Solitary quotients of finite groups

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Abstract: We introduce and study the lattice of normal subgroups of a group $G$ that determine solitary quotients. It is closely connected to the well-known lattice of solitary subgroups of $G$, see [Kaplan G., Levy D., Solitary subgroups, Comm. Algebra, 2009, 37(6), 1873–1883]. A precise description of this lattice is given for some particular classes of finite groups.

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1. Introduction

The relation between the structure of a group and the structure of its lattice of subgroups constitutes an important domain of research in group theory. The topic has enjoyed a rapid development starting with the first half of the 20th century. Many classes of groups determined by different properties of partially ordered subsets of their subgroups (especially lattices of subgroups) have been identified. We refer to Suzuki’s book [9], Schmidt’s book [8] or the more recent book [11] by the author for more information about this theory.

The starting point for our discussion is given by the paper [5], where the lattice $\operatorname{Sol}(G)$ of solitary subgroups of a group $G$ has been introduced. A natural idea is to study the normal subgroups of $G$ that induce solitary quotients. The set of these subgroups also forms a lattice, denoted by $\operatorname{QSol}(G)$, which constitutes a “dual” for $\operatorname{Sol}(G)$. The first steps in studying this new lattice associated to a group is the main goal of the current paper.

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In the following, given a group $G$, we will denote by $L(G)$ the subgroup lattice of $G$. Recall that $L(G)$ is a complete bounded lattice with respect to set inclusion, having initial element the trivial subgroup $\{1\}$ and final element $G$, and its binary operations $\wedge, \vee$ are defined by

$$H \wedge K = H \cap K, \quad H \vee K = (H \cup K), \quad H, K \in L(G).$$

Two important modular sublattices of $L(G)$ are the normal subgroup lattice and the characteristic subgroup lattice of $G$, usually denoted by $N(G)$ and $\text{Char}(G)$, respectively. Note also that $\text{Char}(G)$ is a sublattice of $N(G)$.

Two groups $G_1$ and $G_2$ will be called $L$-isomorphic if their subgroup lattices $L(G_1)$ and $L(G_2)$ are isomorphic. A duality from $G_1$ onto $G_2$ is a bijective map $\delta: L(G_1) \rightarrow L(G_2)$ such that the following equivalent conditions are satisfied:

- for all $H, K \in L(G_1)$, we have $H \leq K$ if and only if $K^\delta \leq H^\delta$;
- $(H \wedge K)^\delta = H^\delta \vee K^\delta$ for all $H, K \in L(G_1)$;
- $(H \vee K)^\delta = H^\delta \wedge K^\delta$ for all $H, K \in L(G_1)$.

We say that a group $G$ has a dual if there exists a duality from $G$ to some group $\tilde{G}$, and that $G$ is self-dual if there exists a duality from $G$ onto $G$.

A subgroup $H \in L(G)$ is called a solitary subgroup of $G$ if $G$ does not contain another isomorphic copy of $H$, that is

for all $K \in L(G)$, $K \cong H \implies K = H$.

By [5, Theorem 25], the set $\text{Sol}(G)$ consisting of all solitary subgroups of $G$ forms a lattice with respect to set inclusion, which is called the lattice of solitary subgroups of $G$.

The paper is organized as follows. In Section 2 we present some basic properties and results on the lattice $Q\text{Sol}(G)$ associated to a finite group $G$. Section 3 deals with this lattice for finite abelian groups. The most important theorem in this section gives a complete description of the lattices $\text{Sol}(G)$ and $Q\text{Sol}(G)$ for such a group $G$. In the final section some conclusions and further research directions are indicated.

Most of our notation is standard and will not be repeated here. Basic definitions and results on lattices and groups can be found in [1, 2] and [3, 4, 10], respectively. For subgroup lattice concepts we refer the reader to [8, 9, 11].

## 2. The lattice $Q\text{Sol}(G)$

Let $G$ be a finite group and $Q\text{Sol}(G)$ be the set of normal subgroups of $G$ that determine solitary quotients, that is

$$Q\text{Sol}(G) = \{H \in N(G) : \text{for all } K \in N(G), G/K \cong G/H \implies K = H\}.$$ 

We remark that all elements of $Q\text{Sol}(G)$ are characteristic subgroups of $G$. It is also obvious that $\{1\}$ and $G$ are contained in $Q\text{Sol}(G)$. If $Q\text{Sol}(G)$ consists only of these two subgroups, then we will call $G$ a quotient solitary free group.

