A method for calculation of the operational availability factor of technical systems using the reliability models based on the continuous-time Markov chains

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Abstract. This scientific paper deals with the reliability models of technical systems based on the continuous-time Markov chains. An offered by the author calculation method for the operational availability factor of technical systems based on the methods for calculation of the probabilities of the system states and probability of system faultless functioning using the reliability models based on the continuous-time Markov chains is discussed. Finally, an example of calculation of the operational availability factor for the fault-tolerant data processing system with two identical and independent nodes is also given.

1. Introduction

In present days, complex technical systems have become an essential part of modern enterprises. In particular, different kind of the specialized computing and data processing systems are used to provide enterprise business processes. Moreover, reliability of these systems directly affects the stability, safety and efficiency of business processes. Therefore, analysis of reliability indices of technical systems is a quite urgent scientific task.

Modern reliability theory [1-6] usually deals with the availability factor, which is well-known reliability index of technical systems, and reliability models based on the continuous-time Markov chains and Kolmogorov-Chapman differential equations systems [7, 8], allowing calculation of the availability factor of technical systems for the given time moment \( t \).

However, in some cases it is important to know not only the availability of system for the given time moment \( t \), but also the operational availability, which equals to the probability of availability of the system at the given time moment \( t \) and its ability to provide faultless functioning during the given time interval \( \Delta t \). In particular, the operational availability factor is important for the data processing and computing systems. For example, if we are planning to start the file archiving task on computer at the time moment \( t = 12:00 \) and this task requires \( \Delta t = 30 \) minutes to be completed, it is important to know the operational availability factor of computer for the given \( t \) and \( \Delta t \).

Within the research work in the field of reliability models of technical systems [9, 10], the author developed a method for calculation of the operational availability factor based on the method for calculation of probabilities of the system states and method for calculation of probability of system faultless functioning using the reliability models based on the Markov chains.
2. Method for calculation of the operational availability factor
The generalized reliability model of technical systems can be represented by the Markov chain (Fig. 1) with the set of states $E$ consisting of the subset of system operable states $E_+$ and subset of non-operable states $E_-$. In turn, the subset $E_+$ contains the border operable states $H_+$, which have direct transition links to the non-operable states. Accordingly, the subset $E_-$ contains the border non-operable states $H_-$, which have direct transition links to the operable states.

One of the operable states $E_+$, designated as $k^*$, is considered as the initial state of system.

![Figure 1. Base Markov chain in the generalized reliability model of technical systems.](image)

Each of the states $i \in E$ (Fig. 2) may have inbound transition links with the given rates $\gamma_{ri}$ from the states $r \in R_i$, which have transition links to the state $i$, as well as outbound transition links with the given rates $\gamma_{is}$ to the states $s \in S_i$, which have transition links from the state $i$.

![Figure 2. Inbound and outbound transition links of the state $i$.](image)
Now, let us discuss the offered by the author two-stage method for calculation of the operational availability factor of technical systems, using the generalized reliability model based on the continuous-time Markov chain described above.

At the first stage, we build and solve the following Kolmogorov-Chapman differential equations system for the base Markov chain (Fig. 1) and given initial state $k^*$:

$$
\begin{align*}
\forall i \in E: \frac{dP_i(t)}{dt} &= \sum_{r \in R_i} (\gamma_{ir}P_r(t)) - P_i(t) \sum_{s \in S_i} \gamma_{is} - \sum_{r \in E} P_i(t) \gamma_{ir} = 1.
\end{align*}
$$

As a result, we obtain the probability functions $P_i(t)$ for all states of the base Markov chain, including the system operable states.

At the second stage, we apply a special reduction of the base Markov chain. The reduction of the base Markov chain is performed by using the following steps:

- Removing of all outbound transition links from all non-operable states $E.$ to any other states. As result, all non-operable states become «dead-end».

- Removing of all non-operable states, which have no transition links from the operable states, because they have no inbound or outbound transition links after carrying out of the first step.

As a result, the reduced Markov chain (Fig. 3) contains the operable states and dead-end non-operable states, which have direct inbound transition links from the operable states.

Next, taking into account that the operable states in the reduced Markov chain have no inbound transition links from the non-operable states, and, accordingly, the probabilities functions of the non-operable states are not used in the Kolmogorov-Chapman differential equations for the operable states, we build the Kolmogorov-Chapman differential equations system only for the operable states of the reduced Markov chain. As for the initial conditions, we consider each of the operable states $k \in E_+$ as the initial state and solve the differential equations system for each of the operable states.

