A bicommutant theorem for dual Banach algebras

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Abstract

A dual Banach algebra is a Banach algebra which is a dual space, with the multiplication being separately weak*-continuous. We show that given a unital dual Banach algebra $A$, we can find a reflexive Banach space $E$, and an isometric, weak*-weak*-continuous homomorphism $\pi : A \to B(E)$ such that $\pi(A)$ equals its own bicommutant.

Keywords: dual Banach algebra, bicommutant, reflexive Banach space.

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1 Introduction

Given a Banach space $E$, we write $B(E)$ for the Banach algebra of operators on $E$. Given a subset $X \subseteq B(E)$, we write $X'$ for the commutant of $X$,

$$X' = \{ T \in B(E) : TS = ST \ (S \in X) \}.$$ 

The von Neumann bicommutant theorem tells us that if $E$ is a Hilbert space, and $X$ is a $\ast$-closed, unital subalgebra, then $X''$ is the strong operator topology closure of $X$ in $B(E)$. If $X$ is not $\ast$-closed, then this result may fail (consider strictly upper-triangular two-by-two matrices). However, a result of Blecher and Solel, [2], shows, in particular, that if $X$ is weak*-closed, that we can find another Hilbert space $K$, and a completely isometric, weak*-weak*-continuous homomorphism $\pi : X \to B(K)$, such that $\pi(X) = \pi(X)''$. That is, if we change the Hilbert space which our algebra acts on, we do have a bicommutant theorem.

A dual Banach algebra is a Banach algebra which is a dual space, such that the multiplication is weak*-continuous. Building on work of Young and Kaiser, the author showed in [5] that given a dual Banach algebra $A$, we can find a reflexive Banach space $E$ and an isometric, weak*-weak*-continuous homomorphism $\pi : A \to B(E)$. In this paper, we show that when $A$ is unital, we can choose $E$ and $\pi$ such that $\pi(A) = \pi(A)''$. The method is similar to that used in [2] (although we follow the presentation of [1]) combined with an idea adapted from [5, Section 6].

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2 Notation and preliminary results

Given a Banach space $E$, let $E^*$ be the dual space to $E$. For $\mu \in E^*$ and $x \in E$, we write $\langle \mu, x \rangle = \mu(x)$. For $X \subseteq E$, let

$$X^\perp = \{ \mu \in E^* : \langle \mu, x \rangle = 0 \ (x \in X) \}.$$
For $Y \subseteq E^*$, let 
\[ \dual{Y} = \{ \mu \in E : \langle \mu, x \rangle = 0 \ (\mu \in Y) \}. \]
Then $\dual{X}$ is the closure of the linear span of $X$, while $(\dual{Y})$ is the weak$^*$-closure of the linear span of $Y$. We may canonically identify $X^*$ with $E/X^*$, and $(E/X)^*$ with $X^*$. In particular, $Y$ is weak$^*$-closed if and only if $Y = (\dual{Y})$, and in this case, the canonical predual of $Y$ is $E^*/\dual{Y}$.

We write $E^* \hat{\otimes} E$ for the projective tensor product of $E^*$ with $E$. This is the completion of the algebraic tensor product $E^* \otimes E$ with respect to the norm
\[ \|\tau\|_{\pi} = \inf \left\{ \sum_{k=1}^{n} \|\mu_k\| \|x_k\| : \tau = \sum_{k=1}^{n} \mu_k \otimes x_k \right\}. \]
Any element of $E^* \hat{\otimes} E$ can be written as $\sum_k \mu_k \otimes x_k$ with $\sum_k \|\mu_k\| \|x_k\| < \infty$. For further details, see [3] or [6], for example.

