^\beta (\lambda_1, \ldots, \lambda_n)$ and $fu^\alpha \sigma_k^\beta (\lambda_1, \ldots, \lambda_n)$ respectively, where $u$ is the support function, $f$ is a smooth positive function on $\mathbb{S}^n$, $\alpha, \beta \in \mathbb{R}$, $\beta > \frac{1}{p}$ and $\sigma_k (\lambda_1, \ldots, \lambda_n)$ is the $k$-th symmetric polynomial of the principal curvature radii of the hypersurface, $k$ is an integer and $1 \leq k \leq n$. Generally, for the flow with the speed of high powers of curvatures, it is required that the initial hypersurface is uniformly convex and satisfies a suitable pinching conditions, so as to preserve the uniformly convexity and converge to a sphere ([1], [6]).

Here we prove the same result without any pinching conditions for the flow at the speed $u^\alpha \sigma_k^\beta$. Moreover, for $\alpha \leq 1 - k\beta$, and $\beta > \frac{1}{k}$ we prove that the solution to the flow which is moving at the speed $fu^\alpha \sigma_k^\beta$ exists for all time and converges smoothly after normalisation to a soliton which is a solution of $fu^{\alpha - 1} \sigma_k^\beta = c$ if $f$ is a smooth positive function on $\mathbb{S}^n$ and satisfies the condition that $(\nabla_i \nabla_j f \frac{1}{\alpha + 1 - k\beta} + \delta_{ij} f \frac{1}{\alpha - k\beta})$ is positive definite. We have the same result for $k = n$ without any constraint on positive smooth function $f$, which recovers the result of Chou and Wang [11]. When $\beta = 1$, the flow has been studied by Ivaki recently [24]. In this case the self-similar solution of the flow $\partial_t u = fu^\alpha \sigma_k^\beta$ is the solution to $fu^{\alpha - 1} \sigma_k = c$ which is just the $L_p$ Christoffel-Minkowski problem for $p = 2 - \alpha$.

Let $M_0$ be a smooth, closed and uniformly convex hypersurface in $\mathbb{R}^{n+1}$, and $M_0$ encloses the origin. We study the following anisotropic expanding curvature flow

$$\begin{cases}
\frac{\partial X}{\partial t} (x, t) = f(\nu) < X, \nu >^\alpha \sigma_k^\beta (x, t) \nu, \\
X(x, 0) = X_0(x),
\end{cases}$$

(1)

where $\sigma_k$ is the $k$-th elementary symmetric function for principal curvature radii, i.e

$$\sigma_k (.., t) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k},$$

$\lambda_i$ is the principal curvature radii of hypersurface $M_t$, parametrized by $X(., t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$, and $\nu$ is the unit outer normal at $X(., t)$.

In this paper, we prove the following

**Theorem 1.1.** Let $M_0$ be a smooth, closed and uniformly convex hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, enclosing the origin. If $f \equiv 1$, $\alpha \in \mathbb{R}$, $\beta > \frac{1}{k}$ and $\alpha \leq 1 - k\beta$, $k$ is an integer and $1 \leq k \leq n$. Then the flow (1) has a unique smooth and uniformly convex solution $M_t$ for all time $t > 0$. After a proper rescaling $X \to \phi^{-1}(t) X$, where

$$\phi(t) = e^{\gamma t}, \quad \text{if} \quad \alpha = 1 - k\beta,$$

$$\phi(t) = (1+(1-k\beta-\alpha)\gamma t)^{\frac{1}{1-k\beta}}, \quad \text{if} \quad \alpha \neq 1-k\beta,$$

(2)

and

$$\gamma = \sigma_k^\beta (1, ..., 1)$$

the hypersurface $\tilde{M}_t = \phi^{-1} M_t$ converges exponentially to a sphere centered at the origin in the $C^\infty$ topology.
We denote by \( u = \langle X, \nu \rangle \) the support function of \( M_t \) at \( X \). When \( f \equiv 1 \), by a direct calculation, we obtain from the flow (1) that the support function \( u \) satisfies

\[
\begin{align*}
\frac{\partial u}{\partial t} (x,t) &= u^{\alpha} \sigma_k^\beta, \\
\left. u \right| (.,0) &= u_0.
\end{align*}
\]

Rescaling the hypersurface \( M_t \) in the way of Theorem 1.1, employing a new parameter \( \tau \), we get the normalised flow

\[
\begin{align*}
\frac{\partial u}{\partial \tau} (x,t) &= u^{\alpha} \sigma_k^\beta - \gamma u, \\
\left. u \right| (.,0) &= u_0,
\end{align*}
\]

where

\[
\gamma = \sigma_k^\beta (1, ..., 1),
\]

\[
\tau = \begin{cases} 
\frac{t}{\log((1 - \alpha - \beta k) \gamma t + 1)} & \text{if } \alpha = 1 - k \beta, \\
\frac{1 - \alpha - \beta k}{1 - \alpha - \beta k} & \text{if } \alpha \neq 1 - k \beta.
\end{cases}
\]

For general function \( f \), under the flow (1), the support function the support function \( u \) satisfies

\[
\begin{align*}
\frac{\partial u}{\partial t} (x,t) &= f u^{\alpha} \sigma_k^\beta, \\
\left. u \right| (.,0) &= u_0.
\end{align*}
\]

We define

\[
\bar{u} = \left( \frac{|S^n|}{V_{k+1}(u,u,\ldots,u)} \right)^\frac{1}{k+1} u,
\]

where the definition of \( V_{k+1}(u,u,\ldots,u) \) see Section 2. Considering the normalised flow of (5) given by

\[
\begin{align*}
\frac{\partial u}{\partial \tau} &= f u^{\alpha} \sigma_k^\beta - u \int_{S^n} f u^{\alpha-1} \sigma_k^\beta \, d\mu, \\
\left. u \right| (.,0) &= u_0.
\end{align*}
\]

where

\[
\tau = \int_0^t \left( \frac{|S^n|}{V_{k+1}(u,u,\ldots,u)} \right)^\frac{1-k \beta-n}{n+1} \, ds.
\]

In (6), we still use \( u \) instead of \( \bar{u} \). Consider the following functional which was introduced by Andrews in [5]

\[
J(u) = \int_{S^n} u \sigma_k (f u^{\alpha-1} \sigma_k^\beta)^{-\frac{1}{4}} \, d\mu.
\]

When \( \alpha \geq 1 - k \beta \), we will prove in Lemma 2.3 that \( J(u) \) is strictly decreasing along the flow (6) unless \( u \) solves the elliptic equation

\[
f u^{\alpha-1} \sigma_k^\beta = c,
\]

where \( c \) is a positive constant.

**Theorem 1.2.** Let \( M_0 \) be a smooth, closed and uniformly convex hypersurface in \( R^{n+1} \), \( n \geq 2 \), enclosing the origin. Suppose \( \alpha, \beta \in R^1 \), \( \beta > \frac{1}{k} \) and \( \alpha \leq 1 - k \beta \), \( k \) is an integer and \( 1 \leq k < n \), \( f \) is a smooth positive function on \( S^n \) and \( \nabla_i \nabla_j f \frac{1}{1+k \beta+n} + \delta_{ij} f \frac{1}{1+k \beta-n} \) is positive definite. Then the flow (1) has a unique smooth and uniformly convex solution \( M_t \) for all time \( t > 0 \). After normalisation,
the rescaled hypersurfaces \( \tilde{M}_t \) converge smoothly to a smooth solution of (8), which is a minimiser of the functional (7).

For \( k = n \), we obtain the same result without constraint on \( f \) as follows.

**Theorem 1.3.** Let \( M_0 \) be a smooth, closed and uniformly convex hypersurface in \( R^{n+1}, n \geq 2 \), enclosing the origin. Suppose \( \alpha \in R^1, \beta > \frac{1}{n} \) and \( \alpha \leq 1 - n\beta. \) Then for any \( f \) smooth positive function on \( S^n \), the flow (1) has a unique smooth and uniformly convex solution \( M_t \) for all time \( t > 0 \). After normalisation, the rescaled hypersurfaces \( \tilde{M}_t \) converge smoothly to a smooth solution of (8), which is a minimiser of the functional (7).

