Aspects of the Thermo-Dynamics of a Black Hole with a Topological Defect

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Abstract

Aspects of the thermo-dynamics of a black hole which is either pierced by a cosmic gauge string or contains a global monopole are investigated. We also make some comments on the physical significance of the fact that the gravitational mass carried by a global monopole is negative. We note in particular that the negative monopole mass implies a gravitational super-radiance effect.

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I. INTRODUCTION

According to most fundamental particle theories, phase transitions have occurred in the early universe. Such transitions may have created various kinds of topological defects such as domain walls, cosmic strings, monopoles and textures [1]. These objects are believed to have contributed to the creation of some of the large scale structures found in the observable part of the universe today. It has further been pointed out that topological defects can be sources for (topological) inflation [2,3]. Hence these objects may also shape the universe on very large scale and give rise to an effective fractal geometry [2].

In this article we will be concerned with some effects produced locally by a cosmic gauge string and a global monopole. We will more specifically be interested in situations where a cosmic string pierce through a black hole and when a global monopole has been swallowed by a black hole [4–7]. It has recently been shown that the rate of change of the energy $E$ of a black hole which exhibits a topological defect in this way and the rate of change of the natural mass $M$ assigned to such a structure do not coincide [4,6,7]. On this ground it is argued that the first law of black hole thermo-dynamics $dM = TdS$, where $T$ is the temperature of the black hole horizon and $S$ its entropy has to be modified when a black hole exhibits a topological defect [3,4]. The primary aim of this work is to show that this conclusion is misleading. We show that the energy flowing into a black hole with a topological defect ($dE$) naturally equals the change in the gravitational mass of the whole structure, i.e. $dE = dMc^2$. From the general thermo-dynamical relation

$$dS = \frac{dE}{T}$$  \hspace{1cm} (1)

it then follows that the entropy carries a similar functional relation to the gravitating energy of the black hole structure as in the Schwarzschild solution $S = 4\pi\alpha^{-2}M^2$ where $\alpha^2$ is a parameter that characterize the topological defect.
We will also consider some general aspects of the gravitational properties of a single
global monopole. Of particular concern will be the fact that a global monopole naturally
can be assigned a negative gravitating mass. It is shown that due to its negative mass a
global monopole will give rise to a gravitational super-radiance effect. It follows that global
monopoles are gravitationally unstable objects.

II. THE CONCEPT OF ENERGY

In pure Einstein theory it has been proved that the total energy (the ADM mass) carried
by an isolated system, i.e. one that generates an asymptotic Minkowski geometry, is positive
[8]. Due to the essential role played by the asymptotic condition this theorem can not be
carried over to solutions of Einsteins theory which does not display a Minkowskian asymp-
totic structure. This problem also arise in Kaluza-Klein and super-string theories where
a system can have negative energy even when the asymptotic behavior approach $M_4 \times K$
where $M_4$ is the four-dimensional Minkowski space and $K$ is a compact internal space [9,10].

In a neighborhood of space-like infinity a black hole pierced by a cosmic gauge string or a
black hole which has swallowed a global monopole gives rise to a non-Minkowskian structure.
This structure arises since strings and monopoles effectively cuts out either a deficit angle
(strings) or a deficit cone (monopoles) from the geometry compared to corresponding space-
times without topological defects of this kind. This means that if one is to measure the
surface area of a sphere $S^2$ with a physical radius $r$ with a monopole in the center the area
will read $A = \alpha^2 A(S^2)$ with $\alpha < 1$. Here $A(S^2)$ denotes the area of a similar sphere in
the Minkowski geometry and $\alpha$ is a constant that characterizes the energy content of the
defect. The appearence of an $\alpha$-factor in the angular part is the only difference compared
to the Minkowski metric in the asymptotic region. This means in particular that redshift
of photon energies for photons propagating from a region near a string or a monopole or a
black hole with a defect to infinity still vanishes there. This allow us to define a meaningful
mass concept in these space-times along similar lines as in asymptotically flat space-times.
Consider a black hole with a topological defect. Let $\xi^a$ be a time translation Killing vector field which is time-like near infinity such that $\xi^a \xi_a = -1$ and which has vanishing norm on the event-horizon. Let $\vec{N}$ be a second Killing vector field orthogonal to the event-horizon with normalization such that $N^a N_a = 1$ near infinity. Let $S$ denote the region outside the event-horizon of the black hole. The boundary $\partial S$ of $S$ is taken to be the event-horizon $\partial B$ and a two-surface $\partial S_\infty$ at infinity. It then follows that the mass $M_\infty$ inside $\partial S_\infty$ as measured by a static observer at infinity can be deduced from \[ \int_{\partial B + \partial S_\infty} \xi^{a:b} d\Sigma_{ab} = - \int_S R^a{}_b \xi^b d\Sigma_a. \] (2) $R^a{}_b$ denotes the Ricci tensor. The integral over $\partial B$ is the surface gravity $\kappa = N_b \xi^a \nabla_a \xi^b$ multiplied with the surface area $A$ of $\partial B$.

