3-connected Planar Graph Isomorphism is in Log-space

Samir Datta\textsuperscript{1}, Nutan Limaye\textsuperscript{2}, and Prajakta Nimbhorkar\textsuperscript{2}

\textsuperscript{1} Chennai Mathematical Institute, Chennai 603 103, India. 
sdatta@cmi.ac.in
\textsuperscript{2} The Institute of Mathematical Sciences, Chennai 600 113, India. 
nutan,prajakta@imsc.res.in

Abstract. We show that the isomorphism of 3-connected planar graphs can be decided in deterministic log-space. This improves the previously known bound $UL \cap coUL$ of [13].

1 Introduction

The general graph isomorphism problem is a well studied problem in computer science. Given two graphs, it deals with finding a bijection between the sets of vertices of these two graphs, such that the adjacencies are preserved. The problem is in $NP$, but it is not known to be complete for $NP$. In fact, it is known that if it is complete for $NP$, then the polynomial hierarchy collapses to its second level. On the other hand, no polynomial time algorithm is known. For general graph isomorphism $NL$ and $PL$ hardness is known [14], whereas for trees, $L$ and $NC^1$ hardness is known, depending on the encoding of the input [6].

In literature, many special cases of this general graph isomorphism problem have been studied. In some cases like trees [8], [3], or graphs with coloured vertices and bounded colour classes [9], NC algorithms are known. We are interested in the case where the graphs under consideration are planar graphs. In [15], Weinberg presented an $O(n^2)$ algorithm for testing isomorphism of 3-connected planar graphs. Hopcroft and Tarjan [5] extended this for general planar graphs, improving the time complexity to $O(n \log n)$. Hopcroft and Wong [4] further improved it to give a linear time algorithm. Its parallel complexity was first considered by Miller and Reif [10] and Ramachandran and Reif [11]. They gave an upper bound of $AC^1$. Recently Thierauf and Wagner [13] improved it to $UL \cap coUL$ for 3-connected planar graphs. They also proved that this problem is hard for $L$. In this paper, we give a log-space algorithm for 3-connected planar graph isomorphism, thereby settling its complexity.

Thierauf and Wagner use shortest paths between nodes of a graph to obtain a canonical spanning tree. A systematic traversal of this tree generates a canonical form for the graph. The best known upper bound for shortest paths in planar graphs is $UL \cap coUL$ [13]. Thus the total complexity of their algorithm goes to $UL \cap coUL$, despite the fact that all other steps can be done in $L$.

We identify that their algorithm hinges on making a systematic traversal of the graph in canonical way. Thus we bypass the step of finding shortest paths and give an orthogonal approach for finding such a traversal. We use the notion of universal exploration sequences (UXS) defined in [7]. Given a graph on $n$ vertices with maximum degree $d$, a UXS is nothing but a polynomial length string over $\{0, \ldots, d-1\}$. Such a sequence can be used to traverse the graph for a chosen combinatorial embedding $\rho$, starting vertex $u$ and a starting edge $e = \{u, v\}$. Reingold [12] proved that such a universal sequence can be constructed in $L$. Using this result, we canonize a 3-connected planar graph in log-space. To our knowledge, this is the best upper bound for this class of graphs.

In Section 2, we give some basic definitions of complexity classes and formally define the notion of universal exploration sequences. In Section 3, we describe our log-space algorithm. We conclude with a discussion of open problems in Section 4.
2 Preliminaries

In this section, we give a brief introduction of the graph isomorphism problem and the notion of universal exploration sequences.

2.1 Universal Exploration Sequences

Let $G = (V, E)$ be a $d$-regular graph, with given combinatorial embedding $\rho$. The edges around any vertex $u$ can be numbered $\{0, 1, \ldots, d - 1\}$ according to $\rho$ arbitrarily in clockwise order. A sequence $\tau_1 \tau_2 \ldots \tau_k \in \{0, 1, \ldots, d - 1\}^k$ and a starting edge $e_0 = (v_{i-1}, v_i) \in E$, define a walk $v_{i-1}, v_i, \ldots, v_k$ as follows: For $0 \leq i \leq k$, if $(v_{i-1}, v_i)$ is the $s$th edge of $v_i$, let $e_i = (v_i, v_{i+1})$ be $(s + \tau_i)$th edge of $v_i$ modulo $d$.

