NONZERO POSITIVE SOLUTIONS OF A MULTI-PARAMETER ELLIPTIC SYSTEM WITH FUNCTIONAL BCS

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Abstract. We prove, by topological methods, new results on the existence of nonzero positive weak solutions for a class of multi-parameter second order elliptic systems subject to functional boundary conditions. The setting is fairly general and covers the case of multi-point, integral and nonlinear boundary conditions. We also present a non-existence result. We provide some examples to illustrate the applicability our theoretical results.

1. Introduction

In this paper we discuss the solvability of the multi-parameter system of second order elliptic equations subject to functional boundary conditions

\begin{align}
\begin{cases}
L_i u_i(x) = \lambda_i f_i(x, u(x)), & x \in \Omega, \quad i = 1, 2, \ldots, n, \\
B_i u_i(x) = \eta_i h_i[u], & x \in \partial \Omega, \quad i = 1, 2, \ldots, n,
\end{cases}
\end{align}

where $\Omega \subset \mathbb{R}^m$ ($m \geq 2$) is a bounded domain with sufficiently regular boundary, $L_i$ is a strongly uniformly elliptic operator, $B_i$ is a first order boundary operator, $u = (u_1, \ldots, u_n)$, $f_i$ is a continuous function, $h_i$ is a suitable compact functional, $\lambda_i, \eta_i$ are parameters.

A motivation for studying this kind of boundary value problems (BVPs) is that they often occur in physical applications. In order to illustrate this fact, take $n = 1$, $m = 2$ and consider the BVP

\begin{align}
\begin{cases}
-\Delta u(x) = f(x, u(x)), & \|x\|_2 < 1, \\
u(x) = \eta u(0), & \|x\|_2 = 1,
\end{cases}
\end{align}

where $\| \cdot \|_2$ is the Euclidean norm. The BVP (1.2) can be used as a model for the steady-states of the temperature of a heated disk of radius 1, where a controller located in the border of the disk adds or removes heat in manner proportional to the temperature registered by a sensor located in the center of the disk. In the context of ODEs, a good reference for this kind of thermostat problems is the recent paper [25].

The assumptions we make on the functionals $h_i$ that occur in (1.1) are fairly weak and allow to cover, for example, the special cases of multi-point boundary conditions (BCs) of

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the form

\begin{equation}
(1.3) \quad h_i[u] = \sum_{k=1}^{n} \sum_{j=1}^{N} \hat{\alpha}_{ijk} u_k(\omega_j),
\end{equation}

where \(\hat{\alpha}_{ijk}\) are non-negative coefficients and \(\omega_j \in \Omega\), or integral BCs of the type

\begin{equation}
(1.4) \quad h_i[u] = \sum_{k=1}^{n} \int_{\Omega} \hat{\alpha}_{ik}(\omega) u_k(\omega) \, d\omega,
\end{equation}

where \(\hat{\alpha}_{ik}\) are non-negative continuous functions on \(\overline{\Omega}\). Note that the functionals \(h_i\) in (1.3) and (1.4) allow an interaction between the components of the solution.

There exists a wide literature on multi-point, integral and, more in general, nonlocal BCs. As far as we know multi-point BCs have been studied firstly by Picone \[22\] in the context of ODEs. For an introduction to nonlocal BCs, we refer the reader to the reviews \[6, 17, 23, 24, 27\] and the papers \[13, 14, 21, 26\].

Note that our approach is not restricted to linear functionals like (1.3) and (1.4), we may also deal with the case of nonlinear BCs. These type of BCs also make physical sense; for example the BVP (1.2) might be modified in order to take into account a nonlinear response of the controller, by having a nonlinear, nonlocal BC of the form

\begin{equation}
(1.5) \quad u(x) = \hat{h}(u(0)), \quad x \in \partial\Omega,
\end{equation}

where \(\hat{h}\) is a continuous function. In the context of radial solutions of PDEs on annular domains, conditions similar to (1.5) have been investigated recently in \[5, 7, 8, 9, 10\]. We stress that nonlinear BCs have been widely studied for different classes of differential equations, nonlinearities and domains, we refer the reader to \[2, 3, 5, 11, 19, 20, 18, 28\] and references therein; in particular, the method of upper and lower solutions has been employed for the System (1.1) in the case of non-homogeneous (not necessarily constant) BCs in \[2\] and in the case of nonlinear BCs (when \(\lambda_i = \eta_i = 1\)) in \[18, 20\].