III. A BLACK HOLE WITH A TOPOLOGICAL DEFECT

The possible presence of global monopoles is indicated by the non-triviality of $\pi_2(M)$ (i.e. $\neq I$) where $M$ is the group manifold of the unbroken global symmetry group of the theory. To be specific we will consider a theory with a Lagrangian on the form (we work in units such that $c = \hbar = k_B = G = 1$)

\[ L = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi_a - \frac{1}{4} \lambda \left( \phi^a \phi_a - \eta^2 \right)^2. \] (3)

Here $\phi^a$ ($a = 1, 2, 3$) is a triplet of scalar fields with global $O(3)$ symmetry which is spontaneously broken down to $U(1)$. Due to the this surviving unbroken symmetry this model will contain solitons (global monopoles or hedgehogs) which carry $U(1)$ charges with masses of the order of the scale of the symmetry break-down ($\sim 10^{16}\text{GeV}$). In Schwarzschild coordinates the metric field outside the core region of a global monopole is to a high degree of approximation given by \[ ds^2 = - \left( \alpha^2 - \frac{2m}{\rho} \right) d\tau^2 + \left( \alpha^2 - \frac{2m}{\rho} \right)^{-1} d\rho^2 + \rho^2 d\Omega_2^2. \] (4)
\( d\Omega^2 \) represents the infinitesimal surface element on the unit sphere and the radius \( \delta \) of the core region is given by \( \delta \sim 2(\lambda \eta^2)^{-1/2} \). The parameters \( m \) and \( \alpha \) in the line-element are similarly [13]

\[
\alpha^2 = 1 - 8\pi\eta^2. \quad (5)
\]

This geometry is not asymptotically flat. In order to see this we perform the coordinate transformation \( \tau \to t = \alpha \tau, \rho \to r = \alpha^{-1} \rho \) along with the replacement of \( m \) by \( M = \alpha^{-3} m \). The metric then takes the form

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + \alpha^2 r^2 d\Omega^2. \quad (6)
\]

At infinity this structure approximates to

\[
ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\Omega^2 \quad (7)
\]

which is trivially seen to describe a curved space. Compared to Minkowski space-time a solid cone is missing from the space when \( \alpha < 1 \). We will always assume that this condition is fulfilled.

A geometry very similar to the geometry generated by a global monopole (or a black hole with a global monopole) is found when a cosmic gauge string pierce through a black hole. The metric of such a structure is given by [4]

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 (d\theta^2 + b^2 \sin^2 \theta d\phi^2). \quad (8)
\]

\( b \) is a constant assumed less than unity. It follows that this geometry has properties at infinity that resembles the effects produced by a monopole. We will in the following therefore mainly concentrate our attention to the situation when a black hole has captured a global monopole.