Definition 1. Universal Exploration sequences (UXS): A sequence $\tau_1 \tau_2 \ldots \tau_l \in \{0, 1, \ldots, d - 1\}^l$ is a universal exploration sequence for $d$-regular graphs of size at most $n$ if for every connected $d$-regular graph on at most $n$ vertices, any numbering of its edges, and any starting edge, the walk obtained visits all the vertices of the graph.

Following lemma suggests that UXS can be constructed in $L$ [12]:

Lemma 1. There exists a log-space algorithm that takes as input $(1^n, 1^D)$ and produces an $(n, D)$-universal exploration sequence.

2.2 The Graph Isomorphism Problem

Definition 2. Graph isomorphism: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there is a bijection $\phi : V_1 \rightarrow V_2$ such that $(u, v) \in E_1$ if and only if $(\phi(u), \phi(v)) \in E_2$.

Let GI be the problem of finding such a bijection $\phi$ given two graphs $G_1, G_2$. Let Planar-GI be the special case of GI when the given graphs are planar. 3-connected planar graph isomorphism problem is a special case of Planar-GI when the graphs are 3-connected planar graphs. We recall the definition of 3-connected planar graphs here:

A graph $G$ is connected if there is a path between any two vertices in $G$. A vertex $v \in V$ is an articulation point if $G - v$ is not connected. A pair of vertices $u, v \in V$ is a separation pair if $G(V \setminus \{u, v\})$ is not connected. A biconnected graph contains no articulation points. A 3-connected graph contains no separation pairs.

3 Log-space Algorithm for 3-connected Planar-GI

In this section, we prove following theorem:

Theorem 1. Given two 3-connected planar graphs $G$ and $H$, deciding whether $G$ is isomorphic to $H$ can be done in $L$.

For general planar graphs, the best known parallel algorithm runs in $AC^1$ [10]. Thierauf and Wagner [13] recently improved the bound for the case of 3-connected planar graphs to $UL \cap coUL$. This case is easier due to a result by Whitney [16] that every planar 3-connected graph has precisely two embeddings on a sphere, where one embedding is the mirror image of the other. Moreover, one can efficiently compute these embeddings.

Using these embeddings, Thierauf and Wagner compute a code for a graph, such that isomorphic graphs will have the same code. A code with this property is called a canonical code.
for the graph. They construct it via a spanning tree, which depends upon the planar embedding of the graph. Bourke, Tewari and Vinodchandran [2] proved that planar reachability is in $\text{UL} \cap \text{coUL}$. Thierauf and Wagner extend their result for computing distances in planar graphs in $\text{UL} \cap \text{coUL}$ and crucially use this in the construction of the spanning tree. Once this spanning tree is constructed, a canonical code can be obtained in $\text{L}$.

Our approach bypasses the spanning tree construction step and thus eliminates distance computations. In that sense, we believe that this is a completely new approach for computing canonical codes for 3-connected planar graphs.

Our algorithm can be outlined as follows:

1. Given a 3-connected planar graph $G = (V, E)$, find its planar embedding $\rho$.
2. Make the graph 3-regular canonically for this embedding $\rho$ to obtain an edge-coloured graph $G'$ as described in algorithm 3.1.
3. Find the canon of $G'$ using algorithm 3.2.

The step 1 is in log-space due to a result by Allender and Mahajan [1]. We prove that steps 2 and 3 can also be done in log-space. Step 3 uses the idea of UXS introduced by Koucký[]. Step 2 essentially does the preprocessing in order to make step 3 applicable.

The canonical code thus constructed is specific to the choice of the combinatorial embedding, the starting edge, and the starting vertex. Given two graphs $G$ and $H$, we fix these arbitrarily for $G$ and cycle through both embeddings and all choices of the starting edge and the starting vertex for $H$, comparing the codes for each of them. As there are only polynomially many choices, this loop runs in $\text{L}$.