**Remark 1.** When \( \beta = 1 \), our second flow (1) is just the one that Ivaki has studied recently [24]. In [24], Ivaki employed the functional in [20] to prove that the flow \( \partial_t u = fu^{\alpha-1} \sigma_k \) has a unique smooth solution, and the rescaled flow converges smoothly to a homothetic self-similar solution which is a solution \( fu^{\alpha-1}\sigma_k = c \) for \( k < n, \alpha \leq 1 - k \) and positive function \( f \) satisfies \((\nabla_i \nabla_j f^{\frac{1}{p-1}} + \delta_{ij} f^{\frac{1}{p-1}}) \) is positive definite. When \( k = n \), he obtained the same result without imposing any condition on \( f \). It has been obtained by Chou and Wang in [11]. Before Ivaki [24], Hu et al. [21] proved the existence result for the \( L_p \) Christoffel Minkowski problem for \( p = 2 - \alpha: fu^{\alpha-1} \sigma_k = 1, \alpha \leq 1 - k, 1 \leq k \leq n - 1 \) for any positive function \( f \) satisfying \((\nabla_i \nabla_j f^{\frac{1}{p-1}} + \delta_{ij} f^{\frac{1}{p-1}}) \geq 0 \). Under the condition that \( f \) is positive even function and satisfies \((\nabla_i \nabla_j f^{\frac{1}{p-1}} + \delta_{ij} f^{\frac{1}{p-1}}) \geq 0 \), Guan and Xia in [20] obtained the existence result for \( 1 - k < \alpha < 1 \).

We still denote by \( u = \langle X, \nu \rangle \) the support function of \( M_t \) at \( X \). We show the condition \( \alpha \leq 1 - k \beta \) is necessary. In fact, by use of the method of [26, 27], we show

**Theorem 1.4.** Suppose \( \alpha > 1 - k \beta, \alpha, \beta \in \mathbb{R} \) and \( \beta > 0 \), \( k \) is an integer and \( 1 \leq k \leq n \). There exist a smooth, closed, uniformly convex hypersurface \( M_0 \), such that under the flow (1),

\[
R(X(\cdot, t)) := \frac{\max_{S^n} u(\cdot, t)}{\min_{S^n} u(\cdot, t)} \rightarrow \infty, \quad \text{as} \quad t \rightarrow T
\]

for some \( T > 0 \).

This paper is organised as follows. In Section 2, we recall some properties of convex hypersurfaces and show that the functional (7) is strictly decreasing along the normalised flow (6) unless \( u \) satisfies the elliptic equation (8). In Section 3, we establish the a priori estimates, which ensure the long time existence of the normalized flows. In Section 4, we show that the flow (1) converges to the unit sphere (i.e. Theorem 1.1) and complete the proofs of Theorems 1.2 and 1.3. Finally, in Section 5, we prove Theorem 1.4.

2. Preliminary. We recall some basic notations at first. Let \( M \) be a smooth, closed, uniformly convex hypersurface in \( R^{n+1} \). Assume that \( M \) is parametrized by the inverse Gauss map \( X : S^n \rightarrow M \subset R^{n+1} \) and encloses origin. The support function \( u : S^n \rightarrow R^1 \) of \( M \) is defined by

\[
u(x) = \sup_{y \in M} \langle x, y \rangle.
\]

The supremum is attained at a point \( y = X(x) \), \( x \) is the outer normal of \( M \) at \( y \). Hence

\[
u(x) = \langle x, X(x) \rangle.
\]
Let $e_1, \cdots, e_n$ be a smooth local orthonormal frame field on $S^n$, and $\nabla$ the covariant derivative with respect to the standard metric $e_i$ on $S^n$. Denote by $g_{ij}$, $g^{ij}$, $h_{ij}$ the metric, the inverse of the metric and the second fundamental form of $M$, respectively. Then the second fundamental form of $M$ is given by (see e.g. [29])

$$h_{ij} = \nabla_i \nabla_j u + u e_{ij},$$

and $h_{ij}$ is symmetric and satisfies the Codazzi equation

$$\nabla_i h_{jk} = \nabla_j h_{ik}.$$

By the Gauss-Weingarten formula

$$\nabla_i x = h_{jk} g^{kl} \nabla_l X,$$

we get

$$e_{ij} = \langle \nabla_i x, \nabla_j x \rangle = h_{ik} g^{kl} h_{jm} g^{ms} \langle \nabla_l X, \nabla_s X \rangle = g^{kl} h_{ik} h_{jl}.$$

Since $M$ is uniformly convex, $h_{ij}$ is invertible. Hence the principal curvature radii are the eigenvalues of the matrix

$$b_{ij} = h^{ik} g_{jk} = \nabla^2 u + u \delta_{ij}.$$

Let $\sigma_k(A)$ be the $k$-th elementary symmetric function defined on the set $\mathcal{M}_n$ of $n \times n$ matrices and $\sigma_k(A_1, \cdots, A_k)$ be the complete polarization of $\sigma_k$ for $A_i \in \mathcal{M}_n$, $i = 1, \cdots, k$, i.e.

$$\sigma_k(A_1, \cdots, A_k) = \frac{1}{k!} \sum_{i_1, \cdots, i_k = 1}^n \delta_{i_1, \cdots, i_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}.$$

Let $\Gamma_k$ be Garding’s cone

$$\Gamma_k = \{ A \in \mathcal{M}_n : \sigma_i(A) > 0, i = 1, \cdots, k \}.$$

For a function $u \in C^2(S^n)$, we denote by $W_u$ the matrix

$$W_u := \nabla^2 u + u I.$$

In the case $W_u$ is positive definite, the eigenvalue of $W_u$ is the principal radii of a strictly convex hypersurface with support function $u$. Let $u^i \in C^2(S^n)$, $i = 1, \cdots, n+1$. Set

$$V(u^1, u^2, \cdots, u^{n+1}) := \int_{S^n} u^1 \sigma_n[W_u, \cdots, W_{u^{n+1}}]d\mu,$$

$$V_{k+1}(u^1, u^2, \cdots, u^{k+1}) := V(u^1, u^2, \cdots, u^{k+1}, 1, \cdots, 1).$$

Therefore we define the $k + 1$-th volume by

$$V_{k+1}(u, \cdots, u) := \int_{S^n} u \sigma_k[W_u, \cdots, W_u]d\mu.$$

Next, we state the well-known Alexandrov-Fenchel inequality.

**Lemma 2.1.** ([28]) Let $u^i \in C^2(S^n)$, $i = 1, 2, \cdots, k$ be such that $u^i > 0$ and $W_{u^i} \in \Gamma_k$ for $i = 1, 2, \cdots, k$. Then for any $v \in C^2(S^n)$, the Alexandrov-Fenchel inequality holds:

$$V_{k+1}(u^1, \cdots, u^k)^2 \geq V_{k+1}(v, v, v^2, \cdots, v^k)V_{k+1}(u^1, u^1, u^2, \cdots, u^k),$$

the equality holds if and only if $v = au^1 + \sum_{l=1}^{n+1} a_l x_l$ for some constants $a, a_1, \cdots, a_{n+1}$. 


