Detecting a radio signal with the unknown parameters and inexact envelope shape

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Abstract. The new quasi-likelihood algorithm is considered for detecting a radio signal with the unknown amplitude, duration, initial phase and inexact envelope shape against Gaussian white noise. The structure and the statistical characteristics of the introduced detection algorithm are found. The influence is studied of the difference in the envelope shapes of the received and reference signals upon detection efficiency.

1. Introduction
The problem of detecting a radio signal against random distortions is relevant for a number of important practical applications of statistical radio engineering. In the case of the unknown signal amplitude and initial phase, this problem is solved and used as a classic example in many textbooks [1]-[3]. However, due to the peculiarities of the propagation and reflection of a radio signal, as a rule, its duration is unknown. This is very common in radar, navigation, seismology, etc. In [4]-[6], there are studied the algorithms for detecting the signals with the unknown amplitude and duration in case when the signal waveform is arbitrary and its harmonic filling is absent, while in [7]-[9] there are tested the algorithms for detecting the rectangular and untailored radio signals with the unknown amplitude, initial phase, and duration. At the same time, the shape of the radio signal envelope is frequently unknown too. This may be the case, for example, when a signal is passed through the unknown nonlinear propagation channel or the interception is implemented of an extraneous signal whose envelope shape is not known exactly. In this regard, it is of interest to synthesize and analyze the algorithms for detecting radio signals with the unknown amplitude and initial phase while the envelope shape is known inexacty.
2. The problem statement
Let the additive mix
\[ \xi(t) = \gamma_0 s(t, a_0, \tau_0, \varphi_0) + n(t) \]  
(1)
is passed to the receiver input over the time interval \([0,T]\). Here
\[ s(t, a_0, \tau_0, \varphi_0) = \begin{cases} 
    a_0 f(t) \cos(\omega t - \varphi_0), & 0 \leq t \leq \tau_0, \\
    0, & t < 0, t > \tau_0
\end{cases} \]  
(2)
is the useful signal with the amplitude \(a_0\), the envelope \(f(t)\), the carrier frequency \(\omega\), the initial phase \(\varphi_0\) and the duration \(\tau_0\) while \(n(t)\) is Gaussian white noise with the one-sided spectral density \(N_0\). The discrete parameter \(\gamma_0\) takes the value of either 1, if the signal (2) is present in the observable realization (1), or 0, otherwise. Let us presuppose that the signal (2) is narrow-band, that is, the condition \(\omega > > \Delta \omega\) is satisfied, where \(\Delta \omega\) is the signal bandwidth. The parameters \(a_0, \tau_0, \varphi_0\) are a priori unknown ones and take the values from the prior intervals \([0, \infty), [T_1, T_2], [0, 2\pi]\), respectively. With the observable realization (1) and available prior information, the decision should be made on the presence or absence of the signal (2).

The specified task can be formulated in terms of the hypothesis tests. Namely, the simple hypothesis \(H_0: \xi(t) = n(t)\) to be verified is that the signal (2) is absent, that is, \(\gamma_0 = 0\). Its complex alternative is \(H_1: \xi(t) = s(t, a_0, \tau_0, \varphi_0) + n(t)\) stating that the signal (2) is present in the observable realization (1), that is, \(\gamma_0 = 0\).

3. Quasi-likelihood detection algorithm

3.1. The synthesis of the algorithm for detecting the radio signal with the inexactly known envelope and the unknown parameters
If the envelope shape \(f(t)\) is known, then, according to the theory of statistical decisions, the logarithm of the functional of the likelihood ratio (FLR) can be found and, on its basis, the maximization of the logarithm of FLR (3) by the variables \(a\) and \(\varphi\) for all the possible duration values \(\tau \in [T_1, T_2]\):
\[ L(a, \tau, \varphi) = \frac{2a}{N_0} \int_0^\tau f(t) \cos(\omega t - \varphi) [\xi(t) - s(t, a_0, \tau_0, \varphi_0)] \ \text{d}t. \]  
(3)
Here \(a, \tau, \varphi\) are the current values of the parameters \(a_0, \tau_0, \varphi_0\).