**Remark.**

Clearly, another important subgroup of $G$ that belongs to $Q\text{Sol}(G)$ is $G'$. We infer that if $G$ is quotient solitary free, then $G' = G$ or $G' = \{1\}$ and therefore $G$ is either perfect or abelian. Observe also that the elementary abelian $p$-groups are quotient solitary free.
Our first result shows that $\text{QSol}(G)$ can naturally be endowed with a lattice structure.

**Proposition 2.1.**
Let $G$ be a finite group. Then $\text{QSol}(G)$ is a lattice with respect to set inclusion.

**Proof.** Let $H_1, H_2 \in \text{QSol}(G)$ and $H \in \text{N}(G)$ be such that $G/H \cong G/H_1 \cap H_2$. Choose an isomorphism $f : G/H_1 \cap H_2 \to G/H$. Then there are two normal subgroups $H'_i, H''_i$ of $G$ with $f((H_1 \cap H_2) / H) = H'_i / H, i = 1, 2$. It is easy to see that $H'_i \cap H''_i = H$. One obtains

\[
\frac{G}{H'_i} \cong \frac{G/H_1 \cap H_2}{H/H_1 \cap H_2} \cong \frac{G/H}{H/H_i} \cong \frac{G}{H''_i}
\]

which implies that $H'_i = H_i, i = 1, 2$, and so $H = H_1 \cap H_2$. Therefore $H_1 \cap H_2$ is the meet of $H_1$ and $H_2$ in $\text{QSol}(G)$.

Obviously, the join of $H_1$ and $H_2$ in $\text{QSol}(G)$ also exists, and consequently $\text{QSol}(G)$ is a lattice. Note that $\{1\}$ and $G$ are, respectively, the initial element and the final element of this lattice. 

An exhaustive description of the above lattice for an important class of finite groups, the dihedral groups

\[
D_{2n} = \langle x, y : x^n = y^2 = 1, yxy = x^{-1} \rangle, \quad n \geq 3,
\]

is indicated in the following.

**Example.**

The structure of $L(D_{2n})$ is well known: for every divisor $r$ of $n$, $D_{2n}$ possesses a unique cyclic subgroup of order $r$ (namely $\langle x^{n/r} \rangle$) and $n/r$ subgroups isomorphic to $D_{2n/r}$ (namely $\langle x^{n/r}, x^{i}y \rangle, i = 0, 1, \ldots, n/r - 1$). Remark that $D_{2n}$ always has a maximal cyclic normal subgroup $M = \langle x \rangle \cong \mathbb{Z}_n$. Clearly, all subgroups of $M$ are normal in $D_{2n}$. On the other hand, if $n$ is even, then $D_{2n}$ has another two maximal normal subgroups of order $n$, namely $M_1 = \langle x^2, y \rangle$ and $M_2 = \langle x^2, xy \rangle$, both isomorphic to $D_n$. In this way, one obtains

\[
N(D_{2n}) = \begin{cases} L(M) \cup \{D_{2n}\}, & n \equiv 1 \pmod{2}, \\ L(M) \cup \{D_{2n}, M_1, M_2\}, & n \equiv 0 \pmod{2}. \end{cases}
\]

We easily infer that

\[
\text{QSol}(D_{2n}) = \begin{cases} L(M) \cup \{D_{2n}\}, & n \equiv 1 \pmod{2}, \\ L(M)^* \cup \{D_{2n}\}, & n \equiv 0 \pmod{2}. \end{cases}
\]

where $L(M)^*$ denotes the set of all proper subgroups of $M$.

Observe that the equality between the lattices $\text{Sol}(G)$ and $\text{QSol}(G)$ associated to a finite group $G$ fails. For example, we have $\langle x^3 \rangle \in \text{QSol}(D_6)$ but $\langle x^3 \rangle \notin \text{Sol}(D_6)$, respectively $\langle x \rangle \in \text{Sol}(D_6)$ but $\langle x \rangle \notin \text{QSol}(D_6)$.