The reason for the curved asymptotic structure in the global monopole space-time is that even though the triplet of scalar fields define a state of lowest energy outside the
monopole core it is not a state of zero energy, i.e. the energy-momentum tensor have non-zero $T_{\tau\tau}$ and $T_{\rho\rho}$ components [13]. This tensor is furthermore Lorentz-invariant in the radial direction. A calculation of the gravitational mass $M^v$ outside the core-region from eq.(2) reveals that this feature of the energy-momentum tensor results in $M^v = 0$ since $T^\tau_\tau - T^\rho_\rho$ vanishes identically. The true vacuum outside the monopole will therefore not give rise to an acceleration field felt by an observer initially at rest relative to the global frame. Of course the presence of this vacuum structure do exert a residual gravitational action since two initially parallel light rays passing on opposite sides of the monopole will converge. This vacuum structure does seemingly also have cosmological implications. Let $R$ denote the expansion factor of the universe in the Robertson-Walker line-element. The “passive” mass $M_p$ of the monopole field outside the core of the monopole inside a volume very much larger than the volume occupied by the core is approximately given by $M_p \sim \alpha^2 \eta^2 r$ [13]. The equation governing the Robertson-Walker line-element then becomes

$$\frac{1}{2} \dot{R}^2 - 4\pi \rho r^3 = -\frac{k}{2}$$

where $k$ is the effective curvature of the universe. With $M_p$ used as a measure of $\rho r^3$, i.e. $\rho r^3 \sim M_p$, we see that the monopole field will give rise to an effective curvature $k_{eff}$ of the universe $k_{eff} = k - 2\pi \alpha^2 \eta^2$. It is unclear whether this will have any significant physical effects.

The core-region of a topological defect is assumed to reside in a state corresponding to the false vacuum state, i.e. the state of lowest energy before the symmetry break-down in the theory. It is found in particular that the monopole core can be modeled by a sphere with $T_{\mu\nu} = -\rho \text{diag}(-1,1,1,1)$ [13]. It follows that the Tolman mass analog from eq.(2) in the core-region is negative. Measured from infinity the global monopole configuration will therefore naturally be assigned a negative inertial mass which is reflected in the negativity of $m$. It also follows that we should not expect such an object to be stable since we would expect that the monopole configuration will seek towards ever decreasing energy states whenever
possible. This follows since the energy steaming from a simple constant re-scaling of the energy \( g_{\mu\nu} \rightarrow \mu^2 g_{\mu\nu} \Rightarrow E \rightarrow \mu E \) of a negative energy object does not have a lower bound as is the case when the energy of the object is positive.

To give another argument in favor of this view we consider a scalar field \( \psi \) propagating in the geometry eq.(6). As in the Schwarzschild geometry we expect that the dynamical development of metric perturbations resembles the dynamics of \( \psi \). In this case the equation governing the evolution of \( \psi \) can be brought to the form

\[
\partial_t^2 \psi = \partial_{r_*}^2 \psi - V(r_*) \psi
\]

where \( r_* \) is as usual a Regge-Wheeler coordinate defined such that \( r = 2M \) is moved to \( r_* = -\infty \). It then follows that the operator \( A \) defined by

\[
A = -\frac{d^2}{dr_*^2} + V(r_*)
\]

is a positive self-adjoint operator on the Hilbert space \( \mathcal{L}^2(r_*) \) of square integrable functions of \( r_* \). Multiply the wave equation with \( \partial_t \psi^* \) (where \( \psi^* \) is the complex conjugate of \( \psi \)) and integrate over \( r_* \). We then obtain

\[
\int |\partial_t \psi|^2 dr_* + \int \psi^* A \psi dr_* = C
\]

where \( C \) is a constant which can be taken positive. Since the last integral is positive due to the positivity of \( A \) in the Schwarzschild geometry \( C \) will represent an upper bound for the first integral. In particular uniform exponential growth in time of initially well behaved perturbations is ruled out. However, in the global monopole geometry it is easily seen that \( A \) is no longer positive since for modes with sufficiently small angular momentum \( V(r_*) \) is negative. In Schwarzschild coordinates eq.(6) the potential reads

\[
\alpha r^3 V(r) = (r + 2|M|)(l(l + 1) - 2\alpha|M|r^{-1})
\]

where \( l \) is the usual angular momentum quantum number. Note that the potential is negative definite for the zero angular momentum mode. In general the potential will be negative whenever
\[ rl(l + 1) < 2\alpha|M| \sim \frac{8\pi\eta^2}{3}\delta. \] (14)

