Revisiting a stability problem of two-component droplets

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ABSTRACT

We study the problem of the stability of a two-component droplet. The standard solution known from the literature is based on a particular form of the mean field energy functional, in particular on distinction of hard mode and soft mode contributions. By imposing the constraint on densities of the two species which minimizes the hard mode energy, the problem is reduced to a stability analysis of a one component system. As opposed to this, we address the issue in full generality. Our analysis is valid for arbitrary forms of energy density. We formulate constraints which correspond to the physically relevant situation of a system which has unconstrained volume and may evaporate particles. For the specific case of a two component Bose-Bose droplet we find approximate analytic solutions and compare them to the standard result. We show that the densities of both components of a stable droplet are limited to a range depending on interaction strength, in contrast to the original unique solution.

Introduction

In his seminal paper D. Petrov\textsuperscript{1} showed the stabilizing role of the Lee-Huang-Young (LHY) energy\textsuperscript{2} in a system of a two component Bose-Bose mixture. The analysis presented in\textsuperscript{1} is based on a particular form of the mean field energy and the observation that a stable droplet can be formed if the densities of both components are chosen in such a way that the hard mode energy (the dominant term) vanishes, so that the instability of the weak mode is suppressed by a small contribution of quantum fluctuations – the LHY term. Soon experiments confirmed this theory\textsuperscript{3–5}. Moreover, a similar mechanism occurred to be responsible for stabilization of elongated dipolar condensates of Dysprosium\textsuperscript{6–9} and Erbium\textsuperscript{10} atoms. Recently, quantum droplets in heteronuclear bosonic mixtures have been reported\textsuperscript{11}. For review of the present status of quantum droplets physics see\textsuperscript{12}.

Standard stability analysis of a two component Bose-Bose mixture is based on a particular form of the mean field energy density. The distinction of a stable hard mode of energetically expensive excitations and a slightly unstable soft mode forms the basis of the entire analysis. In this paper we want to address the issue of stability of a two component droplet going beyond approximations which rely on the distinction between hard and soft modes\textsuperscript{1}. In particular we want to formulate a description of droplets which accounts not only for the ground state and soft mode excitations, but hard mode excitations too. The correct description of hard mode excitations (absent in the former approximate treatment) is crucial when considering collisions of droplets\textsuperscript{13}.

To define a stable solution we have to specify a physical constraint first. Note that the question of a global unconstrained minimum has a simple but trivial answer if analysis is limited to the mean field approach – in an effectively attractive case the system collapses and both densities become infinite, or on the other hand if the system is a repulsive one, the atoms expand to infinity and their densities vanish. The collapse predicted on the mean field level in fact signifies that the description used does not account for physical processes in this situation. Formation of bound molecules, larger complexes or solidification is expected then.

Here we address a typical experimental situation where initially $N_1$ atoms of the first component are mixed with $N_2$ atoms of the second component in an external trap. After tuning the interactions to the region in which a stable droplet is expected, the external potential is removed. Eventually a droplet is formed. This is a scenario which defines plausible physical constraints. Our goal is to find densities of a stable system formed this way and/or the final number of atoms in each component. Note that final number of atoms need not be necessarily the same as the number of atoms mixed initially. Some can evaporate yet.

Constraints

We study a mixture of two species of ultracold atoms of mass $m$. The choice of equal masses has one serious advantage – it allows for (to a large extent) analytical treatment. The entire procedure is also valid for different masses of both species,
We impose on the system. The number of atoms in every component is \( N_1 \) and \( N_2 \) respectively. Intraspecies interactions are repulsive, i.e., corresponding scattering lengths are positive \( a_{11} > 0, a_{22} > 0 \), but interspecies interactions are attractive, \( a_{12} < 0 \).