Looking at the dihedral groups $D_{2n}$ with odd $n$, we also infer that there exist finite groups $G$ such that $\text{QSol}(G) = \text{N}(G)$. Other examples of such groups are the finite groups without normal subgroups of the same order and, in particular, the finite groups $G$ for which $\text{N}(G)$ is a chain (e.g. simple groups, symmetric groups, cyclic $p$-groups or finite groups of order $p^n q^m$ ($p, q$ distinct primes) with cyclic Sylow subgroups and trivial center – see [8, Exercise 3, p. 497]). Note also that for a finite group $G$ the condition $\text{QSol}(G) = \text{N}(G)$ is very close to the conditions of [8, Theorem 9.1.6] which characterize the distributivity of $\text{N}(G)$.

The following proposition shows that the relation "to be quotient solitary" is transitive.

**Proposition 2.2.**
Let $G$ be a finite group and $H \supseteq K$ be two normal subgroups of $G$. If $H \in \text{QSol}(G)$ and $K \in \text{QSol}(H)$, then $K \in \text{QSol}(G)$. 
Proof. Let \( K_i \in N(G) \) such that \( G/K_i \cong G/K \) and take an isomorphism \( f: G/K \to G/K_i \). Set \( H_i/K_i = f(H/K) \), where \( H_i \) is a normal subgroup of \( G \) containing \( K_i \). It follows that

\[
\frac{G}{H} \cong \frac{G/K}{H/K} \cong \frac{G/K_1}{H_1/K_1} \cong \frac{G}{H_1},
\]

which leads to \( H_1 = H \). One obtains \( H/K_1 \cong H/K \) and therefore \( K_1 = K \). Hence \( K \in \text{QSol}(G) \). \( \square \)

Next we study the connections between the lattices \( \text{QSol}(G) \) and \( \text{QSol}(\overline{G}) \), where \( \overline{G} \) is an epimorphic image of \( G \). We mention that a proper subgroup \( H \in \text{QSol}(G) \) will be called \textit{maximal} in \( \text{QSol}(G) \) if it is not properly contained in any proper subgroup of \( \text{QSol}(G) \).

**Proposition 2.3.**

Let \( G \) be a finite group and \( H \) a proper normal subgroup of \( G \). Set \( \overline{G} = G/H \) and denote by \( \pi_2: G \to \overline{G} \) the canonical homomorphism. Then, for every \( K \in \text{QSol}(G) \), we have \( \pi_2(K) \in \text{QSol}(\overline{G}) \). In particular, if \( H \in \text{QSol}(G) \) and \( \overline{G} \) is quotient solitary free, then \( H \) is maximal in \( \text{QSol}(G) \).

**Proof.** Let \( K \in \text{QSol}(G) \) and \( K_1 \) be a normal subgroup of \( G \) which contains \( H \) and satisfies \( \overline{G}/\pi_2(K_1) \cong \overline{G}/\pi_2(K) \). Then \( G/K_1 \cong G/K \) and thus \( K_1 = K \), in view of our hypothesis. Hence \( \pi_2(K_1) = \pi_2(K) \). Suppose now that \( \overline{G} \) is quotient solitary free and let \( K \in \text{QSol}(G) \) with \( K \neq G \) and \( H \subset K \). Then, by what we proved above, \( \overline{K} = \pi_2(K) \in \text{QSol}(\overline{G}) \) and \( \{1\} \subset \overline{K} \subset \overline{G} \), a contradiction. \( \square \)

**Remark 2.4.**

Under the hypotheses of Proposition 2.3, \( \pi_2 \) fails to induce a bijection between the sets \( A = \{ K \in \text{QSol}(G) : H \subseteq K \} \) and \( \text{QSol}(\overline{G}) \). This follows by taking \( G = D_{12} \) and \( H \) the unique normal subgroup of order 3 of \( D_{12} \) (in this case \( A \) consists of three elements, namely \( H, D_{12} \) and the cyclic subgroup of order 6 in \( D_{12} \), while \( D_{12} \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \) is quotient solitary free).

