On the Strongest Three-Valued Paraconsistent Logic Contained in Classical Logic

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Abstract. $\text{LP}^{\supset,F}$ is a three-valued paraconsistent propositional logic which is essentially the same as J3. It has most properties that have been proposed as desirable properties of a reasonable paraconsistent propositional logic. However, it follows easily from already published results that there are exactly 8192 different three-valued paraconsistent propositional logics that have the properties concerned. In this note, properties concerning the logical equivalence relation of a logic are used to distinguish $\text{LP}^{\supset,F}$ from the others. As one of the bonuses of focussing on the logical equivalence relation, it is found that only 32 of the 8192 logics have a logical equivalence relation that satisfies the identity, annihilation, idempotent, and commutative laws for conjunction and disjunction.

Keywords: paraconsistent logic, three-valued logic, logical consequence, logical equivalence.

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1 Introduction

A set of propositions is contradictory if there exists a proposition such that both that proposition and the negation of that proposition can be deduced from it. In classical propositional logic, every proposition can be deduced from every contradictory set of propositions. In a paraconsistent propositional logic, this is not the case.

$\text{LP}^{\supset,F}$ is the three-valued paraconsistent propositional logic LP [10] enriched with an implication connective for which the standard deduction theorem holds and a falsity constant. This logic, which is essentially the same as J3 [9], the propositional fragment of CLuNs [6] without bi-implication, and LF1 [8], has most properties that have been proposed as desirable properties of a reasonable paraconsistent propositional logic. However, it follows easily from results presented in [125] that there are exactly 8192 different three-valued paraconsistent propositional logics that have the properties concerned. In this note, properties concerning the logical equivalence relation of a logic are used to distinguish $\text{LP}^{\supset,F}$ from the others.

It turns out that only 32 of those 8192 logics are logics of which the logical equivalence relation satisfies the identity, annihilation, idempotent, and commutative laws for conjunction and disjunction; and only 16 of them are logics of
which the logical equivalence relation additionally satisfies the double negation
law. LP^{\supset,F} is one of those 16 logics. Three additional classical laws of logical
equivalence turn out to be sufficient to distinguish LP^{\supset,F} completely from the
others.

The desirable properties of reasonable paraconsistent propositional logics re-
ferred to above concern the logical consequence relation of a logic. Unlike in
classical propositional logic, the identity, annihilation, idempotent, and com-
mutative laws for conjunction and disjunction do not follow from those properties in
a three-valued paraconsistent propositional logic. Therefore, if closeness to classi-
cal propositional logic is considered important, it should be a desirable property
of a reasonable paraconsistent propositional logic to have a logical equivalence
relation that satisfies these laws. This would reduce the potentially interesting
three-valued paraconsistent propositional logics from 8192 to 32.

The structure of this note is as follows. First, we give a survey of the para-
consistent propositional logic LP^{\supset,F} (Section 2). Next, we discuss the known
properties of LP^{\supset,F} that have been proposed as desirable properties of a reason-
able paraconsistent propositional logic (Section 3). After that, properties concern-
ing the logical equivalence relation of a logic are used to distinguish LP^{\supset,F}
from the other three-valued paraconsistent propositional logics with the prop-
erties discussed earlier (Section 4). Finally, we make some concluding remarks
(Section 5).

It is relevant to realize that the work presented in this note is restricted to
three-valued paraconsistent propositional logics that are truth-functional three-
valued logics.

There is text overlap between this note and [7]. This note generalizes and
elaborates Section 2 of that paper in such a way that it may be of independent
importance to the area of paraconsistent logics.

2 The Paraconsistent Logic LP^{\supset,F}

A set of propositions \( \Gamma \) is contradictory if there exists a proposition \( A \) such
that both \( A \) and \( \neg A \) can be deduced from \( \Gamma \). In classical propositional logic,
every proposition can be deduced from a contradictory set of propositions. A
paraconsistent propositional logic is a propositional logic in which not every
proposition can be deduced from every contradictory set of propositions.

In [10], Priest proposed the paraconsistent propositional logic LP (Logic of Paradox). The logic introduced in this section is LP enriched with an implication
connective for which the standard deduction theorem holds and a falsity con-
stant. This logic, called LP^{\supset,F}, is in fact the propositional fragment of CLuNs [6]
without bi-implications.