We highlight that the existence of positive solutions of the System (1.1) with homogeneous BCs has been recently discussed in \[15, 16\] (in the sublinear case) and in \[4\] (under monotonicity assumptions on the nonlinearities). Our theory can be applied also in this case, by considering \(h_i[u] \equiv 0\). We do not assume global restrictions on the growth nor we assume monotonicity of the nonlinearities, thus complementing the results in \[4, 15, 16\].

We prove, by means of classical fixed point index, the existence of one nontrivial weak solution of the System (1.1). We also prove, via an elementary argument, a non-existence result. We provide some examples in order to illustrate the applicability of our theoretical results.
2. Existence and non-existence results

We make the following assumptions on the domain \( \Omega \) and the operators \( L_i \) and \( B_i \) that occur in (1.1) (see [2, Section 4 of Chapter 1] and [15, 16]):

1. \( \Omega \subset \mathbb{R}^m, m \geq 2, \) is a bounded domain such that its boundary \( \partial \Omega \) is an \((m - 1)\)-dimensional \( C^{2+\tilde{\mu}} \)-manifold for some \( \tilde{\mu} \in (0, 1) \), such that \( \Omega \) lies locally on one side of \( \partial \Omega \) (see [29, Section 6.2] for more details).

2. \( L_i \) is a the second order elliptic operator given by

\[
L_i u(x) = -\sum_{j,l=1}^{m} a_{ijl}(x) \frac{\partial^2 u}{\partial x_j \partial x_l}(x) + \sum_{j=1}^{m} a_{ij}(x) \frac{\partial u}{\partial x_j}(x) + a_i(x) u(x), \quad \text{for } x \in \Omega,
\]

where \( a_{ijl}, a_{ij}, a_i \in C^{\tilde{\mu}}(\overline{\Omega}) \) for \( j, l = 1, 2, \ldots, m \), \( a_i(x) \geq 0 \) on \( \overline{\Omega} \), \( a_{ijl}(x) = a_{jl}(x) \) on \( \overline{\Omega} \) for \( j, l = 1, 2, \ldots, m \). Moreover \( L_i \) is strongly uniformly elliptic, that is, there exists \( \mu_i > 0 \) such that

\[
\sum_{j,l=1}^{m} a_{ijl}(x) \xi_j \xi_l \geq \mu_i \|\xi\|^2 \quad \text{for } x \in \Omega \text{ and } \xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m.
\]

3. \( B_i \) is a boundary operator given by

\[
B_i u(x) = b_i(x) u(x) + \delta_i \frac{\partial u}{\partial \nu}(x) \quad \text{for } x \in \partial \Omega,
\]

where \( \nu \) is an outward pointing and nowhere tangent vector field on \( \partial \Omega \) of class \( C^{1+\tilde{\mu}} \) (not necessarily a unit vector field), \( \frac{\partial}{\partial \nu} \) is the directional derivative of \( u \) with respect to \( \nu \), \( b_i : \partial \Omega \to \mathbb{R} \) is of class \( C^{1+\tilde{\mu}} \) and moreover one of the following conditions holds:

(a) \( \delta_i = 0 \) and \( b_i(x) \equiv 1 \) (Dirichlet boundary operator).
(b) \( \delta_i = 1, b_i(x) \equiv 0 \) and \( a_i(x) \neq 0 \) (Neumann boundary operator).
(c) \( \delta_i = 1, b_i(x) \geq 0 \) and \( b_i(x) \neq 0 \) (Regular oblique derivative boundary operator).

It is known (see [2, Section 4]) that, under the previous conditions, a strong maximum principle holds and, furthermore, given \( g \in C^{\tilde{\mu}}(\overline{\Omega}) \), the boundary value problem

\[
\begin{cases}
L_i u(x) = g(x), & x \in \Omega, \\
B_i u(x) = 0, & x \in \partial \Omega,
\end{cases}
\]

admits a unique classical solution \( u \in C^{2+\tilde{\mu}}(\overline{\Omega}) \).

