REPRESENTATION AND APPROXIMATION OF THE POLAR FACTOR OF AN OPERATOR ON A HILBERT SPACE

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In memory of our friend Ezzeddine Zahrouni,
who left us very early

Abstract. Let $H$ be a complex Hilbert space and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on $H$. The polar decomposition theorem asserts that every operator $T \in \mathcal{B}(H)$ can be written as the product $T = VP$ of a partial isometry $V \in \mathcal{B}(H)$ and a positive operator $P \in \mathcal{B}(H)$ such that the kernels of $V$ and $P$ coincide. Then this decomposition is unique. $V$ is called the polar factor of $T$. Moreover, we have automatically $P = |T| = (T^*T)^{\frac{1}{2}}$. Unlike $P$, we have no representation formula that is required for $V$.

In this paper, we introduce, for $T \in \mathcal{B}(H)$, a family of functions called a “polar function” for $T$, such that the polar factor of $T$ is obtained as a limit of a net built via continuous functional calculus from this family of functions. We derive several explicit formulas representing different polar factors. These formulas allow new for methods of approximations of the polar factor of $T$.

1. Introduction. Throughout this paper, let $H$ be a complex Hilbert space and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on $H$. For an arbitrary operator $T \in \mathcal{B}(H)$, we denote by $\sigma(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$ and $T^*$ the spectrum, the range, the null subspace and the adjoint operator of $T$, respectively. For any closed subspace $M$ of $H$, let $P_M$ denote the orthogonal projection onto $M$.

An operator $T \in \mathcal{B}(H)$ is a partial isometry when $TT^*T = T$ (or, equivalently, $T^*T$ is an orthogonal projection; in this case, $T^*T = P_{\mathcal{N}(T)^\perp}$). In particular, $T$ is an isometry if $T^*T = I$, and $T$ is unitary if it is a surjective isometry.

The well known polar decomposition of any operator $T \in \mathcal{B}(H)$ is a generalization of the polar decomposition of a nonzero complex number $z = \exp(i\theta)|z|$, $\theta \in \mathbb{R}$, and consists of writing $T$ as a product

$$T = VP,$$

of a partial isometry $V \in \mathcal{B}(H)$ and a positive operator $P \in \mathcal{B}(H)$ such that $\mathcal{N}(V) = \mathcal{N}(P)$. Then $P = |T| := (T^*T)^{\frac{1}{2}}$ is the modulus of $T$ and $\mathcal{N}(V) = \mathcal{N}(T)$. Therefore the decomposition (1) becomes $T = V|T|$ and it is unique. The operator $V$ is called the polar factor of $T$.

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The polar decomposition goes back to von Neumann [10] (see [11, Remark 2.2.11, Page 24]).

In this paper we are interested in the approximation of the polar factor of an operator acting on a Hilbert space. The results obtained are motivated by the work of N.J.Higham. Indeed, in [5], N.J.Higham shows the following: for $A$ a nonsingular matrix, by considering the iteration sequence defined by

$$X_0 = A, \quad X_{k+1} = \frac{1}{2}(X_k + X_k^*)^{-1},$$

then each iterate $X_k$ is nonsingular and $\lim_{k \to \infty} X_k = V$, where $V$ is the unitary polar factor in the polar decomposition of $A$ (for more information on this topic see [6] and references therein). On the other hand, one can show the existence and uniqueness of the polar decomposition by approximation methods in the general context of von Neumann algebras; see, e.g., [11, Proposition 2.2.9] or [13, Theorem 1.12.1].

The paper is organized as follows. In Section 2, we introduce, for $T \in B(H)$, the notion of a “polar-function” for $T$, which consists on a family of functions such that the polar factor of $T$ is obtained as a limit of this family of functions.

In Section 3, we explain several examples of polar-function families so we obtain approximation formulas (by: an integral formula, iterative method, Newton and Hermite interpolation polynomials) for the polar factor of an operator acting on a Hilbert space.

