FINITE DIFFERENCE SCHEMES FOR THE SYMMETRIC KEYFITZ-KRANZER SYSTEM

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Abstract. We are concerned with the convergence of numerical schemes for the initial value problem associated to the Keyfitz-Kranzer system of equations. This system is a toy model for several important models such as in elasticity theory, magnetohydrodynamics, and enhanced oil recovery. In this paper we prove the convergence of three difference schemes. Two of these schemes is shown to converge to the unique entropy solution. Finally, the convergence is illustrated by several examples.

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1. INTRODUCTION

In this paper, we consider difference methods for the Cauchy problem for the $n \times n$ symmetric system of Keyfitz-Kranzer type

$$
\begin{aligned}
&u_t + (u \phi(|u|))_x = 0, & x \in \Omega = \mathbb{R} \times (0, T),
\end{aligned}
\begin{aligned}
&u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{aligned}
$$

(1.1)

where $T > 0$ is fixed, $u = (u^{(1)}, \ldots, u^{(n)}) : \mathbb{R} \times [0, T) \to \mathbb{R}^n$ is the unknown vector map with $|u| = \sqrt{u^{(1)} + \cdots + u^{(n)}} = (u_0^{(1)}, \ldots, u_0^{(n)})$ the initial data, and $\phi : \mathbb{R} \to \mathbb{R}$ is given (sufficiently smooth) scalar function (see Section 2 for the

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complete list of assumptions). Systems of this type was first considered in [10, 12] and later on by several other authors [4], as a prototypical example of a non-strictly hyperbolic system. This type of system is a model system for some phenomena in magnetohydrodynamics, elasticity theory and enhanced oil-recovery. This system also has similarities to a model of chromatography [1] and to a model describing polymer flooding in porous media [18]. Note that since $\phi$ is a function of $|u|$, we call (1.1) a symmetric Keyfitz-Kranzer system. A non-symmetric version of the Keyfitz-Kranzer system reads

$$
\begin{align*}
\begin{cases}
    u_t + (u \phi(u, w_1, w_2, \ldots, w_n))_x = 0, \\
    (u w_i)_t + (u w_i \phi(u, w_1, w_2, \ldots, w_n))_x = 0, 
\end{cases}
\end{align*}
$$

Existence of global bounded weak solutions to (1.2) has been studied by Lu [14] for a specific choice of $\phi$.

For the flux function $F(u) = u \phi(|u|)$, a straightforward calculation shows $B(u)$ is the matrix with entries

$$
B_{i,j}(u) = \phi(|u|) \delta_{i,j} + \phi'(|u|) \frac{u_i u_j}{|u|}, \quad i, j = 1, 2, \ldots, n,
$$

where $\delta_{i,j}$ is the Kronecker delta, given by

$$
\delta_{i,j} = \begin{cases} 
1, & i = j \\
0, & i \neq j.
\end{cases}
$$

The matrix $B(u)$ is symmetric, therefore the system (1.1) is hyperbolic, that is, all the eigenvalues of $B(u)$ are real and the corresponding collection of eigenvectors is complete. It is easy to see that the first eigenvalue of $B(u)$ is $\lambda_1 = \phi(|u|) + \phi'(|u|)|u|$ and other $n - 1$ eigenvalues are $\lambda_i = \phi(|u|)$, $i = 1, 2, \ldots, n - 1$. The presence of repeated eigenvalues shows that the system (1.1) is not strictly hyperbolic.

Due to the nonlinearity, discontinuities in the solution may appear independently of the smoothness of the initial data and weak solution must be sought. A weak solution is defined as follows:

**Definition 1.1.** We say $u(x, t)$ a weak solution to (1.1) if

**D.1** $u(x, t) \in L^\infty(\mathbb{R} \times [0, \infty))$.

**D.2** For all test functions $\psi \in C_0^\infty(\mathbb{R} \times [0, \infty))$

$$
\int_{\mathbb{R} \times [0, \infty)} u \psi_t + u \phi(|u|) \psi_x \, dx \, dt + \int_{\mathbb{R}} u_0 \psi(x, 0) \, dx = 0,
$$

It is well known that weak solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. Therefore the Cauchy problem is viewed in the framework of entropy solutions. For (1.1), an entropy formulation was first introduced by Freistühler [5, 6], and independently, by Panov [16]. An entropy solution to (1.1) is defined as follows:

**Definition 1.2.** A bounded measurable function $u(x, t)$ is called an entropy solution to (1.1) if

**D.1** For all test functions $\psi \in C_0^\infty(\mathbb{R} \times [0, \infty))$

$$
\int_{\mathbb{R} \times [0, \infty)} u \psi_t + u \phi(|u|) \psi_x \, dx \, dt + \int_{\mathbb{R}} u_0 \psi(x, 0) \, dx = 0,
$$
D.2 $r = |u|$ is an entropy solution (in the sense of Kružkov [11]) of the scalar conservation law

$$\begin{align*}
l_t + (r \phi(r))_x &= 0, \quad t > 0, \\
r(x, 0) &= |u_0(x)|.
\end{align*}$$

Regarding the existence, uniqueness of solutions and continuous dependence of solutions on the initial data we have the following result

**Theorem 1.1.** The system (1.1) has the following properties:

1. **(E)** The system has a solution for any $u_0 \in L^\infty(\mathbb{R})$.
2. **(U)** For such $u_0$, there is precisely one solution $u$ with the property that $r = |u|$ satisfies the scalar conservation laws (1.5) and Kružkov’s entropy criterion.
3. **(S)** This solution $u$ depends $L^1_{\text{loc}}(\mathbb{R})$ continuously on the initial data $u_0$.

This theorem was first proved in [6] by using the famous equivalence result of Wagner [19]. The key idea behind this proof is to view the system (1.1) as an extended system, consisting of (1.1) and an additional conservation law satisfied by $r$ (1.5), with Wagner’s transformation theory. On the other hand, in [16], Panov gave a “direct” proof of both existence and uniqueness. The existence was proved by showing the convergence of the singularly perturbed problems

$$u_\varepsilon^t + (u_\varepsilon \phi(|u_\varepsilon|))_x = \varepsilon u_\varepsilon^{xx},$$

and to an entropy solution as $\varepsilon \to 0$. The idea behind the existence proof was first to show the existence of a *measure-valued* solution $\nu(t, x)$ of the Cauchy problem (1.1). Then he showed that indeed $\nu(t, x)$ is regular: $\nu(t, x)(u) = \delta(u - u(t, x)), u(t, x) \in L^\infty(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^n)$ and consequently this gives existence of a solution to (1.1).

In view of the analytic properties of the solutions of (1.1), several different methods for computing the solution suggest themselves. Foremost among these methods is Glimm’s scheme [8]. Regarding other numerical methods, it is tempting to use the equation satisfied by $r$, and view $r$ as an independent variable. Defining $v \in S^{n-1}$ by $vr = u$, we formally have that

(1.6) \hspace{1cm} r_t + (r \phi(r))_x = 0 \\
(1.7) \hspace{1cm} (rv)_t + (r \phi(r)v)_x = 0

or

(1.8) \hspace{1cm} v_t + \phi(r)v_x = 0.

As a strategy, one can then solve (1.6) first, and then either (1.7) or (1.8). These should then hold subject to the constraint $|v| = 1$. Without this constraint, (1.6)–(1.7) is a “triangular” system of conservation laws, see [3]. Using any monotone scheme for (1.6) and (1.7) will ensure the strong convergence of the approximate solutions to (1.6) and the weak-star convergence of the approximate solutions to (1.7). This approach was used in [7]. To show that $u = rv$ is an entropy solution to (1.1), one must show (for the approximations) that $|v| = 1$ in the limit if $|v_0| = 1$.

In this paper, for the semi-discrete scheme, we discretize (1.1) in space and show the convergence of approximate solution to a weak solution of (1.1). But we are unable to extend our analysis to the fully discrete scheme based on discretizing (1.1). To overcome this difficulty, we propose another scheme based on discretizing...
(1.6)-(1.8) and prove the convergence of approximate solution to unique entropy solution of (1.1).

The present paper can be divided into four parts:

(1) In Section 2, we present the mathematical framework used in this paper. In particular, we used a compensated compactness result in the spirit of Tartar [17] but the proof is based on div-curl lemma and does not rely on the Young measure.

(2) In section 3, we propose an upwind semi-discrete finite difference scheme and prove the convergence of the approximate solution to the weak solution of (1.1). The idea behind this proof is to prove first the strong convergence of approximate solution \( r_{\Delta x} = |u_{\Delta x}| \) using the compensated compactness technique [17, 2]. Then prove a \( B.V. \) estimate for \( \tau = u/(u, e) \), where \( e \) is a unit vector in \( \mathbb{R}^n \). Then Kolmogorov’s theorem, combined with the strong convergence of \( r_{\Delta x} \), gives the strong convergence of approximate solution \( u_{\Delta x} \).

(3) In section 4, for a fully discrete scheme, we are only able to conclude that \( u \) is only a distributional solution of

\[
\frac{\partial u}{\partial t} + (u \phi(r))_x = 0
\]

for some \( r \) such that \( |u| \leq r \). We propose another fully discrete scheme relying on explicit decoupling of the variables \( r \) and \( v \) expressed by the “nonconservative” formulation (1.6)-(1.8)

\[
\begin{align*}
    r_t + (r \phi(r))_x &= 0, \\
v_t + \phi(r) v_x &= 0,
\end{align*}
\]

with \( r(0) = |u(0)| \). It is not difficult to show the convergence of \( r_{\Delta x} \) to \( r \), \( r \) being the unique entropy solution of (1.6), and the strong convergence of \( v_{\Delta x} \). In order to conclude that \( u = rv \) is the unique entropy solution of (1.1), one has to show \( |v(x, t)| = 1 \) and this has been achieved in this paper using Wagner transformation [19] (see Section 2 for more details).

(4) Finally, in Section 5, we test our numerical schemes and provide some numerical results.

2. Mathematical Framework

In this section we present some mathematical tools that we shall use in the analysis. To start with the basic assumptions on the initial data and the function \( \phi(r) \), we assume that \( \phi \) is a twice differentiable function \( \phi : [0, \infty) \to [0, \infty) \) so that

A.1 \( \phi(0) = 0, \phi(r) > 0 \) and \( \phi'(r) \geq 0 \) for all relevant \( r \);

A.2 \( \phi(r), \phi'(r) \) and \( \phi''(r) \) are bounded for all relevant \( r \);

A.3 \( \\text{meas} \left\{ r \text{ \mid } 2\phi'(r) + r\phi''(r) = 0 \right\} = 0 \);

A.4 \( |u_0| \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and \( |u_0| \in B(K) \) for any constant \( K \) in \( \mathbb{R} \), where

\[
B(K) := \{ f \text{ \mid } \inf_{x \in A(f, K)} f \geq C_K \}, \text{ and}
\]

\[
A(f, K) := \left\{ x \in (-\infty, K] \text{ \mid } \exists \varepsilon > 0 \text{ with } f(y) > \liminf_{x \to x} f(z), \text{ for a.e } y \in (x - \varepsilon, x) \right\}.
\]

Here \( C_K \) is a positive constant depending on \( K \).
\[ A.5 \quad u_0 \in \Gamma_\delta, \text{ where } \Gamma_\delta \text{ is the cone} \]
\[ \Gamma_\delta := \{ u \in \mathbb{R}^n \mid \delta |u| \leq (e, u) \} \]

for some fixed unit vector (which we without loss of generality choose as \( e = (1, \ldots, 1)/\sqrt{n} \)), and \( \delta \) is a fixed number in the interval \((\sqrt{(n-1)/n}, 1)\).

Next, we recapitulate the results we shall use from the compensated compactness method due to Murat and Tartar [15, 17]. For a nice overview of applications of the compensated compactness method to hyperbolic conservation laws, we refer to Chen [2]. Let \( \mathcal{M}(\mathbb{R}) \) denote the space of bounded Radon measures on \( \mathbb{R} \) and \( C_0^0(\mathbb{R}) = \{ \psi \in C(\mathbb{R}) \mid \lim_{|x| \to \infty} \psi(x) = 0 \} \).