In order to seek solutions of the System (1.1), we work in a suitable cone of positive functions. We recall that a cone \( P \) of a real Banach space \( X \) is a closed set with \( P + P \subset P \), \( \lambda P \subset P \) for all \( \lambda \geq 0 \) and \( P \cap (-P) = \{0\} \). A cone \( P \) induces a partial ordering in \( X \) by means of the relation

\[
x \leq y \quad \text{if and only if } y - x \in P.
\]
The cone $P$ is normal if there exists $d > 0$ such that for all $x, y \in X$ with $0 \leq x \leq y$ then $\|x\| \leq d\|y\|$. Note that every (closed) cone $P$ has the Archimedean property, that is, $nx \leq y$ for all $n \in \mathbb{N}$ and some $y \in X$ implies $x \leq 0$. In what follows, with abuse of notation, we will use the same symbol "\geq" for the different cones appearing in the paper.

Now consider the (normal) cone of non-negative functions $P = C(\bar{\Omega}, \mathbb{R}_+)$, then the so-
olution operator $K_i : C^\mu(\bar{\Omega}) \to C^{2+\mu}(\bar{\Omega})$ defined as $K_i g = u$ is linear, continuous and (due
 to the maximum principle) positive, that is $K_i(P) \subset P$. It is known that $K_i$ can be ex-
tended uniquely to a continuous, linear and compact operator $K_i : C(\bar{\Omega}) \to C(\bar{\Omega})$ (that we
denote again by the same name). The following result (see [1, Lemma 5.3]) provides further
positivity properties of the generalized solution operator.

**Proposition 2.1.** Let $e_i = K_i 1 \in C(\bar{\Omega}, \mathbb{R}_+) \setminus \{0\}$. Then $K_i : C(\bar{\Omega}) \to C^1(\bar{\Omega}) \subset C(\bar{\Omega})$ is
e-positive (and in particular positive), that is for each $g \in C(\bar{\Omega}, \mathbb{R}_+) \setminus \{0\}$ there exist $\alpha_g > 0$
and $\beta_g > 0$ such that $\alpha_g e_i \leq K_i g \leq \beta_g e_i$.

Denote by $r(K_i)$ the spectral radius of $K_i$. As a consequence of Proposition 2.1 and the
Krein-Rutman theorem, it is known (for details see, for example, Lemma 3.3 of [16]) that
$r(K_i) \in (0, +\infty)$ and there exists $\varphi_i \in P \setminus \{0\}$ such that

$$\varphi_i = \mu_i K_i \varphi_i,$$

where $\mu_i = 1/r(K_i)$.

We utilize the space $C(\bar{\Omega}, \mathbb{R}^n)$, endowed with the norm $\|u\| := \max_{i=1,2,\ldots,n} \{|u_i|\}$, where
$\|z\|_\infty = \max_{x \in \bar{\Omega}} |z(x)|$, and consider (with abuse of notation) the cone $P = C(\bar{\Omega}, \mathbb{R}^+_n)$.

Given a nonempty set $D \subset C(\bar{\Omega}, \mathbb{R}^n)$ we define

$$D_I = \{u \in D : u(x) \in I \text{ for all } x \in \Omega\},$$

where $I = \prod_{i=1}^n I_i \subset \mathbb{R}^n$, where each $I_i \subset \mathbb{R}$ is a closed nonempty interval.

Given a function $f_i : \bar{\Omega} \times I \to \mathbb{R}$ we define the Nemytskii (or superposition) operator $F_i$
in the following way

$$F_i(u)(x) := f_i(x, u(x)), \text{ for } u \in C(\bar{\Omega}, I) \text{ and } x \in \Omega.$$
where \( \gamma_i \in C^{2+\beta} (\bar{\Omega}) \) is the unique solution (nonnegative, due to the maximum principle, see [2, Section 4 of Chapter 1]) of the BVP

\[
\begin{align*}
L_i u(x) &= 0, \quad x \in \Omega, \\
B_i u(x) &= 1, \quad x \in \partial \Omega.
\end{align*}
\]

**Definition 2.2.** We say that \( u \in C(\bar{\Omega}, I) \) is a weak solution of the System (1.1) if and only if \( u \) is a fixed point of the operator \( T + \Gamma \), that is,

\[
u = T u + \Gamma u = (\lambda_i K_i F_i (u) + \eta_i \gamma_i h_i [u])_{i=1..n};
\]

if, furthermore, the components of \( u \) are non-negative with \( u_j \neq 0 \) for some \( j \) we say that \( u \) is a nonzero positive solution.