The interaction energy density of the system, for fixed interaction strength, is a function of densities of the two components \( n_1, n_2, \varepsilon = \varepsilon(n_1, n_2) \). The densities are related to corresponding wavefunctions \( n_i = |\psi_i|^2 \) which are normalized to the number of particles \( \int d\mathbf{r} |\psi_i(\mathbf{r})|^2 = N_i \). The total energy density with kinetic energy included is:

\[
\delta' = -\frac{\hbar^2}{2m} \sum_{i=1}^2 \nabla \psi_i^* \nabla \psi_i + \varepsilon(n_1, n_2)
\]

A corresponding time-dependent set of two coupled Gross-Pitaevskii (GP) equations describing the dynamics of both components can be easily obtained by minimizing the action \( S = \int d^3x \int dt L \), where the Lagrangian density is \( L = \hbar \Re (i \sum_j \psi_j^* \partial_t \psi_j) - \delta' \):

\[
\frac{i \hbar}{\partial t} \psi_i(\mathbf{r}) = \left[ -\frac{\hbar^2}{2m} \Delta + \frac{\delta \varepsilon(n_1, n_2)}{\delta n_i} \right] \psi_i(\mathbf{r})
\]

By the standard substitution \( \psi_i(\mathbf{r}, t) = e^{-i\mu_i / \hbar} \psi_i(\mathbf{r}) \) the time dependent equations lead to a set of two stationary GP equations:

\[
\left[ -\frac{\hbar^2}{2m} \Delta + \frac{\delta \varepsilon(n_1, n_2)}{\delta n_i} \right] \psi_i(\mathbf{r}) = \mu_i \psi_i(\mathbf{r})
\]

where \( \mu_i \) are chemical potentials.

We assume that interactions are tuned in such a way that the system is effectively very weakly attractive and is on the collapse side of the stability diagram\(^{14,15} \), \( -\delta a = \sqrt{a_{11}a_{22}} + a_{12} < 0 \), where coupling constants are related to the scattering lengths via \( g_{ij} = 4\pi\hbar^2a_{ij}/m \). As shown by D. Petrov\(^1 \), if the energy of quantum fluctuations is included in addition to the aforementioned mean-field interaction energy, the collapse may be avoided and a liquid droplet of volume \( V \) is formed. This however can only happen if the interactions are appropriately tuned. Moreover, the numbers of available atoms in every component must be in a right proportion.

If interactions are in the region not supporting droplet formation the solutions will correspond to an infinite system with densities vanishing everywhere (the system expands to infinity when repulsive interactions dominate). On the other hand, i.e. for ‘well chosen’ interactions but with wrong initial number of particles only some fraction of them will be bound, while the remaining ones evaporate.

Although its density is small \((\sim 10^{14} - 10^{15}\text{cm}^{-3})\) a quantum droplet behaves like a liquid. This density is fixed by interaction – by adding particles of both kinds the droplet enlarges keeping its density. Moreover, the densities of both components are not independent. The same is true for the number of particles forming a droplet. In general the number of particles forming a droplet is different from the number of atoms \( N_1, N_2 \) prepared initially in the trap and used in the formation process. After the trapping potential is removed the system is free. There are no constraints on volume nor particle number. The volume of the system may change and some particles may evaporate. These excess particles are ejected to infinity and will not contribute to the total energy. We do not assume interaction of the system with any external reservoir of particles, the number of particles forming a droplet may not grow larger than the initial \( N_1 \) and \( N_2 \). These are the only physical constraints we impose on the system.

The question we want to answer here is: which stable system (for fixed interactions) can be formed having at disposal \( N_1 \) atoms of the first kind and \( N_2 \) atoms of the second kind?

Stable solutions of GP equations Eq.(2) should correspond to a minimum of energy. If it is a global minimum the system is absolutely stable. Metastable states correspond to a local minimum of energy. The system has to overcome a potential barrier on a way to a global minimum. The total energy of the system is:

\[
E(N_1, N_2) = \int d\mathbf{r} \delta'(\mathbf{r}).
\]

Chemical potentials \( \mu_i \) appear in Eqs.(2, 3) as eigenenergies of stationary solutions of the GP equations. It is a simple exercise to verify that these eigenenergies \( \mu_i \) as it should be in the case of true chemical potentials, describe a response of the total energy of the system to a change of particle number:

\[
\frac{\partial E}{\partial N_i} = \mu_i
\]
Note that if the system is stable, i.e. its energy \(E(N_1,N_2)\) corresponds to some minimum, then infinitesimally small change of atom number in any component must increase its energy:

\[
dE = \frac{\partial E}{\partial N_1} dN_1 + \frac{\partial E}{\partial N_2} dN_2 = \mu_1 dN_1 + \mu_2 dN_2 > 0
\]  