2. Main results. We start this section with the following definition.

**Definition 2.1.** Let $T \in B(H)$. We say that a net $(F_\alpha)_\alpha$ is polar-function for $T$, if $F_\alpha$ is a continuous real valued function on the compact set $K = \sigma(T^*T) \cup \{0\} = \sigma(TT^*) \cup \{0\}$, such that $(\lambda F_\alpha(\lambda))_\alpha$ is uniformly bounded (i.e. $|\lambda F_\alpha(\lambda)| \leq C$ for all $\lambda \in K$, and all $\alpha$) and

$$\lim_{\alpha} \lambda F_\alpha(\lambda) = 1 \text{ for all } \lambda \in K \setminus \{0\}.$$

**Remark 1.** (1) The term polar-function is justified by the next theorem.

(2) We give, in the next section, several examples of polar-functions.

(3) The limit in the previous definition will be specified for each example.

The following lemma will be useful throughout the rest of this article.

**Lemma 2.2.** For any operator $T \in B(H)$ and any continuous function $f$ on the compact set $\sigma(T^*T) \cup \{0\} = \sigma(TT^*) \cup \{0\}$, we have

$$Tf(T^*T) = f(TT^*)T.$$

In particular,

$$Tf(T^*T)T^* = f(TT^*)TT^* \quad \text{and} \quad T|T| = |T^*|T, \quad (2)$$

**Proof.** Observe that $T(T^*T) = (TT^*)T$. It follows that for any polynomial $P \in \mathbb{C}[X]$, we have $TP(T^*T) = P(TT^*)T$. Using the Stone-Weierstrass theorem, we deduce that

$$Tf(T^*T) = f(TT^*)T.$$

So the first equality is shown. The other equalities follow easily.

The following theorem is one of the two main theorems of this section. He gives a new proof of the theorem of the polar decomposition of an operator $T$ and a new representation of the polar factor of $T$. 
Theorem 2.3. For any operator \( T \in \mathcal{B}(H) \) and any polar-function for \( T \), \((F_\alpha),\) the limit

\[
\lim_{\alpha} F_\alpha(TT^*)|T^*|^T = V \quad \text{exists,}
\]

with convergence in the strong operator topology of \( \mathcal{B}(H) \) and the limit \( V \) is the polar factor of \( T \) (i.e. \( V \) is the unique partial isometry such that \( T = V|T| \) and \( V^*V = P_{\mathcal{N}(T)^\perp} \)).

Proof. We first prove the existence of \( \lim_{\alpha} F_\alpha(TT^*)|T^*|^T \). To see this, put \( A(\alpha) = F_\alpha(TT^*)|T^*|^T \). If \( \alpha' > \alpha \), then we have

\[
(A(\alpha') - A(\alpha))(A(\alpha') - A(\alpha))^* = [(F_{\alpha'}(TT^*) - F_\alpha(TT^*))(TT^*)]^2.
\]

(3)

Since \( TT^* \) is self-adjoint, there exists a unique spectral measure \( \mu \) such that for all \( f \) continuous function on the spectrum of \( TT^* \),

\[
<f(TT^*)x, y> = \int_{\sigma(TT^*)} f(\lambda) d\mu_{x,y}(\lambda), \quad \text{for all } x, y \in H.
\]

In particular, by (3), we have for all \( x \in H \),

\[
\|A(\alpha')x - A(\alpha)x\|^2 = <(A(\alpha') - A(\alpha))(A(\alpha') - A(\alpha))^*x, x> = <(F_{\alpha'}(TT^*) - F_\alpha(TT^*))TT^*|^T|^T, x, x>
\]

\[
= \int_{\sigma(TT^*)} \|F_{\alpha'}(\lambda) - F_\alpha(\lambda)\|^2 d\mu_{x,x}(\lambda).
\]

Now, since, by definition, \((\lambda F_\alpha(\lambda))_{\alpha} \) is uniformly bounded and \( \forall \lambda \in \sigma(TT^*) \setminus \{0\} \), \( \lim_{\alpha} \lambda F_\alpha(\lambda) = 1 \) and \( \alpha = 0 \) if \( \lambda = 0 \), the Lebesgue’s dominated convergence theorem, \( \|A(\alpha')x - A(\alpha)x\| \) converges to zero. It follows that \( A(\alpha) \) is strongly convergent to an element \( V \) in \( \mathcal{B}(H) \).