The Banach algebra $\mathcal{B}(E)$ is a dual Banach algebra with respect to the predual $E^* \hat{\otimes} E$, the dual pairing being given by
\[ \langle T, \mu \otimes x \rangle = \langle \mu, T(x) \rangle \quad (T \in \mathcal{B}(E), \mu \otimes x \in E^* \hat{\otimes} E), \]
and linearity and continuity. Indeed, under many circumstances, this is the unique predual for $\mathcal{B}(E)$, see [5] Theorem 4.4.

It follows that any weak$^*$-closed subalgebra of $\mathcal{B}(E)$ is also a dual Banach algebra: then [5] Corollary 3.8] shows that every dual Banach algebra arises in this way. If $X \subseteq \mathcal{B}(E)$, then $X'$ is a closed subalgebra of $\mathcal{B}(E)$. Notice that $T \in X'$ if and only if $T$ annihilates all $\tau \in E^* \hat{\otimes} E$ of the form
\[ \tau = \mu \otimes S(x) - S^*(\mu) \otimes x \quad (S \in X, \mu \in E^*, x \in E). \]
Hence $X' = Y^* = (E^* \hat{\otimes} E/Y)^*$ is weak$^*$-closed, where $Y$ is the closed linear span of such $\tau$. In particular, $X''$ is a weak$^*$-closed subalgebra of $\mathcal{B}(E)$ containing $X$, and so $X''$ contains the weak$^*$-closed algebra generated by $X$.

We shall follow the ideas of [1, Theorem 3.2.14]; see [2] for a fuller treatment. We first establish some preliminary results. Given a Banach space $E$, we write $\ell^2(E)$ for the Banach space consisting of sequences $(x_n)$ in $E$ with norm $\| (x_n) \|_2 = \left( \sum_n \|x_n\|^2 \right)^{1/2}$. Throughout, we could instead work with $\ell^p(E)$ for $1 < p < \infty$, if we so wished. Then $\ell^2(E)^* = \ell^2(E^*)$, and $\ell^2(E)$ is reflexive if $E$ is. For each $n$, let $\tau_n : E \to \ell^2(E)$ be the injection onto the $n$th co-ordinate, and let $P_n : \ell^2(E) \to E$ be the projection onto the $n$th co-ordinate. For $T \in \mathcal{B}(E)$, let $T^{(\infty)} \in \mathcal{B}(\ell^2(E))$ be the operator given by applying $T$ to each co-ordinate. Notice that $T^{(\infty)} = T_{n}T$ and $P_nT^{(\infty)} = TP_n$, for each $n$.

For $X \subseteq \mathcal{B}(E)$, let $X^{(\infty)} = \{ T^{(\infty)} : T \in X \}$. Given a homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$, let $\pi^{(\infty)} : \mathcal{A} \to \mathcal{B}(\ell^2(E))$ by the homomorphism given by $\pi^{(\infty)}(a) = \pi(a)^{(\infty)}$ for each $a \in \mathcal{A}$.

**Lemma 2.1.** For a Banach space $E$, and $X \subseteq \mathcal{B}(E)$, we have that $(X^{(\infty)})'' = (X'')^{(\infty)}$.

**Proof.** Let $Q \in (X^{(\infty)})'$. For $n, m \in \mathbb{N}$ and $S \in X$, we have that $P_nQ_{\ell m}S = P_nQS_{\ell m} = P_nS_{\ell m}Q_{\ell m} = SP_nQ_{\ell m}$. Thus $P_nQ_{\ell m} \in X'$, for each $n, m$. Similarly, one can show that for $Q \in \mathcal{B}(\ell^2(E))$, if $P_nQ_{\ell m} \in X'$ for all $n, m$, then $Q \in (X^{(\infty)})'$. So, given $T \in X''$ and $Q \in (X^{(\infty)})'$, we have that $TP_nQ_{\ell m} = P_nQ_{\ell m}T$ for all $n, m$. Thus, for all $n, m$, it follows that $P_nT^{(\infty)}Q_{\ell m} = P_nQT^{(\infty)}_{\ell m}$, from which it follows that $T^{(\infty)}Q = QT^{(\infty)}$. Thus $(X'')^{(\infty)} \subseteq (X^{(\infty)})''$.

