A REMARK ON AN ENDPOINT KATO-PONCE INEQUALITY

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Abstract. This note introduces bilinear estimates intended as a step towards an $L^\infty$-endpoint Kato-Ponce inequality. In particular, a bilinear version of the classical Gagliardo-Nirenberg interpolation inequalities for a product of functions is proved.

1. Introduction and main result

The following inequality appears to be missing from the vast literature on a class of inequalities known as Kato-Ponce inequalities or fractional Leibniz rules: For every $s > 0$ there exists $C > 0$, depending only on $s$ and dimension $n$, such that

$$\|D^s (fg)\|_{L^\infty} \leq C (\|D^s f\|_{L^\infty} \|g\|_{L^\infty} + \|D^s g\|_{L^\infty} \|f\|_{L^\infty}),$$

for all $f, g \in S(\mathbb{R}^n)$, where $D^s$ is the $s$-derivative operator defined for $h \in S(\mathbb{R}^n)$ as

$$\hat{D^s h}(\xi) := |\xi|^s \hat{h}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

Inequality (1.1) represents an endpoint case of inequalities of Kato-Ponce type (see [1, 3, 4, 5, 6, 8, 9, 10, 11] and references therein) and we do not know whether it holds true or not. Moreover, the fact that for any $s > 0$ and any $f, g \in S(\mathbb{R}^n)$, both sides of (1.1) are finite, makes it quite difficult to find a counter-example to (1.1). Such counter-example should violate the structure of the right-hand side of (1.1), but not the fact that the left-hand side is finite. As a step towards (1.1) the purpose of this note is to prove the following results

**Theorem 1.** Let $0 \leq r < s < t$ and set

$$\alpha := \frac{t-s}{t-r} \quad \text{and} \quad \beta := \frac{s-r}{t-r}. \quad (1.2)$$

Then, for every $f, g \in S(\mathbb{R}^n)$ we have

$$\|D^s (fg)\|_{L^\infty} \lesssim \|D^r f\|_{B^{\alpha, \infty}_\infty} \|D^t f\|_{B^{\beta, \infty}_\infty^\alpha} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^r g\|_{B^{\alpha, \infty}_\infty^\beta} \|D^t g\|_{B^{\beta, \infty}_\infty^\alpha} \|D^s f\|_{L^\infty} \|D^t g\|_{L^\infty}, \quad (1.3)$$

where the implicit constant depends only on $r, s, t$, and dimension $n$. In particular,

$$\|D^s (fg)\|_{L^\infty} \lesssim \|D^r f\|_{L^\infty} \|D^t f\|_{L^\infty} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^r g\|_{L^\infty} \|D^t g\|_{L^\infty} \|D^s f\|_{L^\infty} \|D^t g\|_{L^\infty}. \quad (1.4)$$

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*The notation $D^s$ seems to be standard for this operator although other notations include $|D|^s$, $|\nabla|^s$ and $(-\Delta)^{s/2}$.\footnote{Note that notation $D^s$ seems to be standard for this operator although other notations include $|D|^s$, $|\nabla|^s$ and $(-\Delta)^{s/2}$.}
Remark 1. Inequality (1.3) can be regarded as a combination of Leibniz-rule and interpolation (or bilinear Gagliardo-Nirenberg) inequalities. Notice that (1.3) is weaker than (1.1). Indeed, given $0 \leq r < s < t$, by the linear Gagliardo-Nirenberg inequality (see, for instance, Theorem 2.44 in \cite{2}), we have

\begin{equation}
\|D^sf\|_{L^\infty} \lesssim \|D^rf\|^\frac{t-s}{L^\infty} \|D^tf\|^\frac{s-r}{L^\infty}, \quad \forall f \in S(\mathbb{R}^n).
\end{equation}

Then, it follows that (1.1), if true, would imply (1.4).

