CODES FROM HALL PLANES OF ODD ORDER

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Abstract. We show explicitly that the dimension of the ternary code of the Hall plane of order 9 is greater than the dimension of the ternary code of the desarguesian plane of order 9. The proof requires finding a word with some defined properties in the dual ternary code of the desarguesian plane of order 9. The idea can be generalised for other orders, provided that words in the dual code of the desarguesian projective plane that have the specified properties can be found.

1. Introduction

Linear \( p \)-ary codes from projective and affine desarguesian planes of characteristic \( p \) have long been studied and many of their most important properties well established: see [1, Chapter 5,6] for some of the results, and for a bibliography. In contrast, very little is known about codes from non-desarguesian planes, beyond what is true in general for codes from planes. The dimensions of the codes are known for small orders computationally, but there is no known general formula for the dimension of any class of non-desarguesian planes, as in the desarguesian case. The Hamada-Sachar conjecture states that their dimension is always strictly greater than that of the desarguesian plane of the same order, and no counter-example is known. In [10] it was proved that the Hall planes of even order satisfy this conjecture, the first infinite class of planes to be shown to confirm the conjecture.

Based on the method of proof in [10], we suggest a method here to extend the result to all Hall planes, which would show that for all \( q \) the Hall plane of order \( q^2 \) satisfies the Hamada-Sachar conjecture, and is tame (see Section 2 for the definition of a tame plane). The idea involves knowledge of suitable words in the dual code of the projective desarguesian plane. We have only managed to use this idea for the Hall plane of order 9, and we give our proof here, as an illustration of what might be possible for the general odd-order case. Note that the dual codes of desarguesian projective planes of odd non-prime order are not polynomial codes in the sense of, for example, [11, Chapter 5], and are not as well known as the generalized Reed-Muller codes, which are polynomial codes.

The dimension of the ternary code of the Hall plane of order 9 has been known for a long time from computational results. We show here without computational help that its dimension is greater than that of the desarguesian plane, and that it is not tame. The idea of the proof is essentially one that is used in the even case in [10].

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2. Terminology and notation

Standard terminology for designs and codes is used as in [1] and for planes as in [7]. An incidence structure \( \mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I}) \), with point set \( \mathcal{P} \), block set \( \mathcal{B} \) and incidence \( \mathcal{I} \) is a \( t(v, k, \lambda) \) design, if \( |\mathcal{P}| = v \), every block \( B \in \mathcal{B} \) is incident with precisely \( k \) points, and every \( t \) distinct points are together incident with precisely \( \lambda \) blocks. A \( 2(n^2 + n + 1, n + 1, 1) \) design, for \( n \geq 2 \), is a finite projective plane of order \( n \). We write \( PG_{2,1}(\mathbb{F}_q) \) or \( PG_2(\mathbb{F}_q) \) for the desarguesian projective plane, i.e. the design of points and 1-dimensional subspaces of the projective space \( PG_2(\mathbb{F}_q) \). Similarly, \( AG_{m,t}(\mathbb{F}_q) \) will denote the design of points of the affine space \( AG_m(\mathbb{F}_q) \) and \( t \)-flats, where \( t \geq 1 \).

If \( \Pi \) is a projective plane of square order \( n^2 \), a subplane \( \pi \) of \( \Pi \) of order \( n \) is called a Baer subplane. Lines of \( \Pi \) meet \( \pi \) in 1 or \( n + 1 \) points. If a line of \( \Pi \) meets \( \pi \) in a set \( \delta \) of \( n + 1 \) points, \( \delta \) is called a Baer segment. An oval or hyperoval in a projective plane of even order \( n \) is a set of \( n + 2 \) points such that lines meet the set in 0 or two points.