LP^{\supset,F} has the following logical constants and connectives: a falsity constant
\( F \), a unary negation connective \( \neg \), a binary conjunction connective \( \land \), a binary
disjunction connective \( \lor \), and a binary implication connective \( \supset \). Truth and bi-
implication are defined as abbreviations: \( T \) stands for \( \neg F \) and \( A \equiv B \) stands for
\( (A \supset B) \land (B \supset A) \).
A Hilbert-style formulation of $\text{LP}^{\supset, F}$ is given in Table 1. In this formulation, which is taken from [4], $A$, $B$, and $C$ are used as meta-variables ranging over all formulas of $\text{LP}^{\supset, F}$. The axiom schemas on the left-hand side of Table 1 and the single inference rule (modus ponens) constitute a Hilbert-style formulation of the positive fragment of classical propositional logic. The first four axiom schemas on the right-hand side of Table 1 allow for the negation connective to be moved inward. The fifth axiom schema on the right-hand side of Table 1 is the law of the excluded middle. This axiom schema can be thought of as saying that, for every proposition, the proposition or its negation is true, while leaving open the possibility that both are true. If we add the axiom schema $\neg A \supset (A \supset B)$, which says that any proposition follows from a contradiction, to the given Hilbert-style formulation of $\text{LP}^{\supset, F}$, then we get a Hilbert-style formulation of classical propositional logic (see e.g. [4]). We use the symbol $\vdash$ without decoration to denote the syntactic logical consequence relation induced by the axiom schemas and inference rule of $\text{LP}^{\supset, F}$.

The following outline of the semantics of $\text{LP}^{\supset, F}$ is based on [4]. Like in the case of classical propositional logic, meanings are assigned to the formulas of $\text{LP}^{\supset, F}$ by means of valuations. However, in addition to the two classical truth values $t$ (true) and $f$ (false), a third meaning $b$ (both true and false) may be assigned. A valuation for $\text{LP}^{\supset, F}$ is a function $\nu$ from the set of all formulas of $\text{LP}^{\supset, F}$ to the set $\{t, f, b\}$ such that for all formulas $A$ and $B$ of $\text{LP}^{\supset, F}$:

$$
\nu(\bot) = f, \\
\nu(\neg A) = \begin{cases} t & \text{if } \nu(A) = f \\ f & \text{if } \nu(A) = t \\ b & \text{otherwise,} \end{cases} \\
\nu(A \land B) = \begin{cases} t & \text{if } \nu(A) = t \text{ and } \nu(B) = t \\ f & \text{if } \nu(A) = f \text{ or } \nu(B) = f \\ b & \text{otherwise,} \end{cases}
$$
The classical truth-conditions and falsehood-conditions for the logical connectives are retained. Except for implications, a formula is classified as both-true-and-false exactly when it cannot be classified as true or false by the classical truth-conditions and falsehood-conditions. The definition of a valuation given above shows that the logical connectives of LP are (three-valued) truth-functional, which means that each n-ary connective represents a function from \(\{t, f, b\}^n\) to \(\{t, f, b\}\).

For LP, the semantic logical consequence relation, denoted by \(\vdash\), is based on the idea that a valuation \(\nu\) satisfies a formula \(A\) if \(\nu(A) \in \{t, b\}\). It is defined as follows: \(\Gamma \vdash A\) iff for every valuation \(\nu\), either \(\nu(A') = t\) for some \(A' \in \Gamma\), or \(\nu(A) \in \{t, b\}\). We have that the Hilbert-style formulation of LP is strongly complete with respect to its semantics, i.e., \(\Gamma \vdash A\) iff \(\Gamma \models A\) (see e.g. [6]).

For all formulas \(A\) of LP in which \(F\) does not occur, for all formulas \(B\) of LP in which no propositional variable occurs that occurs in \(F\), if \(\not\vdash B\) (see e.g. [1])\footnote{On the left-hand side of \(\vdash\), we write \(A\) for \(\{A\}\) and \(\Gamma, \Delta\) for \(\Gamma \cup \Delta\). Moreover, we leave out the left-hand side if it is \(\emptyset\). We also write \(\not\vdash A\) for not \(\Gamma \vdash A\).} Moreover, \(\not\vdash B\) and all formulas \(A\) of LP: (a) \(\nu(\neg A) = t\) if \(\nu(A) = f\), (b) \(\nu(\neg A) = f\) if \(\nu(A) = t\). Hence, LP is a paraconsistent logic.