We consider the flow (6). For convenience we still use \( t \) instead of \( \tau \) to denote the time variable if no confusions arise, and we set
\[
\rho = f u^{\alpha - 1} \sigma_k^\beta, \quad \sigma[f] = \sigma_k[W_f, W_u, \ldots, W_u], \quad \eta(t) = \int_{S^n} f u^{\alpha} \sigma_k^{1 + \beta} \, d\mu = Z_1 |S^n|
\]
and
\[
Z_p(u) = \int_{S^n} u \sigma_k(f u^{\alpha - 1} \sigma_k^\beta)^p \, d\mu = \int_{S^n} u \sigma_k \rho^p \, d\mu,
\]
where \( p \in R^1 \). We mention the fact that \( \int_{S^n} u \sigma_k \, d\mu = |S^n| \) here which comes from the scaling of \( \tilde{u} \). Hence (6) can be written as \( \frac{\partial u}{\partial t} = \rho u - u \frac{Z_1}{|S^n|} \). By a direct calculation, we have
\[
\frac{d}{dt} Z_p(u) = \int_{S^n} (\rho u - u \frac{Z_1}{|S^n|}) \sigma_k \rho^p \, d\mu + \int_{S^n} k u \rho \sigma_k(\rho u - u \frac{Z_1}{|S^n|}) \rho^p \, d\mu
\]
\[
+ \int_{S^n} p \rho^{p-1} u \sigma_k((\alpha - 1)f u^{\alpha - 2}(\rho u - u \frac{Z_1}{|S^n|})) \sigma_k^\beta + k \beta \rho u^{\alpha - 1} \sigma_k^{\beta - 1} \sigma(\rho u - u \frac{Z_1}{|S^n|}) \, d\mu
\]
\[
= Z_{1+p} - Z_1 \frac{Z_p}{|S^n|} + k \int_{S^n} u \sigma_k(\rho u \sigma_k)^p \, d\mu - k \frac{Z_1 Z_p}{|S^n|}
\]
\[
+ p(\alpha - 1) Z_{1+p} - p(\alpha - 1) \frac{Z_1 Z_p^p}{|S^n|} + k \beta p \int_{S^n} u \sigma_k(\rho u \sigma_k)^p \, d\mu - k \frac{Z_1 Z_p}{|S^n|}
\]
\[
= (1 + p(\alpha - 1)) \left( Z_{1+p} - Z_1 \frac{Z_p}{|S^n|} \right) + k(1 + \beta p) \left( \int_{S^n} u \rho^p \sigma_k(\rho u \sigma_k) \, d\mu - \frac{Z_1 Z_p}{|S^n|} \right).
\]

**Lemma 2.2.** Suppose \( \alpha \leq 1 - k \beta, \beta > \frac{1}{k} \), \( \eta(t) \) is uniformly bounded.

**Proof.** Since \( \alpha \leq 1 - k \beta, \beta > \frac{1}{k} \), let \( p = 1 \), we have
\[
\frac{d}{dt} Z_1 = (1 + (\alpha - 1)) \left( Z_2 - \frac{Z_1 Z_1}{|S^n|} \right) + k(1 + \beta) \left( \int_{S^n} u \rho \sigma_k(\rho u \sigma_k) \, d\mu - \frac{Z_1 Z_1}{|S^n|} \right).
\]
Since \( h_{ij} \) satisfies the Codazzi equations, we have \( \sum_i \nabla_i \sigma^{ij} = 0 \) (see [2],[5]), and
\[
\int_{S^n} u \rho \sigma_k[W_{\rho u}, W_u, \ldots, W_u] \, d\mu = \int_{S^n} u \rho \sigma_k \, d\mu
\]
\[
= \int_{S^n} u \rho \sigma_k \nabla_i \nabla_j (u \rho + \delta_{ij} u \rho) \, d\mu
\]
\[
= \int_{S^n} u \rho \sigma_k \nabla_i (h_{ij} \rho + 2 \nabla_i u \nabla_j \rho + u \nabla_i \nabla_j \rho) \, d\mu
\]
\[
= Z_2 - \int_{S^n} u^2 \sigma^{ij} \nabla_i \rho \nabla_j \rho \, d\mu.
\]
By the Alexandrov-Fenchel inequality in Lemma 2.1, we have
\[
\left( \int_{S^n} u \psi \sigma[u] \, d\mu \right)^2 \geq \int_{S^n} u \sigma[u] \, d\mu \int_{S^n} u \psi \sigma[u \psi] \, d\mu
\]
\[
= |S^n| \left( Z_2 - \int_{S^n} u^2 \sigma^{ij} \nabla_i \psi \nabla_j \psi \, d\mu \right).
\]
Set \( \psi = \rho \), the above inequality shows
\[
\int_{S^n} u^2 \sigma^{ij} \nabla_i \rho \nabla_j \rho \, d\mu - Z_2 + \frac{Z_2^2}{|S^n|} \geq 0.
\]
Hence
\[
\frac{d}{dt} Z_1 = \alpha (Z_2 - \frac{Z_2^2}{|S^n|}) - k(1 + \beta) \left( \int_{S^n} u^2 \sigma^2 \nabla_i \rho \nabla_j \rho d\mu - Z_2 + \frac{Z_2^2}{|S^n|} \right) \leq 0,
\]
where \( \alpha \leq 1 - k\beta < 0 \) and the Hölder inequality shows that \( Z_2 \geq \frac{Z_2^2}{|S^n|} \).
\[
\partial_t \eta = \alpha (Z_2 - \frac{Z_2^2}{|S^n|}) - k(1 + \beta) \left( \int_{S^n} u^2 \sigma^2 \nabla_i \rho \nabla_j \rho d\mu - Z_2 + \frac{Z_2^2}{|S^n|} \right) \leq 0.
\]
Hence we obtain the uniform upper bound on \( \eta(t) \). Next we prove the uniform lower bound. Set \( \theta \leq -\frac{1}{\beta} < 0 \), \( \alpha \leq 1 - k\beta \), we have
\[
\frac{d}{dt} Z_\theta(u) = (1 + (\alpha - 1)\theta) \left( Z_{1+\theta} - \frac{Z_1 Z_\theta}{|S^n|} \right) + k(1 + \beta \theta) \left( \int_{S^n} u^2 \rho \sigma [\rho u] d\mu - Z_1 Z_\theta \right)
\]
\[
= [1 + (\alpha - 1)\theta] \left( Z_{\theta+1} - \frac{Z_1 Z_\theta}{|S^n|} \right)
\]
\[
+ k(1 + \theta \beta) \left( Z_{1+\theta} - \frac{4\theta}{(1 + \theta)^2} \int_{S^n} u^2 \sigma^2 \nabla_i (\rho^{\frac{1+\beta}{2}}) \nabla_j (\rho^{\frac{1+\beta}{2}}) - Z_1 Z_\theta \right).
\]
Set \( \psi = \rho^{\frac{1+\beta}{2}} \) in the Alexandrov-Fenchel inequality (10), we obtain
\[
\int_{S^n} u^2 \sigma^2 \nabla_i \rho^{\frac{1+\beta}{2}} \nabla_j \rho^{\frac{1+\beta}{2}} d\mu - Z_{1+\theta} + \frac{Z_1^2 + \theta}{|S^n|} \geq 0.
\]
Thus
\[
\frac{d}{dt} Z_\theta(u) = [1 + (\alpha - 1)\theta + k(1 + \theta \beta)] \left( Z_{\theta+1} - \frac{Z_1 Z_\theta}{|S^n|} \right)
\]
\[
- \frac{4\theta k(1 + \theta \beta)}{(1 + \theta)^2} \left( Z_{1+\theta} - \frac{Z_1^2 + \theta}{|S^n|} \right)
\]
\[
- \frac{4\theta k(1 + \theta \beta)}{(1 + \theta)^2} \left( \int_{S^n} u^2 \sigma^2 \nabla_i \rho^{\frac{1+\beta}{2}} \nabla_j \rho^{\frac{1+\beta}{2}} d\mu - Z_{1+\theta} + \frac{Z_1^2 + \theta}{|S^n|} \right)
\]
\[
\leq 0
\]
since \( \theta < -\frac{1}{\beta} \), and by the Hölder inequality, we obtain \( Z_{1+\theta} \leq \frac{Z_1 Z_\theta}{|S^n|} \) and \( Z_{1+\theta} \geq \frac{Z_1^2 + \theta}{|S^n|} \). Hence, \( Z_\theta(u) \leq Z_\theta(u_0) \). By the Hölder inequality again, we have
\[
|S^n| = \int_{S^n} u^2 \sigma_k d\mu \leq \left( \int_{S^n} f u^2 \sigma_k^{1+\beta} d\mu \right)^{\frac{\gamma}{1+\gamma}} \left( \int_{S^n} u \sigma_k (f u^{\alpha-1} \sigma_k^{\beta}) d\mu \right)^{\frac{1}{\gamma}}
\]
\[
= Z_1^{\frac{\gamma}{1+\gamma}} Z_\theta^{\frac{1}{1+\gamma}}.
\]
Therefore we get the uniform bound on \( \eta(t) \). \( \square \)

**Lemma 2.3.** The functional (7) is non-increasing along the normalised flow (6), and the equality holds if and only if \( M_1 \) satisfies the elliptic equation (8).
Hence Lemma 3.2.