If \( \mu \in \mathcal{M}(\mathbb{R}) \), then
\[ \langle \mu, \psi \rangle = \int_{\mathbb{R}} \psi \, d\mu, \quad \text{for all } \psi \in C_0(\mathbb{R}). \]

Recall that \( \mu \in \mathcal{M}(\mathbb{R}) \) if and only if \( |\langle \mu, \psi \rangle| \leq C \|\psi\|_{L^\infty(\mathbb{R})} \) for all \( \psi \in C_0(\mathbb{R}) \). We define
\[ \|\mu\|_{\mathcal{M}(\mathbb{R})} = \sup \{|\langle \mu, \psi \rangle| : \psi \in C_0(\mathbb{R}), \|\psi\|_{L^\infty(\mathbb{R})} \leq 1\}. \]

The space \( \left( \mathcal{M}(\mathbb{R}), \|\cdot\|_{\mathcal{M}(\mathbb{R})} \right) \) is a Banach space and it is isometrically isomorphic to the dual space of \( \left( C_0(\mathbb{R}), \|\cdot\|_{L^\infty(\mathbb{R})} \right) \), while we define the space of probability measures
\[ \text{Prob}(\mathbb{R}) = \{ \mu \in \mathcal{M}(\mathbb{R}) : \mu \text{ is nonnegative and } \|\mu\|_{\mathcal{M}(\mathbb{R})} = 1 \}. \]

Before we state the compensated compactness theorem, we recall the celebrated div-curl lemma.

**Lemma 2.1 (div-curl lemma).** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^2 \). With \( \varepsilon > 0 \) denoting a parameter taking its value in a sequence which tends to zero, suppose
\[ D^\varepsilon \rightharpoonup D \text{ in } (L^2(\Omega))^2, \quad E^\varepsilon \rightharpoonup E \text{ in } (L^2(\Omega))^2, \]
\[ \{\text{div} \, D^\varepsilon \}_{\varepsilon > 0} \text{ lies in a compact subset of } H_{\text{loc}}^{-1}(\Omega), \]
\[ \{\text{curl} \, E^\varepsilon \}_{\varepsilon > 0} \text{ lies in a compact subset of } H_{\text{loc}}^{-1}(\Omega). \]
Then along a subsequence
\[ D^\varepsilon \cdot E^\varepsilon \rightharpoonup D \cdot E \text{ in } D'(\Omega). \]

We shall use the following compensated compactness result.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R} \times \mathbb{R}^+ \) be a bounded open set, and assume that \( \{u^\varepsilon\} \) is a sequence of uniformly bounded functions such that \( |u^\varepsilon| \leq M \) for all \( \varepsilon \). Also assume that \( f : [-M, M] \to \mathbb{R} \) is a twice differentiable function. Let \( u^\varepsilon \rightharpoonup^* u \) and \( f(u^\varepsilon) \rightharpoonup^* v \), and set
\[ (\eta_1(s), q_1(s)) = (s - k, f(s) - f(k)), \]
\[ (\eta_2(s), q_2(s)) = \left( f(s) - f(k), \int_k^s (f'(\theta))^2 \, d\theta \right), \]
where \( k \) is an arbitrary constant. If
\[ \eta_i(u^\varepsilon)_t + q_i(u^\varepsilon)_x \text{ is in a compact set of } H_{\text{loc}}^{-1}(\Omega) \text{ for } i = 1, 2, \]
then
\[(1)\; v = f(u), \text{ a.e. } (x,t),\]
\[(2)\; u^\varepsilon \to u, \text{ a.e. } (x,t) \text{ if } \text{meas}\{u \mid f''(u) = 0\} = 0.\]

For a proof of this theorem, see the monograph of Lu [13]. A feature of the compensated compactness result above is that it avoids the use of the Young measure by following an approach developed by Chen and Lu [13, 2] for the standard scalar conservation law. This is preferable as the fundamental theorem of Young measures applies most easily to functions that are continuous in all variables.

The following compactness interpolation result (known as Murat’s lemma [15]) is useful in obtaining the $H^{-1}_{loc}$ compactness needed in Theorem 2.1.

**Lemma 2.2.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{\mathcal{L}_\varepsilon\}_{\varepsilon > 0}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that
\[\mathcal{L}_\varepsilon = \mathcal{L}_{1,\varepsilon} + \mathcal{L}_{2,\varepsilon},\]
where $\{\mathcal{L}_{1,\varepsilon}\}_{\varepsilon > 0}$ is in a compact subset of $H^{-1}(\Omega)$ and $\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon > 0}$ is in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_\varepsilon\}_{\varepsilon > 0}$ is in a compact subset of $H^{-1}_{loc}(\Omega)$.

Next, we shall need Kolmogorov’s compactness lemma.

**Lemma 2.3** ($L^1_{loc}$ compactness, see [9]). Let $u^\varepsilon : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a family of functions such that for each positive $T$,
\[|u^\varepsilon(x,t)| \leq C_T, \quad (x,t) \in \mathbb{R} \times [0,T]\]
for a constant $C_T$ independent of $\varepsilon$. Assume in addition that for all compact $B \subset \mathbb{R}$ and for $t \in [0,T]$
\[\sup_{|\xi| \leq |\rho|} \int_B |u^\varepsilon(x + \xi, t) - u^\varepsilon(x, t)| \, dx \leq \nu_{B,T}(|\rho|),\]
for a modulus of continuity $\nu_{B,T}$. Furthermore, assume for $s$ and $t$ in $[0,T]$ that
\[\int_B |u^\varepsilon(x,t) - u^\varepsilon(x,s)| \, dx \leq \omega_{B,T}(|t - s|) \text{ as } \varepsilon \downarrow 0,\]
for some modulus of continuity $\omega_{B,T}$. Then there exists a sequence $\{\varepsilon_j\}$ such that for each $t \in [0,T]$ the function $\{u^{\varepsilon_j}(t)\}$ converges to a function $u(t)$ in $L^1_{loc}(\mathbb{R})$. The convergence is in $C([0,T]; L^1_{loc}(\mathbb{R}))$.

Finally, we state the following result related to Wagner transformation theory.

**Lemma 2.4** (Wagner Transformation, see [19, 5]). For any $n \in \mathbb{N}$, there is a one-to-one correspondence between (equivalence classes of) bounded Lebesgue measurable solutions $(r, rv) : \mathbb{R}^2_+ \to [0, \infty) \times \mathbb{R}^m$ to the system (1.6)–(1.7) which satisfy
\[\int_{-\infty}^0 r(x,t) \, dx = \int_0^\infty r(x,t) \, dx = \infty\]
and (equivalence classes of) weak solutions $(\tau, \tilde{v})$ to the system
\[\begin{align*}
\tau_t - (\phi(1/\tilde{\tau}))_y &= 0, \\
\tilde{v}_t &= 0,
\end{align*}\]
(2.2)
in which $\tau$ is a Radon measure in $\mathbb{R}^2_+$ which dominates Lebesgue (outer) measure $\lambda_2$ (i.e., $\tau \geq k\lambda_2$ for some $k > 0$), $\tilde{\tau}$ is the density of the absolutely continuous part
of $\tau$ with respect to $\lambda_2$, and $\tilde{v} : \mathbb{R}_+^2 \to \mathbb{R}^m$ is bounded and Lebesgue measurable. This correspondence is established through transformations $T : (x,t) \to (y(x,t),t)$ defined relative to any bounded measurable solutions to (1.6) by

$$\frac{\partial y}{\partial x}(x,t) = r(x,t), \quad \frac{\partial y}{\partial t}(x,t) = -\phi(r(x,t))r(x,t), \quad y(0,0) = 0,$$

namely setting

$$\tau = \lambda_2 \circ T^{-1},$$
$$\tilde{v} = v \circ T^{-1}.$$

Observe that using (2.2), (2.3) and (2.4), it is easy to conclude that $|v| = 1$.

3. A SEMI-DISCRETE FINITE DIFFERENCE SCHEME

We start by introducing some notation needed to define the semi-discrete finite difference schemes. Throughout this paper we reserve $\Delta x$ to denote a small positive number that represents the spatial discretization parameter of the numerical schemes. Given $\Delta x > 0$, we set $x_j = j\Delta x$ for $j \in \mathbb{Z}$ and for any function $u = u(x)$ admitting point values we write $u_j = u(x_j)$. Furthermore, let us introduce the spatial grid cells

$$I_j = [x_{j-1/2}, x_{j+1/2}),$$

where $x_{j+1/2} = x_j \pm \Delta x/2$. Let $D_{\pm}$ denote the discrete forward and backward differences, i.e.,

$$D_{\pm}u_j = \pm \frac{u_j - u_{j\pm 1}}{\Delta x}.$$

The discrete Leibnitz rule is given by

$$D_{\pm}(u_jv_j) = u_jD_{\pm}v_j + v_{j\pm 1}D_{\pm}u_j.$$

Furthermore, for any $C^2$ function $f$, using the Taylor expansion on the sequence $f(u_j)$ we obtain

$$D_{\pm}f(u_j) = f'(u_j)D_{\pm}u_j \pm \frac{\Delta x}{2} f''(\xi_{j\pm \frac{1}{2}})(D_{\pm}u_j)^2,$$

for some $\xi_{j\pm \frac{1}{2}}$ between $u_{j\pm 1}$ and $u_j$. We will make frequent use of this, which states that a discrete chain rule holds up to an error term of order $\Delta x(D_{\pm}u_j)^2$. To a sequence $\{u_j\}_{j \in \mathbb{Z}}$ we associate the function $u_{\Delta x}$ defined by

$$u_{\Delta x}(x) = \sum_{j \in \mathbb{Z}} u_j \mathbb{1}_{I_j}(x).$$

Similarly, we also define $r_{\Delta x}$ as

$$r_{\Delta x}(x) = \sum_{j \in \mathbb{Z}} r_j \mathbb{1}_{I_j}(x),$$

where $\mathbb{1}_A$ denotes the characteristic function of the set $A$. Throughout this paper we use the notations $u_{\Delta x}, r_{\Delta x}$ to denote the functions associated with the sequence $\{u_j\}_{j \in \mathbb{Z}}$ and $\{r_j\}_{j \in \mathbb{Z}}$ respectively. For later use, recall that the $L^\infty(\mathbb{R})$ norm, the
Then where
\[ r_j(t) = \sup_{j \in \mathbb{Z}} |u_j(t)|, \]
with initial values
\[ u_j(0) = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x) \, dx, \]
where \( r_j(t) = |u_j(t)| \). We have that \( \{u_j(t)\}_{j \in \mathbb{Z}} \) satisfy the (infinite) system of ordinary differential equations and it is natural to view (3.1) as an ordinary differential equation in \( L^2(\mathbb{R}) \), since the piecewise constant structure of \( u_{\Delta x} \) is preserved by the evolution equation (3.1). To show the local (in time) existence and uniqueness of differentiable solutions we must show that the right hand side of (3.1) is Lipschitz continuous in \( L^2(\mathbb{R}) \). Set
\[ F(u_{\Delta x})_j = D_+ (\phi(r_j)u_j). \]
The infinite system of differential equations (3.1) can then be written
\[ \frac{d}{dt} (u_{\Delta x}(t)) = F(u_{\Delta x})_{\Delta x}. \]
We view \( F(u_{\Delta x})_{\Delta x} \) as an element in \( L^2(\mathbb{R}) \). To establish that this system has a unique solution (at least locally in time) we show that
\[ \| F(u_{\Delta x})_{\Delta x} - F(v_{\Delta x})_{\Delta x} \|_{L^2(\mathbb{R})} \leq \gamma \| u_{\Delta x} - v_{\Delta x} \|_{L^2(\mathbb{R})}, \]
for some locally bounded \( \gamma = \gamma(u_{\Delta x}, v_{\Delta x}) \) and for a fixed \( \Delta x > 0 \). Set \( \tilde{r}_j = |v_j| \), note that
\[ |r_j - \tilde{r}_j| \leq \frac{|u_j + v_j|}{\tilde{r}_j} |u_j - v_j| \leq |u_j - v_j|. \]
Then
\[ \| F(u_{\Delta x}) - F(v_{\Delta x}) \|_{L^2(\mathbb{R})} \leq \frac{2}{\Delta x} \left( \sup_j |u_j| \| \phi' \|_{L^\infty} \| r_{\Delta x} - \tilde{r}_{\Delta x} \|_{L^2(\mathbb{R})} \right) + \| \phi \|_{L^\infty} \| u_{\Delta x} - v_{\Delta x} \|_{L^2(\mathbb{R})}. \]