For LP, the logical equivalence relation \(\leftrightarrow\) is defined as for classical propositional logic: \(A \leftrightarrow B\) iff for every valuation \(\nu\), \(\nu(A) = \nu(B)\). Unlike in classical propositional logic, we do not have that \(A \leftrightarrow B\) iff \(\vdash A \equiv B\).

For LP, the consistency property is defined as to be expected: \(A\) is consistent iff for every valuation \(\nu\), \(\nu(A) \neq b\).

### 3 Known Properties of LP

In this section, the known properties of LP that have been proposed as desirable properties of a reasonable paraconsistent propositional logic are presented. Each of the properties in question has to do with logical consequence relations. Like above, the symbol \(\vdash\) is used to denote the logical consequence relation of LP. The symbol \(\vdash_{\text{CL}}\) is used to denote the logical consequence relation of classical propositional logic.

The known properties of LP that have been proposed as desirable properties of a reasonable paraconsistent propositional logic are:

(a) *containment in classical logic*: \(\vdash \subseteq \vdash_{\text{CL}}\);

(b) *proper basic connectives*: for all sets \(\Gamma\) of formulas of LP and all formulas \(A, B, C\) of LP:

\[
\begin{align*}
\nu(A \lor B) & = \begin{cases} t & \text{if } \nu(A) = t \text{ or } \nu(B) = t \\ f & \text{if } \nu(A) = f \text{ and } \nu(B) = f \\ b & \text{otherwise}, \end{cases} \\
\nu(A \supset B) & = \begin{cases} t & \text{if } \nu(A) = f \\ \nu(B) & \text{otherwise}. \end{cases}
\end{align*}
\]
\[(b_1) \Gamma, A \vdash B \iff \Gamma \vdash A \supset B,\]
\[(b_2) \Gamma \vdash A \land B \iff \Gamma \vdash A \text{ and } \Gamma \vdash B,\]
\[(b_3) \Gamma, A \lor B \vdash C \iff \Gamma, A \vdash C \text{ and } \Gamma, B \vdash C;\]

(c) **weakly maximal paraconsistency relative to classical logic:** for all formulas \(A\) of \(LP^{\supset F}\) with \(\not\vdash A\) and \(\vdash_{CL} A\), for the minimal consequence relation \(\vdash'\) such that \(\vdash \subseteq \vdash'\) and \(\vdash' A\), for all formulas \(B\) of \(LP^{\supset F}\), \(\vdash' B\) iff \(\vdash_{CL} B\);

(d) **strongly maximal absolute paraconsistency:** for all propositional logics \(\mathcal{L}\) with the same logical constants and connectives as \(LP^{\supset F}\) and a consequence relation \(\vdash'\) such that \(\vdash \subset \vdash'\), \(\mathcal{L}\) is not paraconsistent;

(e) **internalized notion of consistency:** \(A\) is consistent iff \(\vdash (A \supset F) \lor (\neg A \supset F)\);

(f) **internalized notion of logical equivalence:** \(A \iff B\) iff \(\vdash (A \equiv B) \land (\neg A \equiv \neg B)\).

Properties (a)–(f) have been mentioned relatively often in the literature (see e.g. [1,2,3,5,6,8]). Properties (a), \((b_1)\), (c), and (d) make \(LP^{\supset F}\) an ideal paraconsistent logic in the sense made precise in [2]. By property (e), \(LP^{\supset F}\) is also a logic of formal inconsistency in the sense made precise in [8].

Properties (a)–(c) indicate that \(LP^{\supset F}\) retains much of classical propositional logic. Actually, property (c) can be strengthened to the following property: for all formulas \(A\) of \(LP^{\supset F}\), \(\vdash A\) iff \(\vdash_{CL} A\). In [2], properties (e) and (f) are considered desirable and essential, respectively, for a paraconsistent propositional logic on which a process algebra that allows for dealing with contradictory states is built.

From Theorem 4.42 in [1], we know that there are exactly 8192 different three-valued paraconsistent propositional logics with properties (a) and (b). From Theorem 2 in [2], we know that properties (c) and (d) are common properties of all three-valued paraconsistent propositional logics with properties (a) and \((b_1)\).

From Fact 103 in [8], we know that property (f) is a common property of all three-valued paraconsistent propositional logics with properties (a), (b) and (e).