In the following we assume that \( G \) is nilpotent and let

\[
G = \prod_{i=1}^{k} G_i
\]

be the decomposition of \( G \) as a direct product of Sylow subgroups. Then the lattice \( N(G) \) is decomposable. More precisely, every \( H \in N(G) \) can be written as \( \prod_{i=1}^{k} H_i \) with \( H_i \in N(G_i), \ i = 1, 2, \ldots, k \). We infer that \( H \in \text{QSol}(G) \) if and only if \( H_i \in \text{QSol}(G_i) \), for all \( i = 1, 2, \ldots, k \). In this way, the lattice \( \text{QSol}(G) \) is also decomposable,

\[
\text{QSol}(G) \cong \prod_{i=1}^{k} \text{QSol}(G_i),
\]

and its study is reduced to \( p \)-groups.

We first remark that for a finite \( p \)-group \( G \), the lattice \( \text{Char}(G) \) is in general strictly contained in \( \text{QSol}(G) \) (take, for example, \( G = S_{p^2} \), the quasi-dihedral group of order \( 2p^2 \), being isomorphic to \( \mathbb{Z}_{p^2-1} \), \( D_{p^2-1} \) and \( S_{p^2-1} \), respectively; the maximal subgroups of \( S_{p^2} \) are characteristic, but clearly they do not belong to \( \text{QSol}(G) \)). In fact, a maximal subgroup \( M \) of \( G \) is contained in \( \text{QSol}(G) \) if and only if \( G \) is cyclic.

The following result will play an essential role in proving the main theorem of Section 3. It illustrates another important element of \( \text{QSol}(G) \) in the particular case of \( p \)-groups.

**Proposition 2.5.**

Let \( G \) be a finite \( p \)-group. Then \( \Phi(G) \) is a maximal element of \( \text{QSol}(G) \).
Proof. If $G/H \cong G/\Phi(G)$ for some $H \in N(G)$, then $G/H$ is elementary abelian. Since $\Phi(G)$ is minimal in $G$ with the property to determine an elementary abelian quotient, we have $\Phi(G) \subseteq H$. On the other hand, we know that $\Phi(G)$ and $H$ are of the same order. These lead to $H = \Phi(G)$, that is $\Phi(G) \in QSol(G)$. We have already seen that $G/\Phi(G)$ is quotient solitary free. According to Proposition 2.3, this implies the maximality of $\Phi(G)$ in $QSol(G)$.\[\]

Obviously, by Propositions 2.2 and 2.5 we infer that the Frattini series of a finite $p$-group $G$ is contained in $QSol(G)$, that is

$$\{\Phi_n(G) : n \in \mathbb{N}\} \subseteq QSol(G),$$

where $\Phi_0(G) = G$ and $\Phi_n(G) = \Phi(\Phi_{n-1}(G))$, for all $n \geq 1$. This can naturally be extended to finite nilpotent groups.

**Corollary 2.6.**

Let $G$ be a finite nilpotent group. Then

$$\{\Phi_n(G) : n \in \mathbb{N}\} \subseteq QSol(G).$$

(1)

Moreover, under the above notation, the maximal elements of $QSol(G)$ are

$$\Phi(G_i) \prod_{j \not= i} G_j, \quad i = 1, 2, \ldots, k.$$  

In particular, we also obtain the following corollary.

**Corollary 2.7.**

Let $G$ be a finite nilpotent group. Then $QSol(G) = N(G)$ if and only if $G$ is cyclic.

### 3. The case of finite abelian groups

In this section we will focus on describing the lattice $QSol(G)$ associated to a finite abelian group $G$. As we have seen above, it suffices to consider finite abelian $p$-groups. In the following our main goal is to prove that for such a group the relation (1) becomes an equality.

Recall first a famous theorem due to Baer (see, for example, [8, Theorem 8.1.4] or [9, Theorem 4.2]) which states that every finite abelian group (and, in particular, every finite abelian $p$-group) $G$ is self-dual. Moreover, by fixing an autoduality $\delta$ of $G$, we have

$$H \cong G/\delta(H) \quad \text{and} \quad \delta(H) \cong G/H \quad \text{for all} \ H \in L(G).$$

These isomorphisms easily lead to the following proposition.