Observe that all the eigenvalues of the system (1.1) are positive by our assumptions. Therefore we consider the following semi-discrete upwind finite difference scheme
\[ u'_j(t) + D_+ (\phi(r_j(t))u_j(t)) = 0, \quad \text{for } j \in \mathbb{Z}, \quad t > 0, \]
with initial values
\[ u_j(0) = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x) \, dx, \]
where \( r_j(t) = |u_j(t)| \). We have that \( \{u_j(t)\}_{j \in \mathbb{Z}} \) satisfy the (infinite) system of ordinary differential equations and it is natural to view (3.1) as an ordinary differential equation in \( L^2(\mathbb{R})^n \), since the piecewise constant structure of \( u_{\Delta x} \) is preserved by the evolution equation (3.1). To show the local (in time) existence and uniqueness of differentiable solutions we must show that the right hand side of (3.1) is Lipschitz continuous in \( L^2(\mathbb{R})^n \). Set
\[ F(u_{\Delta x})_j = D_+ (\phi(r_j)u_j). \]
The infinite system of differential equations (3.1) can then be written
\[ \frac{d}{dt} (u_{\Delta x}(t)) = F(u_{\Delta x})_{\Delta x}. \]
We view \( F(u_{\Delta x})_{\Delta x} \) as an element in \( L^2(\mathbb{R})^n \). To establish that this system has a unique solution (at least locally in time) we show that
\[ \| F(u_{\Delta x})_{\Delta x} - F(v_{\Delta x})_{\Delta x} \|_{L^2(\mathbb{R})^n} \leq \gamma \| u_{\Delta x} - v_{\Delta x} \|_{L^2(\mathbb{R})^n}, \]
for some locally bounded \( \gamma = \gamma(u_{\Delta x}, v_{\Delta x}) \) and for a fixed \( \Delta x > 0 \). Set \( \tilde{r}_j = |v_j| \), note that
\[ |r_j - \tilde{r}_j| \leq \frac{|u_j + v_j|}{\tilde{r}_j} |u_j - v_j| \leq |u_j - v_j|. \]
Then
\[ \| F(u_{\Delta x}) - F(v_{\Delta x}) \|_{L^2(\mathbb{R})^n} \leq \frac{2}{\Delta x} \left( \sup_j |u_j| \| \phi' \|_{L^\infty} \| r_{\Delta x} - \tilde{r}_{\Delta x} \|_{L^2(\mathbb{R})} \right) + \| \phi \|_{L^\infty} \| u_{\Delta x} - v_{\Delta x} \|_{L^2(\mathbb{R})^n}, \]
where we have used Assumption A.2. Therefore $F$ is locally Lipschitz continuous, and there is a $\tau > 0$ so that the initial value problem (3.1) has a unique differentiable solution for $t \in [0, \tau)$, if $\tau < \infty$, then
\[
\lim_{t \uparrow \tau} \| u_\Delta x(t) \|_{L^2(\mathbb{R})} = \infty.
\]
We shall proceed to show that the $L^2$ norm remains bounded if it is bounded initially, so the solution can be defined up to any time.

**Lemma 3.1.** Assume that A.1, A.2 and A.4 hold, and let $\{u_j(t)\}$ be defined by (3.1), and let $r_j(t) = |u_j(t)|$. Then
\[
\| r_\Delta x(t) \|_{L^1(\mathbb{R})} \leq \| r_\Delta x(0) \|_{L^1(\mathbb{R})},
\]
\[
\| r_\Delta x(t) \|_{L^2(\mathbb{R})} \leq \| r_\Delta x(0) \|_{L^2(\mathbb{R})},
\]
\[
\| r_\Delta x(t) \|_{L^\infty(\mathbb{R})} \leq \| r_\Delta x(0) \|_{L^\infty(\mathbb{R})}.
\]
Furthermore, there is a constant $C$, independent of $\Delta x$ and $T$, such that
\[
(3.4) \quad \int_0^T \left( \sum_j \int_{r_{j-1}}^{r_j} (r_j^2 - s^2) \phi'(s) \, ds + \Delta x \sum_j \phi_{j-1} \Delta x |D_- u_j|^2 \right) \, dt \leq C.
\]

**Proof.** Let $\eta$ be a differentiable function $\eta : \mathbb{R}^n \to \mathbb{R}$, take the inner product of (3.1) with $\nabla u \eta(u_j)$ to get
\[
(3.5) \quad \frac{d}{dt} \eta(u_j) + D_- (\phi_j \eta(u_j)) + \left[ (\nabla u \eta(u_j), u_j) - \eta(u_j) \right] D_- \phi_j + \phi_{j-1} \Delta x^2 d^2 \eta_{j-1/2} (D_- u_j, D_- u_j) = 0.
\]
Here $\phi_j = \phi(r_j)$, and $d^2 \eta$ denotes the Hessian matrix of $\eta$, so that
\[
d^2 \eta_{j-1/2} = d^2 \eta(u_{j-1/2})
\]
for some $u_{j-1/2}$ between $u_j$ and $u_{j-1}$. By a limiting argument, the function $\eta(u) = |u|$ can be used. If one approximates by convex smooth functions, this means that
\[
(3.6) \quad \frac{d}{dt} r_j + D_- (r_j \phi(r_j)) \leq 0.
\]
Multiplying by $\Delta x$ and summing over $j$ we get
\[
(3.7) \quad \| r_\Delta x(t) \|_{L^1(\mathbb{R})} \leq \| u_0 \|_{L^1(\mathbb{R})}.
\]
Furthermore, if $j$ is such that $r_j(t) \geq r_{j-1}(t)$, then since $\phi$ is non-decreasing we get $r_j(t) \phi(r_j(t)) \geq r_{j-1}(t) \phi(r_{j-1}(t))$ i.e., $D_- (r_j \phi(r_j)) \geq 0$. Hence, from (3.6), we see that $dr_j(t)/dt \leq 0$. This shows that $0 \leq r_j(t) \leq \sup_j |u_j(0)|$. Hence, if $\| u_0 \|_{L^\infty(\mathbb{R})} < \infty$, then $r_\Delta x$ is bounded independently of $t$ and $\Delta x$.

Choosing $\eta(u) = |u|^2$ in (3.5) we get
\[
\frac{d}{dt} r_j^2(t) + D_- (r_j^2 \phi_j) + r_j^2 D_- \phi_j + \phi_{j-1} \Delta x |D_- u_j|^2 = 0.
\]
We have that
\[
D_- (r_j^2 \phi_j) + r_j^2 D_- \phi_j = \frac{2}{\Delta x} \int_{r_{j-1}}^{r_j} (s \phi(s) + s^2 \phi'(s)) \, ds + \frac{1}{\Delta x} \int_{r_{j-1}}^{r_j} (r_j^2 - s^2) \phi'(s) \, ds
\]
\[
= D_- g(r_j) + \frac{1}{\Delta x} \int_{r_{j-1}}^{r_j} (r_j^2 - s^2) \phi'(s) \, ds,
\]
where
\[
g(r_j) = \frac{2}{\Delta x} \int_{r_{j-1}}^{r_j} (s \phi(s) + s^2 \phi'(s)) \, ds + \frac{1}{\Delta x} \int_{r_{j-1}}^{r_j} \phi'(s) \, ds.
\]
where
\[ g(r) = 2 \int_0^r (s\phi(s) + s^2\phi'(s))\, ds. \]

Using this we find that
\[ \|r_{\Delta x}(t)\|_{L^2(\mathbb{R})} \leq \|\|u_0\|\|_{L^2(\mathbb{R})}, \]

since, by the assumption that \( \phi' \geq 0 \),
\[ \int_{r_{j-1}}^{r_j} (r_j^2 - s^2)\phi'(s)\, ds \geq 0. \]

Hence \( \|u_{\Delta x}(t)\|_{L^2(\mathbb{R})} \) is bounded independently of \( \Delta x \) and \( t \). Therefore, there exists a differentiable solution \( u_{\Delta x}(t) \) to (3.1) for all \( t > 0 \). Furthermore, we have the bound
\[ \int_0^T \left( \sum_j \int_{r_{j-1}}^{r_j} (r_j^2 - s^2)\phi'(s)\, ds + \Delta x \sum_j \phi_{j-1}\Delta x |D^- u_j|^2 \right)\, dt \leq C, \]
for some constant \( C \) which is independent of \( t \) and \( \Delta x \).

Now let \( \delta \) be a positive constant, and let \( e \) be some unit vector in \( \mathbb{R}^n \). Choose
\[ \eta(u) = \max \{ \delta |u| - (e, u), 0 \}. \]

and observe that \( (\nabla \eta(u), u) - \eta(u) = 0 \). Furthermore \( \eta \) is convex, so that
\[ \frac{d}{dt}\eta(u_j) + D^{-}(\eta(u_j))\phi_j \leq 0, \]
which implies that
\[ \sum_j \eta(u_j(t)) \leq \sum_j \eta(u_j(0)). \]

We have that \( \eta(u) = 0 \) if and only if \( u \) is in the cone \( \Gamma_\delta = \{ u \mid \delta |u| \leq (e, u) \} \) for some unit vector \( e \). If \( \theta \) denotes the angle between \( e \) and \( u \), then \( u \in \Gamma_\delta \) if \( \cos(\theta) \geq \delta \), thus if \( \delta < 1 \) this is a cone in \( \mathbb{R}^n \) and this cone is positively invariant for (3.1). Observe that there is no loss of generality in choosing the coordinates such that \( e = (1, \ldots, 1)/\sqrt{n} \). If \( 1 > \delta > (n-1)/\sqrt{n} \), then the invariant cone is in the first \( 2^n \)-tant in \( \mathbb{R}^n \), so that \( u_j^{(i)}(t) \geq 0 \) for all \( t > 0 \) if \( u_0 \in \Gamma_\delta \). Since \( r_j = |u_j| \), it follows that \( u_j^{(i)}(t) = 0 \) if and only if \( r_j(t) = 0 \).

Therefore, if \( u_0 \in \{ u \mid |u| \leq R \} \cap \Gamma_\delta \), then \( u_{\Delta x}(x, t) \) is also in this set. This enables us to deduce the weak*- convergence of a subsequence (which we do not relabel) of \( \{u_{\Delta x}\}_{\Delta x > 0} \).