For the converse, let $T \in (X^{(\infty)})''$. For each $n, m$, notice that $\tau_nP_m \in (X^{(\infty)})'$, so that $T\tau_nP_m = \tau_nP_mT$. Let $r \in \mathbb{N}$, so that
\[ T\tau_n\delta_{m,r} = T\tau_nP_{m,r} = \tau_nP_{m,r}. \]
It follows that $T_{t_r} = t_r R$ for some $R \in \mathcal{B}(E)$, and that $R$ does not depend upon $r$. Thus there must exist $R \in \mathcal{B}(E)$ with $T = R^{(\infty)}$. Now let $S \in X'$, so that $S^{(\infty)} \in (X^{(\infty)})'$, and hence
\[
(RS)^{(\infty)} = TS^{(\infty)} = S^{(\infty)}T = (SR)^{(\infty)}.
\]
It follows that $R \in X''$, and hence that $(X^{(\infty)})'' \subseteq (X'')^{(\infty)}$. □

**Lemma 2.2.** Let $E$ be a reflexive Banach space, and let $X \subseteq \mathcal{B}(E)$ be a subalgebra. Let $X_w$ be the weak∗-closure of $X$ in $\mathcal{B}(E)$, with respect to the predual $E^* \otimes E$. Then $(X_w)^{(\infty)} = (X^{(\infty)})_w$.

**Proof.** Let $T \in (X^{(\infty)})_w$. For $x \in E, \mu \in E^*$ and $n \neq m$, certainly $\langle t_n(\mu) \otimes t_m(x) \rangle \in \downarrow (X^{(\infty)})$, and so
\[
0 = \langle t_n(\mu), Tt_m(x) \rangle = \langle \mu, P_n Tt_m(x) \rangle.
\]
Thus $P_n Tt_m = 0$ whenever $n \neq m$. For any $x, \mu, n$ and $m$, we also have that
\[
\langle t_n(\mu) \otimes t_n(x), t_m(\mu) \otimes t_m(x) \rangle \in \downarrow (X^{(\infty)}).
\]
It follows that $P_n Tt_n = P_m Tt_m$. Combining these results, we conclude that $T = S^{(\infty)}$ for some $S \in \mathcal{B}(E)$.

Let $\tau \in \downarrow X \subseteq E^* \otimes E$, say $\tau = \sum_k \mu_k \otimes x_k$. For $R \in X$ and each $n$, we have that
\[
\langle R^{(\infty)}, \sum_k t_n(\mu_k) \otimes t_n(x_k) \rangle = 0,
\]
so that $\sigma = \sum_k t_n(\mu_k) \otimes t_n(x_k) \in \downarrow (X^{(\infty)})$. So
\[
0 = \langle T, \sigma \rangle = \langle S^{(\infty)}, \sigma \rangle = \langle S, \tau \rangle,
\]
from which it follows that $S \in X_w$. So $(X^{(\infty)})_w \subseteq (X_w)^{(\infty)}$.

For the converse, let $T \in X_w$, and let $\tau \in \downarrow (X^{(\infty)})$, say $\tau = \sum_n \mu_n \otimes x_n$. By rescaling, we may suppose that $\sum_n \|\mu_n\|^2 = \sum_n \|x_n\|^2 < \infty$. For each $n$, we have that $\mu_n = (\mu^{(n)}_k)$, say, where $\|\mu_n\|^2 = \sum_k \|\mu^{(n)}_k\|^2$. Thus $\sum_n \|\mu^{(n)}_k\|^2 < \infty$. Similarly, each $x_n = (x^{(n)}_k)$, and $\sum_n \|x^{(n)}_k\|^2 < \infty$. We can now compute that, for $S \in X$,
\[
0 = \langle S^{(\infty)}, \tau \rangle = \sum_n \langle \mu_n, S^{(\infty)}(x_n) \rangle = \sum_n \langle \mu^{(n)}_k, S(x^{(n)}_k) \rangle,
\]
so that $\sigma = \sum_n \mu^{(n)}_k \otimes x^{(n)}_k \in \downarrow X$ (where this sum converges absolutely by an application of the Cauchy-Schwarz inequality). Then $0 = \langle T, \sigma \rangle = \langle T^{(\infty)}, \tau \rangle$, from which it follows that $T^{(\infty)} \in (X^{(\infty)})_w$. So $(X_w)^{(\infty)} \subseteq (X^{(\infty)})_w$. □