In [5], both synthesis and analysis are carried out of the ML algorithm for the signal (2) detection. According to [5], the ML detector should generate the decision statistics obtained by means of maximization of the logarithm of FLR (3) by the variables \(a\) and \(\varphi\) for all the possible duration values \(\tau \in [T_1, T_2]\):
\[ L(\tau) = \sup_{a, \varphi} L(a, \tau, \varphi) = N_0 \left[ X^2(\tau) + Y^2(\tau) \right] / \int_0^\tau f^2(t) \ \text{d}t. \]  
(4)
In (4), the notations are:
\[ \begin{cases} 
    X(\tau) = \frac{2}{N_0} \int_0^\tau \xi(t) f(t) \cos(\omega t) \ \text{d}t, \\
    Y(\tau) = \frac{2}{N_0} \int_0^\tau \xi(t) f(t) \sin(\omega t) \ \text{d}t.
\end{cases} \]  
(5)
The decision on the presence or absence of the signal (2) in the observable realization (1) is made by comparing the absolute maximum of the logarithm of FLR (4) with the threshold \( h \) chosen according to the accepted optimality criterion:

\[
L = \sup_{\tau} L(\tau) > h.
\]

As the detection algorithm (6) characteristics the false alarm and missing probabilities are usually used [1]-[3]. The expressions for these probabilities are obtained, for example, in [5].

Let us consider now that the function \( f(t) \) is known inexactly. In this case, in the expressions (3)-(5), instead of the unknown function \( f(t) \), a particular expected function \( g(t) \) can be used to describe the envelope. In this case, the decision statistics (3), (4) transform into

\[
L_g(a, \tau, \varphi) = \frac{2a}{N_0}\int_0^{\tau} g(t)\cos(\omega t - \varphi)\left[\frac{\xi(t)}{2} - g(t)\cos(\omega t - \varphi)/2\right] dt,
\]

\[
L_h(\tau) = \sup_{a,\varphi} L_g(a, \tau, \varphi) = N_0\left[X_g^2(\tau) + Y_g^2(\tau)\right]/\left[2\int_0^{\tau} g^2(t) dt\right],
\]

respectively. Here

\[
\begin{bmatrix}
X_g(\tau) \\
Y_g(\tau)
\end{bmatrix} = \frac{2}{N_0}\int_0^{\tau} \xi(t) g(t) \begin{bmatrix}
\cos \\
\sin
\end{bmatrix}(\omega t) dt.
\]

It is easy to see now that the expressions (7) and (8) define the logarithms of FLR for the expected radio signal, the envelope of which is described by the function \( g(t) \). If the envelopes of the expected and received signals are different, then the processing algorithms developed on the basis of the expressions (7), (8) are called the quasi-likelihood (QL) ones [10].

The QL detector of the radio signal with an inexactly known envelope should generate the decision statistics (8) for all the possible signal durations and make the decision on the signal presence or absence by comparing its absolute maximum with the set threshold \( h \):

\[
L_g = \sup_{\tau} L_g(\tau) > h.
\]

3.2. The analysis of the algorithm for detecting the radio signal with the inexactly known envelope and the unknown parameters

In order to find the efficiency characteristics the QL detection algorithm (10), the decision statistics (8) should be studied. By substituting (1) into (9), and then (9) into (8), and taking into account that the hypothesis is true, the logarithm of FLR \( L_g(\tau) \) is transformed into

\[
L_g(\tau) = L_g\left(\tau|Y_0 = j\right) = \left[\frac{jp(\min(\tau, \tau_0))}{N_{g,0}}\right]^2 + \left[\frac{jp(\min(\tau, \tau_0))\sin\varphi_0 + N_{g,0}(\tau)}{N_{g,0}(\tau)}\right]^2 /2N_{g,0}(\tau), \quad j = 0, 1,
\]

where
\[
\begin{align*}
\begin{bmatrix} N_{gc}(\tau) \\ N_{gs}(\tau) \end{bmatrix} &= 2a_0^\tau \int_0^\tau n(t)g(t) \cos(\tau) dt, \\
q_g(\tau) &= \frac{a_0^2}{N_0} \int_0^\tau g^2(t) dt, \\
p(\tau) &= \frac{a_0^2}{N_0} \int_0^\tau f(t)g(t) dt.
\end{align*}
\]

(12)

Hereinafter, there are neglected integrals of the functions oscillating with double frequency, because the signal (2) is narrow-band.