Let now \( \eta_i(r) \) and \( q_i(r) \) be given by (2.1) for \( i = 1, 2 \). We then have that
\[ \frac{d}{dt}\eta_i(u_j) + D^{-}(q_i(r_j)) + e_{1,j} = 0, \]
where
\[ f(r) = r\phi(r), \quad q_1(r) = f(r) - f(k) \quad \text{and} \quad e_{1,j} = \phi_{j-1}\Delta x (D^- u_j)^T \frac{1}{r_{j-1/2}} \left( I - \frac{u_{j-1/2} \otimes u_{j-1/2}}{r_{j-1/2}^2} \right) (D^- u_j). \]
For any vector \( u \), the matrix \( u \otimes u \) is defined as \( (u \otimes u)_{ij} = u_i u_j \). We shall now find an equation satisfied by \( \eta \). First observe that

\[
\frac{d}{dt} r_j + f'(r_j) D_- r_j - \frac{\Delta x}{2} f''(r_{j-1/2}) (D_- r_j)^2 + e_{1,j} = 0.
\]

Multiplying this with \( f'(u_j) \) we get

\[
(3.11) \quad \frac{d}{dt} f(r_j) + q_2(r_j) D_- r_j - f'(r_j) \frac{\Delta x}{2} f''(r_{j-1/2}) (D_- r_j)^2 + f'(r_j) e_{1,j} = 0.
\]

Set

\[
e_{2,j} = \frac{\Delta x}{2} f''(r_{j-1/2}) (D_- r_j)^2.
\]

Then (3.11) can be rewritten as

\[
(3.12) \quad \frac{d}{dt} \eta_2(r_j) + D_- q_2(r_j) + \frac{\Delta x}{2} q_2''(r_{j-1/2}) (D_- r_j)^2 - f'(r_j) (e_{2,j} - e_{1,j}) = 0.
\]

Finally set

\[
e_{3,j} = \frac{\Delta x}{2} q_2''(r_{j-1/2}) (D_- r_j)^2,
\]

and

\[
e_i(x,t) = e_{i,j}(t) \quad \text{for} \quad x \in (x_{j-1/2}, x_{j+1/2}] \quad \text{and} \quad i = 1, 2, 3.
\]

**Lemma 3.2.** Assume that **A.1, A.2 and A.4** hold, then we have that \( e_i \in \mathcal{M}_{\text{loc}}(\mathbb{R} \times [0,T]) \) for \( i = 1, 2, 3 \).

**Proof.** Set \( \Omega = \mathbb{R} \times [0,T] \), and let \( \psi \) be a test function in \( C_0(\Omega) \). Note that from (3.7) and (3.10) it follows that

\[
\int_{\Omega} e_1(x,t) \, dx \, dt \leq C,
\]

where the constant \( C \) does not depend on \( \Delta x \) or \( T \). Since \( e_1 \geq 0 \), this means that

\[
|\langle e_1, \psi \rangle| \leq \int_{\Omega} |\psi| e_1 \, dx \, dt \leq C \|\psi\|_{L^\infty(\Omega)},
\]

and thus \( e_1 \in \mathcal{M}_{\text{loc}}(\Omega) \). To show the same for \( e_2 \) and \( e_3 \) observe that

\[
|D_- r_j| \leq |D_- u_j|.
\]

Since \( \phi'(r) > 0 \), (3.4) implies that

\[
\int_0^T \Delta x \sum_j \Delta x |D_- u_j|^2 \, dt \leq C,
\]

for some constant \( C \) which is independent of \( \Delta x \) and \( T \). We also have that \( f' \) and \( f'' \) are locally bounded, and \( r_{\Delta x} \) is bounded, this means that, for \( i = 2, 3 \),

\[
\int_{\Omega} e_i(x,t) \, dx \, dt \leq C \int_0^T \Delta x \sum_j \Delta x (D_- r_j)^2 \, dt \leq \int_0^T \Delta x \sum_j \Delta x |D_- u_j|^2 \, dt \leq C.
\]

Thus also \( e_2 \) and \( e_3 \) are in \( \mathcal{M}_{\text{loc}}(\Omega) \). \( \Box \)

Observe that, Lemma 3.2 implies that also \( f'(r_j)(e_{1,j} - e_{2,j}) \) is in \( \mathcal{M}_{\text{loc}}(\Omega) \).
Lemma 3.3. Assume that A.1, A.2 and A.4 hold, let \( u_{\Delta x} \) be generated by the scheme (3.1) and set \( r_{\Delta x} = |u_{\Delta x}|. \) Then

\[
\{ \eta_i(r_{\Delta x})_t + q_i(r_{\Delta x}) \}_{\Delta x > 0} \text{ is compact in } H^{-1}_{\text{loc}},
\]

where \( \eta_i \) and \( q_i \) are given by (2.1) for \( i = 1, 2. \)

Proof. Let \( i = 1 \) or \( i = 2, \) and \( \psi \) is a test function in \( H^1_{\text{loc}}(\Omega). \) we define

\[
\langle L_i, \psi \rangle = \langle \eta_i(r_{\Delta x})_t + q_i(r_{\Delta x})_x, \psi \rangle
\]

\[
= \int_0^T \left( \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left( \frac{d}{dt} \eta_i(r_j) \right) \psi(x,t) - q_i(r_j) \psi_x(x,t) \right) dx \right) dt
\]

\[
= \int_0^T \left( \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left( \frac{d}{dt} \eta_i(r_j) \right) \psi(x,t) dx - q_i(r_j) \Delta x D_- \psi(x_{j+1/2},t) \right) dt
\]

\[
= \int_0^T \left( \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left( \frac{d}{dt} \eta_i(r_j) \right) \psi(x,t) + D_- q_i(r_j) \psi(x_{j-1/2},t) \right) dx \right) dt
\]

\[
+ \int_0^T \left( \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left( \psi(x_{j-1/2},t) - \psi(x,t) \right) D_- q_i(r_j) dx \right) dt
\]

\[
= \langle L_{1,i}, \psi \rangle + \langle L_{2,i}, \psi \rangle.
\]

By (3.10), (3.12) and Lemma 3.2 we know that \( L_{1,i} \in \mathcal{M}_{\text{loc}}(\Omega). \) Regarding \( L_{2,i} \) we have

\[
|\langle L_{2,i}, \psi \rangle| = \left| \int_0^T \left( \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} \psi_x(y,t) dy D_- q_i(r_j) dx \right) dt \right|
\]

\[
\leq \int_0^T \left( \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \sqrt{x - x_{j-1/2}} \left( \int_{x_{j-1/2}}^{x_{j+1/2}} \psi_x(y,t)^2 dy \right)^{1/2} dx |D_- q_i(r_j)| dx \right) dt
\]

\[
\leq \int_0^T \left( \sum_j \Delta x^{3/2} \left( \int_{x_{j-1/2}}^{x_{j+1/2}} \psi_x(x,t)^2 dx \right)^{1/2} \| q_i' \|_{L^\infty} |D_- r_j| \right) dt
\]

\[
\leq \| q_i' \|_{L^\infty} \int_0^T \left( \sum_j \Delta x \left( \int_{x_{j-1/2}}^{x_{j+1/2}} \psi_x(x,t)^2 dx \right)^{1/2} \left( \Delta x^2 \sum_j (D_- r_j)^2 \right)^{1/2} \right) dt
\]

\[
\leq \| q_i' \|_{L^\infty} \sqrt{\Delta x} \left( \int_0^T \int_\Omega (\psi_x(x,t))^2 dx dt \right)^{1/2} \left( \int_0^T \Delta x \sum_j (D_- r_j)^2 dx dt \right)^{1/2}
\]

\[
\leq C \sqrt{\Delta x} \| \psi \|_{H^1(\Omega)}.
\]

Therefore the above estimate shows that \( L_{2,i} \) is compact in \( H^{-1}(\Omega). \) By Lemma 2.2, we conclude the sequence \( \{ \eta_i(r_{\Delta x})_t + q_i(r_{\Delta x}) \}_{\Delta x > 0} \) is compact in \( H^{-1}_{\text{loc}}(\Omega). \)
Lemma 3.4. Assume that A.1 – A.4 hold, then there is a subsequence of \( \{\Delta x\} \) (not relabeled) and a function \( r \) such that \( r_{\Delta x} \to r \) a.e. \( (x,t) \in \Omega \). We have that 
\[
\left\{
\begin{array}{ll}
r_t + f(r) x \leq 0, & x \in \mathbb{R}, \ t > 0, \\
r = |u_0|, & x \in \mathbb{R}, \ t = 0,
\end{array}
\right.
\]
in the distributional sense.

Proof. The strong convergence of \( r_{\Delta x} \) follows from the compensated compactness theorem, Theorem 2.1 and the compactness of \( \{\eta_i(r_{\Delta x}) + q_i(r_{\Delta x})\}_{\Delta x > 0} \) for \( i = 1, 2, \) i.e., Lemma 3.3.

To see that \( r \) is a distributional subsolution of the conservation law (1.6), multiply (3.6) with a non-negative test function \( \psi \) and integrate over \( x \) and \( t \) to obtain
\[
\int_0^T \int_{\mathbb{R}} r_{\Delta x} \psi_t + f(r_{\Delta x}) \psi_x \, dx \, dt + \int_0^T \int_{\mathbb{R}} r_{\Delta x}(0,x) \psi(0,x) \, dx \, dt \\
\geq \int_0^T \sum_j f(r_j) \int_{x_j-1/2}^{x_j+1/2} \int_x^{x+\Delta x} (\psi_x(x,t) - \psi_x(z,t)) \, dz \, dx \, dt.
\]
The term on the right tends to 0 as \( \Delta x \to 0 \), which shows that \( r \) is a subsolution. \(\square\)

Let now the vector \( \tau_j \) be defined as
\[
\tau_j = \frac{u_j}{(u_j,e)}, \quad e = \frac{1}{\sqrt{n}} (1, \ldots, 1),
\]
if \( u_j \neq 0 \). If \( u_j = 0 \) set \( \tau_j = \tau_{j+1} \). Observe that this makes sense since \( u_j^{(i)} = 0 \) only if \( r_j = 0 \). Indeed, we have
\[
\frac{d}{dt} u_j^{(i)}(t) + u_j^{(i)}(t) D_{-}\phi_j + \phi_{j-1}(t) D_{-} u_j^{(i)} = 0.
\]
If \( u_j^{(i)}(t) > 0 \) for \( t < t_0 \) and \( u_j^{(i)}(t_0) = 0 \) then \( du_j^{(i)}/dt(t_0) \leq 0 \). If \( u_{j-1}(t_0) > 0 \) then \( du_j^{(i)}/dt(t_0) > 0 \), which is a contradiction. Thus if \( r_{j_0}(t_0) = 0 \), then \( r_j(t) = 0 \) for all \( j < j_0 \) and \( t > t_0 \). Thus the definition of \( \tau_j^{(i)} \) makes sense.

First note that
\[
D_{-}\tau_j = \frac{(D_{-} u_j)(u_j,e) - u_j (D_{-} u_j,e)}{(u_j,e) (u_j-1,e)}.
\]
Using this, we find
\[
\frac{d}{dt} \tau_j = \frac{(du_j/dt - (du_j/dt,e))(u_j,e) - (u_j - (u_j,e))(du_j/dt,e)}{(u_j,e)^2} \\
= -\frac{D_- (u_j \phi_j) (u_j,e) - u_j (D_- (u_j \phi),e)}{(u_j,e)^2} \\
= -\frac{\phi_{j-1} (D_- u_j)(u_j,e) - u_j (D_- u_j,e)}{(u_j,e)^2} \\
= -\frac{(u_j-1,e) D_- \tau_j}{(u_j,e)}.
\]
Now $\tau_j$ satisfies
\begin{equation}
\frac{d}{dt}\tau_j + \lambda_j D_- \tau_j = 0.
\end{equation}

Next, we define for any constant $J > 0$ the set
\[ \mathcal{M}(J; t) := \{ j \mid r_j(t) < r_{j-1}(t) \text{ and } j \Delta x \leq J \}, \]
and
\[ \Gamma_J(t) := \min_{j \in \mathcal{M}(J; t)} r_j(t). \]

**Lemma 3.5.** Assume that the assumption A.4 holds. Then we have that $\Gamma_J(t) \geq \delta \Gamma_J(0) \geq \delta C_J > 0$.

**Proof.** First let $j \in \mathcal{M}(J; t)$. In particular this means that $r_j(t) < r_{j-1}(t)$. Using scheme (3.1), we have
\[
\frac{d}{dt}(u_j, e) = - (D_-(u_j \phi_j), e) \\
= -D_\phi_j(u_j, e) - \phi_{j-1}(D_- u_j, e).
\]

Therefore, if $(u_j, e)$ is a local minima, i.e., if $(u_j, e) < (u_{j-1}, e)$ then $\frac{d}{dt}(u_j, e) \geq 0$. This concludes the proof since $\delta C_J \leq \delta \Gamma_J(0) \leq \delta r_J(0) \leq (u_j(0), e) \leq (u_j(t), e) \leq r_j(t)$. Observe that this proof also implies that the set $\mathcal{M}(J; t)$ can’t increase in time. \qed

**Lemma 3.6.** If the assumptions A.1 - A.5 hold, and if
\begin{equation}
|\tau_{\Delta x}(\cdot, 0)|_{B.V.([0, T]; L_{loc}^1(\mathbb{R}))} \leq C, \quad i = 1, \ldots, n,
\end{equation}
for some constant $C$ which is independent of $\Delta x$, then there is a subsequence of $\{\Delta x\}$ (not relabeled) and functions $\tau^{(i)}$ in $C([0, T]; L_{loc}^1(\mathbb{R}))$ such that $\tau^{(i)}_{\Delta x}(\cdot, t) \rightarrow \tau^{(i)}(\cdot, t)$ in $L_{loc}^1(\mathbb{R})$ for $i = 1, \ldots, n$.