Let A priori estimates.

Proof. We have along the normalised flow (6)

\[
\frac{d}{dt} J(u) = \frac{d}{dt} Z_{-\frac{1}{\beta}}(u) = \frac{1 - \alpha + \beta}{\beta} \left( Z_{1 - \frac{1}{\beta}} - \frac{Z_1 Z_{-\frac{1}{\beta}}}{|S^a|} \right) \leq 0.
\]

The last inequality holds from the Hölder inequality, and the equality holds if and only if \(fu^{\alpha-1}\sigma_k^\beta = c(t)\), where \(c(t)\) is a positive function on \(t\). Indeed, if \(fu^{\alpha-1}\sigma_k^\beta = c(t)\) occurs, then

\[
\eta(t) = \frac{\int_{S^n} fu^{\alpha} \sigma_k^{1+\beta} dx}{|S^a|} = \frac{\int_{S^n} u c(t) dx}{|S^a|} = c(t).
\]

\[\square\]

3. A priori estimates. We first show the \(C^0\)-estimate of the solution to (4).

Lemma 3.1. Let \(u(., t), t \in [0, T]\), be a smooth, uniformly convex solution to (4). If \(\alpha \leq 1 - k\beta\) and \(\beta > \frac{1}{k}\), then there is a positive constant \(C\) depending only on \(\alpha, \beta\) and the lower and upper bounds of \(u(., 0)\) such that

\[
\frac{1}{C} \leq u(., t) \leq C.
\]

Proof. Let \(u_{\min}(t) = \min_{x \in S^n} u(., t) = u(x_t, t)\). For fixed time \(t\), at the point \(x_t\), we have

\[
\nabla_i u = 0 \quad \text{and} \quad \nabla_{ij} u \geq 0,
\]

then

\[
\frac{du_{\min}}{dt} \geq \gamma u_{\min}(u_{\min} + k\beta - 1).
\]

Hence \(u_{\min} \geq \min\{1, u_{\min}(0)\}\). Similarly, we have \(u_{\max} \leq \max\{1, u_{\max}(0)\}\). \(\square\)

Lemma 3.2. Let \(u \geq 2, 1 \leq k \leq n, \alpha \leq 1 - k\beta, \beta > \frac{1}{k}\), and \(X(., t)\) be the solution to the normalised flow (6) which encloses the origin for \(t \in [0, T]\). Then there is a positive constant \(C\) depending on the initial hypersurface and \(f, \alpha, \beta\), such that

\[
C^{-1} \leq u^{\alpha-1}\sigma_k^\beta \leq C.
\]

Proof. Consider the auxiliary function

\[
Q = fu^{\alpha-1}\sigma_k^\beta.
\]

Since

\[
(fu^{\alpha}\sigma_k^\beta)_{ij} = Q_{ij} + Q_i u_j + Q_j u_i + Q_{ij},
\]

we obtain

\[
\partial_t Q = (\alpha - 1)f^2 u^{2\alpha-2}\sigma_k^{2\beta} - \eta(\alpha - 1 + k\beta)fu^{\alpha-1}\sigma_k^\beta + \beta fu^{\alpha-1}\sigma_k^{2\beta-1} \sum_{i} \sigma_k^{ij} + \beta fu^{\alpha-1}\sigma_k^{2\beta-1} \sigma_k^{ij} (fu^{\alpha}\sigma_k^\beta)_{ij}
\]

\[
= (\alpha + k\beta - 1)Q^2 - \eta(\alpha + k\beta - 1)Q + \beta fu^{\alpha}\sigma_k^{\beta-1} \sigma_k^{ij} Q_{ij}
\]

\[
+ 2\beta fu^{\alpha-1}\sigma_k^{\beta-1} \sigma_k^{ij} Q_i u_j.
\]

If \(\alpha + k\beta - 1 \leq 0\), the sign of the coefficient of the highest order term \(Q^2\) is negative. The sign of the coefficient of the lower order term \(Q\) is positive. So it is easy to see \(C^{-1} \leq Q \leq C\), where \(C\) is the positive constant depending on \(f, \alpha, \beta, \min_{S^n \times [0, T]} u\) and \(\max_{S^n \times [0, T]} u\). \(\square\)
When $f = 1$, by use of (11), we can get $C^{-1} \leq Q \leq C$ for the flow (4). Then by Lemma 3.1, we have $C^{-1} \leq \sigma_k \leq C$. That is,

**Corollary 1.** Let $n \geq 2$, $1 \leq k \leq n$, $\alpha \leq 1 - k\beta$, $\beta > \frac{1}{k}$ and $X(\cdot,t)$ be the solution to the normalised flow (4) which encloses the origin for $t \in [0,T)$. Then there is a positive constant $C$ depending on the initial hypersurface and $\alpha, \beta$, such that

$$C^{-1} \leq \sigma_k \leq C.$$

**Lemma 3.3.** Let $n \geq 2$, $1 \leq k \leq n$, $\alpha \leq 1 - k\beta$, $\beta > \frac{1}{k}$ and $X(\cdot,t)$ be the solution to the normalised flow (6) which encloses the origin for $t \in [0,T)$. Then there is a positive constant $C$ depending on the initial hypersurface and $f, \alpha, \beta$, such that $|\nabla \log u| \leq C$.

**Proof.** Let $w = \log u$. Then we have

$$h_{ij} = w_{ij} + u \delta_{ij} = u (w_{ij} + w_i w_j + \delta_{ij}).$$

Assume the auxiliary function $|\nabla u|^2 - Au^2 < 0$, for a positive constant $A > 0$ along the flow. Otherwise there is a point $(x_{0},t_{0})$ where $t_{0}$ is the first time, such that $|\nabla u|^2 - Au^2 = 0$, $A > 0$ is a constant to be decided later. Hence at the point $(x_{0},t_{0})$, $\nabla_i |\nabla w|^2 = 0$. Choosing an orthonormal frame and rotating the coordinate, such that $w_1 = |\nabla w|, w_i = 0$ for $i = 2, \ldots, n$, and $(w_{ij})$ is diagonal. Then we get

$$(a_{ij}) := (w_{ij} + w_i w_j + \delta_{ij}) = \text{diag}(1 + w_1^2, 1 + w_{22}, \ldots, 1 + w_{nn}),$$

and

$$0 \leq \partial_t (|\nabla u|^2 - Au^2) = 2u_i (fu^\alpha)_{\sigma_k} + 2\beta fu^\alpha \sigma_k^{\beta - 1} \sigma_k^{\alpha n} \nabla_i h_{nn} u_i - 2Afu^\alpha + \frac{1}{2}.$$
Lemma 3.4. Let \( u(\cdot, t), \ t \in [0, T) \), be a smooth, uniformly convex solution to (6). If \( 1 \leq k \leq n, \ \alpha \leq 1 - k\beta, \ \beta > \frac{1}{k} \), then there is a positive constant \( C_1 \) and \( C_2 \) depending only on \( f, \alpha, \beta \) and the lower and upper bounds of \( u(\cdot, 0) \) such that
\[
\frac{1}{C_1} \leq u(\cdot, t) \leq C_1, \quad \frac{1}{C_2} \leq \sigma_k \leq C_2.
\]

Proof. Since for the normalised flow (6), \( \int_{S^n} u \sigma_k d\mu = |S^n| \) is constant. From Lemma 3.2, there is a positive constant \( C \), such that \( C^{-1} \leq u^{\sigma_k - 1} \sigma_k^\beta \leq C \). Hence we have
\[
C^{-1} |S^n|^{-\frac{1-a+\beta}{n}} \leq \int_{S^n} u \sigma_k d\mu \leq C |S^n|^{-\frac{1-a+\beta}{n}}.
\]
Hence we obtain the uniform lower and upper bounds on \( u \) from Lemma 3.3. Then by Lemma 3.2, we get the uniform lower and upper bounds on \( \sigma_k \).  \( \square \)

Now we are going to estimate the upper and lower bounds of the principle curvature radii of the hypersurface \( M_t \). We rewrite the equation (6) in the following form
\[
\frac{\partial u}{\partial t} = fu^\alpha \sigma_k^\beta (u) - \eta u = \Phi L - \eta u,
\]
where \( L = F^\beta k, \ \Phi = fu^\alpha, \ \text{and} \ h_{ij} = u_{ij} + \omega_{ij} \).