In the following Proposition we recall the main properties of the classical fixed point index for compact maps, for more details see [2, 12]. In what follows the closure and the boundary of subsets of a cone \( \hat{P} \) are understood to be relative to \( \hat{P} \).

**Proposition 2.3.** Let \( X \) be a real Banach space and let \( \hat{P} \subset X \) be a cone. Let \( D \) be an open bounded set of \( X \) with \( 0 \in D \cap \hat{P} \) and \( D \cap \hat{P} \neq \hat{P} \). Assume that \( T : D \cap \hat{P} \to \hat{P} \) is a compact operator such that \( x \neq T x \) for \( x \in \partial (D \cap \hat{P}) \). Then the fixed point index \( i_{\hat{P}} (T, D \cap \hat{P}) \) has the following properties:

(i) If there exists \( e \in \hat{P} \setminus \{0\} \) such that \( x \neq T x + \lambda e \) for all \( x \in \partial (D \cap \hat{P}) \) and all \( \lambda > 0 \), then \( i_{\hat{P}} (T, D \cap \hat{P}) = 0 \).

(ii) If \( T x \neq \lambda x \) for all \( x \in \partial (D \cap \hat{P}) \) and all \( \lambda > 1 \), then \( i_{\hat{P}} (T, D \cap \hat{P}) = 1 \).

(iii) Let \( D^1 \) be open bounded in \( X \) such that \( \overline{(D^1 \cap \hat{P})} \subset (D \cap \hat{P}) \). If \( i_{\hat{P}} (T, D \cap \hat{P}) = 1 \) and \( i_{\hat{P}} (T, D^1 \cap \hat{P}) = 0 \), then \( T \) has a fixed point in \( (D \cap \hat{P}) \setminus (D^1 \cap \hat{P}) \). The same holds if \( i_{\hat{P}} (T, D \cap \hat{P}) = 0 \) and \( i_{\hat{P}} (T, D^1 \cap \hat{P}) = 1 \).

With these ingredients we can now state a result regarding the existence of positive solutions for the System (1.1).

**Theorem 2.4.** Let \( I = \prod_{i=1}^n [0, \rho_i] \) and assume the following conditions hold.

(a) For every \( i = 1, 2, \ldots, n \), \( f_i \in C(\bar{\Omega} \times I) \) and \( f_i \geq 0 \). Set

\[
M_i := \max_{(x,u) \in \bar{\Omega} \times I} f_i (x, u).
\]

(b) There exist \( \delta \in (0, +\infty) \), \( i_0 \in \{1, 2, \ldots, n\} \) and \( \rho_0 \in (0, \min_{i=1..n} \rho_i) \) such that

\[
f_{i_0} (x, u) \geq \delta u_{i_0}, \quad \text{for every } (x, u) \in \bar{\Omega} \times I_0,
\]

where \( I_0 := \prod_{i=1}^n [0, \rho_0] \).
(c) For every \( i = 1, 2, \ldots, n \), \( h_i : P_I \to [0, +\infty) \) is continuous and
\[
H_i := \sup_{u \in P_I} h_i[u] < +\infty.
\]

(d) For every \( i = 1, 2, \ldots, n \) the following two inequalities are satisfied
\[
(2.4) \quad \frac{\mu_i}{\delta} \leq \lambda_{i_0} \quad \text{and} \quad \lambda_i M_i \| K_i(1) \|_{\infty} + \eta_i H_i \| \gamma_i \|_{\infty} \leq \rho_i.
\]

Then the System (1.1) has a nonzero positive weak solution \( u \) such that
\[
\rho_0 \leq \|u\| \quad \text{and} \quad \|u_i\|_{\infty} \leq \rho_i, \quad \text{for every} \quad i = 1, 2, \ldots, n.
\]

Proof. Take \( P = C(\bar{\Omega}, \mathbb{R}_+^n) \). Due to the assumptions above the operator \( T + \Gamma \) maps \( P_I \) into \( P \) and is compact (the compactness of \( T \) is well-known and \( \Gamma \) is a finite rank operator). If \( T + \Gamma \) has a fixed point either on \( \partial P_I \) or \( \partial P_{i_0} \) we are done.