**Proposition 3.1.**

Let $G$ be a finite abelian $p$-group and $\delta$ an autoduality of $G$. Then $\delta(QSol(G)) = Sol(G)$ and $\delta(Sol(G)) = QSol(G)$, that is, $\delta$ induces an anti-isomorphism between the lattices $QSol(G)$ and $Sol(G)$. Moreover, we have $\delta^2(H) = H$, for all $H \in QSol(G)$.

By Proposition 2.5, we know that $\Phi(G)$ is a maximal element of $QSol(G)$. The $\Phi$-subgroup of a finite abelian $p$-group satisfies some other simple but important properties in $QSol(G)$.

**Lemma 3.2.**

Let $G$ be a finite abelian $p$-group and $G \cong \prod_{i=1}^k \mathbb{Z}_{p^{n_i}}$ be the primary decomposition of $G$. Then, for every proper subgroup $H$ of $QSol(G)$, we have

$$\prod_{i=1}^k \mathbb{Z}_{p^{n_i}} \subseteq QSol(G).$$

where $\mathbb{Z}_{p^{n_i}} = \{0, 1, \ldots, p^{n_i} - 1\}$ for all $i = 1, 2, \ldots, k$. This can naturally be extended to finite nilpotent groups.
a) \( H \subseteq \Phi(G) \);

b) \( H \in \text{QSol}(\Phi(G)) \).

**Proof.**

a) We shall proceed by induction on \( k \). The inclusion is trivial for \( k = 1 \). Assume now that it holds for any abelian \( p \)-group of rank less than \( k \) and put \( G = G_1 \times G_2 \), where \( G_1 \cong \prod_{i=1}^{k-1} Z_{p^{a_i}} \) and \( G_2 \cong Z_{p^{a_k}} \). According to Suzuki [10, vol. I, (4.19)], a subgroup \( H \) of \( G \) is uniquely determined by two subgroups \( H_1 \subseteq H_1' \) of \( G_1 \), two subgroups \( H_2 \subseteq H_2' \) of \( G_2 \) and an isomorphism \( \varphi: H_1'/H_1 \to H_2'/H_2 \) (more precisely, we have \( H = \{ (x_1, x_2) \in H_1' \times H_2' : \varphi(x_1 H_1) = x_2 H_2' \} \)). Mention that \( H_1' = \pi_i(H) \), \( i = 1, 2 \), where \( \pi_1 \) and \( \pi_2 \) are the projections of \( H \) onto \( G_1 \) and \( G_2 \), respectively. Clearly, Proposition 2.3 implies that \( H_1' \) belongs to \( \text{QSol}(G_i) \), \( i = 1, 2 \). Since \( H \in \text{QSol}(G) \), it follows that each \( H_i' \) is properly contained in \( G_i \). Indeed, if \( H_i' = G_i \), then \( H \) induces a surjective homomorphism from \( H \) to \( G_i \), and therefore \( H \) has a quotient isomorphic to \( G_i \). By duality, it also has a subgroup isomorphic to \( G_i \). This implies that \( G/H \) is isomorphic to a quotient of \( G_i \), i.e. it is cyclic. Thus \( \delta(H) \) is a cyclic solitary subgroup of \( G \). In other words, \( G \) contains a unique non-trivial cyclic subgroup of a certain order, a contradiction. Similarly, we have \( H_2' \neq G_2 \). Then, by the inductive hypothesis, one obtains \( H_i' \subseteq \Phi(G_i) \), \( i = 1, 2 \), and so

\[
H \subseteq H_1' \times H_2' \subseteq \Phi(G_1) \times \Phi(G_2) = \Phi(G).
\]

b) Using [8, Lemma 8.1.6], we infer that \( \delta \) induces a bijection between the set of proper subgroups of \( \text{QSol}(G) \) contained in \( \Phi(G) \) and \( \text{Sol}(G) / \delta(\Phi(G)) \). Since the groups \( G / \delta(\Phi(G)) \) and \( \Phi(G) \) are isomorphic, their lattices of solitary subgroups are also isomorphic. Finally, on account of Proposition 3.1, the lattices \( \text{Sol}(\Phi(G)) \) and \( \text{QSol}(\Phi(G)) \) are anti-isomorphic. Hence there is a bijection between the sets \( \{ H \in \text{QSol}(G) : H \subseteq \Phi(G) \} \) and \( \text{QSol}(\Phi(G)) \). On the other hand, by Proposition 2.2 we have

\[
\text{QSol}(\Phi(G)) \subseteq \{ H \in \text{QSol}(G) : H \subseteq \Phi(G) \},
\]

and therefore these sets are equal. This completes the proof.