Lemma 3.5. Let \( 1 \leq k < n, \ \alpha, \beta \in R^1, \ \beta > \frac{1}{k} \) and \( \alpha \leq 1 - k\beta \), \( X(\cdot, t) \) be the solution to the normalised flow (6) for \( t \in [0, T) \), which encloses the origin. Assume \( f \) is a smooth positive function on \( S^n \) and \( (\nabla_i \nabla_j f \frac{1+\beta}{1+\beta}) + \delta_{ij} f \frac{1+\beta}{1+\beta} \) is positive definite. Then there is a constant \( C \) depending only on \( f, \alpha, \beta, \min_{S^n \times [0,T]} u \) and \( \max_{S^n \times [0,T]} u \), such that the principal curvature radii of \( X(\cdot, t) \) are bounded from above and below
\[
\frac{1}{C} \leq \lambda_i(\cdot, t) \leq C
\]
for all \( t \in [0, T) \) and \( i = 1, \ldots, n \).

Proof. Suppose the maximum eigenvalue of the matrix \( \frac{\partial h_{ij}}{\partial u} \) at time \( t \) is attained at the point \( x_t \) with unit eigenvector \( \xi_t \in T_{x_t} S^n \). By rotating the frame \( e_1, \ldots, e_n \) at \( x_t \), assume that at \( x_t \) we have \( \xi_t = e_1 \). At \( (x_t, t) \), we have
\[
\begin{align*}
\frac{\partial h_{ij}}{\partial t} & = \Phi L'_{kl} \nabla_k \nabla_l h_{ij} - \Phi (h_{ii})^2 (2L'_{kl} h_{kl}) + \Phi (\text{tr}L') h_{ij}^1 + \eta h_{ij}^1 \\
& - \Phi (h_{ii})^1 (2L'_{km} h_{kl}) \nabla_k \nabla_l h_{mn} \\
& - (h_{ii})^1 (2\nabla_1 \Phi \nabla_1 L + L \nabla_1 \nabla_1 \Phi).
\end{align*}
\]
Since \( F = \sigma_k^\frac{1}{k} \) is concave and homogeneous of degree one, from [29]
\[
F_{ij} h_{ij} = F, \tag{12}
\]
\[
(2F_{km} h_{nl} + F_{kl, mn}) \nabla_1 h_{kl} \nabla_1 h_{mn} \geq 2F^{-1} F_{kl} F_{mn} \nabla_1 h_{kl} \nabla_1 h_{mn}, \tag{13}
\]
we have
\[
\begin{align*}
\frac{\partial h_{ij}}{\partial t} & = \Phi k \beta F^{k\beta - 1} F_{kl} \nabla_k \nabla_l h_{ij} - \Phi (h_{ii})^2 (F^{k\beta} + k \beta F^{k\beta - 1} F_{kl} h_{kl}) \\
& + \Phi k \beta F^{k\beta - 1} \sum_i F_{ki} h_{ii} - \Phi (h_{ii})^2 \nabla_1 h_{kl} \nabla_1 h_{mn} \left(2k \beta F^{k\beta - 1} F_{km} h_{ml} \right)
\end{align*}
\]
For $\beta > 0$, that is, $h > 0$ since $A \subset (1 + k\beta)(h^{11})^2 + \Phi k\beta F^{k\beta - 1} \sum_i F_i h^{11}$

we have

$$
\partial_t h^{11} \leq \Phi k\beta (h^{11})^2 + \frac{k\beta}{k\beta + 1} F^{k\beta} \frac{(\nabla_1 \Phi)^2}{\Phi} (h^{11})^2 - F^{k\beta} h^{11} \sum_i F_i h^{11}
$$

Since $\nabla_i h_{\alpha} = 0$ and $\nabla_{ij} h_{\alpha} = 0$,

$$
\partial_t h^{11} \leq -\Phi (1 + k\beta) u F^{k\beta} \frac{h^{11}}{u} + \frac{k\beta}{k\beta + 1} u F^{k\beta} \frac{(\nabla_1 \Phi)^2}{\Phi} \frac{h^{11}}{u} \frac{u}{u}
$$

Since $(f_{\alpha}^{\frac{1}{k\beta - \alpha}})_{11} + f_{\alpha}^{\frac{1}{k\beta - \alpha}} > 0$, substituting it into the above inequality, we have

$$
\partial_t h^{11} \leq u F^{k\beta} \frac{h^{11}}{u} \left( - \frac{f^{11}}{f} + \frac{k\beta - \alpha}{1 + k\beta - \alpha} \frac{f_1}{f} - (1 + k\beta - \alpha) \right)
$$

For $\beta > \frac{1}{k}$ and $\alpha \leq 1 - k\beta < 0$, we get $k\beta - 1 - \alpha \geq 2(k\beta - 1) > 0$. Hence

$$
\partial_t h^{11} \leq -C_1 \left( \frac{h^{11}}{u} \right)^2 + C_2 \frac{h^{11}}{u}
$$

that is, $h^{11} \leq C$, where $C$ depends on the initial hypersurface, the minimum eigenvalue of $(\nabla_1 \nabla_j f_{\alpha}^{\frac{1}{k\beta - \alpha}} + \partial_j f_{\alpha}^{\frac{1}{k\beta - \alpha}})$, $f$, $\alpha$, and $\beta$. Now together with Lemma 3.4, we get

$$
C^{-1} \leq \lambda_i \leq C.
$$

We therefore complete the proof.
Lemma 3.6. Let \( 1 \leq k \leq n, \alpha, \beta \in R^1, \beta > \frac{1}{k} \) and \( \alpha \leq 1 - k\beta \). Let \( X(\cdot, t) \) be the solution to the normalised flow (4) for \( t \in [0, T) \), which encloses the origin. Then there is a constant \( C \) depending only on the initial hypersurface and \( \alpha, \beta \), such that the principal curvature radii of \( X(\cdot, t) \) are bounded from above and below

\[
\frac{1}{C} \leq \lambda_i(\cdot, t) \leq C
\]

for all \( t \in [0, T) \) and \( i = 1, \ldots, n \).

Proof. We prove the lemma just as Lemma 3.5. Suppose the maximum eigenvalue of the matrix \([h^{ij}]\) at time \( t \) is attained at the point \( x_t \) with unit eigenvector \( \xi_t \in T_{x_t}S^n \). By rotating the frame \( e_1, \ldots, e_n \) at \( x_t \), assume that at \( x_t \) we have \( \xi_t = e_1 \). At \((x_t, t)\), we have

\[
\partial_th^{11} = \Phi_k\beta F^{k\beta-1}F_{kl} \nabla_k \nabla_l h^{11} - \Phi(h^{11})^2(F^{k\beta} + k\beta F^{k\beta-1}F_{kl}h_{kl})
\]

\[
+ \Phi_k\beta F^{k\beta-1}\sum_{i} F_{ii} h_{i1} - \Phi(h^{11})^2 \left(2k\beta F^{k\beta-1}F_{km} h_{ml}\right)
\]

\[
+ k\beta(k\beta - 1)F^{k\beta-2}F_{mn}F_{kl} + k\beta F^{k\beta-1}F_{kl,mn} \nabla_l h_{kl} \nabla_m h_{mn}
\]

\[
- (h^{11})^2(2\nabla_1 \Phi_k\beta F^{k\beta-1}F_{11} F + F^{k\beta} \nabla_1 \Phi) + \gamma h^{11}
\]

\[
\leq - \Phi(h^{11})^2F^{k\beta} + \gamma h^{11} - \Phi(h^{11})^2k\beta(k\beta + 1)F^{k\beta-2}(\nabla_1 F)^2
\]

\[
- (h^{11})^2(2k\beta F^{k\beta-1}F_{11} F + F^{k\beta} \nabla_1 \Phi).
\]