On the other hand, by Lemma 2.2, \( A(\alpha) = TF_{\alpha}(T^*T)|T| \). So, we have for all \( x \in H \),

\[
\|A(\alpha)|T| x - Tx\|^2 = \|T(F_{\alpha}(T^*T))T^*T - I)|x\|\|^2
\]

\[
= <(F_\alpha(T^*T)^2 - I)T^*Tx, x>
\]

\[
= \int_{\sigma(T^*T)} \lambda(\lambda F_\alpha(\lambda) - 1)^2 d\mu_{x,x}(\lambda),
\]

which tends to zero by the Lebesgue’s dominated convergence theorem. Hence

\[
Tx = \lim_{\alpha} A(\alpha)|T| x = V|T|x \quad \text{for all } x \in H.
\]

Let us now show that \( V \) is a partial isometry such that \( \mathcal{N}(V) = \mathcal{N}(|T|) = \mathcal{N}(T) \). Let \( P = P_{\mathcal{N}(T)^\perp} \) be the orthogonal projection onto \( \mathcal{N}(T)^\perp \). Since \( \mathcal{N}(|T|) = \mathcal{N}(T) \), we have \( |T|(I - P) = 0 \) which implies that

\[
P|T| = |T| = |T|P.
\]

(4)
Therefore \( A(\alpha)P = A(\alpha) \) and hence, \( VP = V \) and \( PV^* = V^* \). It follows that
\[
V^*V = PV^*VP.
\]
(5)

On the other hand, by (4), we have \(|T|P|T| = T^*T = |T|V^*V|T|\). It follows that \(|T|(P - V^*V)|T| = 0\). Since \( \mathcal{N}(|T|) = \mathcal{N}(P) \), we obtain \( P(P - V^*V)P = 0 \) and
\[
P = PV^*VP.
\]
(6)

Now, by (5) and (6), we get \( V^*V = PV \), and hence \( V \) is a partial isometry such that \( \mathcal{N}(V) = \mathcal{N}(P) = \mathcal{N}(T) \). The uniqueness of \( V \) can be deduced immediately.

In the case where \( T \) has closed range, we obtain an additional result which gives more precision on the convergence in Theorem 2.3. To do so, we need to introduce the reduced minimum modulus that measures the closedness of the range of an operator. Recall that the reduced minimum modulus of an operator \( T \in \mathcal{B}(H) \) is defined by
\[
\gamma(T) := \begin{cases} \inf \left\{ \|Tx\|; \|x\| = 1, x \in \mathcal{N}(T)\perp \right\} & \text{if } T \neq 0 \\ +\infty & \text{if } T = 0 \end{cases}
\]
The reduced minimum modulus of an operator \( T \in \mathcal{B}(H) \) has the following properties.

1. \( T \) has closed range if and only if \( \gamma(T) > 0 \).
2. \( \gamma(T) = \inf \sigma(|T|) \setminus \{0\} \);
3. The following equalities are valid:
\[
\gamma(T)^2 = \gamma(T^*T) = \gamma(T T^*) = \gamma(T^*)^2.
\]
4. In particular, if \( T \) has closed range, then
\[
\sigma(T^*T) \cup \sigma(T T^*) \subseteq \{0\} \cup [\gamma(T)^2, \|T\|^2].
\]

For further information, the interested reader may consult [1, 7].

The following theorem gives more precision on the convergence in Theorem 2.3, when the range of \( T \) is closed.

**Theorem 2.4.** Suppose \( T \in \mathcal{B}(H) \) has closed range. With the same assumptions and notations as Theorem 2.3, we have
\[
\|A(\alpha) - V\| \leq \sup_{\lambda \in [\gamma(T)^2, \|T\|^2]} |\lambda F_\alpha(\lambda) - 1|,
\]
where \( A(\alpha) = F_\alpha(T T^*)\).\(T^*\)\(T\).

Therefore, the polar factor of \( T \) is given by
\[
V = \lim_{\alpha} F_\alpha(T T^*)\|T^*\|T,
\]
with convergence in the norm topology of \( \mathcal{B}(H) \).