**Remark 3.1.** If $\tau(\cdot, 0)$ is of bounded variation, or if $u_0$ is of bounded variation, and satisfy A.4 and A.5, then (3.14) holds.

**Proof.** Note that $\lambda_j \geq 0$, and that $\lambda_j$ is bounded in any compact interval. We only want to show that $\lambda_j$ is bounded in any compact interval of $\mathbb{R}$, since Kolmogorov’s compactness theorem gives the convergence in $C([0, T]; L_{loc}^1(\mathbb{R}))$. First observe that
\[
|\phi_{j-1}(u_{j-1}, e) - \phi_{j-1}(u_j, e)| \leq \frac{\phi_{j-1}(r_{j-1}, e)}{r_j}.
\]

Now if $r_j \geq r_{j-1}$, then clearly $\lambda_j$ is bounded. Again if $r_j < r_{j-1}$, thanks to Lemma 3.5, we have $\lambda_j$ is bounded.

Set $\theta_j = D_- \tau^{(i)}_j$. Then $\theta_j$ satisfies
\begin{equation}
\frac{d}{dt}\theta_j + \lambda_j D_- \theta_j + \theta_j D_- \lambda_j = 0.
\end{equation}
Let \( \eta_\alpha(\theta) \) be a smooth approximation to \(|\theta|\) such that
\[
\eta''_\alpha(\theta) \geq 0 \quad \text{and} \quad \lim_{\alpha \to 0} \eta_\alpha(\theta) = \lim_{\alpha \to 0} (\theta \eta'_\alpha(\theta)) = |\theta|.
\]
We multiply (3.15) by \( \eta'_\alpha(\theta_j) \) to get an equation satisfied by \( \eta_\alpha(\theta_j) \). Observe that
\[
\lambda_j - 1\eta''_\alpha(\theta_j) D_\theta \theta_j + \theta_j \eta'_\alpha(\theta_j) D_\theta \lambda_j = \lambda_j - 1 D_\theta \eta_\alpha(\theta_j) + \theta_j \eta'_\alpha(\theta_j) D_\theta \lambda_j + \frac{\Delta x}{2} \lambda_j - 1 \eta''_\alpha(\theta_j - 1/2) (D_\theta \theta_j)^2
\]
\[
\geq D_\tau (\lambda_j \eta_\alpha(\theta_j)) + (\theta_j \eta'_\alpha(\theta_j) - \eta_\alpha(\theta_j)) D_\tau \lambda_j.
\]
Hence
\[
\frac{d}{dt} \eta_\alpha(\theta_j) + D_\tau (\lambda_j \eta_\alpha(\theta_j)) \leq (\eta_\alpha(\theta_j) - \theta_j \eta'_\alpha(\theta_j)) D_\tau \lambda_j.
\]
Now let \( \alpha \to 0 \) to obtain
\[
(3.16) \quad \frac{d}{dt} |\theta_j| + D_\tau (\lambda_j |\theta_j|) \leq 0.
\]
If we multiply this with \( \Delta x \), sum over \( j \) and integrate in \( t \), we find that
\[
(3.17) \quad \left| \tau^{(i)}(\cdot, t) \right|_{B.V.} \leq \left| \tau^{(i)}(\cdot, 0) \right|_{B.V.} \leq C
\]
Note that, since \( \tau^{(i)}_{\Delta x}(\cdot, t) \) has bounded variation and satisfies (3.13), it is \( L^1_{loc} \) Lipschitz continuous in \( t \), that is
\[
(3.18) \quad \left| \tau^{(i)}_{\Delta x}(\cdot, t) - \tau^{(i)}_{\Delta x}(\cdot, s) \right|_{L^1([-J, J])} \leq \sup_{j \Delta x \in [-J, J]} \lambda_j \left| \tau^{(i)}(\cdot, 0) \right|_{B.V.} |t - s|,
\]
for any compact interval \([-J, J]\) of \( \mathbb{R} \). Hence, the above estimates (3.17), (3.18) and an application of Kolmogorov’s compactness criterion (Lemma 2.3) shows that \( \tau^{(i)} = \lim_{\Delta x \to 0} \tau^{(i)}_{\Delta x} \) is continuous in \( t \), with values in \( L^1_{loc}(\mathbb{R}) \). In other words, the convergence is in \( C([0, T]; L^1_{loc}(\mathbb{R})) \).

Now we have the strong convergence of \( \tau_{\Delta x} \) and of \( \tau_{\Delta x} \). This means that also \( u_{\Delta x} \) converges strongly to some function \( u \) in \( C([0, T]; L^1_{loc}(\mathbb{R})) \) since we have
\[
(3.19) \quad u = \tau(u, e) = \frac{\tau}{|\tau|} r.
\]

**Theorem 3.1.** Assume that A1 – A5 and (3.14) hold. Let \( u_{\Delta x} \) be defined by (3.1) – (3.2). Then there exists a function \( u \) in \( L^\infty([0, T]; L^1_{loc}(\mathbb{R})) \) and a subsequence \( \{\Delta x_j\} \) of \( \{\Delta x\} \) such that \( u_{\Delta x_j} \to u \) as \( \Delta x_j \to 0 \). The function \( u \) is a weak solution to (1.1).

**Proof.** We have already established convergence.

It remains to show that \( u \) is a weak solution. To this end, observe that
\[
\int_0^T \int_{\mathbb{R}} D_\tau (u_{\Delta x_j} \phi(r_{\Delta x_j})) \psi(x, t) \, dx \, dt = -\int_0^T \int_{\mathbb{R}} u_{\Delta x_j} \phi(r_{\Delta x_j}) D_\tau \psi(x, t) \, dx \, dt.
\]
As \( \Delta x_j \to 0 \), \( D_\tau \psi \to \psi_x \) for any \( \psi \in C^1_0(\Omega) \). This means that \( u \) is a weak solution. \( \square \)

---

1Here we “extend” the definition of \( D_- \) and \( D_+ \) to arbitrary functions in the obvious manner.
4. Fully Discrete Schemes

In this section, we propose three different fully discrete schemes and show two of them converge to the unique entropy solution of (1.1). We start by introducing some notations needed to define the fully discrete finite difference schemes. We reserve $\Delta t$ to denote a small positive number that represent the temporal discretization parameter of the numerical schemes. For $n = 0, 1, \cdots, N$, where $N\Delta t = T$, for some fixed time horizon $T > 0$, we set $t^n = n\Delta t$. For any function $v(t)$, admitting point values, we let $D^t_+$ denote the discrete forward difference operator in the time direction, i.e.,

$$D^t_+ v(t) = \frac{v(t + \Delta t) - v(t)}{\Delta t}.$$ 

Furthermore, we introduce the spatial-temporal grid cells

$$I^n_j = [x_{j-1/2}, x_{j+1/2}] \times [t^n, t^{n+1}].$$

As before, to a sequence $\{u^n_j\} \in \mathbb{Z}, n \geq 0$ we associate the function $u_{\Delta x}$ defined by

$$u_{\Delta x}(x, t) = \sum_{j \in \mathbb{Z}, n \geq 0} u^n_j \mathbbm{1}_{I^n_j}(x, t),$$

similarly, we also define $r_{\Delta x}$ as

$$r_{\Delta x}(x, t) = \sum_{j \in \mathbb{Z}, n \geq 0} r^n_j \mathbbm{1}_{I^n_j}(x, t).$$

First, we consider the following fully discrete finite difference scheme

$$D^t_+ u^n_j + D_- \left( u^n_j \phi \left( |u^n_j| \right) \right) = 0,$$

with initial values

$$u^n_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x) \, dx.$$

We start by proving the following lemma.

**Lemma 4.1.** Suppose $u_0 \in L^2(\mathbb{R})^n \cap L^\infty(\mathbb{R})^n$, and

$$\lambda \leq \min \left\{ \frac{1}{\|f\|_{L^\infty}}, \frac{1}{C_0 \left( 1 + \|u_0\|_{L^\infty(\mathbb{R})} \right)^2} \right\},$$

hold. Then

$$\|u_{\Delta x}(\cdot, t^n)\|_{L^2(\mathbb{R})^n} \leq \|u_0\|_{L^2(\mathbb{R})^n},$$

$$\|u_{\Delta x}(\cdot, t^n)\|_{L^\infty(\mathbb{R})^n} \leq \|u_0\|_{L^\infty(\mathbb{R})^n},$$

for all $n > 0$, furthermore

$$\Delta t \Delta x \sum_{n=0, j \in \mathbb{Z}}^{N-1} \Delta x \left| D_- u^n_j \right|^2 \leq 2 \|u_0\|_{L^2(\mathbb{R})^n}^2,$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $N\Delta t = T$. 

Proof. For any (differentiable) function \( \eta : \mathbb{R}^n \to \mathbb{R} \), we can take the inner product of (4.1) with \( \nabla_u \eta(u^n_j) \) and obtain

\[
\begin{align*}
(4.4) \quad D^*_t \eta \left( u^n_j \right) &+ D_- \left( \eta \left( u^n_j \phi^n_j \right) \right) \\
&+ \left( \nabla_u \eta \left( u^n_j \right), u^n_j \right) - \eta \left( u^n_j \right) \right) \right) D_- \phi^n_j \\
&+ \frac{\Delta x}{2} \phi^n_{j-1} D_- u^n_j D_- u^n_j - \frac{\Delta t}{2} \left( \int \right) d^2 \eta^{n+1/2} D^*_t u^n_j = 0
\end{align*}
\]

As before, we choose \( \eta(u) = |u|^2 \) to get

\[
D^*_t |u^n_j|^2 + D_- \left( |u^n_j|^2 \phi^n_j \right) + |u^n_j|^2 D_- \phi^n_j \\
+ \phi^n_{j-1} \Delta x \left| D_- u^n_j \right|^2 - \Delta t \left| D_- u^n_j \phi^n_j \right|^2 = 0.
\]

The upper line in the above formula can be rewritten as

\[
D^*_t |u^n_j|^2 + D_- \left( |u^n_j|^2 \phi^n_j \right) + \frac{1}{\Delta x} \int_{r^n_j}^r \left( (r^n_j)^2 - s^2 \right) \phi'(s) ds,
\]

where \( g \) is defined in (3.8). In order to balance the two last terms we proceed as follows.

\[
\begin{align*}
\phi^n_{j-1} \Delta x \left| D_- u^n_j \right|^2 &- \Delta t \left| D_- u^n_j \phi^n_j \right|^2 \\
&\geq \phi(0) \Delta x \left( |D_- u^n_j|^2 - \lambda \left| \phi^n_{j-1} D_- u^n_j + u^n_j D_- \phi^n_j \right|^2 \right) \\
&\geq \phi(0) \Delta x \left( |D_- u^n_j|^2 \\
&- \lambda \left( \left( 1 + |u^n_{j-1}| \right) \max_r \phi'(r) \right)^2 \left| D_- u^n_j \right|^2 + |u^n_j|^2 \max_r \phi'(r) \left| D_- u^n_j \right|^2 \right) \\
&\geq \phi(0) \Delta x \left| D_- u^n_j \right|^2 \left( 1 - C_\phi \lambda \left( 1 + |u^n_j|^2 \right) \right),
\end{align*}
\]

where the constant \( C_\phi \) only depends on \( \phi' \). We have the obvious inequality

\[
|u^n_j|^2 \leq \frac{1}{\Delta x} \| u_{\Delta x} (\cdot, t^n) \|_{L^2(\mathbb{R}^n)}^2.
\]