### 3.1 Making the graph 3-regular

In this section, we describe the procedure to make the graph 3-regular. In Section 3.2, we use Reingold’s construction for UXS [12] to come up with a canonical code. As Reingold’s construction [12] for UXS requires the graph to have constant degree, we do this preprocessing step. In Lemma 2, we prove that two given graphs are isomorphic if and only if they are isomorphic after the preprocessing step. We note that after the preprocessing step, the graph does not remain 3-connected, however, the embedding of the new graph is inherited from the given graph. Hence even the new graph has only two possible embeddings.

We now describe the preprocessing steps in Algorithm 3.1. Note that the new graph thus obtained has $2|E|$ vertices.

**Algorithm 1** Procedure to get a 3-regular planar graph $G'$ from 3-connected planar graph $G$.

**Input:** A 3-connected planar graph $G$ with planar combinatorial embedding $\rho$.

**Output:** A 3-regular planar graph $G'$ on $2m$ vertices, with edges coloured 1 and 2 and planar combinatorial embedding $\rho'$.

```plaintext
for all $v_i \in V$ do
    Replace $v_i$ of by a cycle $\{v_{i1}, \ldots, v_{id_i}\}$ on $d_i$ vertices, where $d_i$ is the degree of $v_i$.
    The $d_i$ edges $\{e_{i1}, \ldots, e_{id_i}\}$ incident to $v_i$ in $G$ are now incident to $\{v_{i1}, \ldots, v_{id_i}\}$ respectively.
    Colour the cycle edges with colour 1.
    Colour $e_{i1}, \ldots, e_{id_i}$ by colour 2.
end for
```

**Lemma 2.** Given two 3-connected planar graphs $G_1, G_2$, $G_1 \cong G_2$ if and only if $G'_1 \cong G'_2$ where the isomorphism between $G'_1$ and $G'_2$ respects colours of the edges.
Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two 3-connected planar graphs with planar combinatorial embeddings $\rho_1$ and $\rho_2$ respectively. Let $\phi : V_1 \to V_2$ be an isomorphism between the oriented graphs $(G_1, \rho_1)$ and $(G_2, \rho_2)$. By isomorphism of oriented graphs we mean that the graphs are isomorphic for the fixed embeddings, in our case $\rho_1$ and $\rho_2$. Construct $G'_1$ and $G'_2$ preserving the orientation of original edges from $G_1$ and $G_2$ respectively. Let the orientations be $\rho'_1$ and $\rho'_2$. By our construction, edges around a vertex in $G_1$ (respectively $G_2$) get the same combinatorial embedding around the corresponding cycle in $G'_1$ ($G'_2$). Consider an edge \( \{v_i, v_j\} \) in $E_1$. Let $\phi(v_i) = u_k$ and $\phi(v_j) = u_i$. \( \{u_k, u_i\} \) $\in E_2$. Let corresponding edge in $G'_1$ be $\{v_{i_p}, v_{i_q}\}$ and that in $G'_2$ be $\{u_{k_p}, u_{k_q}\}$. Then define a map $\phi' : V'_1 \to V'_2$ such that $\phi'(v_{i_p}) = u_{k_p}$ and $\phi'(v_{i_q}) = u_{k_q}$. It is easy to see that $\phi'$ is an isomorphism for edge-coloured oriented graphs $(G'_1, \rho'_1)$ and $(G'_2, \rho'_2)$.

Now let $\phi'$ be an isomorphism between oriented graphs $(G'_1, \rho'_1)$ and $(G'_2, \rho'_2)$. Let $e = \{v_{i_p}, v_{i_q}\} \in E'_1$ where $v_{i_p}$ and $v_{i_q}$ correspond to the same vertex $v_i$ in $G_1$. Then $\text{colour}(e) = 1$ and $\phi'(v_{i_p}), \phi'(v_{i_q}) \in E'_2$ and $\text{colour}(\phi'(e)) = 1$. Thus $\phi'$ maps copies of the same vertex of $G_1$ to copies of a single vertex of $G_2$. Hence a map $\phi$ can be derived from $\phi'$ in a natural way. It is easy to see that $\phi$ is an isomorphism between oriented graphs $(G_1, \rho_1)$ and $(G_2, \rho_2)$.