The code \( C_F \), or \( C_F(\mathcal{D}) \), of the design \( \mathcal{D} \) over the finite field \( F \) is the space spanned by the incidence vectors of the blocks over \( F \). We take \( F \) to be a prime field \( \mathbb{F}_p \), writing \( C_p(\mathcal{D}) \), and the prime must divide the order of the design, i.e. \( n \) for a finite plane of order \( n \). Denoting the incidence vector of a subset \( Q \) of points by \( v^Q \), we have \( C_F = \langle v^B \mid B \in \mathcal{B} \rangle \), and is a subspace of \( F^\mathcal{P} \), the full vector space of functions from \( \mathcal{P} \) to \( F \). For any \( w \in F^\mathcal{P} \) and \( P \in \mathcal{P} \), \( w(P) \) denotes the value of \( w \) at \( P \).

A linear code over \( \mathbb{F}_q \) of length \( n \), dimension \( k \), and minimum weight \( d \), is denoted by \([n, k, d]_q \). If \( c \) is a codeword then the support of \( c \) is the set of non-zero coordinate positions of \( c \), and the weight of a vector is the size of its support.

For any code \( C \), the dual or orthogonal code \( C^\perp \) is the orthogonal under the standard inner product, i.e. \( C^\perp = \{ v \in F^n \mid (v, c) = 0 \text{ for all } c \in C \} \). The hull of a design with code \( C \) is \( C \cap C^\perp \), written Hull(\( \mathcal{D} \)) or Hull(\( C \)).

A proof of the following is in [1] Theorem 6.3.1:

**Result 1.** Let \( \Pi \) be a projective plane of order \( n \) and let \( p \) be a prime dividing \( n \). Then the minimum-weight vectors of \( C_p(\Pi) \) are precisely the scalar multiples of the incidence vectors of the lines. Further, Hull\( _p(\Pi) = \langle v^L - v^M \mid L \text{ and } M \text{ lines of } \Pi \rangle \).

The situation for the codes from desarguesian planes is quoted in [1] Theorem 6.4.2:

**Result 2.** Let \( p \) be any prime, \( q = p^t \), and \( \Pi = PG_2(\mathbb{F}_q) \). Then \( C_p(\Pi) \) is a \([q^2 + q + 1, (q^2 + q + 1)p]_q \) code. The minimum-weight vectors of \( C_p(\Pi) \) and of \( C_p(\Pi) + C_p(\Pi)^\perp \) are the scalar multiples of the incidence vectors of the lines. The minimum weight of \( \text{Hull}_p(\Pi) \) is \( 2q \) and its minimum-weight vectors are the scalar multiples of the differences of the incidence vectors of distinct lines of \( \Pi \).

The notion of a tame plane was introduced in [1] Section 6.9:

**Definition 1.** A projective plane \( \Pi \) of order \( n \) is said to be tame, if, for \( p \) a prime dividing \( n \), \( \text{Hull}_p(\Pi) \) has minimum weight \( 2n \) and the minimum-weight vectors are precisely the scalar multiples of the differences of the incidence vectors of distinct lines of \( \Pi \).

It follows from Result 2 that the desarguesian planes are tame, but at present no other planes have been shown to be tame, and many of small order have been
shown not to be tame, either because the minimum weight of the hull is not 2n
(see [6]) or, more frequently, that there are words of weight 2n that are not scalar
multiples of the differences of the incidence vectors of two lines. We show in [10]
that the Hall planes of even order 2t for t \geq 2 are not tame by exhibiting words
of weight 2^{2t+1} in the binary hull that are not differences of the incidence vectors
of two lines. Using also a result in [8, Corollary 3], this shows that the Hall planes of
even order and their dual planes are not tame for all even orders n > 4.

The Hamada-Sachar conjecture [1, Conjecture 6.9.1] is as follows:

**Conjecture 1.** Every projective plane of order p^s, p a prime, has p-rank at least
\((p^s)^2 + 1\) with equality if and only if it is desarguesian.

In [10] we show that the Hall planes of even order satisfy this conjecture.

The results in [10], and our projected extension to the case of p odd, requires the
following result from [1, Proposition 6.3.3],

**Result 3.** If X is the set of points of a Baer subplane of \(\Pi = PG_2(\mathbb{F}_{q^2})\), then
\(v^X \notin C(\Pi)\).

3. **Hall planes**

We give the definition of a Hall plane using the process of derivation by Baer
segments: see [1, Section 6.10]. The Hall planes are translation planes and the definition
is found in standard texts, for example [1, 7, 11, 12], or [1, Section 6.10].