**Proof.** Using (3), we get
\[
\|A(\alpha') - A(\alpha)\| = \|(F_{\alpha'}(T T^*) - F_\alpha(T T^*))TT^*\|
\leq \sup_{\lambda \in \sigma(T T^*)} \|F_\alpha(\lambda) - F_\alpha(\lambda)\lambda\|
\leq \sup_{\lambda \in [\gamma(T)^2, \|T\|^2]} |(F_\alpha(\lambda) - F_\alpha(\lambda))\lambda| \quad (T \text{ has closed range}).
\]
Now clearly the convergence of $A(\alpha)$ to $V$ is in norm topology and

$$||A(\alpha) - V|| \leq \sup_{\lambda \in \mathcal{N}(T)} |\lambda F_\alpha(\lambda) - 1|.$$  

(7)

The proof of the theorem is complete.  \hfill \Box

3. Approximation of the polar factor. In this section we will apply the theoretical results obtained in the previous section, by giving several examples of a family of polar-function. Let us start with the following observation:

If $T = V|T|$ is the polar decomposition of $T$, then we have

$$T^*V - |T| = 0.$$  

(8)

Indeed, since $\mathcal{N}(V) = \mathcal{N}(T) = \mathcal{N}(|T|)$, $|T|(I - V^*V) = 0$. Therefore

$$|T|V^*V = |T| = V^*V|T|.$$  

Now, if $T = V|T|$, then $T^* = |T|V^*$. It then results $T^*V = |T|V^*V = |T|$ and (8) follows.

Equality (8), will play in the next subsections an essential role.

3.1. Approximation of polar factor by an integral. Equality (8), suggests the problem of minimizing the following function on $H$:

$$F(x) = ||T^*x - |T|y||^2 = <T^*x - |T|y, T^*x - |T|y >,$$

where $y \in H$ fixed. Clearly, for $y \in H$ fixed, the minimum of $F(x)$ is attained at $x = Vy$.

Now, consider $x$ as a derivable function $x(t)$, $t \geq 0$, with $x(0) = 0$. By differentiating the function $F(x(t))$, we obtain

$$\frac{d}{dt} F(x(t)) = 2Re < T^*x(t) - |T|y, T^*x(t) >$$

$$= 2Re < T(T^*x(t) - |T|y), \frac{d}{dt} x(t) > .$$

If we take

$$\frac{d}{dt} x(t) = -T(T^*x(t) - |T|y) = -TT^*x(t) + T|T|y,$$

(9)

we get that

$$\frac{d}{dt} F(x(t)) = -2||T(T^*x(t) - |T|y)||^2 < 0.$$  

It follows that $F(x(t))$ is a decreasing function in $t$, asymptotically approaching its infimum when $t$ tends to infinity. Then $x(t)$ must be approaching $Vy$ as $t \to \infty$ (since, the minimum of $F(x)$ is attained at $Vy$). Therefore by the resolution of the differential equation (9), we obtain

$$x(t) = \int_0^t \exp((t - s)TT^*)T|T|y ds = \int_0^t \exp(-sTT^*)T|T|y ds.$$  

Taking $t \to +\infty$, we get, for arbitrary $y \in H$,

$$Vy = \int_0^\infty \exp(-sTT^*)T|T|y ds.$$  

Now consider the family of functions $F_\alpha(\lambda) = \int_0^\alpha \exp(-s\lambda) ds$, $\alpha \geq 0$, then it is easy to verify that $(F_\alpha)_\alpha$ is a polar-function for each $T \in \mathcal{B}(H)$.

As a direct consequence of Theorems 2.3 and 2.4, we obtain the following theorem.
Theorem 3.1. ([2, 9]) For each $T \in B(H)$, we have
\[ V = \int_0^{\infty} \exp(-sTT^*)|T\| \, ds, \tag{10} \]
where the integral converges in the strong operator topology of $B(H)$ and $V$ is the polar factor of $T$.

If in addition $T$ has a closed range, then
\[ \|A(\alpha) - V\| \leq \exp(-\alpha \gamma(T)^2), \]
where $A(\alpha) = \int_0^\alpha \exp(-sTT^*)|T\| \, ds$.

Therefore, the convergence in (10) is in the norm topology of $B(H)$.

Remark 2. The first part of the previous theorem is in [2] and the second part, the closed range case, is in [9].