(6)

For the purposes of this consideration the number of atoms may only decrease, i.e. \(dN_i < 0\). If any of the chemical potentials were positive the system would decrease its energy by evaporating some particles of the corresponding kind. Therefore the constraints we impose on a stable droplet are:

\[
\mu_1 < 0, \quad \mu_2 < 0.
\]  

(7)\quad (8)

If in a given state both chemical potentials are negative then there is no state of lower energy in its close neighbourhood. We are going to exploit these conditions in the following.

Note, however, that in the above we analyzed only stationary states i.e. states being the solution of stationary GP equation (3). We did not consider stability of the solution against some small perturbation. It is known that there exists stationary localized droplet solutions which are however dynamically unstable - small initial perturbation grows exponentially in time\(^\text{16}\). In what follows we do not consider the issue of dynamical stability.

**Bose-Bose droplets**

In the general approach sketched above a kinetic energy was included. This way we accounted for surface tension providing a necessary pressure to stabilize the system. Unfortunately including kinetic energy leads to differential equations which cannot be treated analytically in more detail in the general case.

To get some better insight into the problem of stability of a droplet we simplify our analysis and assume that the system is large and the surface energy is much smaller than the interaction energy so that it can be neglected. This approximation is known as the Thomas-Fermi approximation. It amounts to assuming that \(\sigma = \varepsilon(n_1,n_2)\). Such a system is uniform, has well defined volume \(V\) and constant densities \(n_i = N_i/V\). The energy density is of the form:

\[
\varepsilon(n_1,n_2)/ (4\pi \hbar^2/m) = \frac{1}{2} (\sqrt{a_{11}n_1} - \sqrt{a_{22}n_2})^2 - \delta \alpha n_1 n_2 + c(a_{11}n_1 + a_{22}n_2)^{5/2}
\]  

(9)

The first two terms are mean field energies, in particular the first term is the 'hard' mode energy. The last term is the LHY energy contribution, \(c = 64\sqrt{\pi}/15\), and \(\delta \alpha = -(a_{12} + \sqrt{a_{11}a_{22}}) > 0\). In order to have an explicit 'minus' sign in front of the attractive term, Eq.(9), we used a non-standard definition of \(\delta \alpha\). We assume that \(\delta \alpha \ll a_{11}, a_{22}\), i.e. that the collapse instability is weak and a small LHY term can balance it. This assumption ensures that the system is weakly interacting. The total energy of the system is \(E_0(N_1,N_2,V) = V \cdot \varepsilon(N_1/V,N_2/V)\). Differential change of energy due to infinitesimal change of volume and particle number is:

\[
dE_u(N_1,N_2,V) = -pdV + \frac{\partial E_u}{\partial N_1} \delta N_1 + \frac{\partial E_u}{\partial N_2} \delta N_2
\]  

(10)

where \(p = -\frac{\partial E_u}{\partial V}\) is a pressure, while \(\mu_{i,u} = \frac{\partial E_u}{\partial N_i} = \frac{\delta \varepsilon}{\delta n_i}\).

For a uniform free system, as opposed to a system with a surface, we get an additional constraint: a droplet will stabilize its volume if internal pressure vanishes:

\[
p = -\frac{\partial E_u}{\partial V} = \mu_1 n_1 + \mu_2 n_2 - \varepsilon(n_1,n_2) = 0
\]  

(11)

Equation Eq. (11) allows to find the volume of a droplet as a function of particle number \(V = V(N_1,N_2)\).

\[
V(N_1,N_2) = \left(\frac{3c(a_{11}n_1 + a_{22}n_2)^{5/2}}{2\delta \alpha N_1 N_2 - (\sqrt{a_{11}}n_1 - \sqrt{a_{22}}n_2)^2}\right)^2
\]  

(12)

Physical solutions of Eq. (11) exist if:

\[
|\sqrt{a_{11}}n_1 - \sqrt{a_{22}}n_2| < \sqrt{2\delta \alpha N_1 N_2}
\]  