The non-existence of an upper bound for \( \psi \) indicates that incident radiation may be scattered with a total cross-section that exceeds unity, i.e. more energy scatters back to infinity than exhibited by the incident radiation. This can be seen as follows [1]. To the field \( \psi \) with an energy-momentum tensor \( T^\mu_\alpha \) we can assign a conserved canonical current \( j^\mu = \xi^\alpha T^\mu_\alpha \) relative to the Killing observer. Let this observer monitor the global monopole in the region between an initial space-like surface \( S_0 \) to a final space-like surface \( S \) in the future of \( S_0 \). Assume that the monopole is contained in a spherical shell with radius \( r \) and that the space-like hyper-surfaces between \( S_0 \) and \( S \) can be parameterized by an affine parameter \( \lambda \). From the conservation equation \( \nabla_\mu j^\mu = 0 \) it then follows that the difference between the energy within the shell in the surface \( S_\lambda, E_\lambda \), and the corresponding energy in \( S_0, E_0 \), is given by

\[
E_\lambda - E_0 = -\lim_{r \to \infty} \int_0^\lambda \xi^\alpha T^\mu_\alpha N_\mu r^2 d\Omega d\lambda = -E_{\text{rad}}. \tag{15}
\]

\( E_{\text{rad}} \) denotes the total radiated energy in the region between \( S_0 \) and \( S \). With the boundary condition that \( E_{\text{rad}} \geq 0 \) it follows that when \( E_0 \) is positive then \( E_\lambda \) either equals or is less than \( E_0 \). If on the other hand \( E_0 \) is negative then \( E_\lambda < 0 \) and \( |E_\lambda| \) will be larger than the initial energy. In \( S_0 \) let a time-dependent \( l = 0 \) mode exist such that no positive energy passes through the surface at \( r \) in the negative radial direction. Since no upper bound exists for \( \psi \) and on the assumption that no time-dependent bound states exist the system must radiate energy to infinity. It follows that \( E_\lambda < 0 \) and energy is effectively extracted from the system. This is a gravitational analog to the well known super-radiance phenomenon in the scattering of charged bosons off large electric potentials. The waves scattered off the monopole will carry a net positive energy to infinity which must be compensated by letting the Tolman mass analog of the monopole grow even more negative. The situation is some-

\[ \text{---} \]

\[ ^1 \text{A detailed investigation will be presented elsewhere.} \]
what similar to the situation when particles are scattered off an isolated rotating star or a rotating black hole. When the star rotates sufficiently fast an ergo-region develops outside the outer surface of the star. Particles coming in from infinity with certain energies (the super-radiant modes) can be scattered off the star and back to infinity with an energy larger than their in-coming energy [14]. A finite amount of energy (the rotational energy) can be extracted from the star in this way. However, in the case of the global monopole an upper bound for this energy does not present itself in a similar way. A similar phenomenon does not occur when particles are scattered off cosmic gauge strings since these carry a vanishing gravitational mass.

We now consider the situation when a black hole has swallowed a global monopole. The metric of such a configuration is still assumed given by eq.(4) but with $m$ taken positive in order to conform with the cosmic censorship hypothesis. Let $\vec{\xi} = \alpha^{-1} \partial_r$ and $\vec{N} = \alpha \partial_\rho$. It follows that $\kappa = \alpha^3 (4m)^{-1} = (4M)^{-1}$ and $A = 16\pi \alpha^{-4} m^2 = 16\pi \alpha^2 M^2$. The mass $M^{\infty}$ of the black hole is then given by

$$M^{\infty} = \alpha^{-1} m = \alpha^2 M.$$  \hfill (16)

Note that neither $m$ nor $M$ has the status as gravitational mass alone. An observer at infinity will thus naturally assign an inertial energy $E = M^{\infty}$ to the black hole. By the use of eq.(1) and $\kappa = 2\pi T$ it then follows that the entropy of the black hole configuration is given via $dS = 4\pi \alpha^{-2} d(M^{\infty})^2$, i.e. $S = 4\pi \alpha^{-2} (M^{\infty})^2$ (up to an additive constant). This relation is rather important since it relates the black hole entropy directly to the gravitating energy “content” of the black hole. Also note that these results are independent of the coordinate system employed. It is furthermore clear that the usual relation between $S$ and event-horizon area holds, i.e. $S = \frac{1}{4} A$.