For the specific initial state $k \in E_+$, the differential equations system is as follows:

$$
\begin{align*}
\forall i \in E_+: \frac{d\tilde{P}_i(t)}{dt} &= \sum_{r \in R_i} (\gamma_{ir}\tilde{P}_r(t)) - \tilde{P}_i(t) \sum_{s \in S_i} \gamma_{is}.
\end{align*}
$$

Solving the differential equations system gives us the probability functions of the operable states of the reduced Markov chain for the specific initial state $k$.

Next, the probability of the system faultless functioning $P_{UA}(\Delta t)$ for the given time interval $\Delta t$ and specific initial state $k$ is calculated as the sum of probabilities of the operable states of the reduced Markov chain, obtained as the result of solving of the differential equations system:

$$
P_{UA,k}(\Delta t) = \sum_{i \in E_+} \tilde{P}_i(\Delta t).
$$

Accordingly, solving the differential equations system (2) for each of the initial states $k \in E_+$ and using the formula (3) we obtain the probabilities of the system faultless functioning $P_{UA}(\Delta t)$ for the given time interval $\Delta t$ and each of the operable states $k \in E_+$.

Finally, the operational availability factor of system for the given time moment $t$ and time interval $\Delta t$ is calculated as the sum of products of the appropriate probability of the operable state $P_k(t)$, obtained using the base Markov chain, and probability of the system faultless functioning $P_{UA}(\Delta t)$, obtained using the reduced Markov chain, for each of the operable states $k \in E_+$:

$$
R(t, \Delta t) = \sum_{k \in E_+} P_k(t)P_{UA,k}(\Delta t).
$$
3. Example of calculation of the operational availability factor

Let us discuss an example of the technical system, which consist of two identical and independent data processing nodes. Each of the nodes may be in one of the following states:

- **Active state** – node is operable and provides the data processing function. From this state node may pass to the failed state with the given rate $\lambda_A$.
- **Passive state** – node is operable, but does not provide the data processing function, because of the software initialization or reconfiguration. From this state node may pass either to the failed state with the given rate $\lambda_P$ or active state with the given rate $\gamma_N$.
- **Failed state** – node is not operable. From this state node may pass to the passive state with the given rate $\mu_N$.

Nodes may fail and become repaired independently. Moreover, nodes may pass from passive state to active state independently from each other. The nodes have identical rates $\lambda_A, \lambda_P, \mu_N$ and $\gamma_N$.

Taking into account all the aforesaid, the reliability model of the fault-tolerant data processing system can be represented by the following Markov chain shown in Figure 4.

State 0 – both of the nodes are in passive state. State 1 – one of the nodes is in active state, another one is in passive state. State 2 – one of the nodes is in failed state, another one is in passive state. State 3 – one of the nodes is in failed state, another one is in active state. State 4 – both of the nodes are in failed state. State 5 – both of the nodes are in active state.

The fault-tolerant system is considered operable only in the states 1, 3 and 5, when at least one of the nodes is in the active state. State 5 is the initial state of the fault-tolerant system.
Figure 4. Base Markov chain in reliability model of the fault-tolerant data processing system.

The failure rate of nodes in the active state is $\lambda_A = 3 / 8760$ hour$^{-1}$ (on average three failures in a year). The failure rate of nodes in the passive state is $\lambda_P = 1 / 8760$ hour$^{-1}$ (on average one failure in a year). The recovery rate of nodes is $\mu_N = 1 / 24$ hour$^{-1}$ (on average one recovery in 24 hours). The activation rate of nodes is $\gamma_N = 20$ hour$^{-1}$ (on average one activation in 3 minutes).

Let us calculate the operational availability factor of the data processing system for the given time moment $t = 720$ hours, time interval $\Delta t = 60$ hours and initial state $k^* = 5$.

At the first stage, we build the Kolmogorov-Chapman differential equations system for the base Markov chain (Fig. 4) and given initial state $k^* = 5$:

\[
\begin{align*}
  P_0(0) &= 0; \quad P_1(0) = 0; \quad P_2(0) = 0; \quad P_3(0) = 0; \quad P_4(0) = 0; \quad P_5(0) = 1; \\
  dP_0(t)/dt &= -(2\lambda_P + 2\gamma_N)P_0(t) + \mu_N P_2(t); \\
  dP_1(t)/dt &= 2\gamma_N P_0(t) - (\gamma_N + \lambda_A + \lambda_P)P_1(t) + \mu_N P_3(t); \\
  dP_2(t)/dt &= 2\lambda_P P_0(t) + \lambda_A P_1(t) - (\mu_N + \lambda_P + \gamma_N)P_2(t) + 2\mu_N P_4(t); \\
  dP_3(t)/dt &= \lambda_P P_1(t) + \gamma_N P_2(t) - (\mu_N + \lambda_A)P_3(t) + 2\lambda_A P_5(t); \\
  dP_4(t)/dt &= \lambda_P P_2(t) + \lambda_A P_3(t) - 2\mu_N P_4(t); \\
  dP_5(t)/dt &= \gamma_N P_3(t) - 2\lambda_A P_5(t).
\end{align*}
\]