(13)
The first important observation is that the right hand of inequality Eq.(13) significantly reduces the possible variation of the ratio $N_1/N_2$, because $\delta a \ll a_{11}, a_{22}$. Thus a term $\sqrt{a_{11}N_1} - \sqrt{a_{22}N_2}$ must be very small. To quantify this difference we introduce a small parameter $\delta b = \frac{\delta a}{\sqrt{a_{11}a_{22}}} \ll 1$, and a variable $\xi$ being a scaled ratio of atom numbers (or atomic densities), $\xi = \frac{n_2 \sqrt{a_{22}}}{n_1 \sqrt{a_{11}}}$. After neglecting corrections of higher order in $\delta b$ Eq.(13) can be brought to the form:

$$\frac{1}{2} \delta \xi^2 < \delta b$$

(14)

where $\delta \xi = \xi - 1$. Obviously $\delta \xi$ is the second small parameter of the theory.

In view of Eq.(14) it is reasonable to assume that at equilibrium the ratio of atom numbers (and therefore the ratio of equilibrium densities too) is approximately equal to:

$$\frac{N_1^0}{N_2^0} = \frac{n_1^0}{n_2^0} = \sqrt{\frac{a_{22}}{a_{11}}} = s$$

(15)

This is a basic assumption of the analysis in1. Note that condition Eq.(15) eliminates the hard mode contribution to the energy density Eq.(9). Only soft mode and LHY energies remain. Using Eqs.(15) and (12) the equilibrium densities $n_i^0$ of a droplet can be well approximated by:

$$n_1^0 a_{11} = \left( \frac{2}{3c} \right)^2 \frac{\delta b^2}{(1+s)^5}$$

(16)

$$n_2^0 a_{22} = \left( \frac{2}{3c} \right)^2 \frac{\delta b^2}{(1+\frac{1}{s})^5}$$

(17)

If $\xi = 1$ then the ratio of densities of the components is equal to the ‘magic’ value $s$ at which the hard mode contribution to the mean field energy vanishes. Therefore $\delta \xi$ measures a deviation of a droplet’s density from this ratio. On the other hand it is easy to check that this parameter equals to fluctuations of density of the hard mode. If we define fluctuations of the densities in each mode as $n_i = n_i^0 (1 + \delta_i)$ then $\delta \xi$ measures the difference between these fluctuations:

$$\delta \xi = \delta_1 - \delta_2$$

(18)

So far we have made use only from the condition that a stable droplet has a vanishing pressure $p = 0$ what stabilizes its volume $V = V(N_1, N_2)$. Total energy of droplets, $E = E_u(N_1, N_2, V(N_1, N_2))$, becomes a function of number of atoms only, $E = E(N_1, N_2)$. We are then on the same footing as in the situation of a droplet having a surface. As we have already discussed, the droplet will stop evaporating atoms if both chemical potentials are negative.

Let us observe that the two functions $E(N_1, N_2, V(N_1, N_2))$ and $E_u(N_1, N_2, V)$ are different because in the latter case $V$ is an independent variable as opposed to $E$ where volume is a well-defined function of $N_1, N_2$, Eq.(12). This leads to two different definitions of $\mu$. One is the $\mu_i$ given by Eq. (5) and the second one is given above $\mu_{ia} = \partial E_u/\partial N_i$. The relation between these two is given by

$$\mu_i = \frac{\partial E}{\partial N_i} = \frac{\partial E_u(N_1, N_2, V)}{\partial N_i} + \frac{\partial E_u(N_1, N_2, V)}{\partial V} \frac{\partial V}{\partial N_i} = \frac{\partial E_u(N_1, N_2, V)}{\partial N_i}$$

(19)

Because pressure in the latter case vanishes, there is no additional energy cost related to change of volume and both chemical potentials are the same, $\mu_i = \mu_{ia}$.