The noise components \( N_{gc}(\tau) \) and \( N_{gs}(\tau) \) are Gaussian, as they are linear transformations of the Gaussian white noise \( n(t) \). They have zero mathematical expectations, and, by direct averaging (12), for their correlation functions the following can be obtained:

\[
\begin{align*}
\langle N_{gc}(\tau_1)N_{gc}(\tau_2) \rangle &= \langle N_{gs}(\tau_1)N_{gs}(\tau_2) \rangle = q_g(\min(\tau_1, \tau_2)), \\
\langle N_{gc}(\tau_1)N_{gs}(\tau_2) \rangle &= 0.
\end{align*}
\]

Let us consider that the functions describing the shape of the envelopes of the reference and received signals are non-negative. In addition, it is assumed that they reach zero values in that part of the interval \([T_1, T_2]\) that has zero measure only. Then the functions \( q_g(\tau) \) and \( p(\tau) \) are monotonically increasing that allows writing \( q_g(\min(\tau, \tau_0)) = \min(q_g(\tau), q_g(\tau_0)), \ p(\min(\tau, \tau_0)) = \min(p(\tau), p(\tau_0)) \).

In the expression (11), one moves to the new variable \( \zeta = q_g(\tau), \ Z_1 \leq \zeta \leq Z_2, \ Z_1 = q_g(T_1), \ Z_2 = q_g(T_2) \). Then the decision statistics (11) as a function of the variable \( \zeta \) can be represented by the expression

\[
L_{g1}(\zeta) = jP^2[q_g^{-1}(\min(\zeta, \zeta_0))]^2 + jp[q_g^{-1}(\min(\zeta, \zeta_0))\] \tag{13}
\]

\[
\text{where} \quad j = 0.1, \quad q_g^{-1}(\zeta) \quad \text{is the function inverse to} \quad q_g(\tau), \quad \zeta_0 = q_g(\tau_0), \quad \text{while} \quad N_{gc}(\zeta) \quad \text{and} \quad N_{gs}(\zeta) \quad \text{are the statistically independent Gaussian random processes with the zero mathematical expectation and the correlation functions}
\]

\[
\begin{align*}
\langle N_{gc}(\zeta_1)N_{gc}(\zeta_2) \rangle &= \langle N_{gs}(\zeta_1)N_{gs}(\zeta_2) \rangle = \min(\zeta_1, \zeta_2), \tag{14}
\end{align*}
\]

\[N_{g1}(\zeta) = N_{gc}(\zeta)\cos \phi_0 + N_{gs}(\zeta)\sin \phi_0 \quad \text{is the Gaussian random process with the zero mathematical expectation and the correlation function (14).}
\]

The main characteristics of the detection efficiency are the false alarm probability \( \alpha_g = P\{L_g > h \mid \gamma_0 = 0\} \) and the missing probability \( \beta_g = P\{L_g < h \mid \gamma_0 = 1\} \) \cite{11-3}. In order to find them, the distribution function of the random variable \( L_g \) (10) should be determined for both hypotheses: \( F_{g0}(x) = P\{L_g < x \mid \gamma_0 = j\} = P\{\sup L_{g0}(\tau) < x, \tau \in [T_1, T_2]\} \), \( j = 0.1 \). Thus, the detection error probabilities can be found using the distribution function of the maxima of the random processes \( L_{g0}(\tau) \) and \( L_{g1}(\tau) \).