Theorem 2. Suppose $s > 2n + 1$. Let $1 < p_1, p_2 < \infty$ and $\varepsilon > 0$ with $n/p := (1/p_1 + 1/p_2)n < \varepsilon < 1$. Then for every $f, g \in S(\mathbb{R}^n)$ we have

\begin{equation}
\|D^s(fg)\|_{L^\infty} \lesssim \|D^s f\|^{1-\frac{n}{p_1}}_{L^{p_1}} \|D^{s+\varepsilon} f\|^{\frac{n}{p_2}}_{L^{p_2}} \|D^sg\|^{\frac{n}{p_2}}_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^sg\|_{L^\infty},
\end{equation}

where the implicit constant depends only on $s$, $\varepsilon$, $p_1$, and $p_2$.

Remark 2. In the case $s > 2n + 1$, the proof of Theorem 2 will be based on a connection between Kato-Ponce inequalities and the bilinear Calderón-Zygmund theory, see Section 4. Notice that the inequality in Theorem 2 involves no derivatives lower than $D^s$. Also, $\varepsilon > 0$ can be arbitrarily small and $p_1, p_2 \in (1, \infty)$ arbitrarily large, as long as $(1/p_1 + 1/p_2)n < \varepsilon$.

2. Preliminaries

Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, non-negative, radial function supported in $\{ \xi \in \mathbb{R}^n : |\xi| \leq 2\}$ with $\Phi \equiv 1$ in $\{ \xi \in \mathbb{R}^n : |\xi| \leq 1\}$. Define $\Psi : \mathbb{R}^n \to \mathbb{R}$ supported in $1/2 \leq |\xi| \leq 2$ as $\Psi(\xi) := \Phi(\xi) - \Phi(2\xi)$ for $\xi \in \mathbb{R}^n$, so that

\begin{equation}
\sum_{j \in \mathbb{Z}} \Delta_j h = h \text{ in } S'(\mathbb{R}^n) \quad \forall h \in S(\mathbb{R}^n),
\end{equation}

where, as usual, $\Delta_j h$ is defined for $h \in S(\mathbb{R}^n)$ as

\[\widehat{\Delta_j h}(\xi) := \Psi(2^{-j}\xi)\hat{h}(\xi) \quad \forall \xi \in \mathbb{R}^n.\]

We recall that the Besov $\dot{B}^{0,\infty}_\infty$-norm is given by

\begin{equation}
\|h\|_{\dot{B}^{0,\infty}_\infty} := \sup_{j \in \mathbb{Z}} \|\Delta_j h\|_{L^\infty} \leq \left\|\Psi\right\|_{L^1} \|h\|_{L^\infty}.
\end{equation}

For $f \in S(\mathbb{R}^n)$ and $\lambda > 0$ set $f_\lambda(x) := f(\lambda x)$ for every $x \in \mathbb{R}^n$. For $s \geq 0$ we have

\begin{equation}
\|D^s(f_\lambda)\|_{\dot{B}^{0,\infty}_\infty} = \lambda^s \|D^s f\|_{\dot{B}^{0,\infty}_\infty} \quad \text{for all } \lambda = 2^{j_0}, \ j_0 \in \mathbb{Z}.
\end{equation}

We note that $\Phi(\xi + \eta)\Phi(\xi)\Psi(\eta) = \Phi(\xi)\Psi(\eta)$ for every $\xi, \eta \in \mathbb{R}^n$, where $\Phi(\cdot) := \Phi(4^{-1} \cdot)$, and write $\Phi(s)(\cdot) := |\cdot|^s \Phi(\cdot)$. Reasoning as in \cite{2}, the absolutely convergent Fourier series for $\Phi(s)(t)\chi_{[-8,8]^n}(t)$,

\begin{equation}
\Phi(s)(t) = \sum_{m \in \mathbb{Z}^n} c_{s,m} e^{2\pi i m \cdot t} \chi_{[-8,8]^n}(t),
\end{equation}
has coefficients $c_{s,m}$ satisfying
\begin{equation}
  c_{s,m} = O(1 + |m|^{-n-s}).
\end{equation}