3.2. Approximation of the polar factor by an iterative method. If we take the differential equation (9)
\[
\frac{d}{dt} x(t) = -TT^* x(t) + T|T|y,
\]
we can use a numerical integration procedure to get an approximate solution:
\[ x(t + a) - x(t) \simeq -aTT^* x(t) + aT|T|y, \quad a > 0 \]
which is equivalent to
\[ x(t + a) \simeq (I - aTT^*) x(t) + aT|T|y. \]

From this, it is easy to obtain
\[ x((n + 1)a) \simeq \sum_{k=0}^{n} (I - aTT^*)^k aT|T|y, \]
setting $t = 0$, we get
\[ x((n + 1)a) \simeq \sum_{k=0}^{n} (I - aTT^*)^k aT|T|y. \]

Therefore, if we consider the family of functions $F_n(\lambda) = a \sum_{k=0}^{n} (1 - a\lambda)^k \ n \geq 0$ then this series converges for all $\lambda \in D_a = \{ \lambda > 0; |1 - a\lambda| < 1 \} = \{ \lambda \in \mathbb{R}; 0 < \lambda < \frac{2}{a} \}$.

Now if $T \in B(H)$ and we choose $a$ such that $0 < a < \frac{2}{\|T\|^2}$, then $\sigma(T^*T) \cup \sigma(TT^*) \subseteq D_a \cup \{0\}$. It follows that the family of functions
\[ F_n(\lambda) = a \sum_{k=0}^{n} (1 - a\lambda)^k \ n \geq 0, \quad \text{and} \ 0 < a < \frac{2}{\|T\|^2}, \]
is a polar-function for $T$.

As direct consequence of Theorems 2.3 and 2.4, we obtain the following theorem.

Theorem 3.2. For each $T \in B(H)$, we have
\[ V = a \sum_{k=0}^{+\infty} (I - aTT^*)^k |T^*|T, \quad 0 < a < \frac{2}{\|T\|^2}, \tag{11} \]
with convergence in strong operator topology of $B(H)$ and $V$ is the polar factor of $T$. 
If in addition $T$ has a closed range, then there exists $C \in [0, 1]$ such that

$$\|A_n - V\| \leq C^{n+1},$$

where $A_n = a \sum_{k=0}^{n}(I - aTT^*)^k|T^*|T$.

Therefore, the convergence in (11) is in the norm topology of $\mathcal{B}(H)$.

**Proof.** The first part of the theorem is deduced directly from Theorem 2.3, since $(F_n)$ is a polar-function for $T$. For the second part of the theorem, by (7), we have

$$\|A_n - V\| \leq \sup_{\lambda \in [\gamma(T)^2, \|T\|^2]} |\lambda F_n(\lambda) - 1|.$$

On the other hand, we have

$$\lambda F_n(\lambda) - 1 = a\lambda \sum_{k=0}^{n}(I - a\lambda)^k - 1$$

$$= (1 - (1 - a\lambda)) \sum_{k=0}^{n}(I - a\lambda)^k - 1$$

$$= \sum_{k=0}^{n}(I - a\lambda)^k - \sum_{k=0}^{n+1}(I - a\lambda)^k + 1$$

$$= \sum_{k=0}^{n}(I - a\lambda)^k - \sum_{k=0}^{n}(I - a\lambda)^k$$

$$= (1 - a\lambda)^{n+1}.$$

Therefore we get

$$\|A_n - V\| \leq \sup_{\lambda \in [\gamma(T)^2, \|T\|^2]} |1 - a\lambda|^{n+1}.$$

A simple calculation shows that,

$$|1 - a\lambda| \leq \max\{|1 - a\|T\|^2|, |1 - a\gamma(T)^2| \} : = C \text{ and } 0 < C < 1.$$

It follows that

$$\|A_n - V\| \leq C^{n+1},$$

and the theorem is proved. $\square$

We can formulate the previous series as a recurring sequence:

$$A_n = a \sum_{k=0}^{n}(I - aTT^*)^k|T^*|T$$

which can be rewritten in the following way:

$$A_0 = a|T^*|T; \quad A_{n+1} = (I - aTT^*)A_n + A_0. \quad (12)$$

**Corollary 1.** If $T \in \mathcal{B}(H)$, then the sequence $(A_n)_{n \geq 0}$ defined by (12), where $0 < a < \frac{2}{\|T\|^2}$, converges to $V \in \mathcal{B}(H)$, the polar factor of $T$, in the strong topology of $\mathcal{B}(H)$.