To start with, it is assumed that the signal (2) is absent in the received realization (1), that is, \( \gamma_0 = 0 \) . Then

\[
\begin{align*}
L_{g0}(\zeta) &= \left[ N_{gc}(\zeta) + N_{gs}(\zeta) \right] / 2\zeta, \tag{15}
F_{g0}(x) &= P\{\sup L_{g0}(\tau) < x, \tau \in [T_1, T_2]\} = P\{\sup L_{g0}(\zeta) < x, \zeta \in [Z_1, Z_2]\},
\end{align*}
\]

The notations

\[
X_{gc}(\zeta) = N_{gc}(\zeta)/\sqrt{\zeta}, \quad X_{gs}(\zeta) = N_{gs}(\zeta)/\sqrt{\zeta}
\]

are introduced and, using them, one can transform the expression (15) in a following way:
\[ L_{g0}(\xi) = \frac{X_{gs}^2(\xi) + X_{gs}^2(\xi)}{2}. \] (17)

The random processes (16) are statistically independent Gaussian random ones with the zero mathematical expectations, unit dispersions and correlation functions
\[ \langle X_{gs}(\xi_1)X_{gs}(\xi_2) \rangle = \langle X_{gs}(\xi_1)X_{gs}(\xi_2) \rangle = \min(\xi_1, \xi_2) / \sqrt{\xi_1 \xi_2}. \] (18)

The approximate expression for the distribution function for the maximum of the random process having the form (17) and the characteristics (18) is determined in [5]:
\[ F_{g0}(x) \approx \begin{cases} \left( Z_1 / Z_2 \right)^{\exp(-x)}, & x \geq 1, \\ 0, & x < 1. \end{cases} \]

Then the false alarm probability for the QL detection algorithm (10) takes the form
\[ \alpha_g \approx \begin{cases} 1 - \left( Z_1 / Z_2 \right)^{\exp(-x)}, & x \geq 1, \\ 0, & x < 1. \end{cases} \] (19)

Let us now find the missing probability assuming that the signal (2) is present in the observable realization. Under \( \gamma_0 = j = 1 \), in the expression (13) the new variable \( l = \xi / \zeta_0 \) is introduced so that \( l \in [L_1, L_2] \), \( L_1 = Z_1 / \zeta_0 \geq 0 \), \( L_2 = Z_2 / \zeta_0 > 0 \) and the decision statistics (13) is represented in the form
\[ L_{g1}(l) = p^2 \left[ q_g^{-1}(\zeta_0 \min(l,1)) \right] / 2l \zeta_0 + p \left[ q_g^{-1}(\zeta_0 \min(l,1)) \right] N_{g1}(l) / l \sqrt{\zeta_0} + \left[ N_{gs}^2(\xi) + N_{gs}^2(\xi) \right] / 2l. \] (20)

Here, the variable change properties are accounted for the realizations of random processes \( N_{g1}(\zeta_0 l) = \sqrt{\zeta_0} N_{g1}(l) \), \( N_{gs}(\xi_0 l) = \sqrt{\zeta_0} N_{gs}(l) \), \( N_{gs}(\xi_0 l) = \sqrt{\zeta_0} N_{gs}(l) \), where \( N_{g1}(l) \), \( N_{gs}(l) \), \( N_{gs}(l) \) are standard Wiener random processes [11].

It is presupposed that the detection is implemented in case when the signal-to-noise ratio (SNR) is big enough while the position of the maximum of the mathematical expectation of the random process \( L_{g1}(l) \) coincides with the value of \( \tau_0 \). Then the position of the maximum of the process \( L_{g1}(l) \) is located within a small neighborhood of the point \( \tau_0 \) with the probability tending to 1 while SNR is increasing.