Since

\[
2k\beta F^{k\beta-1}F_{11} F \leq \Phi_k\beta(k\beta + 1)F^{k\beta} \left(\frac{\nabla_1 F}{F}\right)^2 + \frac{k\beta}{k\beta + 1}F^{k\beta}(\nabla_1 \Phi)^2,
\]

we have

\[
\partial_th^{11} \leq \Phi_k\beta(k\beta + 1)F^{k\beta} \left(\frac{\nabla_1 F}{F}\right)^2 + \frac{k\beta}{k\beta + 1}F^{k\beta}(\nabla_1 \Phi)^2,
\]

\[
\leq (\alpha - 1)u^n \sigma_k^n(h^{11})^2 + (\gamma - \alpha u^{n-1} \sigma_k^n)h^{11}
\]

\[
+ \frac{\alpha(1 + k\beta - \alpha)}{k\beta + 1}u^{n-2}(\nabla_1 u)^2 \sigma_k^n(h^{11})^2.
\]

Since \( \beta > \frac{1}{k} \), \( \alpha \leq 1 - k\beta < 0 \), by (15) we have

\[
\partial_th^{11} \leq -C_0(h^{11})^2 + C_1 h^{11}
\]

where \( C_0 \) and \( C_1 \) are two positive constants which depend only on the initial hypersurface and \( \alpha, \beta \). Therefore we have

\[
h^{11} \leq \max\{C_3, \max h^{11}(0)\}
\]

where \( C_3 \) is also a positive constants depending only on the initial hypersurface and \( \alpha, \beta \). Now together with Lemma 1, we get

\[
C^{-1} \leq \lambda_i \leq C.
\]

\[ \square \]

Now we show Lemma 3.5 holds for any positive smooth function \( f \) when \( k = n, \beta > \frac{1}{n} \) and \( \alpha \leq 1 - n\beta \).

Lemma 3.7. Let \( n \geq 2, k = n \). If \( \alpha \leq 1 - n\beta, \beta > \frac{1}{n} \) and \( X(\cdot, t) \) be the solution to the normalised flow (6) which encloses the origin for \( t \in [0, T) \). Then there is
a constant $C$ depending only on the initial hypersurface and $f$, $\alpha$, $\beta$, such that the principal curvature radii of $X(\cdot, t)$ are bounded from above and below

$$C^{-1} \leq \lambda_i(\cdot, t) \leq C$$

for all $t \in [0, T)$ and $i = 1, \ldots, n$.

Proof. Consider the auxiliary function

$$w(\xi, t, \tau) = \log h^{\tau} - \varepsilon \log u + \frac{M}{2} (u^2 + |\nabla u|^2),$$

where $\tau$ is a unit vector in the tangential space of $S^n$, while $\varepsilon$ and $M$ are large constants to be decided. Assume $w$ achieves its maximum at $(x_0, t_0)$ in the direction $\tau = (1, 0, \cdots, 0)$. By a coordinate rotation, $h_{ij}$ and $h^{ij}$ are diagonal at this point. Then at the point $(x_0, t_0)$,

$$w = \log h^{11} - \varepsilon \log u + \frac{M}{2} (u^2 + |\nabla u|^2),$$

$$0 = \nabla_i w = -h^{11} \nabla_i h_{11} - \varepsilon \frac{\nabla_i u}{u} + Mu_i + M \nabla_k u \nabla_{ki} u,$$

$$0 \geq \nabla_{ij} w = -h^{11} \nabla_{ij} h_{11} + 2h^{11} h^{kk} \nabla_i h_{1k} \nabla_j h_{k1} - (h^{11})^2 \nabla_i h_{11} \nabla_j h_{11}$$

$$- \varepsilon \frac{\nabla_{ij} u}{u} + \frac{\nabla_i u \nabla_j u}{u^2} + Mu_i u_j + Mu_{ij}$$

$$+ M \nabla_{ki} u \nabla_{kj} u + M \nabla_{kk} u \nabla_{kj} u. \quad (17)$$

Set $\Phi = fu^\alpha$, we have

$$0 \leq \partial_t w = -h^{11} \partial_t h_{11} - \varepsilon \frac{\partial_t u}{u} + Mu \nabla_t u + M \nabla_t u \nabla_{kt}$$

$$= -h^{11} \left( \Phi_{11} \sigma_n - 2\beta \sigma_n \beta - 1 \nabla_1 \Phi \nabla_1 \sigma_n + \beta(\beta - 1) \Phi \sigma_n \beta - 2 (\nabla_1 \sigma_n)^2 \right)$$

$$+ \beta \Phi \sigma_n \beta - 1 \nabla_1 \sigma_n \sigma_n - \eta h_{11} + \Phi \sigma_n \beta - \varepsilon \frac{\Phi \sigma_n \beta - \eta u}{u}$$

$$+ Mu(\Phi \sigma_n \beta - \eta u) + Mu_k(\Phi \sigma_n \beta - \eta u)_k$$

$$= -h^{11} \left( \Phi_{11} \sigma_n \beta - 2\beta \sigma_n \beta \nabla_1 \Phi \nabla_1 h_{ij} + \beta(\beta - 1) \Phi \sigma_n \beta (h^{ij} \nabla_1 h_{ij})^2 \right)$$

$$+ \beta \Phi \sigma_n \beta \left( \sum_i h^{ii} - \varepsilon h^{ij} \nabla_i h_{ij} + \eta h_{11} \sum_i h^{ii} - \varepsilon h^{ij} \nabla_i h_{ij} + \eta h_{11} \sum_i h^{ii} - \varepsilon h^{ij} \nabla_i h_{ij} \right)$$

$$- \eta h_{11} + \Phi \sigma_n \beta$$

$$- \varepsilon \frac{\Phi \sigma_n \beta - \eta u}{u} + Mu(\Phi \sigma_n \beta - \eta u) + Mu_k(\Phi \sigma_n \beta - \eta u)_k$$

By (18) and multiplying $\Phi^{-1} \sigma_n^{-\beta}$ the two sides of the above inequality, we obtain

$$0 \leq -h^{11} \frac{\nabla_1 \Phi}{\Phi} + h^{11} \left( (\frac{\nabla_1 \Phi}{\Phi})^2 + \beta^2 (h^{ij} \nabla_1 h_{ij})^2 \right) - h^{11} \beta^2 (h^{ij} \nabla_1 h_{ij})^2$$

$$+ \beta \left( -2h^{11} h^{ij} h^{kk} \nabla_i h_{1k} \nabla_j h_{1k} + (h^{11})^2 h^{ij} \nabla_i h_{11} \nabla_j h_{11} + n \varepsilon u^{-1} \right)$$

$$- \varepsilon \sum_i h^{ii} - \varepsilon h^{ij} u \cdot u \cdot u^{-2} - Mu_{ij} - M h^{ij} \nabla_i h_{1k} h_{1k} + n M u - Mu_{ik} \nabla_k h_{1k}$$

$$- h^{11} (1 + n \beta) + \beta \sum_i h^{ii} + \beta h^{11} h^{ij} (\nabla_i h_{ij})^2 + M u + Mu_k \frac{\nabla_k \Phi}{\Phi}$$
equations (4) and (6) are uniformly parabolic. By the
Lemma 4.1. Let
we require the following better gradient estimate. Theorems 1.1 at first. In order to prove the convergence of the normalized flow (4),
4.
(4) and (6), respectively.
\[\forall i \leq \alpha \leq \beta \leq 1, \quad \forall M \geq \frac{\varepsilon + 1}{\min \sigma} \leq C_1 - C_2 h^{11}.
\]
That is, \(h^{11} \leq C_3\), where \(C_3\) is a constant depending only on \(f, \alpha, \beta, \min_{S_n \times (0,T)} u\) and \(\max_{S_n \times (0,T)} u\). Hence the principal radii \(\lambda_i \geq C_2, i = 1, \ldots, n\). From Lemma 3.4, we know \(\sigma_n = \lambda_1 \cdots \lambda_n \leq C_1\). Therefore we get the \(C^2\) estimate \(C^{-1} \leq \lambda_i \leq C, i = 1, \ldots, n\) for the solutions to the normalised flow (6).