We shall not use this, instead we assume that \( \lambda \) is so small that

\[
(4.5) \quad 1 - C_\phi \lambda \left( 1 + \sup_j r^n_j \right)^2 \geq \frac{1}{2}.
\]

If this holds, setting \( R = r^2 \), then we have that

\[
R^{n+1}_j \leq R^n_j - \lambda \left( g \left( \sqrt{R^n_j} \right) - g \left( \sqrt{R^n_{j-1}} \right) \right) \\
= R^n_j - \lambda \left( \frac{g'}{2 \sqrt{R^n_{j-1/2}}} \right) \left( R^n_j - R^n_{j-1} \right) \\
= R^n_j - \lambda f'(r^n_{j-1/2}) \left( R^n_j - R^n_{j-1} \right) \\
= \left( 1 - \lambda f'(r^n_{j-1/2}) \right) R^n_j + \lambda f'(r^n_{j-1/2}) R^n_{j-1},
\]

\[
= R^n_j + \lambda f'(r^n_{j-1/2}) R^n_{j-1}.
\]
where \( f(r) = r\phi(r) \). Hence if
\[
\lambda \|f\|_{L^\infty} < 1, \tag{4.6}
\]
then \( R_{j+1}^n \) is dominated by a convex combination of \( R^n \) and \( R_{j-1}^n \), and thus
\[
\sup_j \{ |u_{j+1}^n| \} \leq \sup_j \{ |u_j^n| \}. \tag{4.7}
\]
Therefore, if we assume that
\[
\lambda \leq \min \left\{ \frac{1}{\|f\|_{L^\infty}}, \frac{1}{C\phi \left( 1 + \|u_0\|_{L^\infty(\mathbb{R})} \right)^2} \right\},
\]
we have that \( r_j^n \leq r_j^0 \leq \|u_0\|_{L^\infty(\mathbb{R})} \) for all \( n \) and \( j \). Hereafter, (4.7) is always assumed to hold.

Under the CFL-condition, (4.7), the equation for \( R_j^n \) can be written
\[
D_t R_j^n + D_- G(R_j^n) + \frac{\Delta t}{2} |D_- u_j^n|^2 \leq 0,
\]
where \( G(R) = g(\sqrt{R}) \). Summing this over \( n = 0, \ldots, N - 1 \) yields
\[
\Delta x \sum_j R_j^n + \Delta x \Delta t \frac{1}{2} \sum_{n,j} \Delta x |D_- u_j^n|^2 \leq \Delta x \sum_j R_j^n \leq \int_R |u_0|^2 \, dx.
\]
This finishes the proof of the lemma. \( \square \)

Rewriting yet again the scheme for \( R \), we have
\[
D_t R_j^n + D_- G(R_j^n) = e_{1,j} + e_{2,j}^2, \tag{4.8}
\]
where
\[
e_{1,j} = -\frac{1}{\Delta x} \int_{r_j^{n-1}}^{r_j^n} \left( (r_j^n)^2 - s^2 \right) \phi'(s) \, ds
\]
\[
e_{2,j} = -\phi_{j-1}^n \Delta x |D_- u_j^n|^2 + \Delta t \left( |D_- (u_j^n \phi_j^n)|^2 \right).
\]
Let us define
\[
e_{i,\Delta x} = e_{i,j} \quad \text{for} \quad x \in (x_{j-1/2}, x_{j+1/2}) \quad \text{and} \quad t \in [t_n, t_{n+1}], \quad \text{and} \quad i = 1, 2.
\]

**Lemma 4.2.** Assume that A.1, A.2 and A.4 hold, then we have that \( e_{i,\Delta x} \in \mathcal{M}_{loc}(\Omega) \) for \( i = 1, 2 \).

**Proof.** First we observe that there is a constant \( C \) (independent of \( n, j \) and \( \Delta x \)) such that
\[
|e_{1,j}^n| \leq C \Delta x |D_- u_j^n|^2.
\]
We also have that
\[
|e_{2,j}^n| \leq \frac{C}{\Delta x} (r_j^n - r_{j-1}^n)^2 (r_j^n + r_{j-1}^n) \leq C \Delta x |D_- u_j^n|^2,
\]
for some constant \( C \) depending of \( \phi \) and \( \|u_0\|_{L^\infty(\mathbb{R})} \). Then, for a test function \( \psi \in L^\infty(\Omega) \),
\[
\iint_{\Omega} e_{i,\Delta x} \psi \, dx \, dt \leq C \|\psi\|_{L^\infty(\Omega)} \iint_{\Omega} \Delta x |D_- u_{\Delta x}|^2 \, dx \, dt \leq C \|\psi\|_{L^\infty(\Omega)},
\]
for \( i = 1, 2 \). This finishes the proof. \( \square \)
Multiplying (4.8) by \( G'(R_j^n) \) we find

\[
D_t^n G_j^n + D_- q_{2,j}^n = G'(R_j^n) \left( e_{1,j}^n + e_{2,j}^n \right) \\
+ \frac{\Delta t}{2} G'' \left( R_j^{n+1/2} \right) (D_t^n R_j^n)^2 \\
+ \frac{\Delta x}{2} G'' \left( R_{j-1/2}^n \right) (D_- R_j^n)^2 - \frac{\Delta x}{2} q_2'' \left( \bar{R}_{j-1/2}^n \right) (D_- R_j^n)^2,
\]

where

\[
q_2(R) = \int_k^R (G'(\theta))^2 \, d\theta.
\]

We find that

\[
G''(R) = \frac{1}{2} \phi''(r) + \phi'(r) r.
\]

This is potentially unbounded as \( r \downarrow 0 \), hence we shall assume that \( \phi \) is such that

\[
(4.10) \quad \lim_{r \downarrow 0} \frac{\phi'(r)}{r} < \infty.
\]

We are going to show that also the right hand side of (4.9) is in \( M_{\text{loc}}(\Omega) \). First observe that since \( r_j^n \) is bounded,

\[
(D_- R_j^n)^2 = \left[ (r_j^n + r_{j-1}^n) D_- r_j^n \right]^2 \leq C (D_- r_j^n)^2 \leq C |D_- u_j^n|^2,
\]

and similarly \((D_t^n R_j^n)^2 \leq |D_t u_j^n|^2\). Defining

\[
e_{3,j}^n = \frac{\Delta t}{2} G'' \left( R_j^{n+1/2} \right) (D_t^n R_j^n)^2 \\
e_{4,j}^n = + \frac{\Delta x}{2} G'' \left( R_{j-1/2}^n \right) (D_- R_j^n)^2 - q_2'' \left( \bar{R}_{j-1/2}^n \right) (D_- R_j^n)^2,
\]

By the assumption (4.10) \( G'' \) and \( q_2'' \) are locally bounded, hence we have that

\[
|e_{i,j}^n| \leq C \Delta x |D_- u_j^n|^2, \quad \text{for } i = 3, 4,
\]

and for some \( C \) which is independent of \( \Delta x \). By the same argument used to prove Lemma 4.2, also \( e_{3,\Delta x} \) and \( e_{4,\Delta x} \) are in \( M_{\text{loc}}(\Omega) \). We now have established that

\[
D_t^n \eta_{i,j}^n + D_- q_{i,j}^n = E_{i,j}^n, \quad i = 1, 2,
\]

where

\[
\eta_{1,j}^n = (R_j^n - k), \quad \eta_{2,j}^n = G(R_j^n) - G(k),
\]

and \( q_{i,j}^n \) are given by the corresponding fluxes

\[
q_{1,j}^n = G(R_{i,j}^n) - G(k), \quad q_{2,j}^n = \int_k^{R_{i,j}^n} G'(\theta)^2 \, d\theta.
\]

The terms \( E_{i,j}^n \) are such that the corresponding piecewise continuous functions \( E_{i,\Delta x} \) are in \( M_{\text{loc}}(\Omega) \).
Lemma 4.3. Suppose (4.10) and the assumptions A.1 – A.4 hold. Then there is a subsequence of \{R_{\Delta x}\} (not relabeled) and a function \( R \) such that \( R_{\Delta x} \to R \) a.e. \((x,t) \in \Omega\). We have that \( R \in L^\infty([0,T]; L^1(\mathbb{R})) \). Furthermore, \( R \) satisfies

\[
\begin{align*}
R_t + G(R)_x &\leq 0, \quad x \in \mathbb{R}, \quad t > 0, \\
R &\equiv 0, \quad x \in \mathbb{R}, \quad t = 0,
\end{align*}
\]

in the distributional sense.

Proof. The condition A.3 means that if we can show that \((\eta_{i,\Delta x})_t + (q_{i,\Delta x})_x\) for \( i = 1, 2 \) is compact in \( H^{-1}_{\text{loc}}(\Omega) \), \( R_{\Delta x} \) converges strongly. To show this, let \( \psi \) be a smooth function in \( H^1_{\text{loc}}(\Omega) \). We have that

\[
\langle (\eta_{i,\Delta x})_t + (q_{i,\Delta x})_x, \psi \rangle = \sum_{n=0,j=2}^{\infty} \int_{I^*_n} \eta_{i,j}^n \psi_t + q_{i,j}^n \psi_x \, dxdt
\]

\[
= \sum_{n=0,j=2}^{\infty} \int_{I^*_n} \eta_{i,j}^n \left( \psi(t_{n+1},x) - \psi(t_n,x) \right) \, dx
\]

\[
+ \int_{t_n}^{t_{n+1}} q_{i,j}^n \left( \psi(x_{j+1/2},t) - \psi(x_{j-1/2},t) \right) \, dt
\]

\[
= - \sum_{n,j} \left( \eta_{i,j}^{n+1} - \eta_{i,j}^n \right) \int_{I^*_n} \psi(t_{n+1},x) \, dx
\]

\[
+ \left( q_{i,j}^{n+1} - q_{i,j}^n \right) \int_{t_n}^{t_{n+1}} \psi(x_{j+1/2},t) \, dt
\]

\[
- \sum_j \int_{I^*_n} q_{i,j}^0 \psi(x,0) \, dx
\]

\[
= - \sum_{n,j} \left( D_+\eta_{i,j}^n + D_-q_{i,j}^n \right) \int_{I^*_n} \psi(x,t) \, dxdt - \int_R \eta_{i,\Delta x}(x,0)\psi(x,0) \, dx
\]

\[
+ \sum_{n,j} D_+\eta_{i,j}^n \int_{I^*_n} \psi(x,t) - \psi(t_{n+1},x) \, dxdt
\]

\[
+ \sum_{n,j} D_-q_{i,j}^n \int_{I^*_n} \psi(x,t) - \psi(x_{j-1/2},t) \, dxdt
\]

\[
= - \sum_{n,j} E_{i,j}^n \int_{I^*_n} \psi(x,t) \, dxdt - \int_R \eta_{i,\Delta x}(x,0)\psi(x,0) \, dx + \alpha_1 + \alpha_2.
\]

Hence, \((\eta_{i,\Delta x})_t + (q_{i,\Delta x})_x\) consists of the sum of \( \alpha_1 \) and \( \alpha_2 \) and a term which is in \( \mathcal{M}_{\text{loc}}(\Omega) \). We must show that \( \alpha_1 \) and \( \alpha_2 \) are compact in \( H^{-1}(\Omega) \). First, observe that since \( R_{\Delta x}^n \) is uniformly bounded,

\[
(D_+\eta_{i,j}^n)^2 \leq C |D_-u_{j}^n|^2 \quad \text{and} \quad (D_-q_{i,j}^n)^2 \leq C |D_-u_{j}^n|^2.
\]

Using this

\[
\alpha_1 \leq C \sum_{n,j} |D_-u_{j}^n| \int_{I^*_n} \int_{t_n}^{t_{n+1}} |\psi_t(x,\tau)| \, d\tau \, dxdt
\]

\[
\leq C \sum_{n,j} |D_-u_{j}^n| \int_{I^*_n} \sqrt{t_{n+1} - t_{t_n}} \left( \int_{t_n}^{t_{n+1}} (\psi_t(x,\tau))^2 \, d\tau \right)^{1/2} \, dxdt
\]
\[
\leq C \sum_{n,j} |D_u u_j^n| \Delta t^{3/2} \sqrt{\Delta x} \left( \int_t^t (\psi_t(x,t))^2 \, dx \right)^{1/2}
\]
\[
\leq C \sqrt{\Delta t} \left( \Delta t \Delta x \sum_{n,j} \Delta x |D_u u_j^n|^2 \right)^{1/2} \left( \sum_{n,j} \int_t^t (\psi_t(x,t))^2 \, dx \right)^{1/2}
\]

(4.11) \quad \leq C \sqrt{\Delta x} \|\psi\|_{L^1(\Omega)}.