Let \(\Pi = (\mathcal{P}, \mathcal{L})\) be the desarguesian projective plane \(PG_2(\mathbb{F}_{q^2})\) of order \(q^2\),
where \(q = p^s\), p prime. Let \(\delta\) be a Baer segment of \(q + 1\) points of a line \(\ell_{\infty} \in \mathcal{L}\); we use
this as a derivation set to construct the affine Hall plane \(A\mathcal{H}\) of order \(q^2\), and the
projective Hall plane \(\mathcal{H}\).

Let \(\mathcal{B}\) be the set of Baer subplanes of \(\Pi\) that meet \(\ell_{\infty}\) in \(\delta\). Then |\(\mathcal{B}\)| = \(q^2(q + 1)\),
and any two of these Baer subplanes meet in a further one point, or do not intersect
off \(\delta\). Any one of these subplanes, together with all those that are mutually disjoint
from it off \(\delta\), form a set of \(q^2\) subplanes that will form a parallel class of lines in the
new (Hall) plane \(A\mathcal{H}\). There are \(q + 1\) of these parallel classes of subplanes.

Denote the set of lines of \(\Pi\) that meet \(\ell_{\infty}\) in \(\ell_{\infty} \setminus \delta\) by \(\mathcal{C}_c\), and those that meet
\(\ell_{\infty}\) in \(\delta\) by \(\mathcal{L}_o\), so \(\mathcal{L} = \mathcal{C}_c \cup \mathcal{L}_o \cup \{\ell_{\infty}\}\). Then |\(\mathcal{C}_c\)| = \(q^3(q - 1)\). For any line \(m \in \mathcal{C}_c\),
\(m \setminus \ell_{\infty}\) will be a line of \(A\mathcal{H}\). Denote this set of affine lines for both \(\Pi^c\) and \(A\mathcal{H}\),
by \(\mathcal{A}_c\).

The other \(q^2(q + 1)\) lines of \(A\mathcal{H}\) we write as \(A_n\) and each of these consist of the
points of the Baer subplanes in \(\mathcal{B}\) with the points in \(\delta\) removed. To complete the
affine Hall plane \(A\mathcal{H}\) to the projective Hall plane \(\mathcal{H}\), adjoin a line at infinity

\[\ell_{\infty}^h = \{X \mid X \in \ell_{\infty} \setminus \delta\} \cup \{X_i \mid 0 \leq i \leq q\}\]

where the \(X_i\) correspond to the parallel classes of members of \(\mathcal{B}\). The lines of \(\mathcal{H}\)
are \(\ell_{\infty}^h\), \(\mathcal{C}_c\) and

\[\mathcal{L}_n = \{\{(\pi \setminus \delta) \cup \{X_i\} \mid \pi \in \mathcal{B}, \pi \text{ in parallel class of } X_i\} \mid 0 \leq i \leq q\}\].

For \(m \in \mathcal{L}_o\), \(m \setminus \delta\) is an affine Baer subplane of \(A\mathcal{H}\): if \(m_1, m_2 \in \mathcal{L}_o\) and
\(m_1 \cap m_2 \in \delta\) then these planes are disjoint, and in \(\mathcal{H}\) they are Baer subplanes that
share points only on \(\ell_{\infty}^h\).

Let \(E = \{v^\ell \mid \ell \in \mathcal{C}_c\}\). If we identify the members of \(\{X_i \mid 0 \leq i \leq q\}\) with
the points in \(\delta\), it is clear that \(E \subseteq \mathcal{C}_p(\Pi) \cap \mathcal{C}_p(\mathcal{H})\). We aim to prove that for any
\(R \in \delta\), and for any lines \(m_1, m_2 \in \mathcal{L}_o\) such that \(R \in m_1, m_2\), then \(v^{m_1} - v^{m_2} \in E\).
Lemma 1. Let \( \Pi = (P, L) \) be a projective plane with a polarity \( \sigma \). Write \( X^w = X' \) for \( X \in P \cup L \). Let \( p \) be a prime dividing the order of \( \Pi \), and \( C = C_p(\Pi) \). If \( w = \sum_X a_X v^X \in C^\perp \), then \( w' = \sum_X a_X v^{X'} = 0 \).