From the estimates obtained in Lemmata 3.6, 3.5 and 3.7, we know that the equations (4) and (6) are uniformly parabolic. By the \(C^0\) estimates (Lemmata 3.1 and 3.4), the gradient estimates (Lemma 2 and Corollary 3.3) and the \(C^2\) estimates Lammata 3.6, 3.5 and 3.7, and the Krylov’s theory [25], we get the Hölder continuity of \(\nabla^2 u\) and \(u_t\). Then we can get higher order derivation estimates by the regularity theory of the uniformly parabolic equations. Therefore we get the long time existence and the uniqueness of the smooth solution to the normalized flows (4) and (6), respectively.

4. Proofs of Theorems 1.1, 1.2 and 1.3. In this section we give the proof of
Theorems 1.1 at first. In order to prove the convergence of the normalized flow (4),
we require the following better gradient estimate.

Lemma 4.1. Let \(u(\cdot, t)\) be a smooth uniformly convex solution to the flow (4). If \(\alpha \leq 1 - k \beta\), then there exist positive constants \(C\) and \(C_0\), depending only on the initial hypersurface and \(\alpha, \beta\), such that
\[\max_{S_n} \frac{|\nabla u(\cdot, t)|}{u(\cdot, t)} \leq Ce^{-C_0 t}\]
for all $t > 0$.

Proof. Let $w = \log u$. Then we have
\[
h_{ij} = u_{ij} + u \delta_{ij} = e^w (w_{ij} + w_i w_j + \delta_{ij})
\]
and
\[
w_t = (e^w)^{\alpha + k \beta - 1} \sigma_k^\beta ([w_{ij} + w_i w_j + \delta_{ij}]) - \gamma.
\]
Consider the auxiliary function
\[
Q = \frac{1}{2} |\nabla w|^2.
\]
At the point where $Q$ attains its spatial maximum, we have
\[
0 = \nabla_i Q = \sum w_i w_{li},
\]
\[
0 \geq \nabla_{ij} Q = \sum w_i w_{ij} + \sum w_l w_{lij},
\]
and
\[
\partial_t Q = \sum w_i w_{li}
\]
\[
= 2Q(\alpha + k \beta - 1)(e^w)^{\alpha + k \beta - 1} \sigma_k^\beta
\]
\[
+ \beta(e^w)^{\alpha + k \beta - 1} \sigma_k^{\beta - 1} \sigma_{ij}^2 (w_l w_{lij} + w_l w_{li} + w_l w_{lij})
\]
\[
= 2Q(\alpha + k \beta - 1)(e^w)^{\alpha + k \beta - 1} \sigma_k^\beta + \beta(e^w)^{\alpha + k \beta - 1} \sigma_k^{\beta - 1} \sigma_{ij}^2 w_l w_{lij}.
\]
By the Ricci identity,
\[
\nabla_l w_{ij} = \nabla_j w_{li} + \delta_{li} w_j - \delta_{ij} w_l,
\]
we get
\[
\partial_t Q = 2(\alpha + k \beta - 1)(e^w)^{\alpha + k \beta - 1} \sigma_k^\beta Q
\]
\[
+ \beta(e^w)^{\alpha + k \beta - 1} \sigma_k^{\beta - 1} \sigma_{ij}^2 \sum_l w_l w_{lij} + w_l w_j - \delta_{ij} |\nabla w|^2
\]
\[
\leq 2(\alpha + k \beta - 1)(e^w)^{\alpha + k \beta - 1} \sigma_k^\beta Q + 2\beta(e^w)^{\alpha + k \beta - 1} \sigma_k^{\beta - 1} (\max_i \sigma_i^{\alpha i} - \sum_i \sigma_i^{\alpha i}) Q
\]
\[
\leq -C_0 Q,
\]
where the first inequality holds by (20). We therefore have $Q \leq C e^{-C_0 t}$, where $C_0$ and $C$ are two positive constants which depends only on $\alpha$, $\beta$ and the geometry of $M_0$. \(\square\)

Proof of Theorem 1.1. Case (i). $\alpha < 1 - k \beta$. Let $u(\cdot, t)$ be the solution. By rescaling $M_0$ if necessary, we assume
\[
a := \min_{\hat{u}_n} u(\cdot, 0) \leq 1 \leq \max_{\hat{u}_n} u(\cdot, 0) =: b.
\]
We also introduce two time-dependent functions
\[
u_1 = [1 - (1 - a^{-q})e^{\gamma t}]^{-\frac{1}{4}},
\]
\[
u_2 = [1 - (1 - b^{-q})e^{\gamma t}]^{-\frac{1}{4}},
\]
where $q = \alpha + k \beta - 1 < 0$. Both functions $u_1$ and $u_2$ are the solutions of (4), hence by the comparison principle, $u_1(t) \leq u(\cdot, t) \leq u_2(t)$. That is
\[
(a^{-q} - 1)e^{\gamma t} \leq u^{-q} - 1 \leq (b^{-q} - 1)e^{\gamma t}.
\]
Thus $u$ converges to 1 exponentially. By the interpolation and the a priori estimates Lemmas 3.1 and 3.6, we see that $\|u(\cdot, t) - 1\|_{C^k(S^n)} \to 0$ exponentially. Hence $M_t$ converges to the unit sphere centered at the origin.

**Case (ii).** $\alpha = 1 - k\beta$. By Lemma 4.1, we have that $\|\nabla u\| \to 0$ exponentially as $t \to \infty$. Hence by the interpolation and the a priori estimates, we can get that $u$ converges exponentially to a constant in the $C^\infty$ topology as $t \to \infty$.

**Proofs of Theorems 1.2 and 1.3**. Recall Lemma 2.3, we have

$$
\lim_{t \to \infty} J(X(\cdot, t)) - J(X(\cdot, 0)) = -\int_0^\infty |\frac{d}{dt} J(X(\cdot, t))| dt
$$

(21)

By Lemma 3.4, we have

$$
|J(X(\cdot, t))| \leq C_0 \text{ for some positive } C_0.
$$

(22)

By (22), the left hand side of (21) is bounded below by $-2C_0$. Hence there is a sequence $t_i \to \infty$ such that $\frac{d}{dt} J(X(\cdot, t_i)) \to 0$ as $t_i \to \infty$, which implies, by Lemma 2.3 and Lemma 3.4 again, $u(\cdot, t_i)$ converges to a positive and uniformly convex function $u_\infty \in C^\infty(S^n)$ which satisfies (8) with $c$ given by

$$
c = \lim_{t_i \to \infty} \eta(t_i) = \int_{S^n} \frac{f u_\infty^\alpha \sigma_k^{1+\beta}(\nabla^2 u_\infty + u_\infty I)}{|S^n|} dx.
$$

To complete the proof of Theorems 1.2 and 1.3, it suffices to show that the solution of (8) is unique.

**Case 1:** $\alpha < 1 - k\beta$. Let $u_1, u_2$ be two smooth solutions of (8), i.e.

$$
f u_1^{\alpha-1} \sigma_k^\beta(\nabla^2 u_1 + u_1 I) = c, f u_2^{\alpha-1} \sigma_k^\beta(\nabla^2 u_2 + u_2 I) = c.
$$

Suppose $M = \frac{u_1}{u_2}$ attains its maximum at $X_0 \in S^n$, then at $x_0$,

$$
0 = \nabla \log M = \frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2},
$$

$$
0 \geq \nabla^2 \log M = \frac{\nabla^2 u_1}{u_1} - \frac{\nabla^2 u_2}{u_2}.
$$

Hence at $x_0$, we get

$$
1 = \frac{u_1^{\alpha-1} \sigma_k^\beta(\nabla^2 u_1 + u_1 I)}{u_2^{\alpha-1} \sigma_k^\beta(\nabla^2 u_2 + u_2 I)} = \frac{u_1^{\alpha-1+k\beta} \sigma_k^\beta(\nabla^2 u_1 + 1)}{u_2^{\alpha-1+k\beta} \sigma_k^\beta(\nabla^2 u_2 + 1)} \leq M^{\alpha-1+k\beta}.
$$

Since $\alpha < 1 - k\beta$, $M(x_0) = \max_{S^n} M \leq 1$. Similarly one can show $\min_{S^n} M \geq 1$. Therefore $u_1 \equiv u_2$.