3. Proof of Theorem \[ \square \]

**Proof.** Fix $0 \leq r < s < t$. By (2.1), we have
\[
D^s(fg)(x) = \int_{\mathbb{R}^n} |\xi + \eta|^s \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta =: \Pi(f, g)(x) + \tilde{\Pi}(f, g)(x),
\]
with
\[
\Pi(f, g)(x) := \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{k \leq j} |\xi + \eta|^s \Psi(2^{-j} \xi) \Phi(2^{-k} \eta) \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta
\]
and
\[
\tilde{\Pi}(f, g)(x) := \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \sum_{j < k} |\xi + \eta|^s \Psi(2^{-j} \xi) \Phi(2^{-k} \eta) \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta.
\]

Now, we split $\Pi(f, g)$ (and then, similarly, $\tilde{\Pi}$) as follows
\[
\Pi(f, g)(x) = \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |\xi + \eta|^s \Psi(2^{-j} \xi) \Phi(2^{-j} \eta) \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta
\]
\[
= \int_{\mathbb{R}^n} \sum_{j \leq 0} |\xi + \eta|^s \Psi(2^{-j} \xi) \Phi(2^{-j} \eta) \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta
\]
\[
+ \int_{\mathbb{R}^n} \sum_{j > 0} |\xi + \eta|^s \Psi(2^{-j} \xi) \Phi(2^{-j} \eta) \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta
\]
\[
= \int_{\mathbb{R}^n} \sum_{j \leq 0} \frac{|\xi + \eta|^s}{|\xi|^r} \Psi(2^{-j} \xi) \Phi(2^{-j} \eta) \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta
\]
\[
+ \int_{\mathbb{R}^n} \sum_{j > 0} \frac{|\xi + \eta|^s}{|\xi|^r} \Psi(2^{-j} \xi) \Phi(2^{-j} \eta) \hat{f}(\xi) \hat{g}(\eta)e^{2\pi i (\xi + \eta) \cdot x} d\xi d\eta
\]
\[
=: \Pi_1(D^r f, g) + \Pi_2(D^s f, g).
\]

We now look at the bilinear kernel of $\Pi_1$ (the kernel for $\Pi_2$ will be dealt with in a similar way).
\begin{equation}
  \Pi_1(f, g)(x) = \int_{\mathbb{R}^n} K_1(x - y, x - z)f(y)g(z)dydz,
\end{equation}
where, after putting $\Psi_{(-\cdot)}(\cdot) := |\cdot|^{-r}\Psi(\cdot)$ and using that $\Phi(\xi + \eta)\Phi(\xi)\Psi(\eta) = \Phi(\xi)\Psi(\eta)$ for every $\xi, \eta \in \mathbb{R}^n$, $K_1$ is given by
\[
K_1(y, z) = \int_{\mathbb{R}^n} \sum_{j \leq 0} \frac{|\xi + \eta|^s}{|\xi|^r} \Psi(2^{-j} \xi) \Phi(2^{-j} \eta)e^{2\pi i (\xi + \eta \cdot y + \eta \cdot z)} d\xi d\eta
\]
\[
= \int_{\mathbb{R}^n} \sum_{j \leq 0} \frac{2^{js}}{2^{jr}} \Phi(s)(2^{-j}(\xi + \eta))\Psi_{(-\cdot)}(2^{-j} \xi)\Phi(2^{-j} \eta)e^{2\pi i (\xi \cdot y + \eta \cdot z)} d\xi d\eta.
\]
Hence, using the Fourier expansion in (2.3) and noting that the support of \(\psi_{(-r)}(\xi)\phi(\eta)\) is contained in \(\{ (\xi, \eta) : |\xi + \eta| \leq 4 \}\), we get

\[
K_1(y, z) = \int_{\mathbb{R}^{2n}} \sum_{j \leq 0} \sum_{m \in \mathbb{Z}^n} c_{s,m} 2^{j(s-r)} e^{\frac{2\pi i}{16} m \cdot 2^{-j}(\xi + \eta)} \Psi_{(-r)}(2^{-j} \xi) \Phi(2^{-j} \eta) e^{2\pi i (\xi y + \eta z)} d\xi d\eta
\]