Assume now that \( T + \Gamma \) is fixed point free on \( \partial P_I \cup \partial P_{i_0} \), we are going to prove that \( T + \Gamma \) has a fixed point in \( P_I \setminus (\partial P_I \cup P_{i_0}) \).

We firstly prove, by means of (a), (c) and (d), that
\[
\sigma u \neq Tu + \Gamma u \quad \text{for every} \quad u \in \partial P_I \quad \text{and every} \quad \sigma > 1.
\]
If this does not hold, then there exist \( u \in \partial P_I \) and \( \sigma > 1 \) such that \( \sigma u \neq Tu + \Gamma u \). Note that \( \|u_j\|_{\infty} = \rho_j \) for some \( j \) and \( \|u_i\|_{\infty} \leq \rho_i \) for every \( i \). Furthermore for every \( x \in \bar{\Omega} \) we obtain
\[
\sigma u_j(x) = \lambda_j K_j F_j(u)(x) + \eta_j h_j[u] \gamma_j(x) \leq \|\lambda_j K_j F_j(u) + \eta_j h_j[u] \gamma_j\|_{\infty}
\]
\[
\leq \|\lambda_j K_j(M_j)\|_{\infty} + \|\eta_j H_j \gamma_j\|_{\infty} = \lambda_j M_j \|K_j(1)\|_{\infty} + \eta_j H_j \| \gamma_j \|_{\infty} \leq \rho_j.
\]
Taking the supremum over \( \bar{\Omega} \) we obtain \( \sigma \rho_j \leq \rho_j \), a contradiction which yields
\[
i_P(T + \Gamma, P_I \setminus \partial P_I) = 1.
\]

We now consider \( \varphi = (\varphi_1, \ldots, \varphi_n) \) where \( \varphi_i \) is given by (2.2) and use (b) and (d) to show that
\[
u \neq Tu + \Gamma u + \sigma \varphi \quad \text{for every} \quad u \in \partial P_{i_0} \quad \text{and every} \quad \sigma > 0.
\]
If not, there exists \( u \in \partial P_{i_0} \) and \( \sigma > 0 \) such that
\[
u = Tu + \Gamma u + \sigma \varphi.
\]
Then we have \( u \geq \sigma \varphi \) and, in particular, \( u_{i_0} \geq \sigma \varphi_{i_0} \). For every \( x \in \bar{\Omega} \) we have
\[
u_{i_0}(x) = (\lambda_{i_0} K_{i_0} F_{i_0} u)(x) + \eta_{i_0} h_{i_0}[u] \gamma_{i_0}(x) + \sigma \varphi_{i_0}(x)
\]
\[
\geq (\lambda_{i_0} K_{i_0} \delta u_{i_0})(x) + \sigma \varphi_{i_0}(x) \geq (\lambda_{i_0} \delta K_{i_0}(\sigma \varphi_{i_0}))(x) + \sigma \varphi_{i_0}(x)
\]
\[
= \frac{\sigma \lambda_{i_0} \delta}{\mu_{i_0}} \varphi_{i_0}(x) + \sigma \varphi_{i_0}(x) \geq 2\sigma \varphi_{i_0}(x).
\]
By iterating the process, for $x \in \Omega$, we get
\[ u_{i_0}(x) \geq n\sigma \varphi_{i_0}(x) \text{ for every } n \in \mathbb{N}, \]
a contradiction, since $u$ is bounded. Thus we obtain
\[ i_P(T + \Gamma, P_{I_0} \setminus \partial P_{I_0}) = 0. \]
Therefore we have
\[ i_P(T + \Gamma, P_I \setminus (\partial P_I \cup P_{I_0})) = i_P(T + \Gamma, P_I \setminus \partial P_I) - i_P(T + \Gamma, P_{I_0} \setminus \partial P_{I_0}) = 1, \]
which proves the result. \qed