Let us examine the behavior of the functions \( q_g(\tau) \), \( p(\tau) \) (12) in a small neighborhood of the point \( \tau_0 \). One develops these functions into Taylor series within this neighborhood and focus on the terms of expansions that are of the first infinitesimal order:
\[ \zeta = q_g(\tau_0 + 0) + a_g \left( \tau - \tau_0 \right) + a_g^2 \left( \tau - \tau_0 \right)^2 / N_0. \]
\[ p(\tau) \approx p(\tau_0) + p'(\tau_0) \left( \tau - \tau_0 \right) = p(\tau_0) + a_g \left( \tau - \tau_0 \right) f(\tau_0) \left( \tau - \tau_0 / N_0 \right). \]

Taking into account that the expansion into Taylor series for the inverse function takes the form of
\[ q_g^{-1}(\zeta) \approx \tau_0 + (\zeta - \zeta_0) / q_g(\zeta_0) = \tau_0 + N_0 (\zeta - \zeta_0) / a_g \left( \tau - \tau_0 \right)^2 / N_0, \]
the following approximate equality is obtained:
\[ p \left[ q_g^{-1}(\zeta_0 \min(l,1)) \right] \approx \zeta_0 \left[ p(\tau_0) / q(\zeta_0) + f(\tau_0) \min(l - 1,0) / g(\tau_0) \right]. \] (21)

It should be noted that the value \( \zeta_0 = \zeta_g^2 \) is SNR at the output of the ML detector while the radio signal is received having the duration \( \tau_0 \) and the envelope \( g(t) \).
According to (21), the expression \( p \left( g_{g}^{-1} (t \zeta) \right) \) is proportional to \( z_{g}^{2} \). Therefore, the first term in the formula (20) is proportional to \( z_{g}^{2} \), the second term – to \( z_{g}^{1} \), the third term – to \( z_{g}^{0} \). Under sufficiently big SNR (\( z_{g} >> 1 \)), the last term in (20) can be neglected in comparison with the previous ones and the decision statistics can be approximately written as

\[
L_{g1}(t) = p^{2} \left[ q_{g}^{-1} (\zeta_{0} \min(t,l)) / 2 \zeta_{0} + p q_{g}^{-1} (\zeta_{0} \min(l,t)) \right] N_{g1}(t) / \sqrt{\zeta_{0}}.
\]

Reverting to the variable \( \zeta \), so that \( L_{g1}(\zeta) = p^{2} \left[ q_{g}^{-1} (\zeta_{0} \min(\zeta,\zeta_{0})) / 2 \zeta_{0} + p q_{g}^{-1} (\min(\zeta,\zeta_{0})) \right] N_{g1}(\zeta) / \zeta \) and then moving to the variable \( \tau \), one gets

\[
L_{g1}(\tau) = z^{2} p^{2} (\tau, \tau_{0}) / 2u(\tau, \tau) + \left[ p(\tau, \tau) / u(\tau, \tau) \right] \left[ N_{g1} (\tau) \cos \varphi_{0} + N_{g} (\tau) \sin \varphi_{0} \right].
\]

Here the notations are: \( p(\tau_{1}, \tau_{2}) = \int_{0}^{\min(\tau_{1}, \tau_{2})} f(t) g(t) dt \), \( u(\tau_{1}, \tau_{2}) = \int_{0}^{\min(\tau_{1}, \tau_{2})} g^{2}(t) dt \), \( z^{2} = a_{0}^{2} T_{2} / N_{0} \) is SNR at the output of the ML detector while the rectangular radio signal by the duration \( T_{2} \) is received. It can be seen that the decision statistics has the following mathematical expectation and correlation function

\[
S_{g1}(\tau) = \left\langle L_{g1}(\tau) \right\rangle = z^{2} p^{2} (\tau, \tau_{0}) / 2u(\tau, \tau), \quad K_{g} (\tau_{1}, \tau_{2}) = z^{2} F(\tau_{1}) F(\tau_{2}) u(\tau_{1}, \tau_{2}). \tag{22}
\]

where \( F(\tau) = p(\tau, \tau_{0}) / u(\tau, \tau) = \int_{0}^{\min(\tau, \tau_{0})} g(t) f(t) dt / \int_{0}^{\tau} g^{2}(t) dt \).