\[
= \sum_{j \leq 0} \sum_{m \in \mathbb{Z}^n} c_{s,m} 2^{j(s-r)} \int_{\mathbb{R}^{2n}} e^{\frac{2\pi i}{16} m \cdot 2^{-j}(\xi + \eta)} \Psi_{(-r)}(2^{-j} \xi) \Phi(2^{-j} \eta) e^{2\pi i (\xi y + \eta z)} d\xi d\eta
\]

\[
= \sum_{j \leq 0} \sum_{m \in \mathbb{Z}^n} c_{s,m} 2^{j(s-r)} 2^{2jn} \int_{\mathbb{R}^{2n}} e^{\frac{2\pi i}{16} m \cdot (\xi + \eta)} \Psi_{(-r)}(\xi) \Phi(\eta) e^{2\pi i 2^j (\xi y + \eta z)} d\xi d\eta
\]

\[
= \sum_{j \leq 0} \sum_{m \in \mathbb{Z}^n} c_{s,m} 2^{j(s-r)} 2^{2jn} \tilde{\Psi}_{(-r)}(\frac{m}{16} + 2^j y) \tilde{\Phi}(\frac{m}{16} + 2^j z).
\]

Now,

\[
\Pi_1(f, g)(x) = \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} K_1(x - y, x - z) (\Delta_l f)(y) g(z) dy dz
\]

\[
\leq \sum_{j \leq 0} \sum_{m \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}} c_{s,m} 2^{j(s-r)} \int_{\mathbb{R}^{2n}} 2^{2jn} \tilde{\Psi}_{(-r)}(\frac{m}{16} + 2^j (x - y)) \tilde{\Phi}(\frac{m}{16} + 2^j (x - z)) \Delta_l f(y) g(z) dy dz.
\]

For a fixed \(j \in \mathbb{Z}\) we look at the integral in \(y\)

\[
\int_{\mathbb{R}^n} \tilde{\Psi}_{(-r)}(\frac{m}{16} + 2^j (x - y)) \Delta_l f(y) dy = \int_{\mathbb{R}^n} e^{\frac{2\pi i}{16} x \cdot (2^{-j} \frac{m}{16} + x)} \Psi(2^{-j} \xi) \tilde{\Phi}(\xi) d\xi,
\]

which, due to the support conditions on \(\Psi\), vanishes for every \(l \in \mathbb{Z} \setminus \{ j - 1, j, j + 1 \}\). Consequently,

\[
|\Pi_1(f, g)(x)| \leq \sum_{j \leq 0} \sum_{m \in \mathbb{Z}^n} \sum_{l=j-1,j,j+1} |c_{s,m}| 2^{j(s-r)}
\]

\[
\times \int_{\mathbb{R}^{2n}} 2^{2jn} |\tilde{\Psi}_{(-r)}(\frac{m}{16} + 2^j (x - y))| \tilde{\Phi}(\frac{m}{16} + 2^j (x - z)) |\Delta_l f(y)| |g(z)| dy dz
\]

\[
\leq 3 \left( \sum_{j \leq 0} 2^{j(s-r)} \right) \left( \sum_{m \in \mathbb{Z}^n} |c_{s,m}| \right) \left\| \tilde{\Psi}_{(-r)} \right\|_{L^1(\mathbb{R}^n)} \left\| \tilde{\Phi} \right\|_{L^1(\mathbb{R}^n)} \left\| f \right\|_{\dot{B}^{0,\infty}_{\infty}} \left\| g \right\|_{L^\infty}.
\]

Since \(s - r > 0\) we have \(\sum_{j \leq 0} 2^{j(s-r)} < \infty\) and, from (2.5), \(\sum_{m \in \mathbb{Z}^n} |c_{s,m}| < \infty\). Hence,

\[
|\Pi_1(f, g)(x)| \leq C \left\| \tilde{\Psi}_{(-r)} \right\|_{L^1(\mathbb{R}^n)} \left\| \tilde{\Phi} \right\|_{L^1(\mathbb{R}^n)} \left\| f \right\|_{\dot{B}^{0,\infty}_{\infty}} \left\| g \right\|_{L^\infty} \quad \forall x \in \mathbb{R}^n,
\]

where \(C > 0\) depends only on \(r, s,\) and \(n\).