Remark 2.5. Note that, in the applications, sometimes it could be useful to replace the constants $M_i$ and $H_i$ with some majorants, say $\hat{M}_i$ and $\hat{H}_i$, at the cost of having to deal with the condition
\[ \lambda_i \hat{M}_i \| K_i(1) \|_{\infty} + \eta_i \hat{H}_i \| \gamma_i \|_{\infty} \leq \rho_i, \text{ for every } i = 1, 2, \ldots, n, \]
more stringent than the corresponding one occurring in (2.4).

We now illustrate the applicability of Theorem 2.4.

Example 2.6. Take $\Omega = \{ x \in \mathbb{R}^2 : \| x \|_2 < 1 \}$, and consider the system
\[ \begin{cases} -\Delta u_1 = \lambda_1 (\| (u_1, u_2) \|^{\frac{1}{2}} + \tan \| (u_1, u_2) \|), & \text{in } \Omega, \\ -\Delta u_2 = \lambda_2 (1 - \sin(u_2)) \| (u_1, u_2) \|^2, & \text{in } \Omega, \\ u_1 = \eta_1 h_1[u], \ u_2 = \eta_2 h_2[u], & \text{on } \partial \Omega, \end{cases} \tag{2.5} \]
where $\| (u_1, u_2) \| = \max\{ |u_1|, |u_2| \}$,
\[ h_1[u] = (u_1(0))^2 + (u_2(0))^{\frac{1}{2}} \quad \text{and} \quad h_2[u] = (u_1(0))^{\frac{1}{2}} + \left( \int_{\Omega} u_2(\xi) \ d\xi \right)^2. \]

By direct calculation we obtain $K_1(1) = K_2(1) = \frac{1}{4}(1 - x_1^2 - x_2^2)$ and we may take $\gamma_1 = \gamma_2 = 1$, this gives $\| K_i(1) \|_{\infty} = \frac{1}{4}$ and $\| \gamma_i \|_{\infty} = 1$ for $i = 1, 2$.

Fix $\rho_1, \rho_2 = \frac{15}{64}\pi$ and set
\[ f_1(u_1, u_2) = \| (u_1, u_2) \|^{\frac{1}{2}} + \tan \| (u_1, u_2) \| \quad \text{and} \quad f_2(u_1, u_2) = (1 - \sin(u_2)) \| (u_1, u_2) \|^2. \]

First of all note that given $\delta > 0$, $f_1$ satisfies condition (b) in Theorem 2.4 for $\rho_0$ sufficiently small, due to the behaviour near the origin.

In the reminder of this example the numbers are rounded from above to the third decimal place unless exact.

We have $M_1 = f_1\left(\frac{15}{64}\pi, \frac{15}{64}\pi\right) \approx 1.765$ and $M_2 = f_2\left(\frac{15}{64}\pi, 0\right) = \left(\frac{15}{64}\pi\right)^2 \approx 0.543$. Moreover, we can use the estimates $H_1 \leq \left(\frac{15}{64}\pi\right)^2 + \left(\frac{15}{64}\pi\right)^{\frac{1}{2}} \approx 1.401$ and $H_2 \leq \left(\frac{15}{64}\pi\right)^{\frac{1}{2}} + \left(\frac{15}{64}\pi\right)^2 \approx 6.278$. 

By Theorem 2.4, the System (2.5) has a nonzero positive solution \((u_1, u_2)\) such that \(0 < \| (u_1, u_2) \| \leq \frac{15}{64} \pi \) for every \(\lambda_1, \lambda_2, \eta_1, \eta_2 > 0\) with
\[
1.765 \times \frac{\lambda_1}{4} + 1.401 \times \eta_1 \leq \frac{15}{64} \pi \quad \text{and} \quad 0.543 \times \frac{\lambda_2}{4} + 6.278 \times \eta_2 \leq \frac{15}{64} \pi.
\]

We now prove, via an elementary argument, a non-existence result.