Hence \(\alpha_1\) is compact in \(H^{-1}(\Omega)\), that \(\alpha_2\) is compact follows by analogous arguments.

Now we have established the strong convergence (along a subsequence which we do not relabel) of \(R_{\Delta x}\).

By the CFL-condition, (4.7), the right hand side of (4.8) is negative, hence
\[
D_t R_t^n + D_- G(R^n_t) \leq 0.
\]

Multiplying this with a non-negative test function \(\psi\), doing a summation by parts and then sending \(\Delta x\) to zero, using the same arguments that led to (4.11), yields that the limit \(R\) is a subsolution in the distributional sense. \(\Box\)

Since \(R_{\Delta x}\) converges strongly to \(R\), also \(r_{\Delta x}\) will converge strongly to \(r := \sqrt{R}\). The sequence \(\{u_{\Delta x}\}_{\Delta x>0}\) is uniformly bounded, so a subsequence will converge weak-* to some function \(u \in L^\infty(\Omega)\). By using the arguments leading up to (4.11) it is straightforward to show that
\[
\int_\Omega w \psi_t + u \phi(r) \psi_x \, dx + \int_\Omega u_0(x) \psi(x,0) \, dx = 0,
\]
for all \(\psi \in C^\infty_0(\Omega)\). Hence the limit \(u\) is a distributional solution of
\[
u_t + (u \phi(r))_x = 0.
\]

In order to conclude that \(u\) is a weak solution to (1.1), we would have to show that \(|u| = r\). We have not been able to prove this, and merely conclude that \(|u| \leq r\).

The reason for this is that \(v \mapsto |v|\) is convex, and that weak limits of a convex function are not less than the convex function of the weak limit.

To overcome this difficulty, we propose another fully discrete scheme based on explicit decoupling of the variables \(r\) and \(w\).

4.1. **A scheme which enforces the entropy condition.** Define
\[
w_{\Delta x} = \begin{cases} \frac{w_{\Delta x}}{r_{\Delta x}}, & \text{if } r_{\Delta x} \neq 0, \\ 0, & \text{if } r_{\Delta x} = 0, \end{cases}
\]
and let \(r_{\Delta x}\) and \(w_{\Delta x}\) satisfy
\[
\begin{aligned}
r_j^{n+1} &= r_j^n - \Delta t D^- f_j^n, \quad n \geq 0, \\
r_j^0 &= |u_j^0|,
\end{aligned}
\]
and
\[
\begin{aligned}
w_j^{n+1} &= w_j^n - \Delta t \phi_j^n D^- w_j^n, \quad n \geq 0, \\
r_j^0 w_j^0 &= u_j^0.
\end{aligned}
\]

To ensure the convergence of the approximations \(\{r_{\Delta x}\}\) we choose
\[
\Delta t \|f'\|_{L^\infty(\mathbb{R})} \leq \Delta x.
\]

We list some useful properties of \(r_{\Delta x}\) in the next lemma [9].
Lemma 4.4. Assume that the CFL condition (4.14) holds and \( r_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Then for each \( \Delta x > 0 \) we have that

(a) \(-M \leq r_{\Delta x}(x, t) \leq M\), for all \( x \) and \( t > 0 \).

(b) For \( n \geq 0 \) the functions

\[
\begin{align*}
    n &\mapsto \Delta x \sum_{j \in \mathbb{Z}} |r^n_j|, \\
    n &\mapsto \sum_{j \in \mathbb{Z}} |r^n_j - r_{j-1}^n|, \\
    n &\mapsto \sum_{j \in \mathbb{Z}} |r^{n+1}_j - r^n_j|
\end{align*}
\]

are non-increasing. In particular this means that the family \( \{r_{\Delta x}\}_{\Delta x > 0} \) is (uniformly in \( \Delta x \)) bounded in \( L^\infty(\mathbb{R}^+; L^1(\mathbb{R})) \) \( \cap BV(\mathbb{R} \times \mathbb{R}^+) \).

(c) Moreover \( r_{\Delta x}(\cdot, t) \to r(\cdot, t) \) strongly in \( L^1(\mathbb{R}) \) for all \( t \geq 0 \), where \( r \in Lip([0, T]; L^1(\mathbb{R})) \) and is the unique entropy (in the sense of Kružkov) solution of the conservation law

\[
\begin{align*}
    \begin{cases}
        r_t + f(r)_x = 0, \\
        r(x, 0) = r_0.
    \end{cases}
\end{align*}
\]

Observe that, we can write the scheme (4.13) as

\[
\begin{align*}
    w^{n+1}_j &:= (1 - \lambda \phi^n_j) w^n_j + \lambda \phi^n_j w^n_{j-1}.
\end{align*}
\]

If \( \lambda \phi^n_j < 1 \) for all \( j \), then \( w^{n+1}_j \) is a convex combination of \( w^n_j \) and \( w^n_{j-1} \). Thus

\[
\begin{align*}
    \inf_j w^n_j &\leq w^n_j \leq \sup_j w^n_j, \\
    n &> 0.
\end{align*}
\]

and

\[
\begin{align*}
    |w_{\Delta x}(\cdot, t)|_{B.V.(\mathbb{R})^n} &\leq |w_{\Delta x}(\cdot, 0)|_{B.V.(\mathbb{R})^n}.
\end{align*}
\]

Furthermore

\[
\begin{align*}
    \Delta x \sum_j |w^{n+1}_j - w^n_j| &\leq \Delta t \|\phi\|_{L^\infty} \sum_j |w^n_j - w^n_{j-1}| \leq C \Delta t |w_{\Delta x}(\cdot, t_n)|_{B.V.(\mathbb{R})^n}.
\end{align*}
\]

Hence the map \( t \mapsto w_{\Delta x}(\cdot, t) \) is \( L^1 \)-Lipschitz continuous. Finally, the above estimates (4.17), (4.18) and an application of Kolmogorov’s compactness criterion (Lemma 2.3) shows that \( w = \lim_{\Delta x \to 0} w_{\Delta x} \), where the convergence is along a subsequence. Furthermore \( w \in C([0, T]; (L^1_{loc}(\mathbb{R}))^n) \).

Multiply the equation (4.12) for \( r^{n+1}_j \) with that (4.13) for \( w^{n+1}_j \) to get

\[
\begin{align*}
    r^{n+1}_j w^{n+1}_j &:= (r^n_j - \Delta t D_- f^n_j) (w^n_j - \Delta t \phi^n_j D_- w^n_j) \\
    &= r^n_j w^n_j - \Delta t (w^n_j D_- f^n_j + f^n_j D_- w^n_j) + \Delta t^2 \phi^n_j D_- f^n_j D_- w^n_j \\
    &= r^n_j w^n_j - \Delta t (w^n_j D_- f^n_j + f^n_{j-1} D_- w^n_j) - \Delta t (f^n_j - f^n_{j-1}) D_- w^n_j \\
    &\quad + \Delta t^2 \phi^n_j D_- f^n_j D_- w^n_j \\
    &= r^n_j w^n_j - \Delta t D_- (f^n_j w^n_j) + \Delta t (f^n_j - f^n_{j-1}) D_- w^n_j (\lambda \phi^n_j - 1).
\end{align*}
\]

Then we have

\[
D_+ (r^n_j w^n_j) + D_- (f^n_j w^n_j) = \Delta t (\lambda \phi^n_j - 1) (f^n_j - f^n_{j-1}) D_- w^n_j =: e^n_j.
\]

Let now \( \psi \in C^\infty_0(\Omega) \) be a test function, multiply the above equation by \( \psi \) and integrate over \( \Omega \) to get

\[
\sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} r^n_j w^n_j D^T_- \psi + f^n_j w^n_j D^T_+ \psi \, dx \, dt.
\]
\[
\int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} r^n_jw^n_j \psi\, dx dt = \sum_{n,j} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} e^n_j\psi\, dx dt.
\]

Since we have the convergence of \( r_{\Delta x} \) and \( w_{\Delta x} \), the left hand side of this converges to
\[
\int_{\Omega} \int_{t}^{T} rw\psi_t + f(r)w\psi_x\, dx dt + \int_{\mathbb{R}} r(x,0)w(x,0)\psi(x,0)\, dx.
\]

Regarding the right hand side we have
\[
\left| \sum_{n,j} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} e^n_j\psi\, dx dt \right| \leq \Delta t \| \psi \|_{L^\infty(\Omega)} (\lambda \| \phi \|_{L^\infty} + 1) \Delta t \sum_{n,j} |f^n_j - f^n_{j-1}| \| w^n_j - w^n_{j-1} \|
\leq \Delta t C \| \psi \|_{L^\infty(\Omega)} \| w_{\Delta x}(x,0) \|_{L^\infty(\mathbb{R})} T \| r_{\Delta x}(x,0) \|_{B.V.(\mathbb{R})},
\]

where \( T \) is such that \( \text{supp} \psi \subset [0,T] \). Hence
\[
\int_{\Omega} \int_{t}^{T} rw\psi_t + f(r)w\psi_x\, dx dt + \int_{\mathbb{R}} r(x,0)w(x,0)\psi(x,0)\, dx = 0.
\]

Hence, we see that \( rw \) is a weak solution to the Cauchy problem
\[
(rw)_t + (f(r)w)_x = 0.
\]

In other words, \((r,rw)\) is a weak solution to
\[
\begin{align*}
\Delta t r + f(r)_x & = 0, \\
\Delta t r w + (\phi(|r|)r)_x & = 0.
\end{align*}
\]

Next, we shall make use of Lemma 2.4 to conclude that \( |w| = 1 \).

Finally, collecting all the results above, we have proved the following theorem.

**Theorem 4.1.** Assume that \( u_0 \in B.V.(\mathbb{R}) \). If \( \lambda = \Delta t/\Delta x \) satisfies the CFL-condition \( \lambda < \sup_x f'(\|u_0(x)\|) \), and \( u_{\Delta x} \) is defined by (4.12), (4.13), then \( u = \lim_{\Delta x \to 0} u_{\Delta x} \) is the unique entropy (in the sense of Definition 1.2) solution to (1.1).

**Remark 4.1.** Observe that to establish the convergence of the entire sequence \( \{u_{\Delta x}\}_{\Delta x \geq 0} \), we first use Kolmogorov’s compactness criterion to establish convergence of a subsequence, then, since the entropy solution \( u \) is unique, every subsequence of \( \{u_{\Delta x}\} \) will contain a subsequence converging to \( u \). This means that the entire sequence converges.