To summarize the above discussion, a droplet is stable when its densities meet the following conditions: (i) the system has vanishing pressure, $p(n_1, n_2) = 0$, Eq.(11),

$$\frac{p}{(\sqrt{a_{11}a_{22}} n_1 n_2)} = \frac{1}{2} \left( \frac{1}{\sqrt{\xi}} - \sqrt{\xi} \right)^2 - \delta b + \frac{3}{2} c w \left( \frac{1}{\sqrt{\xi}} + \sqrt{\xi} \right)^{\frac{5}{2}} = 0$$

(20)

(ii) both chemical potentials are negative:

$$\frac{\mu_1}{\sqrt{a_{11}a_{22}n_2}} = \left( \frac{1}{\xi} - 1 \right) - \delta b + c \frac{5}{2} w \left( \sqrt{\xi} s + \frac{1}{\sqrt{\xi} s} \right) \frac{1}{\sqrt{\xi} s}$$

(21)

$$\frac{\mu_2}{\sqrt{a_{11}a_{22}n_1}} = -(1 - \xi) - \delta b + c \frac{5}{2} w \left( \sqrt{\xi} s + \frac{1}{\sqrt{\xi} s} \right) \frac{1}{\sqrt{\xi} s}$$

(22)
Figure 1. Solutions of Eq. (11) in the form of contour plots in the $n_1 - n_2$ plane. We show the tip of $p = 0$ isobar where by blue color we indicate the stable region as given by $\mu_1 < 0$ and $\mu_2 < 0$ constraints, Eqs. (21,22). In the inset we show the full zero pressure isobar which has a shape of the elongated loop. By the black dot we indicate the standard solution to the stability problem according to Eq.(15). We consider two cases: (i) $s = \sqrt{2}$ (left panel). The standard solution is located at the border, but inside the stable region. The solution given by Eq.(26) marked in green is at the centre; (ii) $s = 2$ (right panel). In this case the standard solution is located outside the stable region. The solution given by Eq.(26) remains well within the limit of stability. Densities are in units of $n_{10} = 25\frac{a}{\pi^{1/4} \delta_a^2 \xi^5 s^2 (1 + s)^5}$.

where $w = (n_1 a_{11} n_2 a_{22})^{1/4}$. By expanding the above equations to leading order in the small parameter $\delta \xi = \xi - 1$ we get the equation corresponding to the $p = 0$ isobar in the $n_1 - n_2$ plane:

$$w = (n_1 a_{11} n_2 a_{22})^{1/4} = \frac{\delta b - \frac{1}{4} \delta \xi^2}{\frac{3}{2} \left( \frac{1}{\sqrt{s}} + \sqrt{s} \right)^{5/2}}.$$ (23)

Similar expansion allows for approximate but analytic determination of conditions limiting the region of stability of a quantum droplet with respect to evaporation, Eqs. (21,22). The region of corresponding parameters forms a segment of $p = 0$ isobar where the ratio of densities are limited as follows:

$$-1 + \frac{5}{3} \frac{s}{1 + s} < \frac{\delta \xi}{\delta b} < 1 - \frac{5}{3} \frac{1}{1 + s}.$$ (24)

Finally we want to address the solution to the problem of stability of an infinite homogeneous two-component system as presented in $^{17,18}$. In this case only intensive quantities make sense. These are the energy densities and pressure. The stability problem in such a case should be defined as a problem of finding a constrained minimum of the energy density. The imposed constraint is a vanishing pressure. This condition ensures that locally there are no net internal forces acting on a fictitious surface inside the bulk of a droplet. This is the same condition which fixes the volume of a finite homogeneous droplet, Eq.(11).

Additionally, at a minimum of energy density $\varepsilon(n_1, n_2)$ any infinitesimally small variation of densities cannot change the energy density:

$$d\varepsilon = \mu_1 dn_1 + \mu_2 dn_2 = 0$$ (25)

On the contour $p(n_1, n_2) = 0$ variations of both densities $dn_1$ and $dn_2$ are not independent. The following condition is met, $dp = \frac{\partial p}{\partial n_1} dn_1 + \frac{\partial p}{\partial n_2} dn_2 = 0$. Combining this condition with Eq.(25) we get the following equation for the minimum of energy density, $\varepsilon(n_1, n_2)$, on the contour $p = \mu_1 n_1 + \mu_2 n_2 - \varepsilon = 0$:

$$\mu_1 \frac{\partial p}{\partial n_2} - \mu_2 \frac{\partial p}{\partial n_1} = 0$$ (26)
Again, expressing derivatives of pressure contributing to the above equation in terms of $\xi$ and expanding Eq.(26) in the small parameter $\delta \xi$ around the value $\xi = 1$ we get the approximate solution for densities in a stationary droplet state as follows:

$$\frac{\delta \xi}{\delta b} = \frac{1 - s}{1 + s}$$

(27)