If in addition $T$ has a closed range, then the convergence is in the norm topology of $\mathcal{B}(H)$.
3.3. **Approximation of polar factor by a limit.** Equality (8) suggests the problem of minimizing the following function on $H$:

$$f_\alpha(x) = \|T^*x - |T|y\|^2 + \alpha\|x\|^2 \geq 0,$$

where $y \in H$ fixed. Clearly, for $y \in H$ fixed, the minimum of $f_\alpha(x)$ is attained at $x = Vy$ and $\alpha = 0$.

The derivative of $f_\alpha(x)$ with respect to $x$ is given by

$$Df_\alpha(x) = 2(TT^*x - |T|y) + 2\alpha x.$$

Therefore

$$0 = Df_\alpha(x) \iff (\alpha + TT^*)x = |T|y \iff x = (\alpha + TT^*)^{-1}|T|y.$$

It follows that

$$Vy = \lim_{\alpha \to 0} (\alpha + TT^*)^{-1}|T|y.$$

Now, if we consider the family of functions

$$F_\alpha(\lambda) = (\alpha + \lambda)^{-1}, \alpha > 0, \lambda \geq 0,$$

then the family $(F_\alpha)_\alpha$ is a polar-function for any arbitrary operator $T \in B(H)$.

As direct consequence of Theorems 2.3 and 2.4, we have the following result

**Theorem 3.3.** For each $T \in B(H)$, we have

$$V = \lim_{\alpha \to 0} (\alpha + TT^*)^{-1}|T|,$$

with convergence in the strong operator topology of $B(H)$ and $V$ is the polar factor of $T$.

If in addition $T$ has a closed range, then

$$\|A(\alpha) - V\| \leq \frac{1}{\gamma(T)^2} \alpha,$$

where $A(\alpha) = (\alpha + TT^*)^{-1}|T|$.

Therefore, the convergence in (13) is in the norm topology of $B(H)$.

**Proof.** The first part of the theorem is deduced directly from Theorem 2.3, since $(F_\alpha)$ is a polar-function for $T$. For the second part of the theorem, by (7), we have

$$\|A(\alpha) - V\| \leq \sup_{\lambda \in [\gamma(T)^2, \|T\|^2]} |\lambda F_\alpha(\lambda) - 1|.$$

Now, we have for all $\lambda \in [\gamma(T)^2, \|T\|^2]$,

$$|\lambda F_\alpha(\lambda) - 1| = \frac{\alpha}{\lambda + \alpha} \leq \frac{\alpha}{\gamma(T)^2 + \alpha} \leq \frac{1}{\gamma(T)^2} \alpha.$$

Therefore we get

$$\|A(\alpha) - V\| \leq \frac{1}{\gamma(T)^2} \alpha,$$

and the theorem is proved. \qed
3.4. Approximation of polar factor by Newton interpolation polynomials.
In this subsection we shall describe the polar factor by means of Newton interpolation polynomials.

First we shall consider the divided differences interpolation polynomial of \( f(x) = \frac{1}{x} \). For \( i = 1, 2, \ldots, n \), let \( P_n(i) = \sum_{k=0}^{i-1} \frac{(-1)^k}{k+1} \Delta^k f(1) \) where \( \Delta f(i) := f(i+1) - f(i) \), \( \Delta^k f(i) := \Delta(\Delta^{k-1} f(i)) \). In particular, for \( f(i) = \frac{1}{i} \) it is known that \( \Delta^k f(1) = \frac{(-1)^k}{k+1} \). Then, in this case, we obtain
\[
P_n(x) = \sum_{k=0}^{n} \frac{1}{k+1} \prod_{j=0}^{k-1} \left(1 - \frac{x}{j+1}\right)
\]
where, by convention, \( \prod_{j=0}^{0} \left(1 - \frac{x}{j+1}\right) = 1 \), so \( P_0(x) = 1 \) (see [3, 12]).