### 3.2 Obtaining the canonical code

Lemma 2 suggests that for given embeddings $\rho_1$, $\rho_2$ of $G_1$ and $G_2$, it suffices to check the 3-regular oriented graphs $(G'_1, \rho'_1)$ and $(G'_2, \rho'_2)$ for isomorphism. This can be done as follows:

#### Algorithm 2 Procedure canon($G, \rho, v, e = (u, v)$)

| Input: Edge-coloured graph $G = (V, E)$ with maximum degree 3 and combinatorial embedding $\rho$, starting vertex $v$, starting edge $e = (u, v)$ |
| Output: canon of $G$. |
| Construct a $(n, 3)$ universal exploration sequence $U$. |
| With starting vertex $v \in V$ and edge $e = (u, v)$ incident to it, traverse $G$ according to $U$ and $\rho$ and output the labels of the vertices. |
| Give labels to the vertices according to their first occurrence in this output sequence. |
| For every $(i, j)$ in this labelling, output whether $(i, j)$ is an edge or not. If it is an edge, output its colour. This gives a canon for the graph. |

#### Lemma 3. Let $\sigma_1 = \text{canon}(G'_1, \rho'_1, v_1, e_1 = (u_1, v_1))$ and $\sigma_2 = \text{canon}(G'_2, \rho'_2, v_2, e_2 = (u_2, v_2))$. If $\sigma_1 = \sigma_2$ then $G'_1 \cong G'_2$. Further, if $G'_1 \cong G'_2$ then for some choice of $\rho'_2, v_2, e_2$, $\sigma_1 = \sigma_2$. |

Proof. If $G'_1 \cong G'_2$, then there is a bijection $\phi : V'_1 \to V'_2$ for corresponding embeddings $\rho'_1, \rho'_2$. Let $e'_1 = (u, v) \in E'_1$. Then $e_2 = (\phi(u), \phi(v)) \in E'_2$. Let $e_1$ and $e_2$ be chosen as starting edges and $v$ and $\phi(v)$ as starting vertices for traversal using UXS $U$ for $(G'_1, \rho'_1)$ and $(G'_2, \rho'_2)$ respectively. Let $T_1$ and $T_2$ be the output sequence. If a vertex $w \in V'_1$ occurs at position $l$ in $T_1$ then $\phi(w) \in V'_2$ occurs at position $l$ in $T_2$. Thus the sequences are canonical when projected down to the first occurrences and hence $\sigma_1 = \sigma_2$.

Let $\sigma_1 = \sigma_2 = \sigma$. The labels of vertices in $\sigma$ are just a relabelling of vertices of $V'_1$ and $V'_2$. These relabellings are some permutations, say $\pi_1$ and $\pi_2$. Then $\pi_1 \cdot \pi_2^{-1} : V'_1 \to V'_2$ is a bijection.

After constructing canonical code $\sigma'$ for a graph $G'$, it remains to construct canonical code $\sigma$ for the original graph $G$. For each edge $(i, j)$ of colour $2$ in $\sigma'$, traverse along the edges coloured $1$ starting from $i$ and find the minimum among the vertices visited. Let it be $p$. Repeat the process for $j$. Let the minimum vertex visited along edges of colour $1$ be $q$. Output the edge...
The sequence thus obtained contains \( n \) labels for vertices, each between \( \{1, 2, \ldots, 2m\} \). This can further be converted into a sequence with labels for vertices between \( \{1, 2, \ldots, n\} \) by finding the rank of each of the labels. This gives us \( \sigma \). Correctness follows from the fact that vertices connected with edges of colour 1 are copies of the same vertex in \( G \), hence they should get the same number.

Clearly, each of the above steps can be performed in \( L \) and hence the algorithm runs in \( L \). This proves Theorem 1.

4 Conclusion

Our note settles the open question mentioned in [13] by giving a log-space algorithm for 3-connected planar graph isomorphism. The most challenging question is to settle the complexity of the general graph isomorphism problem. The other important goal is to improve upon the \( AC^1 \) upper bound of [10] for planar graph isomorphism.

5 Acknowledgment

We thank Jacobo Toran, V. Arvind, and Meena Mahajan for helpful discussions.

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