Along the same lines, now for \(s < t\) one gets the bound for \(\Pi_2(f, g)\),

\[
|\Pi_2(f, g)(x)| \leq c \left( \sum_{j > 0} 2^{j(s-t)} \right) \left( \sum_{m \in \mathbb{Z}^n} |c_{s,m}| \right) \left\| \tilde{\Psi}_{(-t)} \right\|_{L^1(\mathbb{R}^n)} \left\| \tilde{\Phi} \right\|_{L^1(\mathbb{R}^n)} \left\| f \right\|_{\dot{B}^{0,\infty}_{\infty}} \left\| g \right\|_{L^\infty}.
\]
with $s - t < 0$. Then
\begin{equation}
\|\Pi(f, g)\|_{L^\infty} \leq C(\|D^r f\|_{B^0_{\infty, \infty}} + \|D^t f\|_{B^0_{\infty, \infty}}) \|g\|_{L^\infty}.
\end{equation}

Interchanging the roles of $f$ and $g$ to deal with $\Pi$ yields
\begin{equation}
\|\Pi(f, g)\|_{L^\infty} \leq C(\|D^r g\|_{B^0_{\infty, \infty}} + \|D^t g\|_{B^0_{\infty, \infty}}) \|f\|_{L^\infty}.
\end{equation}

Given a positive dyadic number $\mu$, plugging in $f_\mu$ and $g_\mu$ into (3.2) and (3.3), using the scaling property (2.3) and the fact that $\Pi(\xi, \eta)$ respectively, which belong to the Coifman-Meyer class for every $T$, from which (1.3) follows.

\section{The Case $s > 2n + 1$}

A smooth function $\sigma : \mathbb{R}^{2n} \setminus \{(0, 0)\} \rightarrow \mathbb{C}$ is said to belong to the class of bilinear Coifman-Meyer multipliers if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq 2n + 1$ there exist constants $c_{\alpha, \beta} > 0$ such that
\begin{equation}
|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq c_{\alpha, \beta} (|\xi| + |\eta|)^{-|\alpha| - |\beta|}, \quad \forall (\xi, \eta) \in \mathbb{R}^{2n} \setminus \{(0, 0)\}.
\end{equation}

In [6], the bilinear mapping $(f, g) \mapsto D^s(fg)$ was decomposed into the sum of three bilinear multipliers as follows
\begin{equation}
D^s(fg) = T_{1,s}(D^t f, g) + T_{2,s}(f, D^t g) + T_{3,s}(f, D^s g),
\end{equation}
where, keeping with the notation in Section 3, for $(\xi, \eta) \in \mathbb{R}^{2n} \setminus \{(0, 0)\}$ the bilinear multipliers for $T_{1,s}$ and $T_{2,s}$ are given by
\begin{equation}
\sigma_{1,s}(\xi, \eta) := \sum_{j \in \mathbb{Z}} \Psi(2^{-j} \xi) \Phi(2^{-j+3} \eta) \frac{|\xi + \eta|^s}{|\xi|^s} \quad \text{and} \quad \sigma_{2,s}(\xi, \eta) := \sigma_{1,s}(\xi, \eta),
\end{equation}
respectively, which belong to the Coifman-Meyer class for every $s > 0$. On the other hand, the multiplier for $T_{3,s}$, denoted by $\sigma_{3,s}$, can be expressed as
\begin{equation}
\sigma_{3,s}(\xi, \eta) := \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} c_{s,m} e^{\frac{2^m}{2^k} \xi + \eta} \Psi(2^{-k} \xi) \Psi(-s)(2^{-k} \eta).
\end{equation}