**Remark.**

An alternative way of proving a) of Lemma 3.2 is obtained by using the lattice of characteristic subgroups of \( G \). According to [7, Theorem 3.7] (see also [6]), \( \text{Char}(G) \) has a unique minimal element, say \( M \), and clearly it is solitary in \( G \). It follows that \( \delta(M) \) is the unique maximal element of \( \text{QSol}(G) \) and thus it will coincide with \( \Phi(G) \). In this way, all proper subgroups of \( \text{QSol}(G) \) are contained in \( \Phi(G) \).

Let \( r \) be the length of the Frattini series of \( G \) (note that if \( G \cong \prod_{i=1}^{k} Z_{p^{a_i}} \), then \( a_1 \leq a_2 \leq \ldots \leq a_k \), is the primary decomposition of \( G \), then \( r = a_1 \)). Using Lemma 3.2 and a standard induction on \( r \), we easily come up with the conclusion that \( \text{QSol}(G) \) coincides with this series. The lattice \( \text{Sol}(G) \) is also completely determined in view of Proposition 3.1. Hence we have proved the following theorem.

**Theorem 3.3.**

Let \( G \) be a finite abelian \( p \)-group and \( G \cong \prod_{i=1}^{k} Z_{p^{a_i}} \) be the primary decomposition of \( G \). Then both the lattices \( \text{Sol}(G) \) and \( \text{QSol}(G) \) are chains of length \( a_1 \). More precisely, under the above notation, we have

\[
\text{QSol}(G) : \{ 1 \} = \Phi_{a_1}(G) \subseteq \Phi_{a_2-1}(G) \subseteq \ldots \subseteq \Phi_{1}(G) \subseteq \Phi_{0}(G) = G
\]

and

\[
\text{Sol}(G) : \{ 1 \} = \delta(\Phi_0(G)) \subseteq \delta(\Phi_1(G)) \subseteq \delta(\Phi_2(G)) \subseteq \ldots \subseteq \delta(\Phi_{a_1}(G)) = G.
\]

Notice that an alternative way of proving Theorem 3.3 can be inferred from classification of finite abelian groups and the types of their subgroups. We also observe that for such a group \( G \) the lattice \( \text{Sol}(G) \), as well as \( \text{QSol}(G) \), is isomorphic to the lattice \( \pi_{\alpha}(G) \) of element orders of \( G \), because they are direct product of chains of the same length.
The lattice $Q\text{Sol}(Z_2 \times Z_4)$ and $\text{Sol}(Z_2 \times Z_4)$ associated to the finite abelian 2-group $Z_2 \times Z_4$ consist of the following chains:

$$Q\text{Sol}(Z_2 \times Z_4) : \{1\} \subset \Phi(Z_2 \times Z_4) \triangleq Z_2 \subset Z_2 \times Z_4$$

and

$$\text{Sol}(Z_2 \times Z_4) : \{1\} \subset \delta(\Phi(Z_2 \times Z_4)) \triangleq Z_2 \subset Z_2 \times Z_4,$$

respectively.

**Remark.**

The lattice $Q\text{Sol}(Z_4 \times Z_4)$ is a chain of length 2, too. So, we have $Q\text{Sol}(Z_2 \times Z_4) \cong Q\text{Sol}(Z_4 \times Z_4)$. This shows that there exist non-isomorphic finite groups $G_1$ and $G_2$ such that $Q\text{Sol}(G_1) \cong Q\text{Sol}(G_2)$. Moreover, we remark that none of the conditions $Q\text{Sol}(G_1) \cong Q\text{Sol}(G_2)$ and $\text{Sol}(G_1) \cong \text{Sol}(G_2)$ suffices to assure the lattice isomorphism $L(G_1) \cong L(G_2)$.

**Corollary 3.4.**

The lattice $Q\text{Sol}(G)$ associated to a finite abelian group $G$ is a direct product of chains, and therefore it is decomposable and distributive.