\footnote{This relation still holds since the Euclidean section of the Wick rotated analytic extension of eq.(6) is independent of $\alpha$.}
Relative to the geometry eq.(6) it follows that the quasi-newtonian potential $\Phi$ is given by $\Phi = -M r^{-1}$. The corresponding gravitational acceleration $g$ is $g = |\nabla \Phi| = M r^{-2}$. Let Newton’s gravitational law take the standard form $\nabla^2 \Phi = \gamma \rho$ where $\gamma$ is the coupling constant and $\rho$ the mass density. Integrate this equation over the volume $V$ inside the spherical surface $S$ naturally defined by the observer at infinity

$$\int_S \nabla \Phi \cdot d\vec{S} = \gamma \int_V \rho dV \equiv \gamma M^\infty.$$  \hspace{1cm} (17)

The gravitational acceleration is then given by $g = \gamma M^\infty (4\pi \alpha^2 r^2)^{-1}$ where $r$ is the radial coordinate at which $S$ is positioned. Equality between the two expressions for $g$ is achieved by defining $\gamma \equiv 4\pi$. Hence, again we find that $M^\infty = \alpha^2 M$.

When the black hole either emits radiation or swallows mass it is primary $M^\infty$ that is expected to change. If one assumes that the mass of the hole is changing slowly one can write $m(\tau) = m_0 + \dot{m} \tau$ where $m_0$ and $\dot{m}$ are constants such that $\dot{m} << 1$. The energy flow in the radial direction relative to the global coordinatization to first order in $\dot{m}$ then becomes $T_{\tau \rho} = 2\dot{m} \rho^{-2}$. It follows that the total energy $\dot{E}$ that flow through a surface $\Sigma$ relative to the $\xi^\mu$ observer per coordinate time to first order in $\dot{m}$ becomes

$$\dot{E} = \int \xi^\mu T_{\mu \nu} d\Sigma_\nu = \int \xi^\tau \alpha^2 T_{\tau \rho} d\Sigma_\rho = \dot{m}.$$  \hspace{1cm} (18)

The energy-flow that is related to the change in gravitational mass is a change in energy per unit proper time $\dot{E}$. If we assume that we perform our measurements of the energy-flow in the asymptotic region it follows that the change in the gravitational mass of the black hole with a monopole asymptotically approaches $\dot{E} = \alpha^{-1}\dot{m}$. This is in exact correspondence with eq.(16) since $\dot{E}$ scales as $M^\infty$, i.e. $\dot{E}$ is naturally interpreted as the change in $M^\infty$ per unit proper time. This discussion carry almost unaltered over to the situation when a cosmic gauge string pierce through a black hole (replace $\alpha^2$ with $b$).
IV. DISCUSSION

In this work we noted that due to the negative energy carried by a global monopole the condition for the occurrence of super-radiance is satisfied when particles are scattered off such an object. It implies that the monopole apparently may radiate an infinite amount of energy to infinity \(3\). We also found that when the energy measure was carefully defined no anomalies in the laws of black hole thermo-dynamics of a black hole which has captured a monopole results as indicated in \[3\].

Systems containing black holes and objects with negative energy are certainly very interesting. When a black hole swallows a global monopole the mass of the resulting hole must decrease. This does not break with Hawking's area theorem for black holes since a global monopole breaks the strong energy condition due to its negative mass. What seems more interesting is that the entropy as derived in this work is determined not only by \(M^\infty\), as usual, but also by \(\alpha\). Even though \(M^\infty\) of the black hole decreases when a hole captures a monopole we must also take into account the effects produced by the appearance of \(\alpha\) in \(S\). It might be that \(dS \geq 0\) when one considers such a process since the \(\alpha\)-factor is assumed less than unity.

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\[3\] Of course at some point back-reaction effects must be taken into account.
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