**Theorem 2.7.** Let \(I = \prod_{i=1}^{n} [0, \rho_i]\) and assume that for every \(i = 1, 2, \ldots, n\) we have:

- \(f_i \in C(\bar{\Omega} \times I)\) and there exist \(\tau_i \in (0, +\infty)\) such that
  \[
  0 \leq f_i(x, u) \leq \tau_i u, \quad \text{for every} \quad (x, u) \in \bar{\Omega} \times I,
  \]
- \(h_i : P_I \to [0, +\infty)\) is continuous and there exist \(\xi_i \in (0, +\infty)\) and
  \[
  h_i[u] \leq \xi_i \| u \|, \quad \text{for every} \quad u \in P_I,
  \]
- the following inequality holds
  \[
  (2.6) \quad \lambda_i \tau_i \| K_i(1) \|_{\infty} + \eta_i \xi_i \| \gamma_i \|_{\infty} < 1.
  \]

Then the System (2.5) has at most the zero solution in \(P_I\).

**Proof.** Assume, on the contrary, that there exists \(u \in P_I\), \(\| u \| = \sigma > 0\), such that \(u = Tu + \Gamma u\). Then there exists \(j\) such that \(\| u_j \| = \sigma\). For \(x \in \bar{\Omega}\) we have
\[
u_j(x) = \lambda_j K_j F_j(u)(x) + \eta_j h_j[u] \gamma_j(x) \leq \| \lambda_j K_j F_j(u) + \eta_j h_j[u] \gamma_j \|_{\infty}
\leq \| \lambda_j K_j(\tau_j \sigma) \|_{\infty} + \| \eta_j \xi_j \gamma_j \|_{\infty} = (\lambda_j \tau_j \| K_j(1) \|_{\infty} + \eta_j \xi_j \| \gamma_j \|_{\infty}) \sigma < \sigma.
\]

By taking the supremum over \(\bar{\Omega}\), we obtain \(\sigma < \sigma\), a contradiction. \(\square\)

We conclude by illustrating in the next example the applicability of Theorem 2.7.

**Example 2.8.** Take \(\Omega = \{ x \in \mathbb{R}^2 : \| x \|_2 < 1\} \) and consider the system
\[
(2.7) \quad \begin{cases}
- \Delta u_1 = \lambda_1 u_2^2 \sin(u_2), & \text{in } \Omega, \\
- \Delta u_2 = \lambda_2 u_1^4 \cos(u_1), & \text{in } \Omega, \\
u_1 = \eta_1 h_1[u], \quad u_2 = \eta_2 h_2[u], & \text{on } \partial \Omega,
\end{cases}
\]
where \(h_1[u] = u_1(0) + (u_2(0))^2\) and \(h_2[u] = u_1(0) + (u_2(0))^3\). First of all note that the trivial solution satisfies the System (2.7). Let us fix \(I = [0, \pi^4] \times [0, \pi^2] \) and note that for every \((x, u_1, u_2) \in \bar{\Omega} \times [0, \pi^4] \times [0, \pi^2] \) we have
\[
0 \leq u_1^2 \sin(u_2) \leq \frac{\pi^4}{4} u_1, \quad 0 \leq u_2^4 \cos(u_1) \leq \frac{\pi^3}{8} u_2.
\]
Furthermore for, \(u \in P_I\), we have
\[
0 \leq h_1[u] \leq \left( \frac{\pi^2}{2} + 1 \right) \| u \|, \quad 0 \leq h_2[u] \leq \left( \frac{\pi^2}{4} + 1 \right) \| u \|.
\]
Thus, in this case, the condition (2.6) reads

\begin{equation}
\frac{\pi}{4} \lambda_1 + \left(\frac{\pi}{2} + 1\right) \eta_1 < 1 \quad \text{and} \quad \frac{\pi^3}{8} \lambda_2 + \left(\frac{\pi^2}{4} + 1\right) \eta_2 < 1.
\end{equation}

Therefore if (2.8) is satisfied, by Theorem 2.7 the System (2.7) admits only the trivial solution in $P_I$.

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