The properties of the lattice $Q\text{Sol}(G)$ in Corollary 3.4 are also satisfied by other classes of finite groups $G$. One of them, which is closely connected to abelian groups, is the class of hamiltonian groups, that is the finite nonabelian groups all of whose subgroups are normal. Such a group $H$ can be written as the direct product of the quaternion group

$$Q_8 = \langle x, y : x^4 = y^4 = 1, yxy^{-1} = x^{-1} \rangle,$$

an elementary abelian 2-group and a finite abelian group $A$ of odd order, that is

$$H \cong Q_8 \times Z_2^n \times A.$$

Since $Q_8 \times Z_2^n$ and $A$ are of coprime orders, the structure of $Q\text{Sol}(H)$ can be completely described in view of the above results.

**Corollary 3.5.**

Let $H \cong Q_8 \times Z_2^n \times A$ be a finite hamiltonian group. Then the lattice $Q\text{Sol}(H)$ is distributive. More precisely, it possesses a direct decomposition of type

$$Q\text{Sol}(H) \cong Q\text{Sol}(Q_8 \times Z_2^n) \times Q\text{Sol}(A),$$

where $Q\text{Sol}(Q_8 \times Z_2^n)$ is a chain of length 3 and $Q\text{Sol}(A)$ is a direct product of chains.

As our previous examples show, the lattices $\text{Sol}(G)$ and $Q\text{Sol}(G)$ associated to an arbitrary finite group $G$ are distinct, and this remark remains valid even for finite abelian groups. So, the next question is natural: which are the finite abelian groups $G$ satisfying $\text{Sol}(G) = Q\text{Sol}(G)$? By Theorem 3.3, for an abelian $p$-group $G \cong \prod_{i=1}^{k} Z_{p^{\alpha_i}}$, we have $\text{Sol}(G) = Q\text{Sol}(G)$ if and only if the Frattini series and the dual Frattini series of $G$ coincide, that is $\alpha_1 = \alpha_2 = \ldots = \alpha_k$. Clearly, this leads to the following result.

**Corollary 3.6.**

Let $G$ be a finite abelian group. Then $\text{Sol}(G) = Q\text{Sol}(G)$ if and only if every Sylow subgroup of $G$ is of type $\prod_{i=1}^{k} Z_{p^{\alpha_i}}$ for some positive integer $a$. 

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Finally, Theorem 3.3 can be used to determine the finite groups that are quotient solitary free. We already know that such a group $G$ is either perfect or abelian. Note that we were unable to give a precise description of quotient solitary free perfect groups. For an abelian group $G$, it is clear that $\text{QSol}(G)$ becomes a chain of length 1 if and only if $G$ is elementary abelian. Hence the following theorem holds.

**Theorem 3.7.**
If a finite group is quotient solitary free, then it is perfect or elementary abelian. In particular, a finite nilpotent group is quotient solitary free if and only if it is elementary abelian.

4. Conclusions and further research

All above results show that the study of some new posets of subgroups associated to a group $G$, as $\text{Sol}(G)$ and $\text{QSol}(G)$, is an interesting aspect of subgroup lattice theory. Clearly, it can be continued by investigating some other properties of these lattices and can also be extended to larger classes of groups. This will surely constitute the subject of some further research.

We end this paper by indicating several open problems concerning these two lattices.

4.1 Which are the (finite) groups $G$ such that $\text{Sol}(G)$ and $\text{QSol}(G)$ satisfy a certain lattice-theoretical property: modularity, distributivity, complementation, pseudocomplementation, etc.?

4.2 Determine precisely the (finite) groups $G$ such that $\text{QSol}(G) = N(G)$, $\text{QSol}(G) = \text{Char}(G)$ and $\text{QSol}(G) = \text{Sol}(G)$, respectively.

4.3 Study if some other important subgroups of a group $G$ (such as $\Phi(G)$, $Z(G)$, $F(G)$, etc.) belong to the lattices $\text{Sol}(G)$ and $\text{QSol}(G)$.

4.4 What can be said about two groups $G_1$ and $G_2$ for which we have $\text{Sol}(G_1) \cong \text{Sol}(G_2)$ or $\text{QSol}(G_1) \cong \text{QSol}(G_2)$?

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