We propose another scheme based on discretizing the “conservative” form (1.6)–(1.7). Let \( r_{\Delta x} \) and \( u_{\Delta x} \) satisfy
\[
\begin{align*}
r^n_{j+1} & = r^n_j - \Delta t D_- f(r^n_j), \quad n \geq 0, \\
r^n_0 & = |u^n_0|,
\end{align*}
\]

and
\[
\begin{align*}
u^n_{j+1} & = u^n_j - \Delta t D_- (u^n_j \phi(r^n_j)), \\
\end{align*}
\]

for \( n \geq 0 \) and \( f(r) = r\phi(r) \), with \( u^0_j \) given by (3.2). Again as before, regarding the convergence of the approximations \( \{r_{\Delta x}\} \) we have Lemma 4.4. On the other hand, we can rewrite the scheme for \( u^n_j \) as
\[
\begin{align*}
u^n_{j+1} & = u^n_j + \lambda (u^n_j \phi^n_j - u^n_{j-1} \phi^n_{j-1}) =: F^n_j \left( u^n_j, u^n_{j-1} \right).
\end{align*}
\]
Each component $F_j^{(i),n}(u_j^n, u_j^{n-1}) = F_j^n(u_j^{(i),n}, u_j^{(i),n-1})$ is linear and monotone in both its arguments. Furthermore, the scheme for $r_j^n$ can be written
\[ r_j^{n+1} = F_j^n(r_j^n, r_{j-1}^n). \]
Set $w_j^n = r_j^n - u_j^{(i),n}$. Then, by the linearity of $F_j^n$,
\[ w_j^{n+1} = F(r_j^n, r_{j-1}^n) - F(u_j^{(i),n}, u_{j-1}^{(i),n}) = F(r_j^n - u_j^{(i),n}, r_{j-1}^n - u_{j-1}^{(i),n}) = F(w_j^n, w_{j-1}^n). \]
Therefore $\inf_j w_j^0 \leq w_j^{n+1} \leq \sup_j w_j^0$. We have that $w_j^0 = |u_j^0| - u_j^{(i),0} \geq 0$, therefore
\[ |u_{\Delta x}(x, t)| \leq r_{\Delta x}(x, t) \leq |u_0(x)|. \]
Hence, the sequence $\{u_{\Delta x}\}_{\Delta x > 0}$ is uniformly bounded, and there converges $L^\infty$ weak-* (modulo a subsequence) to some function $u$. A straightforward computation yields that
\[
\left| \int_\Omega u_{\Delta x}\psi_t + u_{\Delta x}\phi(r_{\Delta x}) \phi_t \, dxdt + \int_\mathbb{R} u_0(x)\psi(x, 0) \, dx \right|
\leq \sum_{n=1,3} |u_j^n| \left| \int_{I_j^n} (D_x^+ \psi - \psi_t) \, dxdt \right| + |u_j^n| \phi(r_j^n) \left| \int_{I_j^n} (D_x^- \psi - \psi_x) \, dxdt \right|
+ \frac{1}{\Delta t} \left| \int_0^{\Delta t} \int_\mathbb{R} u_{\Delta x}(x, 0)\psi(x, t) - \Delta t u_0(x)\psi(x, 0) \, dx \right|
\leq C\Delta x \left( \|\psi_t\|_{L^\infty(\Omega)} + \|\psi_{xx}\|_{L^\infty(\Omega)} + \|\psi_t\|_{L^\infty(\Omega)} \right). \]
Sending $\Delta x$ to zero, we see that the limit $u$ is a weak solution to the Cauchy problem
\[ u_t + (u\phi(r))_x = 0. \]
Again we shall make use of Lemma 2.4 to conclude that $|u| = r$.

Finally, we have proved the following theorem.

**Theorem 4.2.** Assume that $u_0 \in B.V.(\mathbb{R})$. If $\lambda = \Delta t/\Delta x$ satisfies the CFL-condition $\lambda < \sup_x f'(|u_0(x)|)$, and $u_{\Delta x}$ is defined by (4.21), (4.22), then $u = \lim_{\Delta x \to 0} u_{\Delta x}$ is the unique entropy (in the sense of Definition 1.2) solution to (1.1).

5. Numerical experiments

We close this paper by demonstrating how these schemes work in practice. We perform all the computations for $2 \times 2$ system with $\phi(r) = r^2$.

5.1. Numerical Experiment 1. In this case we approximate the system (1.1) with initial data
\[
U_0(x) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0. \end{cases}
\]
It is not difficult to find the exact solution of (1.1) in this case. For the sake of completeness we write the explicit form of the exact solutions $U(x, t) = \bar{U}(x/t)$. 
If $|U_l| < |U_r|$, then

$$
\bar{U}(\xi) = \begin{cases} 
U_l, & \xi \leq |U_l|^2, \\
U_m, & |U_l|^2 \leq \xi \leq 3|U_l|^2, \\
(\xi)^{1/2} \frac{U_r}{|U_r|}, & 3|U_l|^2 \leq \xi \leq 3|U_r|^2, \\
U_r, & \xi \geq 3|U_r|^2,
\end{cases}
$$

If $|U_l| > |U_r|$, then

$$
\bar{U}(\xi) = \begin{cases} 
U_l, & \xi \leq |U_l|^2, \\
U_m, & |U_l|^2 \leq \xi \leq |U_l|^2 + |U_l| |U_r| + |U_l|^2, \\
U_r, & \xi \geq |U_l|^2 + |U_l| |U_r| + |U_l|^2,
\end{cases}
$$

with $U_m = \frac{|U_l|}{|U_r|} U_r$ in both cases.

In what follows, we test the fully discrete explicit numerical scheme (4.1) with initial data

$$
U_0(x) = \begin{cases} 
U_-, & x < 0, \\
U_+, & x > 0,
\end{cases}
$$

where

$$
U_- = (0.5, 1.5), \quad U_+ = (1.5, 2.0),
$$

for the first experiment and

$$
U_- = (1.5, 2.0), \quad U_+ = (0.5, 1.5),
$$

for the second experiment. The computations are performed on a computational domain $[-1, 20]$ with 4000 mesh points. To enforce the CFL condition we set the time step $\Delta t = \frac{\text{CFL} \Delta x}{3 \sup |U_0|^2}$, where we use a CFL number 0.75. Although we do not plot the exact solutions, a comparison of the computational results displayed in Figs 5.1 with the exact solution shows good agreement.

5.2. **Numerical experiment 2.** In this case, we test our fully discrete explicit numerical scheme (4.12)–(4.13) with initial data $U_0 = r_0 w_0$, where

$$
r_0(x) = \begin{cases} 
r_-, & x < 0, \\
r_+, & x > 0,
\end{cases}
$$

with

$$
r_- = 1.0, \quad r_+ = 0.75,
$$

for the first and third numerical experiments and

$$
r_- = 0.75, \quad r_+ = 1.0,
$$

for the second and fourth numerical experiments. Similarly, for $w_0$ we take

$$
w_0(x) = \begin{cases} 
(1.0, 0.0), & x < 0.2 \\
(\cos(8\pi(x - 0.2)), \sin(8\pi(x - 0.2))), & 0.2 \leq x \leq 0.7, \\
(1.0, 0.0), & x \geq 0.7,
\end{cases}
$$
For the first and second numerical experiments and

\[ w_0(x) = \begin{cases} (1.0, 0.0), & x \leq 0.2, \\ (-1.0, 0.0), & x \geq 0.2, \end{cases} \]

In this case also, it is easy to find the exact solution. Although we do not plot the exact solutions, we give the explicit form of the exact solution. The exact solution is given by \( U = rw \) with

\[ r(x, t) = \begin{cases} r_-, & x \leq st, \\ r_+, & x \geq st, \end{cases} \quad \text{with} \quad s = r_-^2 + r_- r_+ + r_+^2, \]
and

\[ w(x, t) = \begin{cases} 
    w_0(x - r^2 t), & x \leq r^2 t, \\
    w_0 \left( \frac{x}{r} - r^2 t \right), & r^2 t \leq x \leq st, \\
    w_0(x - r^2 t), & x \geq st, 
\end{cases} \]

for the first and third numerical experiments. Similarly,

\[ r(x, t) = \begin{cases} 
    r_-, & x \leq 3r^2 t, \\
    \left( x/3t \right)^{1/2}, & 3r^2 t \leq x \leq 3r^2 t, \\
    r_+, & x \geq 3r^2 t, 
\end{cases} \]

and

\[ w(x, t) = \begin{cases} 
    w_0(x - r^2 t), & x \leq r^2 t, \\
    w_0 \left( \frac{x}{r} - r^2 t \right), & r^2 t \leq x \leq 3r^2 t, \\
    w_0 \left( \frac{2}{3\sqrt{3r_+}} \left( x/3t - 1/2 \right) \right), & 3r^2 t \leq x \leq 3r^2 t, \\
    w_0(x - r^2 t), & x \geq 3r^2 t, 
\end{cases} \]

for the second and fourth numerical experiments.

In all the experiments computational domain is \([-1, 4]\) and we use Neumann boundary conditions at the left boundary. We also use a CFL number 0.75 and 4000 mesh points for all the experiments. A comparison of the computational results displayed in Figs 5.2–5.3 with the exact solution shows good agreement.

Below we show the computational results for four different qualitatively significant sets of data: a compression or an expansion wave in \(r\) initiated slightly behind a continuous pulse or a discontinuous contact wave in \(w\). Fig 5.2–5.3 display the computed solution at three different times. In the plots, the dot-dash curve represents the first component of \(U\) and the dotted curve represents the second component, while the solid curve represents the \(r\)-component of \((r, U)\).

5.3. Numerical convergence rates. We have not obtained any theoretical convergence rates for the schemes presented here. Never the less, it is interesting to check the possible convergence rate in practice. To this end we have used Riemann initial data (5.1) with \(U_l = (1, 1)\) and \(U_r = (3, 1)\). In this case the exact solution is given by formula (5.2), i.e., a rarefaction wave followed by a contact discontinuity.

We define the relative error for a scheme as

\[ E = 100 \times \frac{\sum_j \left| u_j^N - u(x_j, N\Delta t) \right|}{\sum_j \left| u(x_j, N\Delta t) \right|}, \]

where \(u\) is the exact solution found by (5.2) and \(u_j^N\) is the approximation computed by the numerical scheme. Note that this is a first order accurate approximation to the relative \(L^1\) error since \(u\) is piecewise continuous.

We have computed the errors for the three schemes (4.1) (“scheme1”), the conservative scheme (4.21) – (4.22) (“scheme2”) and the non-conservative scheme (4.12) – (4.13) (“scheme3”). Table 5.1 summarizes the results. We computed the approximations for \(t = 1\), and used \(\Delta x = 40/2^N\) for \(N = 5, \ldots, 14\) for \(x \in [-1, 39]\). From this table it emerges that the three schemes produce very similar errors and numerical convergence rates. This convergence rate is expected to be not higher than 1/2, since the solution contains a contact discontinuity, and the schemes are
Figure 5.2. Left column: Experiment-1: A shock wave initiated behind a continuous rotational wave. The dotted-dashed curve represents the first component of $U$, the dotted curve represents the second component and the solid curve represents $r$. Right column: Experiment-2: An expansion wave initiated behind a continuous rotational wave. The dotted-dashed curve represents the first component of $U$, the dotted curve represents the second component and the solid curve represents $r$. 

(a) $T = 0$  
(b) $T = 0$  
(c) $T = 0.25$  
(d) $T = 0.25$  
(e) $T = 0.75$  
(f) $T = 0.75$
Figure 5.3. Left column: Experiment-3: A shock wave initiated behind a discontinuous rotational wave. The dotted-dashed curve represents the first component of $U$, the dotted curve represents the second component and the solid curve represents $r$. Right column: Experiment-4: An expansion wave initiated behind a discontinuous rotational wave. The dotted-dashed curve represents the first component of $U$, the dotted curve represents the second component and the solid curve represents $r$. 
formally first order. Table 5.1 actually seems to predict that the three schemes have a convergence rate of about 0.6.

Table 5.1. Relative errors and rates.

| N  | scheme1 E rate | scheme2 E rate | scheme3 E rate |
|----|----------------|----------------|----------------|
| 5  | 3.32           | 3.30           | 3.40           |
| 6  | 2.04           | 2.08           | 2.31           |
| 7  | 1.31           | 1.35           | 1.50           |
| 8  | 0.81           | 0.83           | 0.89           |
| 9  | 0.51           | 0.52           | 0.54           |
| 10 | 0.32           | 0.32           | 0.33           |
| 11 | 0.20           | 0.20           | 0.21           |
| 12 | 0.13           | 0.13           | 0.13           |
| 13 | 0.09           | 0.08           | 0.08           |
| 14 | 0.06           | 0.05           | 0.05           |

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