For fixed $\xi, \eta \in \mathbb{R}^{2n} \setminus \{(0, 0)\}$ the condition on the support of $\Psi$ implies that the sum in $k$ has only finitely many terms; namely, those with $2^k \sim |\xi| \sim |\eta|$. When derivatives in $\xi$ and $\eta$ of the product $e^{\frac{2^m}{2^k} \xi + \eta} \Psi(2^{-k} \xi) \Psi(-s)(2^{-k} \eta)$ are taken, after each
derivative a factor $2^{-k} (\sim |\xi|^{-1} \sim |\eta|^{-1} \sim (|\xi| + |\eta|)^{-1})$ appears, producing the right-hand side of (4.11). However, when the derivatives fall on the factor $e^{\frac{2\pi i n 2^{-k} (\xi+\eta) m}{m}}$ also components of $m \in \mathbb{Z}^n$ appear. Since the definition of a Coifman-Meyer multiplier requires at most $2n + 1$ derivatives, the worst case scenario for the sum over $m \in \mathbb{Z}^n$ (i.e., the case in which all $2n + 1$ derivatives fall on $e^{\frac{2\pi i n 2^{-k} (\xi+\eta) m}{m}}$) leads to the sum
\[
\sum_{m \in \mathbb{Z}^n} |c_{s,m}| |m|^{2n+1}.
\]

By (2.3), the sum above will be finite provided that $s > 2n + 1$. That is, whenever $s > 2n + 1$ all three bilinear operators in (4.2), and therefore the mapping $(f, g) \mapsto D^s (fg)$, can be realized as bilinear Coifman-Meyer multipliers. Since the class of Coifman-Meyer multipliers is included in the family of bilinear Calderón-Zygmund operators (see, [7, Section 6]) all the mapping properties of the type
\[
\|T(f, g)\|_{L^1} \lesssim \|f\|_{L^p} \|g\|_{L^q},
\]
that apply to bilinear C-Z operators $T$ on function spaces $X, Y,$ and $Z$ will also apply to $(f, g) \mapsto D^s (fg)$. For example, for a bilinear C-Z operator $T$, given $1 < p_1, p_2 < \infty$ and $1/p := 1/p_1 + 1/p_2$, it holds that
\[
\|T(f, g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}
\]
and (see [7, Proposition 1]) that,
\[
\|T(f, g)\|_{BMO} \lesssim \|f\|_{L^1} \|g\|_{L^1},
\]
with as other end-point estimates such as
\[
\|T(f, g)\|_{L^{1,\infty}} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^1} + \|f\|_{L^1} \|D^s g\|_{L^\infty}.
\]

As a consequence of the results above, we have

**Theorem 3.** If $s > 2n + 1$, then for every $f, g \in S(\mathbb{R}^n)$ we have the endpoint inequalities
\[
\|D^s (fg)\|_{BMO} \lesssim \|D^s f\|_{L^1} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{L^\infty}.
\]
and
\[
\|D^s (fg)\|_{L^{1,\infty}} \lesssim \|D^s f\|_{L^1} \|g\|_{L^1} + \|f\|_{L^1} \|D^s g\|_{L^1}.
\]

**Remark 3.** We note that the conditions (4.11) being satisfied with up to $n + 1$ derivatives (instead of $2n + 1$) are sufficient for the corresponding multiplier operator to be bounded from $L^{p_1} \times L^{p_2}$ into $L^p$ for $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$, as shown in Tomita [13]. The endpoint boundedness $L^\infty \times L^\infty$ into $BMO$ for Coifman-Meyer multipliers, with only up to $n + 1$ derivatives in (4.11), is unknown to us. To pass through the bilinear C-Z theory, as done above, it suffices that the conditions (4.11) be satisfied with up to $2n + 1$ derivatives.