Proof. For \( Y \in P \),
\[
w'(Y) = \sum_{Y \in X'} a_X = \sum_{X \in Y'} a_X = (w, v^{Y'}) = 0
\]
since \( w \in C^\perp \). Thus \( w' = 0 \).

Since \( \Pi = PG_2(\mathbb{F}_q) \) has polarities, if \( q = p^e \) and \( C = C_p(\Pi) \), we need to find a suitable \( w \in C^\perp \) such that:
- \( w = \sum_{i=1}^m a_i v^{R_i} \in C^\perp \) of weight \( m > 0 \), where \( a_i \neq 0 \) for \( 1 \leq i \leq m \);
- \( S = \text{Supp}(w) = \{ P_i \mid 1 \leq i \leq m \} \);
- \( S \) has a 2-secant, \( P_1 P_2 \), and \( R = (P_1 P_2)' = P_1' \cap P_2' \);
- there exists \( P \subseteq P_1 P_2, P \neq P_1, P_2 \) such that there is a Baer segment \( \delta \) with \( R \in \delta \subseteq P' \) with the condition that \( (PP_i)' \notin \delta \) for \( i \geq 3 \), i.e. \( P' \cap P_i' \in P' \setminus \delta \) for \( i \geq 3 \).

Then
\[
a_1(v^{R_1} - v^{R_1'}) = \sum_{i=3}^m a_i v^{R_i'},
\]
and is a word in Hull(\( \mathcal{H} \)) that is not the difference of two lines of \( \mathcal{H} \).

The following lemma is a generalization of Lemmas 3 and 4 from [10] for the case of odd \( p \). The notation is as defined above, with \( q = p^e \).

Lemma 2. If there is a point \( R \in \delta \), and lines \( m_1, m_2 \in L_0 \) such that \( m_1 \cap m_2 = \{ R \} \), and \( v^{n_1} - v^{n_2} \in E \), then
- 1. for every \( Q \in \delta \) and lines \( n_1, n_2 \in L_0 \) such that \( n_1 \cap n_2 = \{ Q \} \), \( v^{n_1} - v^{n_2} \in E \);
- 2. \( \dim(\mathcal{E}) = (p^{e+1})^{2e} - p^e \);
- 3. \( \dim(C_p(\mathcal{H})) \geq \dim(\Pi) + 1 = (p^{e+1})^{2e} + 2 \), and \( \mathcal{H} \) is not tame.

Proof. The proofs of Lemmas 3 and 4 from [10] generalize to \( p \) odd, since the presence of all homologies and elations in Aut(\( \Pi \)), along with the transitivity of the stabilizer in Aut(\( \Pi \)) of \( \delta \) on points of \( \delta \), proves (1).

For (2), first note that to get \( C_p(\Pi) \) we need to add to \( E \) the incidence vector of just one line from \( L_0 \) through each of the points of \( \delta \), so that \( \dim(C_p(\Pi)) = \dim(E) + q + 1 \), and thus \( \dim(E) = (p^{e+1})^{2e} - p^e \).

For (3) and \( C_p(\mathcal{H}) \), start with \( E \subseteq C_p(\mathcal{H}) \), and add \( v^{n_1} \) for one line \( \pi_1 \) of \( \mathcal{H} \) through \( R \in \delta \), this being an affine Baer subplane of \( \Pi \). If for another line \( \pi_2 \) of \( \mathcal{H} \) through \( R \) we have \( v^{n_2} \in (v^{n_1} - v)E \), then \( v^{n_2} = av^{n_1} + v \), where \( v \in E, a \in \mathbb{F}_p \).

Thus the remaining assertions follow.

4. Planes of order 9

We show now that the procedure described can be used for \( q^2 = 9 \). We have
\[ \Pi = PG_2(F_9) \text{ where } F_9 \text{ has primitive generator } u \text{ satisfying } u^2 + 2u + 2 = 0. \] Let \( C = C_3(\Pi) \).