**Case 2:** $\alpha = 1 - k\beta$. We use the same method in [21] to get the solutions for equation (8) differ only by dilation. We omit the proof process here.

Hence we complete the proofs of the Theorems 1.2 and 1.3.
5. **Proof of Theorem 1.4.** In this section we give the proof of Theorem 1.4. The methods for the cases that \( f \) being any smooth positive function and \( f \equiv 1 \) are the same, we just consider the case \( f \equiv 1 \). The calculation for the example of the flow (1) is similar to one in [26, 27], we give the brief proof.

By a simple calculation, we obtain the following expressions of the metric, the unit normal, the second fundamental forms and the support function of the hypersurface, according to the radial function \( r \)

\[
g_{ij} = r^2 \delta_{ij} + r_i r_j, \tag{23}
\]

\[
n = \frac{rz - \nabla r}{\sqrt{r^2 + |\nabla r|^2}}, \tag{24}
\]

\[
h_{ij} = \frac{-rr_{ij} + 2r_i r_j + r^2 \delta_{ij}}{\sqrt{r^2 + |\nabla r|^2}}, \tag{25}
\]

\[
u = \frac{r^2}{\sqrt{r^2 + |\nabla r|^2}}. \tag{26}
\]

For the convex body \( K \) and the hypersurface \( M = \partial K \), consider the dual body \( K^\ast \), by [22] we know \( r^\ast = \frac{1}{u} \), and

\[
K = \frac{\det h_{ij}}{\det g_{ij}}, \quad \frac{1}{\sigma_n} = \frac{\det e_{ij}}{\det g_{ij}^*}, \quad \frac{\det e_{ij}}{\det g_{ij}^*} = \frac{1}{\det g_{ij}} = \frac{1}{2^{n-1}(r^2 + |\nabla r|^2)}. \tag{27}
\]

Since \( r^\ast = \frac{1}{u} \),

\[
\frac{\partial r^\ast}{\partial t} = -r^{2-\alpha} \sigma_k^\beta([r^{\ast-1}\delta_{ij} + 2r^{\ast-3}r_i^\ast r_j^\ast - r^{\ast-2}r_{ij}^\ast]).
\]

Generally, we consider

\[
\begin{cases}
\frac{\partial \sigma}{\partial t}(x, t) = -r^{2-\alpha} \sigma_k^\beta([r^{\ast-1}\delta_{ij} + 2r^{\ast-3}r_i^\ast r_j^\ast - r^{\ast-2}r_{ij}^\ast])(x, t) \\
\sigma(x, 0) = 0
\end{cases}
\]

If \( \alpha + k\beta - 1 > 0 \), that is \( 2 - \alpha < k\beta + 1 \). (9) is equivalent to

\[
R(X(\cdot, t)) = \max_{S^n} r(\cdot, t) \to \infty, \quad \min_{S^n} r(\cdot, t) \to \infty,
\]

as \( t \to T \) for some \( T > 0 \). It is suffice to show that \( \min_{S^n} r \equiv 0 \) in finite time while \( \max_{S^n} r \) remains positive. By the comparison principle, it suffices to construct a sub-solution \( Y(\cdot, t) \)

\[
\begin{cases}
\frac{\partial w}{\partial t}(x, t) \geq -w^{2-\alpha} \sigma_k^\beta([w^{\ast-1}\delta_{ij} + 2w^{\ast-3}w_i w_j - w^{\ast-2}w_{ij}^\ast])(x, t) \\
w(x, 0) = w_0
\end{cases}
\]

such that \( \min_{S^n} w(\cdot, t) \to 0 \) but \( \max_{S^n} w(\cdot, t) \) remains positive, as \( t \to T \) for some finite time \( T > 0 \).

**Lemma 5.1.** There is a sub-solution \( Y(\cdot, t) \), where \( t \in (-1, 0) \), to

\[
\begin{cases}
\frac{\partial r}{\partial t}(x, t) = -ar^{2-\alpha} \sigma_k^\beta([r^{\ast-1}\delta_{ij} + 2r^{\ast-3}r_i^\ast r_j^\ast - r^{\ast-2}r_{ij}^\ast]) \\
r(x, 0) = r_0
\end{cases}
\]

for a sufficiently large constant \( a > 0 \), such that \( \min_{S^n} r(\cdot, t) \to 0 \) but \( \max_{S^n} r(\cdot, t) \) remains positive, as \( t \to 0 \).
Proof. The sub-solution is a family of closed convex hypersurfaces \( \widehat{M}_t := Y(S^n, t) \). Near the origin, let \( \widehat{M}_t \) be the graph of a radial function

\[
\psi(\rho, t) = \left\{ \begin{array}{ll}
-|t|^\theta + |t|^{-\theta+\mu}\rho^2 & \text{if } \rho < |t|^\theta \\
-|t|^\theta - \frac{1}{1+\mu}|t|^\theta(1+\mu) + \frac{2}{1+\mu}\rho^{1+\mu} & \text{if } |t|^\theta \leq \rho \leq 1 
\end{array} \right.
\]

where \( q = k\beta + 1 - \hat{\alpha} > 0, \) \( \hat{\alpha} = 2 - \alpha, \mu = \frac{q-1}{q} \), and \( \theta > \frac{1}{q} \) is a constant. It is easy to verify that \( \psi \) is strictly convex, and \( \psi \in C^{1,1}(B_1(0)) \).

If \( Y(S^n, t) \) is a sub-solution for some \( \hat{\alpha} \), it is also a sub-solution for \( \alpha' < \hat{\alpha} \), so we prove the case when \( q = k\beta + 1 - \hat{\alpha} \) is very small. From the equalities (23), (25), (27) and Newton-Maclaurin inequality, by direct computation, we have

(i) if \( 0 \leq \rho \leq |t|^\theta \), then

\[
\rho^2 \sigma_k((r^{-1}\delta_{ij} + 2r^{-3}r_ir_j - r^{-2}r_{ij}) \geq c|t|^{\theta+1}k\beta(\mu-1) = c|t|^{\theta-1},
\]

where \( p = (x, \psi(|x|, t)) \) is a point on the graph of \( \psi \).

(ii) if \( |t|^\theta \leq \rho \leq 1 \), then

\[
\partial_t Y(p, t) \leq \theta|t|^{\theta-1}.
\]

Extending the graph of \( \psi \) to a closed convex hypersurface \( \widehat{M}_t \), such that it is \( C^{1,1} \) smooth, uniformly convex, rotationally symmetric. Moreover, assume that the ball \( B_1(z) \) is contained in the interior of \( \widehat{M}_t \), for all \( t \in (-1, 0) \), where \( z = (0, ..., 0, 1) \) is a point on the \( x_{n+1} \)-axis.

For a given \( \tau \in (-1, 0) \), let \( B_1(z) \subset M_0 \subset \widehat{M}_\tau \), let \( M_t \) be a solution to the flow (31) with initial data \( M_0 \). \( M_t \) touches the origin at \( t = t_0 \), for some \( t_0 \in (\tau, 0) \). We assume \( \tau \to 0 \) closely.

On the other hand, let \( \widehat{X}(., t) \) be the solution to

\[
\partial_t X = -ba^\beta\sigma_k^\beta \nu
\]

with initial condition \( \widehat{X}(., \tau) = \partial B_1(z), \) where \( b = 2\hat{\alpha} \sup \{|p|^\theta : p \in M_t, \tau < t < t_0\} \) and \( \tau = |X - z| \). We can choose \( \tau \) so small that the ball \( B_1(z) \) is contained in the interior of \( \widehat{X}(., t) \) for all \( t \in (\tau, t_0) \). By the comparison principle, we know that the ball \( B_1(z) \) is contained in the interior of \( M_t \) for all \( t \in (\tau, t_0) \). Hence, as \( t \to t_0 \), we have \( \min r(., \tau), t_0 \to 0 \) and \( \max r(., \tau), r(., t) > |z| = 10 \). Hence (29) is proved for \( M_t \).

For a large constant \( a > 0 \). Making the rescaling \( \widehat{M}_t = a^{-\frac{1}{2}} M_t, \widehat{M}_t \) solve the flow (28). Hence we complete the proof.

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