Moreover, it is easy to see that that property (e) is a common property of all three-valued paraconsistent propositional logics with properties (a) and (b). Hence, each three-valued paraconsistent propositional logic with properties (a) and (b) has properties (c)–(f) as well.

From Corollary 106 in [8], we know that \(LP^{\supset F}\) is the strongest three-valued paraconsistent propositional logic with properties (a) and (b) in the sense that for each three-valued paraconsistent propositional logic with properties (a) and (b), there exists a translation into \(LP^{\supset F}\) that preserves and reflects its logical consequence relation.

### 4 Characterizing \(LP^{\supset F}\) by Laws of Logical Equivalence

There are exactly 8192 different three-valued paraconsistent propositional logics with properties (a) and (b). This means that these properties, which concern the logical consequence relation of a logic, have little discriminating power. Properties (c)–(f), which also concern the logical consequence relation of a logic, do not offer additional discrimination because each of the 8192 three-valued paraconsistent propositional logics with properties (a) and (b) has these properties as well.
Table 2. Distinguishing laws of logical equivalence for LP\(^{2,F}\)

| (1) | \(A \land F \Leftrightarrow F\) | (2) | \(A \lor T \Leftrightarrow T\) |
|-----|--------------------------------|-----|----------------------------|
| (3) | \(A \land T \Leftrightarrow A\) | (4) | \(A \lor F \Leftrightarrow A\) |
| (5) | \(A \land A \Leftrightarrow A\) | (6) | \(A \lor A \Leftrightarrow A\) |
| (7) | \(A \land B \Leftrightarrow B \land A\) | (8) | \(A \lor B \Leftrightarrow B \lor A\) |
| (9) | \(-\neg A \Leftrightarrow A\) | (10) | \((A \lor \neg A) \supset B \Leftrightarrow B\) |
| (11) | \((A \supset B) \land (A \supset C) \Leftrightarrow A \supset (B \land C)\) | (12) | \((A \supset C) \lor (B \lor C) \Leftrightarrow (A \lor B) \lor C\) |

In this section, properties concerning the logical equivalence relation of a logic are used for additional discrimination. It turns out that 12 classical laws of logical equivalence, of which at least 9 are considered to belong to the most basic ones, are sufficient to distinguish LP\(^{2,F}\) completely from the other 8191 three-valued paraconsistent propositional logics with properties (a) and (b).

The logical equivalence relation of LP\(^{2,F}\) satisfies all laws given in Table 2.

Theorem 1. The logical equivalence relation of LP\(^{2,F}\) satisfies laws (1)–(12) from Table 2.

Proof. The proof is very easy by constructing, for each of the laws concerned, truth tables for both sides.

Moreover, among the 8192 three-valued paraconsistent propositional logics with properties (a) and (b), LP\(^{2,F}\) is the only one whose logical equivalence relation satisfies all laws given in Table 2.

Theorem 2. There is exactly one three-valued paraconsistent propositional logic with properties (a) and (b) of which the logical equivalence relation satisfies laws (1)–(12) from Table 2.

Proof. We confine ourselves to a brief outline of the proof. Because ‘non-deterministic truth tables’ that uniquely characterize the 8192 three-valued paraconsistent propositional logics with properties (a) and (b) are given in \[2\] the theorem can be proved by showing that, for each of the connectives, only one of the ordinary truth tables represented by the non-deterministic truth table for that connective is compatible with the laws given in Table 2. It can be shown by short routine case analyses that only one of the 8 ordinary truth tables represented by the non-deterministic truth tables for conjunction is compatible with laws (1), (3), (5), and (7), only one of the 32 ordinary truth tables represented by the non-deterministic truth tables for disjunction is compatible with laws (2), (4), (6), and (8), and only one of the 2 ordinary truth tables represented by the non-deterministic truth table for negation is compatible with law (9). Given the ordinary truth table for conjunction, disjunction, and negation so obtained, it can be shown by slightly longer routine case analyses that only one of the 16 non-deterministic truth table has sets of allowable truth values as results.
Table 3. Additional laws of logical equivalence for LP_{≥F}.

| Law | Expression |
|-----|------------|
| (13) | \((A \land B) \land C \iff A \land (B \land C)\) |
| (15) | \(\neg(A \land B) \iff \neg A \lor \neg B\) |
| (17) | \(\neg A \lor \neg B\) |
| (19) | \(A \lor (B \lor C) \iff (A \lor B) \lor (A \lor C)\) |
| (14) | \((A \lor B) \lor C \iff A \lor (B \lor C)\) |
| (16) | \(A \lor (B \land C) \iff (A \lor B) \land (A \lor C)\) |
| (18) | \(A \lor B \iff \neg A \land \neg B\) |
| (20) | \(A \land (B \land C) \iff (A \land B) \land C\) |

ordinary truth tables represented by the non-deterministic truth table for implication is compatible with laws (10)–(12).