The following lemma is usually stated in terms of “reflexivity” of a subspace of $\mathcal{B}(E)$, but this is a different meaning to that of a reflexive Banach space, so we avoid this terminology.

**Lemma 2.3.** Let $E$ be a reflexive Banach space, and let $X \subseteq \mathcal{B}(E)$ be a weak∗-closed subspace. If $T \in \mathcal{B}(\ell^2(E))$ is such that, for each $x \in \ell^2(E)$, we have that $T(x)$ is in the closure of $\{S^{(\infty)}(x) : S \in X\}$, then actually $T \in X^{(\infty)}$.

**Proof.** Let $T$ be as stated, so for each $n$, we have that the image of $Tt_n$ is a subset of the image of $t_n$. By considering what $T$ maps $(t_1 + \cdots + t_n)(x)$ to, for any $x \in E$, we may conclude that $T = R^{(\infty)}$ for some $R \in \mathcal{B}(E)$.

Let $\tau \in \downarrow X$, say $\tau = \sum_n \mu_n \otimes x_n$, where we may suppose that $\sum_n \|\mu_n\|^2 = \sum_n \|x_n\|^2 < \infty$. Let $\mu = (\mu_n) \in \ell^2(E^*)$ and $x = (x_n) \in \ell^2(E)$, so that
\[
\langle R, \tau \rangle = \langle \mu, R^{(\infty)}(x) \rangle = \langle \mu, T(x) \rangle.
\]
However, notice that $\langle \mu, S^{(\infty)}(x) \rangle = \langle S, \tau \rangle = 0$ for each $S \in X$, so by the assumption on $T$, it follows also that $\langle \mu, T(x) \rangle = 0$, so $\langle R, \tau \rangle = 0$. So $R \in (\downarrow X)^\perp = X$, that is, $T \in X^{(\infty)}$. □
3 The main result

Let us introduce some temporary terminology, motivated by [1]. Let $\mathcal{A}$ be a Banach algebra, and $E$ be a left $\mathcal{A}$-module (which we assume to be a Banach space with contractive actions). In this section, we shall always suppose that $E$ is essential, that is, the linear span of $\{a \cdot x : a \in \mathcal{A}, x \in E\}$ is dense in $E$.

We say that $E$ is cyclic if there exists $x \in E$ with $\mathcal{A} \cdot x = \{a \cdot x : a \in \mathcal{A}\}$ being dense in $E$. We say that $E$ is self-generating if, for each closed cyclic submodule $K \subseteq E$, the linear span of $\{T(E) : T : E \to K$ is an $\mathcal{A}$-module homomorphism} is dense in $K$.

The following is very similar to the presentation in [1], but we check that the details still work for reflexive Banach spaces, and not just Hilbert spaces.

**Theorem 3.1.** Let $\mathcal{A}$ be a unital Banach algebra, and let $E$ be a reflexive Banach space with a bounded homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$. Use $\pi$ to turn $E$ into a left $\mathcal{A}$-module, and suppose that $\ell^2(E)$ is self-generating. Then $\pi(\mathcal{A})''$ agrees with the weak*-closure of $\pi(\mathcal{A})$ in $\mathcal{B}(E)$.