Solving the differential equations system for the given time moment $t = 720$ hours, we obtain probabilities of the operable states 1, 3 and 5:

\[
\begin{align*}
  P_1(t) &\approx 0.000033695953; \\
  P_3(t) &\approx 0.016170973454; \\
  P_5(t) &\approx 0.983728600278.
\end{align*}
\]

At the second stage, we reduce the base Markov chain, using the following steps:

- Removing of all outbound transition links from the non-operable states 0, 2 and 4 to any other states. As a result, all non-operable states become «dead-end».
- Removing of the non-operable state 0, because it has no inbound or outbound transition links after carrying out of the first step.

As a result, we obtain the following reduced Markov chain for the fault-tolerant data processing system, which is shown in Figure 5.
Next, we build the Kolmogorov-Chapman differential equations system for the reduced Markov chain and solve it for each of the operable states $k = 1, 3$ and 5, considering them as the initial state for the reduced Markov chain:

\[
\begin{align*}
\tilde{P}_k(0) &= 1; \quad \forall i \neq k : \tilde{P}_i(0) = 0; \\
d\tilde{P}_1(t)/dt &= -(\gamma_N + \lambda_\Lambda + \lambda_p)\tilde{P}_1(t) + \mu_N\tilde{P}_5(t); \\
d\tilde{P}_3(t)/dt &= \lambda_p\tilde{P}_1(t) - (\mu_N + \lambda_\Lambda)\tilde{P}_5(t) + 2\lambda_\Lambda\tilde{P}_5(t); \\
d\tilde{P}_5(t)/dt &= \gamma_N\tilde{P}_1(t) - 2\lambda_\Lambda\tilde{P}_5(t).
\end{align*}
\]

For each of the initial states $k = 1, 3$ and 5 we calculate probability of the system faultless functioning for the given time interval $\Delta t = 60$ hours as the sum of appropriate probabilities of the operable states 1, 3 and 5, obtained as the result of solving of the differential equations system:

\[
P_{U,1}(\Delta t) = \tilde{P}_1(\Delta t) + \tilde{P}_3(\Delta t) + \tilde{P}_5(\Delta t) \approx 0.999771776479; \quad \text{for } k = 1;
\]

\[
P_{U,3}(\Delta t) = \tilde{P}_1(\Delta t) + \tilde{P}_3(\Delta t) + \tilde{P}_5(\Delta t) \approx 0.992372057650; \quad \text{for } k = 3;
\]

\[
P_{U,5}(\Delta t) = \tilde{P}_1(\Delta t) + \tilde{P}_3(\Delta t) + \tilde{P}_5(\Delta t) \approx 0.999788684143; \quad \text{for } k = 5.
\]

Finally, we calculate the operational availability factor of the fault-tolerant data processing system for the given time moment $t = 720$ hours and time interval $\Delta t = 60$ hours as the sum of products of the appropriate probability of the operable state $P_k(t)$, obtained using the base Markov chain, and probability of the system faultless functioning $P_{U,k}(\Delta t)$, obtained using the reduced Markov chain, for each of the operable states $k = 1, 3$ and 5:

\[
R(t, \Delta t) = P_1(t)P_{U,1}(\Delta t) + P_3(t)P_{U,3}(\Delta t) + P_5(t)P_{U,5}(\Delta t) \approx 0.999602033289.
\]

Now, let us compare obtained value with the traditional availability factor $K(t)$ of the fault-tolerant data processing system for the given time moment $t = 720$ hours, which equals to the sum of the probabilities $P_k(t)$ of the operable states $k = 1, 3$ and 5:

\[
K(t) = P_1(t) + P_3(t) + P_5(t) \approx 0.999933269685.
\]

Obviously, the traditional availability factor $K(t)$ gives us higher value, because it does not take into account the probability of the system faultless functioning during the time interval $\Delta t$.

4. Conclusion

Thus, within the scope of this scientific paper an offered by the author method for calculation of the operational availability factor of technical systems based on the methods for calculation of the probabilities of the system states and probability of system faultless functioning using the reliability models based on the continuous-time Markov chains is discussed.
An example of calculation of the operational availability factor for the fault-tolerant system with two identical and independent data processing nodes is also given.

It is important to mention, that the Kolmogorov-Chapman differential equations systems, used in the offered method for calculation of the operational availability factor, usually cannot be solved analytically and they require application of the numerical solving methods.

Obtained by the author scientific results are used for research work in the field of reliability analysis of technical systems.

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