Let us stress that this solution gives densities which meet the stability criteria defined here. Independently of the value of the parameter $s$ the Eq.(26) predicts droplet densities very close to the center of the stability region. This solution is marked by a green dot in Fig. (1). We rewrite both approximate formulae for ratio of droplet densities, Eq.(15) and Eq.(27), in a form allowing for their direct comparison. The solution based on Eq. (15) gives $n_1 = n_1^0, n_2 = n_2^0$ and:

$$\delta \xi = \left(\frac{n_1}{n_1^0} - \frac{n_2}{n_2^0}\right) = 0.$$  

(28)

Eq.(27) based on the criterion of Ref.\textsuperscript{17,18} is simply:

$$\delta \xi = \left(\frac{n_1}{n_1^0} - \frac{n_2}{n_2^0}\right) = \delta b \left(\frac{1 - s}{1 + s}\right).$$

(29)

Evidently both formulae are equivalent if intraspecies interactions are equal $s = 1$. Note that the right hand side of Eq.(29) confirms the contribution of hard mode excitations to the densities of a stable droplet. This contribution is small as it is proportional to the small parameter $\delta b$. However, the situation is different for $s > 3/2$. Then the standard result $\delta \xi = 0$ given by Eq. (28) is outside the stability region given by Eq.(24). Thus, for sufficiently strong asymmetric intraspecies interaction the standard solution does not support a stable droplet.

Figure 2. The total energy as a function of the number of particles in every component for unequal intraspecies interaction, $s = \sqrt{2}$. Left panel: Coloured region corresponds to such a composition of the mixture for which $p = 0$ condition can be met. The isobar $p = 0$ shown in Fig.(1) becomes here the interior of the angular region given by Eq.(13). White lines indicate the edges of the zone of stable droplets where $\mu_1 < 0, \mu_2 < 0$. The rectangle at the center indicates the region which we zoom-in in the right panel. Right panel: Zoom of the energy landscape in $N_1 - N_2$ plane. It illustrate adiabatic evolution of two initial states $(N_1^{ini}, N_2^{ini})$ marked by black dots. Evolution towards the state of minimal possible energy constrained by initial atoms’ numbers cannot have any positive-valued gradient component of the chemical potential vector $(\mu_1, \mu_2)$. The withe arrows show a trajectories towards the final state $(N_1^{fin}, N_2^{fin})$ (red dots) of lowest possible energy for the assumed arrangement. Note please that only edges of the stability region can be reached. Getting into the interior of this region requires increasing of number of atoms of one kind at least. On the contrary, all systems having initially the number of particles corresponding to the area between white lines is stable against small perturbations. The number of atoms is expressed in convenient units of Ref.\textsuperscript{1}. Therefore, 'the real' number of atoms is equal to $N_r = N \cdot n_{10} \tilde{r}^3 \approx N \cdot 6300$, where $\tilde{r} = \sqrt[4]{\frac{s + 1}{2\pi \delta a n_{10}}}$ is the length unit.

The results are illustrated in Fig.(1) where we show the stability diagram in a plane of atomic densities, $n_1$ and $n_2$. For comparison we present the two cases: $s = \sqrt{2}$ and $s = 2$. The $p = 0$ isobar has the form of a closed loop originating at the
center of the coordinate system – see inset in Fig.(1). The region which is stable with respect to atom losses, ($\mu_1, \mu_2 \leq 0$), Eqs.(21,22), is located close to the tip of the loop which we zoom-in in the main frame. This is the part of the isobar marked in blue. By green dot we mark the solution corresponding to the global minimum of an infinite system as suggested in\textsuperscript{17,18} and given by Eq.(27). This result is well in the stable part of the diagram regardless the interactions. The standard solution of\textsuperscript{1}, Eq.(15), is indicated by a black dot. We stress that when the disproportion of intraspecies interactions is too large ($s=2$) the standard solution of Ref.\textsuperscript{1} is out of the stability region.