The sequence of the polynomial functions \( (P_n) \) admits the following properties, gathered in the two following lemmas

**Lemma 3.4.** ([9]) For all \( \lambda \geq 0 \) and integers \( n \geq 0 \),
\[
\lambda P_n(\lambda) = 1 - \prod_{j=0}^{n} \left(1 - \frac{\lambda}{j+1}\right).
\]

**Lemma 3.5.** ([9]) For every \( m \in \mathbb{N} \),
\[
\lim_{n \to \infty} \prod_{j=0}^{n} \left(1 - \frac{\lambda}{j+1}\right)^m = \begin{cases} 
0 & \text{if } \lambda > 0 \\
1 & \text{if } \lambda = 0.
\end{cases}
\]

The two previous lemmas show that the sequence of functions \( (P_n(x) := P_n(x), n \geq 0) \) is a polar-function for each \( T \in \mathcal{B}(H) \). Direct consequence of Theorem 2.3, we have the following result

**Theorem 3.6.** ([9]) For each \( T \in \mathcal{B}(H) \), we have
\[
V = \sum_{k=0}^{\infty} \frac{T^k}{k+1} \prod_{j=0}^{k-1} (I - \frac{T^* T}{j+1}) |T| = \sum_{k=0}^{\infty} \frac{1}{k+1} \prod_{j=0}^{k-1} (I - \frac{T T^*}{j+1}) |T^* T|,
\]
where the convergence is in the strong topology of \( \mathcal{B}(H) \) and \( V \) is the polar factor of \( T \).

The polynomials \( P_n(x) \) can be rewritten in the following way:
\[
P_0(x) = 1, \quad P_{n+1}(x) = P_n(x) + \frac{1}{n+2} [1 - x P_n(x)], \quad n \geq 1.
\]

In fact, by Lemma 3.4, we have
\[
P_{n+1}(x) = P_n(x) + \frac{1}{n+2} \prod_{j=0}^{n} \left(1 - \frac{x}{j+1}\right) = P_n(x) + \frac{1}{n+2} [1 - x P_n(x)].
\]

Hence, given \( T \in \mathcal{B}(H) \), let us define
\[
V_0 = T |T|, \quad V_{n+1} := V_n + \frac{1}{n+2} (V_0 - T T^* V_n), \quad n \geq 1.
\]

Thus, as an immediate consequence of Theorem 3.6, we obtain the following corollary.

**Corollary 2.** ([9]) If \( T \in \mathcal{B}(H) \), then the sequence \( (V_n)_{n \geq 0} \) defined by (15) converges to \( V \in \mathcal{B}(H) \), the polar factor of \( T \), in the strong topology of \( \mathcal{B}(H) \).
The following theorem gives more precision on the convergence in Theorem 3.6, when the range of $T$ is closed.

**Theorem 3.7.** ([9]) If $T \in \mathcal{B}(H)$ has closed range, then the convergences in Theorem 3.6 and Corollary 2 are in the norm topology of $\mathcal{B}(H)$.

Furthermore, there is a constant $C > 0$, independent of $n$, such that

$$
\|V_n - V\| \leq C \frac{1}{(n + 2)^2(T)} \quad (\text{for } n \text{ sufficiently large}).
$$

3.5. **Approximation of polar factor by Hermite interpolation polynomials.** In this subsection we shall describe the polar factor by means of the Hermite interpolation polynomial of $f(x) = \frac{1}{x}$. This interpolation polynomial is given by

$$
Q_n(x) = \sum_{k=0}^{n} [2(1 + k) - x] \frac{1}{(1 + k)^2} \prod_{j=0}^{k-1} \left(1 - \frac{x}{j + 1}\right)^2 \text{ for } n \geq 0,
$$

where, by convention, $\prod_{j=0}^{-1} (1 - \frac{x}{j + 1})^2 = 1$, so $Q_0(x) = 2 - x$.

The Hermite interpolation polynomial is the unique polynomial of degree $2n + 1$ such that $Q_n(x_i) = f(x_i) = \frac{1}{x_i}$ and $Q'_n(x_i) = f'(x_i) = -\frac{1}{x_i^2}$ for $x_i = i + 1$ with $i = 0, 1, 2, ..., n$ (see [3, 12]).

**Lemma 3.8.** For all $\lambda \geq 0$ and integers $n \geq 0$,

$$
\lambda Q_n(\lambda) = 1 - \prod_{j=0}^{n} (1 - \frac{\lambda}{j + 1})^2.
$$

By lemmas 3.5 and 3.8, we can easily see that the sequence of functions $(F_n = Q_n, n \geq 0)$ is a polar function for each $T \in \mathcal{B}(H)$. As direct consequence of Theorem 2.3, we have the following result.