The next two corollaries follow immediately from the proof of Theorem 2.

**Corollary 1.** There are exactly 16 three-valued paraconsistent propositional logics with properties (a) and (b) of which the logical equivalence relation satisfies laws (1)–(9) from Table 2.

**Corollary 2.** There are exactly 32 three-valued paraconsistent propositional logics with properties (a) and (b) of which the logical equivalence relation satisfies laws (1)–(8) from Table 2.

From a paraconsistent propositional logic with properties (a) and (b), it is only to be expected, because of paraconsistency and property (b), that its negation connective and its implication connective deviate clearly from their counterpart in classical propositional logic. Corollary 2 shows that, among the 8192 three-valued paraconsistent propositional logics with properties (a) and (b), there are 8160 logics whose logical equivalence relation does not even satisfy the identity, annihilation, idempotent, and commutative laws for conjunction and disjunction (laws (1)–(8) from Table 2).

It turns out that the logical equivalence relation of LP_{≥F} does not only satisfy the eight above-mentioned basic classical laws of logical equivalence but also other basic classical laws of logical equivalence, including the associative, distributive, and de Morgan’s laws for conjunction and disjunction (laws (13)–(18) from Table 3). Indeed, the logical equivalence relation of LP_{≥F} satisfies all laws given in Tables 2 and 3.

Laws (10)–(12) and (19)–(20), like laws (1)–(9) and (13)–(18), are also satisfied by the logical equivalence relation of classical propositional logic. We have that \(A' \iff B'\) is satisfied by the logical equivalence relation of classical propositional logic iff it follows from laws (1)–(9) and (13)–(18) and the laws

- (21) \(A \supset B \iff \neg A \lor B\)
- (22) \(A \land \neg A \iff F\)
- (23) \(A \lor \neg A \iff T\)

Laws (10)–(12) and (19)–(20) do not follow from laws (1)–(9) and (13)–(18) alone, but laws (21)–(23) are not satisfied by the logical equivalence relation of LP_{≥F}.

\[^{3}\text{This fact is easy to see because, without law (21), these laws axiomatize Boolean algebras and, in classical propositional logic, law (21) defines } \supset \text{ in terms of } \lor \text{ and } \neg.\]
5 Concluding Remarks

In this note, properties concerning the logical equivalence relation of a logic are used to distinguish the logic \( LP^{3,F} \) from the other logics that belong to the 8192 three-valued paraconsistent propositional logics that have properties (a)–(f) from Section 3. These 8192 logics are considered potentially interesting because properties (a)–(f) are generally considered desirable properties of a reasonable paraconsistent propositional logic.

Properties (a)–(f) concern the logical consequence relation of a logic. Unlike in classical propositional logic, we do not have \( A \iff B \) iff \( A \vdash B \) and \( B \vdash A \) in a three-valued paraconsistent propositional logic. As a consequence, the classical laws of logical equivalence that follow from properties (a) and (b) in classical propositional logic, viz. laws (1)–(8) and (13)–(16) from Section 4, do not follow from properties (a) and (b) in a three-valued paraconsistent propositional logic. Therefore, if closeness to classical propositional logic is considered important, it should be a desirable property of a reasonable paraconsistent propositional logic to have a logical equivalence relation that satisfies laws (1)–(8) and (13)–(16). This would reduce the potentially interesting three-valued paraconsistent propositional logics from 8192 to 32.

In [7], satisfaction of laws (1)–(8) and (10)–(12) is considered essential for a paraconsistent propositional logic on which a process algebra that allows for dealing with contradictory states is built. It follows easily from Theorem 1 and the proof of Theorem 2 that \( LP^{3,F} \) is one of only four three-valued paraconsistent propositional logics with properties (a) and (b) of which the logical equivalence relation satisfies laws (1)–(8) and (10)–(12).

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