We consider the word \( w = v^{\pi_1} - v^{\pi_2} \) in \( C_{\perp} \), where \( \pi_i \) are Baer subplanes (of order 3) of \( \Pi \):

- \( \pi_1 \) is spanned by the quadrangle \( \{(1, u, 0), (1, 2, u^3), (1, 0, 1), (1, u, 2)\} \) and consists of the 13 points:
  \[ S_1 = \{(1, 0, 1), (1, u, 2), (1, u^6, 1), (1, u^7, 1), (1, u^3, u^2), (1, 2, u^3), (0, 1, u^7), (1, u, u^2)\} \]
  and \( T = \{(1, u, 0), (1, u^2, u^6), (1, u^5, 0), (1, u, 1), (1, 1, u^5)\} \).

- \( \pi_2 \) is spanned by the quadrangle \( \{(1, u, 0), (0, 0, 1), (1, 1, u^3), (1, u^2, 2)\} \) and consists of the 13 points:
  \[ S_2 = \{(0, 0, 1), (1, u^2, u^7), (1, u, u^6), (1, 1, u^3), (1, u^2, 2), (1, u^5, u^6), (1, 1, u^6), (1, u^5, u^3)\} \]
  and \( T = \{(1, u, 0), (1, u^2, u^6), (1, u^5, 0), (1, u, 1), (1, 1, u^5)\} \).

The two planes have the set \( T \) of five points in common, i.e. the point \((1, u, 0)\) and four collinear points that are a Baer segment. Thus \( \text{wt}(w) = 16 \).

Recall that the minimum weight of \( C_3(\Pi)_{\perp} \) is 15: see [8].

Thus \( \text{Supp}(w) = S_1 \cup S_2 \) and \( w = \sum_{a \in S_1} v^a - \sum_{a \in S_2} v^a \). Since \( w \in C_{\perp} \), we have, by the earlier arguments, that

\[ \sum_{a \in S_1} v^a' - \sum_{a \in S_2} v^a' = 0. \]

Thus

\[ v^{(0,0,1)'} - v^{(1,0,1)'} = \sum_{a \in S_1 \setminus \{(1,0,1)\}} v^a' - \sum_{a \in S_2 \setminus \{(0,0,1)\}} v^a'. \]

The point \((1, 0, 0)\) is exterior to \( \text{Supp}(w) \) and on the 2-secant through \((1, 0, 1) \in S_1 \) and \((0, 0, 1) \in S_2 \). Taking the line \((1, 0, 0)\)' and the Baer segment \( \delta = (0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 0, 1), \) we see from the tables below that the lines \( a' \) for \( a \in S_1 \cup S_2 \setminus \{(1,0,1), (0,0,1)\} \) meet \((1,0,0)\)' outside of \( \delta \).

In the tables we write \( xyz \) for the point \((x, y, z)\) and \( \ell = (1, 0, 0)' \).

| \( a \in S_1 \) | 101 | 1u2 | 1u^6 | 1u'2 | 1u'6 | 12u^3 | 01u' | 1uu' |
|---|---|---|---|---|---|---|---|---|
| \( a' \cap \ell \) | 010 | 01u | 01u^2 | 01u^6 | 01u' | 01u^3 | 01u^5 | 01u |

| \( a \in S_2 \) | 001 | 11u^3 | 1u^2 | 1u'u^6 | 1u'6 | 11u^6 | 1uu^6 | 1u'u | 1uu |
|---|---|---|---|---|---|---|---|---|---|
| \( a' \cap \ell \) | 010 | 01u | 01u^2 | 01u^6 | 01u' | 01u^3 | 01u^5 | 01u' | 01u |

Now by Lemma 2 we see that \( \dim(\mathcal{H}) \geq 38 \), and that \( \mathcal{H} \) is not tame. From Magma [2, 3], or GAP [5], the actual value for the dimension is 41.