**Proof.** Let $\mathcal{B}$ be the closure of $\pi(\mathcal{A})$ in $\mathcal{B}(E)$, and let $\mathcal{B}_w$ be the weak*-closure of $\mathcal{B}$. We wish to show that $\mathcal{B}_w = \mathcal{B}''$.

Let $T \in (\mathcal{B}'')'(\infty) \subseteq \mathcal{B}(\ell^2(E))$, let $x \in \ell^2(E)$ be non-zero, and let $K$ be the closure of $\mathcal{B}''(\infty)(x)$. As $E$ is essential, it follows that the unit of $\mathcal{A}$ acts as the identity on $E$, and hence also as the identity on $\ell^2(E)$, under $\pi(\infty)$. Thus $x \in K$. We shall show that $T(K) \subseteq K$.

Let $V : \ell^2(E) \to K$ be an $\mathcal{A}$-module homomorphism, and let $\iota : K \to \ell^2(E)$ be the inclusion map. By continuity, and the density of $\mathcal{A}$ in $\mathcal{B}$, we see that $\iota V \in (\mathcal{B}'')(\infty)'$. Hence $T \iota V = \iota T V$, from which it follows that $TV(\ell^2(E)) = V T(\ell^2(E)) \subseteq K$. Let $W$ be the linear span of the images of all such $V$. As $\ell^2(E)$ is self-generating, it follows that $W$ is dense in $K$. However, $T(W) \subseteq K$, and so by continuity, $T(K) \subseteq K$, as required.

So we have shown that for each $x \in \ell^2(E)$, we have that $T(x)$ is in the closed linear span of $\mathcal{B}''(\infty)(x) \subseteq \mathcal{B}_w'(\infty)(x)$. By Lemma 2.3, we conclude that $T \in (\mathcal{B}_w'')'(\infty) \subseteq (\mathcal{B}'')(\infty)$. By Lemma 2.1 and Lemma 2.2, this shows that $(\mathcal{B}'')(\infty)' \subseteq (\mathcal{B}_w'')'(\infty)$. Hence also $(\mathcal{B}_w'(\infty))'' = (\mathcal{B}'')'(\infty)$, from which it follows immediately that $\mathcal{B}_w = \mathcal{B}''$, as required. \[\square\]

By using the Cohen Factorisation theorem, see [3, Corollary 2.9.25], a slightly more subtle argument would show that this theorem also holds for Banach algebras with a bounded approximate identity.

The previous result is only useful if we have a good supply of self-generating modules. The following is similar to an idea we used in [5, Lemma 6.10].

**Proposition 3.2.** Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a reflexive Banach space which is a left $\mathcal{A}$-module. There exists a reflexive left $\mathcal{A}$-module $F$ such that:

1. $E$ is isomorphic to a one-complemented submodule of $F$;
2. each closed, cyclic submodule of $\ell^2(F)$ is isomorphic to a one-complemented submodule of $F$;

In particular, $\ell^2(F)$ is self-generating.

**Proof.** Let $\mathcal{E}_0 = \{E\}$. We use transfinite induction to define $\mathcal{E}_\alpha$ to be a set of reflexive left $\mathcal{A}$-modules, for each ordinal $\alpha \leq \aleph_1$. If $\alpha$ is a limit ordinal, we simply define $\mathcal{E}_\alpha = \bigcup_{\beta < \alpha} \mathcal{E}_\beta$.