**Proof of Theorem 2** By hypothesis, $1/p := 1/p_1 + 1/p_2$, so that $n/p < \varepsilon < 1$. It was proved in [12, pp.193–198] that a function $F$ with $\|D^s F\|_{L^p} + \|F\|_{BMO} + \|F\|_{L^p} < \infty$ can be written as $F = F_0 + G + F_1$ where
\[
\|F_0\|_{L^\infty} \lesssim \|D^s F\|_{L^p}, \quad \|G\|_{L^\infty} \lesssim \|F\|_{BMO}, \quad \text{and} \quad \|F_1\|_{L^\infty} \lesssim \|F\|_{L^p}.
\]
Now, with $T_{1,s}$ as in the decomposition (4.2), let us first choose $F := T_{1,s}(D^s f, g)$, so that from (4.11) we get
\[ \|T_{1,s}(D^s f, g)\|_{L^\infty} \lesssim \|T_{1,s}(D^s f, g)\|_{L^p} + \|T_{1,s}(D^s f, g)\|_{BMO} + \|D^\varepsilon(T_{1,s}(D^s f, g))\|_{L^p}. \]
The fact that $T_{1,s}$ is a bilinear C-Z operator and (4.6) yield
\[ \|T_{1,s}(D^s f, g)\|_{L^p} \lesssim \|D^s f\|_{L^p} \|g\|_{L^{p_2}}. \]
Also, from (4.7), it follows that
\[ \|T_{1,s}(D^s f, g)\|_{BMO} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^\infty}. \]
On the other hand, notice that
\[ D^\varepsilon(T_{1,s}(D^s f, g)) =: T_{1,s+\varepsilon}(D^{s+\varepsilon} f, g), \]
where the bilinear symbol for the operator $T_{1,s+\varepsilon}$ equals $\sigma_{1,s+\varepsilon}(\xi, \eta)$ (using the notation in (4.3)), also a Coifman-Meyer multiplier. Hence, (4.6) gives
\[ \|D^\varepsilon(T_{1,s}(D^s f, g))\|_{L^p} \lesssim \|D^{s+\varepsilon} f\|_{L^p} \|g\|_{L^{p_2}}. \]
Putting all together, for $T_{1,s}(f, g)$ we have
\[ \|T_{1,s}(D^s f, g)\|_{L^\infty} \lesssim (\|D^s f\|_{L^p} + \|D^{s+\varepsilon} f\|_{L^p}) \|g\|_{L^{p_2}}, \]
Given a positive dyadic number $\mu$, by replacing $f$ and $g$ in (4.12) with $f_\mu$ and $g_\mu$ and using the facts that
\[ \|D^s(f_\mu)\|_{L^q} = \mu^{\frac{n}{q}} \|D^s f\|_{L^q}, \quad \forall q \in [1, \infty], \]
that $1/p = 1/p_1 + 1/p_2$, and that $T_{1,s}(D^s f_\mu, g_\mu) = \mu^s T_{1,s}(D^s f, g)_\mu$, we obtain
\[ \|T_{1,s}(D^s f, g)\|_{L^\infty} \lesssim (\lambda^{-\frac{n}{p}} \|D^s f\|_{L^{p_1}} + \lambda^{\varepsilon^{-\frac{n}{p}}} \|D^{s+\varepsilon} f\|_{L^{p_1}}) \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}, \]
for every positive number $\lambda$. Minimization over $\lambda$ then implies
\[ \|T_{1,s}(D^s f, g)\|_{L^\infty} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}. \]
And, by an analogous argument based on $T_{2,s},$
\[ \|T_{2,s}(f, D^s g)\|_{L^\infty} \lesssim \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}} + \|f\|_{L^\infty} \|D^s g\|_{L^\infty}. \]
It only remains to consider $T_{3,s}$. Since $s > 2n + 1$, again from (4.6) and (4.7), we have
\[ \|T_{3,s}(D^s f, g)\|_{L^p} + \|T_{3,s}(D^s f, g)\|_{BMO} \lesssim \|D^s f\|_{L^p} \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}. \]
Now,
\[ D^\varepsilon(T_{3,s}(D^s f, g)) =: T_{3,s+\varepsilon}(D^{s+\varepsilon} f, g) \]
where the bilinear symbol for $T_{3,s+\varepsilon}$ is similar to $\sigma_{3,s}$ in (4.4) but with $c_{s,m}$ replaced by $c_{s+\varepsilon,m}$, the Fourier coefficients for $\Phi_{(s+\varepsilon)}$ which will satisfy $c_{s+\varepsilon,m} = O(1 + |m|^{-n-s-\varepsilon})$. Consequently,
\[ \|D^\varepsilon(T_{3,s}(D^s f, g))\|_{L^p} \lesssim \|D^{s+\varepsilon} f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|D^s f\|_{L^\infty} \|g\|_{L^\infty}. \]
Finally, Theorem 2 follows from (4.2), (4.13), (4.14), and (4.15).

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