Let us develop the functions \( p(\tau, \tau_{0}) \) and \( u(\tau, \tau_{0}) \) into Taylor series by the first argument \( \tau \) within a neighborhood of the point \( \tau_{0} \):

\[
p(\tau, \tau_{0}) \approx p_{0} + g_{0} f_{0} \min(\tau - \tau_{0}, 0) / T_{2}, \quad u(\tau, \tau_{0}) \approx u_{0} + g_{0}^{2} (\tau - \tau_{0}) / T_{2}. \tag{23}
\]

In (23), the notations are: \( p_{0} = p(\tau_{0}, \tau_{0}), \quad u_{0} = u(\tau_{0}, \tau_{0}), \quad f_{0} = f(\tau_{0}), \quad g_{0} = g(\tau_{0}) \). After applying the expansions (23), the expressions (22) can be represented in the following way:

\[
S_{g1}(\tau) \approx \left( z^{2} p_{0}^{2} / 2u_{0} \right) \left[ 1 + 2 g_{0} f_{0} \min(\tau - \tau_{0}, 0) / T_{2} + g_{0}^{2} (\tau - \tau_{0}) / u_{0} T_{2} \right],
\]

\[
K_{g} (\tau_{1}, \tau_{2}) \approx \left( z^{2} p_{0}^{2} / 2u_{0} \right) \left[ 1 - g_{0}^{2} (\tau_{1} + \tau_{2} - 2 \tau_{0}) / u_{0} T_{2} + g_{0}^{2} \min(\tau_{1} - \tau_{0}, 0) / u_{0} T_{2} + \right.
\]

\[
\left. + g_{0} f_{0} \min(\tau_{1} - \tau_{0}, 0) + \min(\tau_{2} - \tau_{0}, 0) / u_{0} T_{2} \right]. \tag{24}
\]

Thus, in order to find the approximate expression for the missing probability \( \beta_{g} \), it is necessary to obtain the distribution function of the greatest maximum of the Gaussian random process \( L_{g1}(\tau) \). Its mathematical expectation and correlation function in a neighborhood of the point \( \tau_{0} \) are described by the expressions (24) and (25), respectively. For this purpose, the local Markov approximation method [12], [13] can be used. One represents the desired probability in the form of \( \beta_{g} = F_{g1}(h) = \left\{ \text{sup} L_{g1}(\tau) < h \right\} = \left\{ \text{sup} \chi_{g}(\tau) < (h - L_{g1}(x)) / \zeta \right\} \), where \( x \) is an arbitrary value from the interval \( (T_{1}, T_{2}) \), \( \chi_{g}(\tau) = \left[ L_{g1}(\tau) - L_{g1}(x) \right] / \zeta \) is the realization of the Gaussian random process with the mathematical expectation \( S_{\chi}(\tau) \) and the correlation function \( K_{\chi}(\tau_{1}, \tau_{2}) \):

\[
S_{\chi}(\tau) = \left( z^{2} p_{0}^{2} / 2u_{0} \right) \left[ 2 g_{0} f_{0} \left[ (\tau - \tau_{0}) - (\tau - \tau_{0}) + \tau - x \right] / T_{2} + g_{0}^{2} (\tau - x) / u_{0} T_{2} \right].
\]
\[ K_x(\tau_1, \tau_2) = \frac{p_0^2 g_0^2}{u_0^2 T_0} \min(\tau_1 - x|, |\tau_2 - x|, 0 , \begin{cases} 0 , & (\tau_1 - x)(\tau_2 - x) \geq 0, \\ (\tau_1 - x)(\tau_2 - x) < 0. \end{cases} \] (26)

It follows from the formula (26) and the study [12] that \( \chi_{\xi}(\tau) \) is the Markov random process and that its values in the non-overlapping intervals \([T_1, x]\) and \((x, T_2]\) are not correlated, and therefore they are statistically independent, as being Gaussian. Therefore, according to [12], one gets

\[ \beta_\xi = \int_{-\infty}^{u} F^+(u-y)F^-(u-y)W_x(y)dy, \] (27)

where \( F^-(x) = P\{\sup L_{g1}(\tau) < x, \tau \in [T_1, x]\} \), \( F^+(x) = P\{\sup L_{g1}(\tau) < x, \tau \in (x, T_2]\} \), \( u = h/z \) is the normalized threshold, \( W_x(y) \) is the probability density of the random variable \( Y = L_{g1}(x)/z \).