Otherwise, we let $E_\alpha$ to be the $\ell^2$ direct sum of each module in $\mathcal{E}_\alpha$, so that $E_\alpha$ is a reflexive left $\mathcal{A}$-module in the obvious way. Let $\mathcal{E}_{\alpha+1}$ be $\mathcal{E}_\alpha$ unioned with the set of all closed cyclic submodules of $\ell^2(E_\alpha)$.
Let \( F \) be the \( \ell^2 \) direct sum of all the modules in \( \mathcal{E}_{\mathcal{R}_1} \). As \( \{E\} = \mathcal{E}_0 \subseteq \mathcal{E}_{\mathcal{R}_1} \), condition (1) follows. Let \( K \) be a closed, cyclic submodule of \( \ell^2(F) \), say \( K \) is the closure of \( A \cdot x \). Thus

\[
x \in \ell^2(F) \cong \ell^2 - \bigoplus_{G \in \mathcal{E}_{\mathcal{R}_1}} \ell^2(G).
\]

Say \( x = (x_G)_{G \in \mathcal{E}_{\mathcal{R}_1}} \), where each \( x_G \in \ell^2(G) \). As \( \|x\|^2 = \sum_G \|x_G\|^2 < \infty \), it follows that \( x_G \neq 0 \) for at most countably many \( G \). As \( \aleph_1 \) is uncountable, we must actually have that there exists \( \alpha < \aleph_1 \) with \( x \in \ell^2 - \bigoplus_{G \in \mathcal{E}_{\alpha}} \ell^2(G) \cong \ell^2(E_\alpha) \). Then, by construction, \( K \subseteq \mathcal{E}_{\alpha+1} \), and so \( K \) is a one-complemented submodule of \( F \).

Let \( \mathcal{A} \) be a Banach algebra. Recall, for example from [5], that \( \text{WAP}(\mathcal{A}^*) \) is the closed submodule of \( \mathcal{A}^* \) consisting of those functionals \( \phi \in \mathcal{A}^* \) such that

\[
\mathcal{A} \rightarrow \mathcal{A}^*; \quad a \mapsto a \cdot \phi
\]

is weakly-compact. Young’s result, [5], shows that for each \( \phi \in \text{WAP}(\mathcal{A}^*) \), there exists a reflexive Banach space \( E \), a contractive homomorphism \( \pi : \mathcal{A} \rightarrow \mathcal{B}(E) \), and \( x \in E, \mu \in E^* \) with \( \|\phi\| = \|x\|\|\mu\| \) and such that

\[
\langle \phi, a \rangle = \langle \mu, \pi(a)(x) \rangle \quad (a \in \mathcal{A}).
\]

Let \( \mathcal{A} \) be a dual Banach algebra with predual \( \mathcal{A}_* \). It is easy to show (see [5] for example) that \( \mathcal{A}_* \subseteq \text{WAP}(\mathcal{A}^*) \). We showed in [5, Section 3] that Young’s result holds for \( \phi \in \mathcal{A}_* \), with the additional condition that for any \( \lambda \in E^* \) and \( y \in E \), the functional \( \pi^*(\lambda \otimes y) \) is in \( \mathcal{A}_* \), where

\[
\langle \pi^*(\lambda \otimes y), a \rangle = \langle \lambda, \pi(a)(y) \rangle \quad (a \in \mathcal{A}).
\]

Note that, a priori, Young’s result only shows that \( \pi^*(\lambda \otimes y) \in \text{WAP}(\mathcal{A}^*) \).

**Proposition 3.3.** With the notation of Proposition 3.2, we have that \( \pi^*(F^* \hat{\otimes} F) \) is a subset of the closed submodule generated by \( \pi^*(E^* \hat{\otimes} E) \).

**Proof.** The module \( F \) is generated from \( E \) by two constructions: (i) taking submodules; and (ii) taking \( \ell^2 \)-direct sums. For (i), let \( K \) be a submodule of \( E \). The Hahn-Banach theorem shows that \( \pi^*(K^* \hat{\otimes} K) \subseteq \pi^*(E^* \hat{\otimes} E) \). For (ii), let \( (K_i) \) be a family of submodules of \( E \) with \( \pi^*(K_i^* \hat{\otimes} K_i) \subseteq \pi^*(E^* \hat{\otimes} E) \) for each \( i \), and let \( F = \ell^2 - \bigoplus_i K_i \). Let \( \sum_n \mu_n \otimes x_n \in F^* \hat{\otimes} F \), with, say, \( \sum_n \|\mu_n\|^2 = \sum_n \|x_n\|^2 < \infty \). For each \( n \), we have \( \mu_n = (\mu_i^{(n)}) \) with \( \|\mu_n\|^2 = \sum_i \|\mu_i^{(n)}\|^2 \), and \( x_n = (x_i^{(n)}) \) with \( \|x_n\|^2 = \sum_i \|x_i^{(n)}\|^2 \). Then