In the last part of the paper we go back to the problem which was the inspiration for our study. We address the question asked at the beginning of this work, i.e. we are going to show which is a minimal energy state which can be reached having to the disposal $N_1^{ini}$ atoms of the first kind and $N_2^{ini}$ atoms of the second kind allowing for throwing away some of them.

The solution to this problem is illustrated in Fig.(2) which shows the total energy of the system $E(N_1, N_2, V(N_1, N_2)) = V(N_1, N_2)/\varepsilon(N_1/V, N_2/V)$ in the plane of extensive quantities $N_1, N_2$. If one has initially the two component mixture with $(N_1^{ini}, N_2^{ini})$ atoms then the droplet formed would be in general a mixture of $(N_1^{fin}, N_2^{fin})$ atoms of both kinds. To find the droplet of the lowest energy among all possible final states all droplets composed with number of atoms limited by the initial values, $N_1^{fin} \leq N_1^{ini}$ and $N_2^{fin} \leq N_2^{ini}$ we directly examine the region of energies in the relevant rectangular domain in $N_1 - N_2$ plane:

\begin{align}
0 \leq N_1 & \leq N_1^{ini}, \\
0 \leq N_2 & \leq N_2^{ini}.
\end{align}

In Fig. (2) the initial composition of droplet is marked by a black dot. White vertical and horizontal arrows point to the final states $(N_1^{fin}, N_2^{fin})$ which minimize the energy constrained according to the previous discussion. We consider two situations. The first one is that $N_2^{fin}/N_1^{fin}$ is so large that $-\delta \xi / \delta b > 1 - 5/3(1/(1+s))$, i.e. the second component of the mixture is strongly excessive one. In such a case the atoms simply evaporate until the system reaches the boundary of the stable region, vertical arrow in figure. It is worth mentioning that the number of minority atoms is conserved. Further evaporation stops when the border of the stability sector is reached. This is because total energy is an extensive quantity and although chemical potential (inside this area) becomes more negative, total energy grows.

The second case shown in Fig.(2) relates to the situation where the first component dominates, i.e. if $-\delta \xi / \delta b < -1 + 5/3(1/(1+s))$. The the scenario described above repeats. Excessive atoms of the first component evaporate, while the number of atoms in the second component remains constant (horizontal white arrow in figure). This process stops while reaching the border of the stable sector.

If initially the system is prepared in the stable zone, i.e. in the area limited by the two white lines in Fig.(2) it will not evaporate atoms at all.

We have to add that all presented discussion is based on the stability analysis and no any time dynamics was considered at all. Therefore all our conclusions, in particular these invoking dynamic processes such as evaporation, implicitly assume that the system remains at equilibrium and adiabatically follows the state determined by external parameters and temporal number of atoms. For the same reason we are not able to discuss such a situation when initial number of atoms is outside the entire coloured angular sector in Fig.(2). This is the unstable sector and the way the instability develops depends on the details of dynamics. Only dynamical studies of the process of formation of the droplet might give the state of droplet formed eventually.

Conclusions

In this paper we specified stability conditions for a self-bound two-component droplet. They are not related to any particular form of the energy density functional, and are therefore valid not only for Bose-Bose but also Bose-Fermi mixtures. The case of Bose-Bose droplets was studied carefully. In contrast to the standard solution of Ref.\textsuperscript{1} we show that the stable droplets’ densities, for fixed values of interactions strengths, can take values from some finite range of parameters, thus there is no unique droplet solution. This regime of allowed densities is however rather small and deviations from the standard solution are limited particularly for similar strengths of intraspecies interactions. In a limit of large droplets, when kinetic energy can be neglected, we found very useful analytic expression for the boundaries of the stability zone. We have shown that if the intraspecies interactions are very different from each other then the prediction of Ref.\textsuperscript{1} is out of the stability sector.

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Author contributions statement

P.Z., M.P., and M.G. equally contributed to the idea of the paper and discussions of results. P.Z found analytic results. M.P. led numerical calculations and prepared the figures and M.G formulated the problem and supervised the project. All authors participated in writing and reviewed the manuscript.

Additional information

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