Using the Markov properties of the random process, one can find the expressions for the functions \( F^-(x) \), \( F^+(x) \), and after their substitution into (27), the following expression for the missing probability is obtained:

\[
\begin{align*}
\beta_\xi & \approx \frac{1}{p_0} \exp\left[-u_0 \left( \frac{x-zp_0^2/2u_0}{2p_0^2} \right)^2 \right] \left[ \Phi\left( \frac{a_1}{b} (\tau_0 - T_1) + \frac{\xi}{\sqrt{b(b(\tau_0 - T_1))}} \right) - \Phi\left( \frac{a_2}{b} (T_2 - \tau_0) + \frac{\xi}{\sqrt{b(b(T_2 - \tau_0))}} \right) \\
& \hspace{1cm} - \exp\left(-\frac{2a_1^2}{b} \xi \right) \Phi\left( \frac{a_1^2}{b} (\tau_0 - T_1) - \frac{\xi}{\sqrt{b(b(\tau_0 - T_1))}} \right) \right] d\xi,
\end{align*}
\] (28)

where

\[ a_1 = zp_0^2 \left( 2g_0 f_0/p_0 - g_0^2/u_0 \right)/2u_0 T_0, \quad a_2 = zp_0^2 g_0^2 / 2u_0^2 T_2, \quad b = p_0^2 g_0^2 / u_0^2 T_2, \quad \Phi(x) = \int_{-\infty}^{x} \exp(-t^2/2)dt/\sqrt{2\pi} \] is the probability integral.

Let us specify the results obtained for the case when the received signal is the rectangular radio pulse \( f(t) = 1 \), while the envelope of the expected signal \( g(t) \) is changing exponentially:

\[ g(t) = \exp(-\kappa t/T_2)\sqrt{2\kappa/[1-\exp(-2\kappa)]}. \] (29)

The multiplier \( \sqrt{2\kappa/[1-\exp(-2\kappa)]} \) in (29) is chosen so that the energy of the signal of the maximum duration \( T_2 \) does not depend on the rate of change of the exponent \( \kappa \) and is equal to the energy of a rectangular pulse of the same duration. The dynamic range of the possible duration values is characterized by the value \( k = T_2/T_1 \).

In figure 1, there are shown the dependences of the false alarm probability (19) upon the threshold value \( h \), while in figure 2 – the dependences of the missing probability (28) upon the SNR under \( k = 10 \) and varying values of the parameter \( \kappa \). The solid curve corresponds to the case when the functions coincide so that \( f(t) = g(t) = 1 \), the dashed curve is plotted for the case when \( \kappa = 2 \), and the dash-dotted curve is for the case when \( \kappa = 5 \).
The threshold value $h$ while calculating the missing probability is defined from (19) by the Neumann-Pearson criterion, according to the specified level of the false alarm probability $\alpha_s = 0.01$.

It can be seen from these figures that for the signal considered as an example, the deviation of the envelope shape of the input radio signal from the envelope shape of the reference signal leads to the increasing of the missing probability while the false alarm probability remains fixed.

4. Conclusion
In order to detect a radio pulse while its envelope and some parameters are unknown, the quasi-likelihood approach can be used. The application of this approach allows us to obtain rather simple block diagrams of detectors that implement the adaptation to the unknown parameters of the received signal.

If the deviations of the envelope shape of the received signal from the envelope shape of the reference signal are small, the proposed detector is effective enough. However, with increasing differences in the shapes of the specified envelopes, the detector characteristics of the can deteriorate significantly. The results obtained make it possible to calculate the losses in detection efficiency resulting from the a priori ignorance of the shape of the received radio signal in each of the specific cases.

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