\[
\sum_n \langle \mu_n, a \cdot x_n \rangle = \sum_{n,i} \langle \mu_i^{(n)}, a \cdot x_i^{(n)} \rangle \quad (a \in \mathcal{A}).
\]

Hence

\[
\pi^* \left( \sum_n \mu_n \otimes x_n \right) = \pi^* \left( \sum_{n,i} \mu_i^{(n)} \otimes x_i^{(n)} \right) \subseteq \pi^*(E^* \otimes E).
\]

Again, the Cauchy-Schwarz inequality shows that the sum on the right converges. \( \square \)

**Theorem 3.4.** Let \( \mathcal{A} \) be a unital dual Banach algebra. There exists a reflexive Banach space \( E \) and an isometric, weak*-weak*-continuous homomorphism \( \pi : \mathcal{A} \rightarrow \mathcal{B}(E) \) such that \( \pi(\mathcal{A})'' = \pi(\mathcal{A}) \).

**Proof.** By [5, Corollary 3.8], we may suppose that \( \mathcal{A} \subseteq \mathcal{B}(E_0) \), for some reflexive Banach space \( E_0 \). By Proposition 3.2, we can find a self-generating, reflexive Banach space \( E \) and a contractive representation \( \pi : \mathcal{A} \rightarrow \mathcal{B}(E) \). As \( E_0 \subseteq E \), it follows that \( \pi \) is an isometry. By Proposition 3.3, \( \pi \) is weak*-weak*-continuous. The result now follows from Theorem 3.1. \( \square \)
It is well-known that for any Banach algebra $\mathcal{A}$, we have that $\text{WAP}(\mathcal{A}^\ast)^\ast$ is a dual Banach algebra (see, for example, [5, Proposition 2.4]). When $\mathcal{A}$ has a bounded approximate identity, a weak$^*$-limit point in $\text{WAP}(\mathcal{A}^\ast)^\ast$ will be a unit for $\text{WAP}(\mathcal{A}^\ast)^\ast$.

**Corollary 3.5.** Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. There exists a reflexive Banach space $E$ and a contractive homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$ such that $\pi(\mathcal{A})''$ is isometrically, weak$^*$-weak$^*$-continuously isomorphic to $\text{WAP}(\mathcal{A}^\ast)^\ast$.

Finally, we remark that Uygul showed in [7] that given a dual, completely contractive Banach algebra $\mathcal{A}$, we can find a reflexive operator space and a completely isometric, weak$^*$-weak$^*$-continuous homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$. Using this result, we can easily prove a version of Theorem 3.4 for completely contractive Banach algebras. Indeed, the only thing to do is to equip $\ell^2$ direct sums with an Operator Space structure such that the inclusion and projection maps are complete contractions. This is worked out in detail in [8] (see also [7]).

Finally, we remark that the space constructed in Theorem 3.4 is very abstract. For a group measure space convolution algebra $M(G)$, Young showed in [9] that $M(G)$ can be weak$^*$-represented on a direct sum of $L^p(G)$ spaces; the analogous result for the Fourier algebra was shown by the author in [4]. For such concrete Banach algebras $\mathcal{A}$, it would be interesting to know if “nice” reflexive Banach spaces $E$ could be found with $\pi : \mathcal{A} \to \mathcal{B}(E)$ such that $\pi(\mathcal{A})'' = \pi(\mathcal{A})$.

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