5. Conclusion

Clearly we need suitable words in the dual code of the desarguesian projective plane of odd square order in order to use this method. Unfortunately not a lot is known about these codes, and even their minimum weight is unknown: see [9] for some remarks. Candidates for words in the dual code could be of the form \( v^X - v^Y \) where \( X, Y \) are subsets of points that are met in 1 mod \( p \) points by any line; for example Baer subplanes, lines, unitals in the plane (see also [11] Chapter 6). Clearly words of the form \( v^\ell_1 - v^\ell_2 \) where \( \ell_1, \ell_2 \) are lines cannot give the sum we need, but some other combinations might work. Words of the form \( v^\pi - v^\pi' \) where \( \pi, \pi' \) is a Baer
was the incidence vector of a regular hyperoval. In that paper, in $PG_2(F_{q^2})$ where $q$ is even, the regular hyperoval with equation $X^2 = YZ$, and nucleus $(1, 0, 0)$ gave the required word for the Baer segment $\{(1, 1, t+z) \mid t \in F_q \} \cup \{(0, 0, 1)\}$ on the line $(1,1,0)'$, where $z \in F_{q^2} \setminus \{y^2 + y \mid y \in F_{q^2}\}$. This word is not in $\text{Hull}(C_2(\Pi))$, and the word we found for $q = 9$ is also not in $\text{Hull}(C_2(\Pi))$; there is no obvious reason why the word $w$ should not be in the hull.

By using the method described in Lemma 2 of adding incidence vectors of lines through points on $\delta$ to the code $E$, a possible formula for the dimension of the $p$-ary code of the Hall plane of order $p^2$, where $p$ is a prime, was found to hold through computations up to $p = 11$:

**Observation 1.** For $p$ prime, $\mathcal{H}_{p^2}$ the projective Hall plane of order $p^2$, $C = C_p(\mathcal{H}_{p^2})$,

$$\dim(C) = \left(\frac{p+1}{2}\right)^2 - p + \sum_{r=2}^{p} \binom{r}{2} + \binom{p}{2} + 1 = \left(\frac{p+1}{2}\right)^2 + \binom{p+1}{3} + \binom{p}{2} + 1 - p.$$  

For $p = 2$ this gives 10, i.e. $\dim(C_2(PG_2(F_4)))$, since $\mathcal{H}_4 = PG_2(F_4)$, and for $p = 3$ it gives $\dim(\mathcal{H}_9) = 41$.

The idea of Lemma 1 to show that a plane is not tame can be applied to other planes obtained by the process of derivation from the desarguesian plane, i.e. not using Baer subplanes for the derivation (see [1] Section 6.10), and using a derivation set instead of a Baer segment. We have not tried this, but we have shown computationally that some non-desarguesian planes of order 27 have words of the form $v^{S_1} - v^{S_2}$ in their hulls, where the $S_i$ are sets of size 27 that are not lines, for example the Sherk and Figueroa planes.

A rather strange observation that we made computationally that we have not proved involves the code of the dual plane in the case of non-desarguesian planes of order $p^2$. If $\Pi$ is a projective plane and we denote by $\Pi'$ its dual plane, and for a set $X$ of points and line of $\Pi$, we write $X'$ for the dual structure of lines and points in $\Pi'$, we found the following which has not been contradicted for small values of $p \geq 3$:

**Observation 2.** If the projective plane $\Pi$ has order $p^2$ and $\pi_1$ and $\pi_2$ are (projective) Baer subplanes (of order $p$) of $\Pi$ which are such that the word $v^{\pi_1} - v^{\pi_2}$ has weight $2p^2$ and is in $\text{Hull}(\Pi)$, then the word $v^{\pi_1'} - v^{\pi_2'}$ is in $\text{Hull}(\Pi')$.

Thus both $\Pi$ and $\Pi'$ are not tame. Here the two Baer subplanes have a line $\ell$ in common and $p+1$ points of $\ell$ in common, and also they have a point $P$ in common with $p+1$ lines through $P$ in common, i.e. a self-dual condition, so that the dual structure is similar. Remove the line $\ell$ and the two affine parts are disjoint pointwise. A different type of intersection of planes was found for the Hughes plane of order 9, where the two Baer subplanes intersect in four points, only three of which are collinear.

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