**Theorem 3.9.** ([9]) For each $T \in \mathcal{B}(H)$, we have

$$
V = \sum_{k=0}^{\infty} [2(1 + k)I - TT^*] \frac{T}{(1 + k)^2} \prod_{j=0}^{k-1} (I - \frac{T^*T}{1 + j})^2 |T|,
$$

with convergence in the strong topology of $\mathcal{B}(H)$ and $V$ is the polar factor of $T$.

The Hermite polynomials can also be defined as

$$
Q_0(x) = 2 - x, \quad Q_{n+1}(x) = Q_n(x) + \frac{1}{n + 2} (2 - \frac{x}{n + 2}) [1 - xQ_n(x)], \quad n \geq 1.
$$

In fact,

$$
Q_{n+1}(x) = Q_n(x) + \frac{1}{(n + 2)^2} (2(n + 2) - x) \prod_{j=0}^{n} (1 - \frac{x}{1 + j})^2
$$

$$
= Q_n(x) + \frac{1}{(n + 2)^2} (2 - \frac{x}{n + 2}) [1 - xQ_n(x)].
$$

Hence, given $T \in \mathcal{B}(H)$, let us define

$$
V_0 = (2 - TT^*)T |T|, V_{n+1}
$$

$$
= V_n + \frac{1}{n + 2} (2I - \frac{T^*T}{n + 2}) |T| - TT^*V_n, \quad n \geq 1.
$$

Thus, as an immediate consequence of Theorem 3.9, we obtain the following corollary.
Corollary 3. If $T \in \mathcal{B}(H)$, then the sequence $(V_n)_{n \geq 0}$ defined by (17) converges to $V \in \mathcal{B}(H)$, the polar factor of $T$, in the strong topology of $\mathcal{B}(H)$.

In the case where $T$ has closed range, we obtain the following result which gives more precision on the convergence.

Theorem 3.10. ([9]) If $T \in \mathcal{B}(H)$ has closed range, then the convergence in Theorem 3.9 and Corollary 3 are in the norm topology of $\mathcal{B}(H)$.

Furthermore, there is a constant $C > 0$, independent of $n$, such that

$$\|V_n - V\| \leq C \frac{1}{(n + 2)^{2\gamma(T)^2}} \quad \text{(for } n \text{ sufficiently large)}.$$  

4. Remark and problem. For $T \in \mathcal{B}(H)$, note by $V_T$ the polar factor of $T$. The following question:

Question 1: Under which conditions the application $T \mapsto V_T$ is Lipschitzian?

Has interested several authors (see [8, 4] and references therein). The results obtained are based on the famous Theorem which gives an estimate of the solution of the equation $AX - XB = Y$ where $A, B$ are positive operators and $Y$ are given.

More precisely if $\delta = \text{dist}(\sigma(A), \sigma(B)) > 0$, then $\|X\| \leq \frac{1}{\delta}\|Y\|$.

For example, we have the following result in finite dimension. Its extension in infinite dimension, remains true.

Theorem 4.1. ([8, Theorem 1]) If $T, S \in \mathcal{B}(H)$ are invertible, then

$$\|V_T - V_S\| \leq \frac{2}{\gamma(T) + \gamma(S)} \|T - S\|.$$  

Note that since $T, S$ are invertible, $V_T, V_S$ are unitary and $\gamma(T) = \frac{1}{\|T^{-1}\|}$, $\gamma(S) = \frac{1}{\|S^{-1}\|}$.

In the same spirit as Theorem 4.1, in the case of closed range operators, we ask the following question:

Question 2: Can we use one of the representations of the polar factor obtained above, to weaken the assumptions of the previous theorem?

We end this paper with the following conjecture. First, for $\alpha > 0$, let us note by

$$\Gamma_\alpha(H) = \{T \in \mathcal{B}(H) \setminus \{0\}; \; \gamma(T) \geq \alpha\}.$$  

Conjecture 1. If $T, S \in \Gamma_\alpha$, then

$$\|V_T - V_S\| \leq \frac{1}{\alpha} \|T - S\|.$$  

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